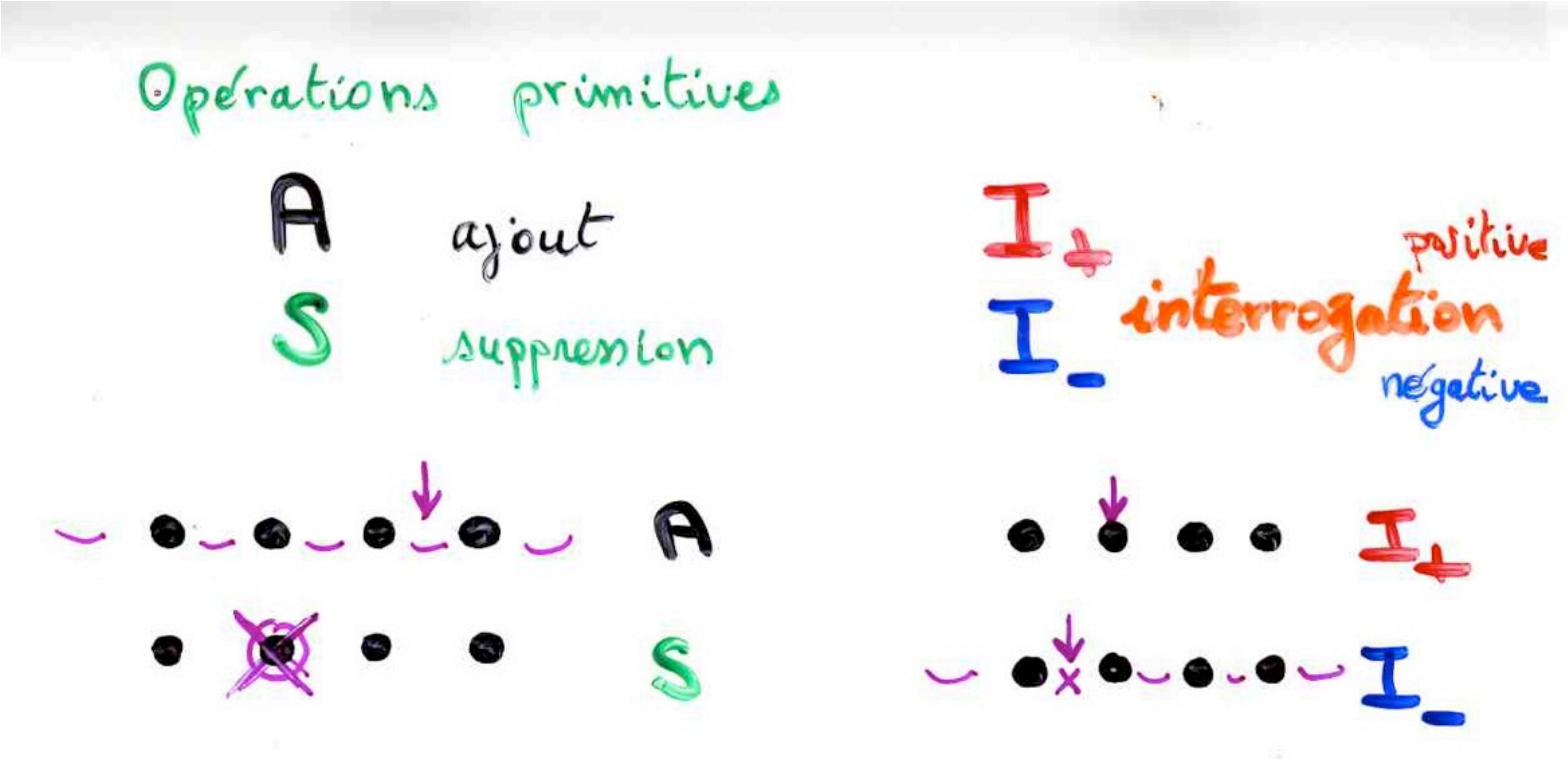
Algèbres d'opérateurs
et
Physique combinatoire

(part 3)

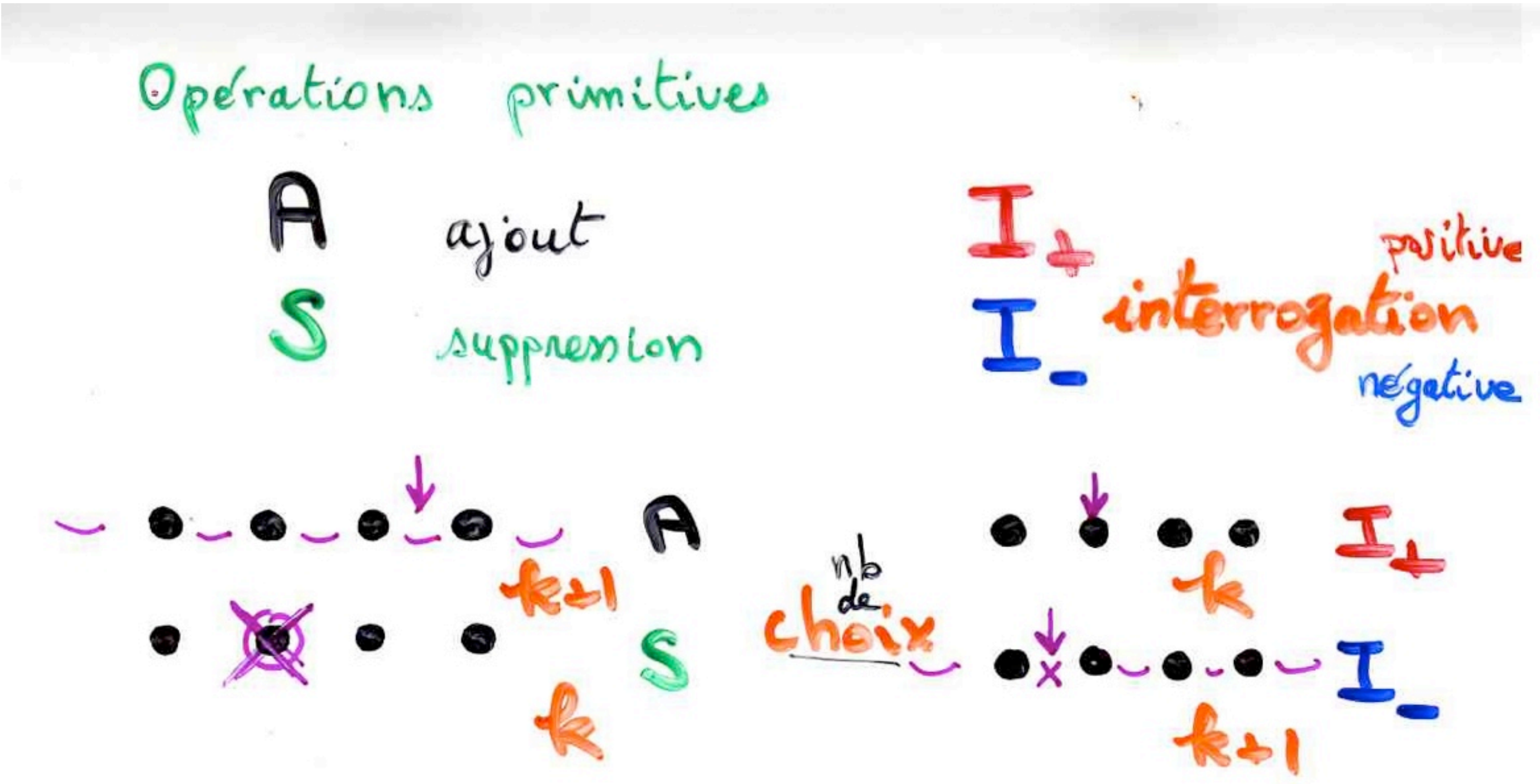
12 Avril 2012
colloquium de l'IMJ
Institut mathématiques de Jussieu

Xavier Viennot
CNRS, LaBRI, Bordeaux

quadratic algebra
operators
data structures
and orthogonal polynomials



Primitive operations
for "dictionnaries" data structure:
add or delete any elements, asking questions (with positive or negative answer)



number of choices for each
primitive operations

(formal) orthogonal polynomials

Orthogonal polynomials

Def. $\{P_n(x)\}_{n\geq 0}$ $\quad P_n(x) \in \mathbb{K}[x]$

orthogonal iff $\exists\ f : \mathbb{K}[x] \longrightarrow \mathbb{K}$

linear functional

$$\begin{cases} \text{(i)} & \deg(P_n(x)) = n & (\forall n \geq 0) \\ \text{(ii)} & f(P_k P_l) = 0 & \text{for } k \neq l \geq 0 \\ \text{(iii)} & f(P_k^2) \neq 0 & \text{for } k \geq 0 \end{cases}$$

$$\mathcal{f}(x^n) = \mu_n \qquad (n \geq 0)$$

moments

$$\mathcal{f}(PQ) = \int_a^b P(x)\,Q(x)\,d\mu$$

mesure

combinatorial interpretation of the moments

Thm. (Favard)

- $\{P_n(x)\}_{n\geq 0}$ sequence of monic polynomials, $\deg(P_n)=n$
- $\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ coeff. in $\mathbb{K}$

orthogonality $\Longleftrightarrow$

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

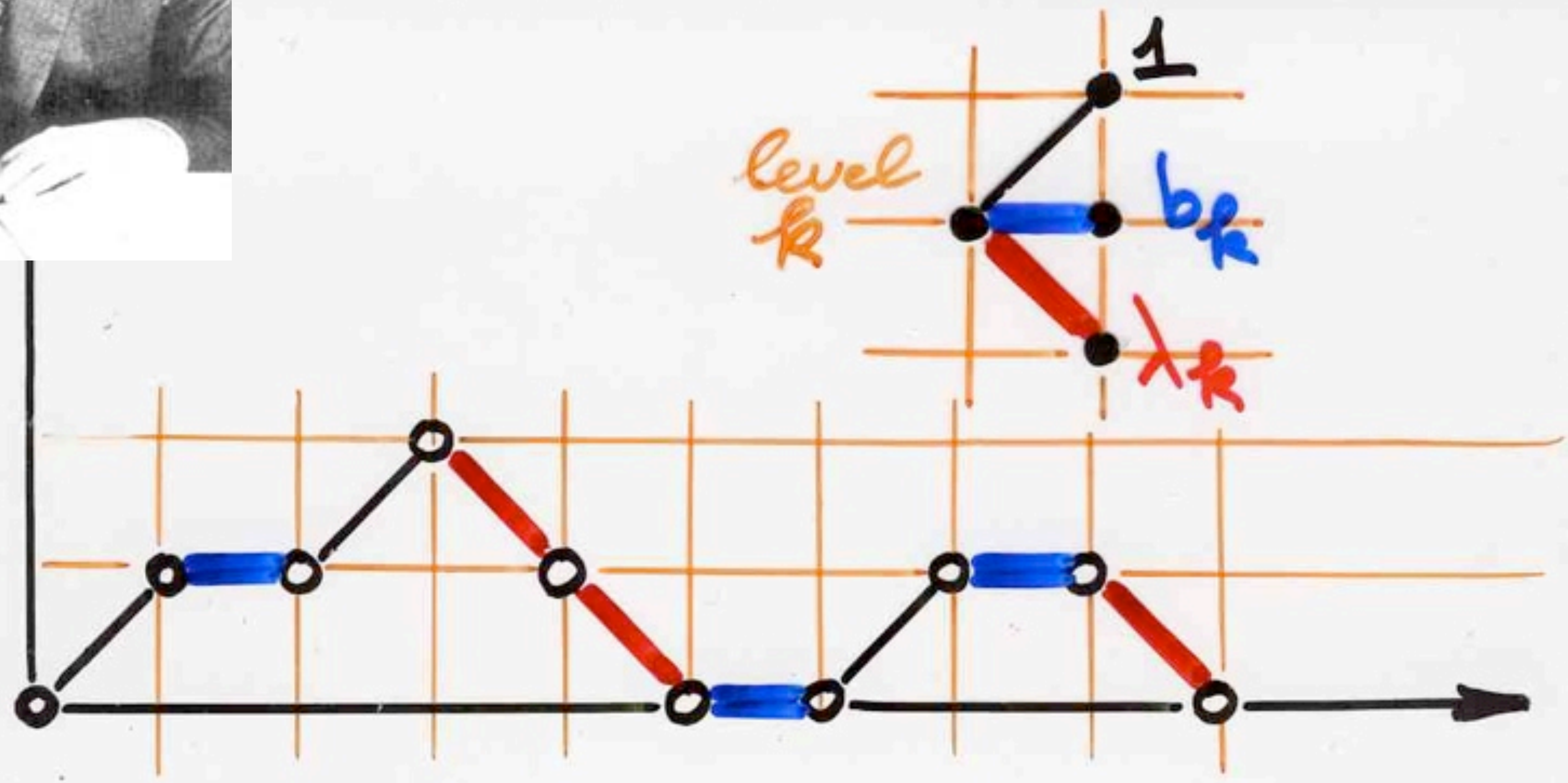
$(\forall k\geq 1)$

3 terms linear recurrence relation

$\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$

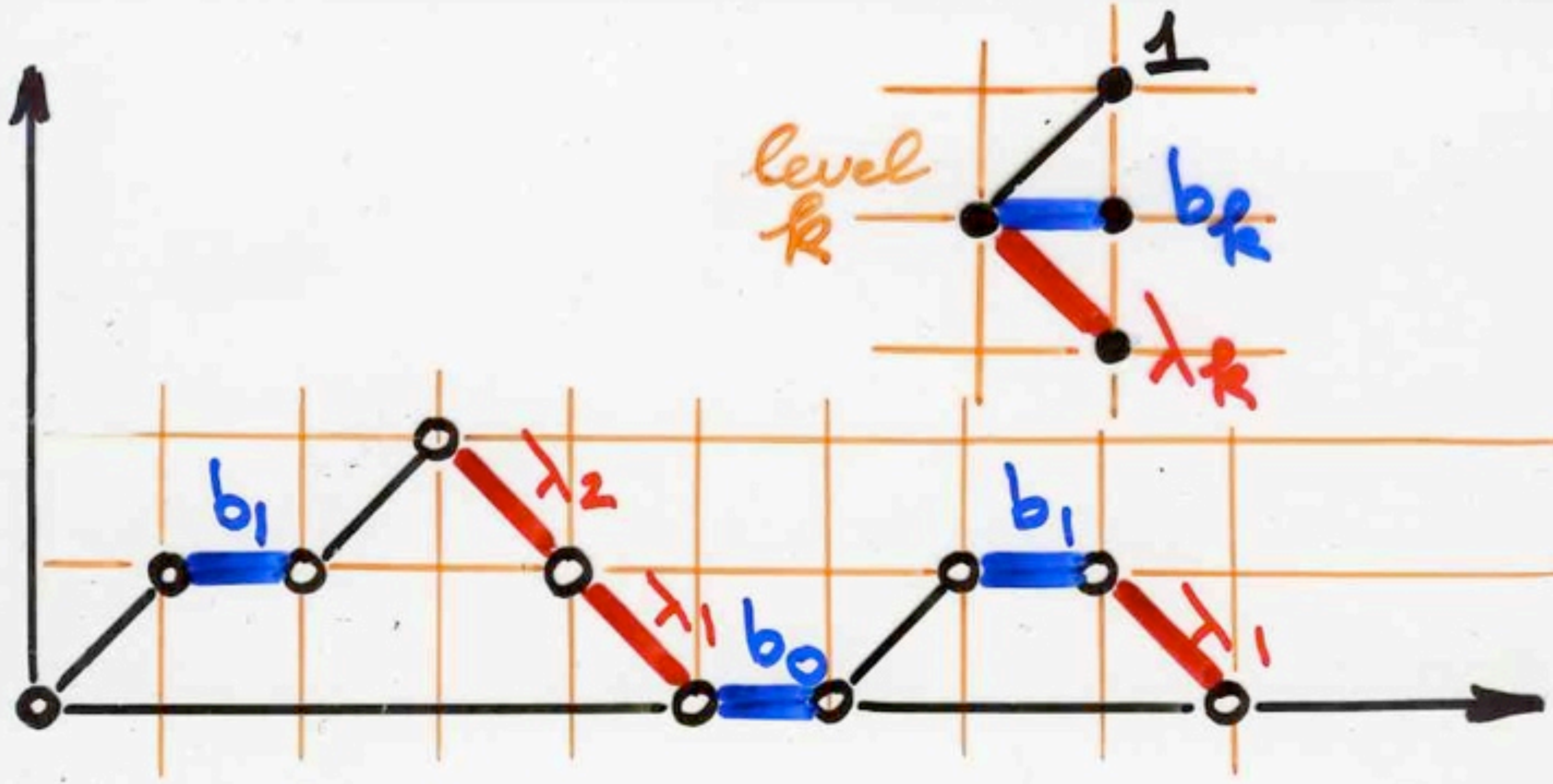
$b_k, \lambda_k \in \mathbb{K}$ ring

valuation v



ω Motzkin path

valuation v



ω Motzkin path

$$v(\omega) = b_0 \, b_1^2 \, \lambda_1^2 \, \lambda_2$$

$$f(x^n) = \mu_n \qquad (n \geq 0)$$

moments

$$\mu_n = \sum_{\substack{\omega \\ \text{Motzkin path} \\ |\omega| = n}} v(\omega)$$

continued
fractions
1980

orthogonal
polynomials

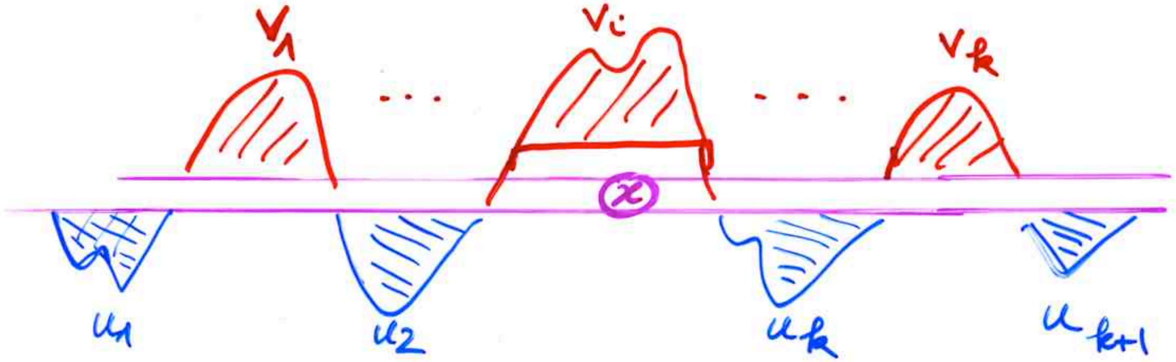
P. Flajolet

Lecture Note
X.V. 1983

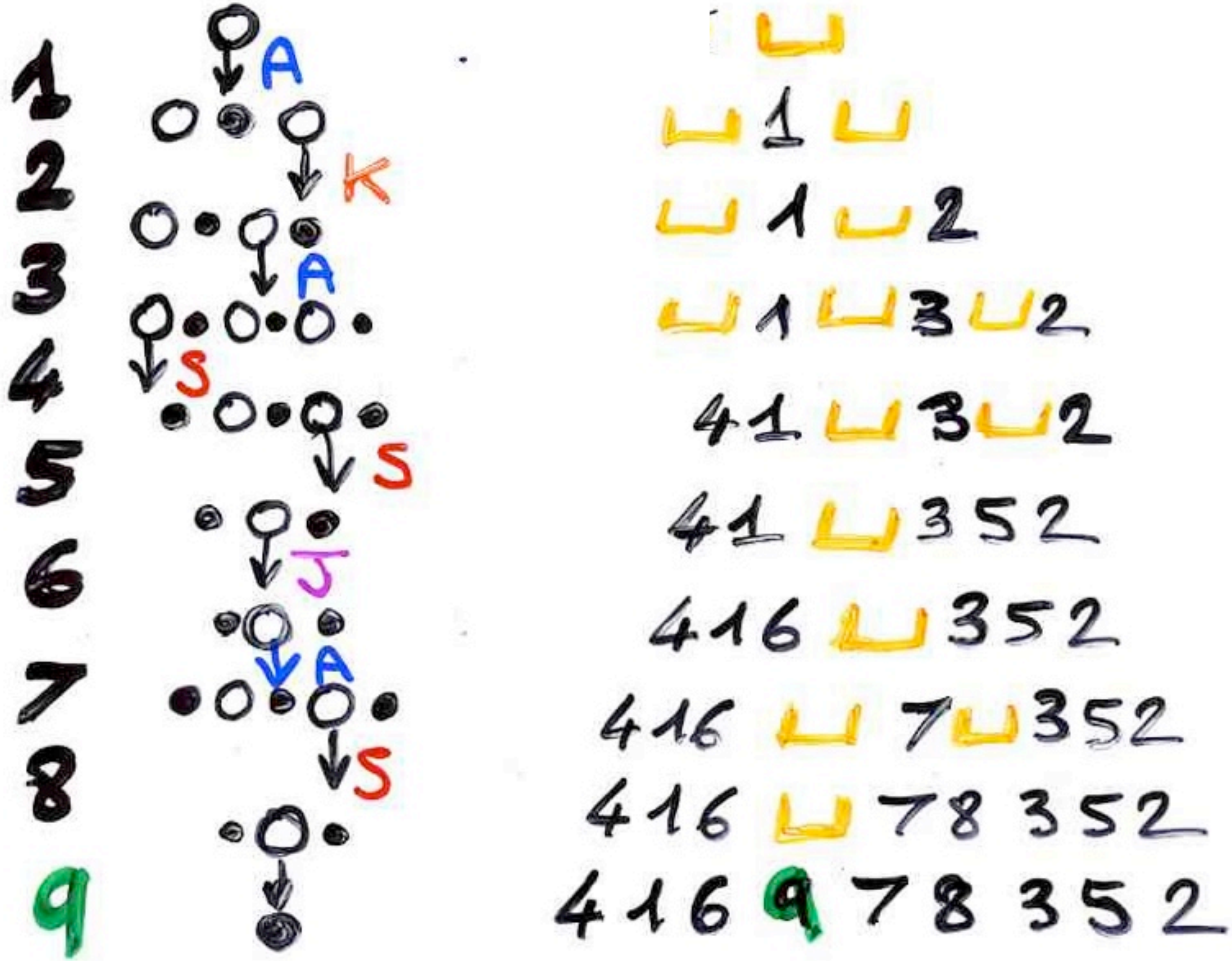
PASEP algebra

$DE = qED + E + D$

q-Laguerre polynomials



v1
vi
vk
x
u1
u2
uk
uk+1

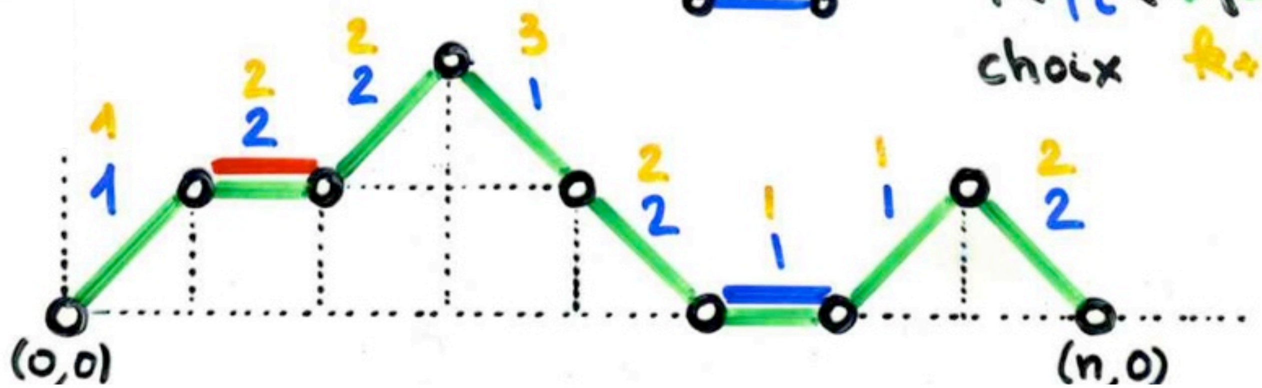


1
2
3
4
5
6
7
8
9
A
K
A
S
S
J
A
S
1
1 2
1 3 2
4 1 3 2
4 1 3 5 2
4 1 6 3 5 2
4 1 6 7 3 5 2
4 1 6 7 8 3 5 2
4 1 6 9 7 8 3 5 2

$h = (\omega_c ; (p_1, \dots, p_n))$

$1 \leq p_i \leq v(\omega_i)$

choix $k+1$



x	ω_c	pos	v
1	↗	1	1
2	→	2	2
3	↗	2	2
4	↘	1	3
5	↘	2	2
6	→	1	1
7	↗	1	1
$n=$ 8	↘	2	2
9	•		

␣

␣ 1 ␣

␣ 1 ␣ 2

␣ 1 ␣ 3 ␣ 2

4 1 ␣ 3 ␣ 2

4 1 ␣ 3 5 2

4 1 6 ␣ 3 5 2

4 1 6 ␣ 7 ␣ 3 5 2

4 1 6 ␣ 7 8 3 5 2

4 1 6 9 7 8 3 5 2 $= \sigma \in \mathfrak{S}_{n+1}$

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$P_0 = 1 \qquad P_1 = x - b_0$$

$$\mu_n = (n+1)!$$

$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

Laguerre

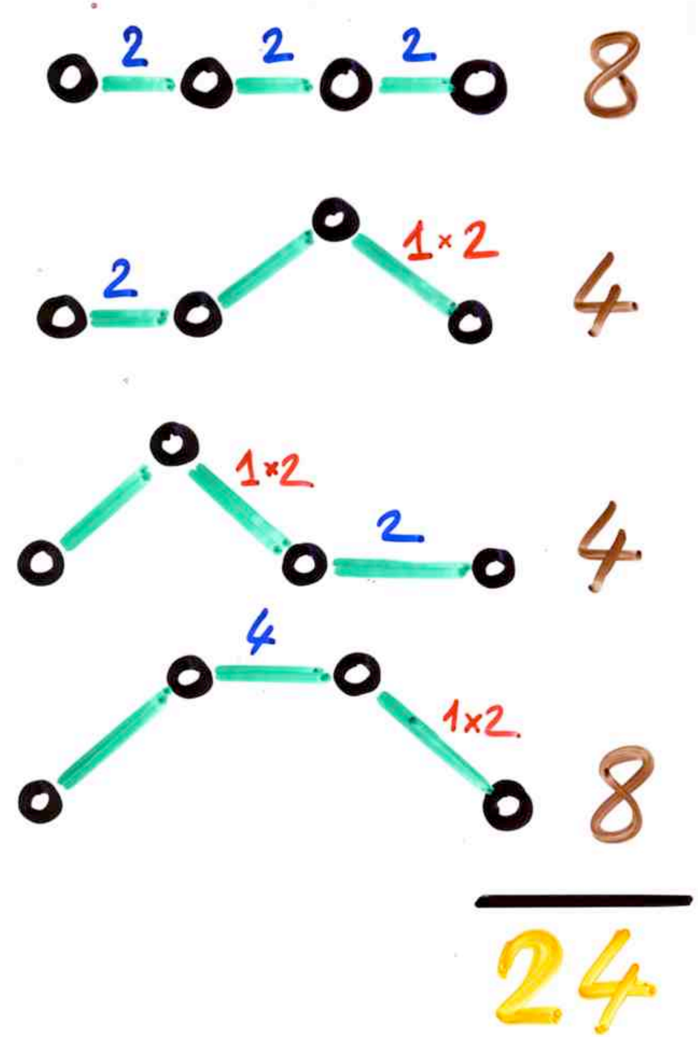
polynomial

Laguerre $L_n^{(1)}(x)$

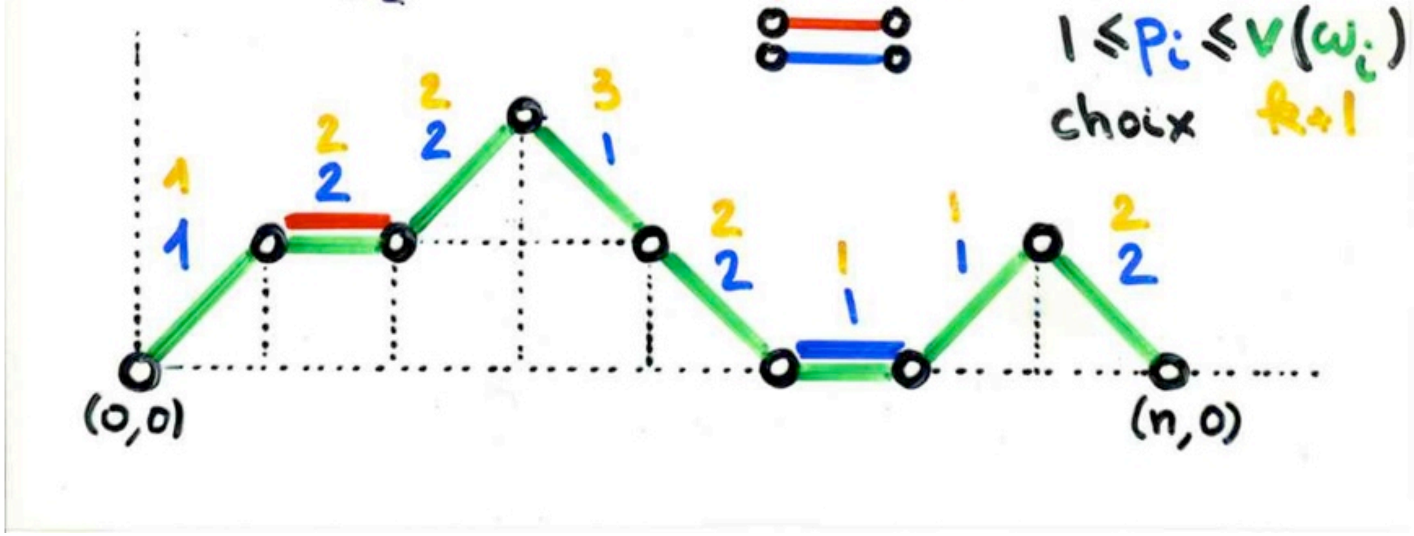
moment $\mu_n = (n+1)!$

$$b_k = 2k+2$$

$$\lambda_k = k(k+1)$$



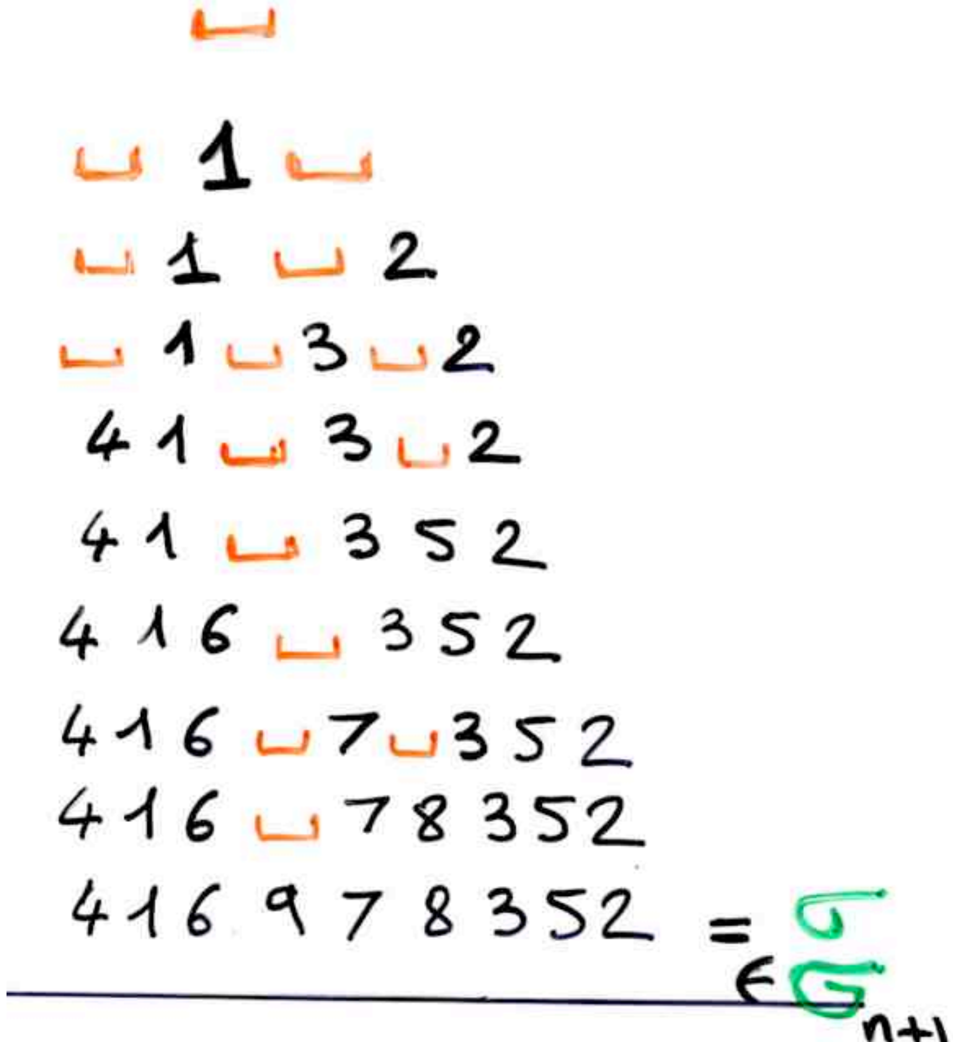
"q-analogue" of Laguerre histories



choices function

1	2	3	4	5	6	7	8
1	2	2	1	2	1	1	2
0	1	1	0	1	0	0	1

q-Laguerre : q^4



q-Laguerre

$$L_n^{I\,\beta}(x;q)$$

$\beta = \alpha + 1$

$$\begin{cases} b_{k,q}^{(\beta)} = [k]_q + [k+1;\beta]_q \\ \lambda_{k,q}^{(\beta)} = [k]_q \cdot [k;\beta]_q \end{cases}$$

$$[k;\beta]_q = \beta + q + q^2 + \cdots + q^{k-1}$$

$$\mu_n = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\binom{2n}{n-k} - \binom{2n}{n-k-2} \right) \left(\sum_{i=0}^{k} q^{i(k+1-i)} \right)$$

Corteel, Josuat-Vergès, Prellberg, Rubey (2008) y

general PASEP

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q- Hermite polynomial

α, β, q $\gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}} \hat{a}$$
$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}} \hat{a}^+$$

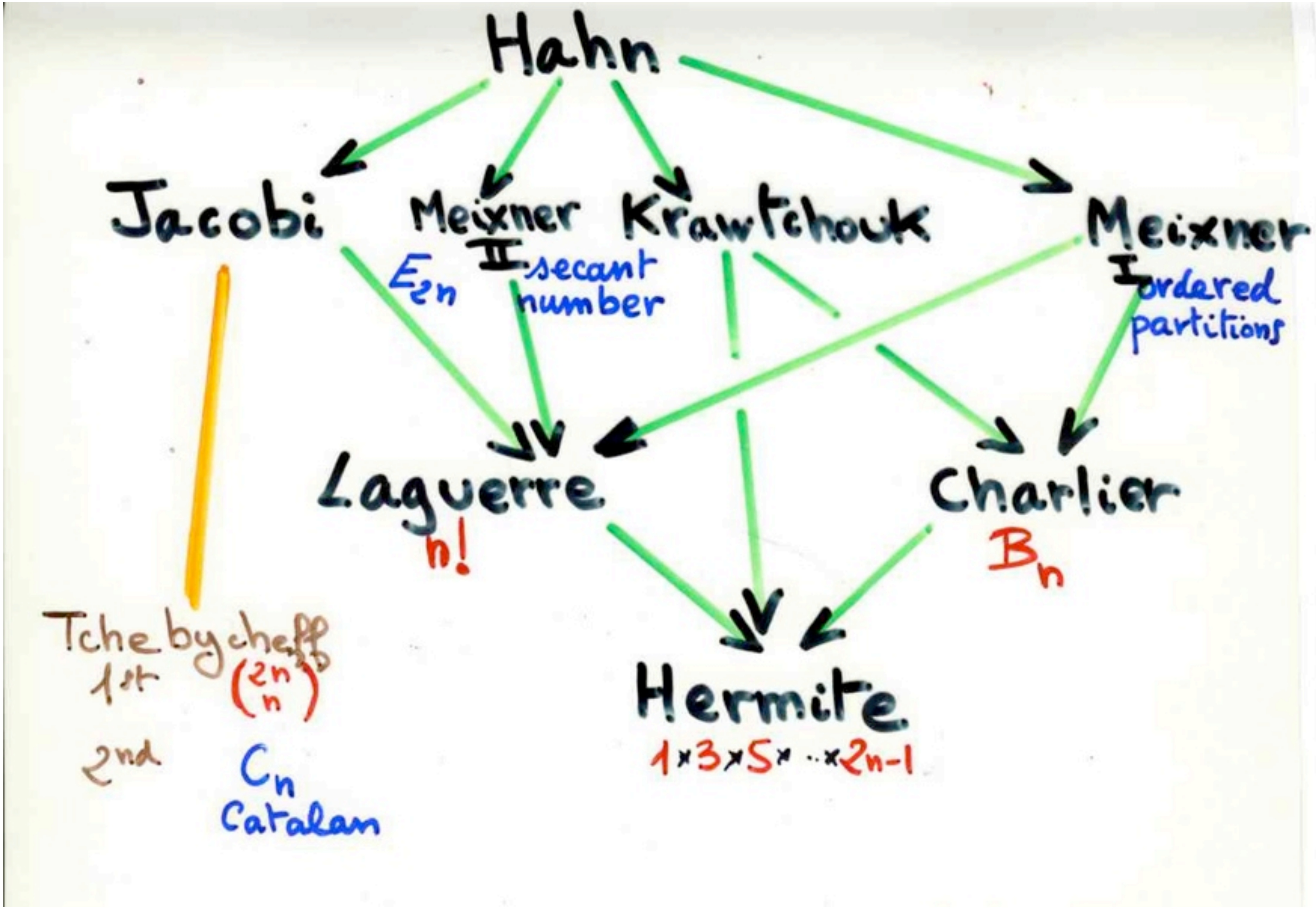
$$\hat{a} \hat{a}^+ - q \hat{a}^+ \hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta,$ q

Askey-Wilson polynomials

Askey-Wilson



steady state probability PASEP

$$\frac{1}{Z_n} Z_\tau(\alpha, \beta, \gamma, \delta ; q \;)$$

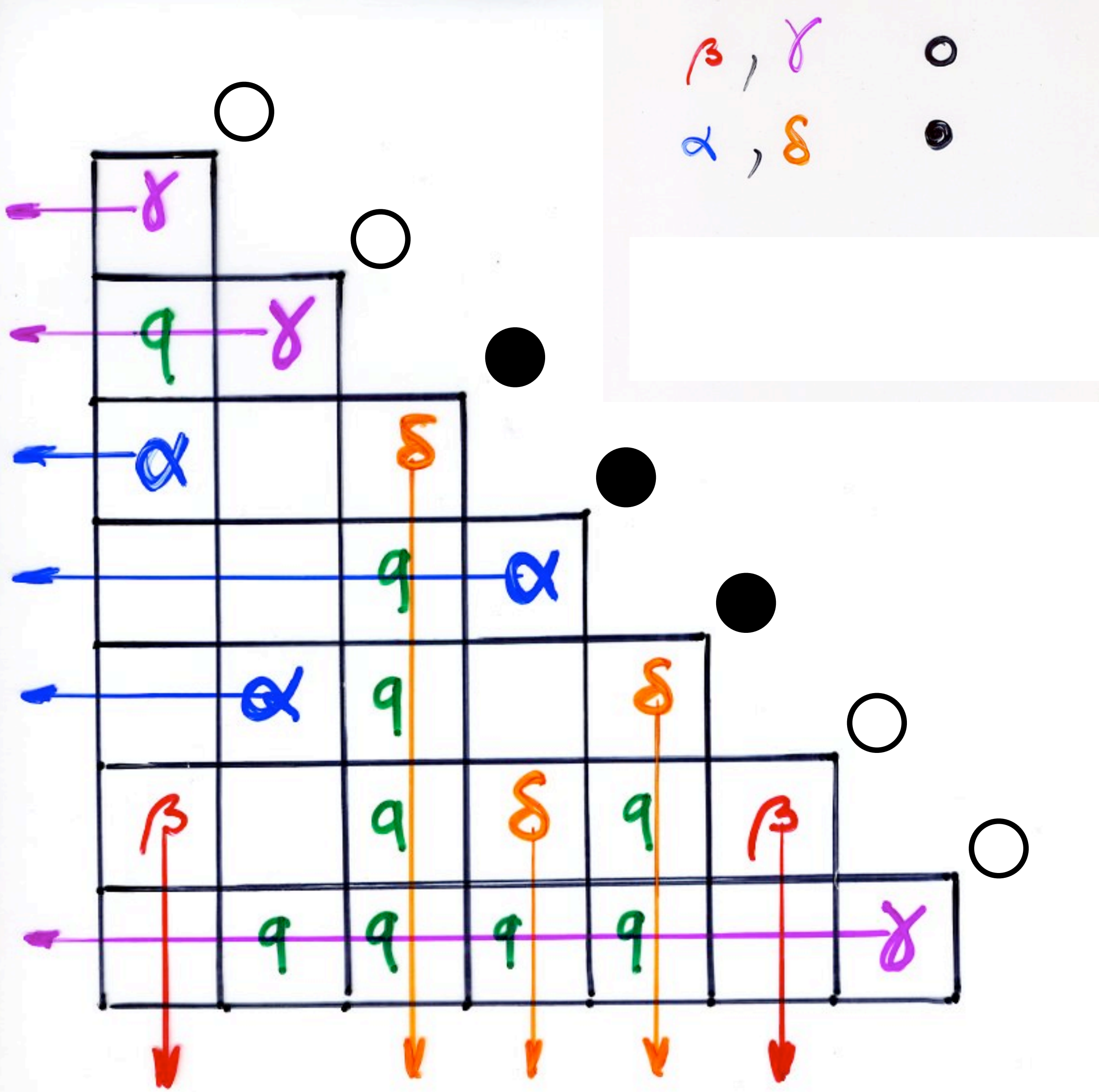
$$Z_n = \sum_{\tau} Z_\tau$$

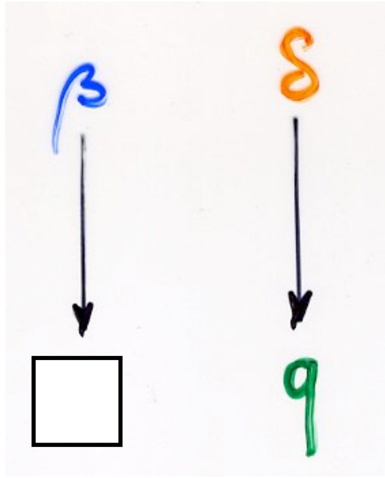
$\tau = (\tau_1, \ldots, \tau_n)$ state

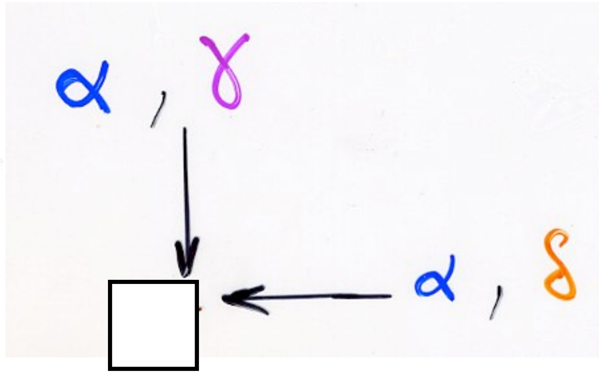
relation with moments of Askey-Wilson polynomials

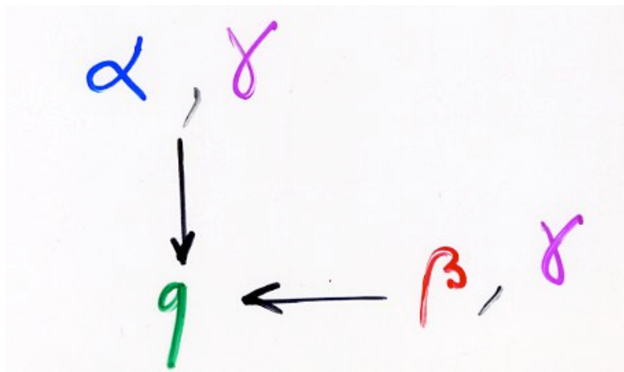
Corteel, Williams, 2009

Corteel, Stanley, Stanton, Williams, 2010


β , γ
α , δ


β
δ
q


α , γ
α , δ


α , γ
q
β , γ

The cellular Ansatz

From quadratic algebra Q
to combinatorial objects (Q-tableaux)
and bijections

"The cellular Ansatz"

combinatorial
objects
on a 2d lattice

representation
by operators

Physics

data structures
"histories"

bijections

orthogonal
polynomials

"normal ordering"

UD = DU + Id

Weyl-Heisenberg

DE = qED + E + D

PASEP

towers placements

permutations ⟷ RSK ⟷ pairs of Tableaux Young

tableaux alternatifs ⟷ permutations

Laguerre histories

Q-tableaux

ex: ASM,
(alternating sign matrices)
FPL(fully packed loops)
tilings, 8-vertex

quadratic algebra Q

commutations
rewriting rules
planarisation

planar
automata

Koszul algebras
duality

The 8-vertex algebra
(or XYZ - algebra)
(or Z - algebra)

The quadratic algebra $\mathbf{Z}$

4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$\left\{ \begin{array}{l} BA = q_{00}\, AB + t_{00}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, AB \\ B_\bullet A = q_{\bullet 0}\, A B_\bullet + t_{\bullet 0}\, A_\bullet B \\ B A_\bullet = q_{0\bullet}\, A_\bullet B + t_{0\bullet}\, A B_\bullet \end{array} \right.$$

$$\begin{cases} t_{\bullet\circ} = t_{\circ\bullet} = 0 \\ q_{\bullet\bullet} = 0 \end{cases} \qquad \text{(ASM)}$$

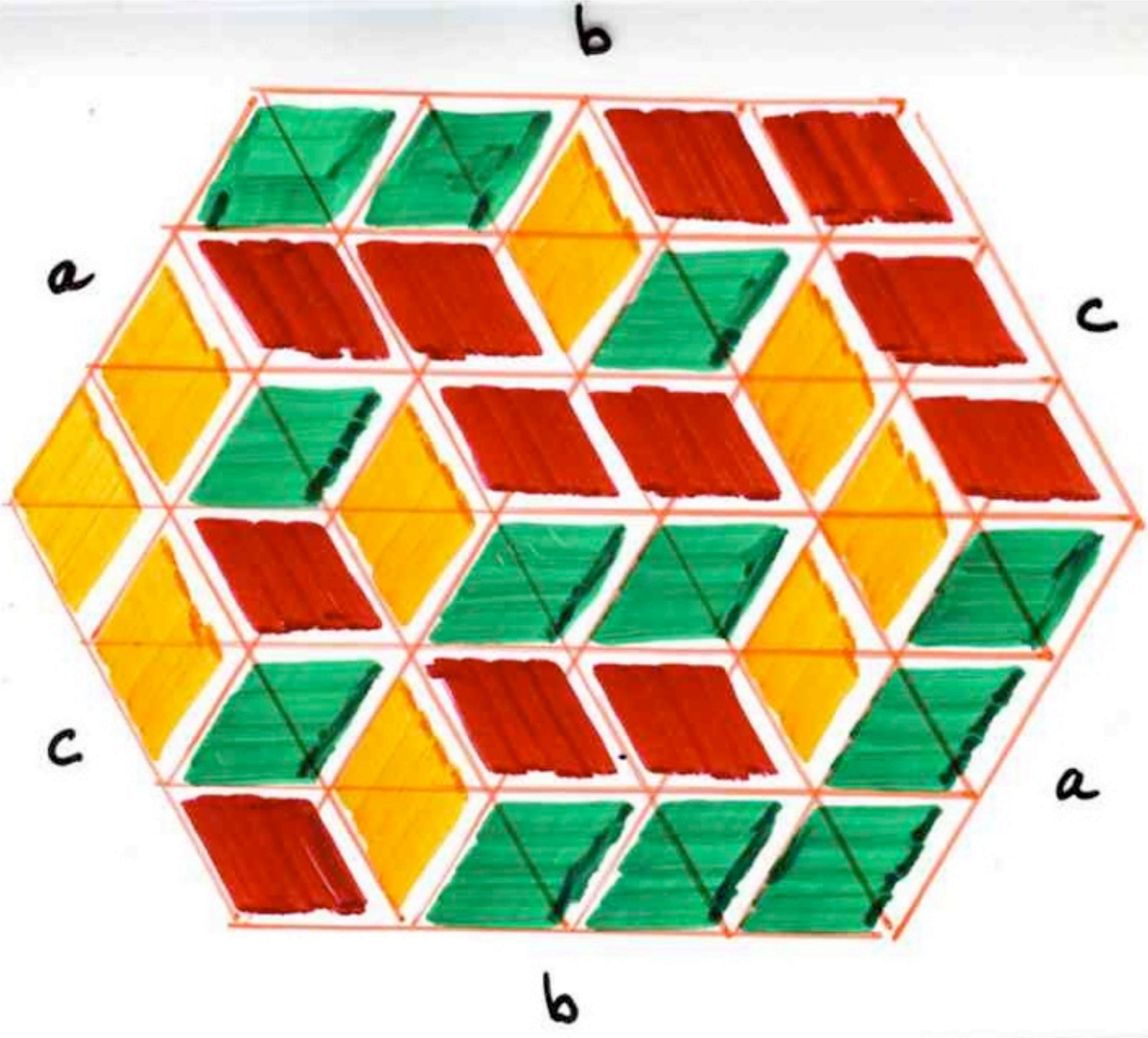
Rhombus tilings

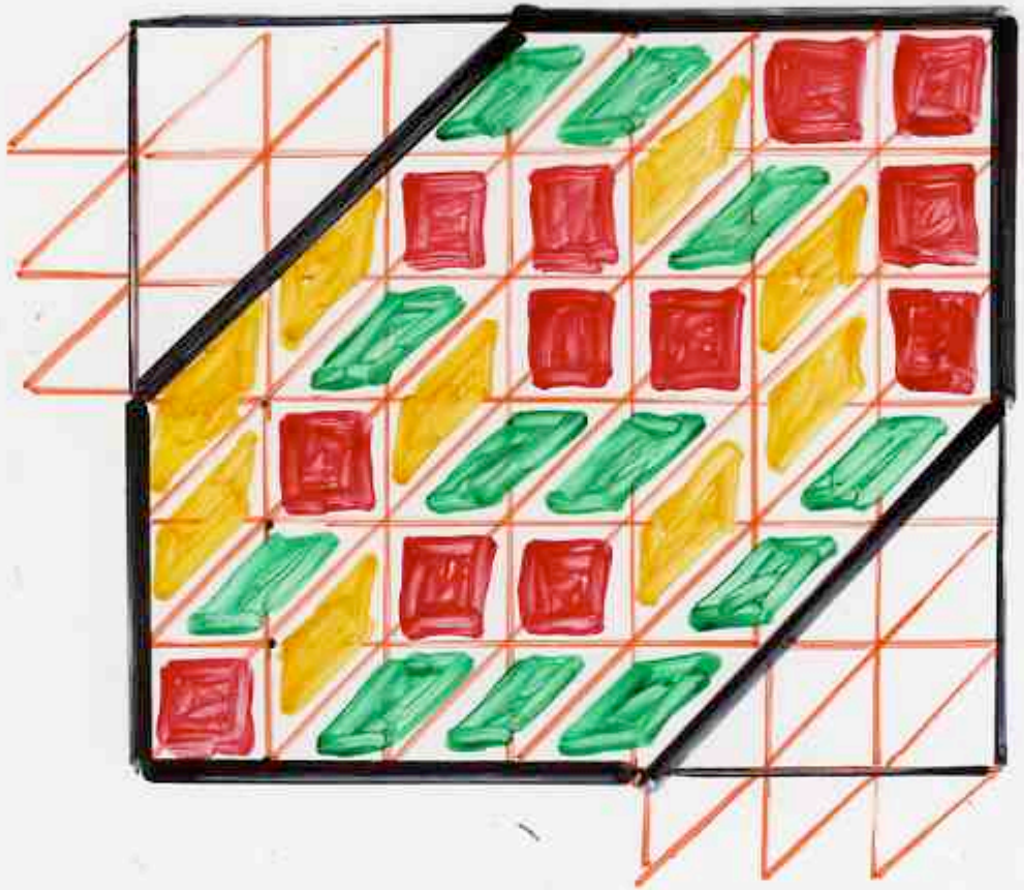
The quadratic algebra Z

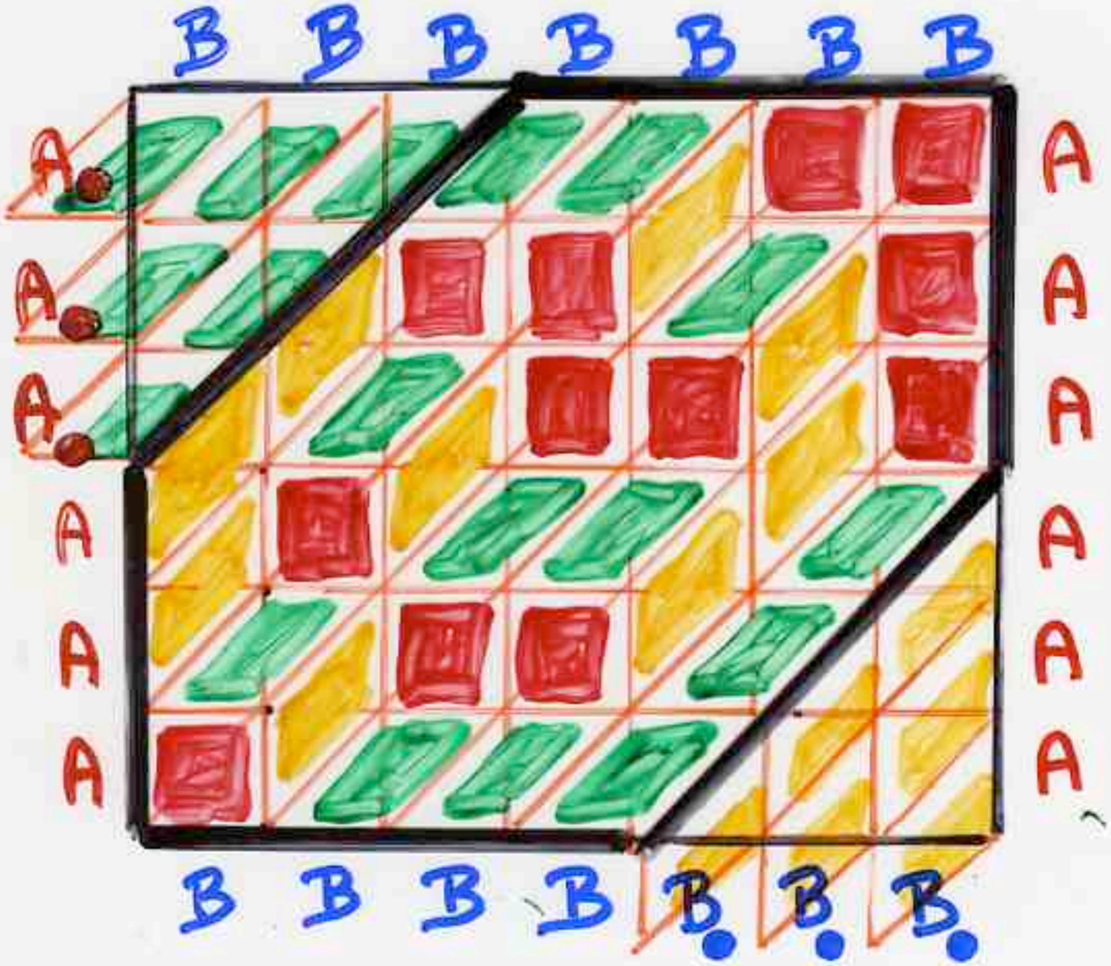
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} B A = q_{\circ\circ} \, A B + t_{\circ\circ} \, A_\bullet B_\bullet \\ B_\bullet A_\bullet = \bigcirc \, A_\bullet B_\bullet + t_{\bullet\bullet} \, A B \\ B_\bullet A = q_{\bullet\circ} \, A B_\bullet + \bigcirc \, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} \, A_\bullet B + \bigcirc \, A B_\bullet \end{cases}$$


b
a
c
c
a
b





B B B B B B B
A A A A A A
A A A A A A
B B B B B B B

Aztec tilings

$$t_{\bullet\circ} = t_{\circ\bullet} = 0 \qquad \text{(ASM)}$$

$$t_{\bullet\bullet} = 2 \qquad \text{(nb of } -1 \text{ in ASM)}$$

The quadratic algebra Z

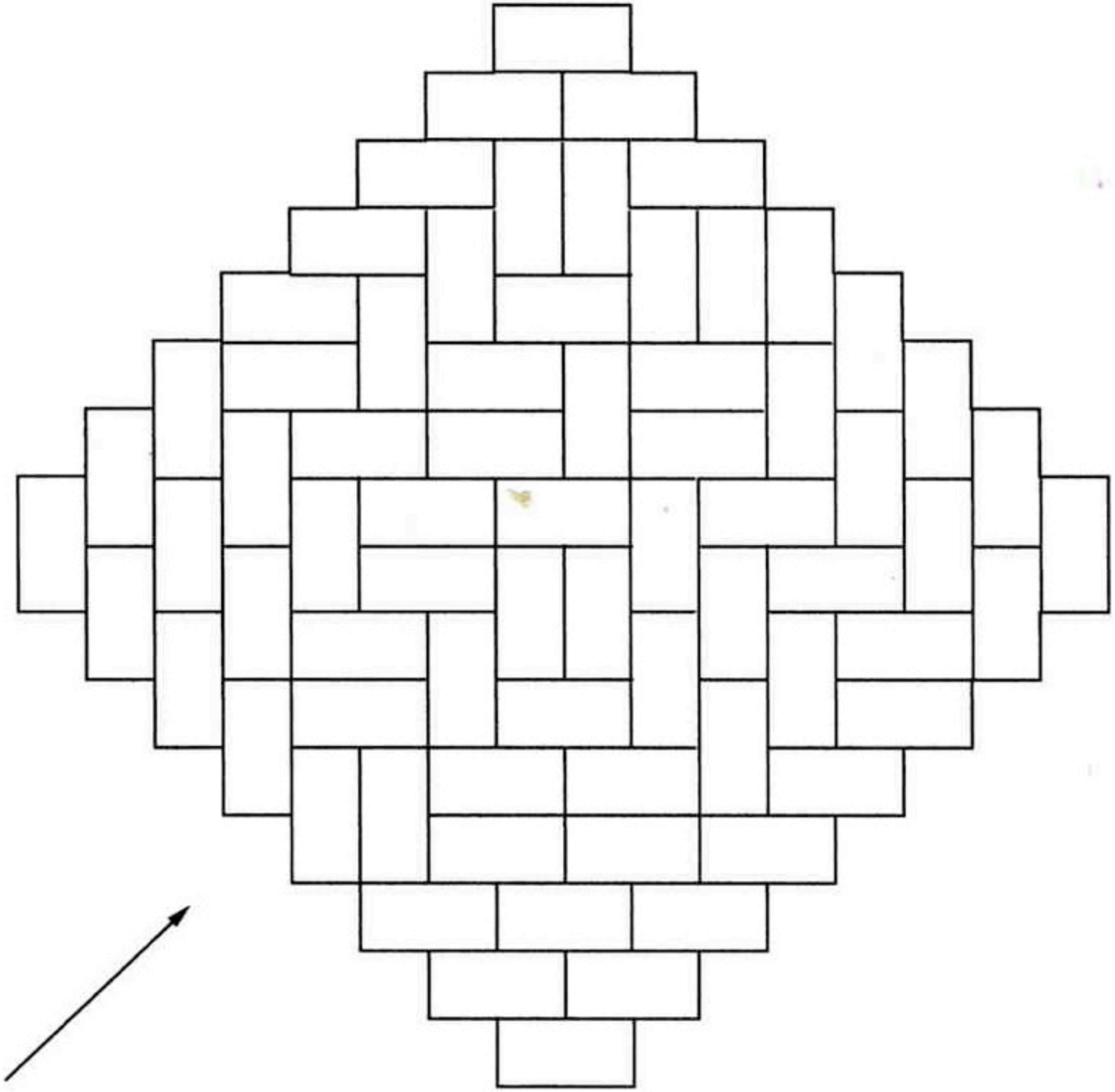
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} BA = q_{\circ\circ} AB + t_{\circ\circ} A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet} A_\bullet B_\bullet + 2\, AB \\ B_\bullet A = q_{\bullet\circ} A B_\bullet + \bigcirc\, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} A_\bullet B + \bigcirc\, A B_\bullet \end{cases}$$

$2^{n(n-1)/2}$

$A_n(2)$

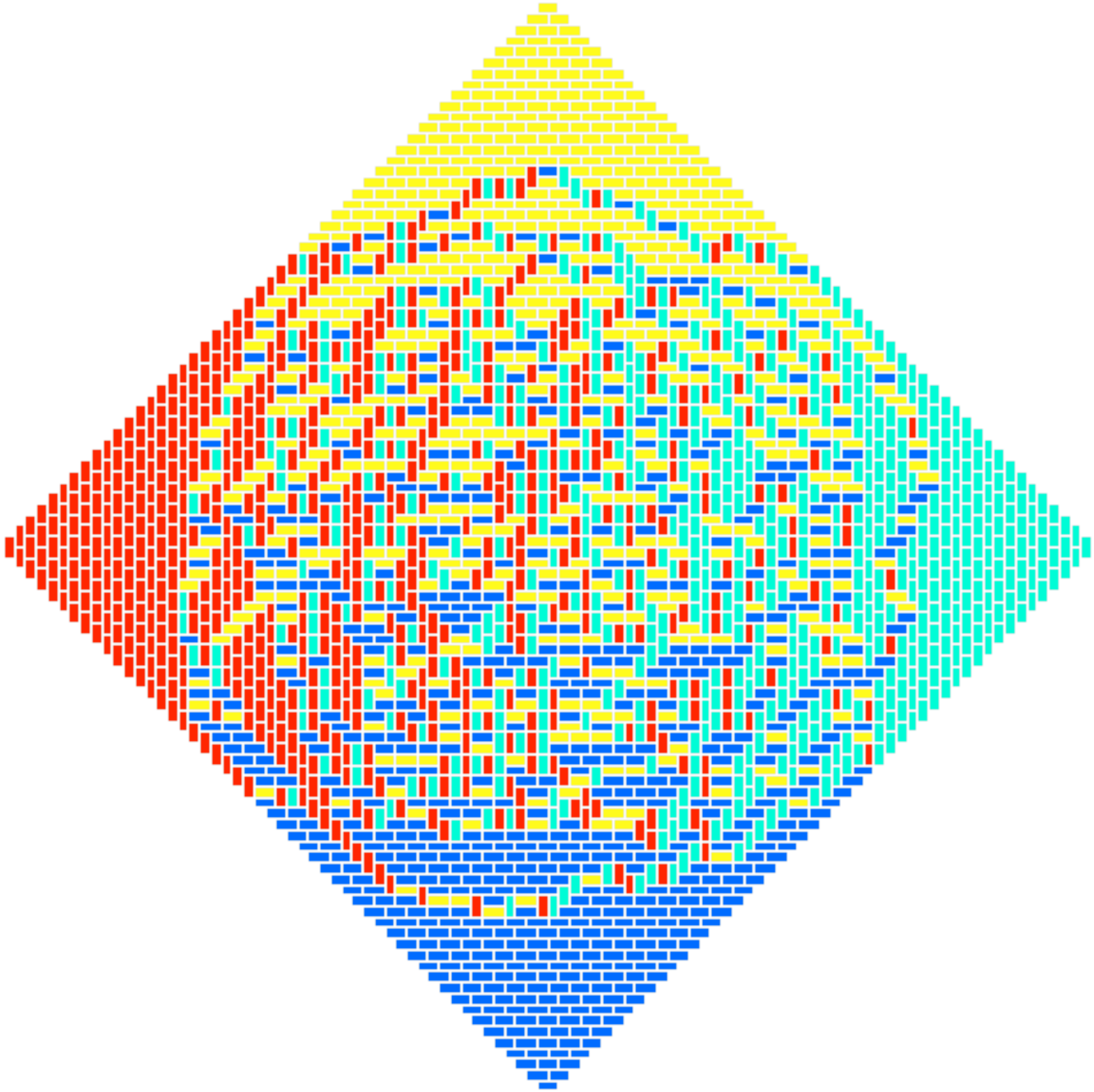


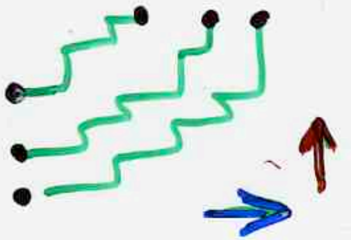
Elkies, Kuperberg, Larsen, Propp (1992)

random

Aztec

tilings





$$\begin{cases} t_{\circ\circ} = 0 \\ q_{\bullet\bullet} = t_{\bullet\bullet} = 0 \end{cases}$$

$A \leftrightarrow A_\bullet$

exchanging

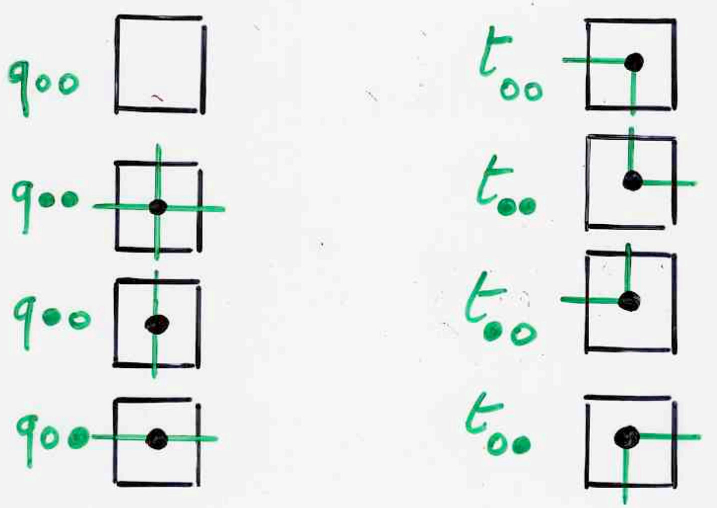
$$\begin{cases} t_{\circ\bullet} = 0 \\ q_{\bullet\circ} = t_{\bullet\circ} = 0 \end{cases}$$

The quadratic algebra $\mathbb{Z}$

4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots} \, , \; t_{\dots}$

$$\begin{cases} B A = q_{\circ\circ} \, A B + \bigcirc \, A_\bullet B_\bullet \\ B_\bullet A_\bullet = \bigcirc \, A_\bullet B_\bullet + \bigcirc \, A B \\ B_\bullet A = q_{\bullet\circ} \, A B_\bullet + t_{\bullet\circ} \, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} \, A_\bullet B + t_{\circ\bullet} \, A B_\bullet \end{cases}$$

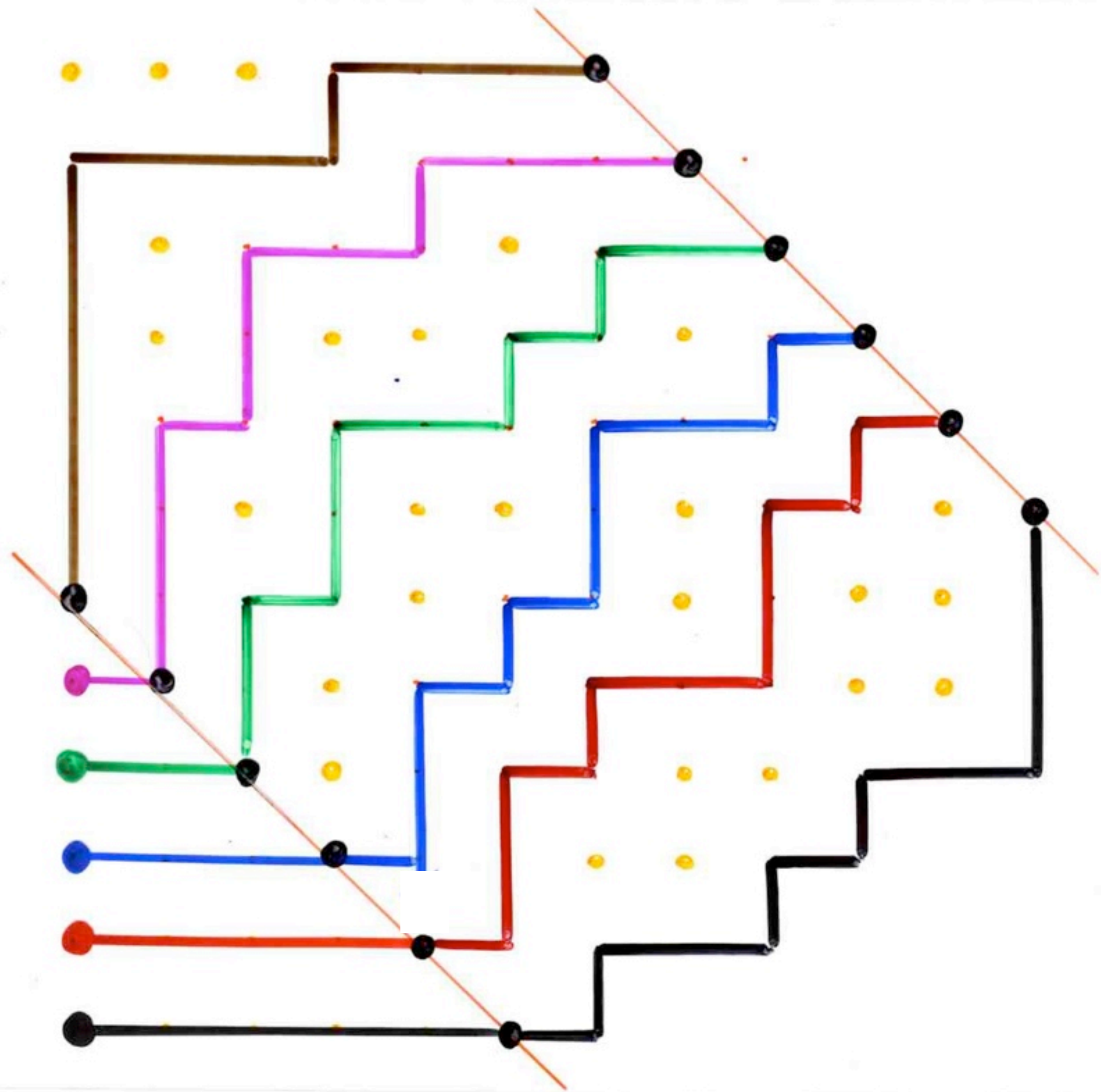


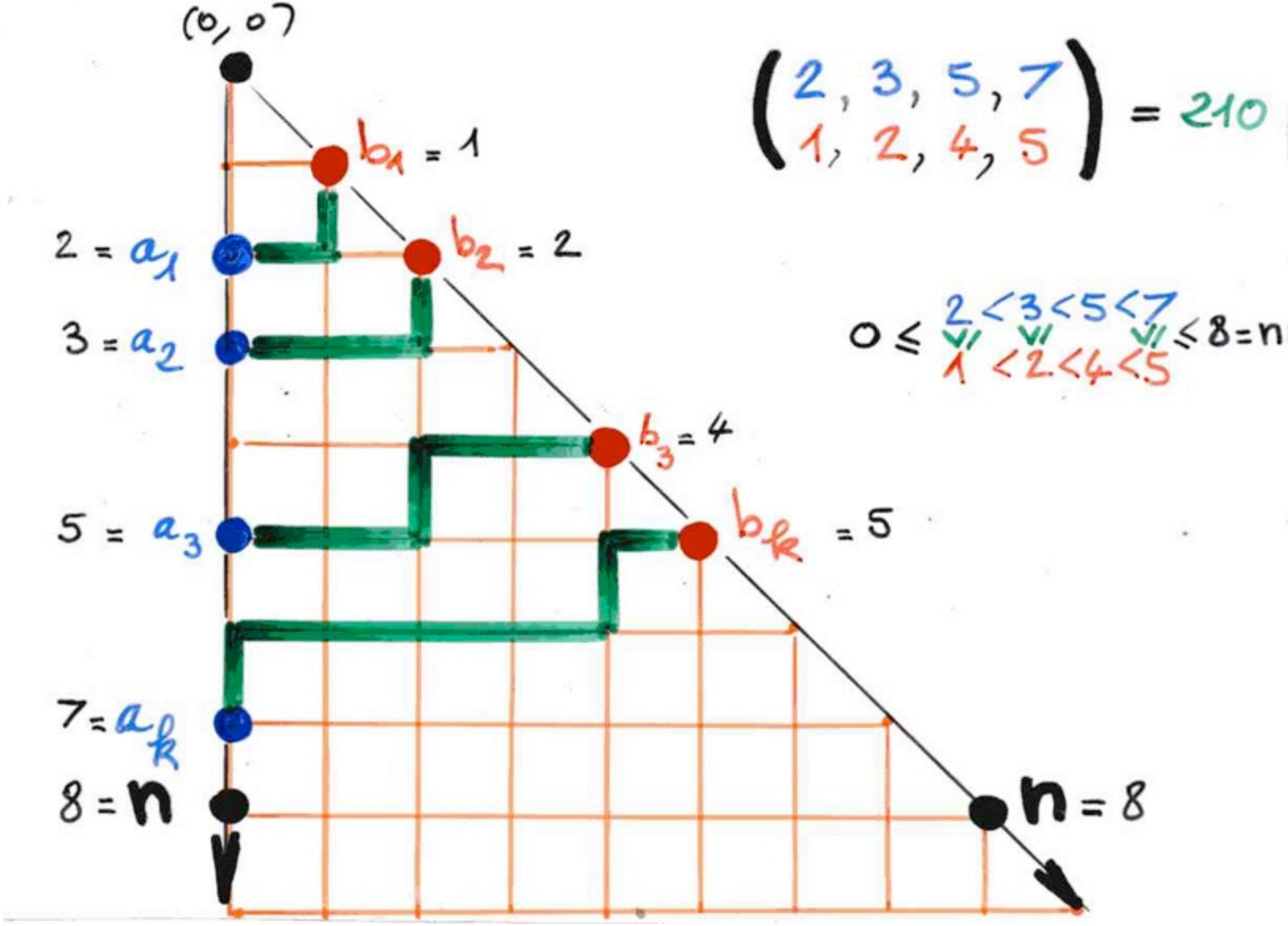
The quadratic algebra $\mathbb{Z}$

4 generators $B_\bullet \; A_\bullet \; B \; A$

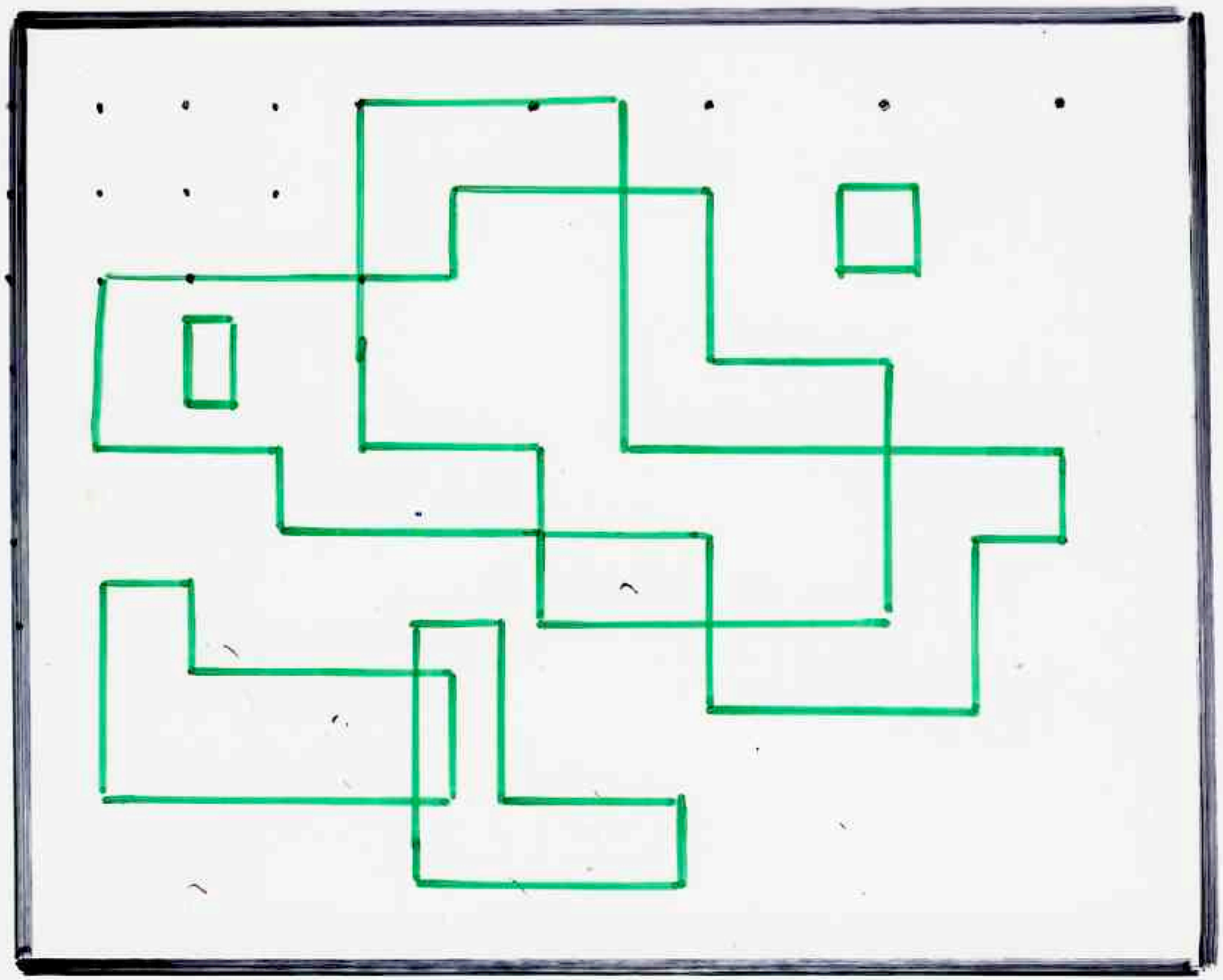
8 parameters $q_{\dots} \, , \; t_{\dots}$

$$\begin{cases} B A = q_{\circ\circ} \, A B + t_{\circ\circ} \, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet} \, A_\bullet B_\bullet + t_{\bullet\bullet} \, A B \\ B_\bullet A = \bigcirc \, A B_\bullet + \bigcirc \, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} \, A_\bullet B + \bigcirc \, A B_\bullet \end{cases}$$





example: binomial determinant



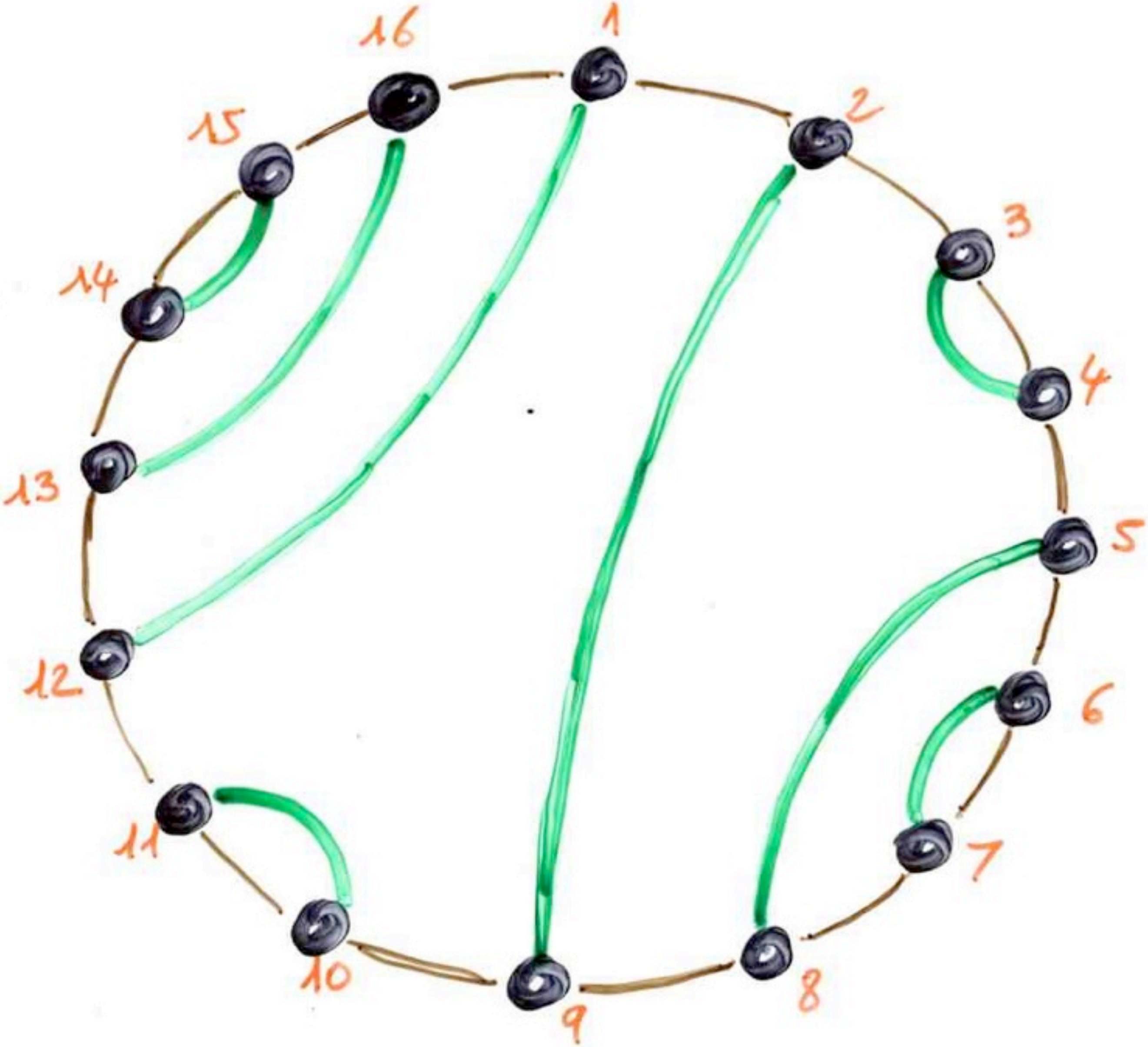
"closed" graph

Ising model

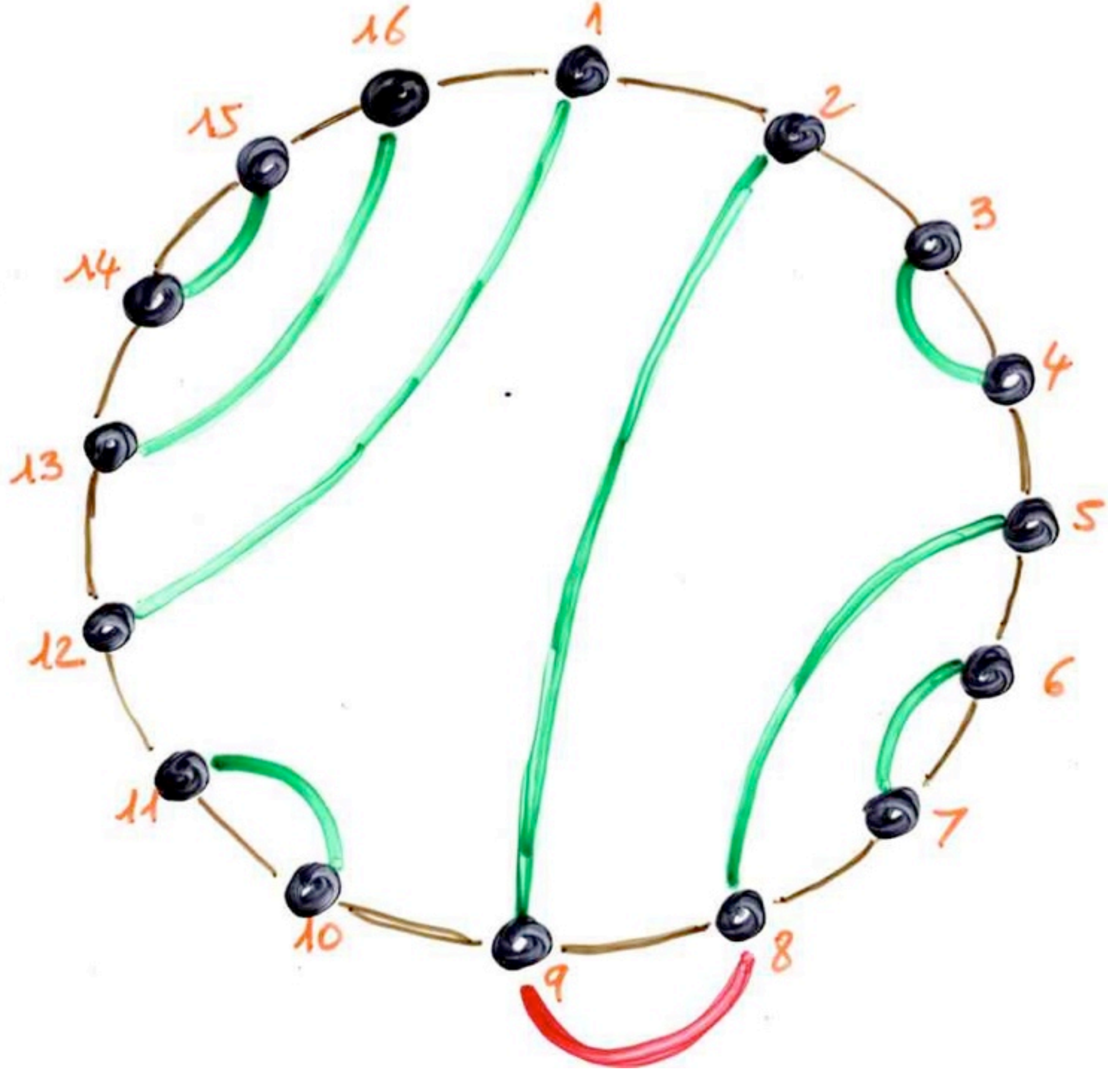
$$w = B^m A^n$$

$$uv = A^n B^m$$

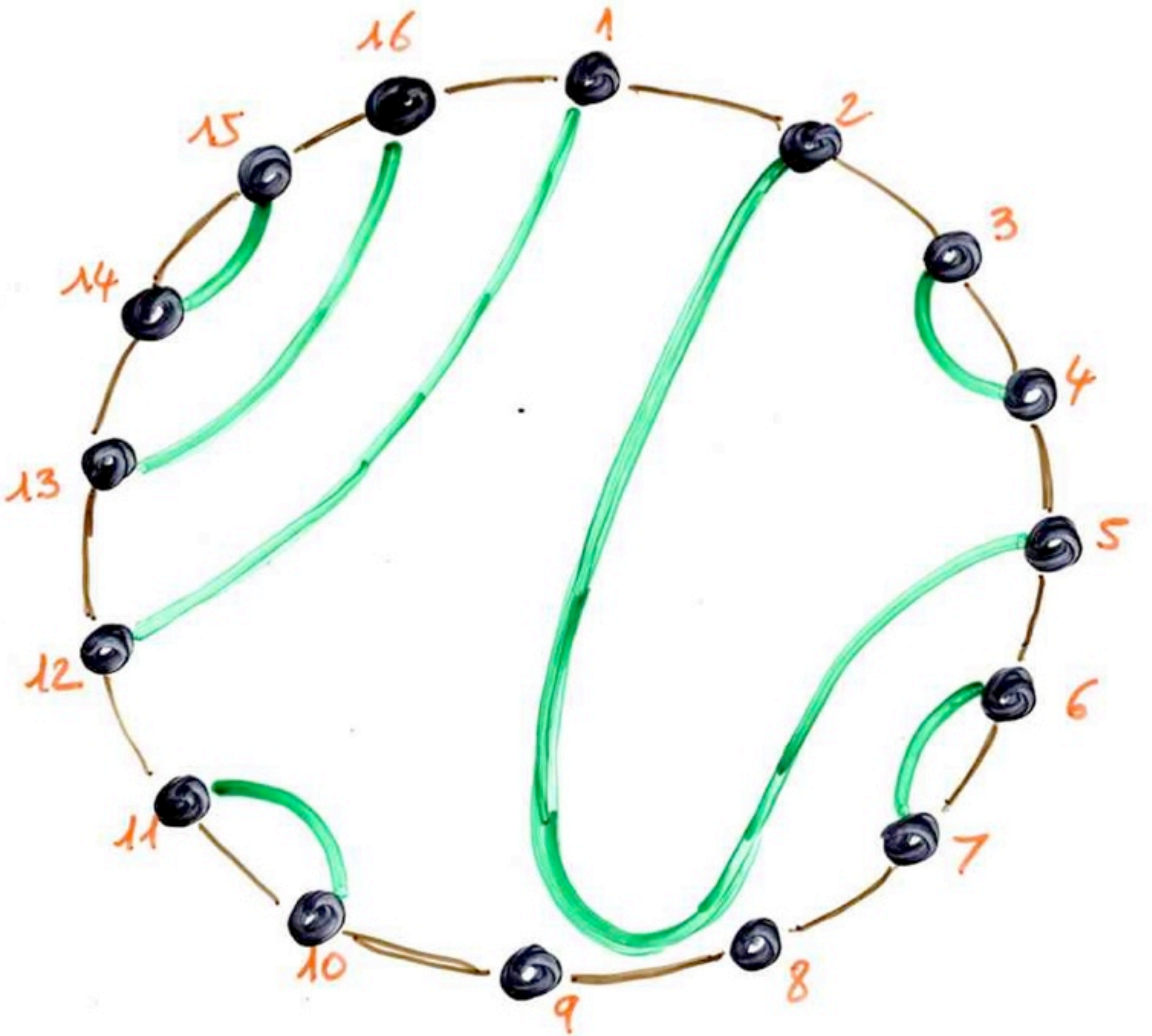
Razumov - Stroganov
(ex)- conjecture 2000-2001

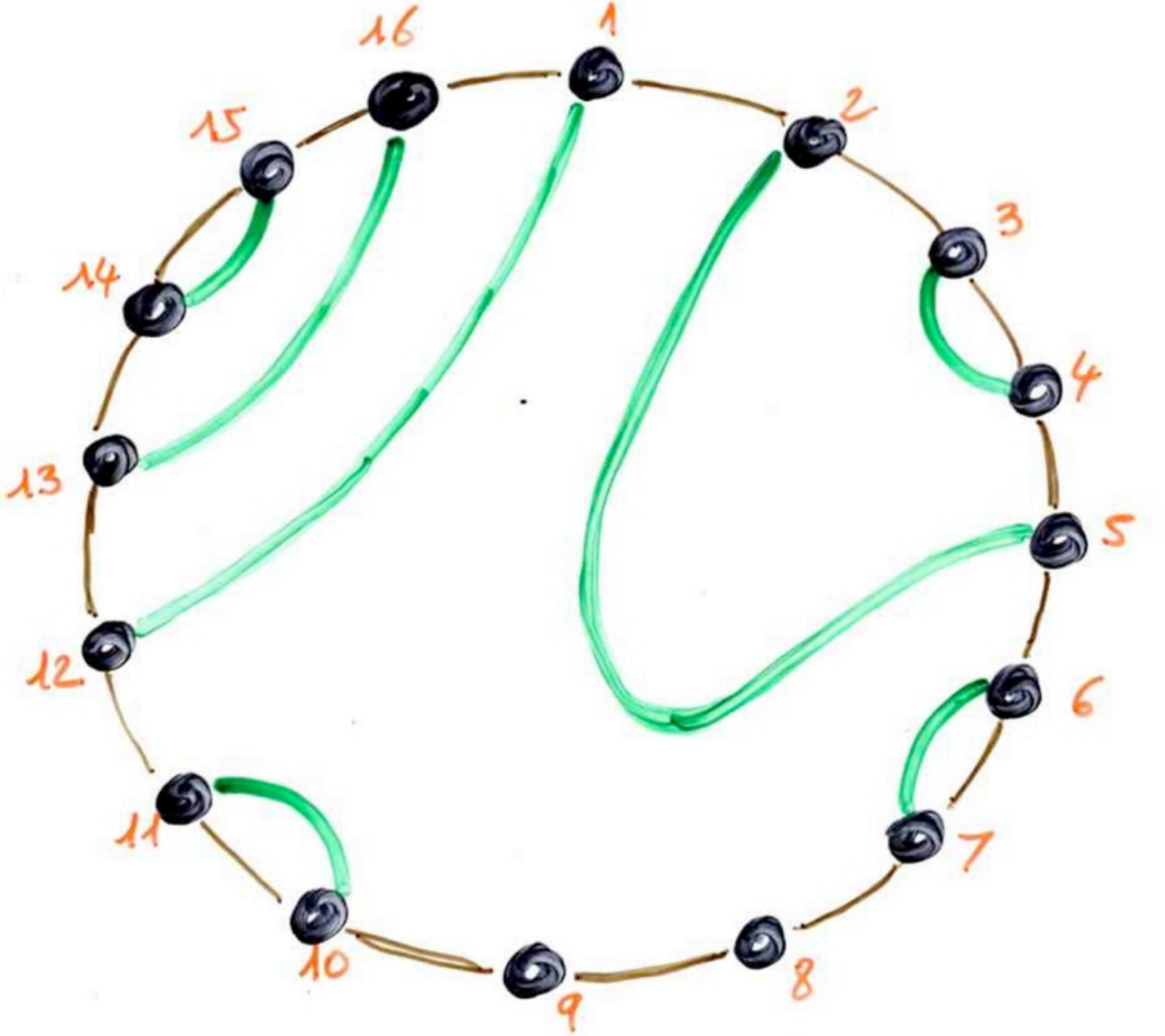


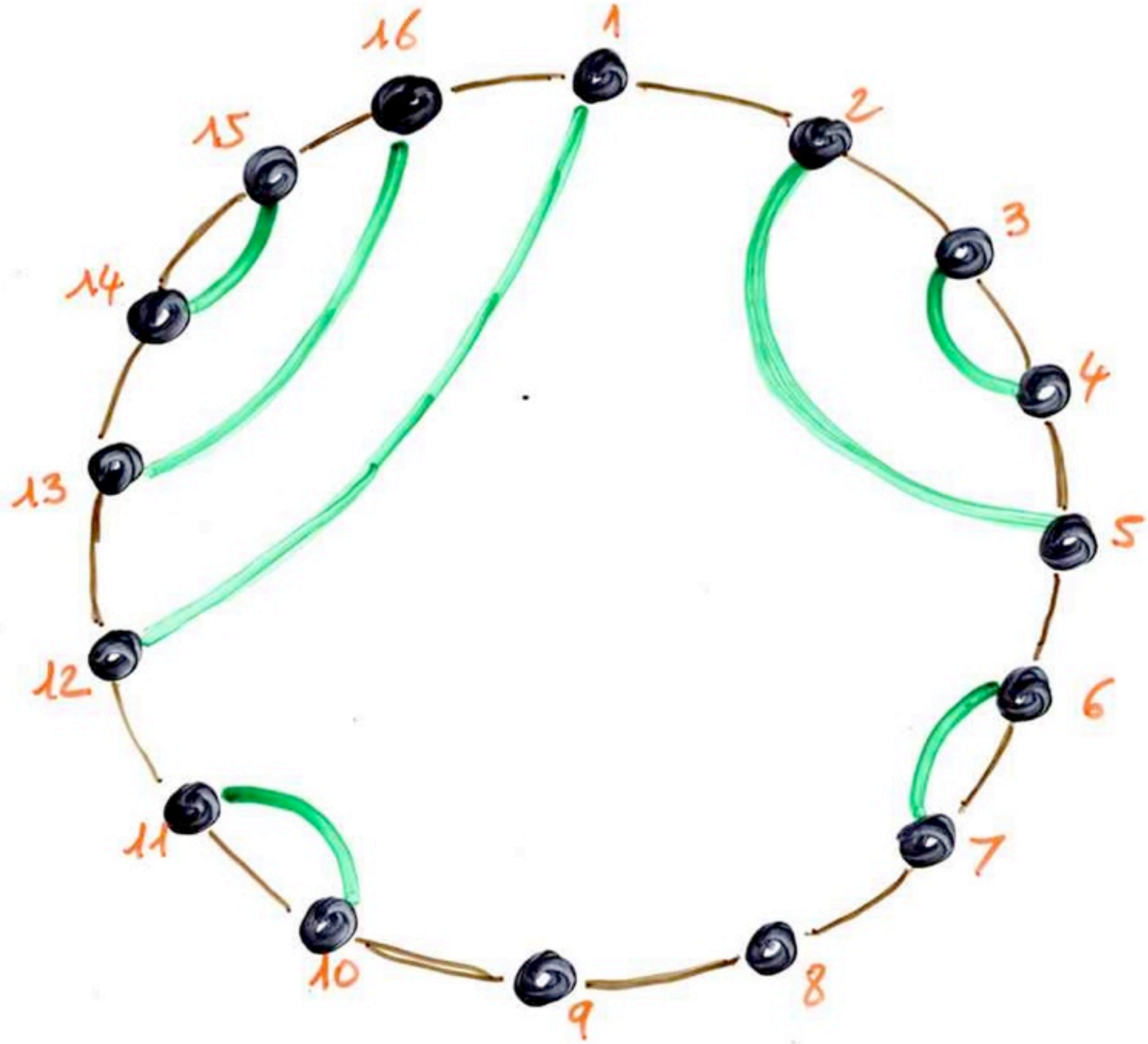
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16

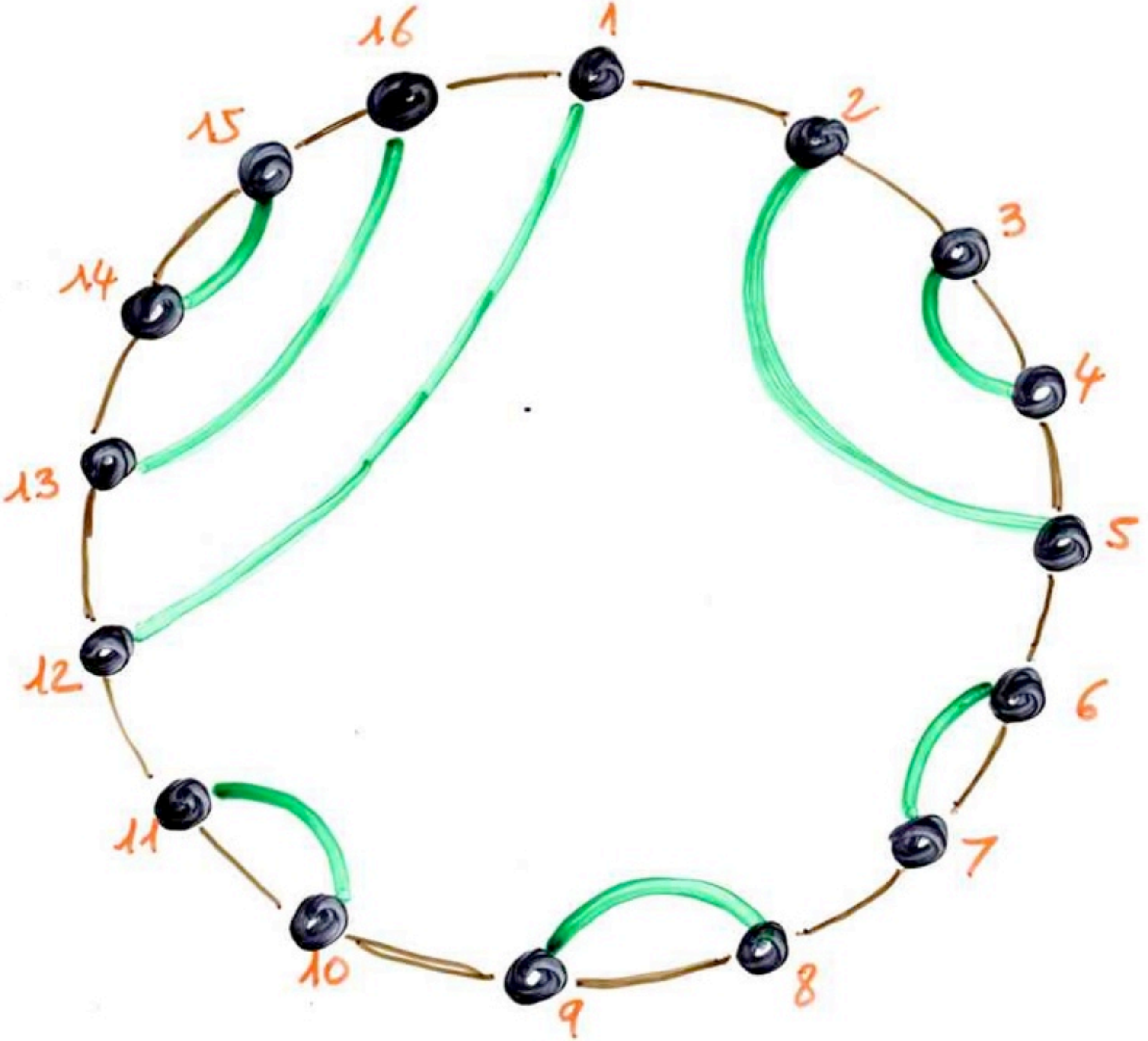


1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16


16
1
15
2
3
14
4
13
5
12
6
11
7
10
9
8


1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16


1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16



1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16

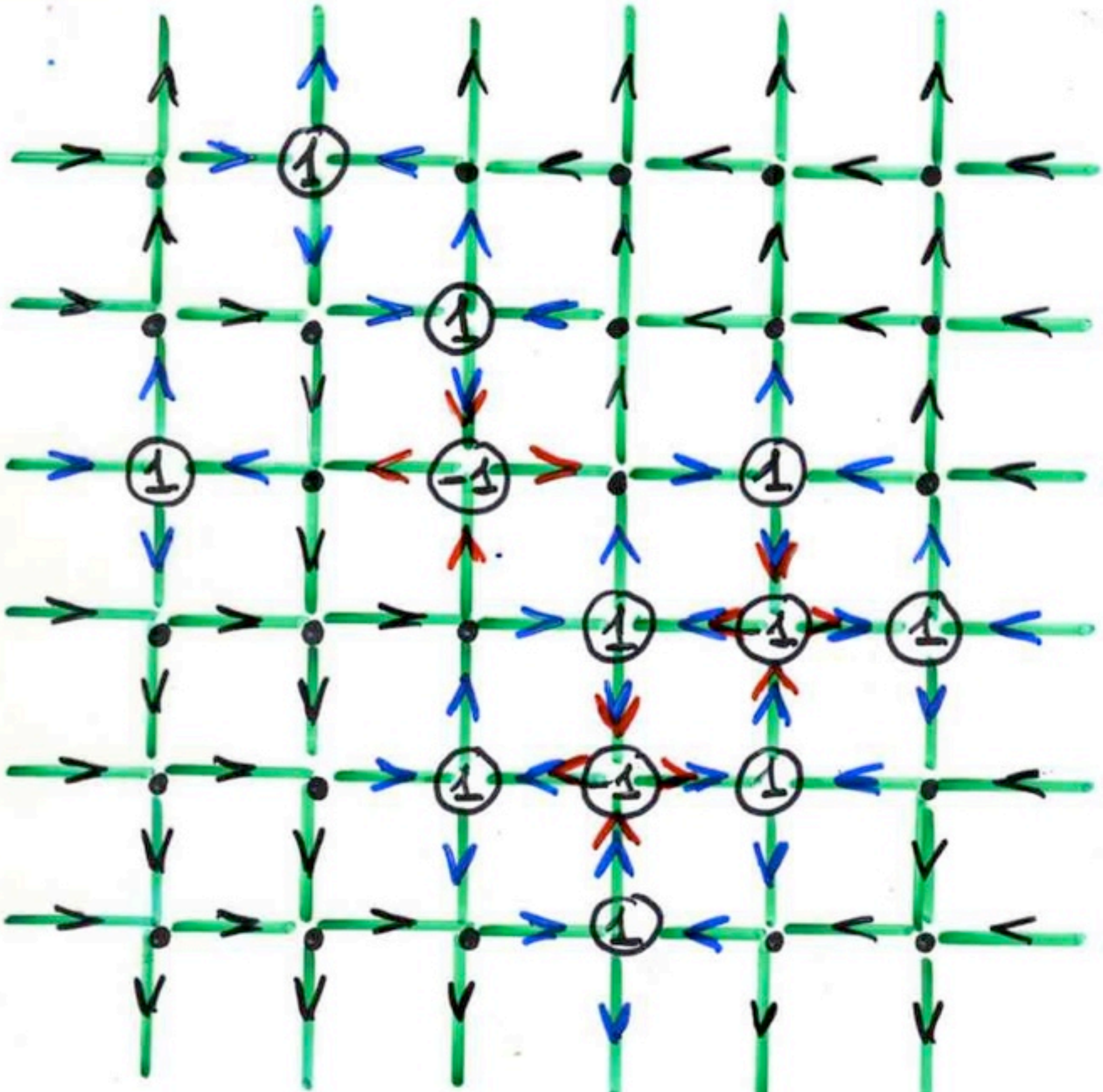
FPL

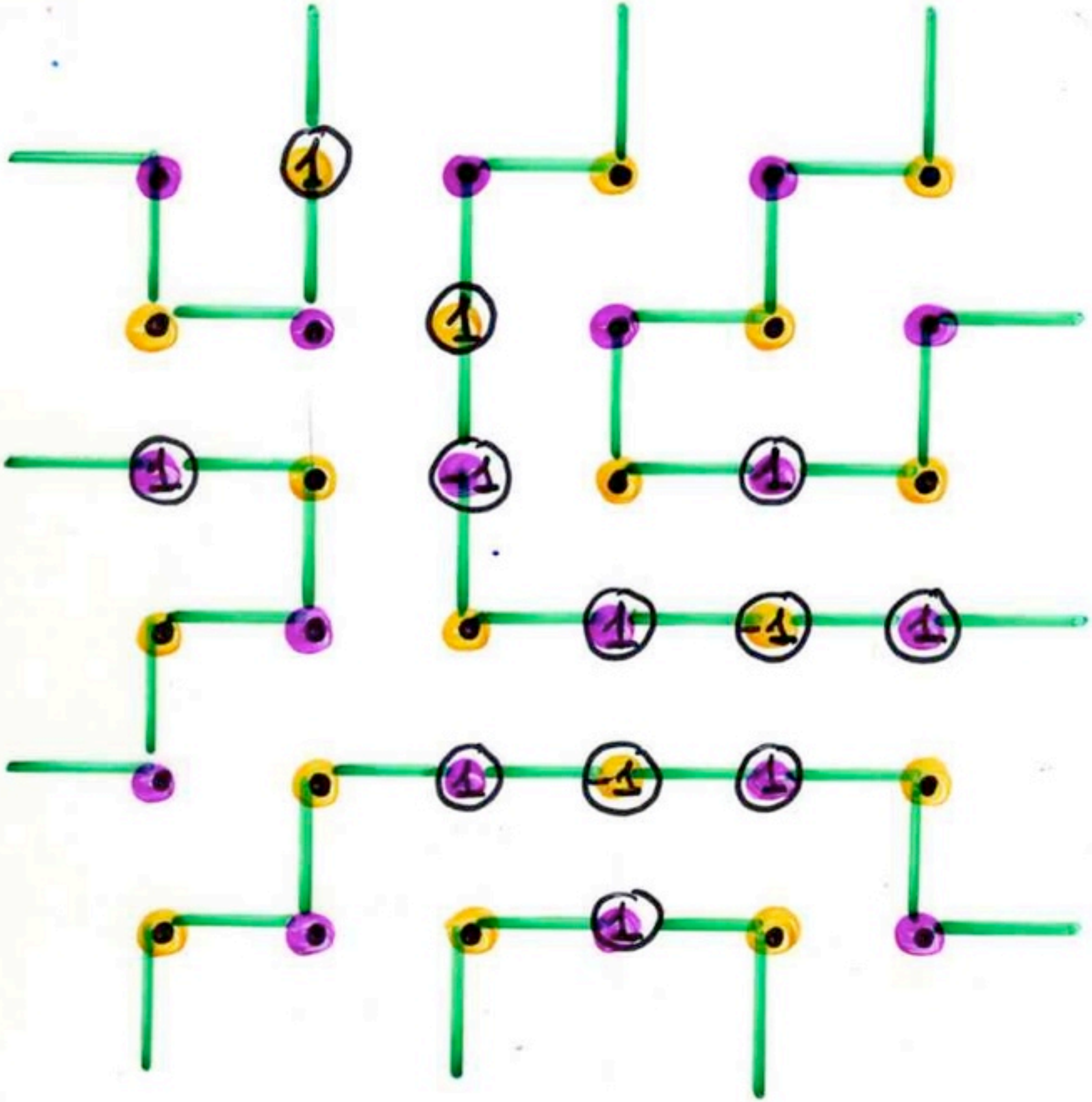
"Fully packed loop configurations"

The
bijection
AMS
FPL

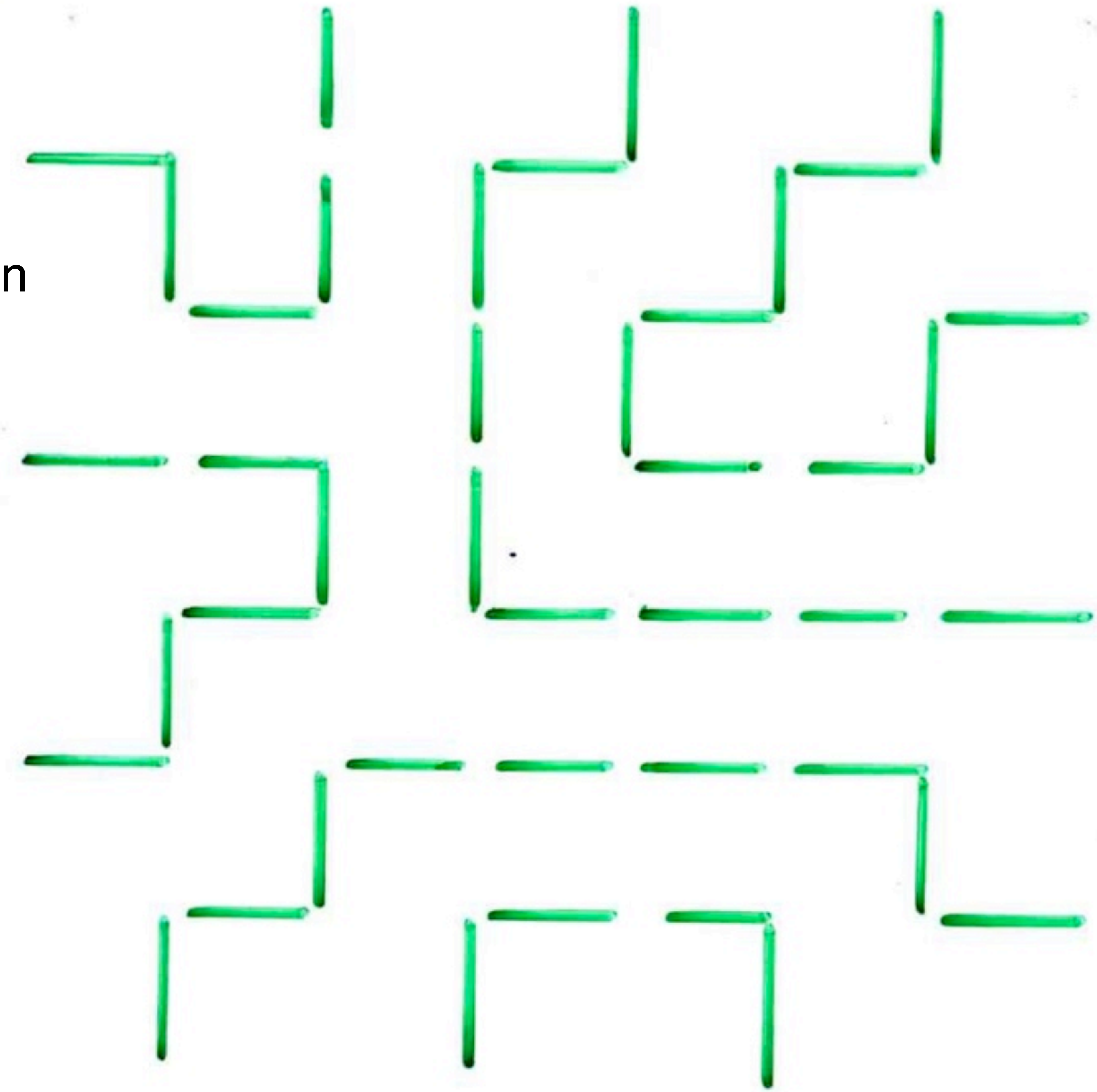
·	1	·	·	·	·
·	·	1	·	·	·
1	·	-1	·	1	·
·	·	·	1	-1	1
·	·	1	-1	1	·
·	·	·	1	·	·

The 6-vertex model

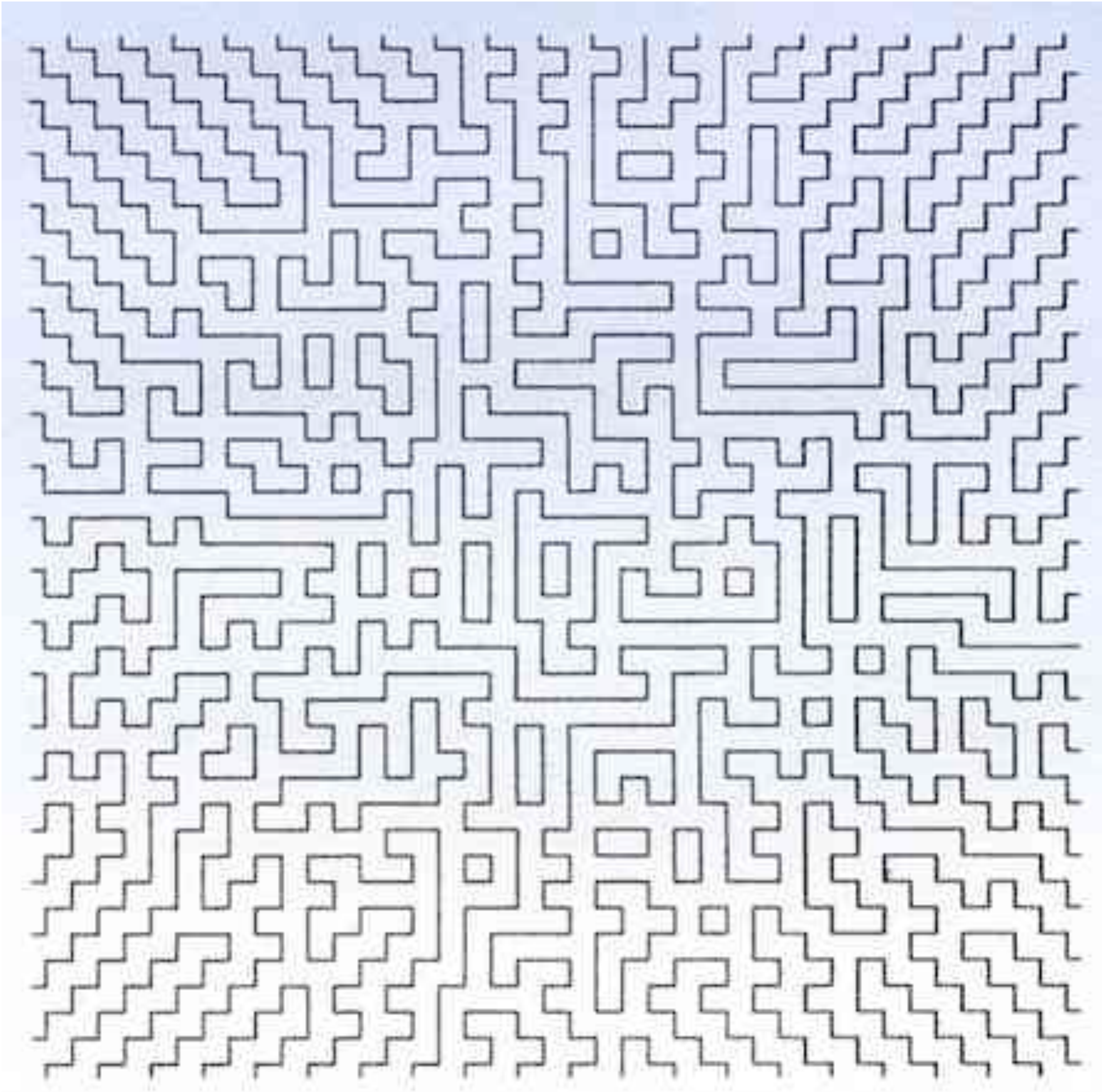




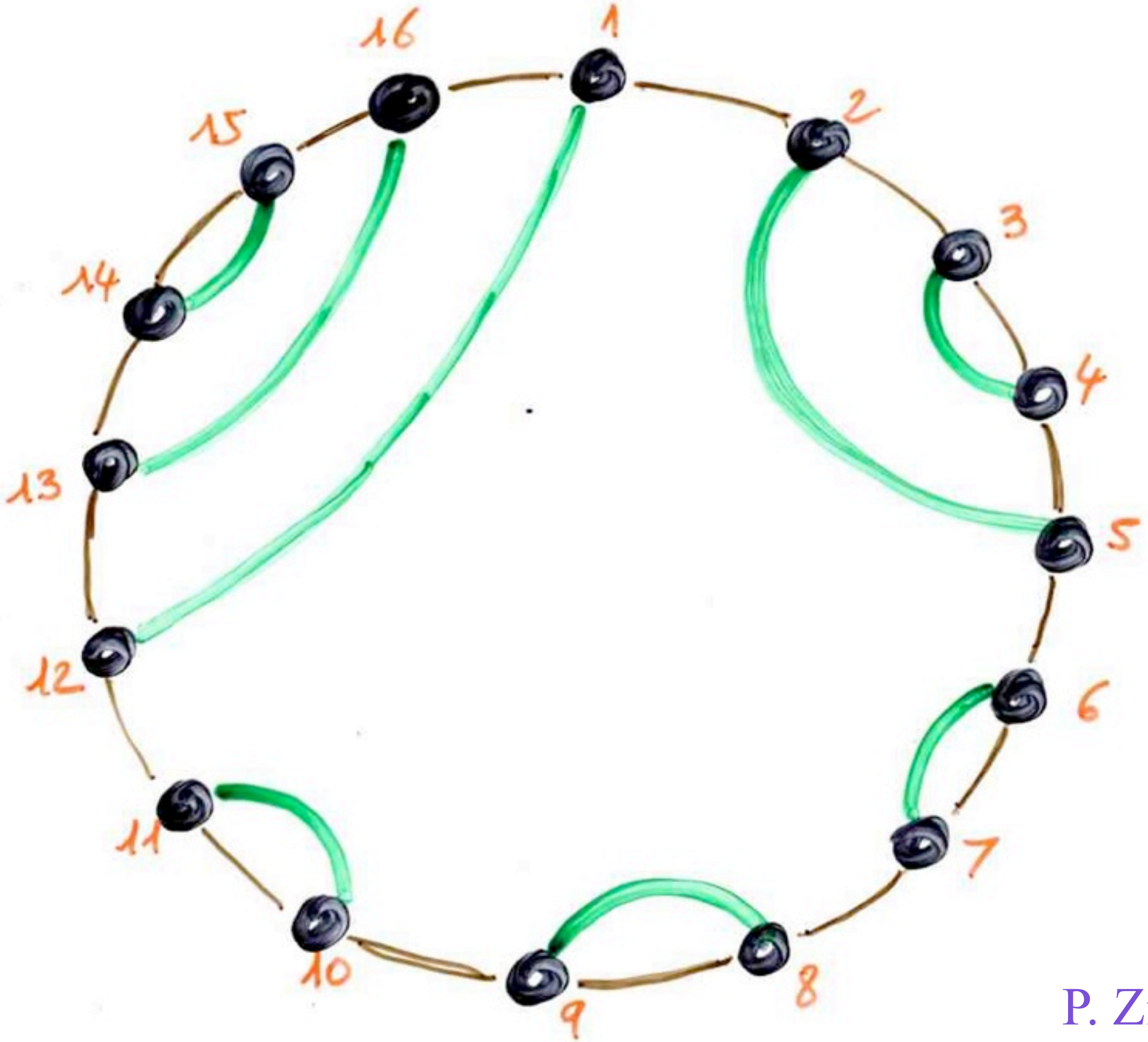
FPL
"Fully
Packed
Loop"
configuration



random
FPL



Razumov-Stroganov conjecture



stationary probabilities

Di Francesco, P. Zinn-Justin (2005)

Around the Razumov-Stroganov conjecture

Philippe Di Francesco, Paul Zinn-Justin (2005 - 2009)

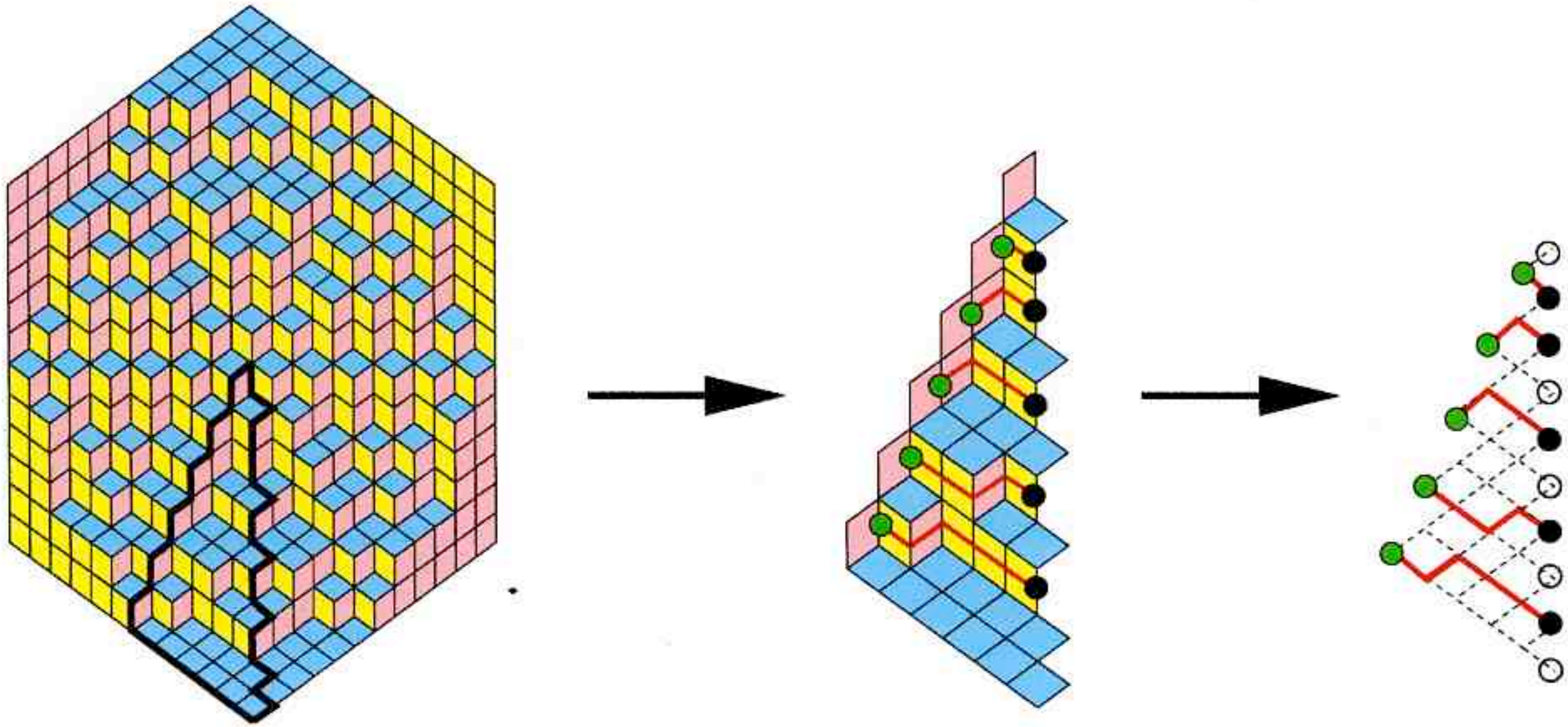
De Gier, Pyatov (2007)

Knizhnik - Zamolodchikov
equation

qKZ

TSSCPP

ASM



Di Francesco (2006)

ASM

1-, 2-, 3- enumeration $A_n(x)$

Colomo, Pronco, (2004)

Hankel determinants

(continuous) Hahn, Meixner-Pollaczek,
(continuous) dual Hahn orthogonal polynomials

Ismail, Lin, Roan (2004)
XXZ spin chains and Askey-Wilson operator

Schubert and Grothendick polynomials
Lascoux, Schützenberger

Razumov - Stroganov
(ex)- conjecture 2000-2001

proof by :
L. Cantini and A.Sportiello (March 2010)
arXiv: 1003.3376 [math.CO]
completely combinatorial proof

"The cellular Ansatz"

combinatorial
objects
on a 2d lattice

representation
by operators

Physics

data structures
"histories"

bijections

orthogonal
polynomials

"normal ordering"

UD = DU + Id

Weyl-Heisenberg

DE = qED + E + D

PASEP

towers placements
permutations ⟷ RSK pairs of Tableaux Young

tableaux alternatifs ⟷ permutations
Laguerre histories

Q-tableaux
ex: ASM,
(alternating sign matrices)
FPL(fully packed loops)
tilings, 8-vertex

quadratic algebra Q

?

commutations
rewriting rules
planarisation

planar
automata

Koszul algebras
duality

more details, see the diaporamas:

Cours XGV, Universidad de Talca «The Cellular Ansatz»
(December 2010 - January 2011)
Combinatorics and interactions (with physics) (24h)

also: **4 talks in India**
January-March 2012

or «**Petite Ecole de Combinatoire**»
LaBRI, Fall 2011, Spring 2012

and **2 videos:**
Newton Institute 23 April 2008
http://www.newton.cam.ac.uk/webseminars/pg+ws/2008/csm/csmw04/0423/viennot/
IMSc, Chennai, 1 March 2012
http://youtu.be/u44ZmEU8svY

accessible from the websites:

http://www.labri.fr/perso/viennot/

Recherche, cv, publications, exposés, diaporamas, livres, petite école, photos: voir ma page personnelle ici

Vulgarisation scientifique voir la page de l'association Cont'Science

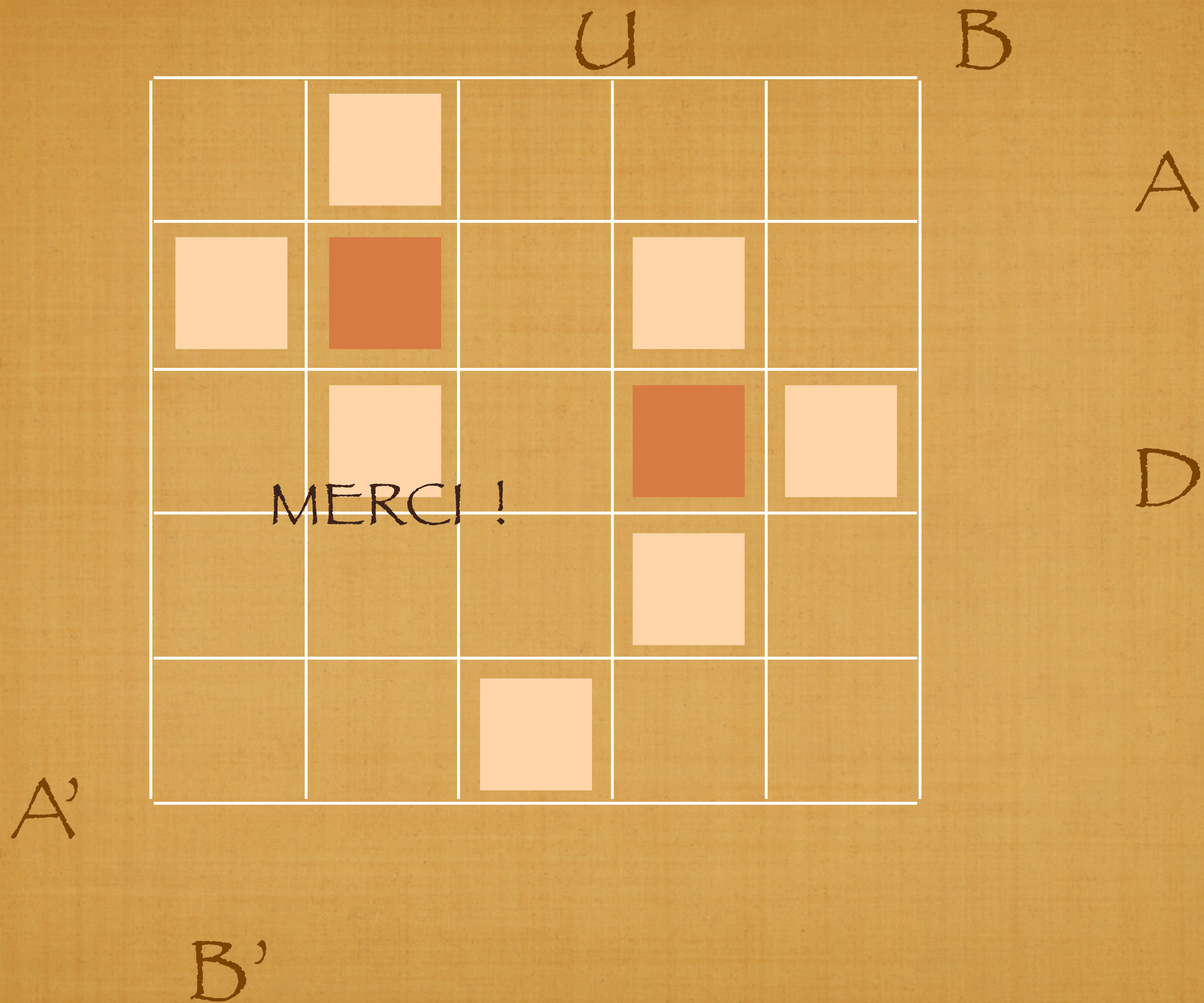
http://web.me.com/xgviennot/Xavier_Viennot/

http://web.me.com/xgviennot/Xavier_Viennot2/

Cours XGV
Universidad de Talca

(December 2010 - January 2011)
24 h

Combinatorics and interactions
(with physics)
«The Cellular Ansatz»


U
B
A
MERCI !
D
A'
B'