Hommage à mon ami
Pierre Leroux

7 April 2008
Isaac Newton Institute

12th FPSAC (Formal Power Series and algebraic combinatorics)
Moscow 2000

C’est l’hiver, le soleil joue avec les cristaux de neige qui virevoltent au vent

Les épinettes chantent, se courbent et nous regardent tendrement

Les skis martèlent la trace gelée

Tu es l’ami Pierre qui glisse et nous entraîne sur le chemin poudré.

C'est le soir, la bougie illumine les ombres

L'amitié des coeurs autour des bols fumant

Le velours de la nuit glaciale coule sur les étoiles immaculées

Tu es l'ami Pierre qui veille et nous réveille.

Dans ma tête les délicieux souvenirs dansent et s'entrechoquent

Monts-Valin, Saguenay, Charlevoix, Chic-Chocs ...

Et bien d'autres des Rocheuses aux Pyrénées

Tu es là, le sourire paisible, entre neige et cosmos.

C’est l’été, appel, écart, R2, appel, R3, bac avant, chaque seconde compte

La peur s’alourdit, l’attention redoublée, le rapide gronde et se rebelle

Le canot fend les vagues et les souvenirs se gravent à jamais

Tu es l’ami Pierre qui guide, apaise et conduit sur l’écume.

C’est la nuit, les flammes dansent sur la plage

Un rayon de Jupiter émerge de la rivière des Inuits

Entends-tu le loup hurler dans la nuit du grand Nord ?

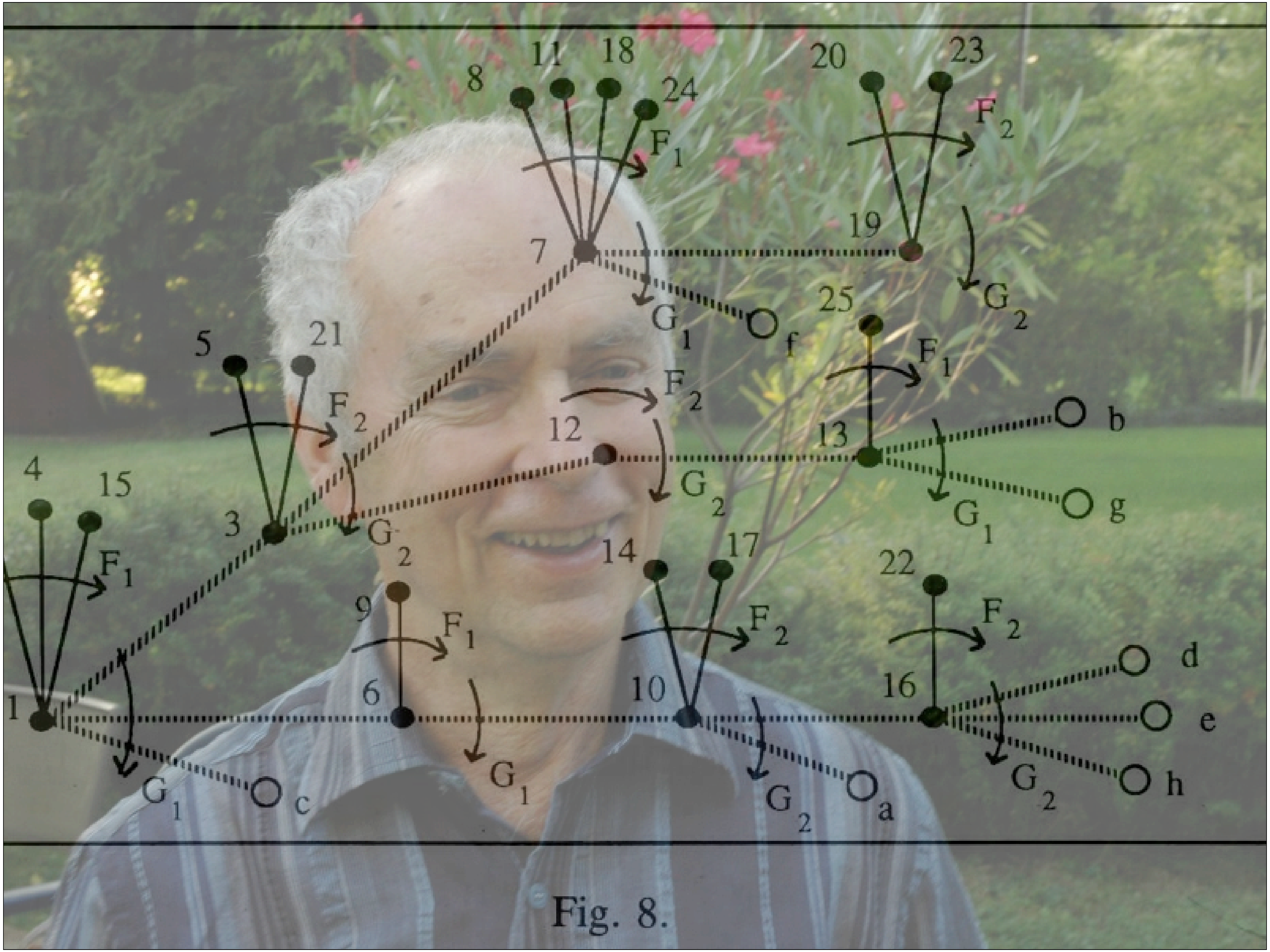
Tu es l’ami Pierre dans les draperies de l’aurore boréale.

C’est l’automne, les couleurs des arborescences rivalisent d’élégance

Arborescences différentielles, noyau des intégrales itérées

Entend tu bruisser les bourgeons des éclosions combinatoires ?

Tu es l’ami Pierre qui navigue sur les cimes mathématiques.



8
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g
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9
F1
F2
F2
d
1
6
10
16
e
G1
c
G1
G2
a
G2
h

Fig. 8.

C'est l'hiver, la neige est abondante et la douleur s'installe

De ta grande fenêtre, une dernière éclipse de lune

Une dernière embrassade, une heure intense et poignante

Le moment du départ approche, un dernier au revoir

Tu es mon ami Pierre qui est gravé dans mon coeur.
Je ne t'oublierai jamais.


Avec infiniment
de reconnaissance,
ton ami xavier,
“LACIM 2000”,Montreal

Introduction to the theory of heaps of pieces with applications to statistical mechanics and quantum gravity

dedicated to my dear friend Pierre Leroux

7 April 2008

Isaac Newton Institute

xavier viennot

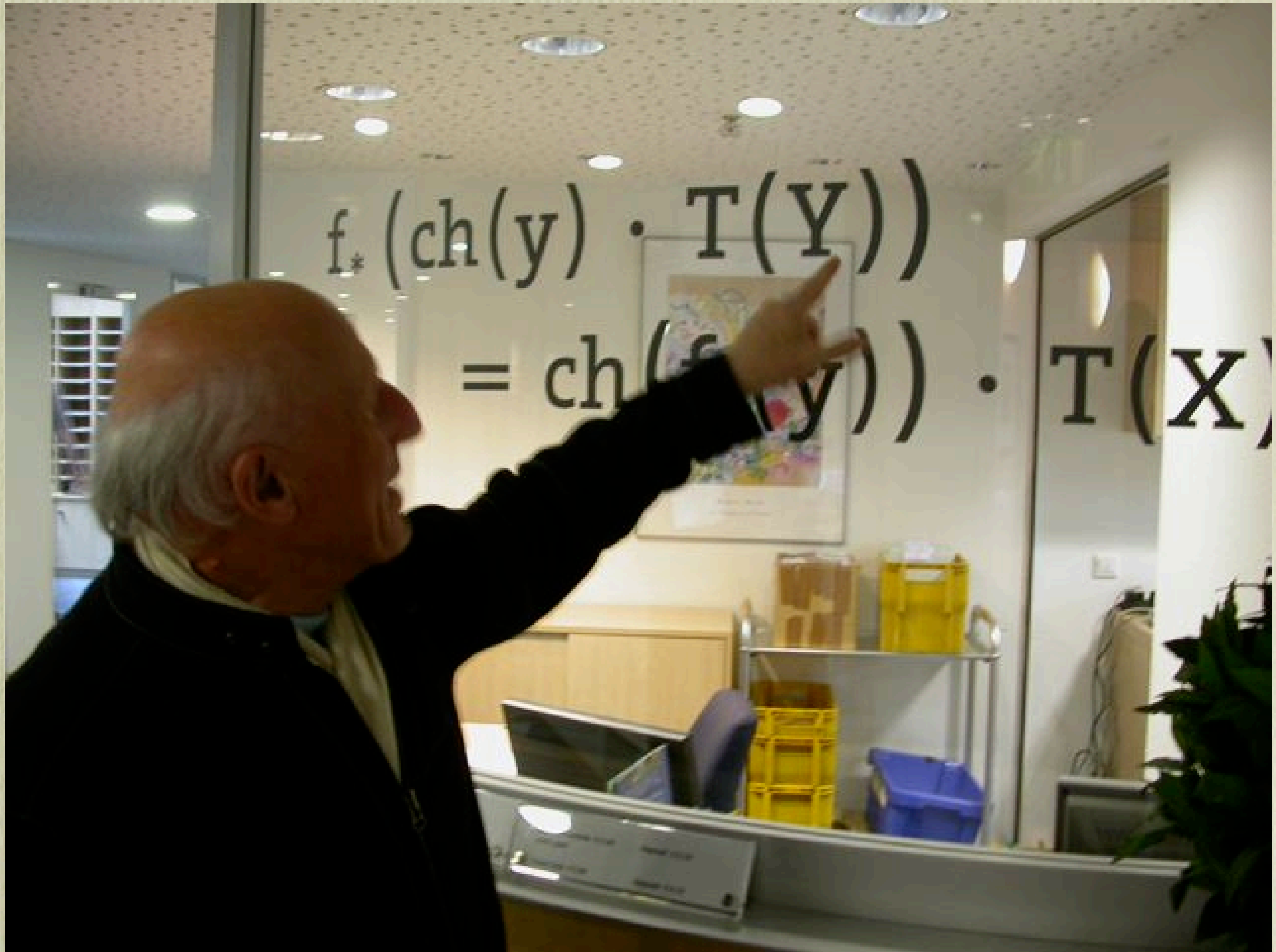
LaBRI, CNRS, Bordeaux

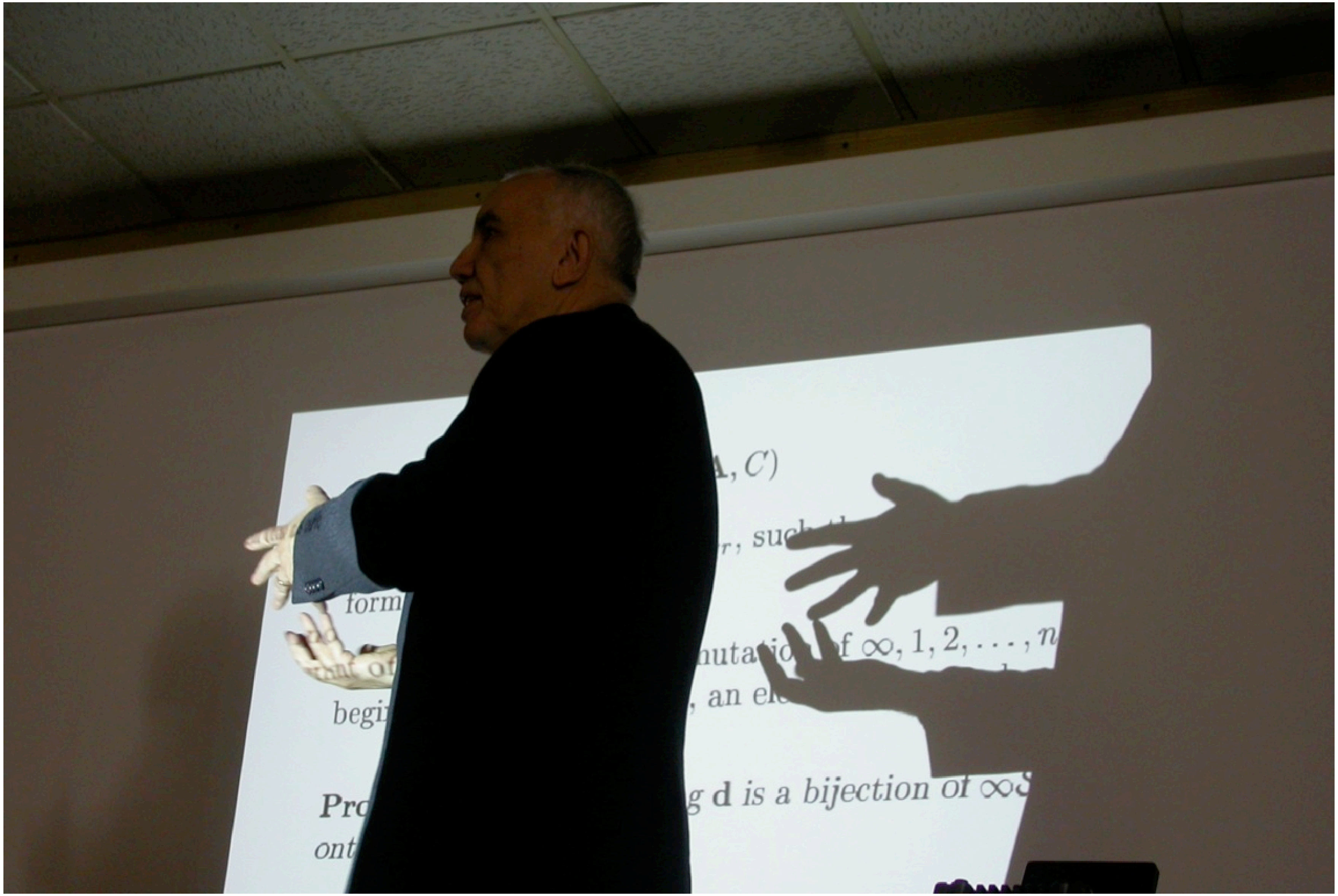
§1 Commutation monoids

Cartier-Foata **commutation** **monoid**

Lecture Note in Maths n° 85 (1969)

"Problèmes combinatoires de
commutation et réarrangements"


$f_*(ch(y) \cdot T(Y))$
$= ch($
$y)) \cdot T(X$


such
$\infty, 1, 2, \ldots, n$
an
d is a bijection of

alphabet A

free monoid A^*

words $w = a_1 a_2 \cdots a_p$

product : concatenation

$\left. \begin{array}{l} u = a_1 \cdots a_p \\ v = b_1 \cdots b_q \end{array} \right\} uv = a_1 \cdots a_p b_1 \cdots b_q$

empty word

commutation relation C antireflexive symmetric

$\equiv_C$ congruence of A^* generated by the commutations

$ab \equiv ba$ iff $a C b$

ex: $\mathcal{A} = \{a, b, c, d\}$

$$\mathcal{C} \quad \begin{cases} ad = da \\ bc = cb \\ cd = dc \end{cases}$$

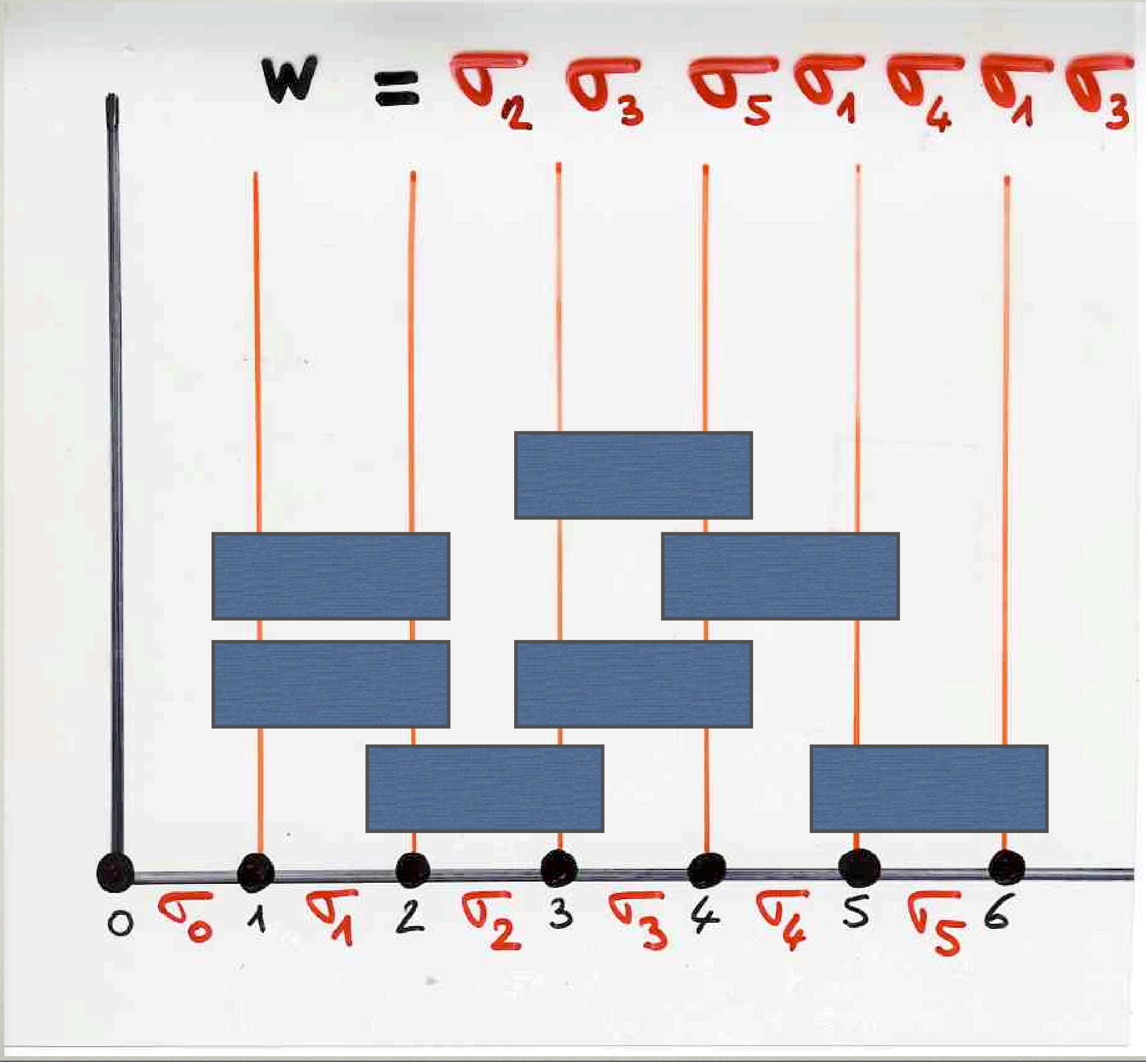
equivalence class

$w =$ abcad abcda

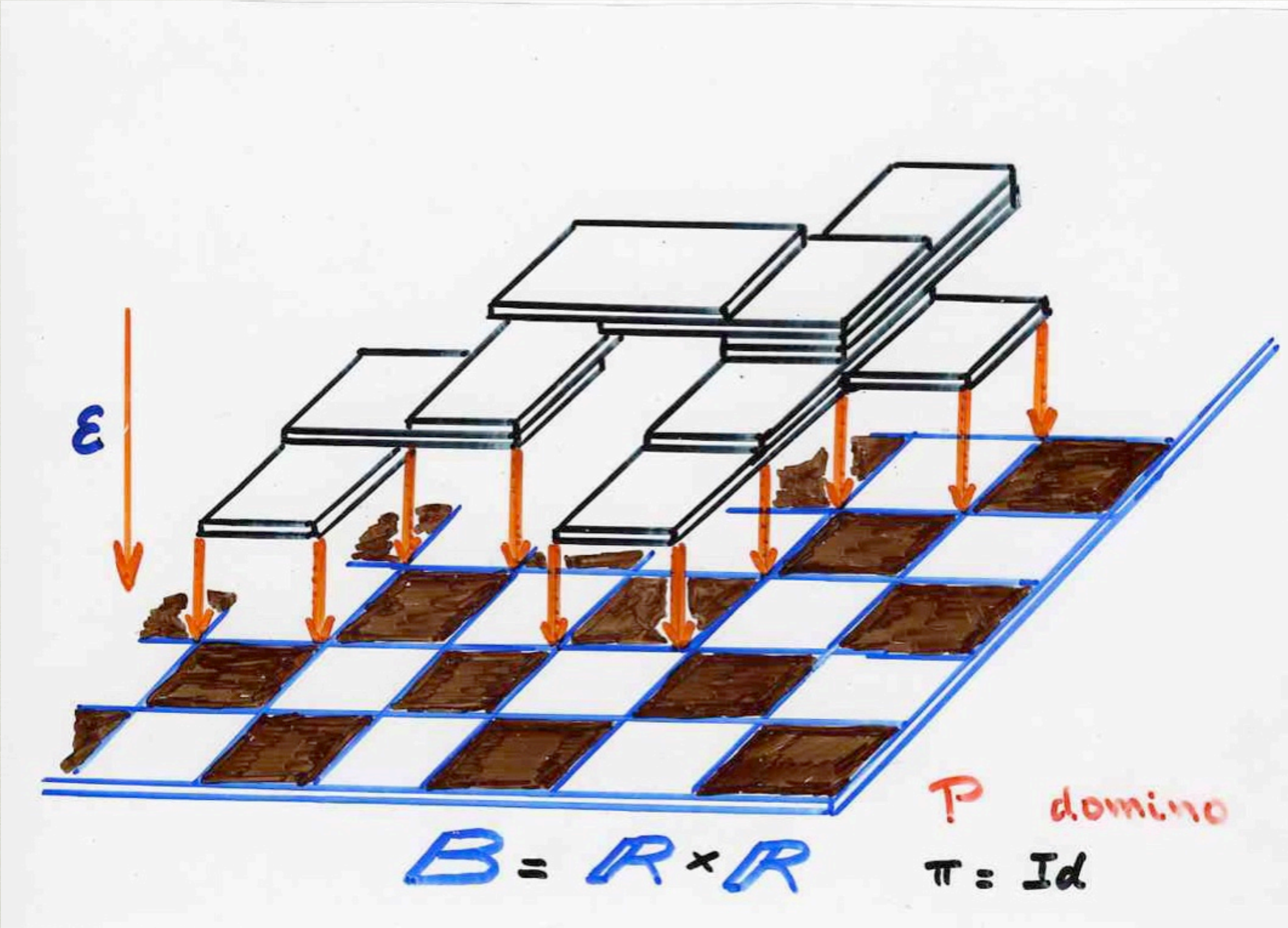
acbad abdca

acbda

§2 Heaps of pieces: basic definitions



$W = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$
0 σ_0 1 σ_1 2 σ_2 3 σ_3 4 σ_4 5 σ_5 6



ε
P
domino
B = ℝ×ℝ
π = Id

heap definition

- $\mathbf{P}$ set (of basic pieces)
- $\mathcal{C}$ binary relation on $\mathbf{P}$ $\begin{cases} \text{symmetric} \\ \text{reflexive} \end{cases}$

  (**dependency** relation)

- heap E, finite set of pairs

  (α, i) $\alpha \in P$, $i \in \mathbb{N}$ (called pieces)

  projection level

(i)

(ii)

heap definition

- P set (of basic pieces)
- $\mathcal{C}$ binary relation on P $\left\{\begin{array}{l}\text{symmetric}\\ \text{reflexive}\end{array}\right.$

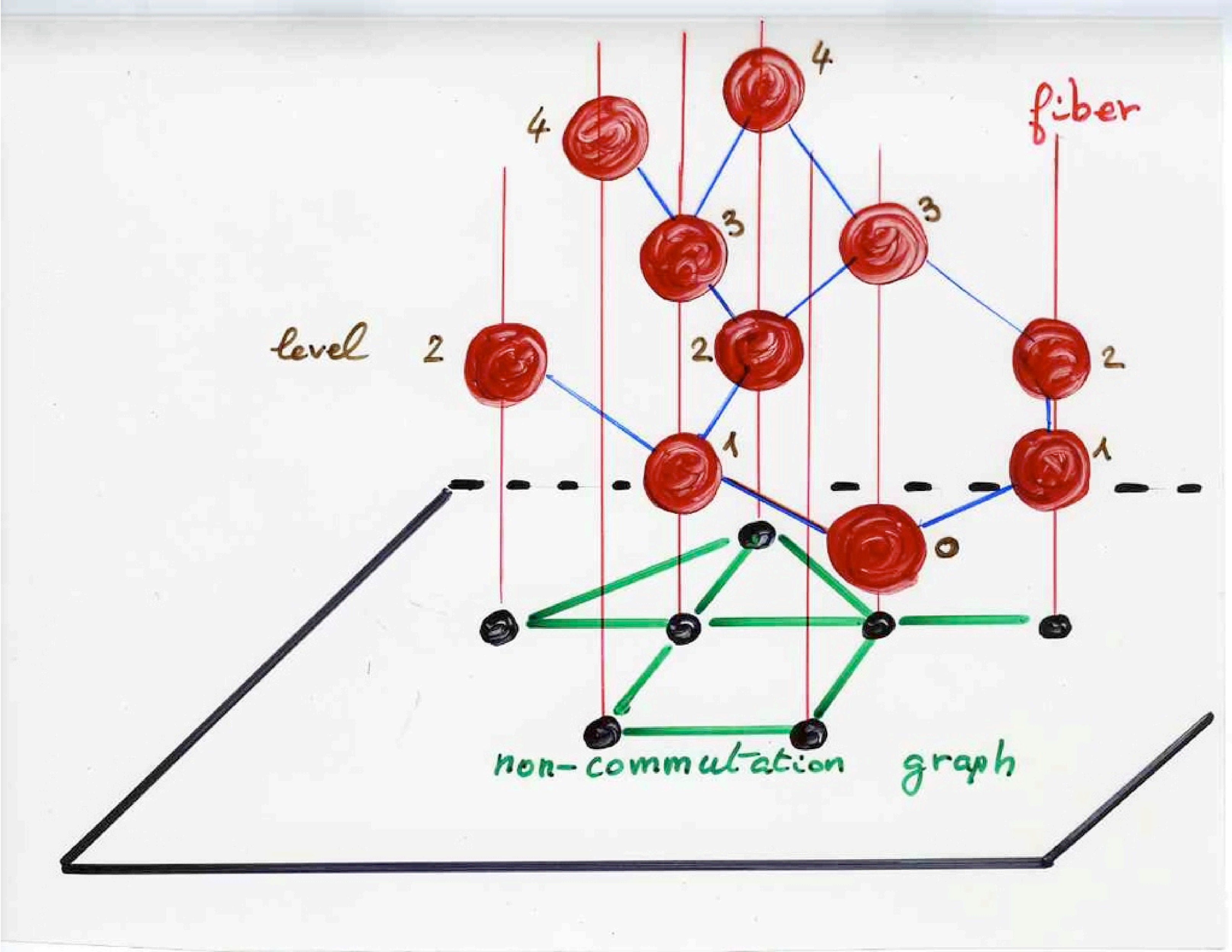
  (dependency relation)
- heap E, finite set of pairs (α, i) $\alpha \in P$, $i \in \mathbb{N}$ (called pieces)

projection level

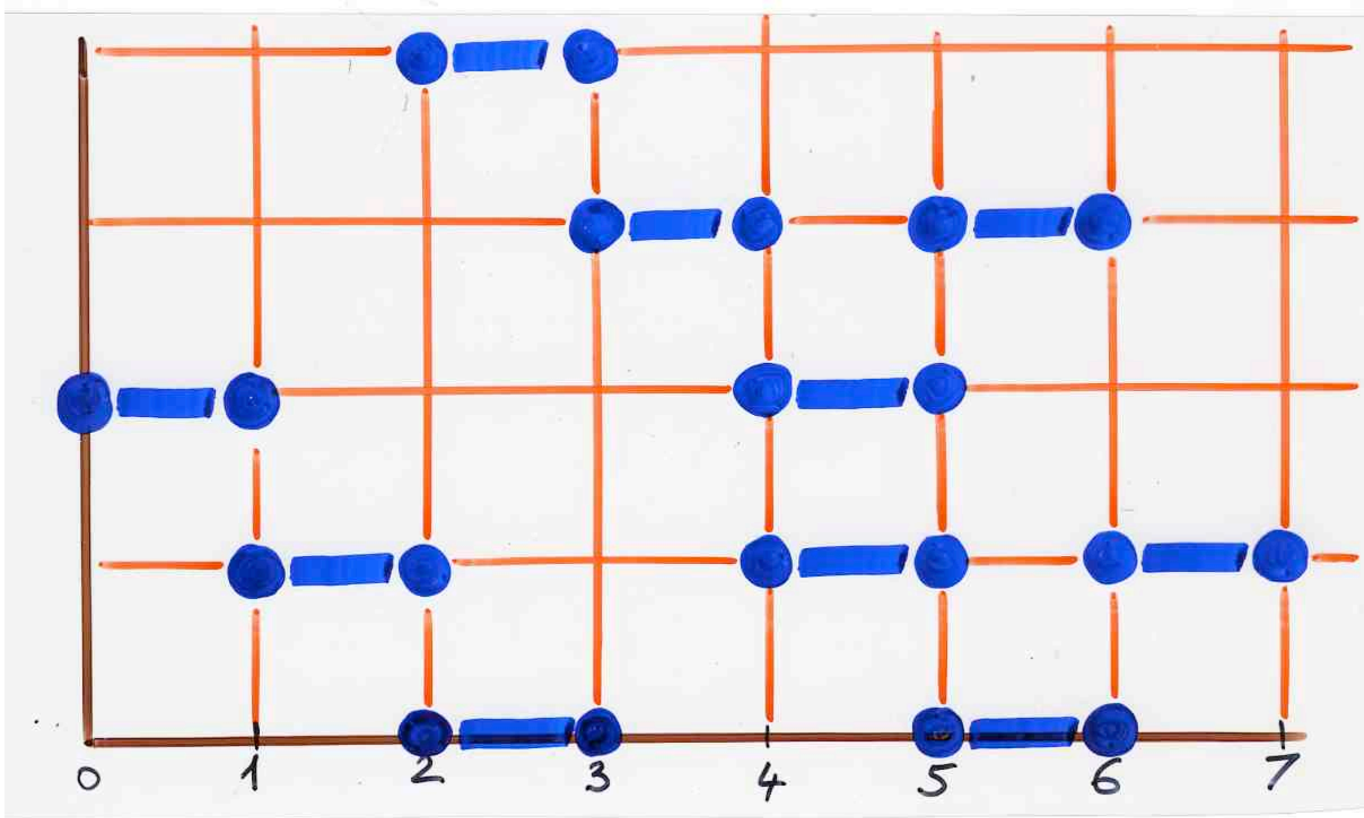
(i) $(\alpha, i), (\beta, j) \in E$, $\alpha \,\mathcal{C}\, \beta \implies i \neq j$

(ii) $(\alpha, i) \in E$, $i > 0 \implies \exists \beta \in P$, $\alpha \,\mathcal{C}\, \beta$,

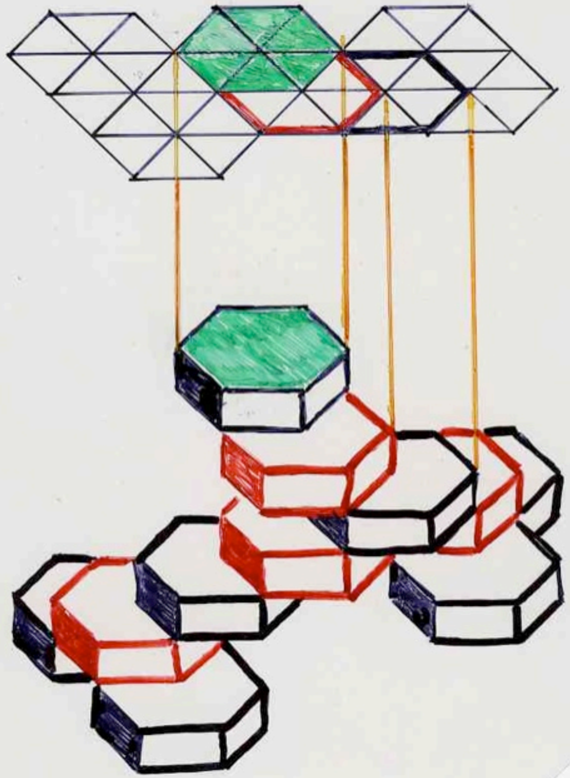
$(\beta, i-1) \in E$


fiber
level
4
4
3
3
2
2
2
1
1
0
non-commutation graph

Heap of dimers over [1, n]



$$-p(-t) = y$$



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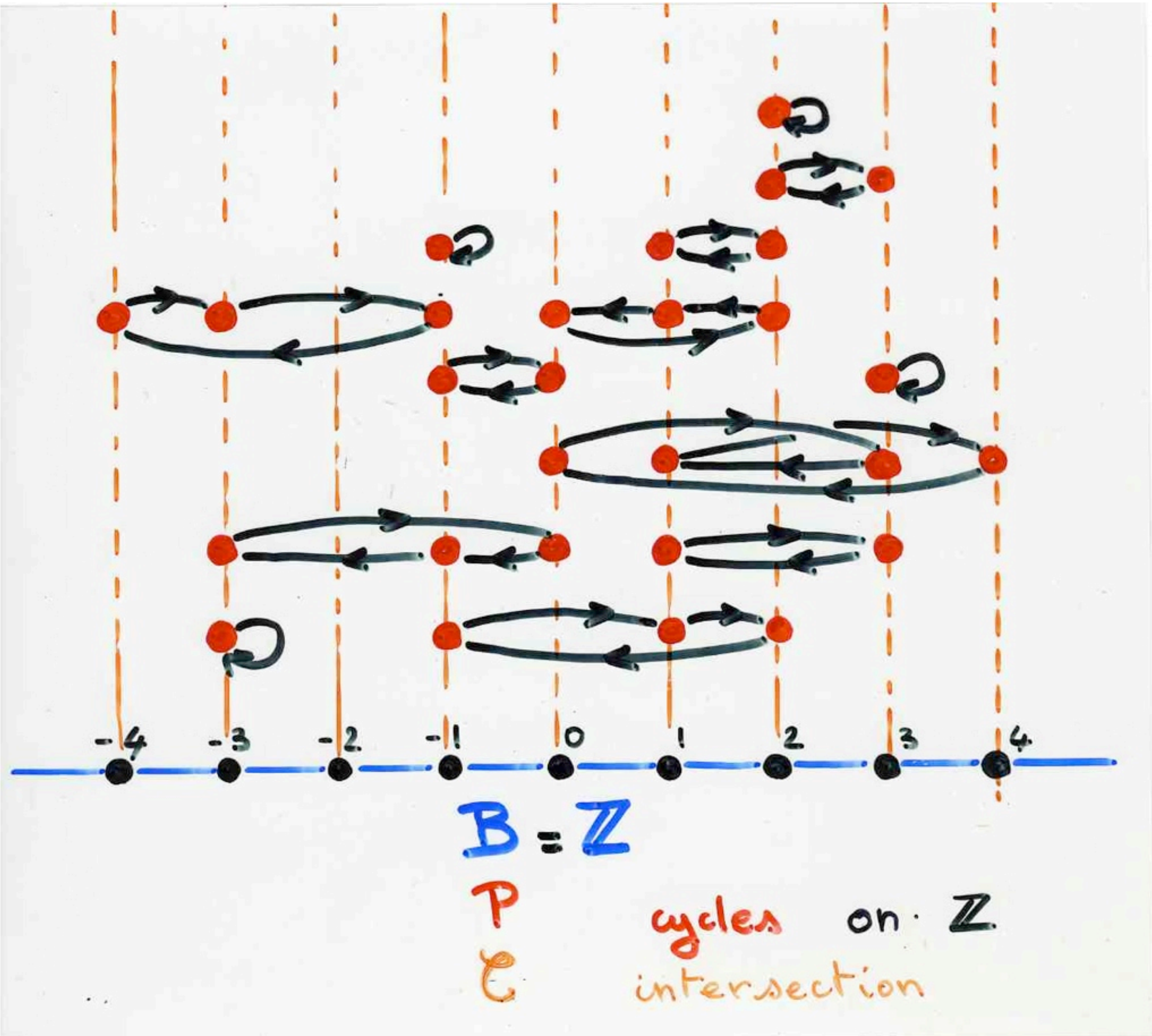

-4 -3 -2 -1 0 1 2 3 4
B = ℤ
P cycles on ℤ
C intersection

Proposition

$$\mathrm{Heap}(P, \mathcal{E}) \simeq P^{*} / \equiv_{C}$$

commutation

monoid

$$C = \overline{\mathcal{E}}$$

complementary relation

heaps of **dimers** $(i, i+1)$

on $\{0, 1, \dots, n-1\}$

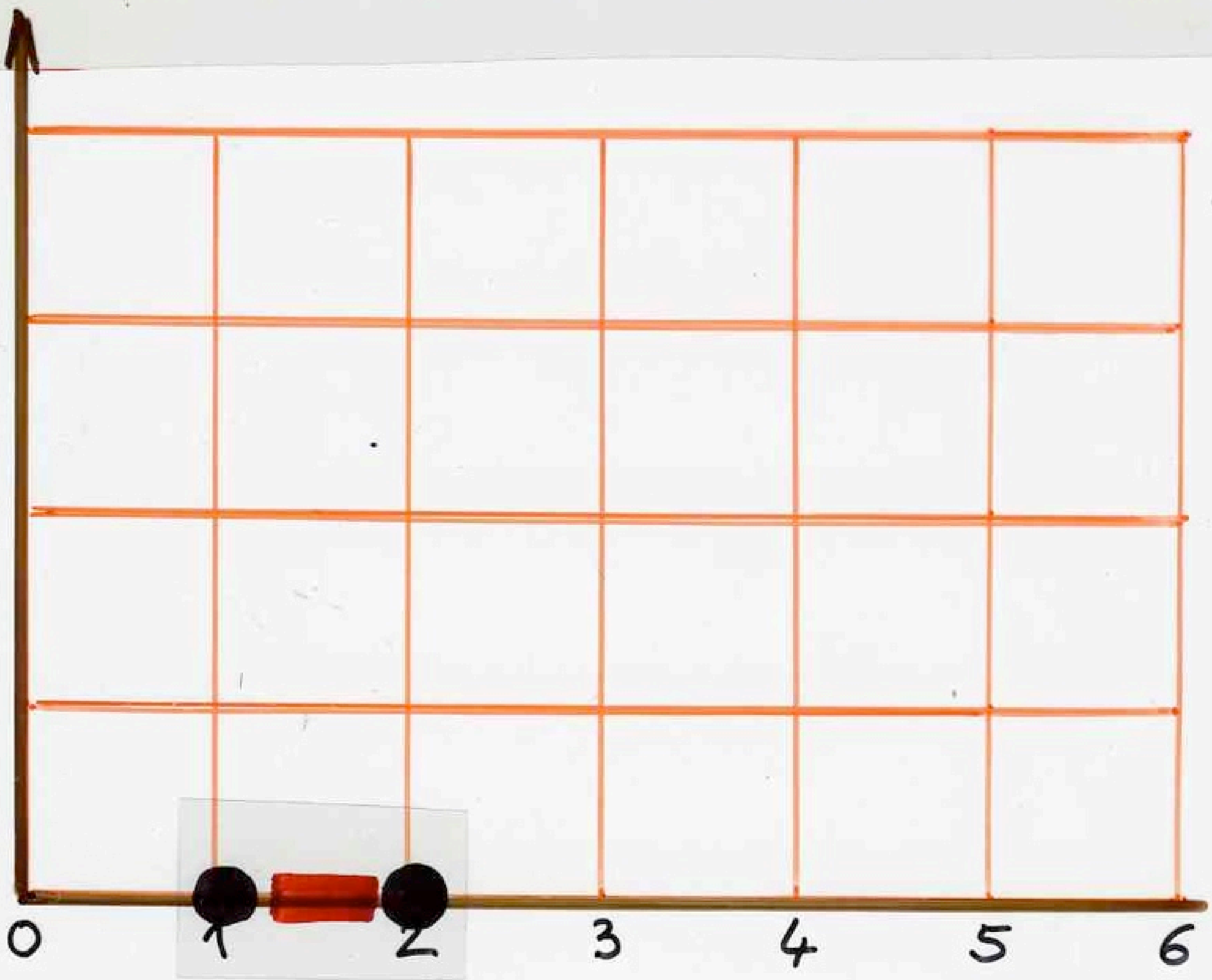
generators $\{\sigma_0, \sigma_1, \dots, \sigma_{n-1}\}$

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

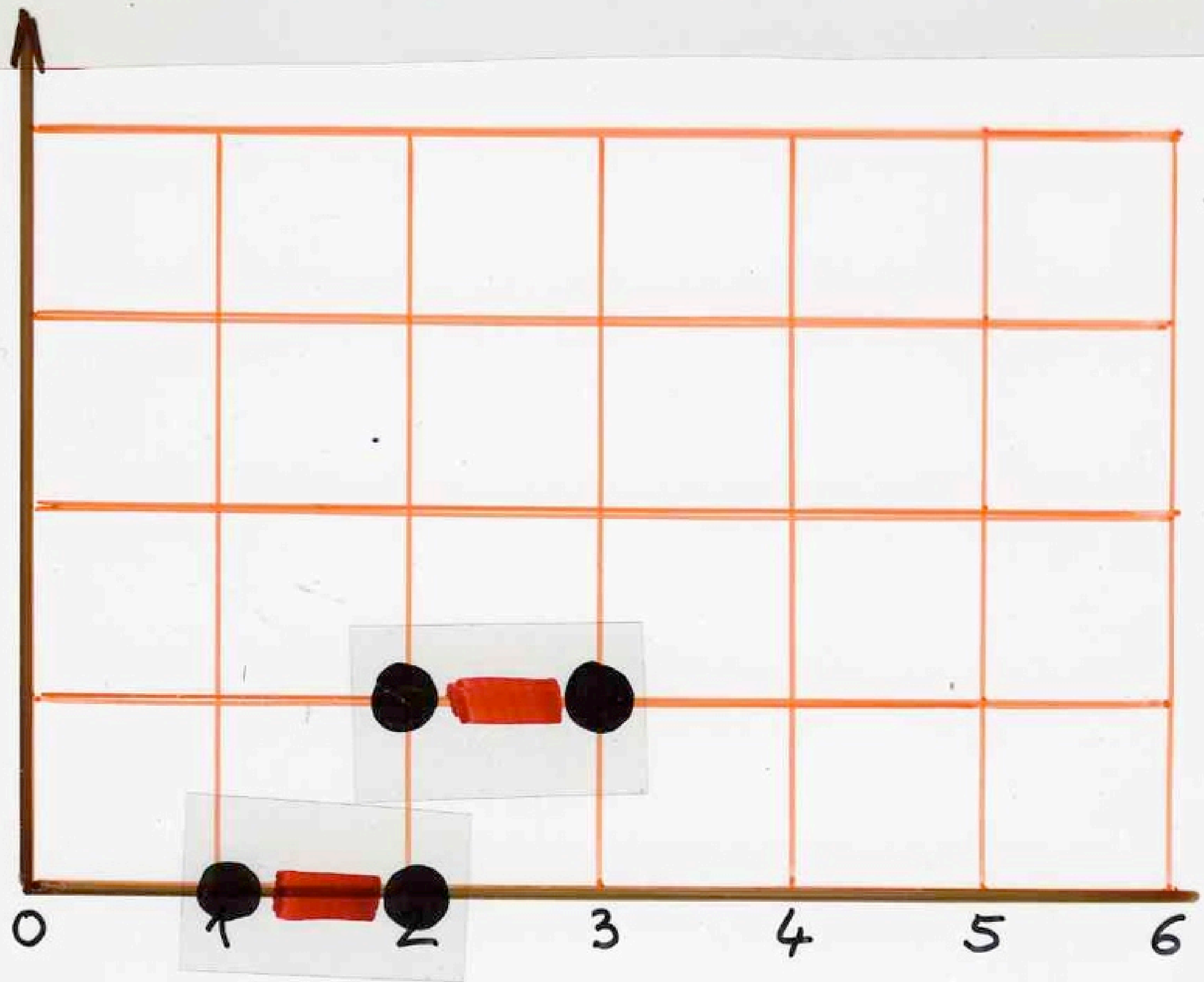
iff $|i-j| \geq 2$

$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$

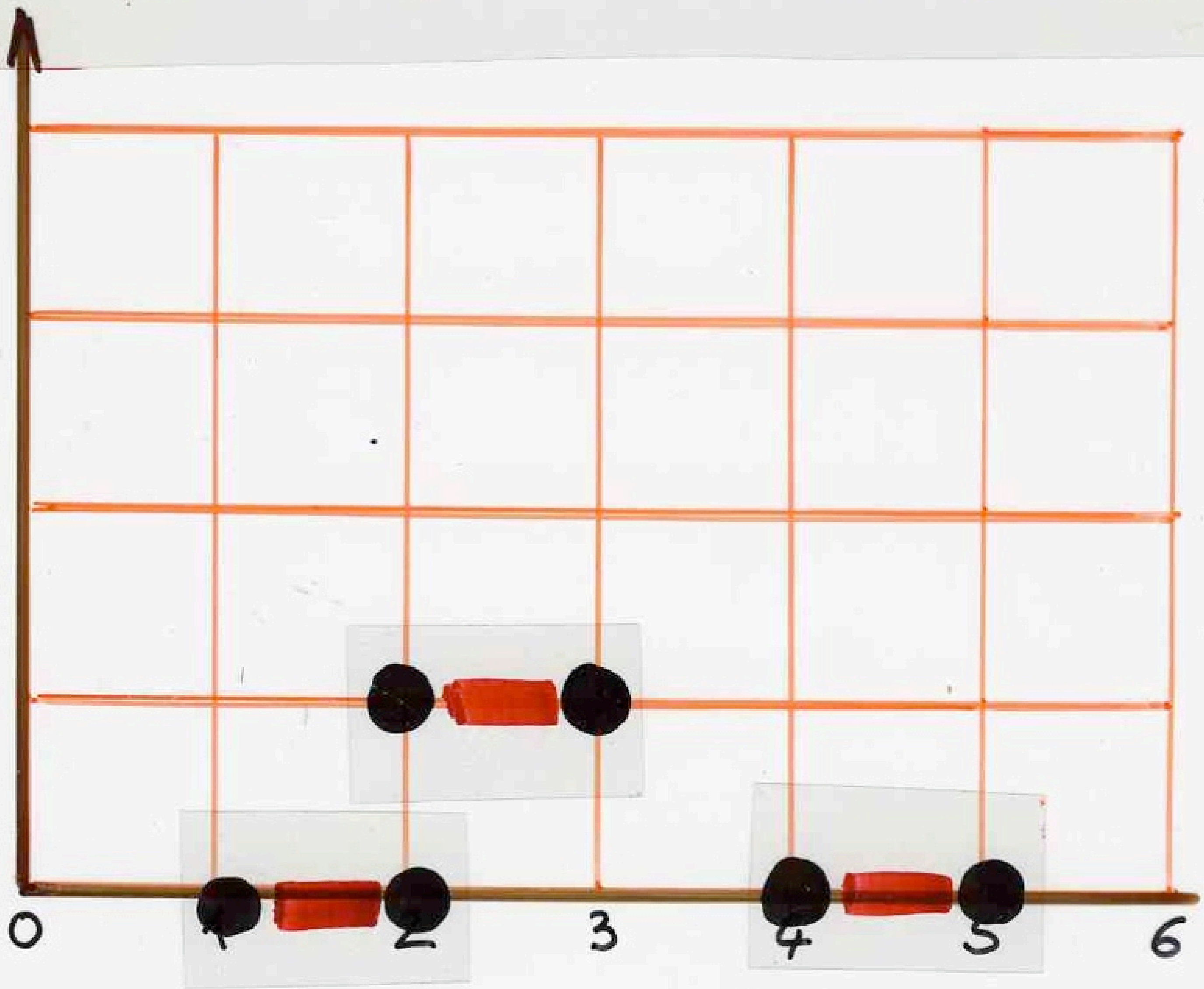
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



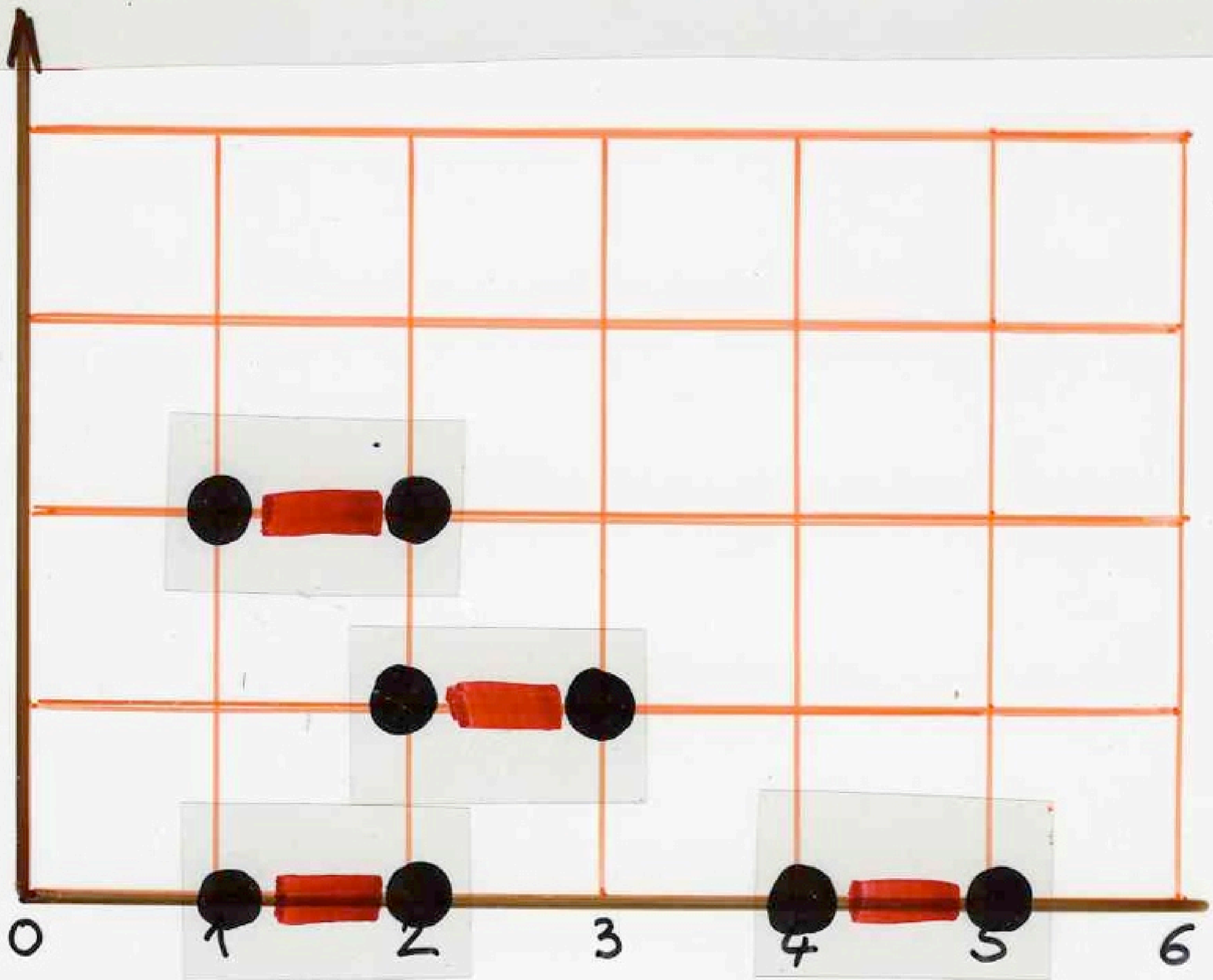
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



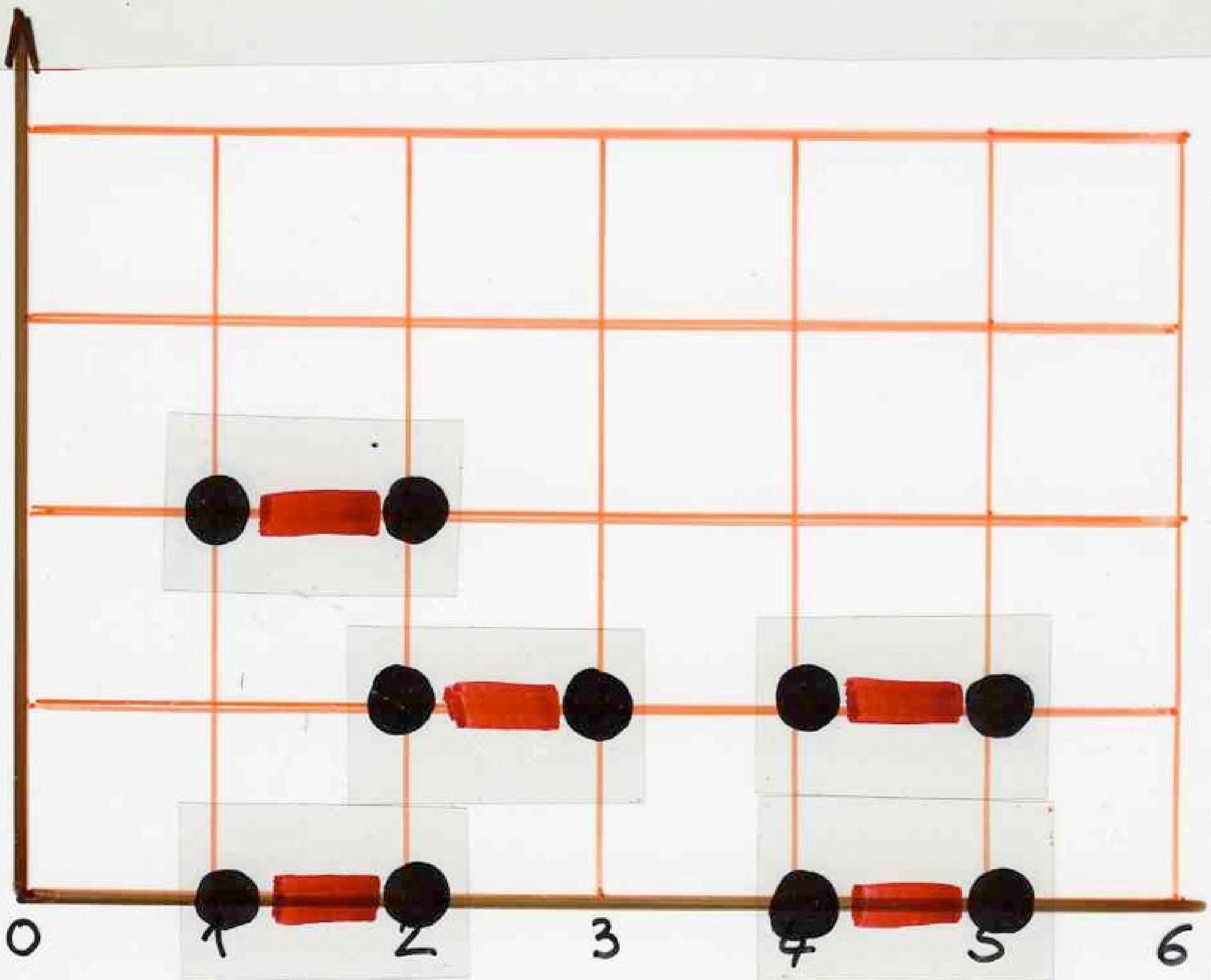
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



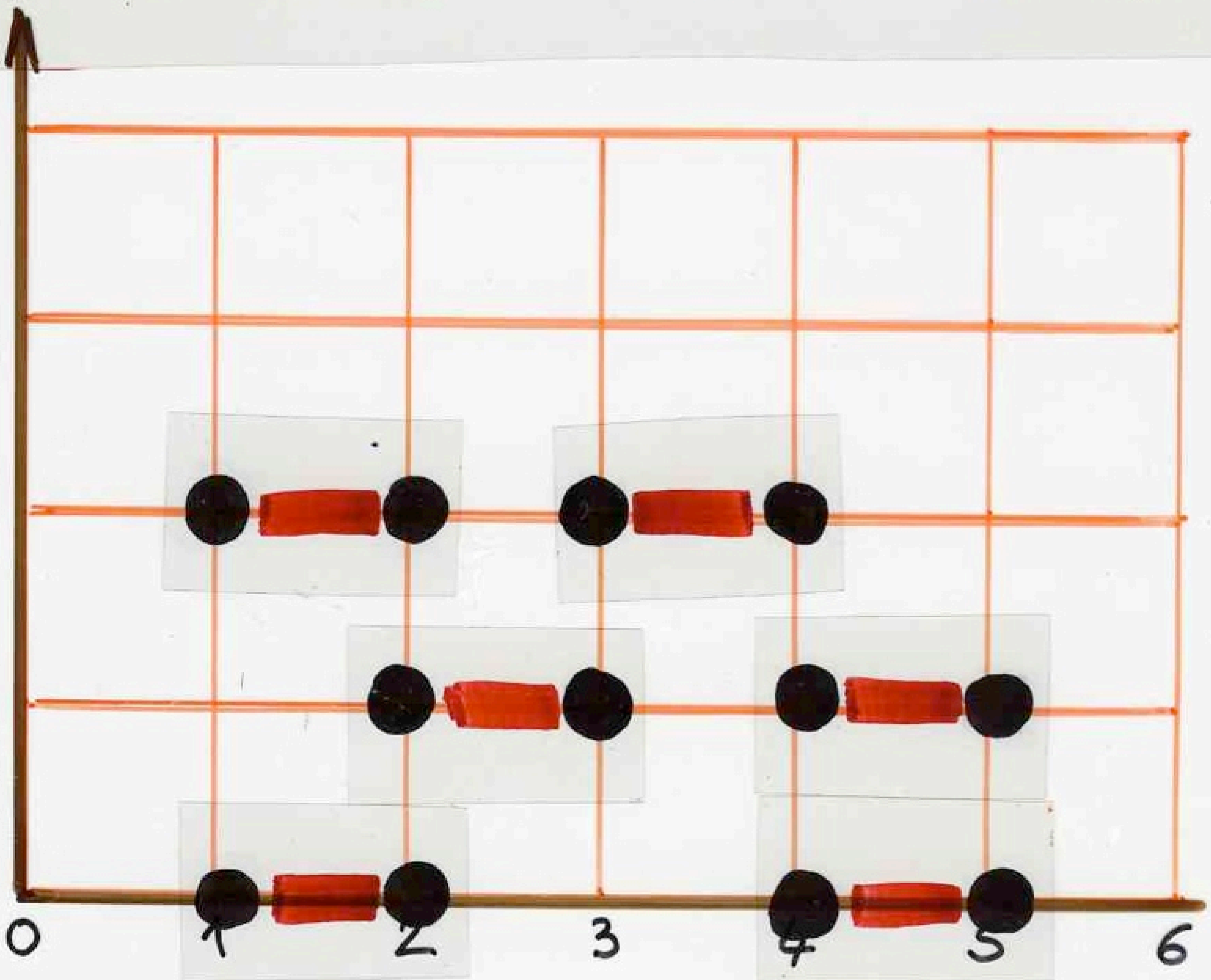
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



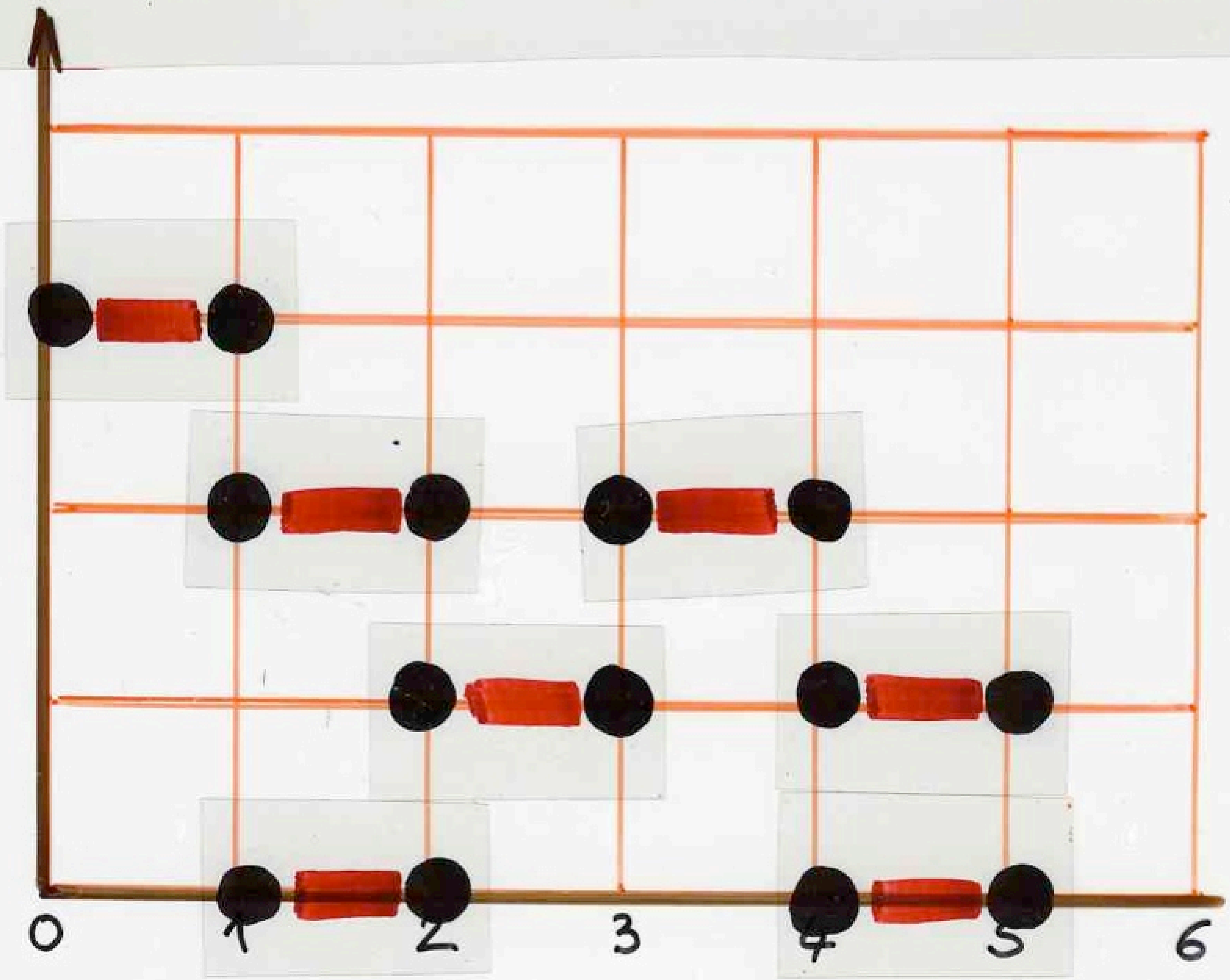
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



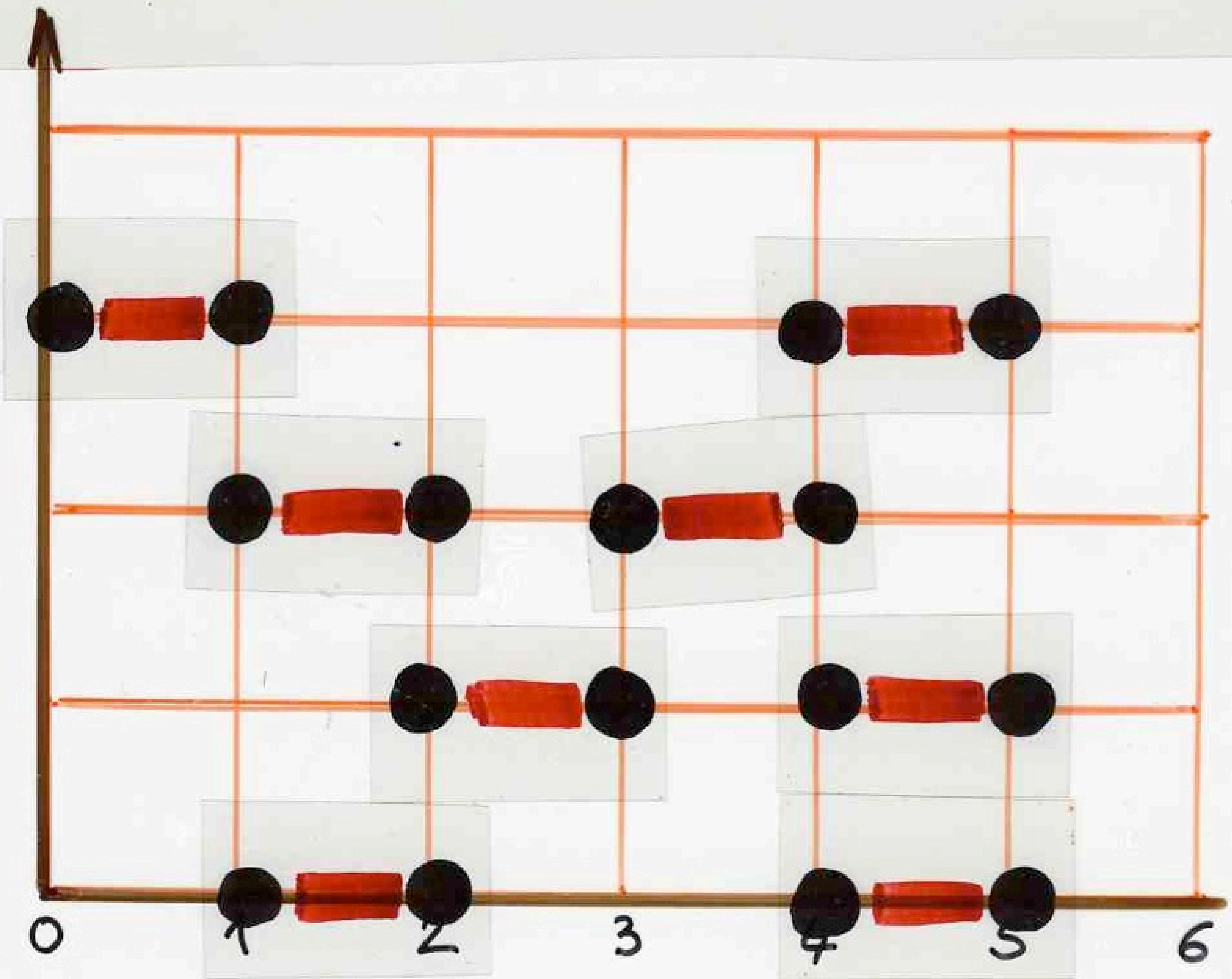
$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$

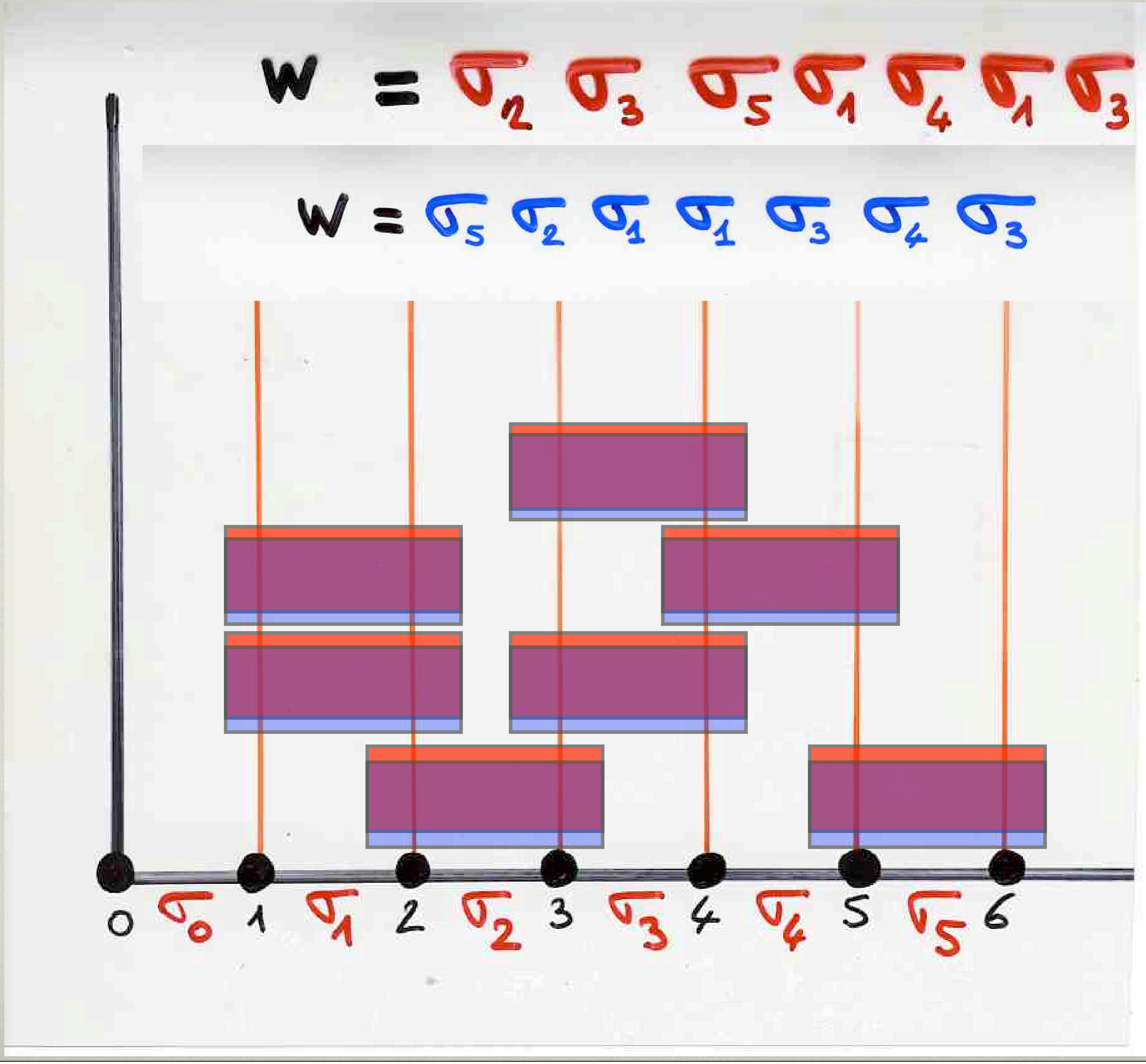


$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



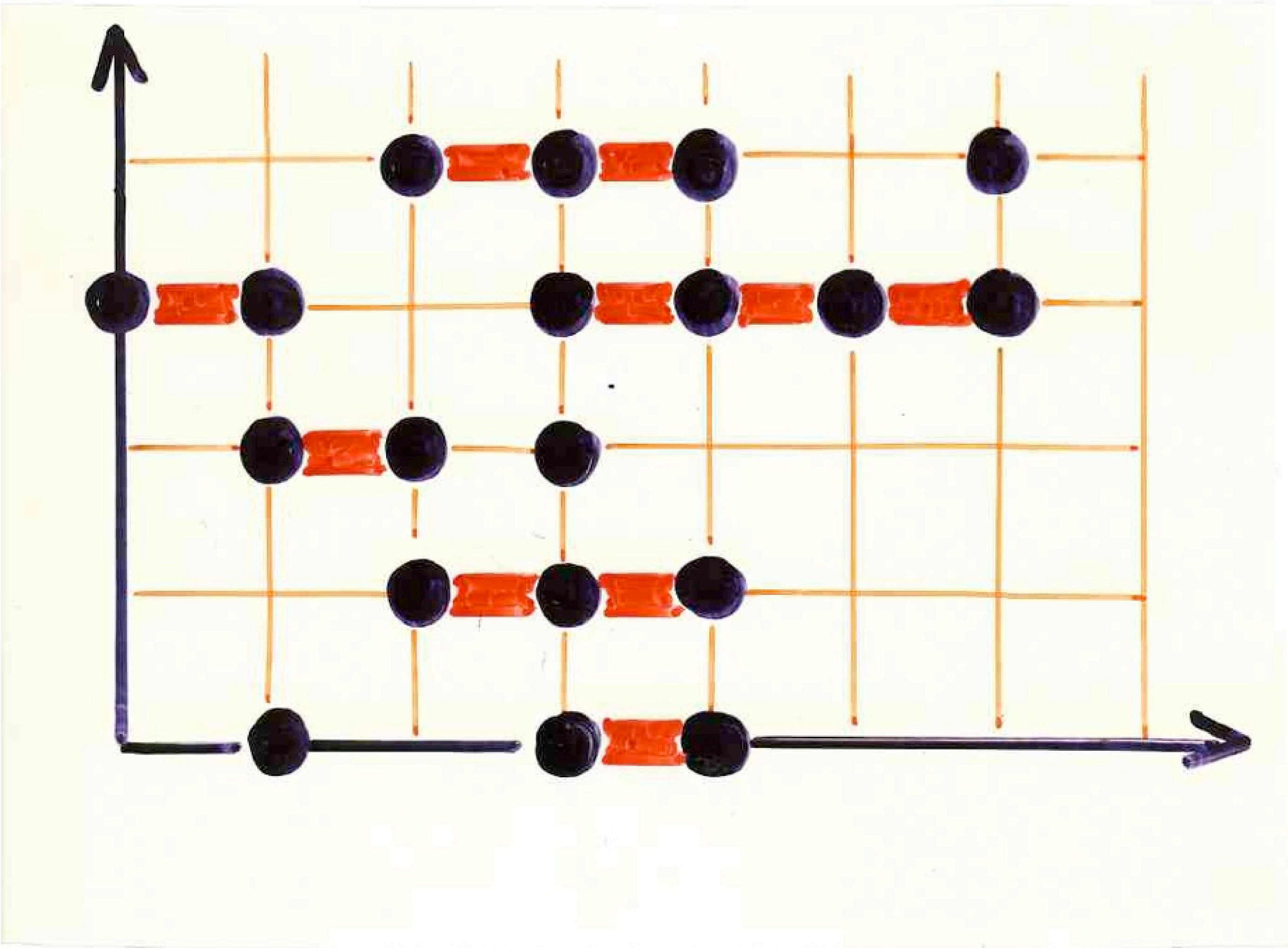
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$

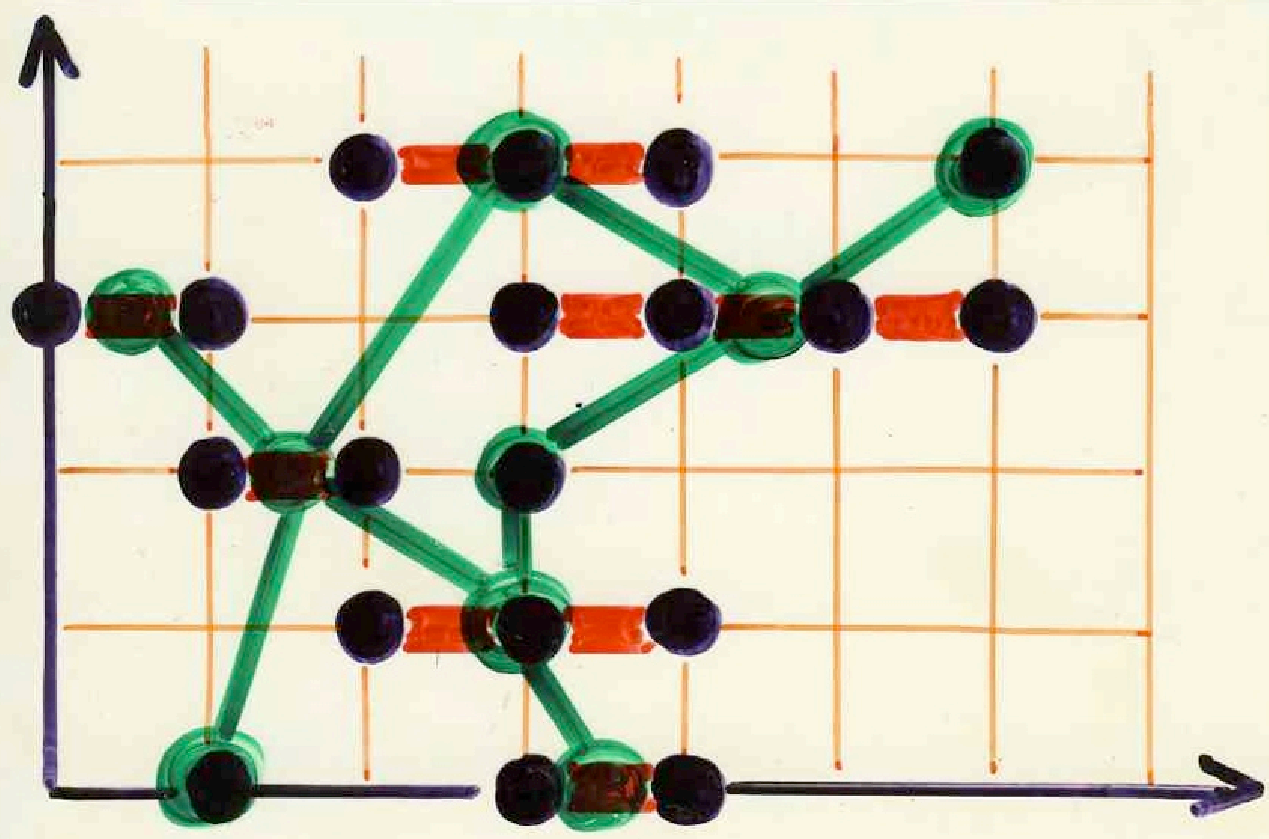


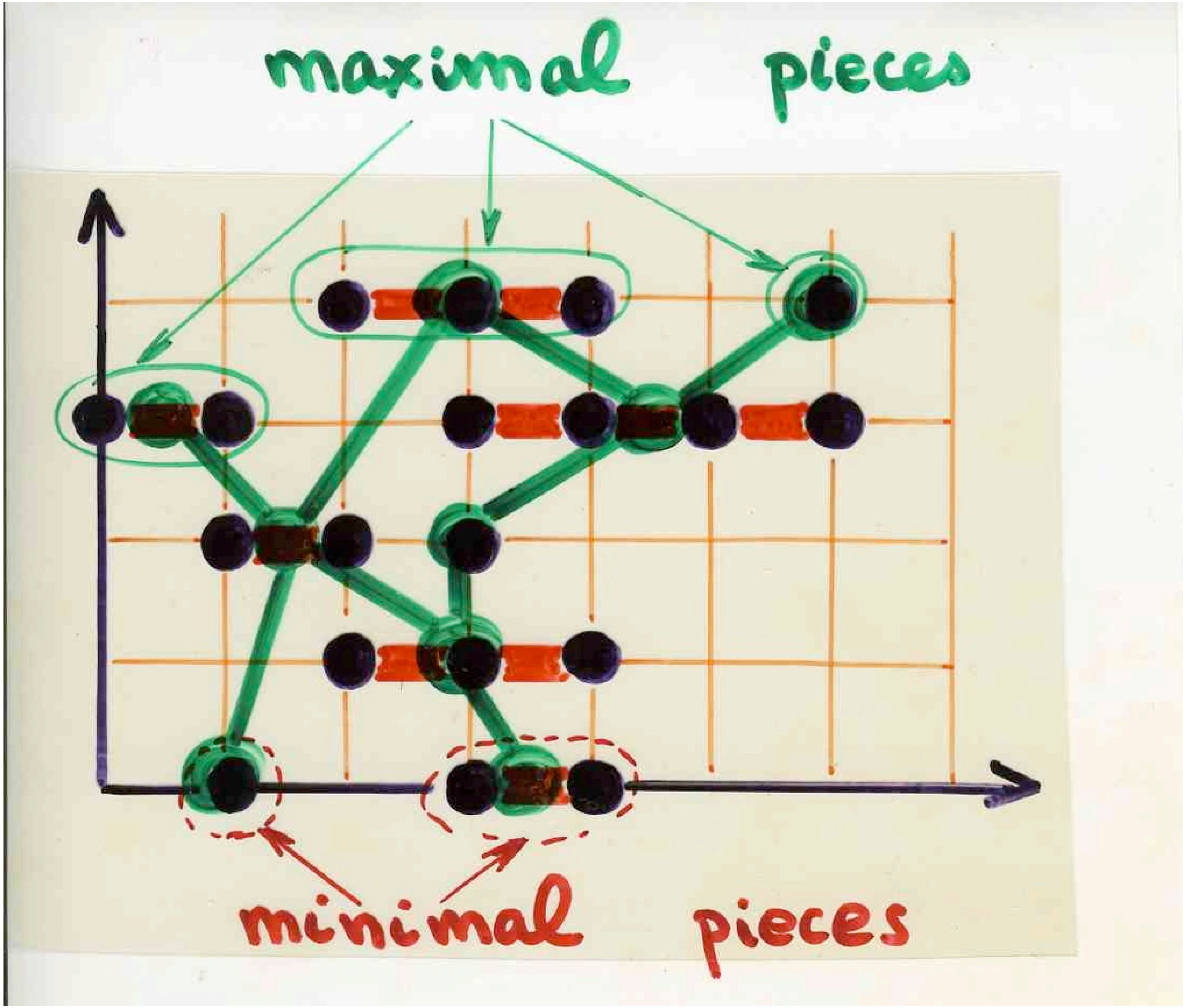


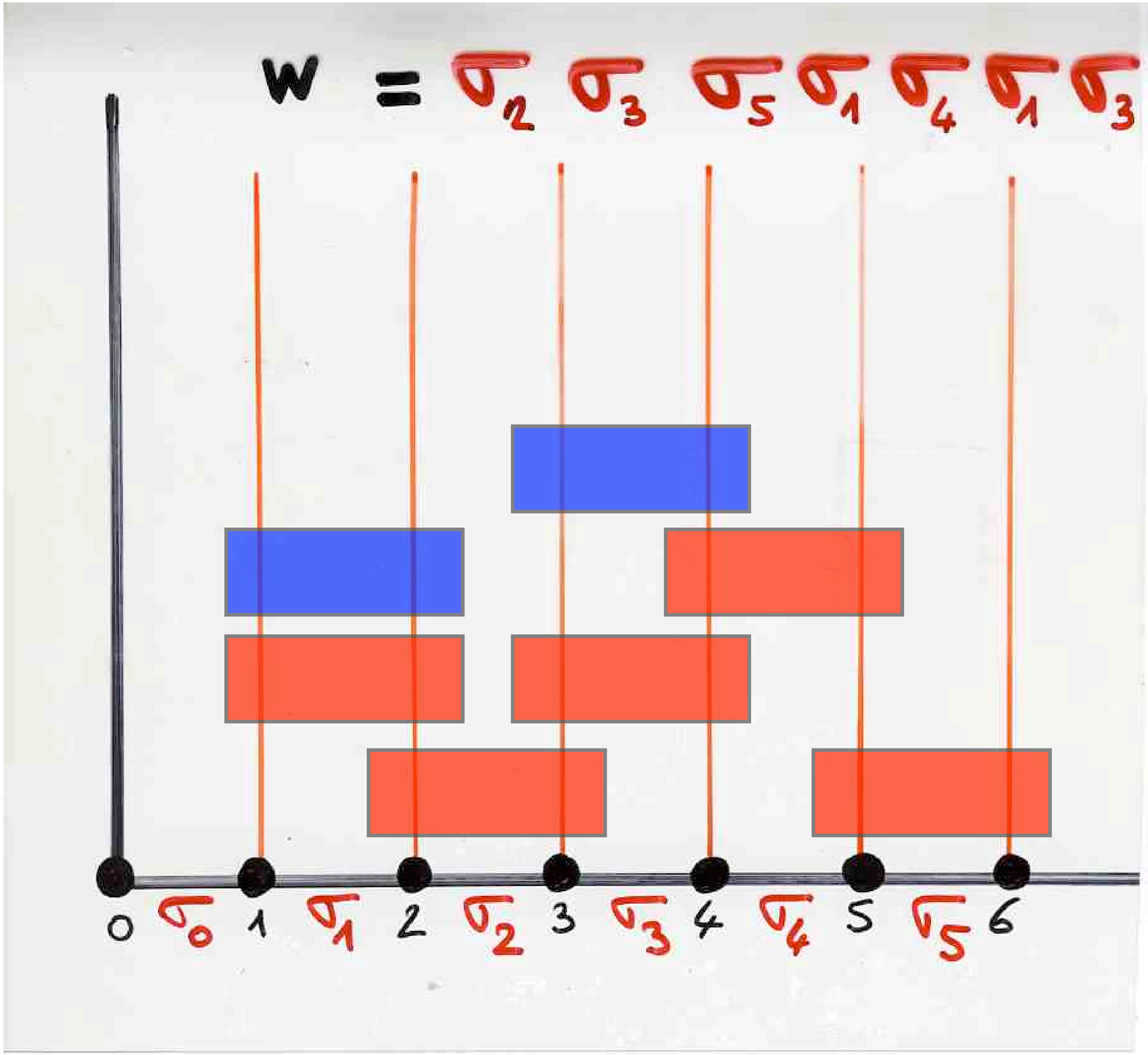
w = σ2 σ3 σ5 σ1 σ4 σ1 σ3
w = σ5 σ2 σ1 σ1 σ3 σ4 σ3
0 σ0 1 σ1 2 σ2 3 σ3 4 σ4 5 σ5 6

§3 Heaps and posets






maximal pieces
minimal pieces

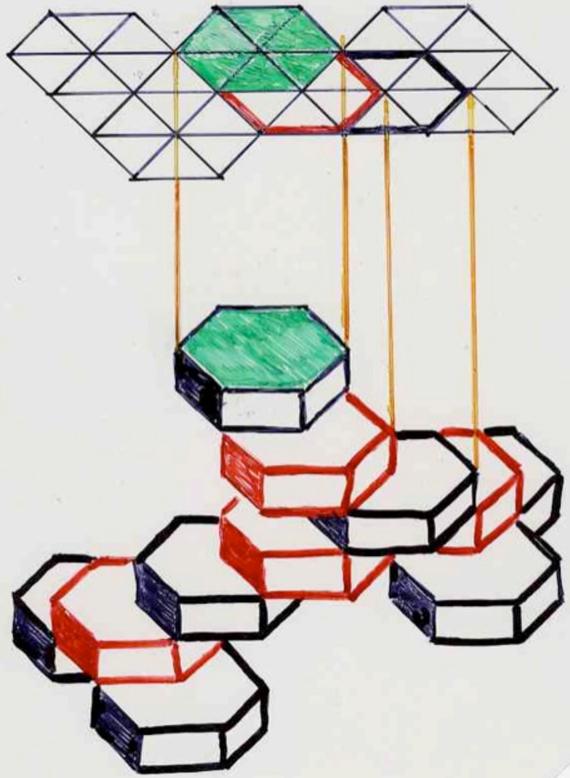


$w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$
0 σ_0 1 σ_1 2 σ_2 3 σ_3 4 σ_4 5 σ_5 6

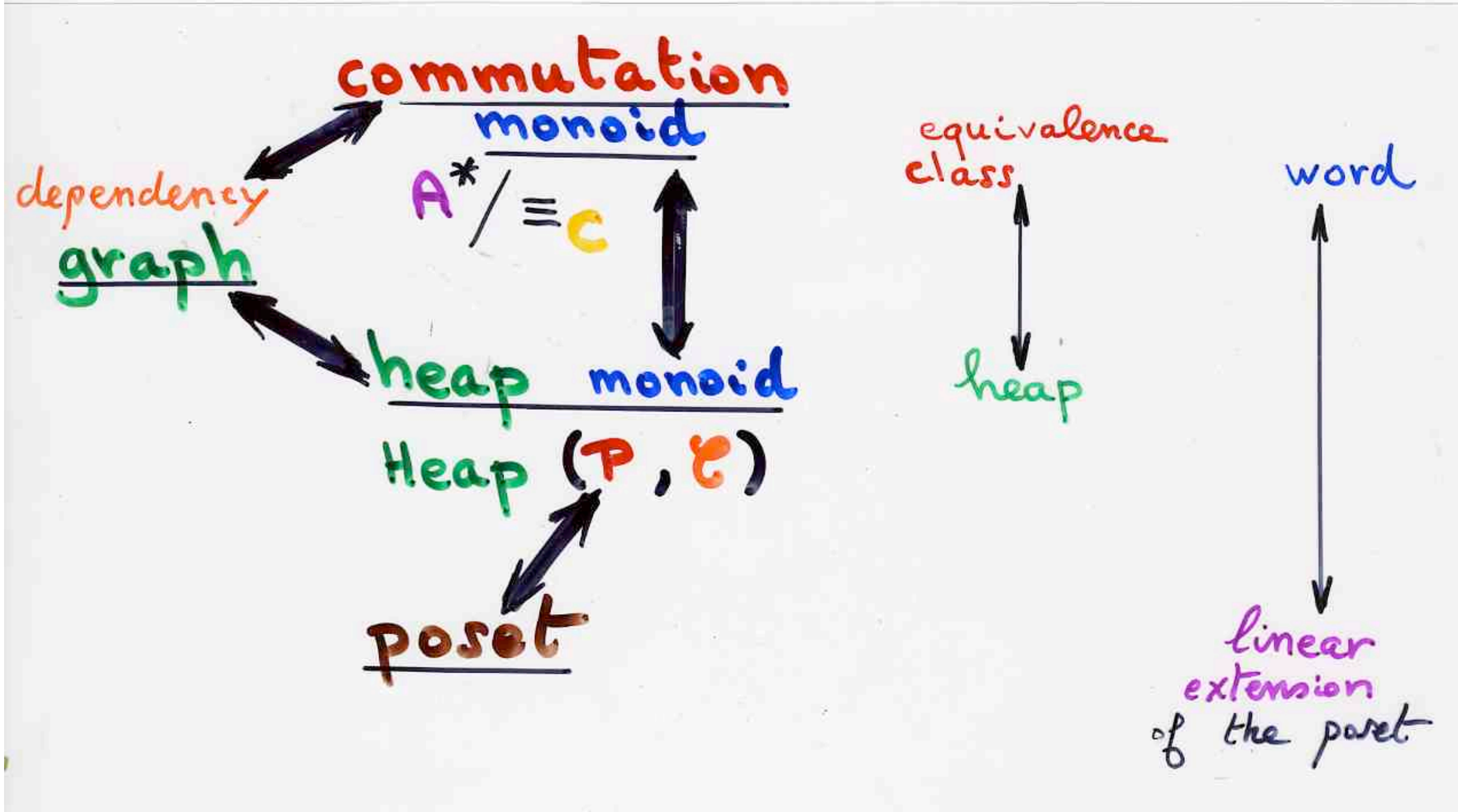
Pyramid

Def- Heap having only one maximal piece

$-p(-t) = y$



C.


commutation
monoid
$A^*/\equiv_C$
dependency
graph
heap monoid
Heap (P, E)
poset
equivalence
class
heap
word
linear
extension
of the poset

3 basic lemma

- $\displaystyle (\text{heaps}) = \frac{1}{(\underset{\text{heaps}}{\text{trivial}})}$
- $\log(\text{heaps}) = \text{pyramids}$
- path = heap

§4 Heaps of pieces: generating functions

weight
valuation

$v(E)$

- $v : \mathcal{P} \longrightarrow \mathbb{K}[x, y, \ldots]$

  basic piece

- $v(\alpha, i) = v(\alpha)$

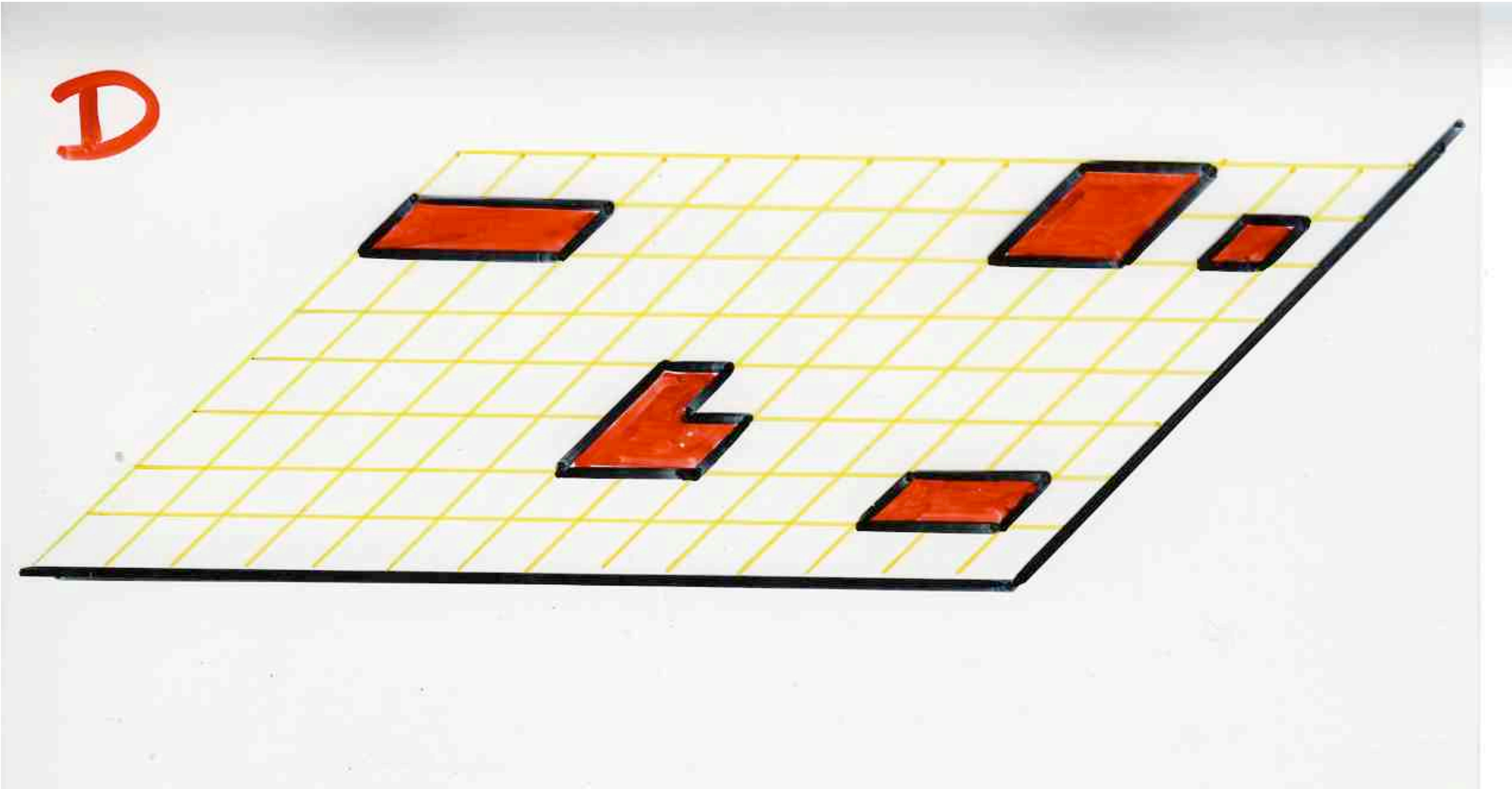
  piece

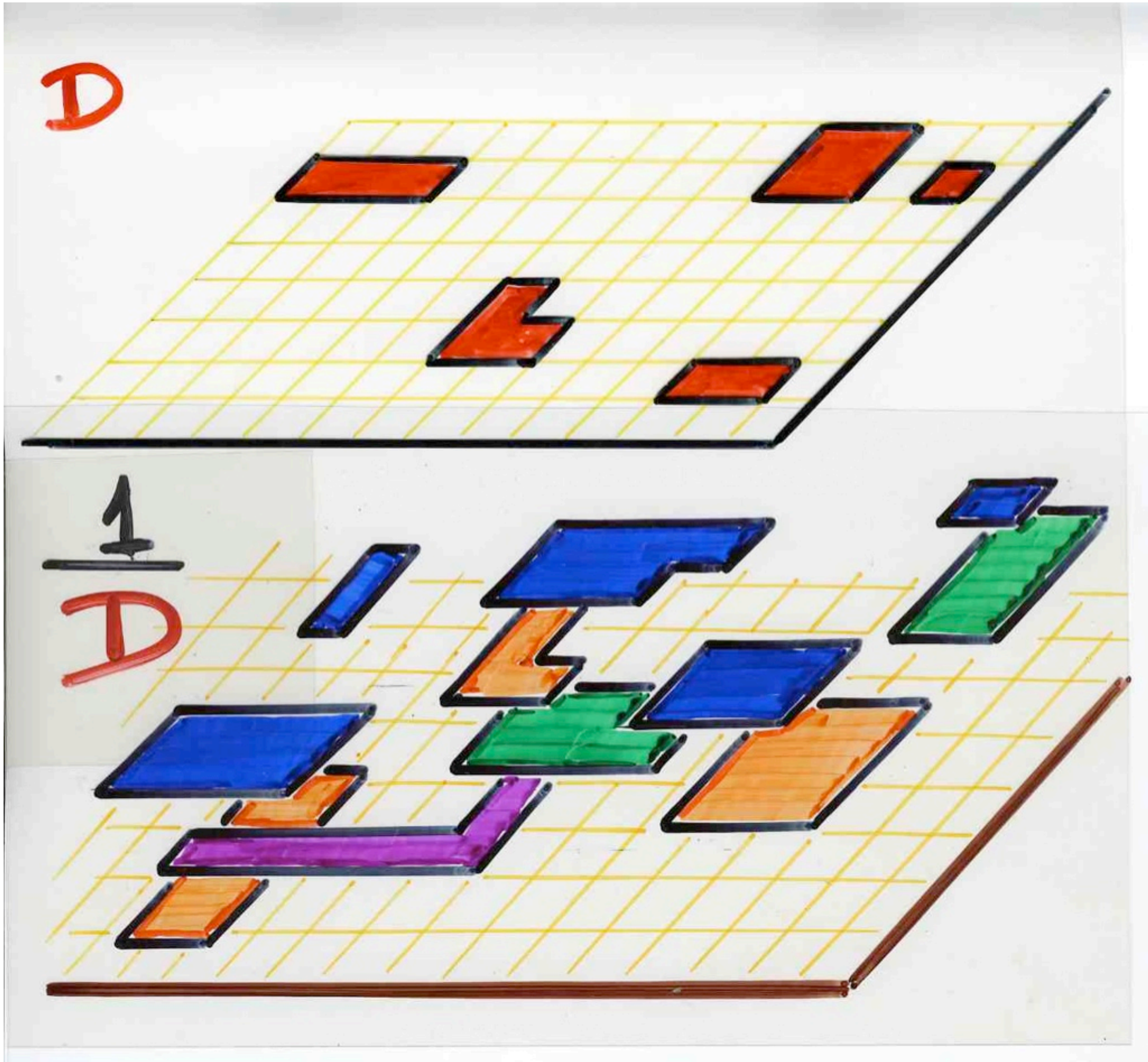
- $v(E) = \prod_{(\alpha, i) \in E} v(\alpha\ i)$

  heap

Inversion lemma

$$\sum_{\substack{E \\ \text{heaps}}} v(E) = \frac{1}{\displaystyle\sum_{\substack{F \\ \text{trivial} \\ \text{heaps}}} (-1)^{|F|} v(F)}$$


D


D
1
D

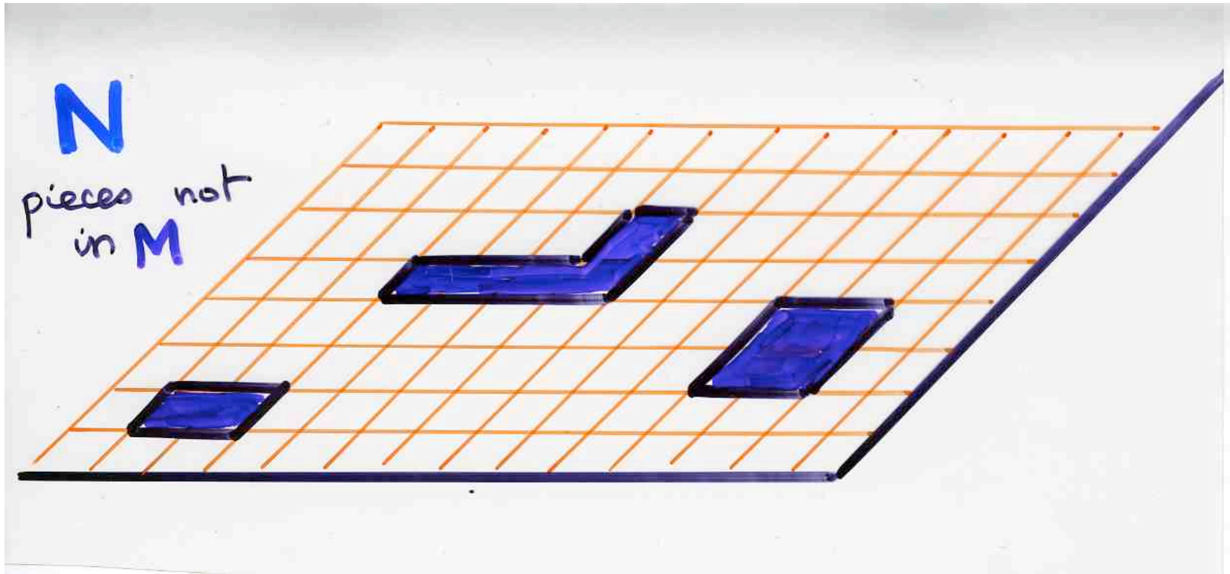
extension of the inversion lemma

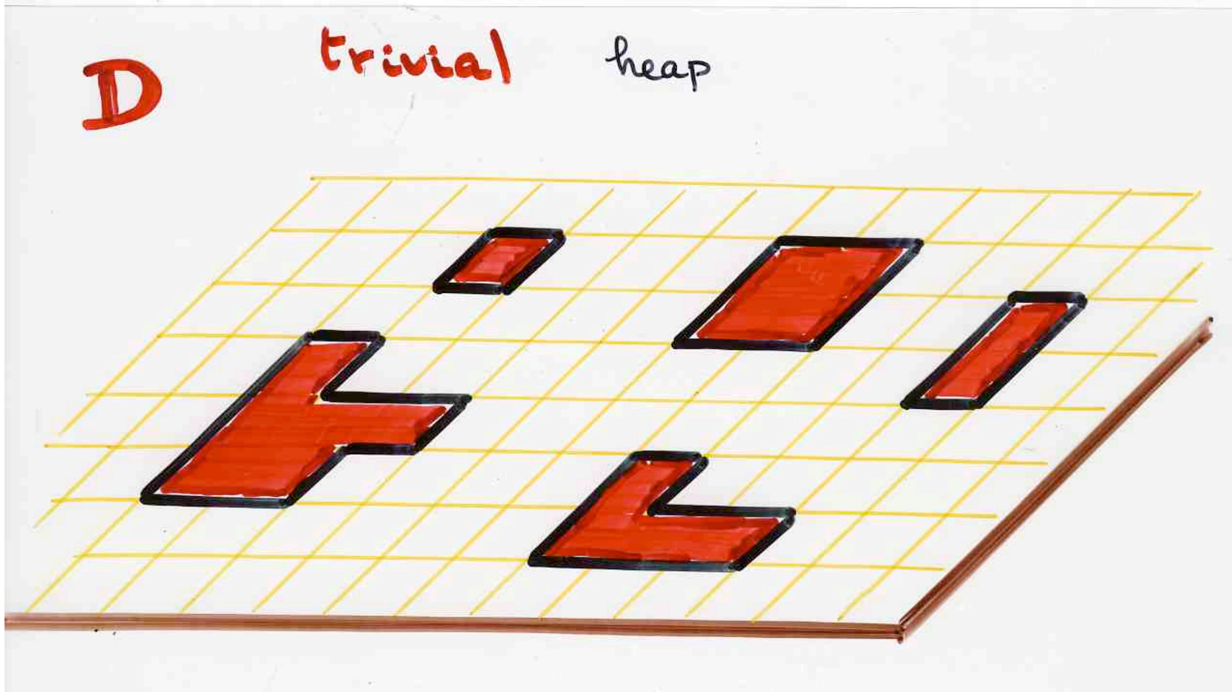
$M \subseteq P$

$$\sum_{\substack{E \\ \Pi(\text{maximal pieces}) \in M}} v(E) = \frac{N}{D}$$

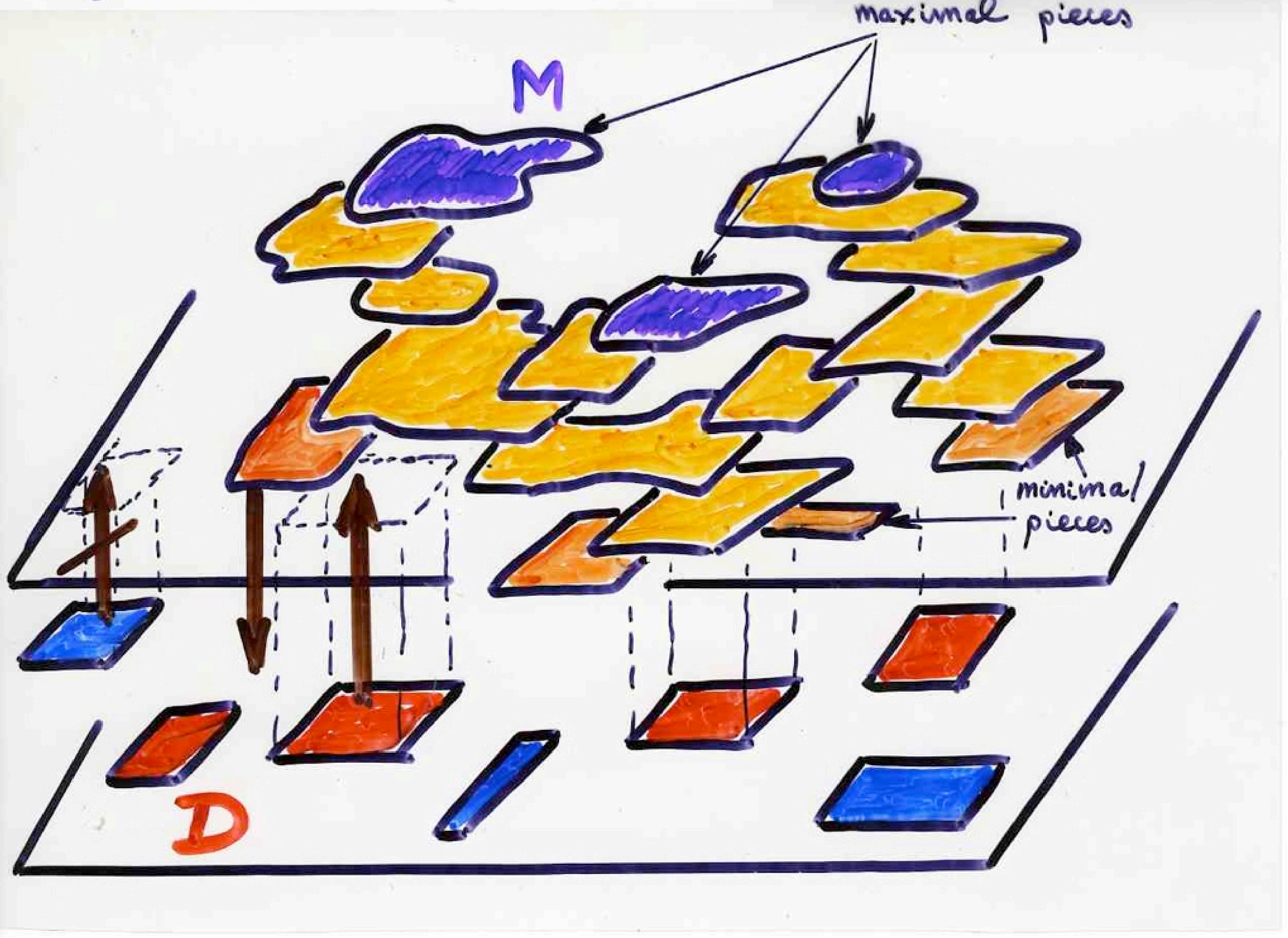
$$D = \sum_{\substack{F \\ \text{trivial heaps}}} (-1)^{|F|} v(F)$$

$$N = \sum_{\substack{F \\ \text{trivial heaps} \\ \text{pieces} \notin M}} (-1)^{|F|} v(F)$$


N
pieces not in M


D
trivial heap

Proof by involution

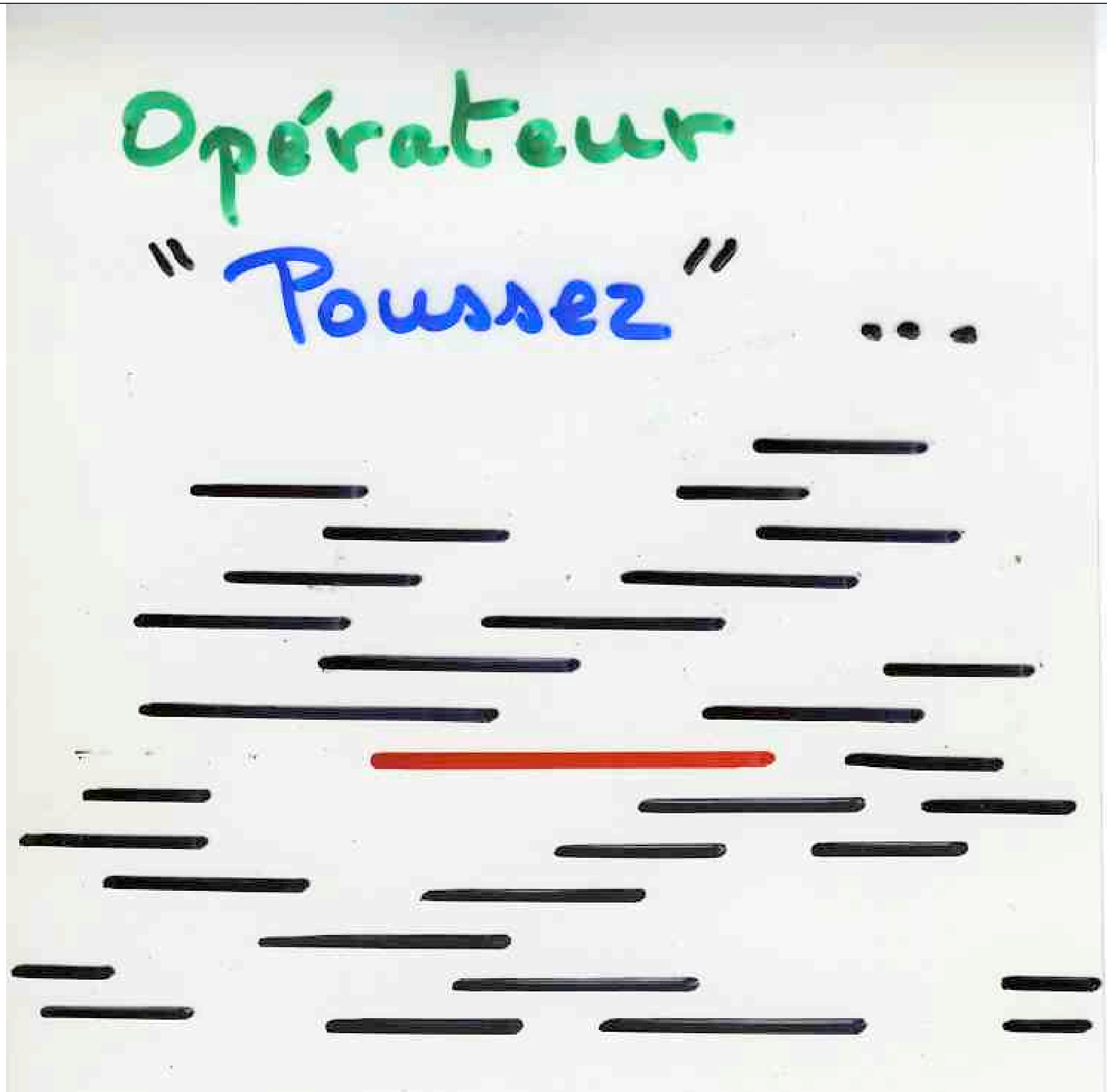


- logarithmic lemma

$$v(\text{piece}) = t\, \underbrace{w(\text{piece})}_{\substack{\text{polynomial} \\ \text{not containing } t}}$$

$$t \frac{d}{dt} \log\Big(\sum_{\text{heap } E} v(E) \Big) = \sum_{\substack{P \\ \text{pyramid}}} v(P)$$

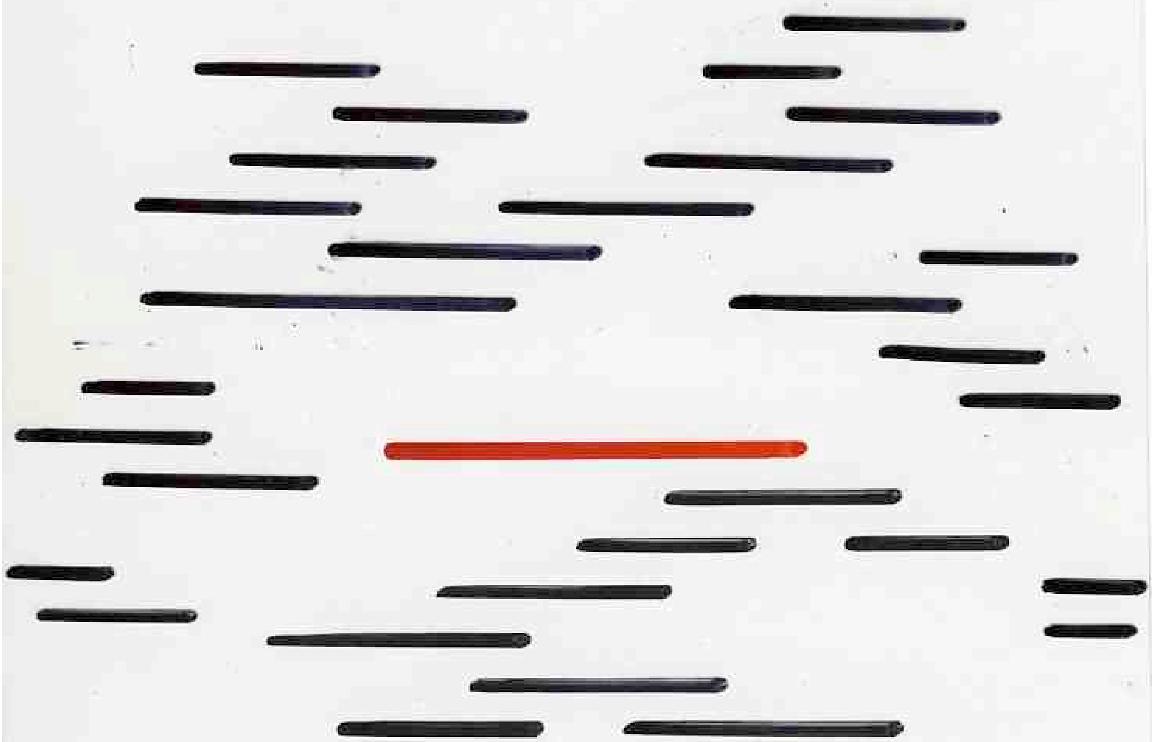
Pointed Heap = Pyramid x Heap



Push operator

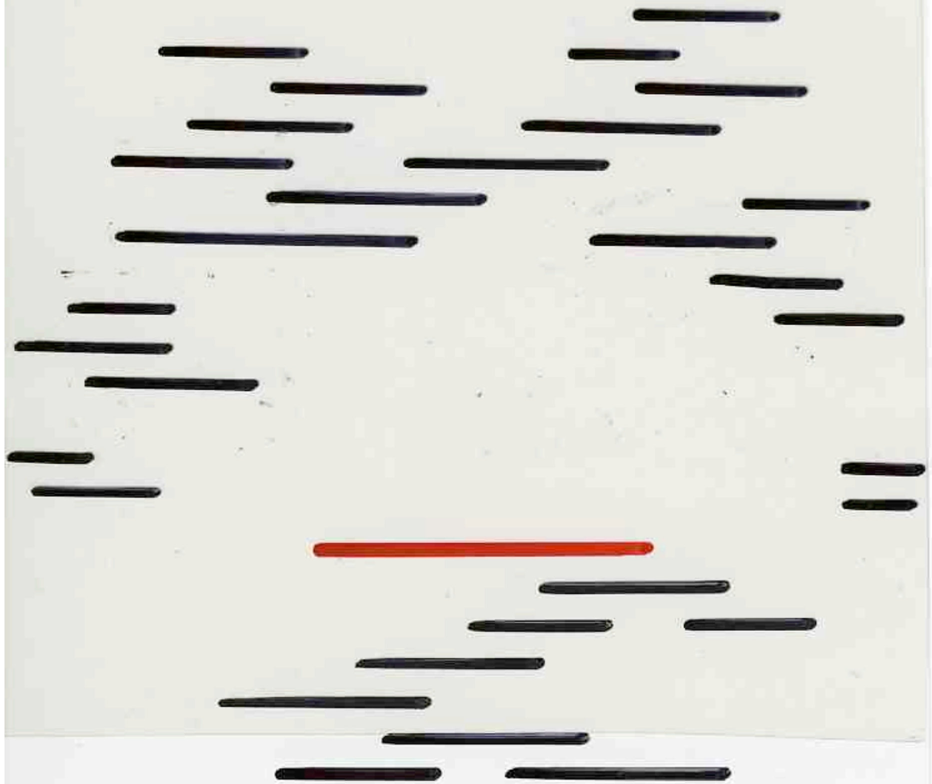
Opérateur

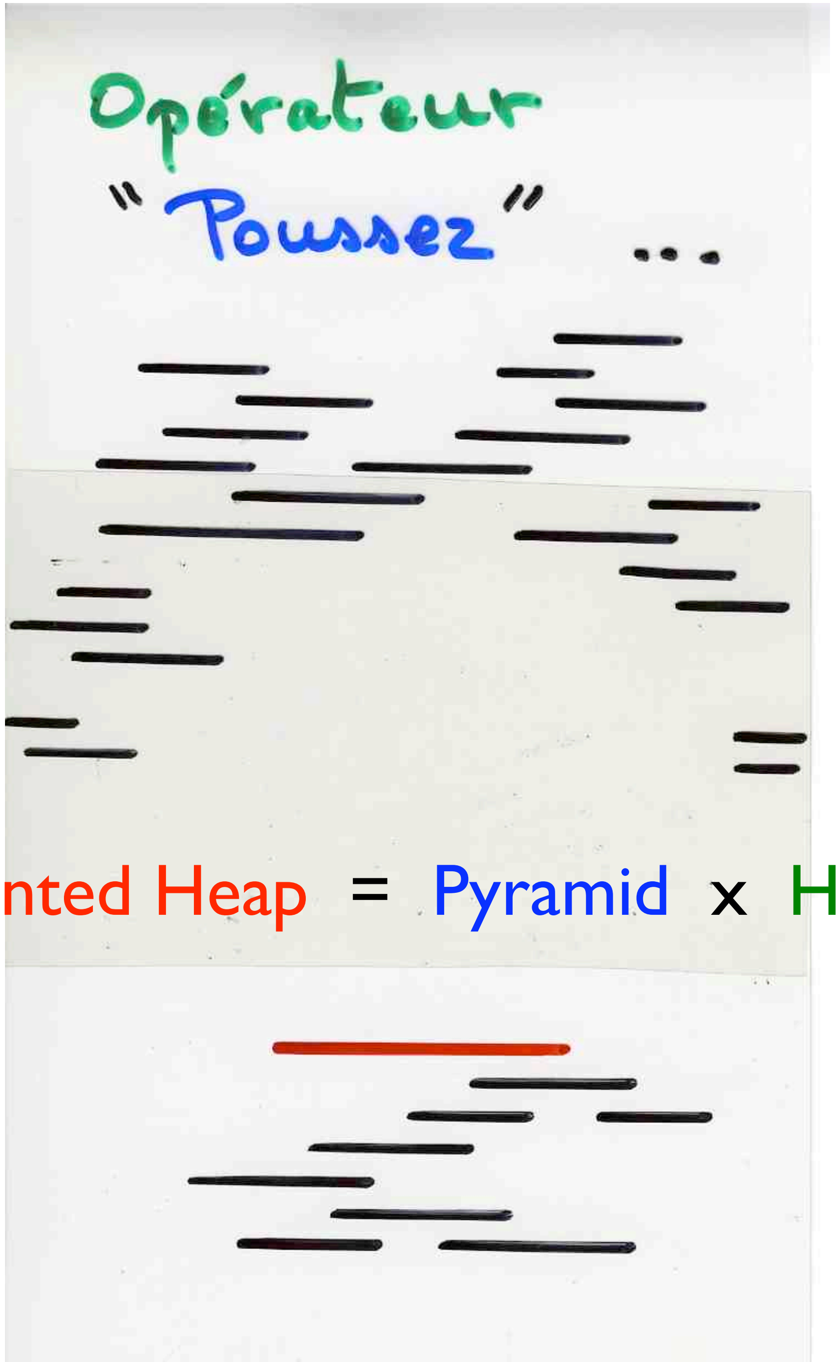
"Poussez" ...



Opérateur

"Poussez" ...





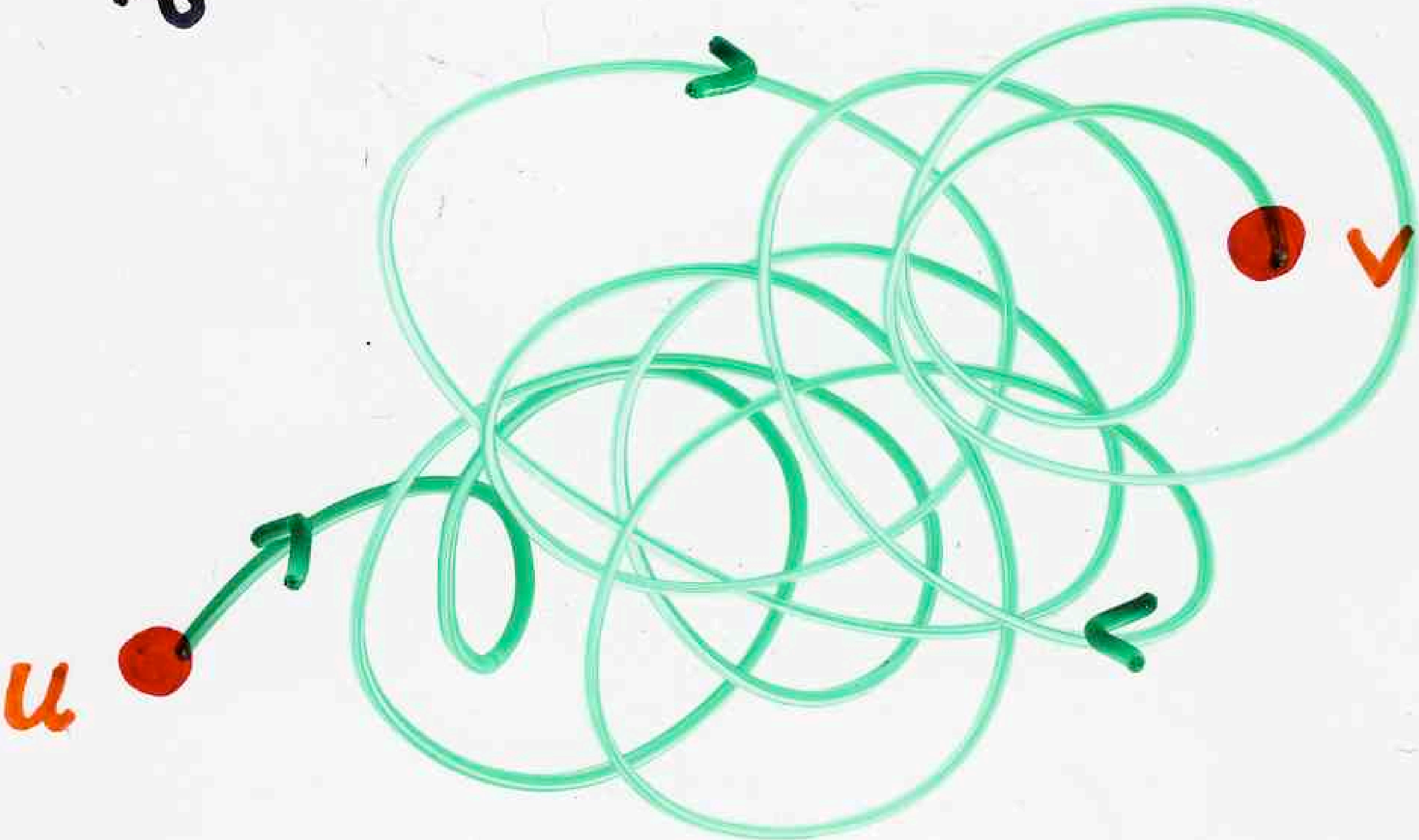
Pointed Heap = Pyramid x Heap

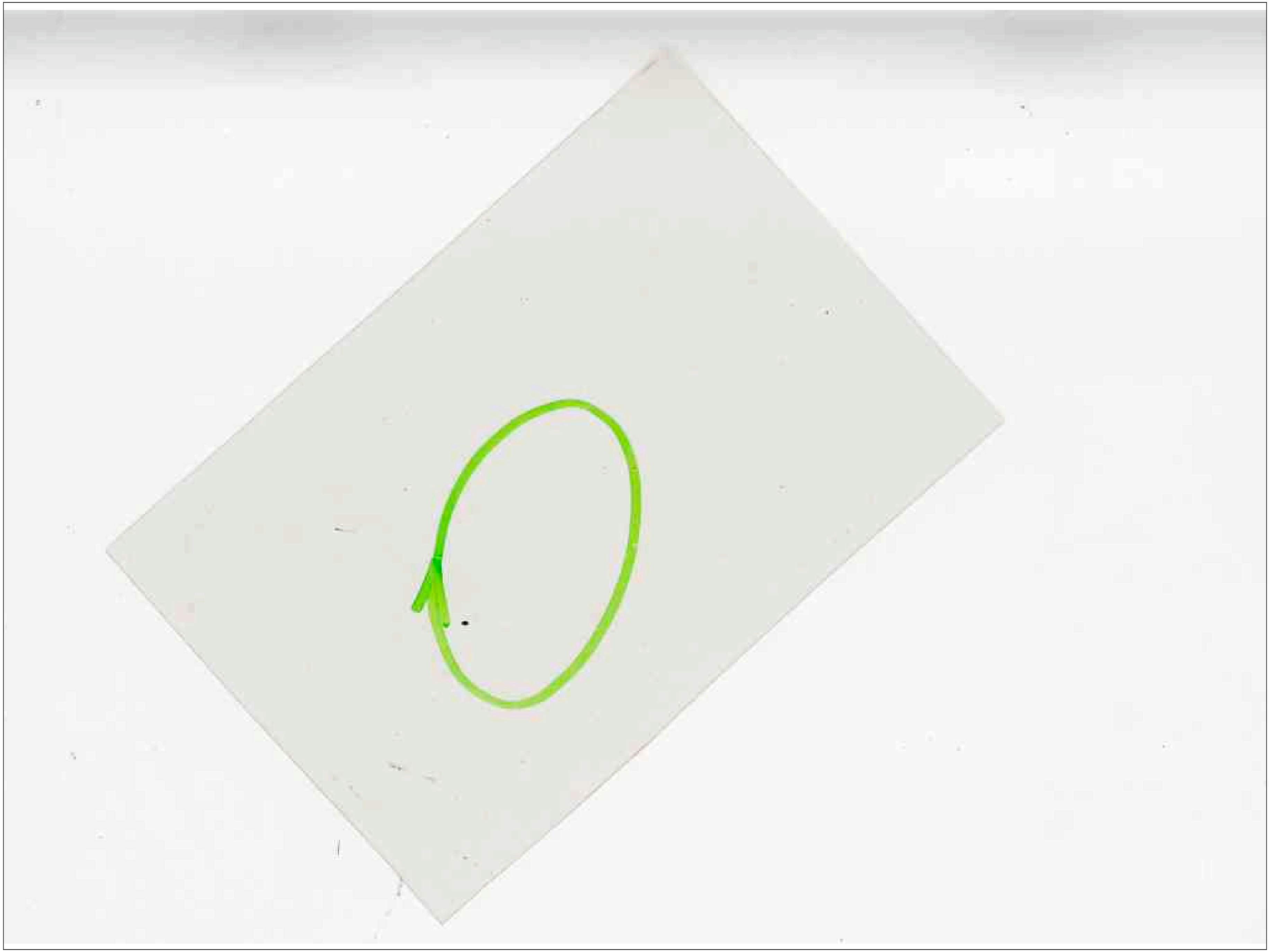
§5 Paths and heaps of cycles

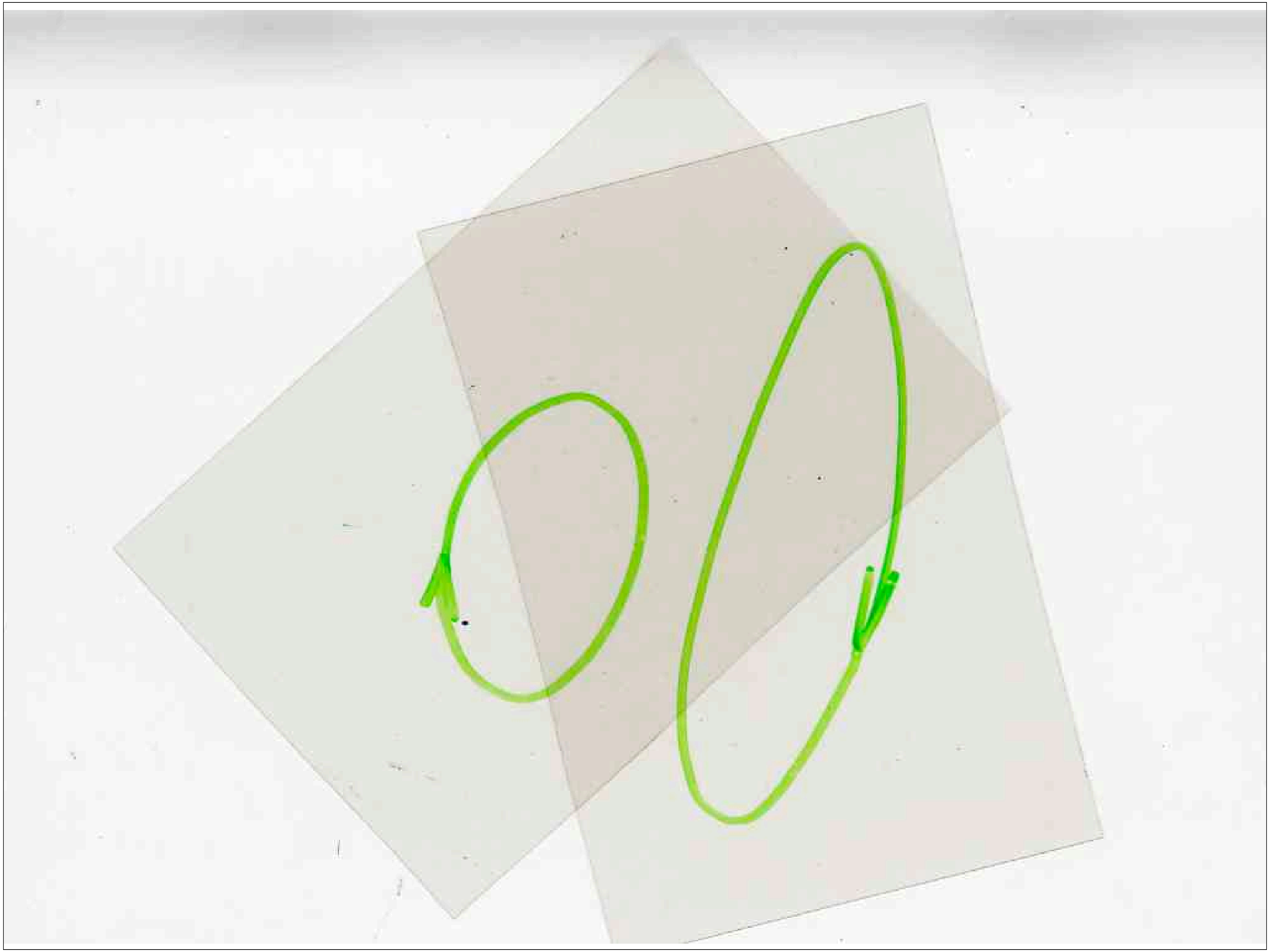
Path = Heap

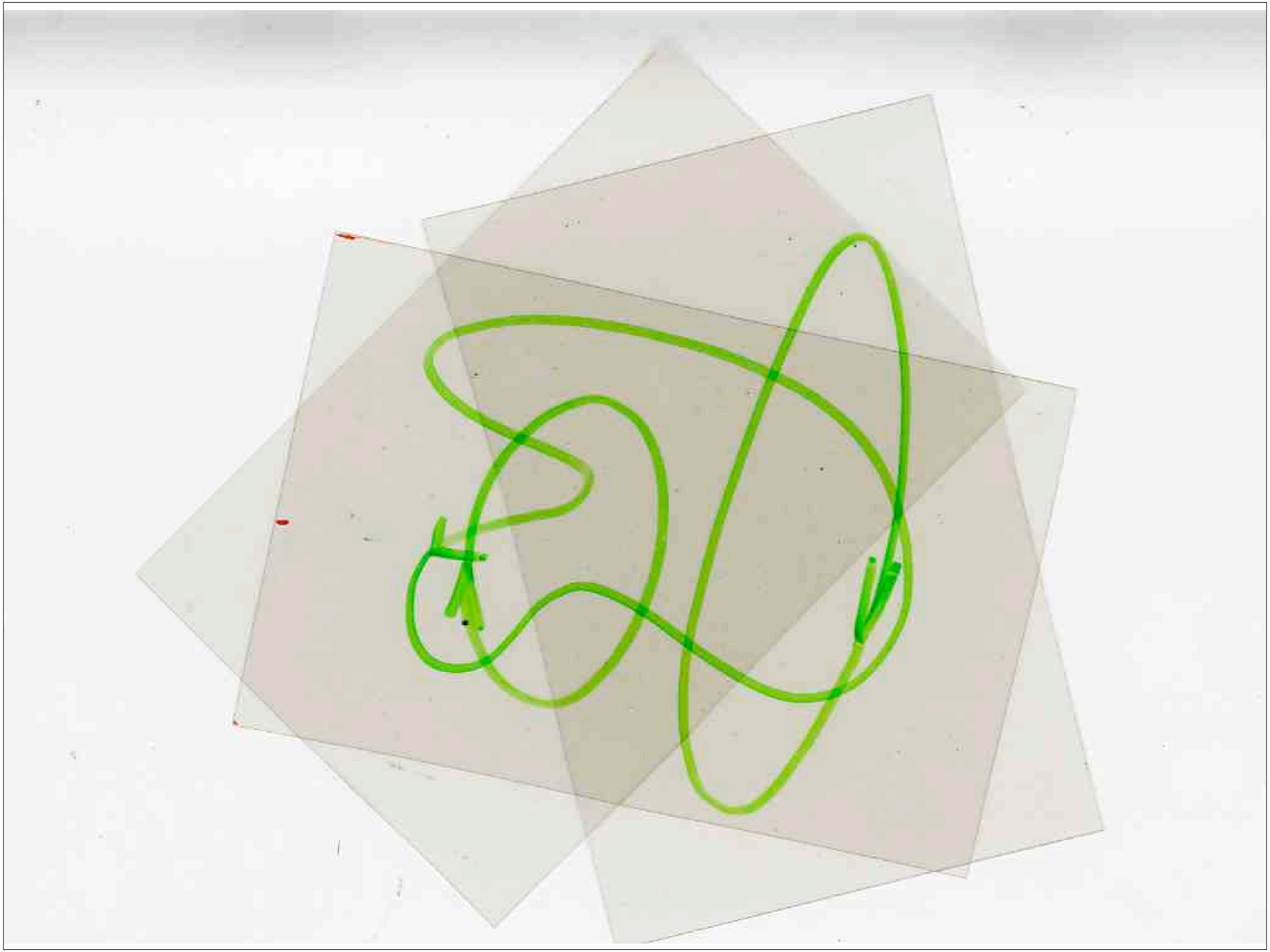
(of cycles)

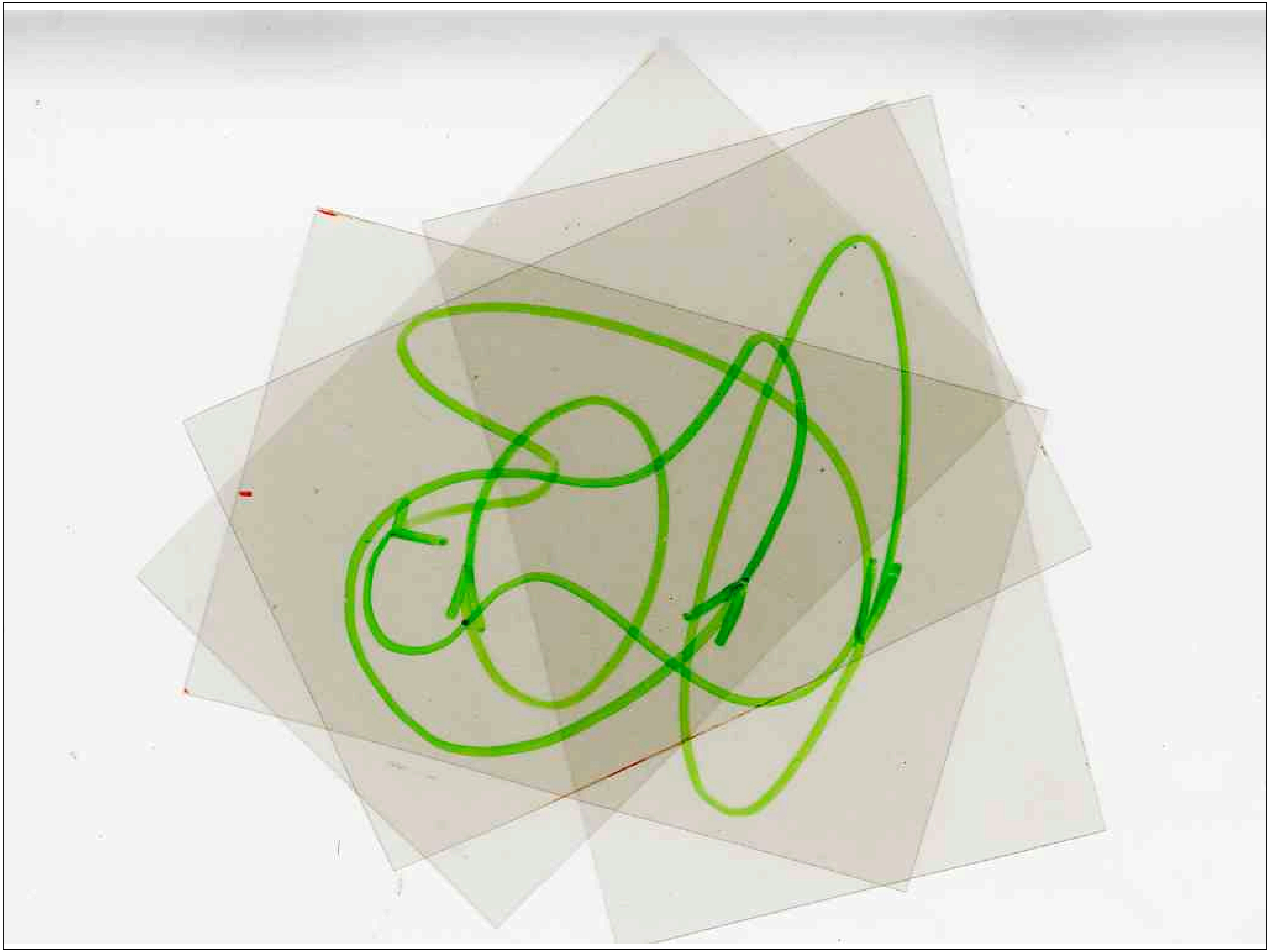
Proof:

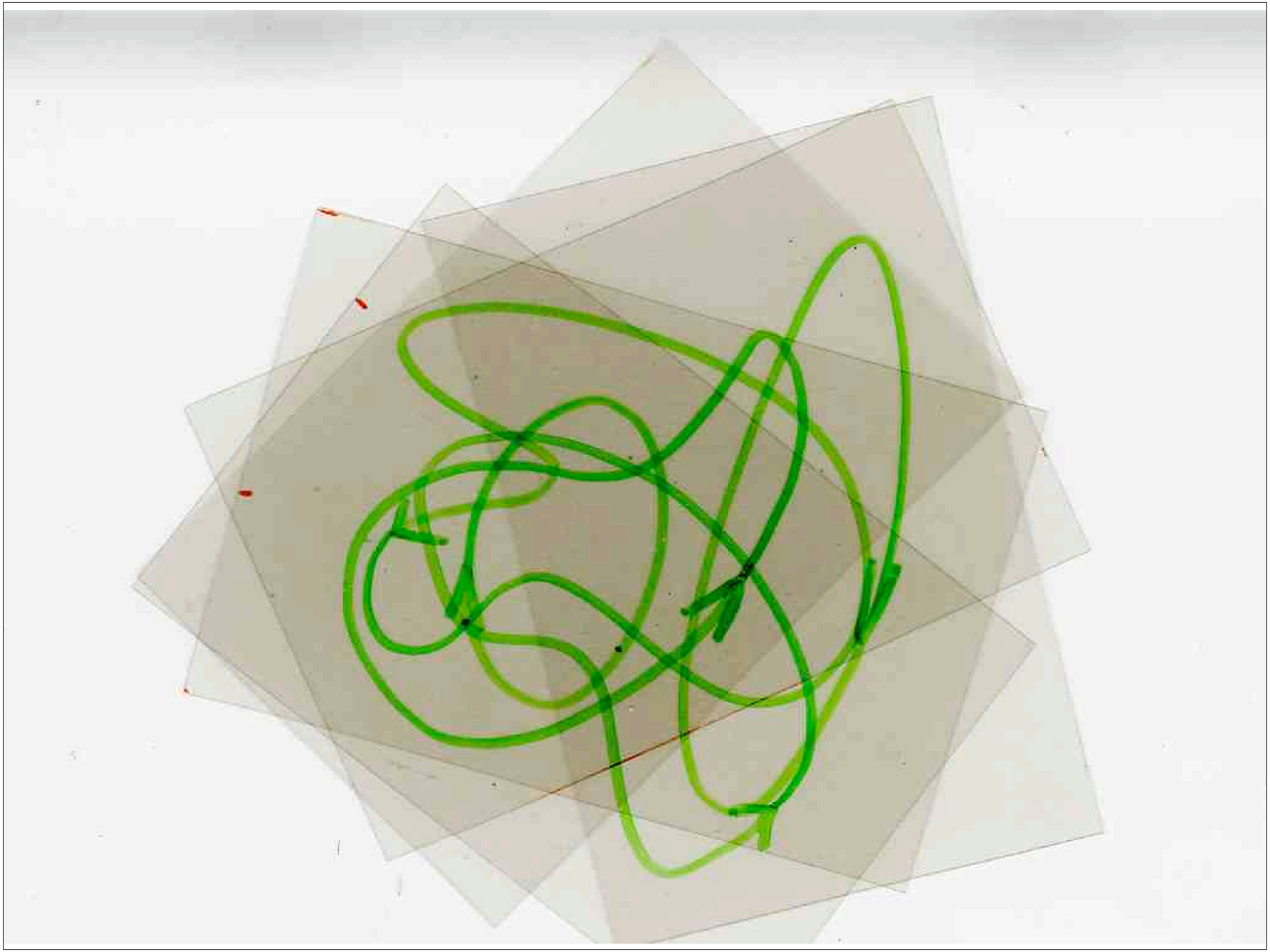


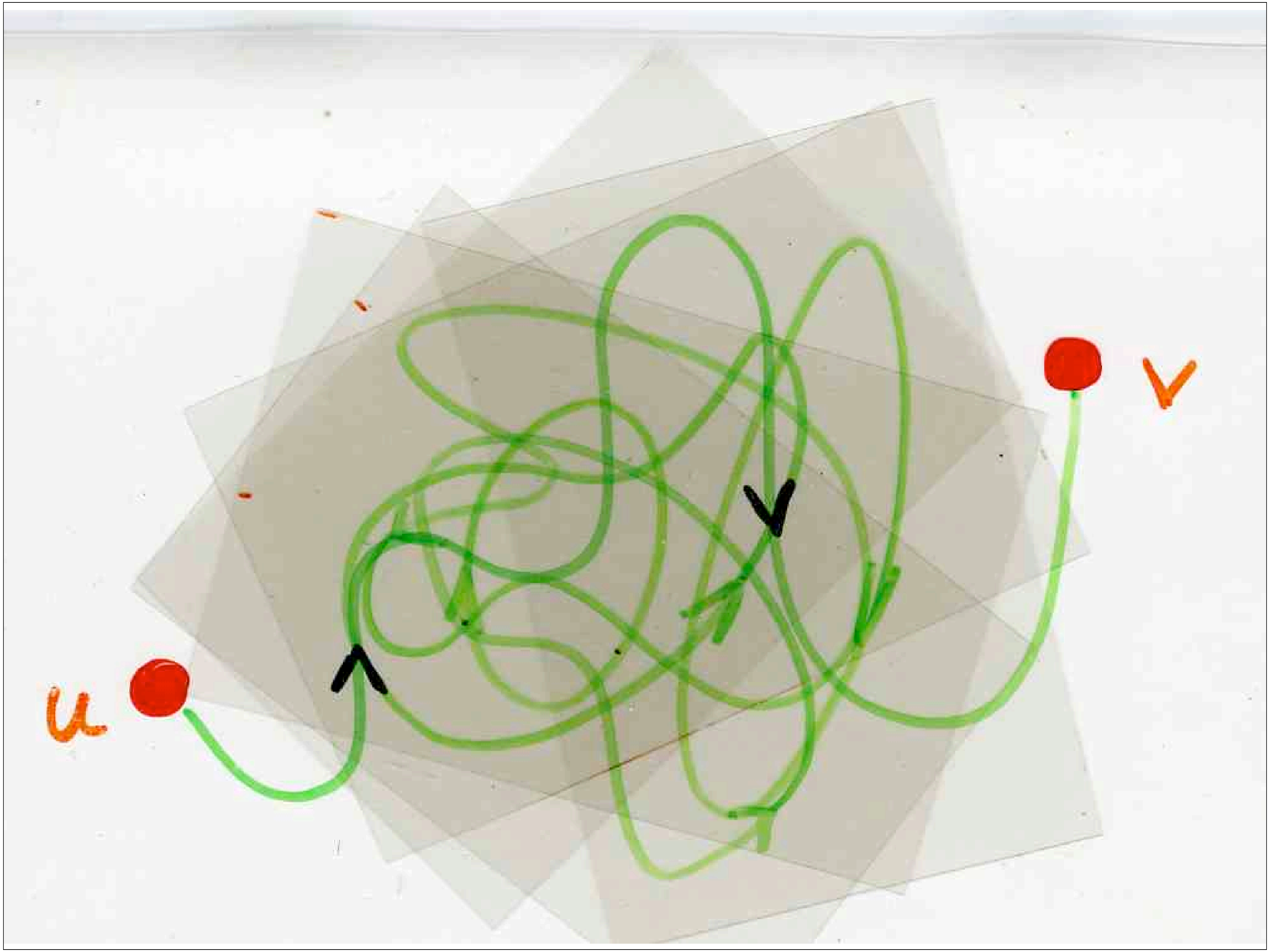


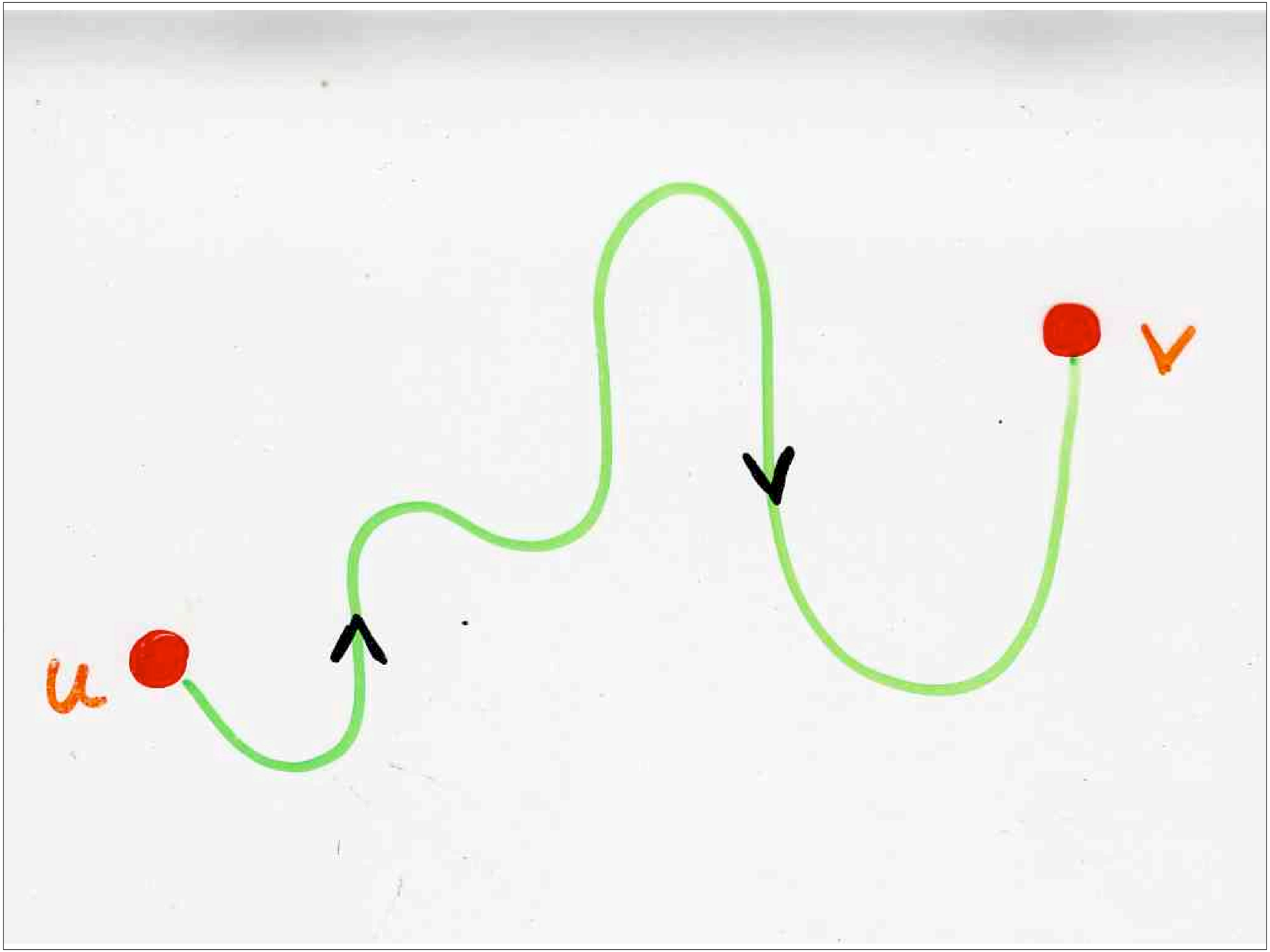









u
v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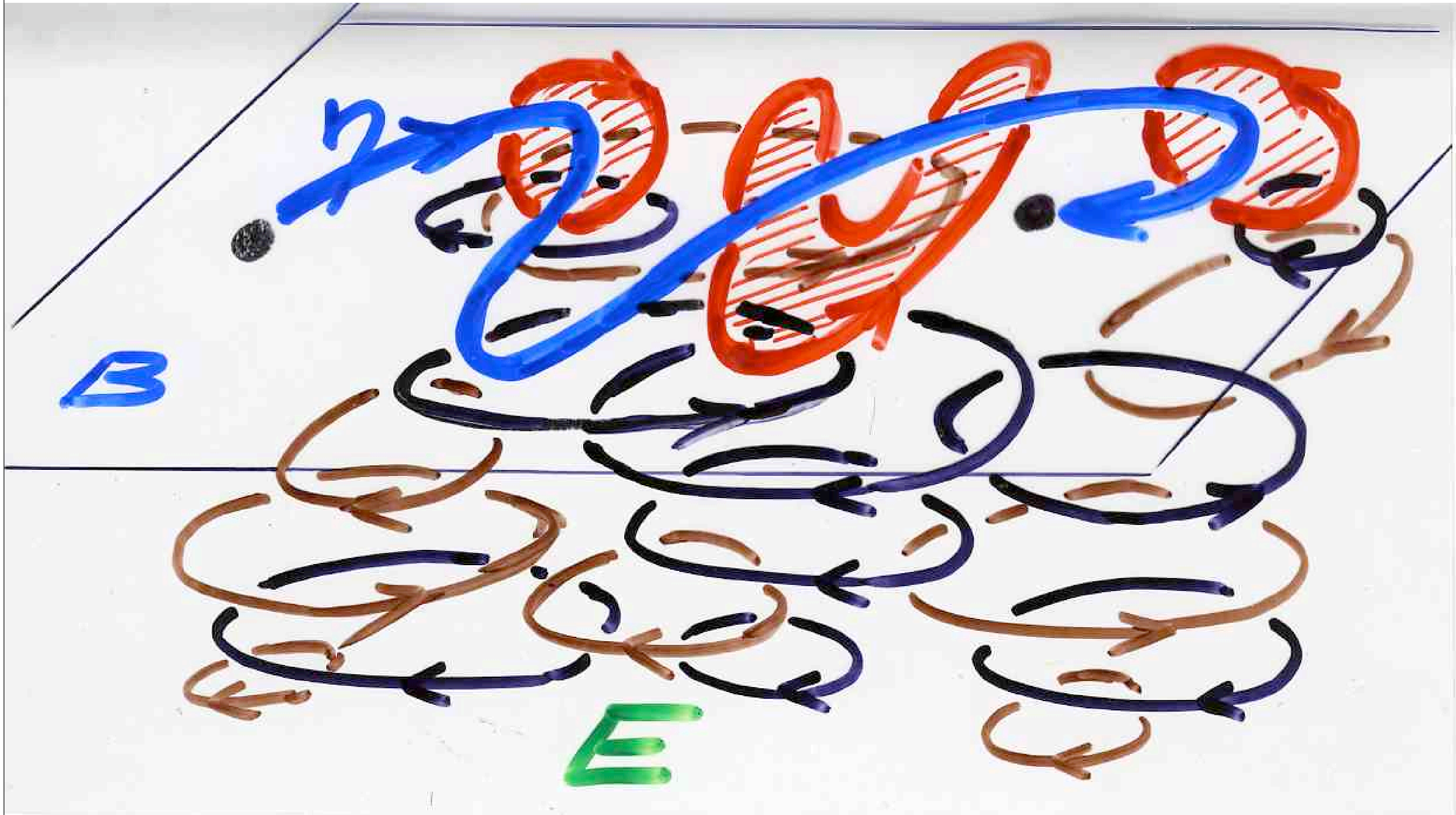

B
E

Bijection

$$\text{Paths}_{u \rightsquigarrow v}\ \omega \longrightarrow (\eta, E)$$

- η self-avoiding path going from u to v
- E heap of cycles, $\pi(\alpha)$, $\alpha \in \max(E)$ intersects η

$\omega = (s_0 = u, \ldots, s_n = v)$ path on B
$u \rightsquigarrow v$

$$\omega \longrightarrow (\eta\ ;\ \{\gamma_1, \ldots, \gamma_{r_n}\})$$

self-avoiding path $u \rightsquigarrow v$ ("coupe")

sequence of cycles

$$\omega \longrightarrow (\eta \; ; \; \{\dot{\gamma}_1, \ldots, \dot{\gamma}_{r_n}\})$$

coupe(ω) suite(ω)

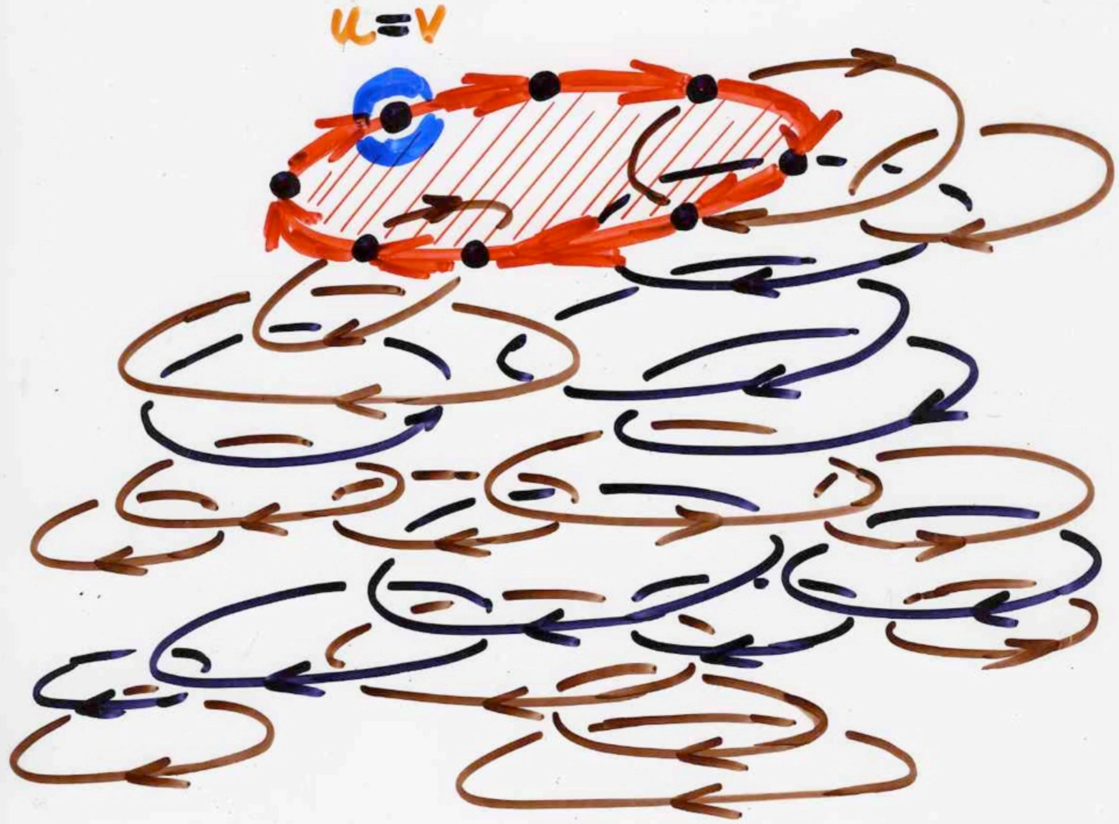
$$\downarrow$$

cycles heap

$$(\eta \; ; \; (\gamma_1 \bullet \cdots \bullet \gamma_{r_n}))$$

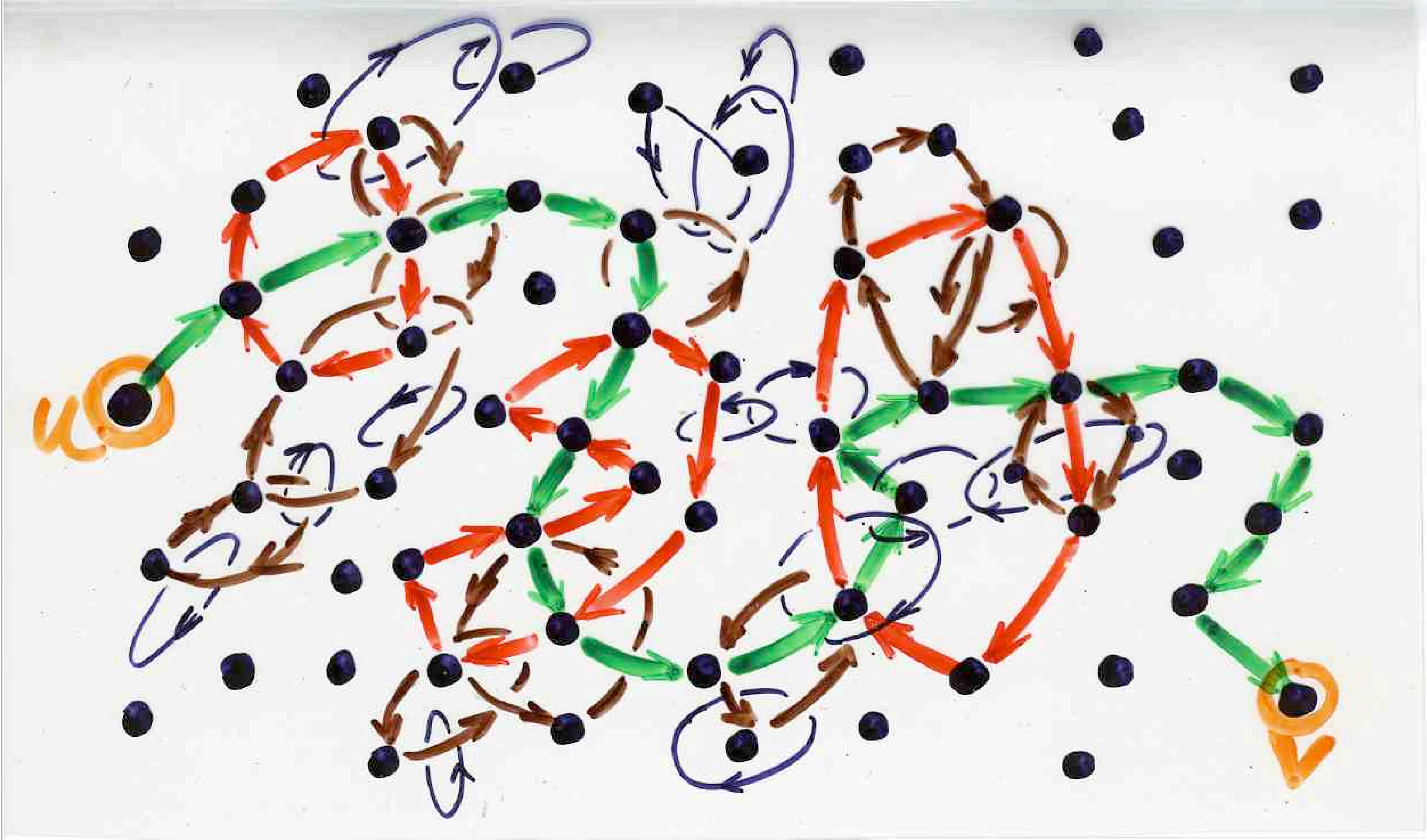
or pyramid $(\gamma_1 \bullet \ldots \bullet \gamma_{r_n} \bullet \eta) = \mathrm{Pyr}(\omega)$

$\omega \longrightarrow \mathrm{Pyr}(\omega)$ **bijection**



Rooted cycles pyramid

path -- heap of cycles:
inverse bijection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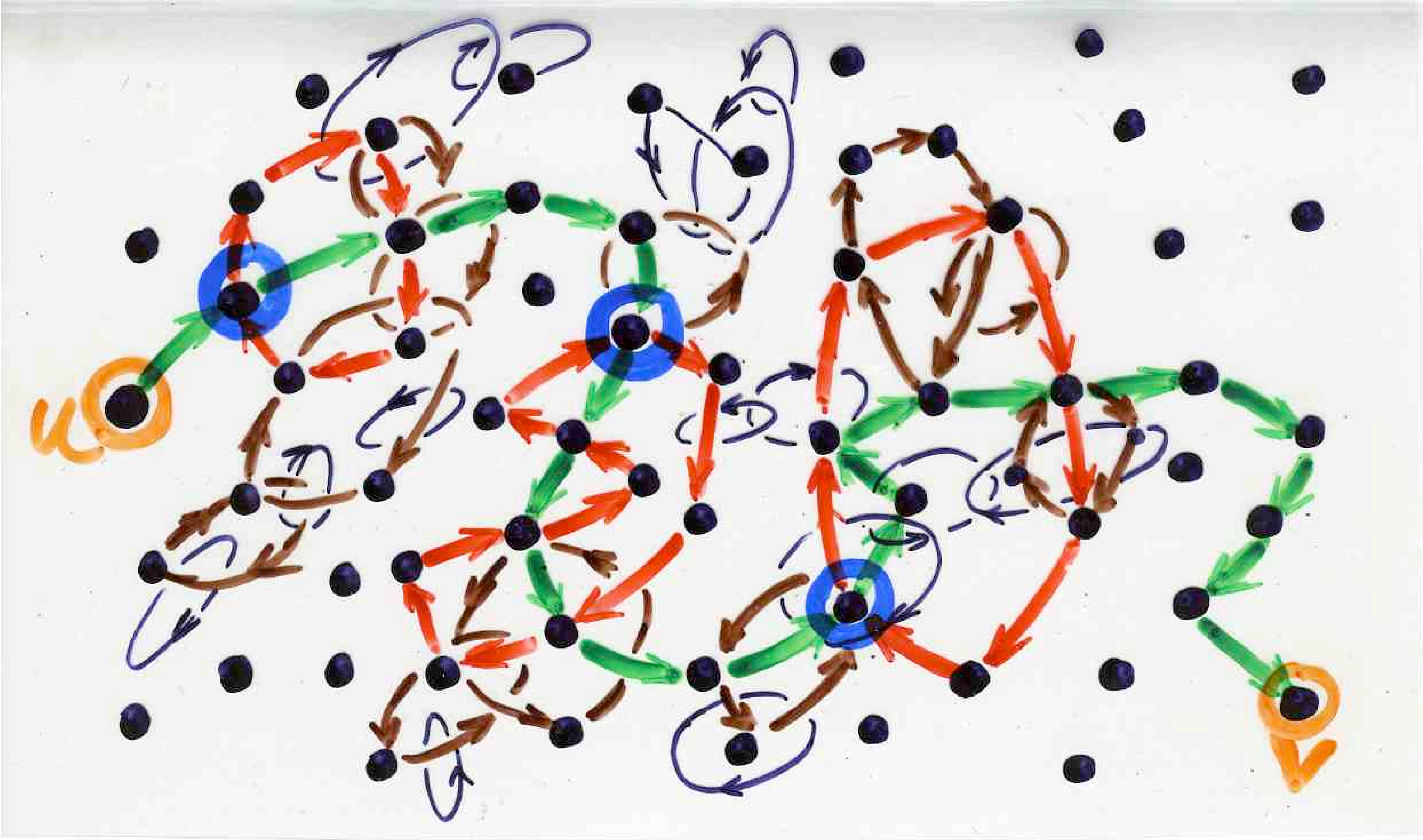
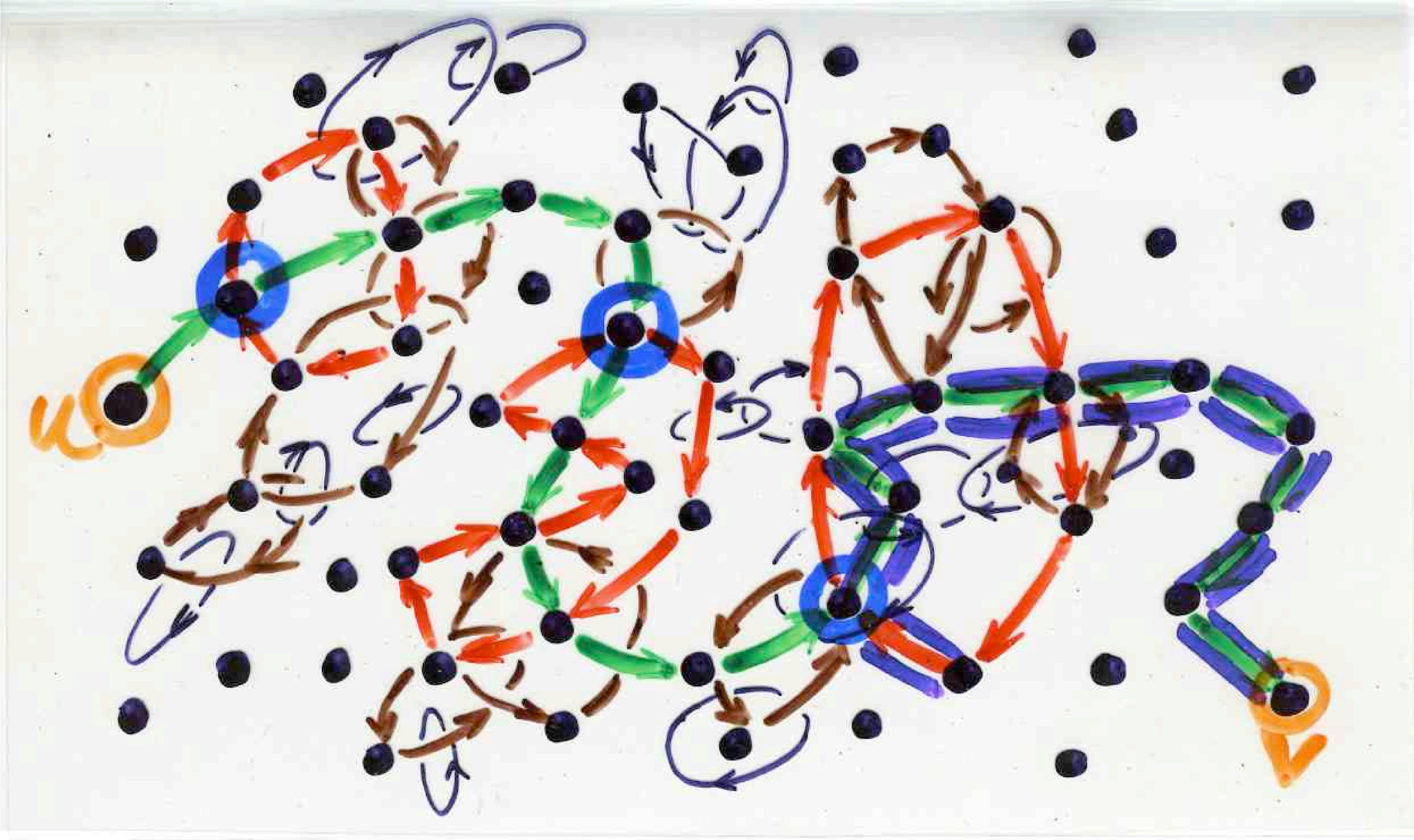
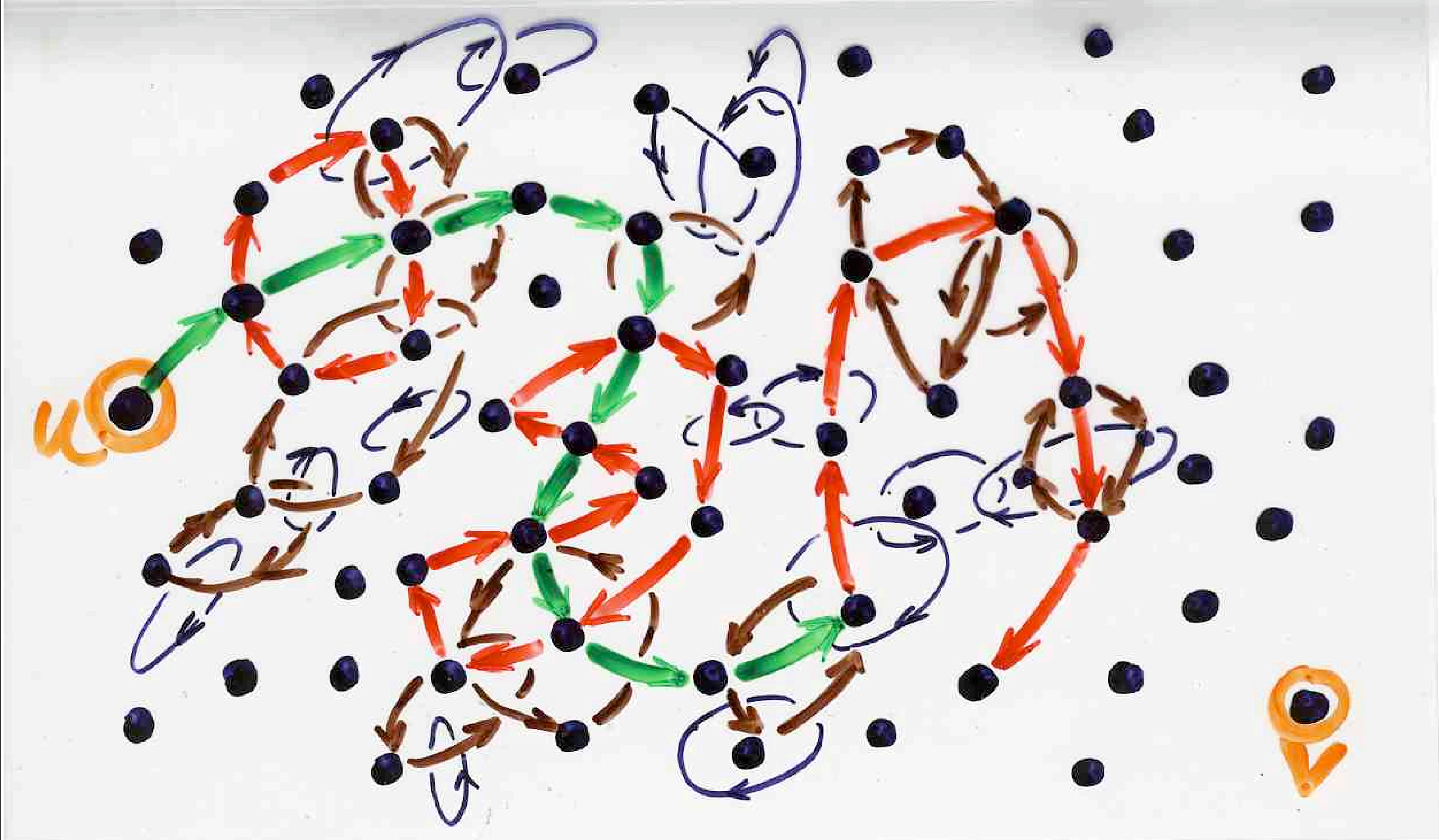
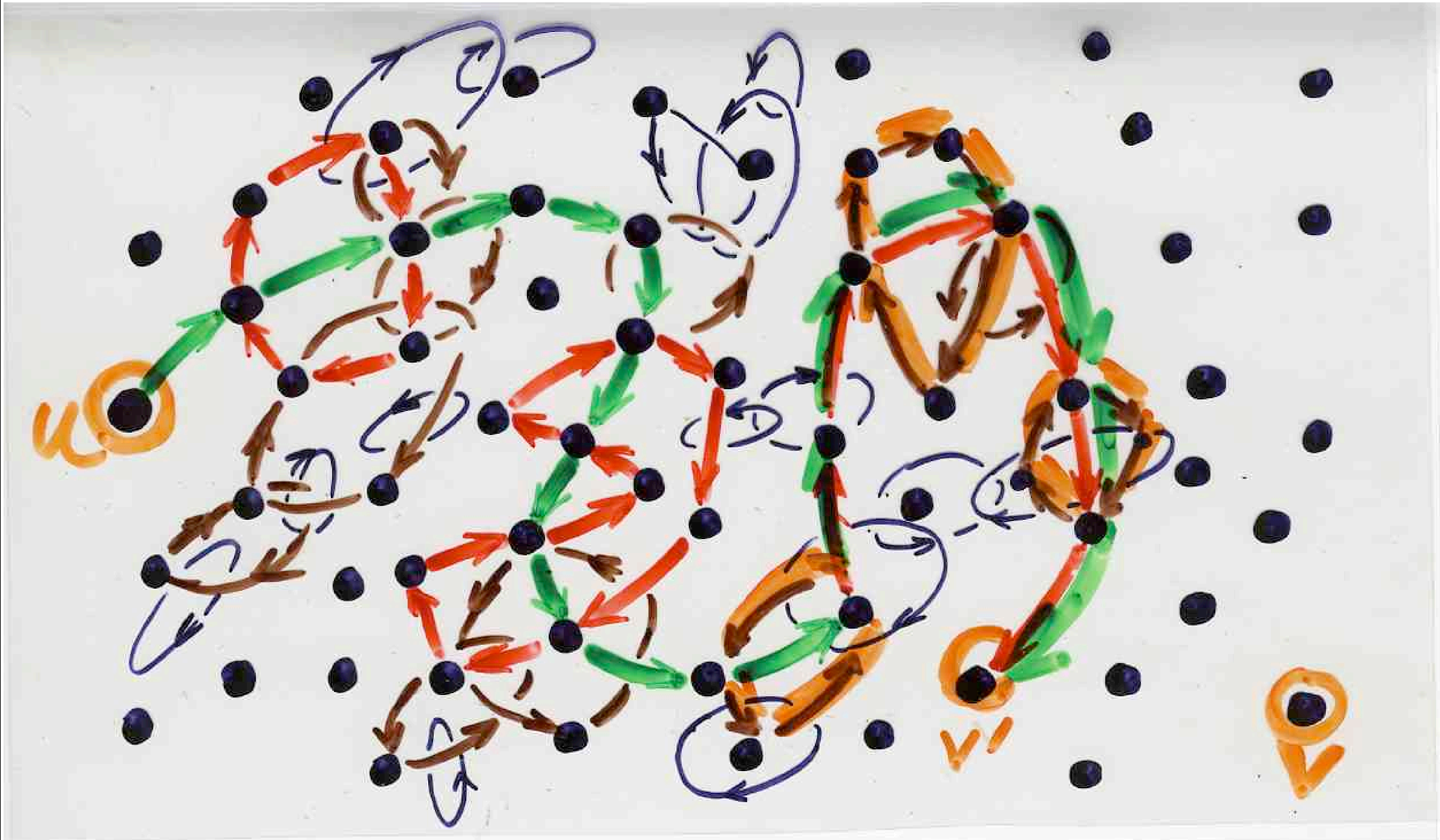

u
v


u
v

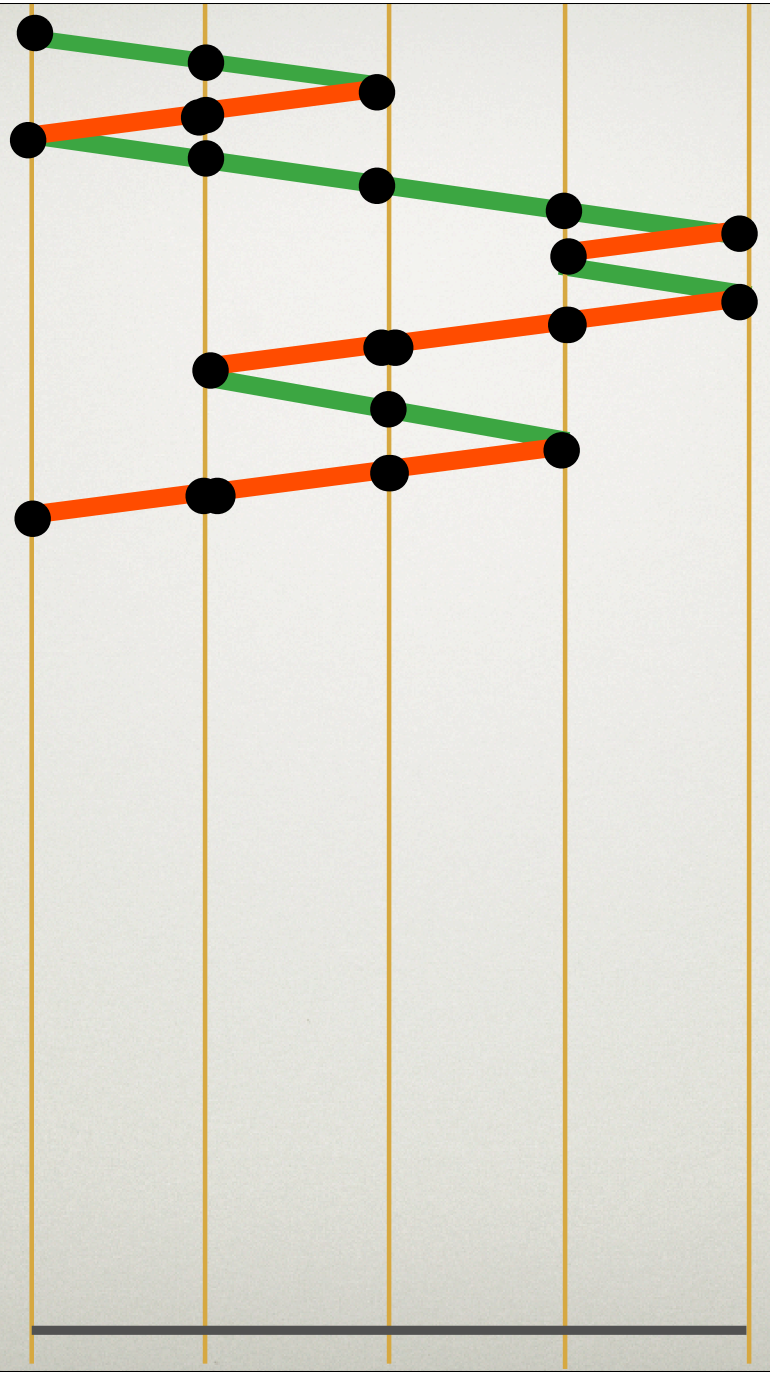

u
v

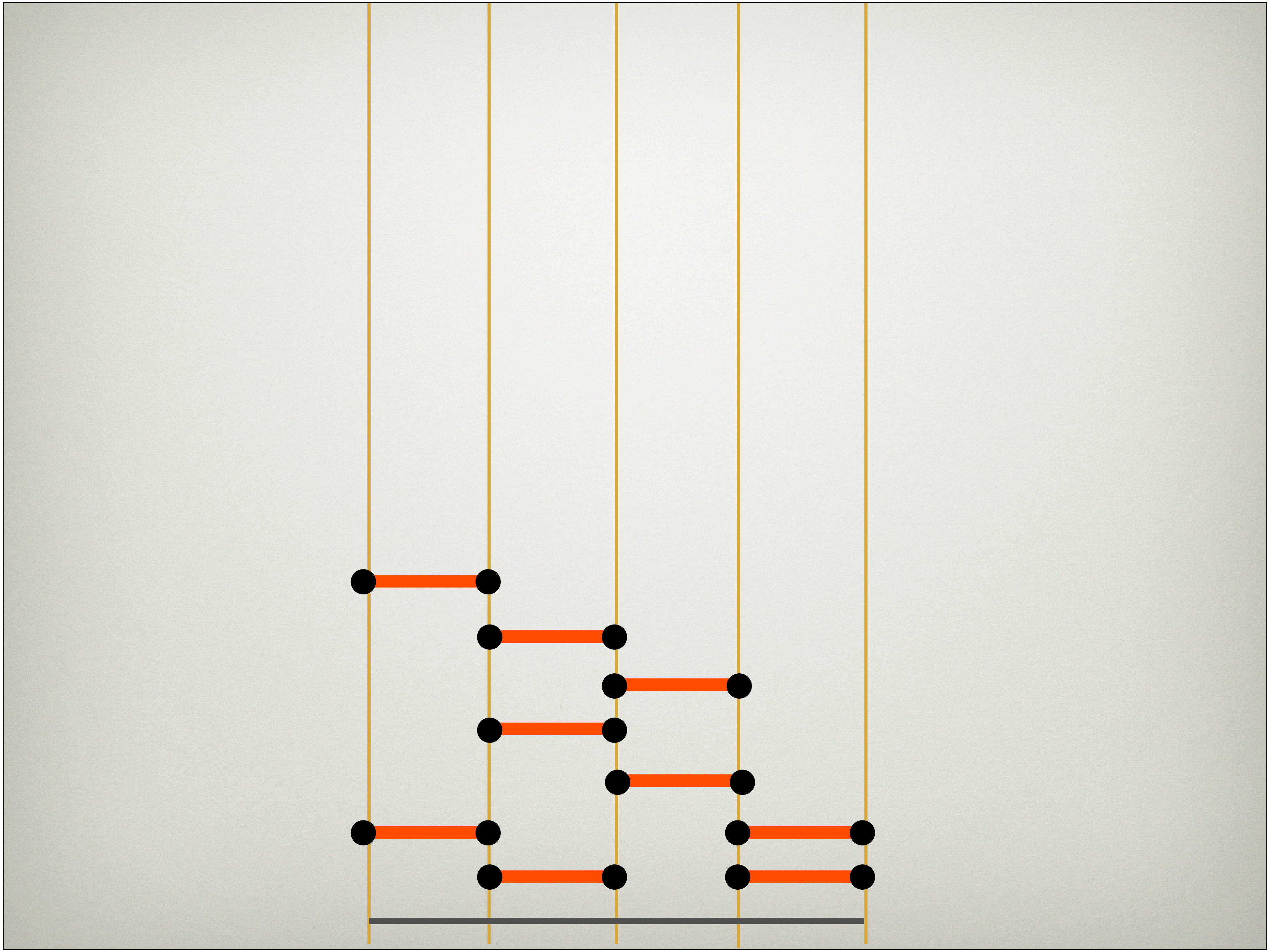

u
v'
v

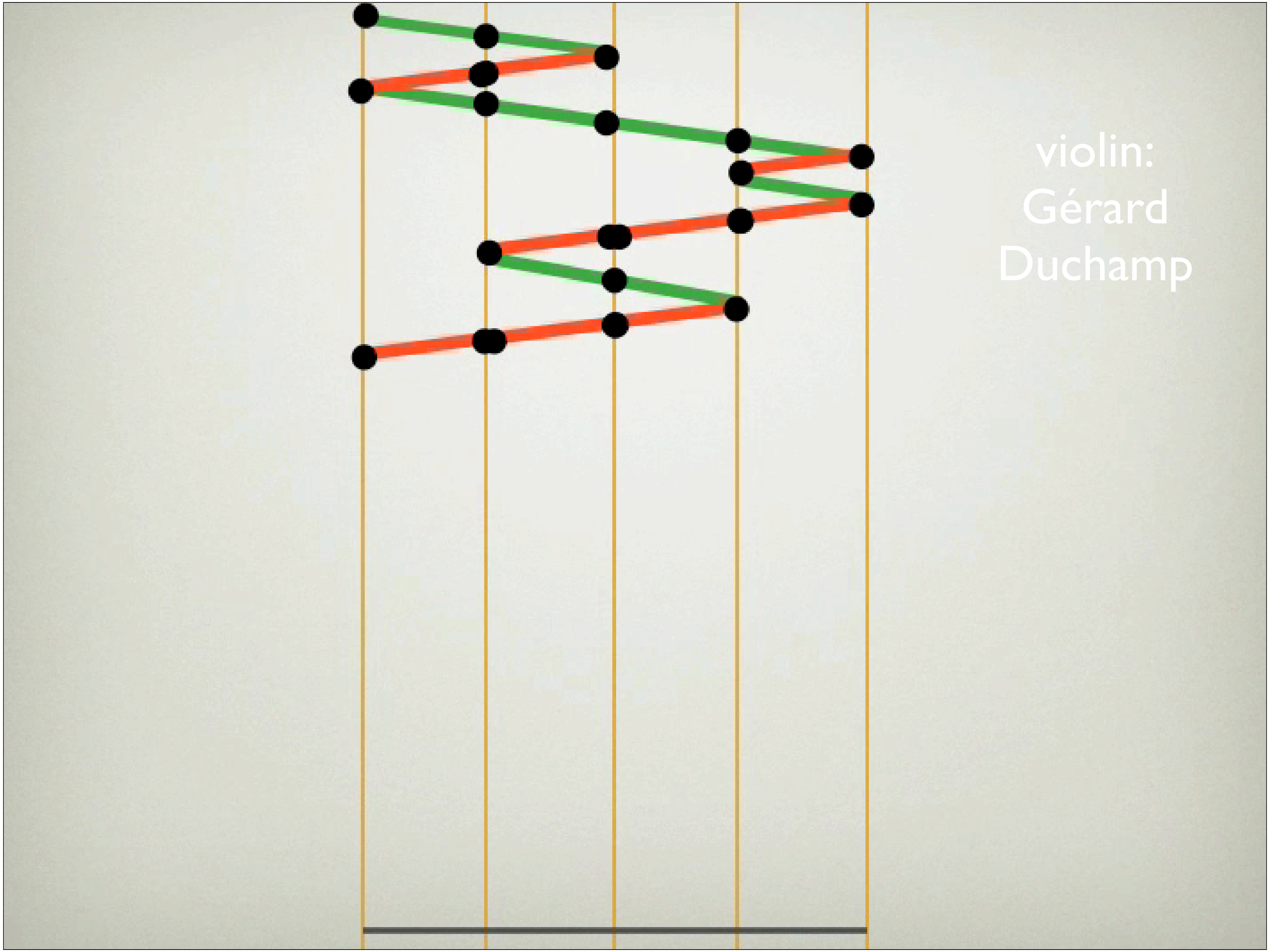
example: bijection

Dyck paths

semi-pyramid of dimers






violin:
Gérard
Duchamp

§6 Transition matrix

(Cramer's rule)

Path (or walk)

$$\omega = (s_0, s_1, \dots, s_n) \qquad s_i \in S$$

s_0 starting, s_n ending point

length n

(s_i, s_{i+1}) elementary step

valuation (weight)

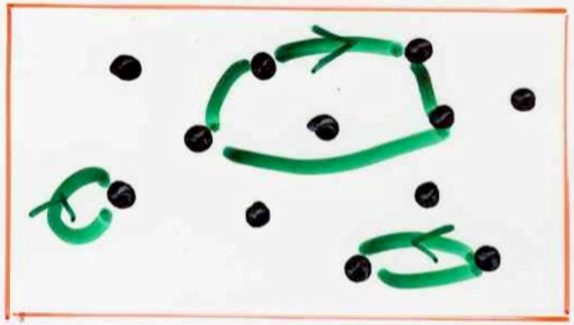
$$v(\omega) = \prod_{i=1}^{n} v(s_{i-1}, s_i)$$

$$v : S \times S \to \mathbb{K}[x]$$

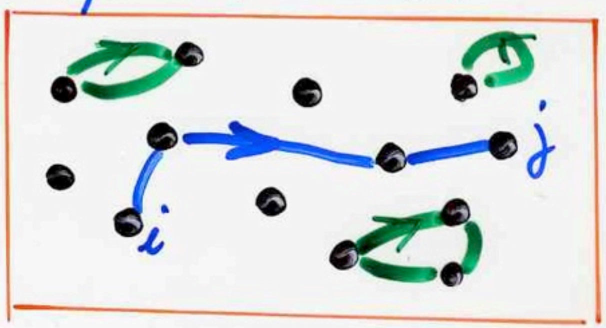
Prop. $$\sum_{\substack{\omega \\ i \rightsquigarrow j}} v(\omega) = \frac{N_{ij}}{D}$$

$$D = \sum_{\{\gamma_1, \dots, \gamma_r\}} (-1)^r \, v(\gamma_1) \cdots v(\gamma_r)$$

$\{\gamma_1, \dots, \gamma_r\}$ 2 by 2 disjoint cycles



$$N_{ij} = \sum_{\{\eta;\, \gamma_1, \dots, \gamma_r\}} (-1)^r \, v(\eta)\, v(\gamma_1) \cdots v(\gamma_r)$$

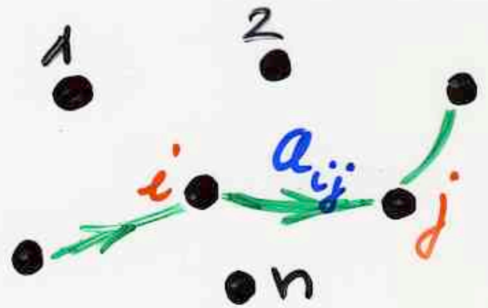


$$(I_n - A)^{-1} = \frac{\mathrm{cof}_{ji}\,(I_n - A)}{\det\,(I_n - A)}$$

ij

$$I_n + A + A^2 + \dots + A^n + \dots$$

$$A = (a_{ij})$$



Abdesselam, Brydges (2006)

loop ensembles — Cramer's rule

Mayer expansion

interactions with physics

- loop-erased walks

- the directed animal model
  and hard particule gas model

- q-Bessel functions in statistical physics:
  staircase polygons, SOS model, ...

- Lorentzian triangulations in 2D quantum gravity

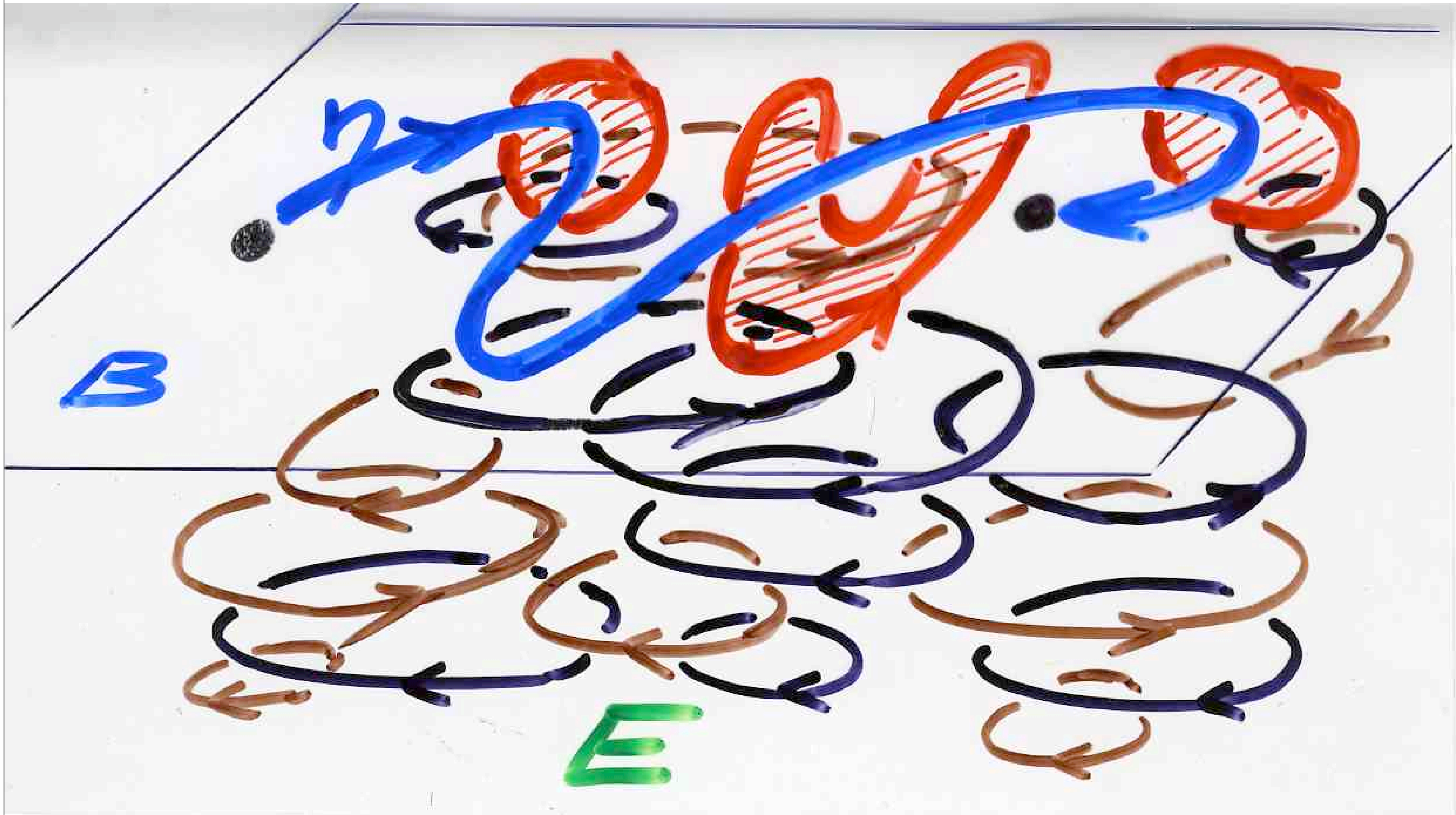
§7 Bauer identity

Bauer identity

path ω $\xrightarrow{\text{erasing loops}}$ path η

$u \rightsquigarrow v$

$$v(\eta) = \sum_{\omega \to \eta} v(\omega)$$



B
E

Prop. Bauer (2007)

$\eta = (s_0 = u, s_1, \dots, s_n = v)$

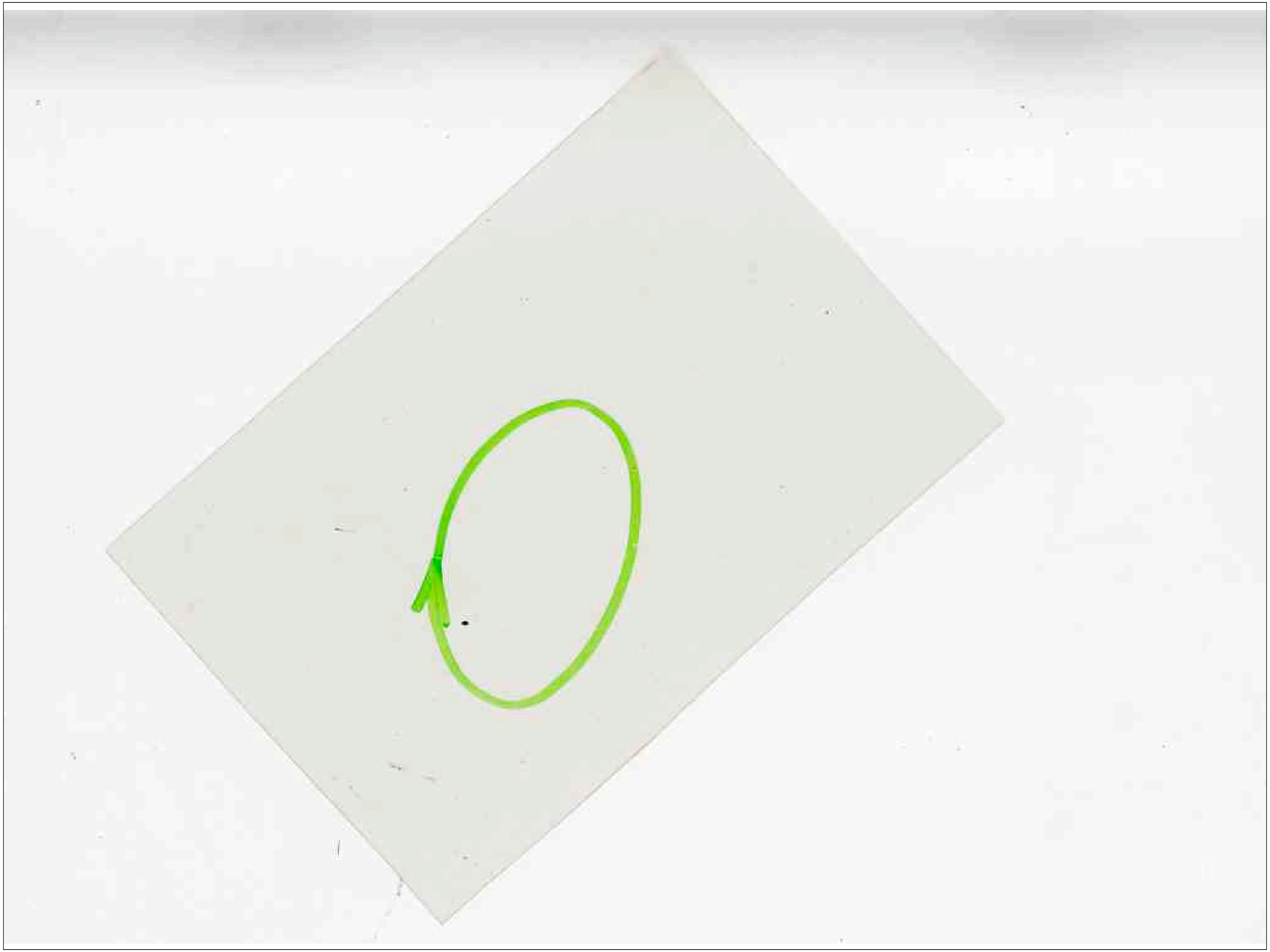
$$\nu(\eta) = \frac{1}{\det(I - K_{ij})}$$

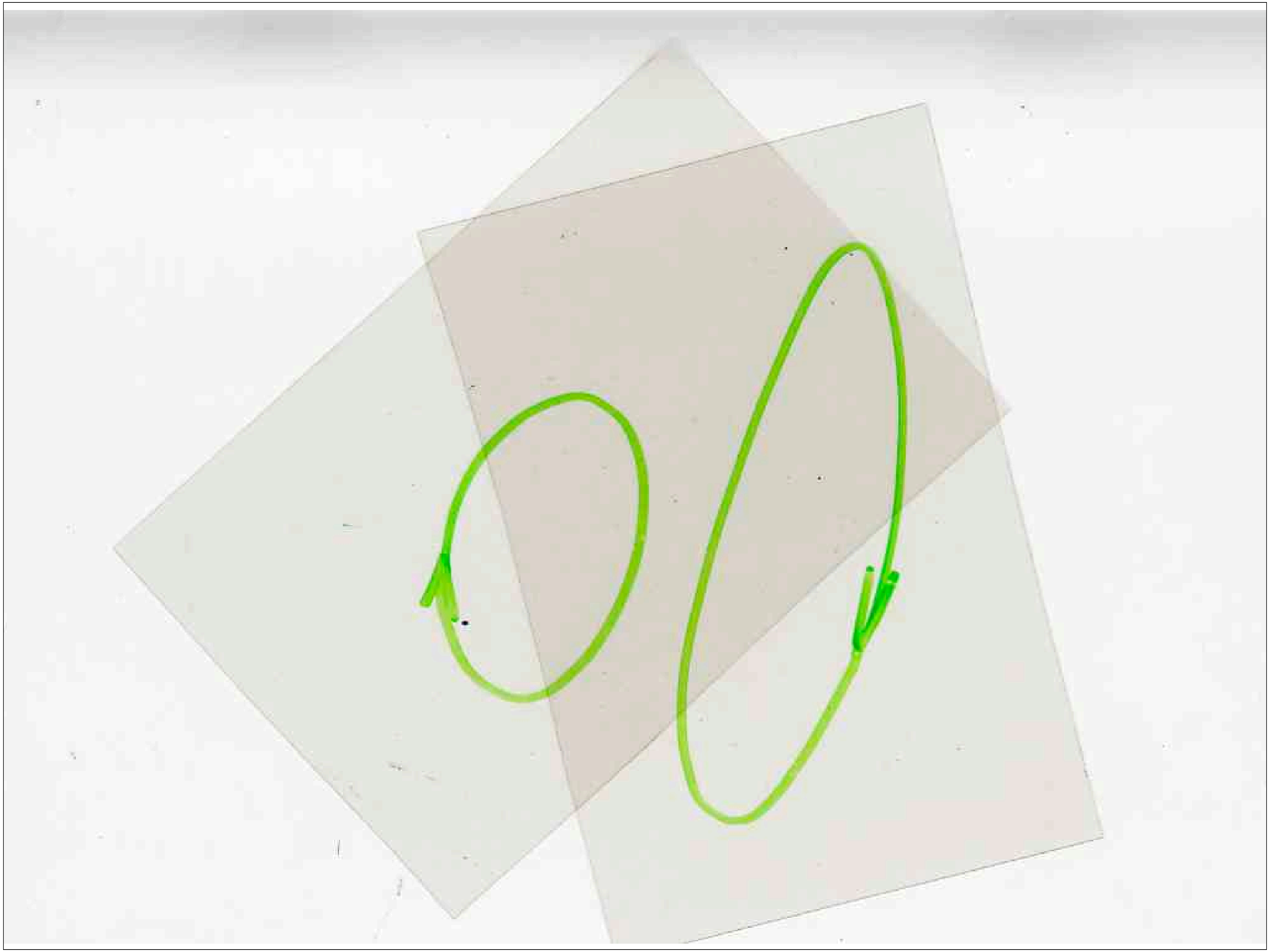
$$K_{ij} = \sum_{\substack{\omega \\ s_i \rightsquigarrow s_j \\ \text{avoiding } \eta}} \nu(\omega)$$

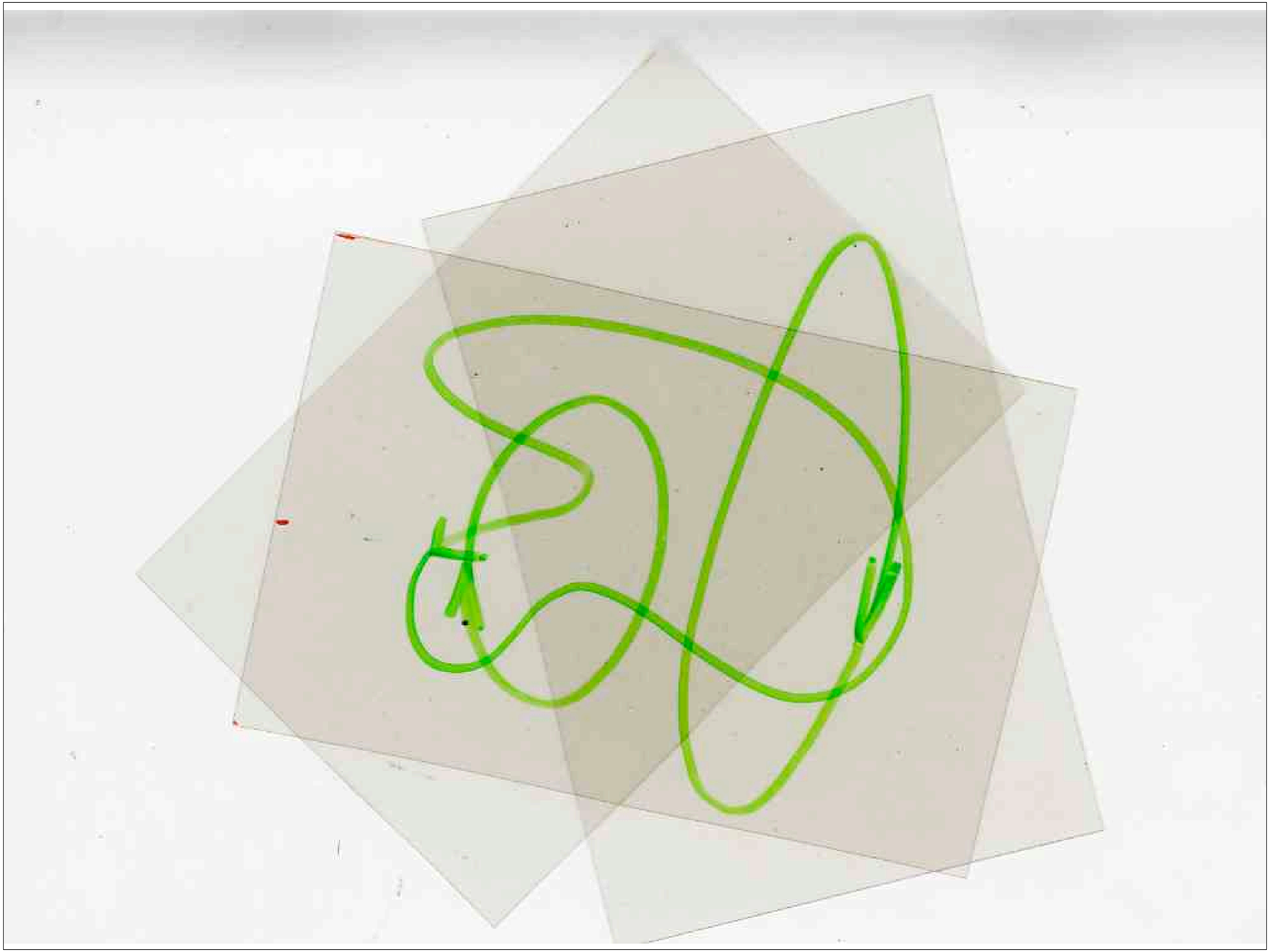
$0 \le i, j \le n$

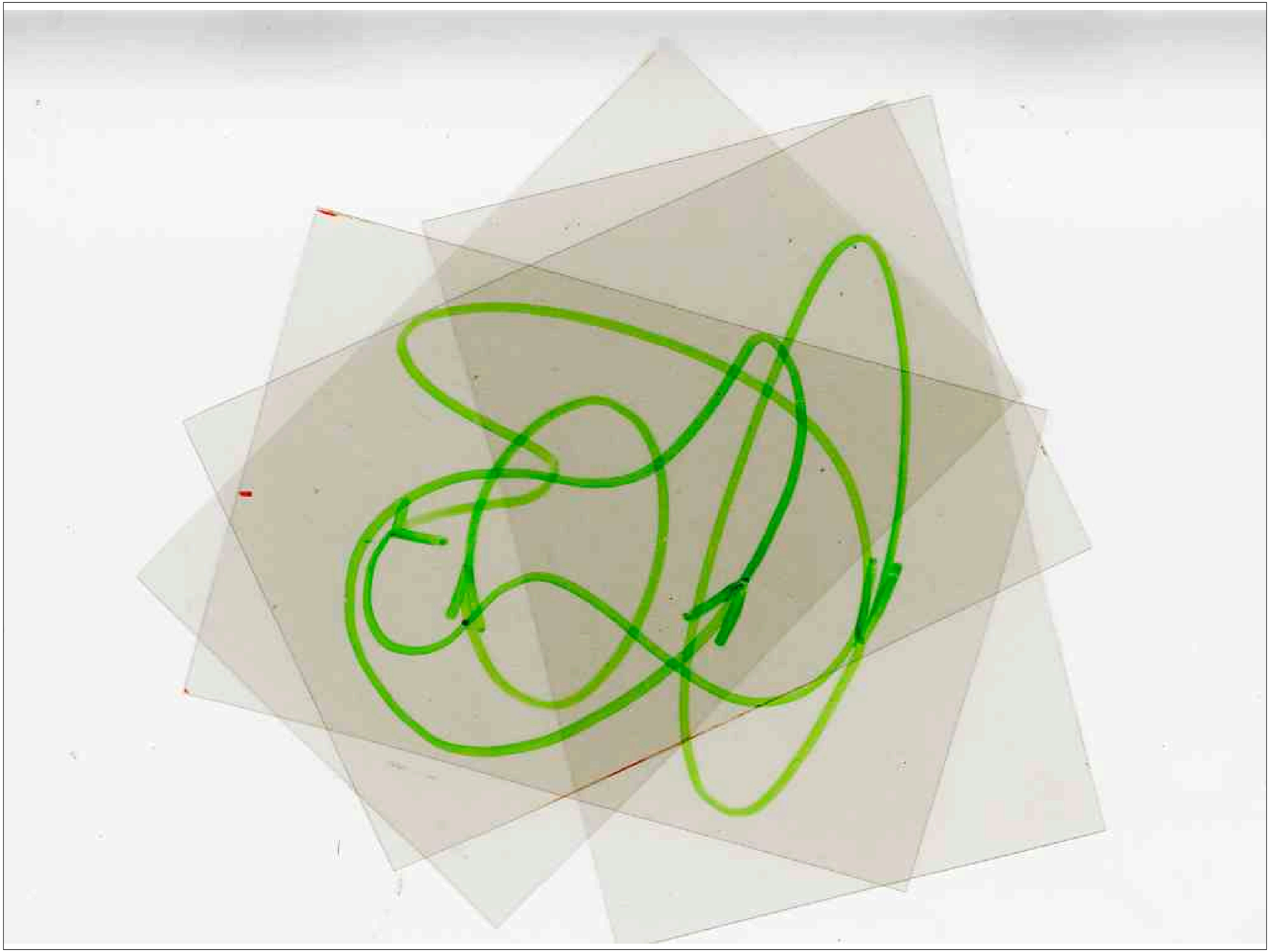
MacMahon master theorem

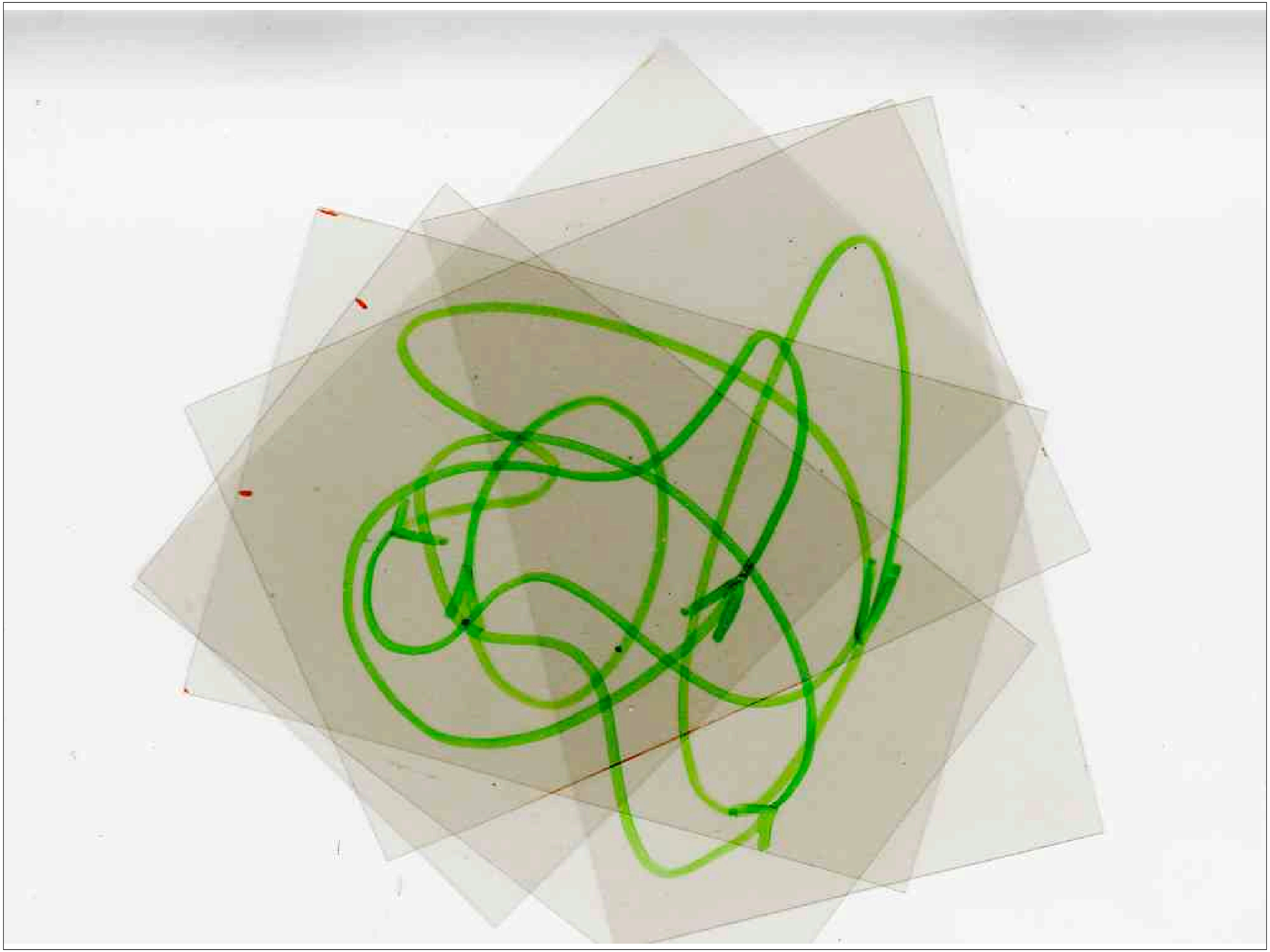
$$\frac{1}{\det(I - A)}$$









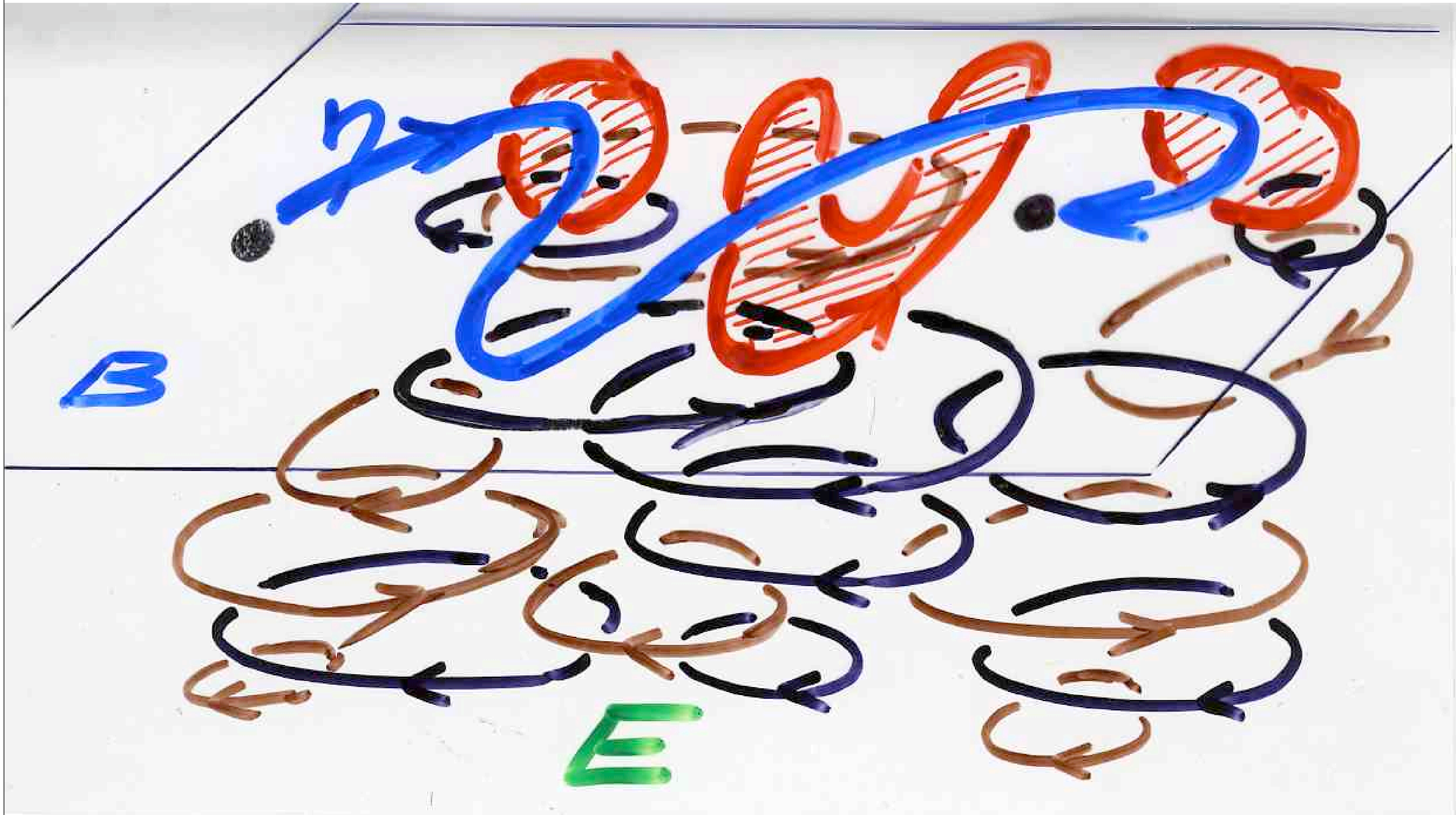


Prop. Bauer (2007) $\quad \eta = (s_0 = u, s_1, \dots, s_n = v)$

$$\nu(\eta) = \frac{1}{\det(I - K_{ij})}$$

$$K_{ij} = \sum_{\substack{\omega \\ s_i \rightsquigarrow s_j \\ \text{avoiding } \eta}} \nu(\omega)$$

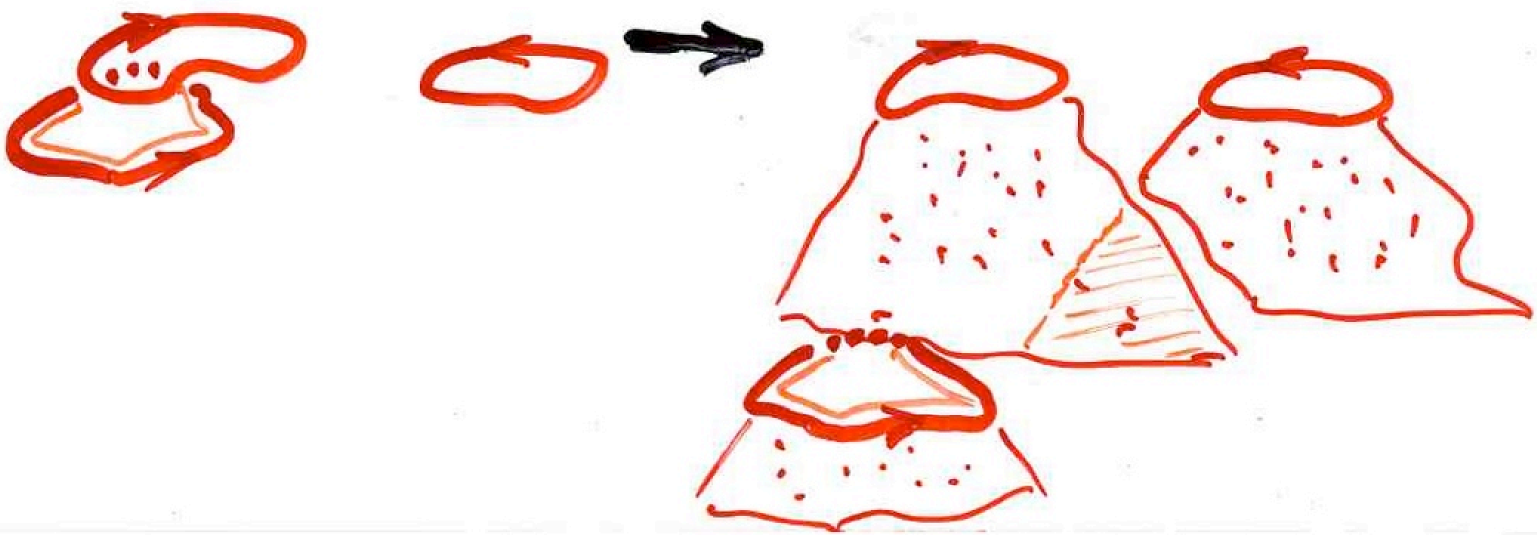
$0 \leq i, j \leq n$

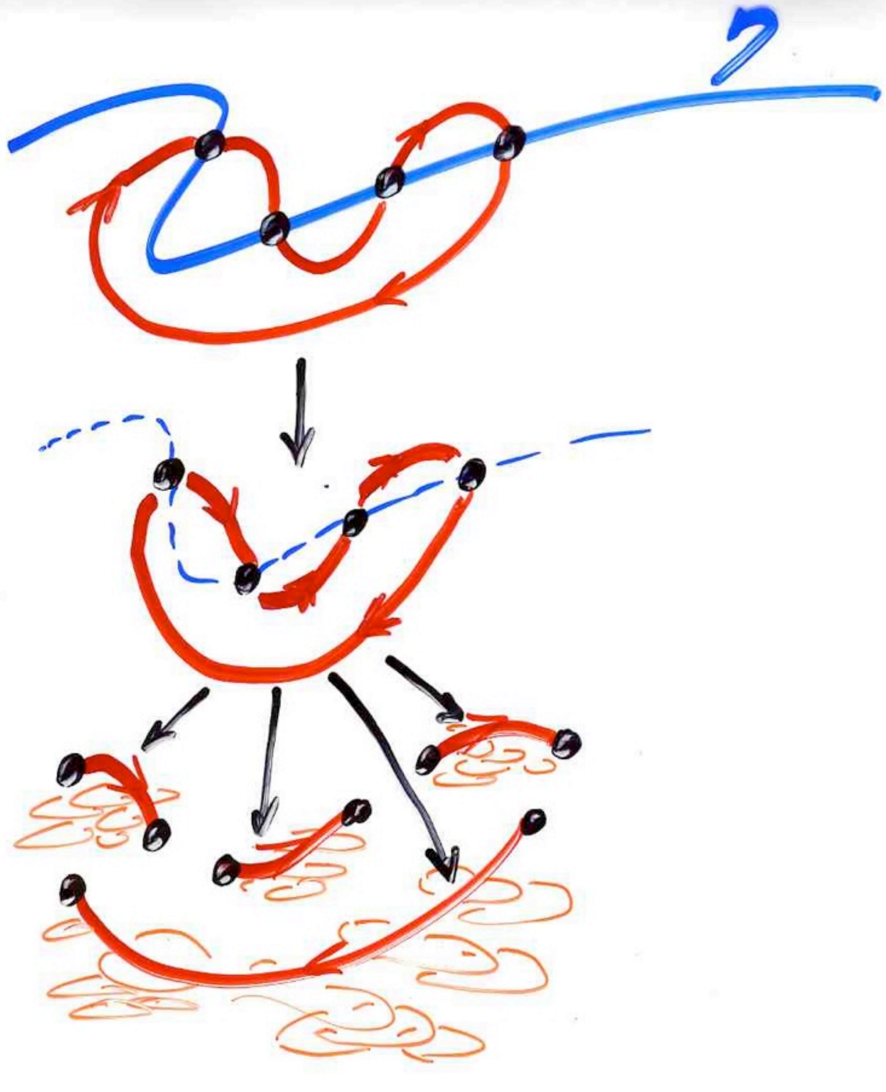


B
E

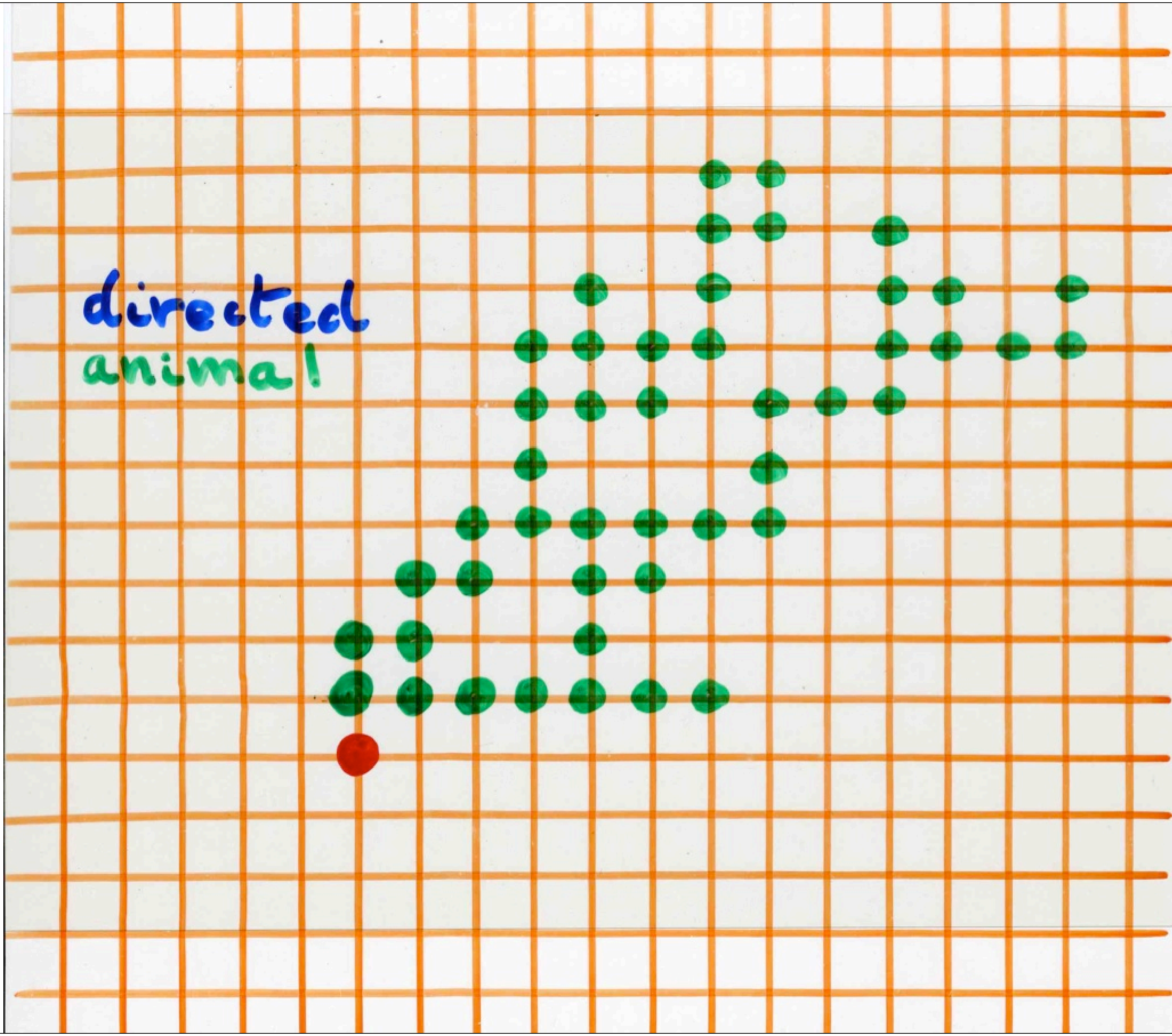
"substitution"

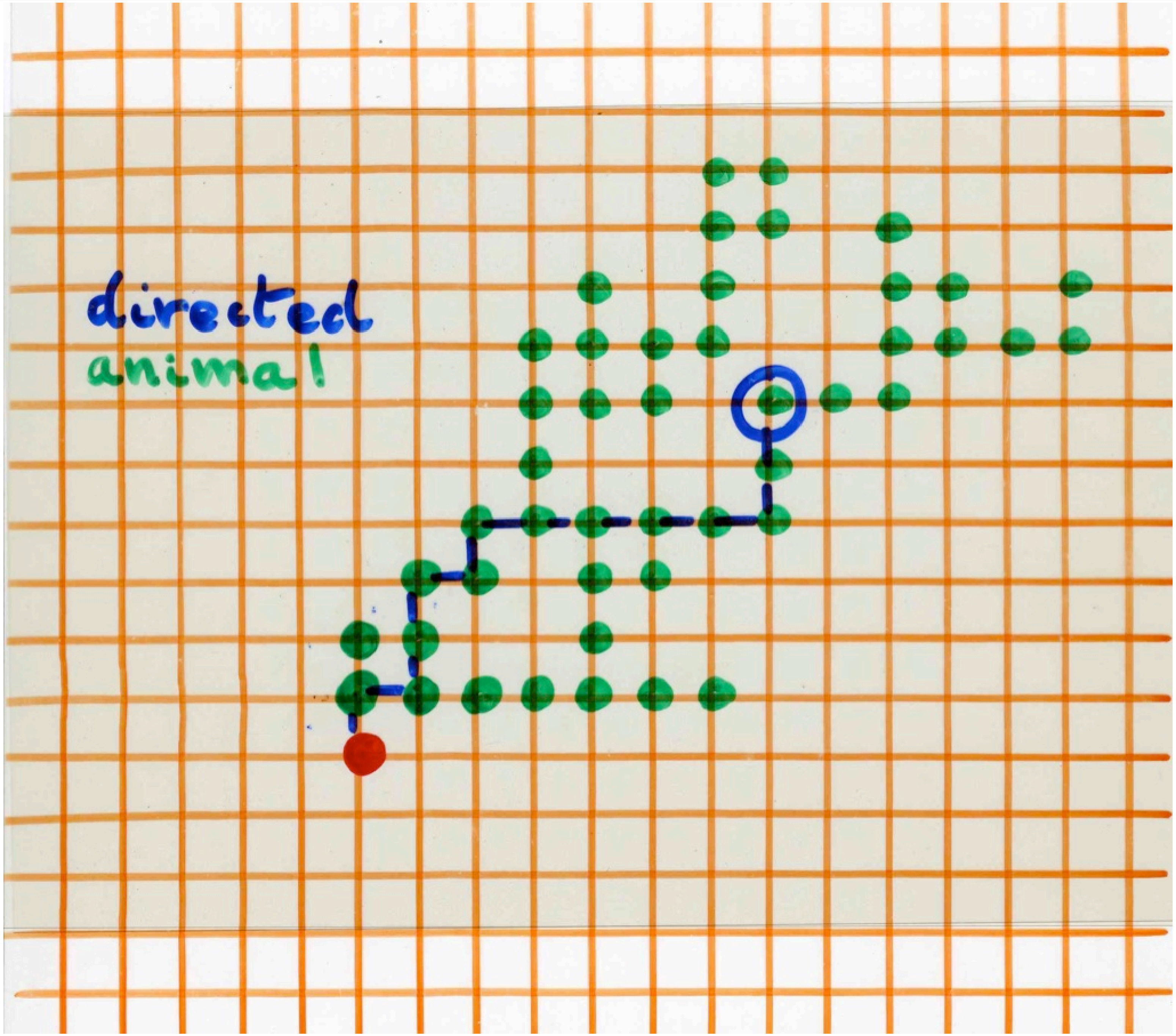
in heaps of pieces





§ 8 The directed animal model


directed
animal


directed
animal

$$a_n \sim \mu^n \, n^{-\theta}$$

number of directed animals n points

$$\ell_n \sim n^{\nu_\perp}$$

average width

$$L_n \sim n^{\nu_\parallel}$$

average length

$$a_n \sim \mu^n \, n^{-\theta}$$

number of directed animals n points

$$\ell_n \sim n^{\nu_\perp}$$

average width

$$L_n \sim n^{\nu_\parallel}$$

average length

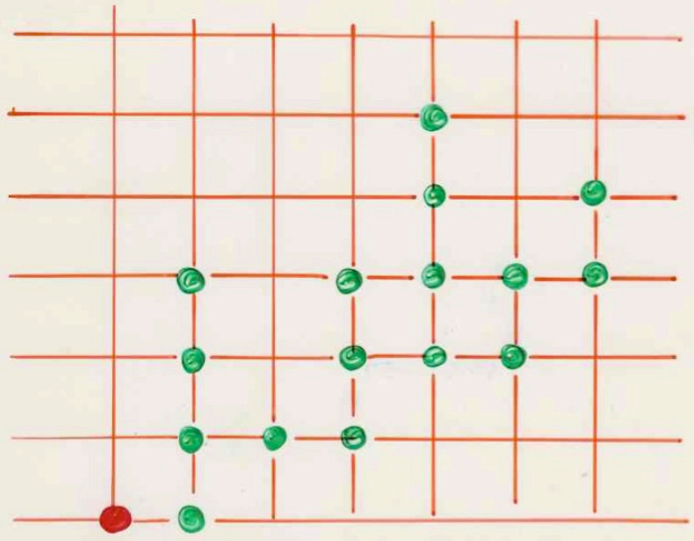
Critical exponents

$$a_n \sim \mu^n \, n^{-\theta}$$

$$\mu = 3 \qquad \theta = 1/2$$

$$\nu_\perp = 1/2$$

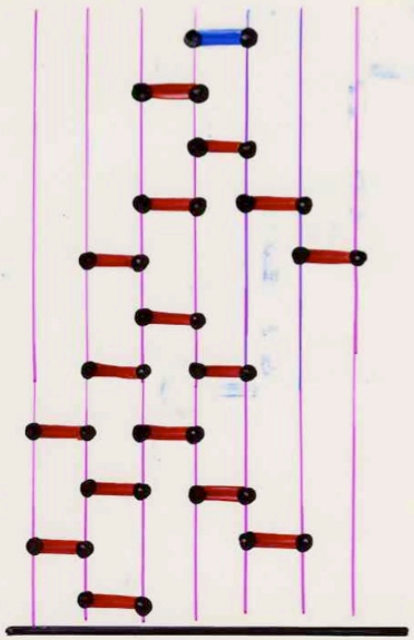
$$\nu_\parallel = \cancel{9/11} \; ?$$



animal

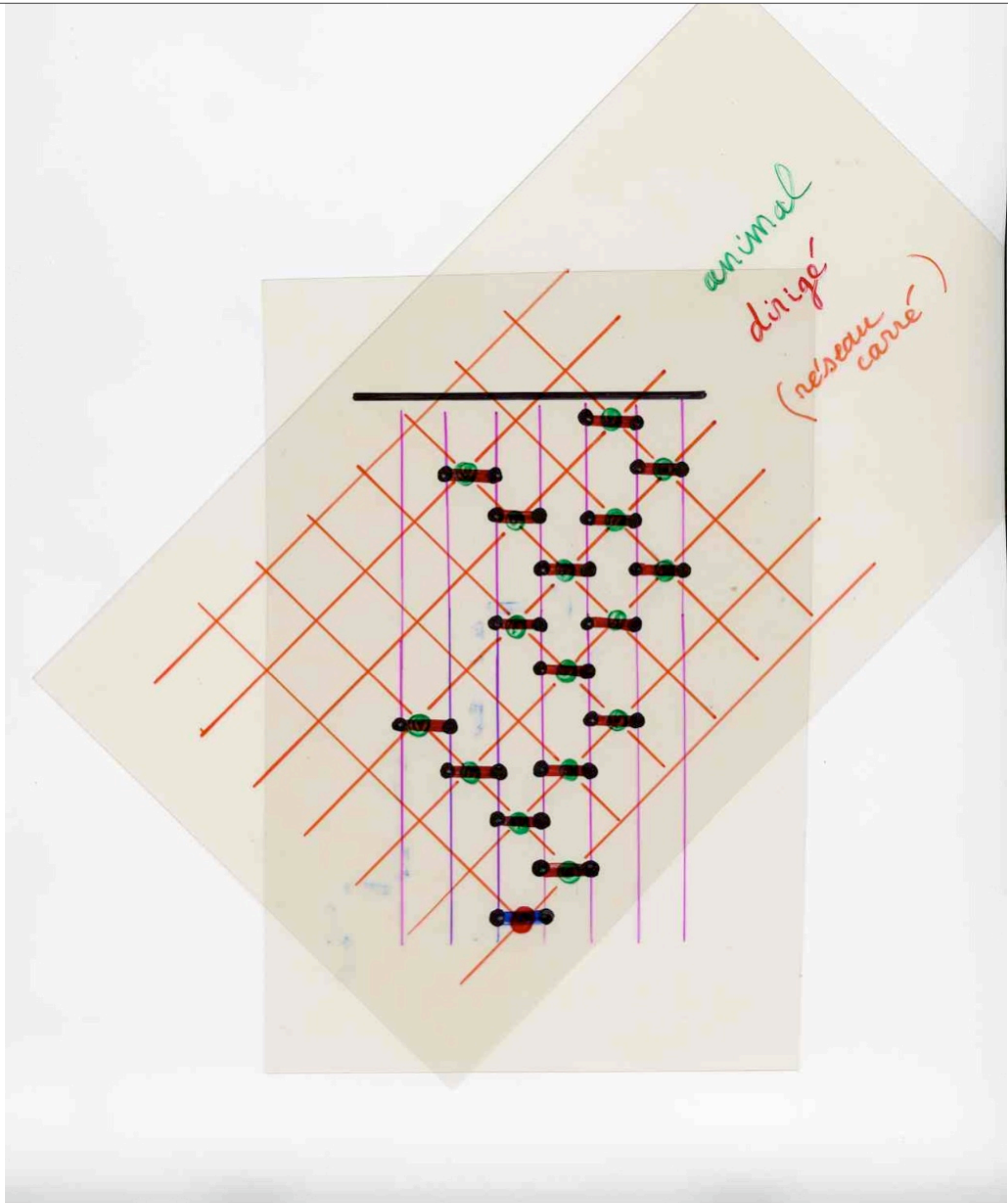
dirigé

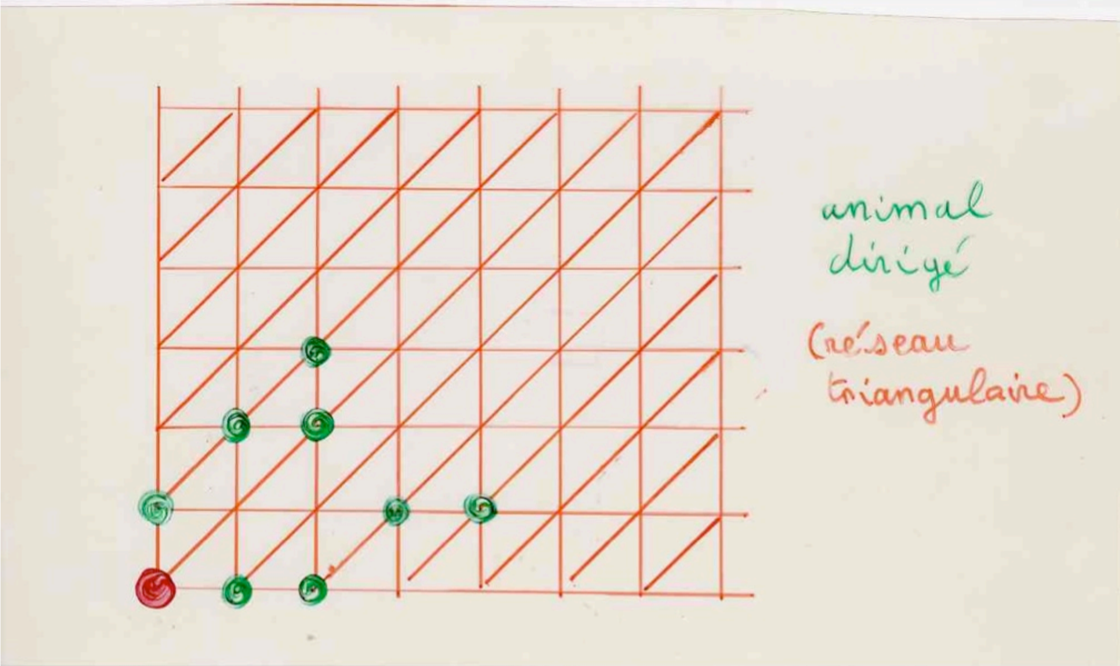
(réseau carré)

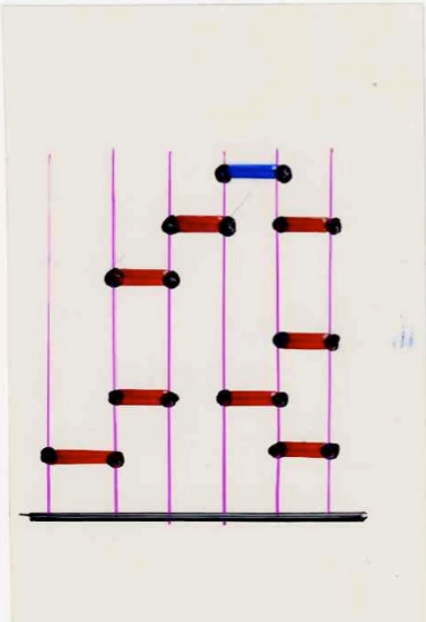


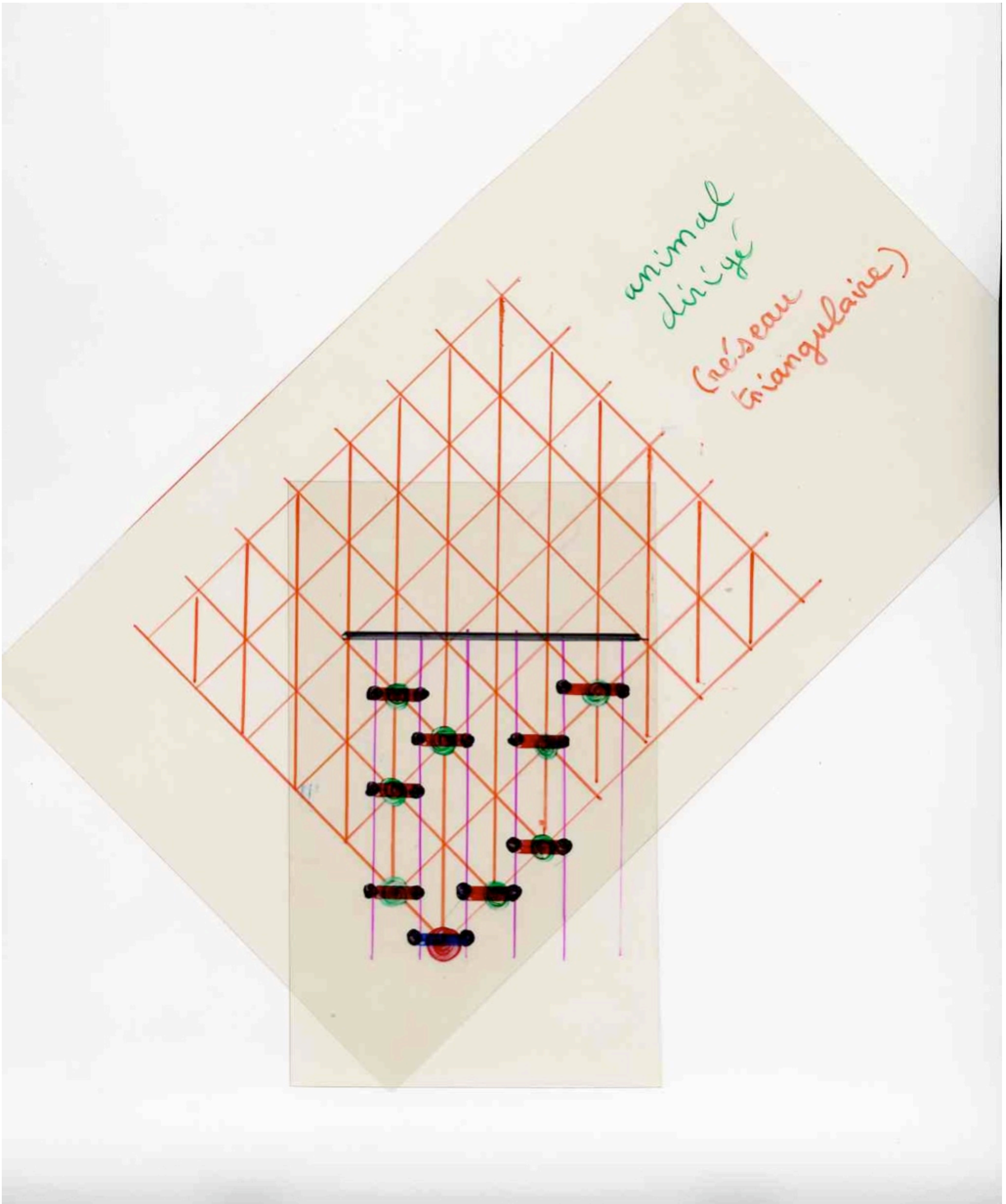
strict

heap


animal
dirigé
(réseau carré)


animal
dirigé
(réseau
triangulaire)

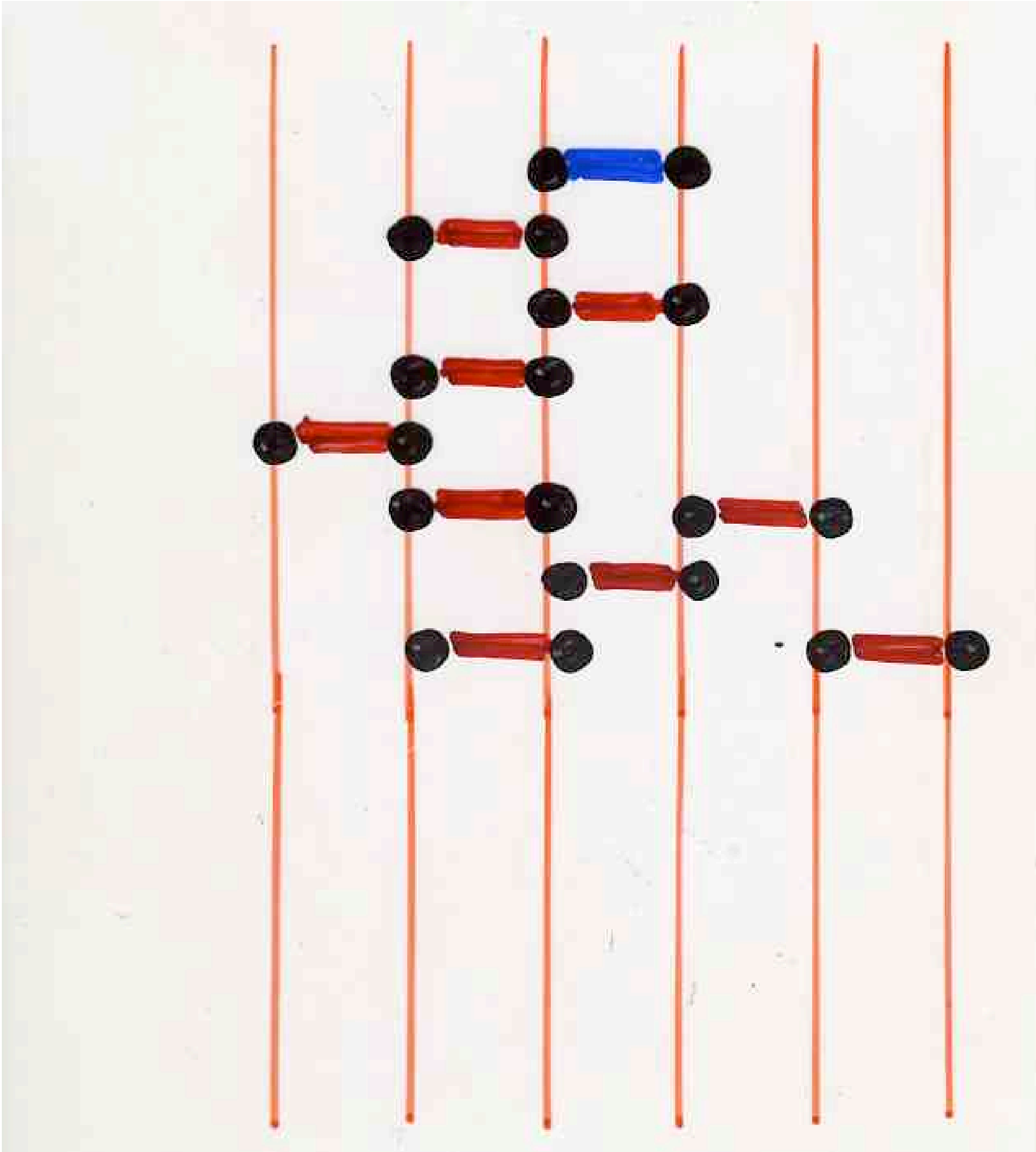


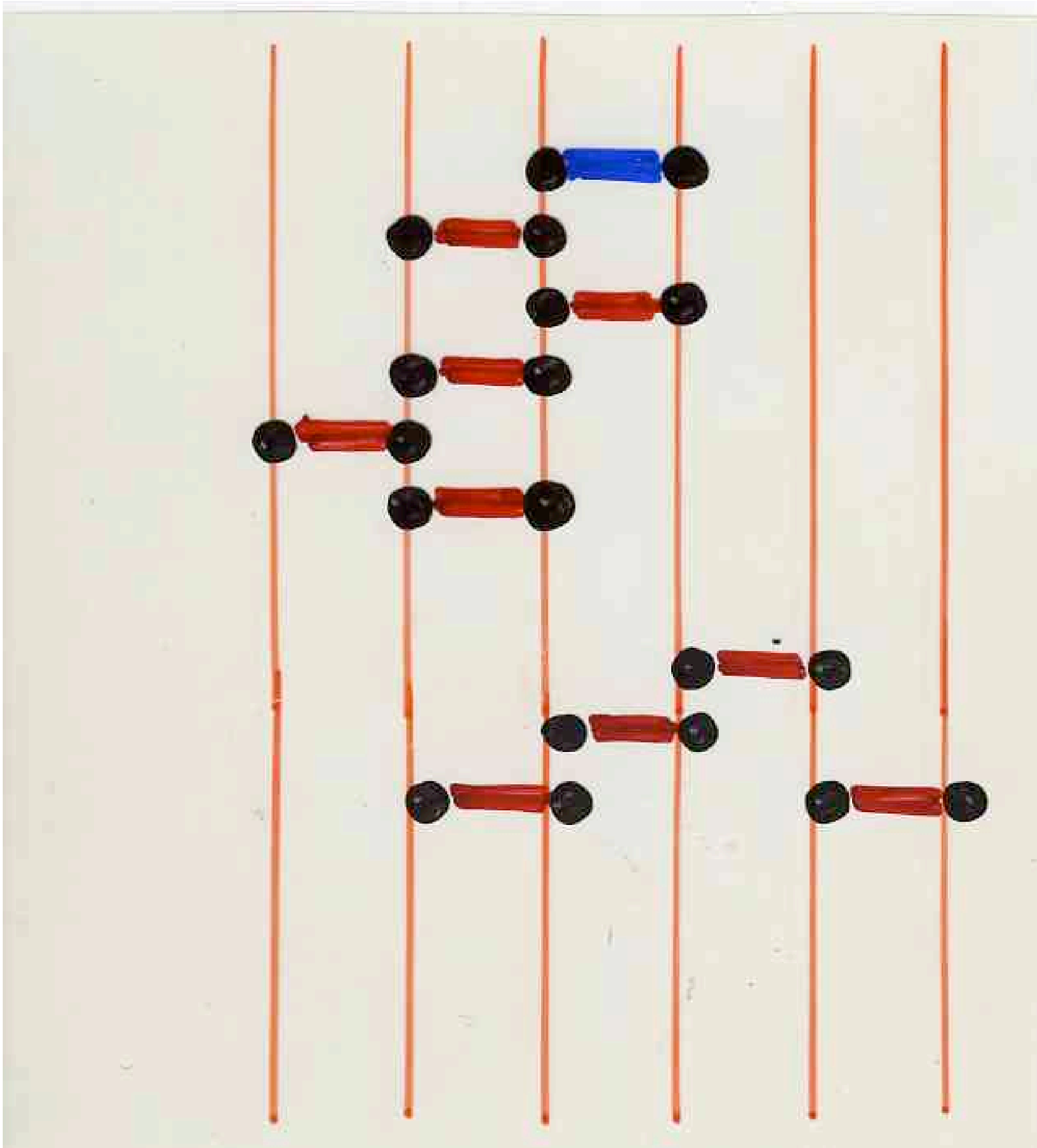

animal dirigé
(réseau triangulaire)

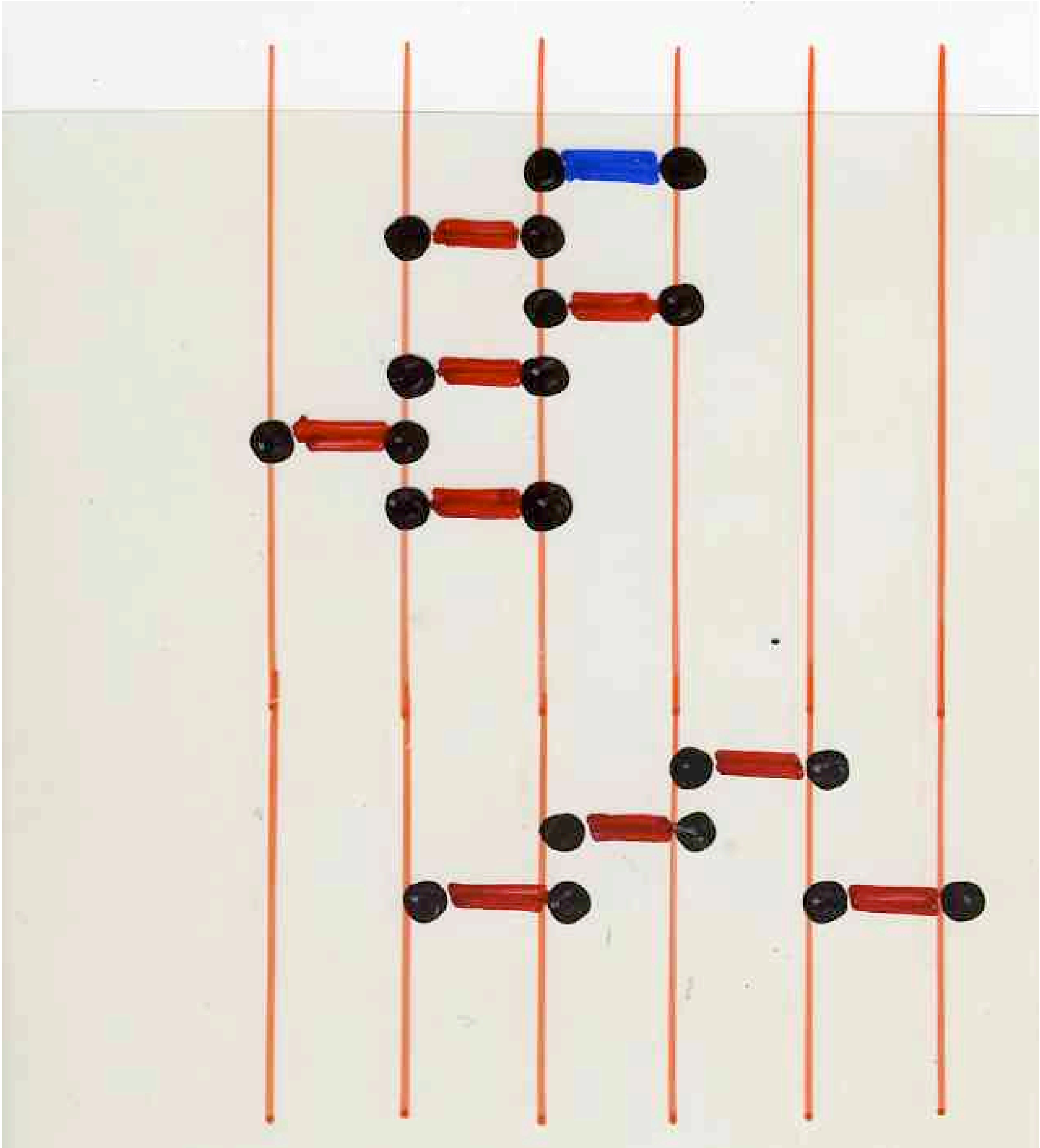
generating functions
for
directed animals

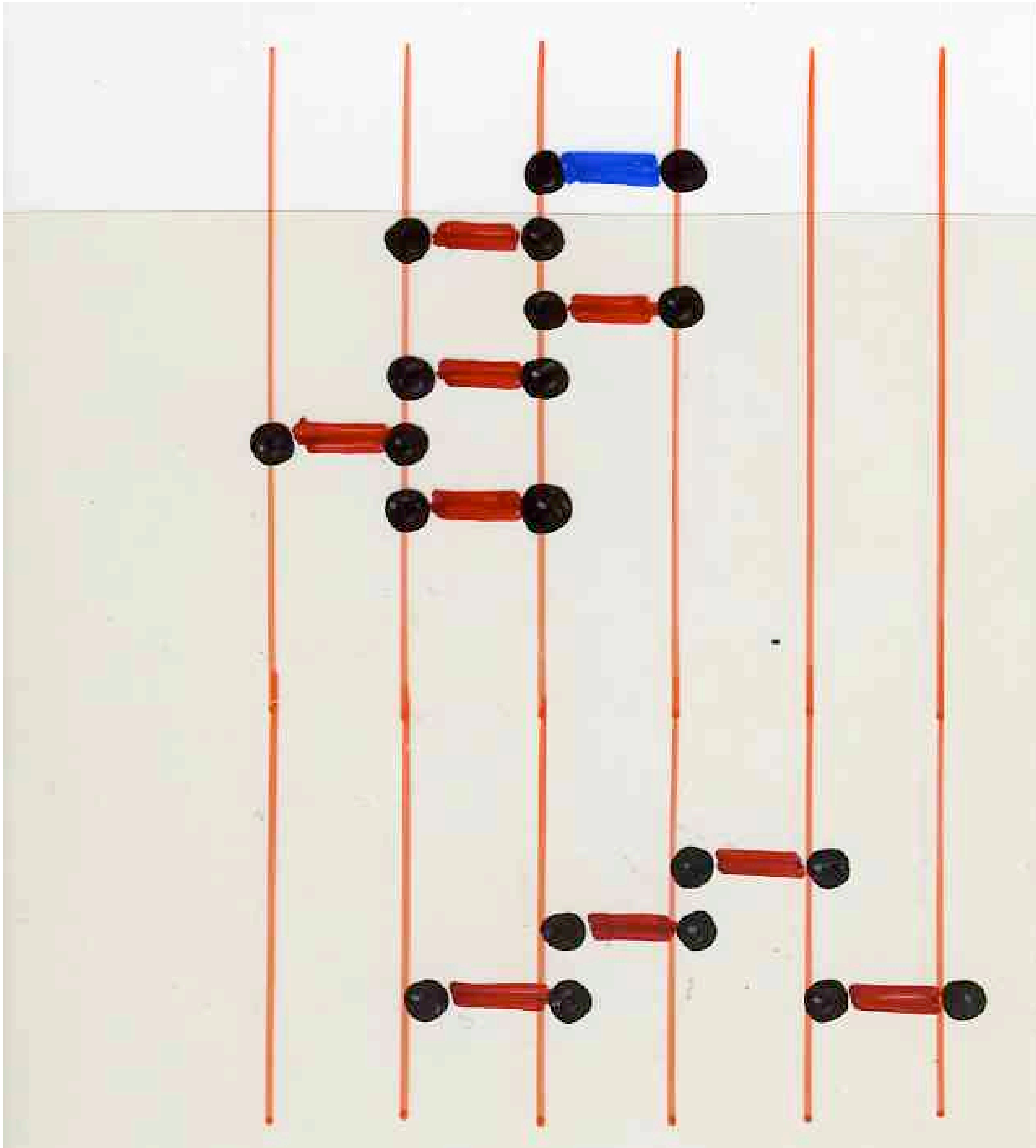
- strict pyramid = $\left\{ \begin{array}{l} \bullet \text{ strict semi-pyramid} \\ \text{or} \\ \bullet \text{ (strict pyramid)} \times \text{(strict semi-pyramid)} \end{array} \right.$

$$y = z + yz$$



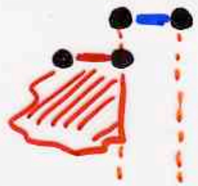




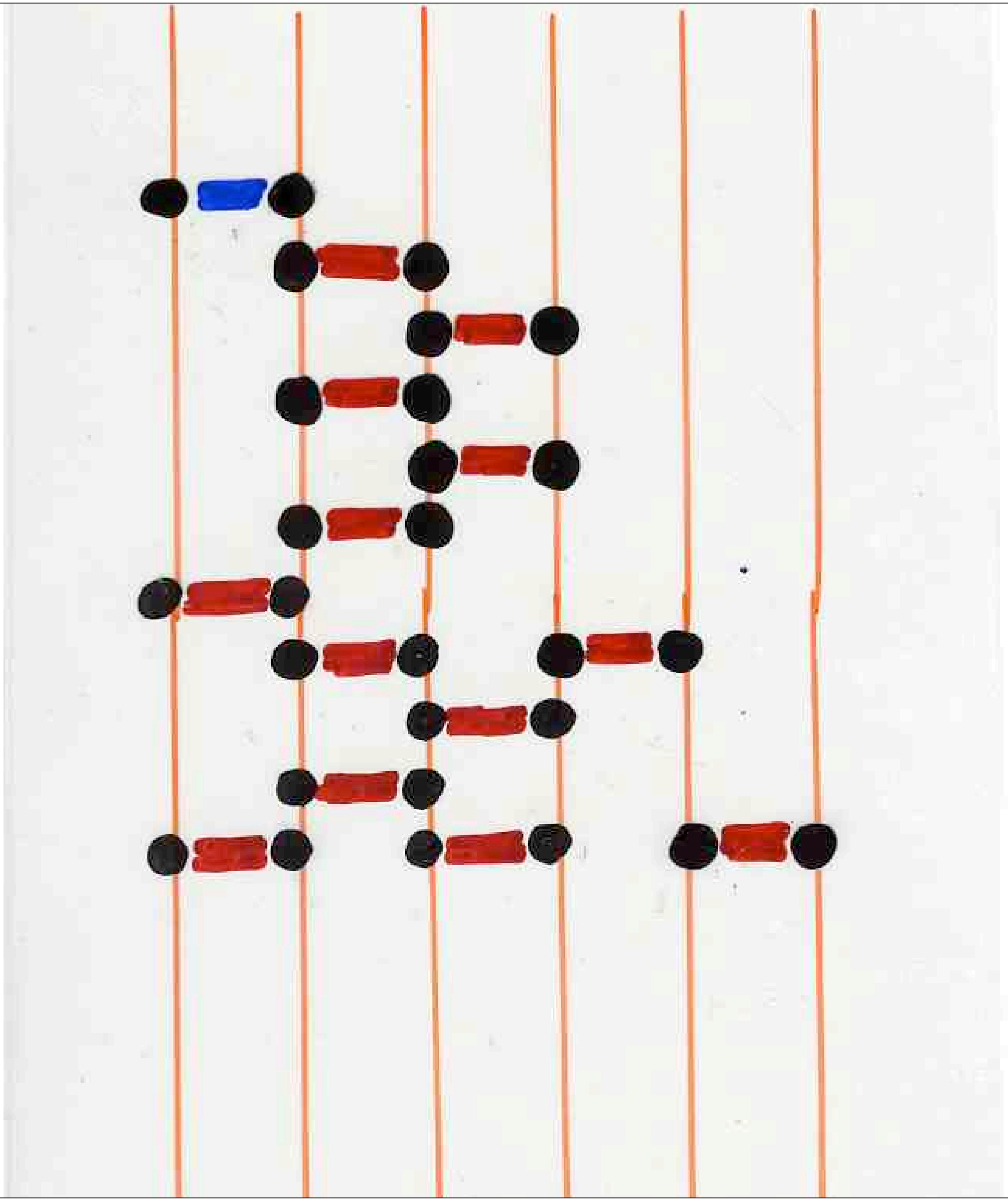


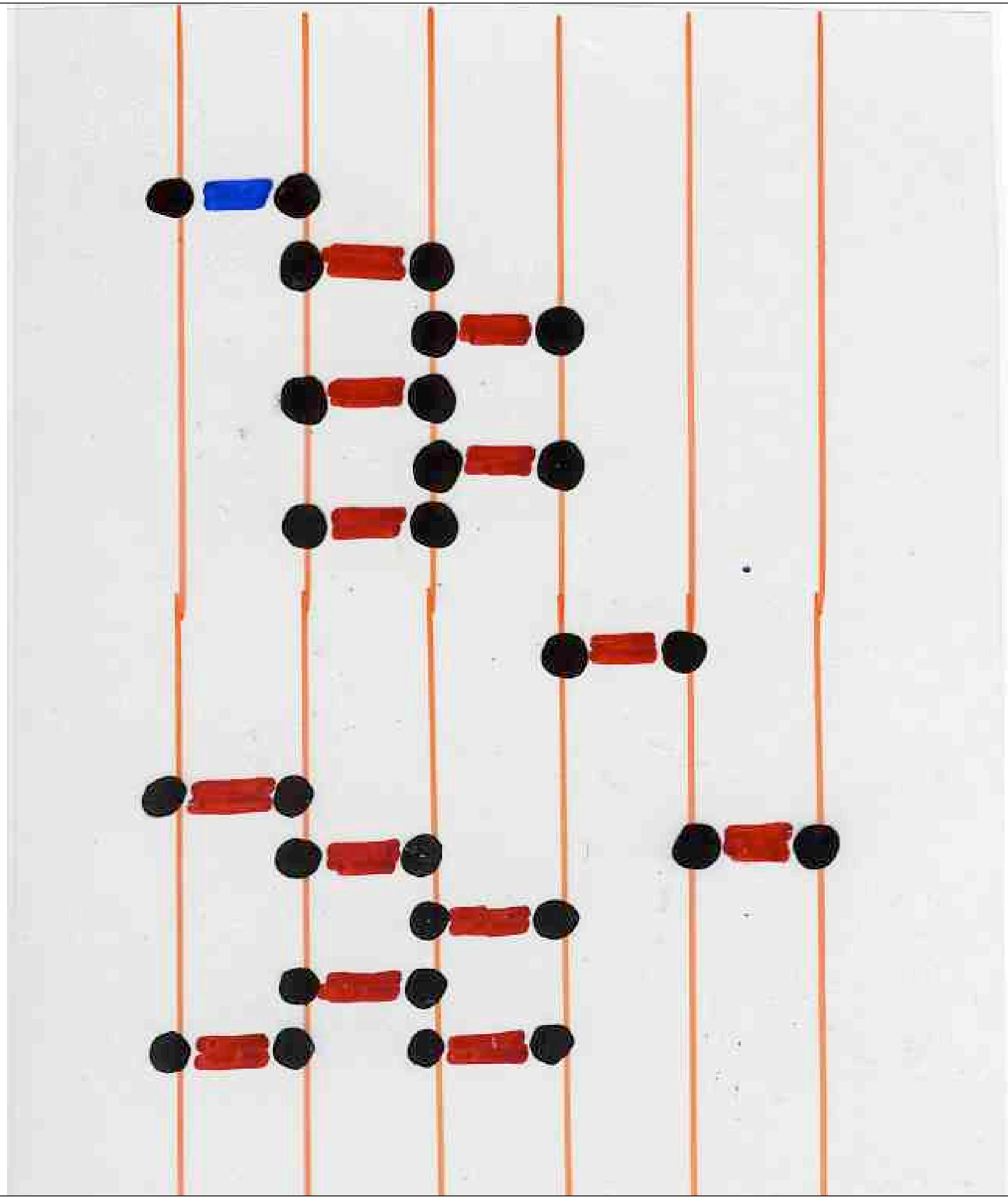
- strict pyramid =
  - • strict semi-pyramid
  - or
  - • (strict pyramid) × (strict semi-pyramid)

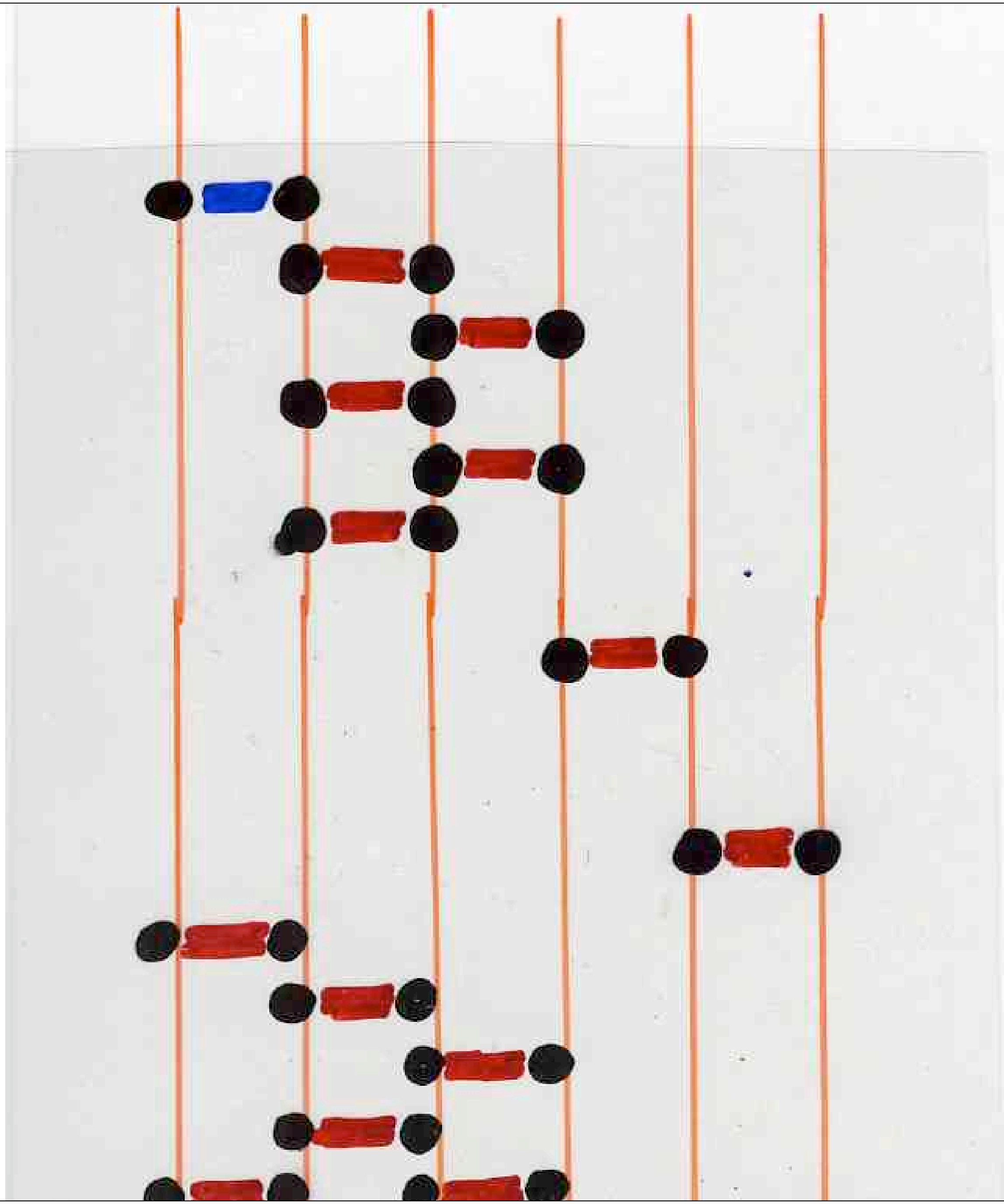
$$y = z + y z$$

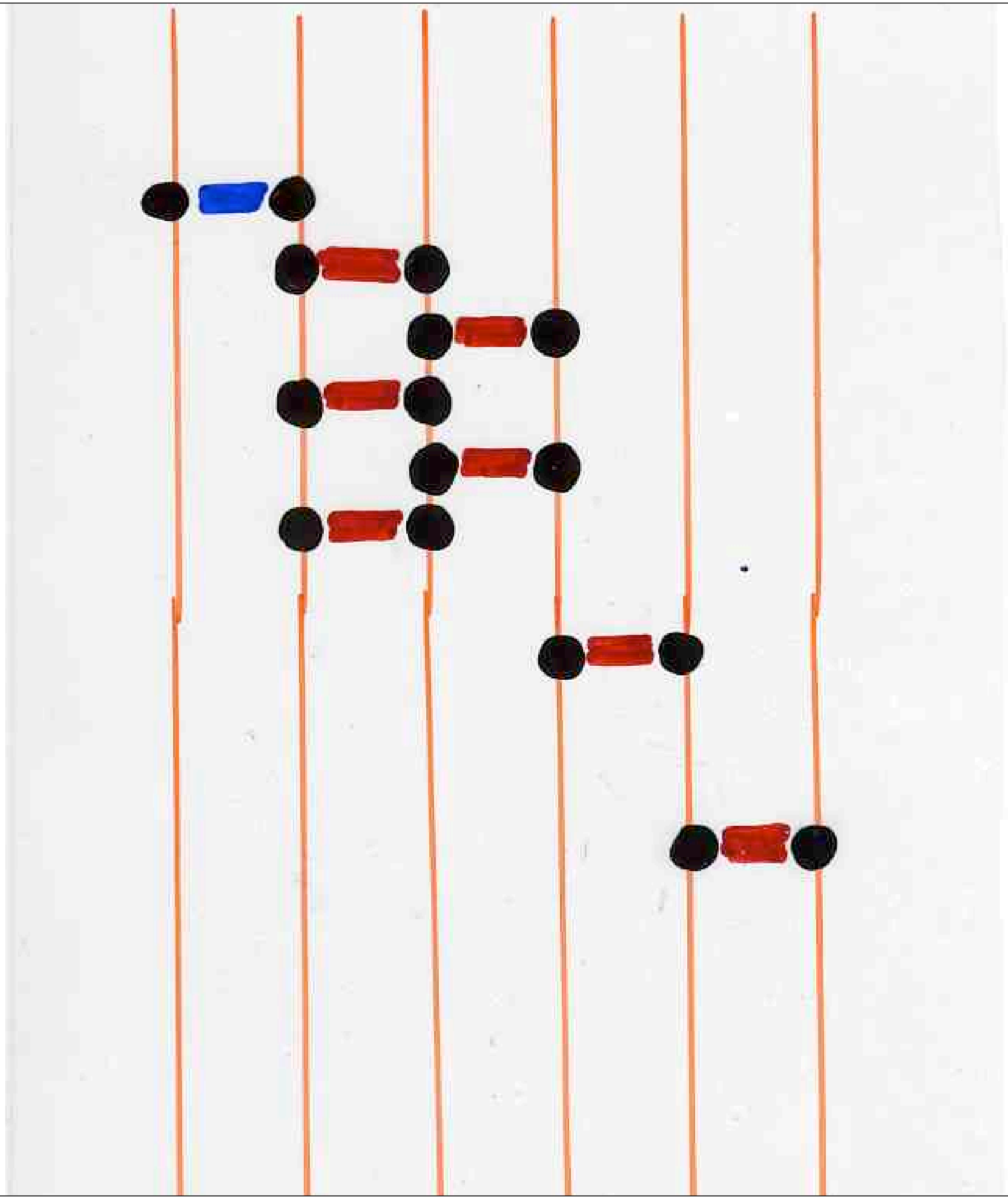
- strict semi-pyramid =
  - • 
  - or
  - • 
  - or
  - • (strict semi-pyramid)^2 × ()

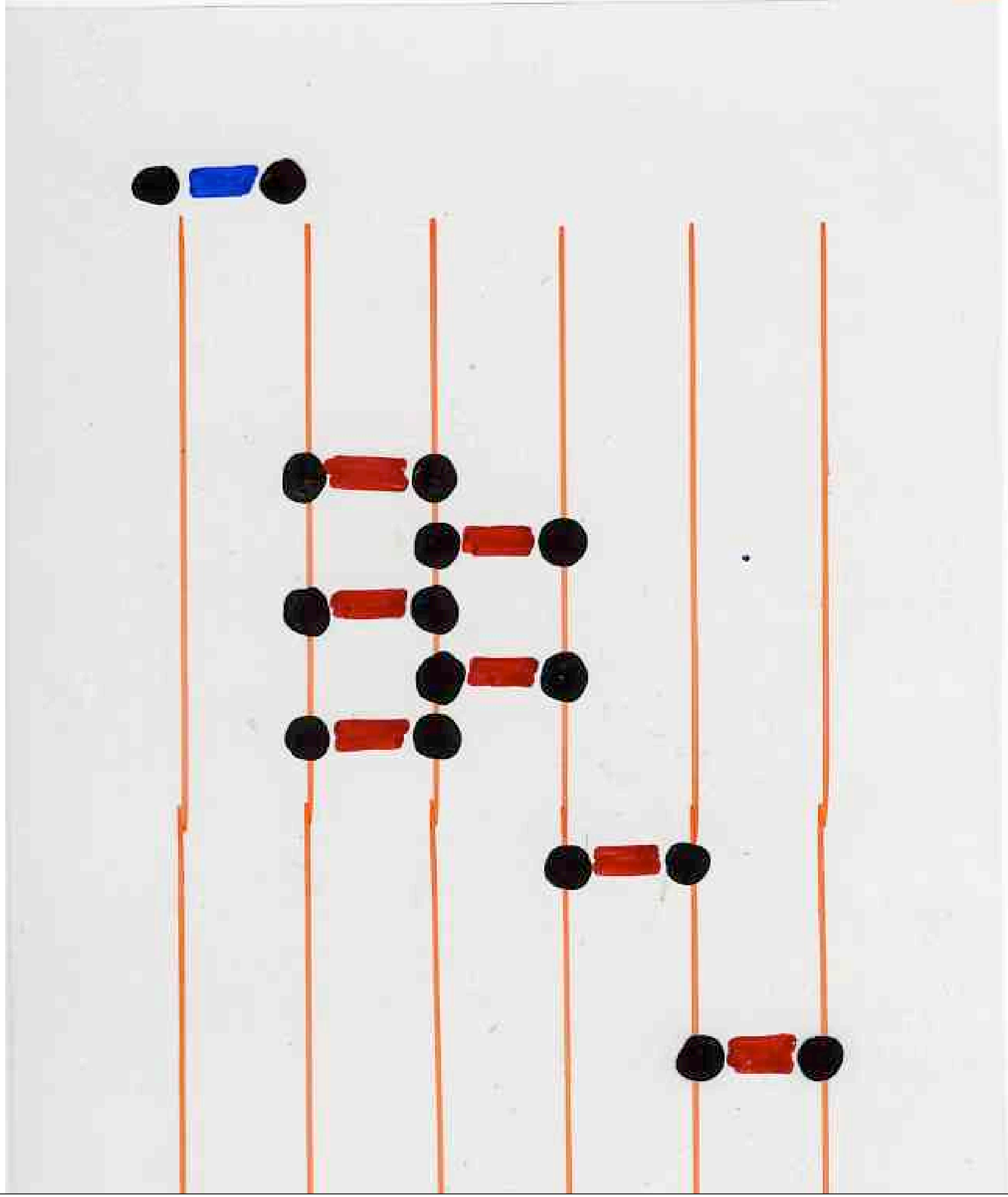
$$z = t + t z + t z^2$$

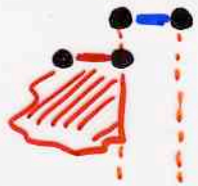






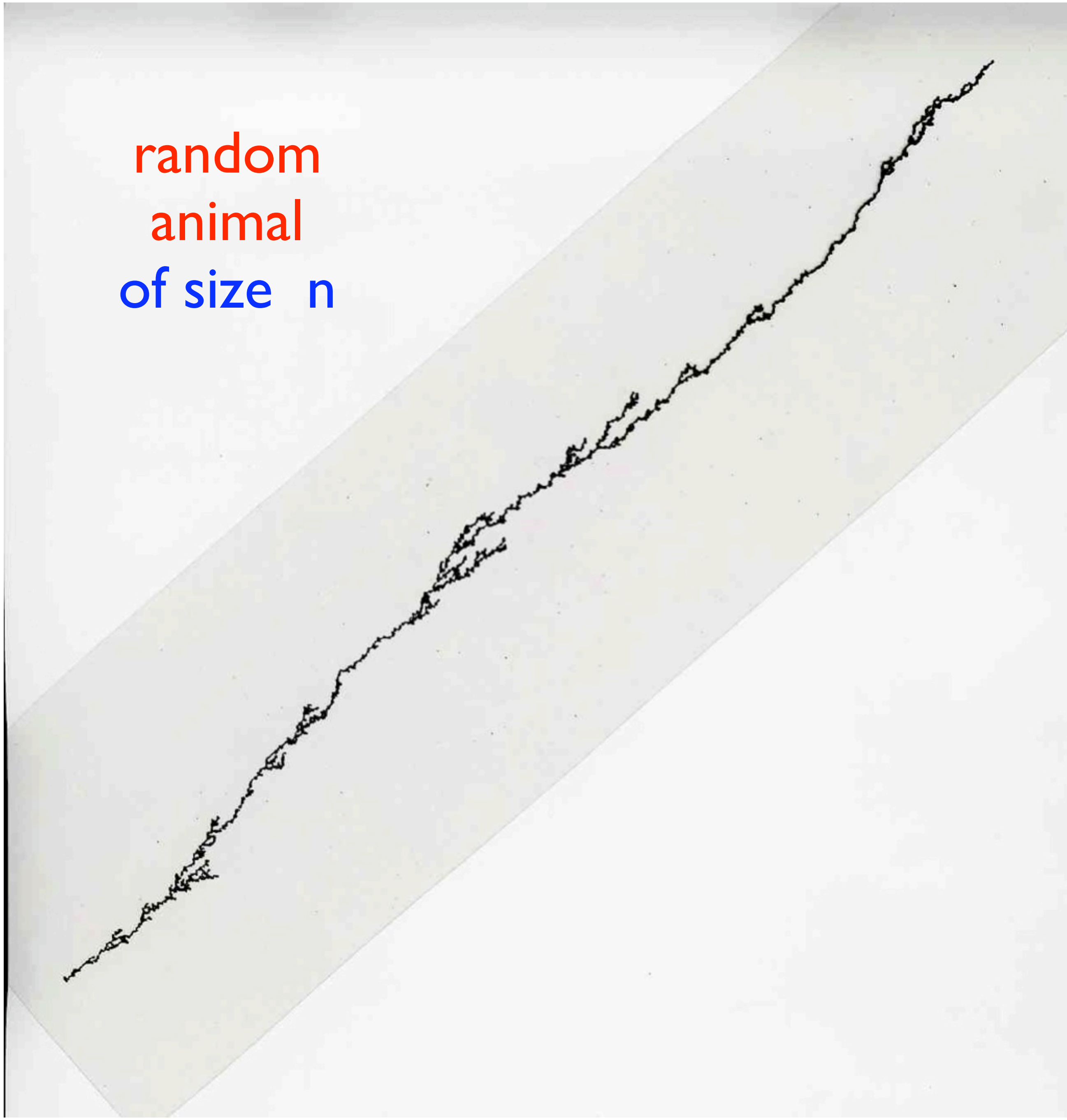


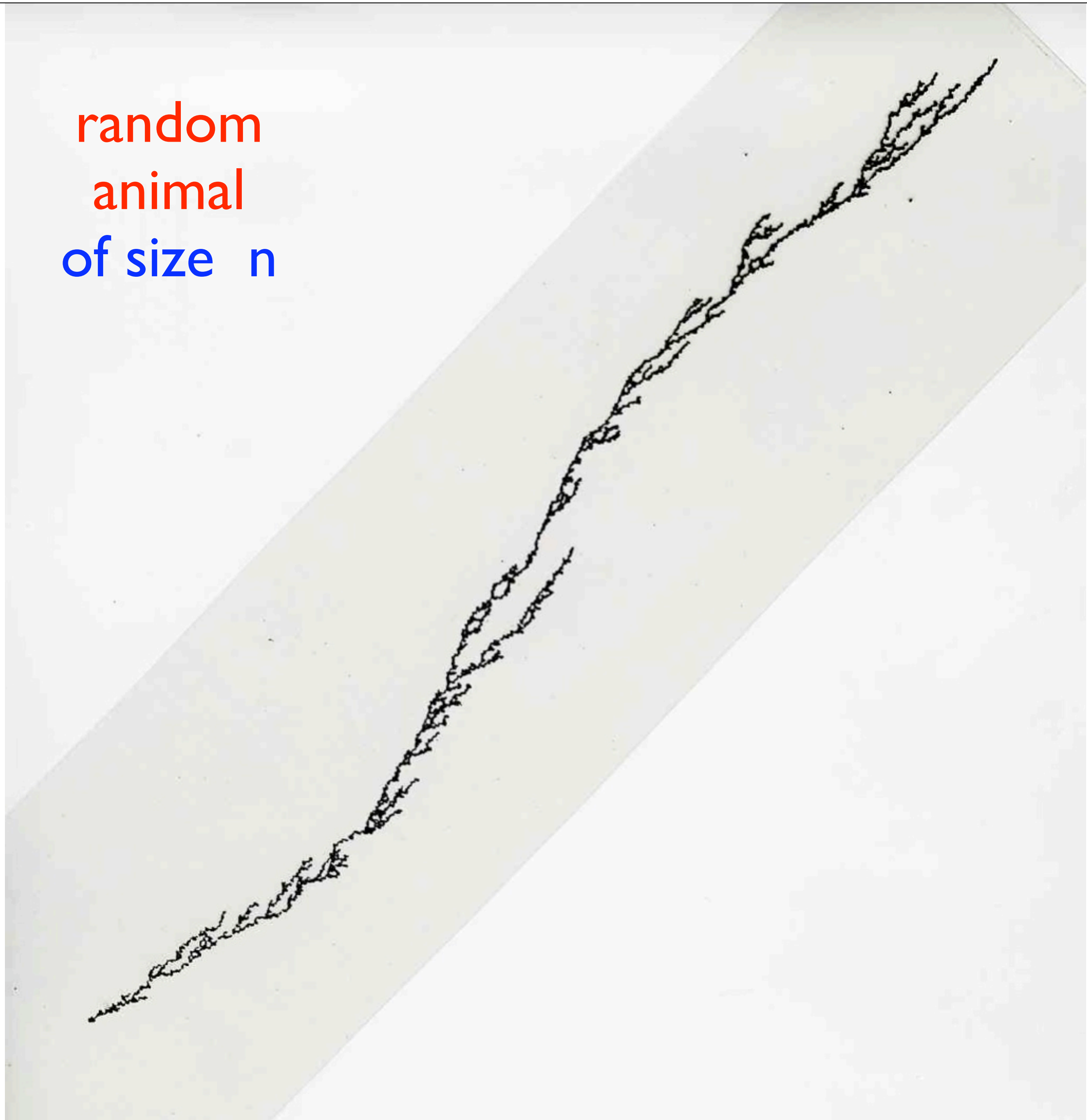


• strict semi-pyramid = { • or •  or • (strict semi-pyramid)^2 × ( )

$$z = t + tz + tz^2$$

random
animal
of size n




random
animal
of size n

§ 9

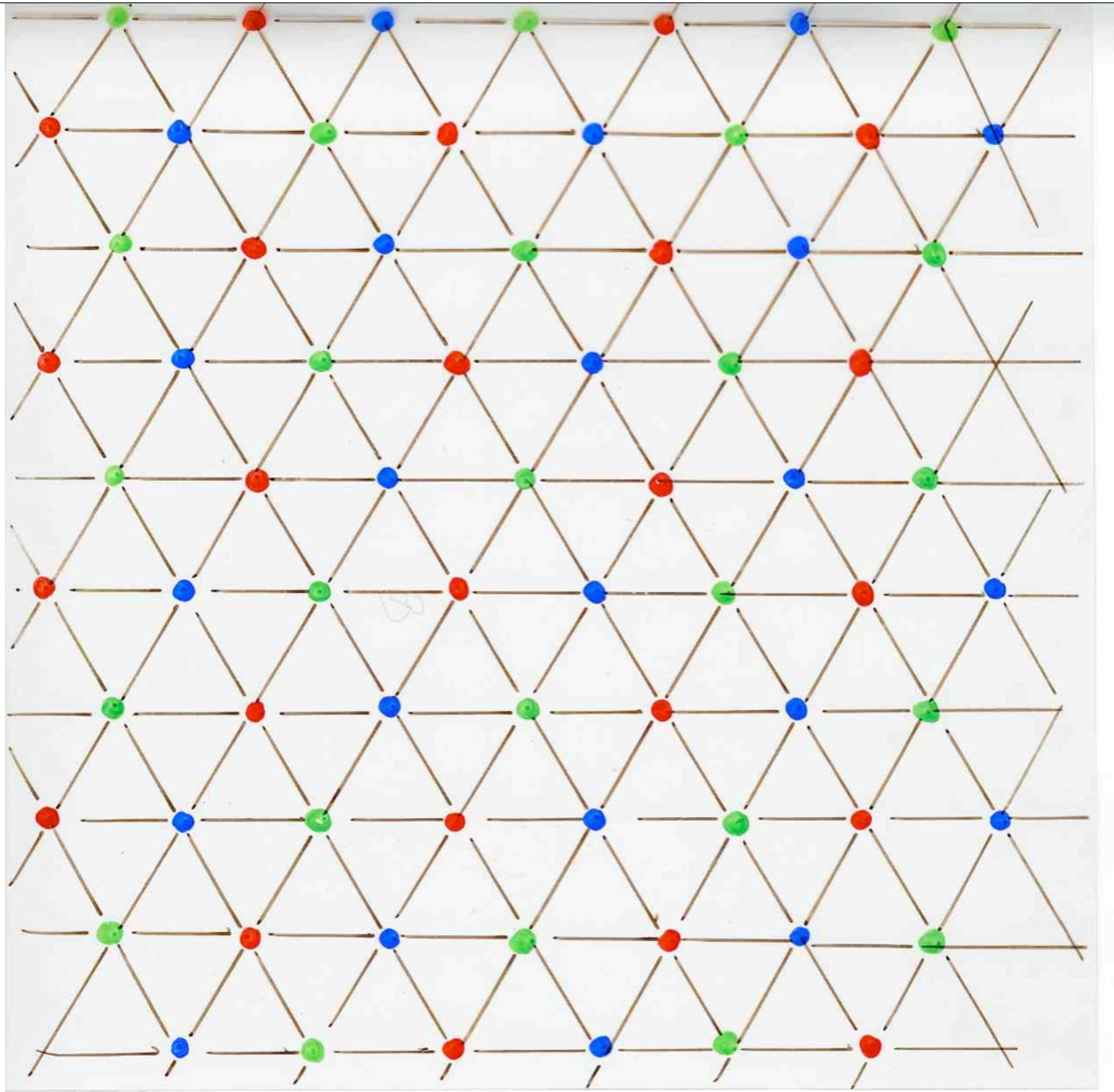
Baxter

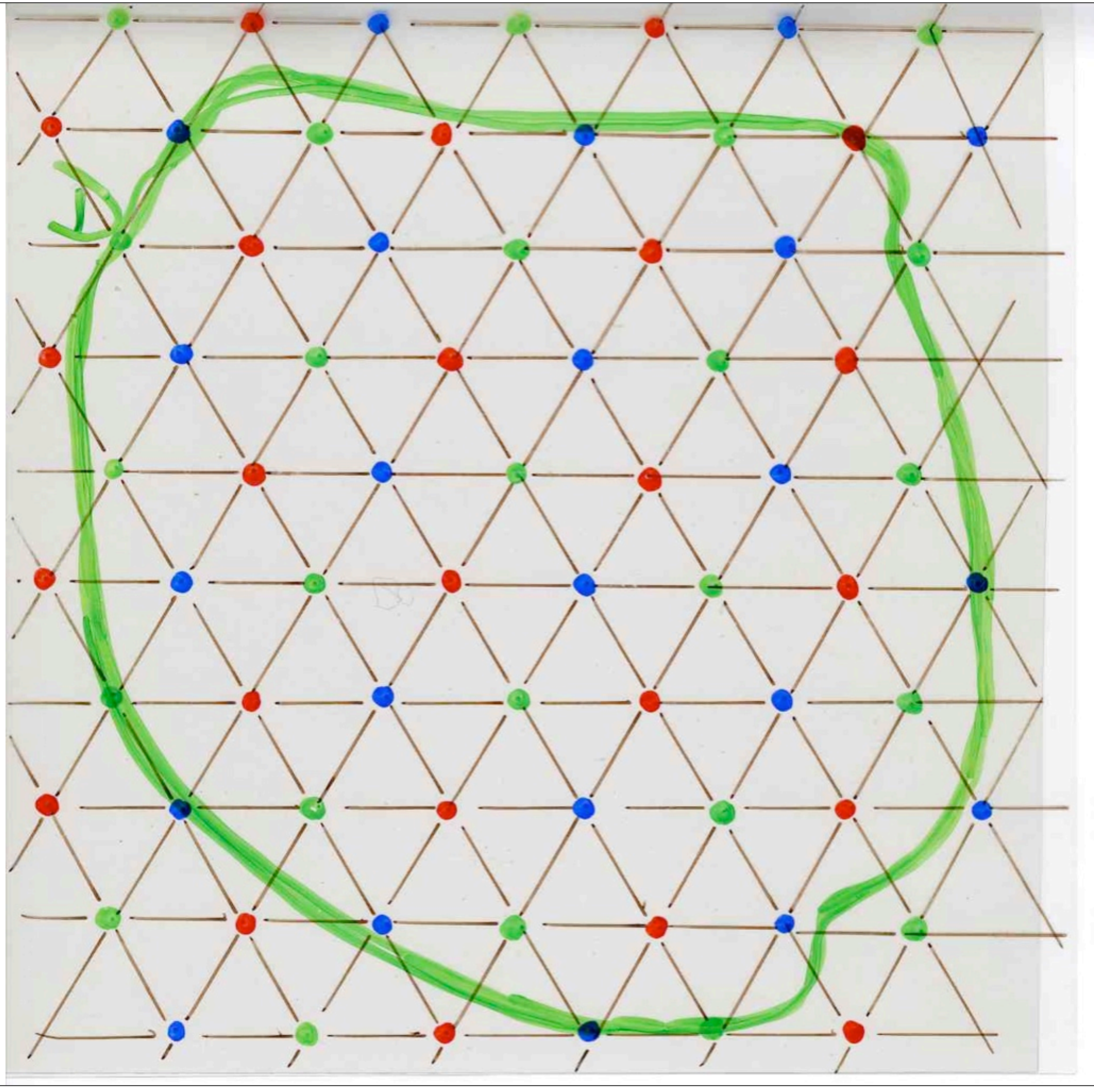
1980
1982

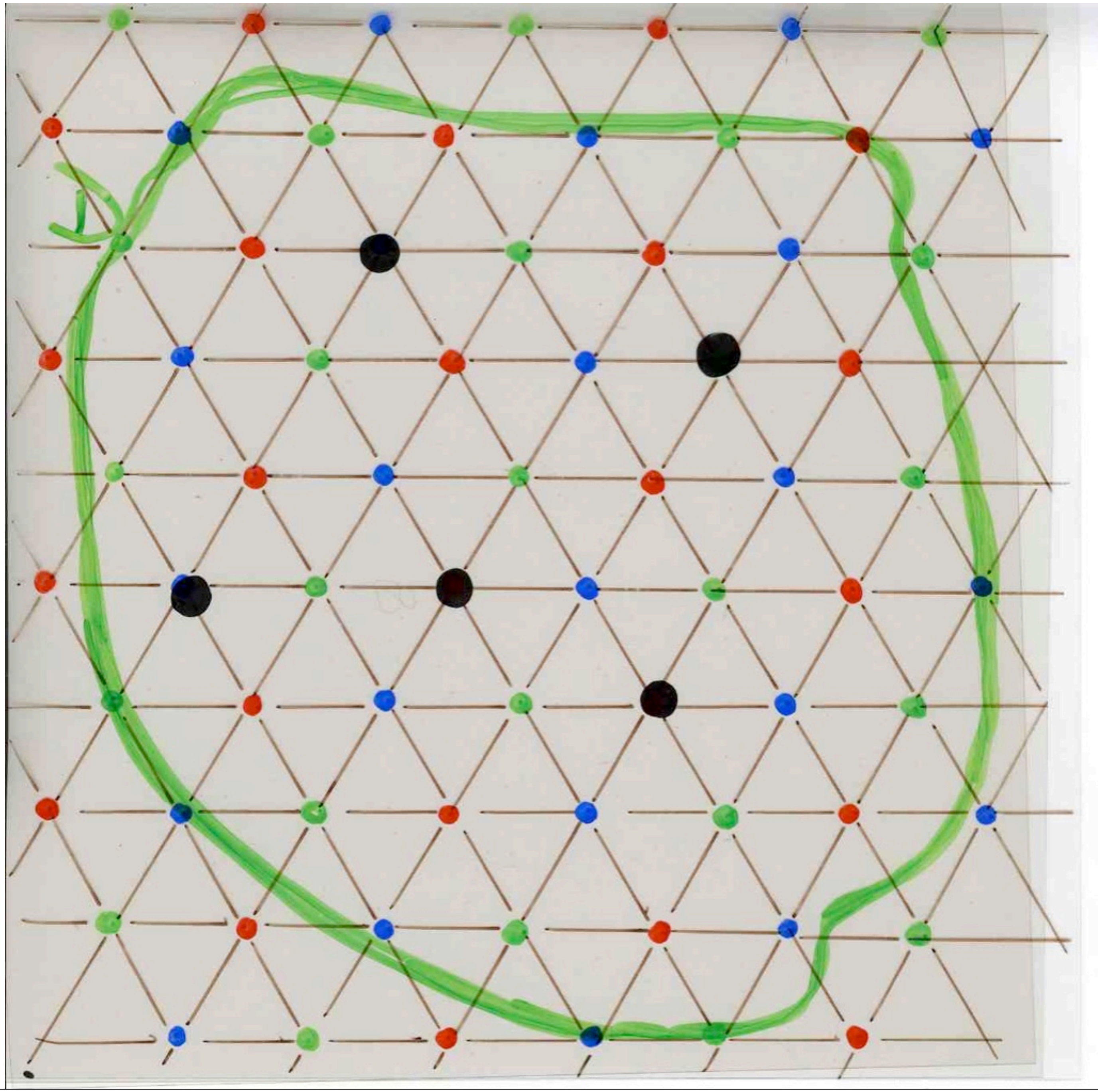
Hard
Hexagons

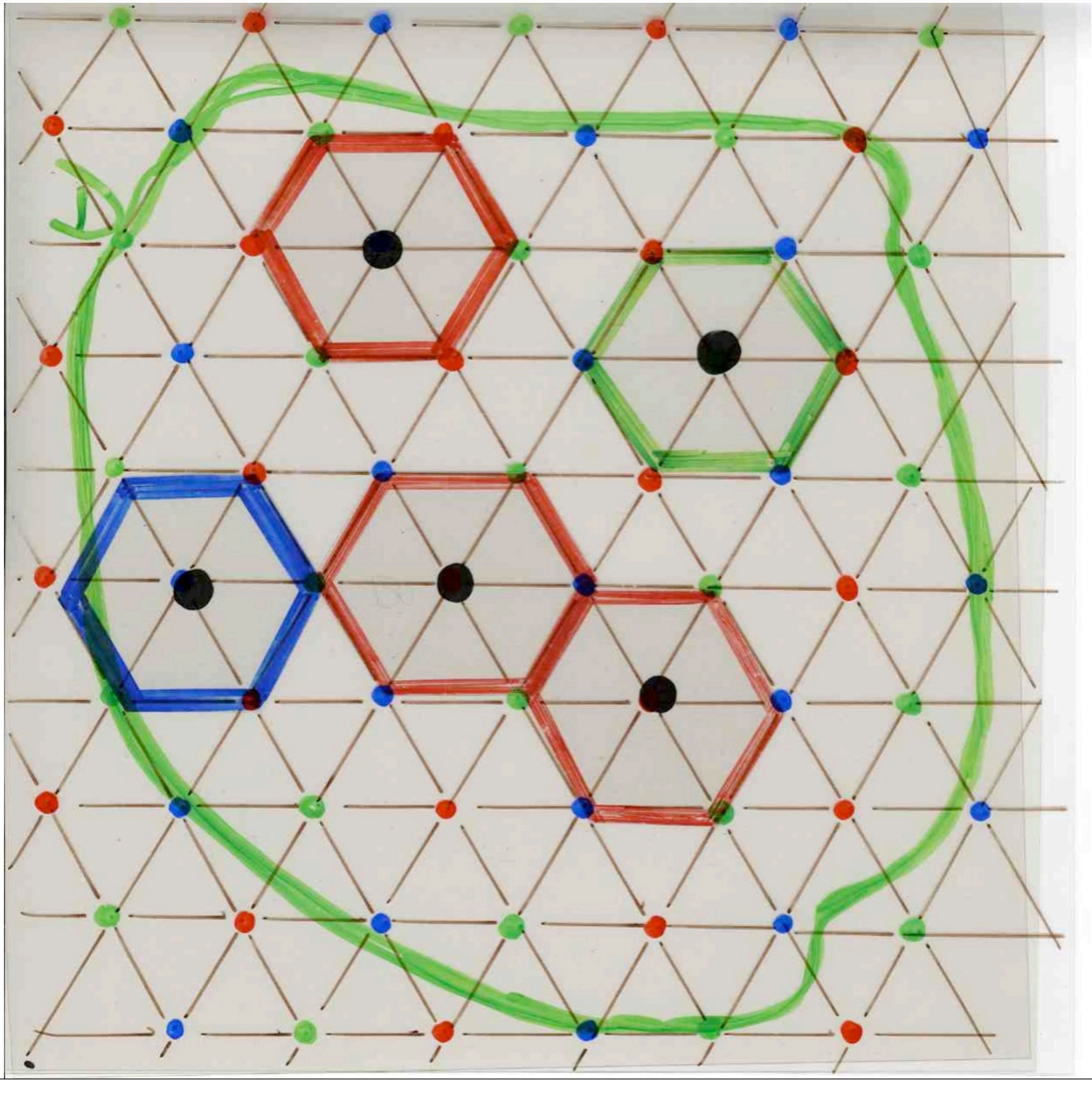
gas model

with "hardcore interactions"









partition function

$$Z_D(t) = \sum_{n \geq 0} a_{n,D} \, t^n$$

$$Z(t) = \lim_{\text{"}D \to \infty\text{"}} \left(Z_D(t) \right)^{1/|D|}$$

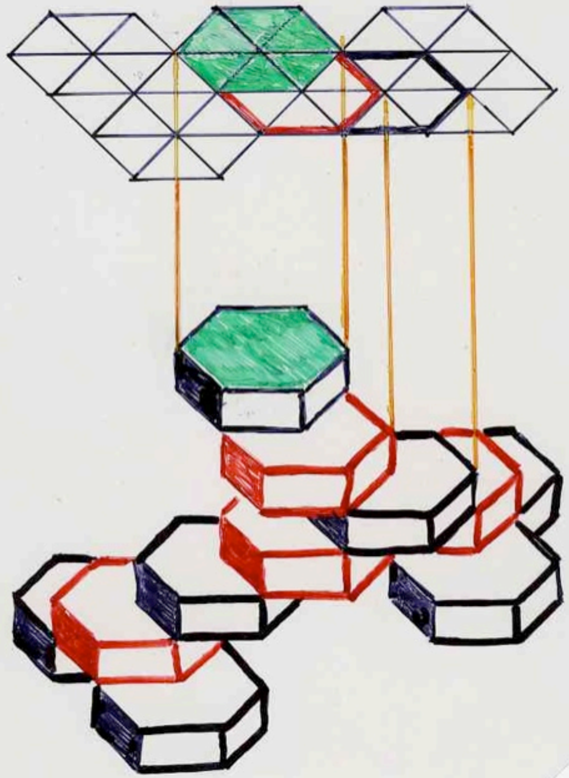
thermodynamic limit

$$\rho(t) = t \frac{d}{dt} \log Z(t)$$

density of a "hard-core" lattice gas model

t is the "activity" of the gas

$$-p(-t) = y$$



KC.

thermodynamic limit

partition function

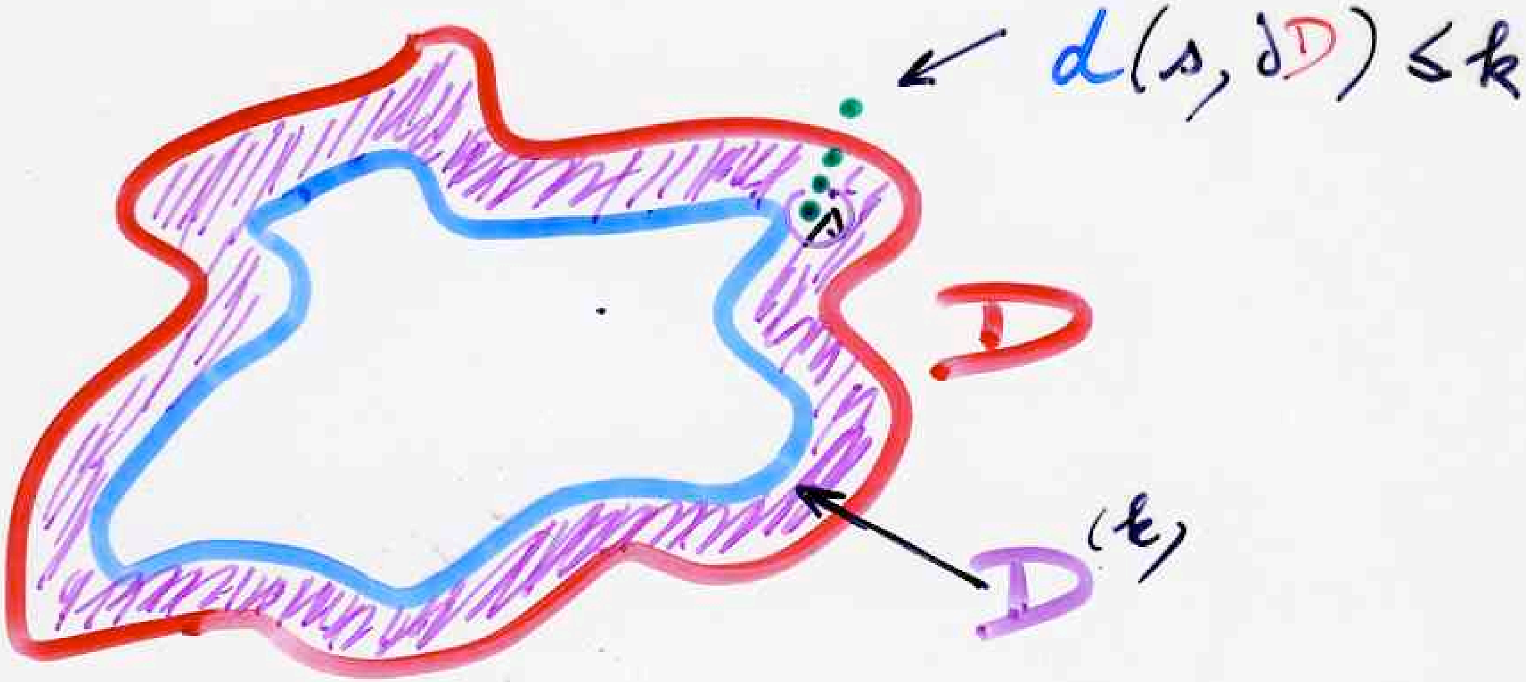
$$Z_D(t) = \sum_{n \geq 0} a_{n,D}\, t^n$$

$$Z(t) = \lim_{\text{"}D \to \infty\text{"}} \left(Z_D(t) \right)^{1/|D|}$$

thermodynamic limit

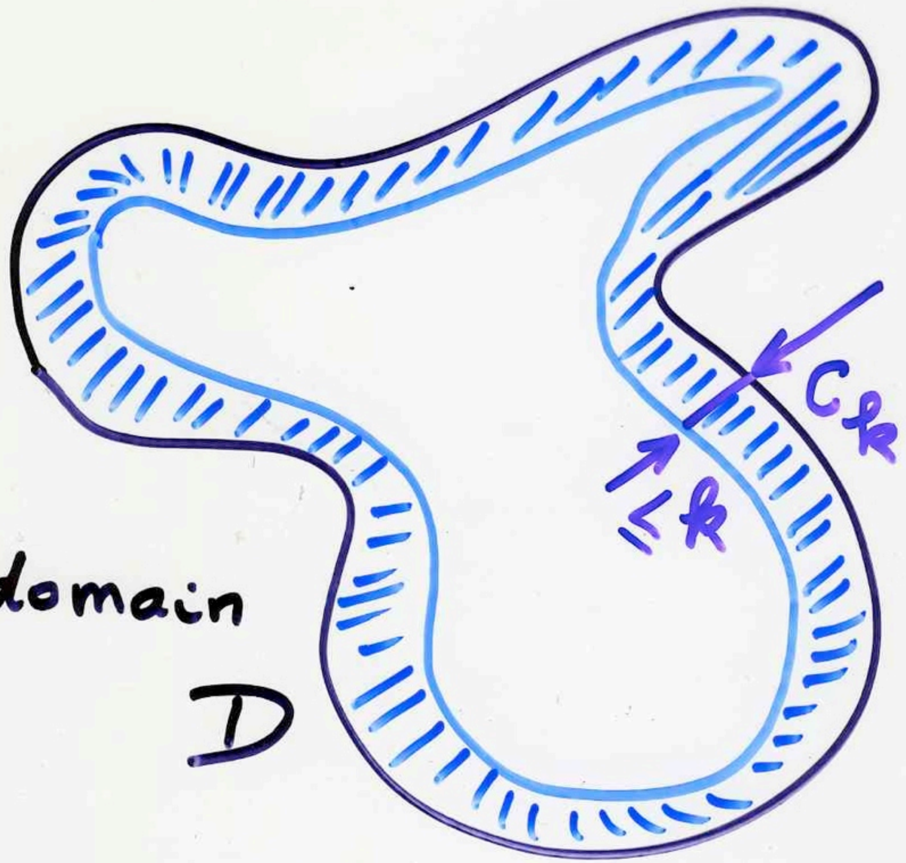
Def. $d(s, \partial D)$

smallest length of paths (on Hex) to go from s to the outside of D



domain

D

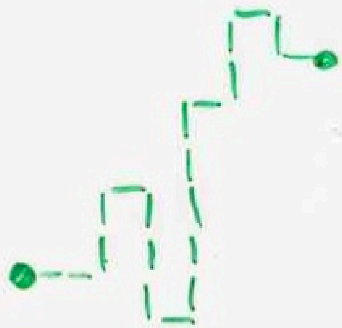


$$\frac{|C_k|}{|D|} \to 0$$

§ 10 q-Bessel functions in statistical physics

9- Bessel

self-avoiding
walks



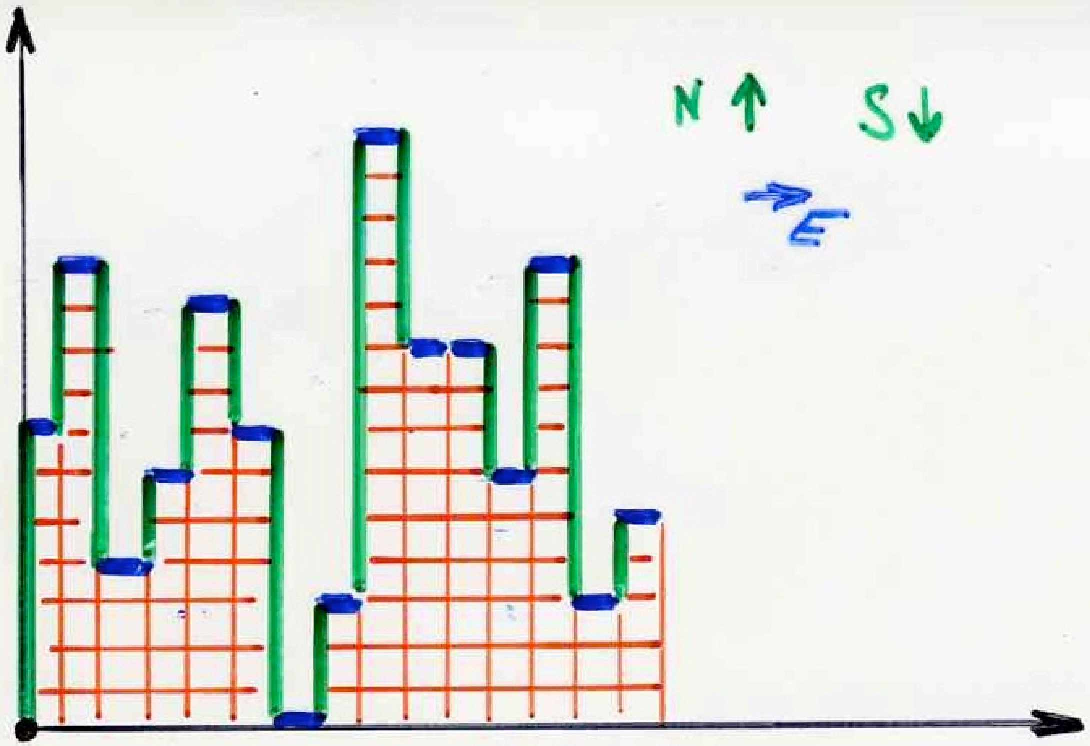
self-avoiding
polygons
(vesicles)

some subclasses :

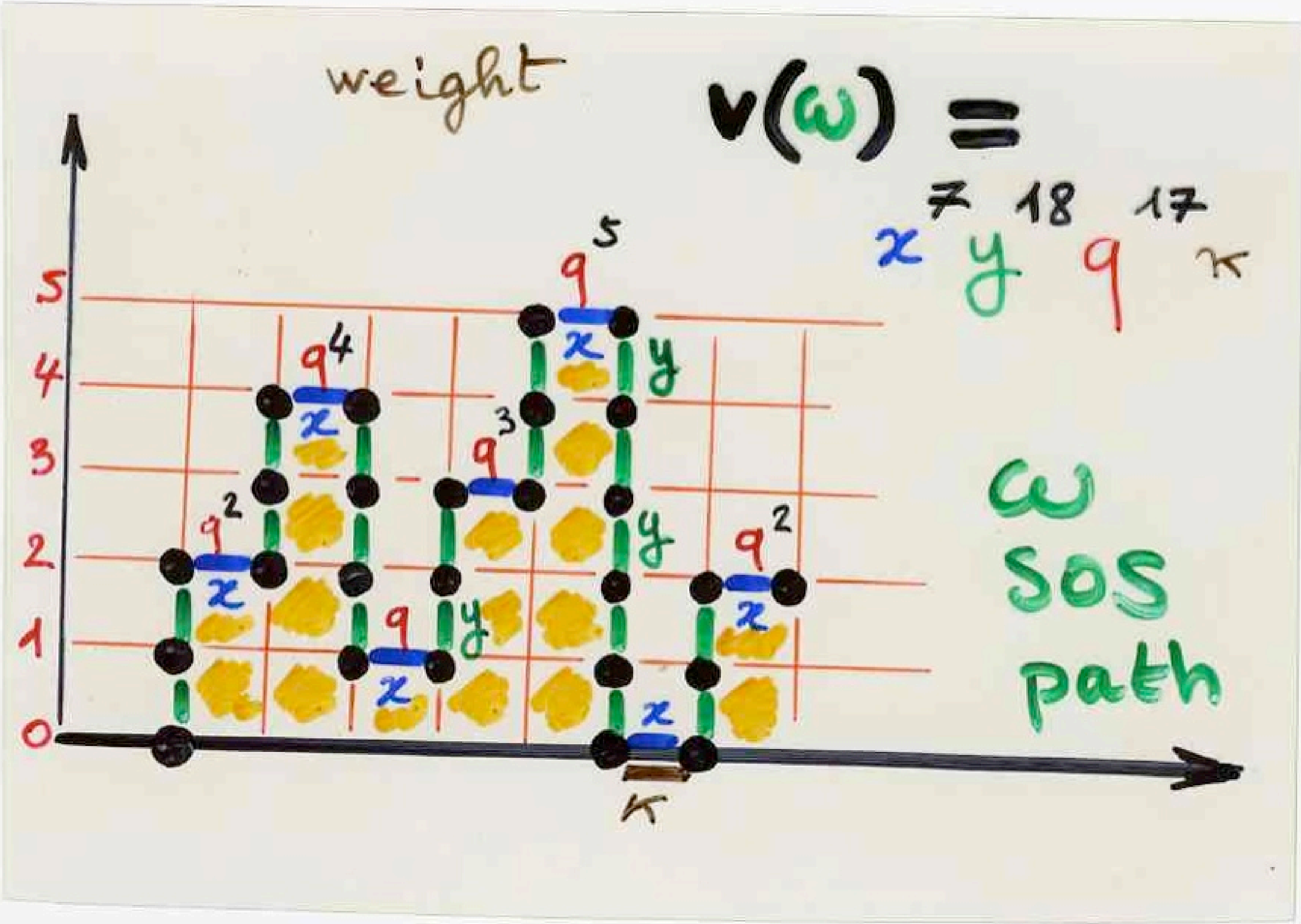
exactly solved

explicit formulae for **generating**

partition function **function**



Partially directed
self-avoiding walks
(paths)



weight
$v(\omega) =$
$x^7 y^{18} q^{17} \kappa$
ω
SOS
path
q^5
q^4
q^3
q^2
q
q^2
x
y
κ
0
1
2
3
4
5

ou encore :

$$\sum_{\substack{\omega \\ \text{chemins SOS} \\ \text{arrivant au} \\ \text{niveau } 0}} v(\omega) = x \frac{H(qy^2, q, x(1-y^2)q)}{H(y^2, q, x(1-y^2))}$$

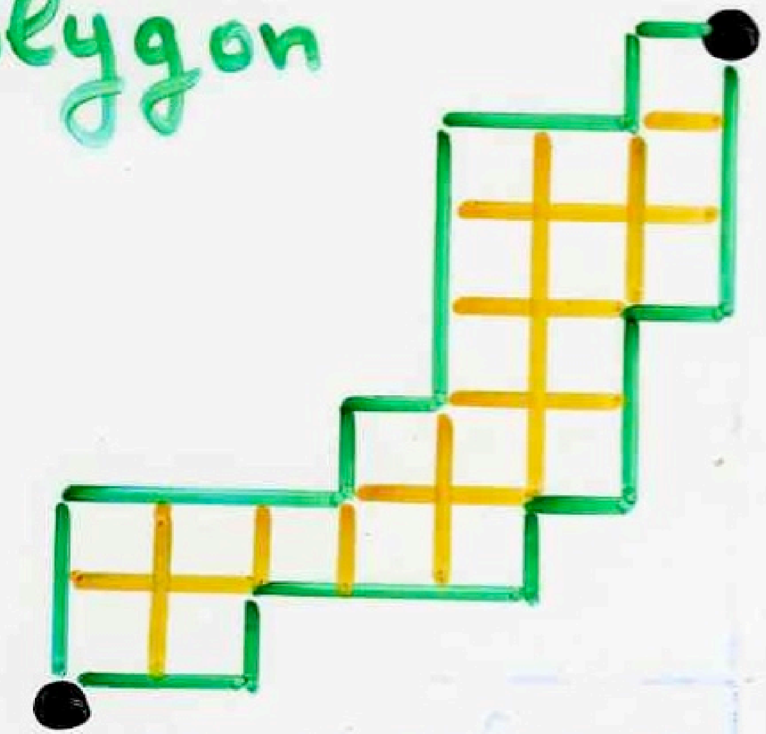
Owczarek, Prellberg (1993)

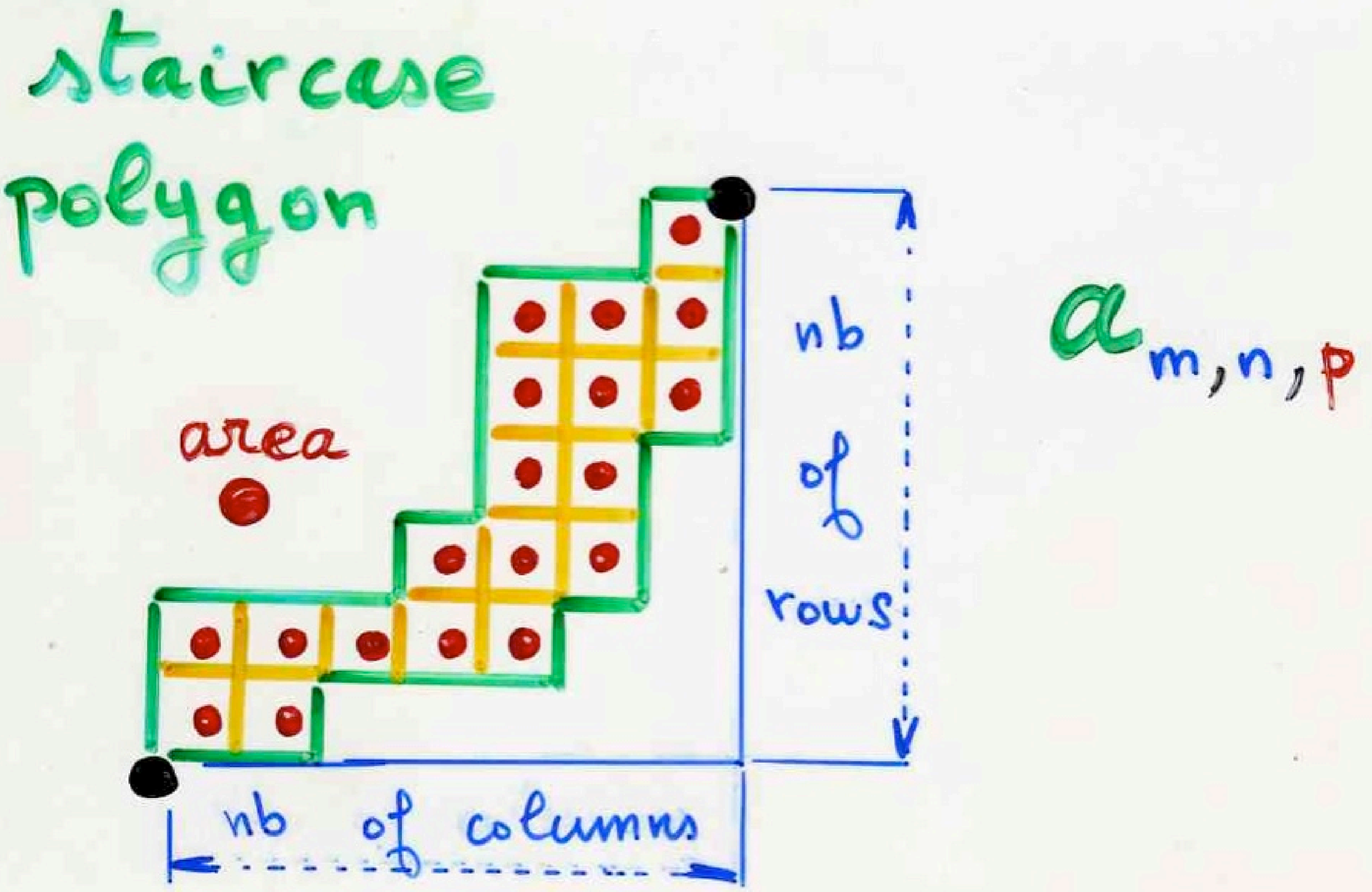
notations : $H(u, q, t) = \sum_{n \geq 0} \frac{(-t)^n q^{\binom{n}{2}}}{(u,q)_n (q,q)_n}$

avec $(u,q)_n = (1-u)(1-uq) \dots (1-uq^{n-1})$
$(u)_n$

staircase polygons

staircase polygon





staircase
polygon
area
nb
of
rows
nb of columns
$a_{m,n,p}$

generating function

$$f(x,y;q) = \sum_{m,n,p} a_{m,n,p}\, x^m y^n q^p$$

$$= \sum_{P} x^{c(P)}\, y^{r(P)}\, q^{a(P)}$$

staircase polygons

nb of columns

nb of rows

area

parallelogram polyominoes

$$\begin{cases} x & \text{length} \quad (\text{nb of columns}) \\ y & \text{height} \quad (\text{" rows "}) \\ q & \text{area} \end{cases}$$

Klarner, Rivest (1974)
Bender

Delest, Fedou (1989)

Brak, Guttmann (1990)

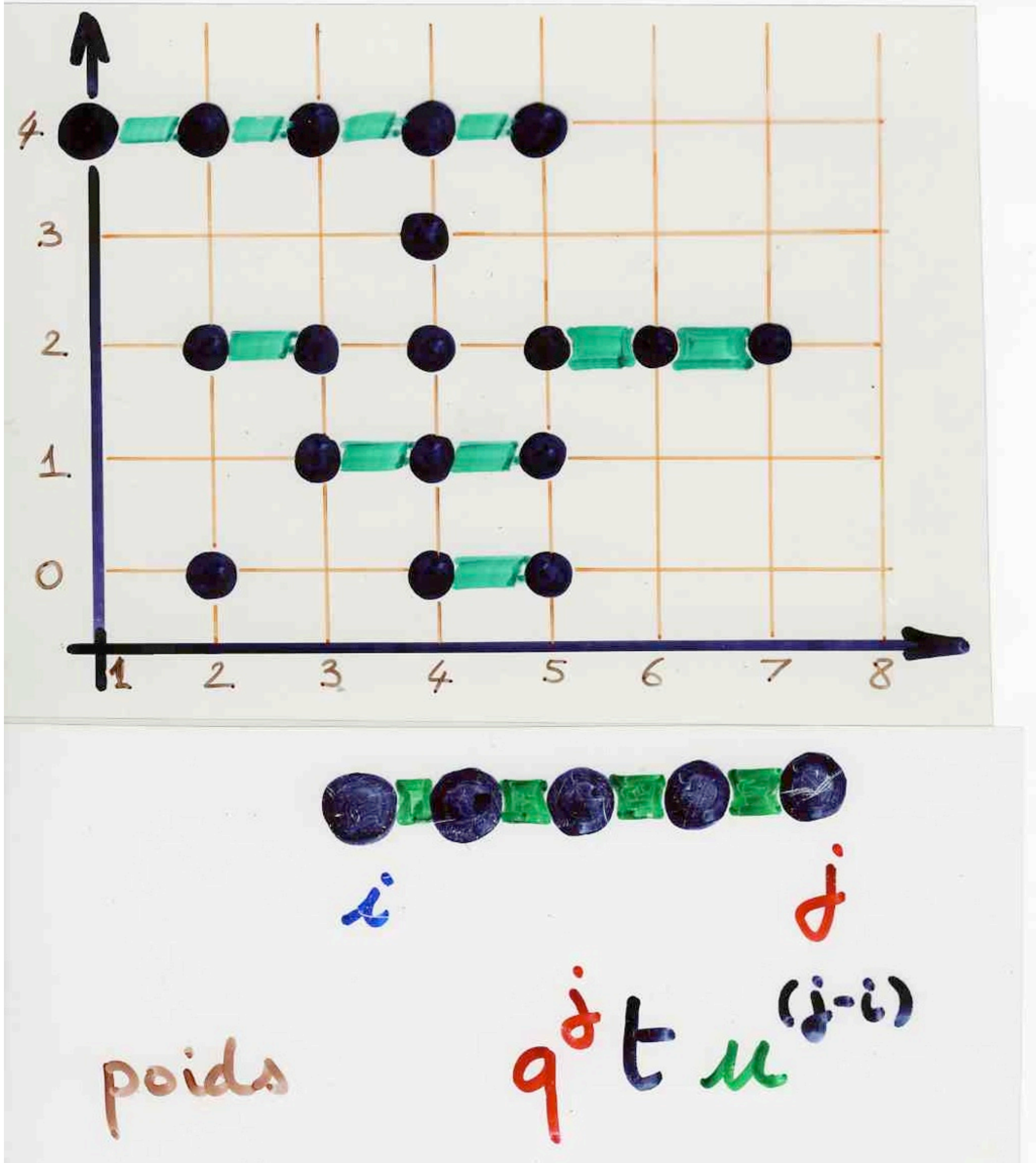
Bousquet-Mélou, Viennot (1990)

$$y \, \frac{J_1(x,y,q)}{J_0(x,y,q)}$$

$$J_0 = \sum_{n \geq 0} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n (yq)_n}$$

$$J_1 = \sum_{n \geq 1} \frac{(-1)^{n-1} x^n q^{\binom{n+1}{2}}}{(q)_{n-1} (yq)_n}$$

notation $\quad (a)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$

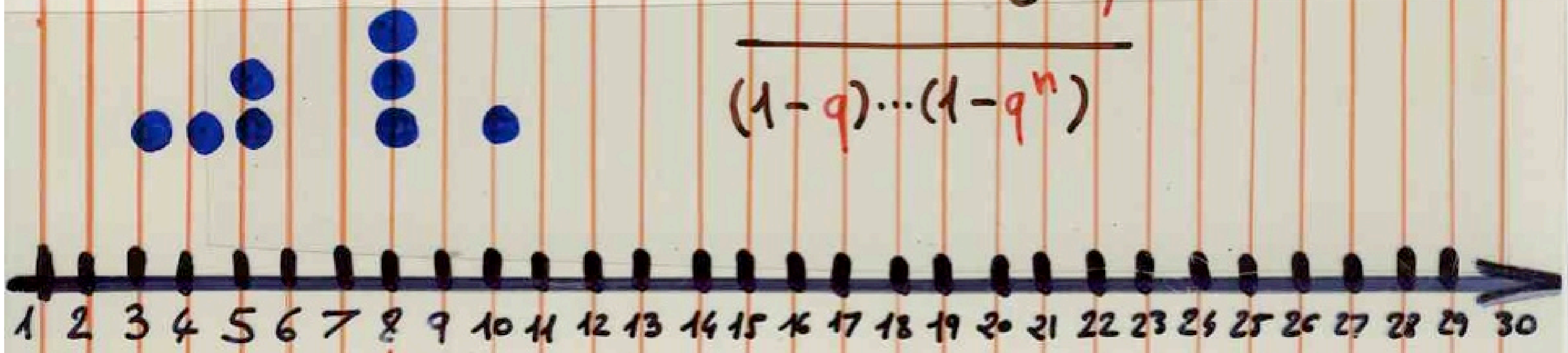


4
3
2
1
0
1 2 3 4 5 6 7 8
i
j
poids
$q^{j}\,t\,u^{(j-i)}$

$$D = \sum_{n\geq 0} \frac{(-1)^n \, t^n q^n \, q^{\binom{n}{2}}}{(1-q)\cdots(1-q^n)(1-uq)\cdots(1-uq^n)}$$

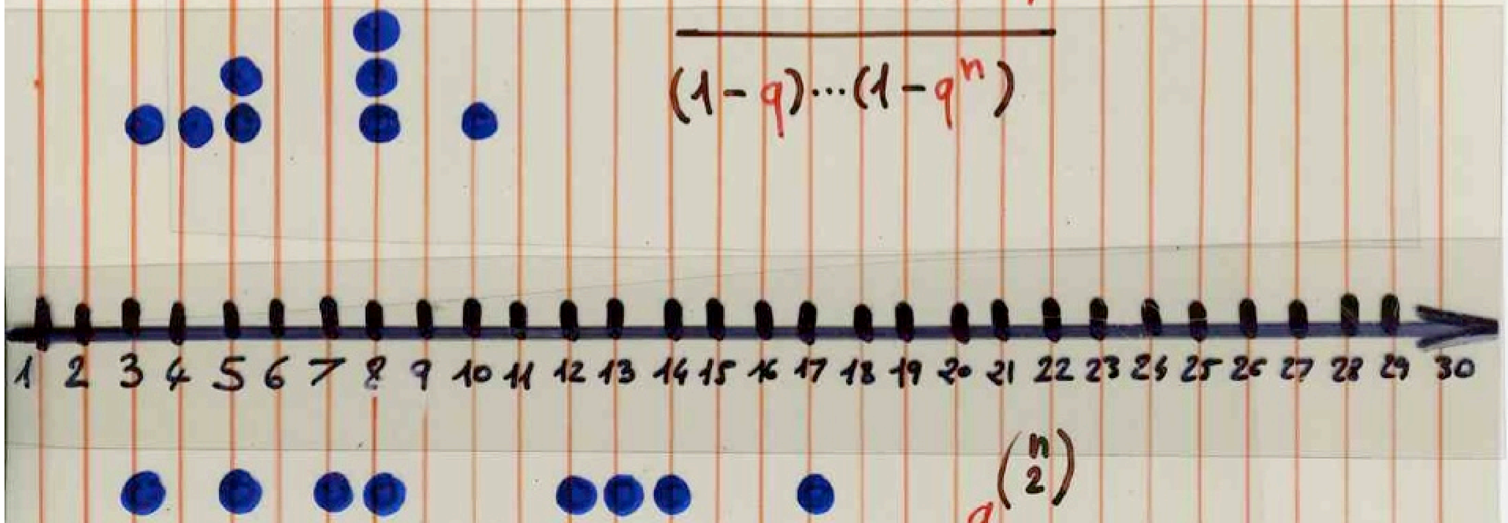
$$D = \sum_{n \geq 0} \frac{(-1)^n \, q^{\binom{n}{2}}}{(1-uq)\cdots(1-uq^n)}$$

$$\frac{t^n q^n}{(1-q)\cdots(1-q^n)}$$

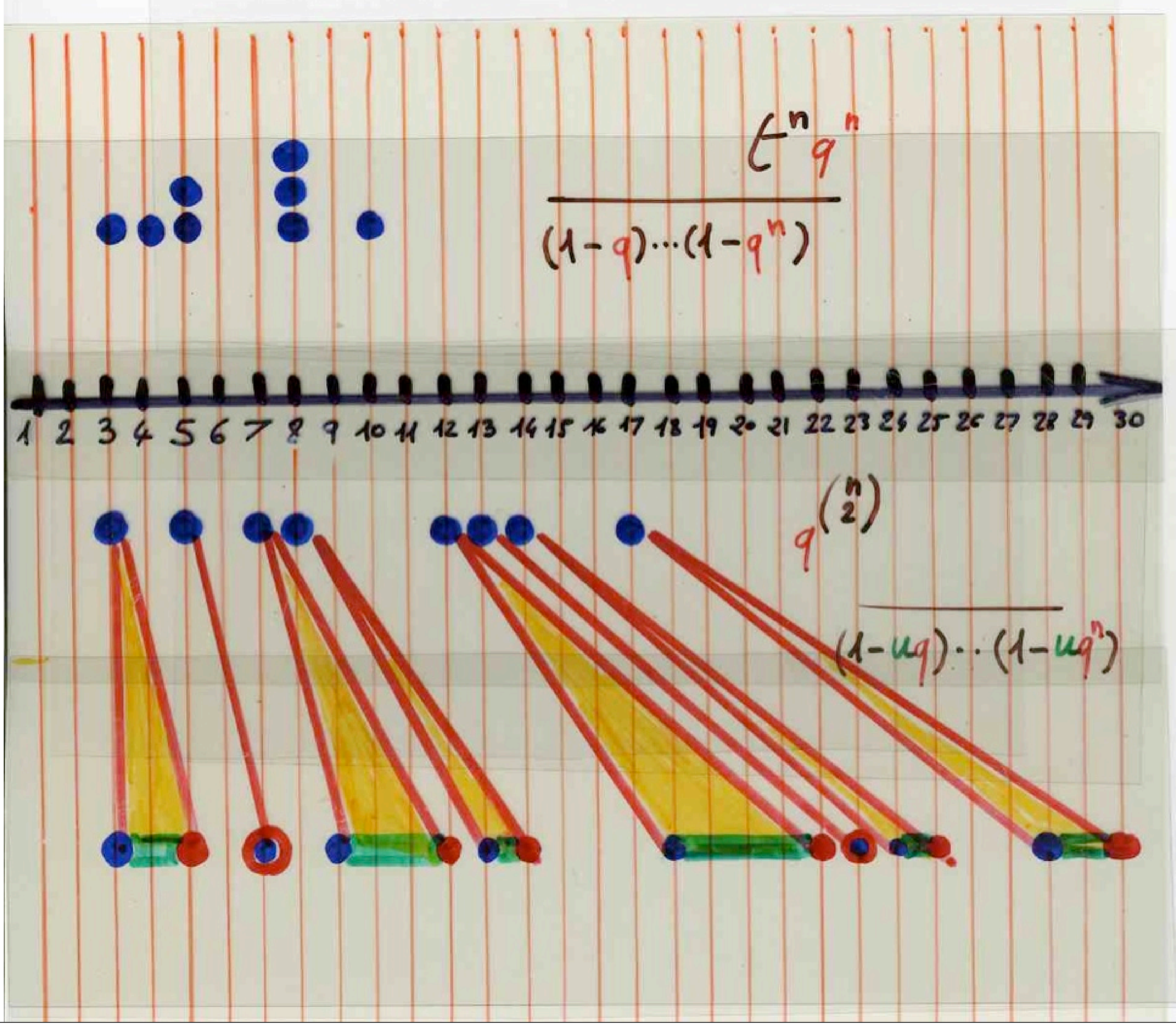


$$D = \sum_{n \geq 0} \frac{(-1)^n}{(1-uq)\cdots(1-uq^n)}$$

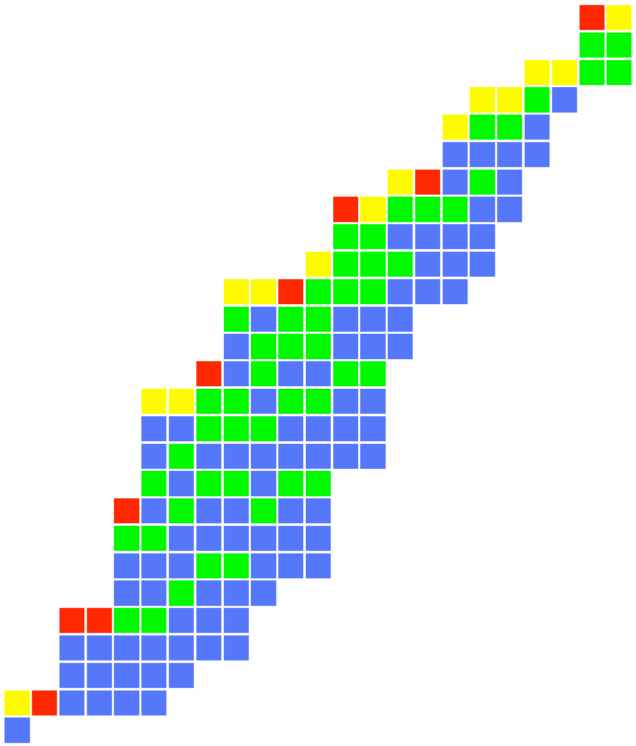
$$\frac{t^n q^n}{(1-q)\cdots(1-q^n)}$$



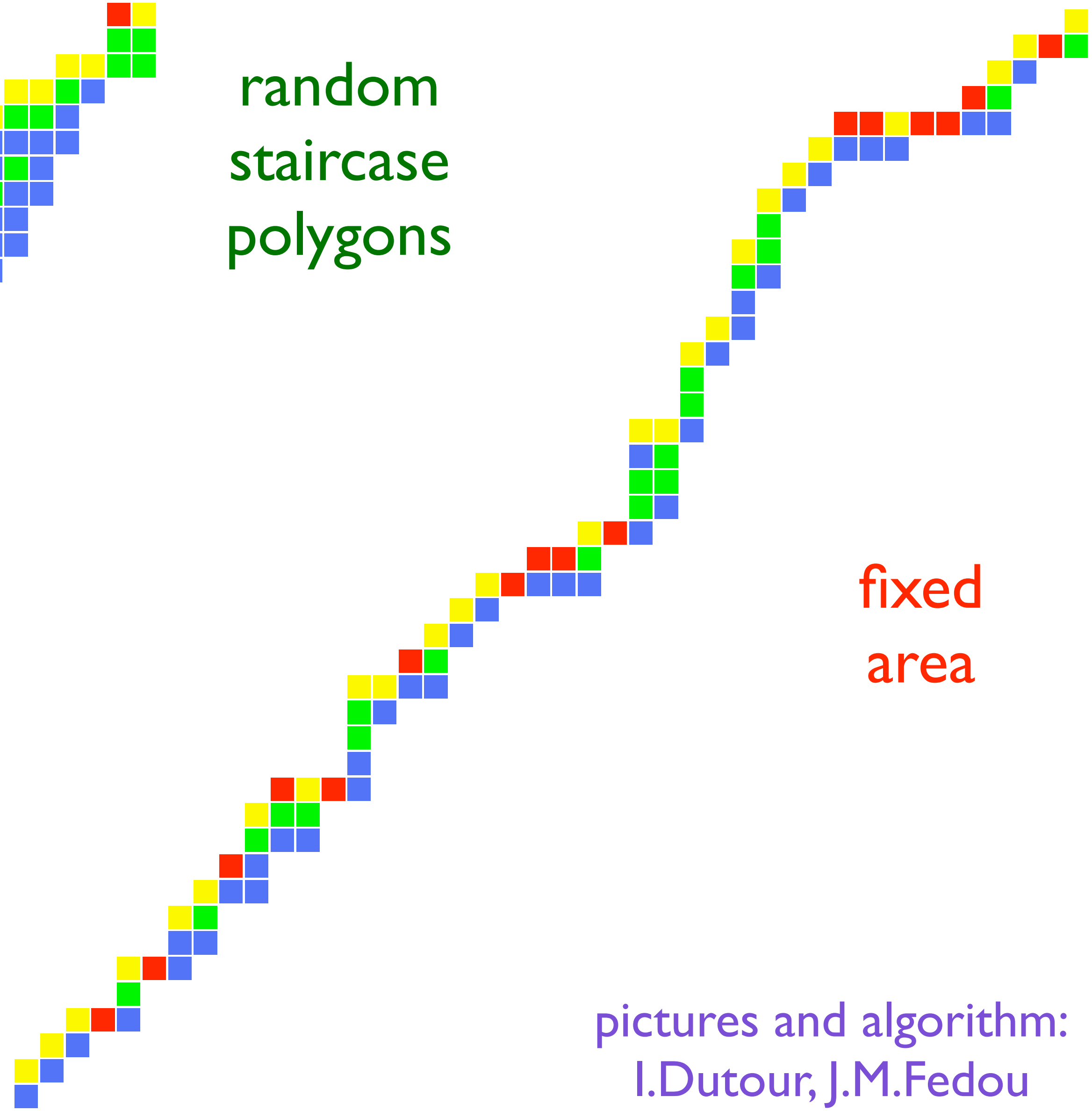
$$q^{\binom{n}{2}}$$


$\frac{t^n q^n}{(1-q)\cdots(1-q^n)}$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30
$\frac{q^{\binom{n}{2}}}{(1-uq)\cdots(1-uq^n)}$

random
staircase
polygons



fixed
perimeter



fixed
area

pictures and algorithm:
I.Dutour, J.M.Fedou

§ 11 Lorentzian triangulations in 2D qantum gravity

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  arXiv: hep-th/9907084

gravitation quantique

(p5)

.... In quantum gravity we are instructed to sum over all geometries connecting, say, two spatial boundaries of length ℓ_1 and ℓ_2, with the weight of each geometry g given by

$$(5) \quad e^{iS[g]} \qquad S[g] = \Lambda \int \sqrt{-g}, \quad \text{(in 2d)}$$

where Λ is the cosmological constant.

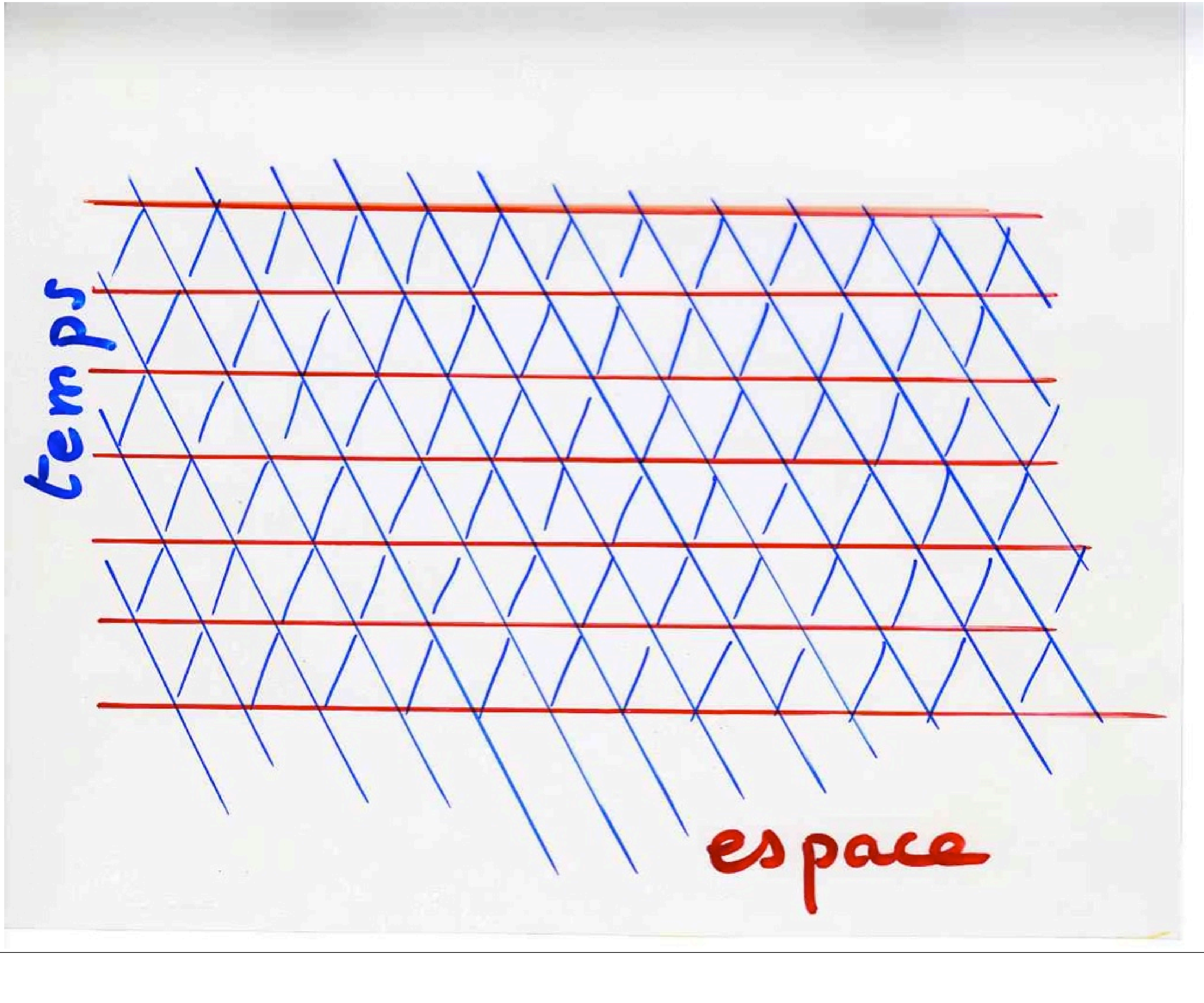
(p7)

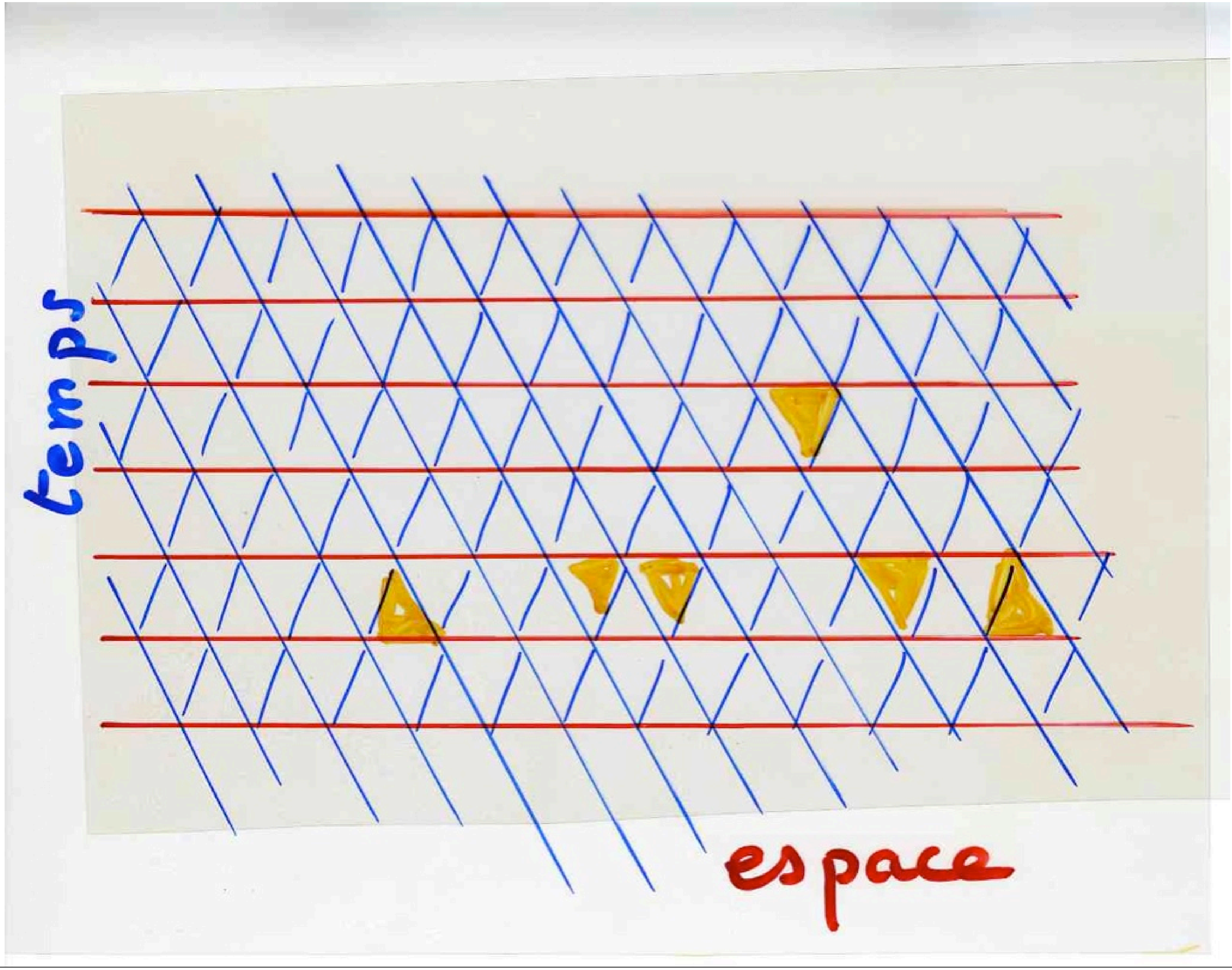
(21) $$F_t(x) = F \frac{1 - xF + F^{2t-1}(x-F)}{1 - xF + F^{2t-1}(x-F)}$$

$$F = \frac{1 - \sqrt{1-4g^2}}{2g}$$

Catalan number !

$$C_n = \frac{1}{(n+1)} \binom{2n}{n}$$


temps
espace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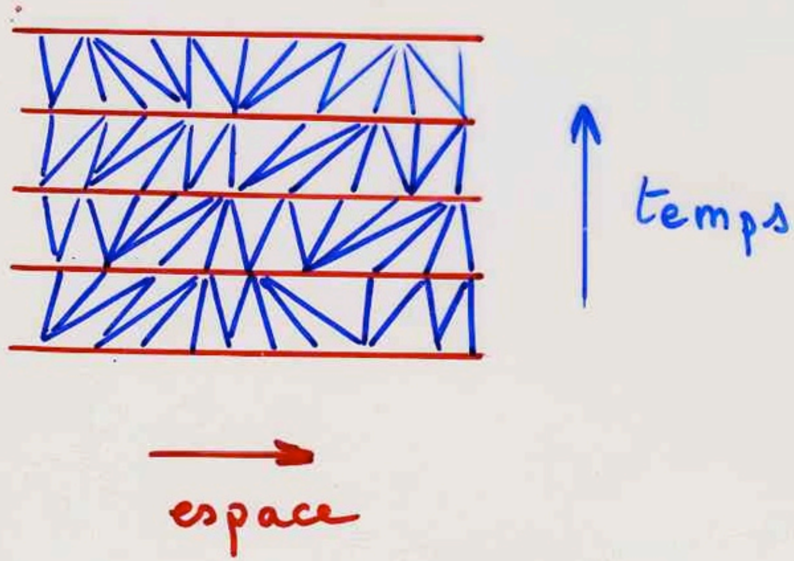
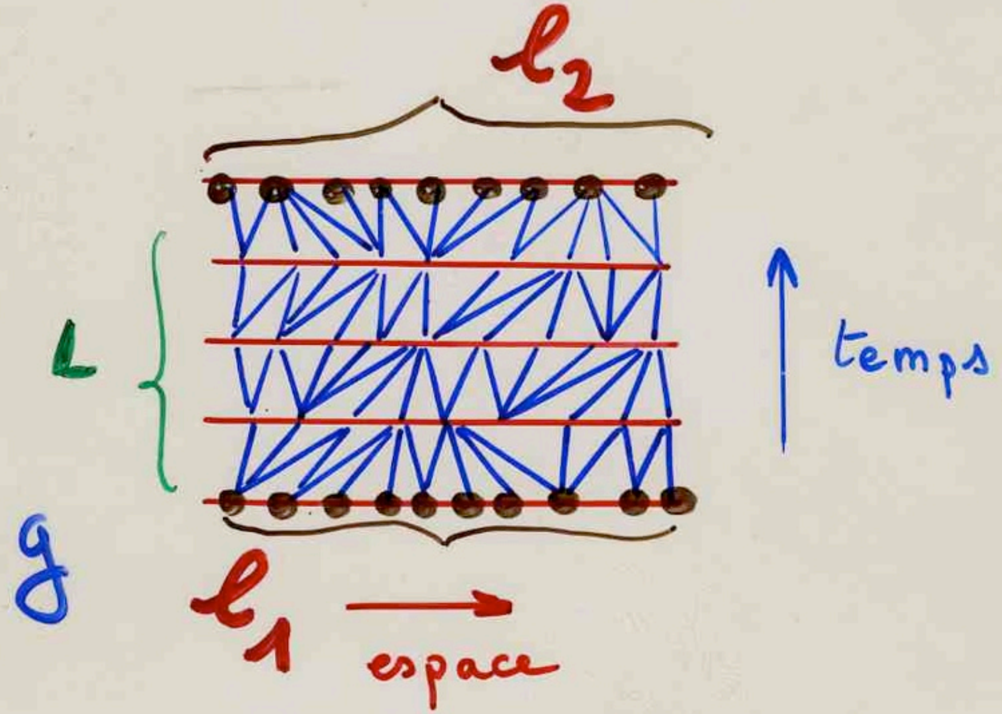
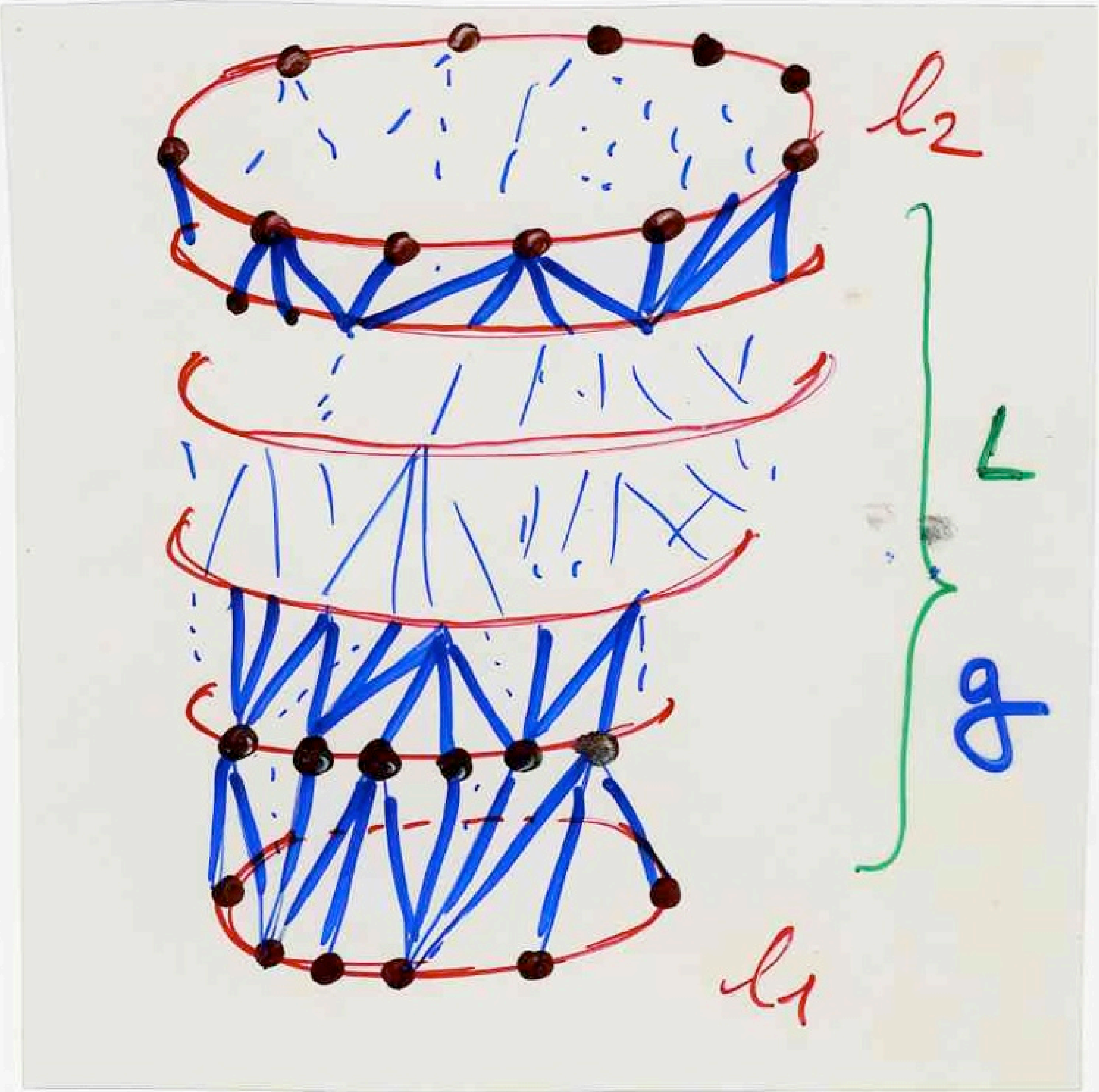


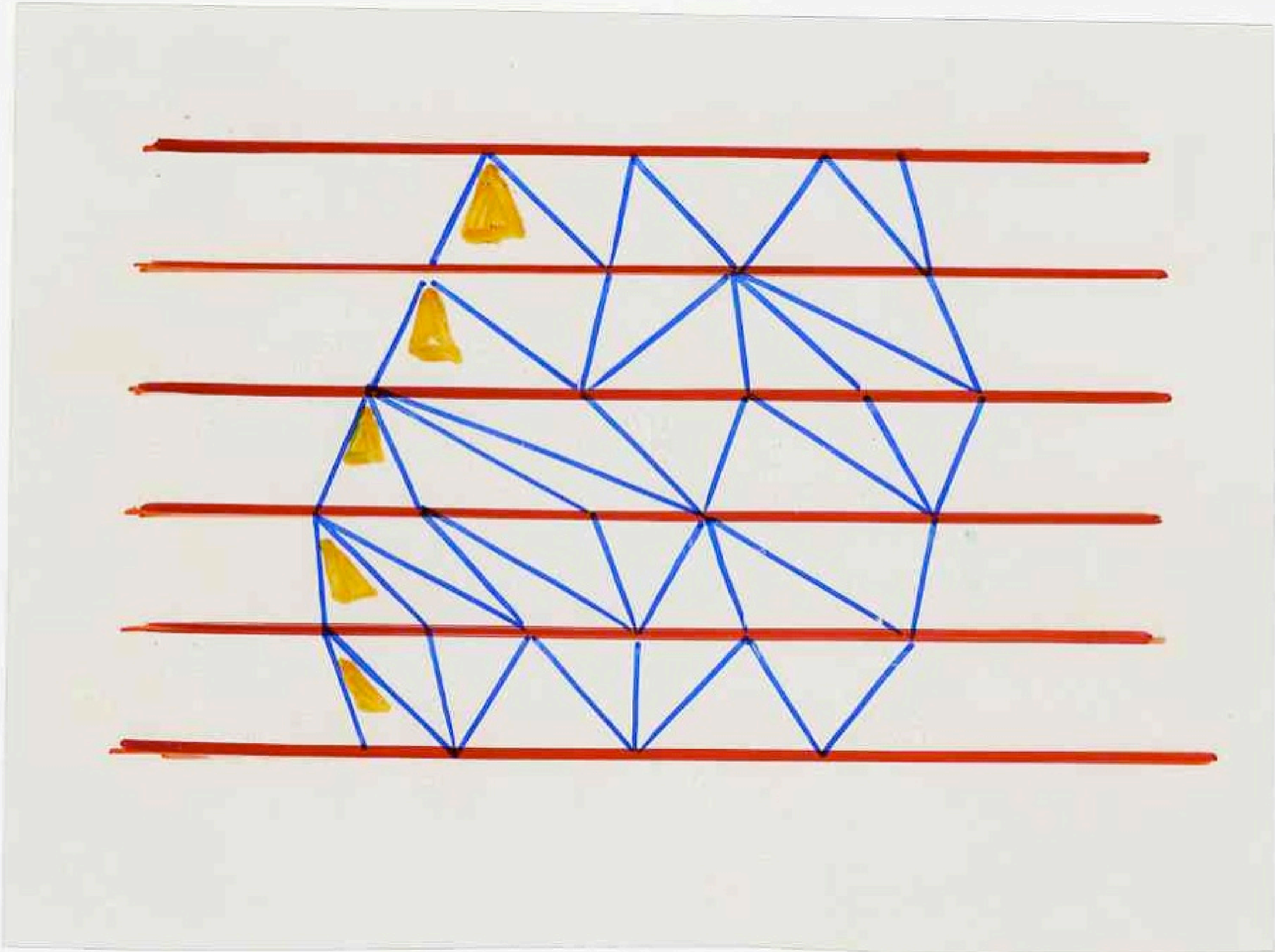
Lorentzian
triangulation

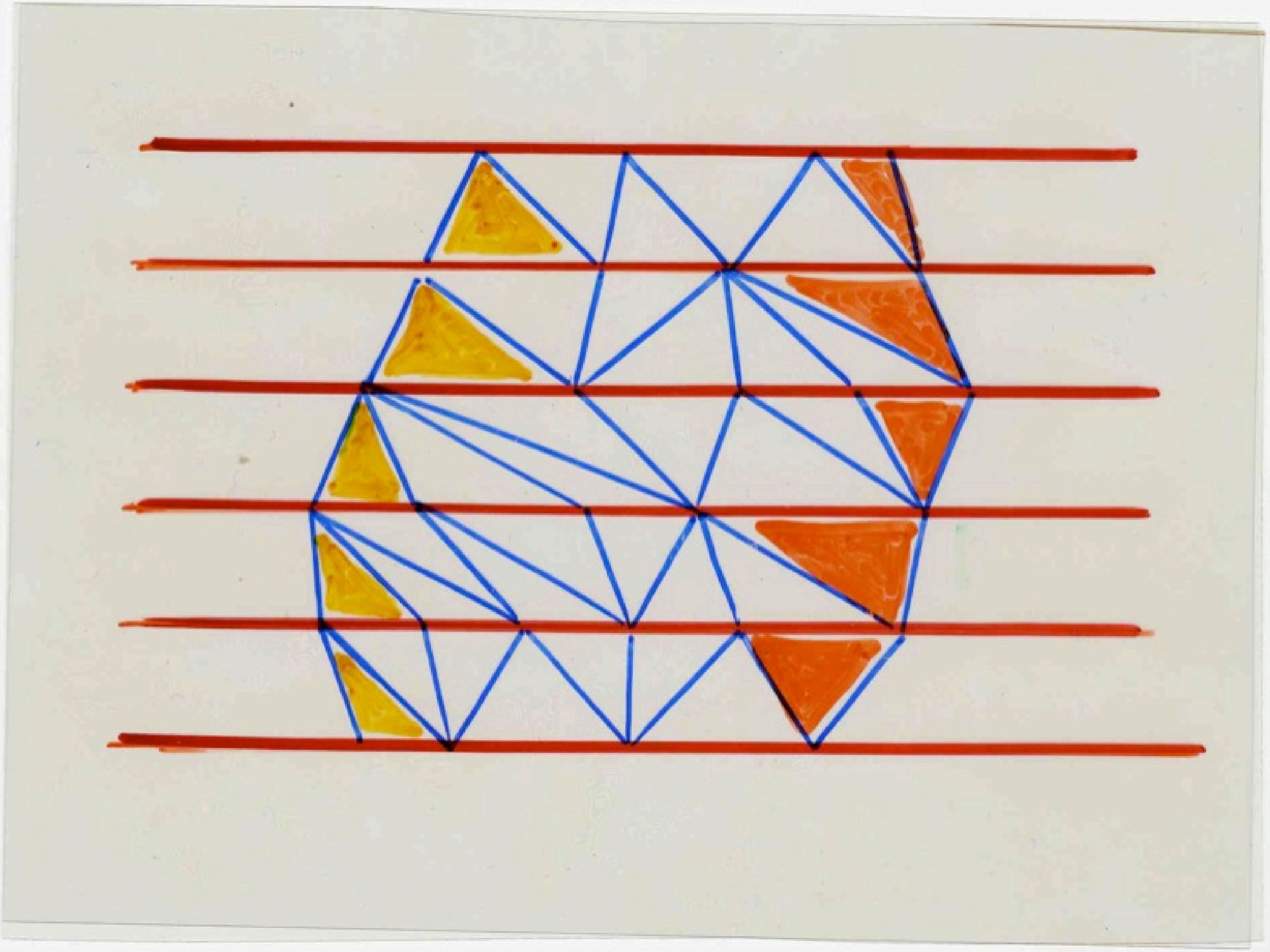


Path integral amplitude for the propagation from geometry ℓ_1 to ℓ_2



l_2
L
g
l_1





Dyck paths

$\updownarrow$

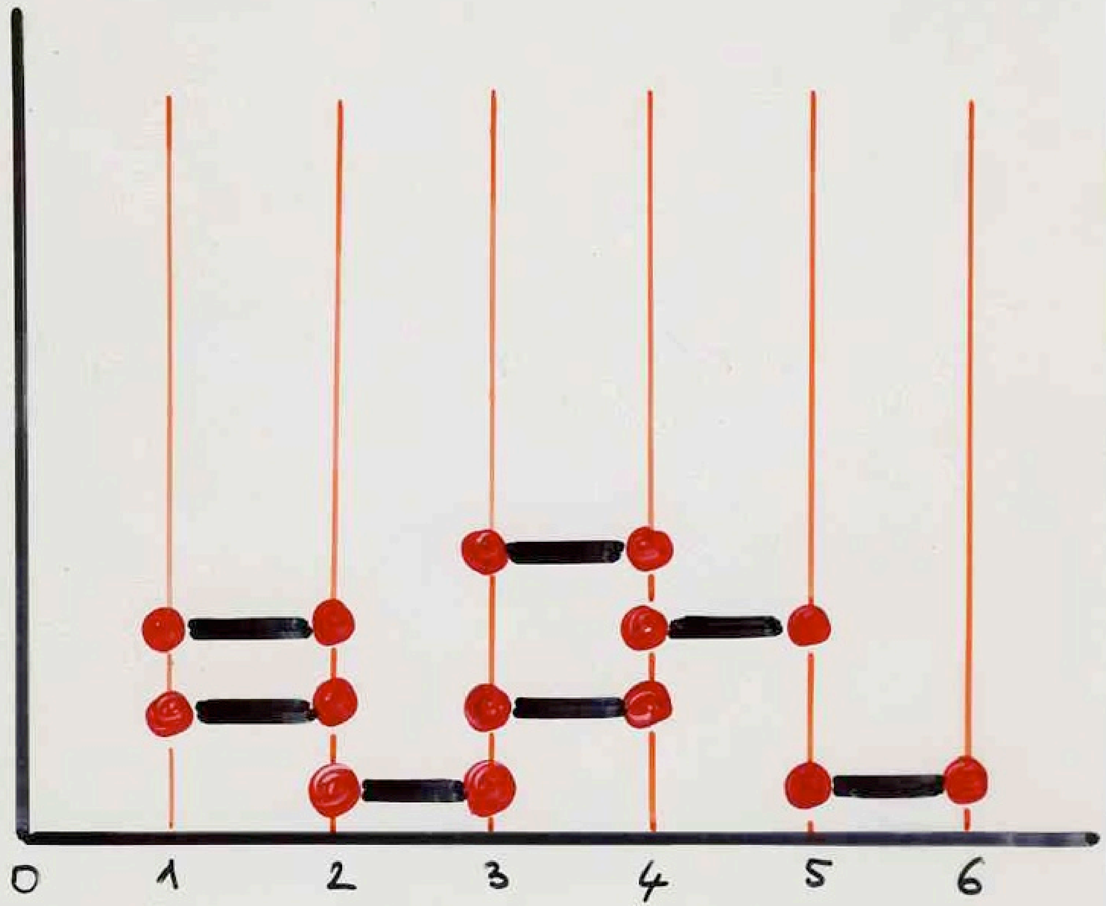
Heaps of dimers

(Pyramids)

$\updownarrow$

Lorentzian triangulations

(*) border condition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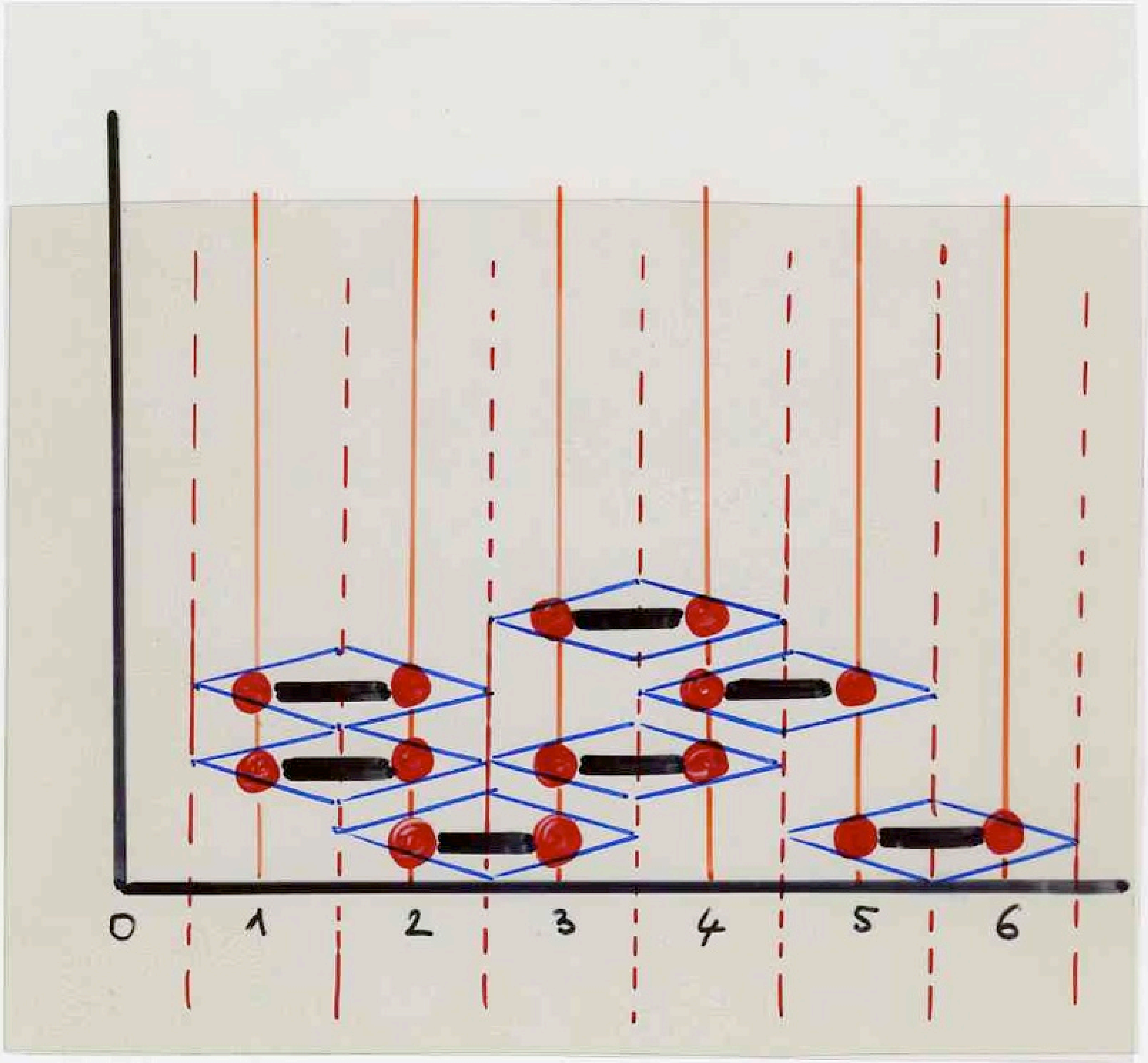

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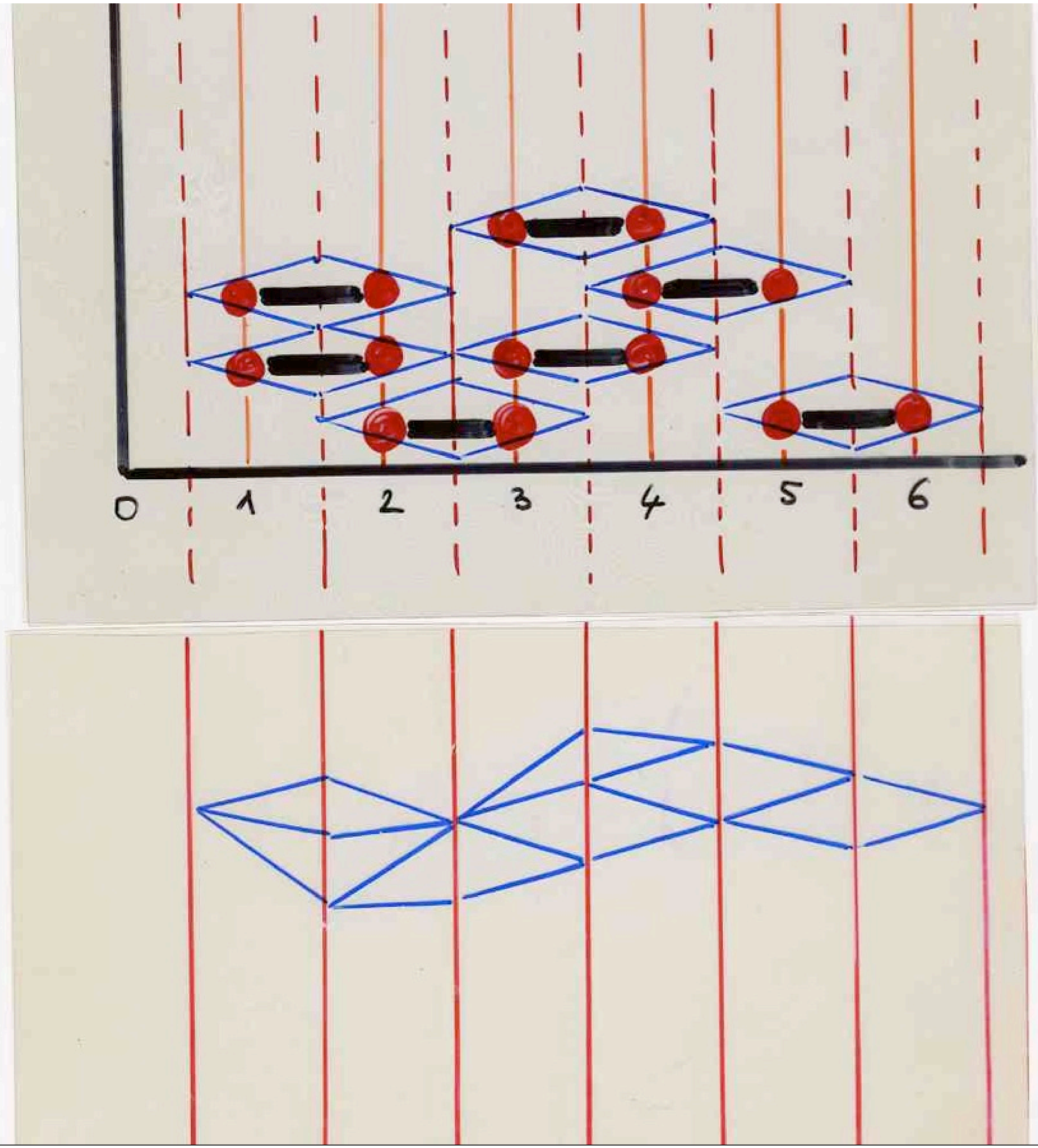
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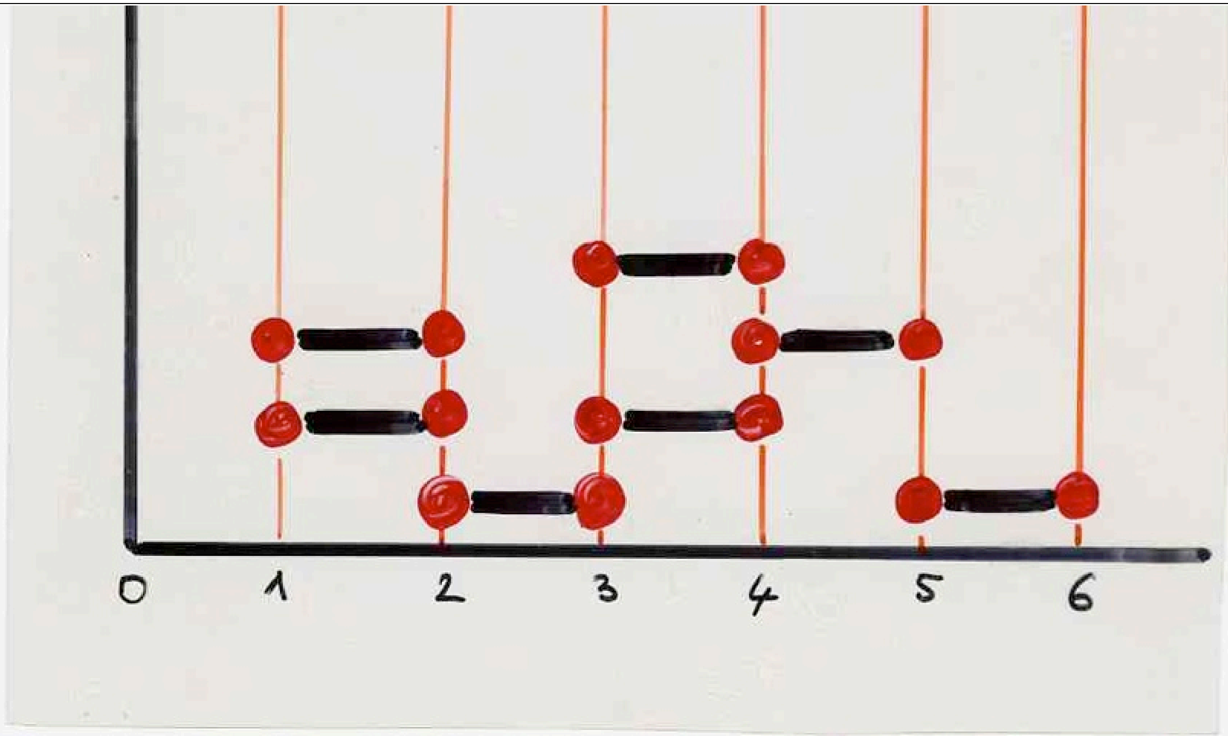
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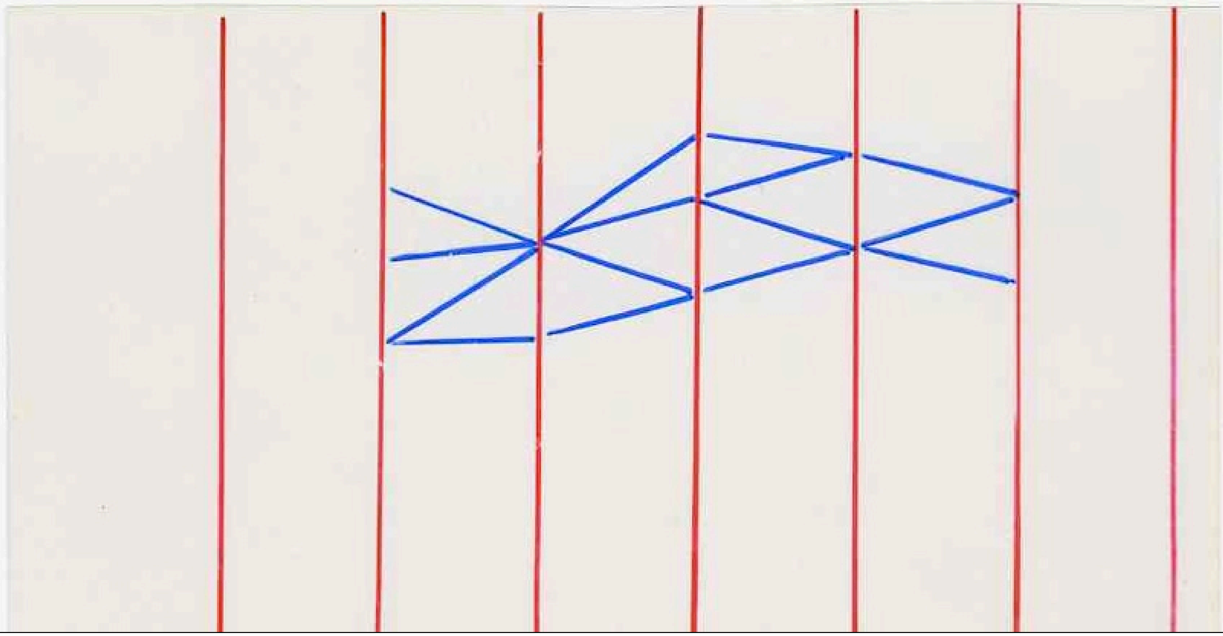

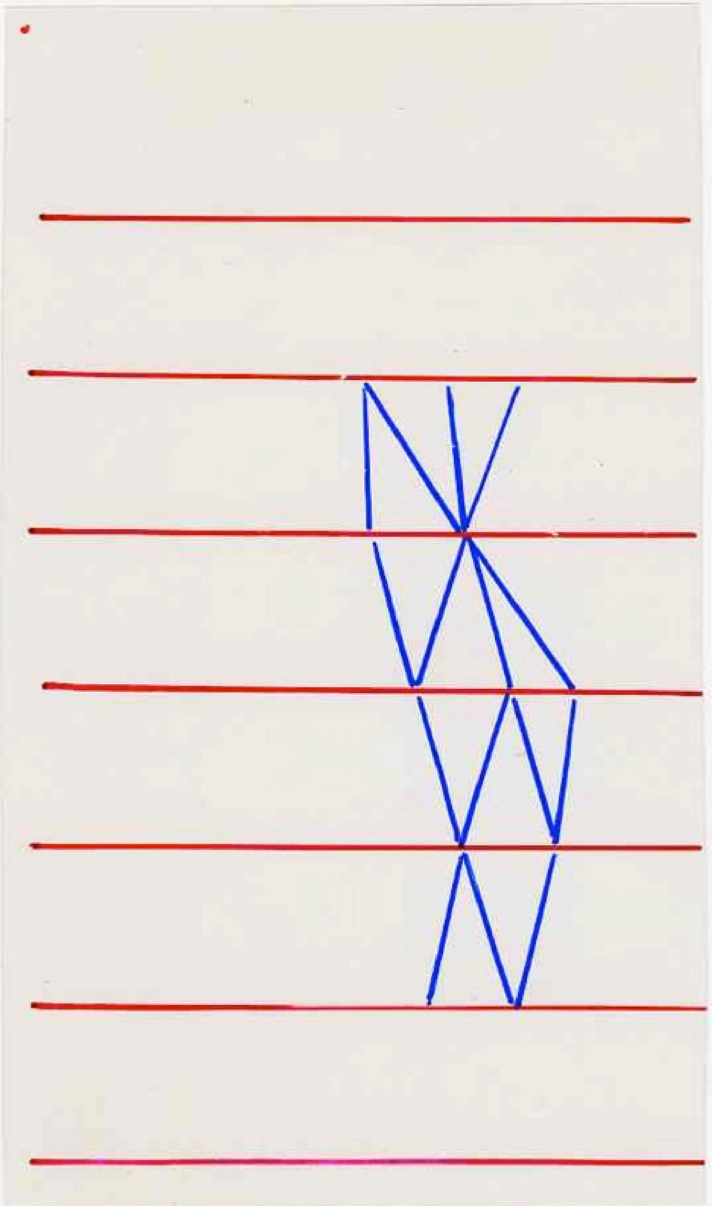

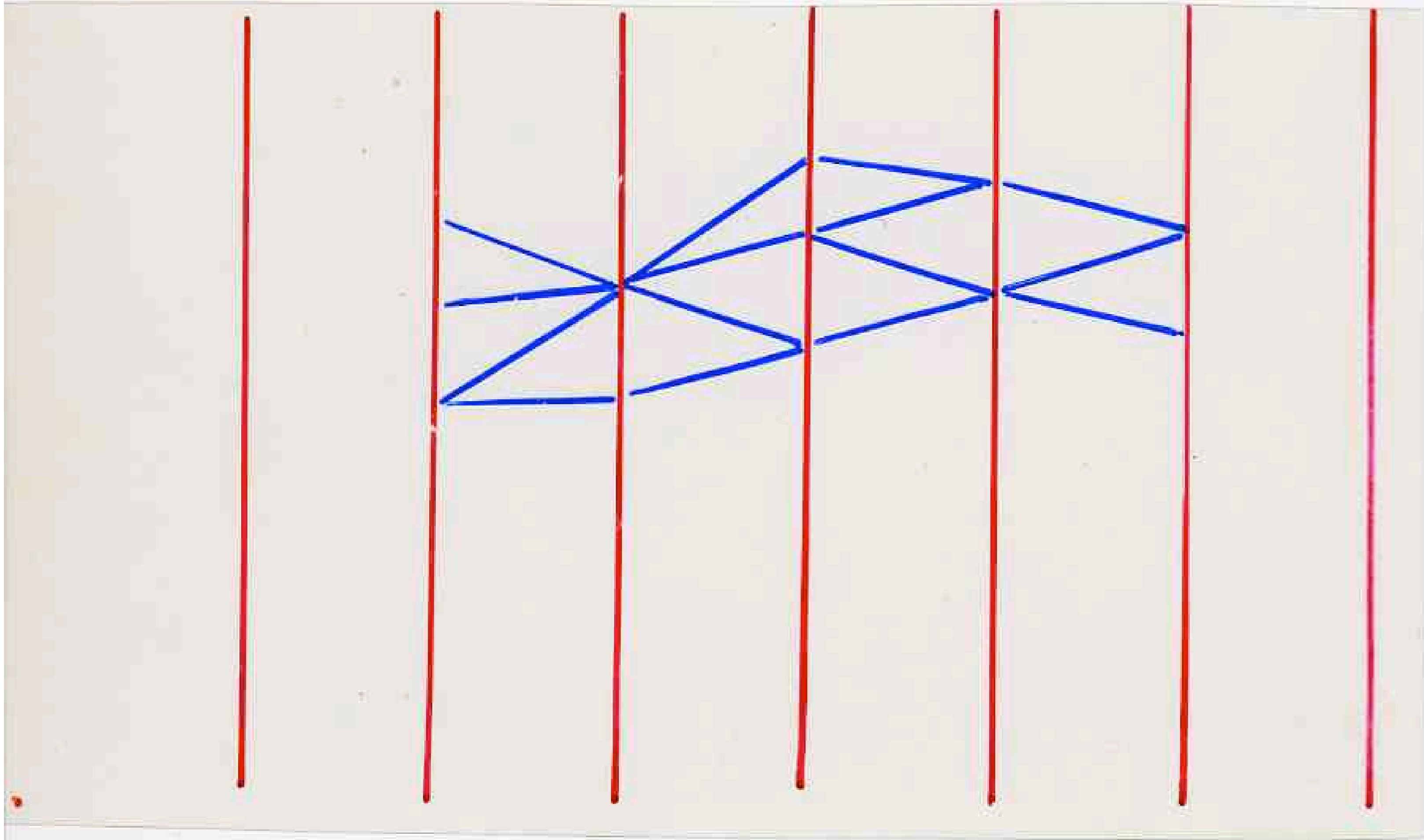

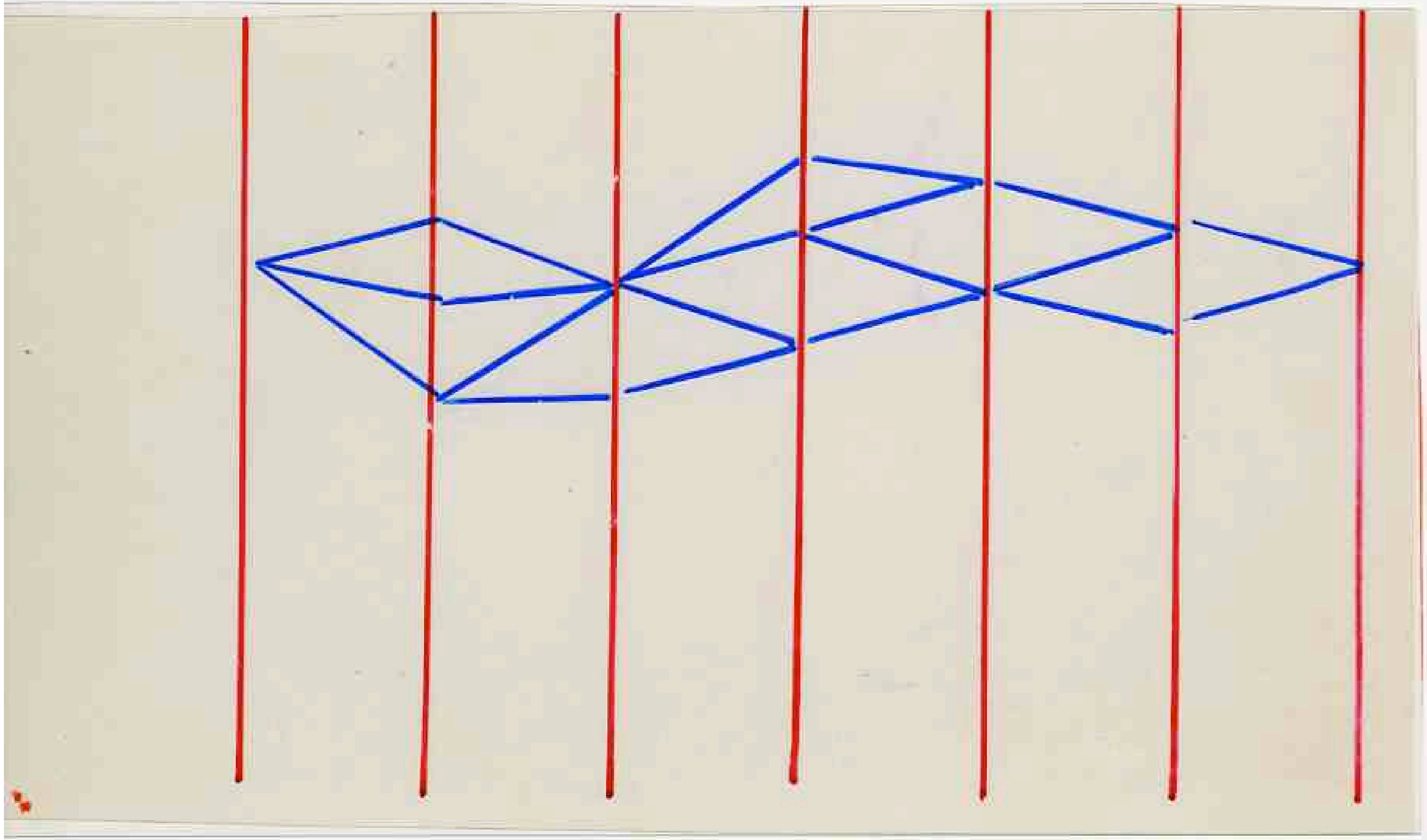

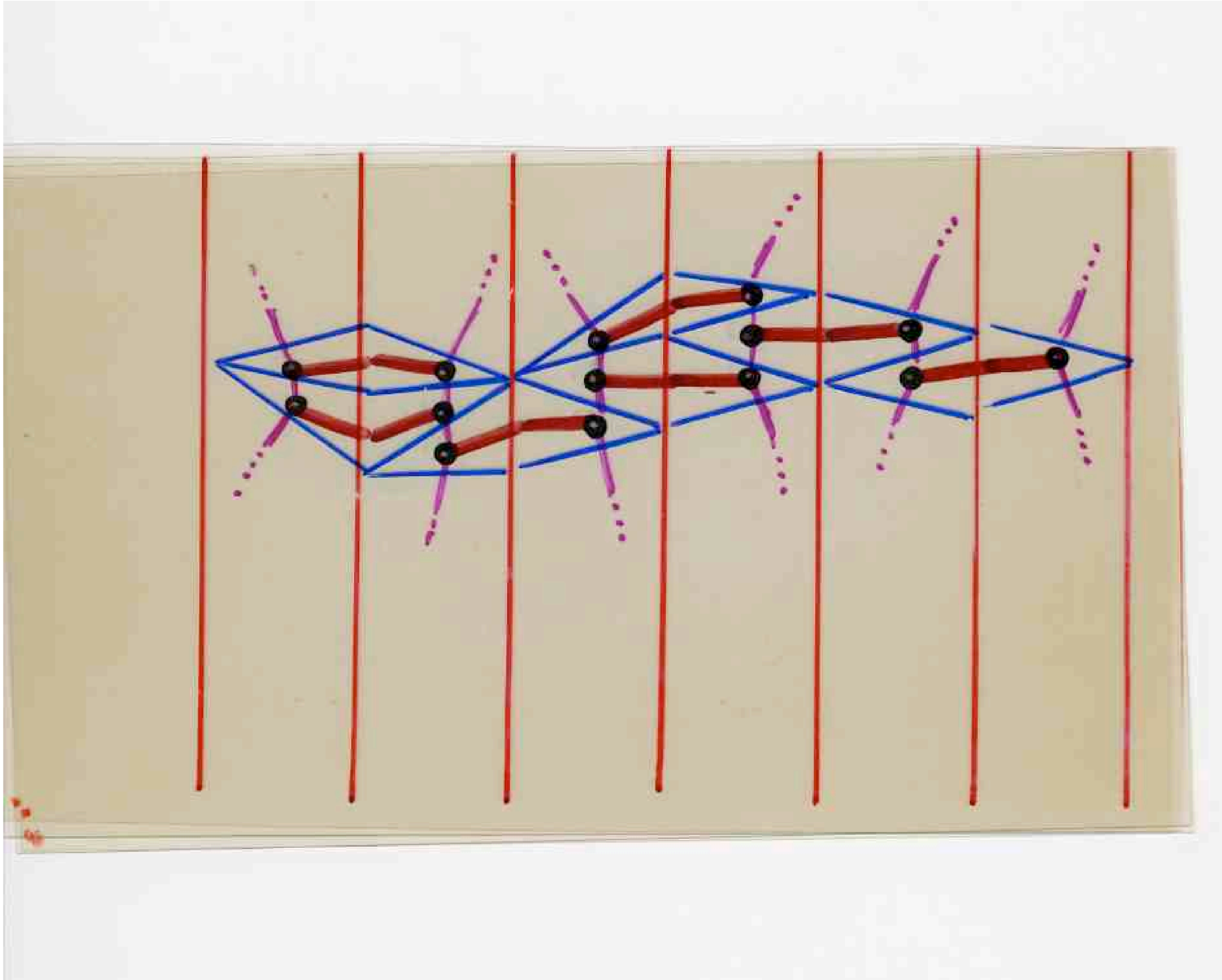

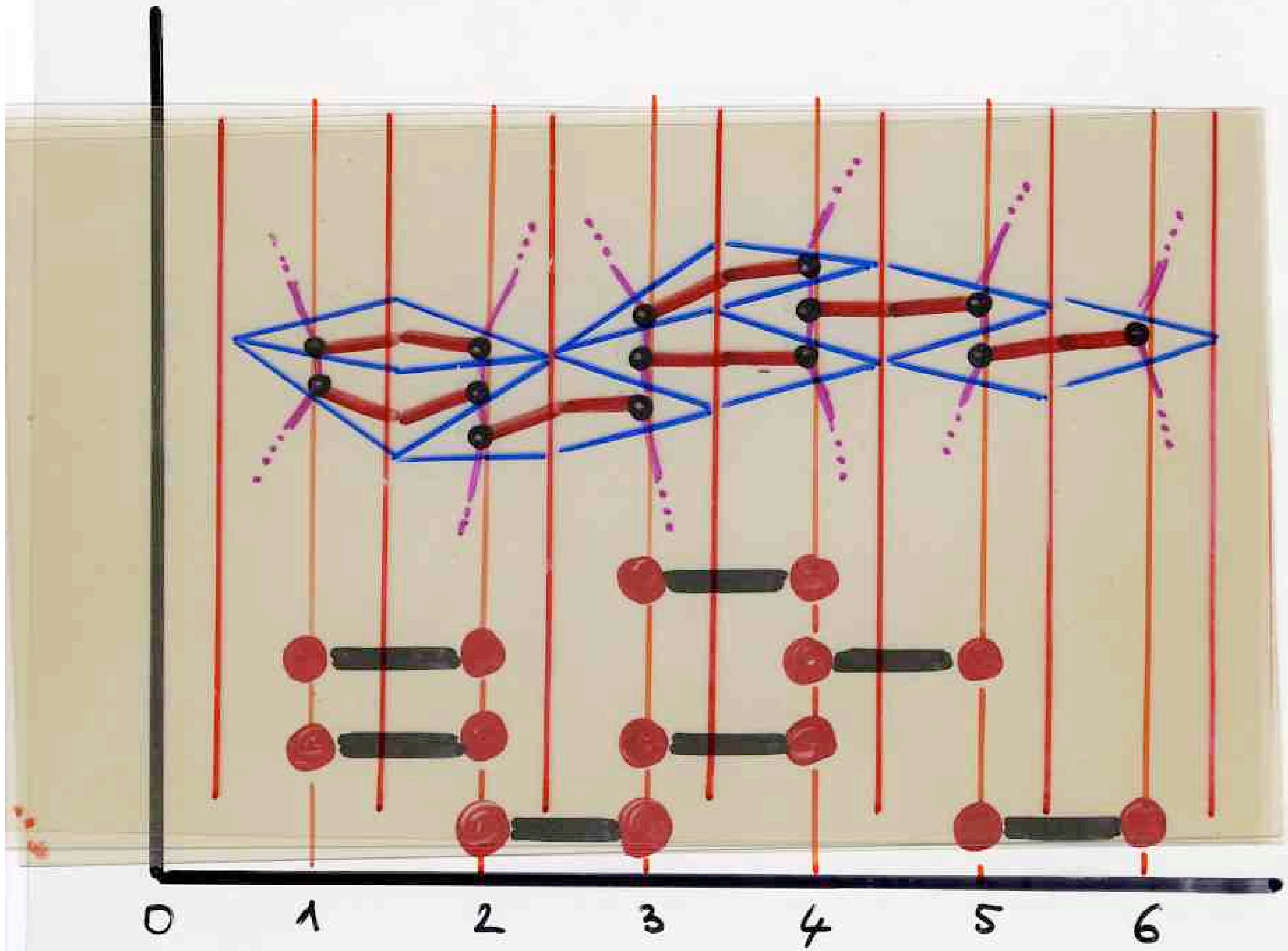
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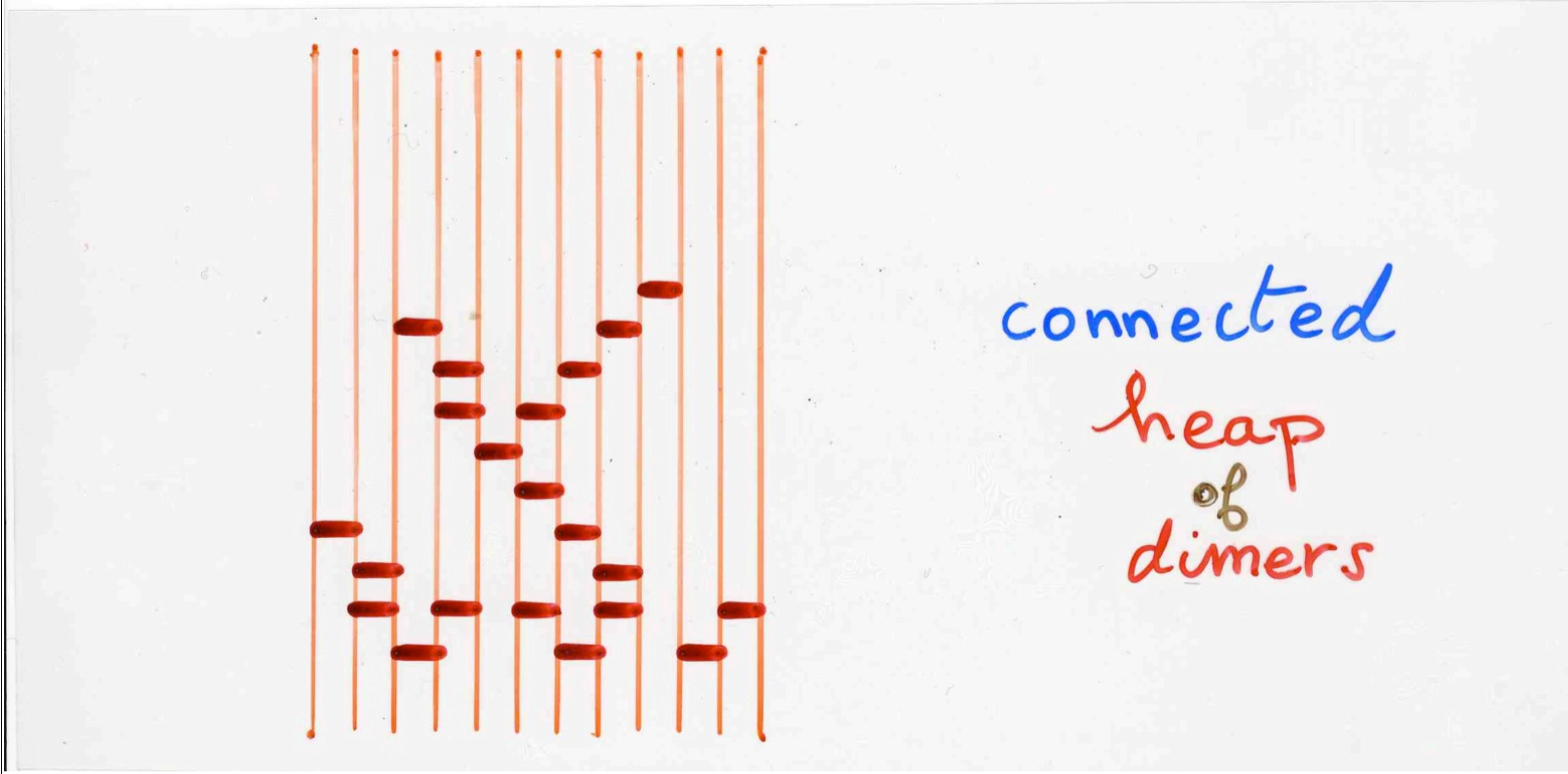
§ 12 extensions :

free border conditions



Bousquet-Mélou, Rechnitzer (2002)

$$C = \frac{Q}{(1-Q)\left[1-\boxed{\sum_{k\geq 1}\frac{Q^{k+1}(1+Q)^{-k-1}}{1-Q^{k}}}\right]}$$

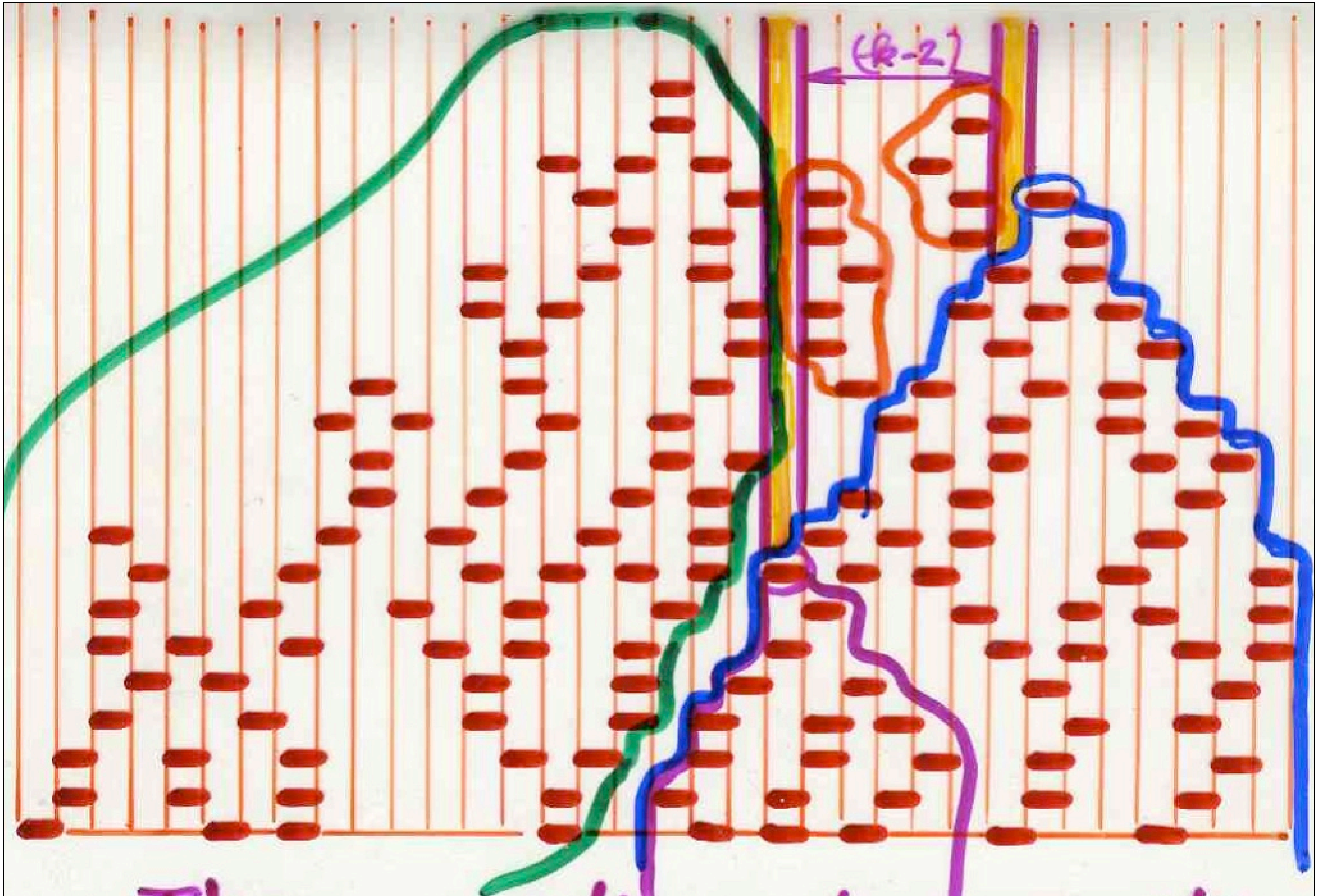
$$= \sum_{k\geq 1}\frac{Q^{k+1}}{1-Q^{k}(1+Q)}$$

Bousquet-Mélou, Rechnitzer (2002)

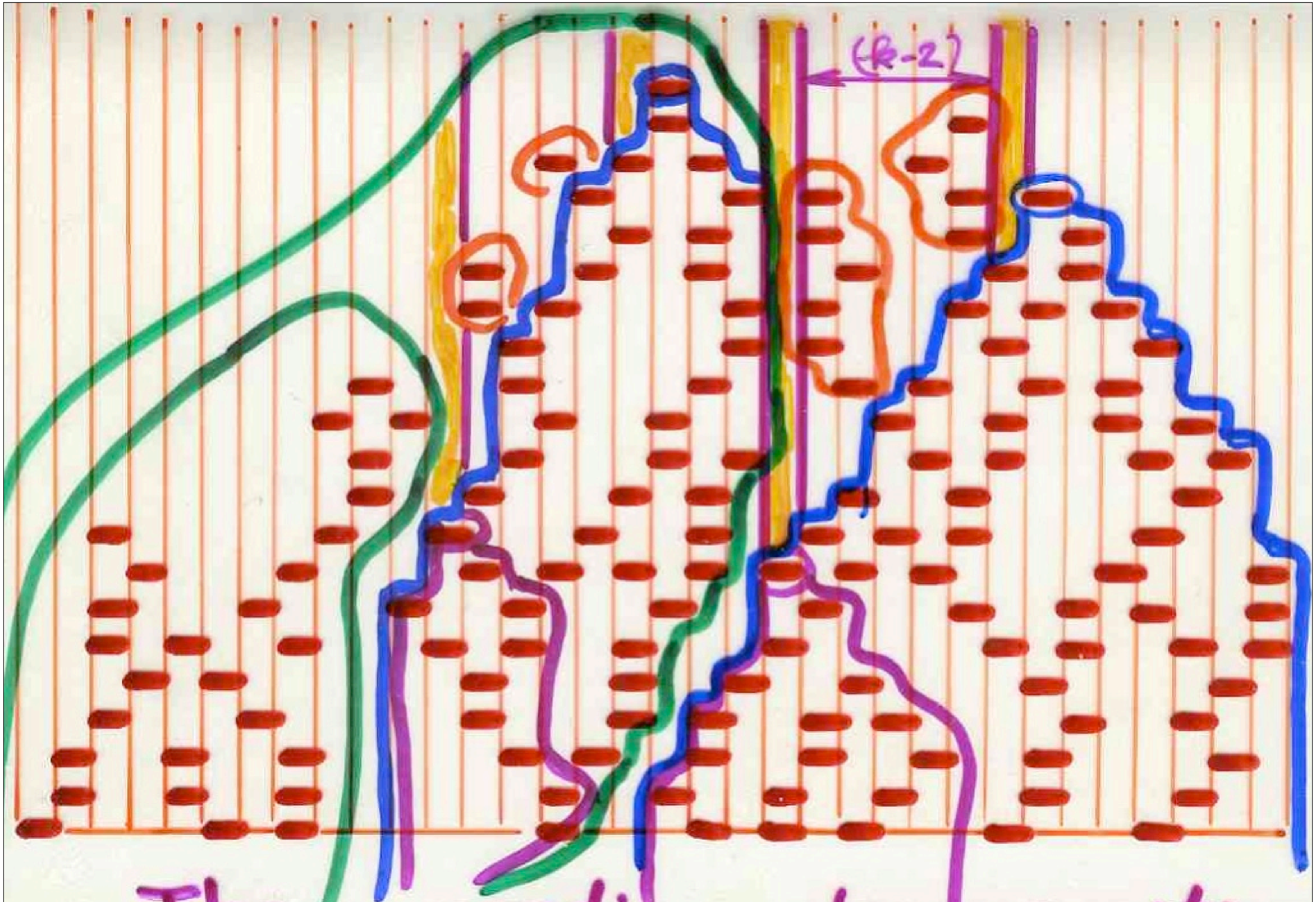
$$C = \frac{Q}{1-Q} \times \frac{1}{\left[1 - \left(\sum_{k \geqslant 1} \frac{Q}{1-Q} \times Q^{k} \times \frac{1}{F_{k-1}}\right)\right]}$$

connected heap

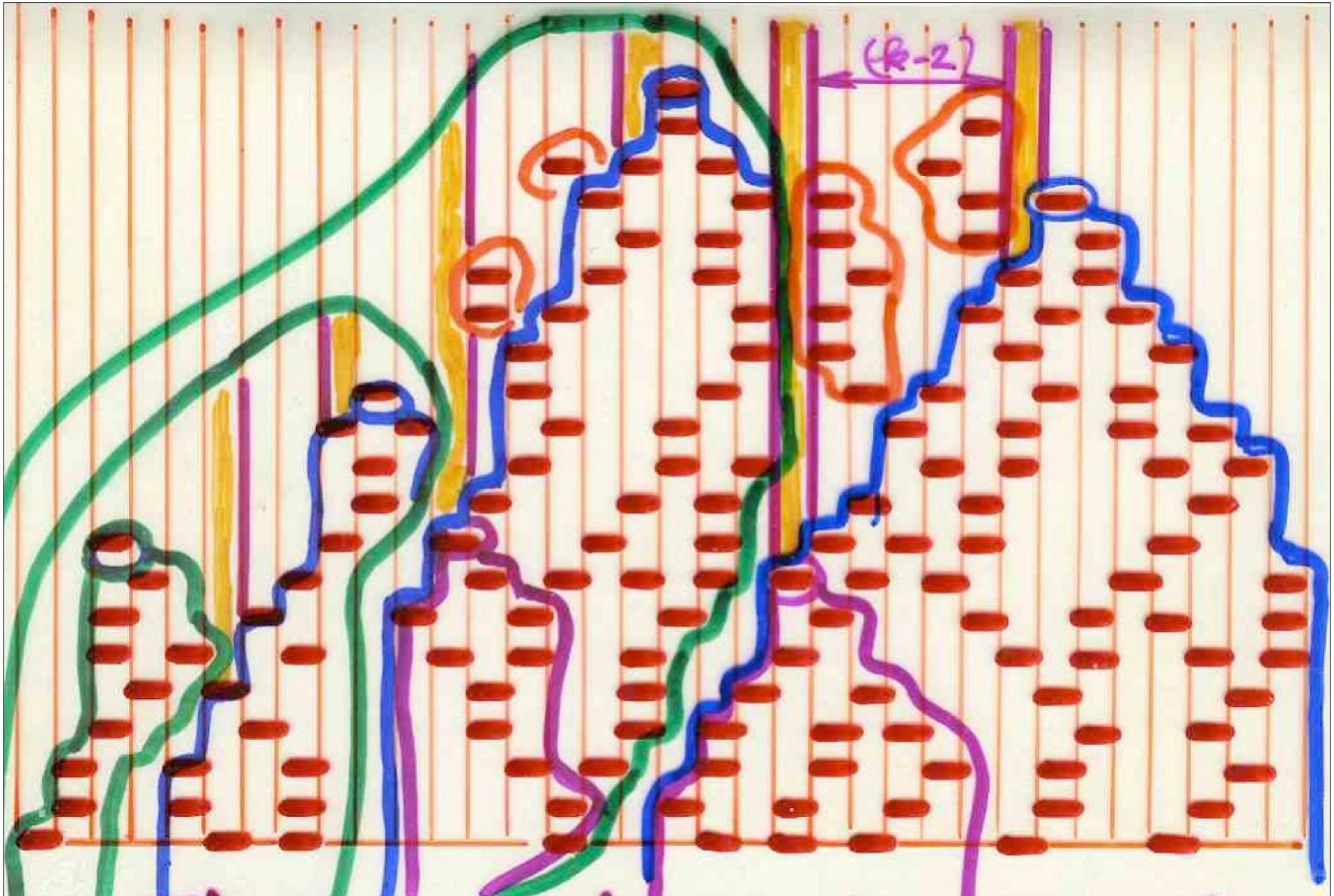
bijective proof viennot (2002)



The nordic decomposition



The nordic decomposition



The nordic decomposition

Lorentzian quantum gravity

(1+1) +1 dimension

Benedetti, Loll, Zamponi (2007)
arXiv:0704.3214

Benedetti, thesis (2007)
arXiv: 0707.3070

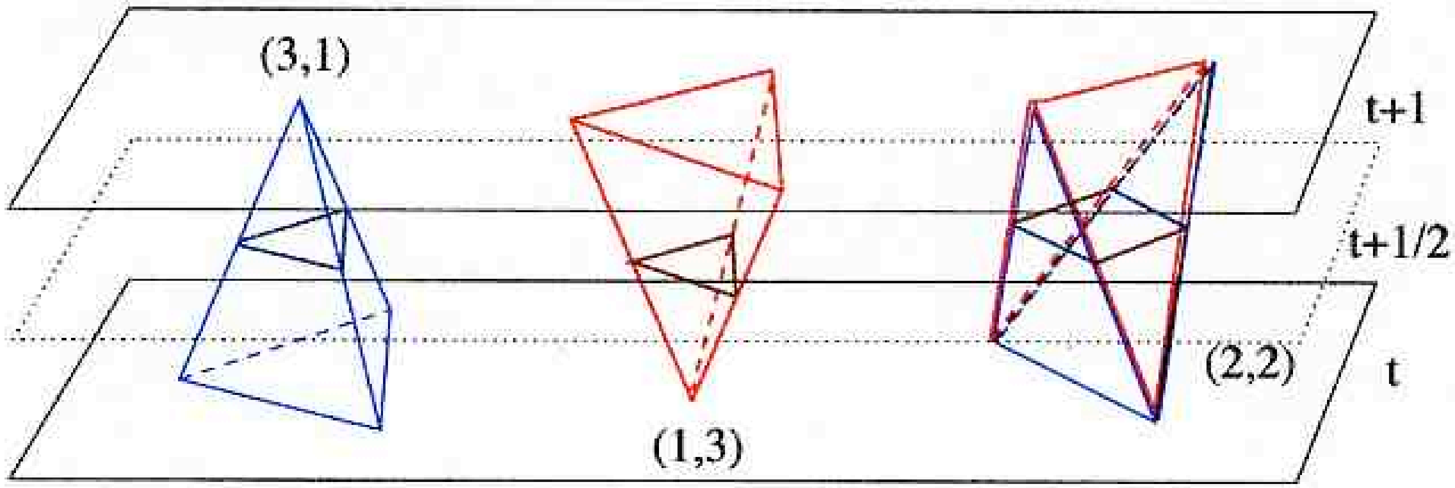
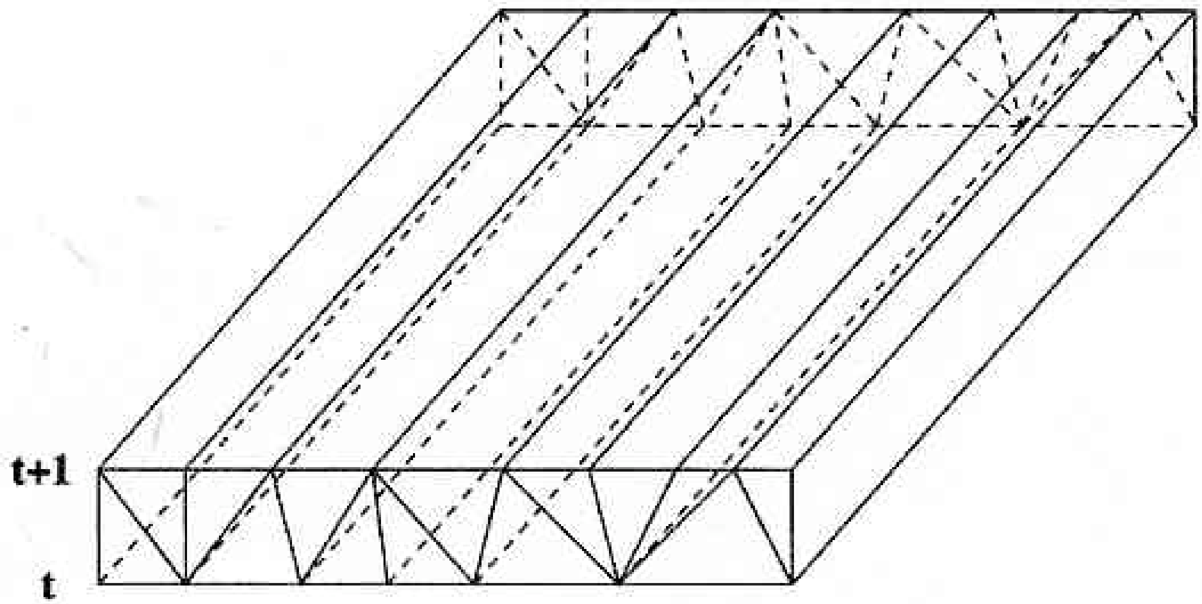


Figure 4: The three types of tetrahedral building blocks and their intersections at time $t + 1/2$.


t+1
t

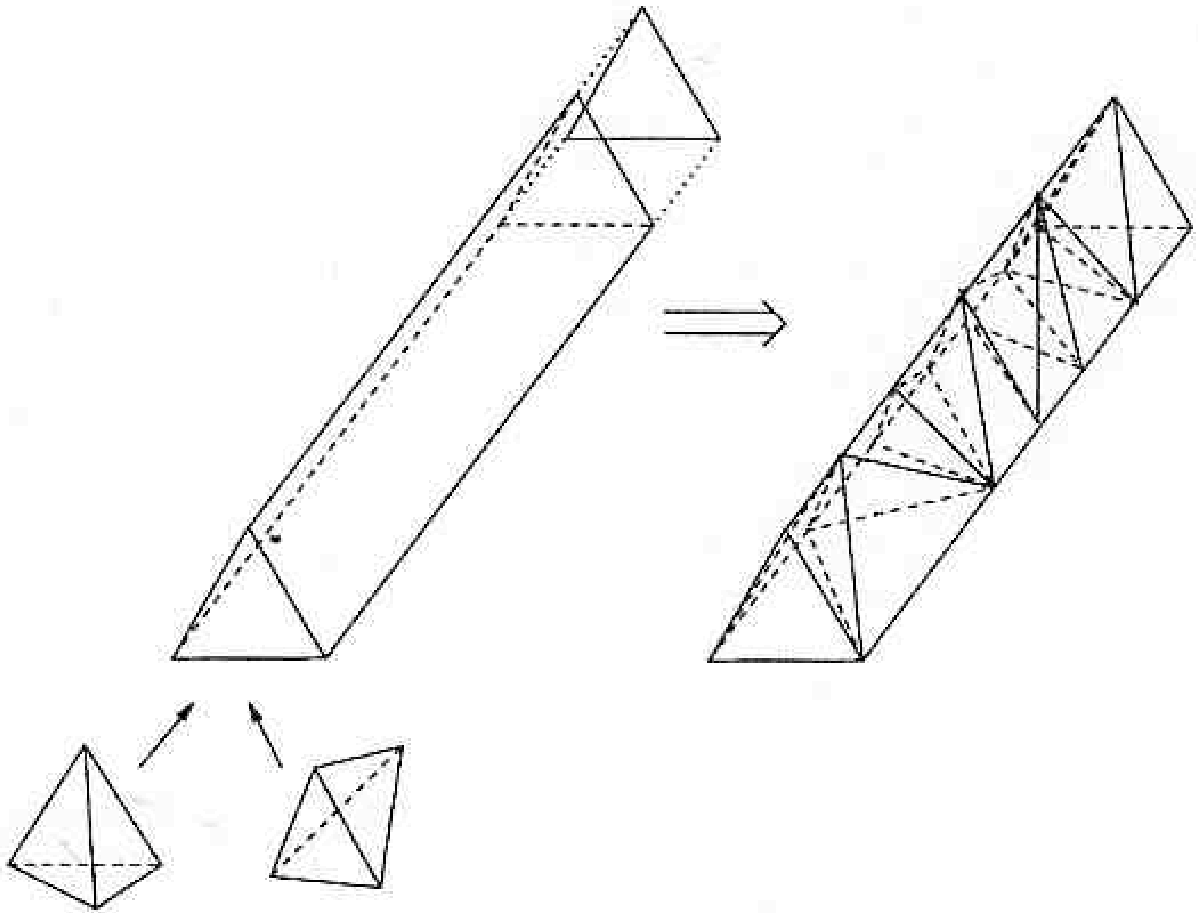
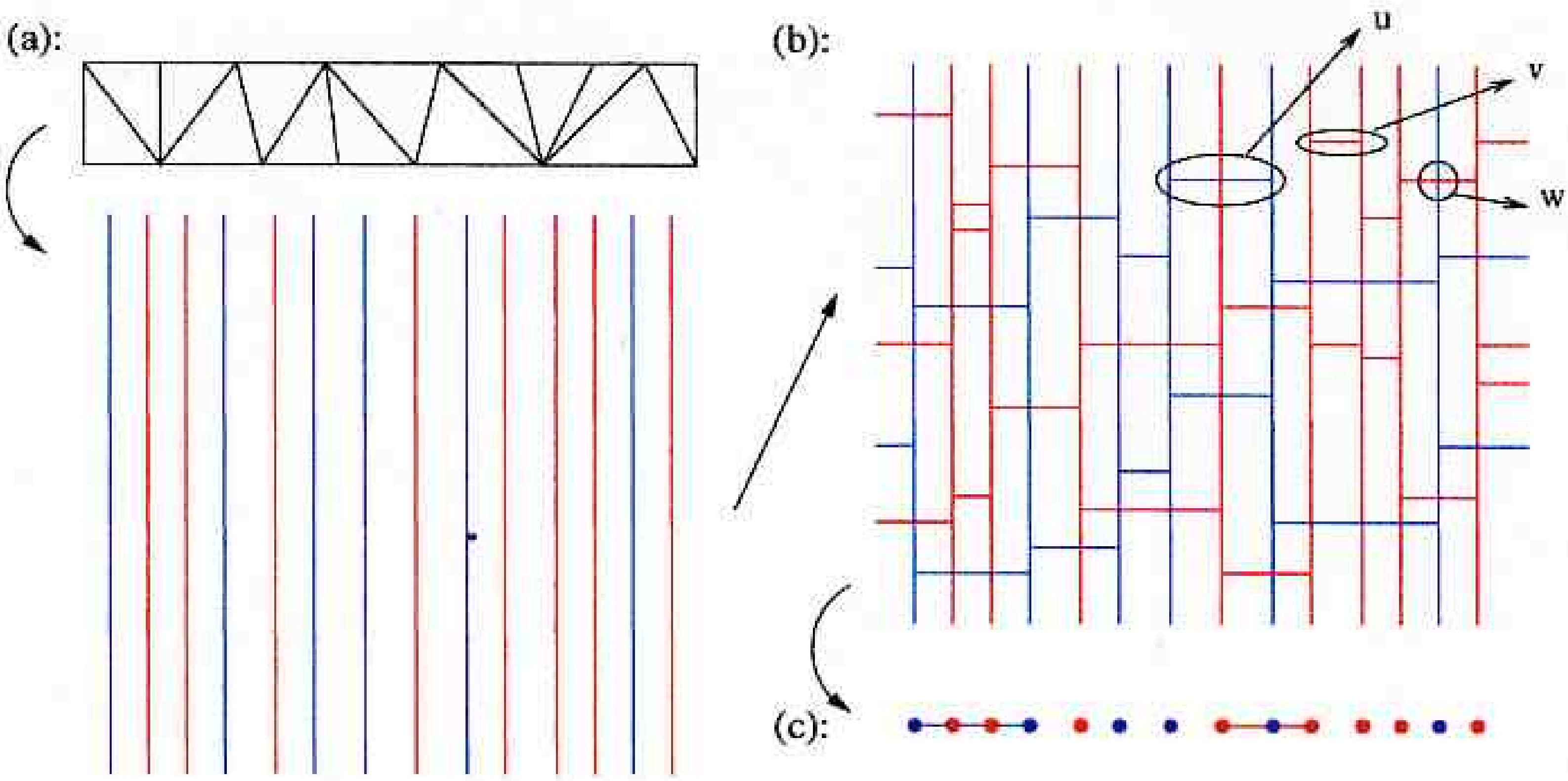


Figure 2: A triangulated prism constructed as a tower over a two-dimensional triangle.


(a):
(b):
u
v
w
(c):

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http://www.labri.fr/perso/viennot

papers [37], [41], [44], [67], [84]
also at the page "cours", Combinatorics course given at Physics Institute, CEA Saclay, Sept-Oct 2007, chapter 3 more slides on "heaps and physics" (15+15 Mo)

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see papers of M. Bousquet-Melou on directed animals and multidirected

Pierre Lalonde, "Contribution à l'étude des empilements", thesis, Monographies du LACIM n° 4, LACIM, UQAM, Montréal, 1991

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"book", Cartier-Foata monography
addition by Krattenthaler on heaps of pieces

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Christian Krattenthaler, go to "papers", "other papers"
survey paper (55 p) for "Encyclopedia for theoretical physics"

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Merci
Pierre !

§1 Commutation monoids

§2 Heaps of pieces: basic definitions

§3 Heaps and posets

§4 Heaps of pieces: generating functions

§5 Paths and heaps of cycles

path -- heap of cycles: inverse bijection

example: bijection Dyck paths

§6 Transition matrix

§7 Bauer identity

§ 8 The directed animal model

generating functions for directed animals

§ 9 hard hexagons

thermodynamic limit

§ 10 q-Bessel functions in statistical physics

staircase polygons

§ 11 Lorentzian triangulations in 2D qantum gravity

§ 12 extensions : free border conditions

(1+1) + 1 dimension

references