

Algèbres d'opérateurs  
et  
Physique combinatoire  
  
(part 3)

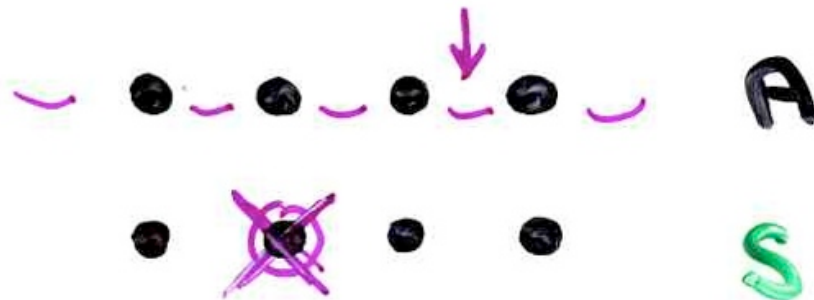
12 Avril 2012  
colloquium de l'IMJ  
Institut mathématiques de Jussieu

Xavier Viennot  
CNRS, LaBRI, Bordeaux

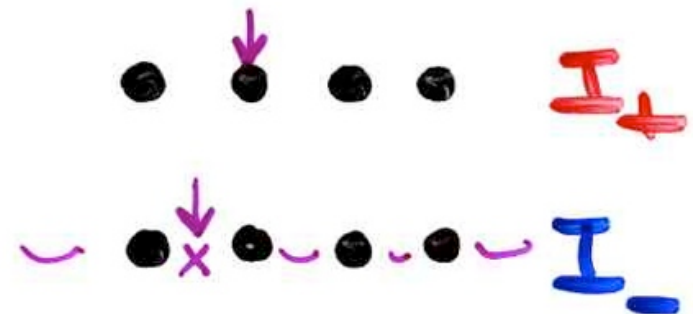
quadratic algebra  
operators  
data structures  
and orthogonal polynomials

## Opérations primitives

A ajout  
S suppression



I<sub>+</sub> interrogation positive  
I<sub>-</sub> interrogation negative

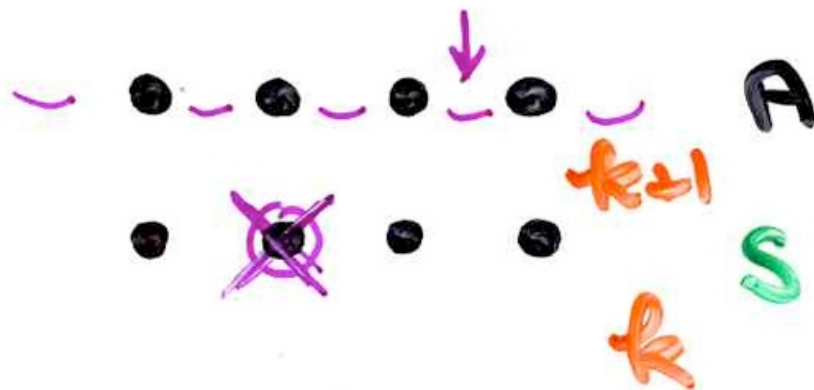


Primitive operations  
for "dictionnaires" data structure:  
add or delete any elements, asking questions (with  
positive or negative answer)

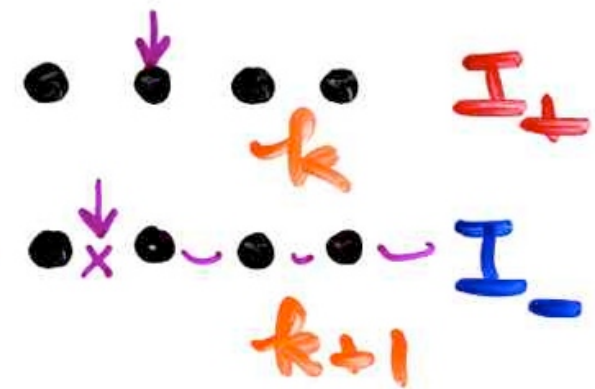
# Opérations primitives

A ajout  
S suppression

$I_+$  interrogation positive  
 $I_-$  interrogation négative



nb de choix



number of choices for each  
primitive operations

(formal) orthogonal  
polynomials

# Orthogonal polynomials

Def.  $\{P_n(x)\}_{n \geq 0}$

orthogonal iff

$$P_n(x) \in \mathbb{K}[x]$$

$$\exists \ell: \mathbb{K}[x] \rightarrow \mathbb{K}$$

linear functional

$$\left\{ \begin{array}{ll} \text{(i)} & \deg(P_n(x)) = n \\ \text{(ii)} & \ell(P_h P_l) = 0 \\ \text{(iii)} & \ell(P_h^2) \neq 0 \end{array} \right.$$

$$(\forall n \geq 0)$$

$$\text{for } h \neq l \geq 0$$

$$\text{for } h \geq 0$$

---

$$f(x^n) = \mu_n \quad (n \geq 0)$$

moments

$$f(PQ) = \int_a^b P(x) Q(x) d\mu$$

measure

combinatorial interpretation  
of the moments

## Thm. (Favard)

- $\{P_n(x)\}_{n \geq 0}$  sequence of **monic** polynomials,  $\deg(P_n) = n$
- $\{b_k\}_{k \geq 0}$ ,  $\{\lambda_k\}_{k \geq 1}$  coeff. in  $\mathbb{K}$

orthogonality  $\iff$

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

( $\forall k \geq 1$ )

3 terms linear recurrence relation

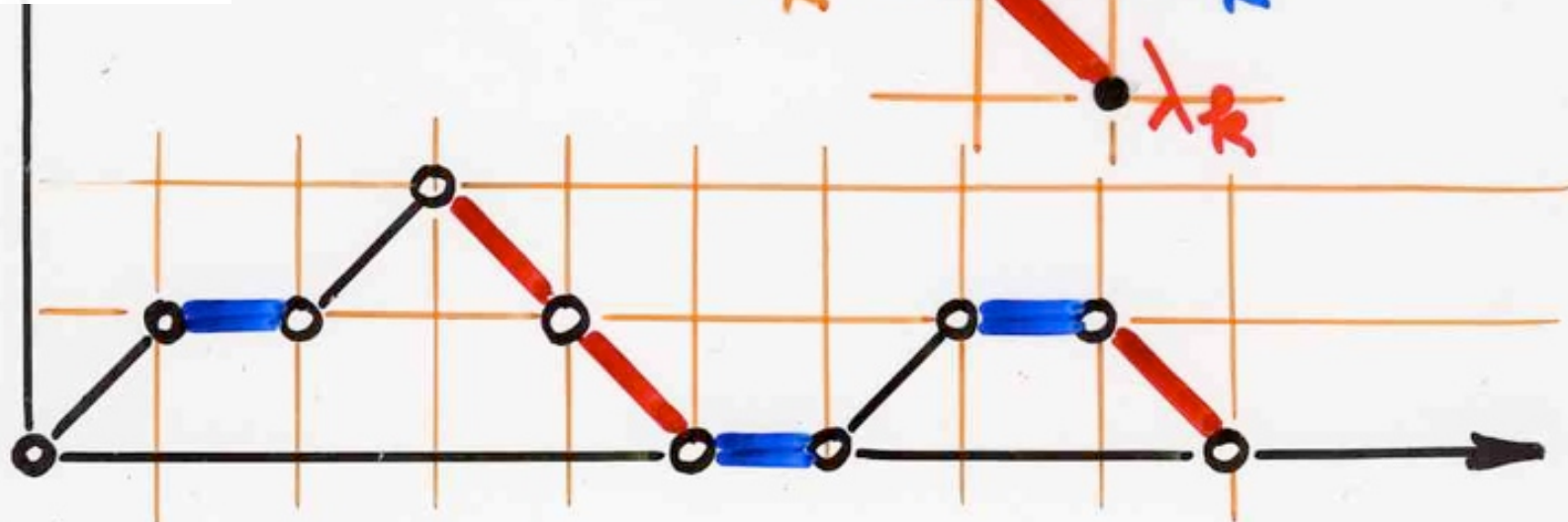
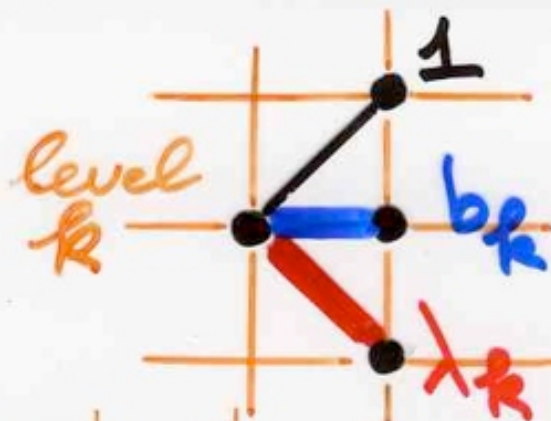


$$\{b_k\}_{k \geq 0}$$

$$\{\lambda_k\}_{k \geq 1}$$

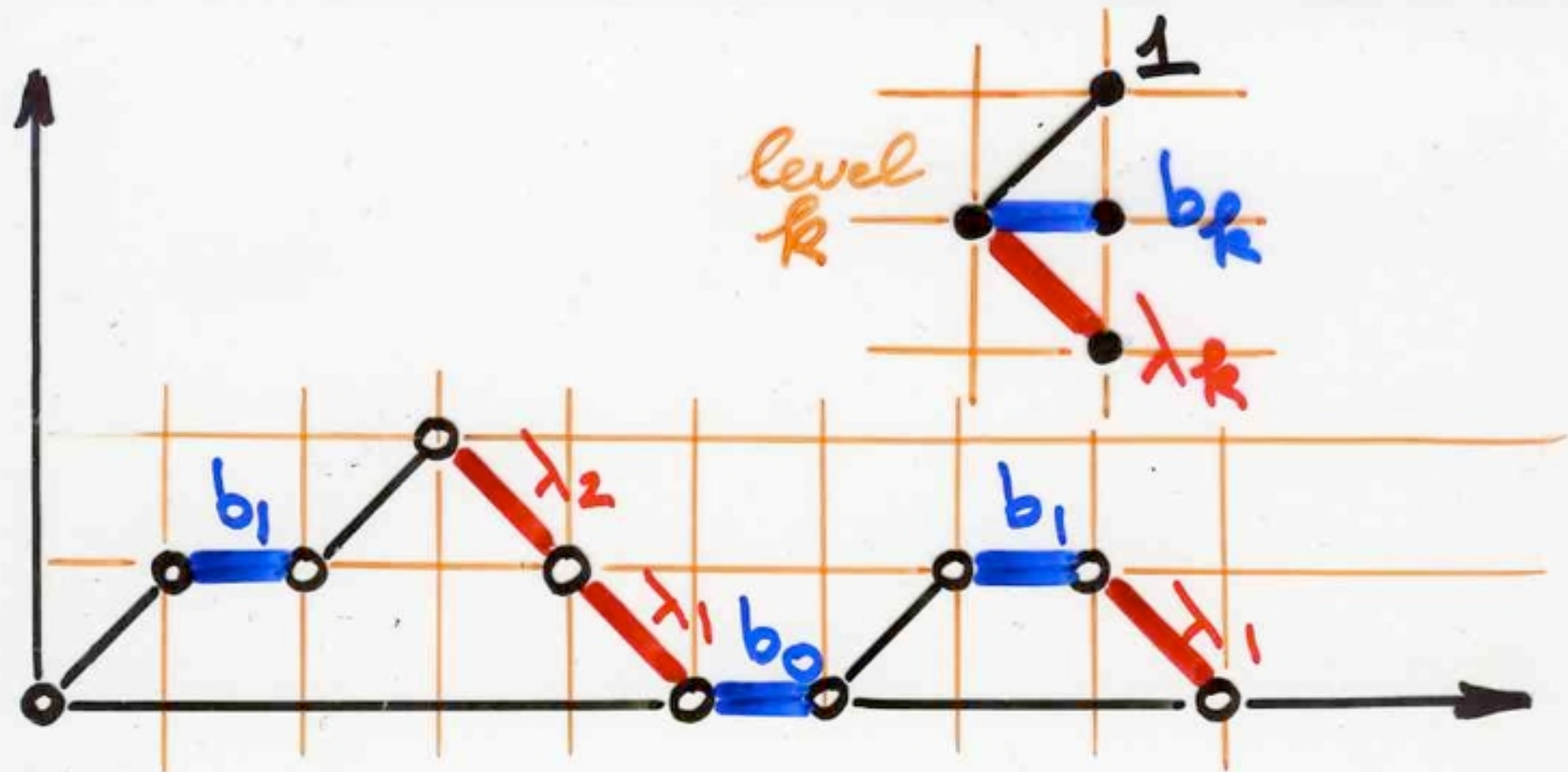
$$b_k, \lambda_k \in \mathbb{K} \text{ ring}$$

valuation



$\omega$  Motzkin path

valuation ✓



$\omega$  Motzkin path

$$v(\omega) = b_0 b_1^2 \lambda_1^2 \lambda_2$$

---

$$f(x^n) = \mu_n \quad (n \geq 0)$$

moments

$$\mu_n = \sum_{\omega} v(\omega)$$

$\omega$   
Motzkin  
path  
 $|\omega| = n$



P. Flajolet

continued  
fractions

1980

orthogonal  
polynomials

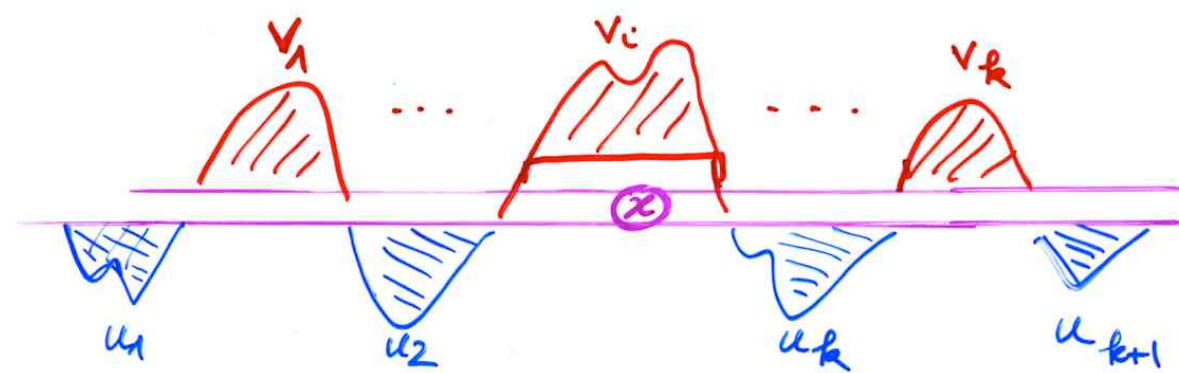
Lecture Note

X.V. 1983

PASEP algebra

$$DE = qED + E + D$$

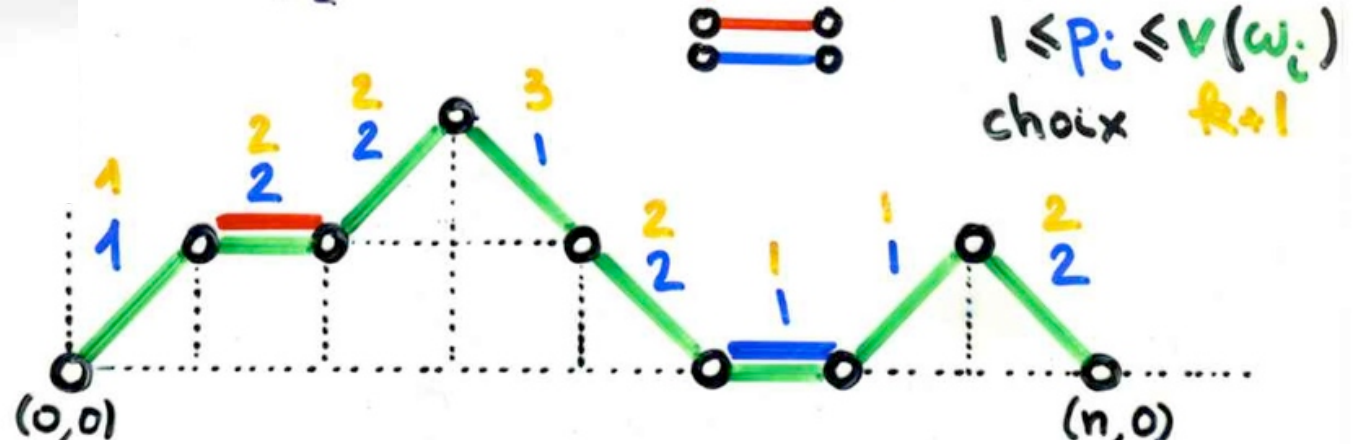
$q$ -Laguerre polynomials



1  
 2  
 3  
 4  
 5  
 6  
 7  
 8  
 9

1  
 1 1  
 1 1 2  
 1 1 3 2  
 4 1 3 2  
 4 1 3 5 2  
 4 1 6 3 5 2  
 4 1 6 7 3 5 2  
 4 1 6 7 8 3 5 2  
 4 1 6 9 7 8 3 5 2

$$h = (\omega_c; (p_1, \dots, p_n))$$



| $x$ | $\omega_c$ | pos | $v$ |
|-----|------------|-----|-----|
| 1   |            | 1   | 1   |
| 2   |            | 2   | 2   |
| 3   |            | 2   | 2   |
| 4   |            | 1   | 3   |
| 5   |            | 2   | 2   |
| 6   |            | 1   | 1   |
| 7   |            | 1   | 1   |
| 8   |            | 2   | 2   |
| 9   |            |     |     |

1   
 1 2  
 1 3 2  
4 1 3 2  
4 1 3 5 2  
4 1 6 3 5 2  
4 1 6 7 3 5 2  
4 1 6 7 8 3 5 2  
4 1 6 9 7 8 3 5 2

$= \text{GG}_{n+1}$

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$P_0 = 1$$

$$P_1 = x - b_0$$

$$\mu_n = (n+1)!$$

$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

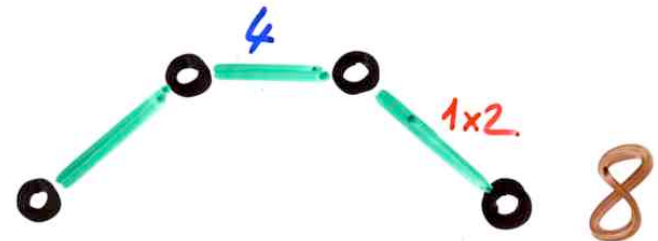
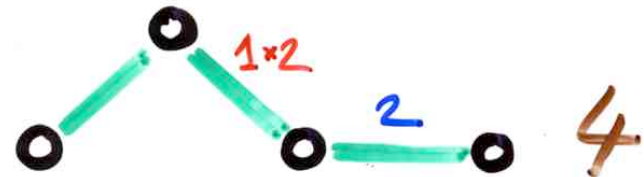
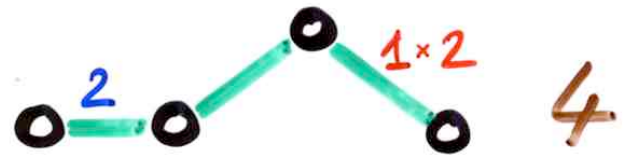
Laguerre  
polynomial

# Laguerre $L_n^{(1)}(x)$

moment  $\mu_n = (n+1)!$

$$b_k = 2k+2$$

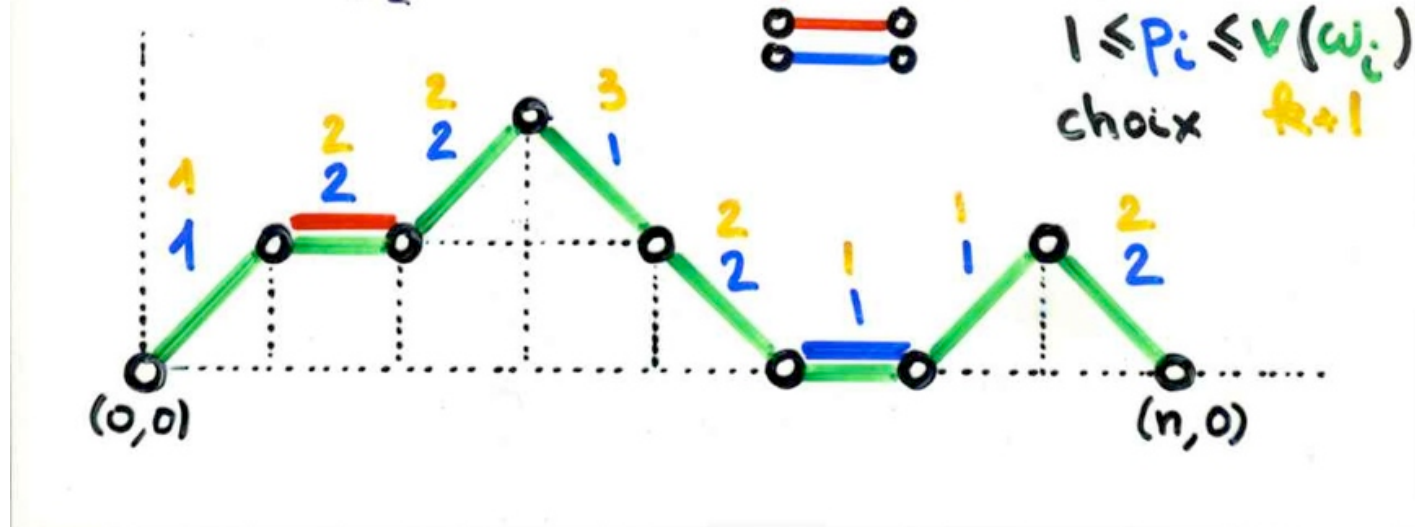
$$\lambda_k = k(k+1)$$




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“q-analogue”  
of Laguerre  
histories



choices function

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 2 | 2 | 1 | 2 | 1 | 1 | 2 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |

q-Laguerre :  $q^4$

$\lceil$  1  $\rfloor$   
 $\lceil$  1  $\rfloor$  2  
 $\lceil$  1  $\rfloor$  3  $\rfloor$  2  
4 1  $\lceil$  3  $\rfloor$  2  
4 1  $\lceil$  3 5 2  
4 1 6  $\lceil$  3 5 2  
4 1 6  $\lceil$  7  $\rfloor$  3 5 2  
4 1 6  $\lceil$  7 8 3 5 2  
4 1 6 9 7 8 3 5 2 =  $\in \mathcal{GL}_{n+1}$

# q-Laguerre

$$\tilde{L}_n^{\beta}(x; q)$$

$$\beta = \alpha + 1$$

$$\left\{ \begin{array}{l} b_{k,q}^{(\beta)} = [k]_q + [k+1; \beta]_q \\ \lambda_{k,q}^{(\beta)} = [k]_q \cdot [k; \beta]_q \end{array} \right.$$

$$[k; \beta]_q = \beta + q + q^2 + \dots + q^{k-1}$$

$$\mu_n = \frac{1}{(1-q)^n} \sum_{k=0}^n (-1)^k \left( \binom{2n}{n-k} - \binom{2n}{n-k-2} \right) \left( \sum_{i=0}^k q^{i(k+i)} \right)$$

Cortez, Josuat-Vergès  
 Pnellberg, Rubey (2008)  $y$

general PASEP

# • Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

$q$ -Hermite polynomial  
 $\alpha, \beta, q$   $\gamma = \delta = 1$

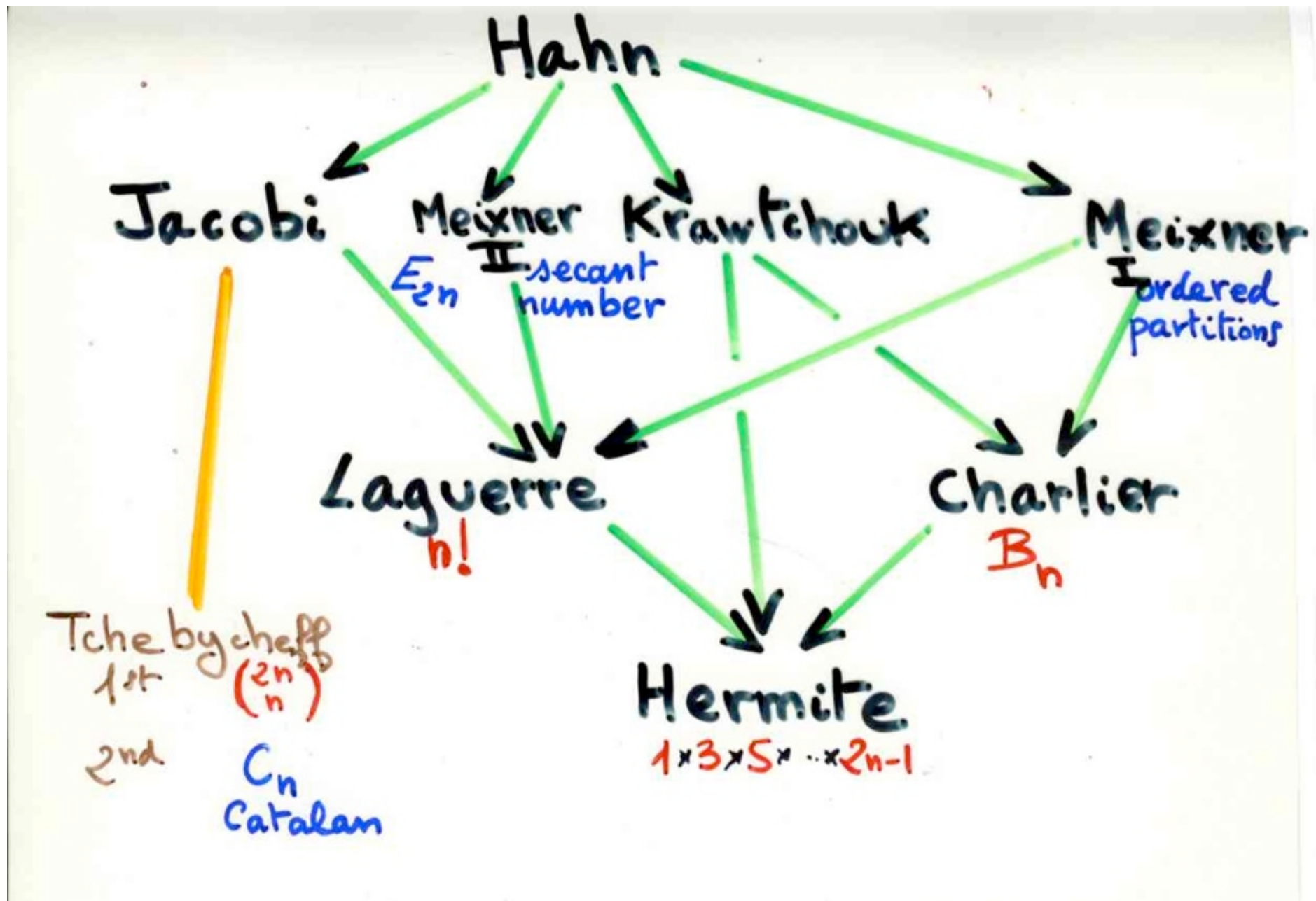
$$\begin{aligned} D &= \frac{1}{1-q} + \frac{1}{\sqrt{1-q}} \hat{a} \\ E &= \frac{1}{1-q} + \frac{1}{\sqrt{1-q}} \hat{a}^+ \\ \hat{a} \hat{a}^+ - q \hat{a}^+ \hat{a} &= 1 \end{aligned}$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, q$

Askey-Wilson polynomials

# Askey-Wilson



steady state  
probability  
PASEP

$$\frac{1}{Z_n} Z_{\tau}(\alpha, \beta, \gamma, \delta; q)$$

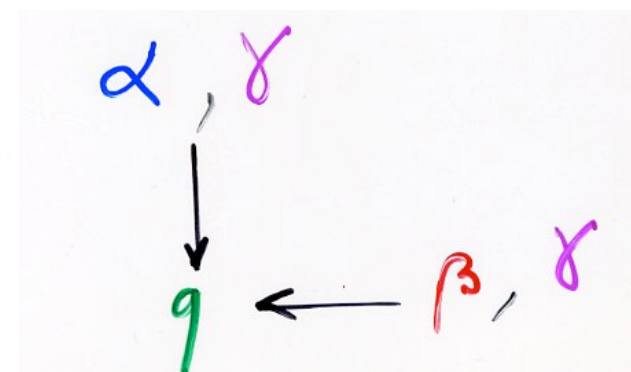
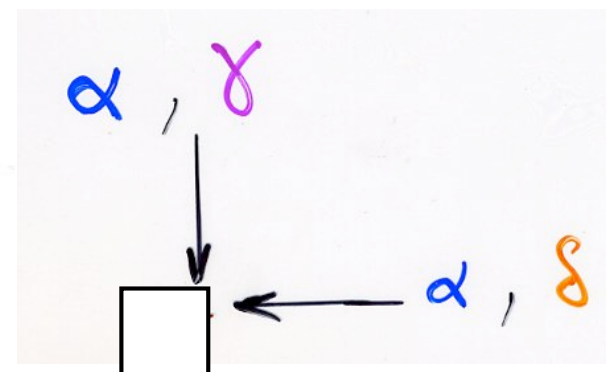
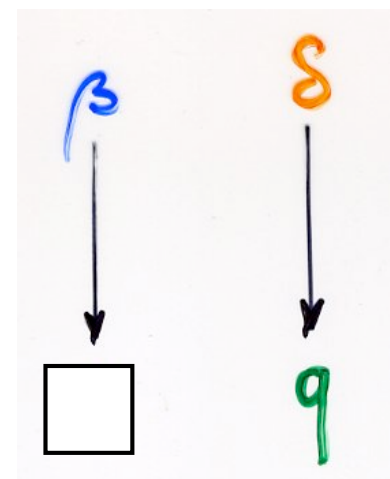
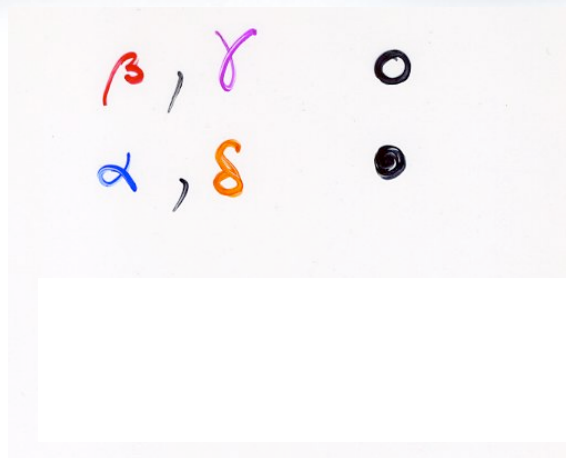
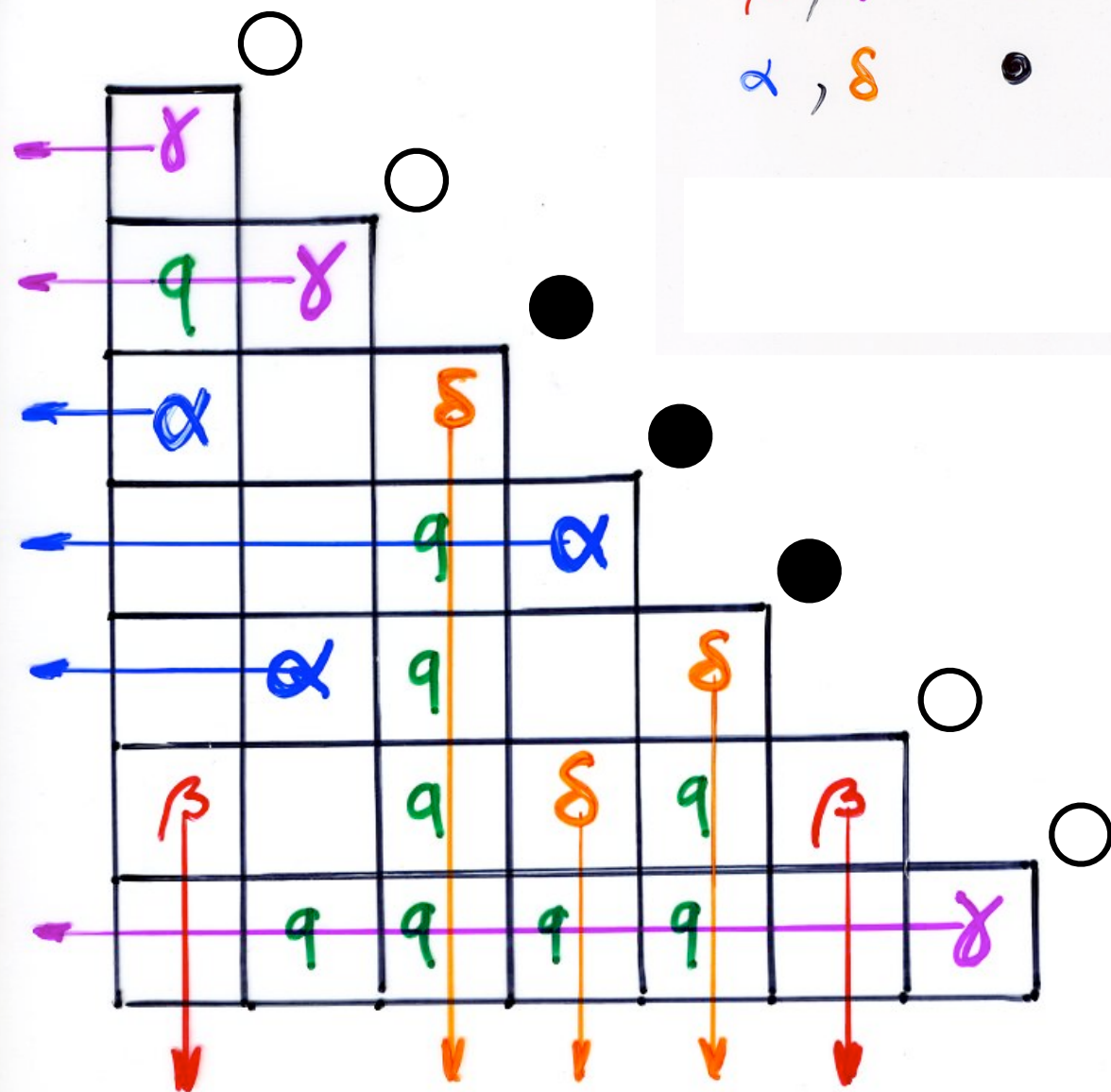
$$Z_n = \sum_{\tau} Z_{\tau}$$

$\tau = (\tau_1, \dots, \tau_n)$   
state

relation with moments of Askey-Wilson polynomials

Corteel, Williams, 2009

Corteel, Stanley, Stanton, Williams, 2010



The cellular Ansatz

From quadratic algebra  $Q$   
to combinatorial objects ( $Q$ -tableaux)  
and bijections

# "The cellular Ansatz"

Physics

"normal ordering"

$$UD = DU + Id$$

Weyl-Heisenberg

$$DE = qED + E + D$$

PASEP

quadratic algebra  $Q$

commutations

rewriting rules

planarisation

combinatorial  
objects  
on a 2d lattice

towers placements

permutations

tableaux alternatifs

Q-tableaux

ex: ASM,

(alternating sign matrices)

FPL(fully packed loops)

tilings, 8-vertex

planar  
automata

representation  
by operators

data structures  
"histories"

orthogonal  
polynomials

bijections

RSK



pairs of Tableaux Young



permutations

Laguerre histories

Koszul algebras  
duality

The  $\delta$ -vertex algebra  
(or XYZ - algebra)  
(or Z - algebra)

# The quadratic algebra $\mathbb{Z}$

4 generators  $B, A, B, A$   
8 parameters  $q, \dots, t, \dots$

$$\left\{ \begin{array}{l} BA = q_{00} AB + t_{00} A \cdot B \\ B \cdot A = q_{\cdot\cdot} A \cdot B + t_{\cdot\cdot} AB \\ B \cdot A = q_{\cdot\cdot} AB + t_{\cdot\cdot} A \cdot B \\ BA \cdot = q_{\cdot\cdot} A \cdot B + t_{\cdot\cdot} AB \end{array} \right.$$

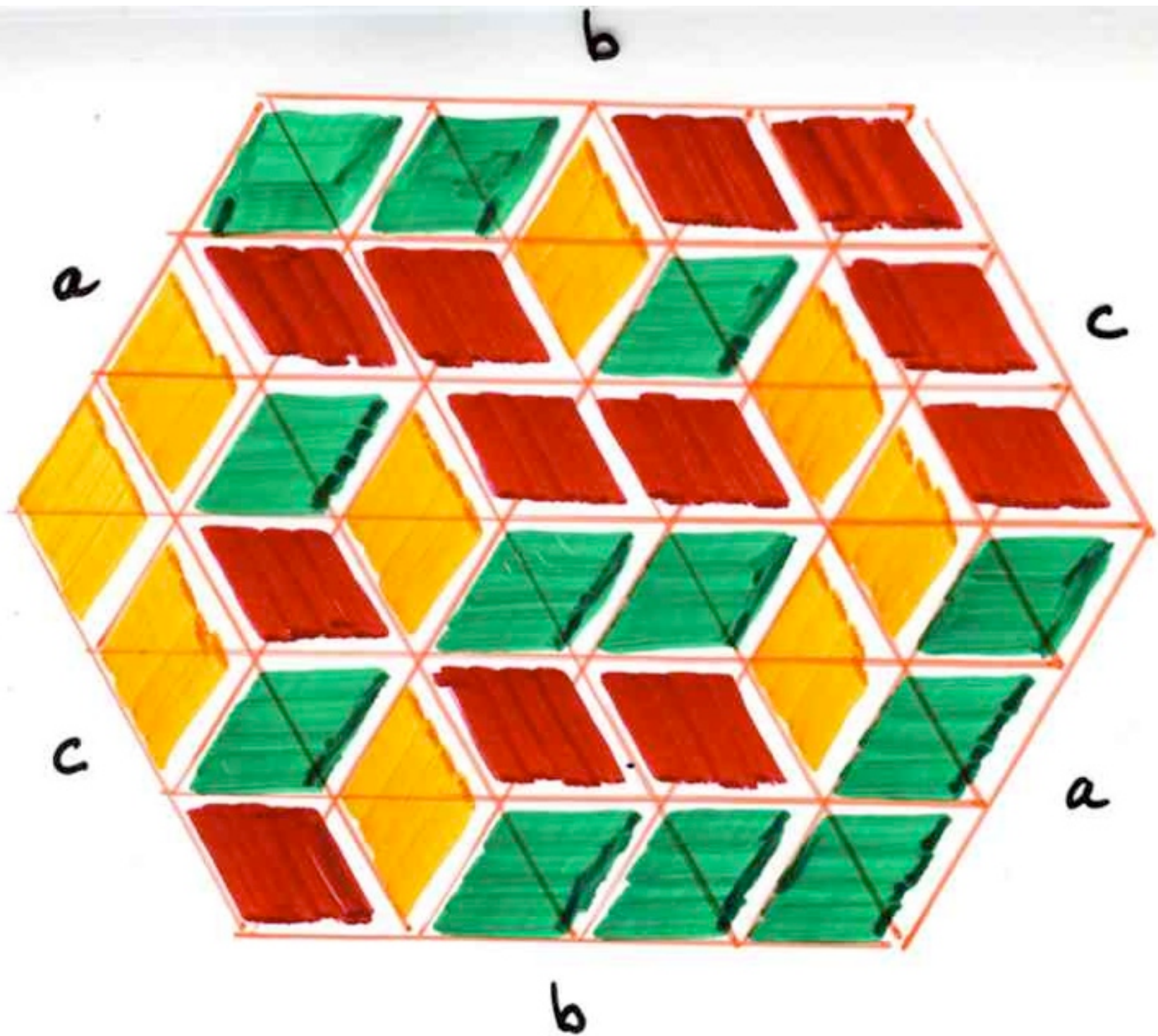
$$\begin{cases} t_{\bullet\bullet} = t_{\bullet\bullet} = \bigcirc \\ q_{\bullet\bullet} = \bigcirc \end{cases} \quad (\text{ASM})$$

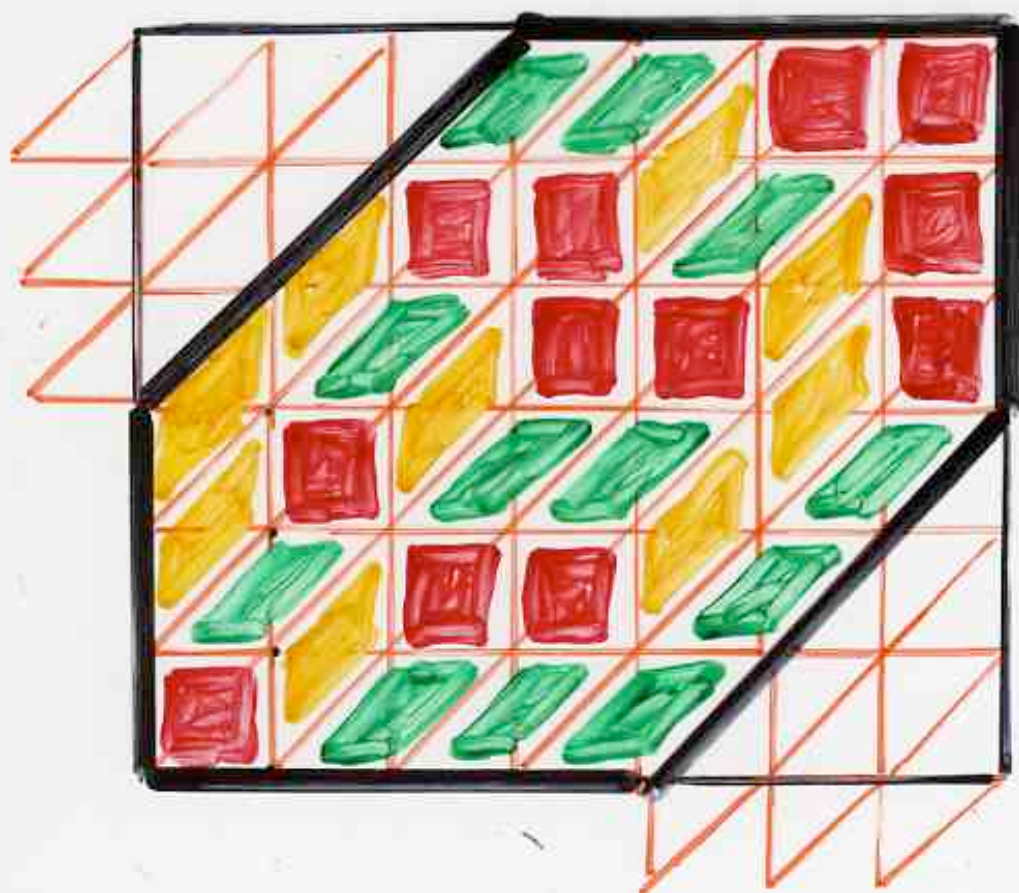
Rhombus tilings

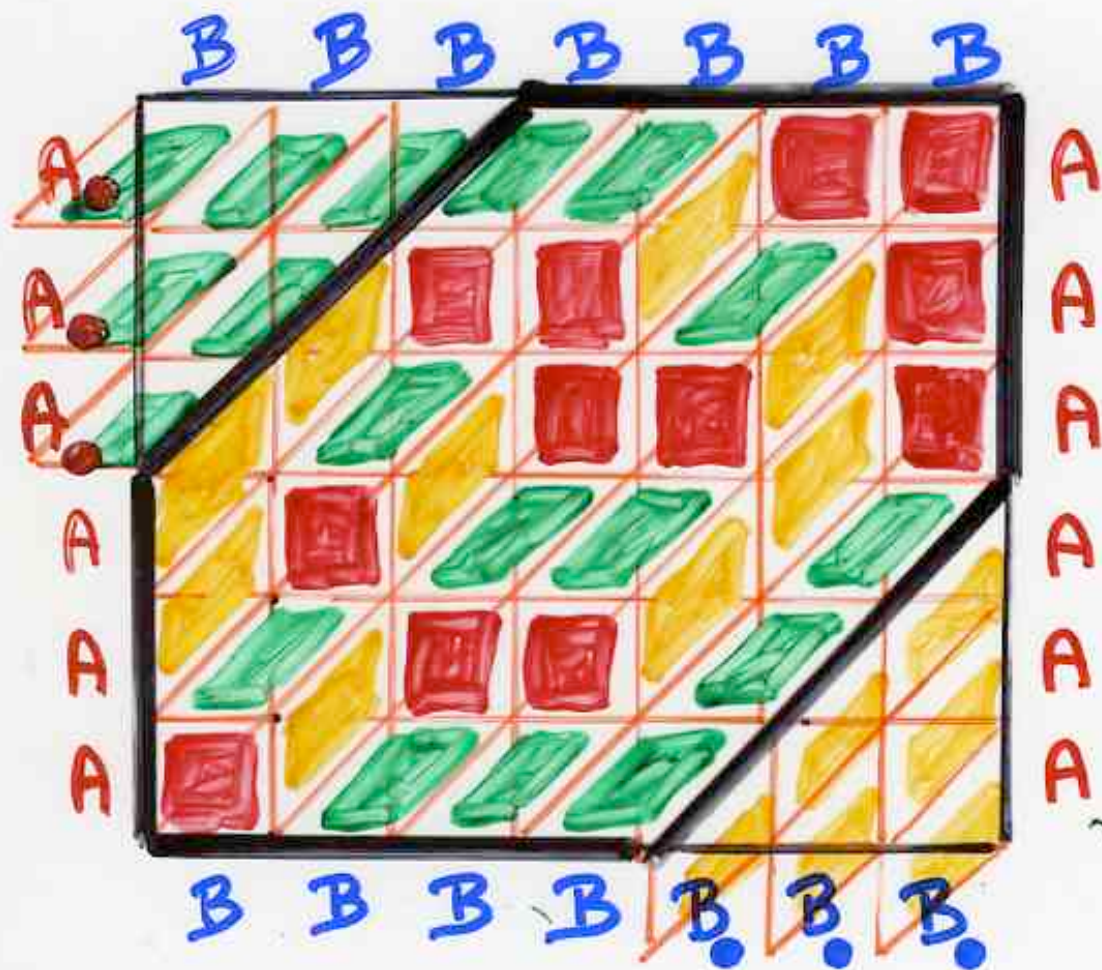
The quadratic algebra  $\mathbb{Z}$

4 generators  $B, A, B, A$   
 8 parameters  $q, \dots, t, \dots$

$$\begin{cases} BA = q_{\bullet\bullet} AB + t_{\bullet\bullet} A \cdot B \\ B \cdot A = \bigcirc A \cdot B + t_{\bullet\bullet} A B \\ B \cdot A = q_{\bullet\bullet} A B + \bigcirc A \cdot B \\ BA = q_{\bullet\bullet} A \cdot B + \bigcirc A B \end{cases}$$







## Aztec tilings

$$t_{\bullet\bullet} = t_{\bullet\bullet} = 0 \quad (\text{ASM})$$

$$t_{\bullet\bullet} = 2 \quad (\text{nb of } -1 \text{ in ASM})$$

The quadratic algebra  $\mathbb{Z}$

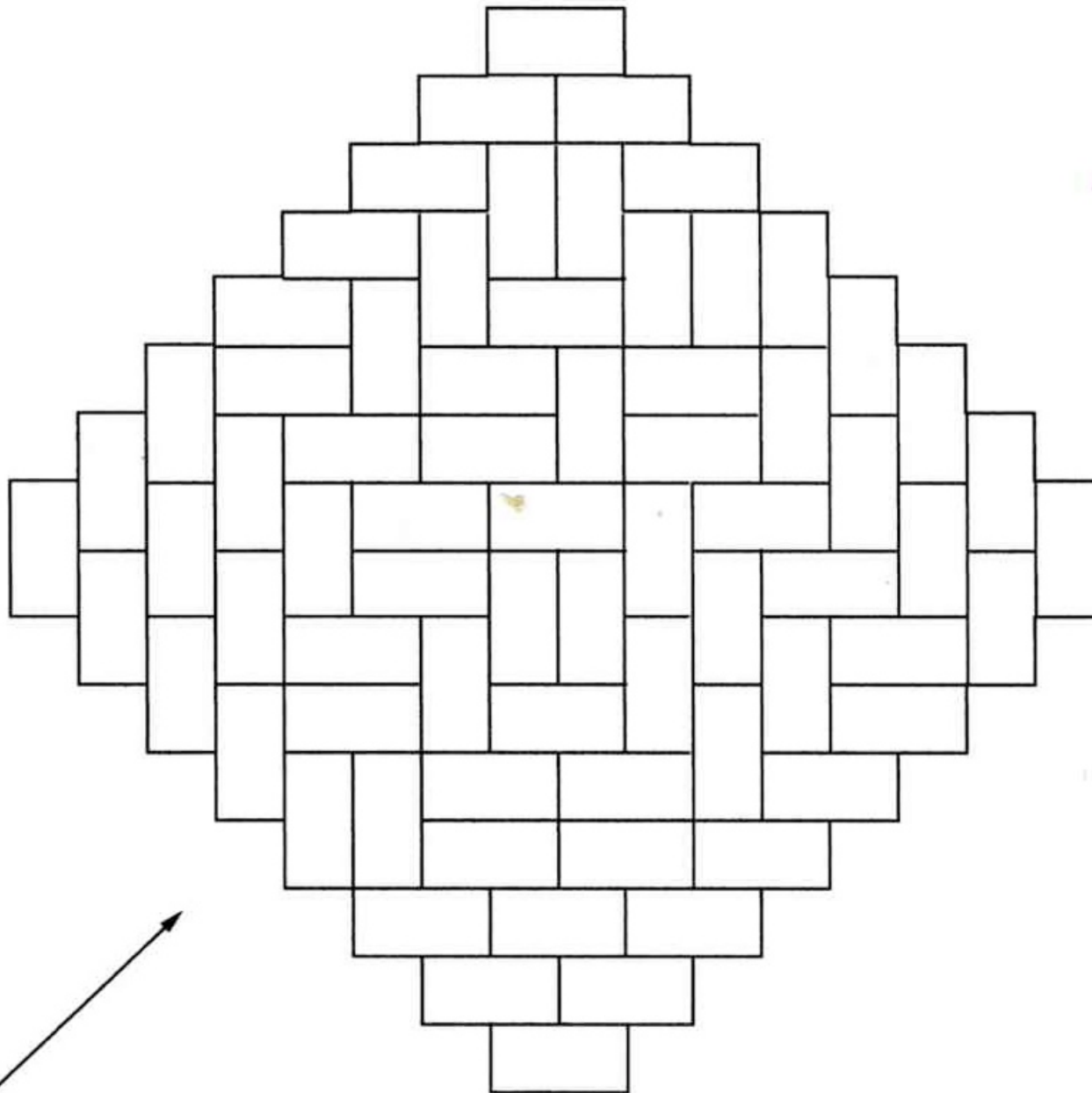
4 generators  $B, A, B, A$

8 parameters  $q, \dots, t, \dots$

$$\left\{ \begin{array}{l} BA = q_{00} AB + t_{00} A.B \\ B.A = q_{\bullet\bullet} A.B + 2 AB \\ B.A = q_{\bullet\bullet} AB + \bigcirc A.B \\ BA = q_{\bullet\bullet} A.B + \bigcirc AB \end{array} \right.$$

$$2^{n(n-1)/2}$$

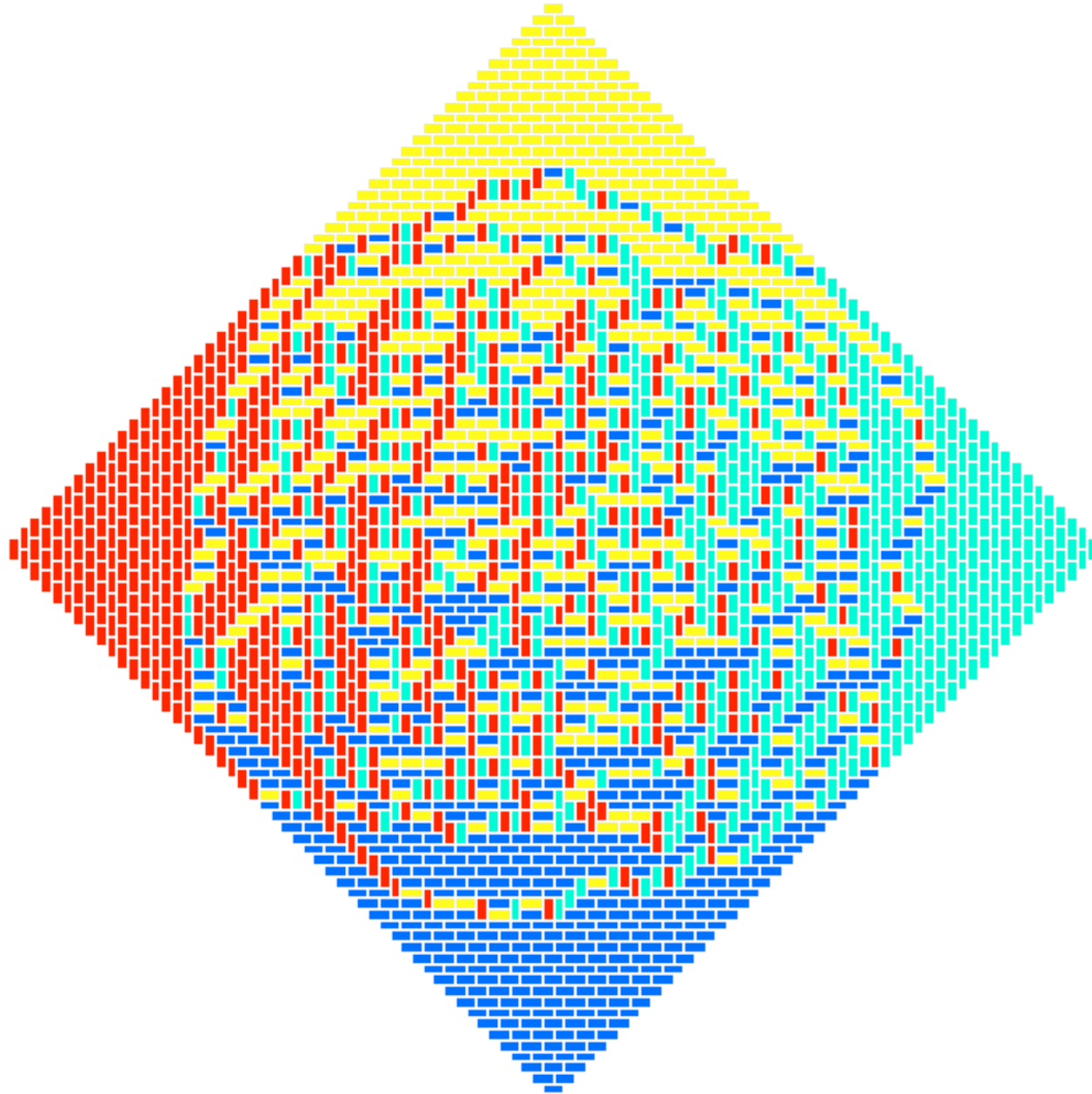
$$A_n(2)$$

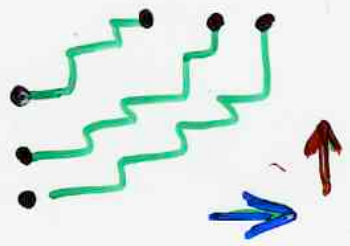


Elkies,  
Kuperberg,  
Larsen,  
Propp  
(1992)



random  
Aztec  
tilings





$$\begin{cases} t_{00} = 0 \\ q_{00} = t_{00} = 0 \end{cases}$$

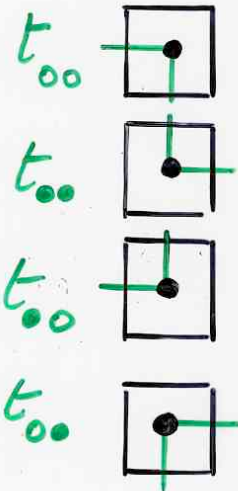
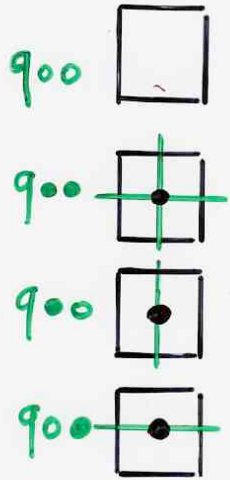
$A \leftrightarrow A_0$   
exchanging

$$\begin{cases} t_{00} = 0 \\ q_{00} = t_{00} = 0 \end{cases}$$

The quadratic algebra  $\mathbb{Z}$

4 generators  $B, A, B, A$   
8 parameters  $q, \dots, t, \dots$

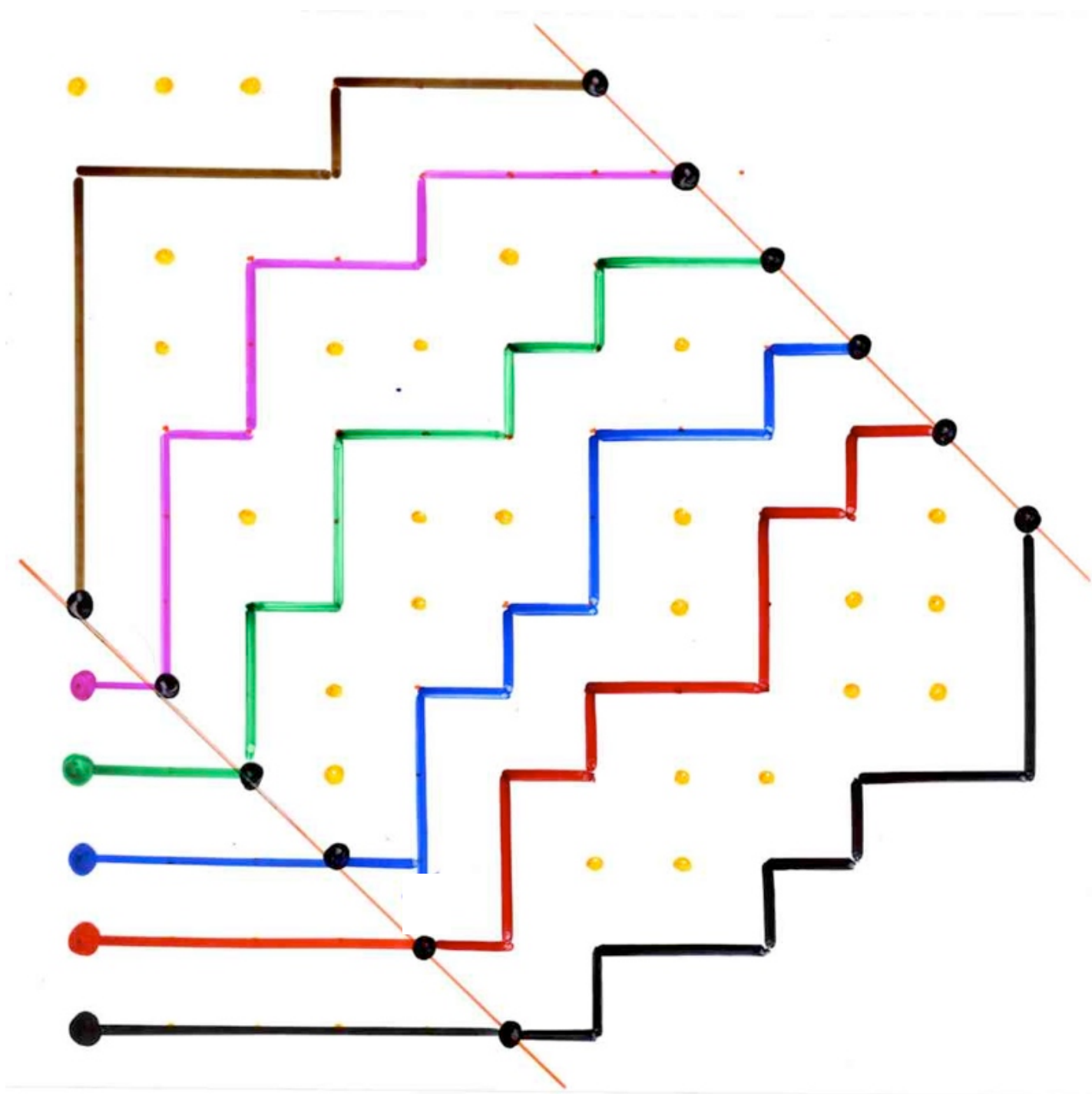
$$\begin{cases} BA = q_{00} AB + \bigcirc A_0 B_0 \\ B_0 A_0 = \bigcirc A_0 B_0 + \bigcirc AB \\ B_0 A = q_{00} AB + t_{00} A_0 B \\ BA_0 = q_{00} A_0 B + t_{00} AB \end{cases}$$



The quadratic algebra  $\mathbb{Z}$

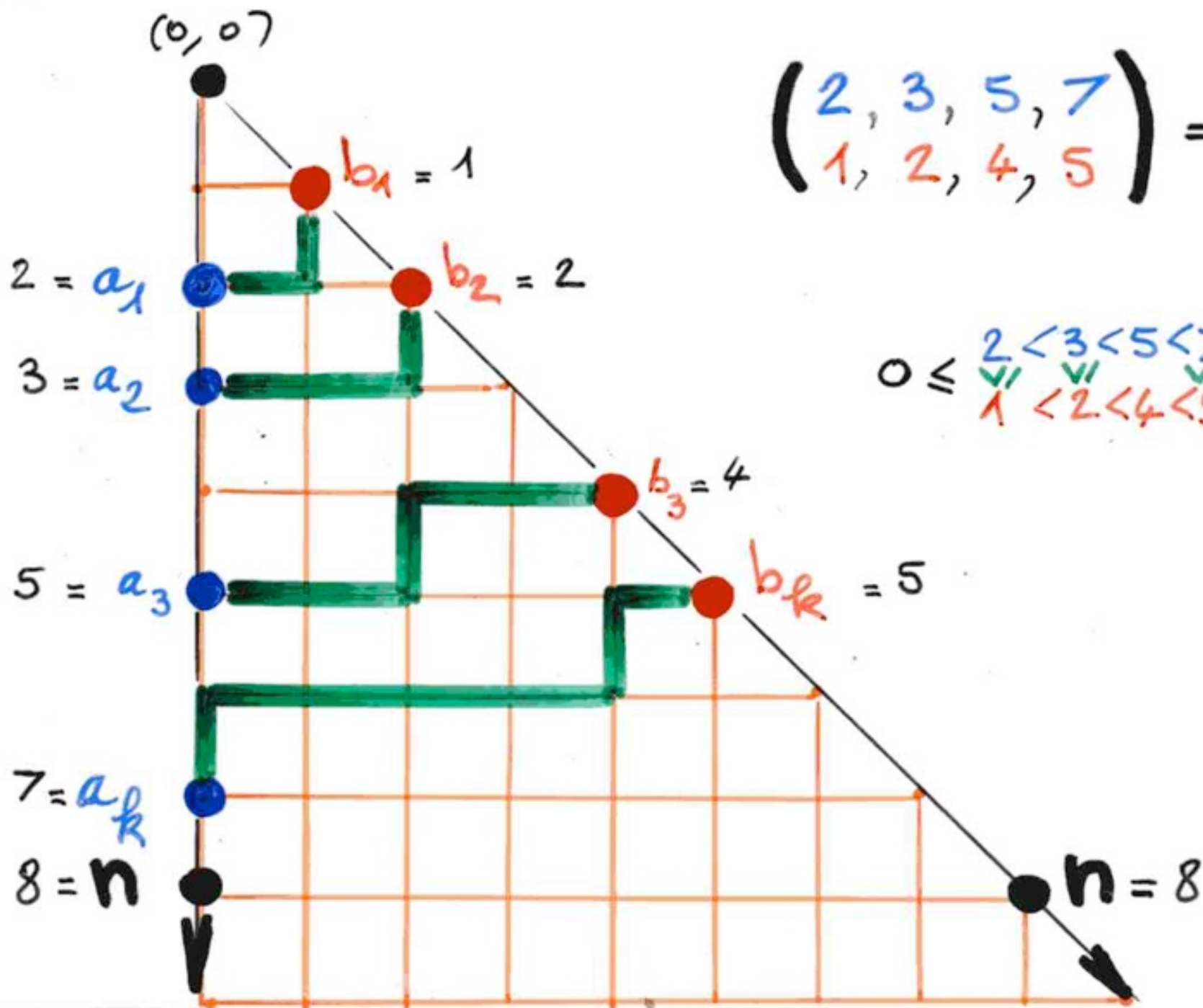
4 generators  $B, A, B, A$   
8 parameters  $q, \dots, t, \dots$

$$\begin{cases} BA = q_{00} AB + t_{00} A_0 B_0 \\ B_0 A_0 = q_{00} A_0 B_0 + t_{00} AB \\ B_0 A = \bigcirc A_0 B_0 + \bigcirc A_0 B \\ BA_0 = q_{00} A_0 B + \bigcirc AB \end{cases}$$

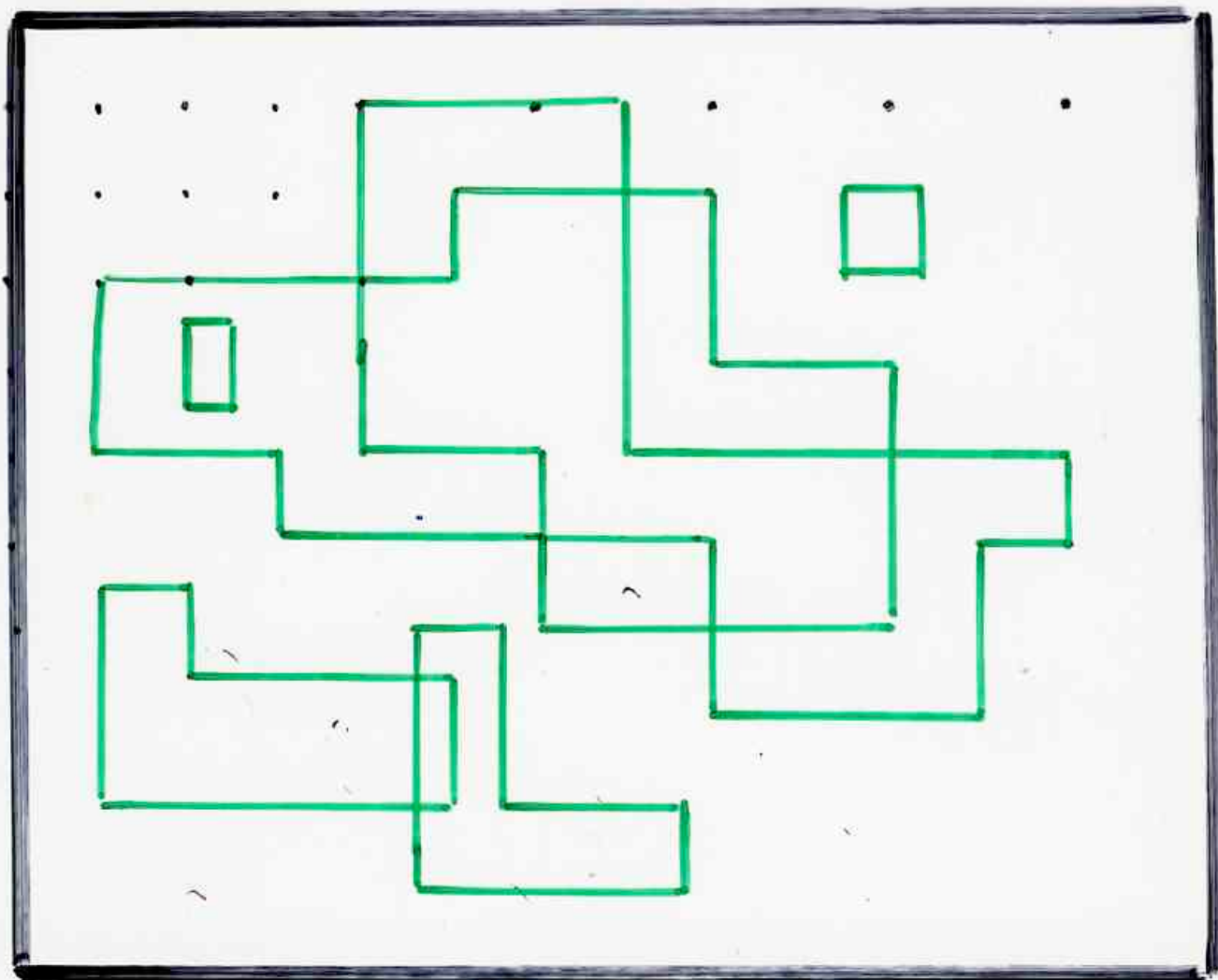


$$\begin{pmatrix} 2, 3, 5, 7 \\ 1, 2, 4, 5 \end{pmatrix} = 210$$

$$0 \leq \begin{matrix} 2 < 3 < 5 < 7 \\ \checkmark & \checkmark & \checkmark \\ 1 < 2 < 4 < 5 \end{matrix} \leq 8 = n$$



example: binomial determinant

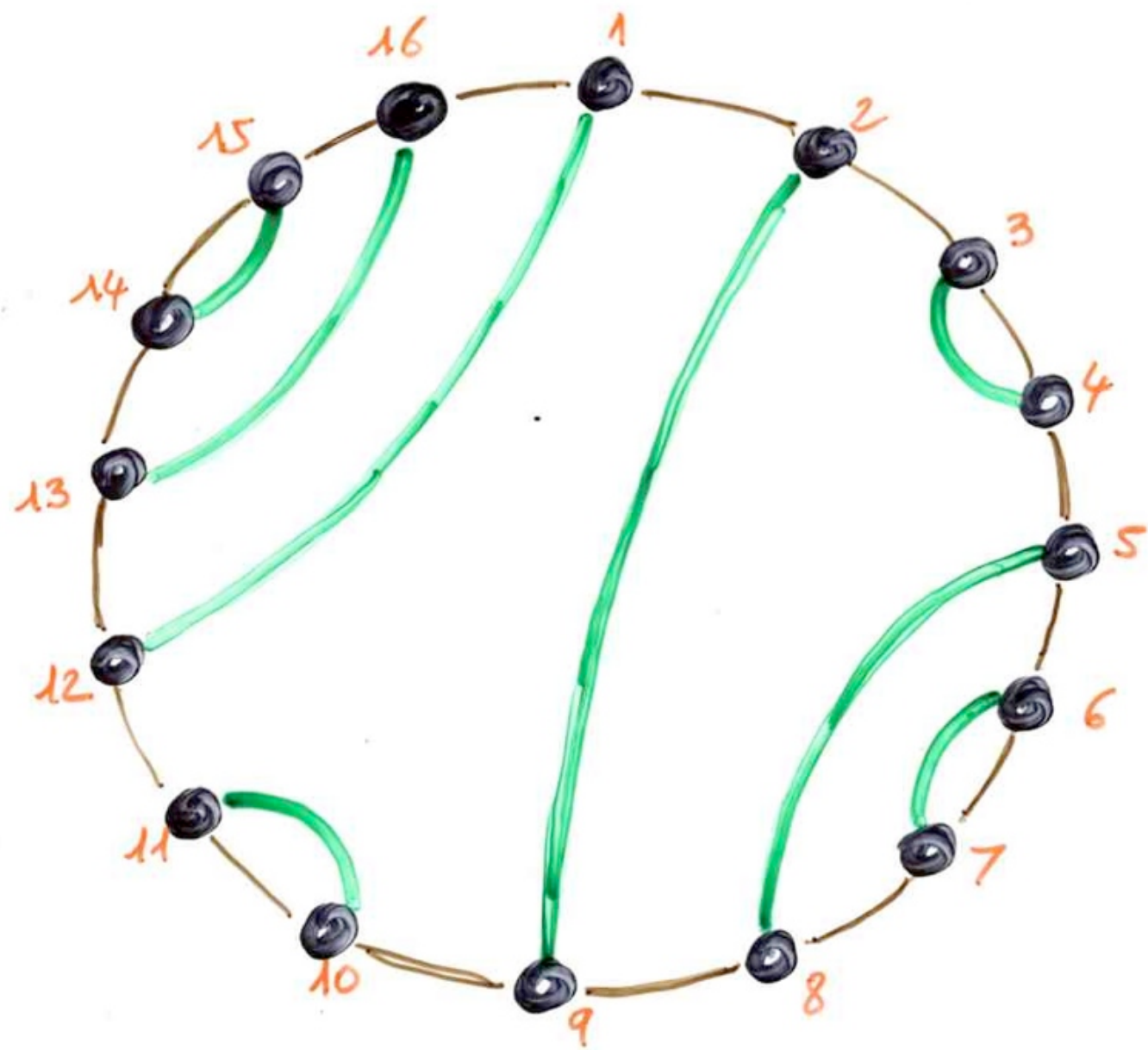


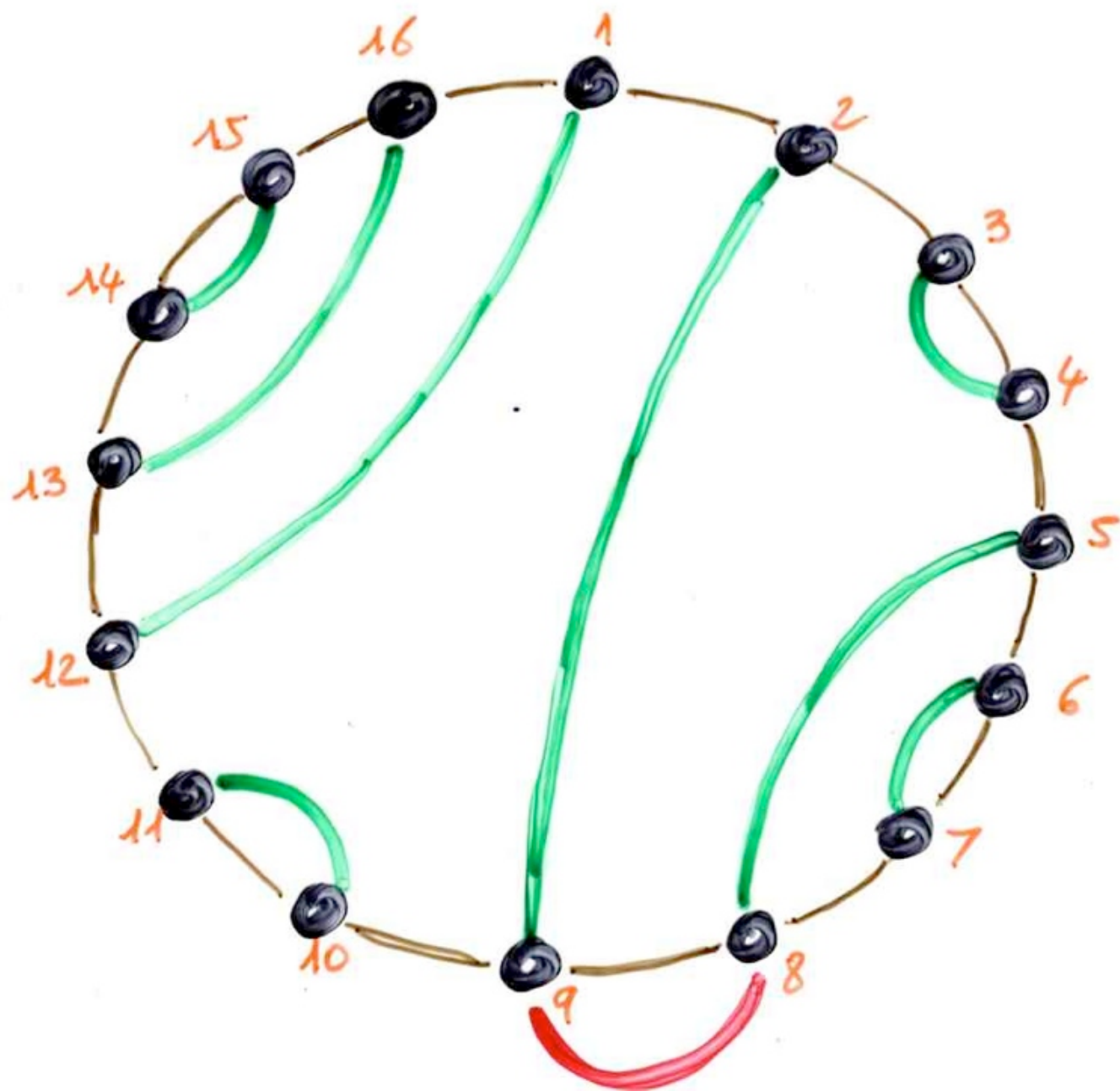
"closed" graph

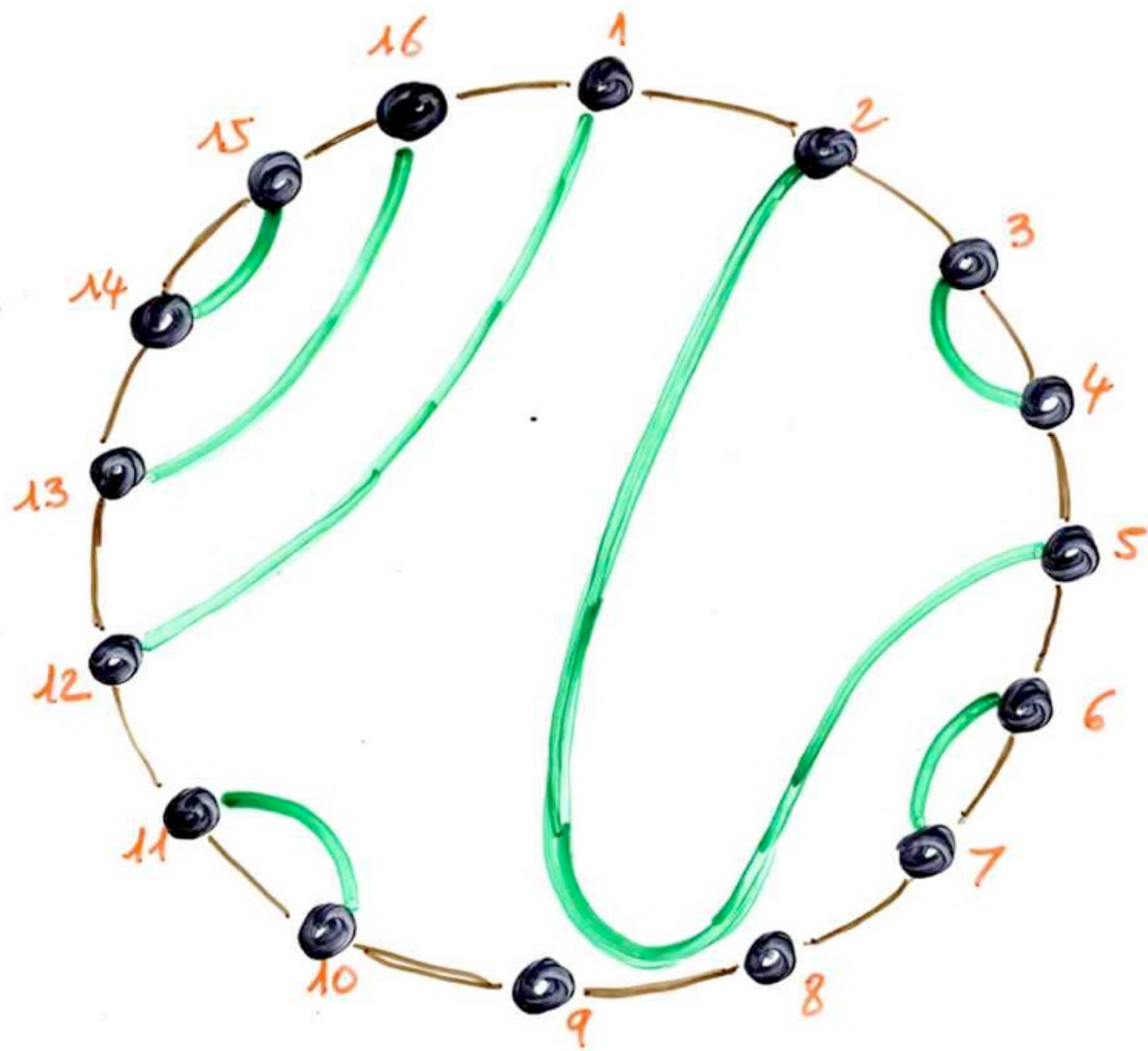
Ising model

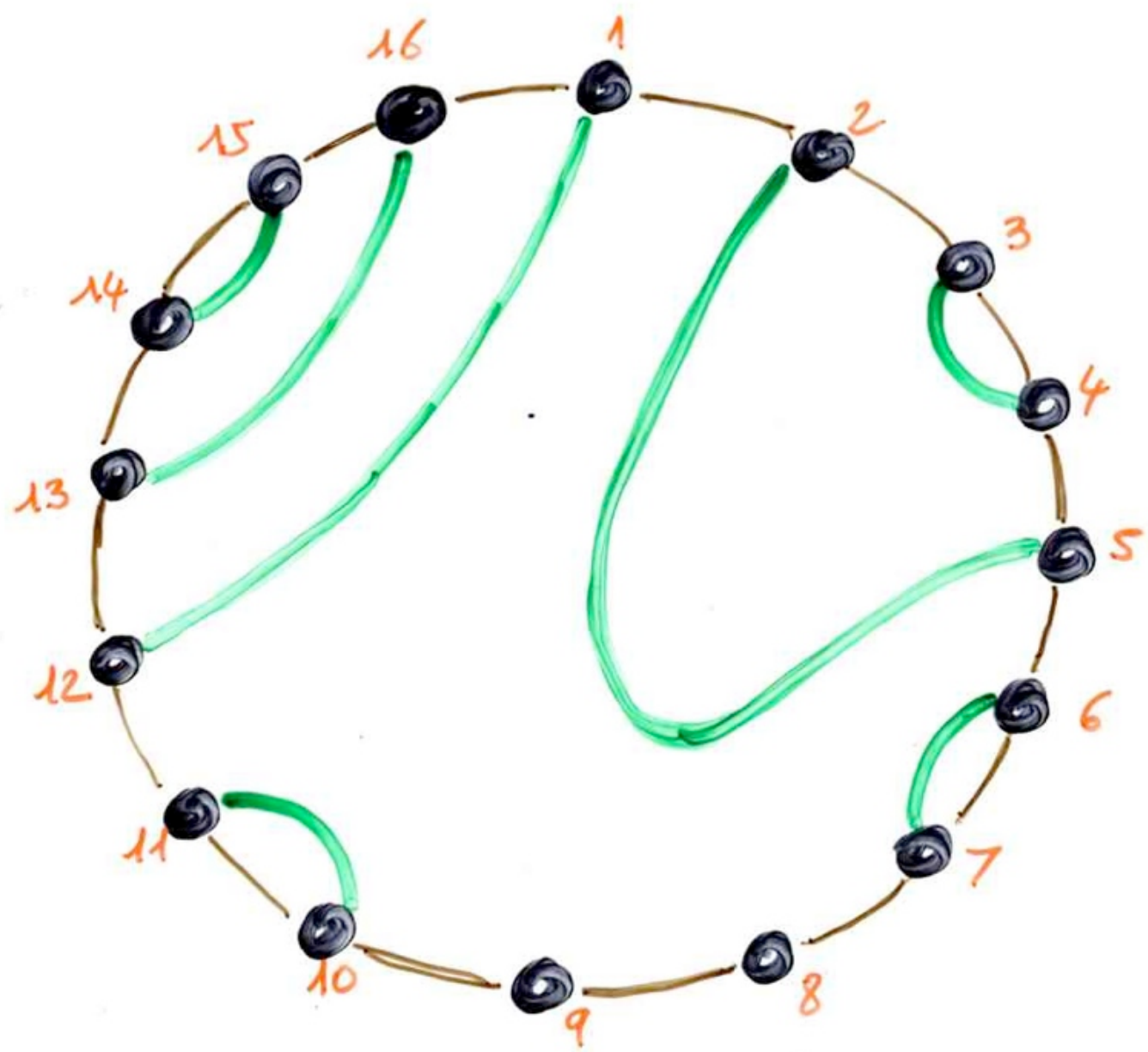
$$\begin{array}{lcl}
 w & = & B^m A^n \\
 uv & = & A^n B^m
 \end{array}$$

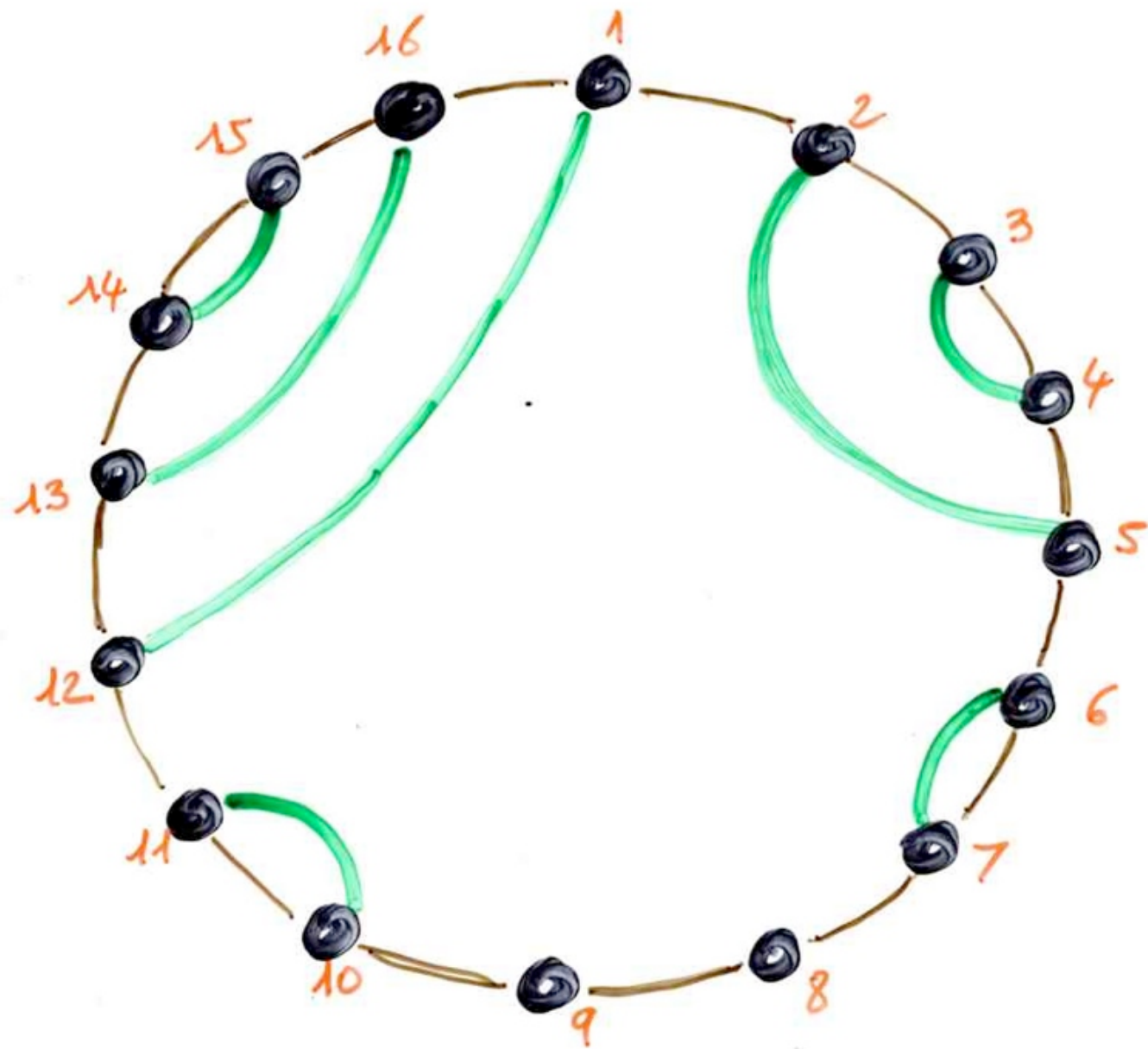
Razumov - Stroganov  
(ex) - conjecture 2000-2001

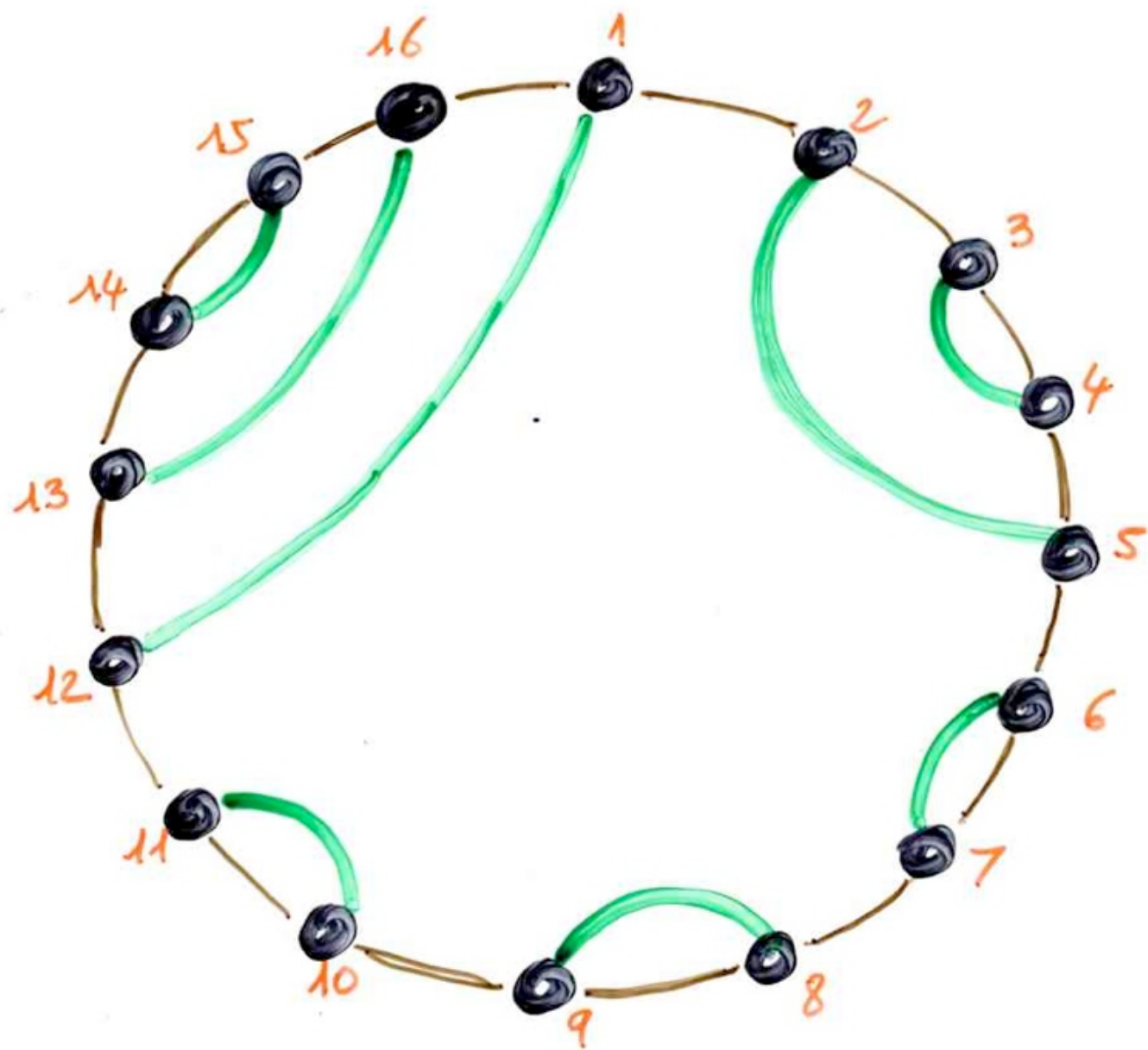








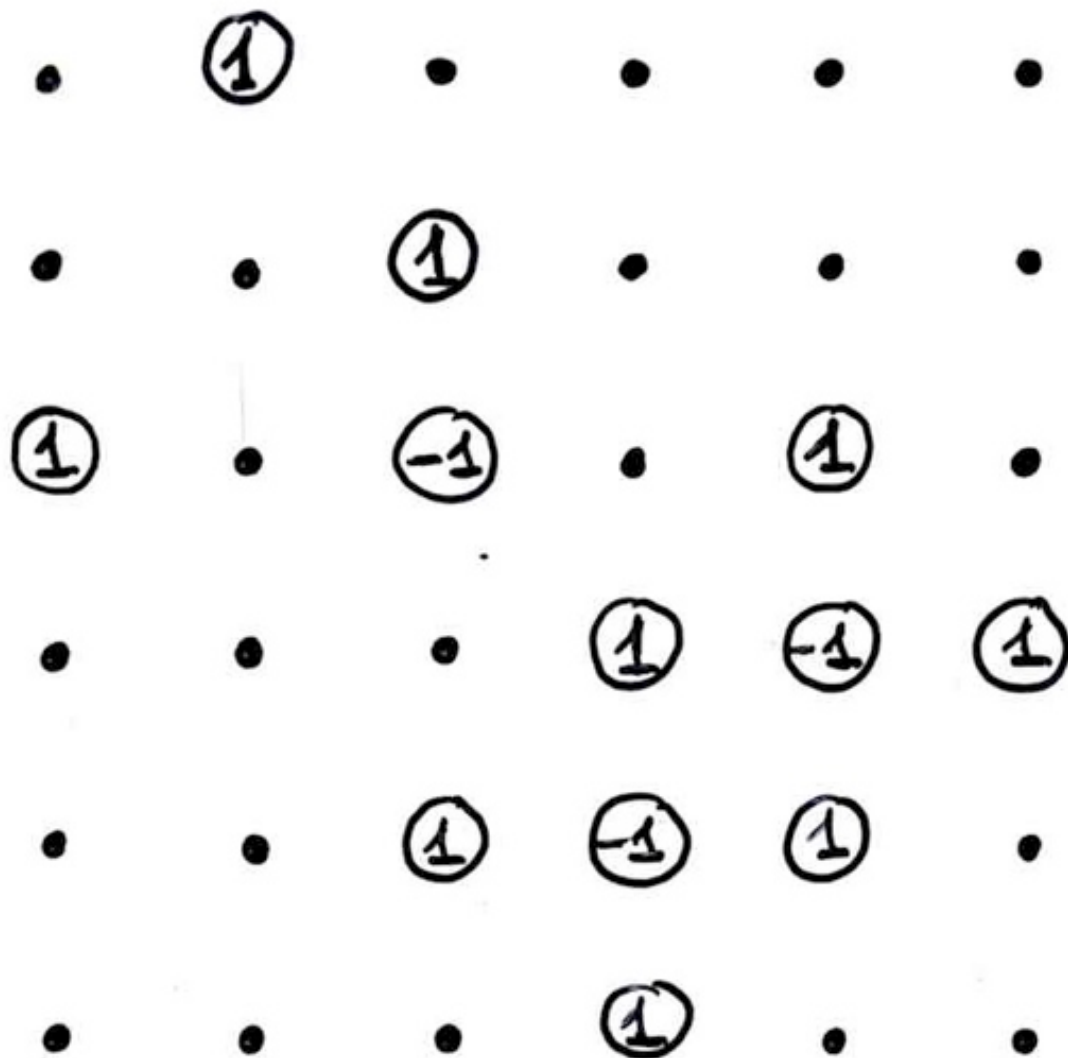




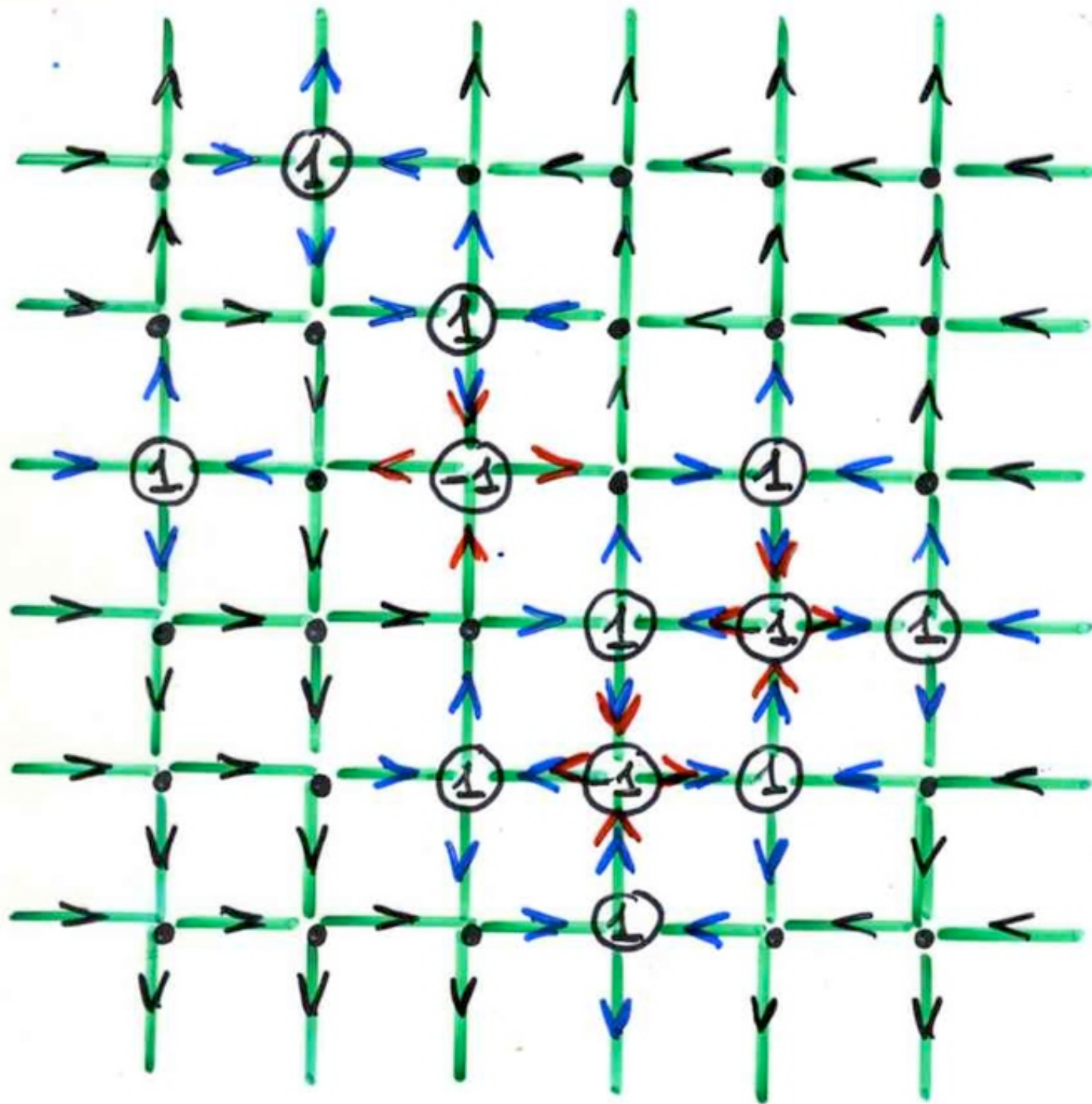
FPL

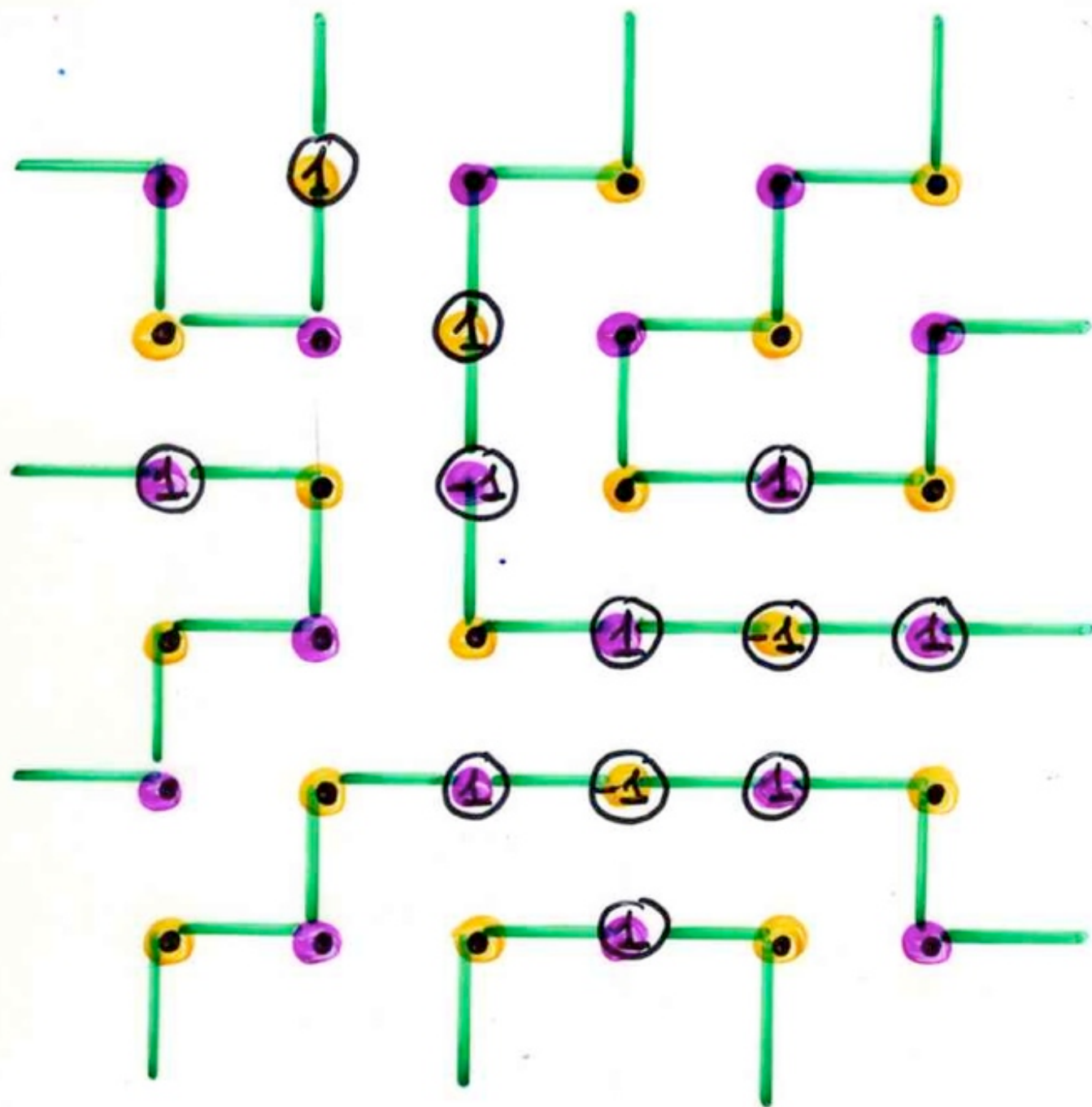
“Fully packed loop configurations”

The  
bijection  
AMS  
FPL

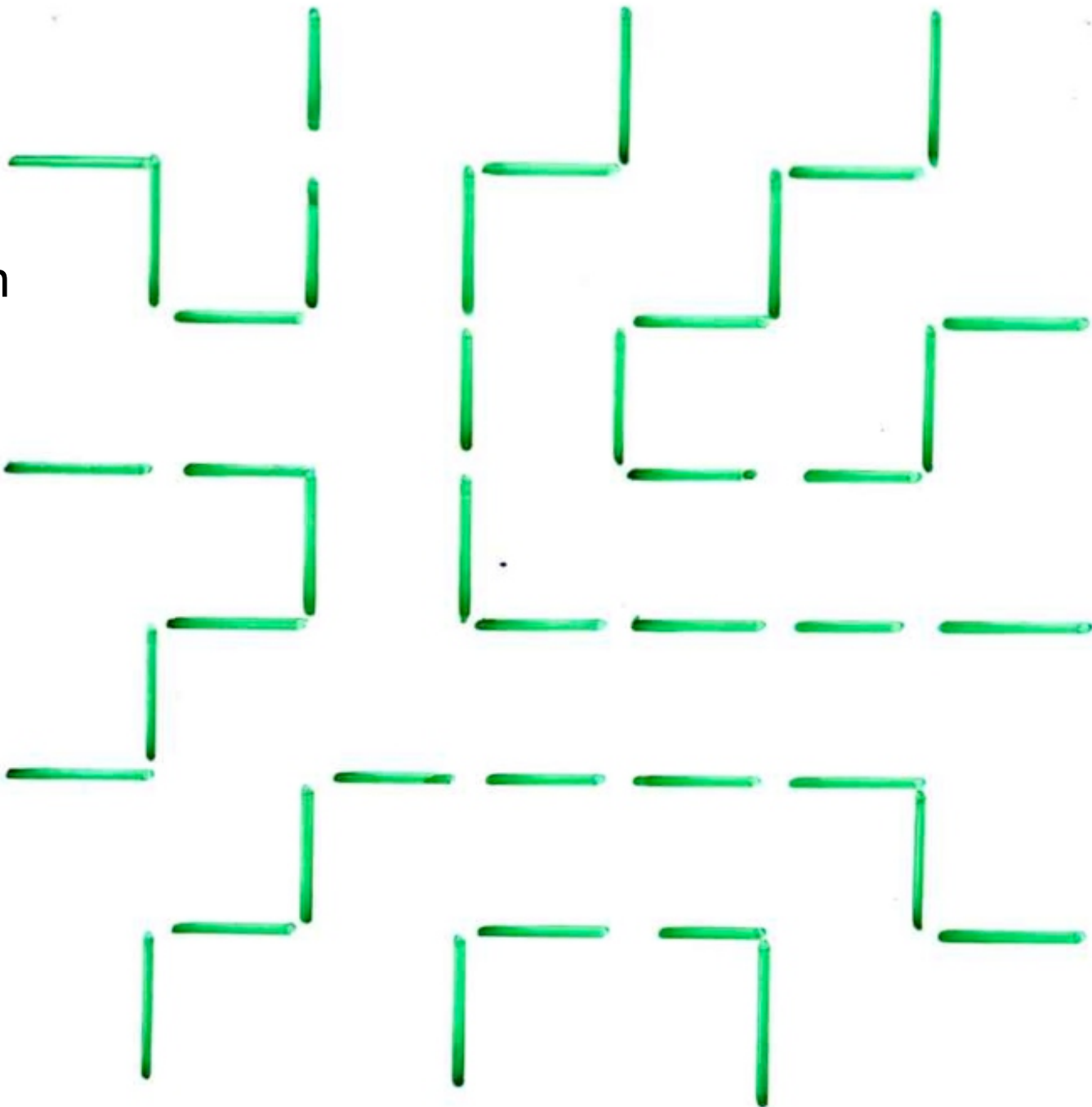


# The 6-vertex model



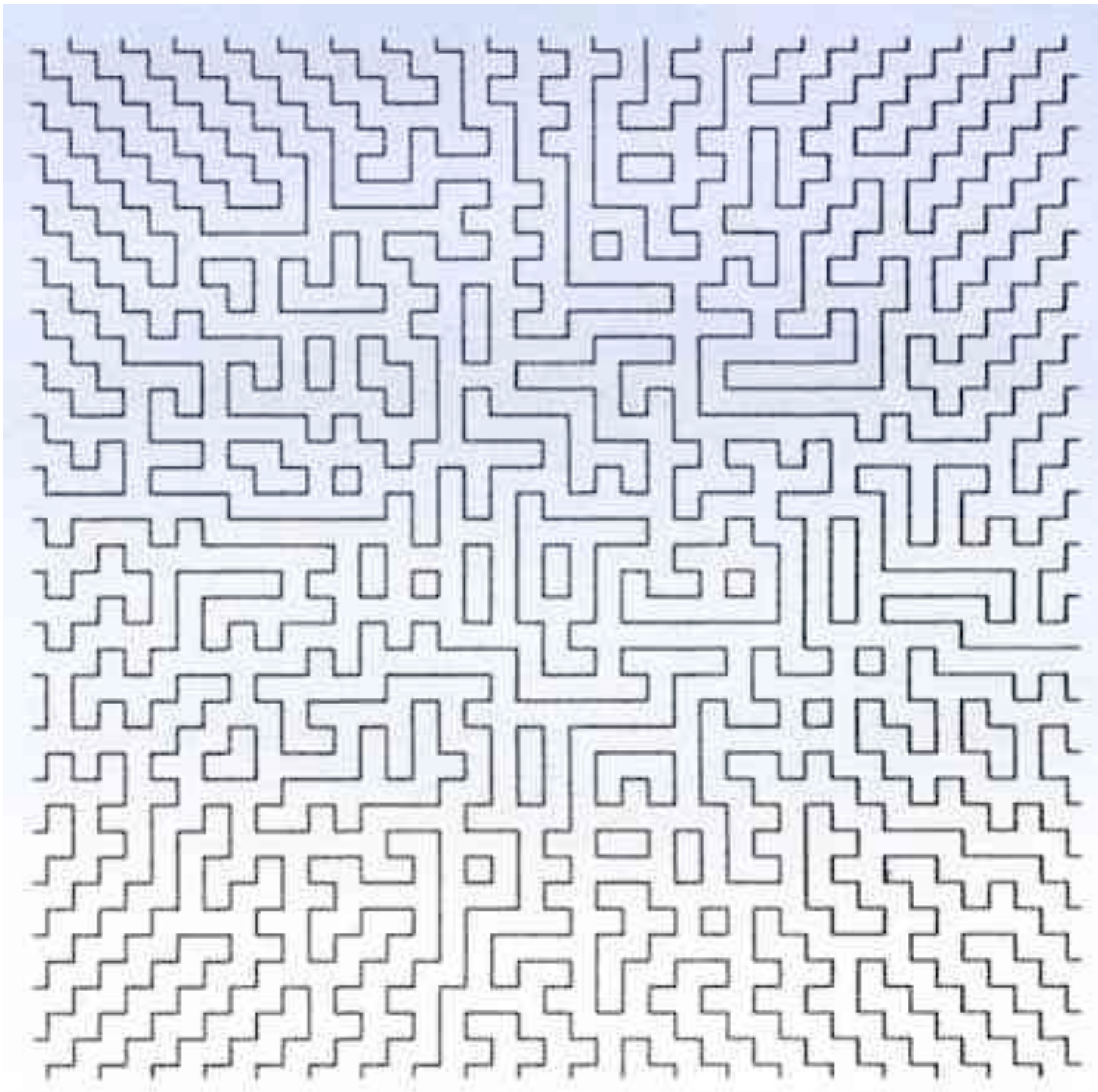


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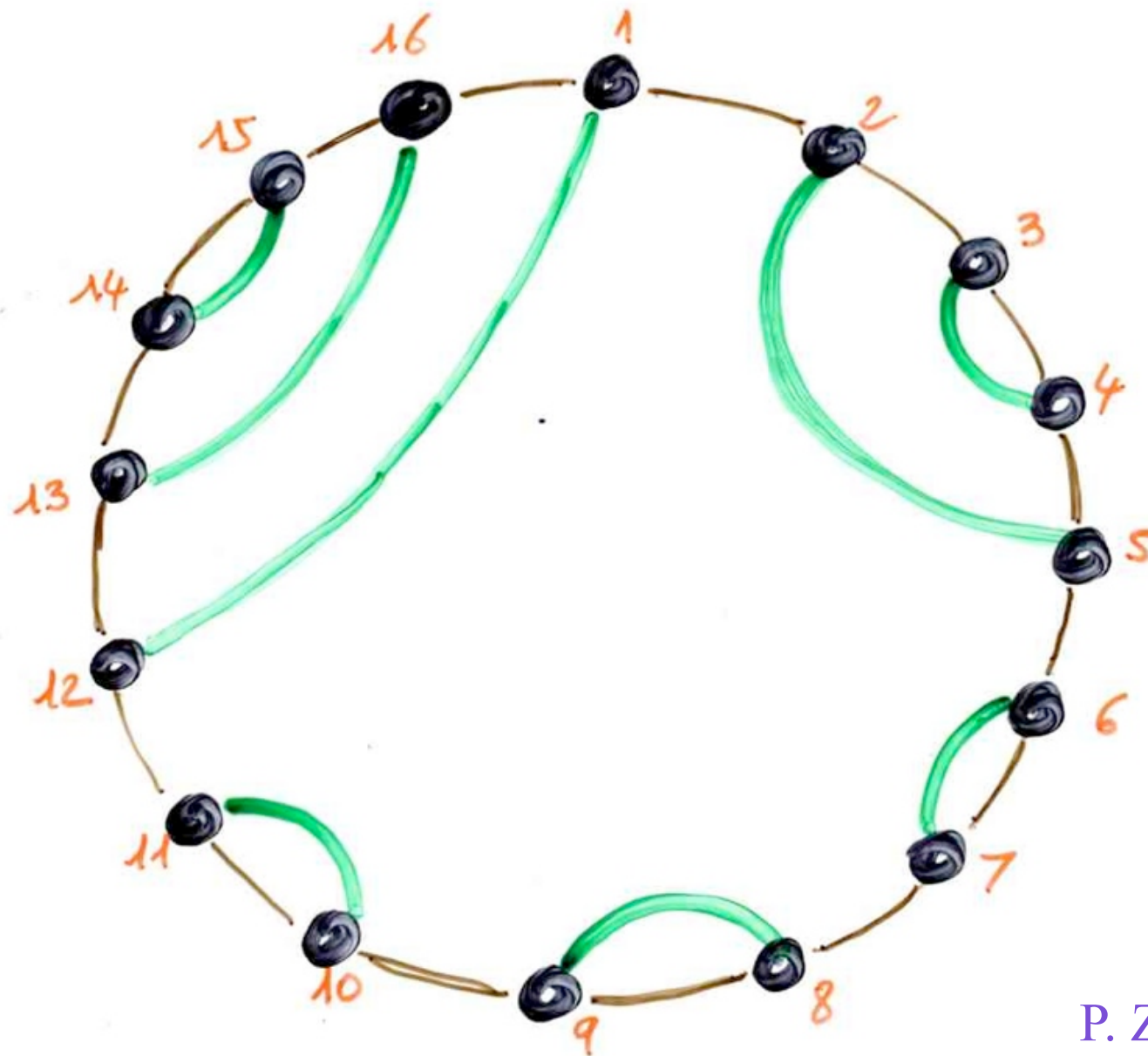


A 5x5 grid of green sticks forming a maze-like pattern. The sticks are arranged in a way that creates a complex, interconnected network of paths and dead ends. The pattern is symmetrical and resembles a stylized, abstract representation of a maze or a complex circuit board layout. The sticks are arranged in a way that creates a complex, interconnected network of paths and dead ends. The pattern is symmetrical and resembles a stylized, abstract representation of a maze or a complex circuit board layout.

random  
FPL



# Razumov-Stroganov conjecture



stationary  
probabilities

Di Francesco,  
P. Zinn-Justin (2005)

# Around the Razumov-Stroganov conjecture

Philippe Di Francesco, Paul Zinn-Justin (2005 - 2009)

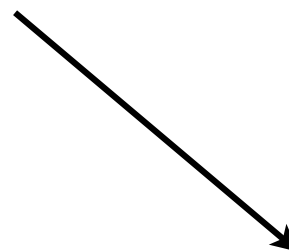
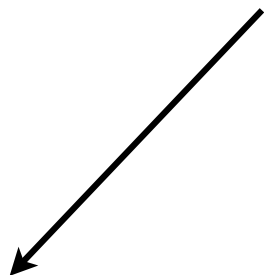
De Gier, Pyatov (2007)

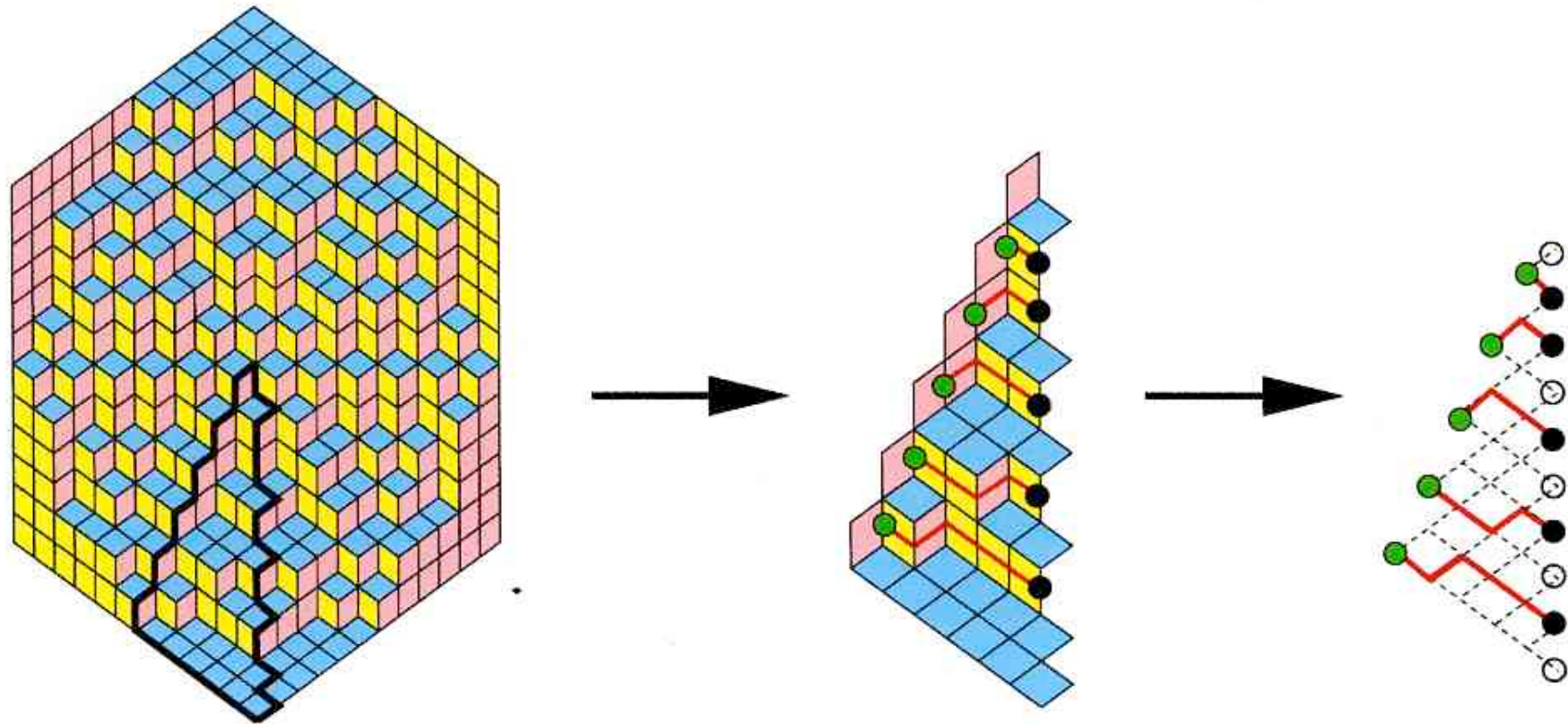
Knizhnik - Zamolodchikov  
equation

$q$ KZ

TSSCPP

ASM





Di Francesco (2006)

# ASM

1-, 2-, 3- enumeration  $A_n(x)$

Colomo, Pronco, (2004)

Hankel determinants

(continuous) Hahn, Meixner-Pollaczek,  
(continuous) dual Hahn orthogonal polynomials

Ismail, Lin, Roan (2004)

XXZ spin chains and Askey-Wilson operator

Schubert and Grothendick polynomials

Lascoux, Schützenberger

# Razumov - Stroganov (ex) - conjecture 2000-2001

proof by :

L. Cantini and A.Sportiello (March 2010)

arXiv: 1003.3376 [math.CO]

completely combinatorial proof

# "The cellular Ansatz"

Physics

"normal ordering"

$$UD = DU + Id$$

Weyl-Heisenberg

$$DE = qED + E + D$$

PASEP

quadratic algebra  $Q$

commutations  
rewriting rules  
planarisation

combinatorial  
objects  
on a 2d lattice

representation  
by operators

bijections

data structures  
"histories"

orthogonal  
polynomials

towers placements

permutations

tableaux alternatifs

RSK



pairs of Tableaux Young



permutations

Laguerre histories

Q-tableaux

ex: ASM,

(alternating sign matrices)

FPL(fully packed loops)

tilings, 8-vertex

planar  
automata

?

Koszul algebras  
duality

more details, see the diaporamas:

**Cours XGV, Universidad de Talca «The Cellular Ansatz»**

(December 2010 - January 2011)

**Combinatorics and interactions (with physics) (24h)**

also: **4 talks in India**  
January-March 2012

or **«Petite Ecole de Combinatoire»**

LaBRI, Fall 2011, Spring 2012

and **2 videos:**

**Newton Institute 23 April 2008**

[http://www.newton.cam.ac.uk/webseminars/pg+ws/  
2008/csm/csmw04/0423/viennot/](http://www.newton.cam.ac.uk/webseminars/pg+ws/2008/csm/csmw04/0423/viennot/)

**IMSc, Chennai, 1 March 2012**

<http://youtu.be/u44ZmEU8svY>

accessible from the websites:

<http://www.labri.fr/perso/viennot/>

Recherche, cv, publications, exposés, diaporamas, livres, petite école, photos: voir ma page personnelle [ici](#)

Vulgarisation scientifique voir la page de l'association [Cont'Science](#)

[http://web.me.com/xgviennot/Xavier\\_Viennot/](http://web.me.com/xgviennot/Xavier_Viennot/)

[http://web.me.com/xgviennot/Xavier\\_Viennot2/](http://web.me.com/xgviennot/Xavier_Viennot2/)

## Ch 0 Introduction

## Ch 1 Ordinary generating function, the Catalan garden

Ch1a (1/12/2010, 54 p.)

Ch 1b (7/12/2010, 81 p.)

Ch 1c (7/12/2010, 30 p.) **algebraic complements in relation with physics**

## Ch 2 Exponential generating functions, permutations

Ch 2a (22/12/2010, 40 p.)

Ch 2b (4/01/2010, 63 p.)

Ch 2c (4/01/2010, 33 p.) **Permutations: Laguerre histories**

## Ch 3 Permutations and Young tableaux, the Robinson-Schensted correspondence (RSK)

Ch 3a (6/01/2011, 117 p.)

Ch 3b (6, 11/01/2011, 121 p.) **RSK and operators**

## Ch 4 Alternative tableaux and the PASEP (partially asymmetric exclusion process)

Ch 4a (13/01/2011, 98p.)

Ch 4b (13, 18/01/2011, 102 p.) **alternative tableaux and the PASEP**

Ch 4c (18/01/2011, 81 p.) **complements**

## Ch 5 Combinatorial theory of orthogonal polynomials

(20/01/2011, 110 p.)

## Ch 6 "jeu de taquin" for binary trees, Catalan tableaux and the TASEP

Ch 6a (24/01/2011, 98 p.)

Ch 6b (24/01/2011, 111 p.) **alternative tableaux and increasing/alternative binary trees**

Ch 6c (24/01/2011, 21 p.) **Catalan tableaux and the Loday-Ronco algebra**

## Ch 7 The cellular Ansatz

Ch 7a (25/01/2011, 117 p.)

Ch 7b (25/01/2011, 49 p.) **complements**

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**Combinatorics and interactions**

(with physics)

«The Cellular Ansatz»

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