Random matrix theory and its applications

Combinatorics of orthogonal polynomials and exclusion processes in physics

ICTS, TIFR, Bangalore

30 January 2012

Xavier G. Viennot

CNRS, Bordeaux, France

Random
matrix
theory

Combinatorics

orthogonal
polynomials

tridiagonal
matrices

PASEP
physics

Random
matrix
theory

↔

orthogonal
polynomials

Random
matrix
theory

$\updownarrow$

tridiagonal
matrices

Tridiagonal matrix

$$A = \begin{bmatrix} b_0 & 1 & & & & \\ \lambda_1 & b_1 & 1 & & & \\ & \lambda_2 & b_2 & 1 & & \\ & & \lambda_3 & b_3 & 1 & \\ & & & \lambda_4 & \ddots & \ddots \\ & & & & \ddots & \ddots \end{bmatrix}$$

Tridiagonal matrix

$$A_n = \begin{bmatrix} b_0 & 1 & & & & \\ \lambda_1 & b_1 & 1 & & & \\ & \lambda_2 & b_2 & 1 & & \\ & & \lambda_3 & b_3 & 1 & \\ & & & \lambda_4 & \ddots & \ddots \\ & & & & \ddots & \ddots \end{bmatrix}$$

density of eigenvalues of A_n

GOE_n

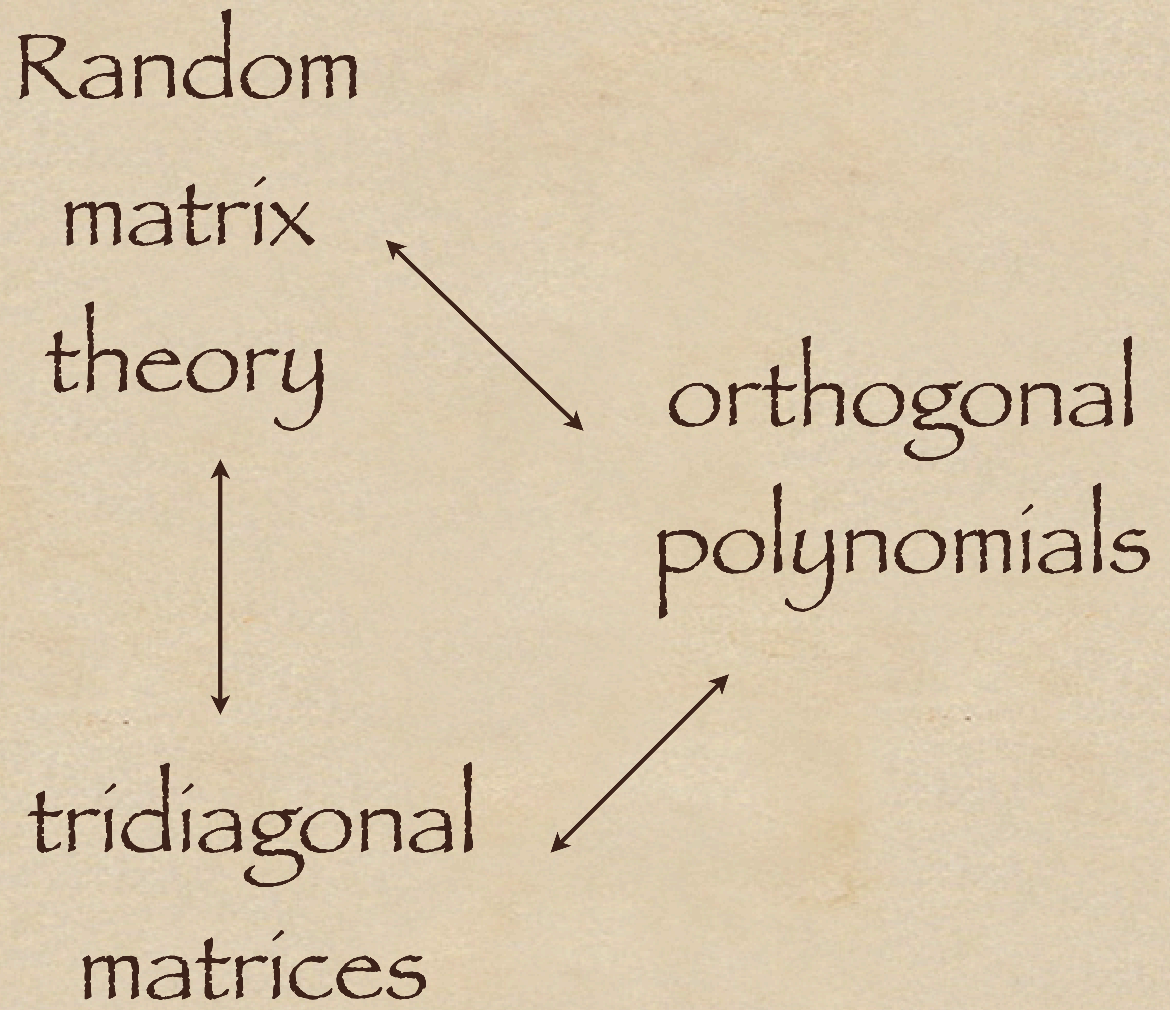
GUE_n

course by Manjunath Krishnapur

Tridiagonal matrix

$$A_n = \begin{bmatrix} b_0 & 1 & & & & \\ \lambda_1 & b_1 & 1 & & & \\ & \lambda_2 & b_2 & 1 & & \\ & & \lambda_3 & b_3 & 1 & \\ & & & \lambda_4 & \ddots & \ddots \\ & & & & \ddots & \ddots \end{bmatrix}$$

eigenvalues of A_n = zeros of polynomials $P_n(x)$


Random matrix theory
orthogonal polynomials
tridiagonal matrices

$$\det(A_n - xI)$$

Tridiagonal matrix

$$A = \begin{bmatrix} b_0 & 1 & & & & \\ \lambda_1 & b_1 & 1 & & & \\ & \lambda_2 & b_2 & 1 & & \\ & & \lambda_3 & b_3 & 1 & \\ & & & \lambda_4 & \ddots & \ddots \\ & & & & \ddots & \ddots \end{bmatrix}$$

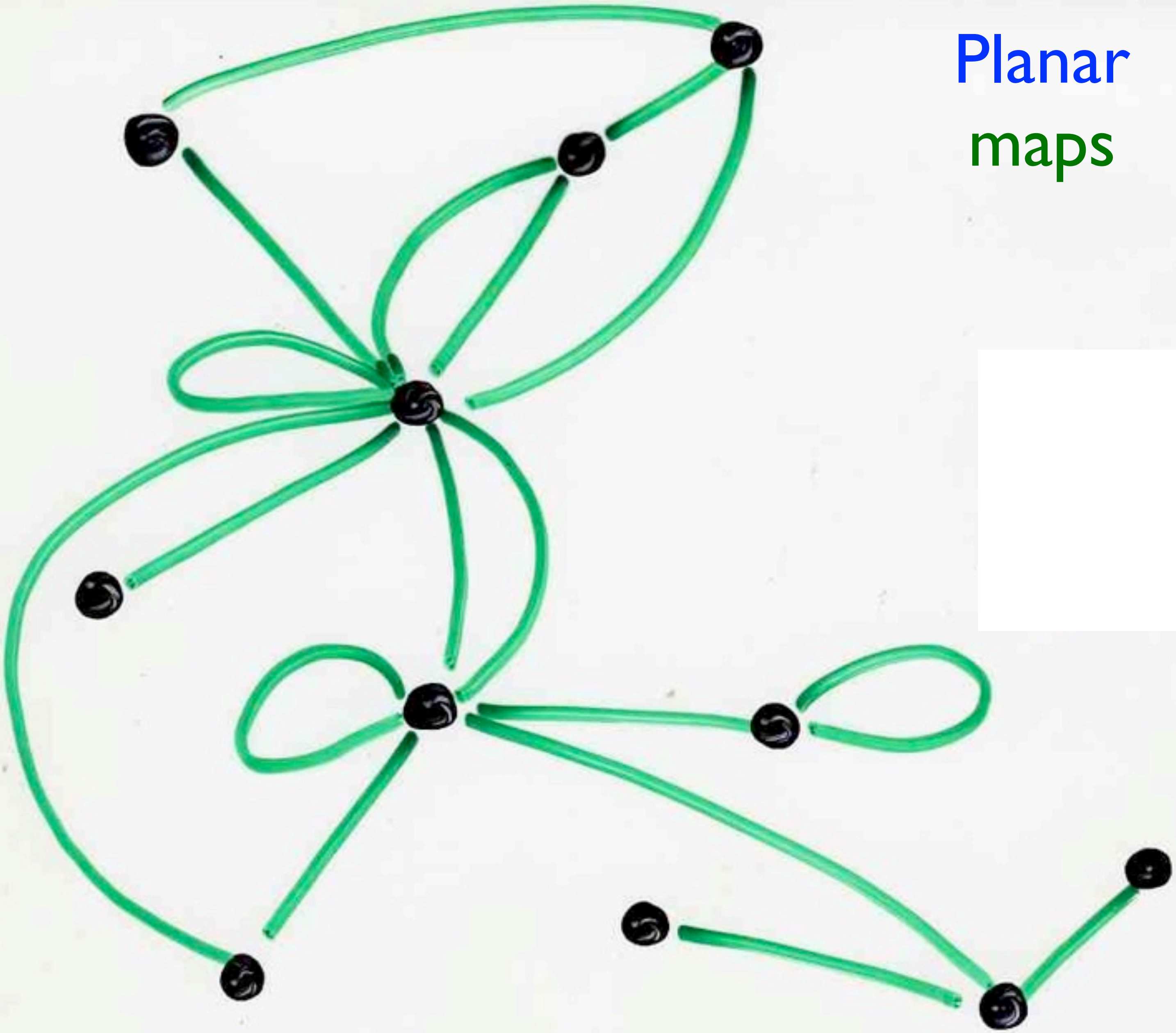
$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

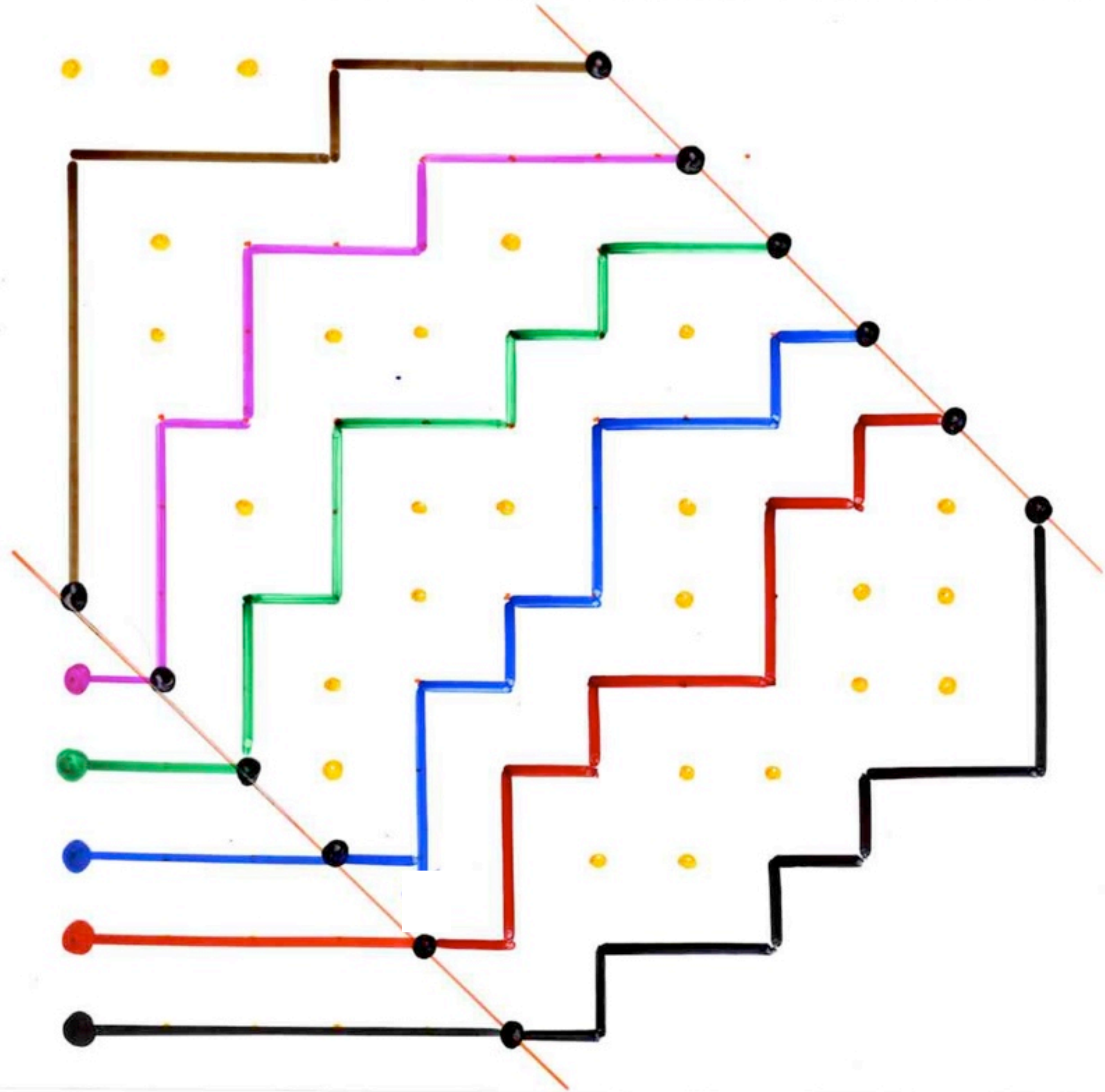
$(\forall k \geq 1)$

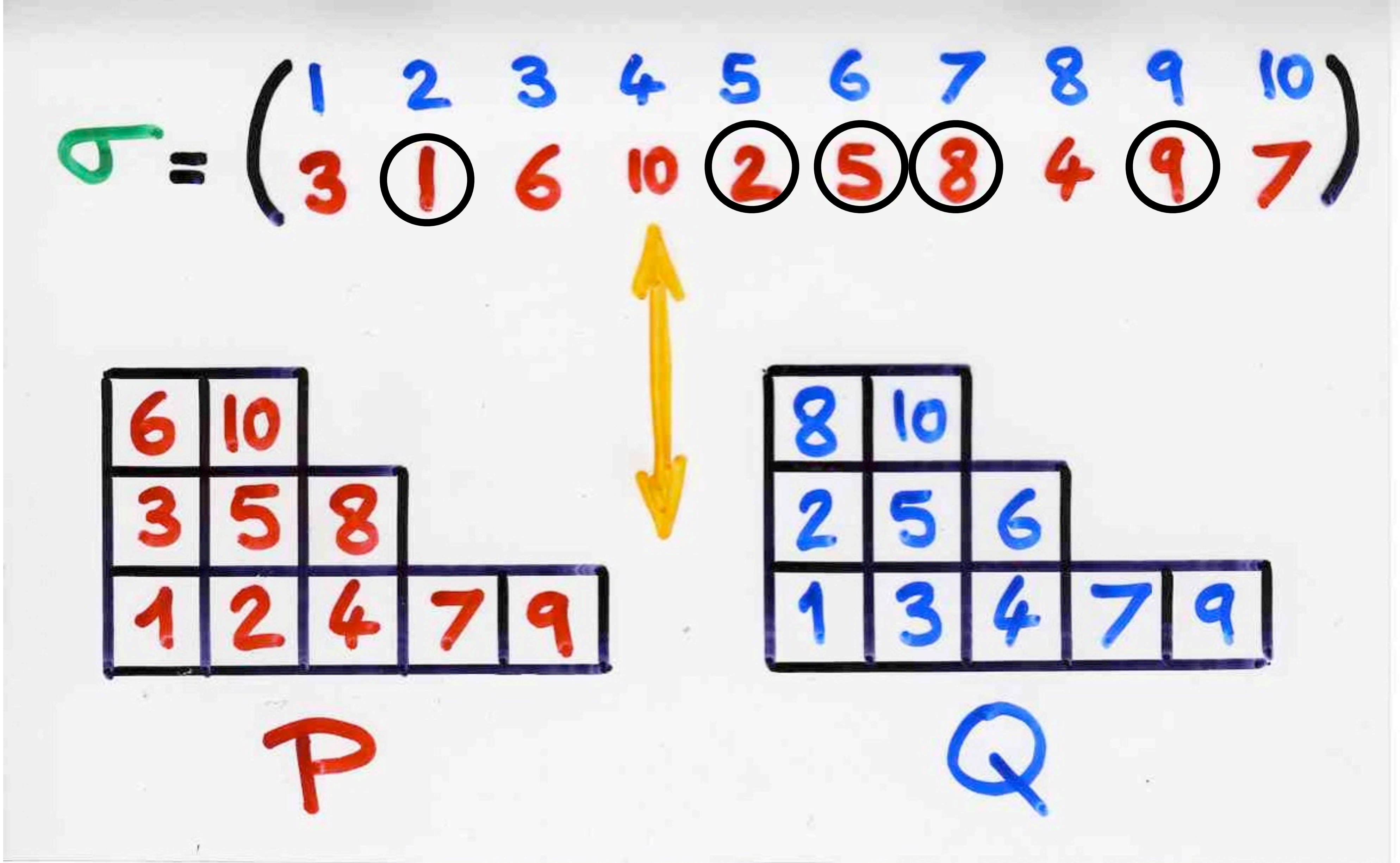
3 terms linear recurrence relation

Random matrix theory ⟷ Combinatorics

Planar maps







The Robinson-Schensted correspondence (RSK) enumerative

algebraic

bijective

analytic

existentialist

experimental

quantum

magic ....

combinatorics

combinatorial physics

or

integrable combinatorics

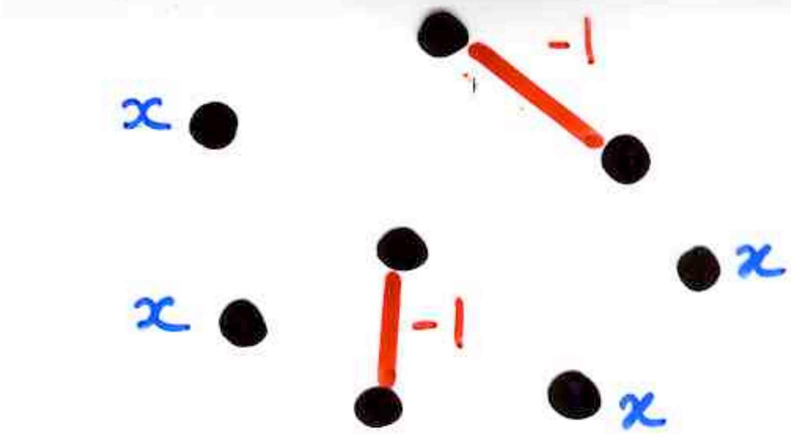
Combinatorics

↕

orthogonal
polynomials

$$H_n(x) = \sum_{0 \leq 2k \leq n} (-1)^k \frac{n!}{2^k\, k!\, (n-2k)!} x^{n-2k}$$

Hermite polynomials



matching

moments

Hermite polynomials

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

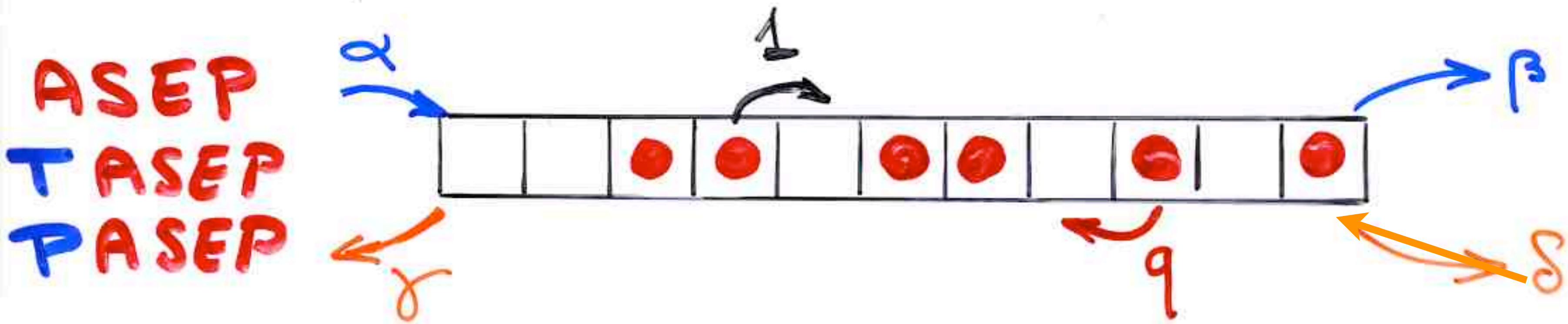
chord diagrams

perfect matching

orthogonal polynomials

↔

PASEP physics


ASEP
TASEP
PASEP
α
1
β
γ
q
δ

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

$\alpha, \beta, q \qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^+$$

$$\hat{a}\,\hat{a}^+ - q\,\hat{a}^+\,\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

Askey-Wilson polynomials

Combinatorics

↕

PASEP
physics

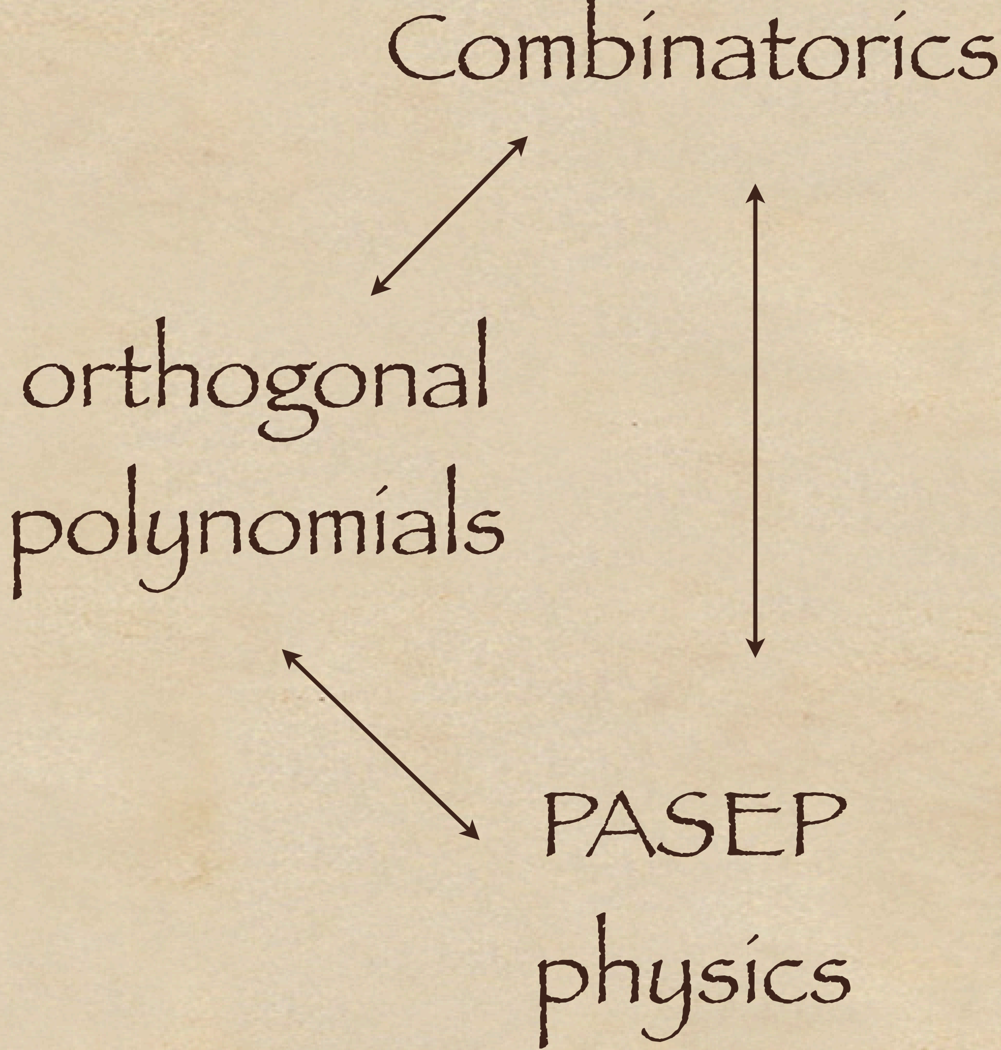
TASEP

Shapiro, Zeilberger (1982)
Brak, Essam (2003), Duchi, Schaeffer, (2004),
Angel (2005), XGV (2007)

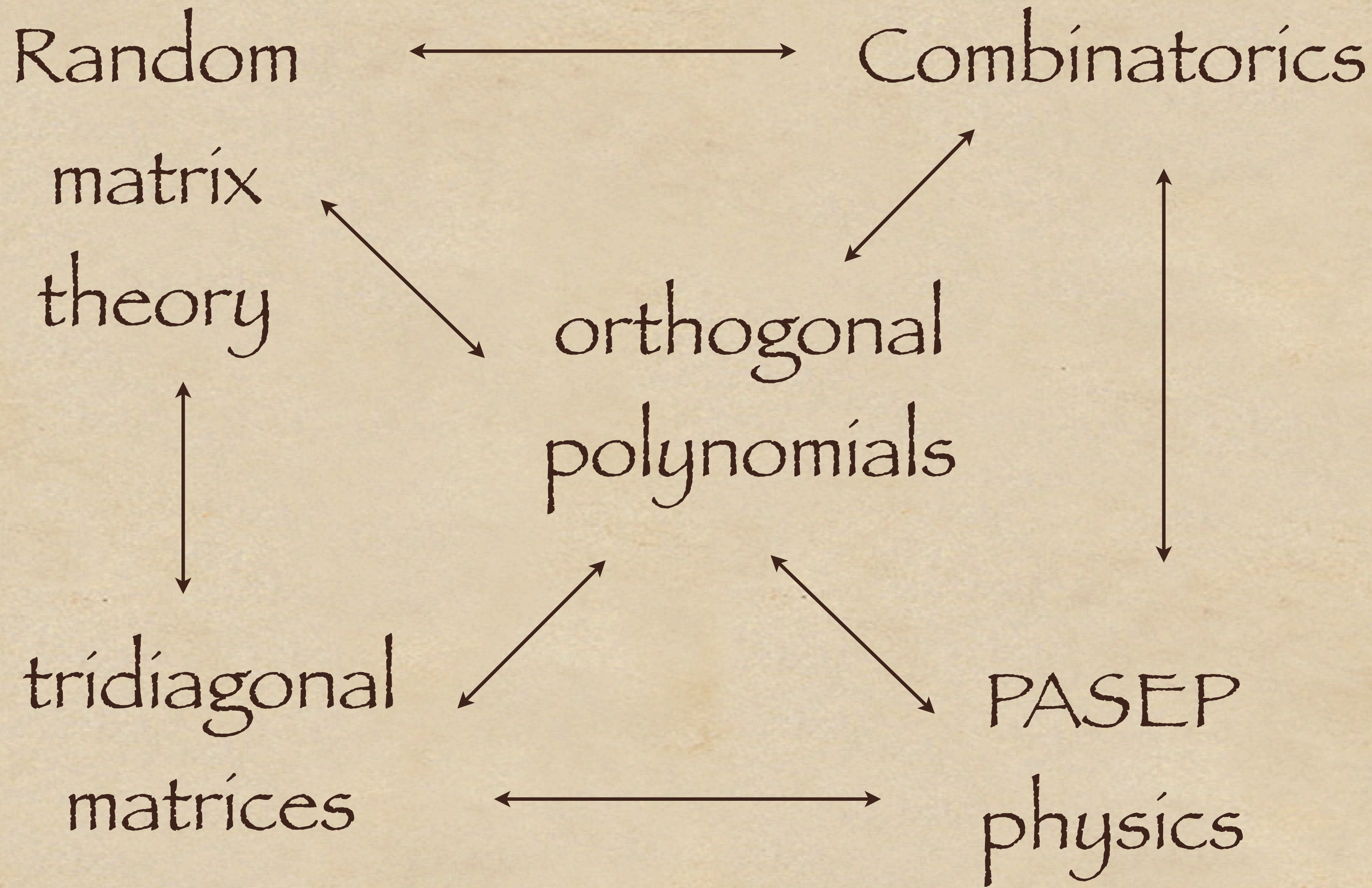
(P) ASEP

Brak, Corteel, Essam, Parviainen, Rechnitzer (2006)
Corteel, Williams (2006,..., 2010)
Corteel, Stanton, Stanley, Williams (2011)
Josuat-Vergès (2008,..., 2010)

Derrida, ...
Mallick, ... Golinelli, Mallick (2006)



Combinatorics
orthogonal polynomials
PASEP
physics


Random
matrix
theory
Combinatorics
orthogonal
polynomials
tridiagonal
matrices
PASEP
physics

(formal) orthogonal polynomials

Orthogonal polynomials

Def. $\{P_n(x)\}_{n\geq 0}$ $\quad P_n(x) \in \mathbb{K}[x]$

orthogonal iff $\exists\ f : \mathbb{K}[x] \to \mathbb{K}$

linear functional

$$\begin{cases} \text{(i)} & \deg(P_n(x)) = n \quad (\forall n \geq 0) \\ \text{(ii)} & f(P_k P_l) = 0 \quad \text{for } k \neq l \geq 0 \\ \text{(iii)} & f(P_k^2) \neq 0 \quad \text{for } k \geq 0 \end{cases}$$

$$\mathcal{L}(x^n) = \mu_n \quad (n \geqslant 0)$$

moments

$$\mathcal{L}(PQ) = \int_a^b P(x)\,Q(x)\,d\mu$$

mesure

Combinatorial approach to orthogonal polynomials

First step:

combinatorial intrepretation of

- polynomials (coefficients ....)
- moments

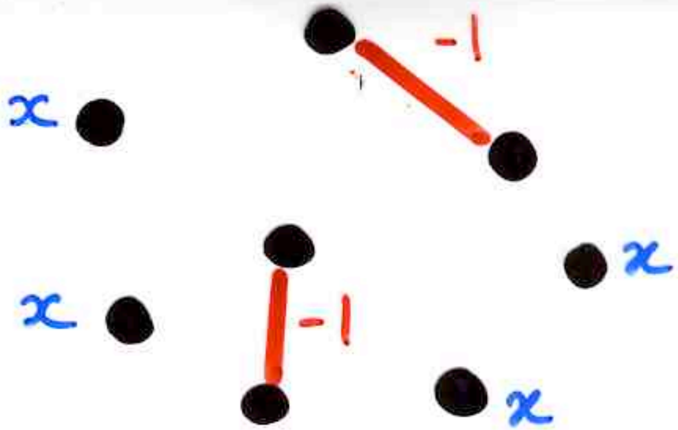
two examples:

- Hermite
- Laguerre

$$H_n(x) = \sum_{0 \leq 2k \leq n} (-1)^k \frac{n!}{2^k\, k!\, (n-2k)!} x^{n-2k}$$

$$\int_{-\infty}^{+\infty} H_n(x) H_m(x) e^{-x^2/2} dx = \sqrt{\pi}\; n!\; \delta_{n,m}$$

Hermite



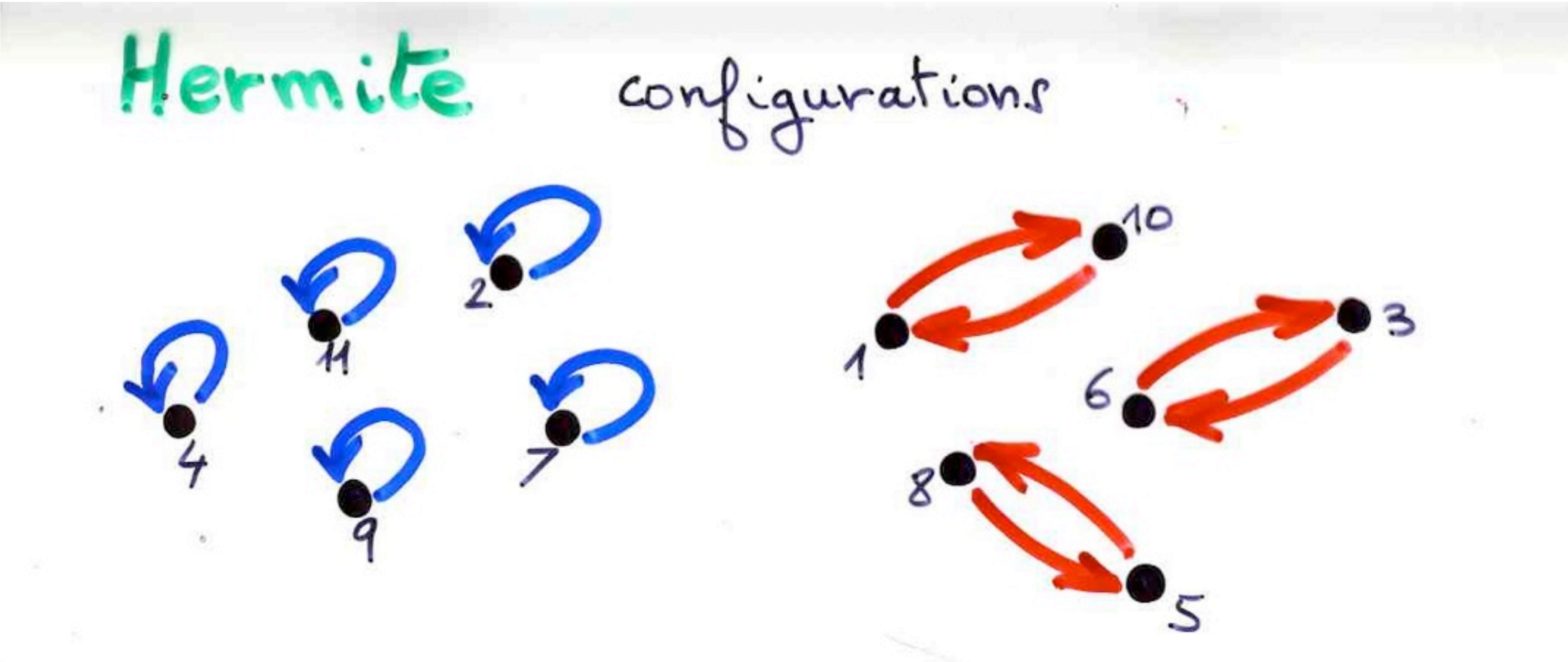
matching

ex: Hermite

$$H_n(x) = \sum_{\substack{\text{matching } \gamma \\ \text{of } K_n}} (-1)^{|\gamma|} x^{\text{fix}(\gamma)}$$

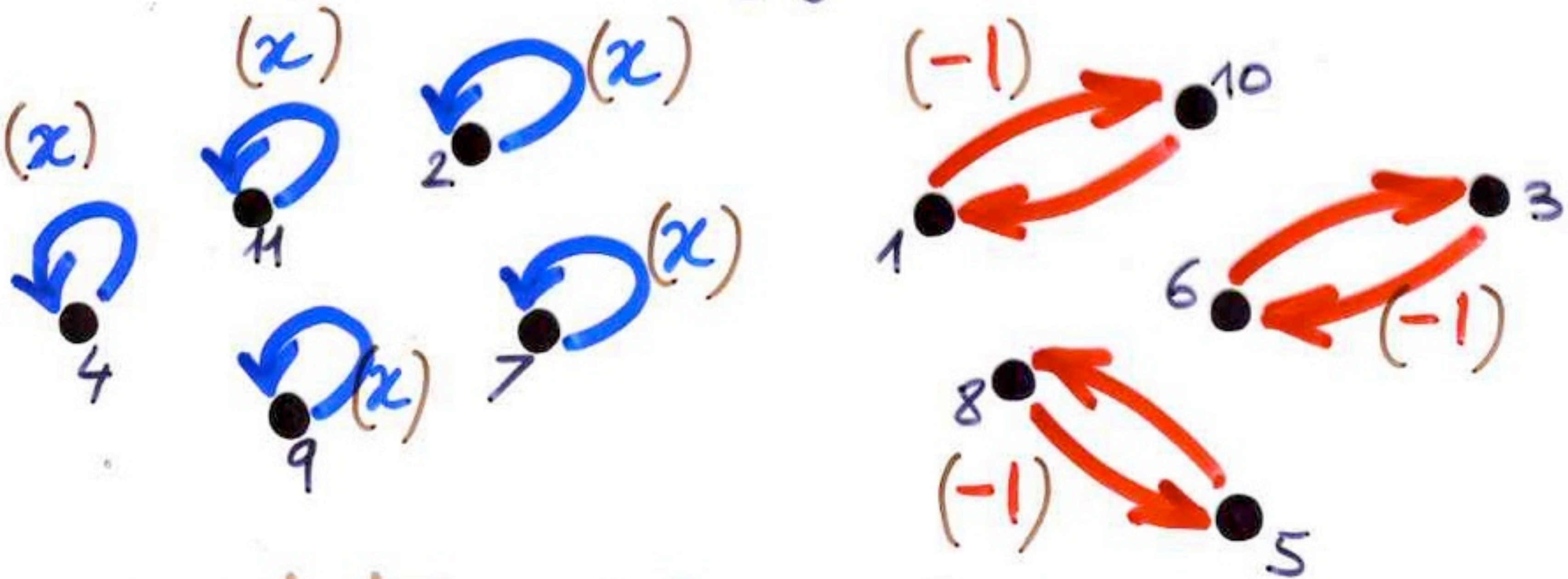
$$\mu_{2n+1} = 0 \qquad \mu_{2n} = 1 \times 3 \times \dots \times (2n-1)$$

number of perfect matchings of K_{2n}



Involutions on (1, 2, ..., n)

Hermite configurations



weight

(x)

(−1)

Laguerre polynomial

Laguerre $L_n^{(\alpha)}(x)$

$$\sum_{n\geq 0} L_n^{(\alpha)}(x)\,\frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}}\,\exp\left(\frac{-xt}{1-t}\right)$$

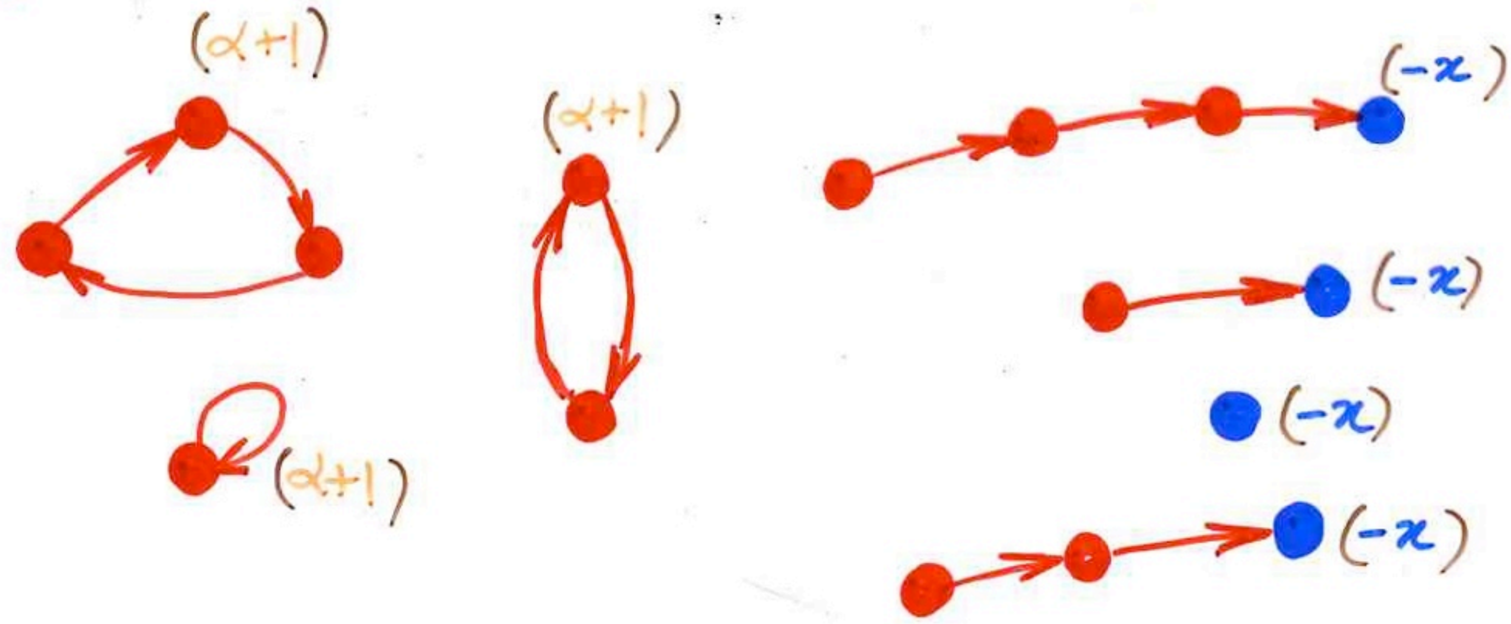
$$L_n^{(\alpha)}(x) = (\alpha+1)_n \; {}_1F_1\left[\begin{matrix}-n\\ \alpha+1\end{matrix};x\right]$$

$$= \sum_{i+j=n} \binom{n}{i} (\alpha+1+j)_i \, (-x)_j$$

$$\int_0^\infty L_m^{(\alpha)}(x) L_n^{(\alpha)}(x)\; x^\alpha e^{-x}\,dx = n!\,\Gamma(n+\alpha+1)\,\delta_{n,m}$$

Laguerre

Laguerre configuration



Askey tableau

Hermite $H_n(x) = (2x)^n \, {}_2F_0\left(\begin{matrix} -n/2, \frac{1-n}{2} \\ \text{—} \end{matrix} ; \frac{-1}{x^2}\right)$

Laguerre $n! L_n^{(\alpha)}(x) = (\alpha+1)_n \, {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix} ; x\right)$

Charlier $C_n^{(a)}(x) = {}_2F_0\left(\begin{matrix} -n, -x \\ \text{—} \end{matrix} ; -\frac{1}{a}\right)$

Jacobi $n! P_n^{(\alpha,\beta)}(x) = (\alpha+1)_n \, {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} ; \frac{1-x}{2}\right)$

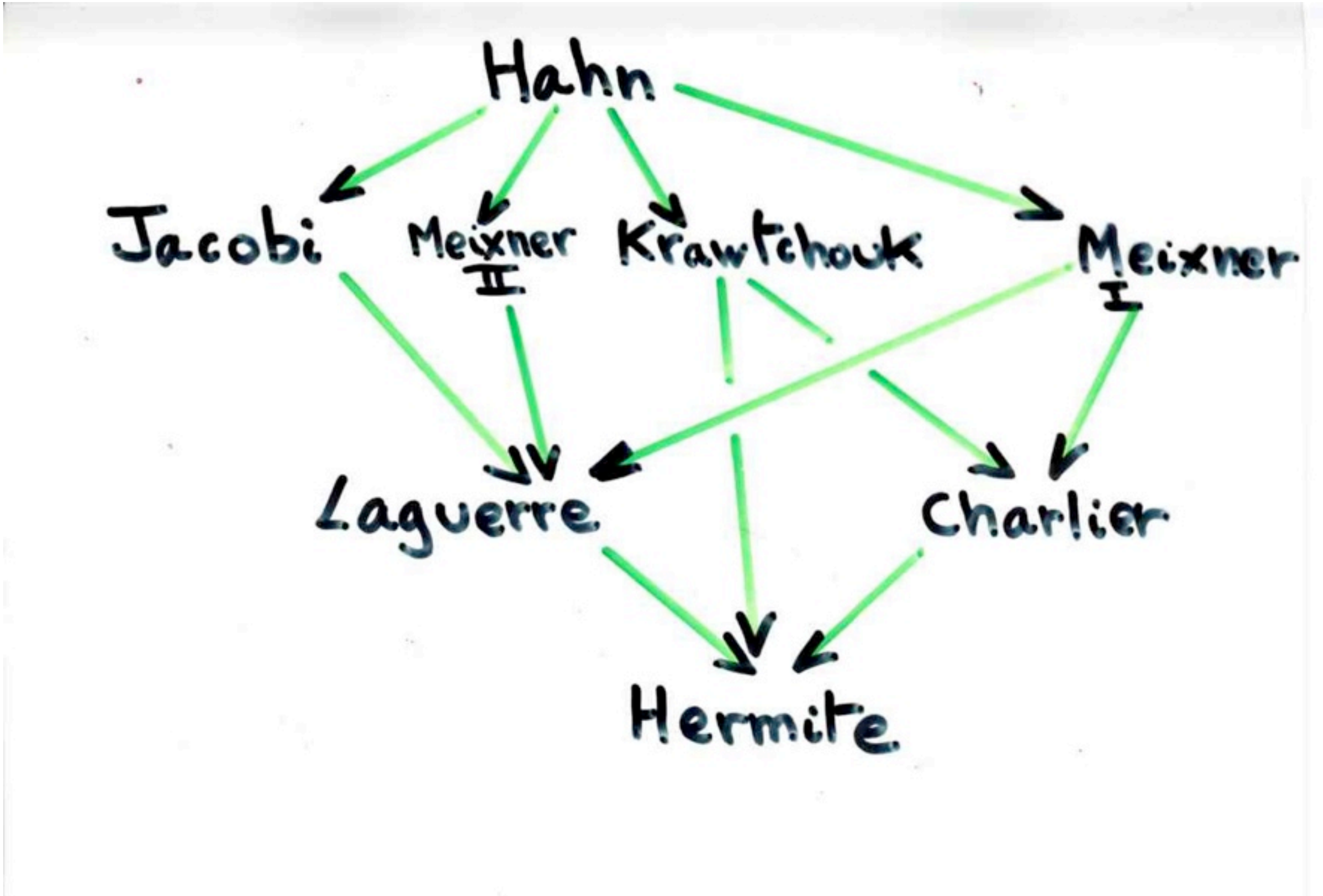
Meixner $m_n(x;\beta,c) = (\beta)_n \, {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix} ; 1-c^{-1}\right)$

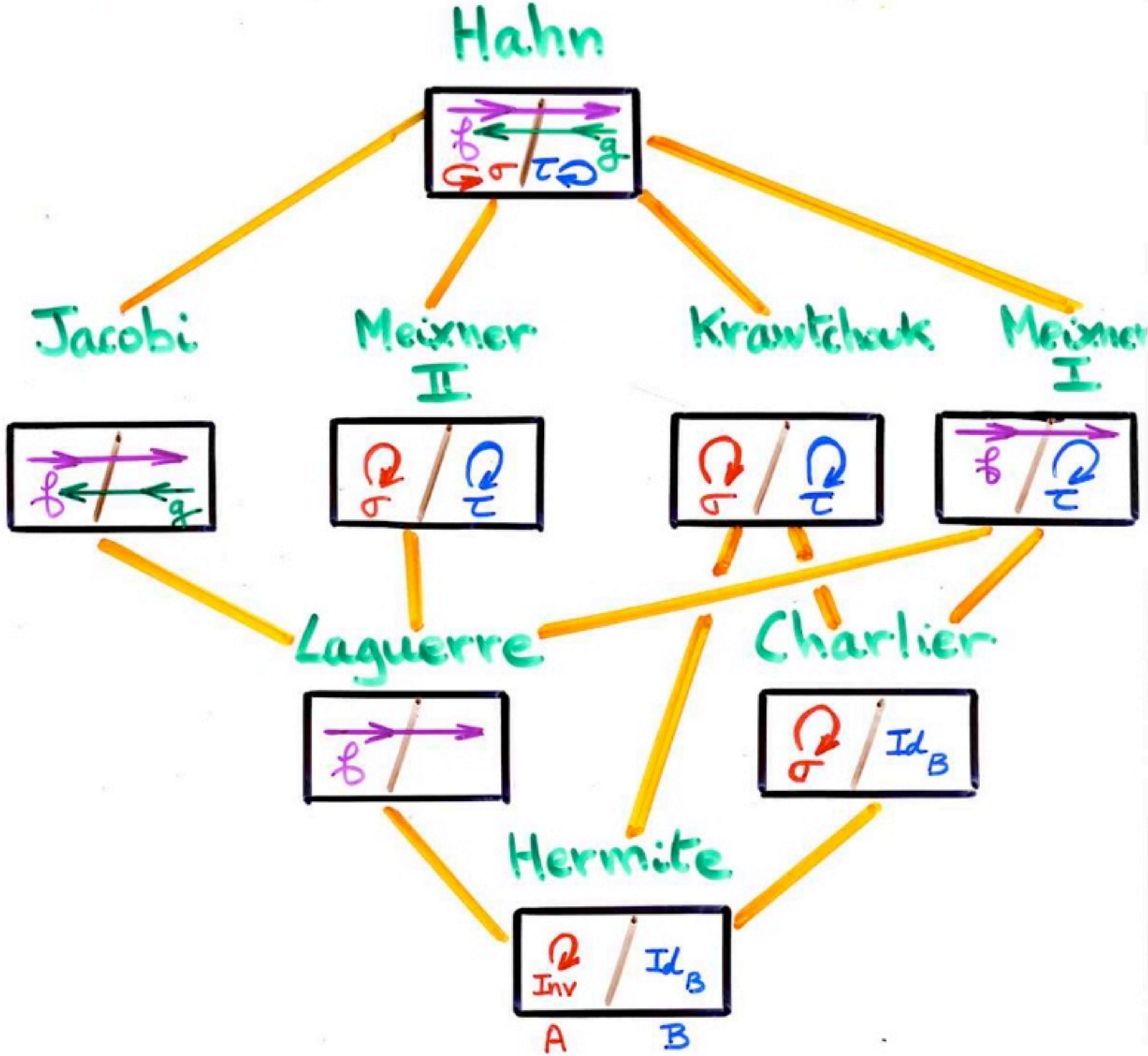
Krawtchouk $K_n(x;p,N) = {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix} ; p^{-1}\right)$

Meixner-Pollaczek $P_n^{a}(x;\varphi) = e^{in\varphi}\frac{(2a)_n}{n!} \, {}_2F_1\left(\begin{matrix} -n, a+ix \\ 2a \end{matrix} ; 1-e^{-2i\varphi}\right)$

Hahn $Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -N \end{matrix} ; 1\right)$

Askey-Wilson





Hahn
f
g
σ
τ
Jacobi
f
g
Meixner II
σ
τ
Krawtchouk
σ
τ
Meixner I
f
τ
Laguerre
f
Charlier
σ
Id_B
Hermite
Inv
Id_B
A
B

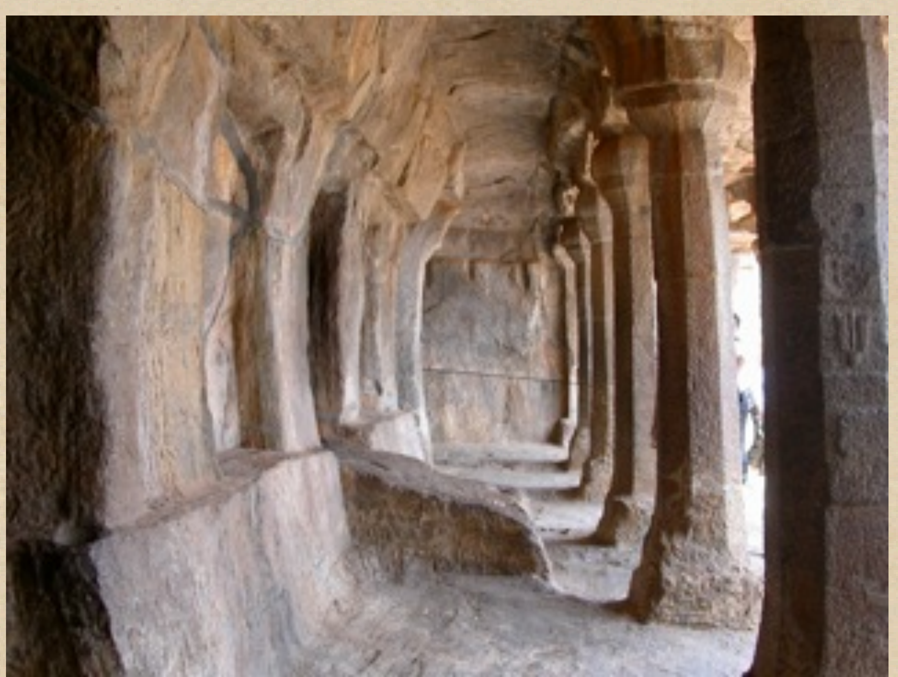
Yeh, Labelle, Strehl

Limit Formulas

ex: Jacobi ⟶ Laguerre

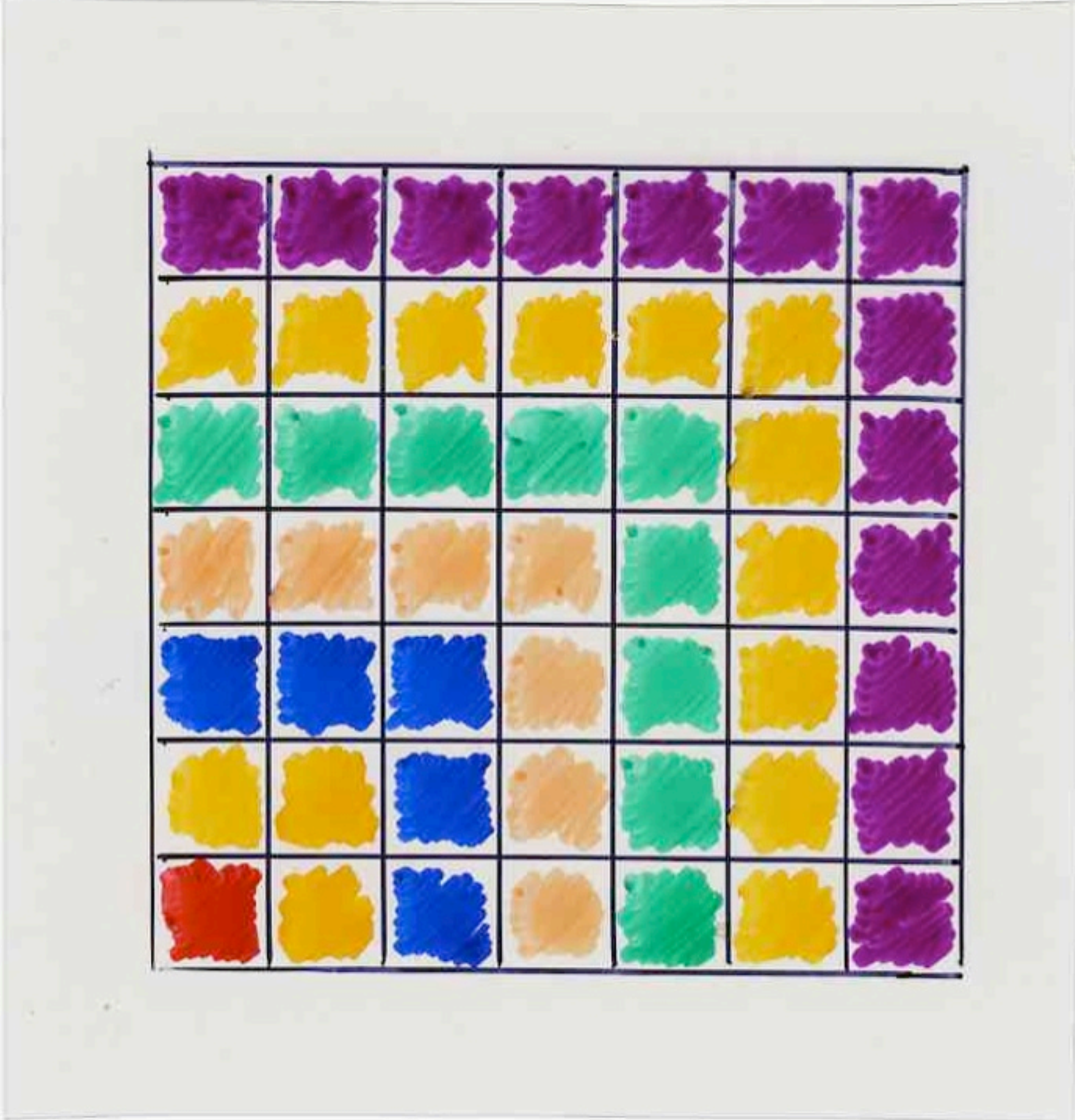
$$\lim_{\beta \to \infty} P_n^{(\alpha,\beta)}(1 - 2x\beta^{-1}) = L_n^{(\alpha)}(x)$$

Combinatorial proof of identities



an example:
Mehler identity
for Hermite
polynomials

Bijective proof of an identity



$$n^2 = 1 + 3 + \dots + (2n-1)$$

$$\sum_{n\geqslant 0} H_n(x)\,H(y)\,\frac{t^n}{n!} = (1-4t^2)^{-1/2} \exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

The "bijective" paradigm

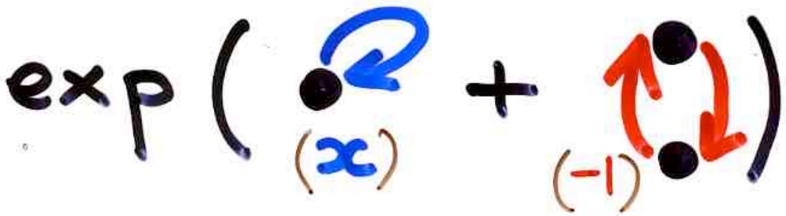
"drawing calculus"

each "pieces" of an identity

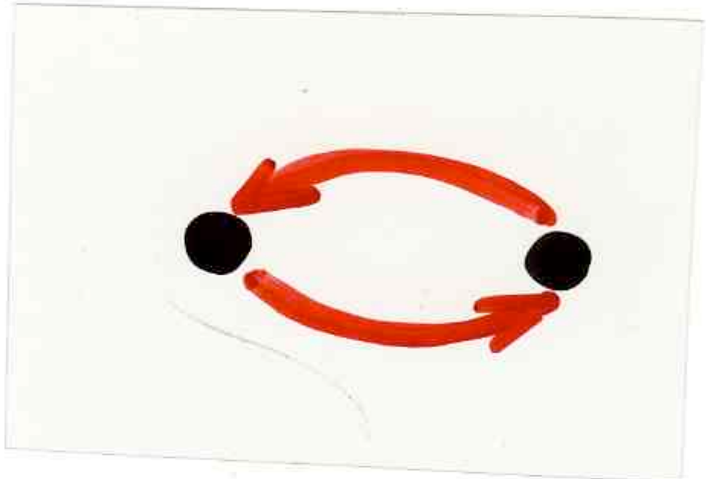
↕

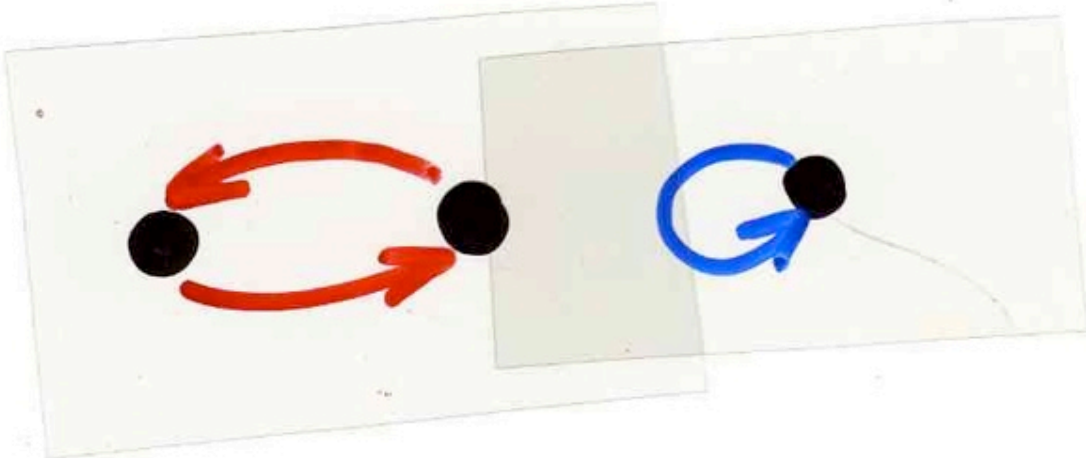
combinatorial object

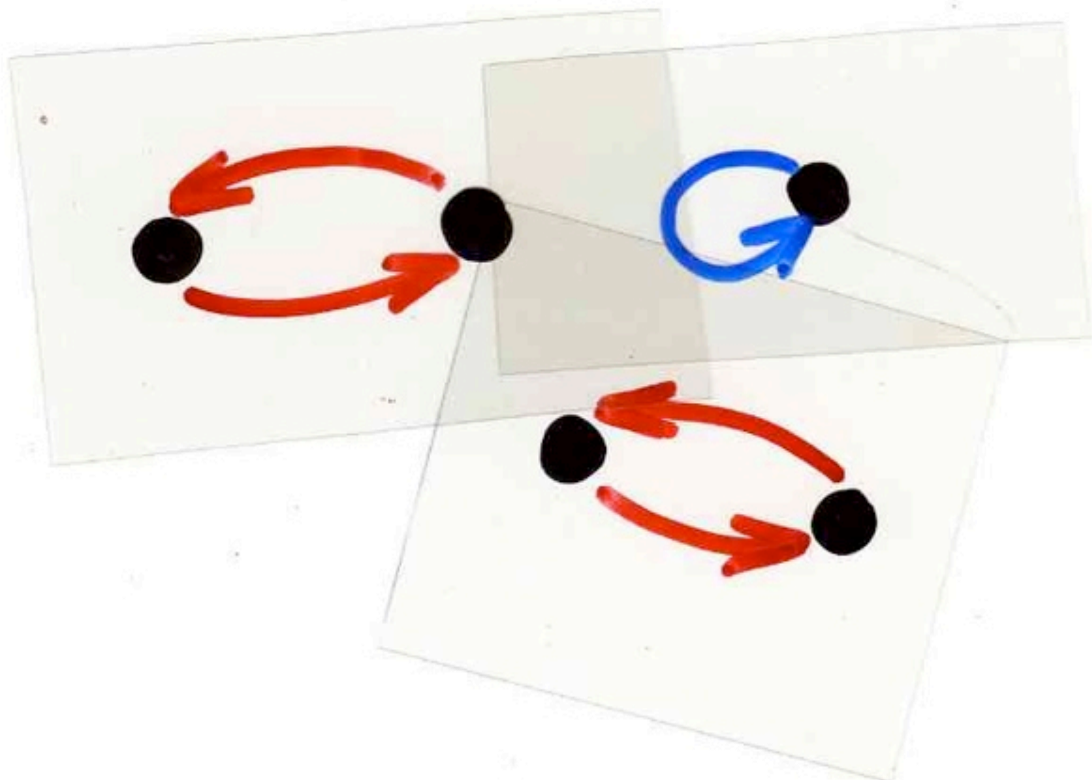
identities ⟷ correspondences
combinatorial construction
bijections

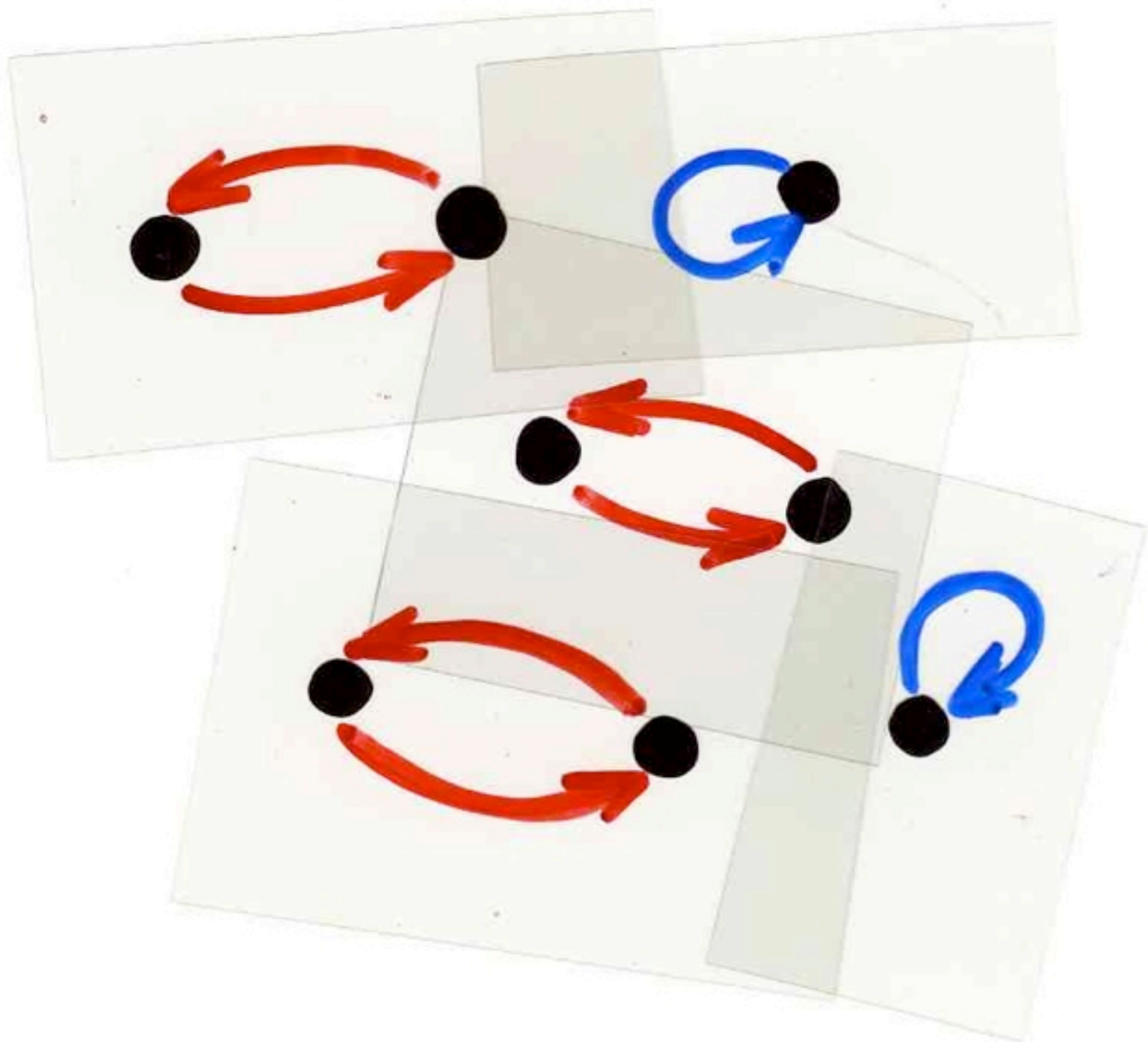


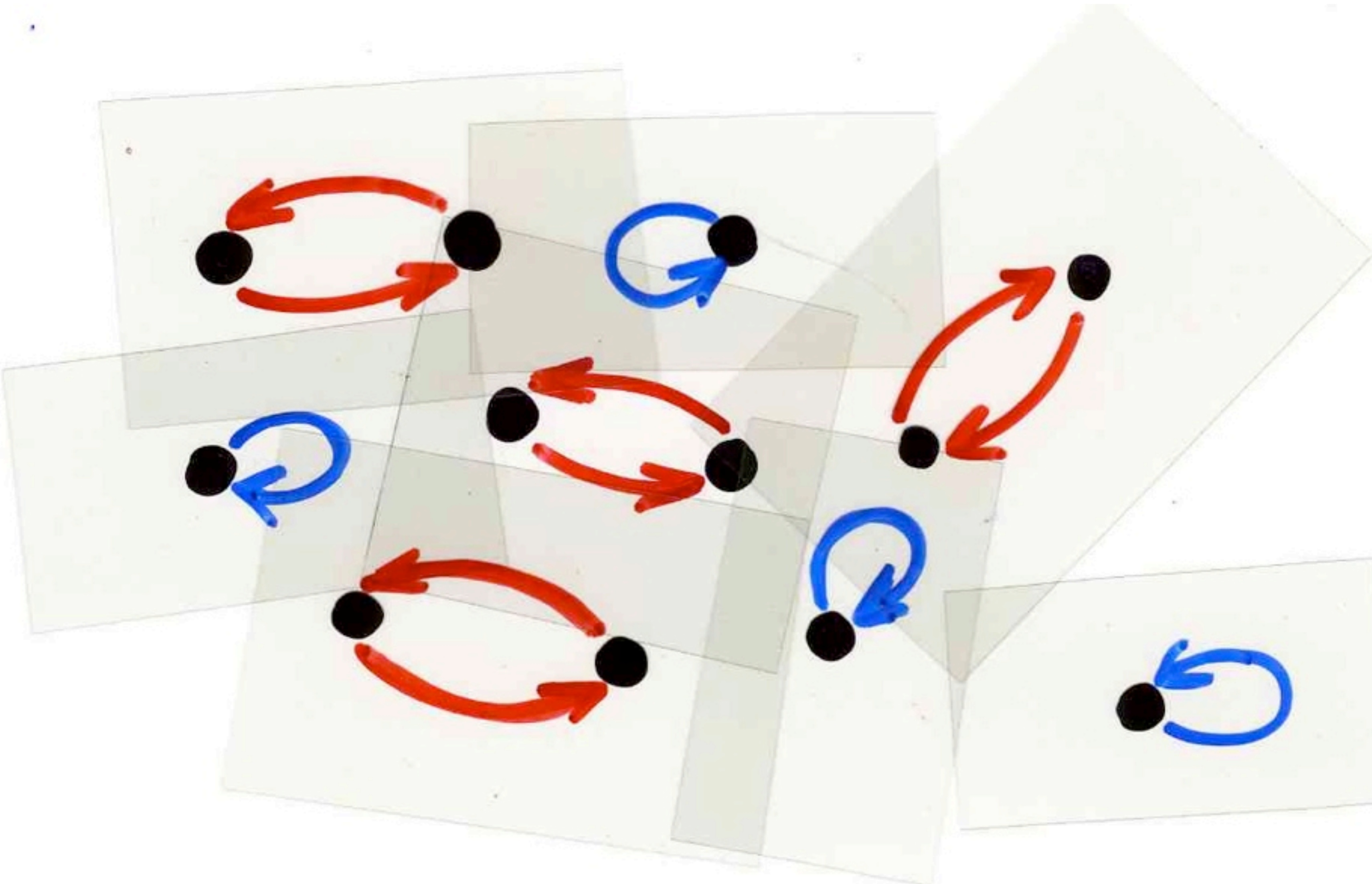
$$\sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = \exp\left(x t - \frac{t^2}{2}\right)$$

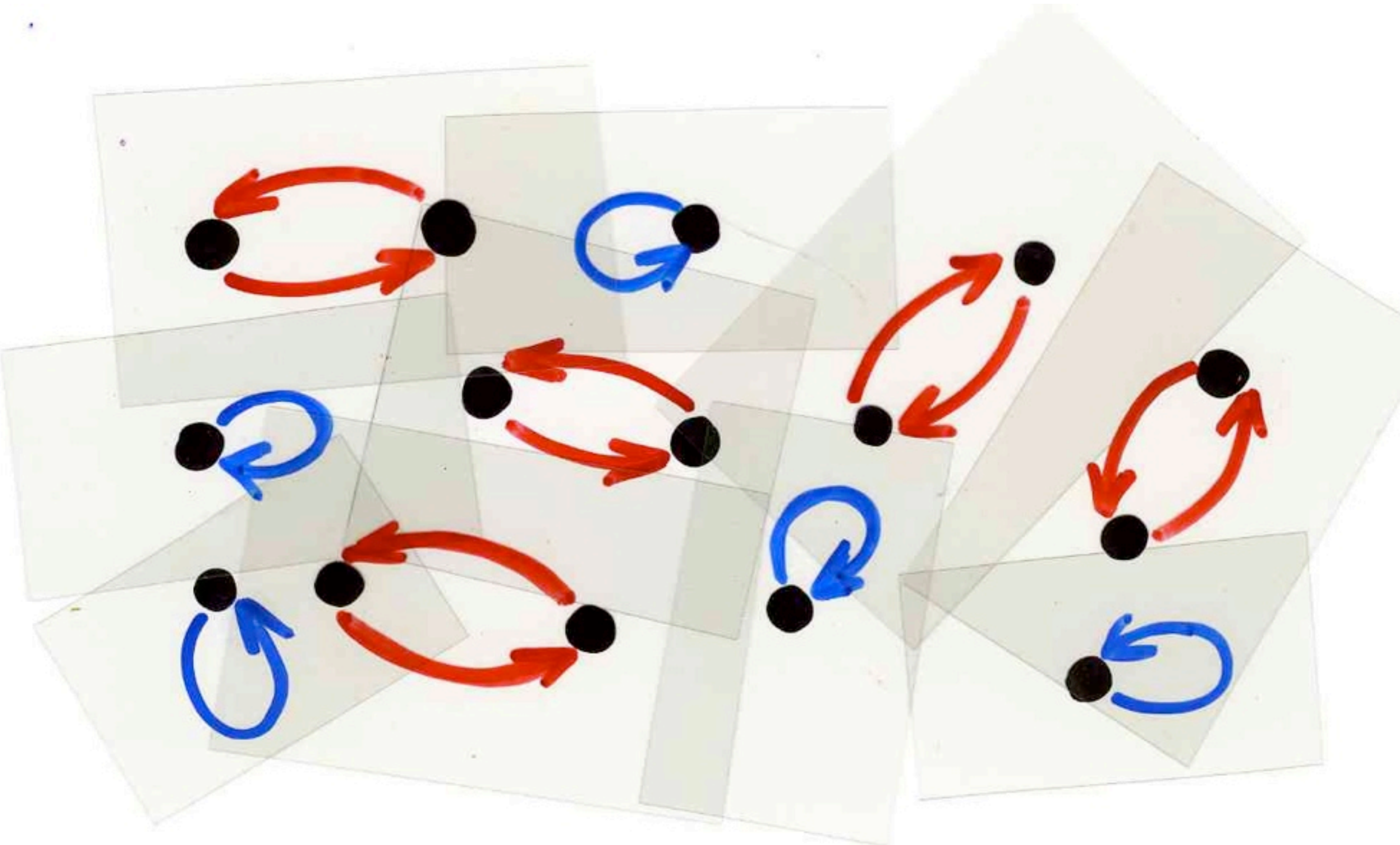




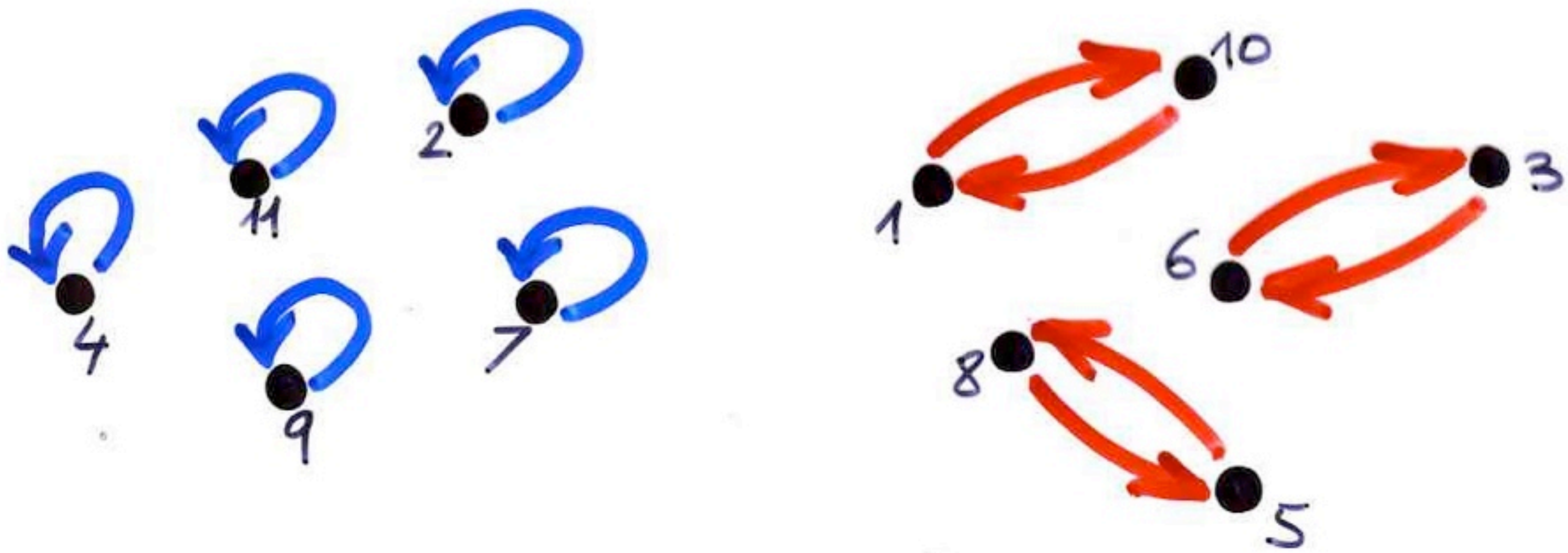




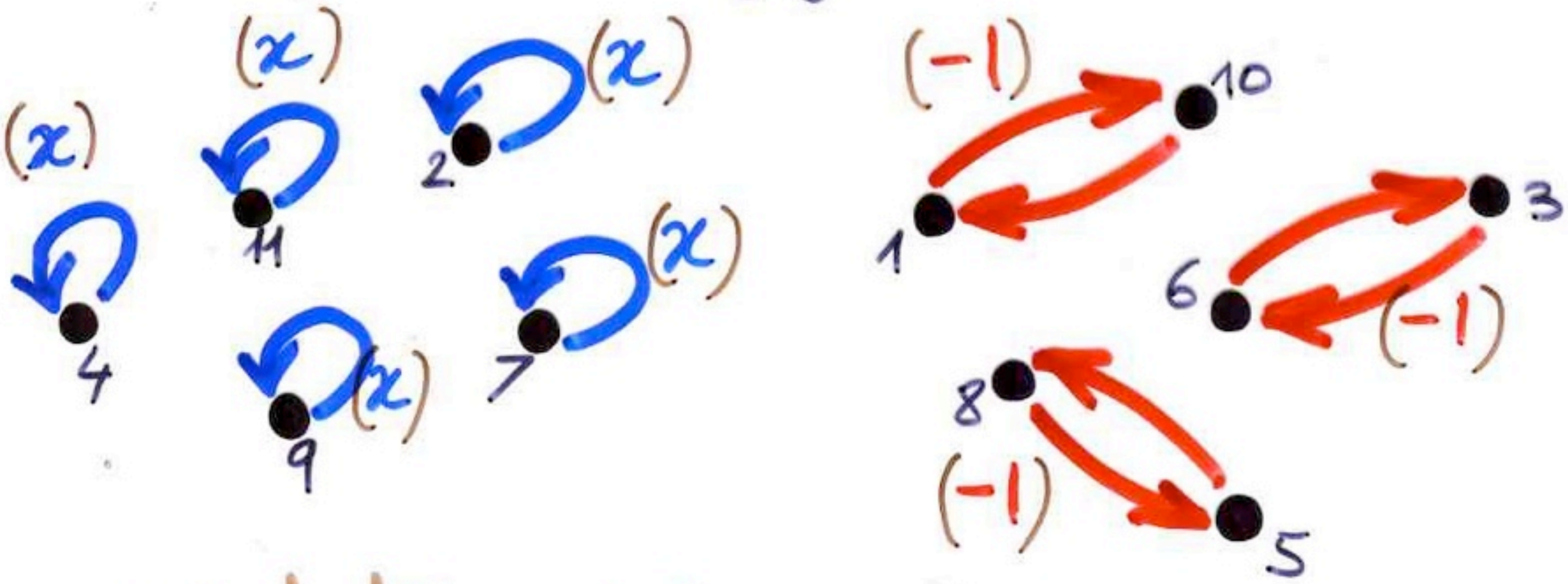




Hermite configurations



Hermite configurations



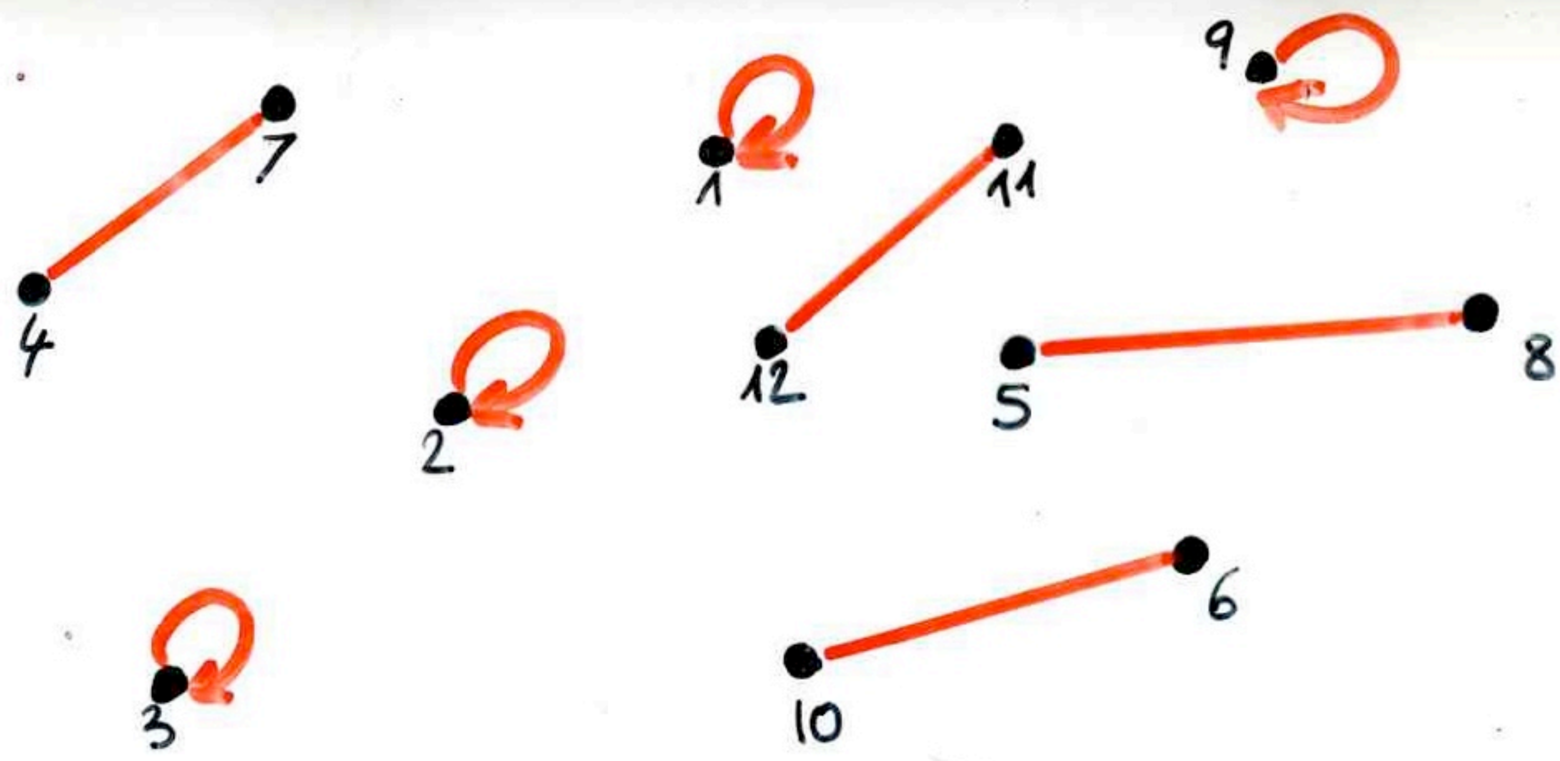
weight (x)

(-1)

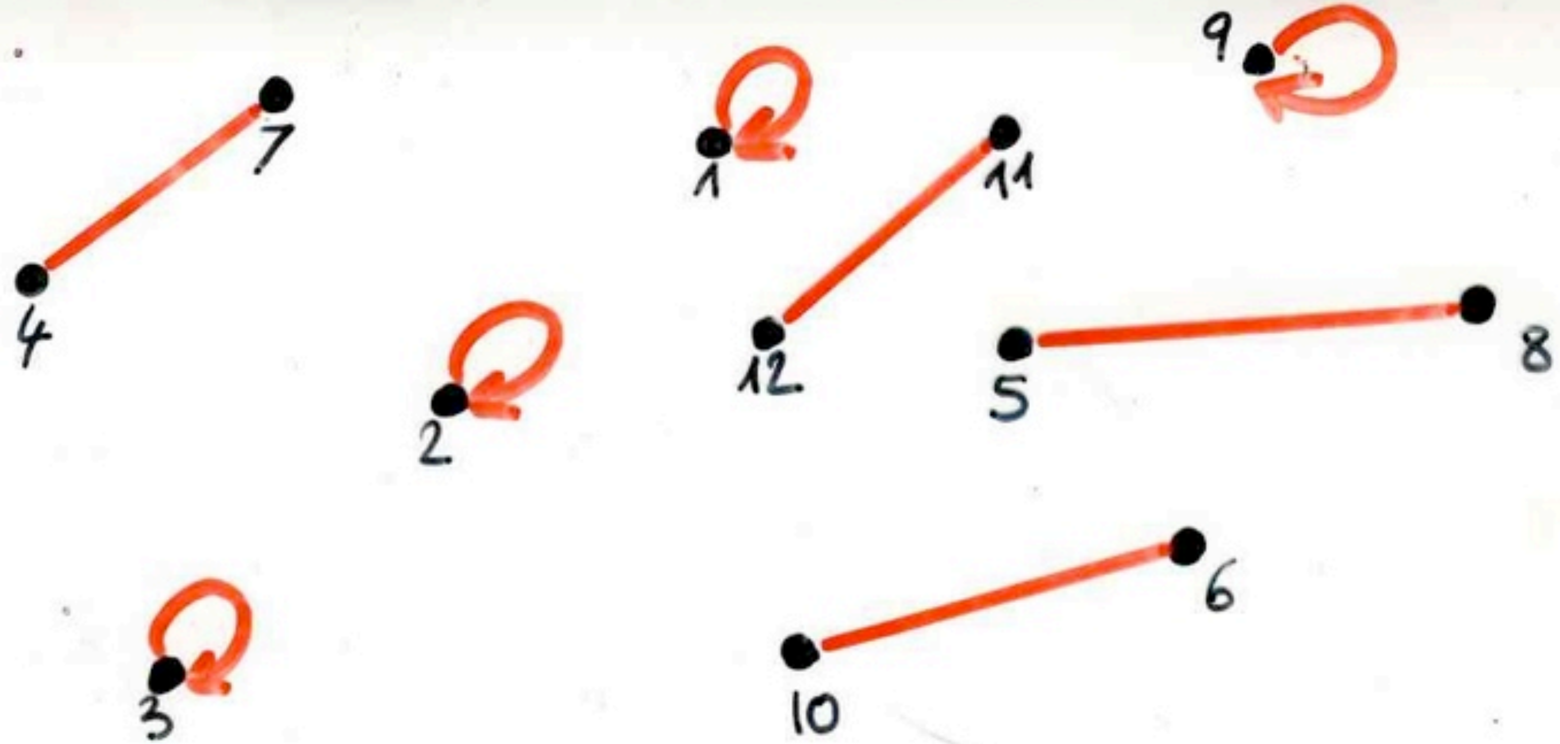
$$\sum_{n\geq 0} H_n(x) H_n(y) \frac{t^n}{n!} = (1-4t^2)^{-1/2} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\sum_{n\geq 0} H_n(x) \qquad \frac{t^n}{n!} =$$

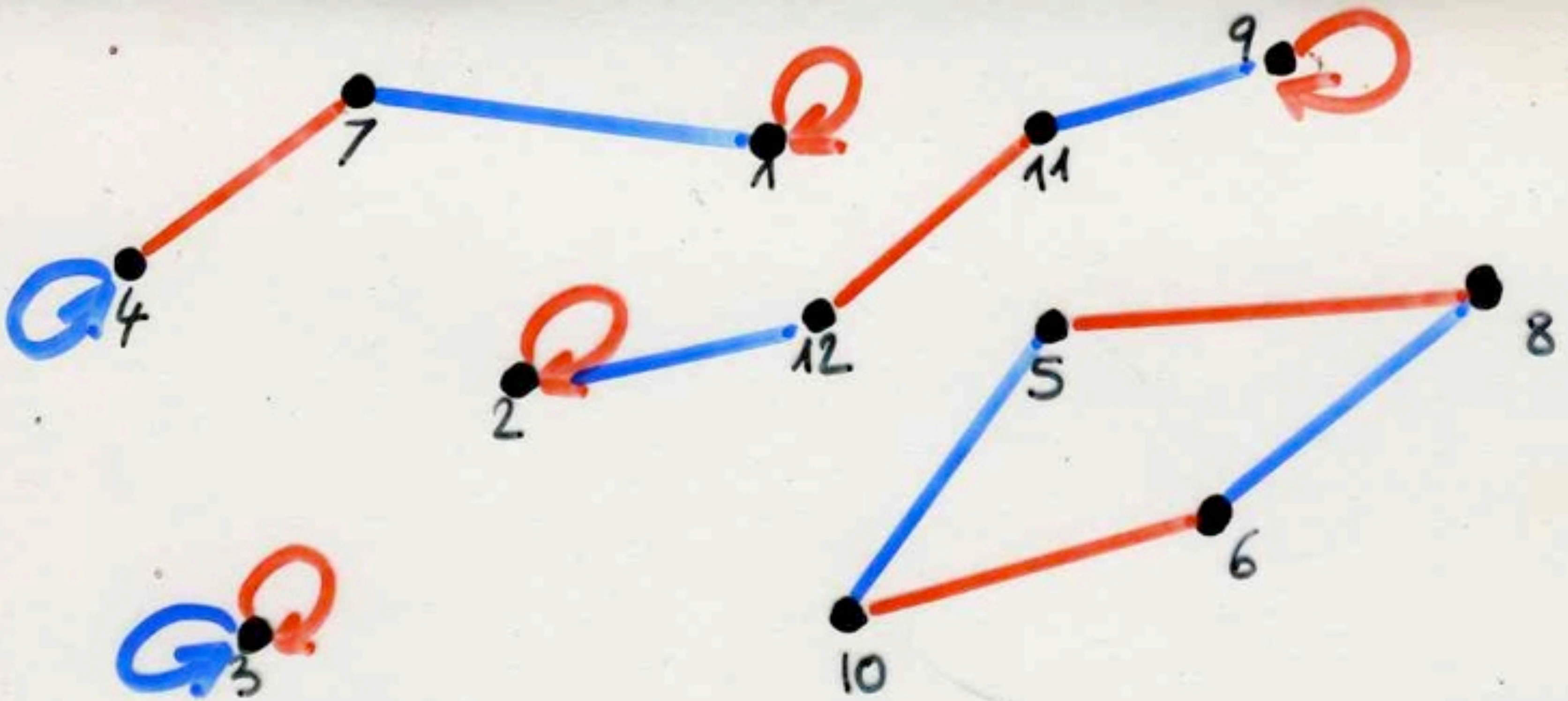
$$H_n(y)$$



9
7
1
11
4
2
12
5
8
6
3
10



$$\sum_{n\geq 0} H_n(x) \frac{t^n}{n!} =$$



$$\sum_{n\geq 0} H_n(x) H_n(y) \frac{t^n}{n!} =$$

$$(1-4t^2)^{-1/2}\exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\exp\left[\frac{1}{2}\log\frac{1}{(1-4t^2)}+\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\exp\left[\frac{1}{2}\log\frac{1}{(1-4t^2)} + \frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\frac{-4\,y^2\,t^2}{1-4t^2}$$

$$\frac{-4\,x^2\,t^2}{1-4t^2}$$

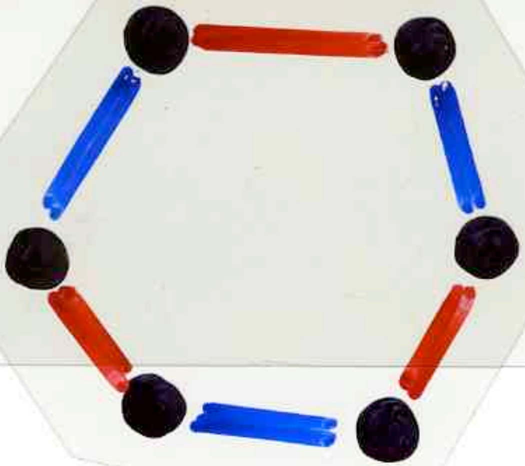
$$\frac{1}{2}\log\frac{1}{(1-4t^2)}$$

$$\frac{4xyt}{1-4t^2}$$

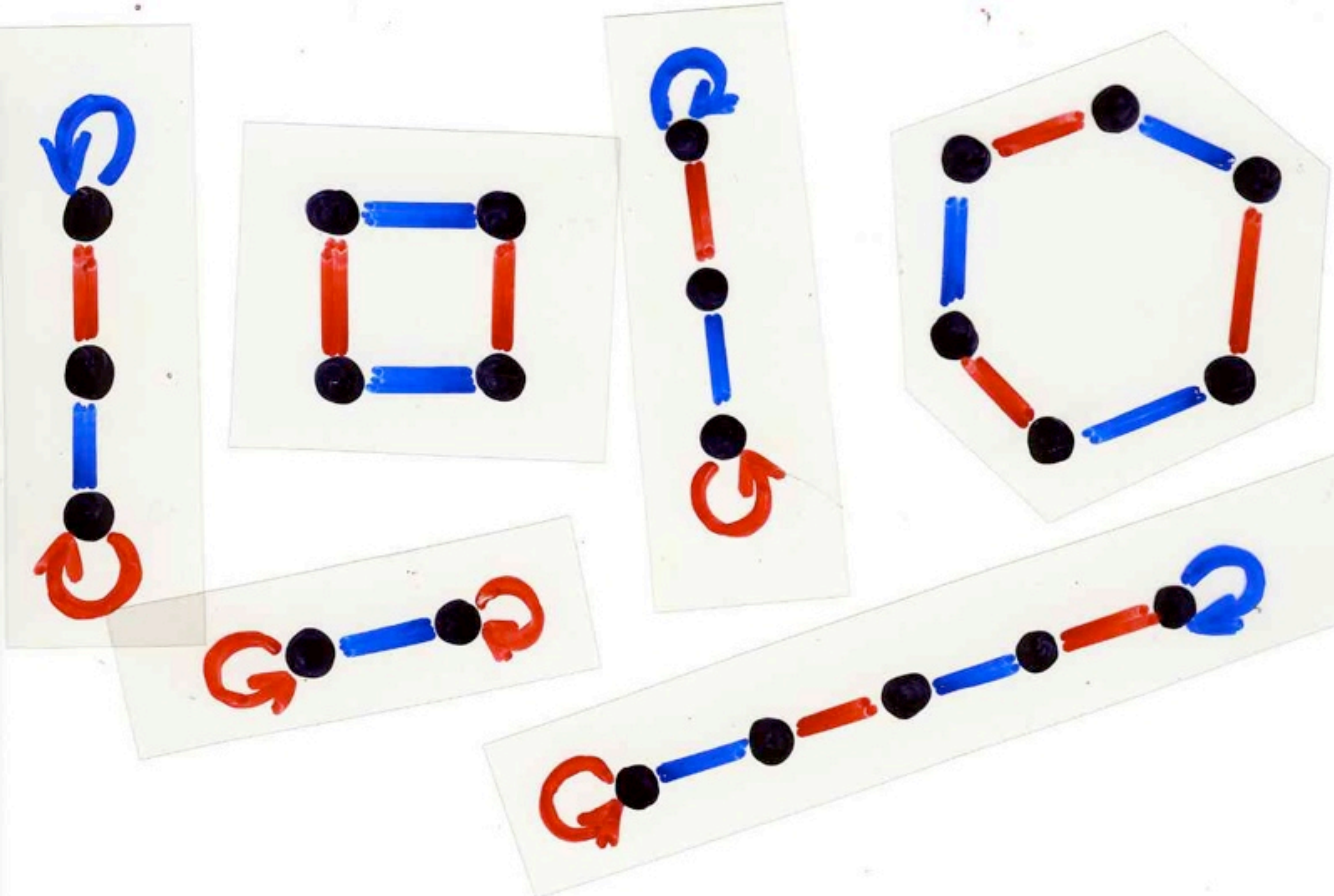
$$\frac{-4x^2\,t^2}{1-4t^2}$$

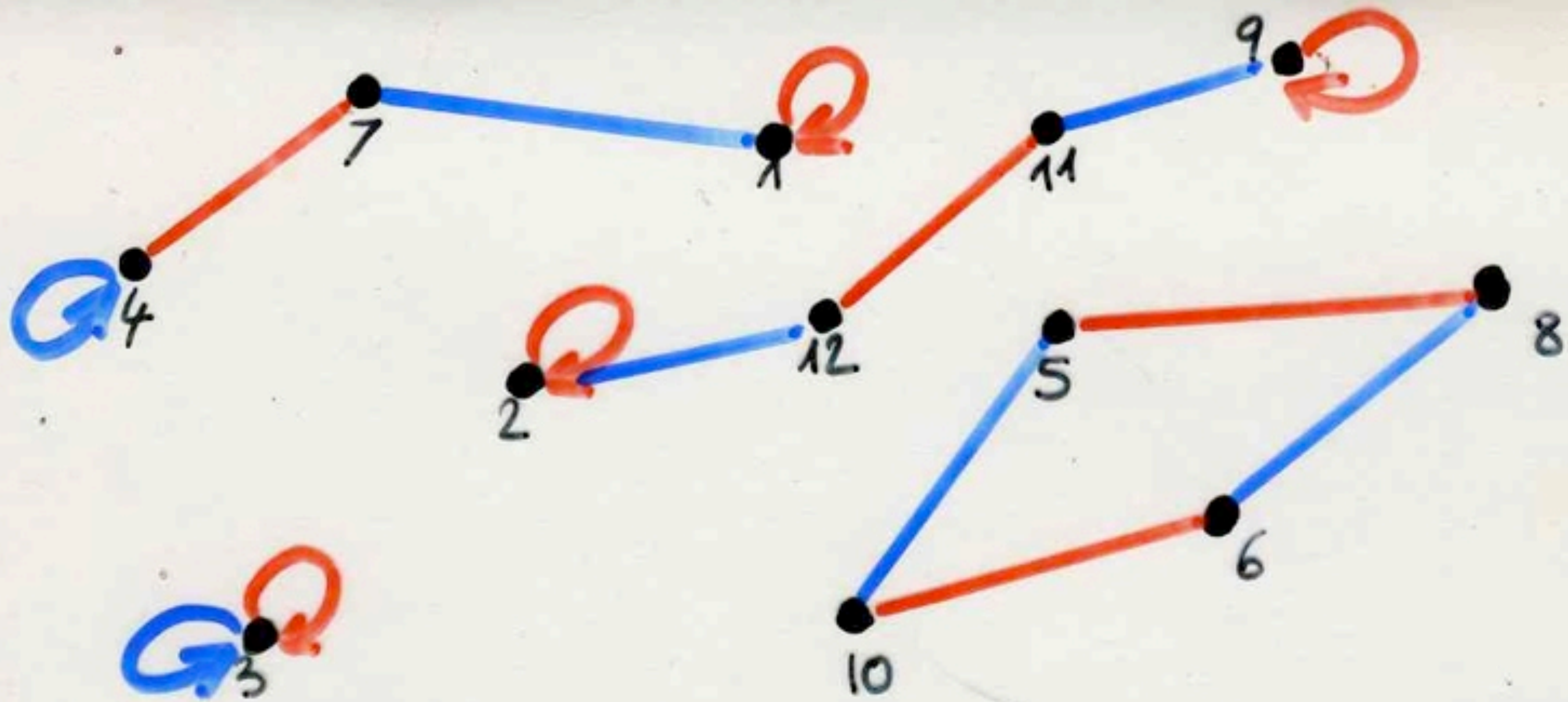
$$\frac{-4\,y^2t^2}{1-4t^2}$$

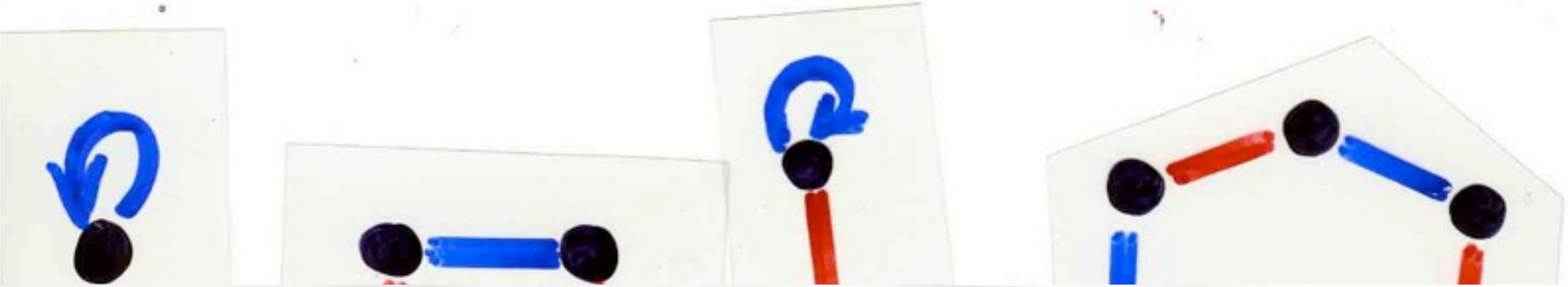
$$\frac{4xyt}{1-4t^2}$$



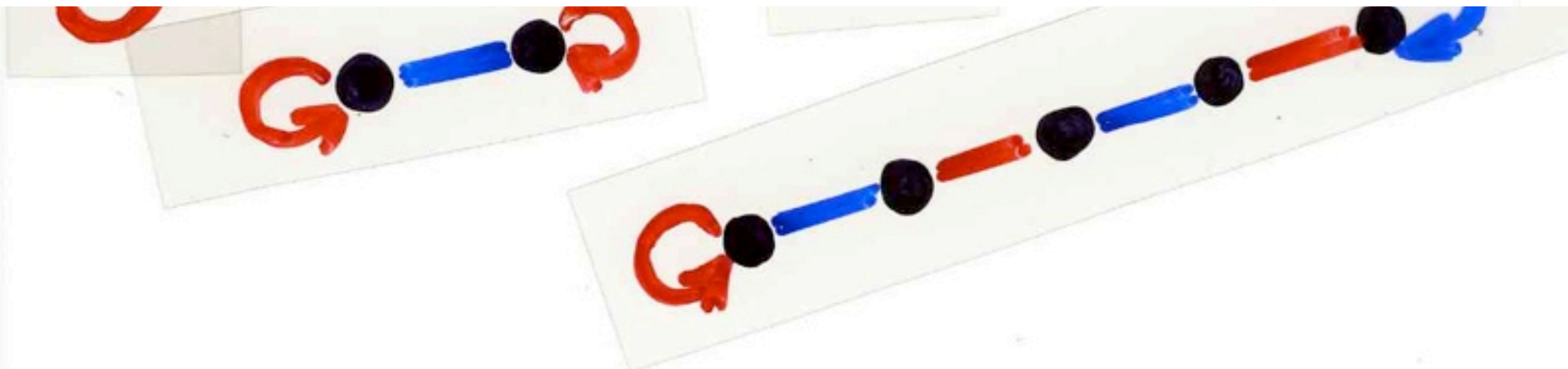
$$\frac{1}{2}\log\frac{1}{(1-4t^2)}$$




7
1
9
11
4
12
2
5
8
6
3
10



$$\sum_{n\geq 0} H_n(x)\, H(y)\, \frac{t^n}{n!} = \left(1-4t^2\right)^{-1/2} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$



the bijective paradigm in physics

random planar maps, geodesics in quadrangulations

P.Di Francesco, J. Bouttier, E. Guitter (2007) (2010)

Razumov - Stroganov (ex) - conjecture 2000-2001

on quantum spin chains XXZ

proof by :

L. Cantini and A.Sportiello (2010)

completely combinatorial proof

combinatorial interpretation of the moments

Thm. (Favard)

- $\{P_n(x)\}_{n\geq 0}$ sequence of monic polynomials, $\deg(P_n)=n$
- $\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ coeff. in $\mathbb{K}$

orthogonality $\Longleftrightarrow$

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

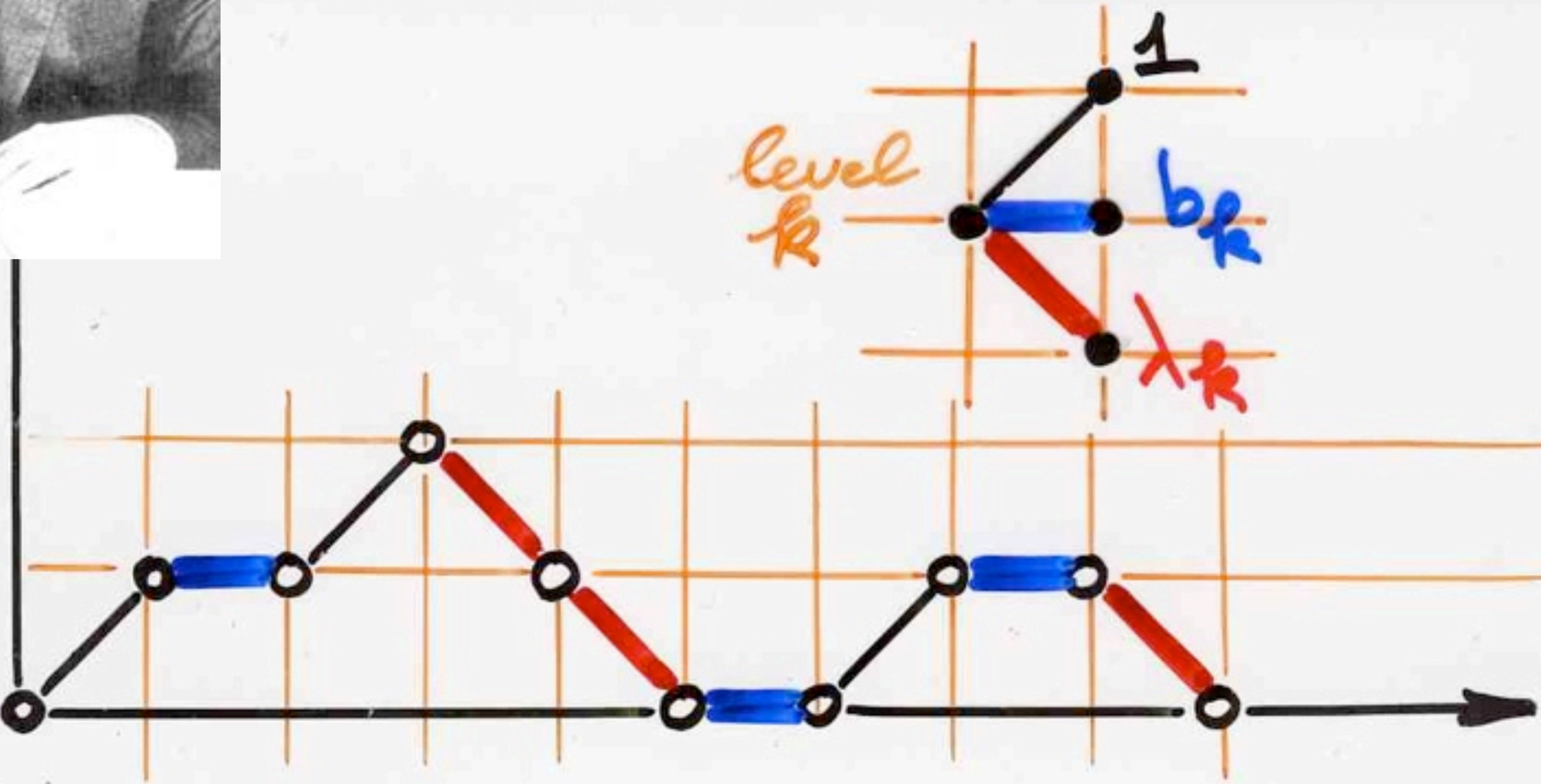
$(\forall k \geq 1)$

3 terms linear recurrence relation

$\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$

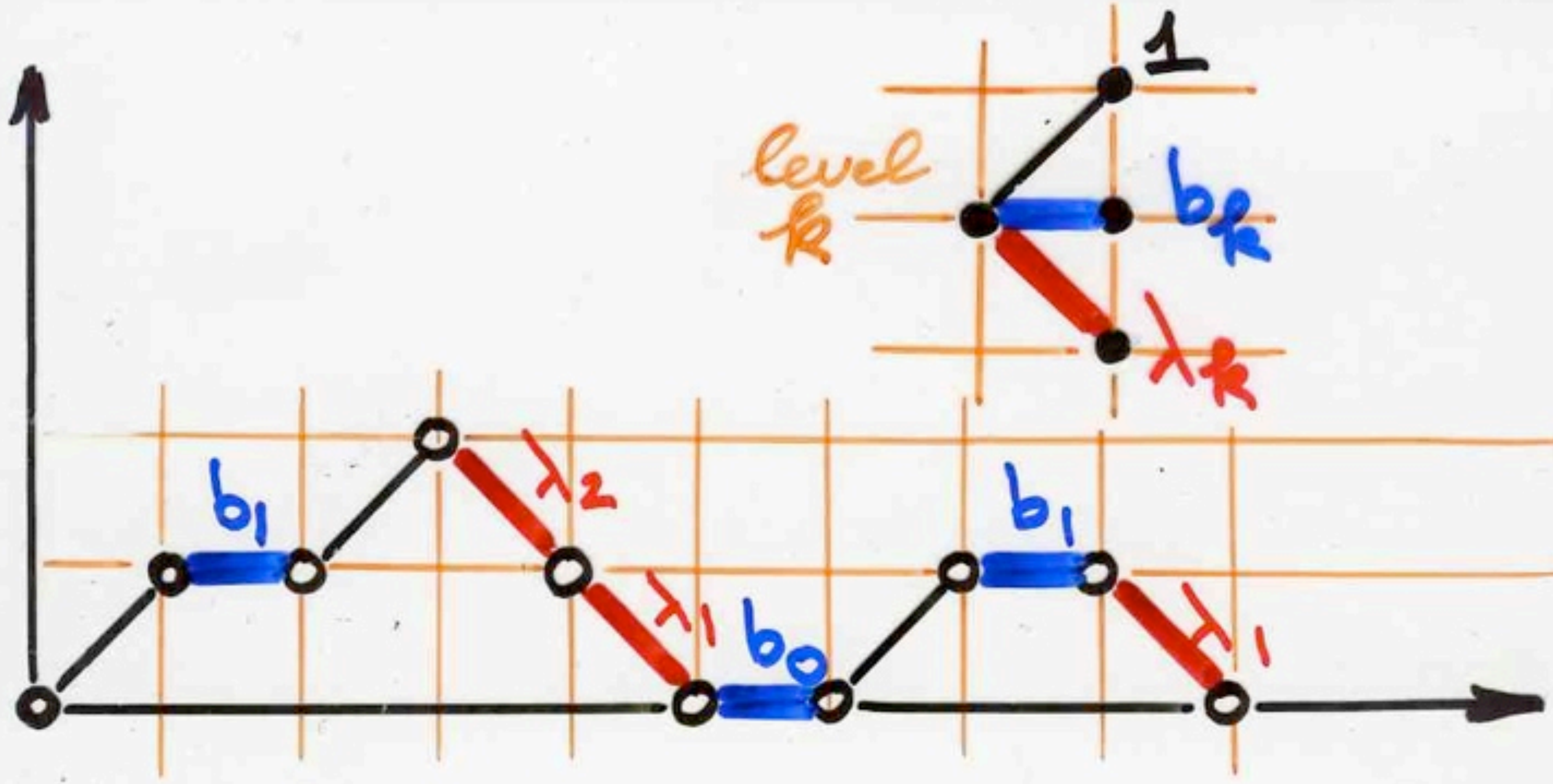
$b_k , \lambda_k \in \mathbb{K}$ ring

valuation v



ω Motzkin path

valuation v



ω Motzkin path

$$v(\omega) = b_0 \, b_1^2 \, \lambda_1^2 \, \lambda_2$$

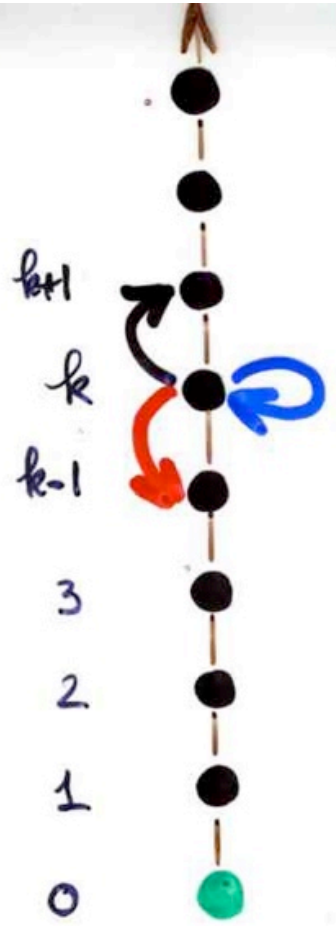
$$f(x^n) = \mu_n \quad (n \geq 0)$$

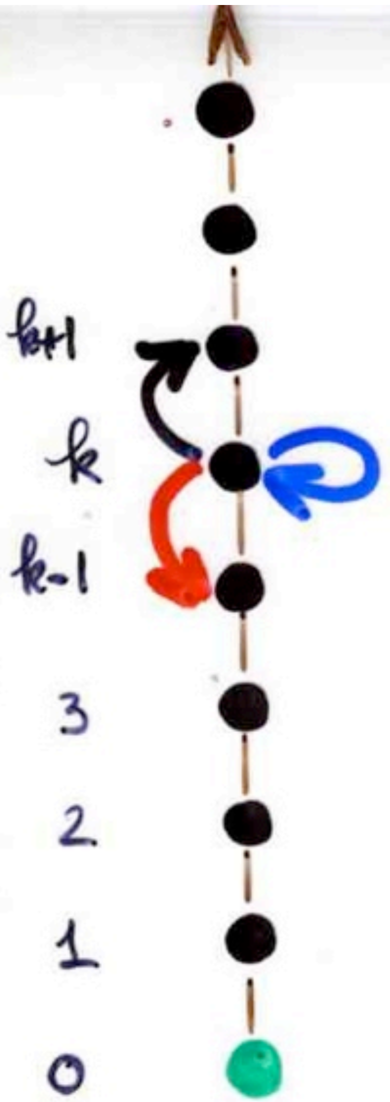
moments

$$\mu_n = \sum_{\omega} v(\omega)$$

ω Motzkin path

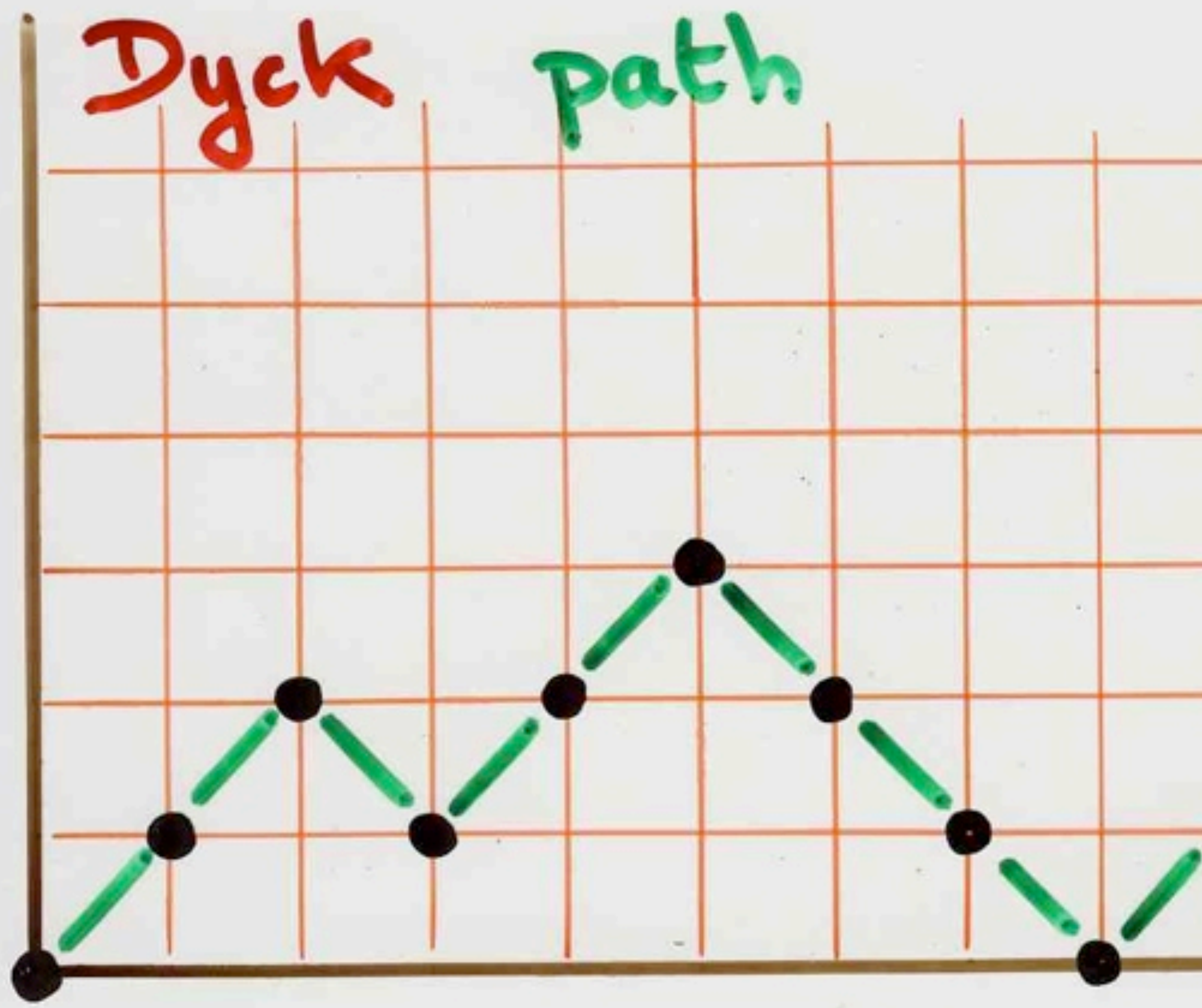
$|\omega| = n$

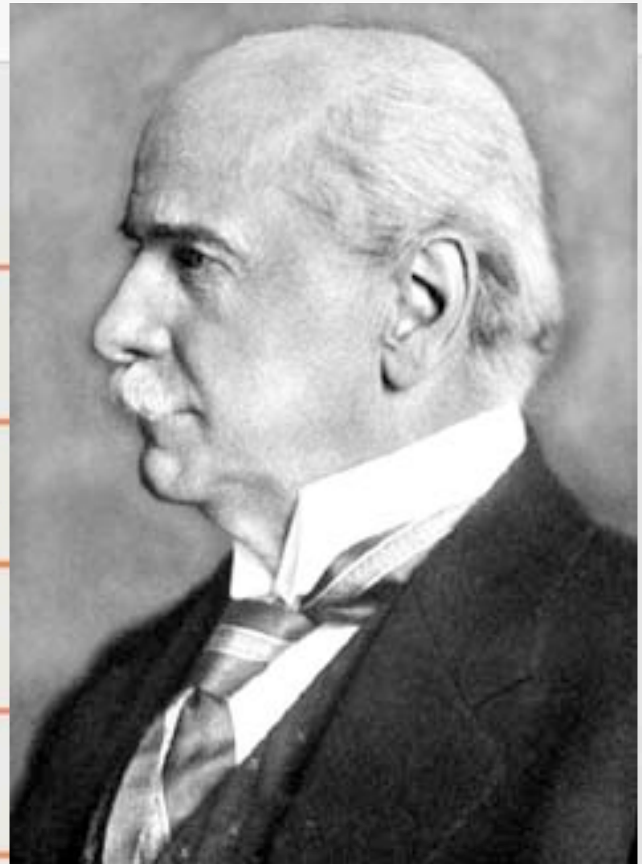

k+1
k
k-1
3
2
1
0

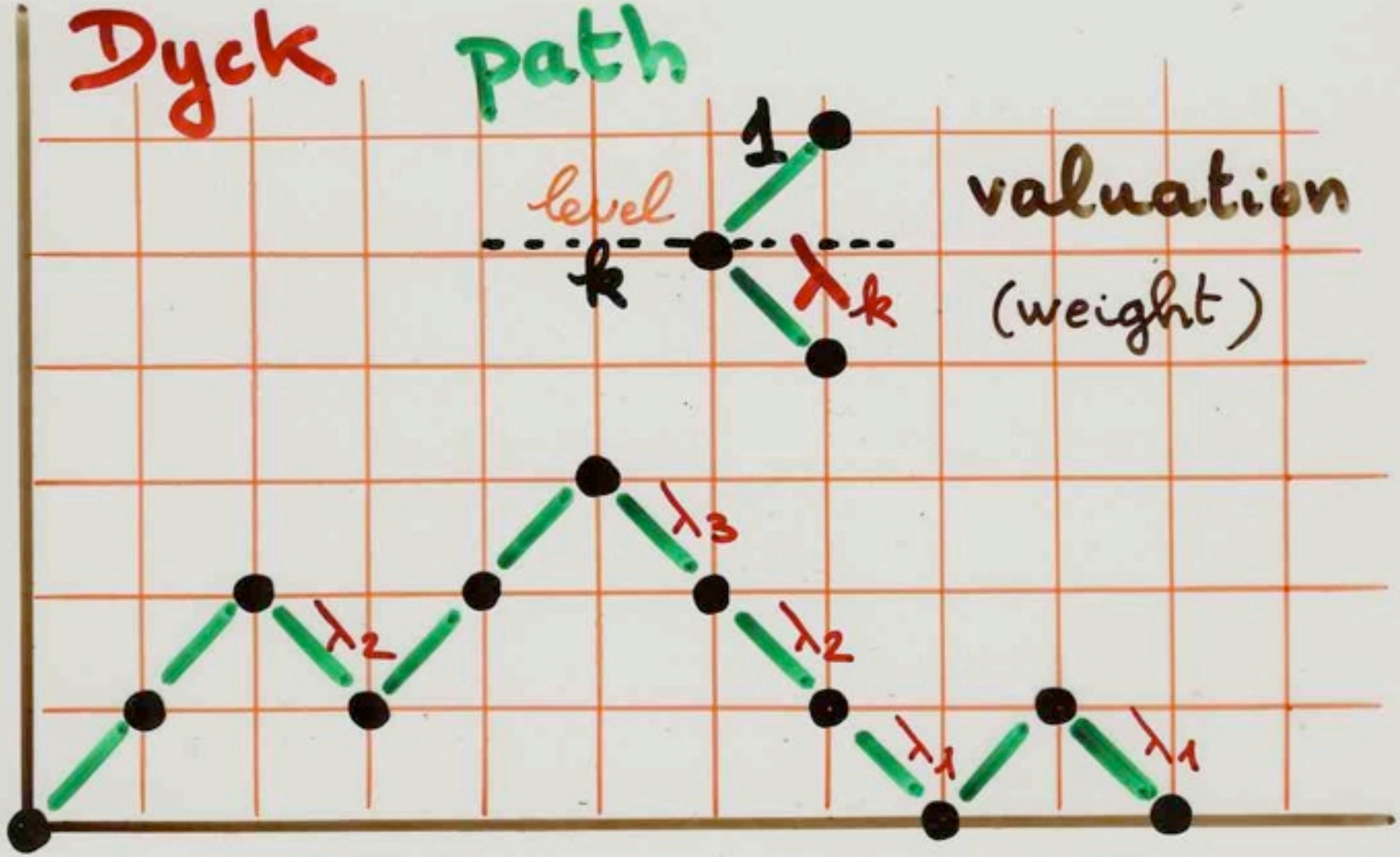


Tridiagonal matrix

$$A = \begin{bmatrix} b_0 & 1 & & & & \\ \lambda_1 & b_1 & 1 & & & \\ & \lambda_2 & b_2 & 1 & & \\ & & \lambda_3 & b_3 & 1 & \\ & & & \lambda_4 & \ddots & \ddots \\ & & & & \ddots & \ddots \end{bmatrix}$$


Dyck path





weight $v(\omega) = \lambda_1^2 \lambda_2^2 \lambda_3$

combinatorial

theory of

orthogonal polynomials

and

continued fractions

Flajolet (1980) Viennot (1983, ...)

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots\cdots}}}}}$$

$J(t; b, \lambda)$

Jacobi continued fraction

$b = \{b_k\}_{k\geq 0}$ $\lambda = \{\lambda_k\}_{k\geq 1}$

continued fractions

$$\sum_{n \geq 0} \mu_n t^n = \cfrac{1}{1 - \cfrac{\lambda_1 t}{1 - \cfrac{\lambda_2 t}{\cdots\cdots\cdot \cfrac{}{1 - \cfrac{\lambda_k t}{\cdots\cdots}}}}}$$

$\mu_0 = 1$

$$\underbrace{\qquad\qquad}_{S(t;\lambda)}$$

Stieltjès continued fraction

classical theory

continued fractions

orthogonal polynomials

J-fraction

$$J(t) = \cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots}}}}$$

$$P_{k+1}(x) = (x-b_k)P_k(x)-\lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

convergents

$$J_k(t) = \frac{\delta P_k^*(x)}{P_{k+1}^*(x)}$$

classical theory

continued fractions | **orthogonal polynomials**

J-fraction

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-b_0 t-\cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1-b_k t-\cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

moments generating function

$$P_{k+1}(x) = (x-b_k)P_k(x)-\lambda_k P_{k-1}(x)$$

$$f(x^n)=\mu_n$$

moments

convergents

$$J_k(t) = \frac{\delta P_k^*(x)}{P_{k+1}^*(x)}$$

$$\mu_n = \sum_{\omega} v(\omega)$$

ω Motzkin path

$|\omega| = n$

combinatorial theory of orthogonal polynomials and continued fractions

Flajolet (1980) Viennot (1983, ...)

Françon, XGV (1978)

- théorie combinatoire des polynômes orthogonaux généraux (x.g.v.)

Publications du LACIM
UQAM (Université du Québec à Montréal)
Notes de conférences (1984) 214 p.
réédition : n° hors série

http://www.labri.fr/perso/viennot/

example:
Hermite polynomials

moments
Hermite
polynomials

Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of
involutions
no fixed point
on $\{1, 2, \ldots, 2n\}$

chord diagrams
perfect matching

Hermite history

$$h = (\omega \; ; \; f)$$

ω: Dyck path

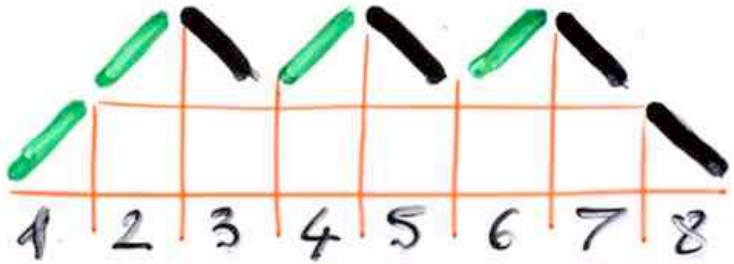
f: choice function

$\omega = \omega_1 \ldots \omega_{2n}$

$f = (p_1, \ldots, p_{2n})$

$p_i = 1$

$1 \le p_i \le v(\omega_i) = \lambda_{k_i}$



pos	1	1	1	1	2	1	1	1

Hermite history

$$h = (\omega \; ; \; f)$$

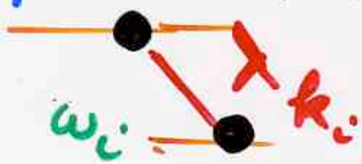
ω: Dyck path ; f: choice function

$\omega = \omega_1 \dots \omega_{2n}$

$f = (p_1, \dots, p_{2n})$

$p_i = 1$

$1 \le p_i \le v(\omega_i) = \lambda_{k_i}$



$\mu_{2n+1} = 0$

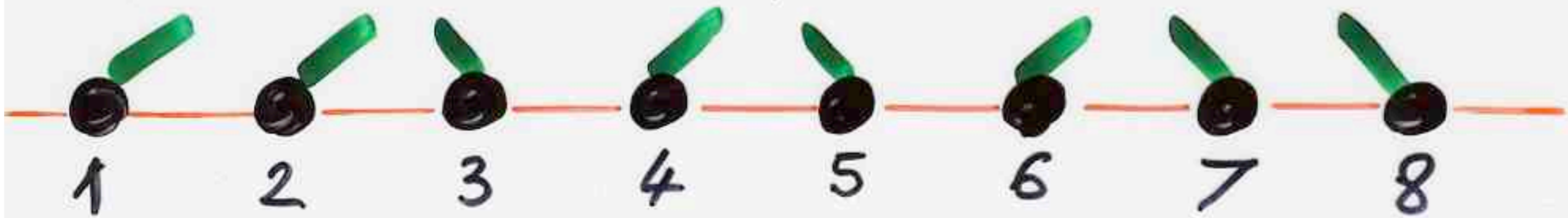
$\mu_{2n} = 1 \cdot 3 \cdot \dots \cdot (2n-1)$

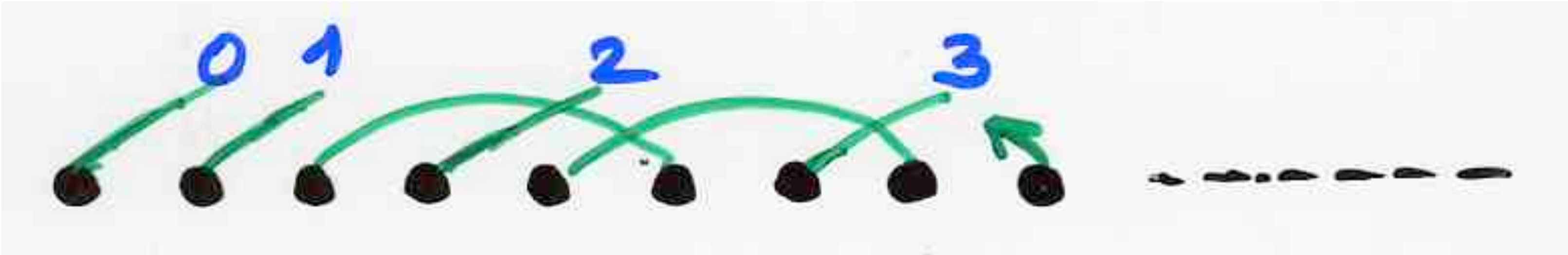
number of involutions no fixed point on $\{1, 2, \dots, 2n\}$

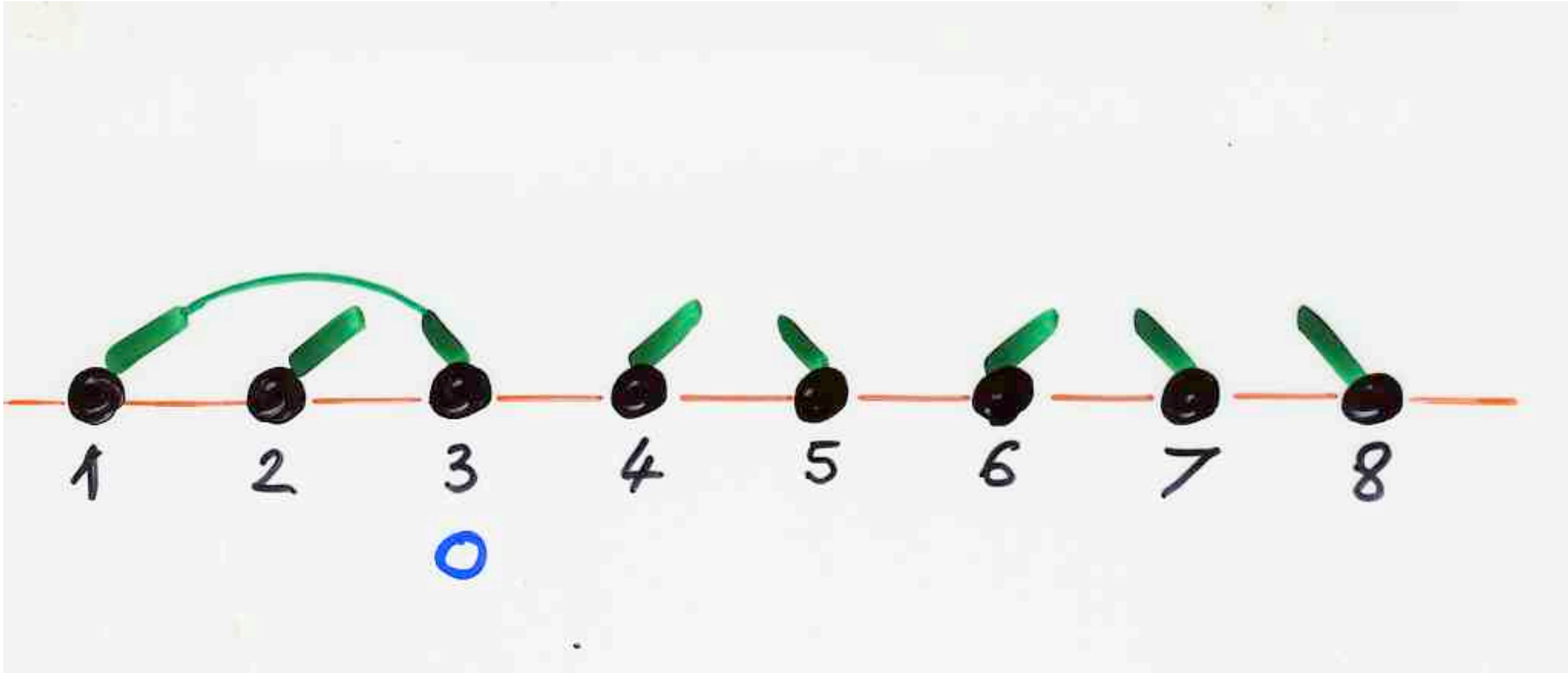
chord diagrams

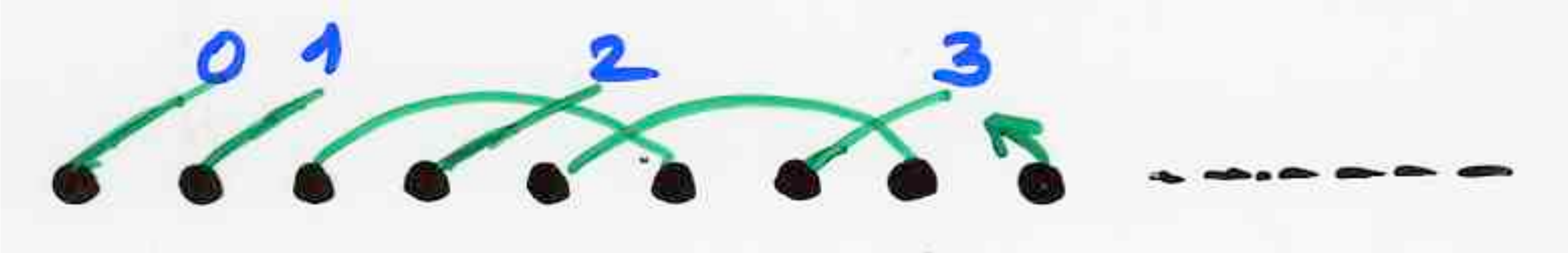
perfect matching

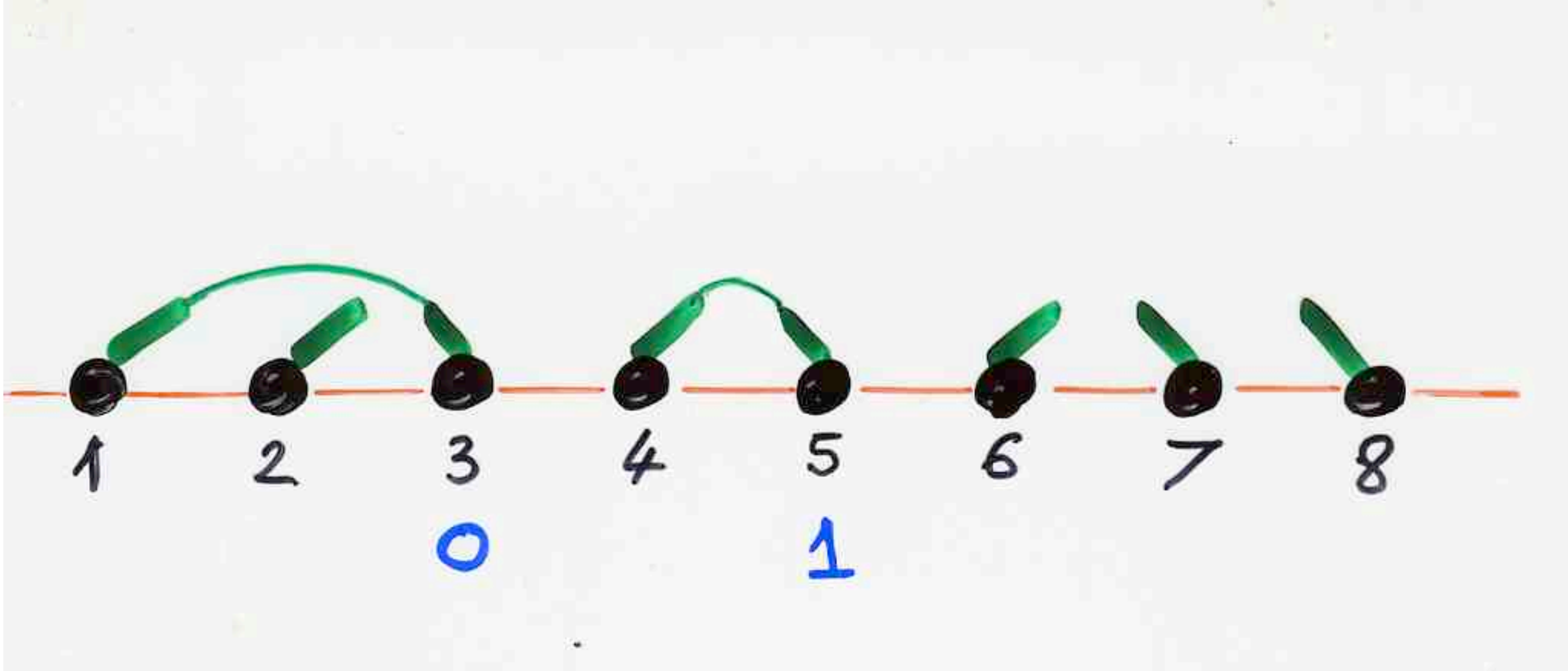
"histories"

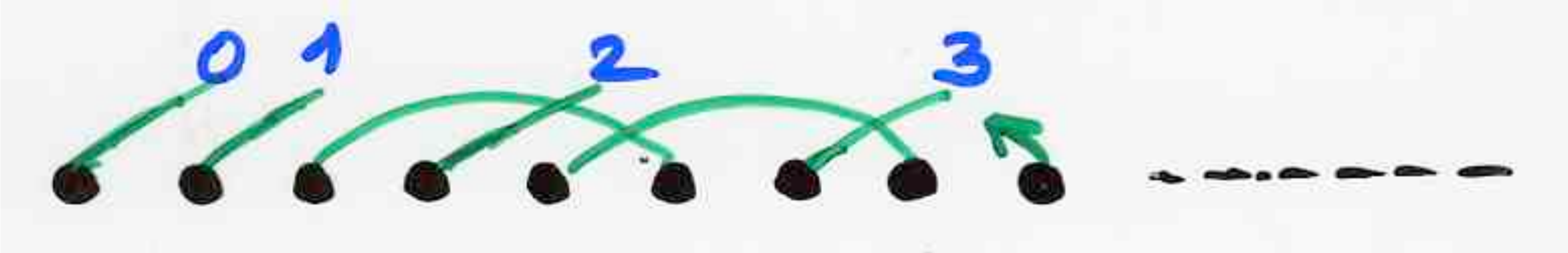

1
2
3
4
5
6
7
8

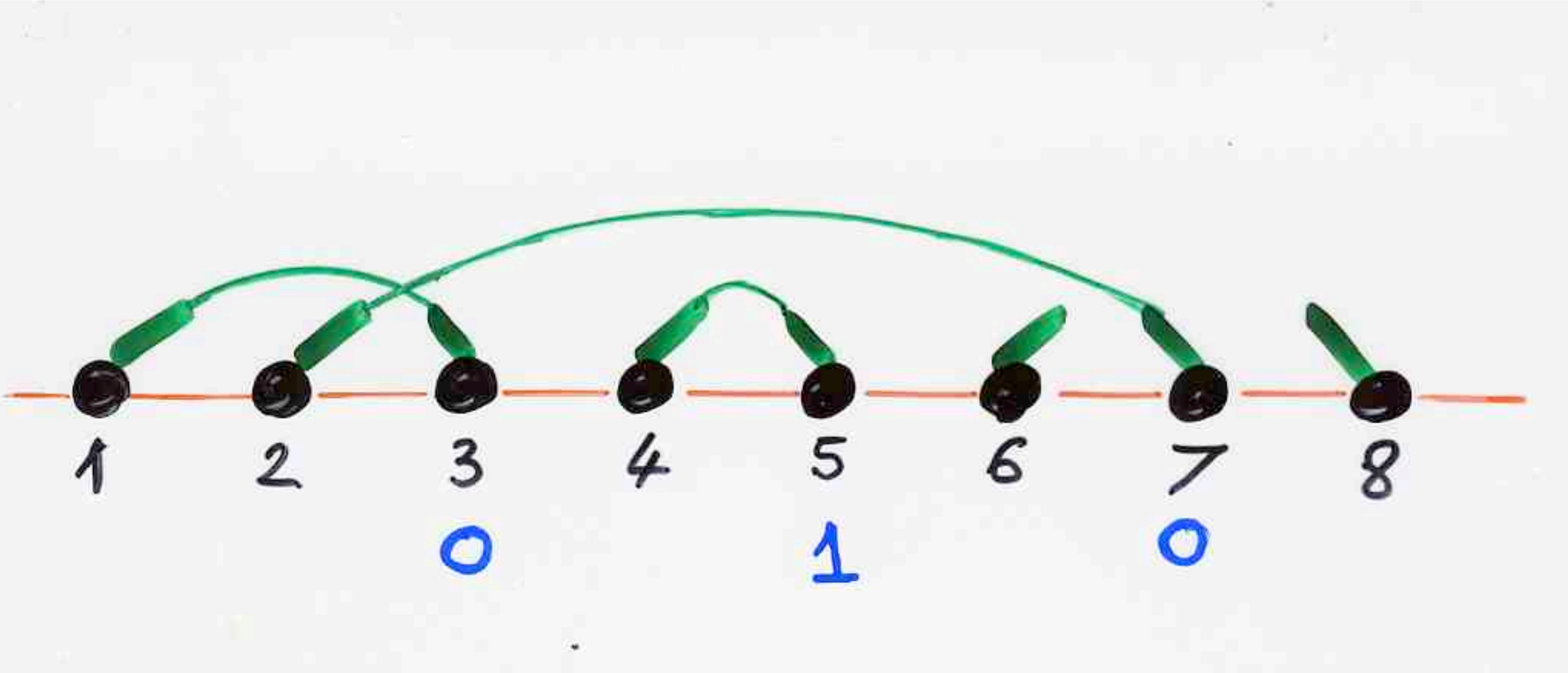

0
1
2
3

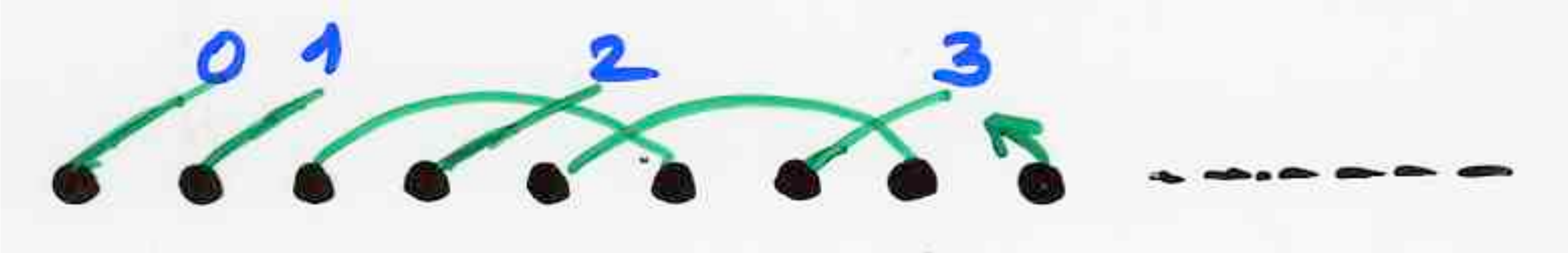

1
2
3
4
5
6
7
8
0


0 1 2 3

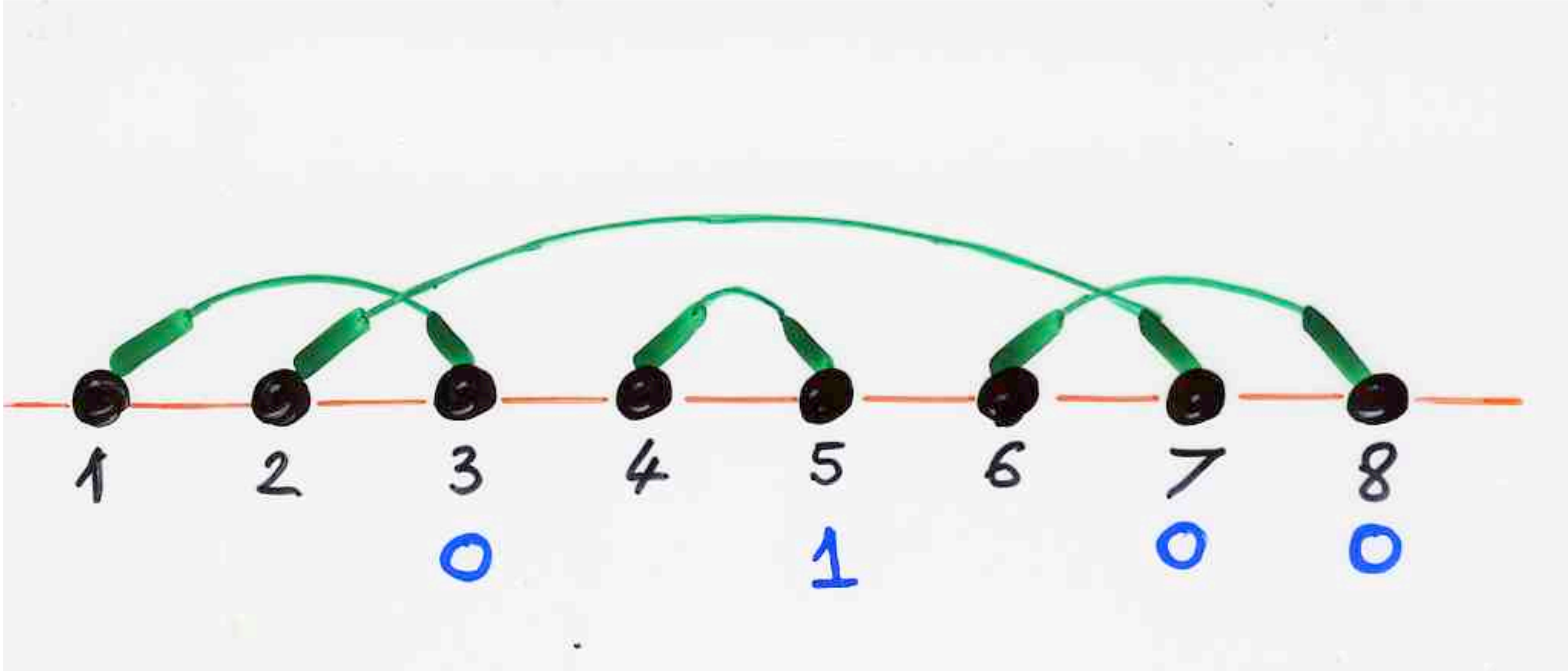

1 2 3 4 5 6 7 8
0 1


0
1
2
3


1
2
3
4
5
6
7
8
0
1
0



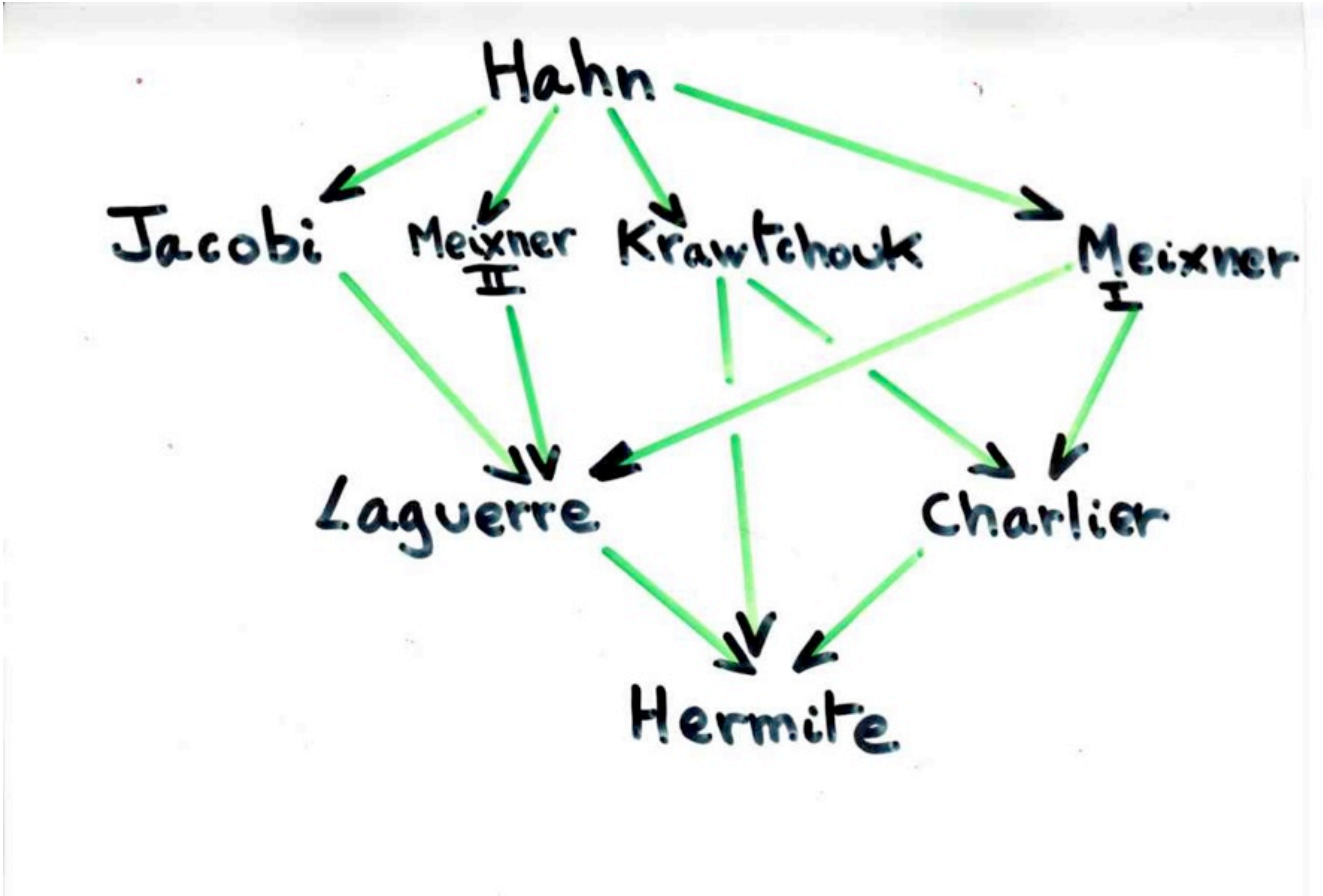
0
1
2
3



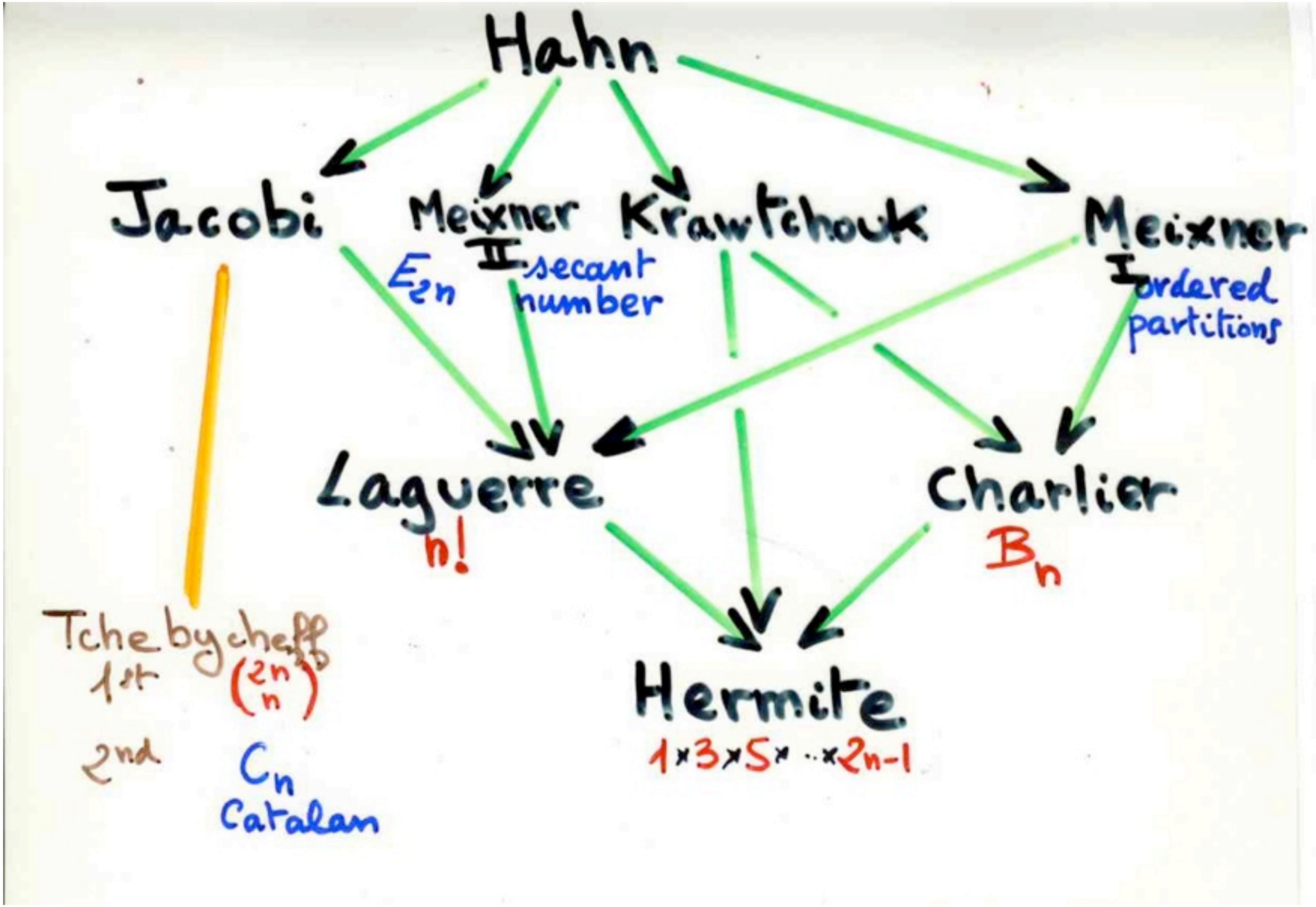
1
2
3
4
5
6
7
8
0
1
0
0

Askey tableau

Askey-Wilson



Askey-Wilson



Laguerre histories

The FV bijection

Françon-XV 1978

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$P_0 = 1 \qquad P_1 = x - b_0$$

$$\mu_n = (n+1)!$$

$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

Laguerre polynomial

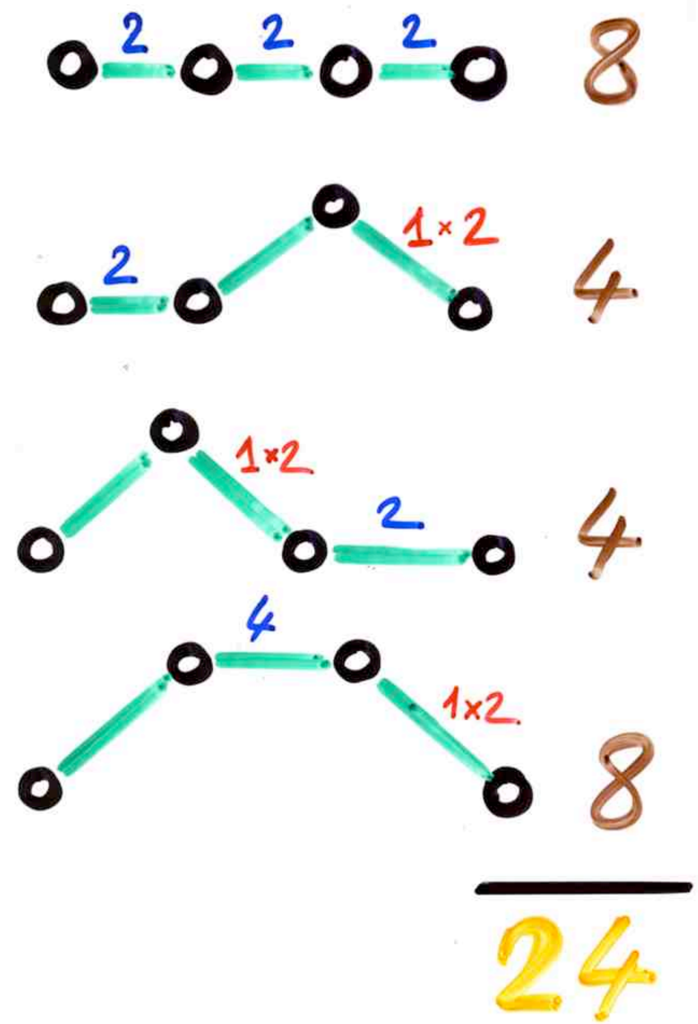
$$J(t) = \cfrac{1}{1 - 2t - \cfrac{1\cdot 2t^2}{1 - 4t - \cfrac{2\cdot 3t^2}{\cdots}}}$$

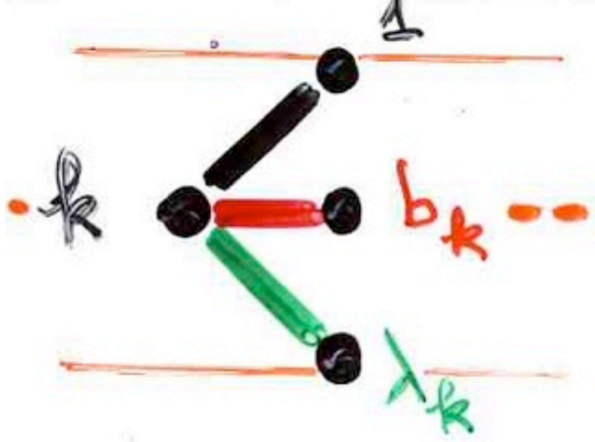
Laguerre $L_n^{(1)}(x)$

moment $\mu_n = (n+1)!$

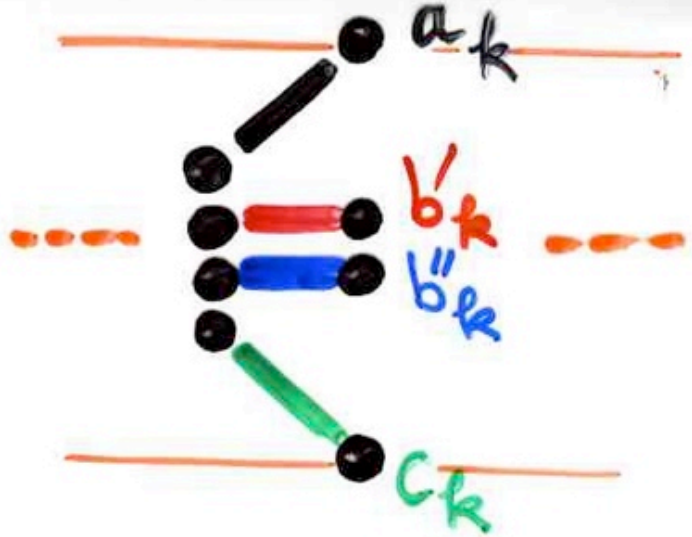
$b_k = 2k+2$

$\lambda_k = k(k+1)$

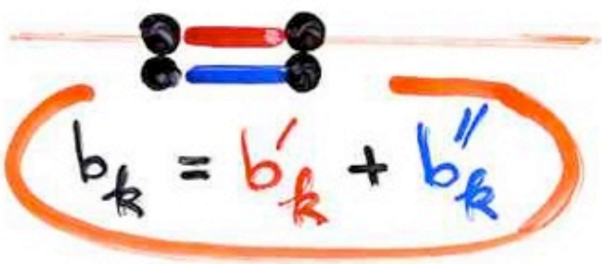




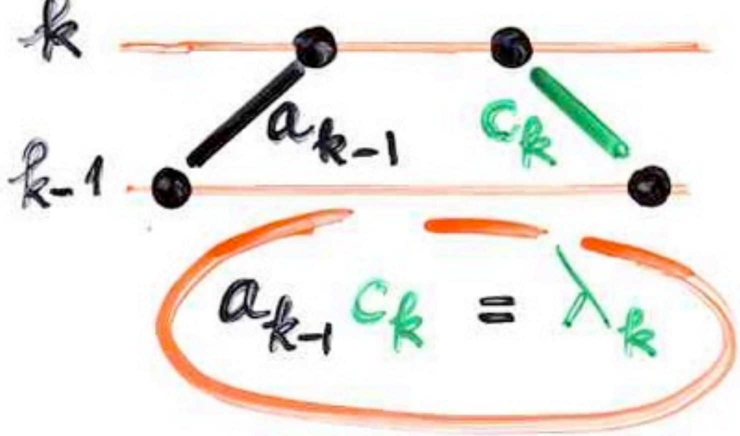
1
k
b_k
λ_k



a_k
b'_k
b''_k
c_k



$b_k = b'_k + b''_k$



k
k-1
a_{k-1}
c_k
$a_{k-1} c_k = \lambda_k$

Laguerre $L_n^{(1)}(x)$

$$\mu_n = (n+1)!$$

$$a_k = k+1$$
$$b'_k = k+1$$
$$b''_k = k+1$$
$$c_k = k+1$$

Bijection Permutations $n+1$

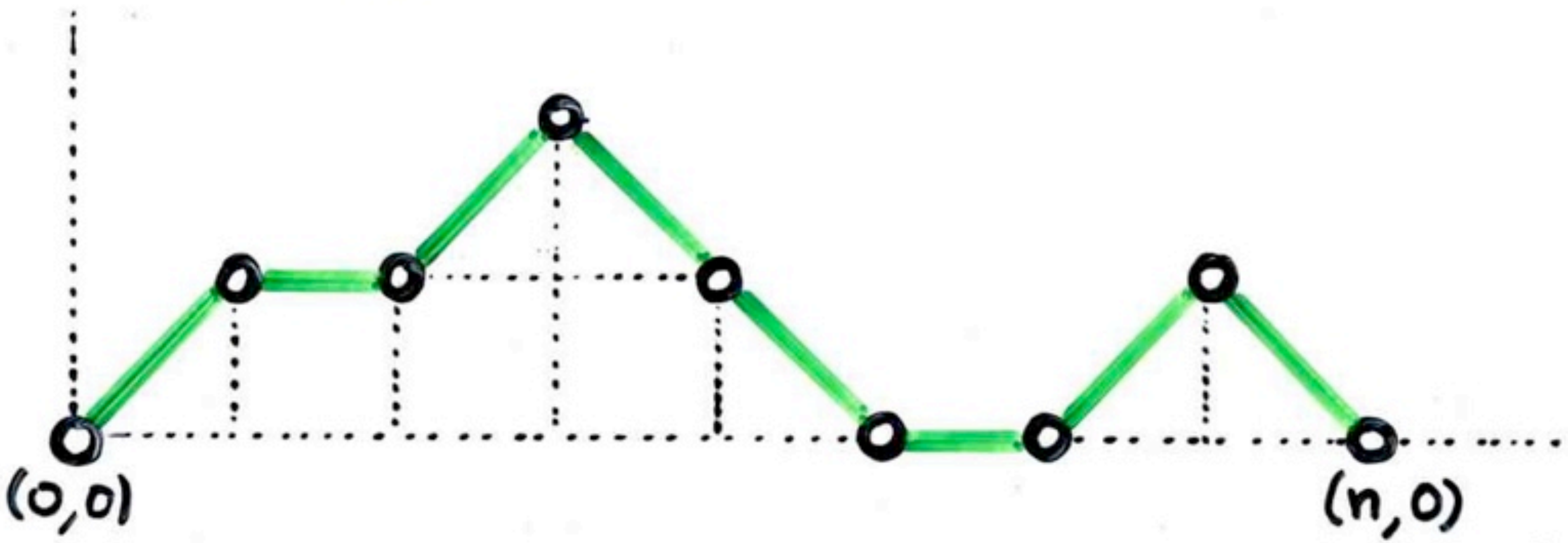
Histoires de Laguerre $(\gamma_c ; f)$ n

Bijection **Permutations** $n+1$

Histoires de Laguerre $(\gamma_c ; f)$ n

Chemin de **Motzkin**

n[illegible]

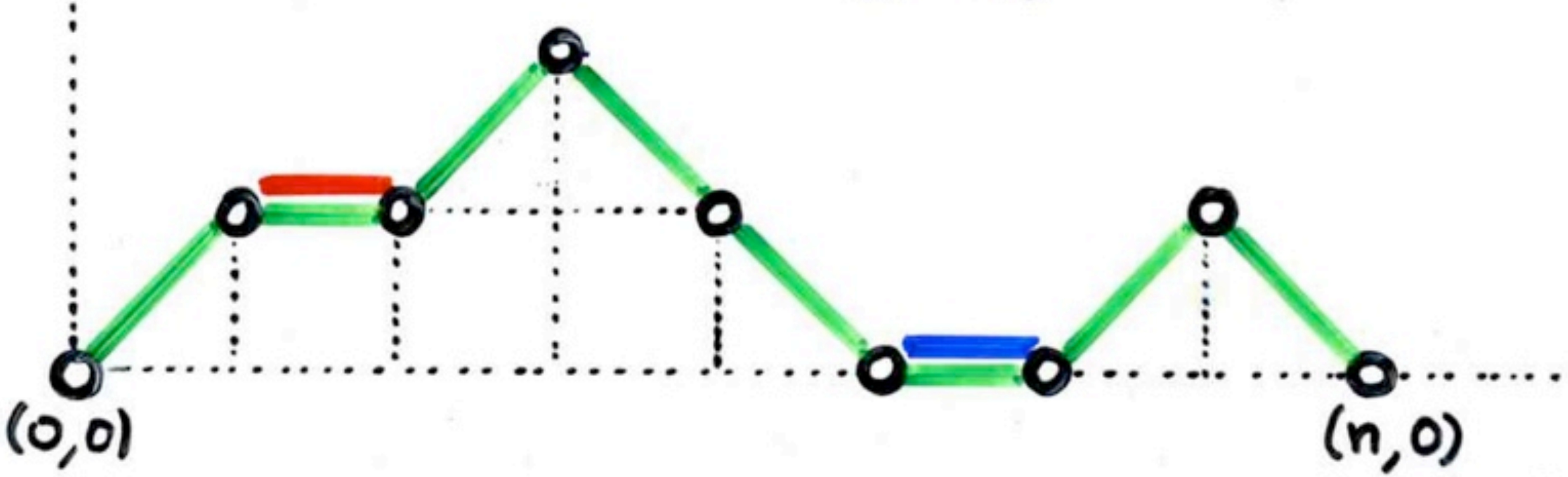


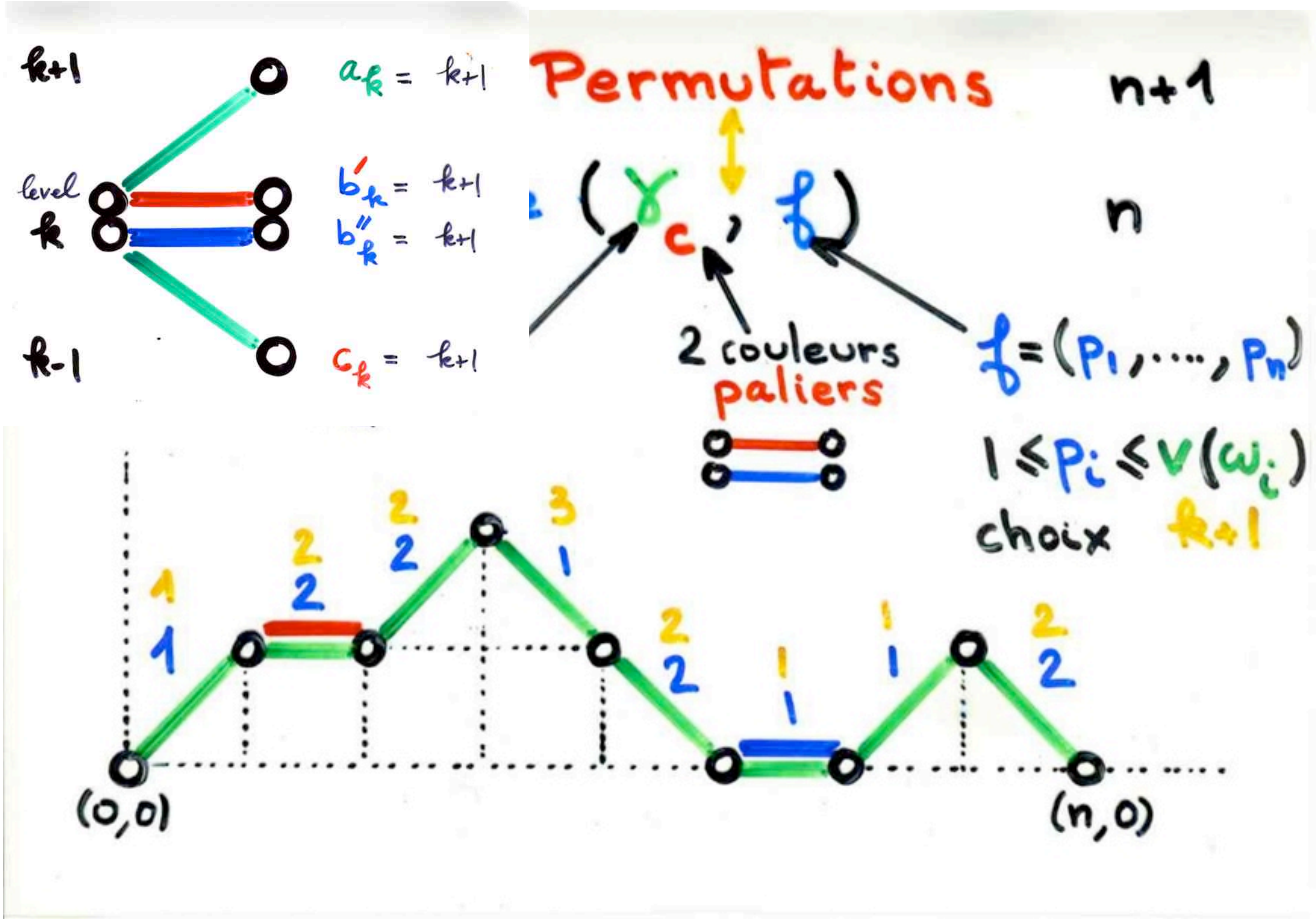
Bijection **Permutations** $n+1$

Histoires de Laguerre $(\gamma_c \, ; f)$ n

Chemin de Motzkin $n-1$

2 couleurs paliers

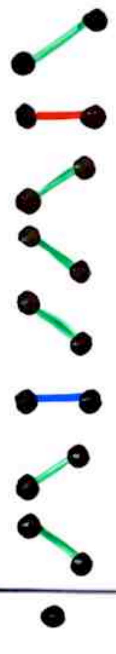




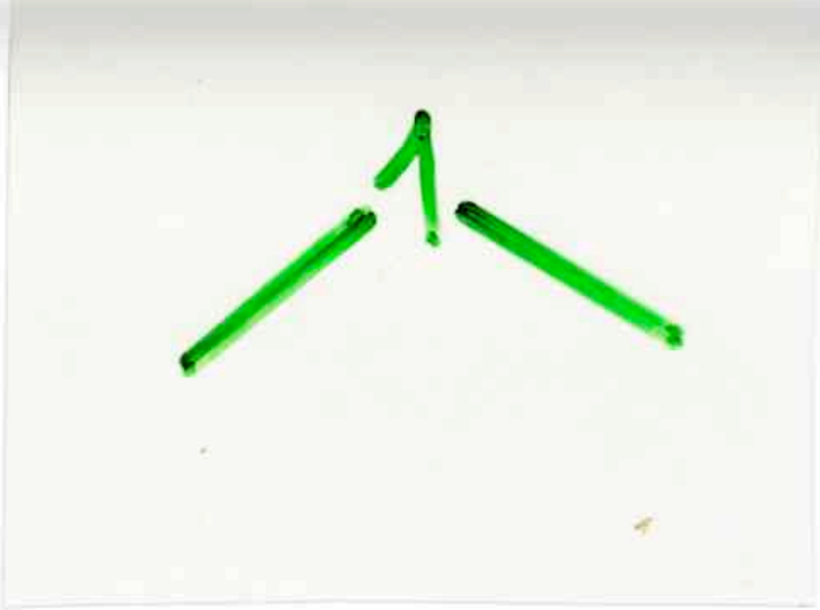
k+1
level
k
k-1
a_k = k+1
b'_k = k+1
b''_k = k+1
c_k = k+1
Permutations
n+1
(γ_c , f)
n
2 couleurs
paliers
f = (p_1, ..., p_n)
1 ≤ p_i ≤ v(ω_i)
choix k+1
(0,0)
(n,0)

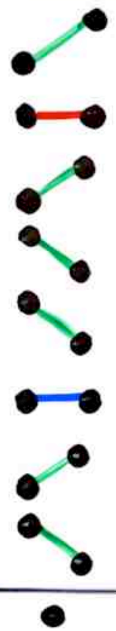
Bijection

Histoires de Laguerre $(\omega; (p_1, \ldots, p_n)) \longleftrightarrow$ permutations $(n+1)!$

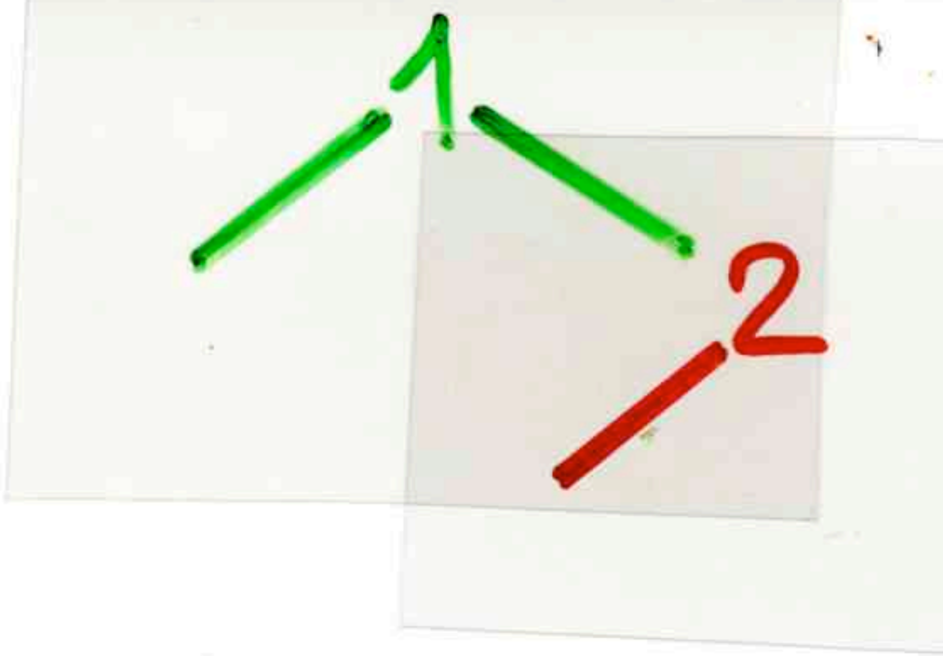


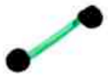
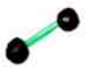
x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9			

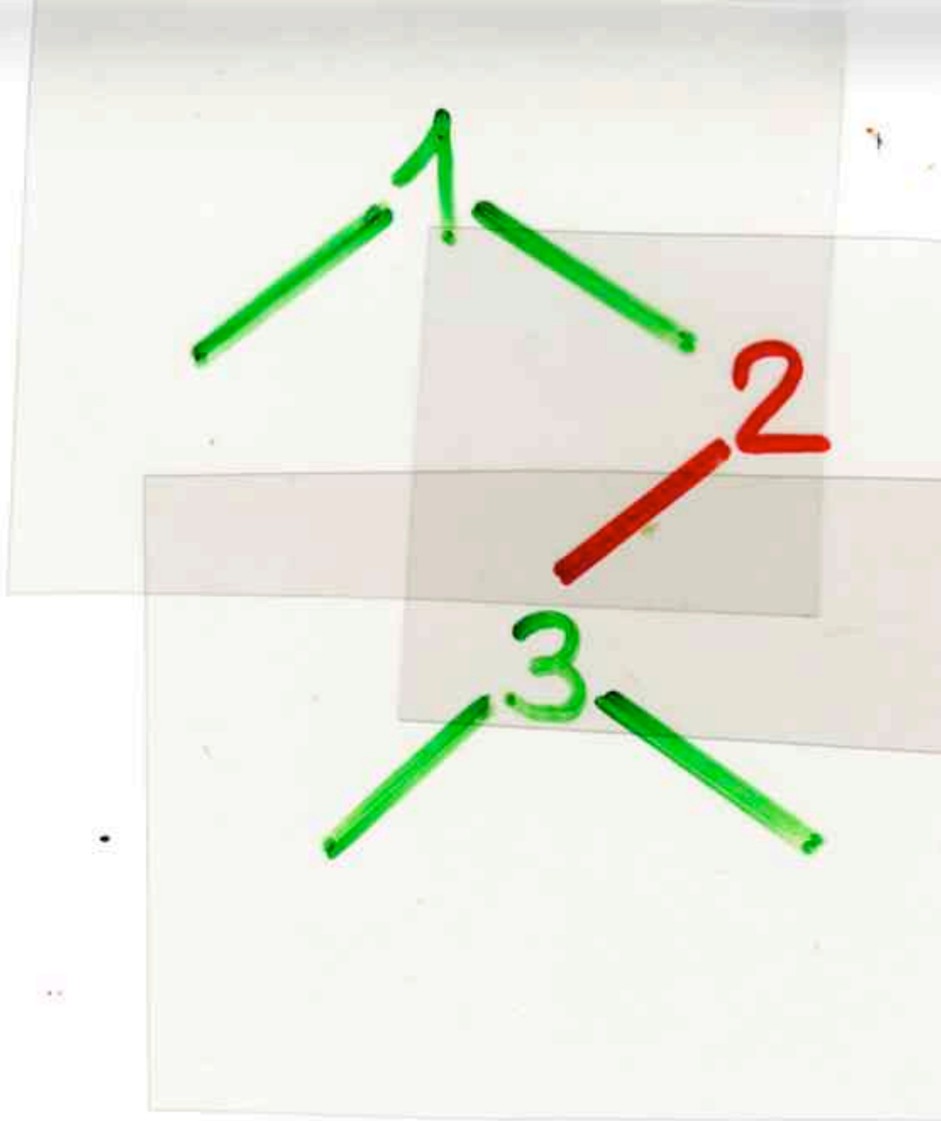


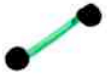
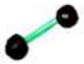


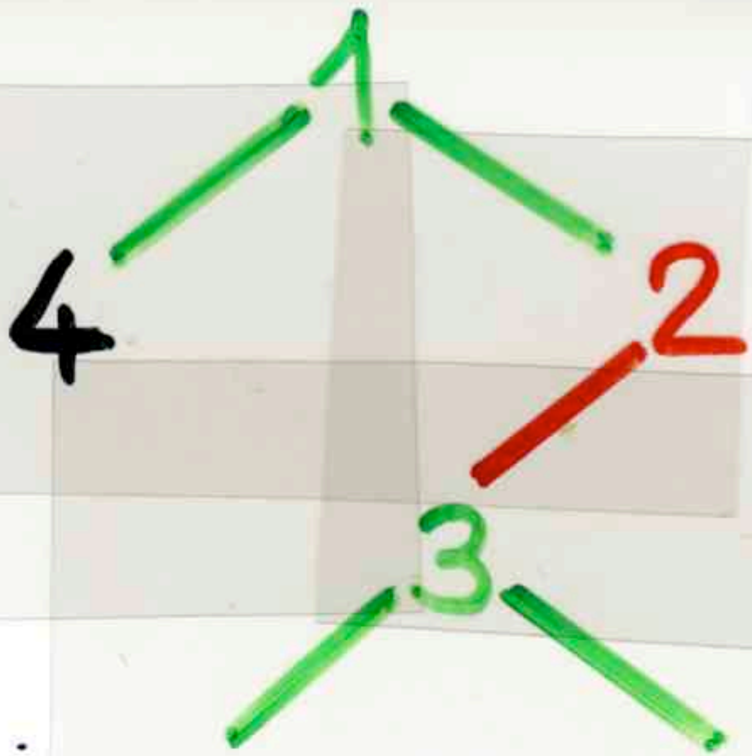
x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9			



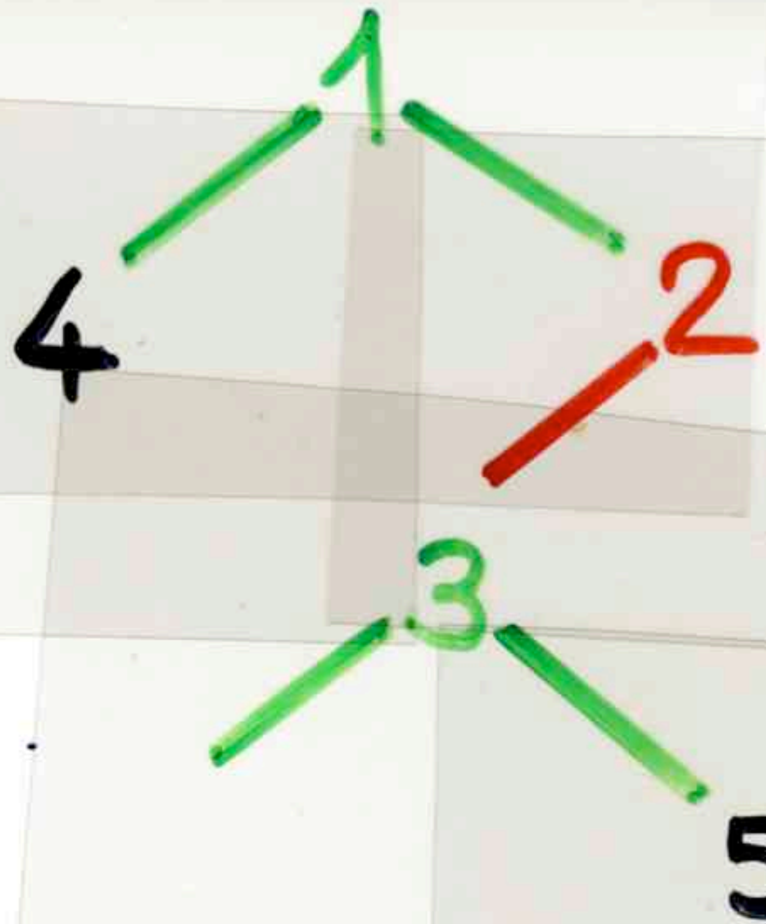
x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9			



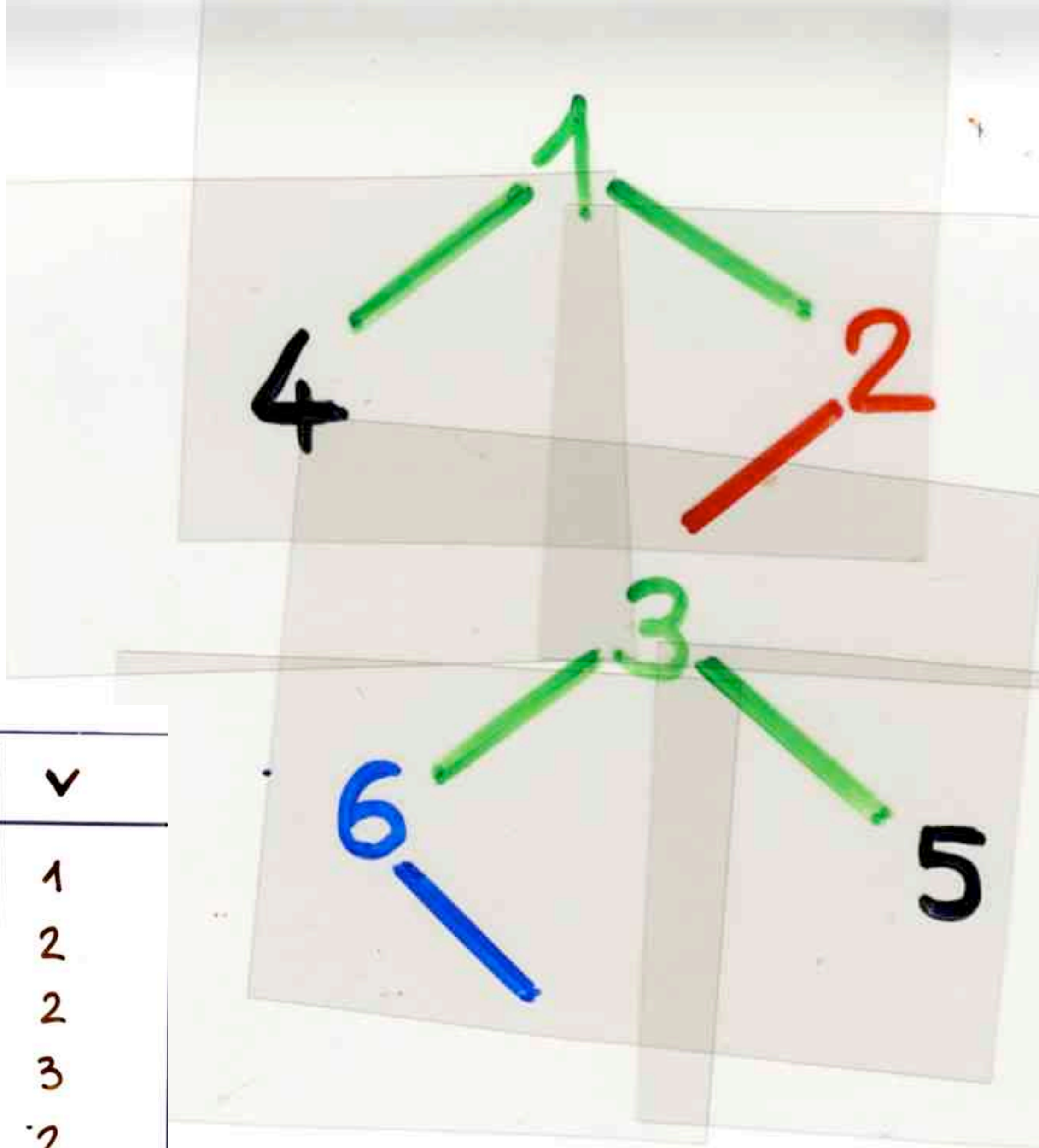
x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9			



x	ω_c	pos	v
1	╱	1	1
2	—	2	2
3	╱	2	2
4	╲	1	3
5	╲	2	2
6	—	1	1
7	╱	1	1
$n=$ 8	╲	2	2
9	•		

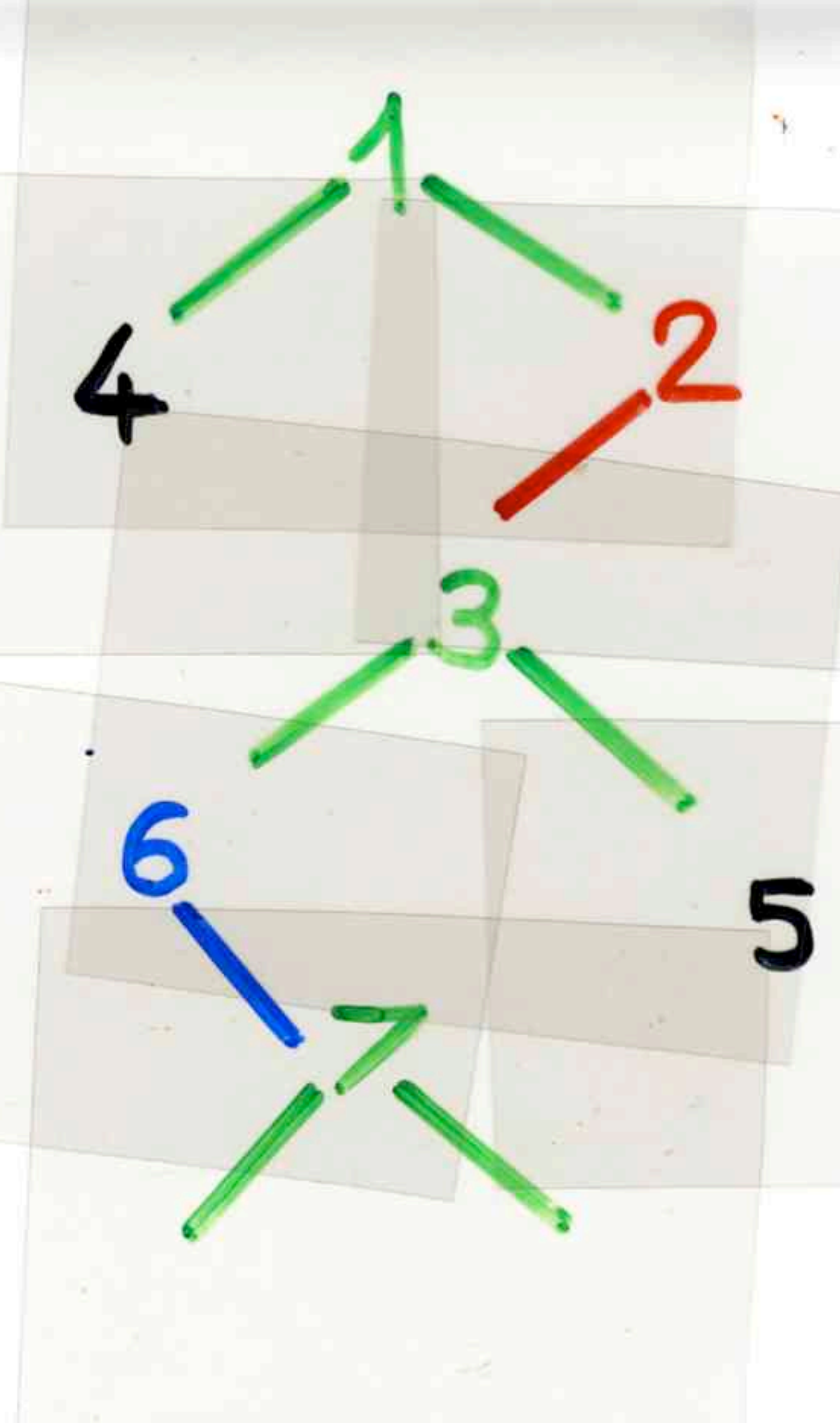


x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9	•		

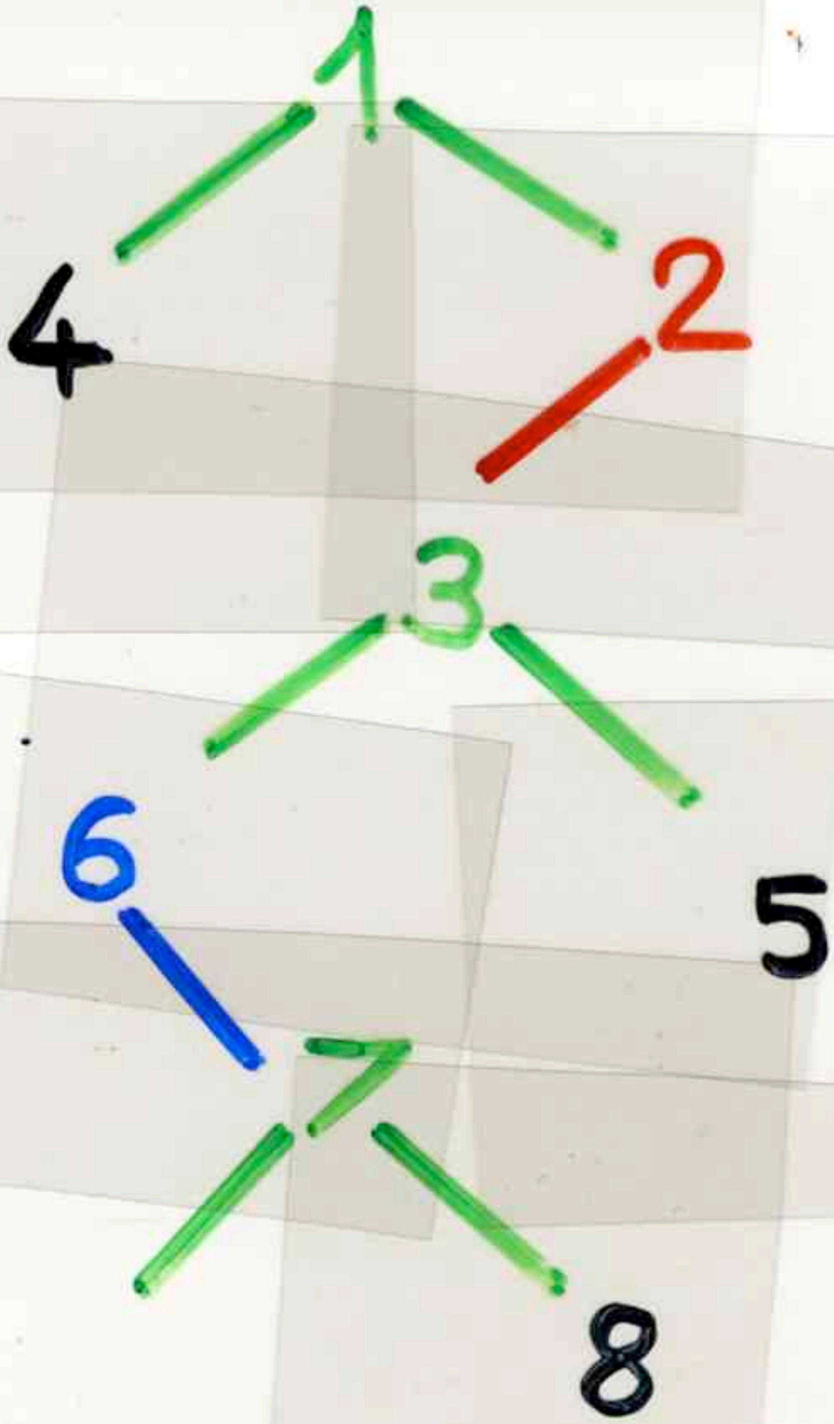


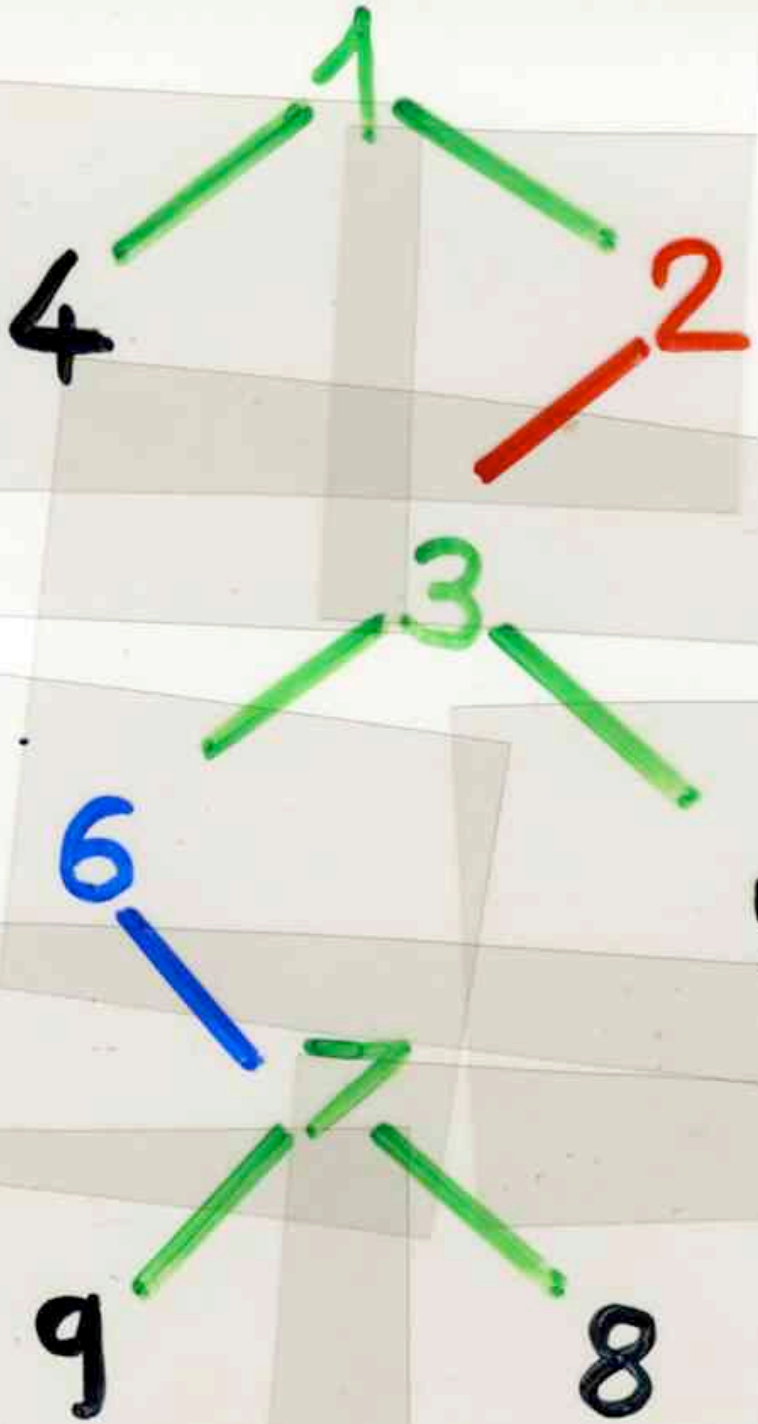
x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9			

x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9	•		



x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9	•		





x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9	•		

4 1 6 9 7 8 3 5 2

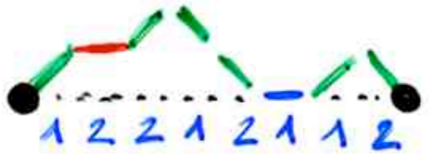
$$\mathcal{L}_n \xrightarrow{\ \theta\ } \mathcal{T}_{n+1} \xrightarrow[\text{projection}]{\ \pi\ } \mathfrak{S}_{n+1}$$

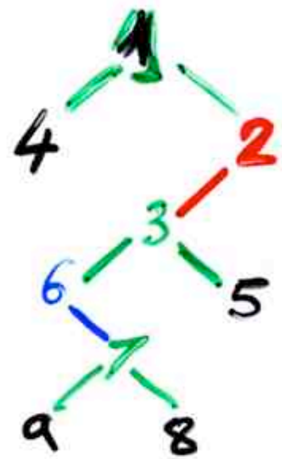
histoires de Laguerre — arbres binaires croissants — permutations

$$h = (\omega_c \,;\, (p_1, \dots, p_n))$$

ω_c : chemin Motzkin coloré

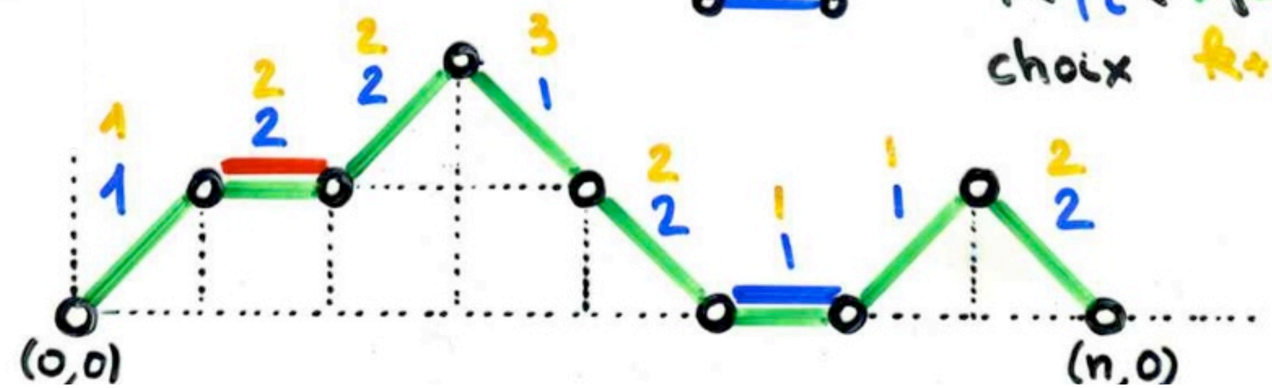
$(p_1, \dots, p_n)$: fonction de possibilité





4 1 6 9 7 8 3 5 2

$h = (\omega_c ; (p_1, \dots, p_n))$



$1 \leq p_i \leq v(\omega_i)$

choix k+1

x	ω_c	pos	v
1	↗	1	1
2	— (red)	2	2
3	↗	2	2
4	↘	1	3
5	↘	2	2
6	— (blue)	1	1
7	↗	1	1
$n=$ 8	↘	2	2
9	•		

␣

␣ 1 ␣

␣ 1 ␣ 2

␣ 1 ␣ 3 ␣ 2

4 1 ␣ 3 ␣ 2

4 1 ␣ 3 5 2

4 1 6 ␣ 3 5 2

4 1 6 ␣ 7 ␣ 3 5 2

4 1 6 ␣ 7 8 3 5 2

4 1 6 9 7 8 3 5 2 $= \sigma \in \mathfrak{S}_{n+1}$

P. Biane

cycle structure

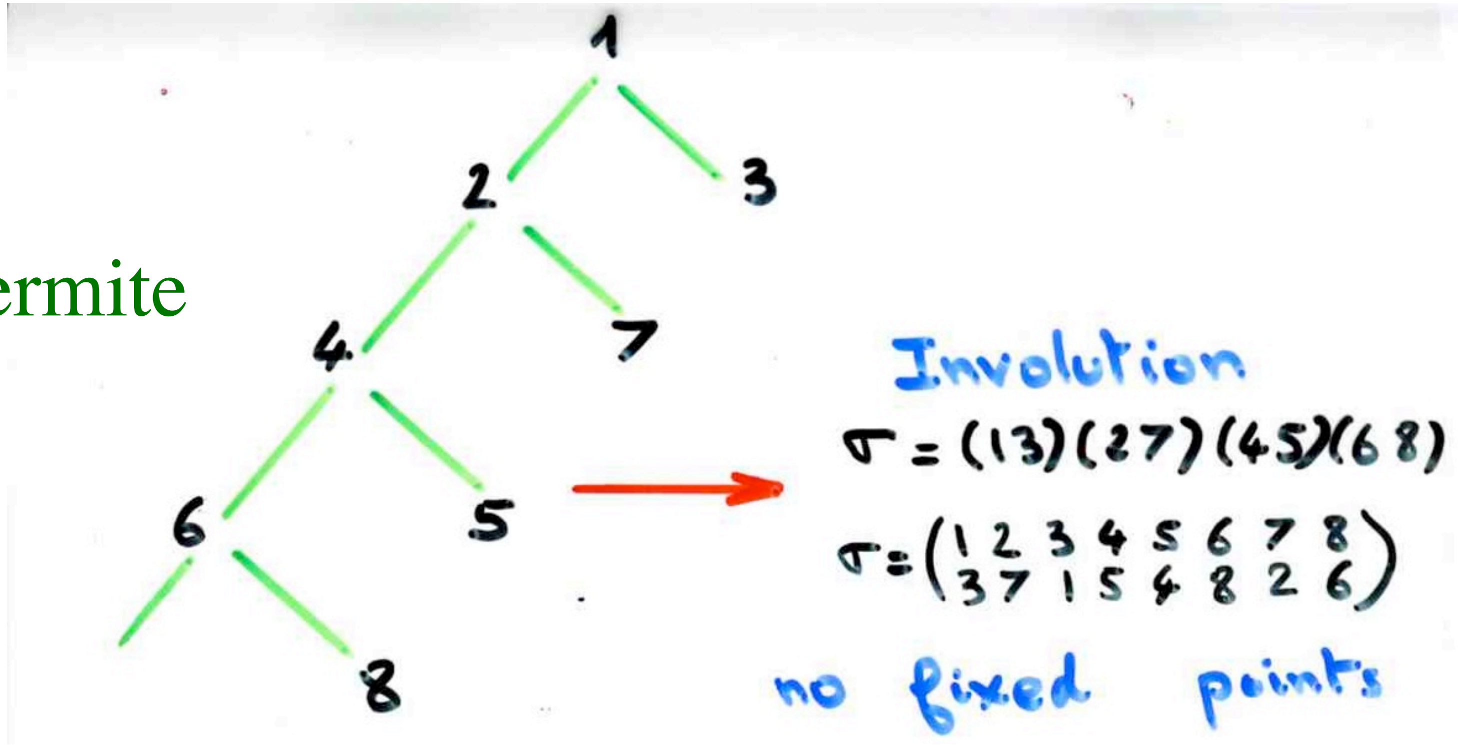
Foata, Zeilberger
de Médicis, X.V.

q-analog

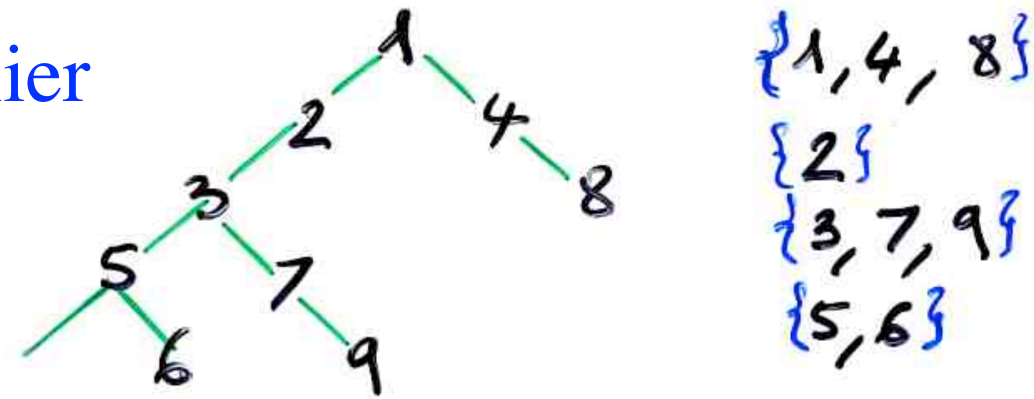
Stieljés
continued fraction

combinatorial proof for:
moments of orthogonal polynomials

Hermite



Charlier



orthogonal

polynomials

(binomial type)

Scheffer type

$$\sum P_n(x) \frac{t^n}{n!} = g(t) e^{x f(t)}$$

orthogonal polynomials

(binomial type) Scheffer type

$$\sum P_n(x)\frac{t^n}{n!} = g(t)e^{xf(t)}$$

- Hermite H_n
- Laguerre $L_n^{(\alpha)}$
- Charlier $C_n^{(a)}$
- Meixner I $M_n^{I\,(\alpha,)}$
- Meixner II $M_n^{II\,(\delta,\eta)}$

weighted histories

Laguerre $L_n^{(\alpha)}$

$b_k = 2k+\alpha+1$; $\lambda_k = k(k+\alpha)$

 $a_k = k+1$ $(k\geq 0)$

 $\begin{cases} b'_k = k+\alpha \\ b''_k = k+1 \end{cases}$ $(k\geq 0)$

; $c_k = k+\alpha$  $(k\geq 1)$

<u>Cor</u>. moments des polynômes $\{L_n^{(\alpha)}\}_{n\geq 0}$ Laguerre

$$\mu_n = (\alpha+1)(\alpha+2)\cdots(\alpha+n)$$

Polynômes	$b_k = b'_k + b''_k$	$\lambda_k = a_{k-1}\, c_k$	Moments
Tchebycheff unitaires $U_n(x)$	0	$1/4$	$\frac{1}{4^n} C_n$ Catalan
$T_n(x)$	0	$1/4$ $\quad \lambda_0 = 1/2$	$\frac{1}{4^n} \binom{2n}{n}$
Laguerre $L_n^1(x)$	$2k+2$	$k(k+1)$	$(n+1)!$
$L_n^\alpha(x)$	$2k+\alpha+1$	$k(k+\alpha)$	$(\alpha+1)\ldots(\alpha+n) = (\alpha+1)_n$
Hermite $H_n(x)$	0	k	$\mu_{2n} = 1.3\ldots(2n-1)$ $\mu_{2n+1} = 0$
Charlier $C_n^a(x)$	$k+a$	ak	$\sum S(n,k)\, a^k$
Meixner I $\hat{m}_n(x;\beta,c)$	$\frac{(1+c)k+\beta c}{1-c}$	$\frac{c\,k(k-1+\beta)}{(1-c)^2}$	$\frac{\sum_{\sigma\in\mathfrak{S}_n} \beta^{s(\sigma)}\, c^{1+d(\sigma)}}{(1-c)^n}$
Kreweras $\beta=1$ $\quad c=1/2$	$3k+1$	$2k^2$	$=(1-c)^\beta \sum_{k\geq 0} k^n c^k \frac{(\beta)_k}{k!}$
Meixner II $M_n(x;\delta,\eta)$	$(2k+\eta)\,\delta$	$(\delta^2+1)\,k(k-1+\eta)$	$\delta^n \sum_{\sigma\in\mathfrak{S}_n} \eta^{s(\sigma)} \left(1+\frac{1}{\delta^2}\right)^{p(\sigma)}$
$\delta=0$ $\quad \eta=1$	0	k^2	E_{2n} Sécant

weighted histories

q-analog of
Hermite histories

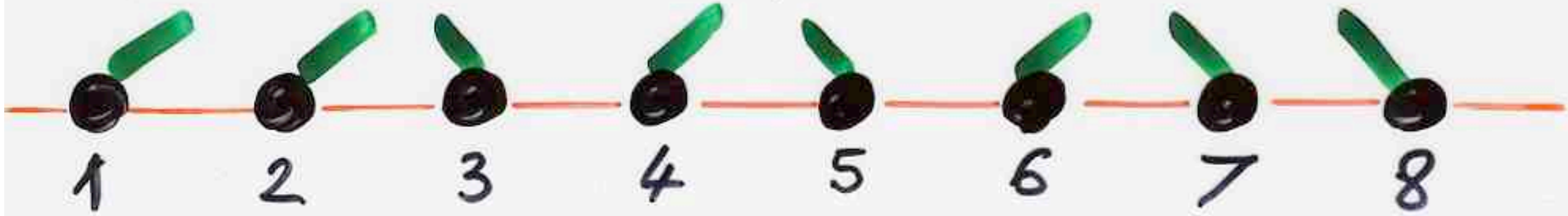
q-Hermite

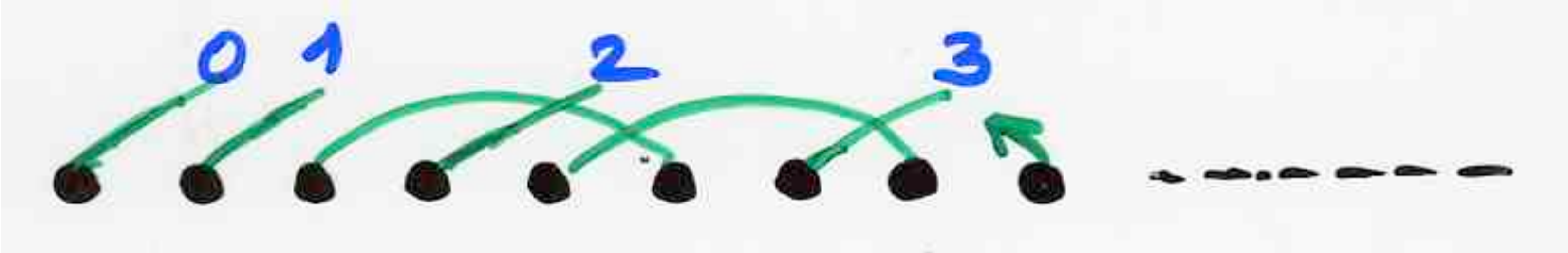
$H^{I}_{n}(x;q)$

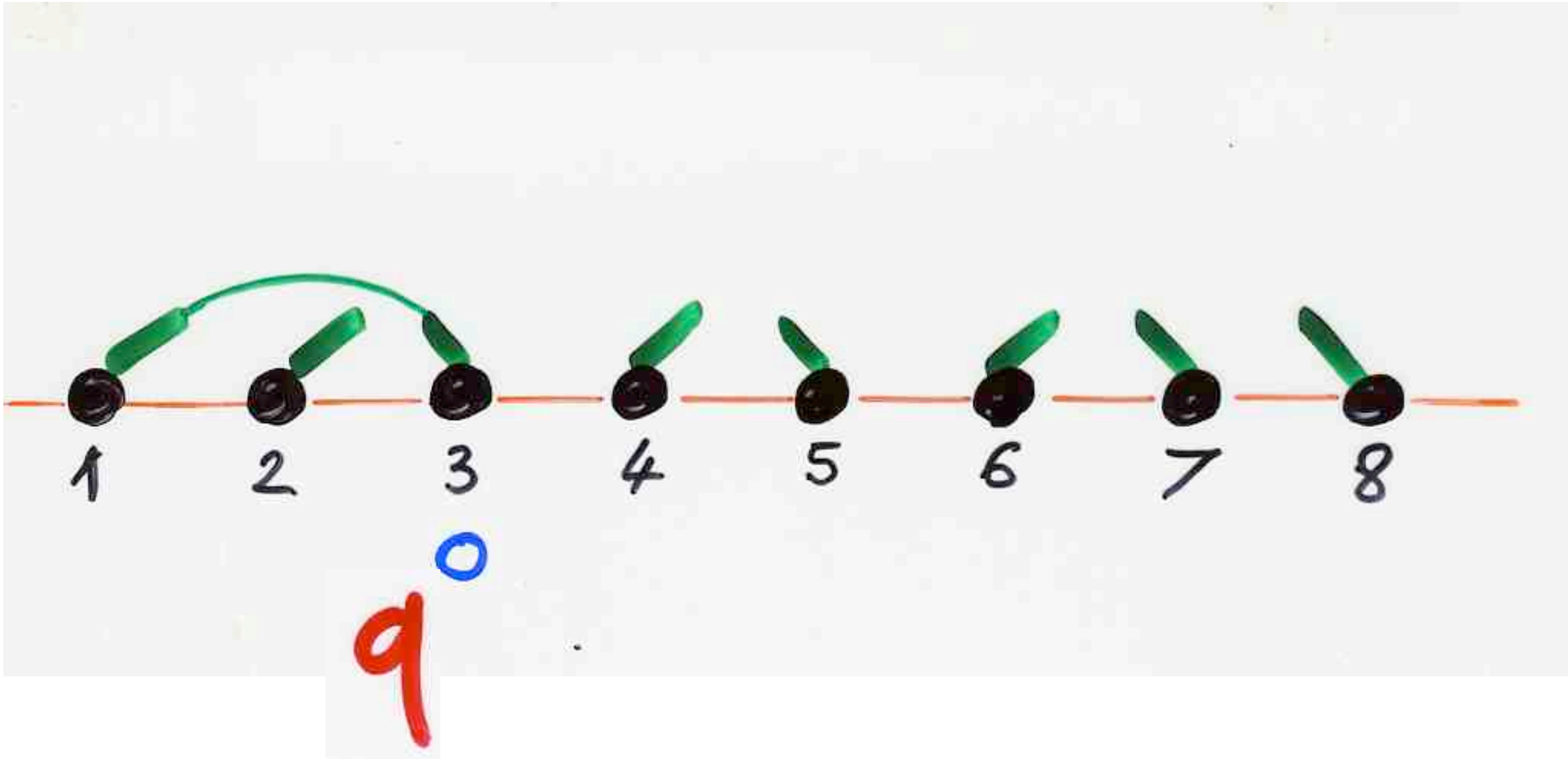
$b_k = 0$

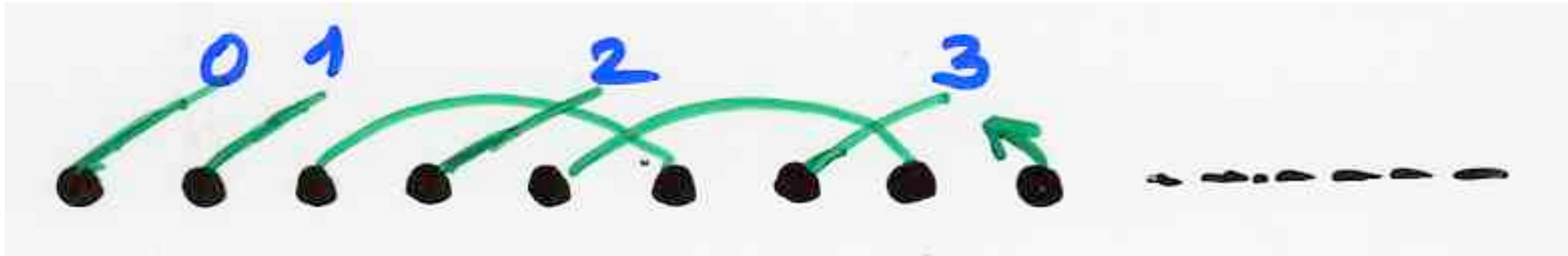
$\lambda_k = [k]_q$

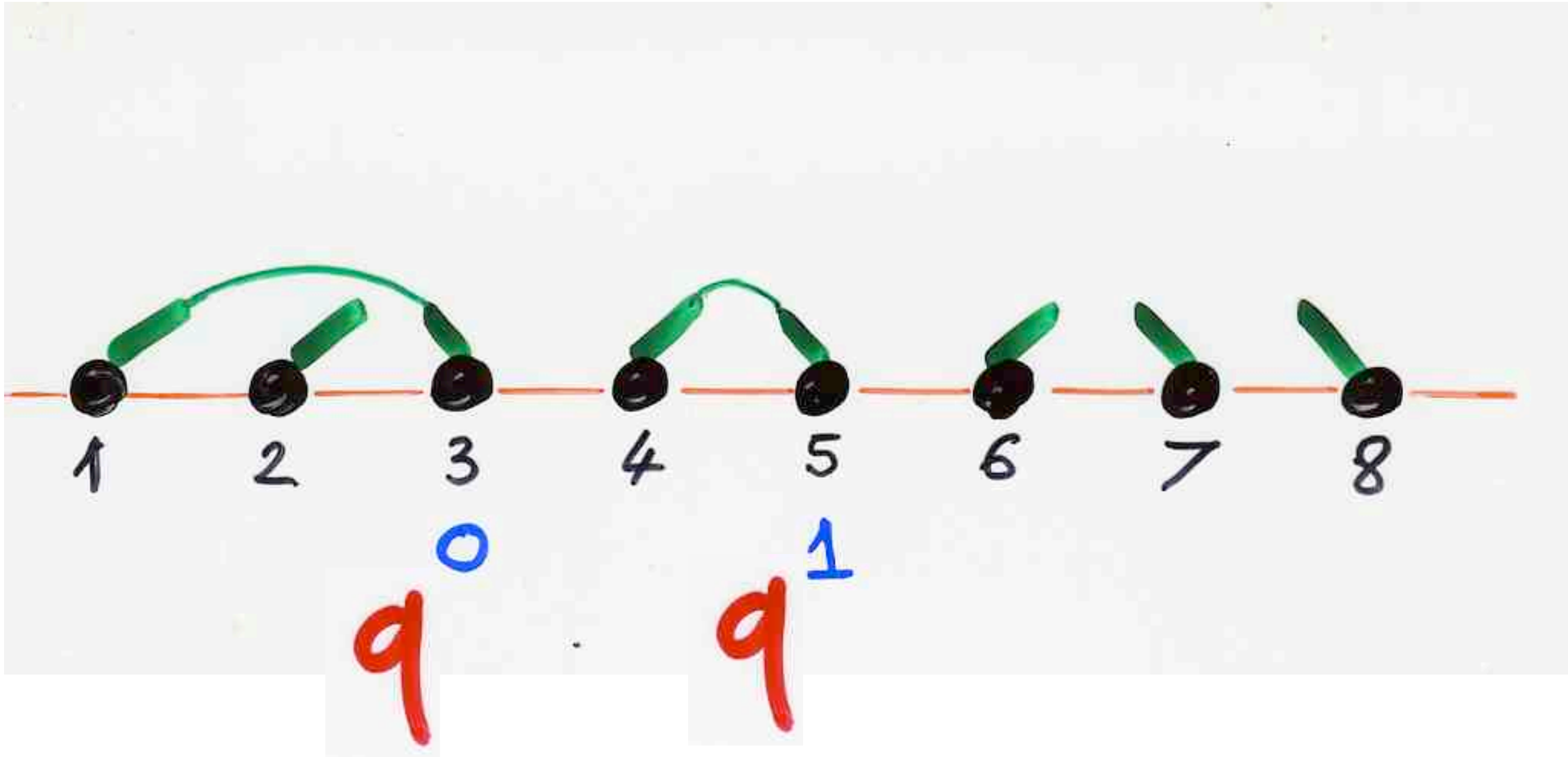
$= 1 + q + \dots + q^{k-1}$

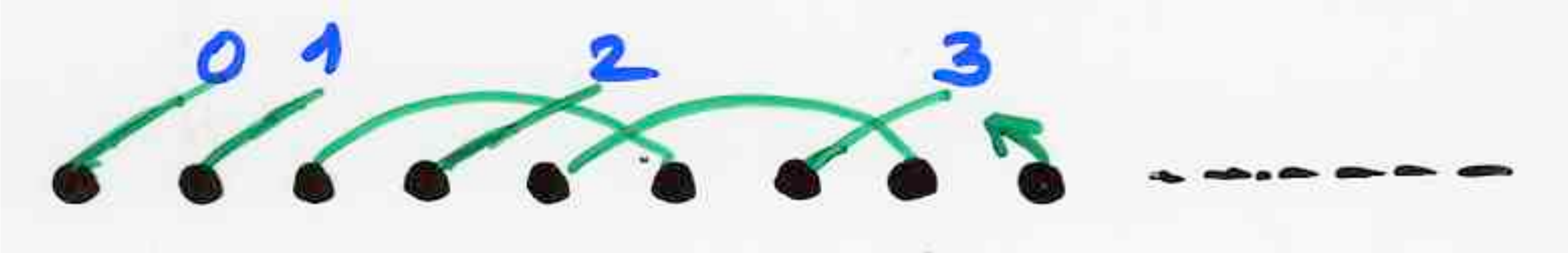

1
2
3
4
5
6
7
8

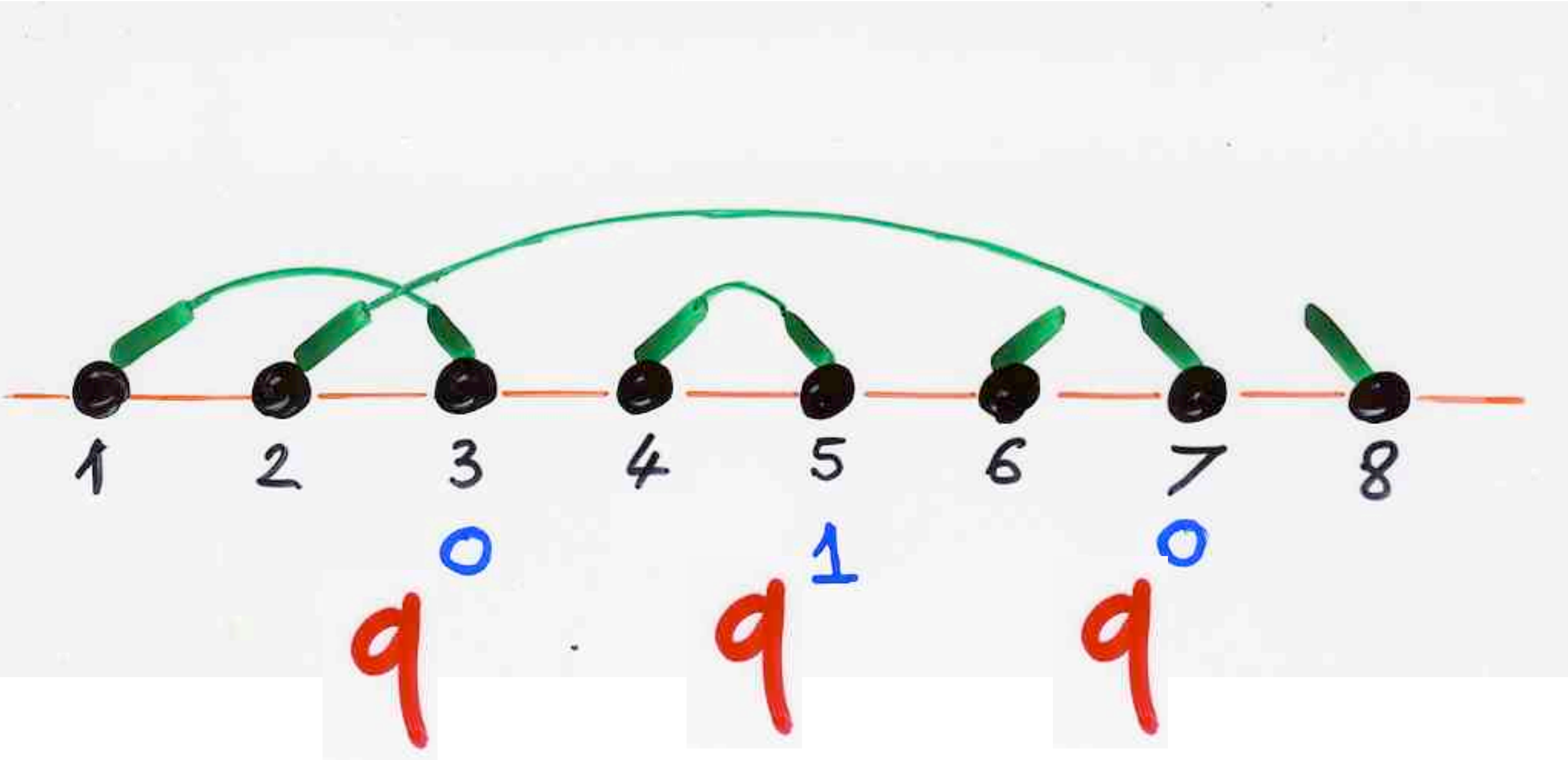

0 1 2 3


1 2 3 4 5 6 7 8
9
0

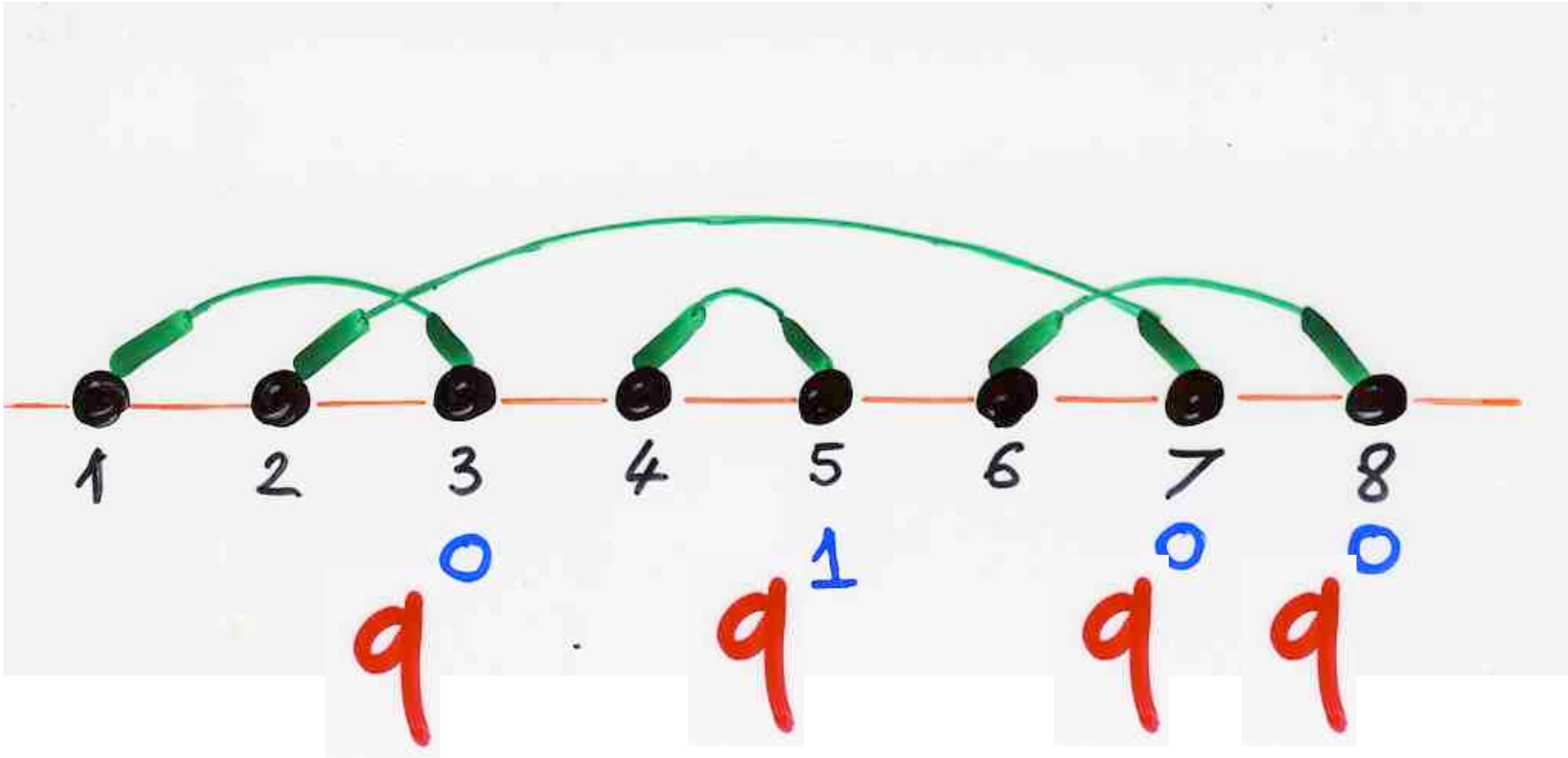

0 1 2 3

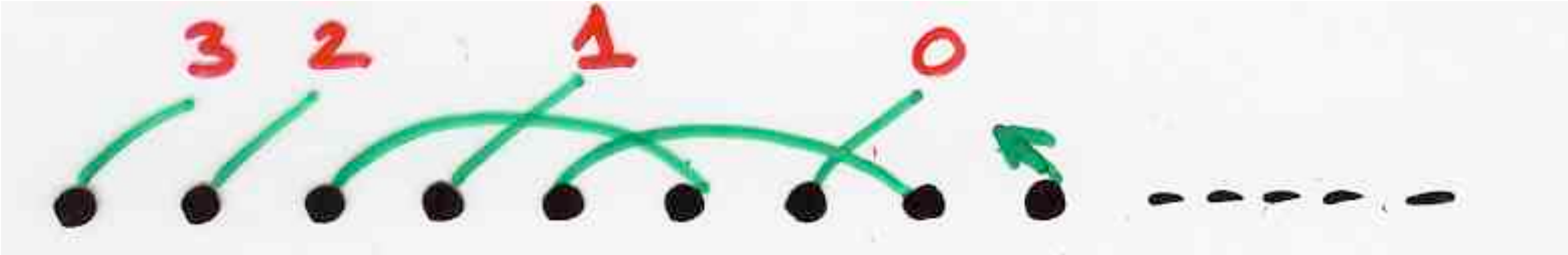

1 2 3 4 5 6 7 8
q^0
q^1

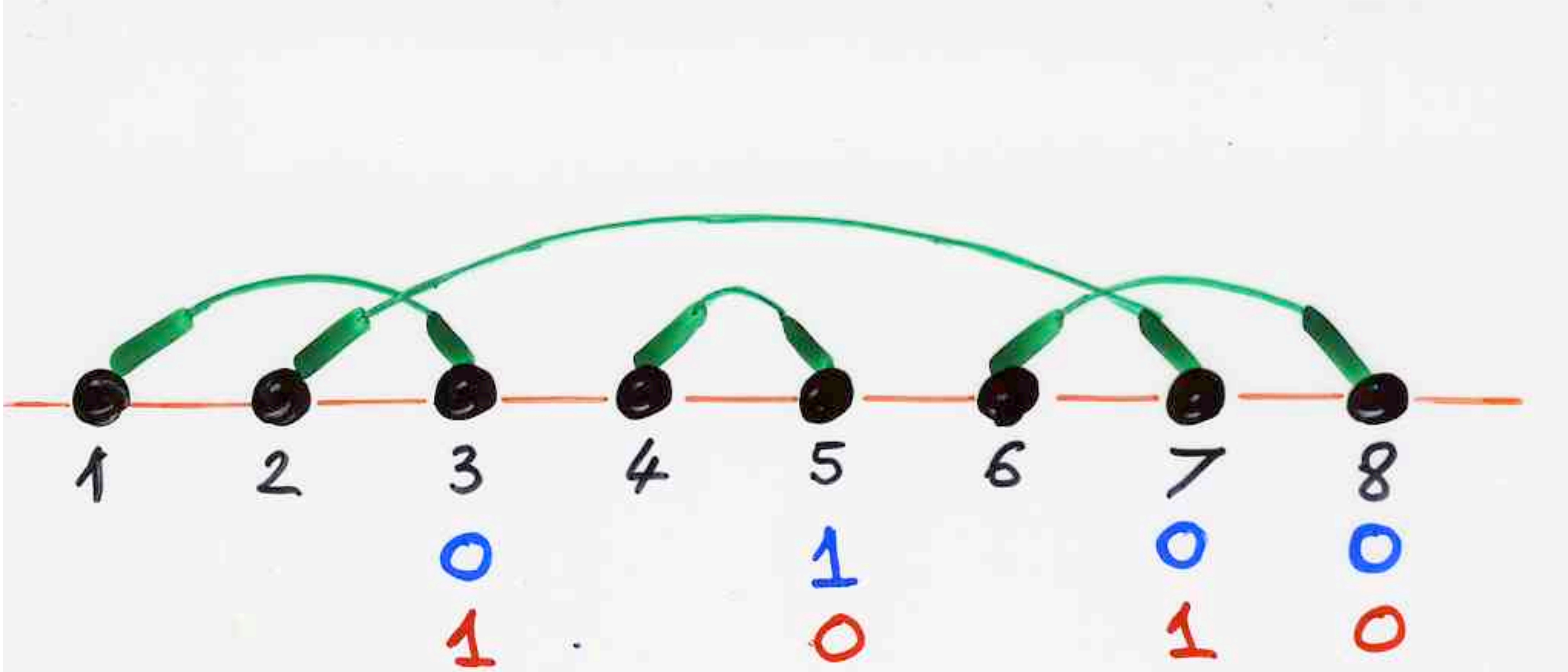

0 1 2 3

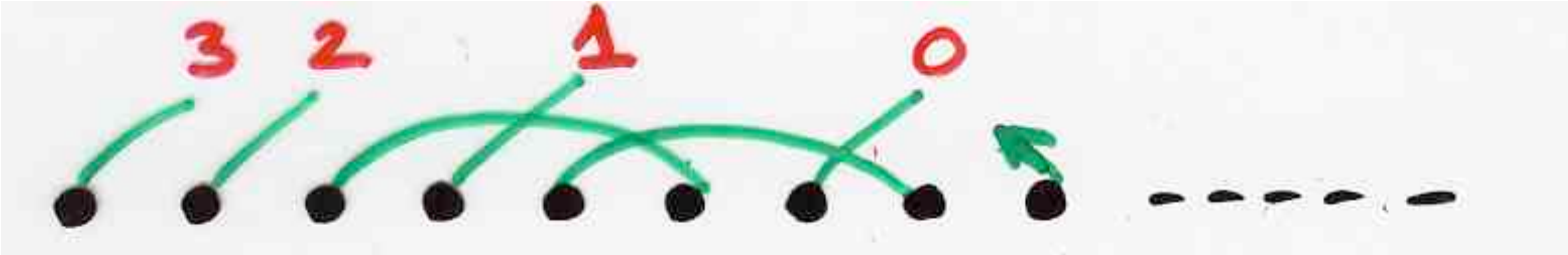

1 2 3 4 5 6 7 8
0 1 0
q q q

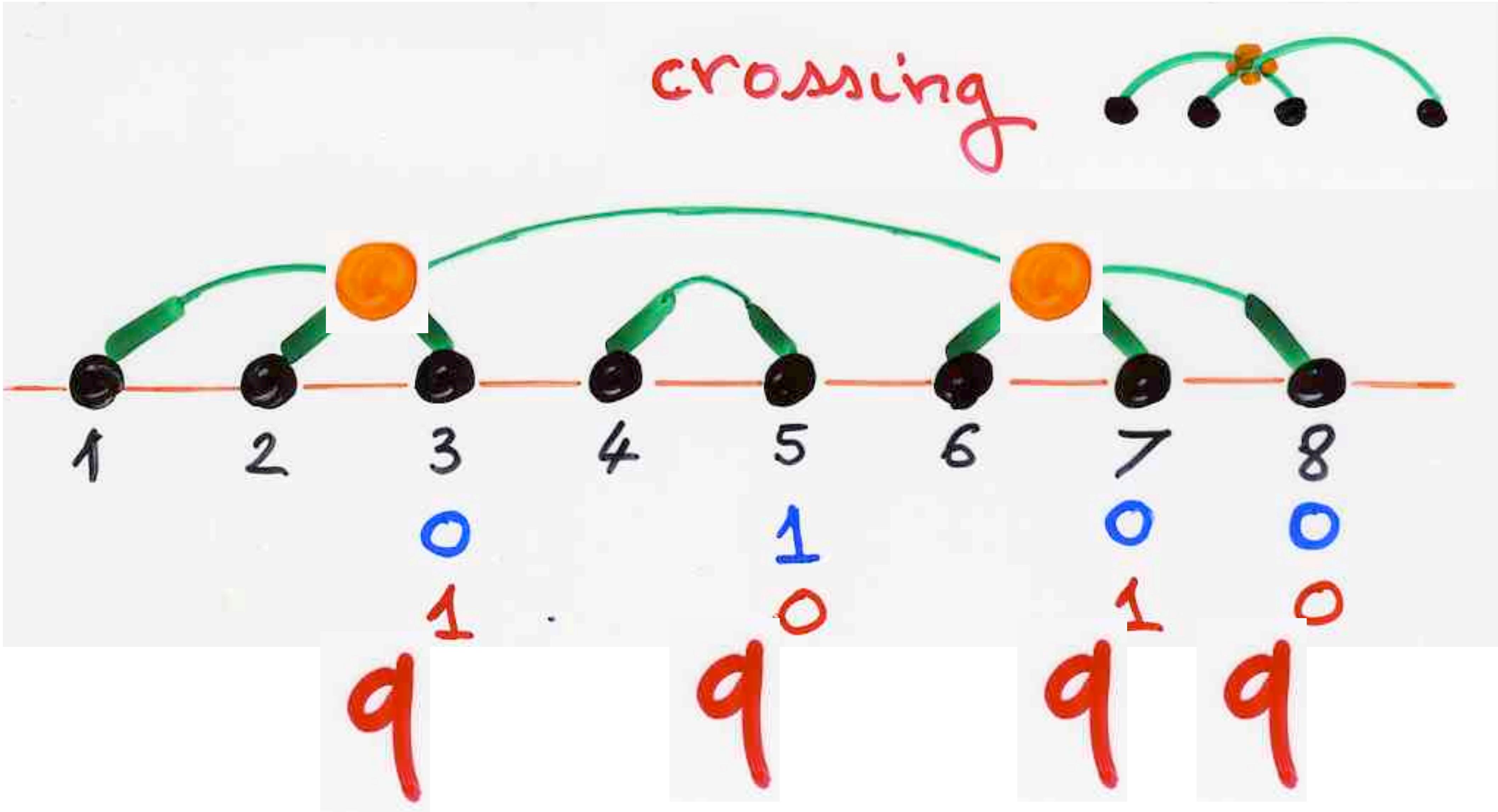

nesting


1
2
3
4
5
6
7
8
0
1
0
0
q
q
q
q


3 2 1 0


1 2 3 4 5 6 7 8
0 1 0 0
1 0 1 0


3
2
1
0


crossing
1
2
3
4
5
6
7
8
0
1
0
0
1
0
1
0
9
9
9
9

Linearisation coefficients

Linearization coefficients

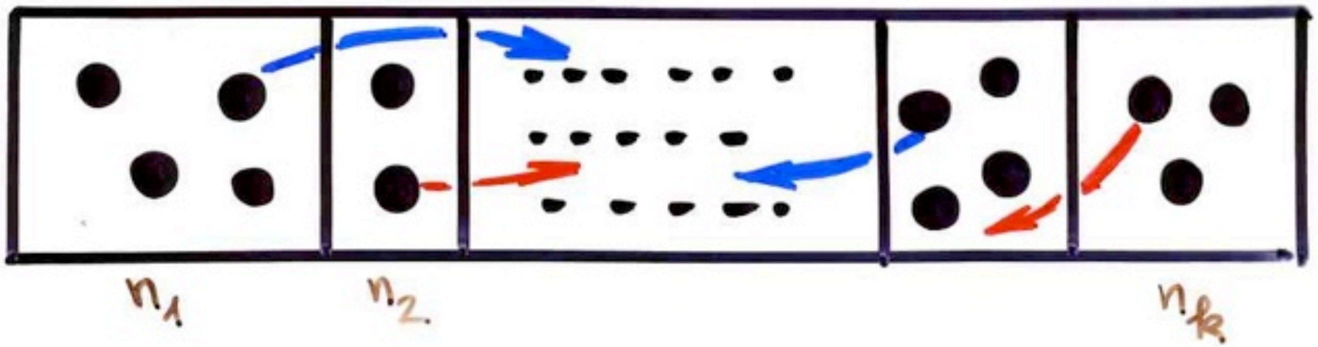
positivity orthogonality

$$P_k(x) P_\ell(x) = \sum_n c^n_{k,\ell} P_n(x)$$

$$\int_a^b P_k(x) P_\ell(x) \, d\mu = c_{kk} \delta_{k\ell}$$

$$I_{n_1, n_2, \dots, n_k} = \int_a^b P_{n_1}(x) P_{n_2}(x) \cdots P_{n_k}(x) \, d\mu$$

derangements



- Hermite (matching) Azor, Gillis, Victor, Godsil
  Askey, Koorwinder, Ismail, Tamhankar
- Laguerre Even, Gillis, Jackson, Foata, Zeilberger
  DeSainte-Catherine, Viennot, Kaplansky
- Jacobi Rahman, Foata, Zeilberger
  Legendre Gillis, Jedwab, Zeilberger Hsu
  Tchebycheff DeSainte-Catherine, Viennot
- Meixner, Charlier, Krawtchouk Zeng Askey
  (rook polynomials) Gessel

- (completely) **bijective**

$$\int_a^b x^n \, d\omega = f(x^n) = \mu_n$$

moment

f linear functional

$f(P(x))$

$$I_{n_1,\dots,n_k} = f(P_{n_1} P_{n_2} \cdots P_{n_k})$$

Viennot
Desainte-Catherine

Askey-Wilson integral

L'intégrale de Askey-Wilson

$$(a)_\infty = \prod_i (1 - aq^i)$$

$$w(\cos\theta, a, b, c, d \mid q) = \frac{(e^{2i\theta})_\infty (e^{-2i\theta})_\infty}{(ae^{i\theta})_\infty (ae^{-i\theta})_\infty (be^{i\theta})_\infty (be^{-i\theta})_\infty (ce^{i\theta})_\infty (ce^{-i\theta})_\infty (de^{i\theta})_\infty (de^{-i\theta})_\infty}$$

$$\frac{(q)_\infty}{2\pi} \int_0^\pi w(\cos\theta, a, b, c, d \mid q)\, d\theta = \frac{(abcd)_\infty}{(ab)_\infty (ac)_\infty (ad)_\infty (bc)_\infty (bd)_\infty (cd)_\infty}$$

Askey, Wilson (1985)
Ismail, Stanton, Viennot (1986)
Rahman (1984), Ismail, Stanton (1989)
Gasper, Rahman (1989)

Integral of the product of

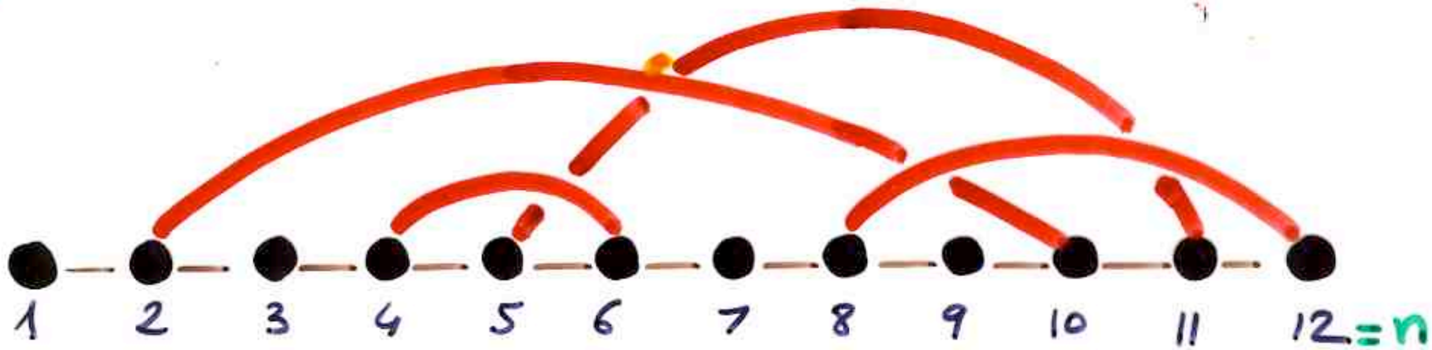
4 q-Hermite polynomials

$$\frac{(q)_\infty}{2\pi}\int_0^\pi H_n(\cos\theta|q)\,H_m(\cos\theta|q)\,(e^{2i\theta})_\infty(e^{-2i\theta})_\infty = (q)_n\delta_{nm}$$

q-moments

perfect matchings

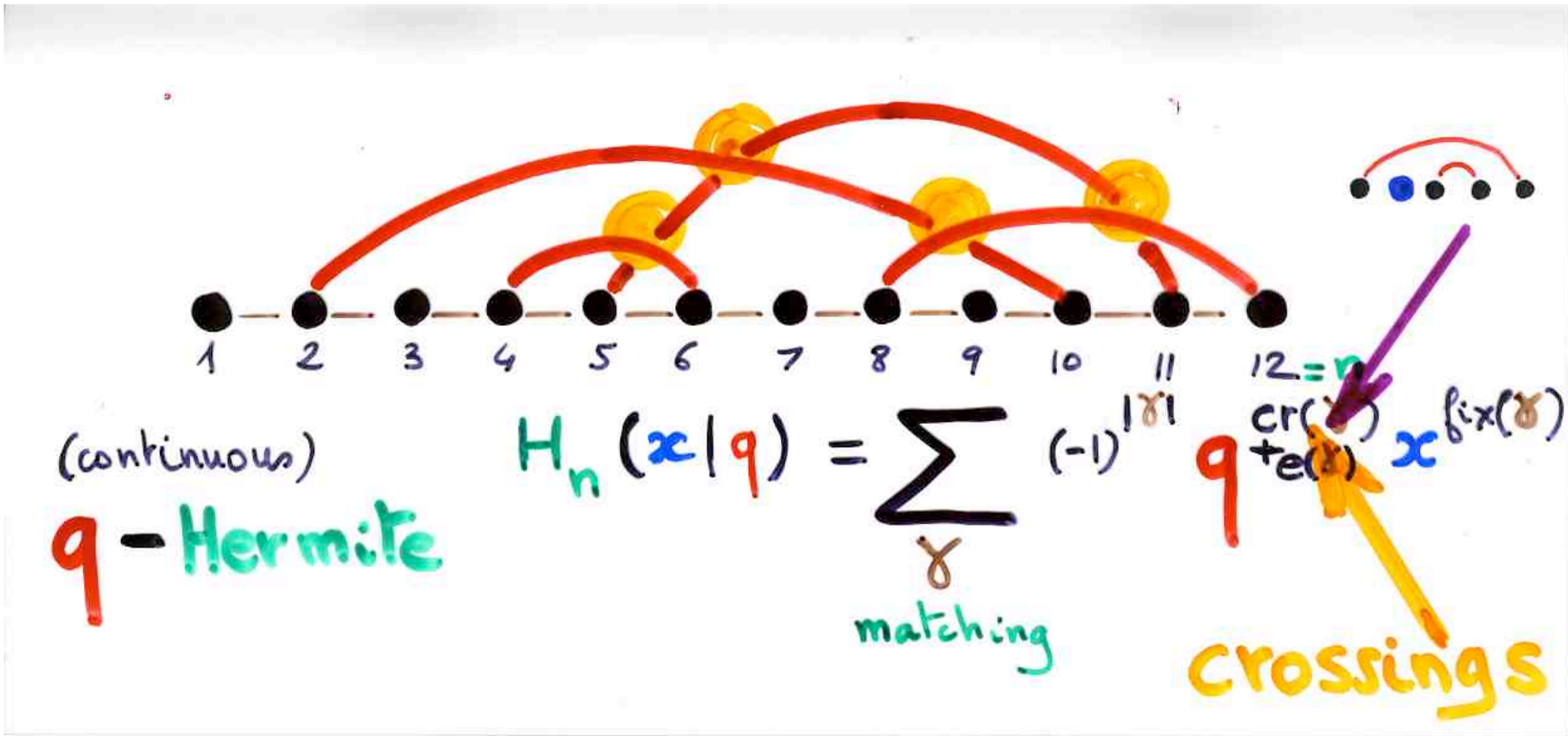
number of crossings



(continuous)

q-Hermite

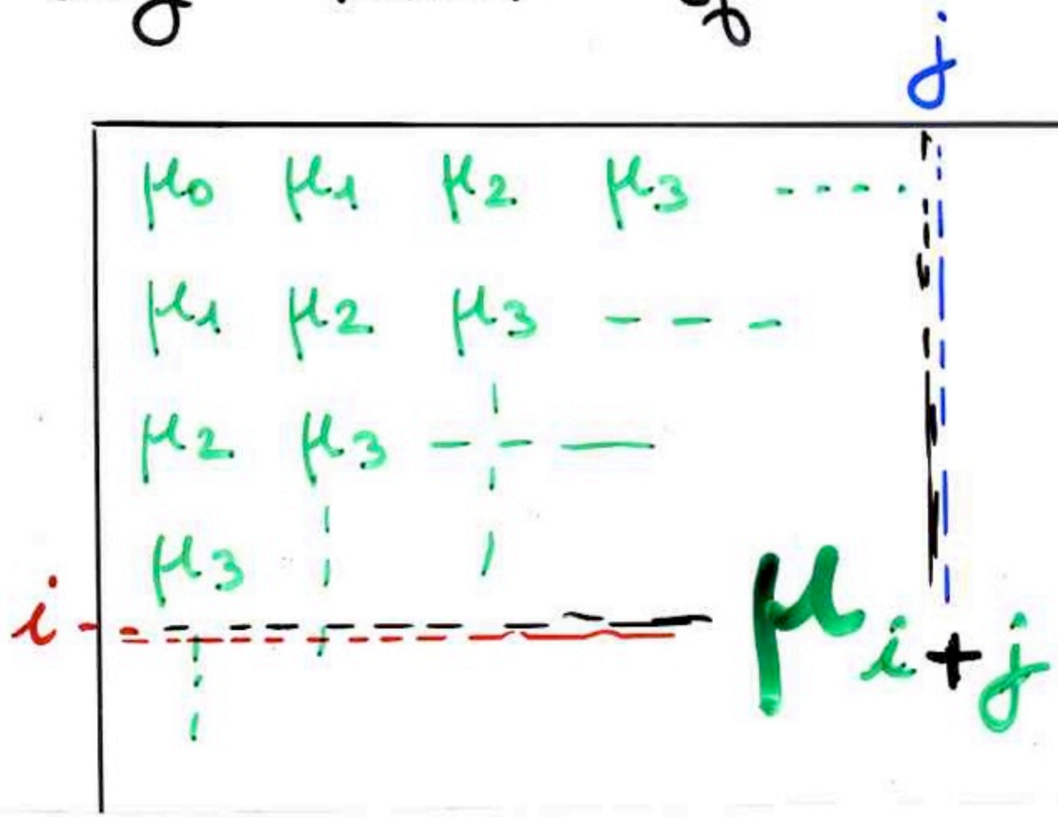
$$H_n(x|q) = \sum_{\substack{\gamma \\ \text{matching}}} (-1)^{|\gamma|} \, q^{cr(\gamma)+e(\gamma)} \, x^{fix(\gamma)}$$


1 2 3 4 5 6 7 8 9 10 11 12 = n
(continuous)
q - Hermite
H_n (x | q) = Σ_γ (-1)^|γ| q^(cr(γ) + e(γ)) x^fix(γ)
matching
crossings

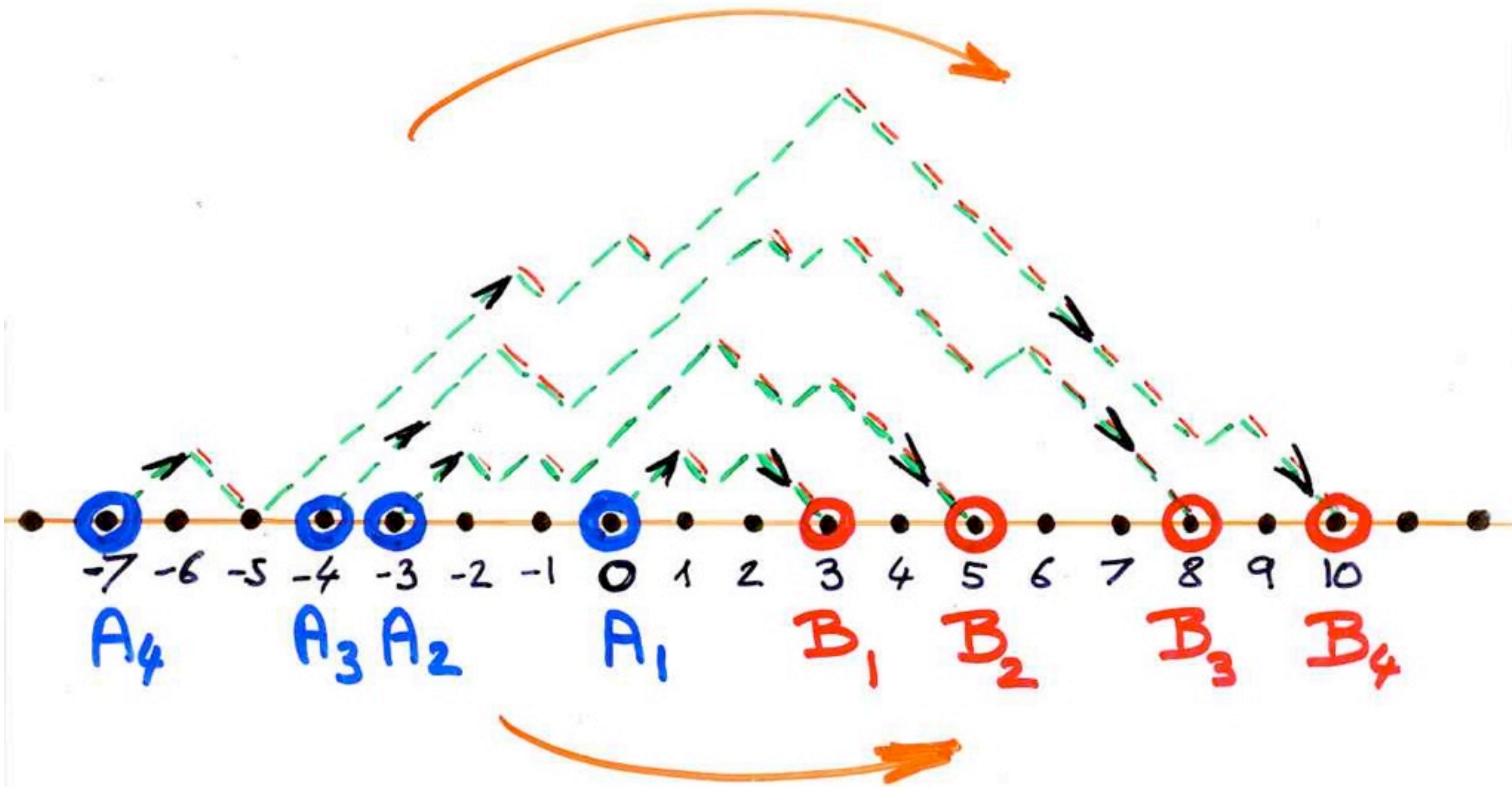
Hankel determinants

Hankel determinant

any minor of

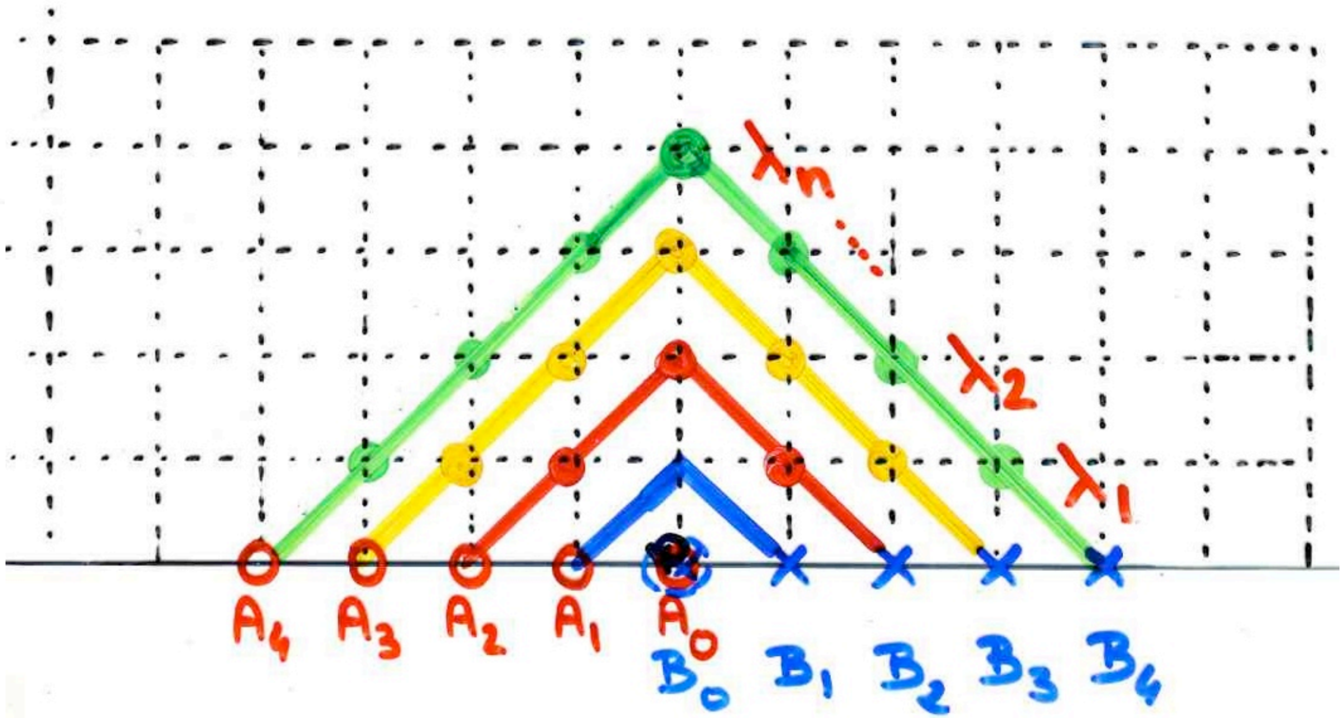


$$\begin{vmatrix} \mu_3 & \mu_5 & \mu_8 & \mu_{10} \\ \mu_6 & \mu_8 & \mu_{11} & \mu_{13} \\ \mu_7 & \mu_9 & \mu_{12} & \mu_{14} \\ \mu_{10} & \mu_{12} & \mu_{15} & \mu_{17} \end{vmatrix}$$



Hankel

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & & & \\ \mu_n & \cdots & \cdots & \mu_{n+n} \end{vmatrix}$$



$$\frac{\Delta_n}{\Delta_{n-1}} : \frac{\Delta_{n-1}}{\Delta_{n-2}} = \lambda_n$$

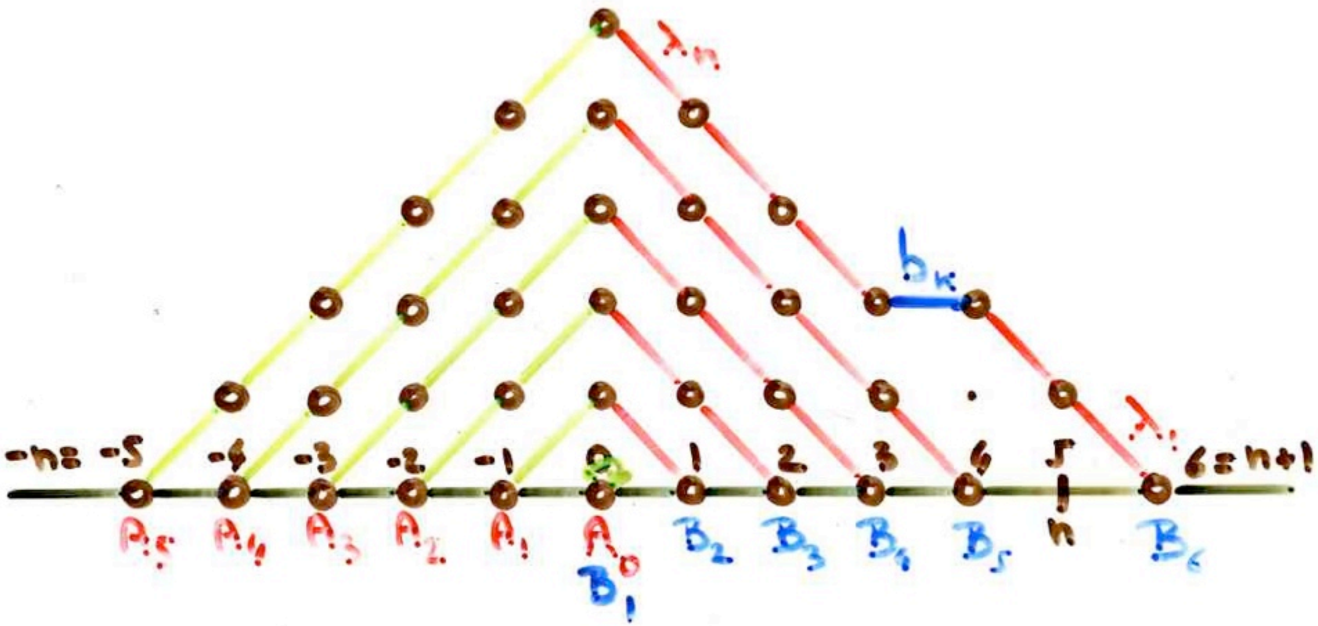


Fig 8. χ_n

$$b_n = \frac{\chi_n}{\Delta_n} - \frac{\chi_{n-1}}{\Delta_{n-1}}$$