Alternative tableaux, permutations, a Robinson-Schensted like bijection and the asymmetric exclusion process in physics

dédié à la mémoire de Pierre Leroux (1942 - 2008)

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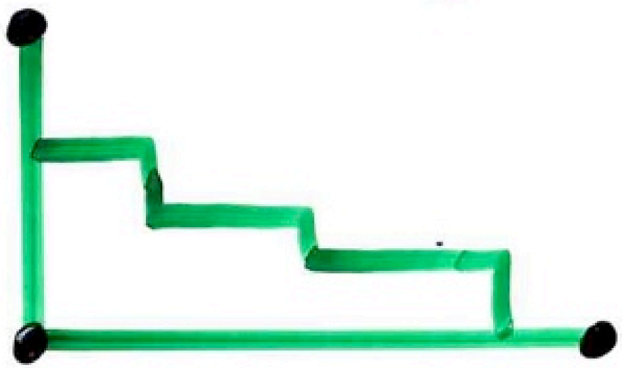
SLC 61
Curia, Portugal, Sept 2008

§1

alternative tableau: definition

alternative tableau

- Ferrers diagram **F**

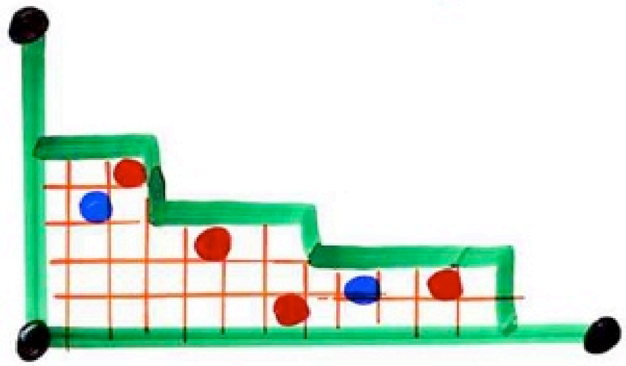


(possibly empty rows or column)

(nb of rows) + (nb of columns) $= n$

alternative tableau

- Ferrers diagram F



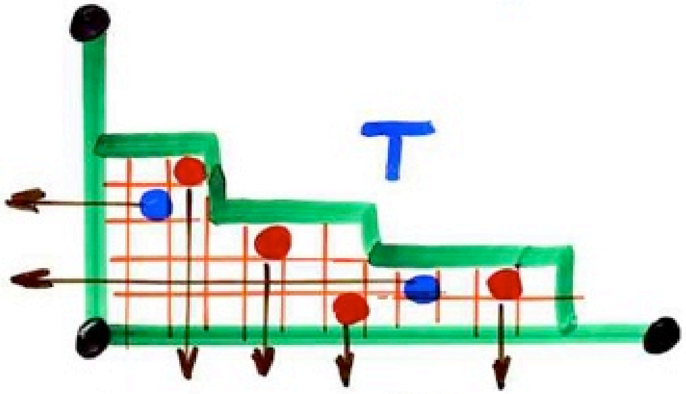
(possibly empty rows or column)

(nb of rows) + (nb of columns) = n

- some cells are coloured **red** or **blue**

alternative tableau T

- Ferrers diagram F (possibly empty rows or column)



(nb of rows) + (nb of columns) = n

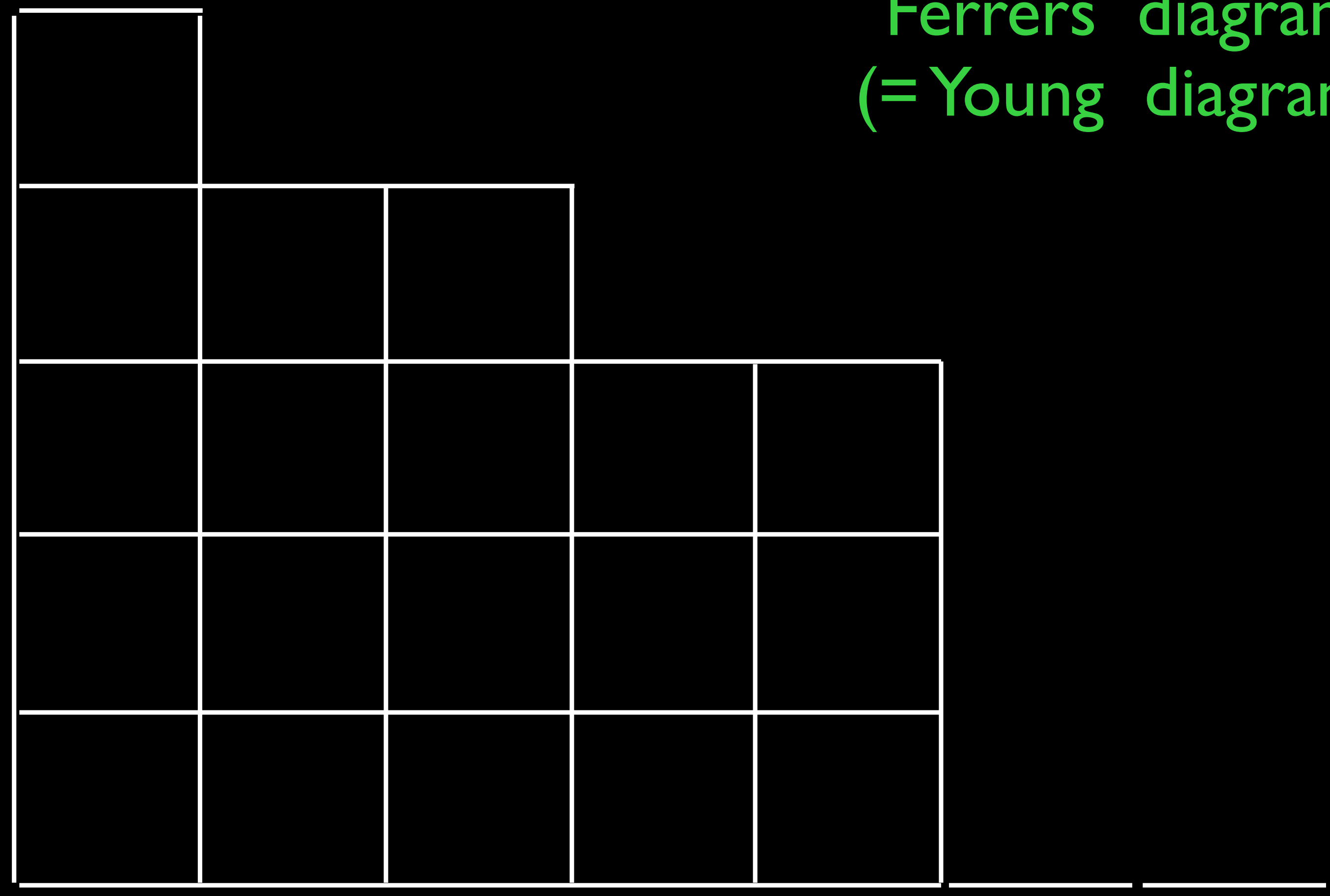
- some cells are coloured **red** or **blue**

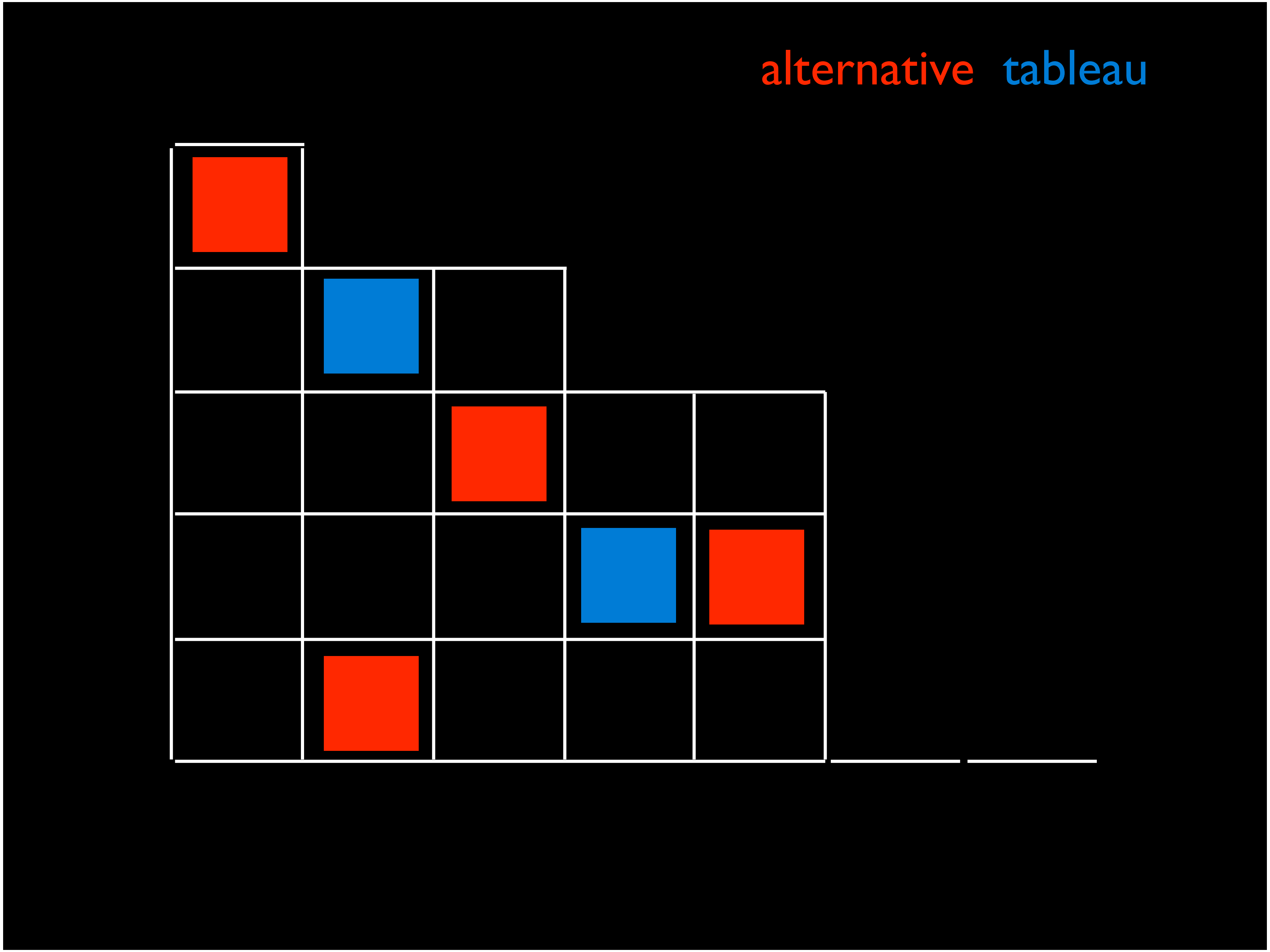
- { no coloured cell at the left of [blue cell]
  { no coloured cell below [red cell]

n size of T

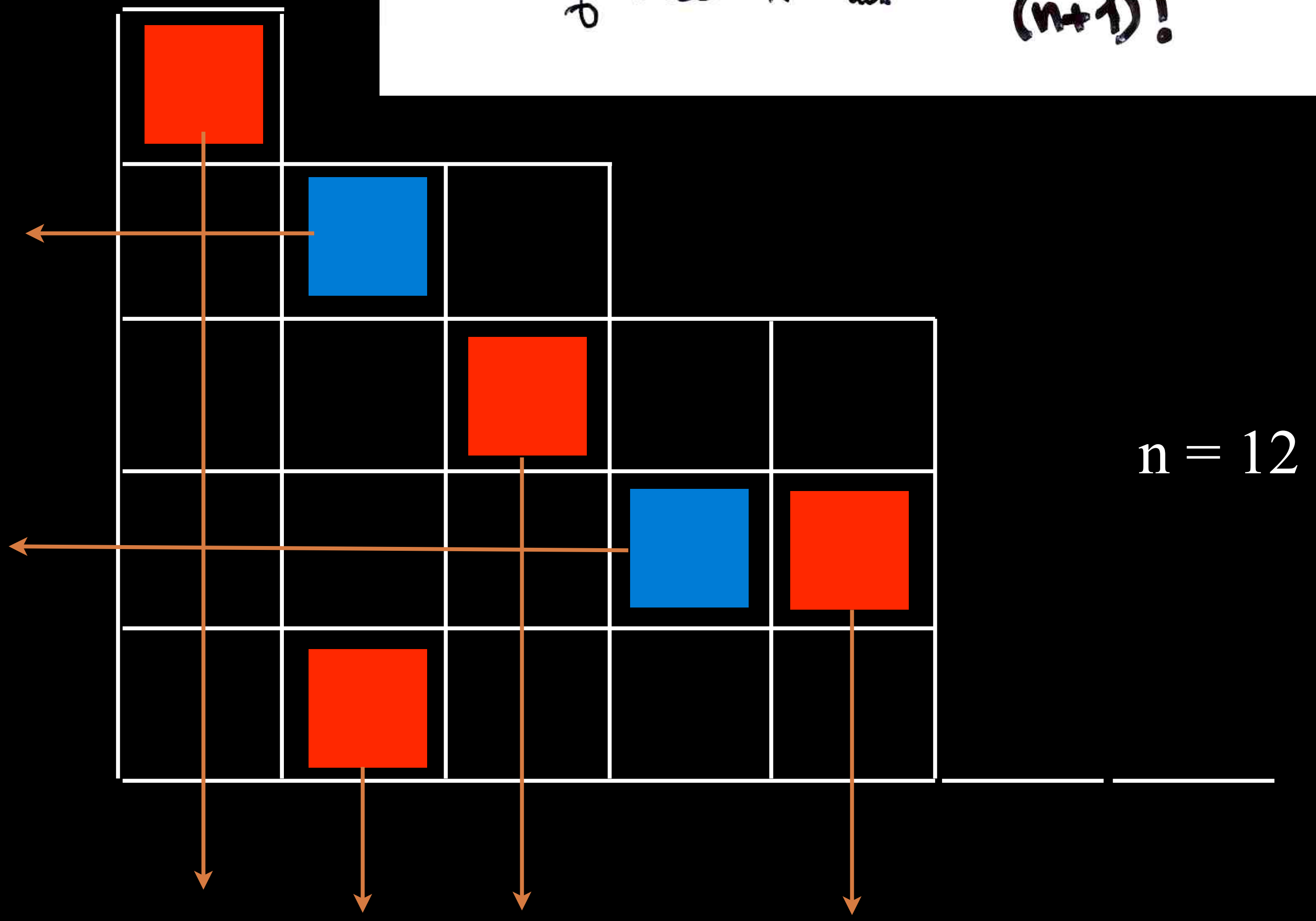
alternative tableau

Ferrers diagram
(= Young diagram)




alternative tableau

Prop. The number of alternative tableaux of size n is (n+1)!



ex: n=2

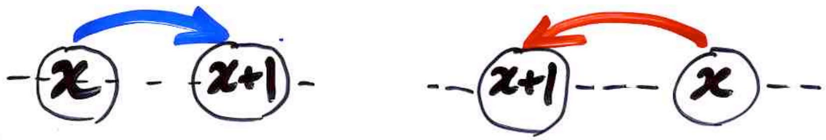
§2 The alternative bijection

Déf – Permutation $\sigma = \sigma(1) \cdots \sigma(n)$

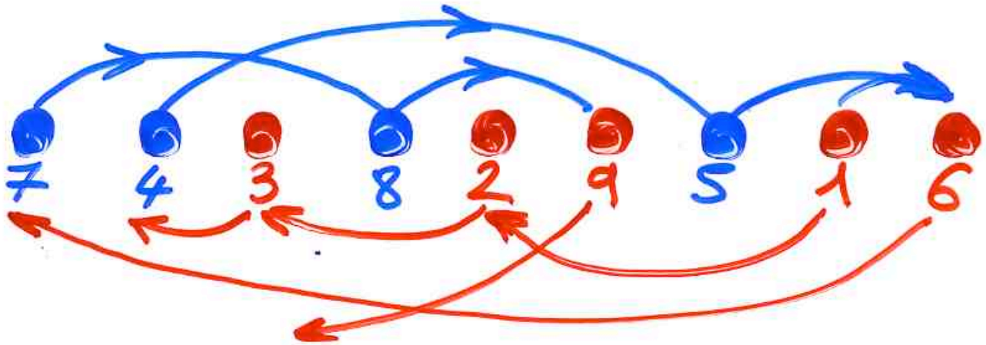
$x = \sigma(i)$, $1 \leq x < n$

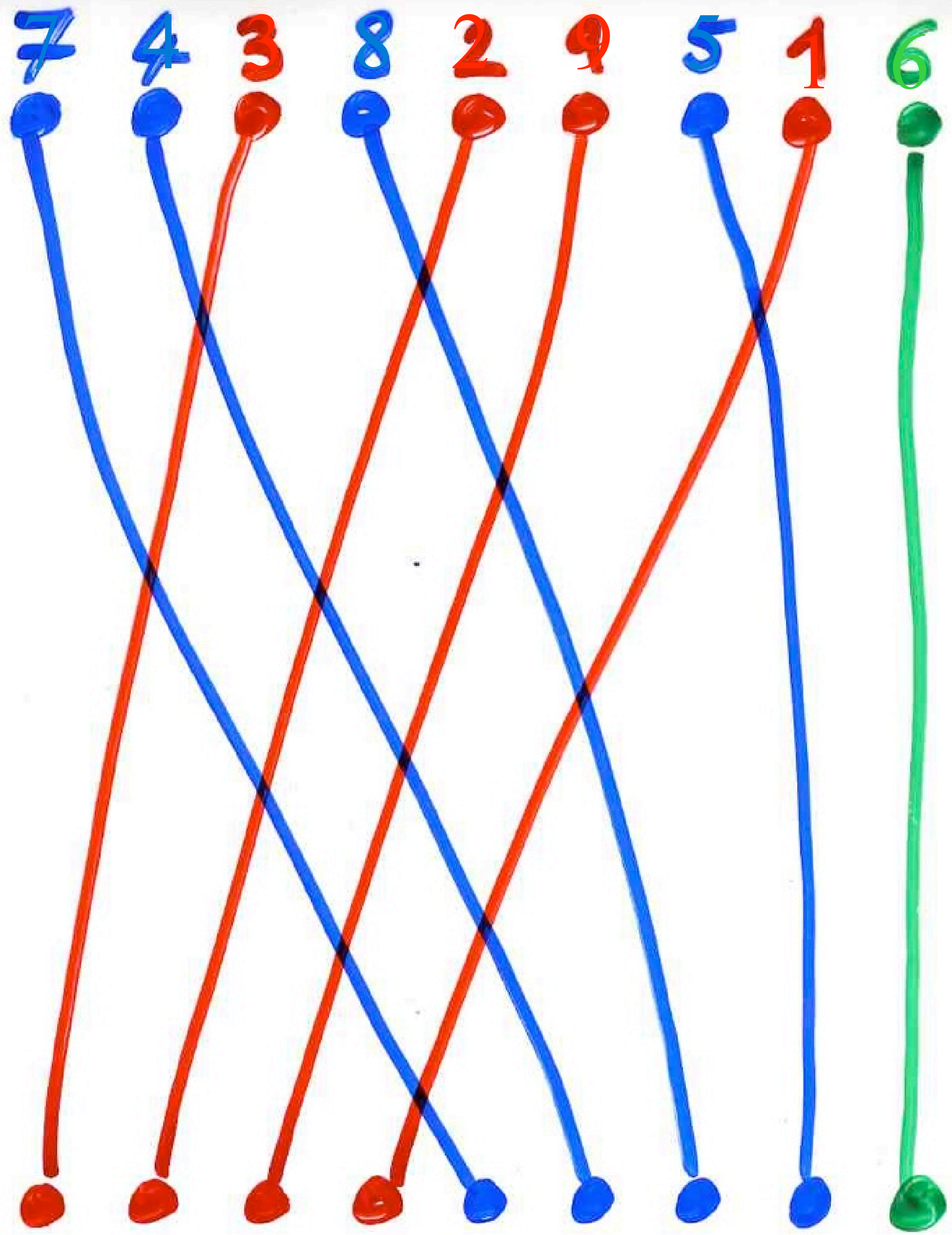
(valeur) x $\begin{cases} \text{avance} \\ \text{recul} \end{cases}$ $x+1 = \sigma(j)$, $\begin{cases} i < j \\ j < i \end{cases}$

- convention $x = n$ est un recul



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$

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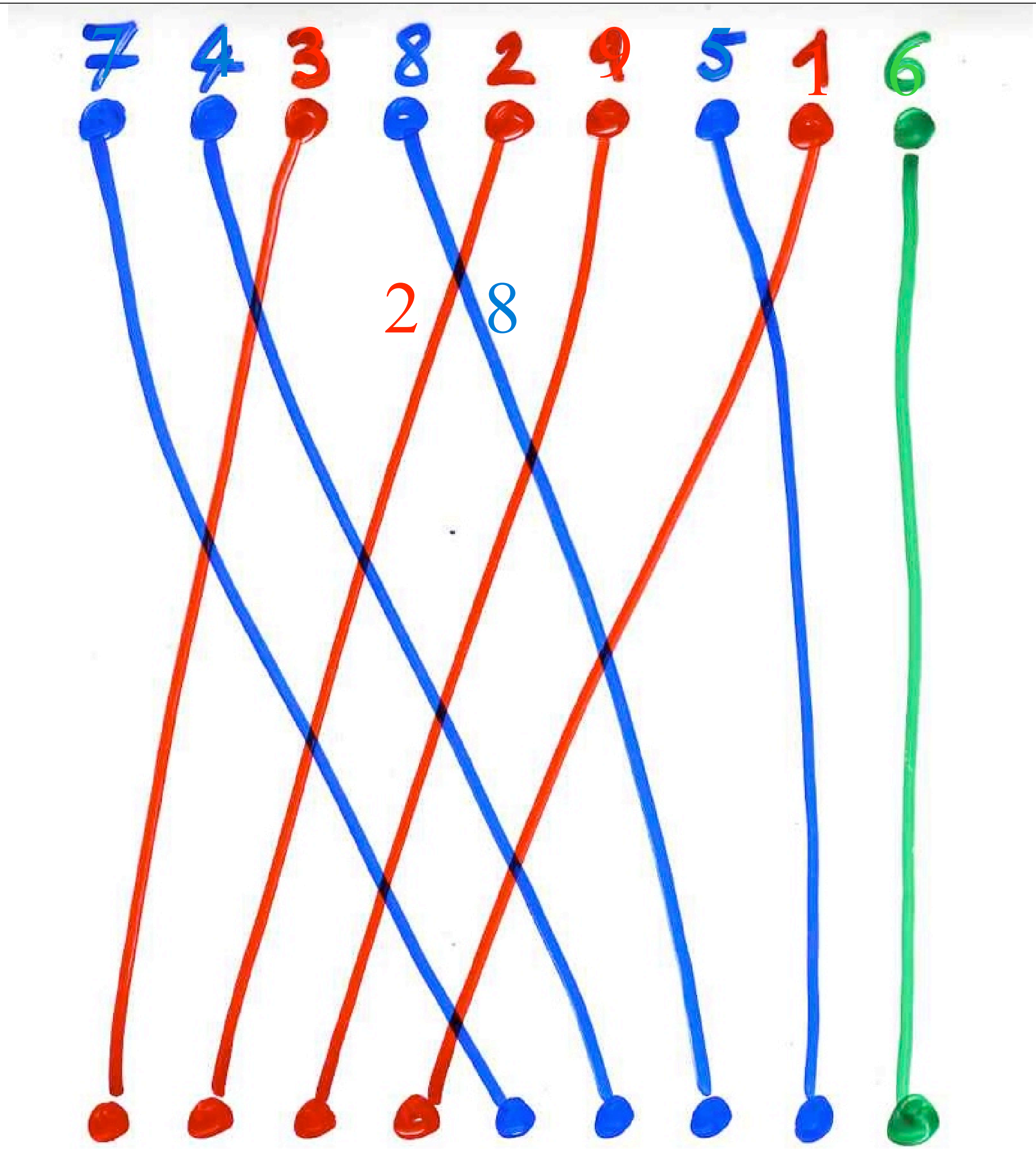
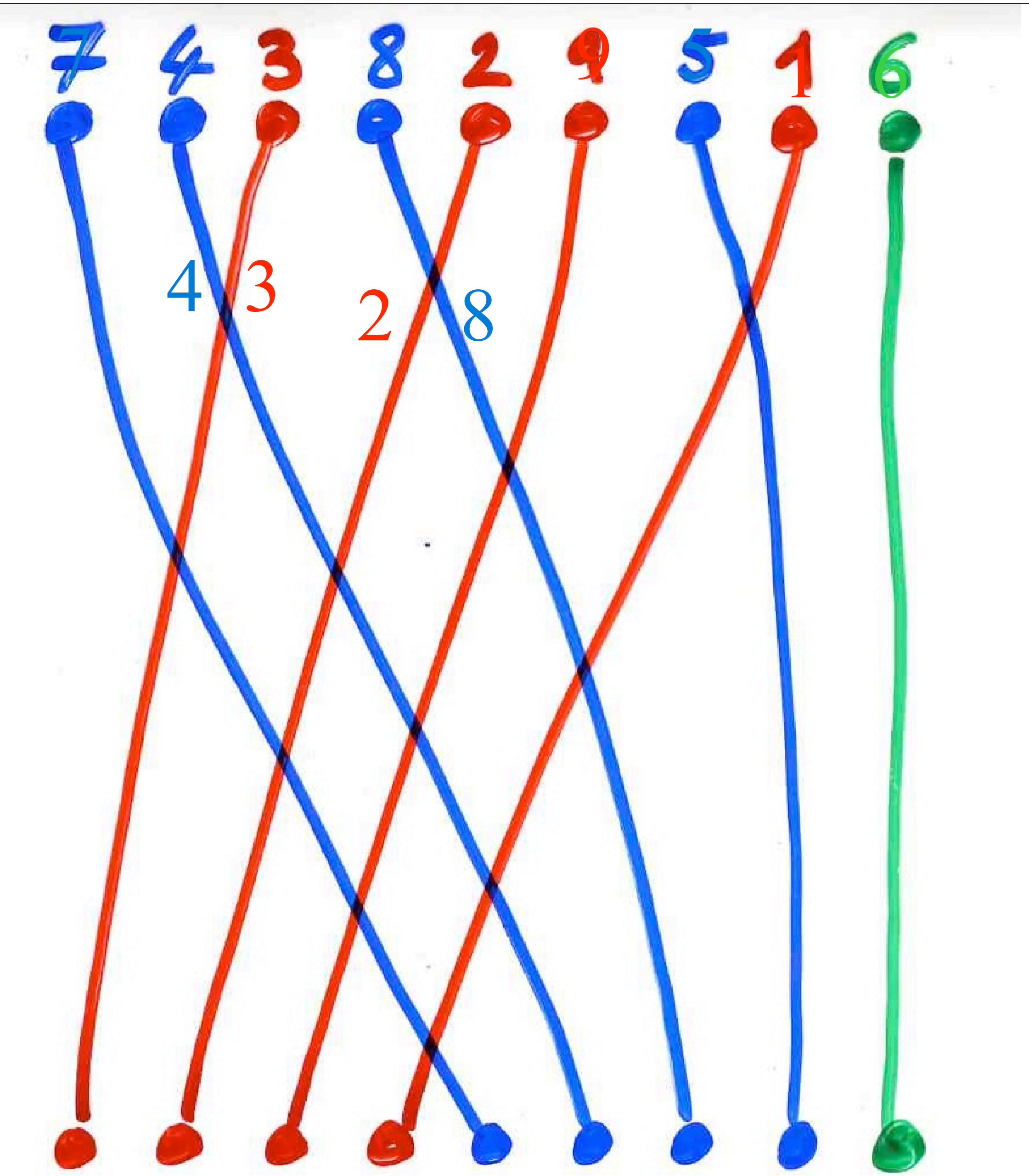


7 4 3 8 2 9 5 1 6
2 8



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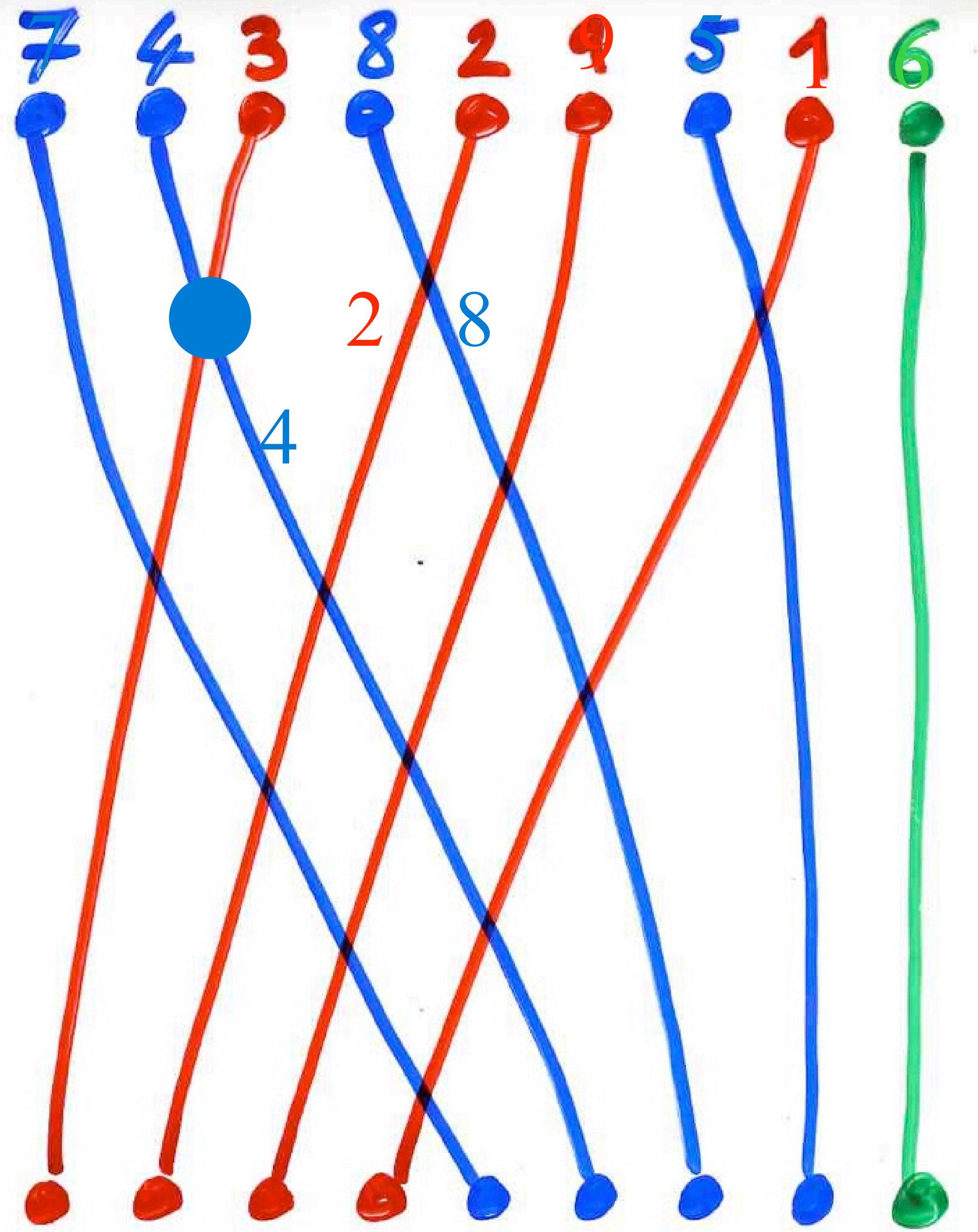
4 3
2 8



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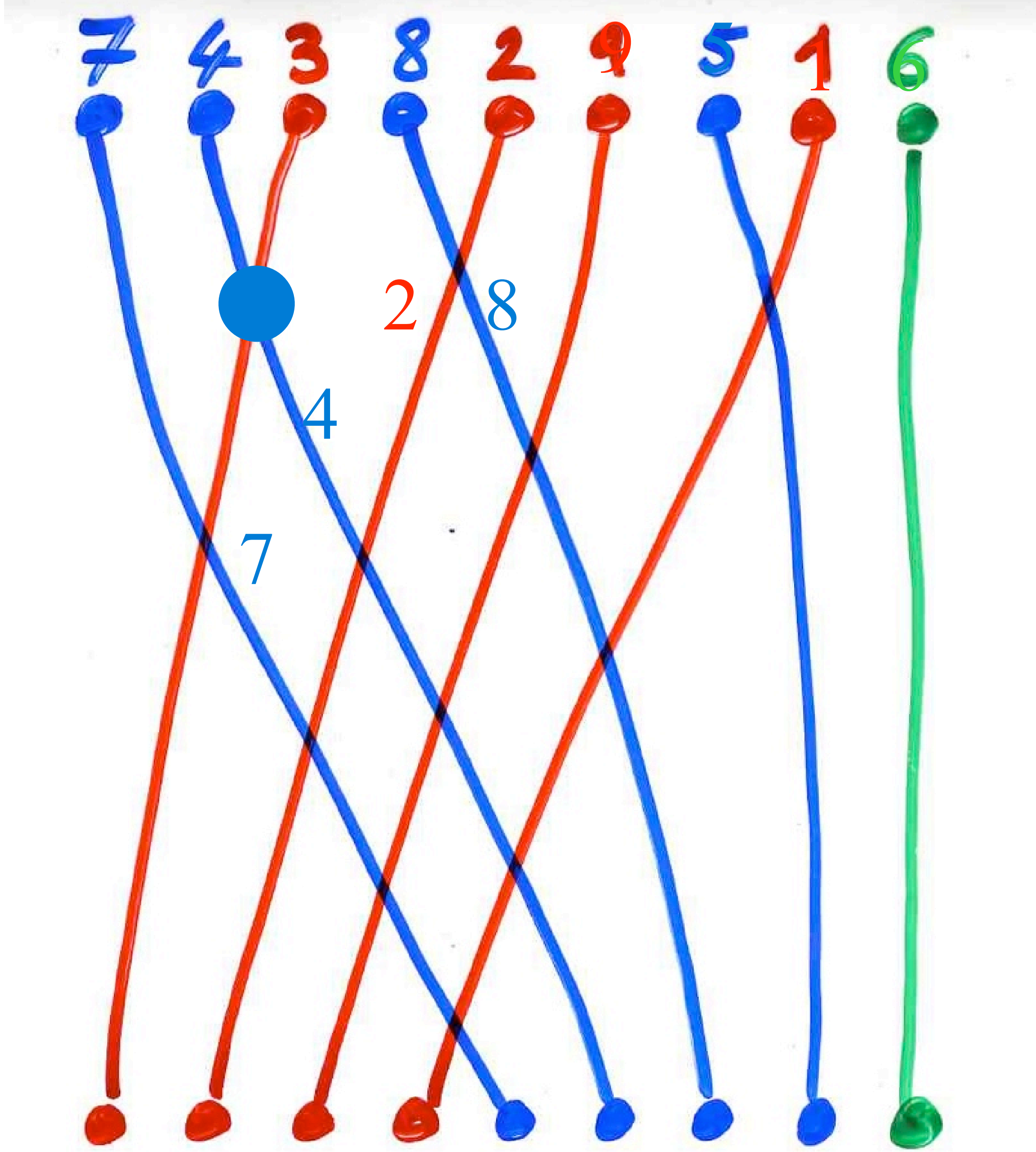
2
8
4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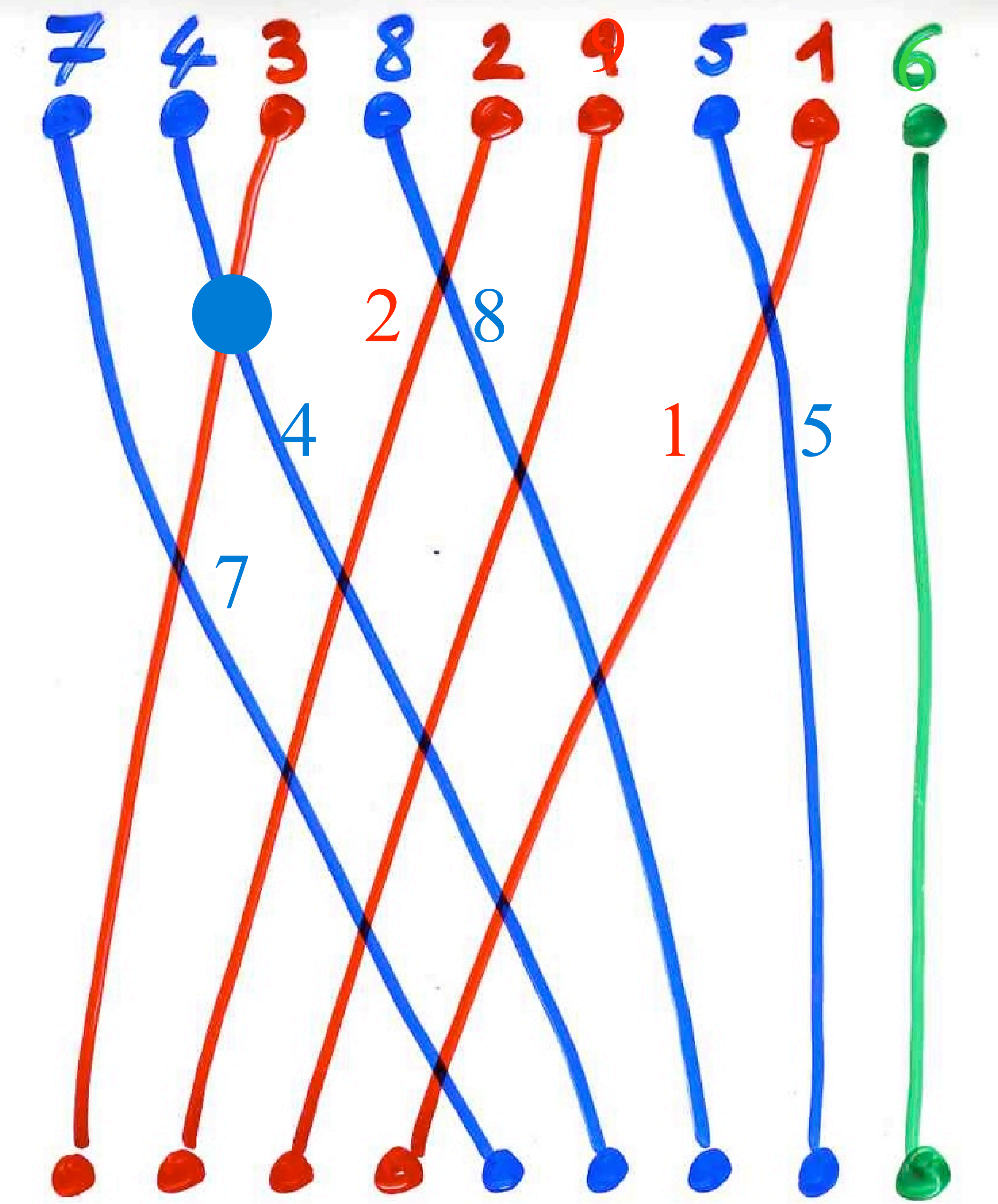
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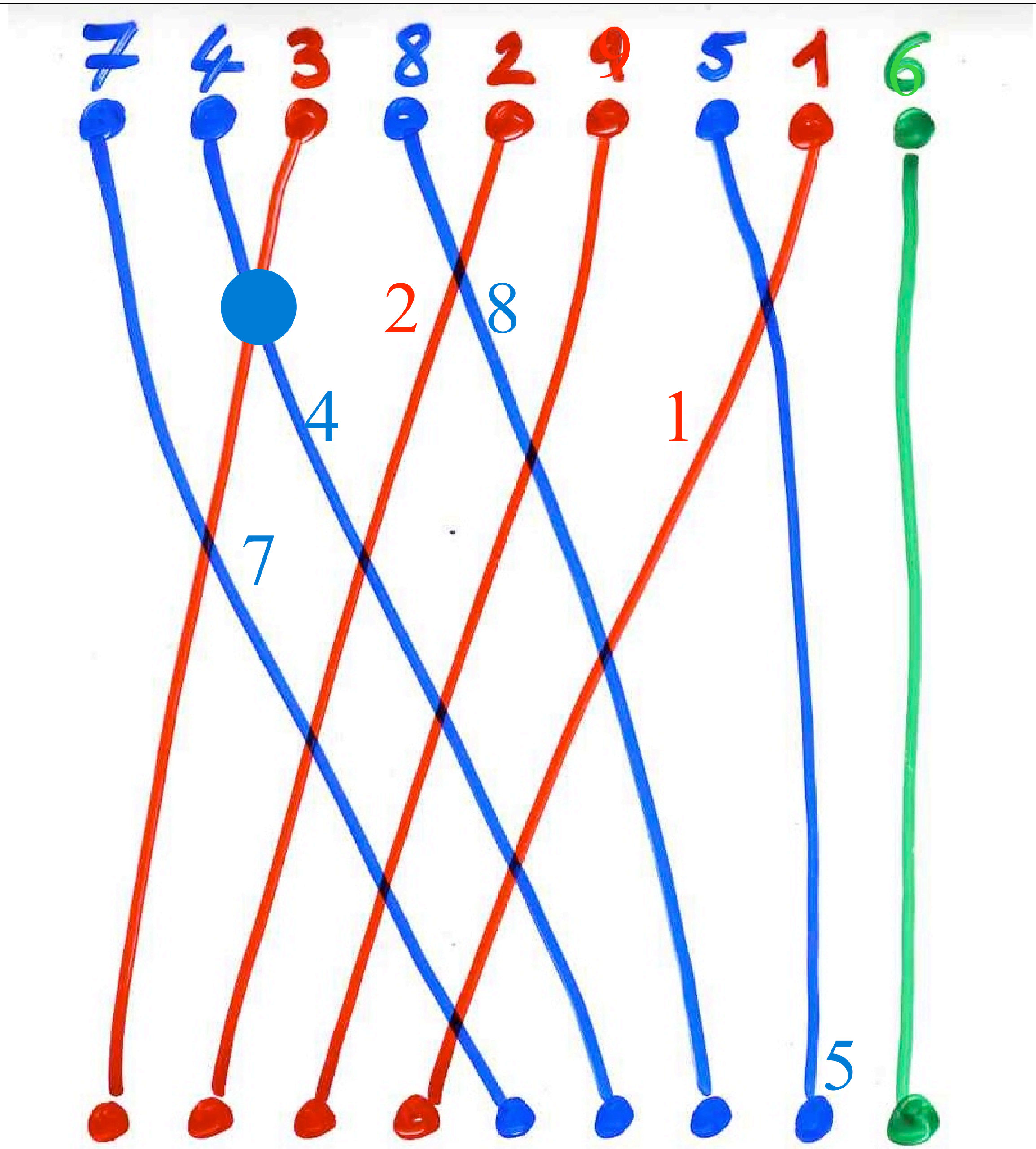
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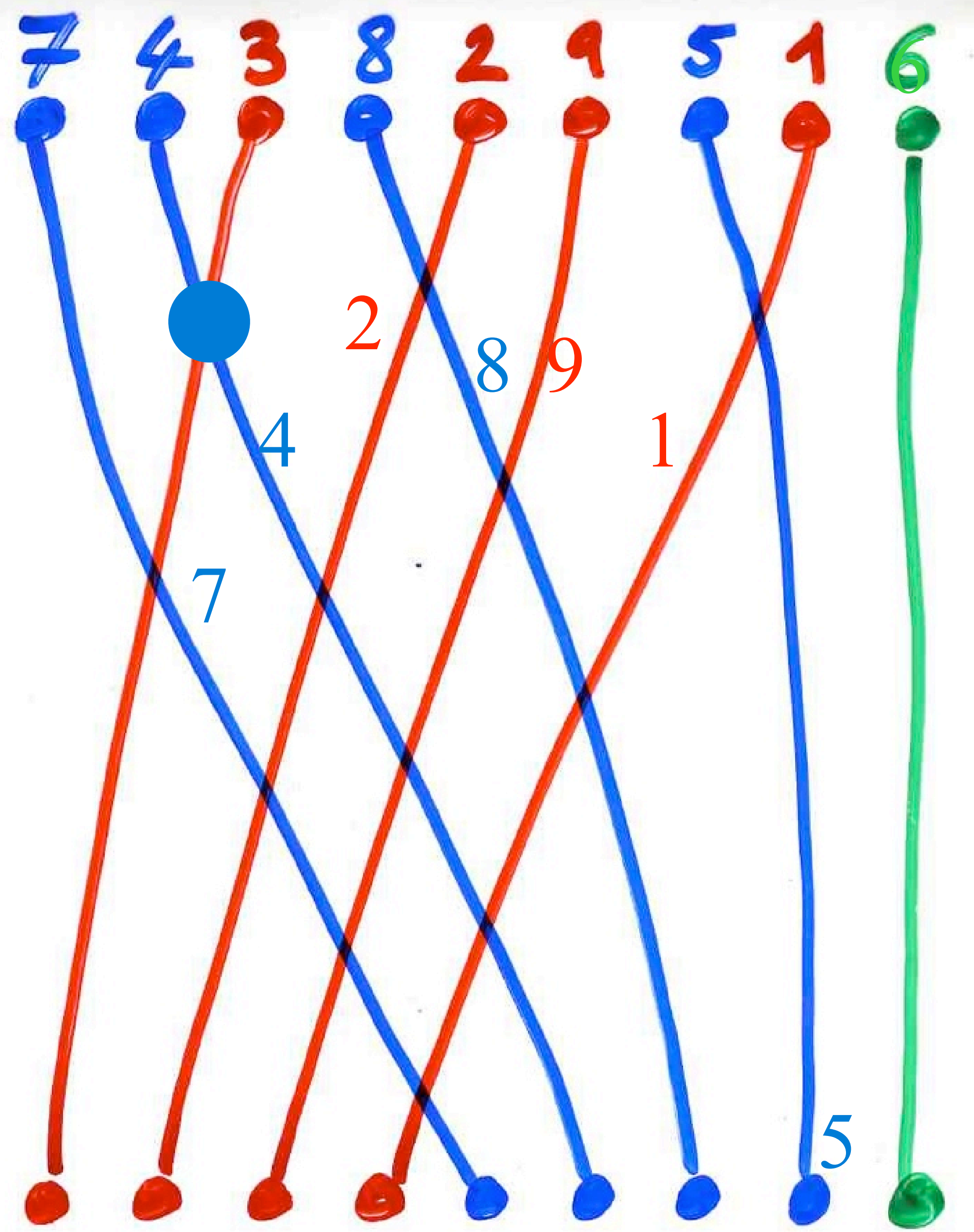
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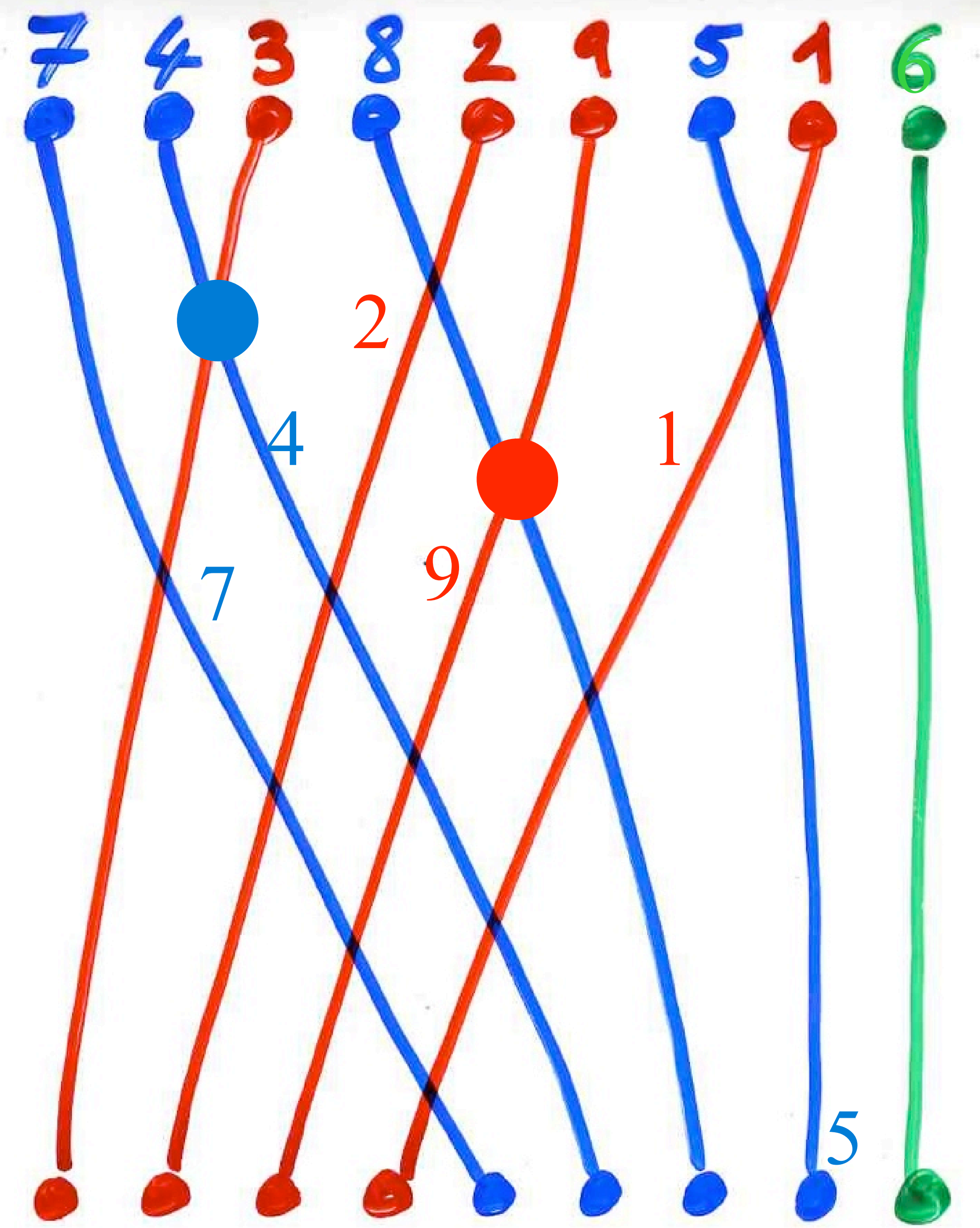

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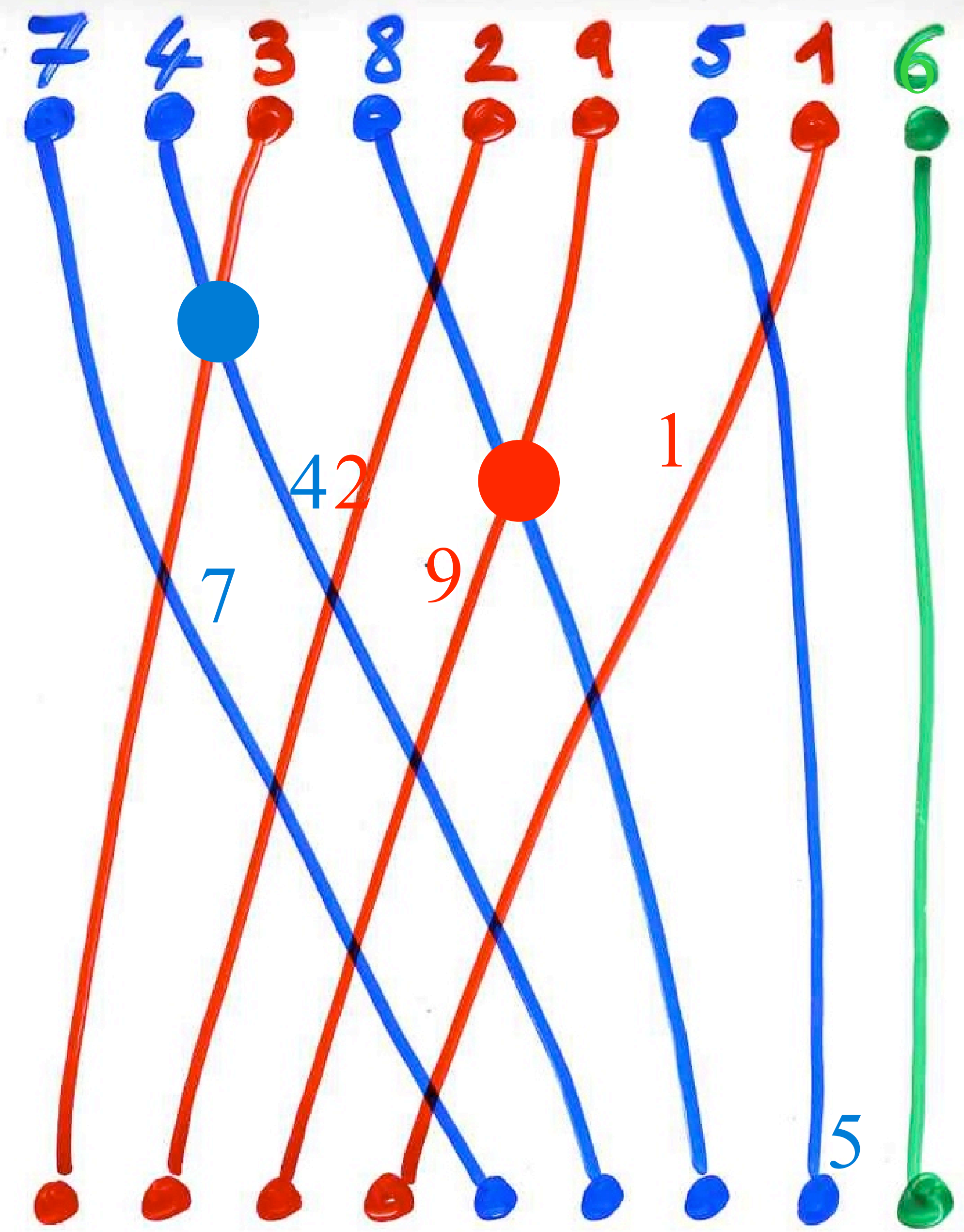

7 4 3 8 2 9 5 1 6
4 2
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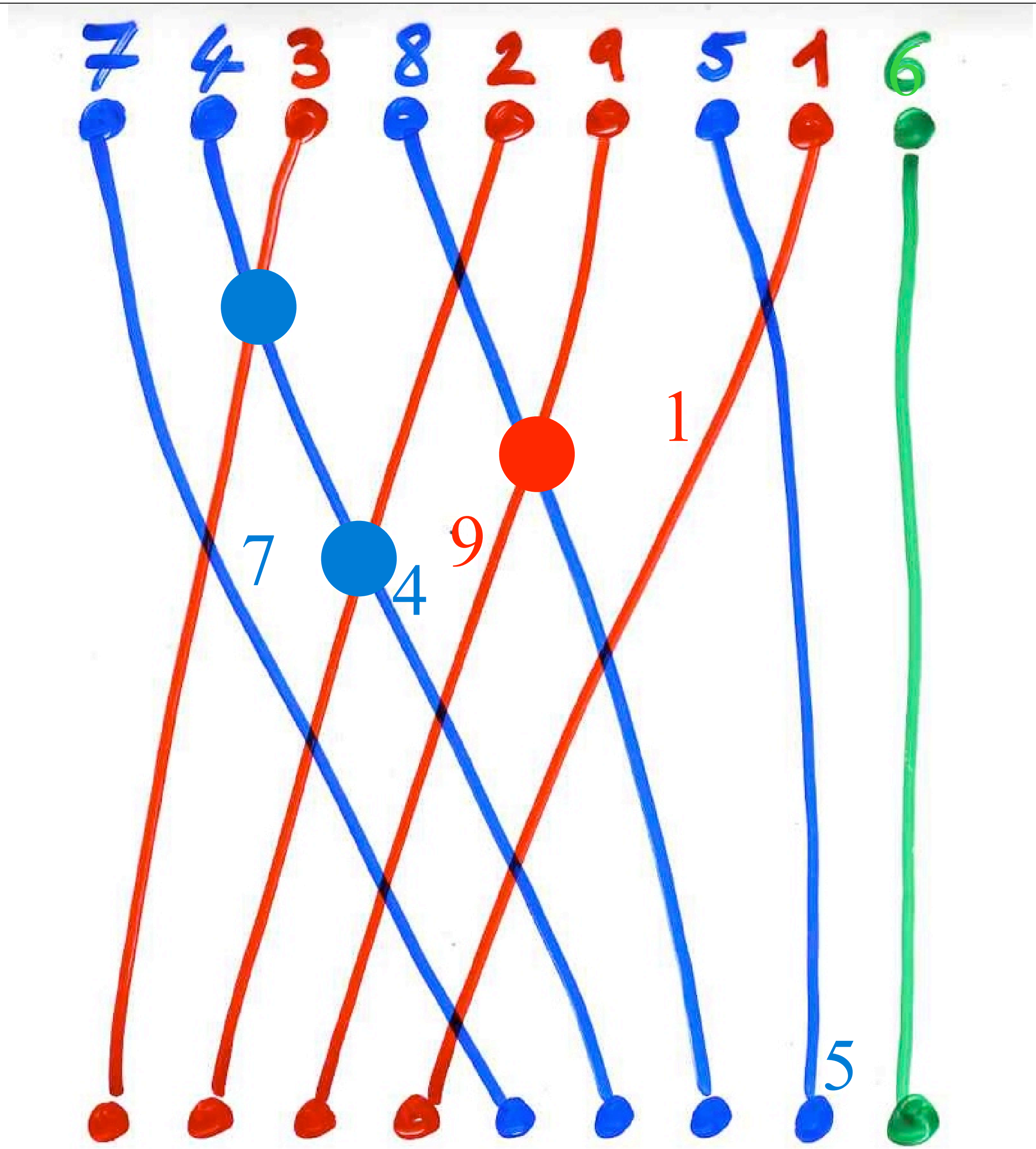
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1
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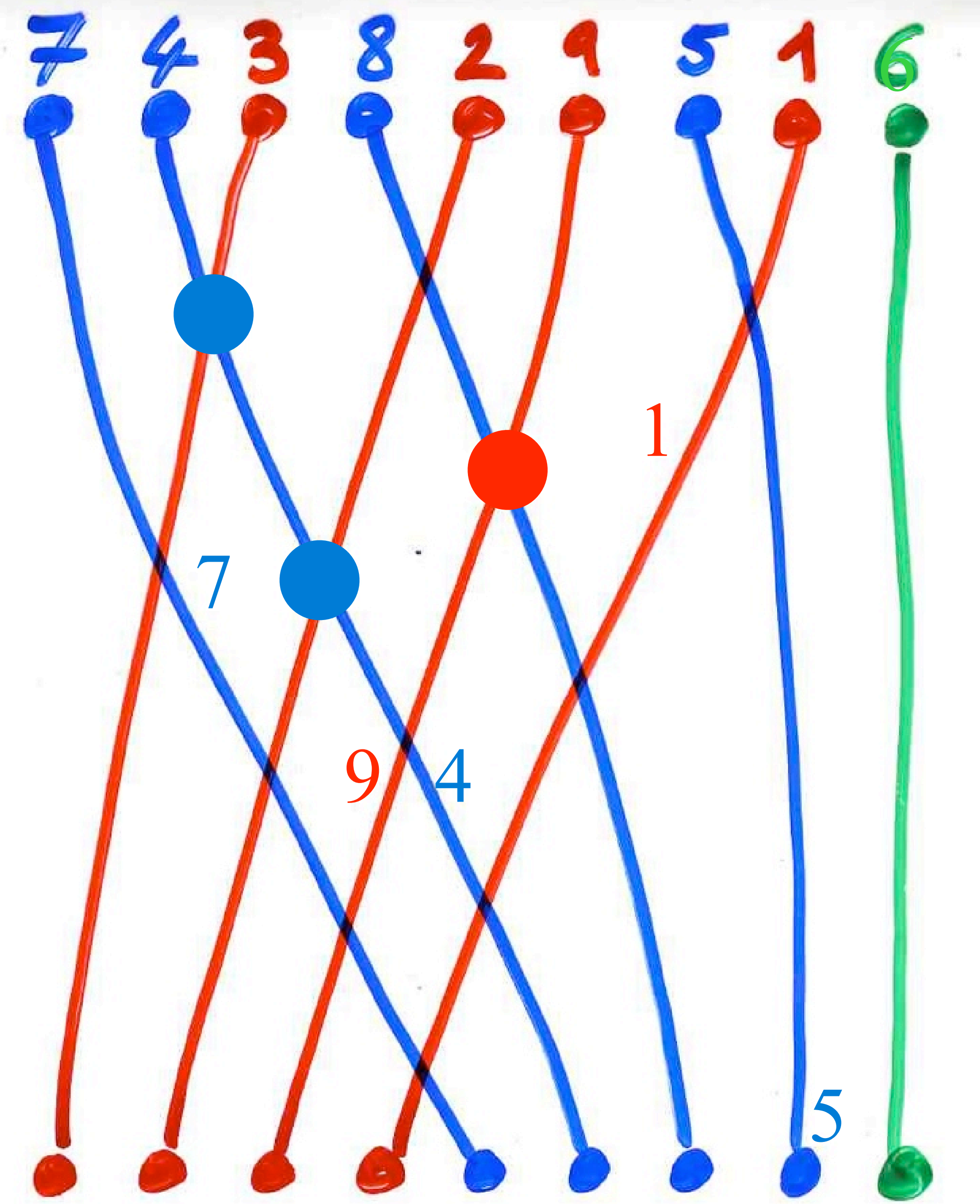

7 4 3 8 2 9 5 1 6
1
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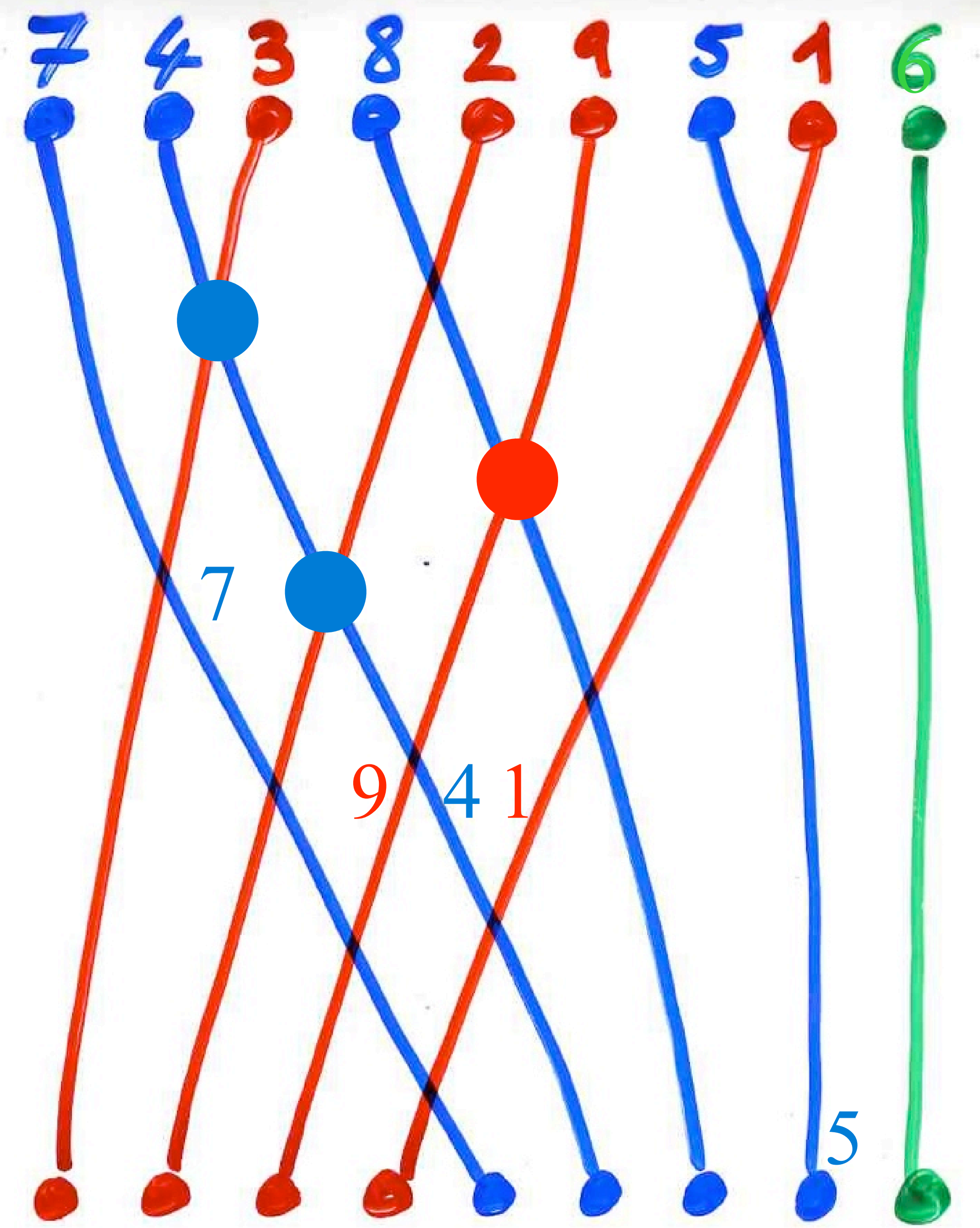
9 4
5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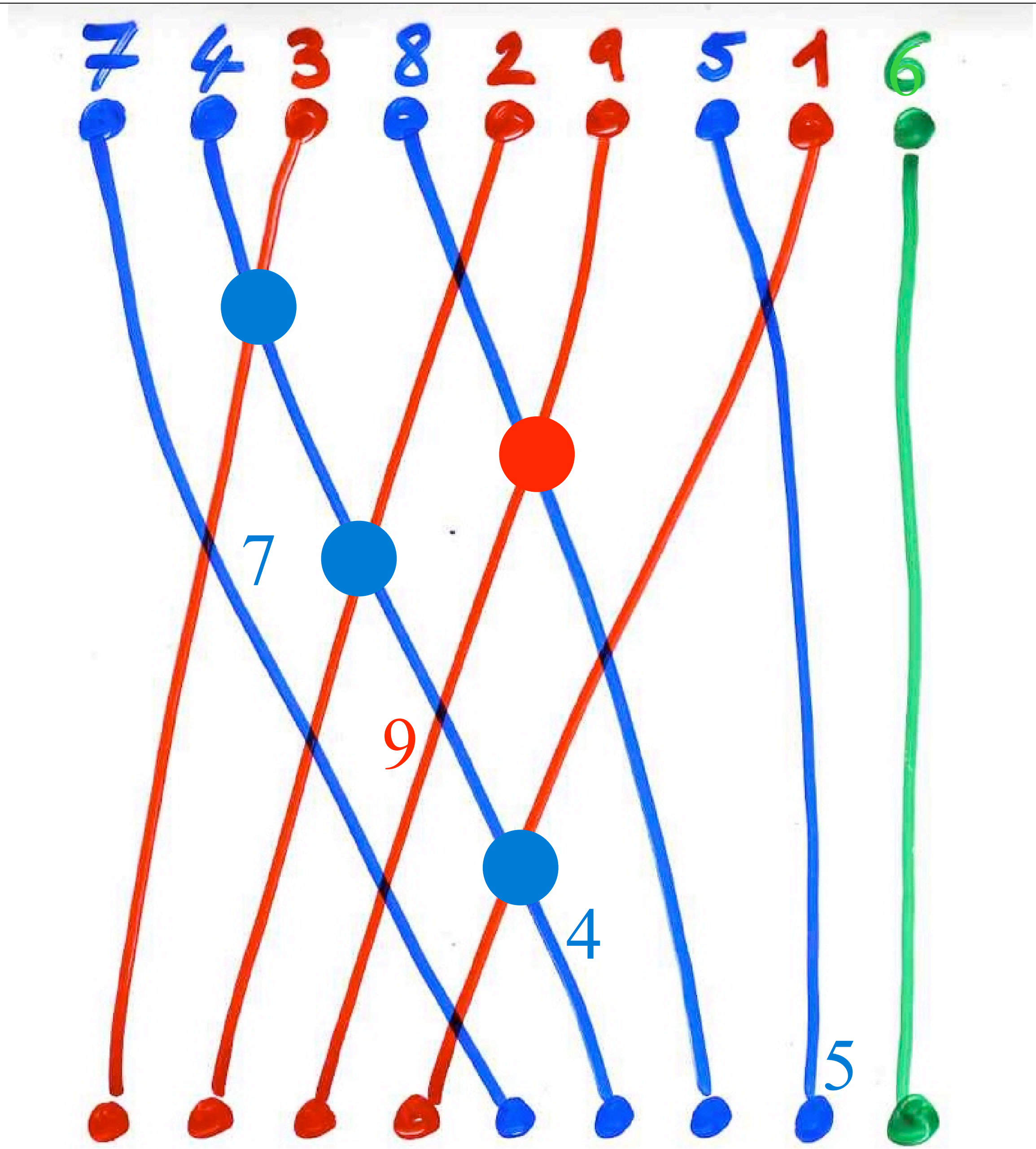
7
9 4 1
5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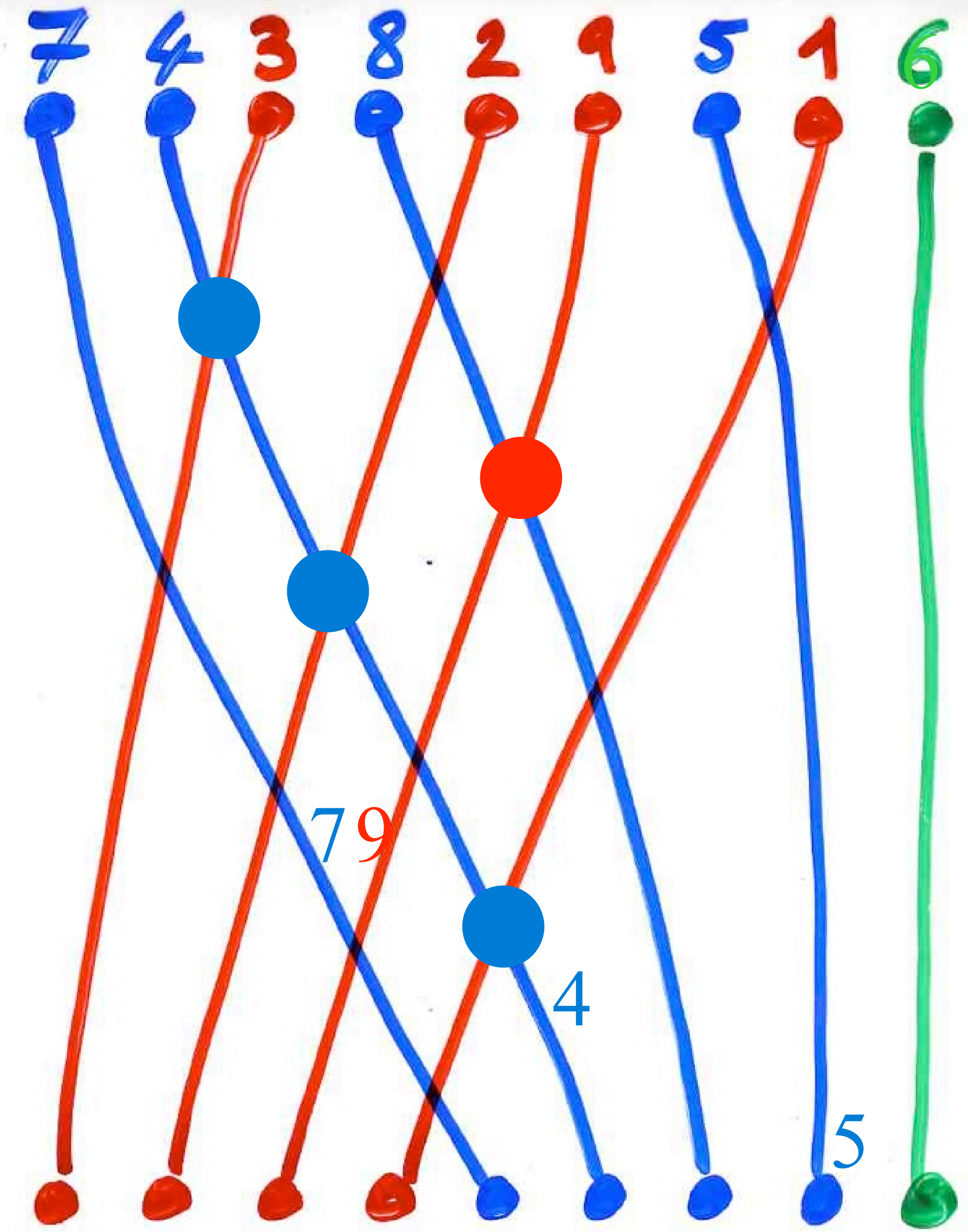
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79
4
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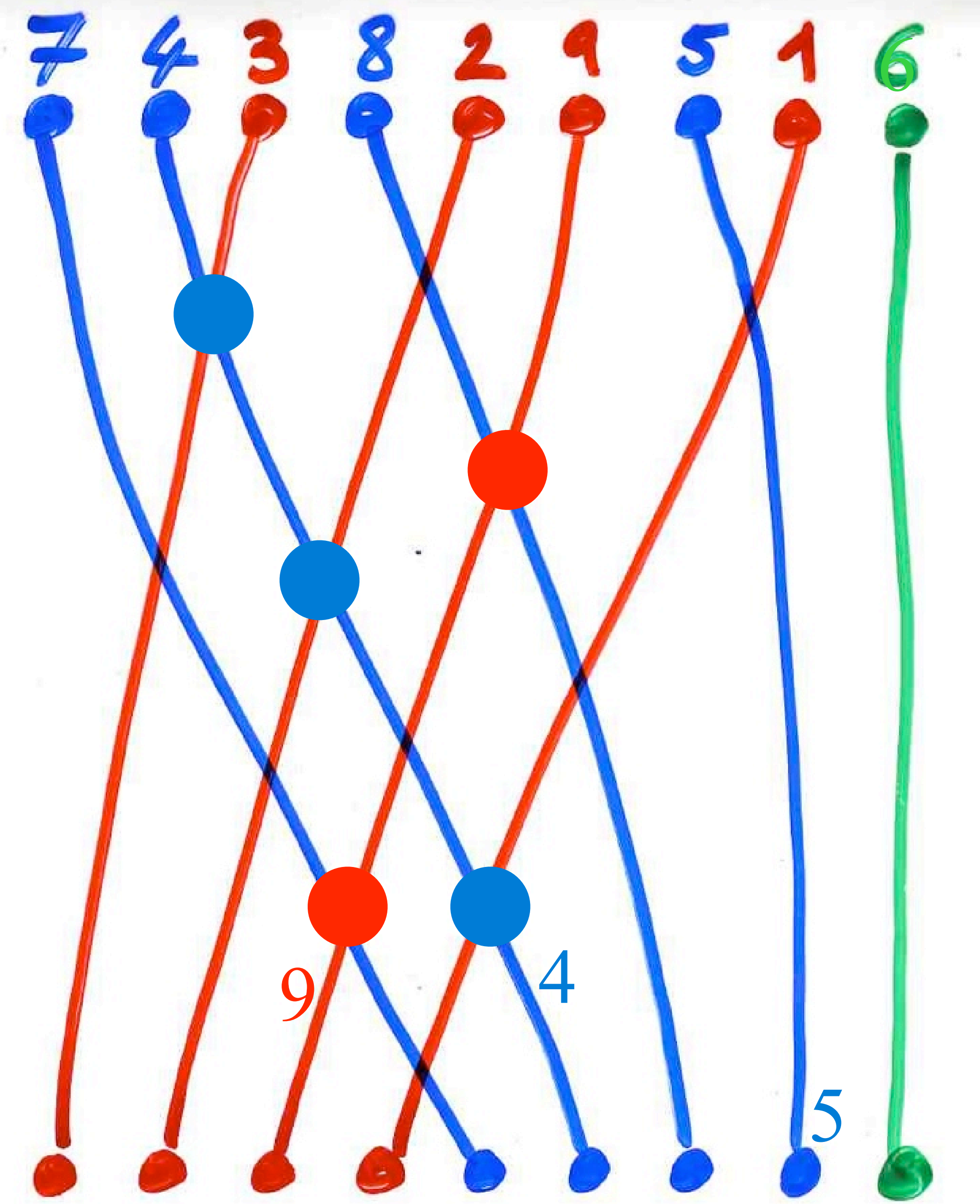
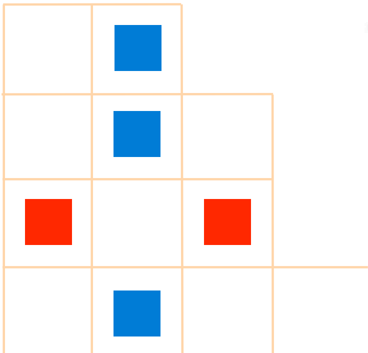
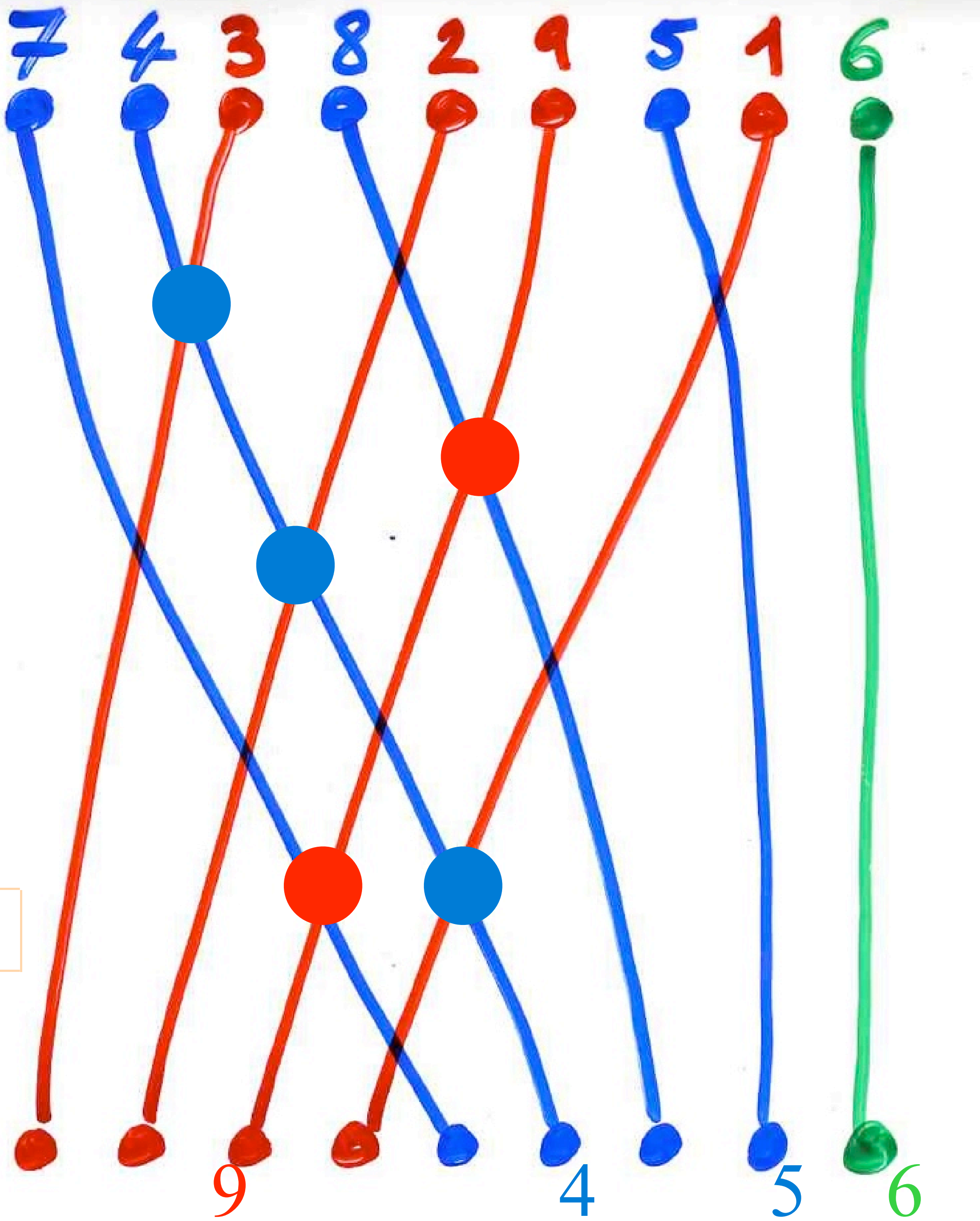


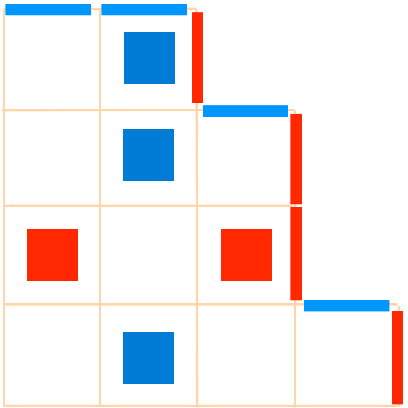
7 4 3 8 2 9 5 1 6
9
4
5

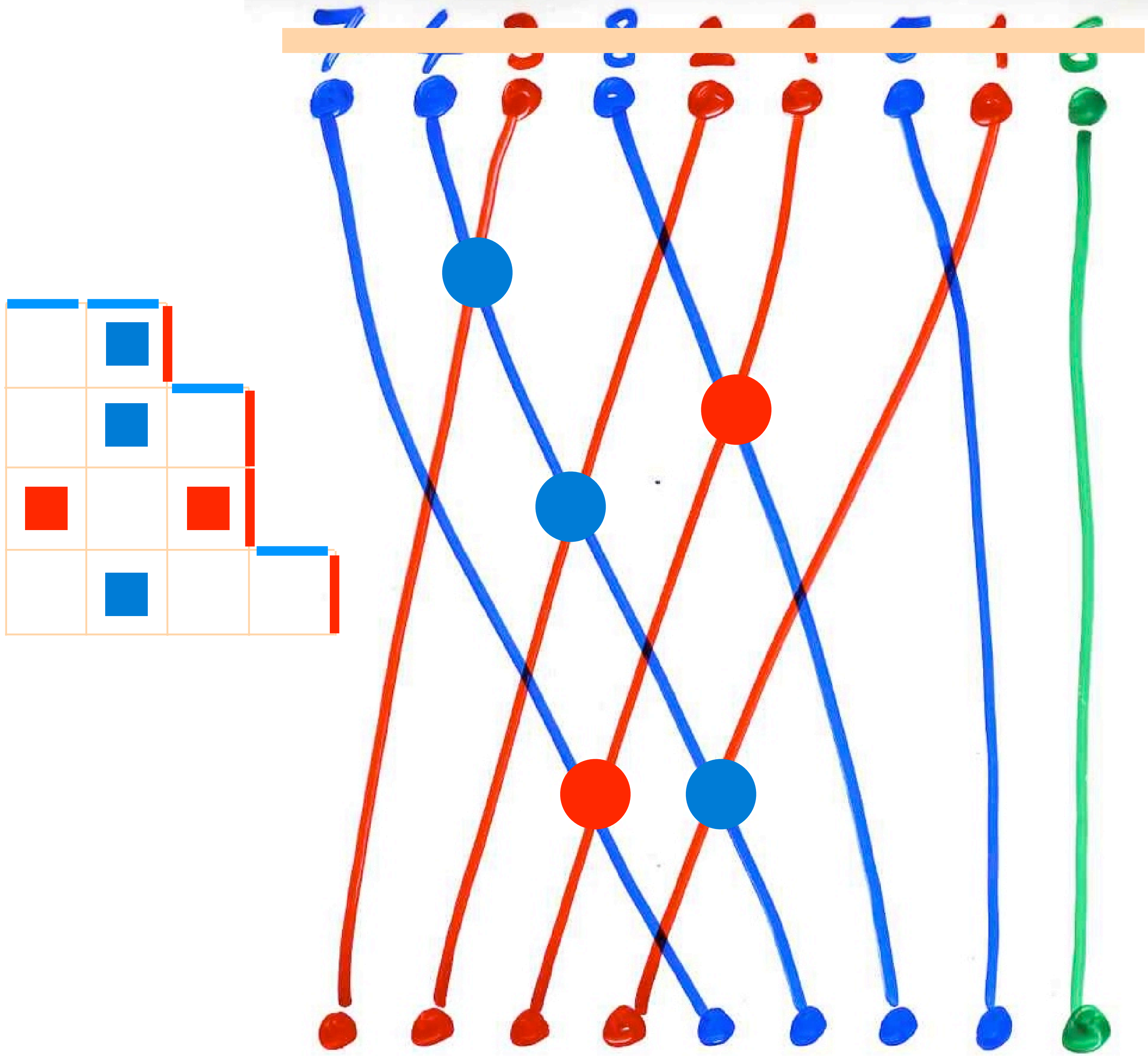
“exchange-deletion” algorithm

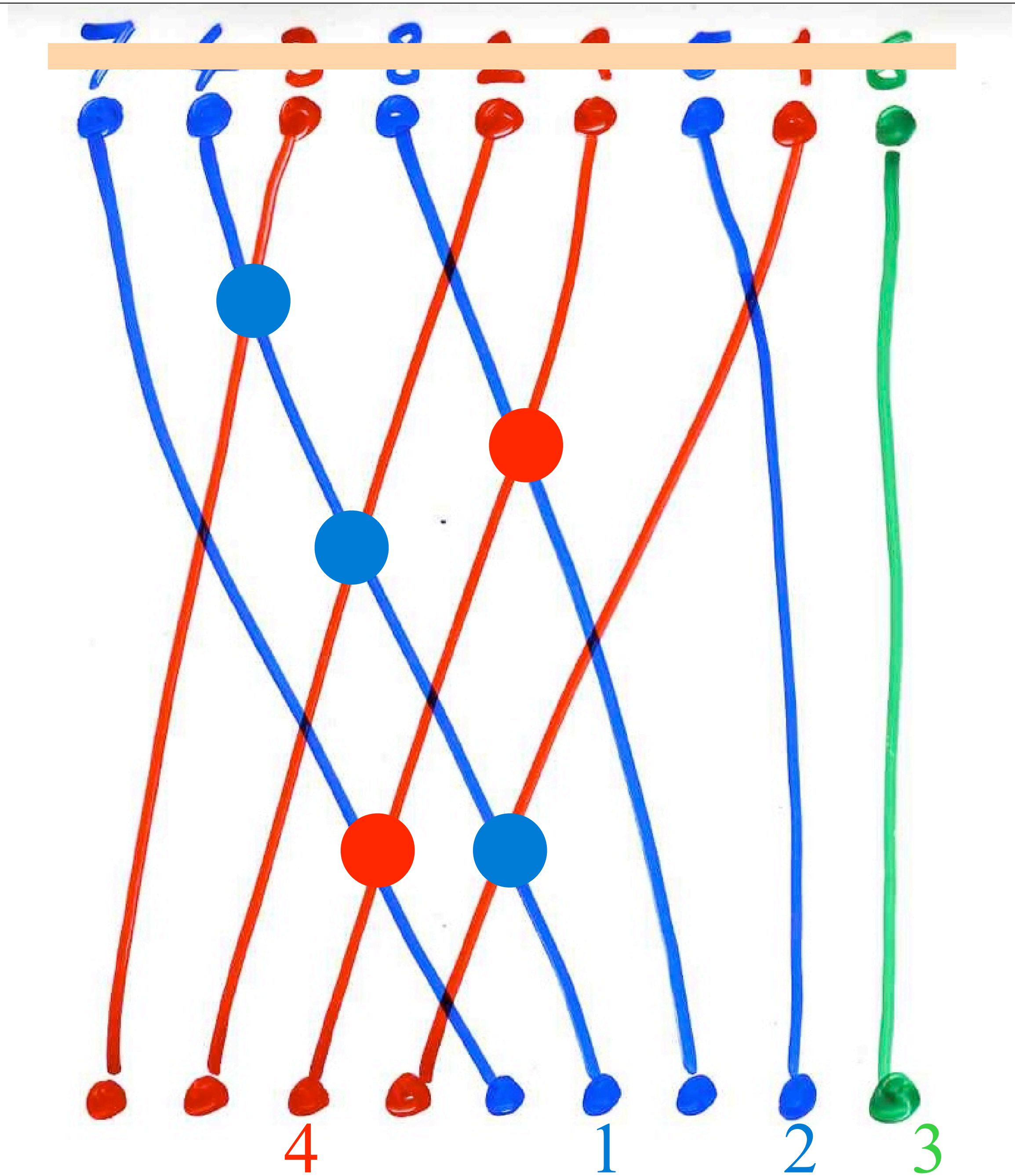


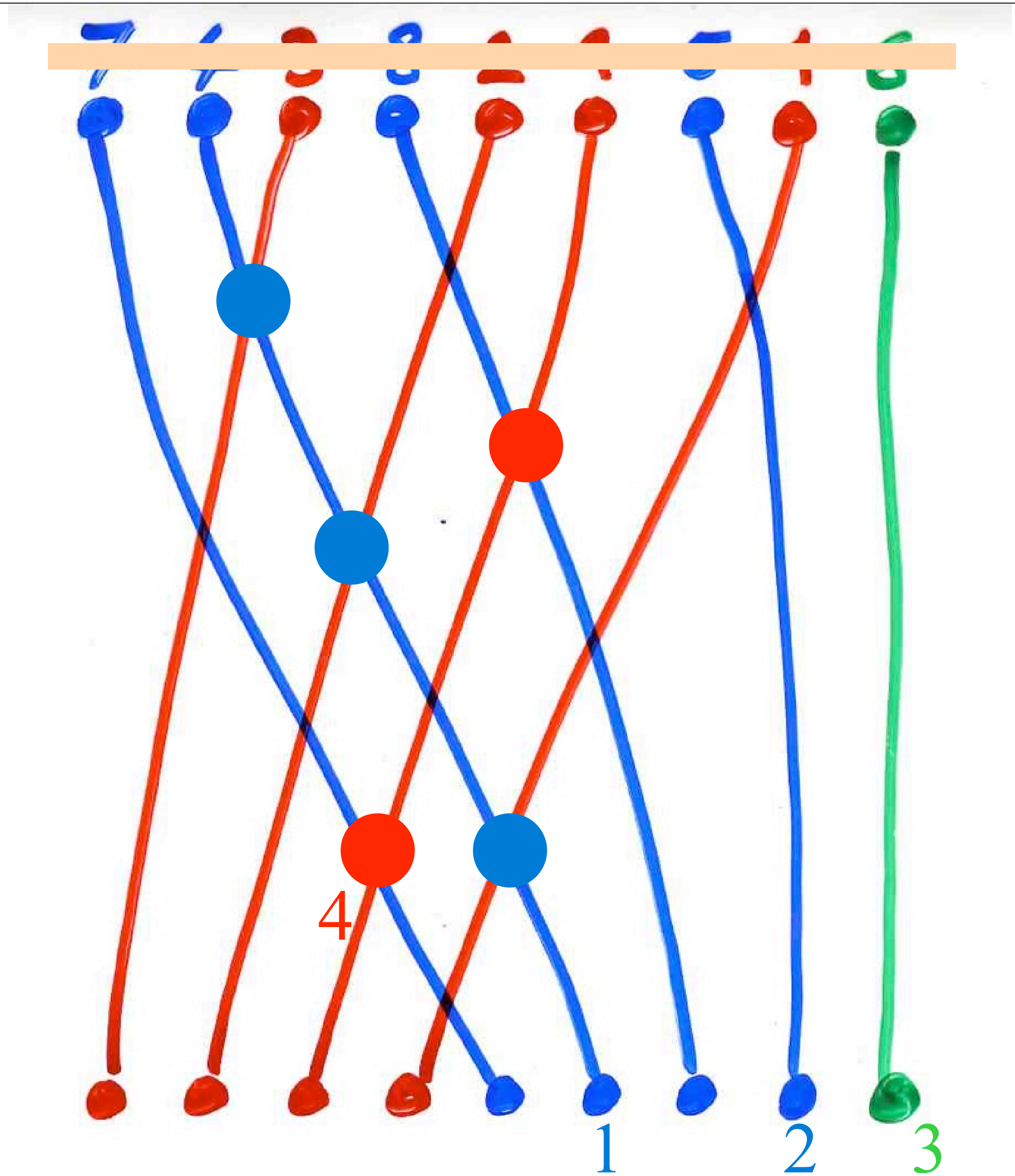


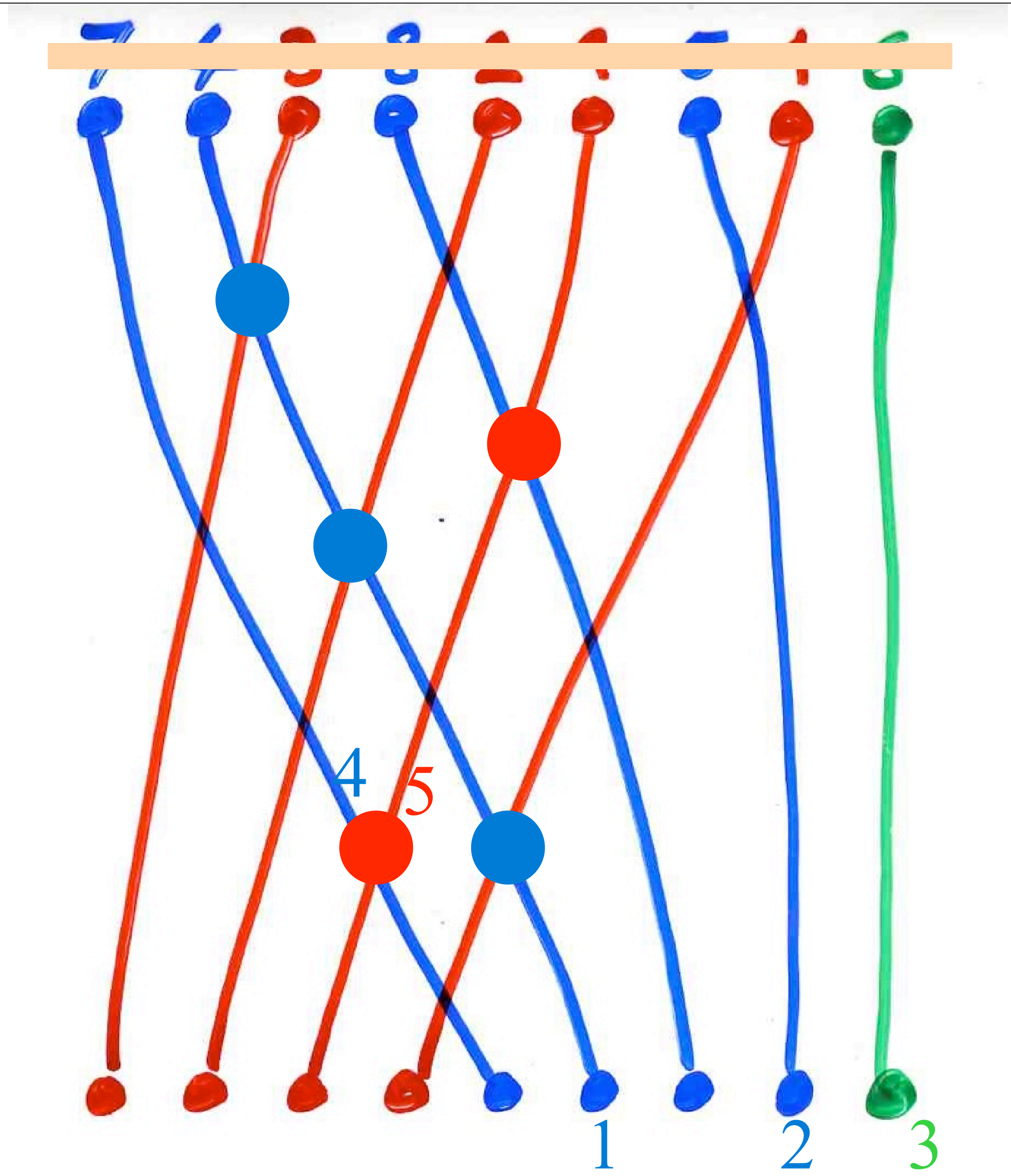
The inverse exchange-deletion bijection

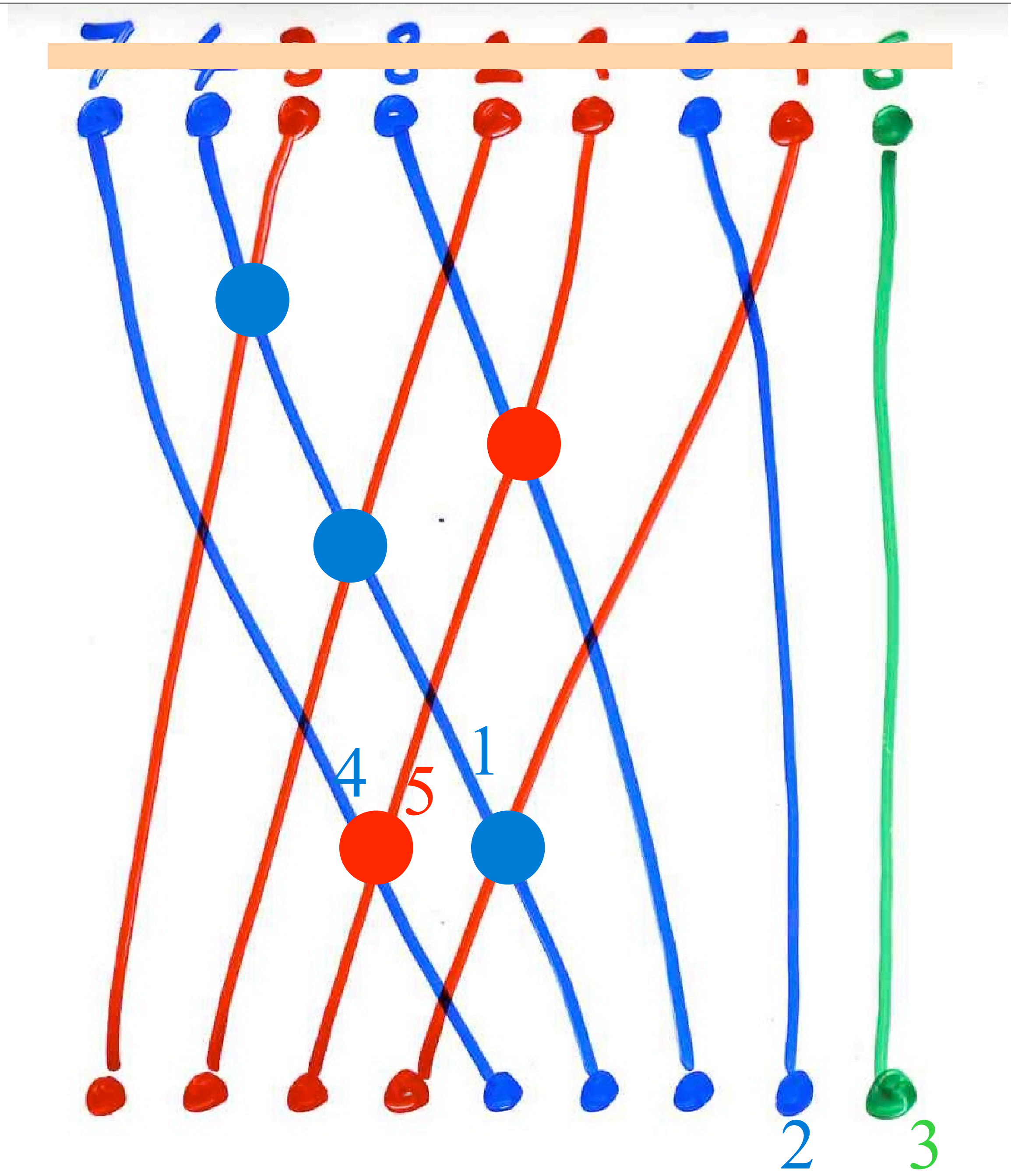


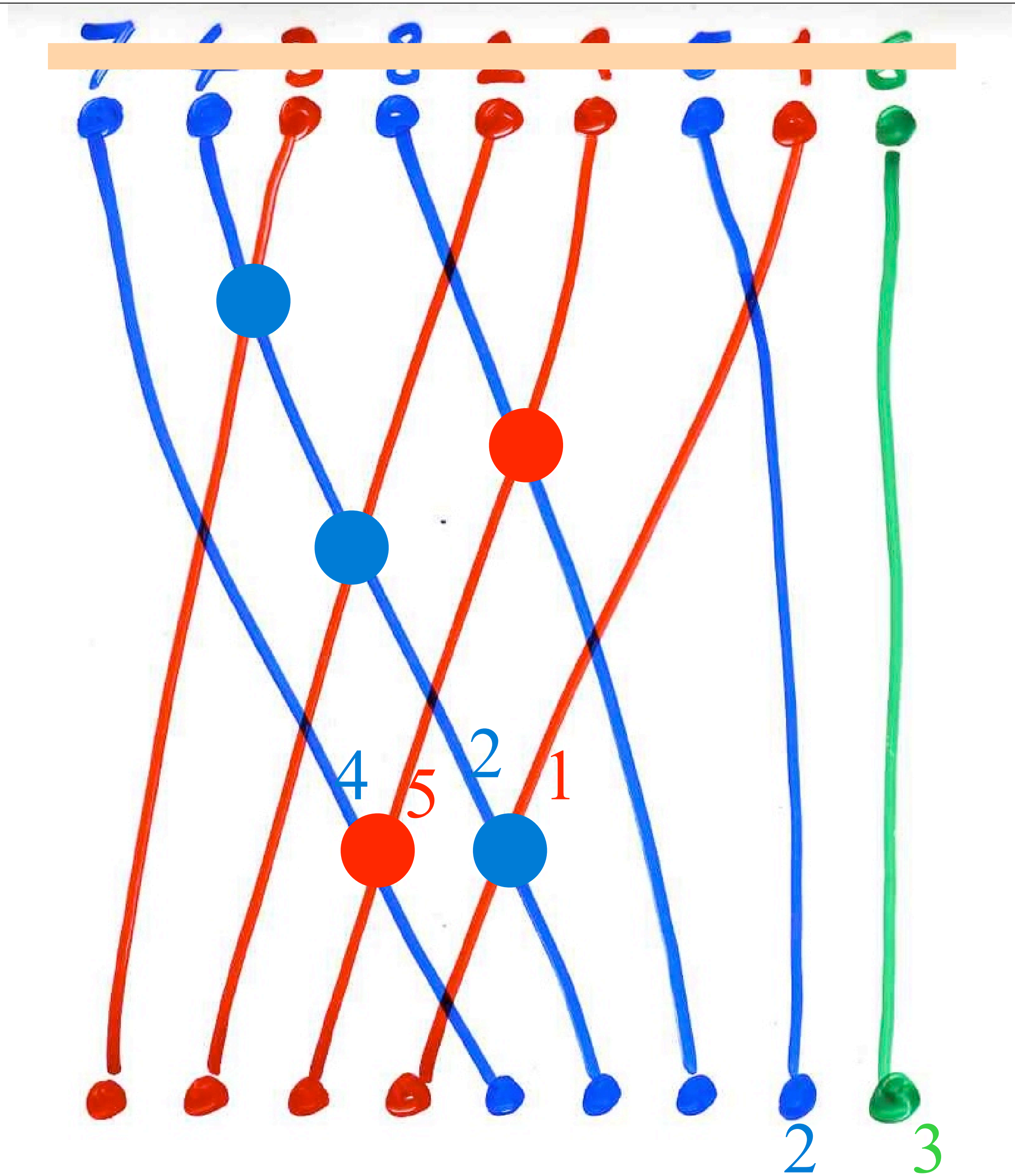


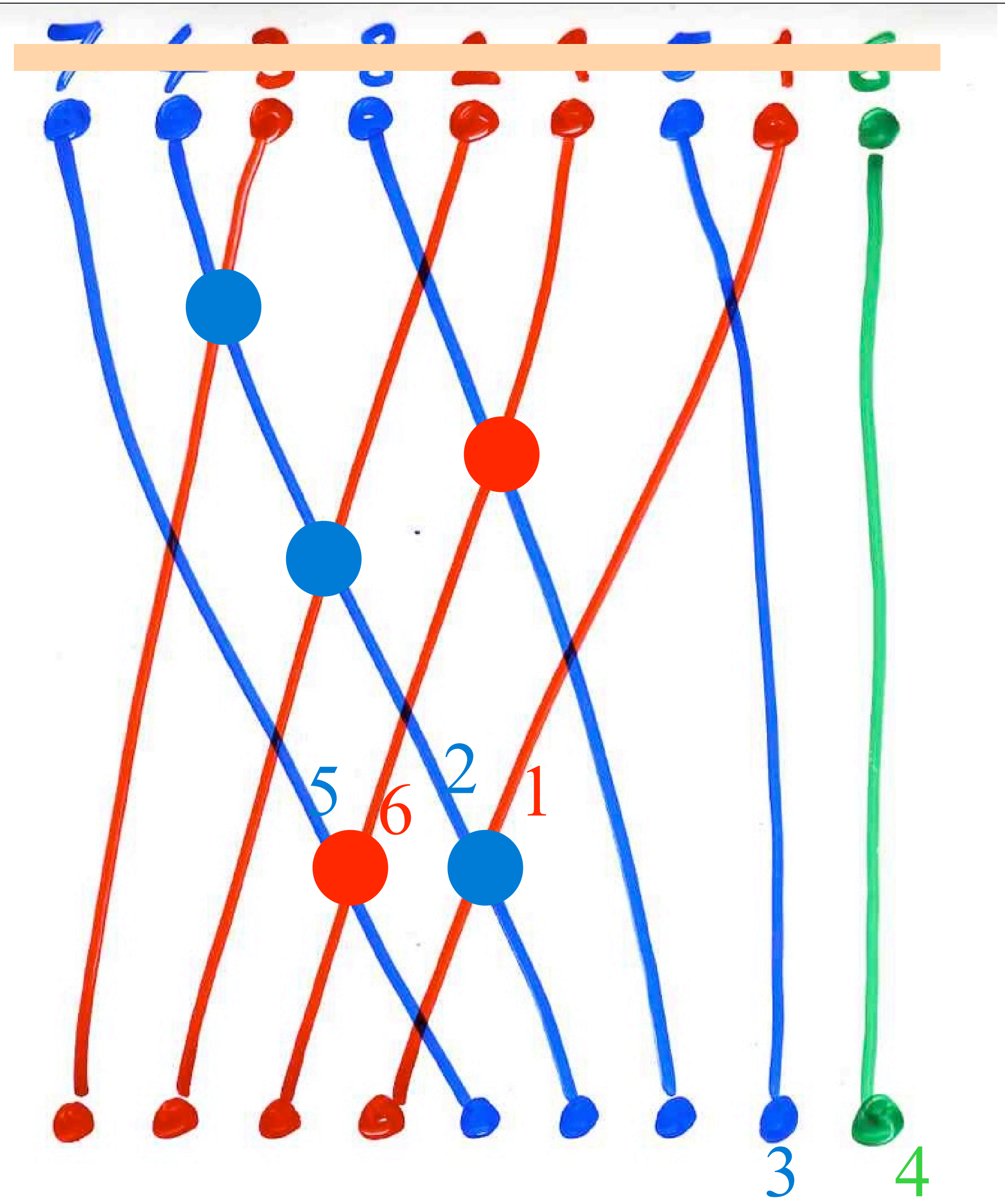

4
1
2
3


4
1
2
3

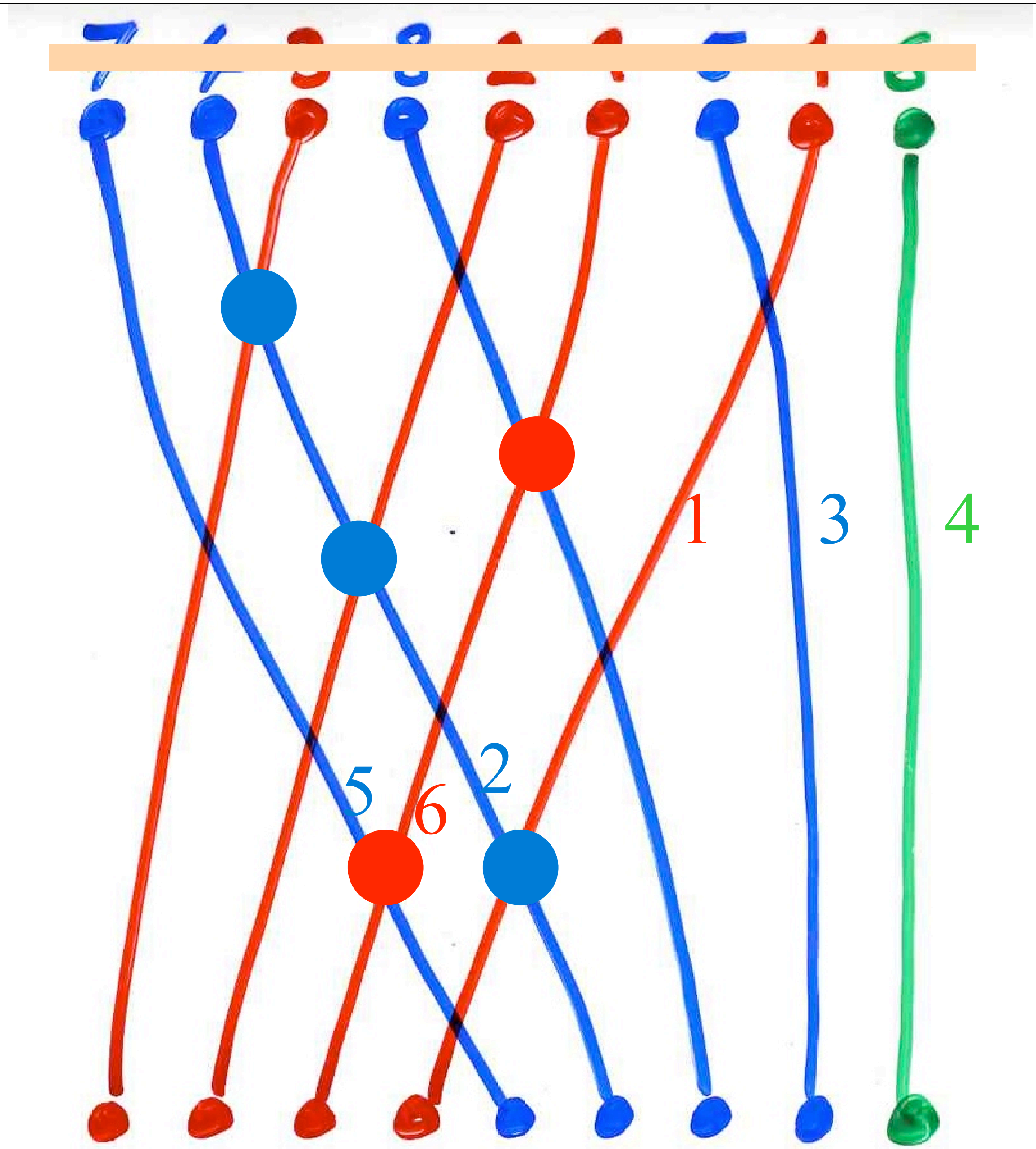

4
5
1
2
3

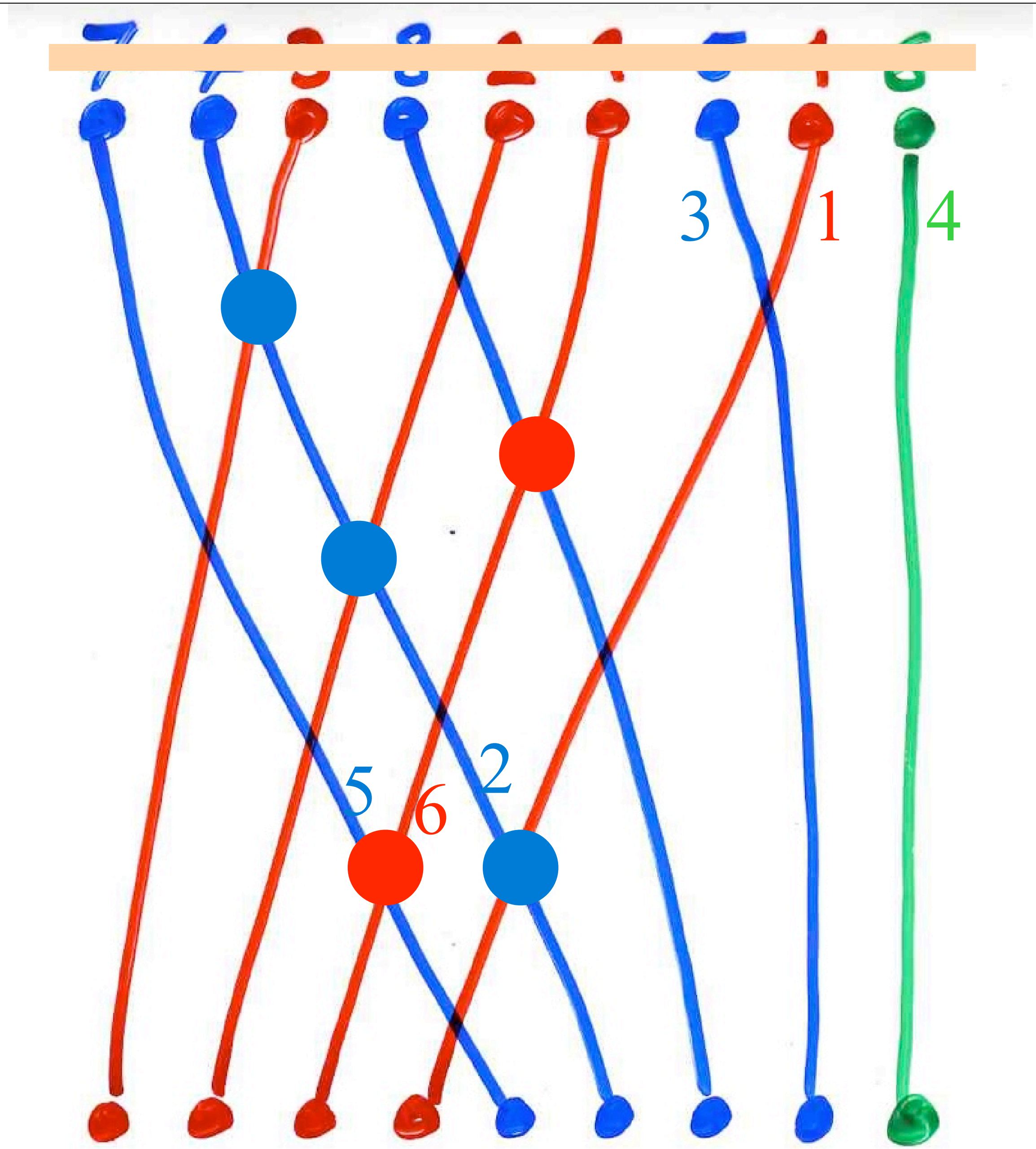

1
4
5
2
3


4
5
2
1
2
3



5
6
2
1
3
4


1
3
4
5
6
2


3
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4
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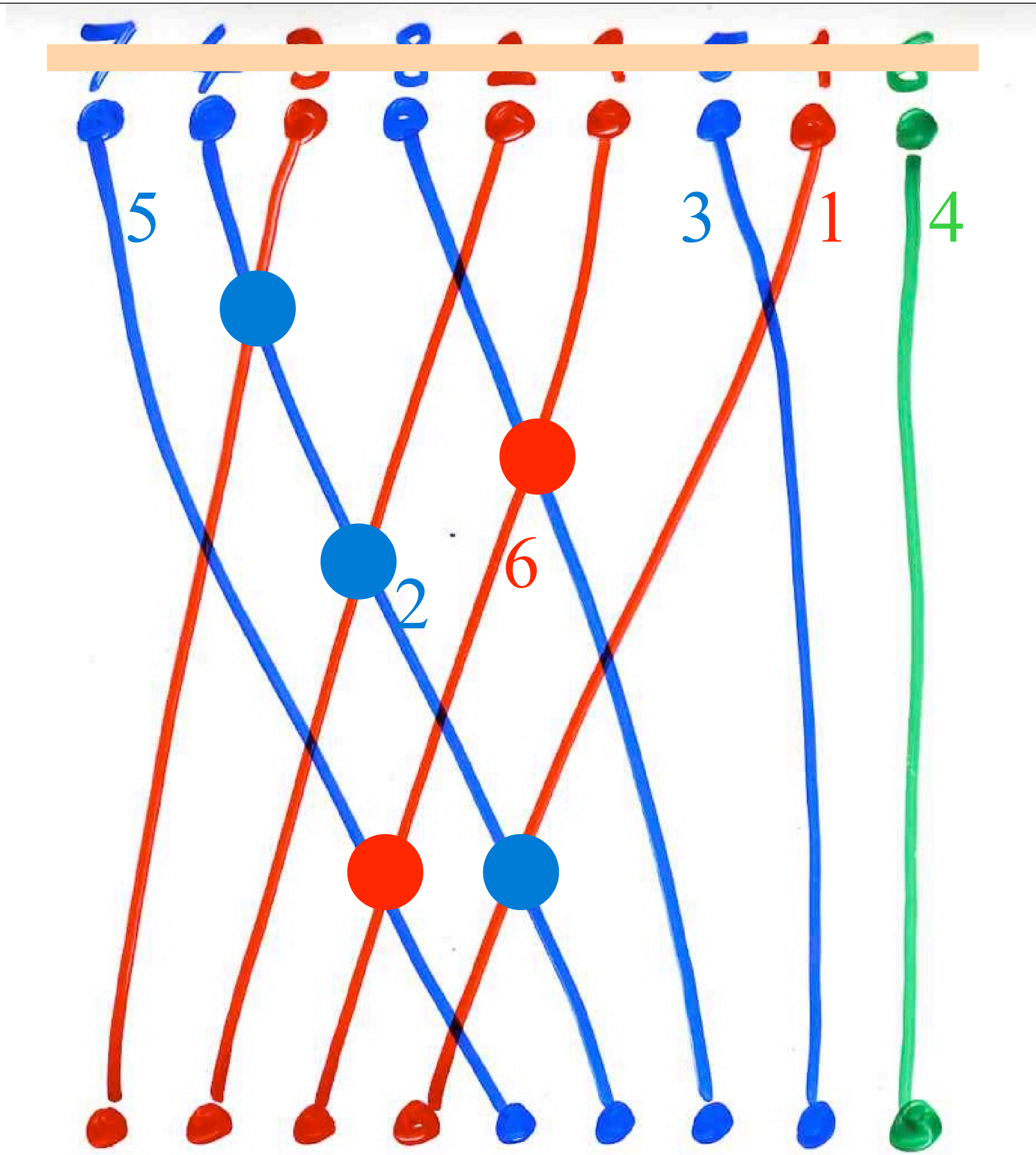
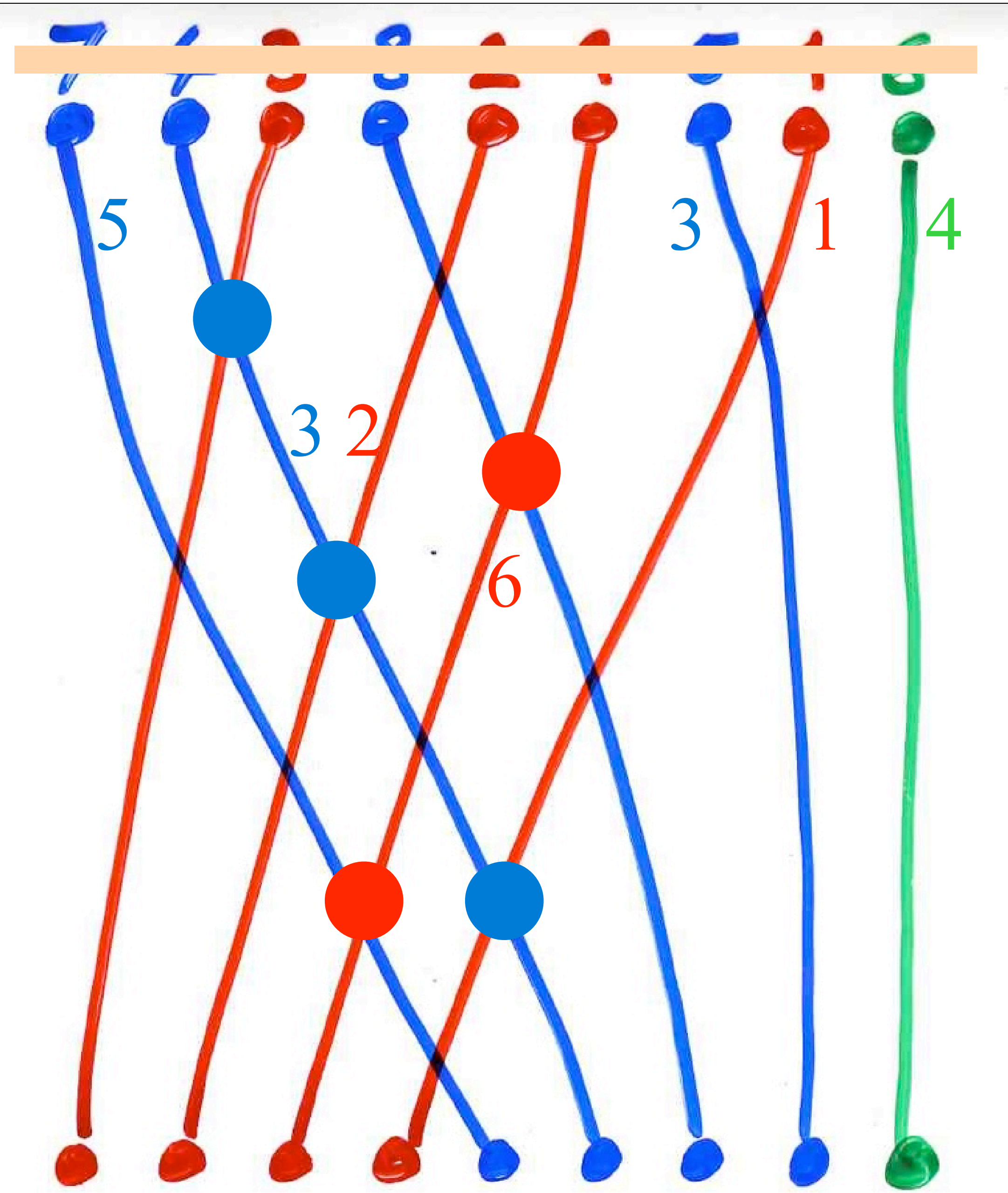
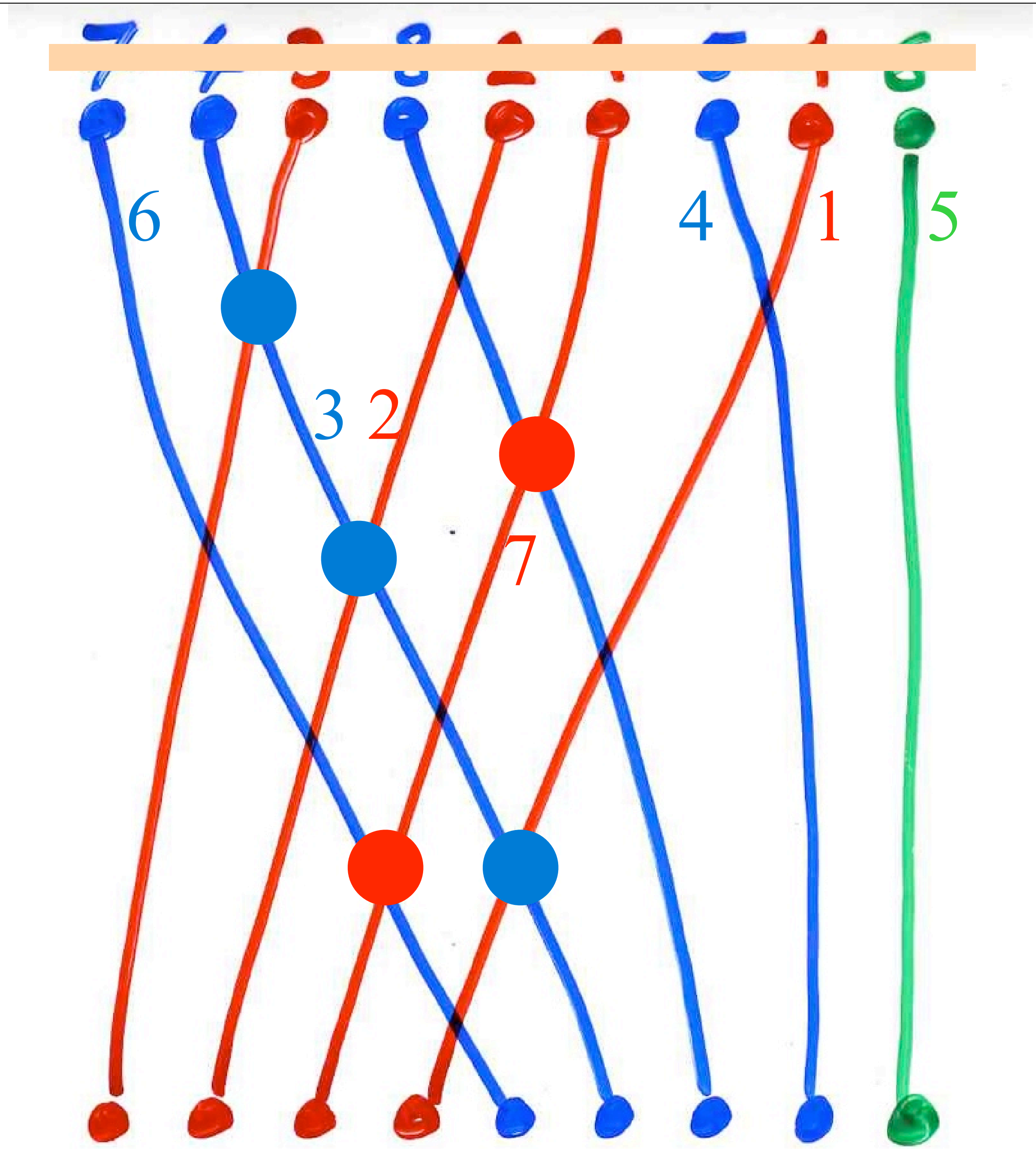
6
2


5
3
1
4
2
6


5
3
1
4
3 2
6


6
4
1
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3 2
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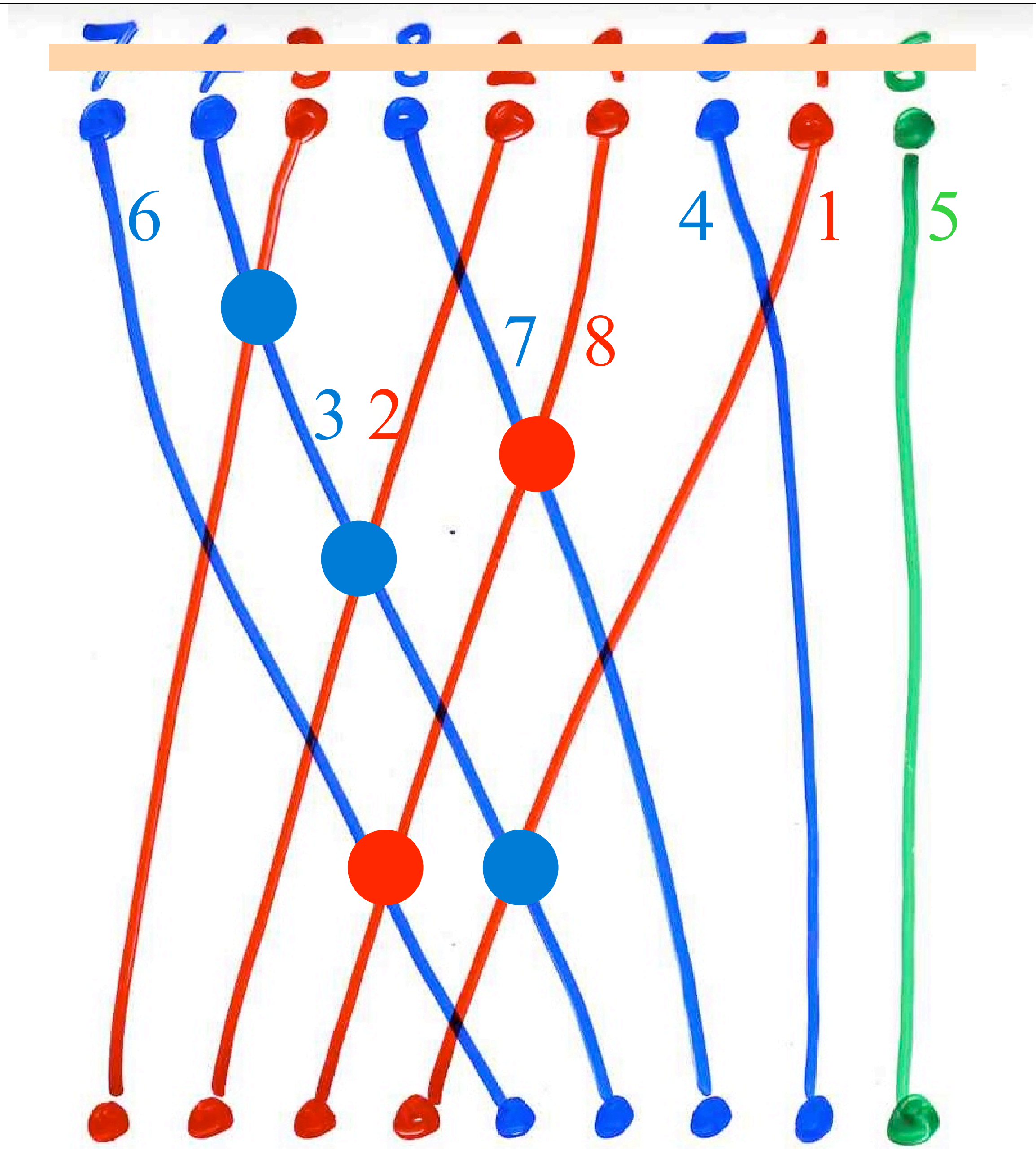
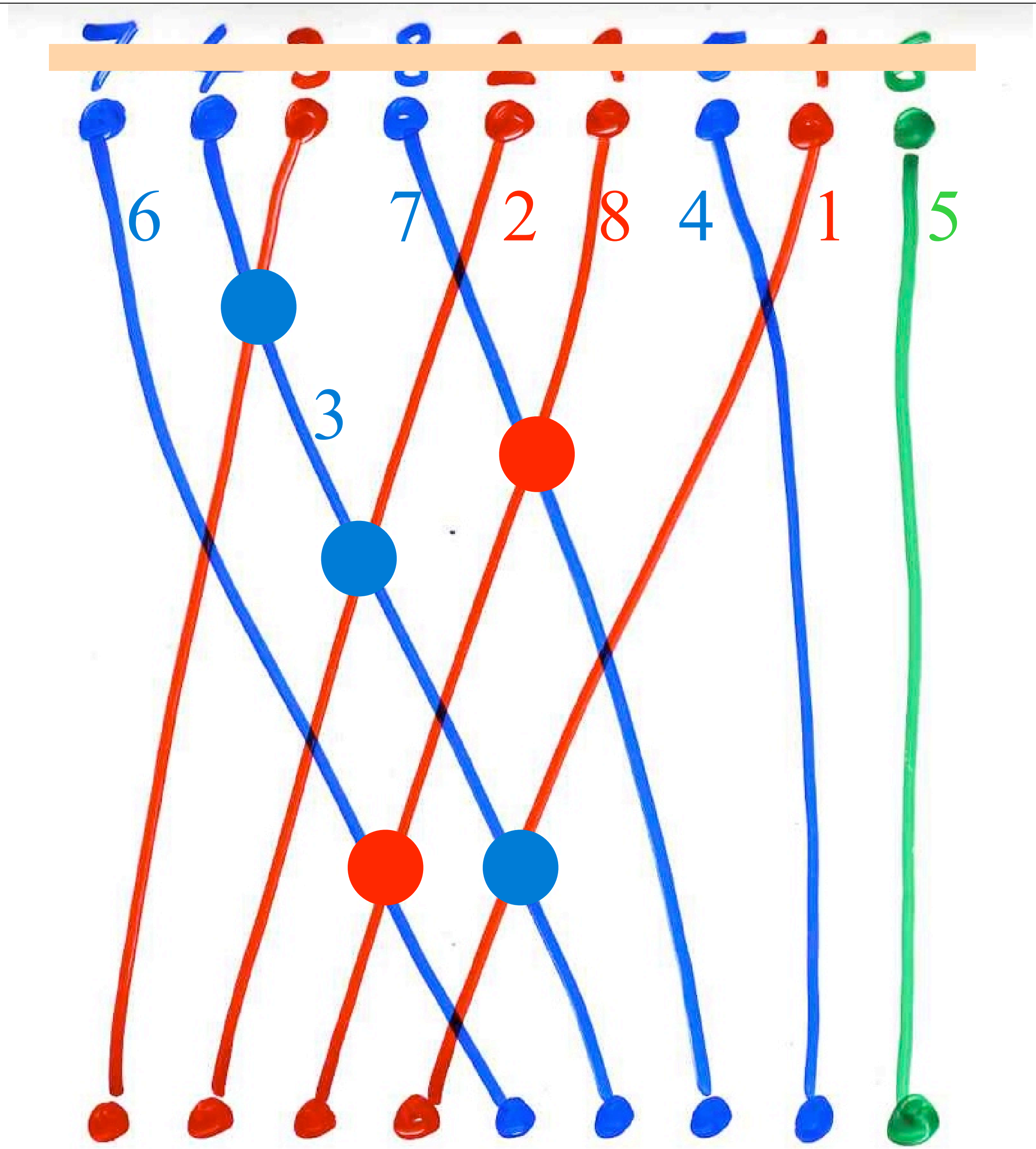


6
4
1
5
7
8
3
2



6
7
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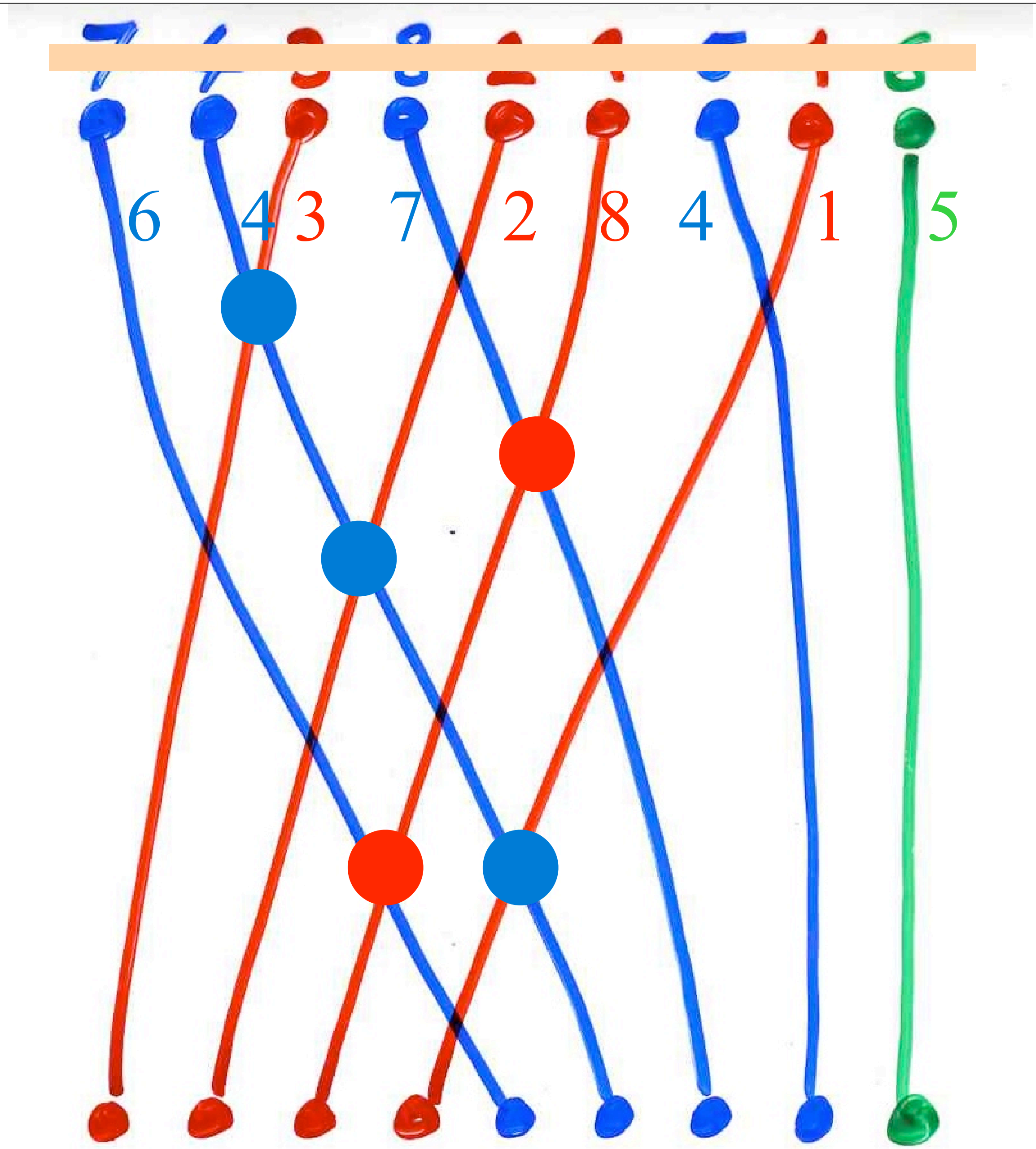
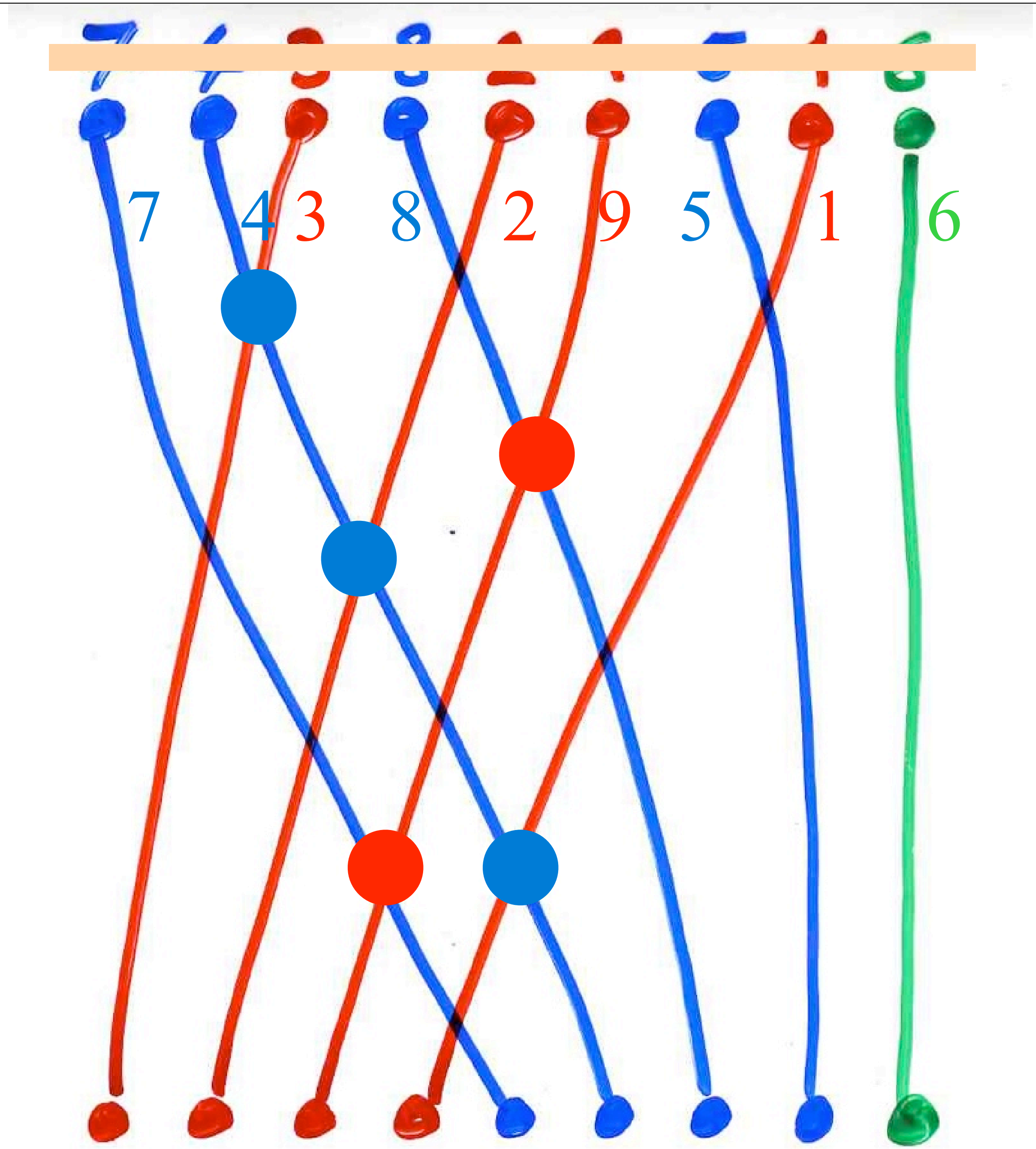

6 4 3 7 2 8 4 1 5


7 4 3 8 2 9 5 1 6

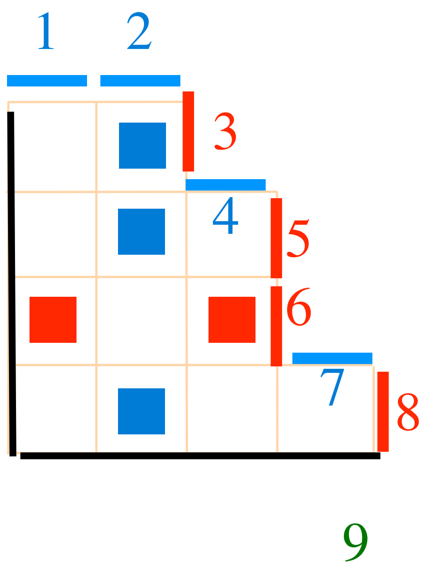
σ permutation

(valeur) x $\begin{cases}\text{avance}\\ \text{recul}\end{cases}$ ssi (indice) x $\begin{cases}\text{montée}\\ \text{descente}\end{cases}$

$\sigma^{-1}(x) < \sigma^{-1}(x+1)$

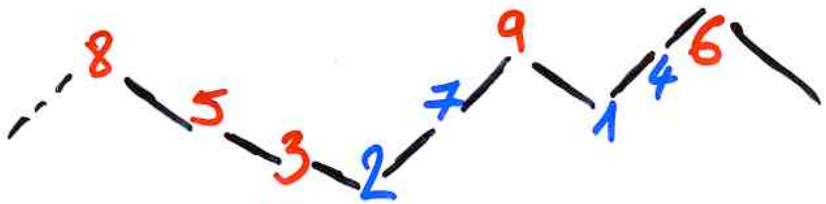
$\sigma^{-1}(x) > \sigma^{-1}(x+1)$

convention : $\sigma(n)$ descente



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 3 & 8 & 2 & 9 & 5 & 1 & 6 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \end{pmatrix}$$



“Genocchi shape” of a permutation

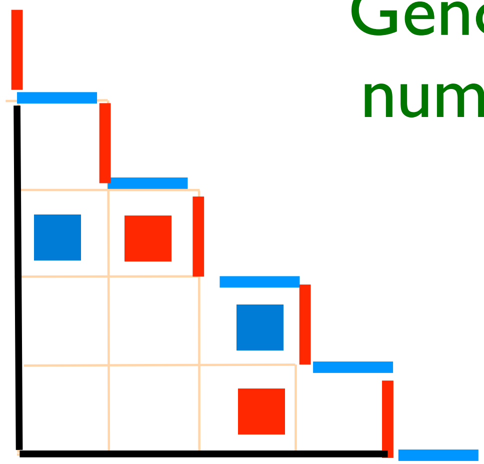
nombres de Genocchi

$$G_{2n} = 2(2^{2n}-1)\,B_{2n}$$

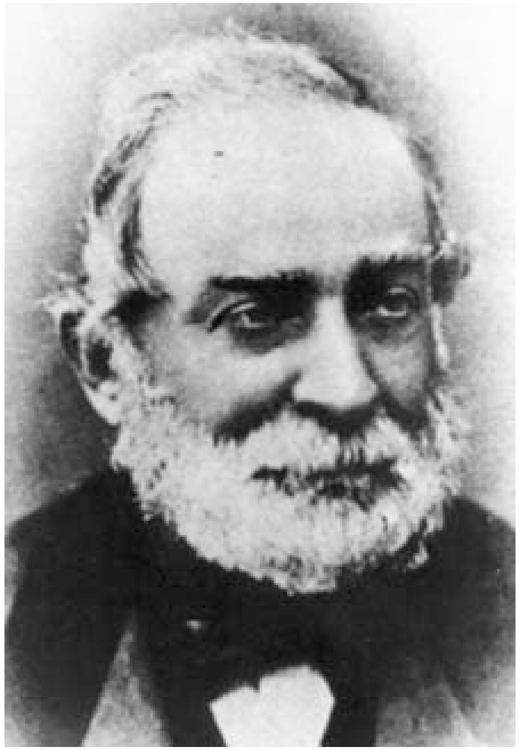
Bernoulli

$$2^{2n}\,G_{2n+2} = (n+1)\,T_{2n+1}$$

Genocchi numbers



alternating shape



Angelo Genocchi
1817 - 1889

Hinc igitur calculo instituto reperietur:

$$
\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= 3 \\
D &= 17 \\
E &= 155 = 5.31 \\
F &= 2073 = 691.3 \\
G &= 38227 = 7.5461 = 7.\frac{127.129}{3} \\
H &= 929569 = 3617.257 \\
I &= 28820619 = 43867.9.73 \quad \&c.
\end{aligned}
$$


EULER

UNIVERSITÉ

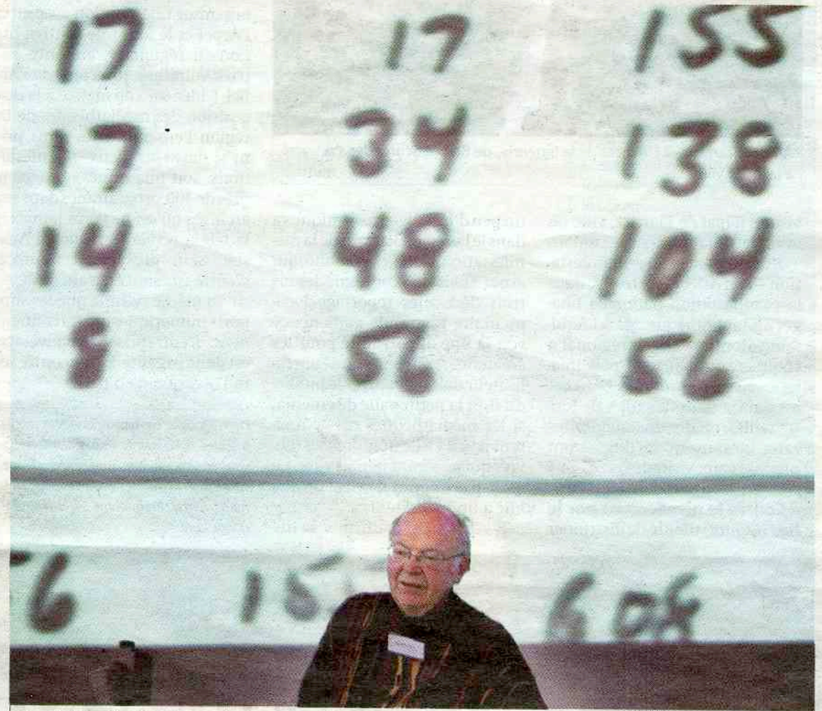
BORDEAUX 1. Le professeur Donald Knuth consacre sa vie à la programmation informatique, considérée comme un art. Il vient d'être sacré docteur honoris causa à Bordeaux

L'ermite de l'informatique

: Bernard Broustet

Une sommité de l'informatique mondiale a séjourné en Gironde ces derniers jours. Donald Knuth, 69 ans, a été sacré mardi docteur honoris causa de l'université Bordeaux 1, après avoir été lundi au centre d'une journée d'échanges qui réunissait une bonne partie du gratin français et européen de la recherche en informatique (1).

Depuis son premier contact, il y a un demi-siècle, avec un monumental et dinosaurien IBM 650, Donald Knuth n'a cessé d'être habité par la passion de l'informatique. Physicien, puis mathématicien de formation, ce géant affable et modeste a voué sa vie à ce qu'il appelle « l'art de la programmation informatique ». Car, à ses yeux, plus qu'une technique, c'est une forme d'activité qui requiert à la fois rigueur, intuition et sens esthétique. Les programmes informatiques réussis ont une sorte de beauté à laquelle même les non-spécialistes peuvent être sensibles.



Donald Knuth, à Bordeaux, le 29 octobre. À 69 ans, il animait une journée d'échanges avec le gratin européen de la recherche en informatique

PHOTO LAURENT THEILLET

Une encyclopédie. Au long de sa carrière académique (pour l'essentiel à l'université californienne de Stanford), Donald Knuth a fait preuve d'une grande fécondité, en jouant notamment un rôle essentiel dans le développement de langages toujours utilisés par la communauté des mathématiciens. Mais, à 55 ans, le professeur Knuth a décidé de prendre sa retraite de Stanford. Il trouve que les fonctions administratives sont trop absorbantes pour lui permettre de mener à bien l'œuvre entamée à la fin des années 60 sous le titre de « Art of computer programming », sorte d'encyclopédie de l'algorithmique et de la programmation informatique. Donald Knuth a publié, il y a quelque temps déjà, les trois premiers volumes de cette gigantesque somme, traduite en russe, en japonais, en polonais, etc. mais pas en français. Le quatrième tome est pour bientôt. Et Donald Knuth se dit décidé à poursuivre sa tâche tant qu'il en aura la force. Ses ouvrages, dont les ventes cumulées au fil des ans approchent le million d'exemplaires, visent essentiellement les informaticiens et créateurs de programmes. Une communauté certes minoritaire à travers le monde, mais qui se trouve investie d'une mission considérable. En quelques décennies, l'écriture informatique a aidé à résoudre d'innombrables problèmes. « Mais il y en a tant d'autres qui attendent des solutions, notamment dans le domaine médical », affirme le professeur émérite de Stanford.

Un chèque de 2,56 dollars. Pour mener à bien sa tâche, Donald Knuth s'est imposé une vie d'ermite. D'ordinaire, sa journée débute par la bibliothèque ou la piscine. Après quoi, il passe tout le reste de son temps à sa table de travail, dimanche compris. Il n'a plus d'e-mail depuis le début des années 90, considérant que le courrier électronique représente une perte de temps, dès lors qu'on veut aller au fond des choses et non pas rester à leur surface. Une secrétaire lui fait passer les messages considérés comme les plus urgents. Pour le reste, Donald Knuth demande qu'on lui écrive par courrier ordinaire ou par fax, dont il prend parfois connaissance avec des mois de retard. Il s'oblige, en revanche, à tenir aussi scrupuleusement que possible sa promesse d'envoyer un chèque de 2,56 dollars à tout lecteur ayant détecté une erreur dans un de ses livres. Par ailleurs, pour se détendre, il pratique l'orgue, appris dans sa prime jeunesse auprès de son père qui partagea sa vie entre la musique et l'enseignement.

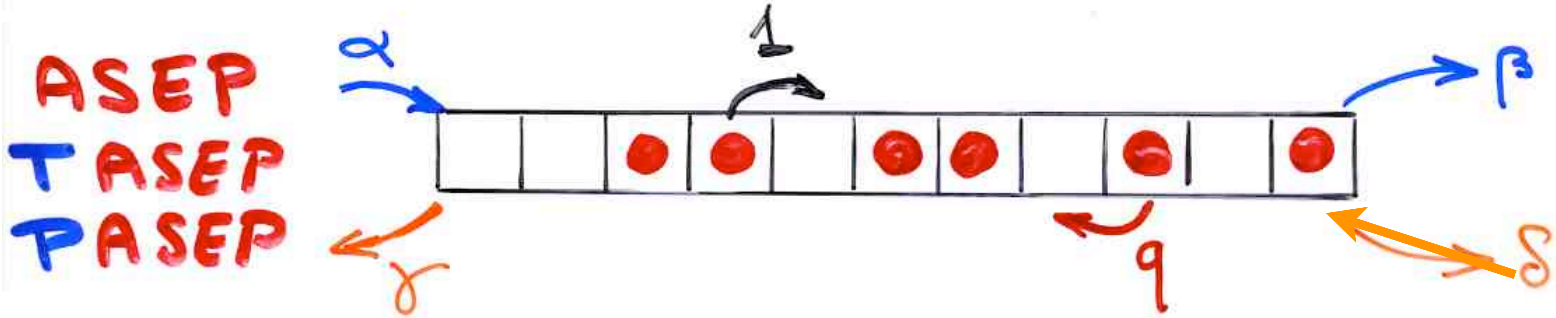
L'orgue de Sainte-Croix. Donald Knuth n'est pas fermé aux choses de ce monde. Sur son site Internet, à la rubrique « Questions qui ne me sont pas fréquemment posées », il demande entre autres : « Pourquoi mon pays a-t-il le droit d'occuper l'Irak ? », « Pourquoi mon pays ne soutient-il pas une Cour internationale de justice ? » Mais cet homme de conscience ne se veut pas militant, pas plus qu'il n'aspire au vedettariat et à la richesse. « Beaucoup de gens, dit-il, ont tendance à considérer que l'informatique, c'est surtout des histoires de business, d'entreprise. Ce n'est pas mon cas. » Sortant de sa semi-réclusion, Donald Knuth s'est donc laissé convaincre d'accepter les hommages de l'université de Bordeaux, après celles de Harvard, d'Oxford, de Tübingen. Il a eu le coup de foudre pour la beauté et l'agrément de la ville. Et il n'oubliera sans doute pas de sitôt l'orgue illustre de l'église Sainte-Croix (2), sur lequel il a eu le bonheur d'exercer son talent.

(1) Ces journées étaient organisées par le Laboratoire bordelais de recherche en informatique (Labri).

(2) Thierry Semenoux, professeur d'orgue au conservatoire de Bordeaux, a joué dans ce domaine un rôle de cicérone auprès de Donald Knuth.

§ 3

The PASEP



ASEP
TASEP
PASEP
α
1
β
γ
q
δ

boundary induced phase transitions

molecular diffusion
linear array of enzymes
biopolymers
traffic flow
------ formation of shocks

$$P_n(\tau_1, \dots, \tau_n) = f_n(\tau_1, \dots, \tau_n) / Z_n$$

$$Z_n = \sum_{\tau} f_n(\tau_1, \dots, \tau_n)$$

partition function

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

$$\begin{cases} DE = q\,ED + D + E \\ (\beta D - \delta E)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \gamma D) = \langle W| \end{cases}$$

Then

$f_n(\tau_1, \dots, \tau_n)$

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

TASEP

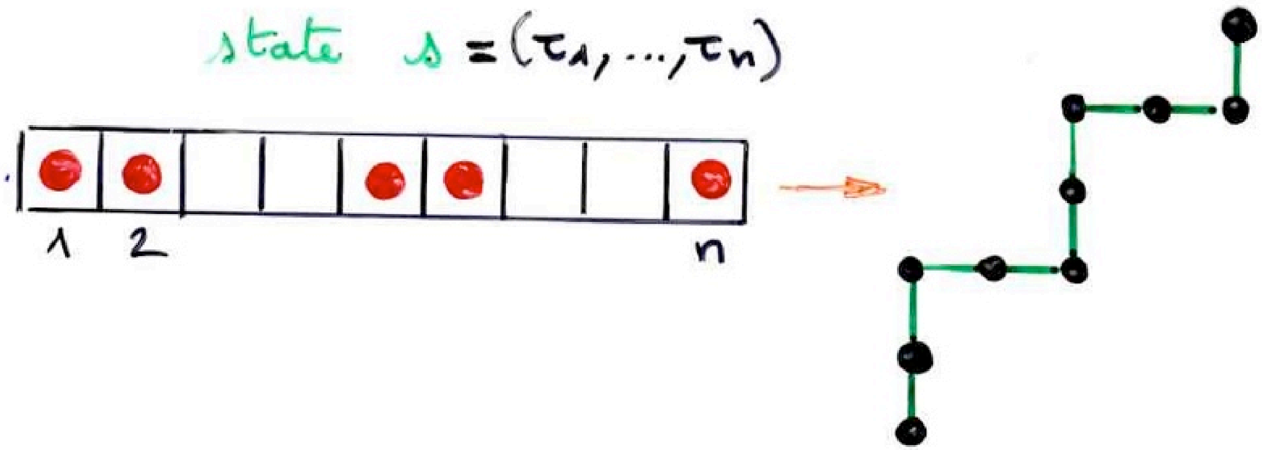
q=0

$$\begin{cases} DE = \quad\quad + D + E \\ (\beta D - \text{[redacted]})|V\rangle = |V\rangle \\ \langle W|(\alpha E - \text{[redacted]}) = \langle W| \end{cases}$$

Then

$$f_n(\tau_1, \dots, \tau_n) = \langle W| \prod_{i=1}^{n} (\tau_i D + (1-\tau_i) E) |V\rangle$$

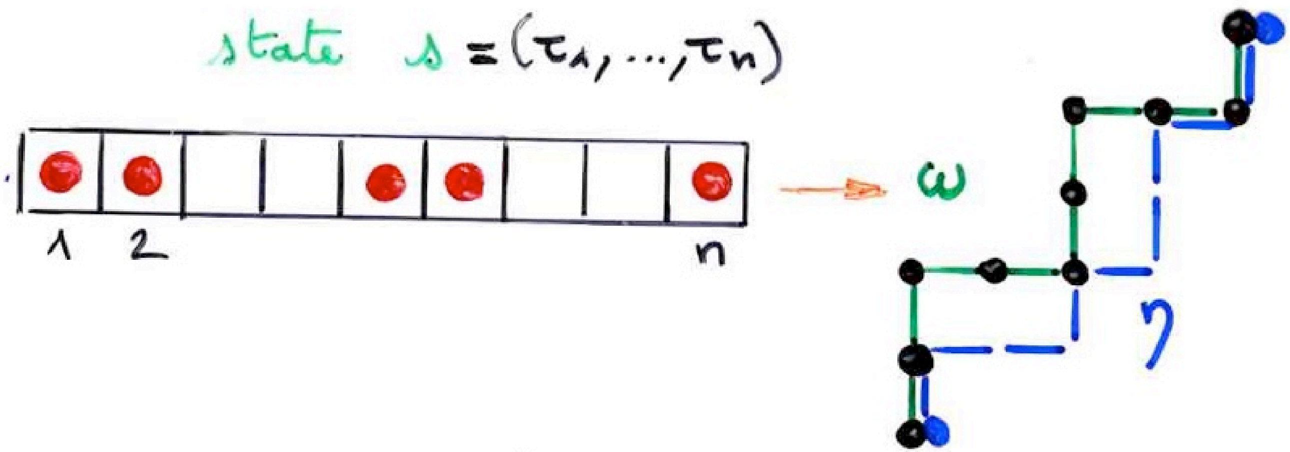
state $s = (\tau_1, \ldots, \tau_n)$



$\mathcal{P}_n(s) =$

Shapiro, Zeilberger, 1982

state $s = (\tau_1, \dots, \tau_n)$



$$P_n(s) = \frac{1}{C_{n+1}} \left(\text{number of paths } \eta \text{ below the path } \omega \text{ associated to } s \right)$$

Shapiro, Zeilberger, 1982

TASEP

Brak, Essam (2003), Duchi, Schaeffer, (2004),
Angel (2005), xgv, (2007)

(P) ASEP

Brak, Corteel, Essam, Parviainen, Rechnitzer (2006)
Corteel, Williams (2006)
Josuat-Vergès (2008)

Derrida, ...
Mallick, ... Golinelli, Mallick (2006)

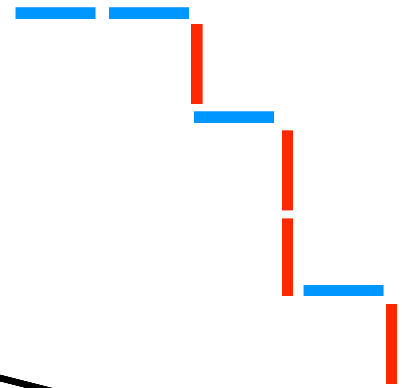
§4 Stationary probability with alternative tableaux

Def- profile of an alternative tableau

word $w \in \{E,D\}^*$

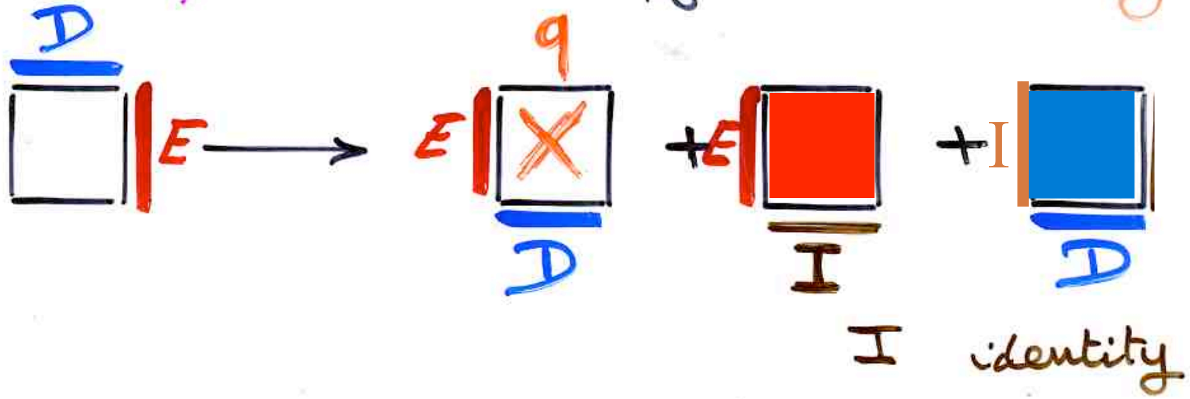
D D E D E E D E

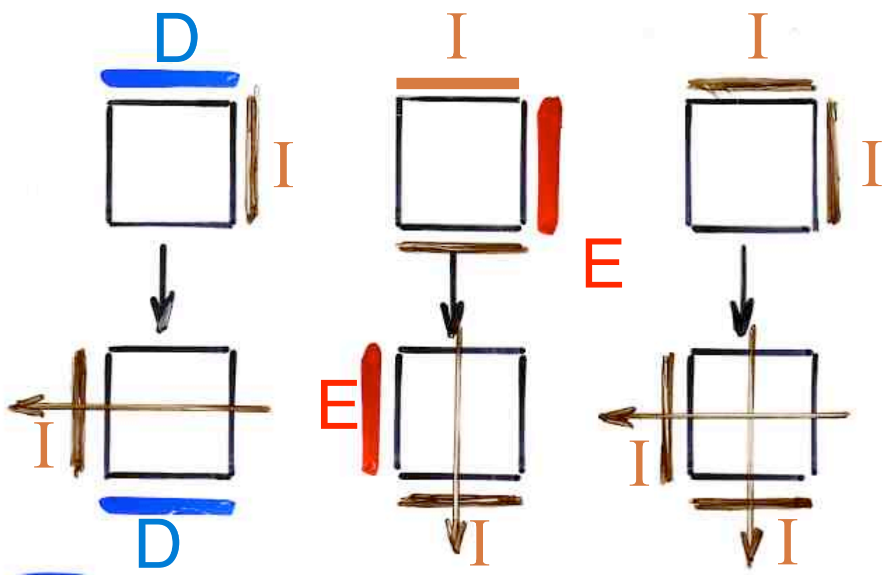
D DE (D E) E D E

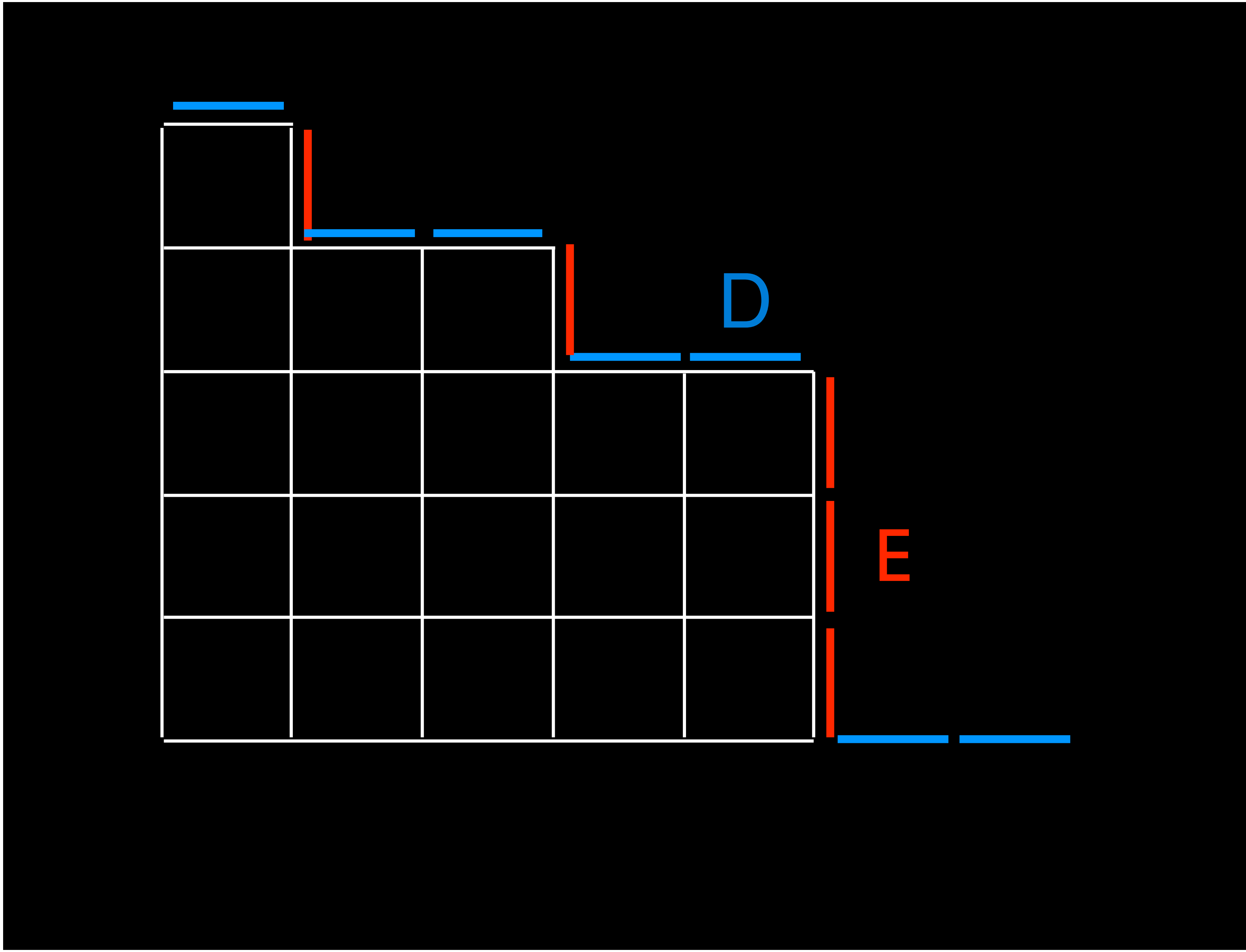


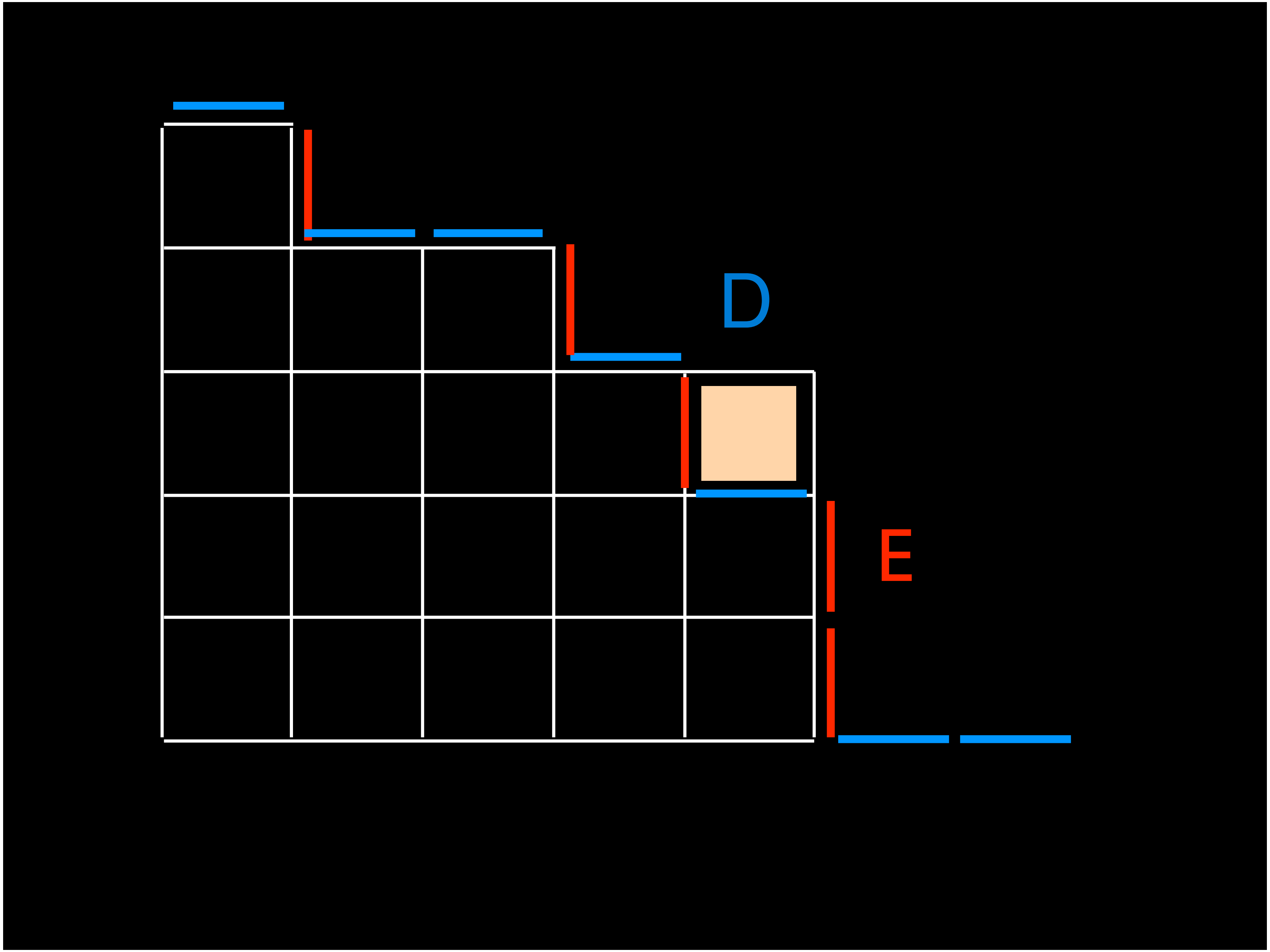
DDE(E)EDE + DDE(ED)EDE + DDE(D)EDE

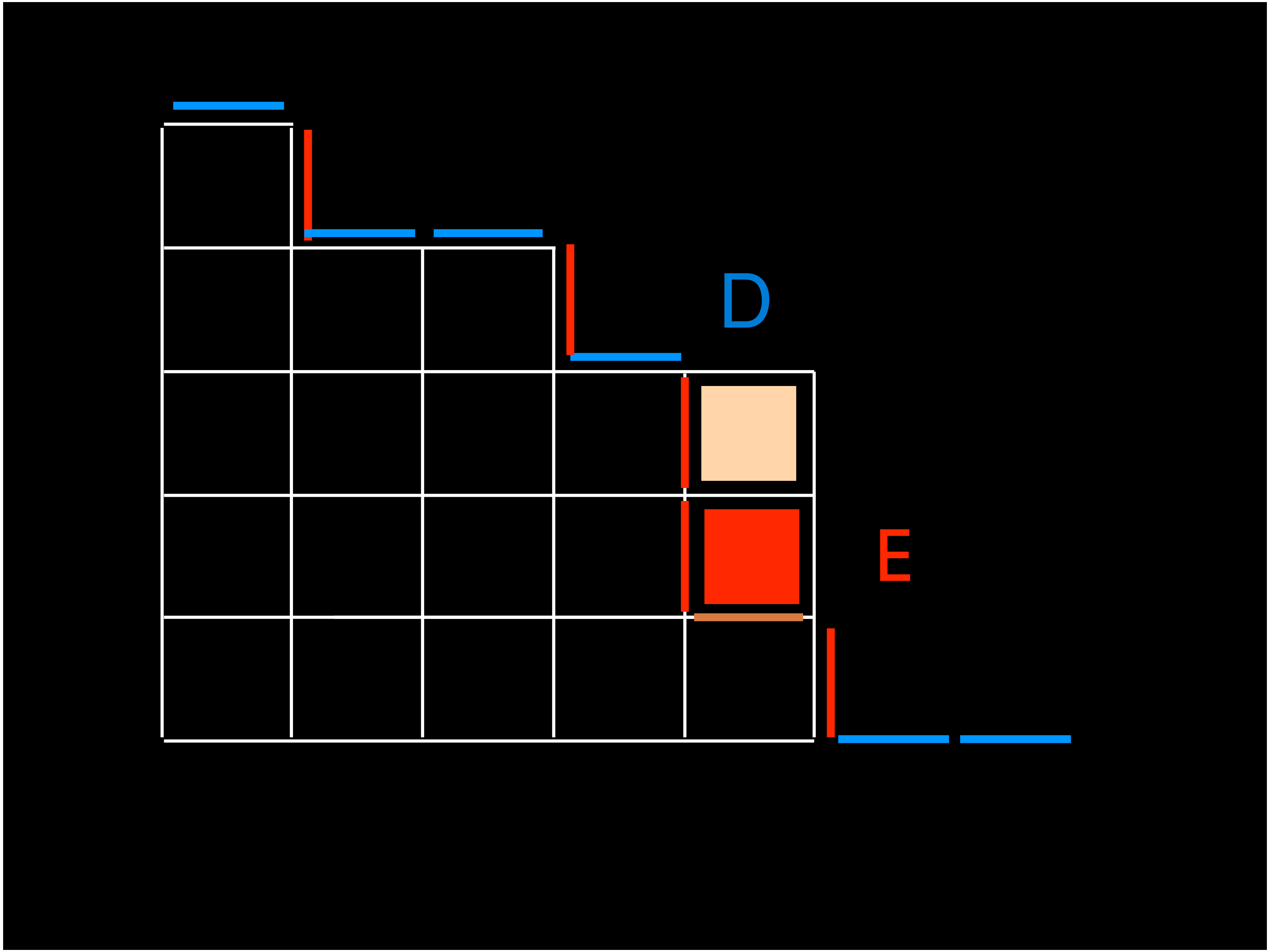
Proof: "planarization" of the rewriting rules






D
E


D
E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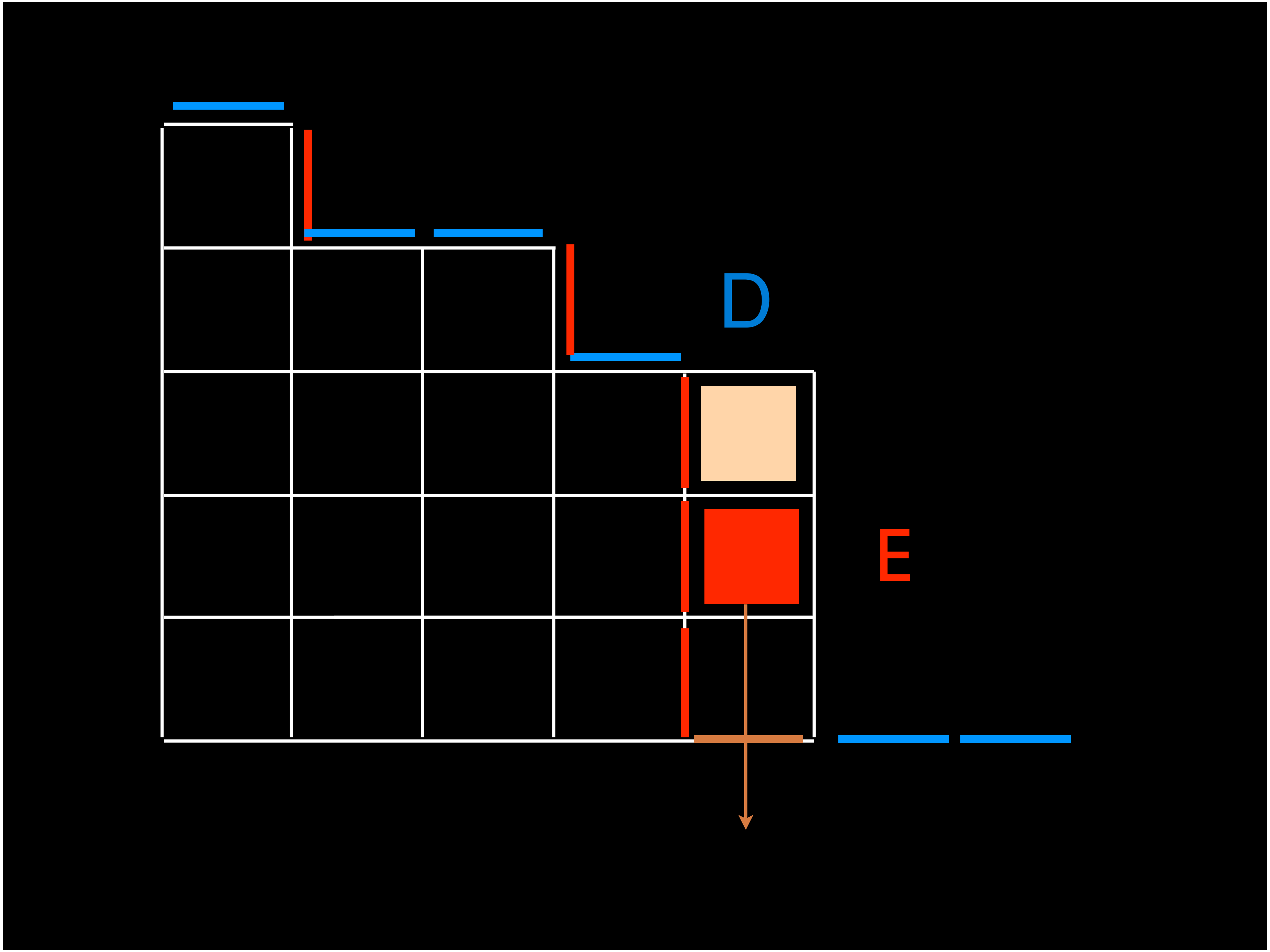
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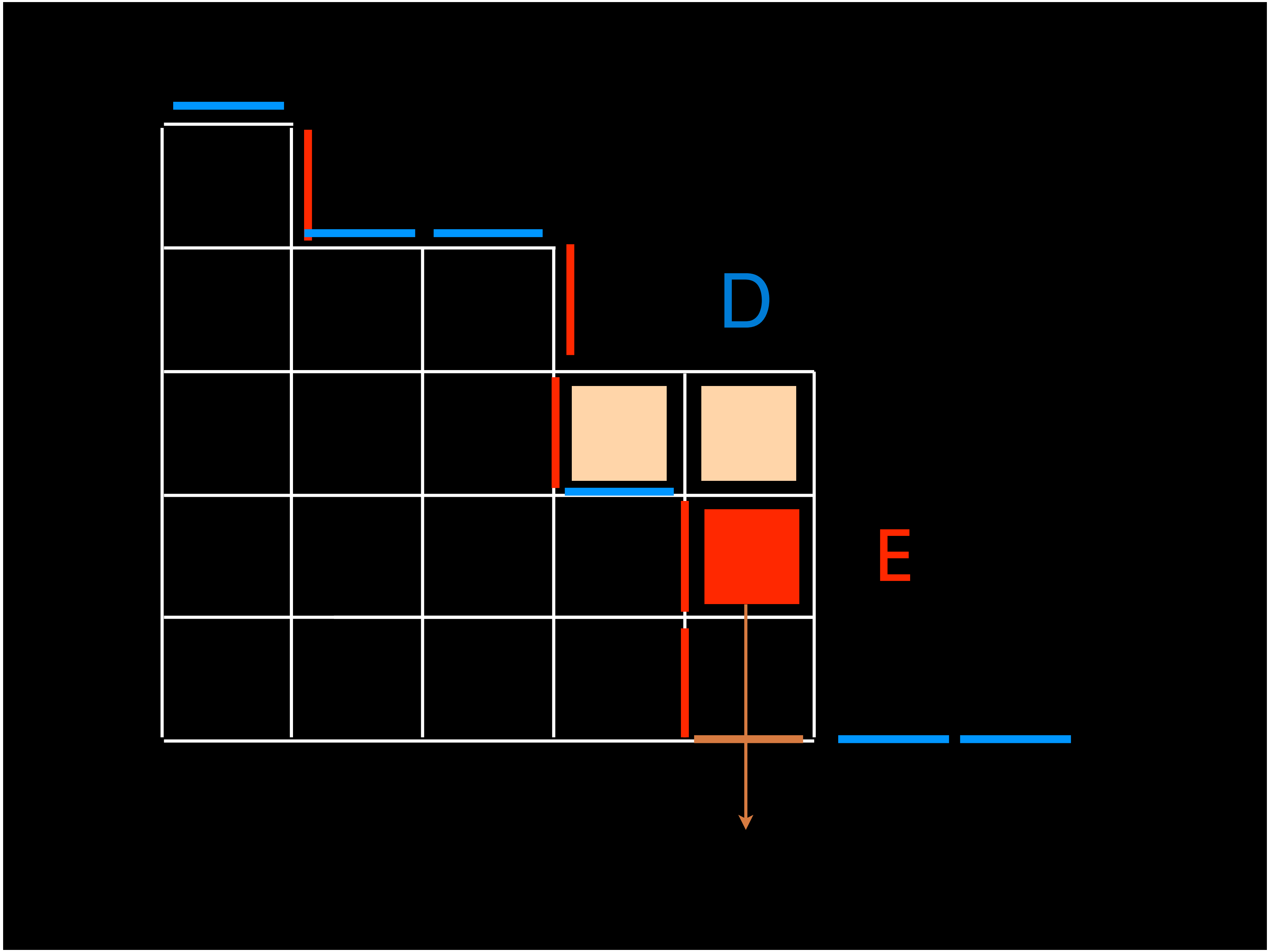
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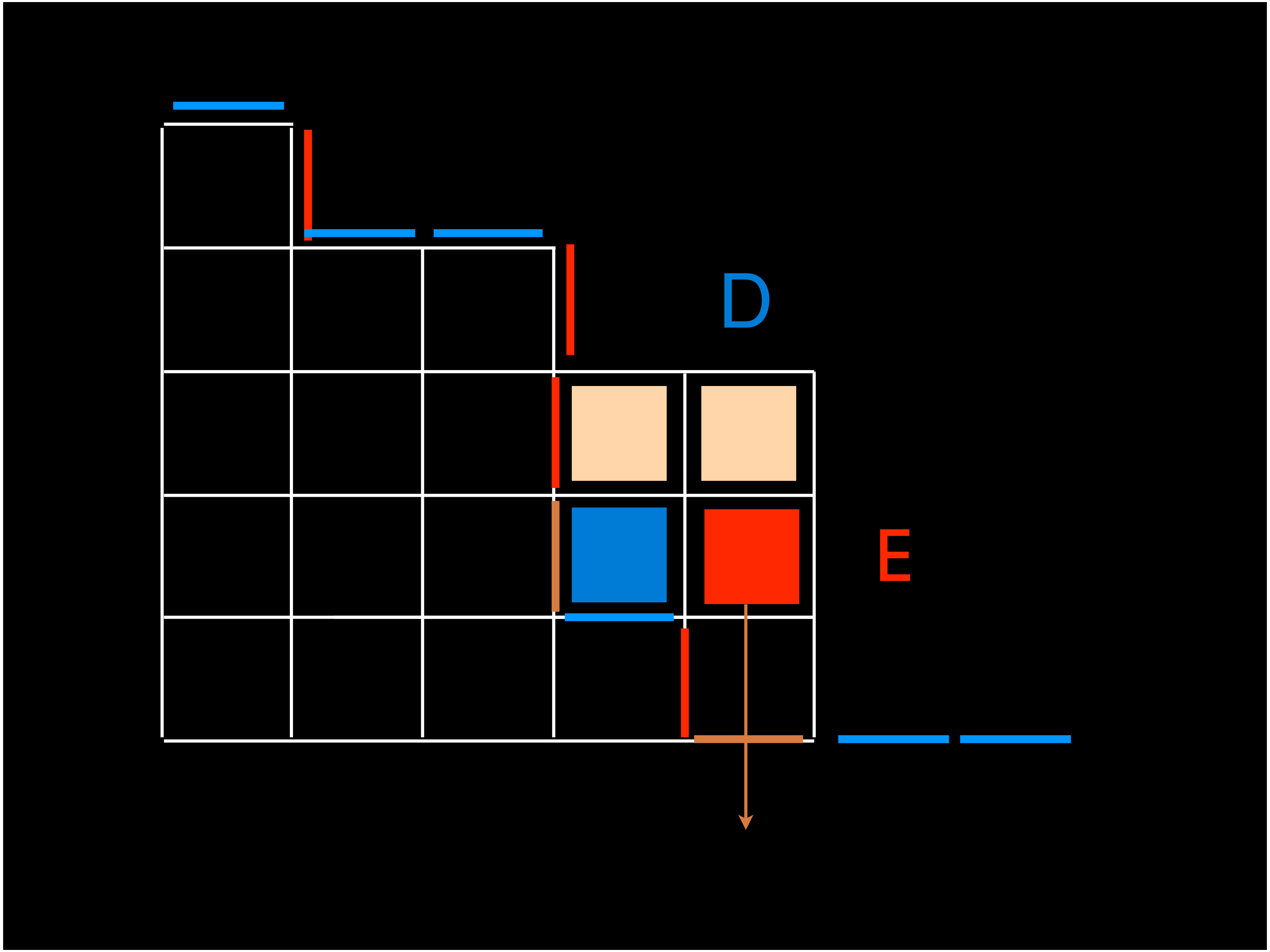
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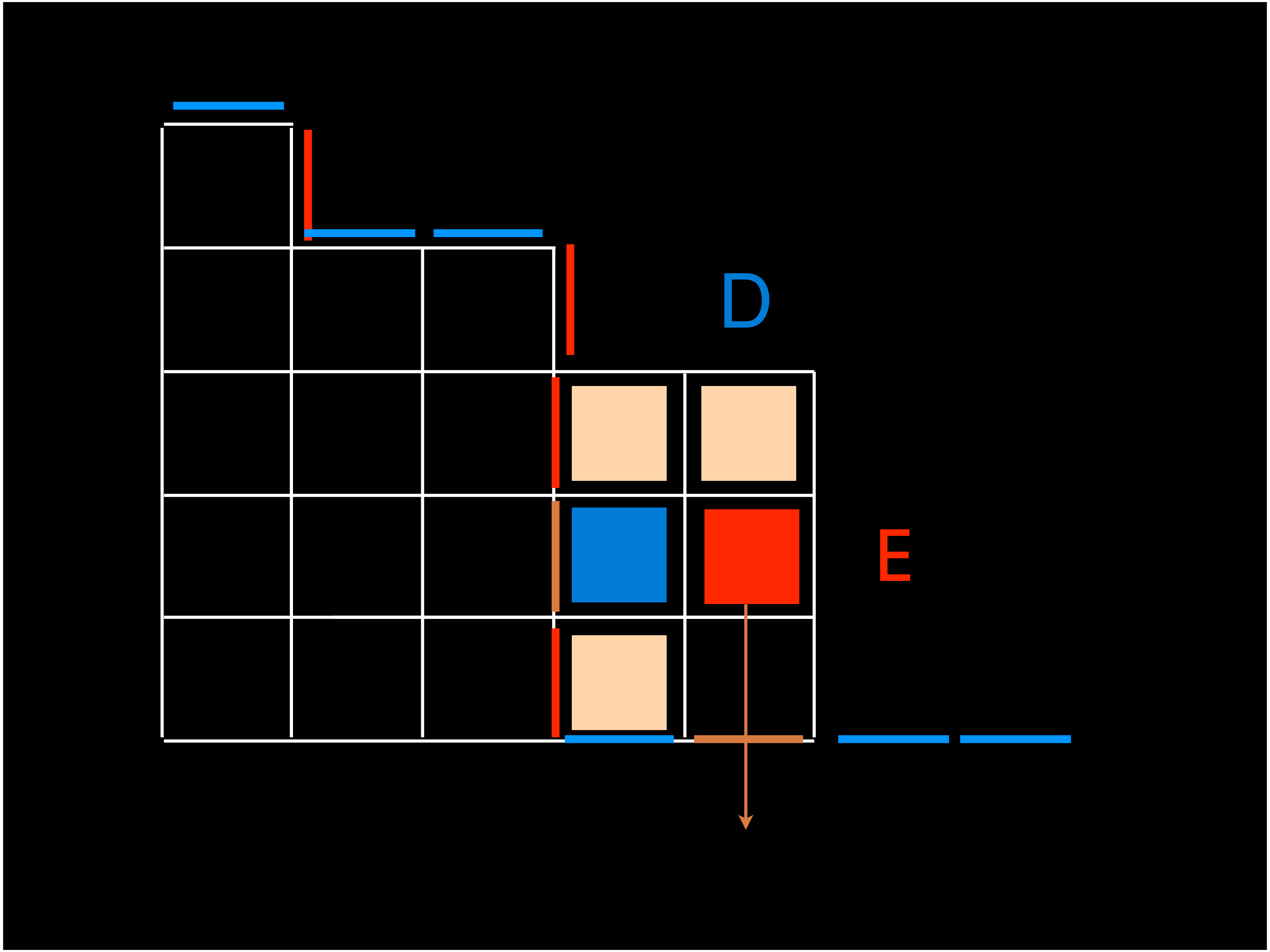
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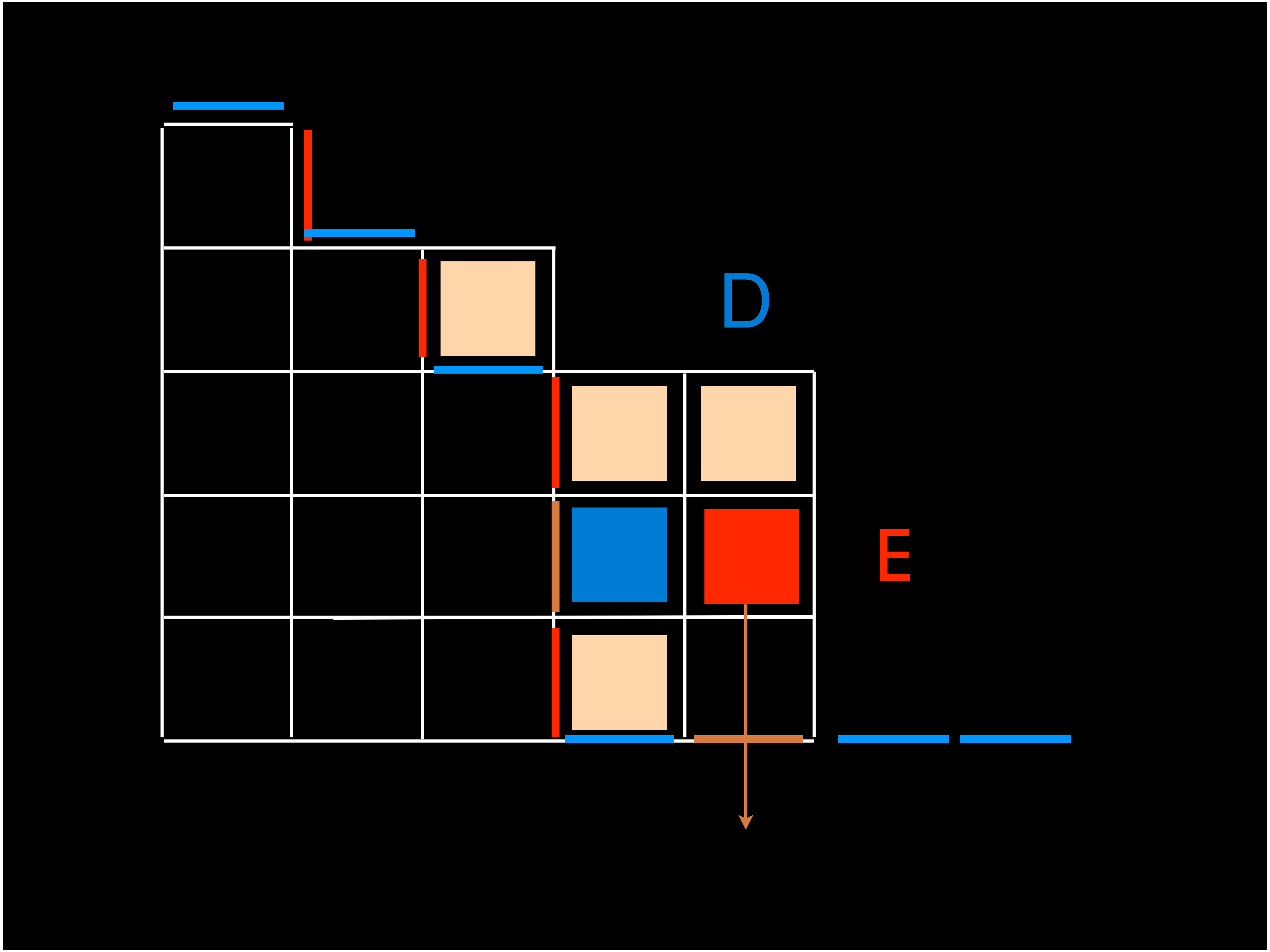
E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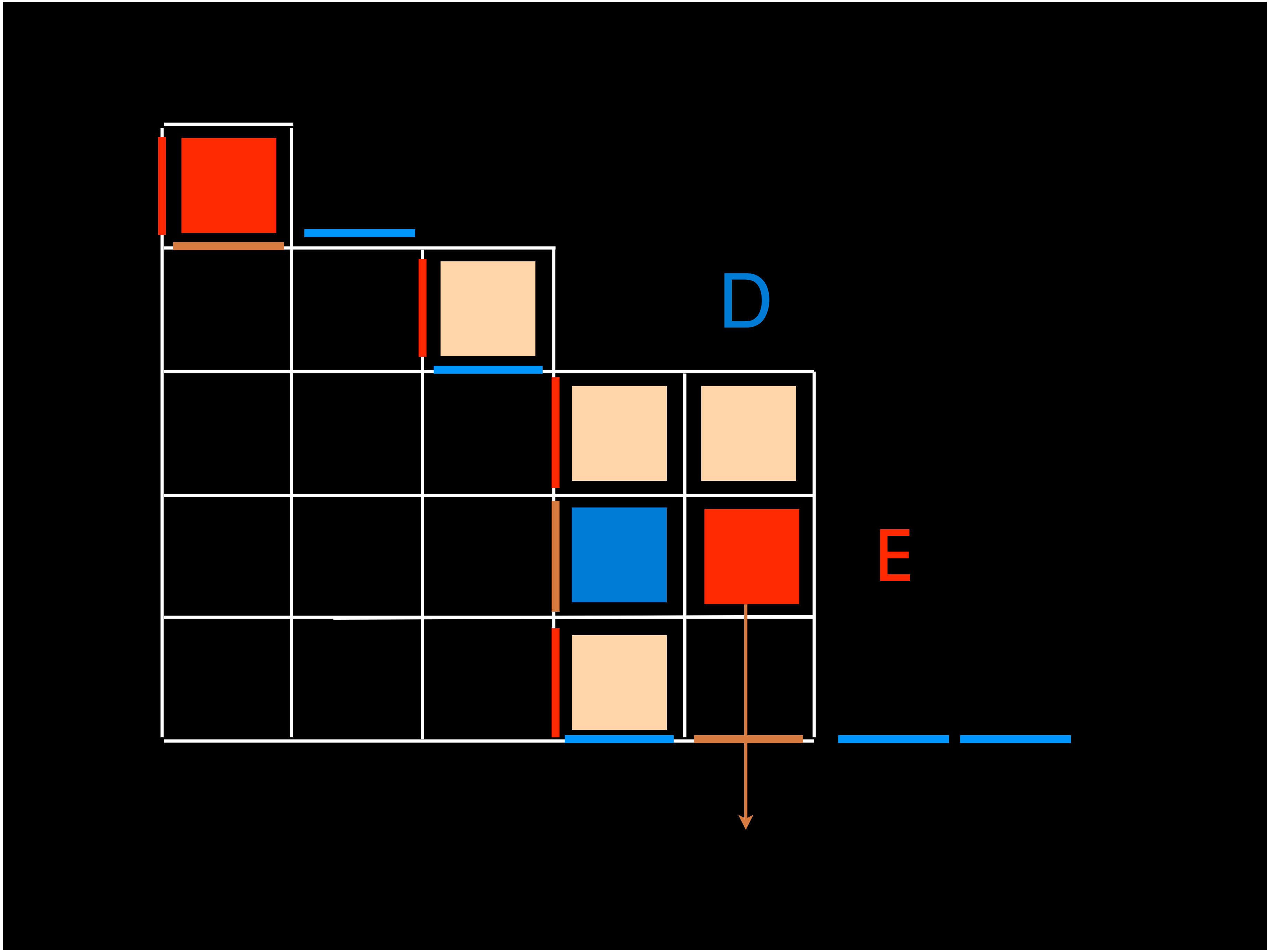
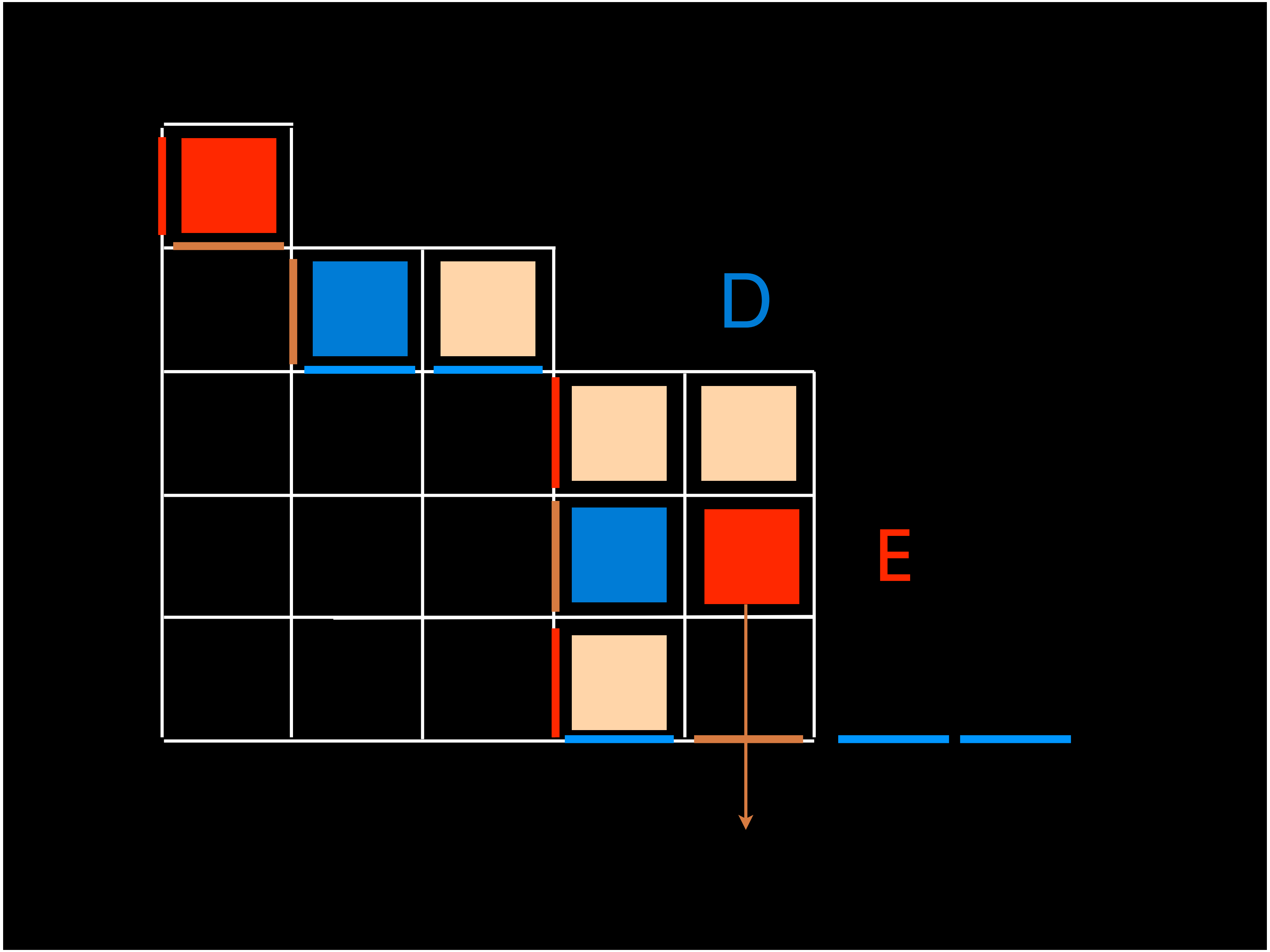
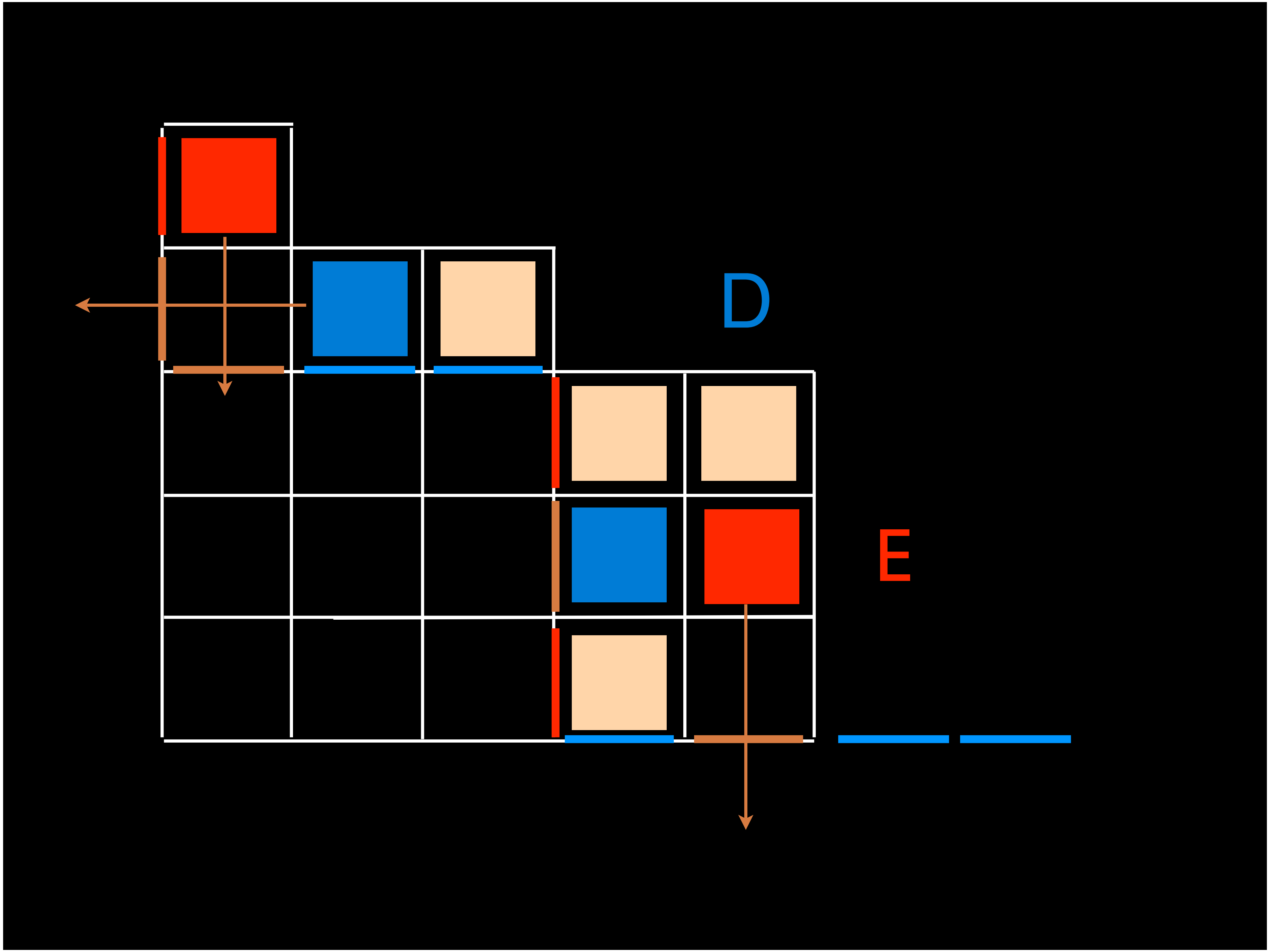

D
E


D
E


D
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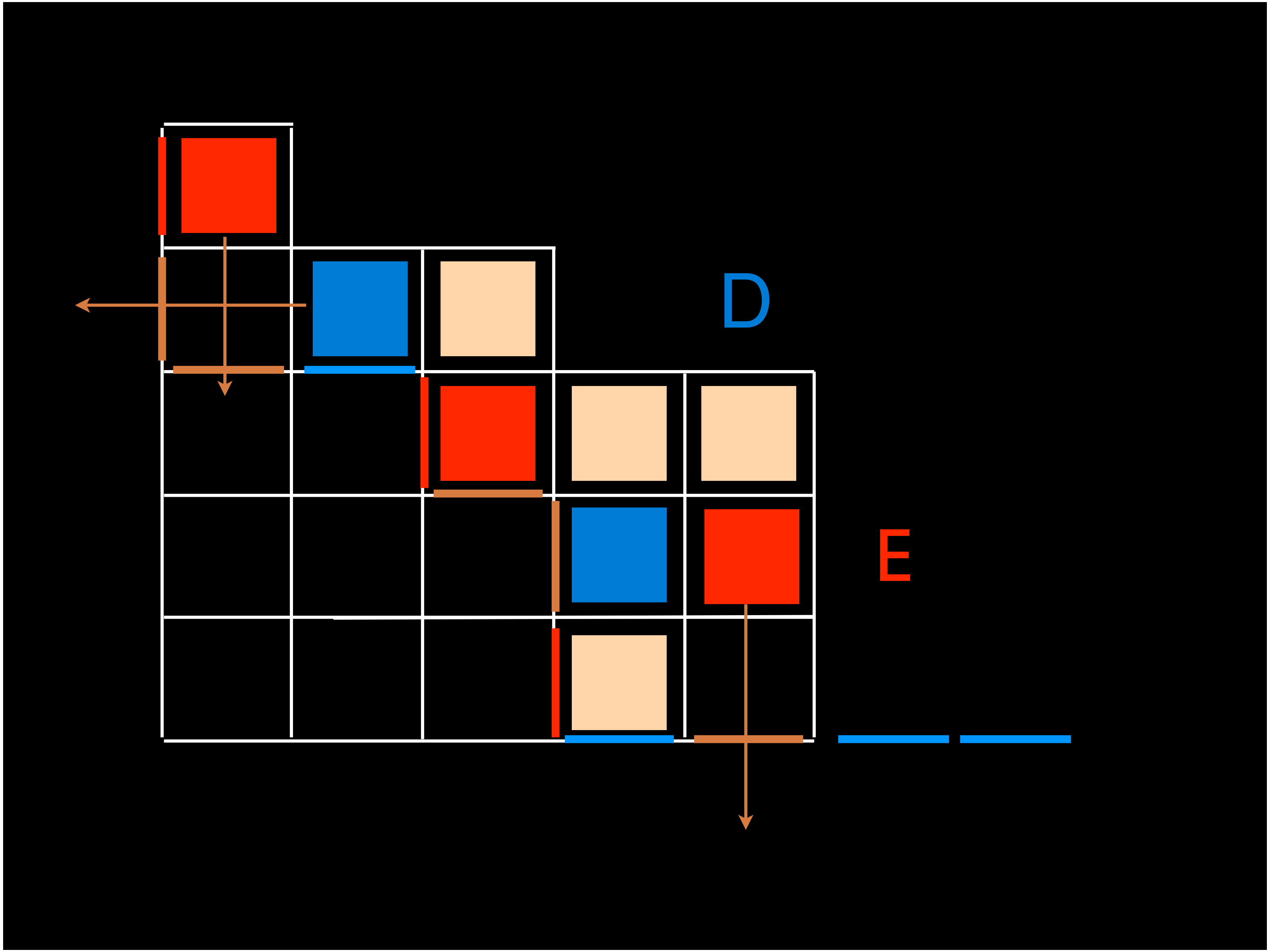
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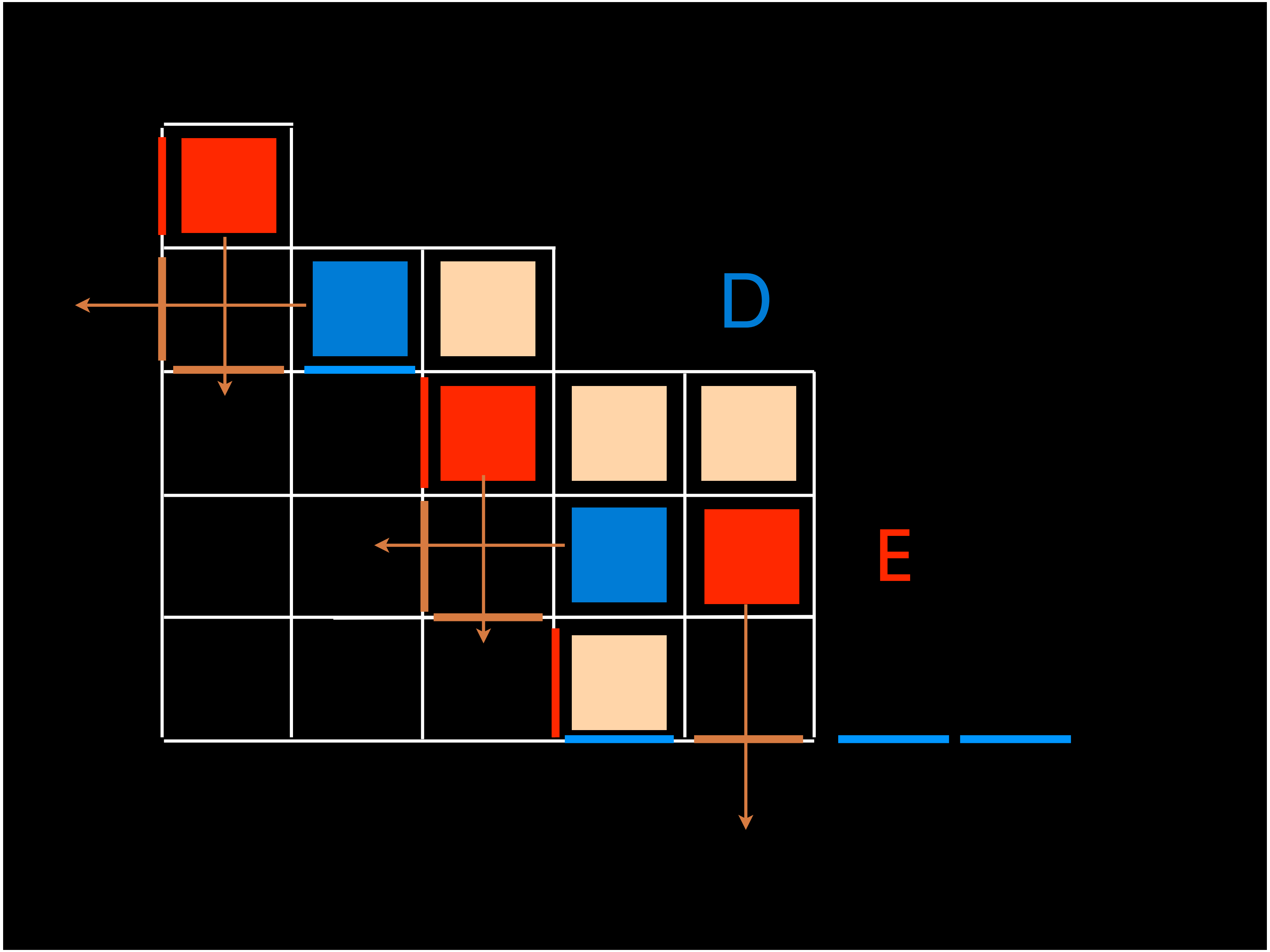
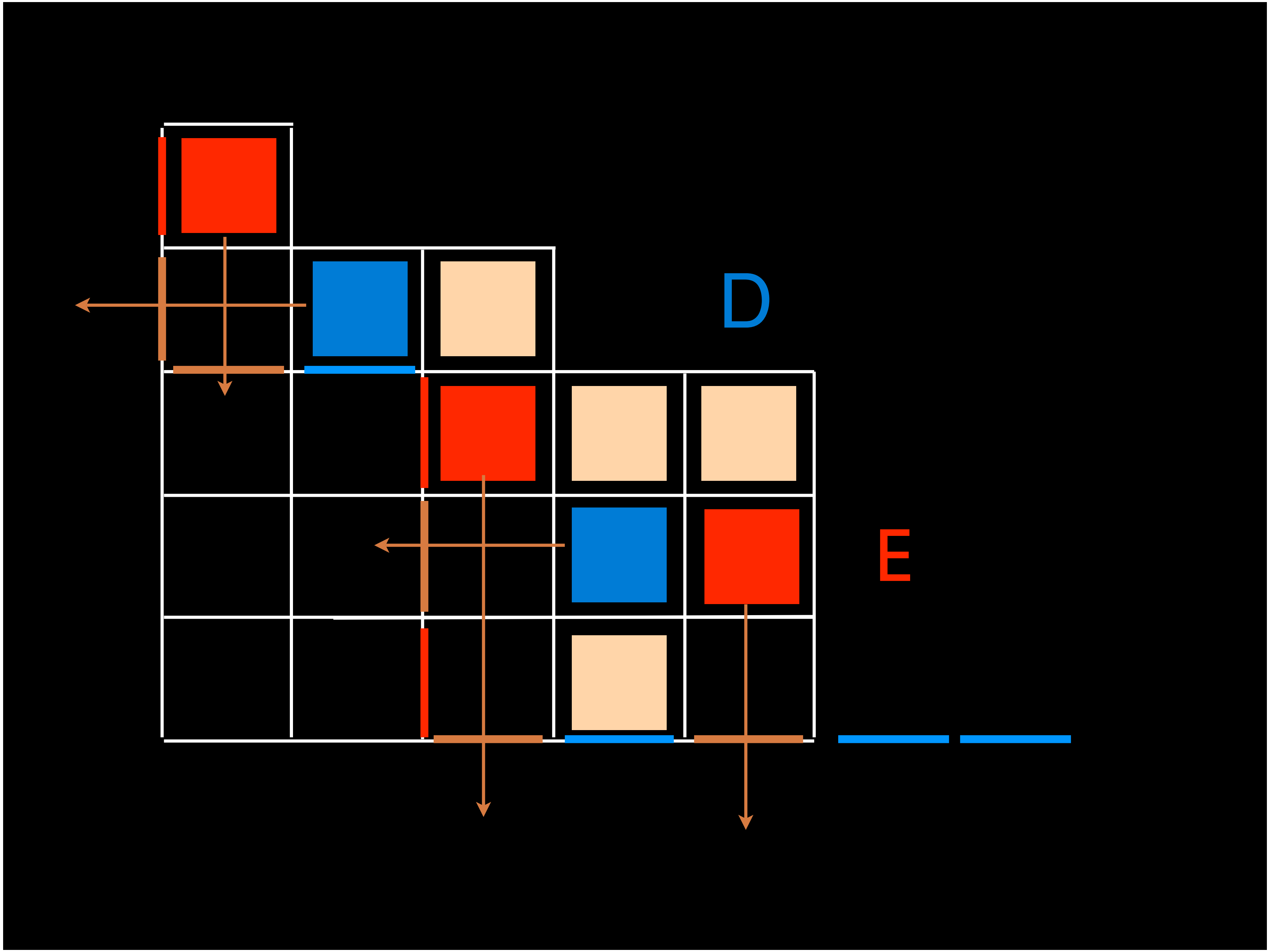
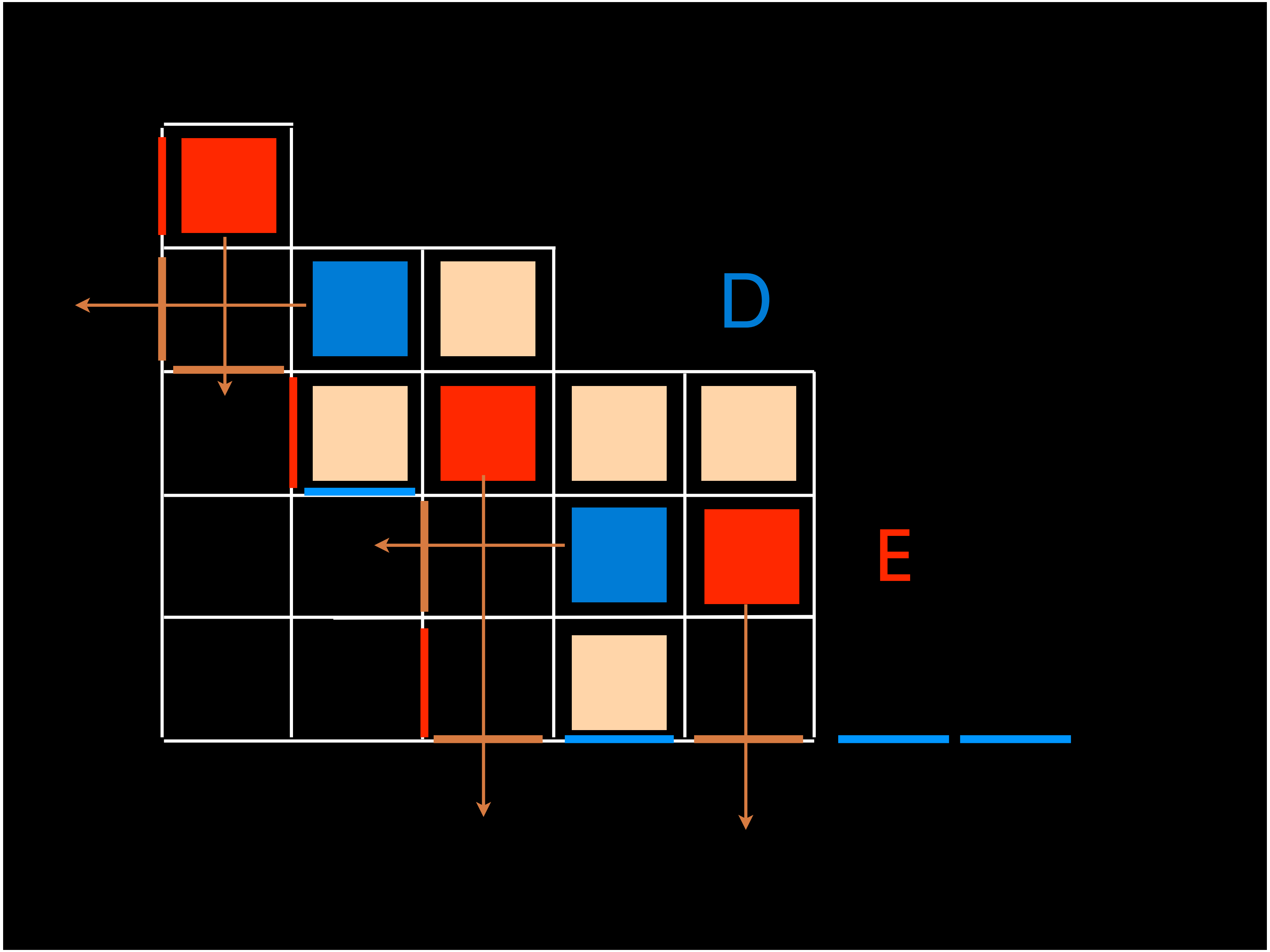
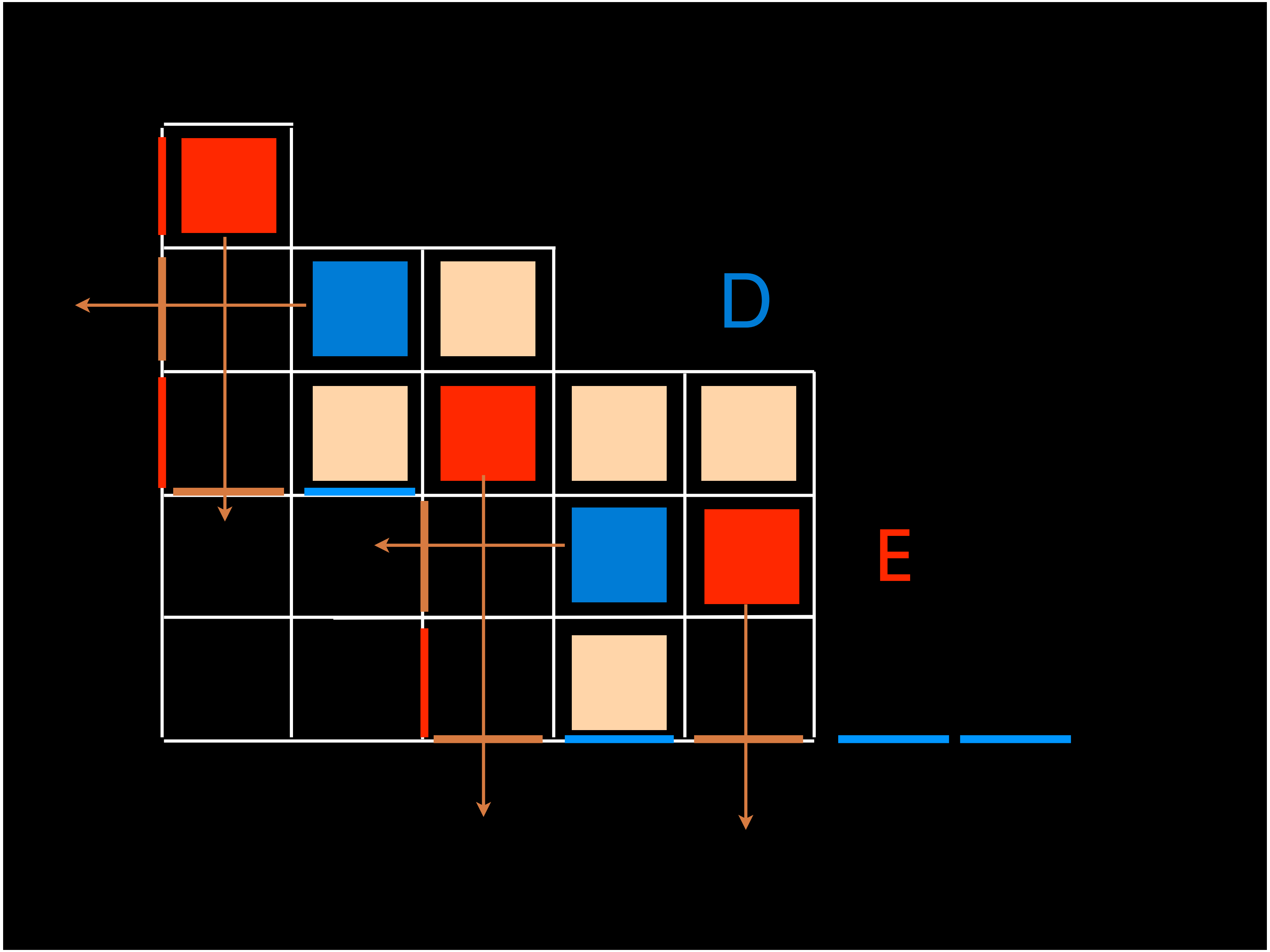

D
E


D
E


D
E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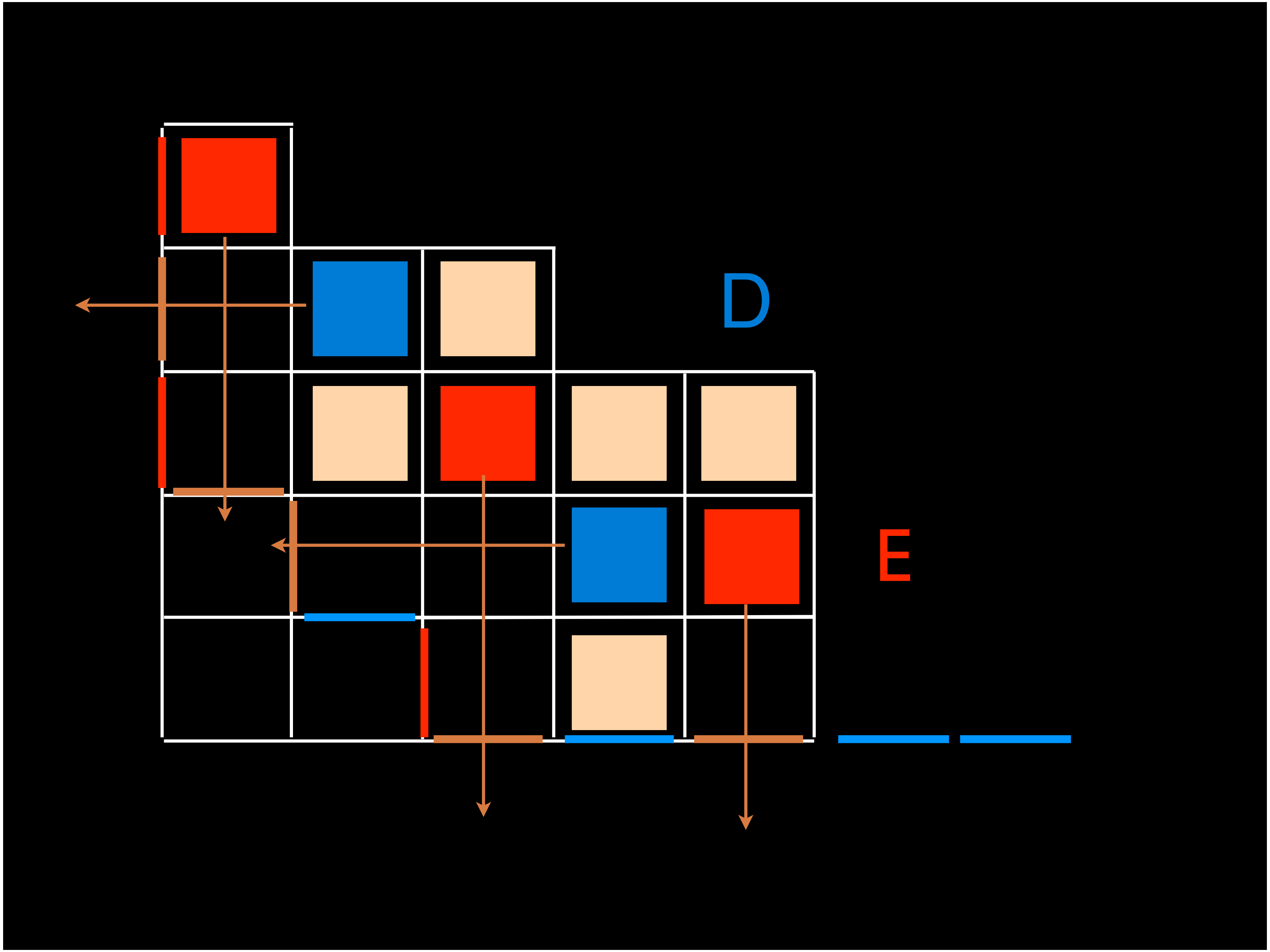
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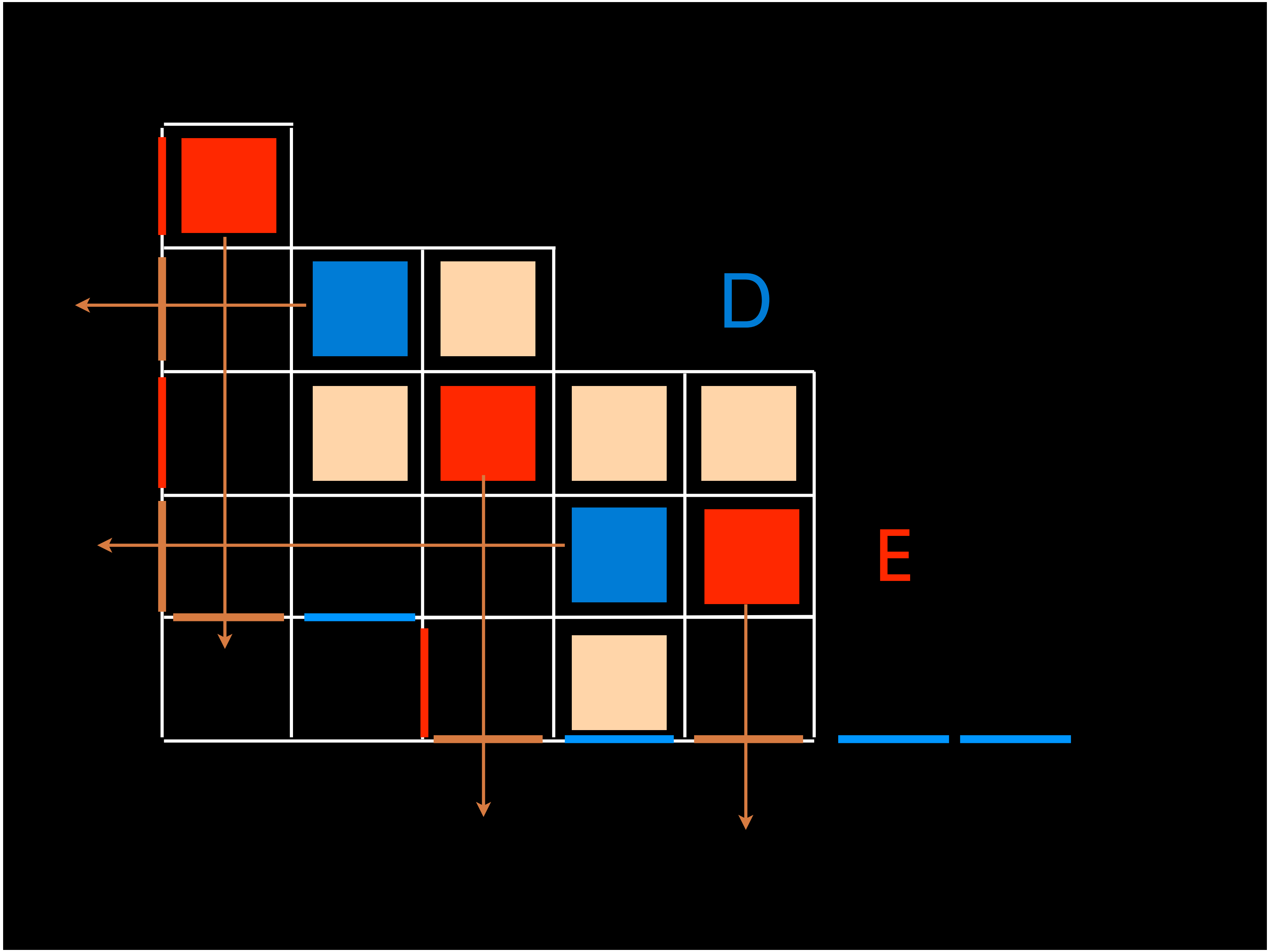
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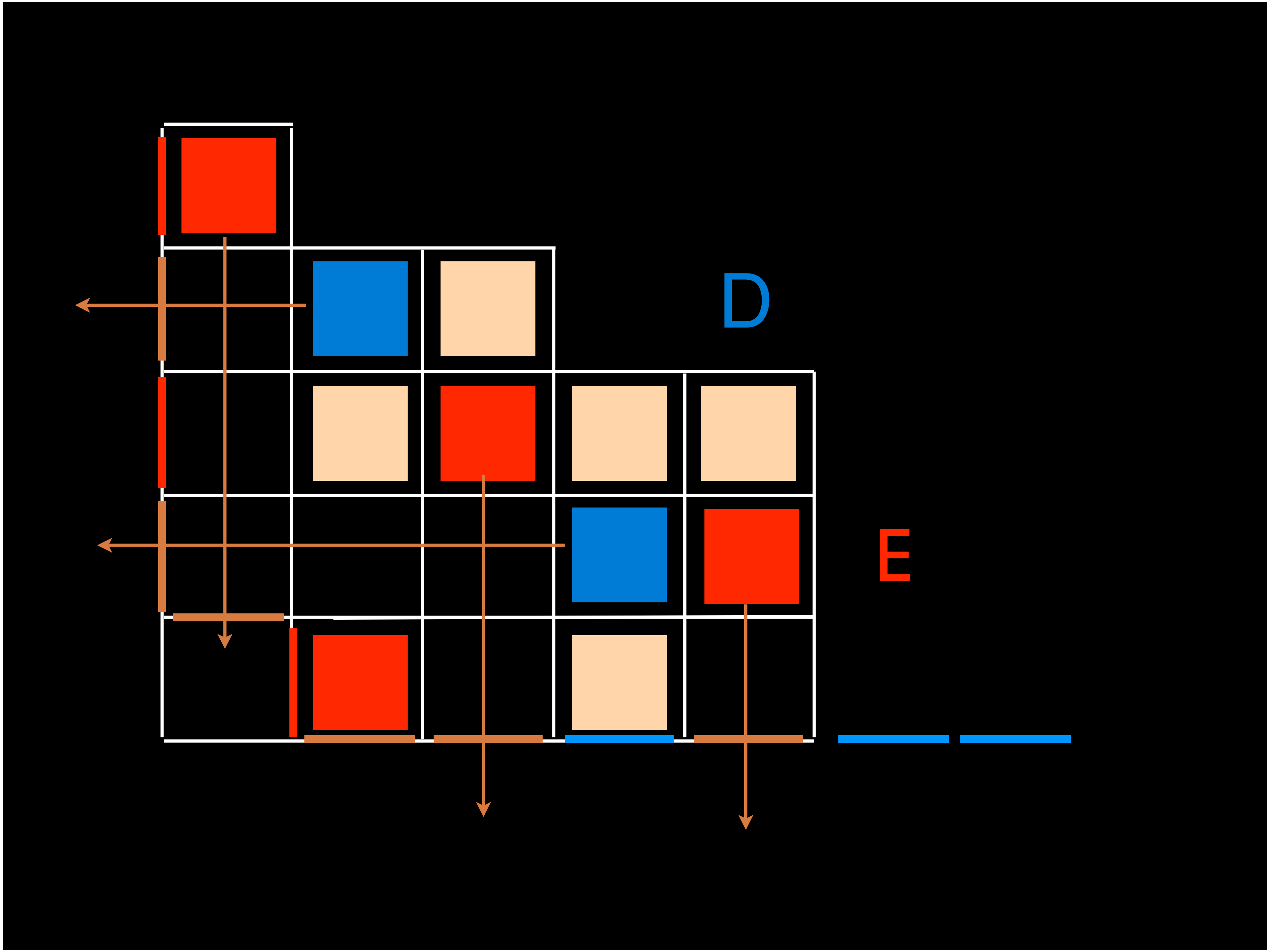
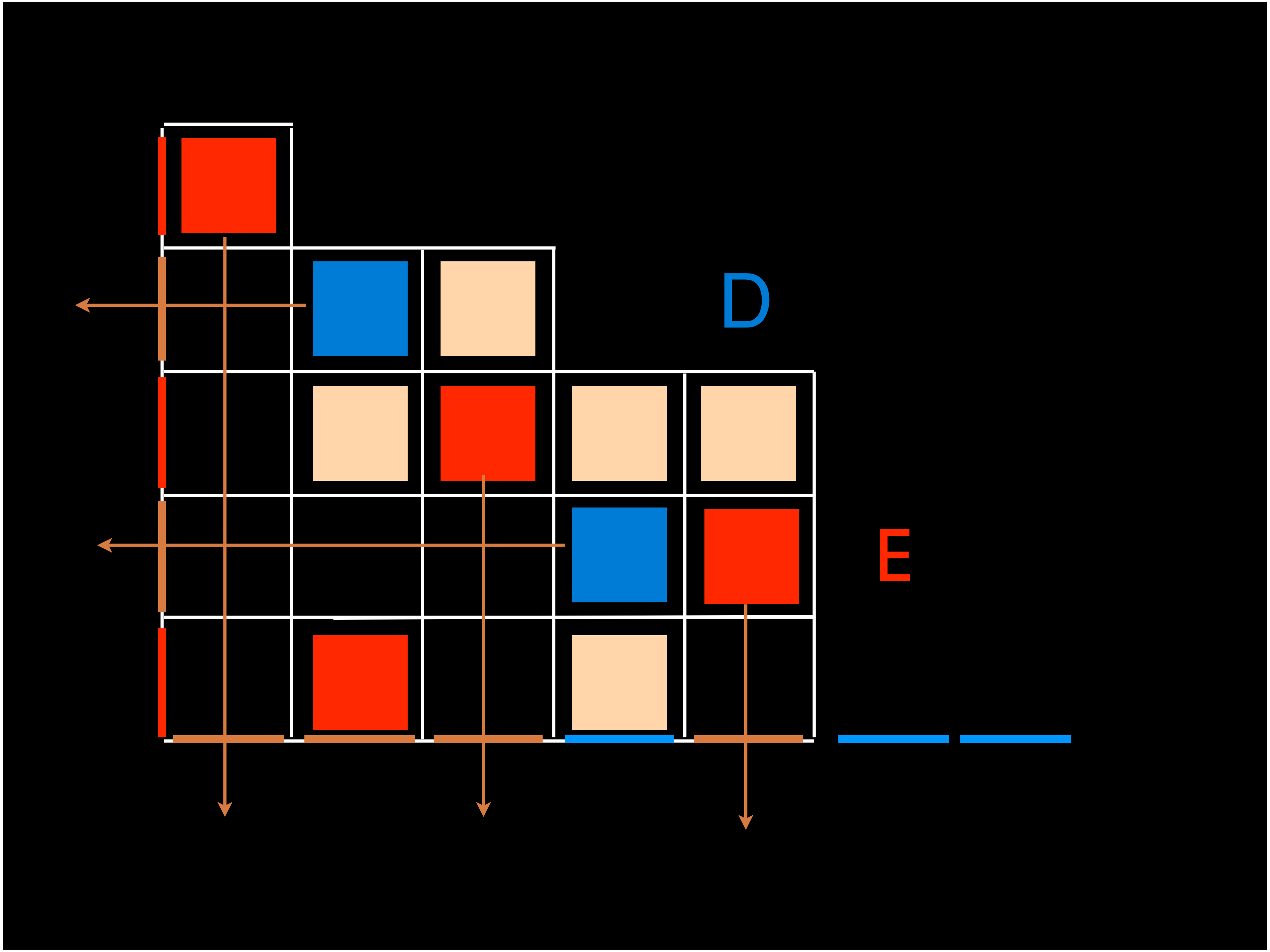
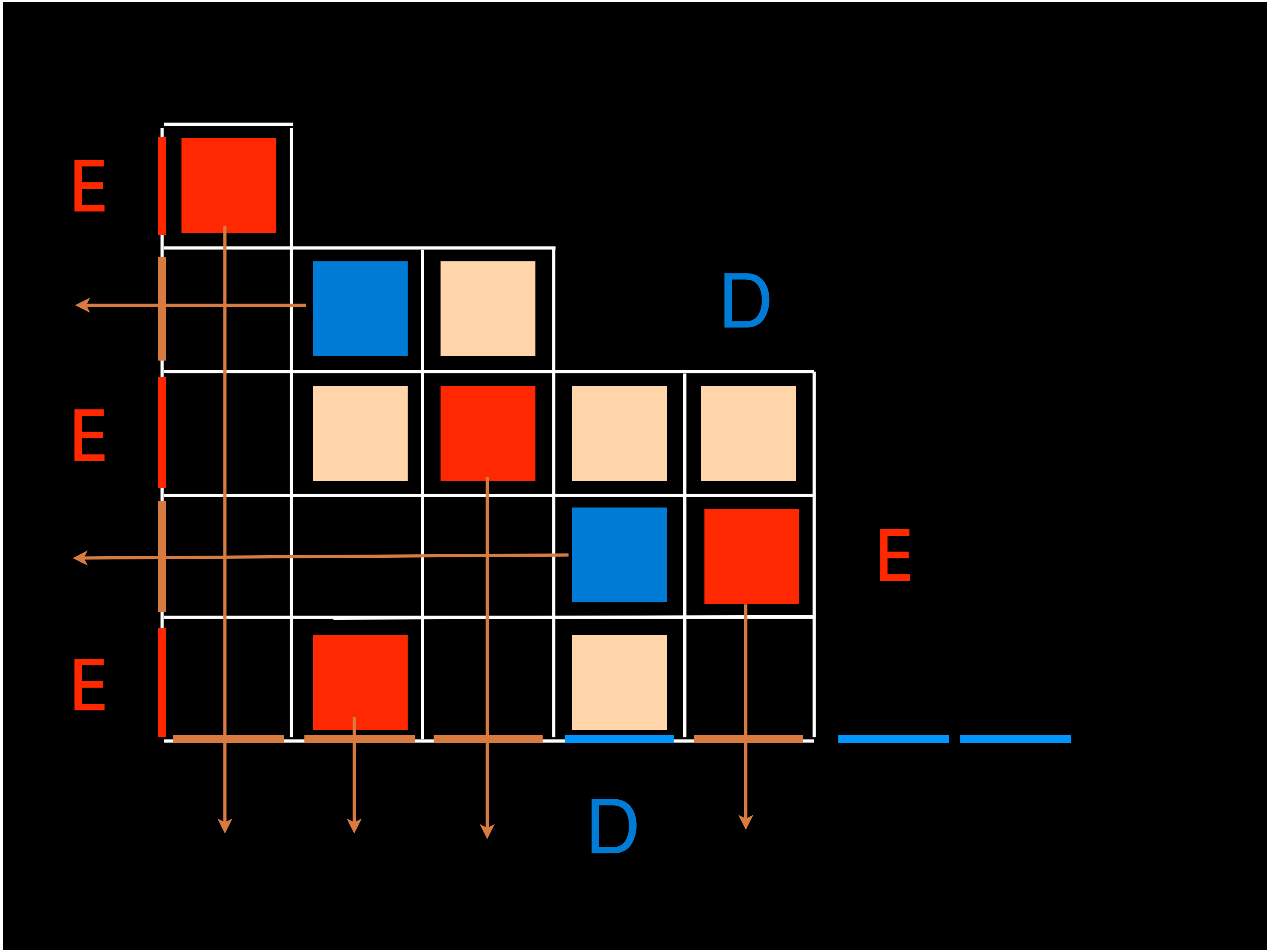
E


D
E


D
E


E
D
E
E
E
D

q-analog

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T \text{ alternative tableau with profile } w} q^{k(T)} E^{i(T)} D^{j(T)}$$

$k(T)$ = nb of $\boxtimes$

$i(T)$ = nb of columns without red cell

$j(T)$ = nb of rows without blue cell

$$\begin{cases} DE = qED + D + E & \\ DV = \bar{\beta} V & \bar{\beta} = 1/\beta \\ WE = \bar{\alpha} W & \bar{\alpha} = 1/\alpha \end{cases}$$

$$W E^{i} D^{j} V = \bar{\alpha}^{i} \bar{\beta}^{j} \underbrace{WV}_{1}$$

Cor. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ (PASEP) is

$$\text{proba}_{\tau}(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{T} q^{L(T)} \alpha^{-f(T)} \beta^{-u(T)}$$

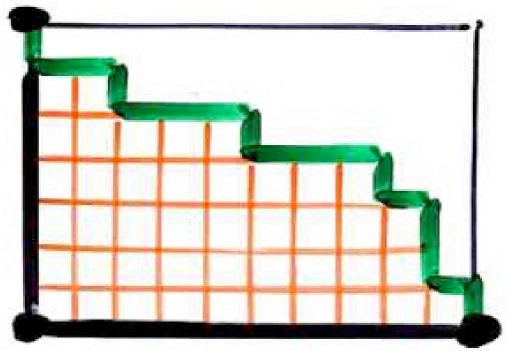
T alternative tableaux profile τ

$$\begin{cases} f(T) & \text{nb of rows without red cell} \\ u(T) & \text{nb of columns without blue cell} \\ L(T) & \text{nb of cells} \end{cases}$$

§ 5 Permutation tableaux

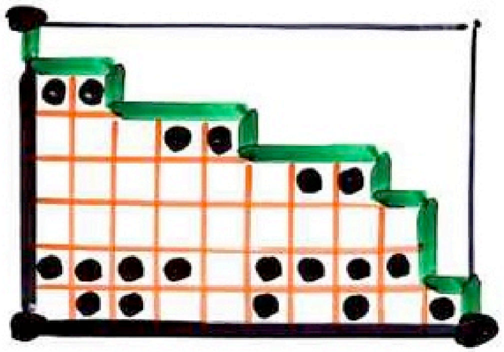
Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



Permutation Tableau

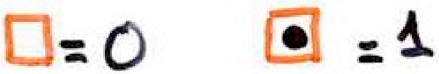
Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

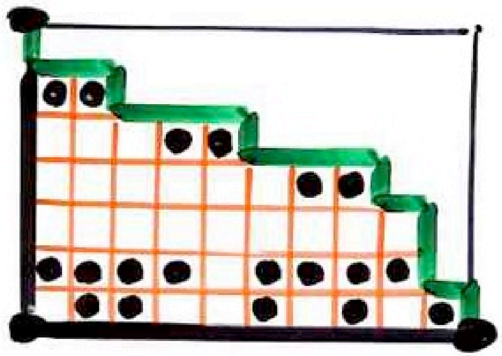
(i)

(ii)



Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

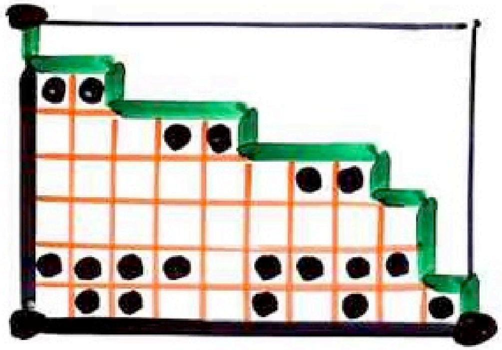
(i) in each column : at least one 1

□ = 0   ◙ = 1

(ii)

Permutation Tableau

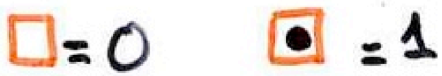
Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

(i) in each column : at least one 1

(ii)  forbidden



permutation tableau

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)

Corteel, Williams (2006) PASEP

Partially Asymmetric Exclusion Process

M. Josuat-Vergès (2007)

The total number of permutation tableaux (n fixed, $1 \leq k \leq n$) is

$$n!$$

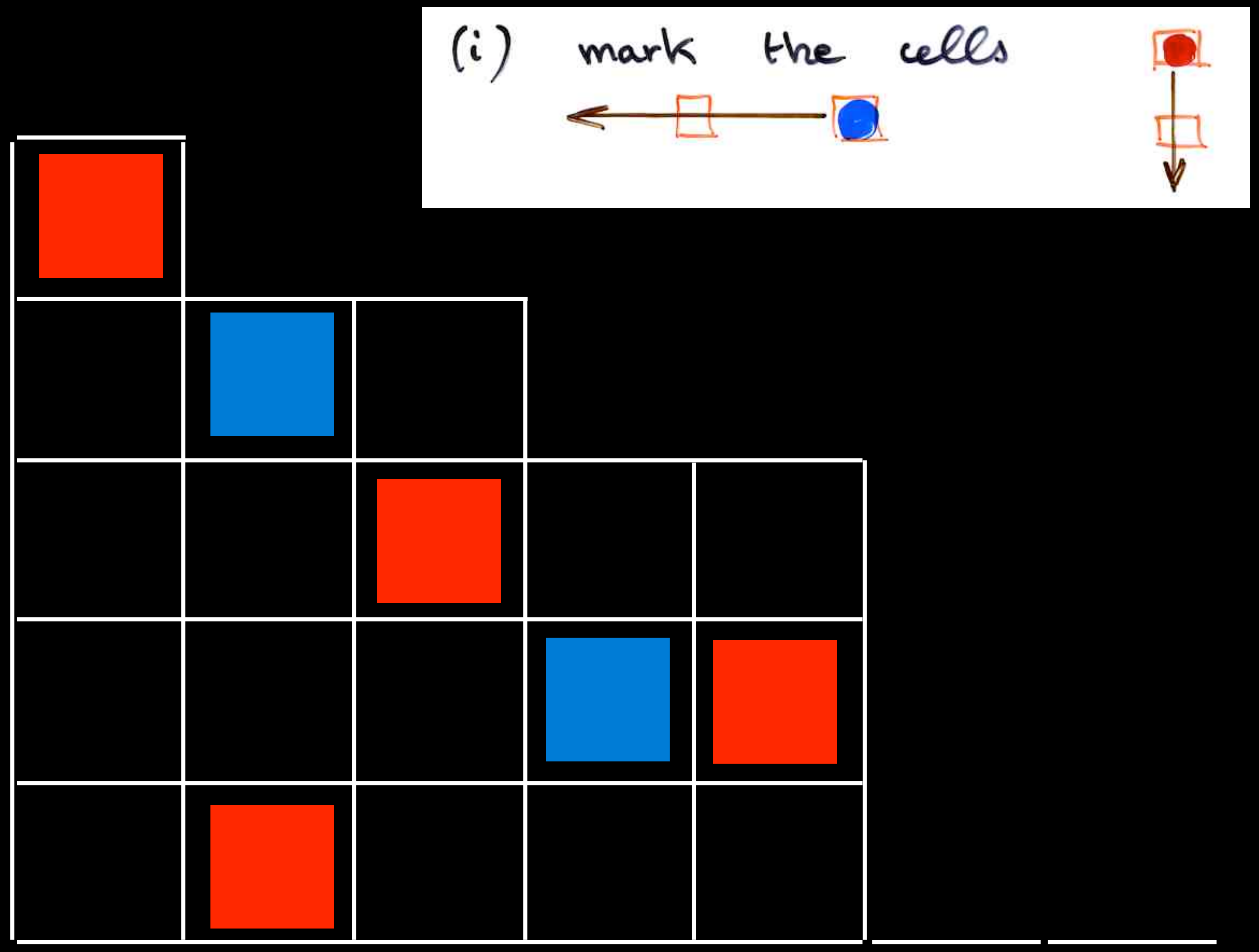
bijection

permutations $\longleftrightarrow$ permutation tableaux

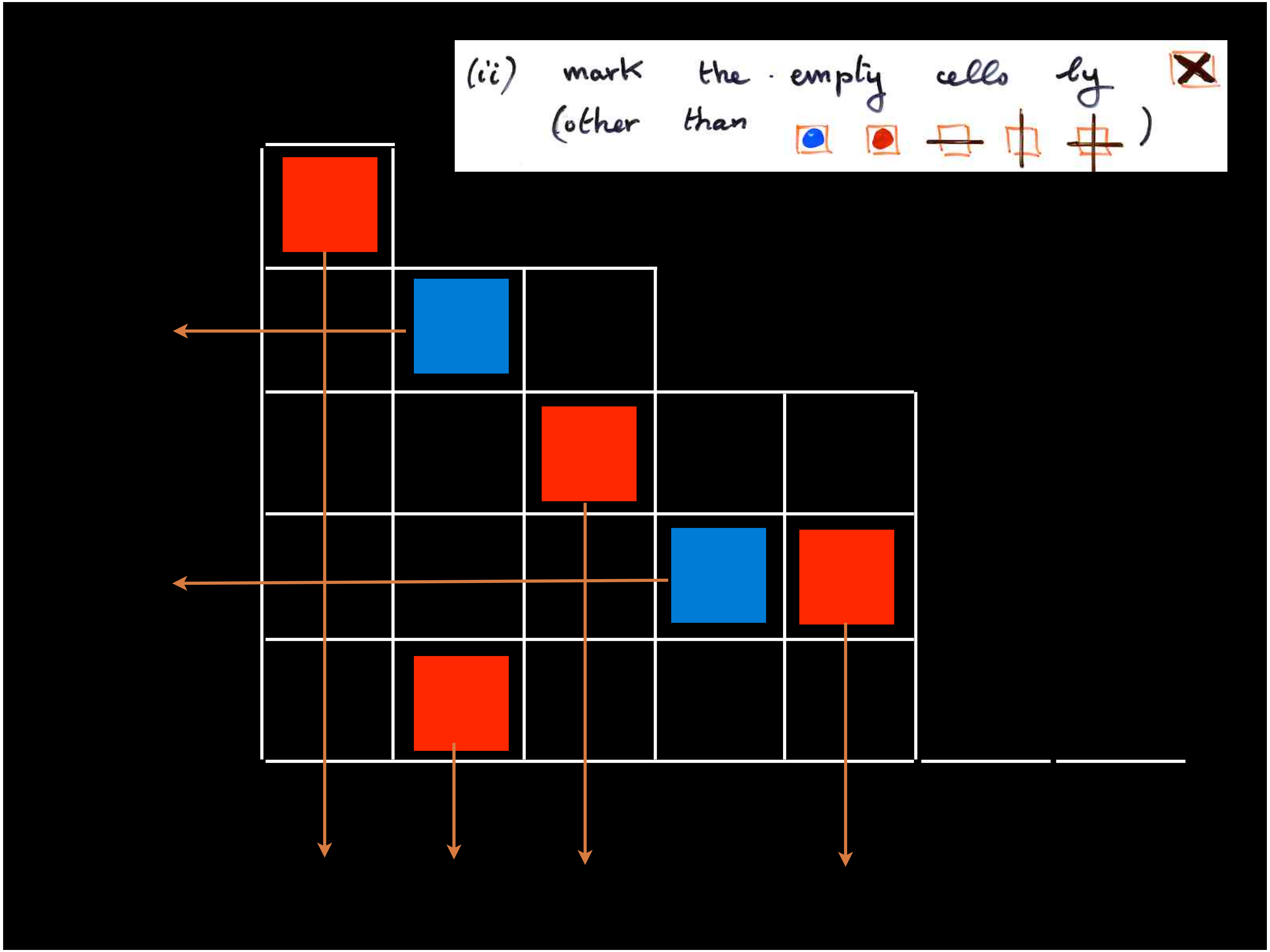
- Postnikov, Steingrímsson, Williams (2005)
- Corteel (2006)
- Corteel, Nadeau (2007)

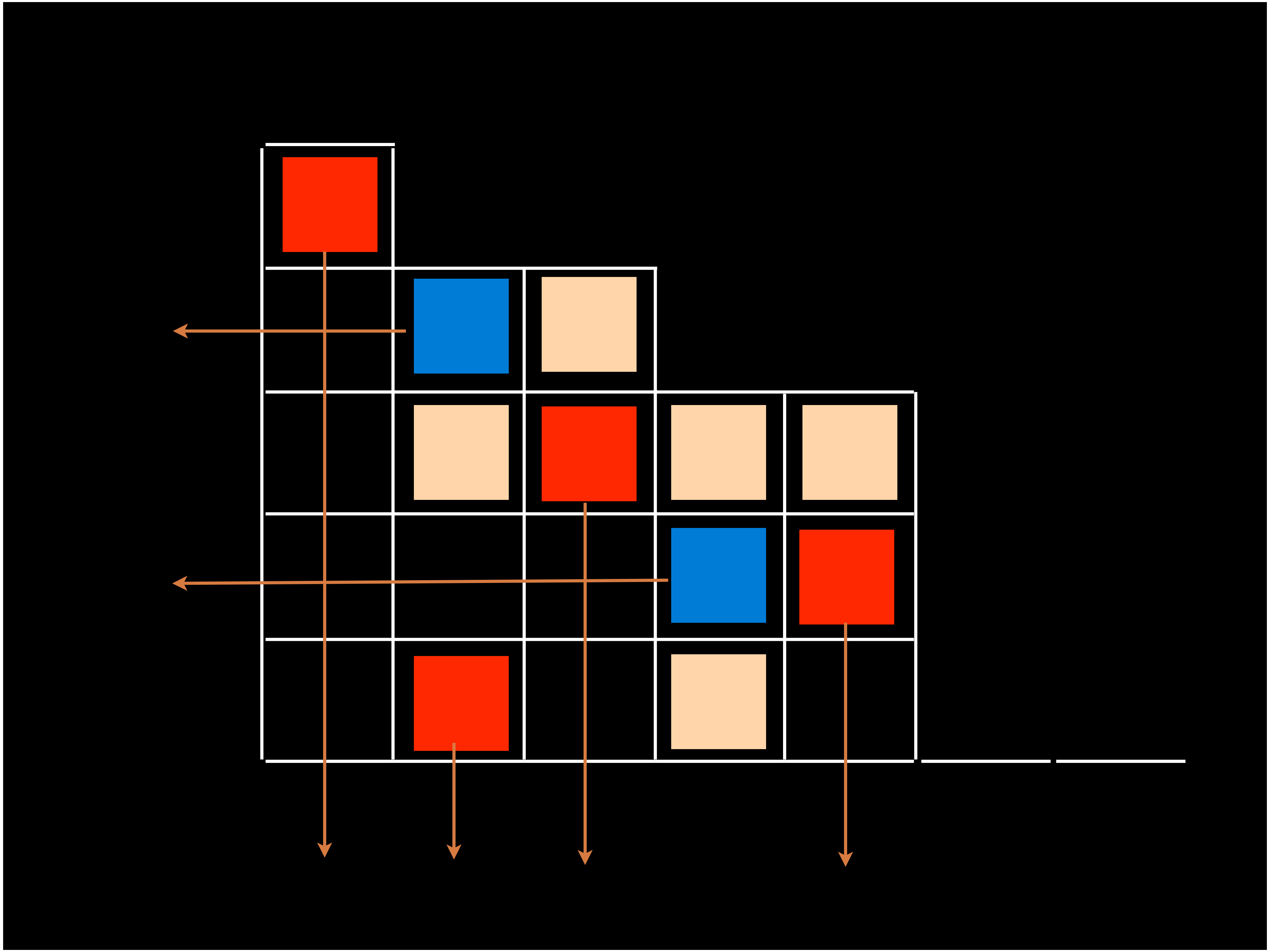
bijection $\updownarrow$ alternative tableaux size n

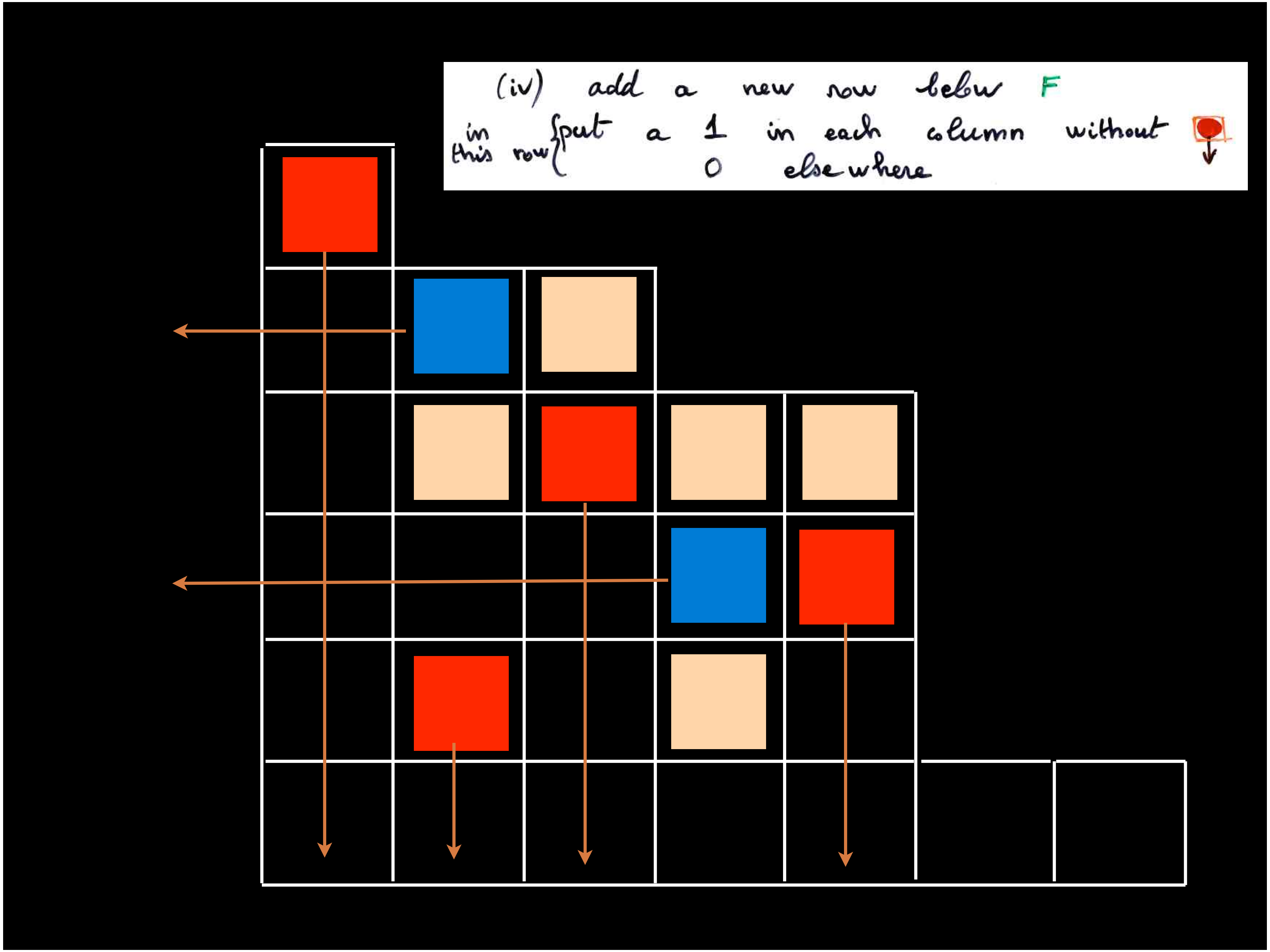
permutation tableaux size $(n+1)$

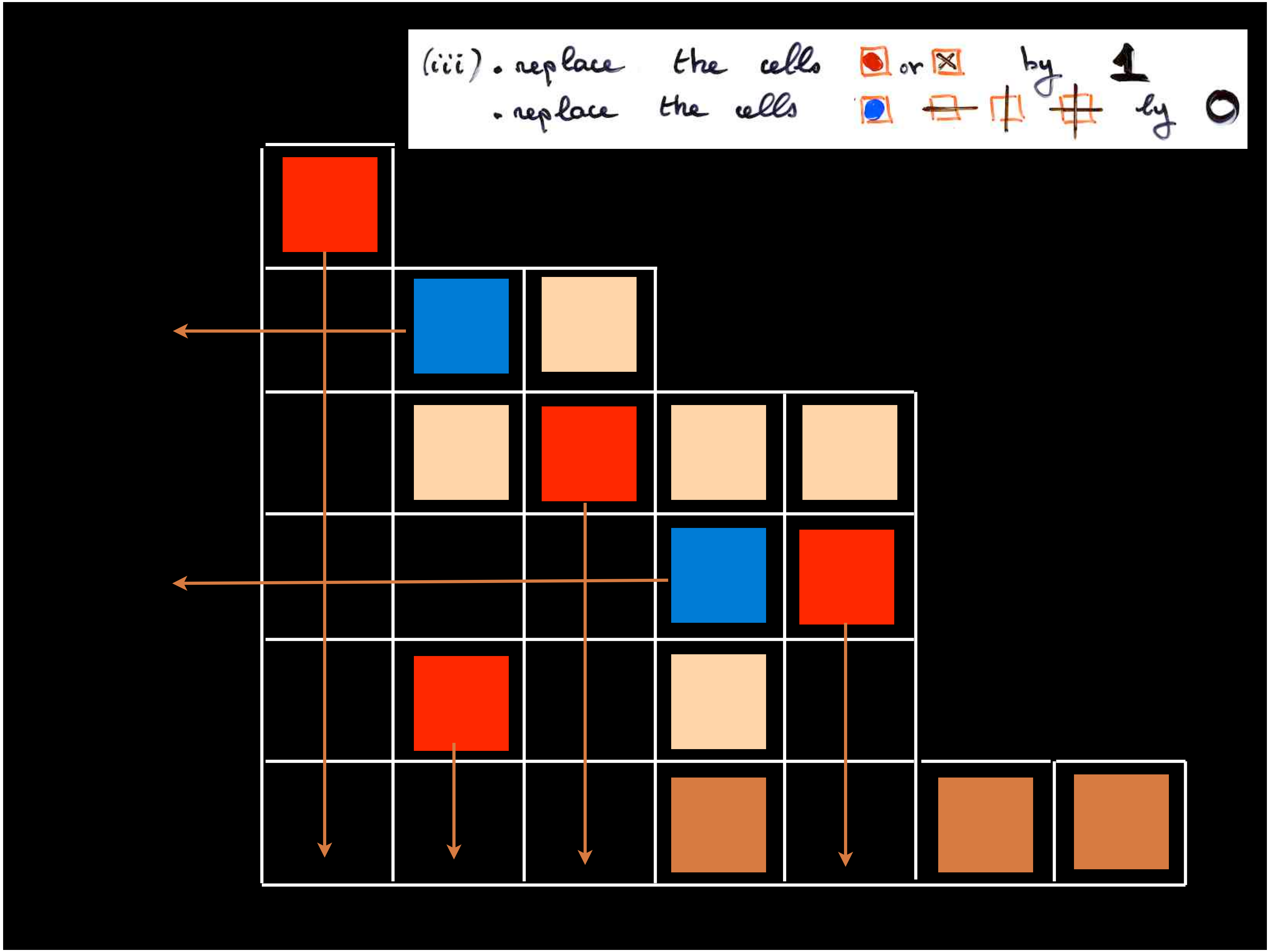


alternative tableau

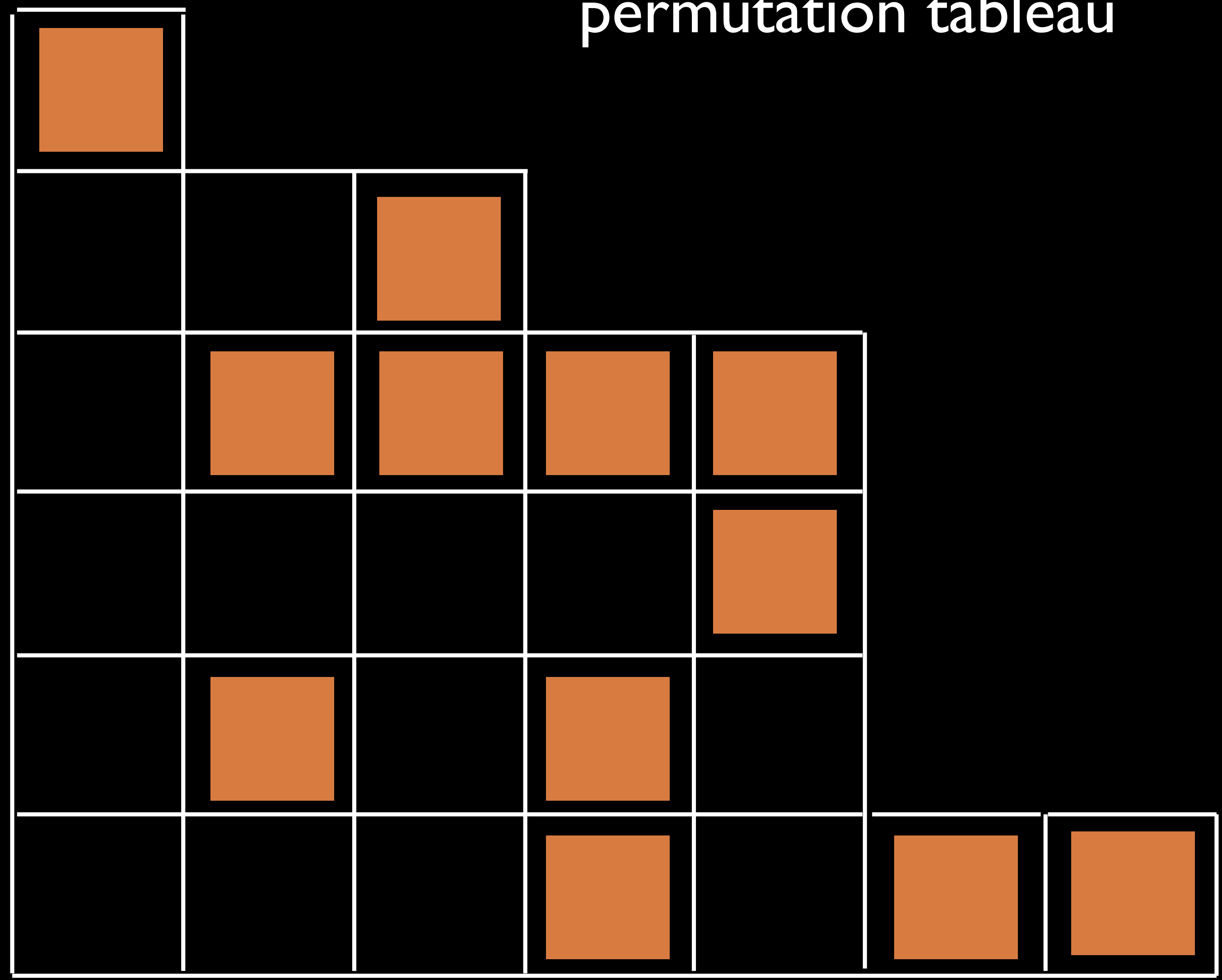

(ii) mark the empty cells by
(other than)




(iv) add a new row below F
in this row {put a 1 in each column without
0 elsewhere

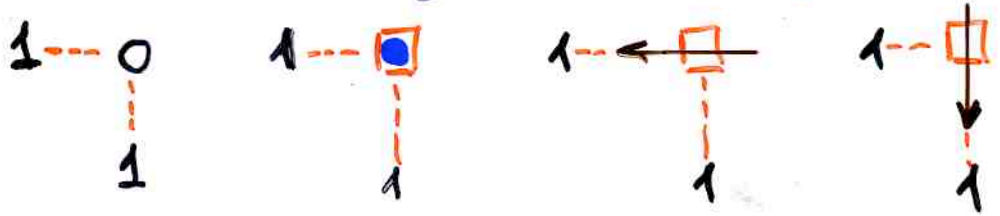

(iii) • replace the cells … or … by 1
• replace the cells … by 0

permutation tableau



check: AT $\xrightarrow{\varphi}$ PT size $(n+1)$

- the exist at least a 1 in each column of PT $= \varphi(AT)$

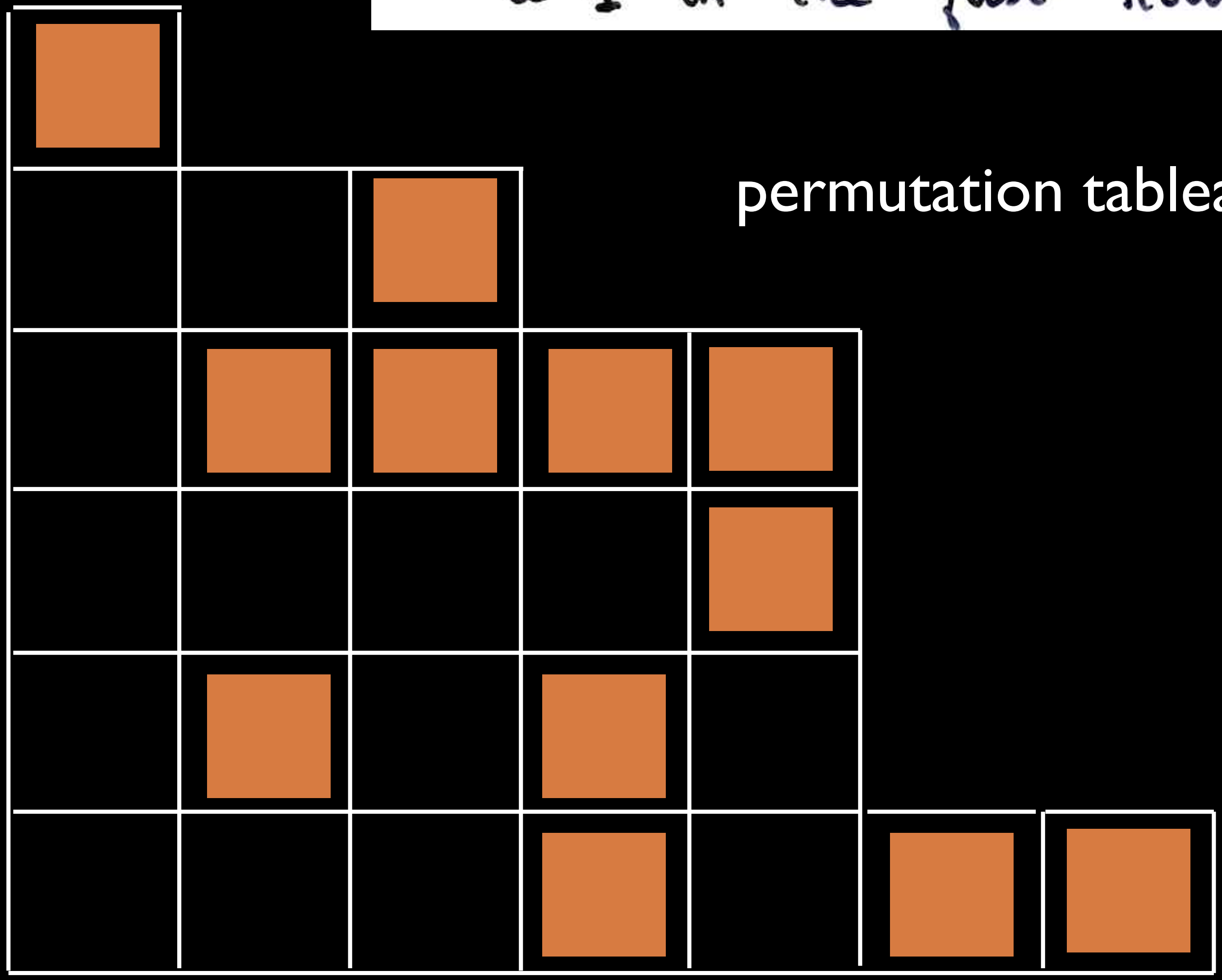


impossible

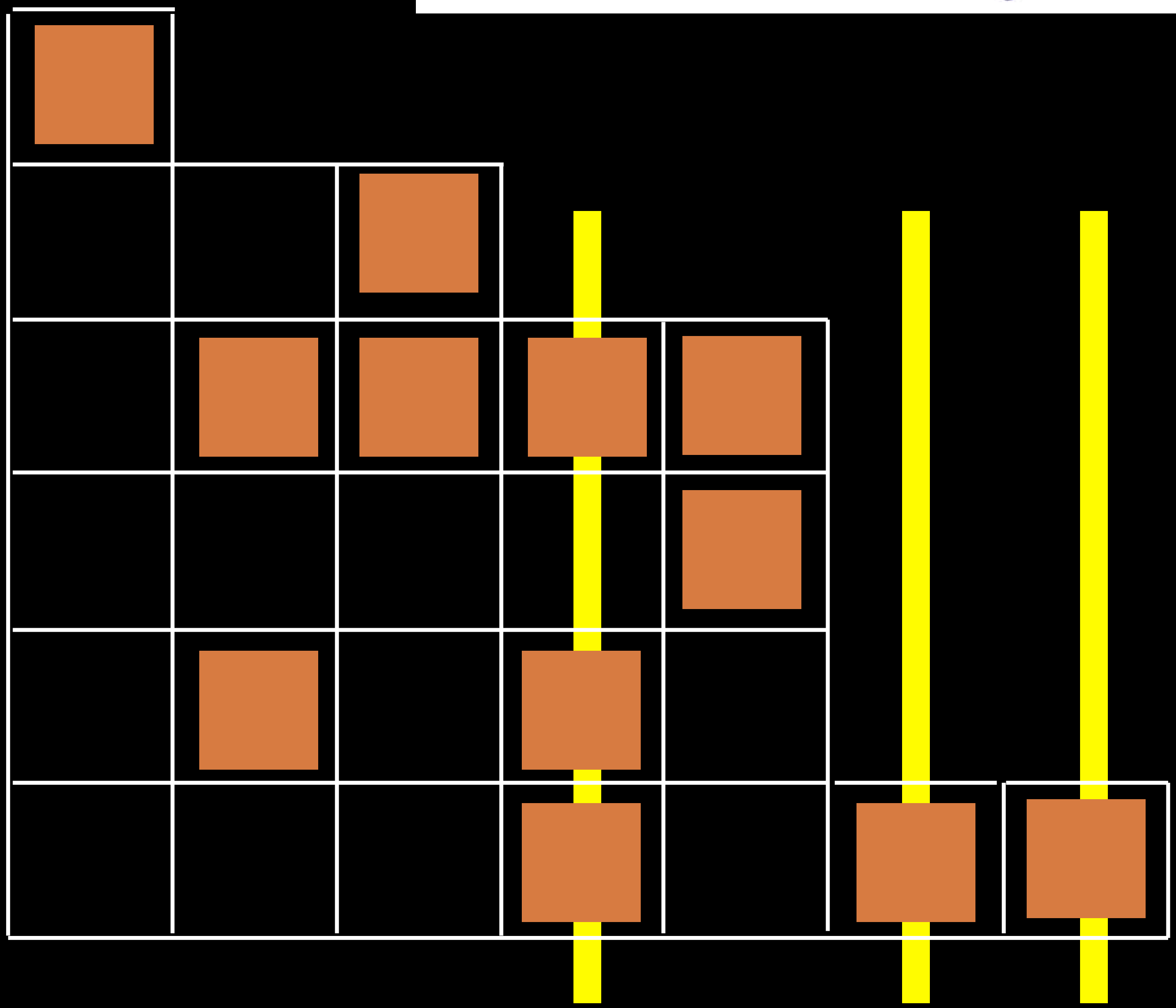
inverse bijection $\psi = \varphi^{-1}$

(i) mark the columns with a 1 in the first row

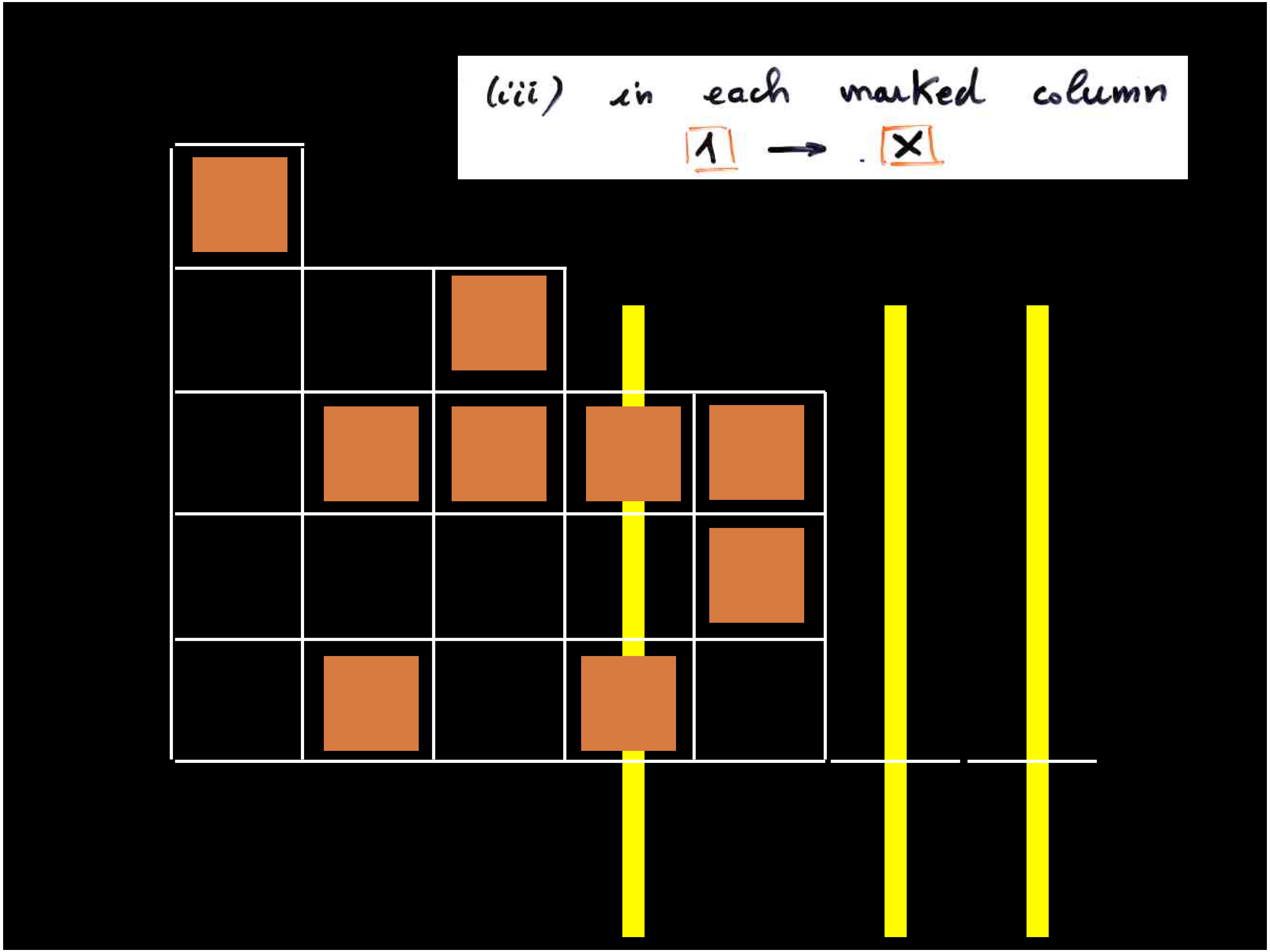
permutation tableau



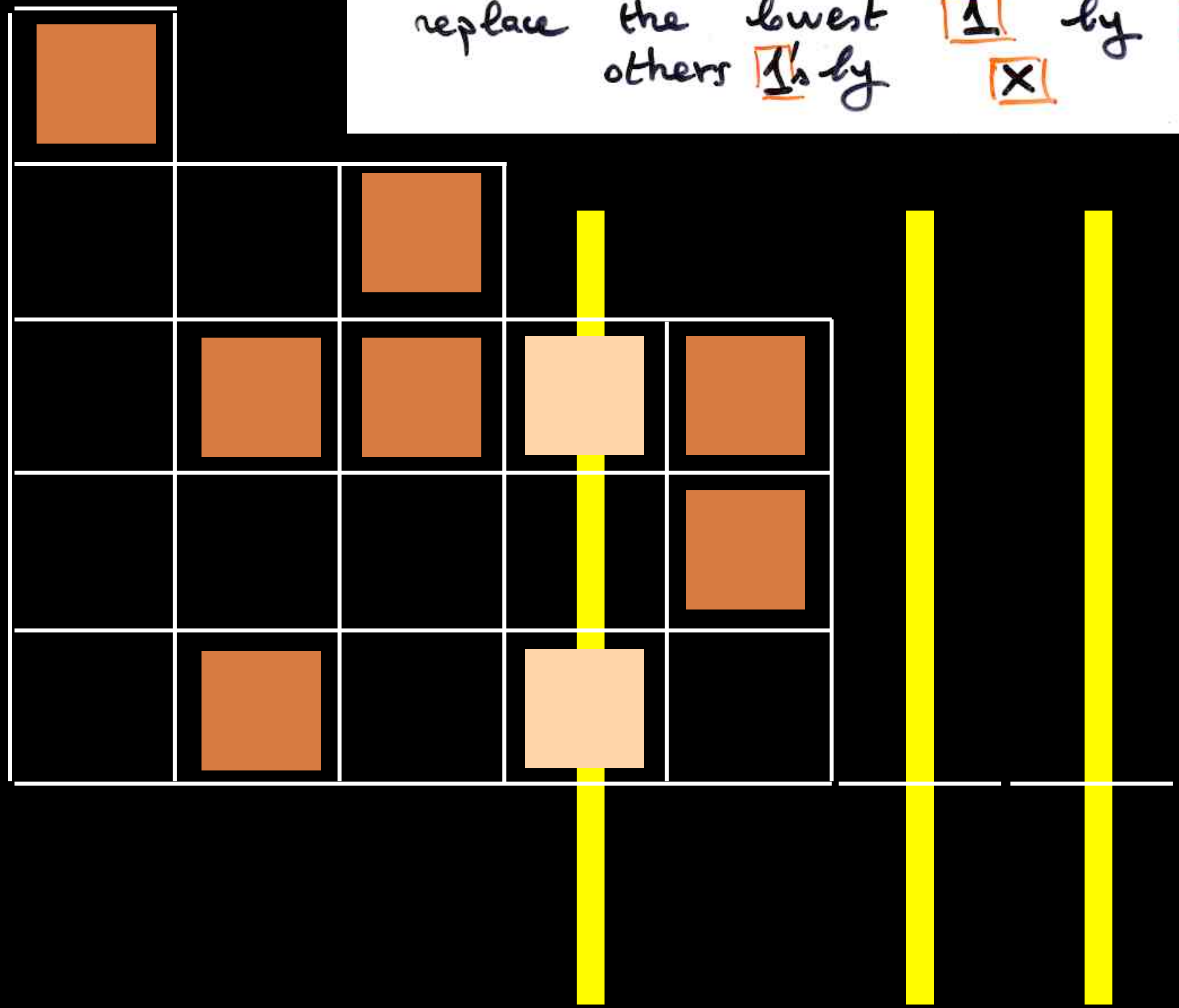
(ii) delete the first row

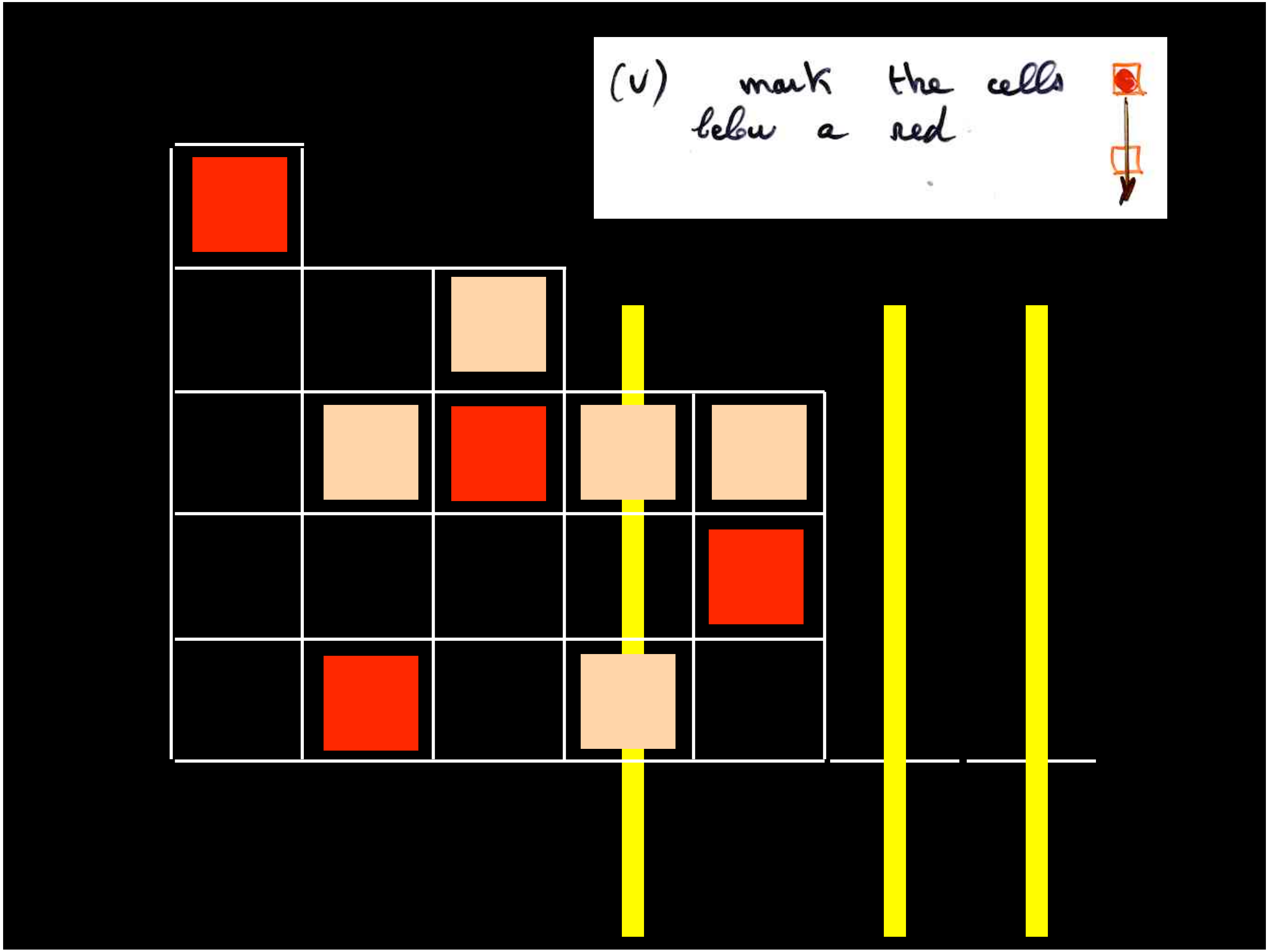


(iii) in each marked column

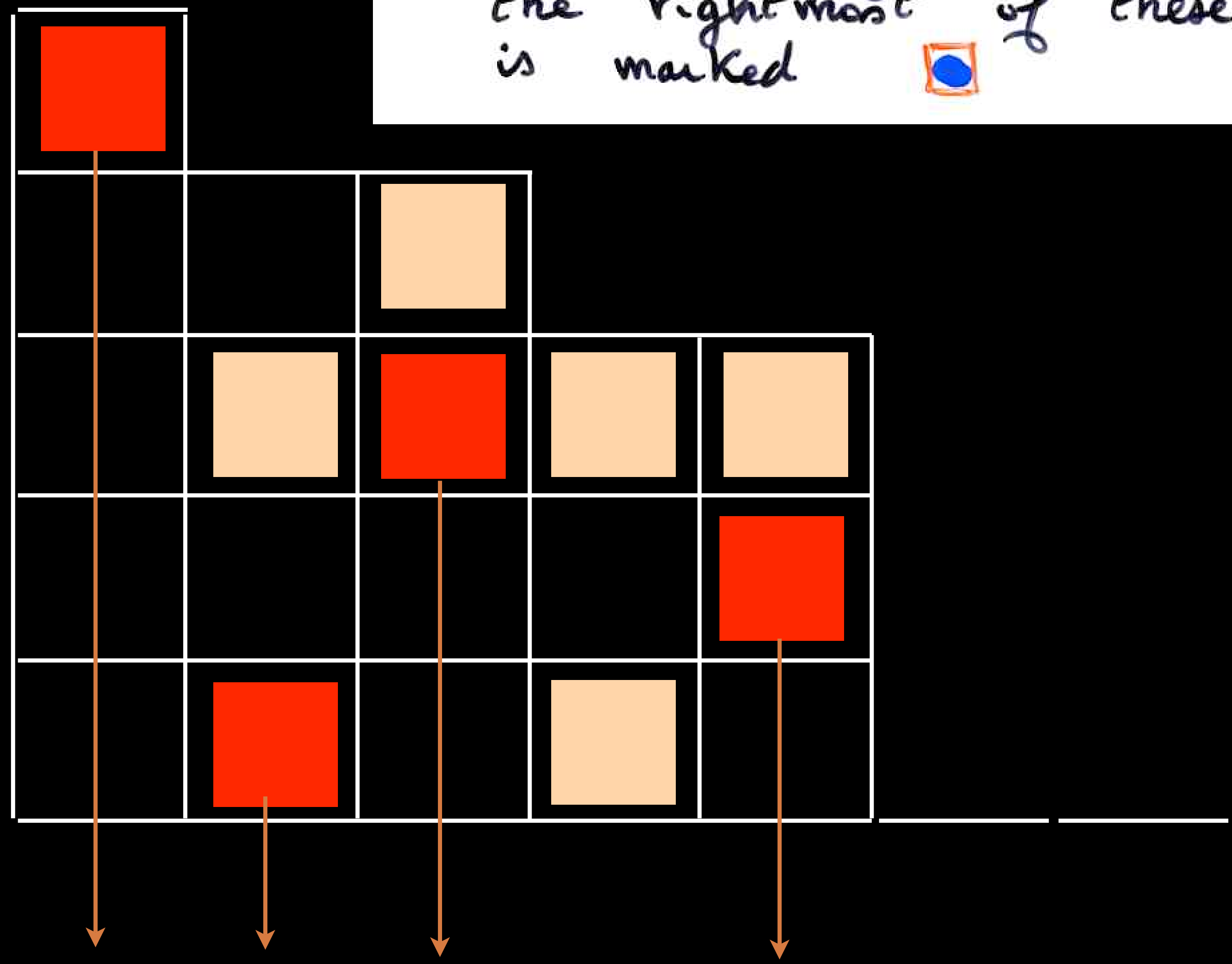


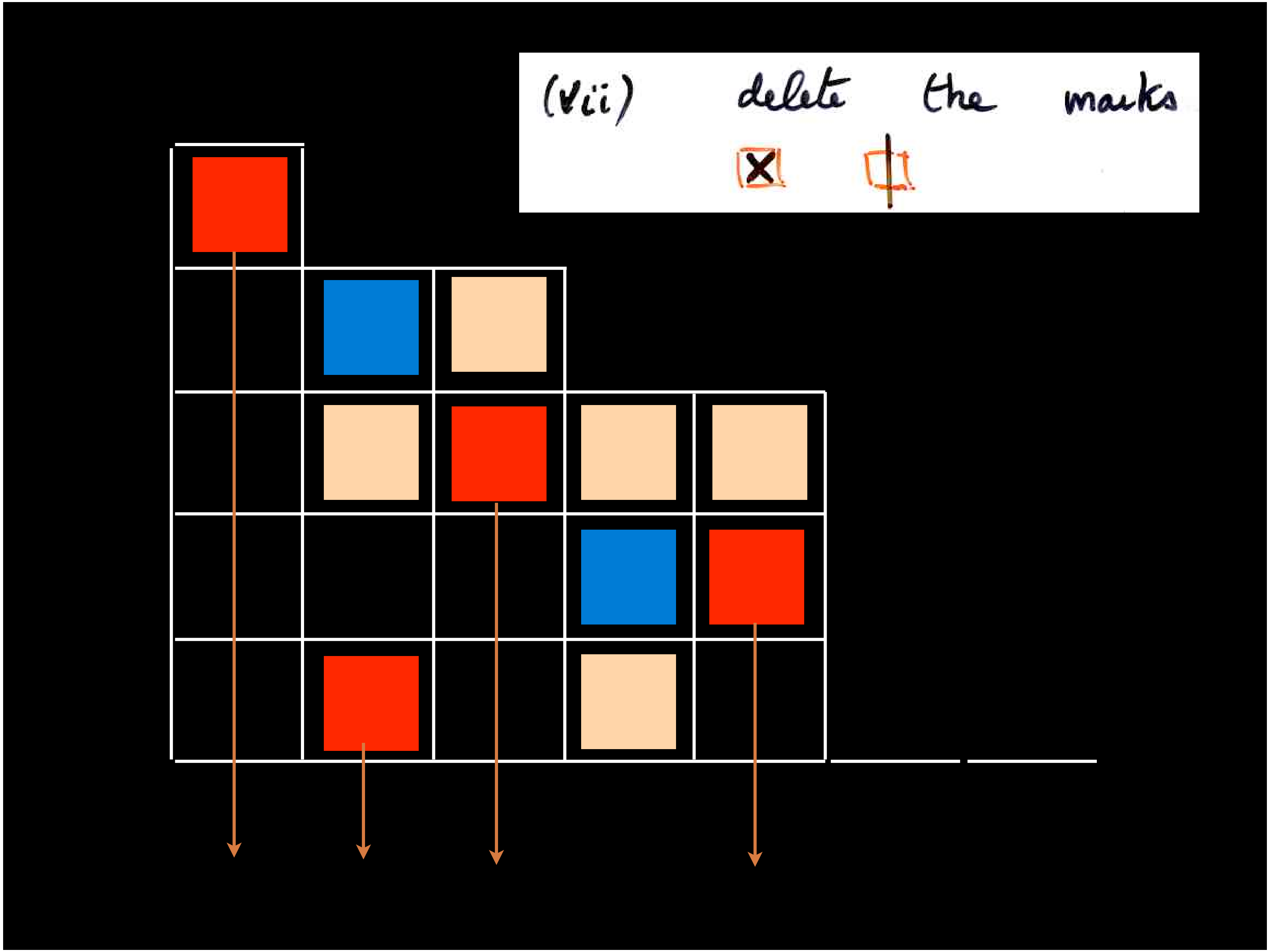
(iv) in each non marked column (∃ some cells with 1)
replace the lowest [1] by ●
others 1's by [X]



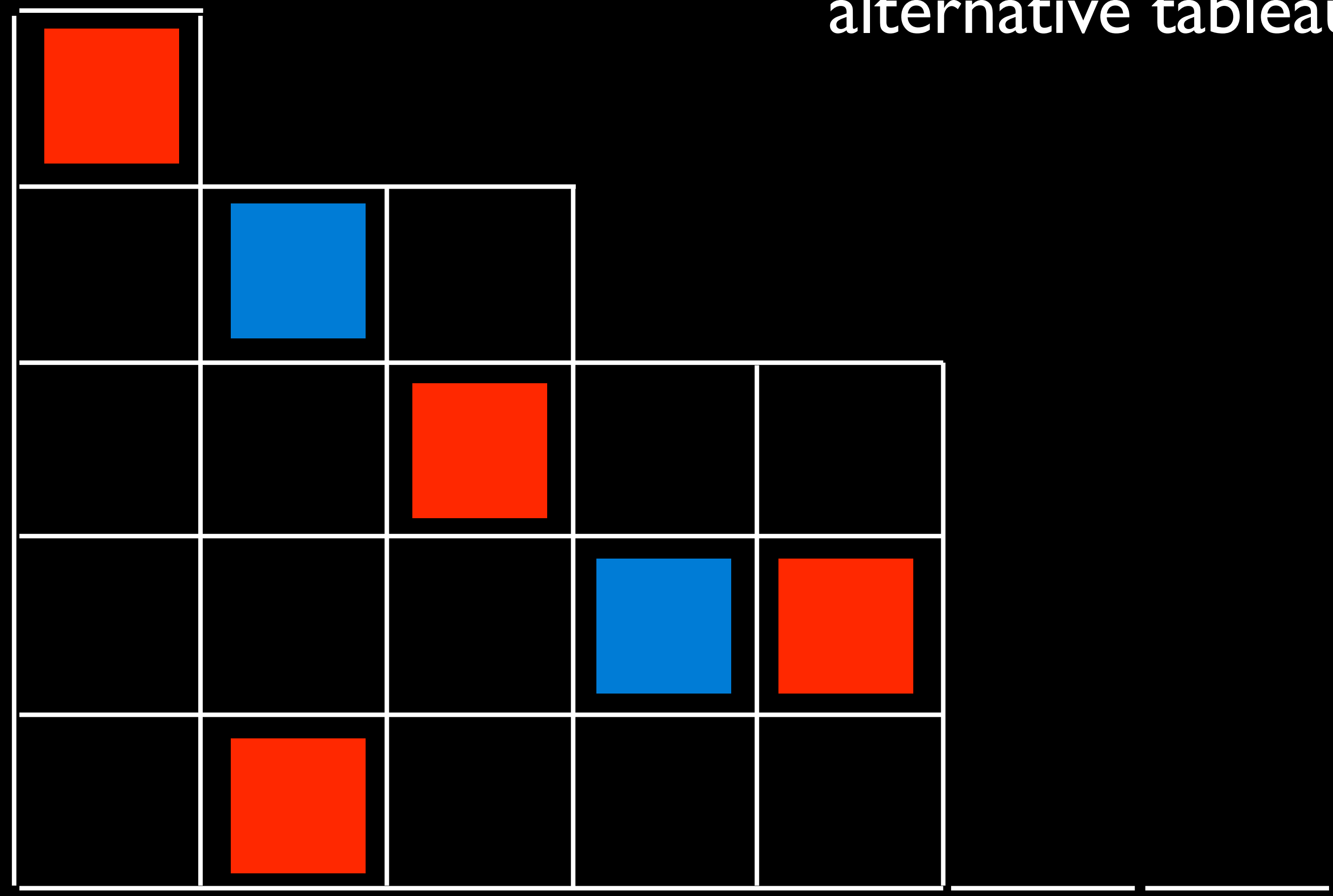

(v) mark the cells below a red

(vi) in each rows where there exist empty cells, the rightmost of these cells is marked




(vii) delete the marks

alternative tableau



notations. T tableau de permutations

- $wt(T)$ = (nb total de 1) − (nb de colonnes)
- $f(T)$ = (nb de 1 sur la 1ère ligne)
- $u(T)$ = (nb de lignes non restreintes)

Def − ligne restreinte : ssi elle a une case restreinte, c.à.d une case contenant un 0 et située au dessus d'un 1.

Corteel, Williams (2006)

Cor. La probabilité stationnaire associée à l'état $\tau = (\tau_1, \dots, \tau_n)$ (PASEP) est

$$P_\tau(q) = \frac{1}{Z_n} \sum_{T} q^{wt(T)} \alpha^{-f(T)} \beta^{-u(T)}$$

T tableau de permutation forme F associée à τ

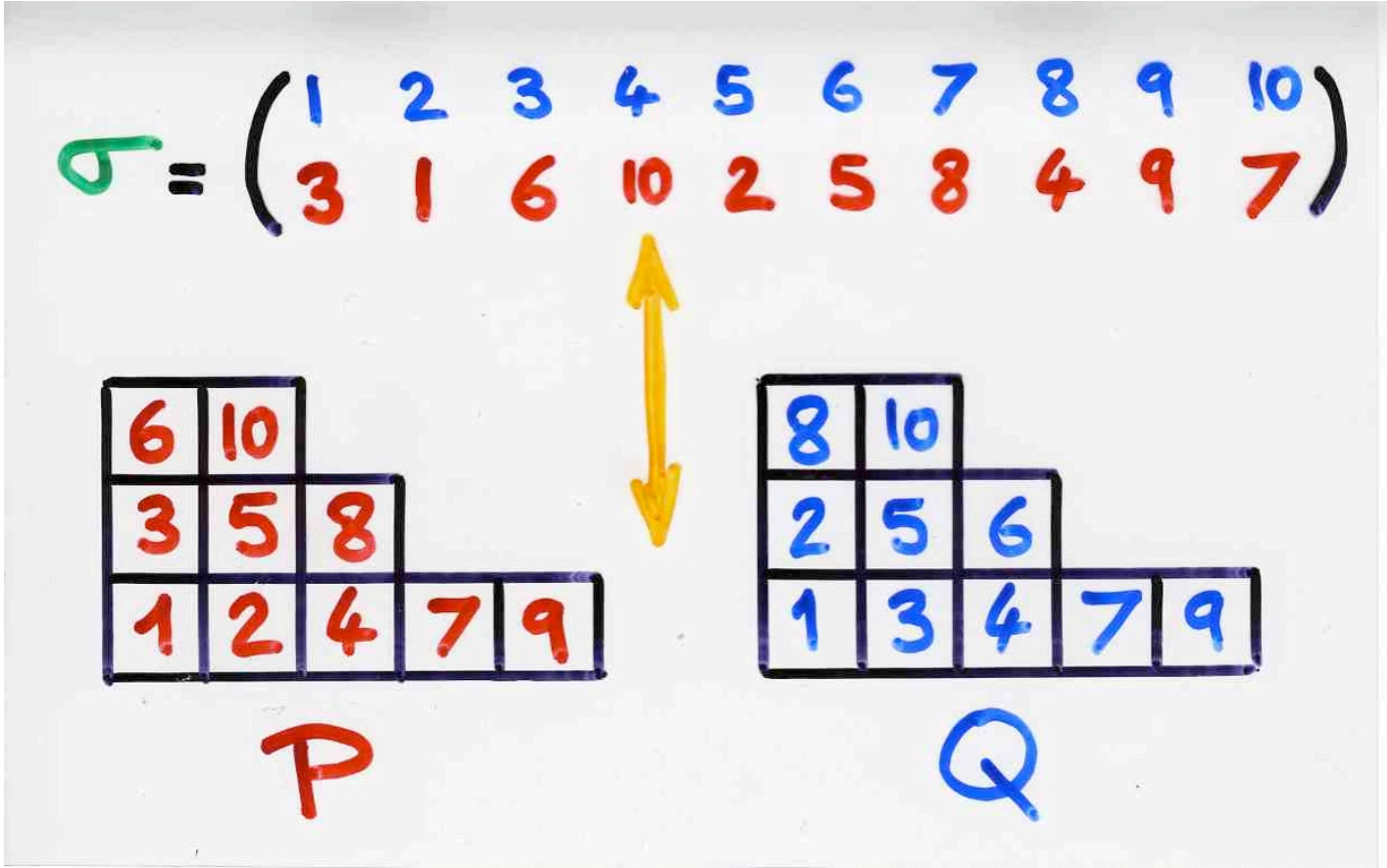
From algebra $DE = ED + D + E$
to bijections

$UD = DU + 1$

RSK correspondence

Combinatorial theory
of orthogonal polynomials

§ 6 RSK with operators, commutations and local rules



σ =
1 2 3 4 5 6 7 8 9 10
3 1 6 10 2 5 8 4 9 7
6 10
3 5 8
1 2 4 7 9
8 10
2 5 6
1 3 4 7 9
P
Q

Heisenberg operators

U, D

UD = DU + I

differential poset

Fomin, Stanley

$U^n D^n =$

$U \cdots U U D D \cdots D$

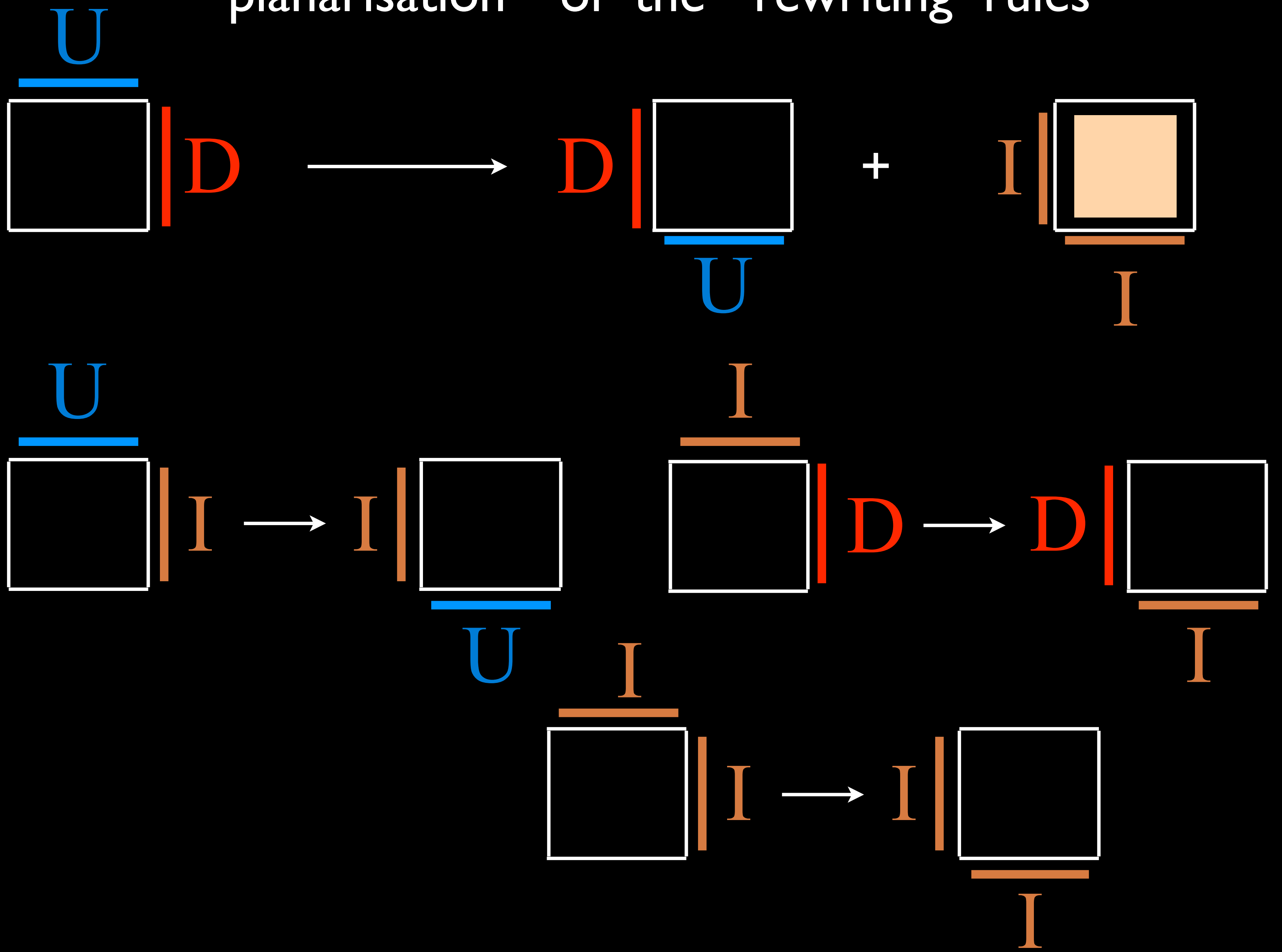
$(DU + I)$

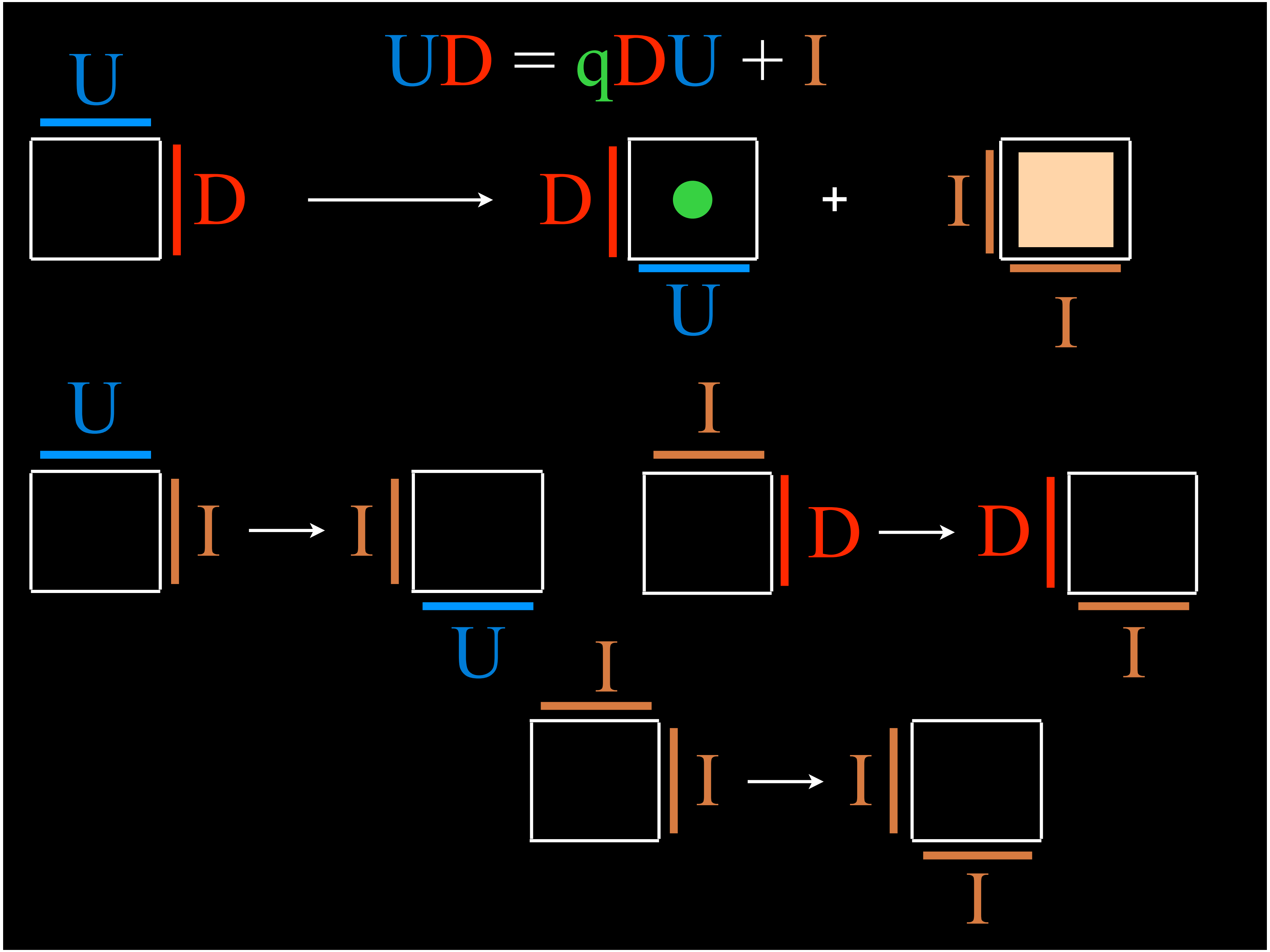
. . . .

Robinson - Schensted - Schützenberger

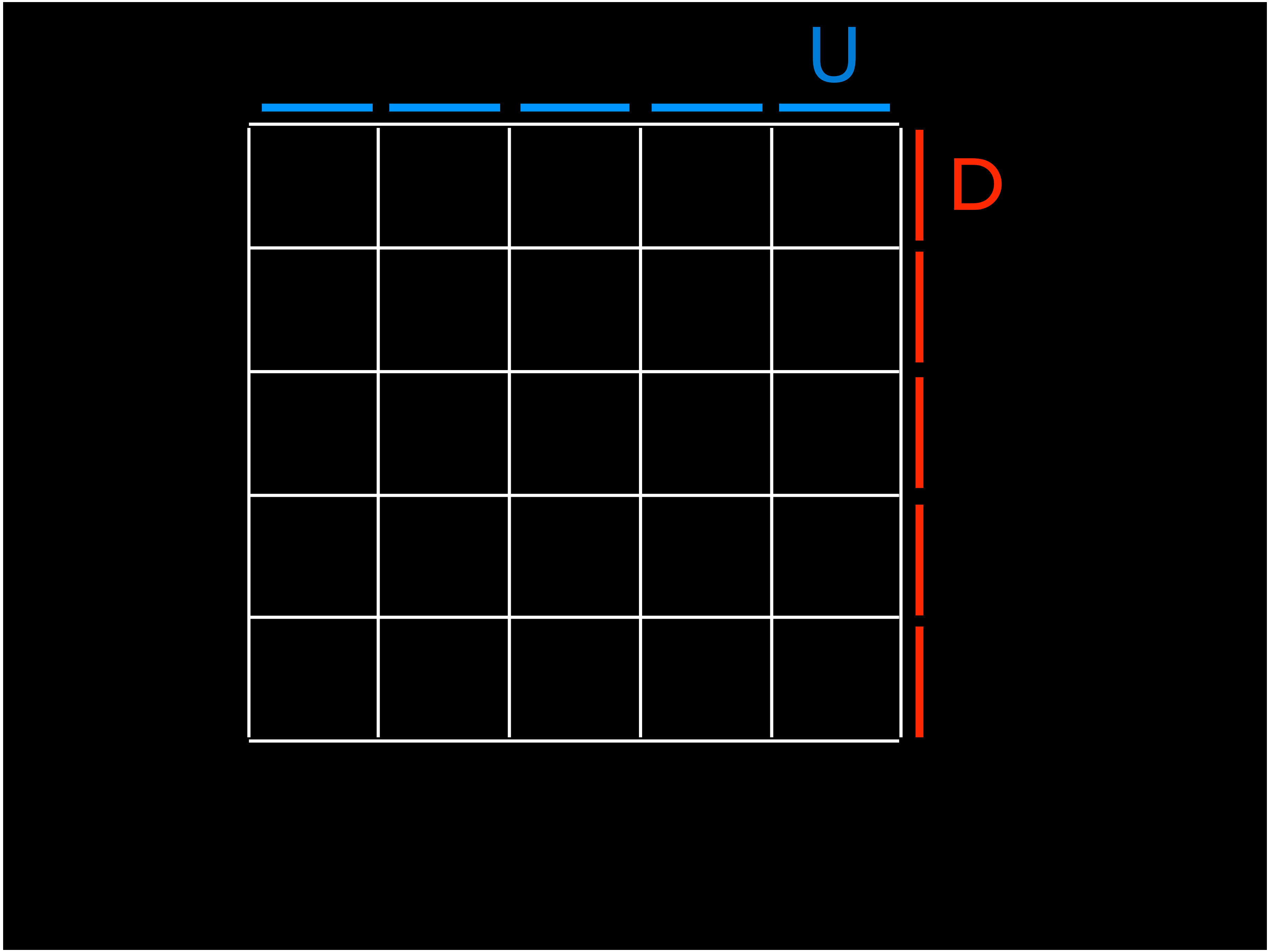
bijection

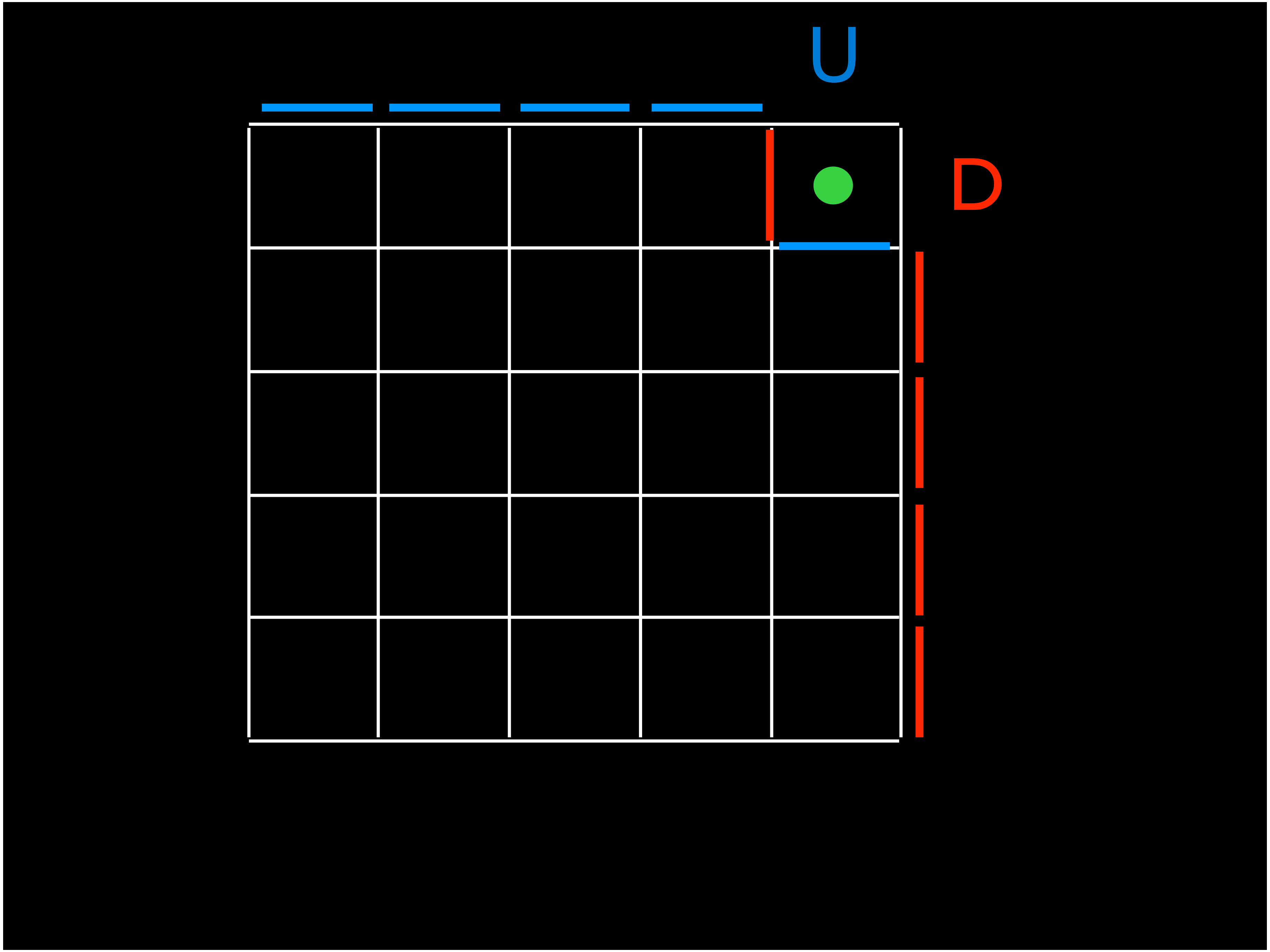
“planarisation” of the “rewriting rules”

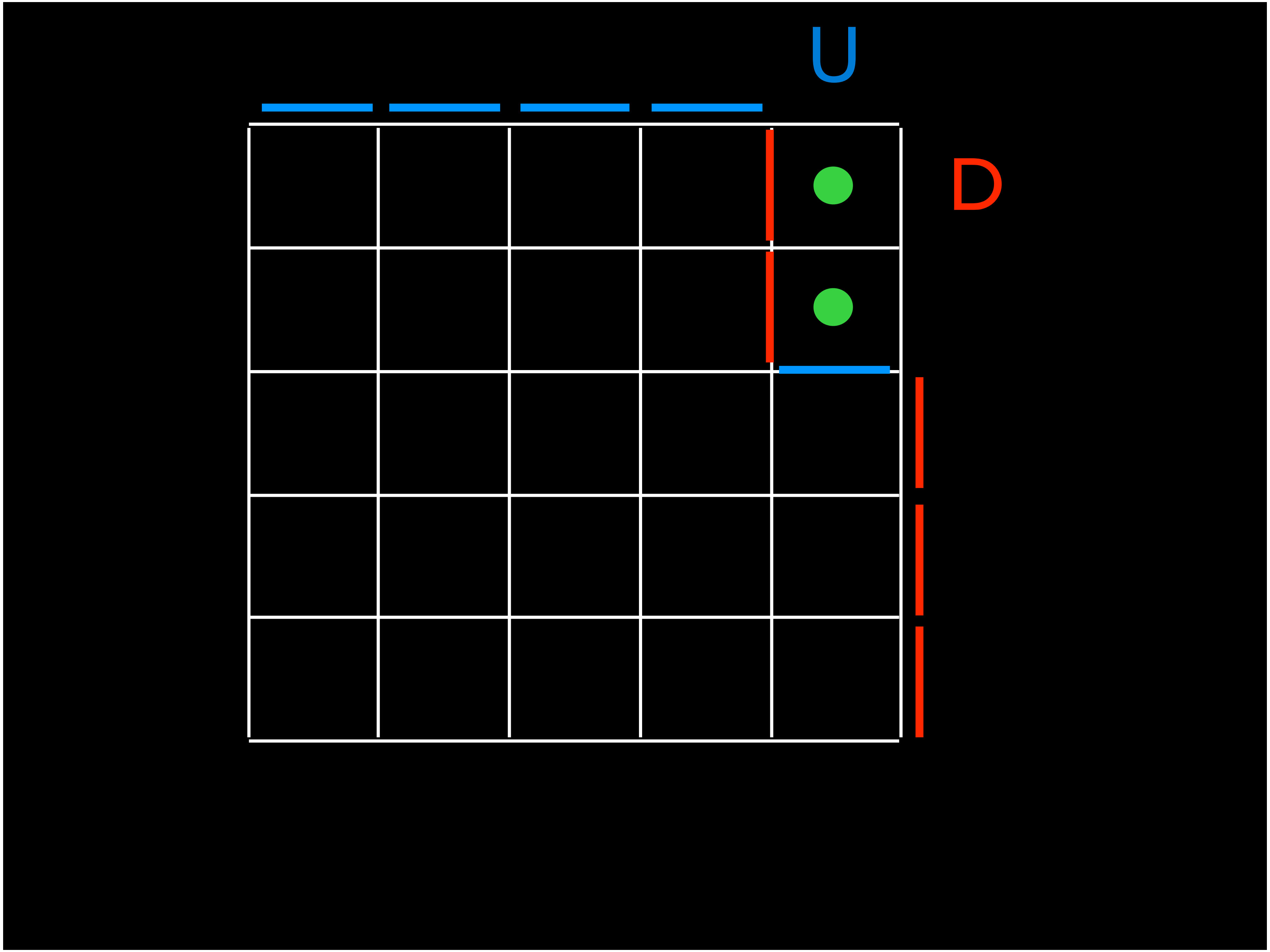


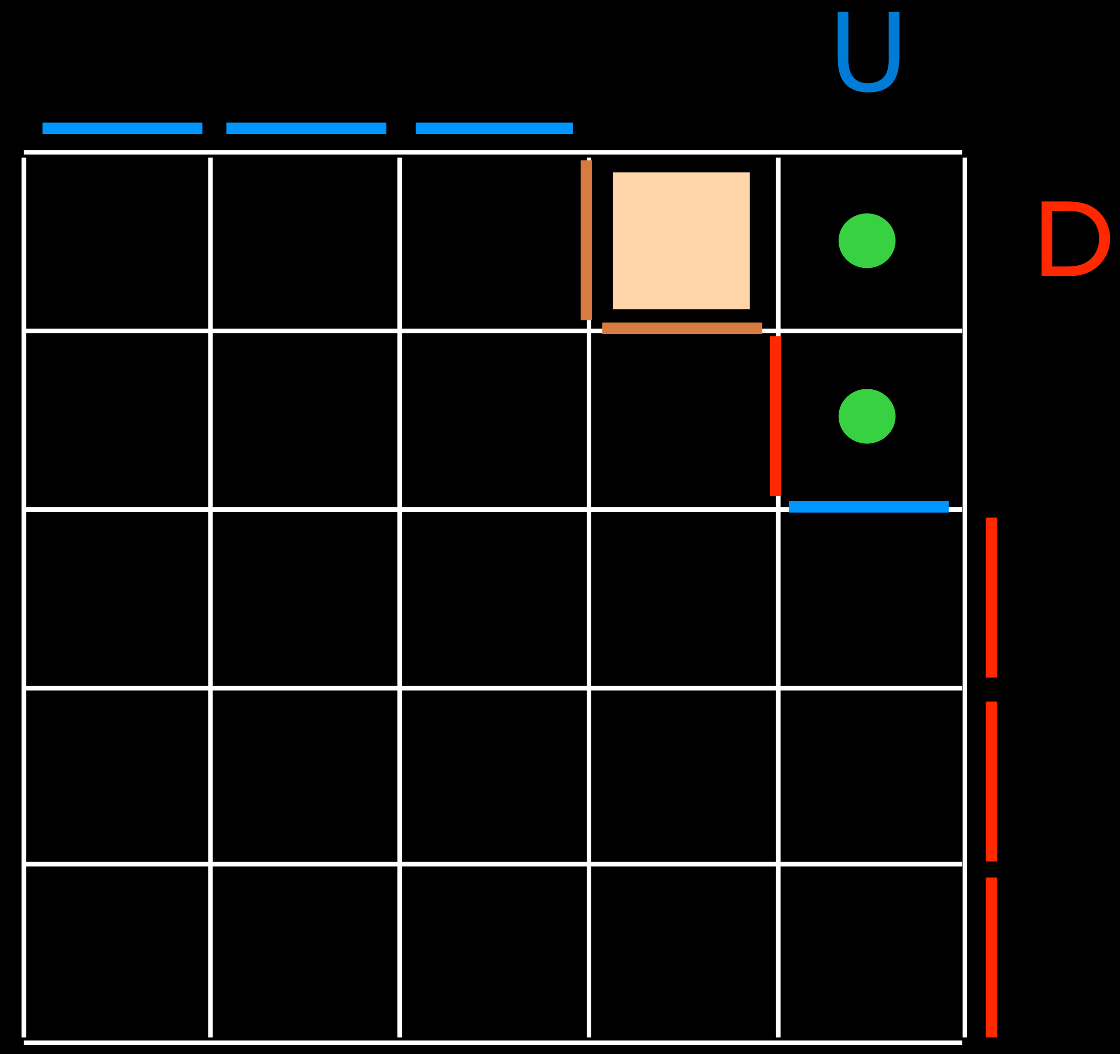


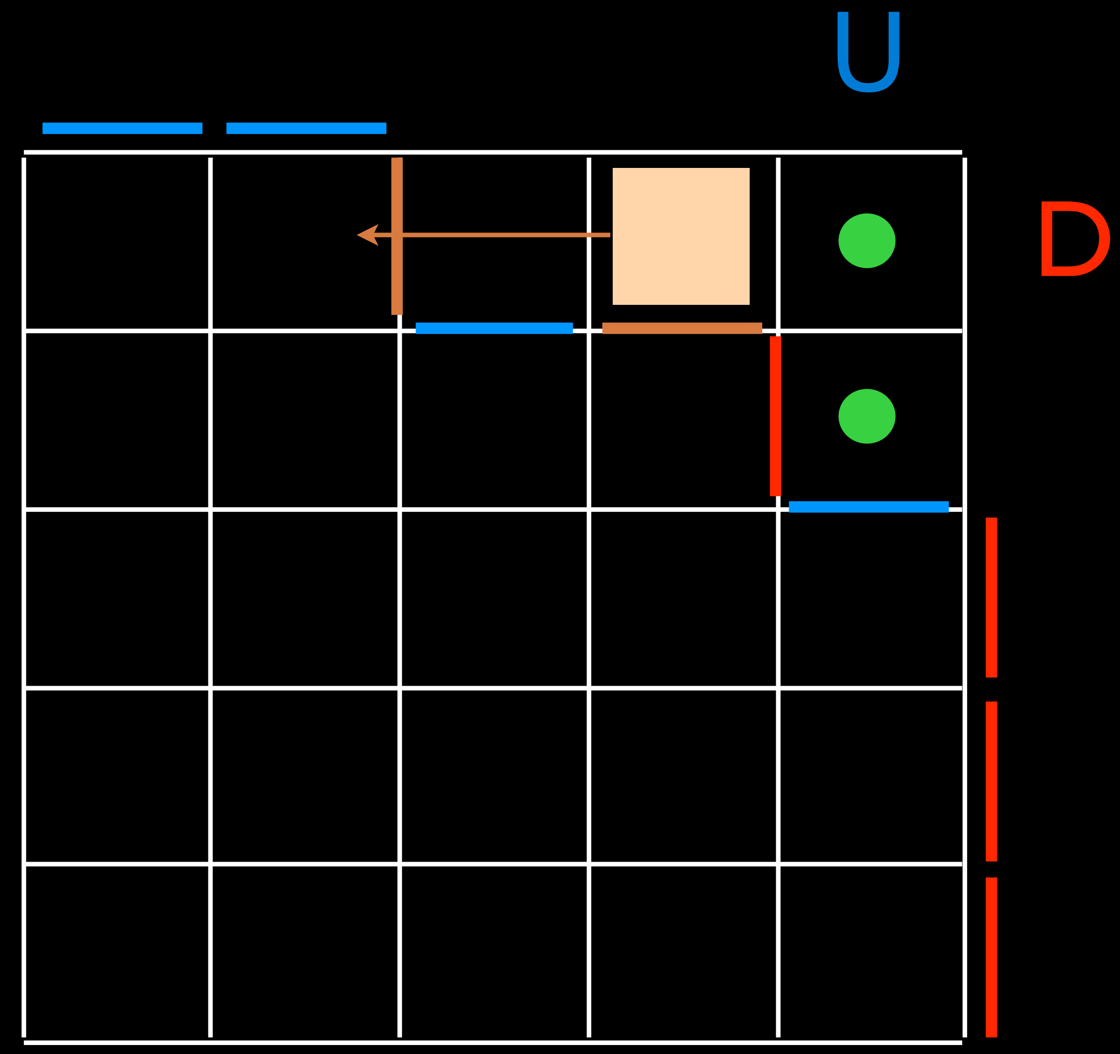
UD = qDU + I
U
D
D
U
+
I
I
U
I
I
U
I
D
D
I
I
I
I
I


U
D


U
D


U
D


U
D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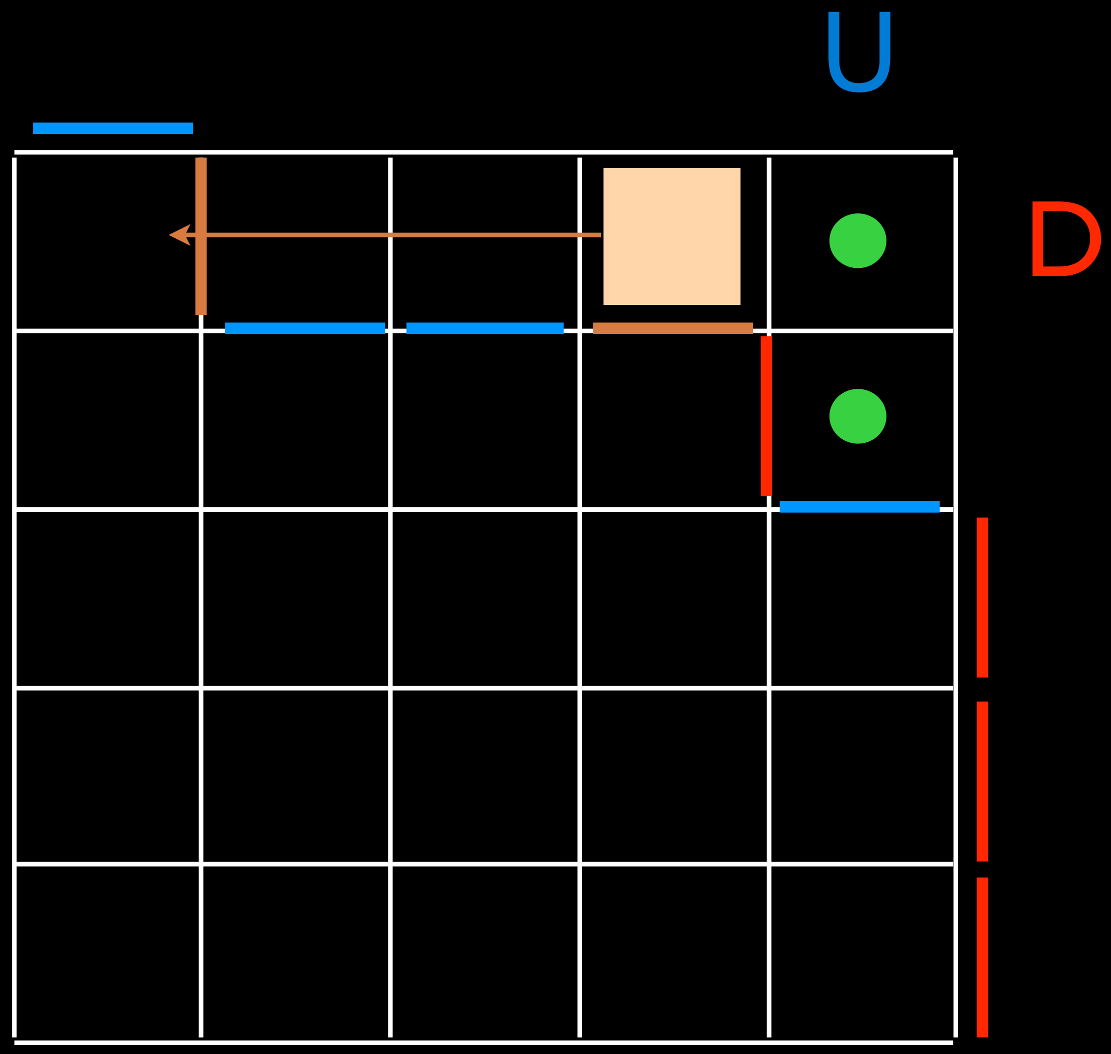
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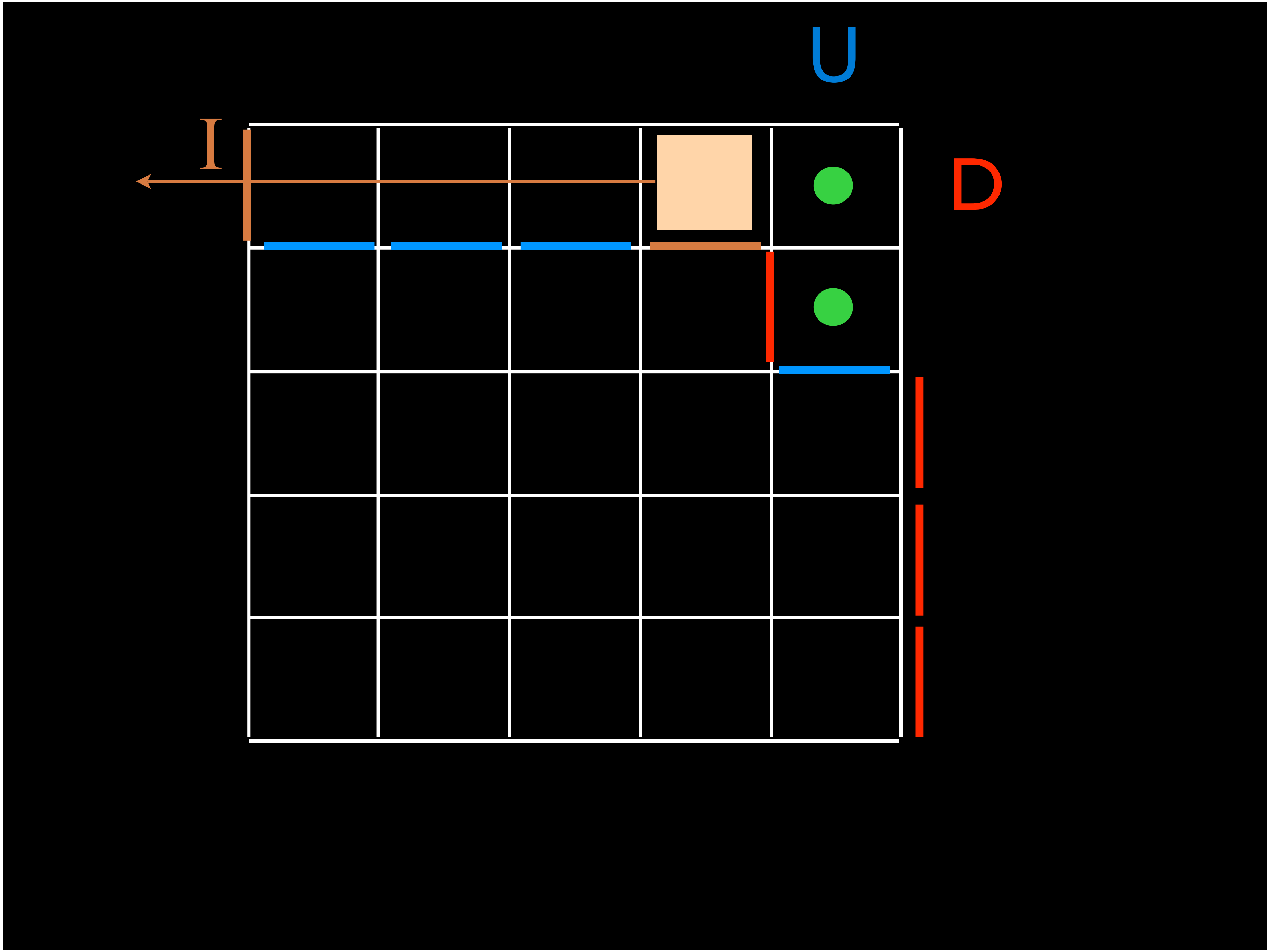
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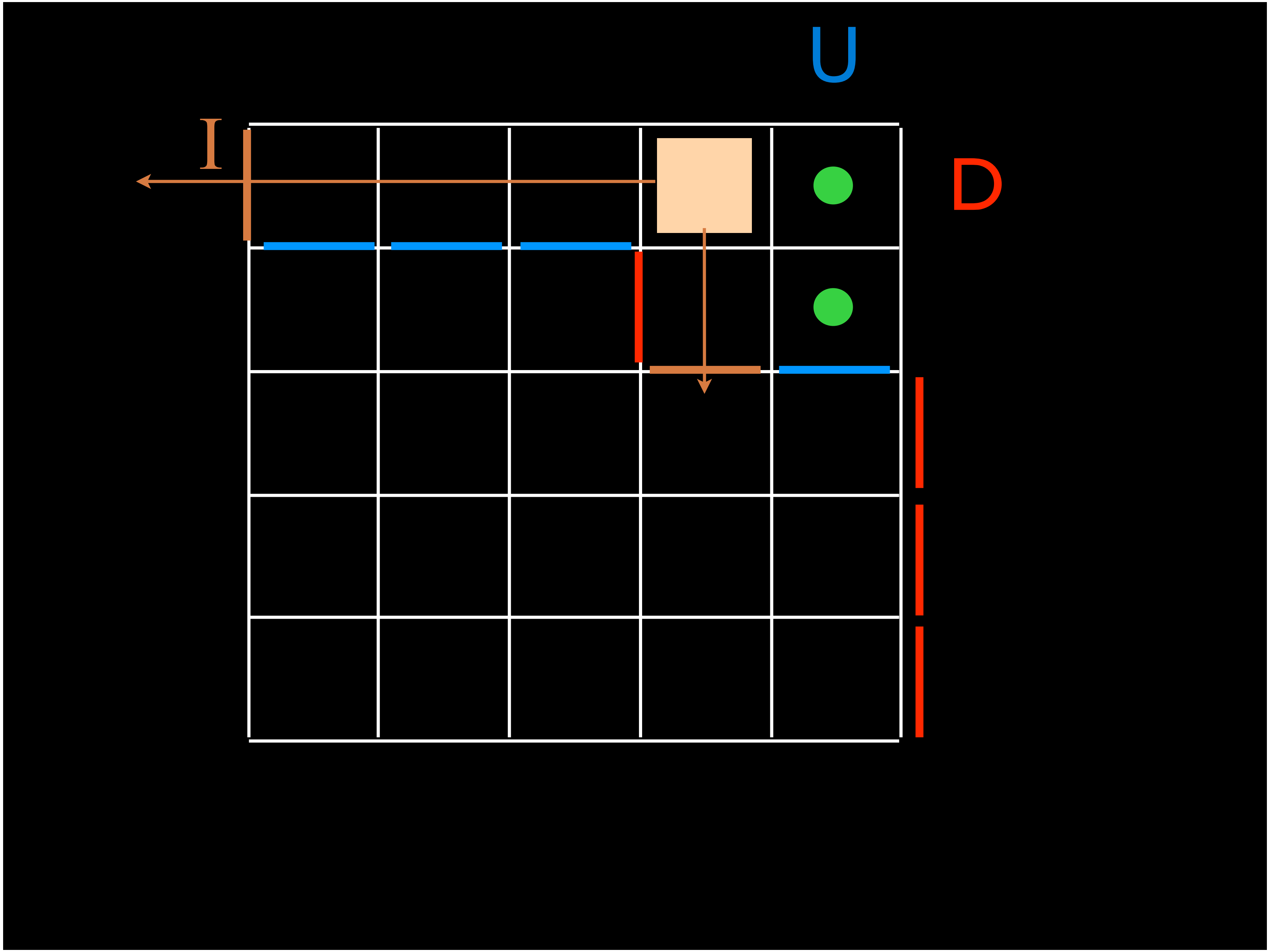

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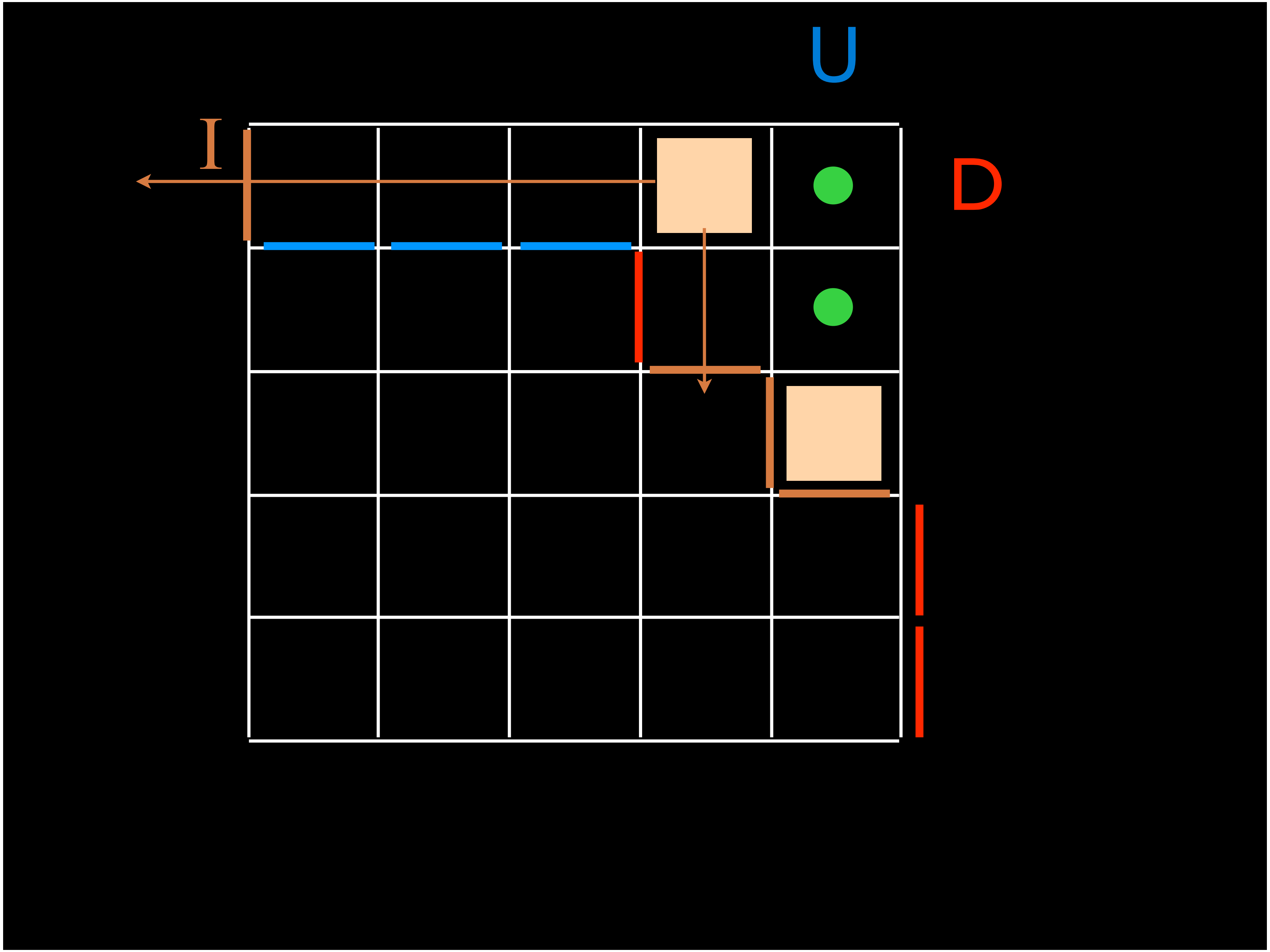
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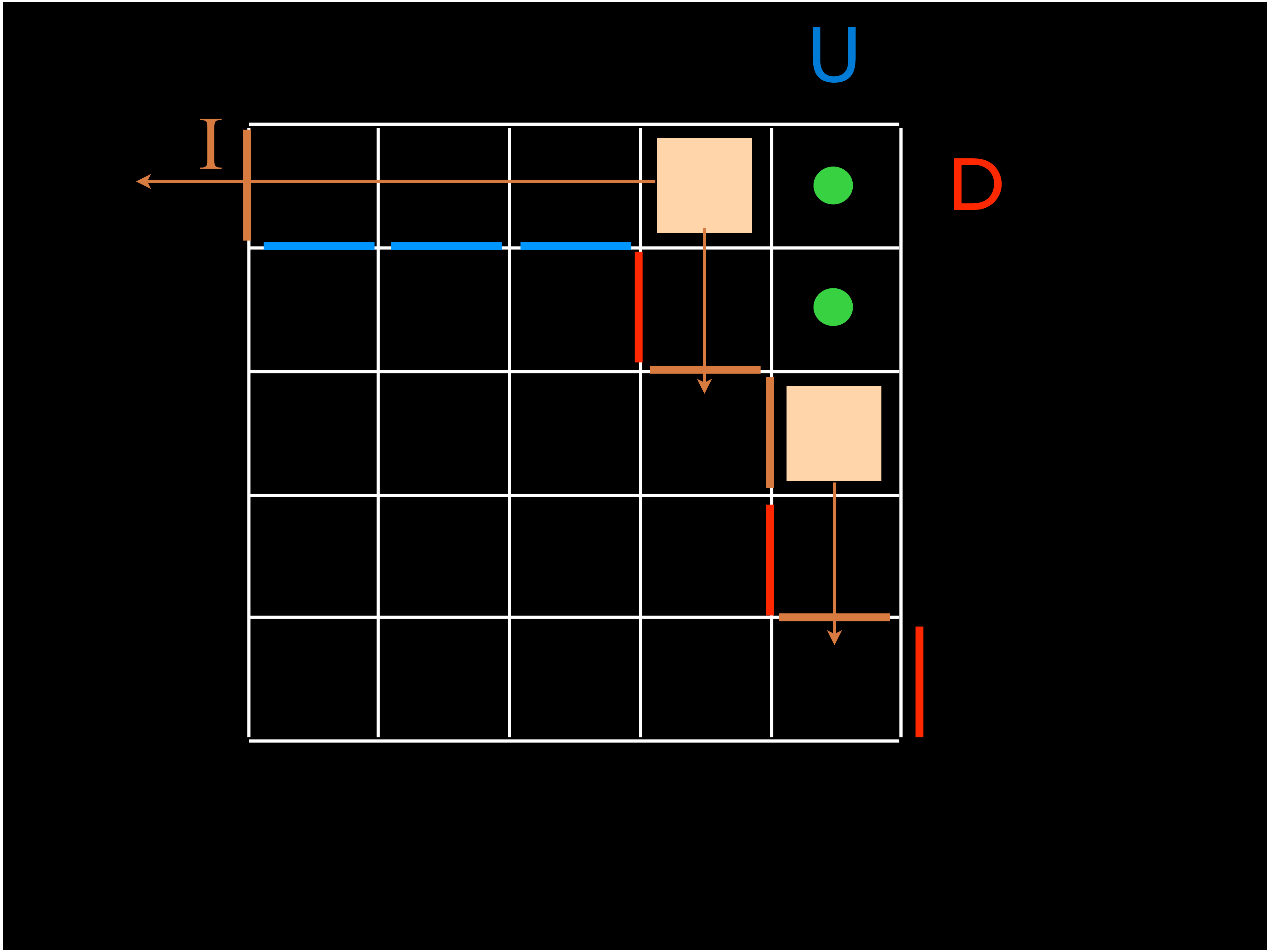

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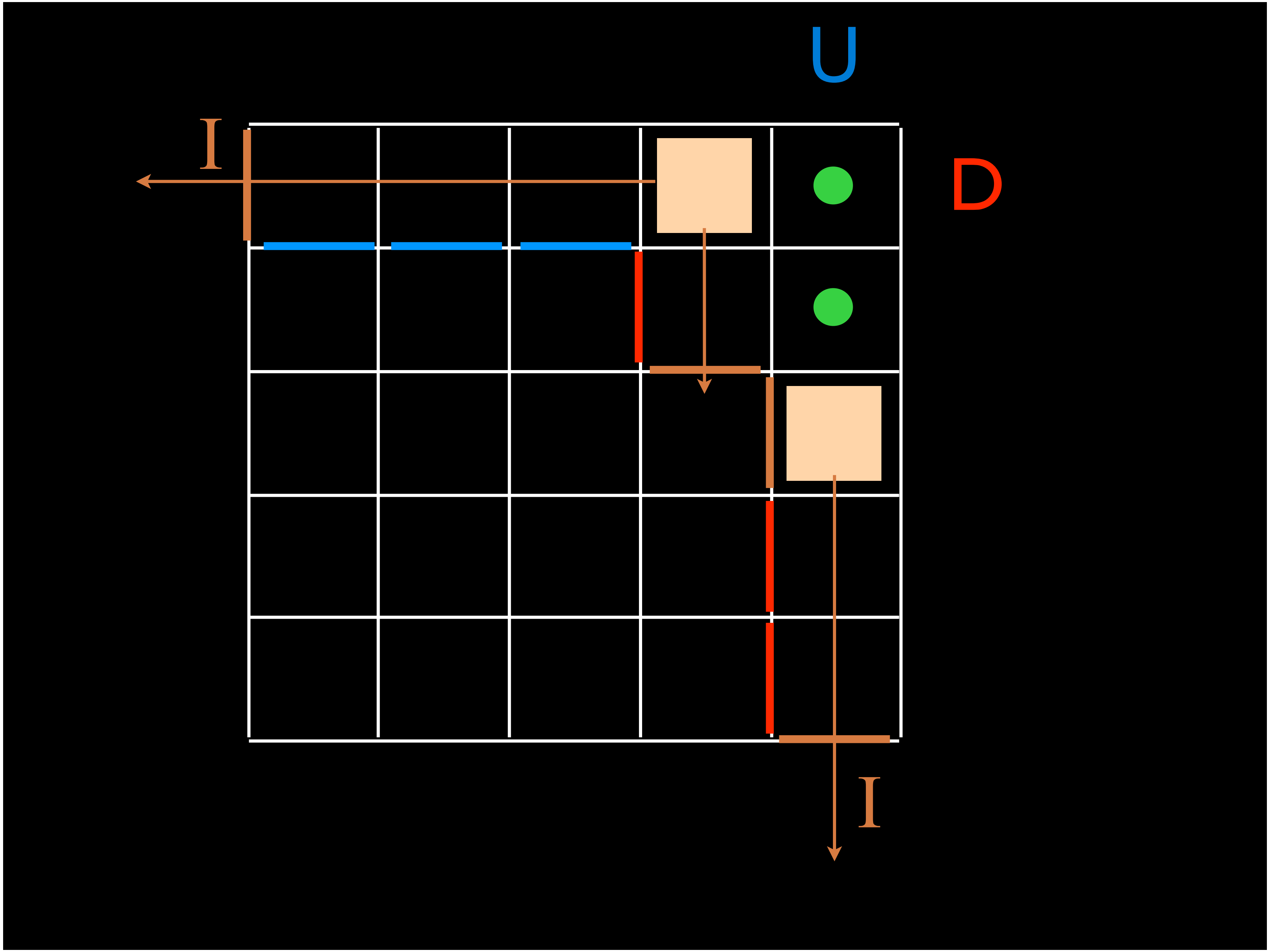
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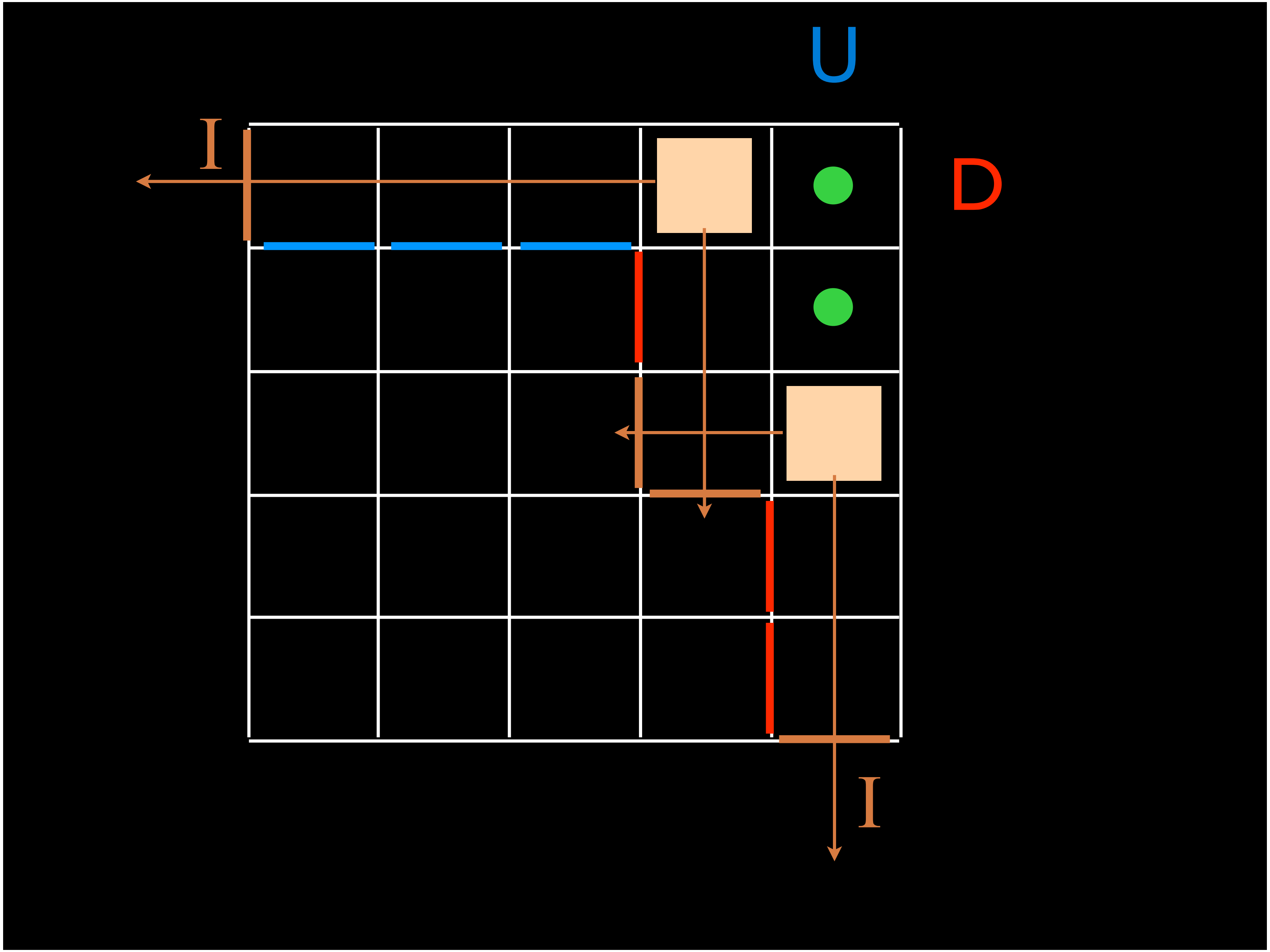
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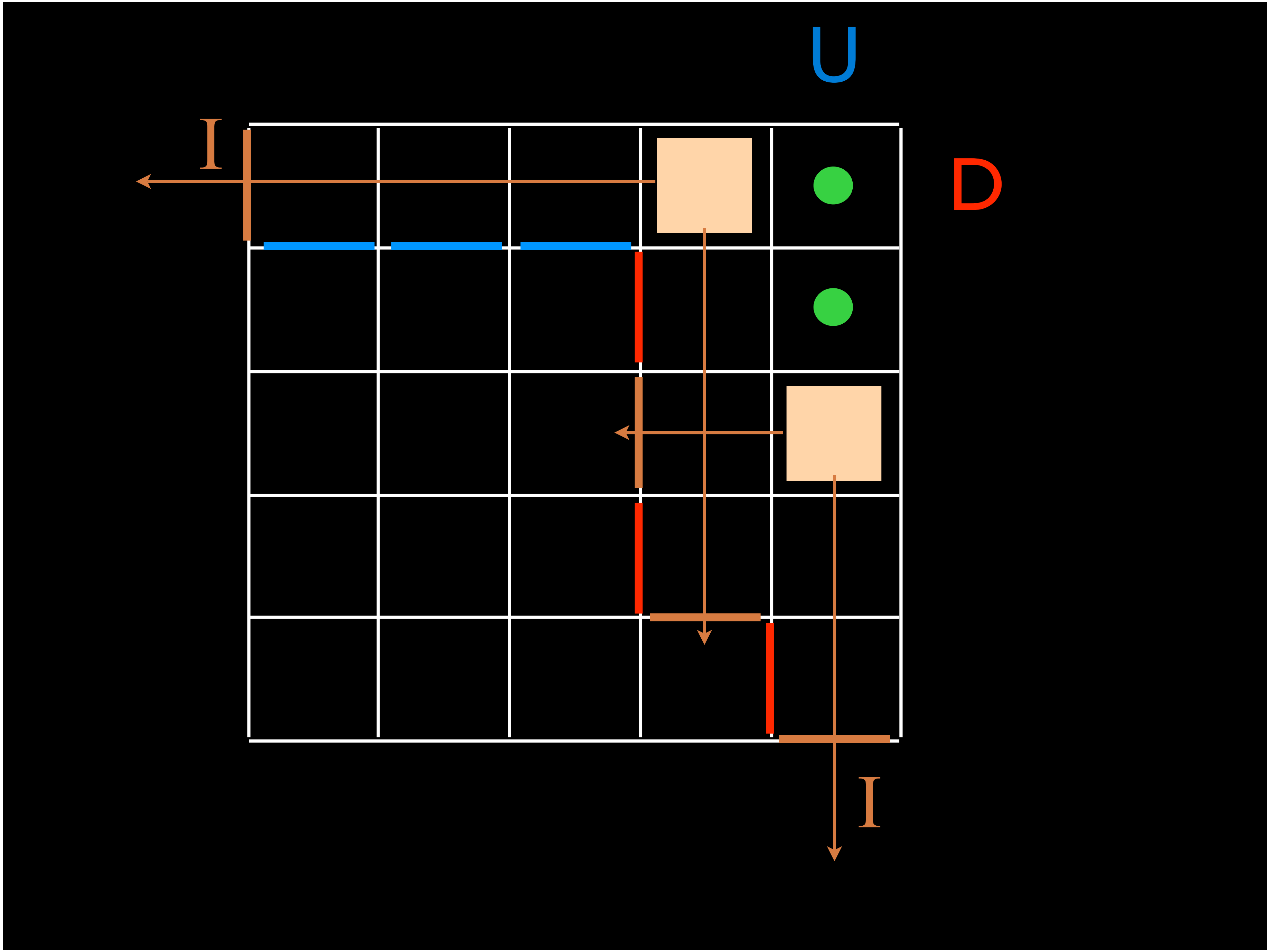
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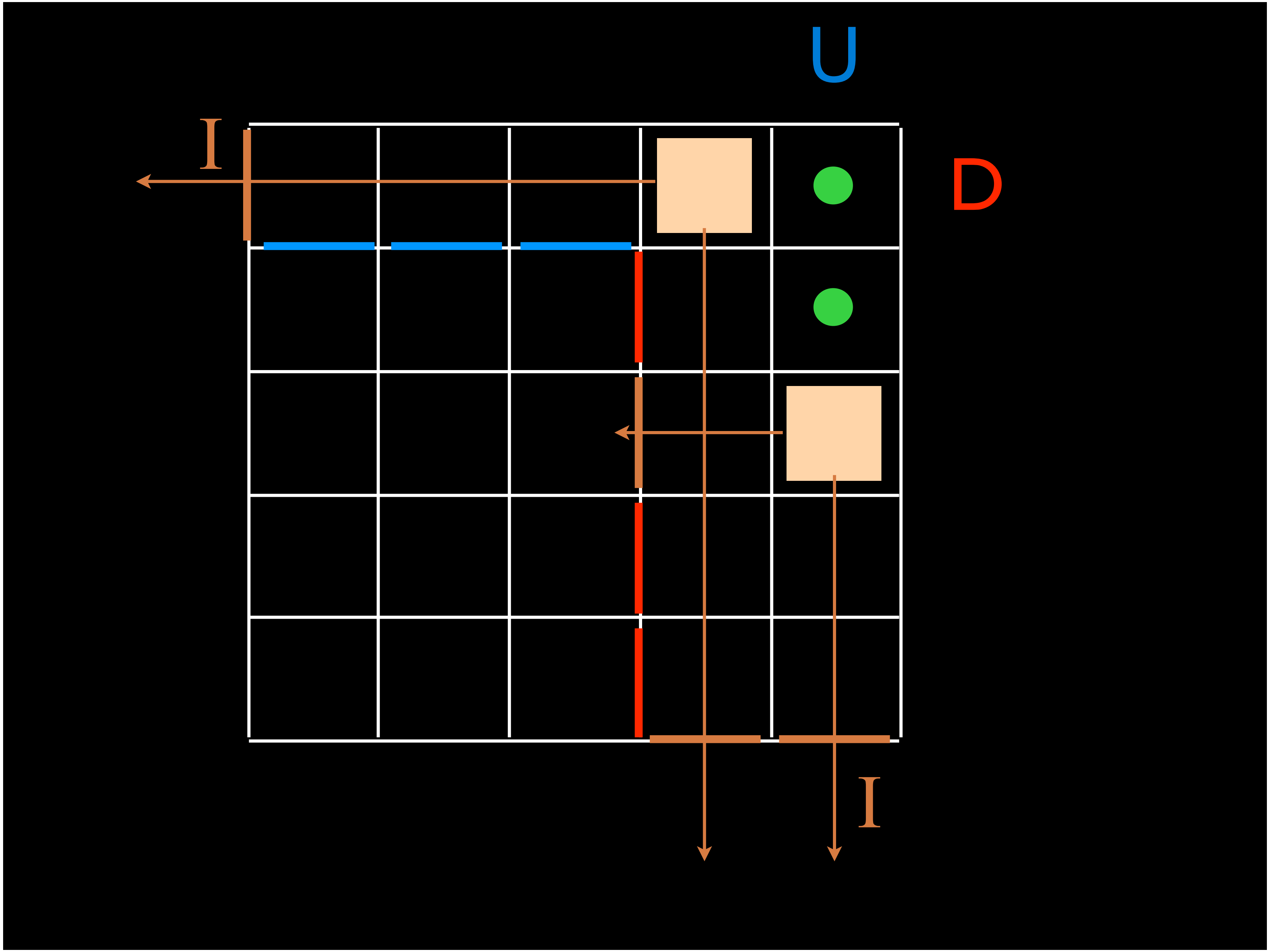
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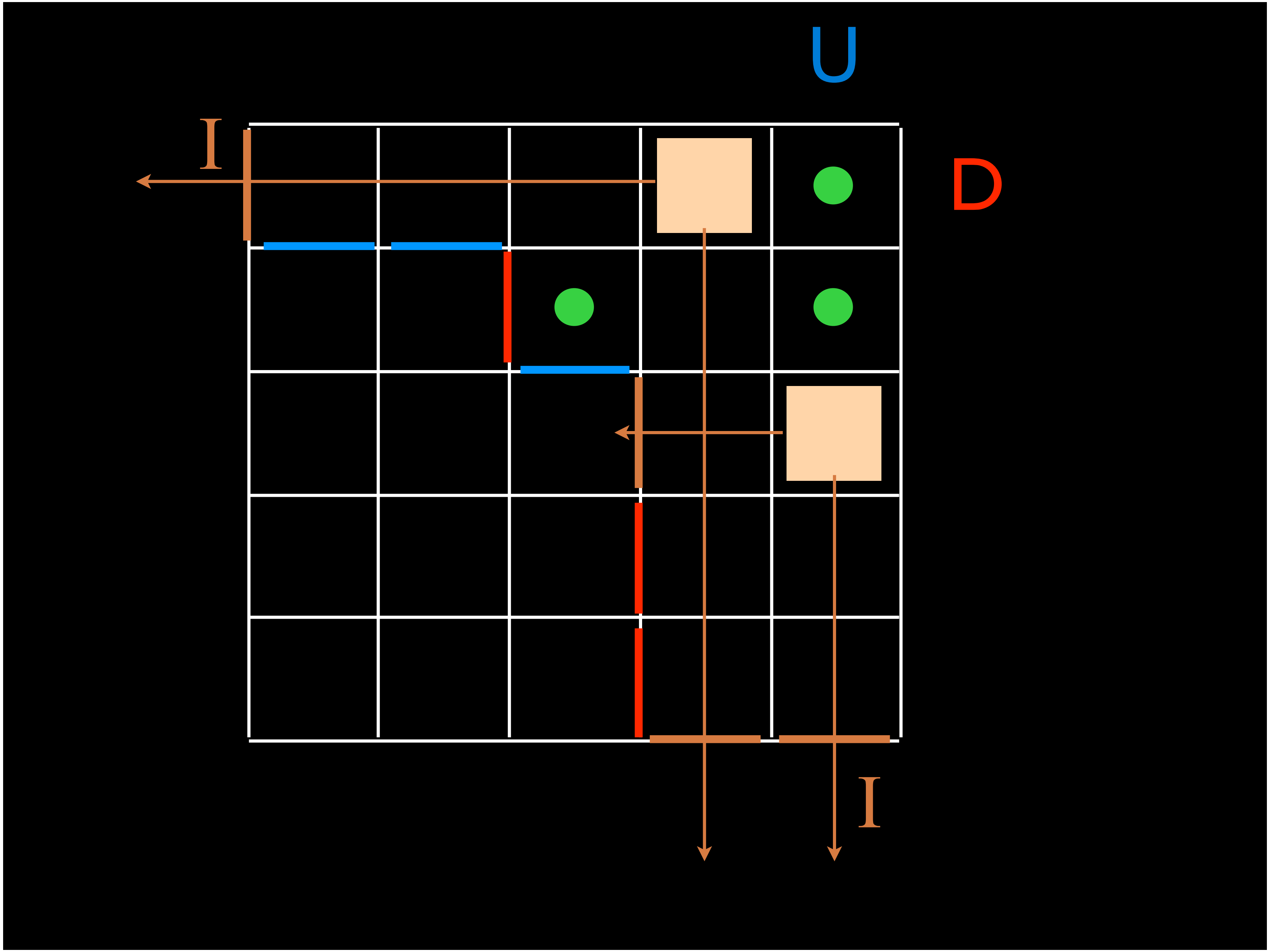
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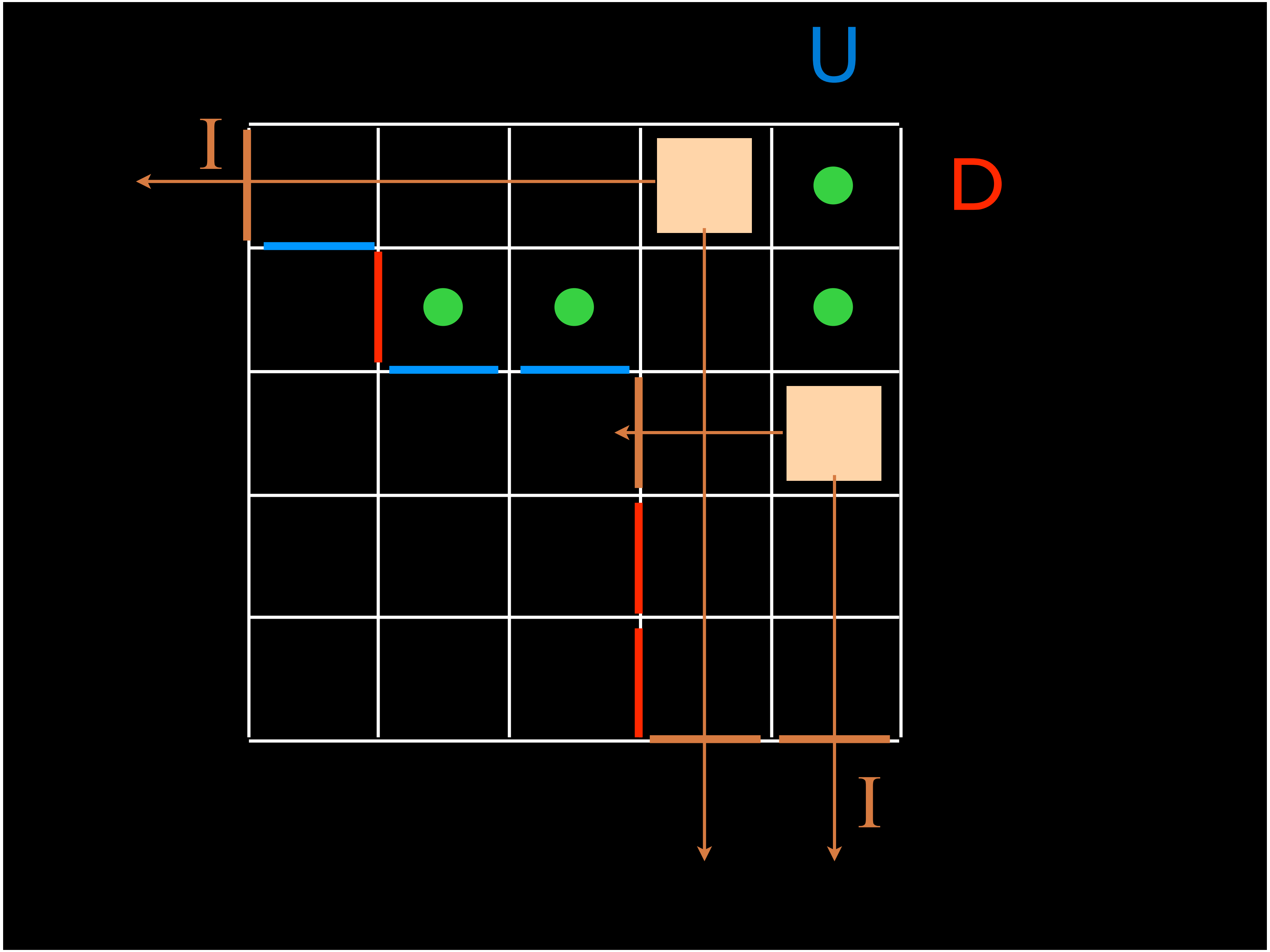
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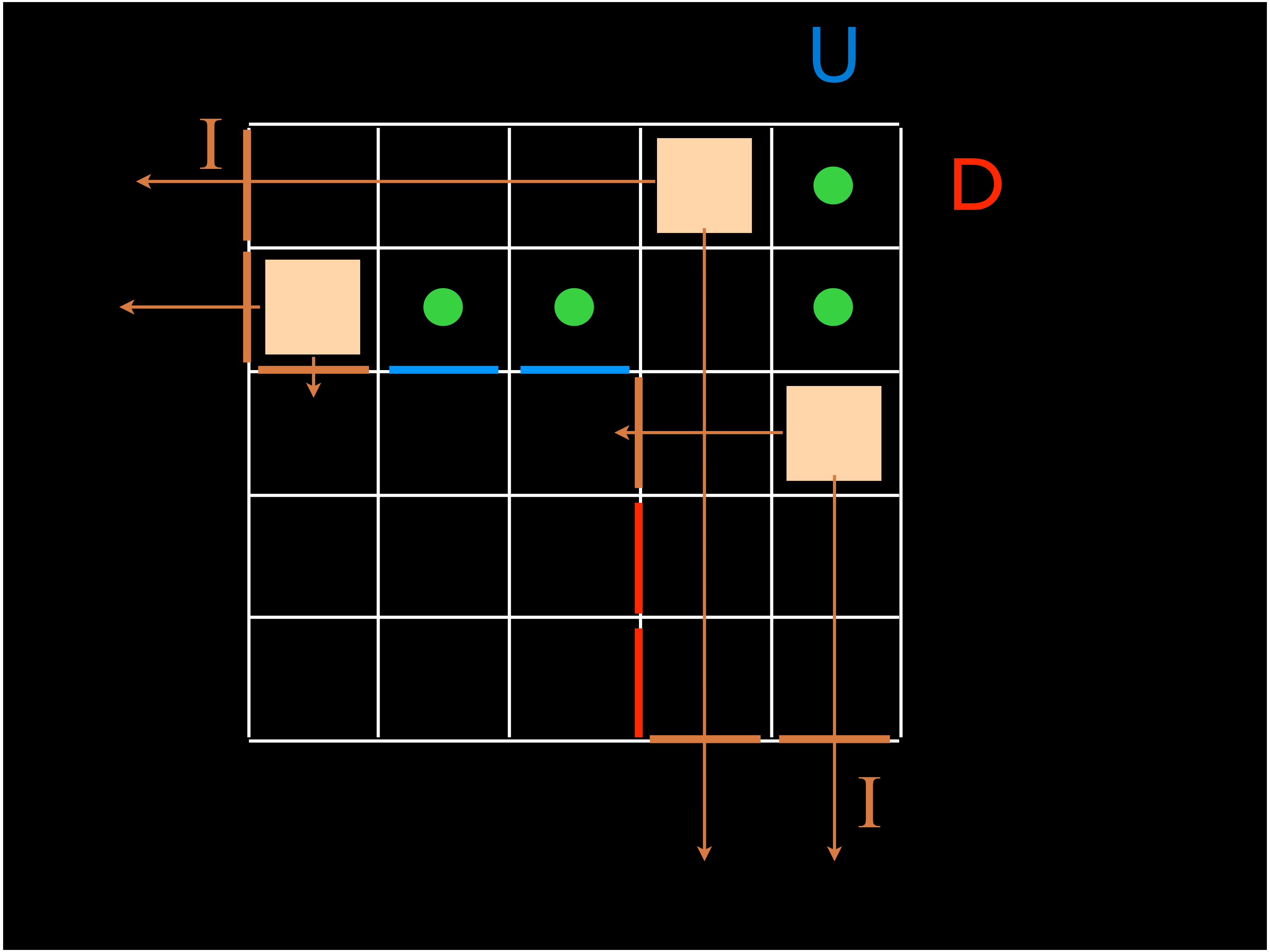
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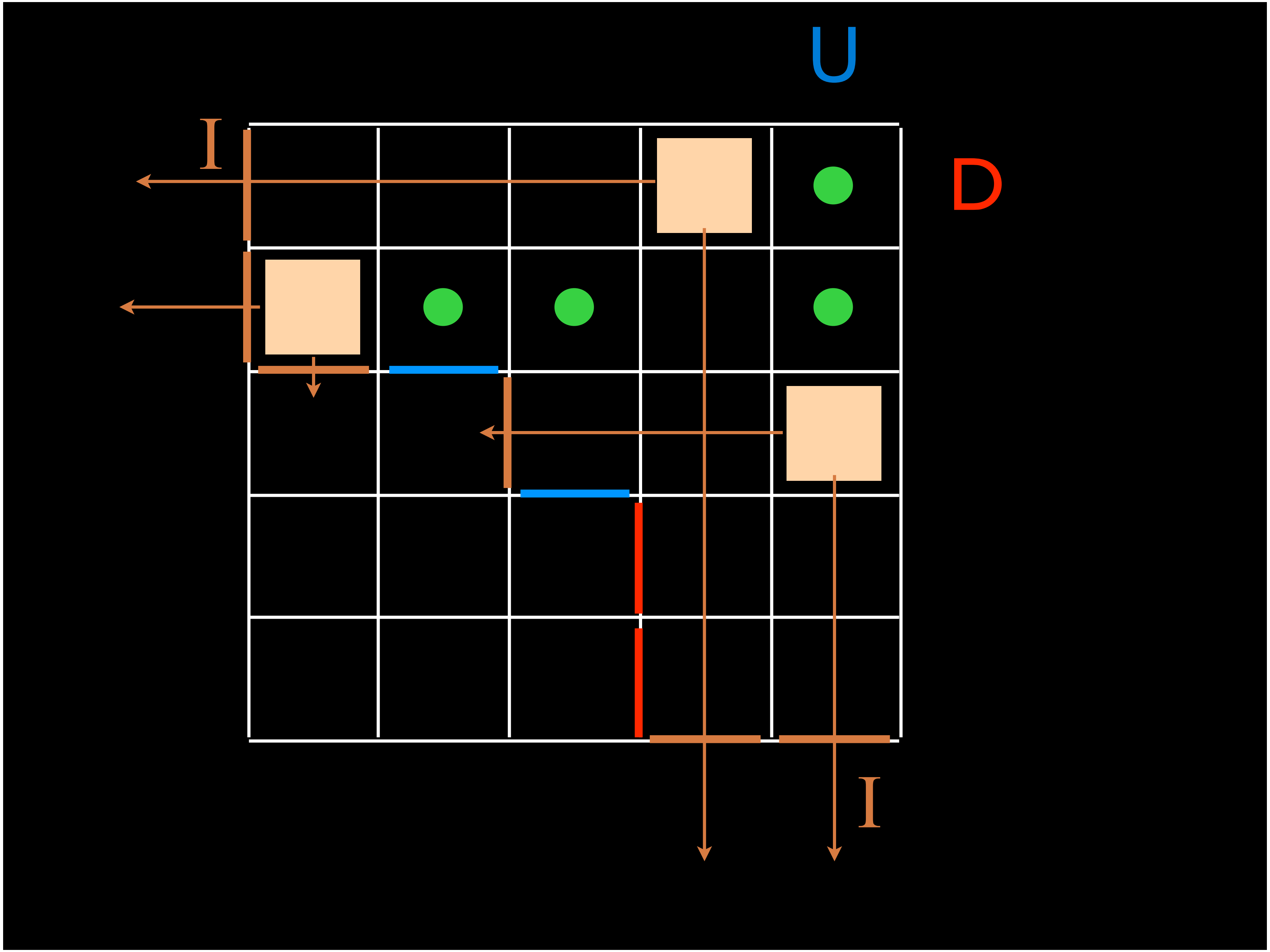
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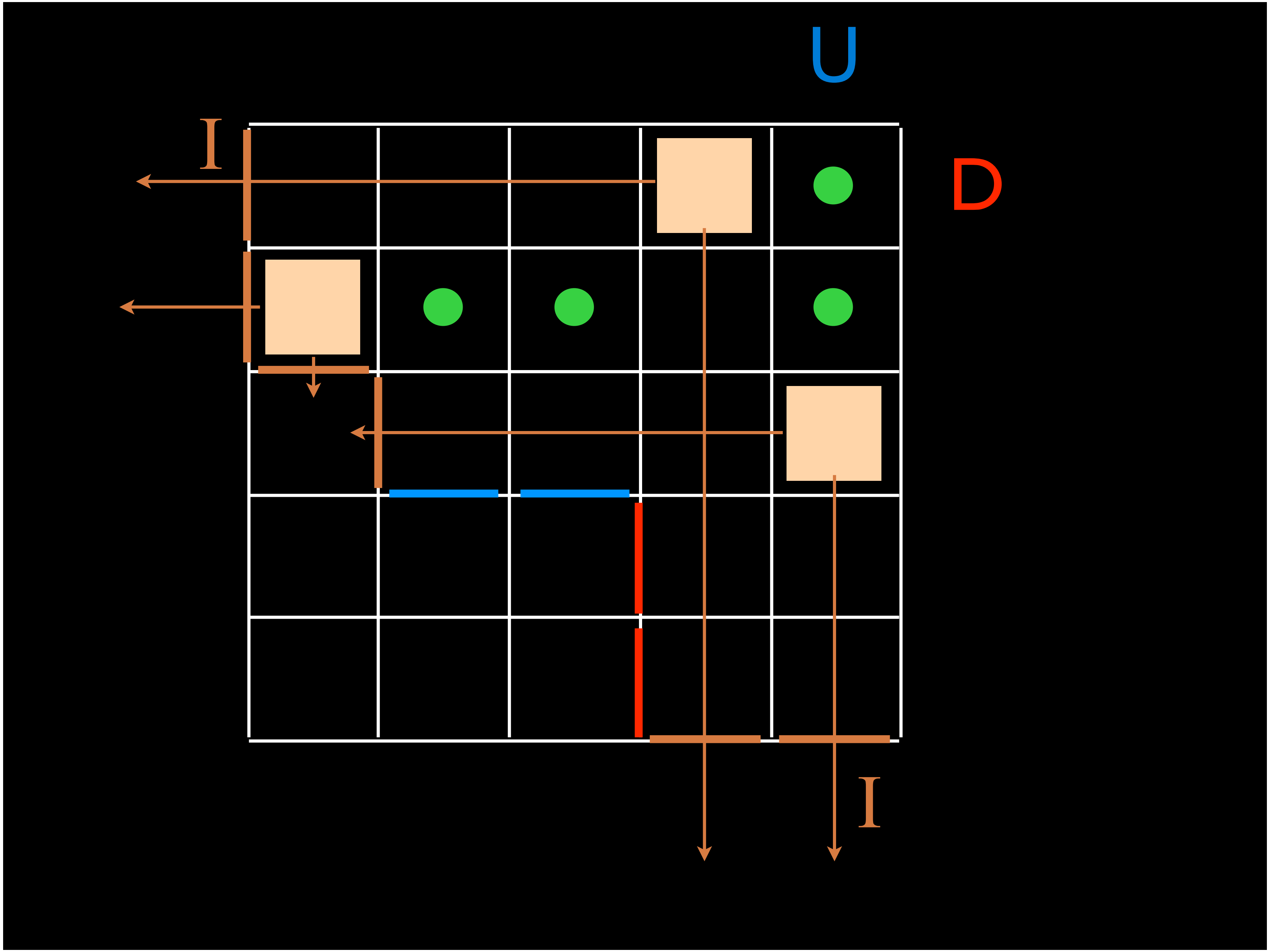
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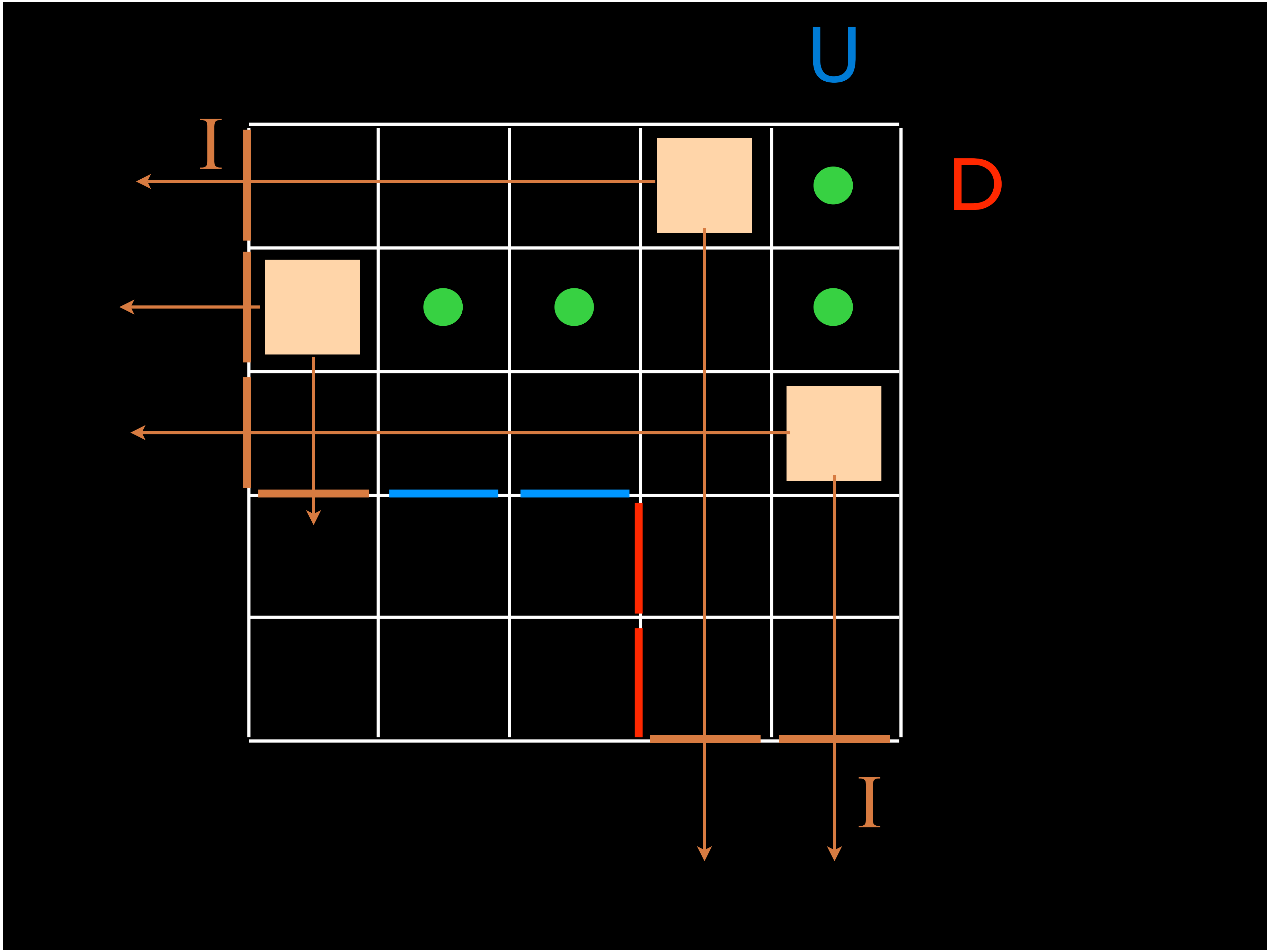
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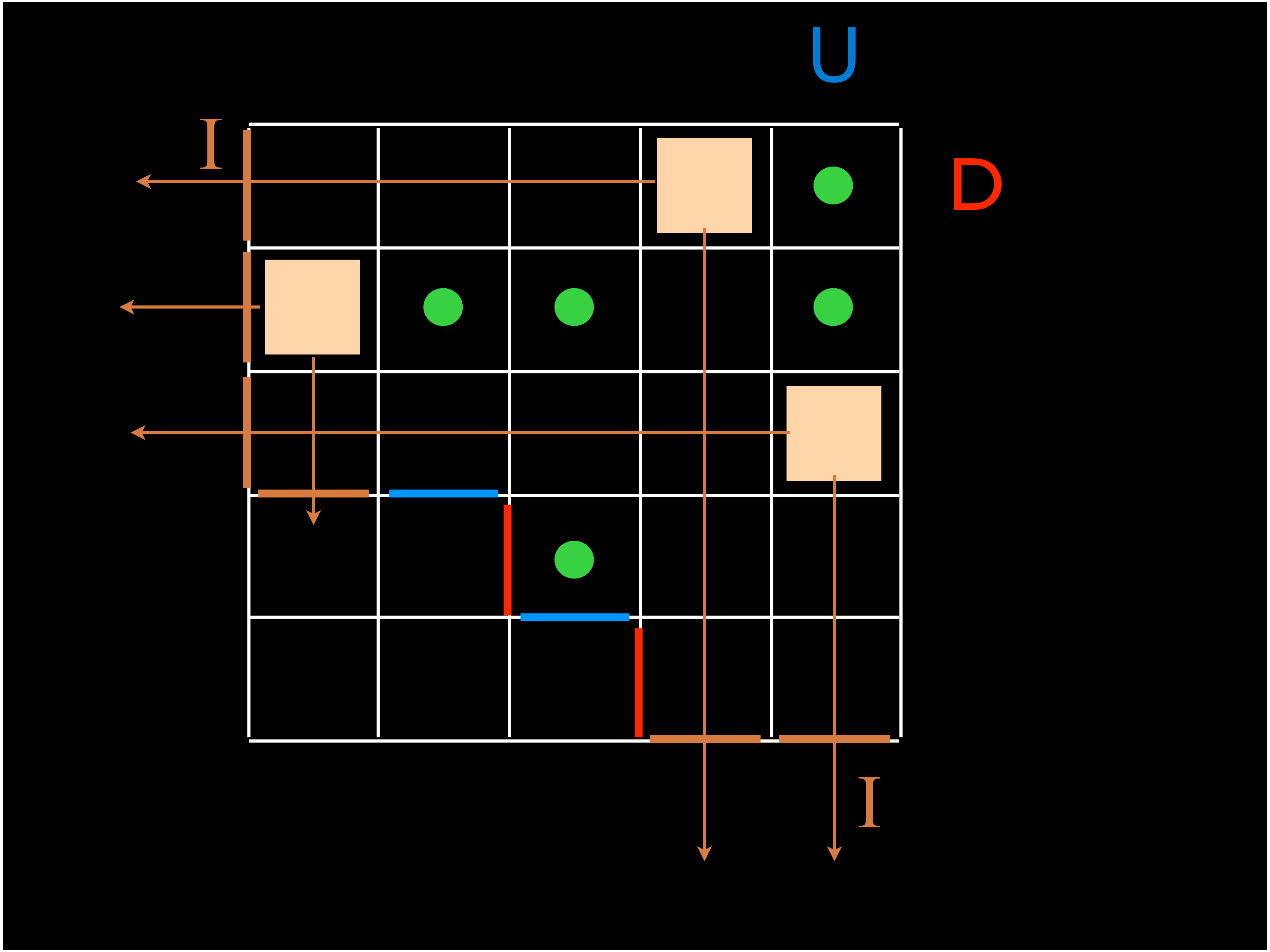
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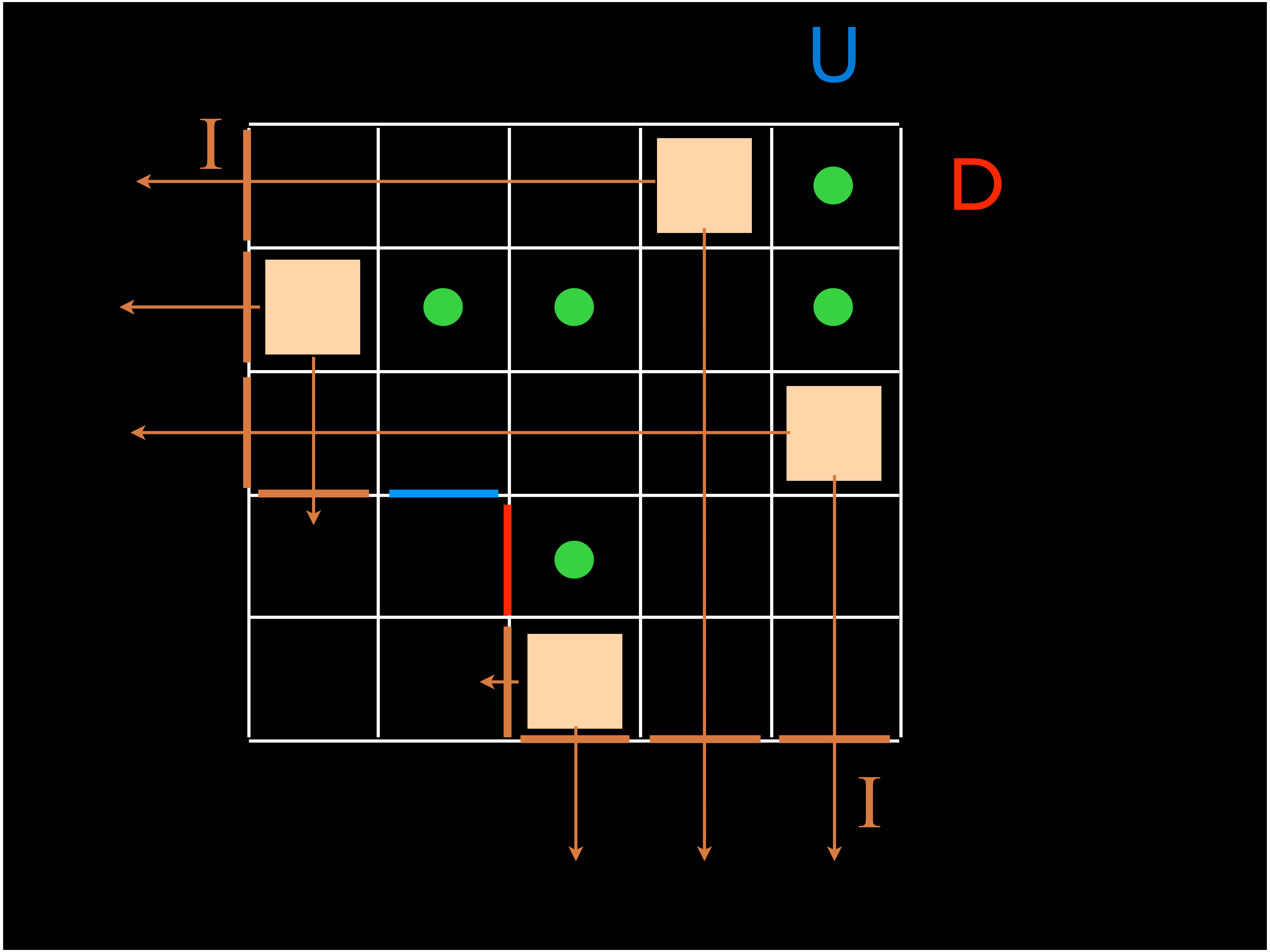

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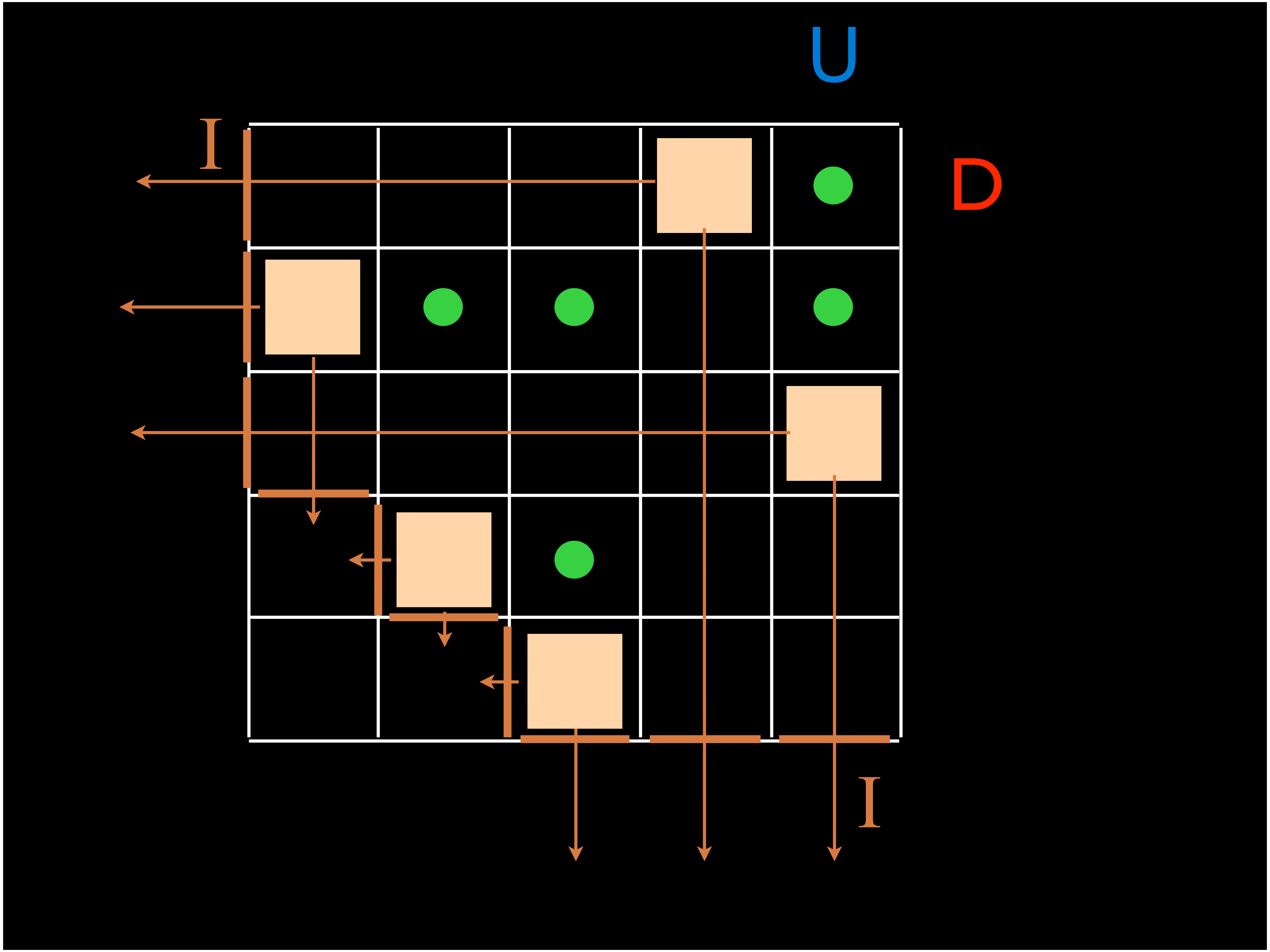

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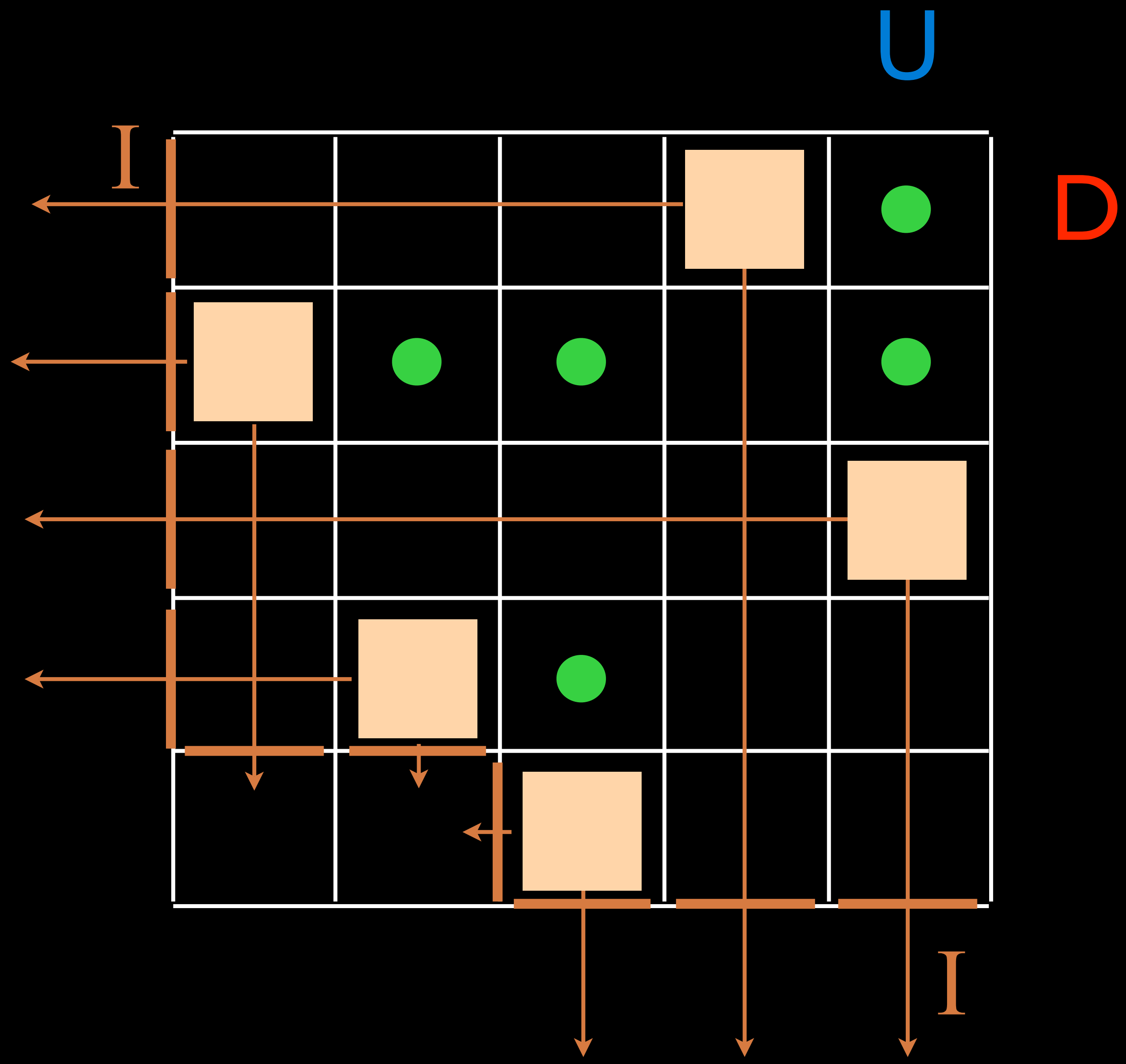

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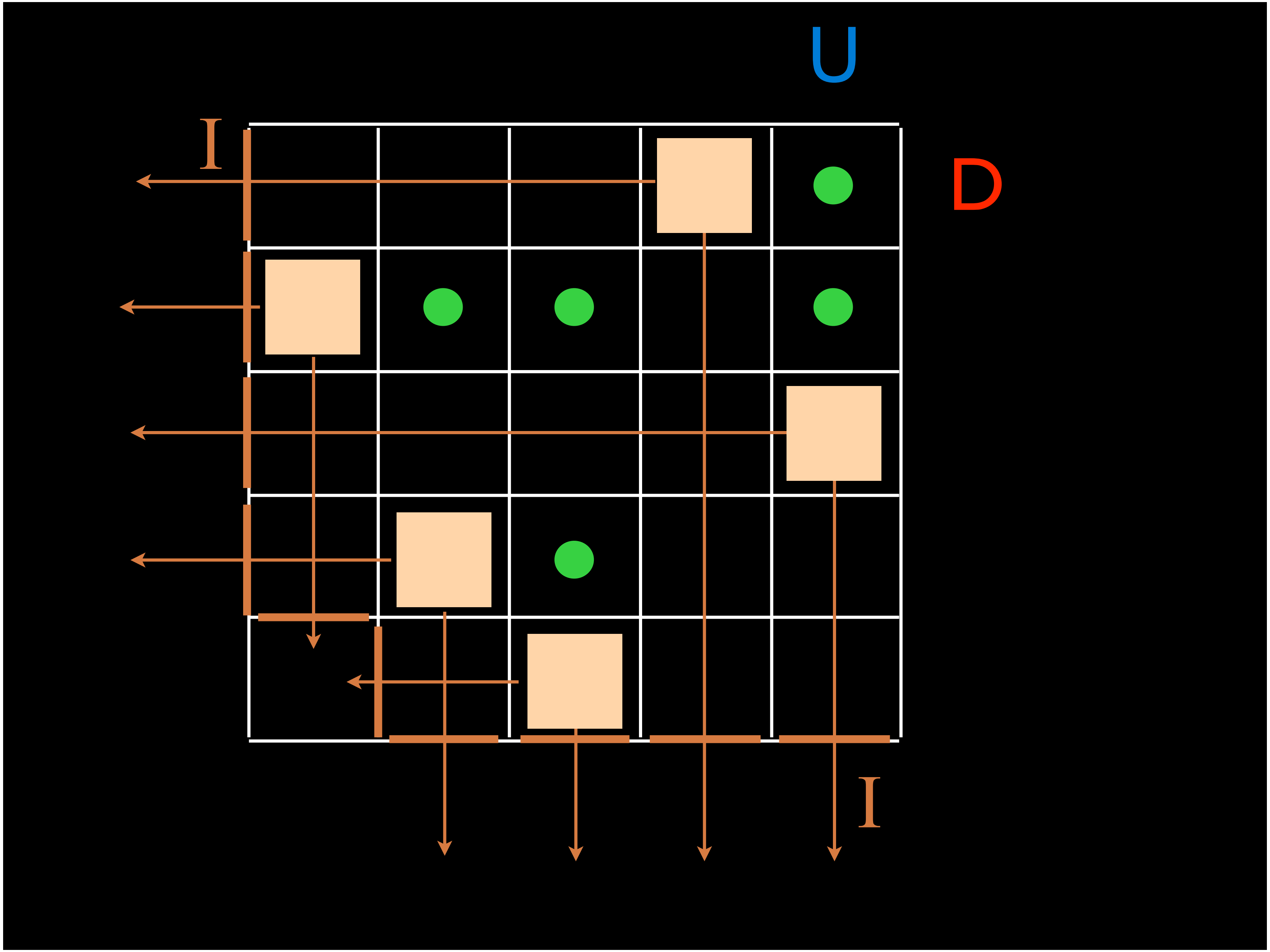
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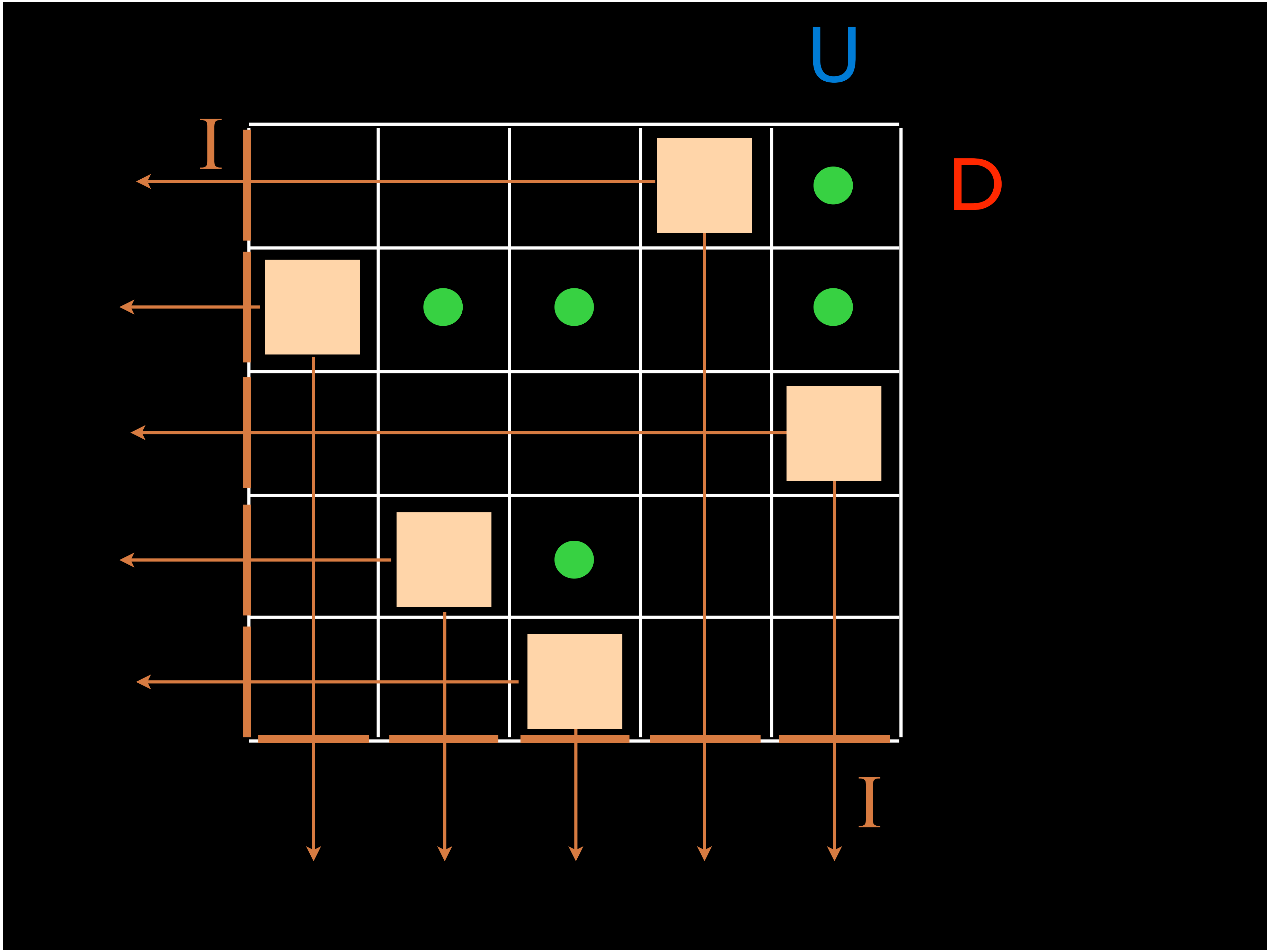
U
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I
I

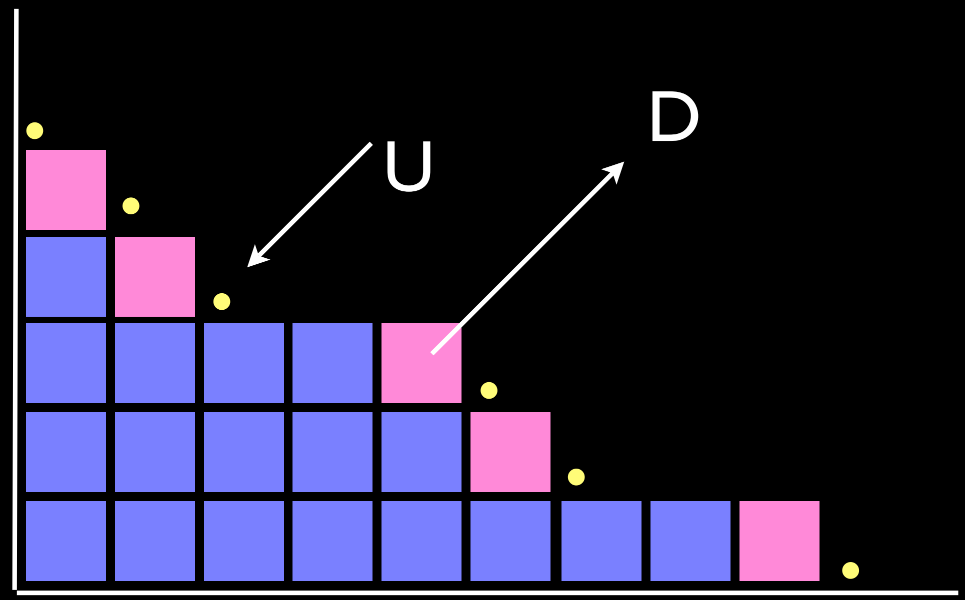
representation

of

operators

U, D

Operators U and D

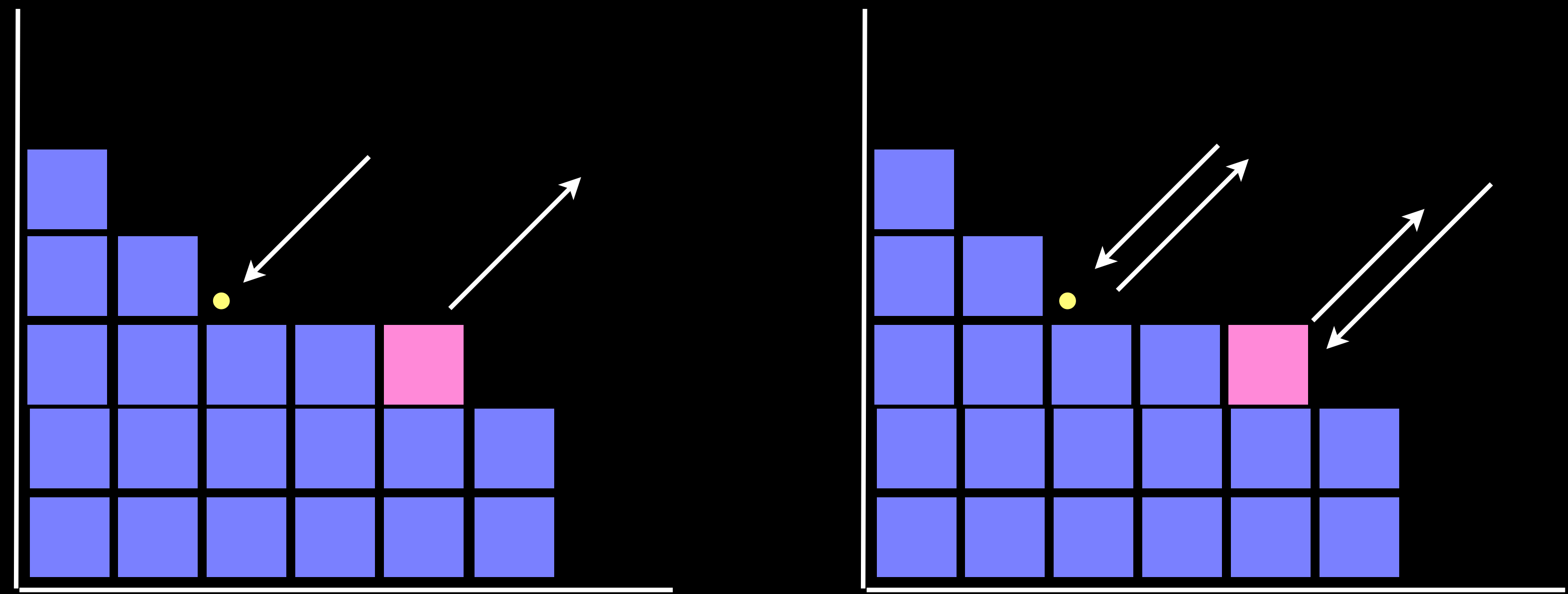


adding
or deleting
a cell in
a Ferrers
diagram

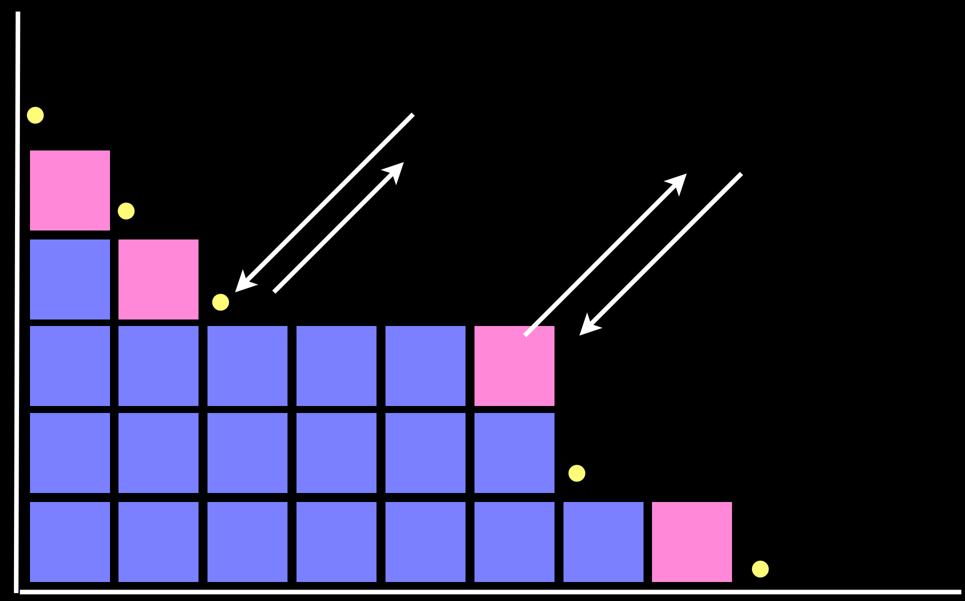
Young lattice

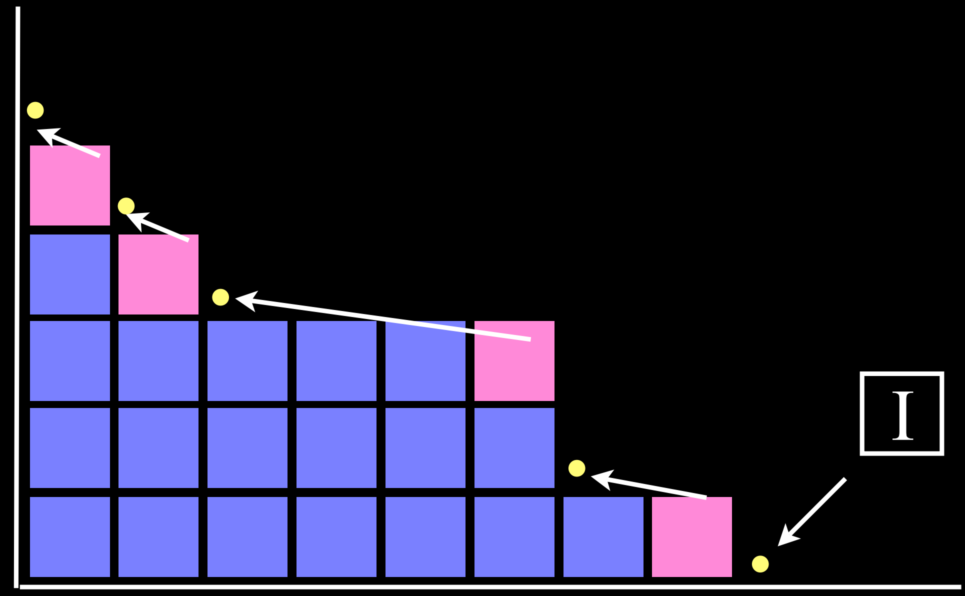
Heisenberg commutation relation

$$UD = DU + I$$



$$UD = DU + I$$



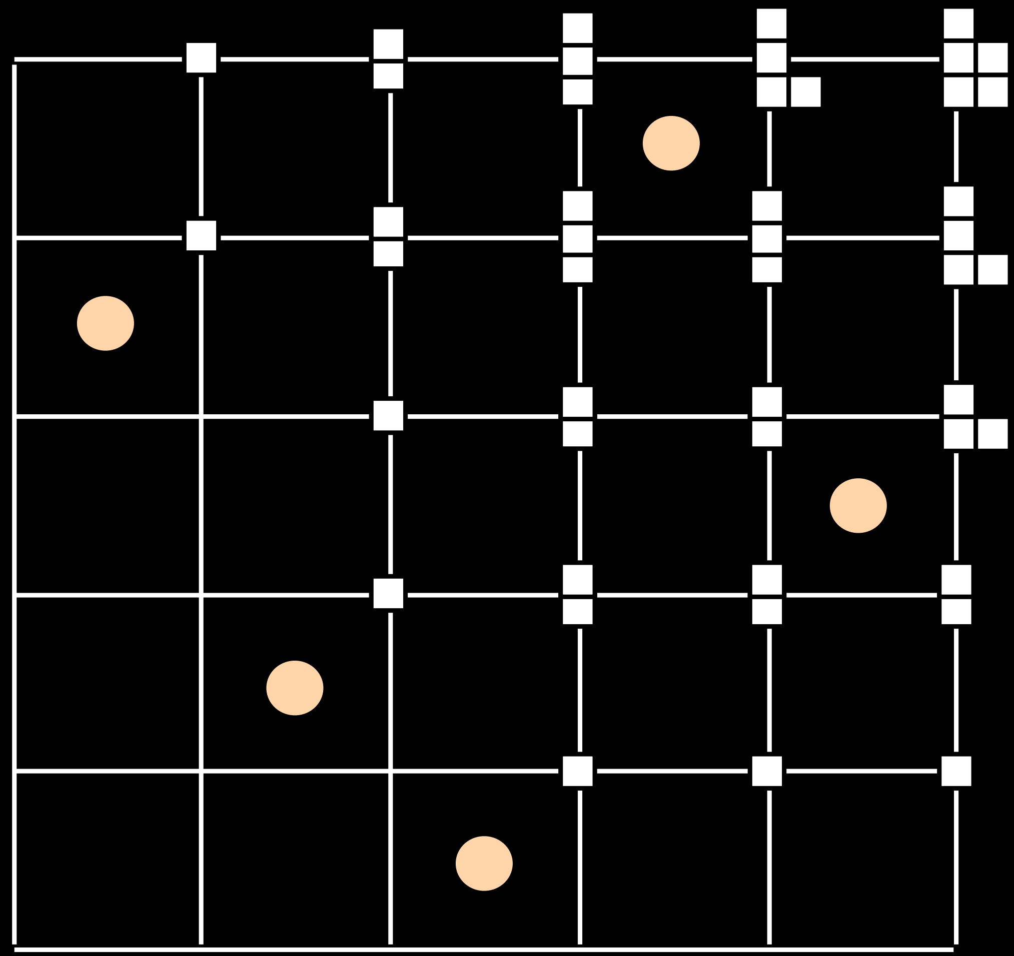


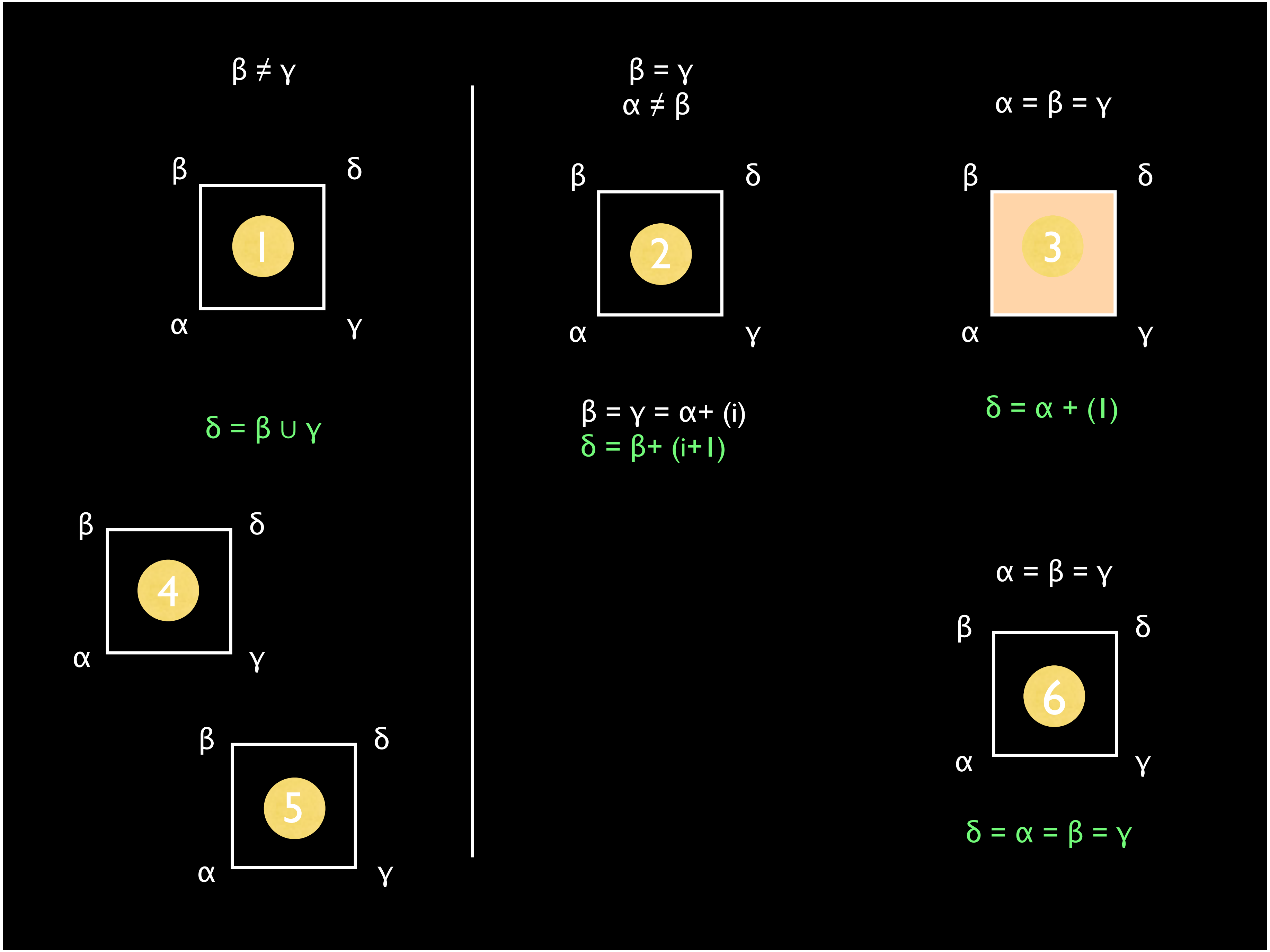
combinatorial “representation” of the
commutation relation $UD = DU + I$

RSK with Fomin's "local rules"

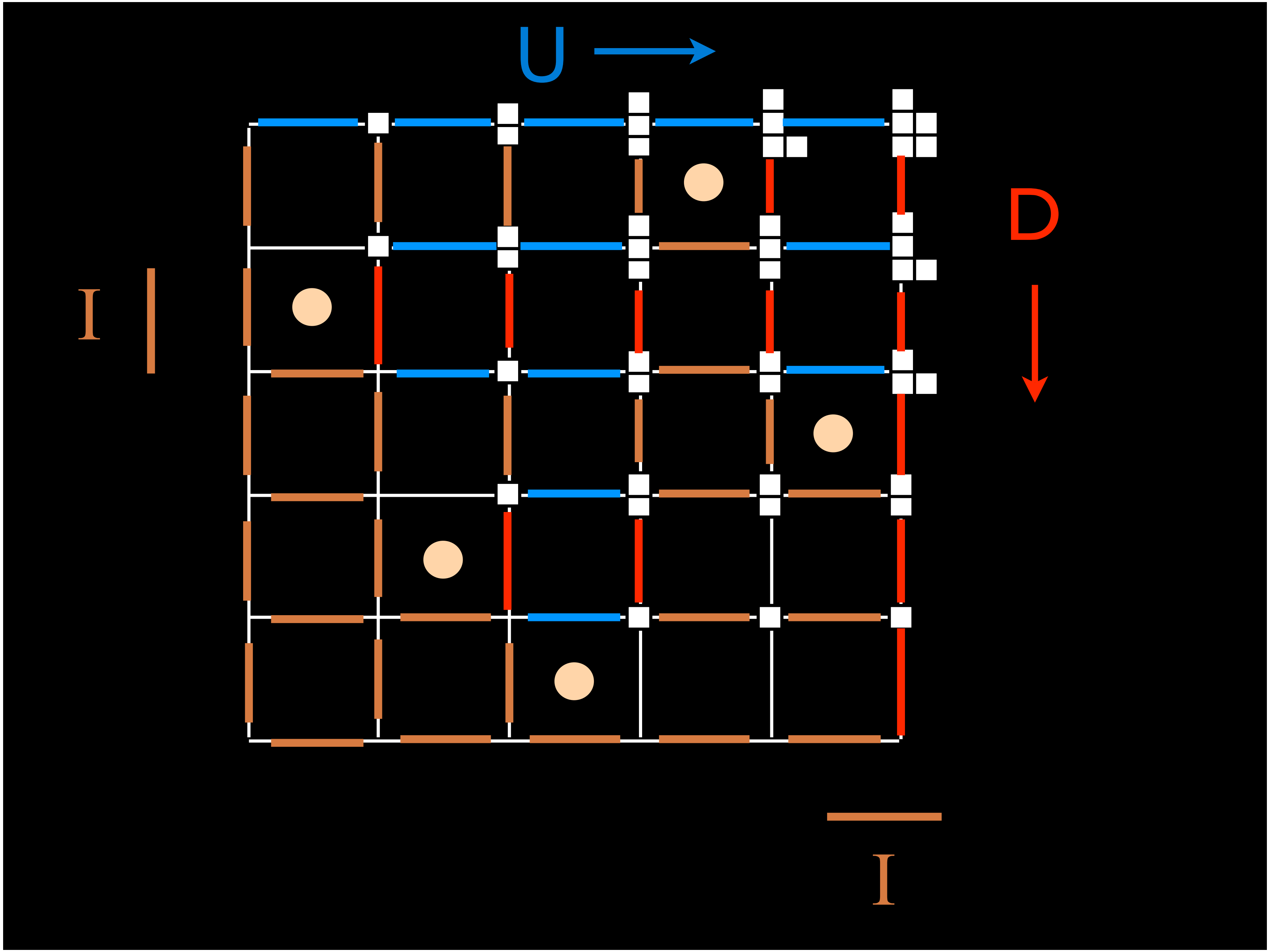
$UD = q\,DU + 1$

Sergey Fomin
(with C. K.)





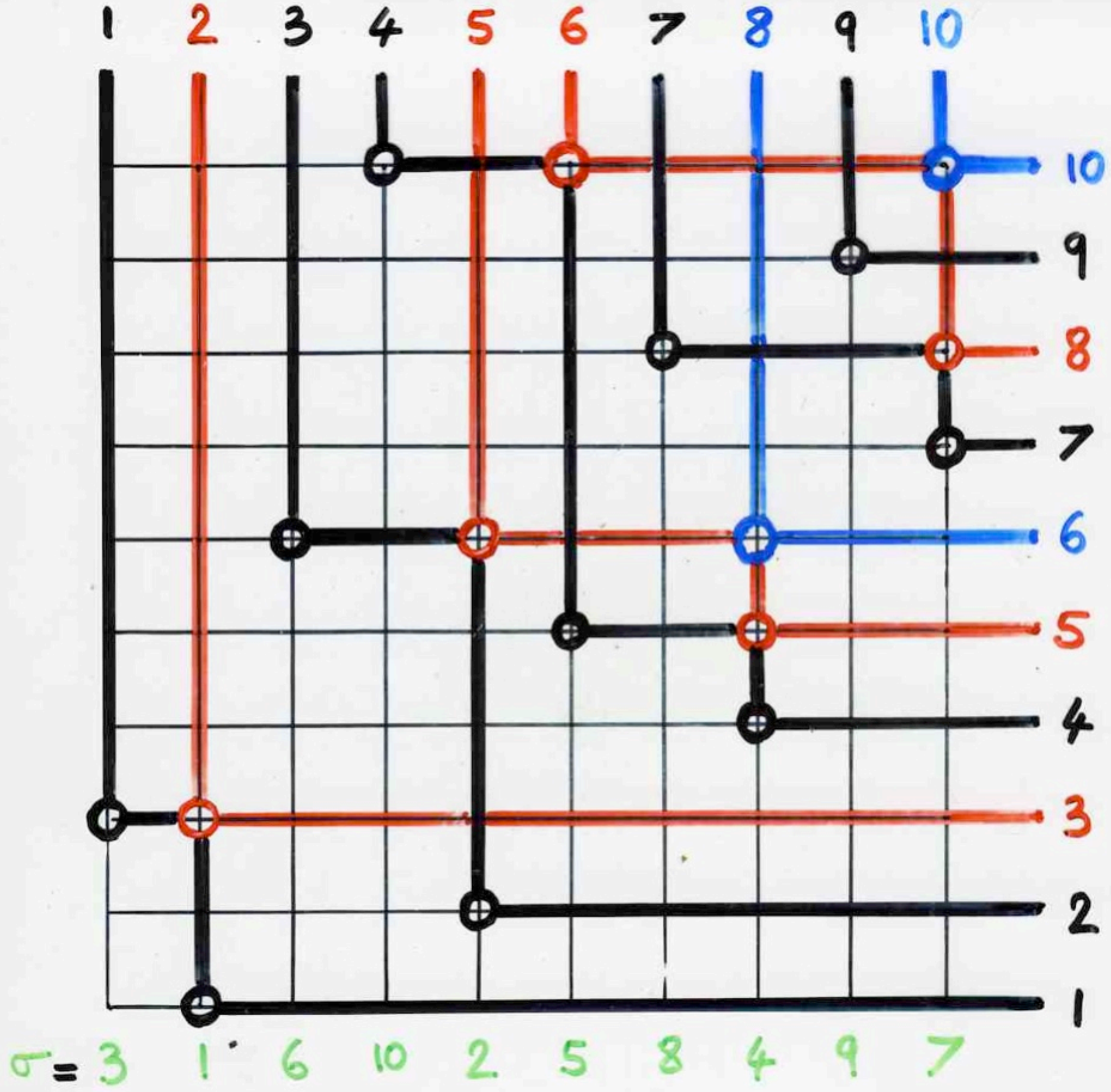
β ≠ γ
β δ
1
α γ
δ = β ∪ γ
β δ
4
α γ
β δ
5
α γ
β = γ
α ≠ β
β δ
2
α γ
β = γ = α+ (i)
δ = β+ (i+1)
α = β = γ
β δ
3
α γ
δ = α + (1)
α = β = γ
β δ
6
α γ
δ = α = β = γ



U
D
I
I

local RSK and geometric RSK

(the geometric construction with "light" and "shadow" for RSK leads to a simple proof of the fact that RSK and the "local rules" give the same bijection)


1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
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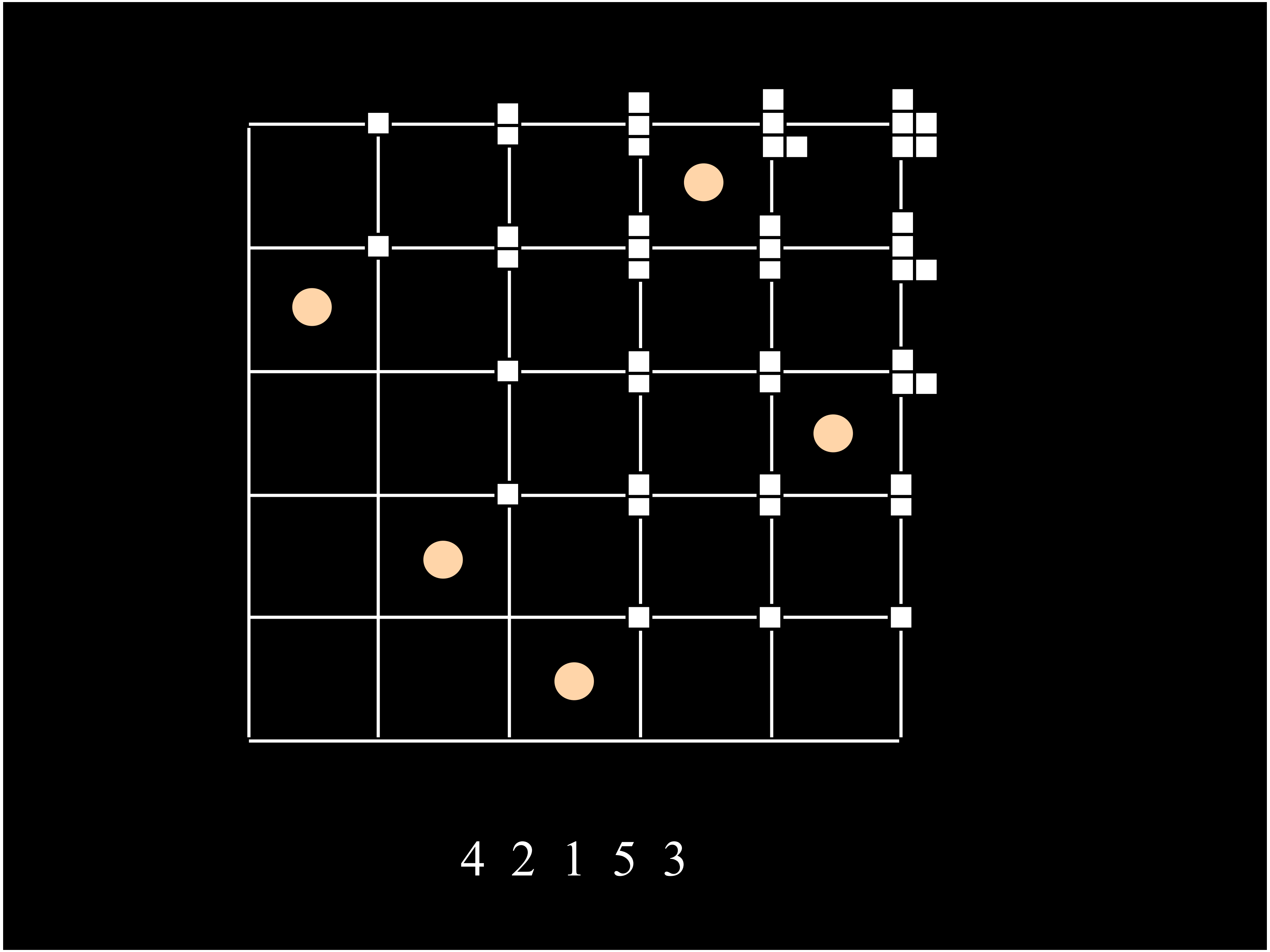
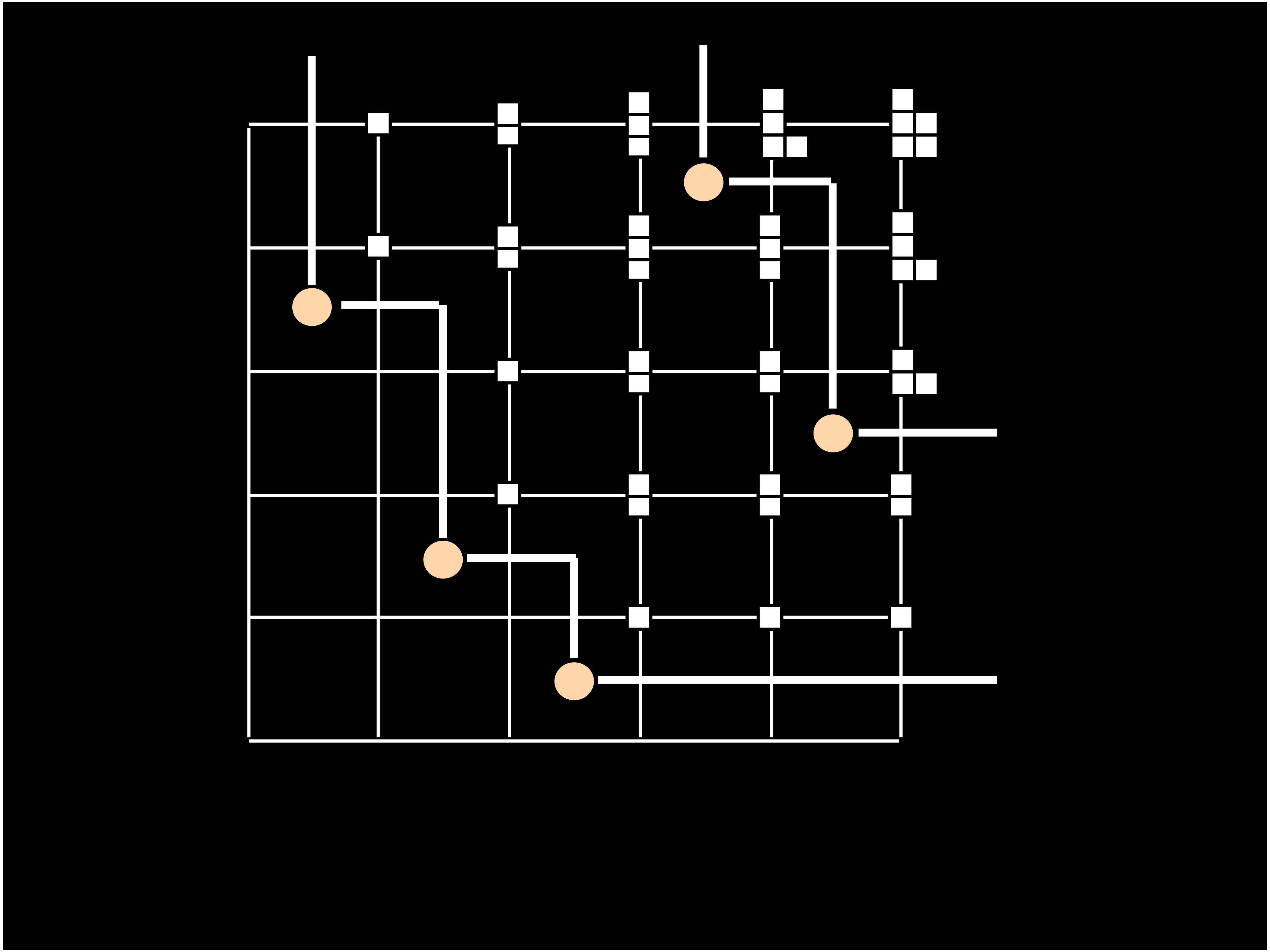
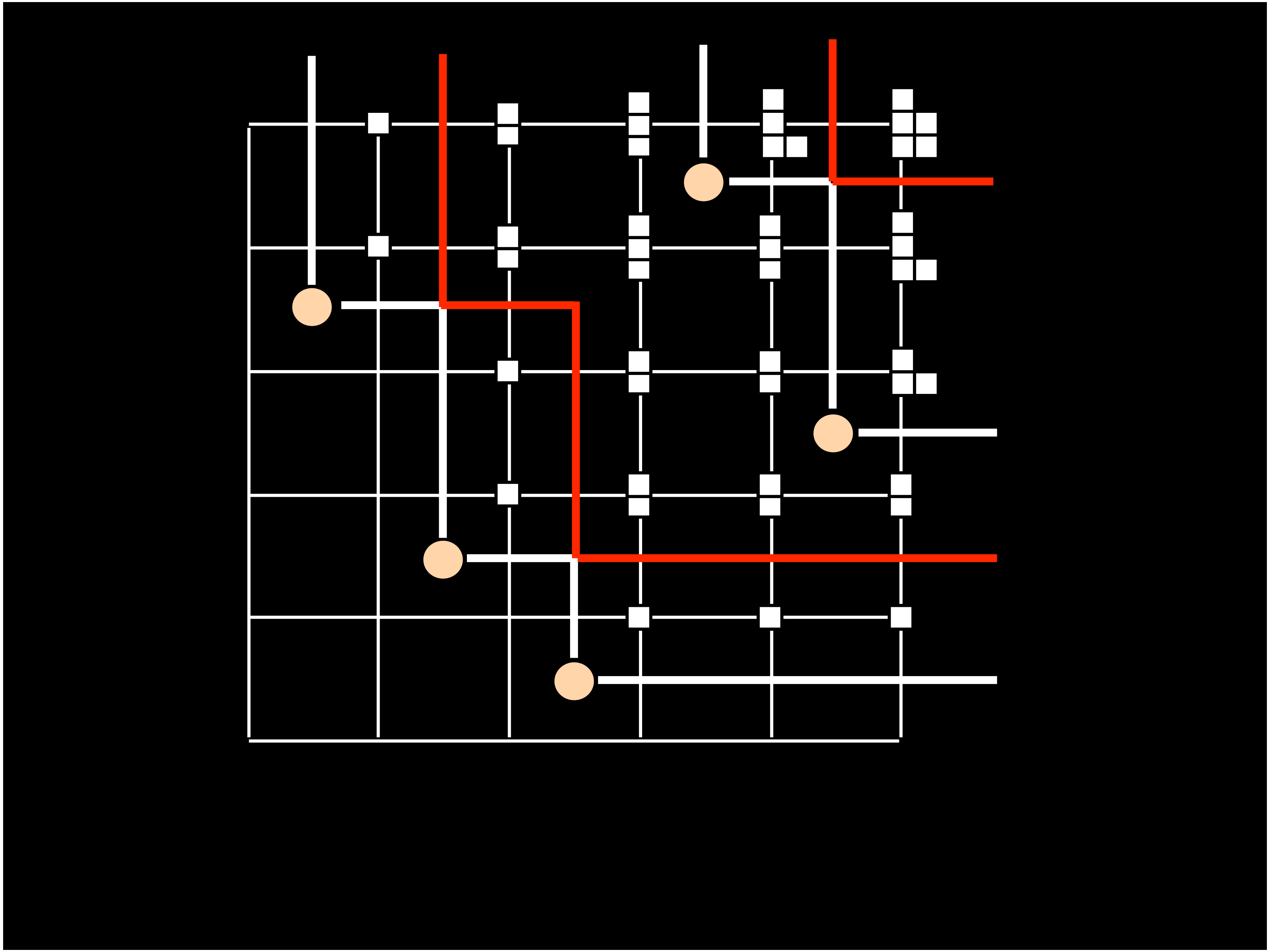
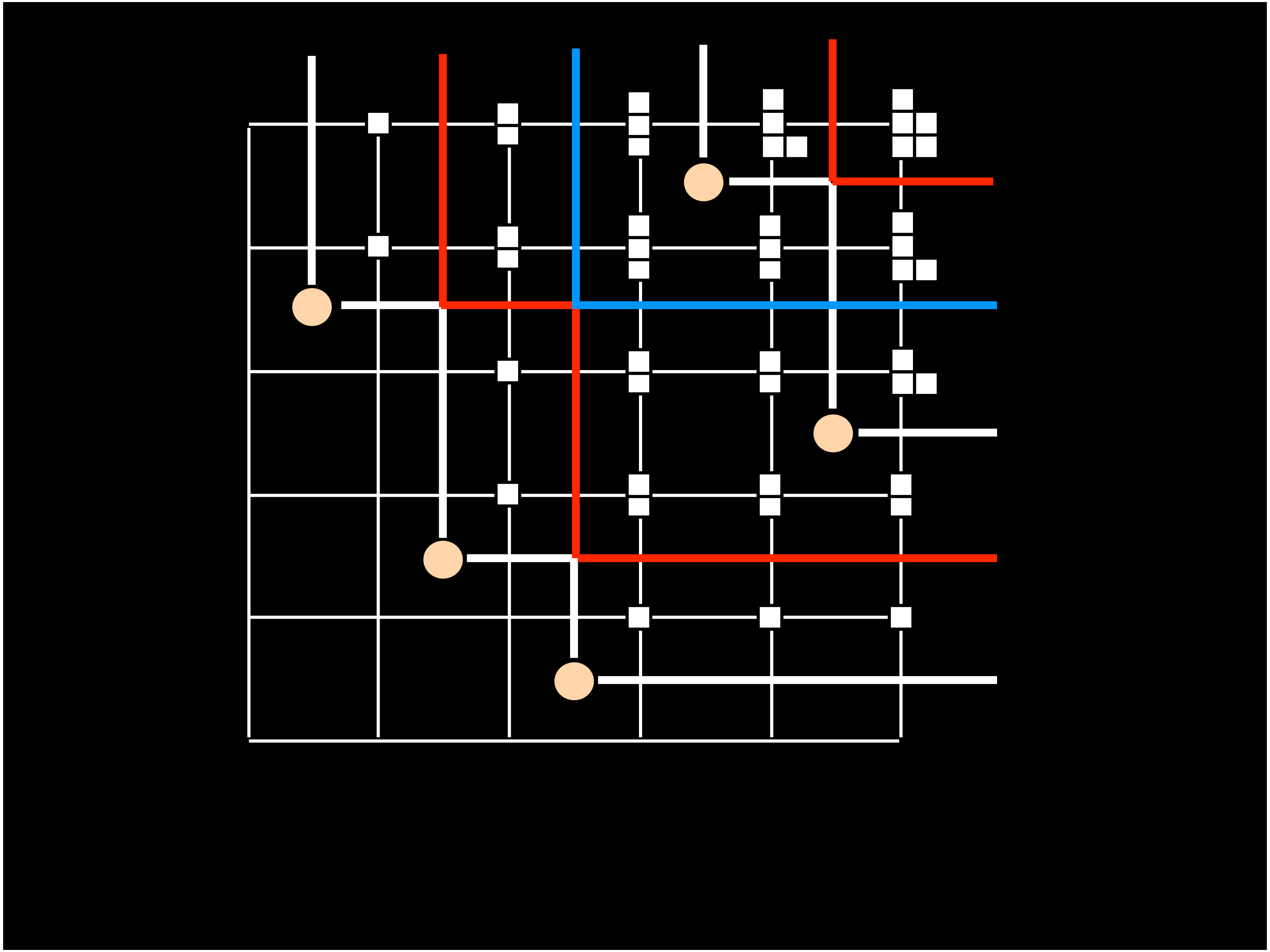
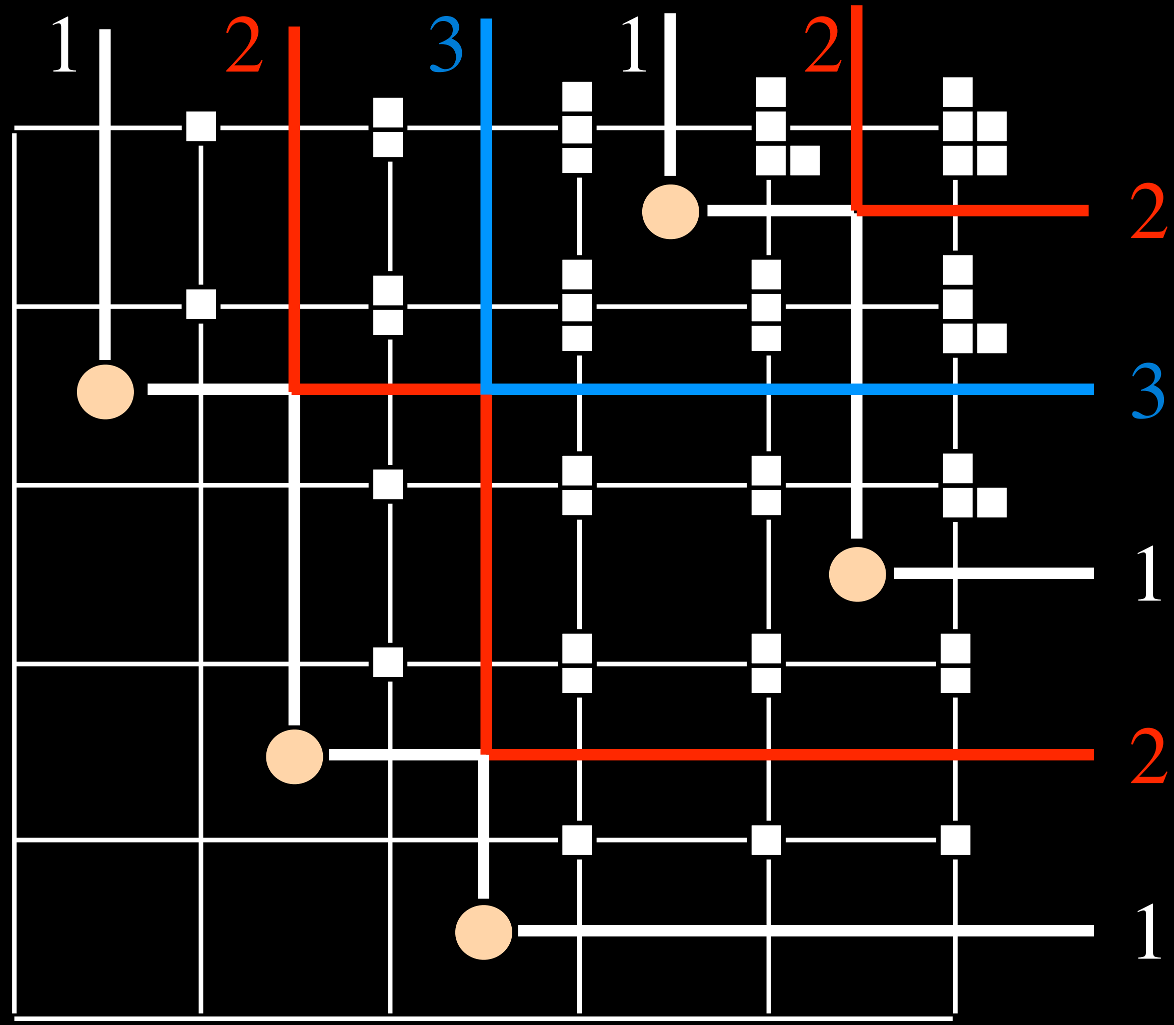
4
3
2
1
σ = 3 1 6 10 2 5 8 4 9 7


4 2 1 5 3

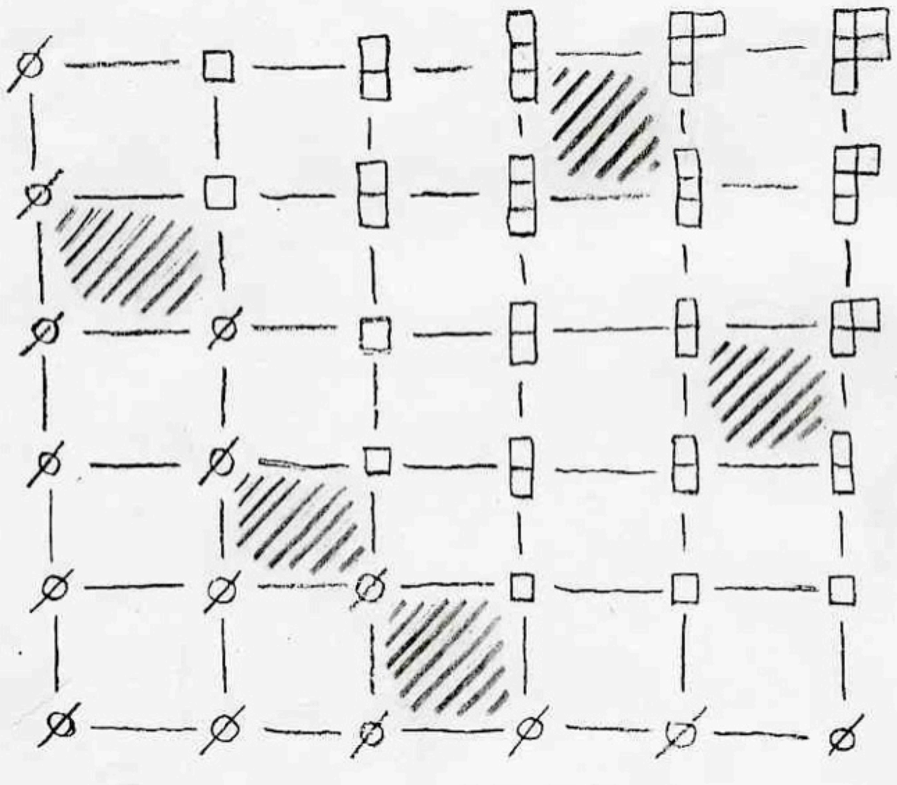







1
2
3
1
2
2
3
1
2
1

dessin fait par S. FOMIN



$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

permutation associée

Sergey Fomin

- **Schur operators and Knuth correspondences**, *Journal of Combinatorial Theory, Ser.A* **72** (1995), 277-292.
- **Duality of graded graphs**, *Journal of Algebraic Combinatorics* **3** (1994), 357-404.
- **Schensted algorithms for dual graded graphs**, *Journal of Algebraic Combinatorics* **4** (1995), 5-45.
- **Dual graphs and Schensted correspondences**, *Series formelles et combinatoire algebrique*, P.Leroux and C.Reutenauer, Ed., Montreal, LACIM, UQAM, 1992, 221-236.

- **Finite posets and Ferrers shapes** (with T.Britz, 41 pages)
Advances in Mathematics **158** (2000), 86-127.
A survey on the Greene-Kleitman correspondence; many proofs are new.

- **Knuth equivalence, jeu de taquin, and the Littlewood-Richardson rule** (30 pages)
Appendix 1 to Chapter 7 in: R.P.Stanley, *Enumerative Combinatorics, vol.2*, Cambridge University Press, 1999.

Richard P. Stanley

- **Differential posets, *J. Amer. Math. Soc.* 1 (1988), 919-961.**
- **Variations on differential posets, in *Invariant Theory and Tableaux* (D. Stanton, ed.),** The IMA Volumes in Mathematics and Its Applications, vol. 19, Springer-Verlag, New York, 1990, pp. 145-165.

Christian Krattenthaler

- **Growth diagram and increasing and decreasing chains in filling of Ferrers shapes,** arXiv math.CO/0510676

Xavier Gérard Viennot

- Une forme géométrique de la correspondance de Robinson-Schensted, in “Combinatoire et Représentation du groupe symétrique” (D. Foata ed.) Lecture Notes in Mathematics n° 579, pp 29-68, 1976

Marc van Leeuwen

- The Robinson-Schensted and Schützenberger algorithms, an elementary approach
(a 272 Kb dvi file) Electronic Journal of Combinatorics, Foata Festschrift, Vol 3(no.2), R15 (1996)

Guoniu Han

http://math.u-strasbg.fr/~guoniu/software/rsk/index.html
Autour de la correspondance de Robinson-Schensted
Exposé au SLC 52 et LascouxFest, 29/03/2004

§ 7 Laguerre histories

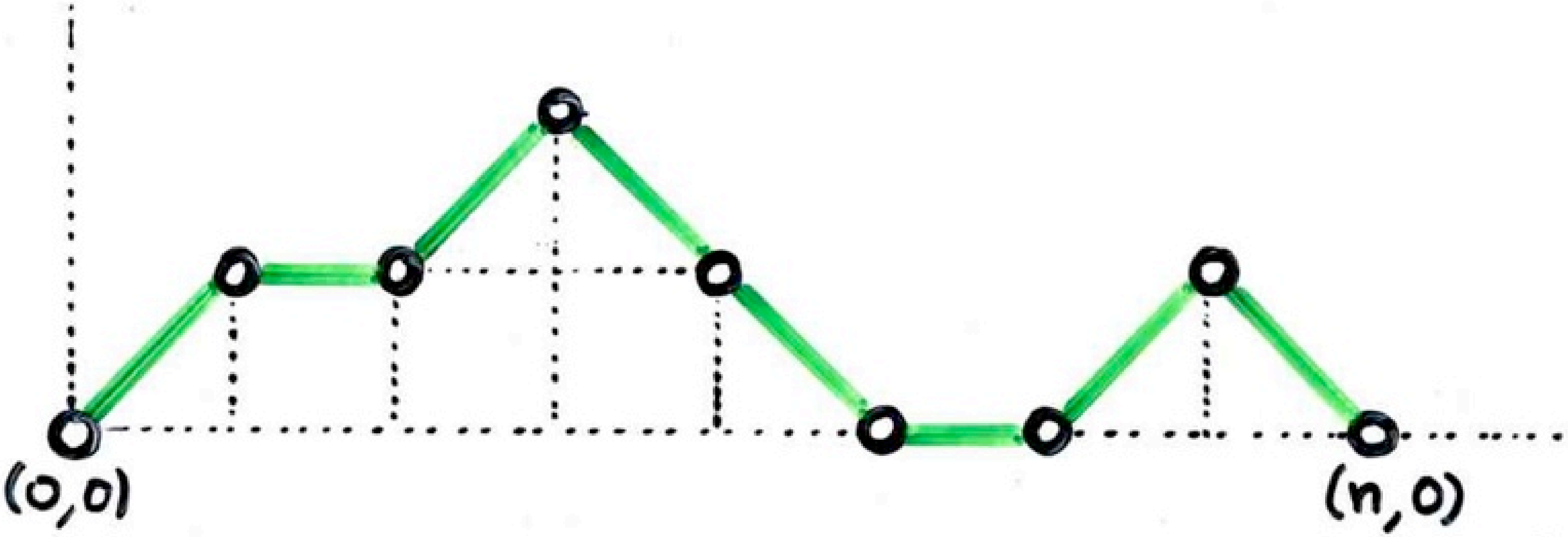
Bijection Permutations $n+1$

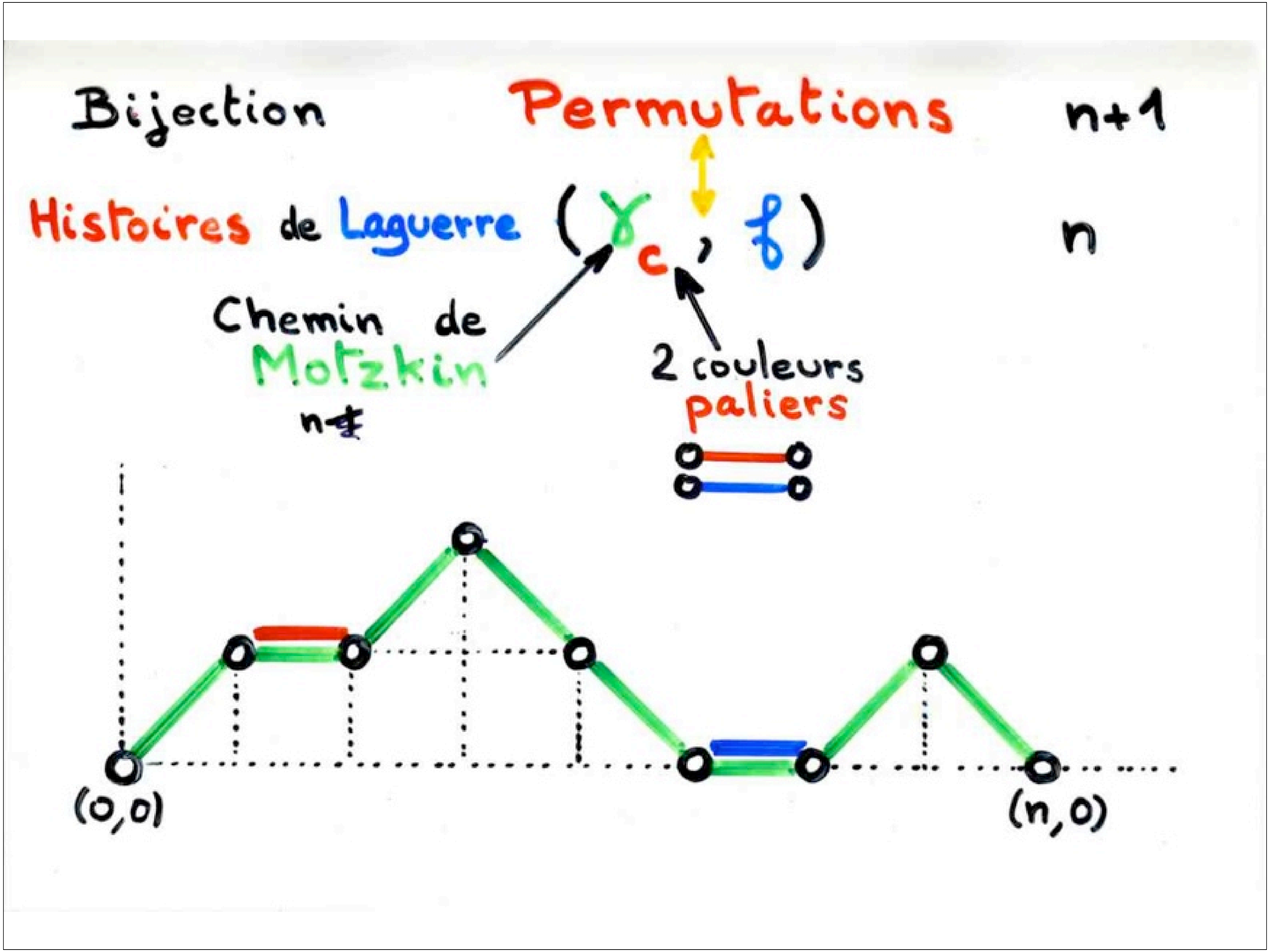
Histoires de Laguerre $(\gamma_c \,;\, f)$ $\updownarrow$ n

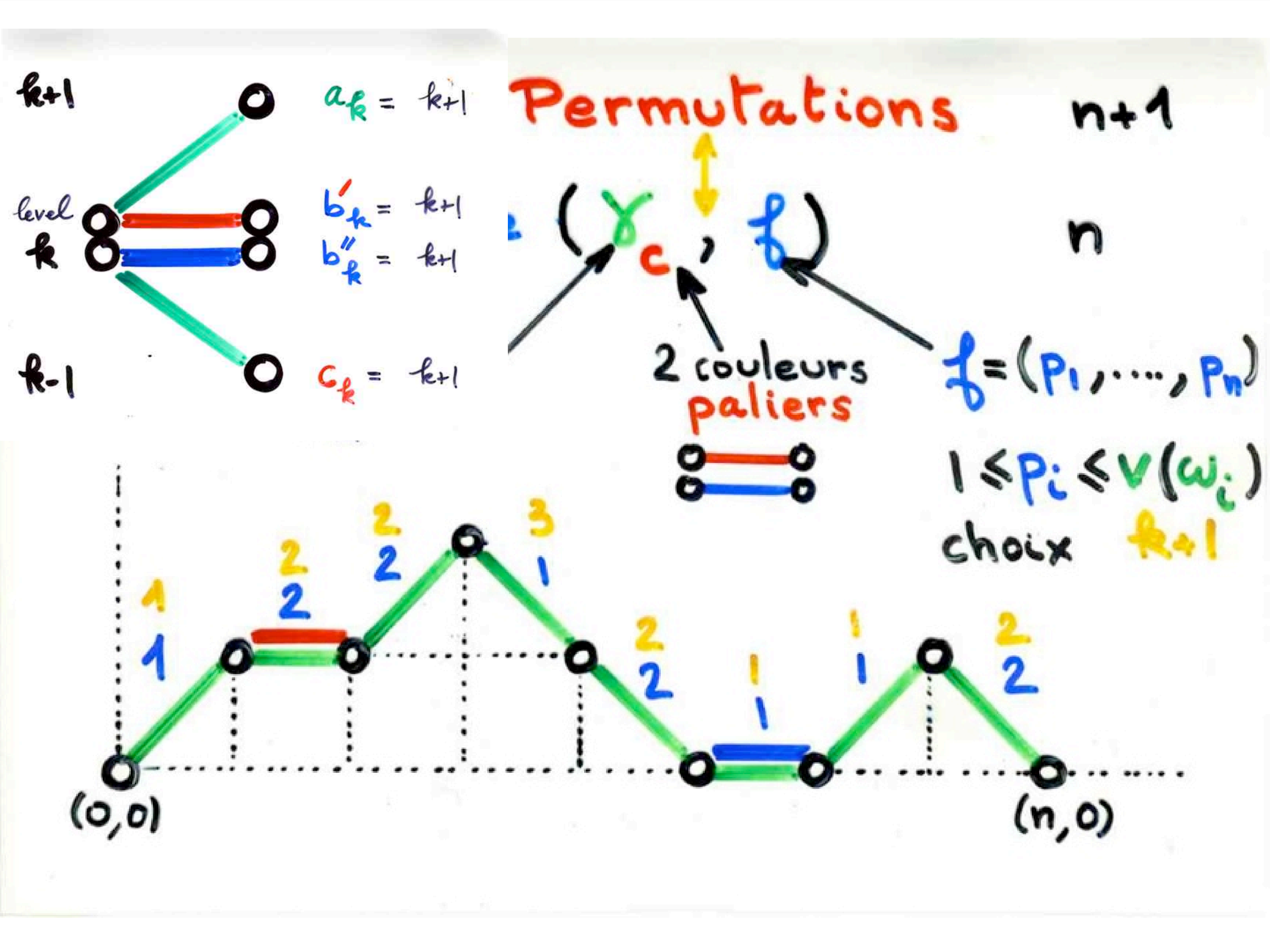
Bijection **Permutations** $n+1$

Histoires de Laguerre $(\gamma_c \, ; f)$ n

Chemin de Motzkin $n-1$




Bijection
Permutations
n+1
Histoires de Laguerre
$(\gamma_c ; f)$
n
Chemin de Motzkin
2 couleurs paliers
(0,0)
(n,0)



k+1
level
k
k-1
a_k = k+1
b'_k = k+1
b''_k = k+1
c_k = k+1
Permutations
n+1
(γ
c
, f)
n
2 couleurs
paliers
f=(p1,...,pn)
1 ≤ pi ≤ v(ωi)
choix
k+1
(0,0)
(n,0)

Laguerre
polynomial

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

$$P_0 = 1 \qquad P_1 = x - b_0$$

$$\mu_n = (n+1)!$$

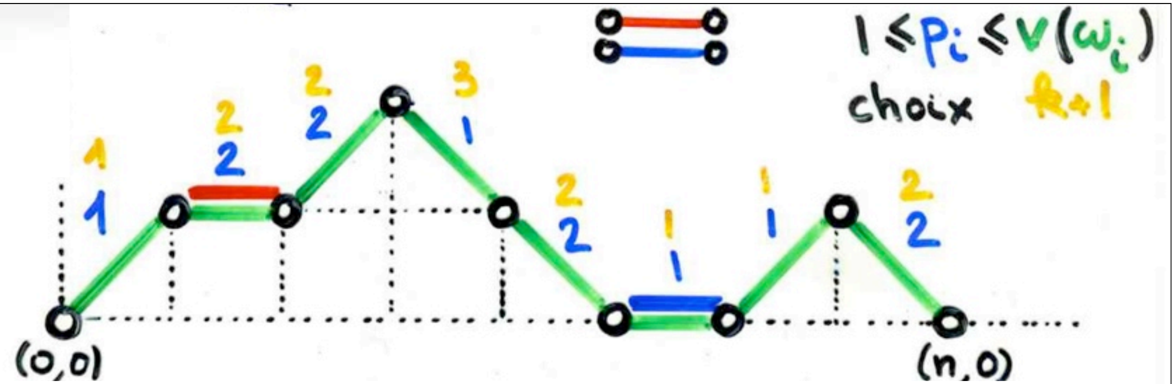
$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

$$J(t) = \cfrac{1}{1 - 2t - \cfrac{1 \cdot 2t^2}{1 - 4t - \cfrac{2 \cdot 3t^2}{\cdots}}}$$

Bijection Laguerre histories permutations

Françon-xgv., 1978

$h = (\omega_c \,;\, (p_1, \dots, p_n))$



x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
n= 8		2	2
9	•		

␣

␣ 1 ␣

␣ 1 ␣ 2

␣ 1 ␣ 3 ␣ 2

4 1 ␣ 3 ␣ 2

4 1 ␣ 3 5 2

4 1 6 ␣ 3 5 2

4 1 6 ␣ 7 ␣ 3 5 2

4 1 6 ␣ 7 8 3 5 2

4 1 6 9 7 8 3 5 2 $= \sigma \in \mathfrak{S}_{n+1}$

parameter “q-Laguerre”

Lemme. $h = (\omega_c \,;\, (p_1, \dots, p_n)) \in \mathcal{L}_n$ Laguerre history

$\Pi \circ \Theta_a \downarrow$

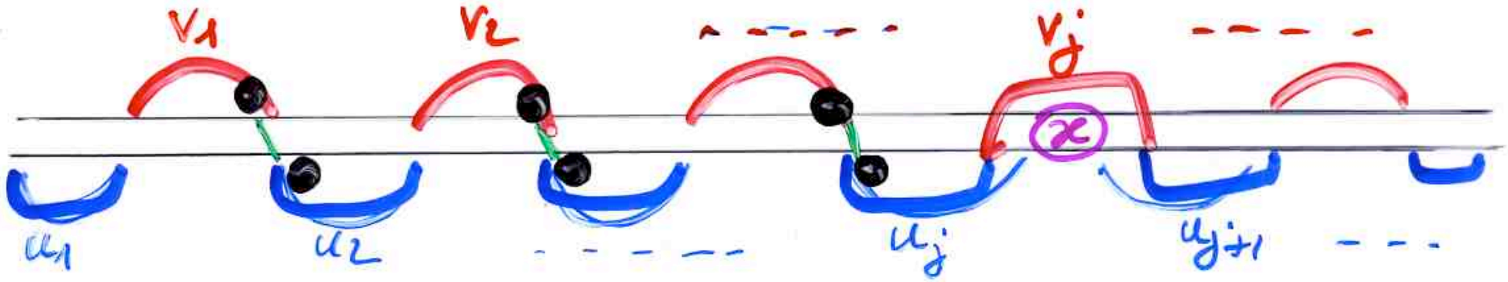
permutation $\sigma \in \mathfrak{S}_{n+1}$

$p_x = j$ est aussi :

$j = 1 +$ nb de triplets (a, b, x) ayant le "motif" $(31\text{-}2)$ c.à.d :

$a = \sigma(i)$, $b = \sigma(i+1)$, $x = \sigma(\ell)$

$i < i+1 < \ell$ $\quad$ $b < x < a$



"q-analogue" of Laguerre histories

§ 8

representation of the operators E and D

$DE = ED + E + D$

V vector space generated by B basis

B alternating words two letters $\{○, ●\}$

(no occurrence of ○○ or ●●)

4 operators A, S, J, K

4 operators A, S, J, K, $u \in B$

$$\langle u | A = \sum_{\text{letter } \circ \text{ of } u} v$$

, v obtained by:

○ → ○●○

$$\langle u | S = \sum_{\circ \text{ of } u} v$$

v obtained by:

○ → ●

(and ●● → ● ●●● → ●)

$$\langle u | J = \sum_{\circ \text{ of } u} v$$

v, ○ → ●○ (and ●● → ●)

$$\langle u | K = \sum_{\circ \text{ of } u} v$$

v, ○ → ○● (and ●● → ●)

$$\circ \bullet \circ \bullet \, | S = \bullet \circ \bullet + \circ \bullet$$

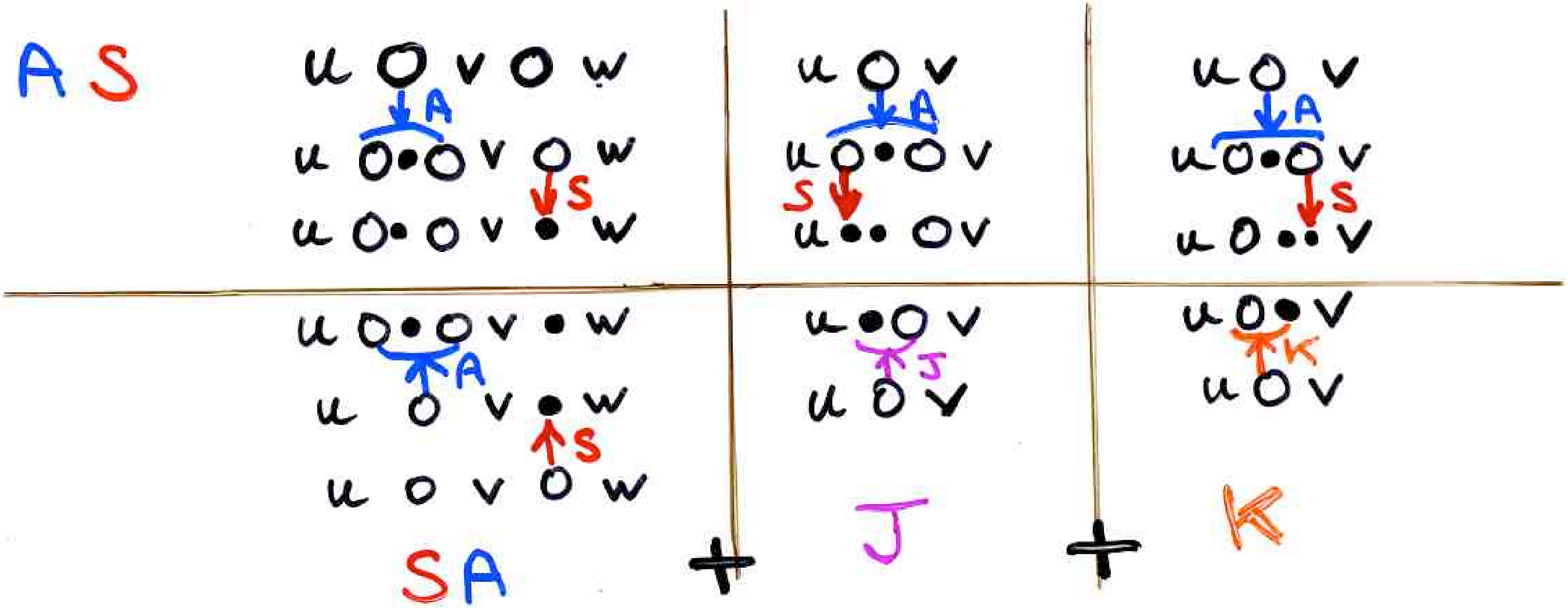
Lemma.

$$AS = SA + J + K$$

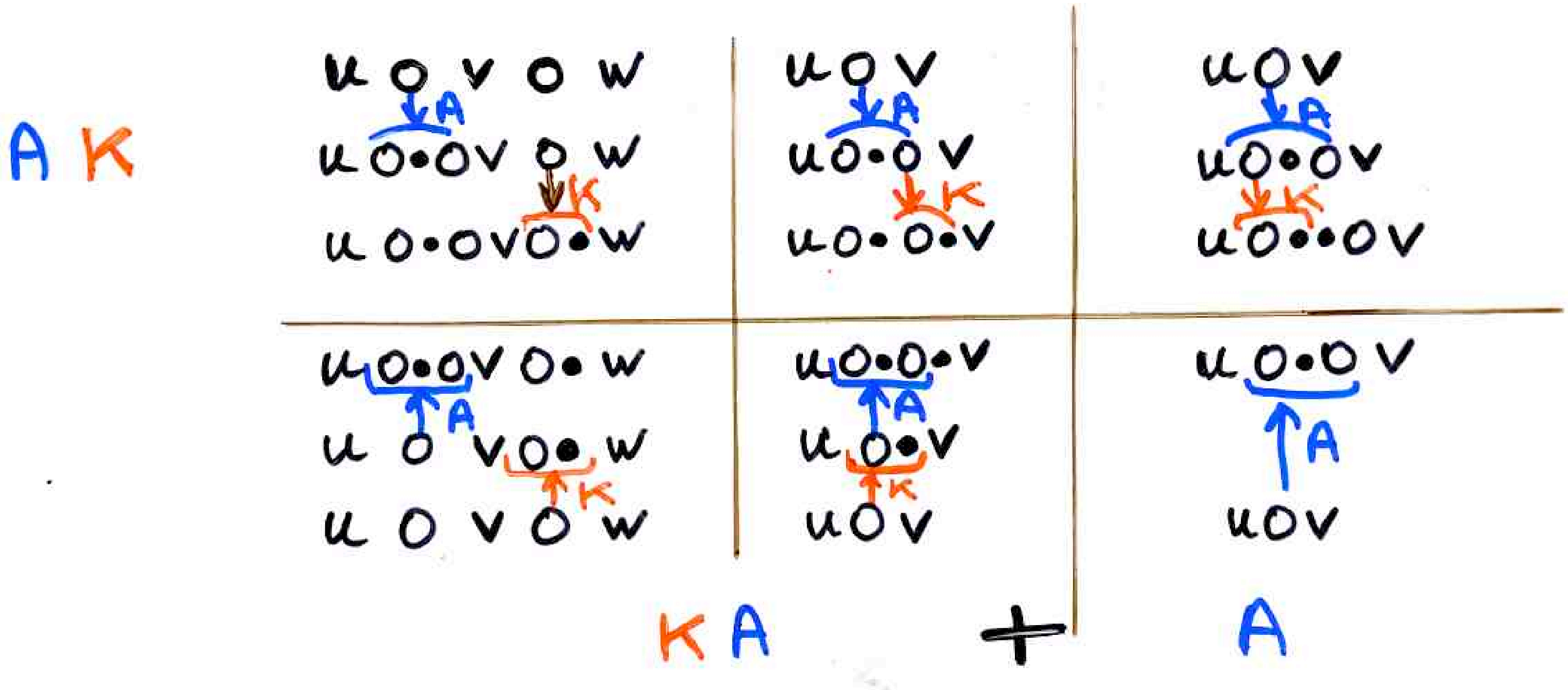
$$AK = KA + A$$

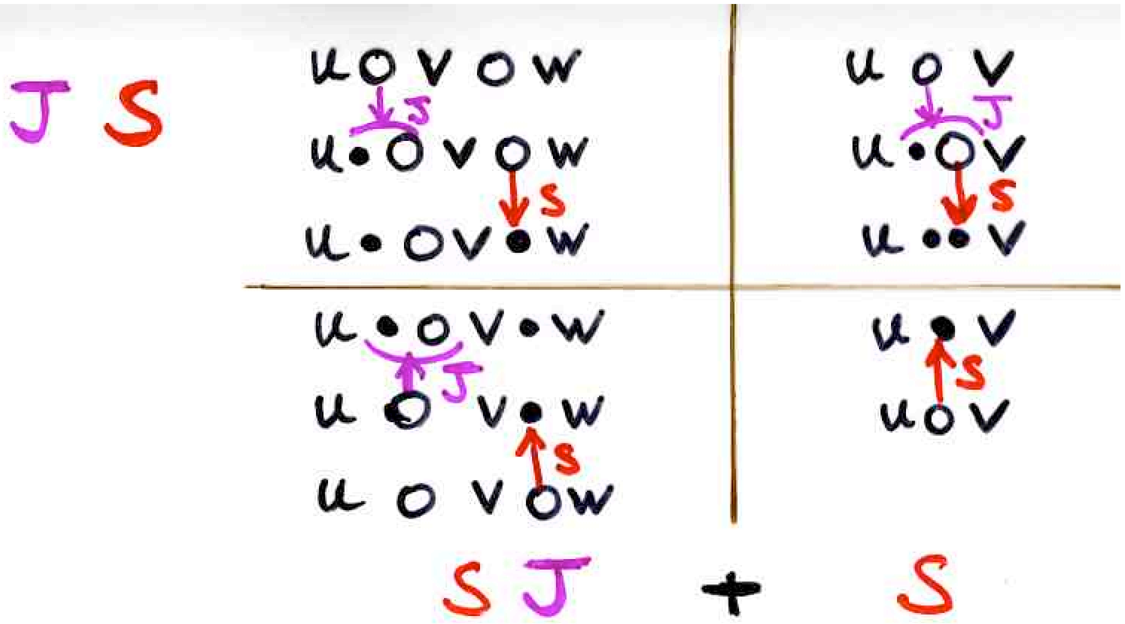
$$JS = SJ + S$$

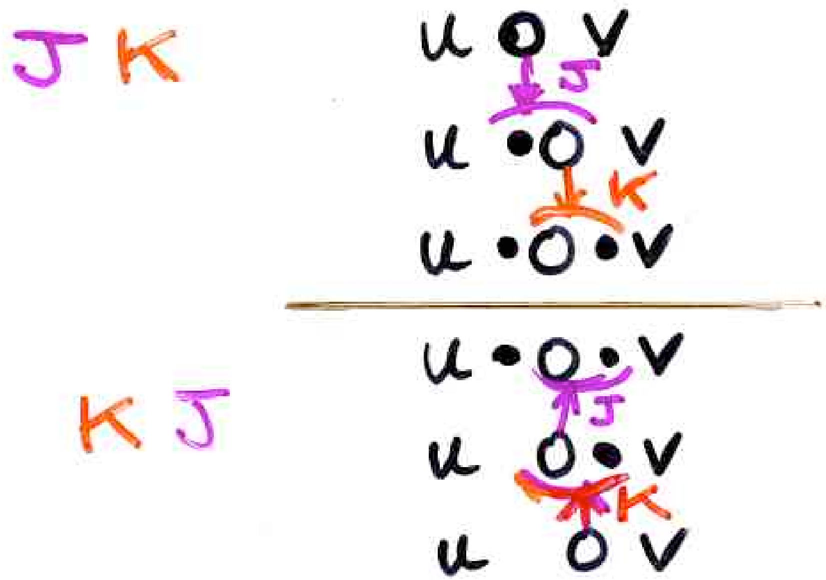
$$JK = KJ$$



AS
SA
+
J
+
K


A K
K A
+
A


J S
u o v o w
u • o v o w
u • o v • w
u • o v • w
u o v • w
u o v o w
u o v
u • o v
u • • v
u • v
u o v
J
S
SJ + S


J K
u o v
u • o v
u • o • v
K J
u • o • v
u o • v
u o v
J
K

Lemma.

$$AS = SA + J + K$$

$$AK = KA + A$$

$$JS = SJ + S$$

$$JK = KJ$$

$$D = A + J$$

$$E = S + K$$

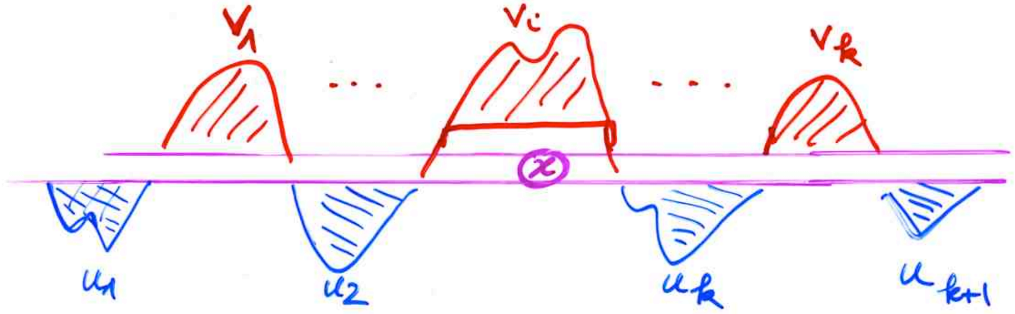
$$D = A + J$$

$$E = S + K$$

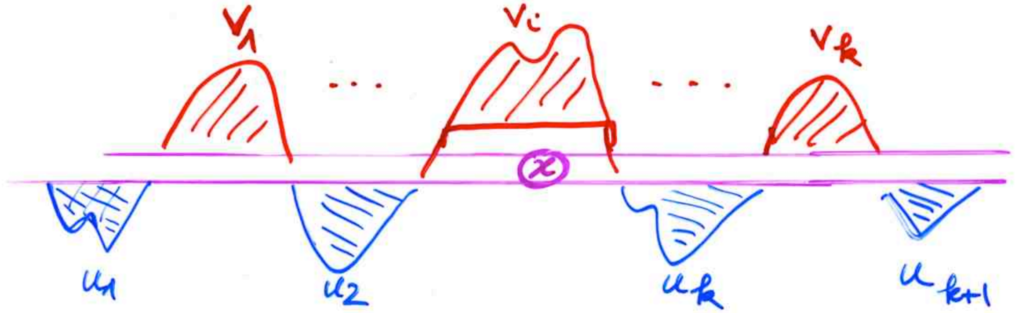
$$DE = (A+J)(S+K)$$

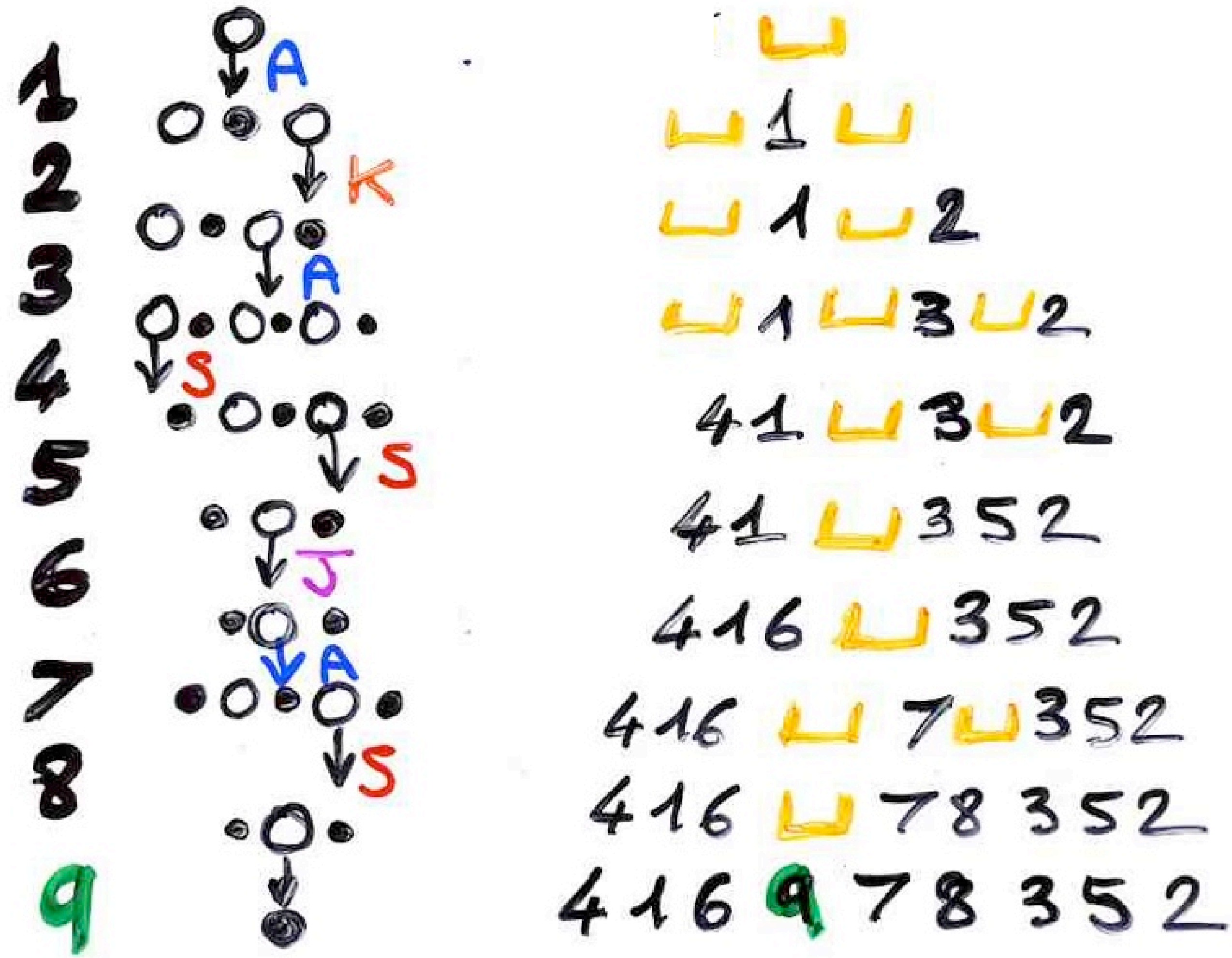
$$= AS + AK + JS + JK$$

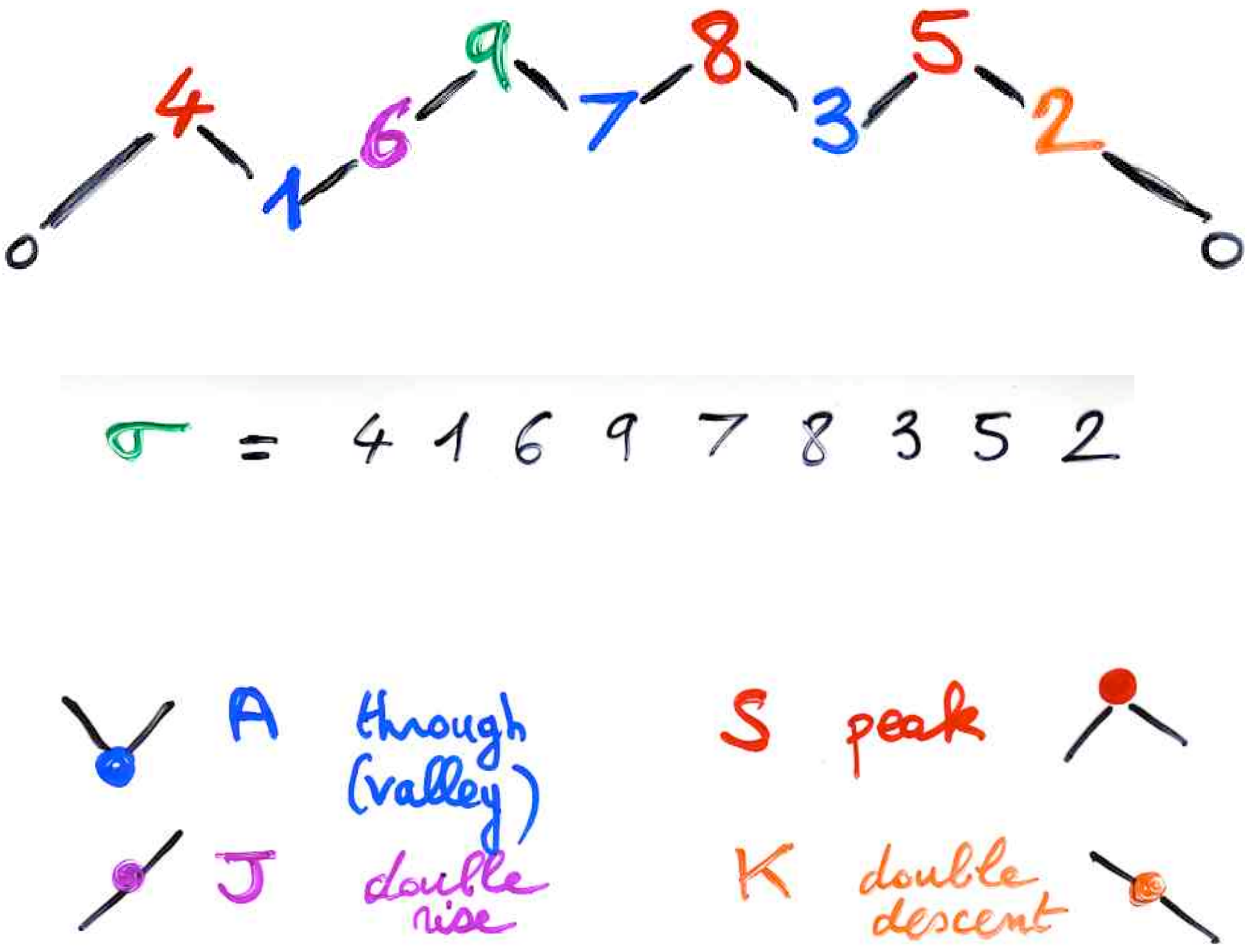
$$= \underbrace{(SA + KA + SJ + KJ)}_{\underbrace{(S+K)(A+J)}_{ED}} + \underbrace{J + K + A + S}_{E + D}$$



	␣
1	␣ 1 ␣
2	␣ 1 ␣ 2
3	␣ 1 ␣ 3 ␣ 2
4	4 1 ␣ 3 ␣ 2
5	4 1 ␣ 3 5 2
6	4 1 6 ␣ 3 5 2
7	4 1 6 ␣ 7 ␣ 3 5 2
8	4 1 6 ␣ 7 8 3 5 2
9	4 1 6 9 7 8 3 5 2


v_1
v_i
v_k
x
u_1
u_2
u_k
u_{k+1}


1
2
3
4
5
6
7
8
9
A
K
A
S
S
J
A
S
1
1 2
1 3 2
4 1 3 2
4 1 3 5 2
4 1 6 3 5 2
4 1 6 7 3 5 2
4 1 6 7 8 3 5 2
4 1 6 9 7 8 3 5 2


0 4 1 6 9 7 8 3 5 2 0
σ = 4 1 6 9 7 8 3 5 2
A through (valley)
S peak
J double rise
K double descent

data

structures

histories

Representation of the operators D and E

and

"Data structure histories"

- Computer Science

Computing the average cost of a data structure integrated on a sequence of primitive operations

ex: stack

- priority queue
- dictionnary
- list
- symbol table

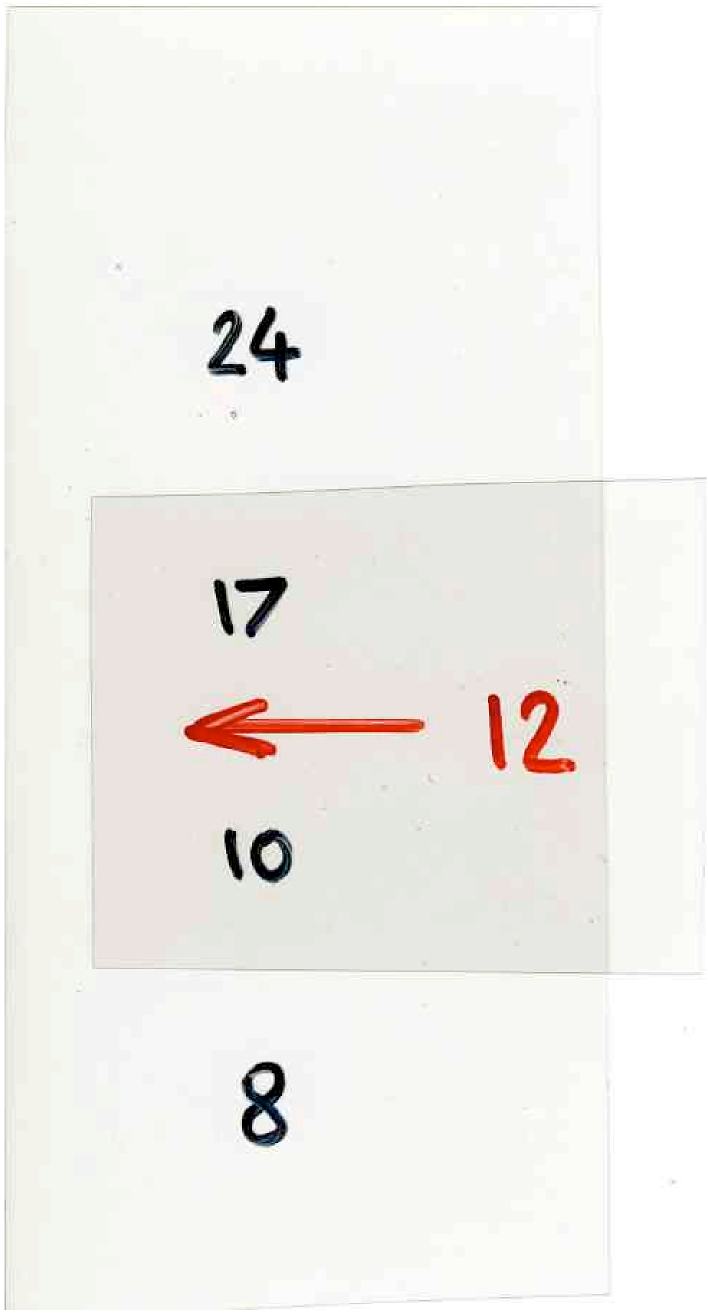
Flajolet, Françon, Vuillemin

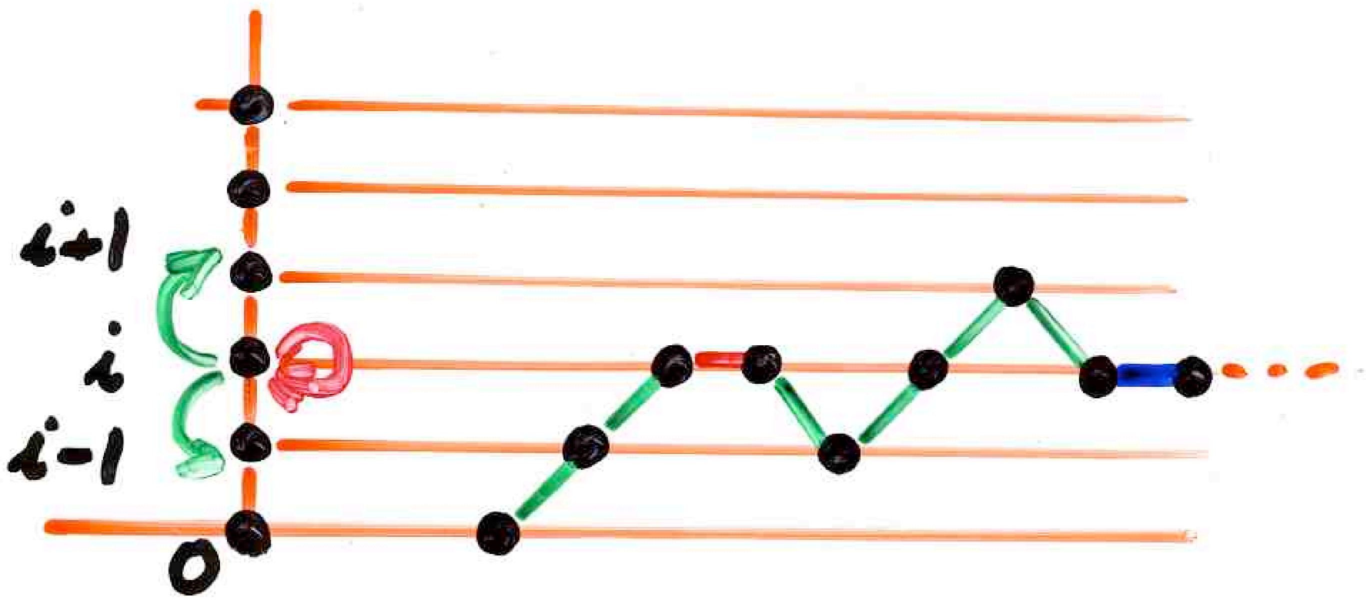
24

17

10

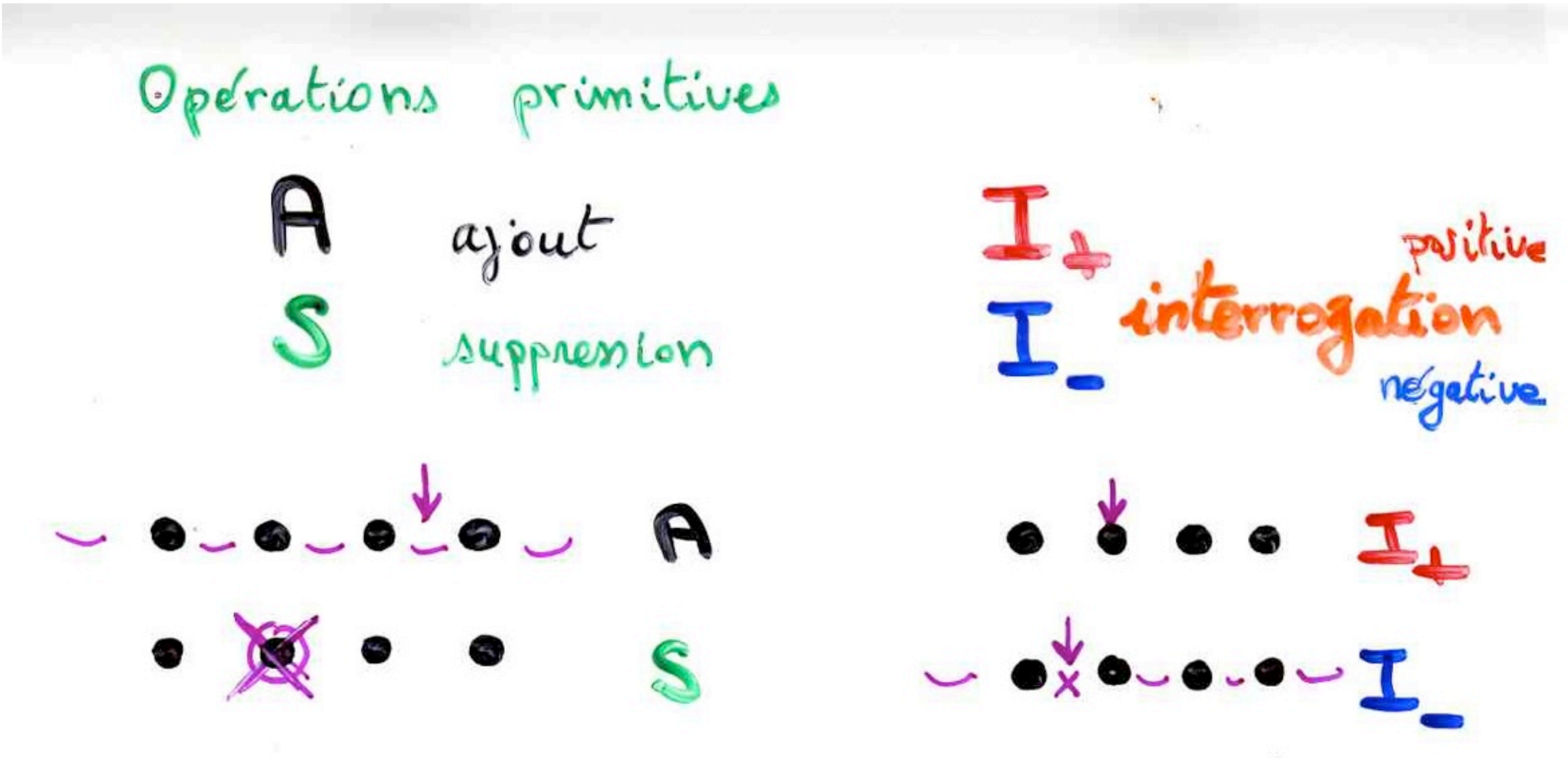
8


24
17
12
10
8

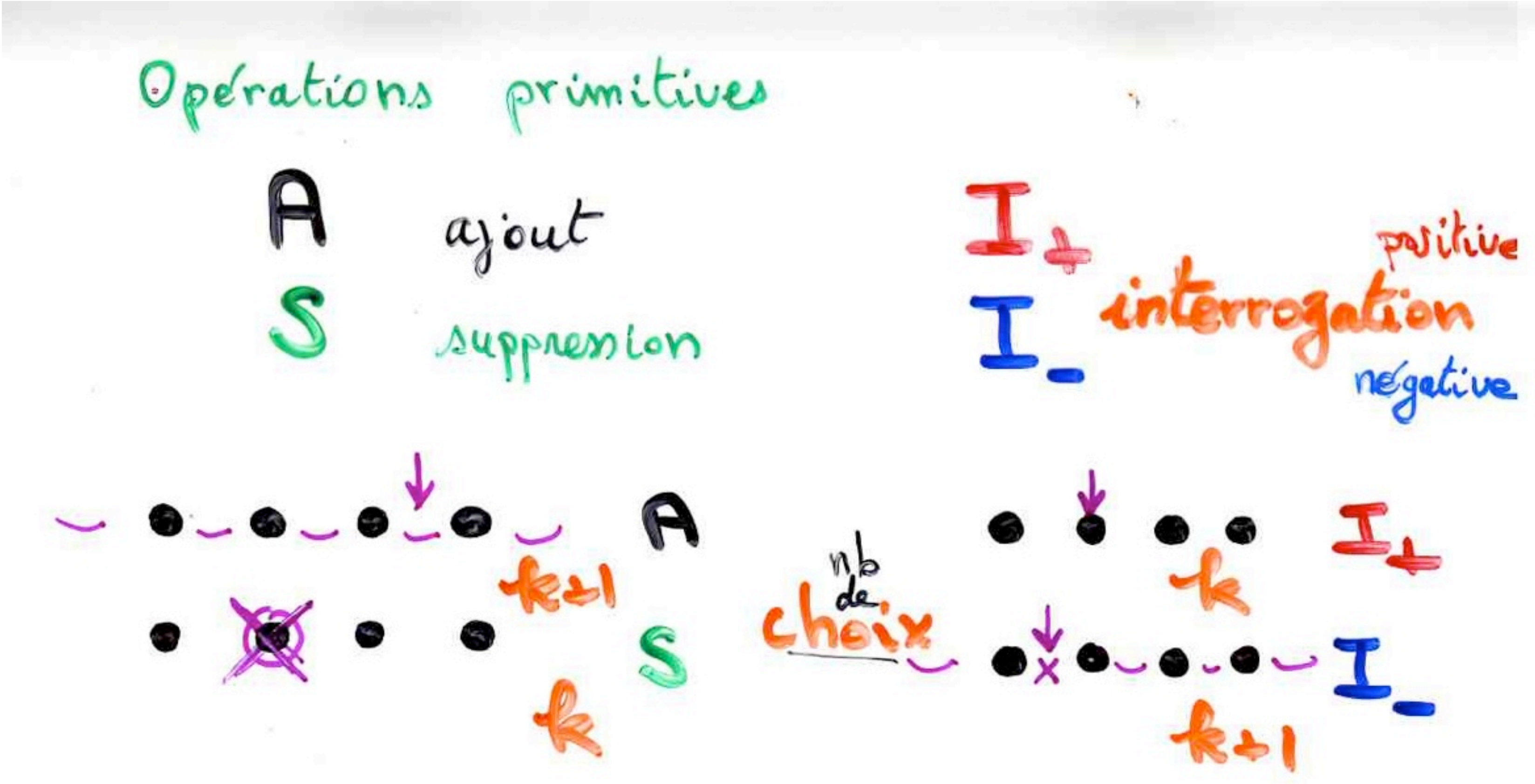


histoires de fichiers Françon, (1976)

Data structure histories

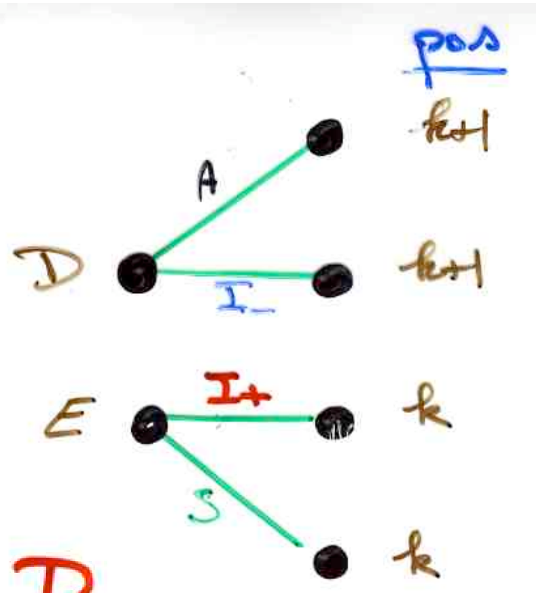


Primitive operations
for “dictionnaries” data structure:
add or delete any elements, asking questions
(with positive or negative answer)



number of choices for each
primitive operations

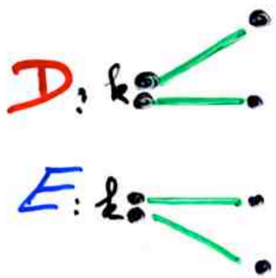
$$\begin{cases} D = A + I_- \\ E = S + I_+ \end{cases}$$



this corresponds to the n!
"restricted Laguerre histories"

$$DE = ED + E + D$$

aussi :

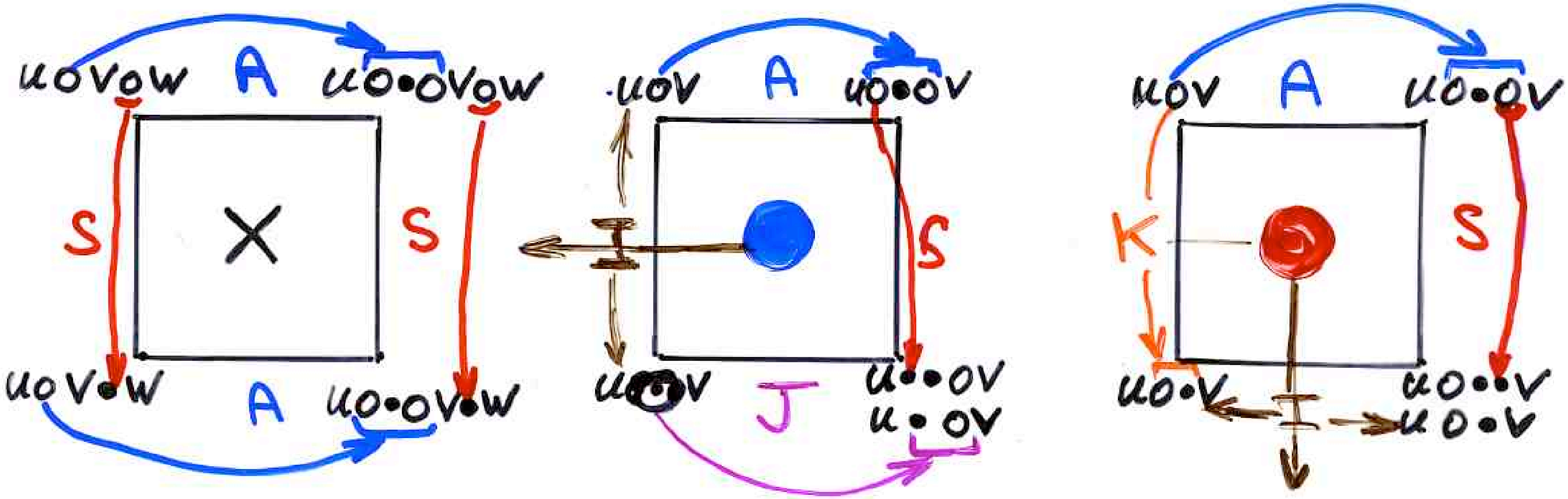


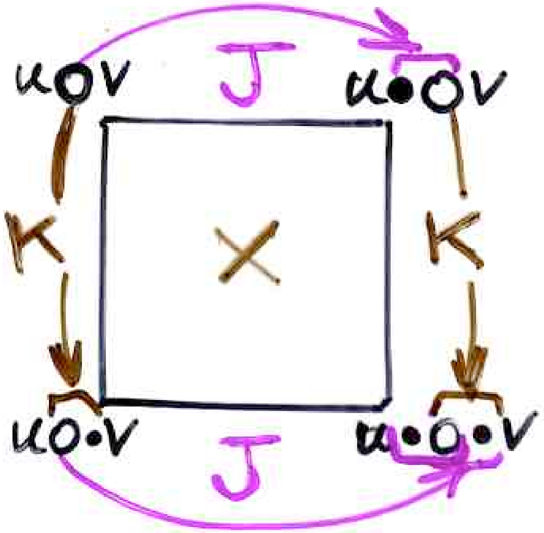
(k+1) possibilités partout

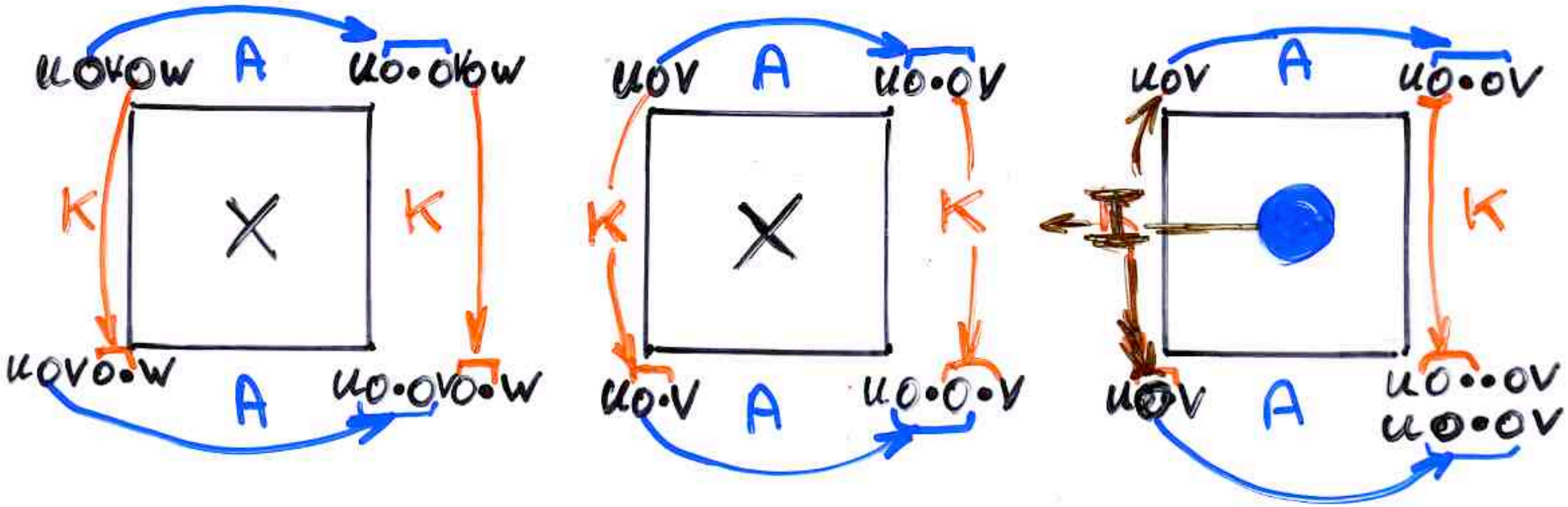
(histoires de Laguerre) "larges"

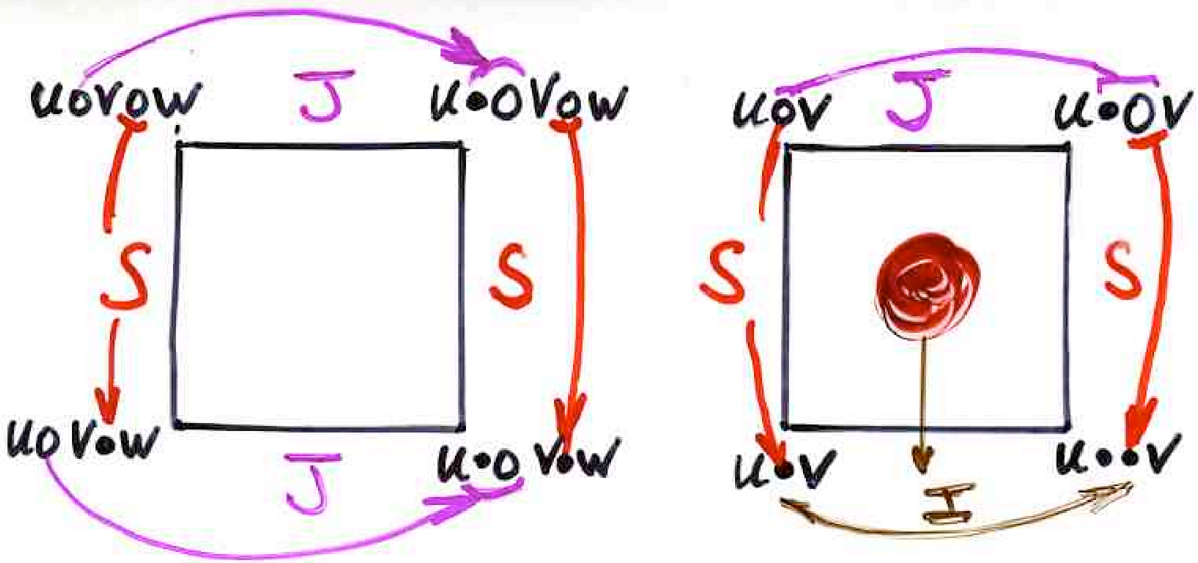
this valuation corresponds to the (n+1)!
"enlarged Laguerre histories"

§ 9 another bijection permutations alternative tableaux


A
S
S
A
A
S
J
A
K
S


J
K
K
J


A
K
A
K
A
K
I


J
S
J
S
I

[redacted]

416978352

●

● 1 ●

●1● 2 ●

●1●3●2

41● 3 ●2

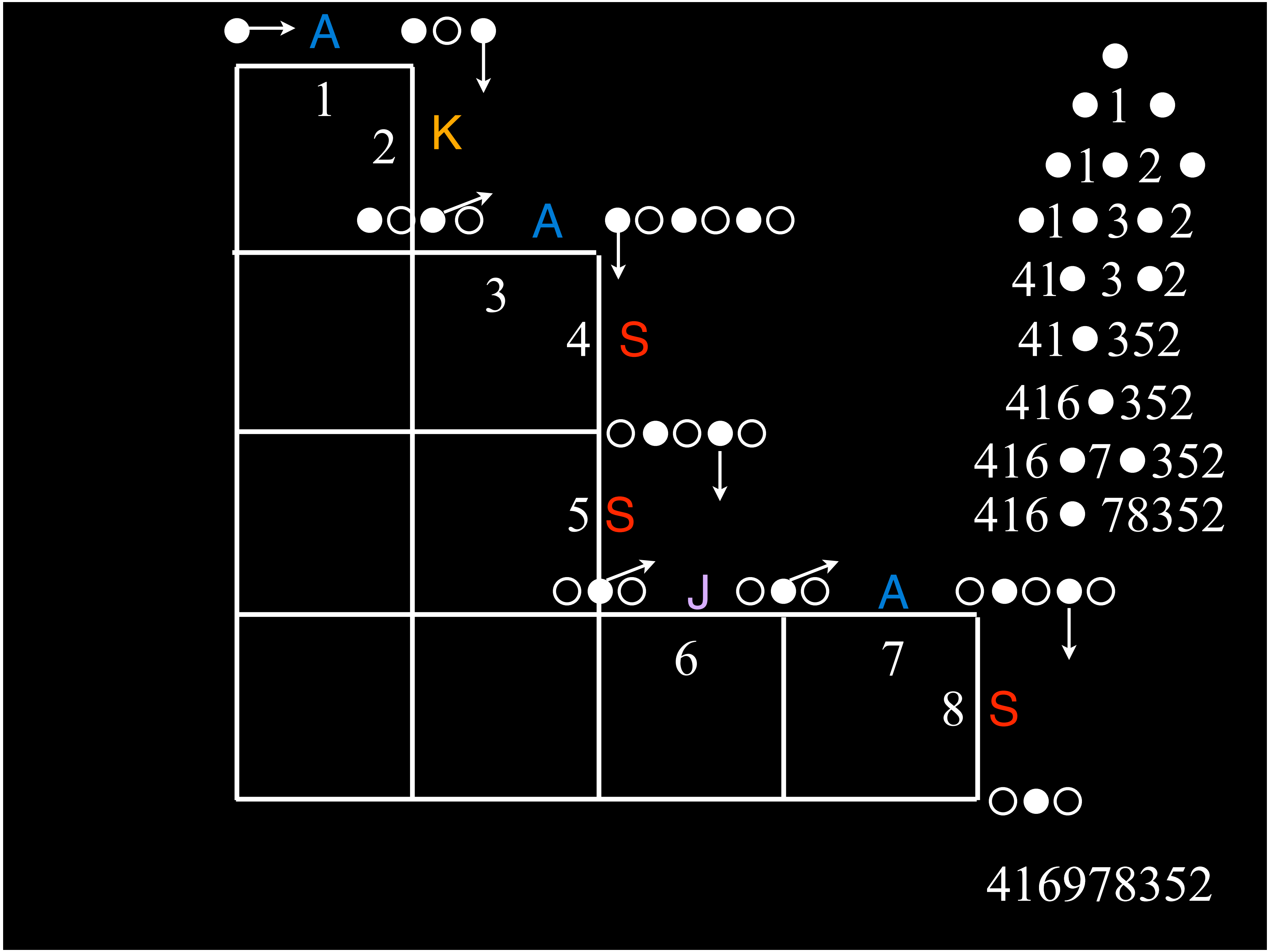
41●352

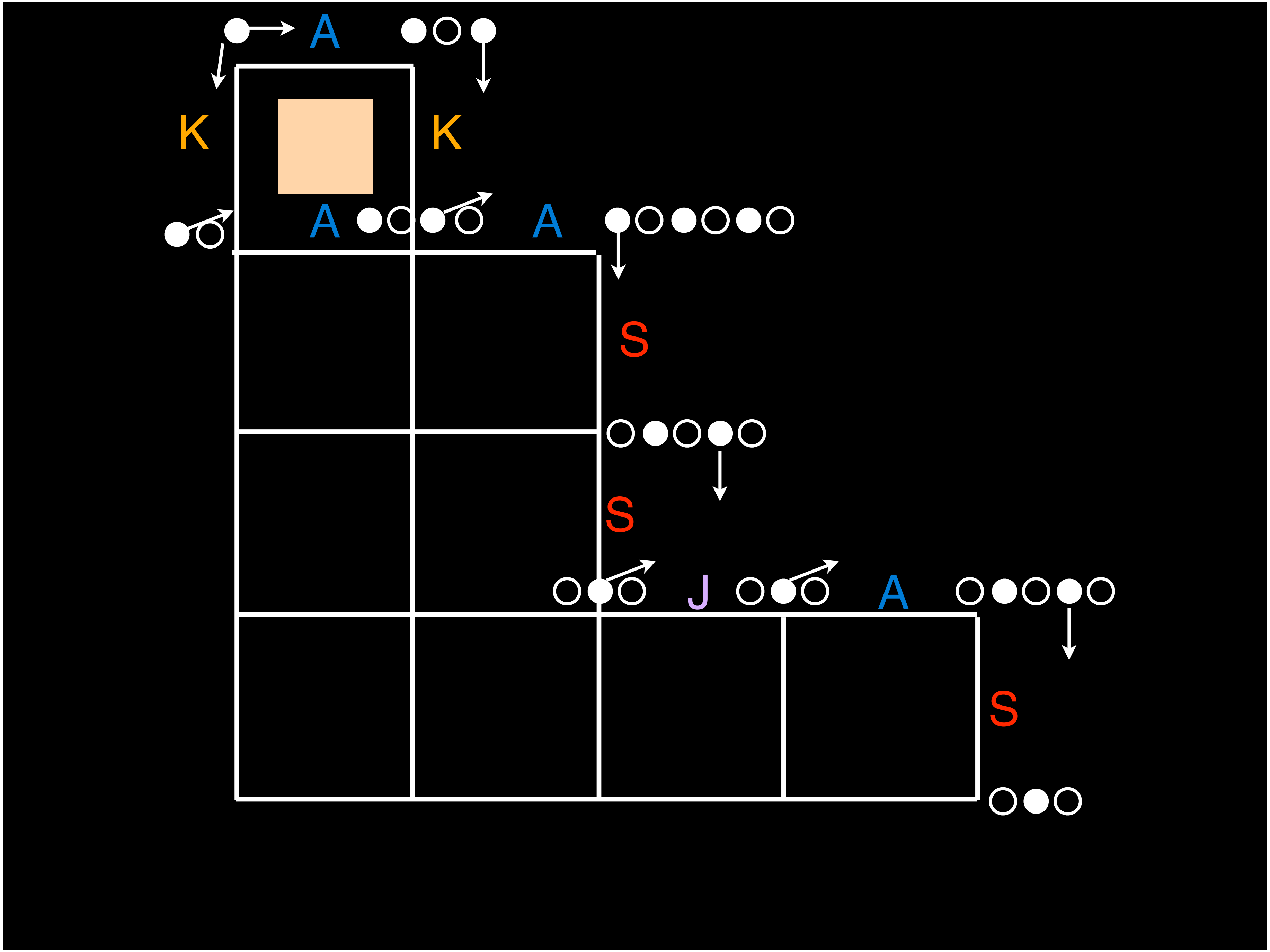
416●352

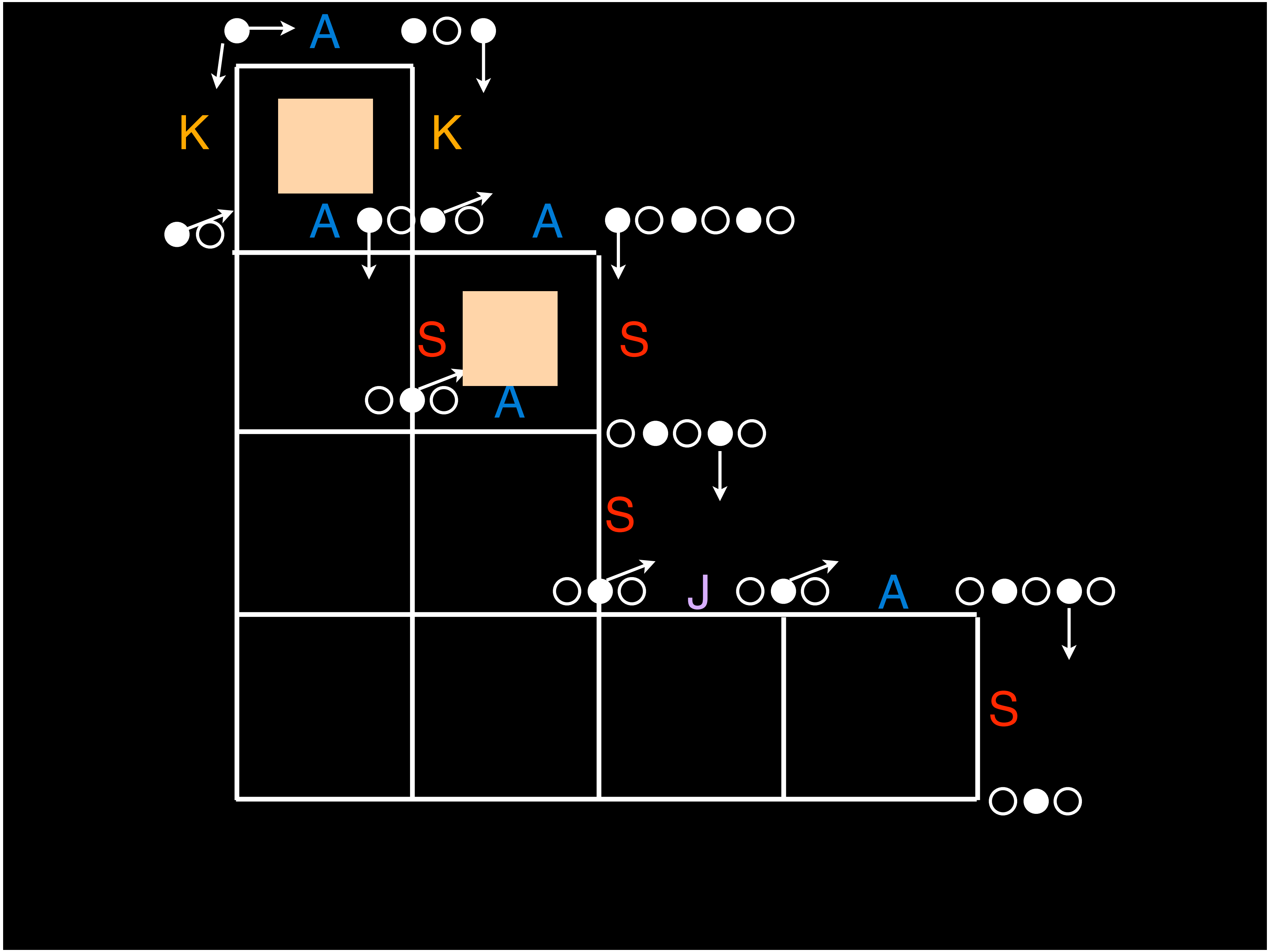
416 ●7●352

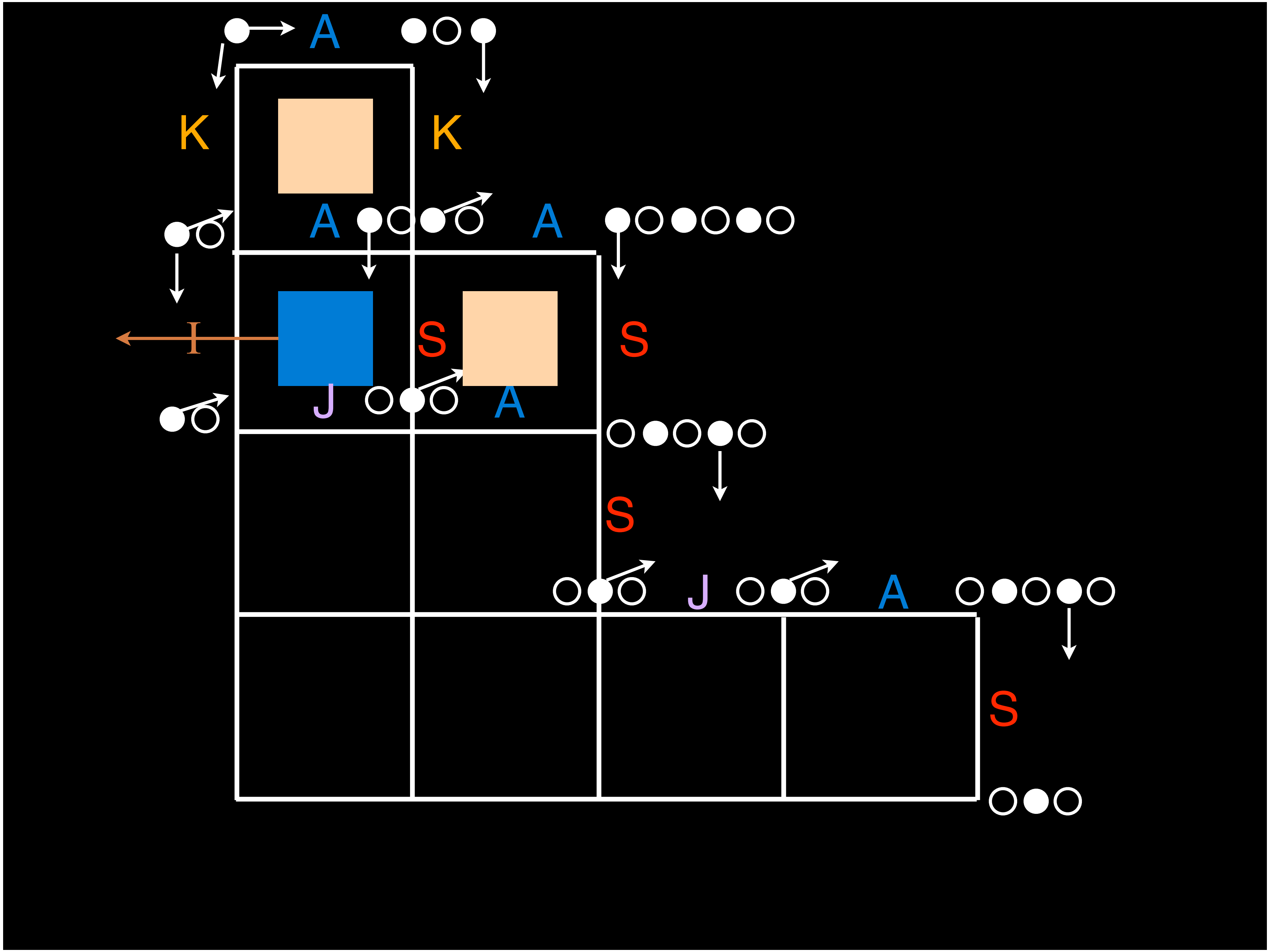
416 ● 78352

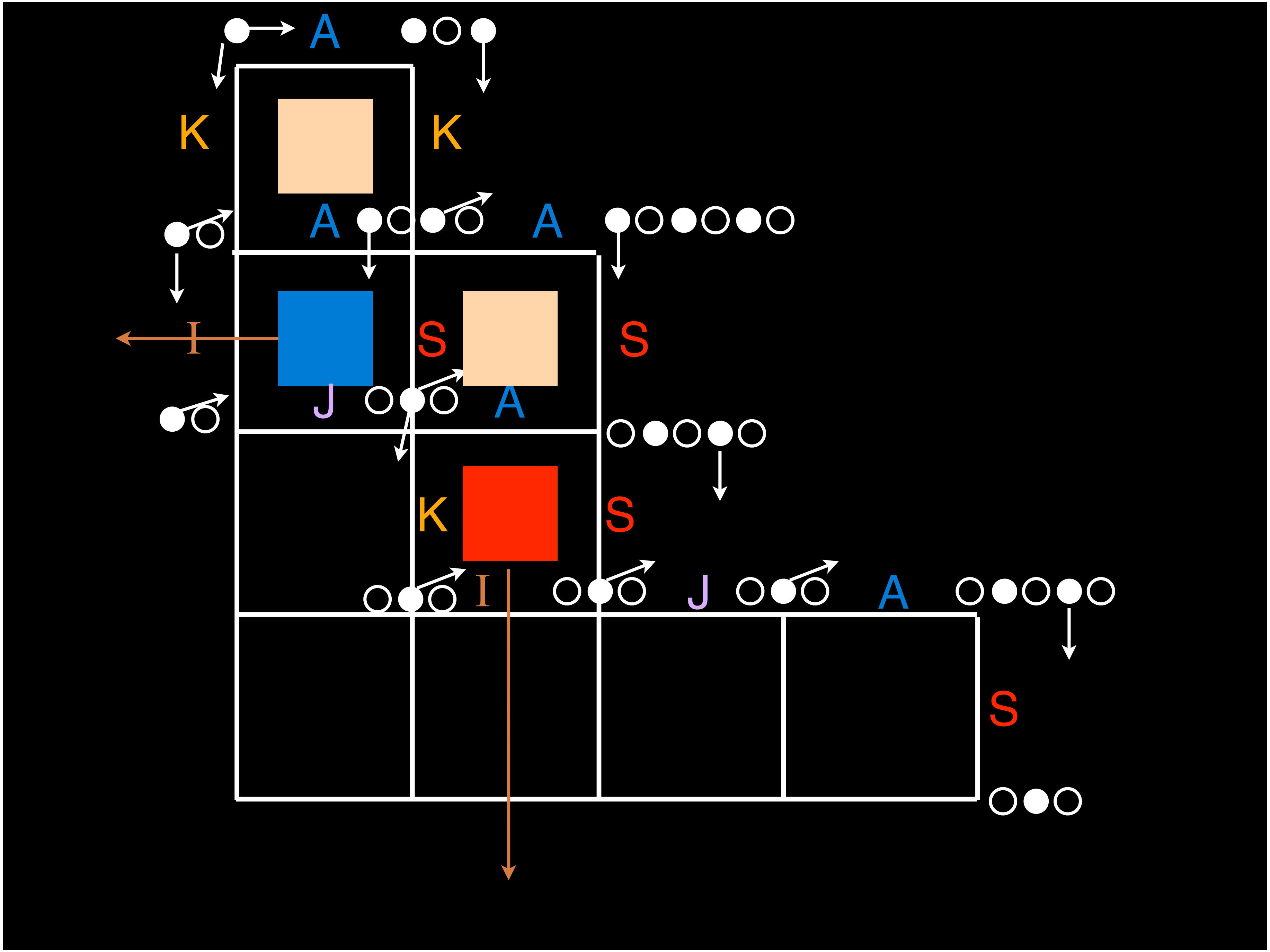
416978352


A
1
2
K
A
3
4
S
5
S
J
A
6
7
8
S
1
1 2
1 3 2
41 3 2
41 352
416 352
416 7 352
416 78352
416978352

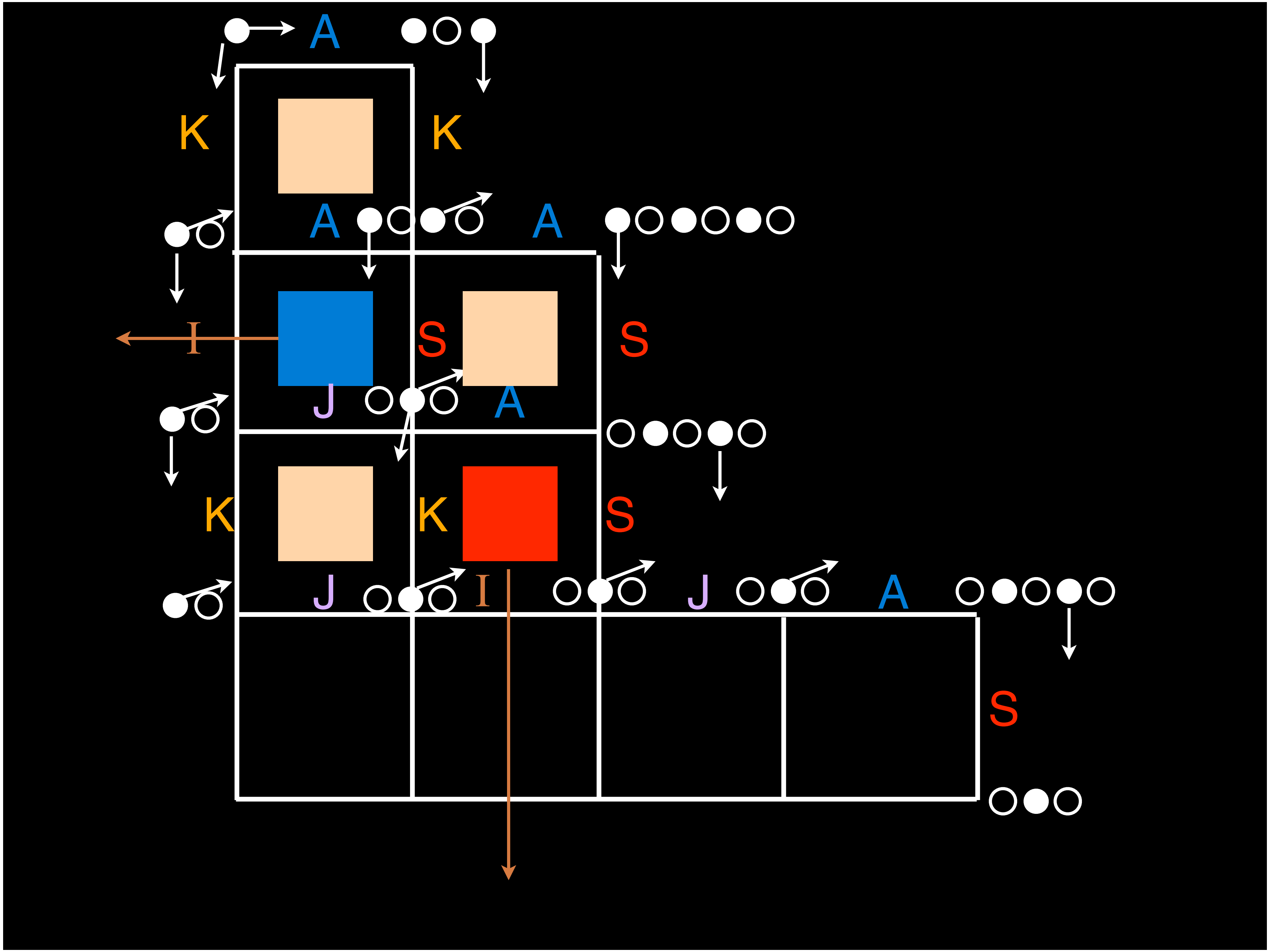

A
K
K
A
A
S
S
J
A
S


A
K
K
A
A
S
S
A
S
J
A
S


A
K
K
A
A
S
S
J
A
S
J
A
S



A
K
K
A
A
I
S
S
J
A
K
S
I
J
A
S



A
K
K
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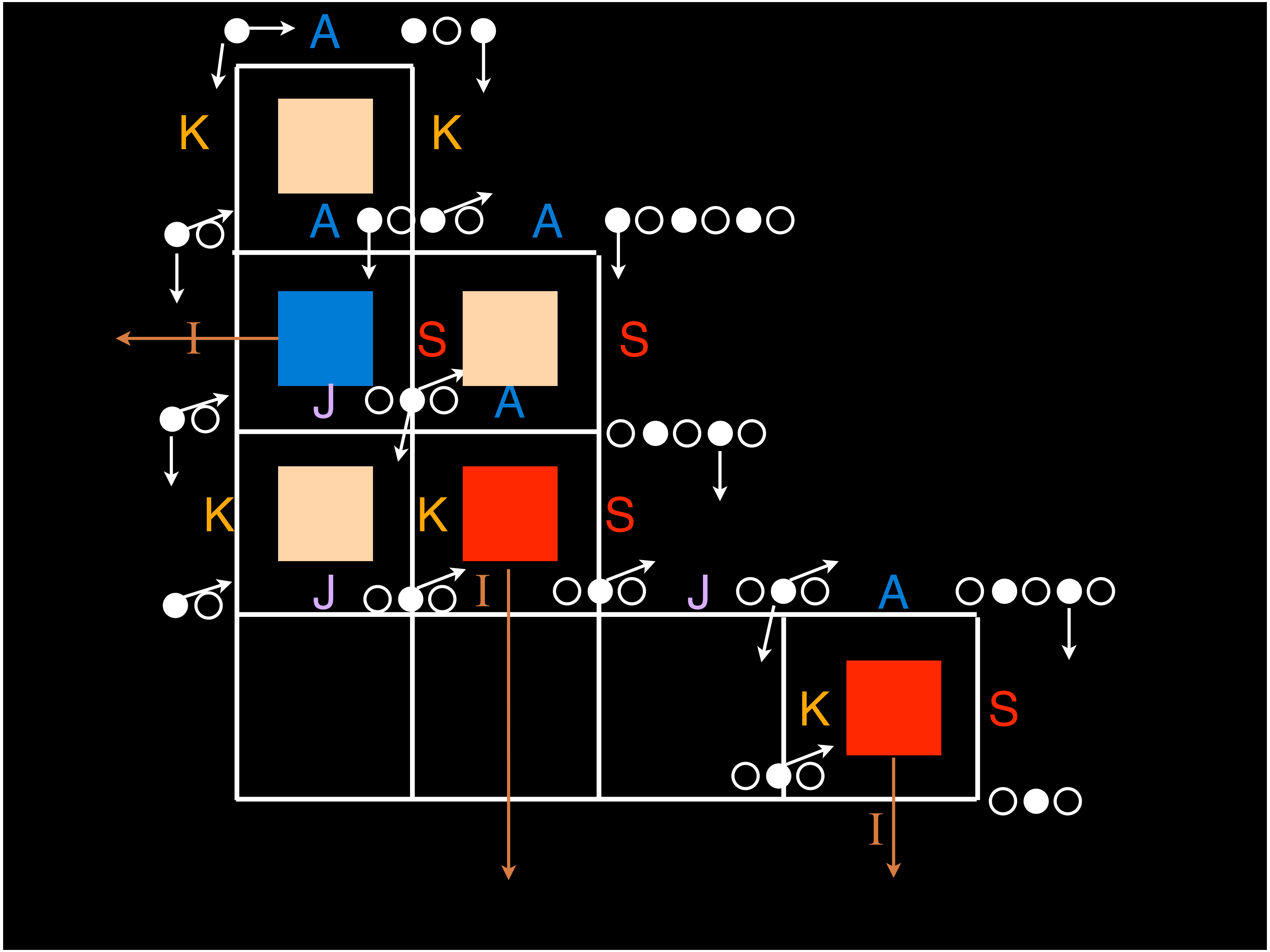
A
I
S
S
J
A
K
K
S
J
I
J
A
S


A
K
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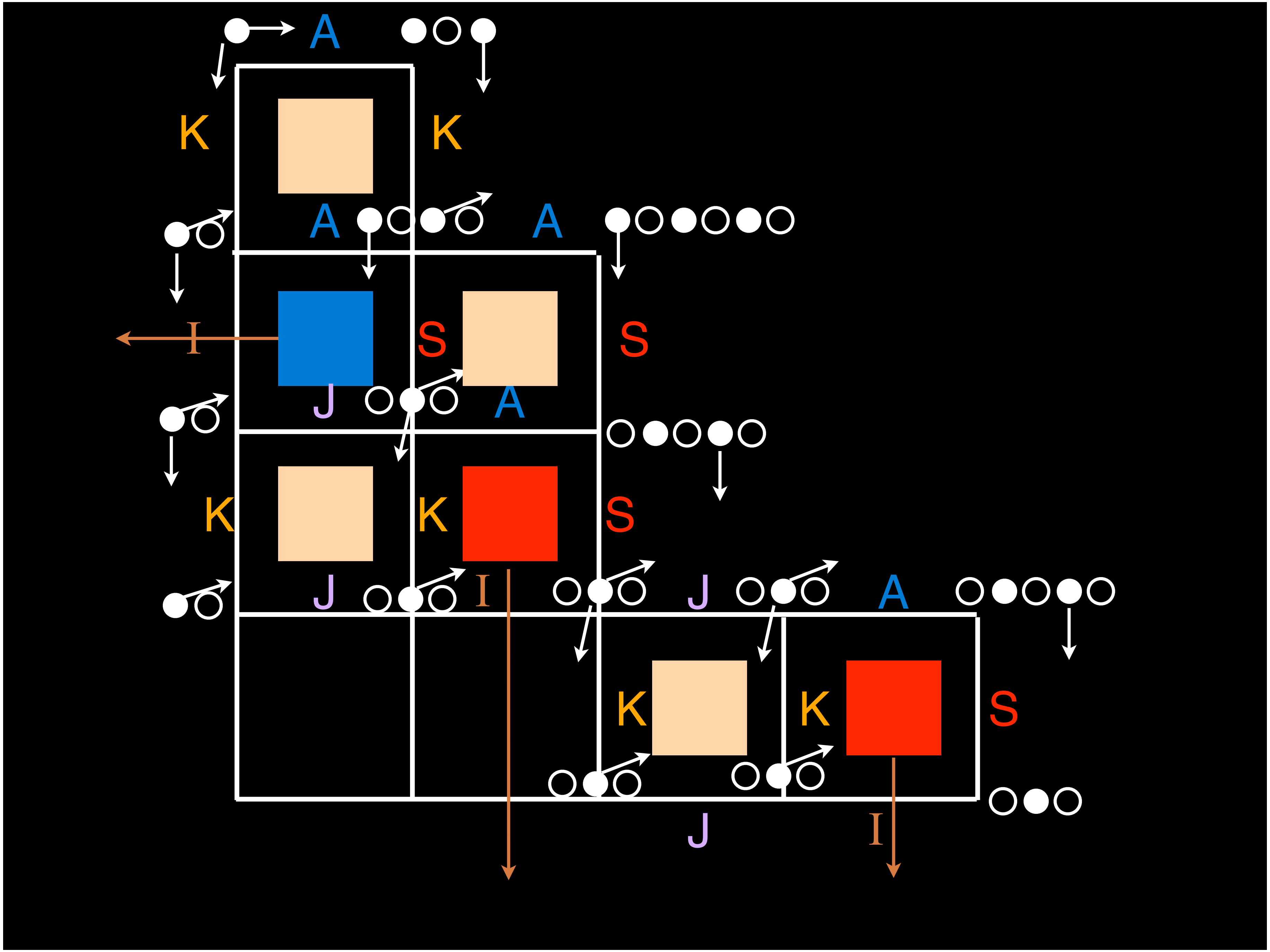
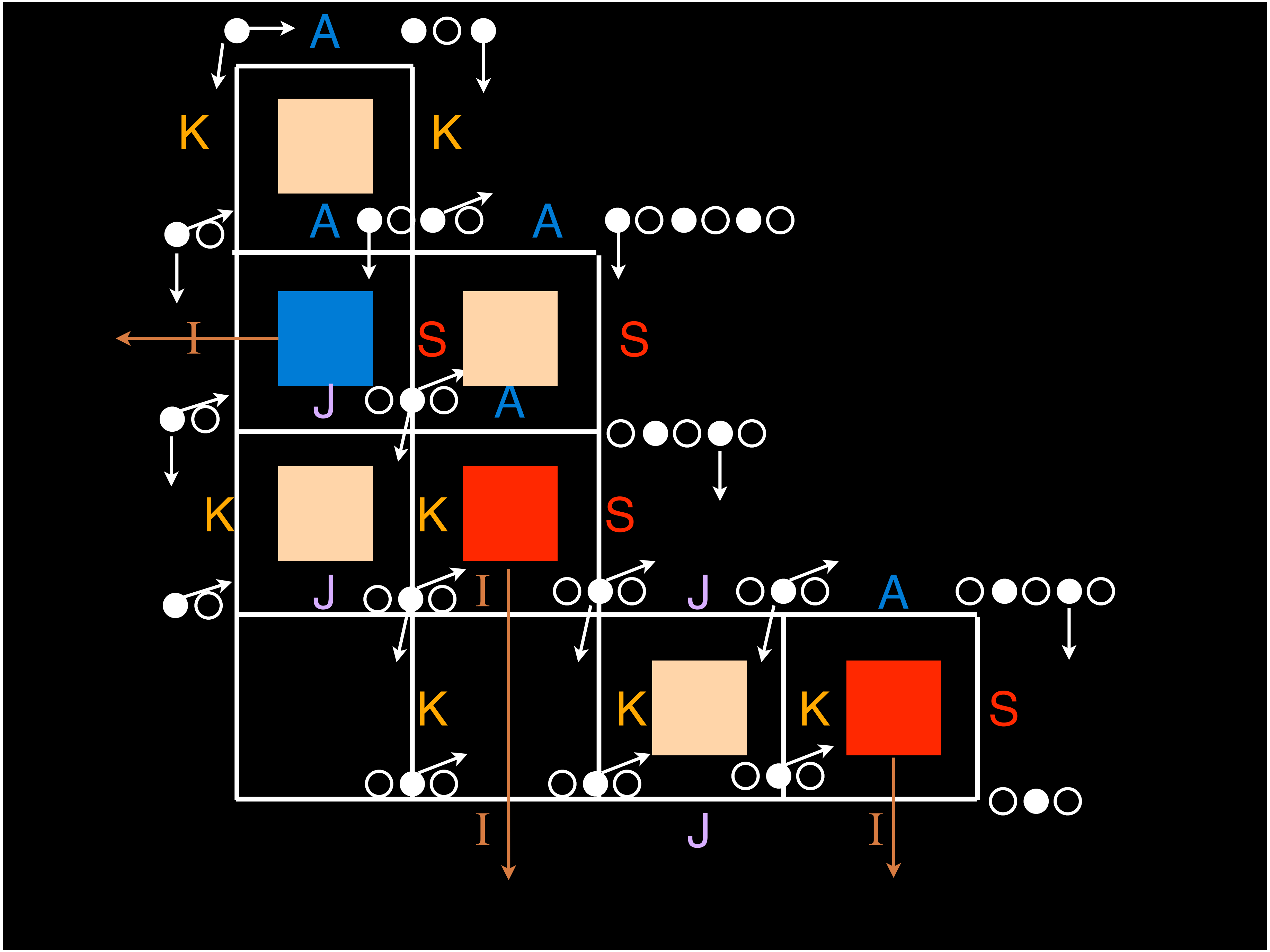
A
I
S
S
J
A
K
K
S
J
I
J
A
K
S
I


A
K
K
A
A
I
S
S
J
A
K
K
S
J
I
J
A
K
K
S
J
I


A
K
K
A
A
I
S
S
J
A
K
K
S
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I
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A
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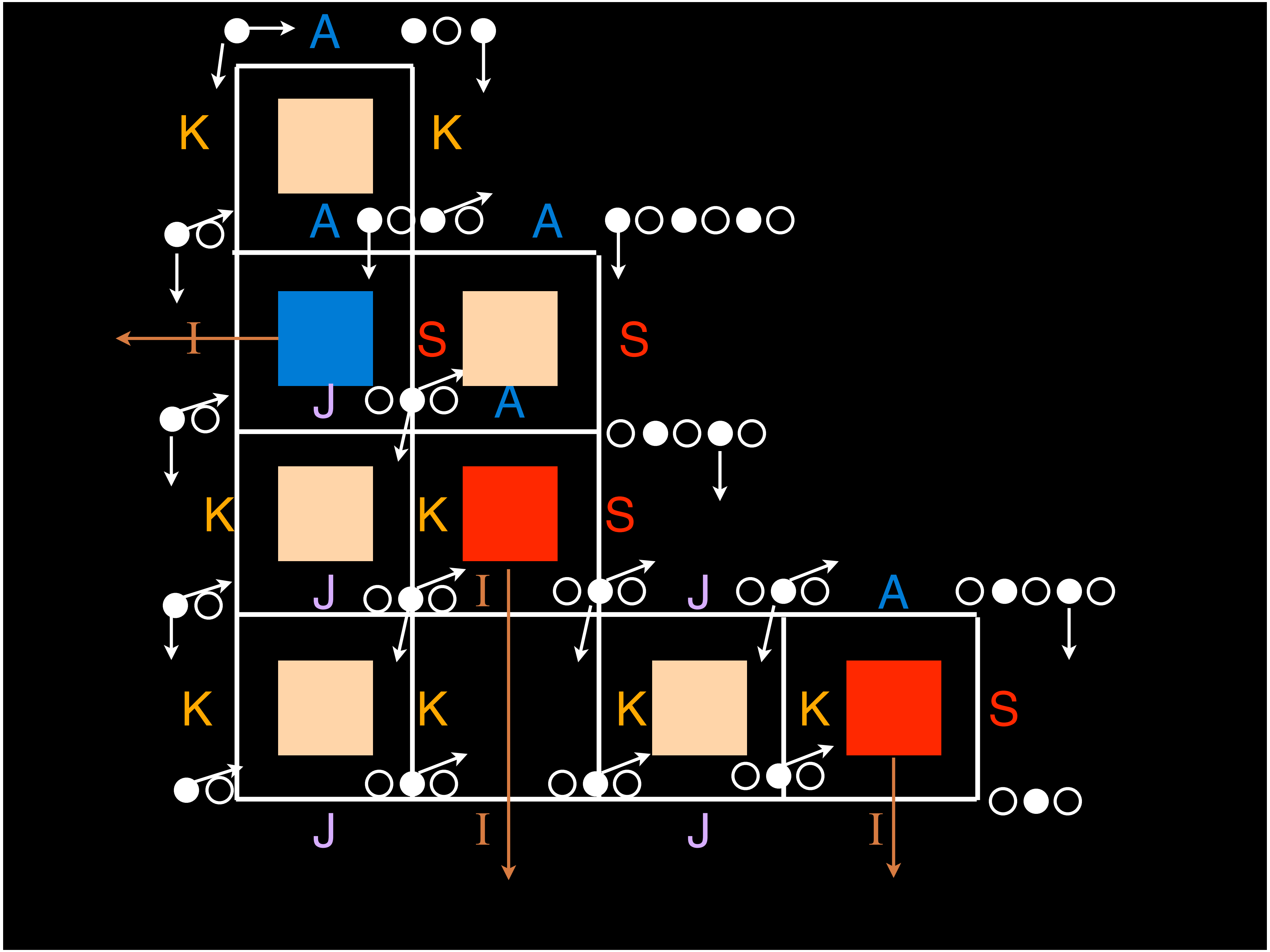
K
S
I
J
I


A
K
K
A
A
I
S
S
J
A
K
K
S
J
I
J
A
K
K
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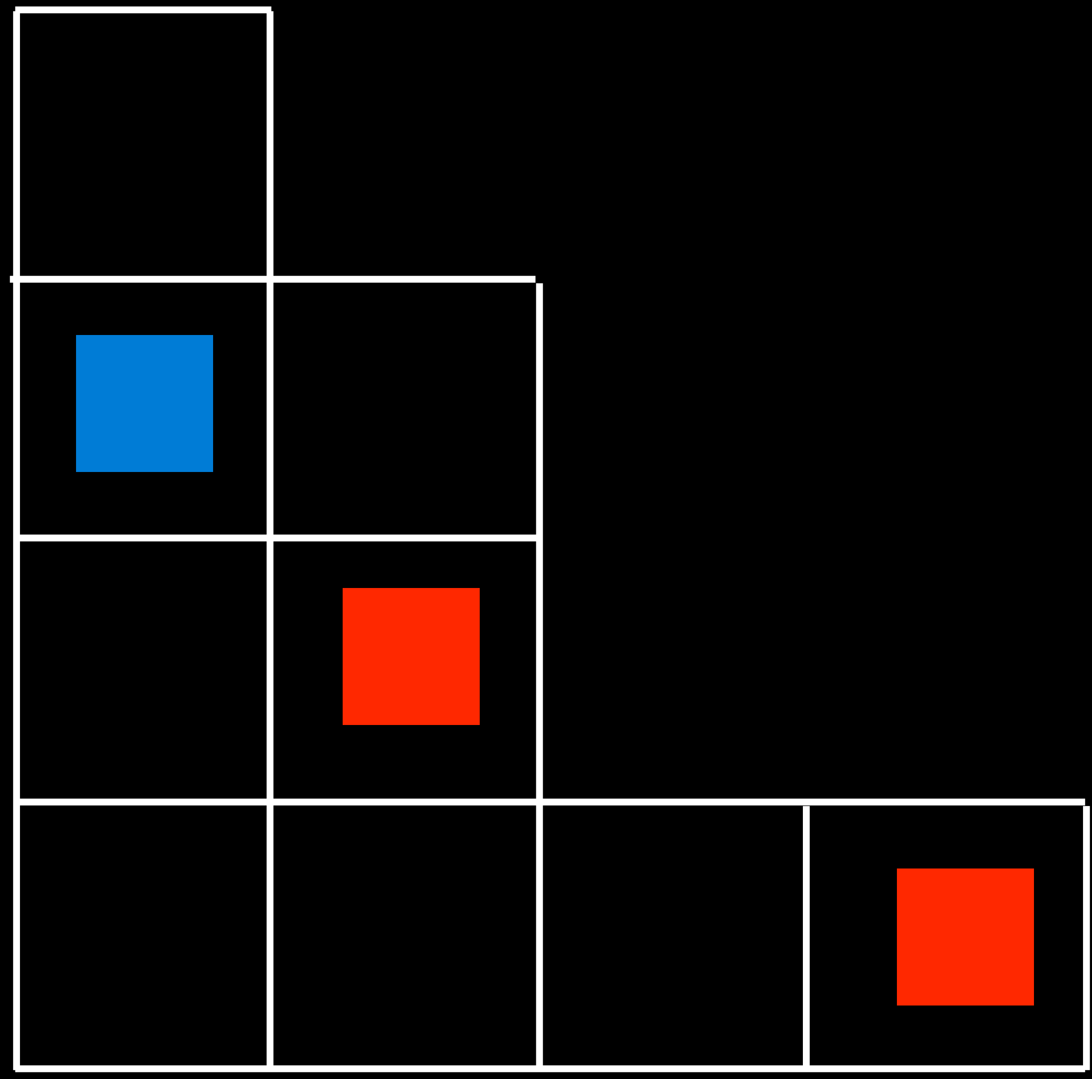
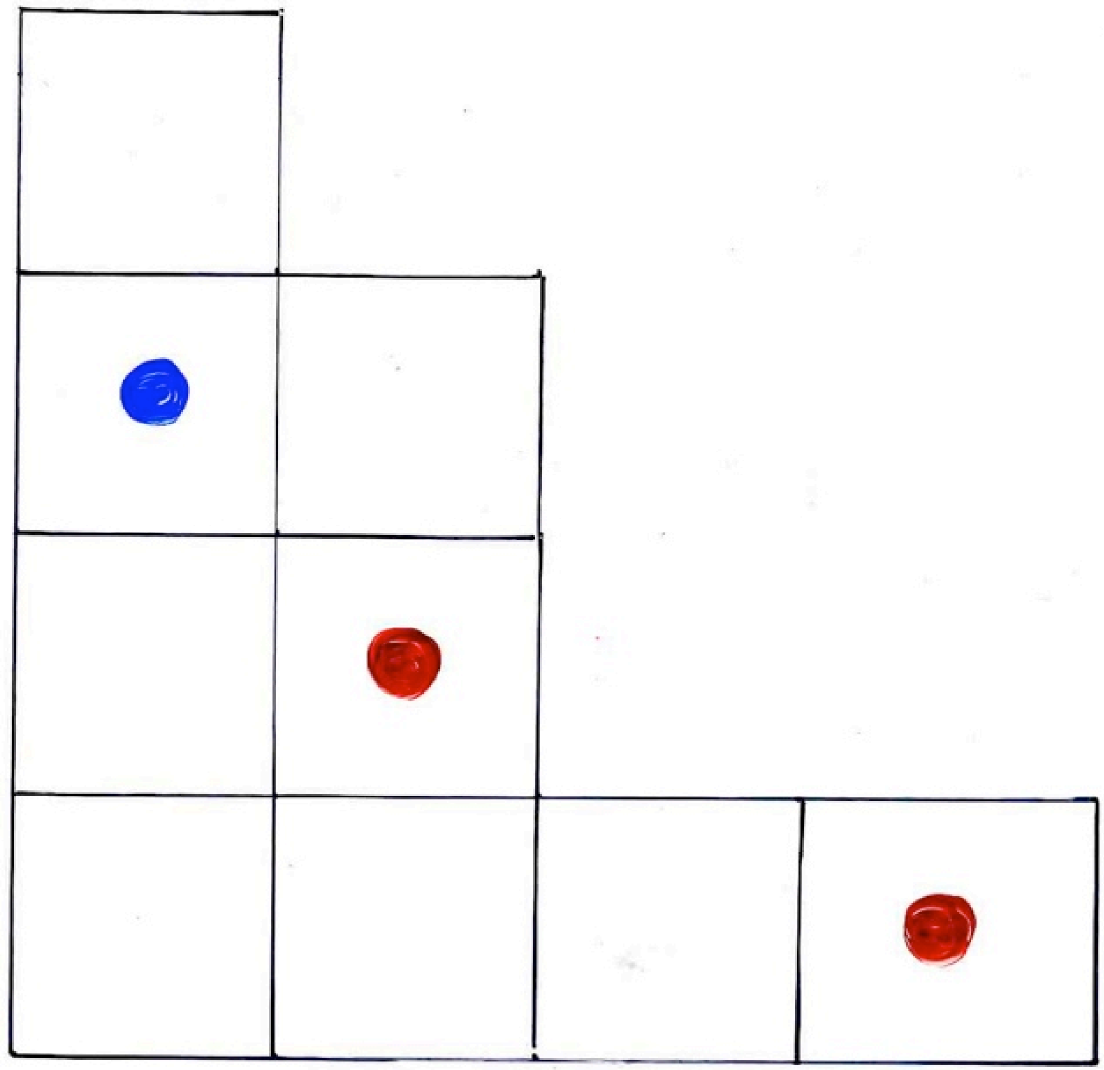
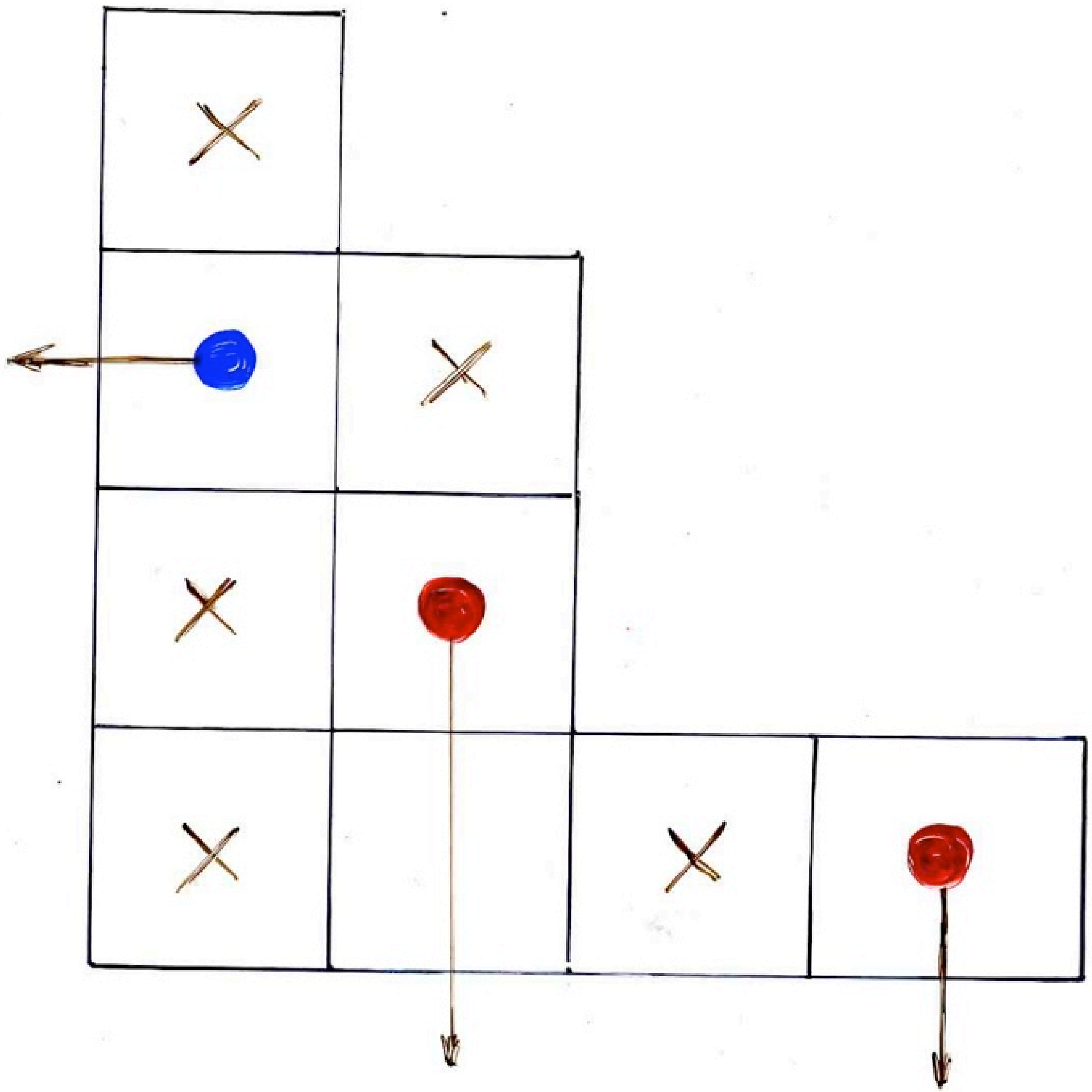
K
S
J
I
J
I

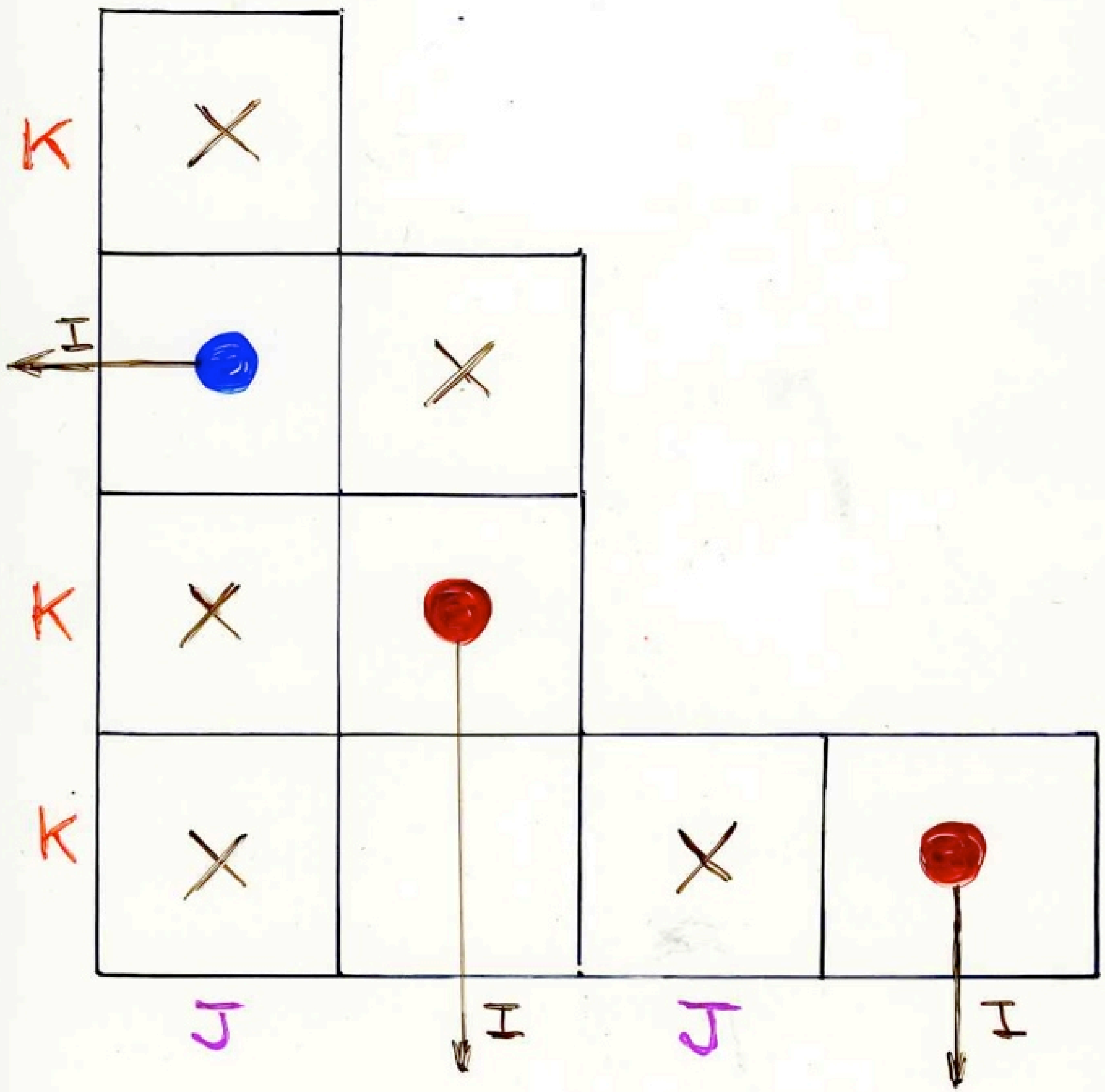


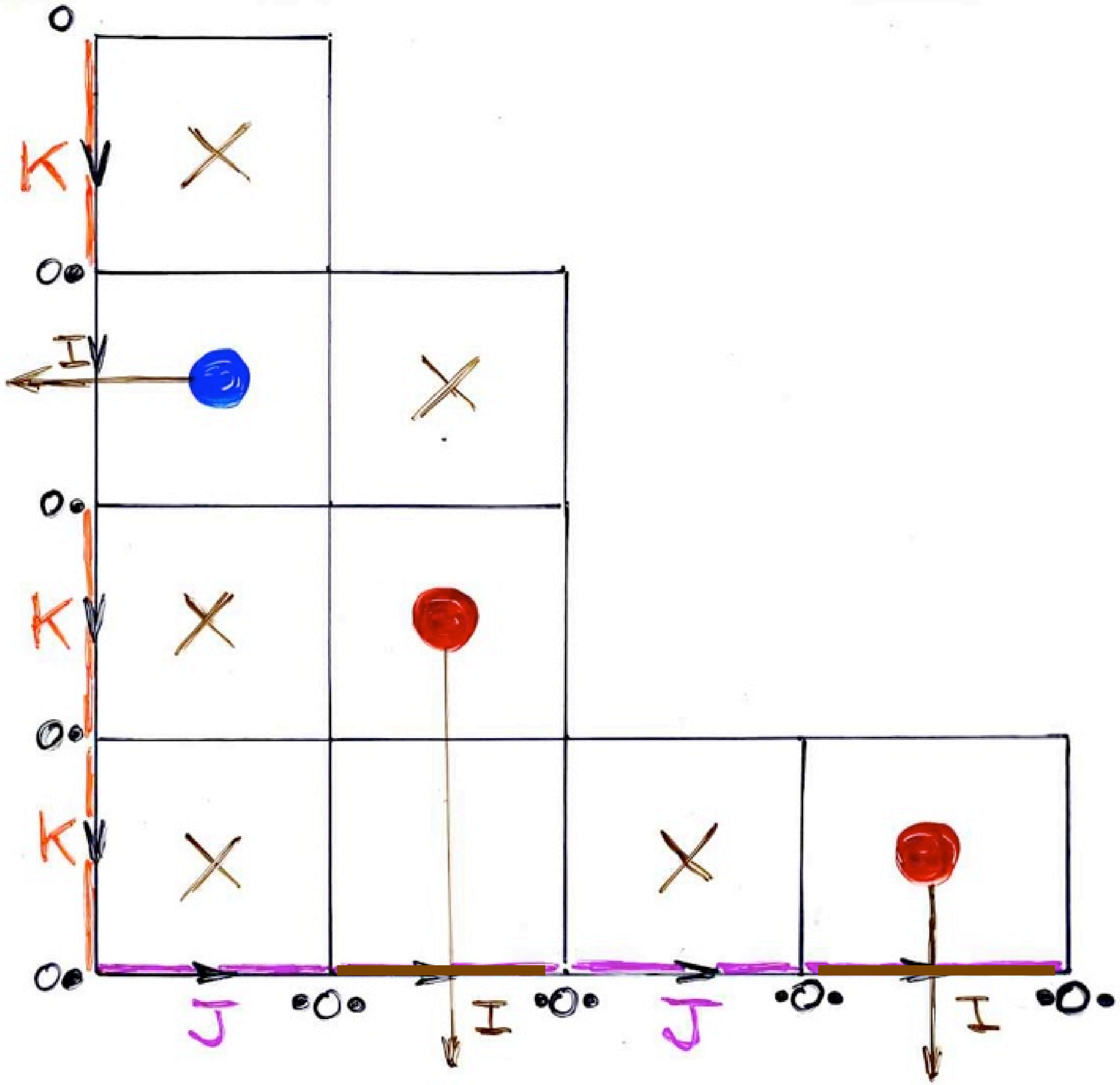
416978352

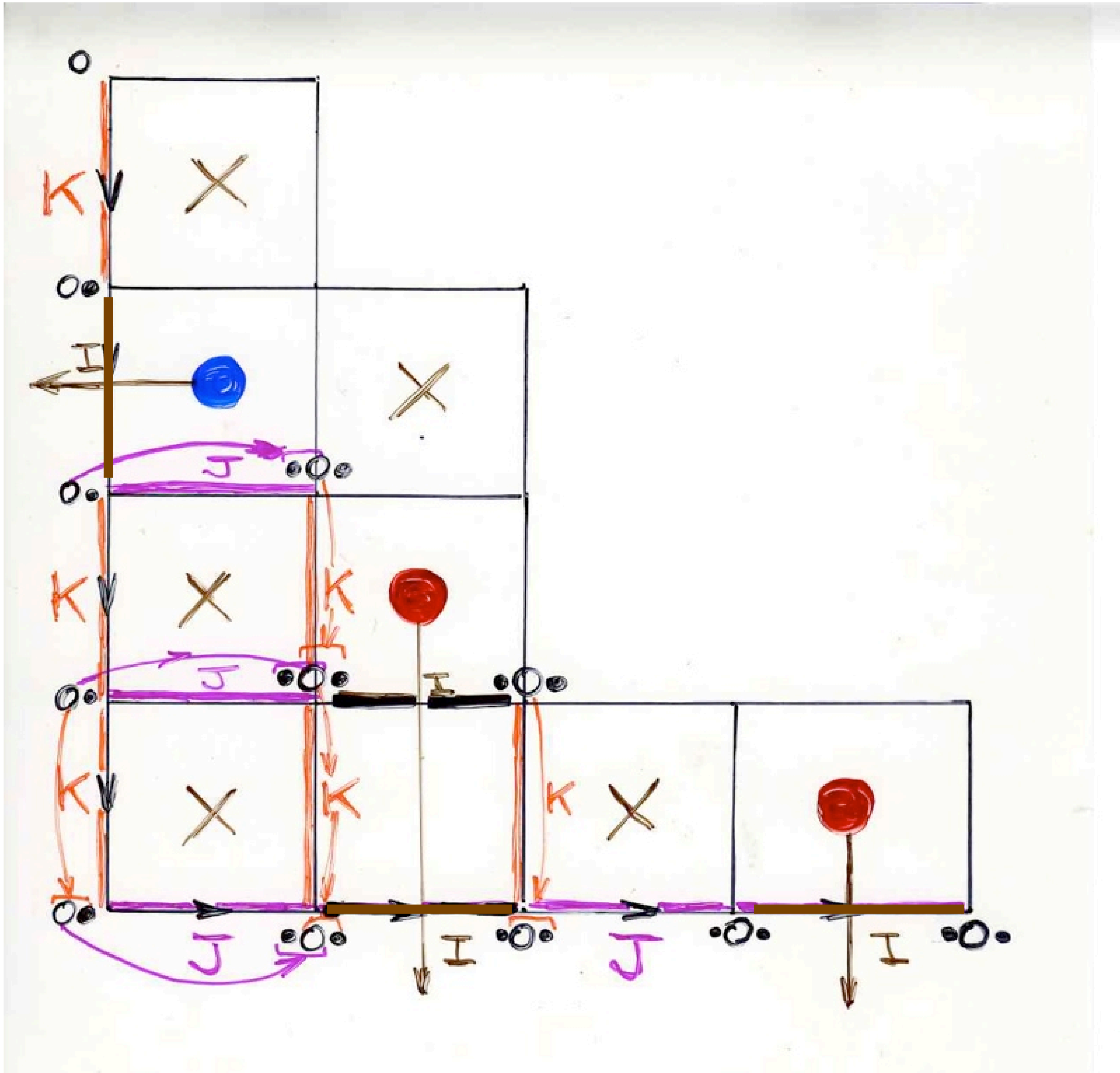
inverse bijection



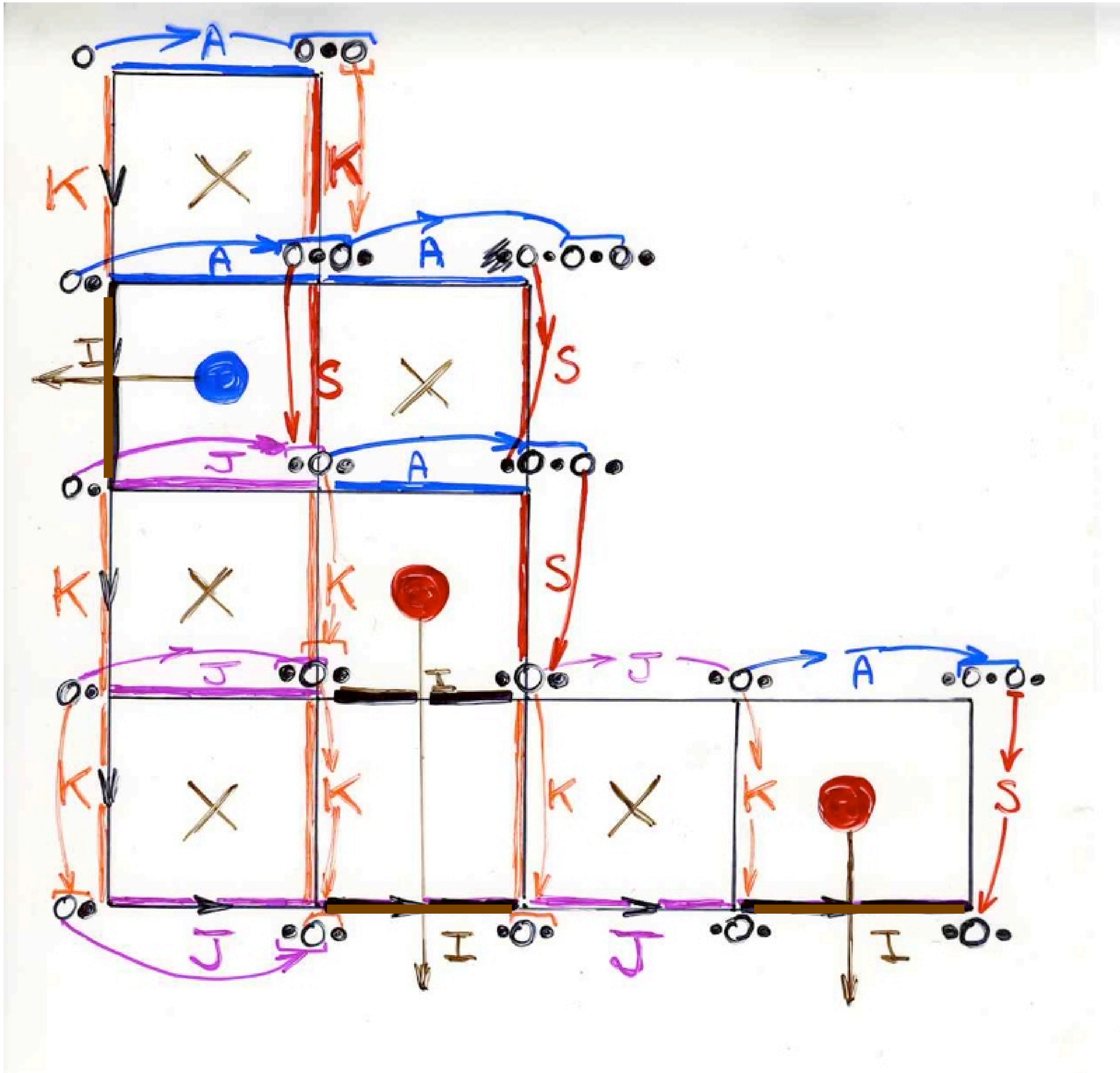



K
I
K
K
J
I
J
I

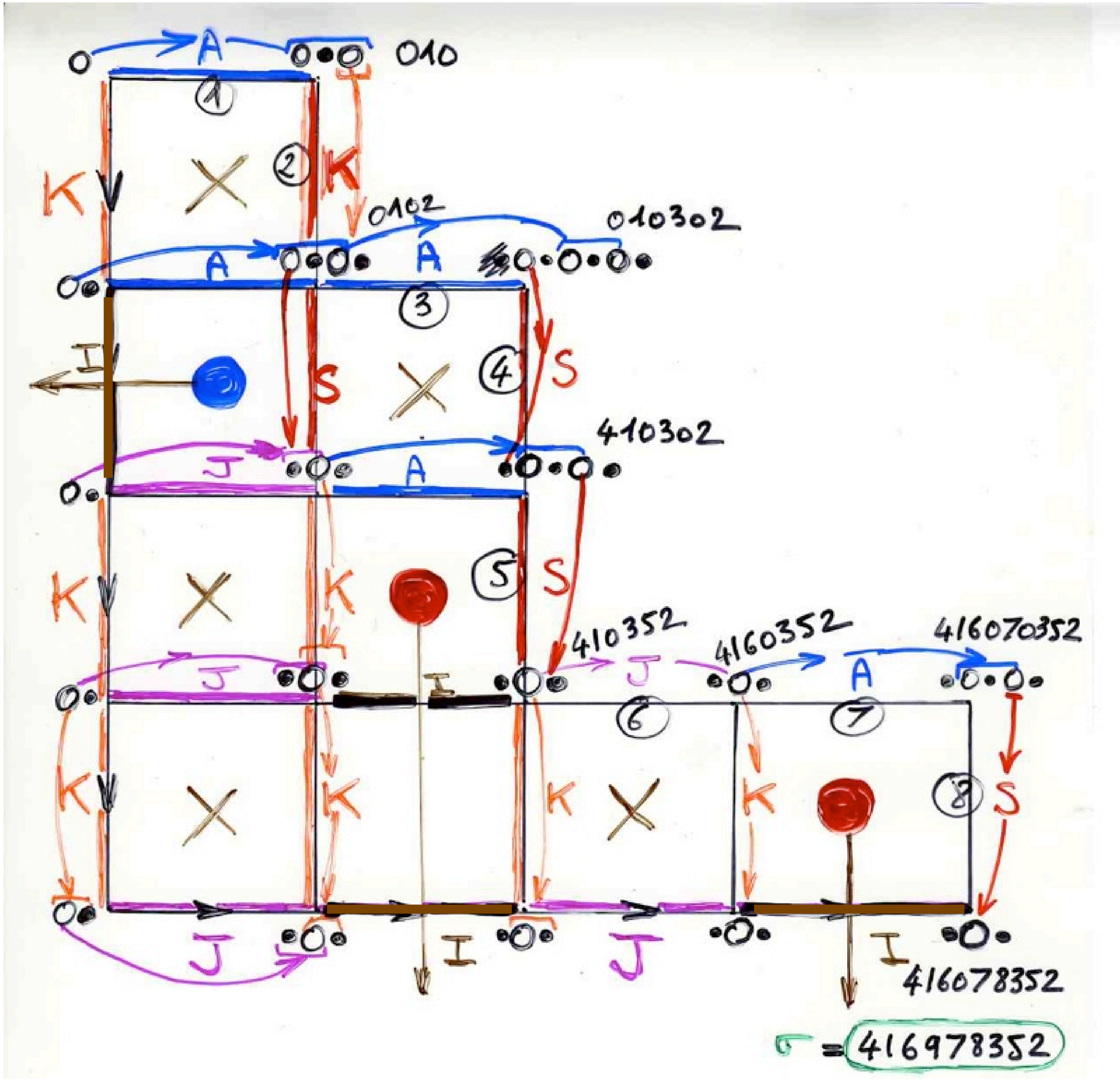

K
I
K
K
J
I
J
I



O
K
I
J
K
K
J
I
K
K
K
J
I
J
I



A
K
K
A
A
I
S
S
J
A
K
K
S
J
J
A
K
K
K
K
S
J
I
J
I


A
010
K
K
1
2
0102
010302
A
A
3
I
S
4
S
410302
J
A
K
K
5
S
410352
4160352
416070352
J
J
A
I
6
7
K
K
K
K
K
S
8
J
I
J
I
416078352
σ = 416978352

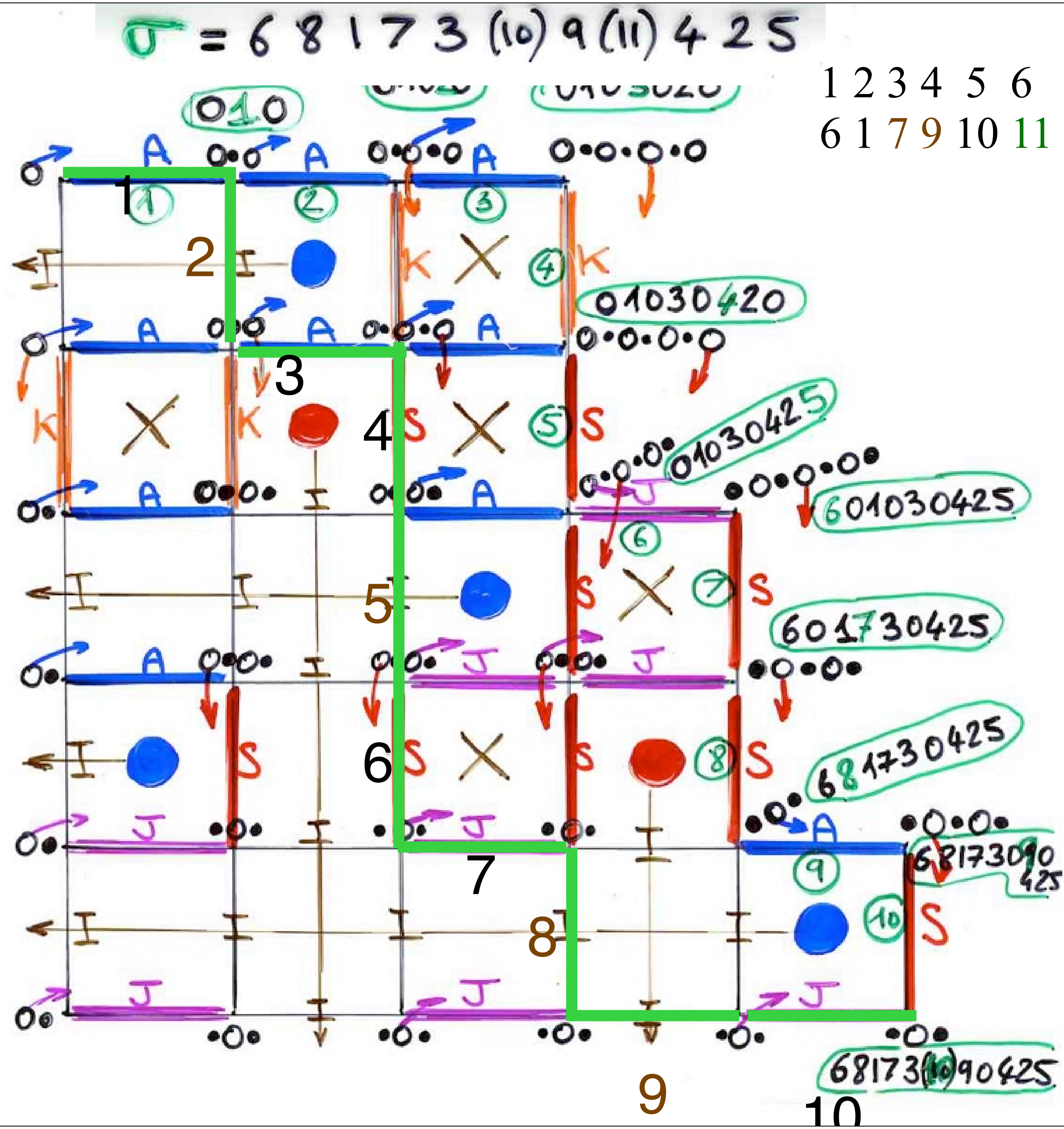
Two bijections
one theorem

Prop.

T alternative tableau → σ "exchange-fusion" inverse algorithm

T alternative tableau → τ "local" algorithm from $DE = ED + E + D$

$$\sigma = \tau^{-1}$$



1 2 3 4 5 6 7 8 9 10 11

6 1 7 9 10 11 8 5 3 4 2 = S

P. Nadeau notice that , the (first) bijection described by him and S.Corteel (published in European J. of Combinatorics) between permutation tableaux and permutations, is equivalent to a “column insertion” in the algorithm presented here with “local rules”, up to transforming permutation tableaux into alternating tableaux and taking complements mirror image of the permutation constructed by “local rules” (which is the inverse of the permutation used in the “exchange-delete” algorithm).

§ 10

some

parameters

permutation tableaux

- nb of unrestricted rows
- nb of 1's in the first row

$\longleftrightarrow$

alternative tableaux

- nb of rows without ●
- nb of columns without ●

Corteel (2006)

$$T_n(x,y) = \prod_{i=0}^{n-1} (x+y+i)$$

- RL- minima
- LR- minima

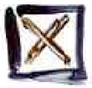
bijection Corteel-Nadeau (2007)

permutation tableaux ⟷ permutation

- profile ⟷ • (rises, descents)
- ~~nb~~ of unrestricted rows ⟷ RL- minimum
- nb of "superfluous" 1 ⟷ nb of occurences of (31-2)

↕

alternative tableaux

- profile
- nb of rows without 
- nb of cells ~~without~~ 

The “exchange-fusion” algorithm

An alternative description of the bijection

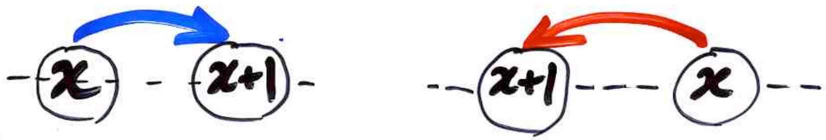
alternative tableaux -- permutations

Déf – Permutation $\sigma = \sigma(1) \cdots \sigma(n)$

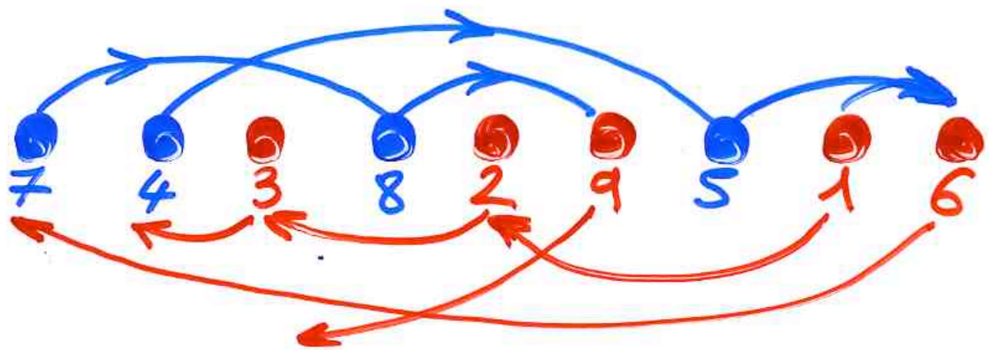
$x = \sigma(i)$, $1 \leq x < n$

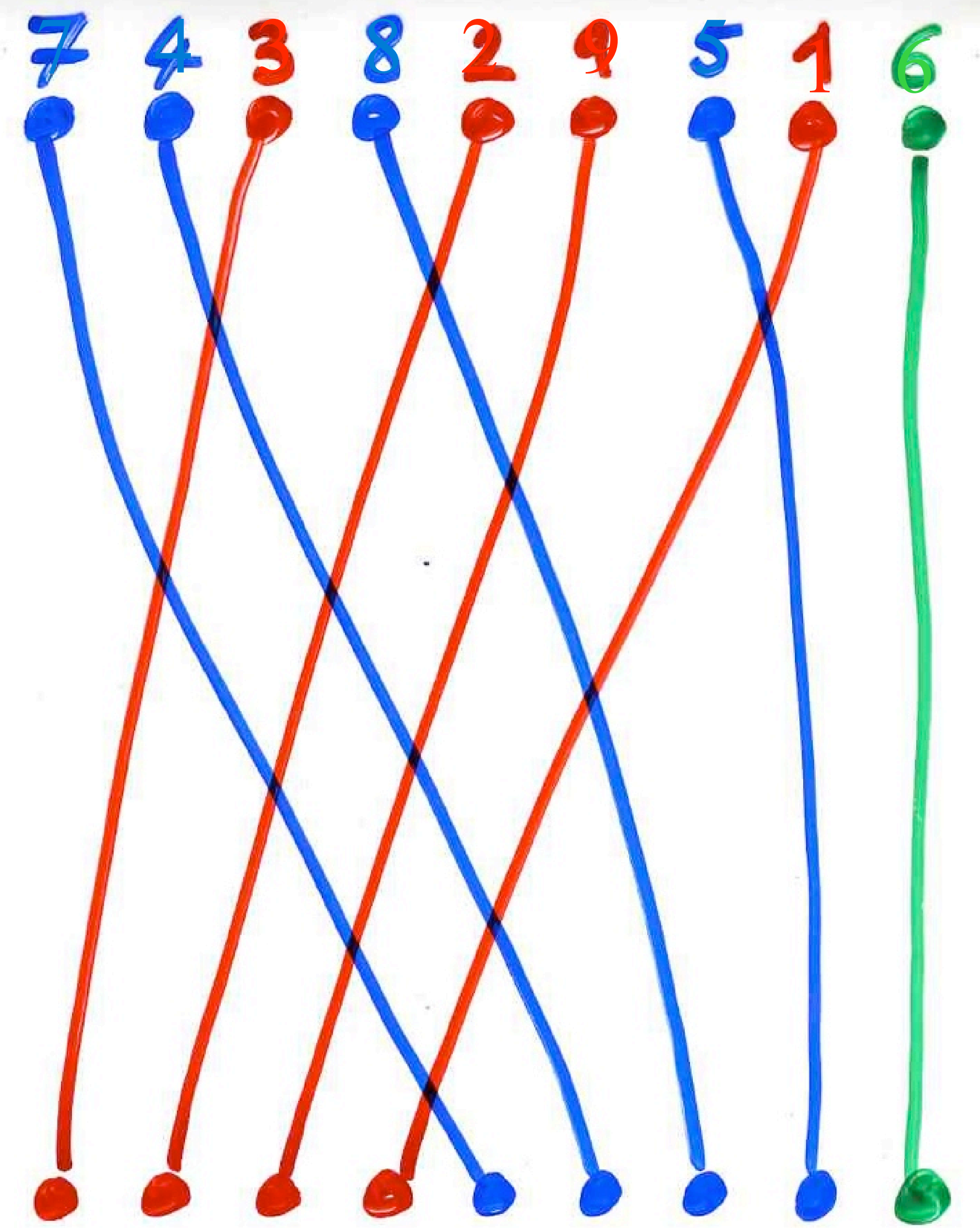
(valeur) x $\begin{cases} \text{avance} \\ \text{recul} \end{cases}$ $x+1 = \sigma(j)$, $\begin{cases} i<j \\ j<i \end{cases}$

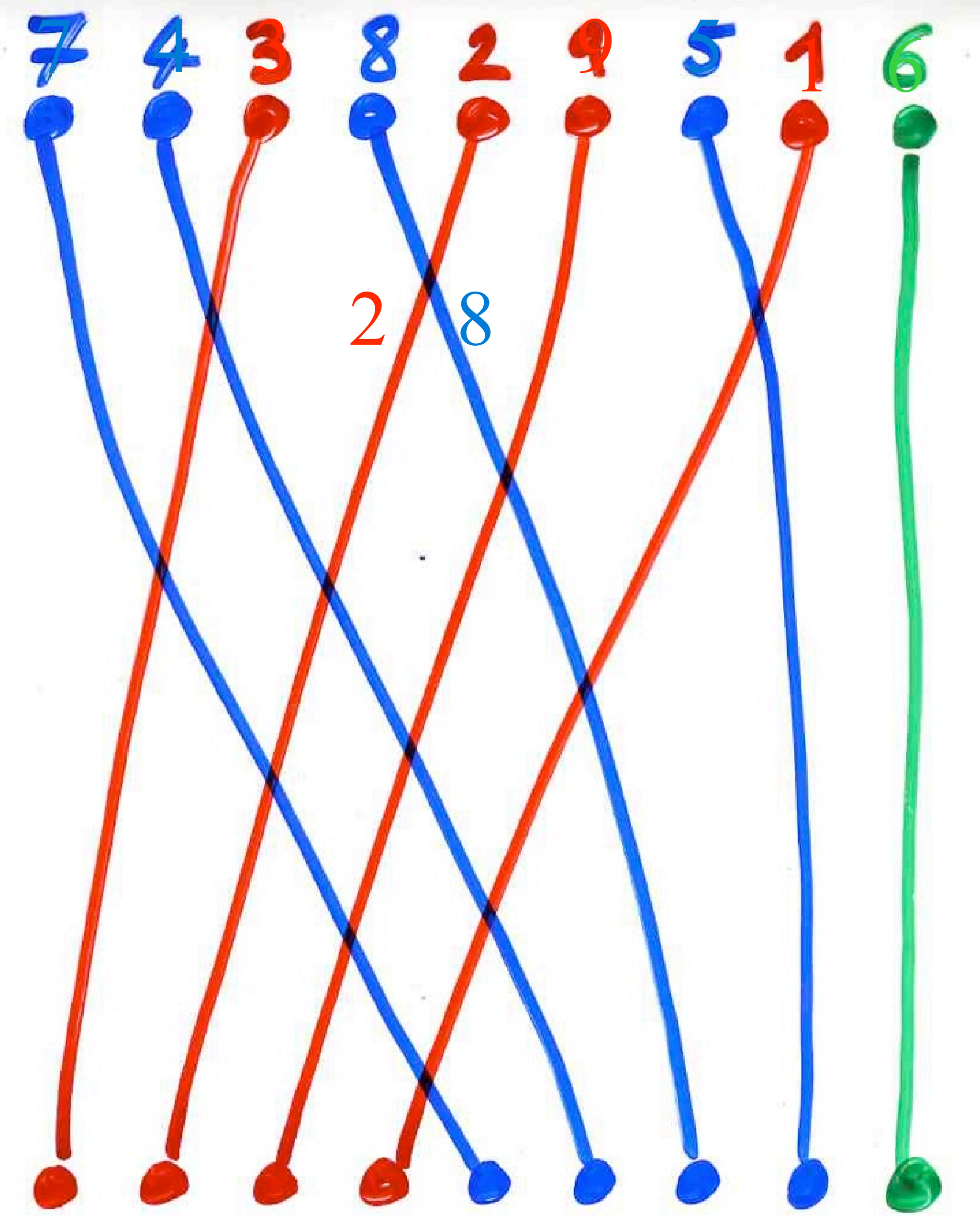
- convention $x = n$ est un recul

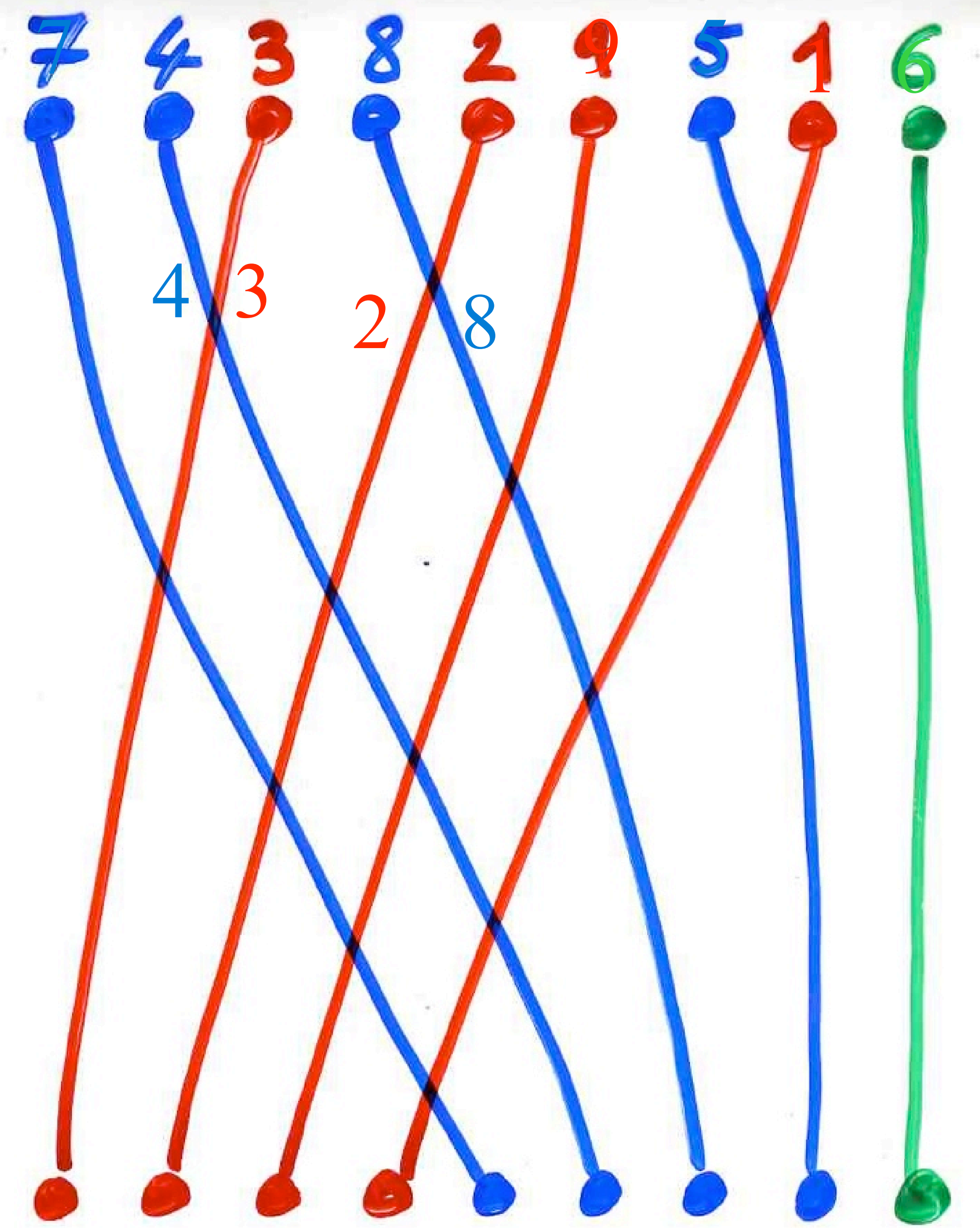


$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$

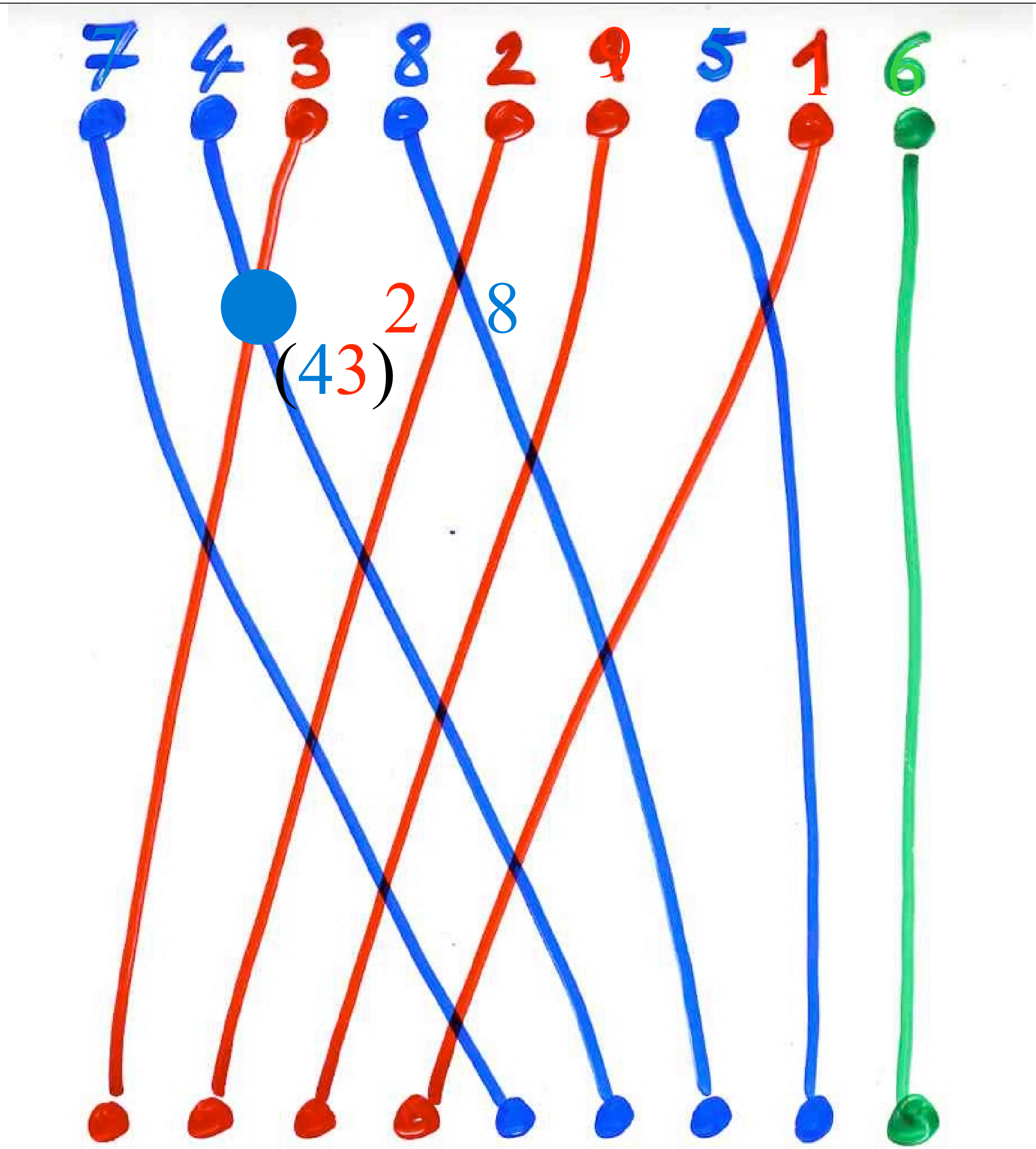


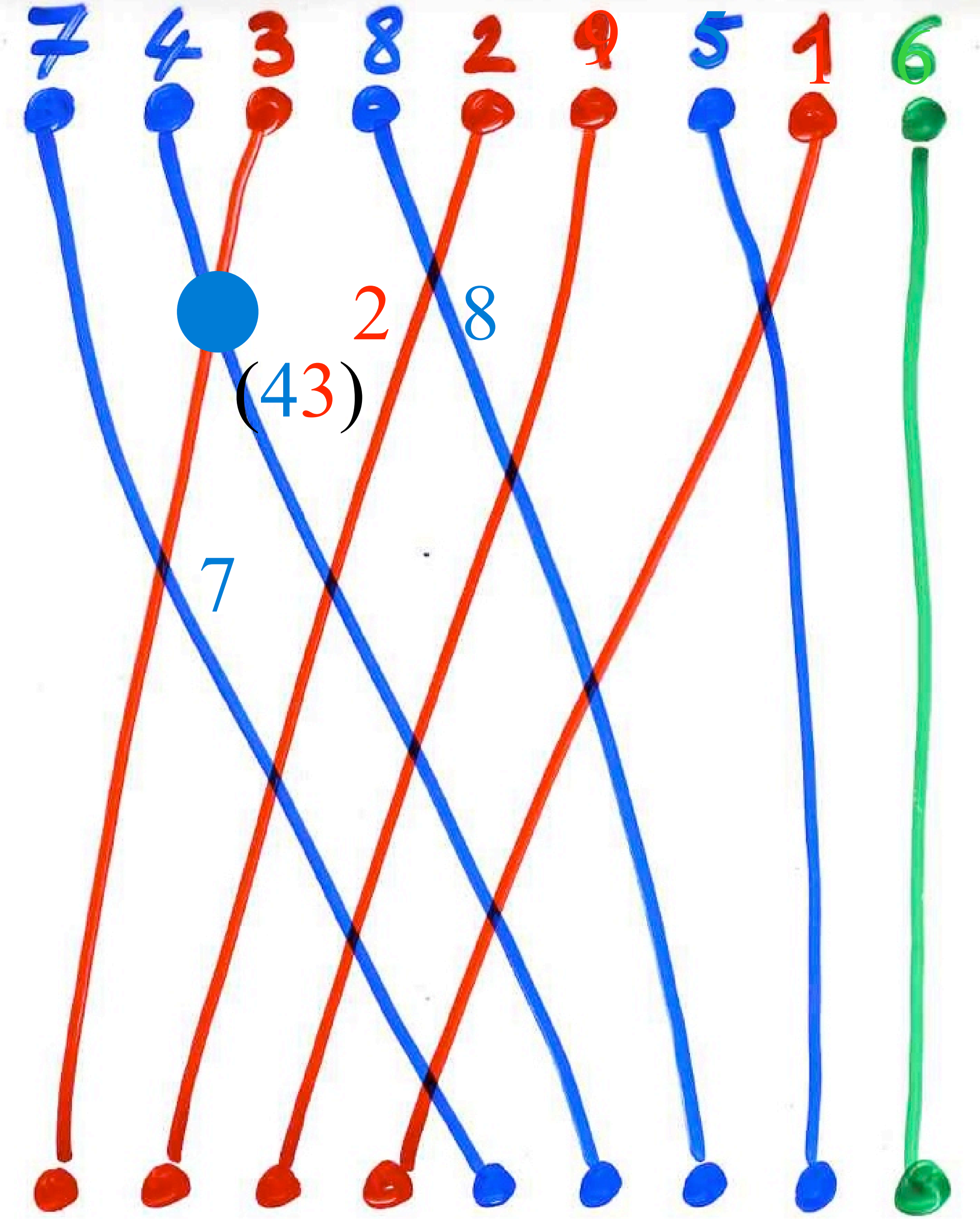

7 4 3 8 2 9 5 1 6


7 4 3 8 2 9 5 1 6
2 8

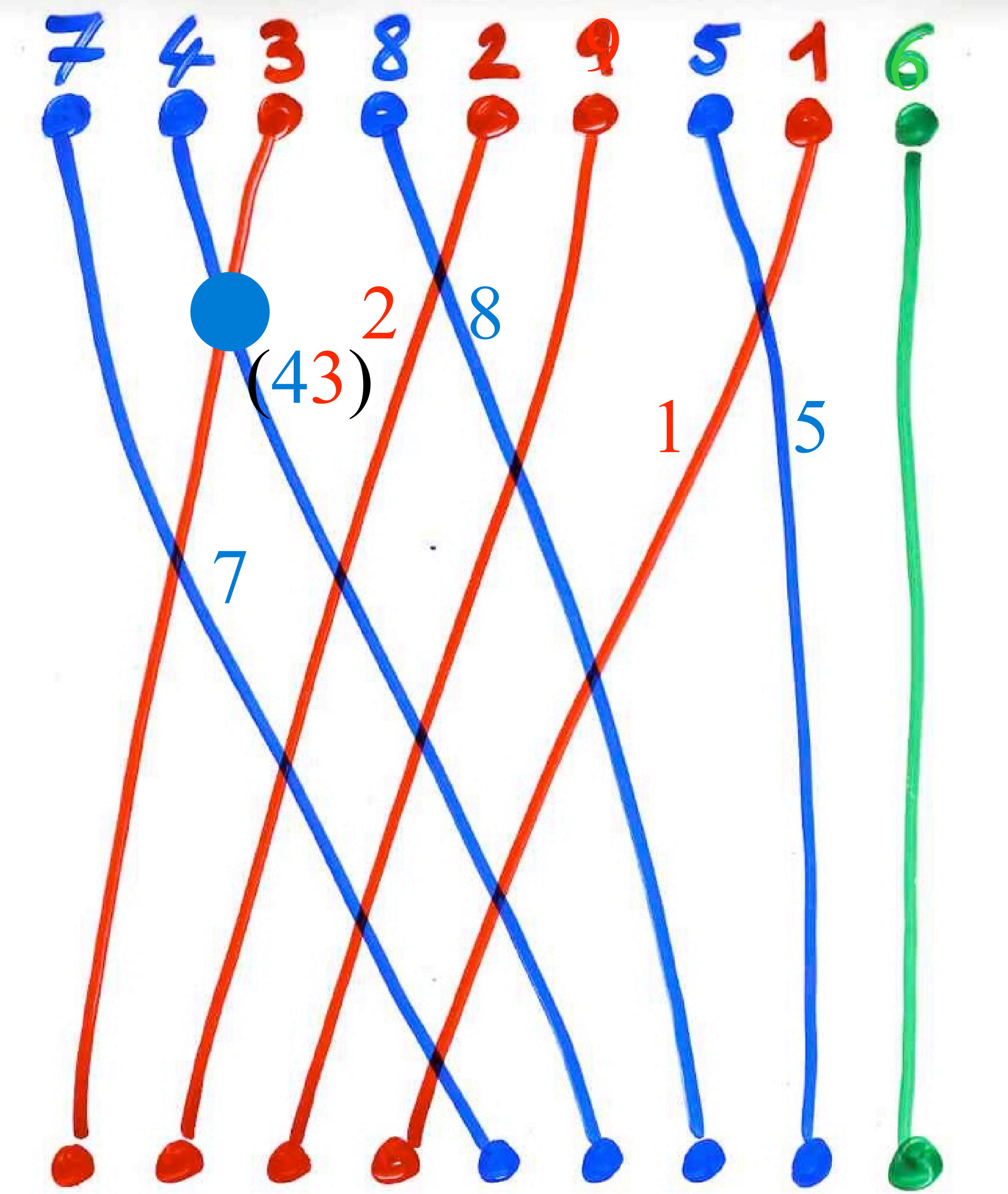


7 4 3 8 2 9 5 1 6
4 3 2 8

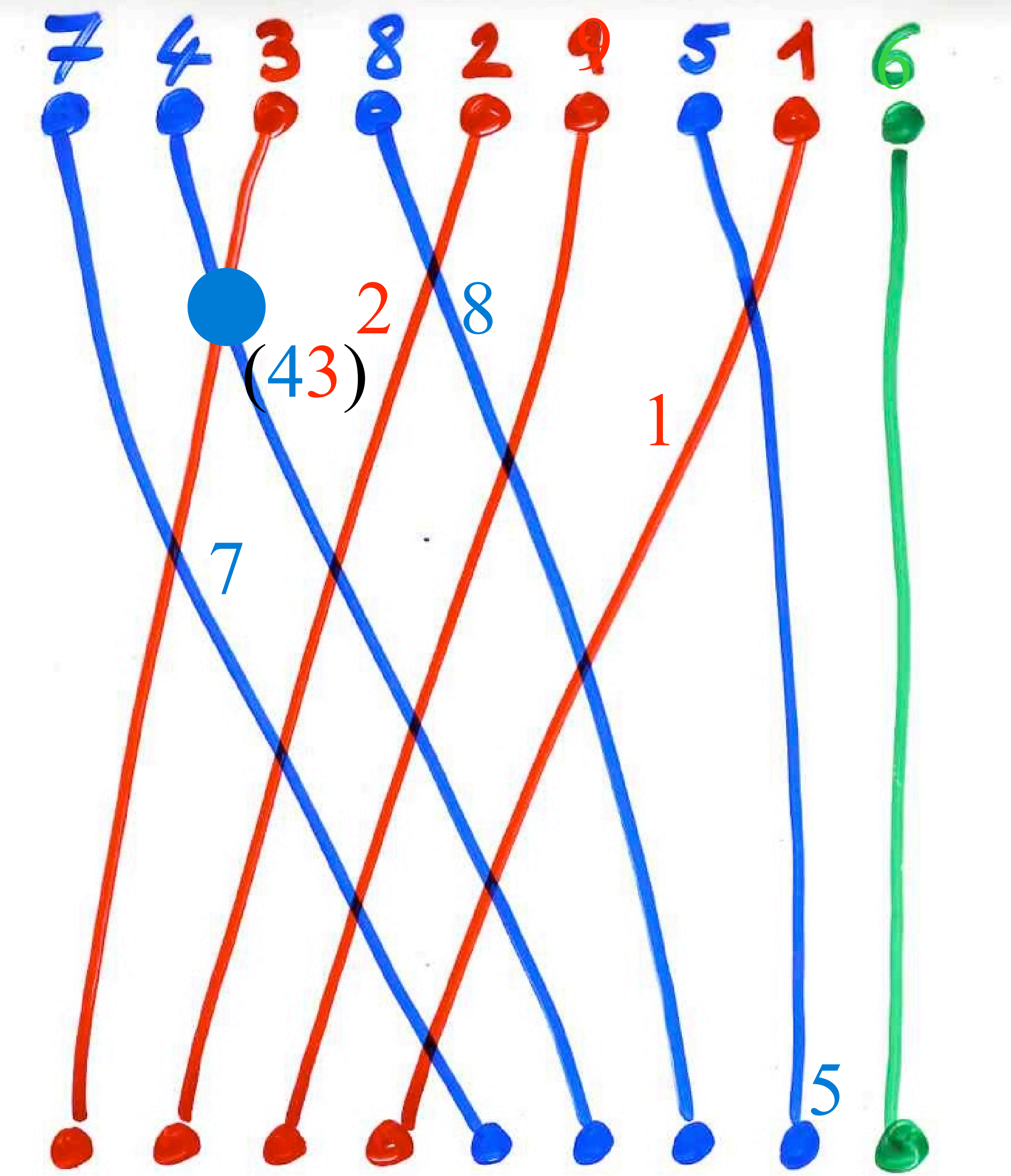

7 4 3 8 2 9 5 1 6
2 8
(43)



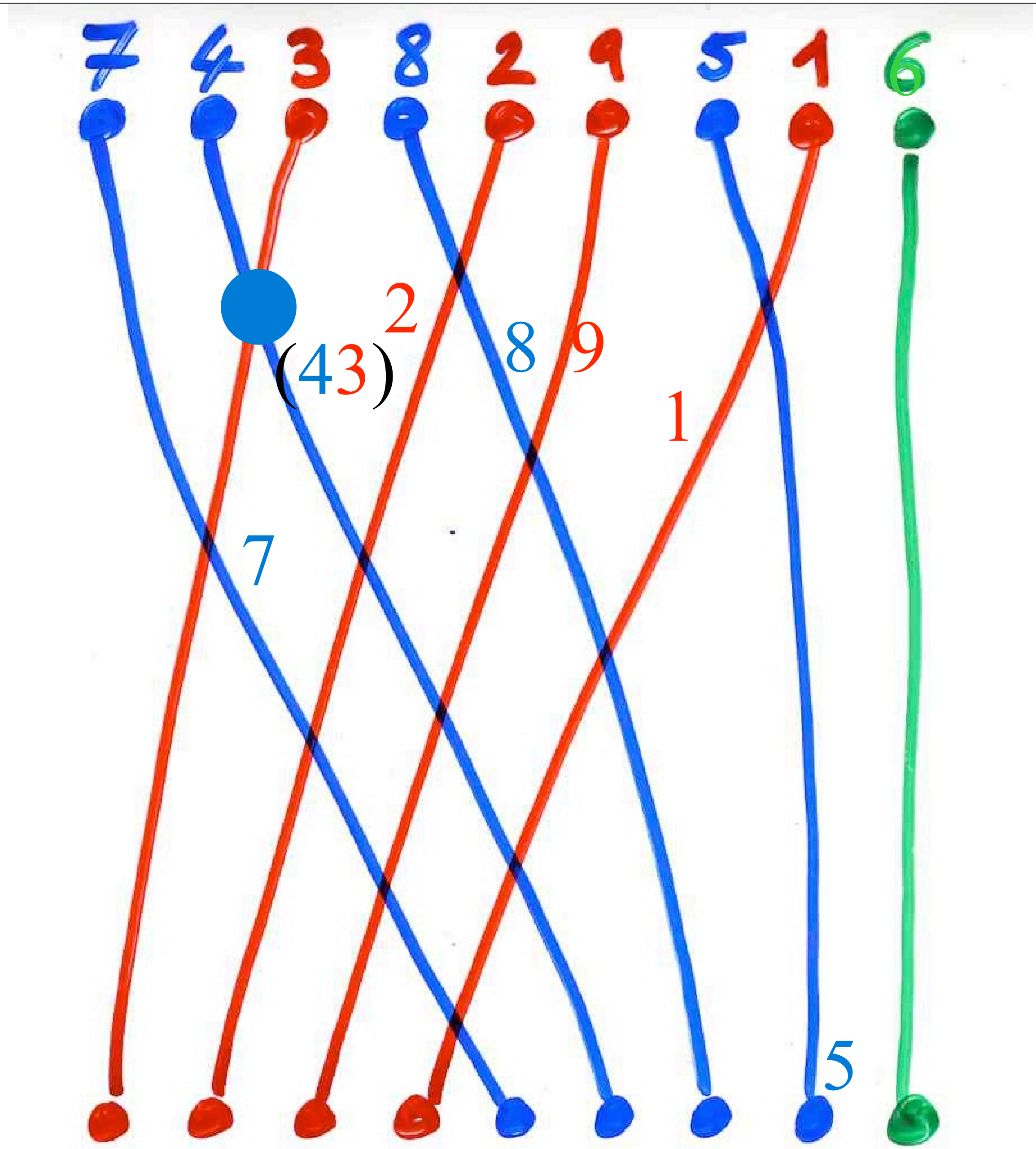
7 4 3 8 2 9 5 1 6
2
8
(43)
7

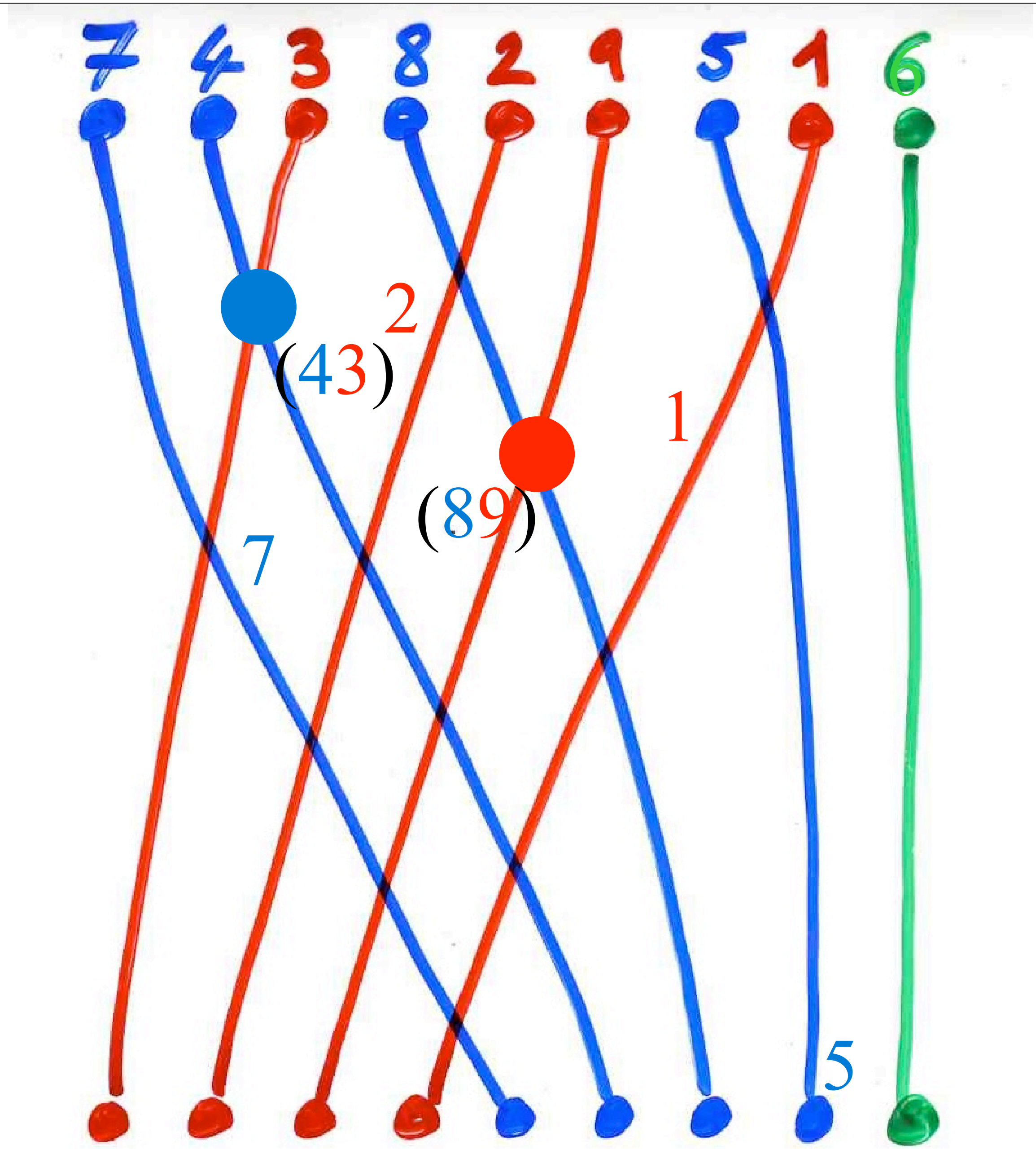


7 4 3 8 2 9 5 1 6
2
8
(43)
1
5
7

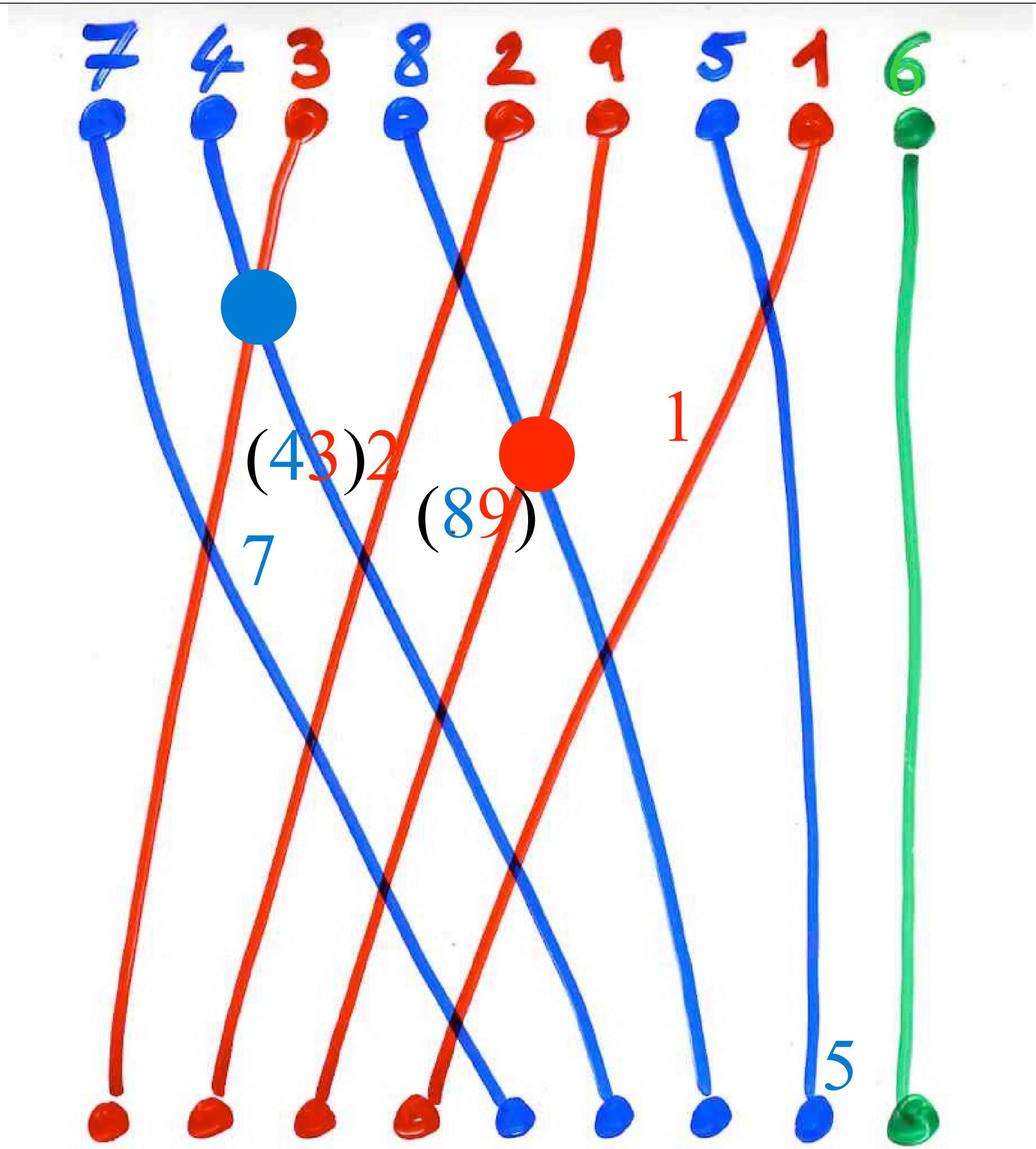


7 4 3 8 2 9 5 1 6
(43)
2
8
1
7
5

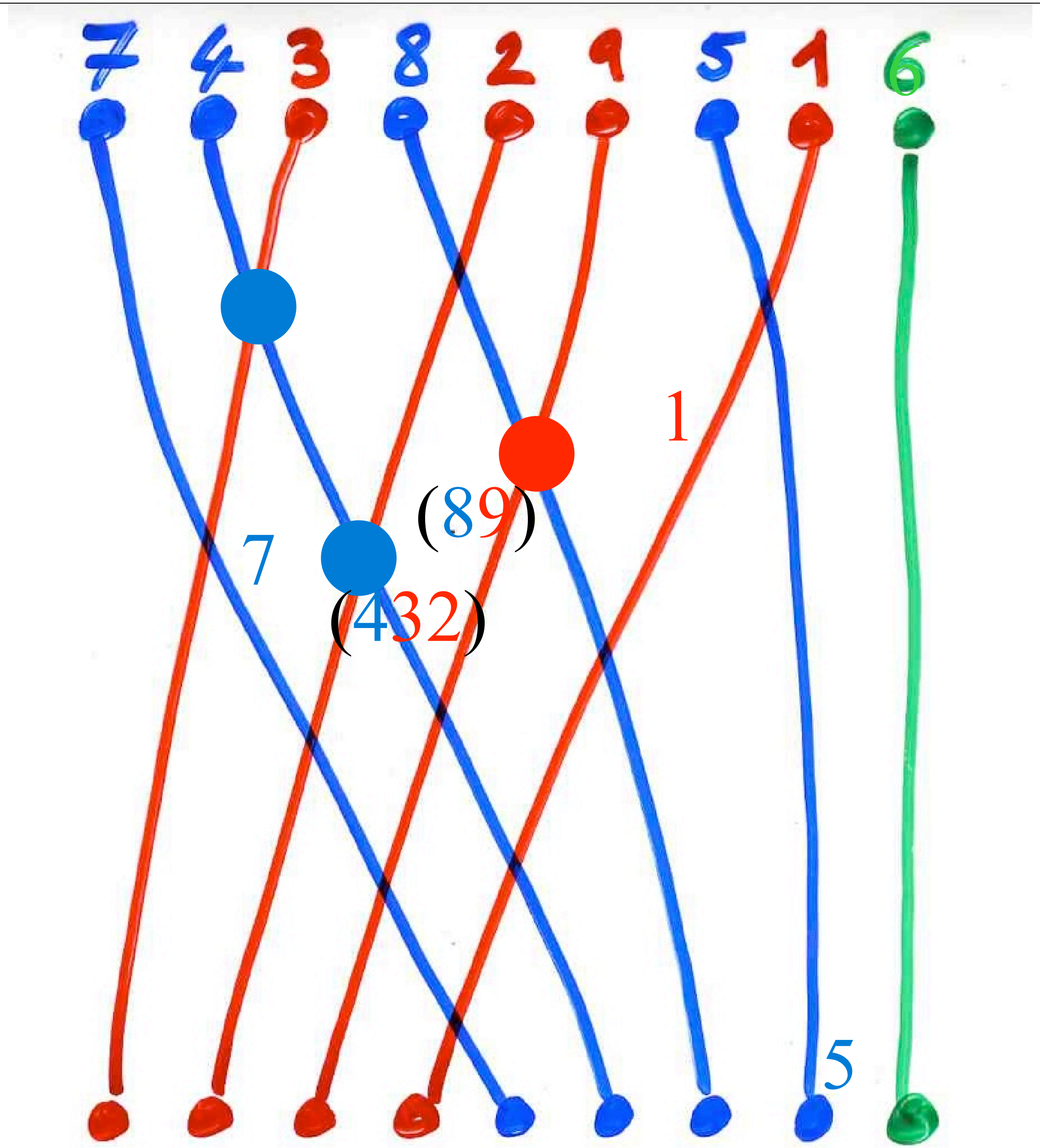

7 4 3 8 2 9 5 1 6
(43)
2
8
9
1
7
5



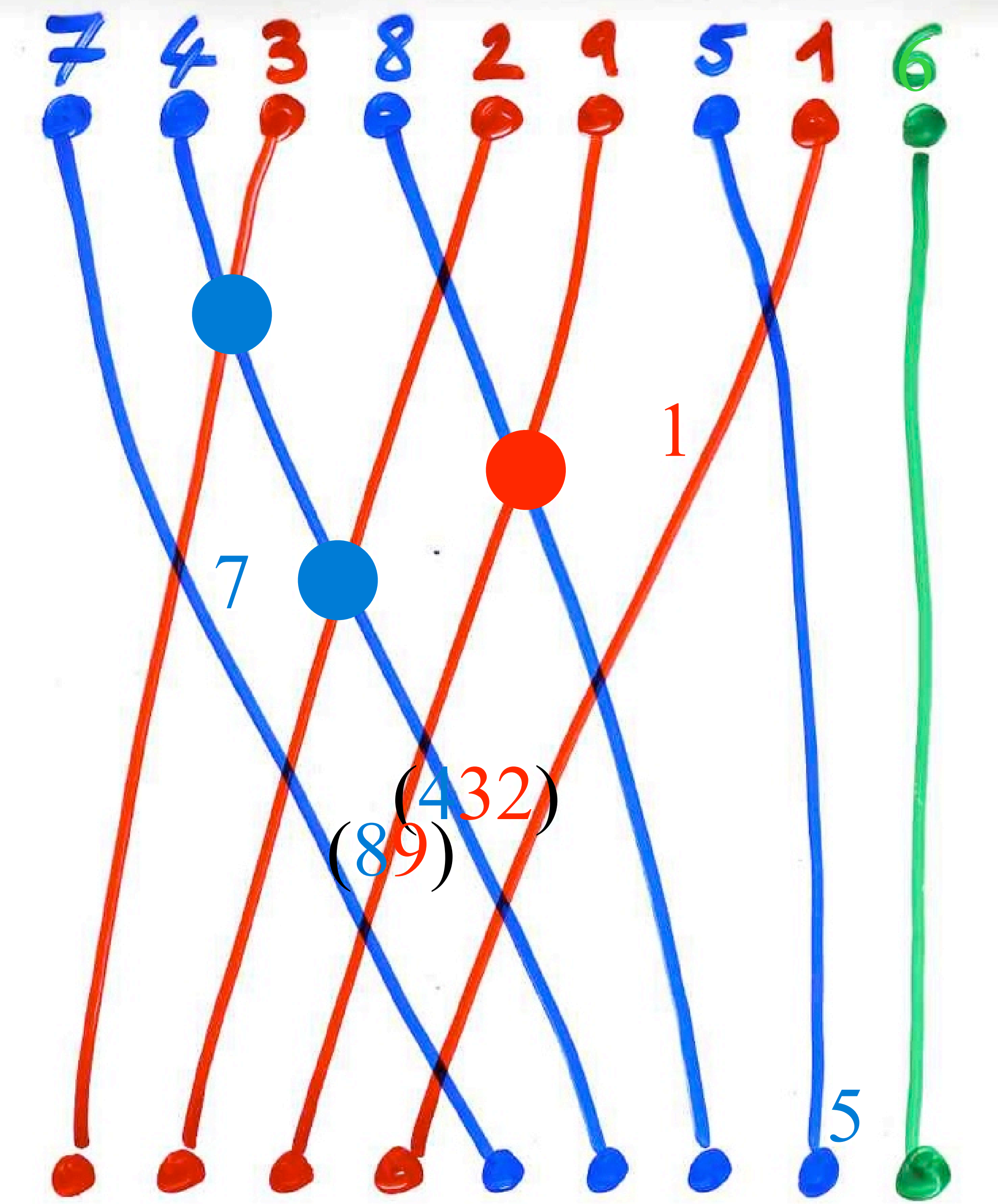
7 4 3 8 2 9 5 1 6
2
(43)
1
(89)
7
5

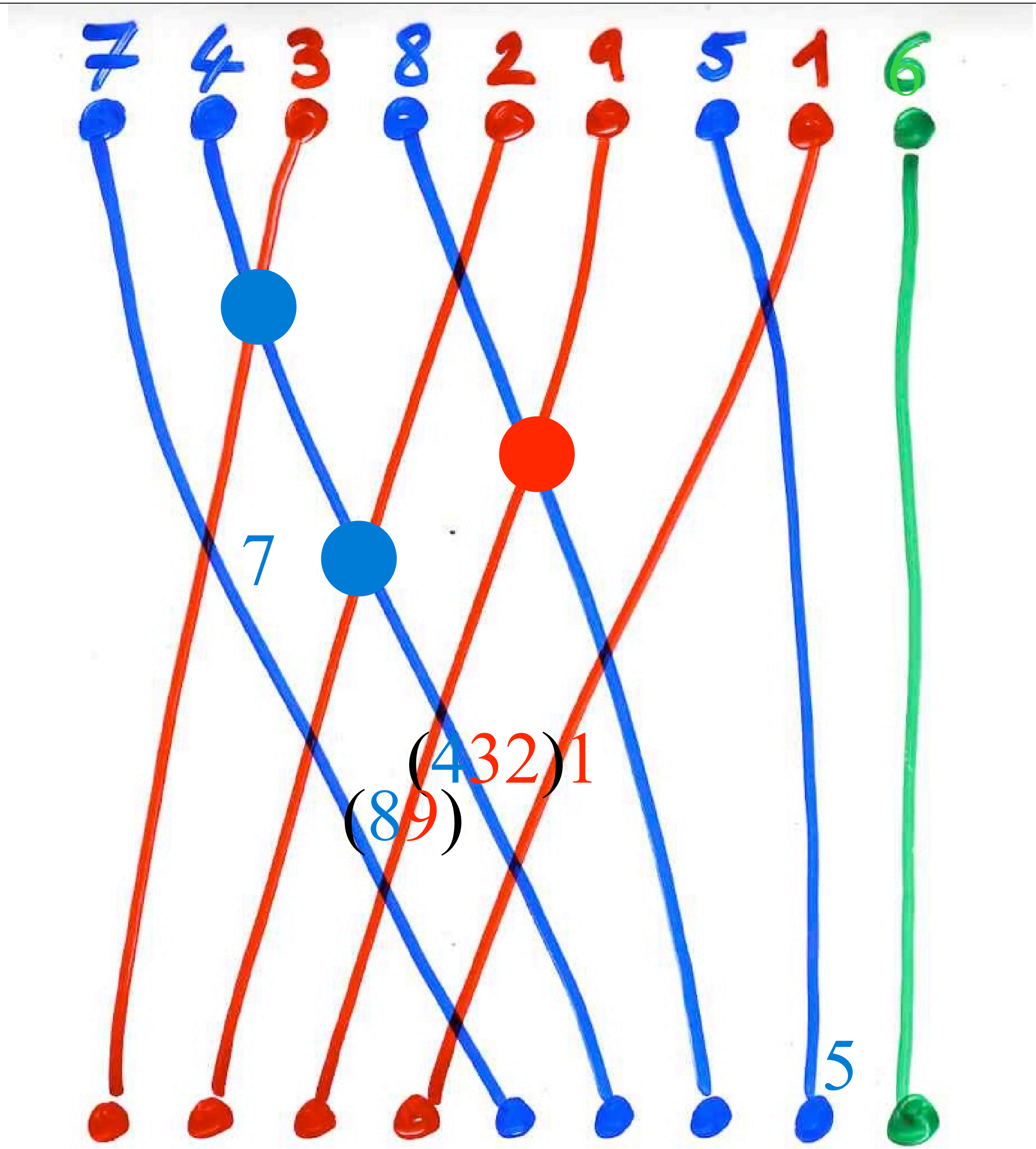


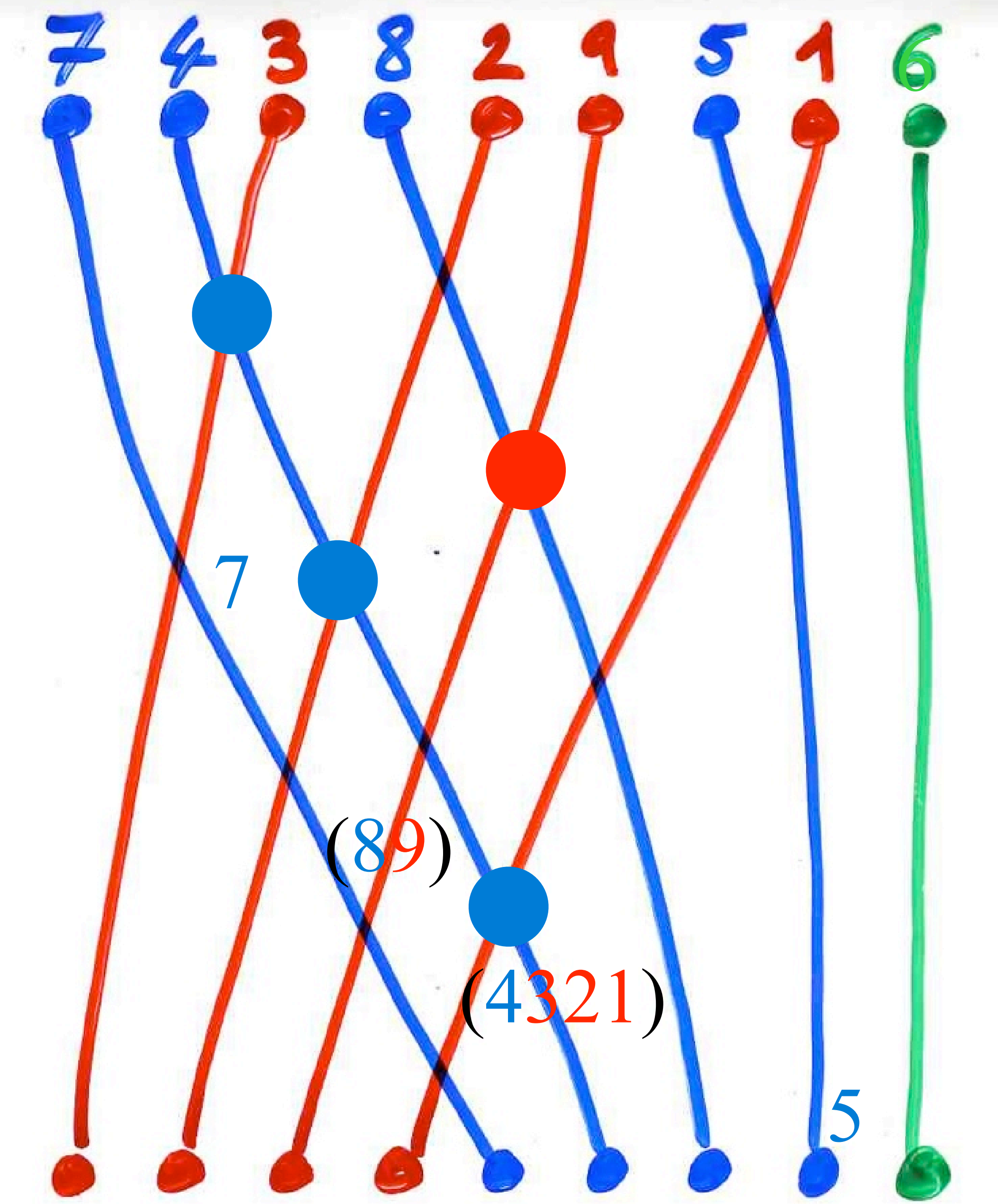
7 4 3 8 2 9 5 1 6
(43)2
(89)
1
7
5

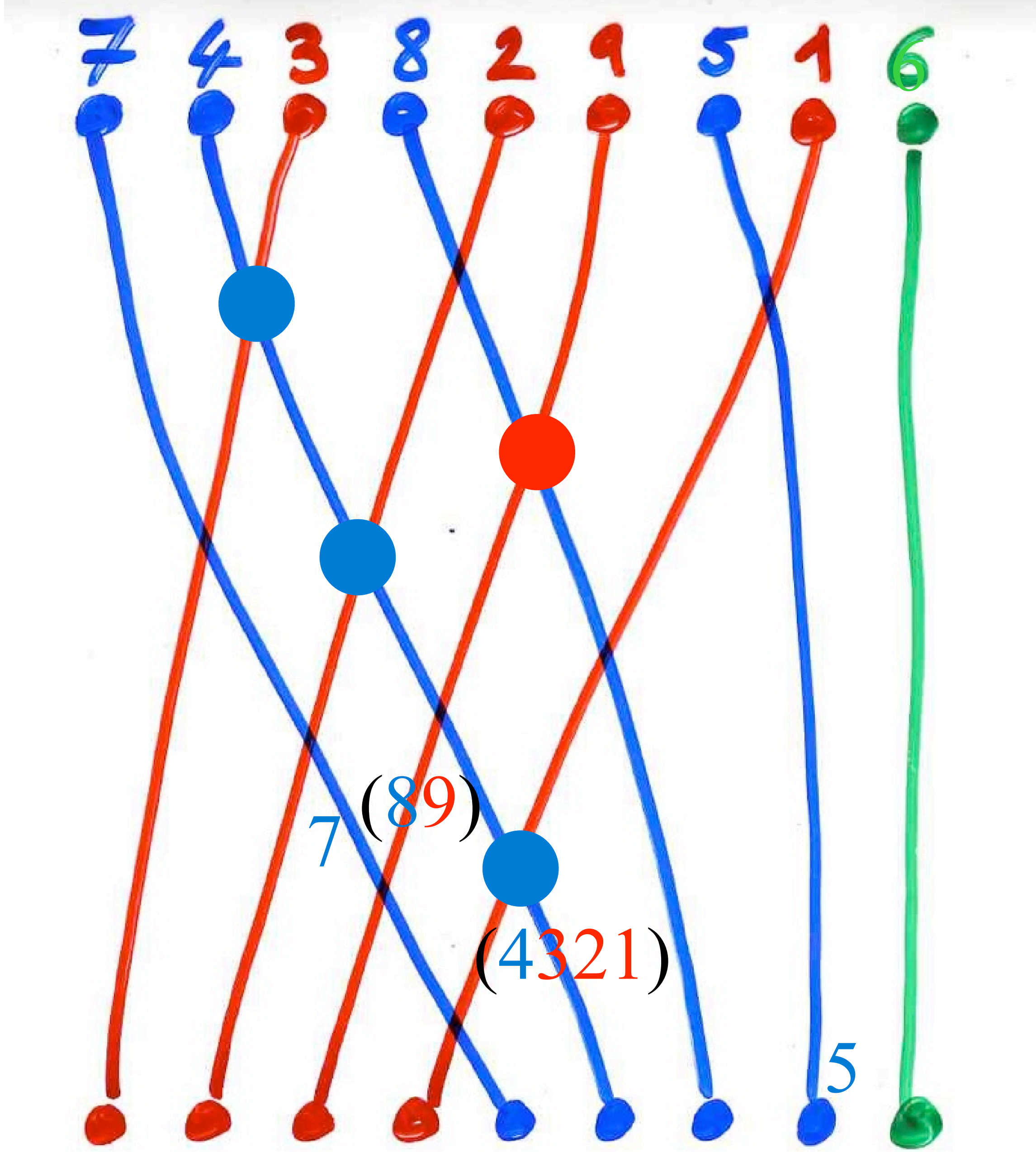


7 4 3 8 2 9 5 1 6
1
(89)
7
(432)
5

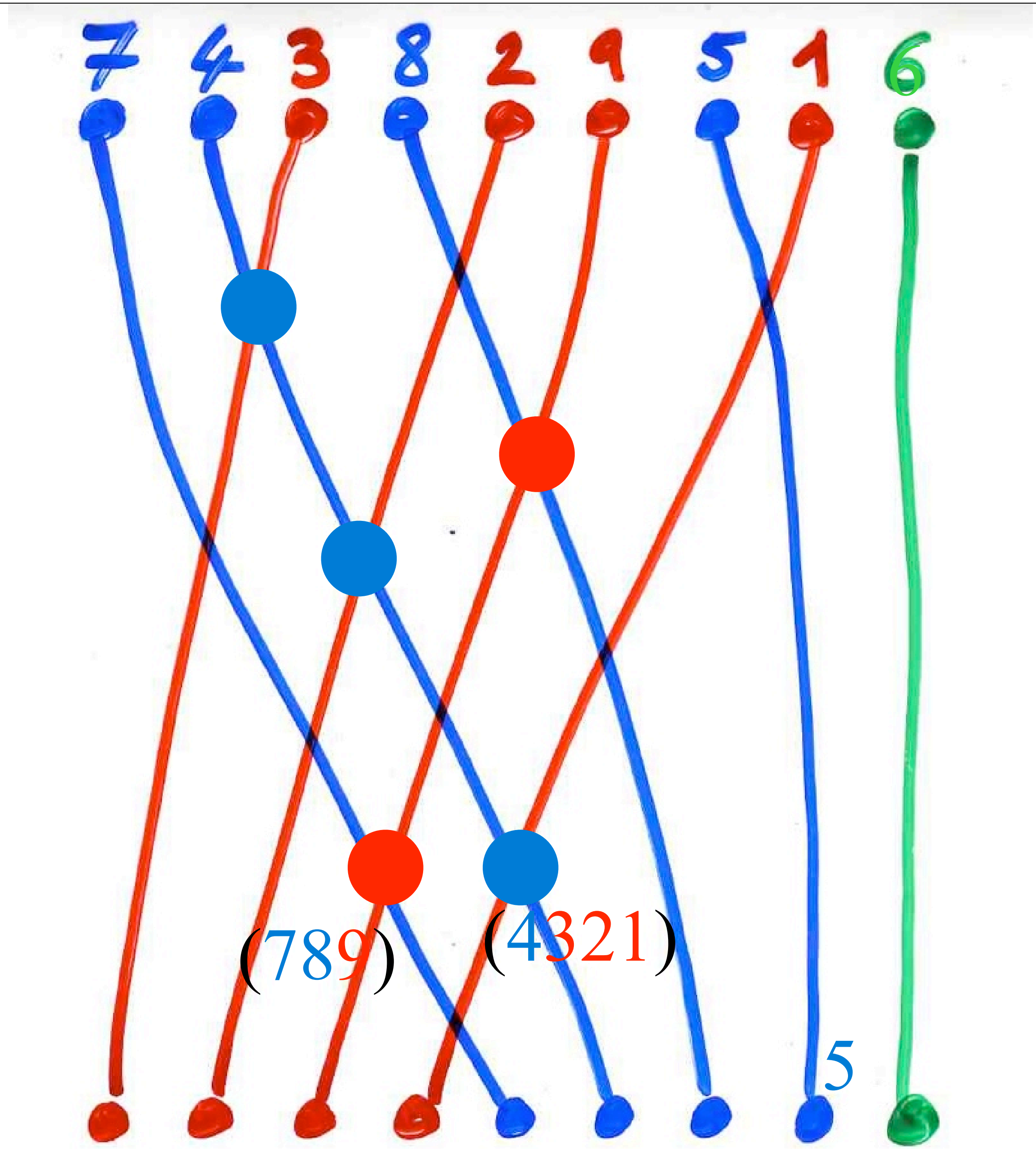

7 4 3 8 2 9 5 1 6
1
7
(432)
(89)
5


7 4 3 8 2 9 5 1 6
7
(432)1
(89)
5


7 4 3 8 2 9 5 1 6
7
(89)
(4321)
5

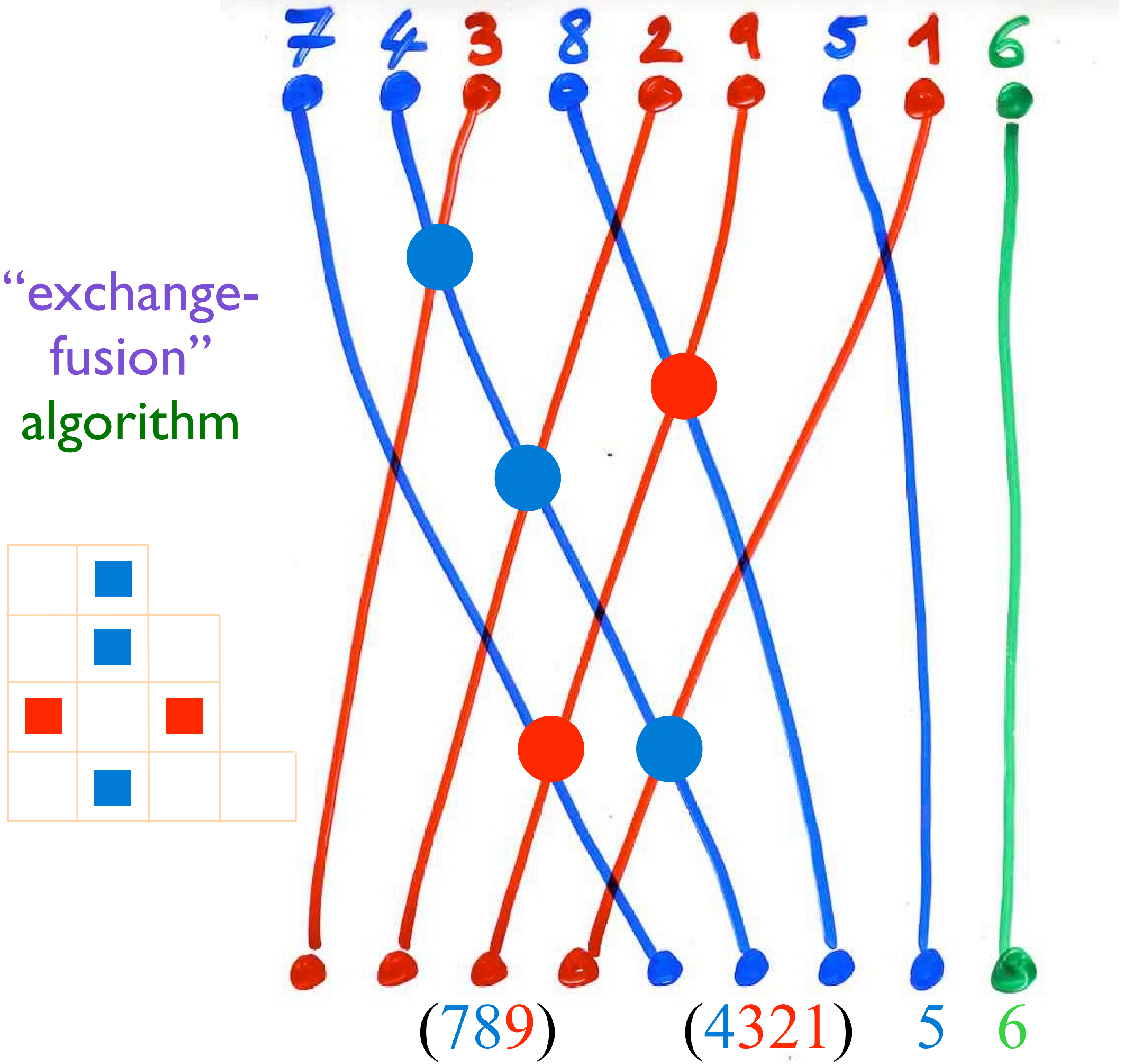


7 4 3 8 2 9 5 1 6
7
(89)
(4321)
5



7 4 3 8 2 9 5 1 6
(789)
(4321)
5

"exchange-fusion" algorithm



Description of the “exchange-fusion” algorithm

In the “exchange-fusion” algorithm, the red and blue blocks are falling down, starting at the beginning where all the blocks have only one letter. Each blocks is formed of consecutive letters.

- When two blocks meet at the crossing of a blue and red thread, if the union of the two blocks is formed with consecutive letters, then the two blocks form a single block by concatenation, and the new block follows the thread of the block having the biggest letters.
- If not, then the two blocks cross and follow their own colored thread.

The proof of the fact that the two algorithms “exchange-delete” and “exchange-fusion” produce exactly the same alternating tableau is based on the following observation:

(key) **observation**
In the “exchange-delete” algorithm, when a blue or a red dot is put on a crossing, that is when the two values x and y which are going to cross are “consecutive”, then all the intermediate values between x and y (which have disappeared) belong to one of the corresponding blocks in the analog crossing which will appears in the “exchange-fusion” algorithm.

A consequence of that is to give an interpretation of the number of red or blue blocks falling on the ground level, that is the number of columns having no red cells and numbers of rows having no blue cells. We call such row or column **“open”.**

Some Parameters

The maximum letter of the blocks of letters reaching the ground level are:

- for the columns of T (red threads), the left-to-right maximum elements of the values of the permutation s less than the last letter s(n+1),
- for the rows of T (blue threads), the right-to-left maximum elements of the values of the permutation s bigger than the last letter

(3 proofs comming 3 different methodologies: by P. Nadeau , O.Bernardi and xgv)

This gives an interpretation of the two parameters on alternative tableaux:
- number of "open" columns (i.e. columns without a red cell)
- number of "open" rows (i.e. rows without a blue cell)

In fact, each block falling on the ground level in the "exchange-fusion" algorithm (corresponding to an open column or row), has an underlying binary tree structure coming from the different fusions (or equivalently the deletions of the "exchange-delete" algorithm)

(see a forthcoming paper of P. Nadeau on "alternative trees" and alternative tableaux).

Number of “crossings” in the alternative tableaux

This parameter is the number of crossing occurring in the “exchange-delete”, or equivalently of the “exchange-fusion” algorithm.Each crossing corresponds to a cell in the alternative tableau (colored) which is above a red cell and at the right of a blue cell. It has the same distribution as the parameter “number of occurrences of the pattern (31-2)” in permutations. (from the bijection of S. Corteel and P. Nadeau or from Steingrimsson and Williams)

This parameter is the natural q-analogue of Laguerre histories, that is the parameter obtained by taking the sum of all the “possibilities choices decreased by one”. In other words, if at each step 1, 2, ..., x, ... , n+1, of the construction of the permutation, the (k+1) free positions available to insert the value x are labeled (in a certain way) 0, 1, ..., k, then we put the weight q^i when value x is inserted at position i , and the weight of the Laguerre history is the product of the weight of each individual step. If the labeling is always from left to right, then the q-analogue becomes the number of occurrence of (31-2). (see the next section).

The number of crossings of the alternative tableau has been be characterized by O.Bernardi on the corresponding permutation s.

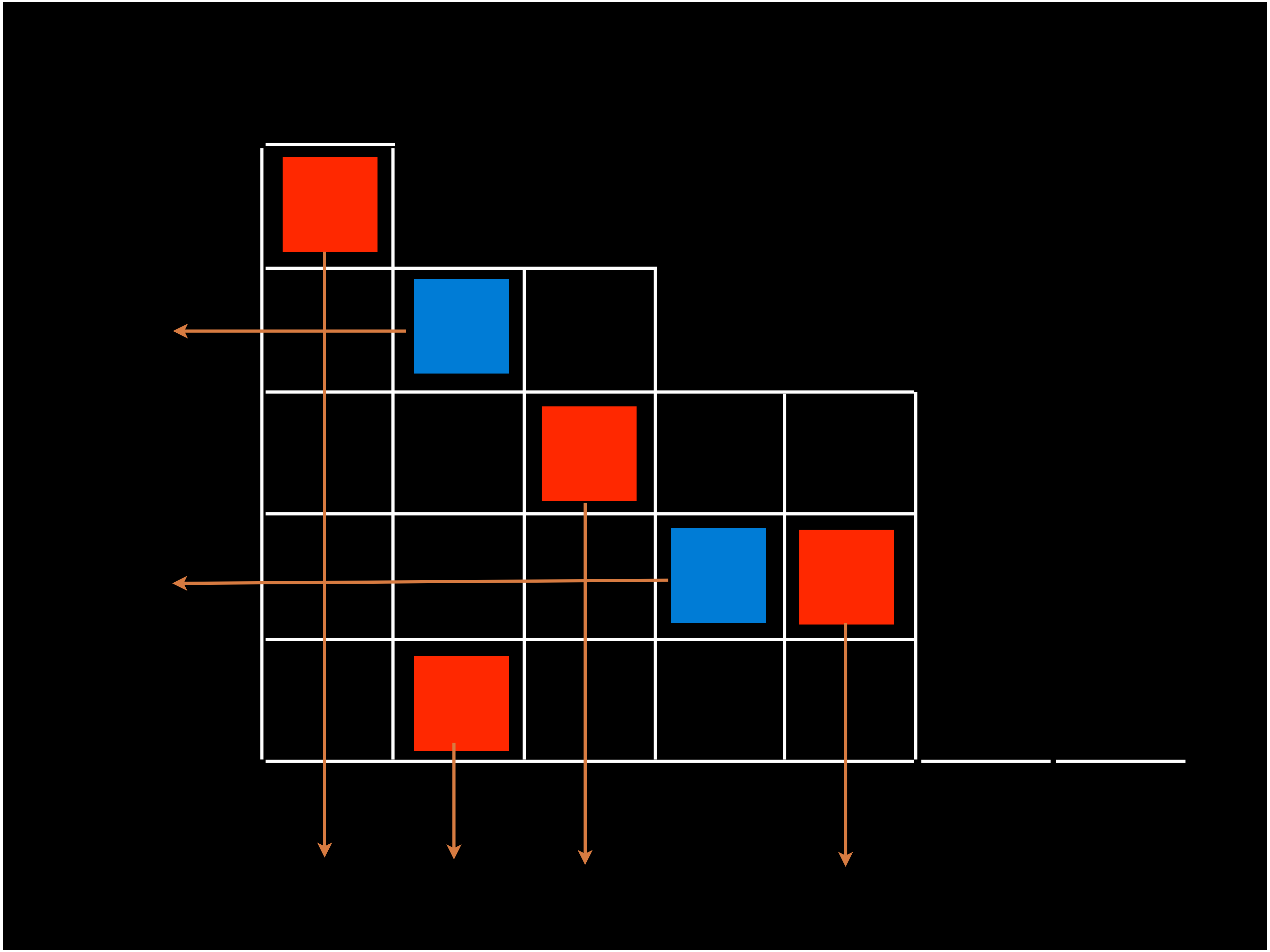
It is the number of pairs (x,y), x=s(i), y=s(j), 1≤ i<j≤n+1, such that there exist two integers k, l≥ 0 such that:
the set of the values x+1, x+2, ..., x+k, y+1, .., y+l are located between x and y (in the word s), and x+k+1 is located (in s) at the right of y and y+l+1 is located (in s) at the left of x (with the convention of n+2 at the left of all the values).

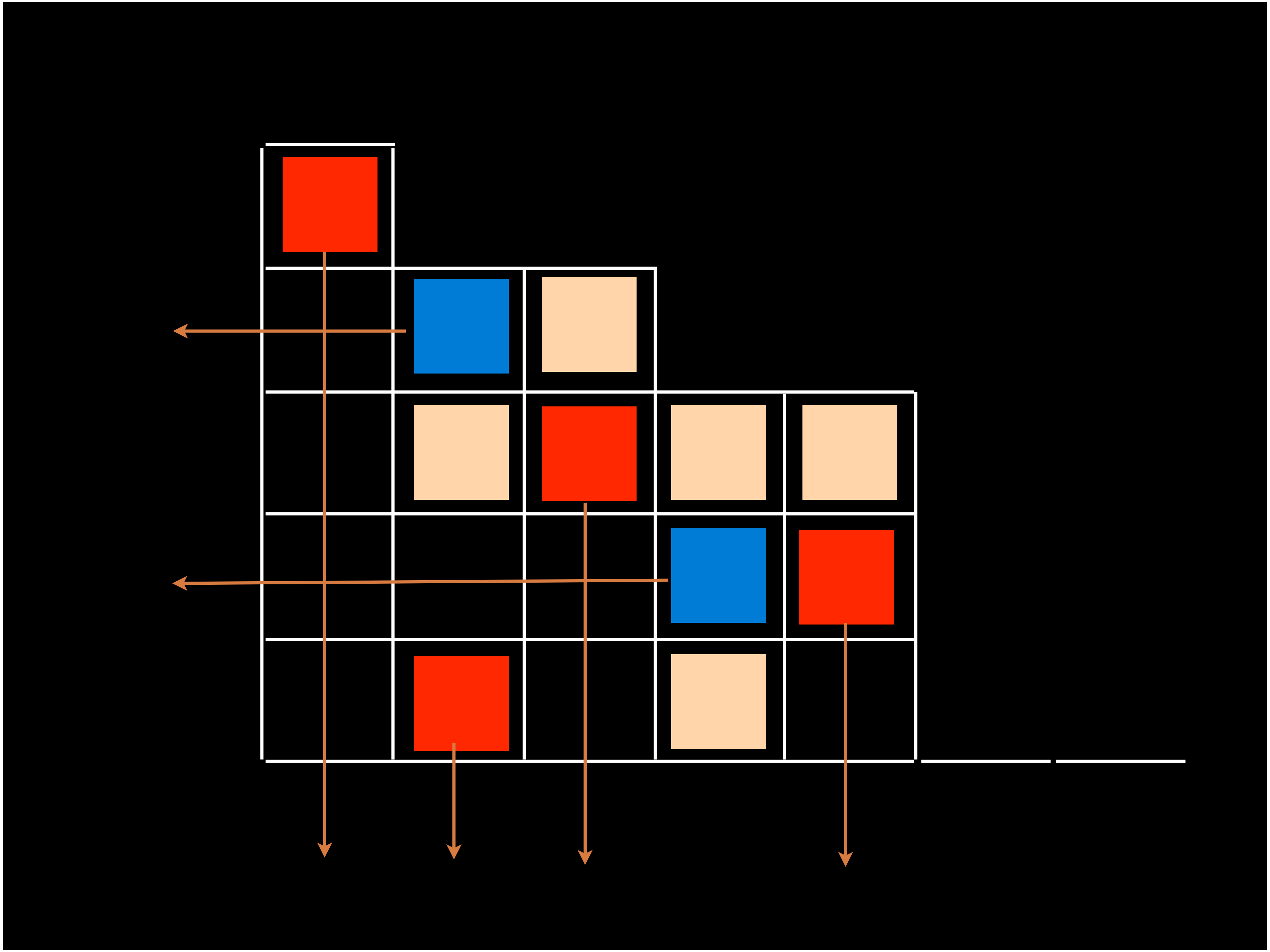
O.B. deduce the nice corollary:
The permutations s coming from tableaux with no crossing (counted by Catalan numbers) are characterised by the following condition

there is no pair of values (x, y) such that the four values (x,x+1,y,y+1) appear in the following order in the permutation:

s = ... y+1 ..... x ..... y .... x+1 .....

§ 11
Catalan alternative tableaux





Def Catalan alternative tableau T
alt. tab. without cells ☒

i.e: every empty cell is below a red cell or
on the left of a blue cell

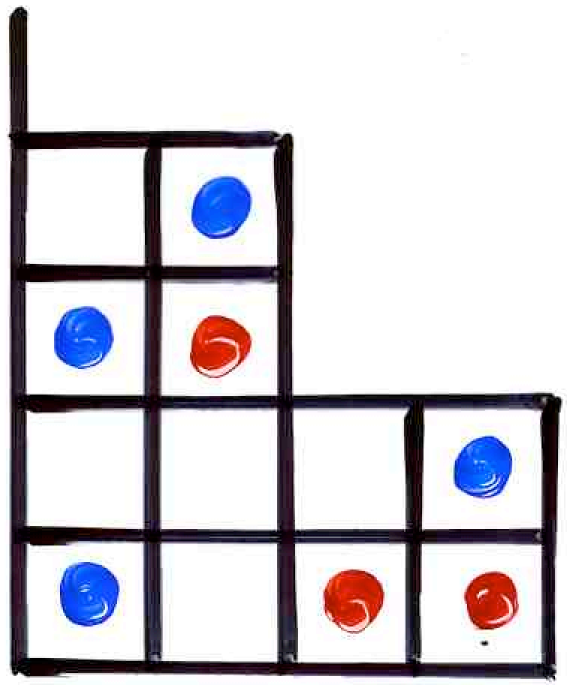


tableau
alternatif
de Catalan

Def Catalan alternative tableau T
alt. tab. without cells ☒
i.e: every empty cell is below a red cell or on the left of a blue cell

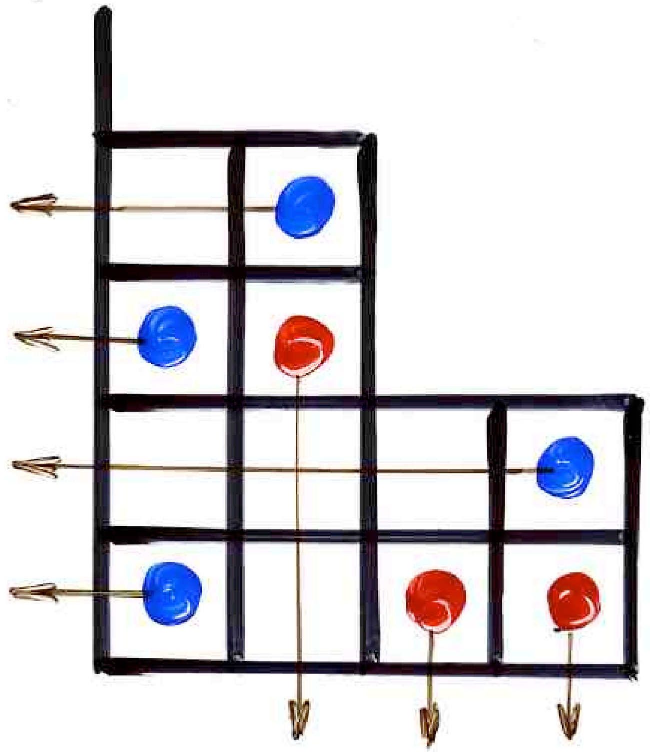


tableau
alternatif
de Catalan

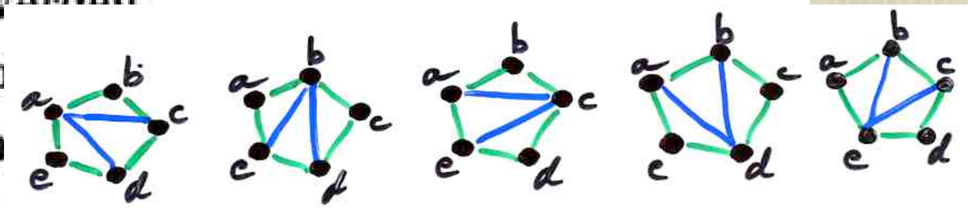
Une lettre d'Euler
à Christian Goldbach ....

Berlin, 4 Septembre 1751

Ich bin neulich auf eine Betrachtung gefallen, welche mir nicht wenig merkwürdig vorkam. Dieselbe betrifft, auf wie vielerley Arten ein gegebenes polygonum durch Diagonallinien in triangula zerschnitten werden könne.

Also ein quadrilaterum *abcd* kann entweder durch die diagonalem *ac*, oder durch *bd*, und also auf zweyerley Art in zwey triangula resolvirt w...

Ein Fünfeck *abcde* wi...
drey triangula getheilet, un...
verschiedene Arten geschehen, nemlich durch die diagonales



I. *ac*, *ad*. II. *bd*, *be*. III. *ca*; *ce*. IV. *db*, *da*, V. *ec*, *eb*.

Ferner wird ein Sechseck durch drey diagonales in vier triangula zertheilet, und dieses kann auf 14 verschiedene Arten geschehen.

Nun ist die Frage generaliter: da ein polygonum von n Seiten durch $n-3$ diagonales in $n-2$ triangula zerschnitten wird, auf wie vielerley verschiedene Arten solches geschehen könne. Setze ich nun die Anzahl dieser verschiedenen Arten $= x$, so habe ich per inductionem gefunden

wenn $n = 3, 4, 5, 6, 7, 8, 9, 10$

so ist $x = 1, 2, 5, 14, 42, 132, 429, 1430$.

Hieraus habe ich nun den Schluss gemacht, dass generaliter sey

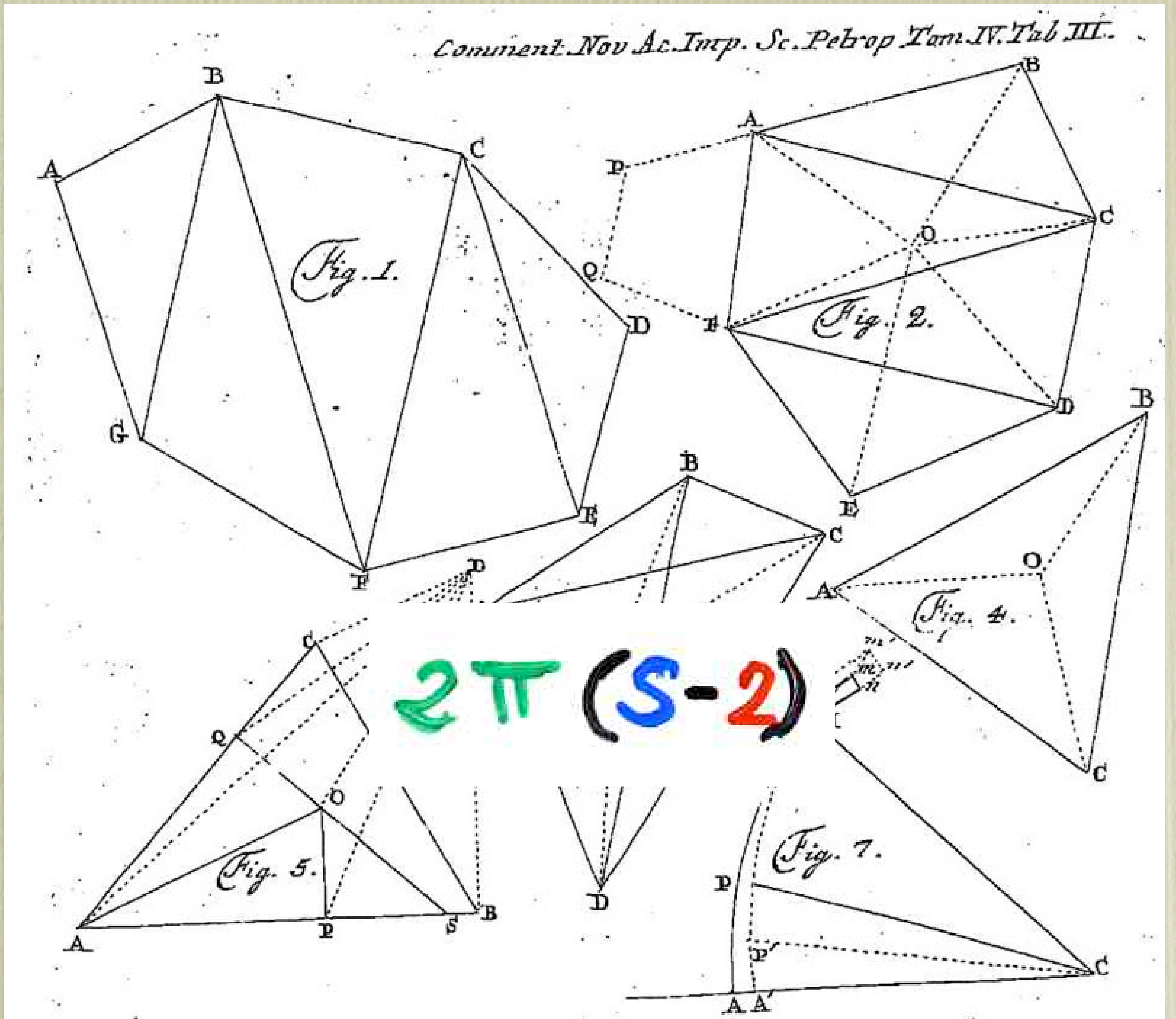
$$x = \frac{2.6.10.14.18.22\ldots.(4n-10)}{2.3.4.5.6.7\ldots\ldots(n-1)}$$

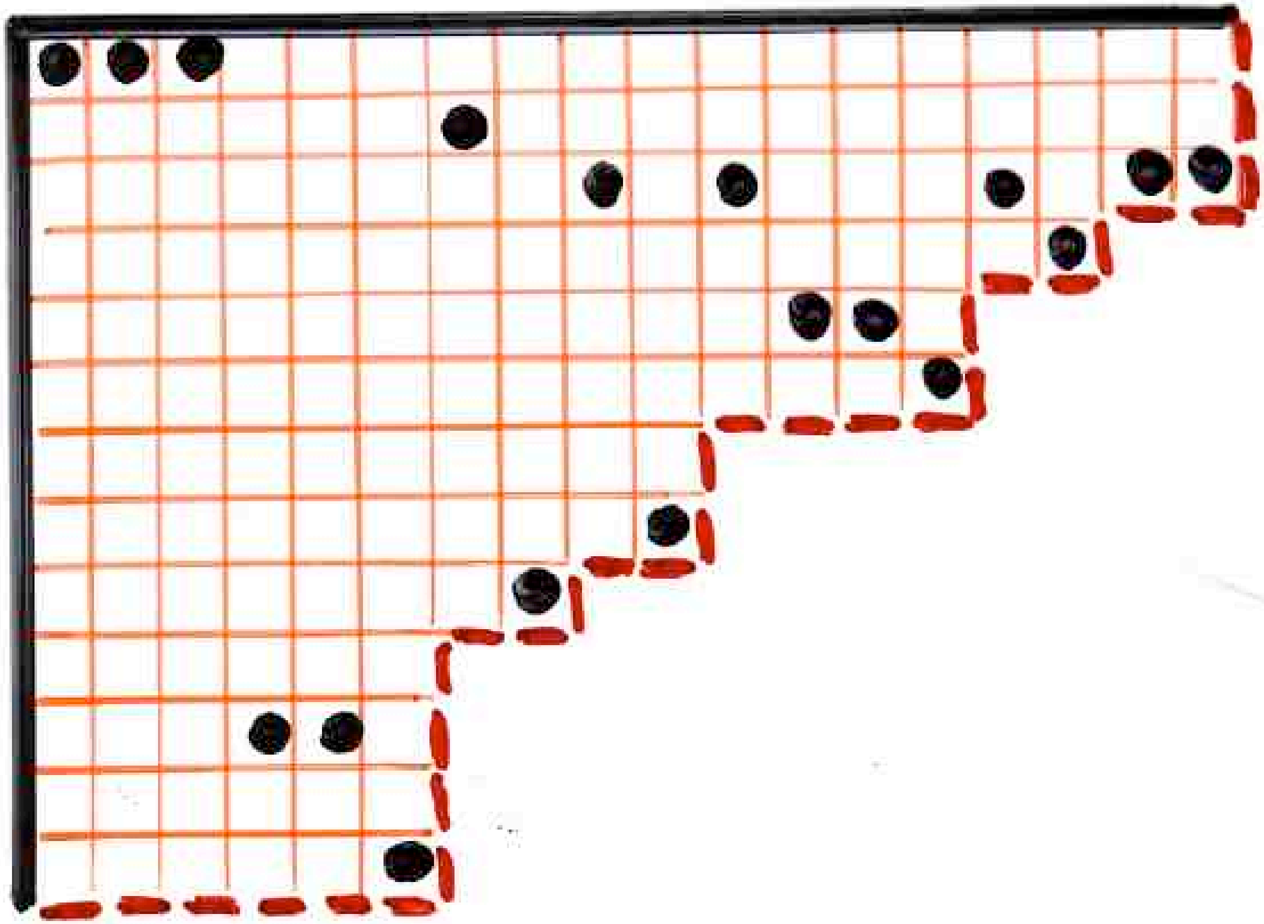
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

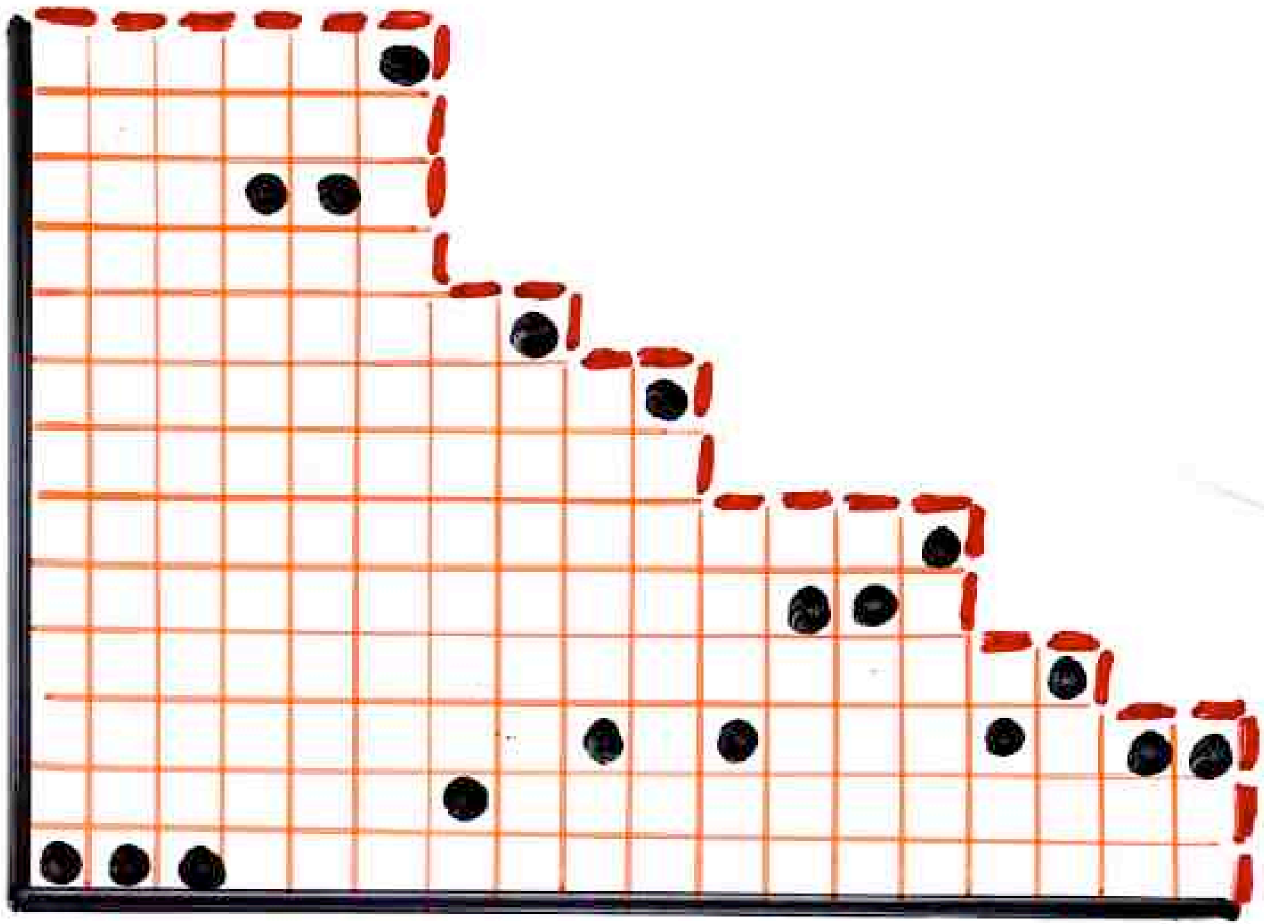
ist $1 = \frac{2}{2}$, $2 = 1.\frac{6}{3}$, $5 = 2.\frac{10}{4}$, $14 = 5.\frac{14}{5}$, $42 = 14.\frac{18}{6}$, $132 = 42.\frac{22}{7}$; dass also aus einer jeden Zahl die folgende leicht gefunden wird. Die Induction aber, so ich gebraucht, war ziemlich mühsam, doch zweifle ich nicht, dass diese Sach nicht sollte weit leichter entwickelt werden können. Ueber die Progression der Zahlen 1, 2, 5, 14, 42, 132, etc. habe ich auch diese Eigenschaft angemerket, dass $1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc.} = \frac{1 - 2a - \sqrt{(1-4a)}}{2aa}$. Also wenn $a = \frac{1}{4}$, so ist

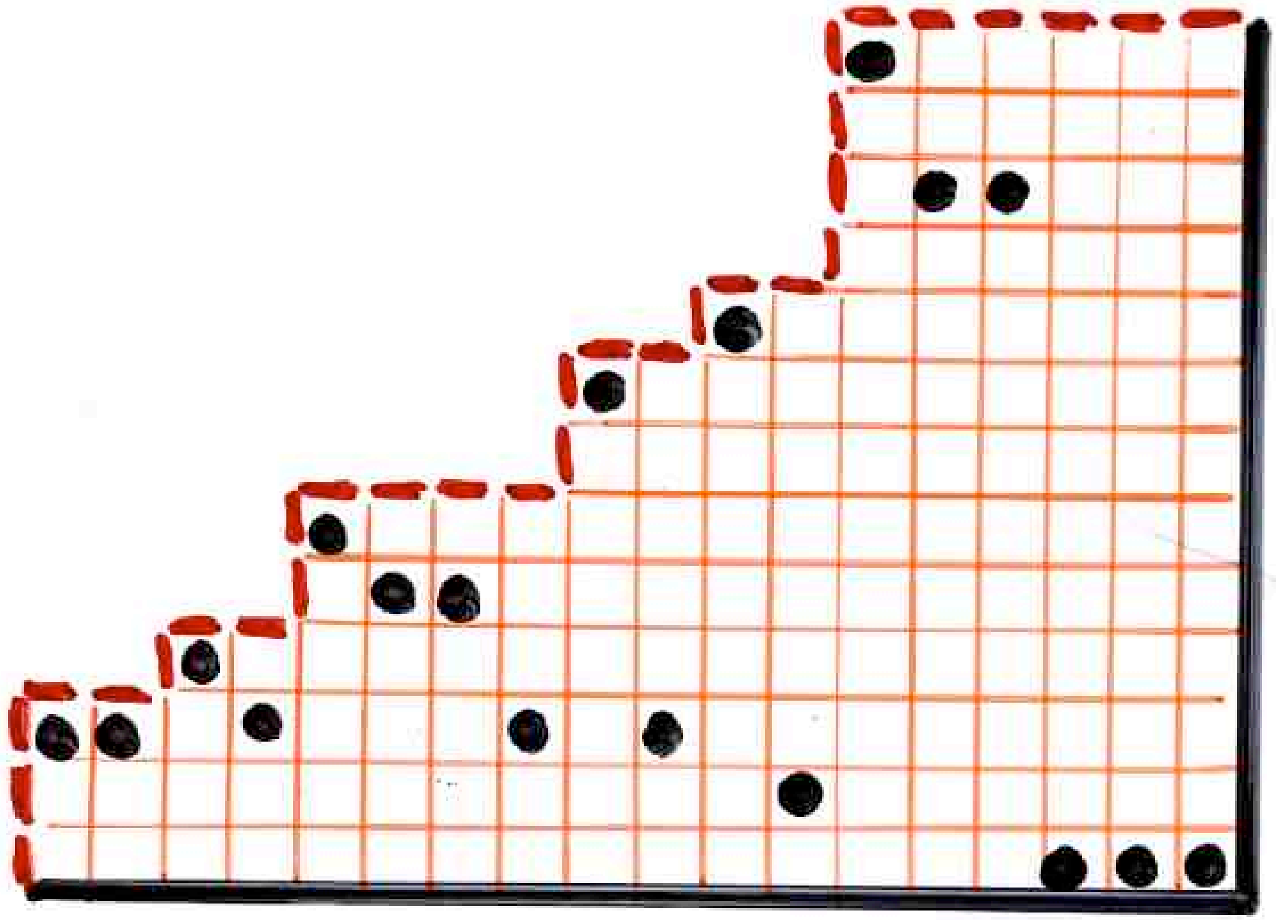
$$1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} + \text{etc.} = 4.$$

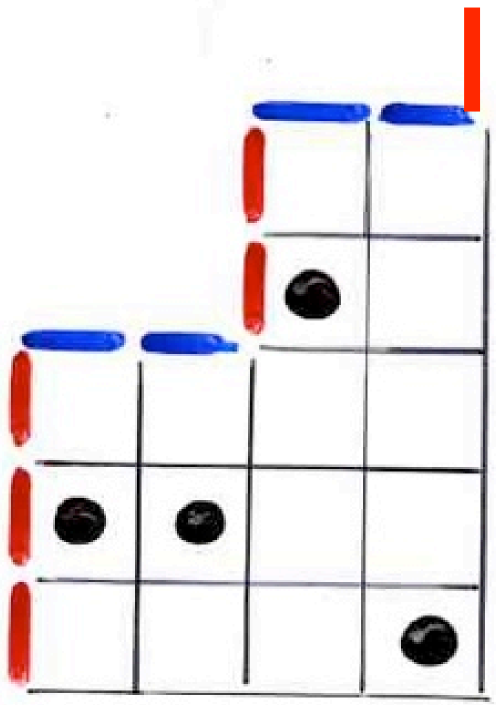
Euler.

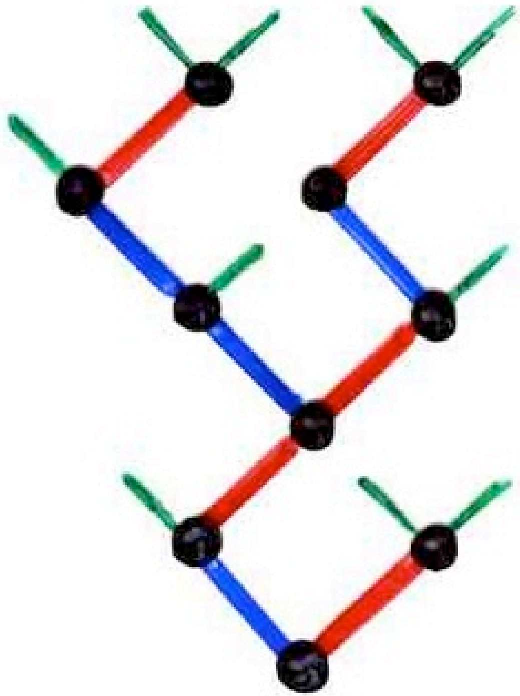

Comment. Nov. Ac. Imp. Sc. Petrop. Tom. IV. Tab. III.
Fig. 1.
Fig. 2.
Fig. 4.
Fig. 5.
Fig. 7.
2π (S-2)











Journées Pierre Leroux

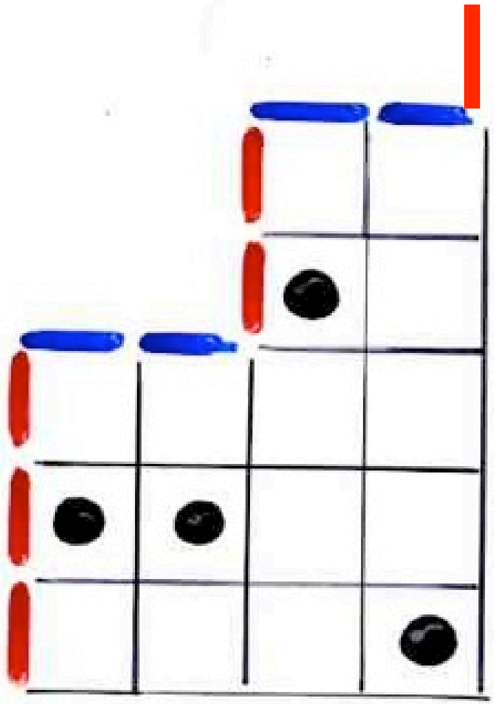
Montréal, UQAM, 8-9 Septembre 2006

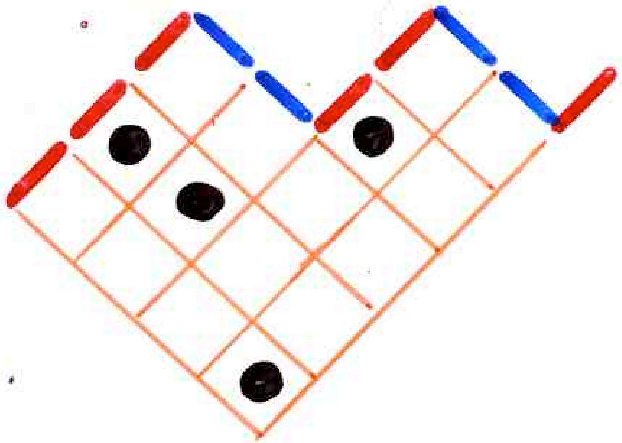
the "binary trees sliding" algorithm

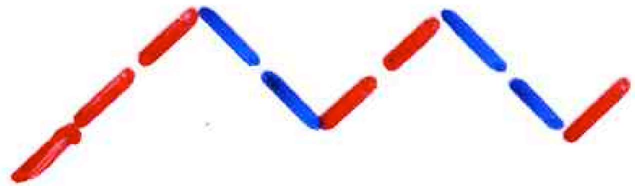
Catalan tableaux ⟷ binary trees

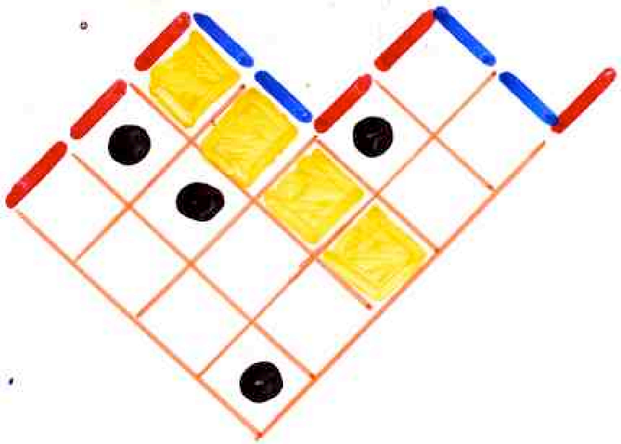
in Proc. FPSAC'07, Tienjin

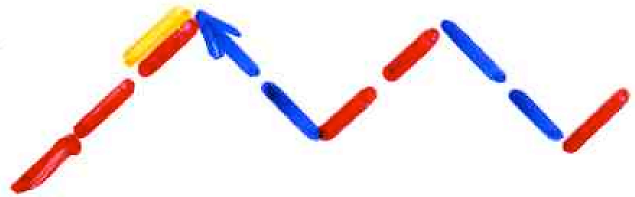
(described in term of permutation tableaux)

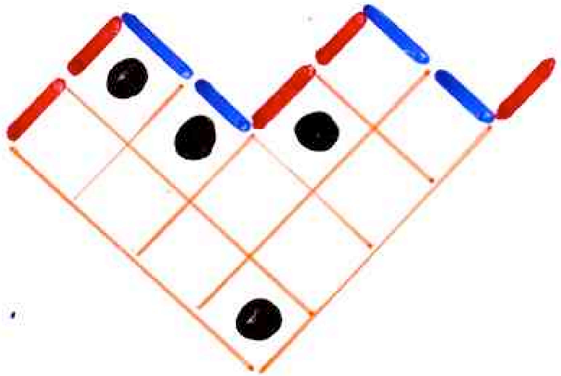


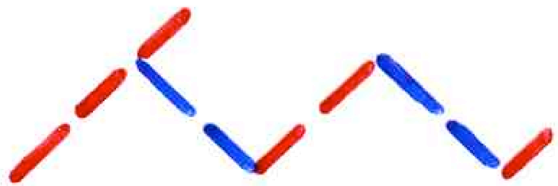


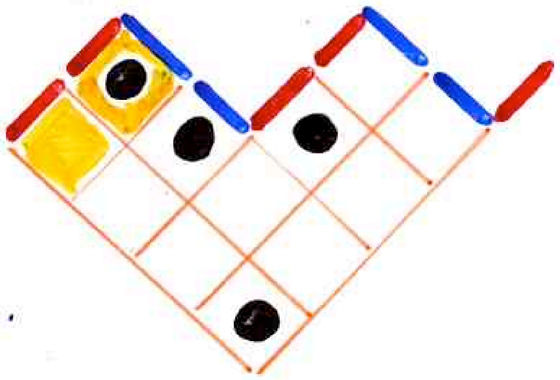


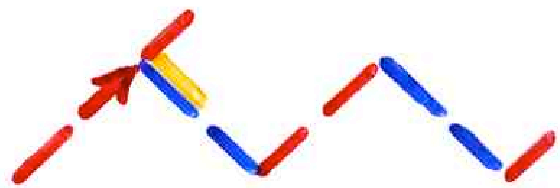


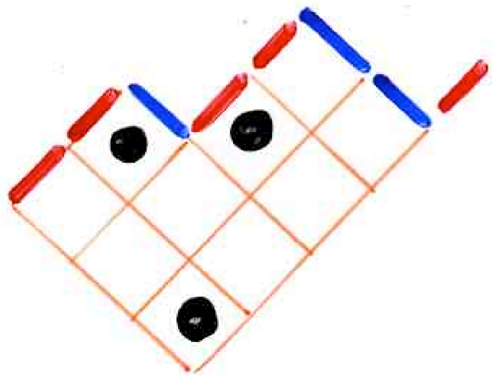


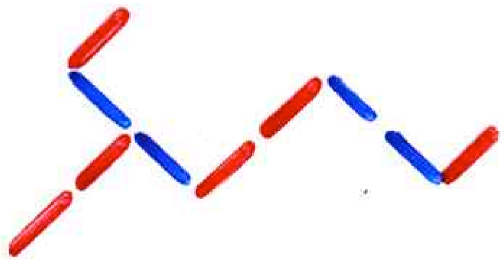


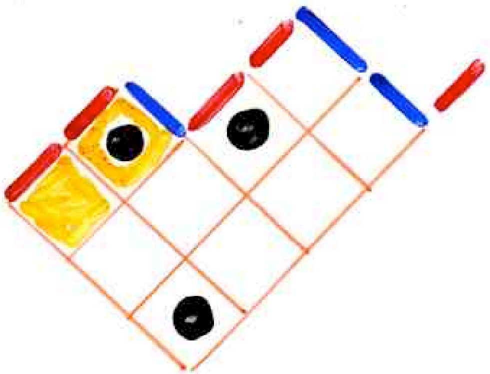


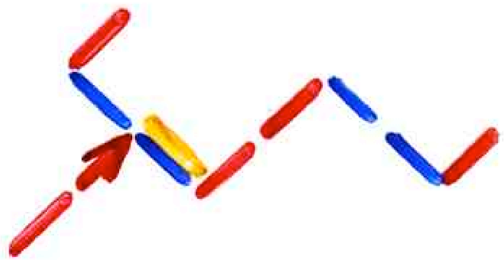


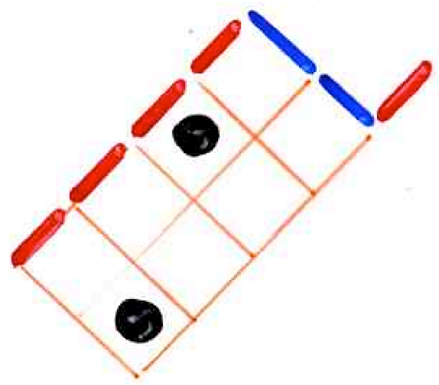


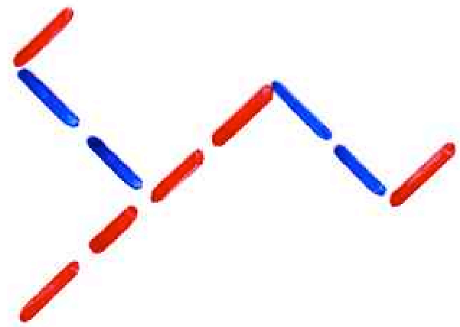


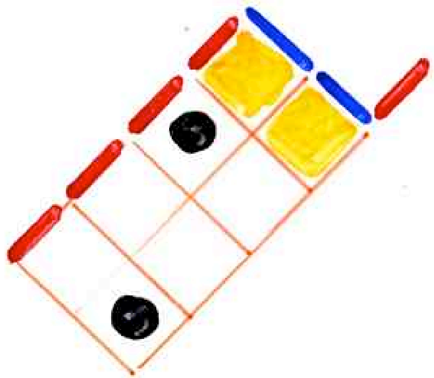


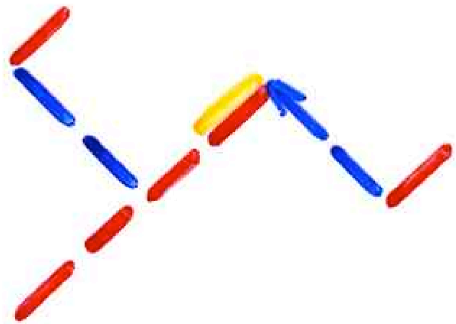


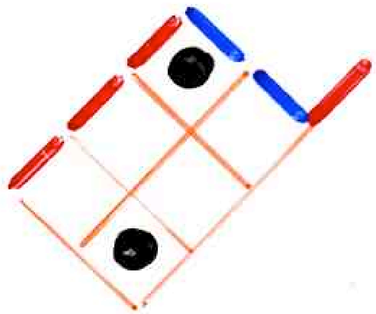


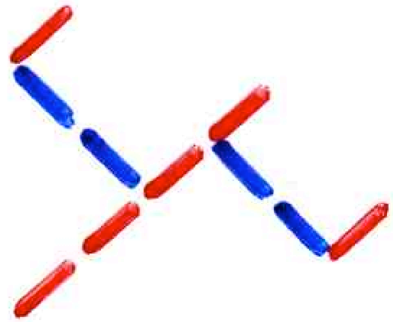


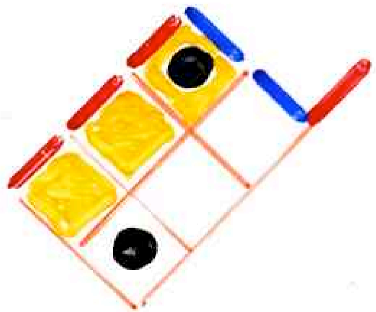


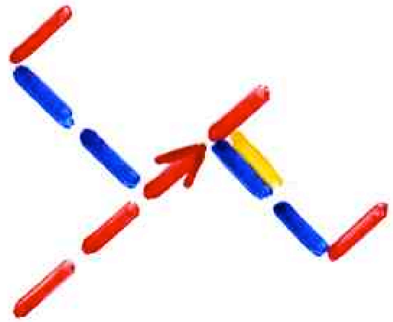


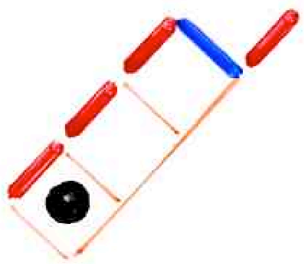


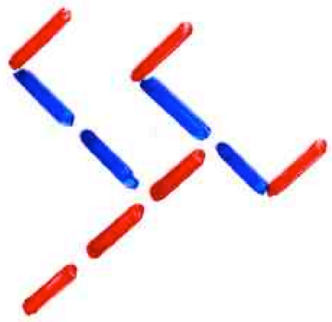


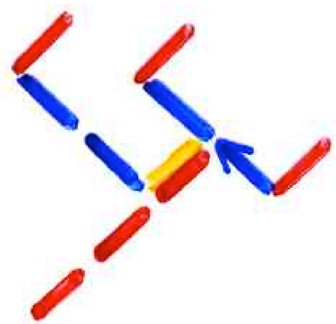


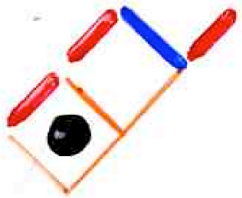


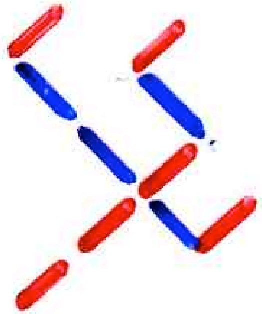


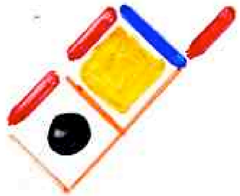


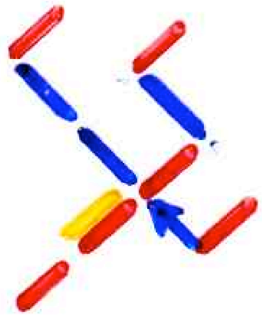


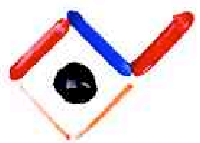


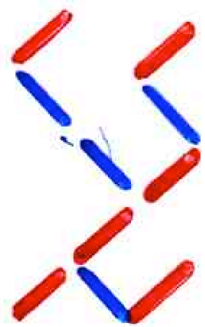


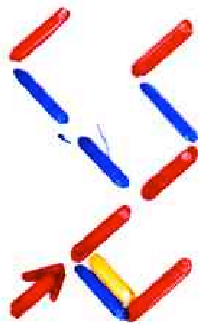


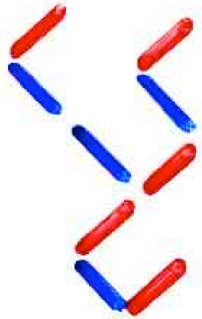


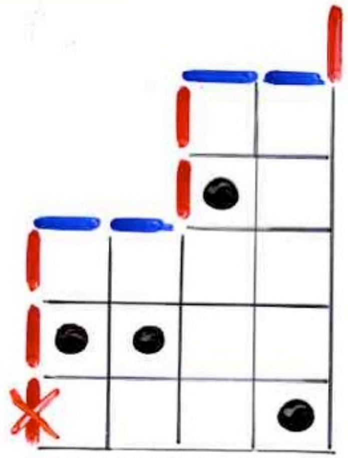


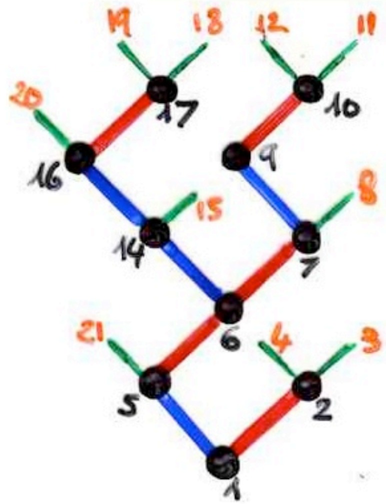


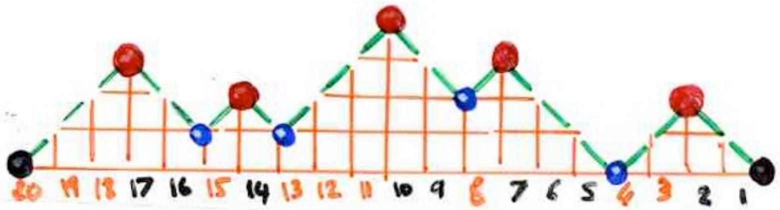


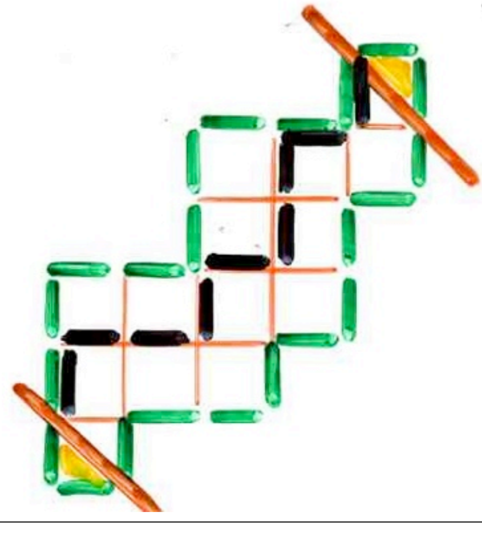







19 18 12 11
20 17 10
16 9 8
15 14 7
6
21 4 3
5 2
1


20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1



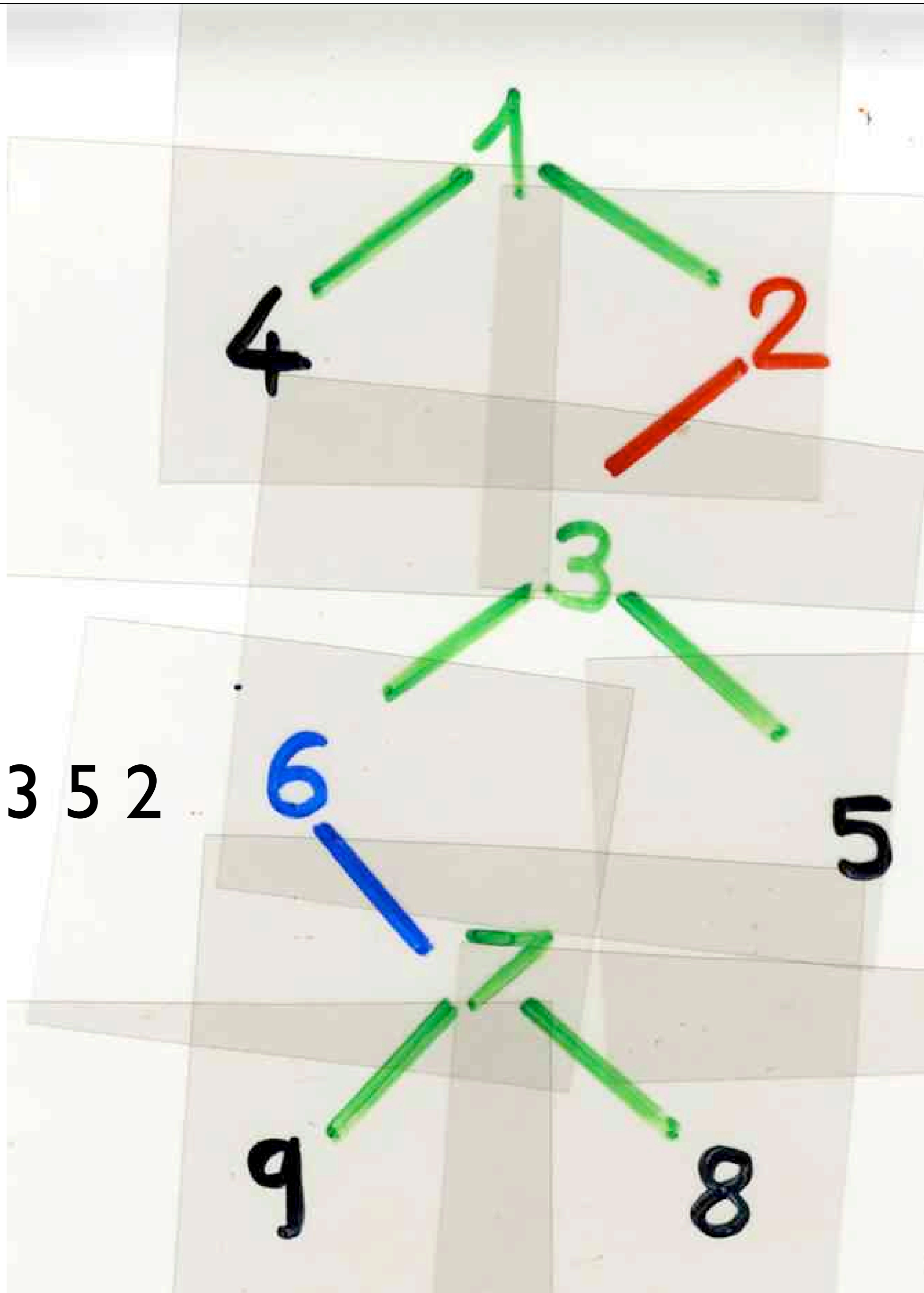
The bijection presented at Tienjin FPSAC'07 between binary trees and "Catalan permutation tableaux", once rewritten in term in terms of "Catalan alternating tableaux" (which is immediate to do), can be viewed as a particular case of the inverse of the "exchange-fusion" algorithm.

This "binary tree sliding algorithm" can be extended to permutations and gives a bijection between alternative tableaux and a new kind of binary trees introduced by P. Nadeau in his forthcoming paper under the name of "alternative binary tree"

P. Nadeau "alternative binary tree"

"increasing binary tree" and associated permutation

4 1 6 9 7 8 3 5 2



§12
an alternative approach to alternating sign matrices

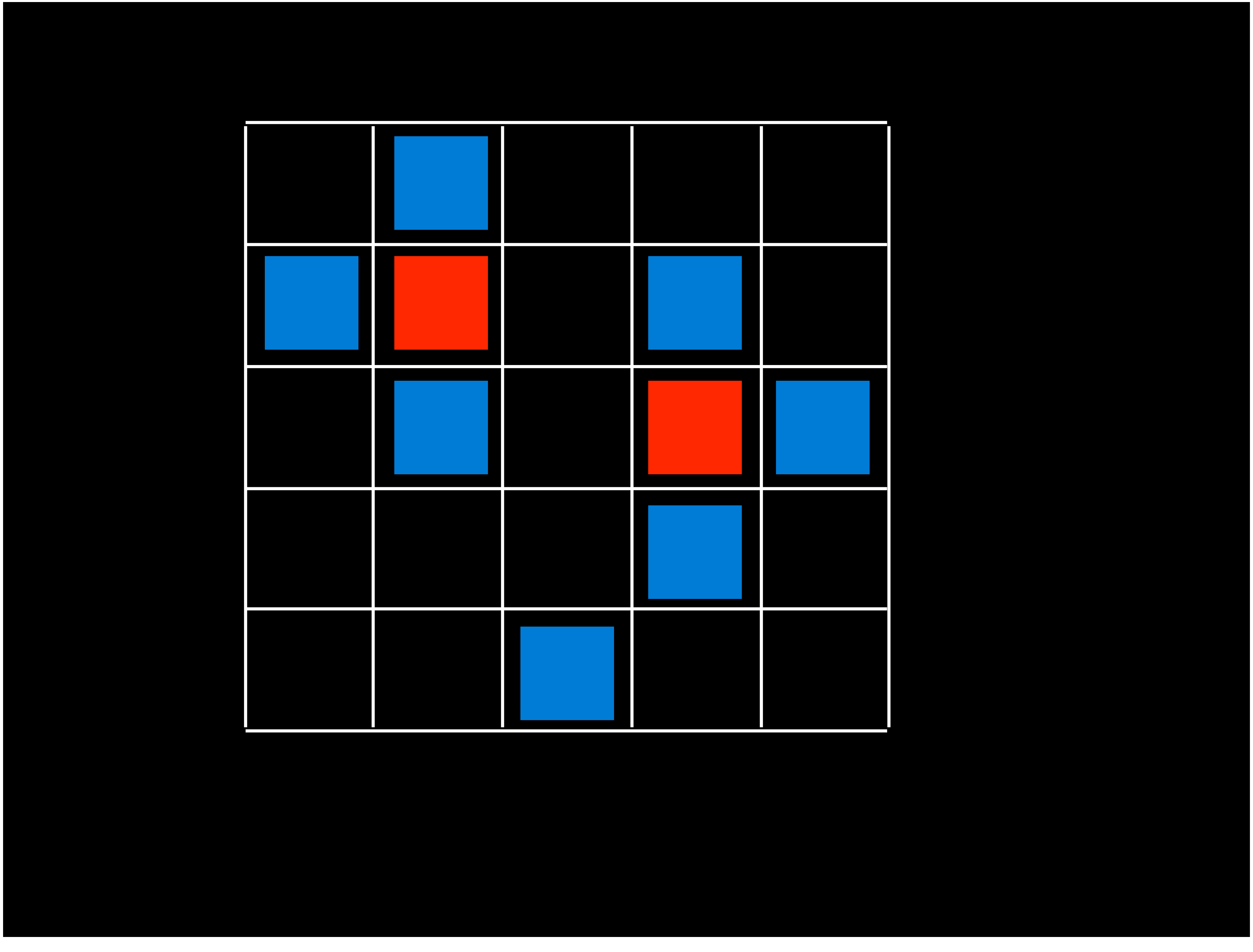
Def- **ASM** alternating sign matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(i) entries: $0, 1, -1$

(ii) sum of entries in each (row / column) $= 1$

(iii) non-zero entries alternate in each { row / column



Alernating sign matrices: at the crossroads of algebra, combinatorics and physics",

colloquium au CMUC (Centro de Matematica da Universidade do Coimbra), Portugal,
26 Sept 2008, 17 h

A, A', B, B',

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

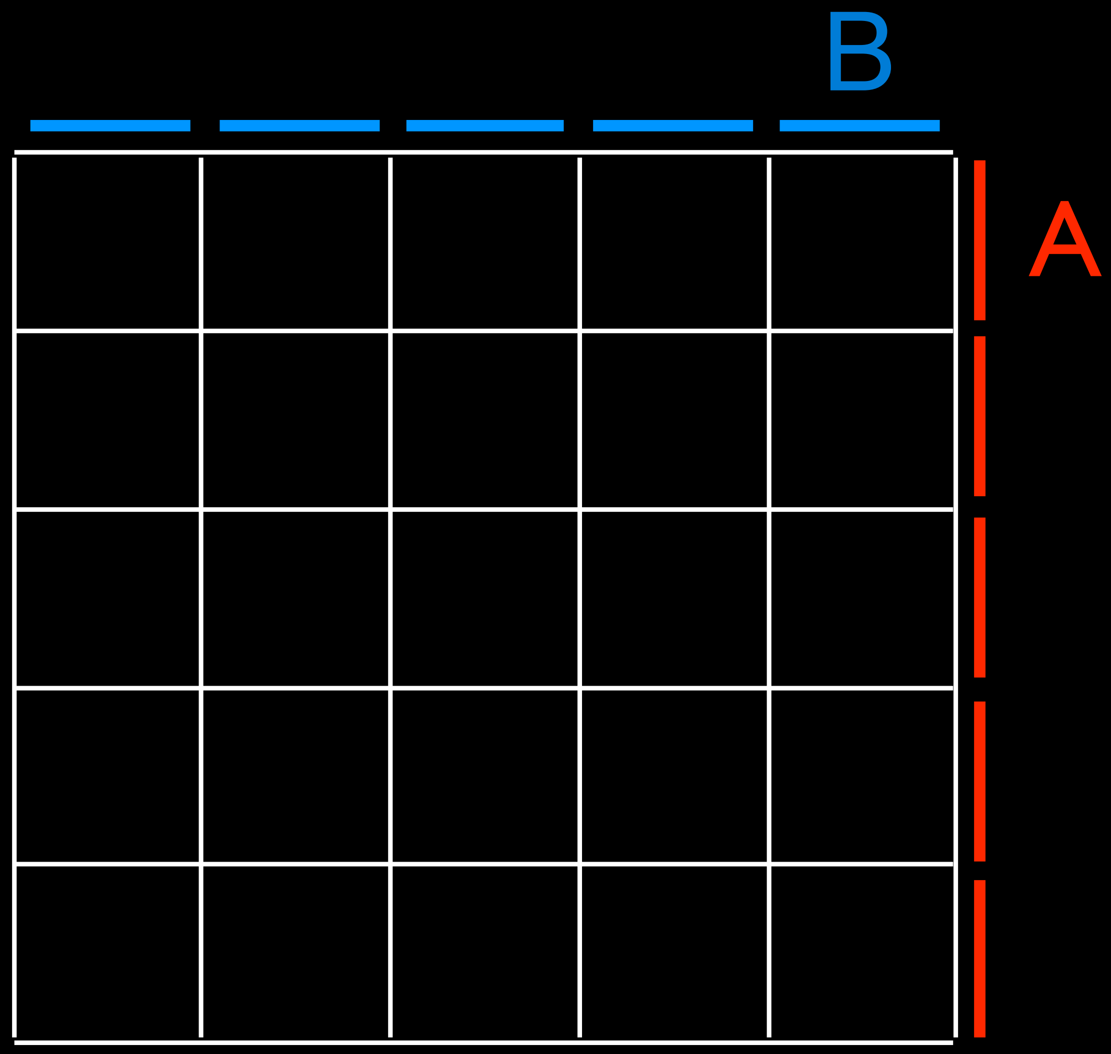
Lemma. Any word $w(A, A', B, B')$ in letters A, A', B, B', can be uniquely written

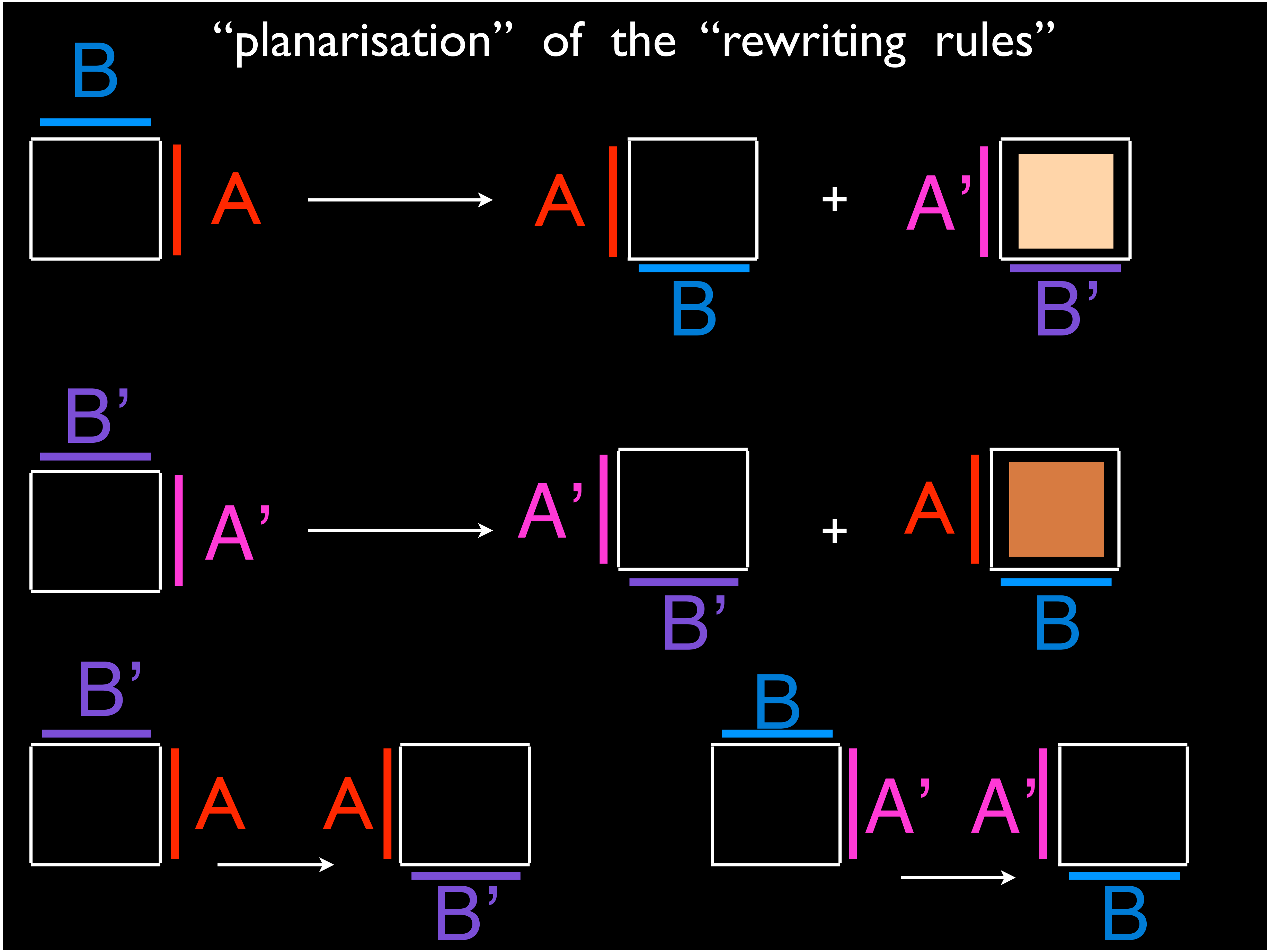
$$\sum C(u,v;w)\ \underbrace{u(A,A')}_{\text{word in } A,\, A'}\ \underbrace{v(B,B')}_{\text{word in } B,\, B'}$$

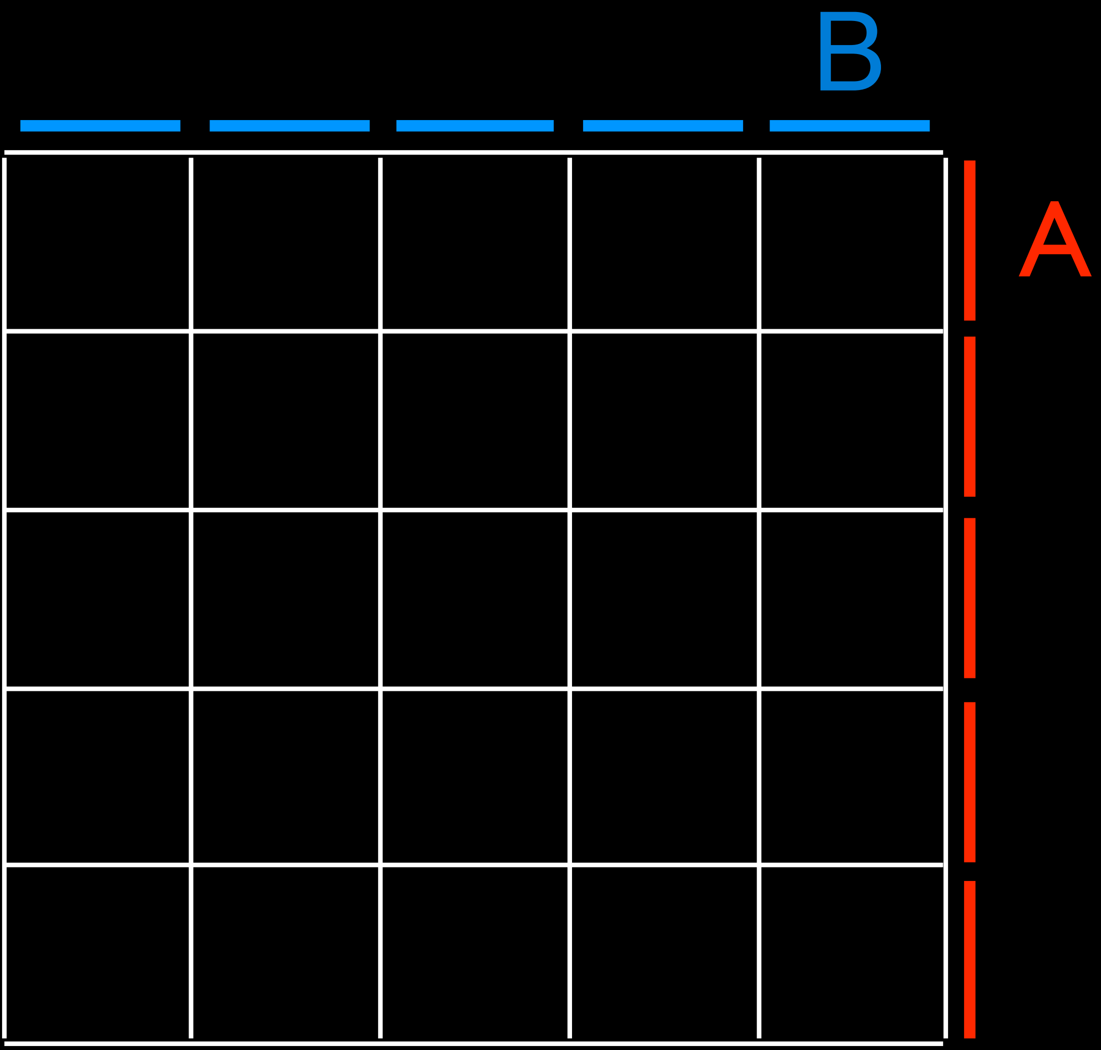
Prop. For $w = B^n A^m$

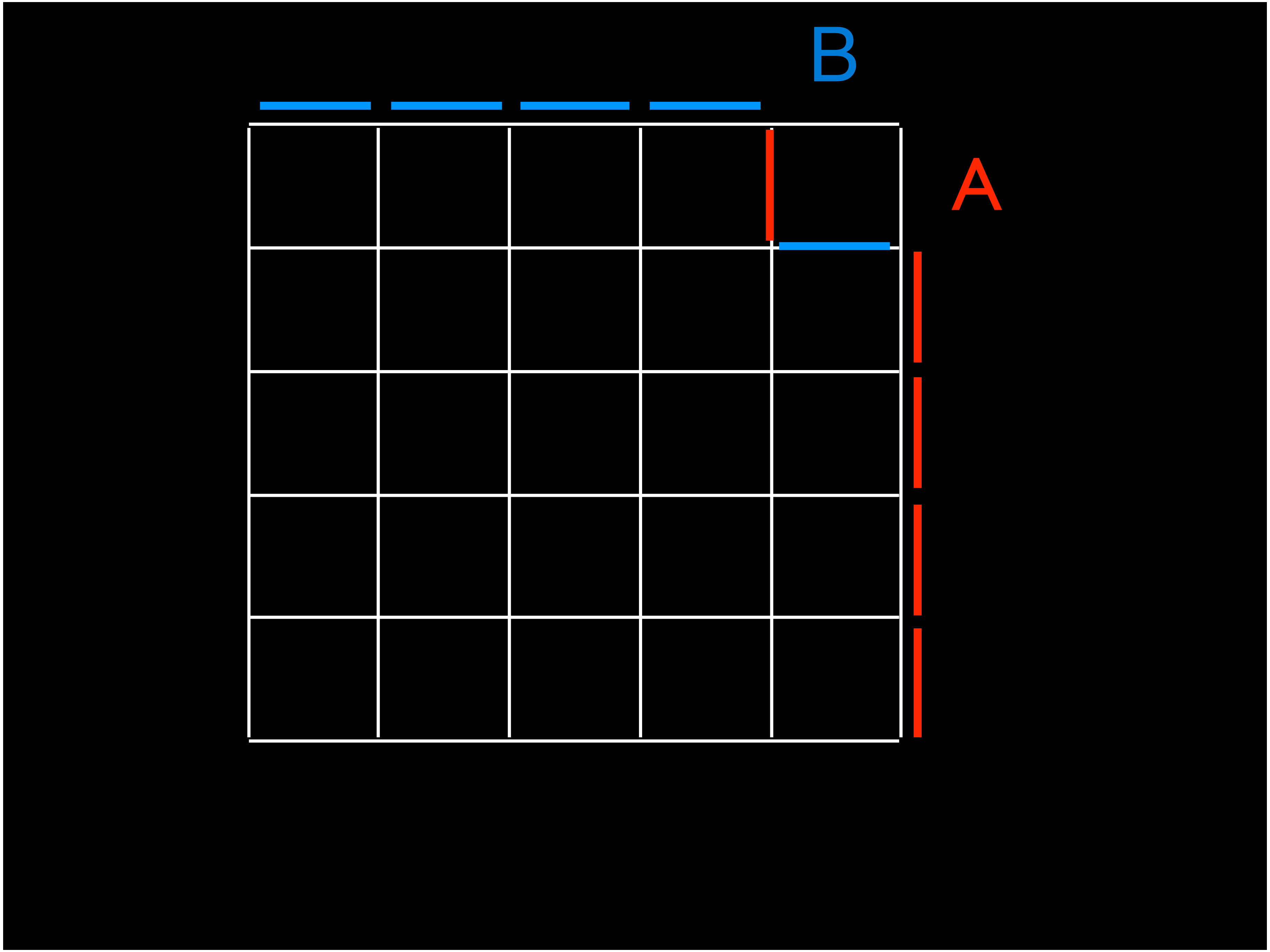
$u = A'^n \;,\; v = B'^n$

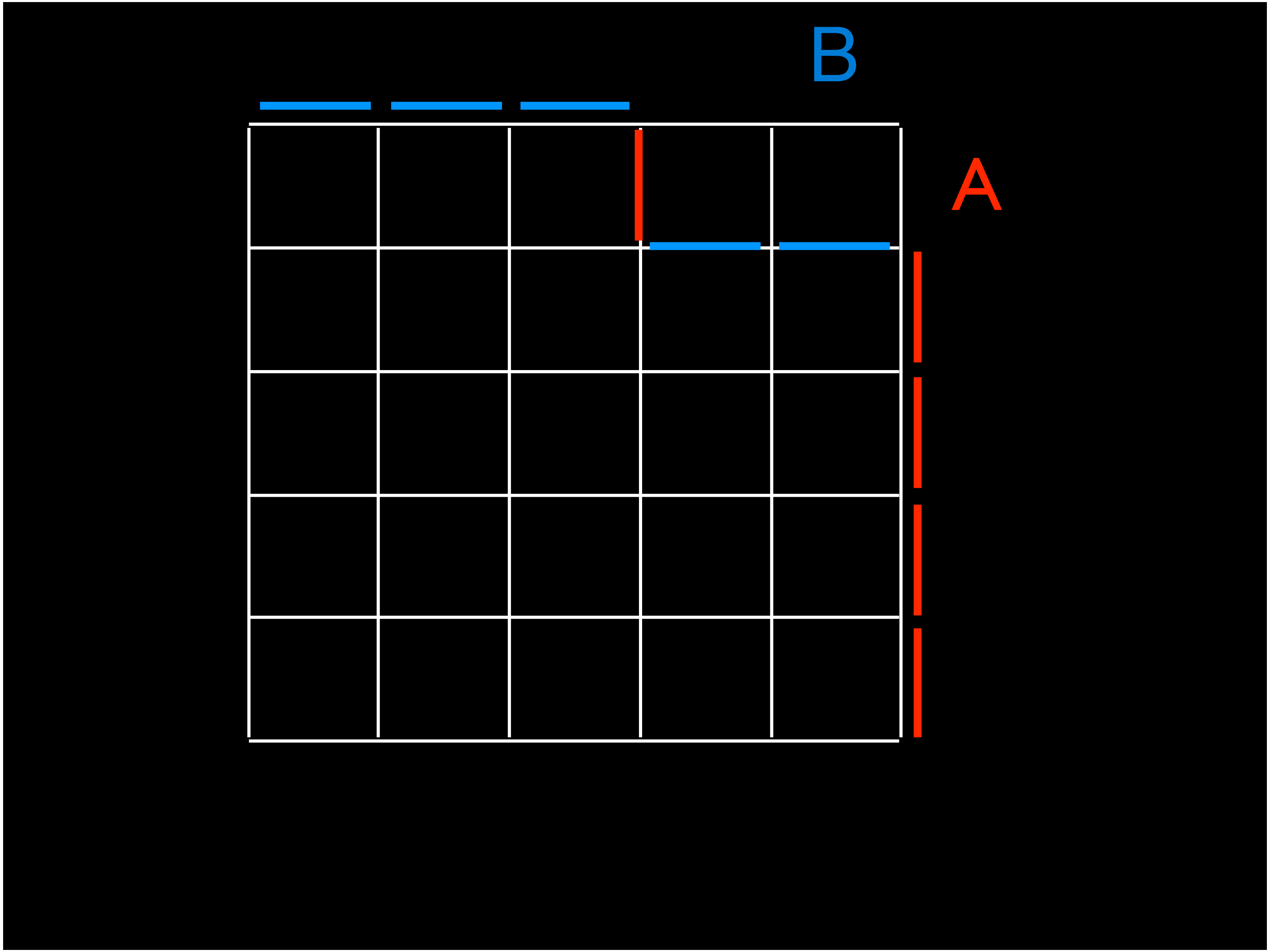
$C(u,v;w)$ = the number of $n \times n$ ASM (alternating sign matrices) “planar” proof:

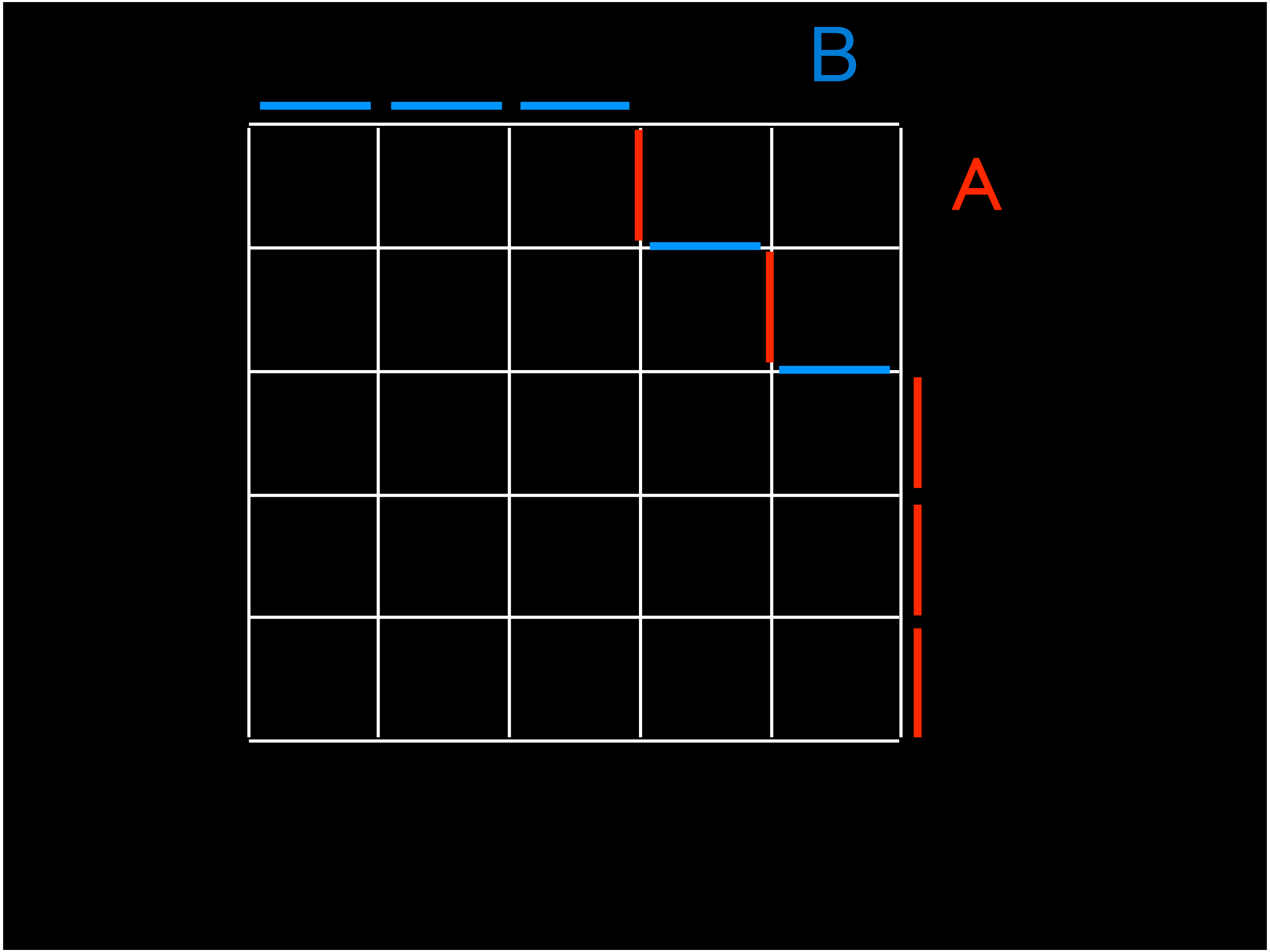



"planarisation" of the "rewriting rules"
B
A
A
B
+
A'
B'
B'
A'
A'
B'
+
A
B
B'
A
A
B'
B
A'
A'
B


B
A


B
A


B
A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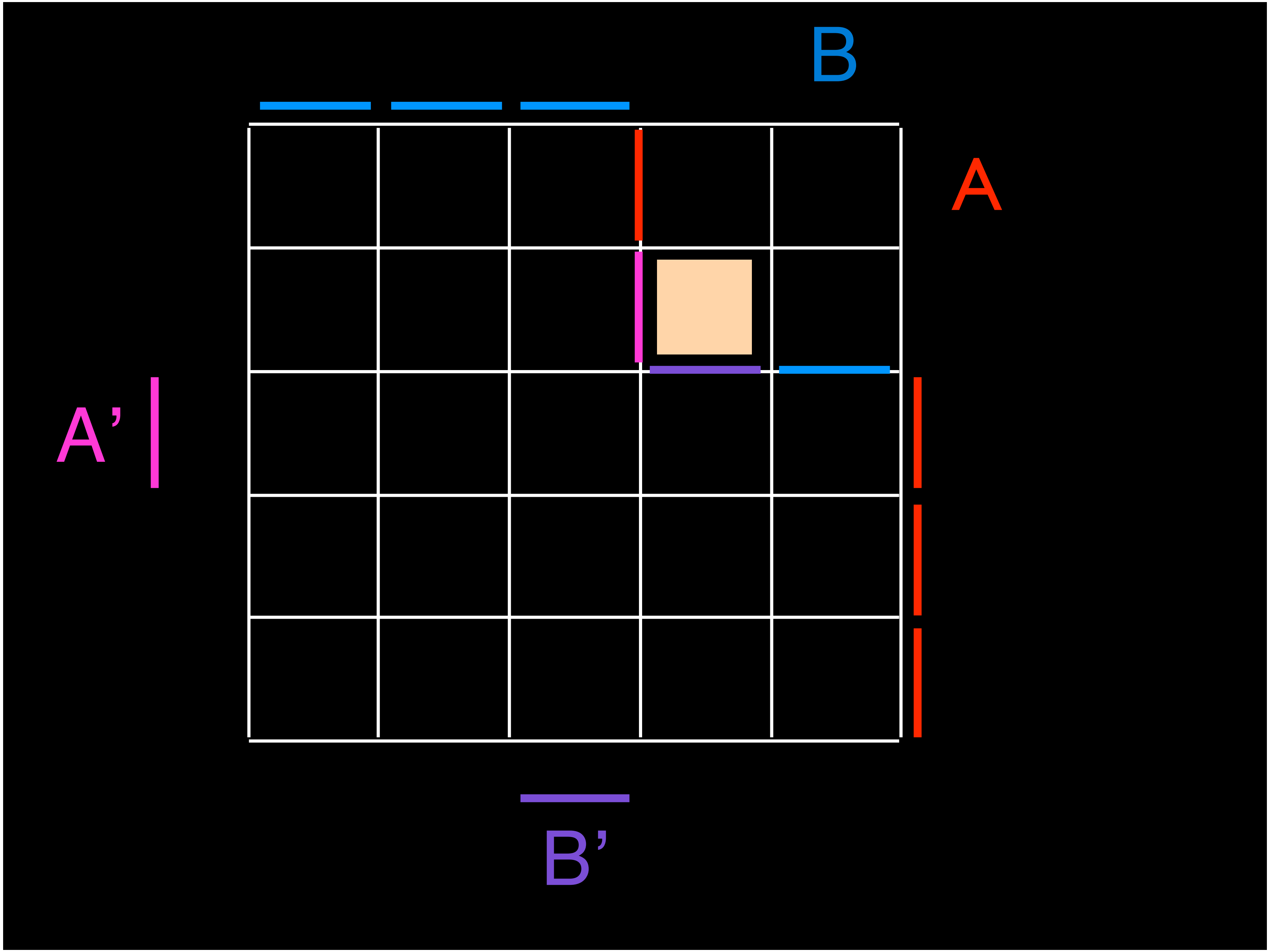

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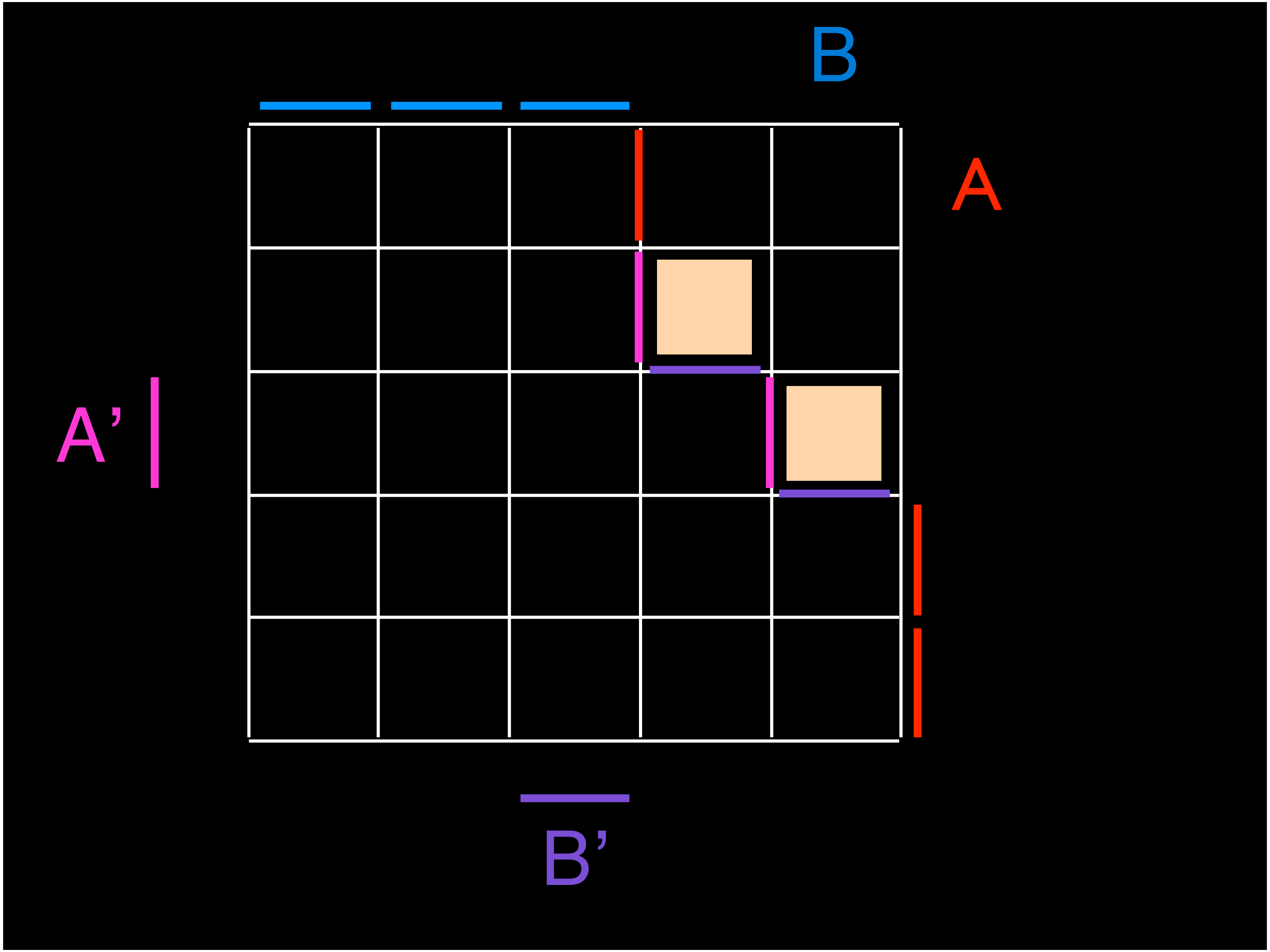
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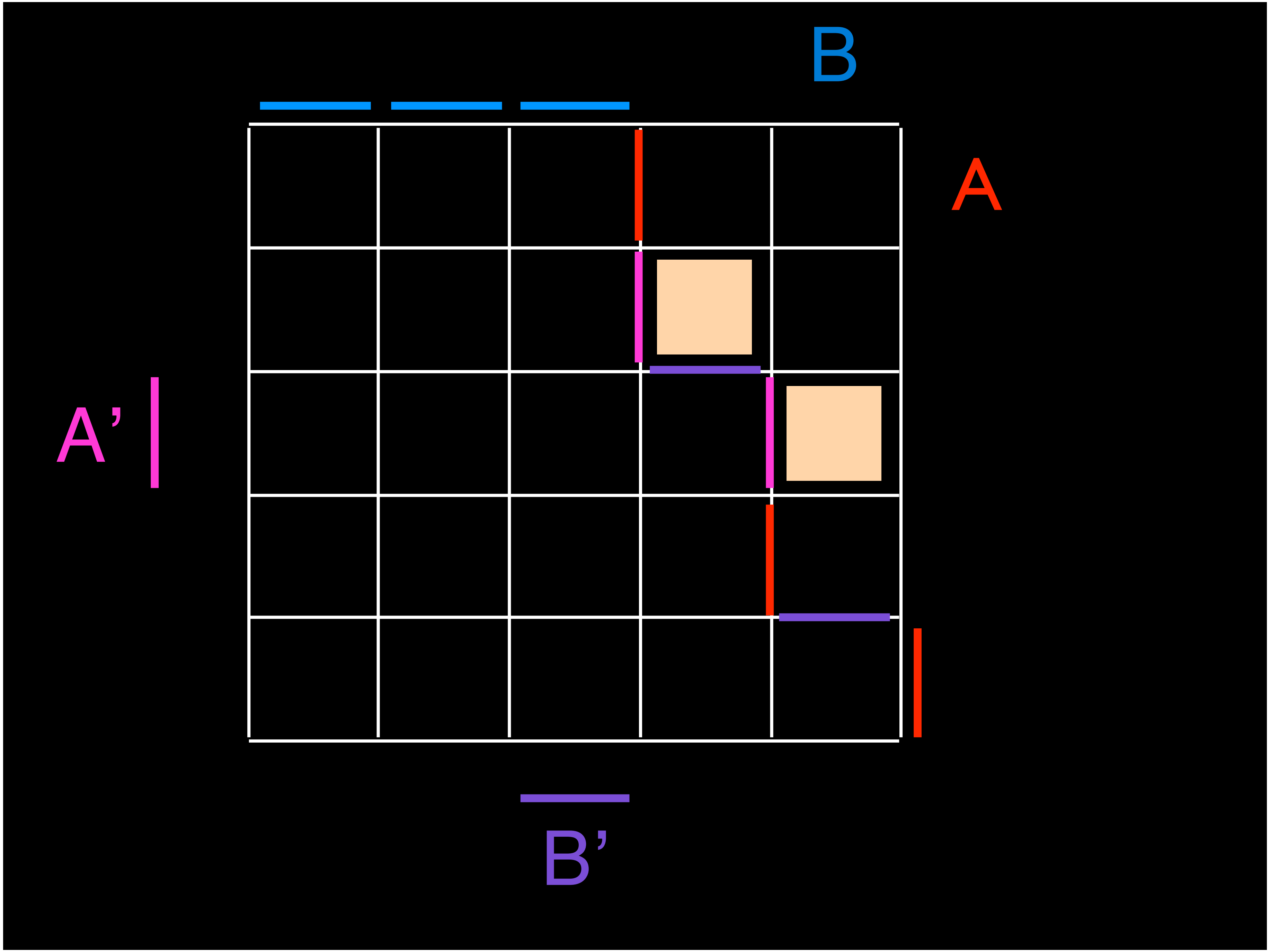
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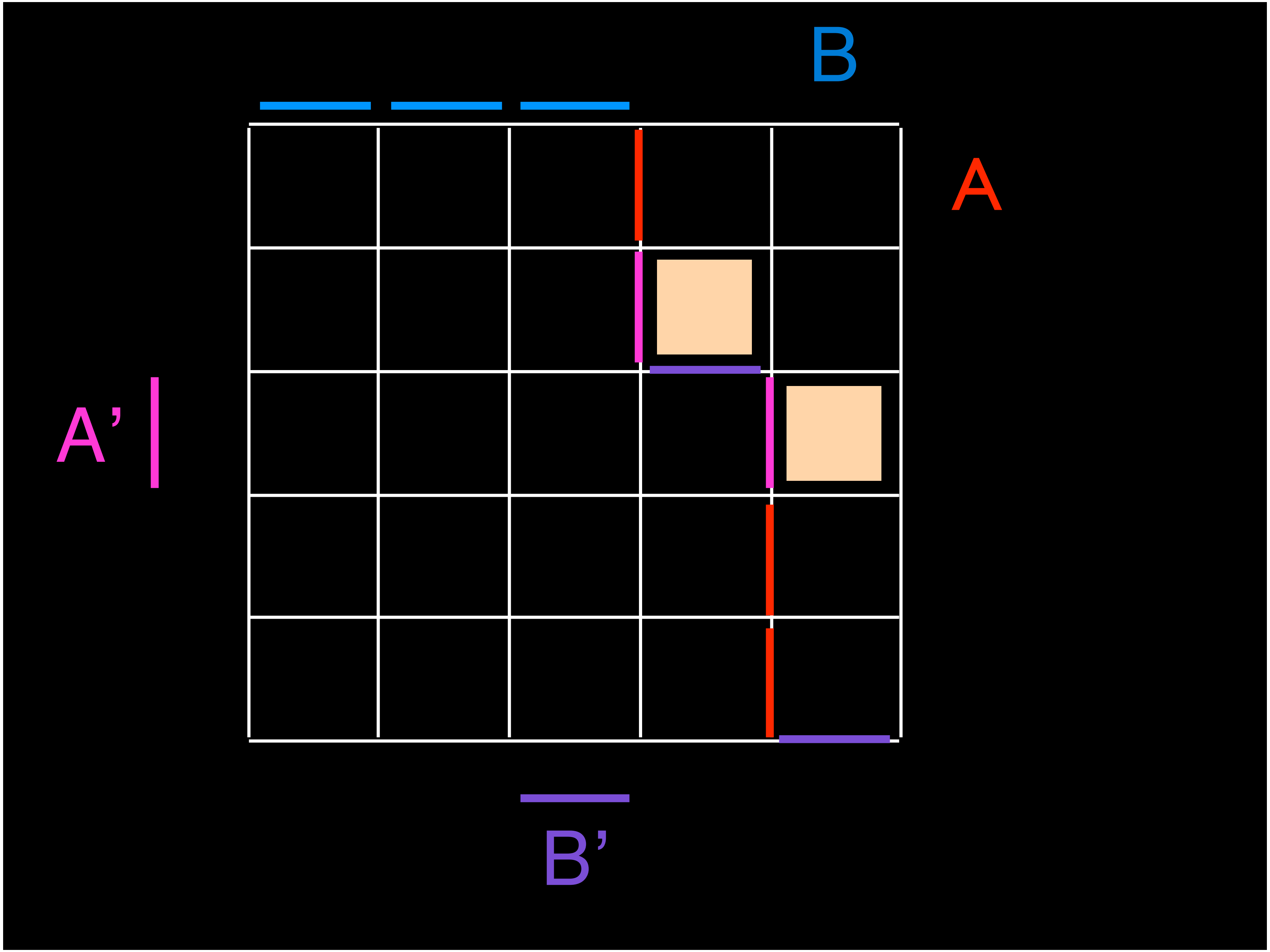

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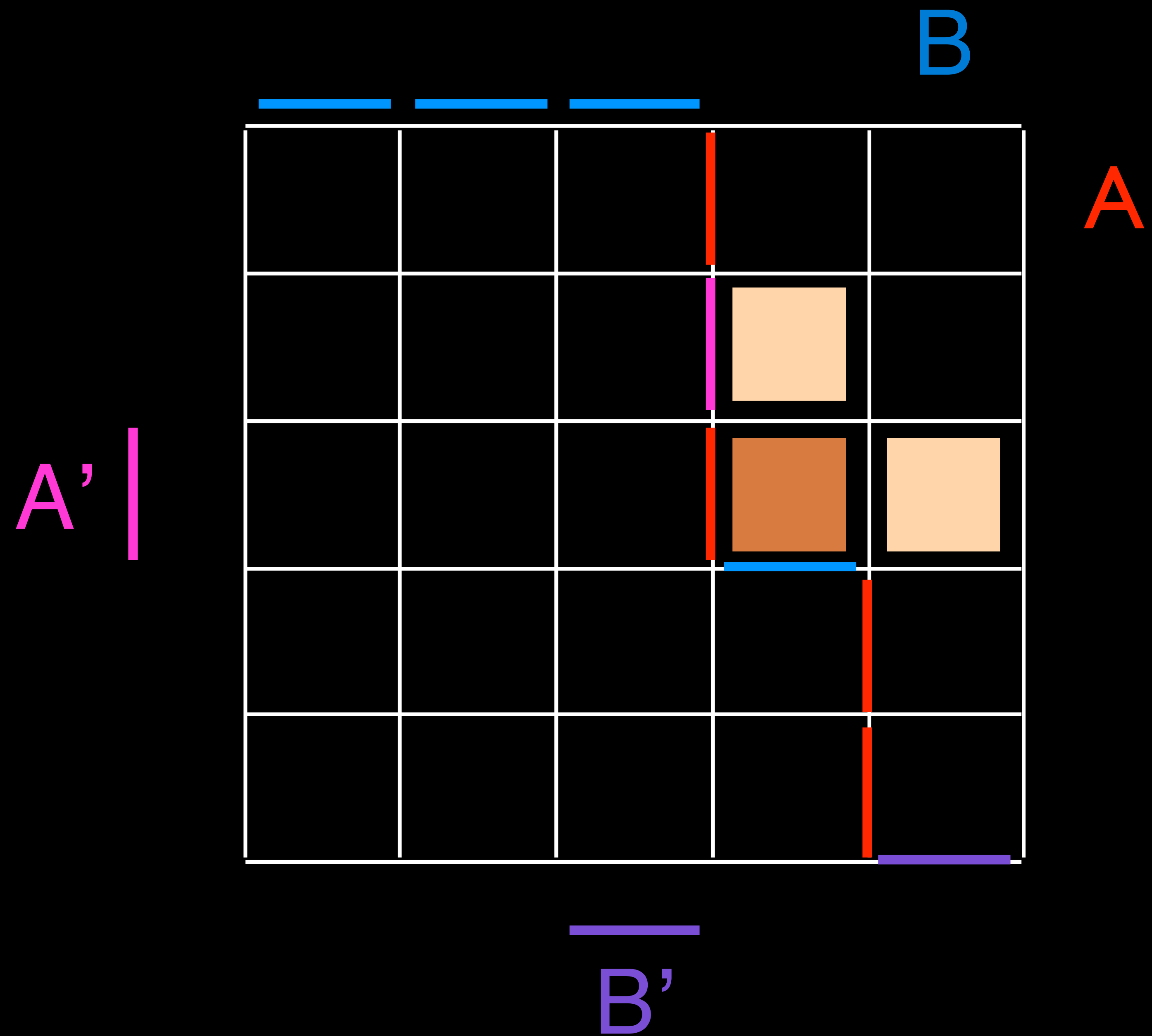
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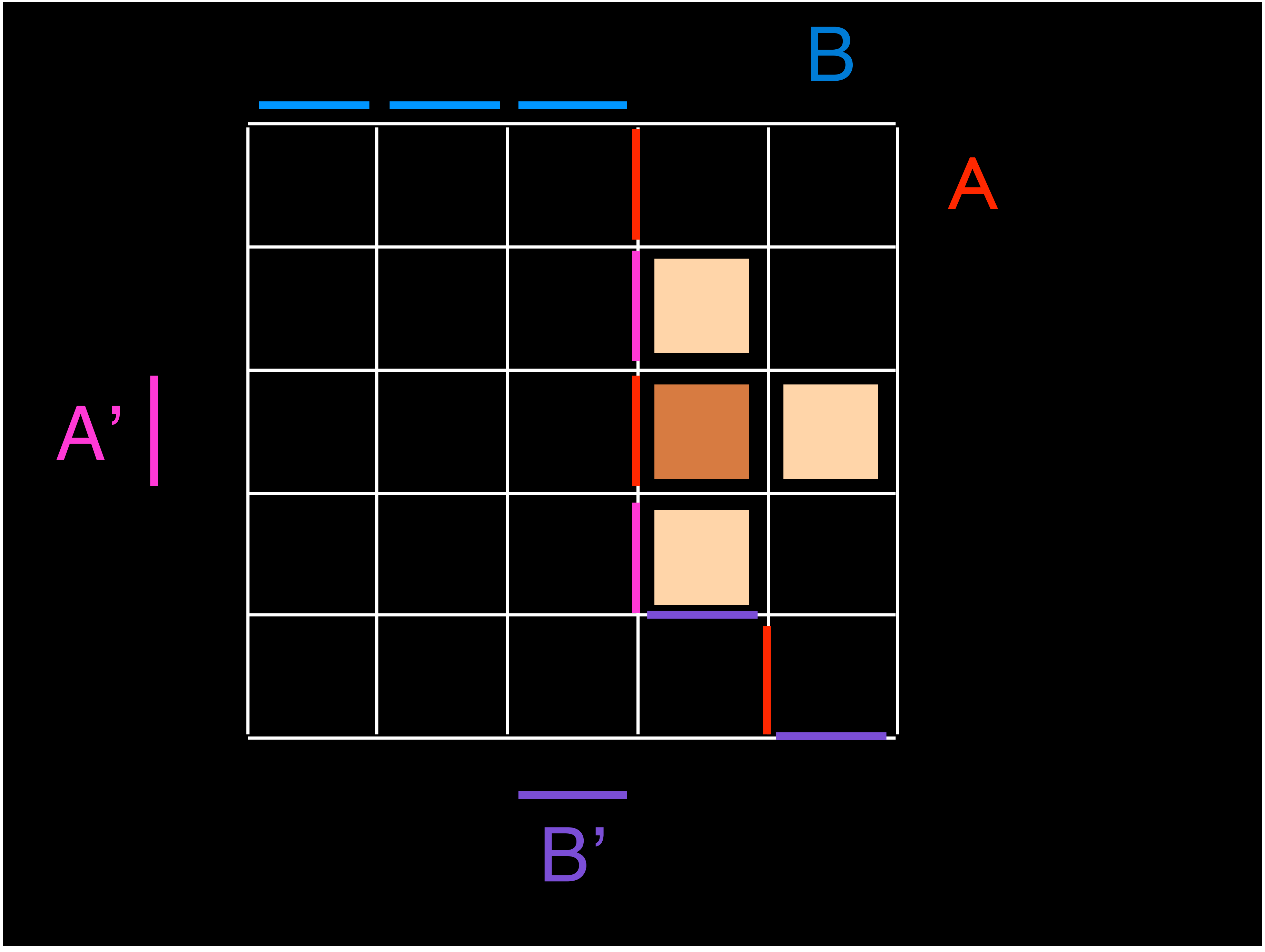

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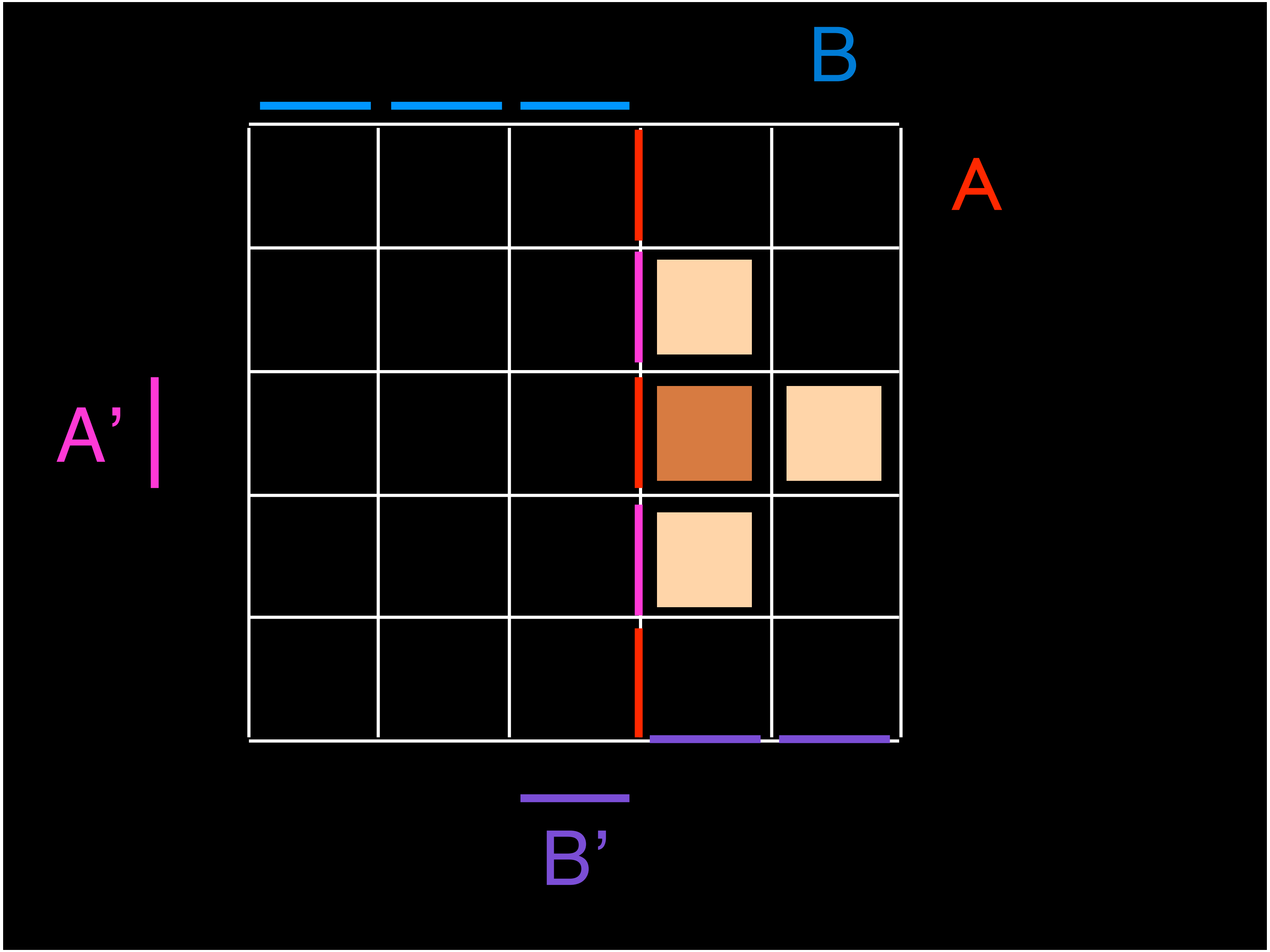

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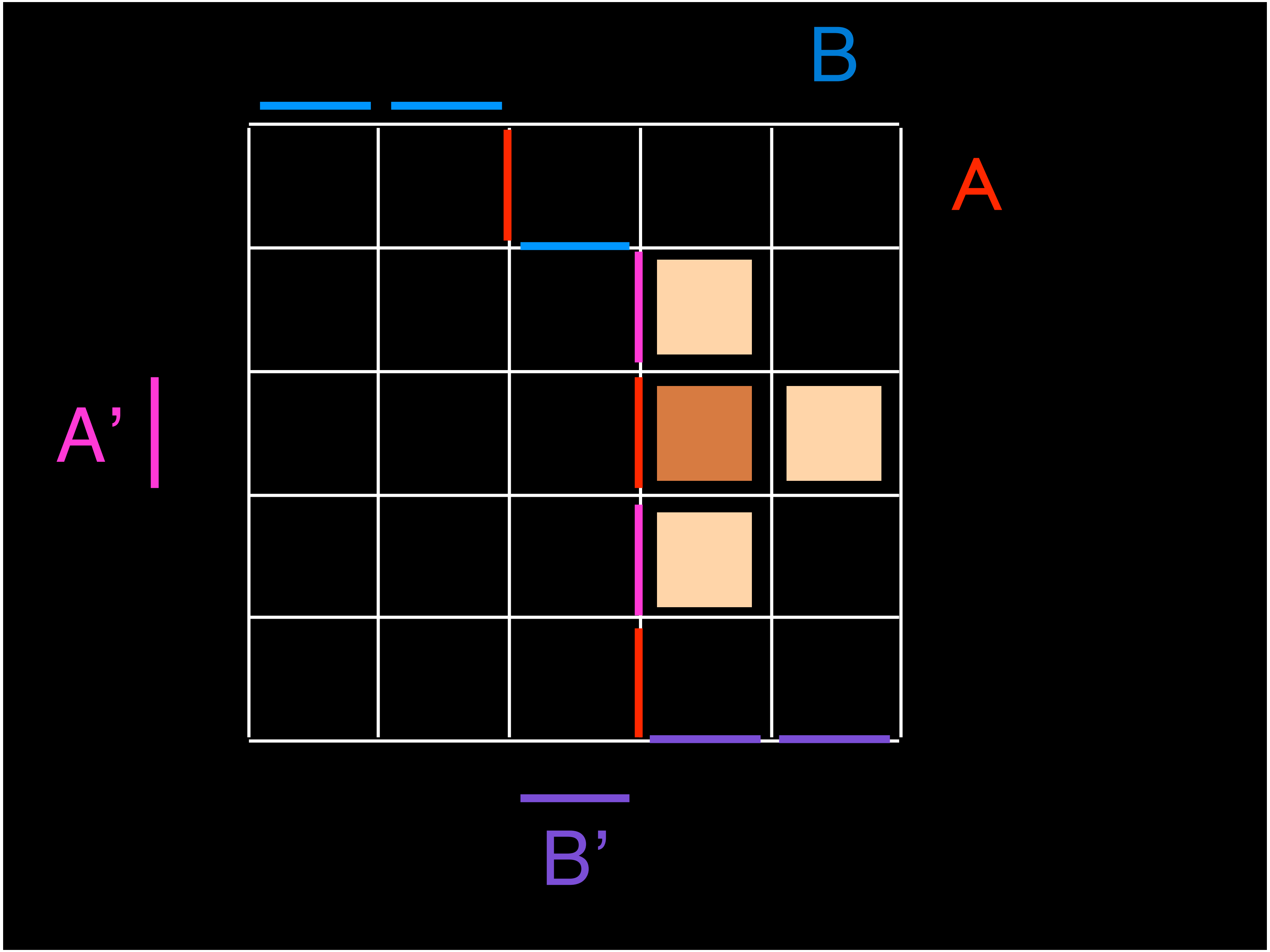

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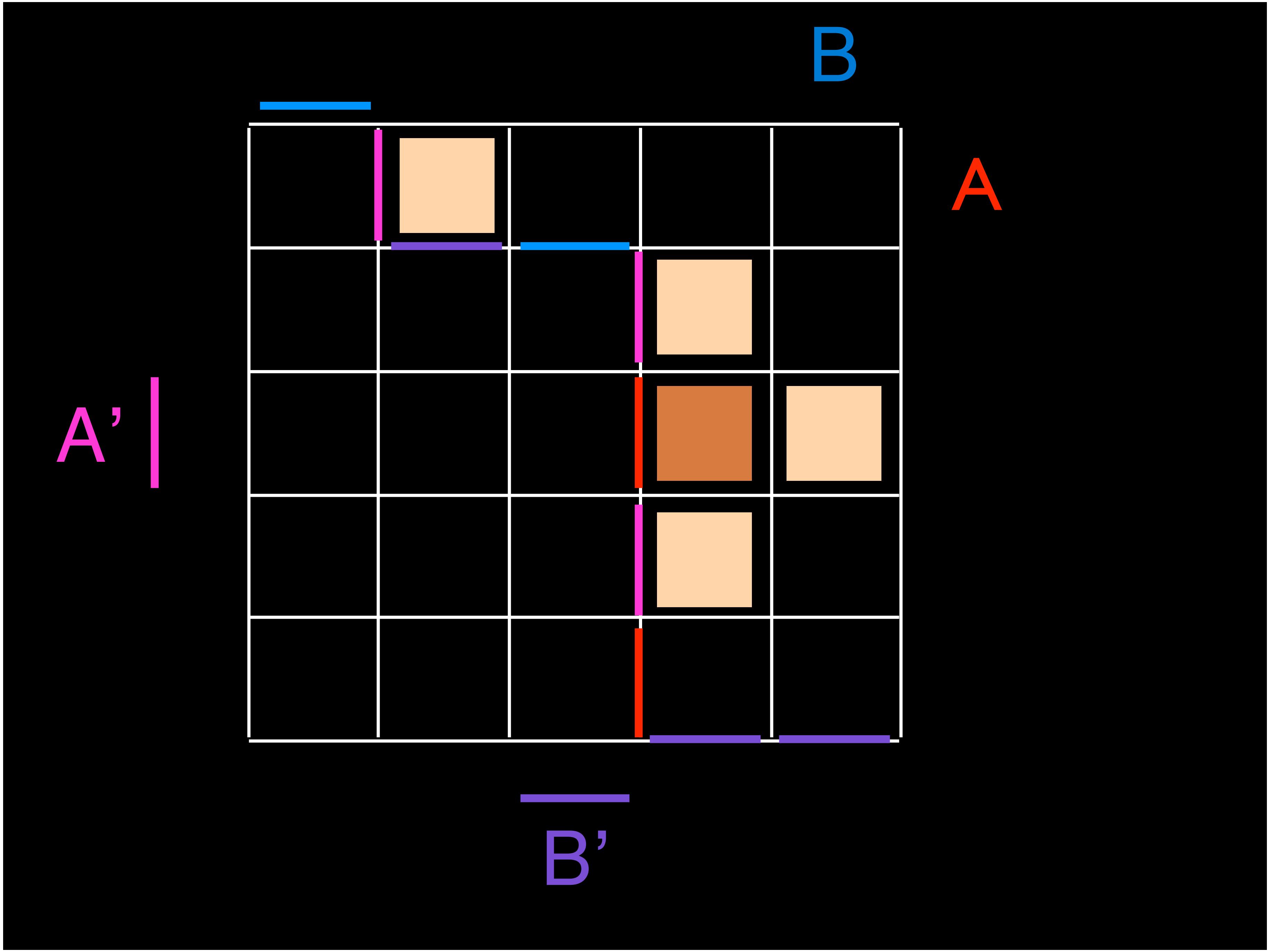
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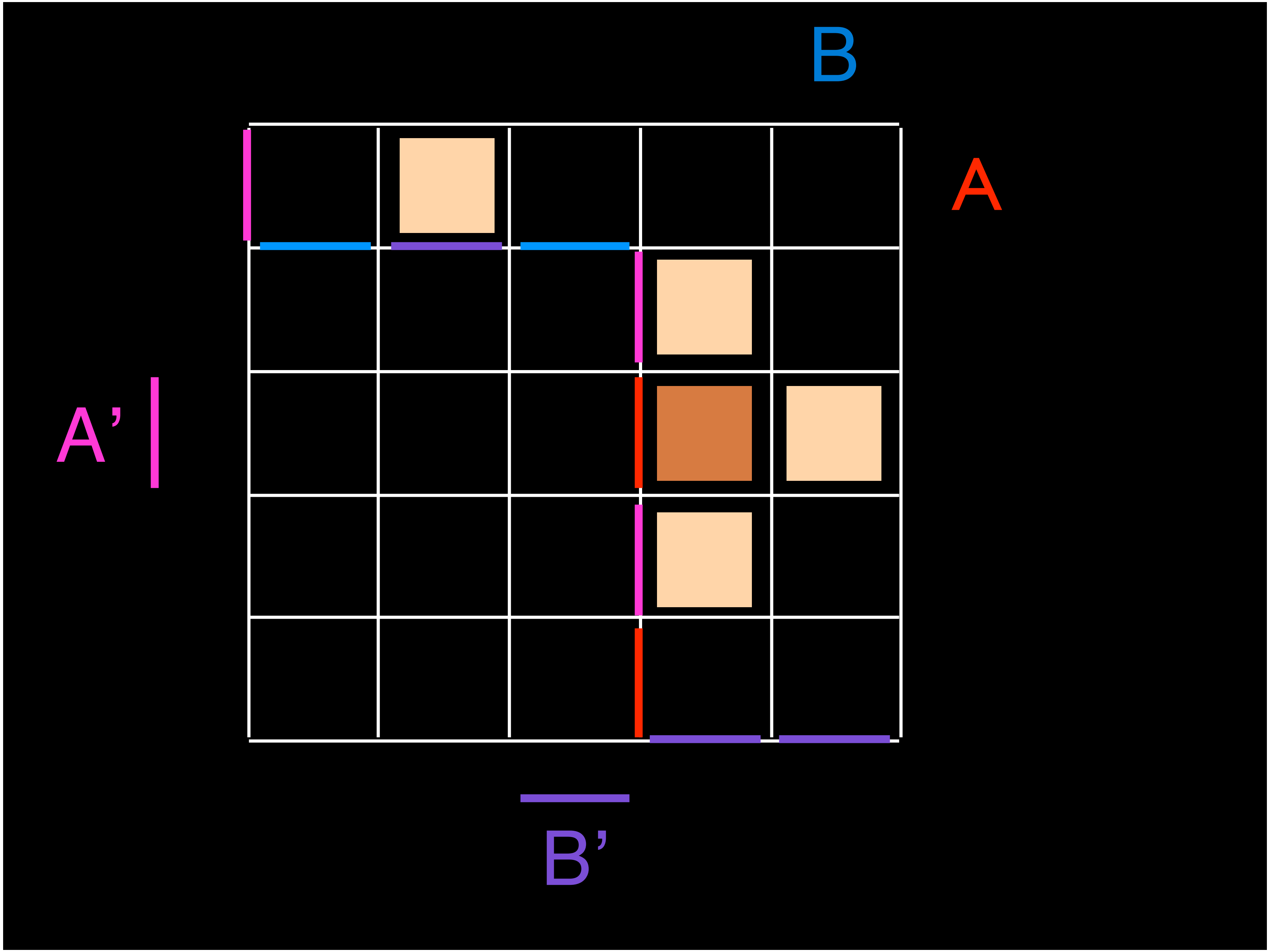
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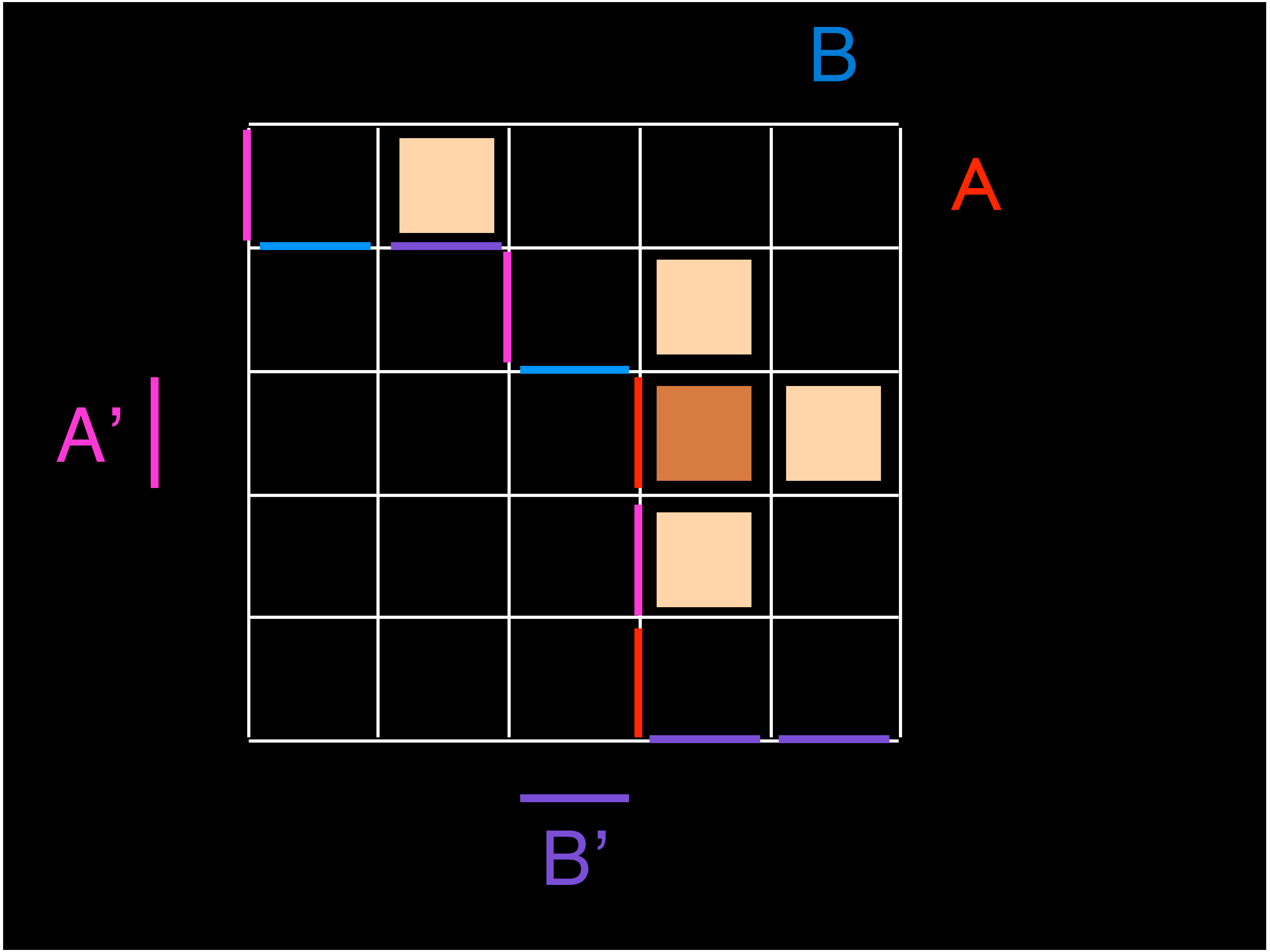
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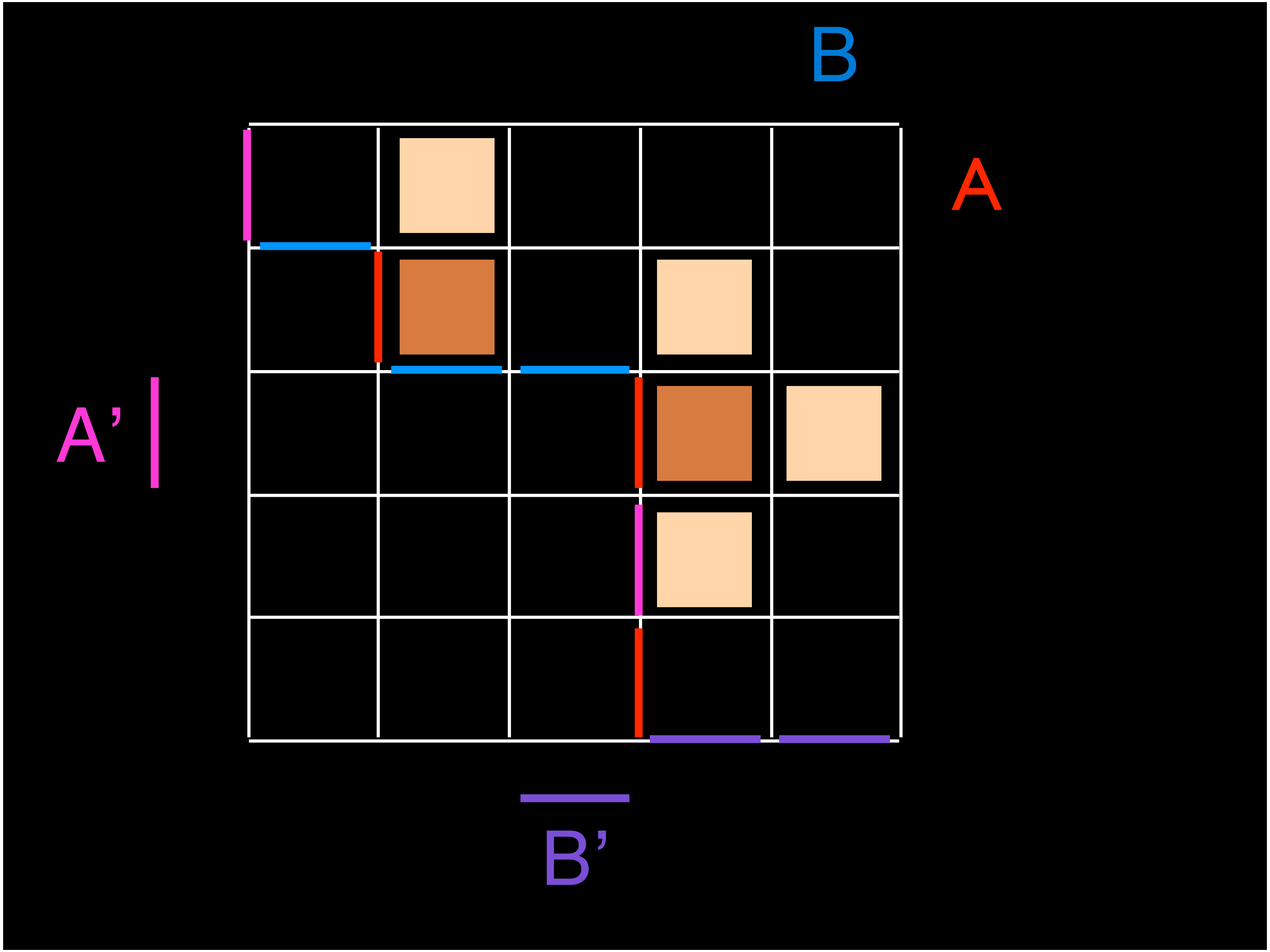
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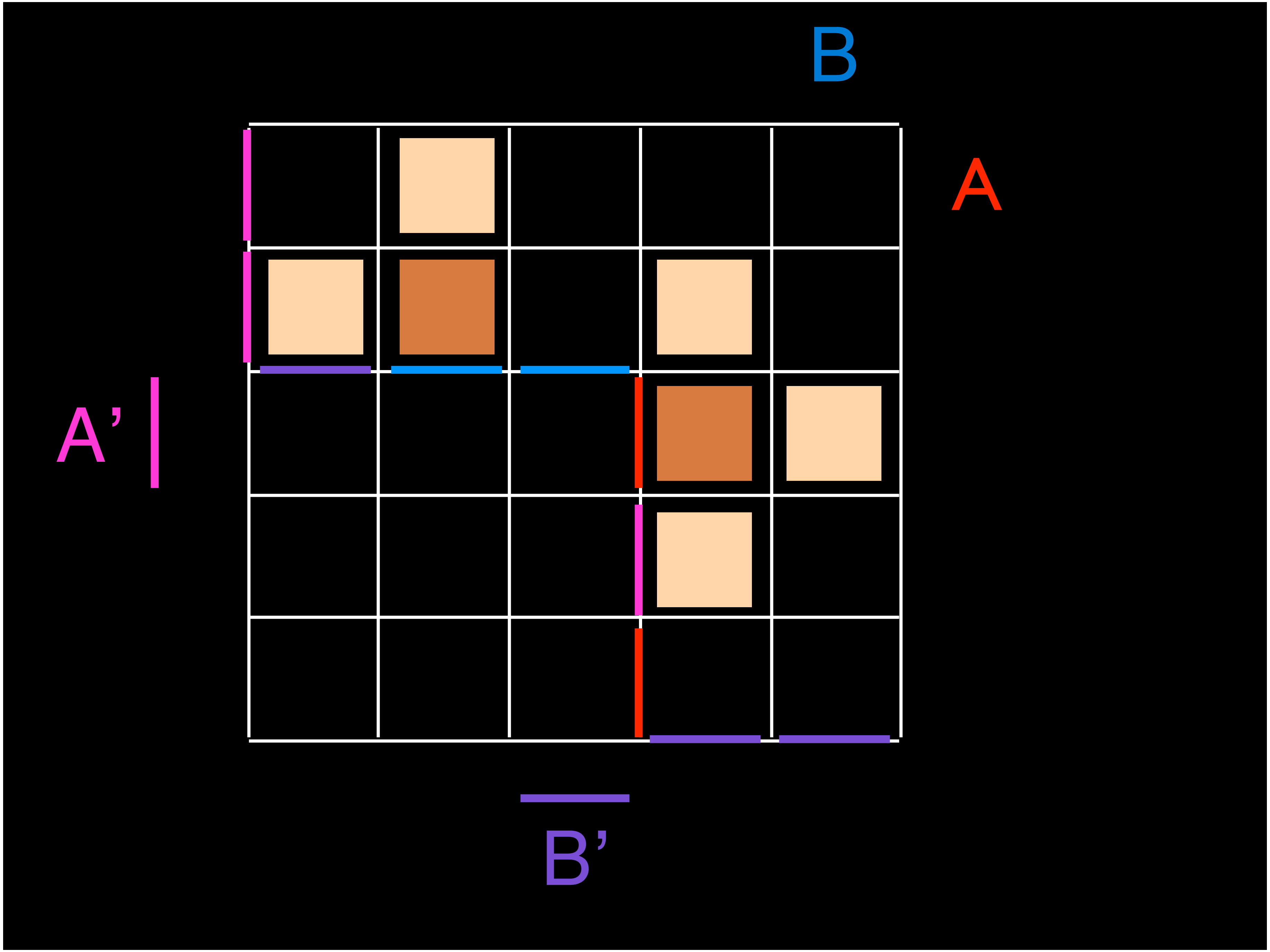

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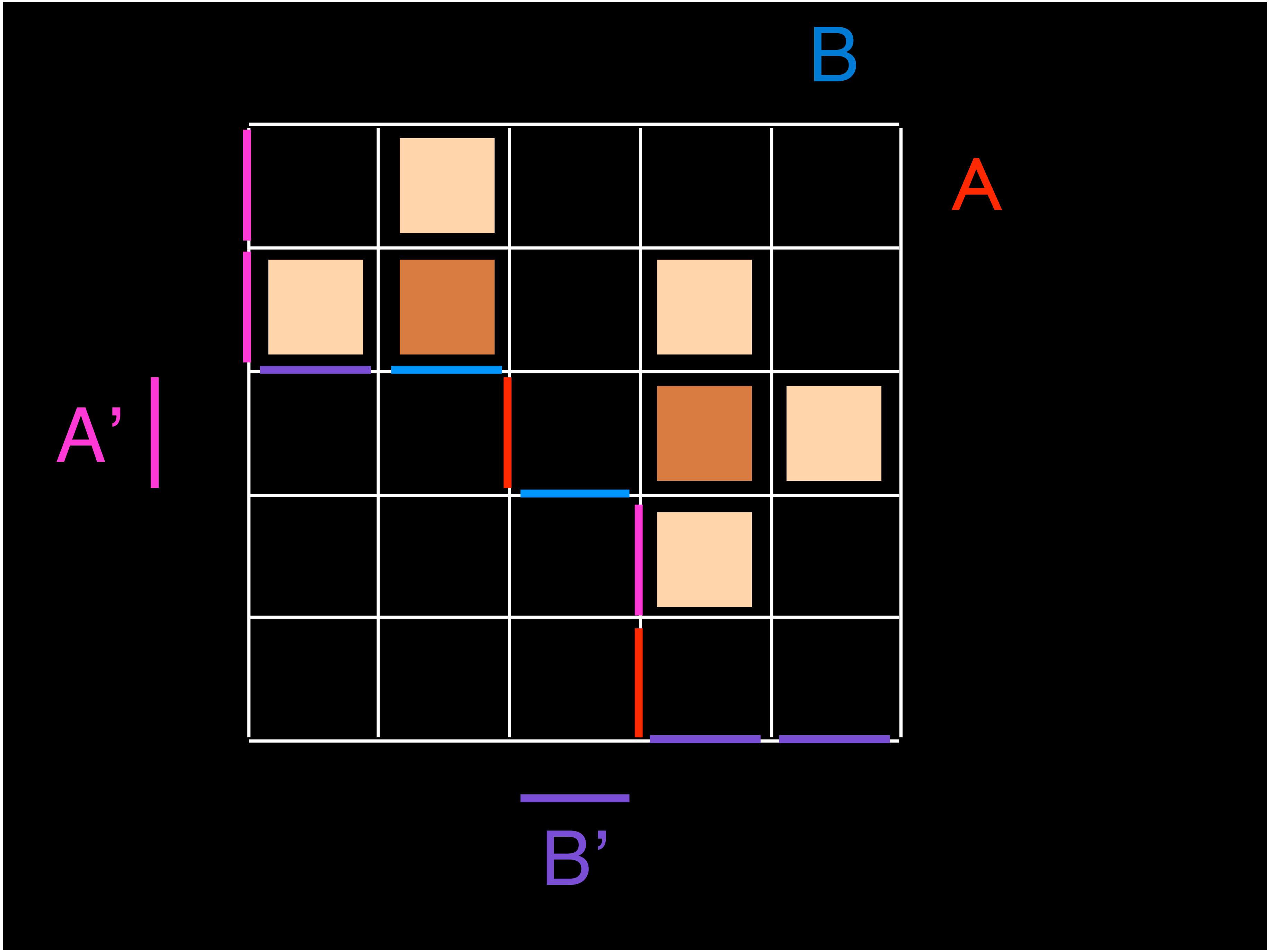

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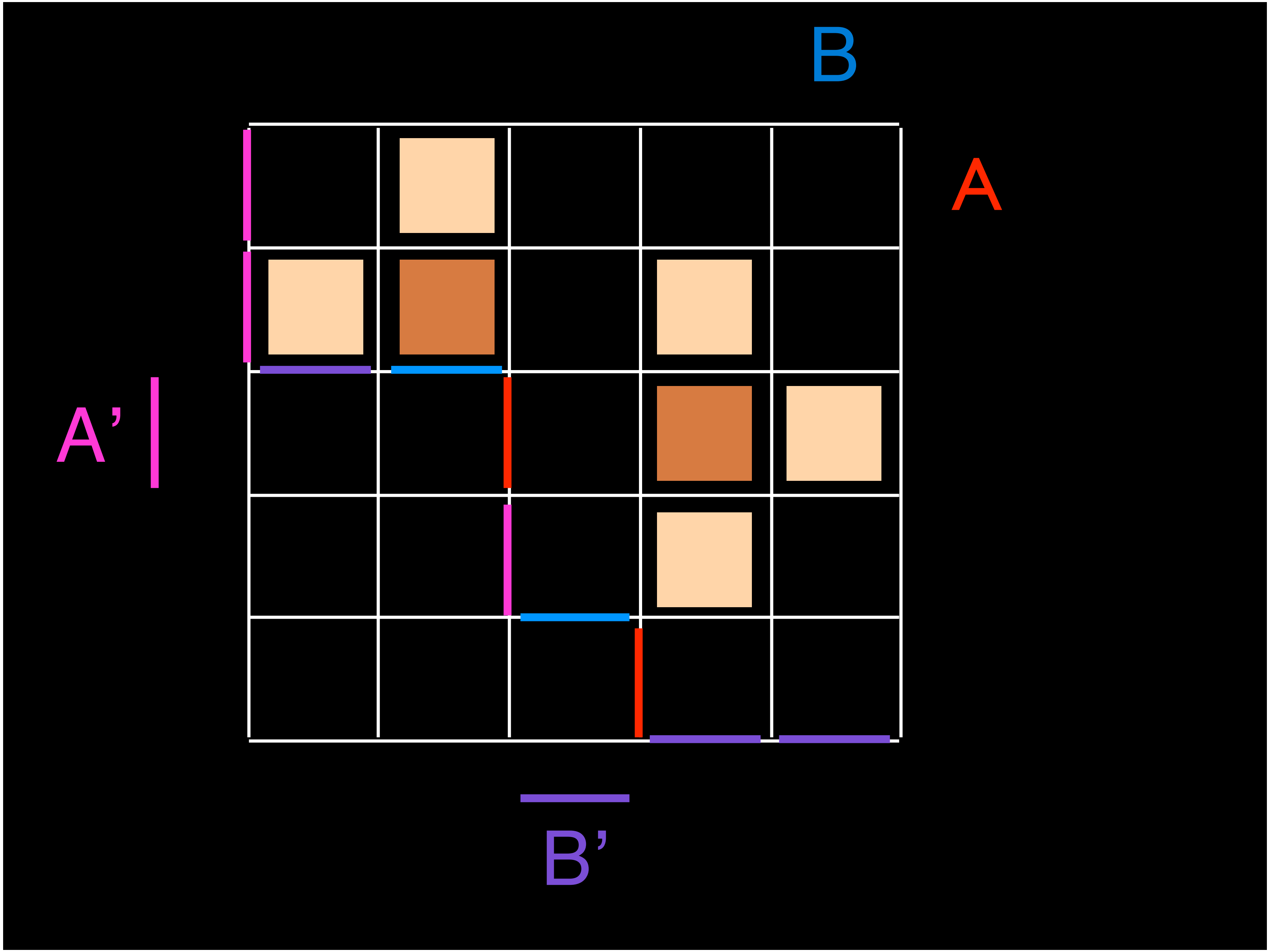
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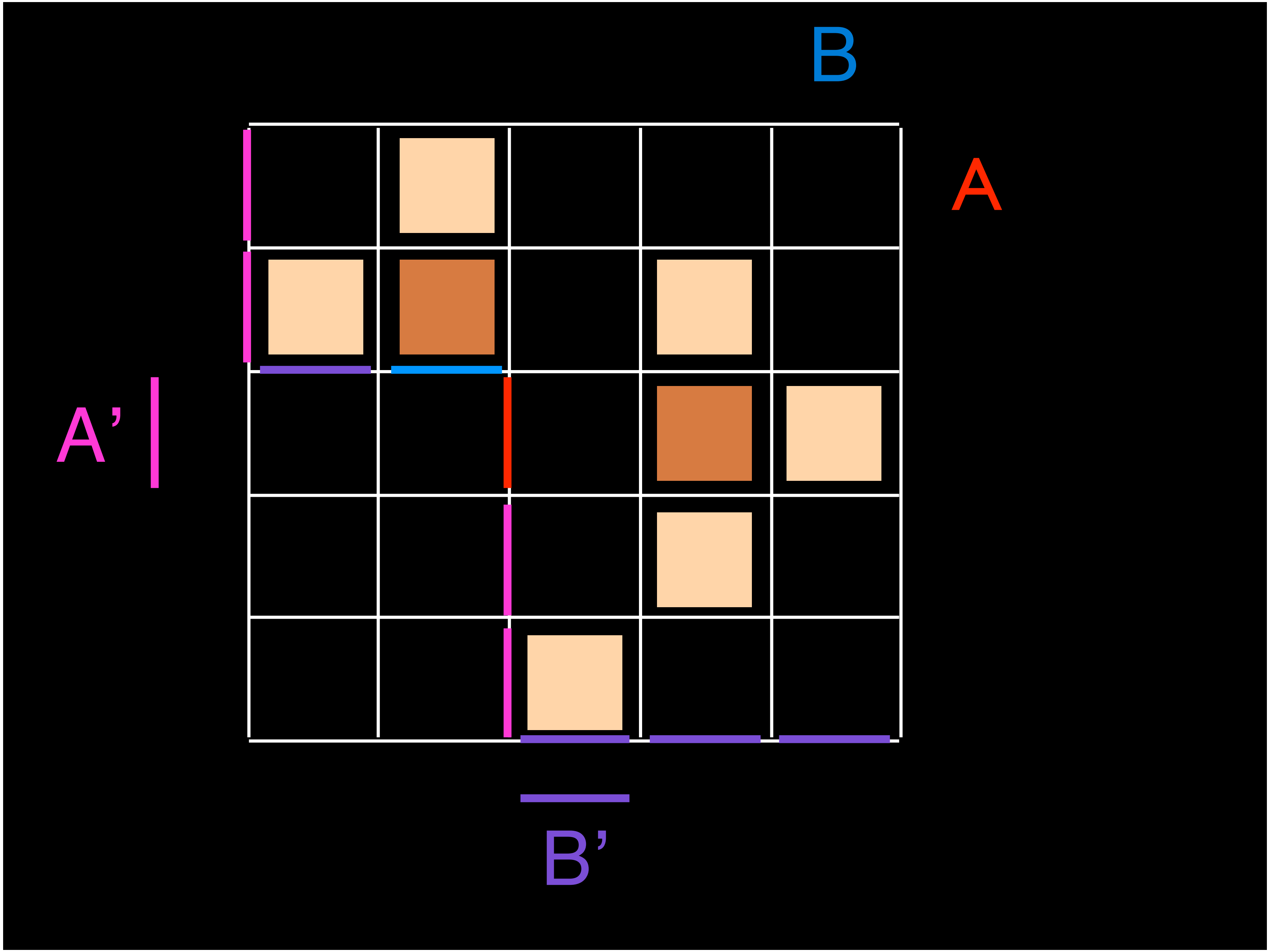
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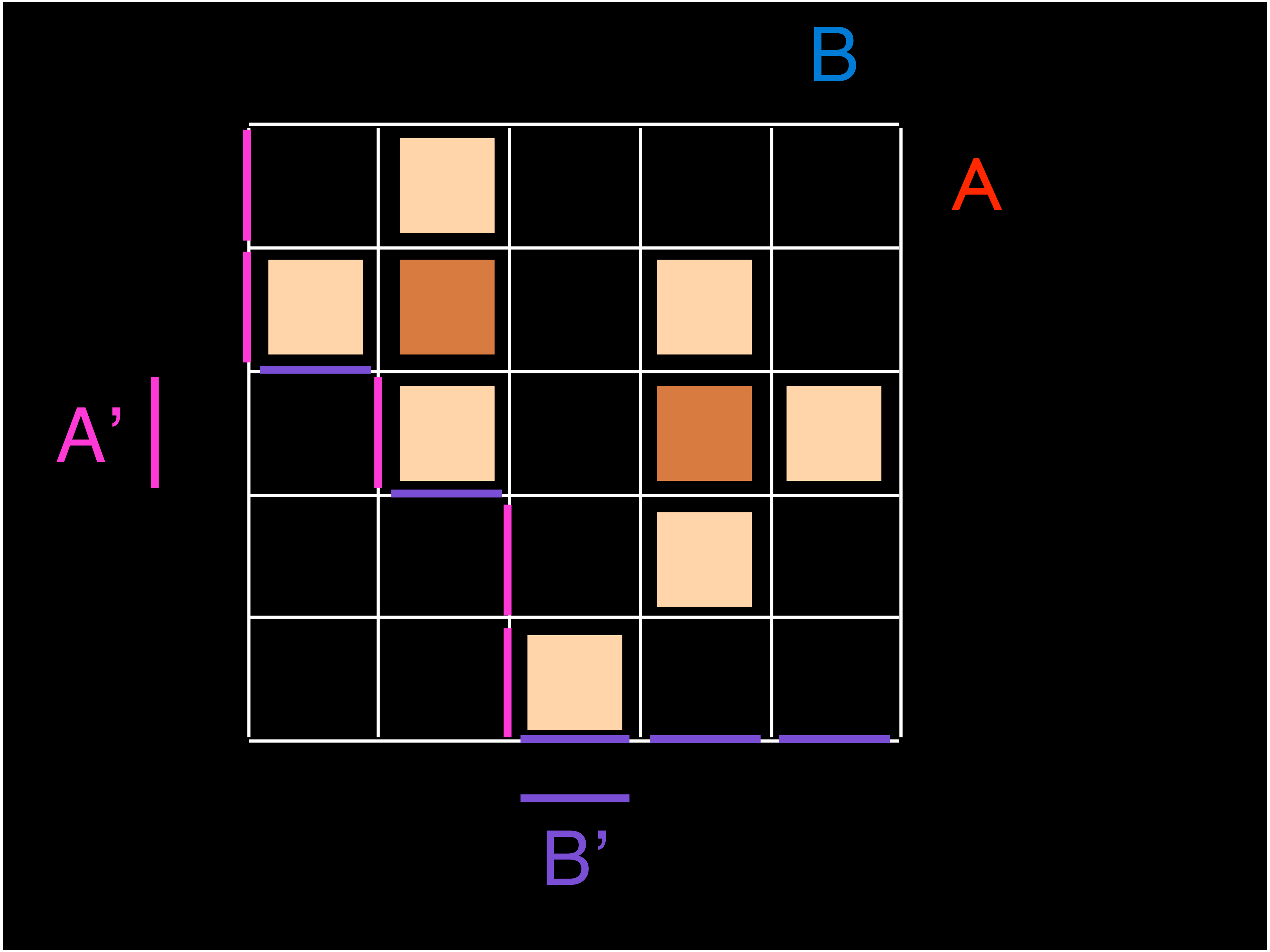
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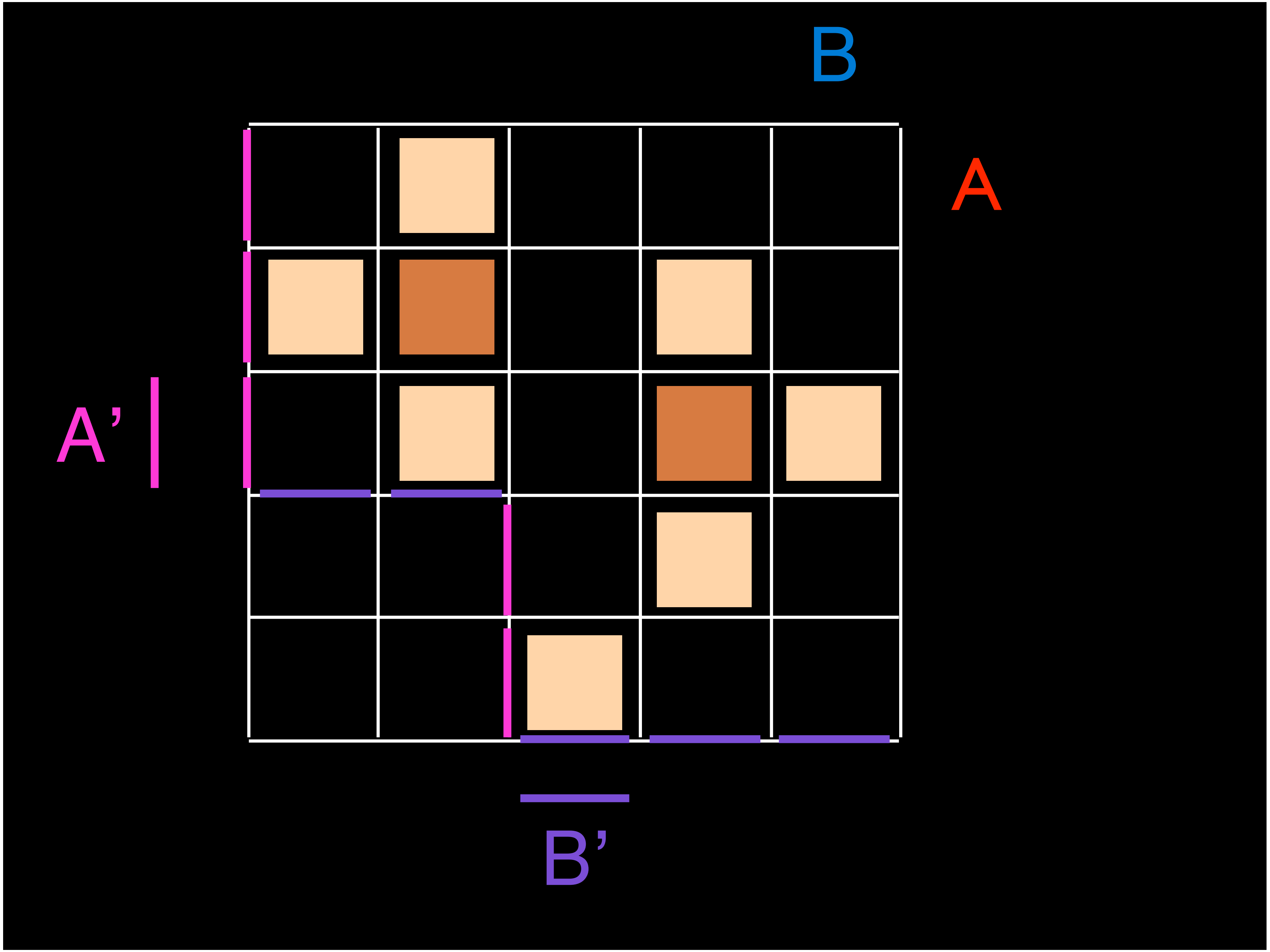
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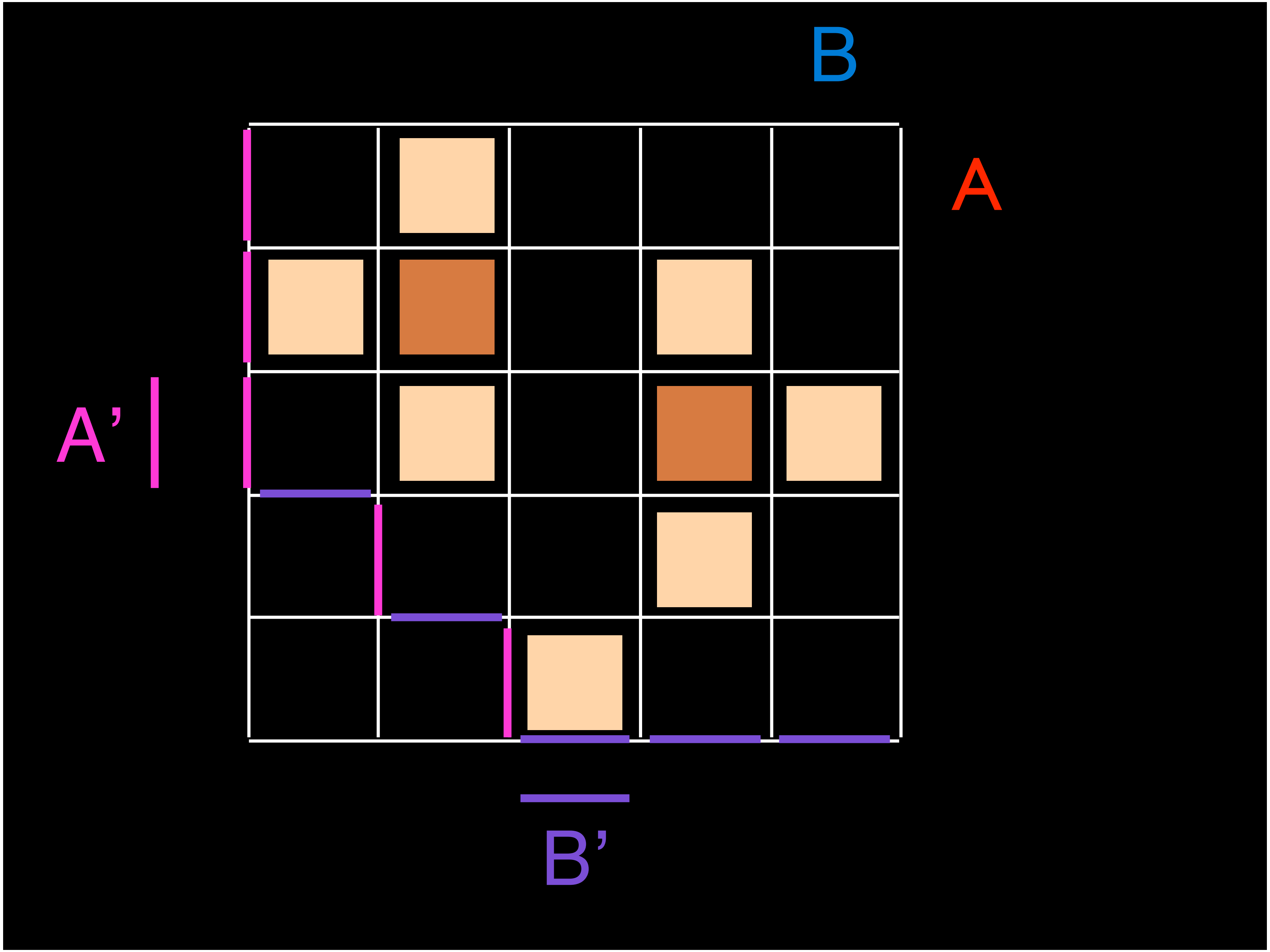
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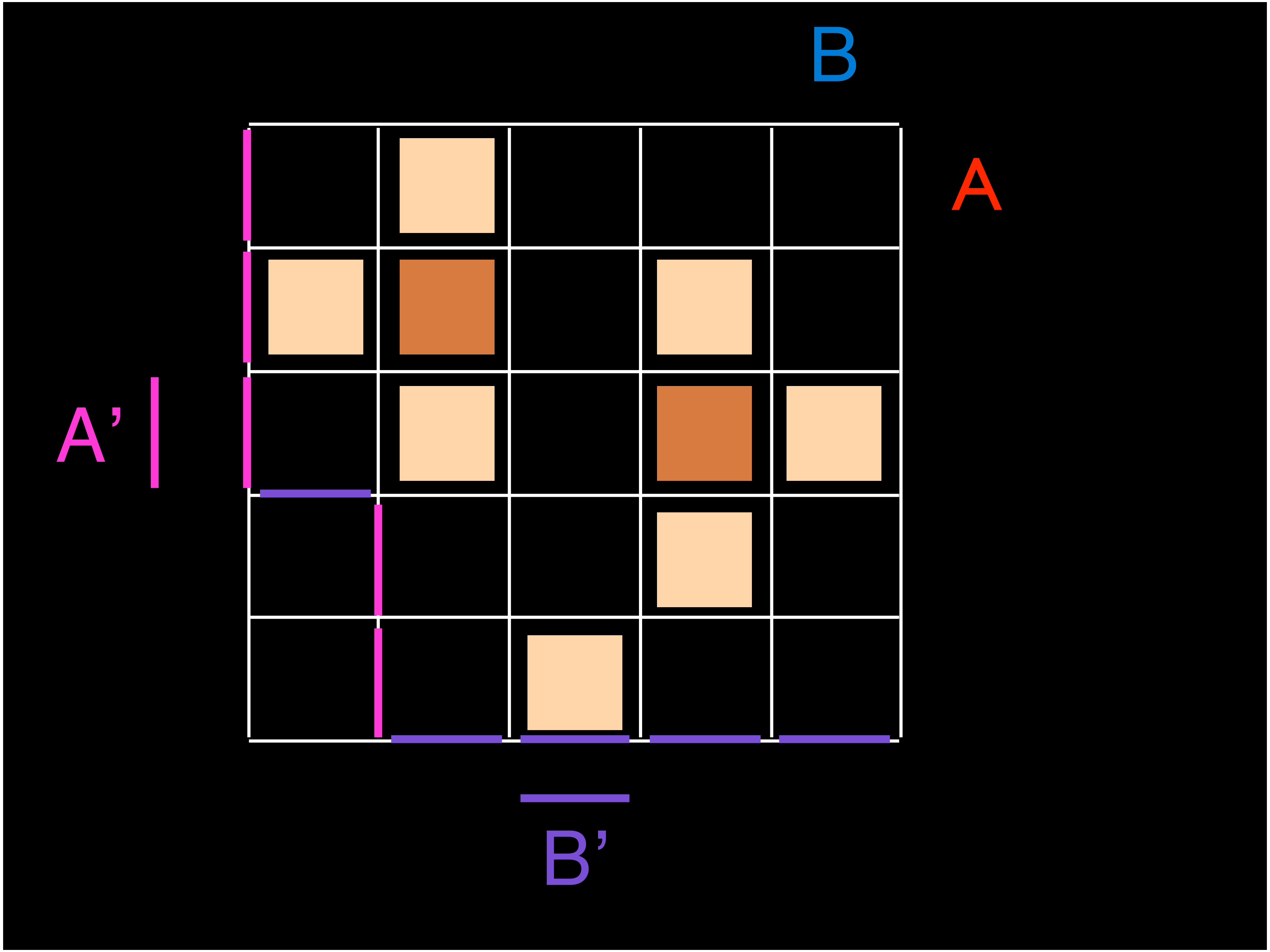

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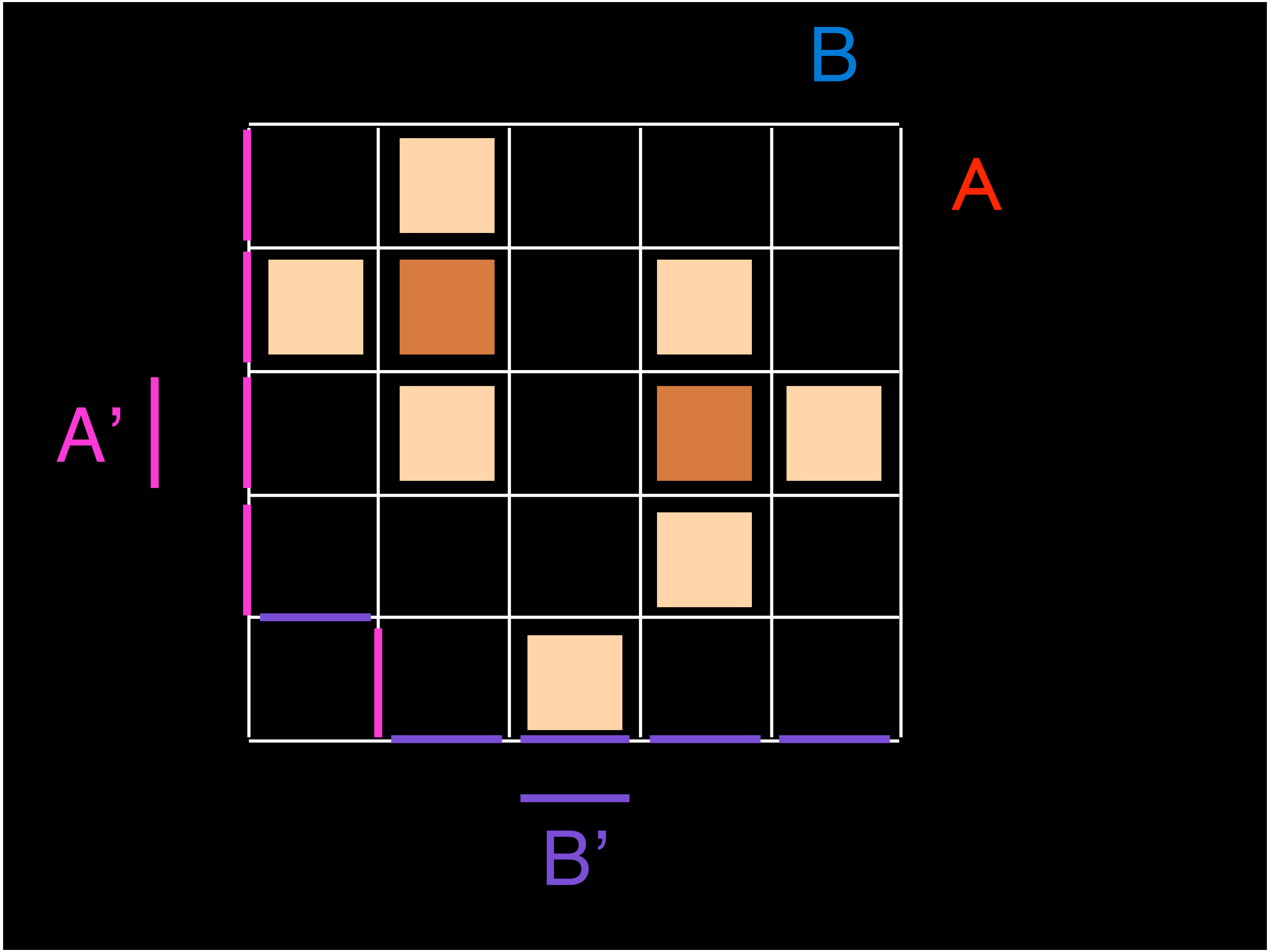
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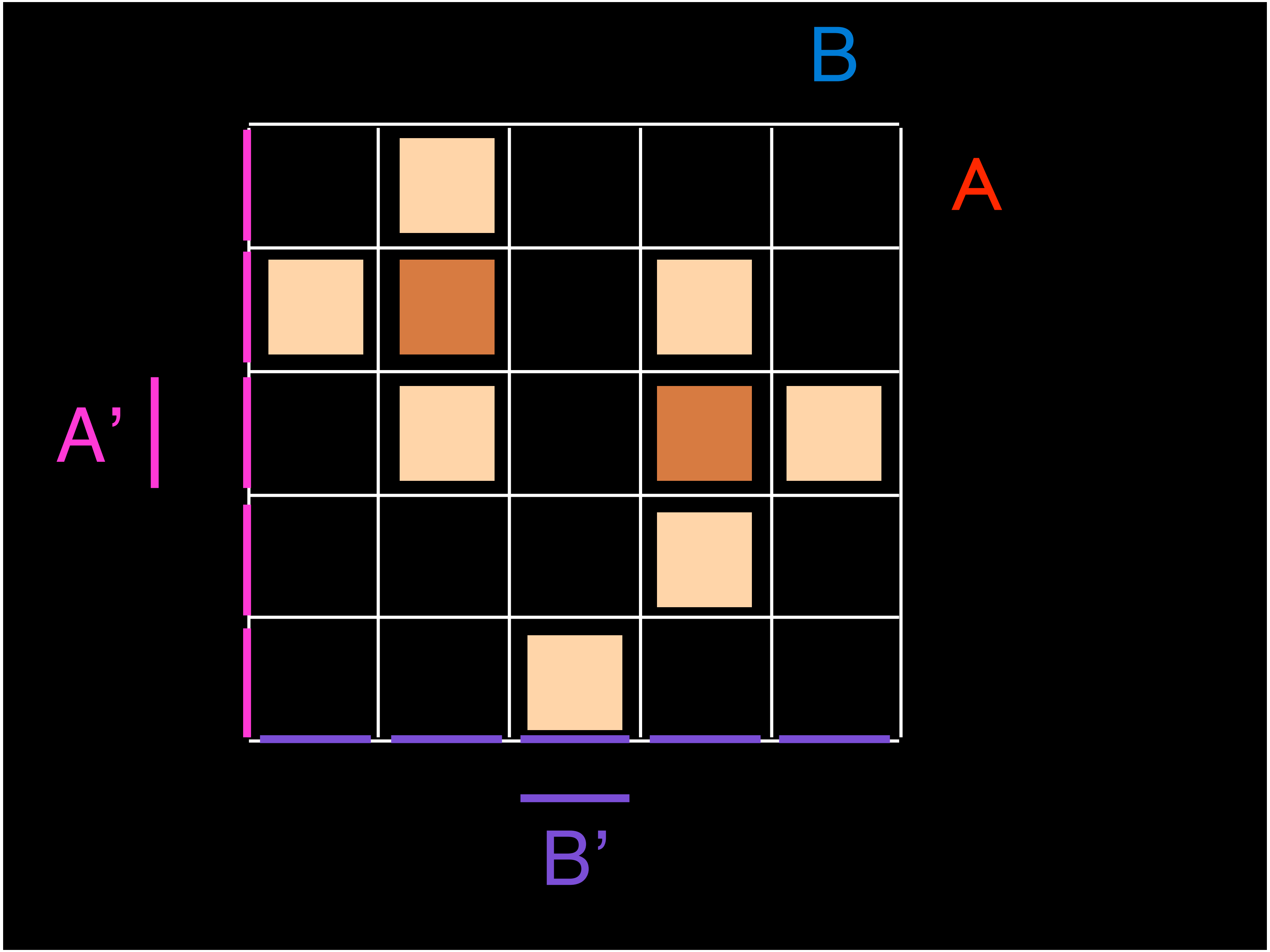
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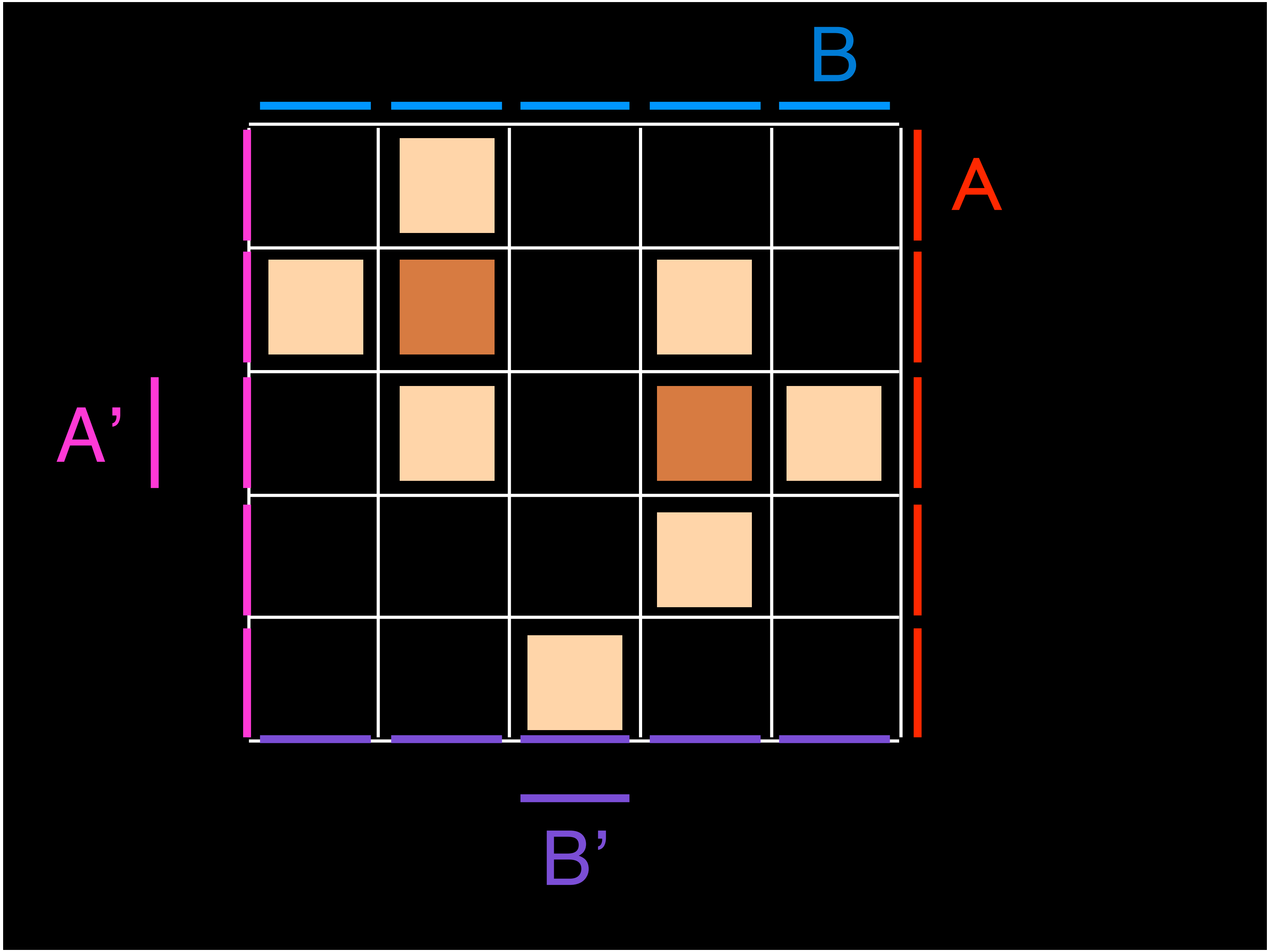
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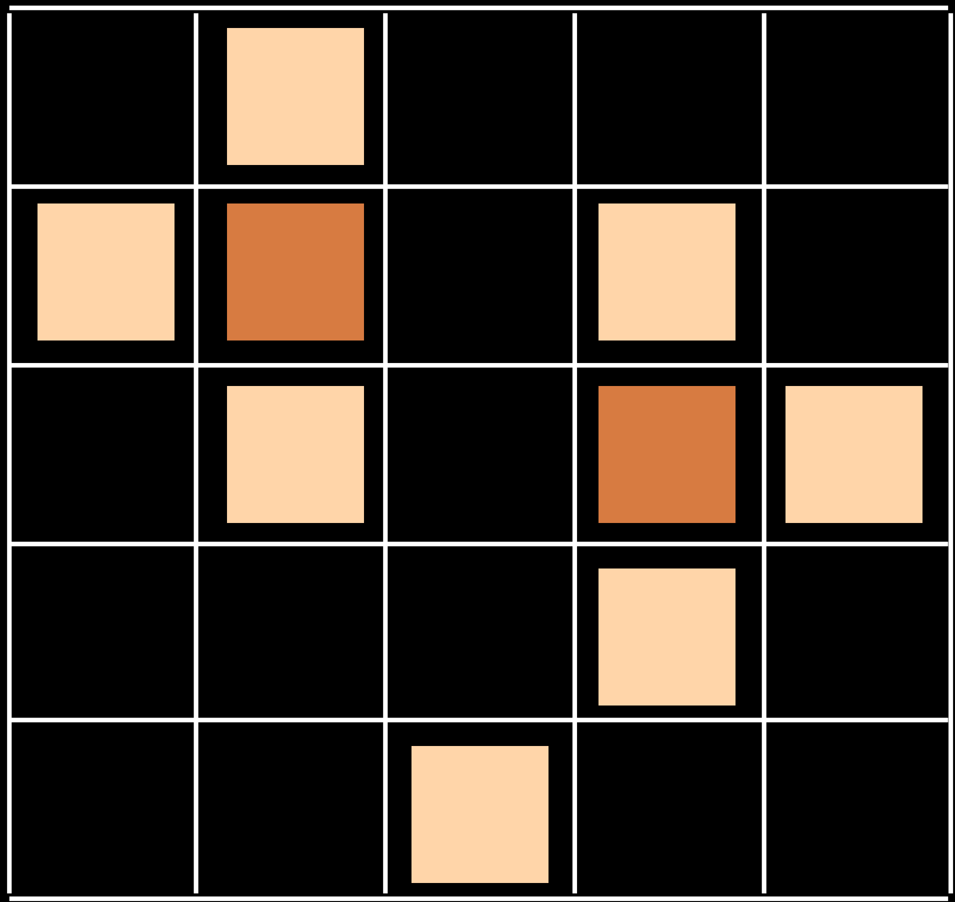

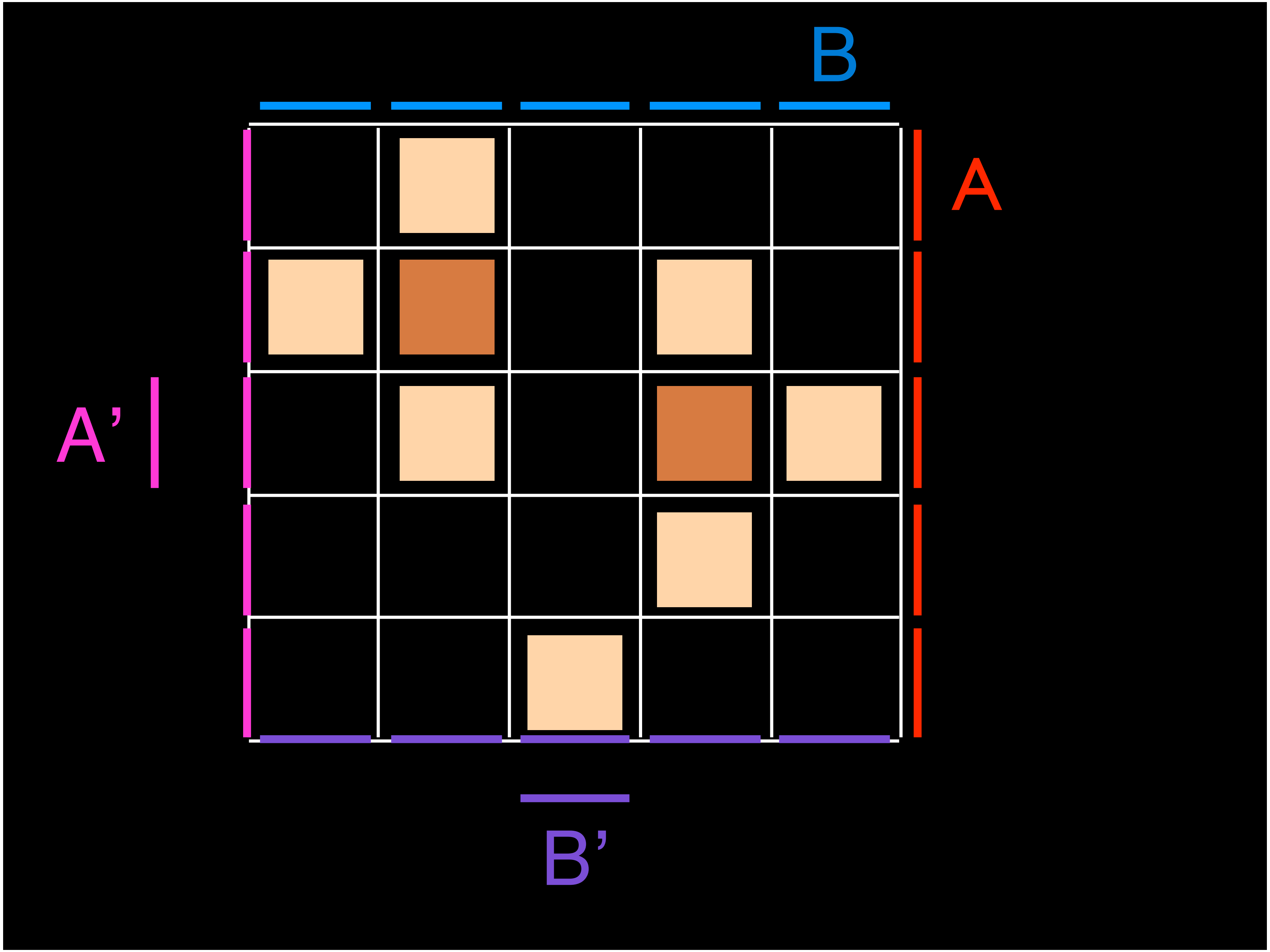
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8- parameters quadratic algebra

commutations

$$\begin{cases} BA = q_1 AB + q_2 A'B' \\ B'A' = q_3 A'B' + q_4 AB \end{cases}$$

$$\begin{cases} B'A = q_5 AB' + q_6 A'B \\ BA' = q_7 A'B + q_8 AB' \end{cases}$$

Conclusion: In this talk I have presented a sort of

"cellular ansatz"

- Some (formal) operators satisfying some commutation relations are given and generate a certain quadratic algebra.

- The computations in this algebra are made by some (oriented) rewriting rules which are visualized in a planar way on a (square) elementary cell of a grid. May be the operator identity I has to be introduced as another formal operator.

- The rewriting of a word of the algebra is visualized by a kind of a 2D cellular oriented expansion. The edges of the grid are labeled by the operators, the cells are labeled by each of the possible rewriting rules.

- The grid with the final labeling of the cells is in bijection with a class P of combinatorial objects (Permutations, Alernative tableaux, ASM, FPL, Tilings, etc ...).

- If the operators can be represented as combinatorial operators acting on a certain class F of combinatorial objects, then a simple combinatorial explanation of the commutation rules can be "attached" to each labeled cell of the grid. The vertices of the grid becomes labeled by the objects of F and "local rules" should be defined. In the case (as in the two examples of RSK and Alternating tableaux) when only the labels of the cells, and not those of the edges, are needed for defining the local rules, then from the cellular propagation of these local rules across the grid, one obtain a bijection between the objects of P and some other objects coded by the sequence of the F-labels on the border of the grid.

some perspectives

Questions.

- find a "combinatorial representation" for operators A, A', B, B'.
- analogue of RSK (Robinson-Schensted-Knuth) for ASM ?
- analogue of "local rules" (Fomin)
- direct proof of the formula

$$A_n = \prod_{j=1}^{n} \frac{(3j-2)!}{(n+j-1)!}$$

(nb of ASM of size n)

$= 1, 2, 7, 42, \ldots$

?

(with P. Nadeau)
another representation of operators D and E
with triangulations of regular polygons

hypercube -- associahedron -- permutohedron -- alternohedron
(Loday-Ronco) (Lascoux-Schützenberger)

Razumov-Stroganov conjecture

spin chain Heisenberg XXZ model

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

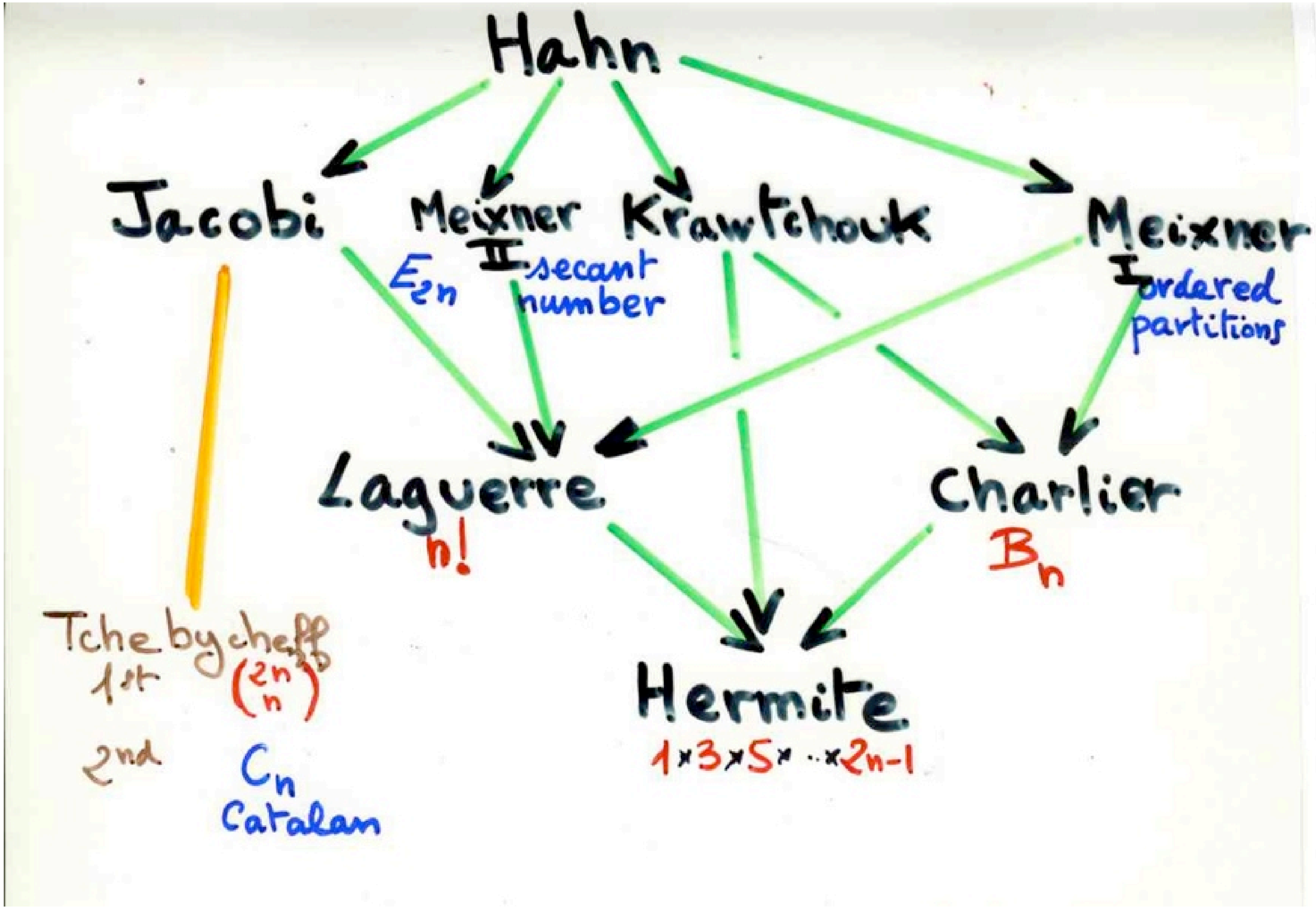
$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\,\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

Askey-Wilson polynomials

Askey-Wilson



Novelli, Thibon, Williams (April 2008)

Hall-Littlewood functions, Tevlin' bases (2007)

conjectures

references:

xgv website :

http://www.labri.fr/perso/viennot/

Recherche, cv, publications, exposés, diaporamas, livres, petite école, photos: voir ma page personnelle ici
Vulgarisation scientifique voir la page de l'association Cont'Science

downloadable papers, slides and lecture notes, etc ... here
(the summary on page "recherches" and most slides are in english)

→ **page "video"**

"Alternative tableaux, permutations and asymmetric exclusion process"
conference 23 April 2008,
Isaac Newton Institute for Mathematical science

or http://www.newton.cam.ac.uk/ (page "web seminar")

page “exposés”

An alternative approach to alternating sign matrices, (pdf 9,3 Mo) Workshop on **“Combinatorics and Statistical Physics”,** The Erwin Schrödinger International Institute for Mathematicals Physics (**ESI**), Vienna, 20 May 2008.

Growth diagrams for Young tableaux, Robinson-Schensted correspondance and some quadratic algebra coming from physics, exposé au CMUP (Centro de Matematica da Universidade do Porto), Portugal, 17 Sept 2008 **slides** (13,1 Mo)

Alernating sign matrices: at the crossroads of algebra, combinatorics and physics", exposé au CMUC (Centro de Matematica da Universidade do Coimbra), Portugal, 26 Sept 2008

TASEP:

→ **page "exposés"**

Catalan numbers, permutation tableaux and asymmetric exclusion process (pdf, 4,8 Mo)
GASCOM'06, Dijon, Septembre 2006, aussi: Journées Pierre Leroux, Montréal, Septembre 2006

Robinson-Schensted-Knuth: RSK1 (pdf, 9,1 Mo)
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survey paper on Robinson-Schensted correspondence:
[30] *Chain and antichain families, grids and Young tableaux*,
Annals of Discrete Maths., 23 (1984) 409-464.

from xgv website :

→ A Combinatorial theory of orthogonal polynomials

[4] *Une théorie combinatoire des polynômes orthogonaux*, Lecture Notes UQAM, 219p., Publication du LACIM, Université du Québec à Montréal, 1984, rééd. 1991.

→ page **"petite école"**

Petite école de combinatoire LaBRI, année 2006/07
"Une théorie combinatoire des polynômes orthogonaux,
ses extensions, interactions et applications"

Chapitre 2, Histoires et moments, (17, 23 Nov , 1, 8, 15 Dec 2006)

Chapitre X Histoires et opérateurs (10 and 12 January 2007)

→ page **"cours"**

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"Eléments de combinatoire algébrique"

Ch 4 - (9,4 Mo) théorie combinatoire des polynômes orthogonaux et fractions continues *from xgv website :*

→ **Paper:** FV bijection

[21] (avec J. Françon) *Permutations selon les pics, creux, doubles montées et doubles descentes, nombres d'Euler et nombres de Genocchi*, Discrete Maths., 28 (1979) 21-35

survey paper: Genocchi, Euler (tangent and secant numbers), Jacobi elliptic functions

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