Alternative tableaux, permutations and Partially ASymmetric Exclusion Process

Isaac Newton Institute
23 April 2008

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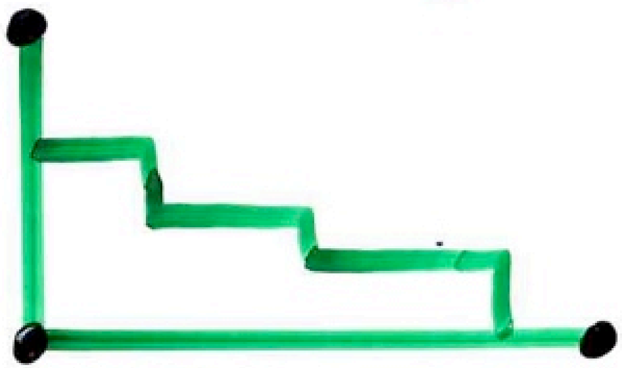
§1

alternative tableau: definition

alternative tableau

- Ferrers diagram F

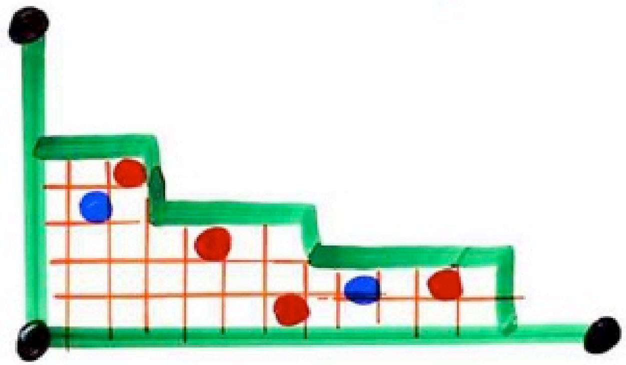


(possibly empty rows or column)

(nb of rows) + (nb of columns) $= n$

alternative tableau

- Ferrers diagram F



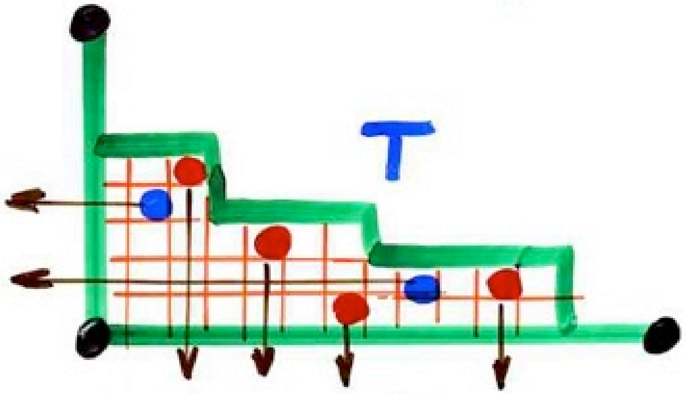
(possibly empty rows or column)

(nb of rows) + (nb of columns) $= n$

- some cells are coloured **red** or **blue**

alternative tableau T

- Ferrers diagram F (possibly empty rows or column)



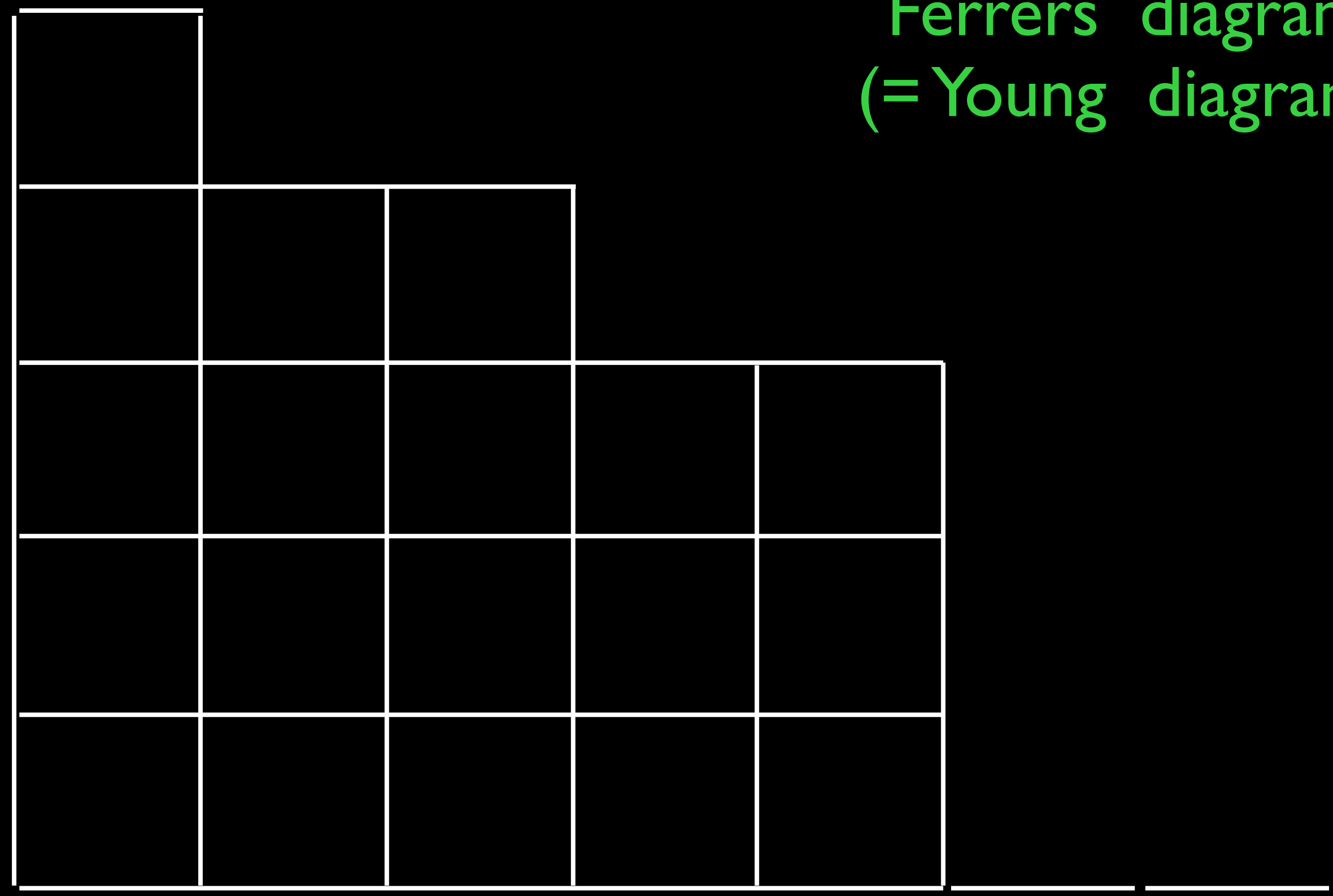
(nb of rows) + (nb of columns) = n

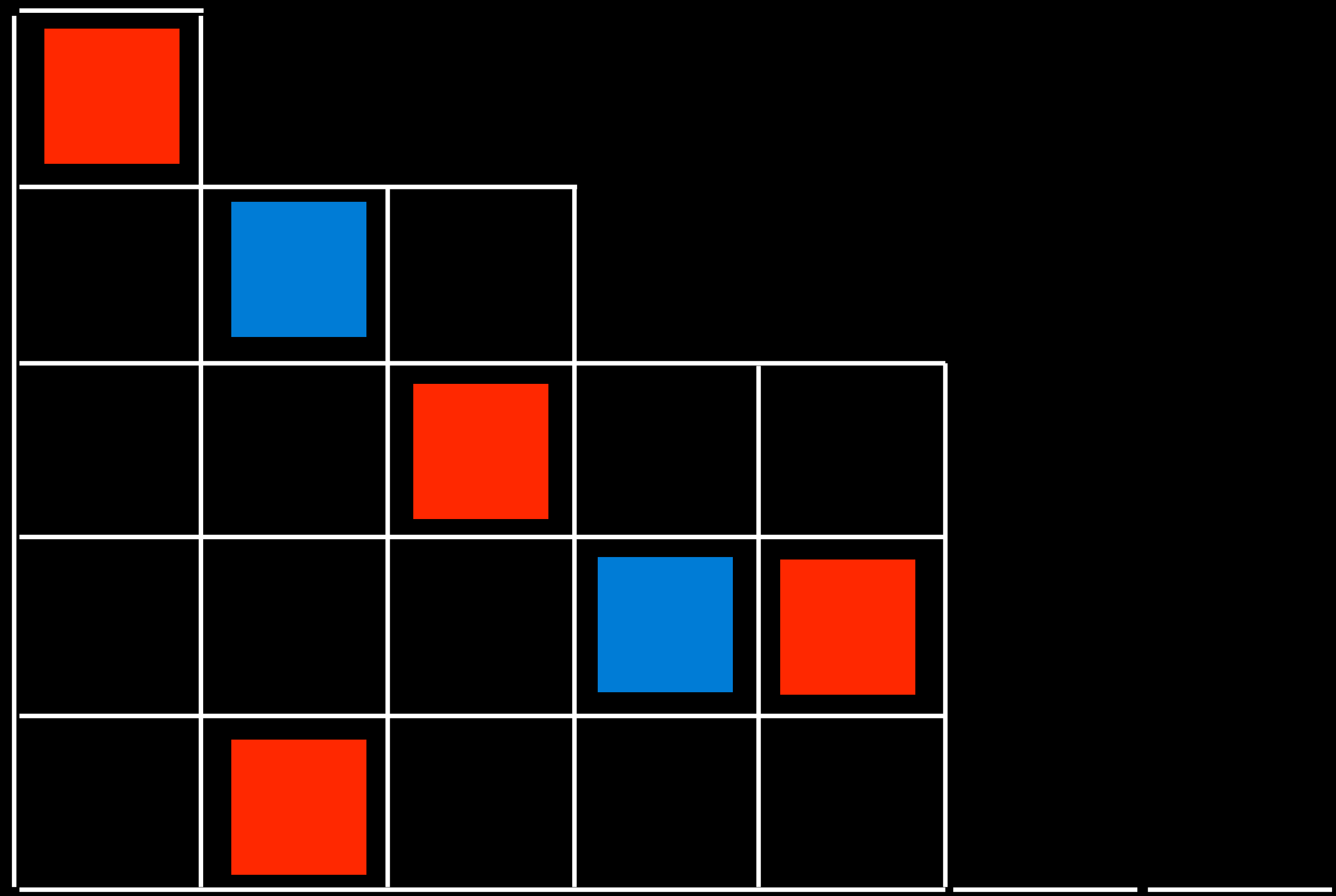
- some cells are coloured **red** or **blue**
- { no coloured cell at the left of [blue cell]
  { no coloured cell below [red cell]

n size of T

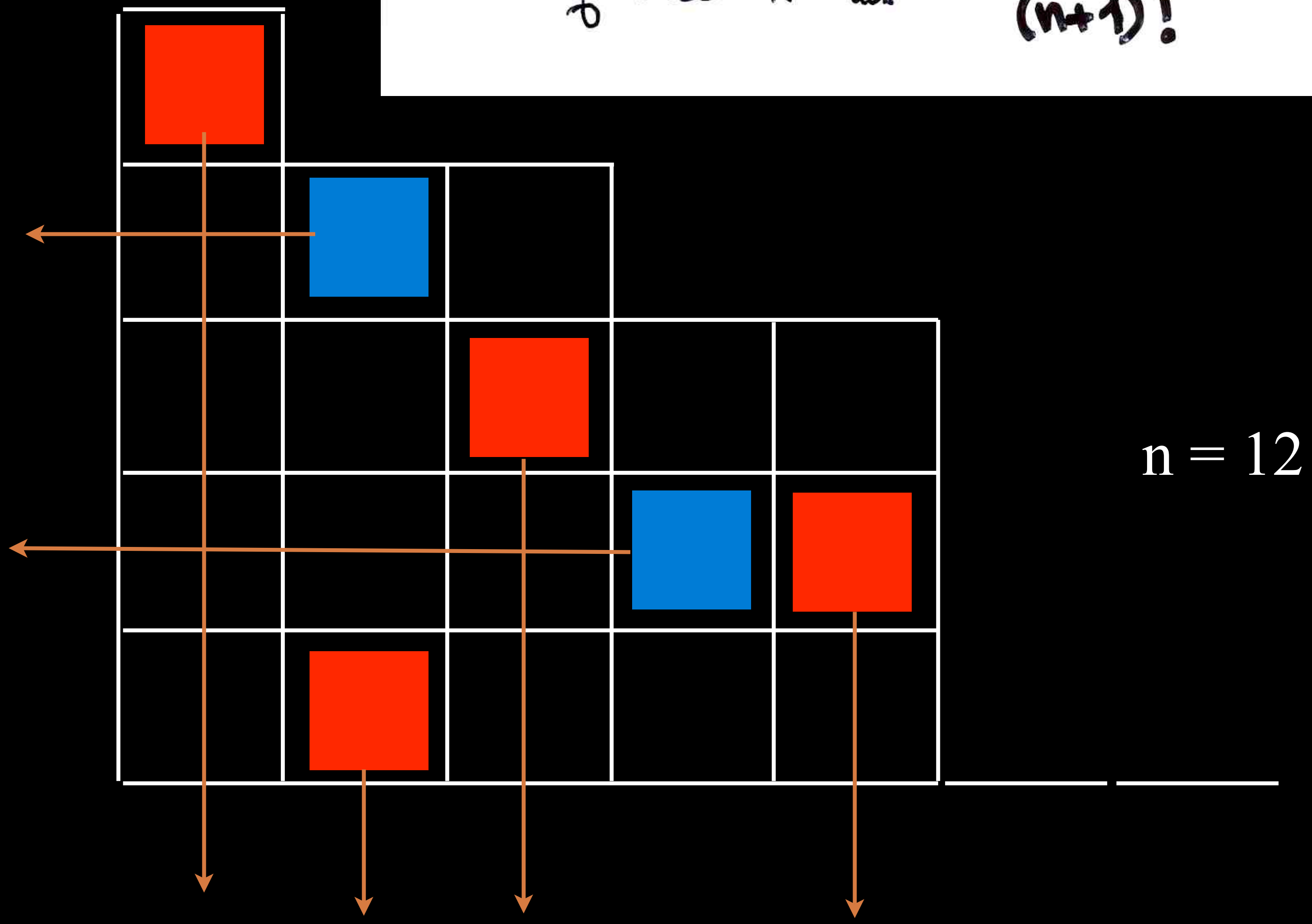
alternative tableau

Ferrers diagram
(= Young diagram)




alternative tableau

Prop. The number of alternative tableaux of size n is (n+1)!



ex: n=2

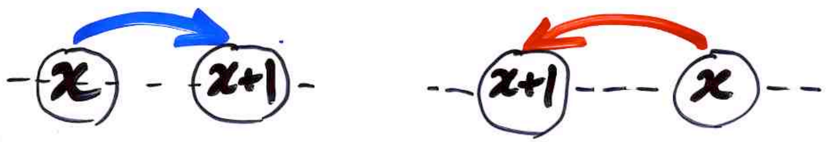
§2 The alternative bijection

Def - Permutation $\sigma = \sigma(1) \cdots \sigma(n)$

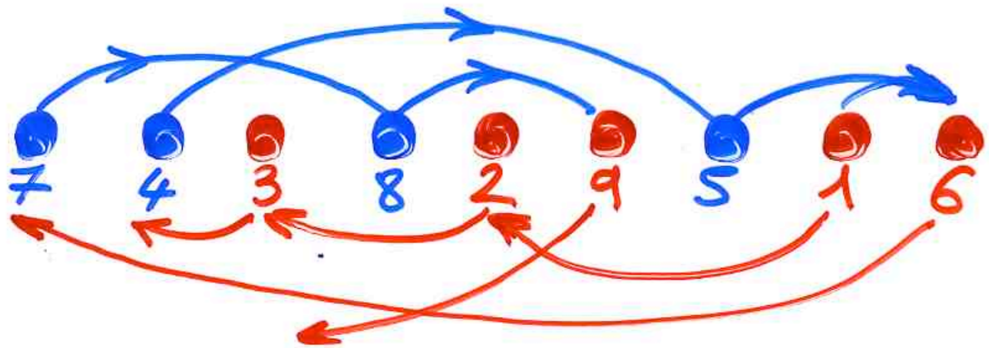
$x = \sigma(i)$, $1 \leq x < n$

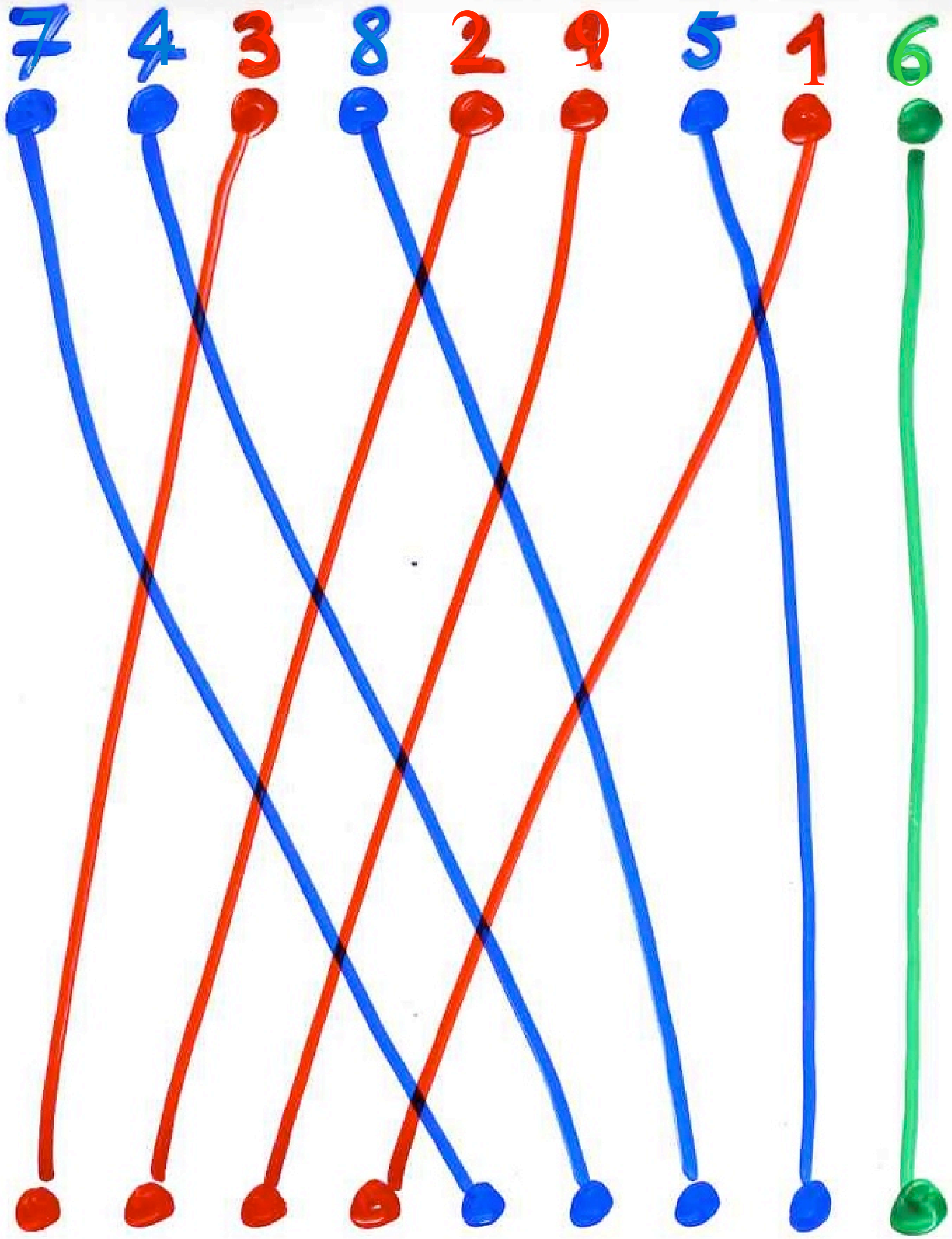
(valeur) x $\begin{cases} \text{avance} \\ \text{recul} \end{cases}$ $x+1 = \sigma(j)$, $\begin{cases} i < j \\ j < i \end{cases}$

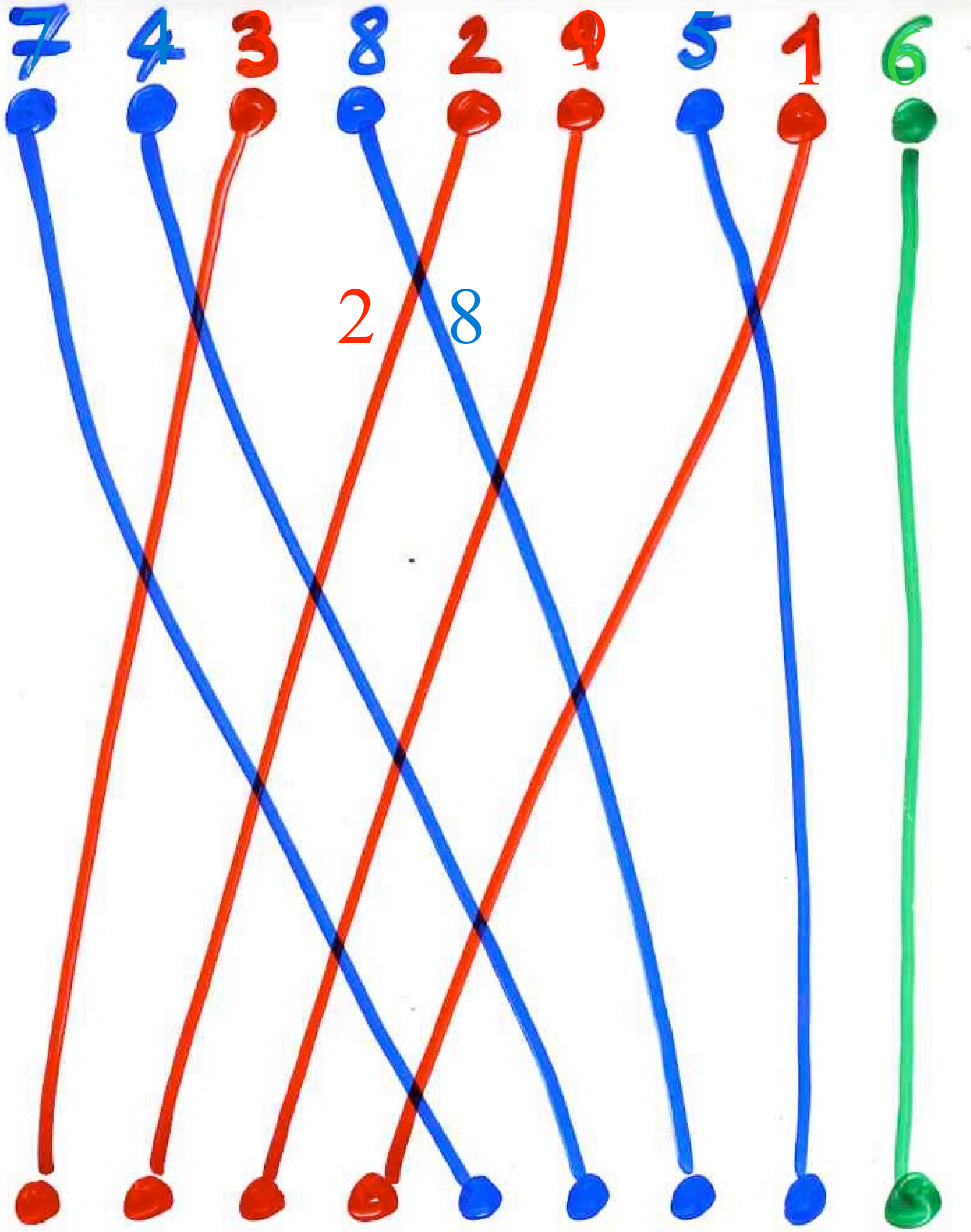
- convention $x = n$ est un recul

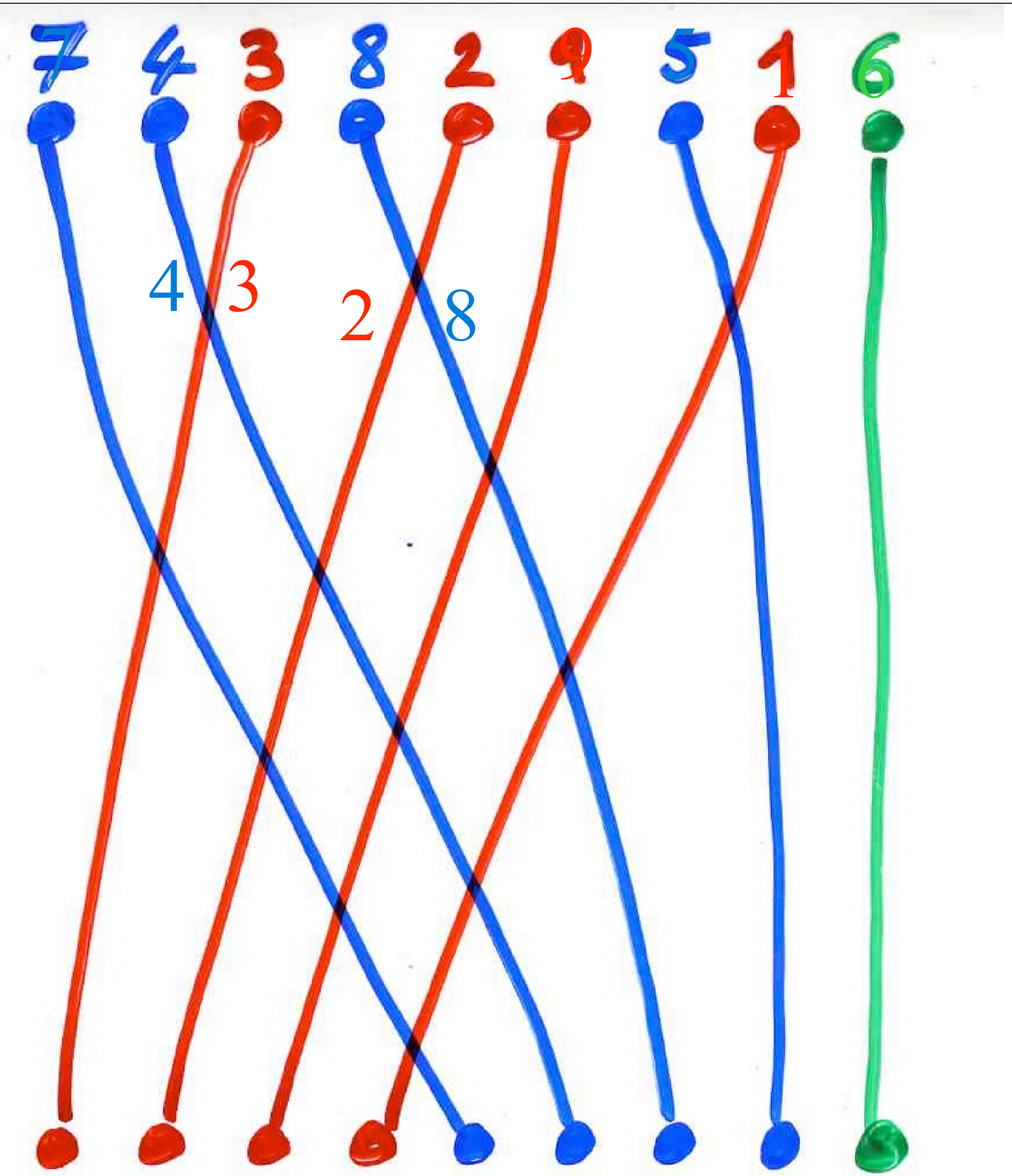


$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$

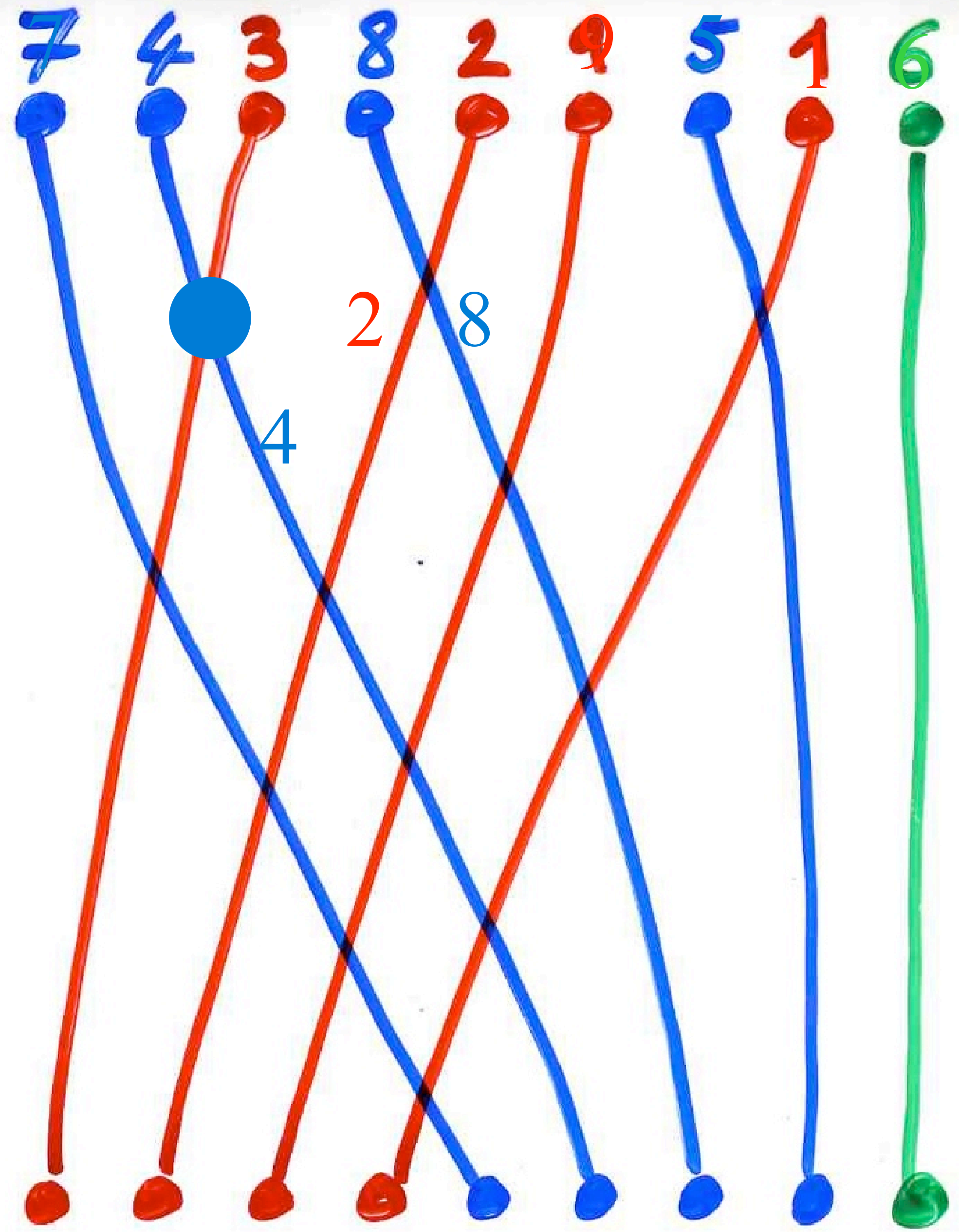


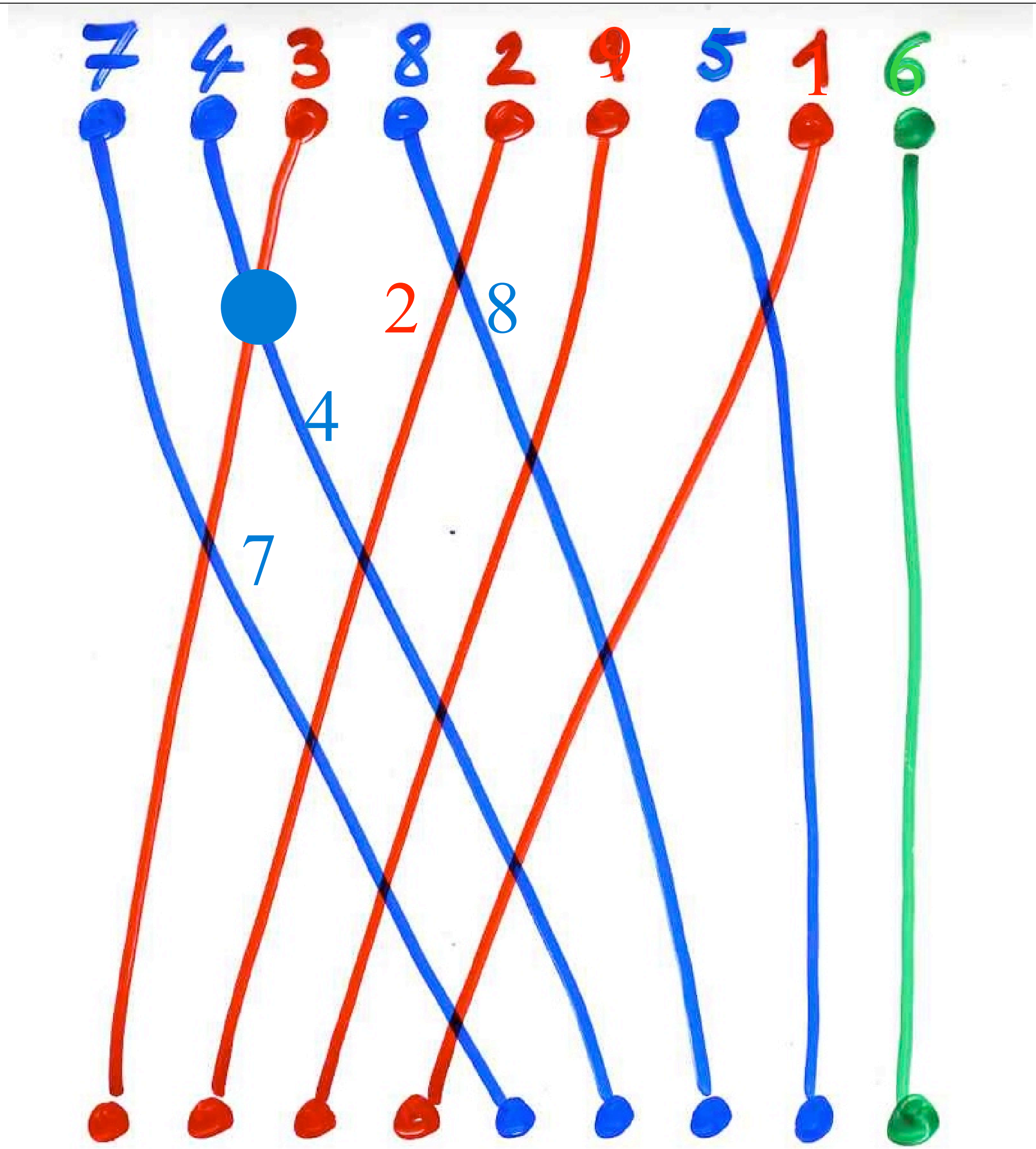

7 4 3 8 2 9 5 1 6

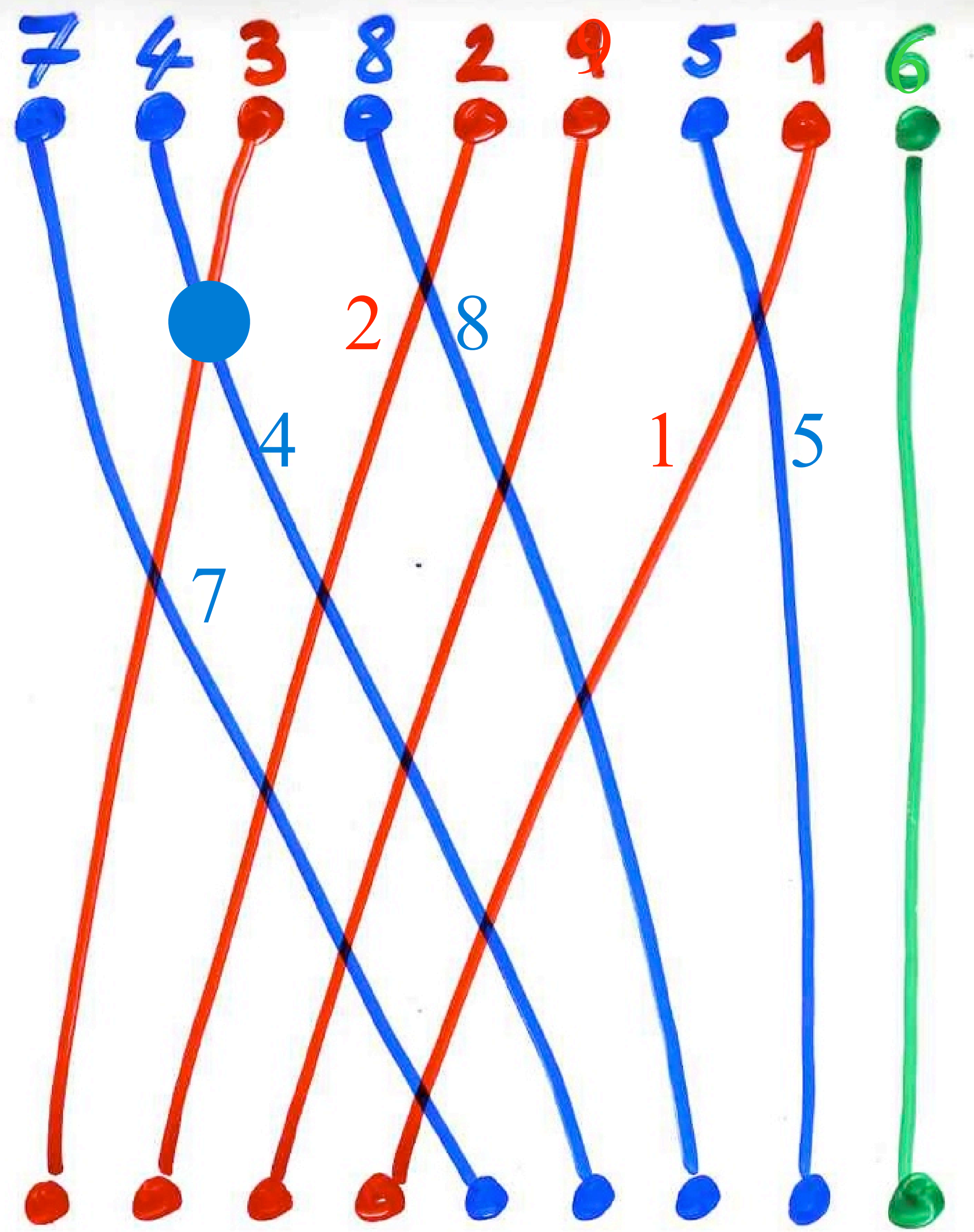

7 4 3 8 2 9 5 1 6
2 8



7 4 3 8 2 9 5 1 6
4 3
2 8


7 4 3 8 2 9 5 1 6
2
8
4


7 4 3 8 2 9 5 1 6
2
8
4
7



7 4 3 8 2 9 5 1 6
2
8
4
1
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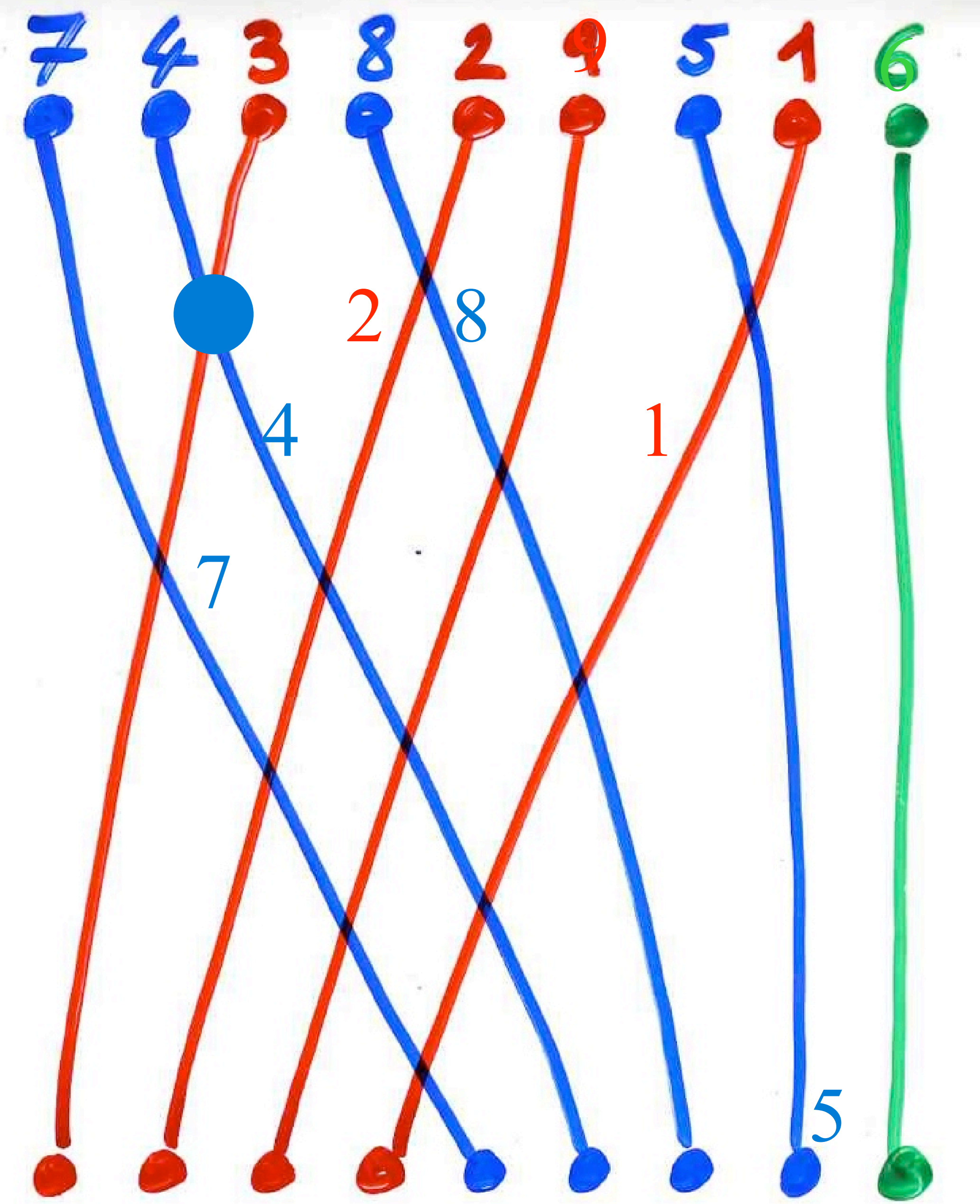
7


7 4 3 8 2 9 5 1 6
2
8
4
1
7
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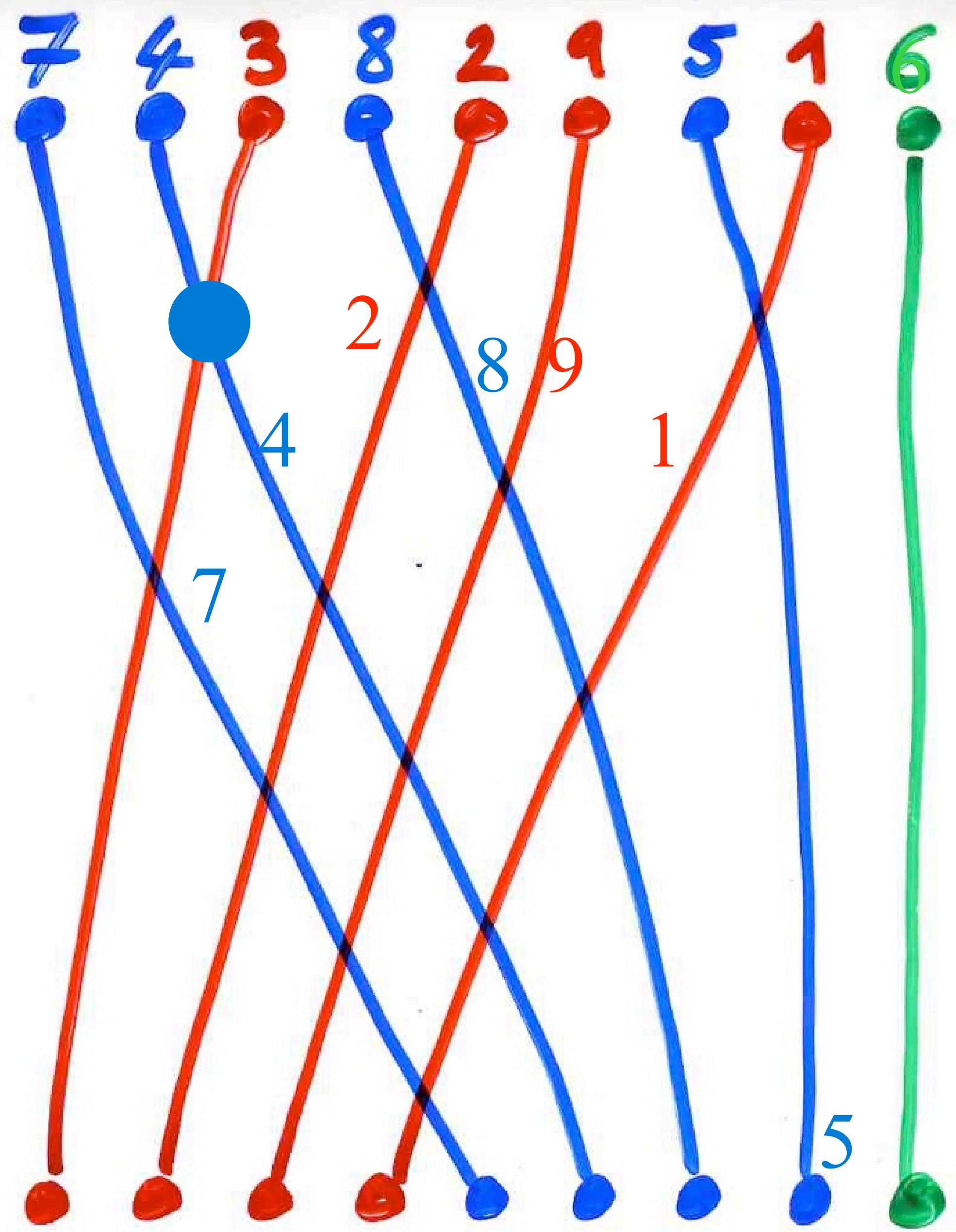
7 4 3 8 2 9 5 1 6
2
8 9
4
1
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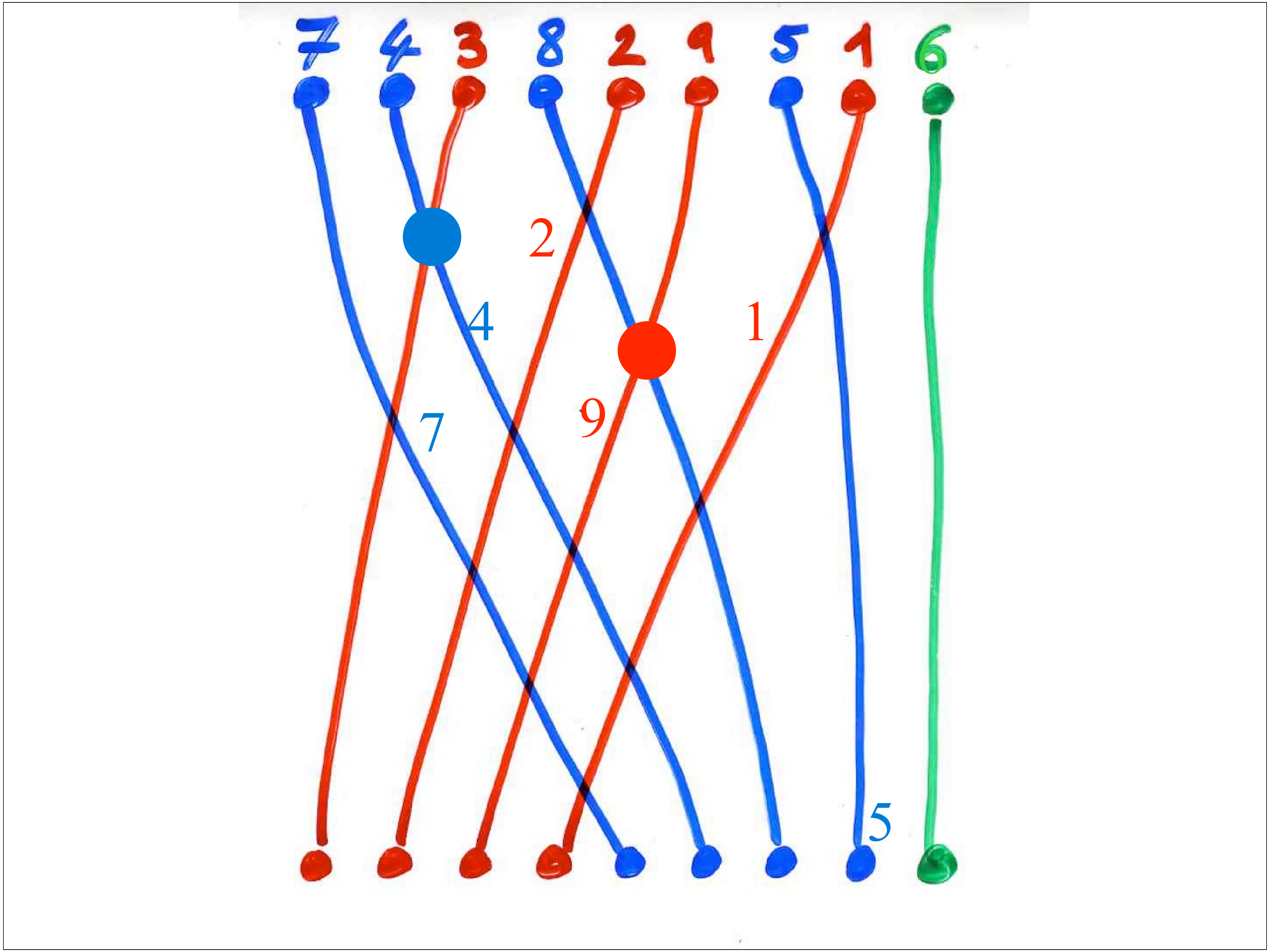
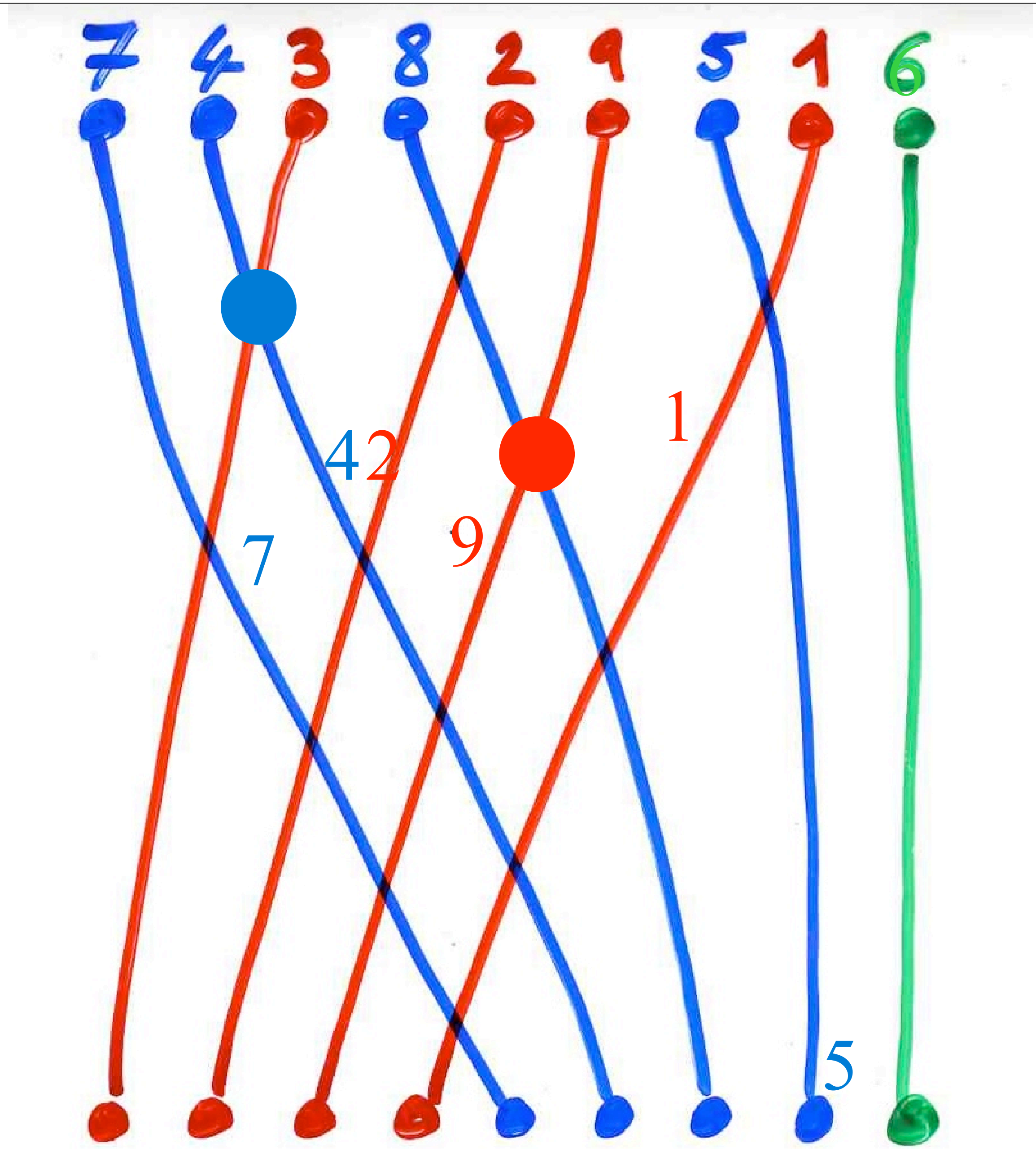
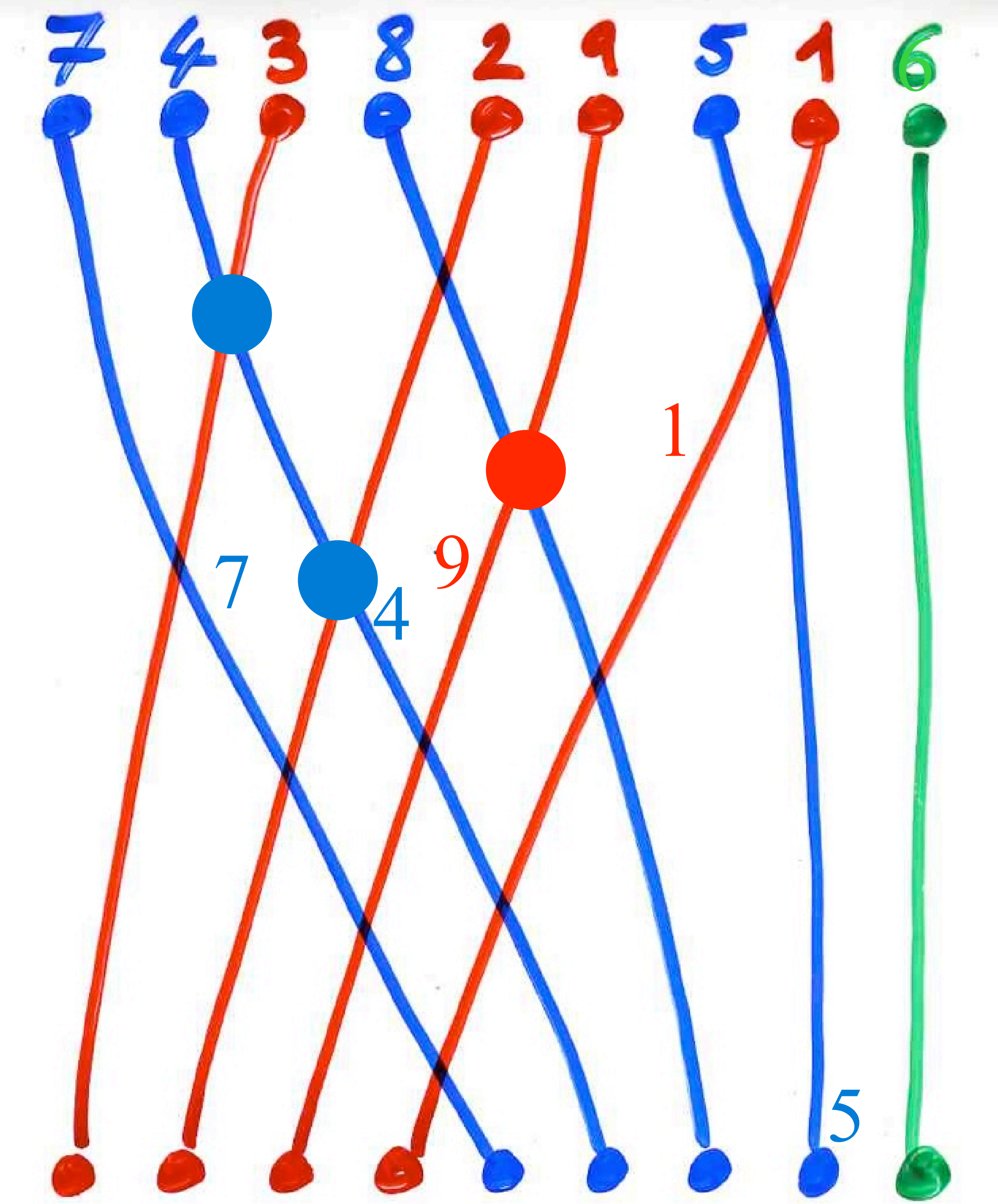
5



7 4 3 8 2 9 5 1 6
2
4
1
7
9
5


7 4 3 8 2 9 5 1 6
42
1
9
7
5



7 4 3 8 2 9 5 1 6
1
7
4
9
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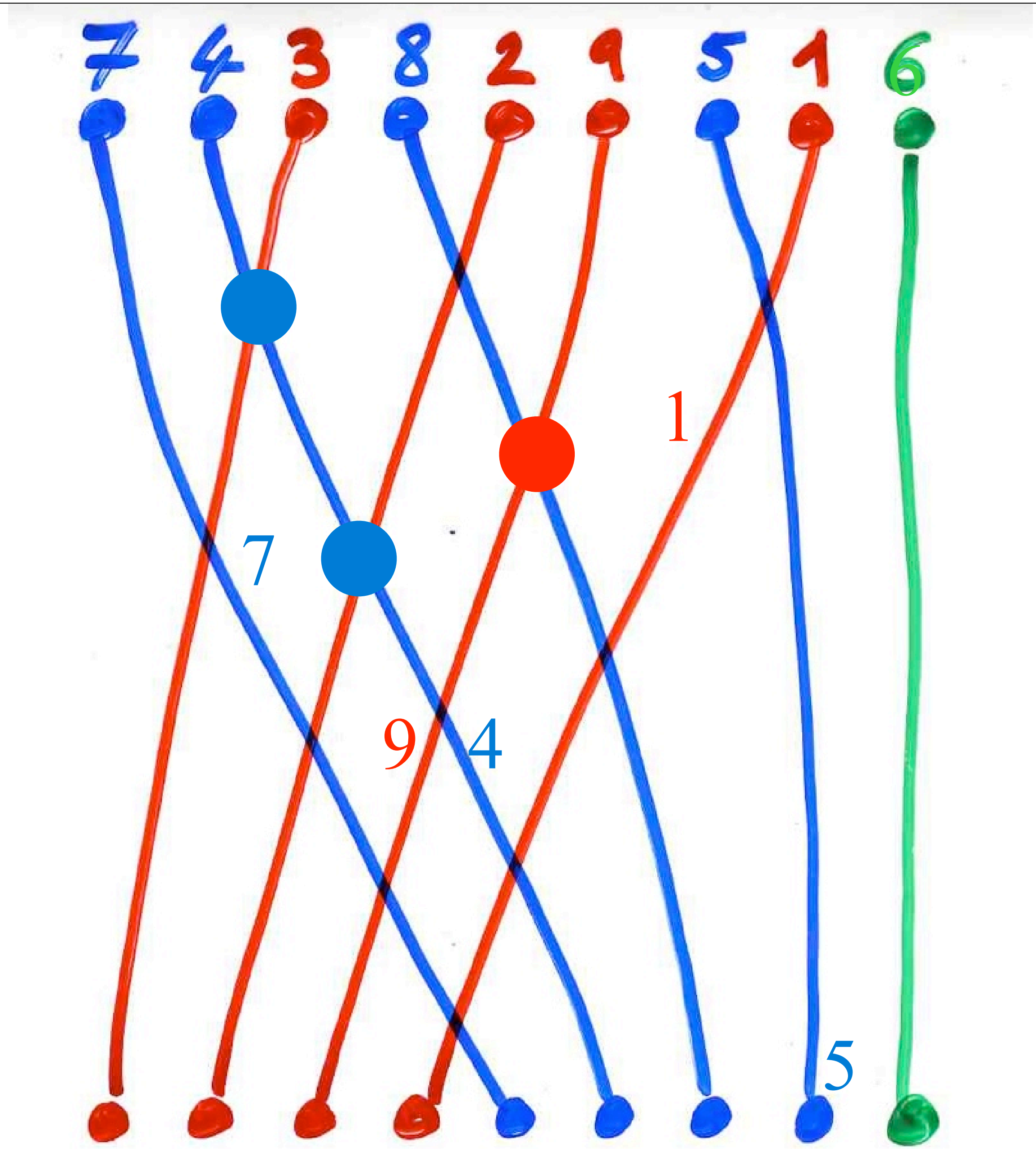
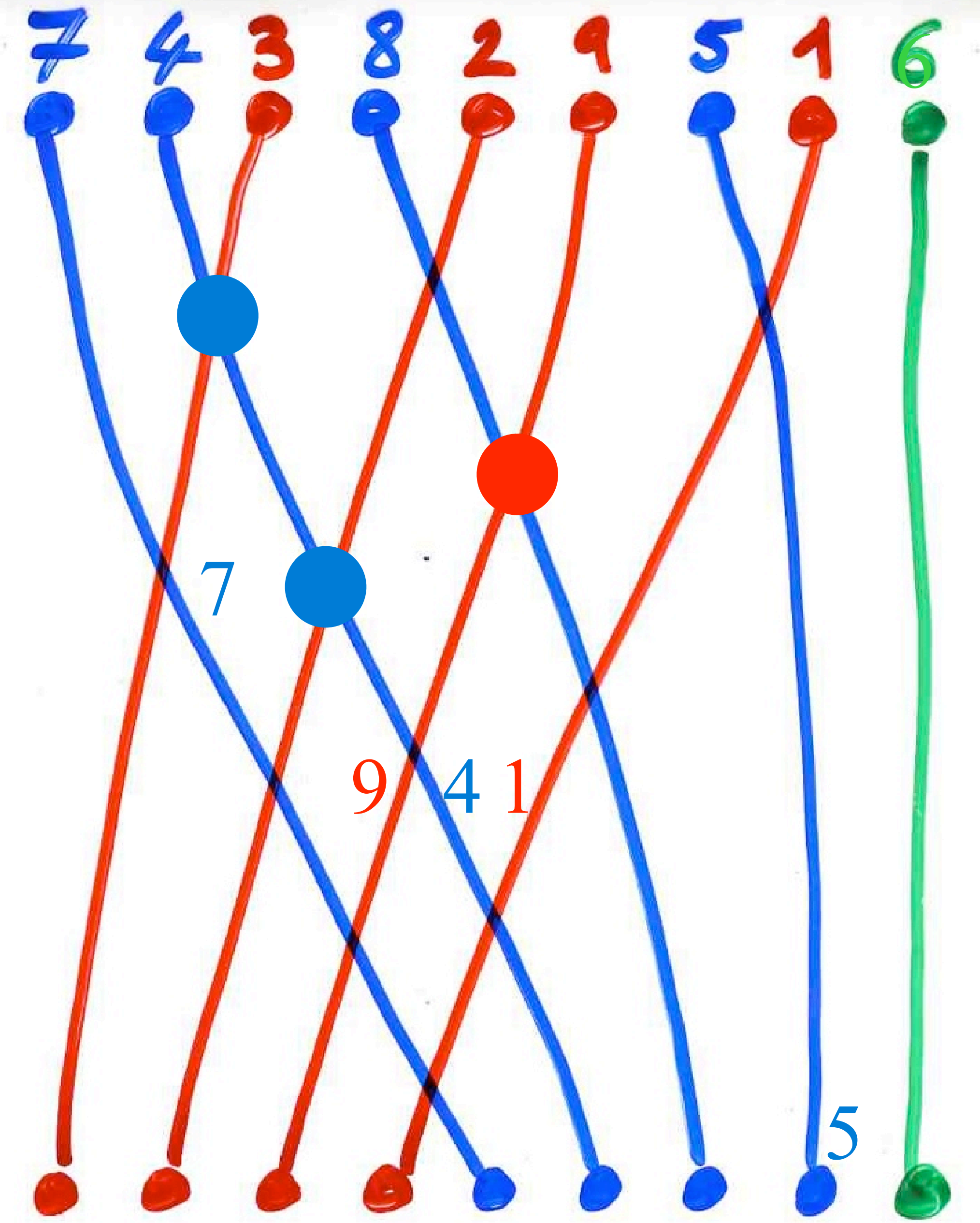
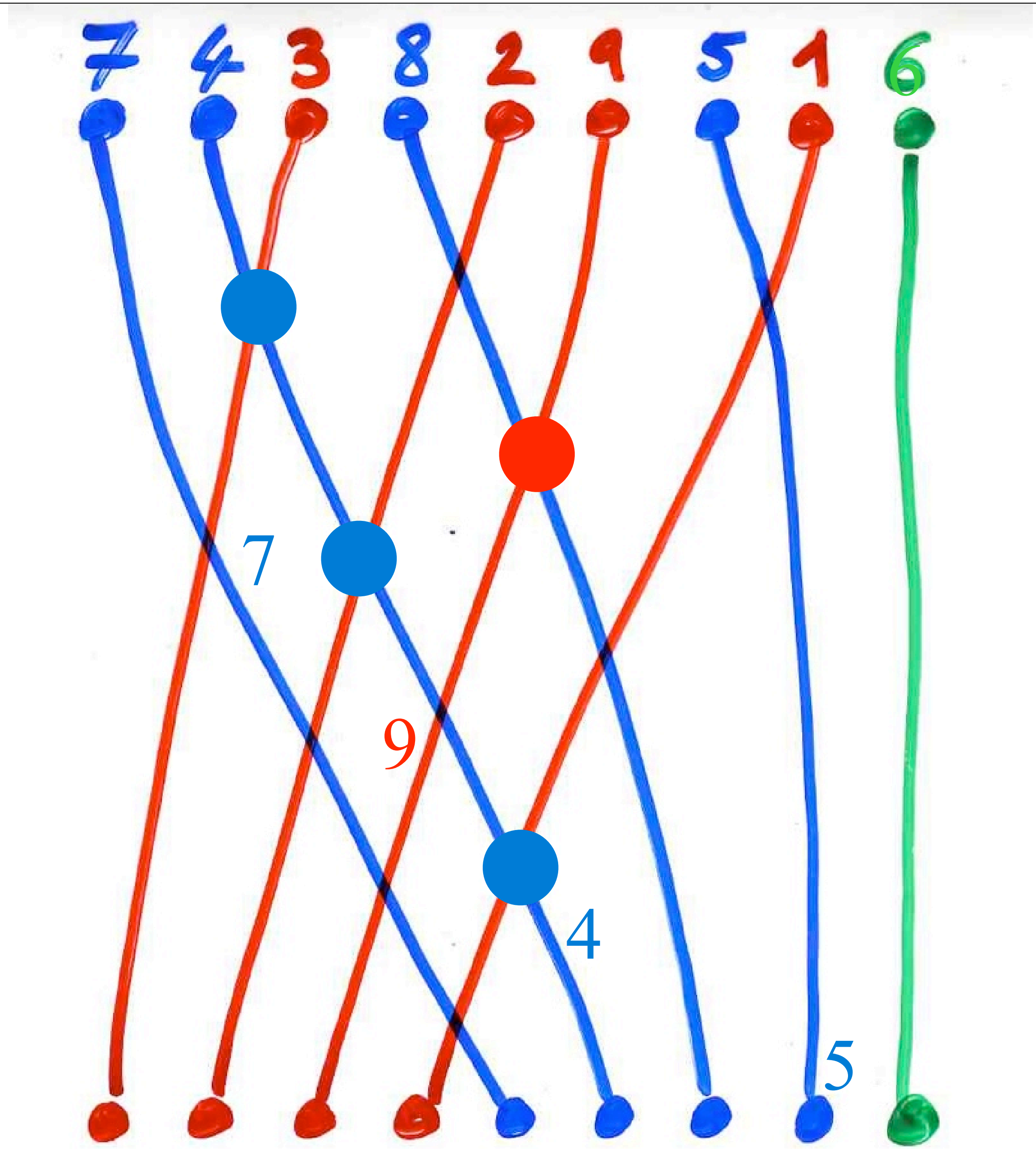


7 4 3 8 2 9 5 1 6
1
7
9 4
5



7 4 3 8 2 9 5 1 6
7
9 4 1
5


7 4 3 8 2 9 5 1 6
7
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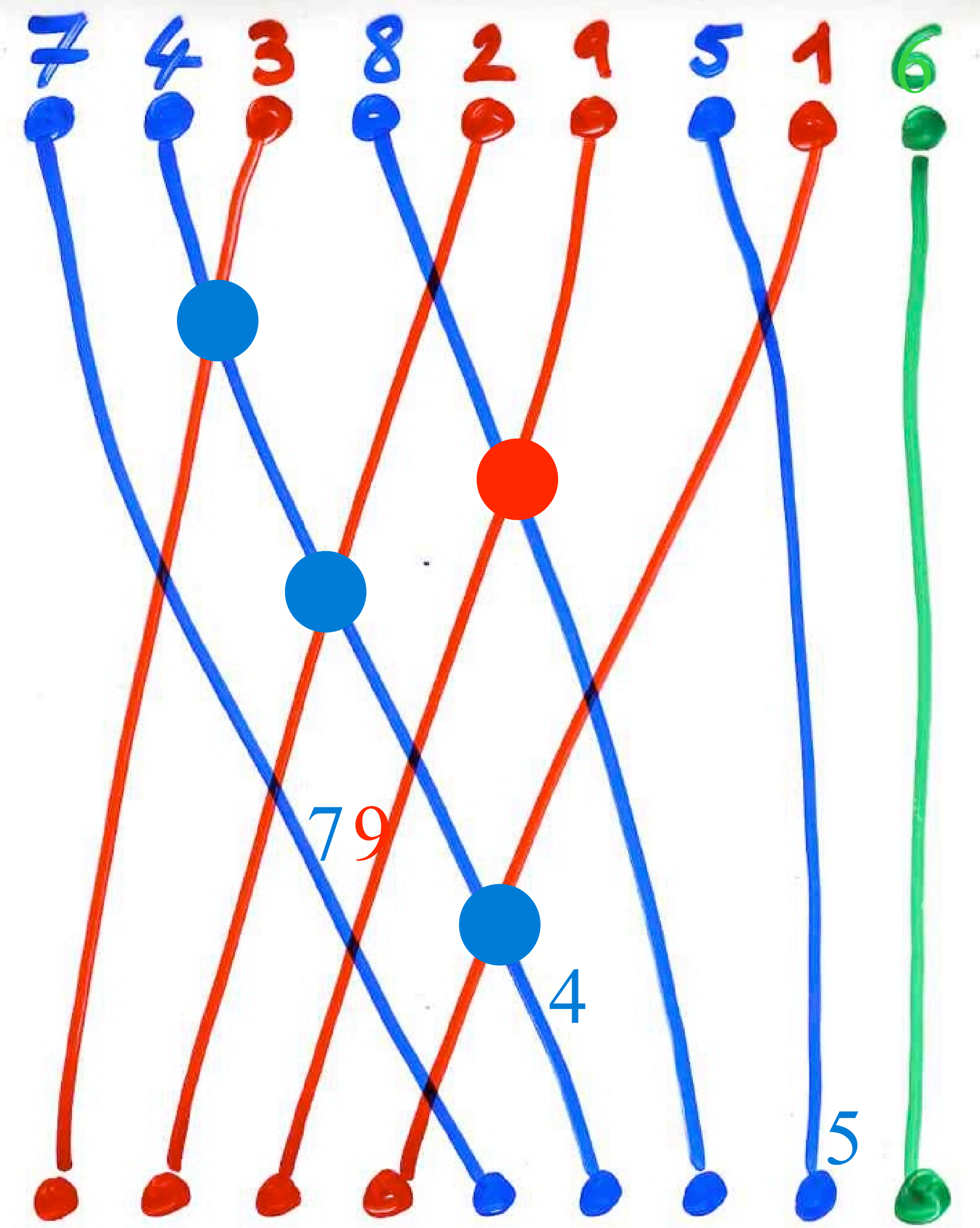
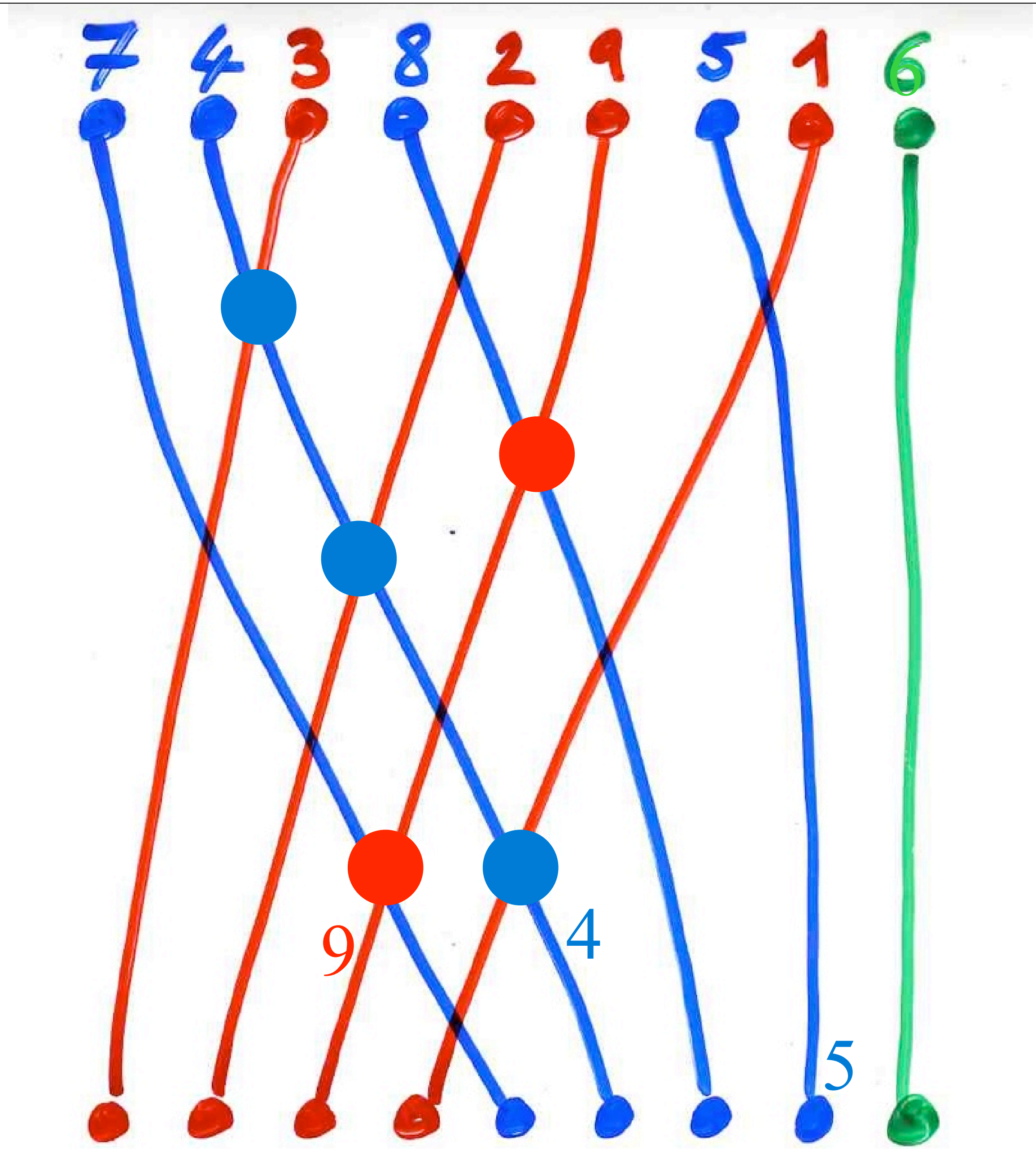
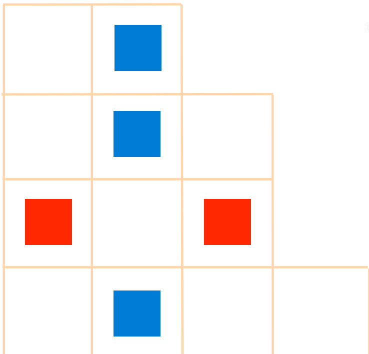


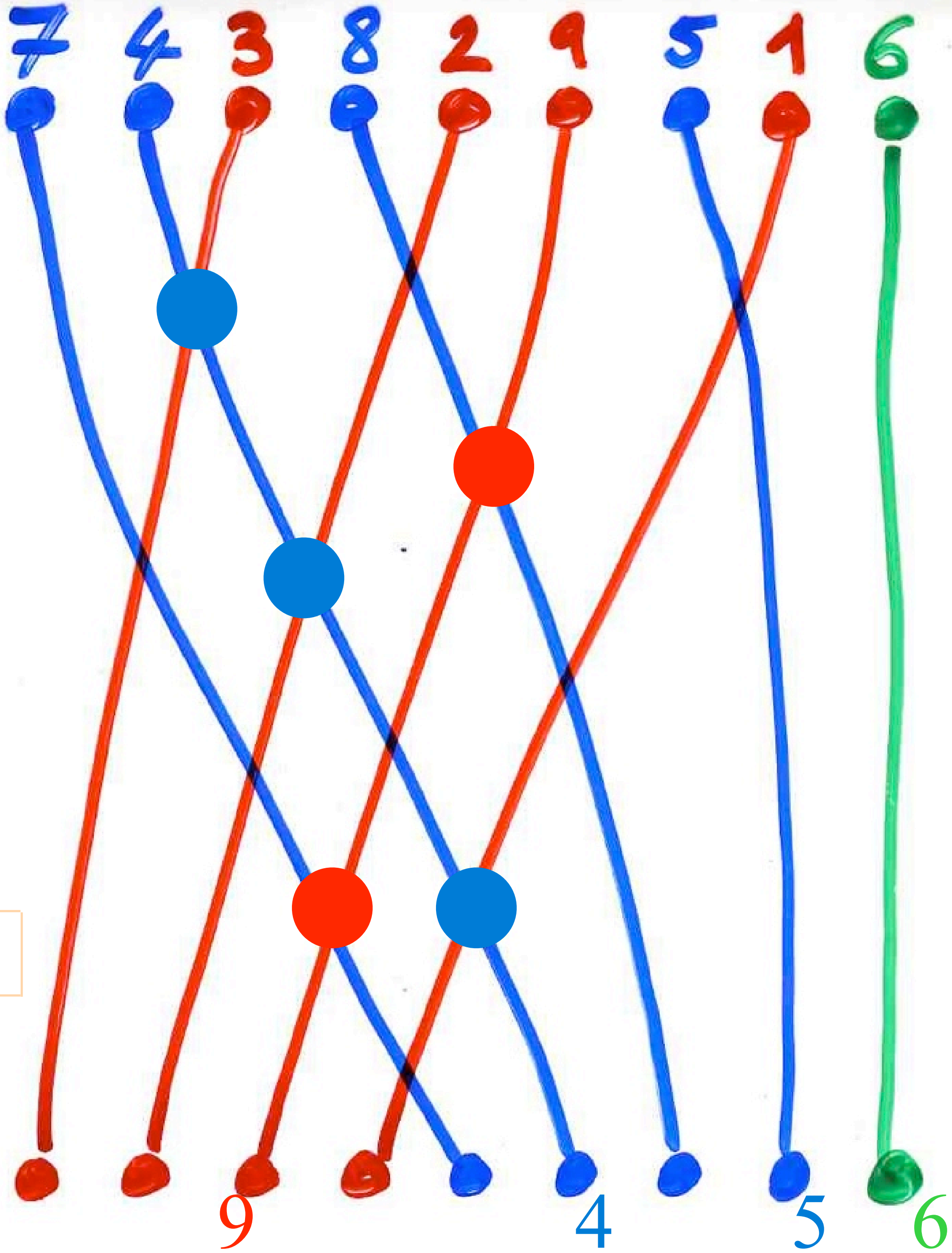
7 4 3 8 2 9 5 1 6
7 9
4
5



7 4 3 8 2 9 5 1 6
9
4
5

“exchange-deletion” algorithm





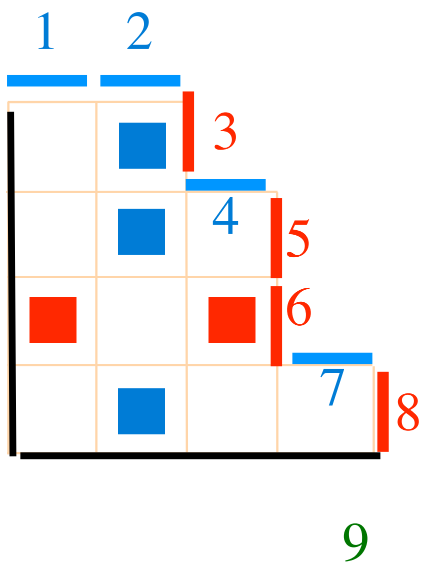
σ permutation

(valeur) x $\begin{cases}\text{avance}\\ \text{recul}\end{cases}$ ssi (indice) x $\begin{cases}\text{montée}\\ \text{descente}\end{cases}$

$\sigma^{-1}(x) < \sigma^{-1}(x+1)$

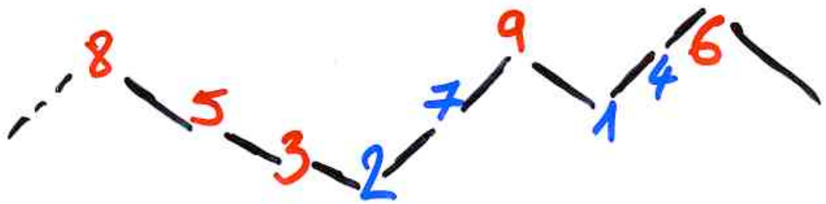
$\sigma^{-1}(x) > \sigma^{-1}(x+1)$

convention : $\sigma(n)$ descente



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 3 & 8 & 2 & 9 & 5 & 1 & 6 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \end{pmatrix}$$



"Genocchi shape" of a permutation

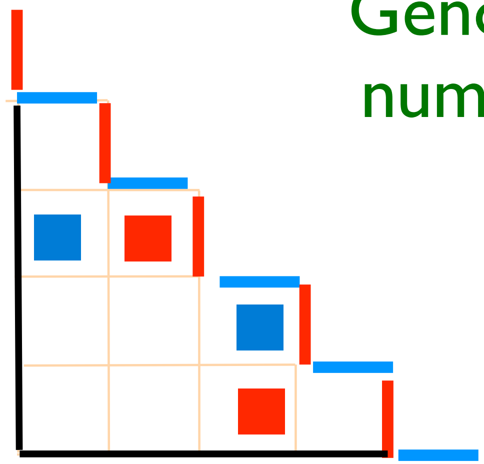
nombres de Genocchi

$$G_{2n} = 2(2^{2n}-1)\,B_{2n}$$

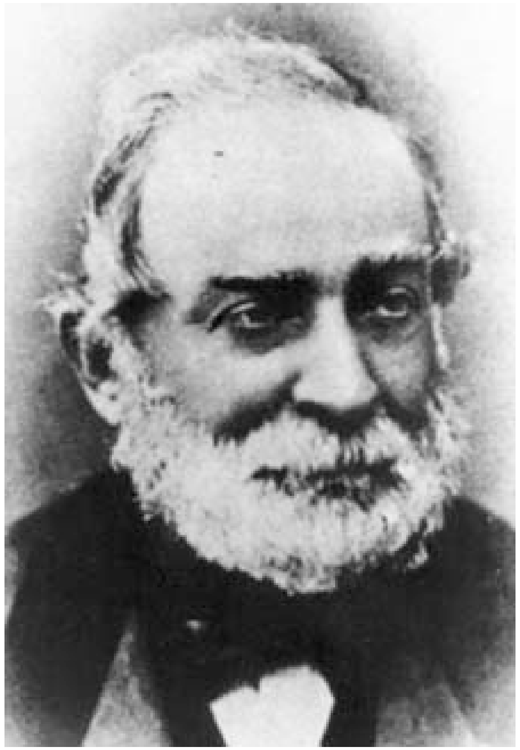
Bernoulli

$$2^{2n}G_{2n+2} = (n+1)\,T_{2n+1}$$

Genocchi numbers



alternating shape

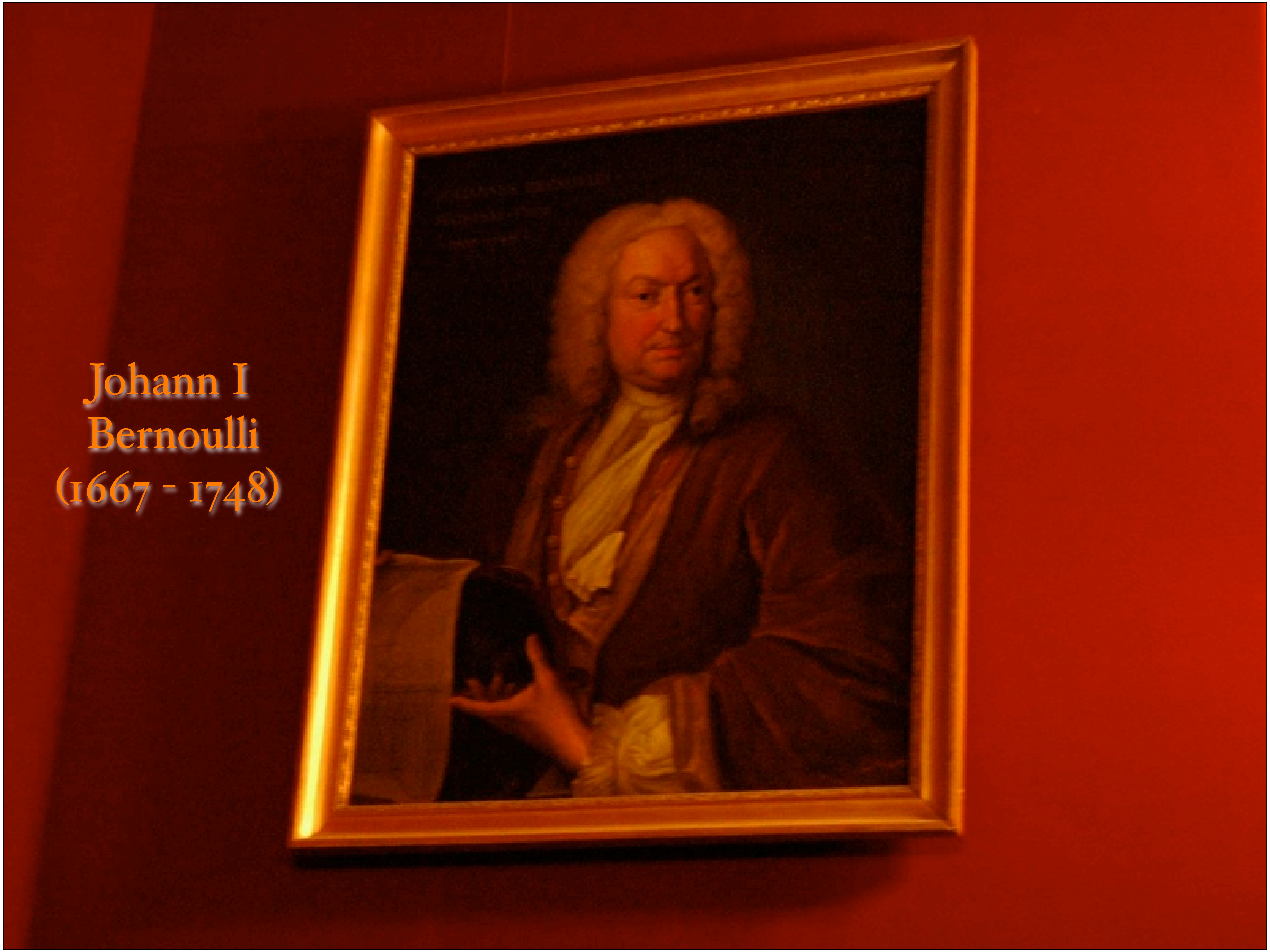


Angelo Genocchi
1817 - 1889

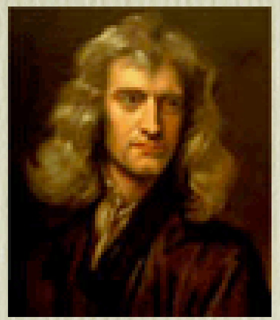
a drop
of
history

Jakob I
Bernoulli
(1654 - 1705)

peinture par
Niklaus Bernoulli


Johann I
Bernoulli
(1667 - 1748)

Isaac Newton
1643 - 1727



erit hanc ſeriem ab illa ſubtrahendo :

$$\operatorname{tg} x = \frac{2^2(2^2-1)\mathfrak{A}x}{1.2} + \frac{2^4(2^4-1)\mathfrak{B}x^3}{1.2.3.4} + \frac{2^6(2^6-1)\mathfrak{C}x^5}{1.2\ldots 6} + \frac{2^8(2^8-1)\mathfrak{D}x^7}{1.2\ldots 8} + \&c.$$

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1.2} - \frac{2^4\mathfrak{B}x^3}{1.2.3.4} - \frac{2^6\mathfrak{C}x^5}{1.2.3\ldots 6} - \frac{2^8\mathfrak{D}x^7}{1.2\ldots 8} - \&c.$$

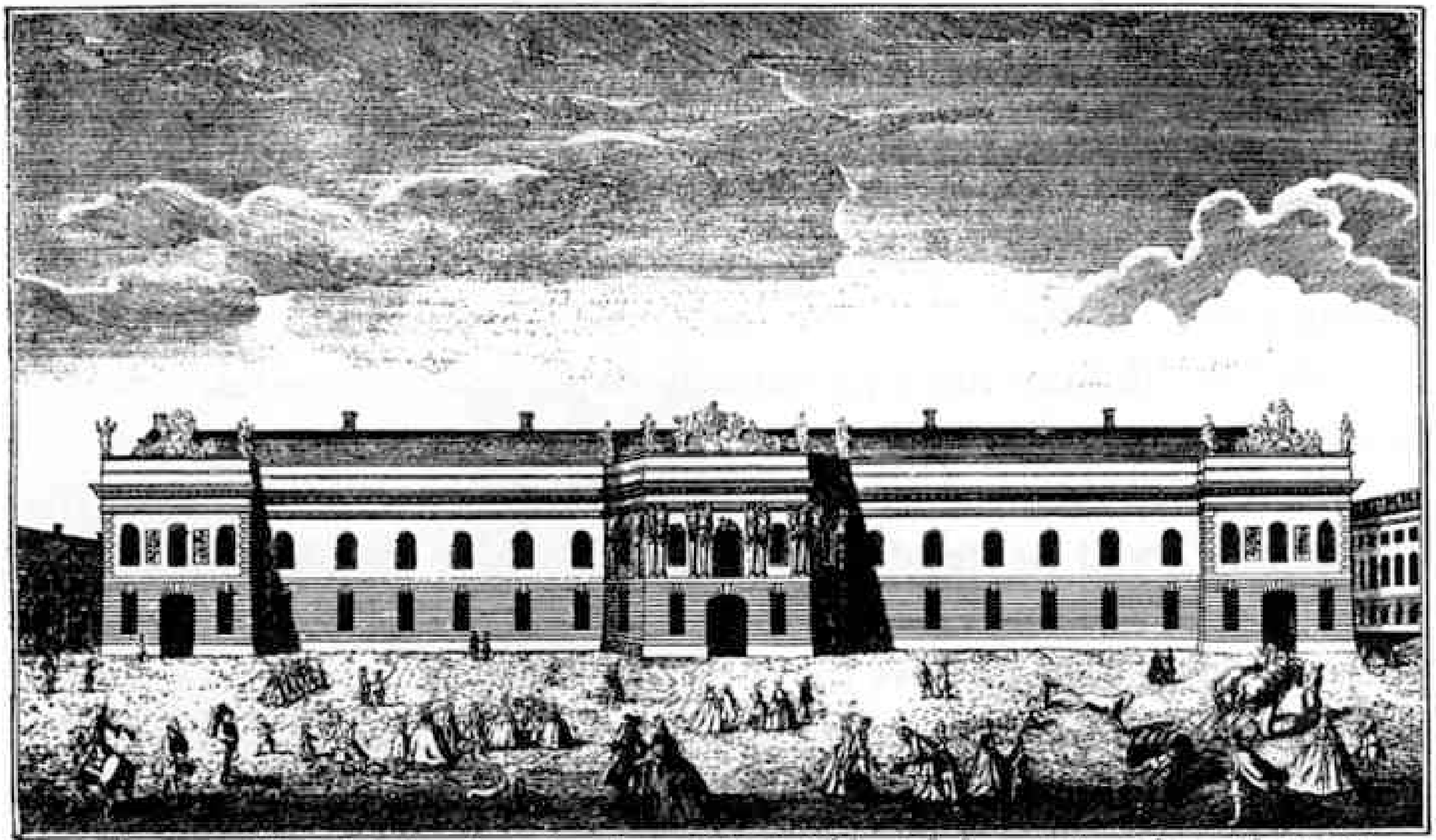
Si ergo hic introducantur numeri A, B, C, &c. §. 182. inventi; erit: $\operatorname{tang} x = \frac{2Ax}{1.2} + \frac{2^3Bx^3}{1.2.3.4} + \frac{2^5Cx^5}{1.2\ldots 6} + \frac{2^7Dx^7}{1.2\ldots 8} + \&c.$

Hinc igitur calculo instituto reperietur:

$$\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= 3 \\
D &= 17 \\
E &= 155 = 5.31 \\
F &= 2073 = 691.3 \\
G &= 38227 = 7.5461 = 7.\frac{127.129}{3} \\
H &= 929569 = 3617.257 \\
I &= 28820619 = 43867.9.73 \quad \&c.
\end{aligned}$$

Leonhard
Euler

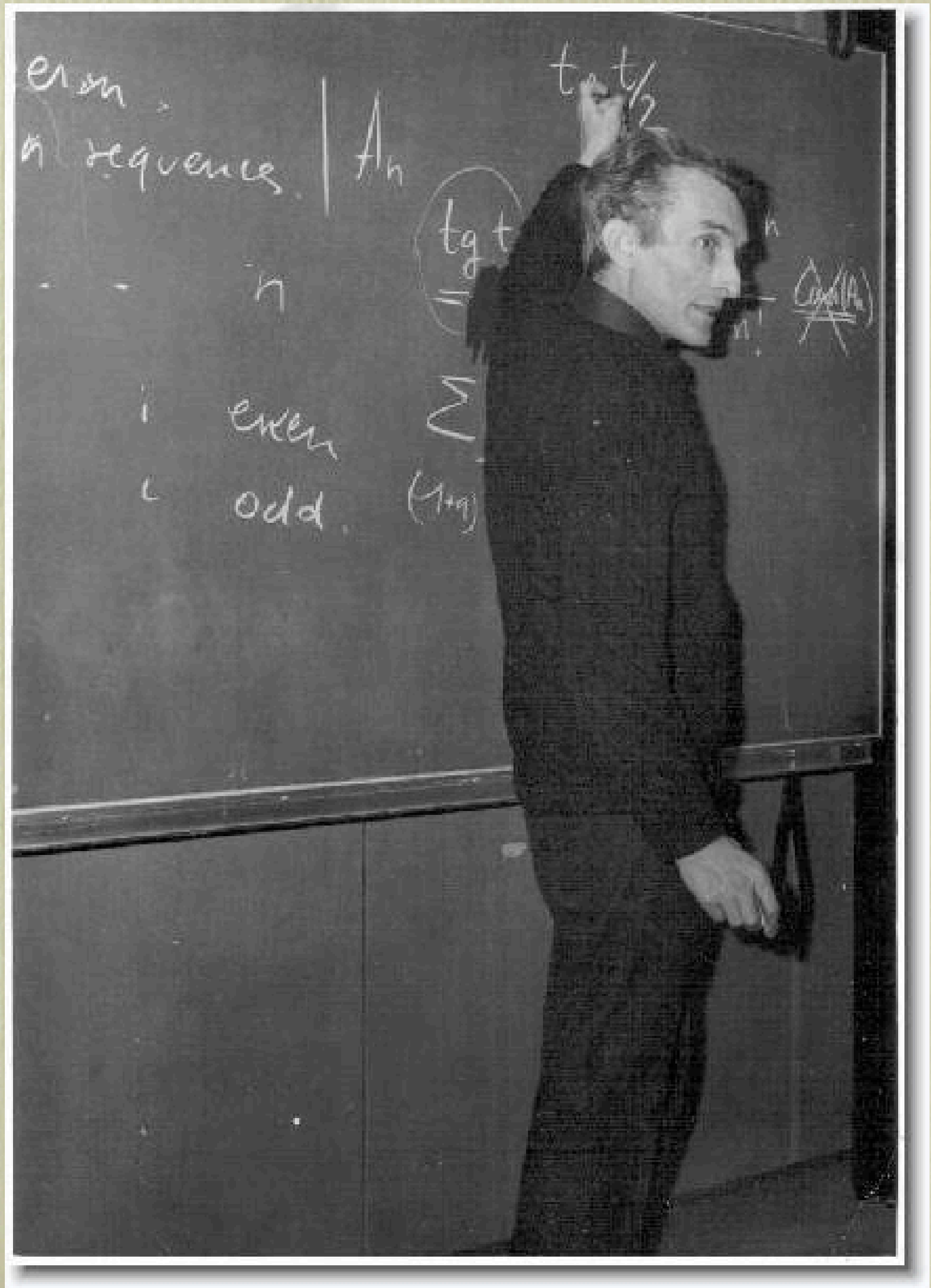
(1707- 1783)



Prospect der vor einigen Jahren abgebrannten, und nunmehro gantz neu aufgeführten Fördergebäudes des grossen Königl. Stalls auf der Dorotheen-Stadt zu Berlin. Welches überaus ansehnliche Gebäude für die Königl. Academie der Wissenschaften und freyen Künste, und für die Academie der Künste und [illegible] Wissenschaften bestimmet ist.

123. Die Akademie der Wissenschaften. 1752

Nach einem Stiche von Schleuen

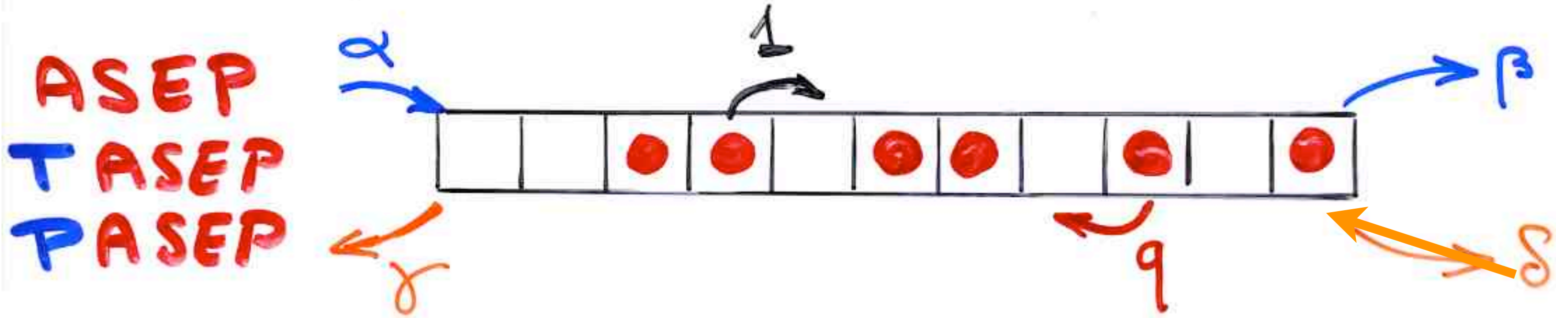


notre “bon” Maître

Marcel Paul
Schützenberger

1920 - 1996

§ 3
The
PASEP



ASEP
TASEP
PASEP
α
1
β
γ
q
δ

non-equilibrium statistical mechanics

... relaxation → stationary state

states $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$

$$\tau_i = \begin{cases} 1 & \text{site } i \text{ occupied} \\ 0 & \text{site } i \text{ empty} \end{cases}$$

unique stationary state

$$\frac{d}{dt} P_n(\tau_1, \ldots, \tau_n) = 0$$

Derrida, Evans, Hakim, Pasquier (1993)

boundary induced phase transitions

molecular diffusion
linear array of enzymes
biopolymers
traffic flow
------ formation of shocks

$$P_n(\tau_1, \dots, \tau_n) = f_n(\tau_1, \dots, \tau_n) / Z_n$$

$$Z_n = \sum_{\tau} f_n(\tau_1, \dots, \tau_n)$$

partition function

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

$$\begin{cases} DE = q\,ED + D + E \\ (\beta D - \delta E)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \gamma D) = \langle W| \end{cases}$$

Then

$f_n(\tau_1, \dots, \tau_n)$

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

$$\begin{cases} DE = q\,ED + D + E \\ (\beta D - \delta E)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \gamma D) = \langle W| \end{cases}$$

Then

$$f_n(\tau_1, \dots, \tau_n) = \langle W| \prod_{i=1}^{n} (\tau_i D + (1-\tau_i) E) |V\rangle$$

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

$$\begin{cases} DE = qED + D + E \\ (\beta D - \text{[redacted]})|V\rangle = |V\rangle \\ \langle W|(\alpha E - \text{[redacted]}) = \langle W| \end{cases}$$

Then

$$f_n(\tau_1, \dots, \tau_n) = \langle W| \prod_{i=1}^{n} (\tau_i D + (1-\tau_i) E) |V\rangle$$

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D, E matrices,

V column vector, W row vector

TASEP

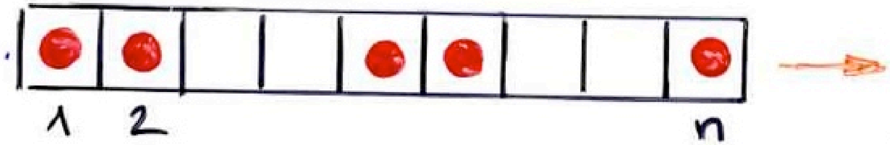
q=0

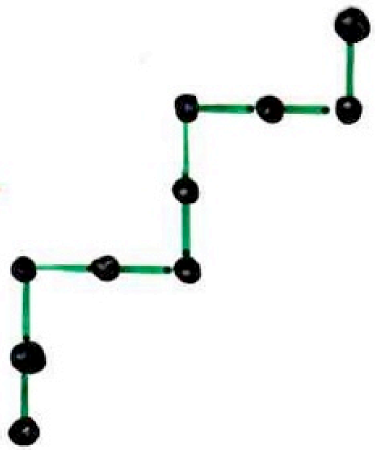
$$\begin{cases} DE = \boxed{} + D + E \\ (\beta D - \boxed{})|V\rangle = |V\rangle \\ \langle W|(\alpha E - \boxed{}) = \langle W| \end{cases}$$

Then

$$f_n(\tau_1, \dots, \tau_n) = \langle W| \prod_{i=1}^{n} (\tau_i D + (1-\tau_i)E) |V\rangle$$

state $s = (\tau_1, \dots, \tau_n)$

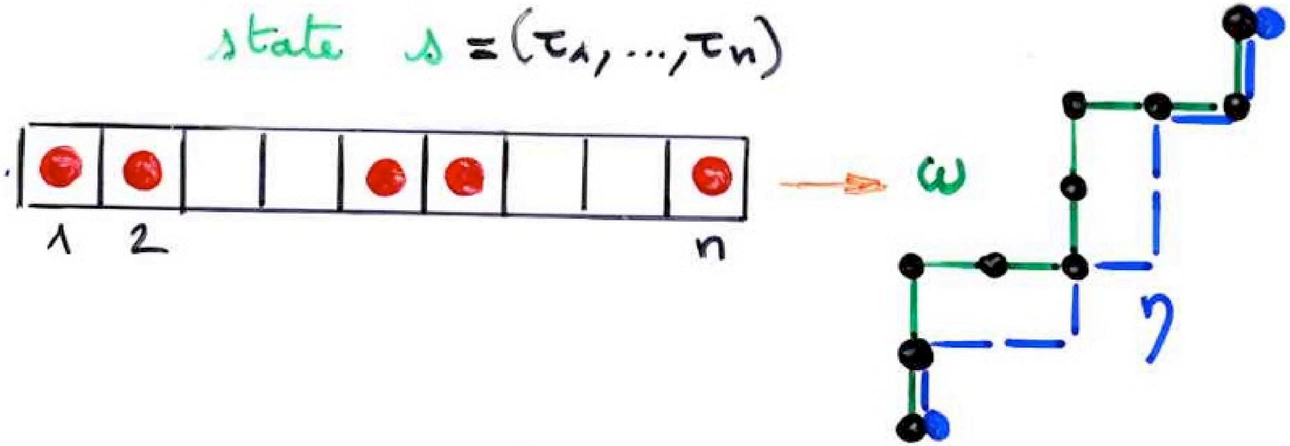




$\mathcal{P}_n(s) =$

Shapiro, Zeilberger, 1982

state $s = (\tau_1, \ldots, \tau_n)$



$$\mathcal{P}_n(s) = \frac{1}{C_{n+1}} \left(\begin{array}{c} \text{number of paths } \eta \\ \text{below the path } \omega \\ \text{associated to } s \end{array} \right)$$

Shapiro, Zeilberger, 1982

TASEP

Brak, Essam (2003), Duchi, Schaeffer, (2004),
Angel (2005), xgv, (2007)

(P) ASEP

Brak, Corteel, Essam, Parviainen, Rechnitzer (2006)
Corteel, Williams (2006)

Derrida, ...
Mallick, ... Golinelli, Mallick (2006)

§4 Stationary probability with alternative tableaux

q-analog

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T \text{ alternative tableau with profile } w} q^{k(T)} E^{i(T)} D^{j(T)}$$

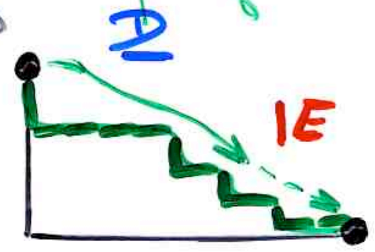
$k(T)$ = nb of $\boxtimes$

$i(T)$ = nb of columns without red cell

$j(T)$ = nb of rows without blue cell

Def- profile of an alternative tableau

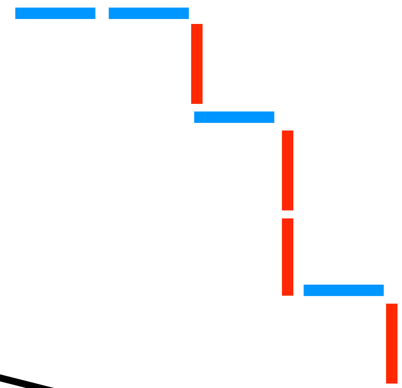
word $w \in \{E,D\}^*$



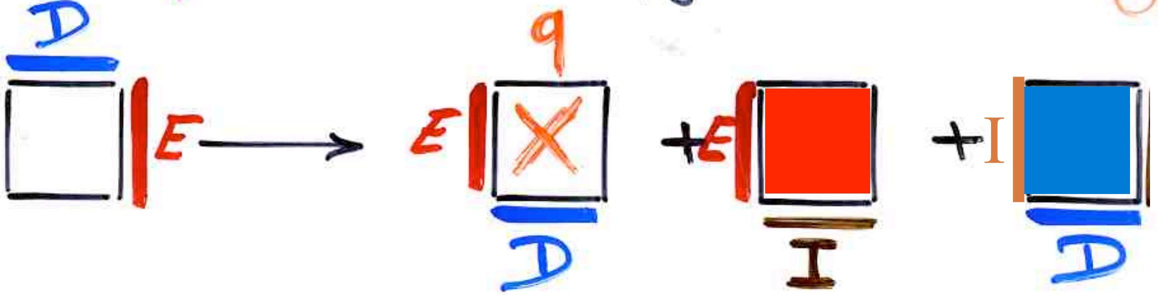
D D E D E E D E

D DE (D E) E D E

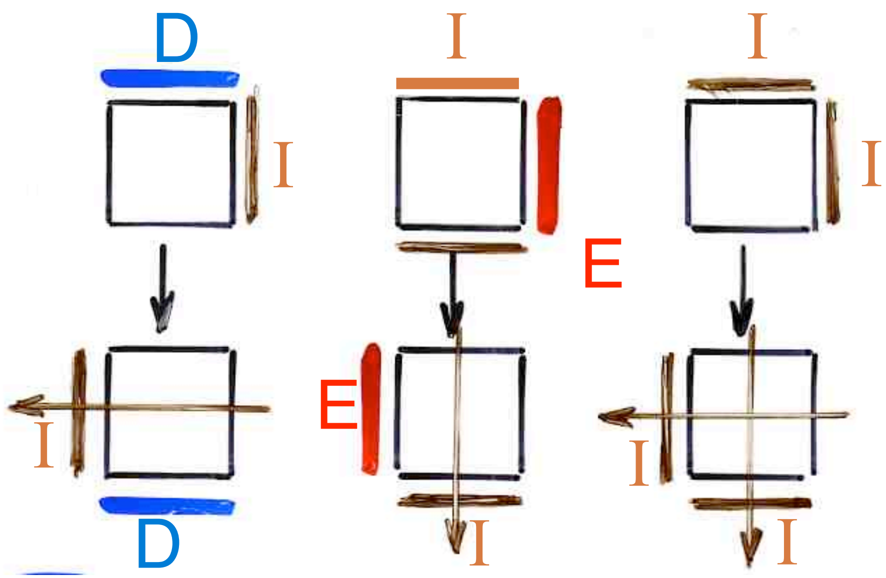
DDE(E)EDE + DDE(ED)EDE + DDE(D)EDE

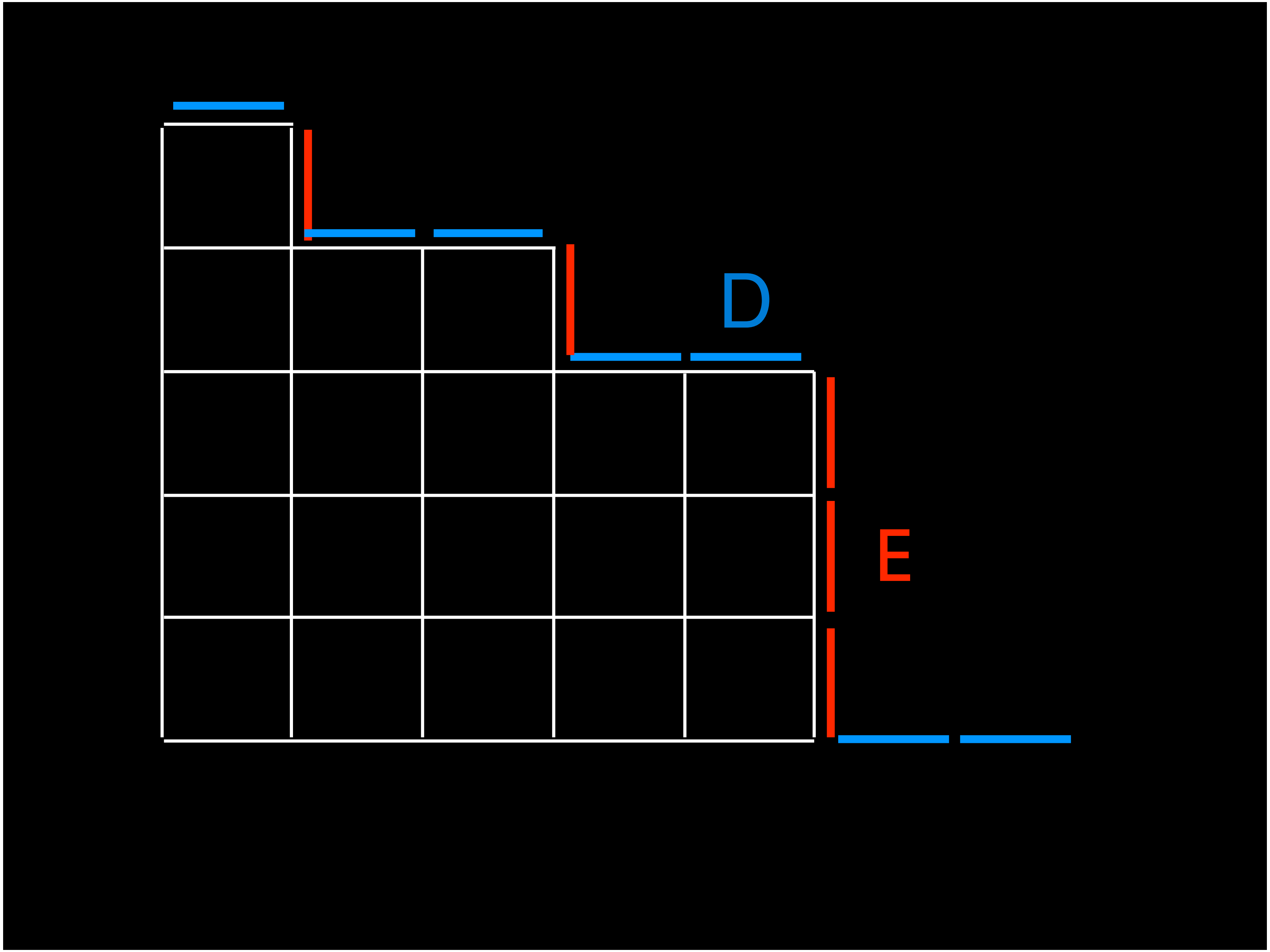


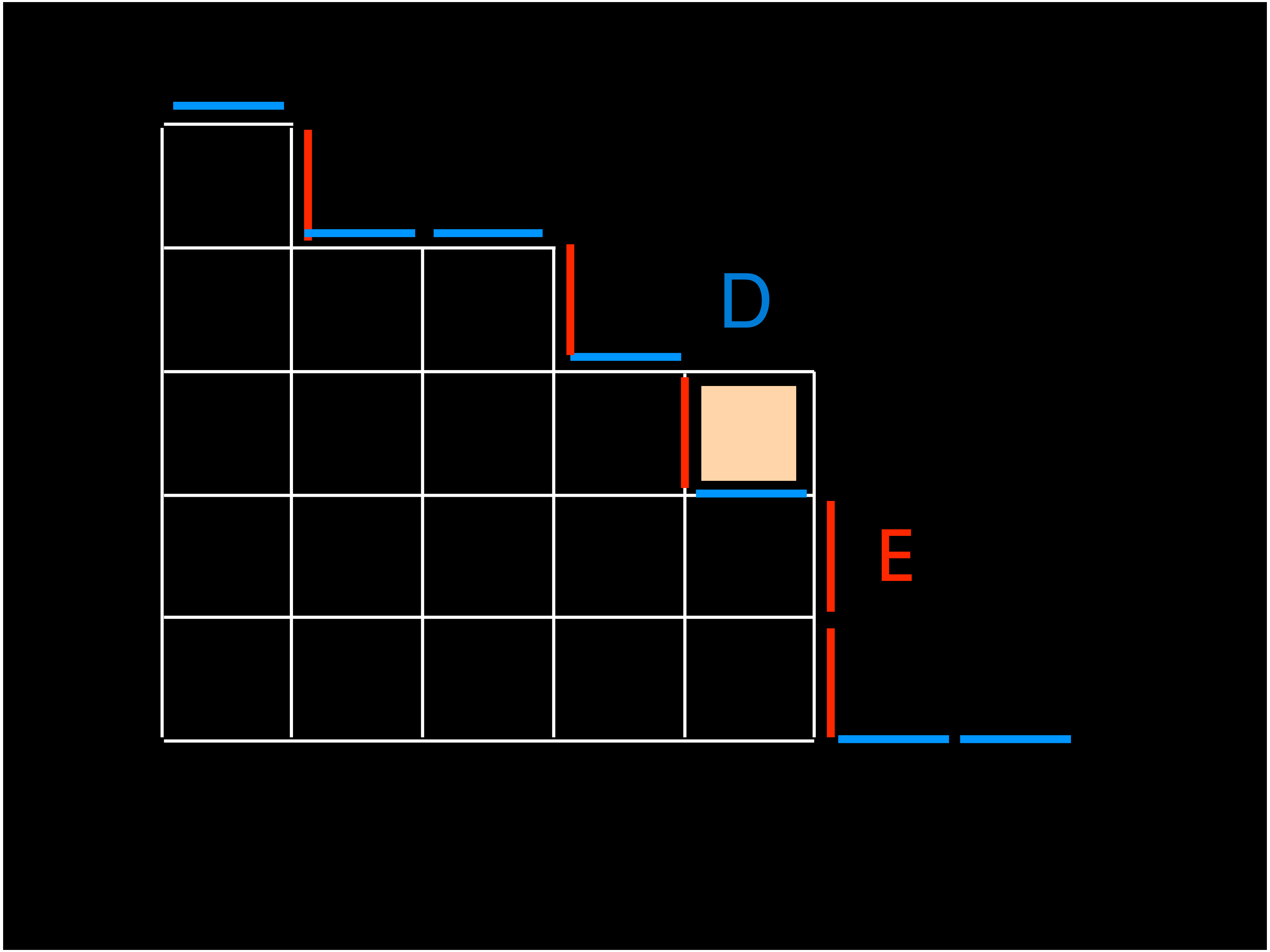
Proof: "planarization" of the rewriting rules

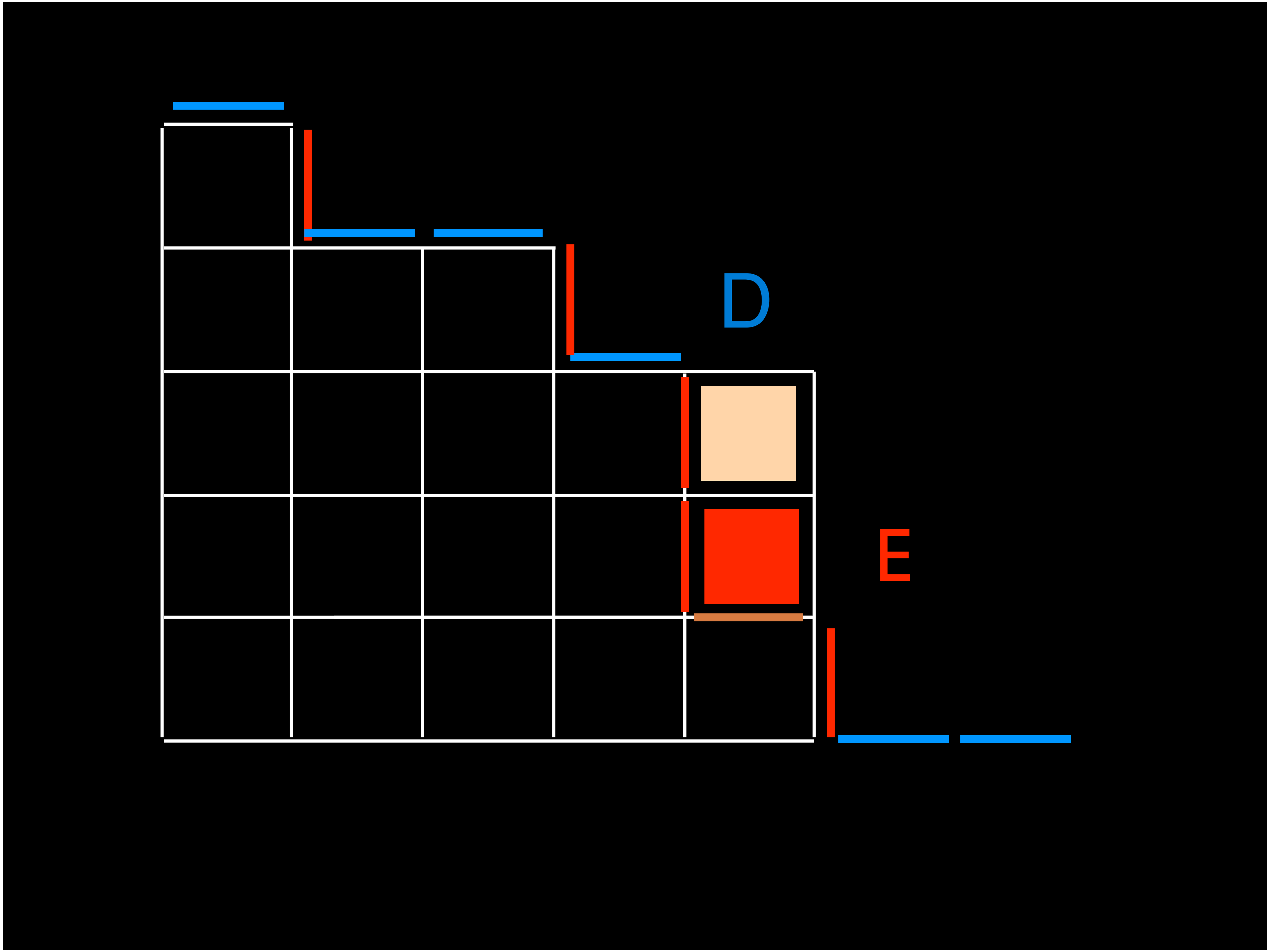


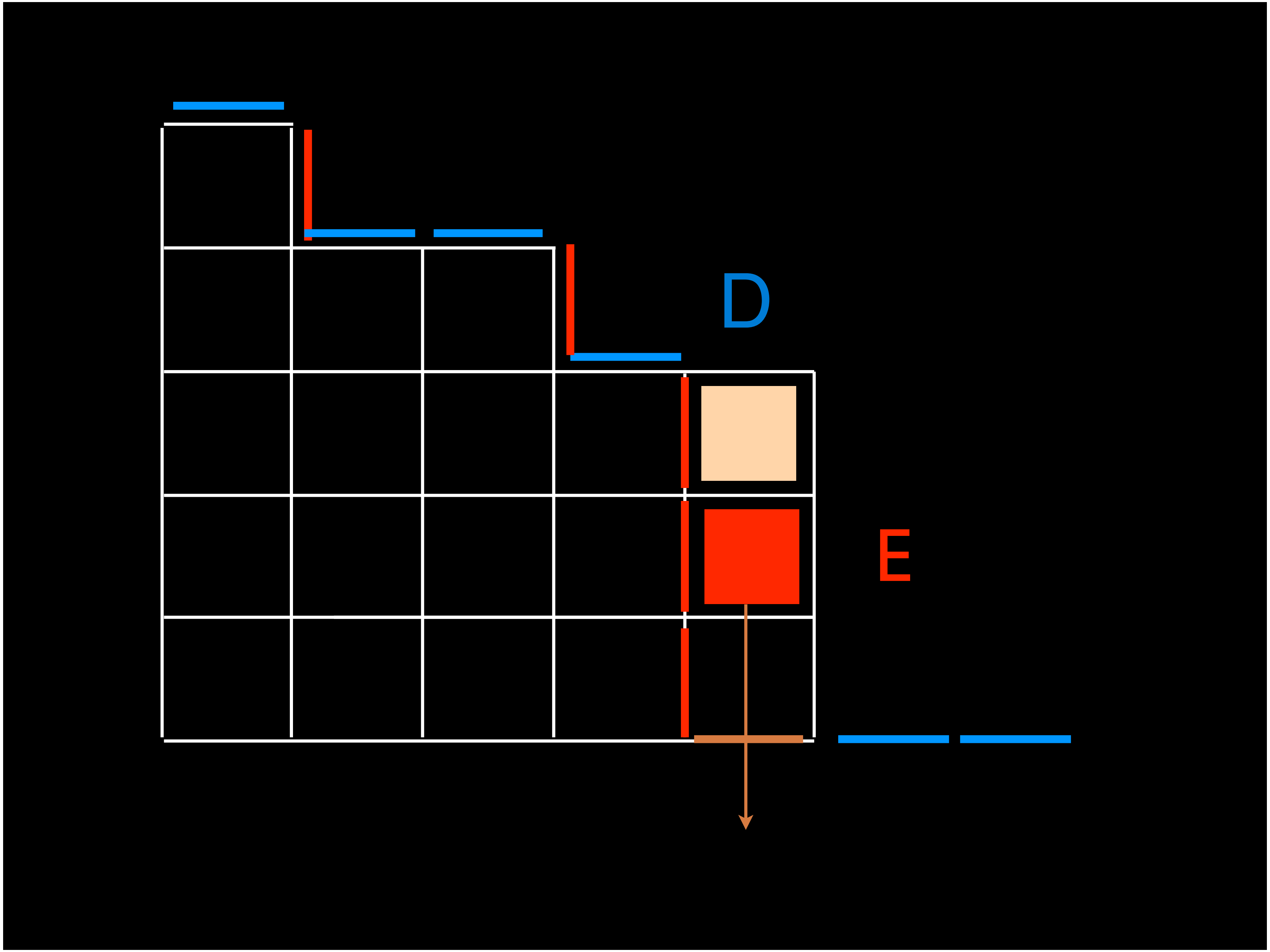
I identity

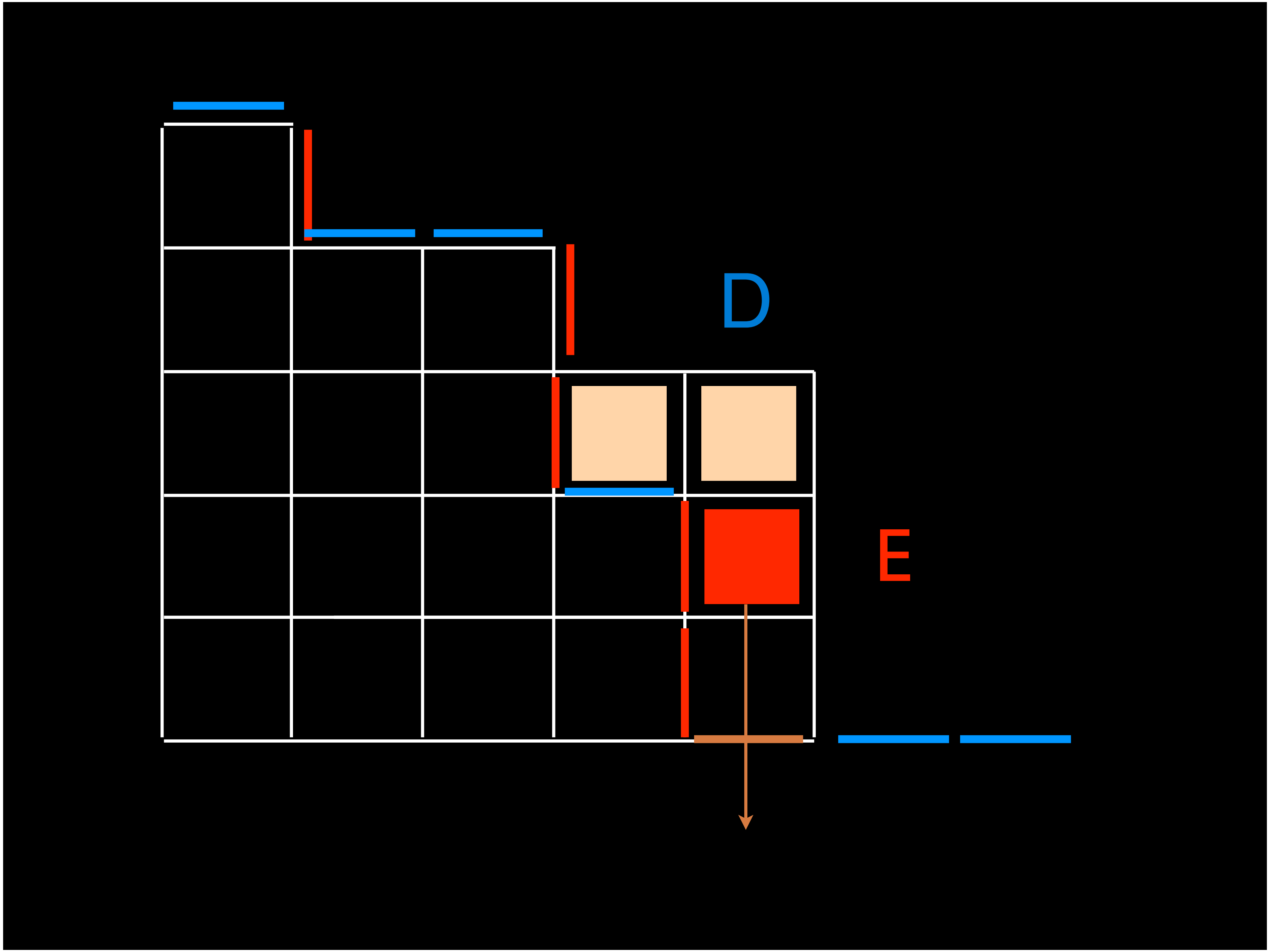


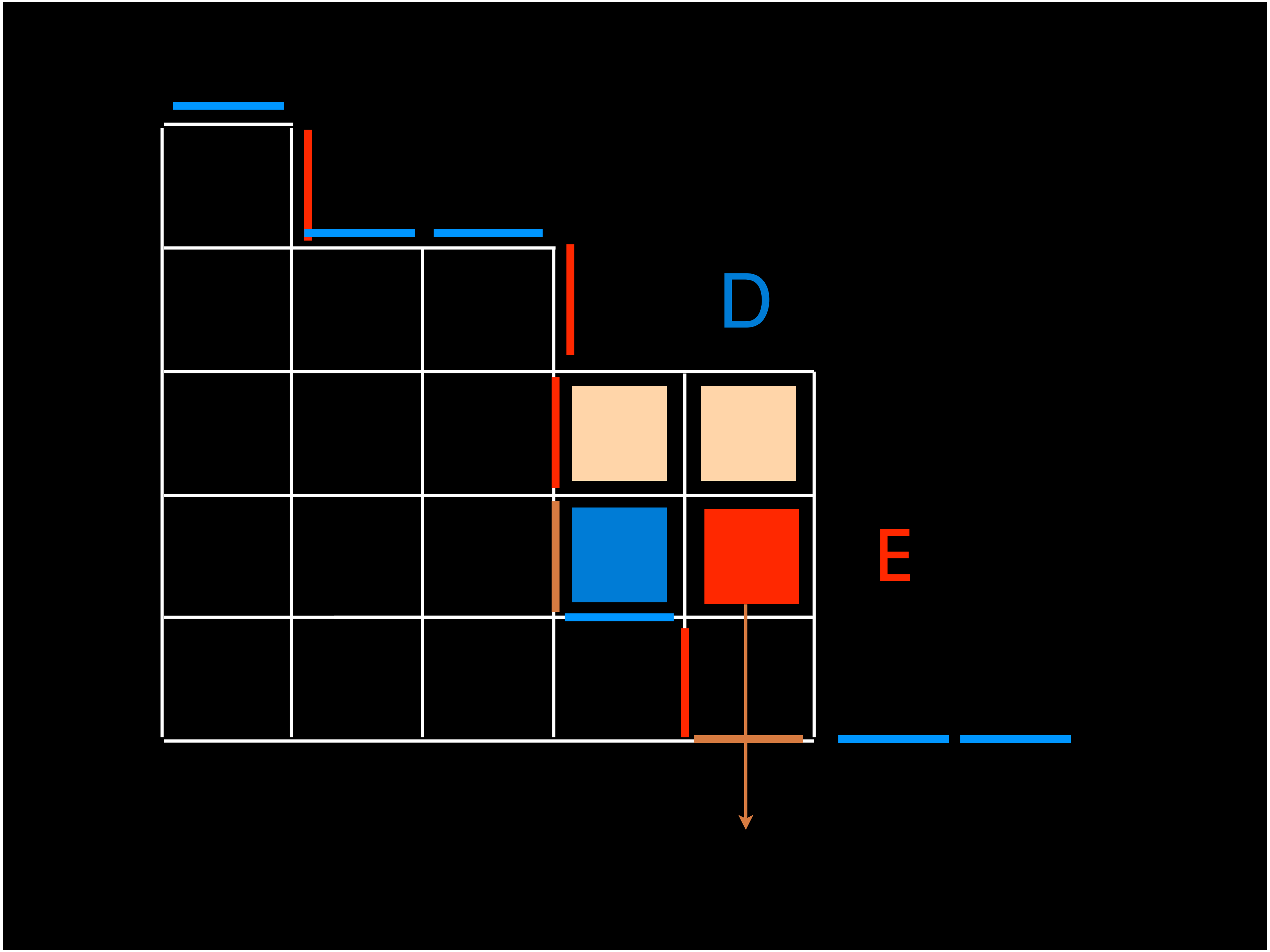

D
E

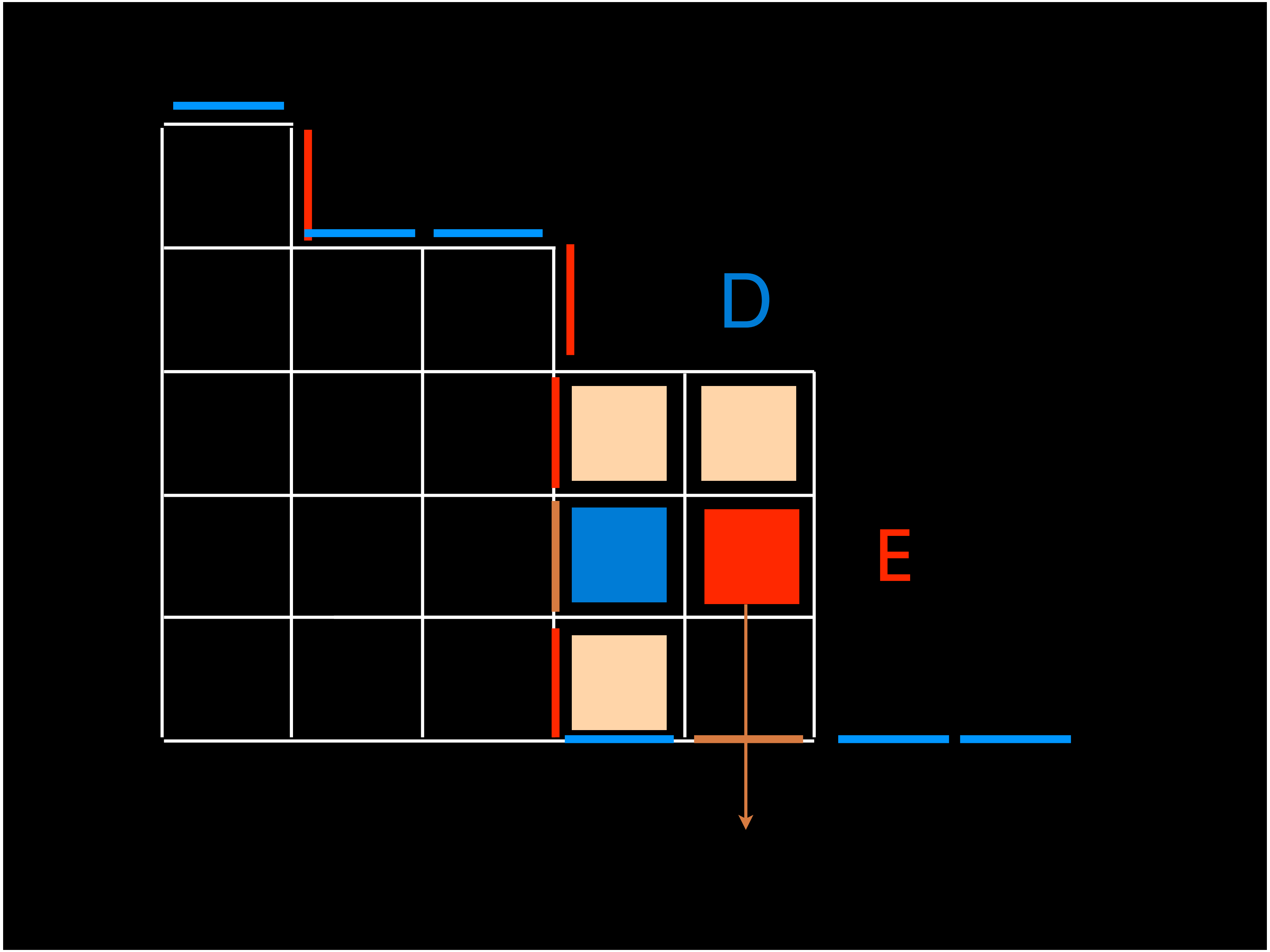

D
E


D
E


D
E


D
E


D
E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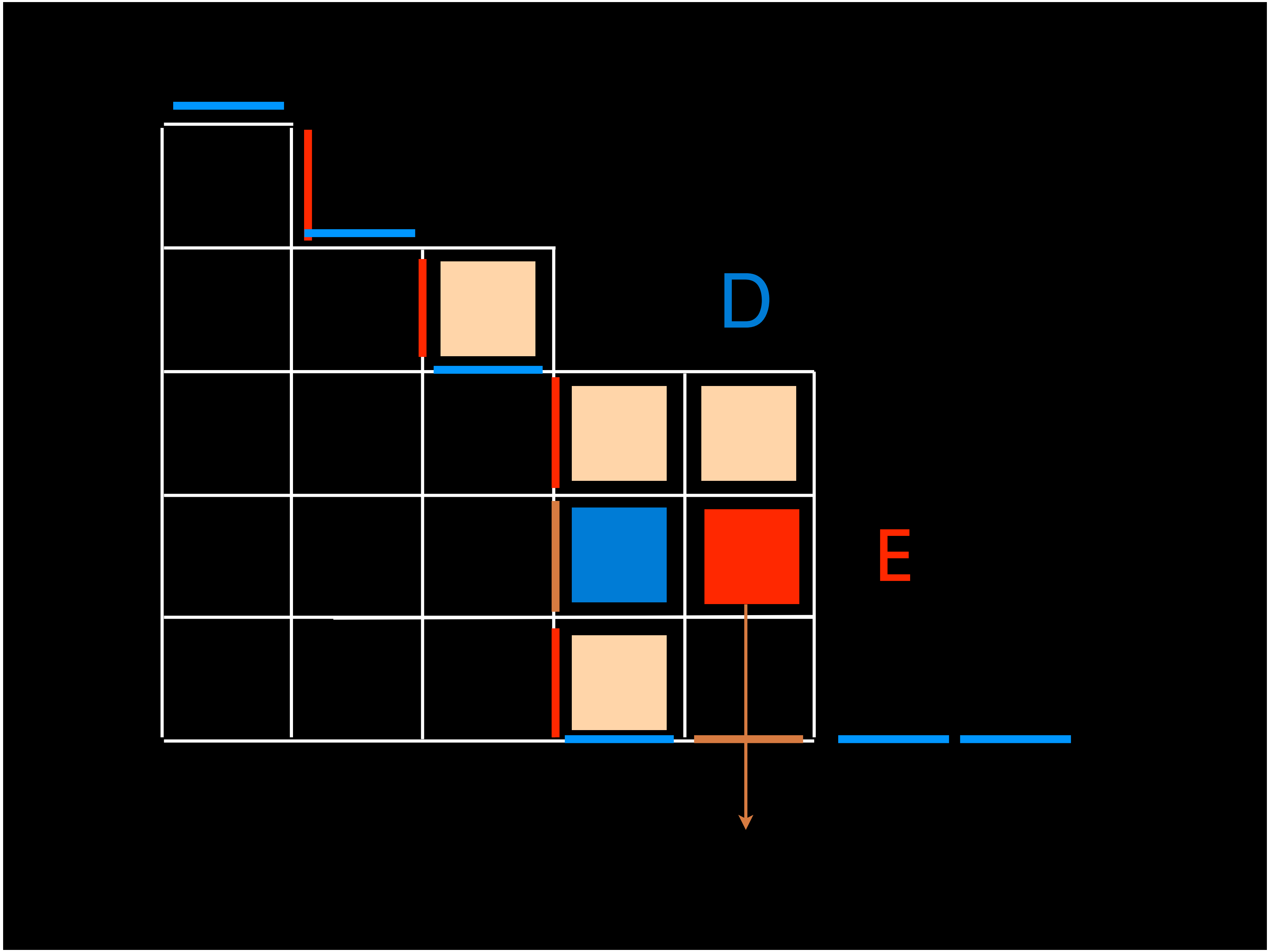
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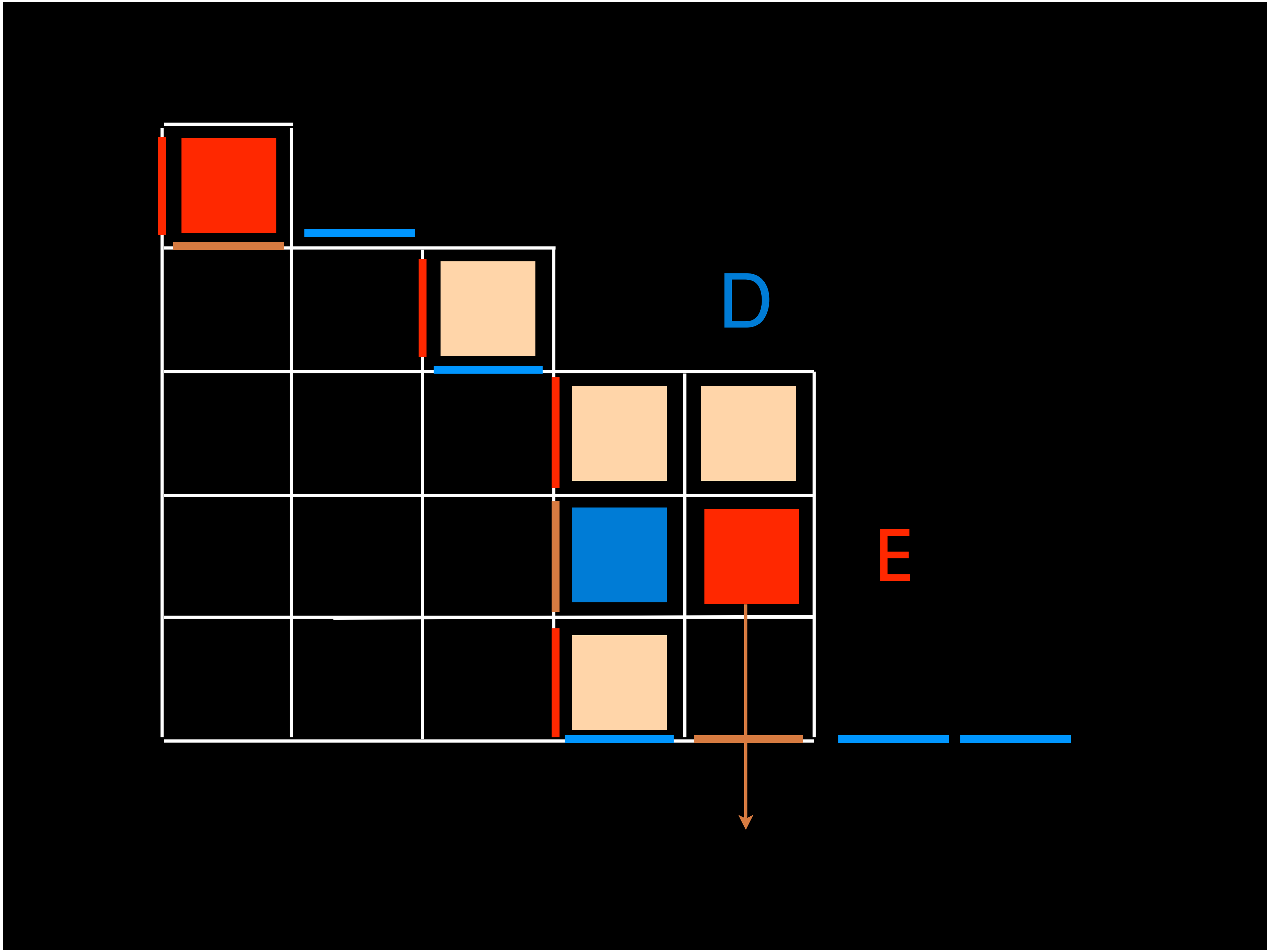
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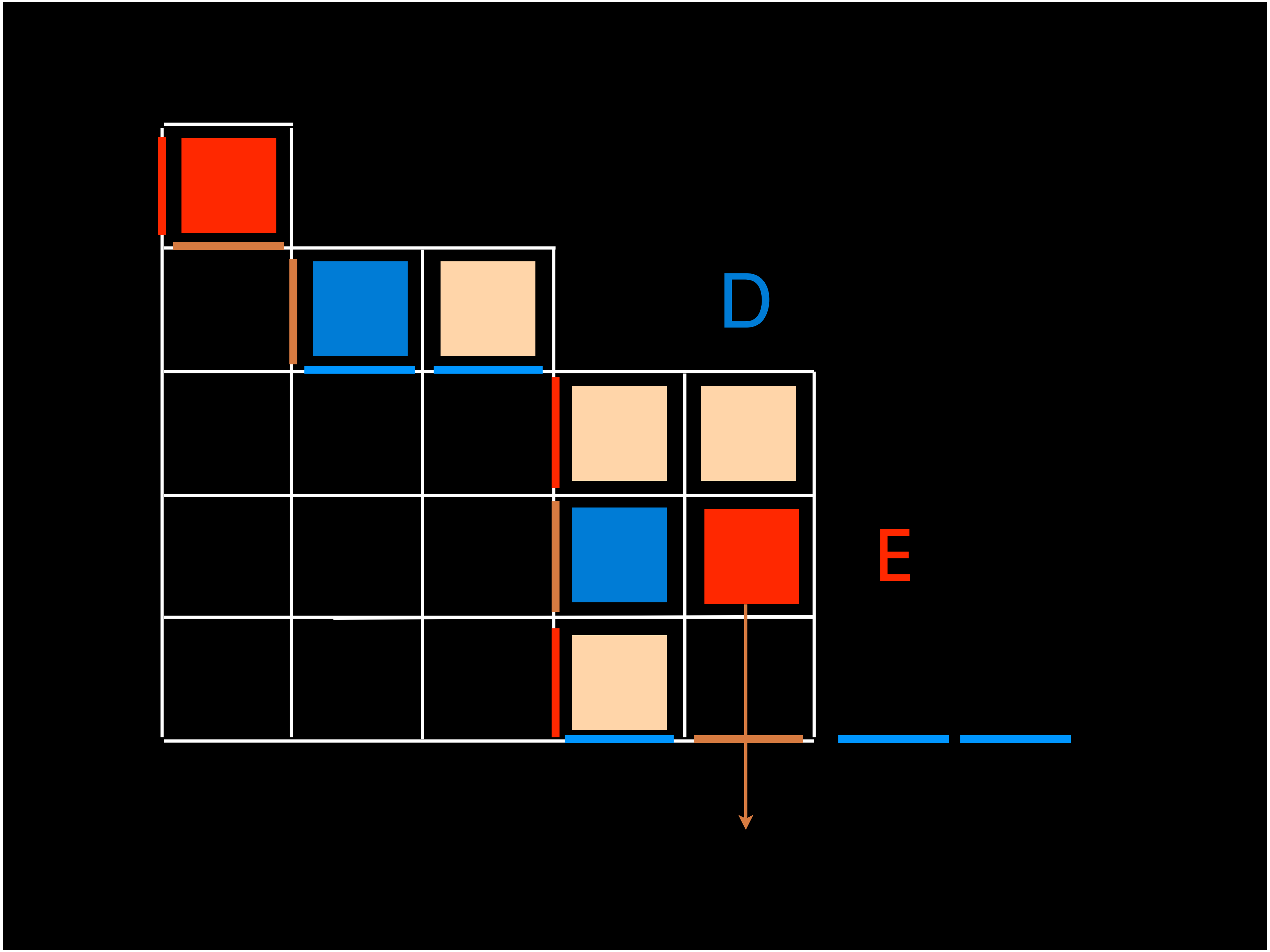
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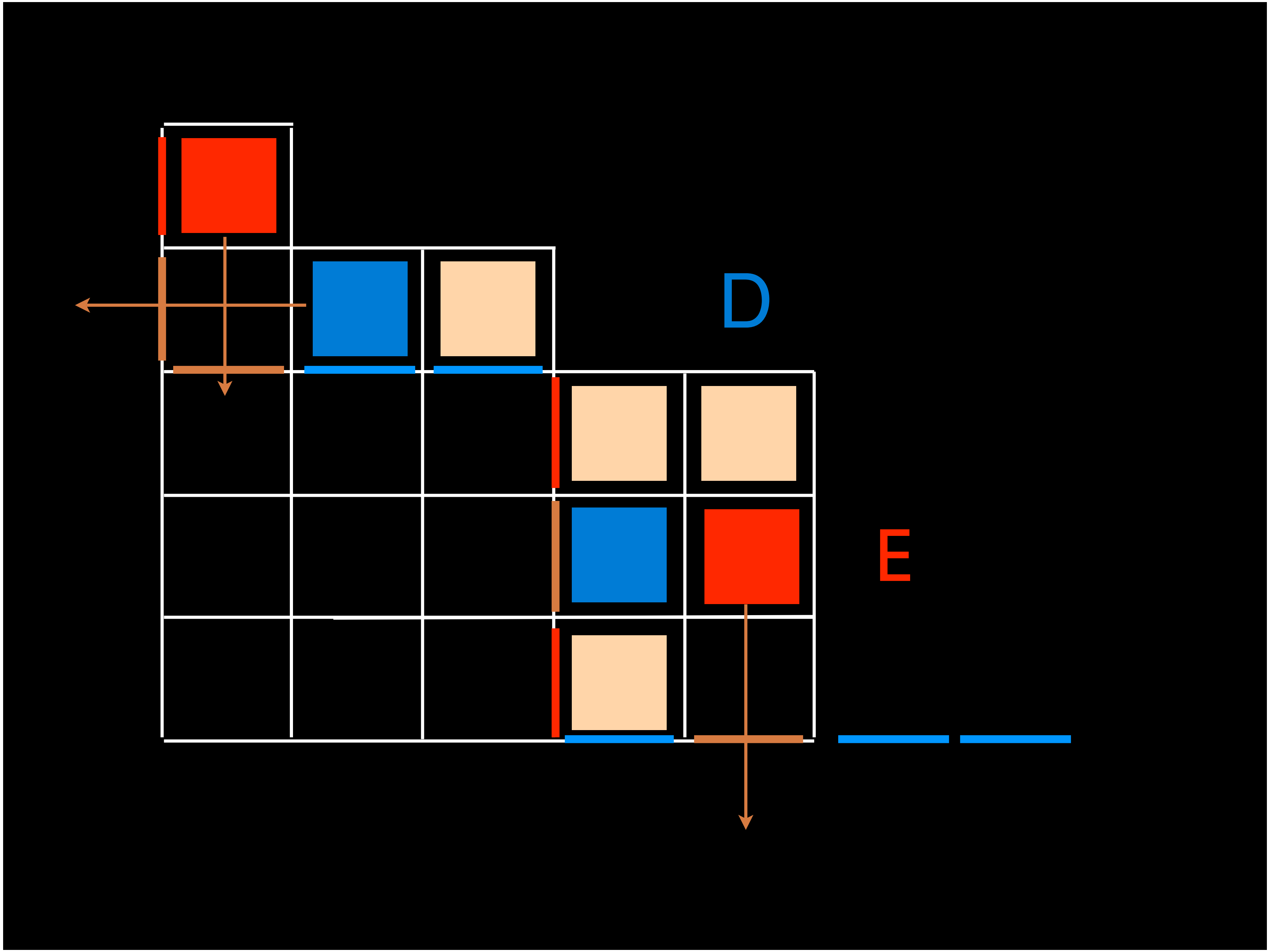
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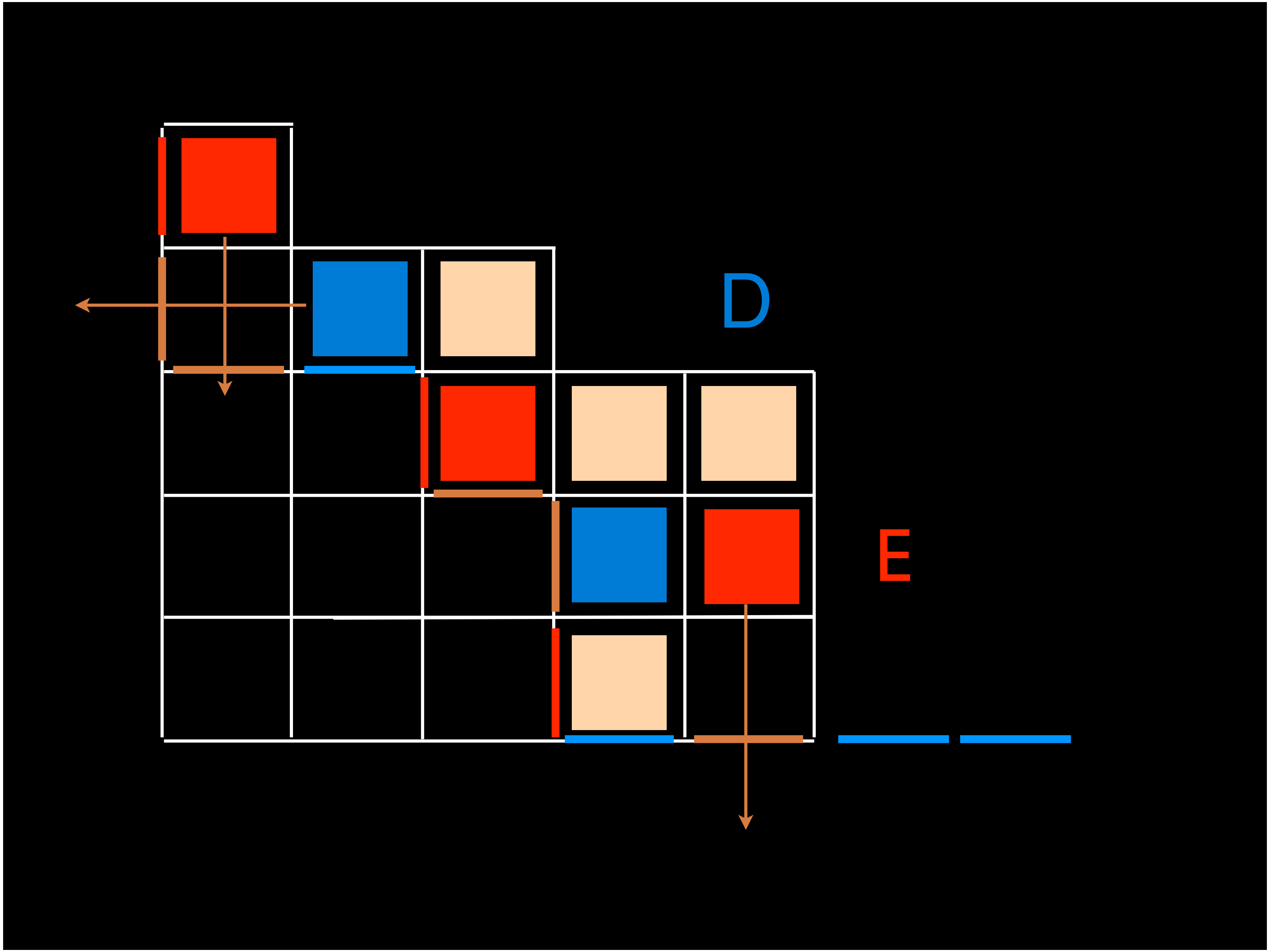
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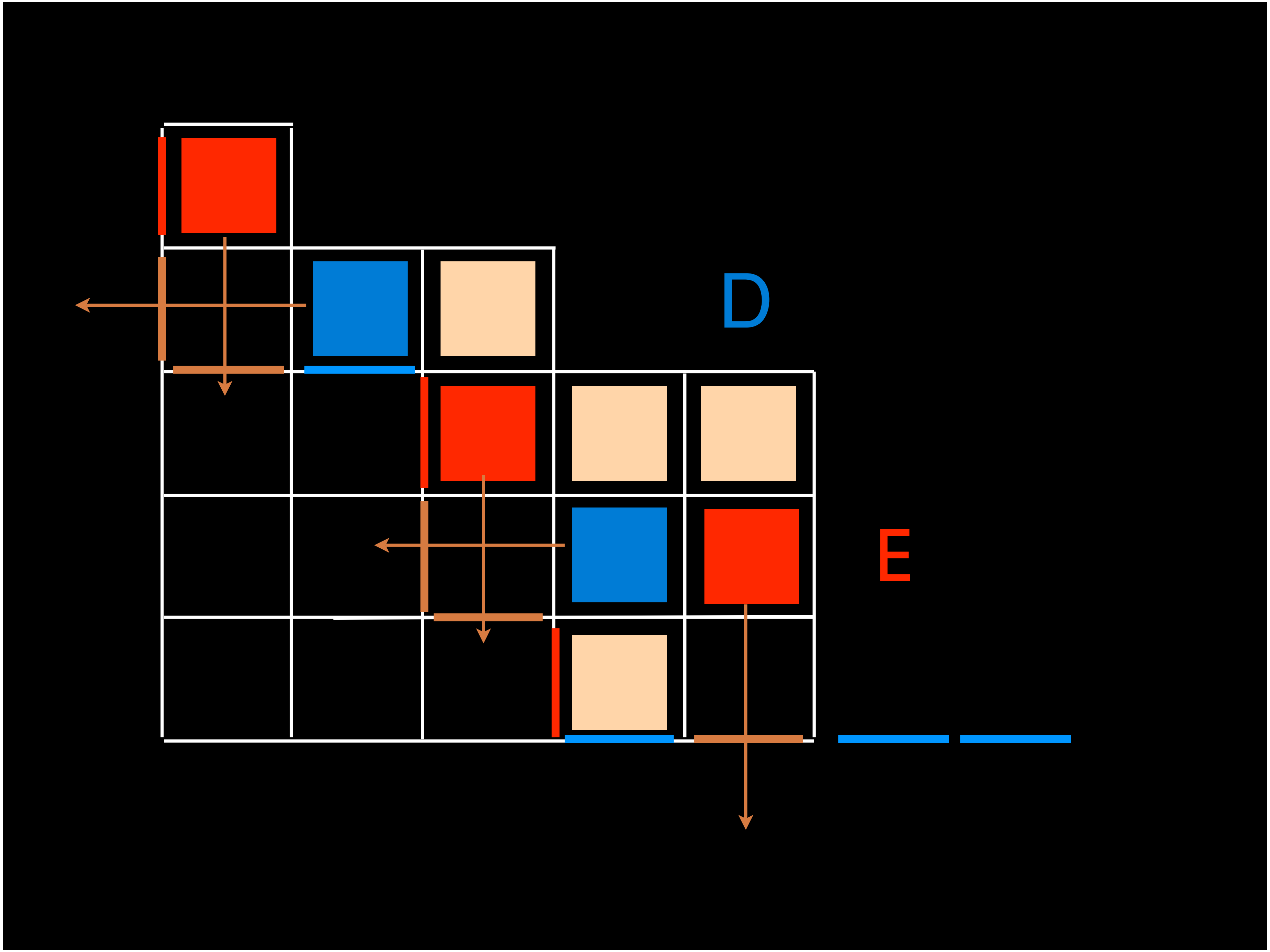
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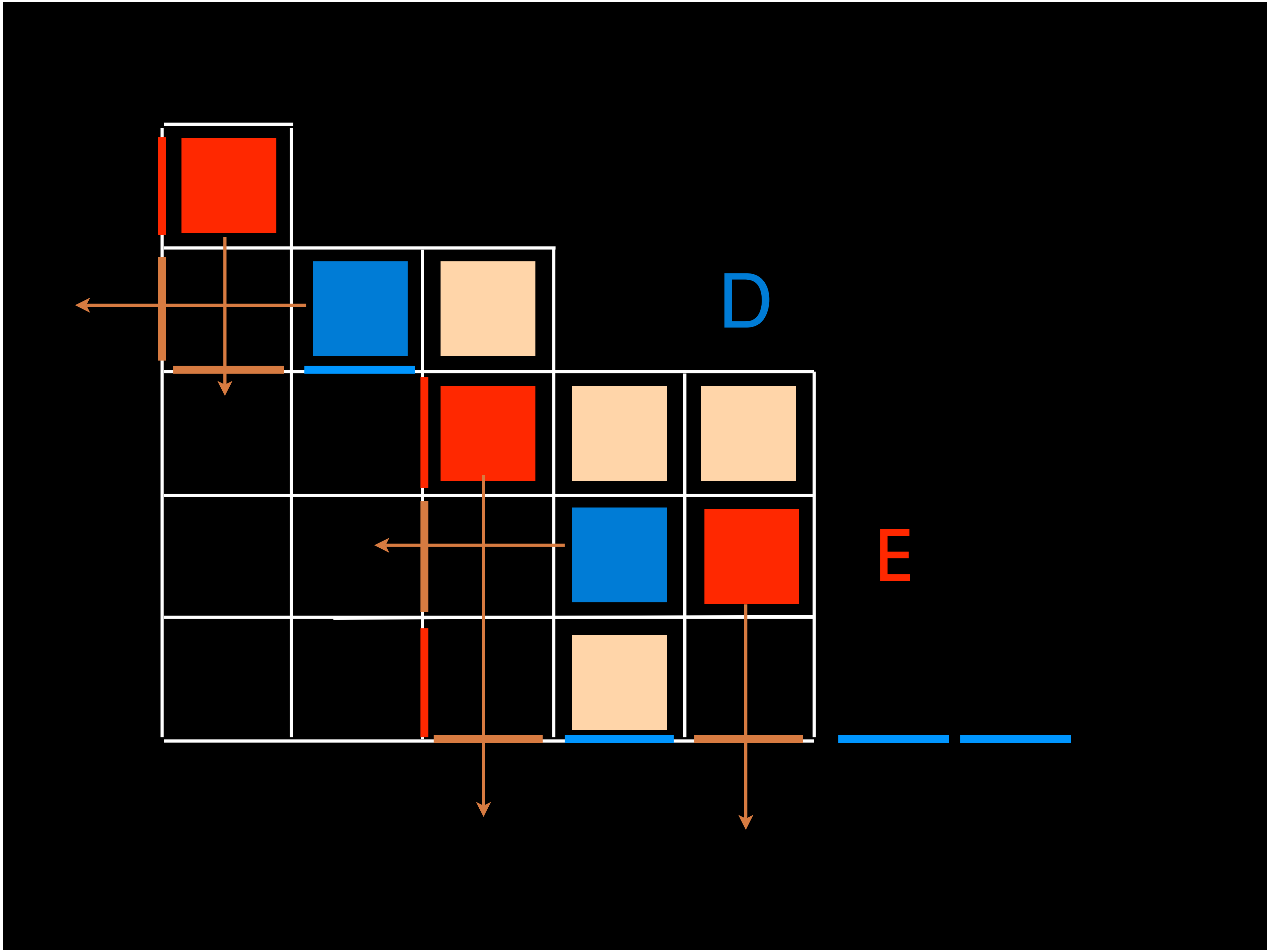
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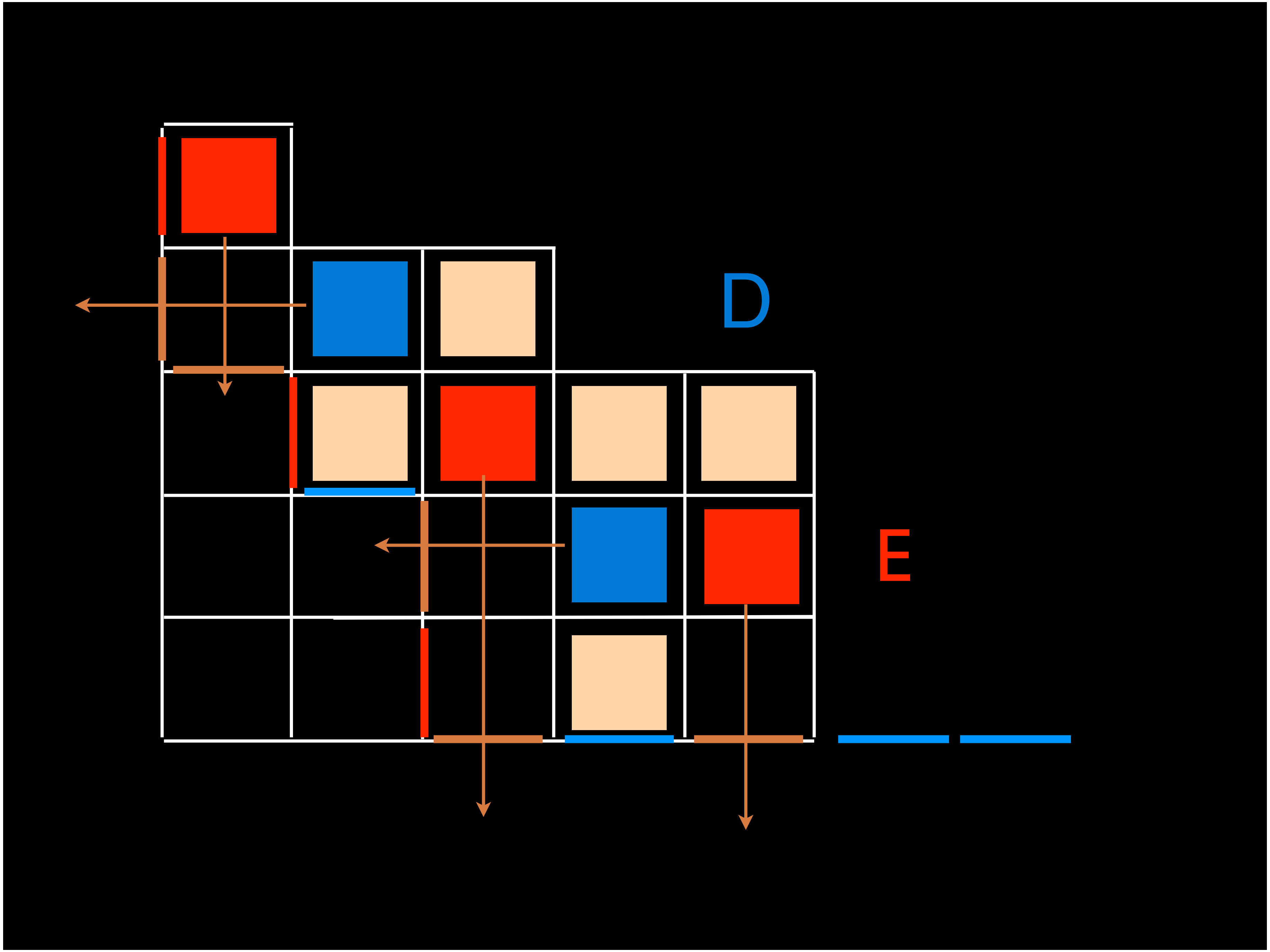
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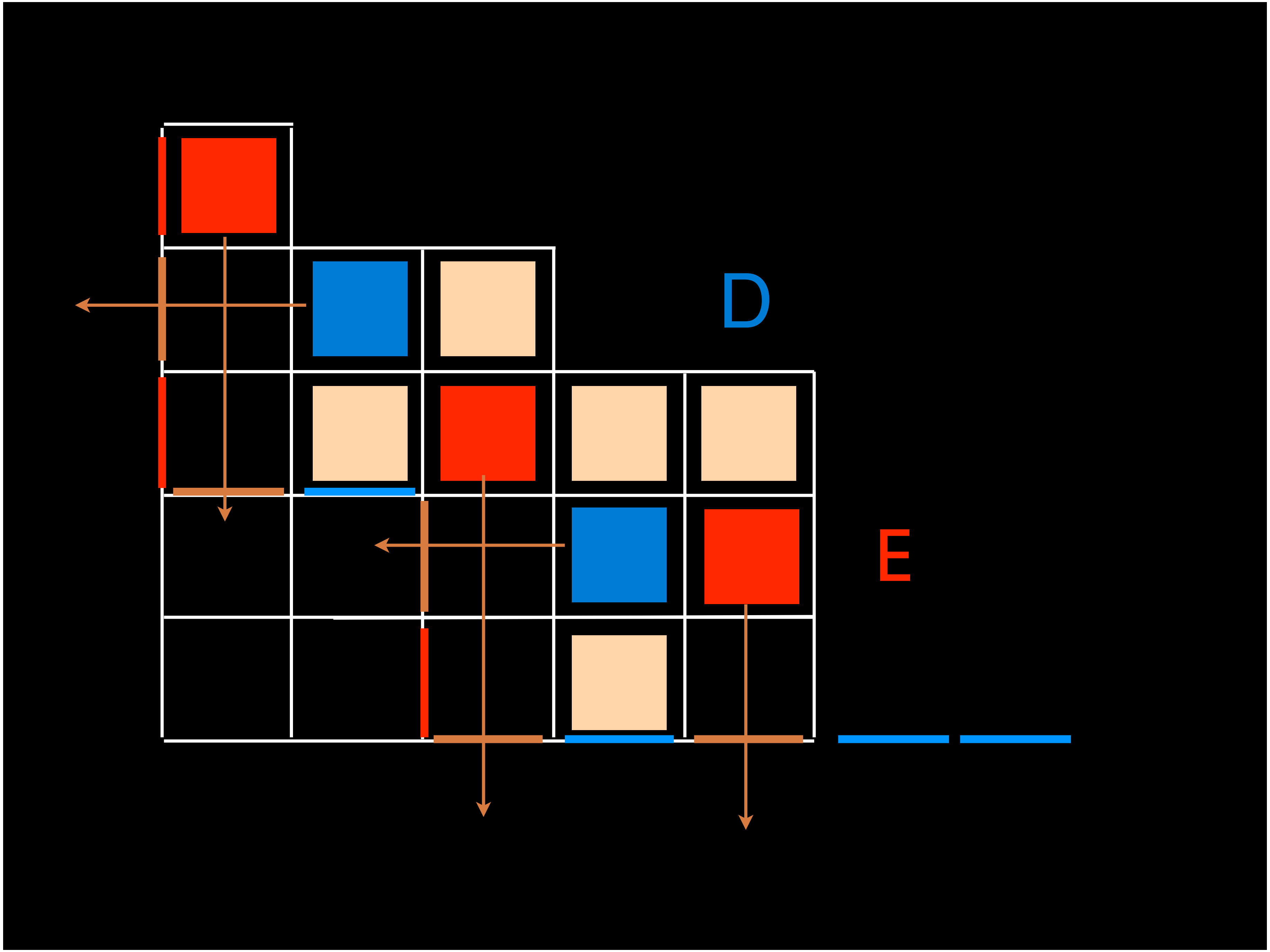

D
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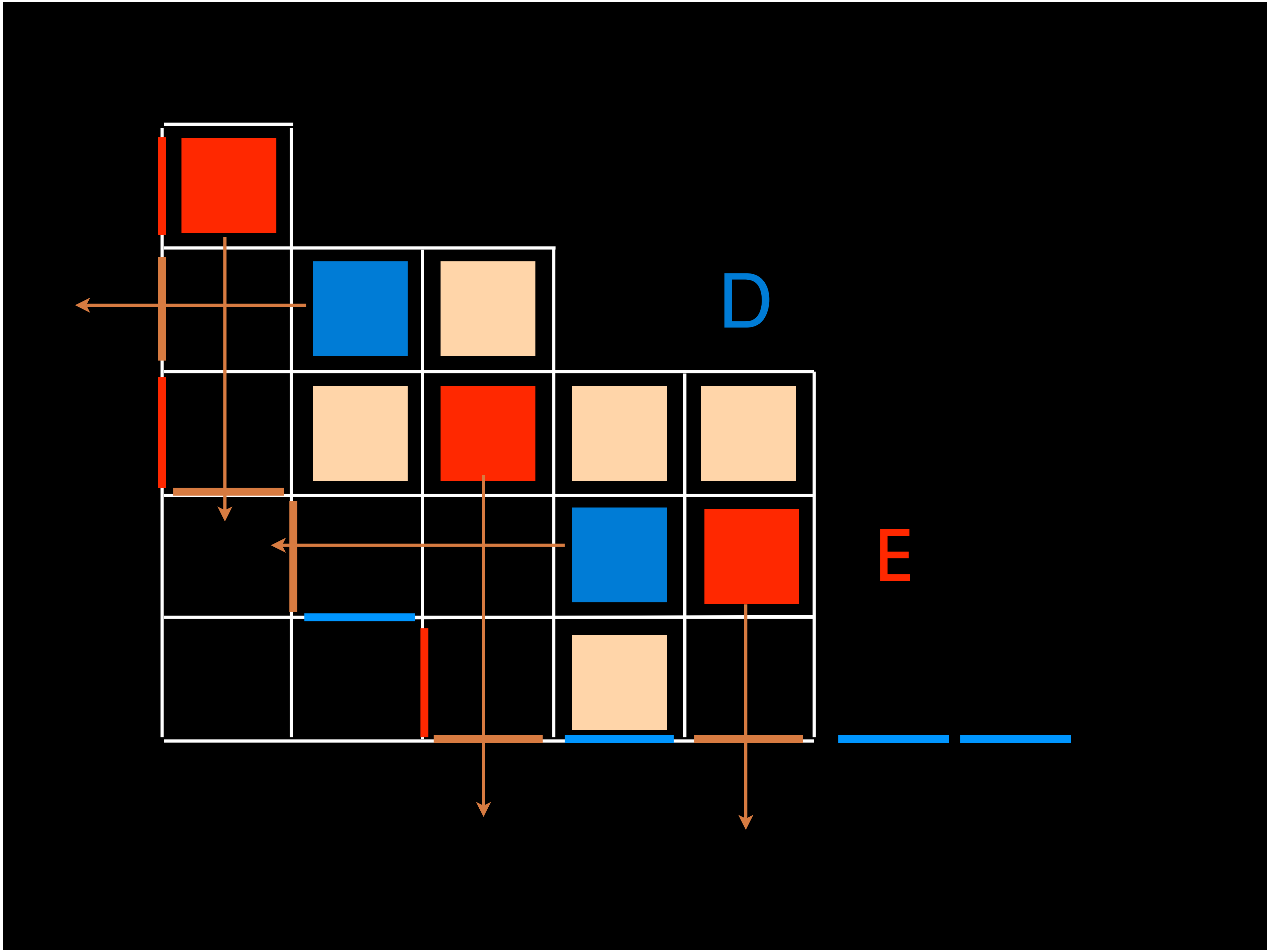
D
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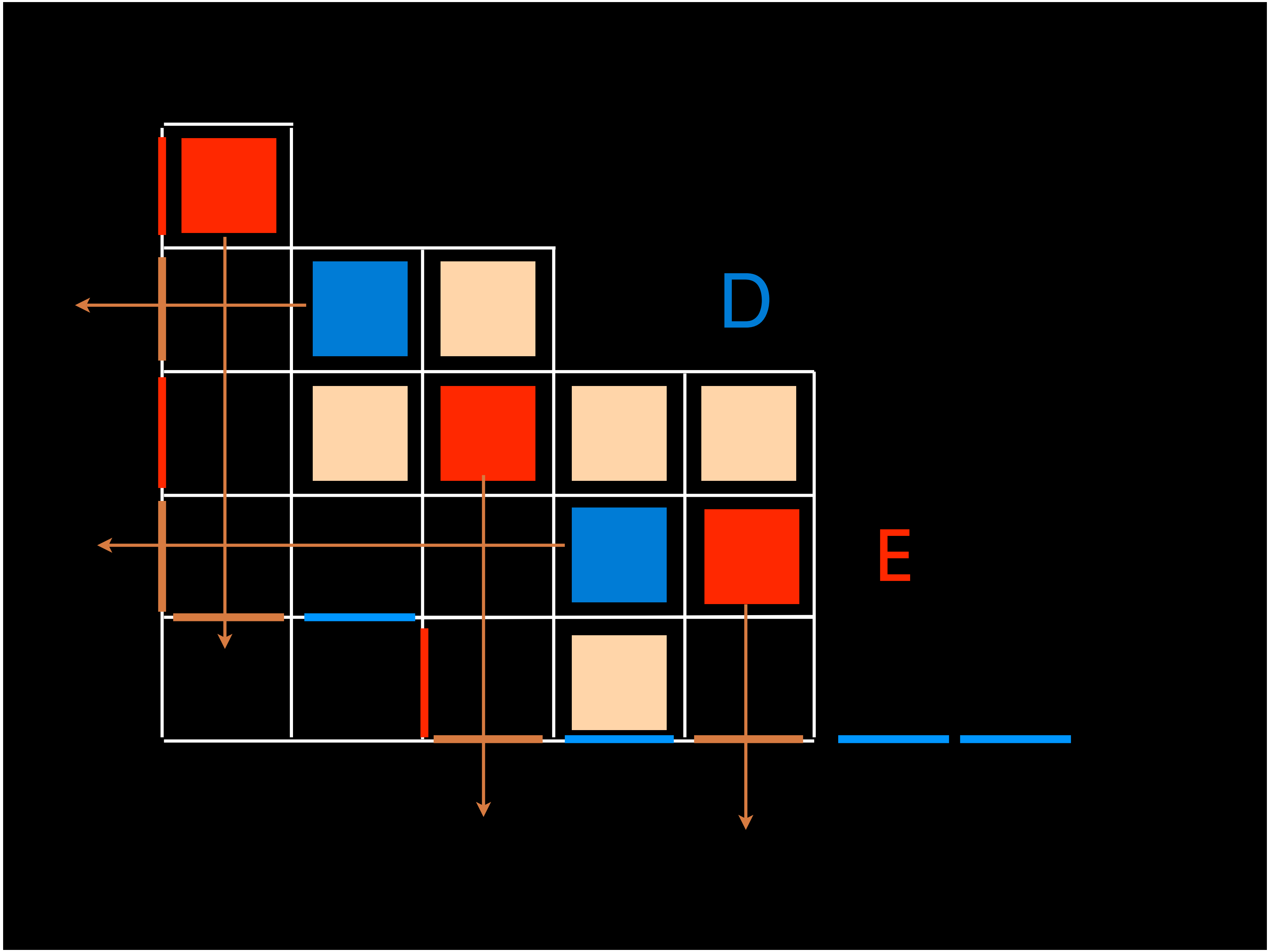
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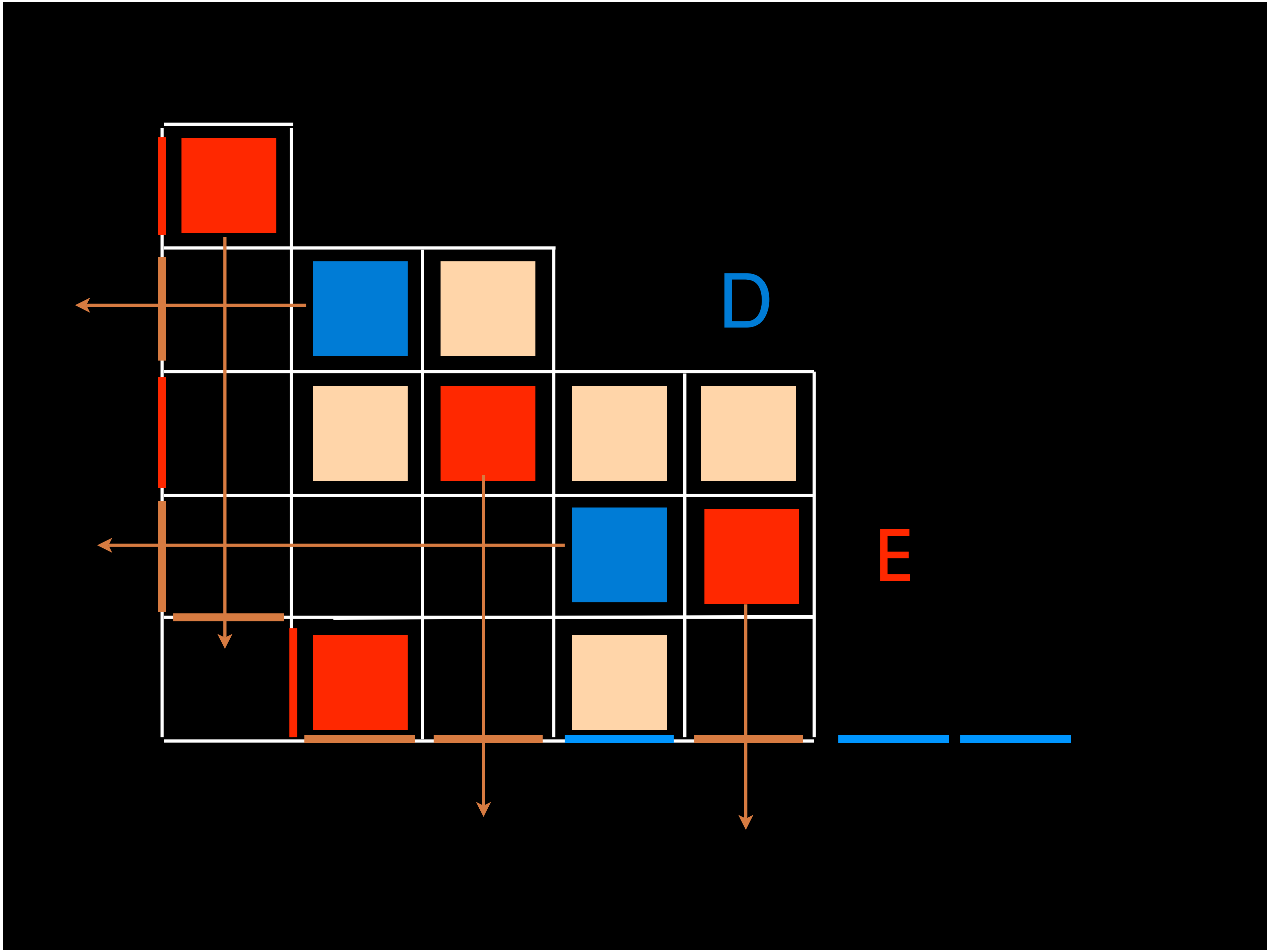
E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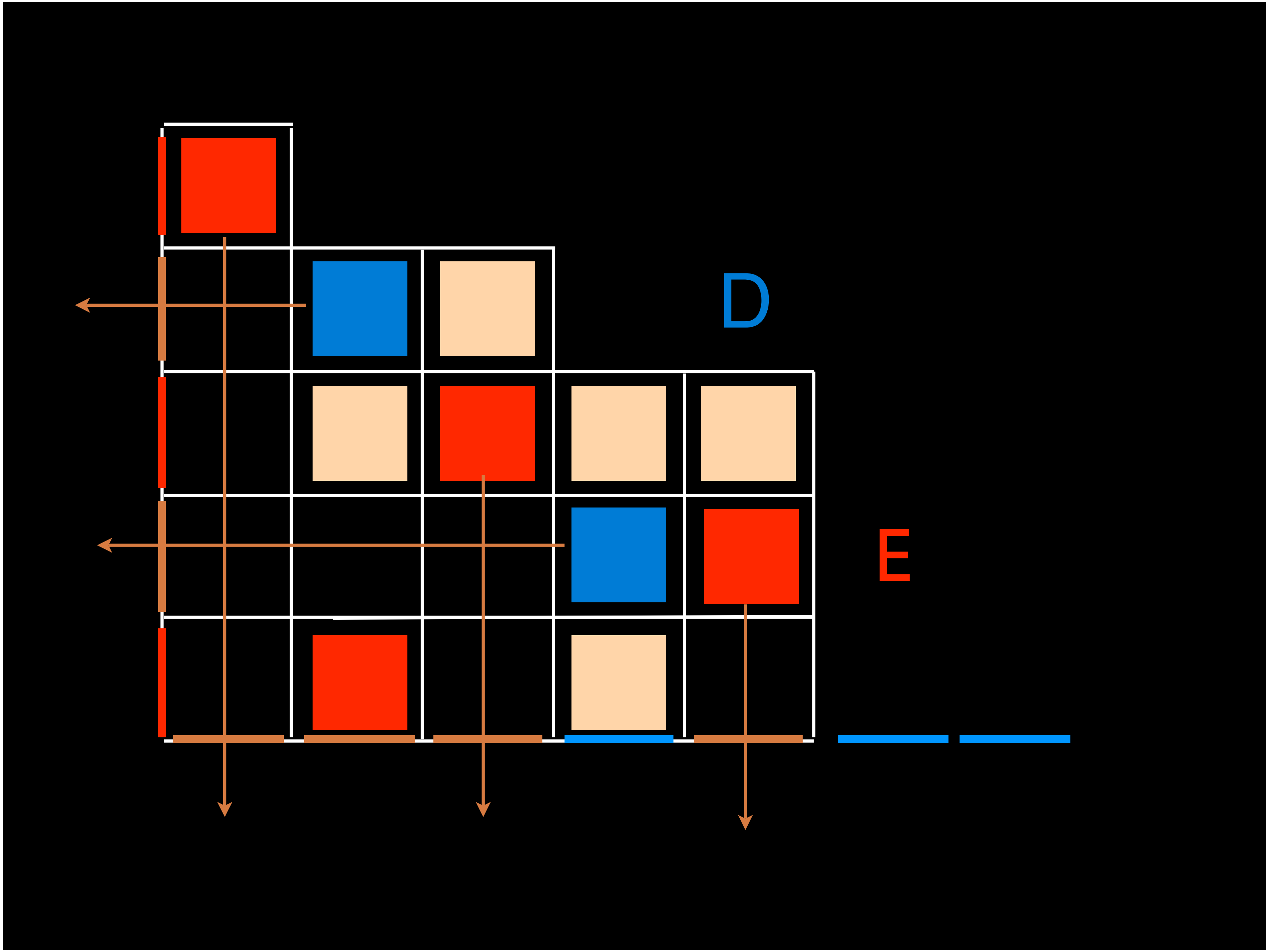
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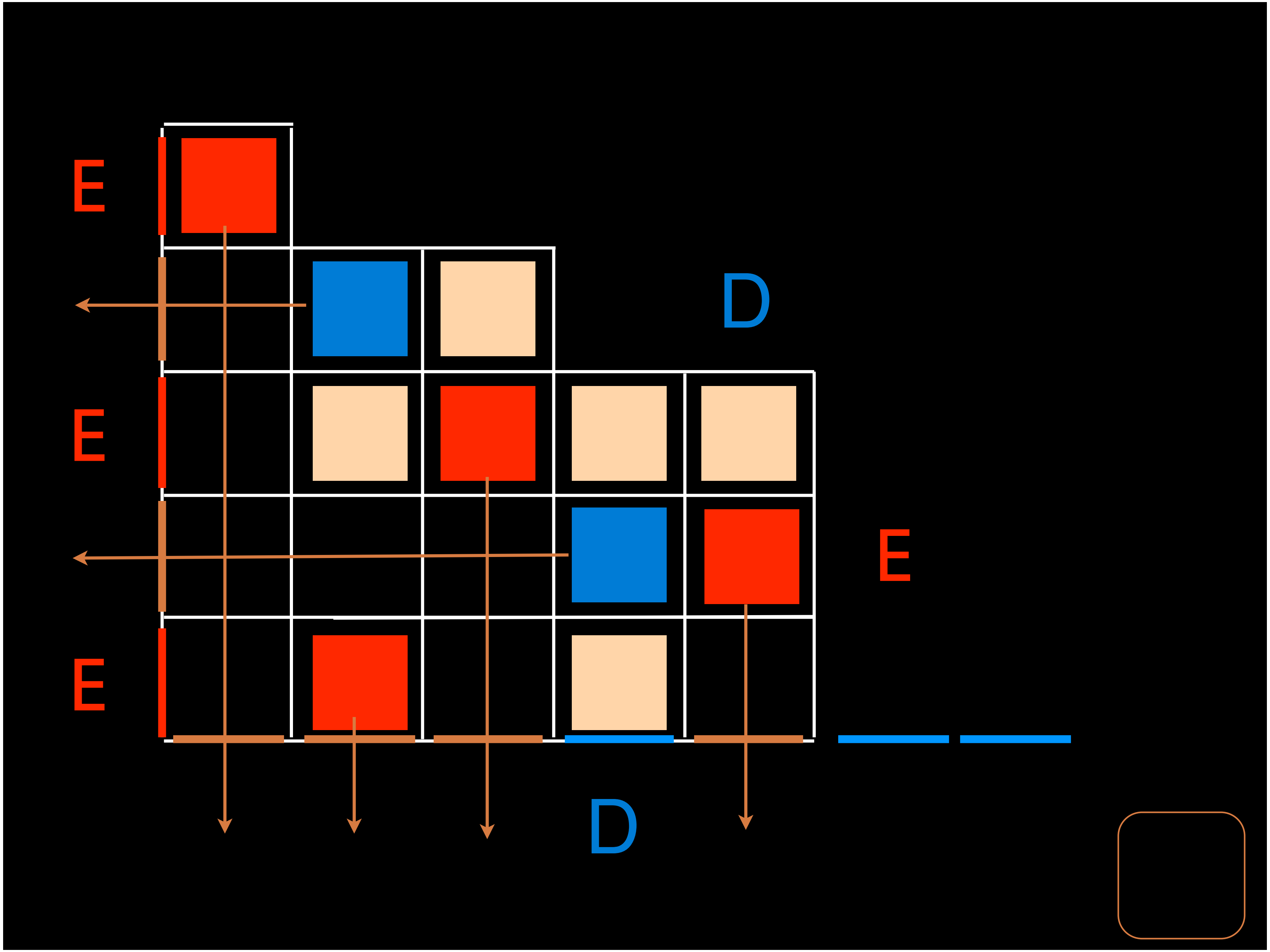

E
D
E
E
E
D

q-analog

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T \text{ alternative tableau with profile } w} q^{k(T)} E^{i(T)} D^{j(T)}$$

$k(T)$ = nb of $\boxtimes$

$i(T)$ = nb of columns without red cell

$j(T)$ = nb of rows without blue cell

$$\begin{cases} DE = qED + D + E \\ DV = \bar{\beta} V & \bar{\beta} = 1/\beta \\ WE = \bar{\alpha} W & \bar{\alpha} = 1/\alpha \end{cases}$$

$$W E^i D^j V = \bar{\alpha}^i \bar{\beta}^j \underbrace{WV}_{1}$$

<u>Cor.</u> The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ (PASEP) is

$$\text{proba}_\tau(q; \alpha, \beta) = \frac{1}{Z_n} \sum_{T} q^{L(T)} \alpha^{-f(T)} \beta^{-u(T)}$$

T: alternative tableaux, profile τ

$\begin{cases} f(T) \\ u(T) \end{cases}$ nb of $\begin{cases} \text{rows} \\ \text{columns} \end{cases}$ without  cell

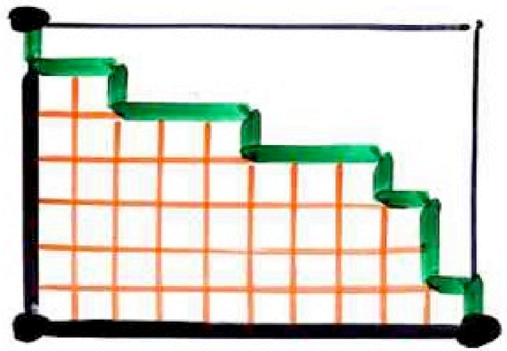
$L(T)$ nb of cells 

§5

Permutation tableaux

Permutation Tableau

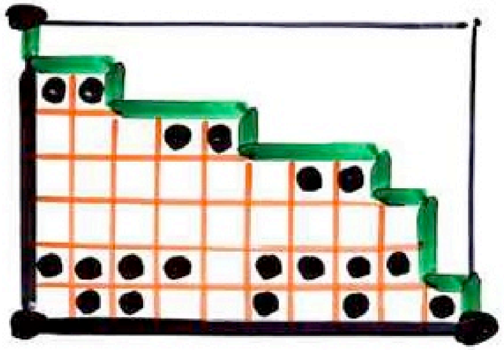
Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle

filling of the cells with 0 and 1



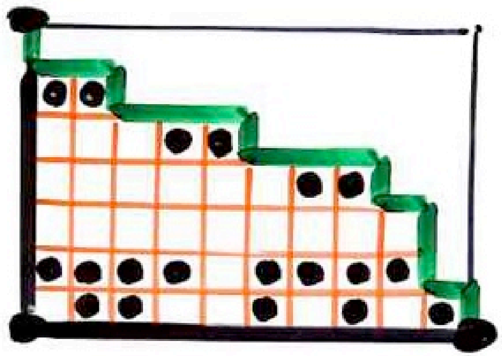
□ = 0 ⊡ = 1

(i)

(ii)

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



□ = 0   ▣ = 1

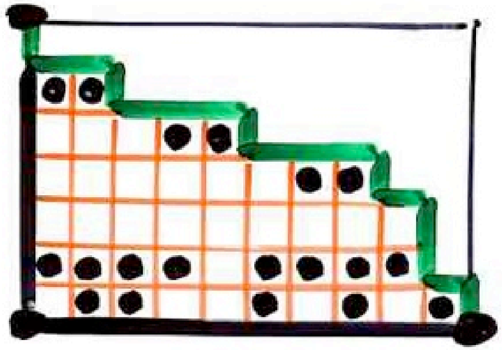
filling of the cells with 0 and 1

(i) in each column: at least one 1

(ii)

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



□ = 0 ■ = 1

filling of the cells with 0 and 1

(i) in each column : at least one 1

(ii)  forbidden

permutation tableau

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)

Corteel, Williams (2006) PASEP

Partially Asymmetric Exclusion Process

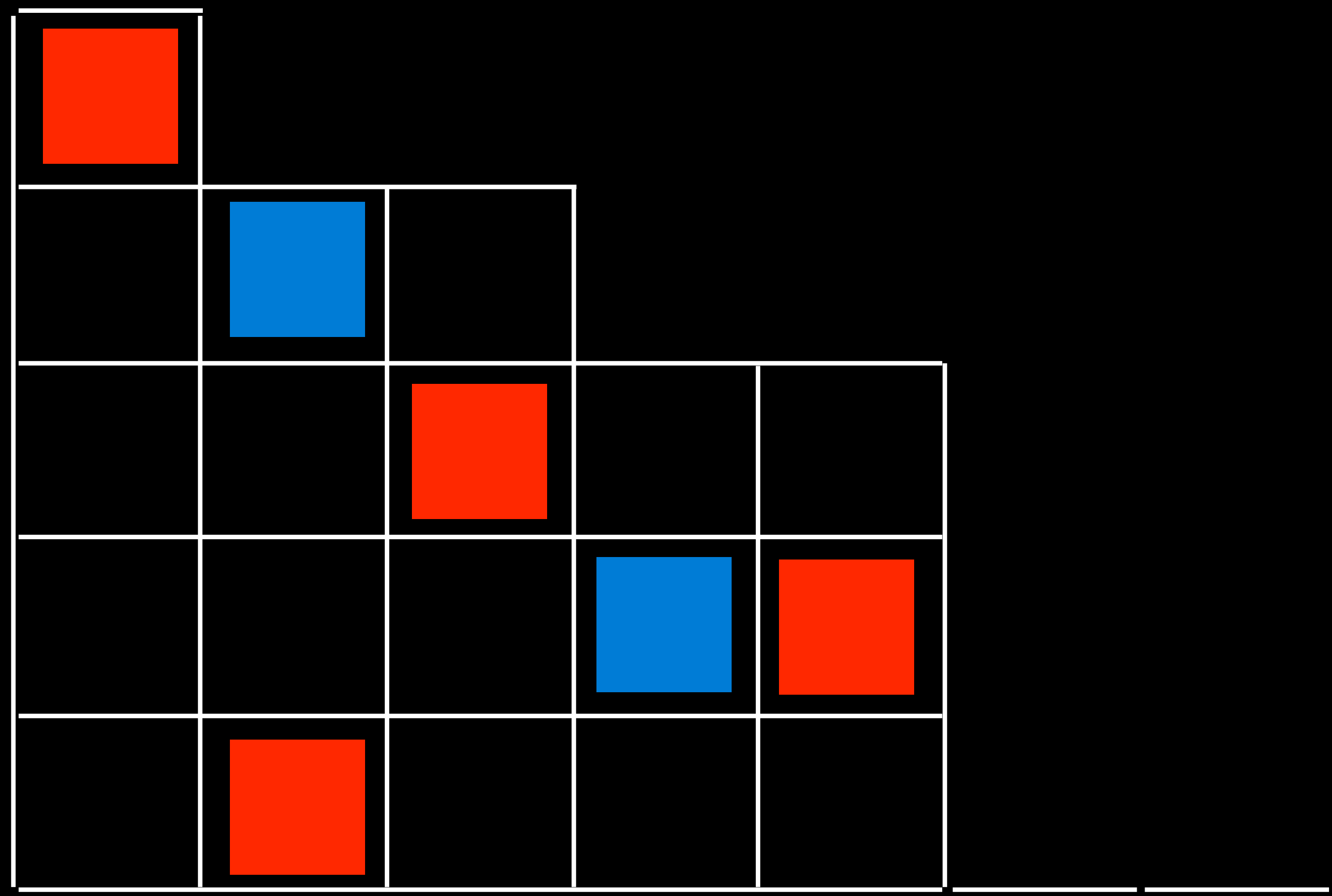
The total number of permutation tableaux (n fixed, $1 \leqslant k \leqslant n$) is

$$n!$$

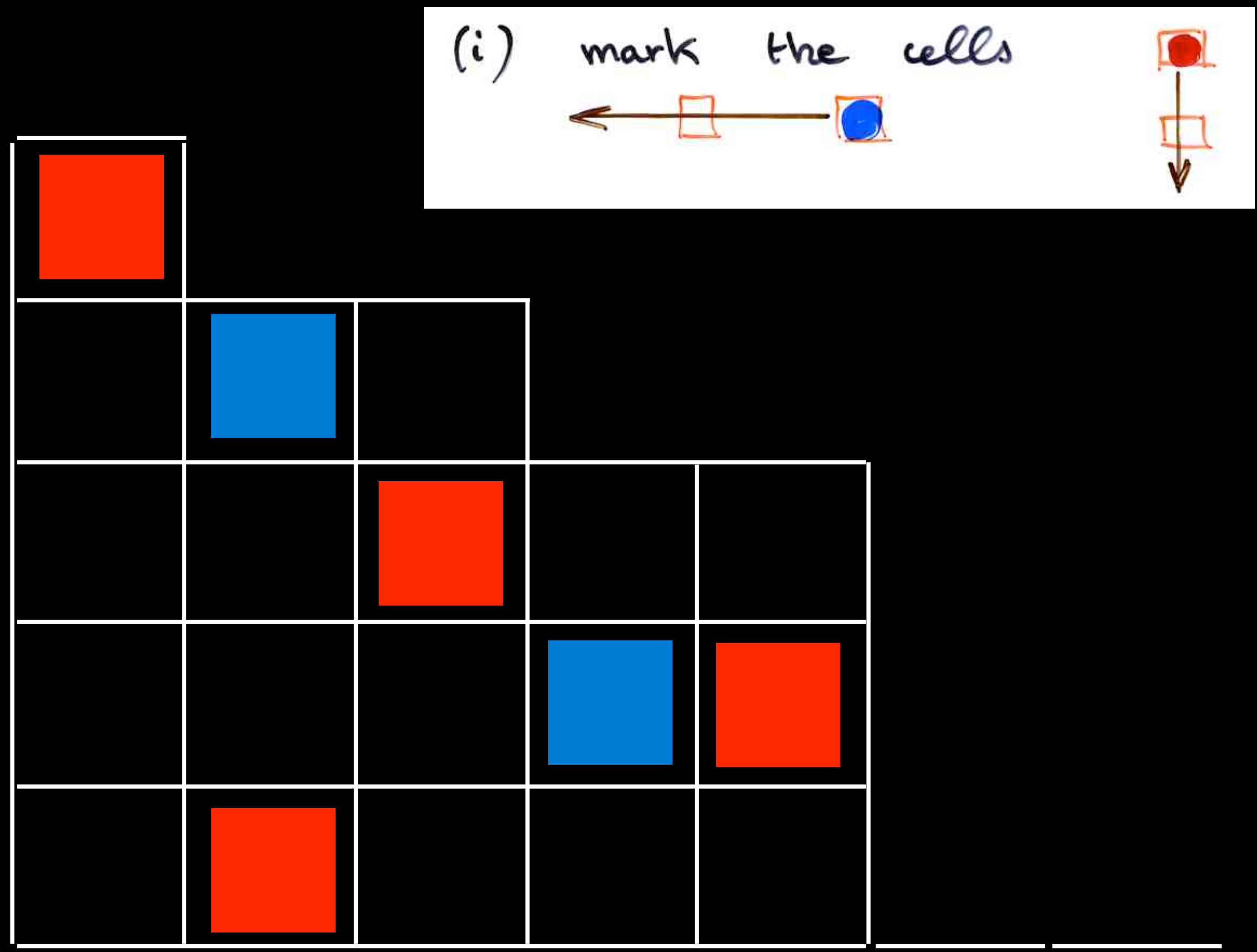
bijection permutations $\longleftrightarrow$ permutation tableaux

- Postnikov, Steingrímsson, Williams (2005)
- Corteel (2006)
- Corteel, Nadeau (2007)

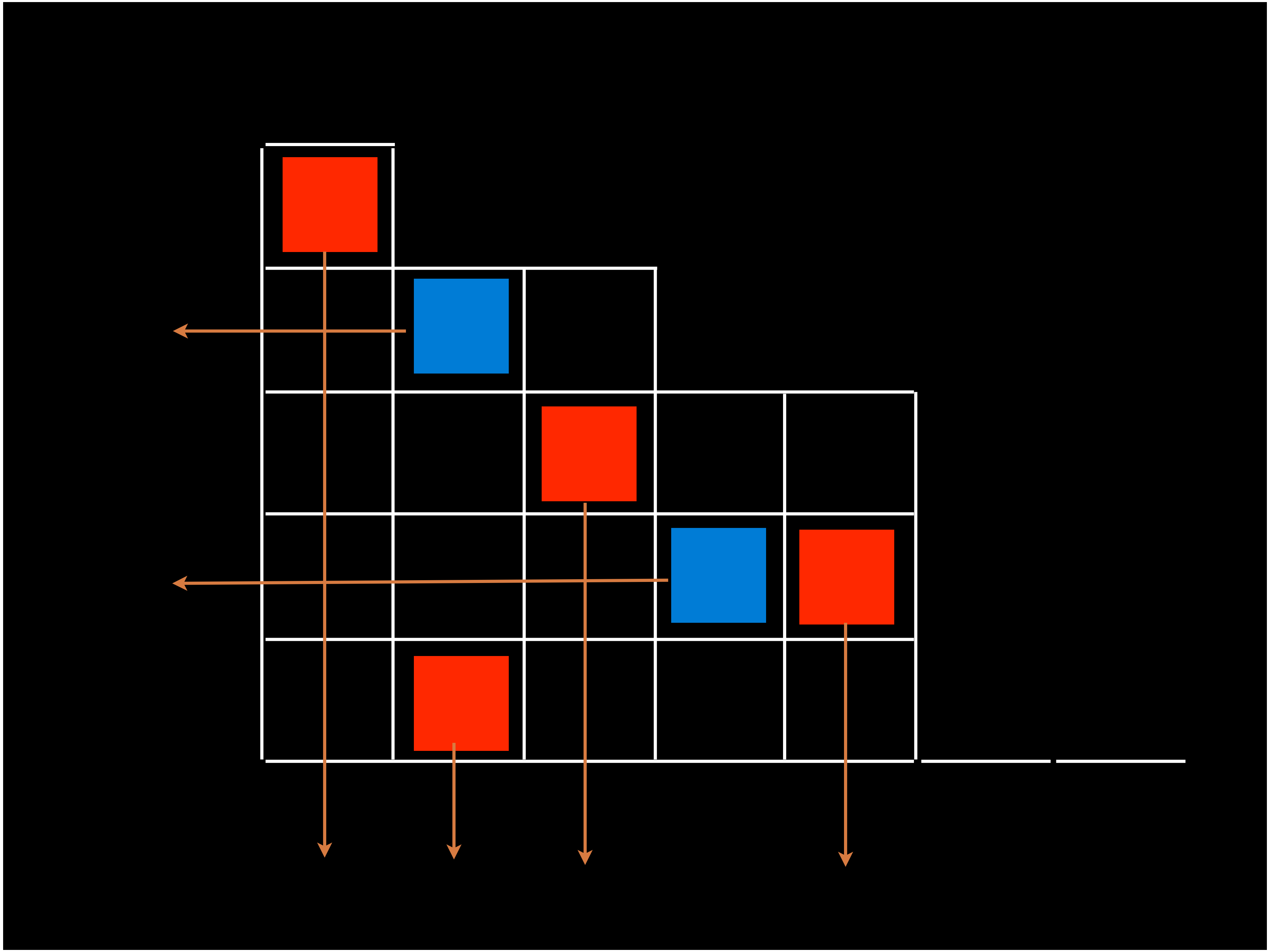
bijection: alternative tableaux size n $\updownarrow$ permutation tableaux size $(n+1)$

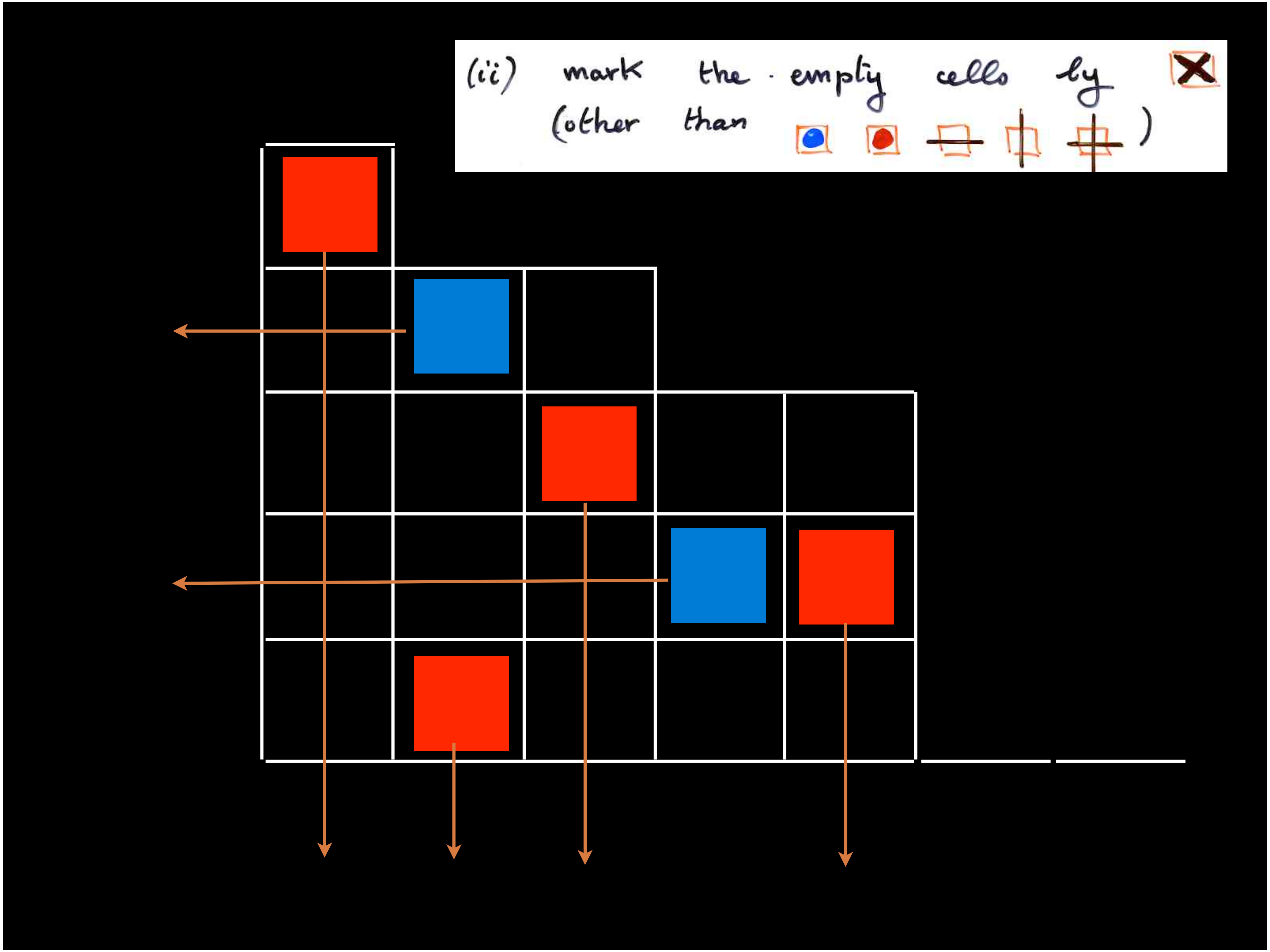


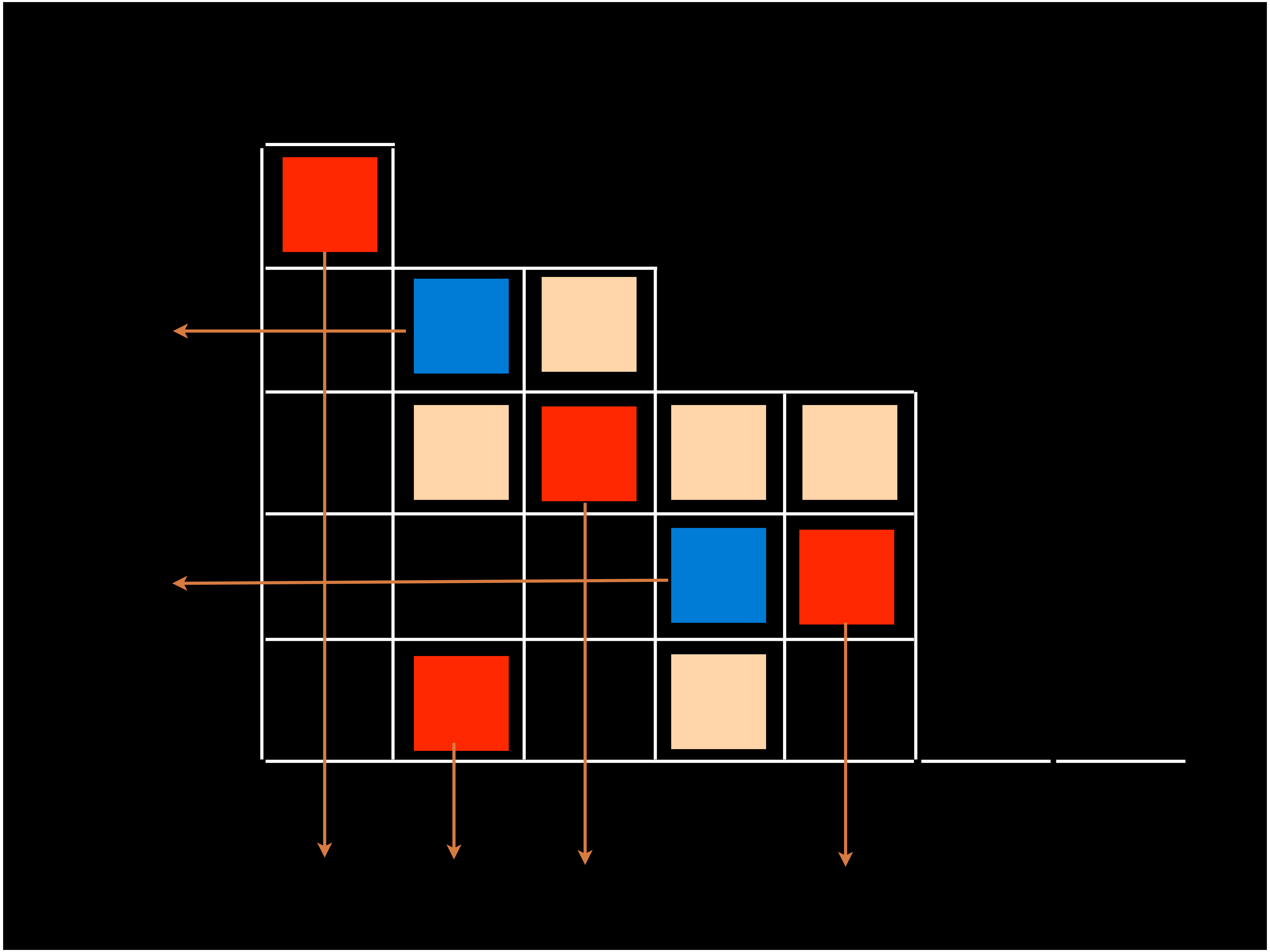
alternative tableau



alternative tableau

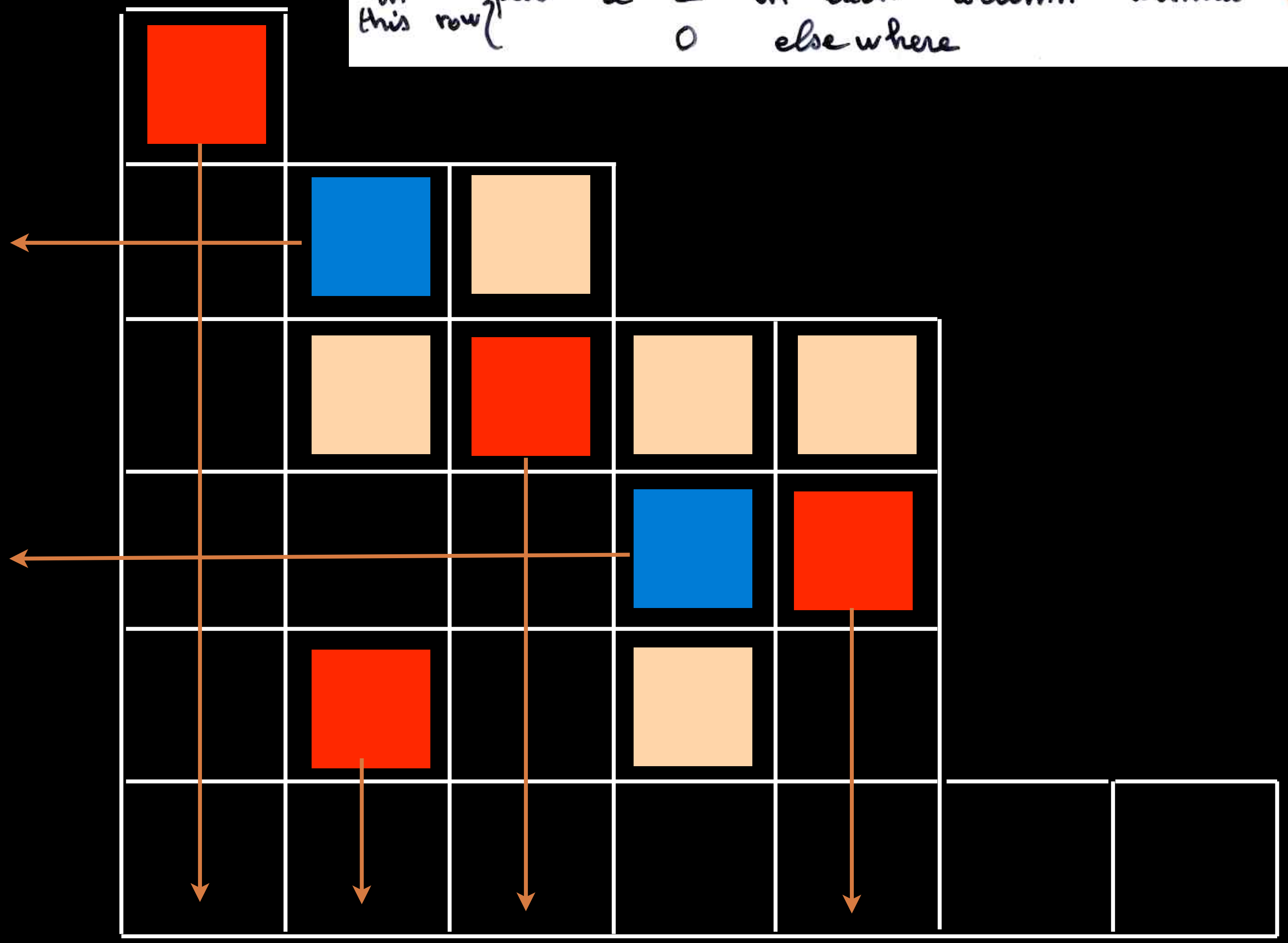


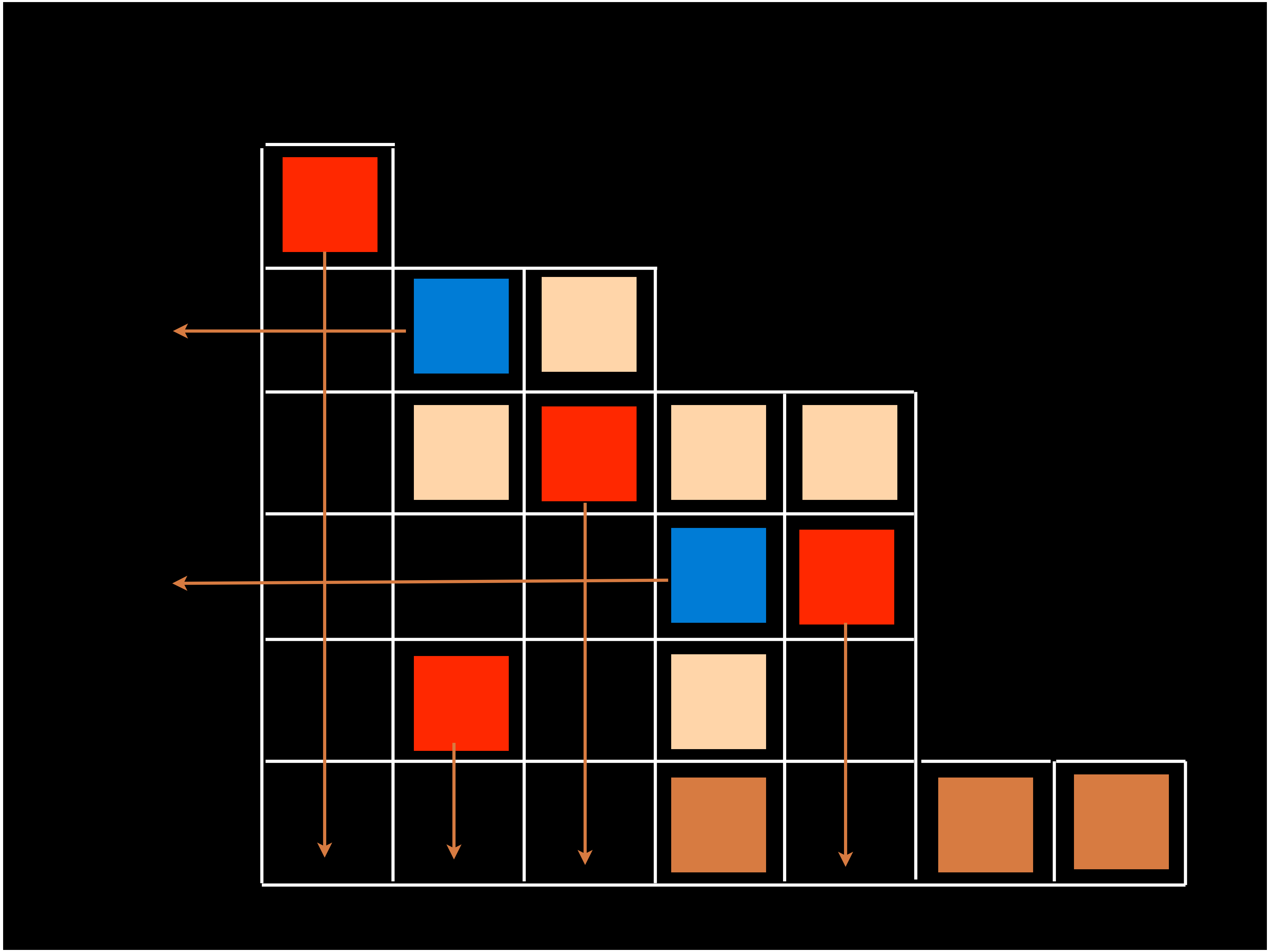

(ii) mark the empty cells by
(other than)

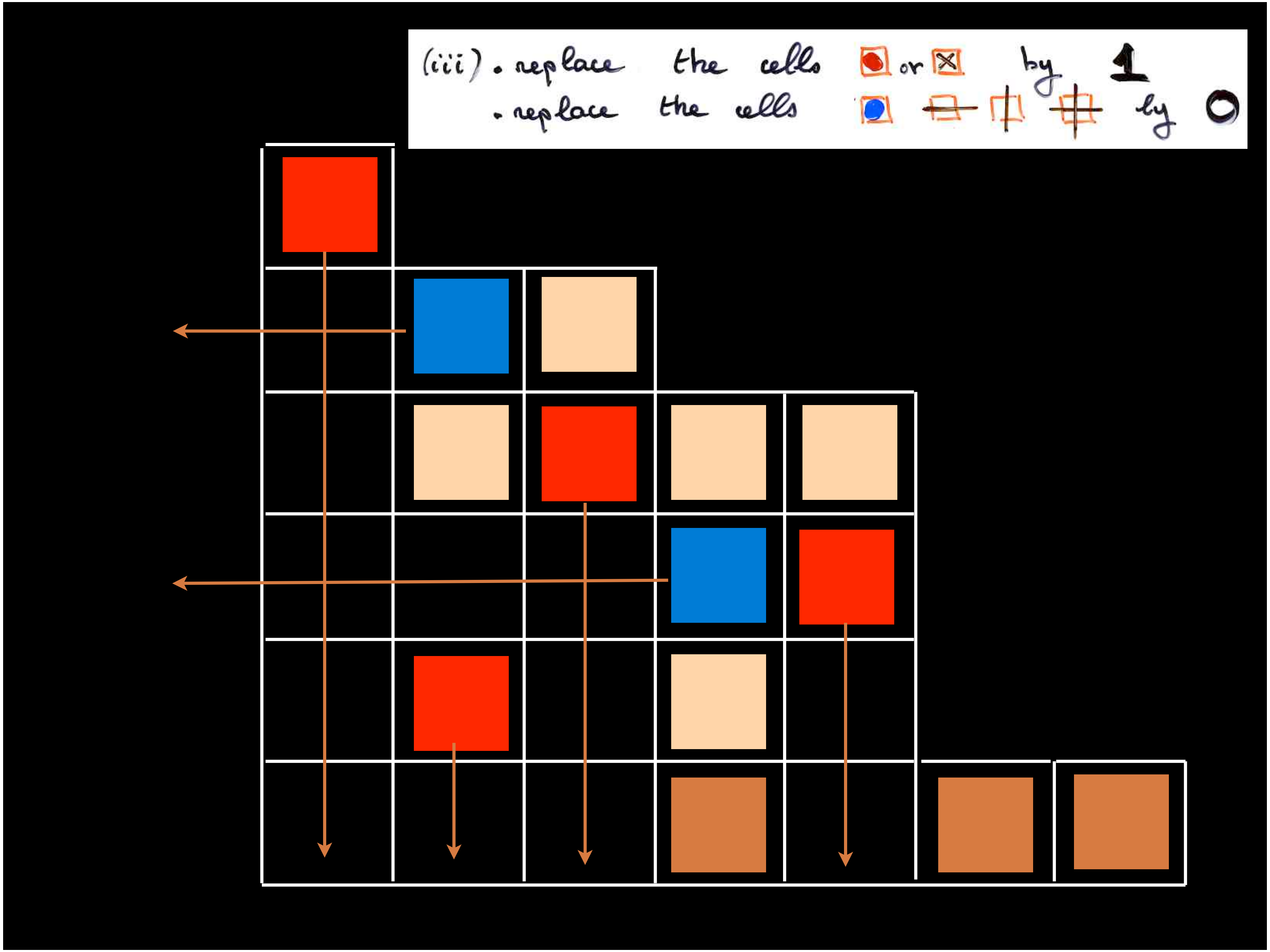


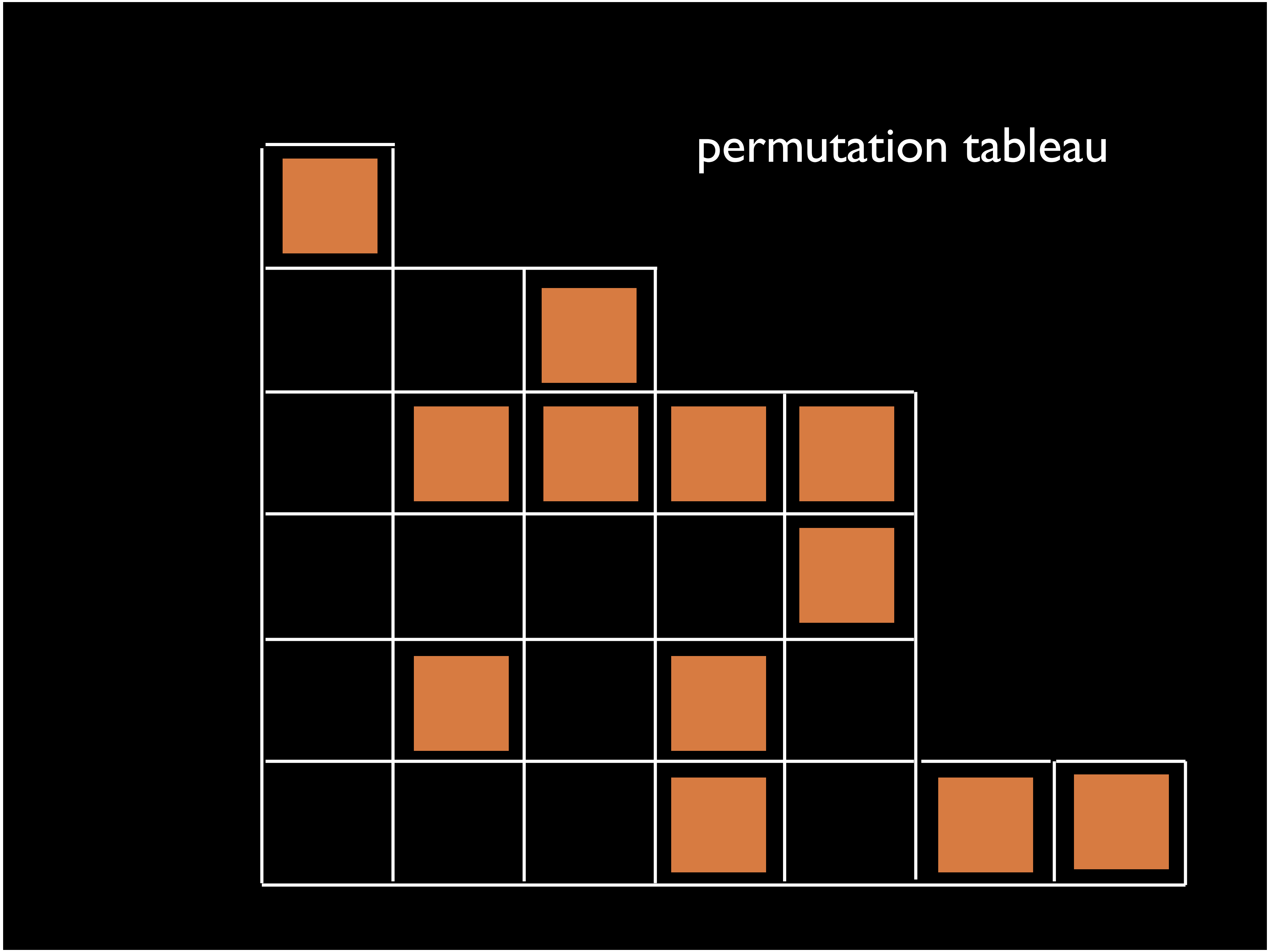
(iv) add a new row below F

in this row { put a 1 in each column without [red cell with arrow]
0 elsewhere



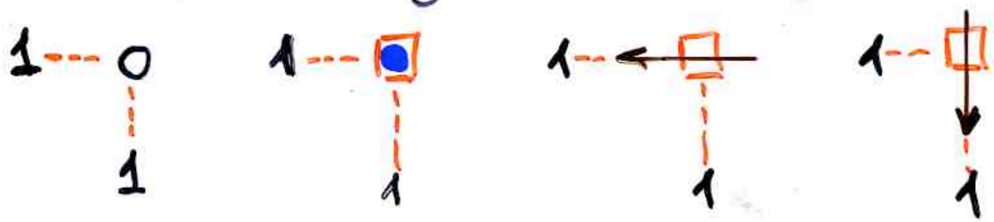



(iii) • replace the cells or by 1
• replace the cells by 0


permutation tableau

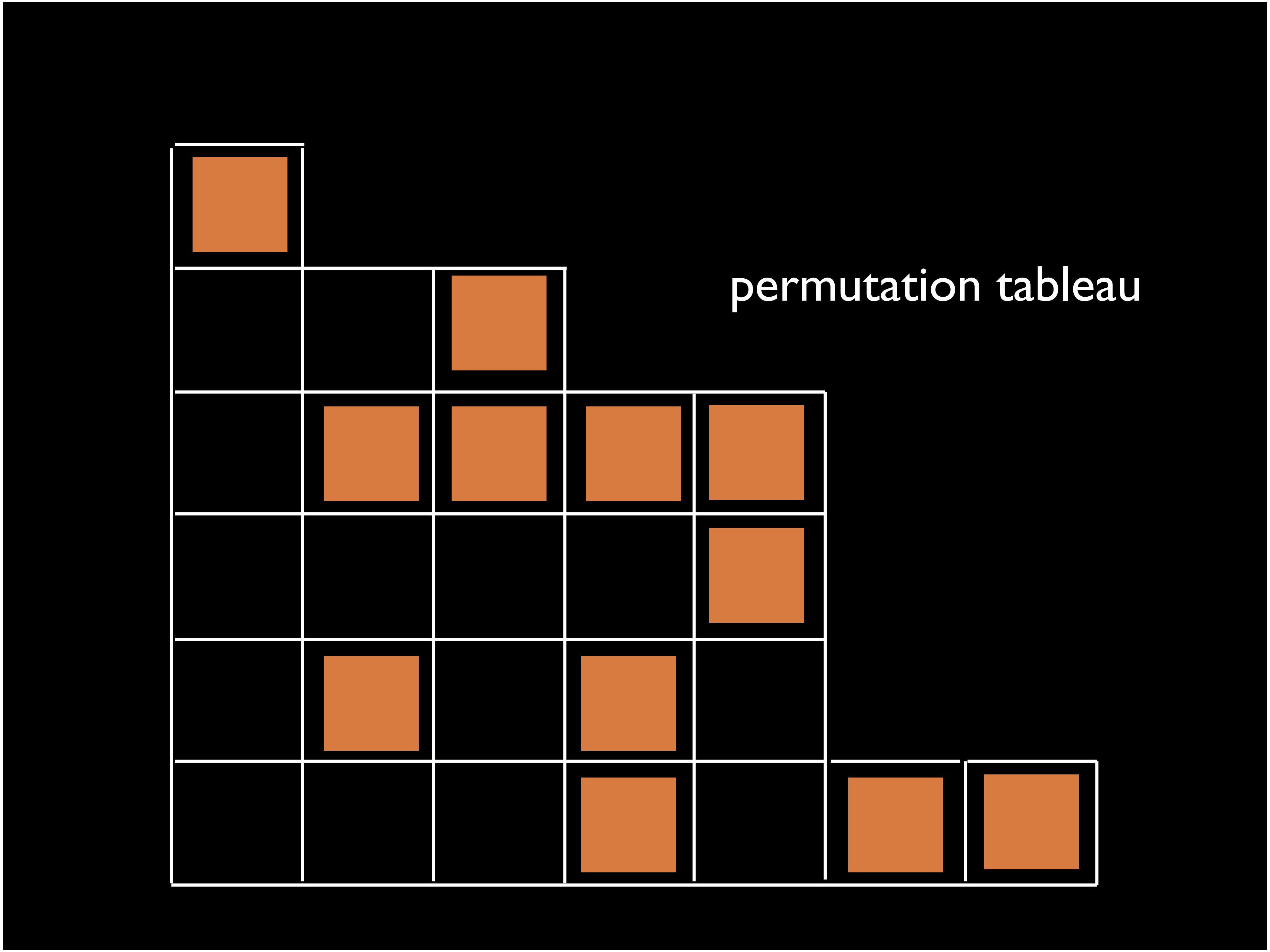
check: AT $\xrightarrow{\varphi}$ PT size $(n+1)$

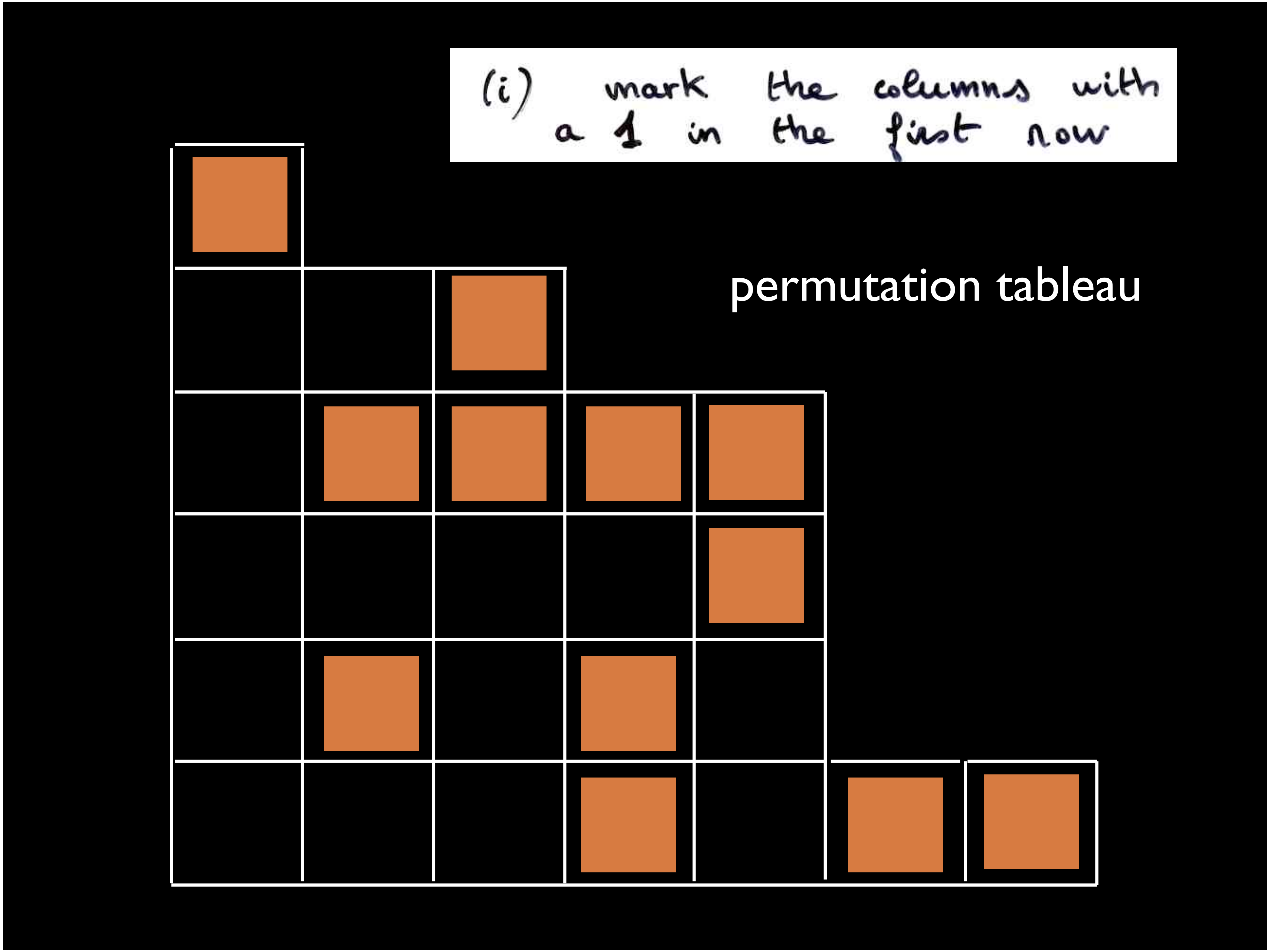
- there exist at least a 1 in each column of PT $= \varphi(\text{AT})$

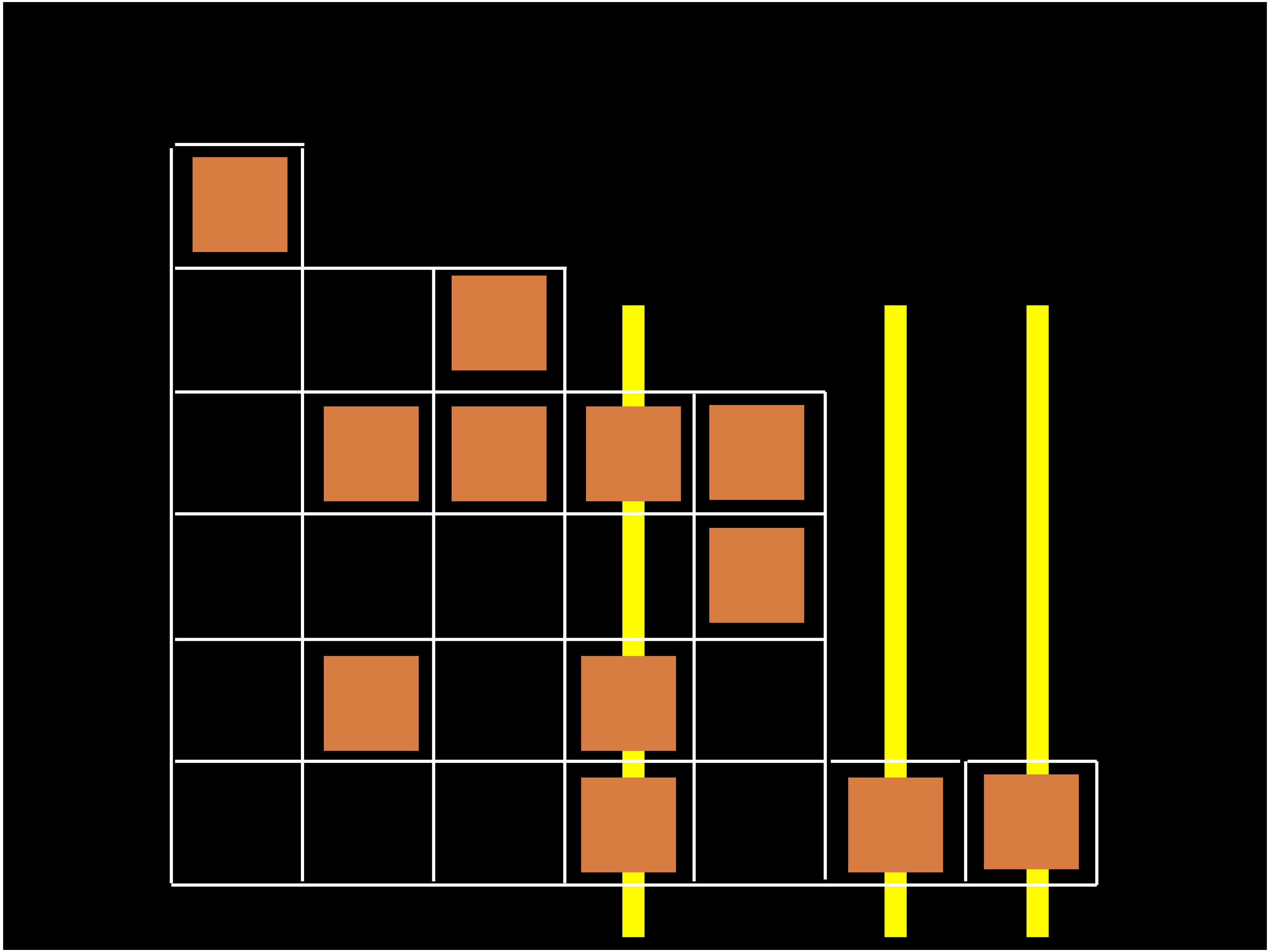


impossible

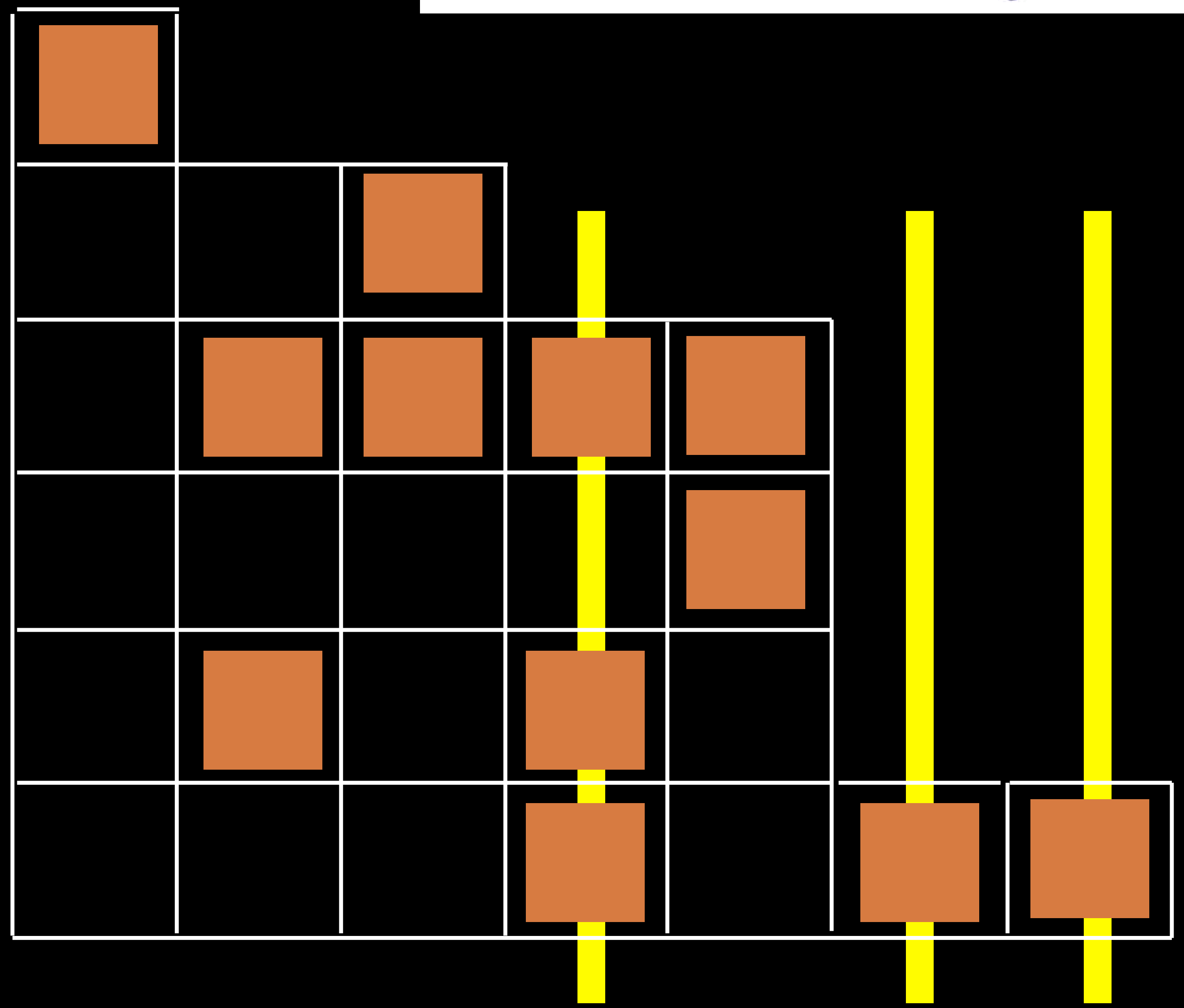
inverse bijection $\psi = \varphi^{-1}$

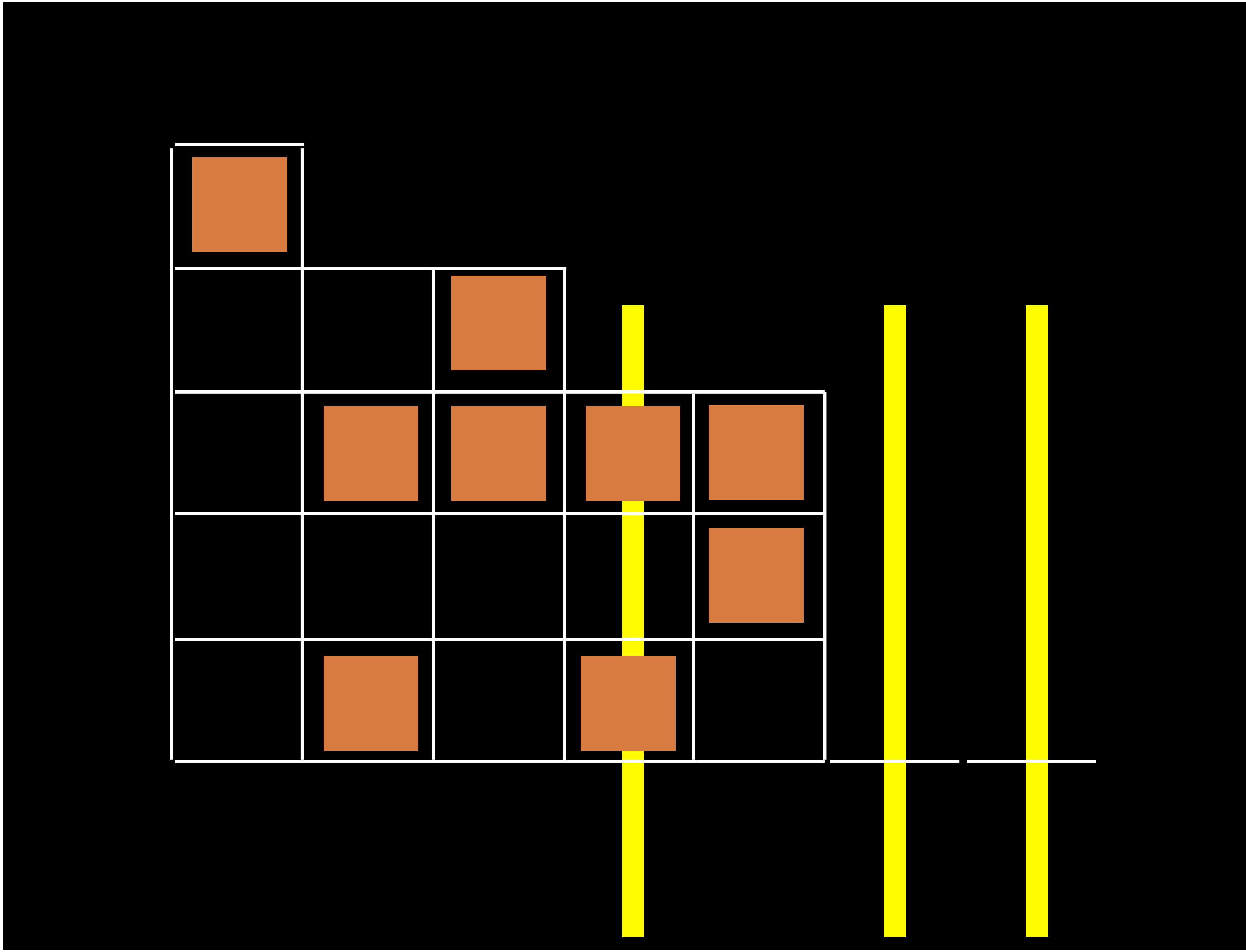

permutation tableau


(i) mark the columns with a 1 in the first row
permutation tableau



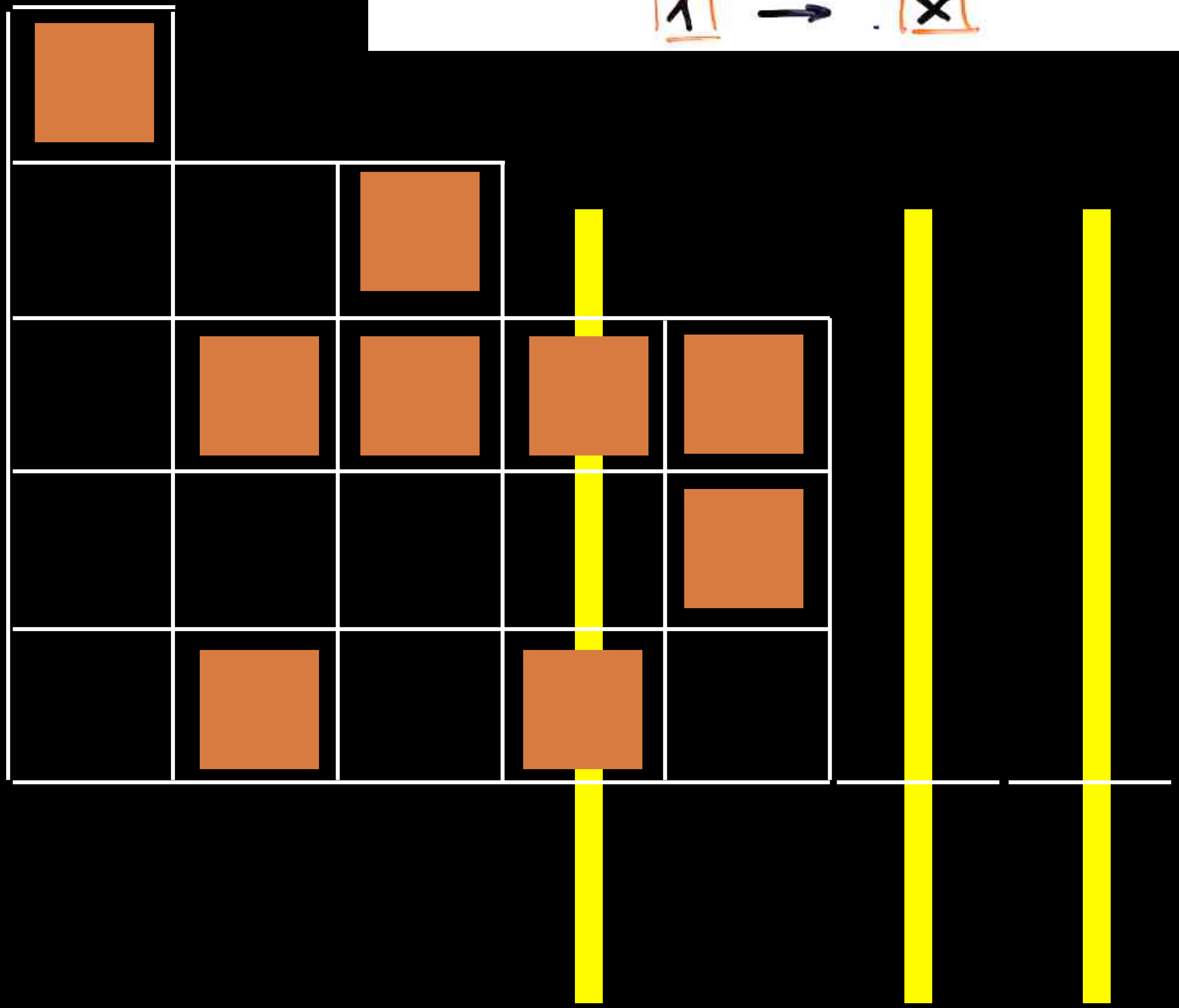
(ii) delete the first row

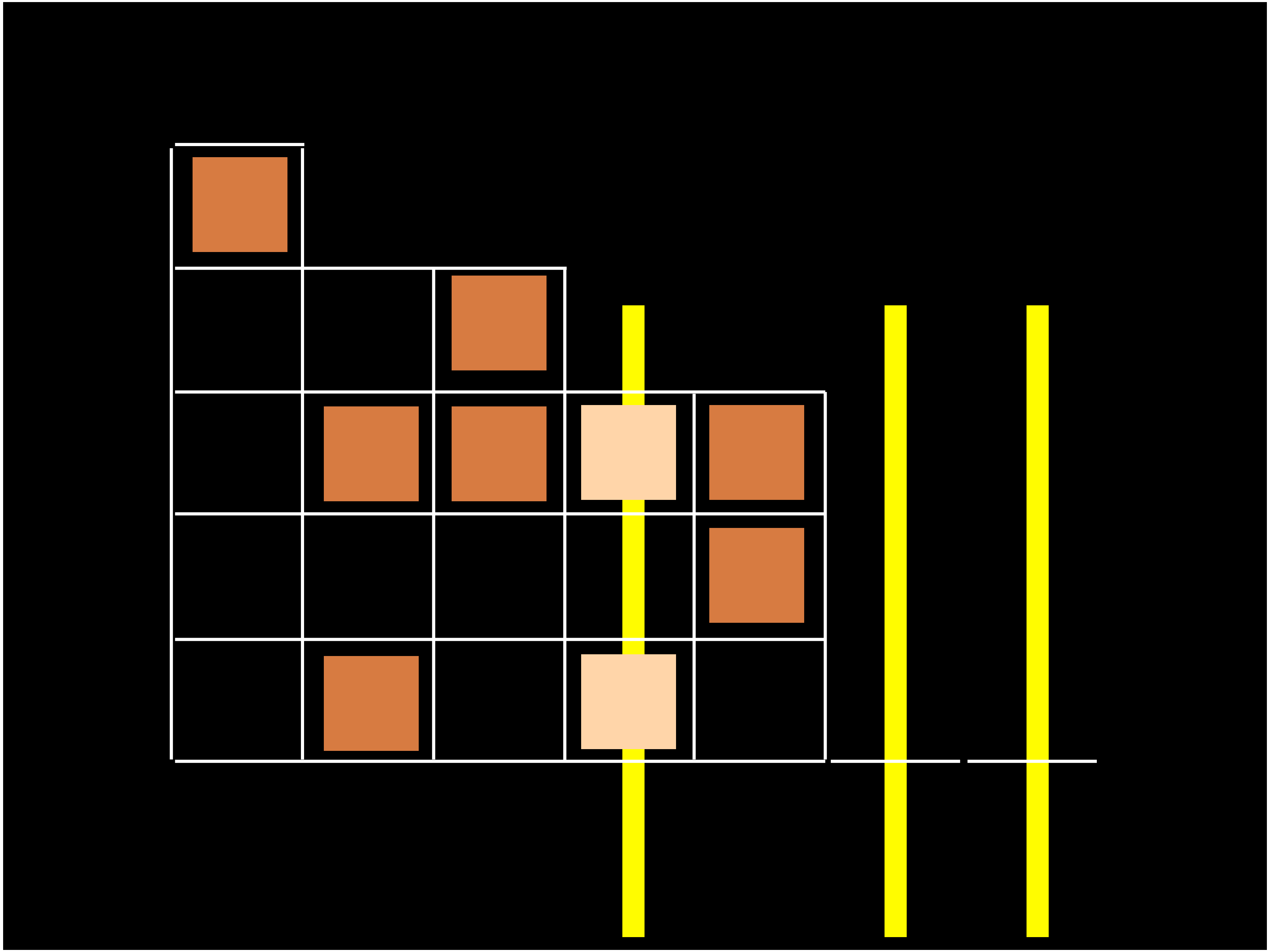




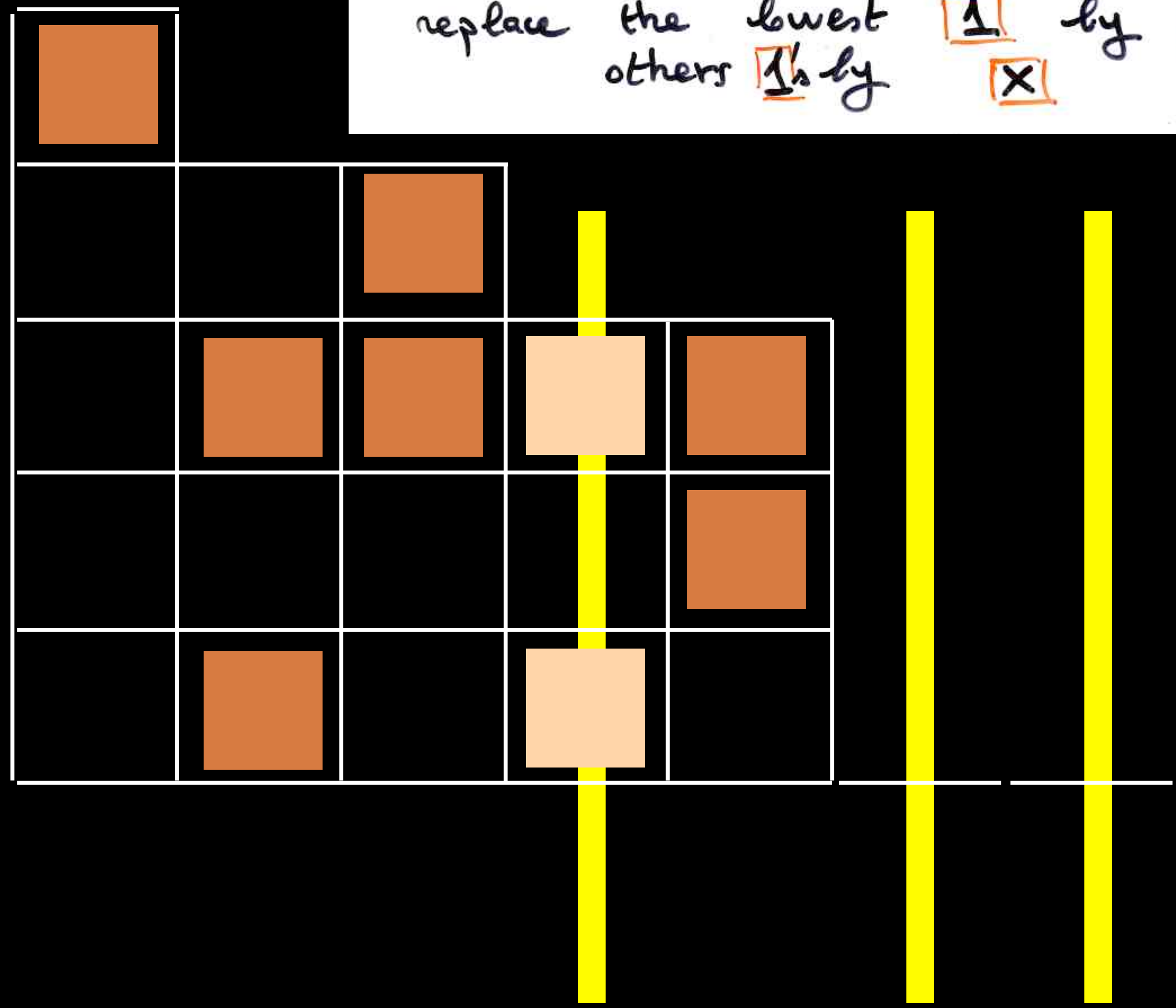
(iii) in each marked column

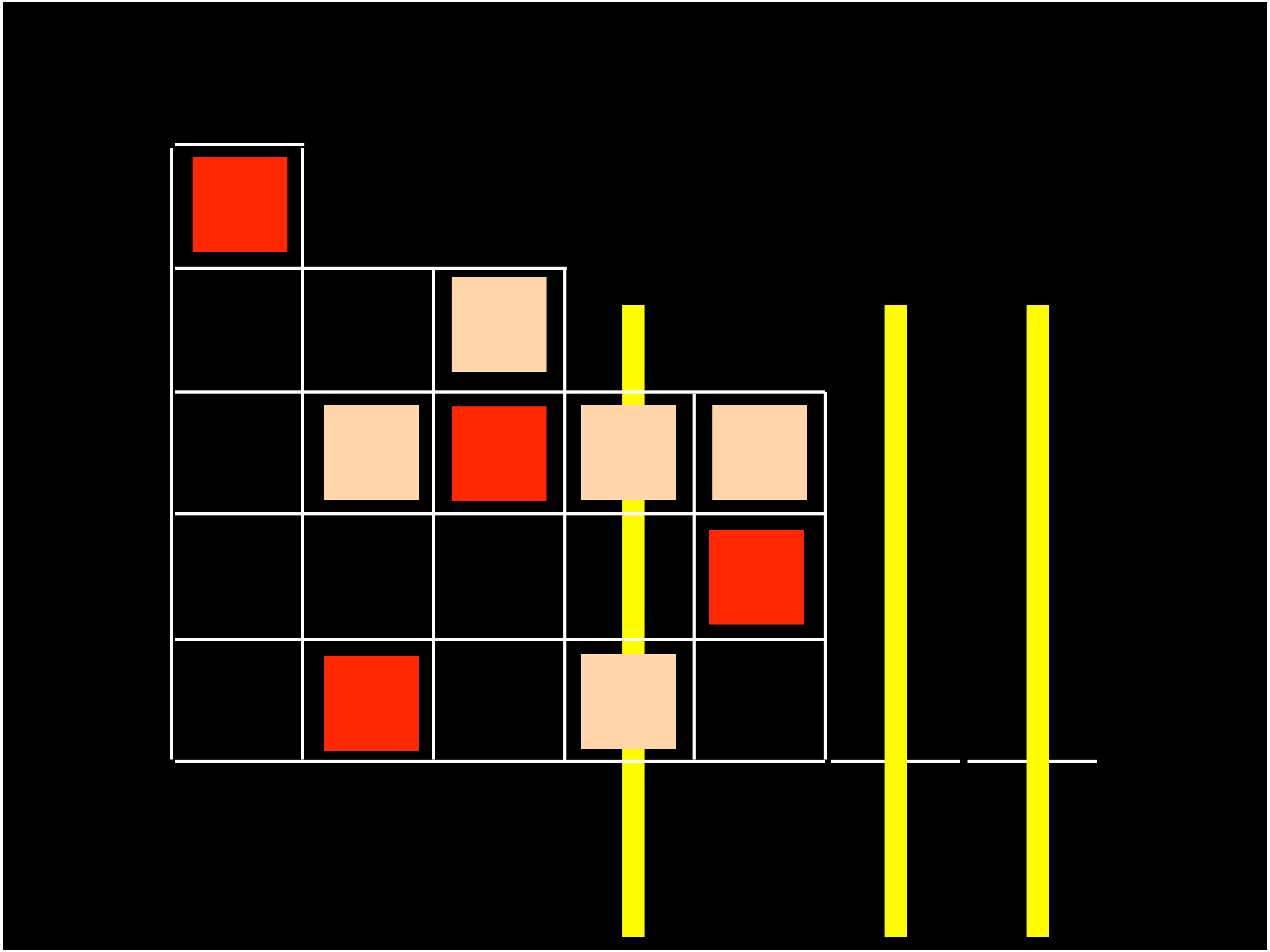
1 → X

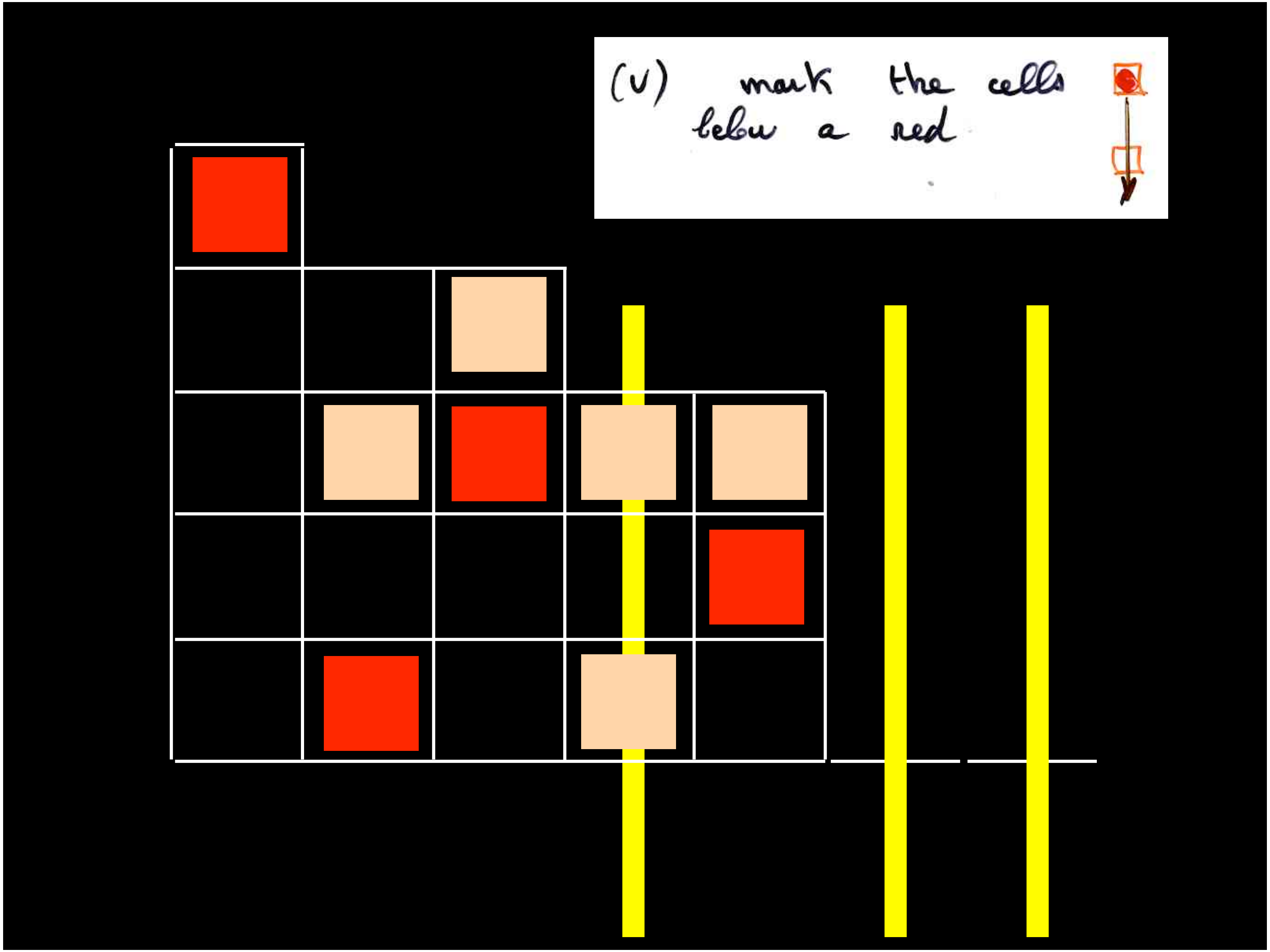


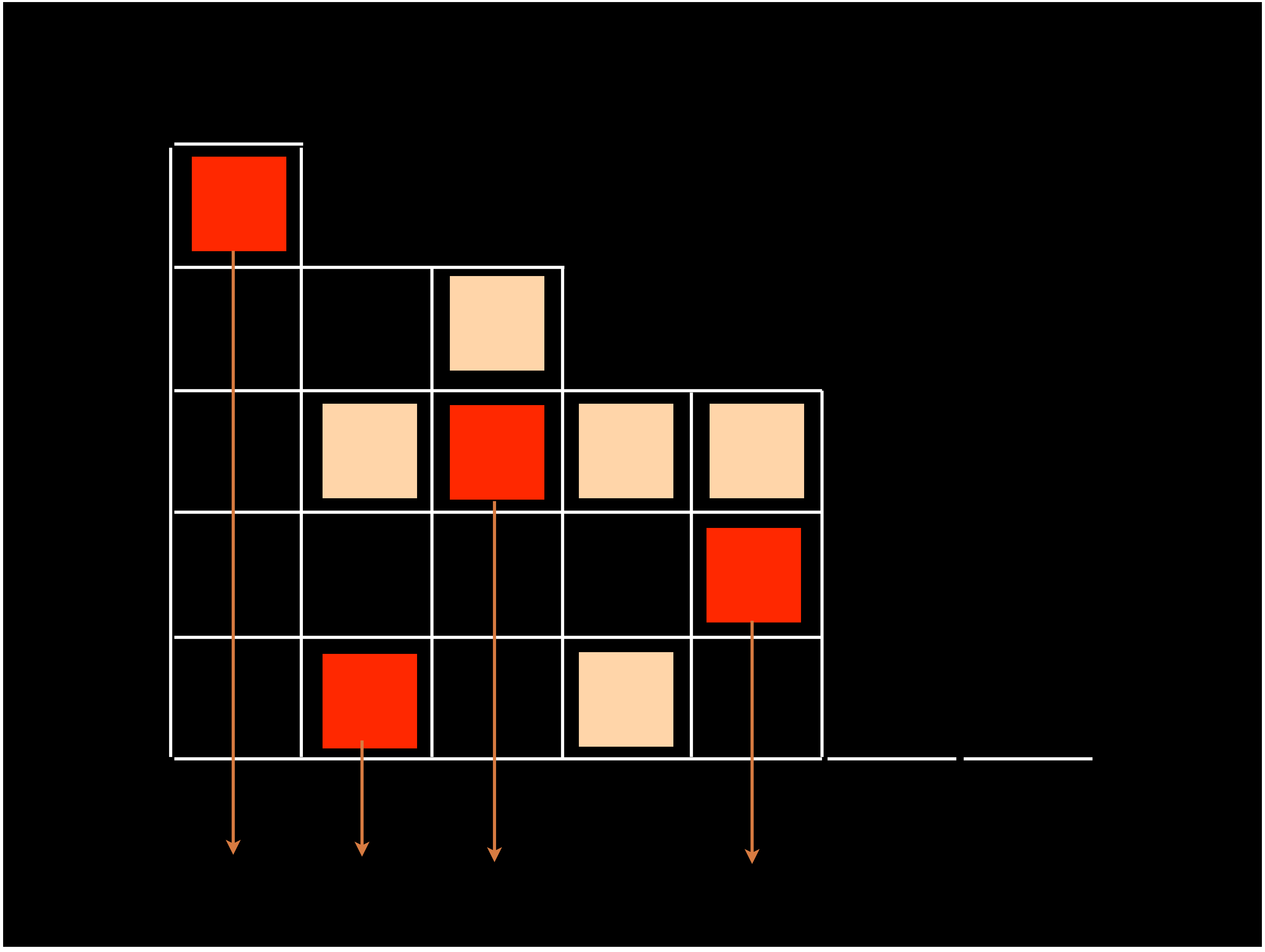


(iv) in each non marked column
(∃ some cells with 1)
replace the lowest [1] by [●]
others [1]'s by [X]

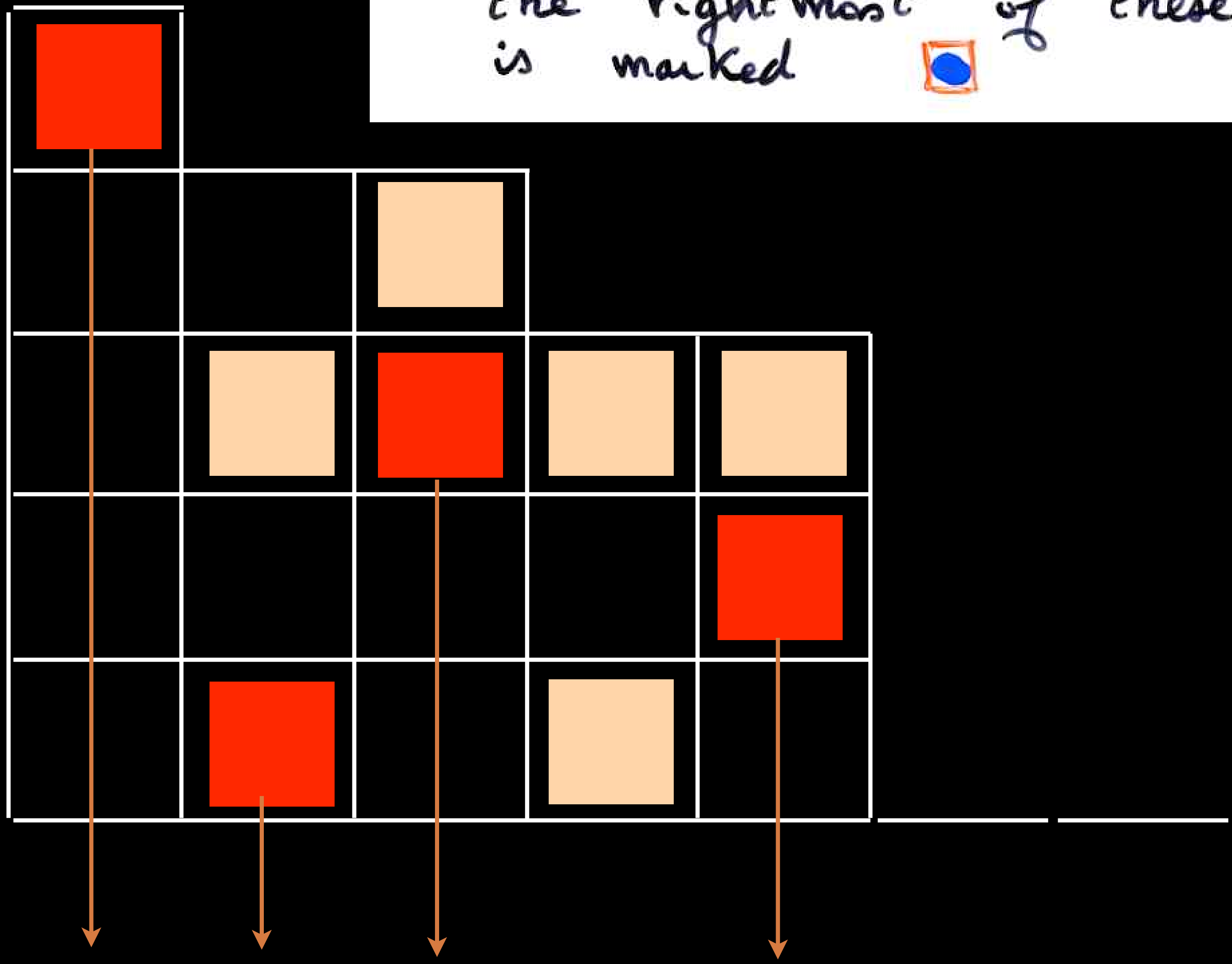


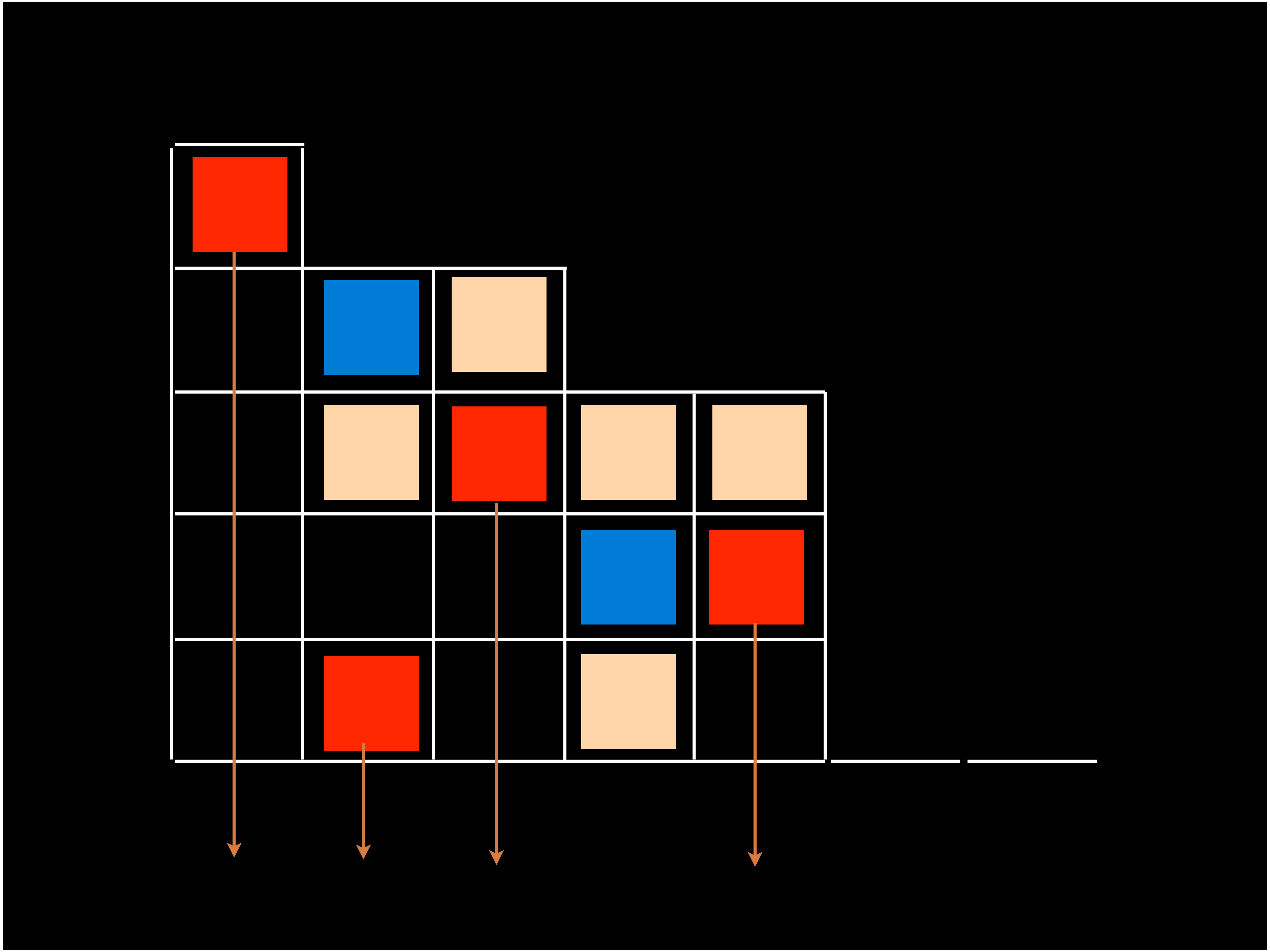



(v) mark the cells below a red

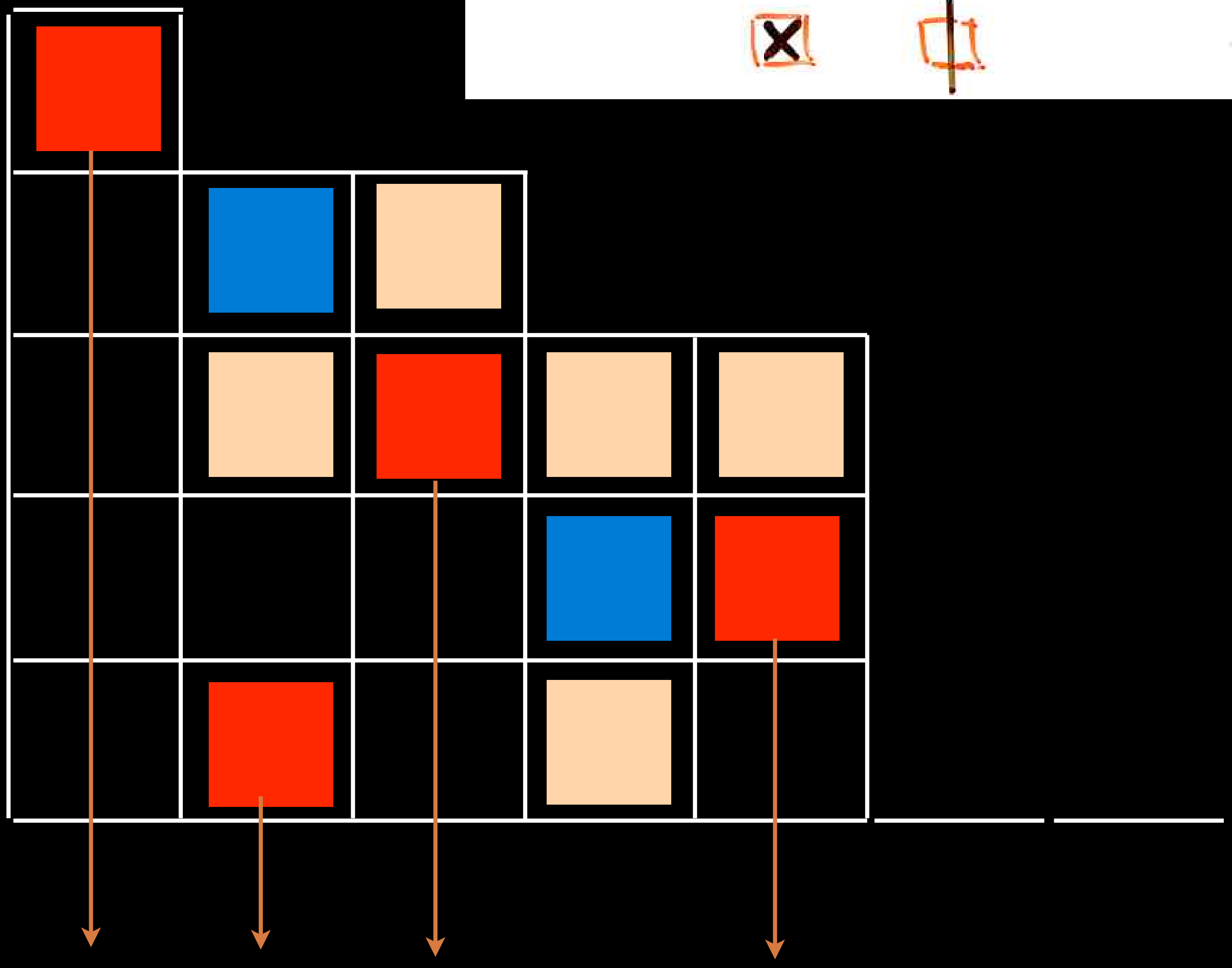


(vi) in each row where there exist empty cells, the rightmost of these cells is marked

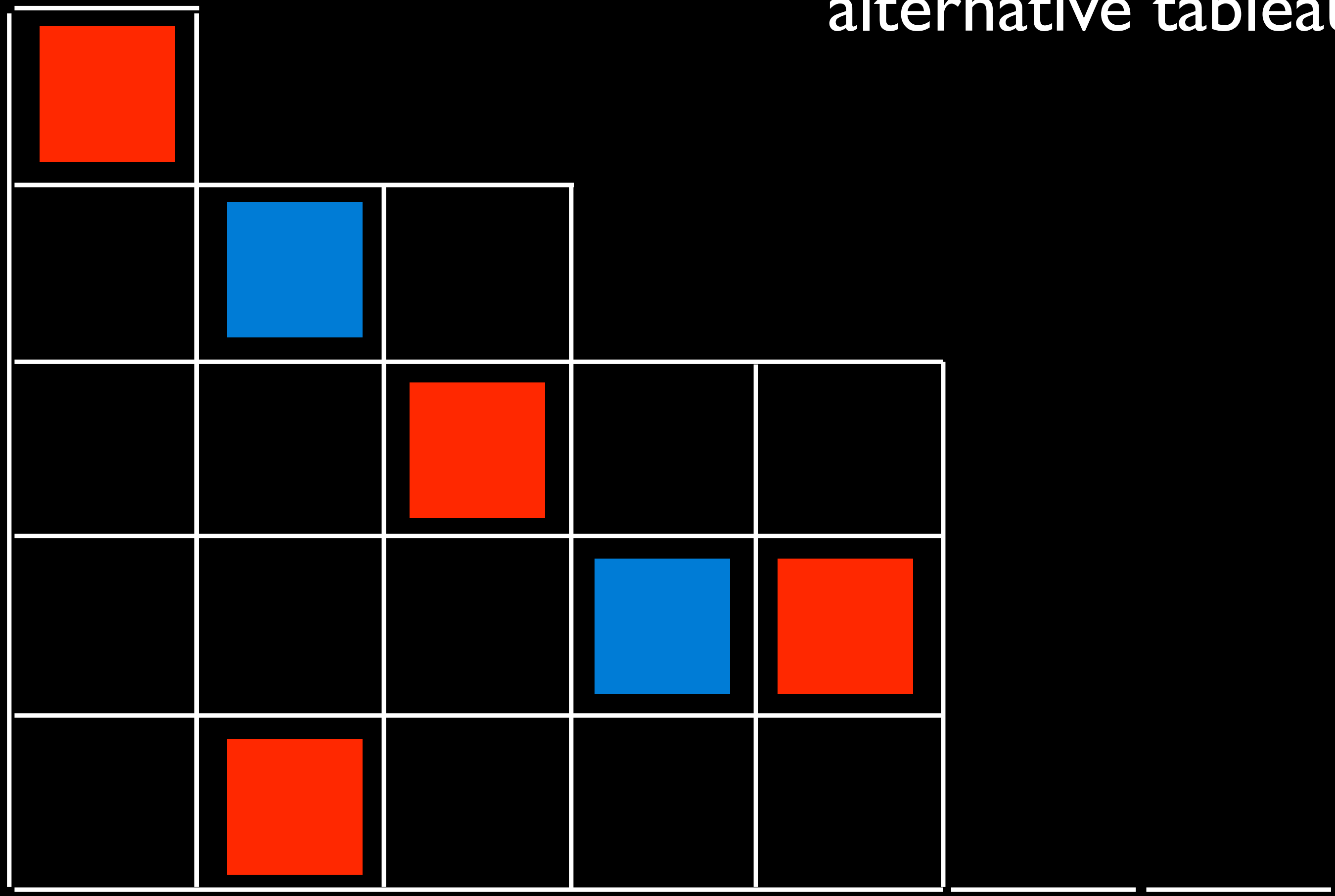




(vii) delete the marks



alternative tableau



check

- $\psi(PT)$ is an alternative tableau
- $\psi = \varphi^{-1}$.

<u>notations.</u> T tableau de permutations

- $wt(T)$ = (nb total de 1) − (nb de colonnes)
- $f(T)$ = (nb de 1 sur la 1ère ligne)
- $u(T)$ = (nb de lignes non restreintes)

<u>Def</u>– ligne restreinte : ssi elle a une case restreinte, c.à.d une case contenant un 0 et située au dessus d'un 1.

Corteel, Williams (2006)

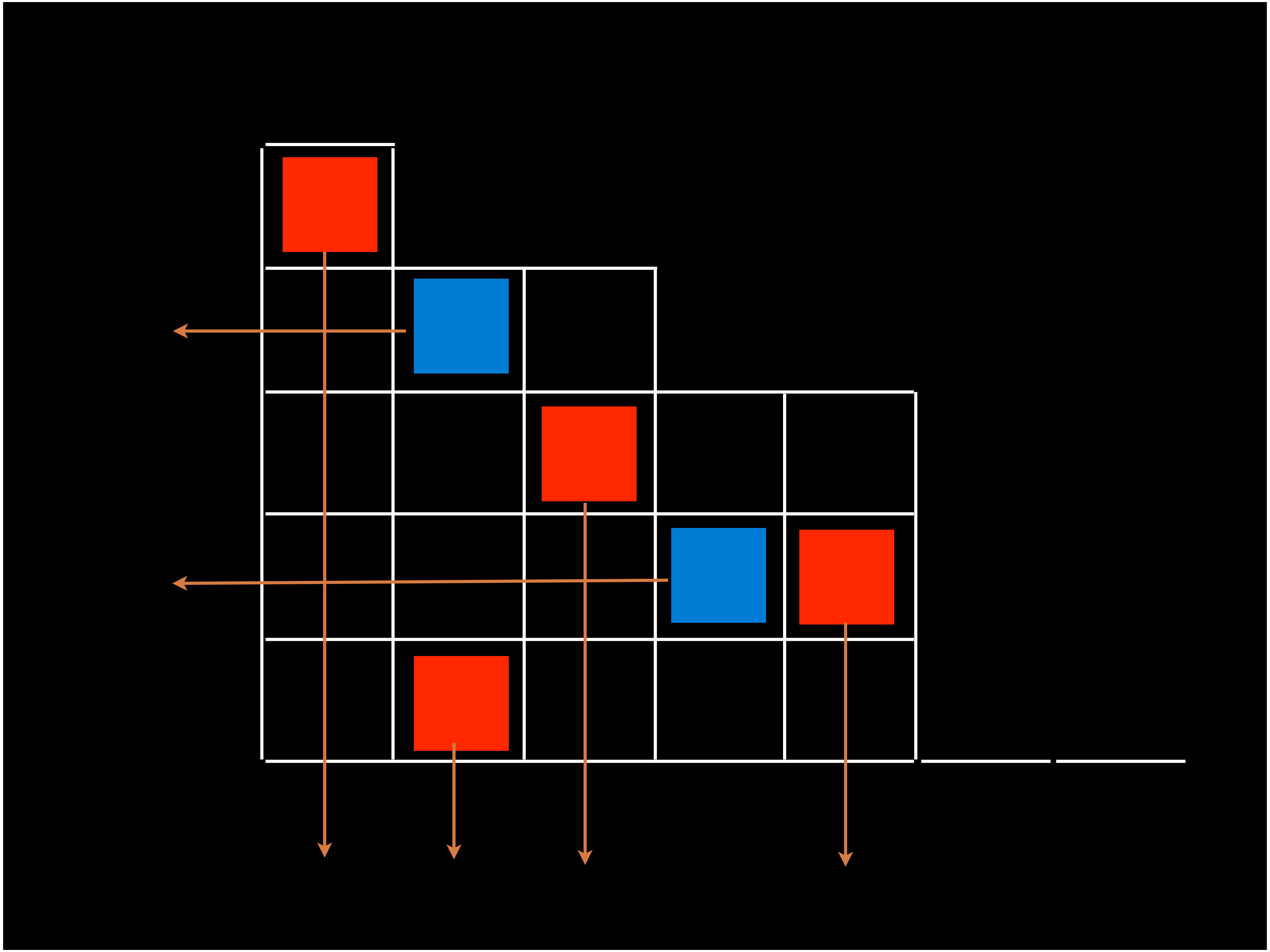
<u>Cor</u>– La probabilité stationnaire associée à l'état $\tau = (\tau_1, \dots, \tau_n)$ (PASEP)

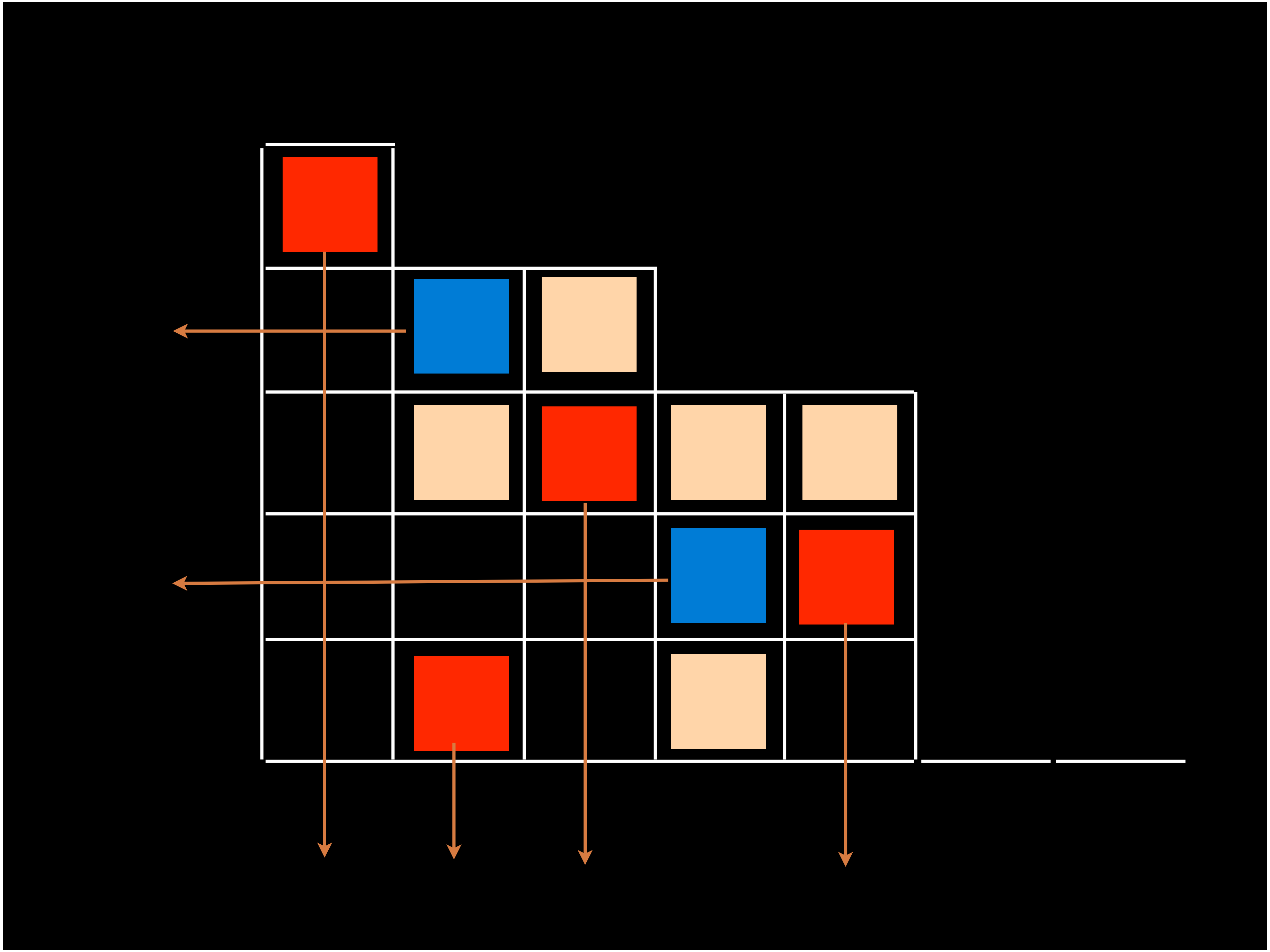
est

$$P_\tau(q) = \frac{1}{Z_n} \sum_{T} q^{wt(T)} \alpha^{-f(T)} \beta^{-u(T)}$$

T tableau de permutation forme F associé à τ

§ 6 Catalan alternative tableaux





Def Catalan alternative tableau T
alt. tab. without cells ☒
i.e: every empty cell is below a red cell or
on the left of a blue cell

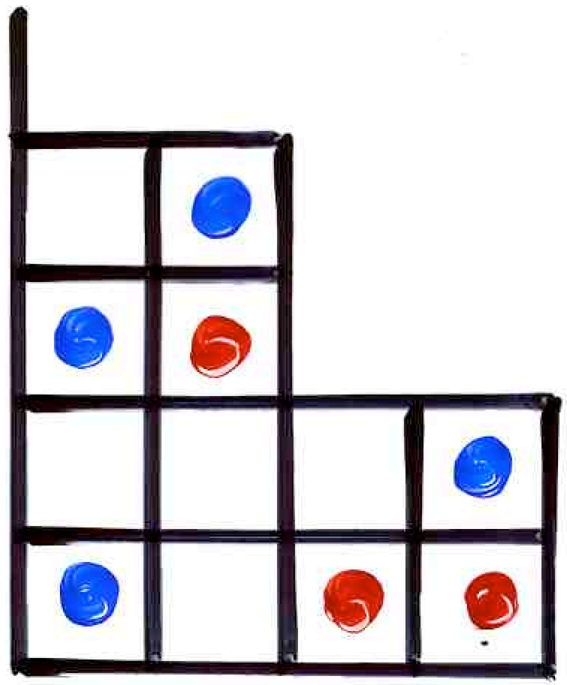


tableau
alternatif
de Catalan

Def Catalan alternative tableau T
alt. tab. without cells ☒
i.e: every empty cell is below a red cell or on the left of a blue cell

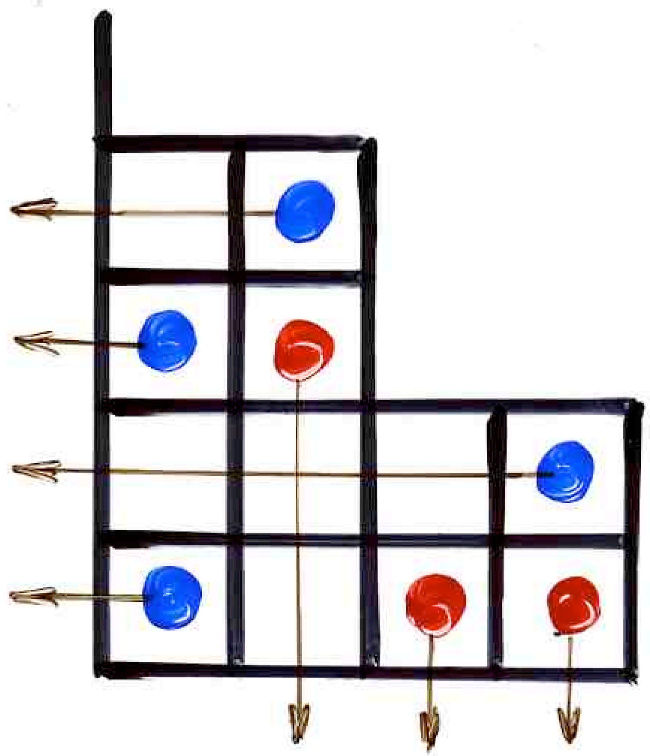
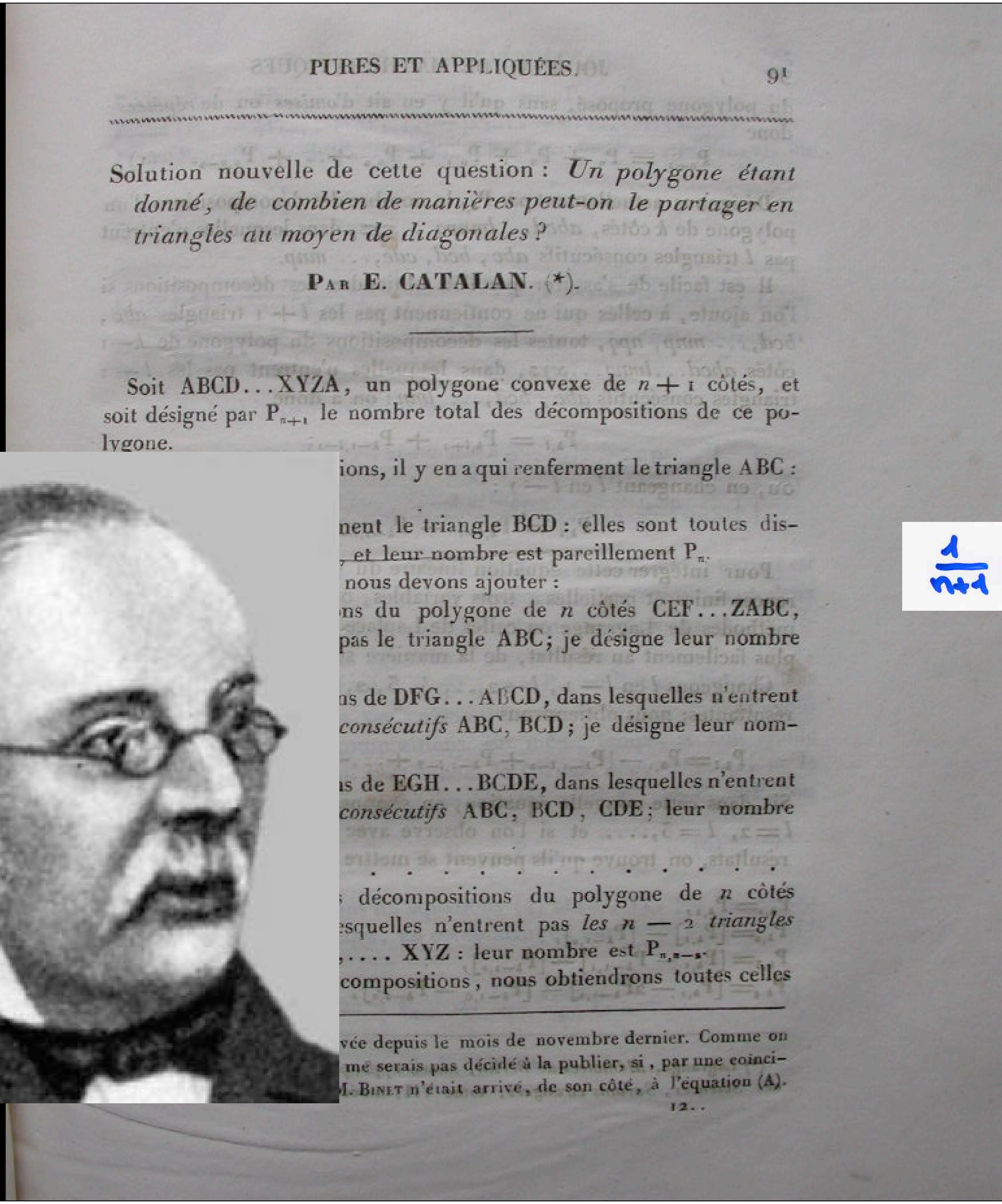


tableau alternatif de Catalan



Solution nouvelle de cette question : *Un polygone étant donné, de combien de manières peut-on le partager en triangles au moyen de diagonales ?*

PAR E. CATALAN. (*).

Soit ABCD...XYZA, un polygone convexe de $n+1$ côtés, et soit désigné par P_{n+1} le nombre total des décompositions de ce polygone.

...ions, il y en a qui renferment le triangle ABC :

...nent le triangle BCD : elles sont toutes dis-
..., et leur nombre est pareillement P_n.
... nous devons ajouter :
...ns du polygone de n côtés CEF...ZABC,
...pas le triangle ABC; je désigne leur nombre

...ns de DFG...ABCD, dans lesquelles n'entrent
... *consécutifs* ABC, BCD; je désigne leur nom-

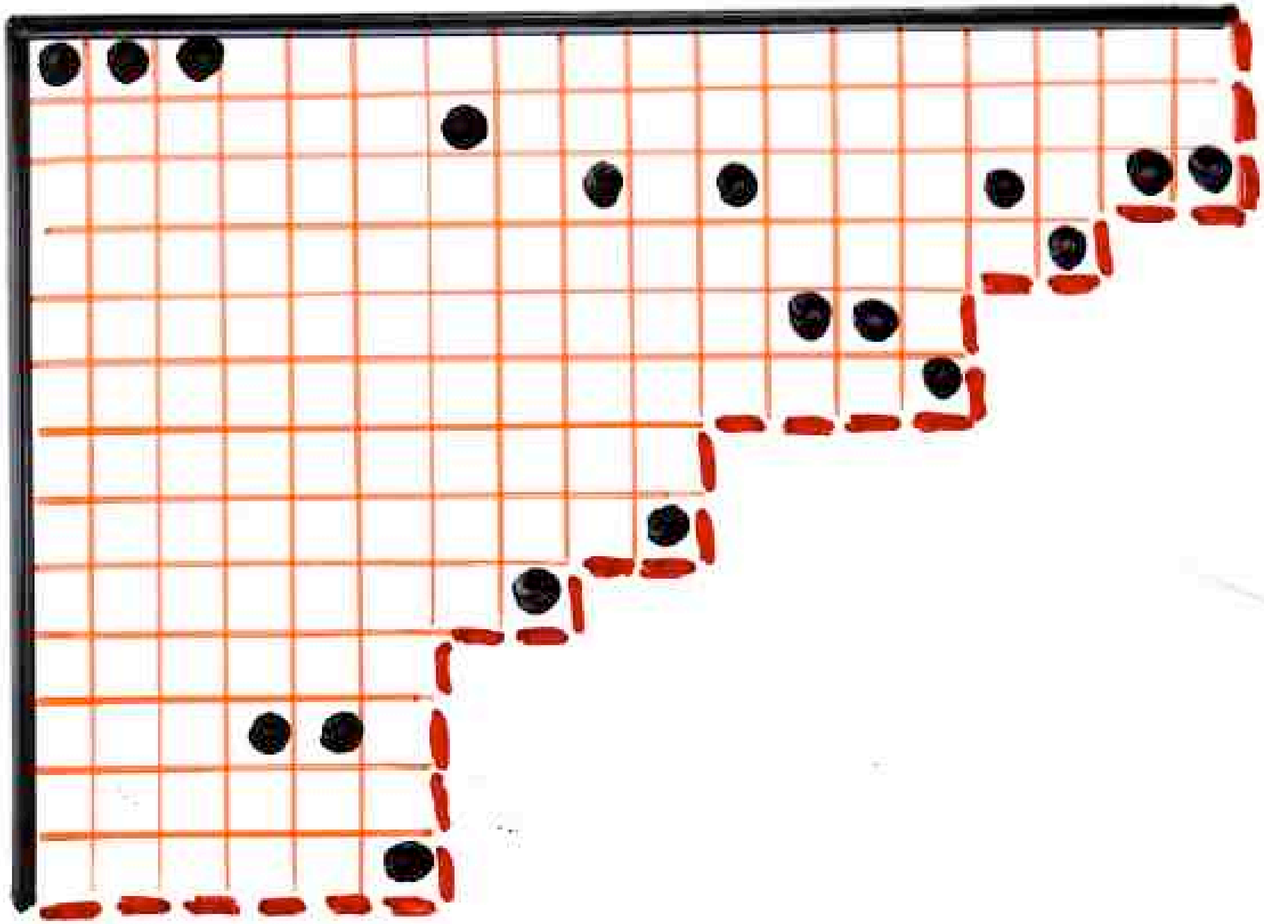
...s de EGH...BCDE, dans lesquelles n'entrent
... *consécutifs* ABC, BCD, CDE; leur nombre

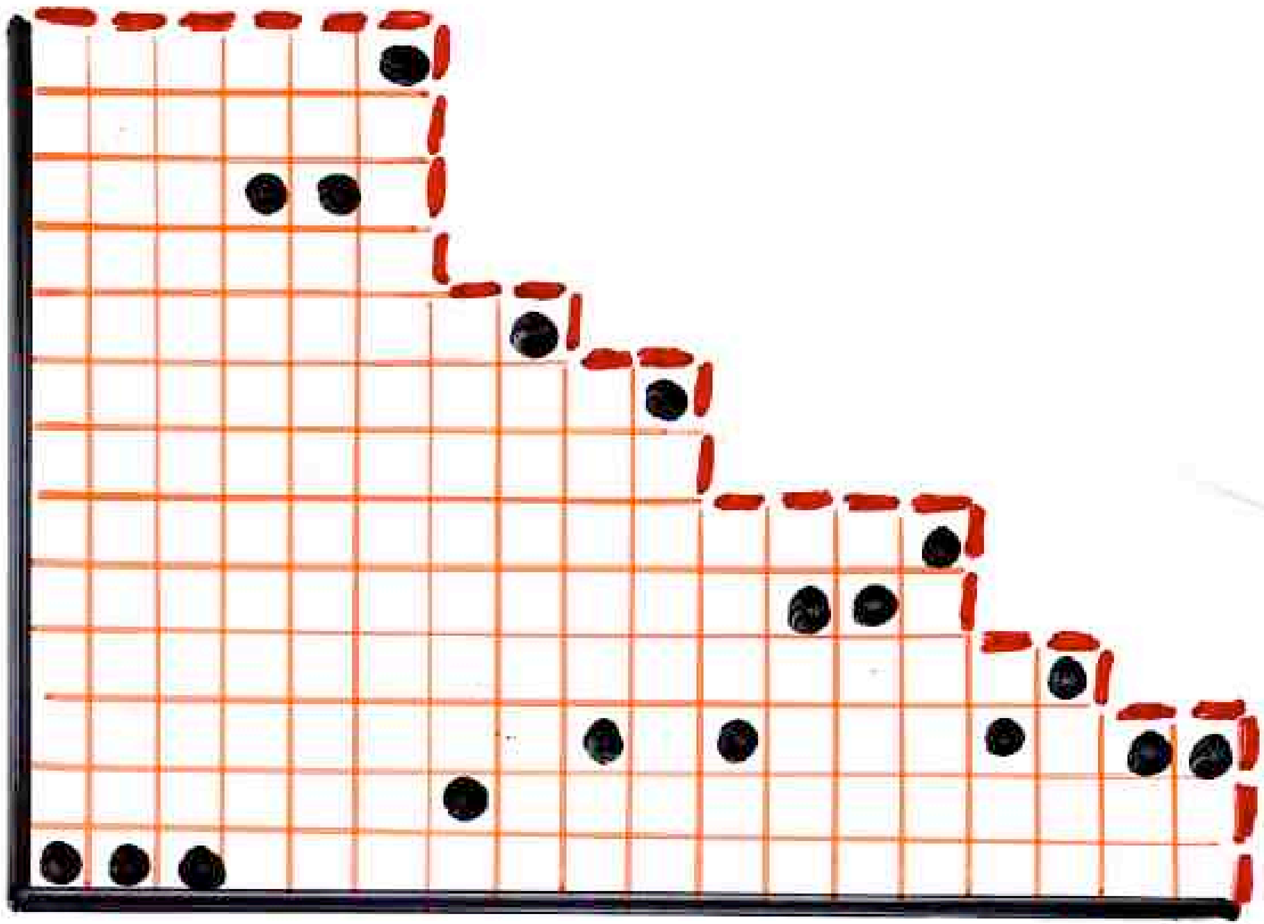
. . . .

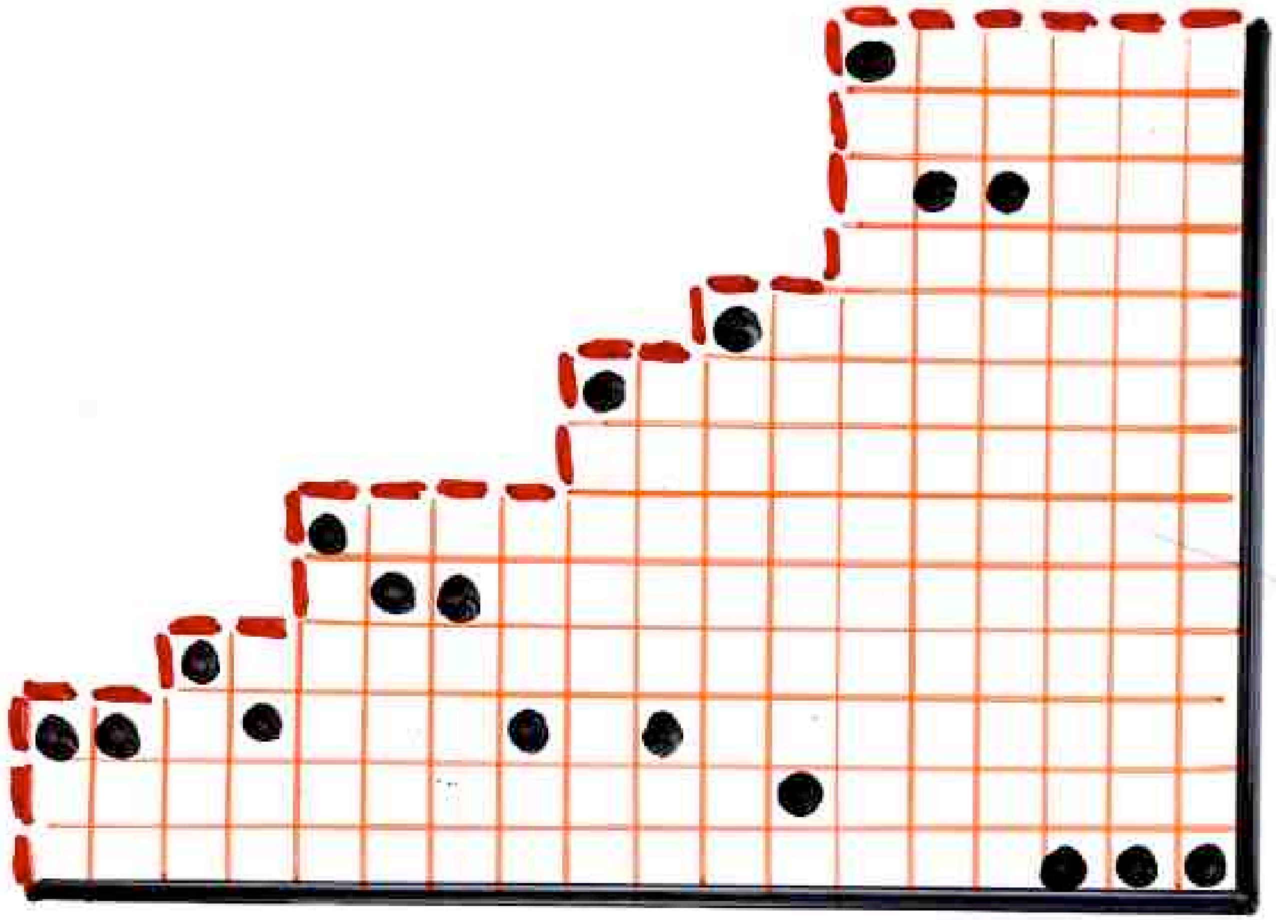
... décompositions du polygone de n côtés
...squelles n'entrent pas *les* $n-2$ *triangles*
...,.... XYZ : leur nombre est $P_{n,n-2}$.
...compositions, nous obtiendrons toutes celles

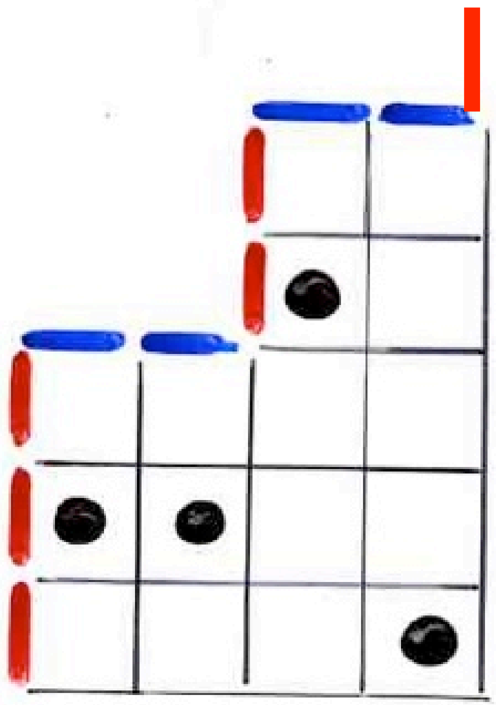
(*) ...vée depuis le mois de novembre dernier. Comme on
... me serais pas décidé à la publier, si, par une coïnci-
...M. BINET n'était arrivé, de son côté, à l'équation (A).

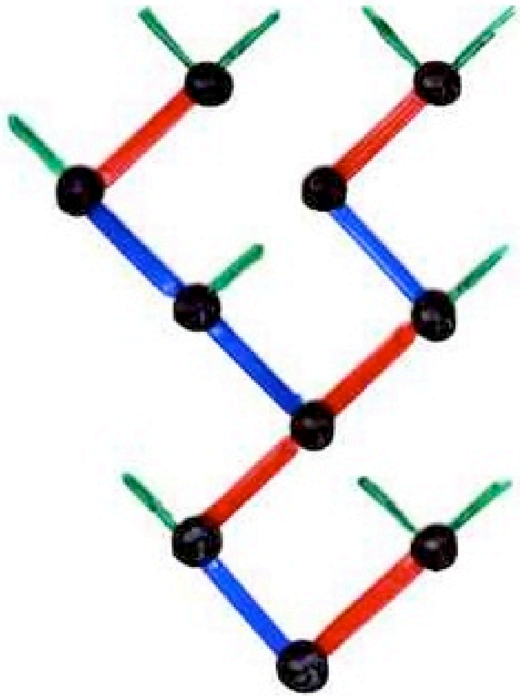
$$\frac{1}{n+1}\binom{2n}{n}$$











bijection permutation Catalan tableaux ---- binary trees

xgv, FPSAC'07

§7 Laguerre histories

Bijection Permutations $n+1$

$\updownarrow$

Histoires de Laguerre $(\gamma_c ; f)$ n

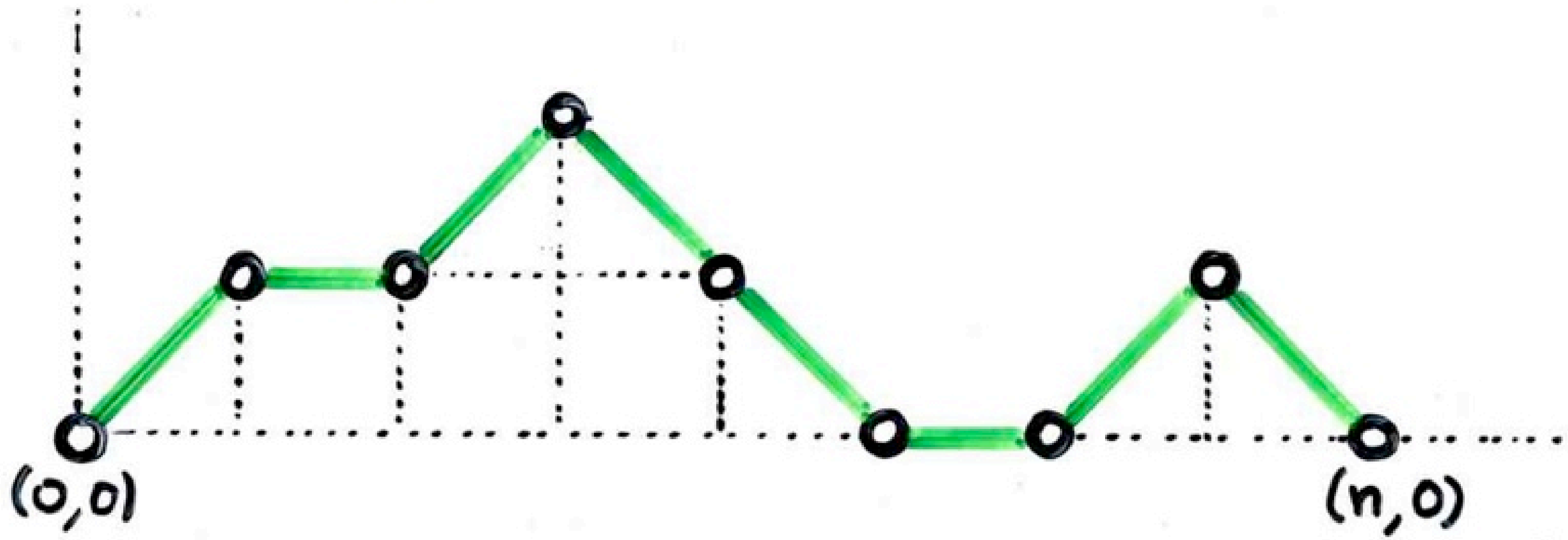
Bijection **Permutations** n+1

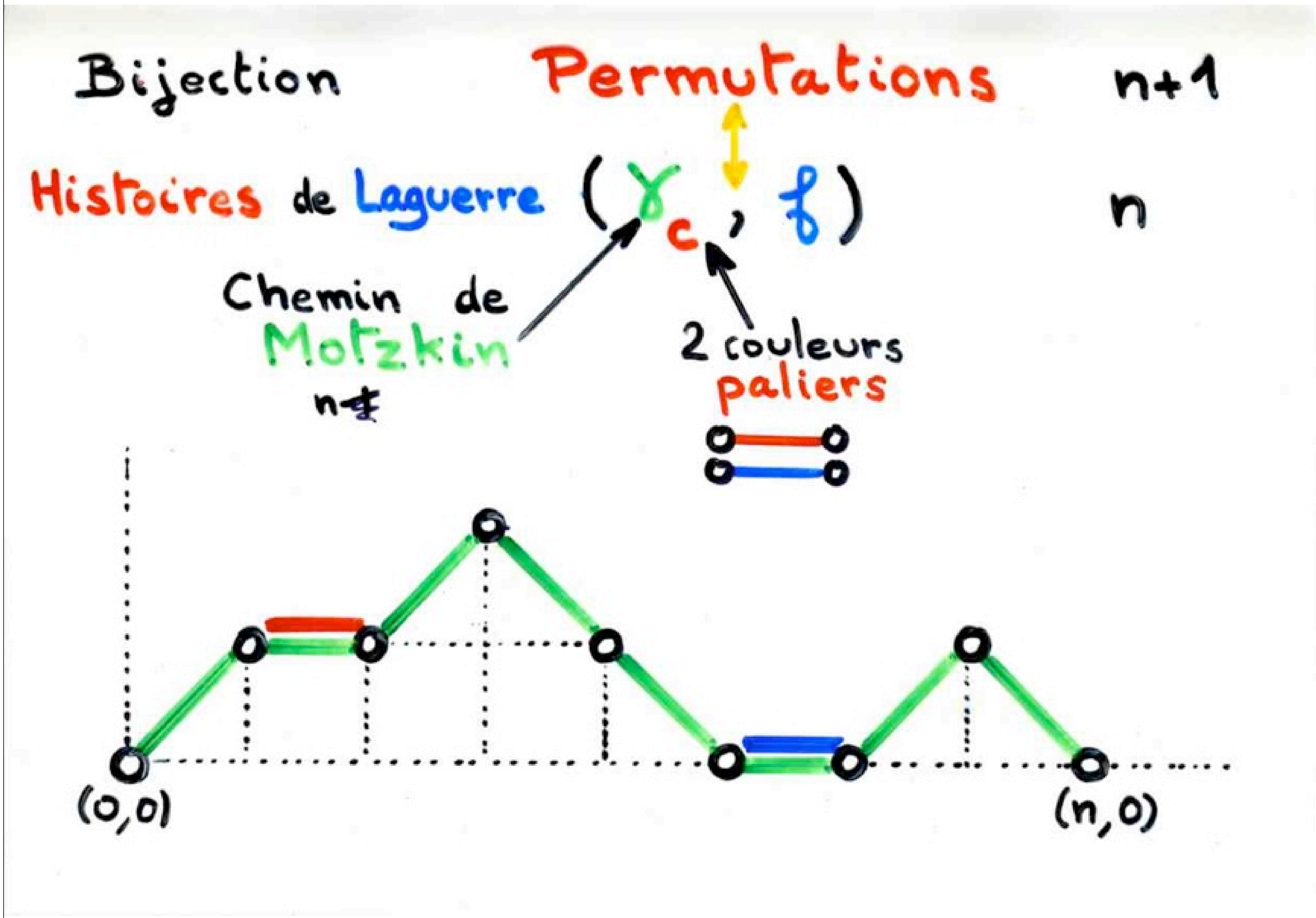
↕

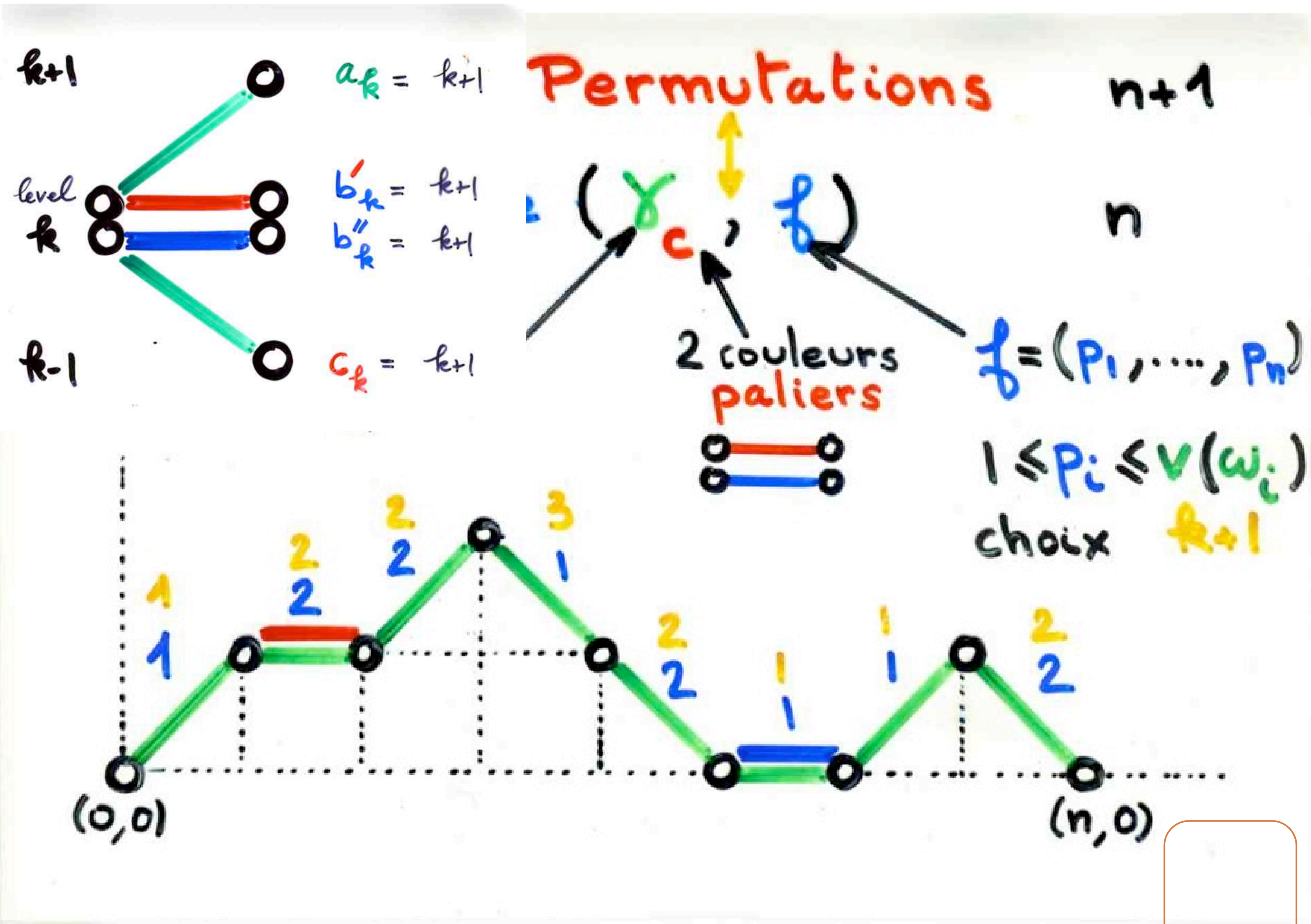
Histoires de Laguerre $(\gamma_c\ ;\ f)$ n

Chemin de Motzkin

n




Bijection
Permutations
n+1
Histoires de Laguerre
(γc , f)
n
Chemin de Motzkin
2 couleurs paliers
(0,0)
(n,0)


k+1
level k
k-1
a_k = k+1
b'_k = k+1
b''_k = k+1
c_k = k+1
Permutations
n+1
n
2 couleurs
paliers
f=(p1, ..., pn)
1 ≤ pi ≤ v(ωi)
choix k+1
(0,0)
(n,0)

Laguerre polynomial

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

$$P_0 = 1 \qquad P_1 = x - b_0$$

$$\mu_n = (n+1)!$$

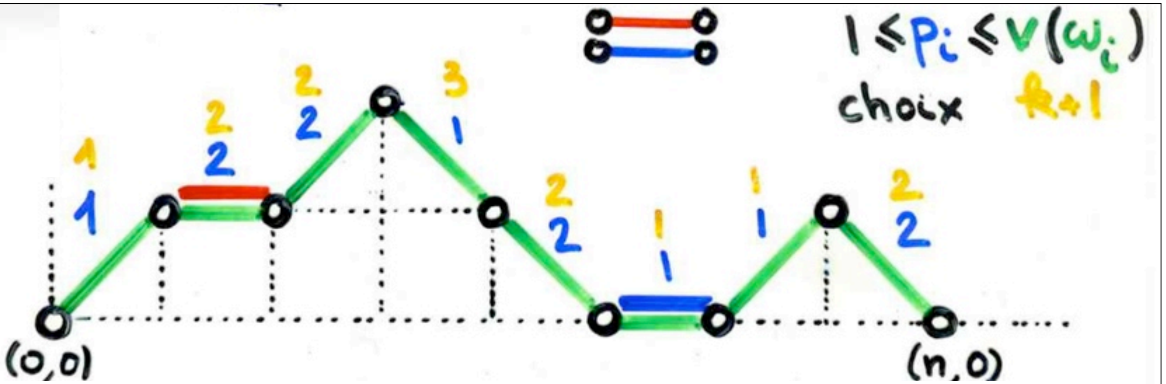
$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

$$J(t) = \cfrac{1}{1 - 2t - \cfrac{1\cdot 2t^2}{1 - 4t - \cfrac{2\cdot 3t^2}{\cdots}}}$$

Bijection
Laguerre histories
permutations

Françon-xgv., 1978

$h = (\omega_c ; (p_1, \dots, p_n))$



x	ω_c	pos	v
1	↗	1	1
2	→	2	2
3	↗	2	2
4	↘	1	3
5	↘	2	2
6	→	1	1
7	↗	1	1
n= 8	↘	2	2
9	•		

⊔

⊔ 1 ⊔

⊔ 1 ⊔ 2

⊔ 1 ⊔ 3 ⊔ 2

4 1 ⊔ 3 ⊔ 2

4 1 ⊔ 3 5 2

4 1 6 ⊔ 3 5 2

4 1 6 ⊔ 7 ⊔ 3 5 2

4 1 6 ⊔ 7 8 3 5 2

4 1 6 9 7 8 3 5 2 $= \sigma \in \mathfrak{S}_{n+1}$

§ 8

representation of the operators E and D

$$DE = ED + E + D$$

V vector space generated by B basis

B alternating words two letters {○, ●}

(no occurrence of ○○ or ●●)

4 operators A, S, J, K

4 operators A, S, J, K, $u \in B$

$$\langle u | A = \sum_{\substack{\text{letter } ○ \\ \text{of } u}} v$$, v obtained by: ○ → ○●○

$$\langle u | S = \sum_{○ \text{ of } u} v$$ v obtained by: ○ → ● (and ●● → ● ●●● → ●)

$$\langle u | J = \sum_{○ \text{ of } u} v$$ v, ○ → ●○ (and ●● → ●)

$$\langle u | K = \sum_{\substack{○ \\ \text{of } u}} v$$ v, ○ → ○● (and ●● → ●)

○●○●|S = ●○● + ○●

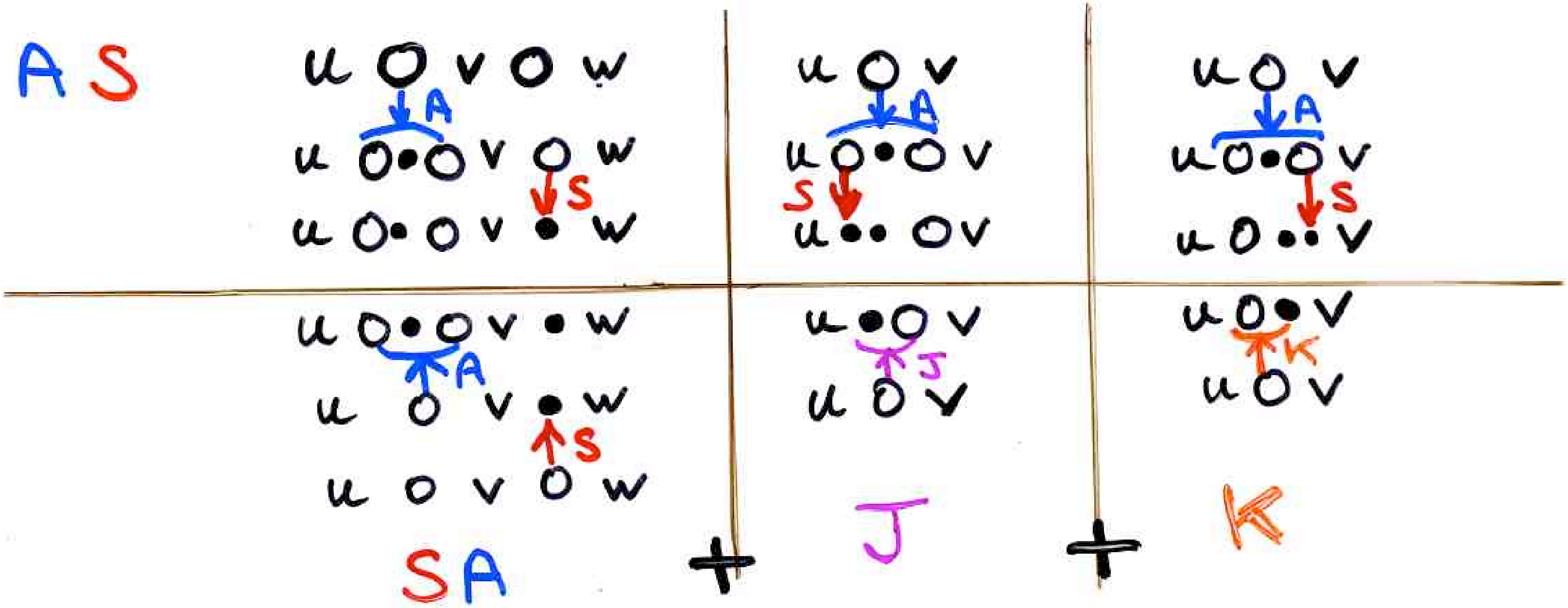
Lemma.

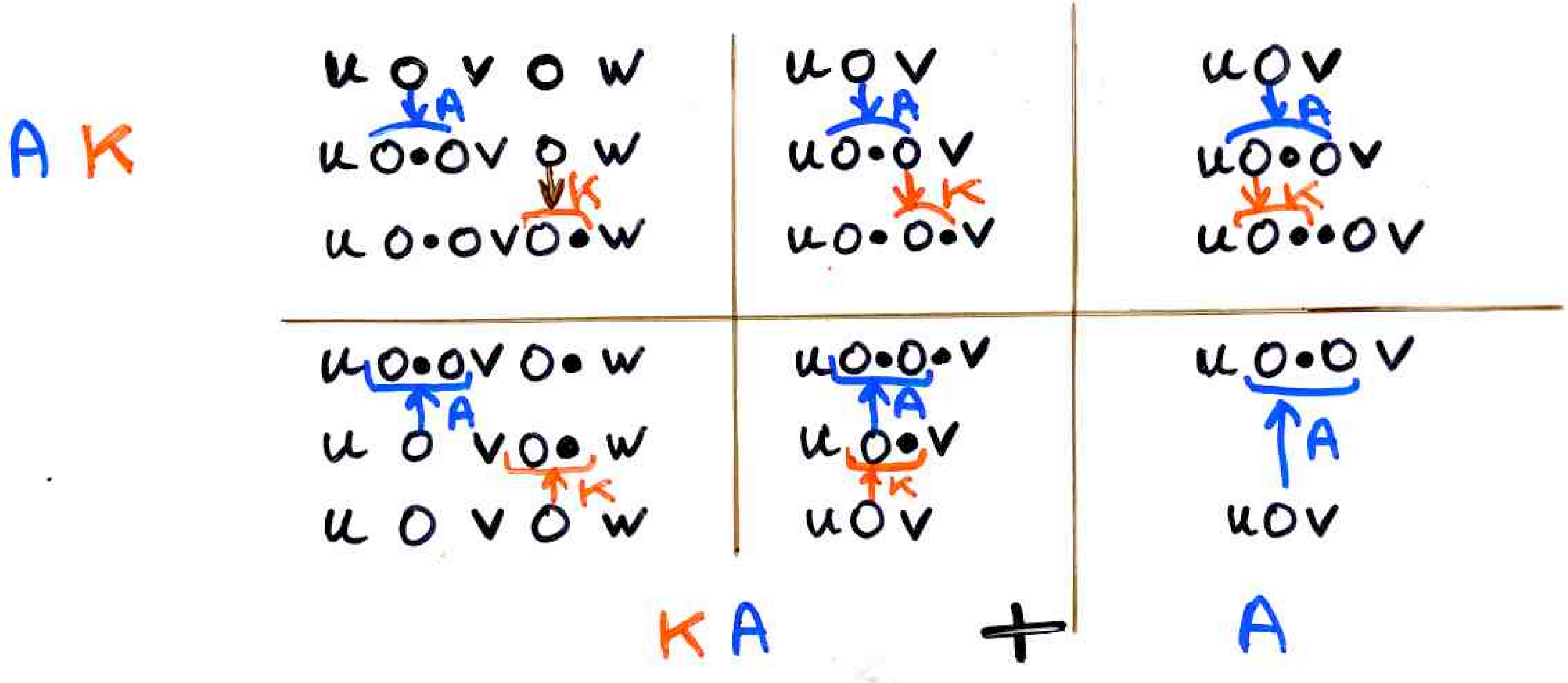
$$AS = SA + J + K$$

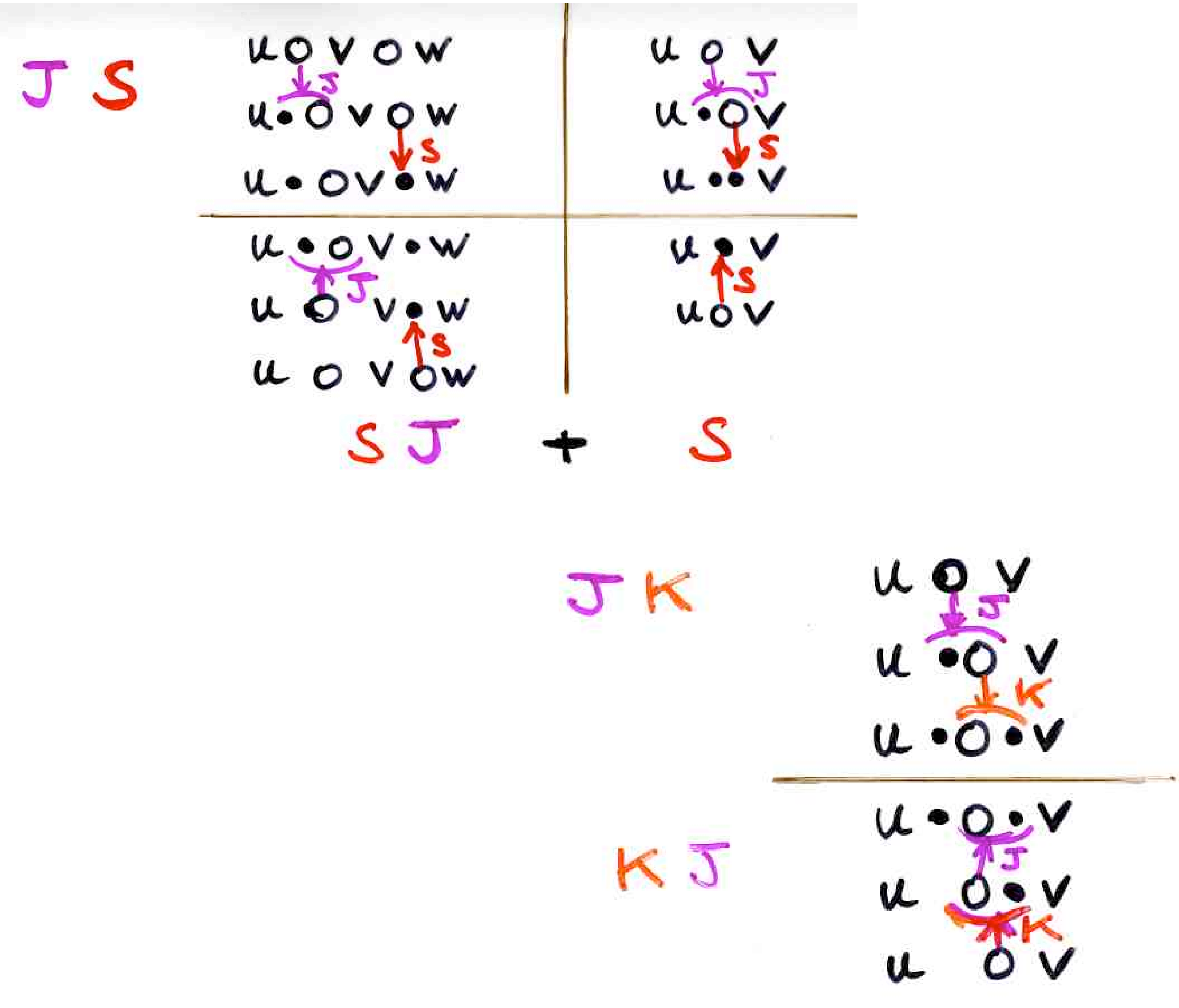
$$AK = KA + A$$

$$JS = SJ + S$$

$$JK = KJ$$


AS
SA
+
J
+
K


A K
u o v o w
u o•o v o w
u o•o v o• w
u o v
u o•o v
u o• o• v
u o v
u o•o v
u o••o v
u o••o v o• w
u o v o• w
u o v o w
u o•o• v
u o• v
u o v
u o•o v
u o v
K A + A



J S
SJ + S
J K
K J

Lemma.

$$AS = SA + J + K$$
$$AK = KA + A$$
$$JS = SJ + S$$
$$JK = KJ$$

$$D = A + J$$
$$E = S + K$$

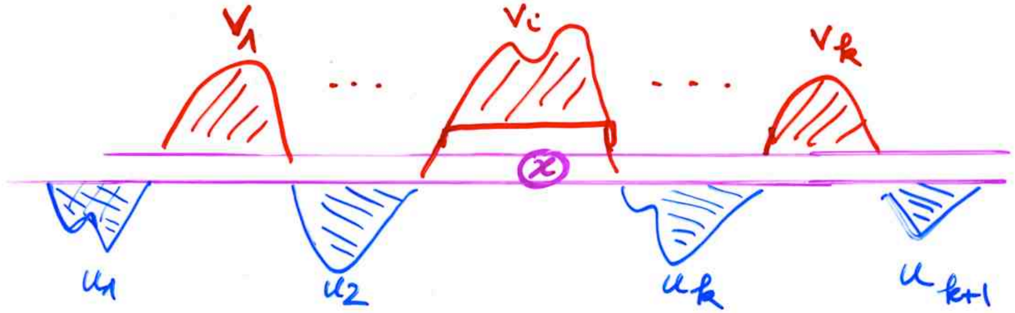
$$D = A + J$$

$$E = S + K$$

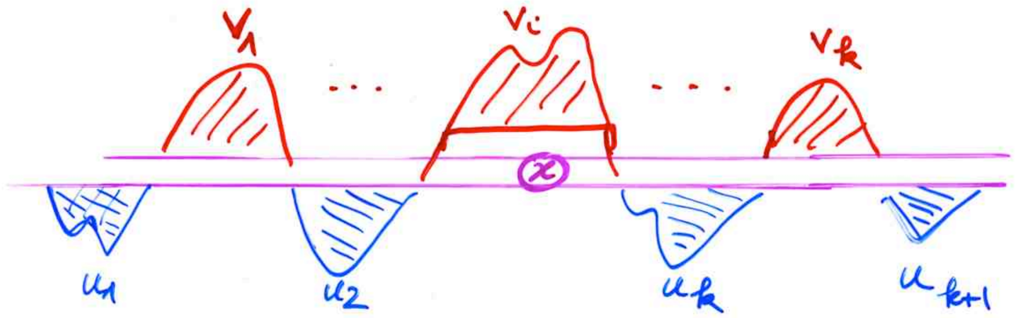
$$DE = (A+J)(S+K)$$

$$= AS + AK + JS + JK$$

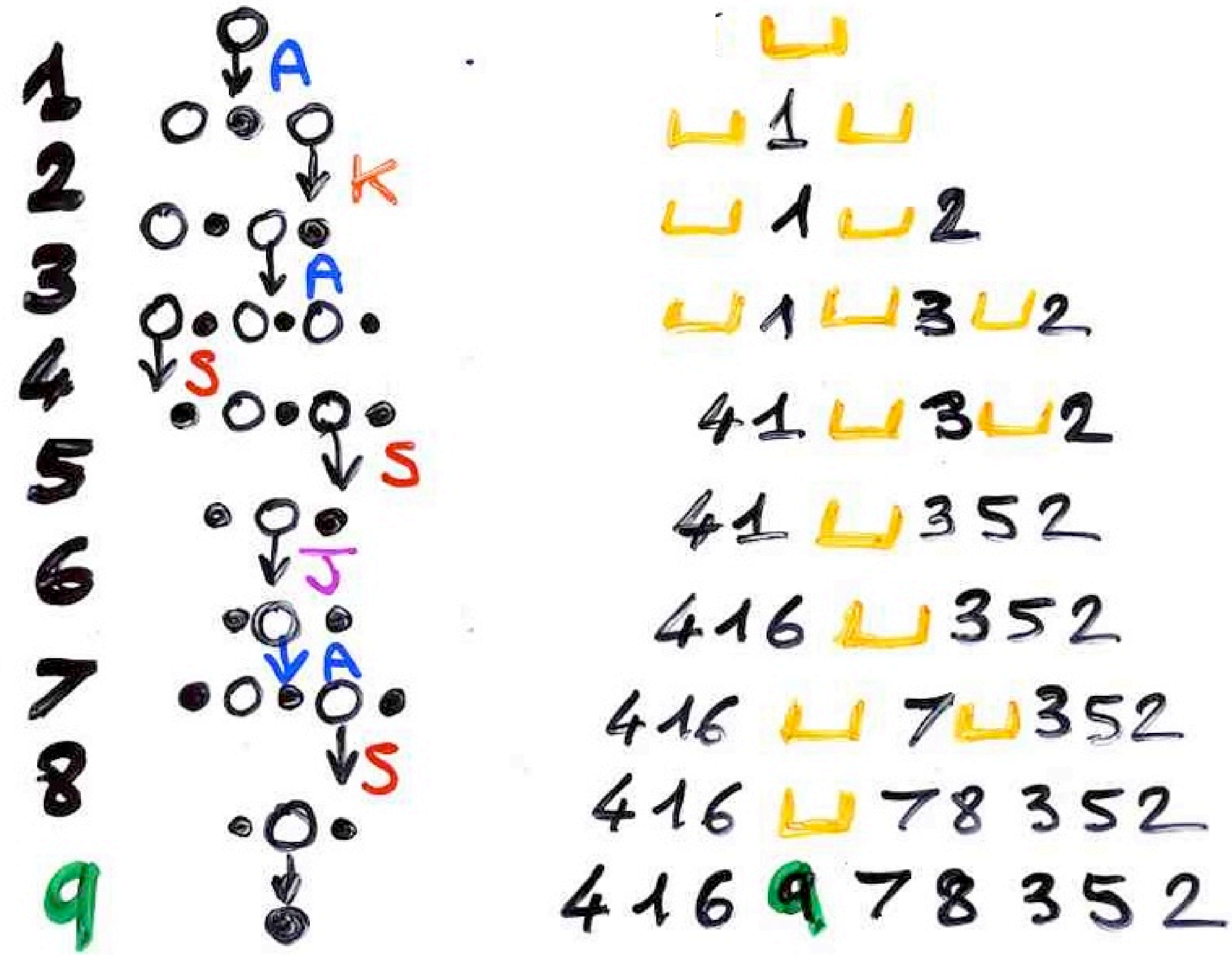
$$= \underbrace{(SA + KA + SJ + KJ)}_{\underbrace{(S+K)(A+J)}_{ED}} + \underbrace{J + K + A + S}_{E + D}$$



	⊔
1	⊔ 1 ⊔
2	⊔ 1 ⊔ 2
3	⊔ 1 ⊔ 3 ⊔ 2
4	4 1 ⊔ 3 ⊔ 2
5	4 1 ⊔ 3 5 2
6	4 1 6 ⊔ 3 5 2
7	4 1 6 ⊔ 7 ⊔ 3 5 2
8	4 1 6 ⊔ 7 8 3 5 2
9	4 1 6 9 7 8 3 5 2

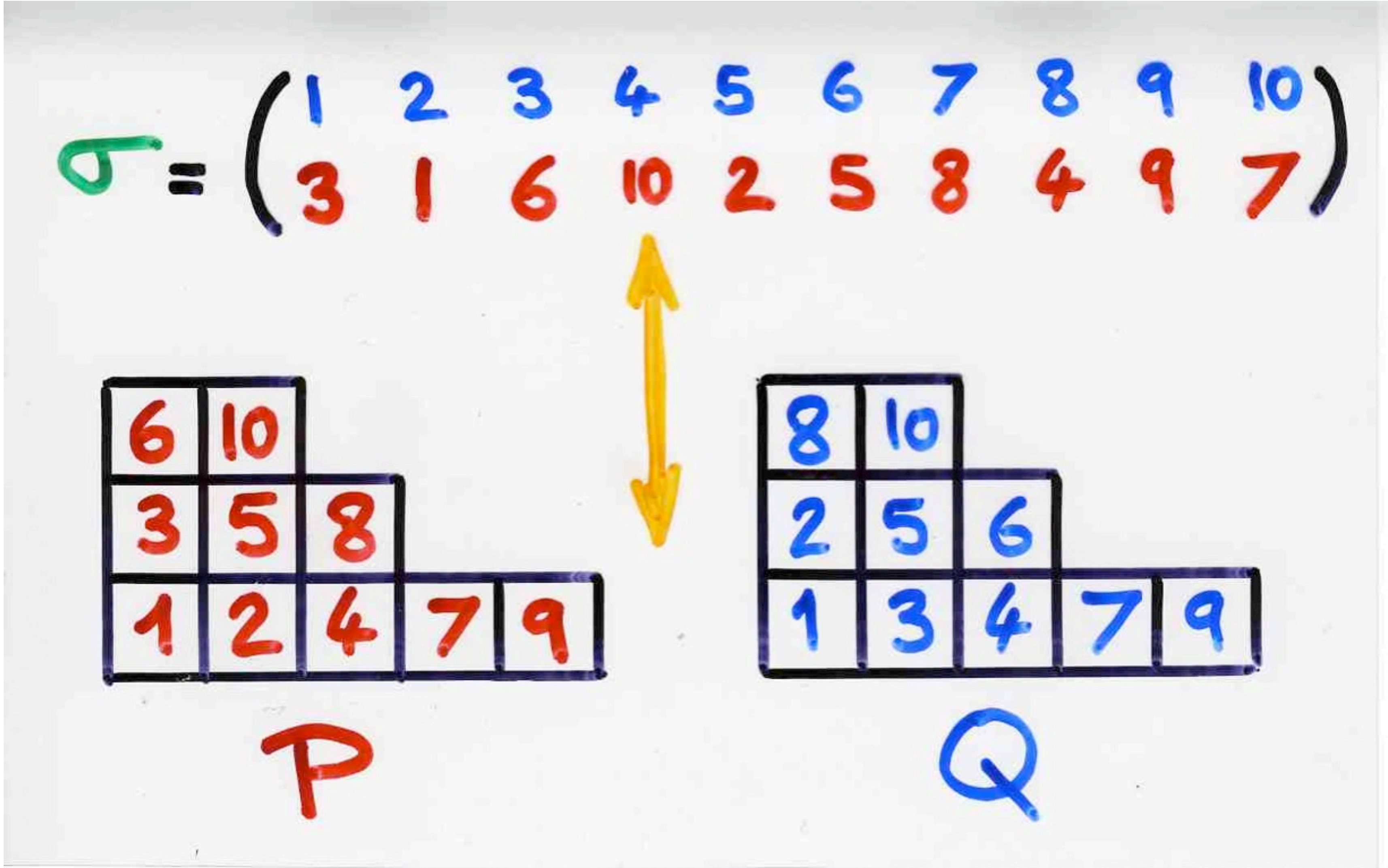


v1
vi
vk
x
u1
u2
uk
uk+1

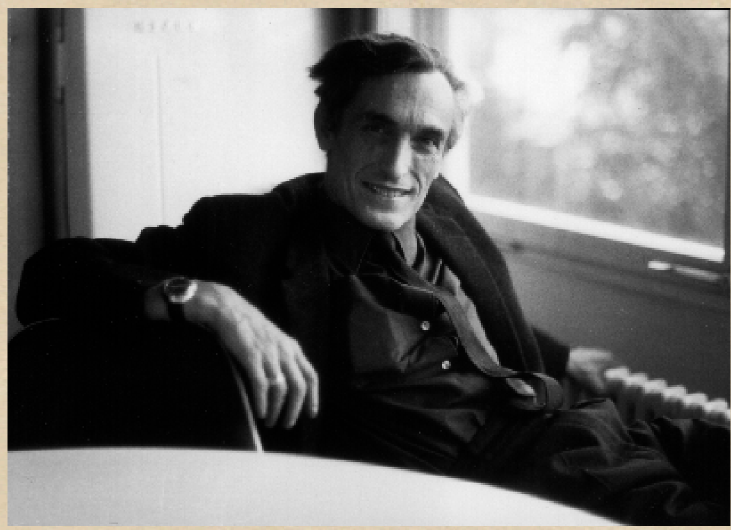


1
2
3
4
5
6
7
8
9
A
K
A
S
S
J
A
S
⊔
⊔ 1 ⊔
⊔ 1 ⊔ 2
⊔ 1 ⊔ 3 ⊔ 2
4 1 ⊔ 3 ⊔ 2
4 1 ⊔ 3 5 2
4 1 6 ⊔ 3 5 2
4 1 6 ⊔ 7 ⊔ 3 5 2
4 1 6 ⊔ 7 8 3 5 2
4 1 6 9 7 8 3 5 2

§9 RSK with operators, commutations and local rules

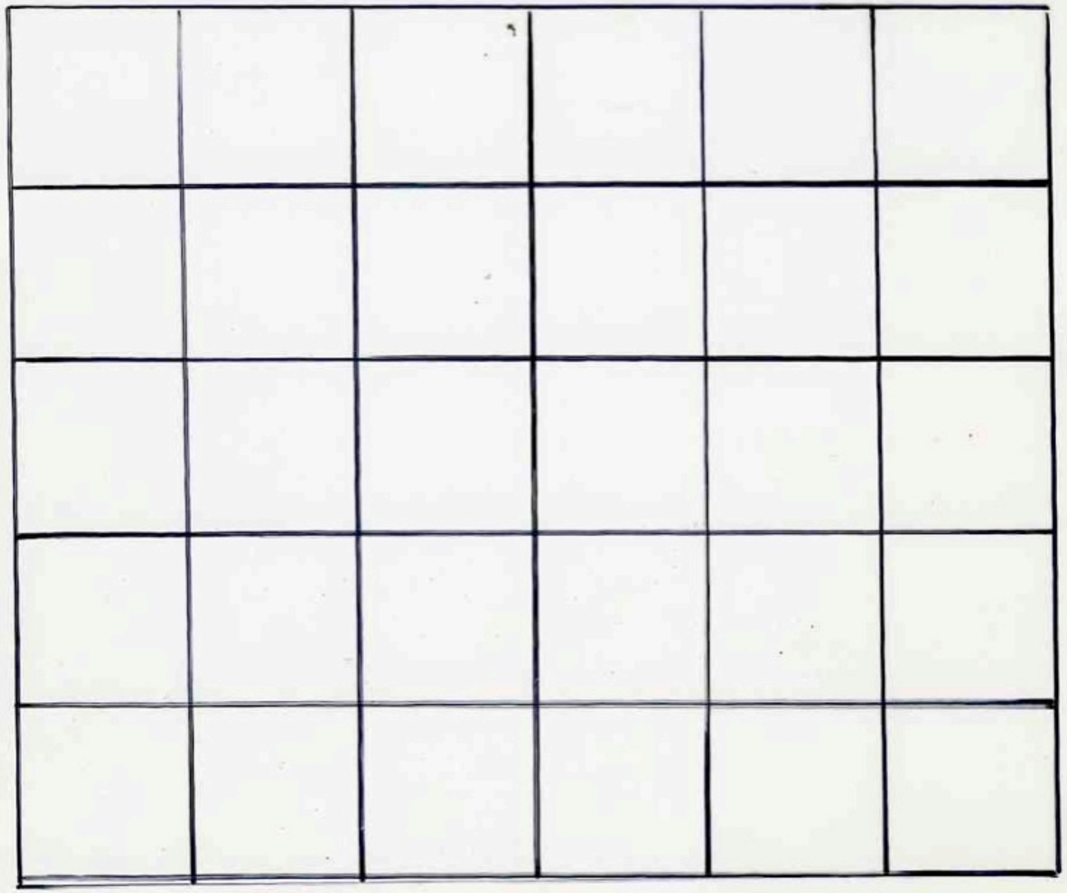

σ =
1 2 3 4 5 6 7 8 9 10
3 1 6 10 2 5 8 4 9 7
6 10
3 5 8
1 2 4 7 9
8 10
2 5 6
1 3 4 7 9
P
Q

Jeu de taquin



M.P. Schützenberger

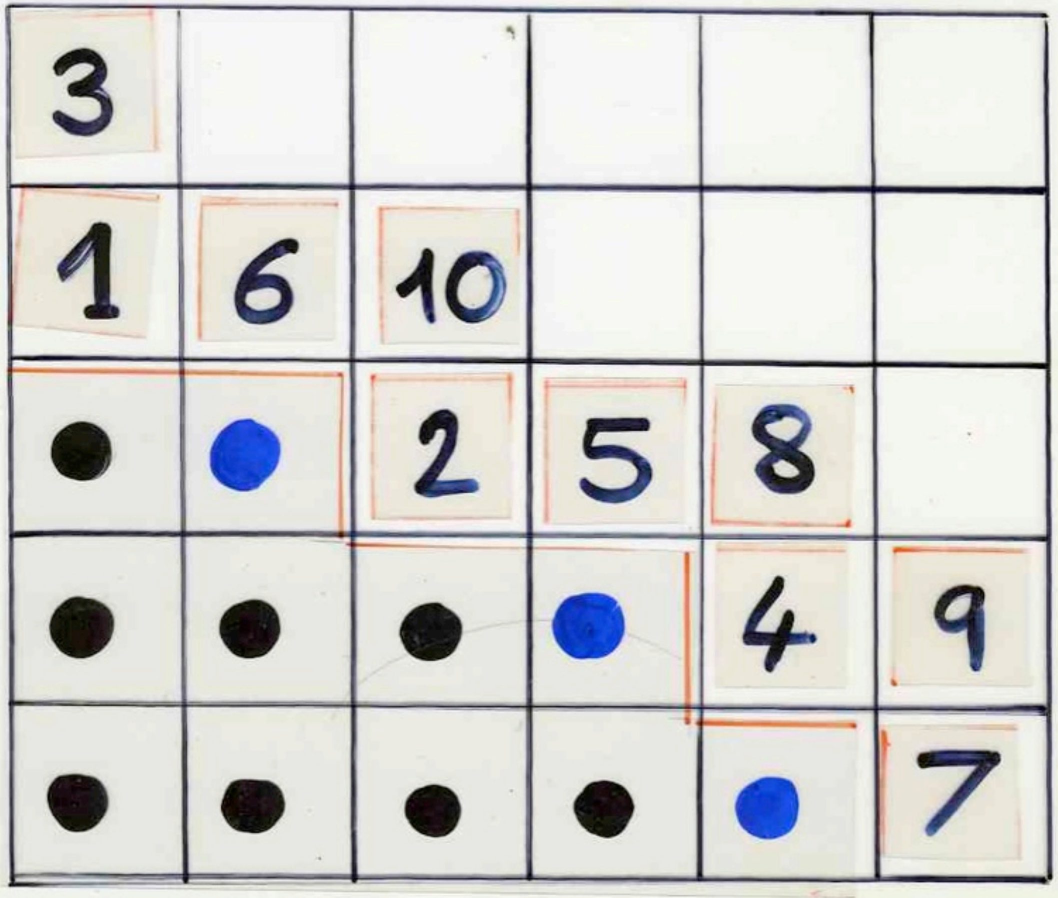
$$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$$



$$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$$

3					
1	6	10			
		2	5	8	
				4	9
					7

$$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$$



$$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$$

6	10				
3	5	8			
1	2	4	7	9	

RSK "local"

Sergey Fomin

- operator algebra

$$DE - qED = D + E$$

$\rightarrow$ $n!$

$$UD - DU = 1 \qquad n!$$

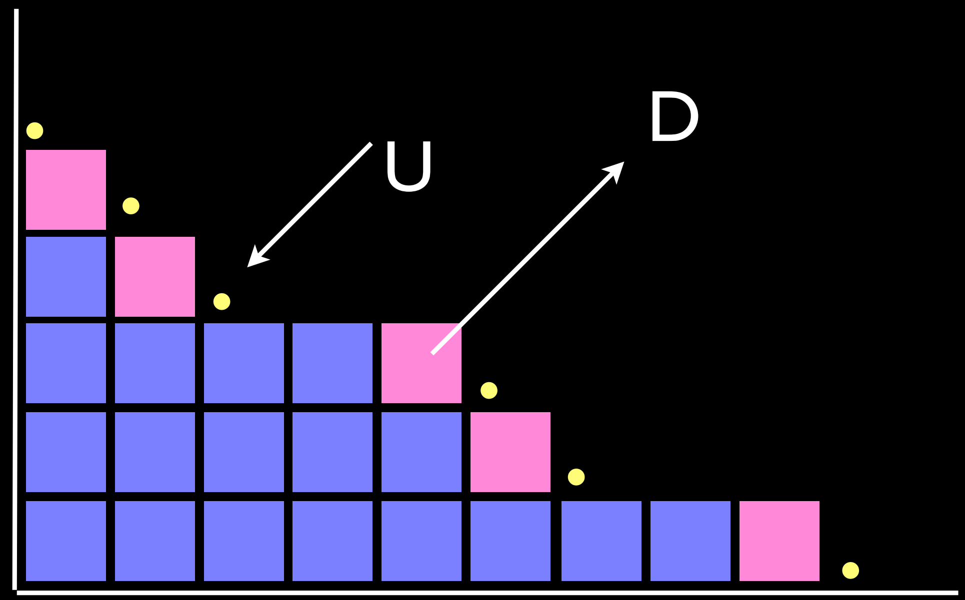
$\rightarrow$ Robinson-Schensted

Fomin

normal ordering

$$UUU\ldots\underline{UD}D\ldots D \rightarrow$$

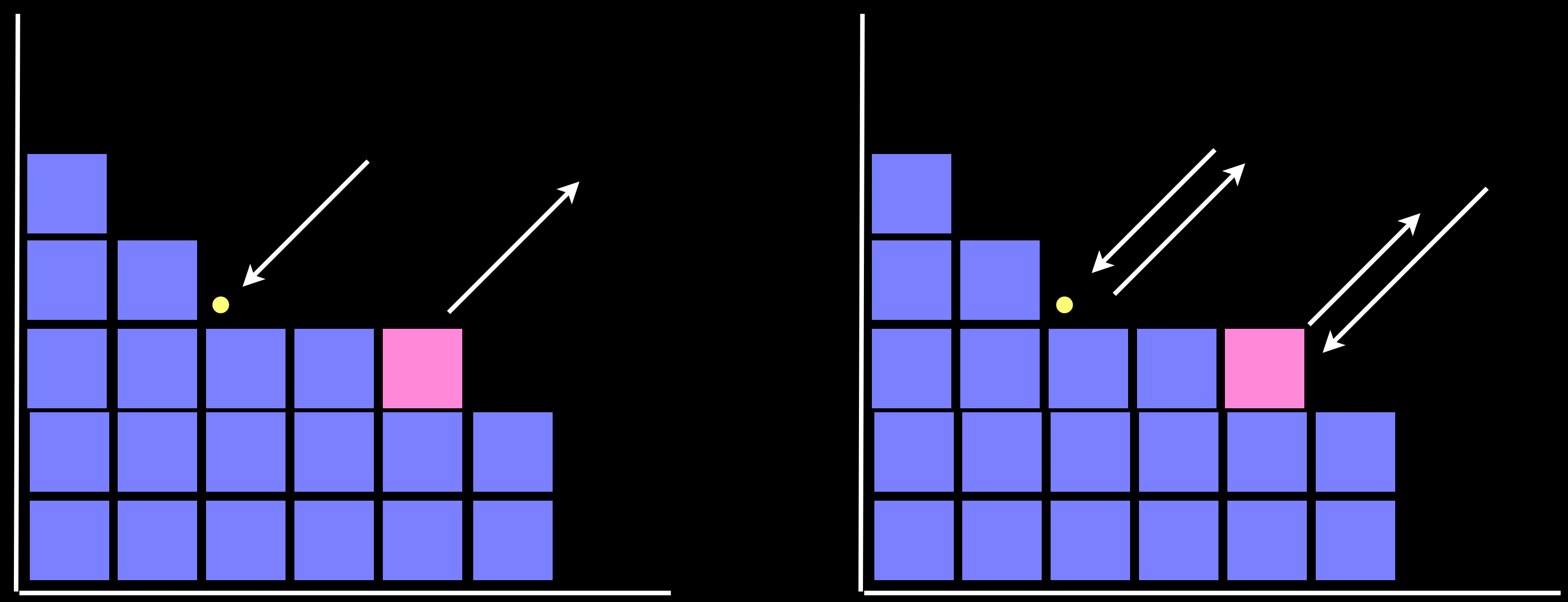
Operators U and D



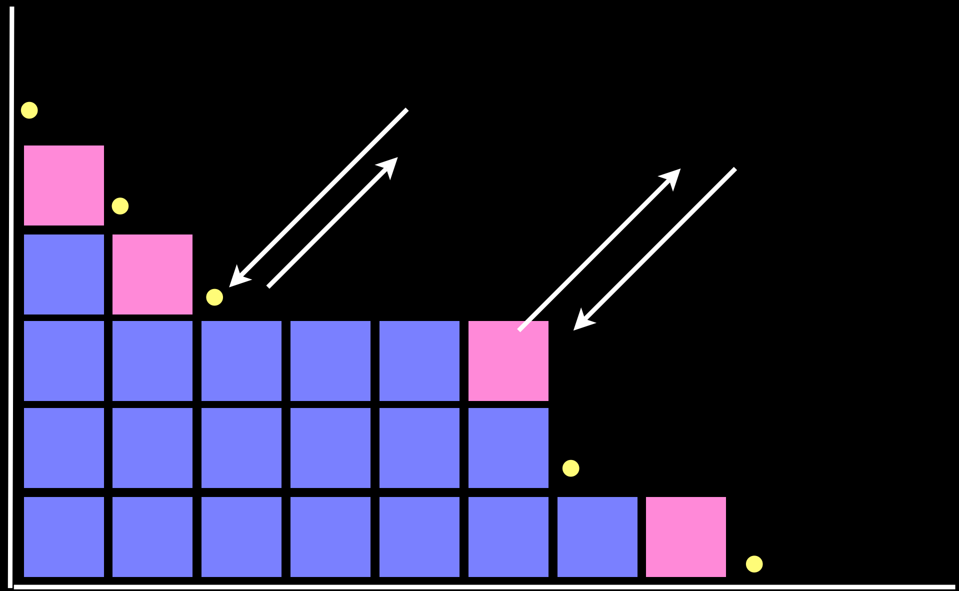
Young lattice

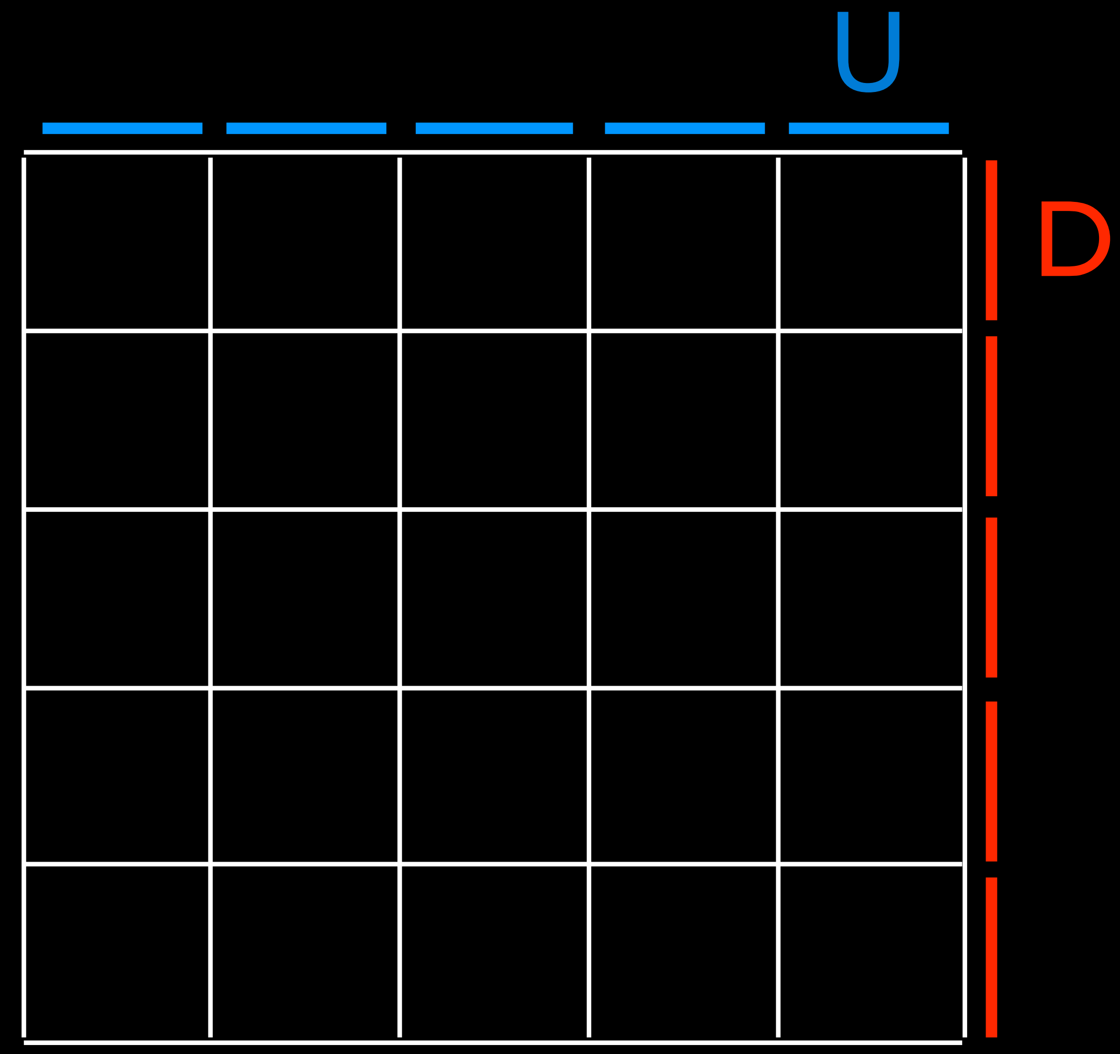
Heisenberg commutation relation

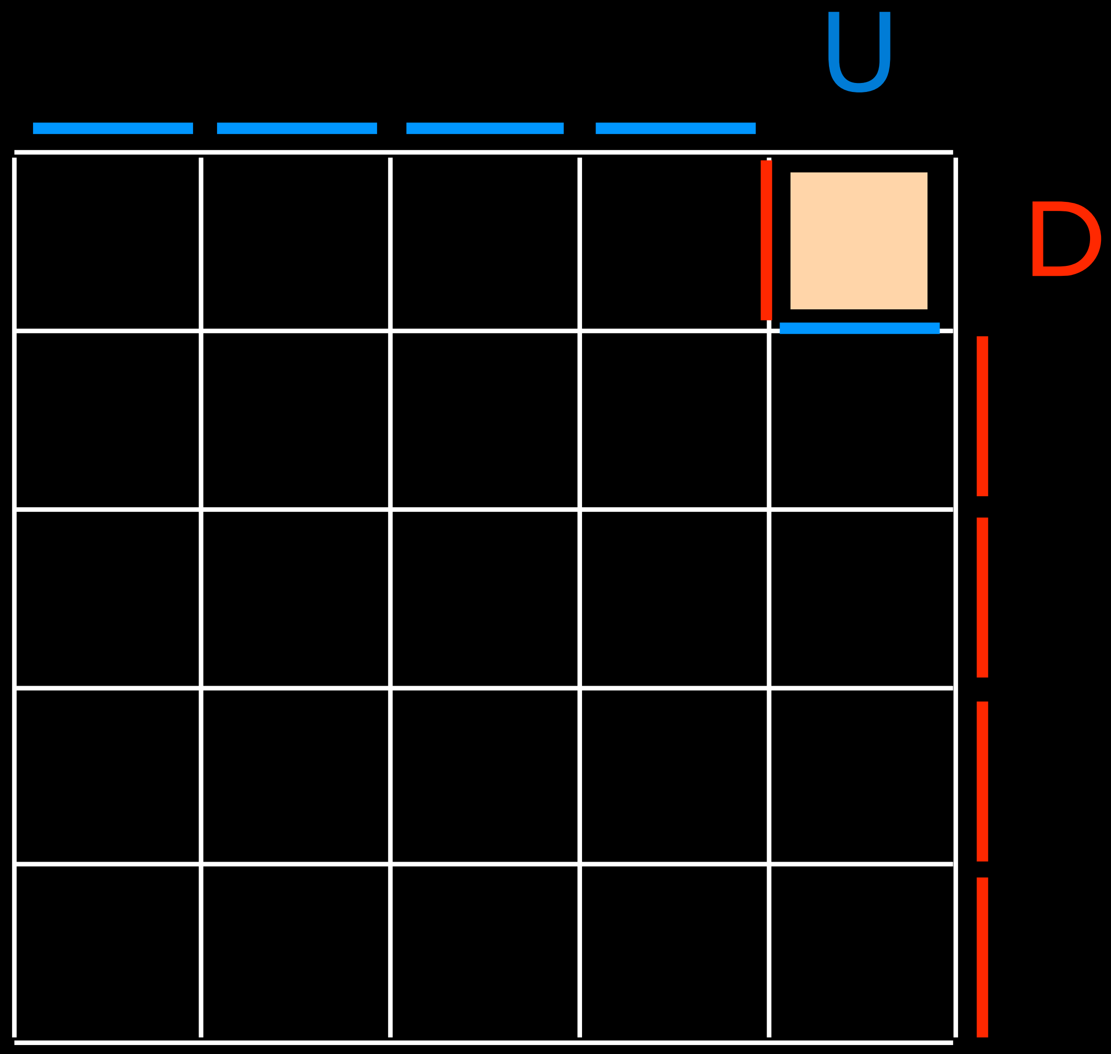
$$UD = DU + I$$

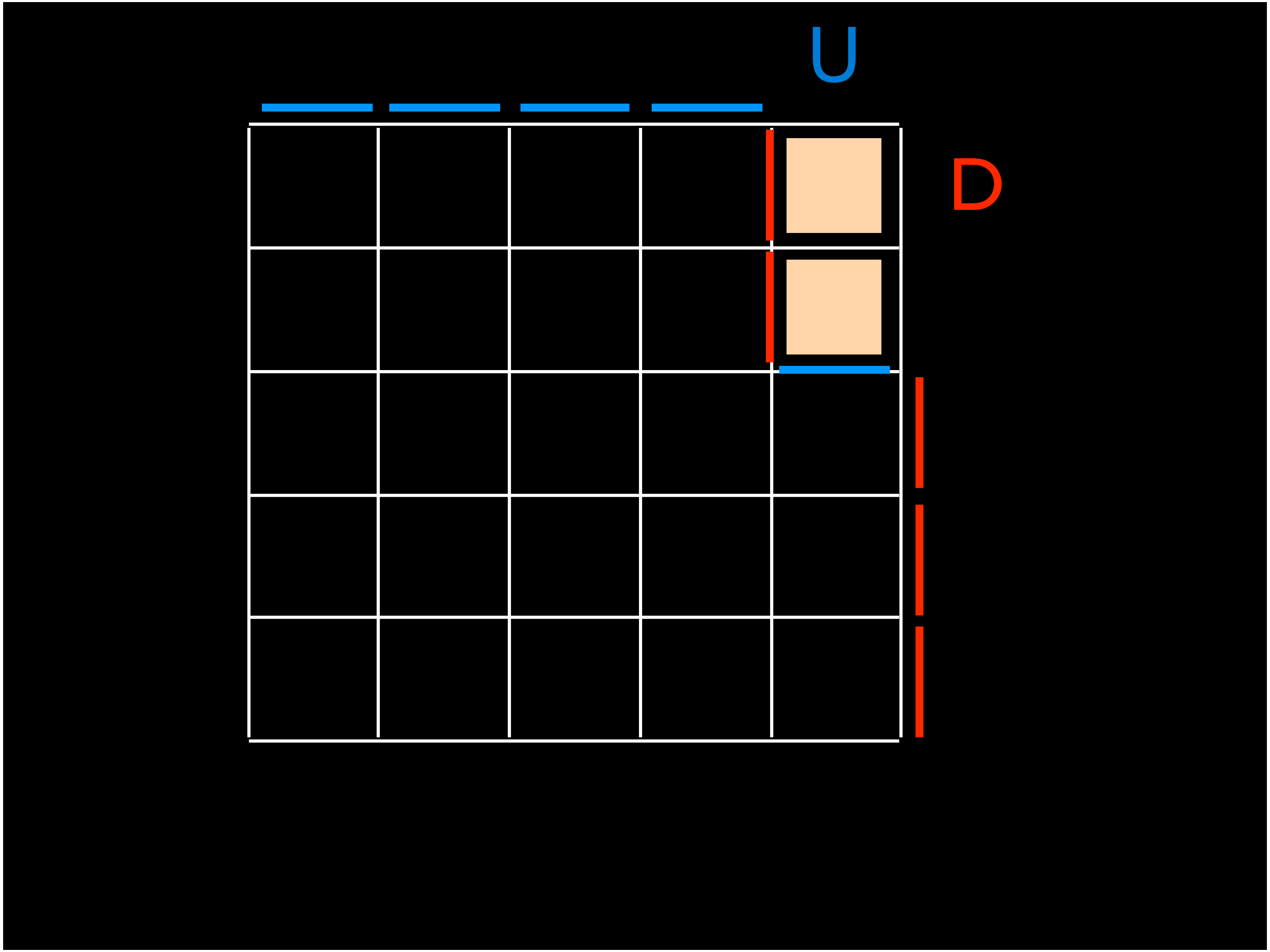


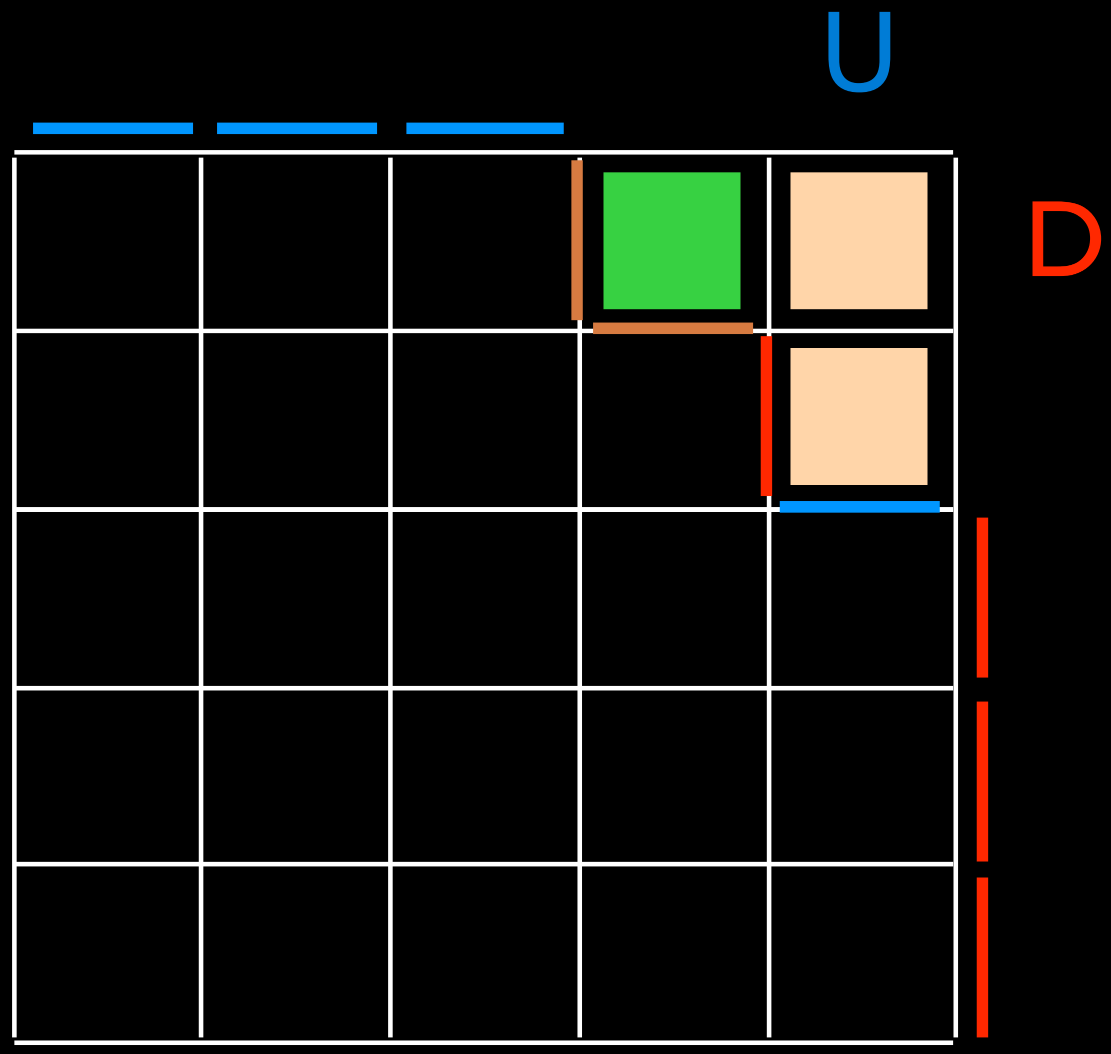
UD = DU + I




U
D


U
D


U
D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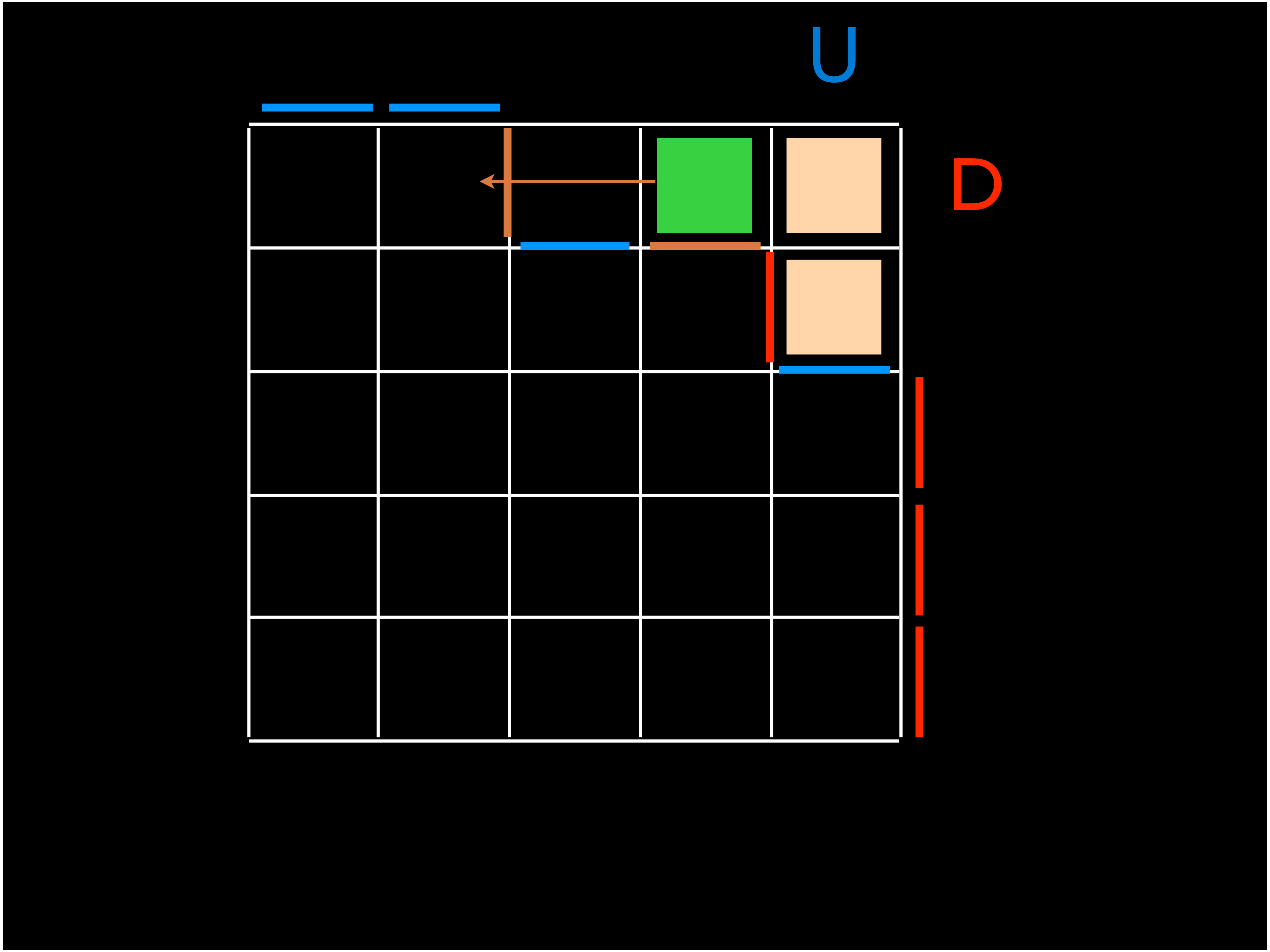
U
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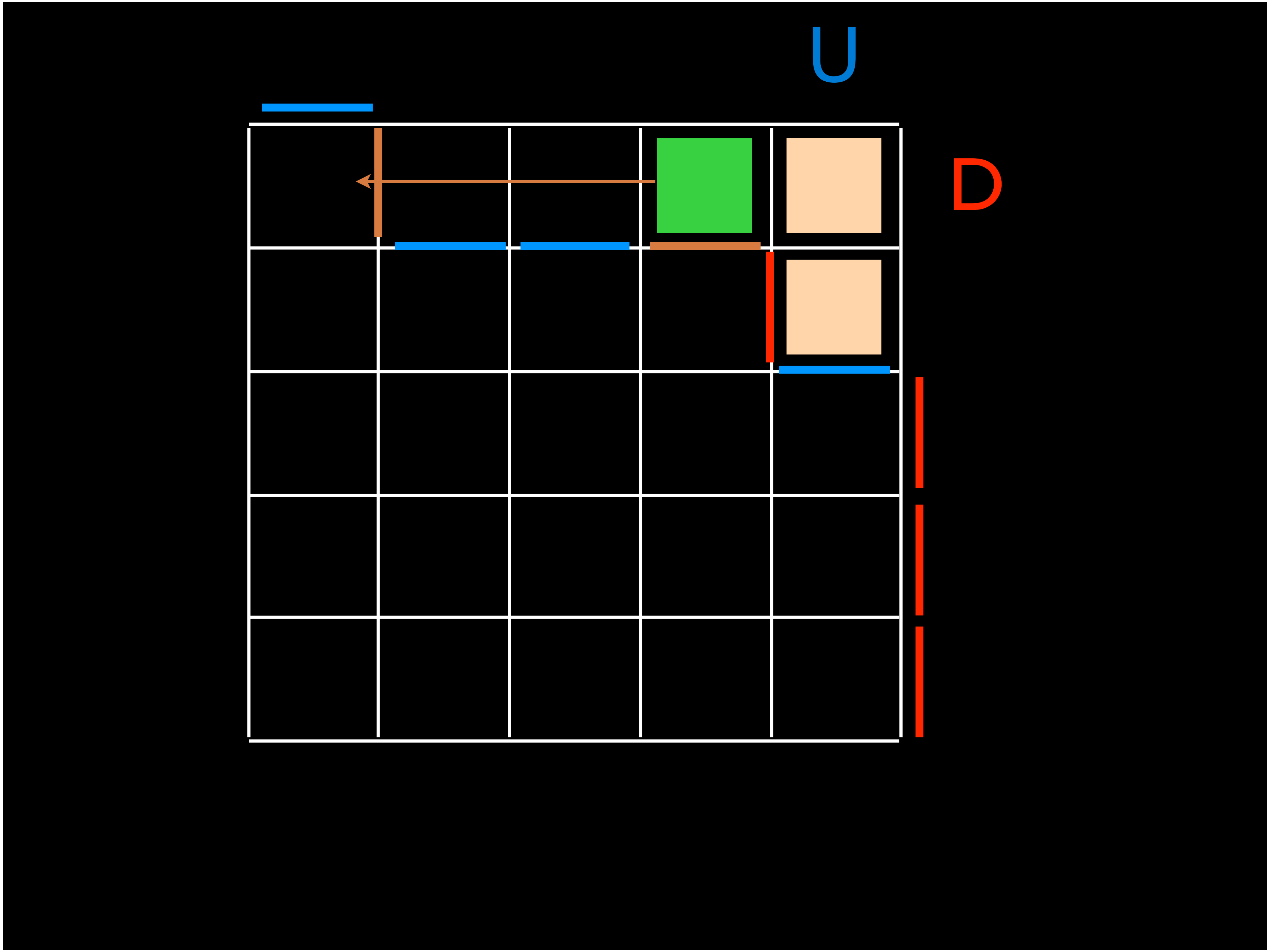
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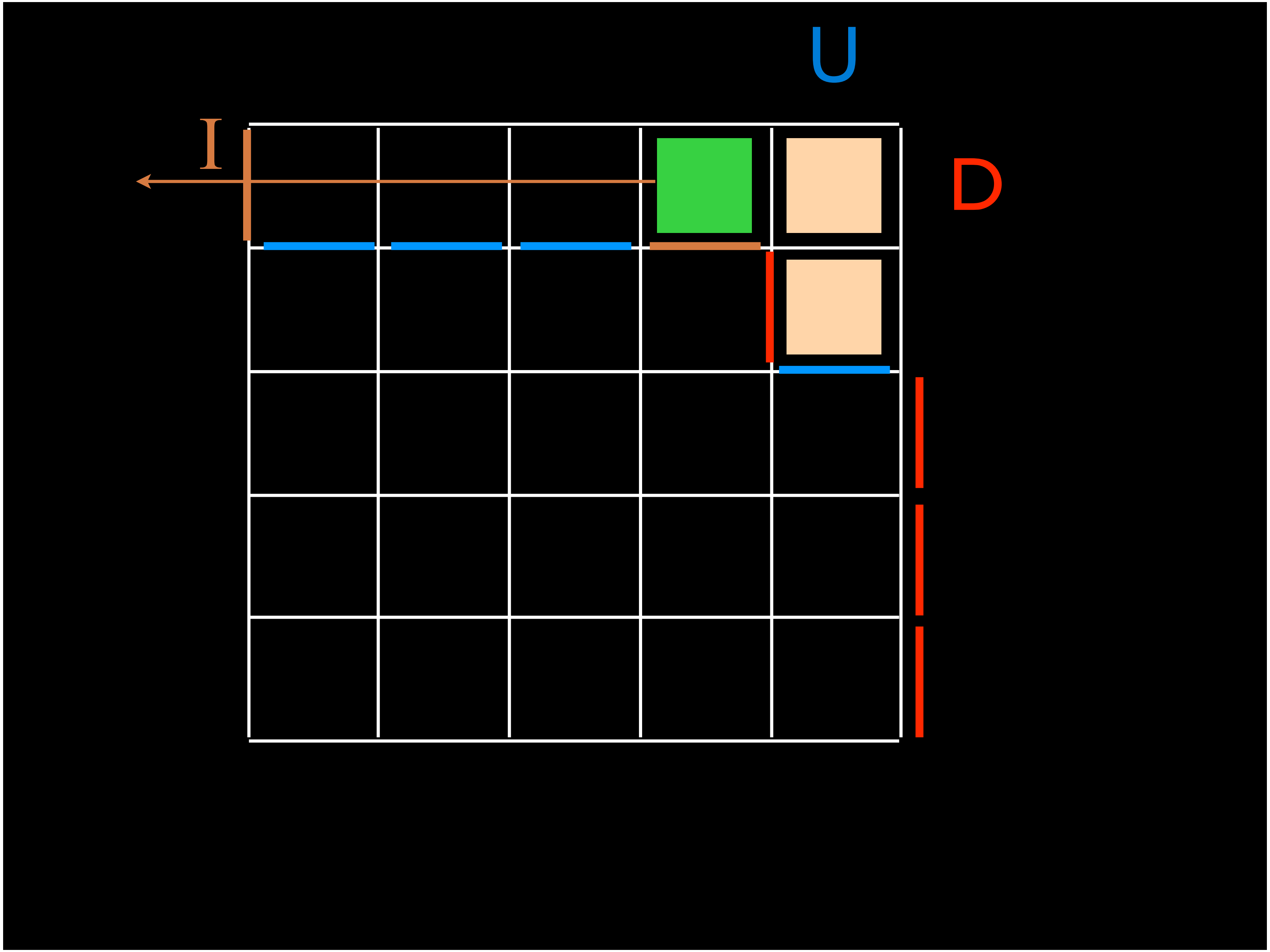
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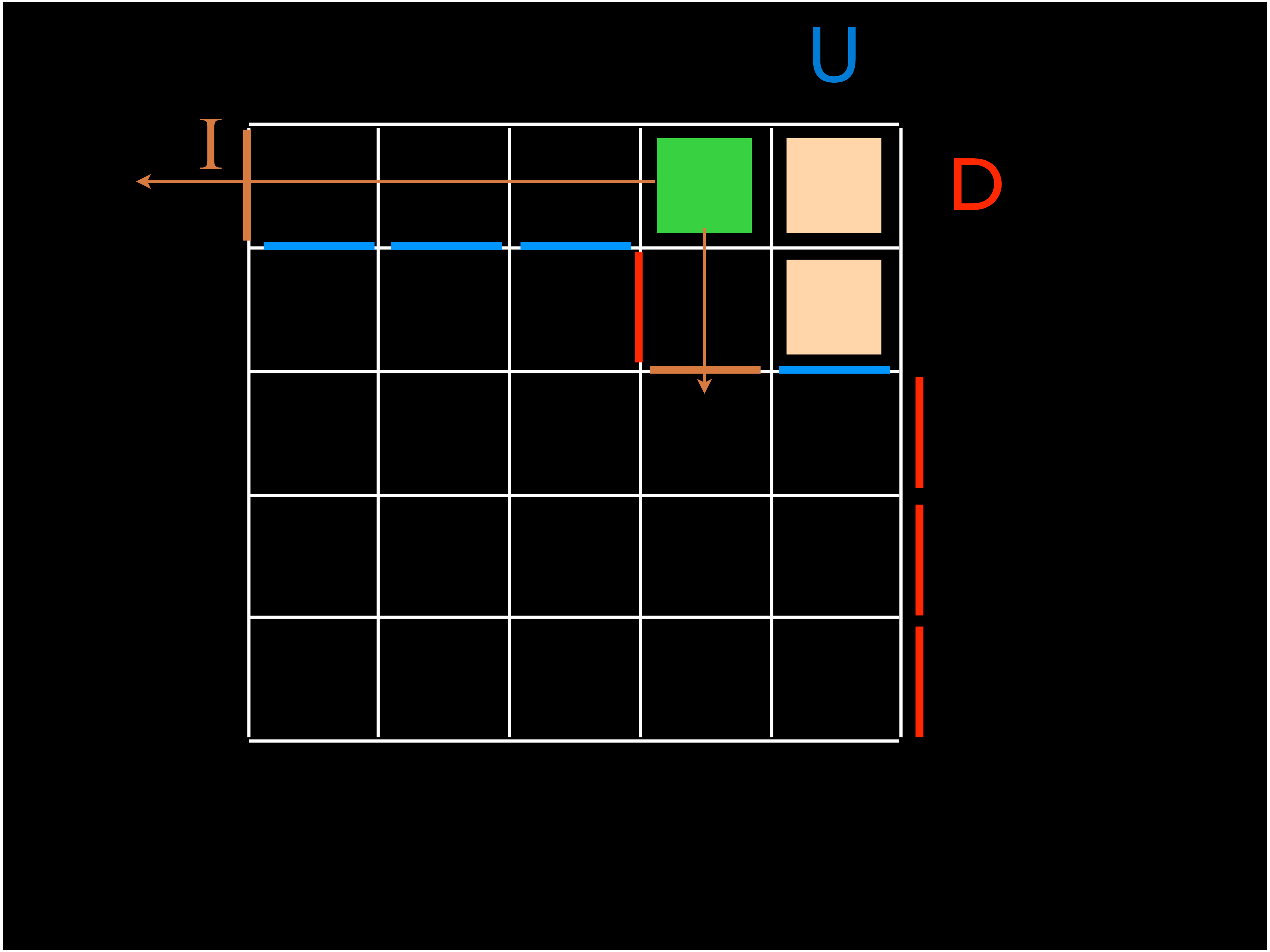
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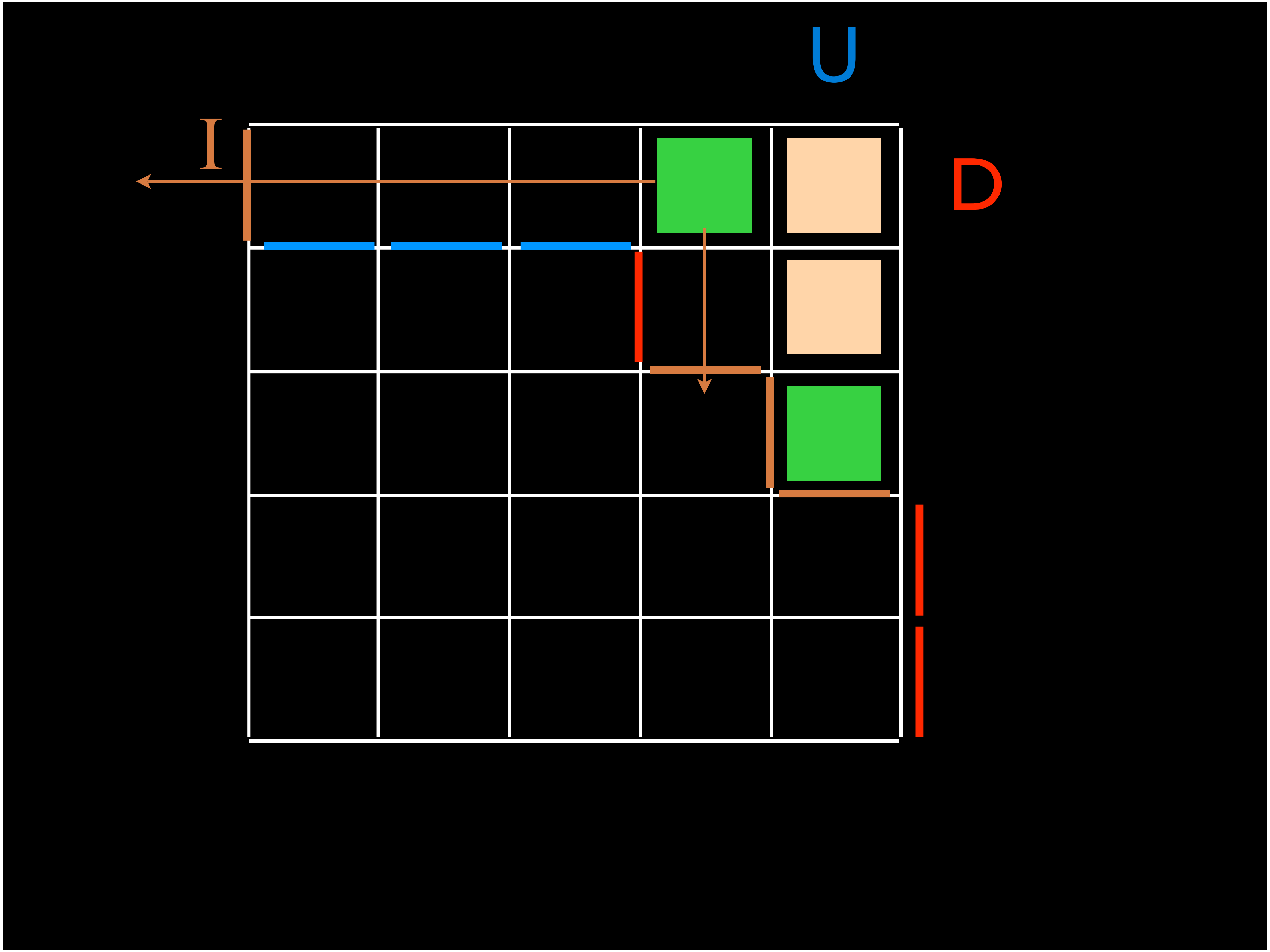

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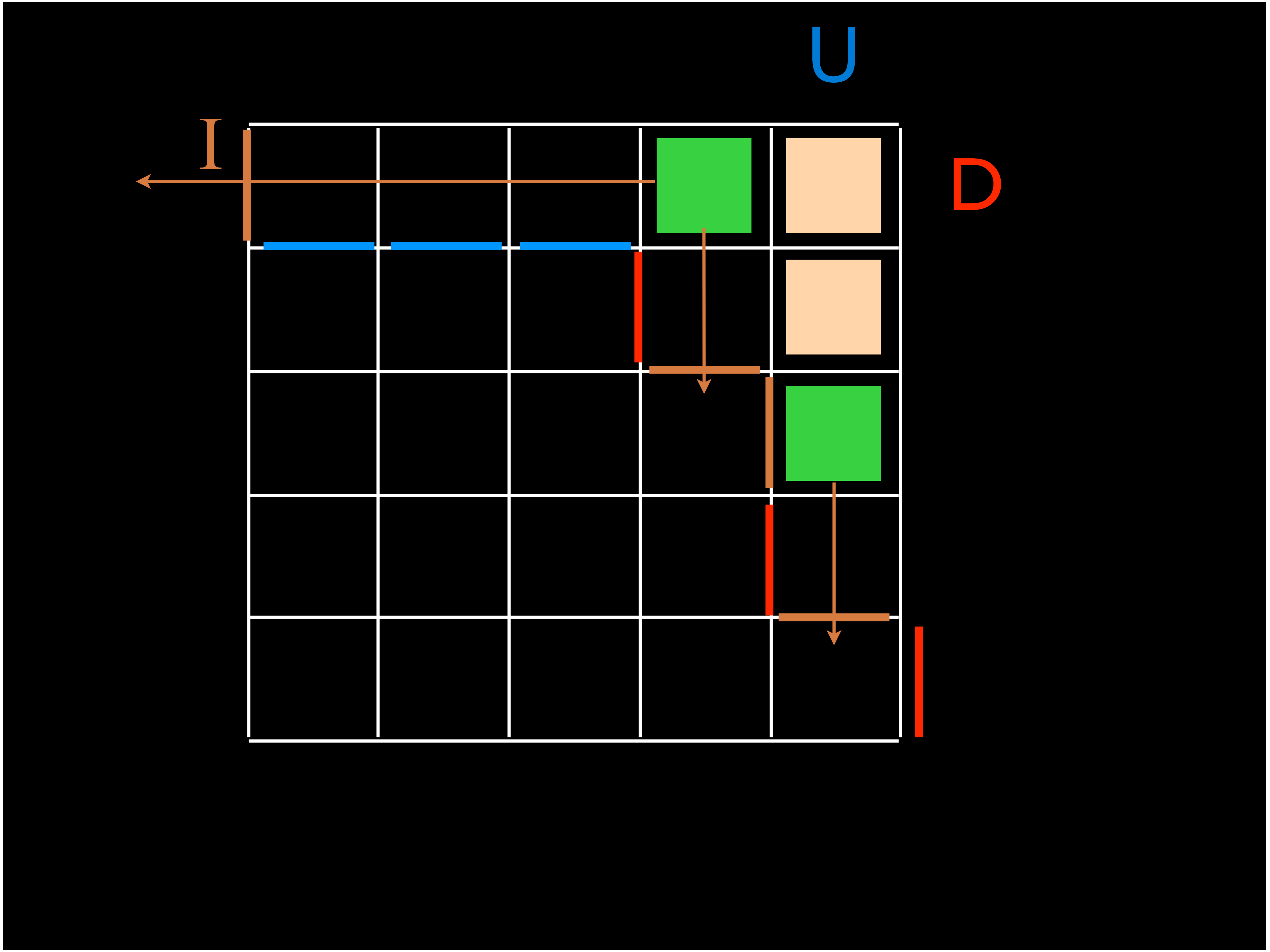
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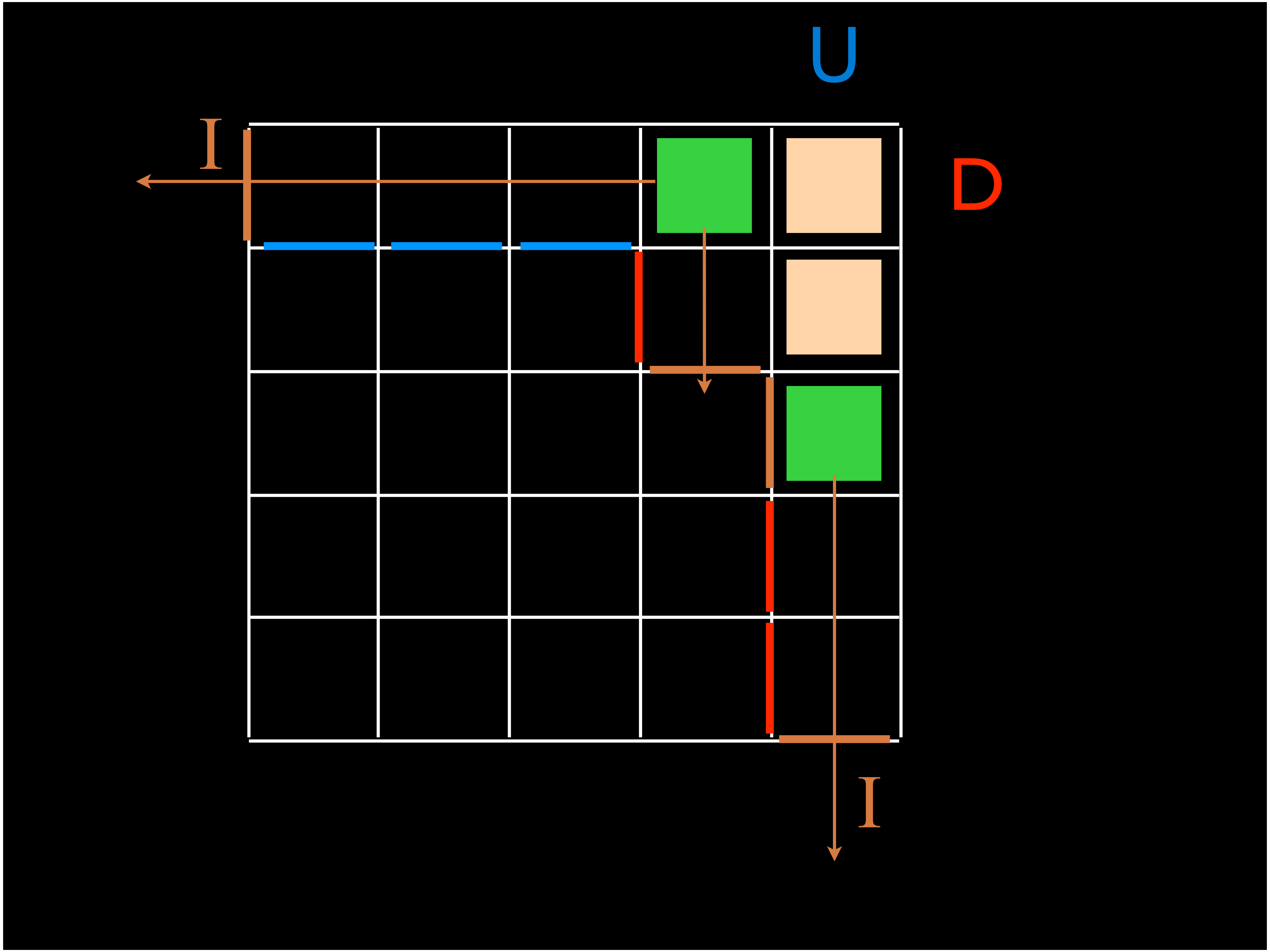
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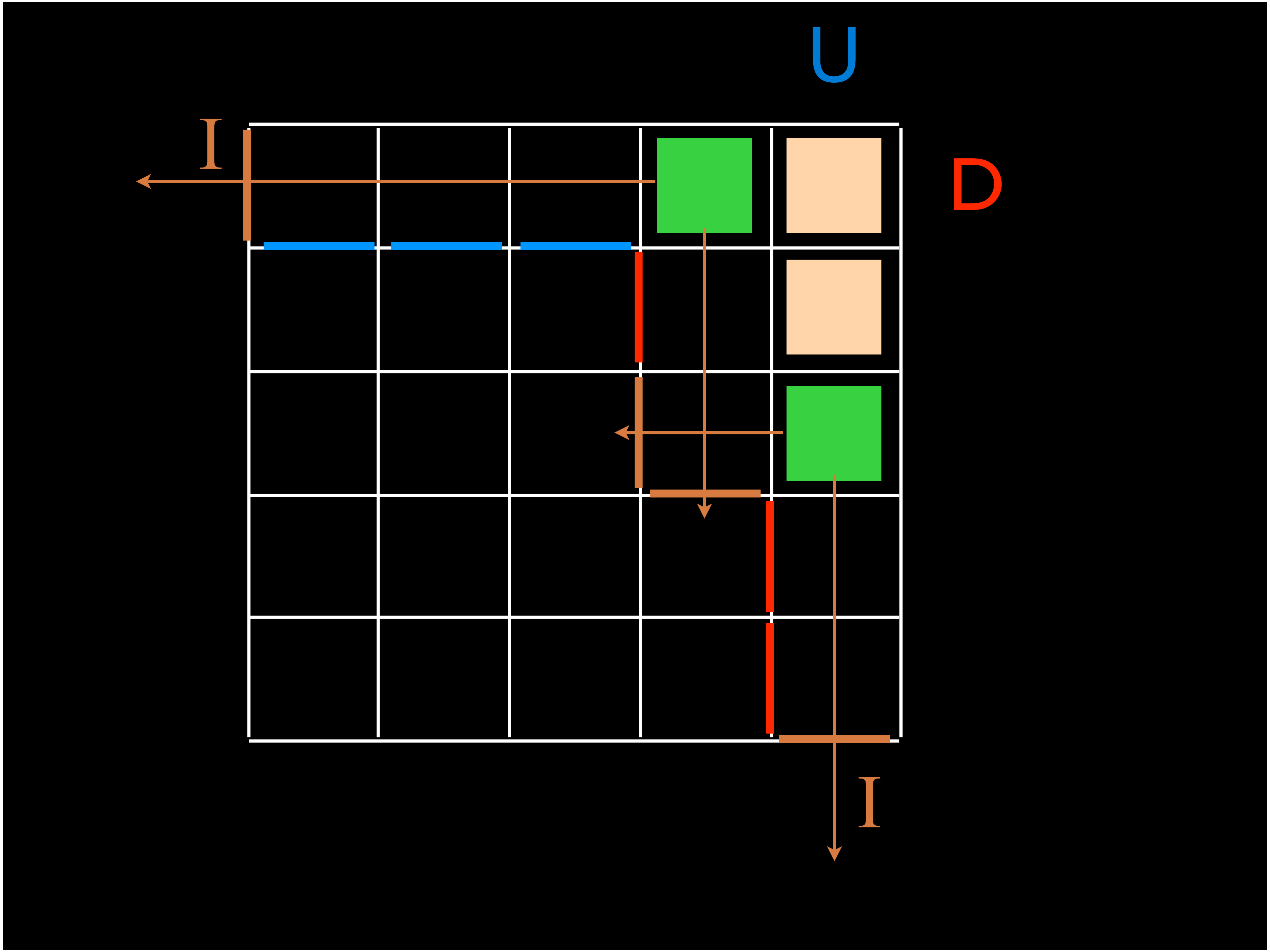
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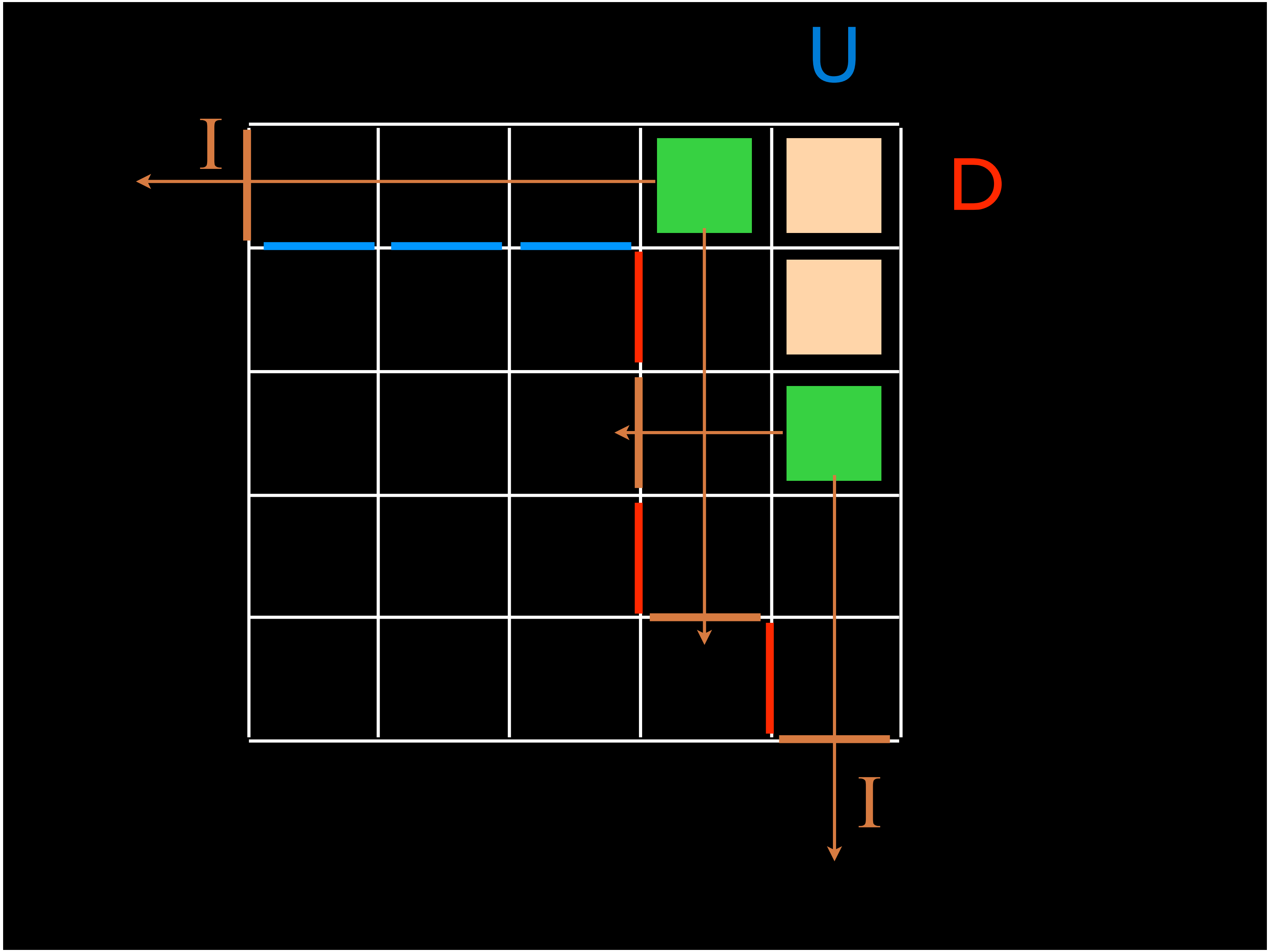

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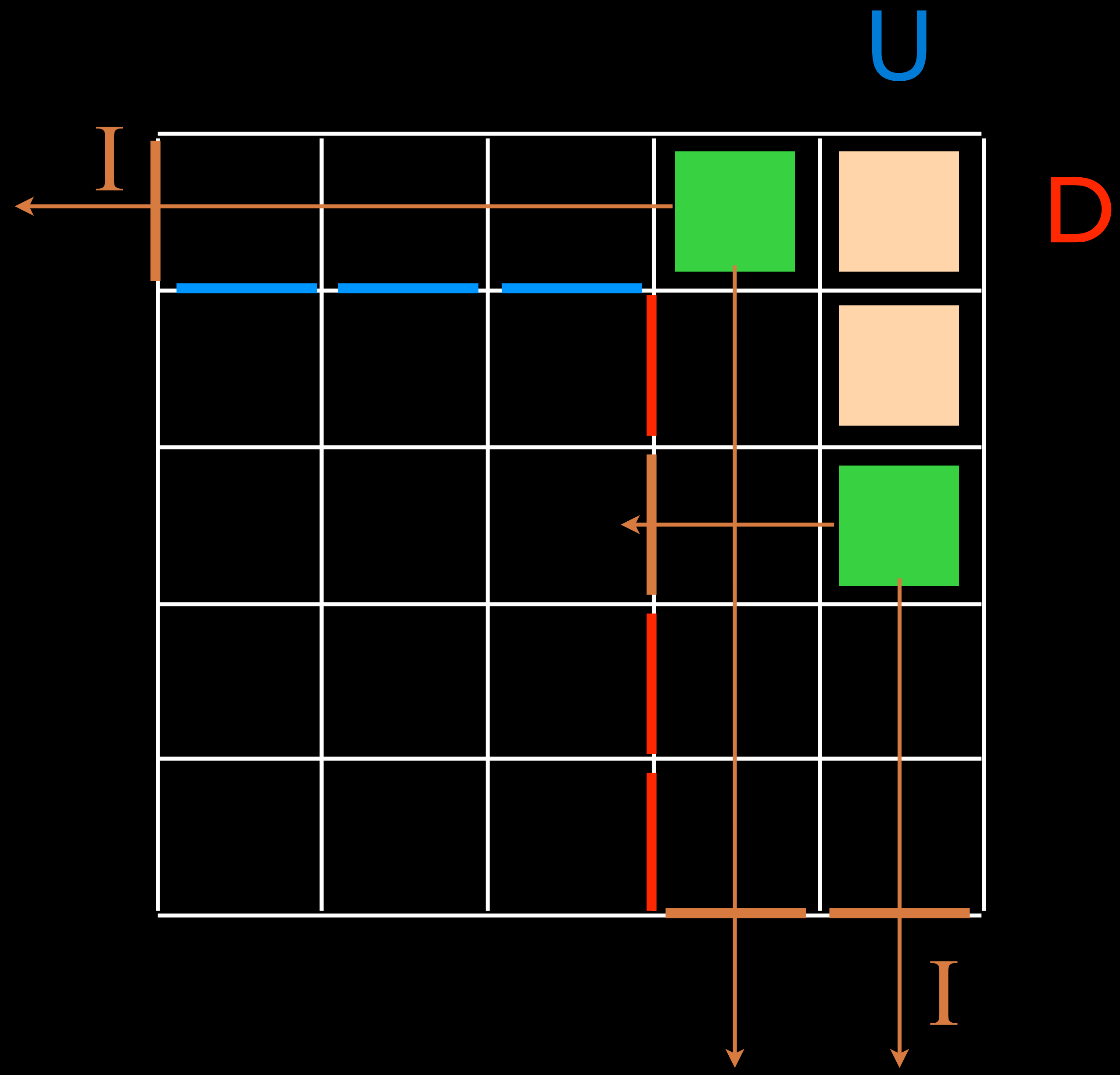

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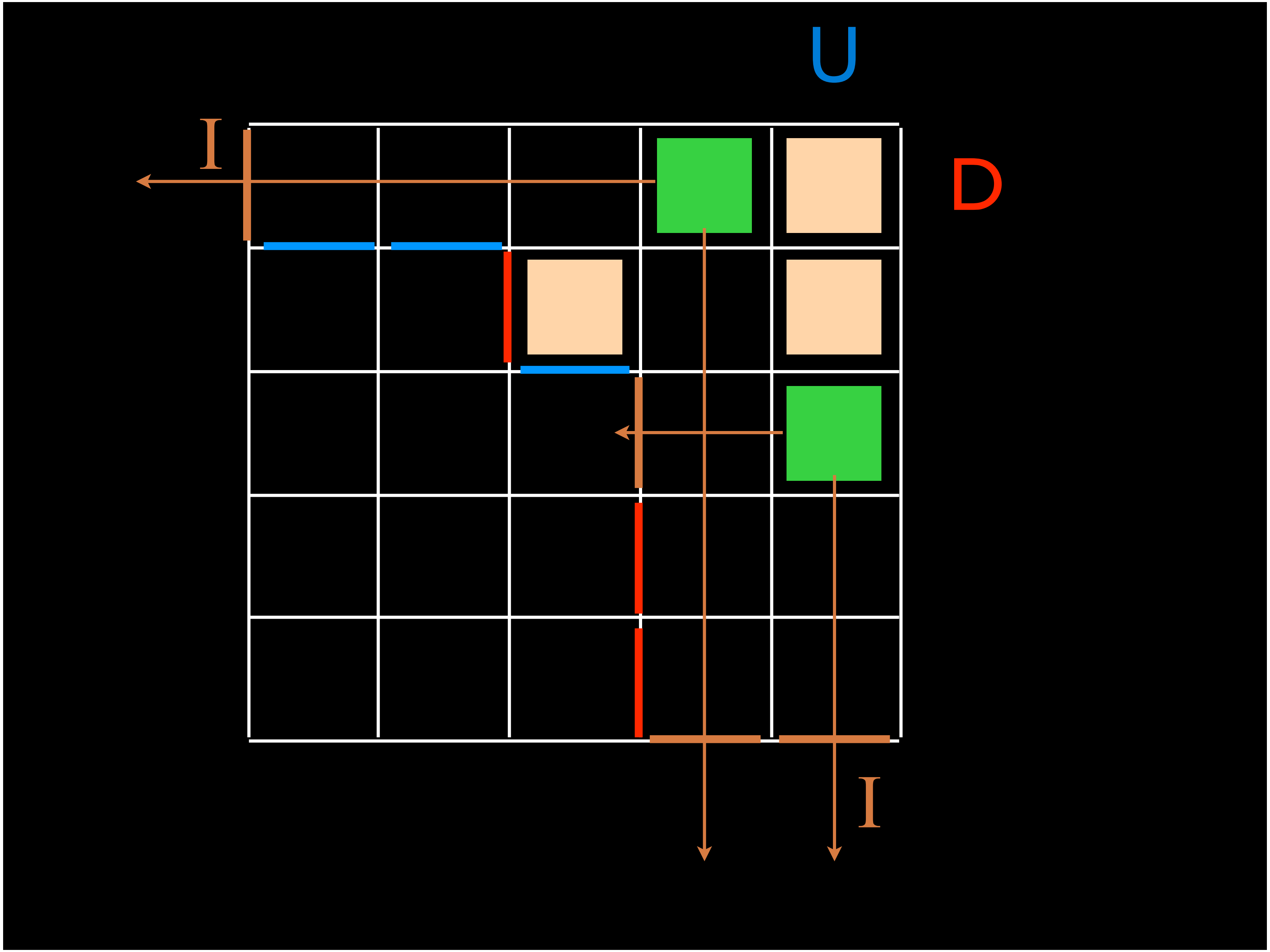

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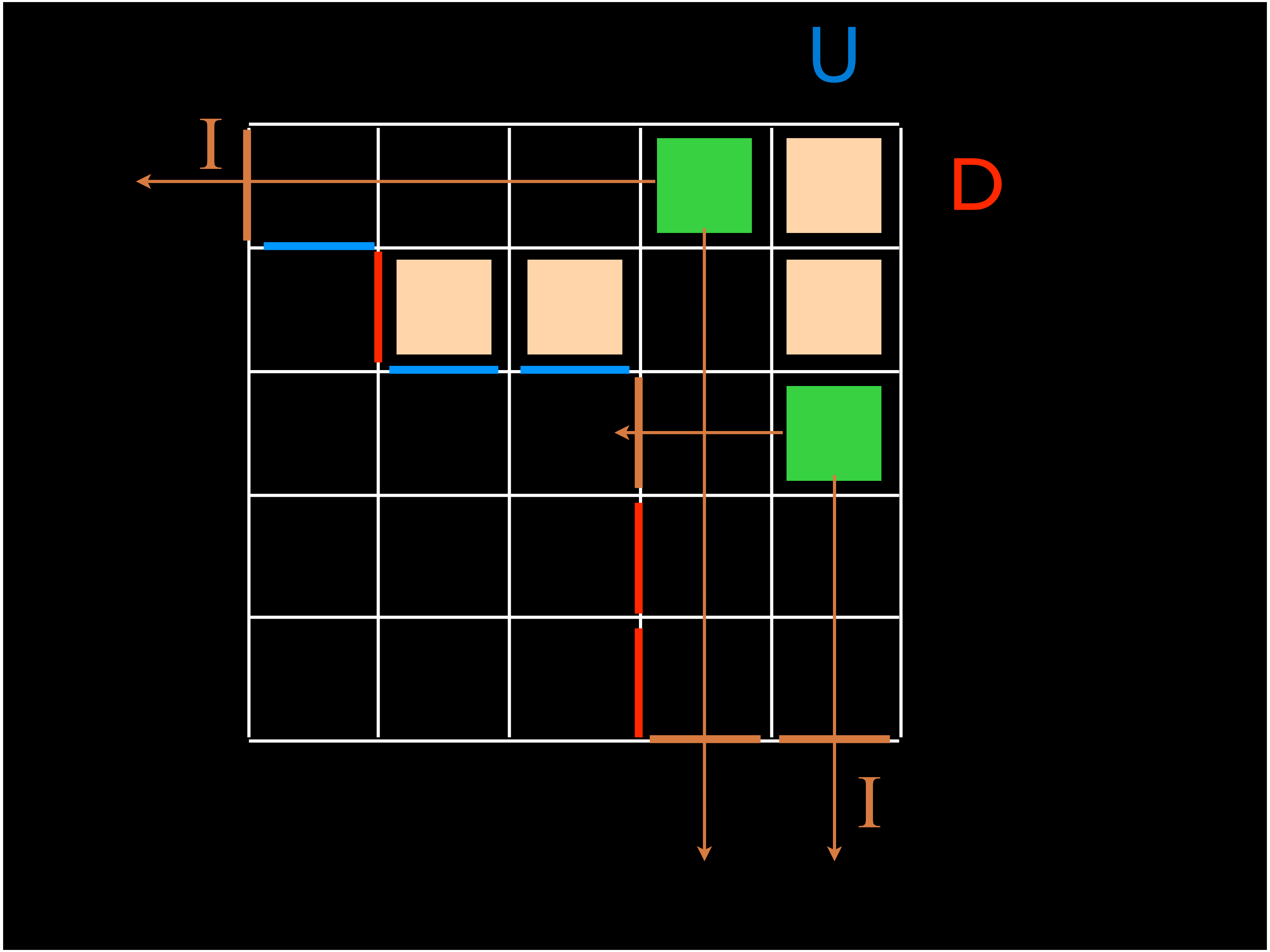
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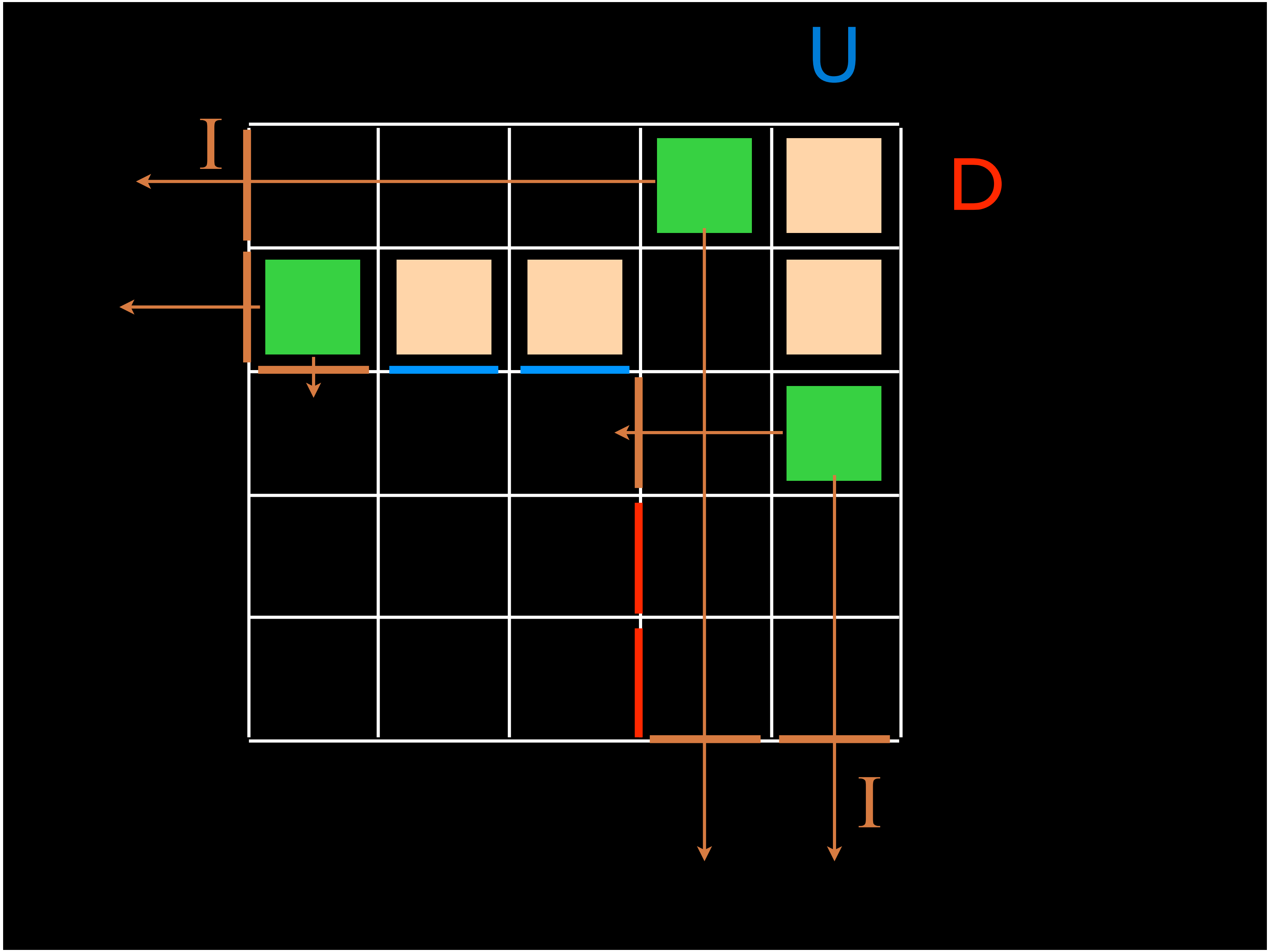
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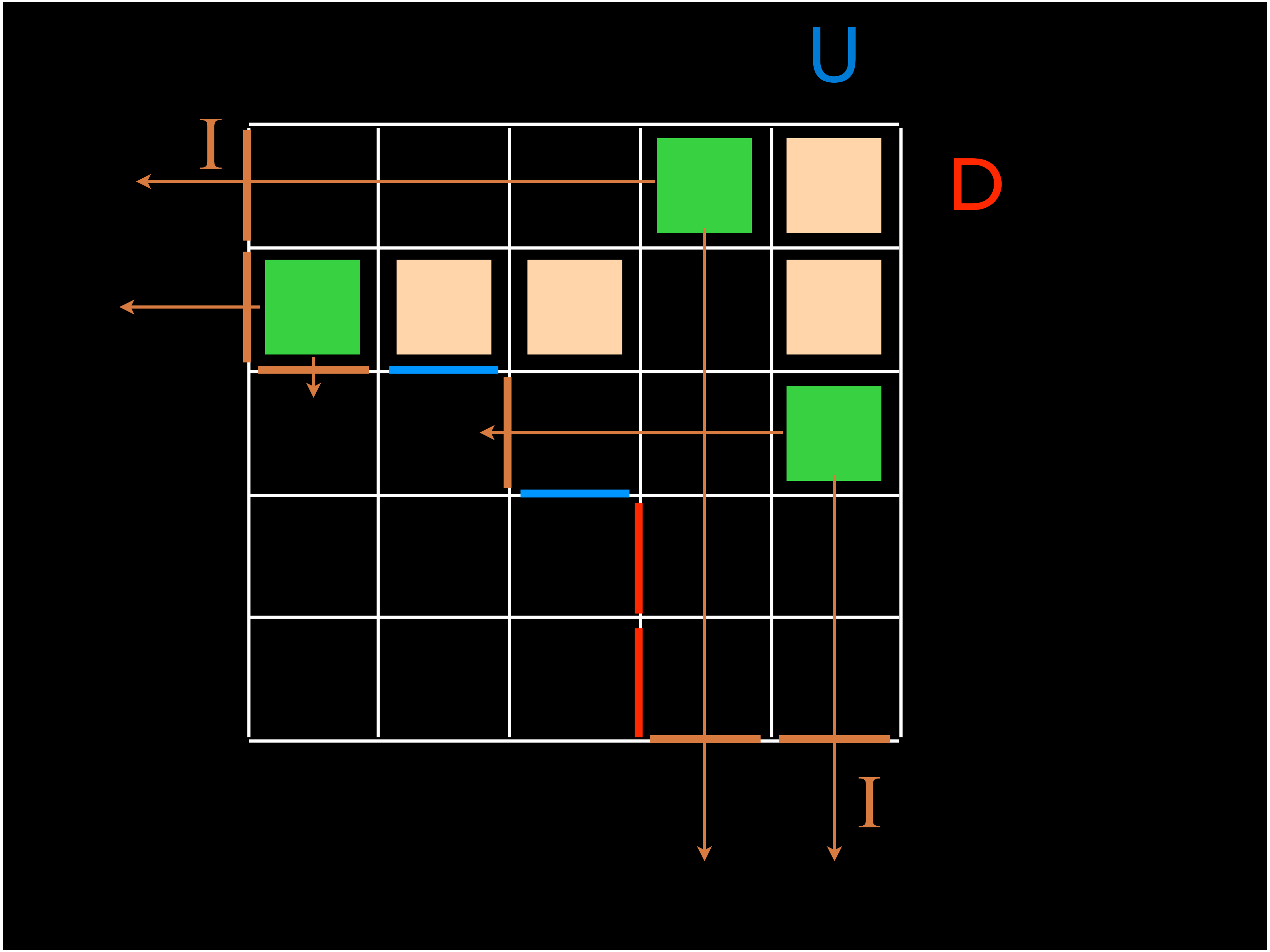
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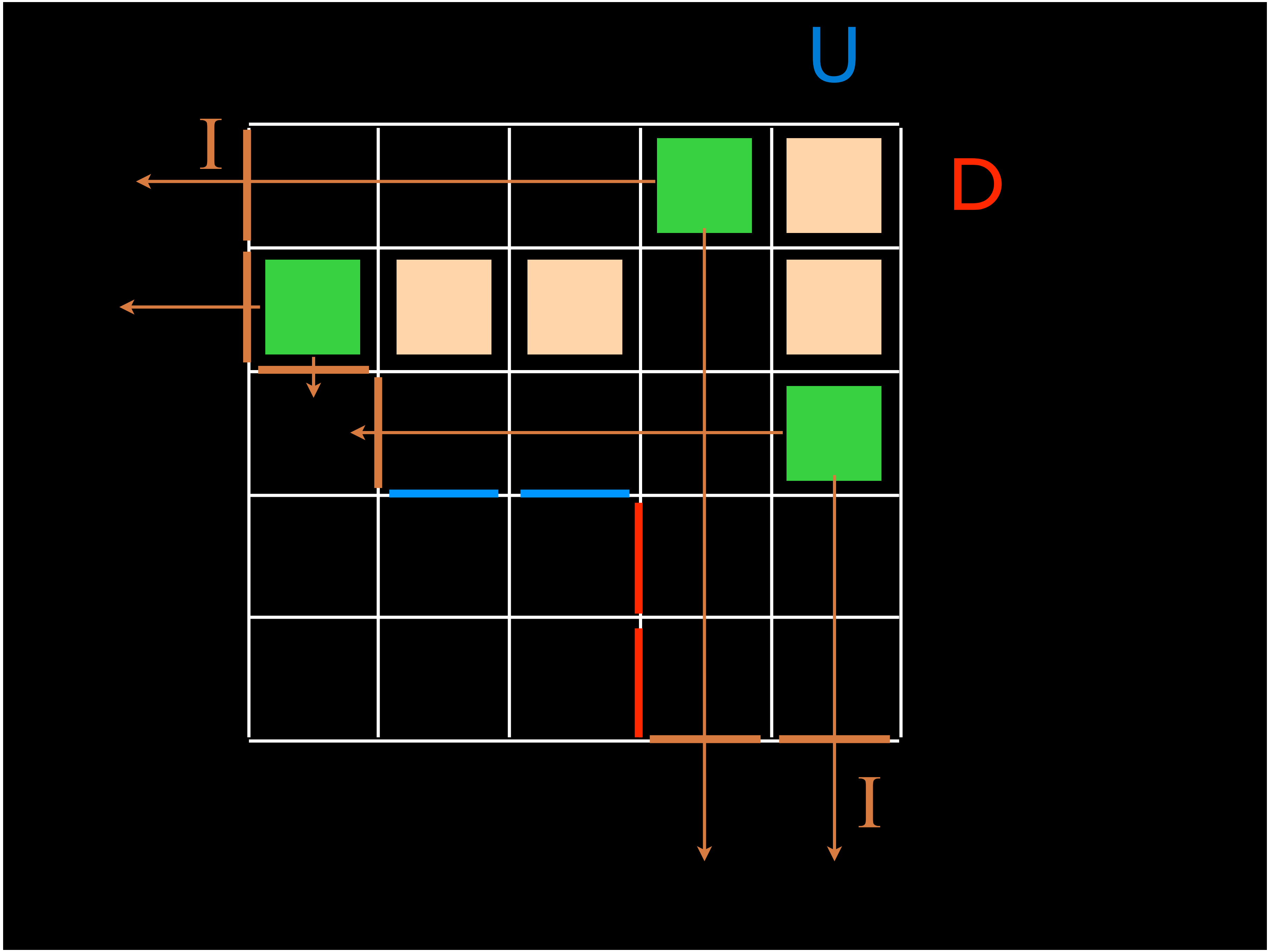
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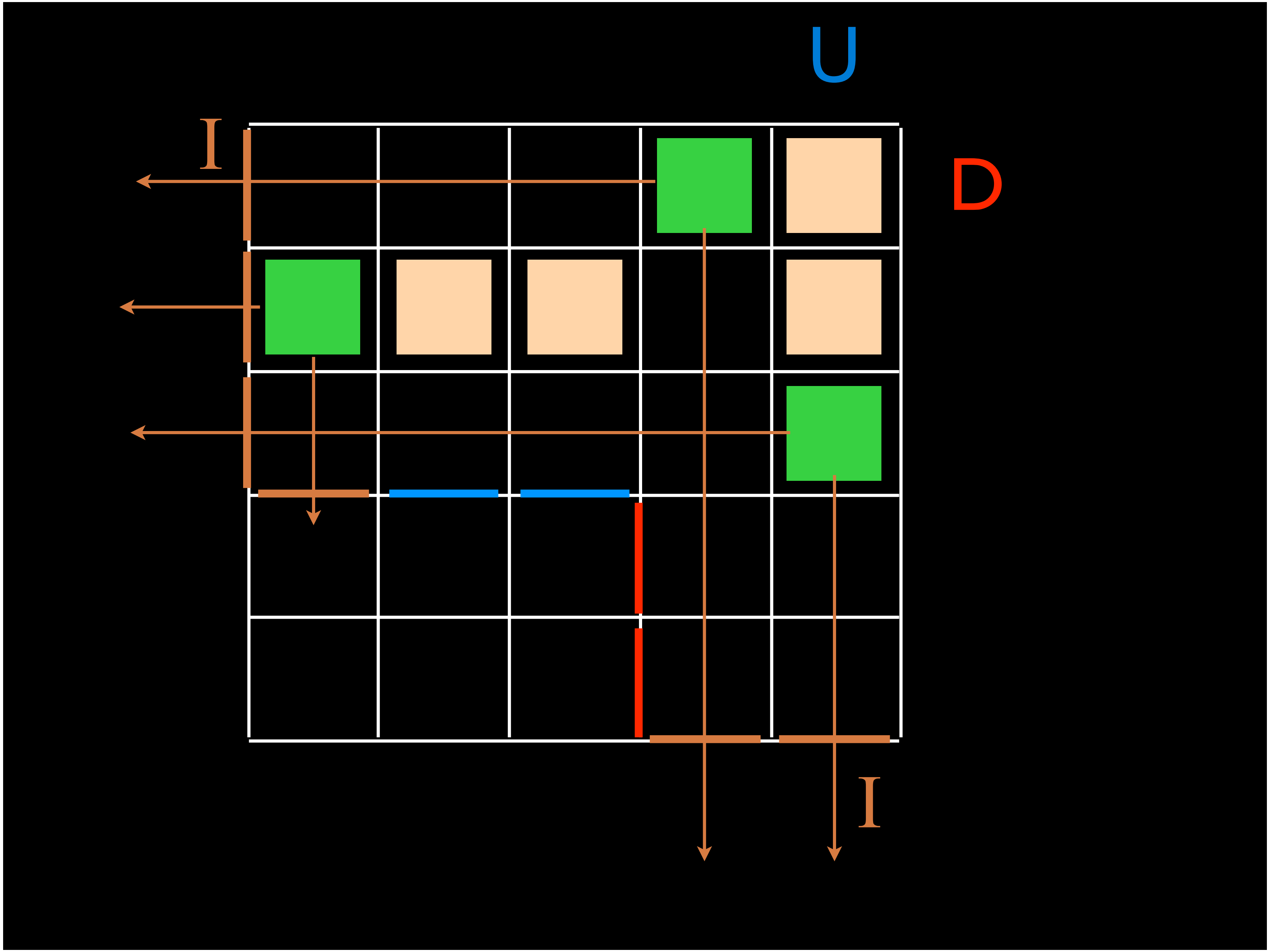

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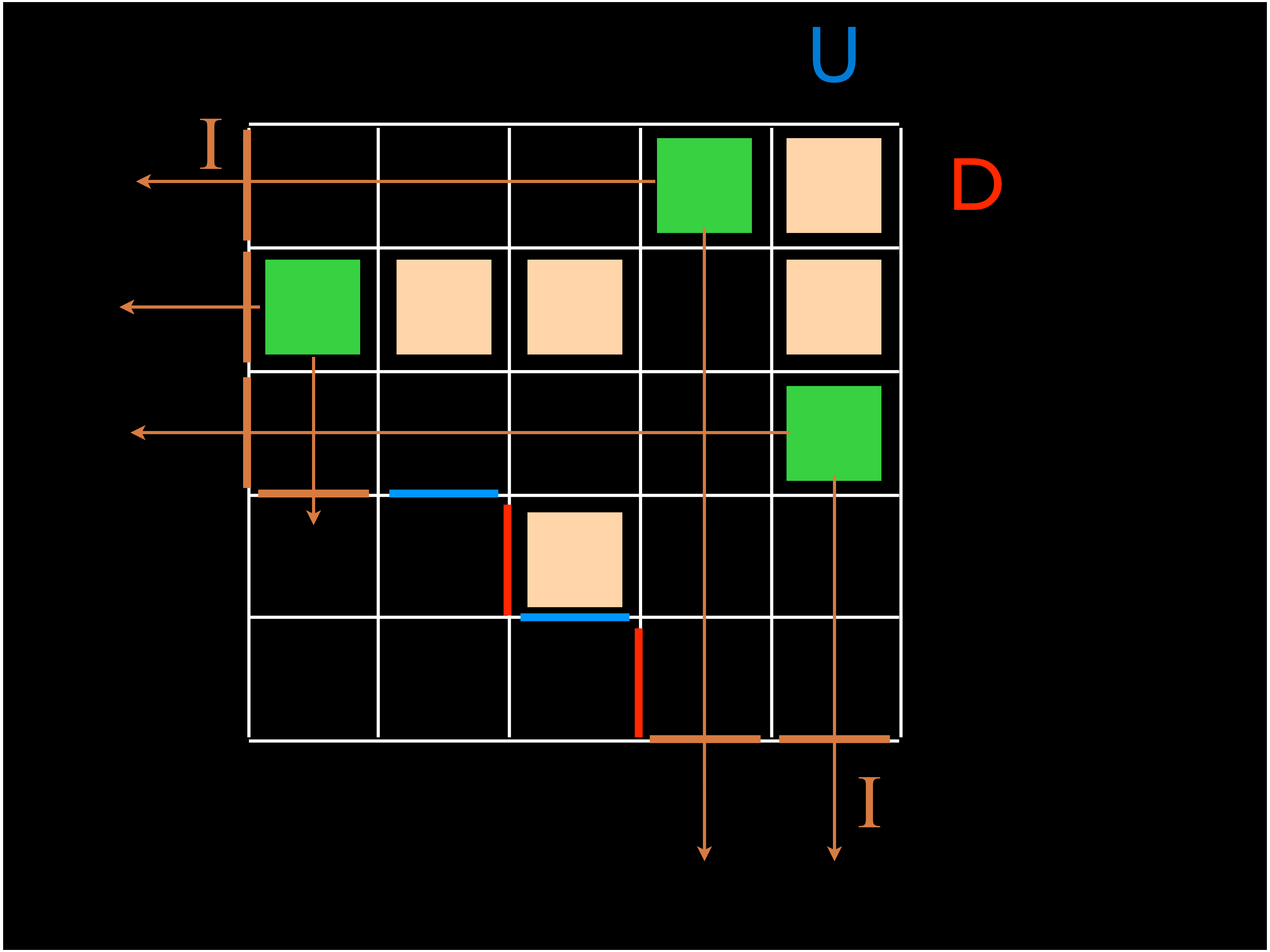
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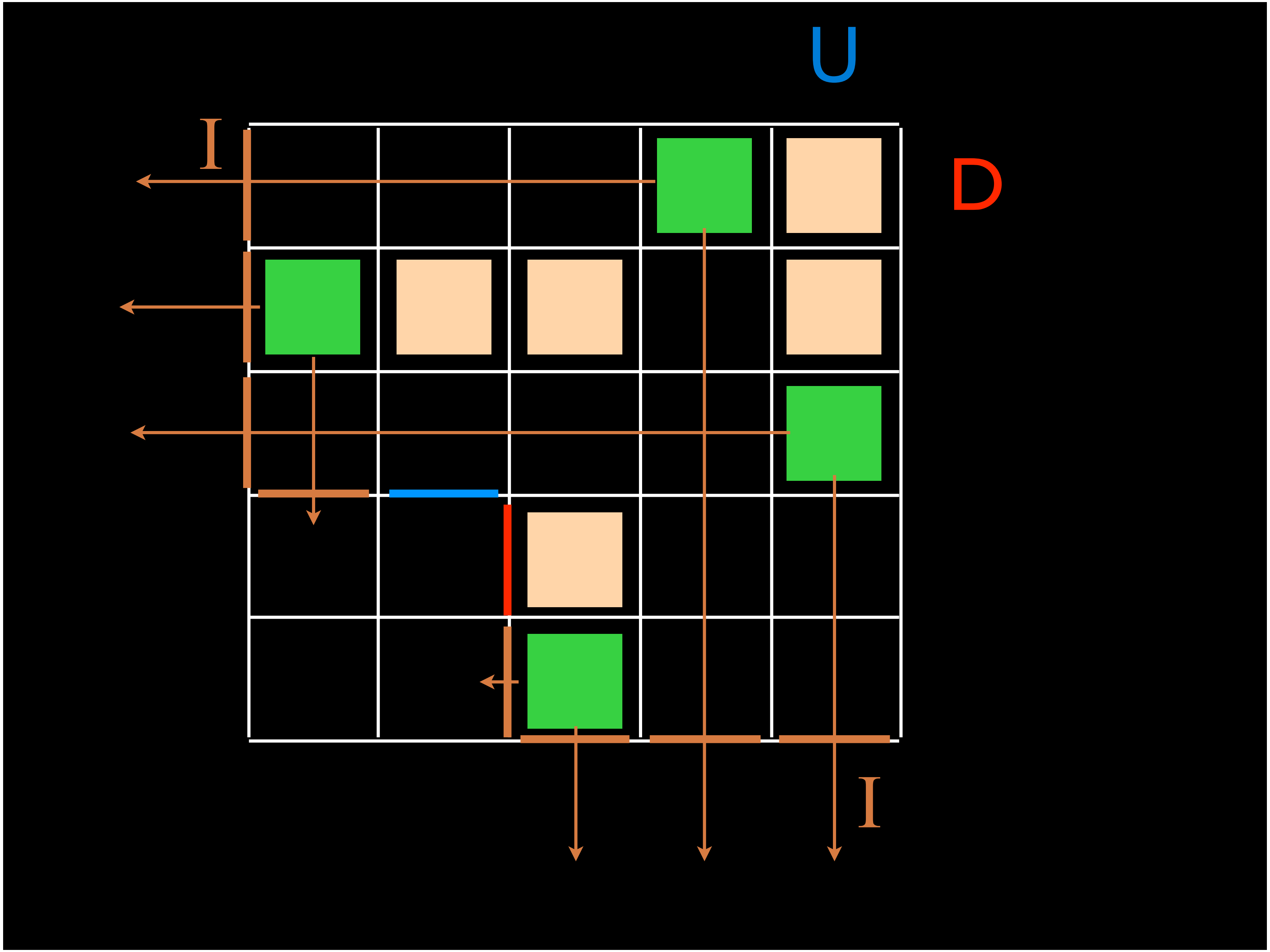

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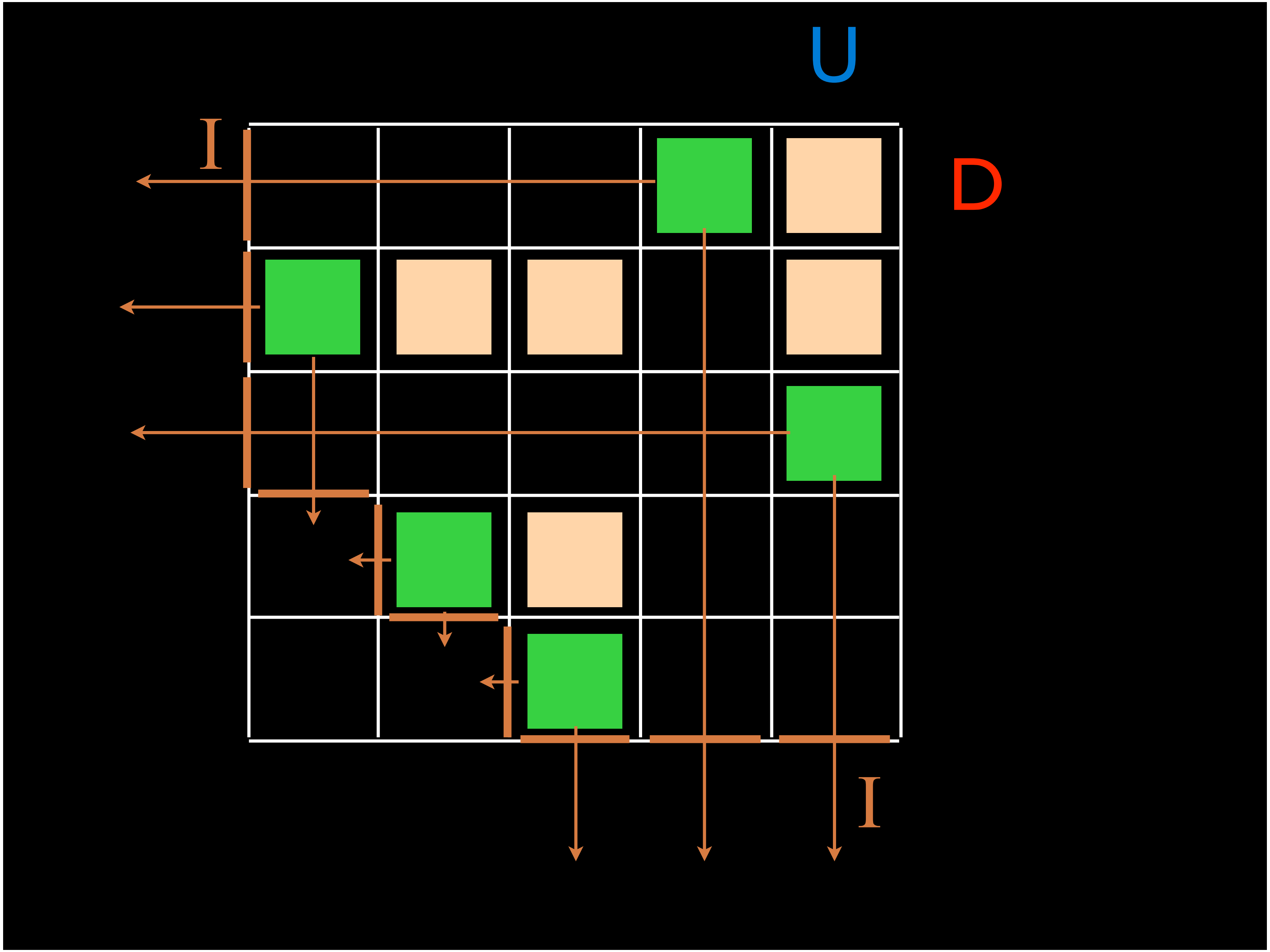
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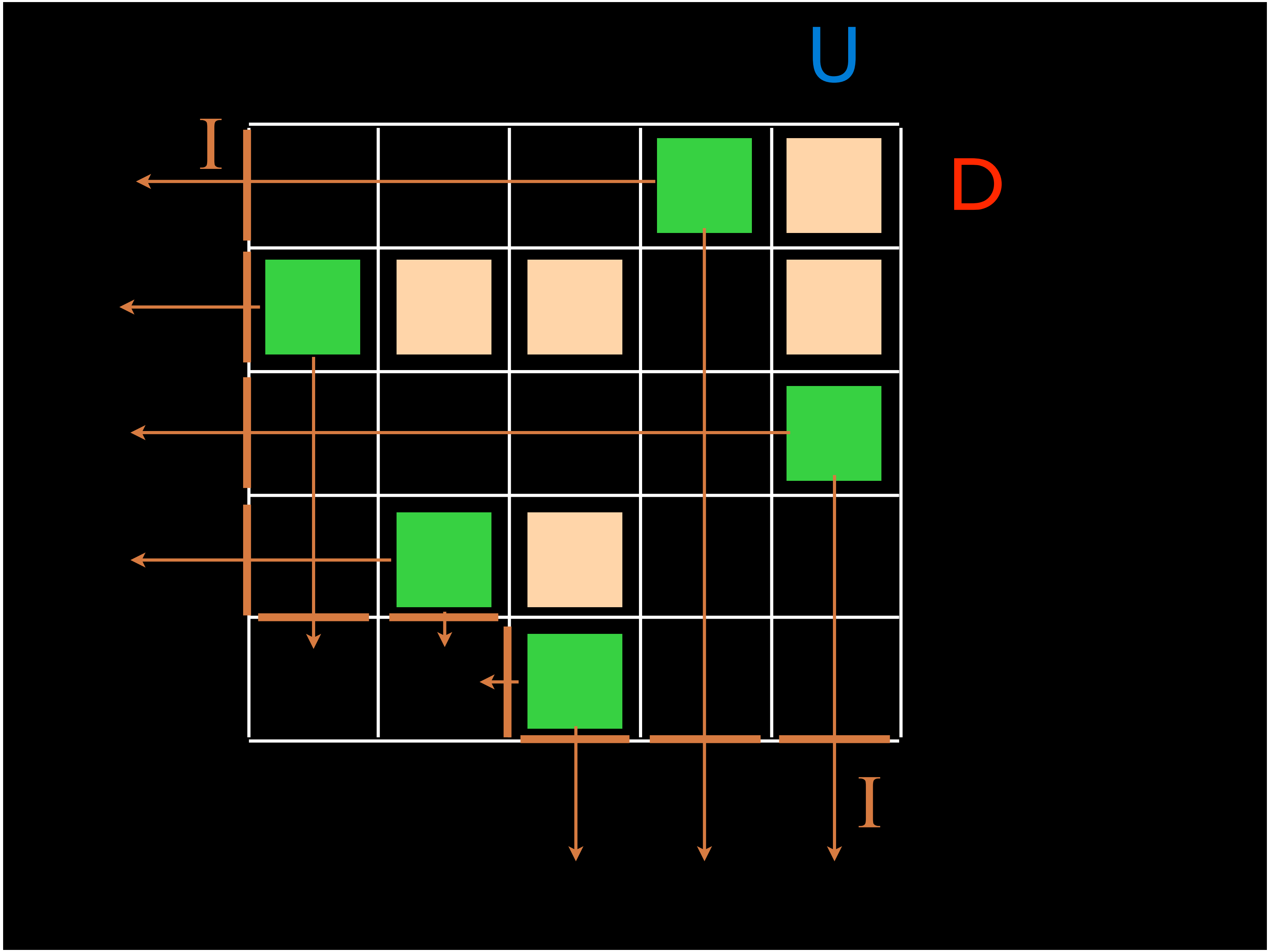

U
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I
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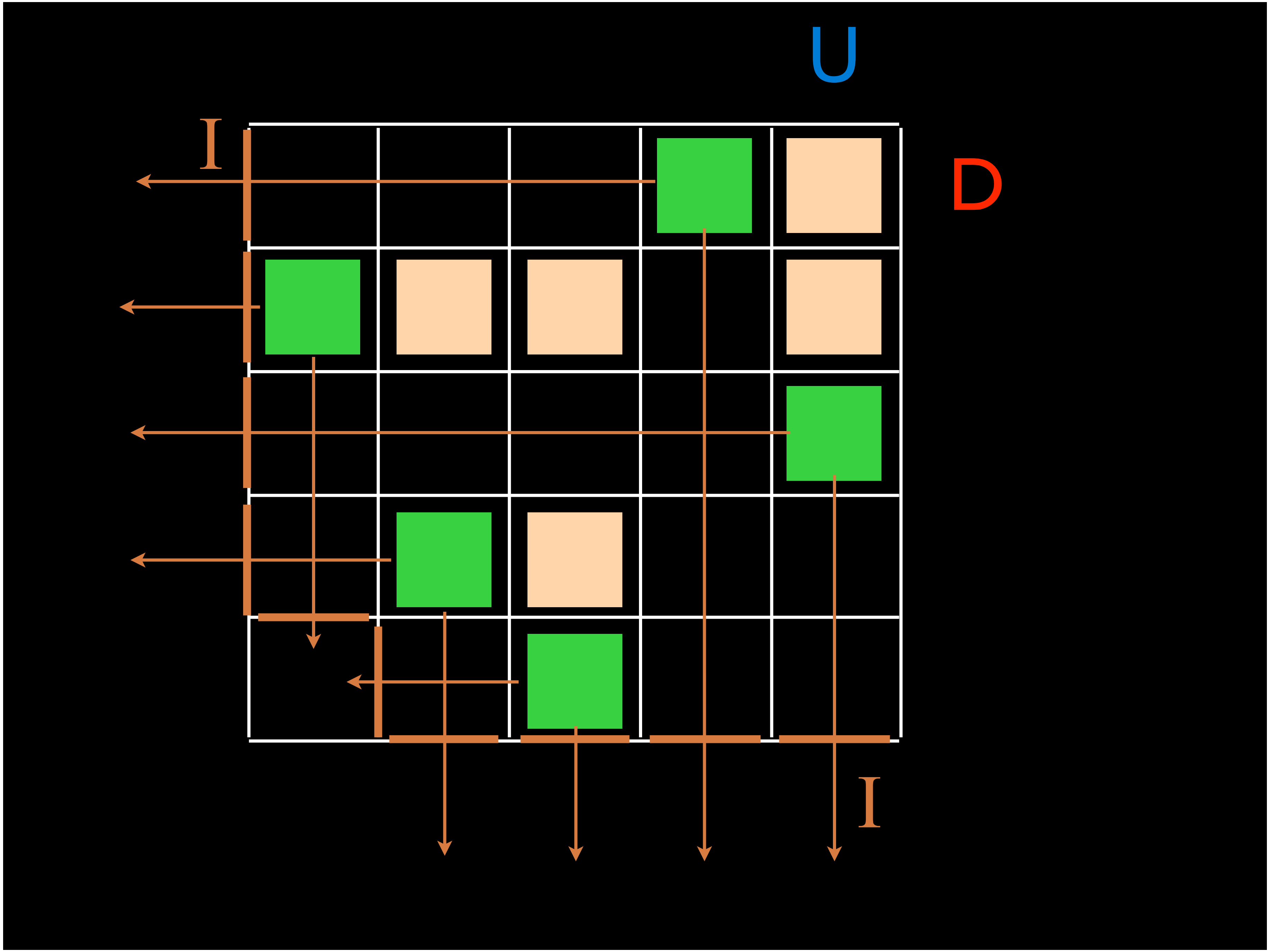

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D
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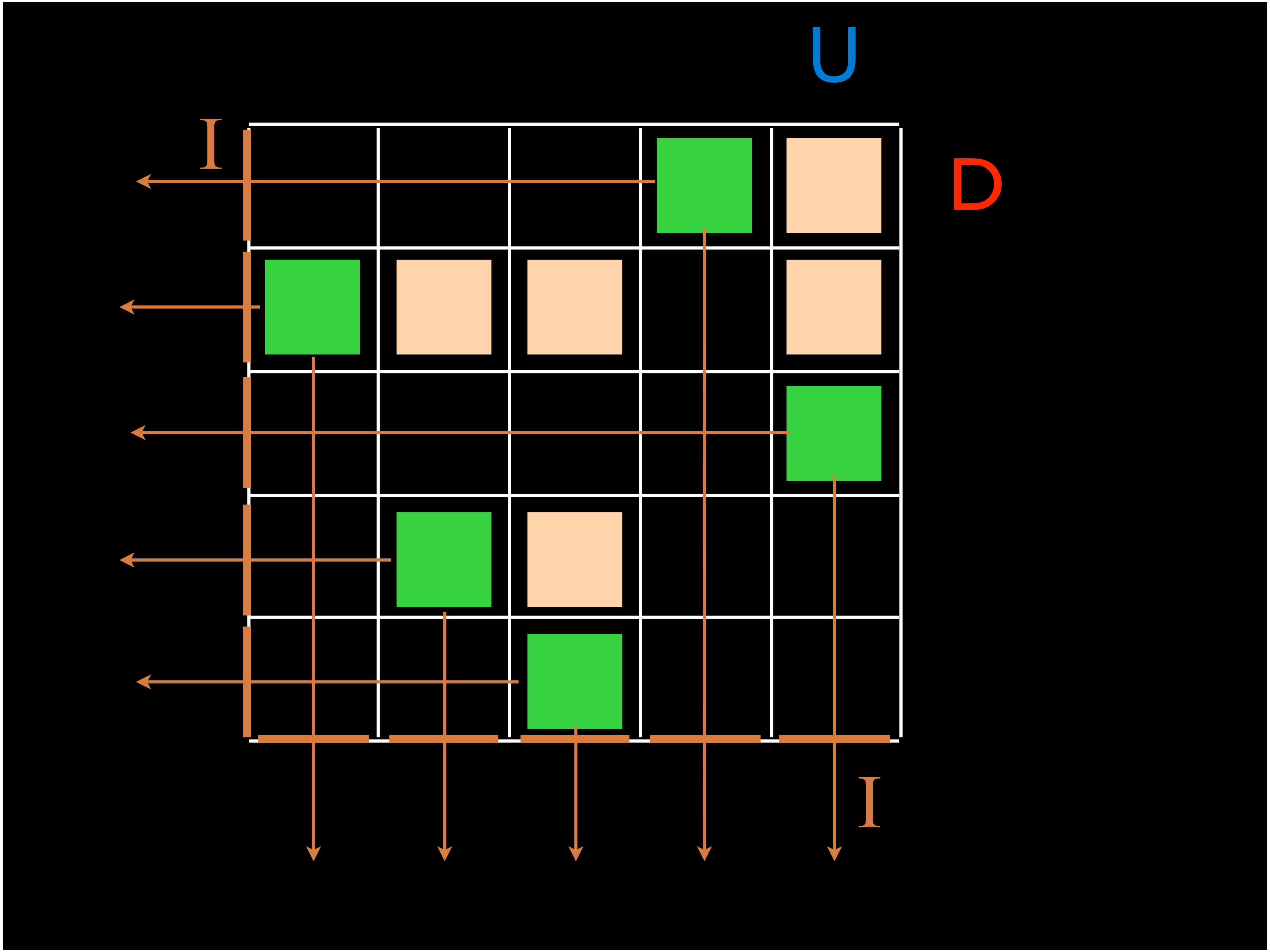
U
D
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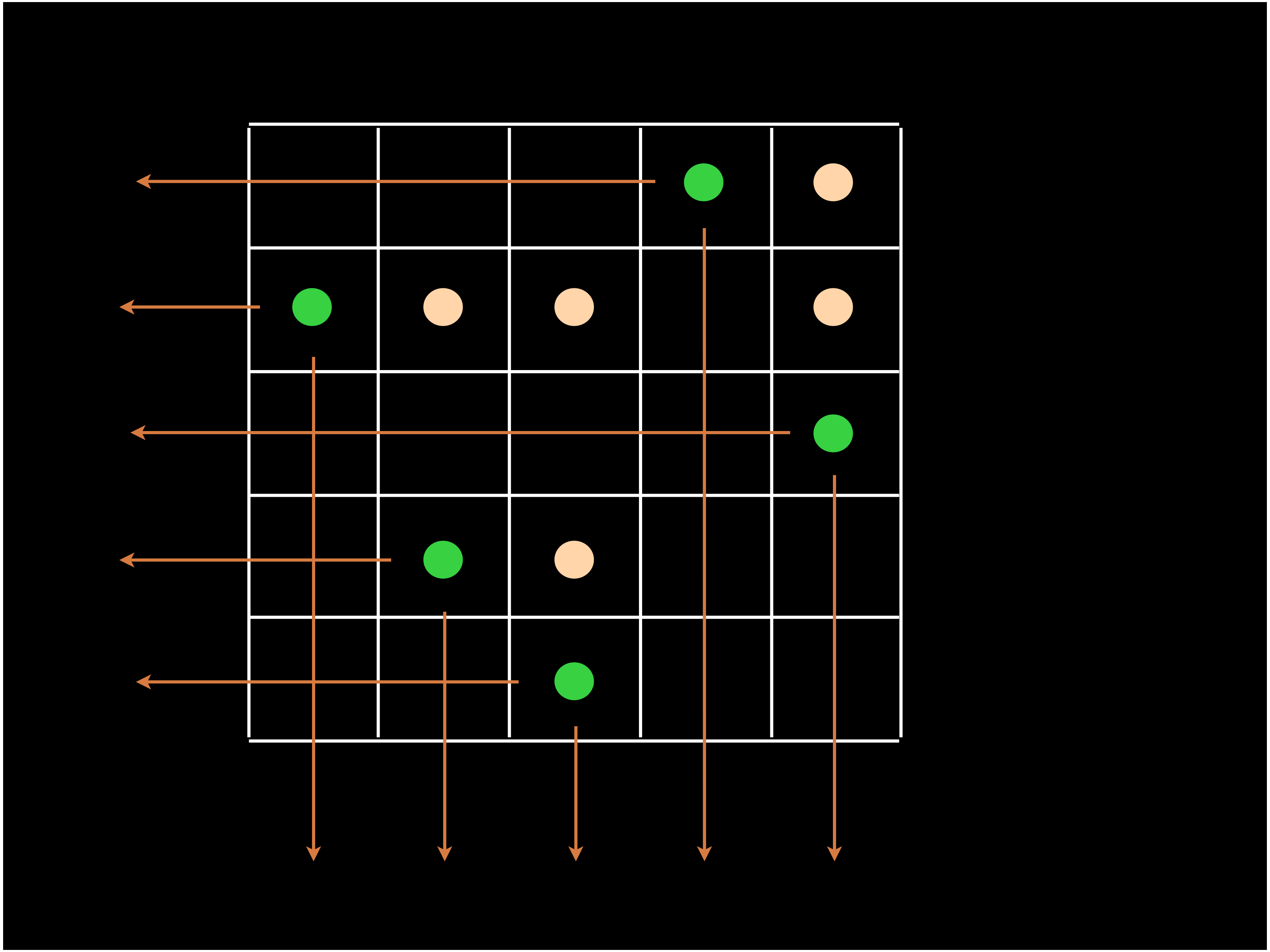
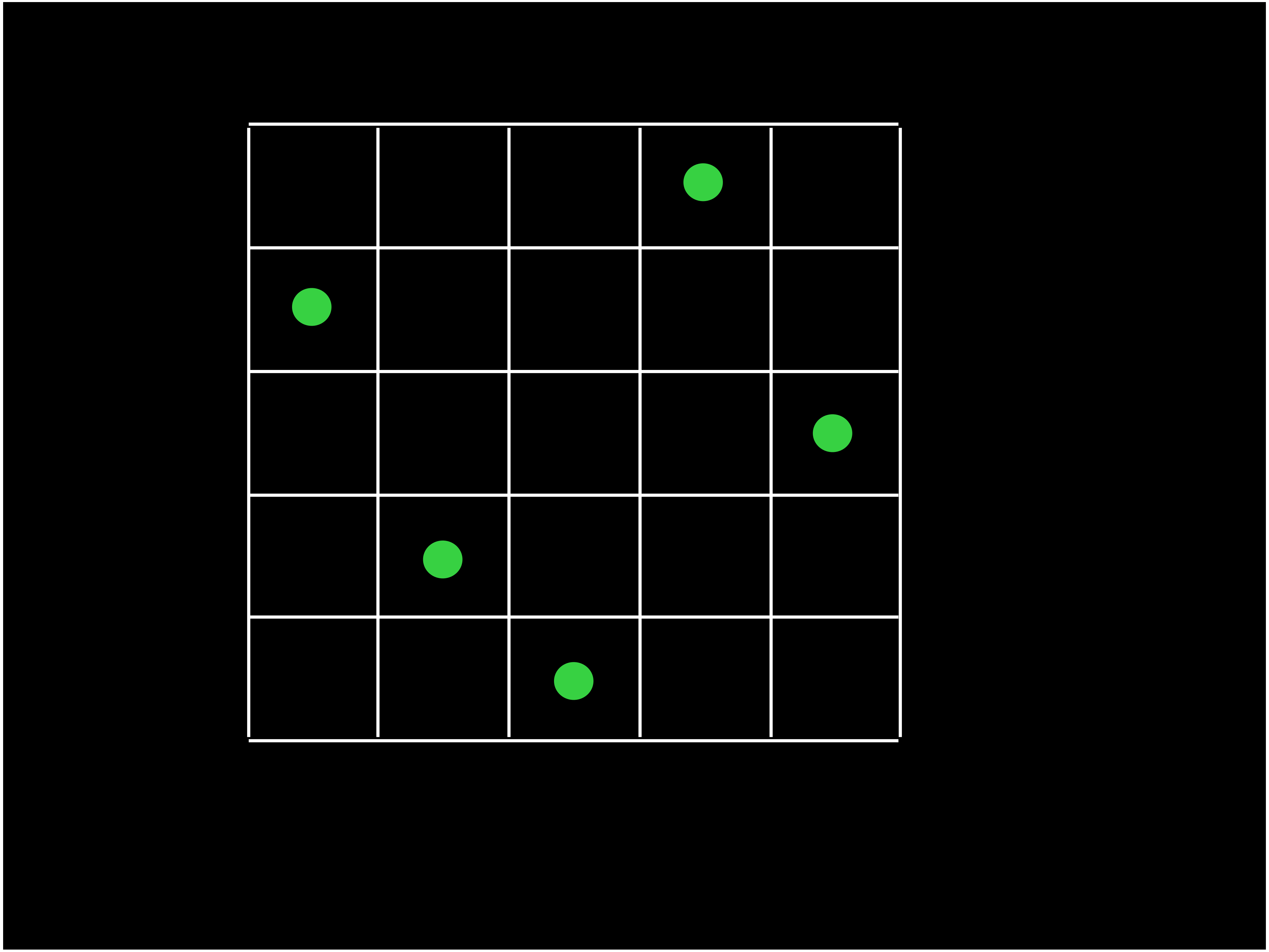
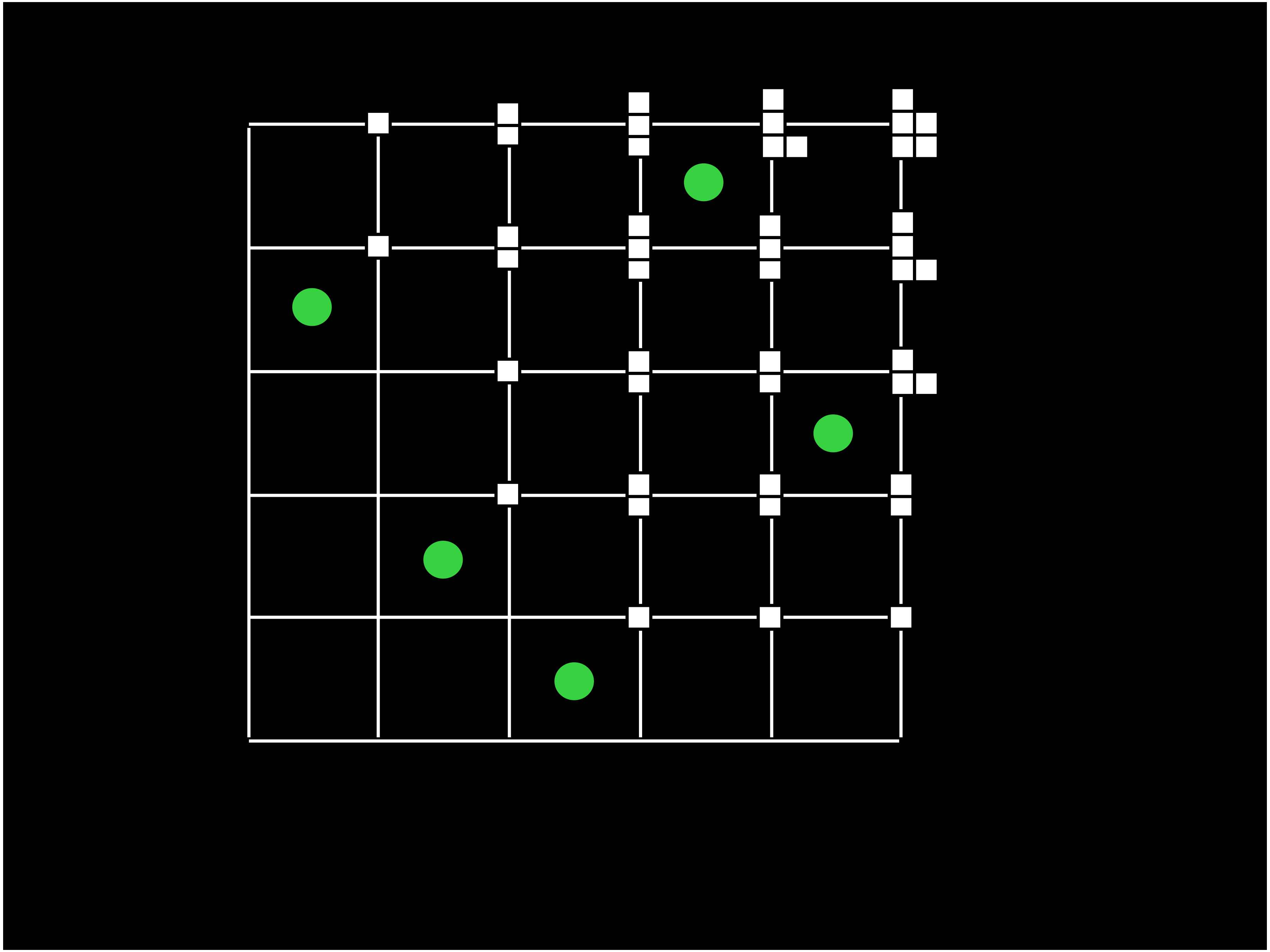
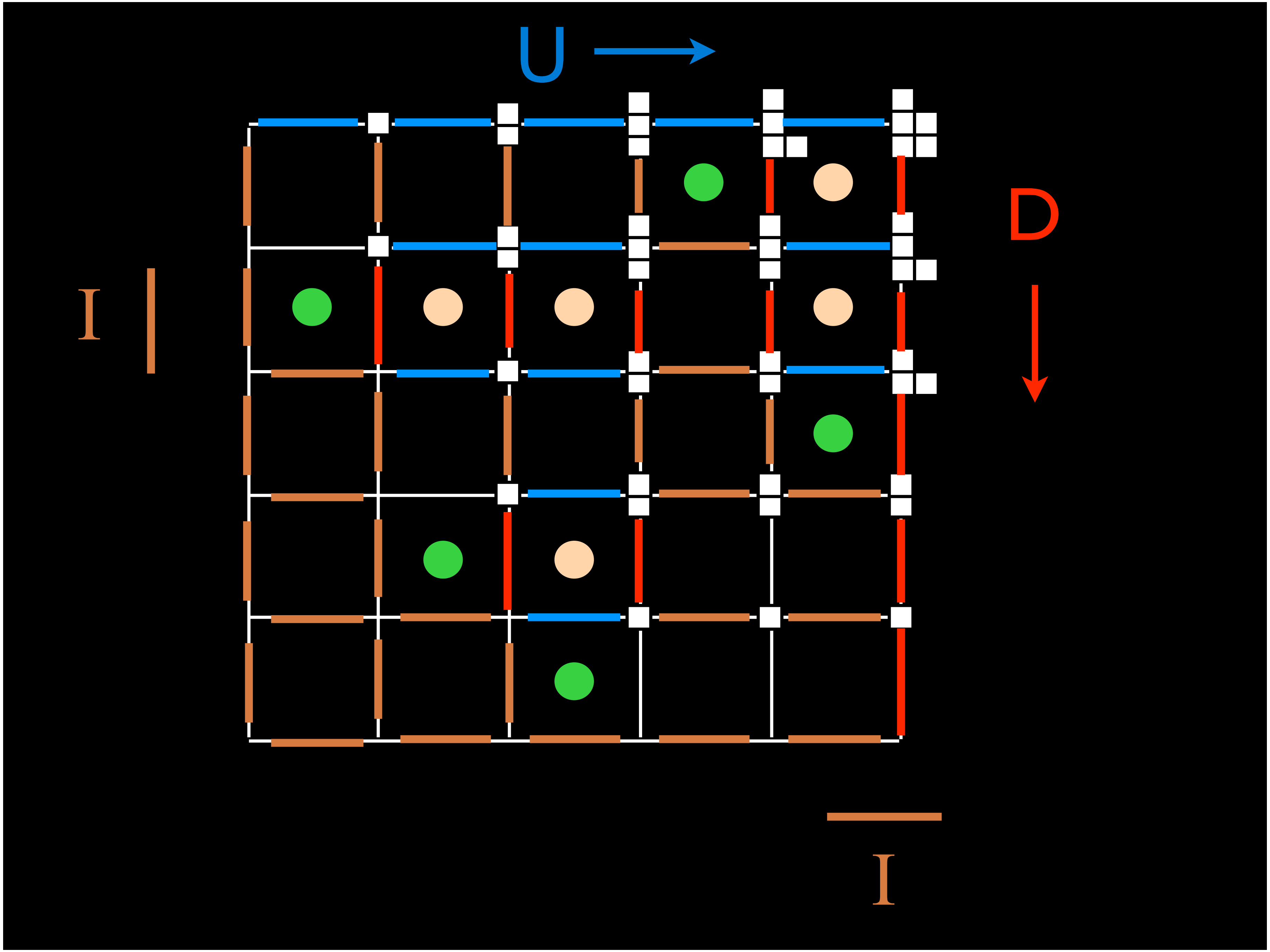
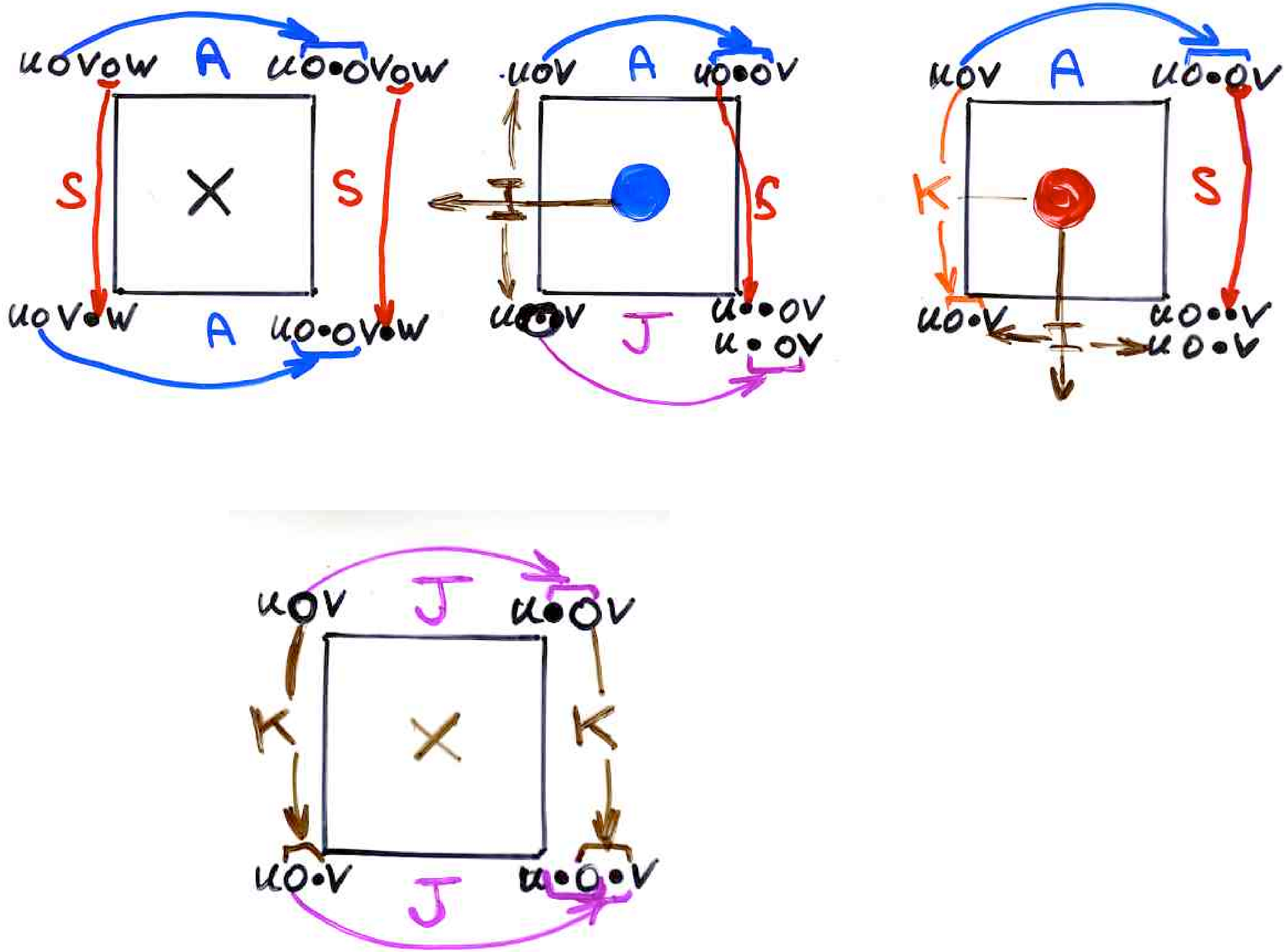




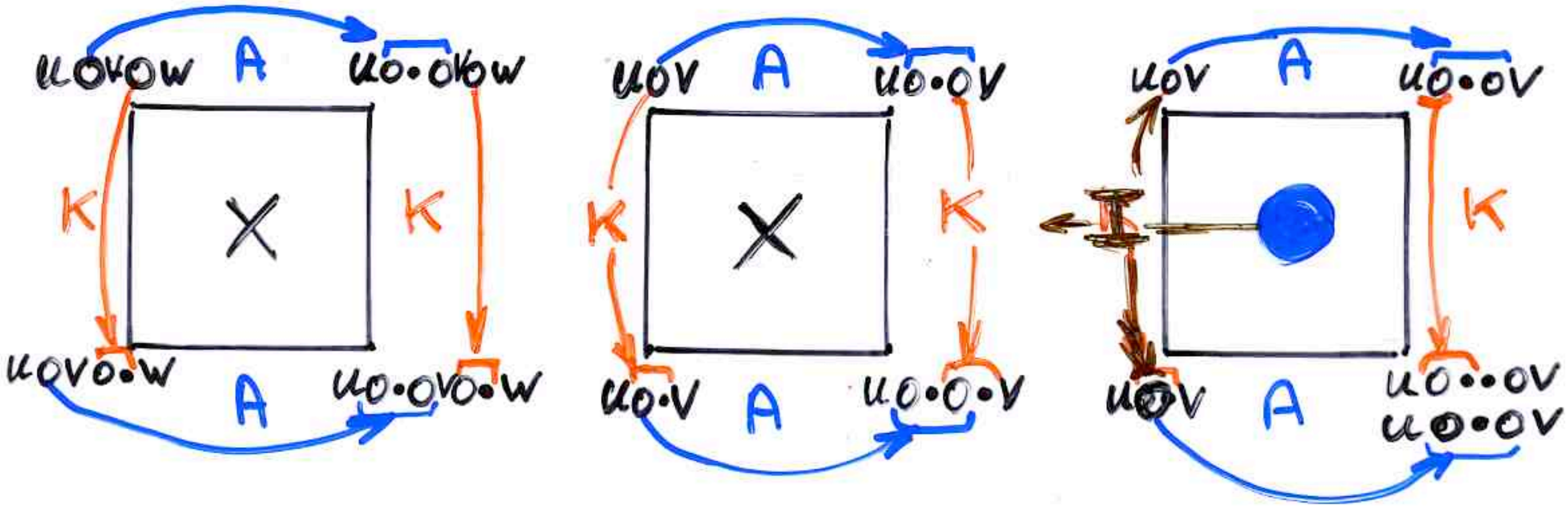


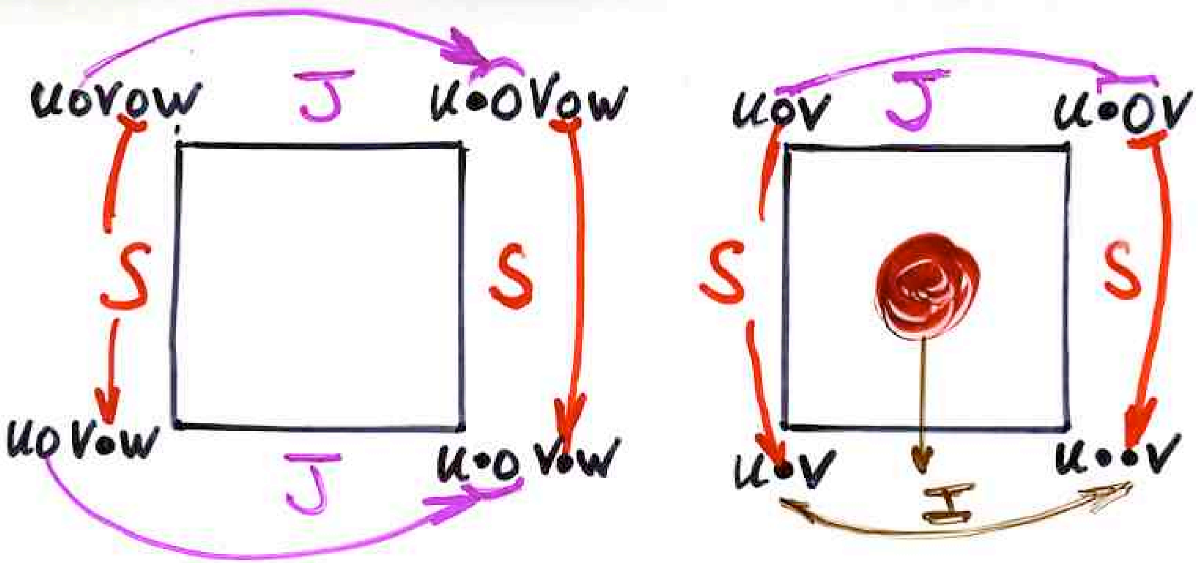

U
D
I
I

§ 10 another bijection permutations alternative tableaux



A
S
S
A
A
S
J
A
K
S
J
K
K
J


A
K
A
K
A
K
A
A
K
A
K


J
S
S
J
J
S
S

[redacted]

416978352

●

● 1 ●

●1● 2 ●

●1●3●2

41● 3 ●2

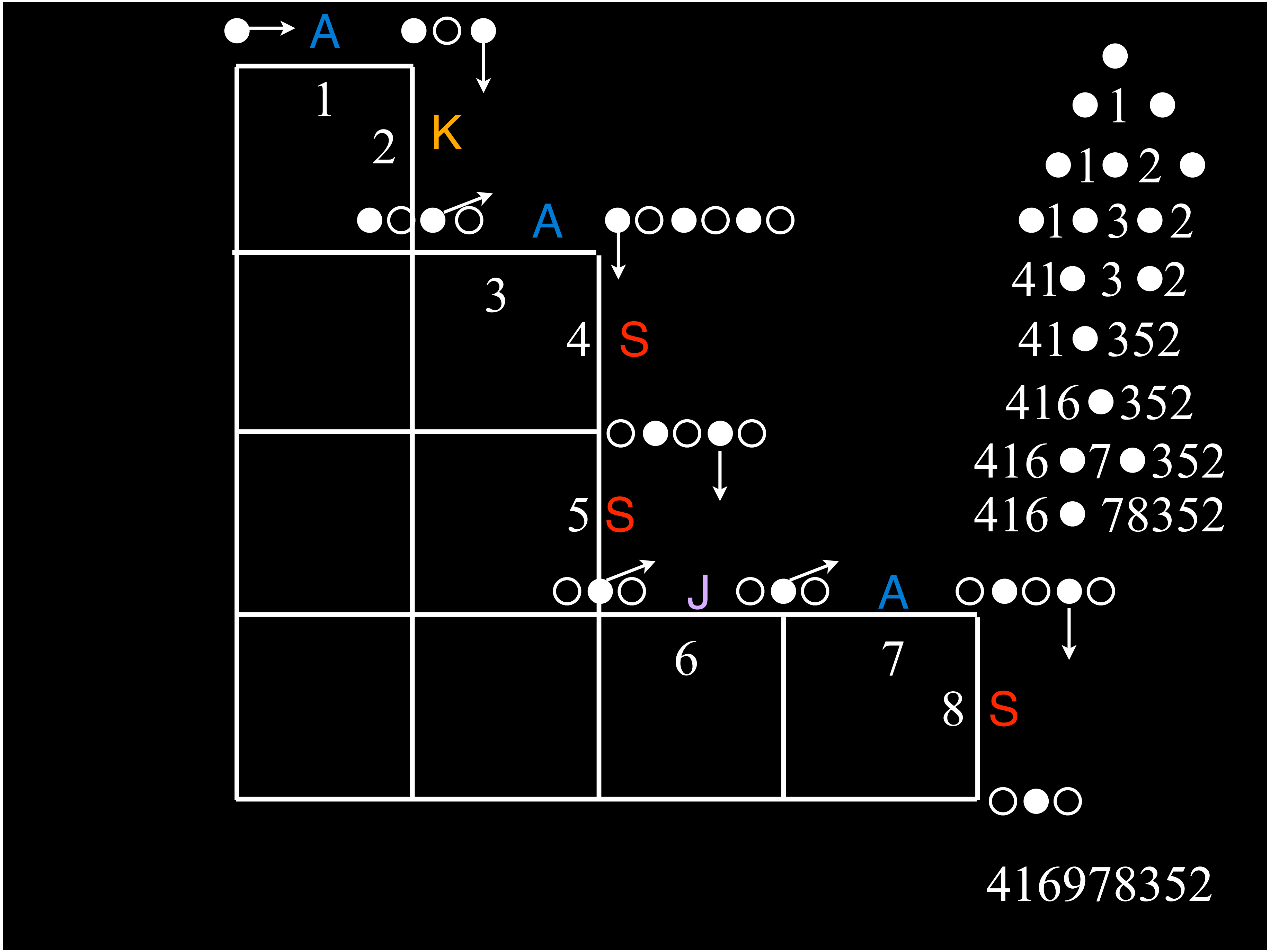
41●352

416●352

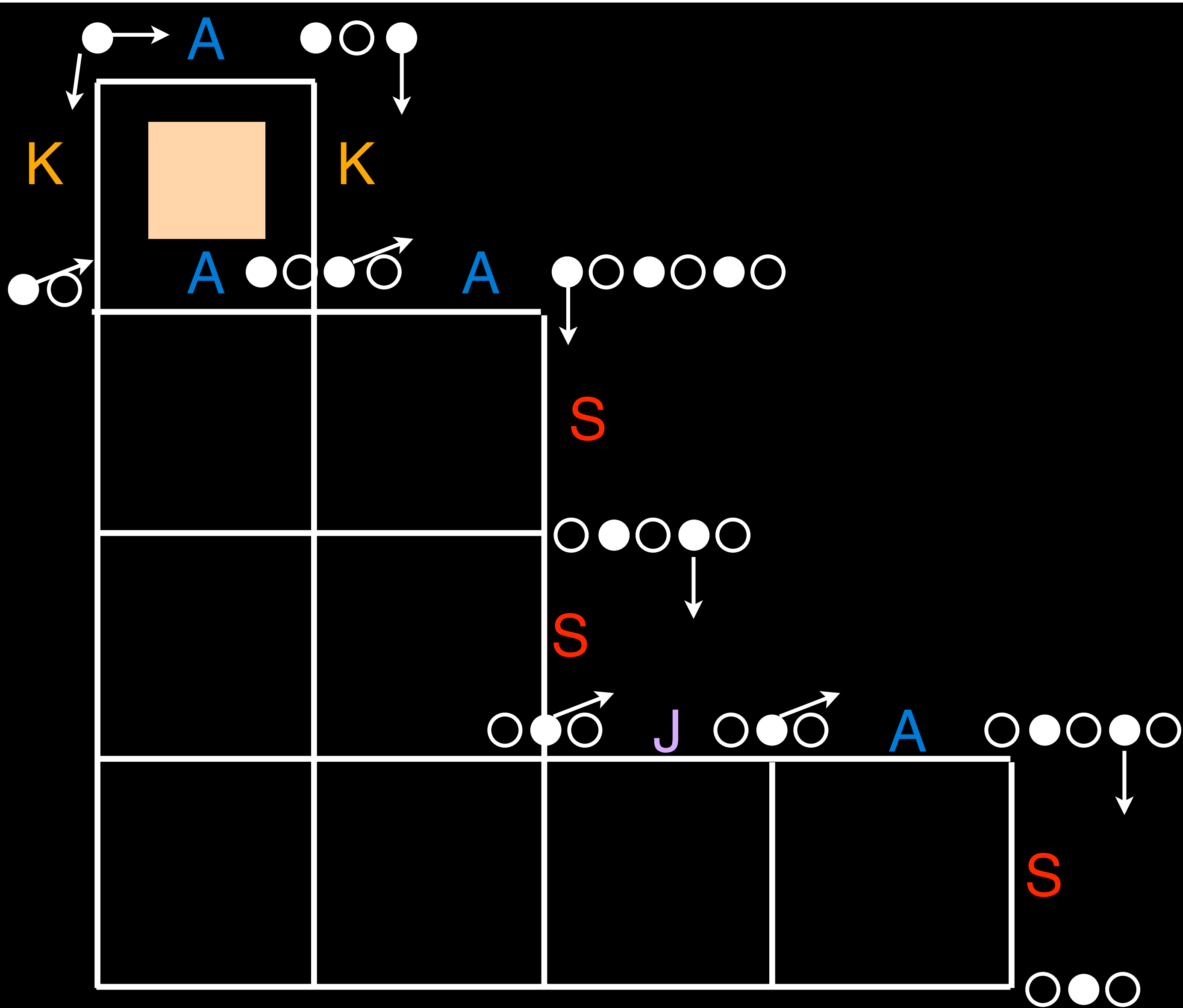
416 ●7●352

416 ● 78352

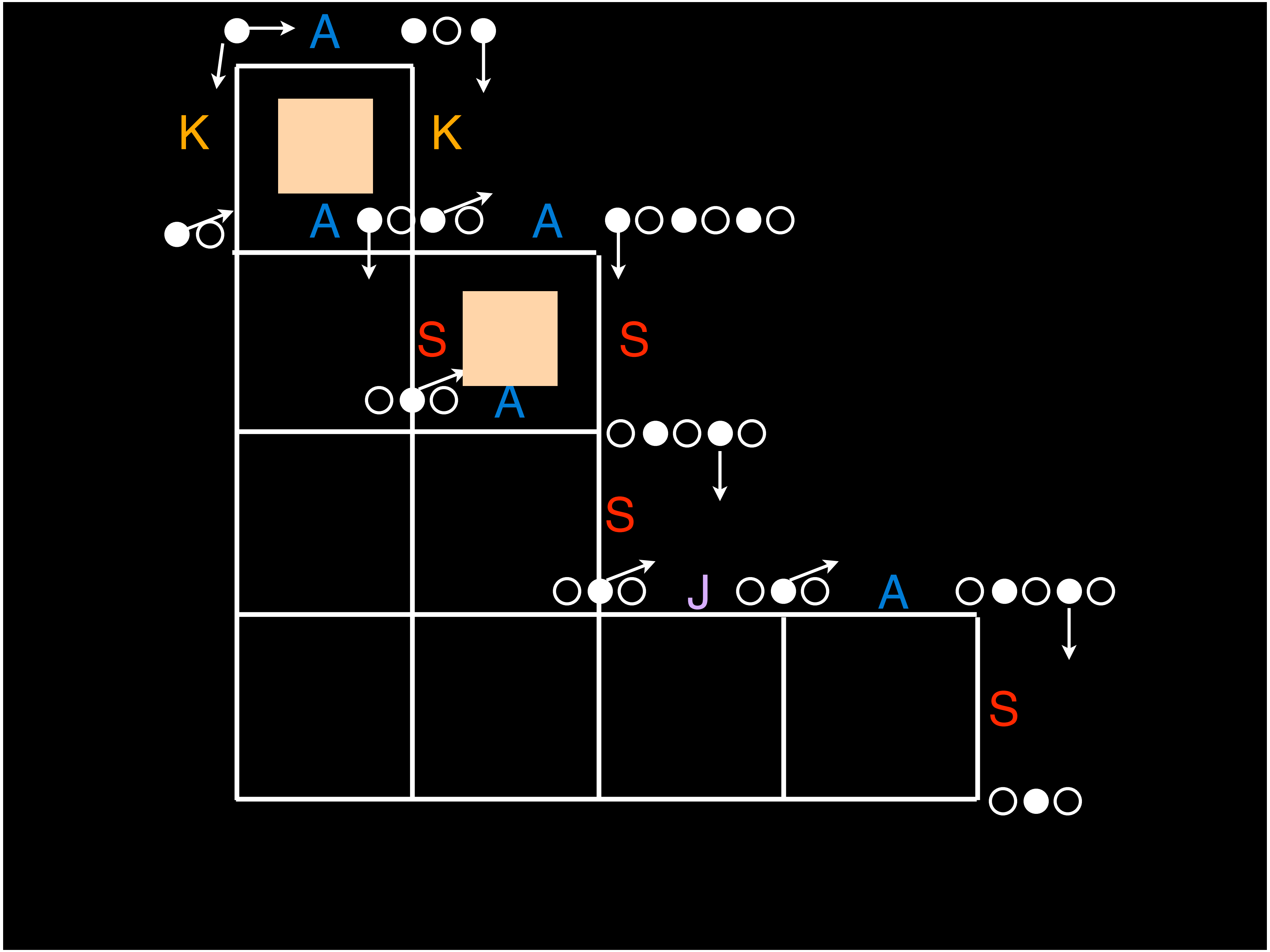
416978352



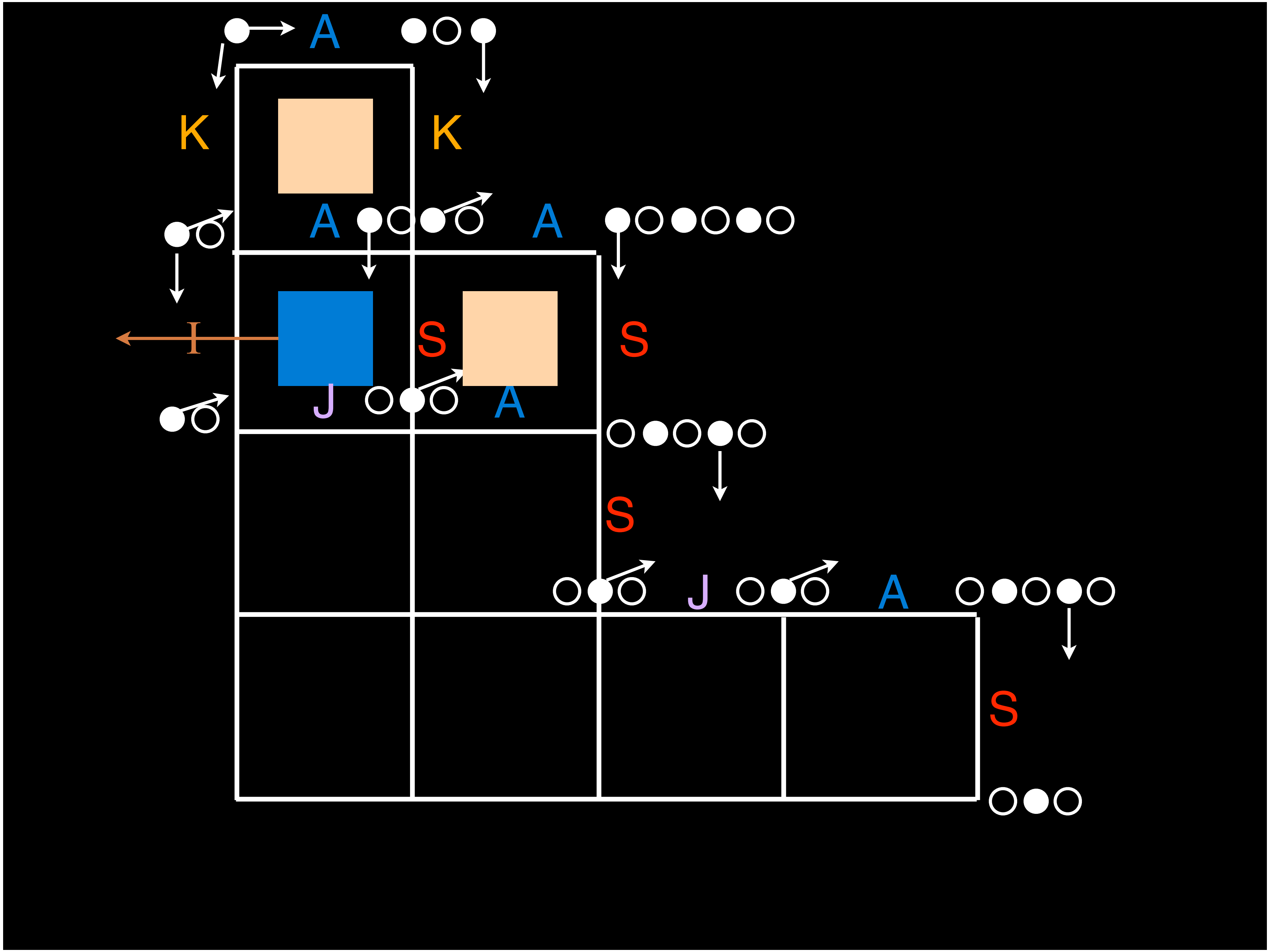
A
1
2
K
A
3
4
S
5
S
J
A
6
7
8
S
1
1 2
1 3 2
41 3 2
41 352
416 352
416 7 352
416 78352
416978352



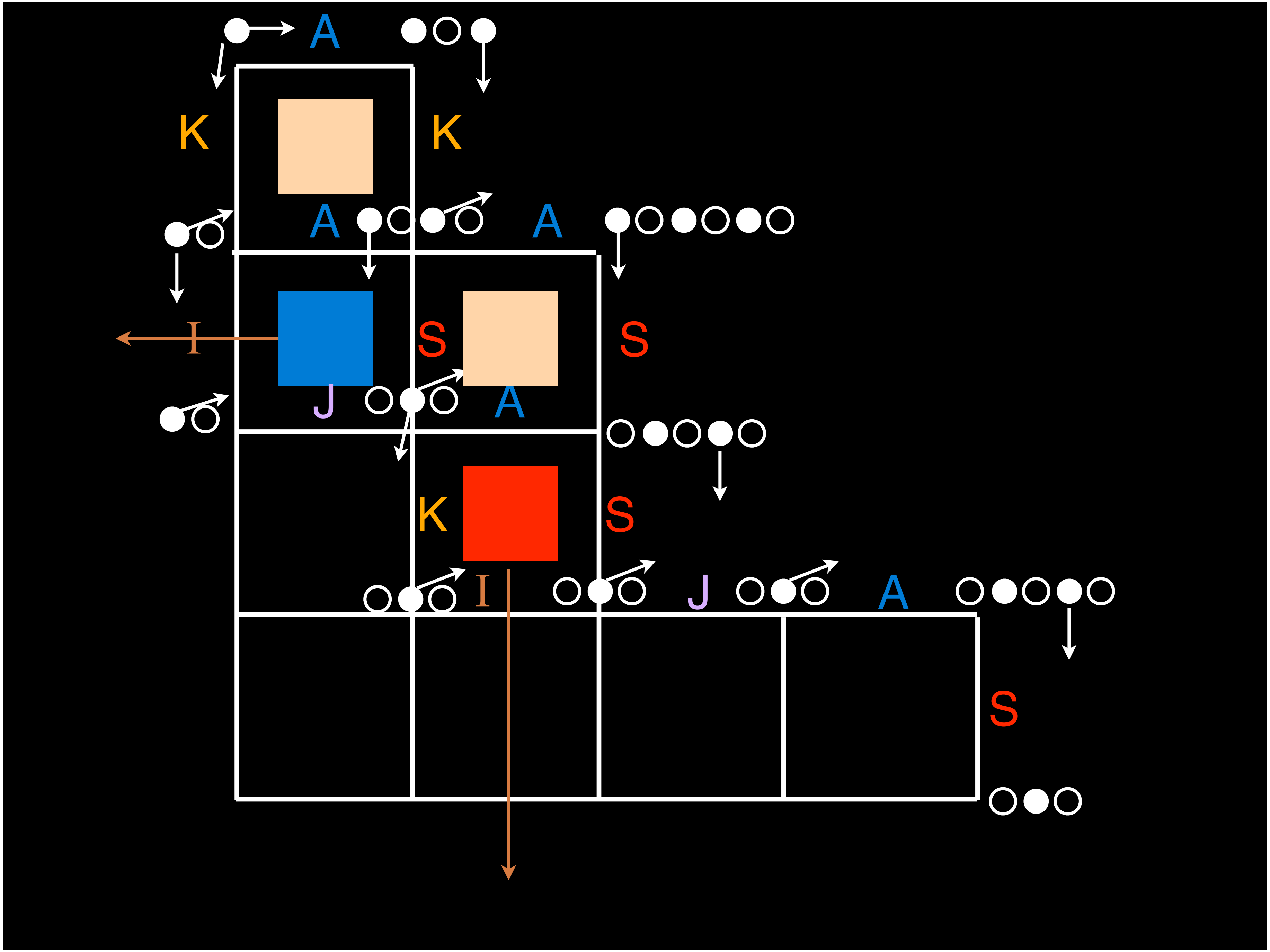
A
K
K
A
A
S
S
J
A
S

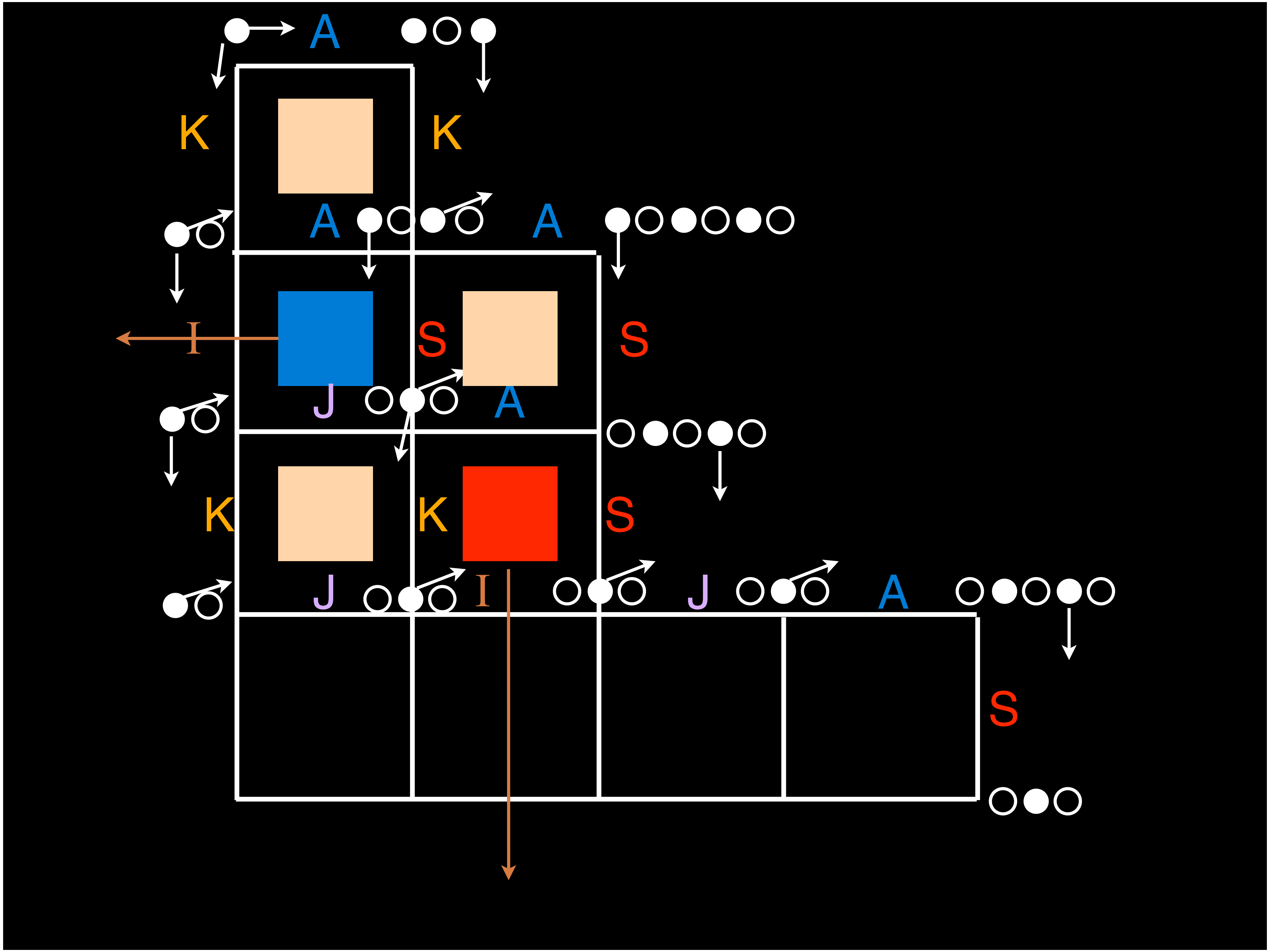


A
K
K
A
A
S
S
A
S
J
A
S



A
K
K
A
A
S
S
J
A
S
J
A
S


A
K
K
A
A
I
S
S
J
A
K
S
I
J
A
S


A
K
K
A
A
I
S
S
J
A
K
K
S
J
I
J
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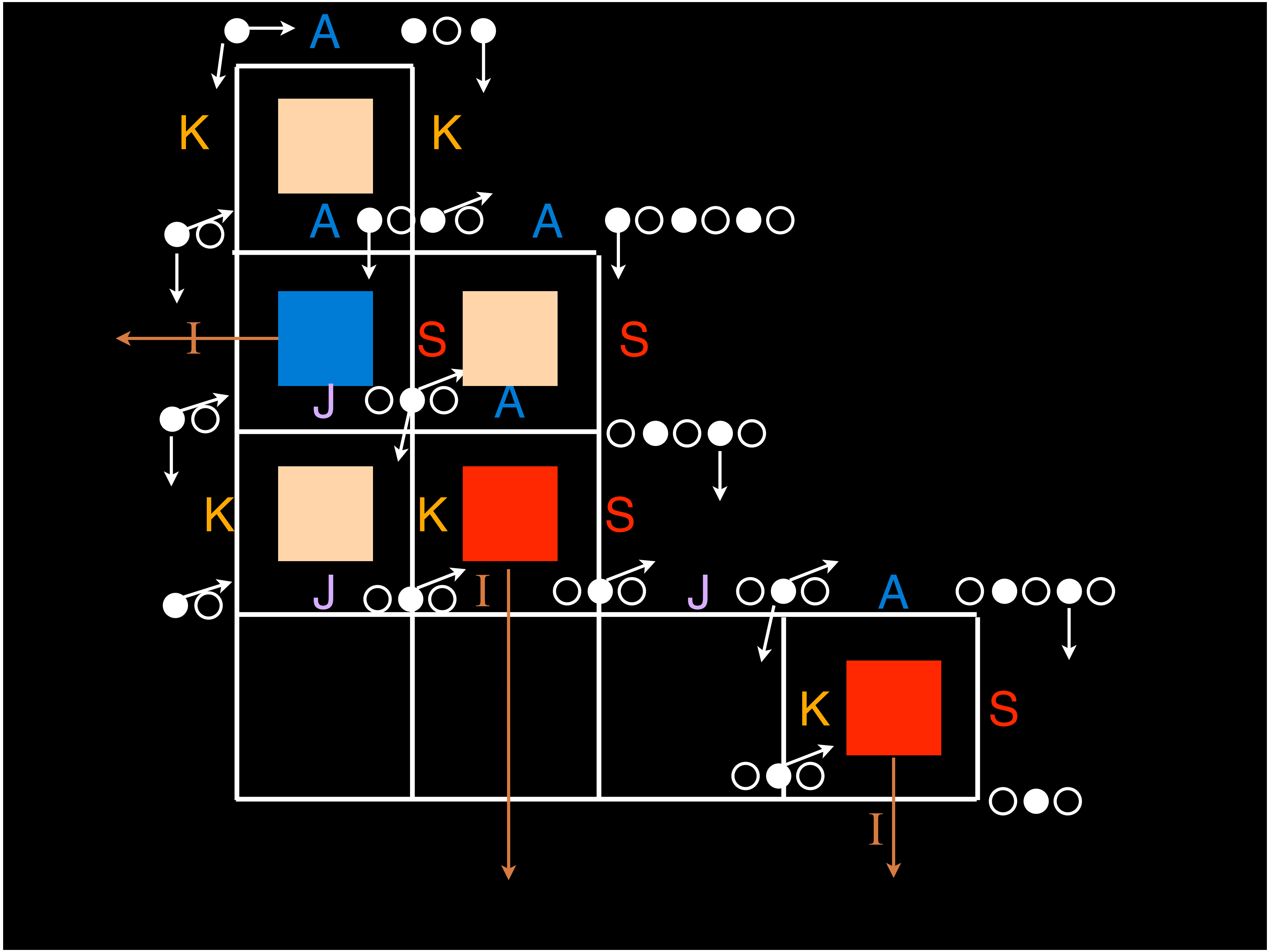

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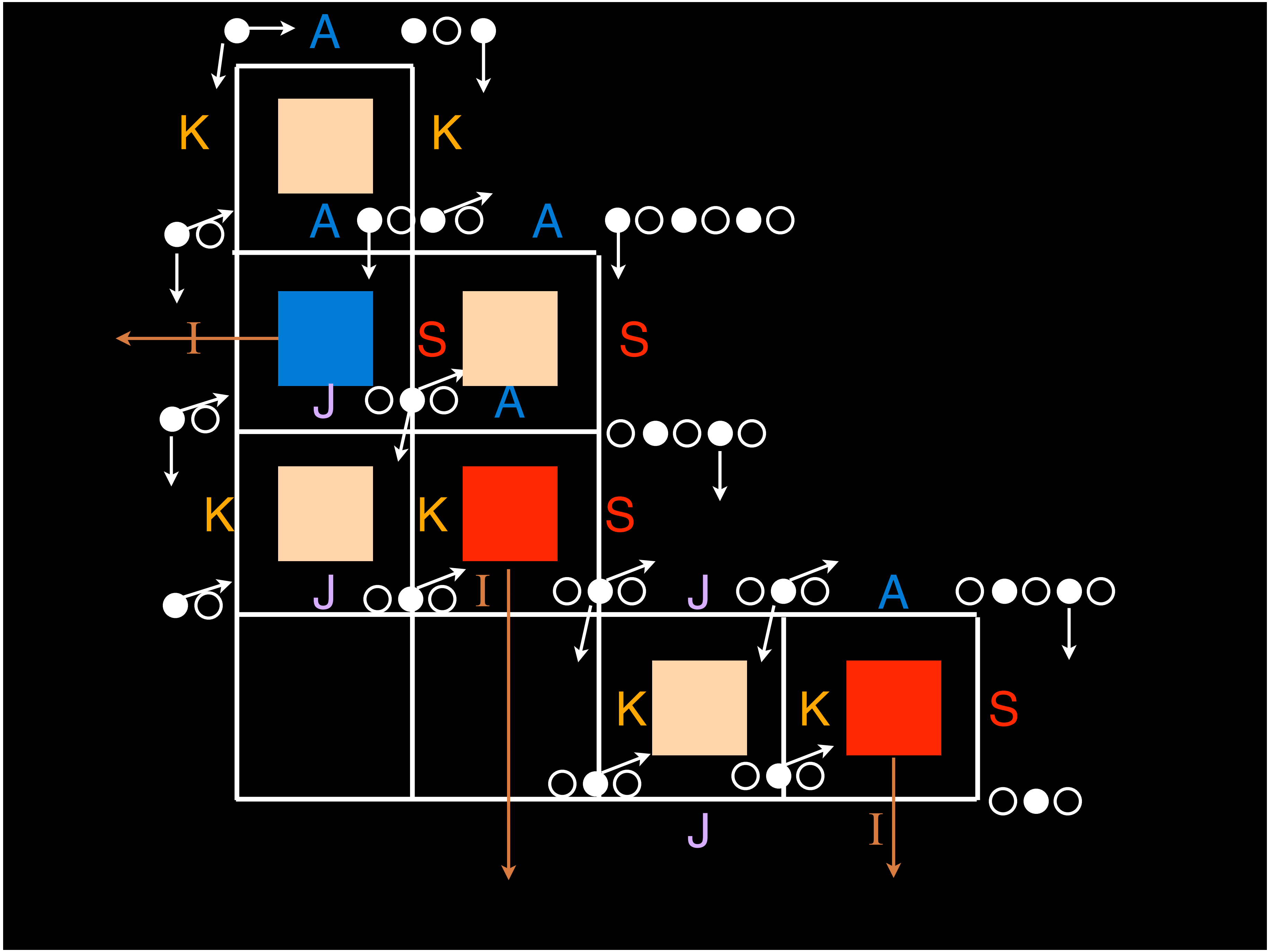

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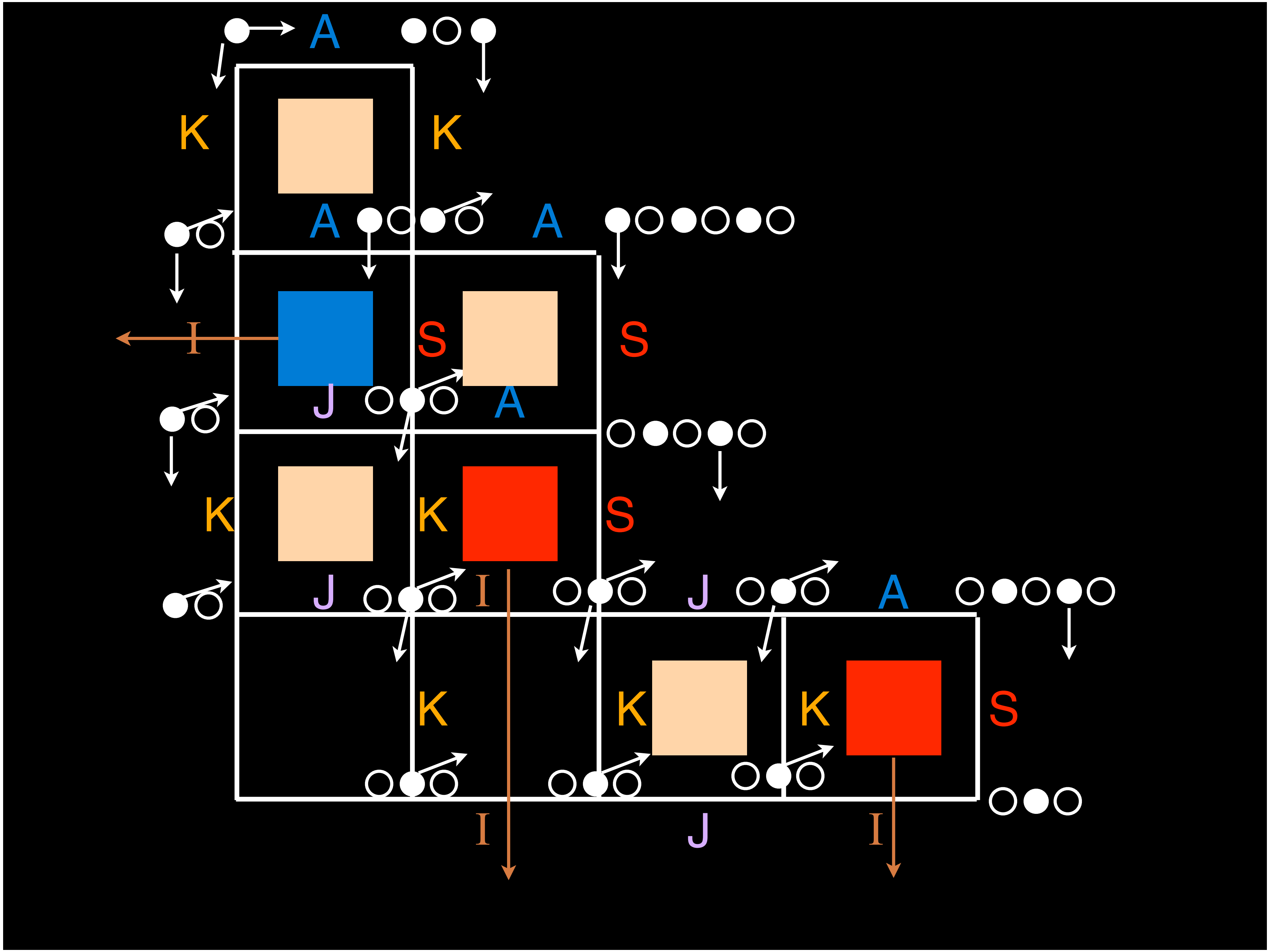
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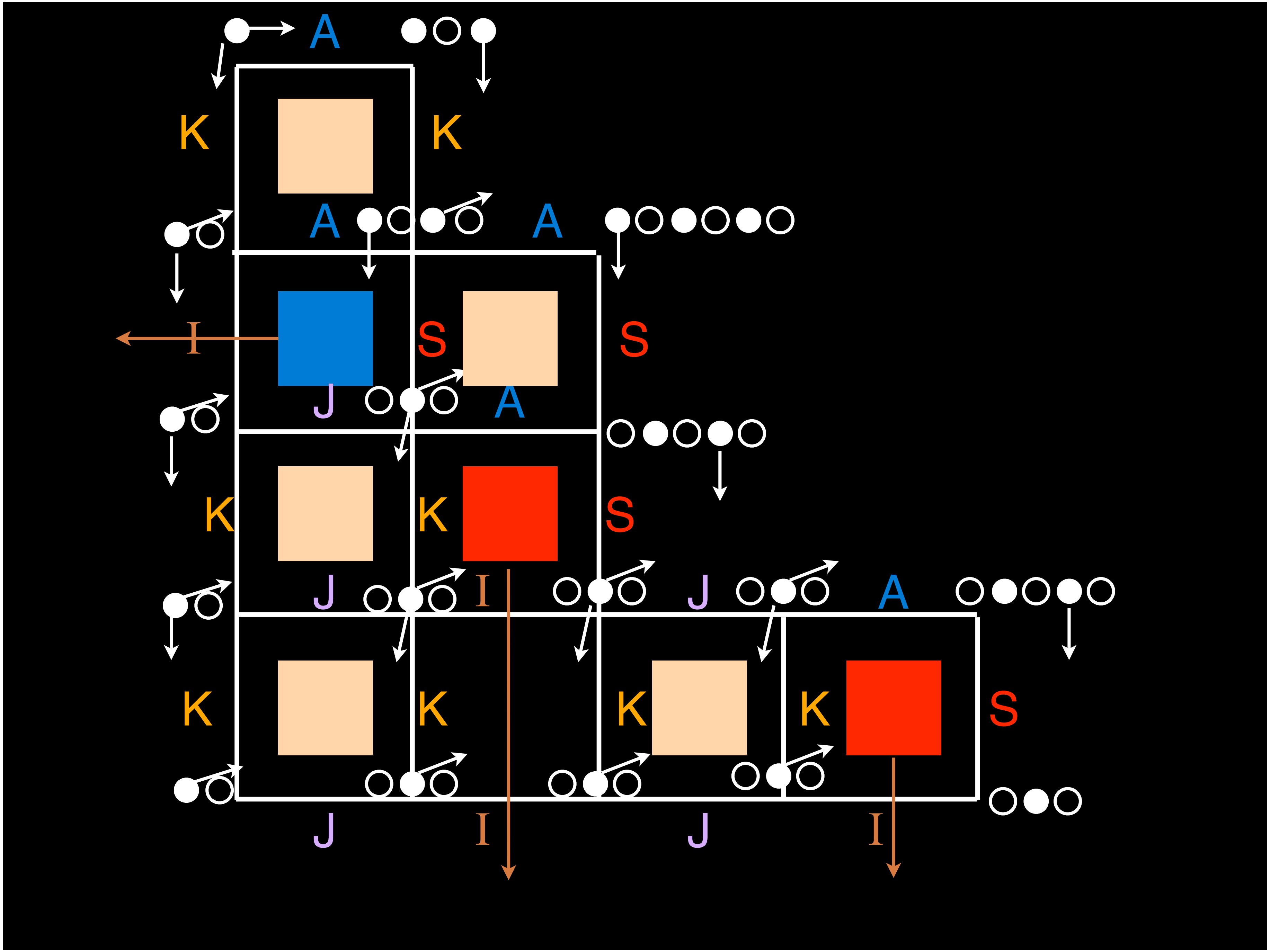
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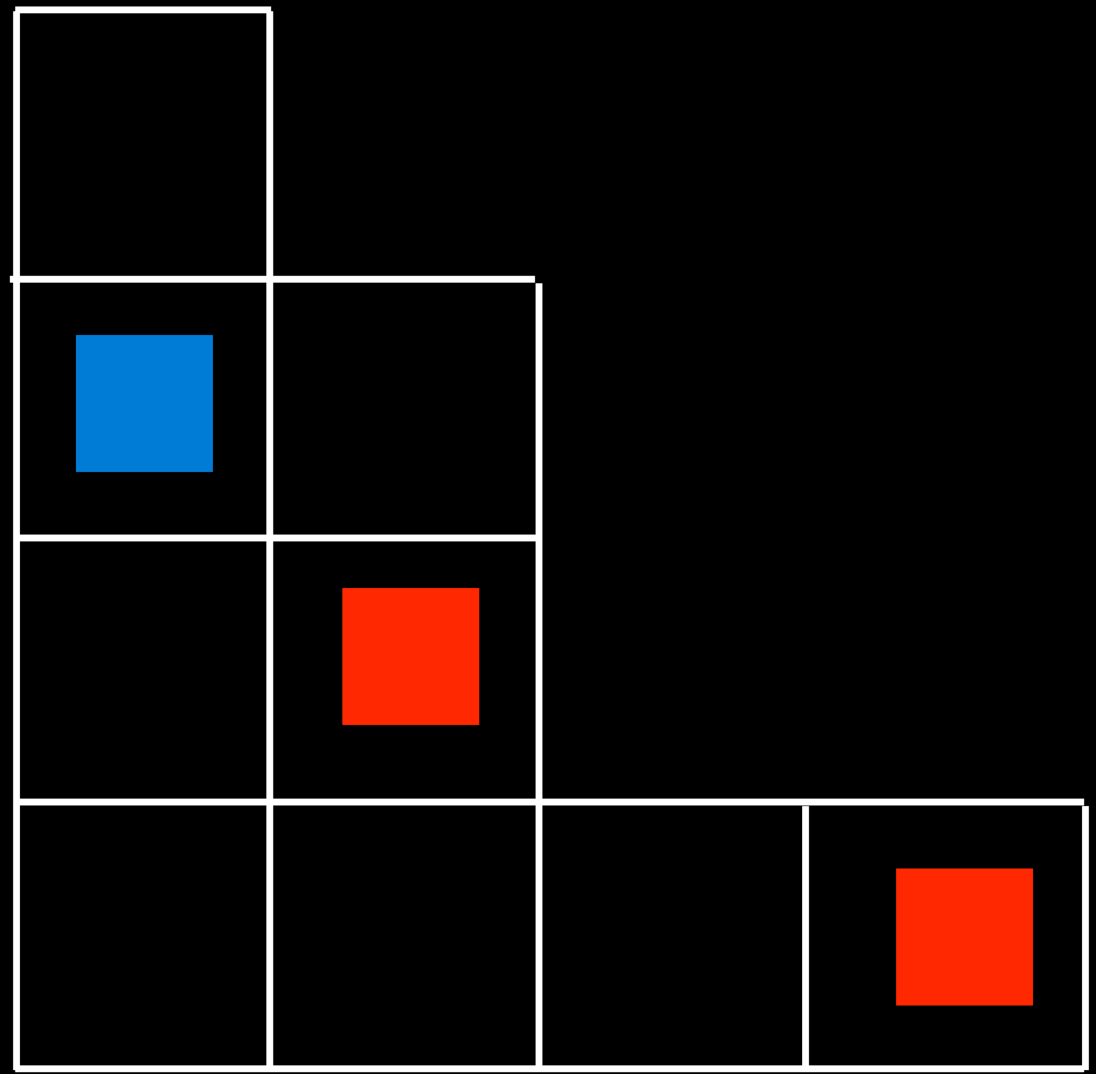

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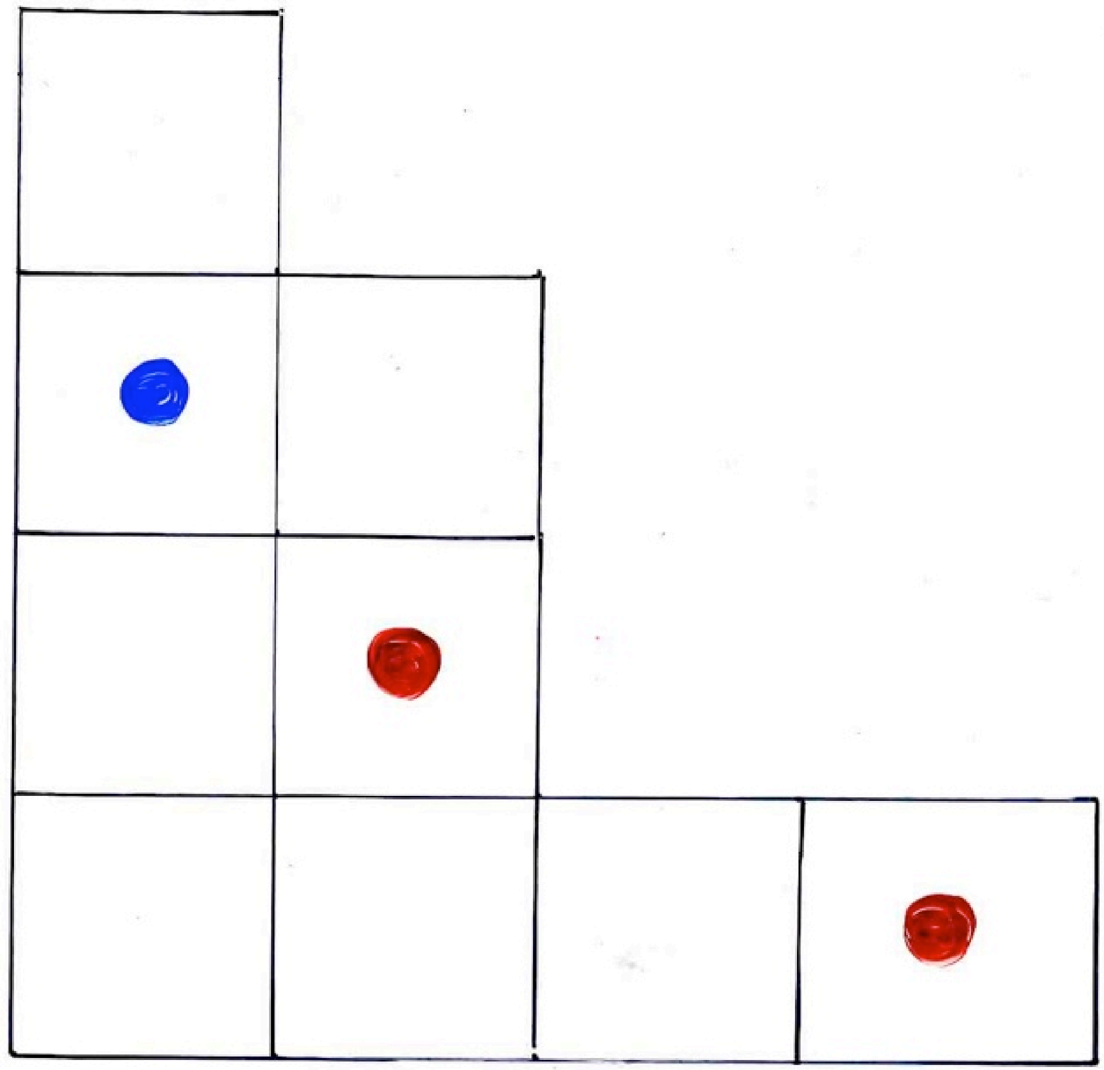
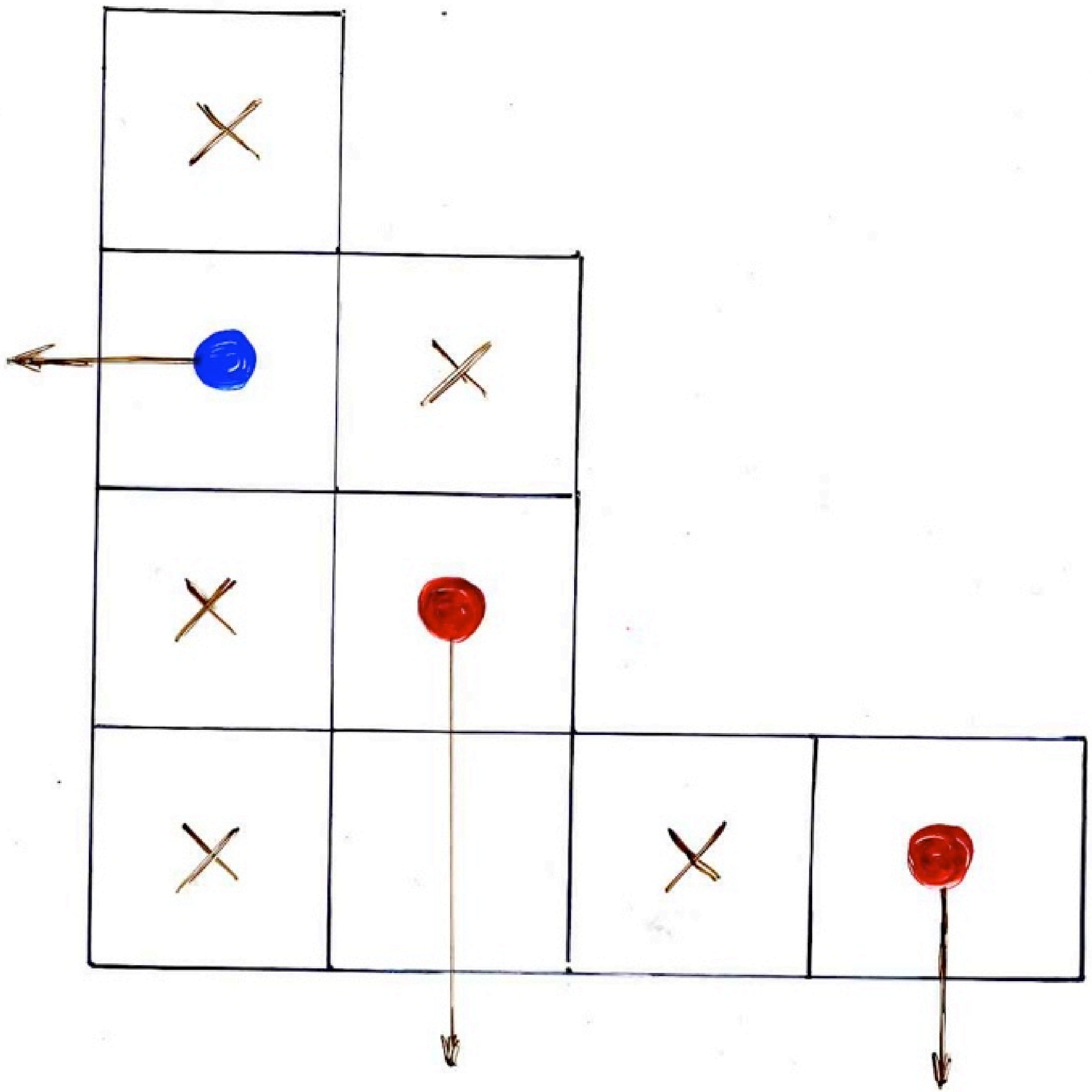
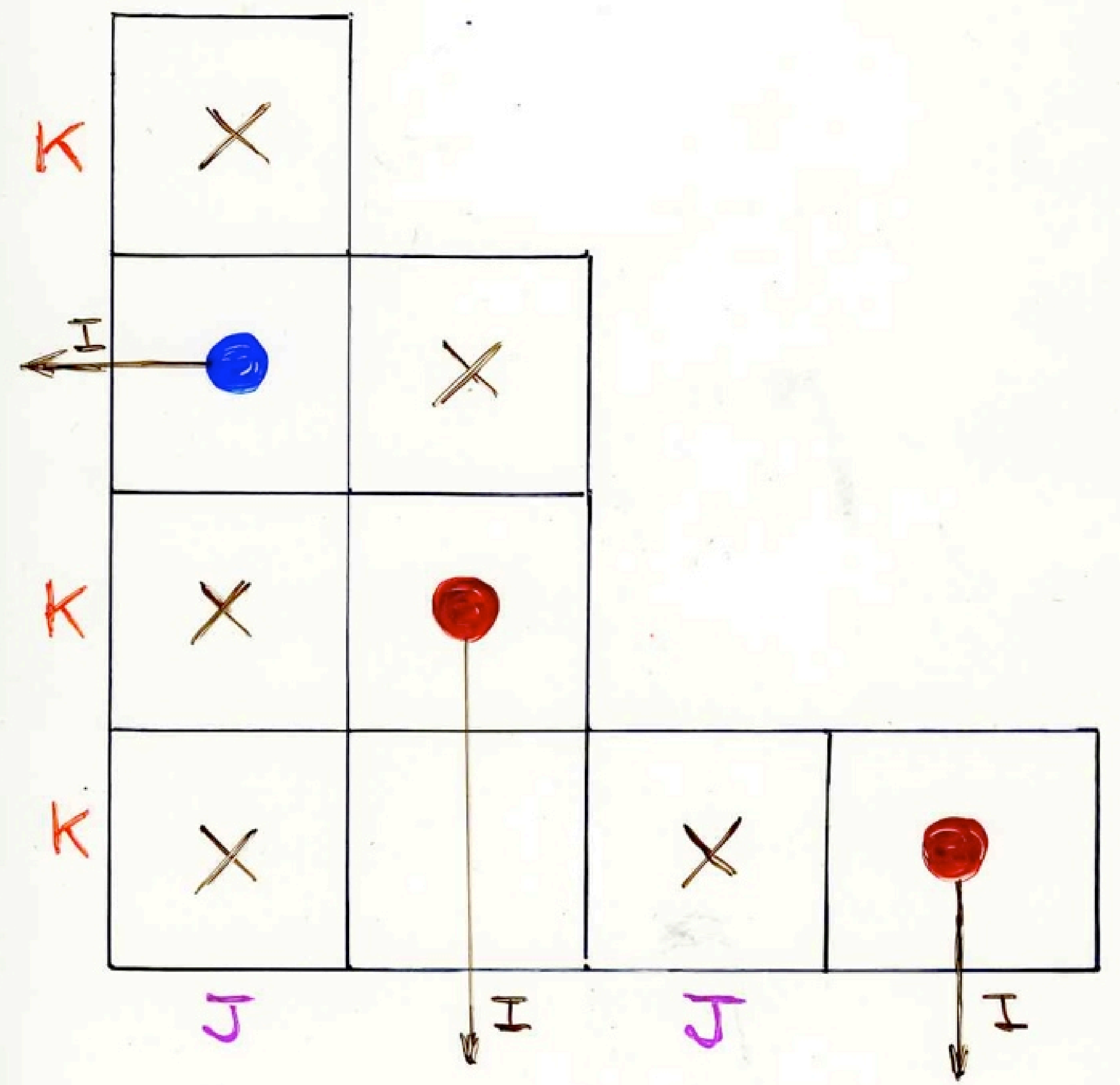
inverse bijection







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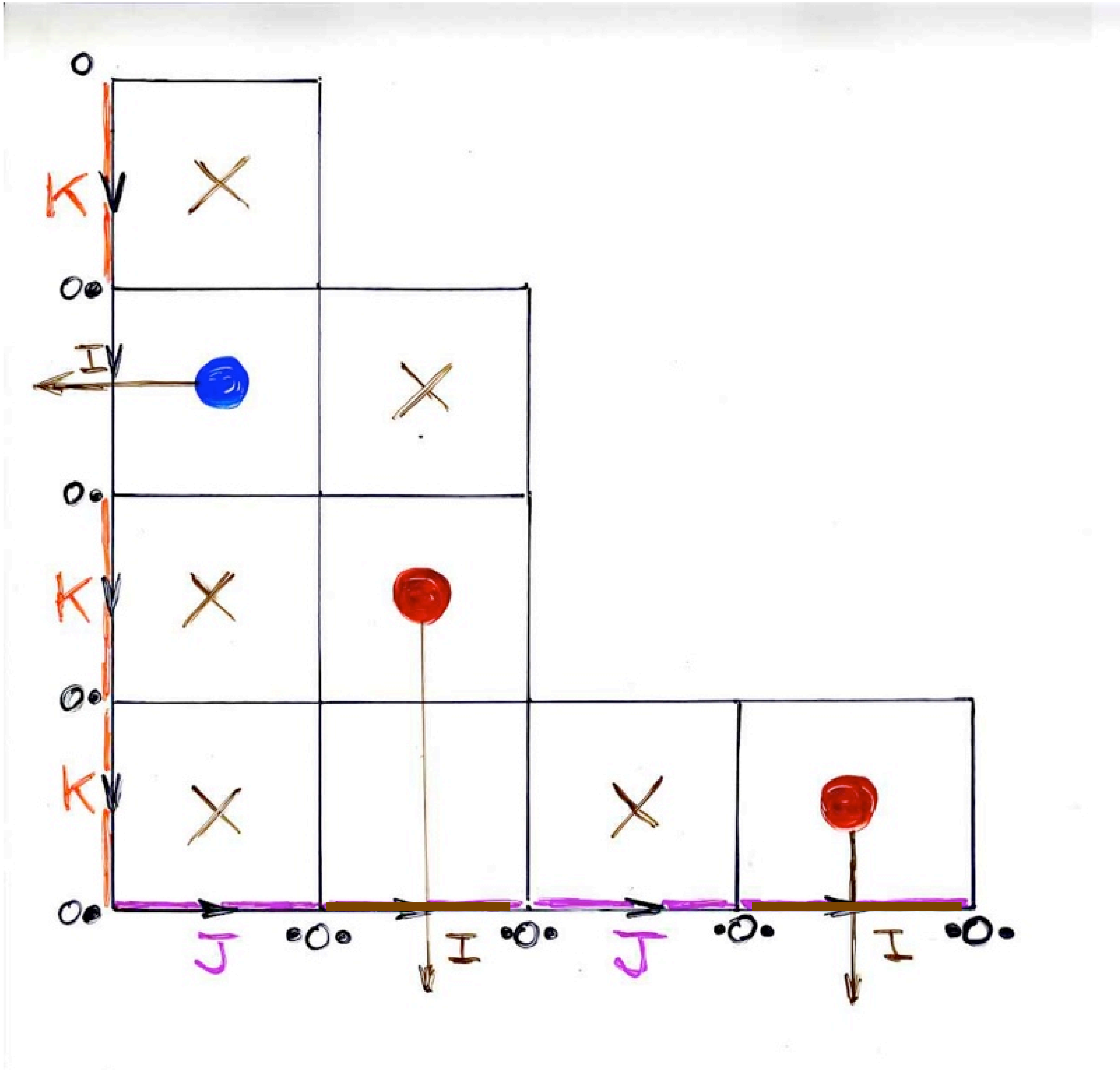
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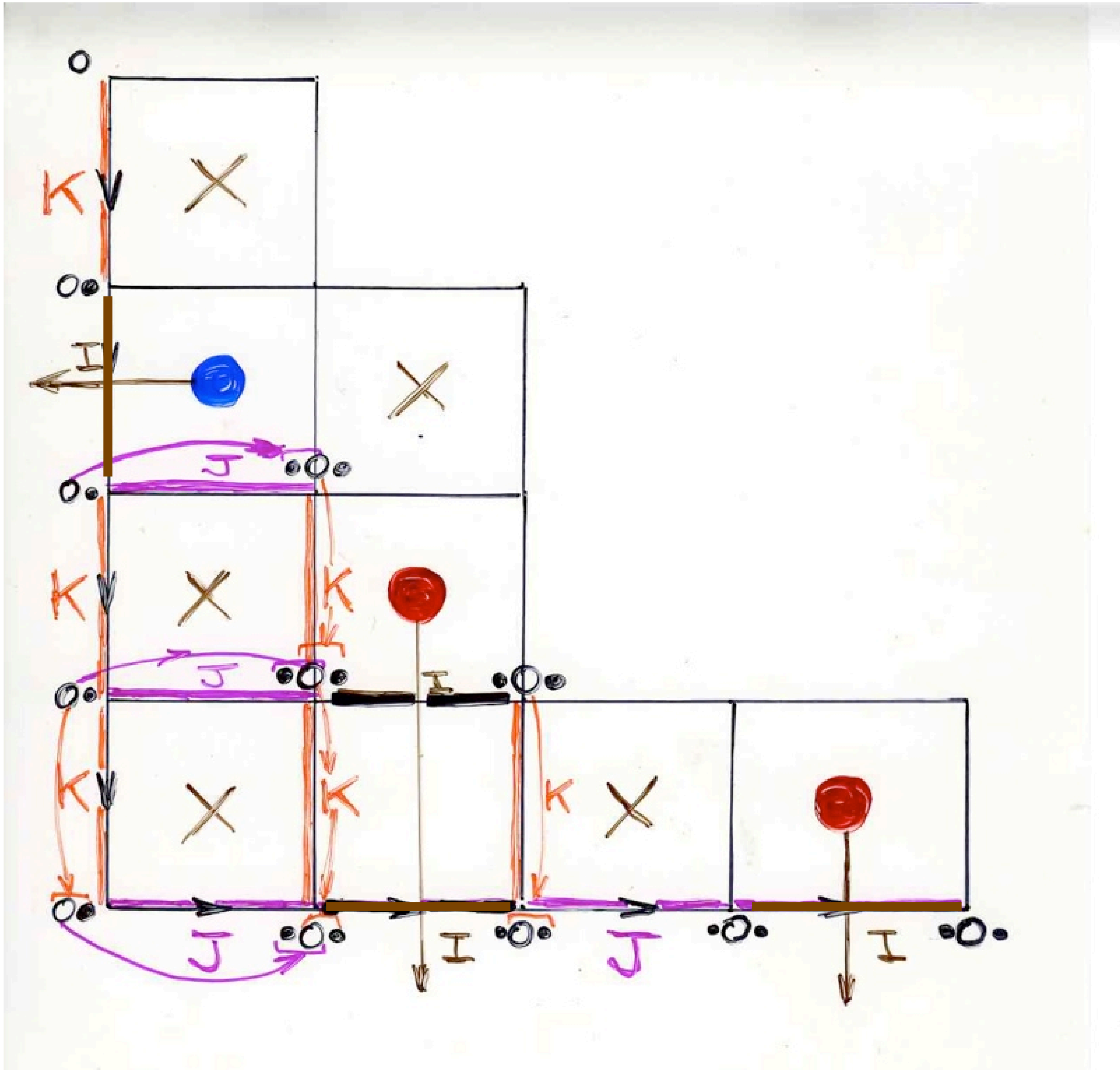

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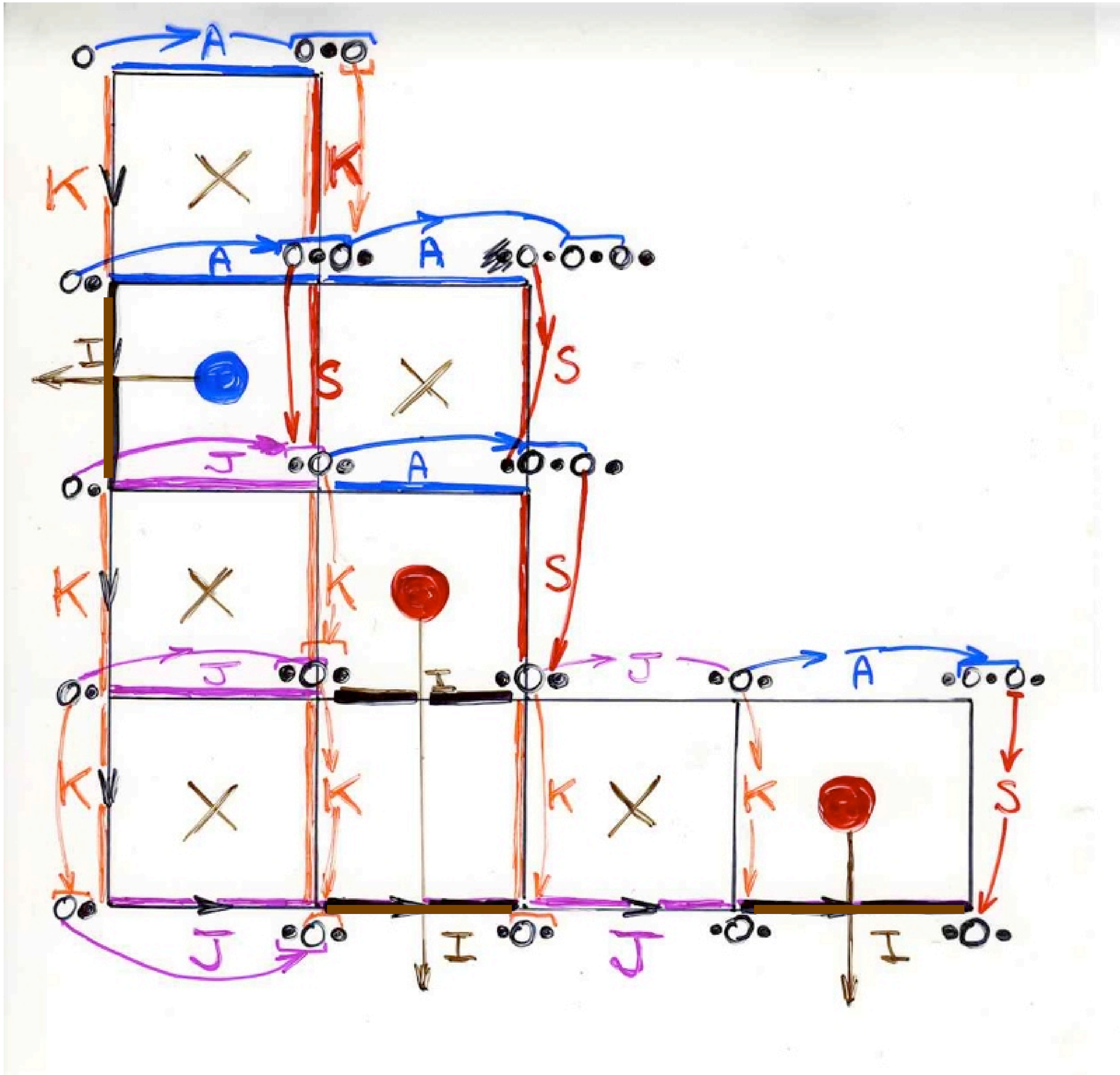

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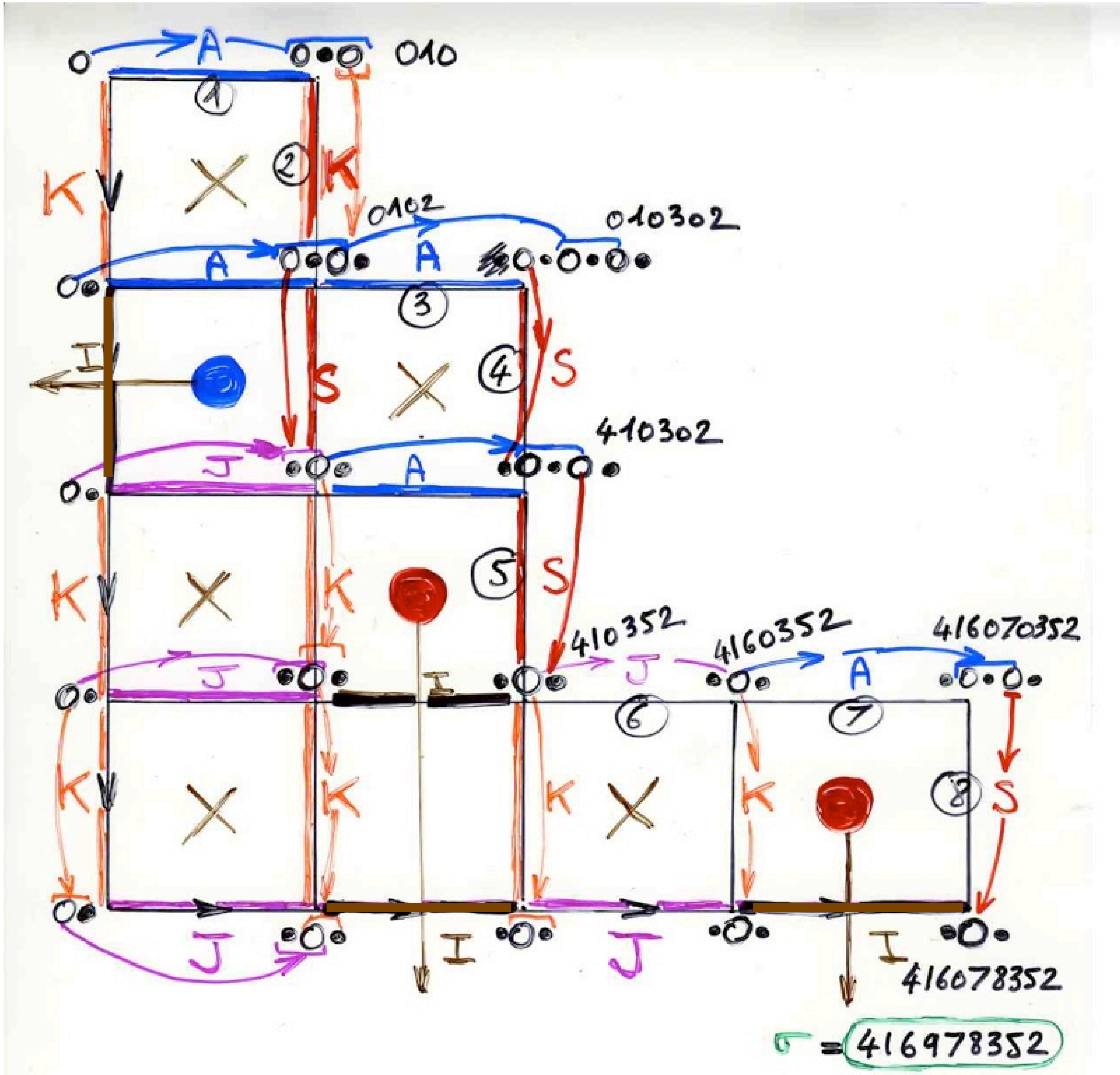

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Two bijections
one theorem

Prop.

T alternative tableau → σ "exchange-fusion" inverse algorithm

T alternative tableau → τ "local" algorithm from $DE = ED + E + D$

Prop.

T alternative tableau → σ "exchange-fusion" inverse algorithm

T alternative tableau → τ "local" algorithm from $DE = ED + E + D$

$$\sigma = \tau^{-1}$$

some perspectives

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\quad\quad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

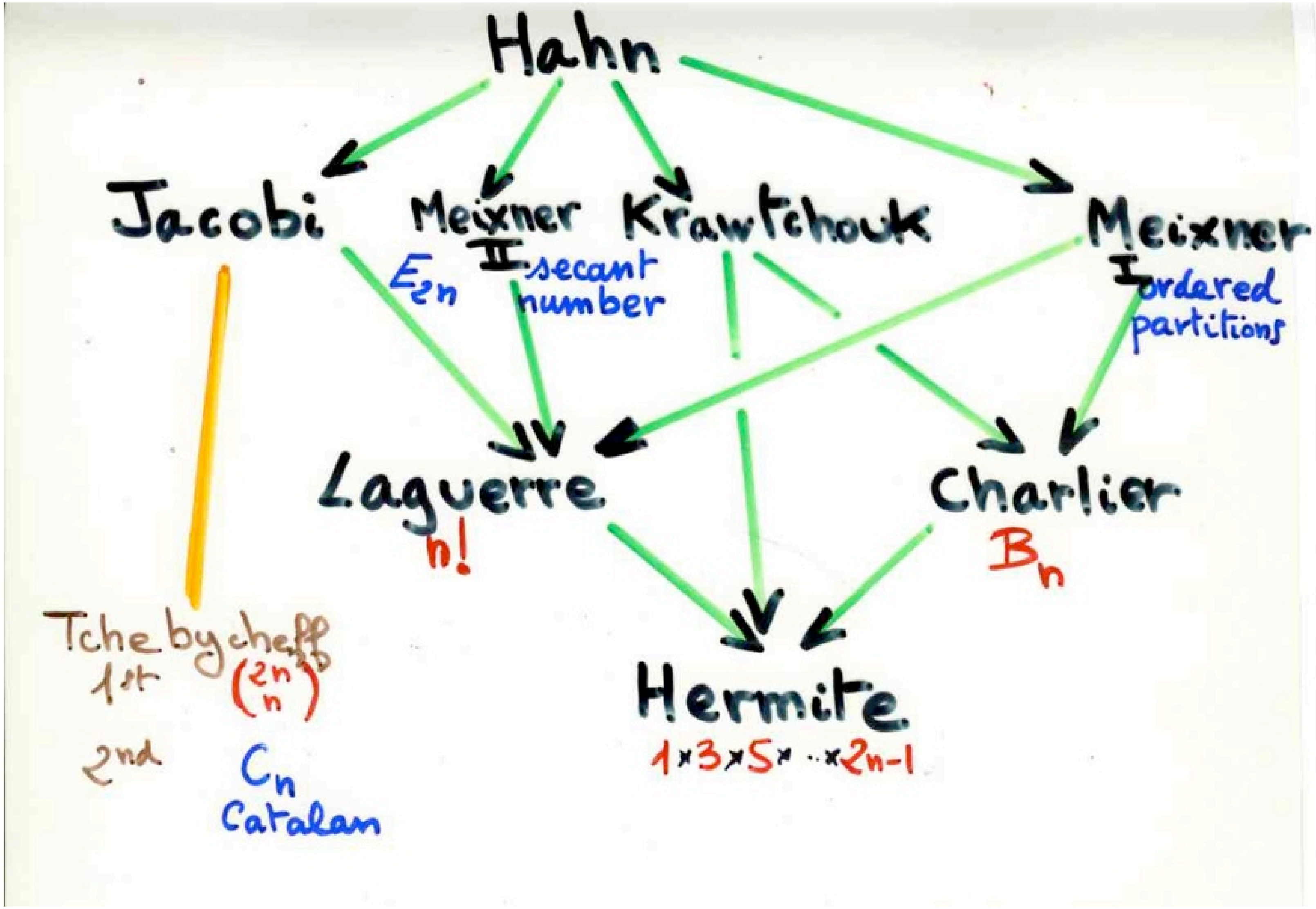
$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\,\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

Askey-Wilson polynomials

Askey-Wilson



Novelli, Thibon, Williams (April 2008)

Hall-Littlewood functions, Tevlin' bases (2007)

conjectures

alternative
tableaux

alternating
sign
matrices

next talk:

an alternative tableaux approach to alternating sign matrices

Wien, May (2008)
Erwin Schrödinger Institute

project ANR BLAN06-2-134516 MARS

xgv website :

http://www.labri.fr/perso/viennot/

Recherche, cv, publications, exposés, diaporamas, livres, petite école, photos: voir ma page personnelle ici
Vulgarisation scientifique voir la page de l'association Cont'Science

downloadable papers, slides and lecture notes, etc ... here
(the summary on page "recherches" and most slides are in english)

TASEP:

→ **page "exposés"**

Catalan numbers, permutation tableaux and asymmetric exclusion process (pdf, 4,8 Mo)
GASCOM'06, Dijon, Septembre 2006, aussi: Journées Pierre Leroux, Montréal, Septembre 2006

Robinson-Schensted-Knuth: RSK1 (pdf, 9,1 Mo)
groupe de travail de combinatoire, Bordeaux, LaBRI, Février 2005
Robinson-Schensted-Knuth: RSK2 (pdf,10,8Mo)
groupe de travail de combinatoire, Bordeaux, LaBRI, Février 2005

survey paper on Robinson-Schensted correspondence:
[30] *Chain and antichain families, grids and Young tableaux*,
Annals of Discrete Maths., 23 (1984) 409-464.

from xgv website :

→ **A Combinatorial theory of orthogonal polynomials**

[4] *Une théorie combinatoire des polynômes orthogonaux*, Lecture Notes UQAM, 219p., Publication du LACIM, Université du Québec à Montréal, 1984, rééd. 1991.

→ page **"petite école"**

Petite école de combinatoire LaBRI, année 2006/07
"Une théorie combinatoire des polynômes orthogonaux, ses extensions, interactions et applications"

Chapitre 2, Histoires et moments, (17, 23 Nov , 1, 8, 15 Dec 2006)

Chapitre X Histoires et opératerus (10 and 12 January 2007)

→ page **"cours"**

Cours au Service de Physique Théorique du CEA, Saclay Sept-Oct 2007
"Eléments de combinatoire algébrique"

Ch 4 - (9,4 Mo) théorie combinatoire des polynômes orthogonaux et fractions continues *from xgv website :*

→ **Paper:** FV bijection

[21] (avec J. Françon) *Permutations selon les pics, creux, doubles montées et doubles descentes, nombres d'Euler et nombres de Genocchi*, Discrete Maths., 28 (1979) 21-35

survey paper: Genocchi, Euler (tangent and secant numbers), Jacobi elliptic functions

[6] *Interprétations combinatoires des nombres d'Euler et de Genocchi* (pdf, 9,2 Mo) Séminaire de Théorie des nombres de Bordeaux, Publi. de l'Université Bordeaux I, 1982-83, 94p.

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more references:

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X.G.Viennot, Alternative tableaux and partially asymmetric exclusion process, in preparation

X.G.Viennot, A Robinson-Schensted like bijection for alternative tableaux, in preparation

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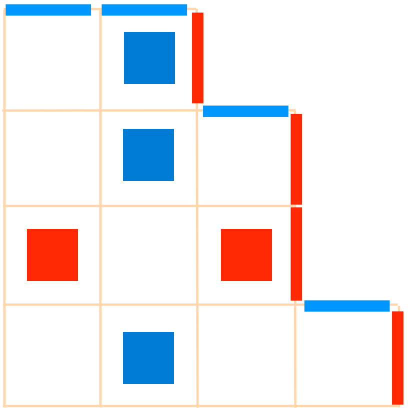
http://www.fpsac.cn/PDF-Proceedings/Talks/87.pdf

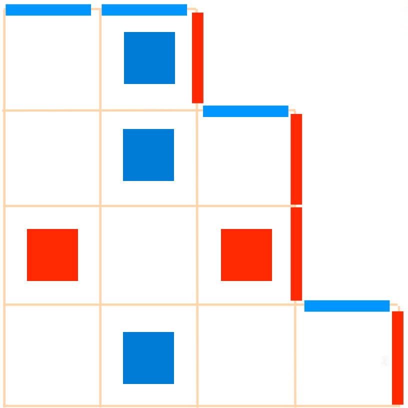
A complementary set of slides of this talk will be put in a near future on the page of Isaac Newton Web seminars

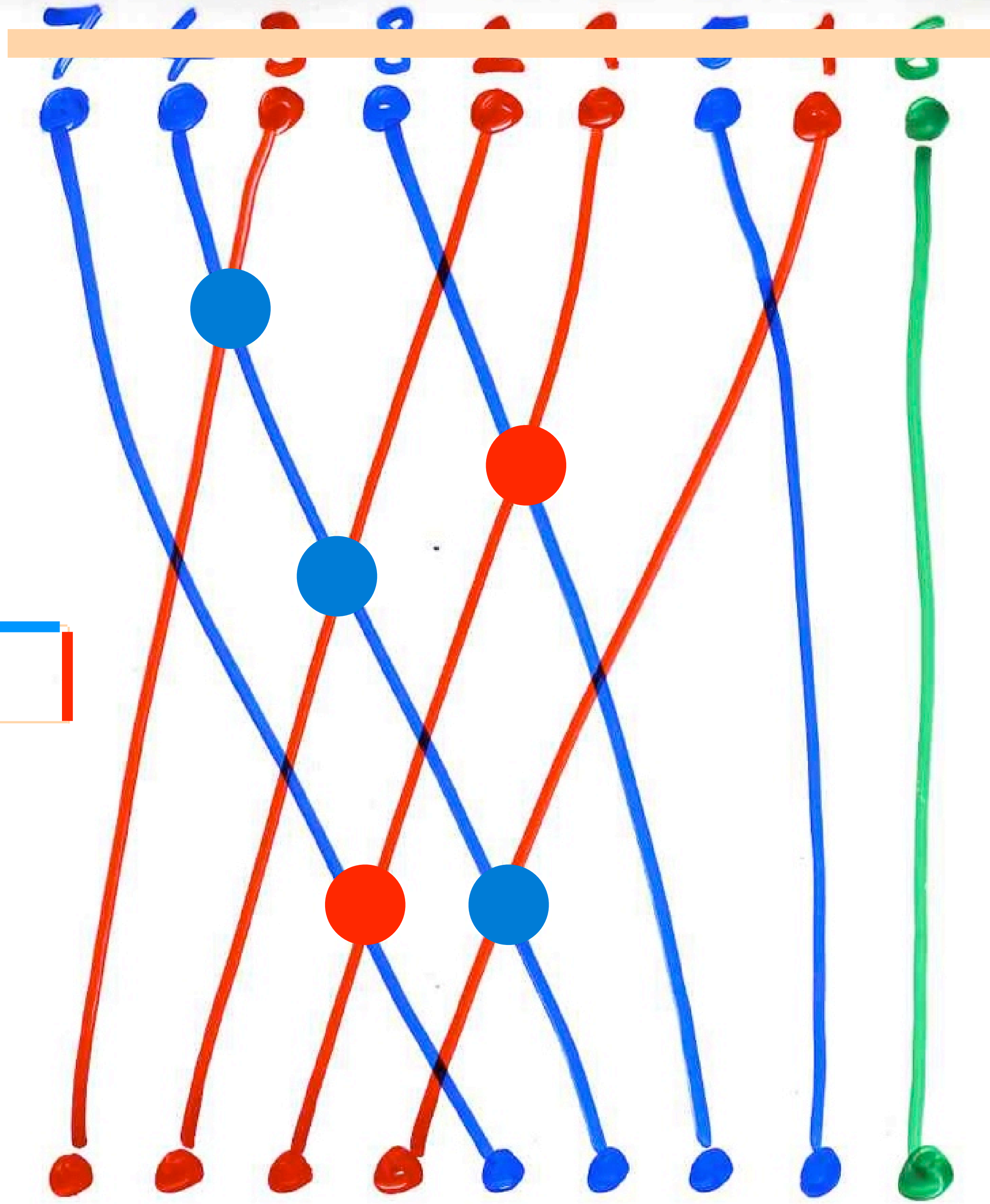
Thank you !

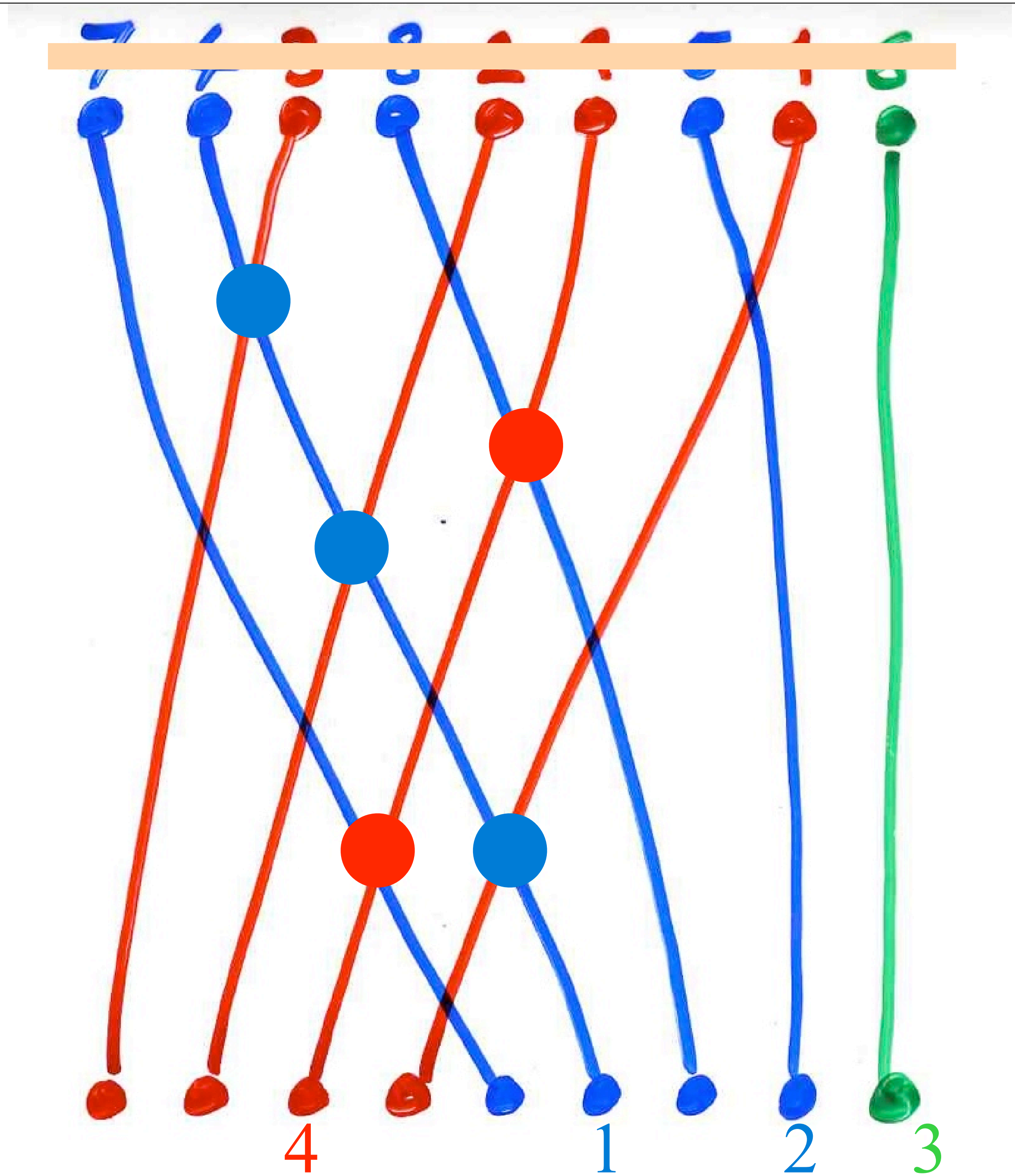
photos by
Hélène
Decoste

The inverse exchange-deletion bijection





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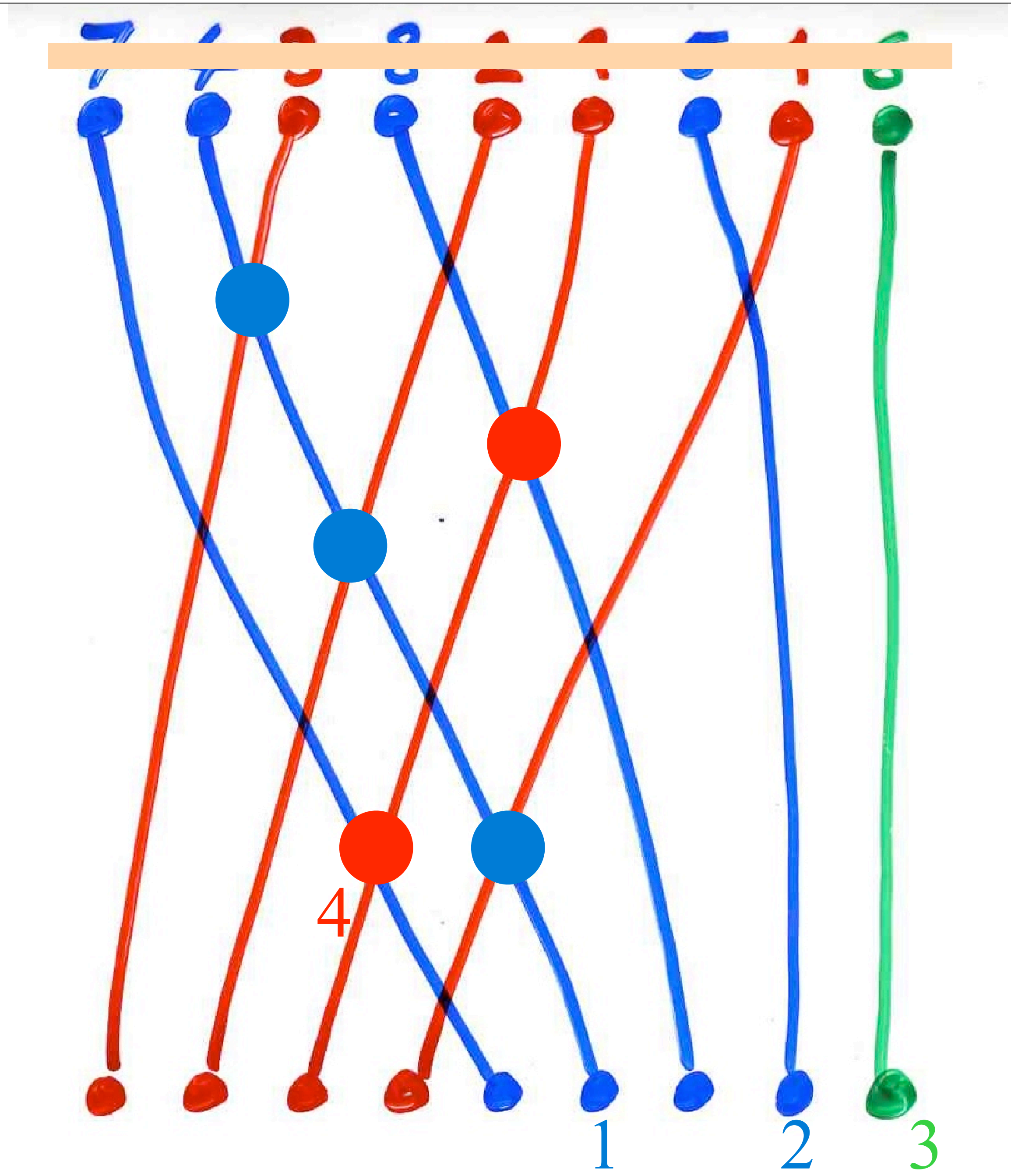
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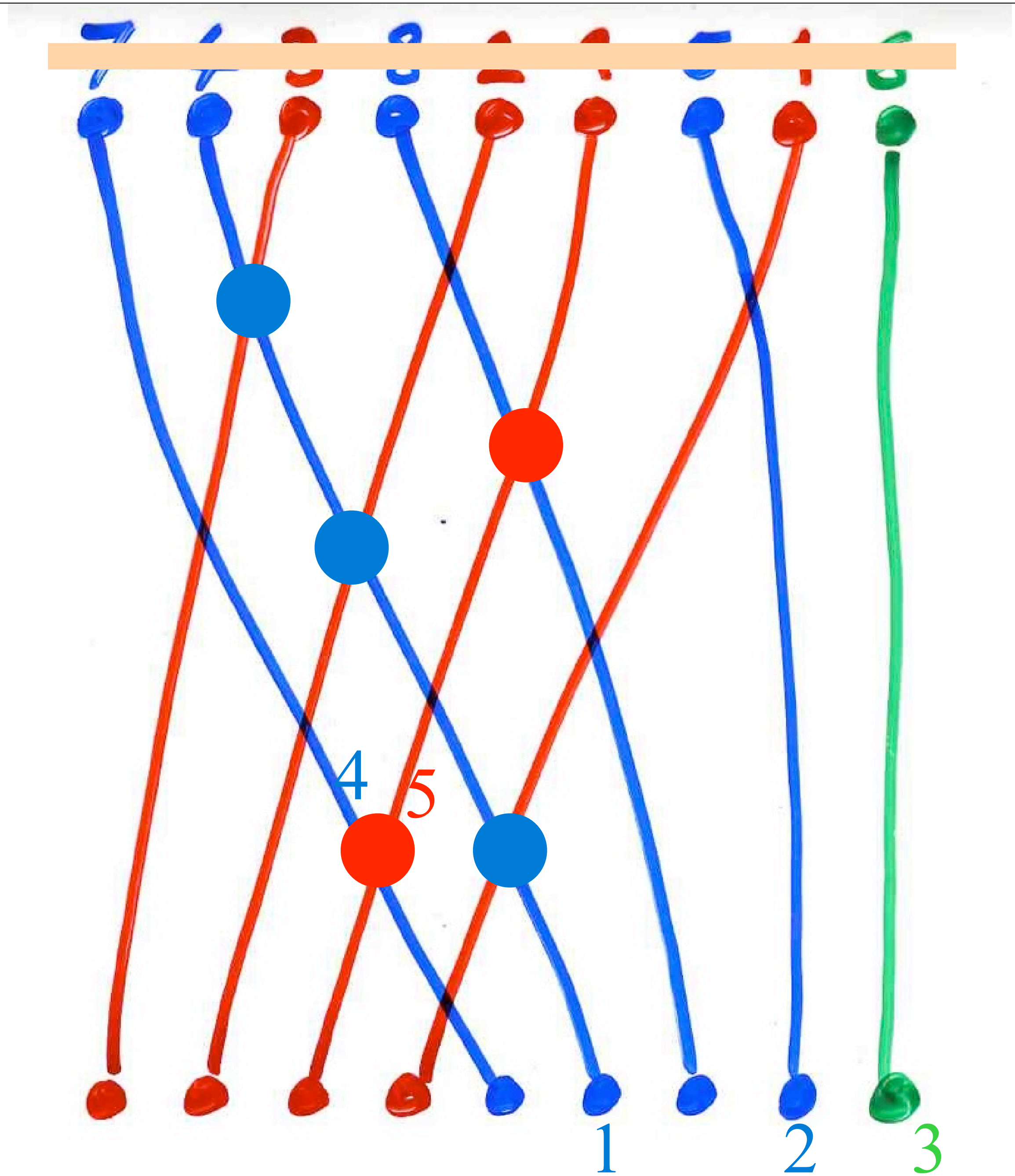

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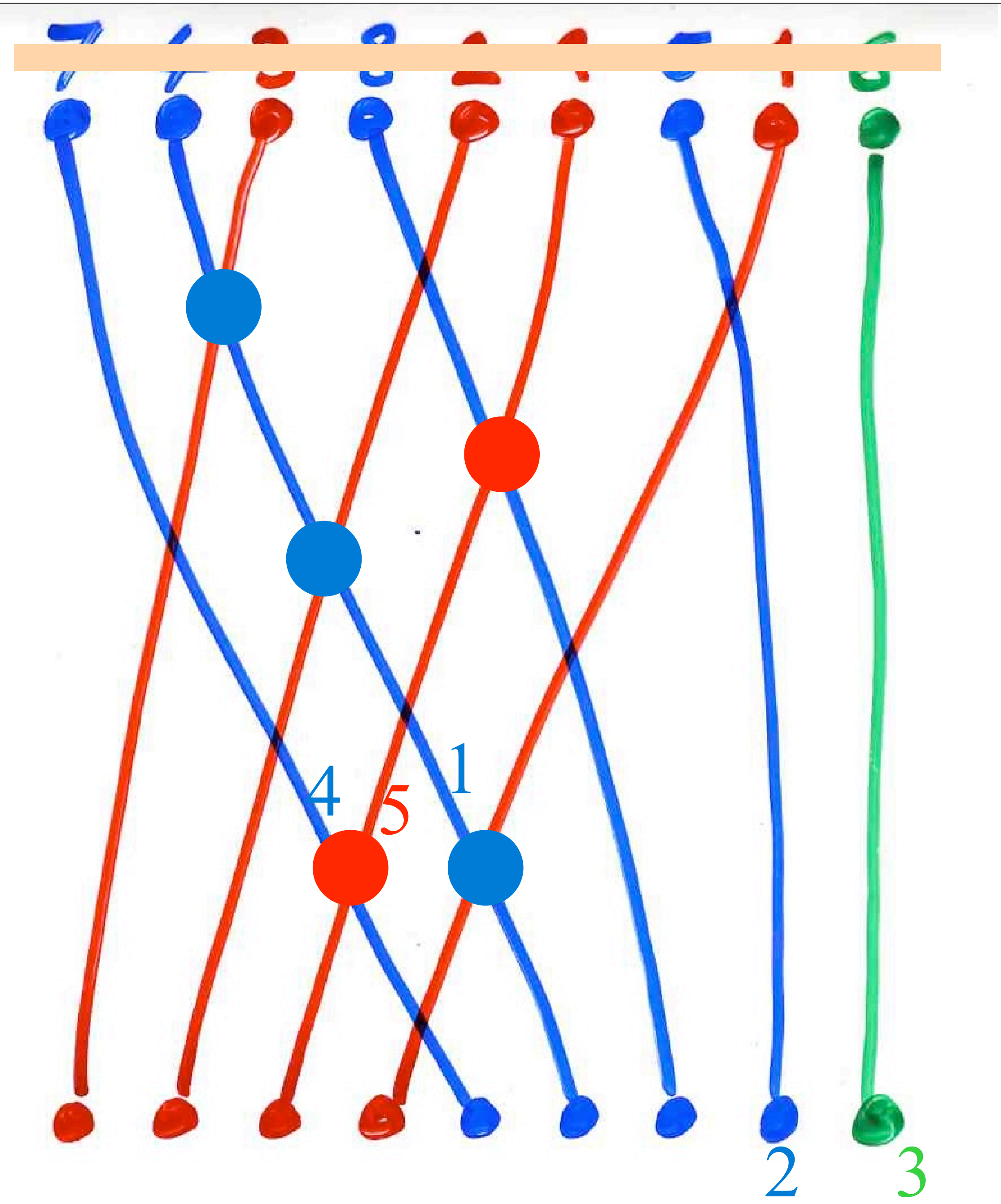
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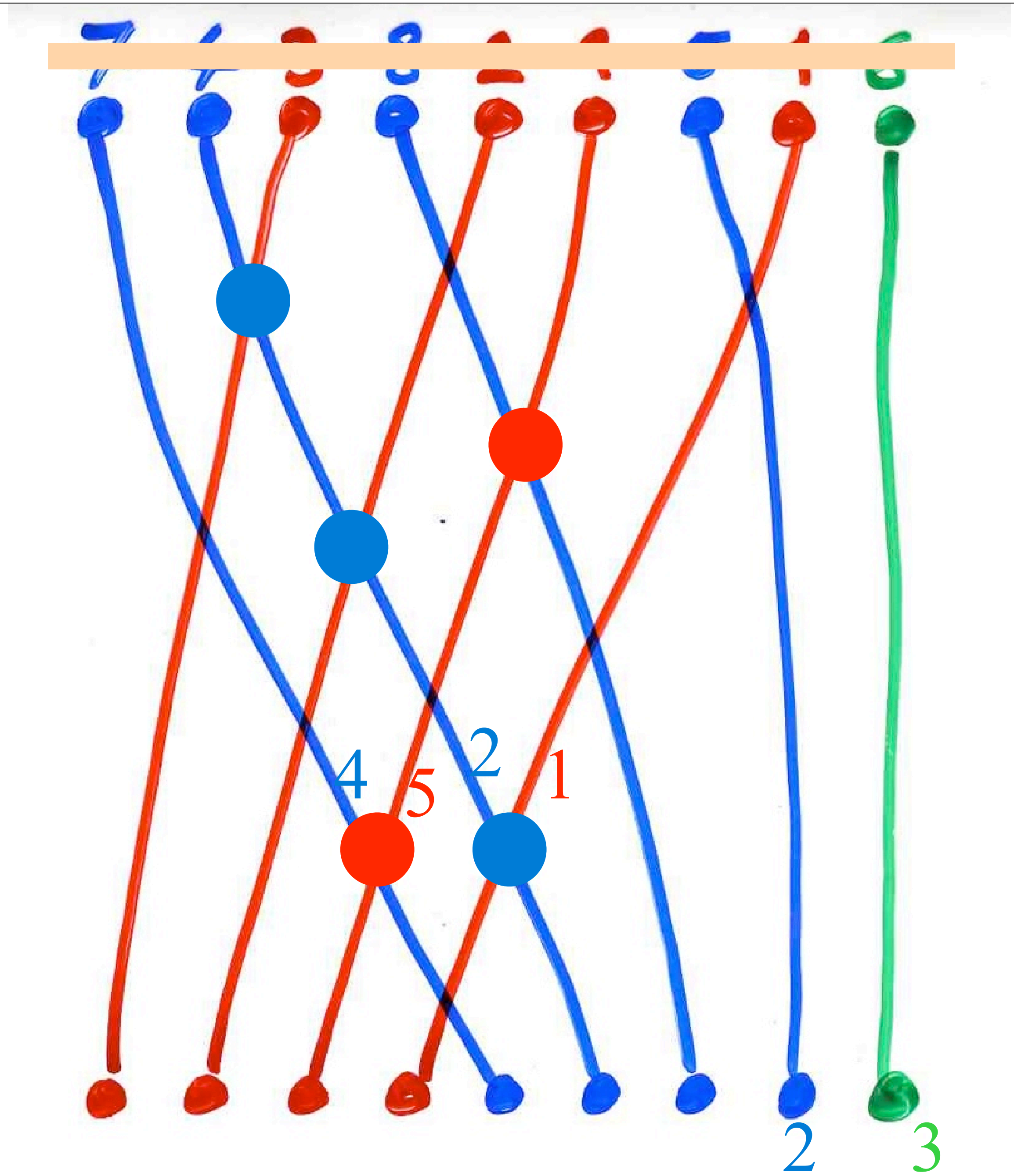
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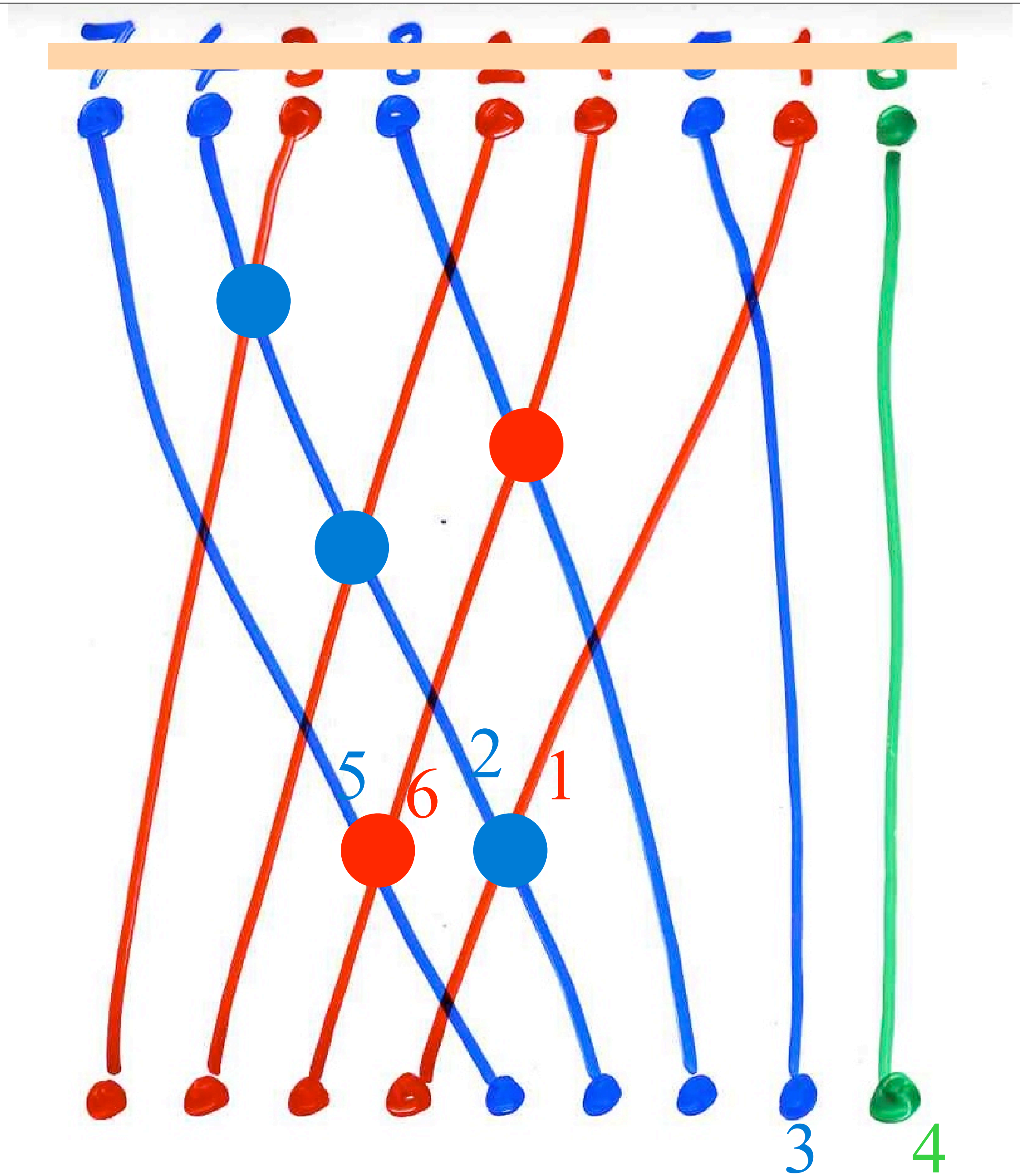
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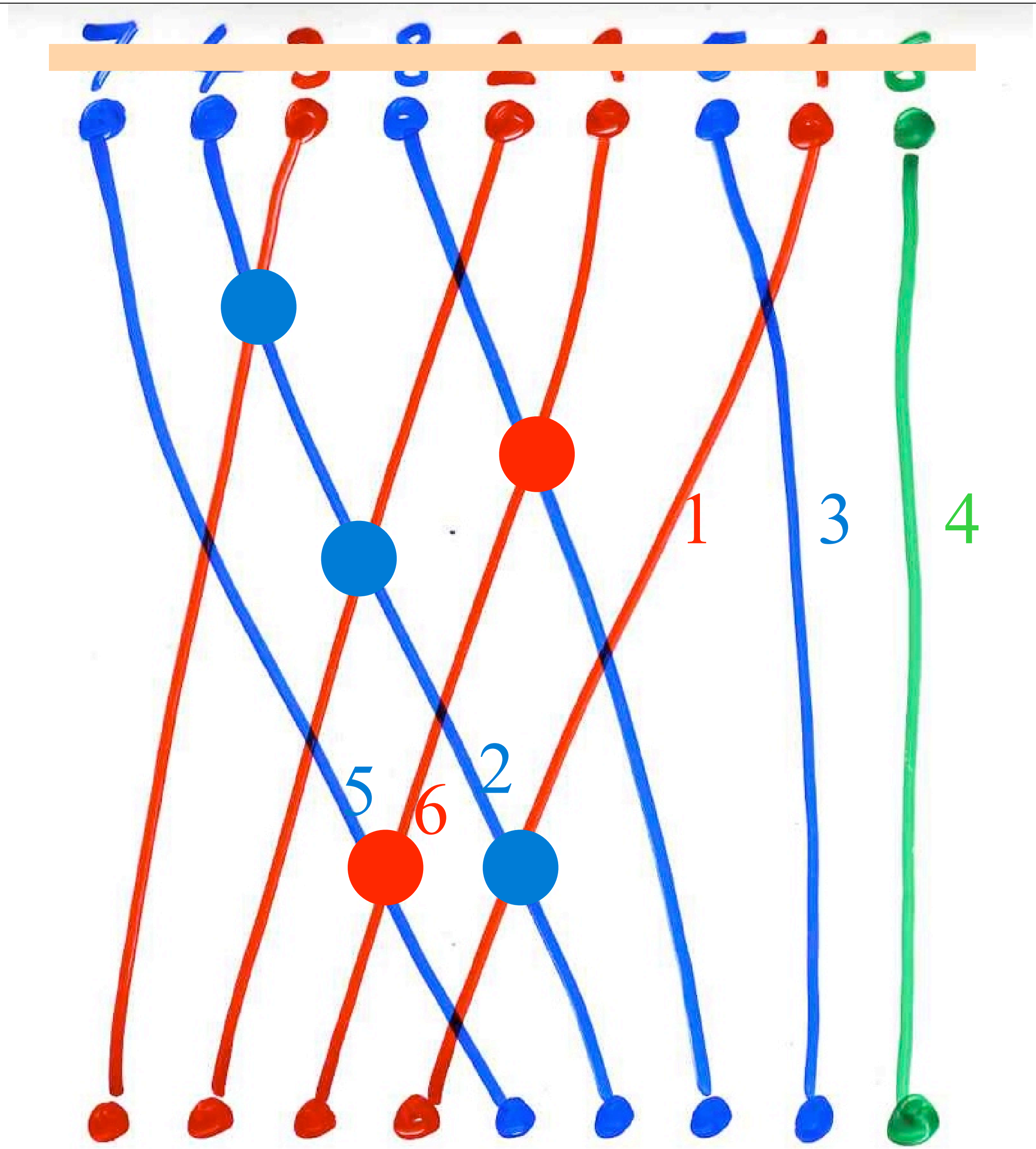
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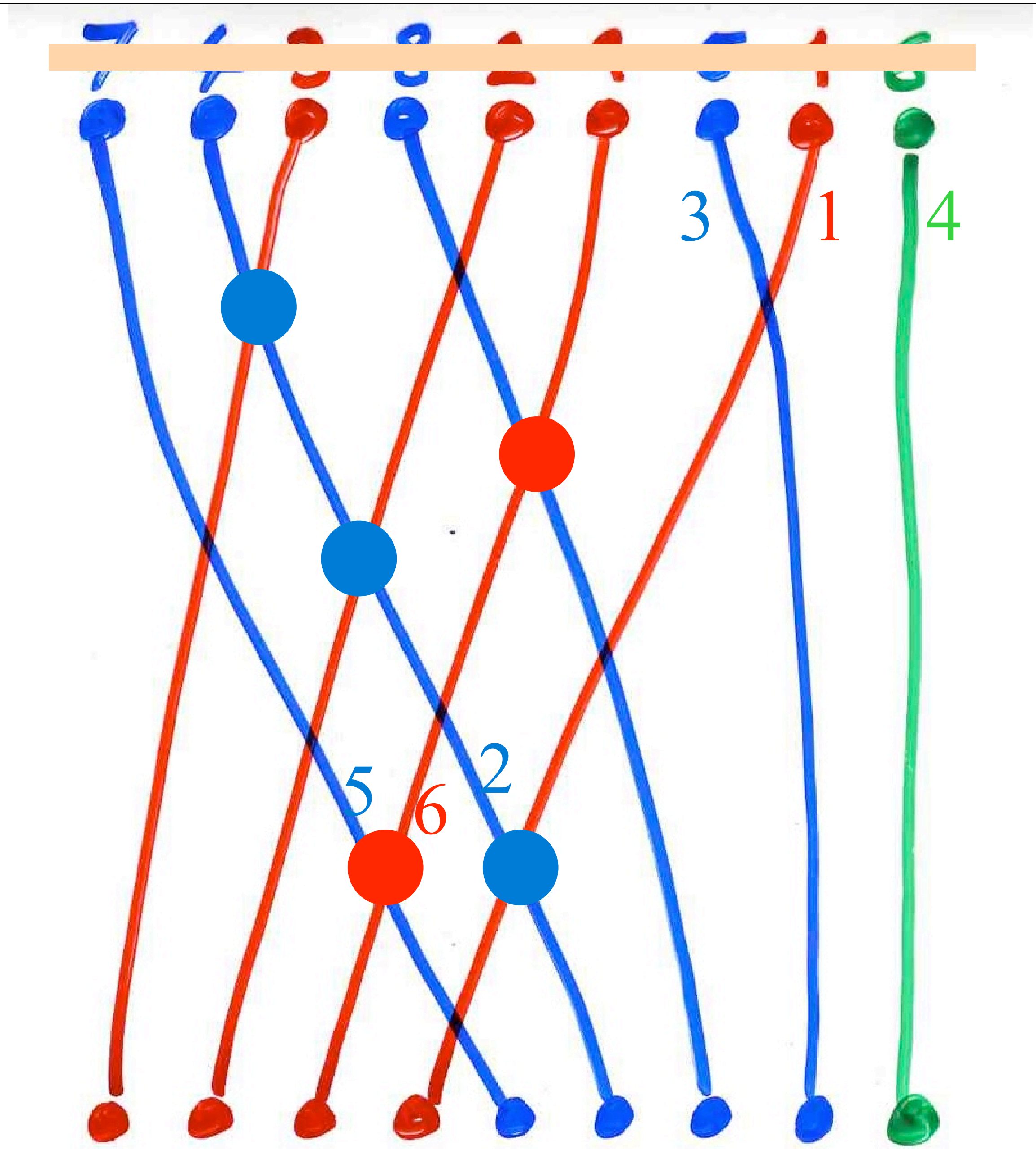

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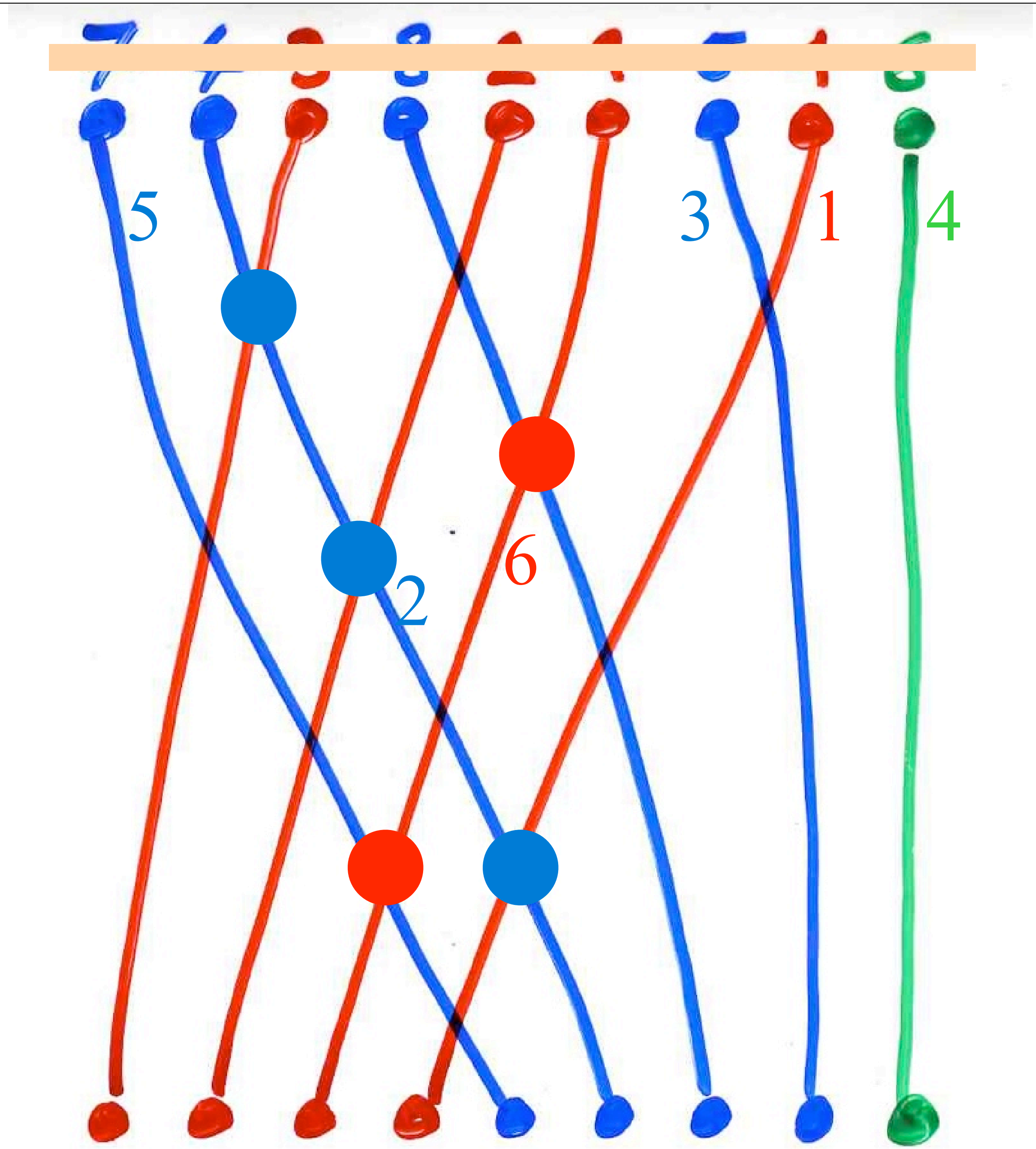

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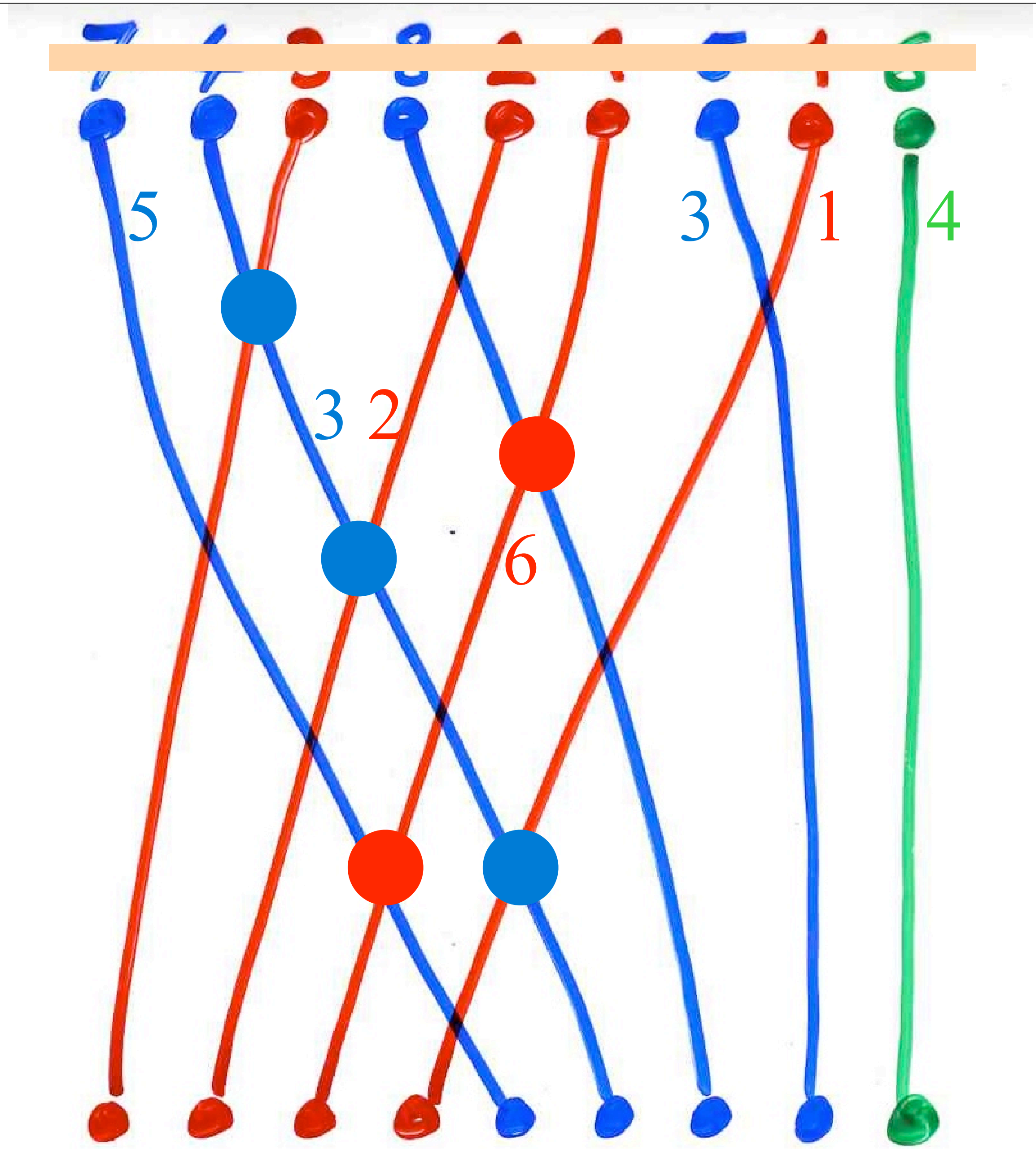
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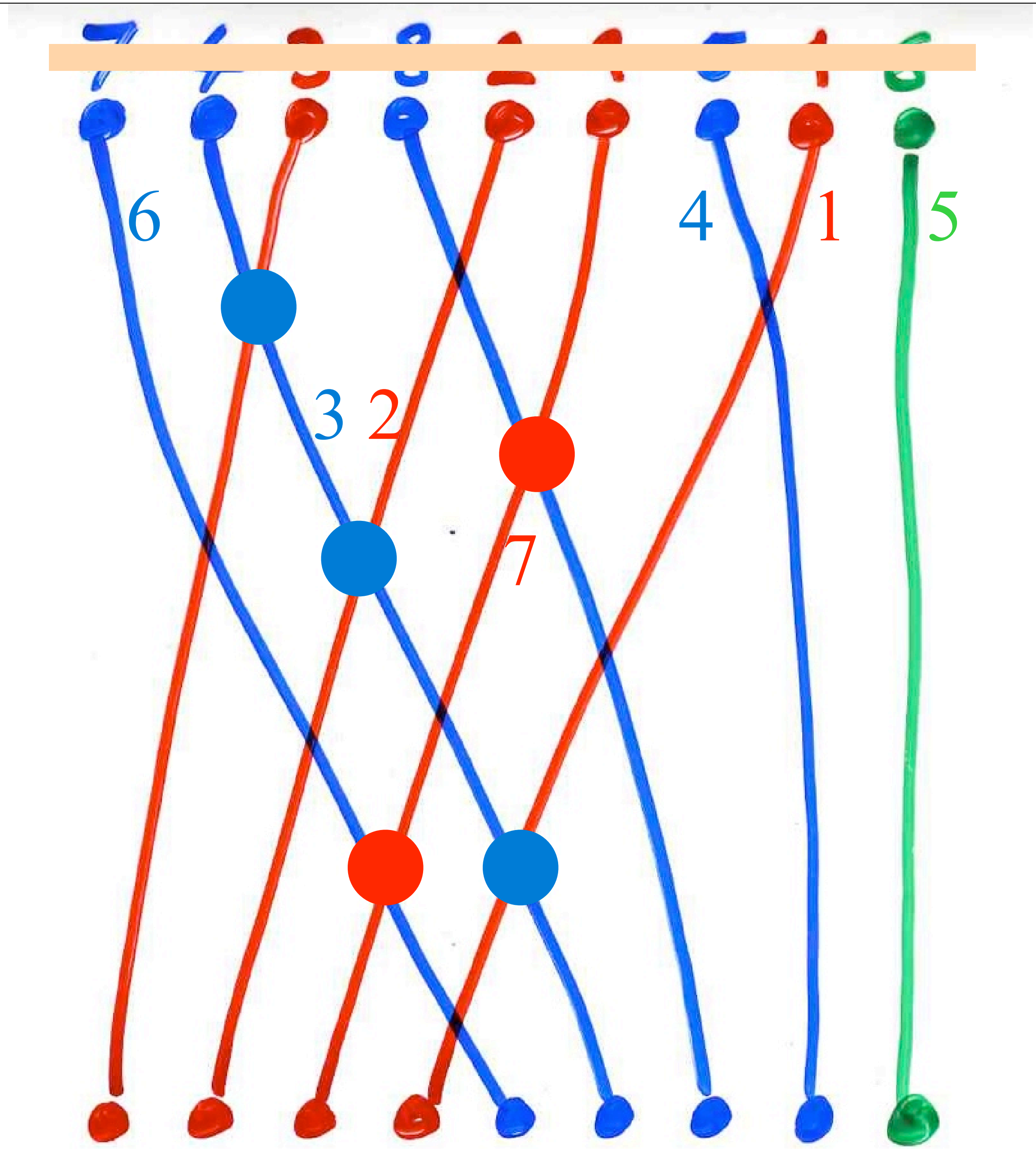

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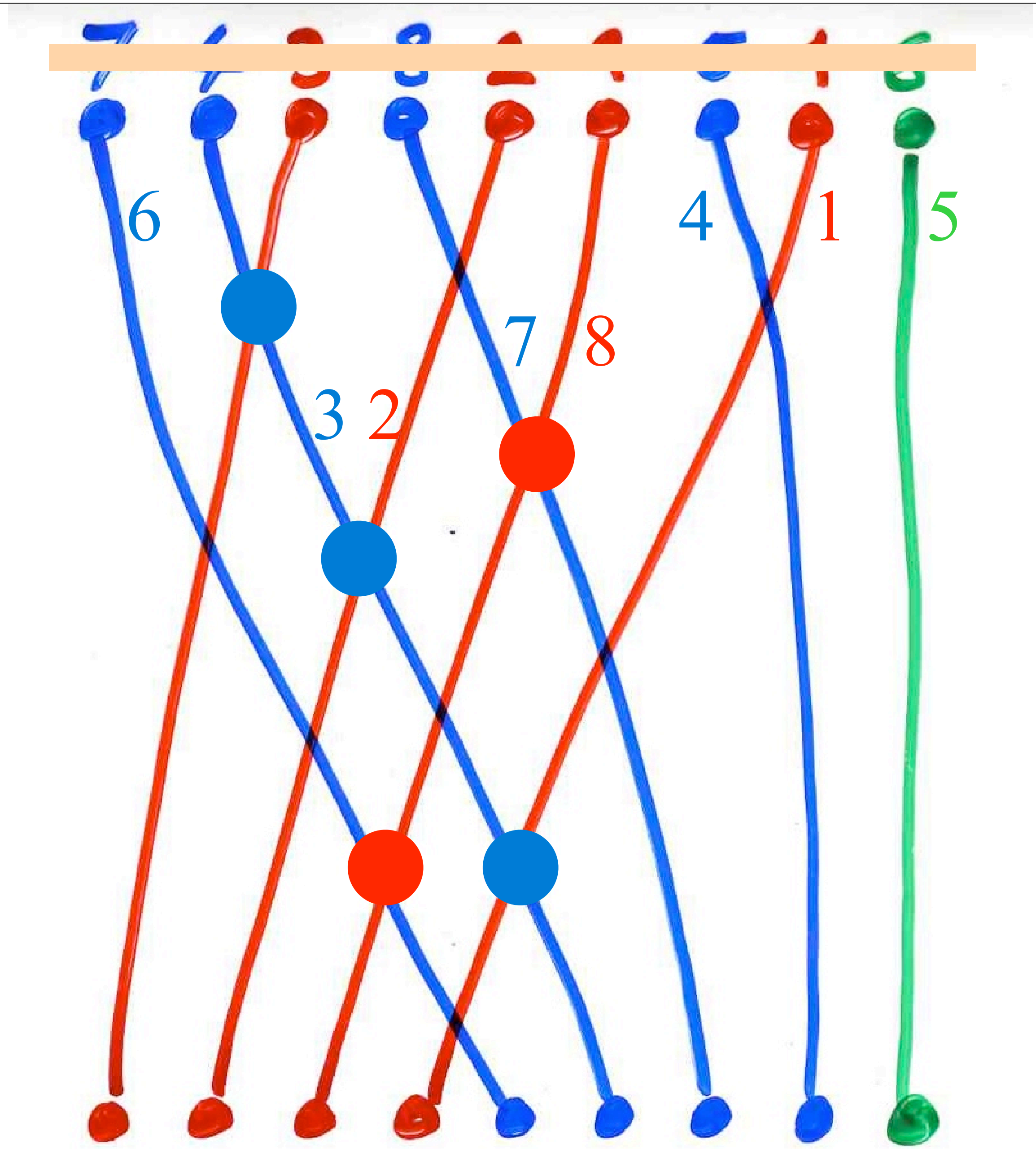
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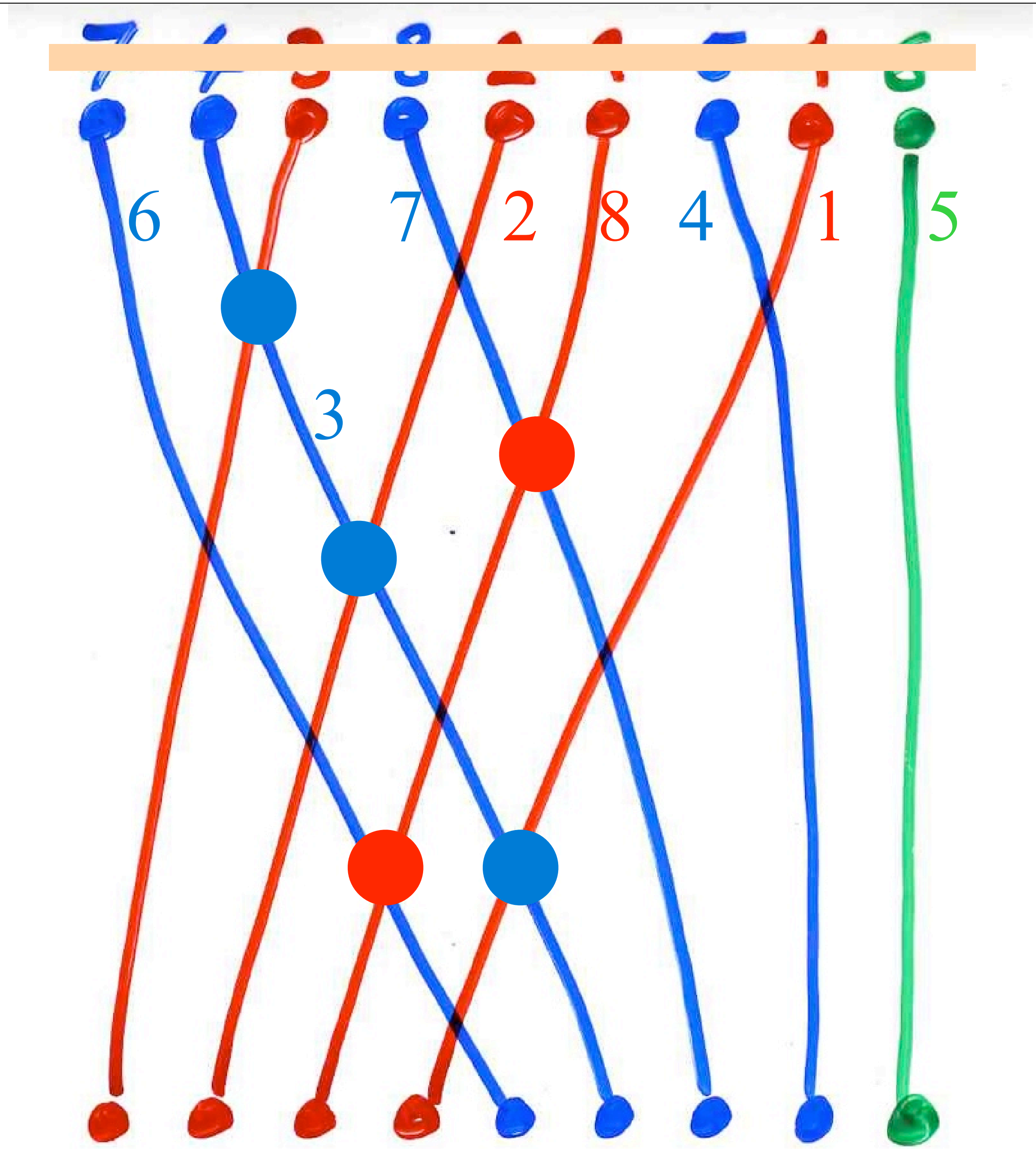
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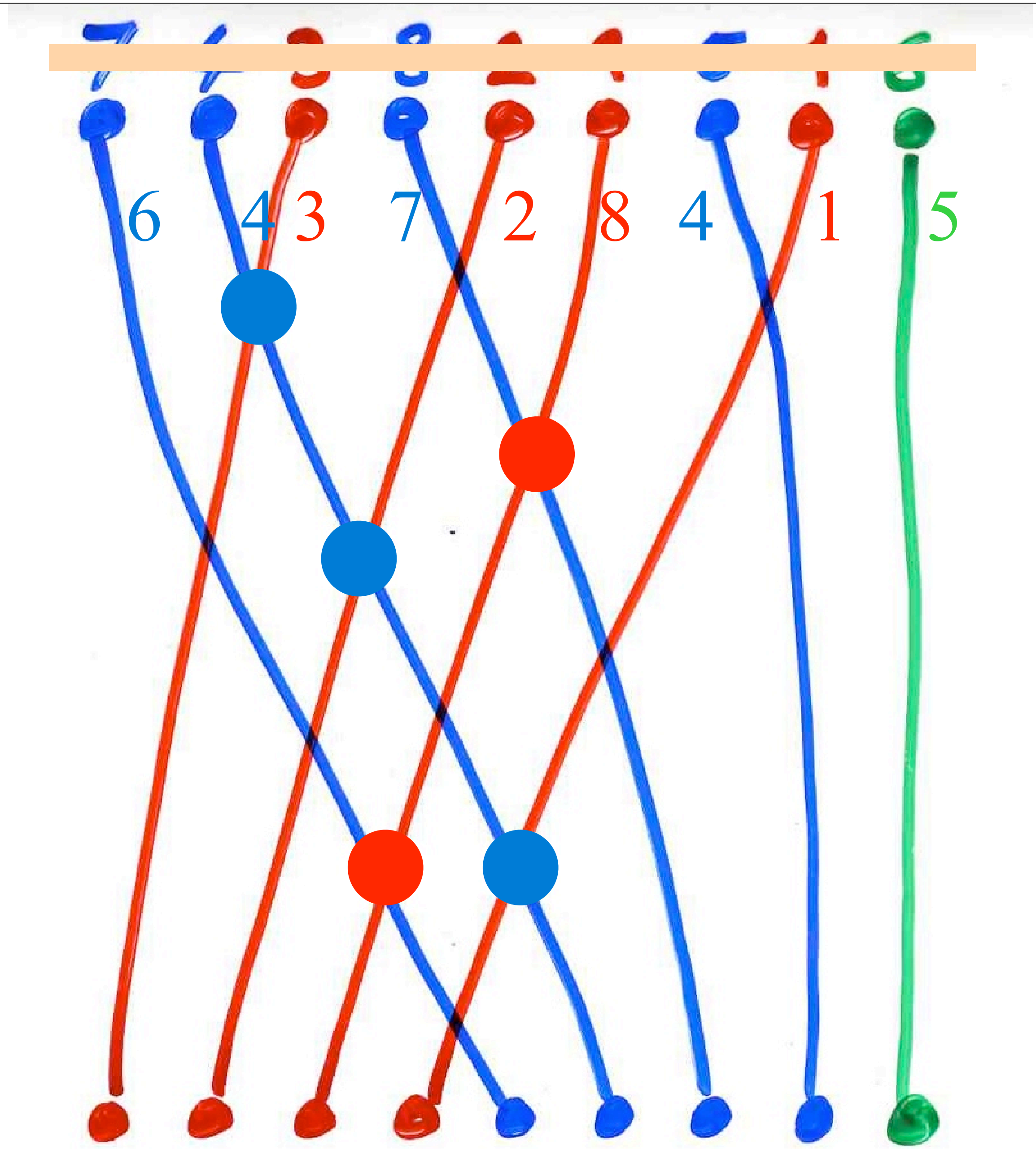
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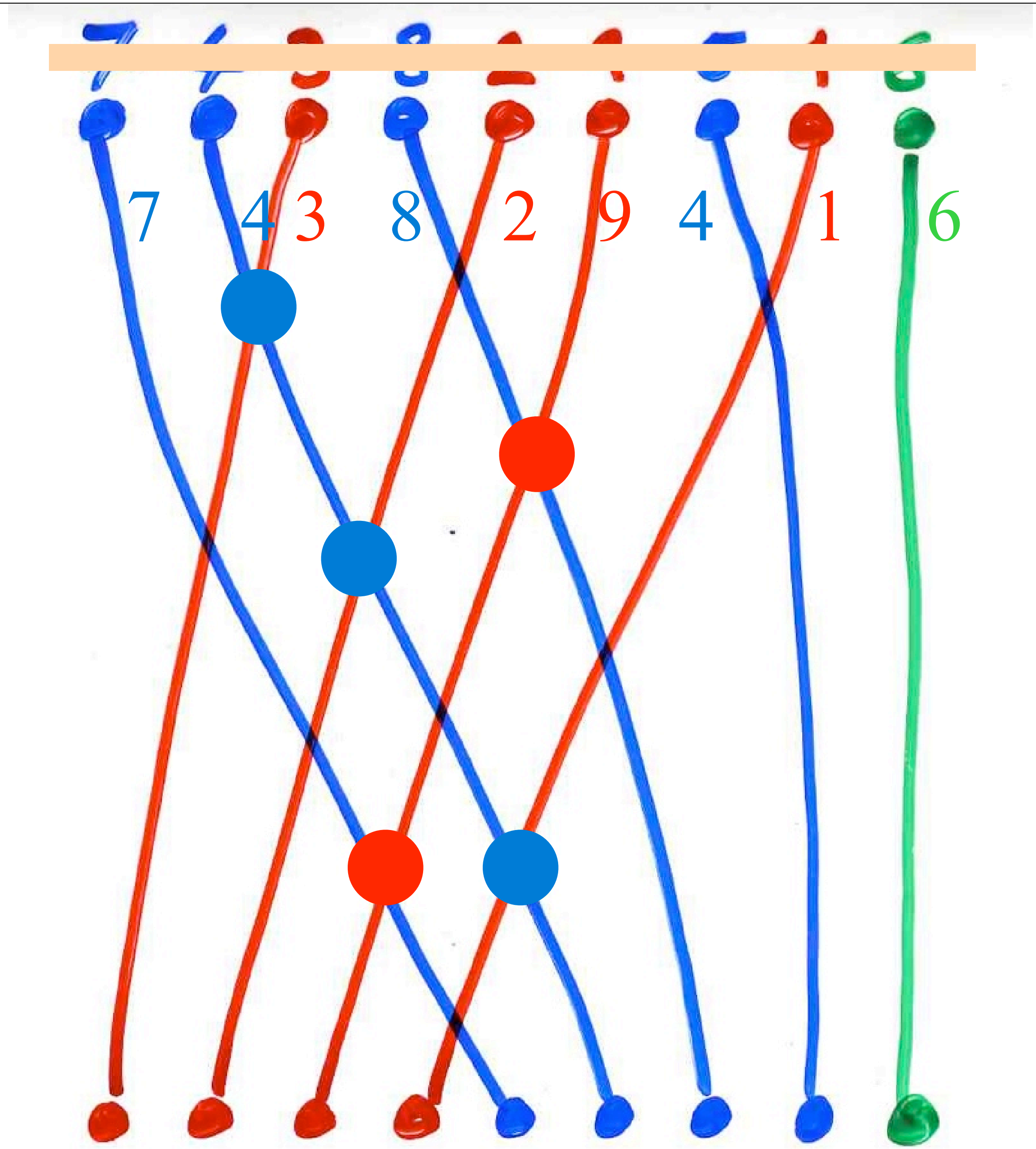
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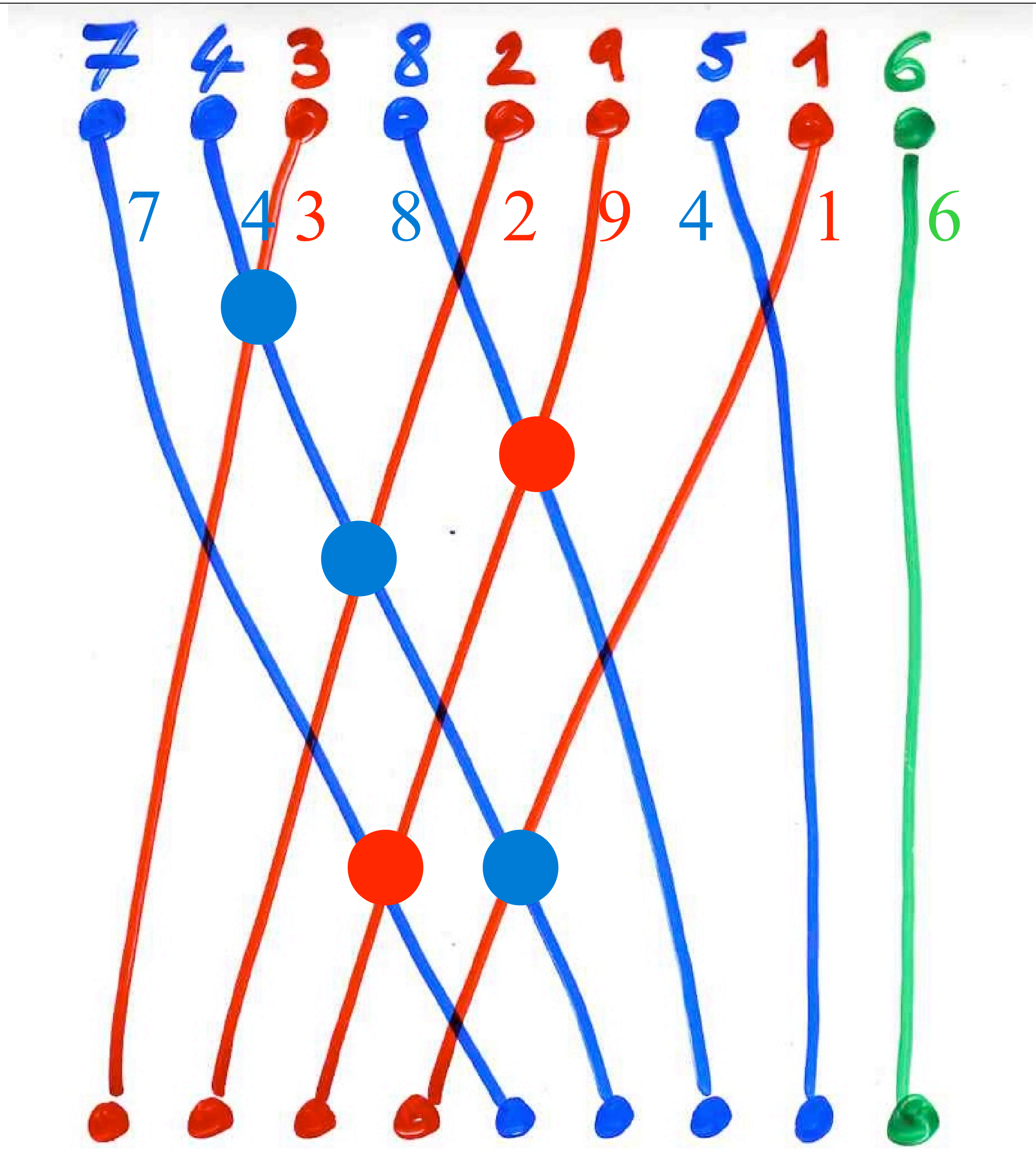

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