At the crossroad of algebra, combinatorics and physics: the 2-species PASEP

Indo-french conference
IMSc, Chennai, 2016

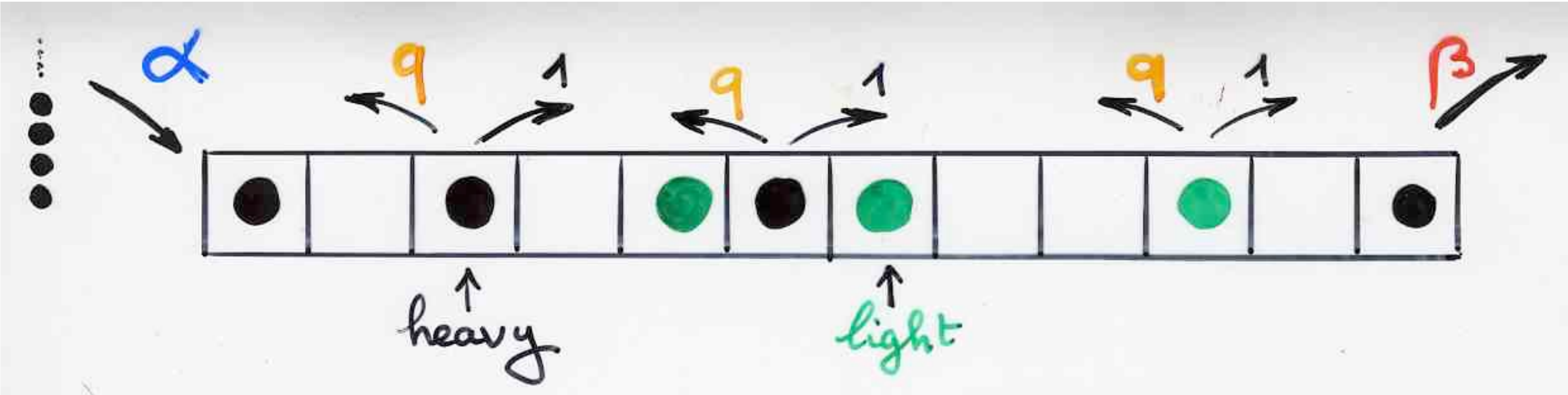
Xavier Viennot
CNRS, LaBRI, Bordeaux
IMSc, Chennai
www.xavierviennot.org

second part

- The 2-species PASEP

(join work with O. Mandelshtam, Berkeley)

The 2-species PASEP



Matrix Ansatz

for the 2-species PASEP

Matrix Ansatz (Uchiyama, 2008)

$X = X_1 \dots X_n \qquad X_i \in \{ \bullet, \bullet, \circ \}$

D, E, A matrices

W row vector $\qquad V$ column vector

$$\begin{cases} DE = qED + D + E \\ DA = qAD + A \\ AE = qEA + A \end{cases}$$

$$\langle W | E = \frac{1}{\alpha} \langle W |$$

$$D | V \rangle = \frac{1}{\beta} | V \rangle$$

Matrix Ansatz (Uchiyama, 2008)

$X = X_1 \dots X_n \qquad X_i \in \{ \bullet , {\color{green}\bullet} , \circ \}$

D, E, A matrices

W row vector $\qquad V$ column vector

$$\mathrm{Prob}(X) = \frac{1}{Z_{n,r}} \langle W | \prod_{i=1}^{n} D\, \mathbb{1}_{(X_i = \bullet)} + A\, \mathbb{1}_{(X_i = {\color{green}\bullet})} + E\, \mathbb{1}_{(X_i = \circ)} | V \rangle$$

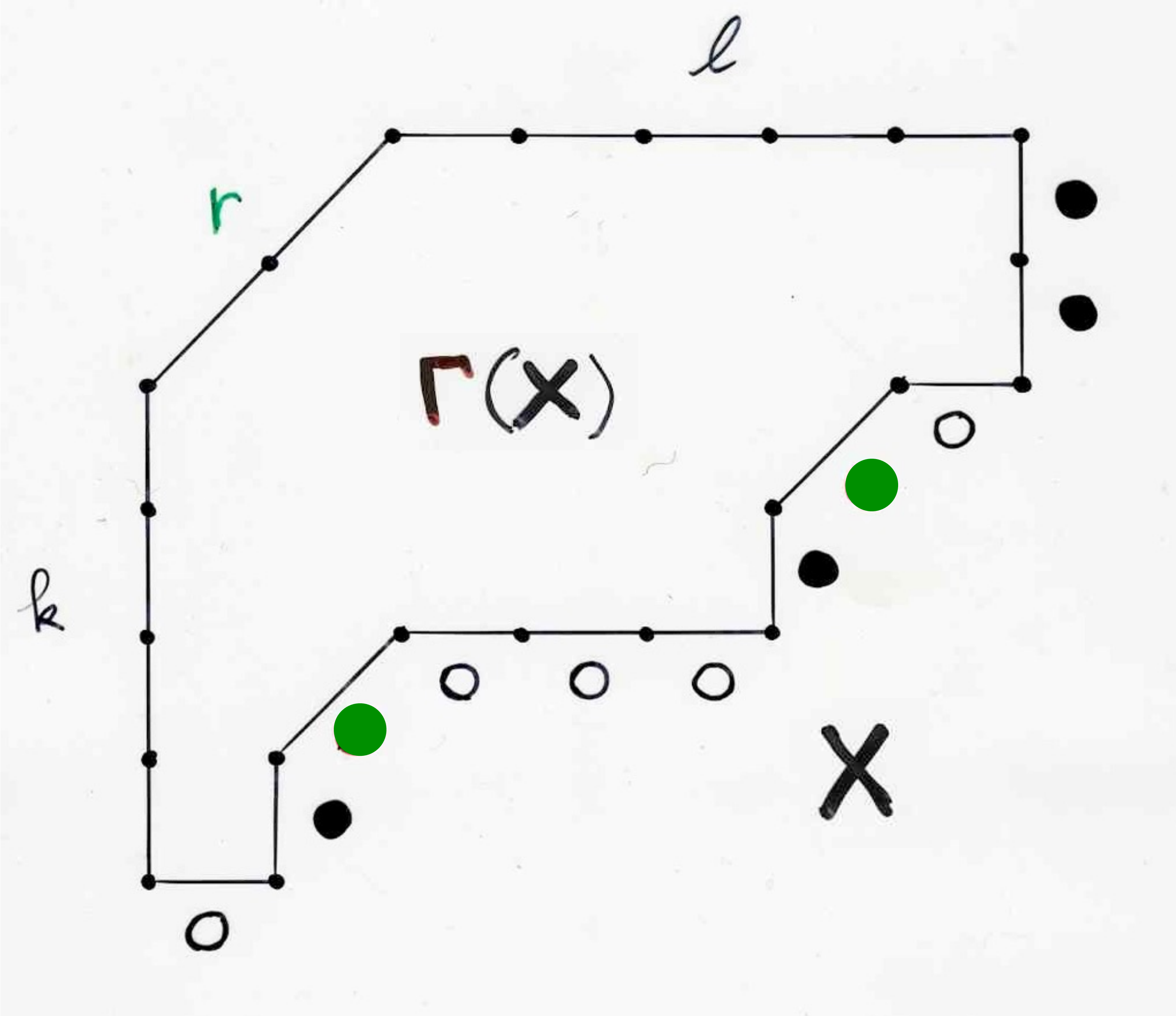
$$Z_{n,r} = \text{coeff. of } y^r \text{ in } \langle W | (D + yA + E)^n | V \rangle$$

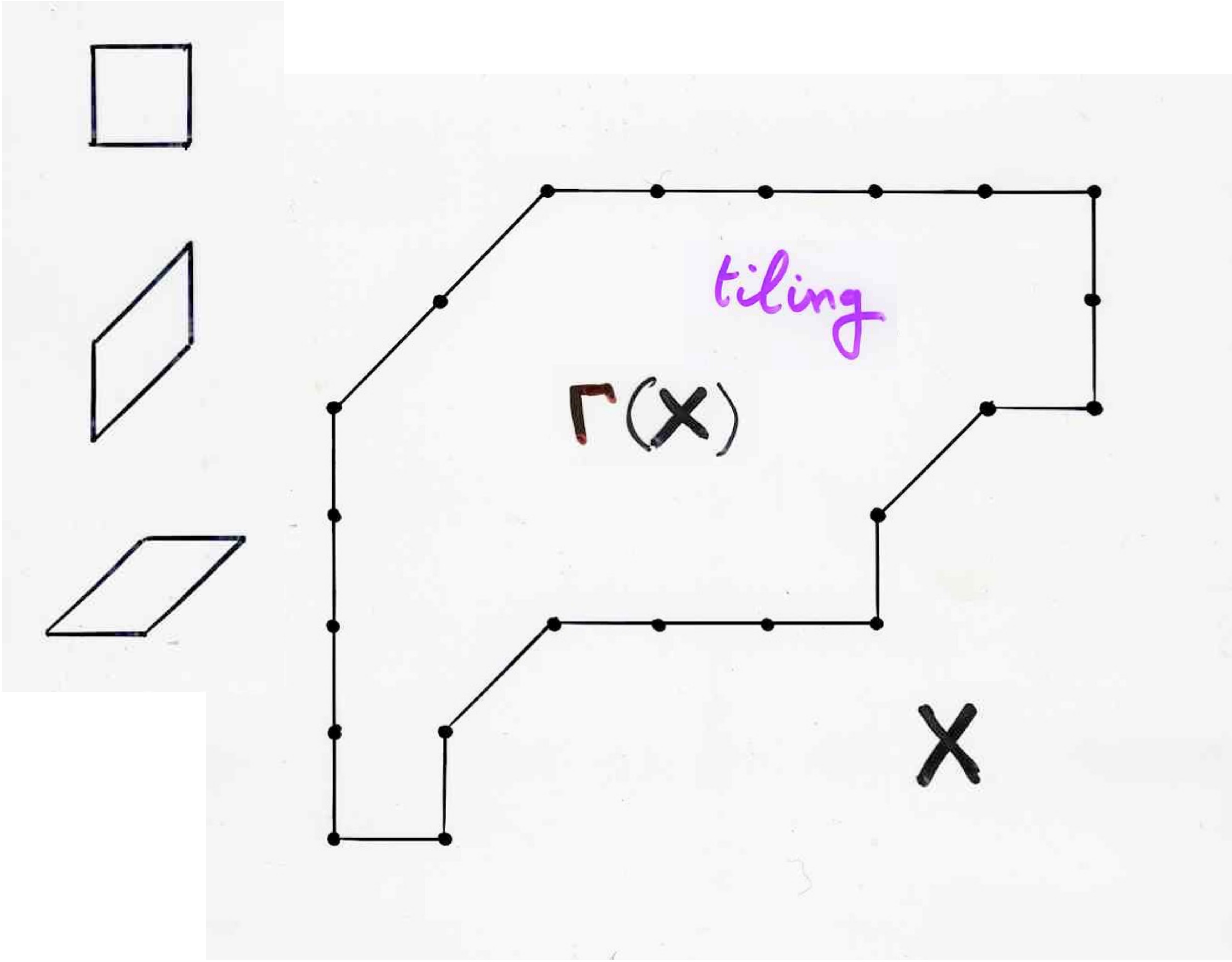
Rhombic alternative tableaux

(RAT)

Rhombic alternative tableaux

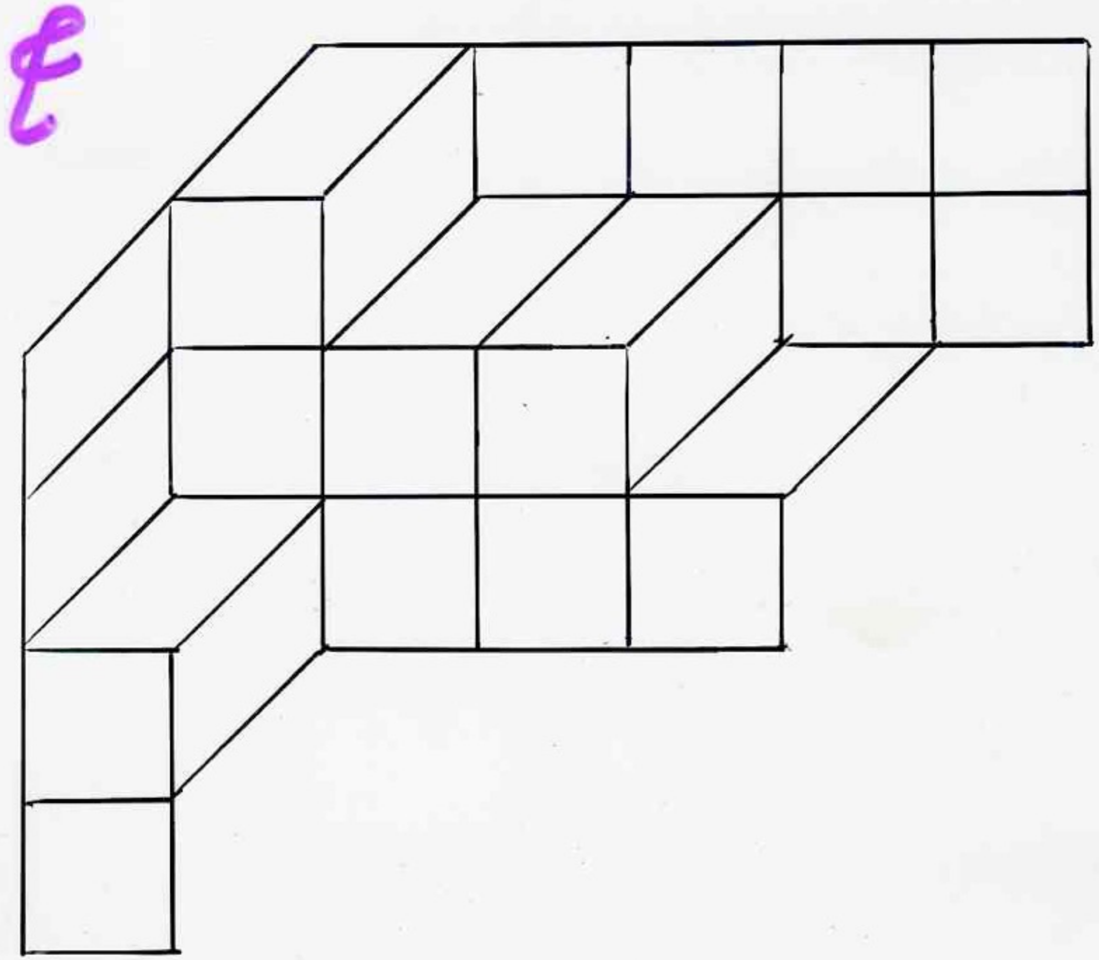
(tableaux alternatifs rhomboïdaux)


ℓ
r
Γ(X)
k
X



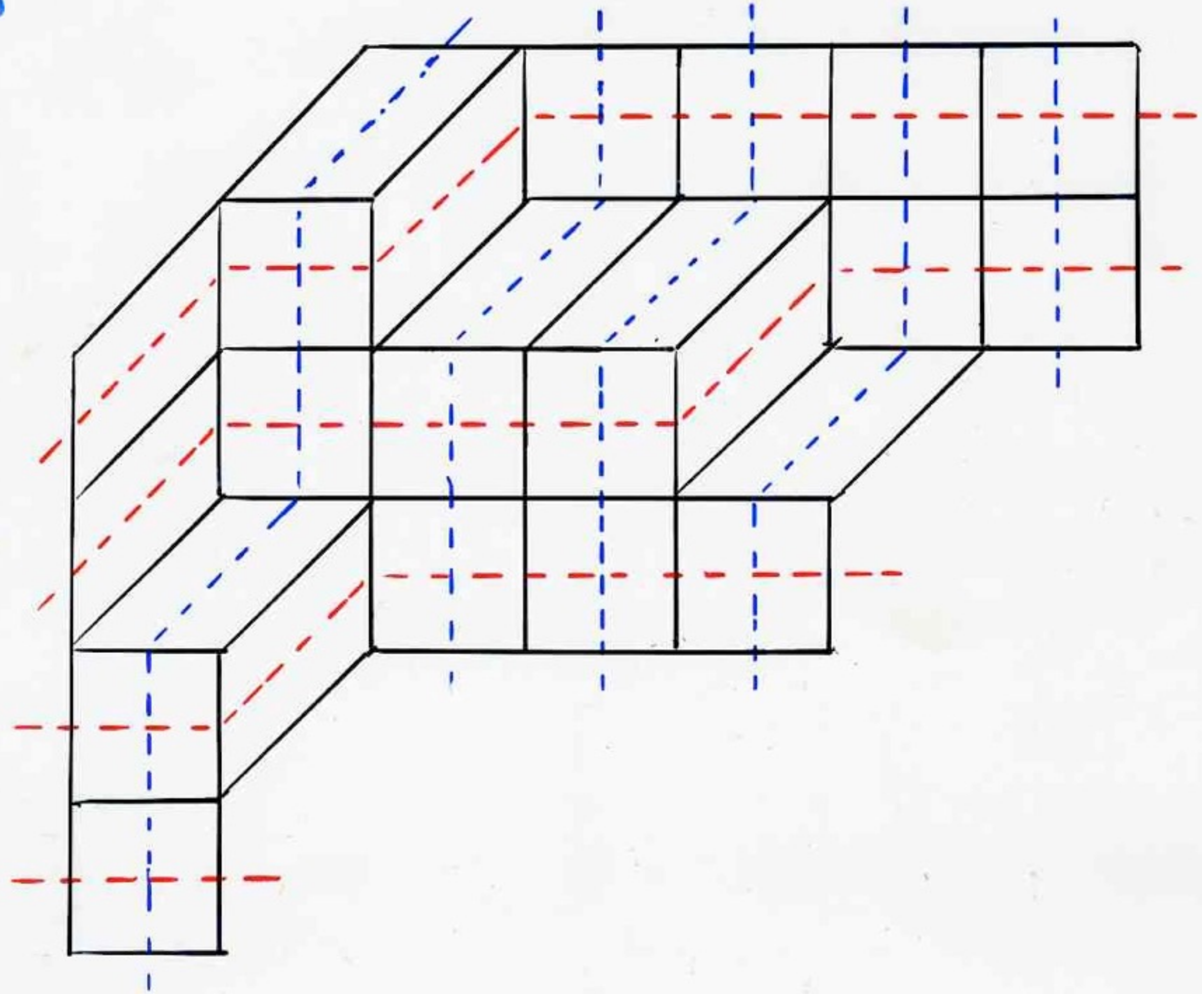
tiling
Γ(X)
X

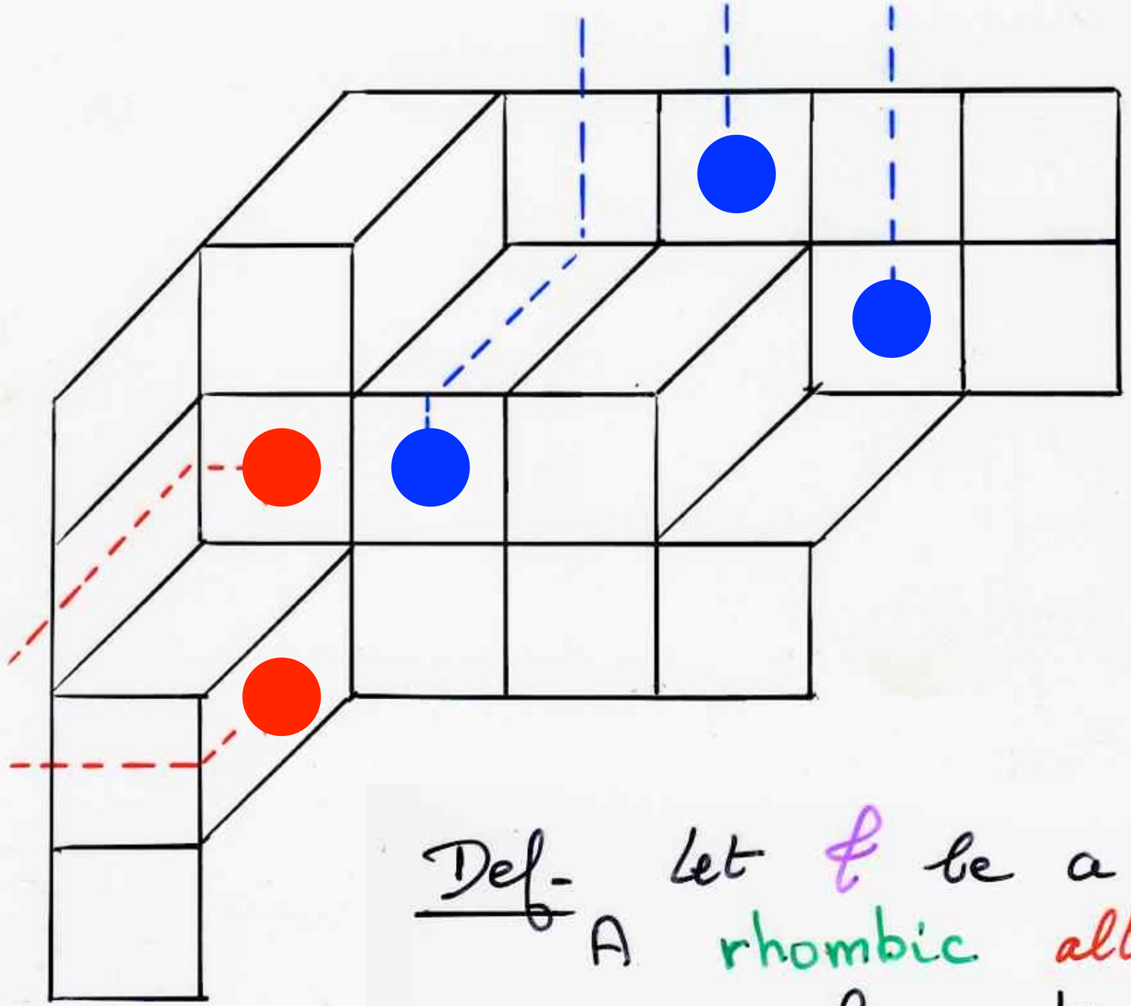
tiling



west- strips

north-strips





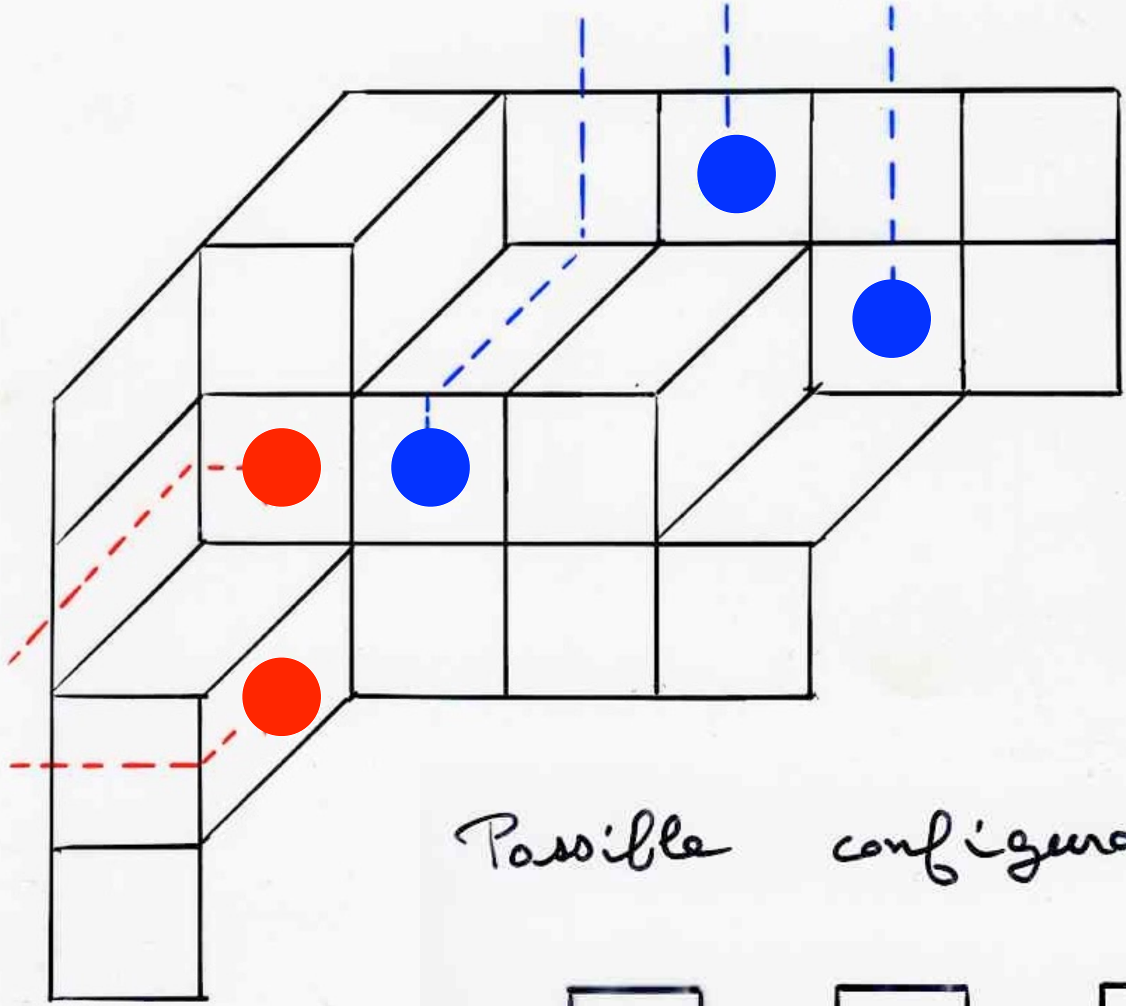
west-strips

north-strips

<u>Def-</u> Let f be a tiling of $\Gamma(X)$

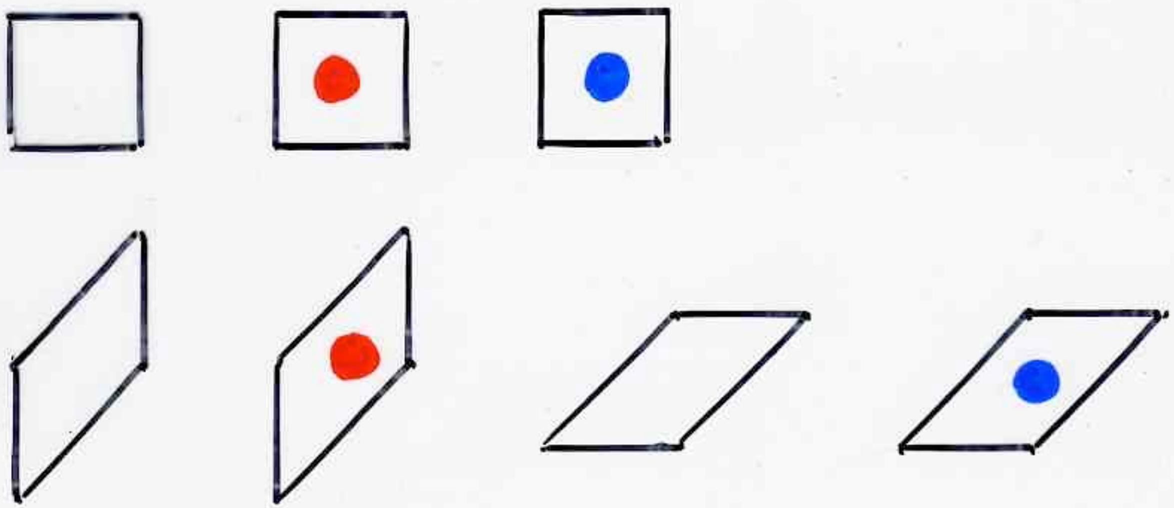
A rhombic alternative tableau T is a placement of 🔵, 🔴 in the tiles such that:

- a 🔴 is on a west-strip and any tile left of this 🔴 is empty.
- a 🔵 is on a north-strip and any tile north of this 🔵 is empty



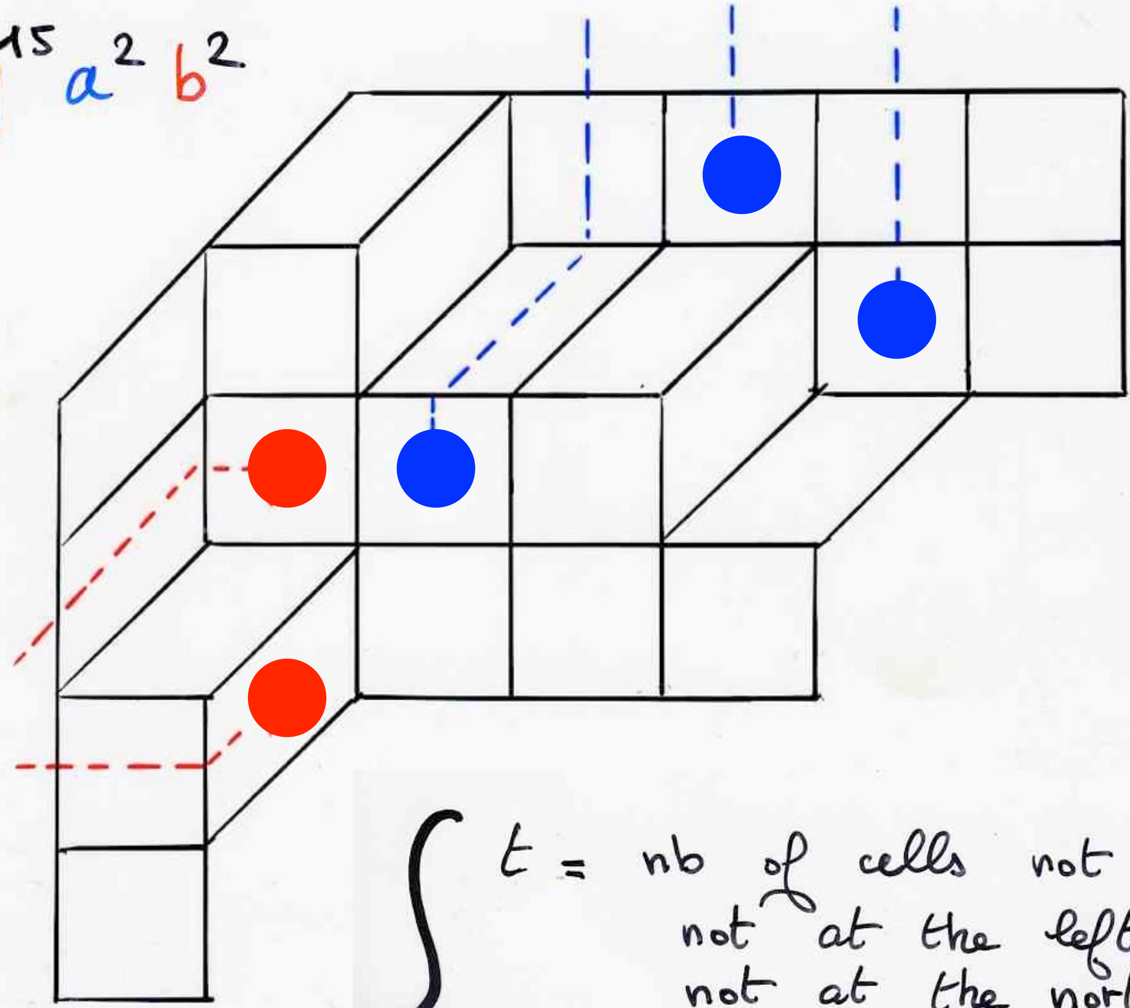
west- strips
north-strips

Possible configurations for a tile :



weight $\mathrm{wt}(T) = q^{t} a^{i} b^{j}$

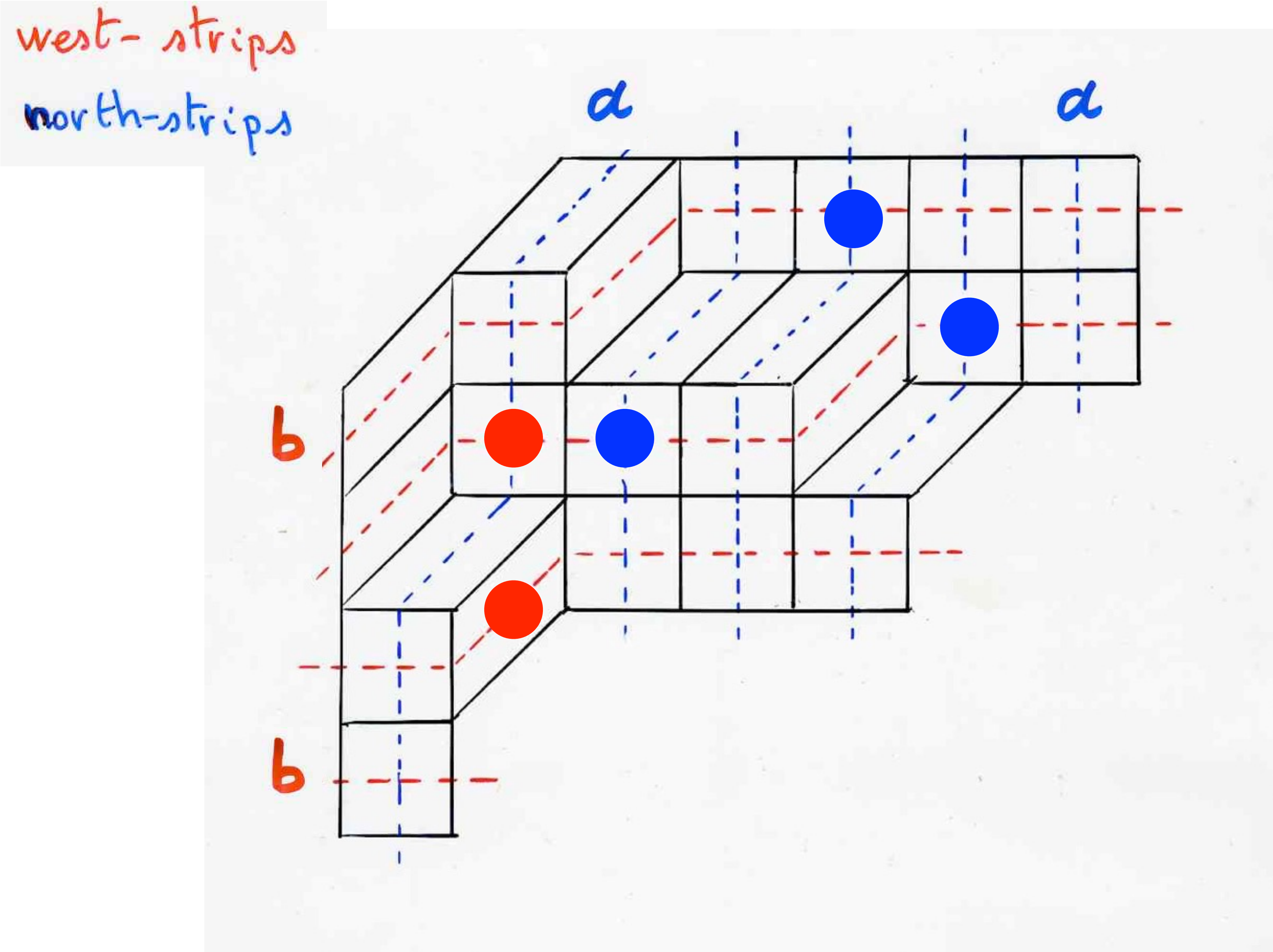
$\mathrm{wt}(T) = q^{15} a^{2} b^{2}$



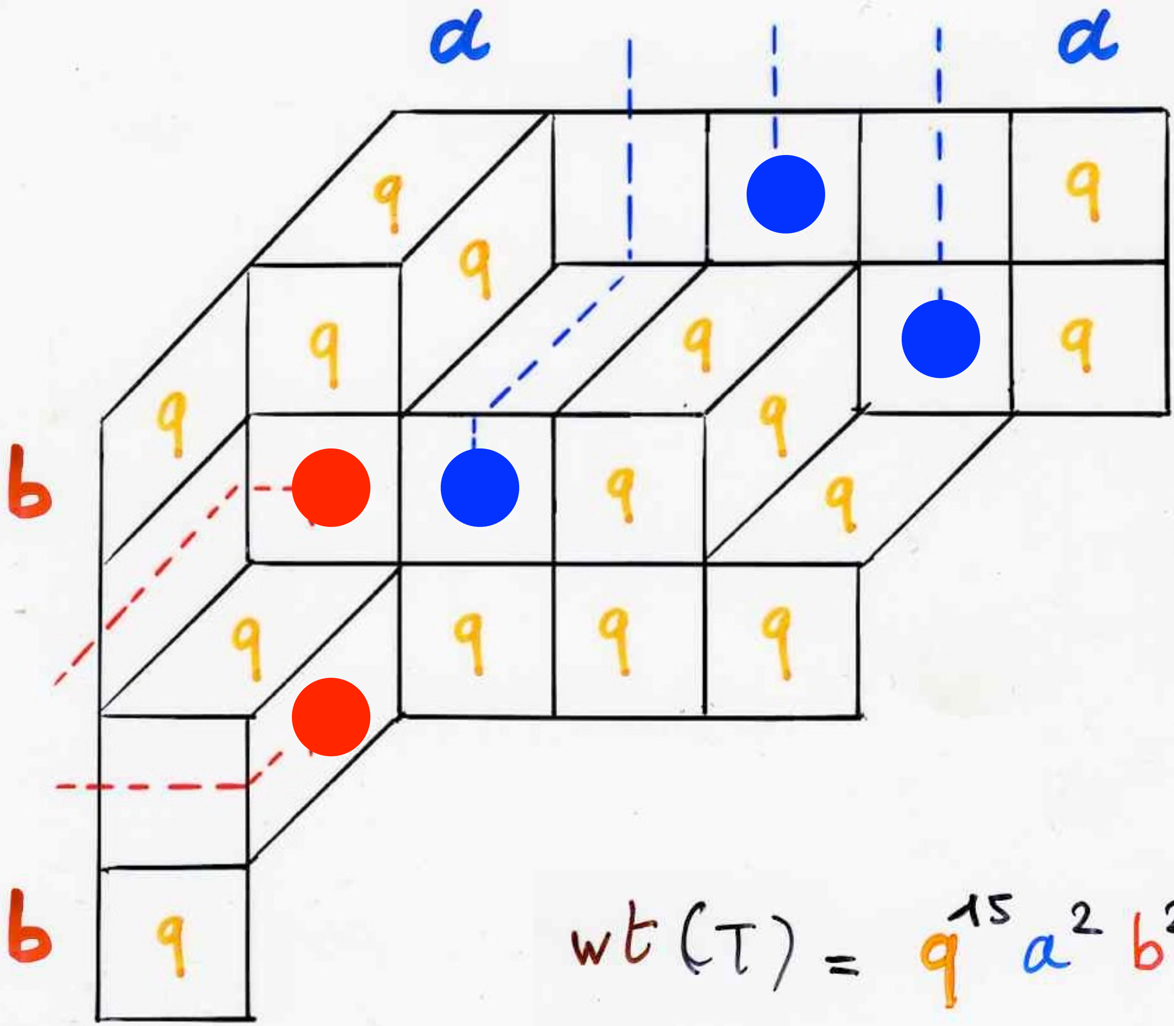
t = nb of cells not ●, not ●, not at the left of a ●, not at the north of a ●

i = nb of north-strips without a ●

j = nb of west-strips without a ●



west- strips
north-strips
a
a
b
b



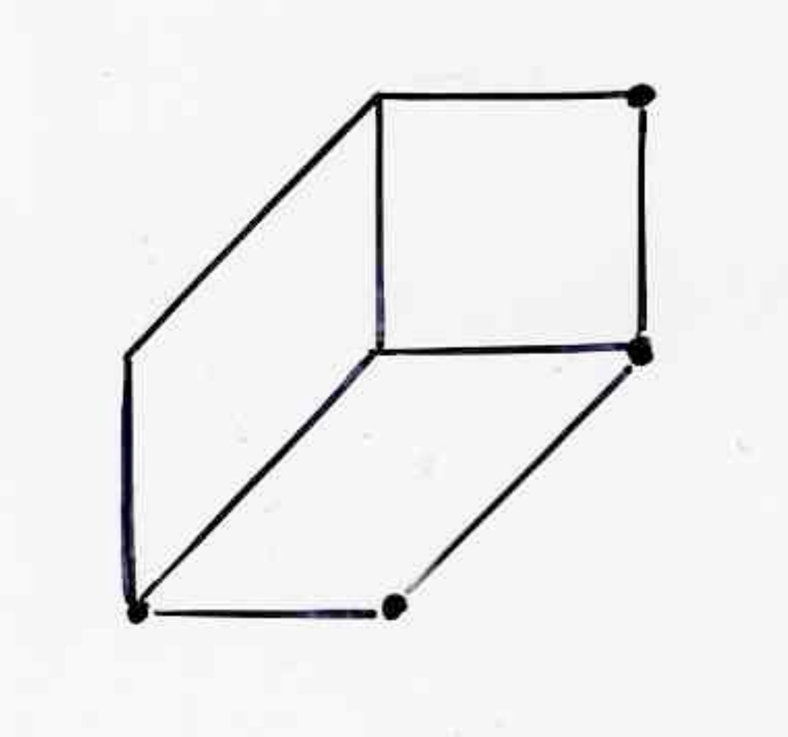
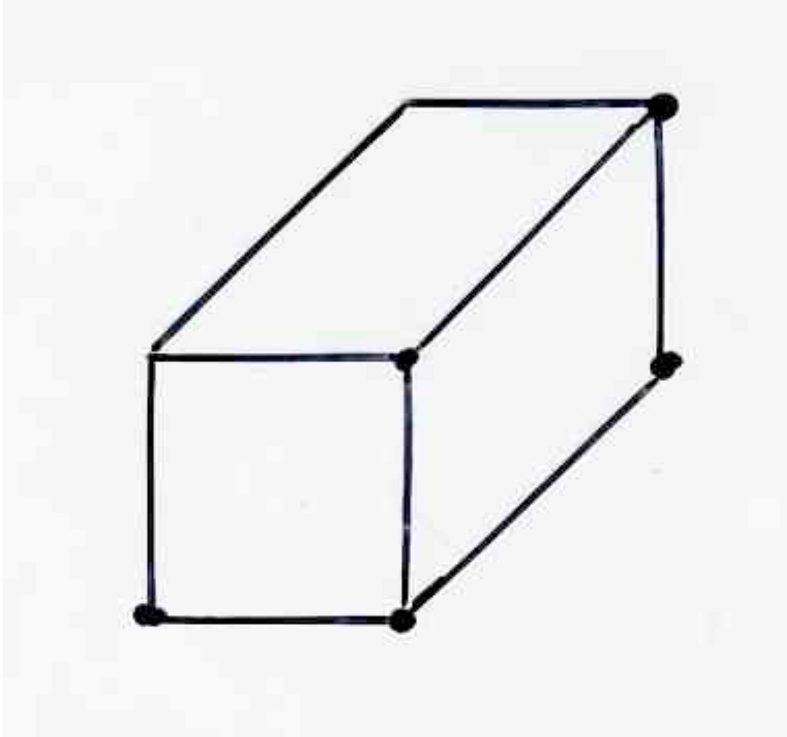
$wt(T) = q^{15} a^2 b^2$

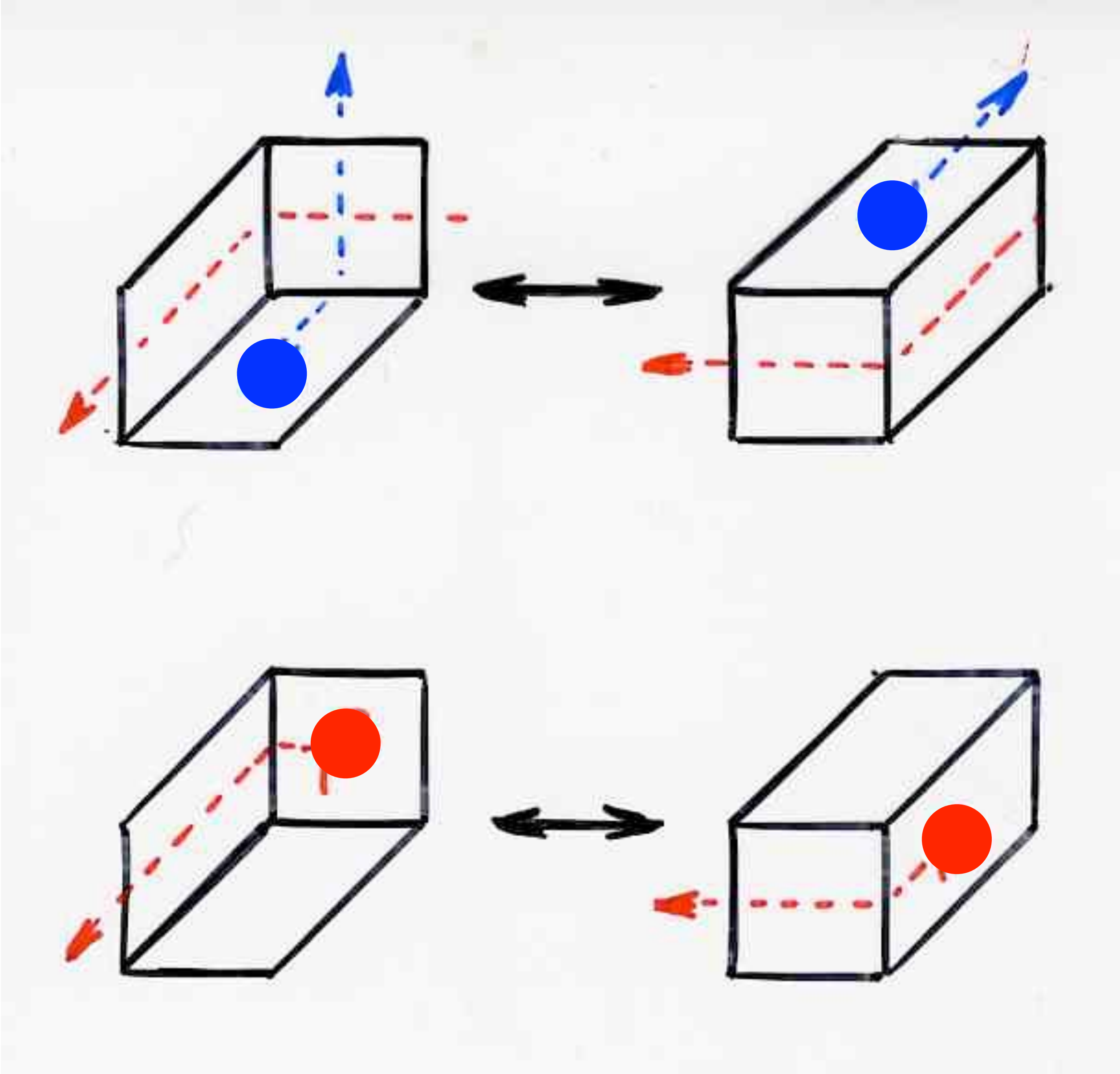
$R(X, \mathcal{t})$ set of rhombic alternative tableaux related to X, with the tiling $\mathcal{t}$ of $\Gamma(X)$

<u>Prop</u> X, $\Gamma(X)$ diagram

$\mathcal{t}$, $\mathcal{t}'$ tiling of $\Gamma(X)$

$$\sum_{T \in R(X, \mathcal{t})} \mathrm{wt}(T) = \sum_{T \in R(X, \mathcal{t}')} \mathrm{wt}(T)$$



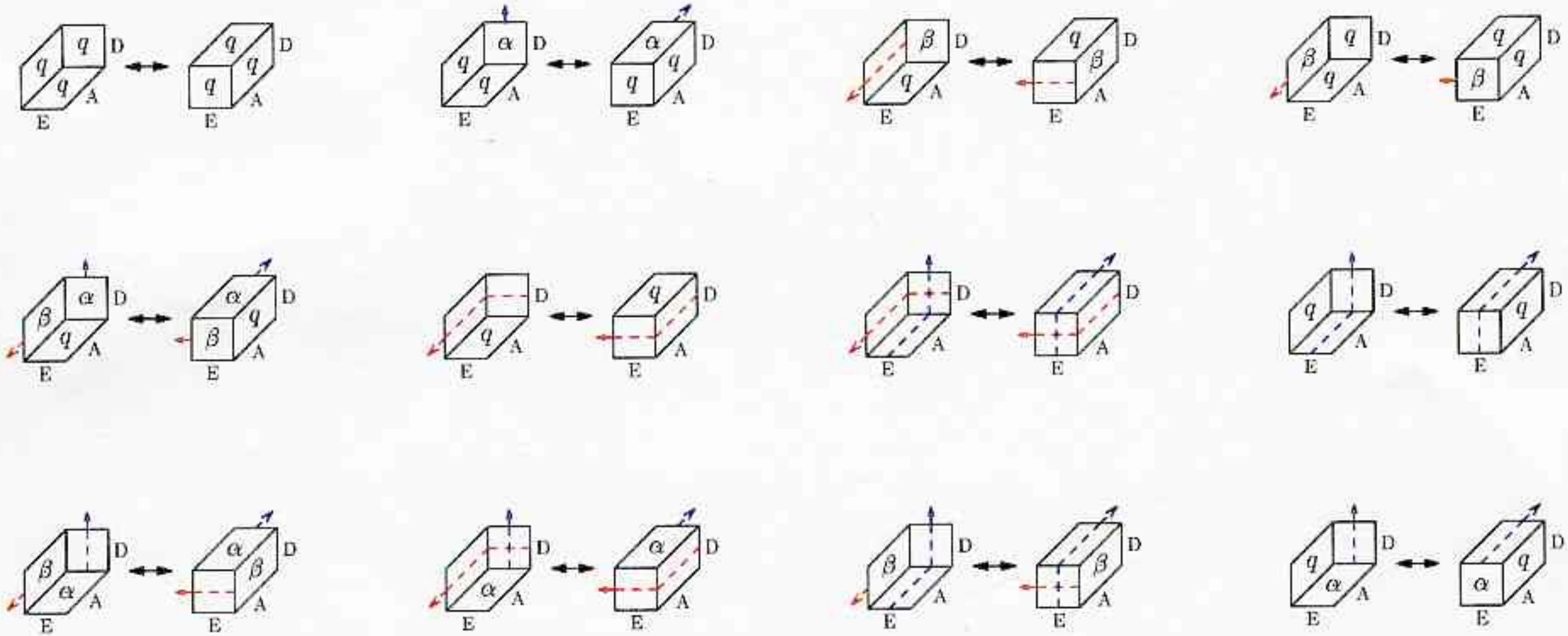


Figure 11: The involution ϕ from each possible filling of a minimal hexagon (left) to a maximal hexagon (right). The arrows imply compatibility requirements.

combinatorial interpretation
of
stationary probabilities

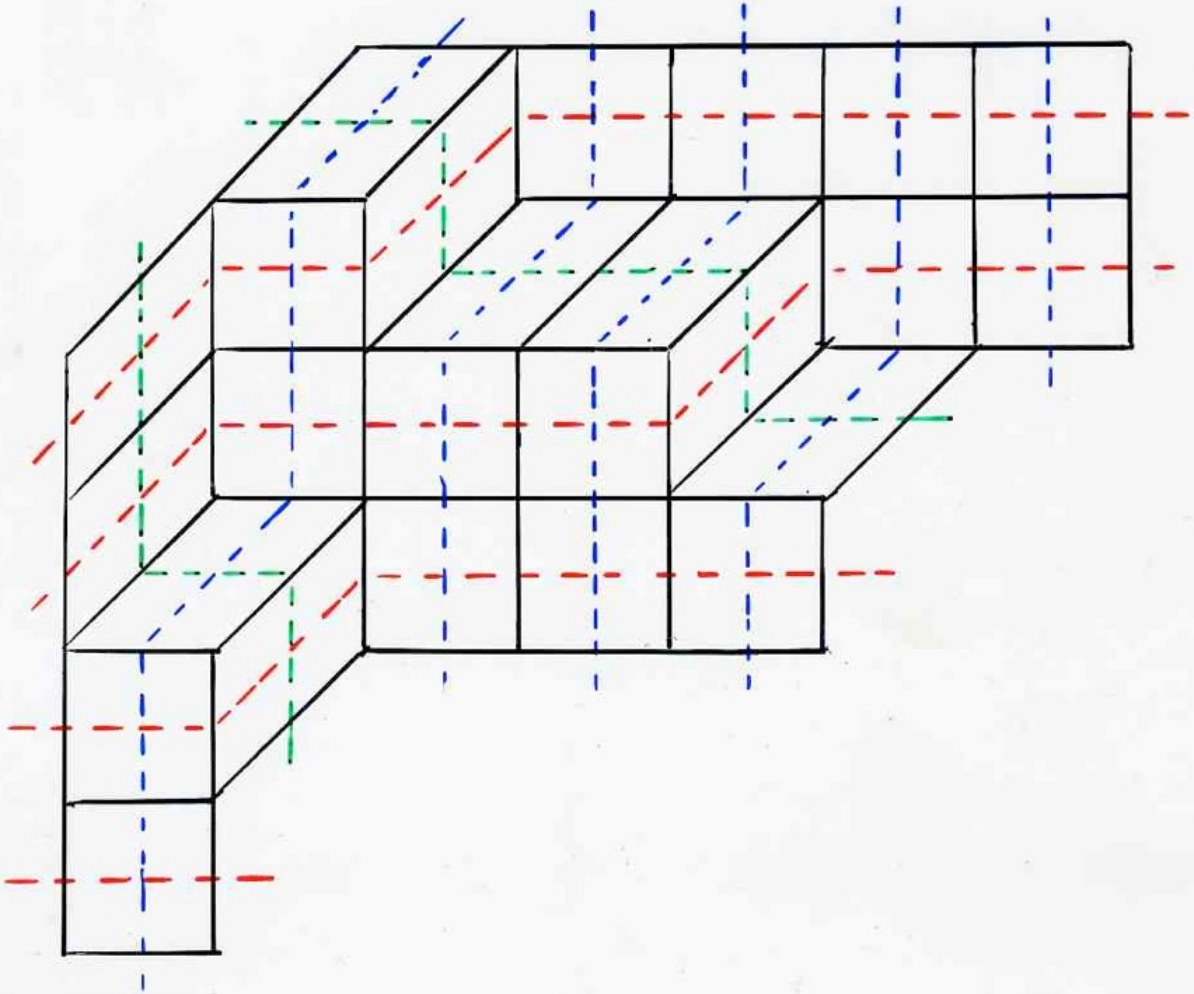
$$\mathrm{Prob}(X) = \frac{1}{Z^*_{n,r}} \sum_{T \in R(X,\tau_x)} \underbrace{q^t \left(\frac{1}{\alpha}\right)^i \left(\frac{1}{\beta}\right)^j}_{wt(T)}$$

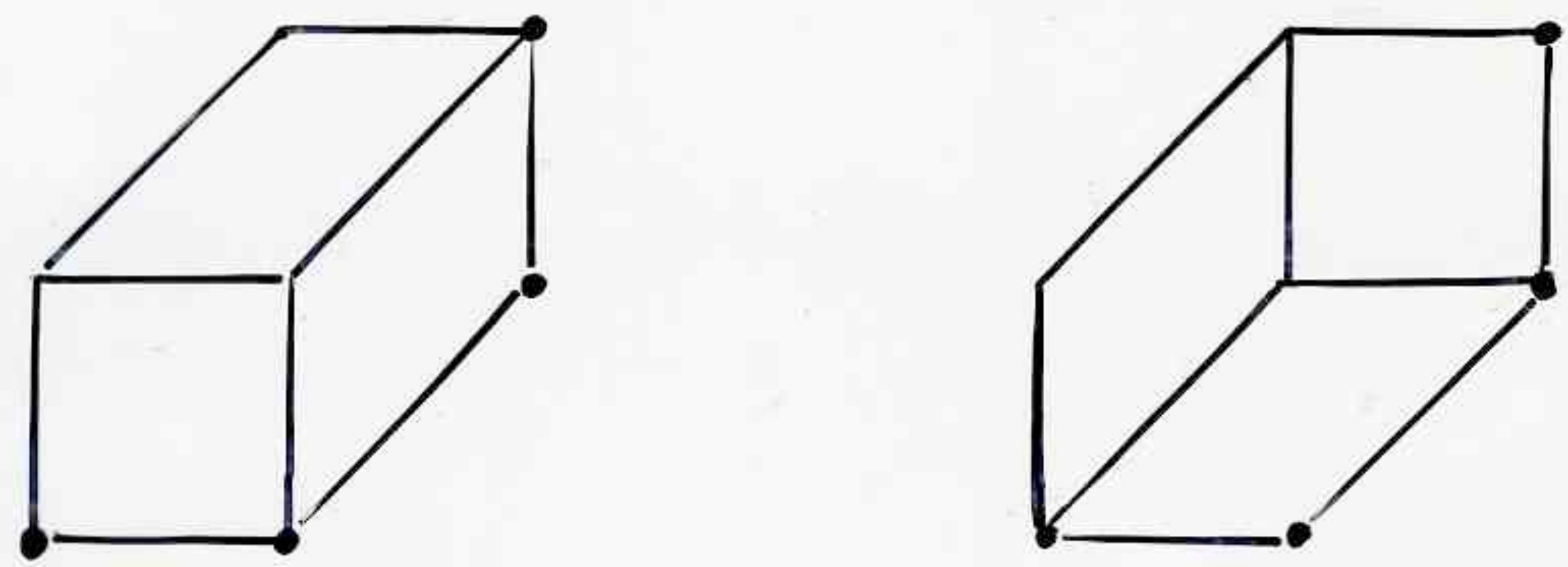
$$a = \frac{1}{\alpha}$$

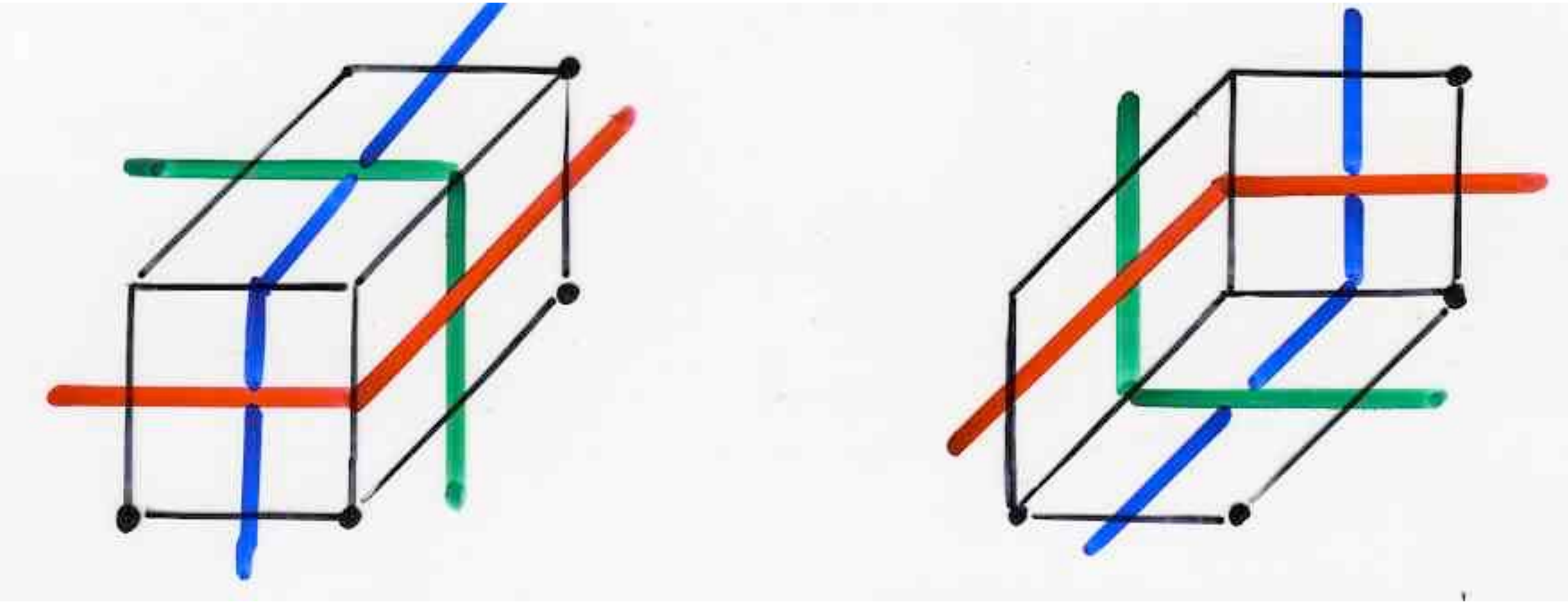
$$b = \frac{1}{\beta}$$

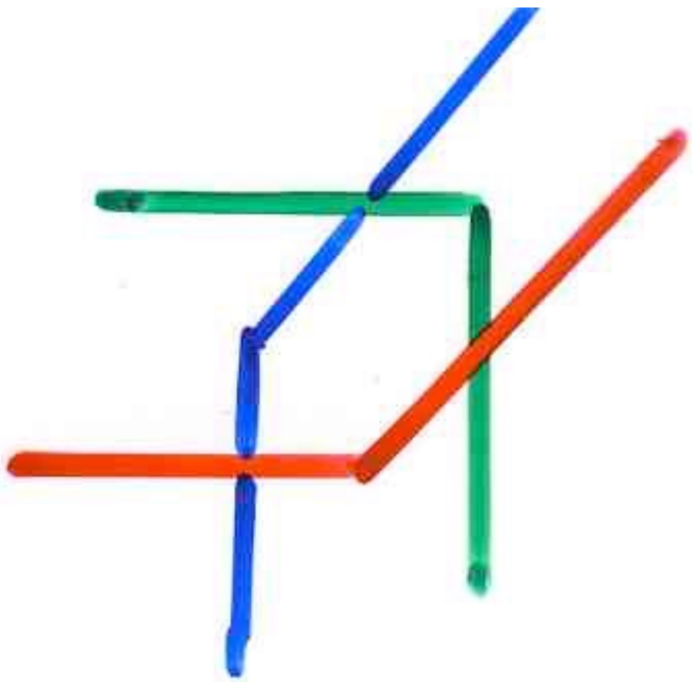
$$Z^*_{n,r} = \sum_{X} \sum_{T \in R(X,\tau_x)} q^t \left(\frac{1}{\alpha}\right)^i \left(\frac{1}{\beta}\right)^j$$

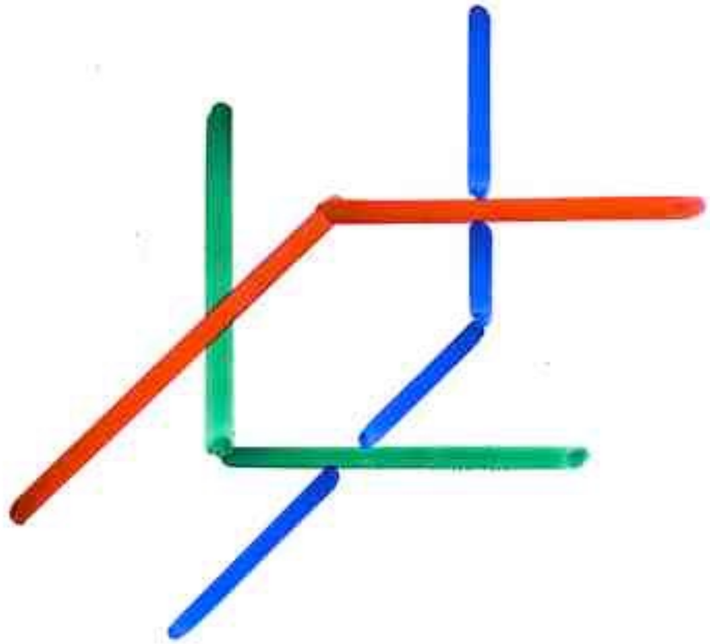
quelques remarques







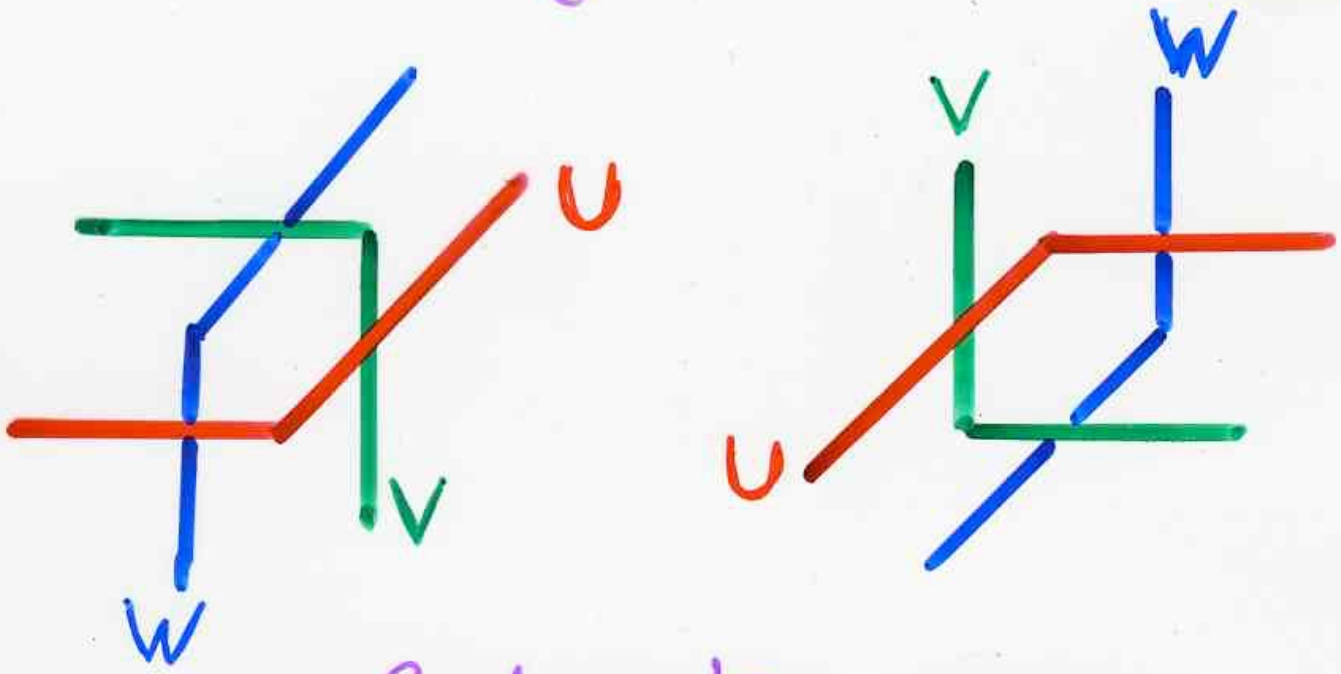




Yang-Baxter equation

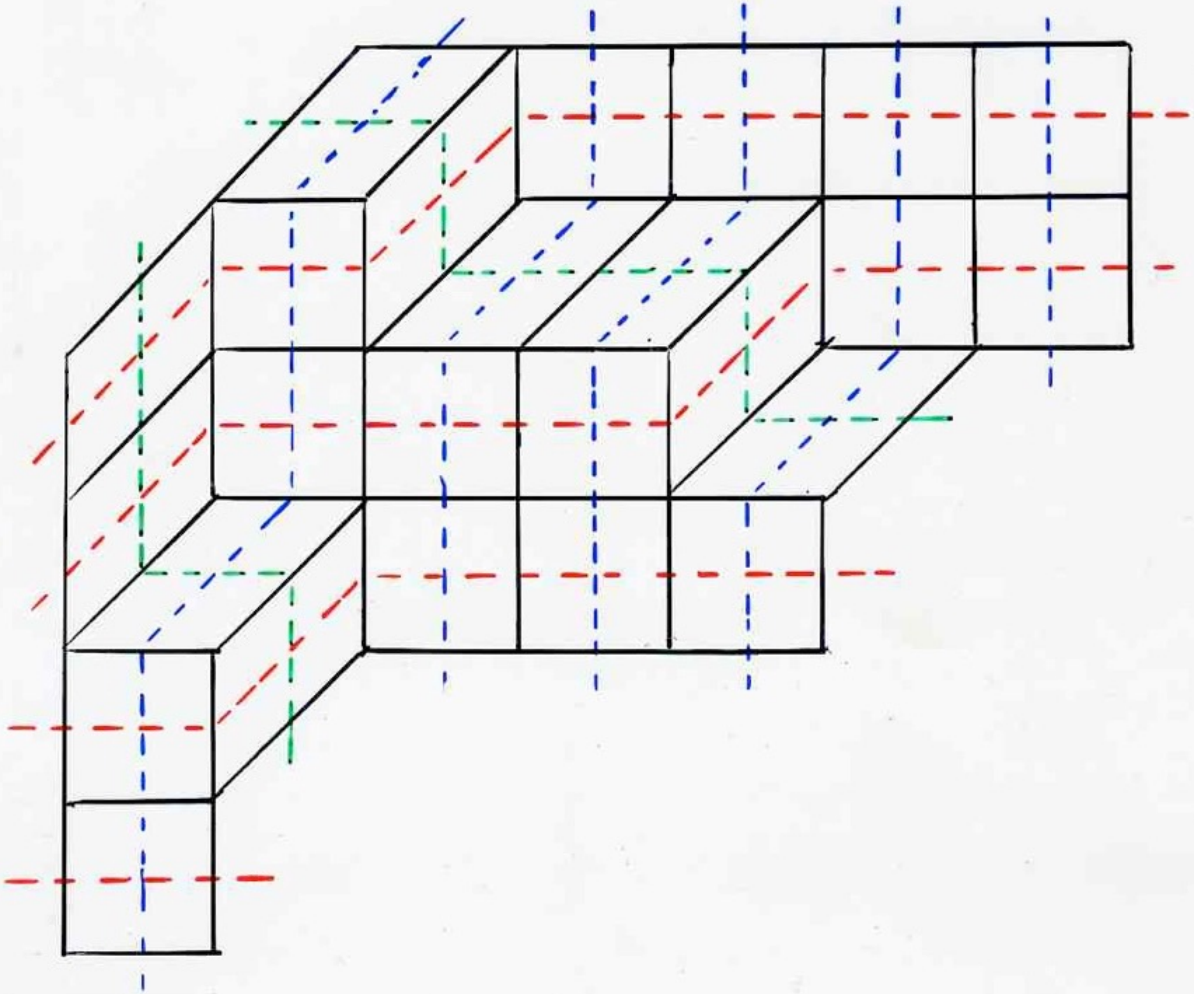
$UVW = WVU$

Yang-Baxter moves



Reidemester moves

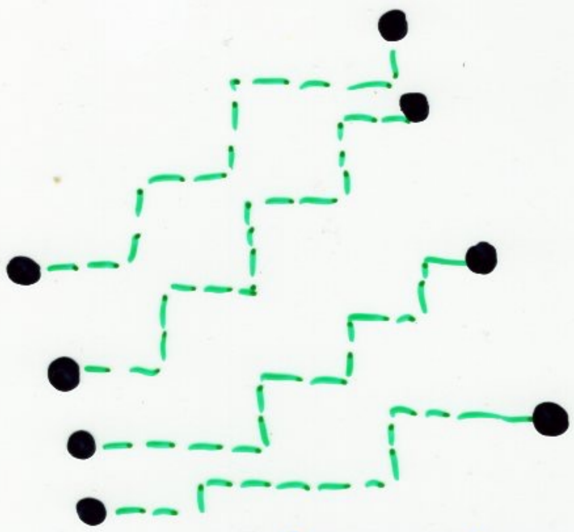
(knot theory)



LGV- Lemma

determinant =

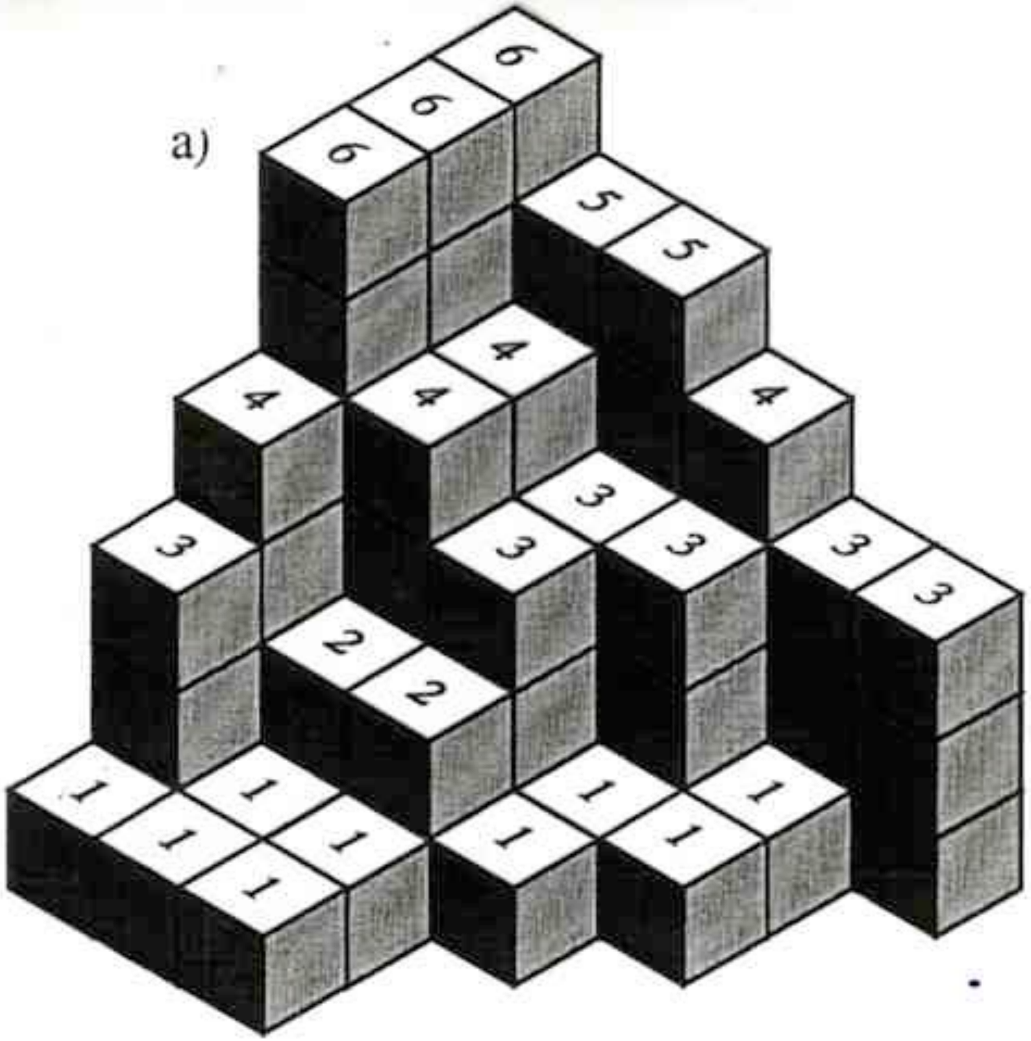
(under certain conditions)



non-intersecting

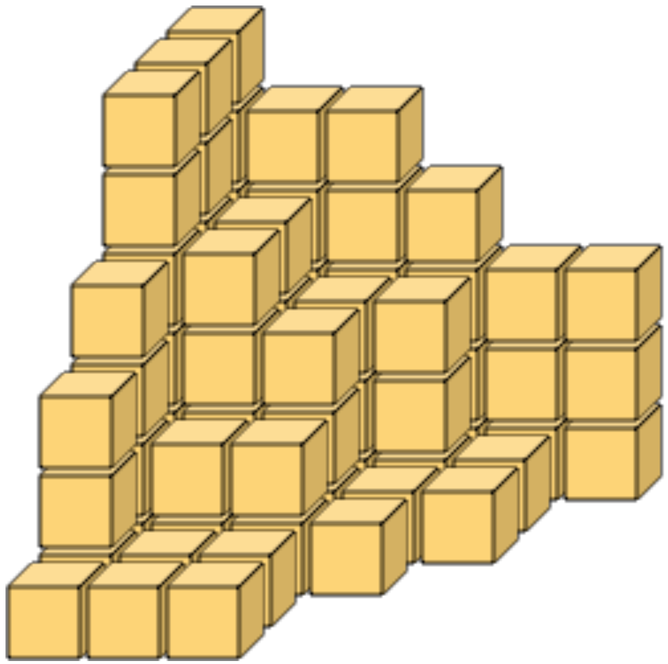
paths

a)



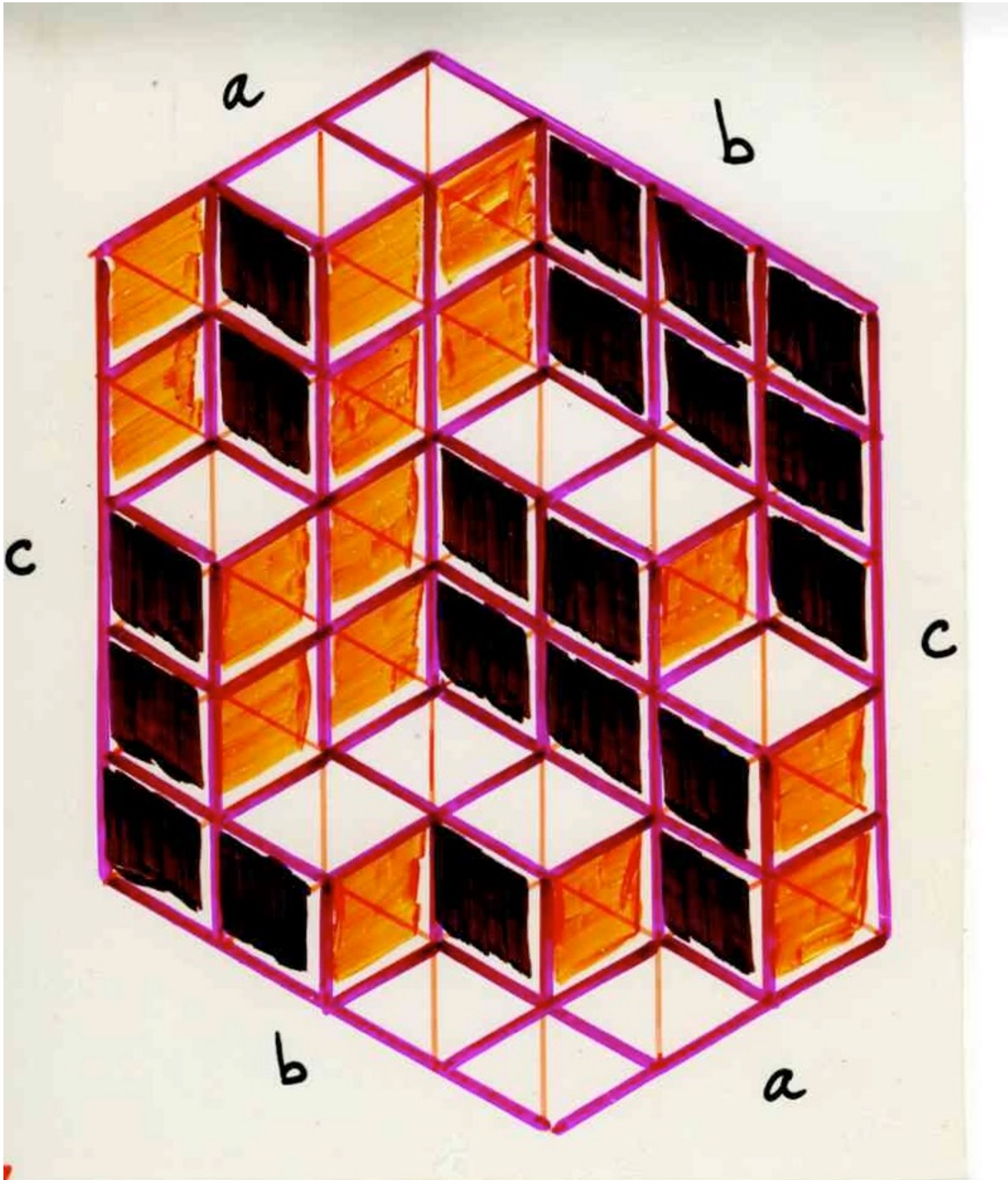
b)

6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			



example:
plane
partitions
in a box

(MacMahon
formula)

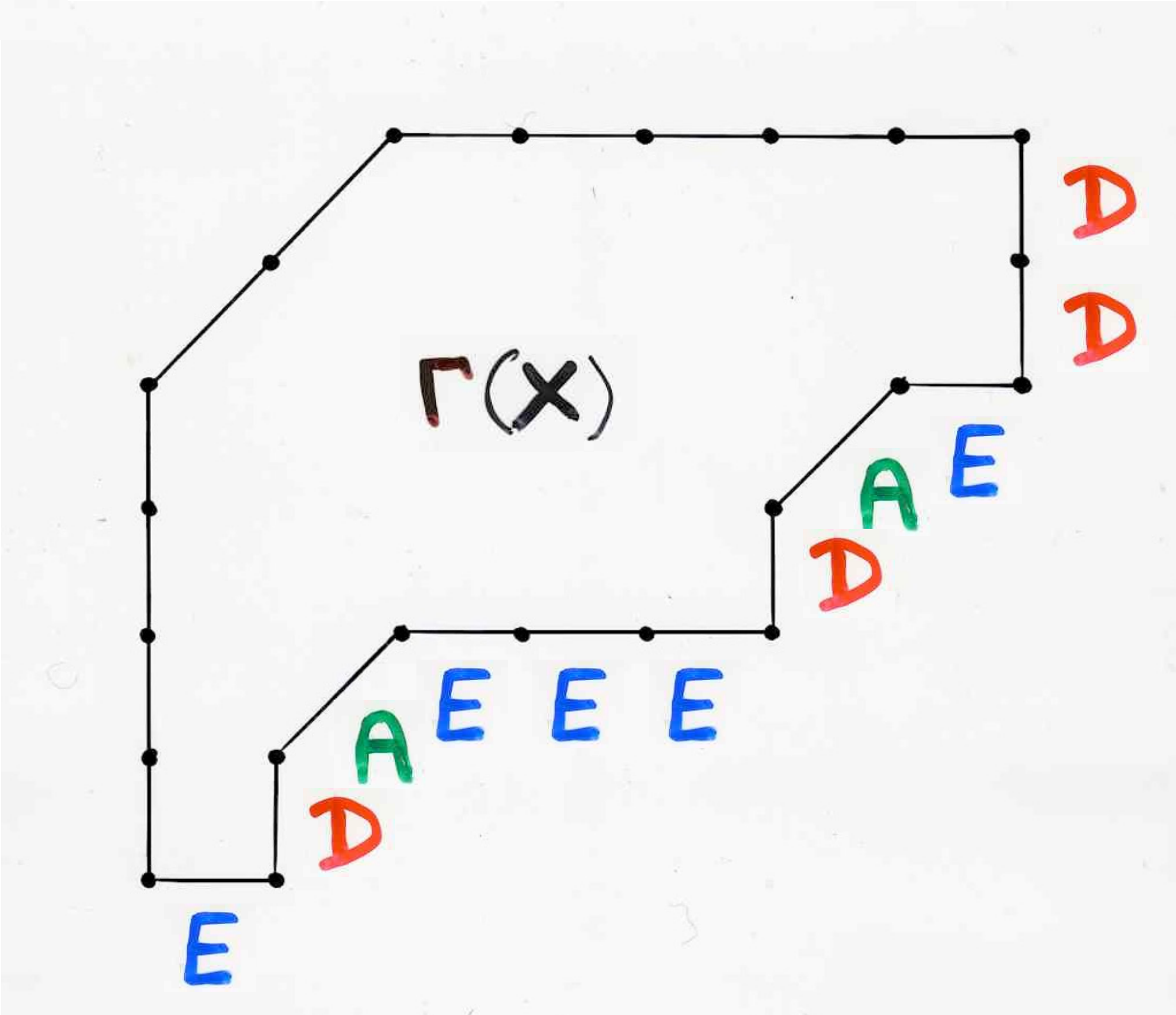


$$\prod_{\substack{1 \leq i \leq a \\ 1 \leq j \leq b \\ 1 \leq k \leq c}} \frac{i+j+k-1}{i+j+k-2}$$

2-PASEP algebra

$$
\begin{cases}
DE = qED + D + E \\
DA = qAD + A \\
AE = qEA + A
\end{cases}
$$

X = D D E A D E E E A D E



In the 2-PASEP algebra

Every word $X \in \{D, E, A\}^*$ can be expressed in a unique way

$$X = \sum_{T \in R(X, \mathcal{T})} q^t E^i A^r D^j$$

where :

$r = |X|_A$ (nb of A in the word X)

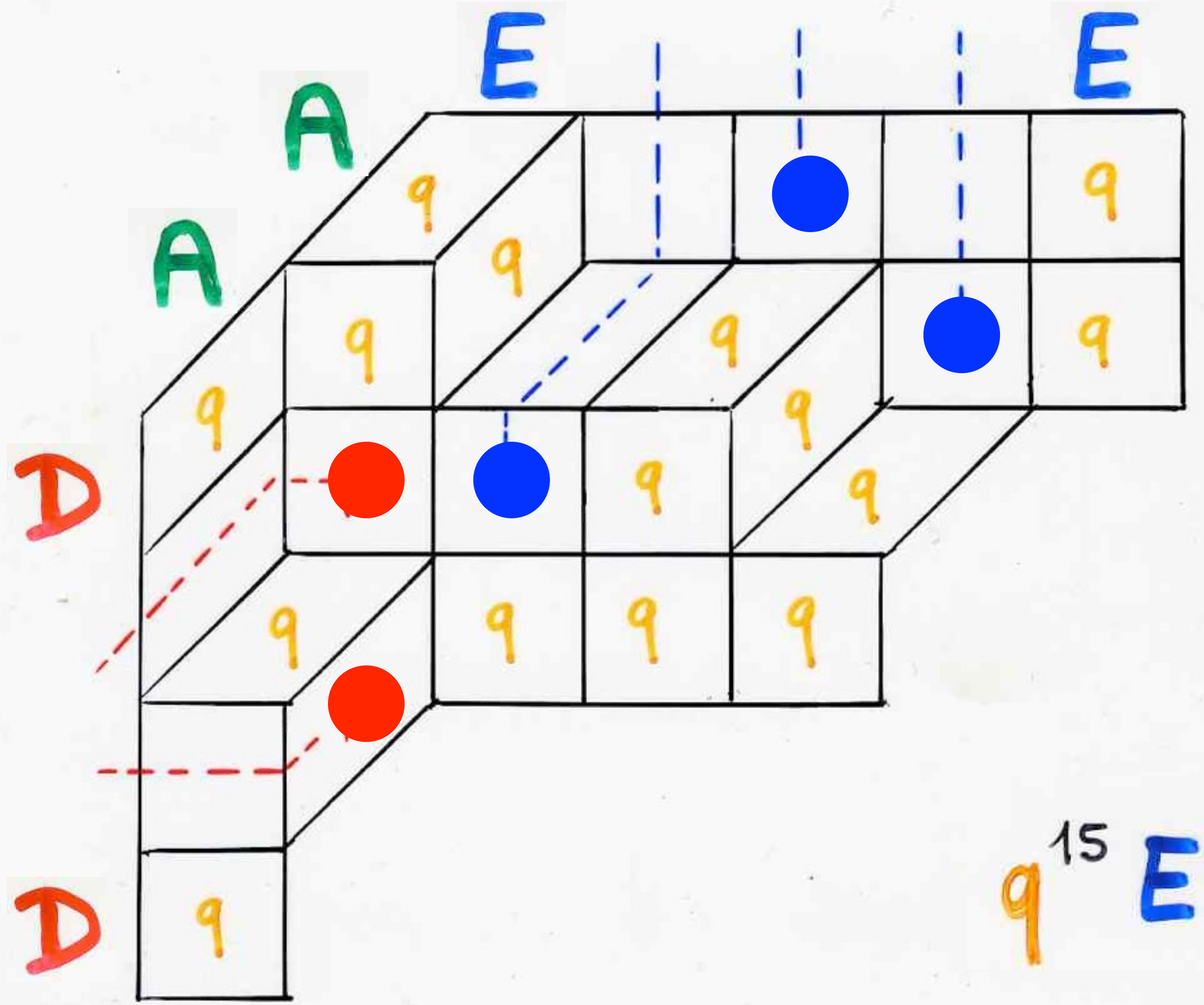
$\mathcal{T}$ a fixed tiling of $\Gamma(X)$

i = nb of free north-strips in T (not containing an )

j = nb of free south-strips in T (=not containing a )

t = nb of cells labeled q in T

D D E A D E E E A D E



$q^{15}E^2A^2D^2$

combinatorial interpretation
of
stationary probabilities

$$\mathrm{Prob}(X) = \frac{1}{Z_{n,r}} \langle W | \prod_{i=1}^{n} D\,\mathbb{1}_{(x_i=\bullet)} + A\,\mathbb{1}_{(x_i=\bullet)} + E\,\mathbb{1}_{(x_i=0)} | V \rangle$$

$$\langle W | X | V \rangle = \sum_{T \in R(X,\tau)} q^{t} \left(\frac{1}{\alpha}\right)^{i} \left(\frac{1}{\beta}\right)^{j} \langle W | A^{r} | V \rangle$$

i = nb of free north-strips in T
(=not containing an ●)

j = nb of free south-strips in T
(=not containing a ●)

t = nb of cells labeled q in T

$$Z_{n,r} = \text{coeff. of } y^r \text{ in } \langle W | (D + yA + E)^n | V \rangle$$

$$Z_{n,r} = Z^*_{n,r} \langle W | A^r | V \rangle$$

$$Z^*_{n,r} = \sum_{X} \sum_{T \in R(X, T_X)} q^t \left(\frac{1}{\alpha}\right)^i \left(\frac{1}{\beta}\right)^j$$

$$\mathrm{Prob}(X) = \frac{1}{Z^*_{n,r}} \sum_{T \in R(X, T_X)} \underbrace{q^t \left(\frac{1}{\alpha}\right)^i \left(\frac{1}{\beta}\right)^j}_{wt(T)}$$

enumeration
of
rhombic alternative tableaux

$$Z^{*}_{n,r}(\alpha = \beta = q = 1) = \binom{n}{r} \frac{(n+1)!}{(r+1)!}$$

Lah numbers

nb of "assemblées" of permutations

$$\left\{ [7, 10, 5, 8] \; [3, 1, 4] \; [9, 2, 11, 6] \right\}$$

$$\exp\left(\frac{x\,t}{1-t}\right)$$

$$Z^{*}_{n,r}(\alpha, \beta, q=1) = \binom{n}{r} \prod_{i=r}^{n-1} (\alpha^{-1} + \beta^{-1} + i)$$

arXiv : 1506.01980
[math.CO]

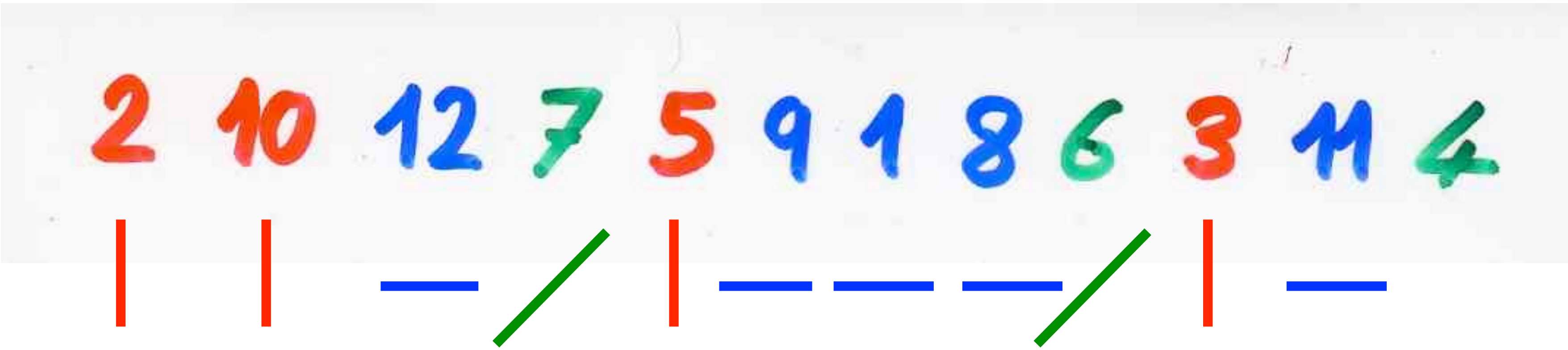
from «assemblées» of permutations

to

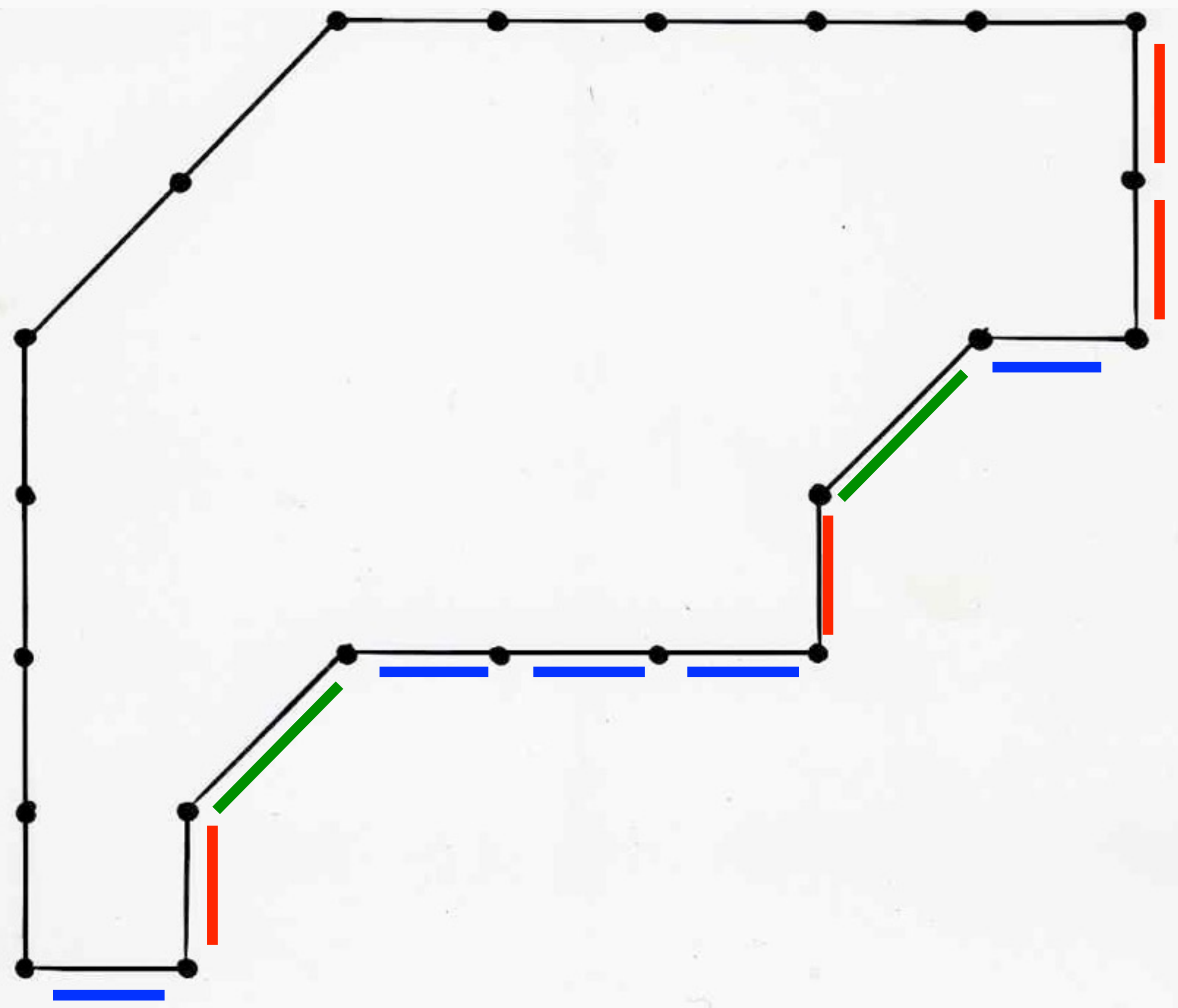
rhombic alternative tableaux

for an assemblée $\alpha \to$ permutation σ

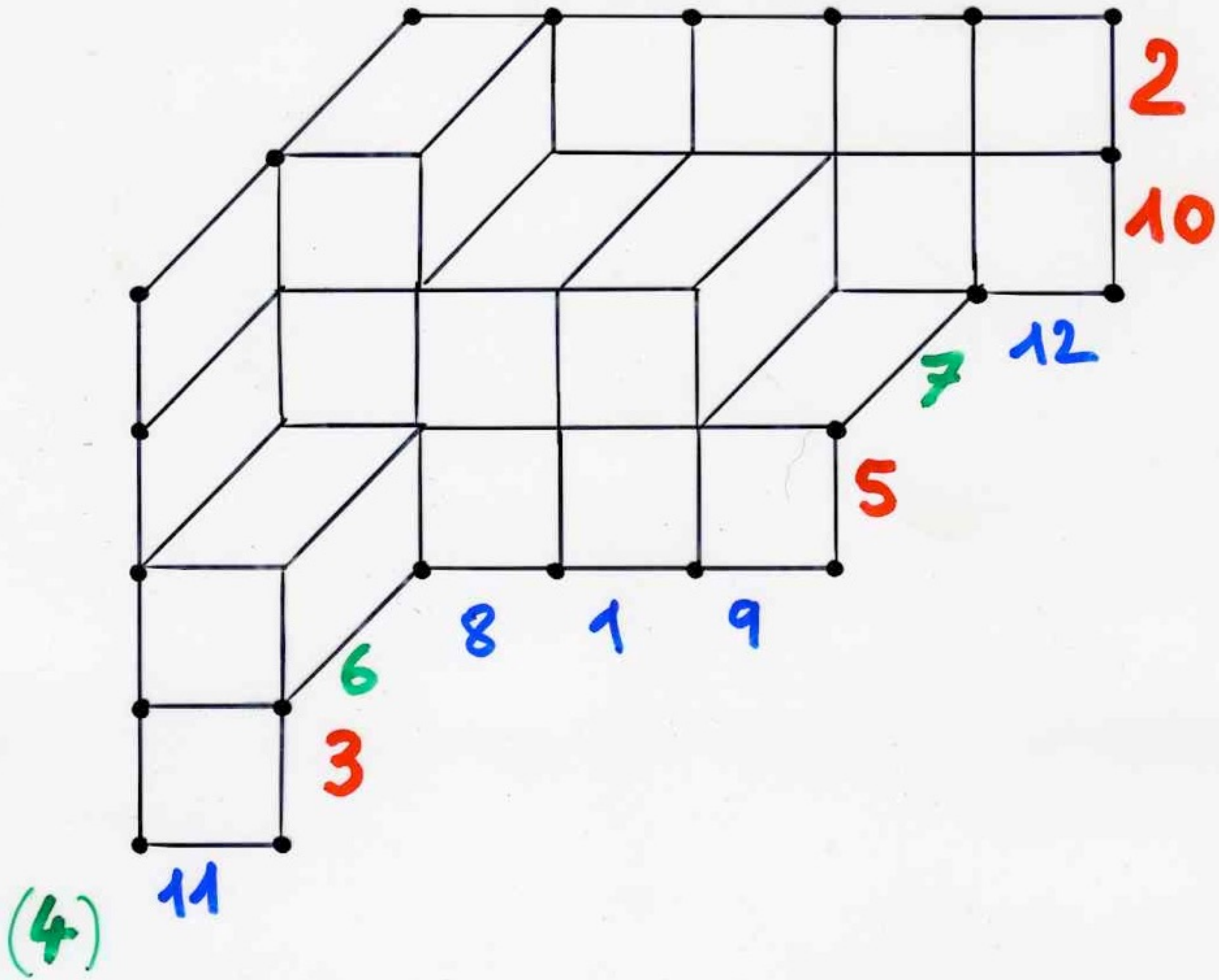
$\alpha \to \sigma \to$ X word $\{\bullet, \bullet, \bullet\} \to \Gamma(X)$ diagram

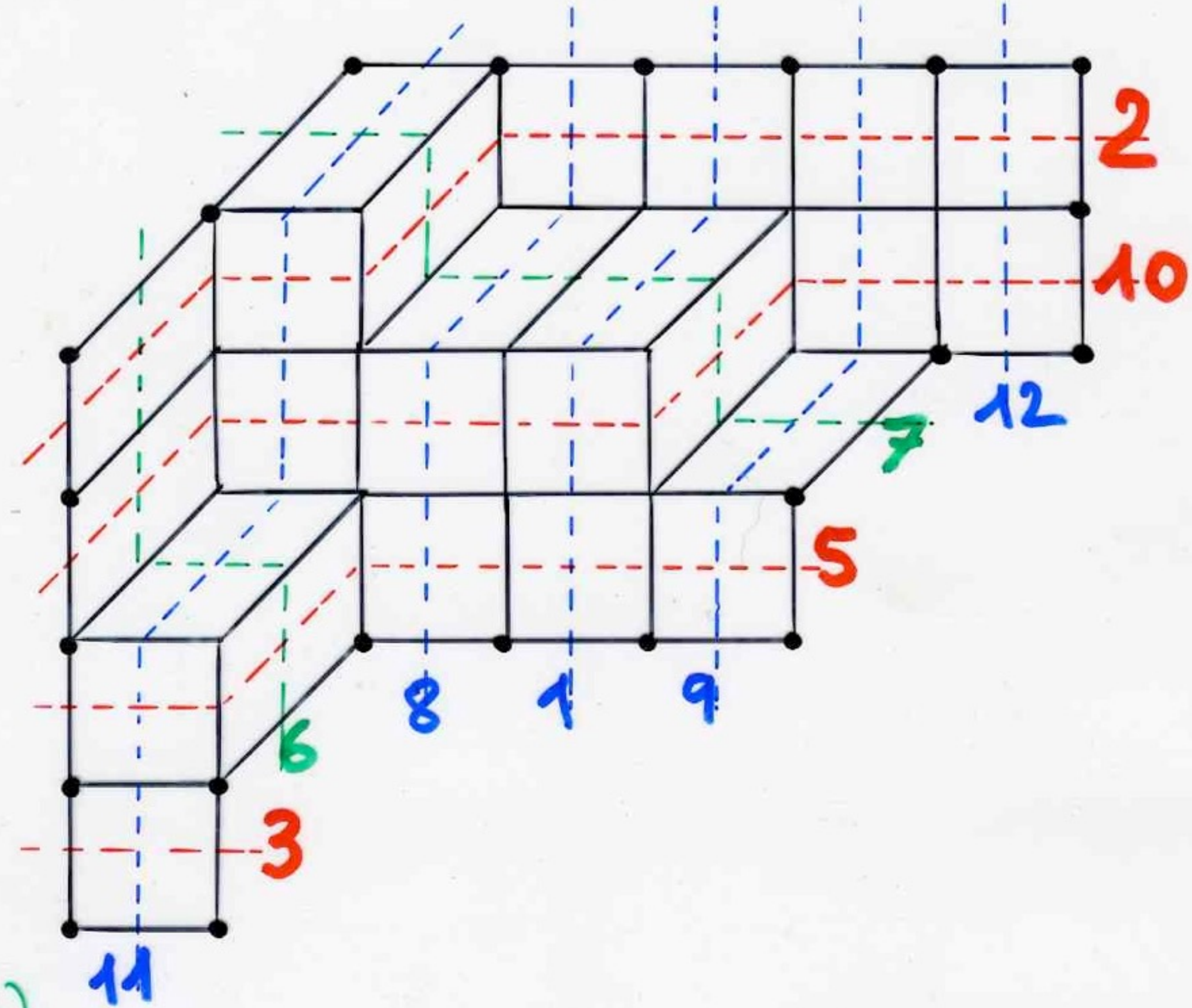


2 10 12 7 5 9 1 8 6 3 11 4

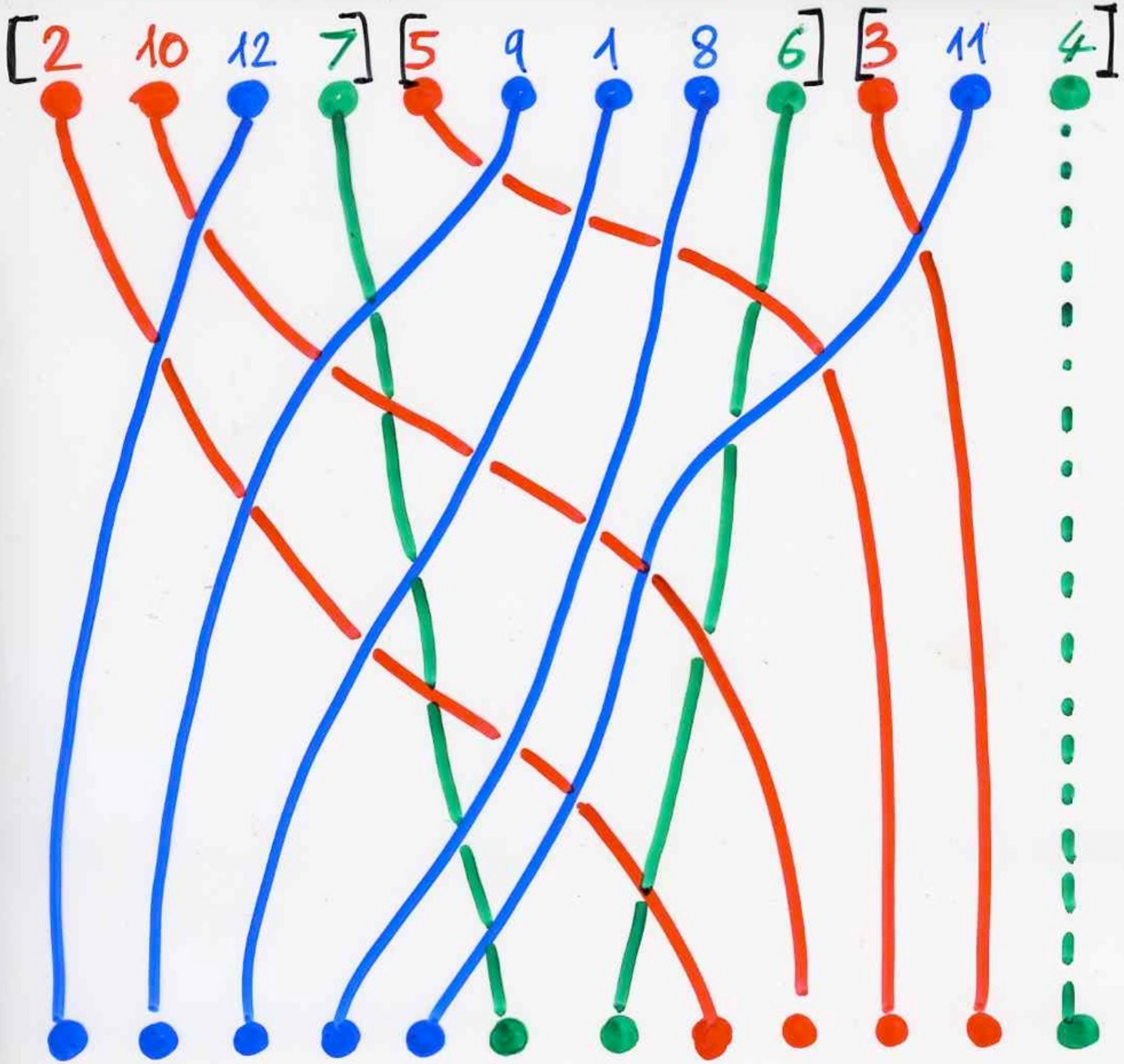


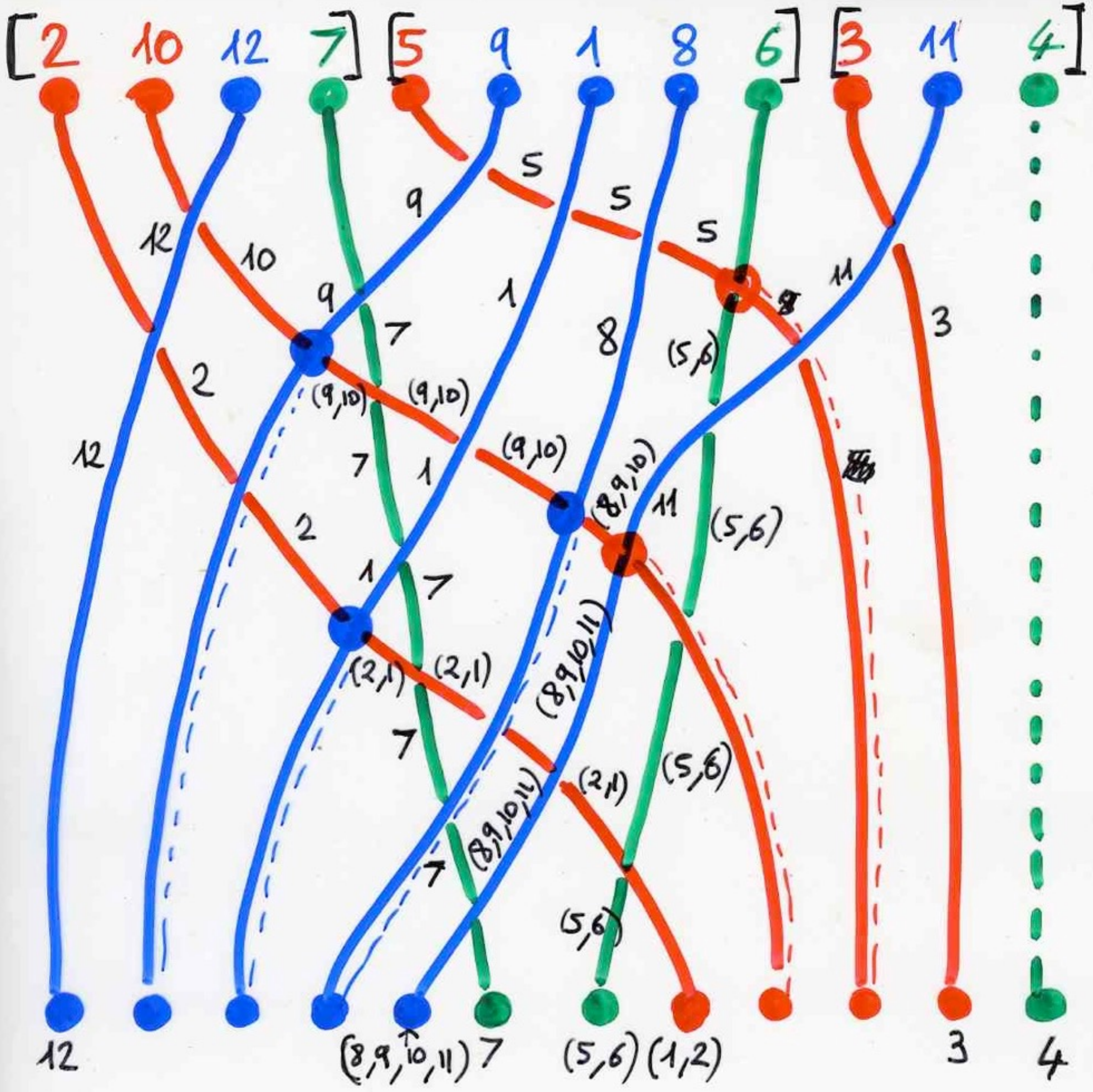
2 10 12 7 5 9 1 8 6 3 11 4



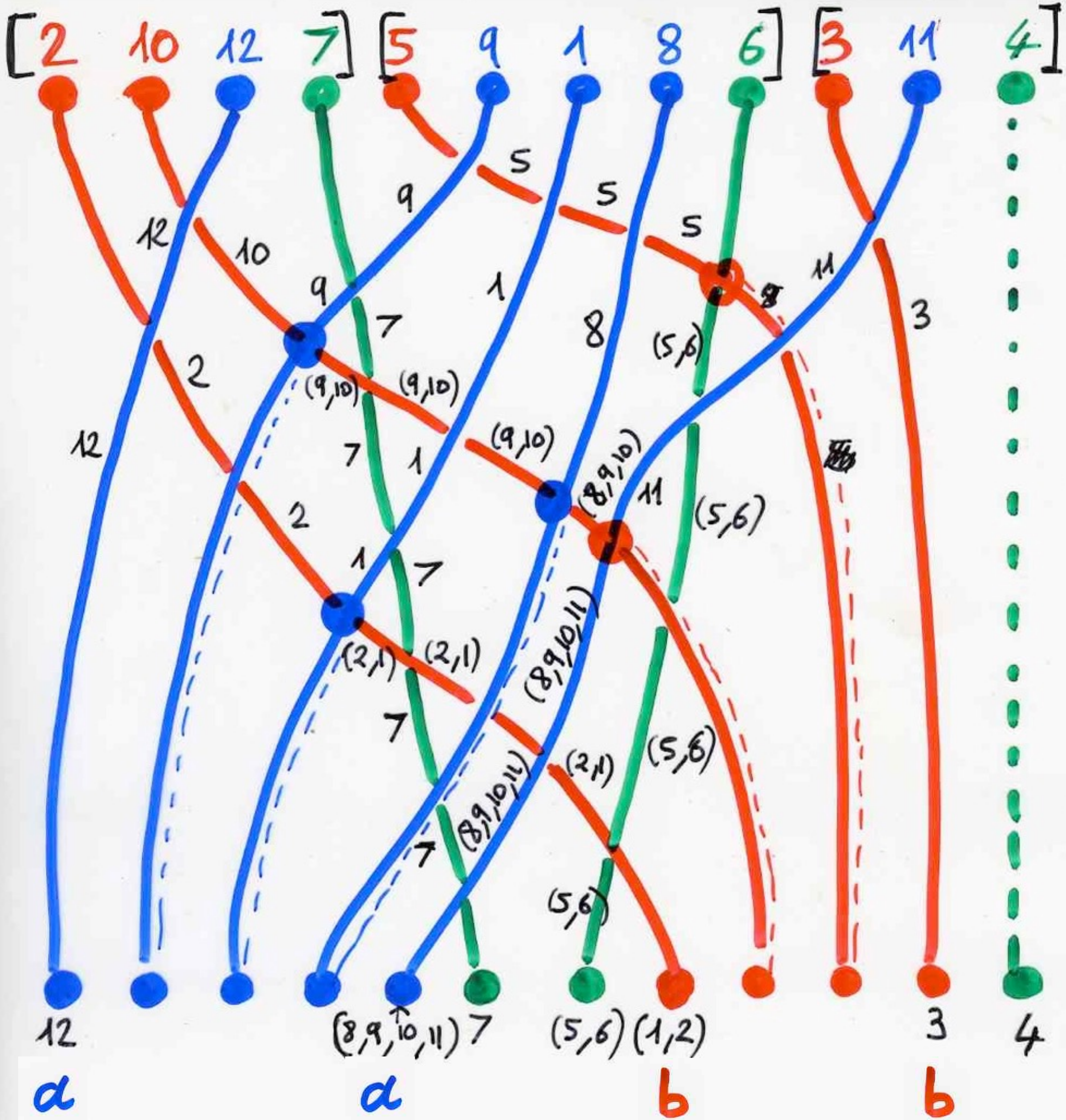


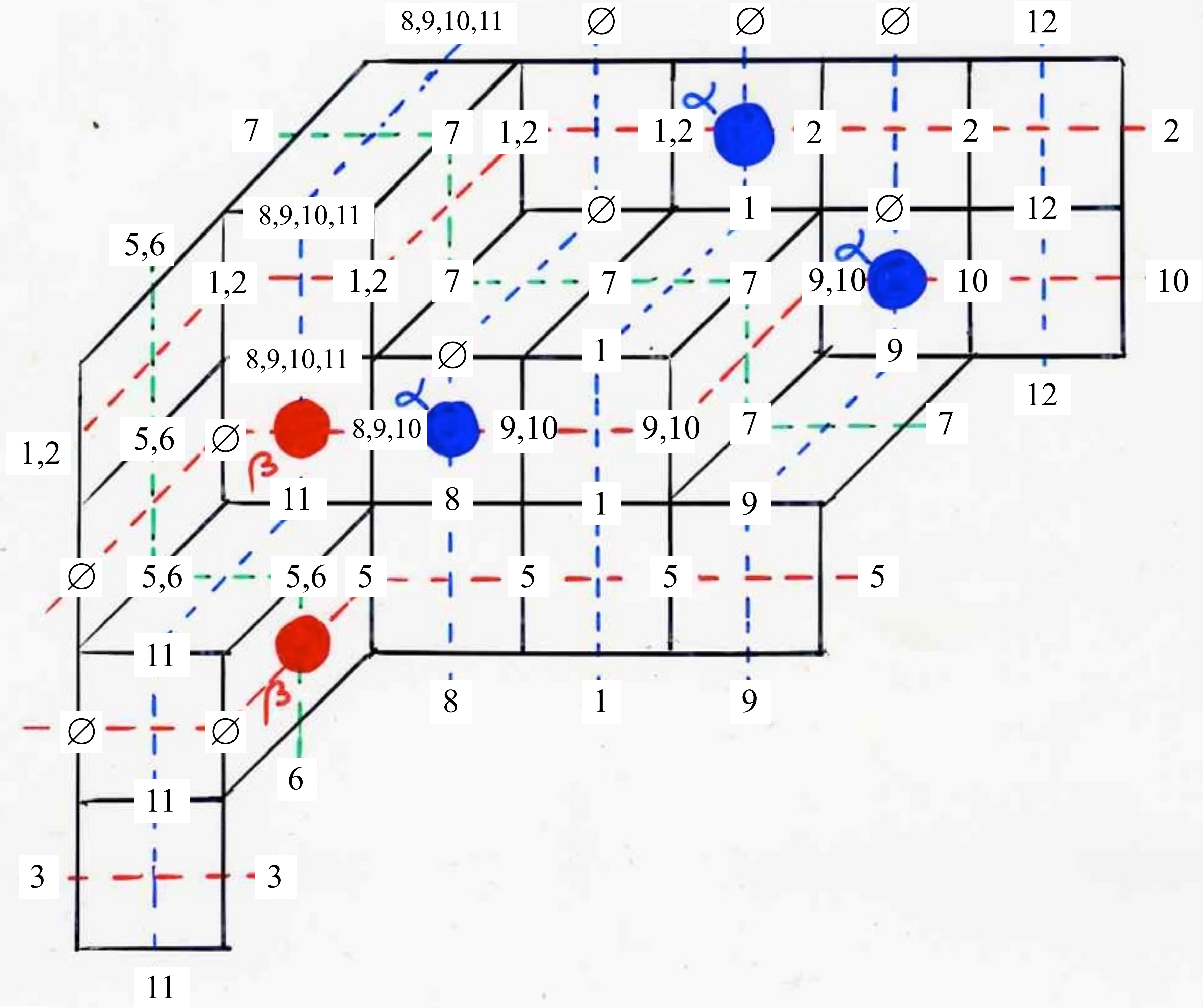
(4)


[2 10 12 7] [5 9 1 8 6] [3 11 4]



[2 10 12 7] [5 9 1 8 6] [3 11 4]
5
5
5
9
12
10
9
7
1
8
(5,6)
11
3
2
(9,10)
(9,10)
(9,10)
12
7
1
(8,9,10)
11
(5,6)
2
1
7
(2,1)
(2,1)
(8,9,10,11)
7
(2,1)
(5,6)
(8,9,10,11)
7
(5,6)
12
(8,9,10,11) 7
(5,6) (1,2)
3
4


[2 10 12 7] [5 9 1 8 6] [3 11 4]
12
(8,9,10,11) 7
(5,6) (1,2)
3
4
a
a
b
b



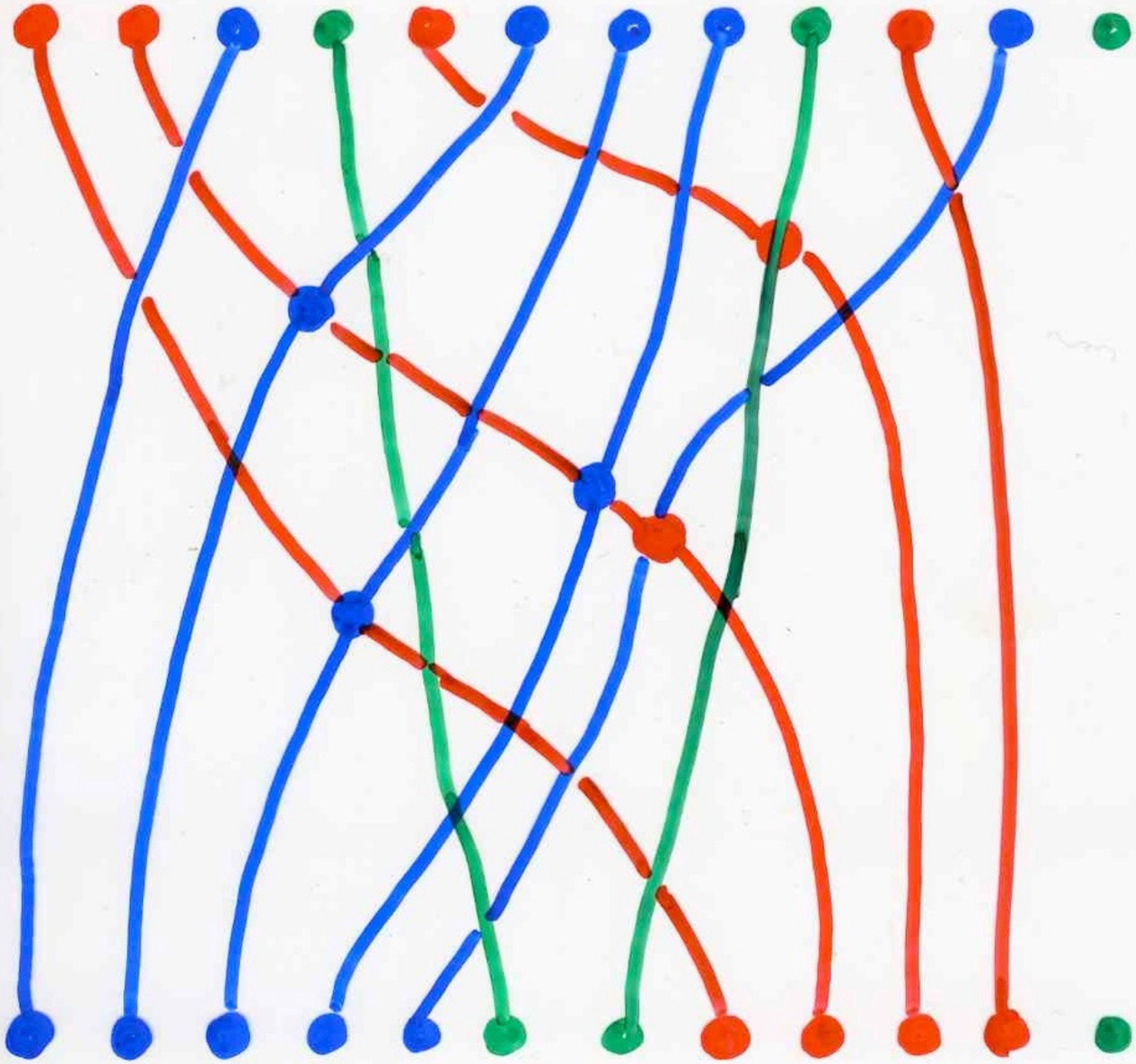
8,9,10,11
∅
∅
∅
12
7
7
1,2
1,2
α
2
2
2
8,9,10,11
∅
1
∅
12
5,6
1,2
1,2
7
7
7
α
9,10
10
10
8,9,10,11
∅
1
9
12
α
1,2
5,6
∅
8,9,10
9,10
9,10
7
7
β
11
8
1
9
∅
5,6
5,6
5
5
5
5
11
β
8
1
9
∅
∅
6
11
3
3
11

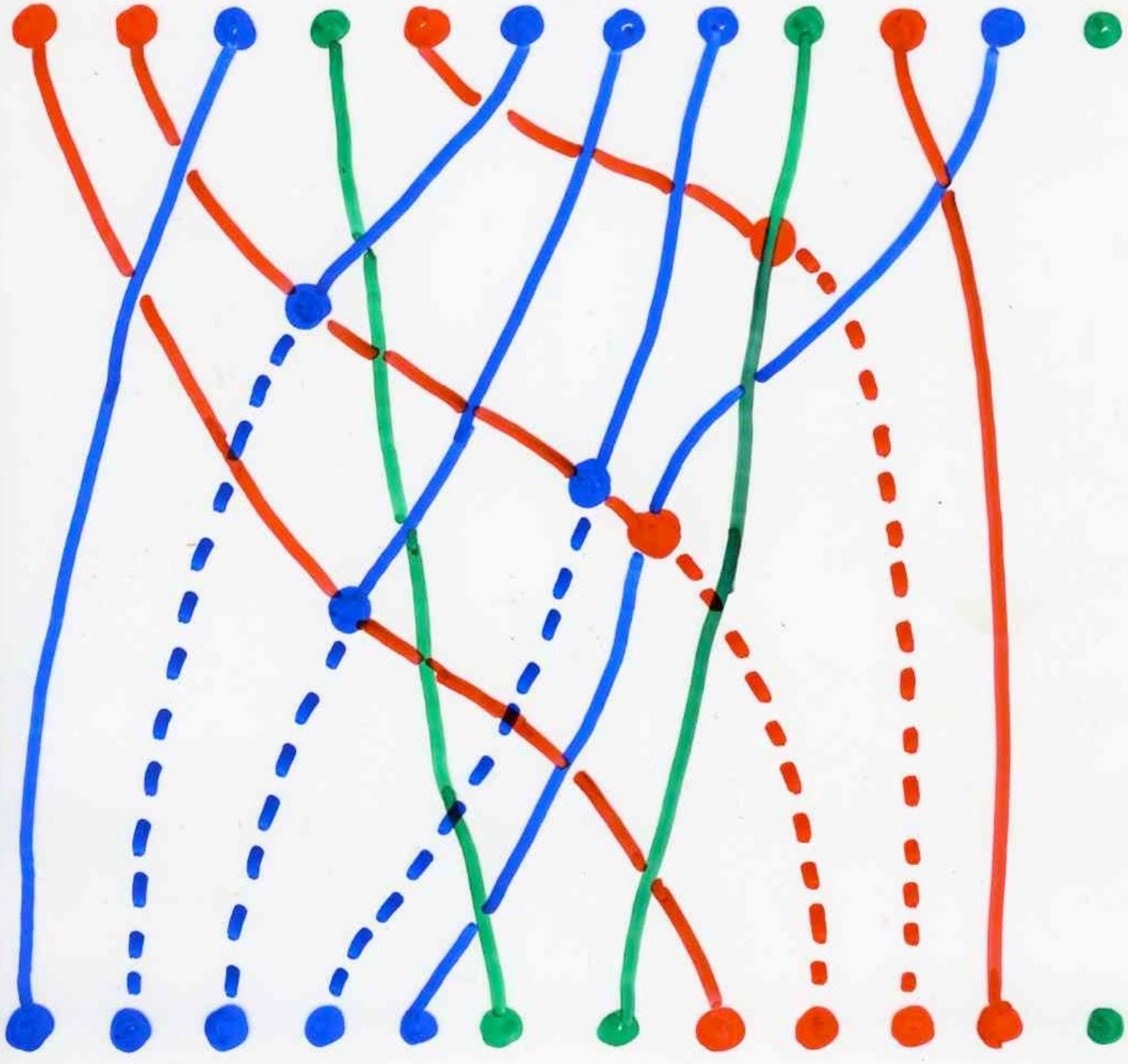
(4)

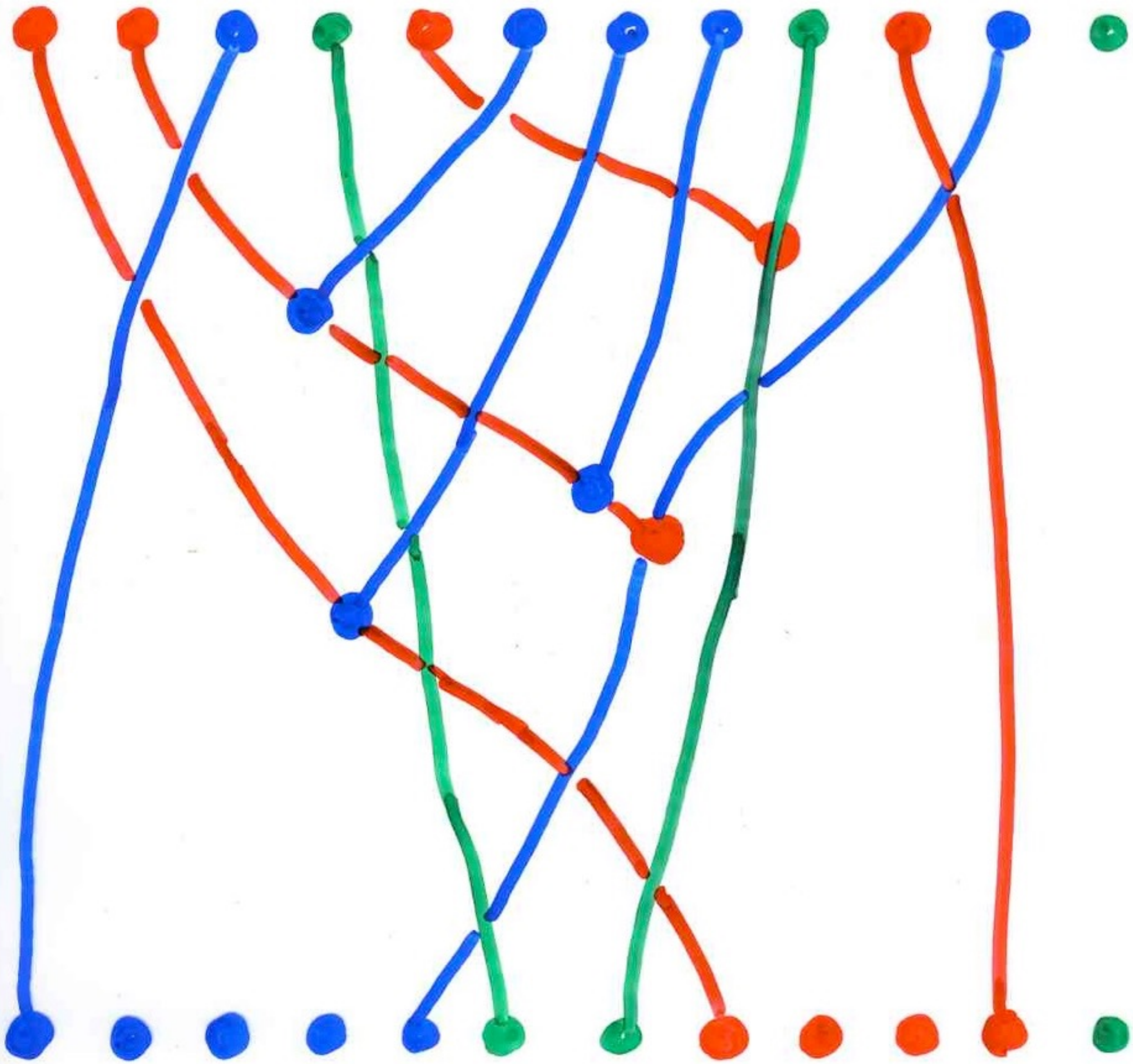
from rhombic alternative tableaux

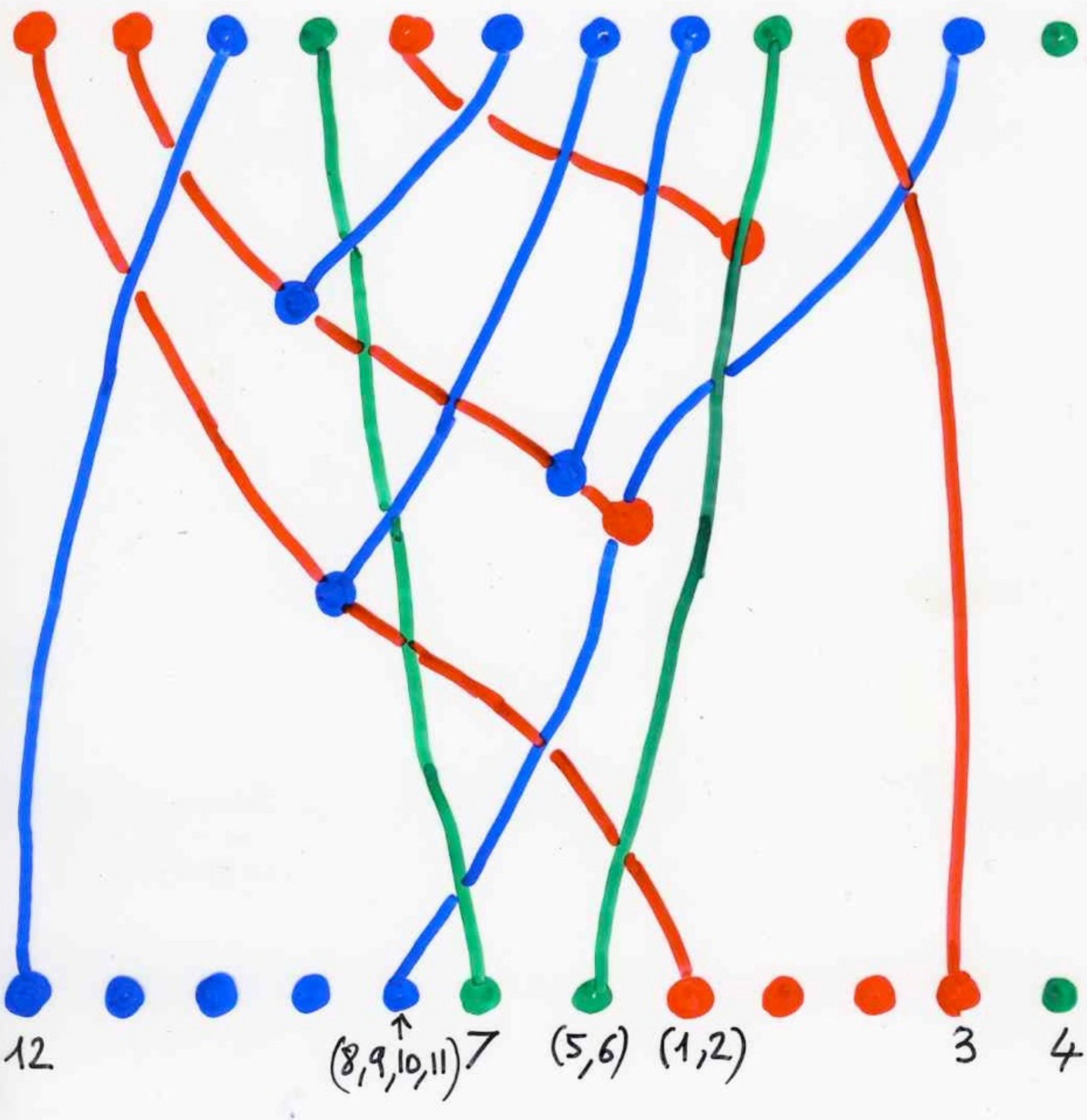
to

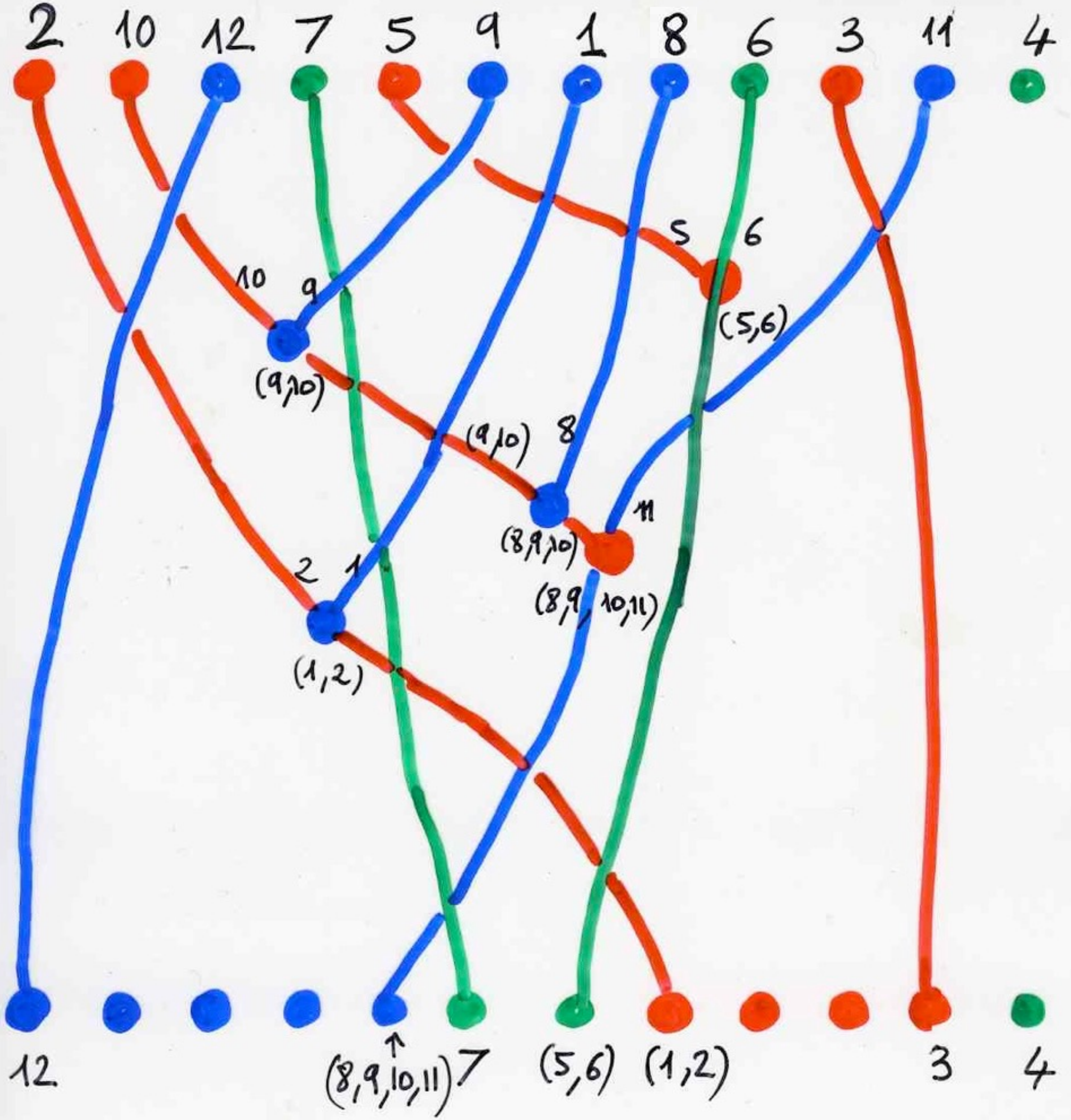
assemblée of permutations



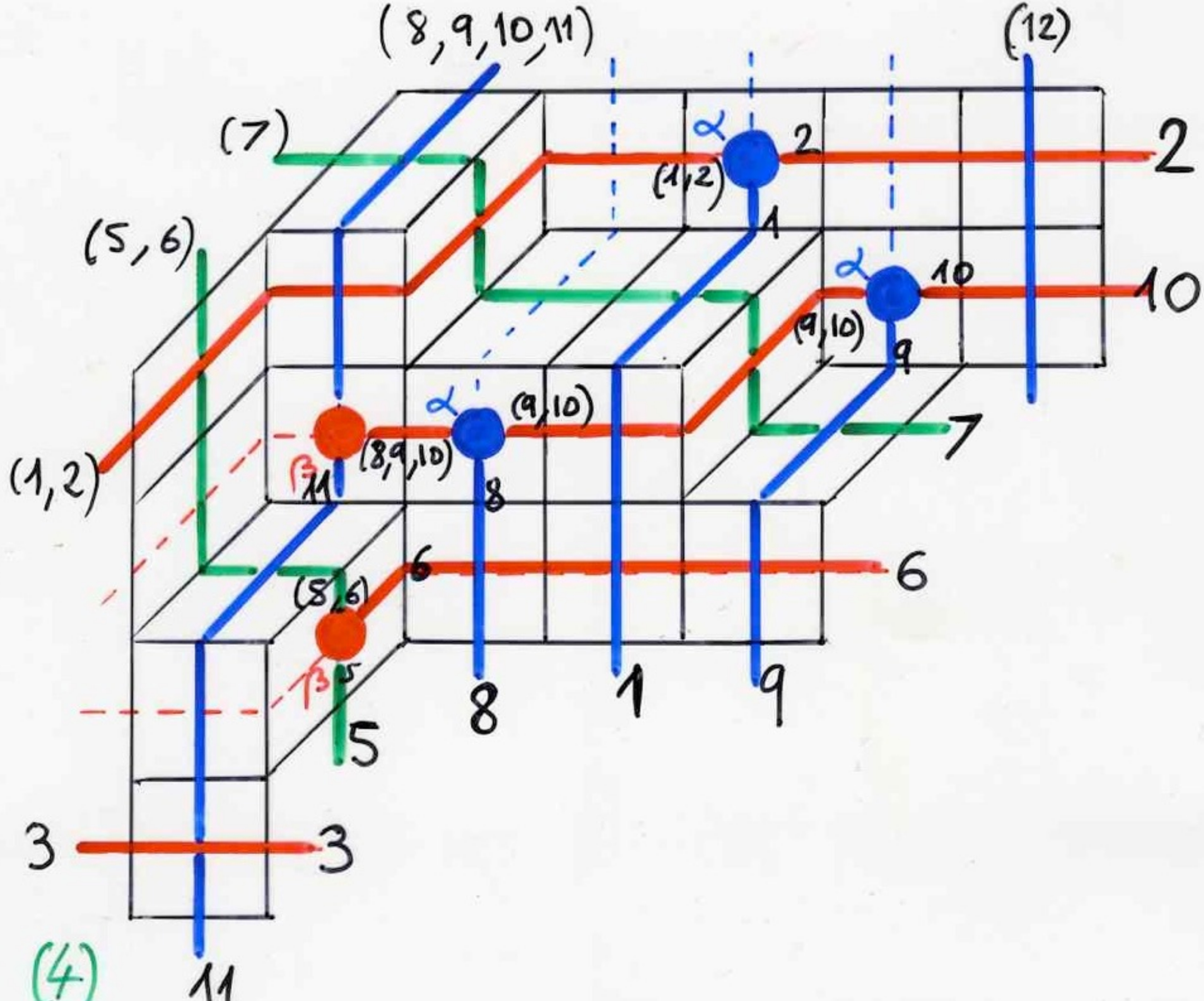





12
(8,9,10,11)
7
(5,6)
(1,2)
3
4



2 10 12 7 5 9 1 8 6 3 11 4
5 6
(5,6)
10 9
(9,10)
(9,10) 8
(8,9,10)
11
(8,9, 10,11)
2 1
(1,2)
12
(8,9,10,11) 7
(5,6)
(1,2)
3 4


(8,9,10,11)
(12)
(7)
α
2
(1,2)
1
2
(5,6)
α
10
10
(9,10)
9
α
(9,10)
(1,2)
(8,9,10)
β
11
8
7
6
6
(8,6)
β
5
5
8
1
9
3
3
(4)
11

"The cellular Ansatz"

Physics

combinatorial objects on a 2d lattice

representation by operators

bijections

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

$DE = qED + E + D$

PASEP

permutations

alternative tableaux

RSK

⟷

pairs of Tableaux Young

permutations

quadratic algebra Q

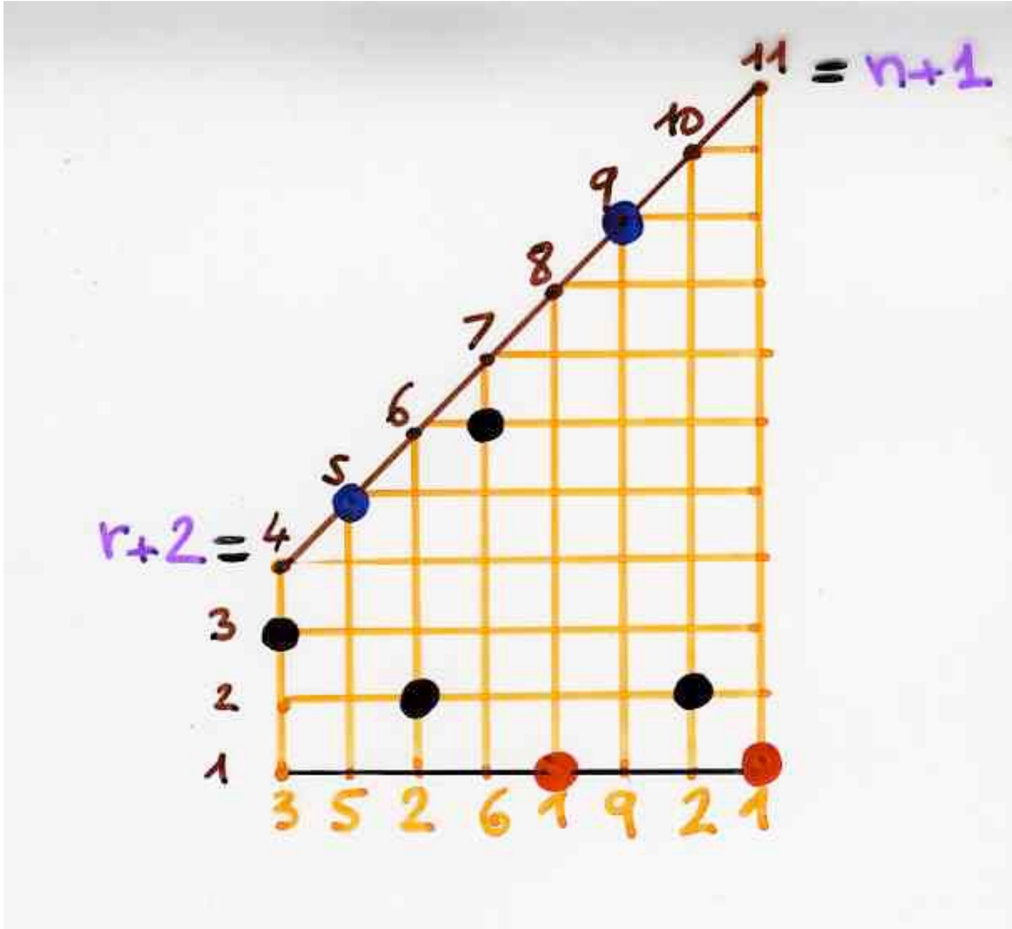
$$\begin{cases} DE = qED + D + E \\ DA = qAD + A \\ AE = qEA + A \end{cases}$$

Q-tableaux

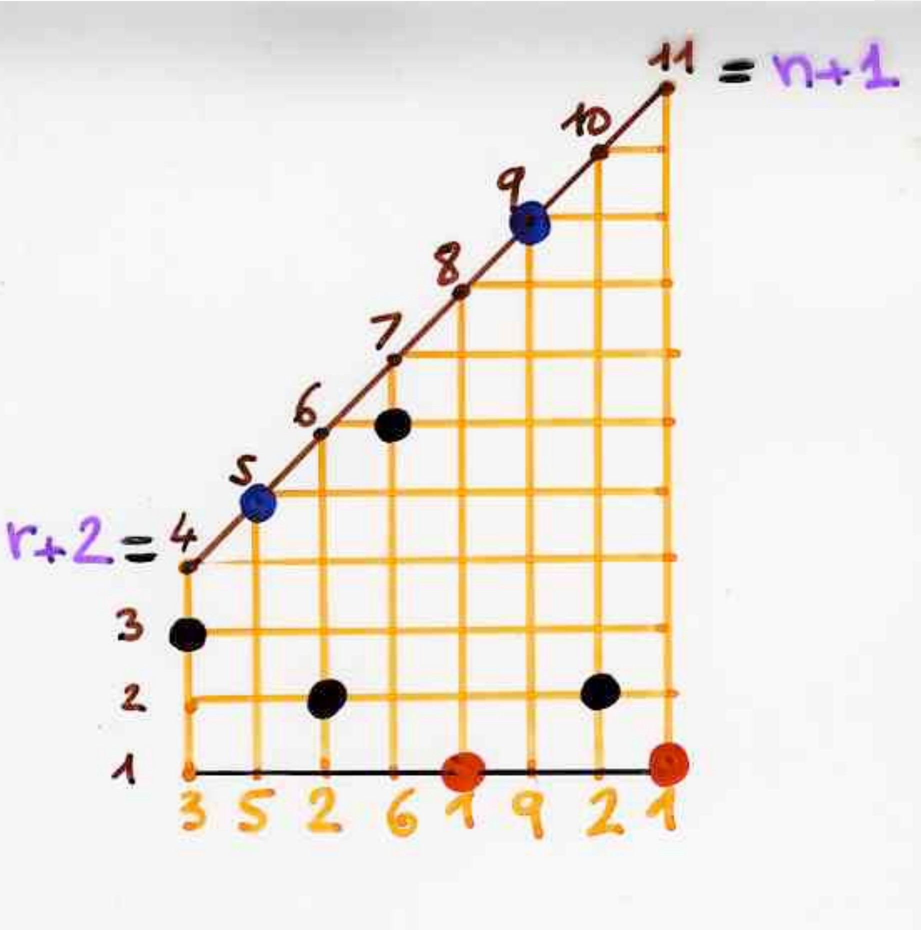
RAT ⟷ assemblée of permutations

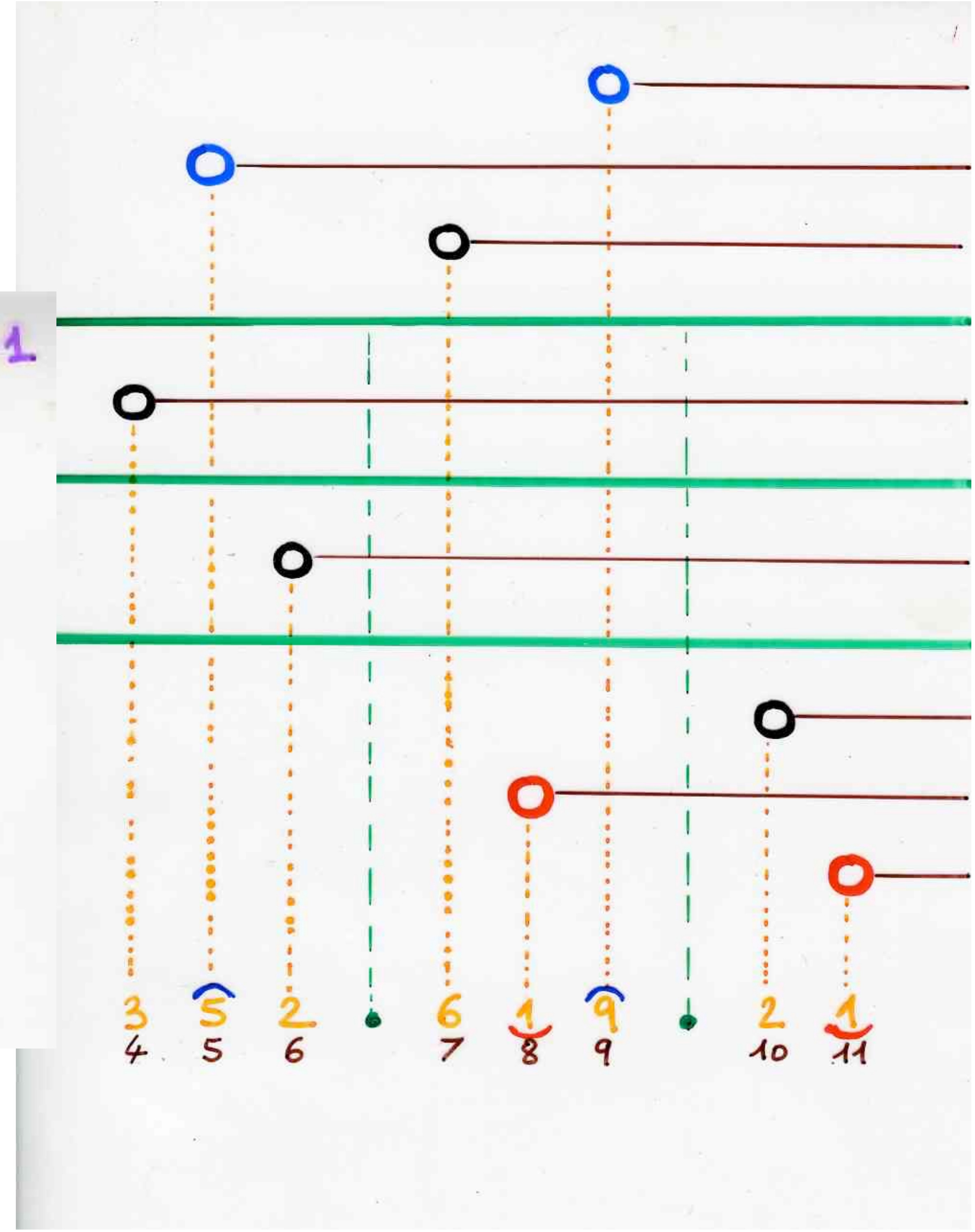
$$Z^{*}_{n,r}(\alpha, \beta, q=1) = \binom{n}{r} \prod_{i=r}^{n-1} \left(\alpha^{-1} + \beta^{-1} + i\right)$$

$$\binom{n}{r}$$

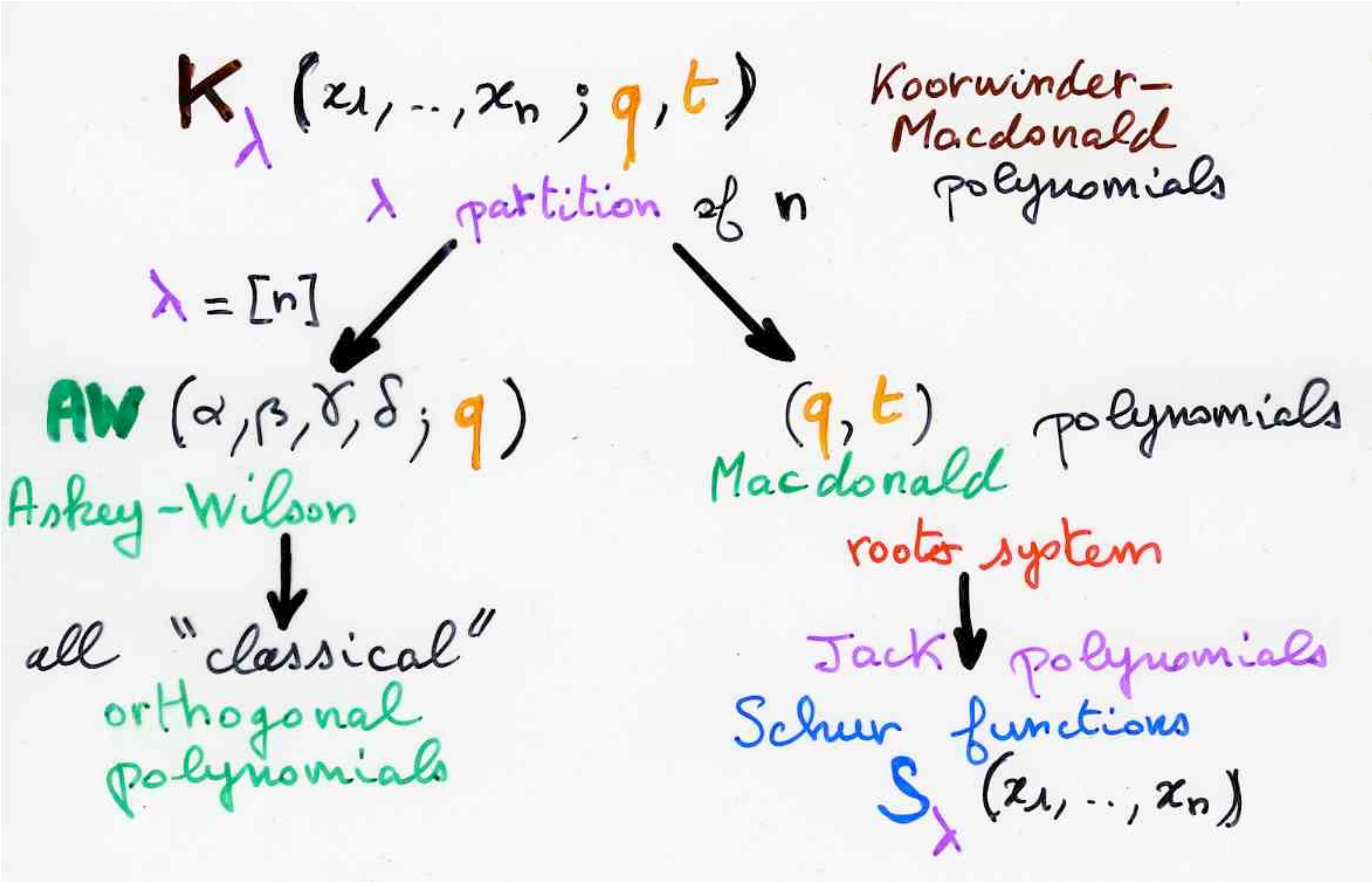


$$\binom{n}{r}\prod_{i=r}^{n-1}\left(\alpha^{-1}+\beta^{-1}+i\right)$$





relation with
Koorwinder-Macdonald polynomials

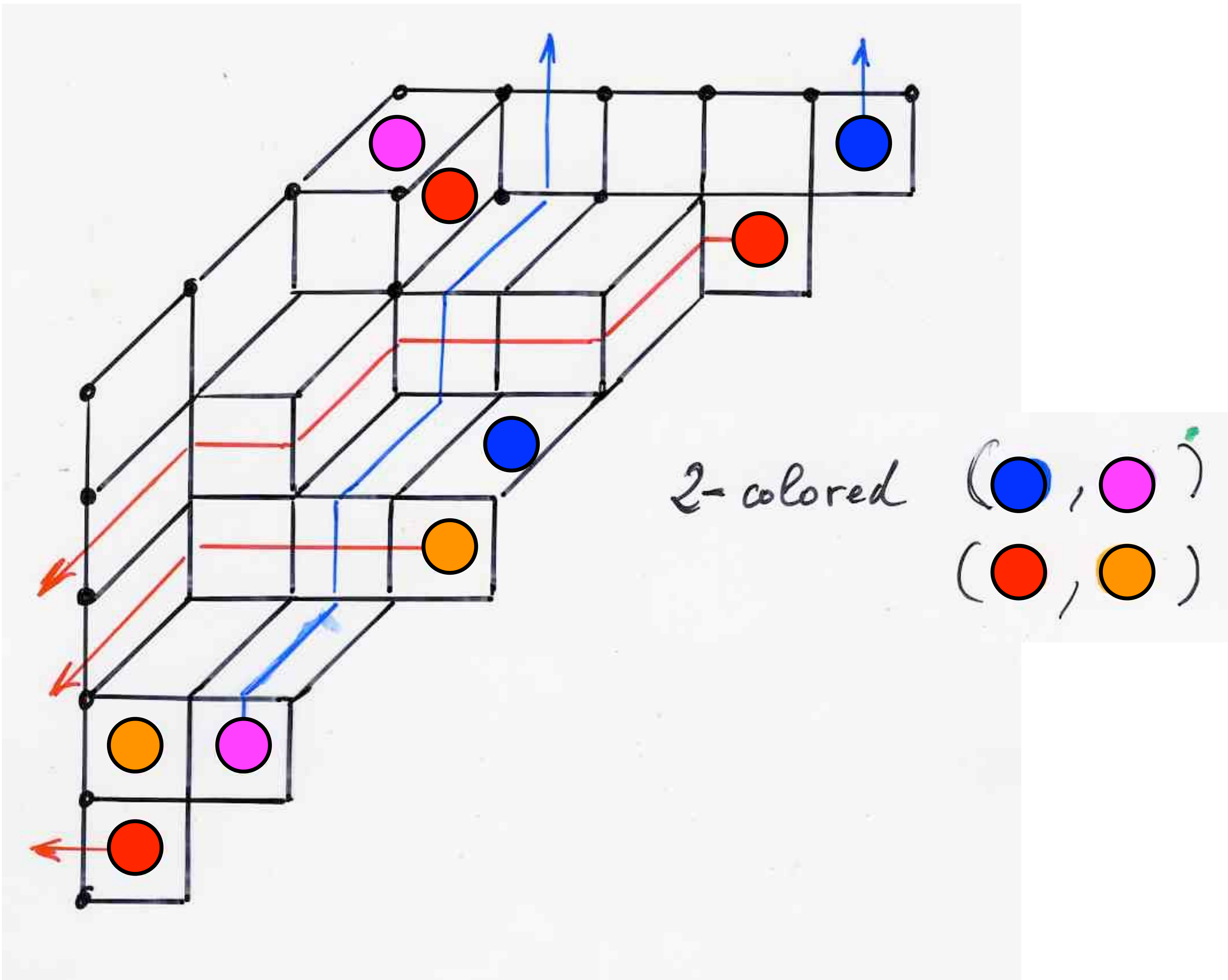


Kλ (x1, .., xn ; q, t)
Koorwinder-
Macdonald
polynomials
λ partition of n
λ = [n]
AW (α, β, γ, δ ; q)
Askey-Wilson
all "classical"
orthogonal
polynomials
(q, t)
Macdonald
polynomials
roots system
Jack polynomials
Schur functions
Sλ (x1, .., xn)

Luigi Cantini
arXiv: 1506.00284

Sylvie Corteel, Olya Mandelshtam,
Lauren Williams, arXiv: 1510.05023

Koornwinder moments (for $q=t$) $\lambda=[n]$
rhombic staircase tableaux


2-colored

Thank you !

www.xavierviennot.org