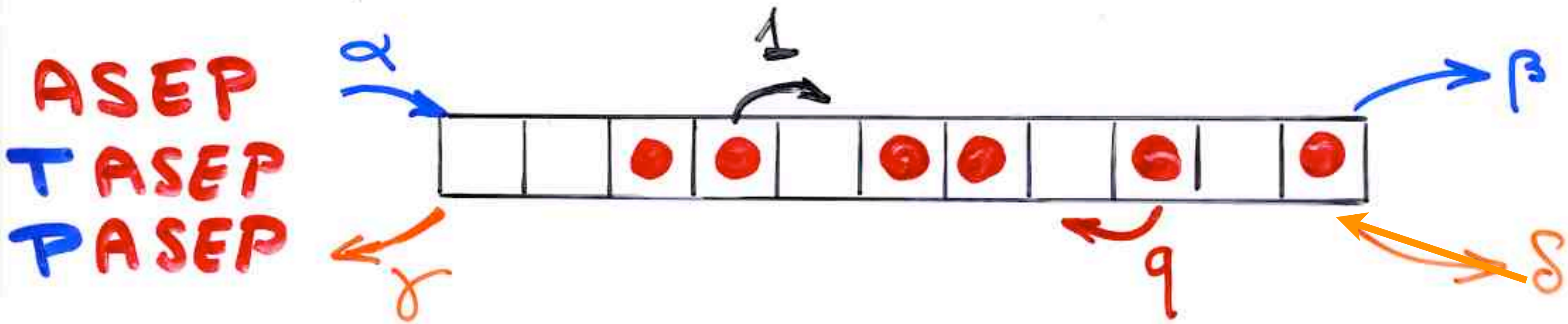
The PASEP

Partialy asymmetric exclusion process


ASEP
TASEP
PASEP
α
1
β
γ
q
δ

non-equilibrium statistical mechanics

... relaxation → stationary state

states $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$

$$\tau_i = \begin{cases} 1 & \text{site } i \text{ occupied} \\ 0 & \text{site } i \text{ empty} \end{cases}$$

unique stationary state $\quad \frac{d}{dt} P_n(\tau_1, \ldots, \tau_n) = 0$

Derrida, Evans, Hakim, Pasquier (1993)

boundary induced phase transitions

molecular diffusion
linear array of enzymes
biopolymers
traffic flow
------
formation of shocks

$$\mathcal{P}_n(\tau_1, \dots, \tau_n) = f_n(\tau_1, \dots, \tau_n) / Z_n$$

$$Z_n = \sum_{\tau} f_n(\tau_1, \dots, \tau_n)$$

partition function

The Matrix Ansatz

Derrida, Evans, Hakim, Pasquier (1993)

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

$$\begin{cases} DE = q\,ED + D + E \\ (\beta D - \delta E)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \gamma D) = \langle W| \end{cases}$$

Then

$f_n(\tau_1, \ldots, \tau_n)$

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

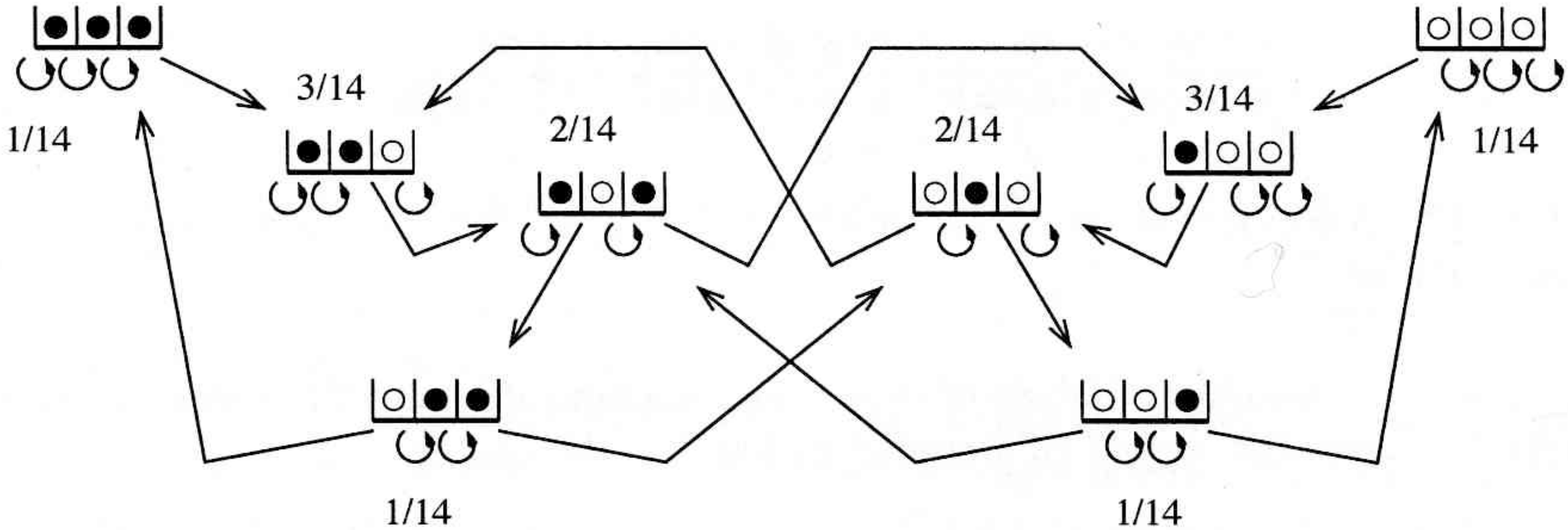
TASEP

q=0

$$
\begin{cases}
DE = \square + D + E \\
(\beta D - \square)|V\rangle = |V\rangle \\
\langle W|(\alpha E - \square) = \langle W|
\end{cases}
$$

Then

$$f_n(\tau_1, \dots, \tau_n) = \langle W| \prod_{i=1}^{n} (\tau_i D + (1-\tau_i) E) |V\rangle$$


3/14
1/14
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3/14
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1/14
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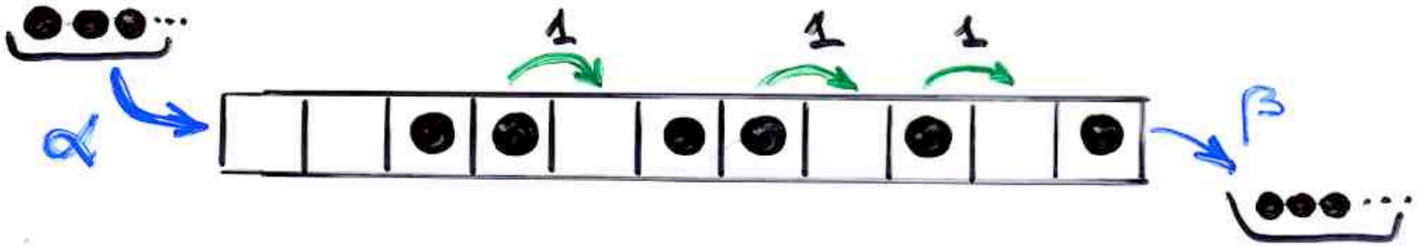
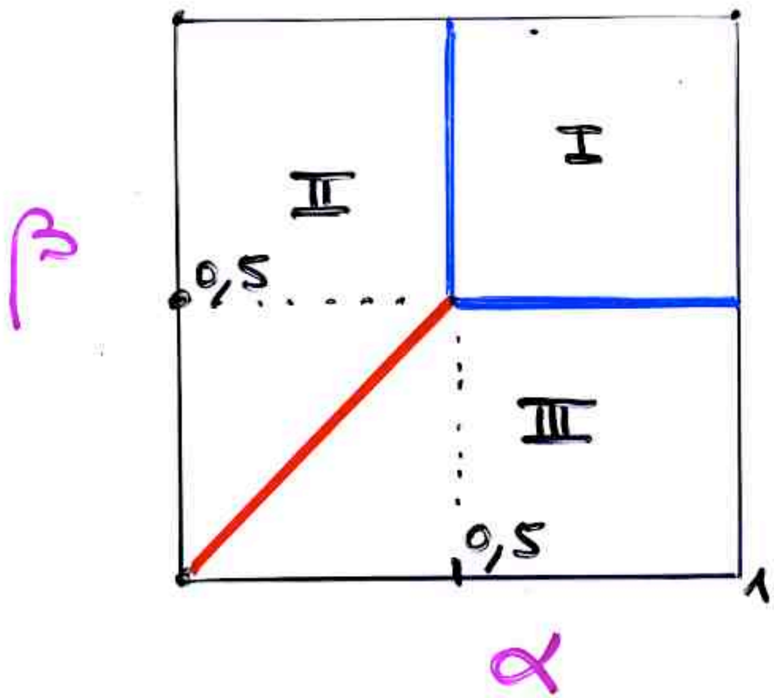
TASEP

"totally asymmetric exclusion process"





$n \to \infty$

$\rho = \langle \tau_i \rangle$ = taux moyen d'occupation

i loin des bords

(I) $\rho = 1/2$

(II) $\rho = \alpha$

(III) $\rho = 1-\beta$

phase transition

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q- Hermite polynomial

$\alpha, \beta, q \qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\hat{a} = 1$$

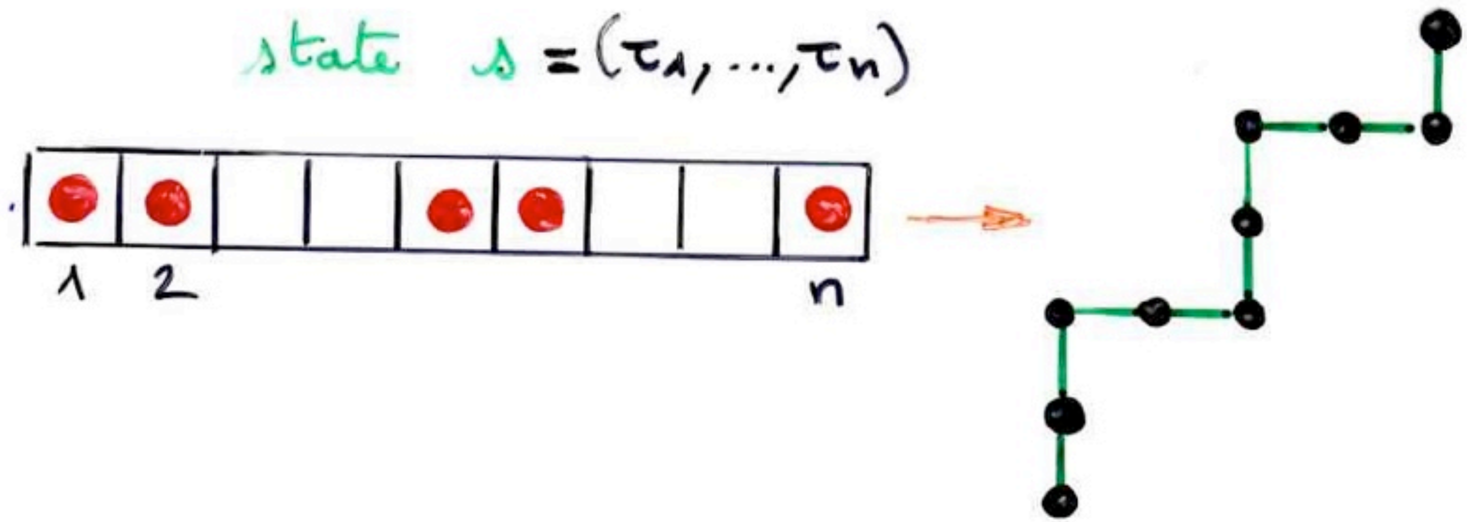
→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

Askey-Wilson polynomials

Combinatorics of the TASEP

state $s = (\tau_1, \dots, \tau_n)$

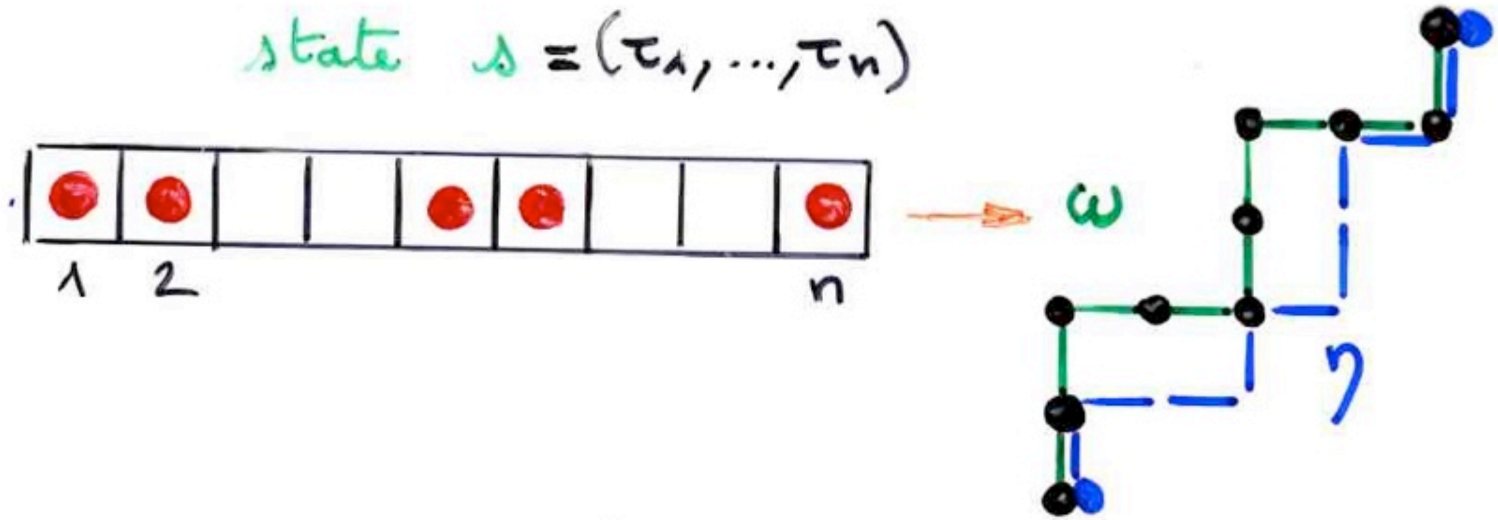


$\mathcal{P}_n(s) =$

Shapiro, Zeilberger, 1982

Combinatorics of the TASEP

state $s = (\tau_1, \dots, \tau_n)$



$$P_n(s) = \frac{1}{C_{n+1}} \left(\begin{array}{c} \text{number of paths } \eta \\ \text{below the path } \omega \\ \text{associated to } s \end{array} \right)$$

Shapiro, Zeilberger, 1982

TASEP

Shapiro, Zeilberger (1982)
Brak, Essam (2003), Duchi, Schaeffer, (2004),
Angel (2005), XGV (2007)

(P) ASEP

Brak, Corteel, Essam, Parviainen, Rechnitzer (2006)
Corteel, Williams (2006,..., 2010)
Corteel, Stanton, Stanley, Williams (2011)
Josuat-Vergès (2008,..., 2010)

Derrida, ...
Mallick, ... Golinelli, Mallick (2006)

"normal ordering"

Heisenberg
operators
U, D

UD = DU + I

$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} \, D^i U^i$$

normal ordering

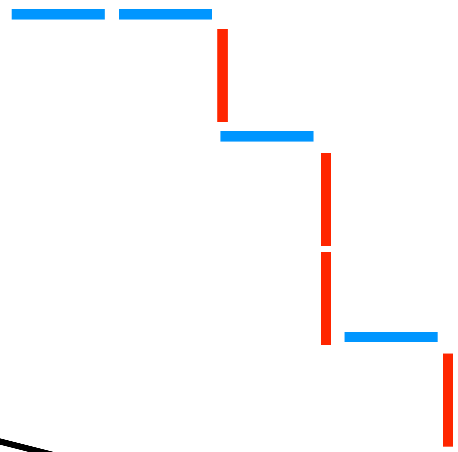
$$c_{n,0} = n!$$

The PASEP algebra

$$DE = qED + E + D$$

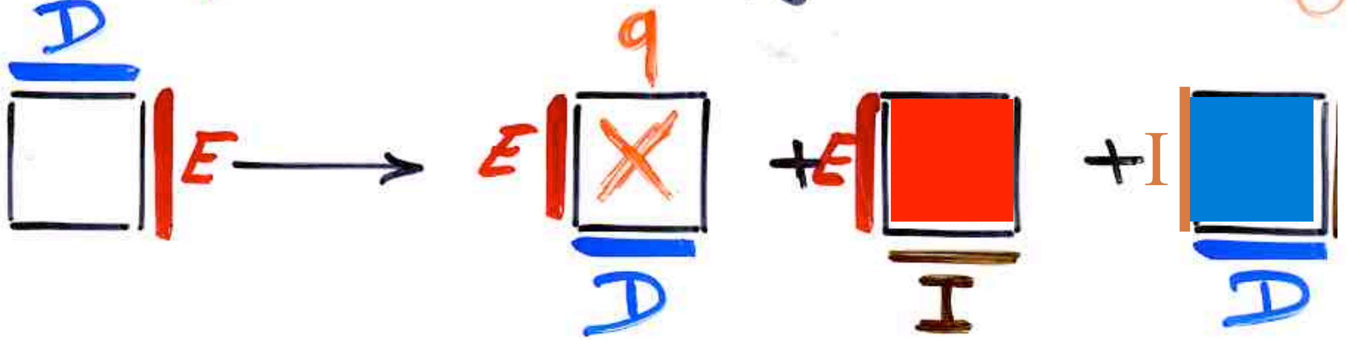
D D E D E E D E

D DE (D E) E D E

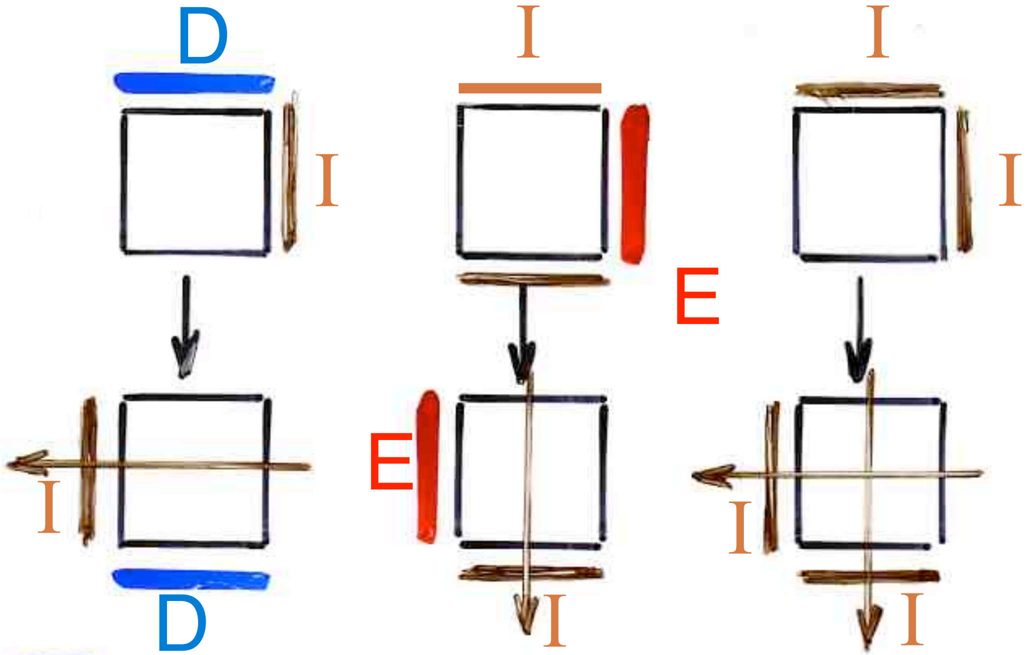


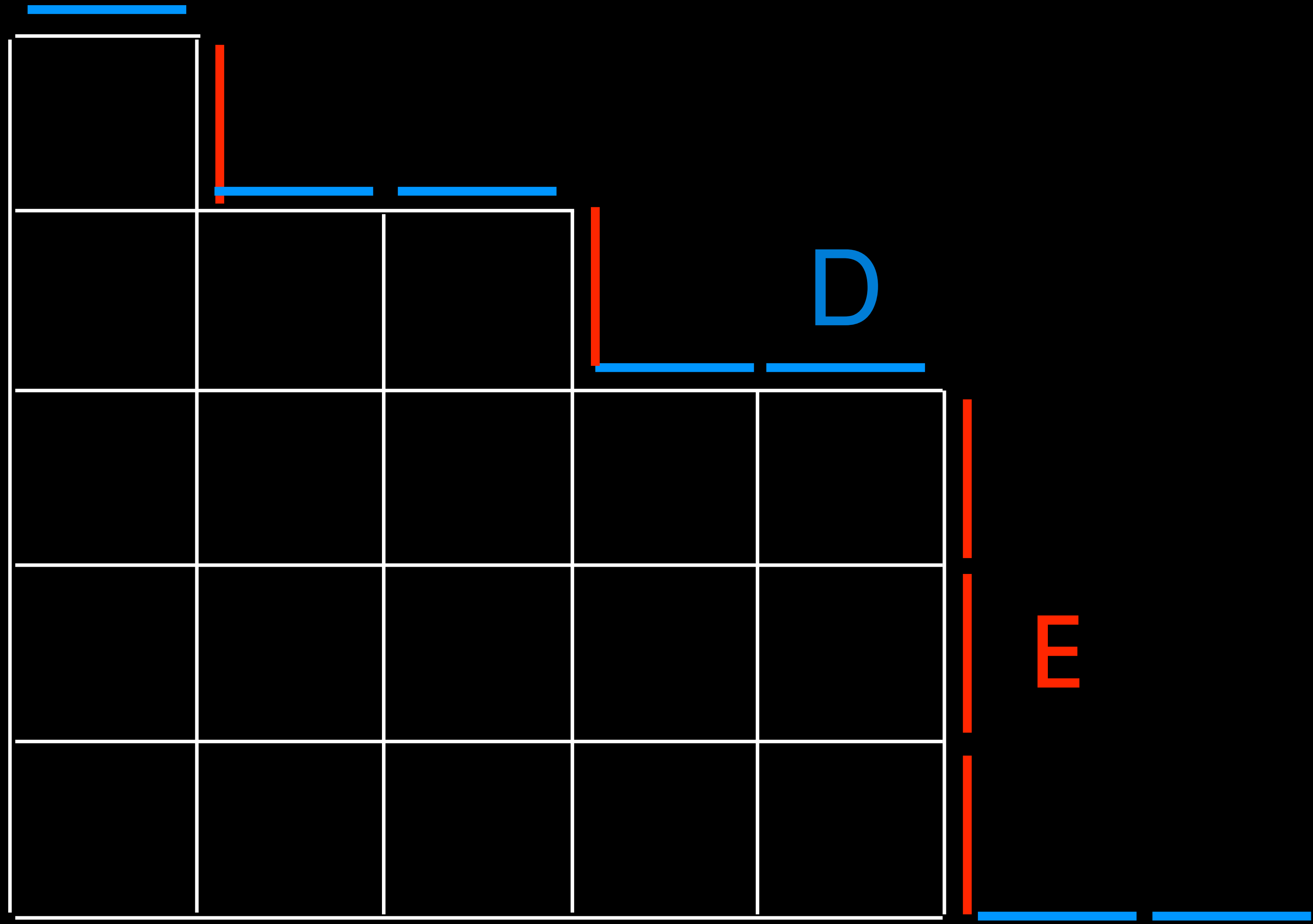
DDE(E)EDE + DDE(ED)EDE + DDE(D)EDE

Proof: "planarization" of the rewriting rules



I identity




D
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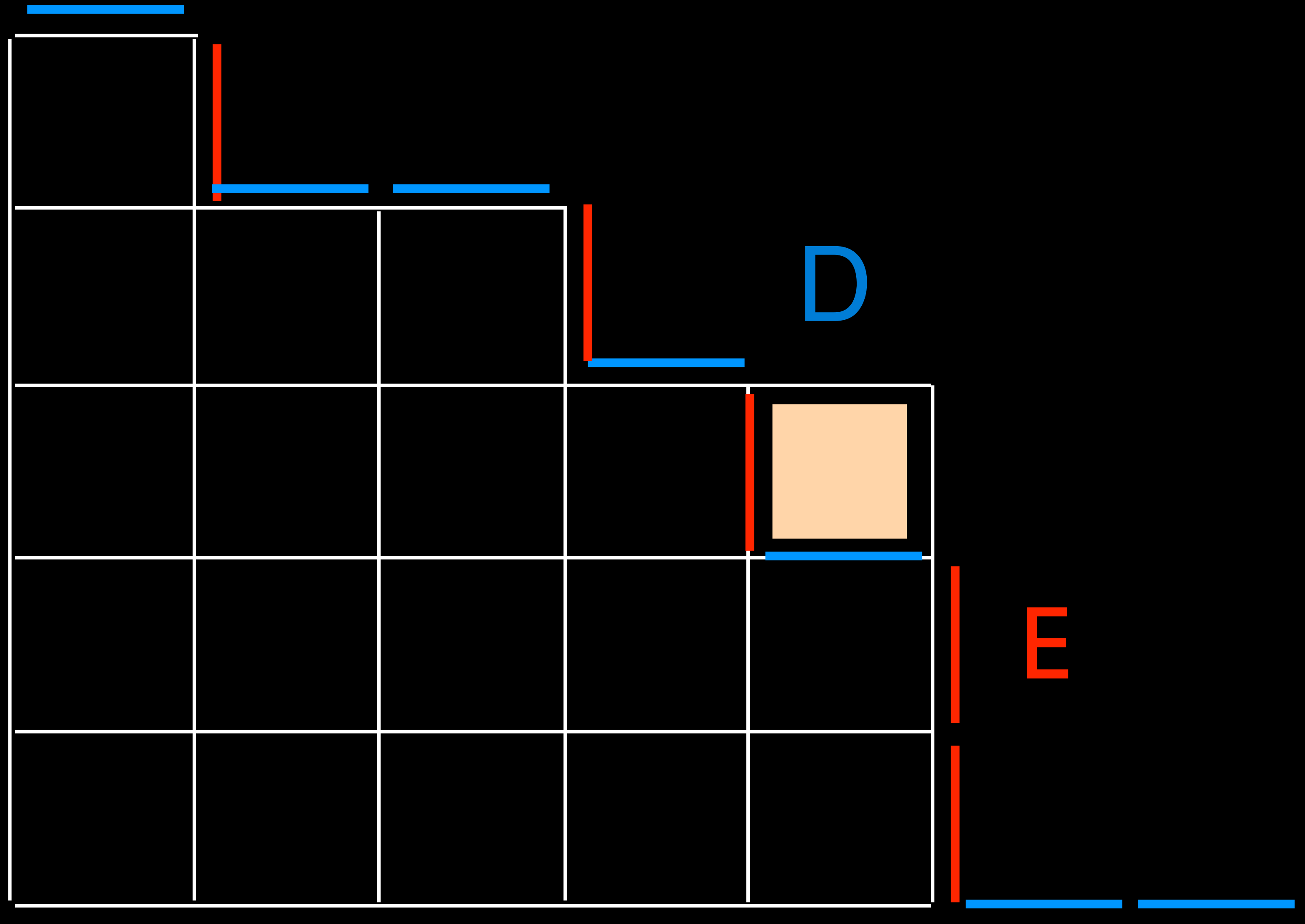
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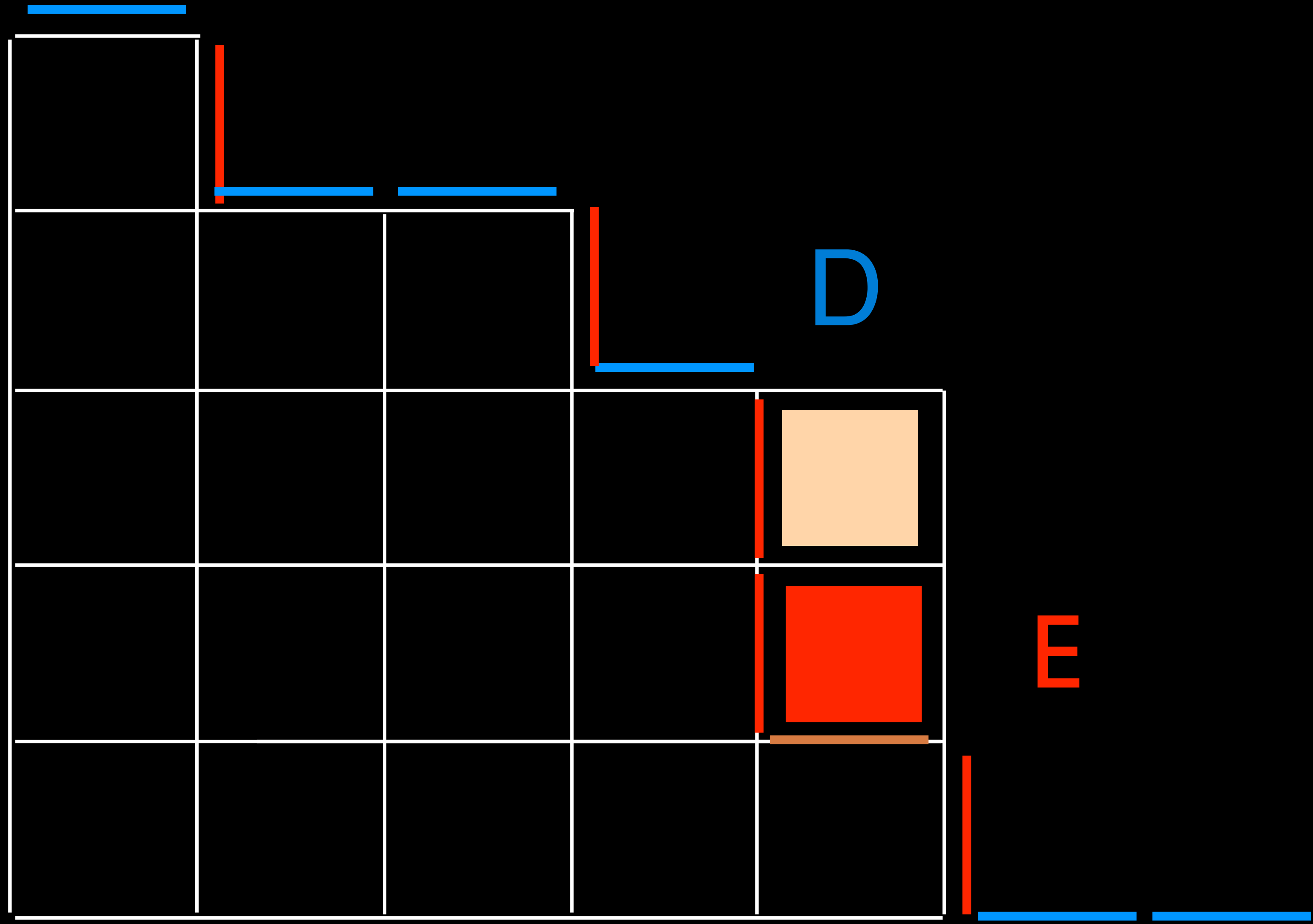
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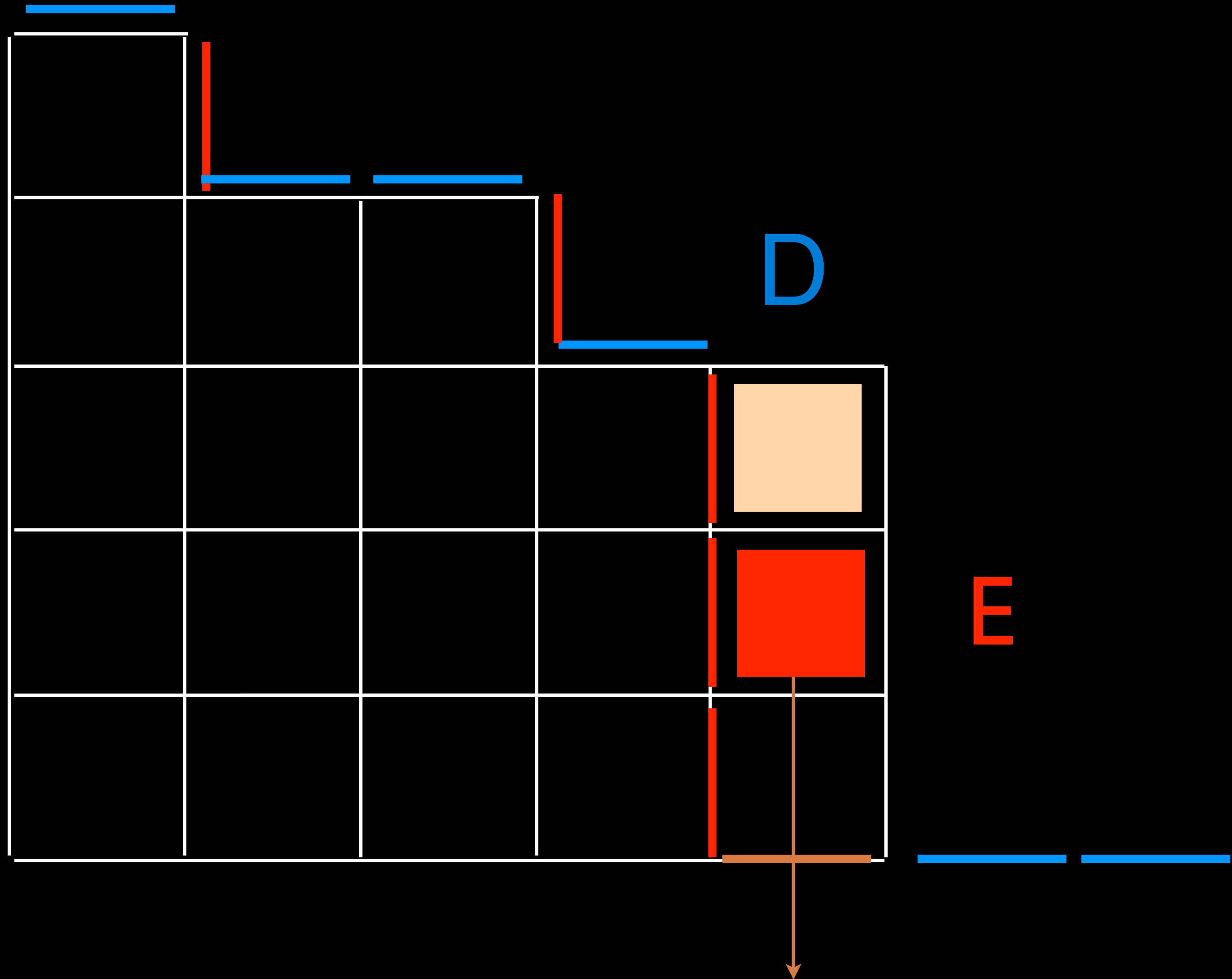
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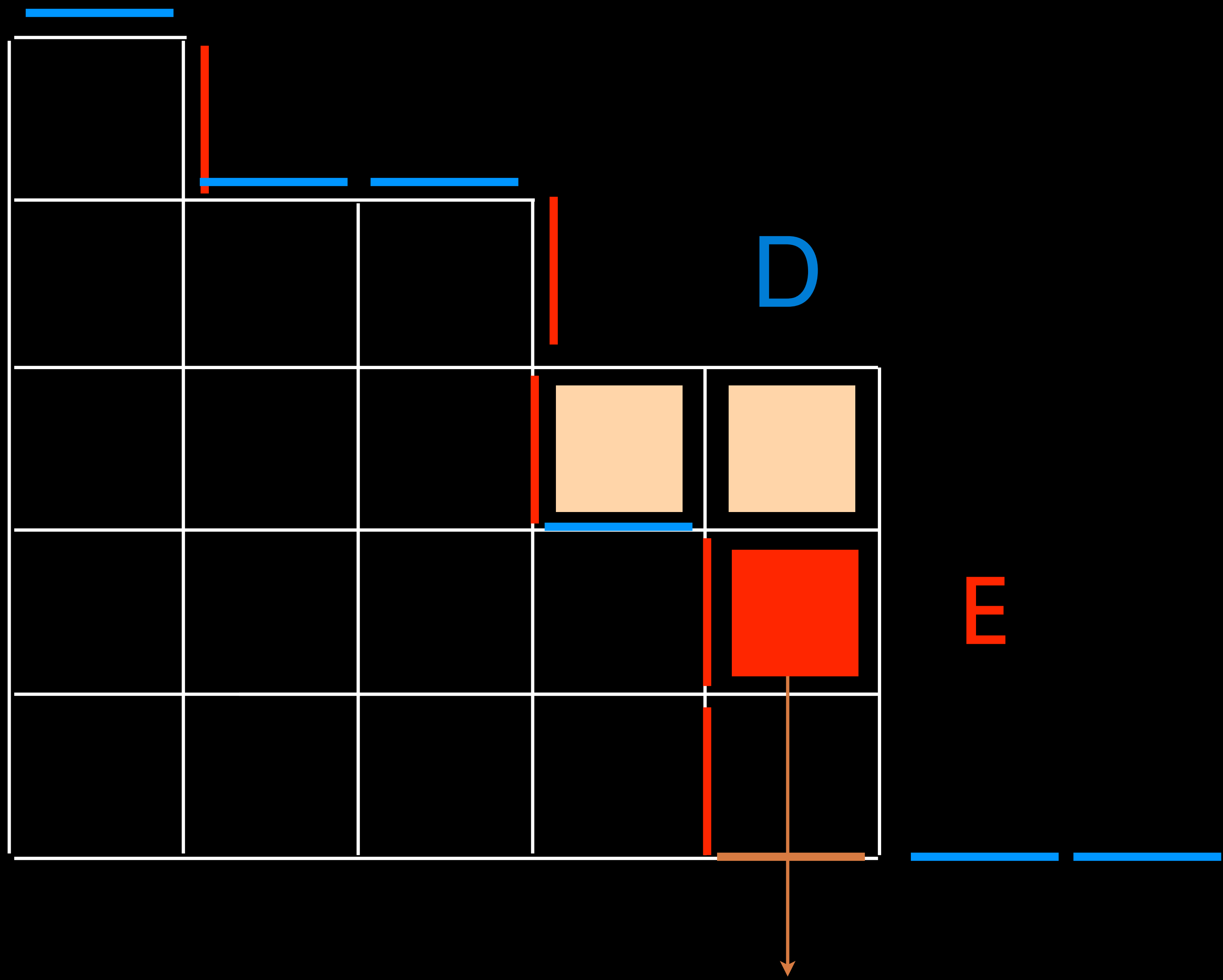
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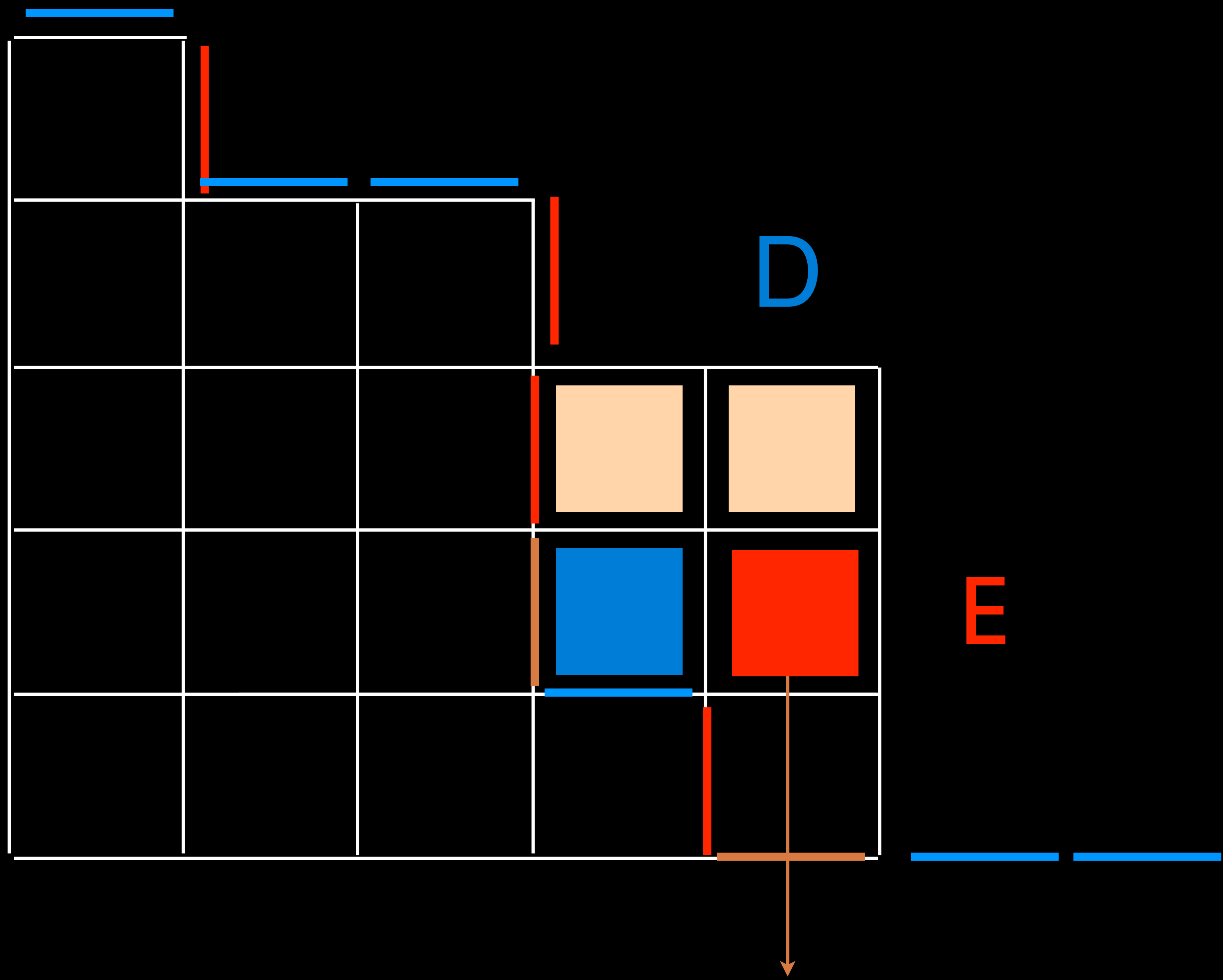
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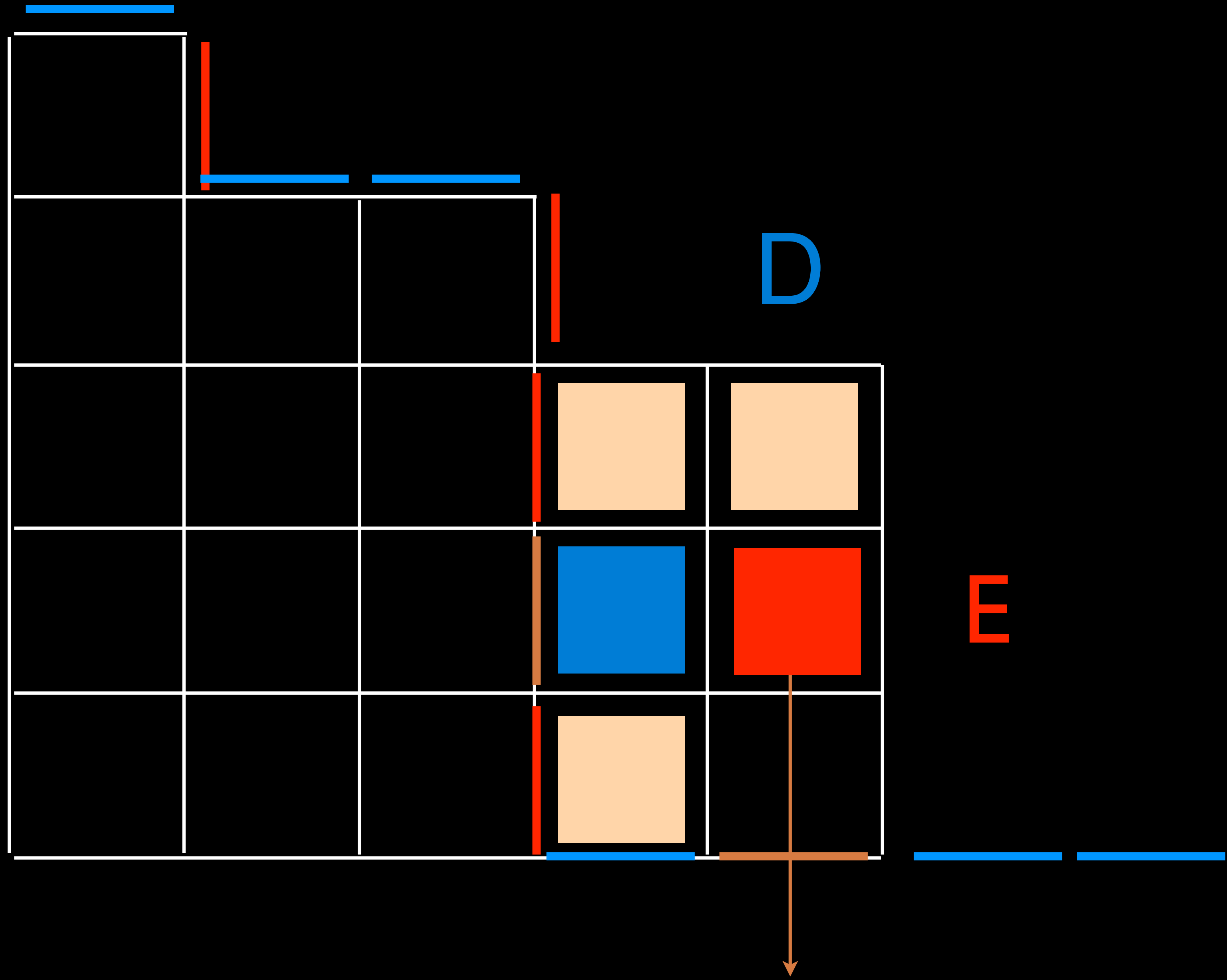
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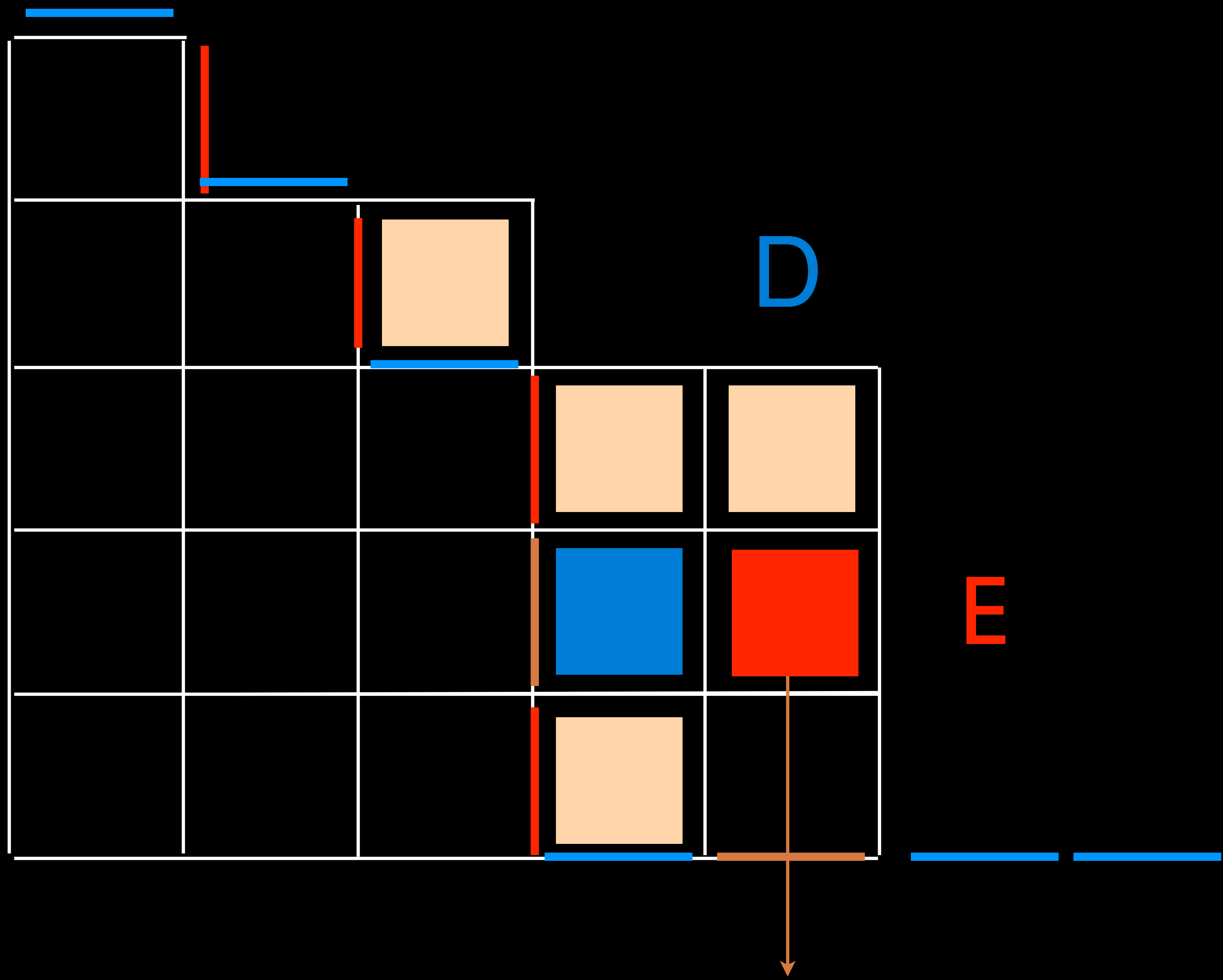

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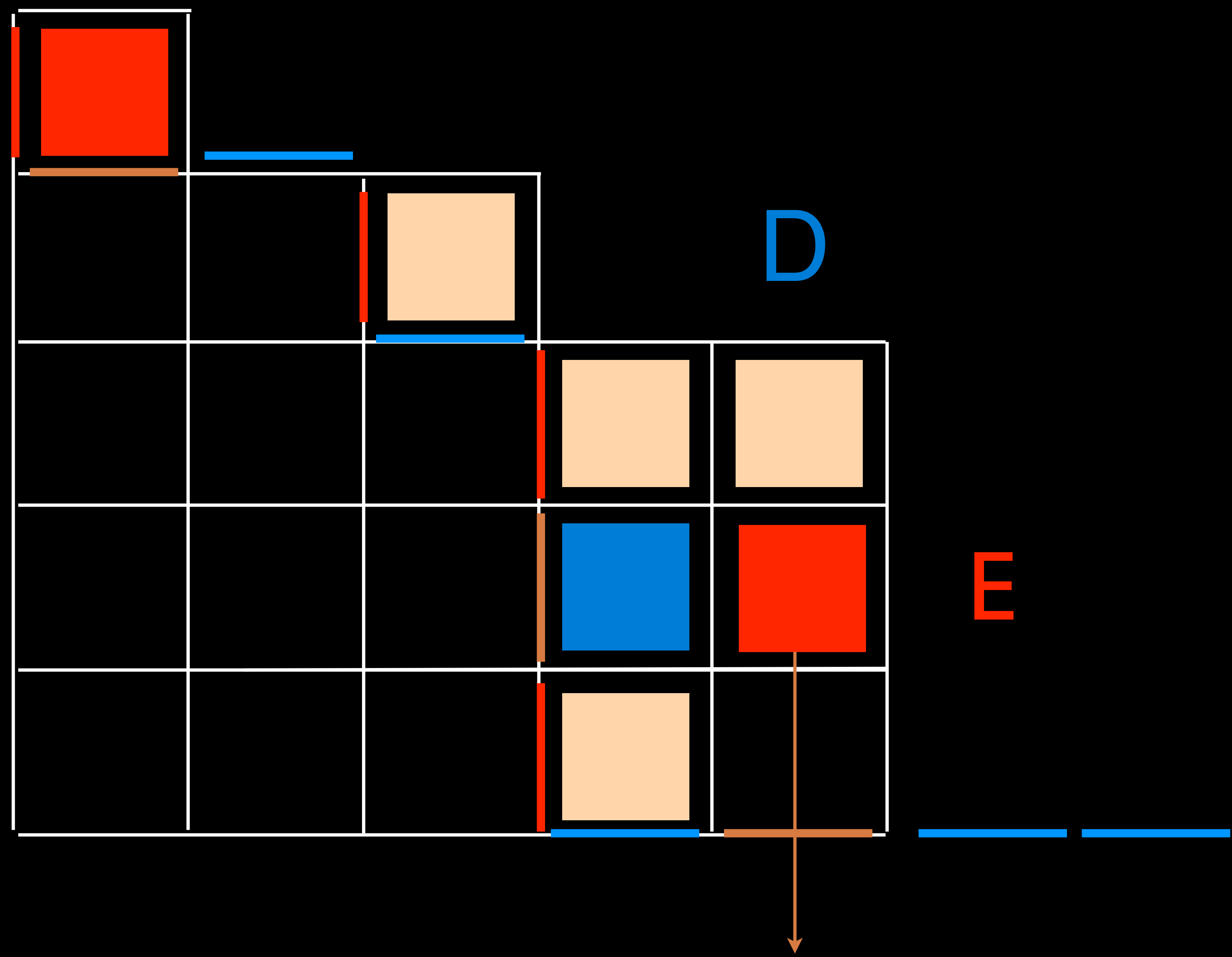
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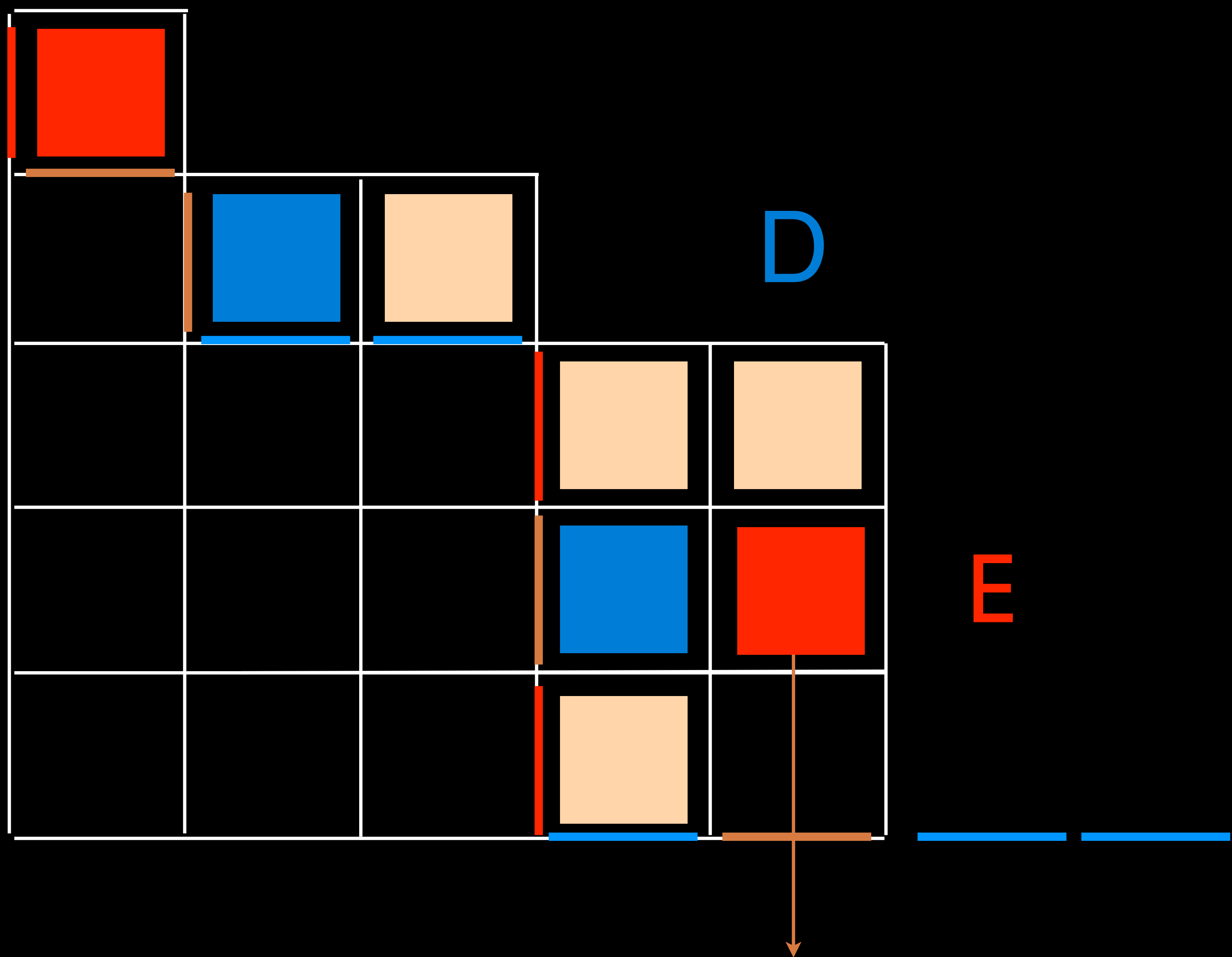
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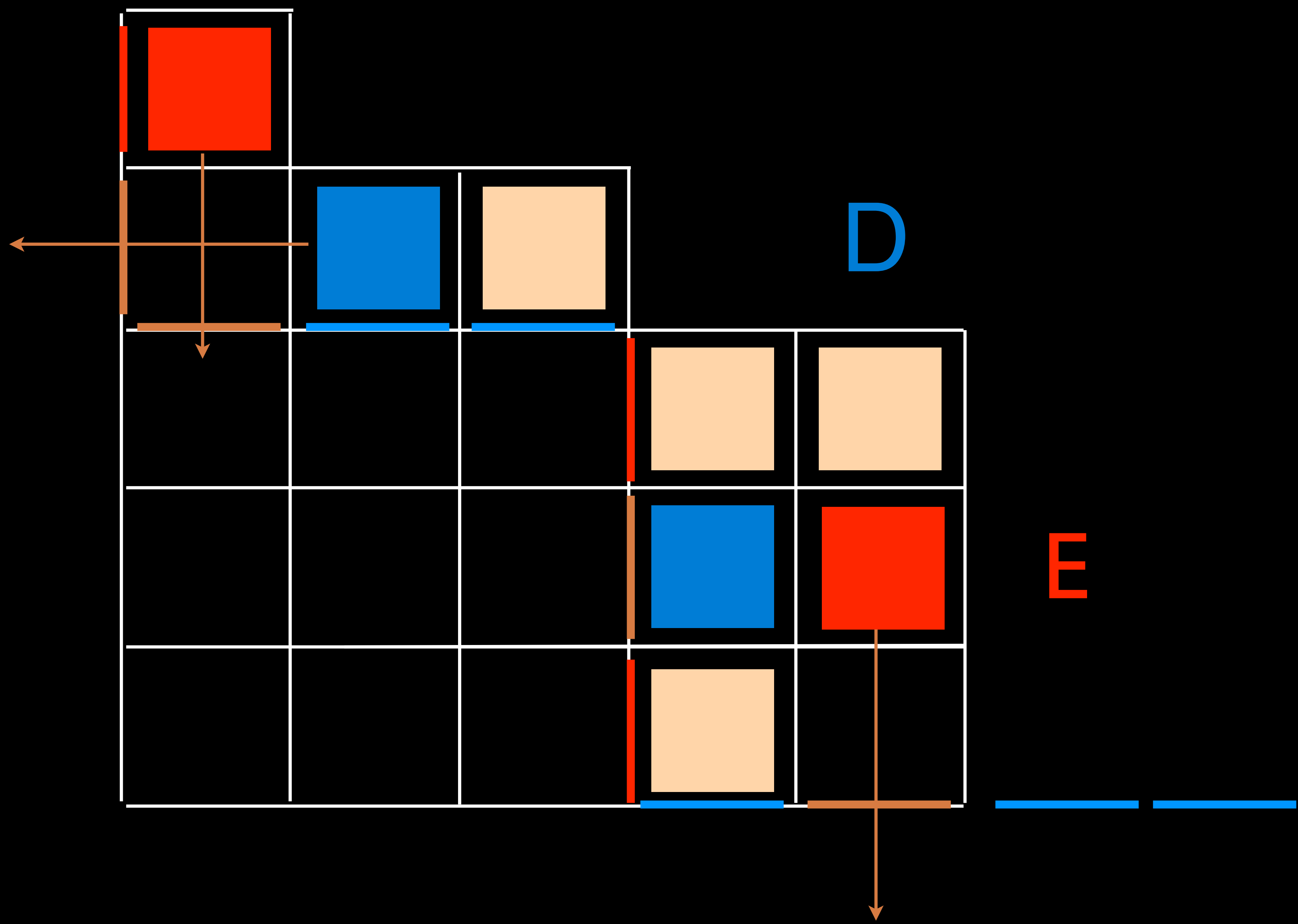
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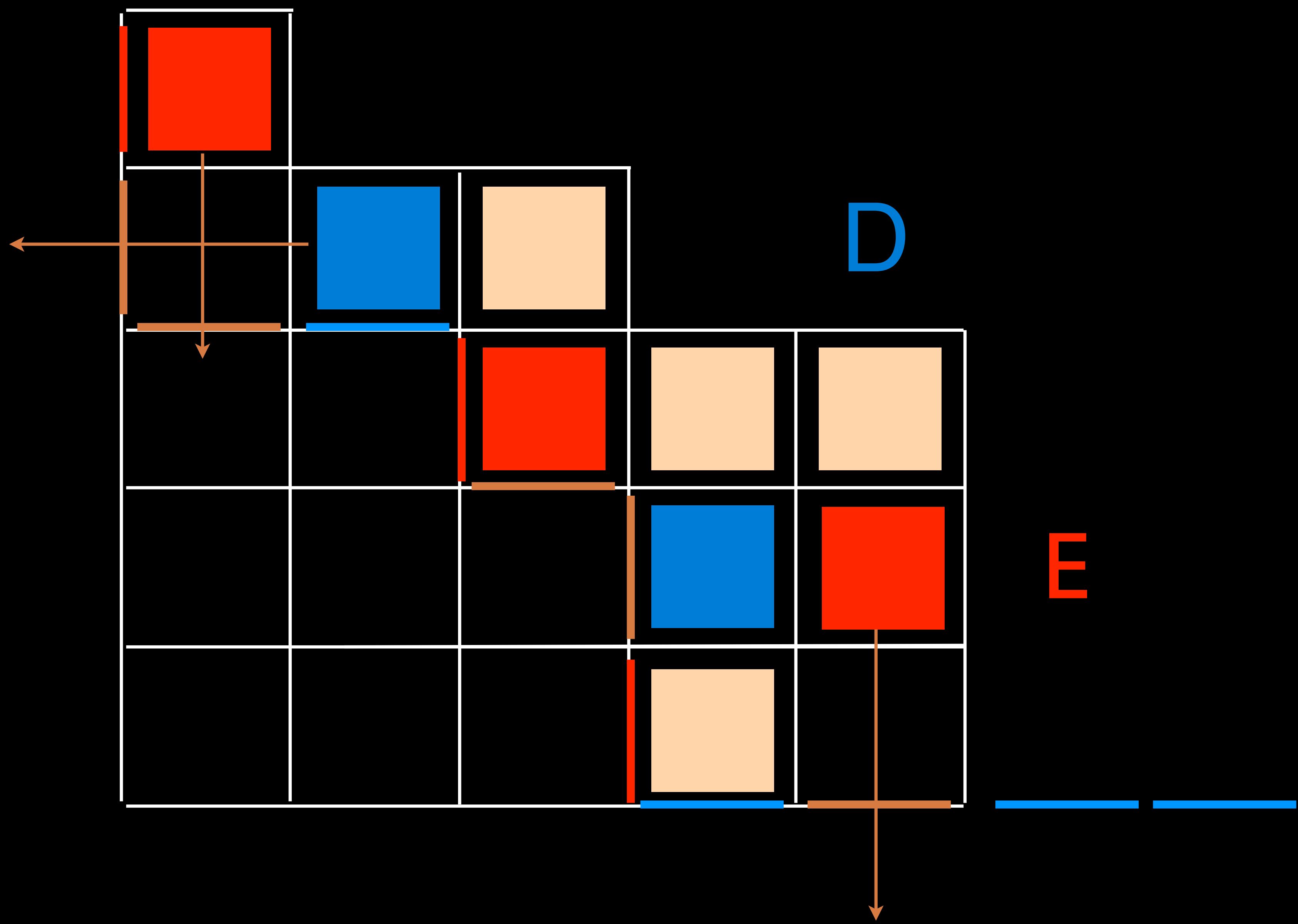
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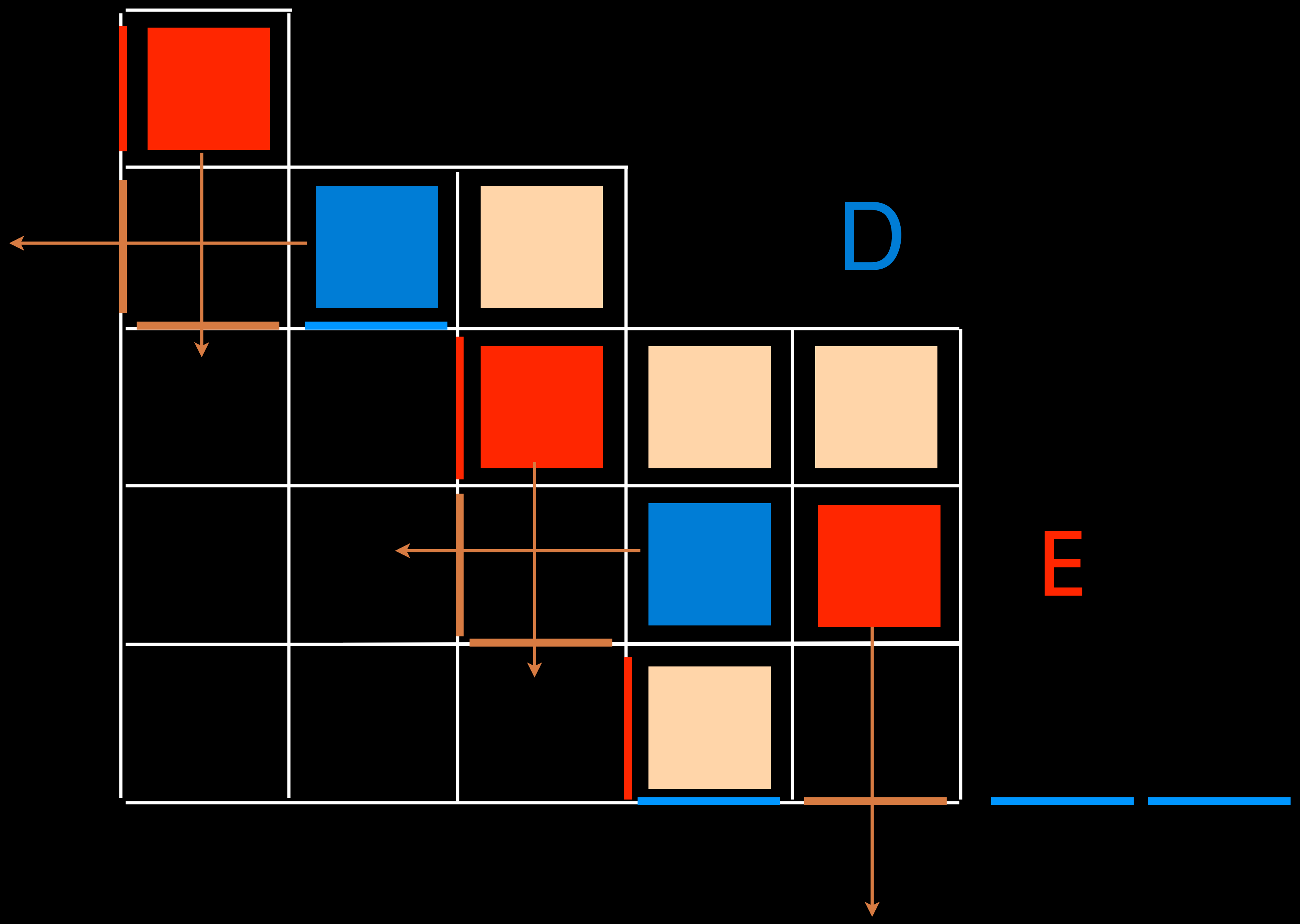
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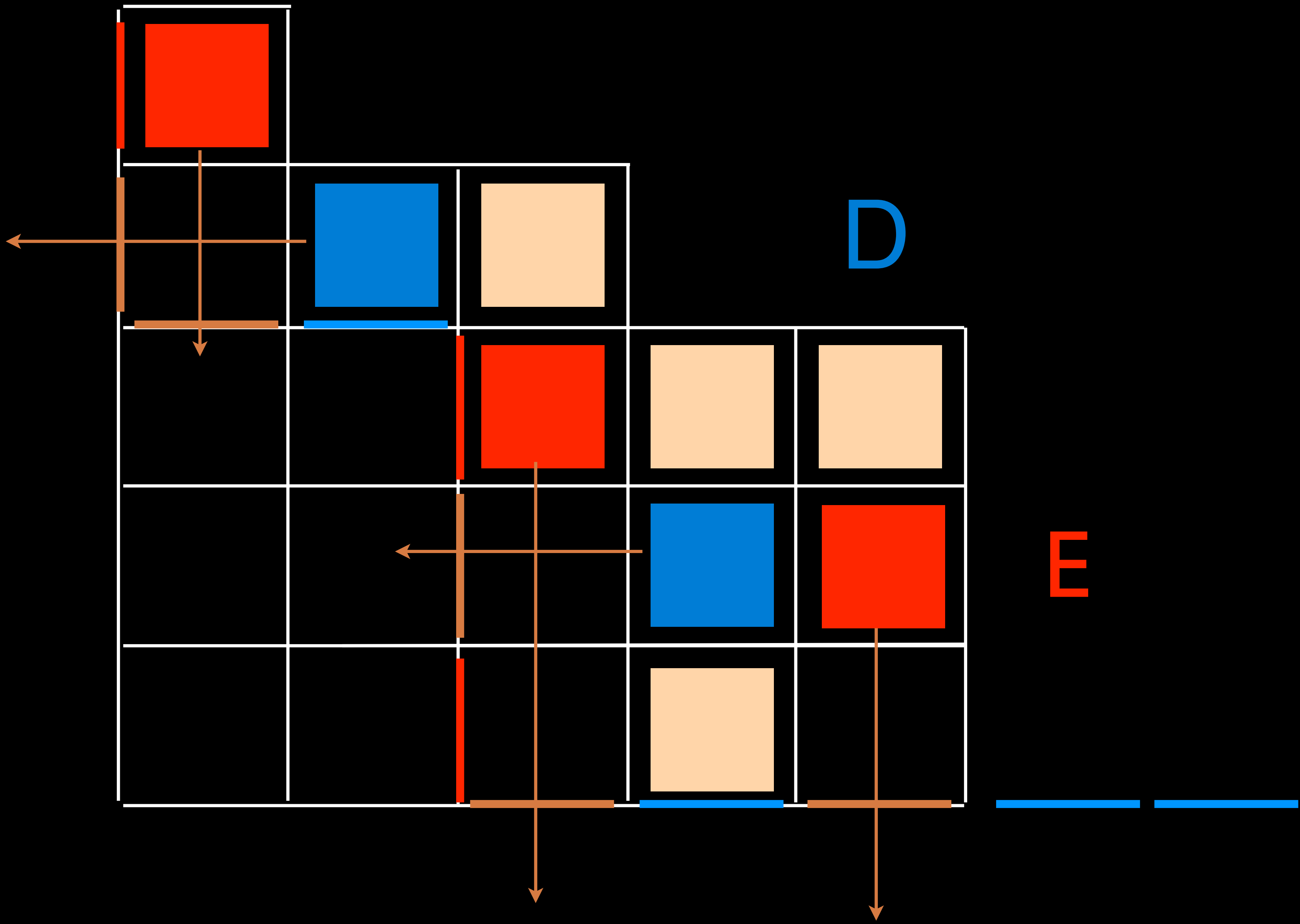
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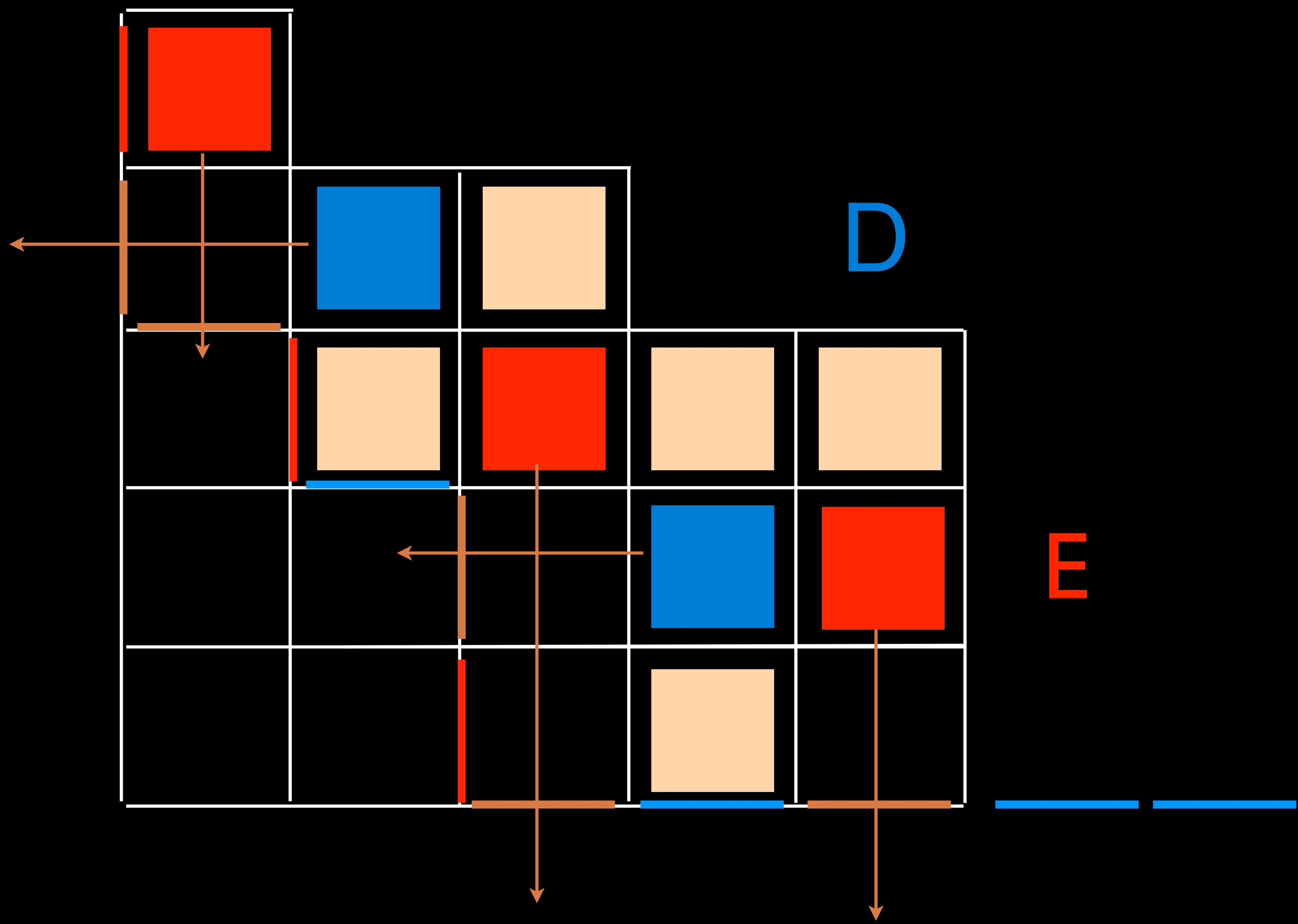
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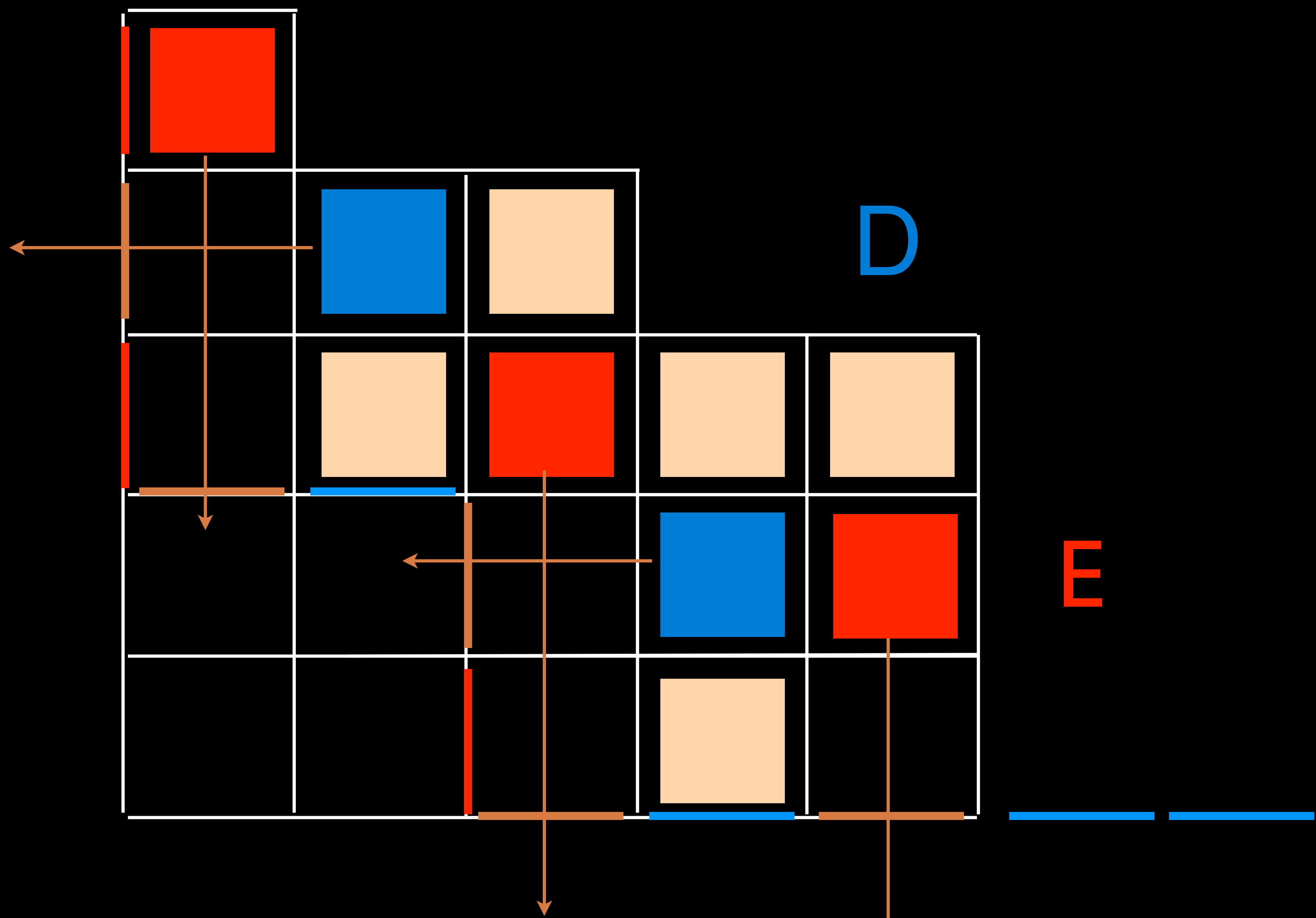
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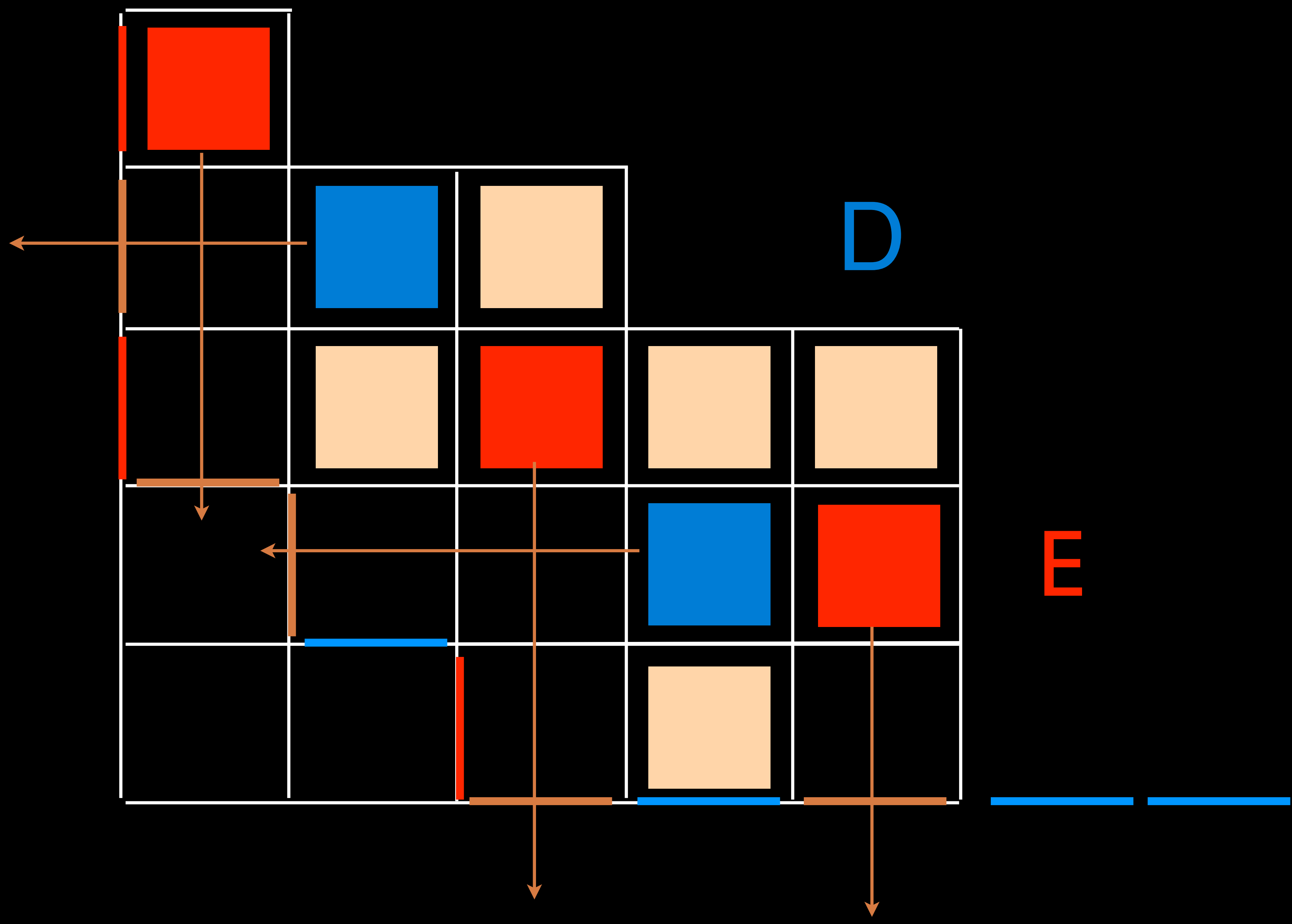
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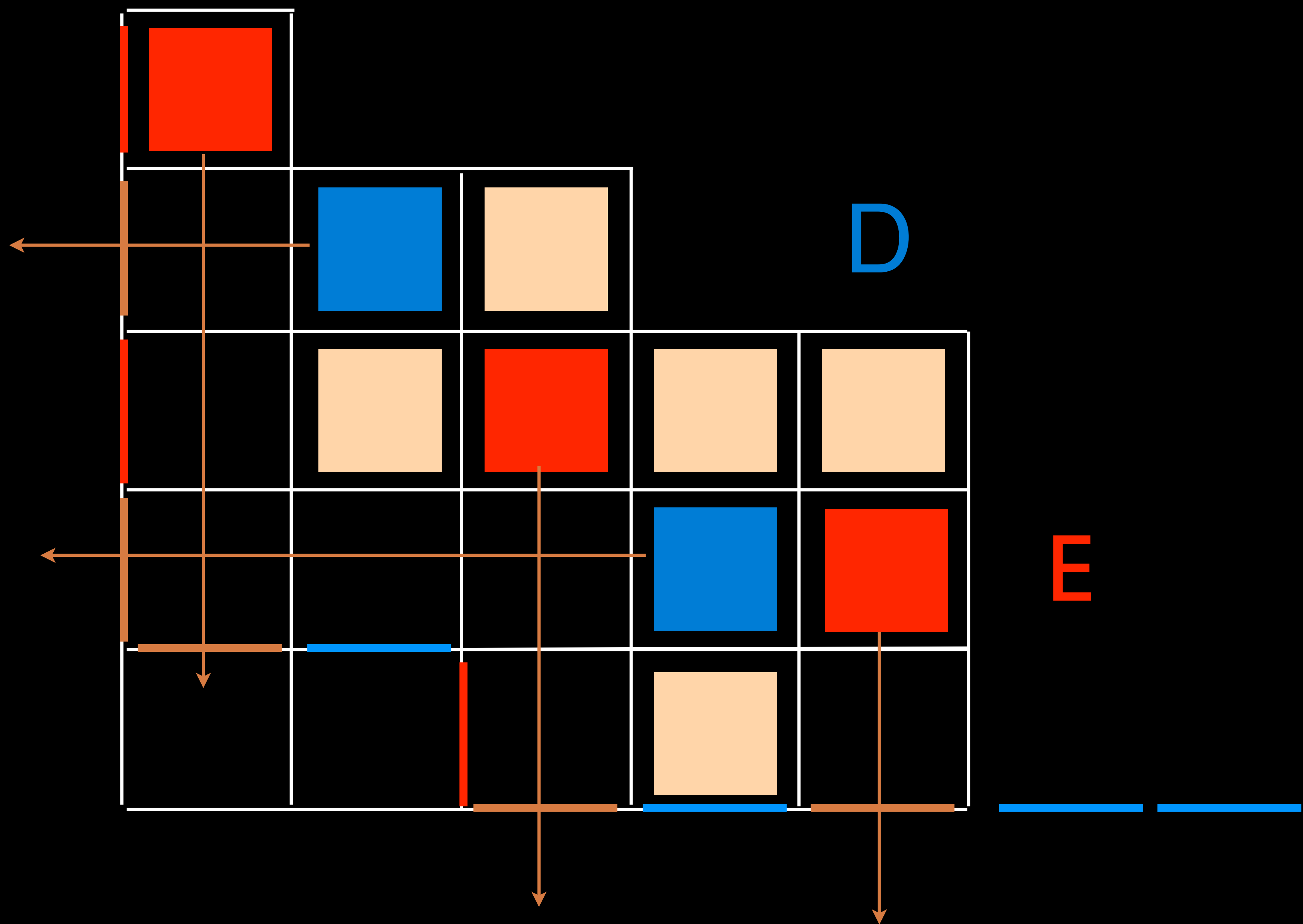
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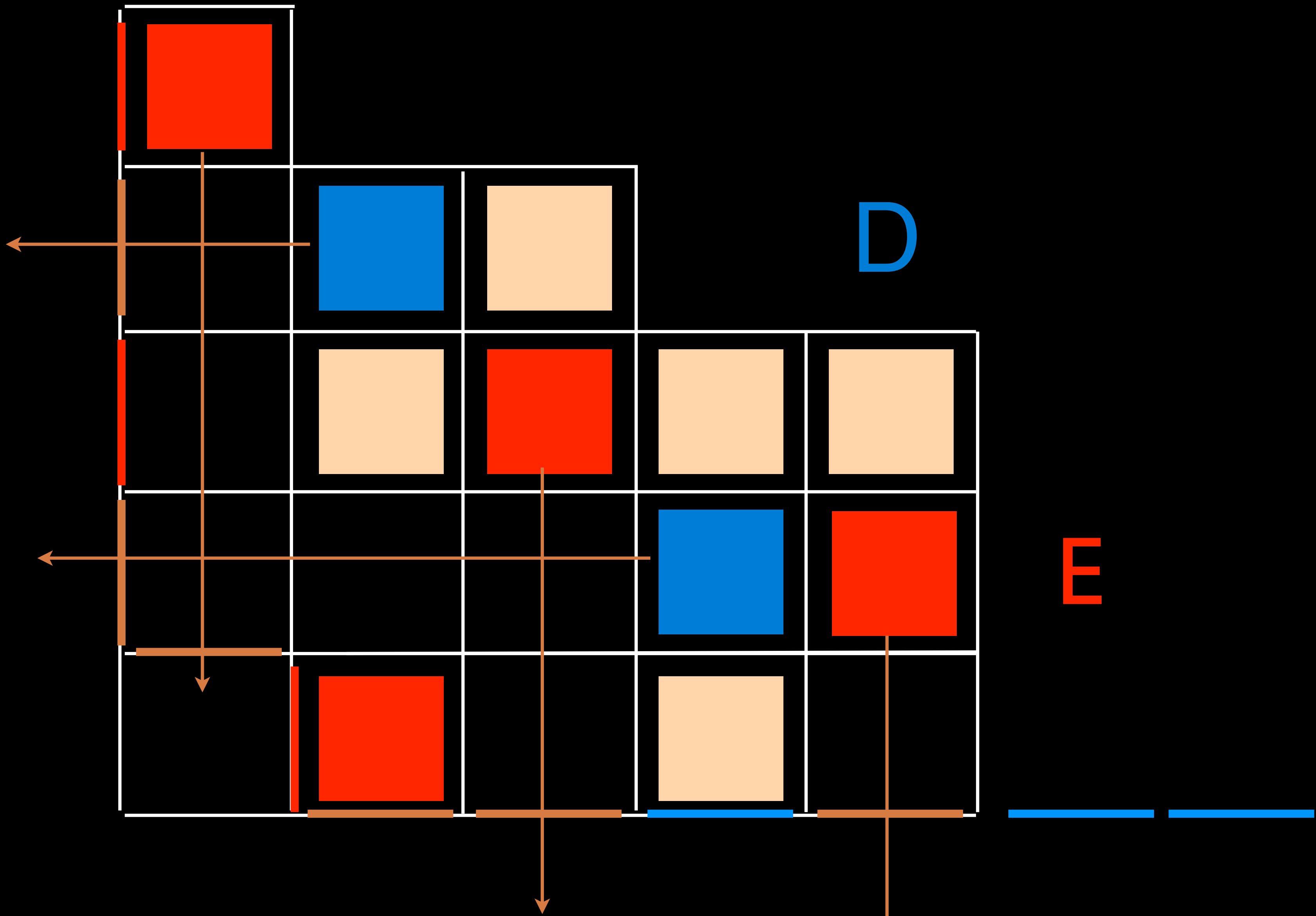
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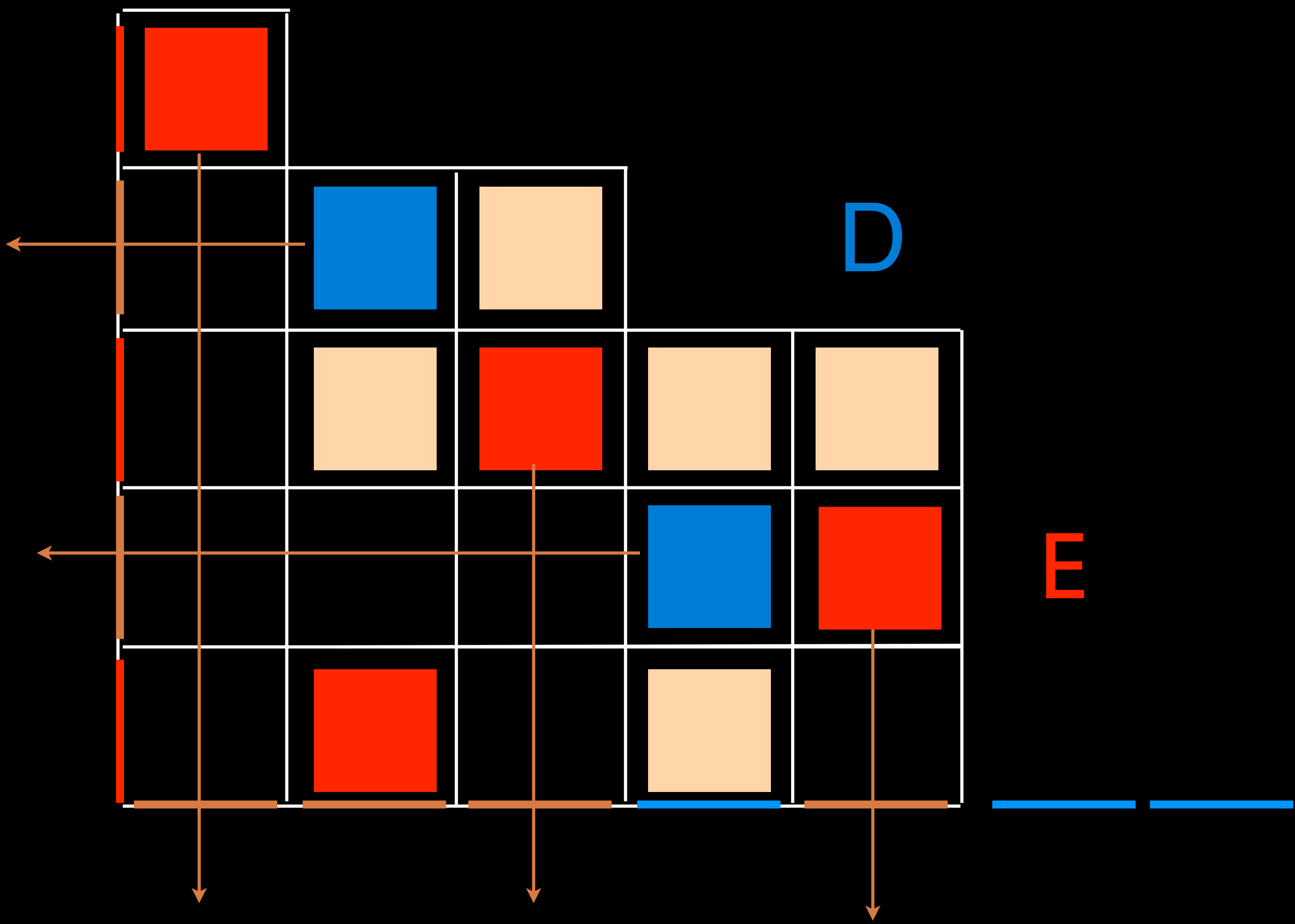
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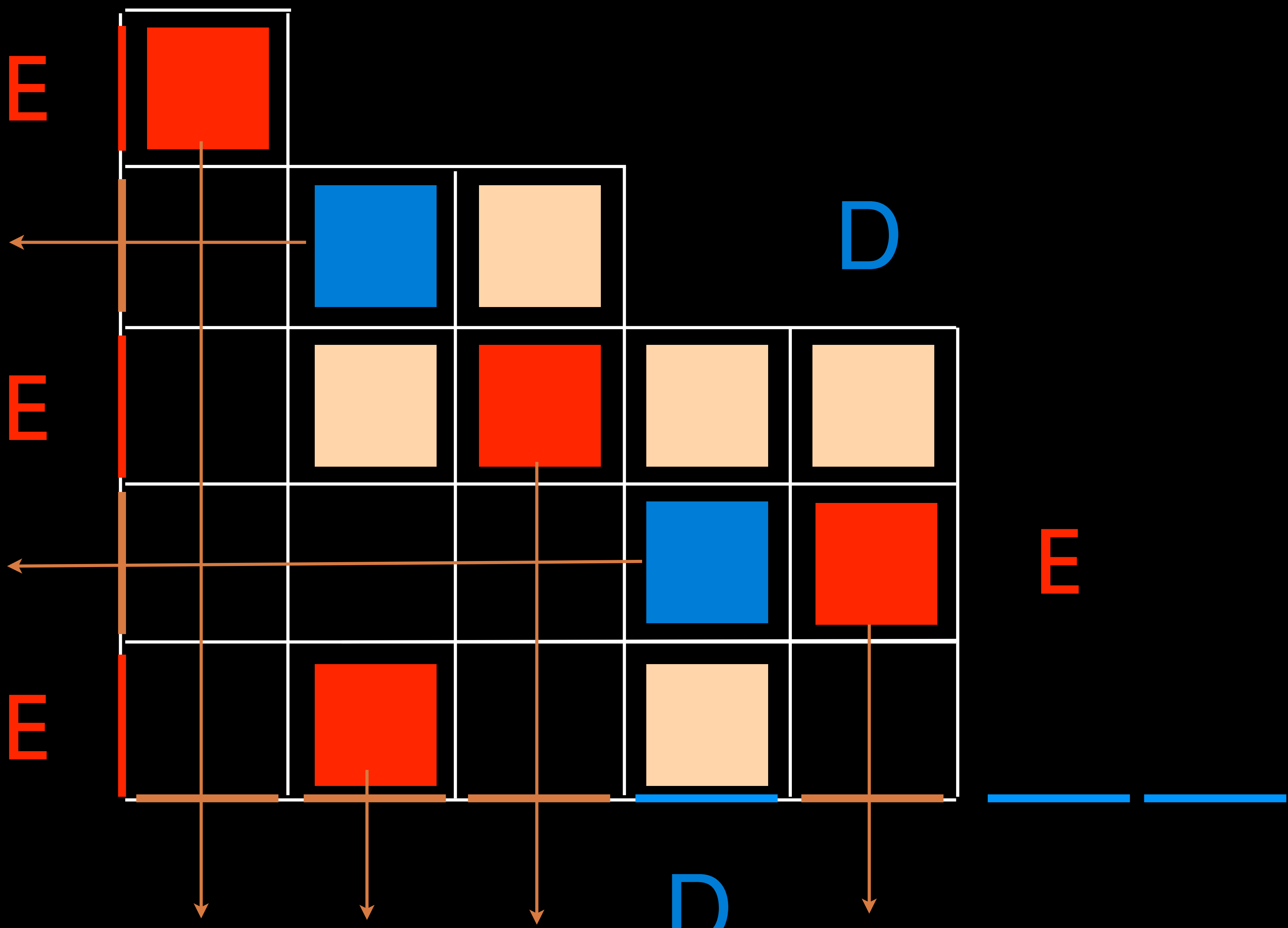
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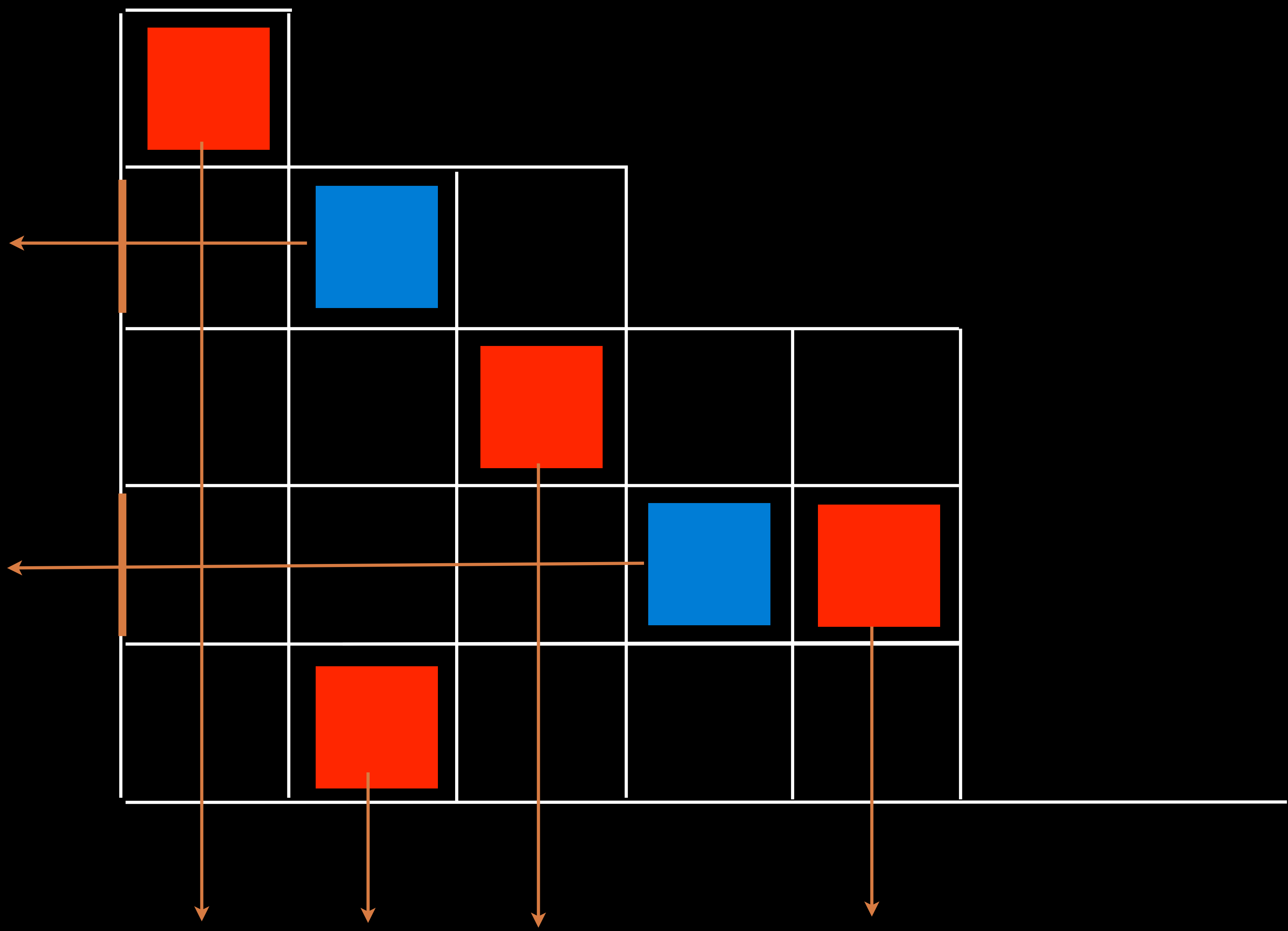
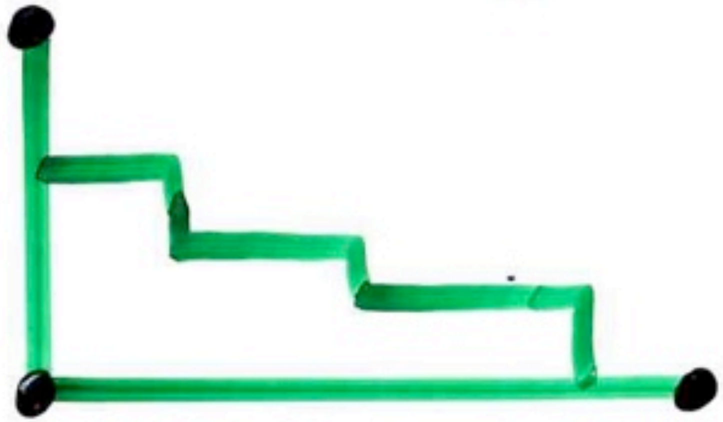
E
E
D



alternative tableau

- Ferrers diagram **F**

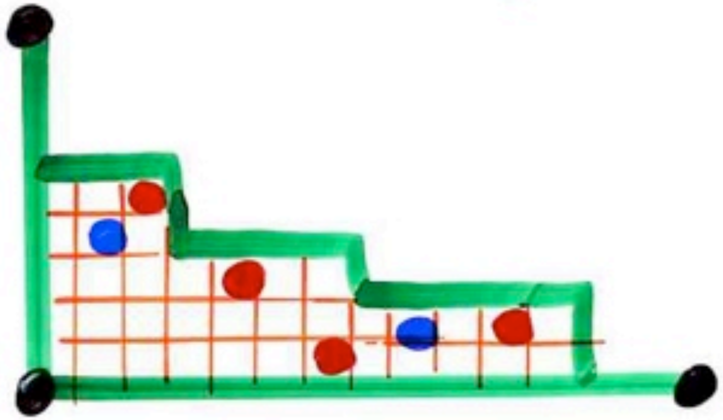


(possibly empty rows or column)

(nb of rows) + (nb of columns) = n

alternative tableau

- Ferrers diagram **F**



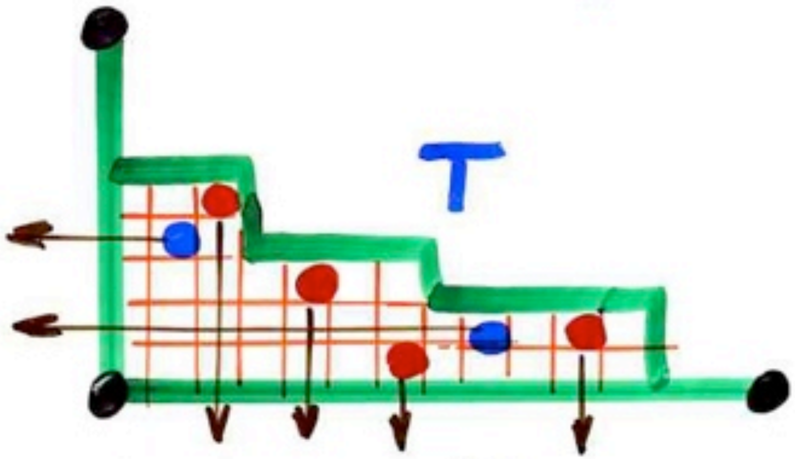
(possibly empty rows or column)

(nb of rows) + (nb of columns) = n

- some cells are coloured **red** or **blue**

alternative tableau T

- Ferrers diagram F (possibly empty rows or column)



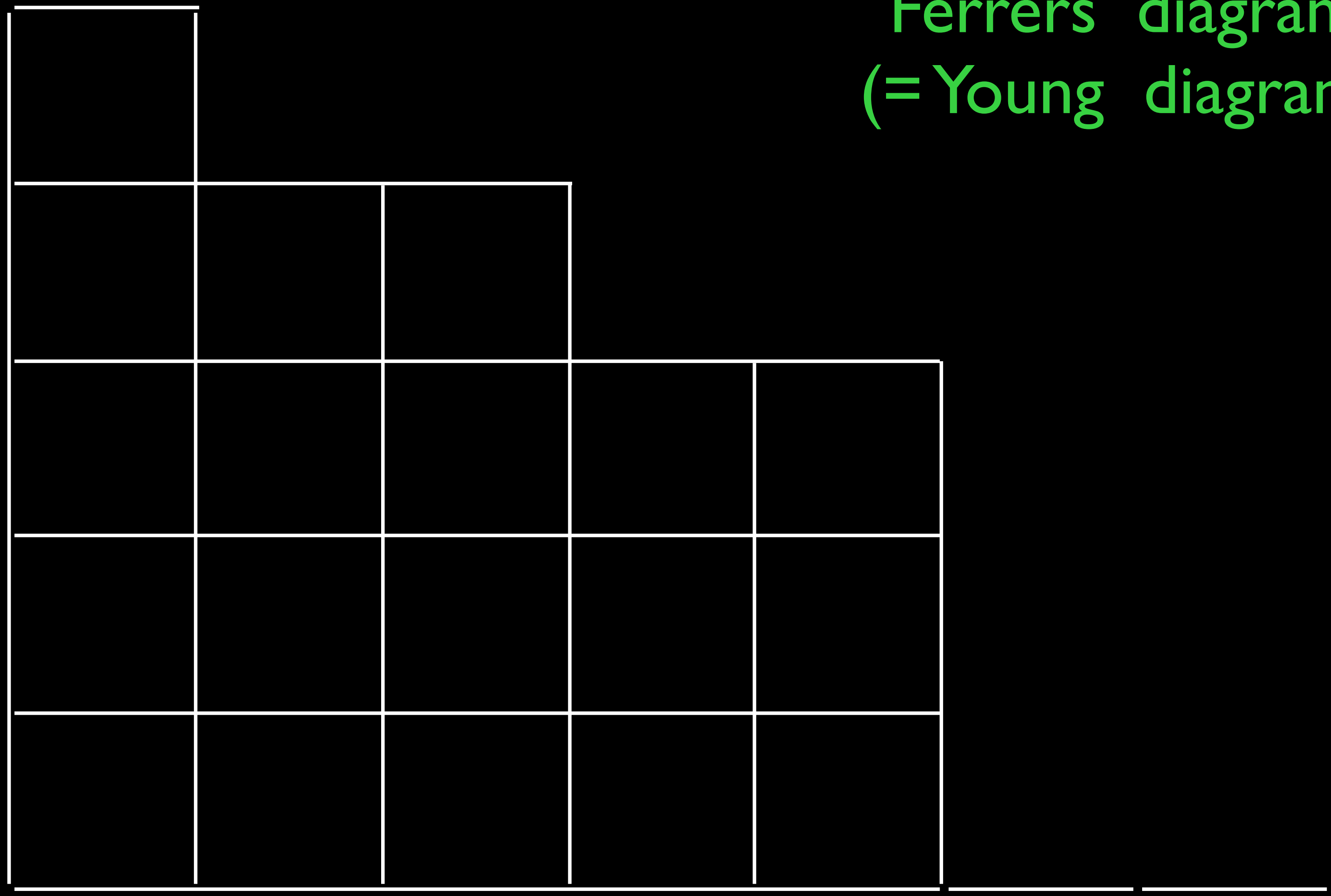
(nb of rows) + (nb of columns) = n

- some cells are coloured **red** or **blue**
- no coloured cell at the left of 
  no coloured cell below 

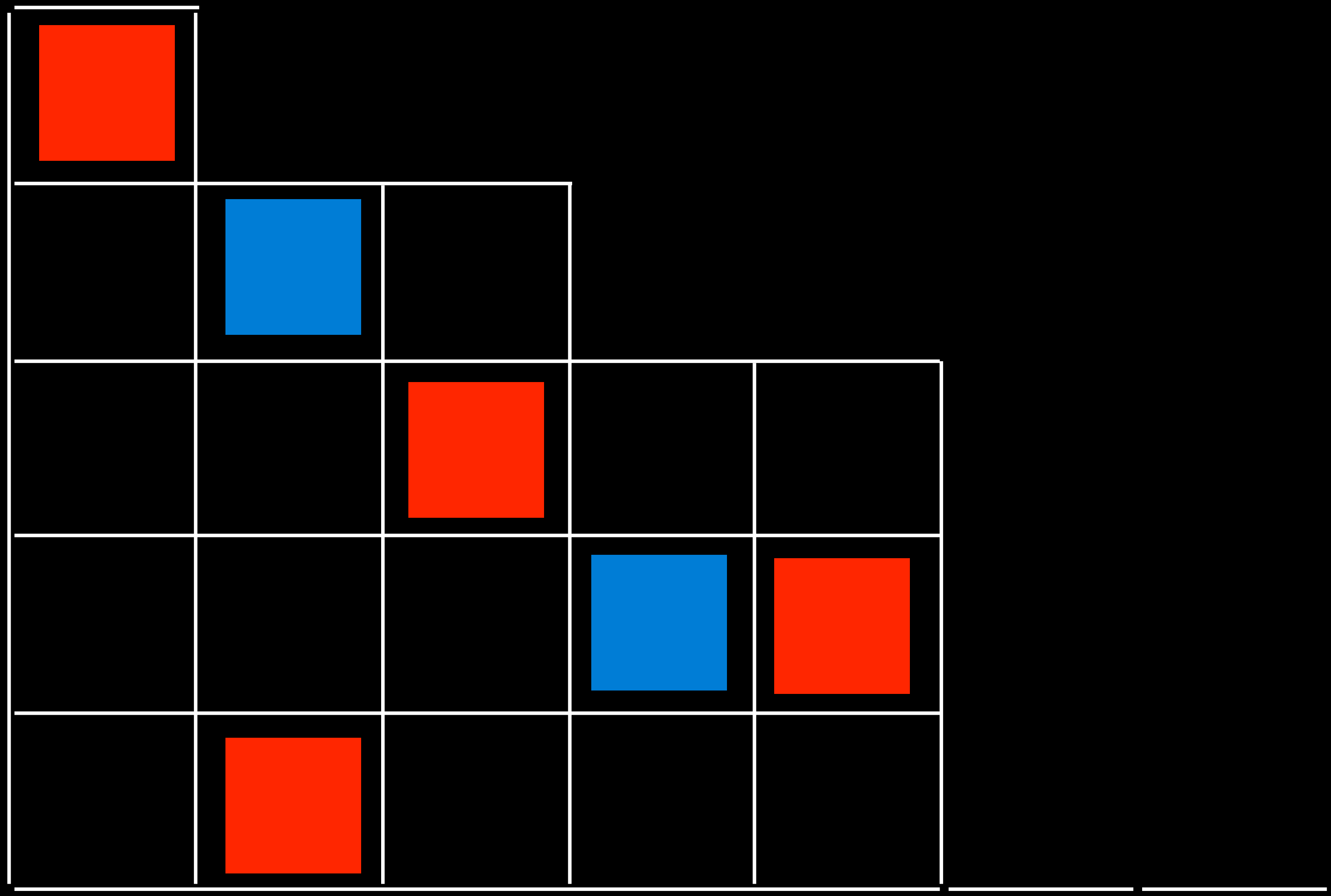
n size of T

alternative tableau

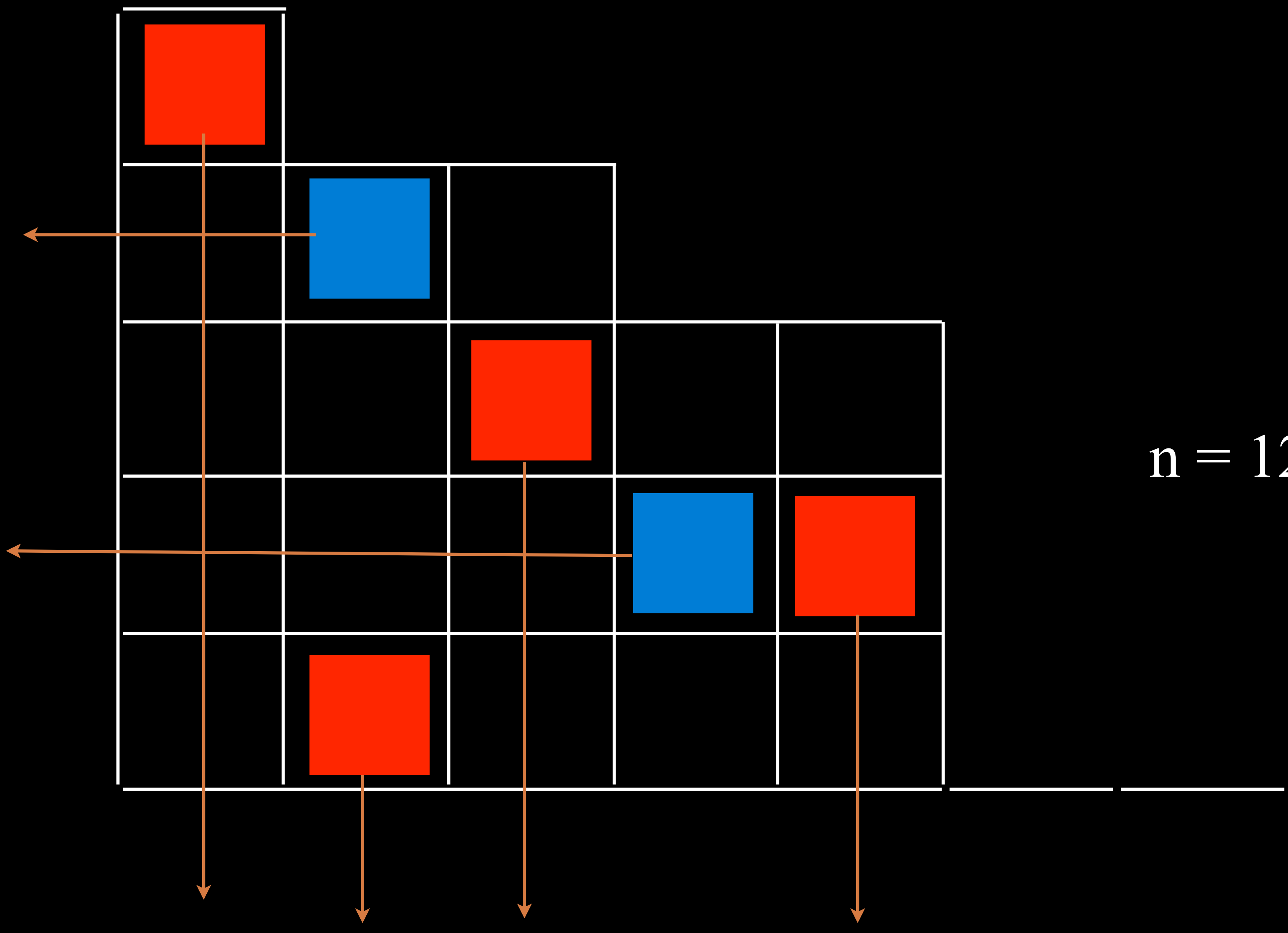
Ferrers diagram
(= Young diagram)



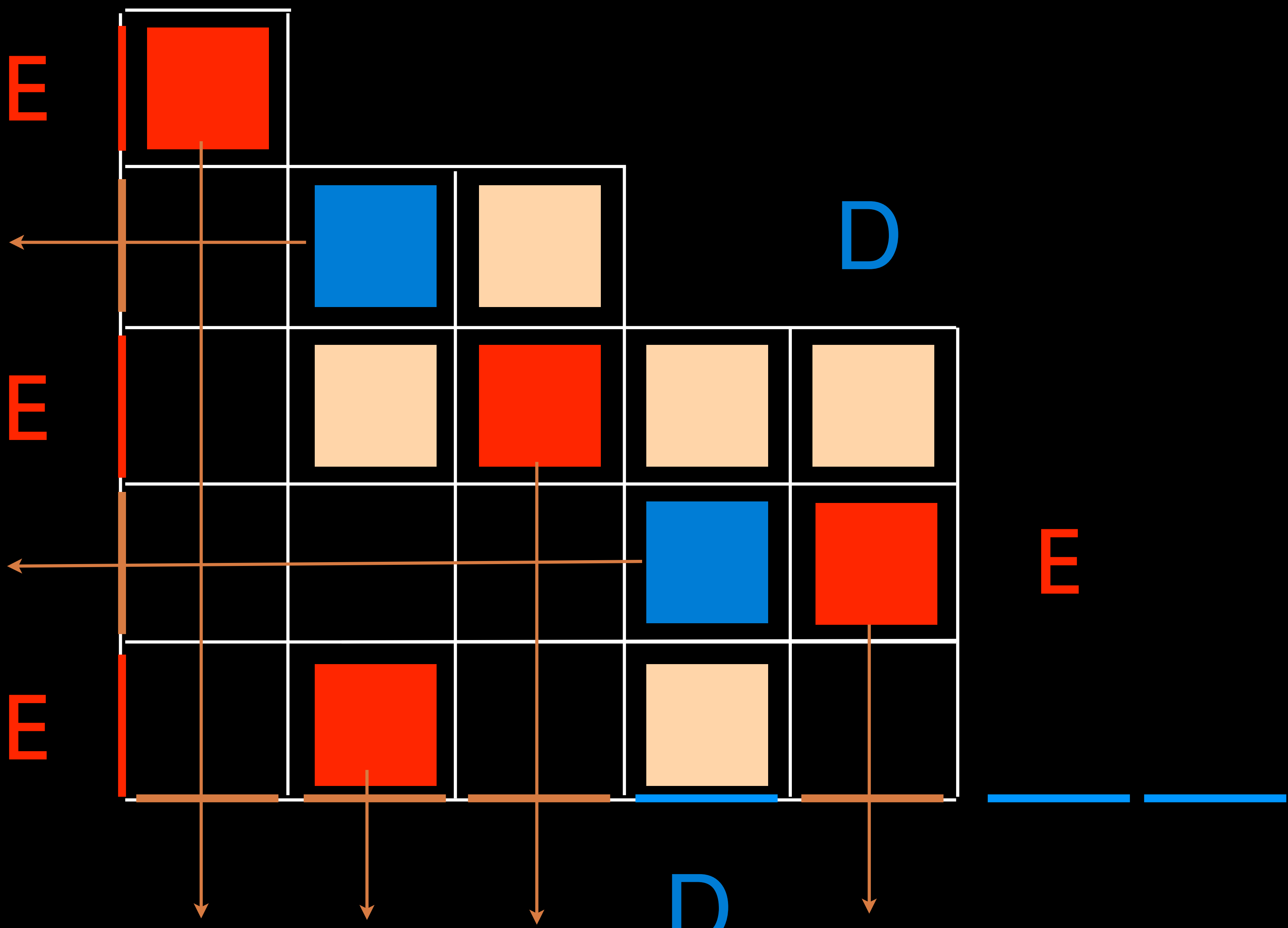
alternative tableau



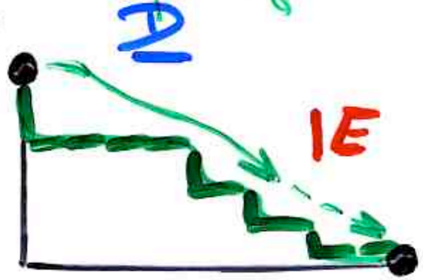
alternative tableau



n = 12


E
D
E
E
E
D

Def- profile of an alternative tableau

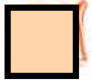


word $w \in \{E, D\}^*$

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

T alternative tableau with profile w

$k(T)$ = nb of 

$i(T)$ = nb of rows without blue cell

$j(T)$ = nb of columns without red cell

stationary probabilities for the PASEP

Def - profile of an alternative tableau

word $w \in \{E, D\}^*$

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

T alternative tableau with profile w

$k(T)$ = nb of □

$i(T)$ = nb of rows without blue cell

$j(T)$ = nb of columns without red cell

stationary probabilities

$$\begin{cases} DE = qED + D + E \\ DV = \bar{\beta} V & \bar{\beta} = 1/\beta \\ WE = \bar{\alpha} W & \bar{\alpha} = 1/\alpha \end{cases}$$

$$W E^{i} D^{j} V = \bar{\alpha}^{i} \bar{\beta}^{j} \underbrace{WV}_{1}$$

Cor. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ (PASEP) is

$$\mathrm{proba}_\tau(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{T} q^{L(T)} \alpha^{-f(T)} \beta^{-u(T)}$$

T: alternative tableaux, profile τ

- $f(T)$: nb of rows without [blue] cell
- $u(T)$: nb of columns without [red] cell
- $L(T)$: nb of cells [square]

permutation tableau

S. Corteel, L. Williams
(2007) (2008) (2009)

permutation tableau

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)

Corteel, Williams (2006) PASEP

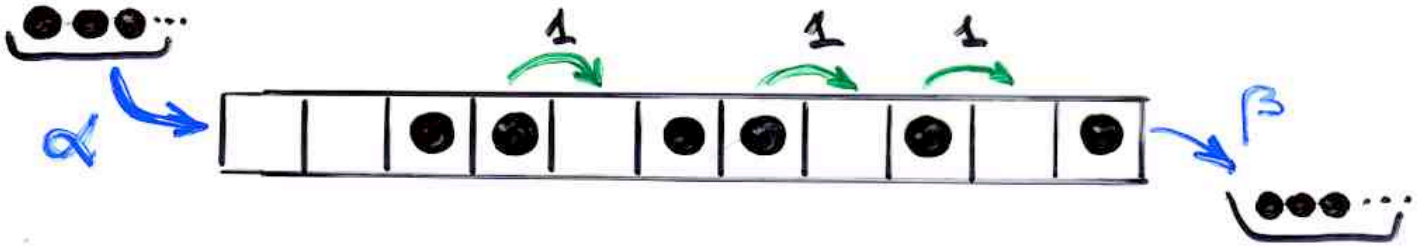
Partially Asymmetric Exclusion Process

M. Josuat-Vergès (2007)

TASEP

TASEP

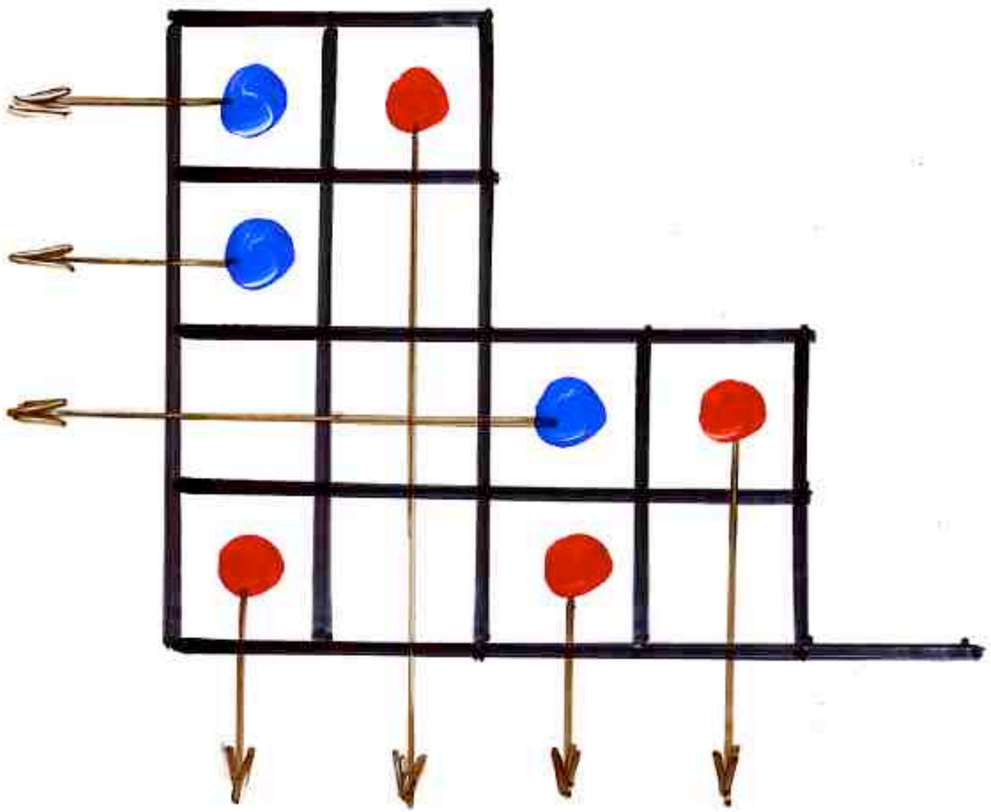
"totally asymmetric exclusion process"



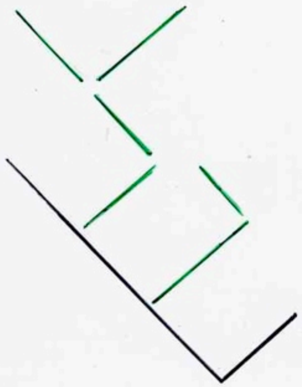
Def Catalan alternative tableau T

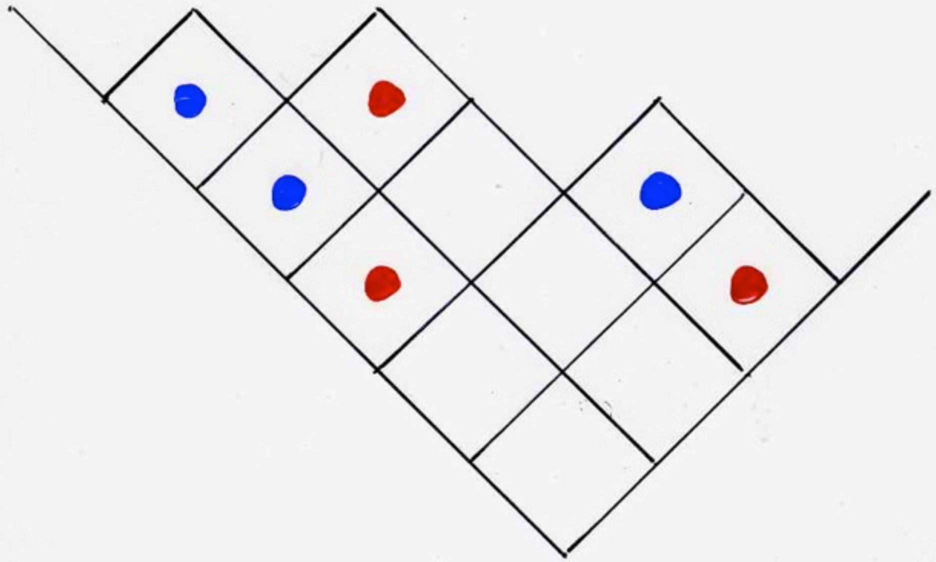
alt. tab. without cells ☒

i.e: every empty cell is below a red cell or on the left of a blue cell

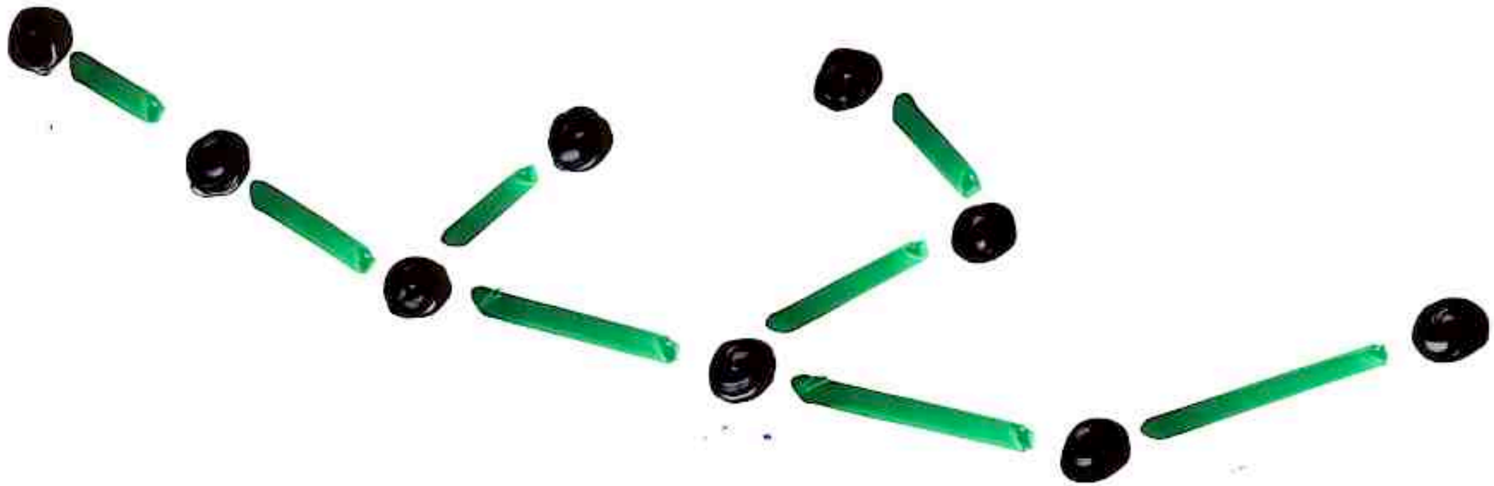


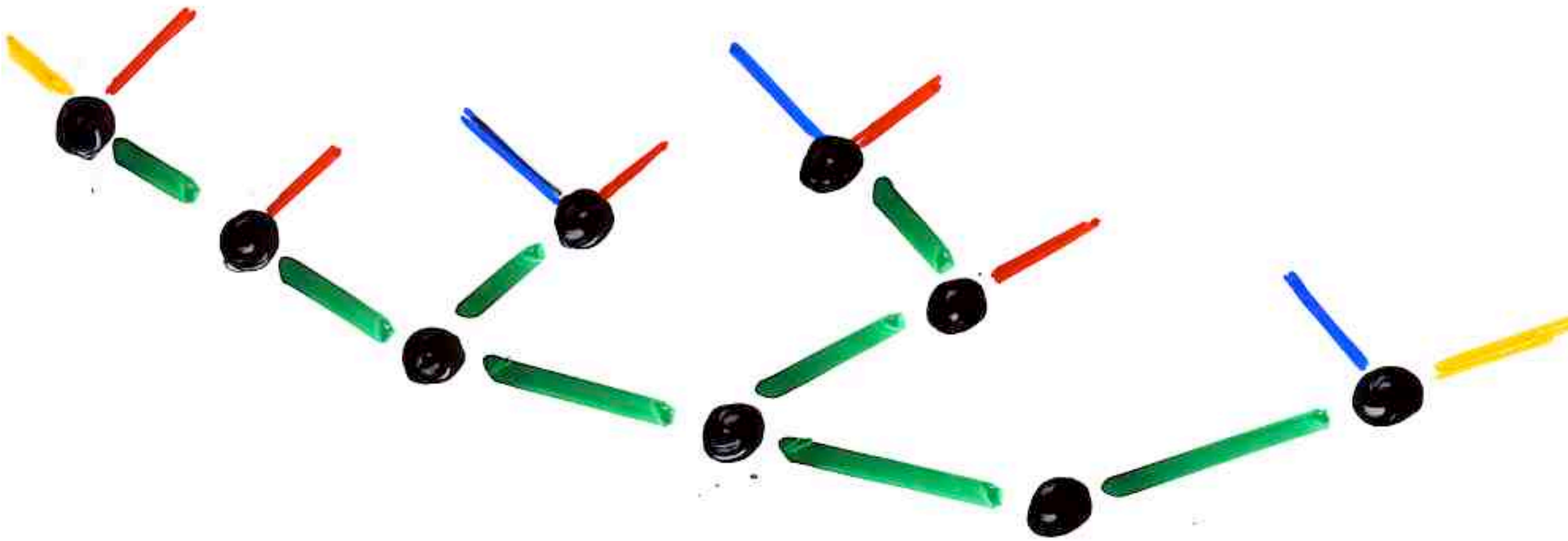
Bijection tableaux alternatifs de Catalan taille n ⟷ arbres binaires n arêtes





profil (bord) du diagramme de Ferrers ←---→ canopée



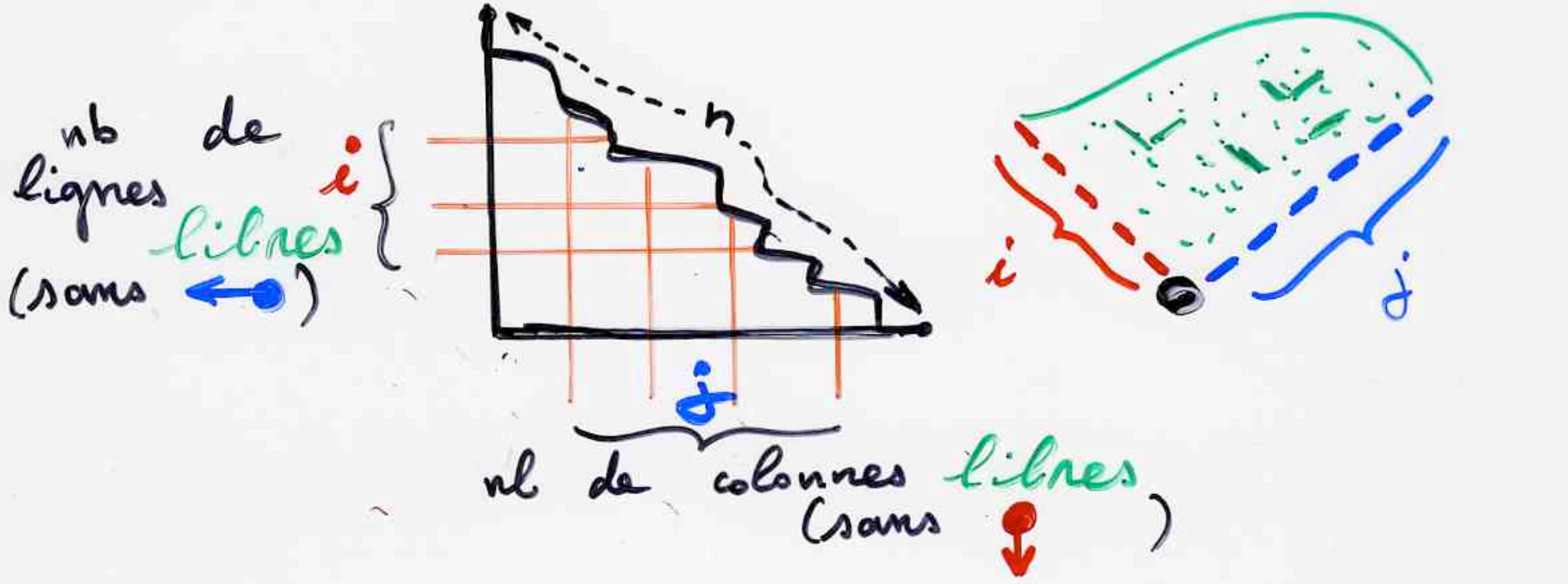


canopy of a binary tree

$c(B) = - - + - + - - +$

Bijection tableaux alternatifs de Catalan taille n ⟷ arbres binaires n arêtes

profil (bord) du diagramme de Ferrers ⟵---⟶ canopée



$$s = (\tau_1, \ldots, \tau_n)$$

$$P_n(s; \alpha, \beta) = \frac{1}{Z_n} \sum_{B} \bar{\alpha}^{\,lb(B)} \, \bar{\beta}^{\,rb(B)}$$

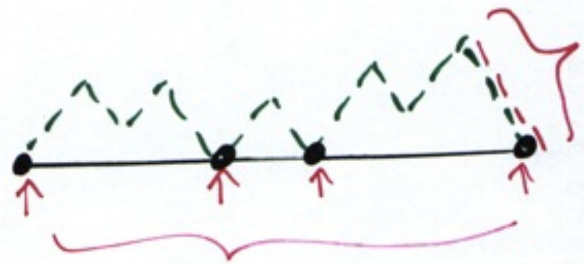
binary trees

canopy s

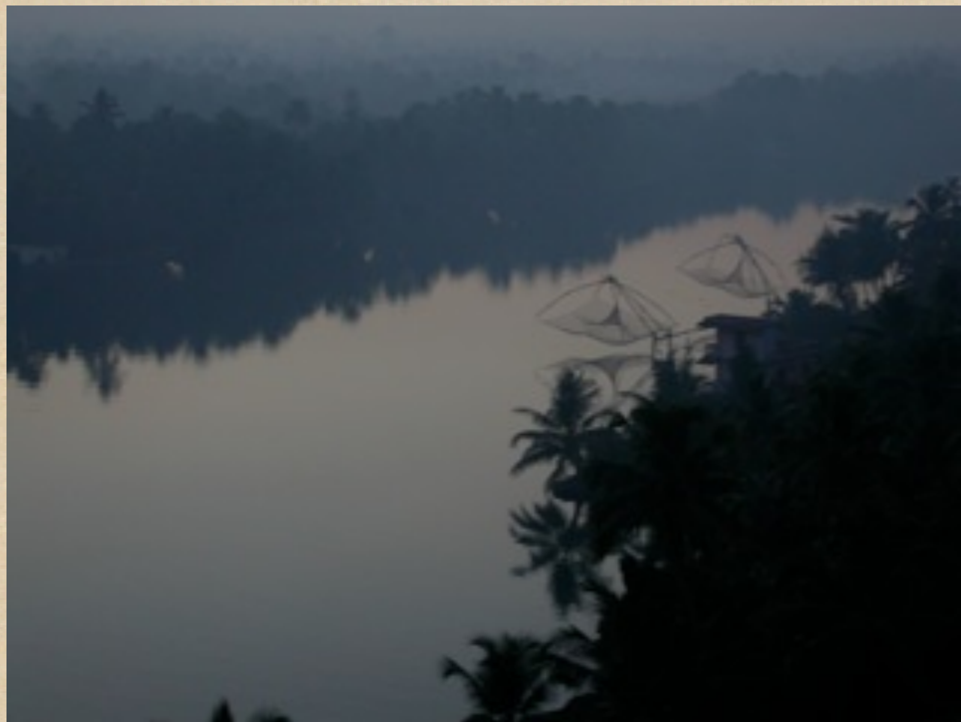
$$Z_n = \sum_{i=1}^{n} \frac{i}{2n-i} \binom{2n-i}{n} \frac{\bar{\alpha}^{(i+1)} - \bar{\beta}^{(i+1)}}{\bar{\alpha} - \bar{\beta}}$$

partition function

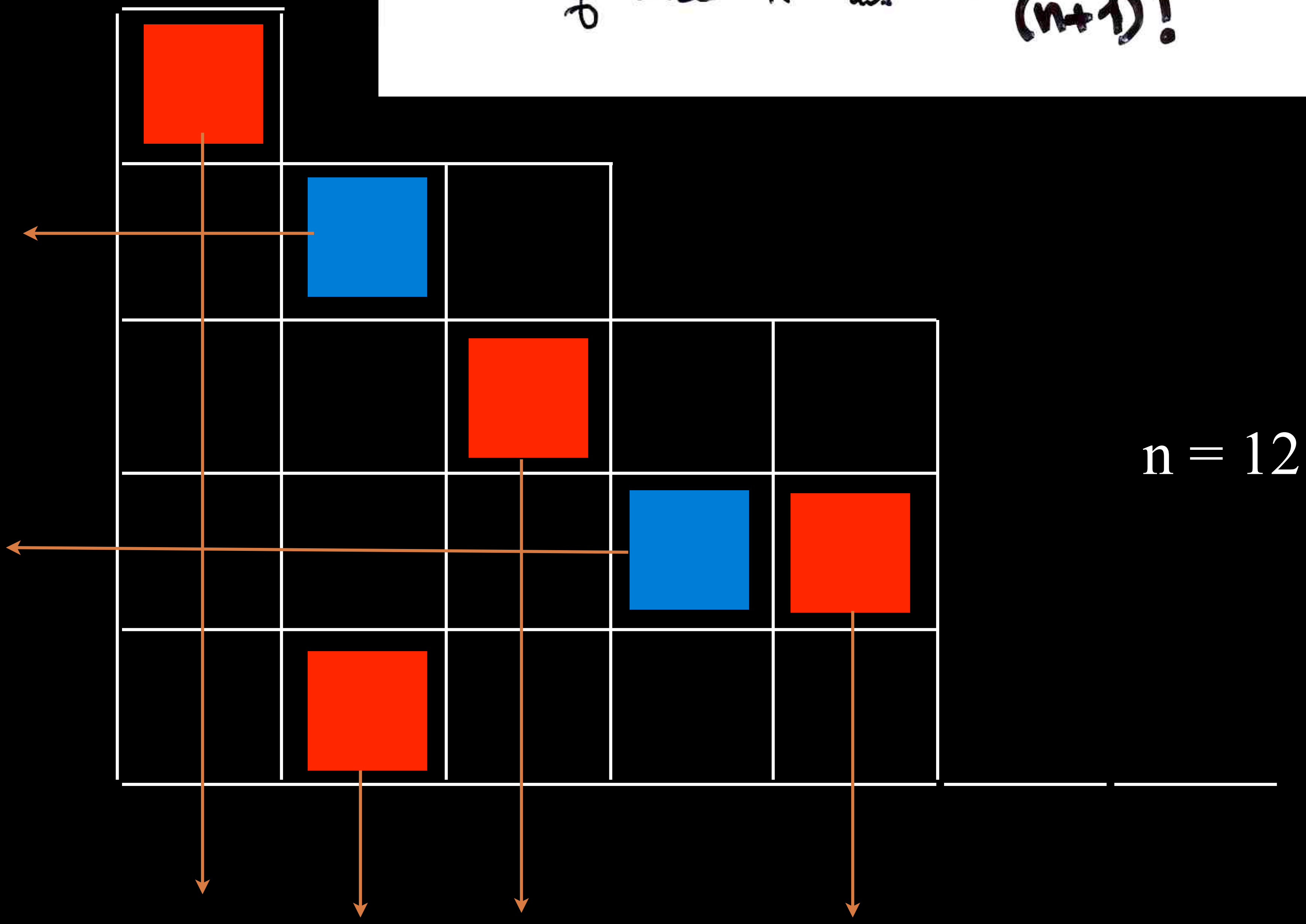
"ballot" numbers



number of alternative tableaux (or permutation tableaux)



Prop. The number of alternative tableaux of size n is $(n+1)!$



ex: $n=2$

combinatorial representation of the operators E and D

$DE = ED + E + D$

V vector space generated by B basis

B alternating words two letters {○, ●}
(no occurrence of ○○ or ●●)

4 operators A, S, J, K

4 operators $A, S, J, K,$ $\quad u \in B$

$$\langle u | A = \sum_{\substack{\text{letter } \circ \\ \text{of } u}} v \quad , \quad v \text{ obtained by: } \circ \to \circ\bullet\circ$$

$$\langle u | S = \sum_{\circ \text{ of } u} v \quad \quad v \text{ obtained by: } \circ \to \bullet$$

(and $\bullet\bullet \to \bullet$ $\quad \bullet\bullet\bullet \to \bullet$)

$$\langle u | J = \sum_{\circ \text{ of } u} v \quad \quad v, \quad \circ \to \bullet\circ$$

(and $\bullet\bullet \to \bullet$)

$$\langle u | K = \sum_{\substack{\circ \\ \text{of } u}} v \quad \quad v, \quad \circ \to \circ\bullet$$

(and $\bullet\bullet \to \bullet$)

$$\circ\bullet\circ\bullet \, | S = \bullet\circ\bullet + \circ\bullet$$

Lemma.

$$AS = SA + J + K$$

$$AK = KA + A$$

$$JS = SJ + S$$

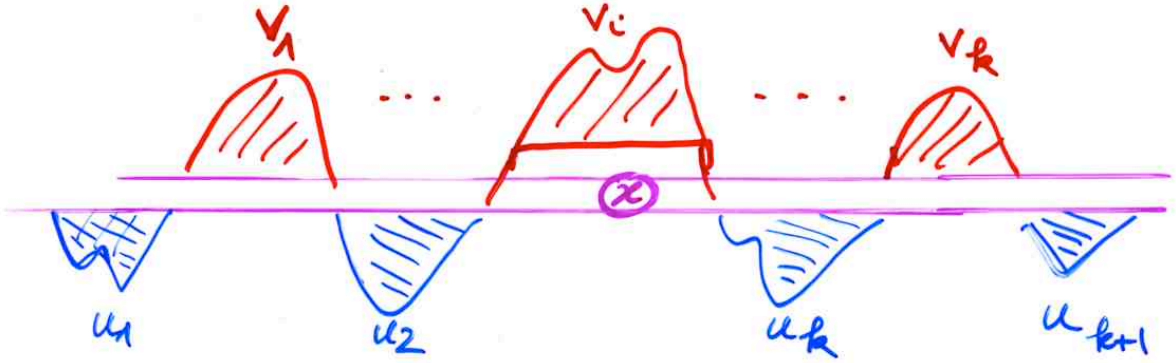
$$JK = KJ$$

$$D = A + J$$

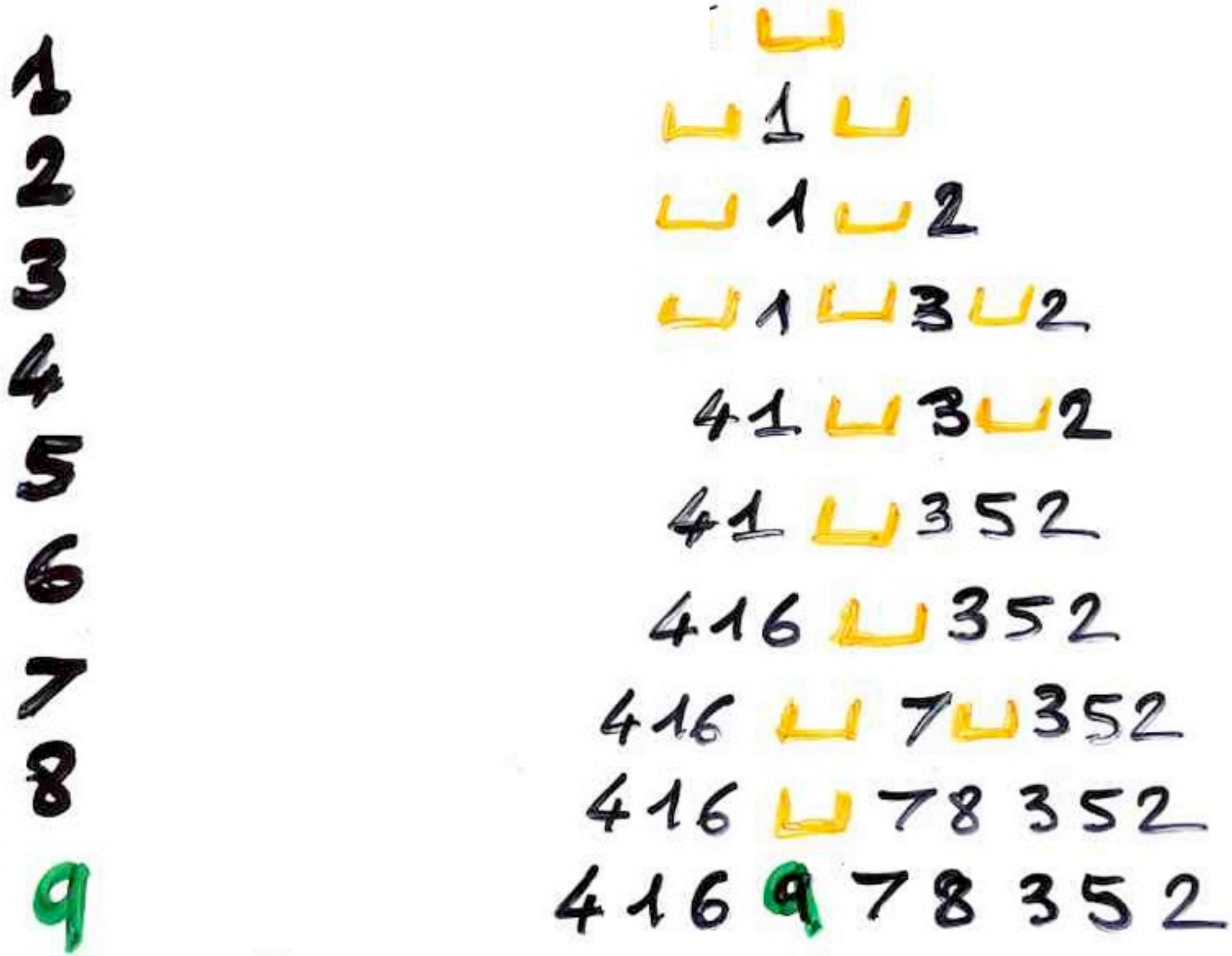
$$E = S + K$$

$$D = A + J$$
$$E = S + K$$

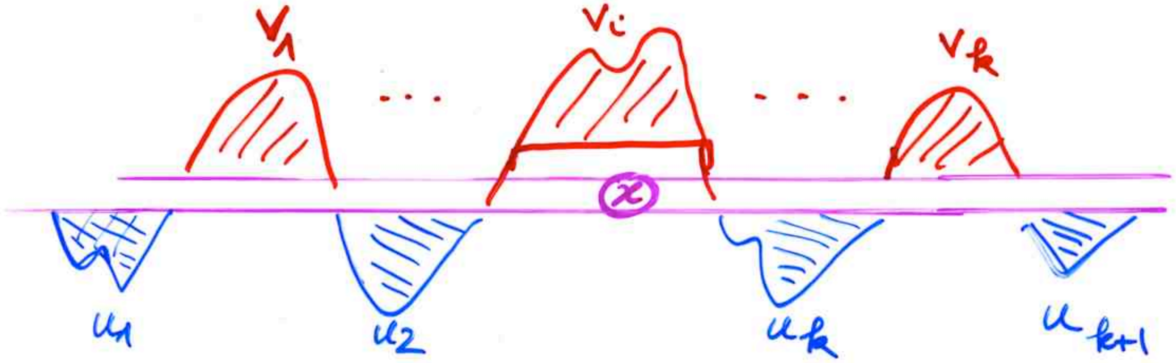
$$DE = (A+J)(S+K)$$
$$= AS + AK + JS + JK$$
$$= \underbrace{(SA + KA + SJ + KJ)}_{\underbrace{(S+K)(A+J)}_{ED}} + \underbrace{J + K + A + S}_{E + D}$$



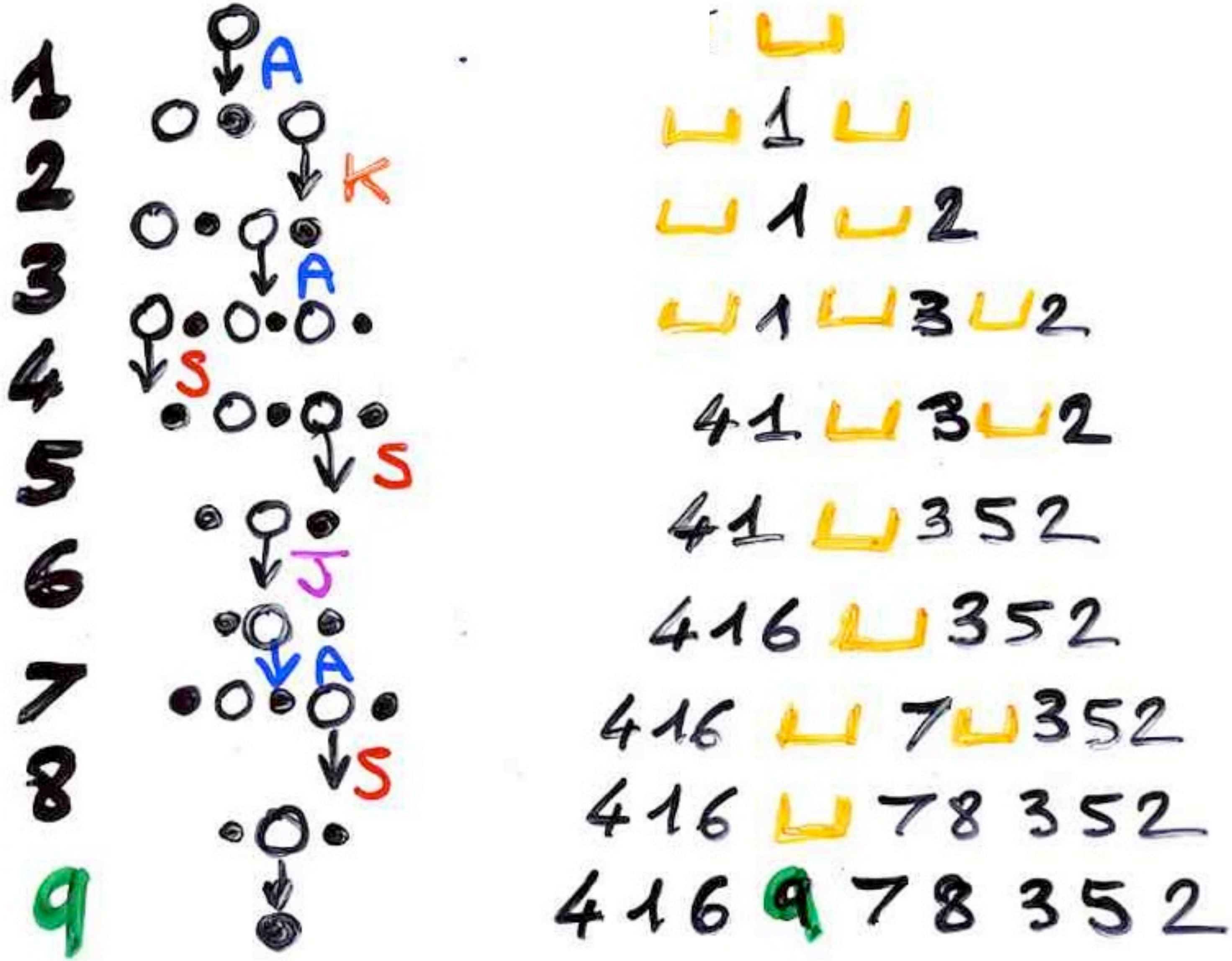
v_1
v_i
v_k
x
u_1
u_2
u_k
u_{k+1}



1 ⊔1⊔
2 ⊔1⊔2
3 ⊔1⊔3⊔2
4 41⊔3⊔2
5 41⊔352
6 416⊔352
7 416⊔7⊔352
8 416⊔78352
9 416978352
⊔



v1
vi
vk
x
u1
u2
uk
uk+1



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9
A
K
A
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S
J
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S
1
1 2
1 3 2
41 3 2
41 352
416 352
416 7 352
416 78 352
416 9 78 352

bijection
permutations
alternative
tableaux

416978352

●

●1 ●

●1●2 ●

●1●3●2

41● 3 ●2

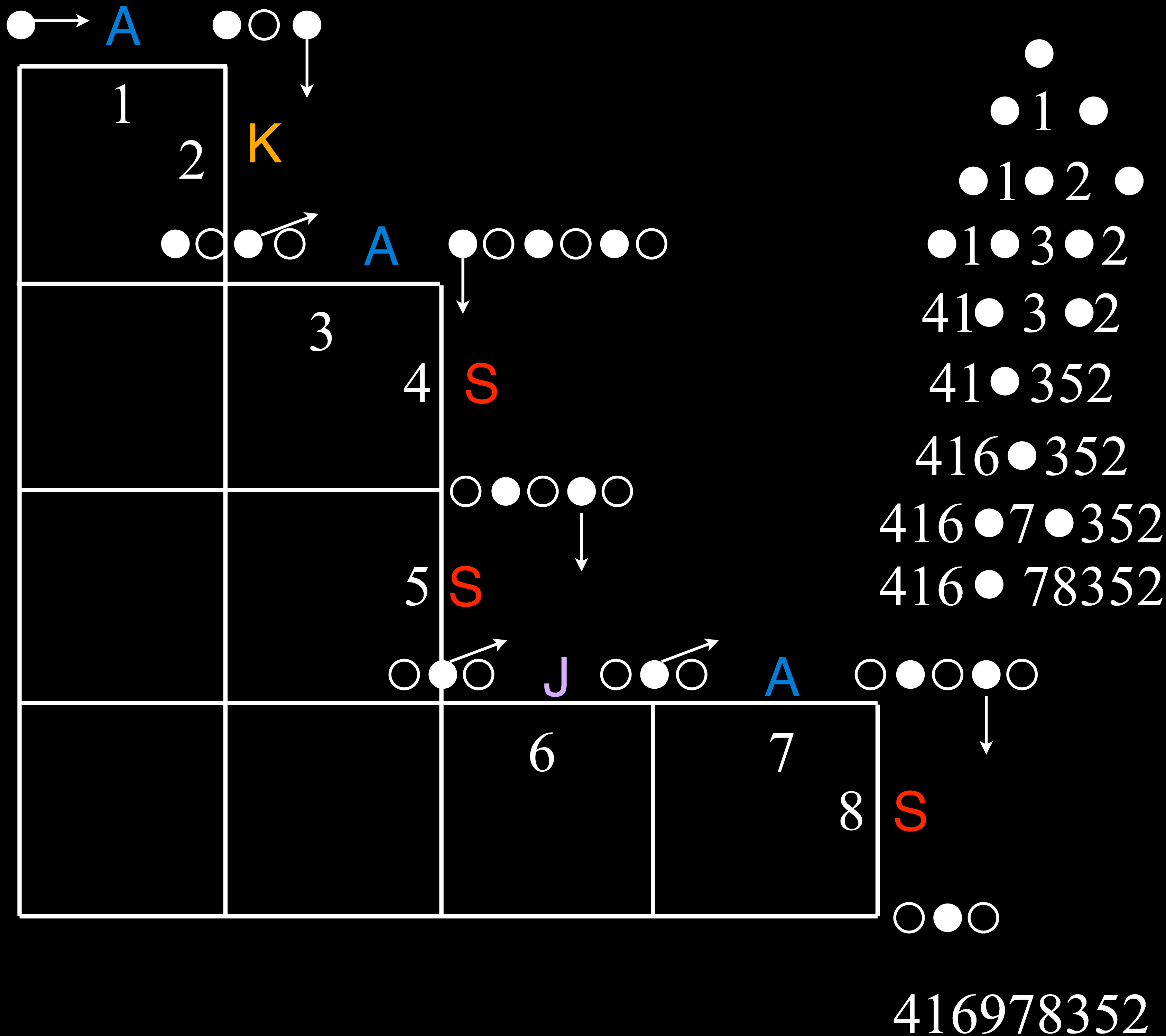
41●352

416●352

416●7●352

416● 78352

416978352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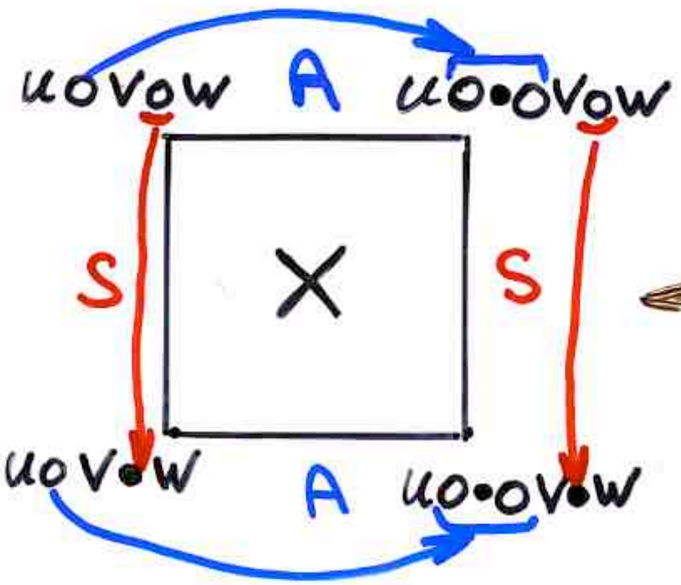
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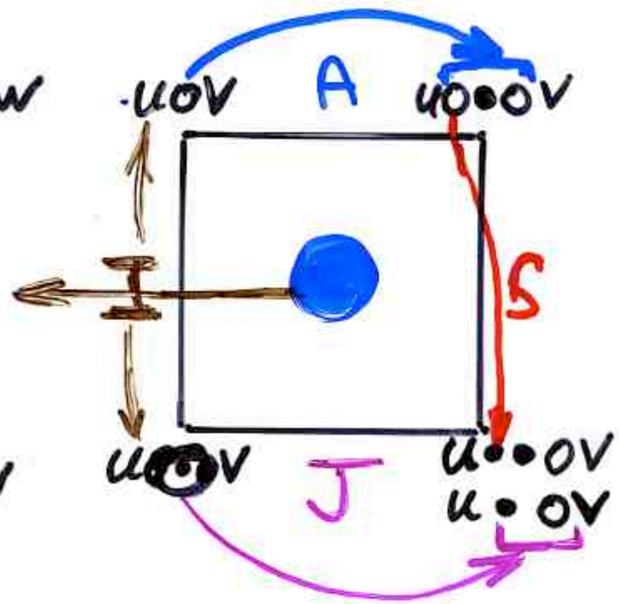
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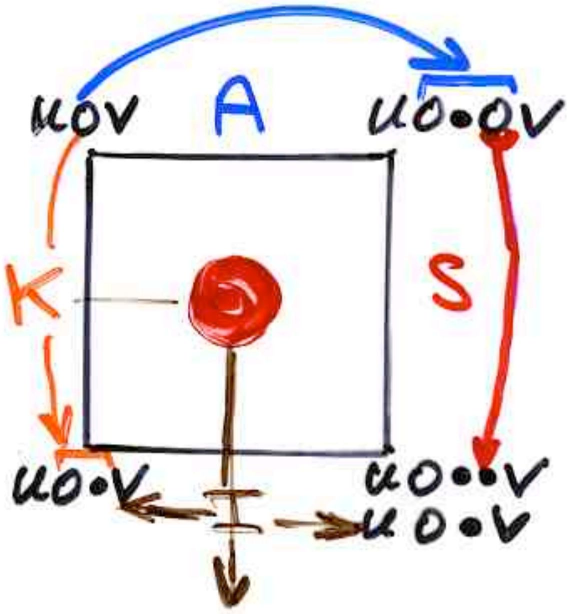
A
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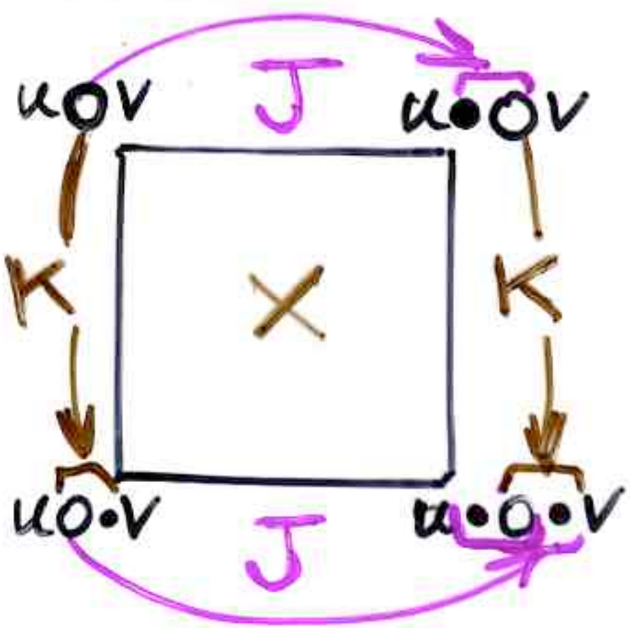
J
K
K
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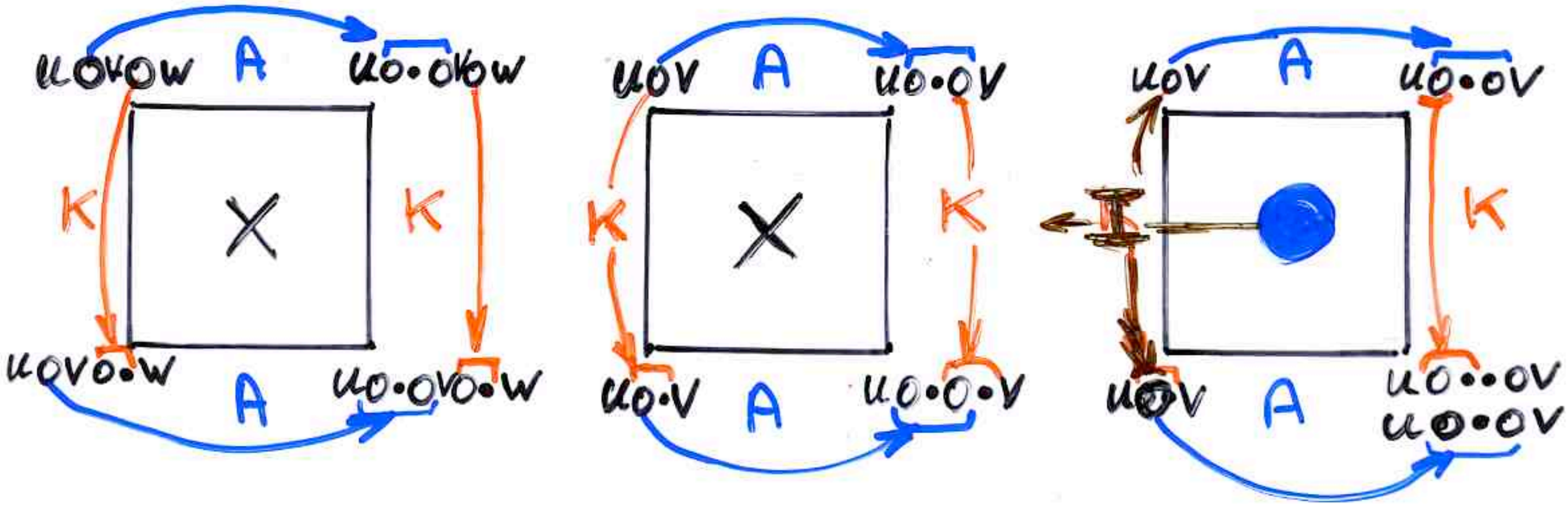
A
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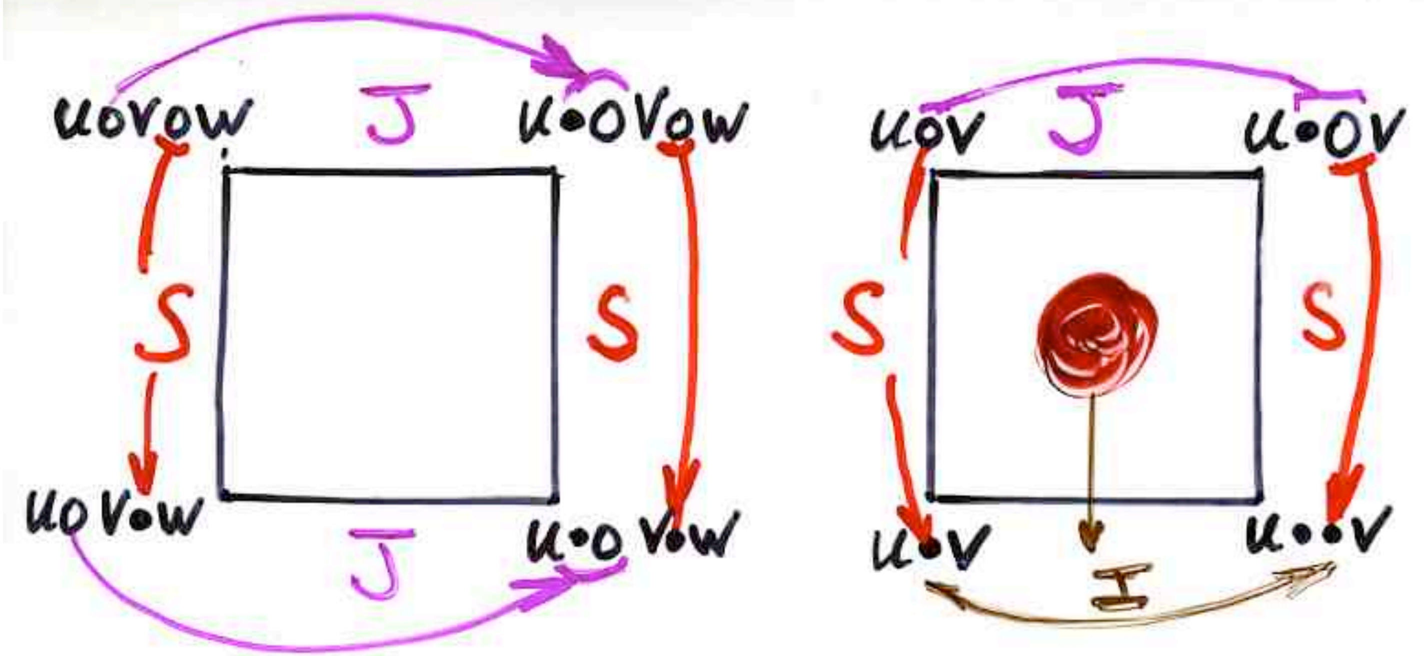
A
K


J
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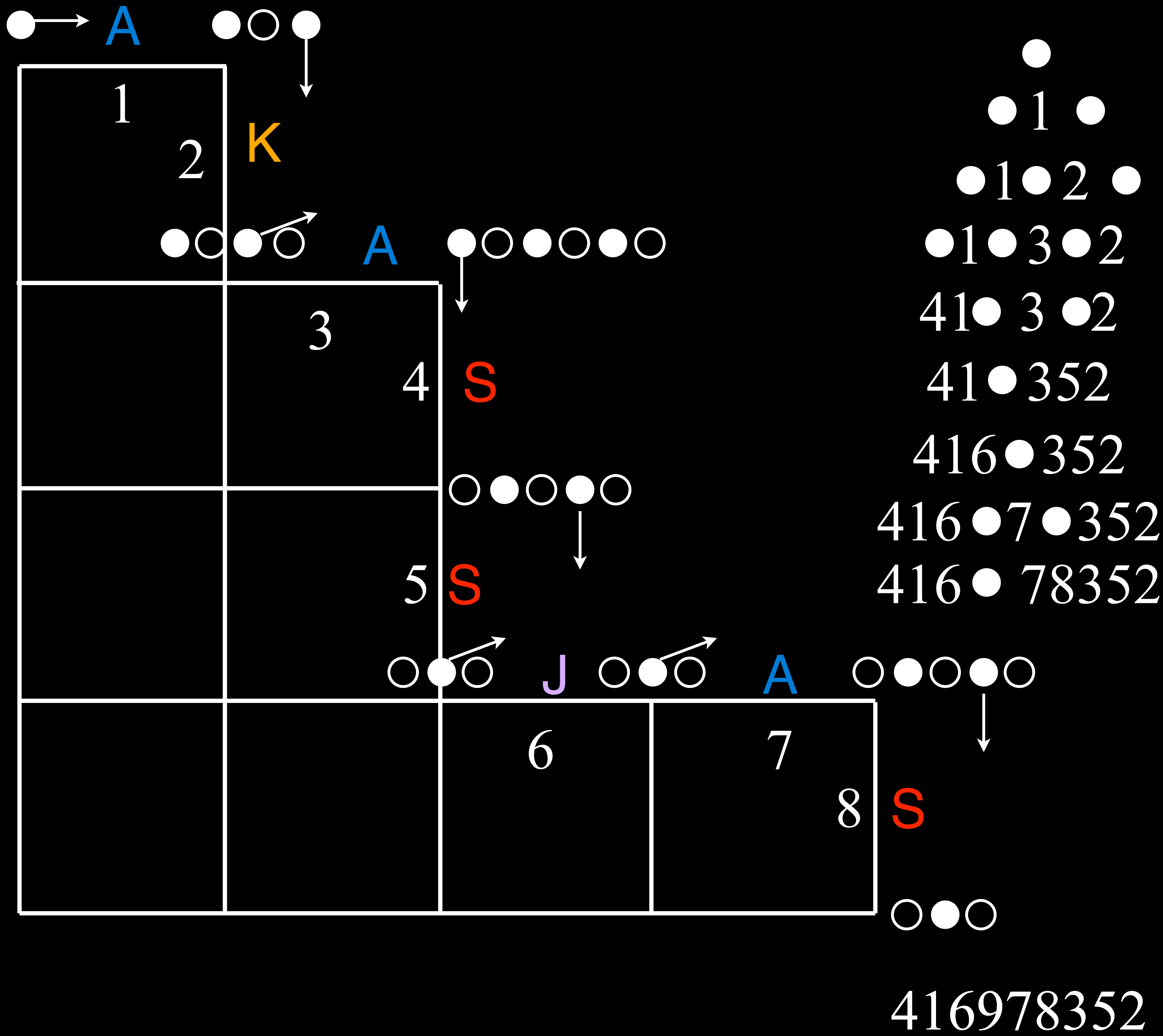
S


A
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J
A
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7
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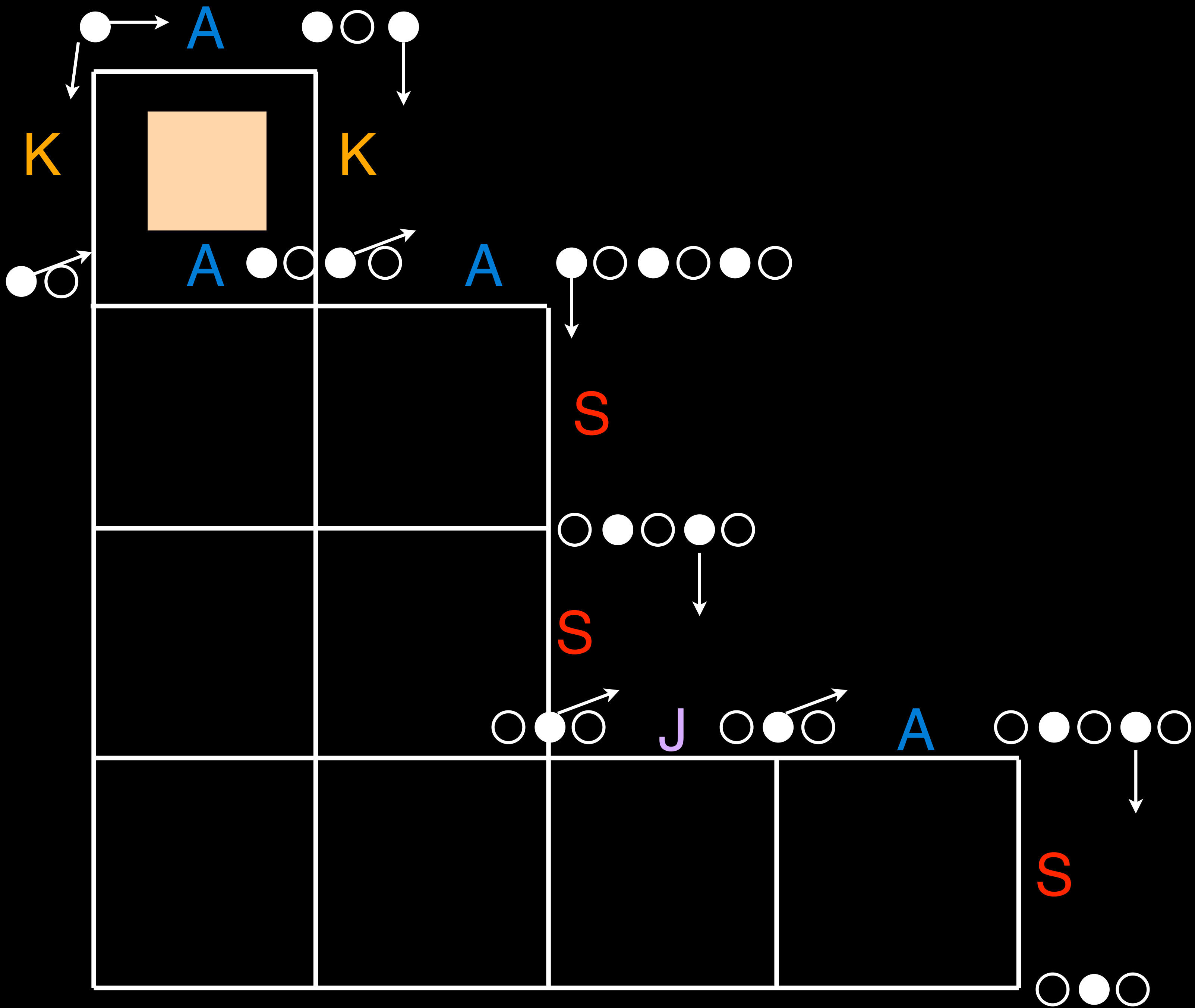
●
●1●
●1●2●
●1●3●2
41●3●2
41●352
416●352
416●7●352
416●78352
416978352


A
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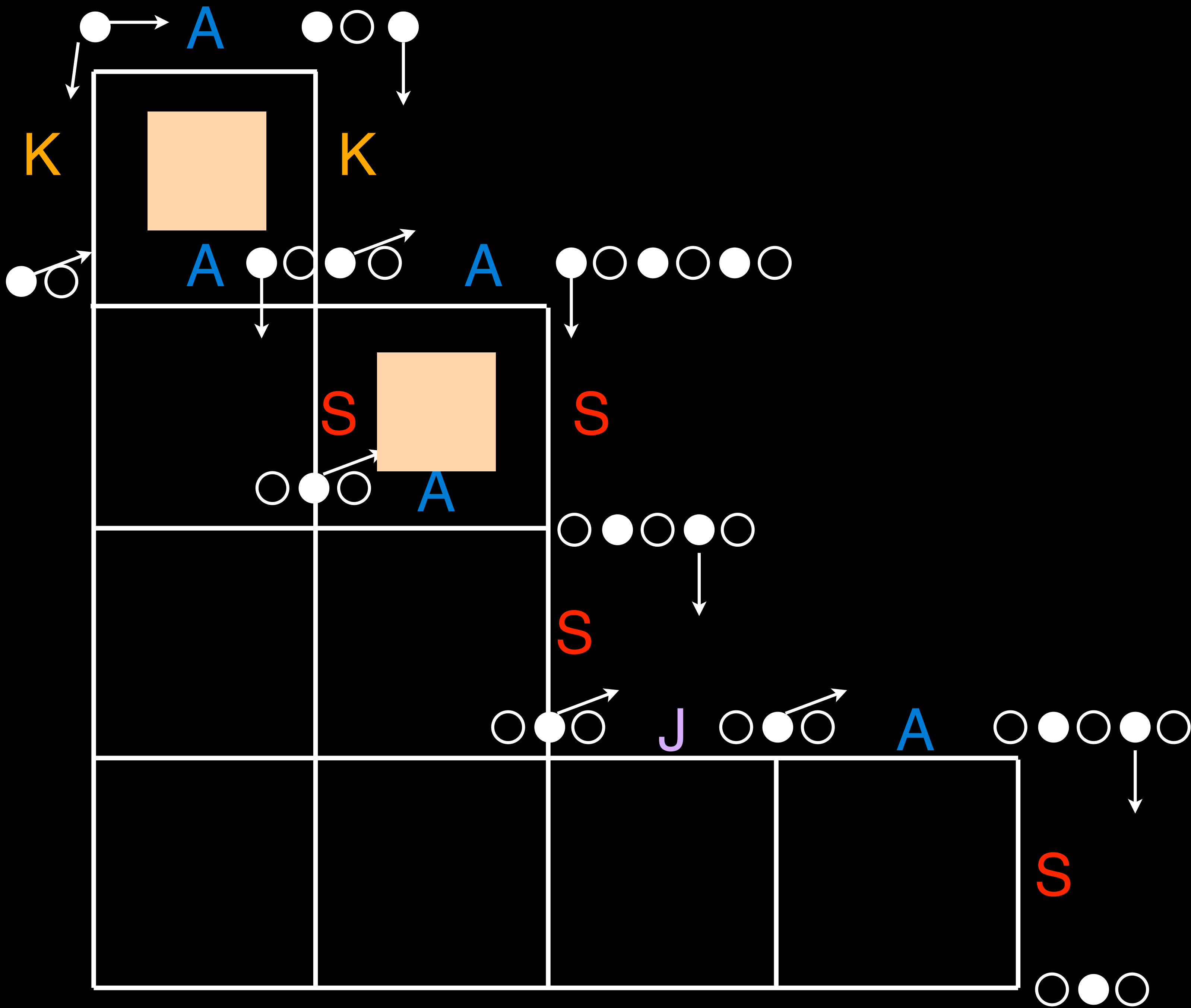
A
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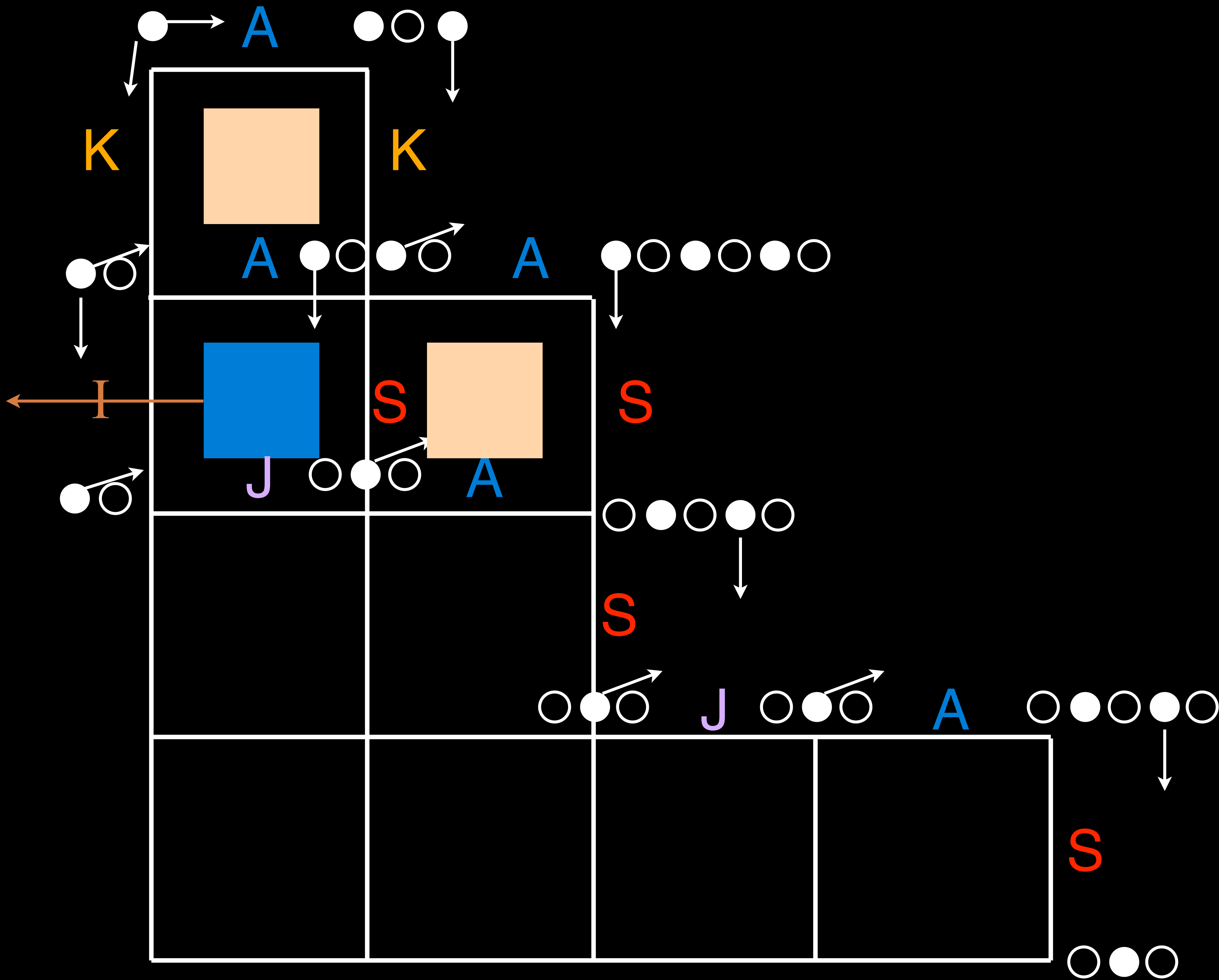
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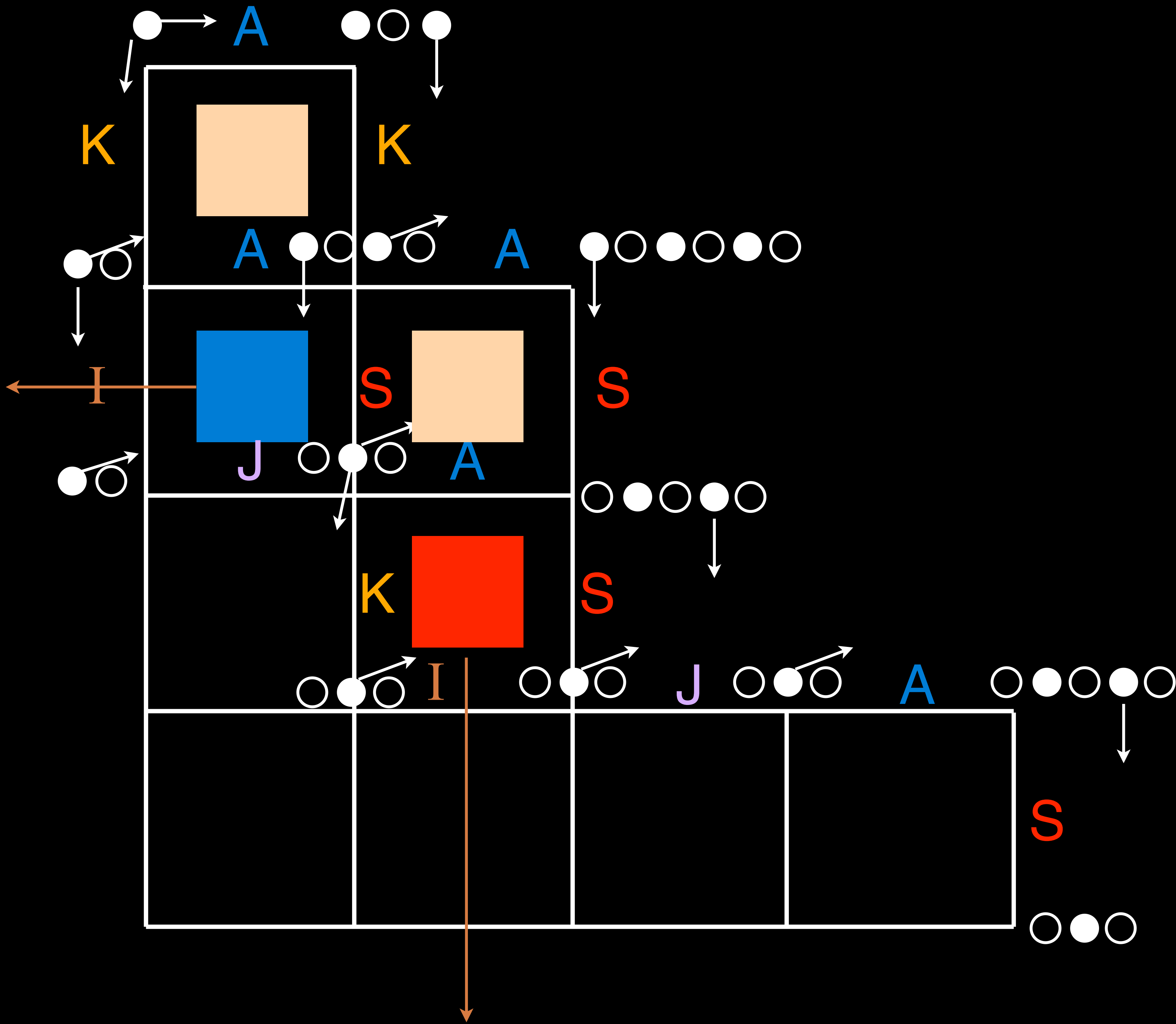
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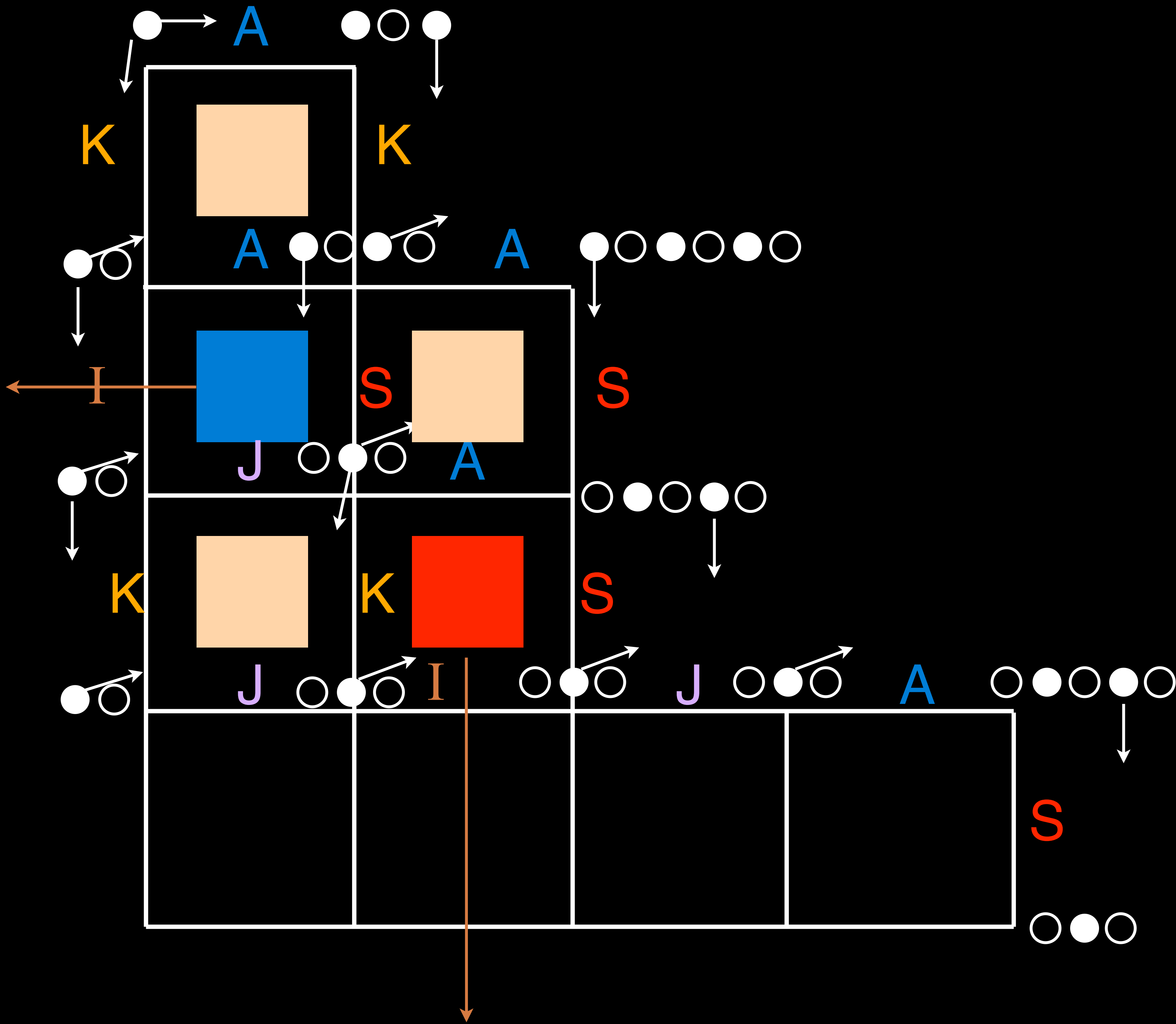
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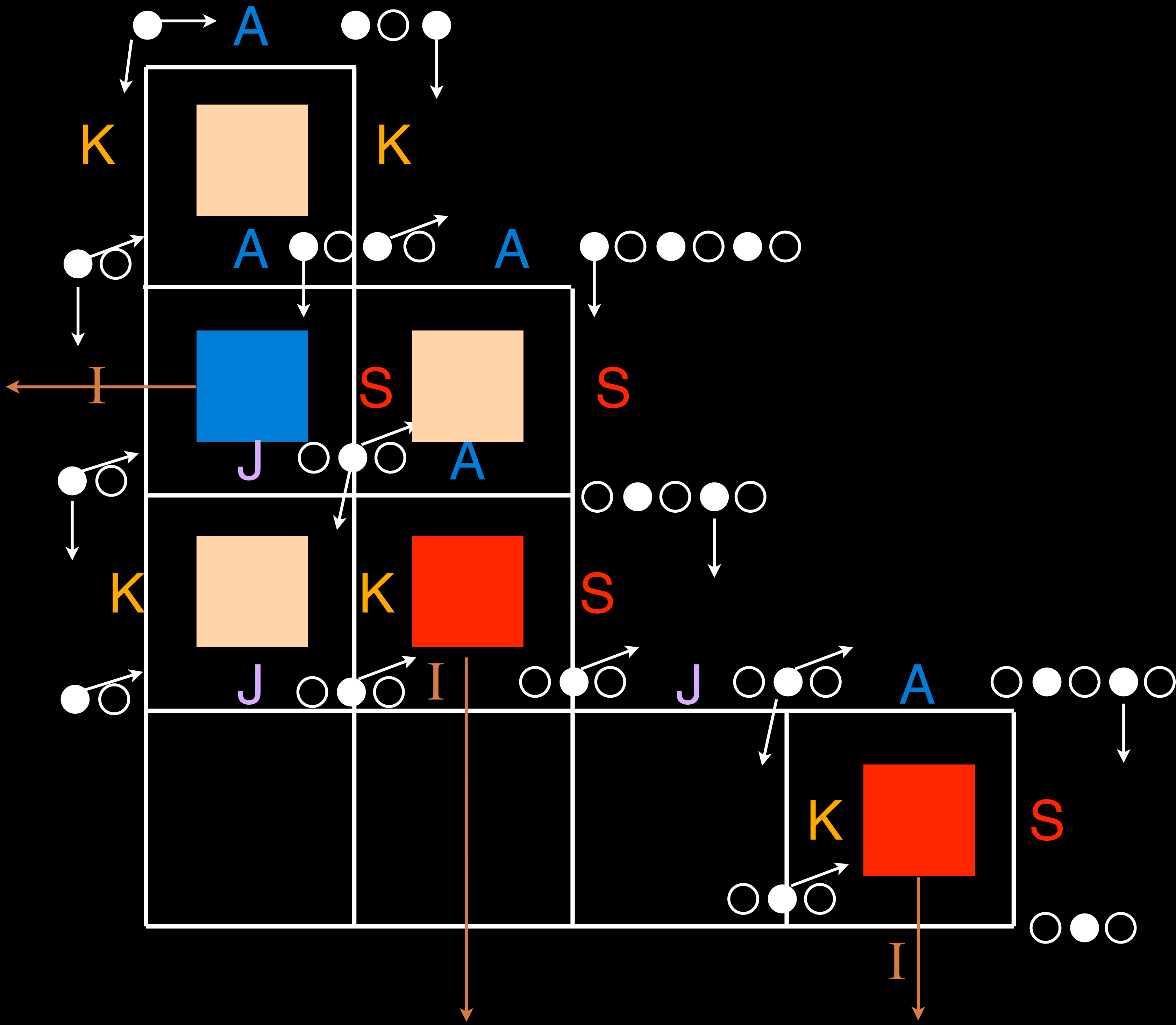

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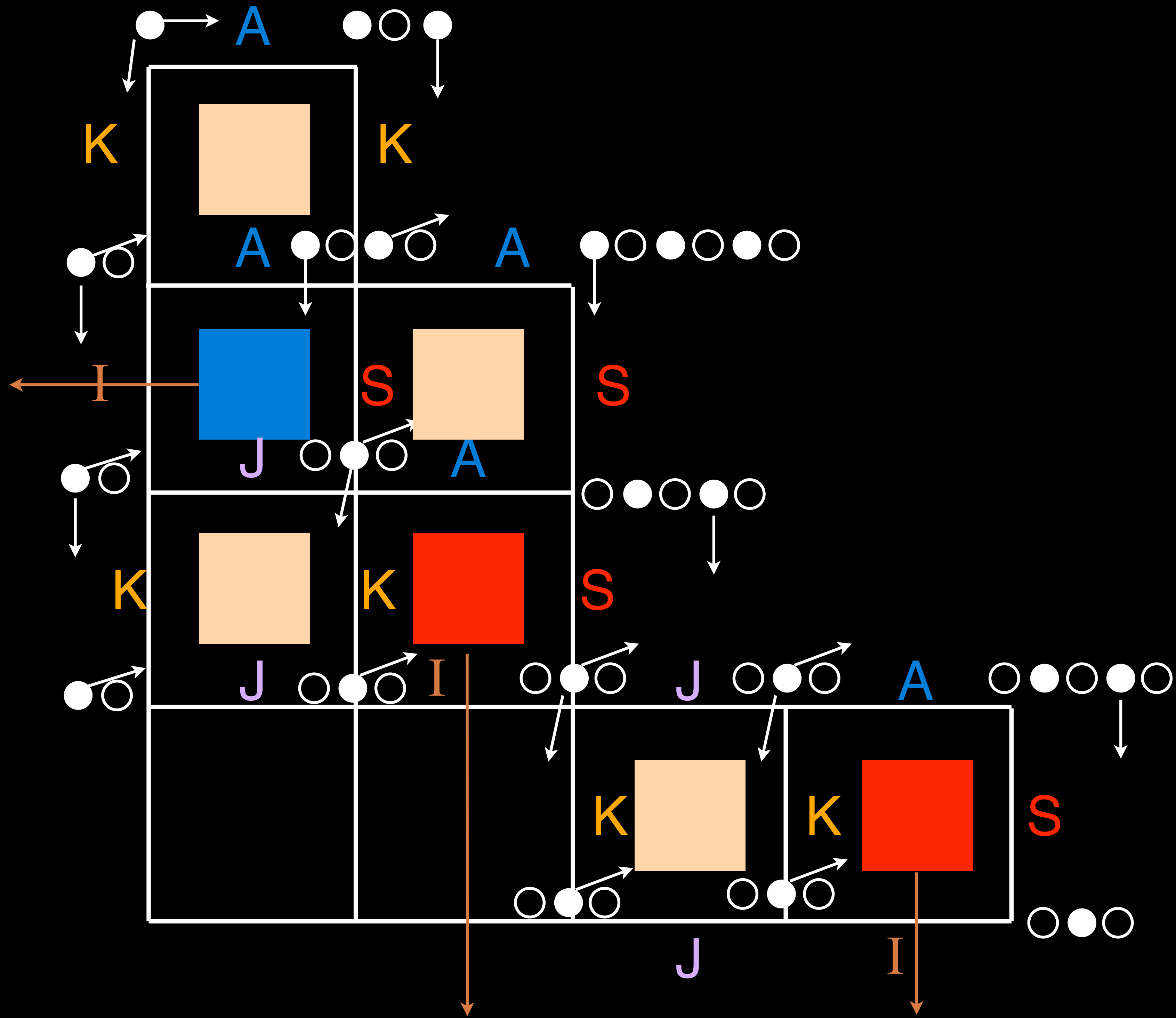
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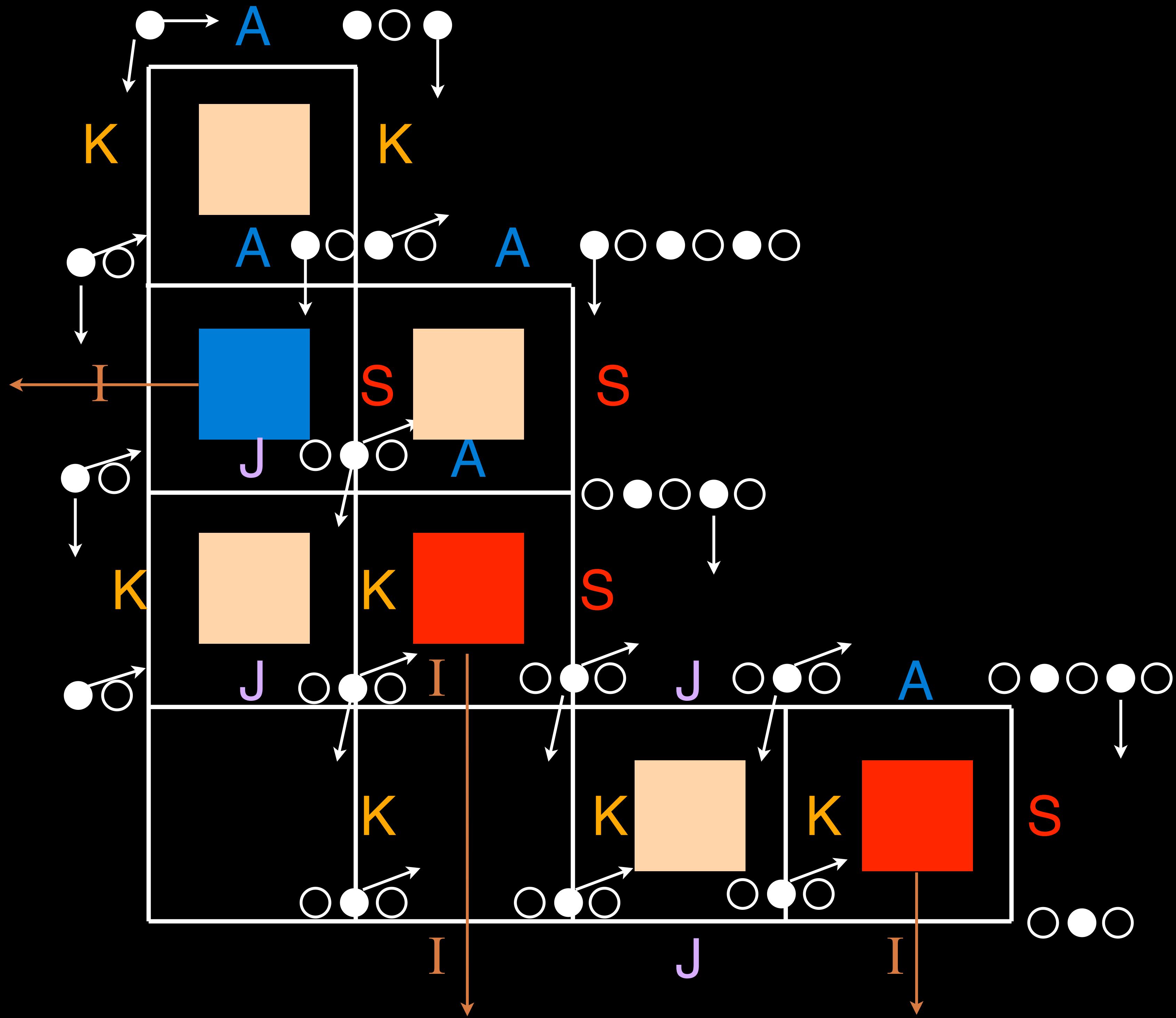

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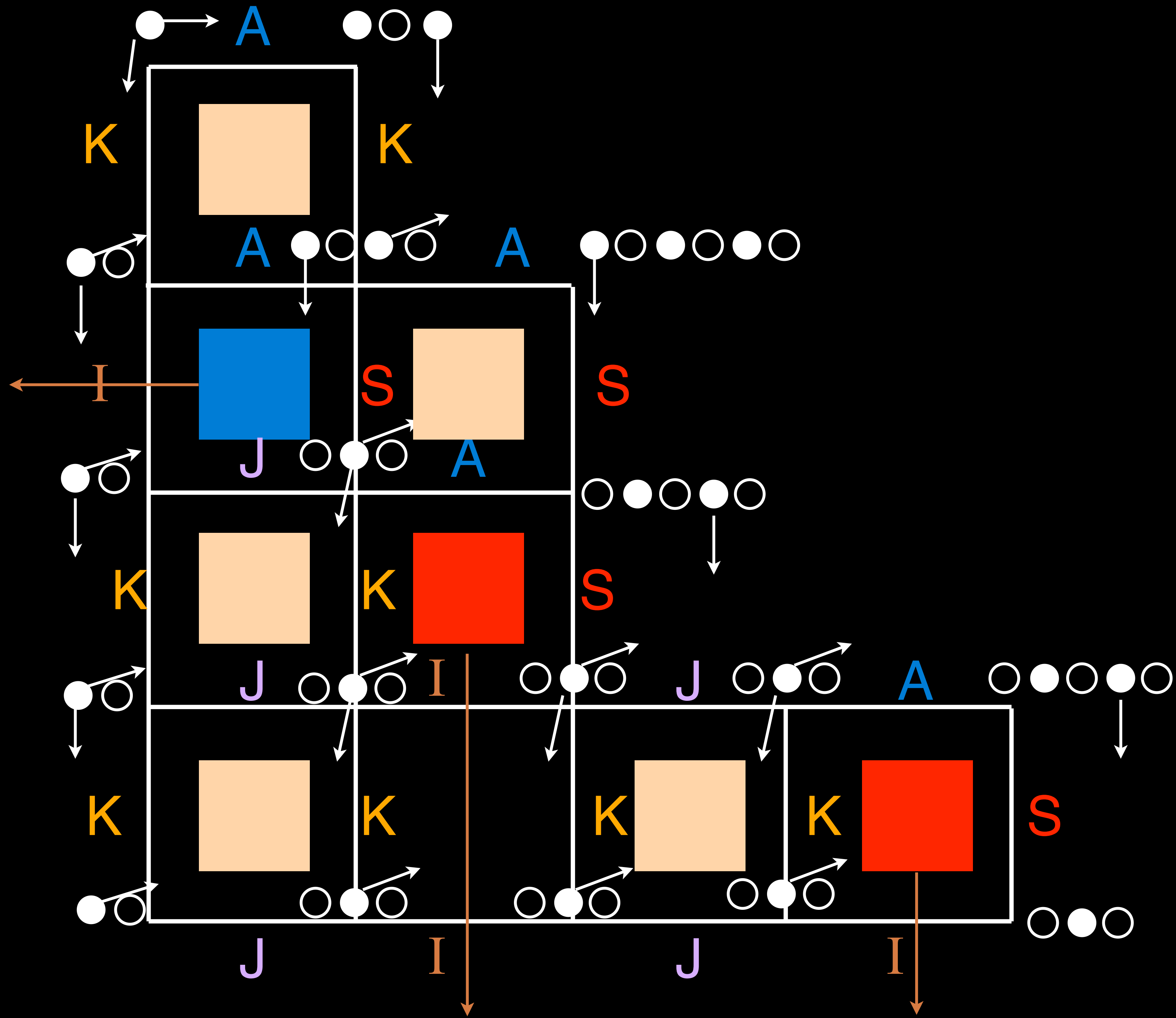
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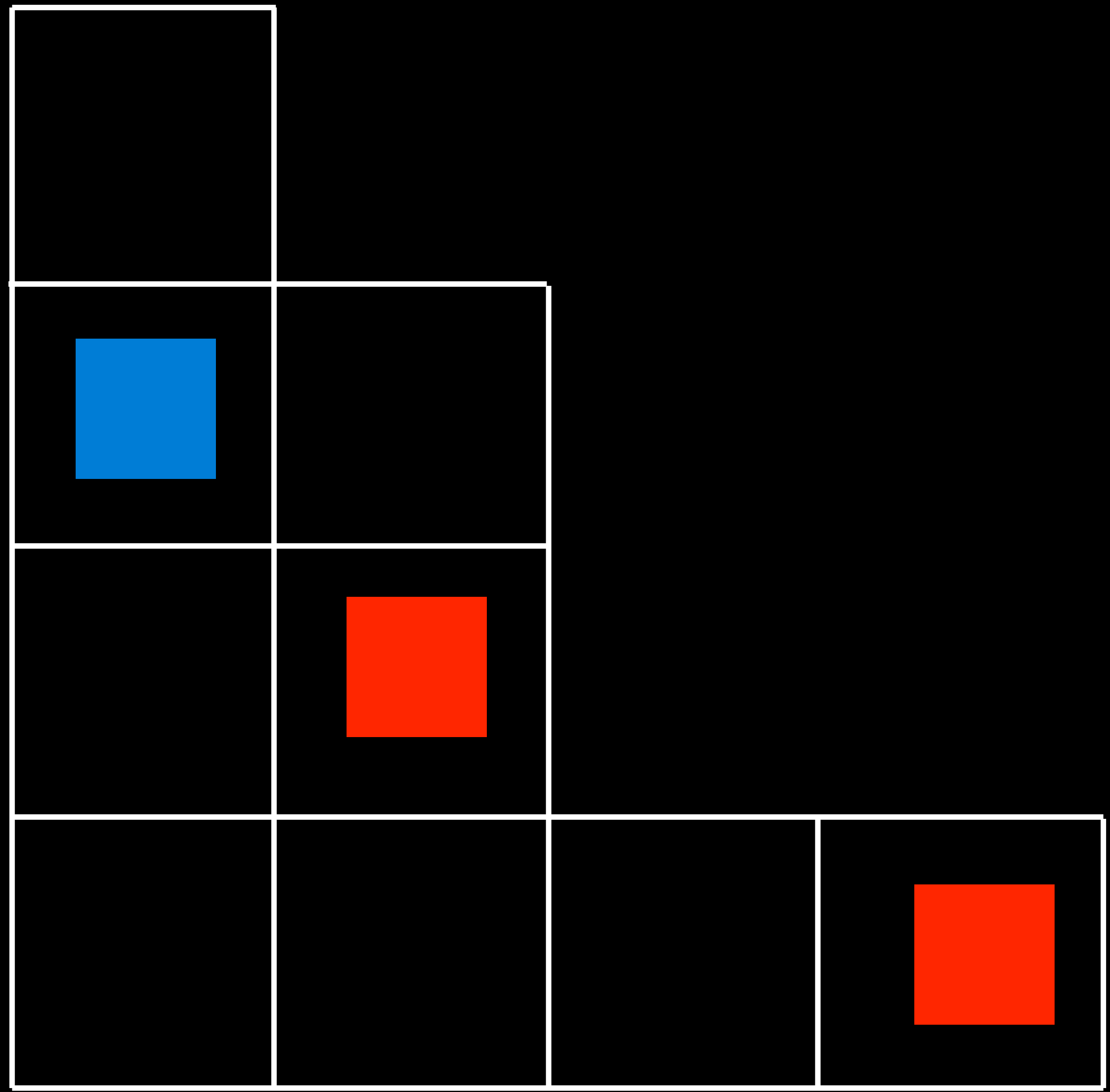

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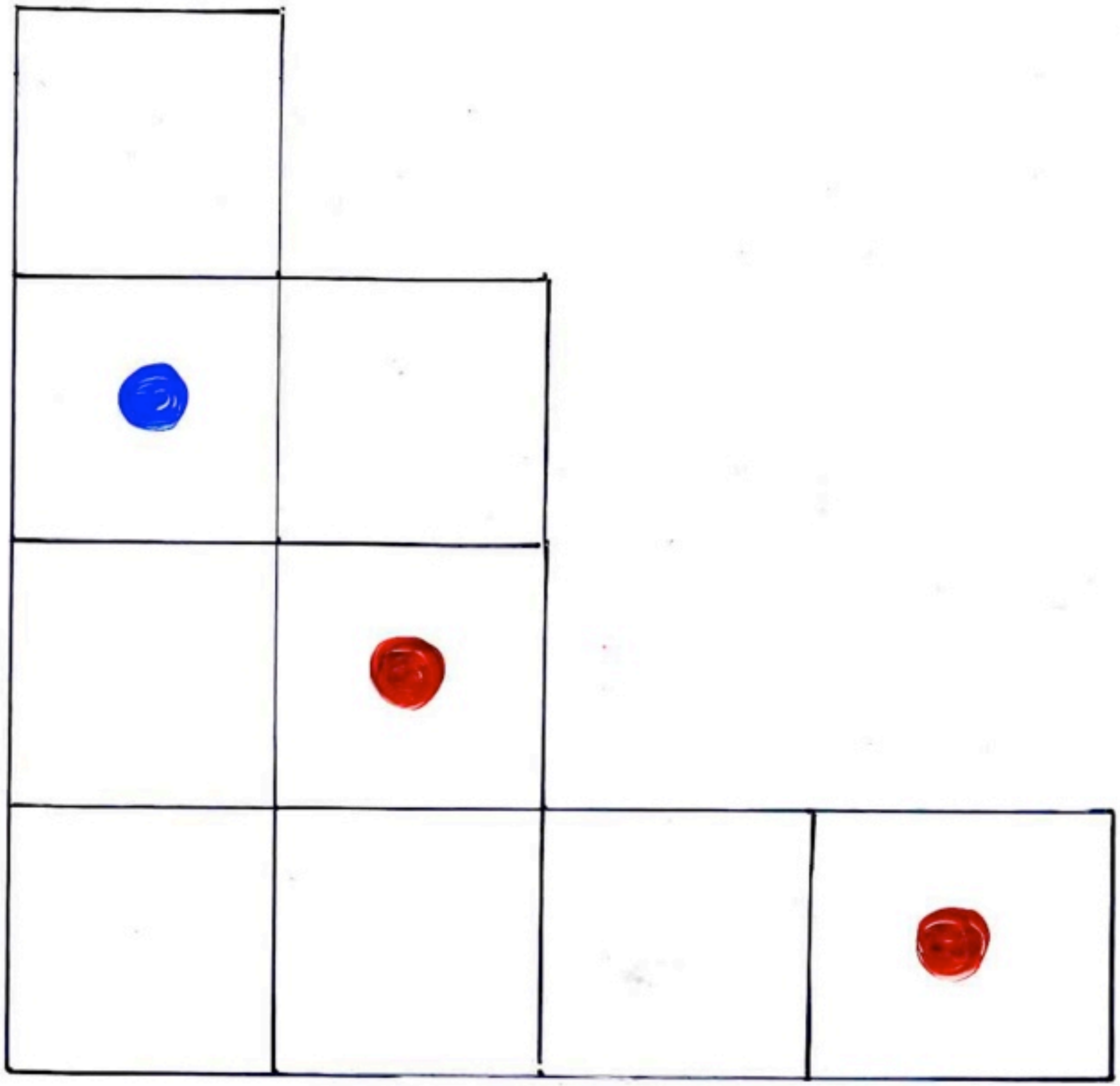
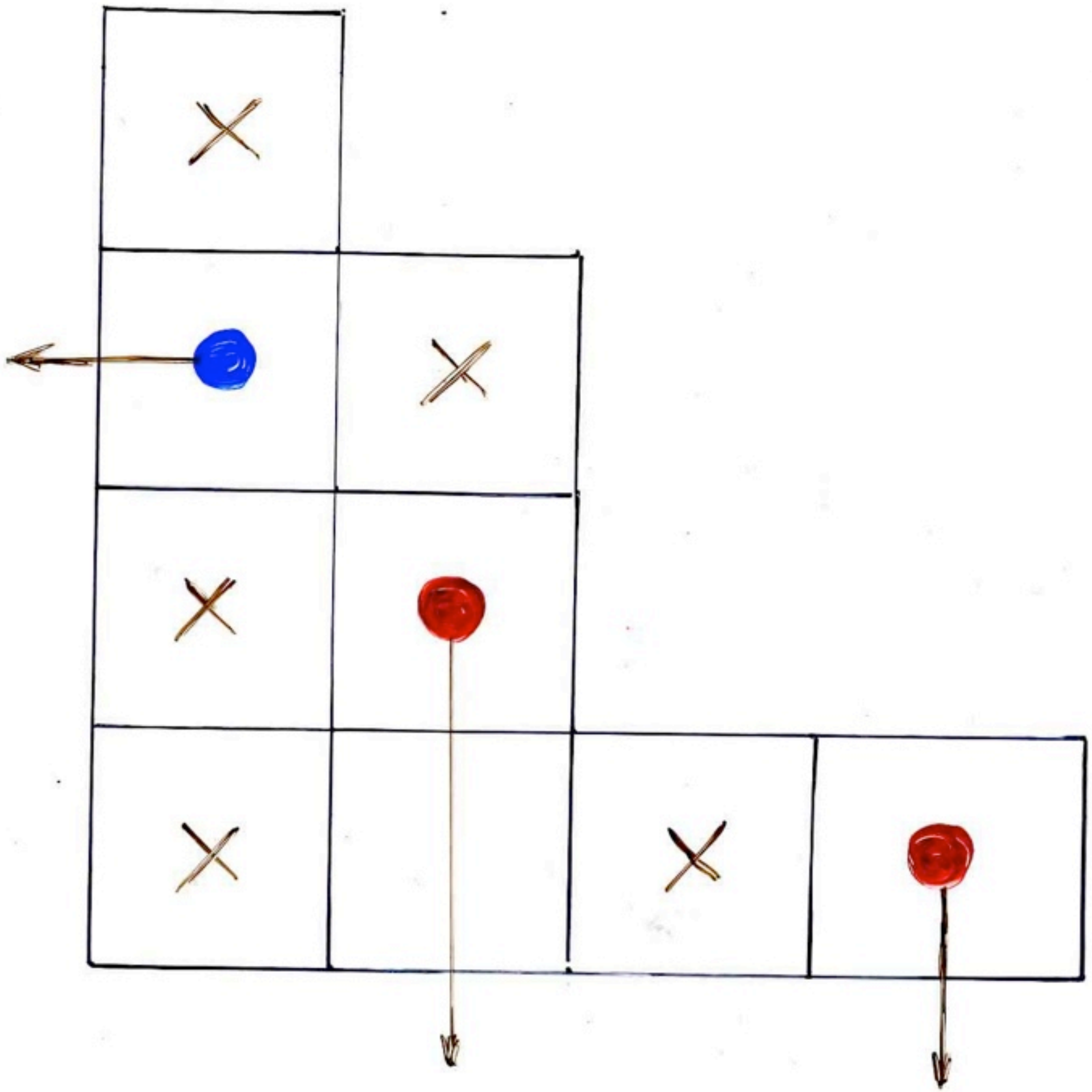
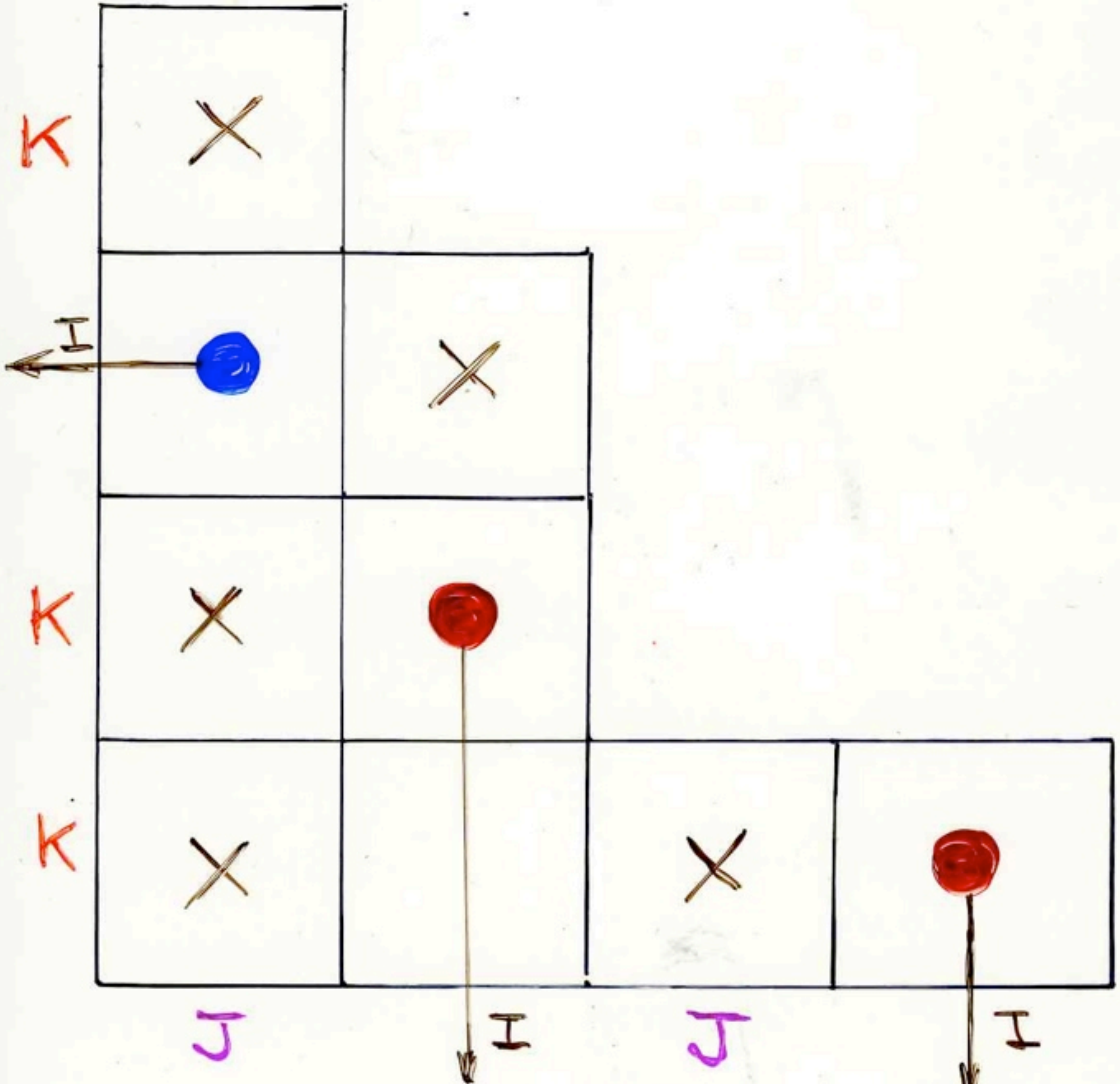
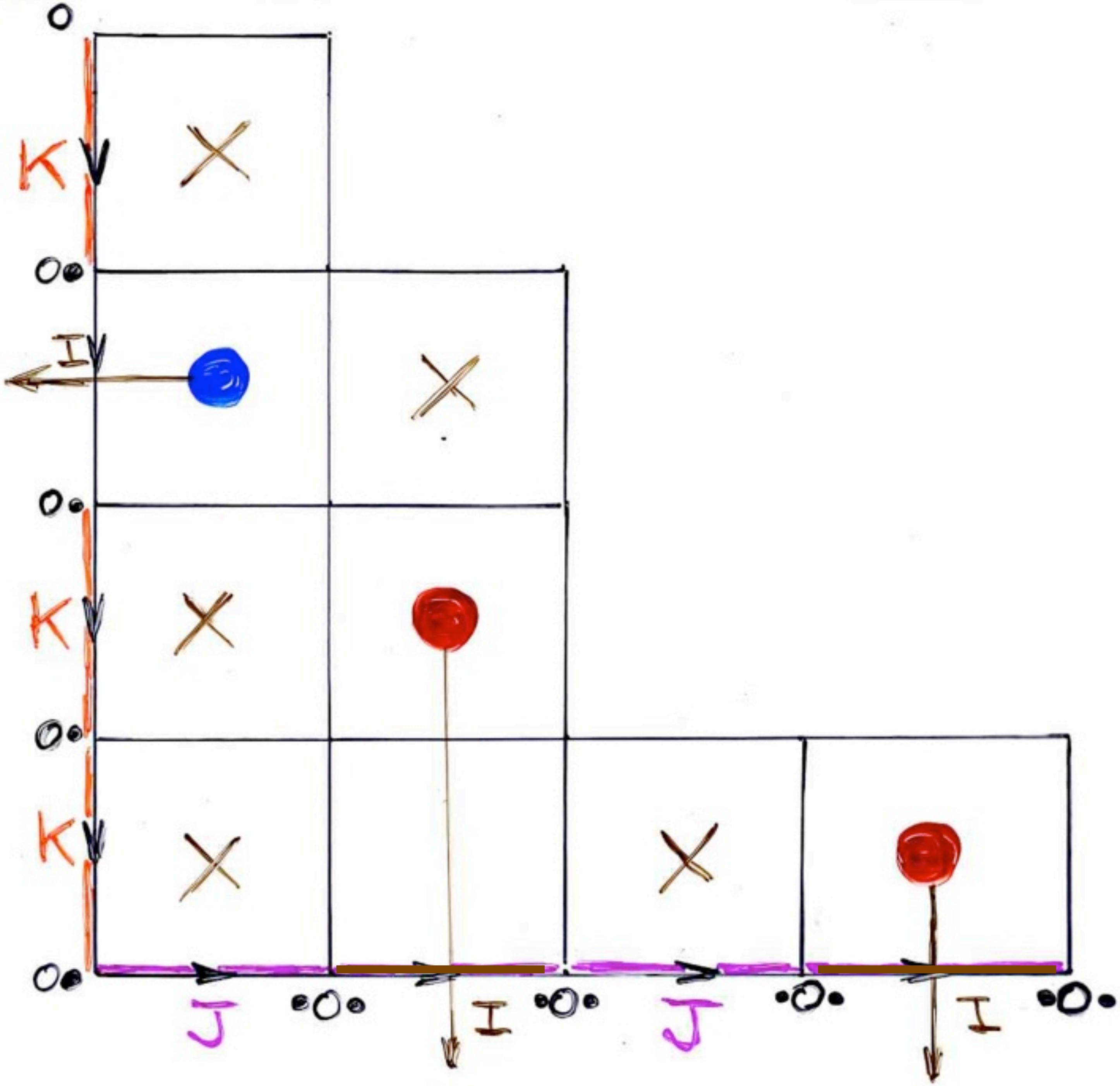
the reverse bijection
permutations --- alternative tableaux






K
I
K
K
J
I
J
I



0
K
O•
I
O•
K
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K
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J
•O•
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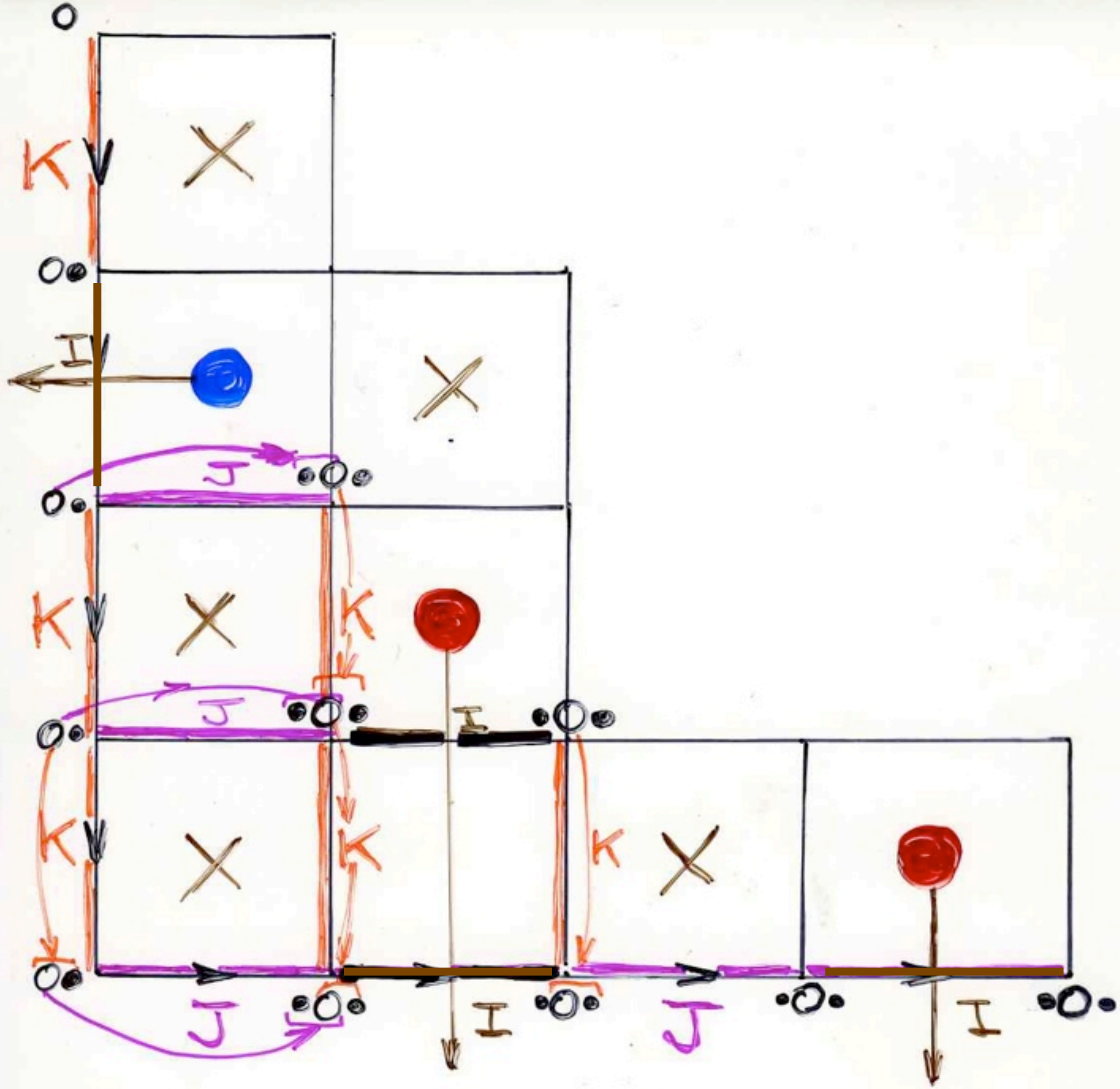
I
•O•



K
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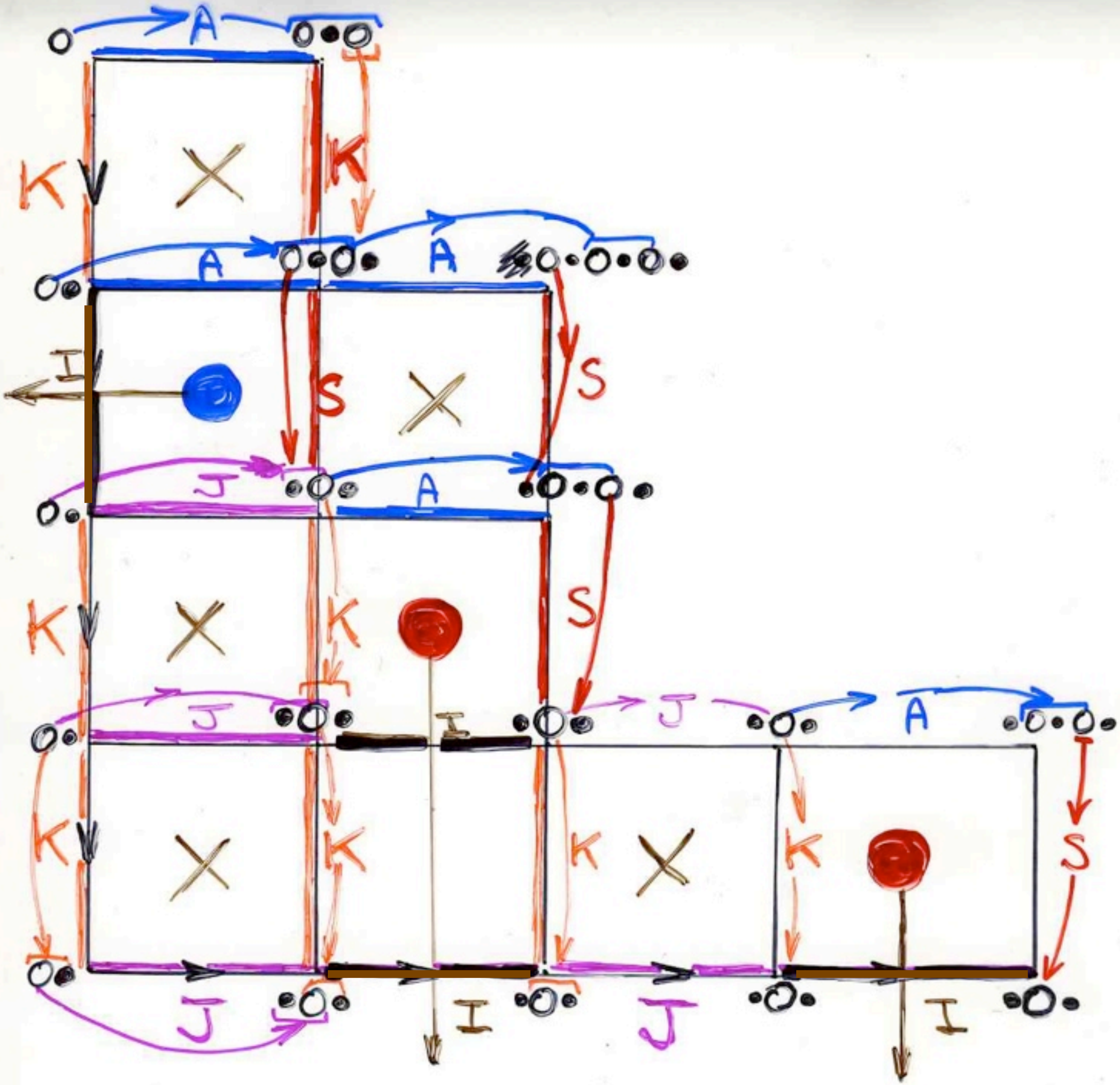
I



A
K
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K
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J
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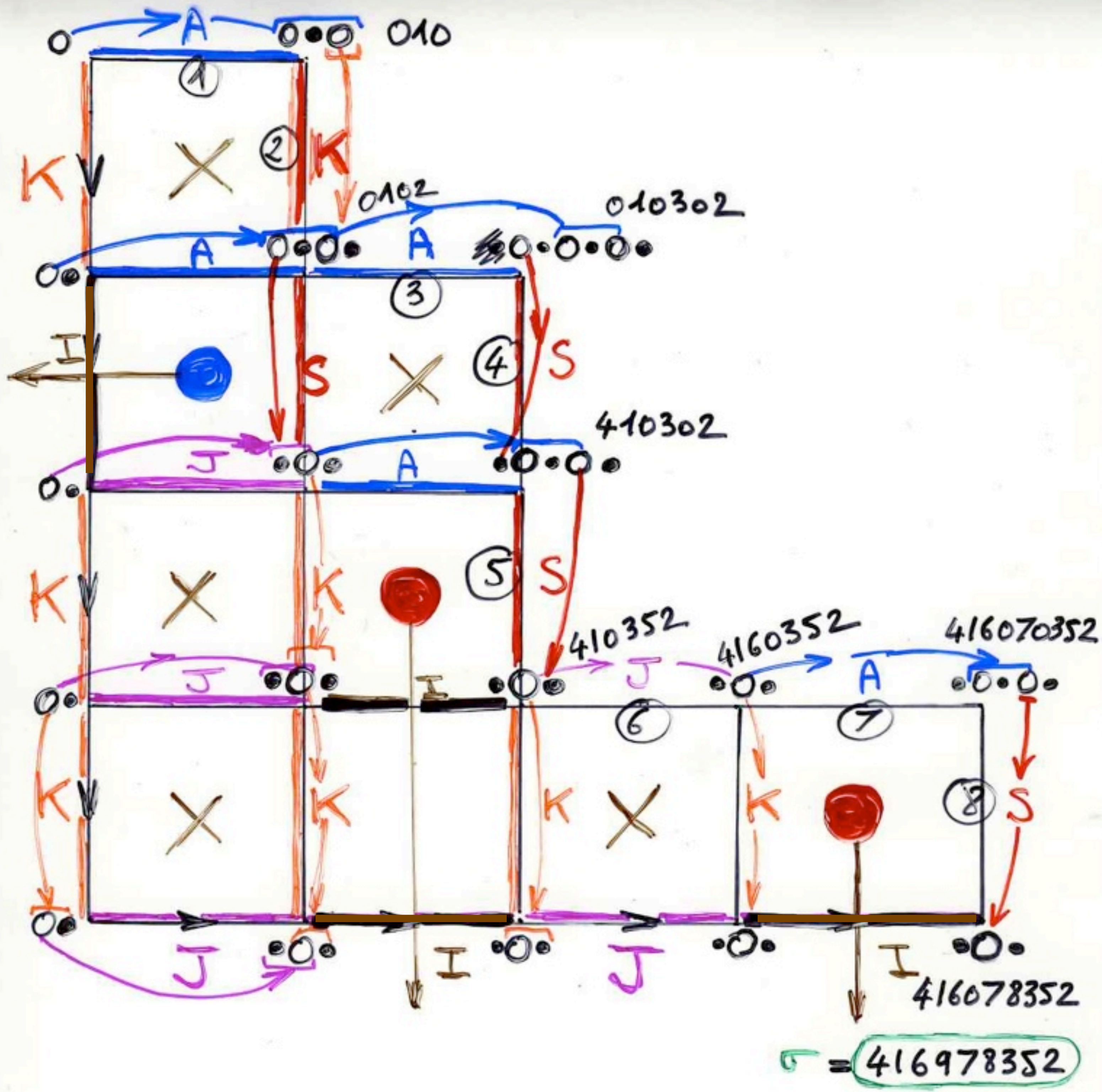
J
I



010
0102
010302
410302
410352
4160352
416070352
416078352
σ = 416978352

3 parameters

Cor. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ (PASEP) is

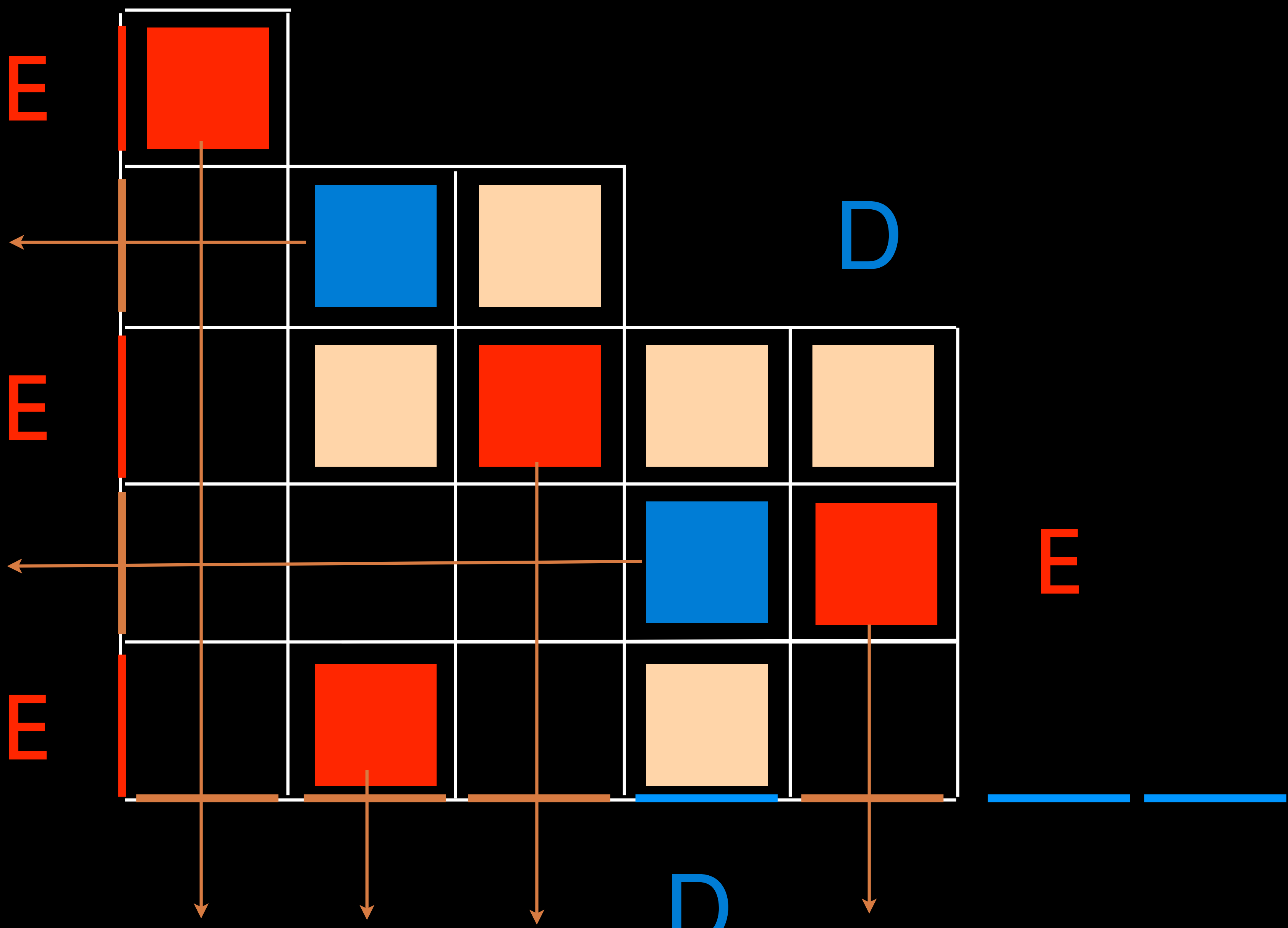
$$\text{prob}_\tau(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{T} q^{L(T)} \alpha^{-f(T)} \beta^{-u(T)}$$

T: alternative tableaux, profile τ

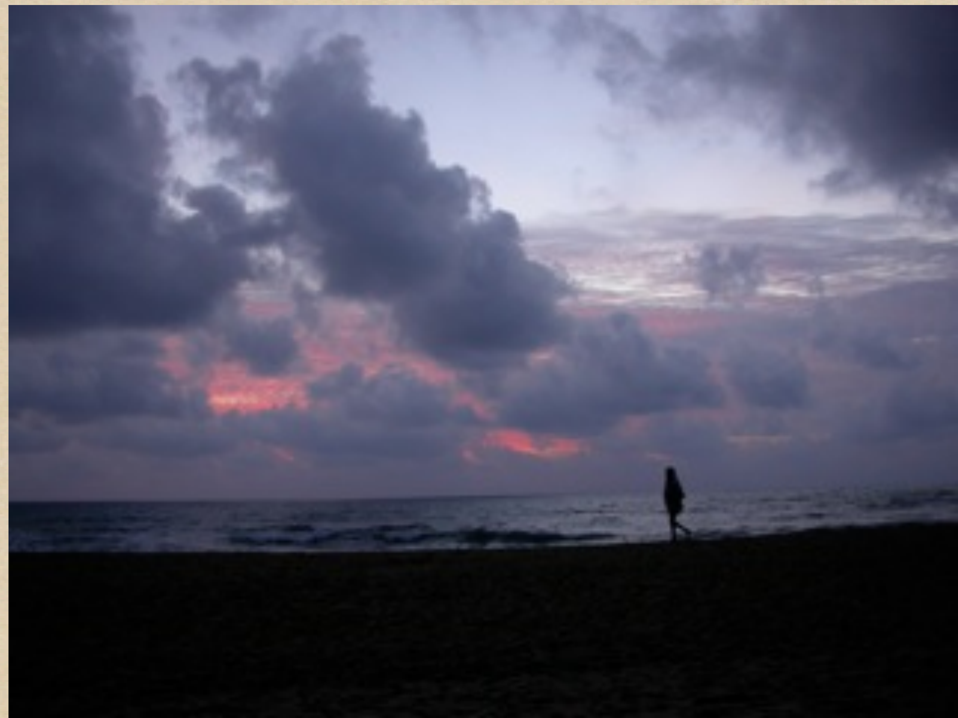
$f(T)$ nb of rows without cell
$u(T)$ nb of columns without cell
$L(T)$ nb of cells

permutation tableau

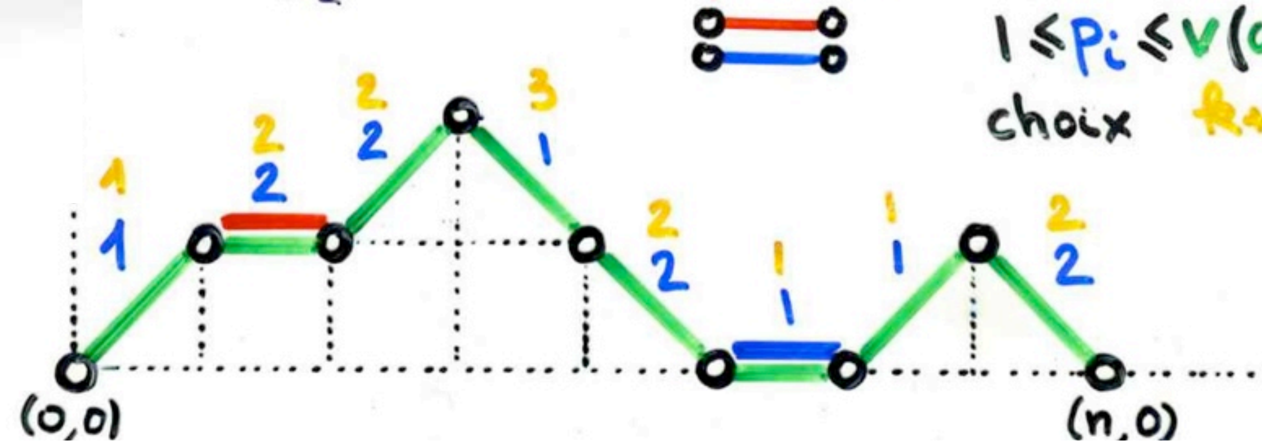
S. Corteel, L. Williams (2007) (2008) (2009)


E
D
E
E
E
D

q-analog of Laguerre histories



$h = (\omega_c \, ; (p_1, \ldots, p_n))$



$1 \leq p_i \leq v(\omega_i)$

choix $k+1$

x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9	•		

␣

␣ 1 ␣

␣ 1 ␣ 2

␣ 1 ␣ 3 ␣ 2

4 1 ␣ 3 ␣ 2

4 1 ␣ 3 5 2

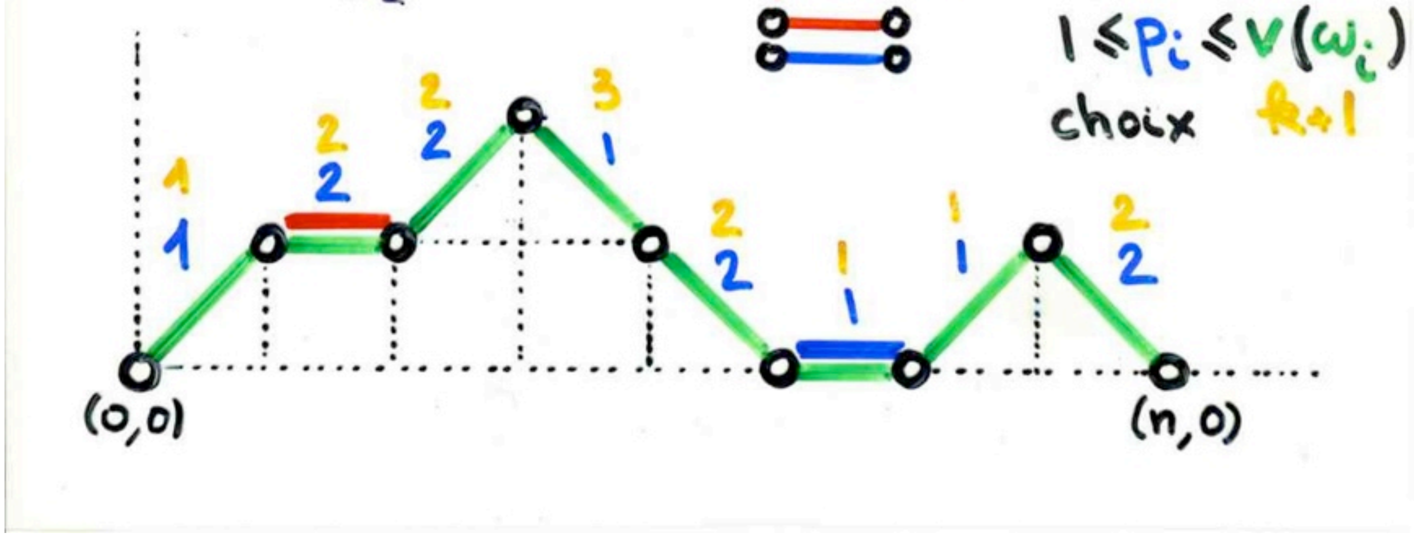
4 1 6 ␣ 3 5 2

4 1 6 ␣ 7 ␣ 3 5 2

4 1 6 ␣ 7 8 3 5 2

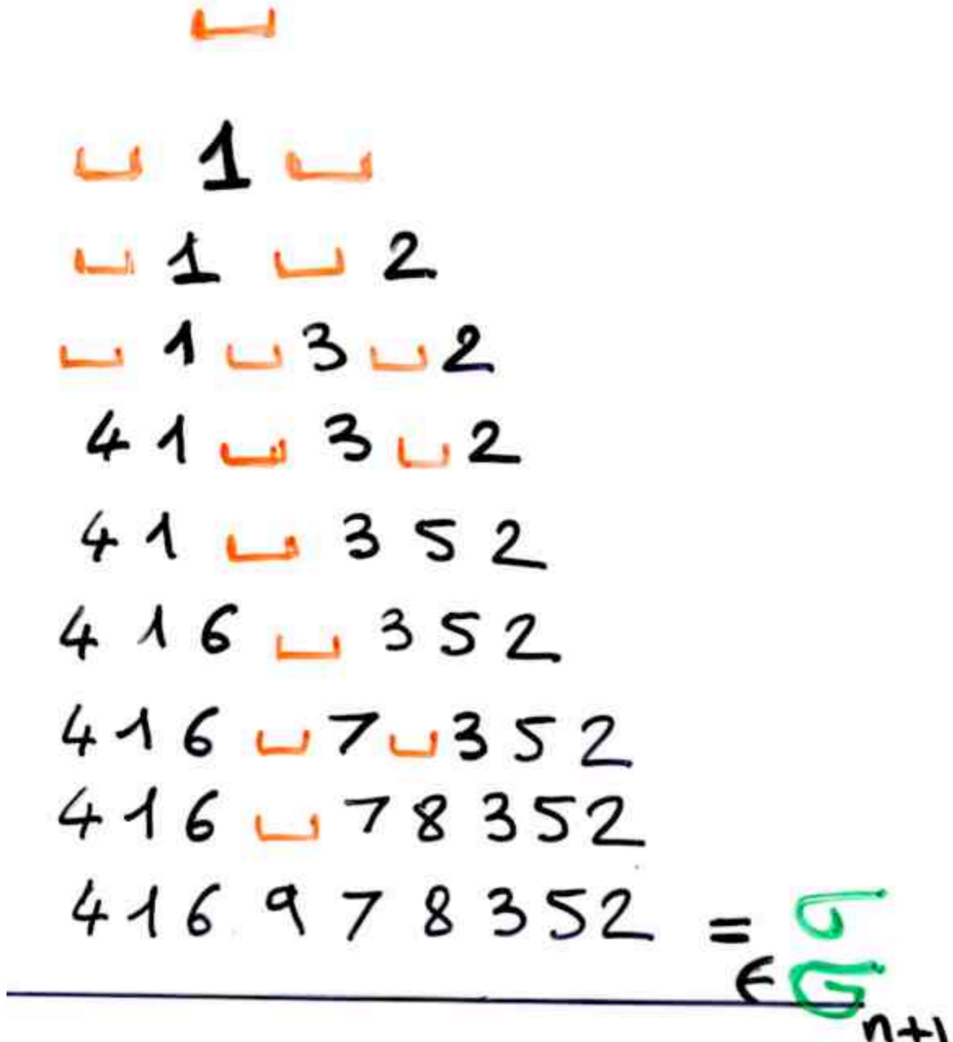
4 1 6 9 7 8 3 5 2 $= \sigma \in \mathfrak{S}_{n+1}$

"q-analogue" of Laguerre histories



choices function

1	2	3	4	5	6	7	8
1	2	2	1	2	1	1	2
0	1	1	0	1	0	0	1



q-Laguerre : q^4

q-Laguerre

$L_n^{\beta}(x;q)$ (I)

$\beta = \alpha + 1$

$$\begin{cases} b_{k,q}^{(\beta)} = [k]_q + [k+1;\beta]_q \\ \lambda_{k,q}^{(\beta)} = [k]_q \cdot [k;\beta]_q \end{cases}$$

$$[k;\beta]_q = \beta + q + q^2 + \cdots + q^{k-1}$$

$$\mu_n = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\binom{2n}{n-k} - \binom{2n}{n-k-2} \right) \left(\sum_{i=0}^{k} q^{i(k+1-i)} \right)$$

Corteel, Josuat-Vergès
Prellberg, Rubey (2008) y

$$L^{II\,\beta}_{n}(x;q)$$

$$\begin{cases} b^{(\beta)}_{k,q} = q^{k}\left([k]_q + [k+1;\beta]_q\right) \\ \lambda^{(\beta)}_{k,q} = q^{2k-1}[k]_q \cdot [k;\beta]_q \end{cases}$$

$$\mu^{(\beta)}_{n,q} = [n;\beta]_q\,!$$

$$= [1;\beta]_q\,[2;\beta]_q \cdots [n;\beta]_q$$

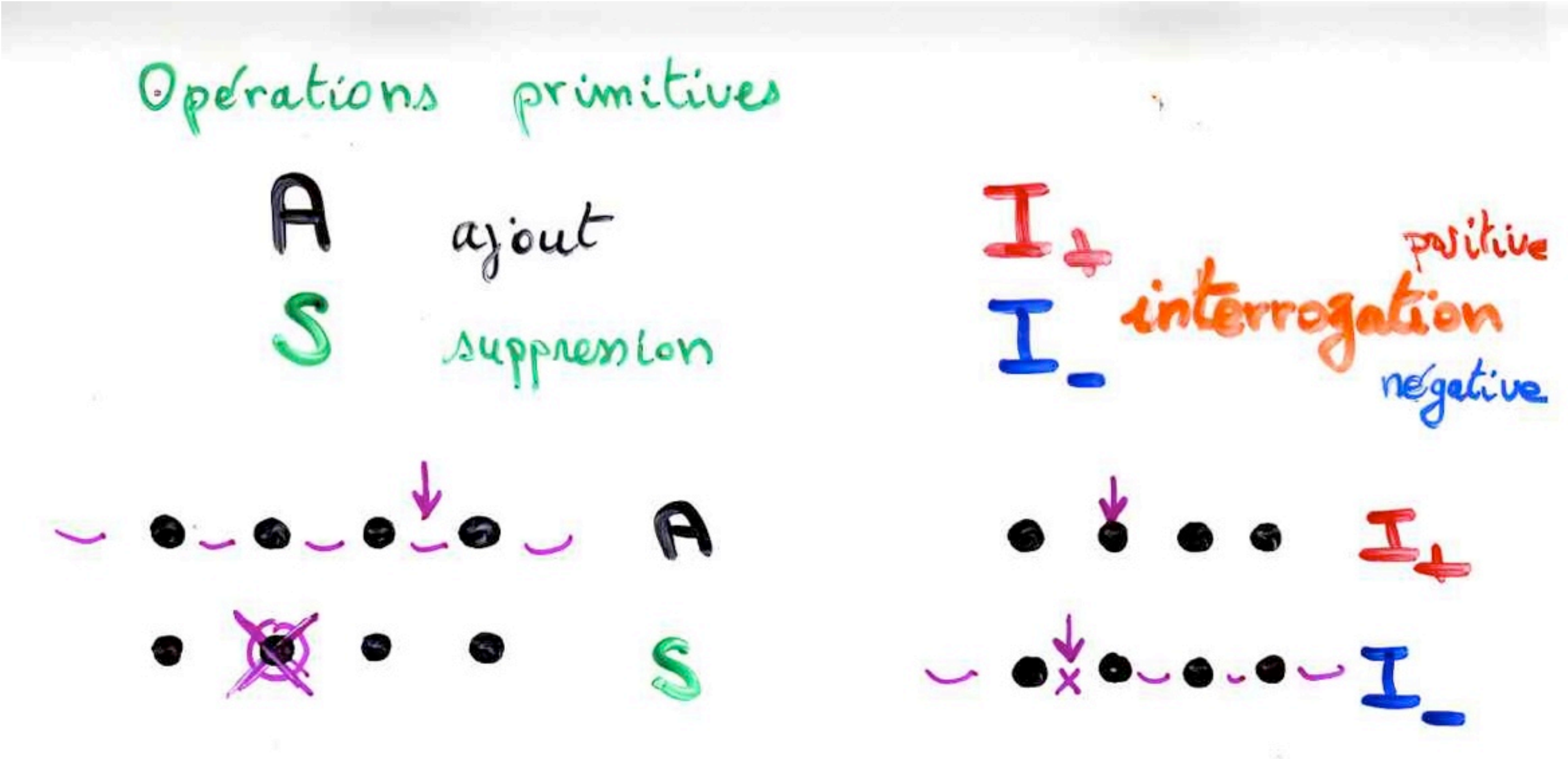
quadratic algebra

operators

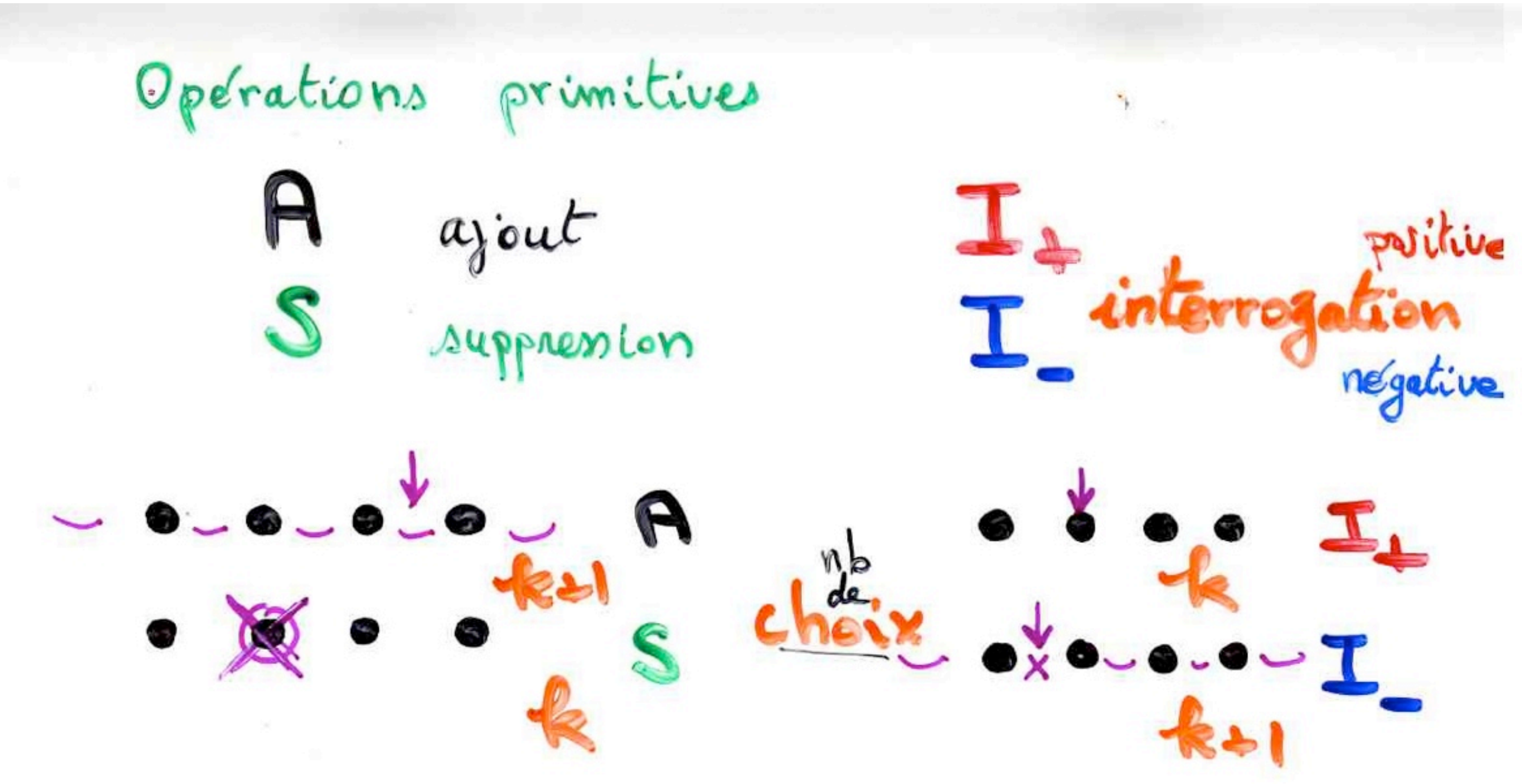
data structures

and

orthogonal polynomials

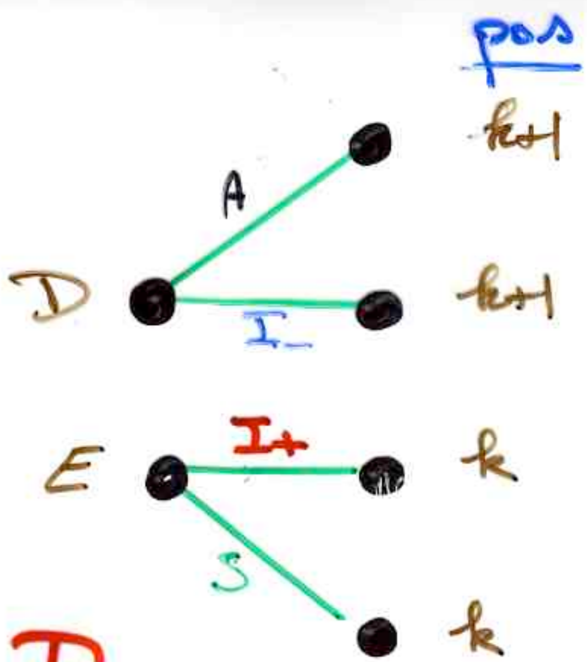


Primitive operations
for “dictionnaries” data structure:
add or delete any elements, asking questions (with
positive or negative answer)



number of choices for each
primitive operations

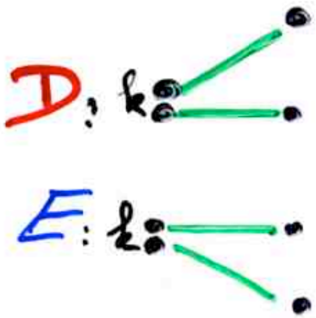
$$\begin{cases} D = A + I_- \\ E = S + I_+ \end{cases}$$



this corresponds to the n!
"restricted Laguerre histories"

$$DE = ED + E + D$$

aussi :



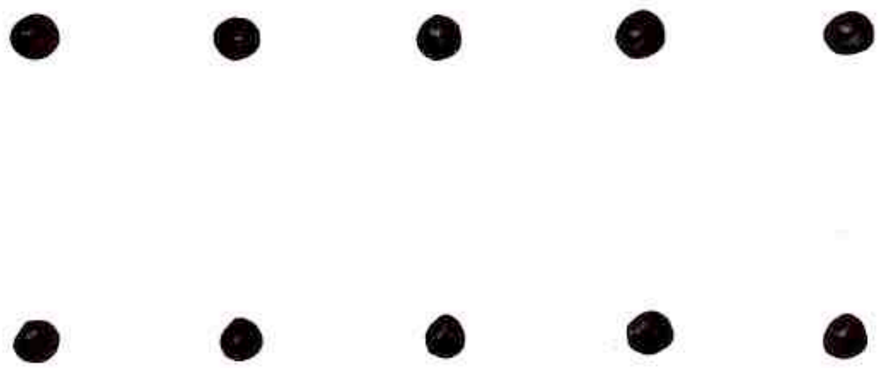
(k+1) possibilités partout

(histoires de Laguerre "larges")

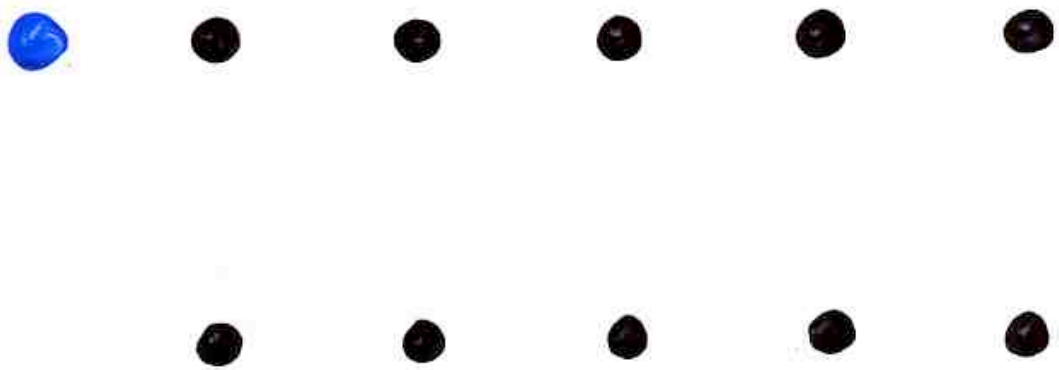
this valuation corresponds to the (n+1)!
"enlarged Laguerre histories"

priority queue

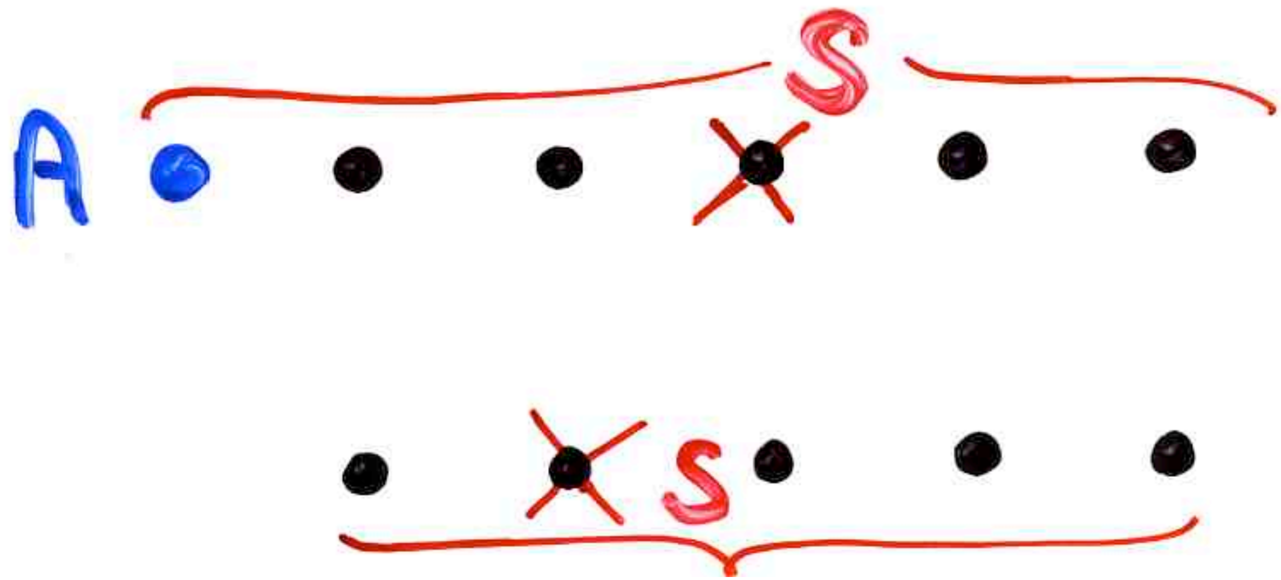
Polya urn



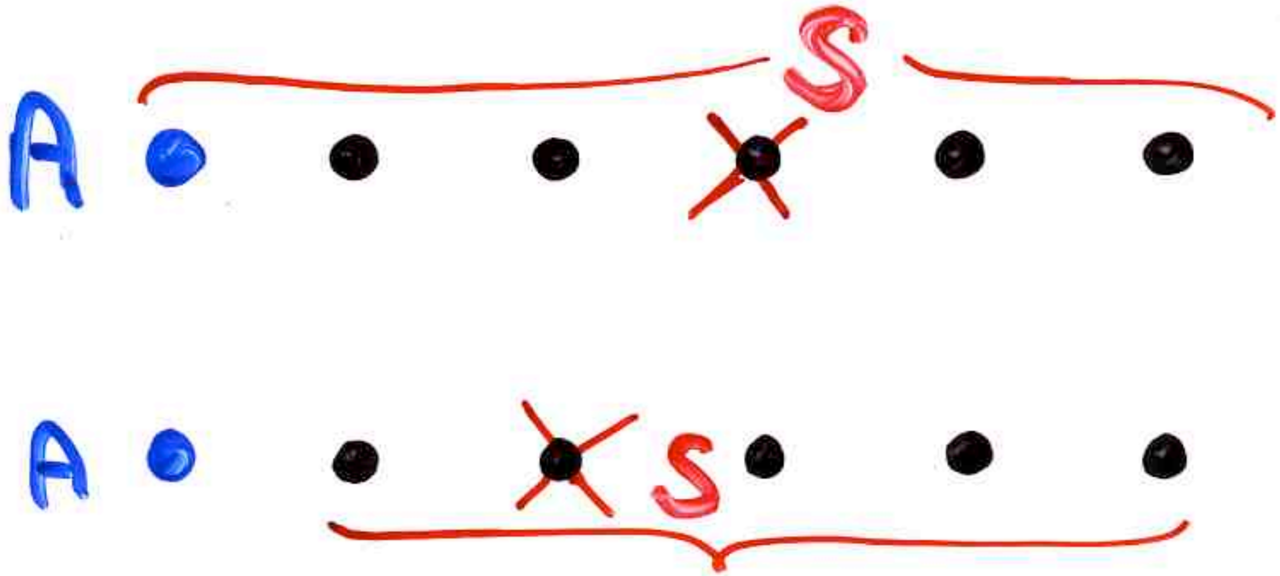
A




A
S


A
S
S

$AS - SA = I$



A produit par x

S $\frac{d}{dx}(\ )$

polynôme d'Hermite $H_n(x)$

$\lambda_k = k$; $b_k = 0$

$(k \geq 1)$ $(k \geq 0)$

$a_k = 1$ $\begin{cases} b'_k = 0 \\ b''_k = 0 \end{cases}$ $c_k = k$

Histoires d'Hermite

general PASEP

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q- Hermite polynomial

$\alpha, \beta, q \qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\,\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

Askey-Wilson polynomials

$$D = \frac{1}{(1-q)}(1+\mathrm{d}) \qquad\qquad E = \frac{1}{(1-q)}(1+\mathrm{e})$$

$$\mathrm{de}\text{-}q\mathrm{ed} = (1-q)1$$

$$\langle W | [\mathrm{e} - (a+c)1 + ac\mathrm{d}] = 0$$

$$[\mathrm{d} - (b+d)1 + bd\mathrm{e} | V \rangle] = 0$$

$$\mathrm{d} = \begin{pmatrix} d_0^0 & d_0^{+1} & 0 & \\ d_0^{-1} & \ddots & d_{n-1}^{+1} & \cdots\cdots \\ 0 & d_{n-1}^{-1} & d_n^0 & \\ \cdots\cdots & & & \end{pmatrix} \qquad \mathrm{e} = \begin{pmatrix} e_0^0 & e_0^{+1} & 0 & \\ e_0^{-1} & \ddots & e_{n-1}^{+1} & \cdots\cdots \\ 0 & e_{n-1}^{-1} & e_n^0 & \\ \cdots\cdots & & & \end{pmatrix}$$

$$d_n^0 = \frac{q^{n-1}}{(1-q^{2n-2}abcd)(1-q^{2n}abcd)} \times$$

$$\begin{bmatrix} bd(a+c)+(b+d)q-abcd(b+d)q^{n-1}-\{bd(a+c)+abcd(b+d)\}q^n \\ -bd(a+c)q^{n+1}+ab^2cd^2(a+c)q^{2n-1}+abcd(b+d)q^{2n} \end{bmatrix}$$

$$e_n^0 = \frac{q^{n-1}}{(1-q^{2n-2}abcd)(1-q^{2n}abcd)} \times$$

$$\begin{bmatrix} ac(b+d)+(a+c)q-abcd(a+c)q^{n-1}-\{ac(b+d)+abcd(a+c)\}q^n \\ -ac(b+d)q^{n+1}+a^2bc^2d(b+d)q^{2n-1}+abcd(a+c)q^{2n} \end{bmatrix}$$

$$d_n^{+1} = \frac{1}{1-q^n ac} A_n \qquad e_n^{+1} = -\frac{q^n ac}{1-q^n ac} A_n$$

$$d_n^{-1} = -\frac{q^n bd}{1-q^n bd} A_n \qquad e_n^{-1} = \frac{1}{1-q^n bd} A_n$$

$$A_n = \left[\frac{(1-q^{n-1}abcd)(1-q^{n+1})(1-q^n ab)(1-q^n ac)(1-q^n ad)(1-q^n bc)(1-q^n bd)(1-q^n cd)}{(1-q^{2n-1}abcd)(1-q^{2n}abcd)^2(1-q^{2n+1}abcd)} \right]^{1/2}$$

$$\langle W | = h_0^{1/2}(1,0,0,\ldots) \qquad | V \rangle = h_0^{1/2}(1,0,0,\ldots)^T$$

$$h_0 = \frac{(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}$$

$$(a_1,a_2,\ldots,a_s;q)_\infty = \prod_{r=1}^{s}\prod_{k\geq 0}(1-a_r q^k)$$

Askey-Wilson integral

L'intégrale de Askey-Wilson

$$(a)_\infty = \prod_i (1 - aq^i)$$

$$w(\cos\theta, a, b, c, d \mid q) = \frac{(e^{2i\theta})_\infty (e^{-2i\theta})_\infty}{(ae^{i\theta})_\infty (ae^{-i\theta})_\infty (be^{i\theta})_\infty (be^{-i\theta})_\infty (ce^{i\theta})_\infty (ce^{-i\theta})_\infty (de^{i\theta})_\infty (de^{-i\theta})_\infty}$$

$$\frac{(q)_\infty}{2\pi} \int_0^\pi w(\cos\theta, a, b, c, d \mid q)\, d\theta = \frac{(abcd)_\infty}{(ab)_\infty (ac)_\infty (ad)_\infty (bc)_\infty (bd)_\infty (cd)_\infty}$$

Askey, Wilson (1985)
Ismail, Stanton, Viennot (1986)
Rahman (1984), Ismail, Stanton (1989)
Gasper, Rahman (1989)

Integral of the product of

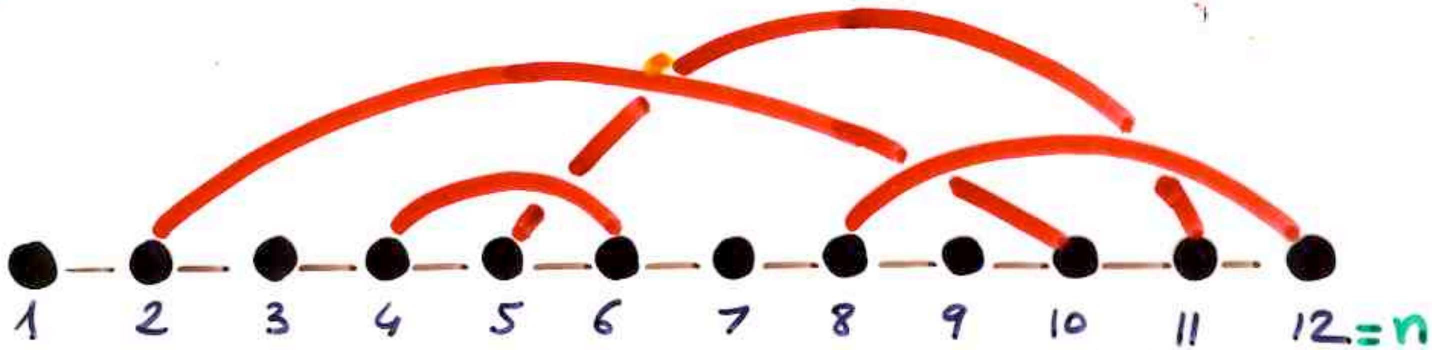
4 q-Hermite polynomials

$$\frac{(q)_\infty}{2\pi}\int_0^{\pi} H_n(\cos\theta|q)\, H_m(\cos\theta|q)\,(e^{2i\theta})_\infty (e^{-2i\theta})_\infty = (q)_n \delta_{nm}$$

q-moments

perfect matchings

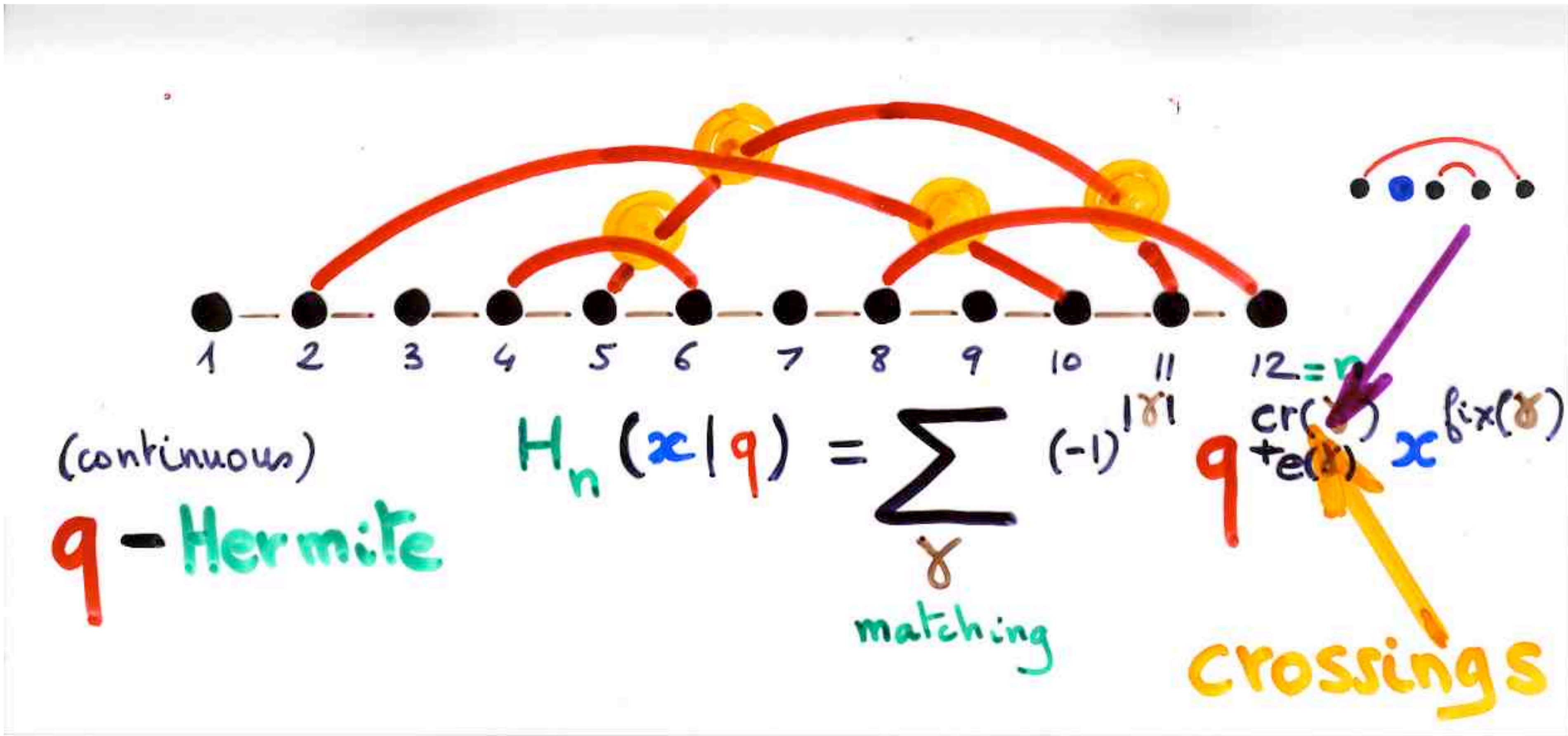
number of crossings



(continuous)

q-Hermite

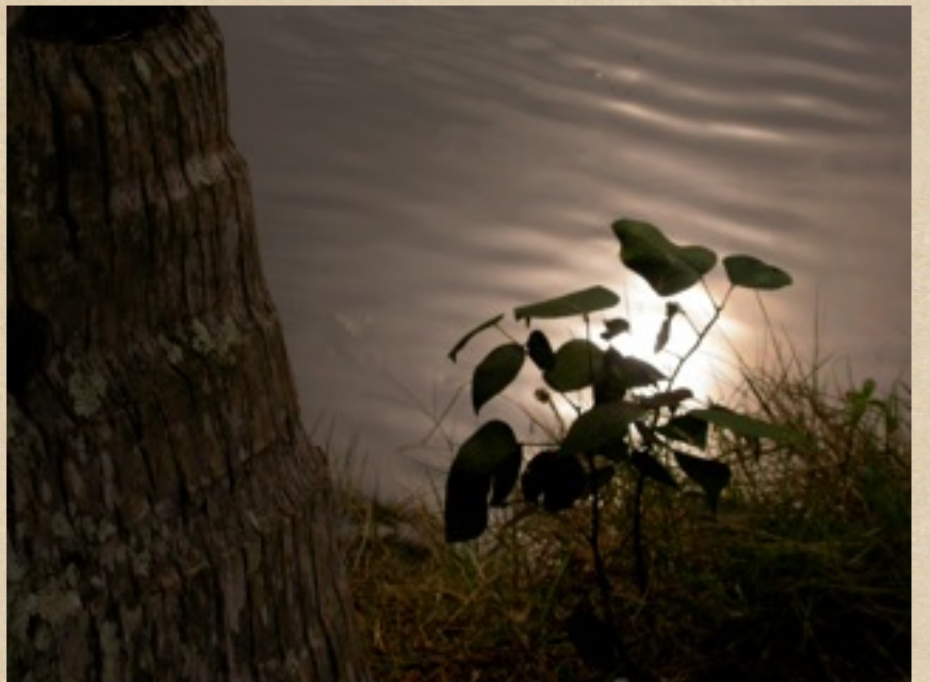
$$H_n(x|q) = \sum_{\substack{\gamma \\ \text{matching}}} (-1)^{|\gamma|} \, q^{cr(\gamma)+e(\gamma)} \, x^{fix(\gamma)}$$

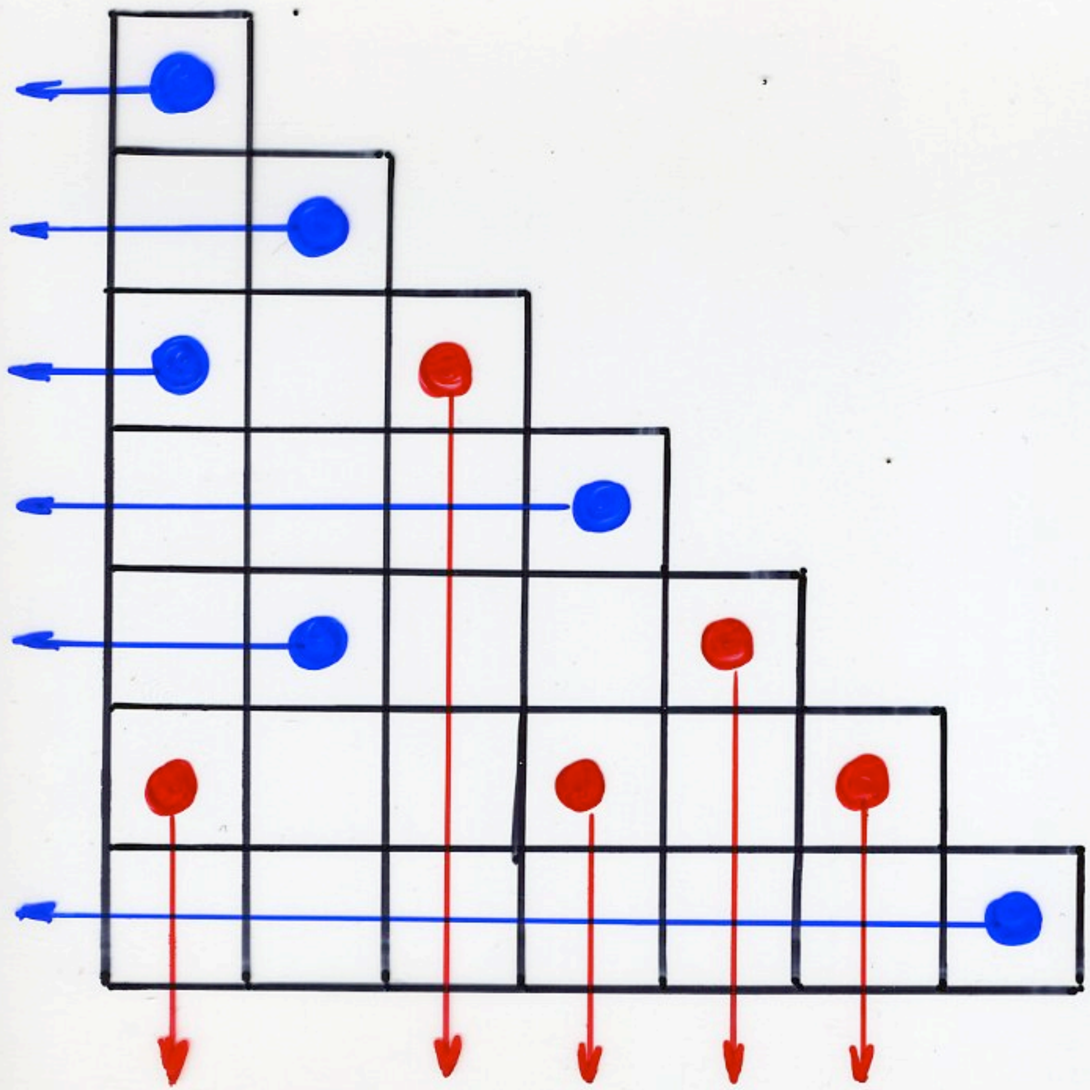

1 2 3 4 5 6 7 8 9 10 11 12=n
(continuous)
q-Hermite
$H_n(x|q) = \sum_{\gamma} (-1)^{|\gamma|} q^{cr(\gamma)+e(\gamma)} x^{fix(\gamma)}$
matching
crossings

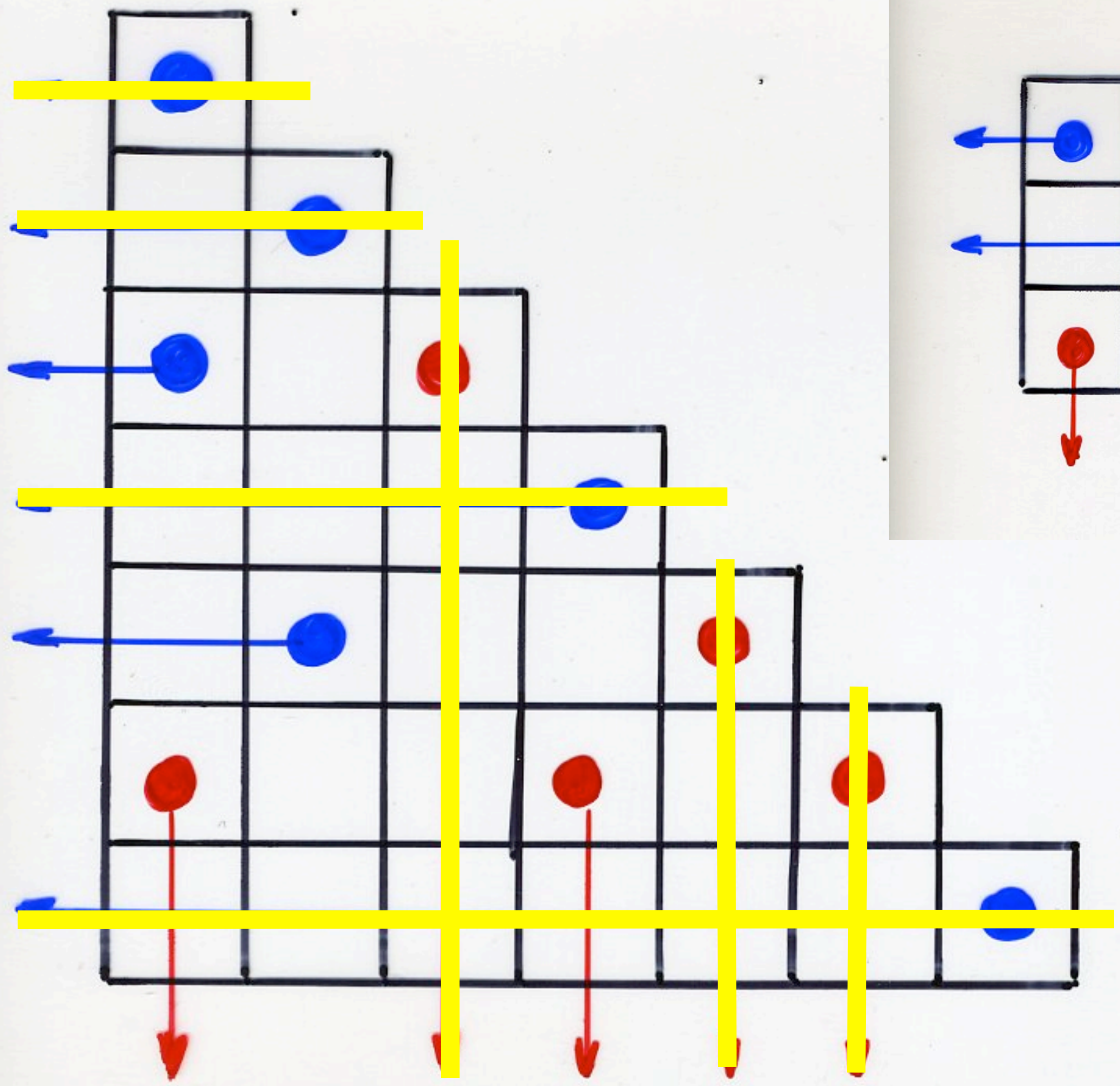
staircase tableaux

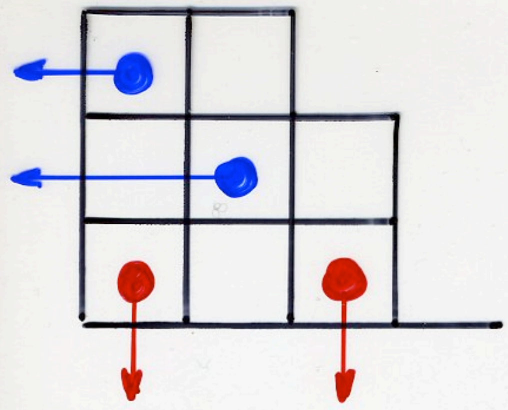
Corteel, Williams, 2009

Corteel, Stanley, Stanton, Williams, 2010

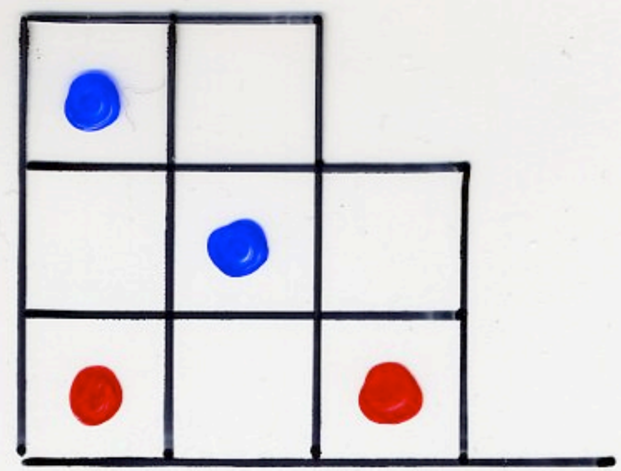




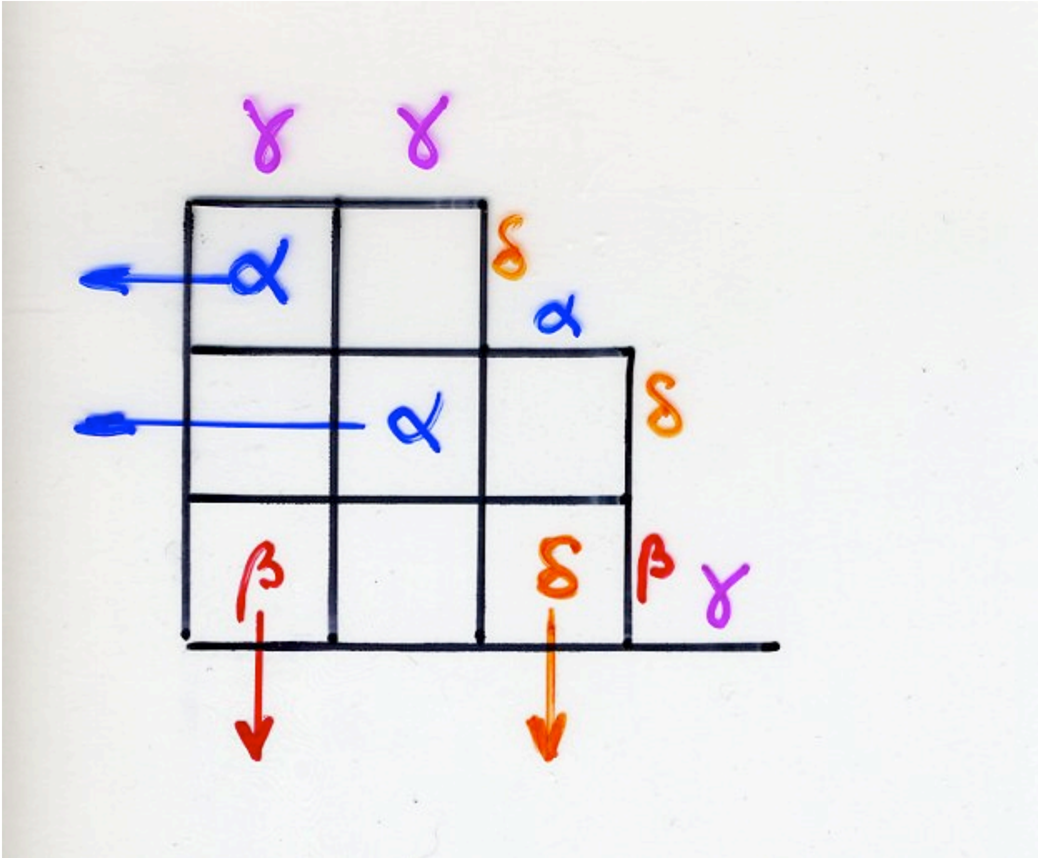




nb of 2-colored alternative tableaux $= 2^n \cdot n!$

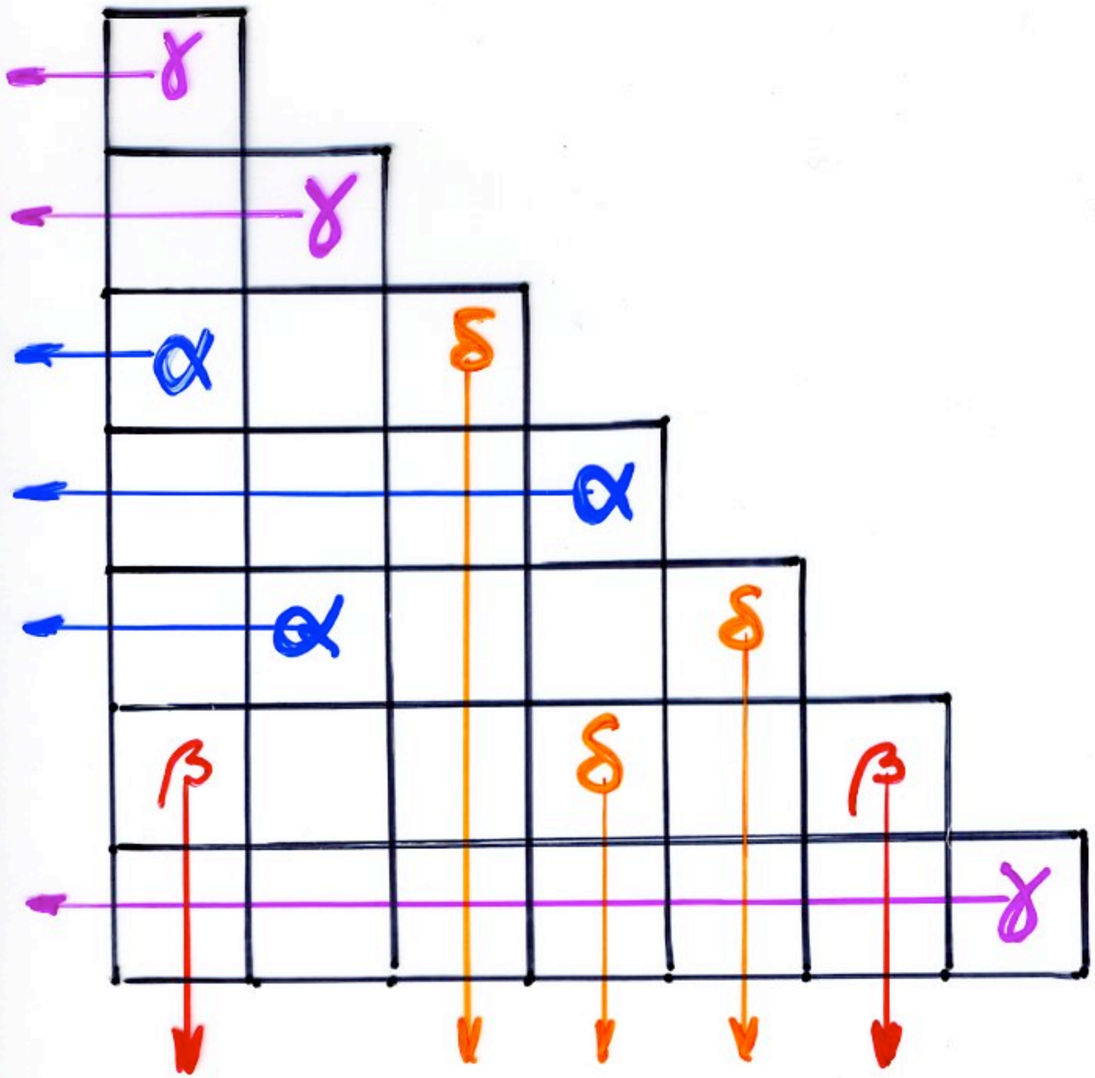


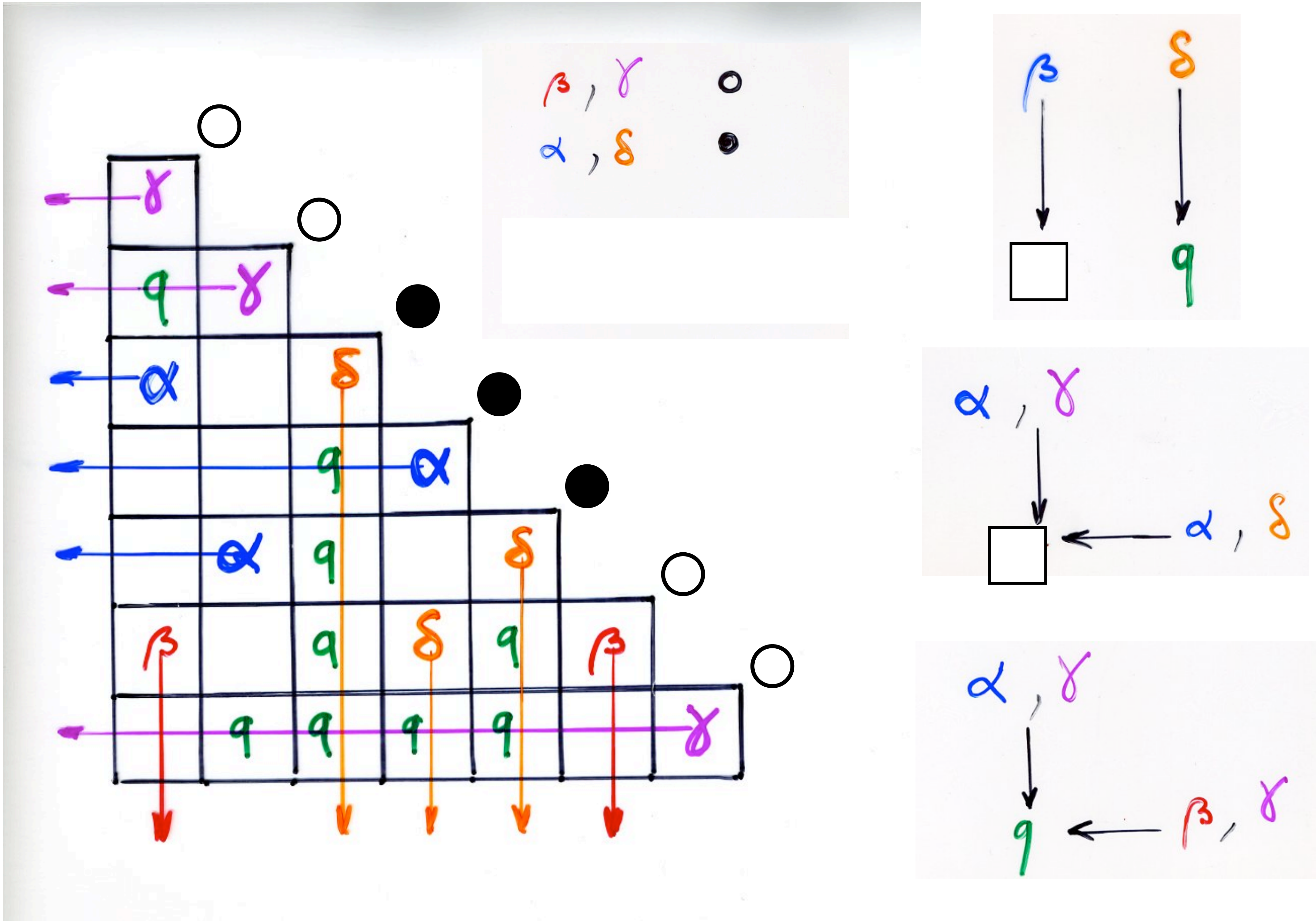
nb of 2-colored alternative tableaux $= 2^n \cdot n!$



nb of staircase tableaux $= 4^n \cdot n!$

staircase tableaux





β , γ
α , δ
β
δ
q
α , γ
α , δ
α , γ
q
β , γ

steady state probability PASEP

$$\frac{1}{Z_n} Z_\tau(\alpha, \beta, \gamma, \delta ; q)$$

$$Z_n = \sum_{\tau} Z_\tau$$

$\tau = (\tau_1, \ldots, \tau_n)$ state

relation with moments of Askey-Wilson polynomials

Corteel, Williams, 2009

Corteel, Stanley, Stanton, Williams, 2010

The cellular Ansatz

From quadratic algebra Q
to combinatorial objects (Q-tableaux)
and bijections

"The cellular Ansatz"

combinatorial
objects
on a 2d lattice

representation
by operators

Physics

data structures
"histories"

bijections

orthogonal
polynomials

"normal ordering"

UD = DU + Id

Weyl-Heisenberg

DE = qED + E + D

PASEP

towers placements
permutations
tableaux alternatifs

RSK

↔ pairs of Tableaux Young

↔ permutations
Laguerre histories

Q-tableaux
ex: ASM,
(alternating sign matrices)
FPL(fully packed loops)
tilings, 8-vertex

quadratic algebra Q

?

commutations
rewriting rules
planarisation

planar
automata

Koszul algebras
duality
J.L.Loday

ॐ सरस्वत्यै नमः।

Thank you !