At the crossroad of algebra, combinatorics and physics: the 2-species PASEP

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IMSc, Chennai, 2016

2 parts:

- The PASEP (ASEP)

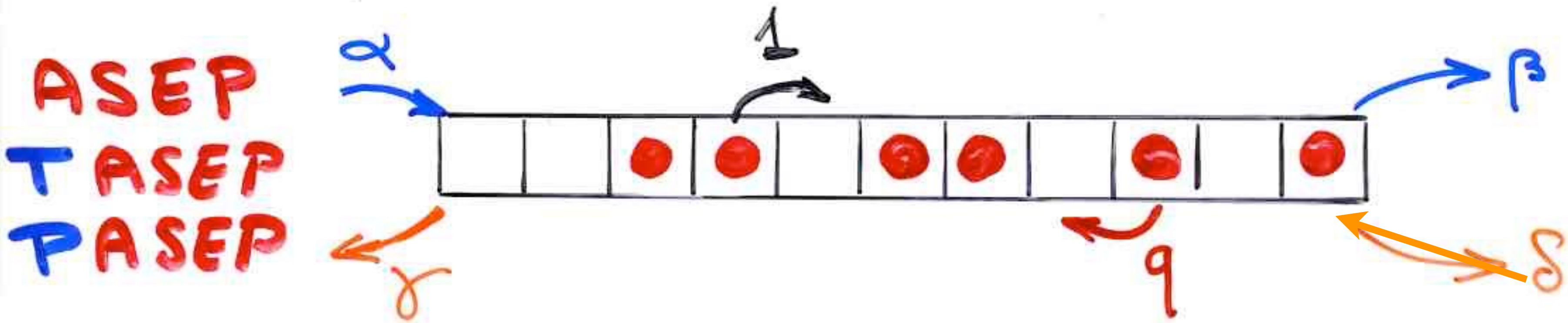
(Partially) ASymmetric Exclusion Process

- The 2-species PASEP

(join work with O. Mandelshtam, Berkeley)

The PASEP
(ASEP)

(Partially) ASymmetric Exclusion Process



ASEP
TASEP
PASEP
α
1
β
γ
q
δ

boundary induced phase transitions

molecular diffusion

linear array of enzymes

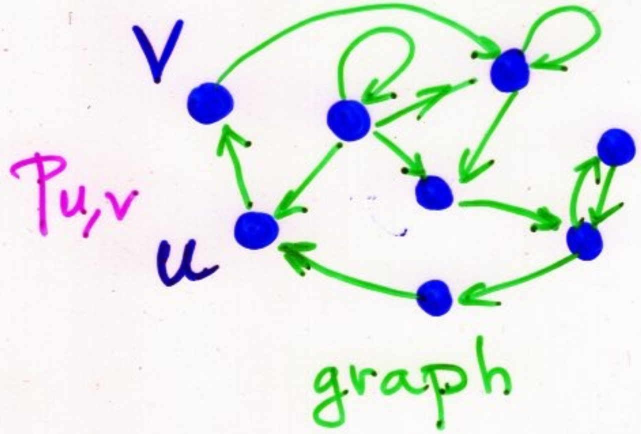
biopolymers

traffic flow

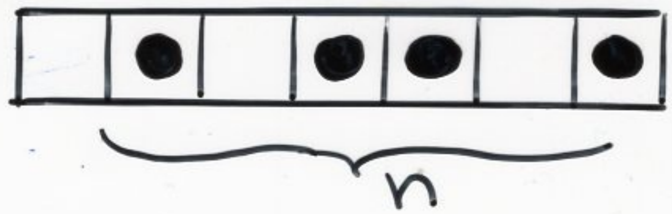
------ formation of shocks

Markov chain

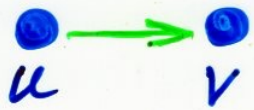
2^n states



S set of states
(vertices of the graph)



$p_{u,v}$ probability

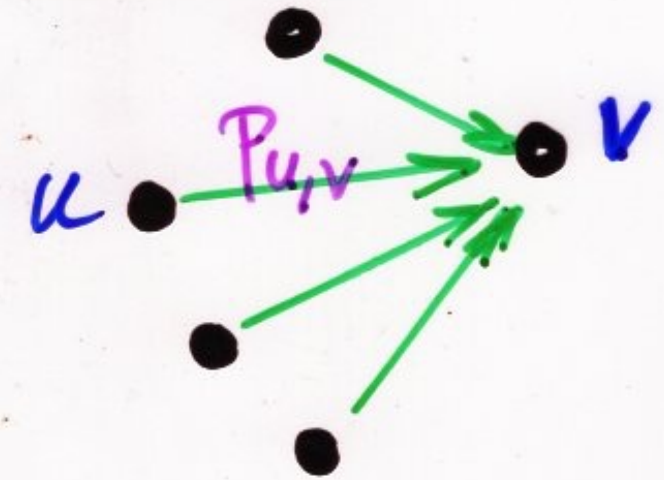


$$T = (p_{u,v})_{u,v \in S}$$

(stochastic)
transition matrix

time t $\quad V_t = (p_u^t, \dots)_{u \in S}$ probability vector at time t

time $t+1$ $\quad V_{t+1} = V_t T$



$$p_v^{(t+1)} = \sum_u p_u^{(t)} p_{u,v}$$

time $(t+1)$ — time t

$$V_t = V_{t+1}$$

$$V = (p_u^\infty, \ldots)_{u \in S}$$

$$V = VT$$

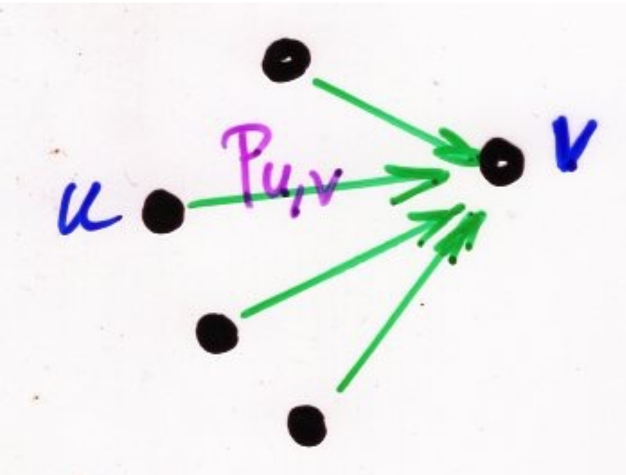
eigenvector of T^T

eigenvalue 1

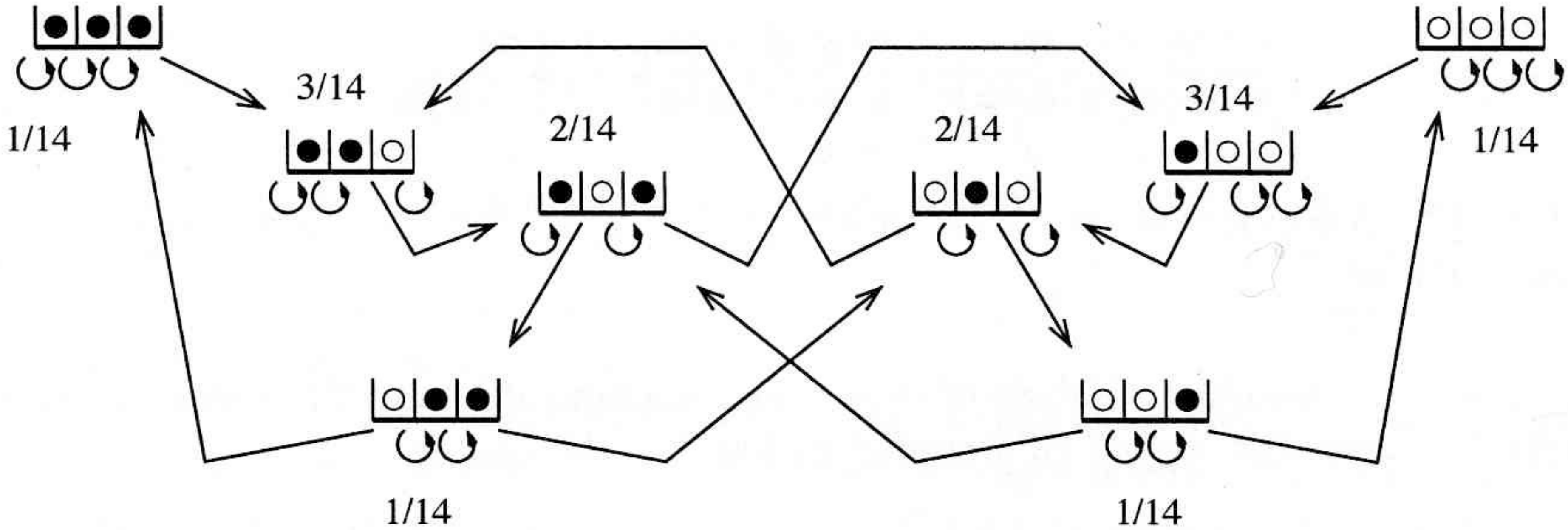
unique

stationnary probabilities

time $\longrightarrow \infty$



$$p_v^\infty = \sum_{u \in S} p_u^\infty \, p_{u,v}$$



1/14
3/14
2/14
2/14
3/14
1/14
1/14
1/14

The Matrix Ansatz

Derrida, Evans, Hakim, Pasquier 1993

state $(\tau_1, \ldots, \tau_n)$

$\tau_i = 0$

$= 1$

$$P_n(\tau_1, \ldots, \tau_n) = f_n(\tau_1, \ldots, \tau_n) / Z_n$$

$$Z_n = \sum_{\tau} f_n(\tau_1, \ldots, \tau_n)$$

partition function

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

$$\begin{cases} DE = qED + D + E \\ (\beta D - \delta E)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \gamma D) = \langle W| \end{cases}$$

Then

$$f_n(\tau_1, \dots, \tau_n)$$

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

TASEP

D E matrices,

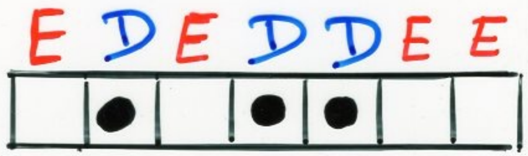
V column vector, W row vector

$q=0$

$$\begin{cases} DE = \qquad + D + E \\ (\beta D - \quad) \qquad |V\rangle \\ \langle W|(\alpha E - \quad) = \langle W| \end{cases}$$

Then

$$f_n(\tau_1, \ldots, \tau_n) = \langle W| \prod_{i=1}^{n} \left(\tau_i D + (1-\tau_i) E\right) |V\rangle$$



examples:

$$\begin{cases} DE = D + E \\ D|V\rangle = \bar{\beta}\,|V\rangle \\ \langle W|E = \bar{\alpha}\,\langle W| \end{cases}$$

TASEP

$$D = \begin{bmatrix} 0 & \bar{\beta} & & 0 \\ & 0 & \ddots & \\ & & \ddots & \bar{\beta} \\ 0 & & & 0 \end{bmatrix} \qquad E = \begin{bmatrix} \bar{\alpha} & & & 0 \\ \bar{\alpha}\beta & 1 & & \\ \bar{\alpha}\beta^2 & \beta & 1 & \\ \bar{\alpha}\beta^3 & \beta^2 & \beta & 1 \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}$$

$$\bar{\beta} = \frac{1}{\beta}, \quad \bar{\alpha} = \frac{1}{\alpha}$$

$$\langle W| = (1, 0, \ldots, 0, \ldots) \qquad |V\rangle = (1, 1, \ldots, 1, \ldots)^T$$

(infinite matrices)

orthogonal polynomials

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

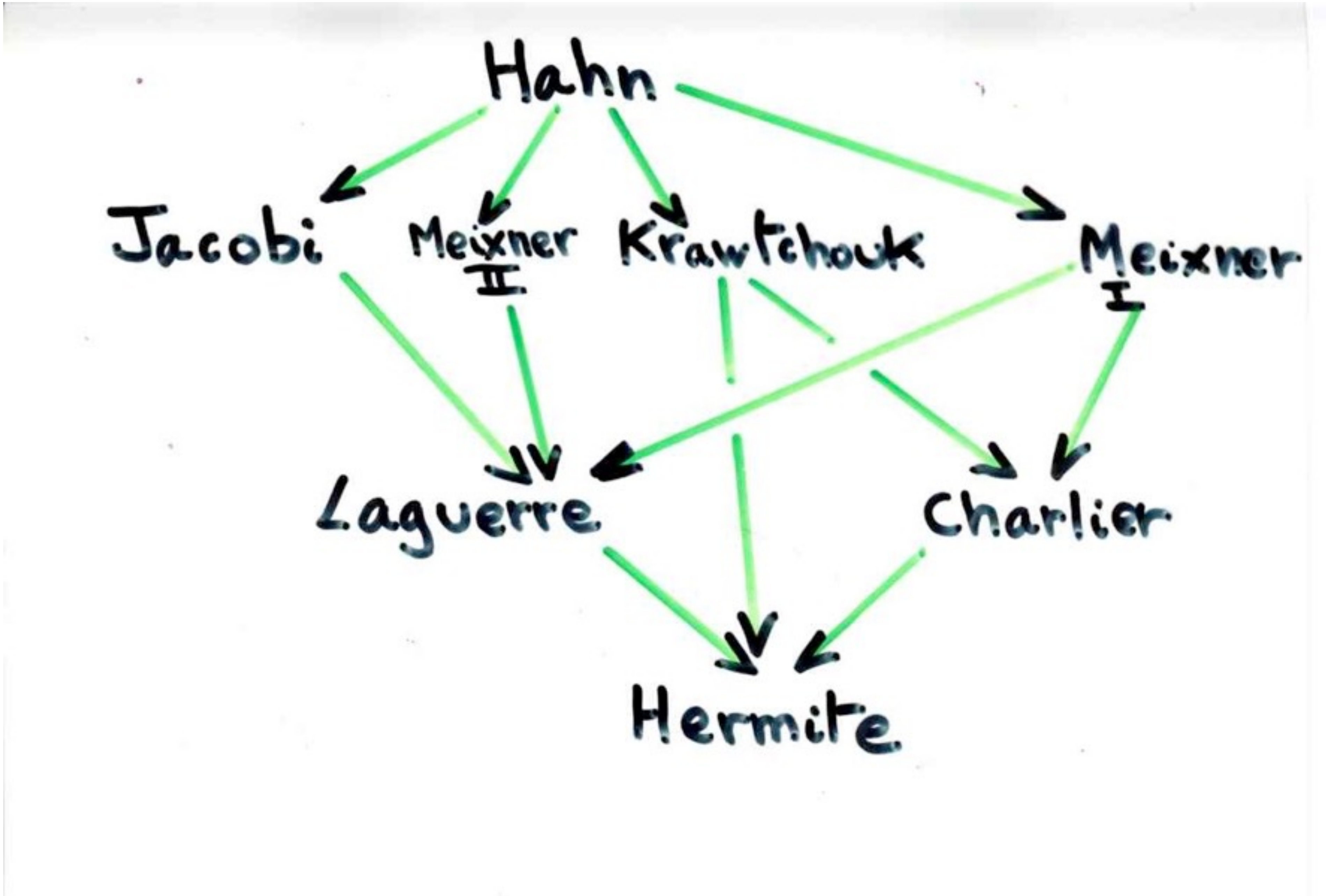
$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

Askey-Wilson polynomials

Askey-Wilson



combinatorics

TASEP

Brak, Essam (2003) Duchi, Schaeffer (2004)

Angel (2005) X.V (2007)

PASEP

M. Josuat-Vergès (2007)

Brak, Corteel, Essam, Parviainen, Rechnitzer (2006)

Corteel, Williams (2006, 7, 8) X.V. (2008)

Corteel, Stanton (2011), Stanley, Williams (2010)

Dasse-Hartaut (2011, 12, ...)

Aval, Boussicault, Nadeau (2013)

Corteel, Williams, Mandelshtam, X.V. (2015)

Phys

Derrida, Mallick, Golinelli, ...

Cantini (2015)

The PASEP algebra

$$DE = qED + E + D$$

$$DE = qED + E + D$$

In the PASEP algebra

an word $w(E,D)$ can be uniquely written

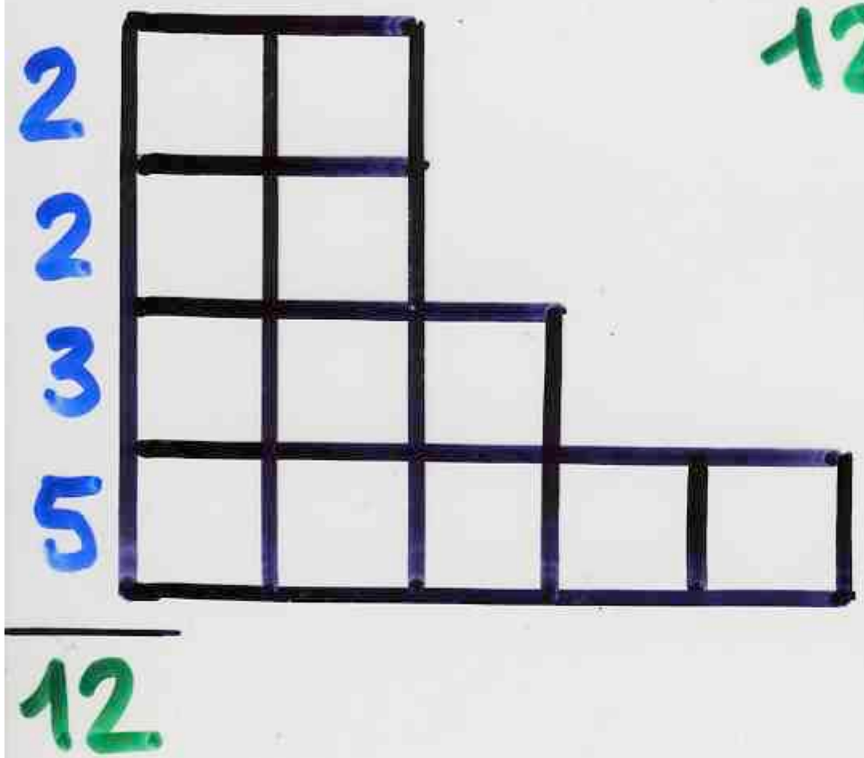
$$w(E,D) = \sum_{k,i,j} c_{k,i,j}\, q^k E^i D^j$$

$$= \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

alternative tableaux with profile w

alternative tableaux

$12 = n = 5+3+2+2$

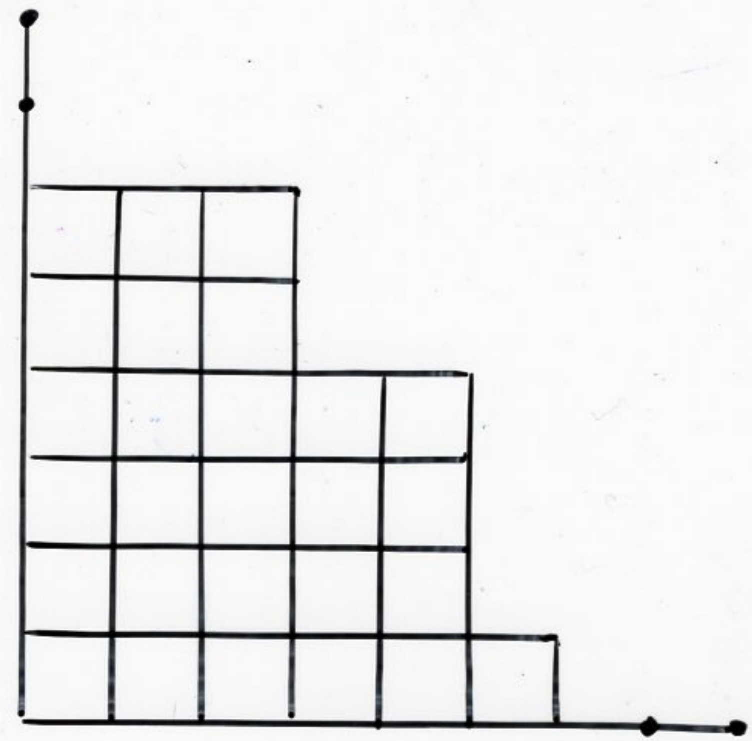


Ferrers

diagram

Partition of n

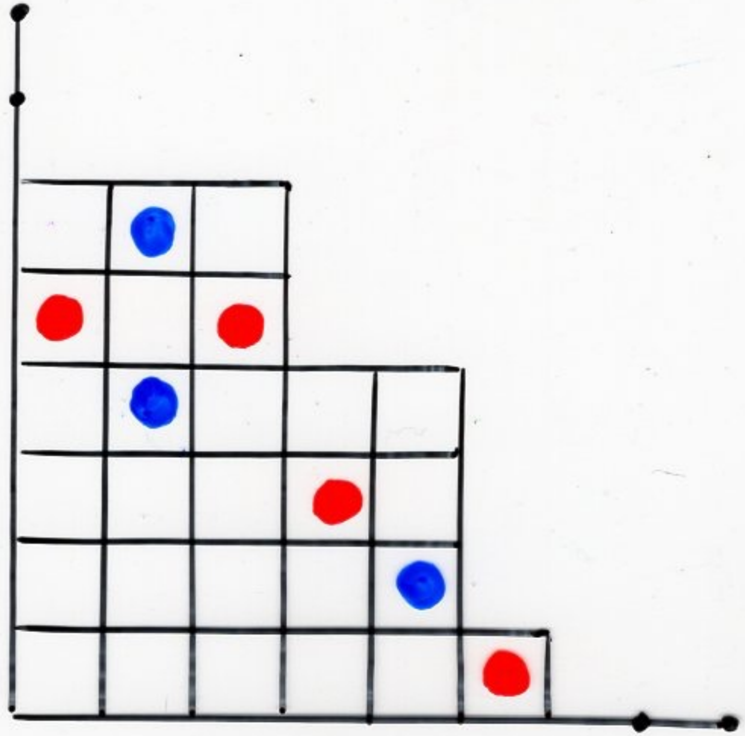
λ



Ferrers diagram

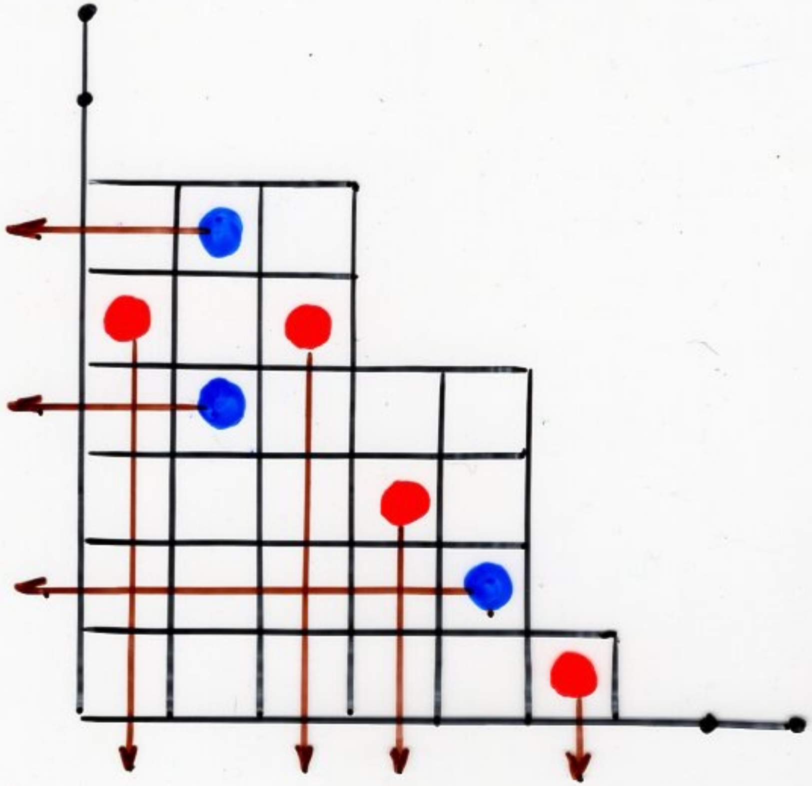
(possibly empty rows and columns)

n = total nb of rows and columns



alternative
tableau
T

– some cells are coloured ● or ●



alternative tableau T

– some cells are coloured 🔴 or 🔵

– { no coloured cell at the left of a 🔵
  { no coloured cell below a 🔴

$$DE = qED + E + D$$

In the **PASEP** algebra

any word $w(E,D)$ can be uniquely written

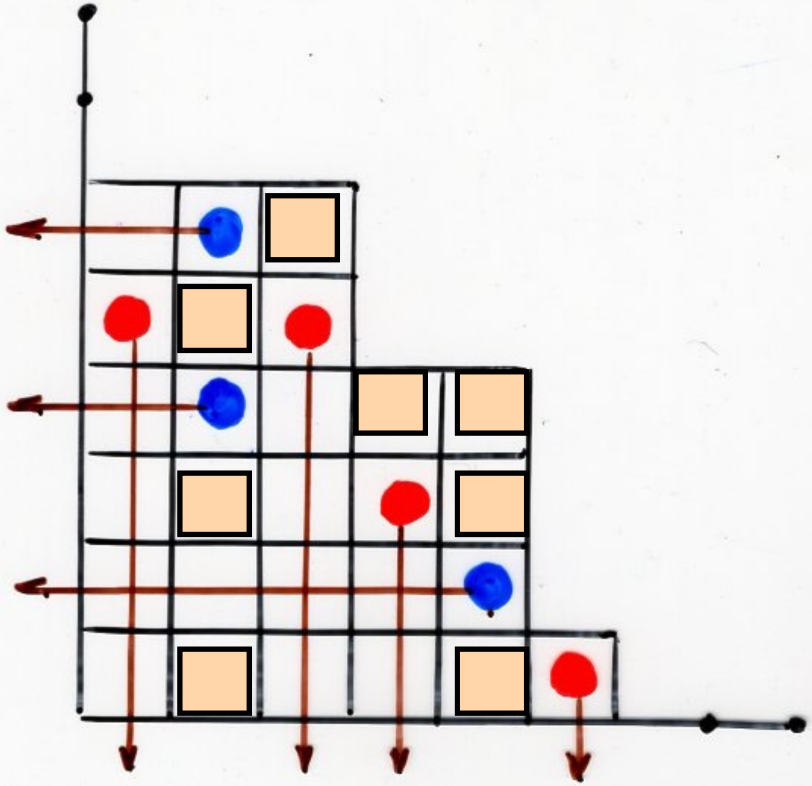
$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

alternative tableaux profile w

$k(T)$ = nb of cells

$i(T)$ = nb of rows without

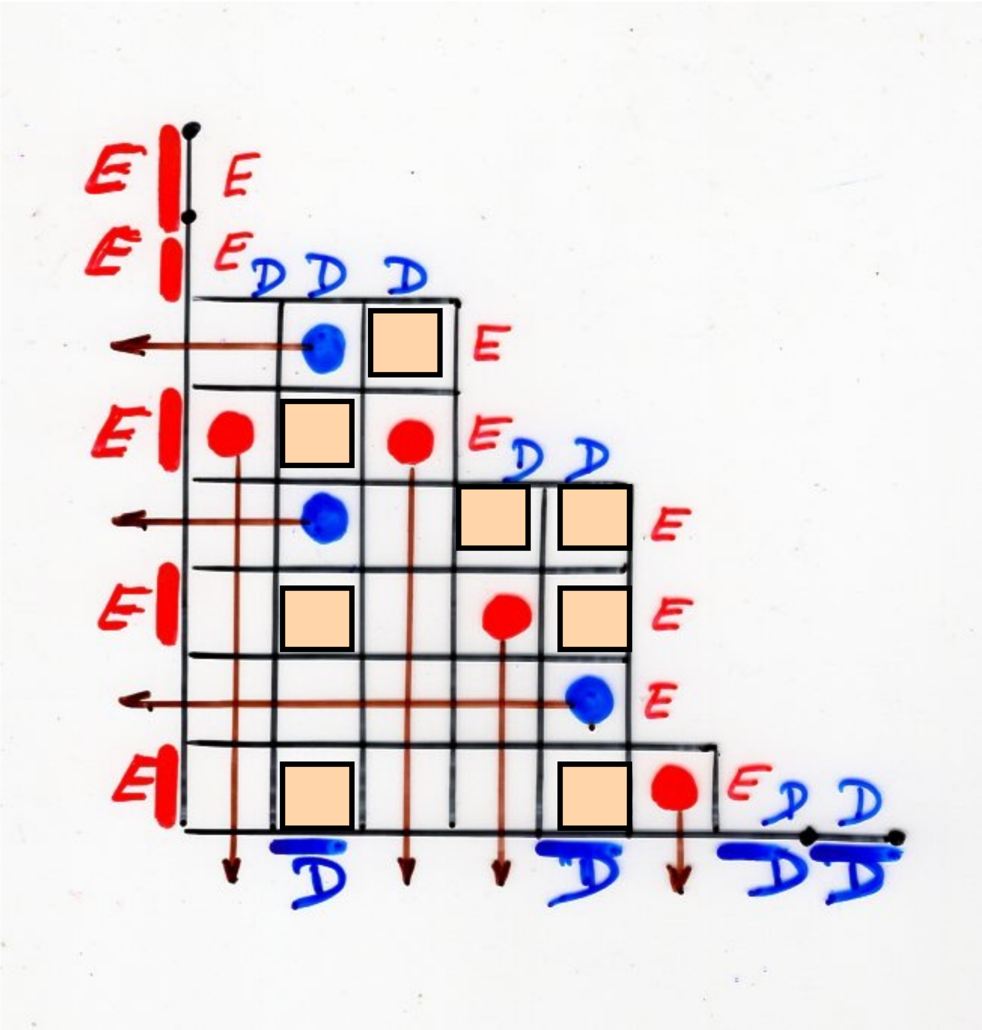
$j(T)$ = nb of columns without



$k(T)$ = nb of cells

$i(T)$ = nb of rows without

$j(T)$ = nb of columns without



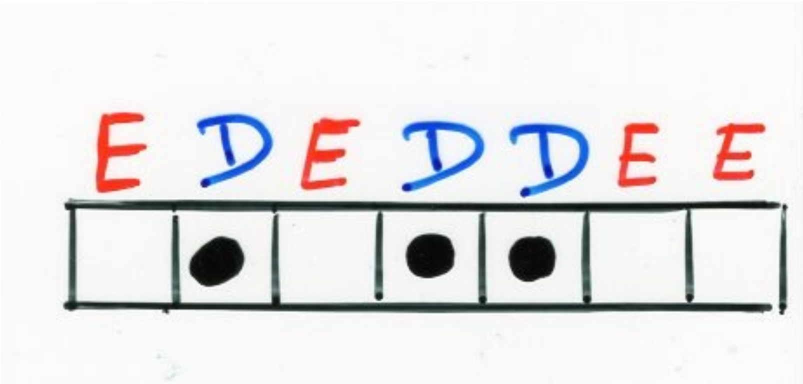
$$q^8 E^5 D^4$$

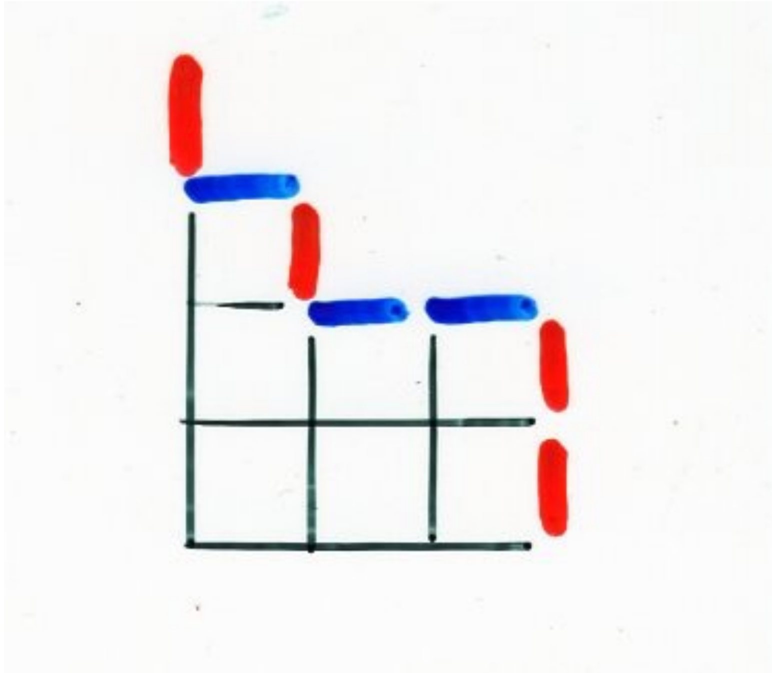
$k(T)$ = nb of cells □

$i(T)$ = nb of rows without ● (blue)

$j(T)$ = nb of columns without ● (red)

$$DE = qED + E + D$$





Def - profile of an alternative tableau

word $w \in \{E, D\}^*$

stationary probabilities for the PASEP

Corollary. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ is

$$\text{proba}_\tau(q; \alpha, \beta) = \frac{1}{Z_n} \sum_{T} q^{k(T)} \alpha^{-i(T)} \beta^{-j(T)}$$

T: alternative tableaux, profile τ

$k(T)$ = nb of cells ☐

$i(T)$ = nb of rows without ●

$j(T)$ = nb of columns without ●

permutation tableau

S. Corteel, L. Williams (2007) (2008) (2009)

permutation tableau

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)

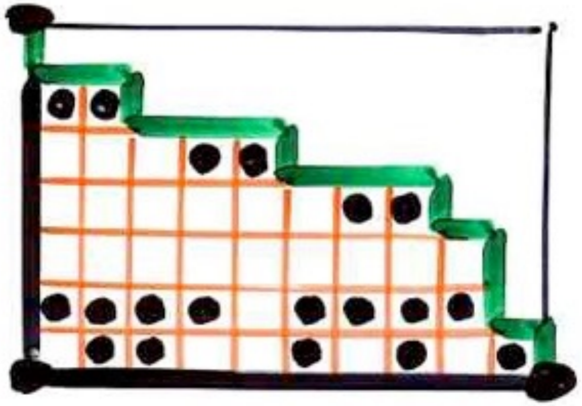
Corteel, Williams (2006) PASEP

Partially Asymmetric Exclusion Process

M. Josuat-Vergès (2007)

Permutation Tableau

Ferrers diagram F $\subseteq k \times (n-k)$
rectangle



filling of the cells with 0 and 1

(i) in each column : at least one 1

(ii) 

forbidden

permutations tableaux A. Postnikov (2001)
E. Steingrímsson (2005) + L. W.

S. Corteel, L. Williams (2007)

alternative tableaux

X.V. (2008)

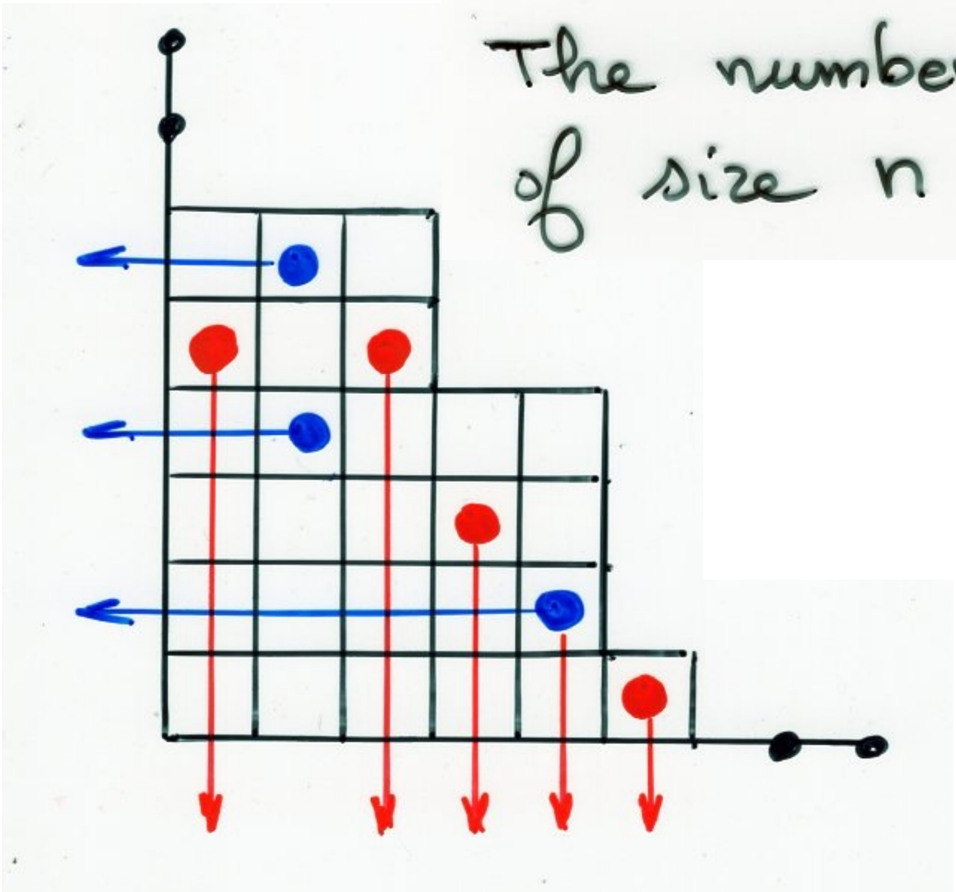
tree-like tableaux

J.-C. Aval, A. Boussicault, P. Nadeau (2013)

staircase tableaux

S. Corteel, L. Williams (2011)

number of
alternative tableaux



The number of alternative tableaux of size n is

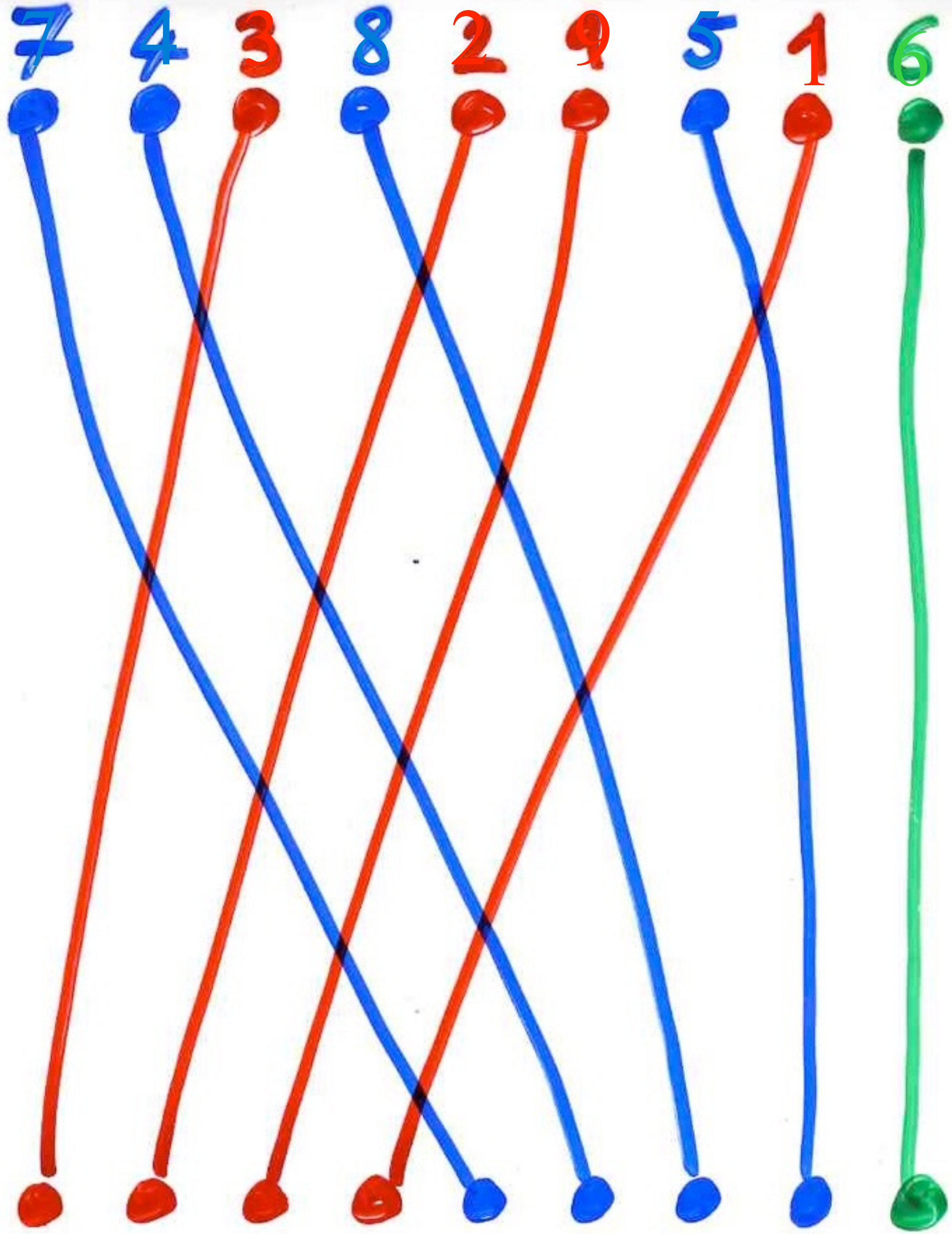
$$(n+1)!$$

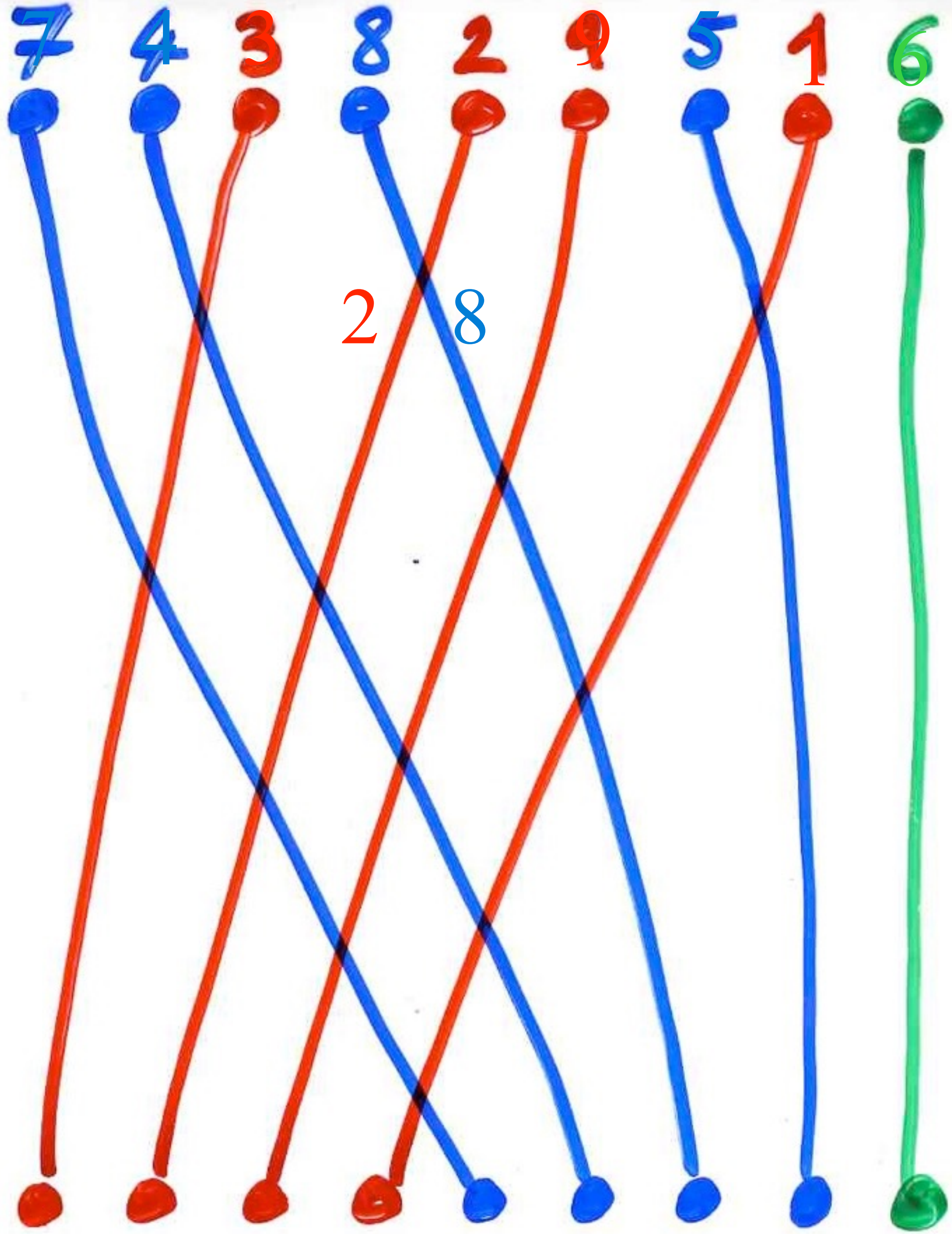
ex: $n=2$

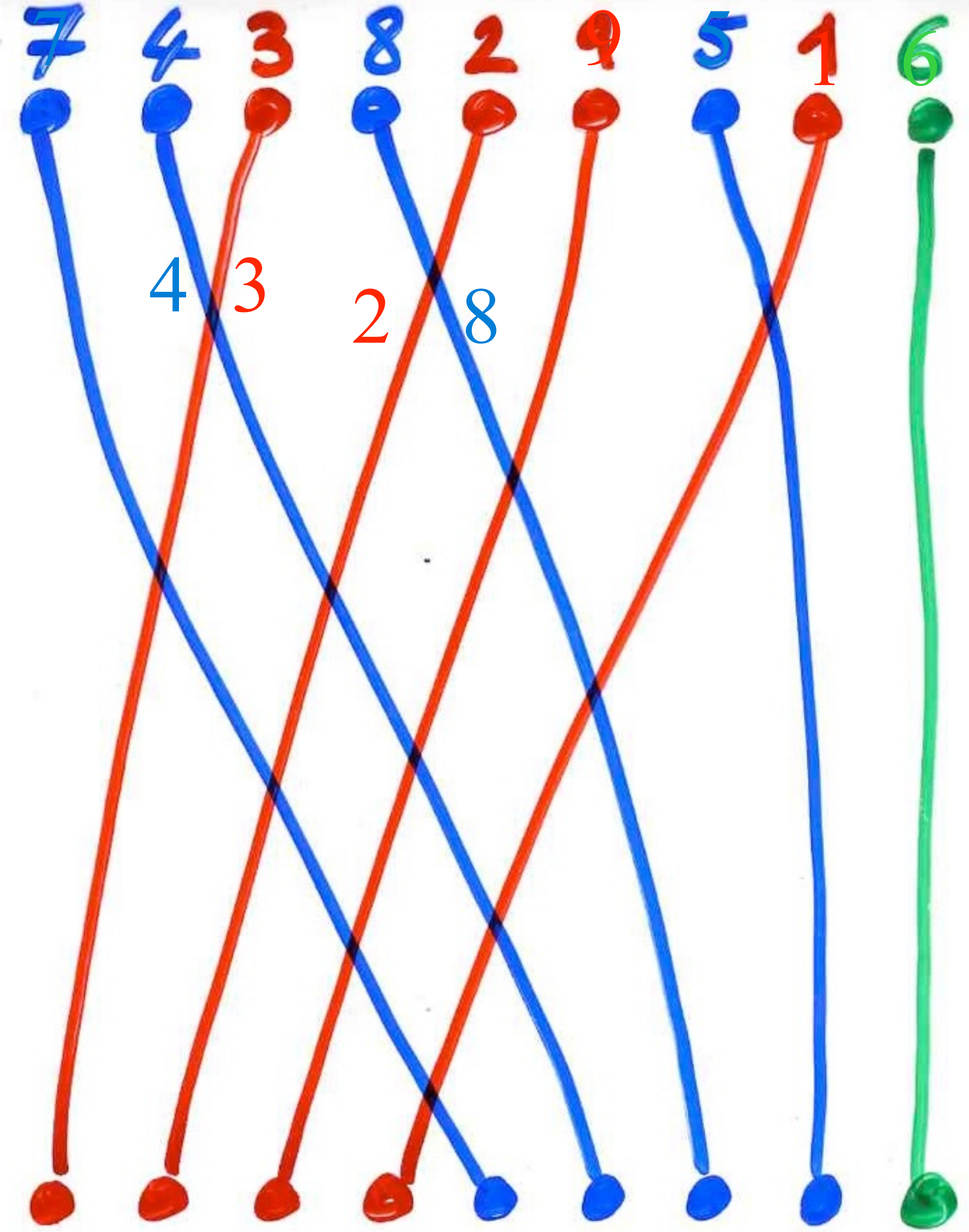
bijection

permutations --- alternative tableaux

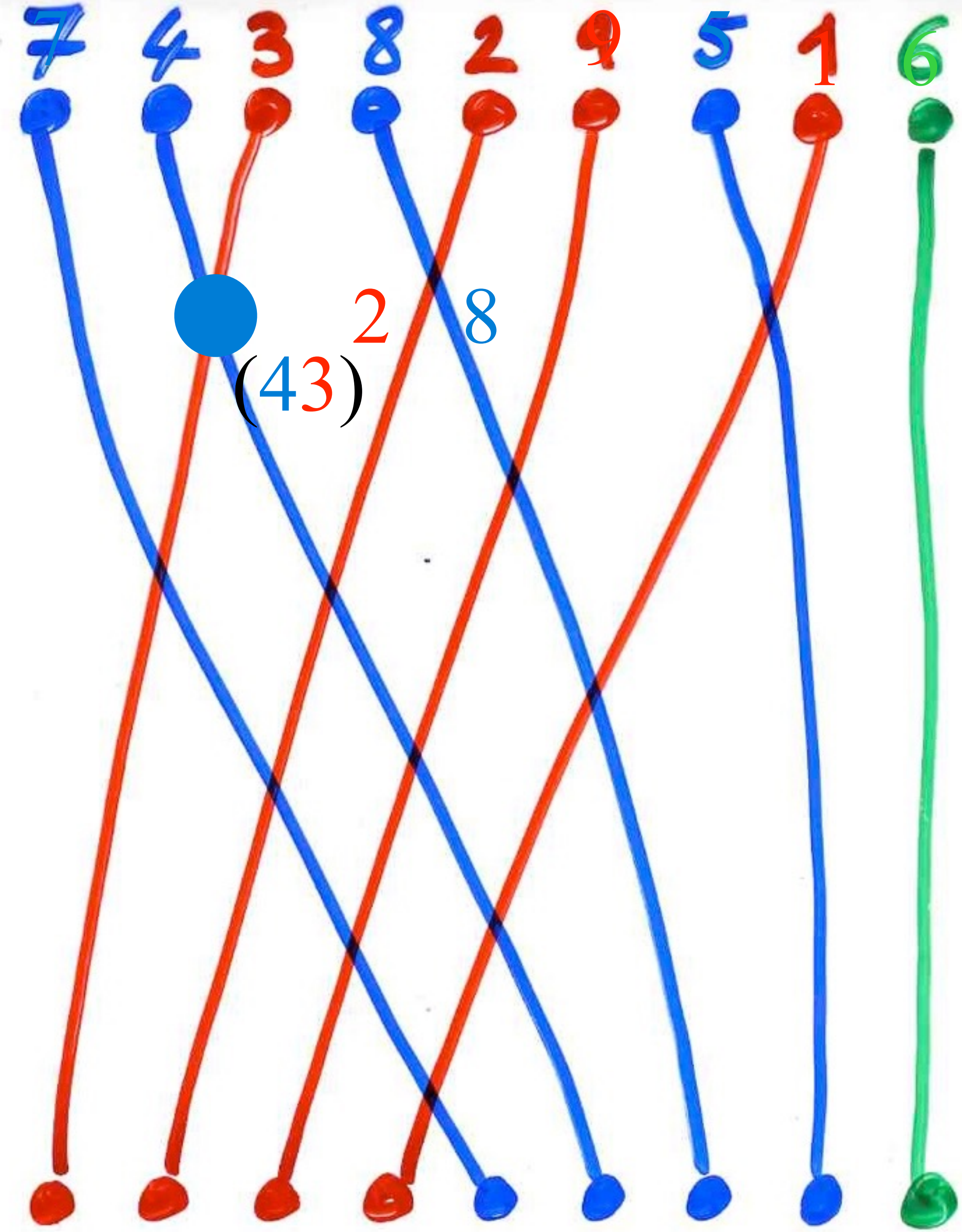
The "exchange-fusion" algorithm


7 4 3 8 2 9 5 1 6

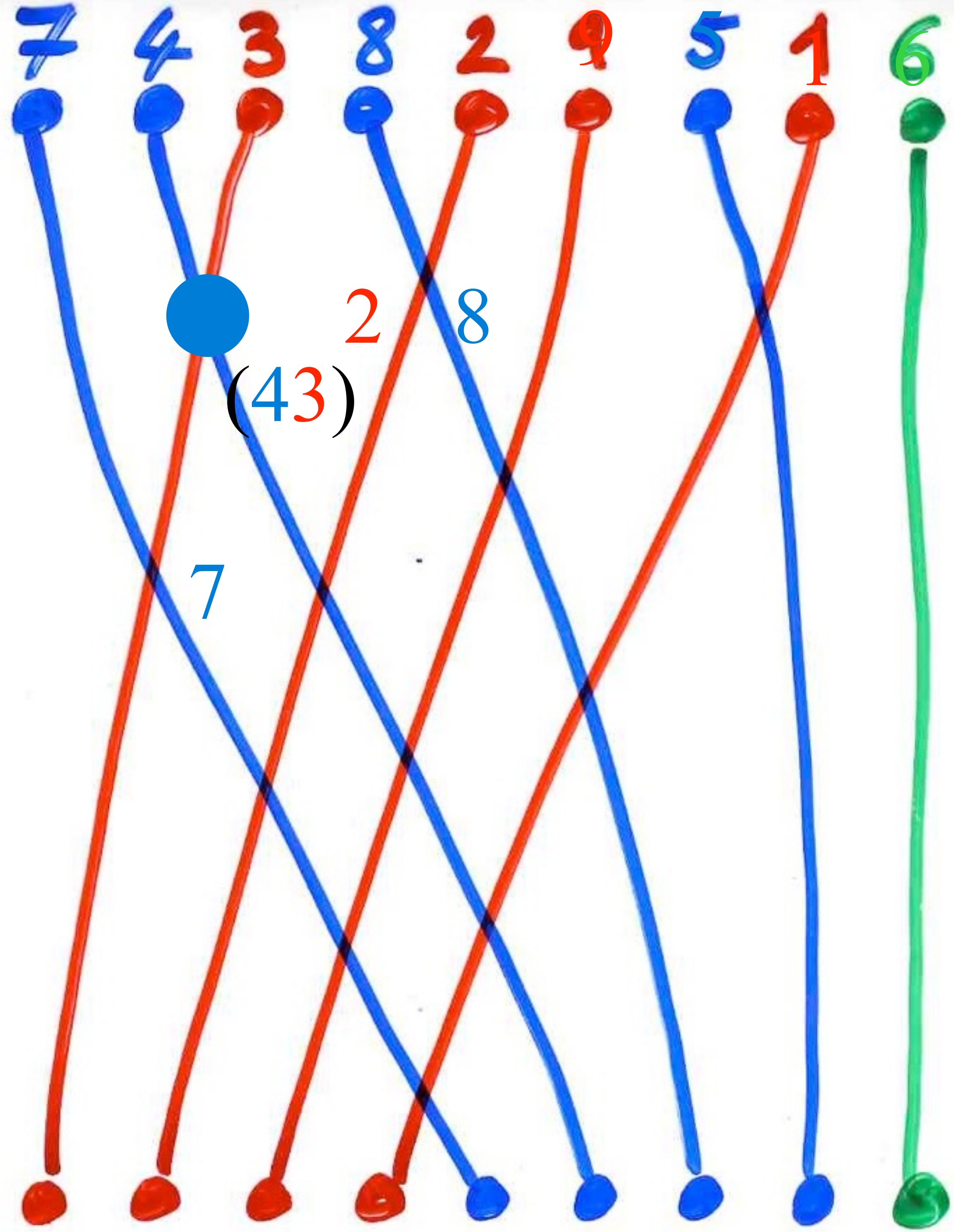

7 4 3 8 2 9 5 1 6
2 8



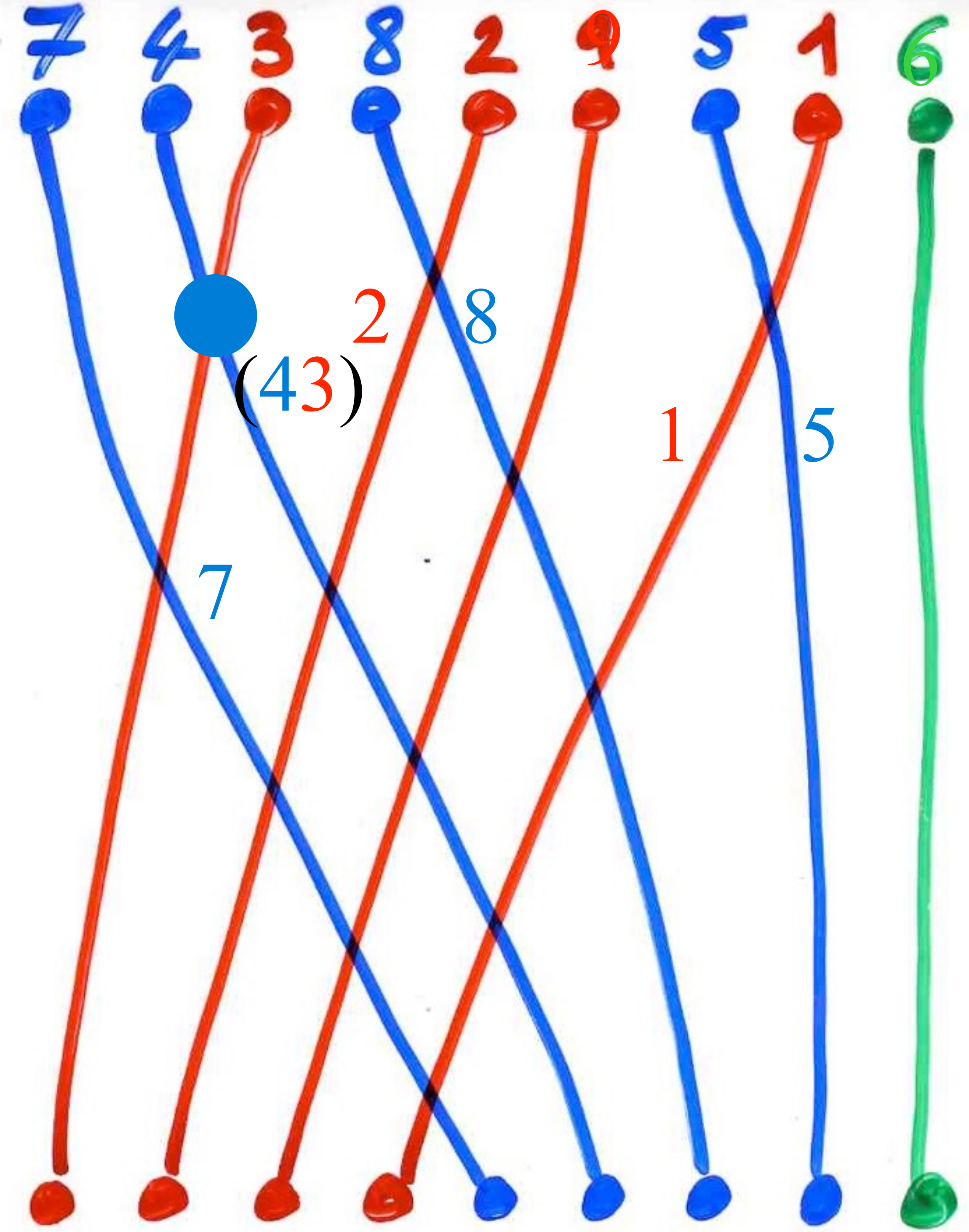
7 4 3 8 2 9 5 1 6
4 3
2 8



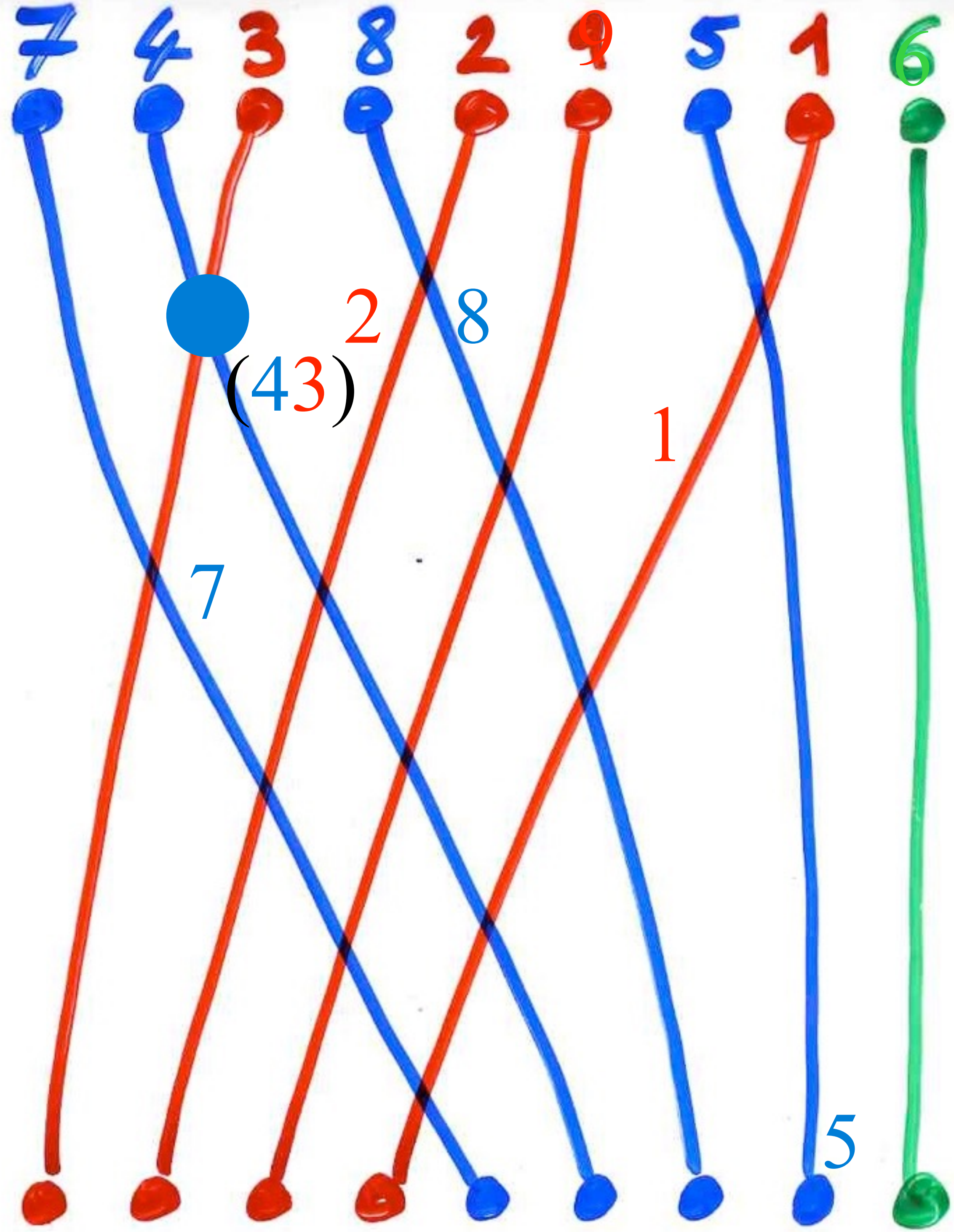
7 4 3 8 2 9 5 1 6
2 8
(43)



7 4 3 8 2 9 5 1 6
2
8
(43)
7



7 4 3 8 2 9 5 1 6
2
8
(43)
1
5
7



7 4 3 8 2 9 5 1 6
(43)
2
8
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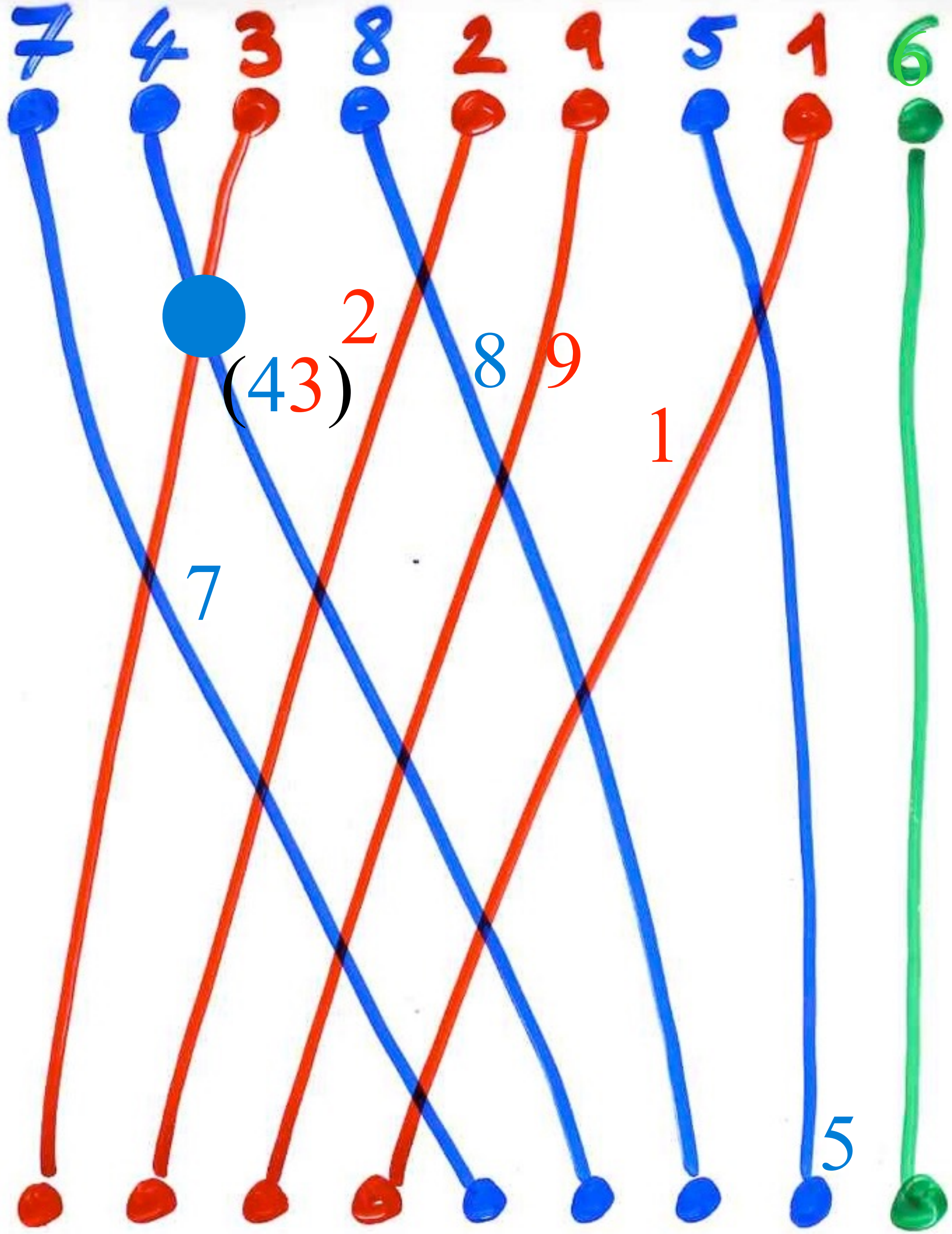
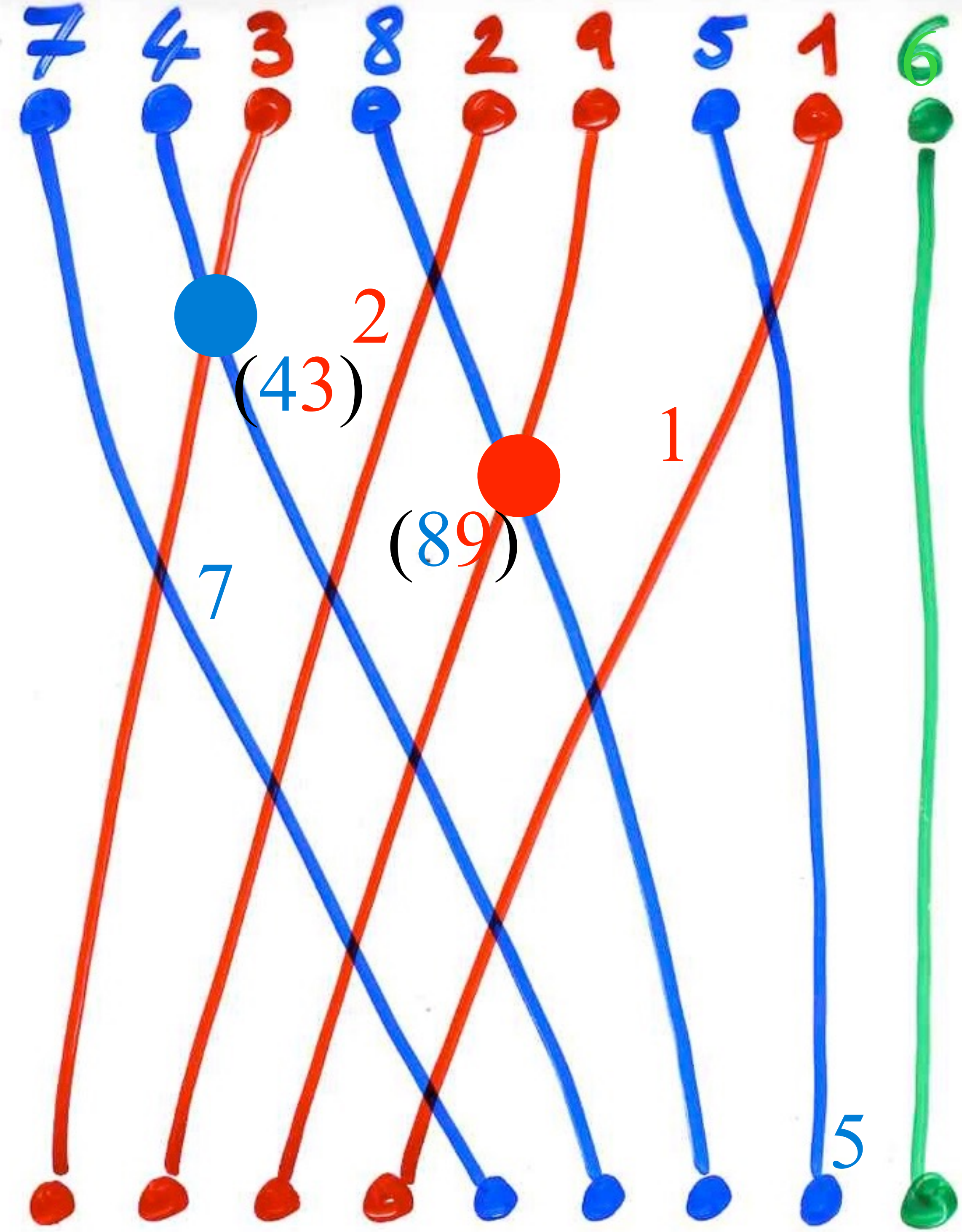
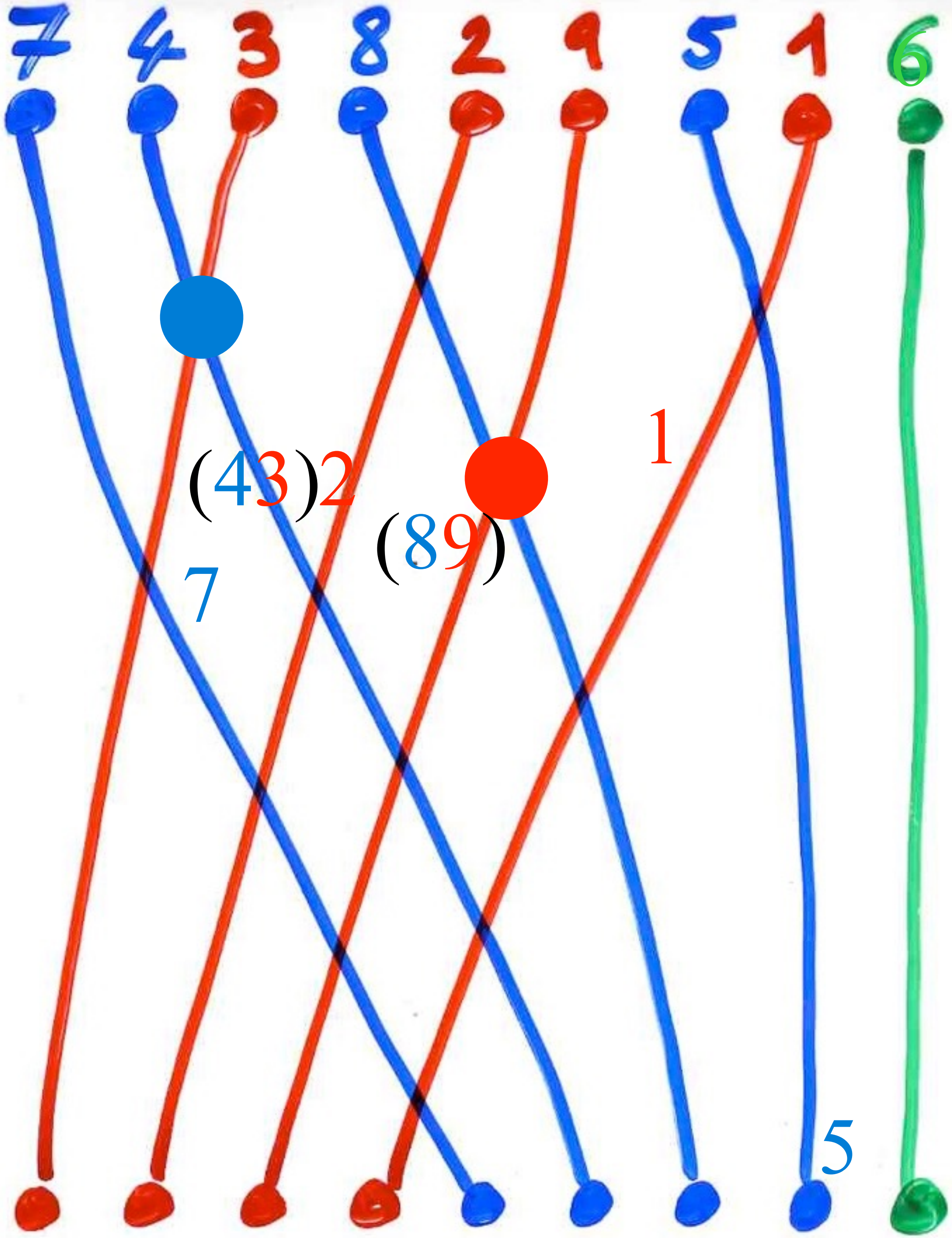
5



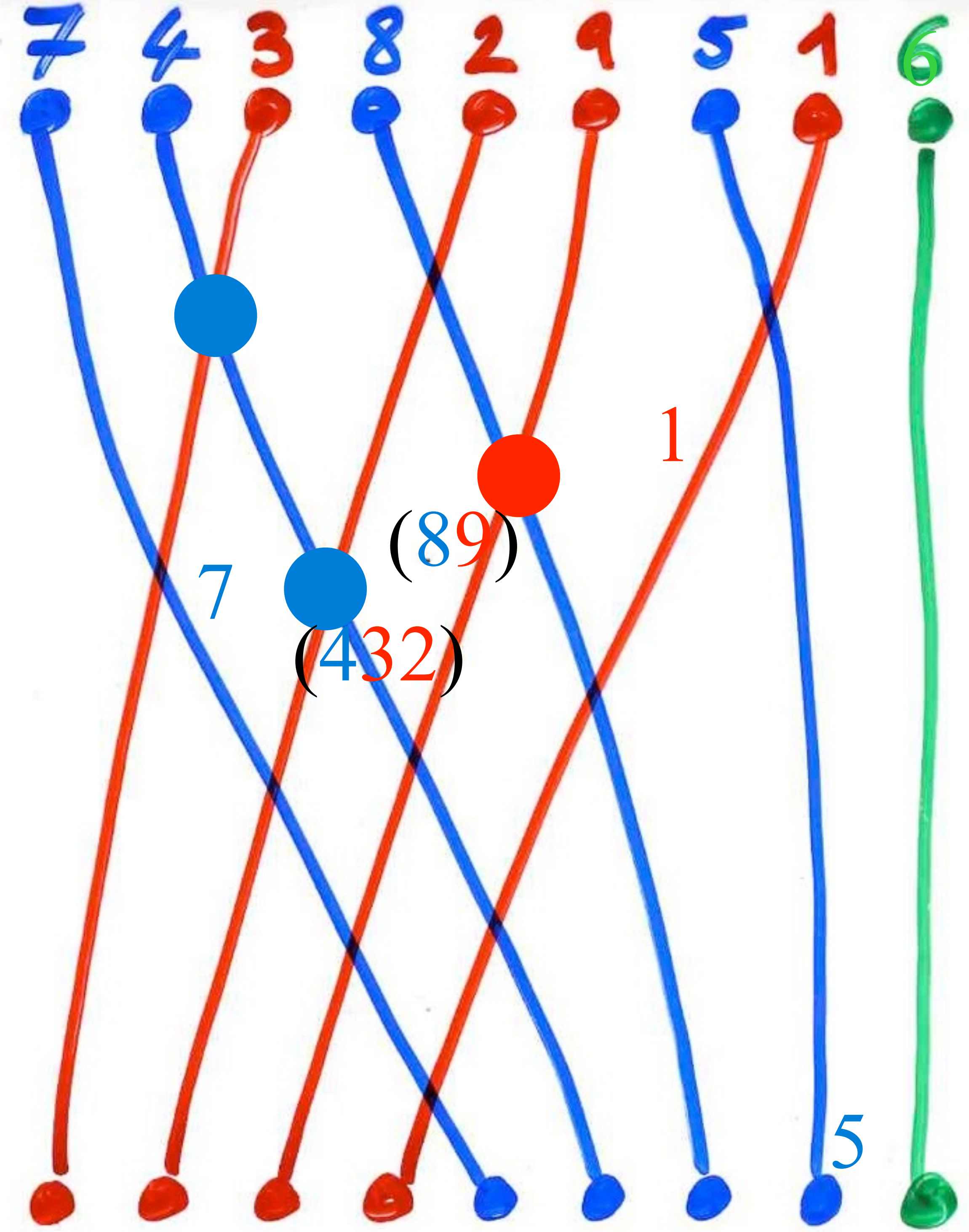
7 4 3 8 2 9 5 1 6
(43)
2
8
9
1
7
5

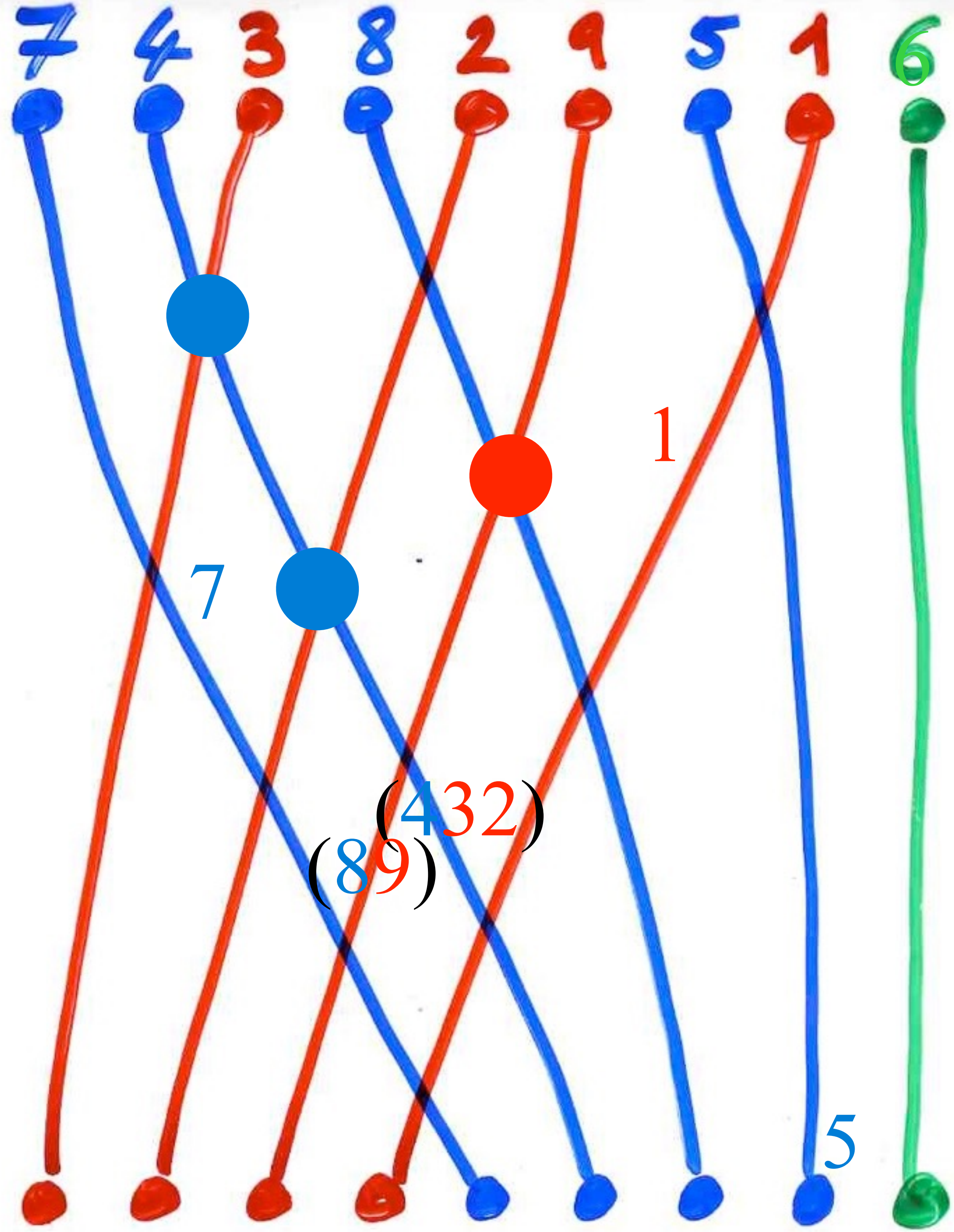


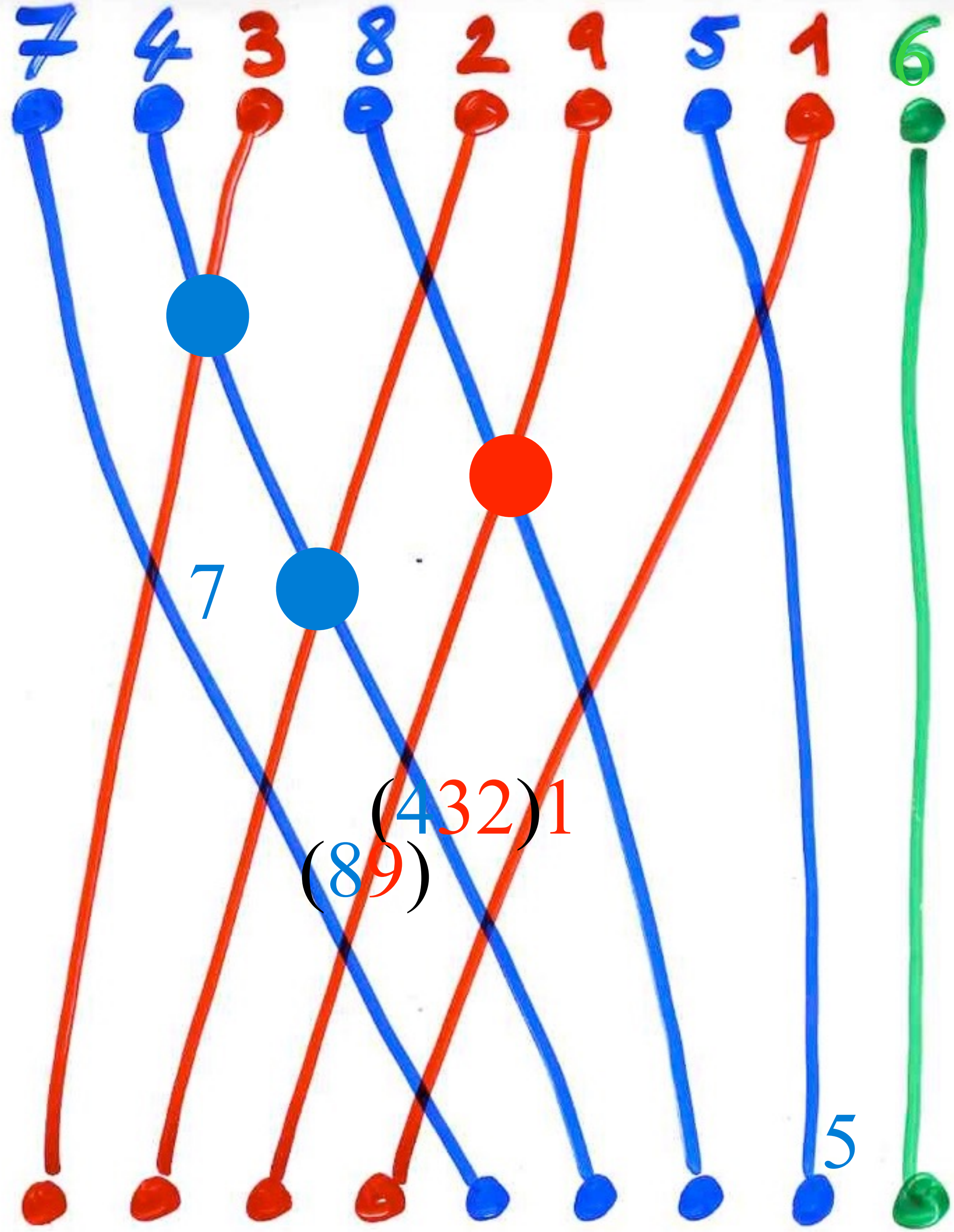
7 4 3 8 2 9 5 1 6
2
(43)
1
(89)
7
5



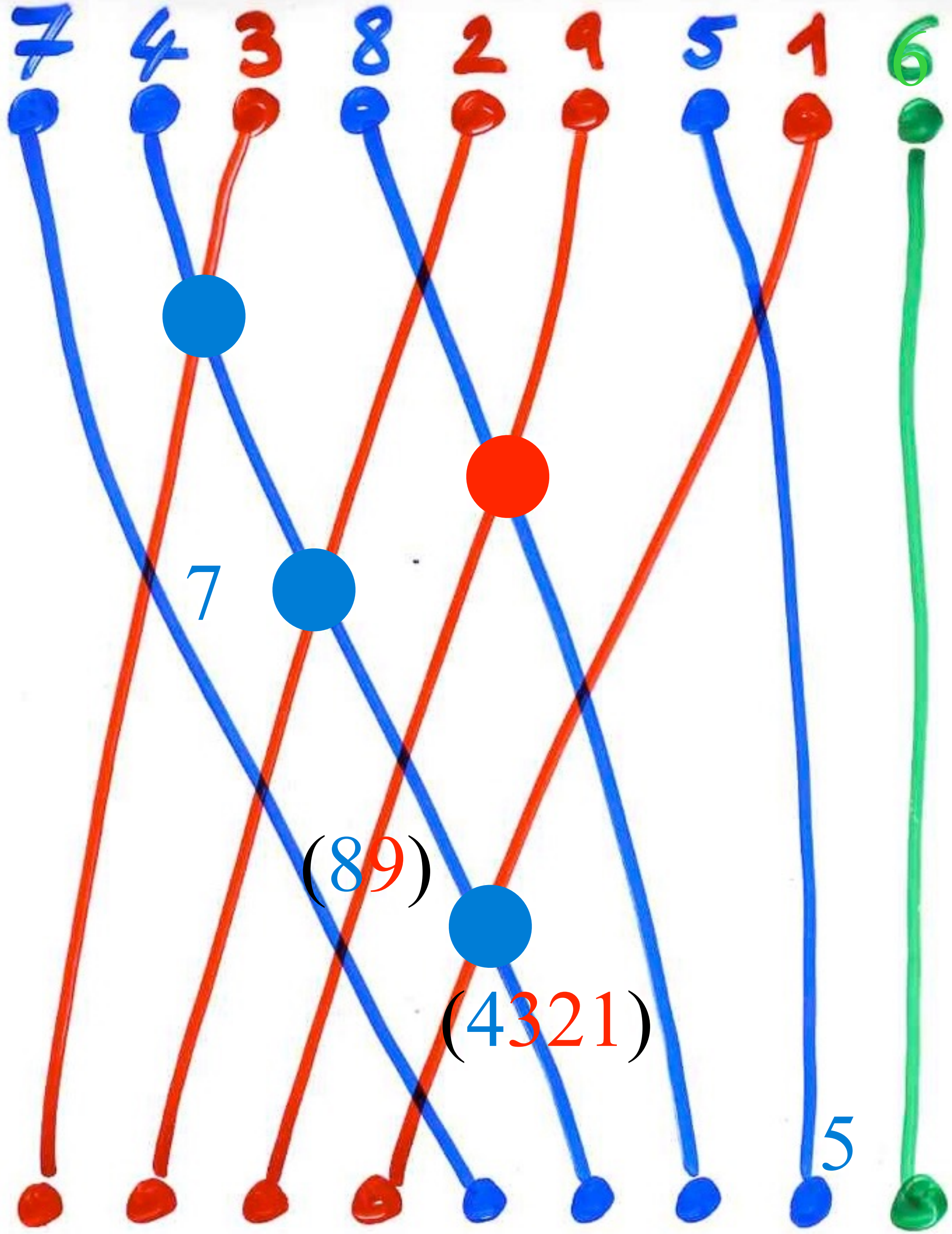
7 4 3 8 2 9 5 1 6
(43)2
(89)
1
7
5


7 4 3 8 2 9 5 1 6
1
(89)
7
(432)
5

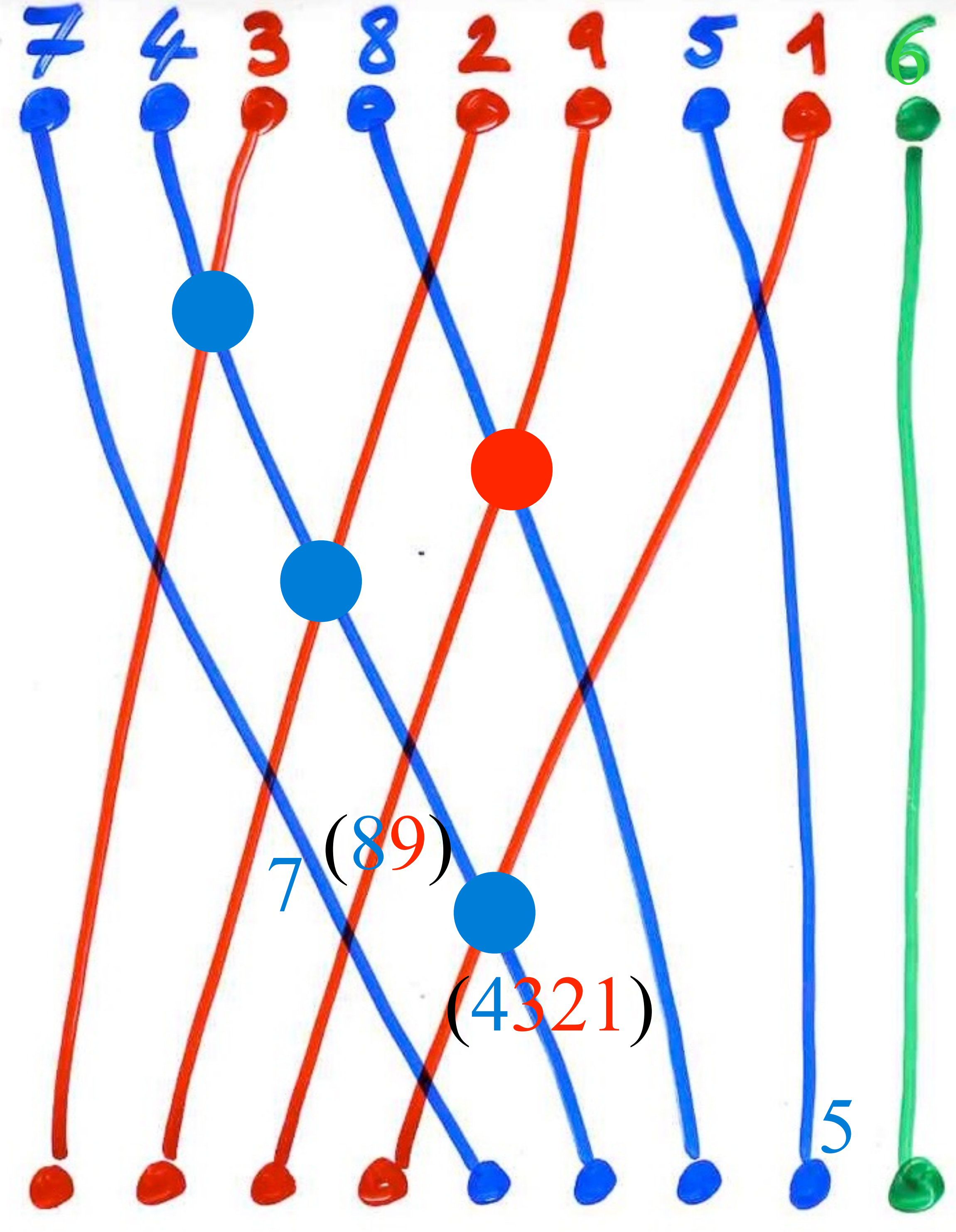

7 4 3 8 2 9 5 1 6
1
7
(432)
(89)
5

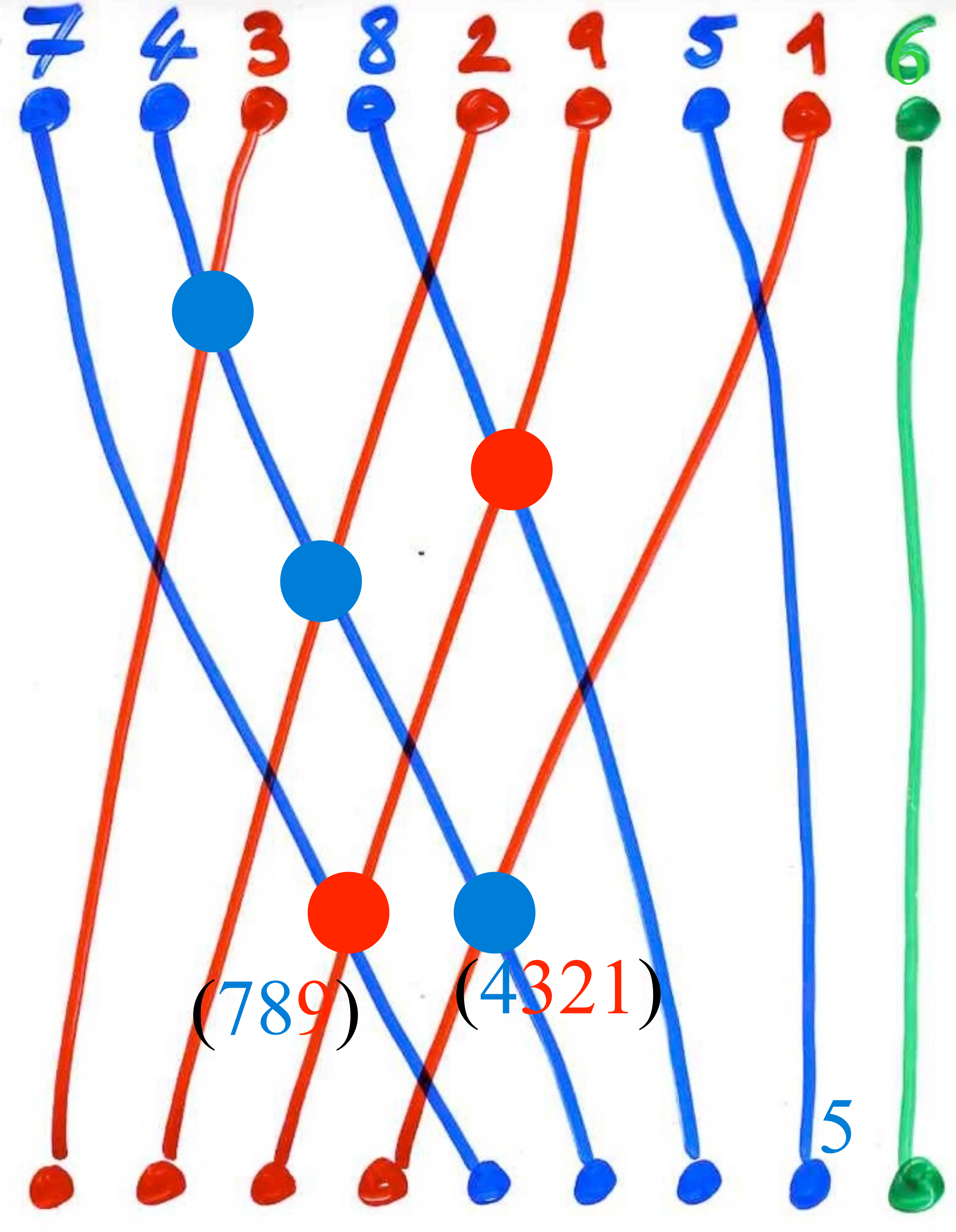


7 4 3 8 2 9 5 1 6
7
(432)1
(89)
5



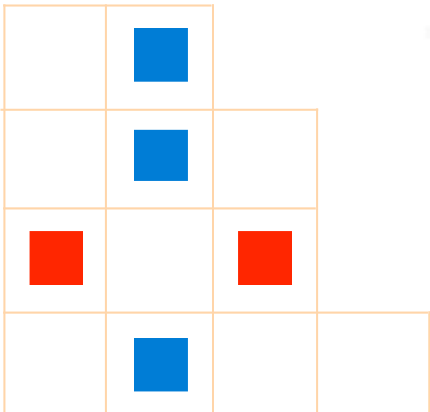
7 4 3 8 2 9 5 1 6
7
(89)
(4321)
5

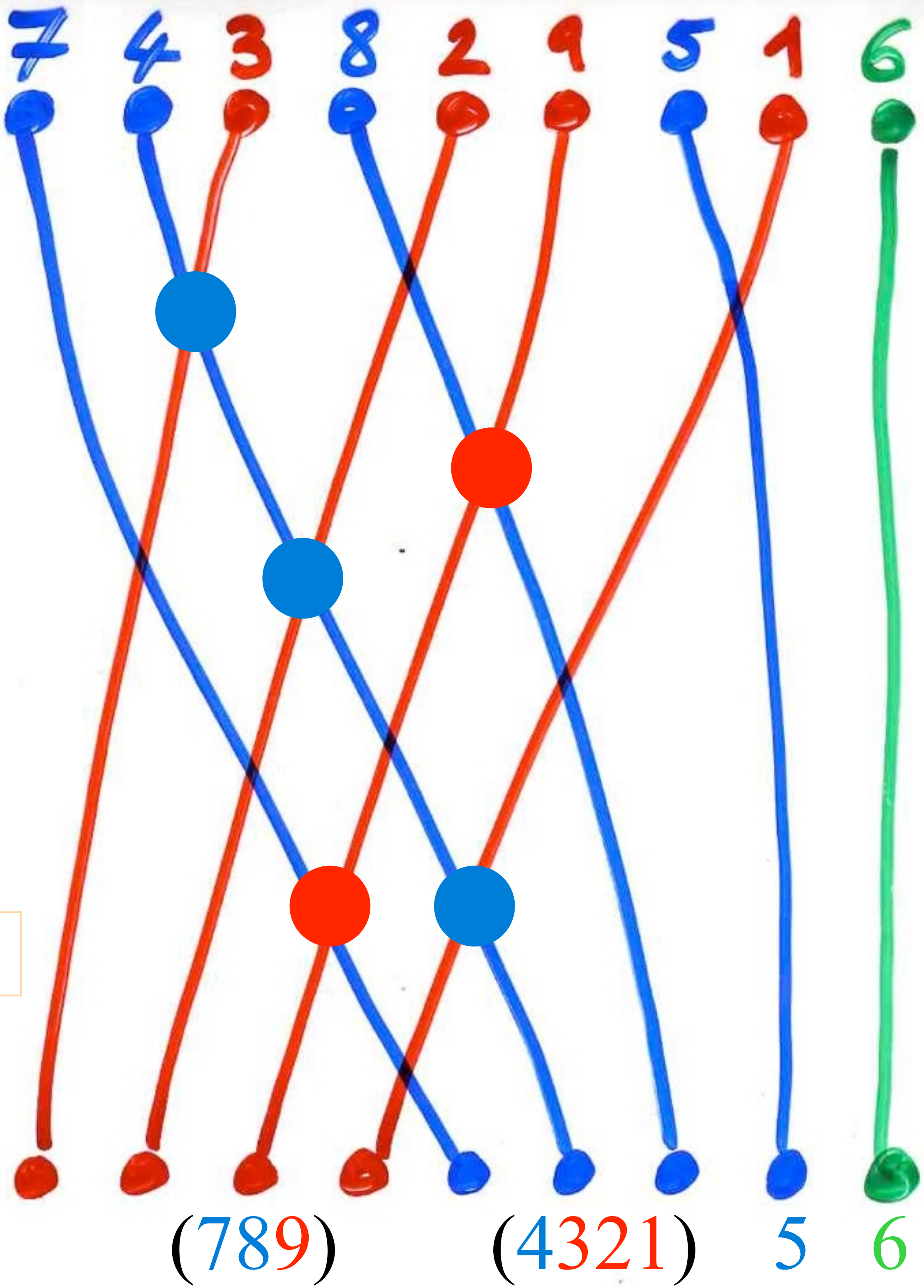

7 4 3 8 2 9 5 1 6
(89)
7
(4321)
5



7 4 3 8 2 9 5 1 6
(789)
(4321)
5

“exchange-fusion” algorithm





this bijection can be constructed
from a combinatorial representation
of the PASEP algebra

and using the methodology
of the «Cellular Ansatz»

"The cellular Ansatz"

Physics

combinatorial
objects
on a 2d lattice

representation
by operators

bijections

"normal ordering"

$UD = DU + Id$

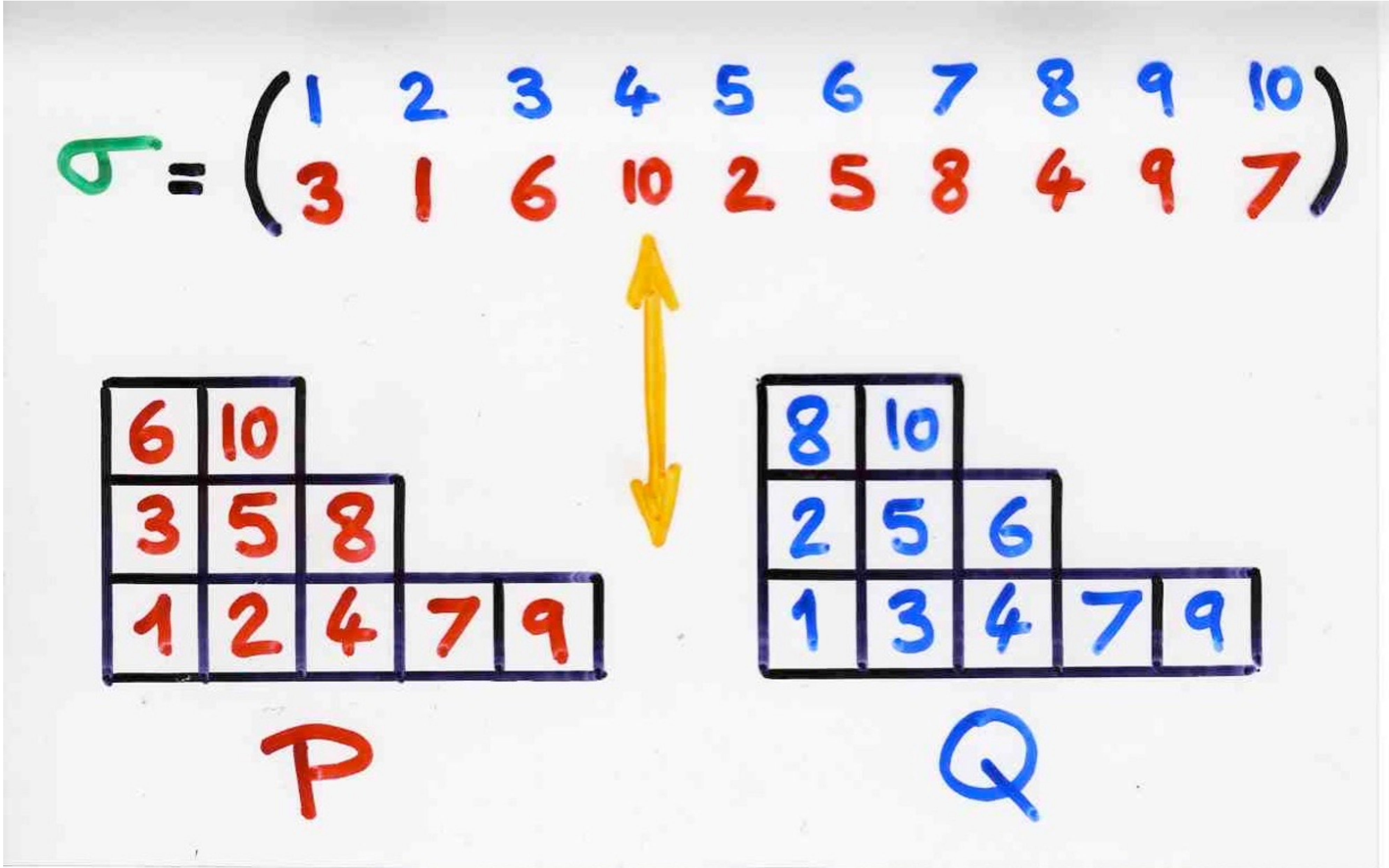
Weyl-Heisenberg

RSK

permutations ⟷ pairs of Tableaux Young

quadratic algebra Q

Q-tableaux



The Robinson-Schensted correspondence (RSK) between permutations and pair of (standard) Young tableaux with the same shape in (finite) group theory :

$$|G| = \sum_{R} (\deg R)^2$$

order of the group

R irreducible representation

for the symmetric group $\mathfrak{S}_n$:

$$n! = \sum_{\lambda} (f_\lambda)^2$$

λ partitions of n

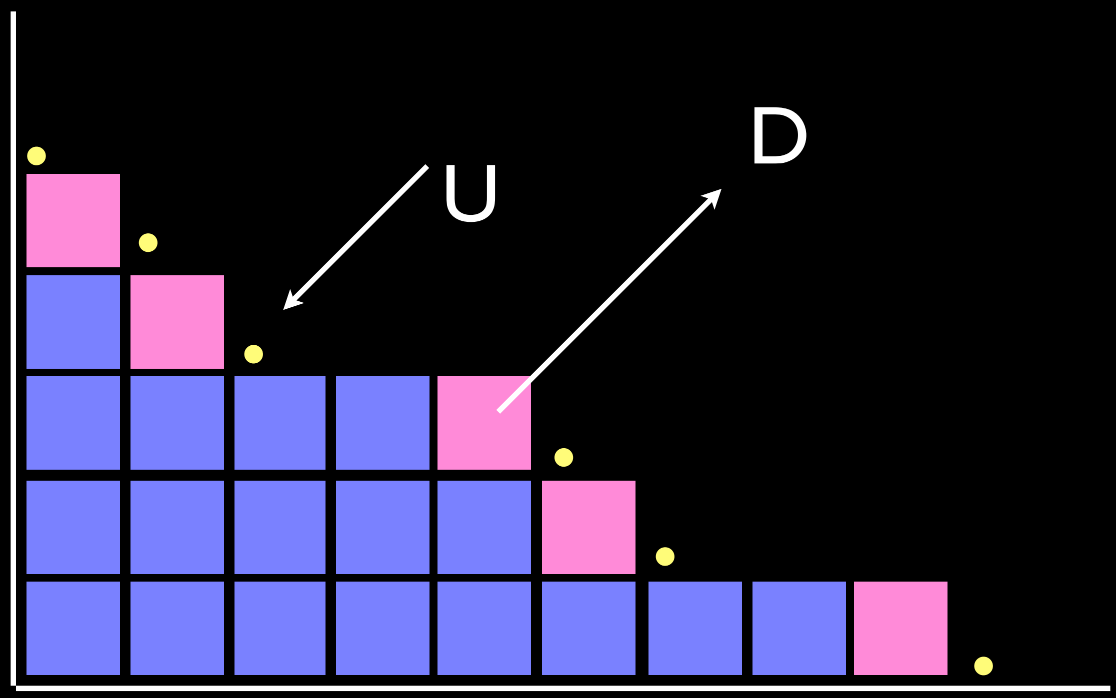
representation of the operators U, D

$UD = DU + I$

Sergey Fomin

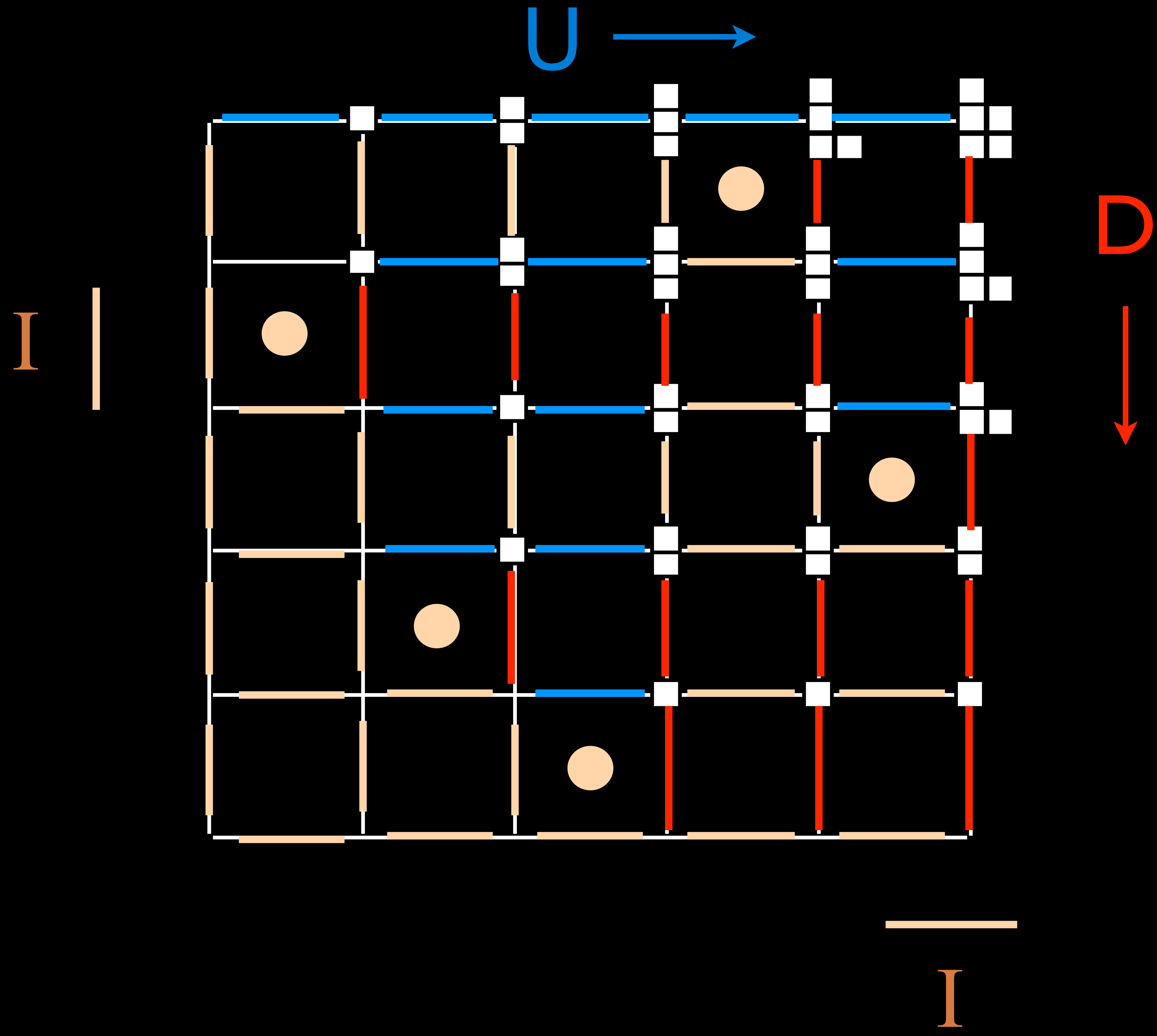
(with C. K.)

Operators U and D



adding
or deleting
a cell in
a Ferrers
diagram

Young lattice


U
D
I
I

The cellular Ansatz

guided construction
of a bijection
(from a representation of the quadratic
algebra Q with "combinatorial operators")

"The cellular Ansatz"

Physics

combinatorial
objects
on a 2d lattice

representation
by operators

bijections

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

$DE = qED + E + D$

PASEP

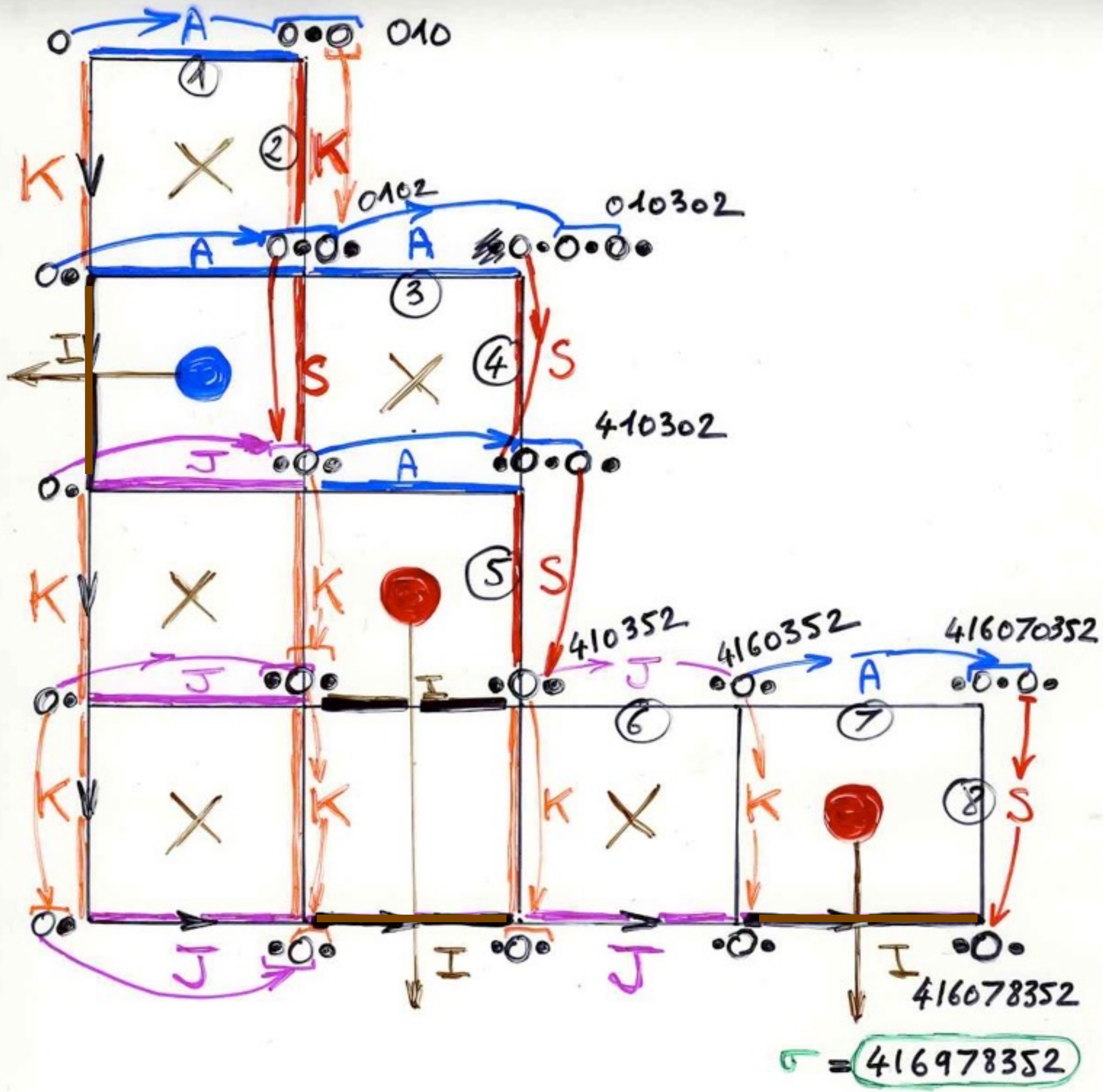
RSK

permutations ⟷ pairs of Tableaux Young

alternative tableaux ⟷ permutations

quadratic algebra Q

Q-tableaux



010
0102
010302
410302
410352
4160352
416070352
416078352
σ = 416978352

relation with orthogonal polynomials

$$\sum_{T} q^{k(T)} \alpha^{i(T)} \beta^{j(T)}$$

alternative tableaux size n

moments

q-analogue of Laguerre polynomials

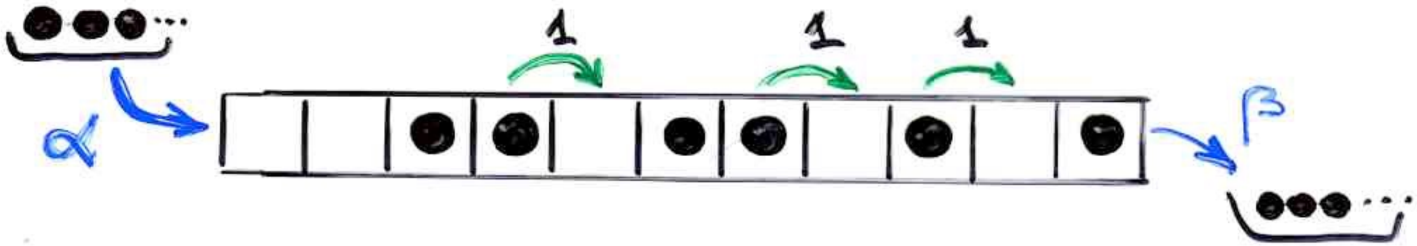
$$q\text{-}L_n^{(\alpha,\beta)}(x)$$

TASEP

Totally asymmetric exclusion process

TASEP

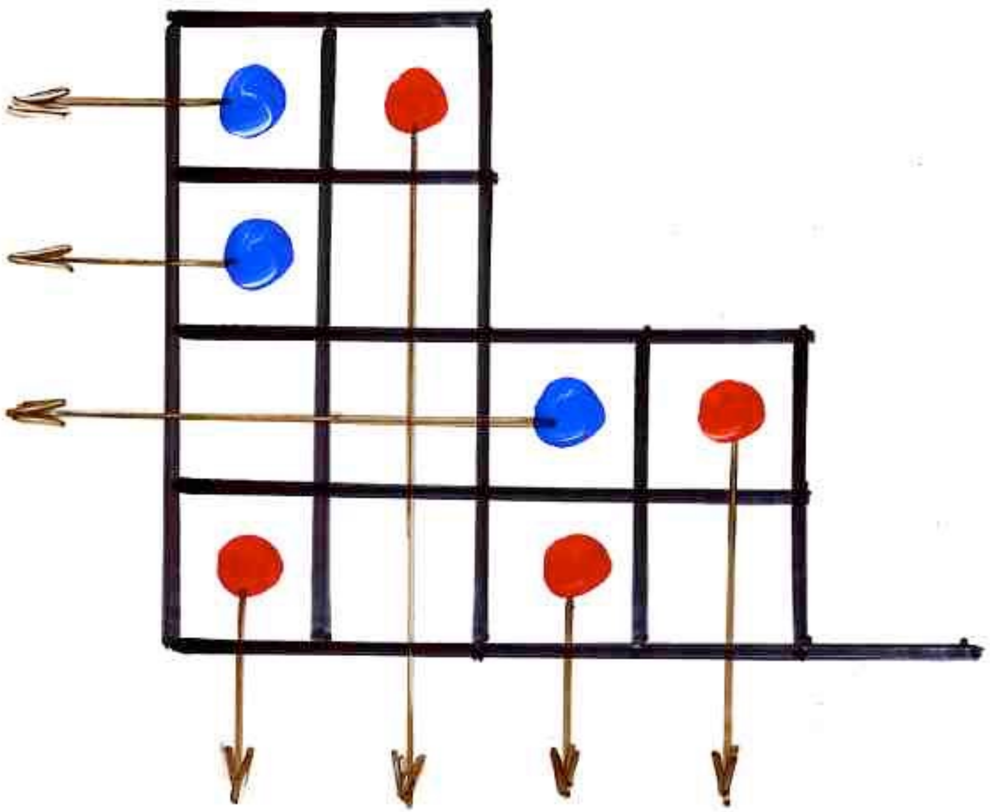
"totally asymmetric exclusion process"



Def Catalan alternative tableau T

alt. tab. without cells ▢

i.e: every empty cell is below a red cell or on the left of a blue cell

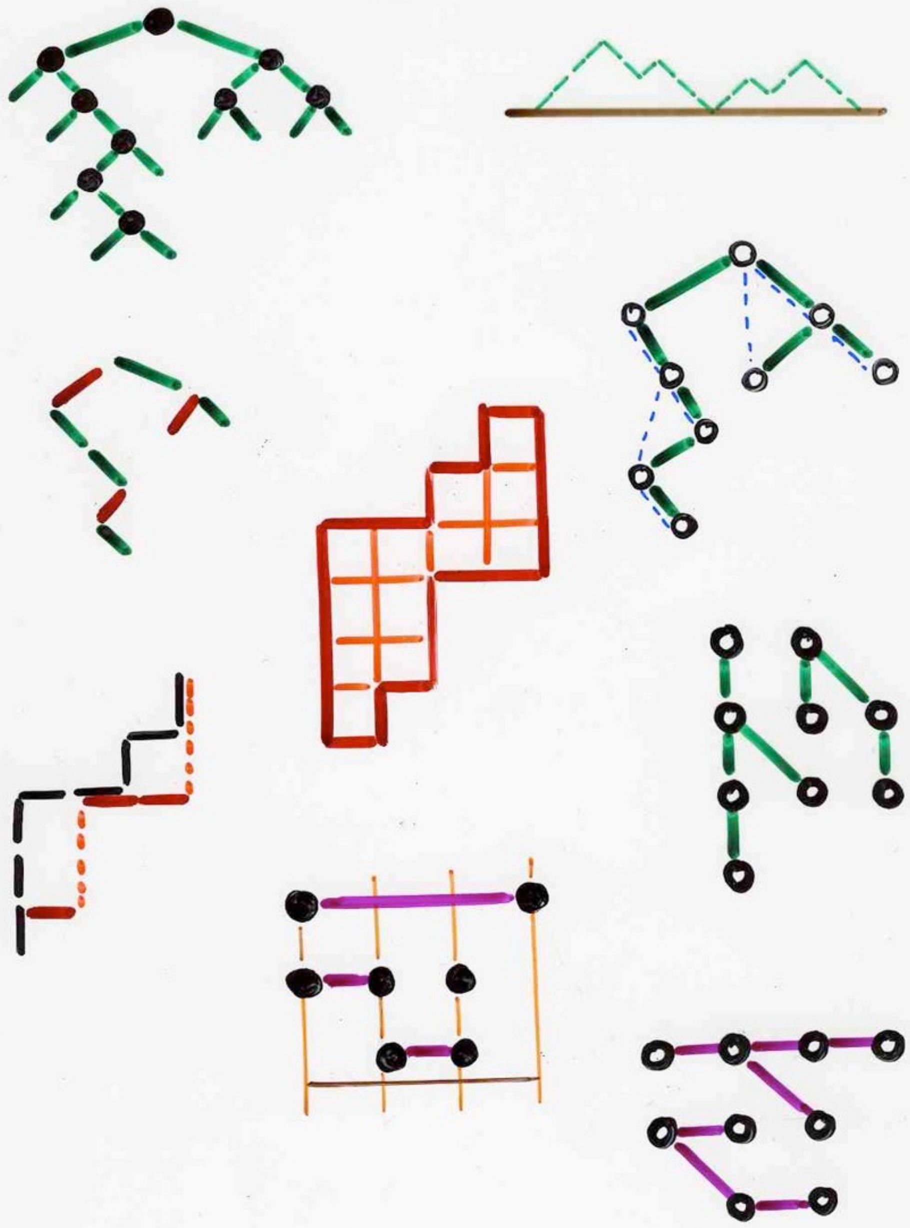


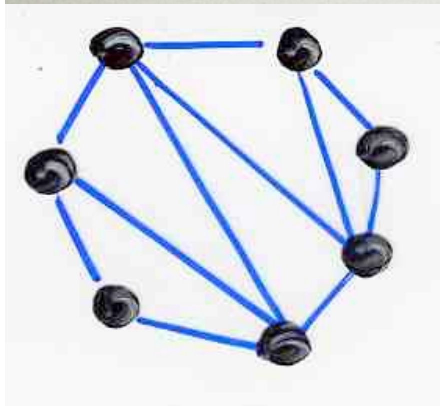
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\ n!}$$

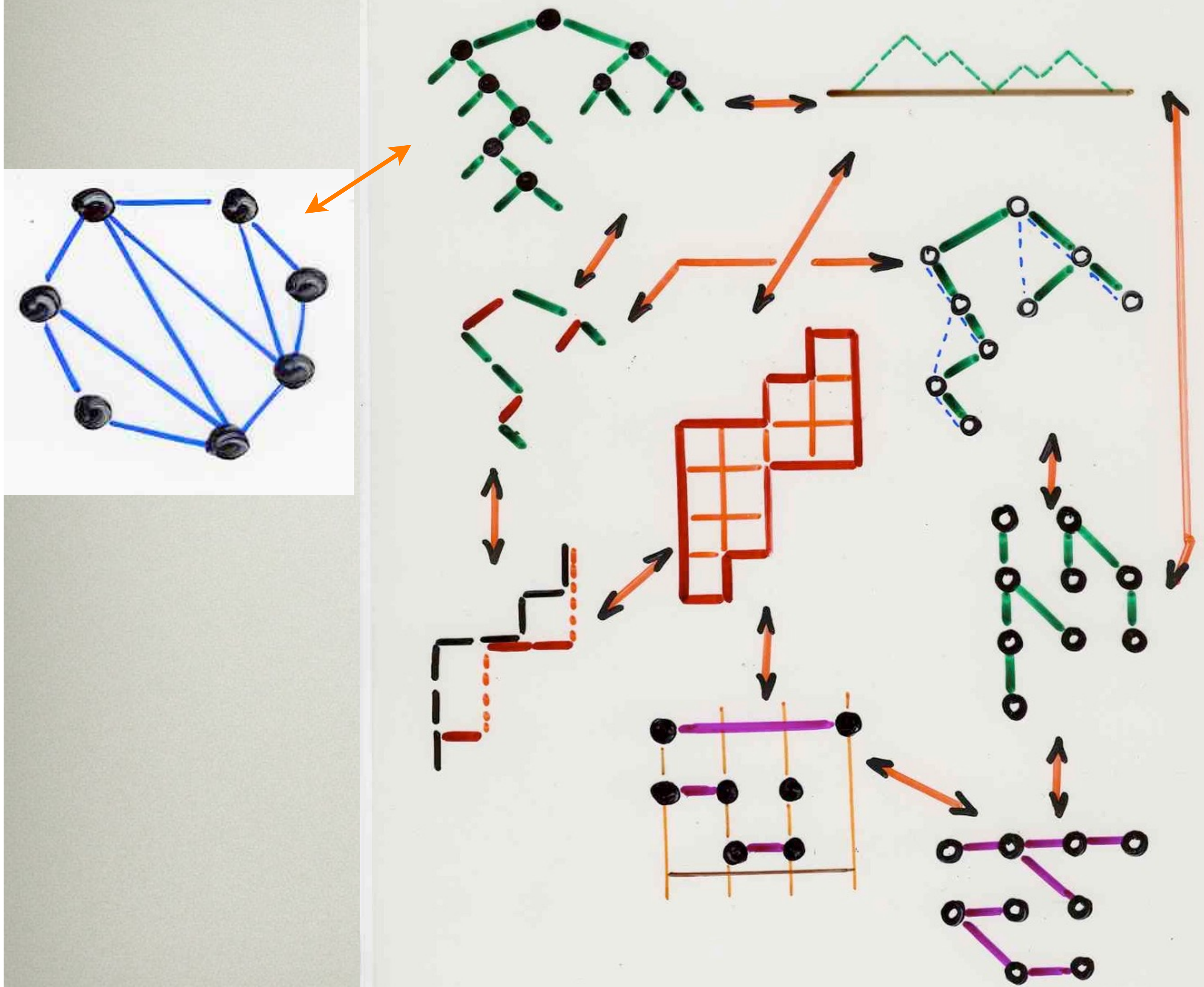
$$n! = 1 \times 2 \times \dots \times n$$

the Catalan garden





the Catalan garden



steady state probability PASEP

$$\frac{1}{Z_n} Z_\tau(\alpha, \beta, \gamma, \delta; q)$$

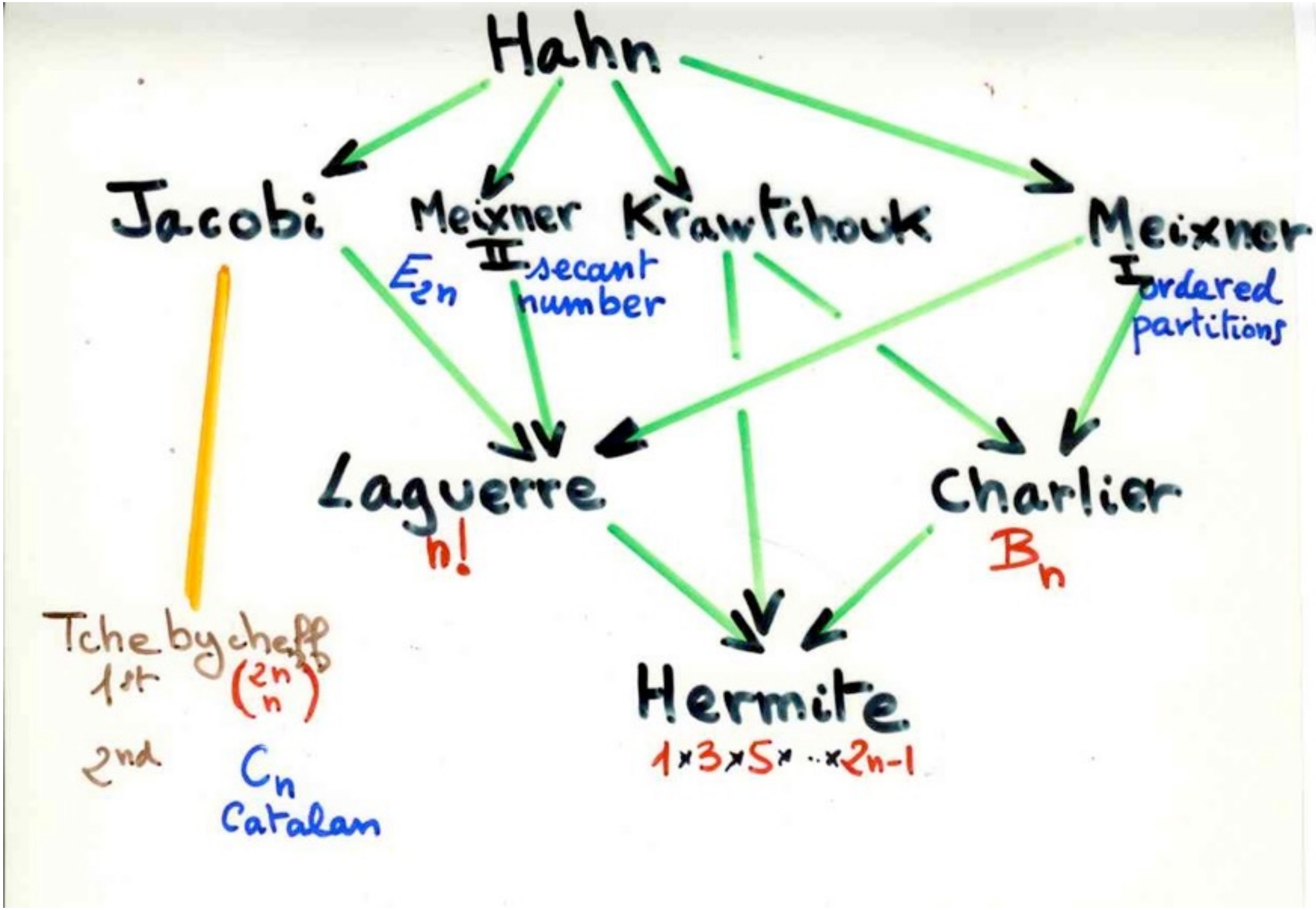
$$Z_n = \sum_\tau Z_\tau$$

$\tau = (\tau_1, \dots, \tau_n)$ state

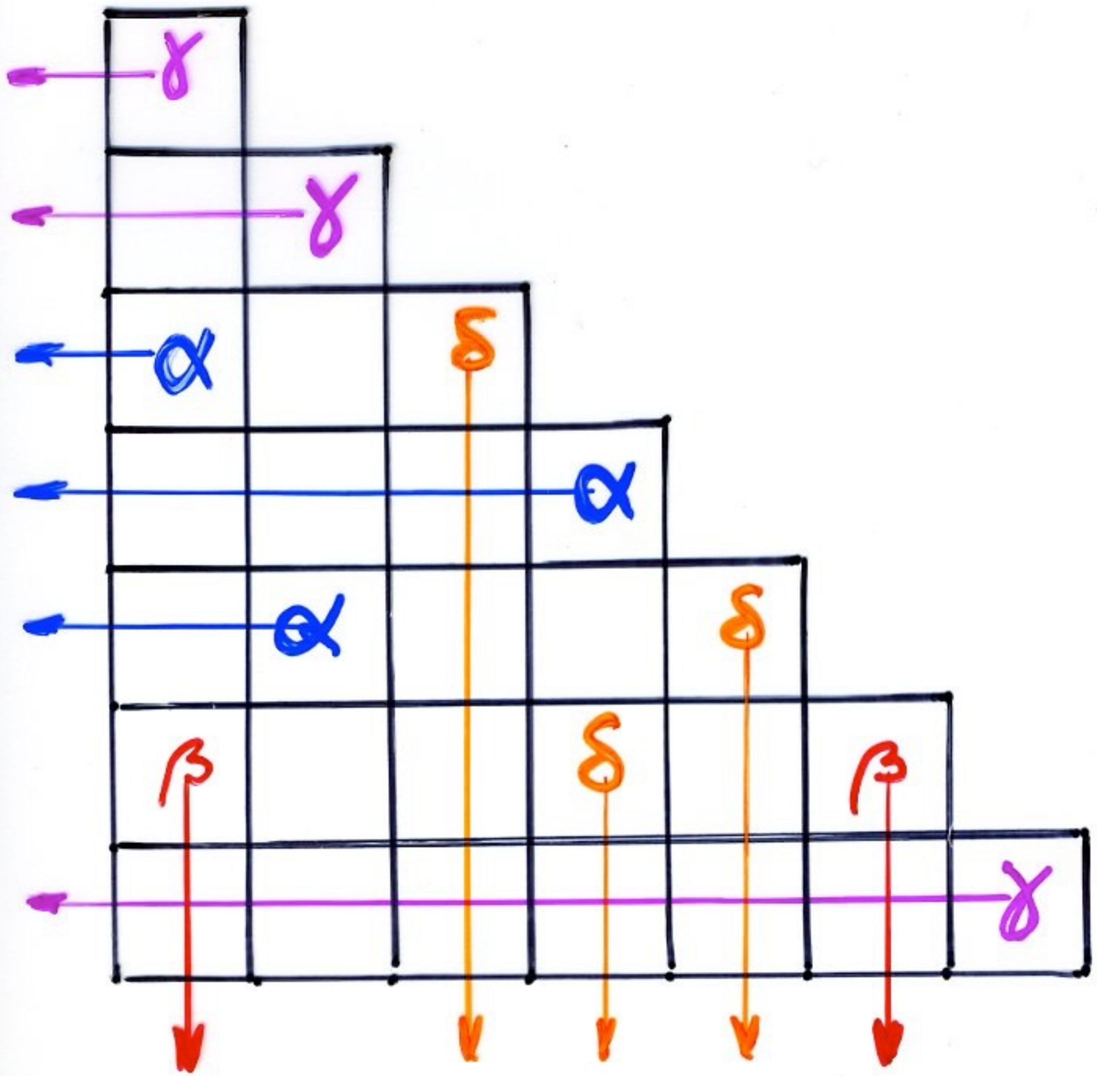
relation with moments of Askey-Wilson polynomials

Corteel, Williams, 2009
Corteel, Stanley, Stanton, Williams, 2010

Askey-Wilson



staircase tableaux



number of
alternative tableaux

with alternating shape

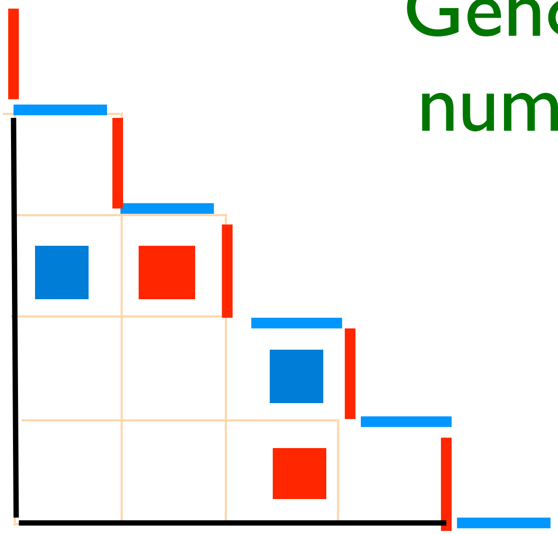
nombres de Genocchi

$$G_{2n} = 2(2^{2n} - 1) B_{2n}$$

Bernoulli

$$2^{2n} G_{2n+2} = (n+1) T_{2n+1}$$

Genocchi numbers



alternating shape

Angelo Genocchi
1817 - 1889

Hinc igitur calculo inſtituto reperietur:

$$A = 1$$
$$B = 1$$
$$C = 3$$
$$D = 17$$
$$E = 155 = 5.31$$
$$F = 2073 = 691.3$$
$$G = 38227 = 7.5461 = 7.\frac{127.129}{3}.$$
$$H = 929569 = 3617.257$$
$$I = 28820619 = 43867.9.73 \quad \&c.$$