Algèbres d'opérateurs
et
Physique combinatoire

(part 2)

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colloquium de l'IMJ
Institut mathématiques de Jussieu

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CNRS, LaBRI, Bordeaux

The cellular Ansatz
second part:

guided construction of a bijection
from a combinatorial representation
of the algebra Q

The algebra UD=DU+Id

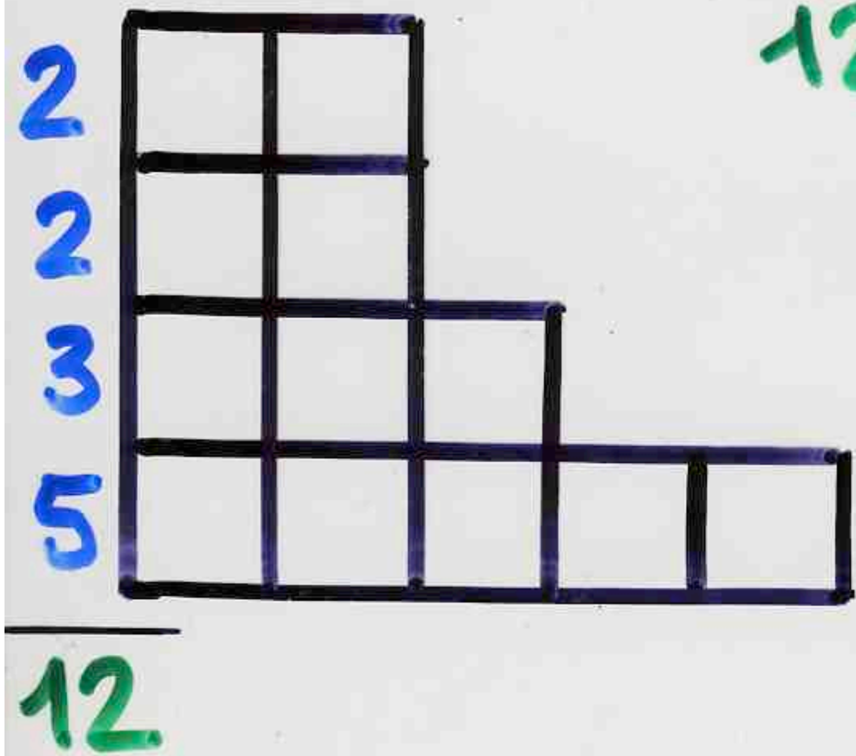
The RSK correspondence

Robinson-Schensted-Knuth

G. de B. Robinson, 1938

C. Schensted, 1961

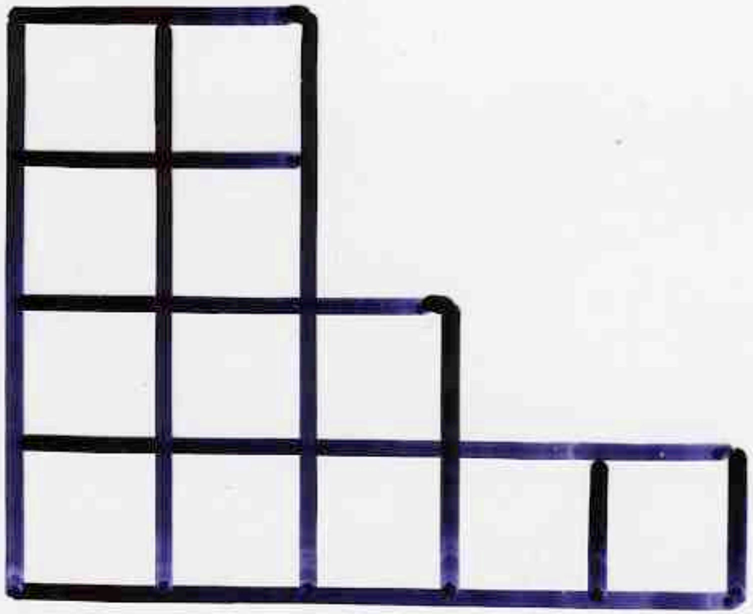
$$12 = n = 5 + 3 + 2 + 2$$

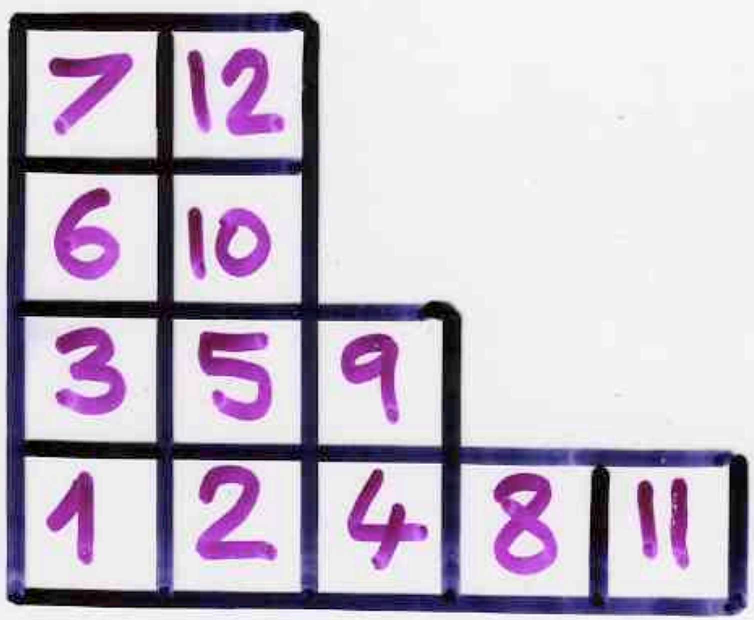


Ferrers

diagram

Partition of n





Young tableau

G fini

$$|G| = \sum_{\varphi} \deg^2(\varphi)$$

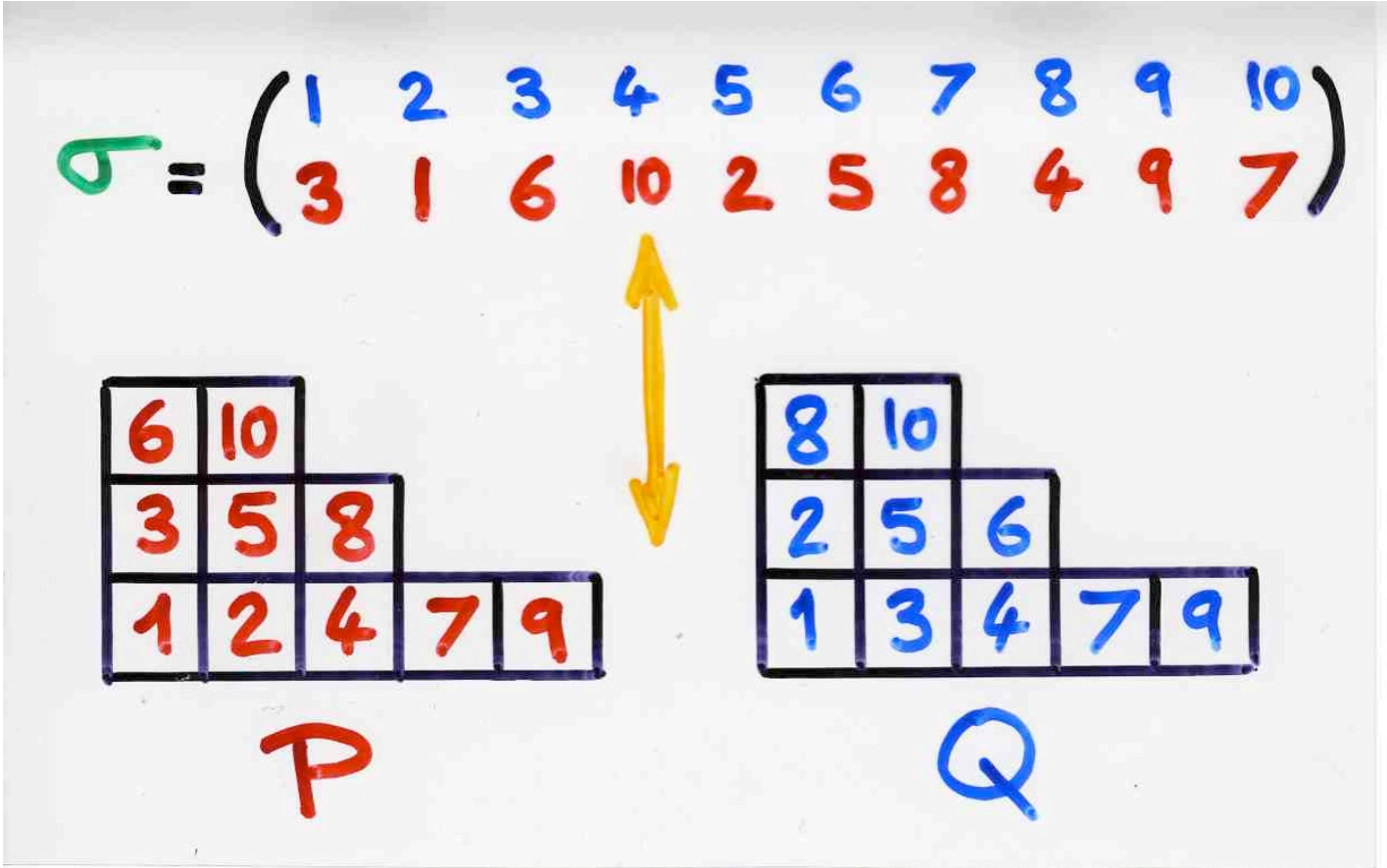
φ représentation irréductible

$$n! = \sum_{\lambda} f_{\lambda}^2$$

$n!$: ordre groupe fini $\mathfrak{S}_n$

λ : représentations irréductibles

f_{λ} : degré



The Robinson-Schensted correspondence between permutations and pair of (standard) Young tableaux with the same shape

RSK with Schensted's insertions

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1					

3					

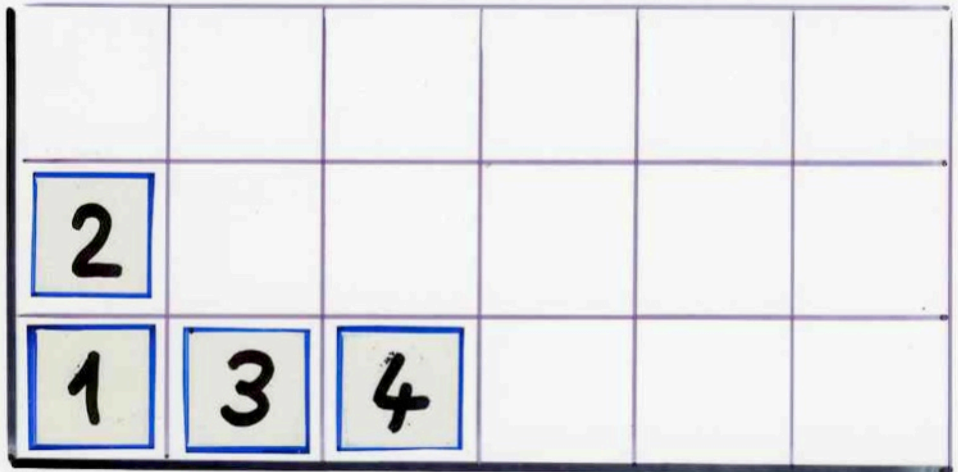
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

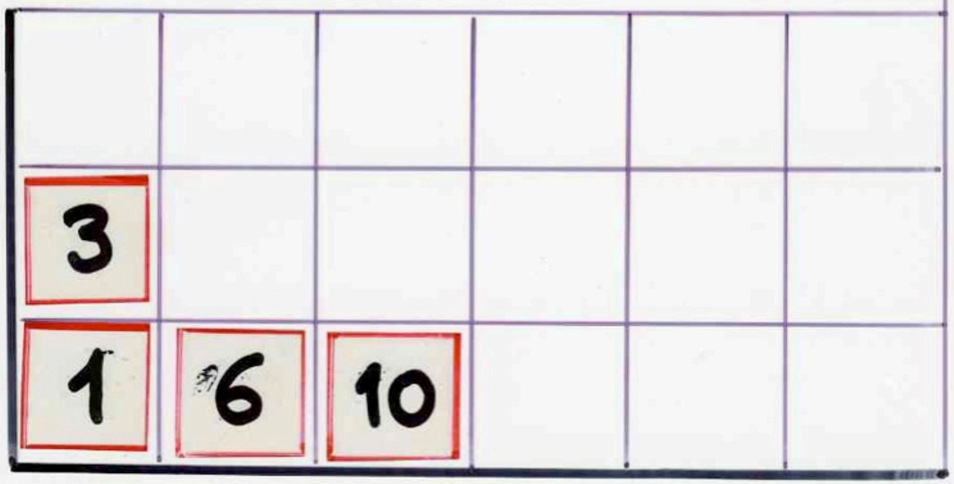
2					
1					

3					
1					

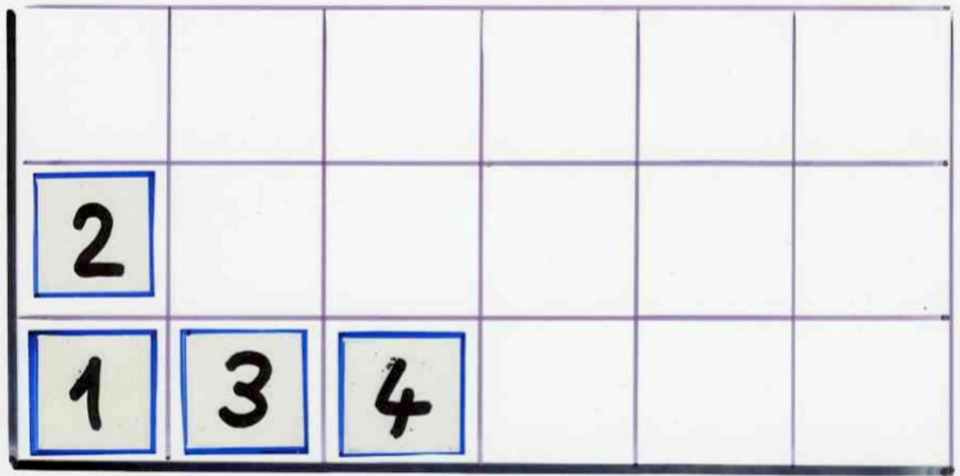
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

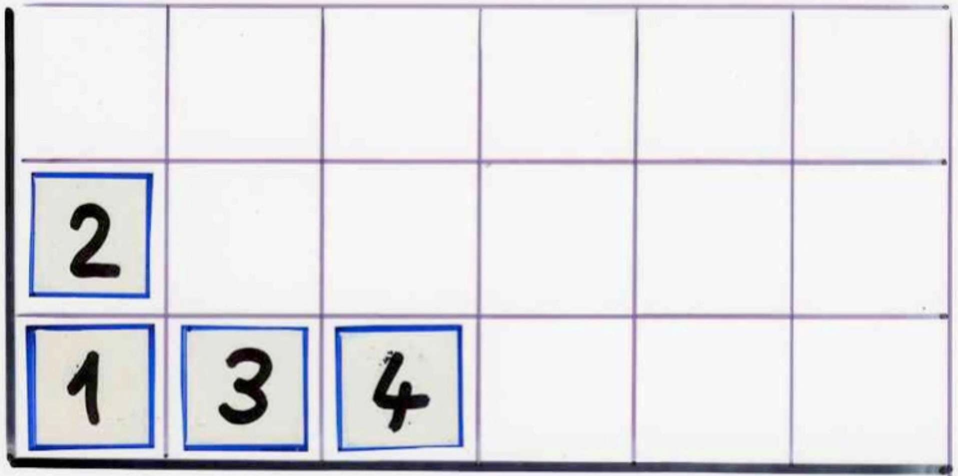


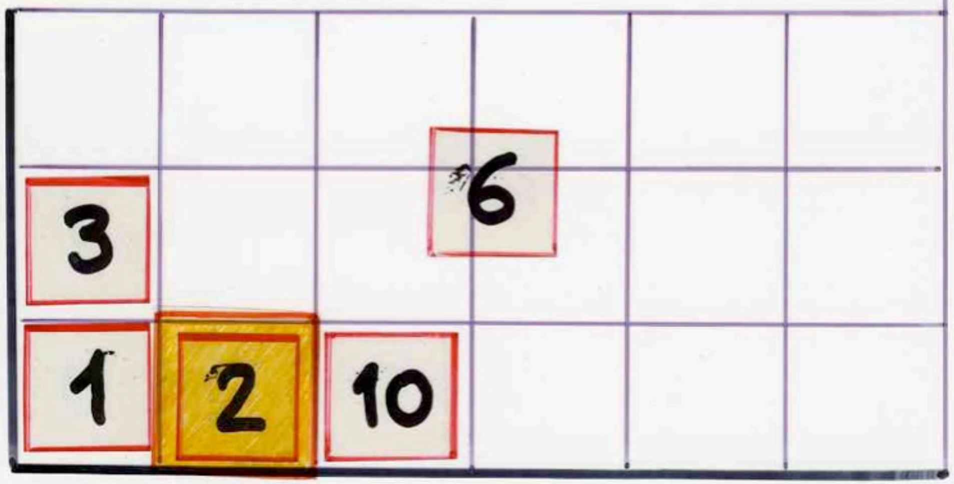


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

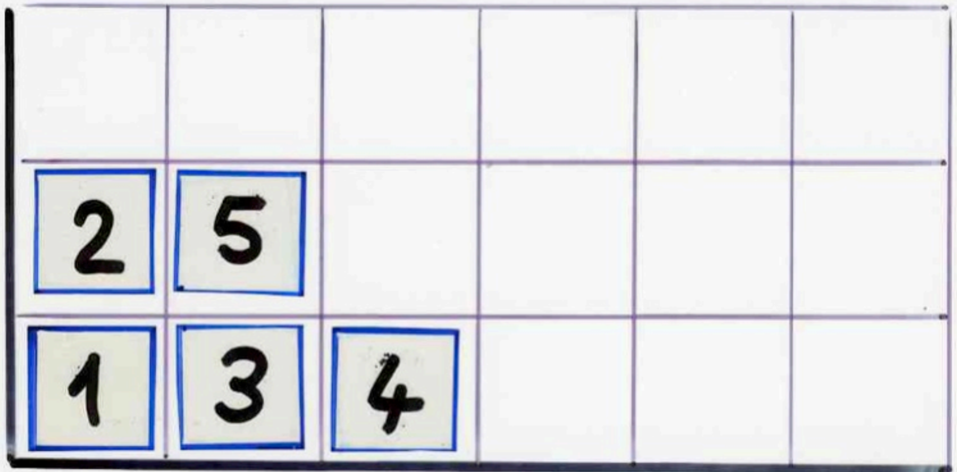


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

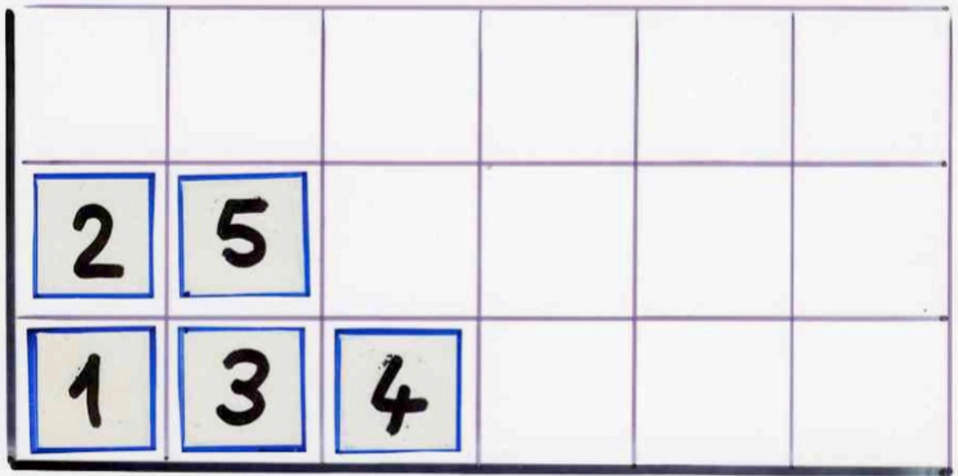


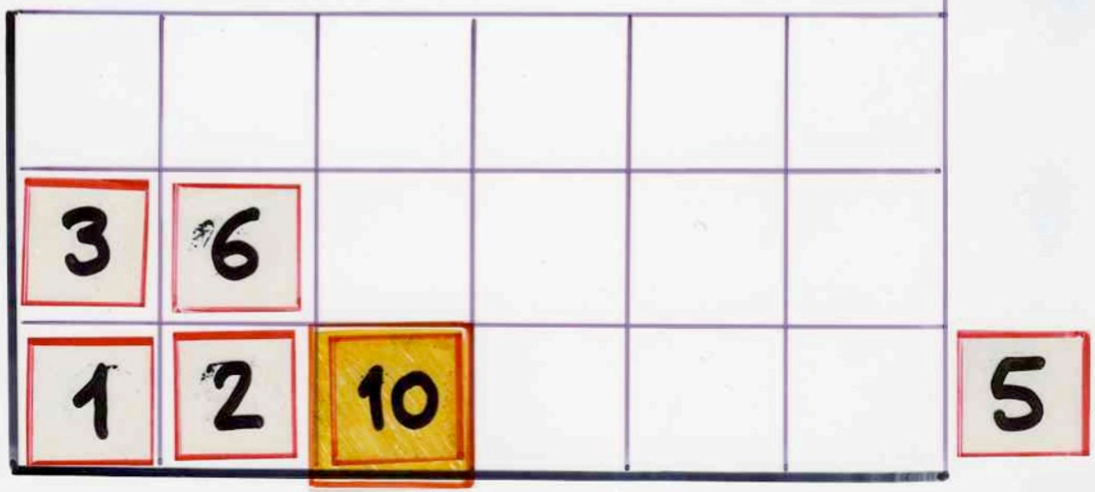


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

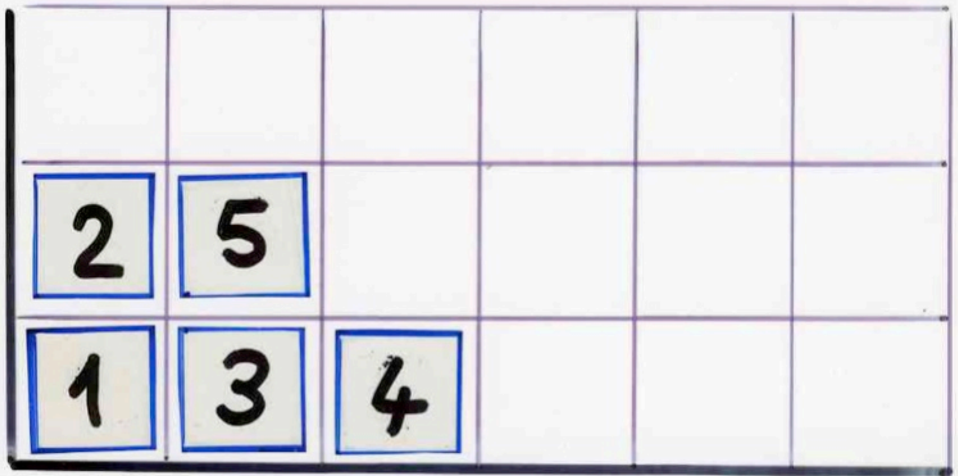


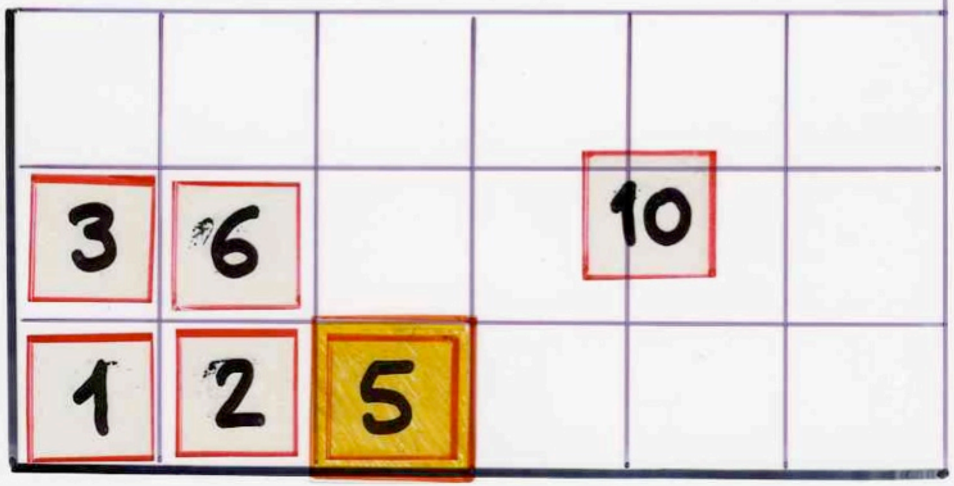
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4			

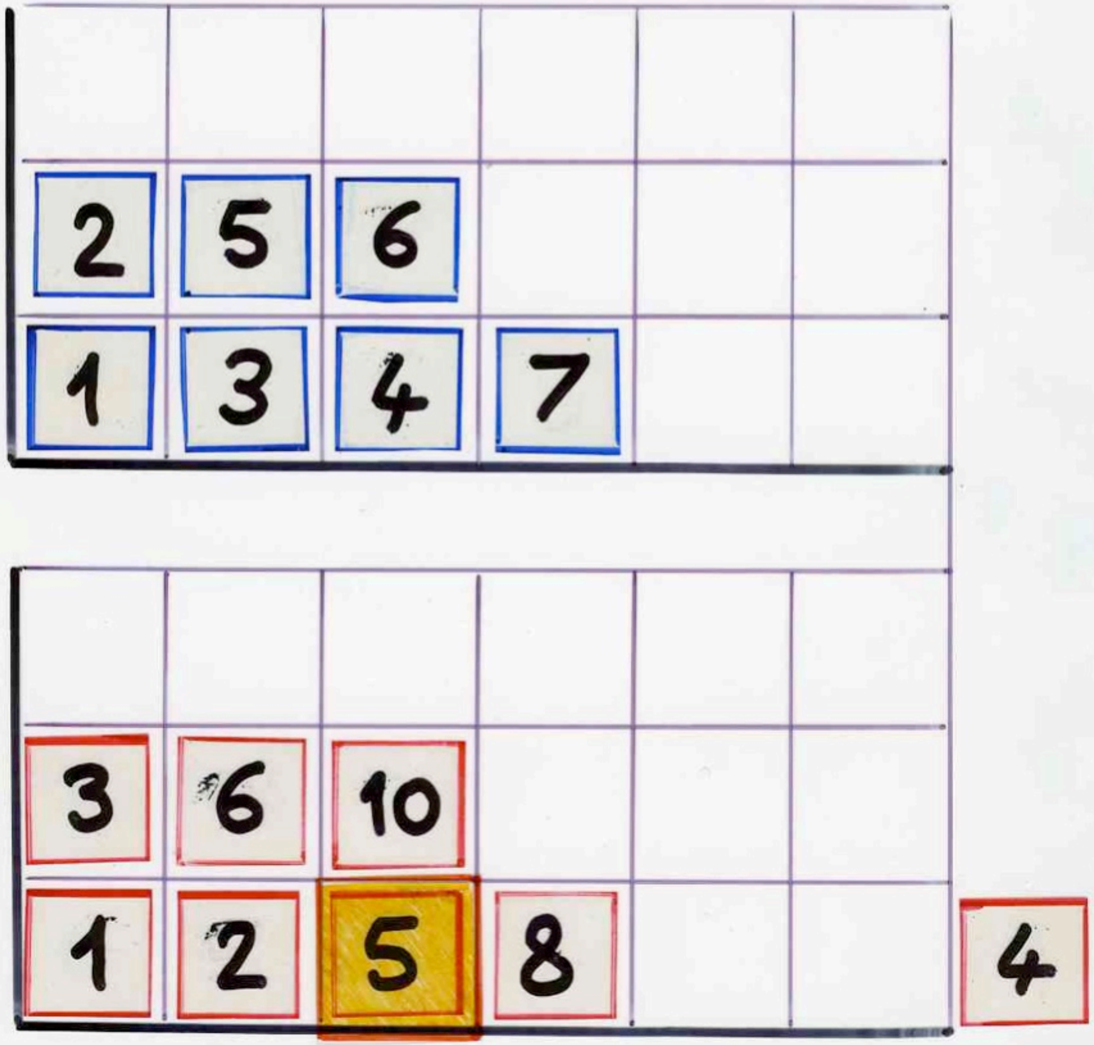
3	6	10			
1	2	5			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

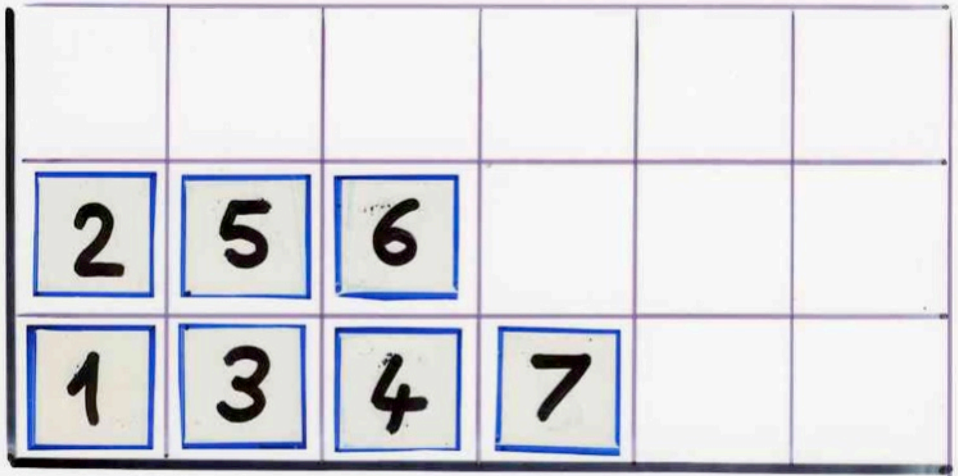
2	5	6			
1	3	4	7		

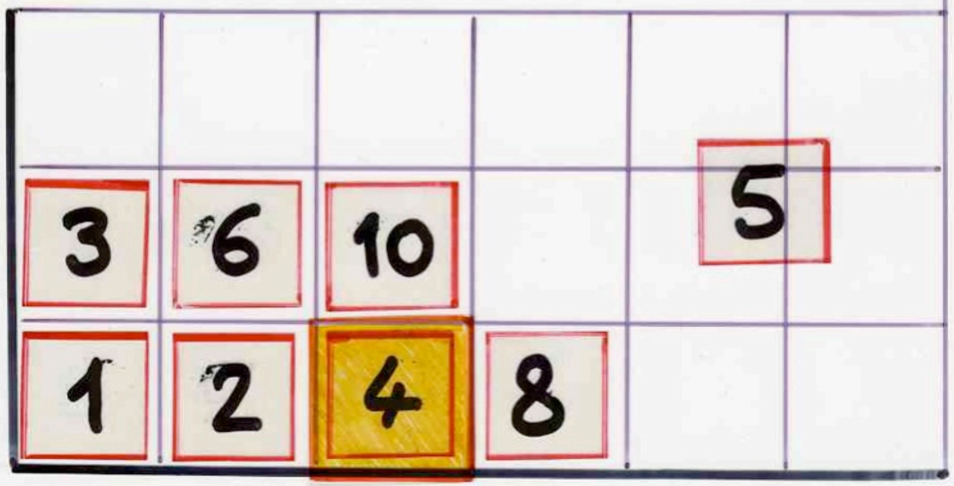
3	6	10			
1	2	5	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

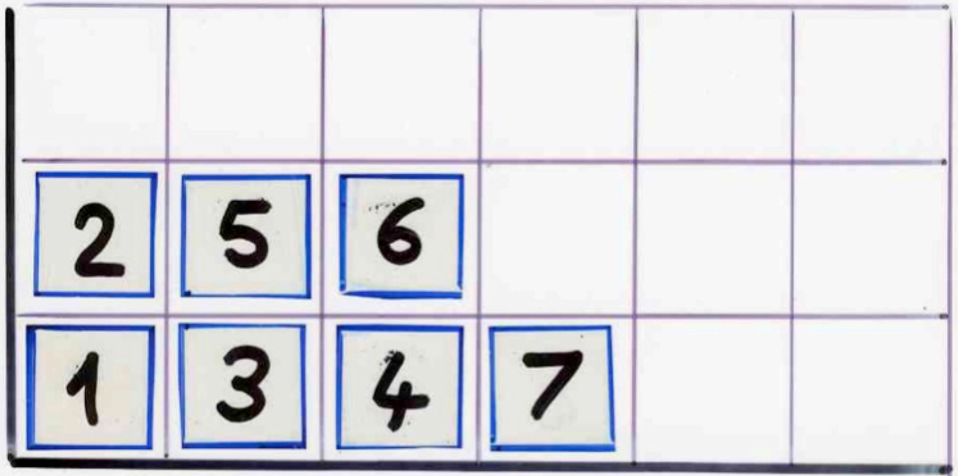


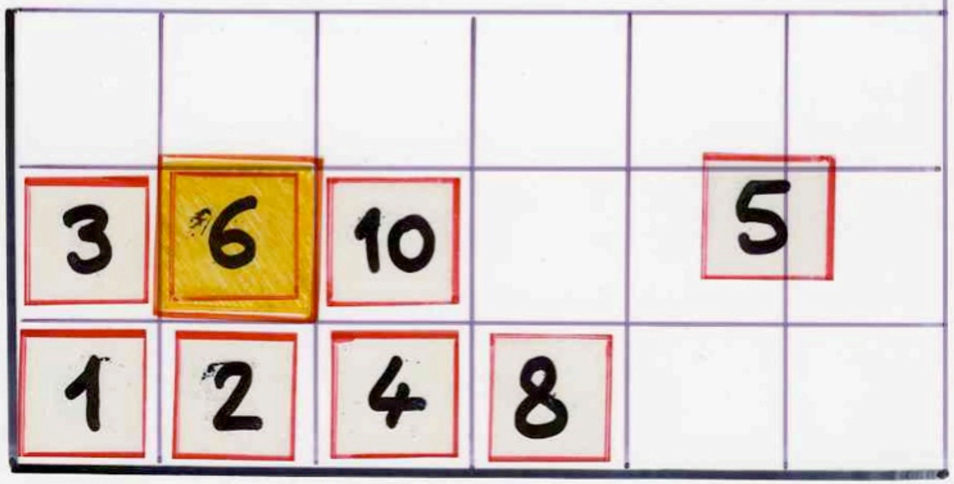
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



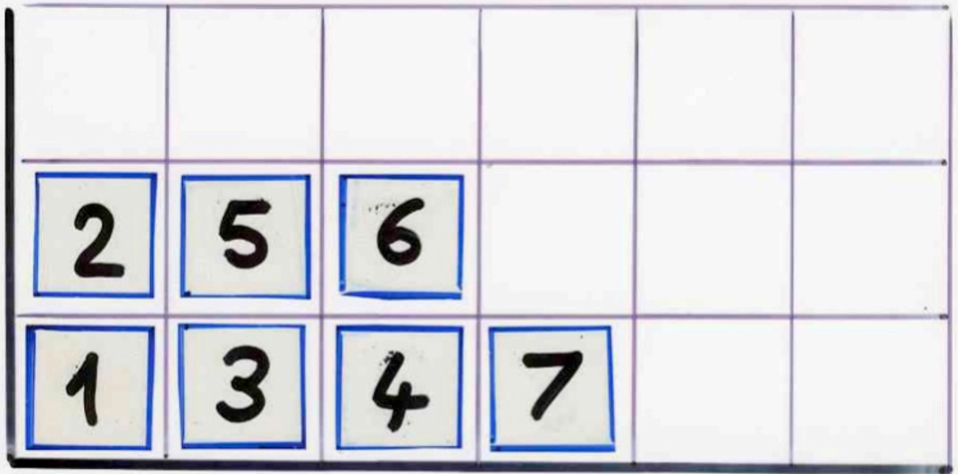


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7		

6					
3	5	10			
1	2	4	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

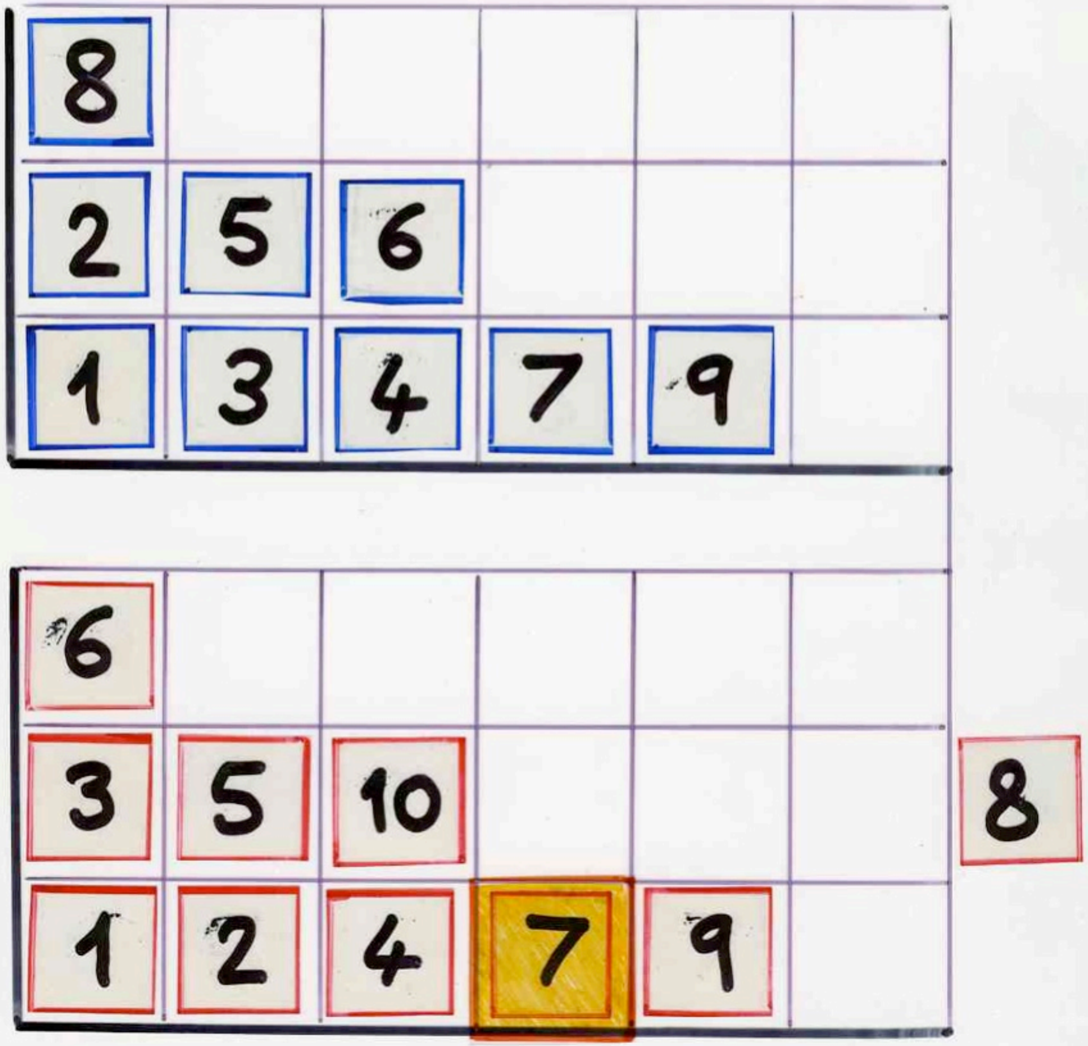
6					
3	5	10			
1	2	4	8	9	

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

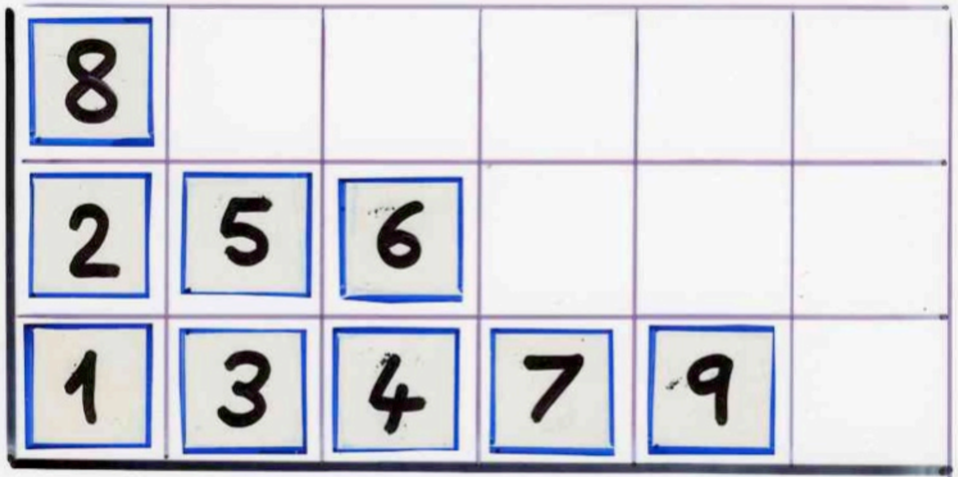
8					
2	5	6			
1	3	4	7	9	

6						
3	5	10				
1	2	4	8	9		7

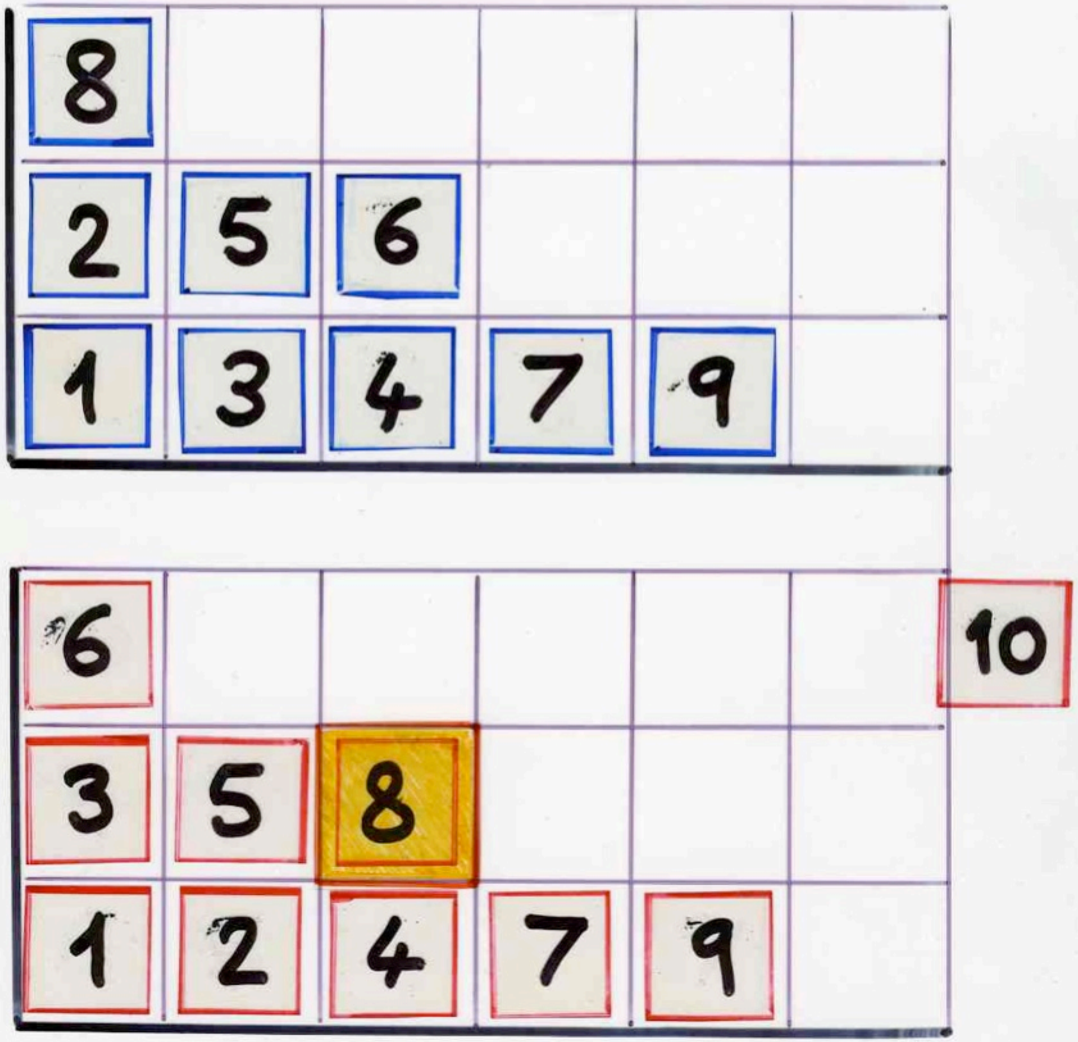
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

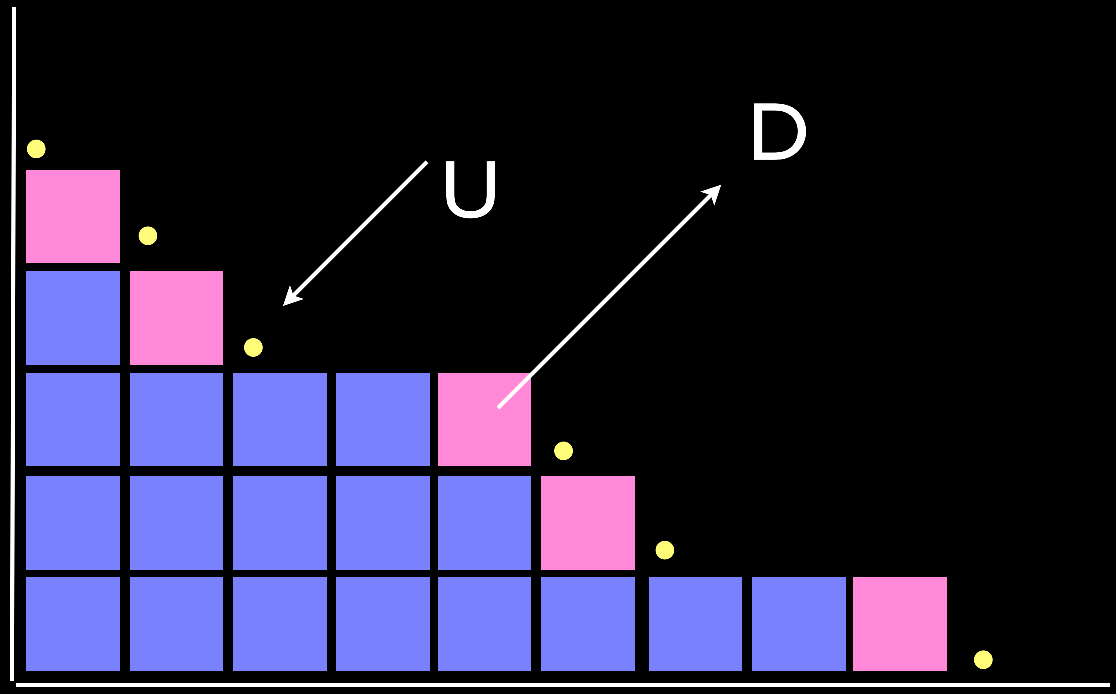
6	10				
3	5	8			
1	2	4	7	9	

representation of the operators U, D

Sergey Fomin

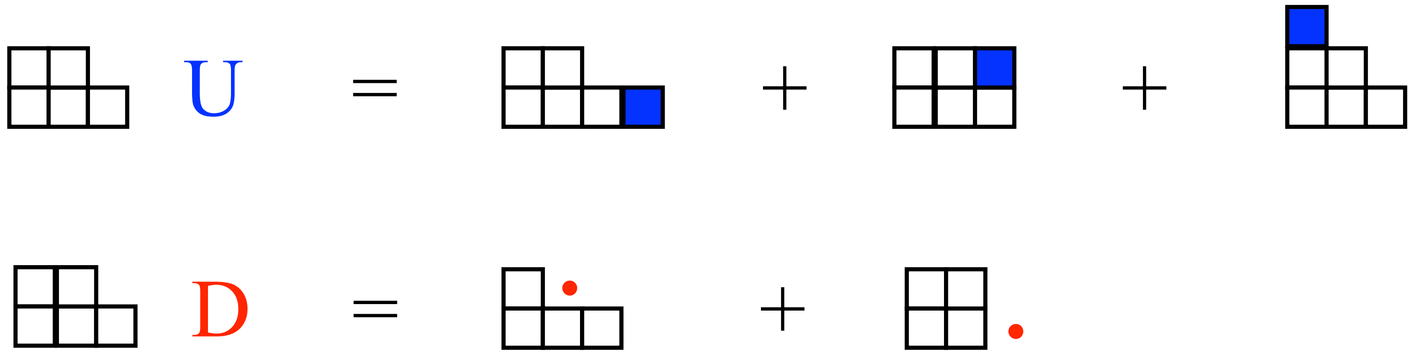
(with C. K.)

Operators U and D



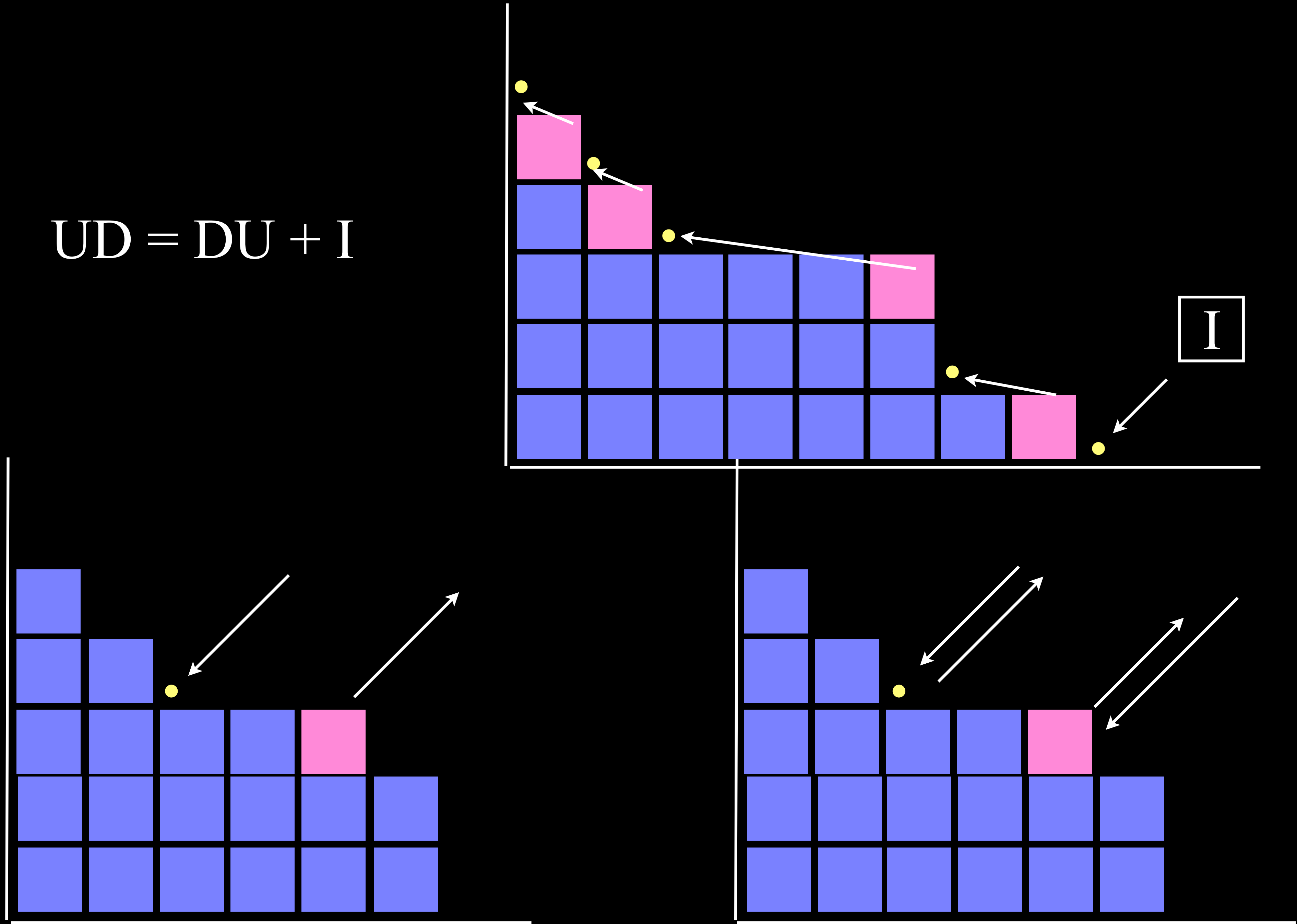
adding
or deleting
a cell in
a Ferrers
diagram

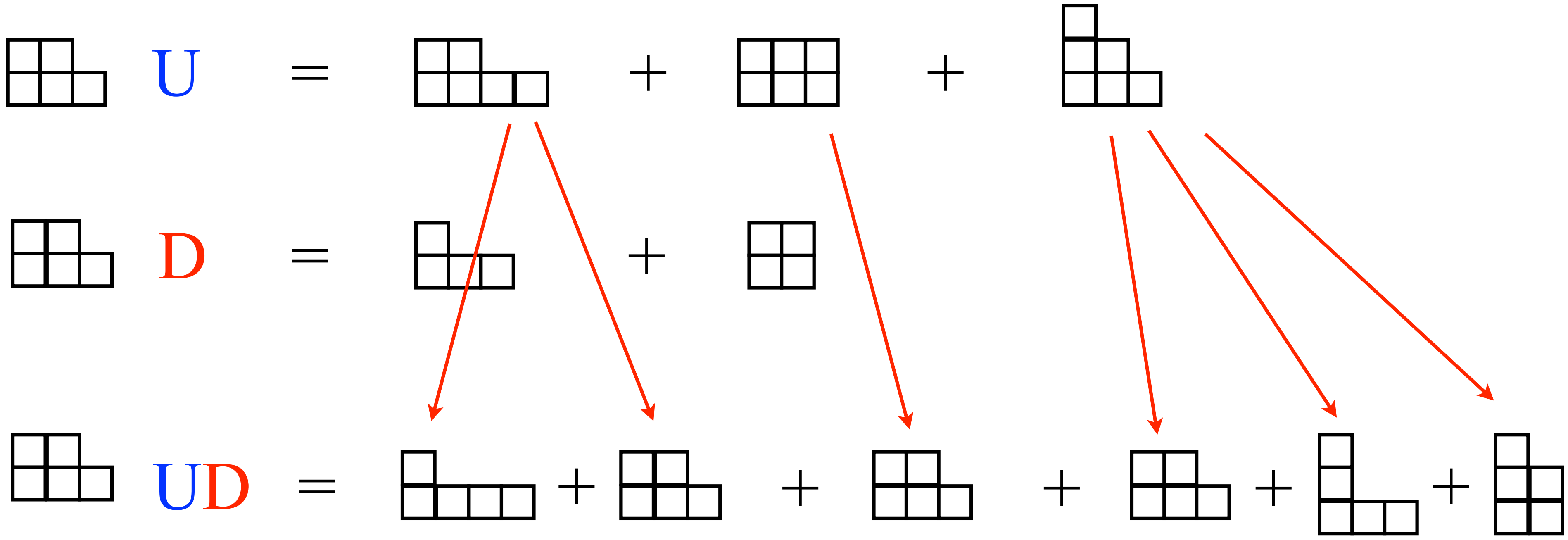
Young lattice

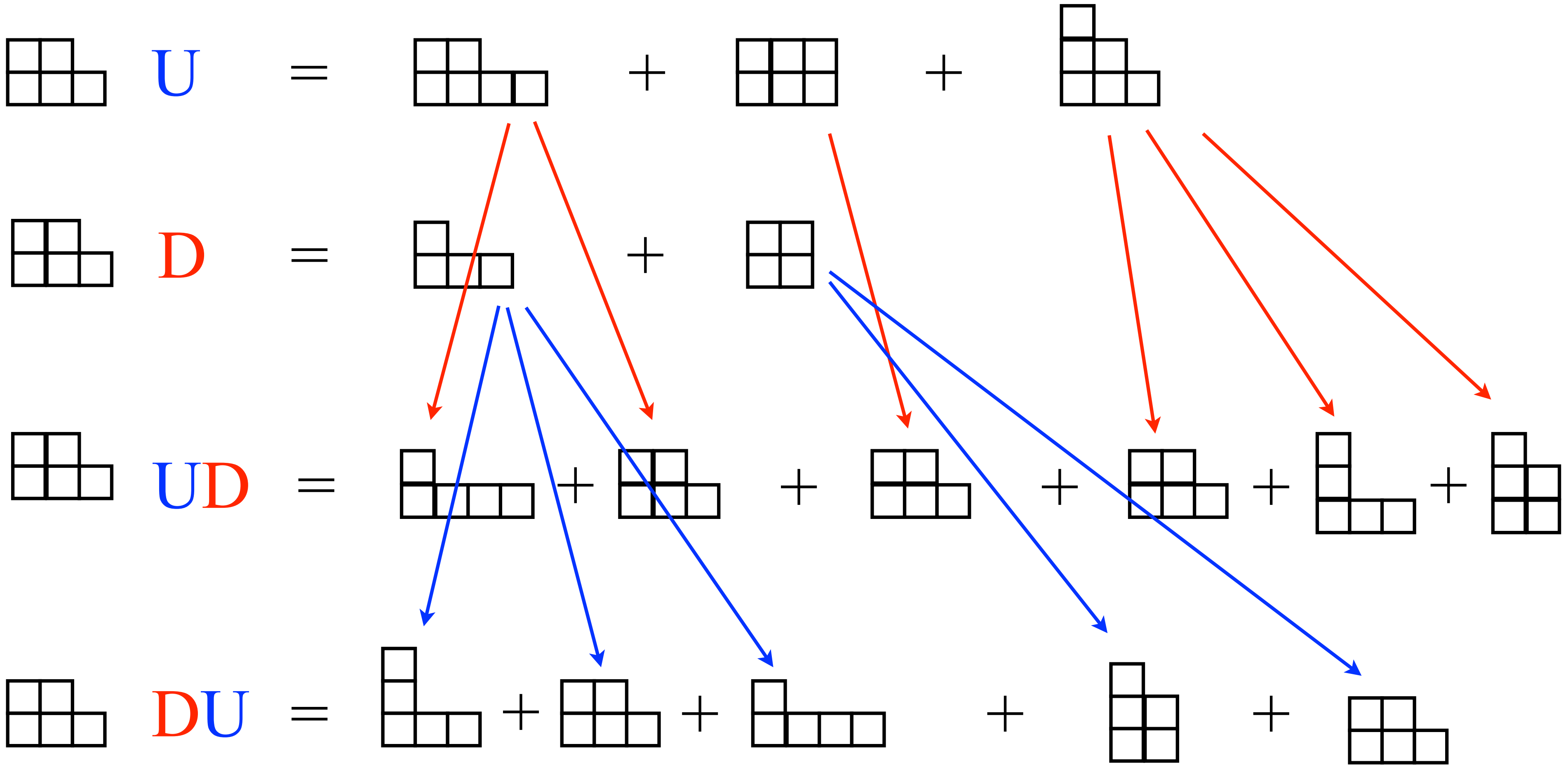

U
=
+
+
D
=
+

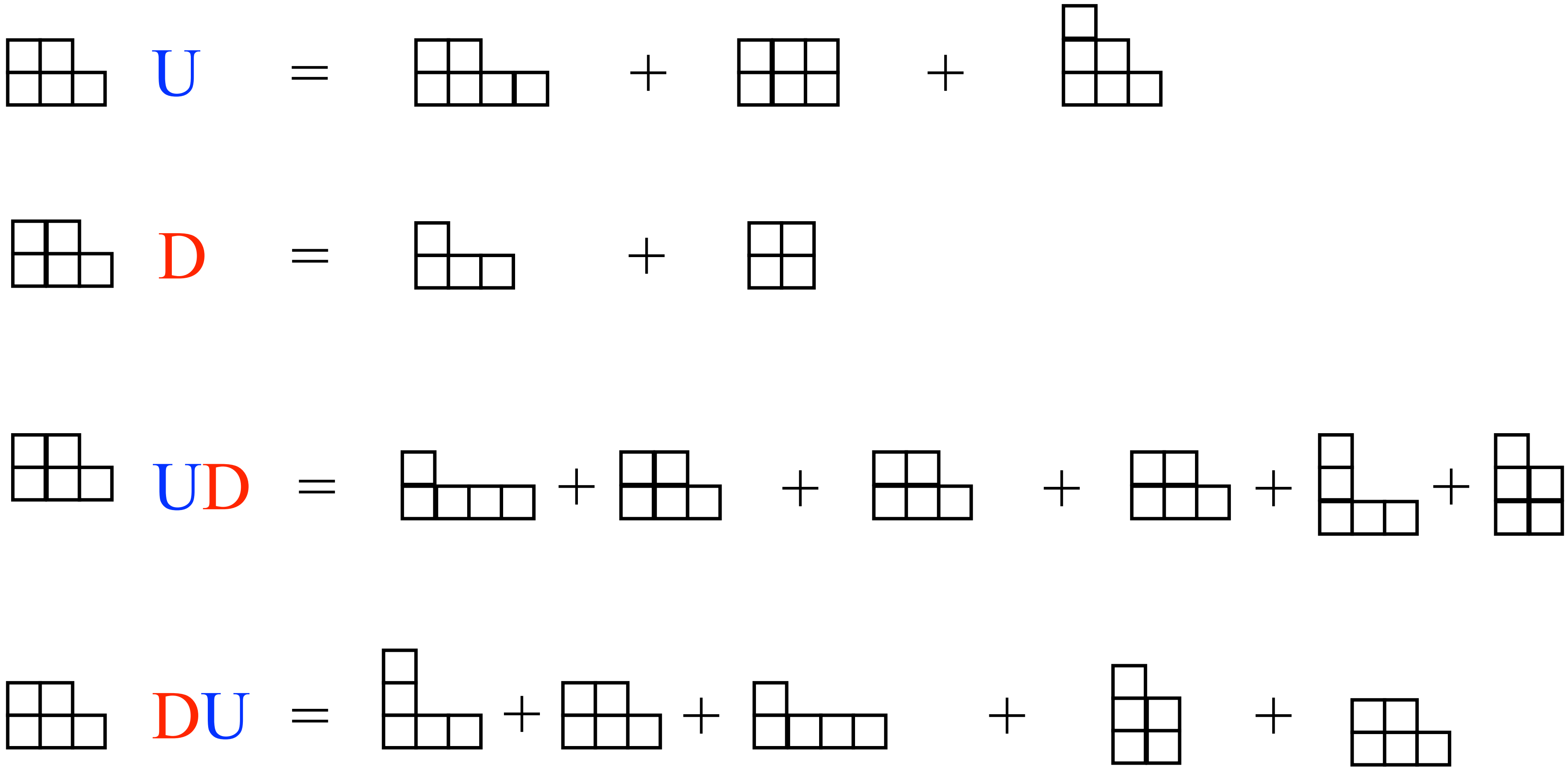
combinatorial “representation” of the
commutation relation $UD = DU + I$

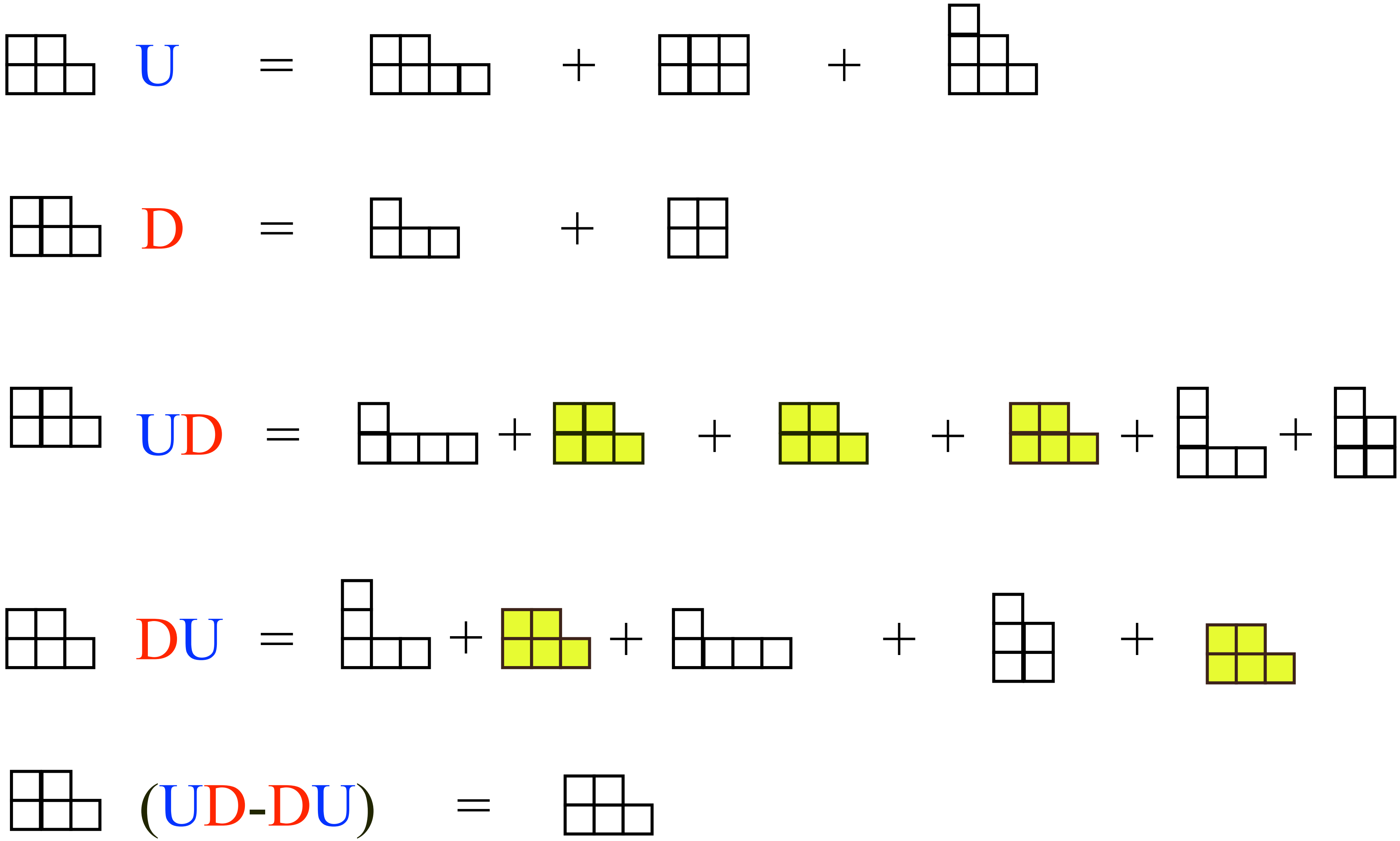
$$UD = DU + I$$



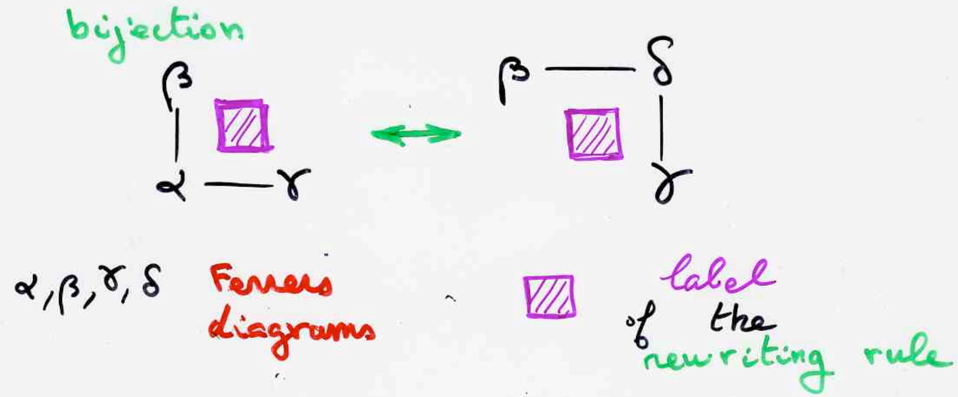

U
D
UD


U
D
UD
DU

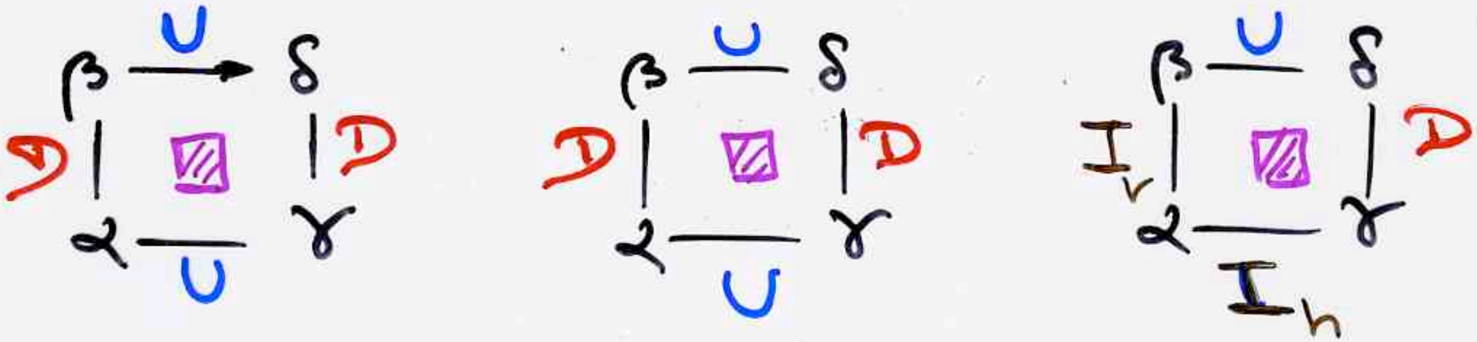

U
D
UD
DU


U
D
UD
DU
(UD-DU)

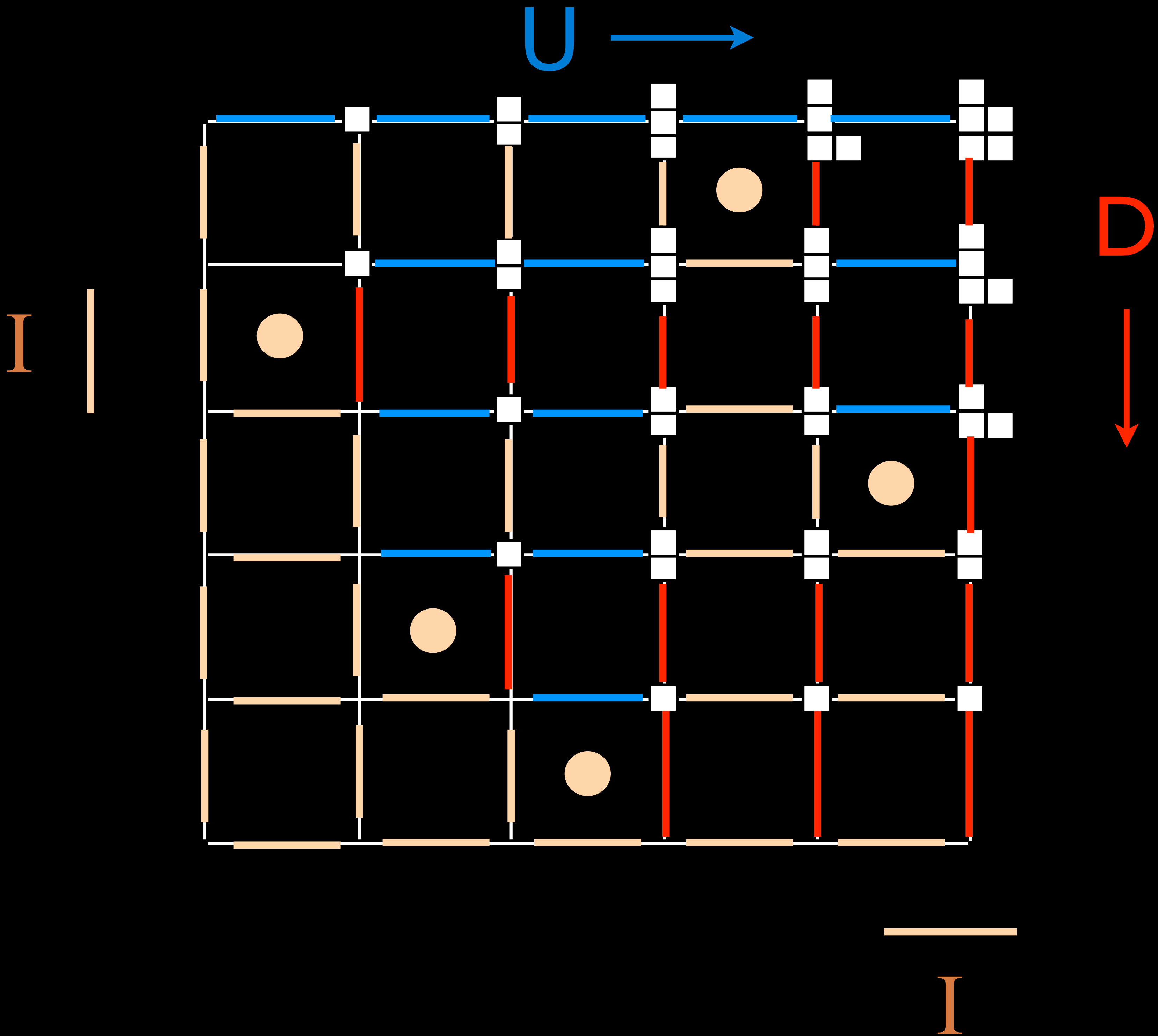
Commutation diagrams

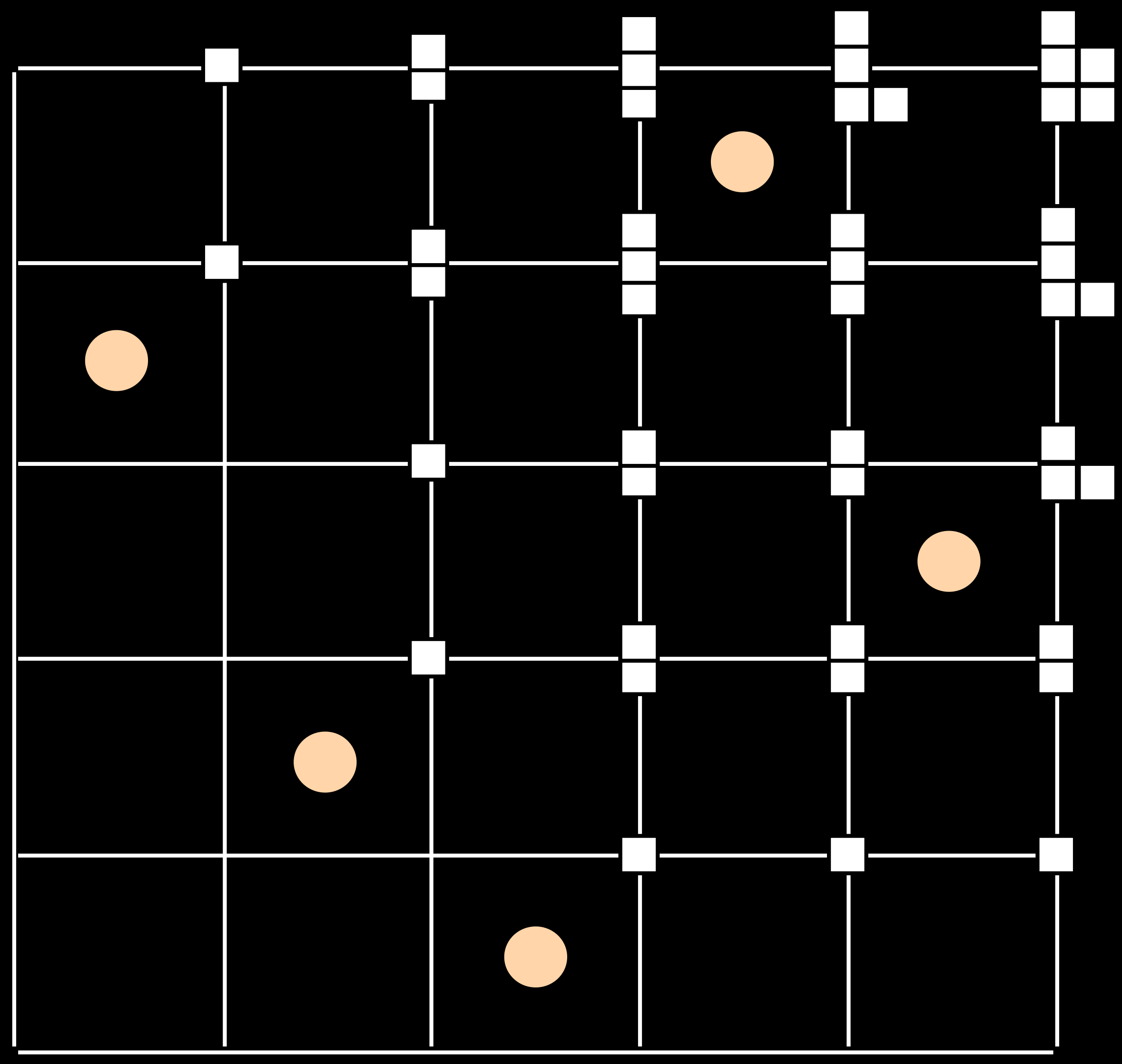


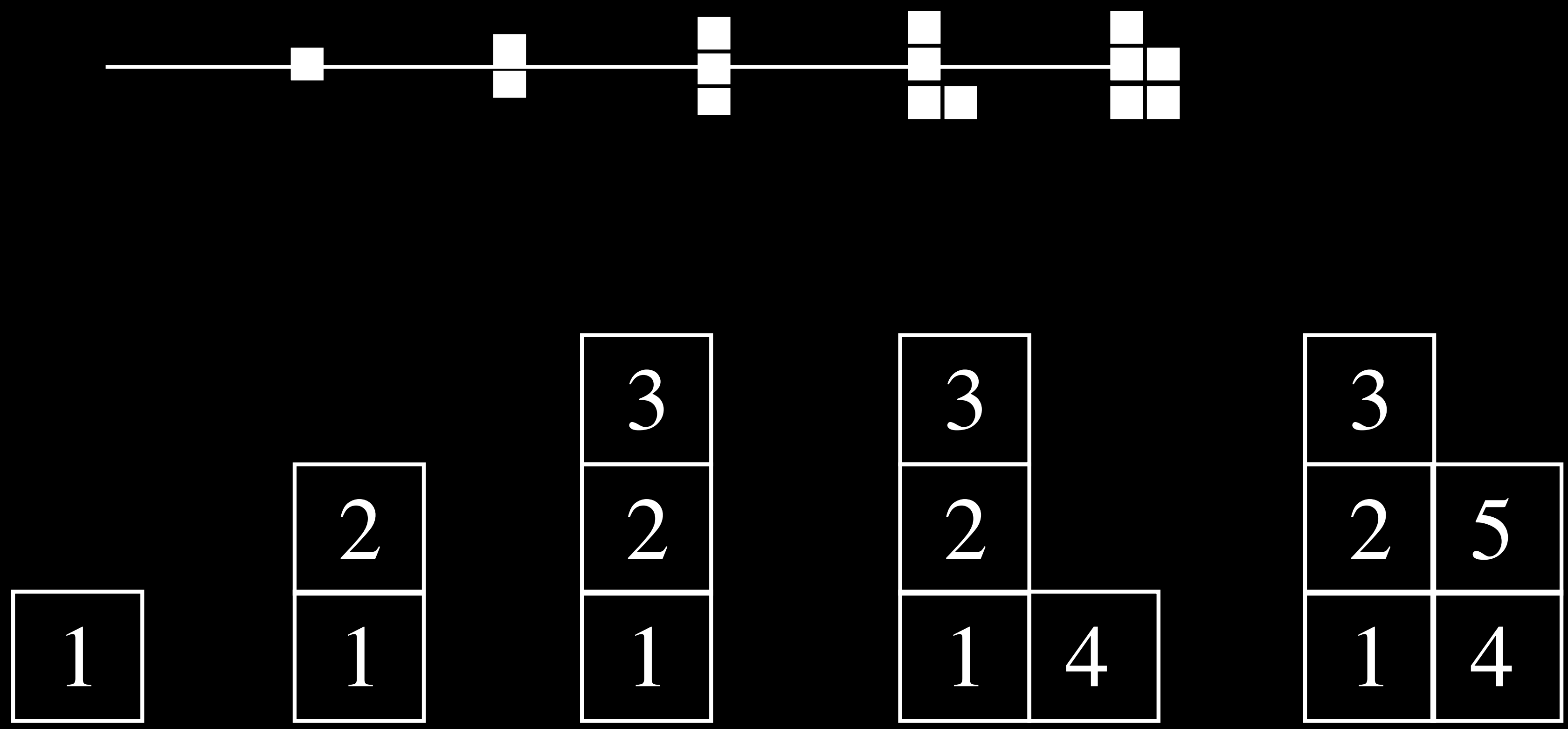
$$UD = DU + I_v I_h$$

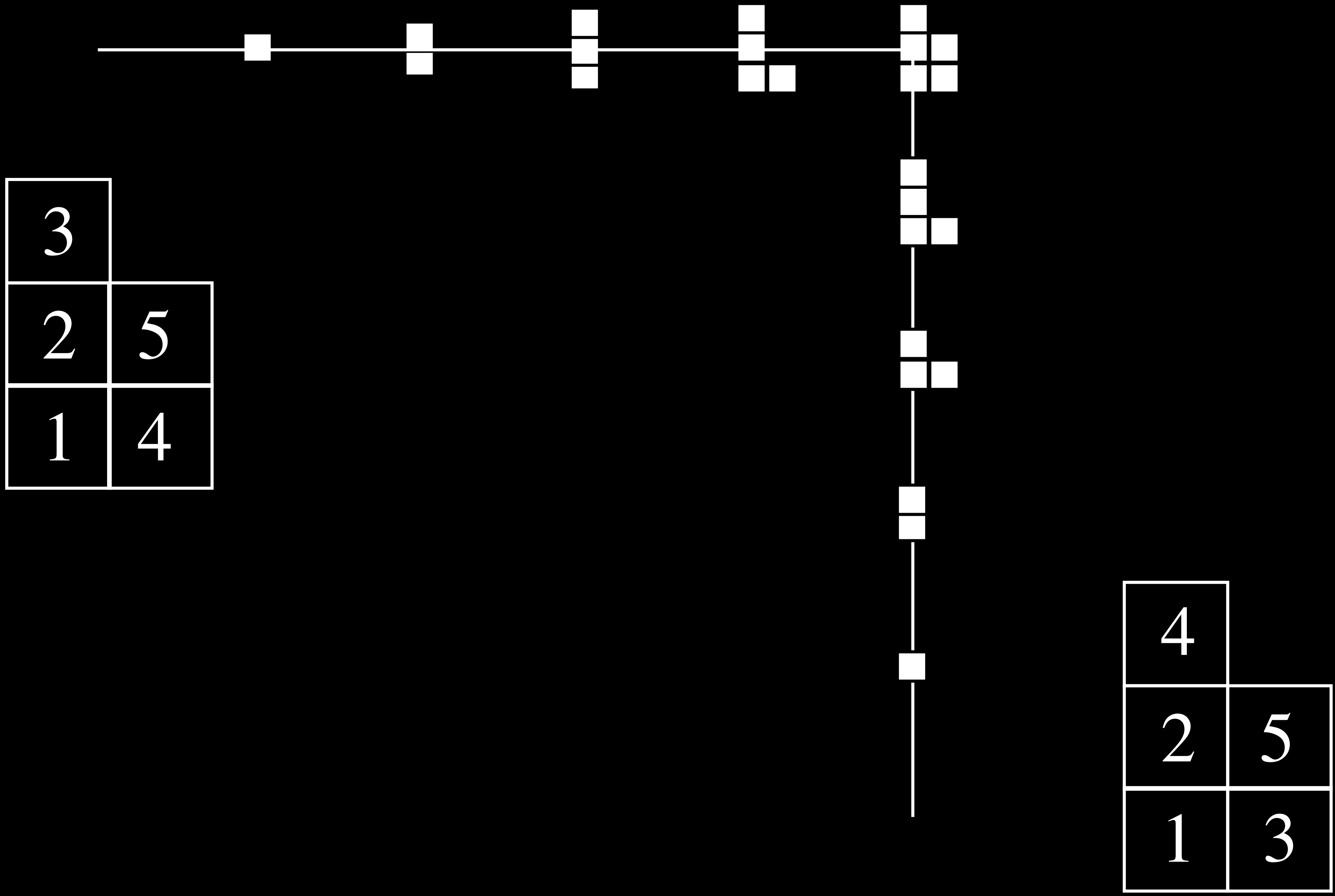


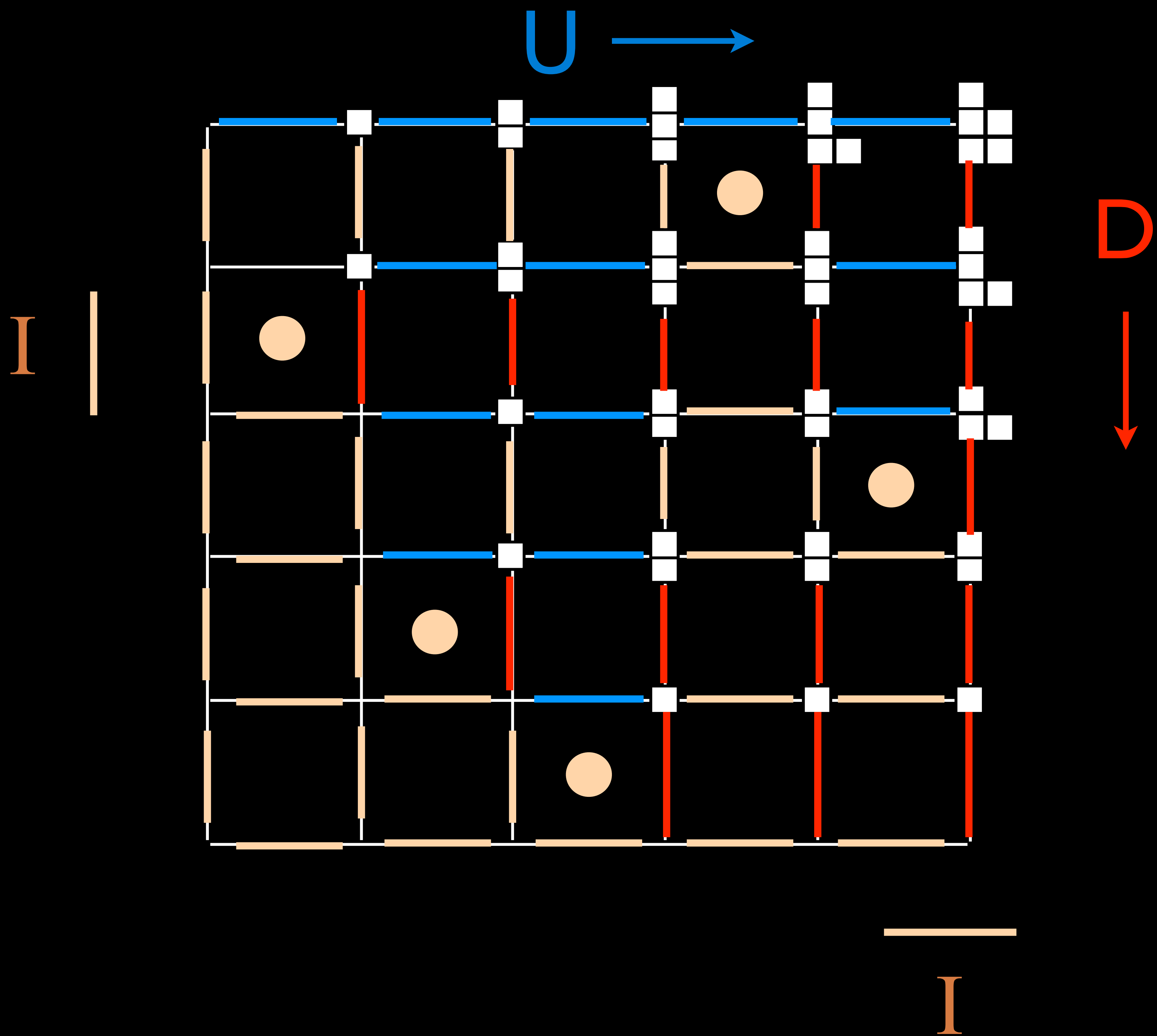
$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$


U
D
I
I



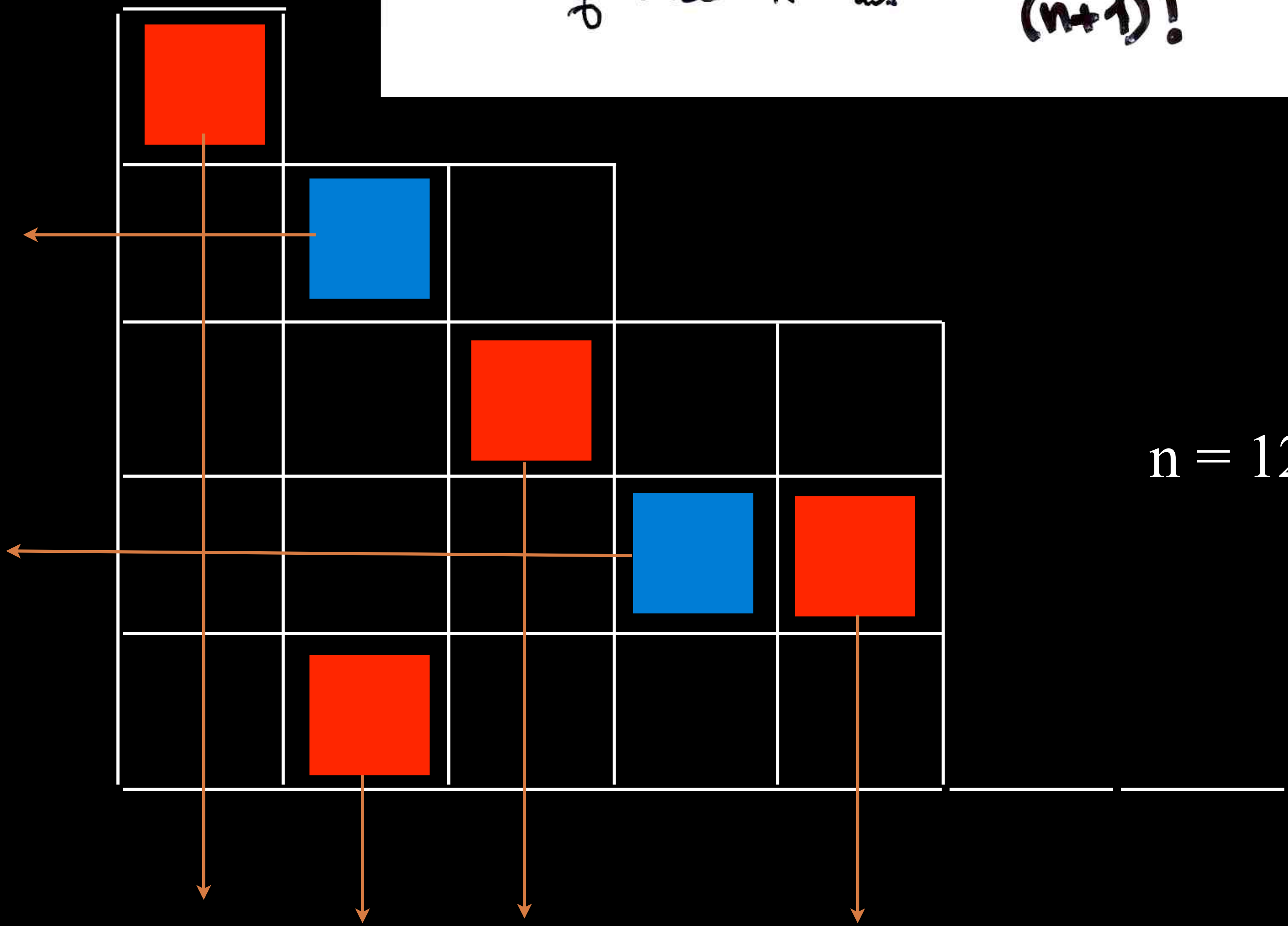

1
2
1
3
2
1
3
2
1 4
3
2 5
1 4


3
2 5
1 4
4
2 5
1 3


U
D
I
I

number of alternative tableaux

Prop. The number of alternative tableaux of size n is (n+1)!



n = 12

ex: n=2

combinatorial representation of the operators E and D

PASEP algebra

$$DE = qED + E + D$$

V vector space generated by B basis

B alternating words two letters {○, ●}

(no occurrence of ○○ or ●●)

4 operators A, S, J, K

4 operators A, S, J, K, $u \in B$

$$\langle u | A = \sum_{\text{letter } \circ \text{ of } u} v$$, v obtained by: $\circ \to \circ\bullet\circ$

$$\langle u | S = \sum_{\circ \text{ of } u} v$$ v obtained by: $\circ \to \bullet$ (and $\bullet\bullet \to \bullet$ $\bullet\bullet\bullet \to \bullet$)

$$\langle u | J = \sum_{\circ \text{ of } u} v$$ v, $\circ \to \bullet\circ$ (and $\bullet\bullet \to \bullet$)

$$\langle u | K = \sum_{\circ \text{ of } u} v$$ v, $\circ \to \circ\bullet$ (and $\bullet\bullet \to \bullet$)

$$\circ\bullet\circ\bullet \,|\, S = \bullet\circ\bullet + \circ\bullet$$

Lemma.

$$AS = SA + J + K$$

$$AK = KA + A$$

$$JS = SJ + S$$

$$JK = KJ$$

$$D = A + J$$

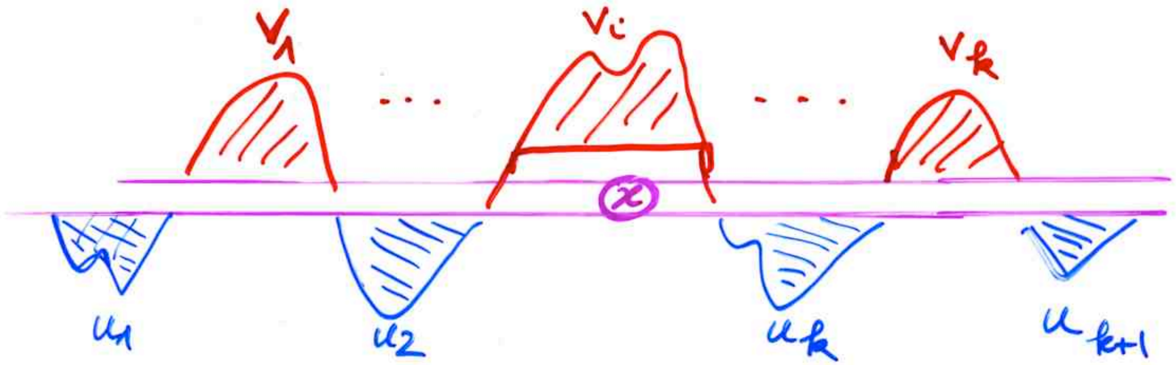
$$E = S + K$$

$$D = A + J$$
$$E = S + K$$

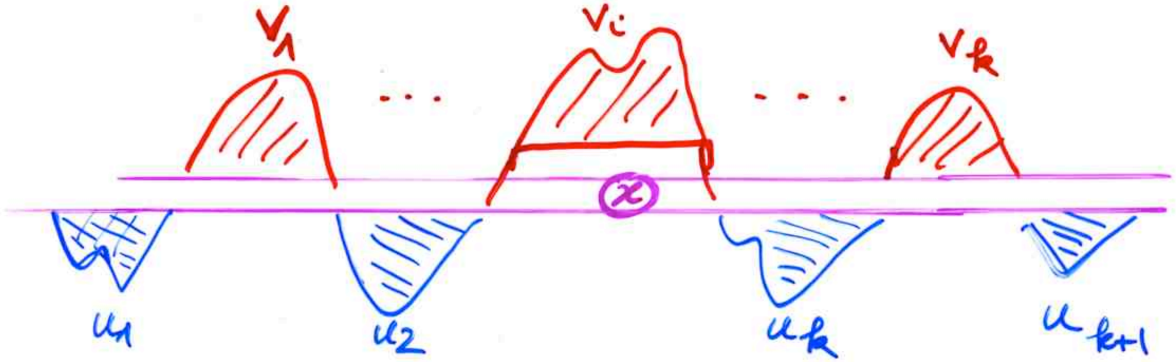
$$DE = (A+J)(S+K)$$
$$= AS + AK + JS + JK$$
$$= \underbrace{\underbrace{(SA + KA + SJ + KJ)}_{(S+K)(A+J)}}_{ED} + \underbrace{J + K + A + S}_{E + D}$$

Bijection
Laguerre histories
permutations

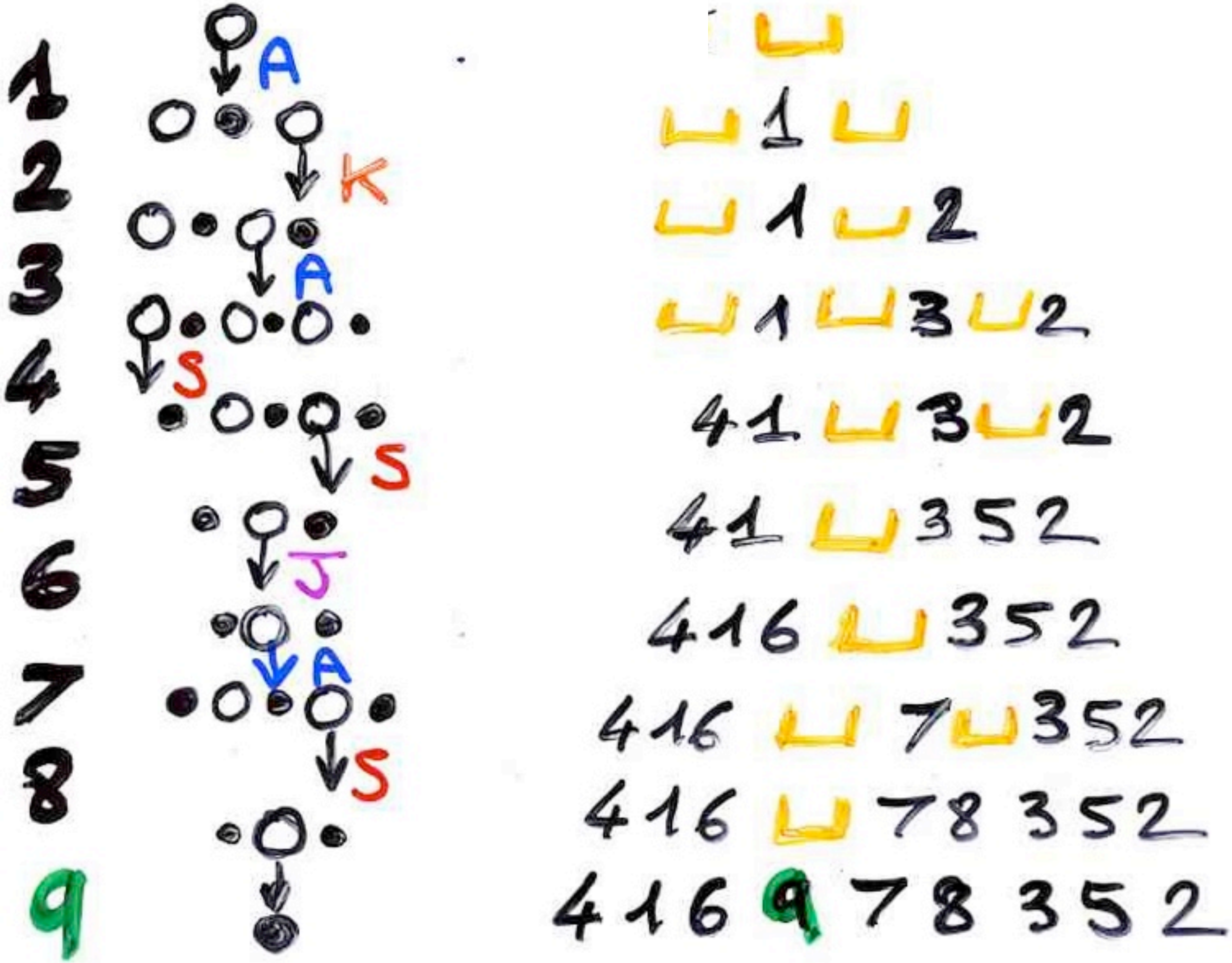
Françon-X.V., 1978



	␣
1	␣ 1 ␣
2	␣ 1 ␣ 2
3	␣ 1 ␣ 3 ␣ 2
4	4 1 ␣ 3 ␣ 2
5	4 1 ␣ 3 5 2
6	4 1 6 ␣ 3 5 2
7	4 1 6 ␣ 7 ␣ 3 5 2
8	4 1 6 ␣ 7 8 3 5 2
9	4 1 6 9 7 8 3 5 2



v1
vi
vk
x
u1
u2
uk
uk+1



1
2
3
4
5
6
7
8
9
A
K
A
S
S
J
A
S
1
1 2
1 3 2
4 1 3 2
4 1 3 5 2
4 1 6 3 5 2
4 1 6 7 3 5 2
4 1 6 7 8 3 5 2
4 1 6 9 7 8 3 5 2

bijection
permutations
alternative
tableaux

416978352

●

●1 ●

●1●2 ●

●1●3●2

41● 3 ●2

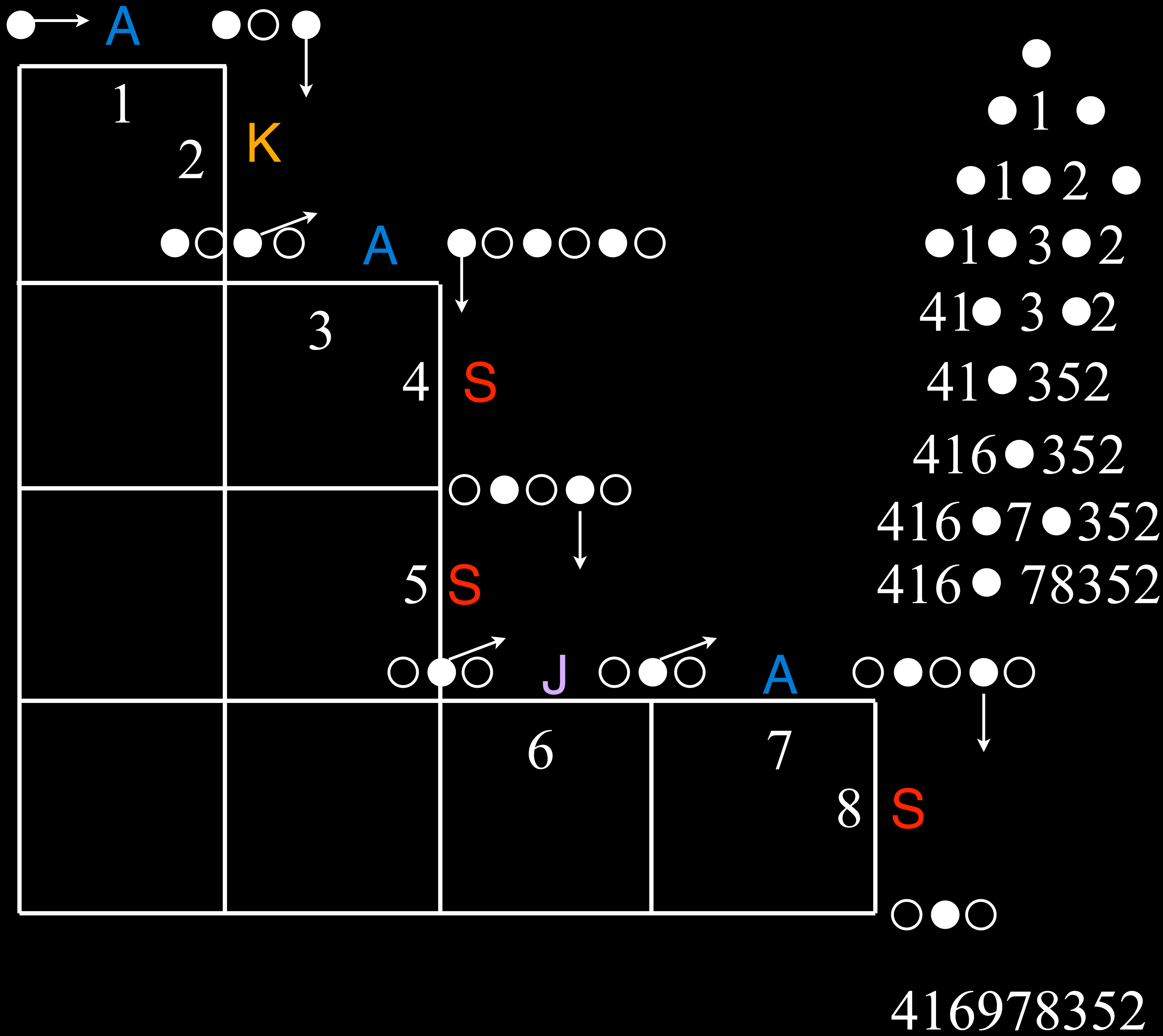
41●352

416●352

416●7●352

416● 78352

416978352


A
1
2
K
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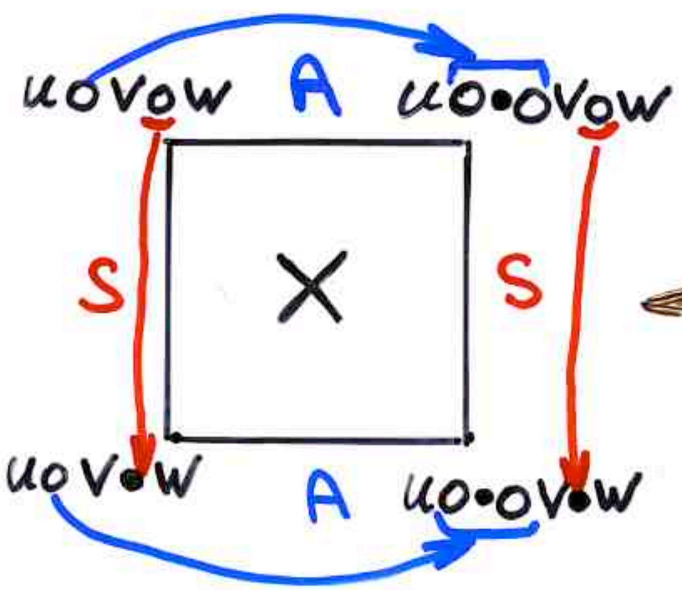
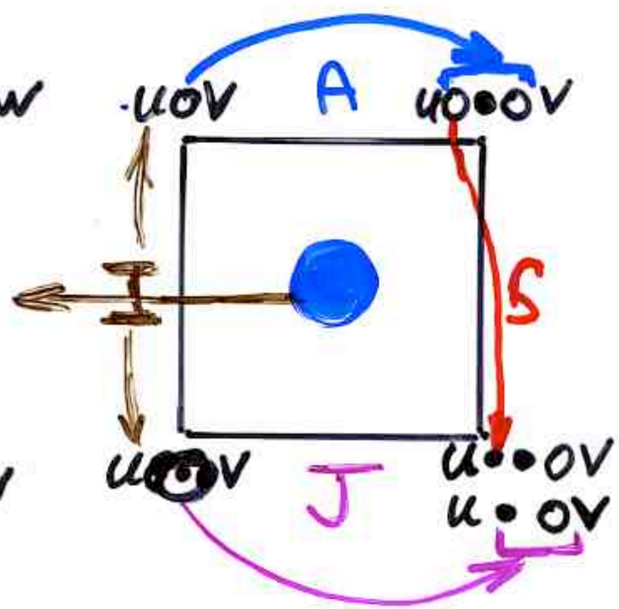
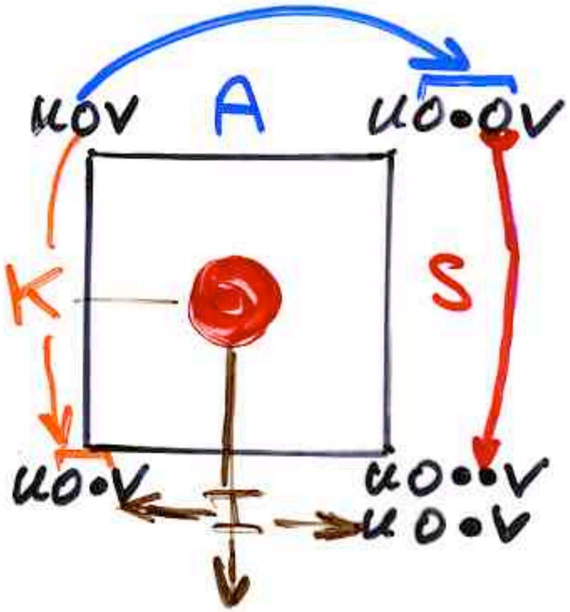
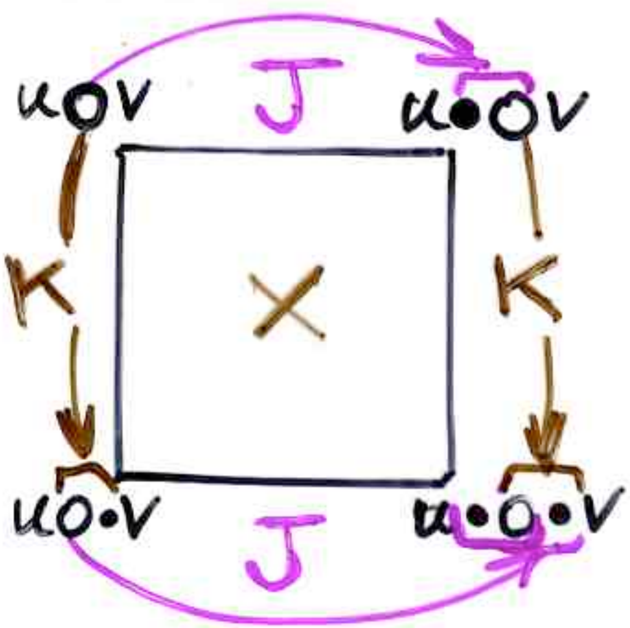
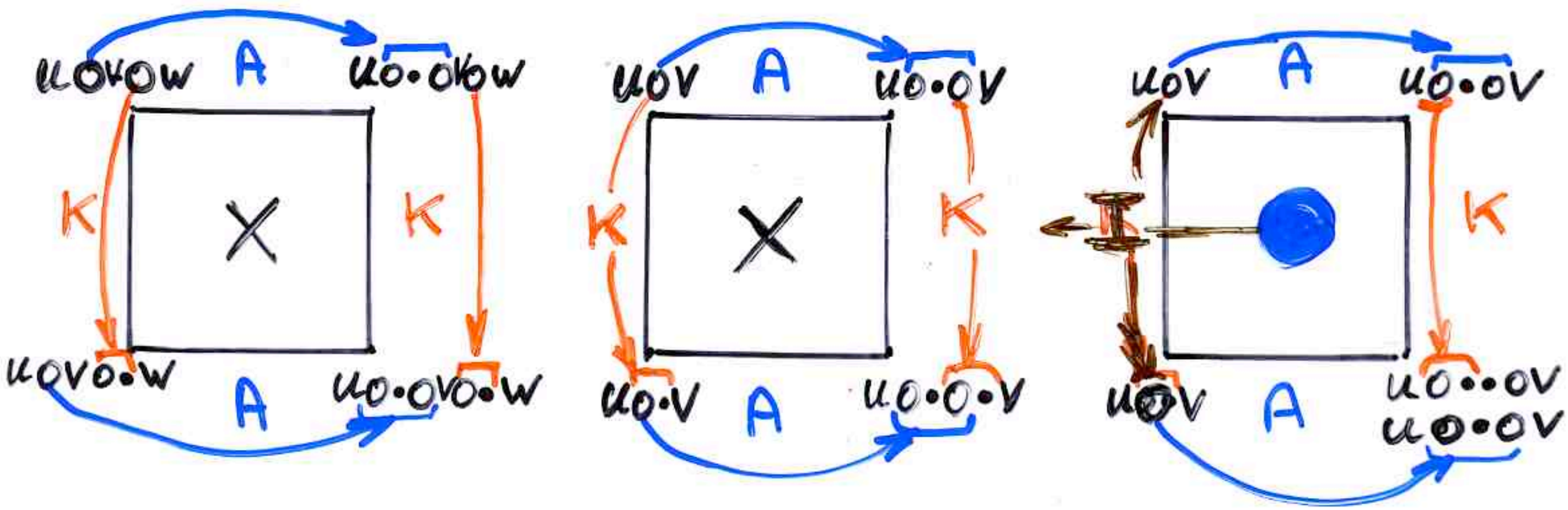
3
4
S
5
S
J
A
6
7
8
S
1
1 2
1 3 2
41 3 2
41 352
416 352
416 7 352
416 78352
416978352


A
S
S
A


A
S
J


A
K
S


J
K
K
J


A
K
K
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A
K
K
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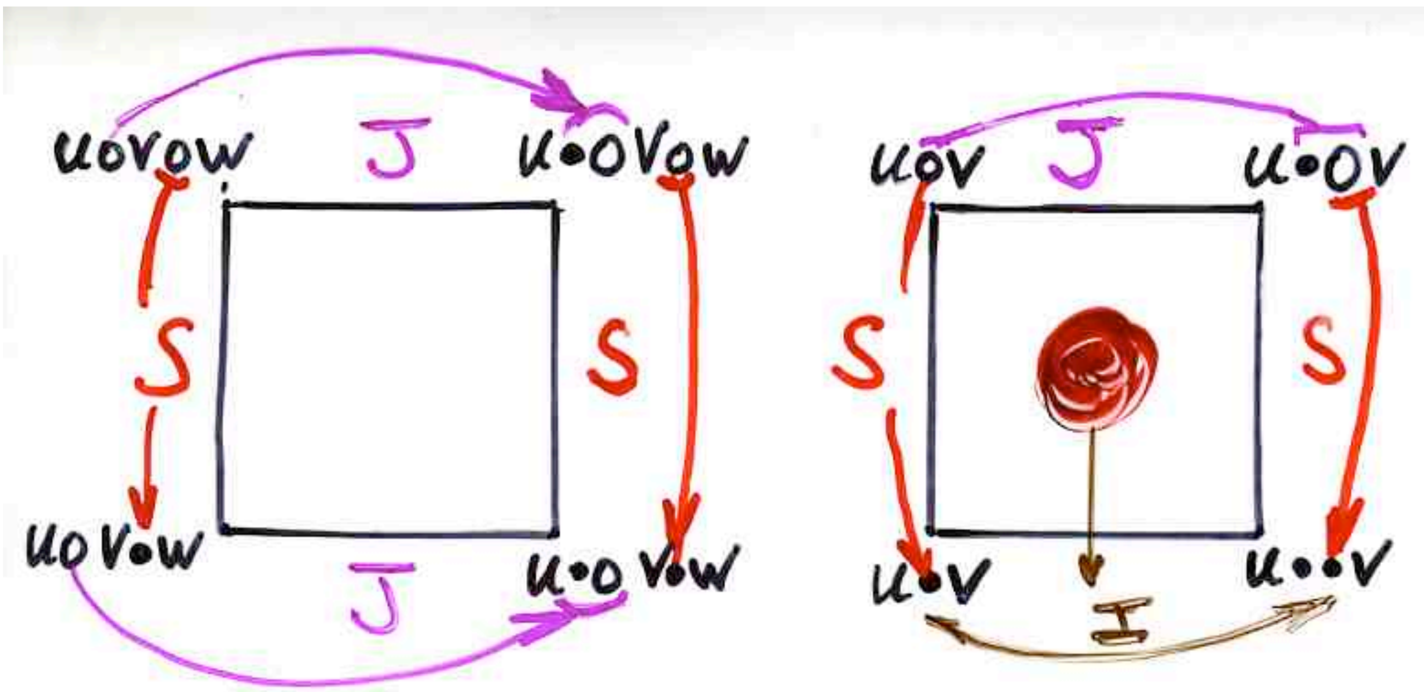
A
K
I
A


J
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J
J
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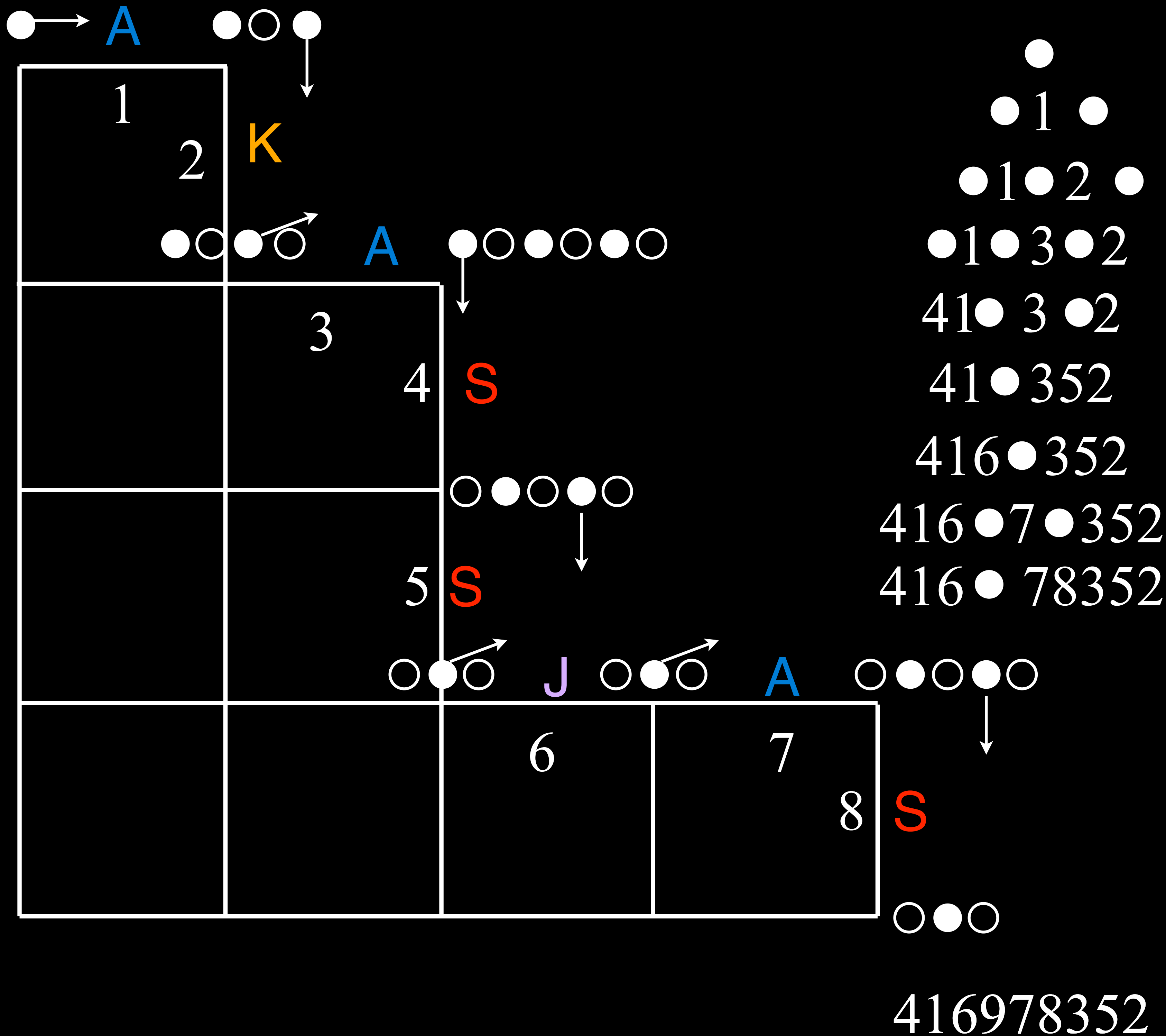
S
I


A
1
2
K
A
3
4
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J
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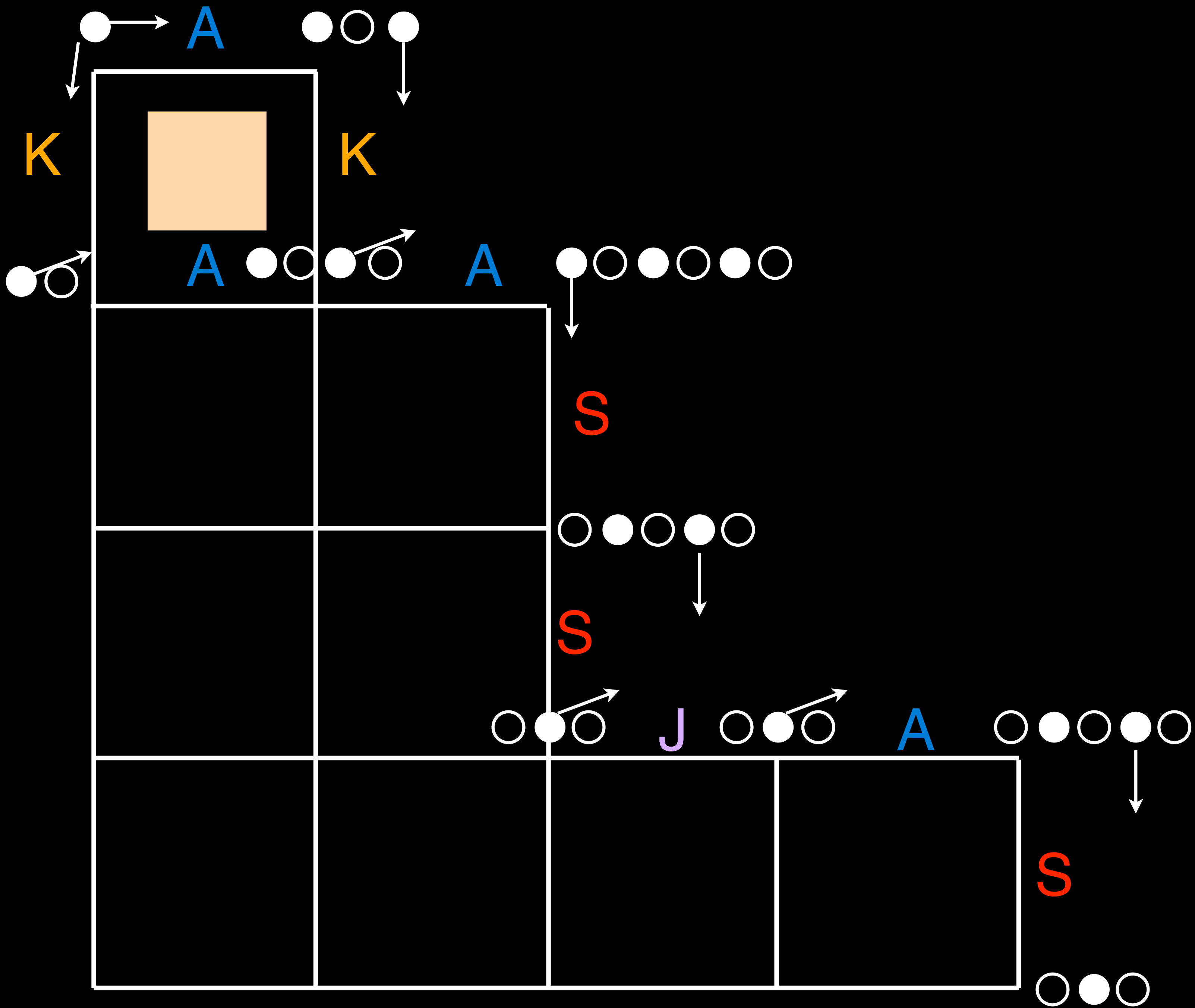
12
132
41 3 2
41 352
416 352
416 7 352
416 78352
416978352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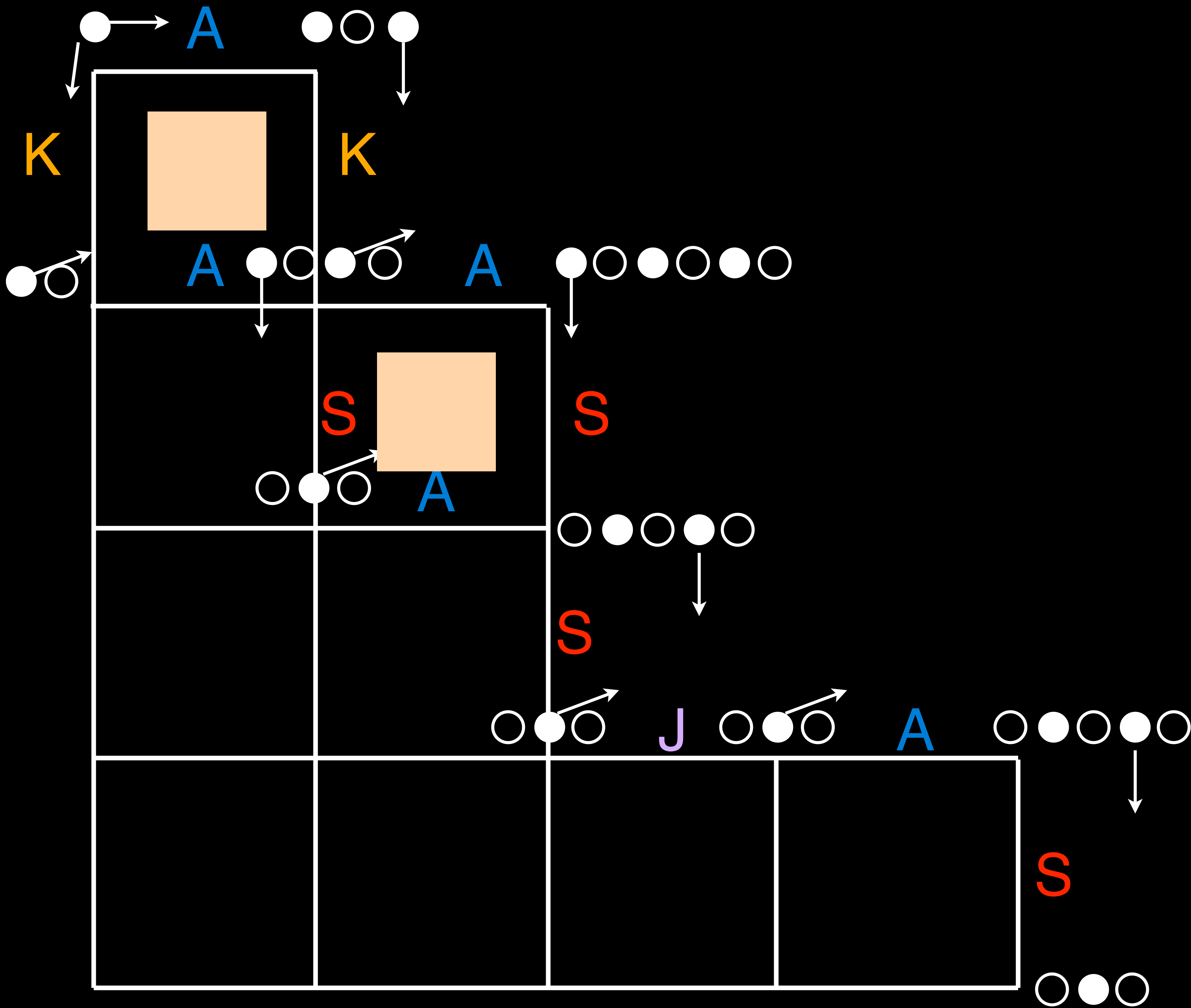

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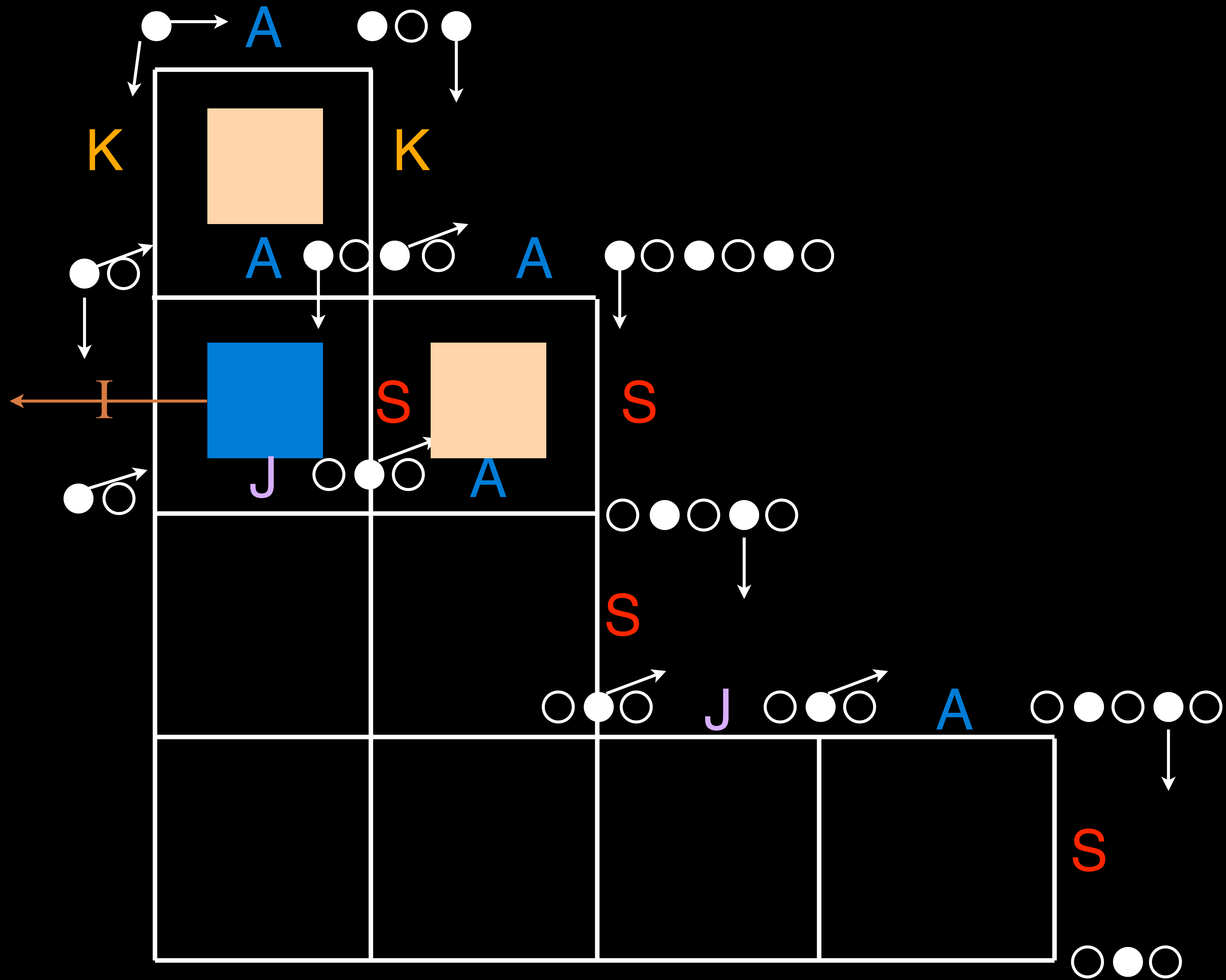
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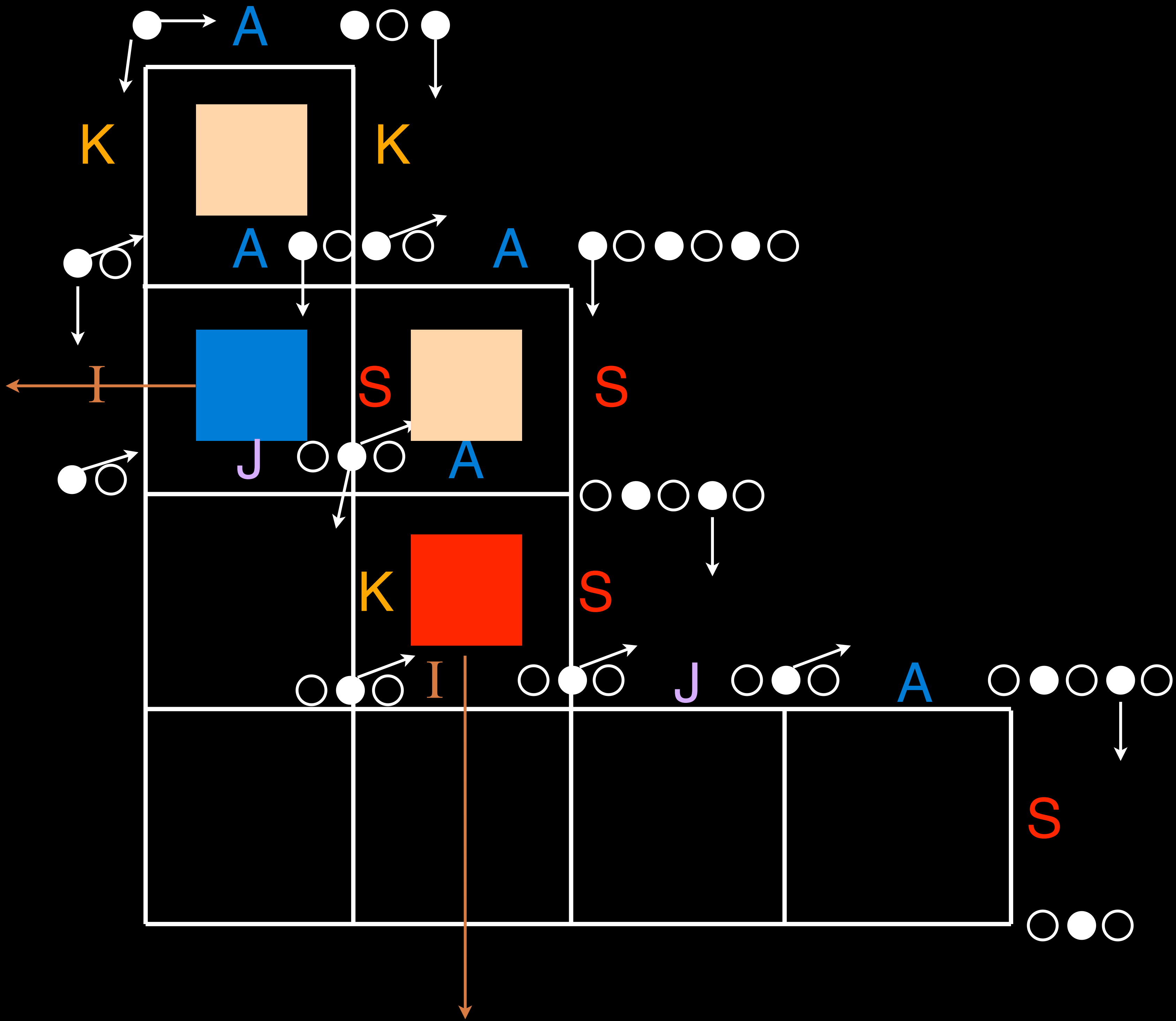
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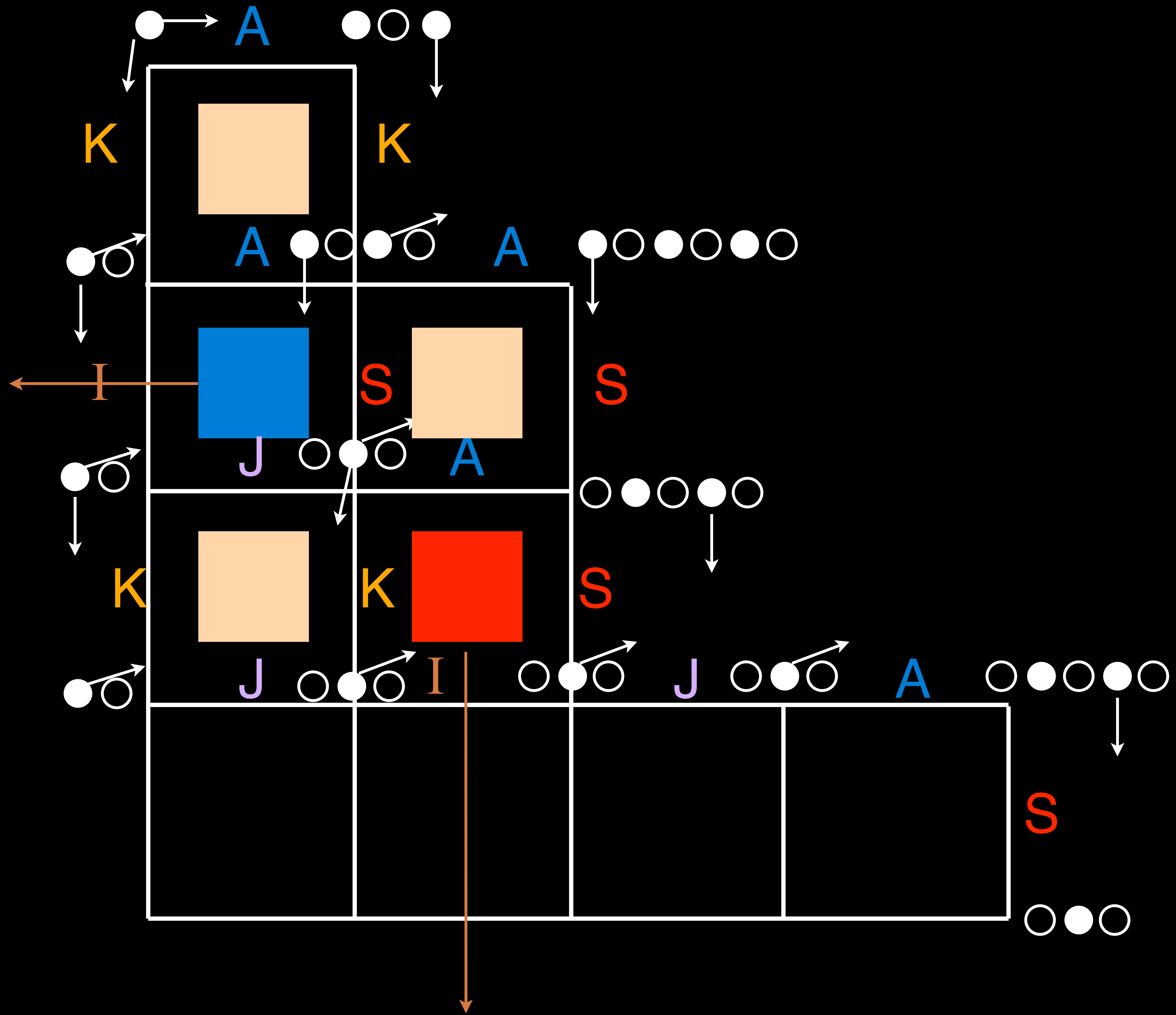

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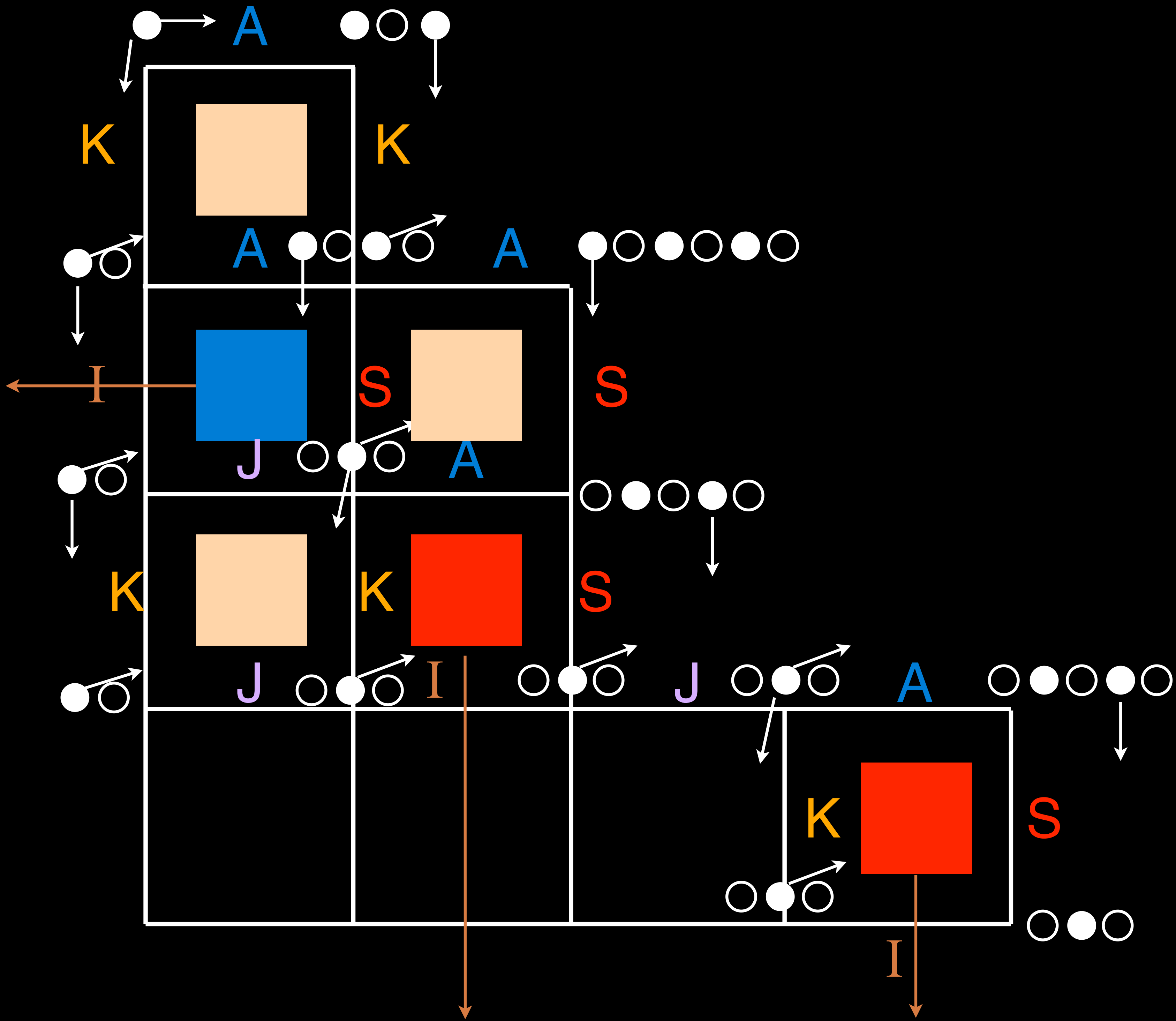
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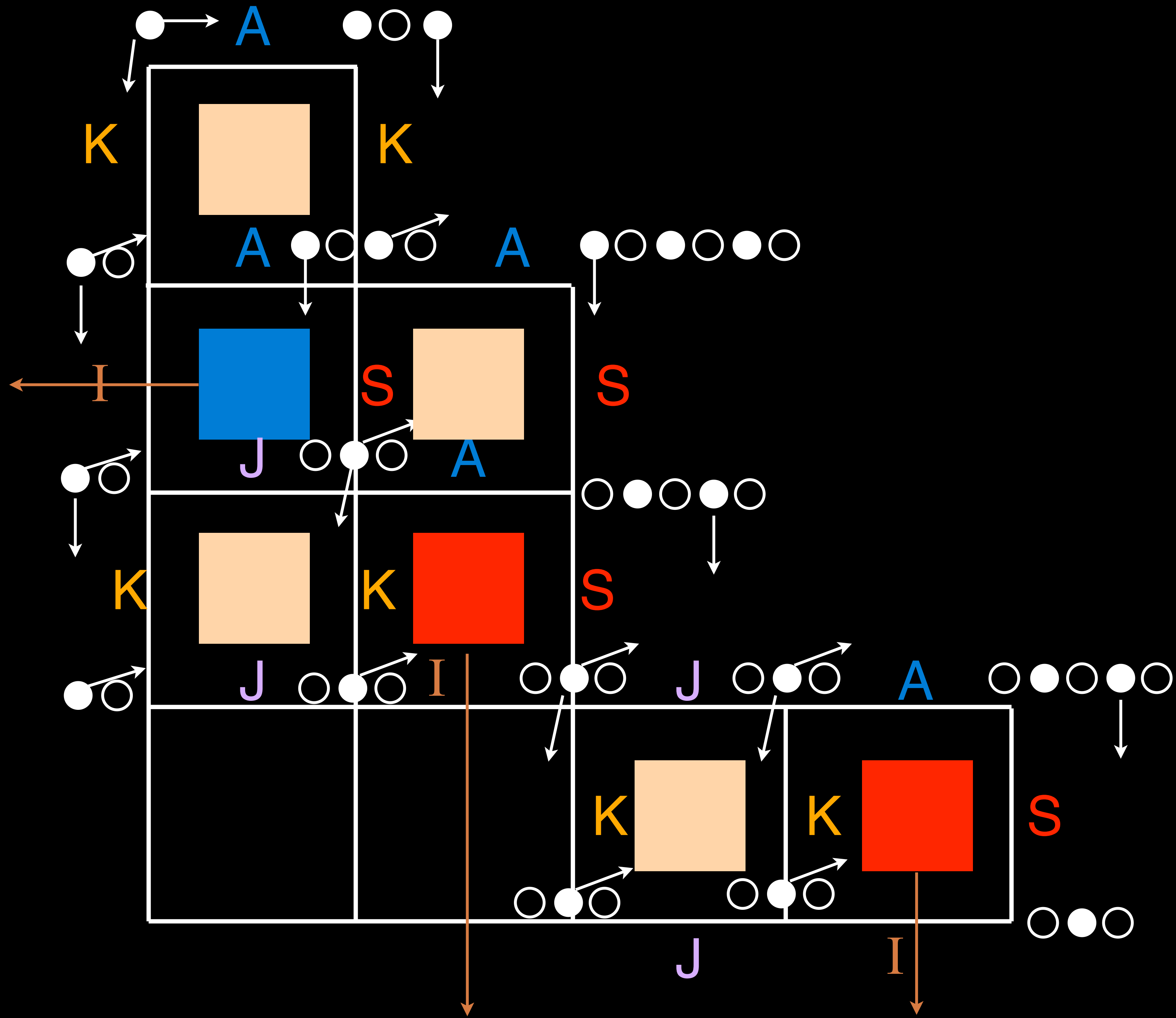

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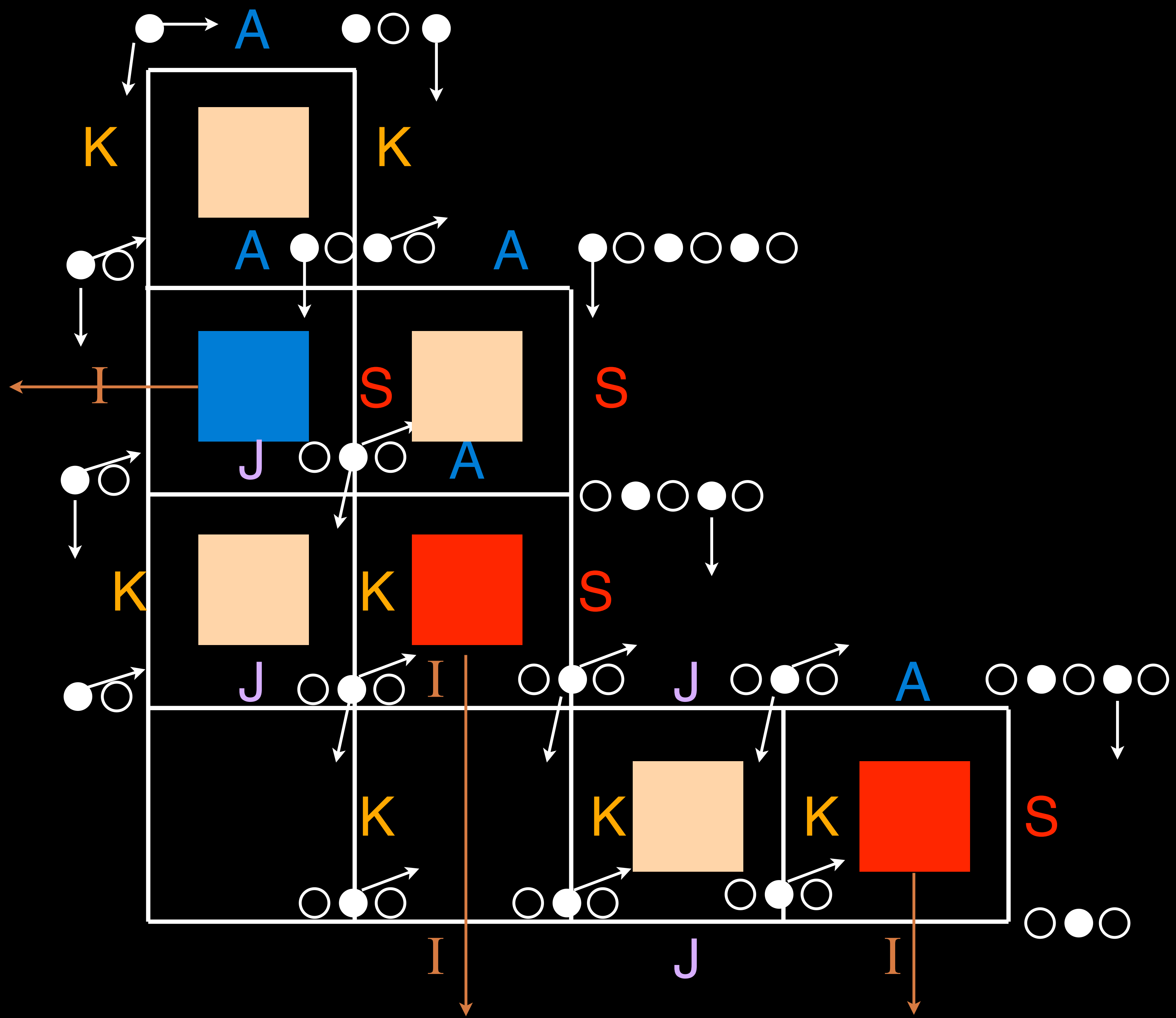
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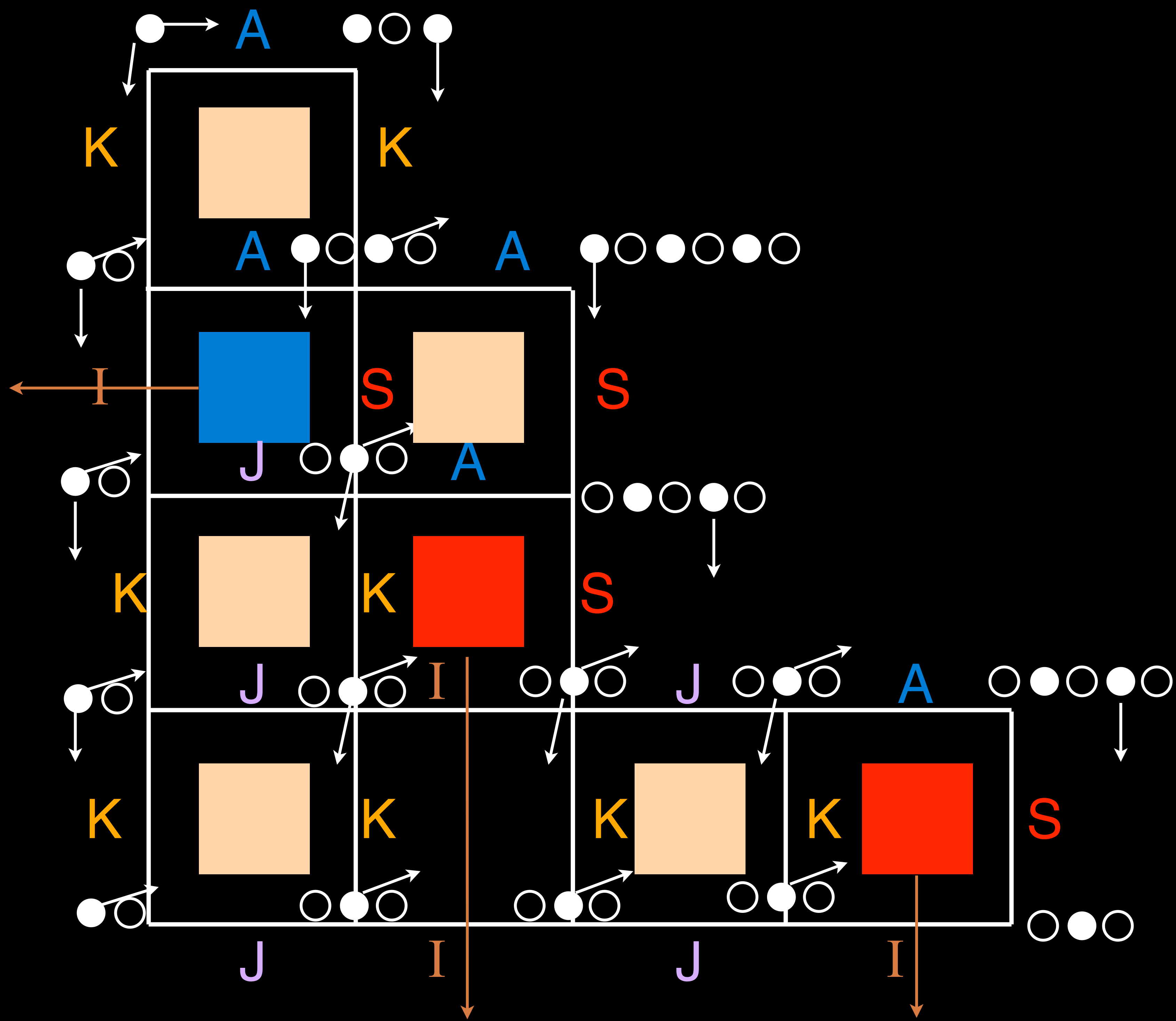
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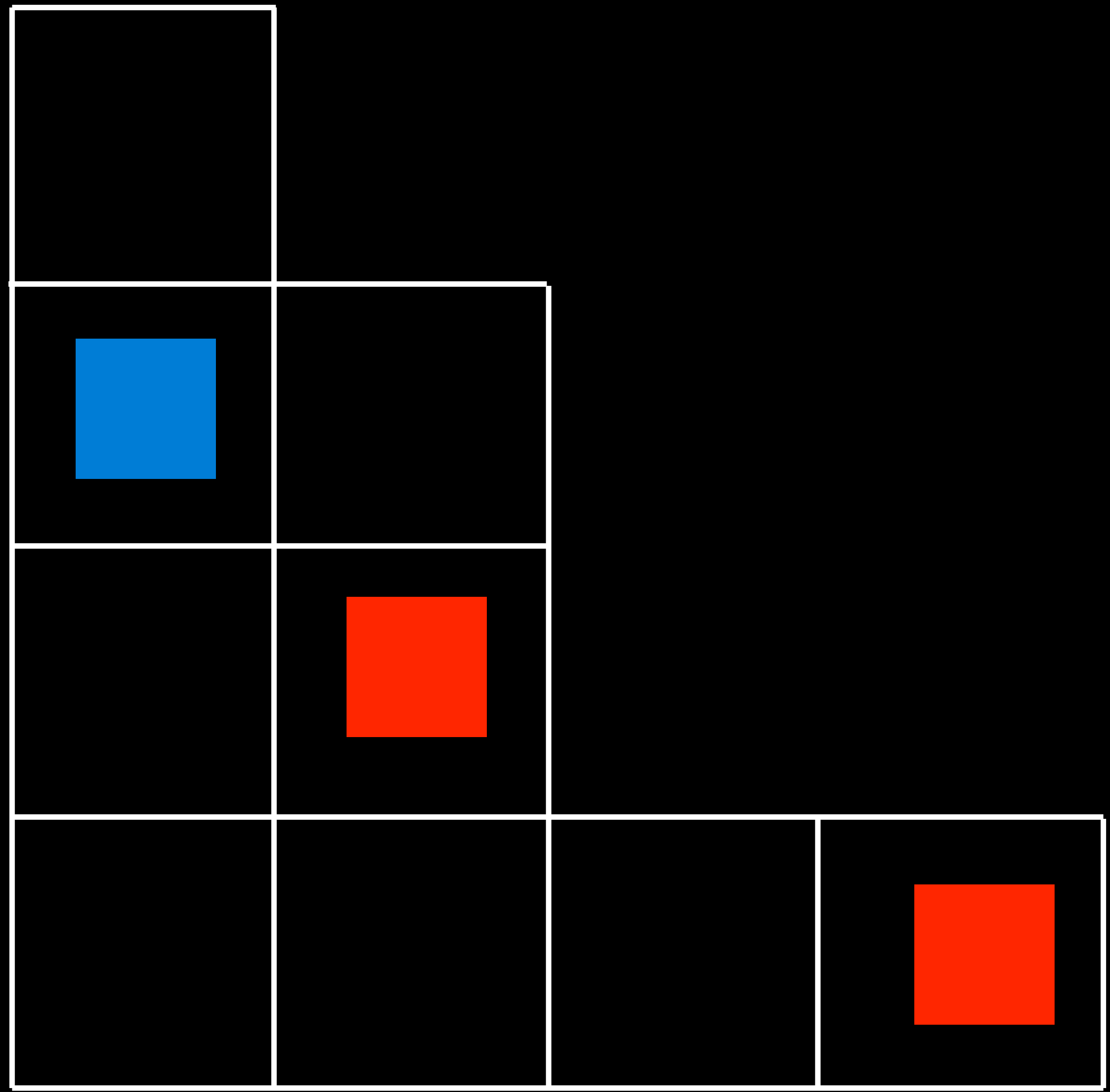
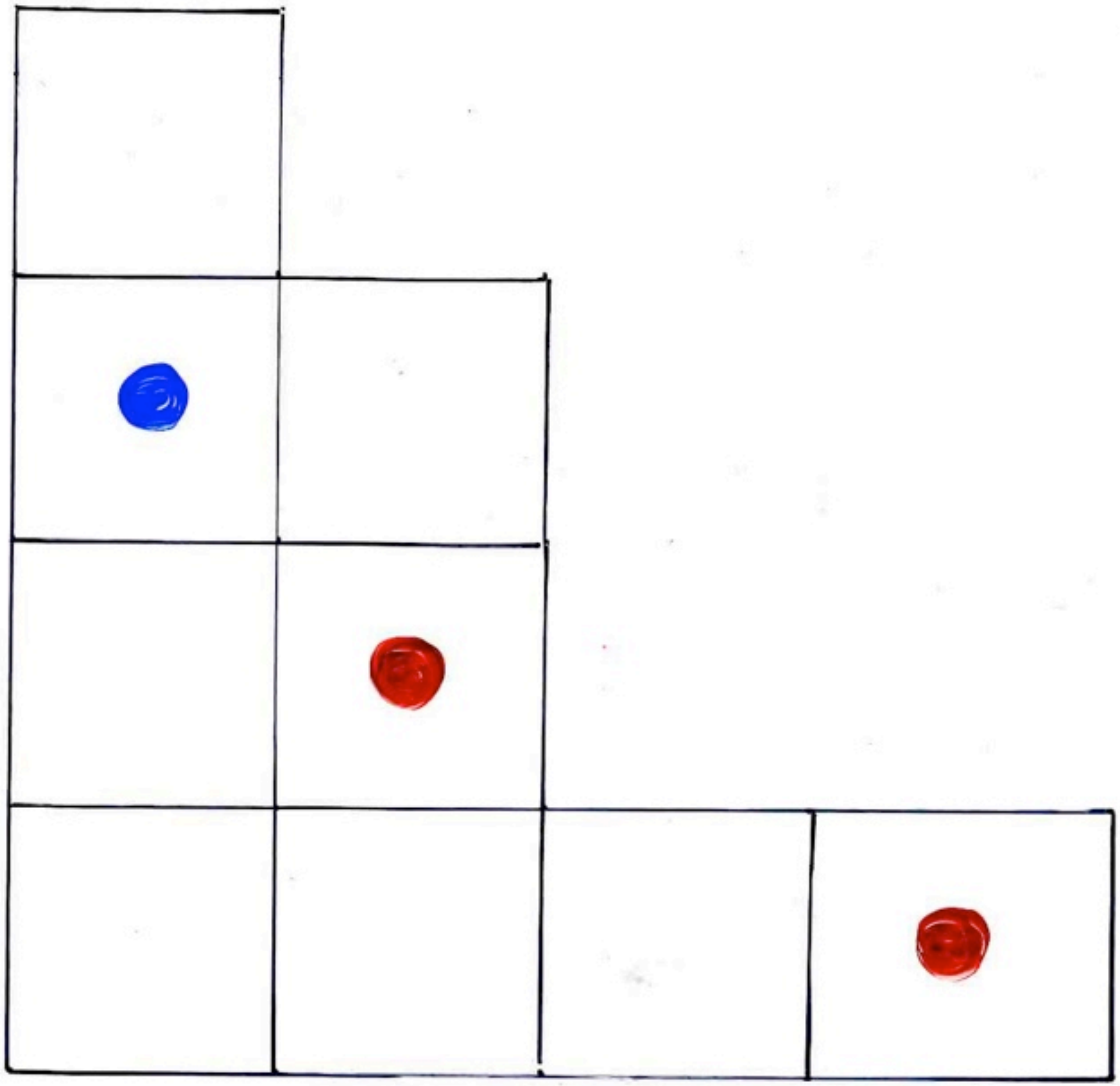
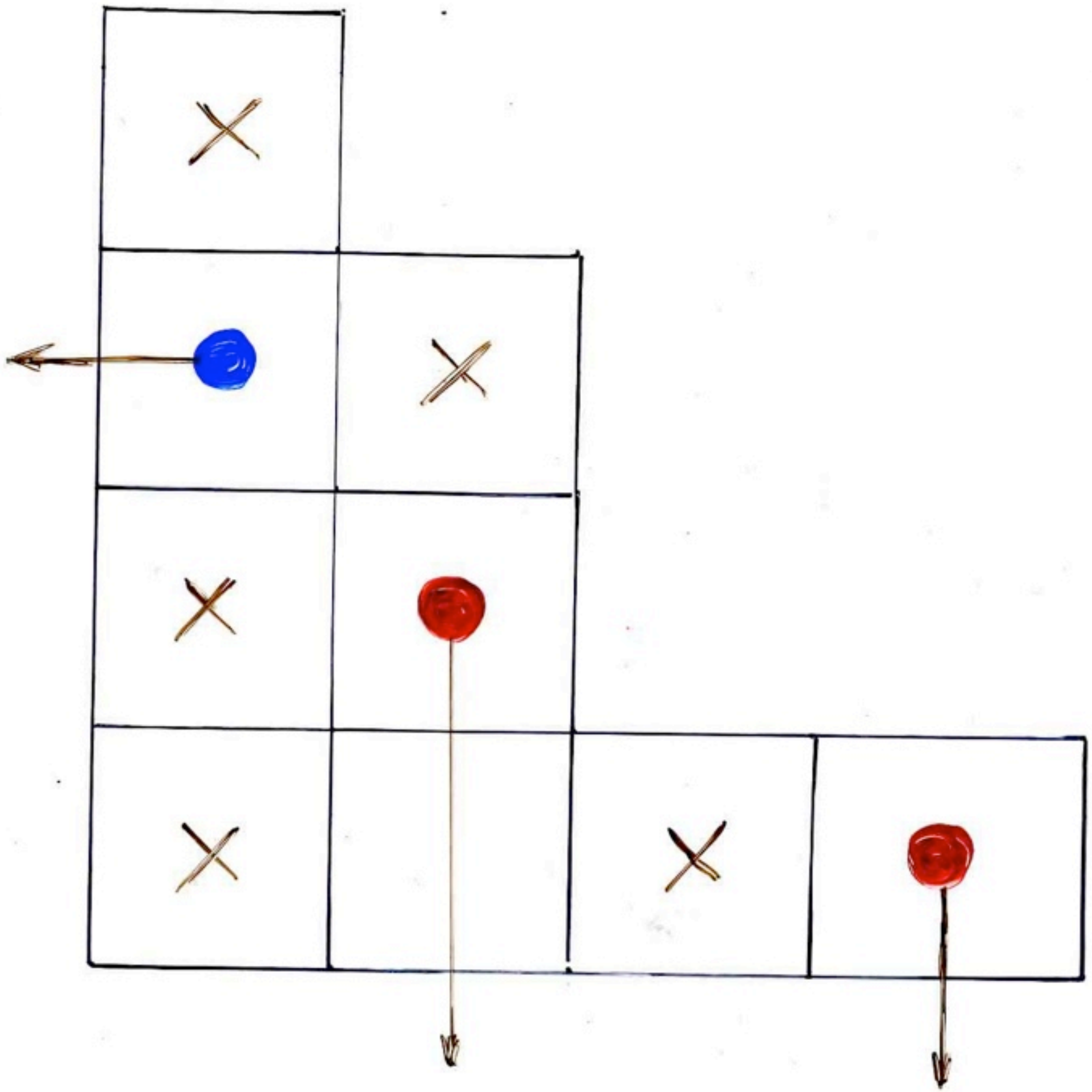
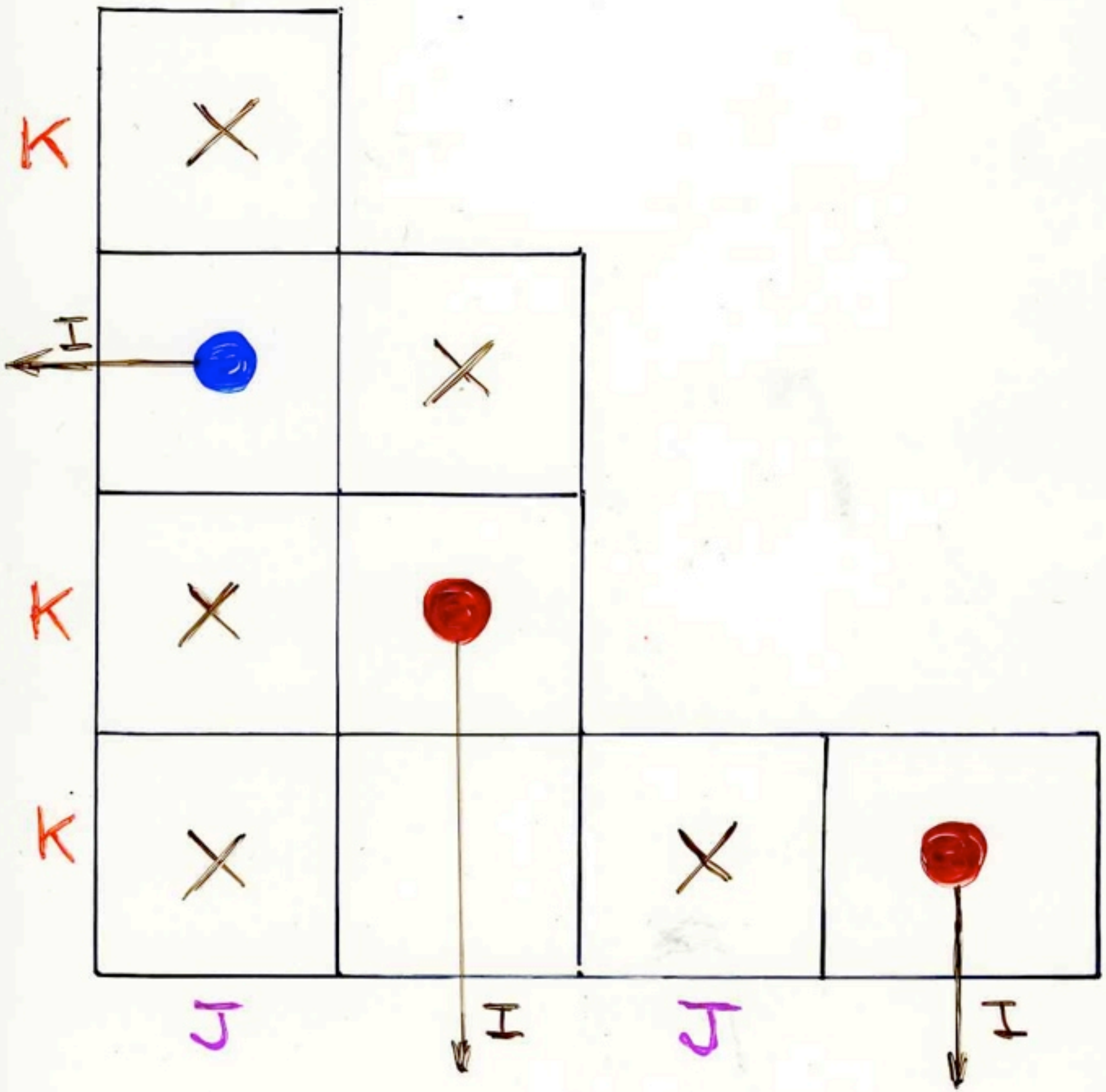


416978352

inverse bijection







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J
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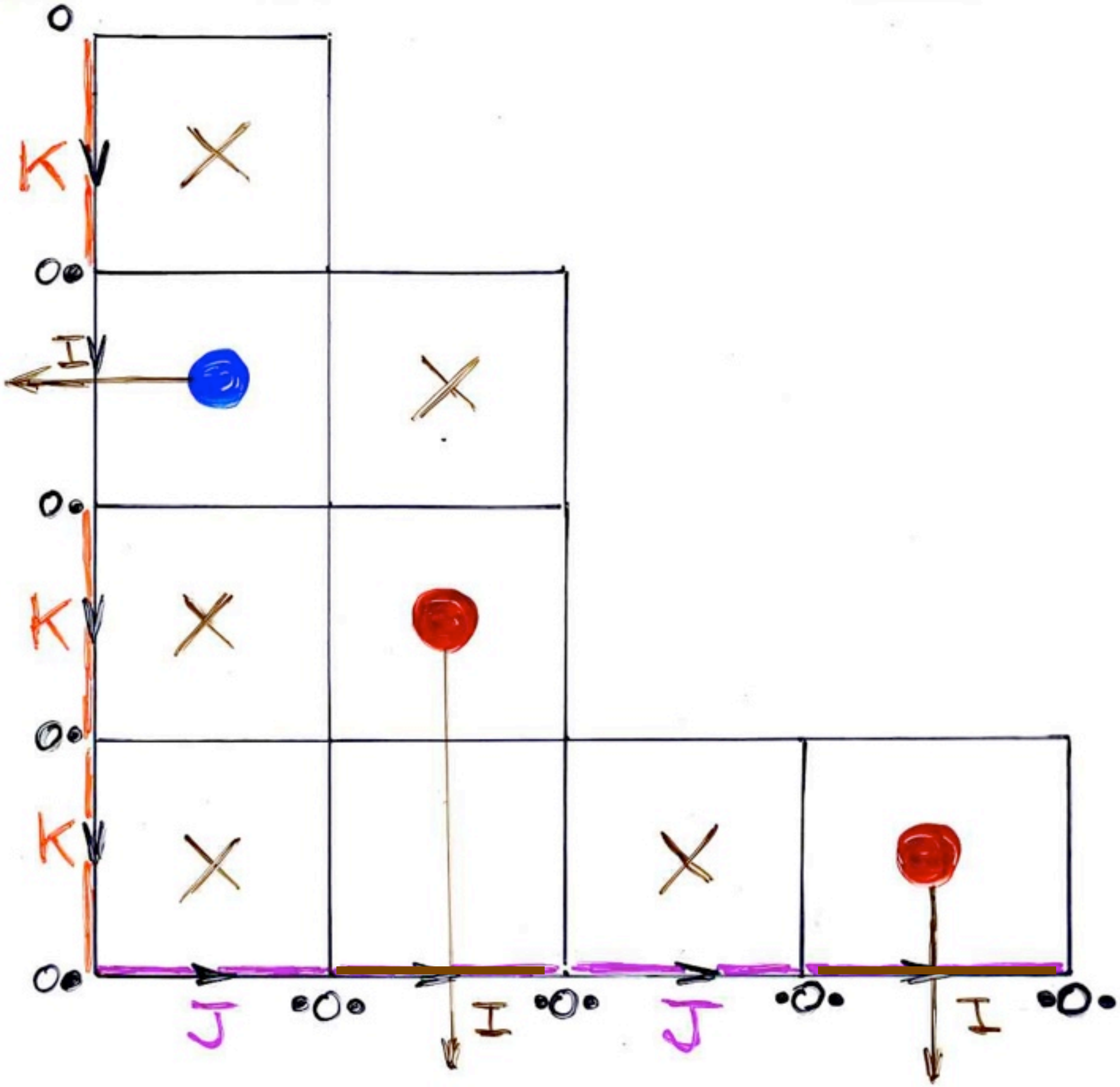

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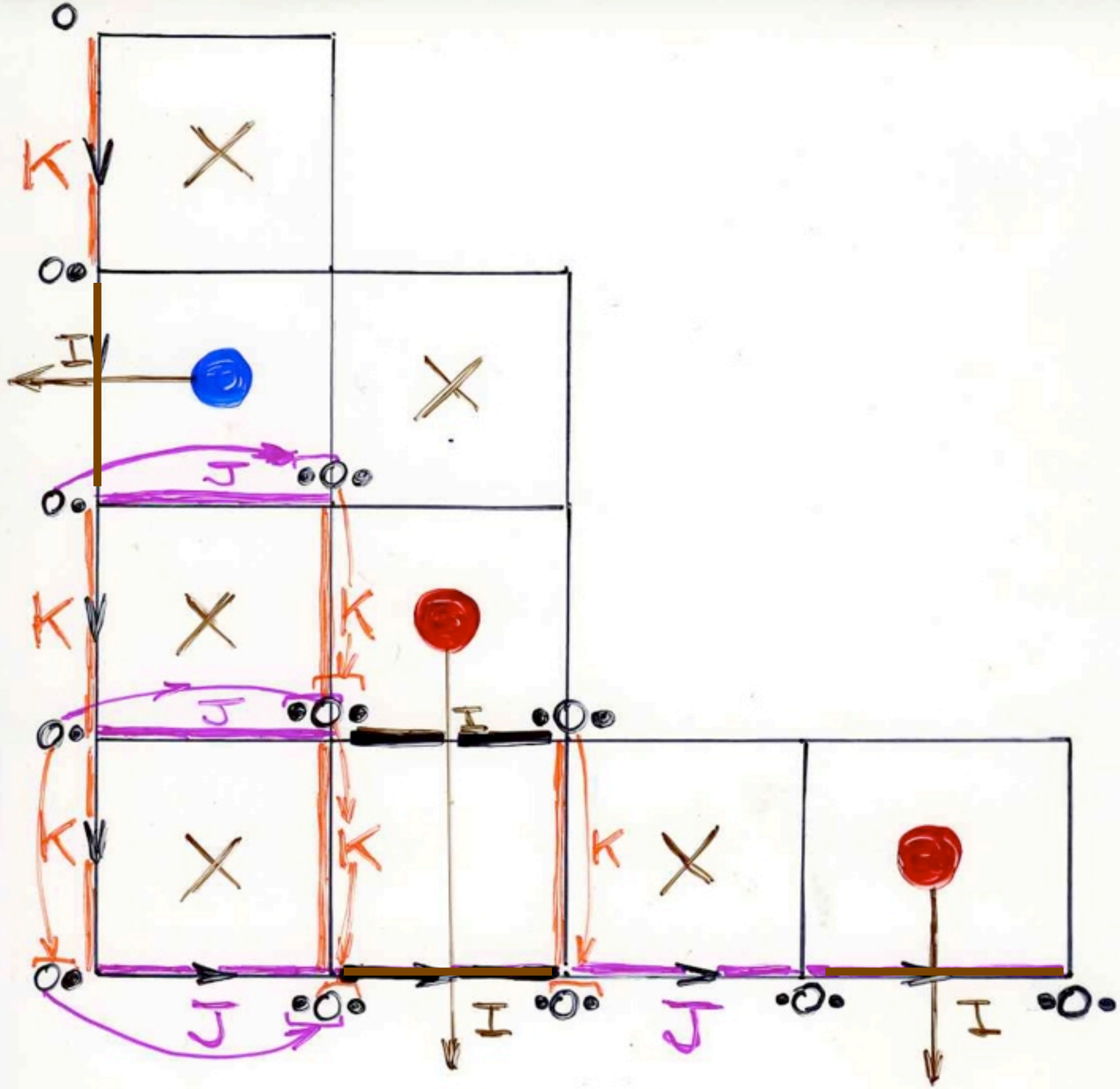
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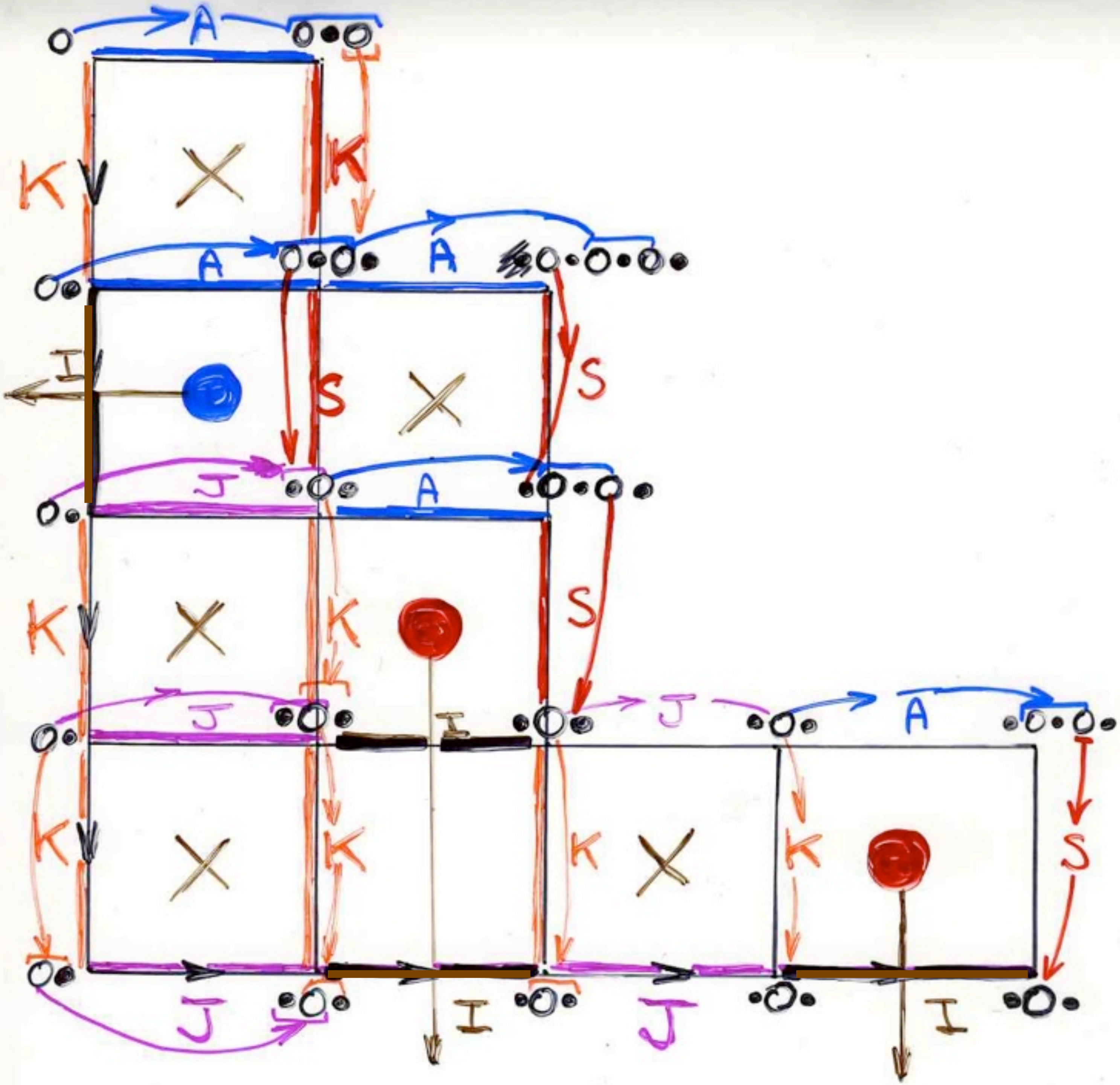
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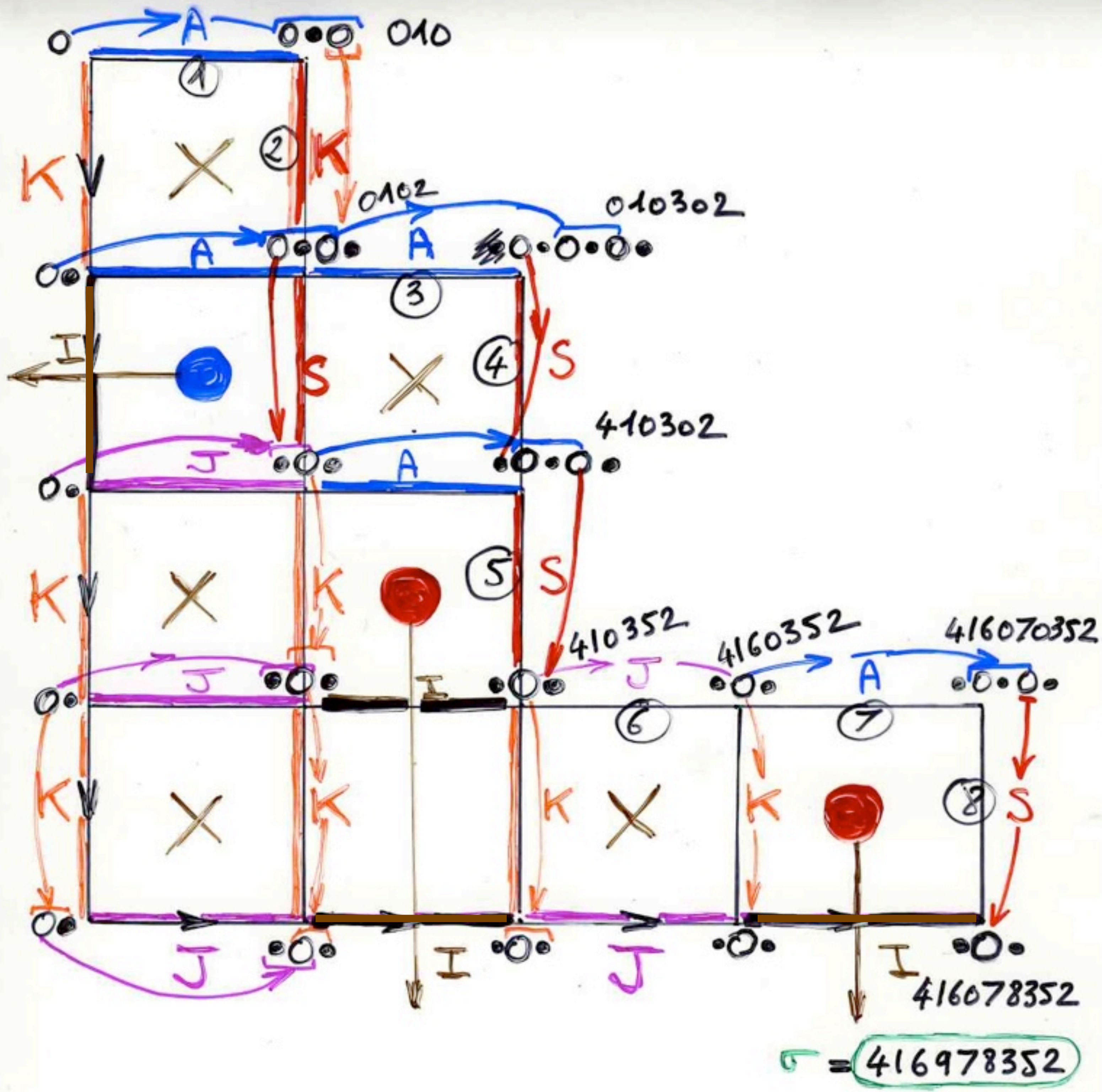


010
0102
010302
410302
410352
4160352
416070352
416078352
σ = 416978352

equivalent bijection:

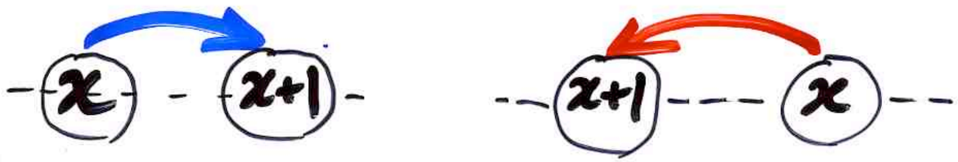
The “exchange-fusion” algorithm

Déf. Permutation $\sigma = \sigma(1) \cdots \sigma(n)$

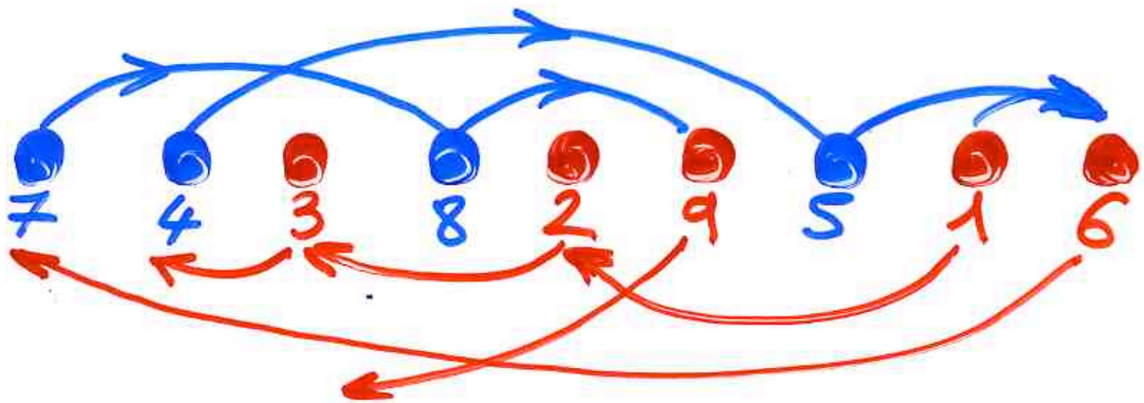
$x = \sigma(i)$, $1 \leq x < n$

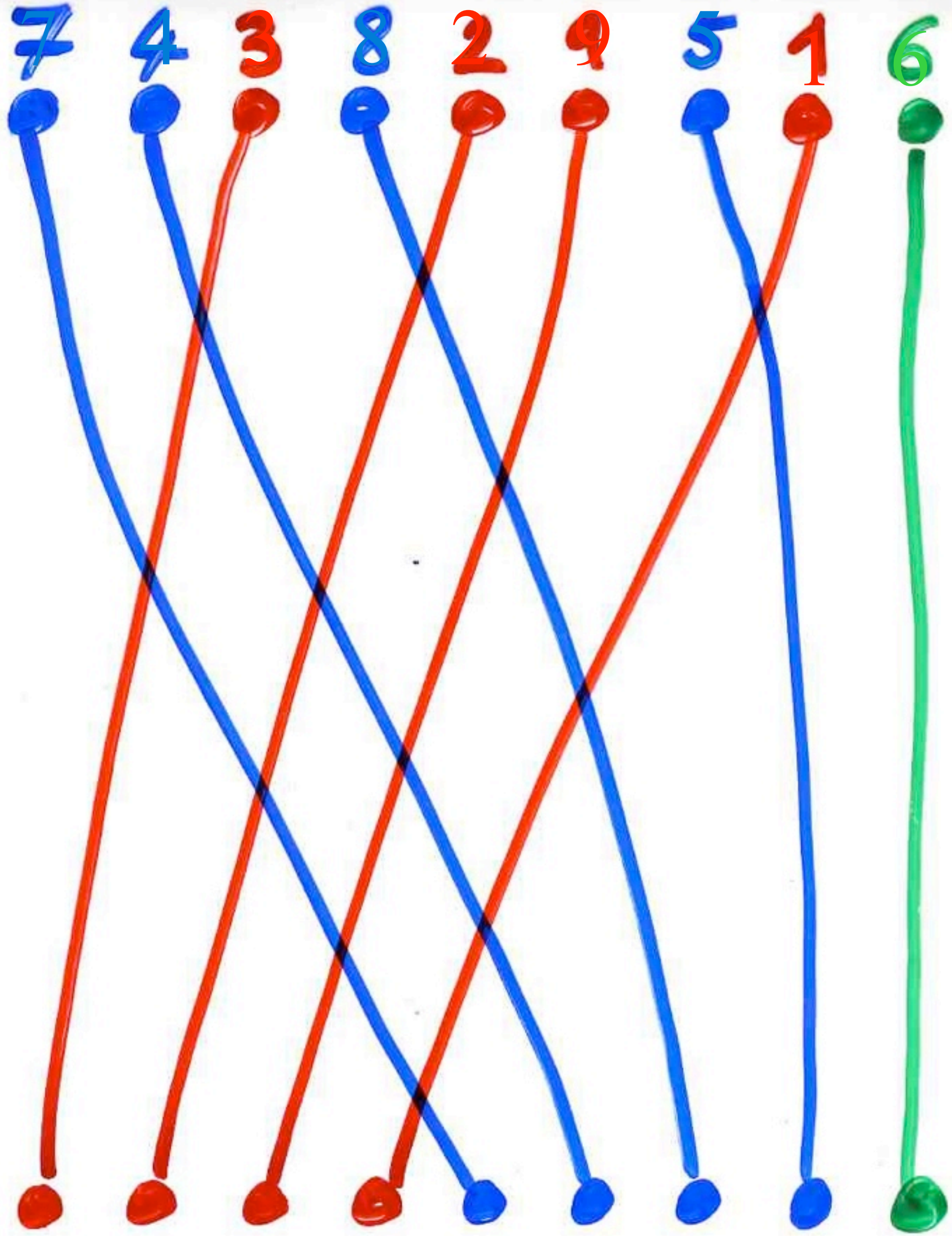
(valeur) x $\begin{cases} \text{avance} \\ \text{recul} \end{cases}$ $x+1 = \sigma(j)$, $\begin{cases} i < j \\ j < i \end{cases}$

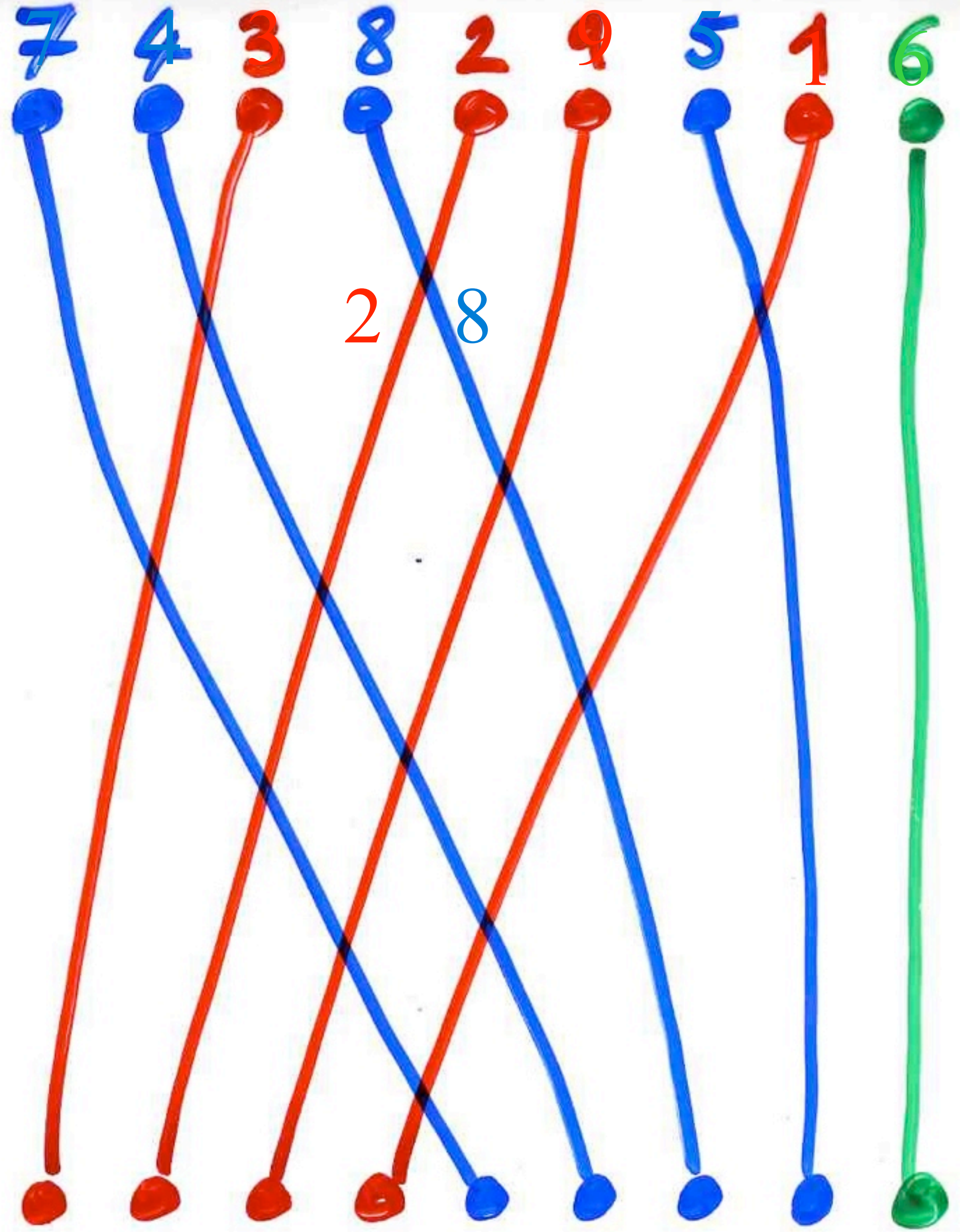
- convention $x = n$ est un recul



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$




7 4 3 8 2 9 5 1 6



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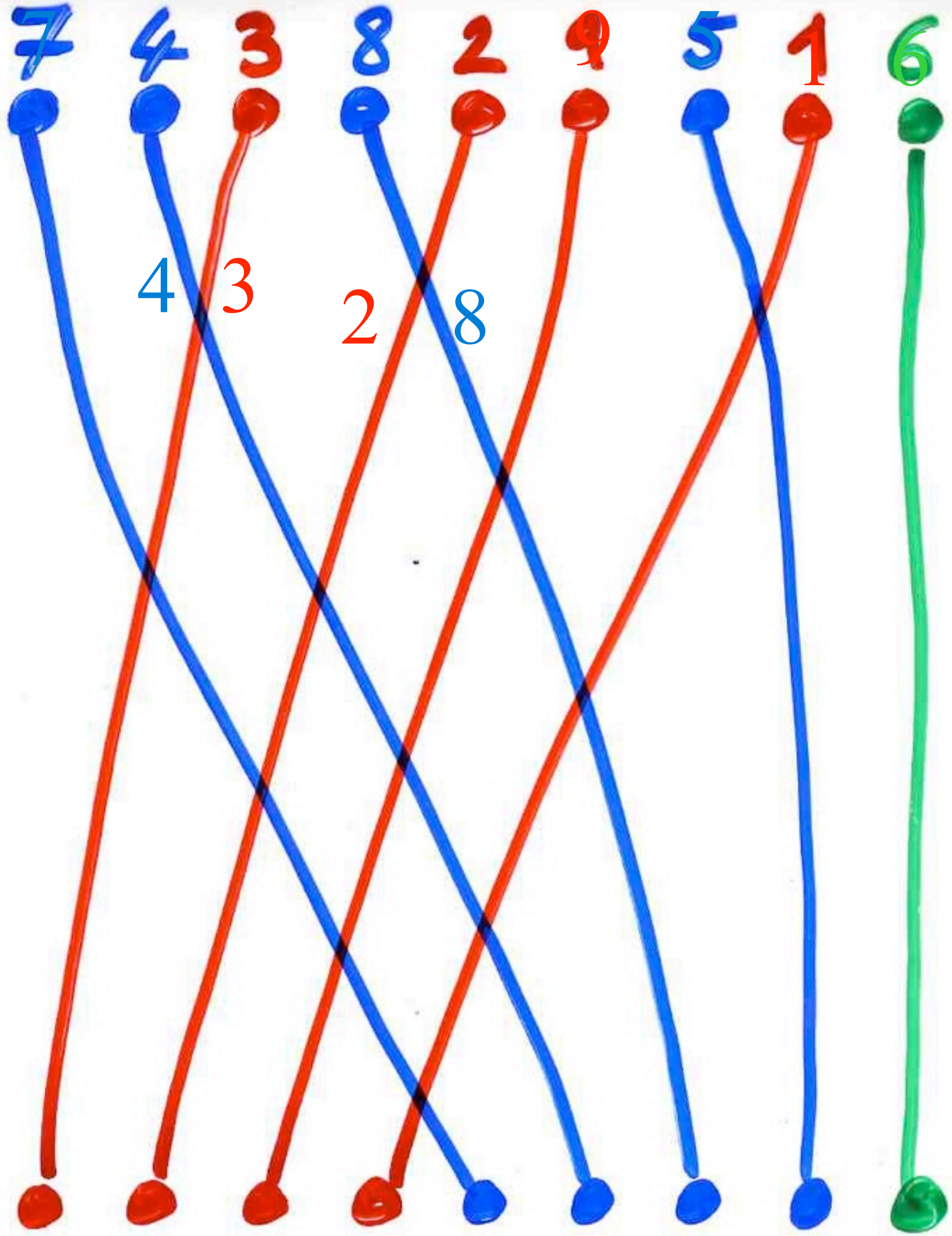
2 8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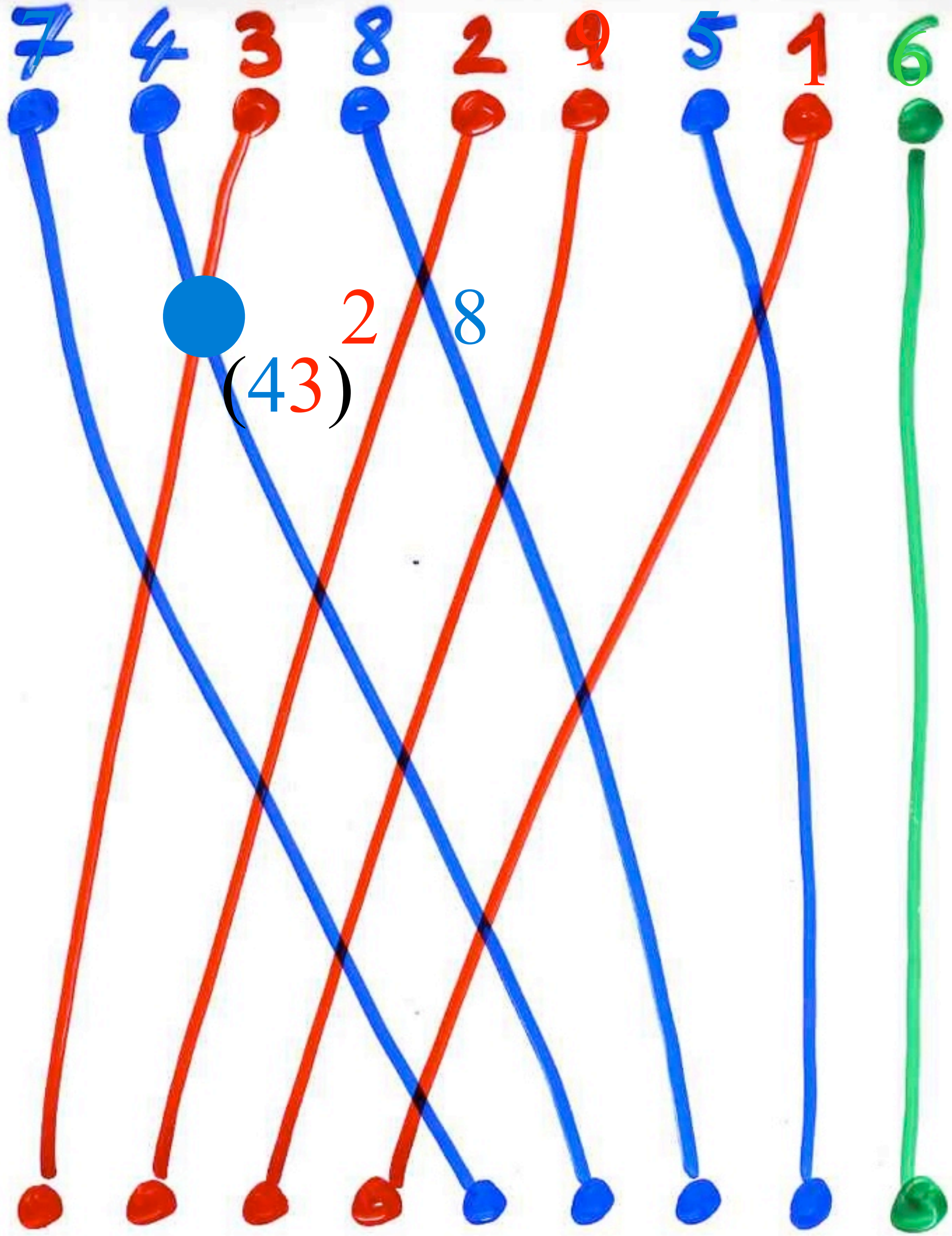
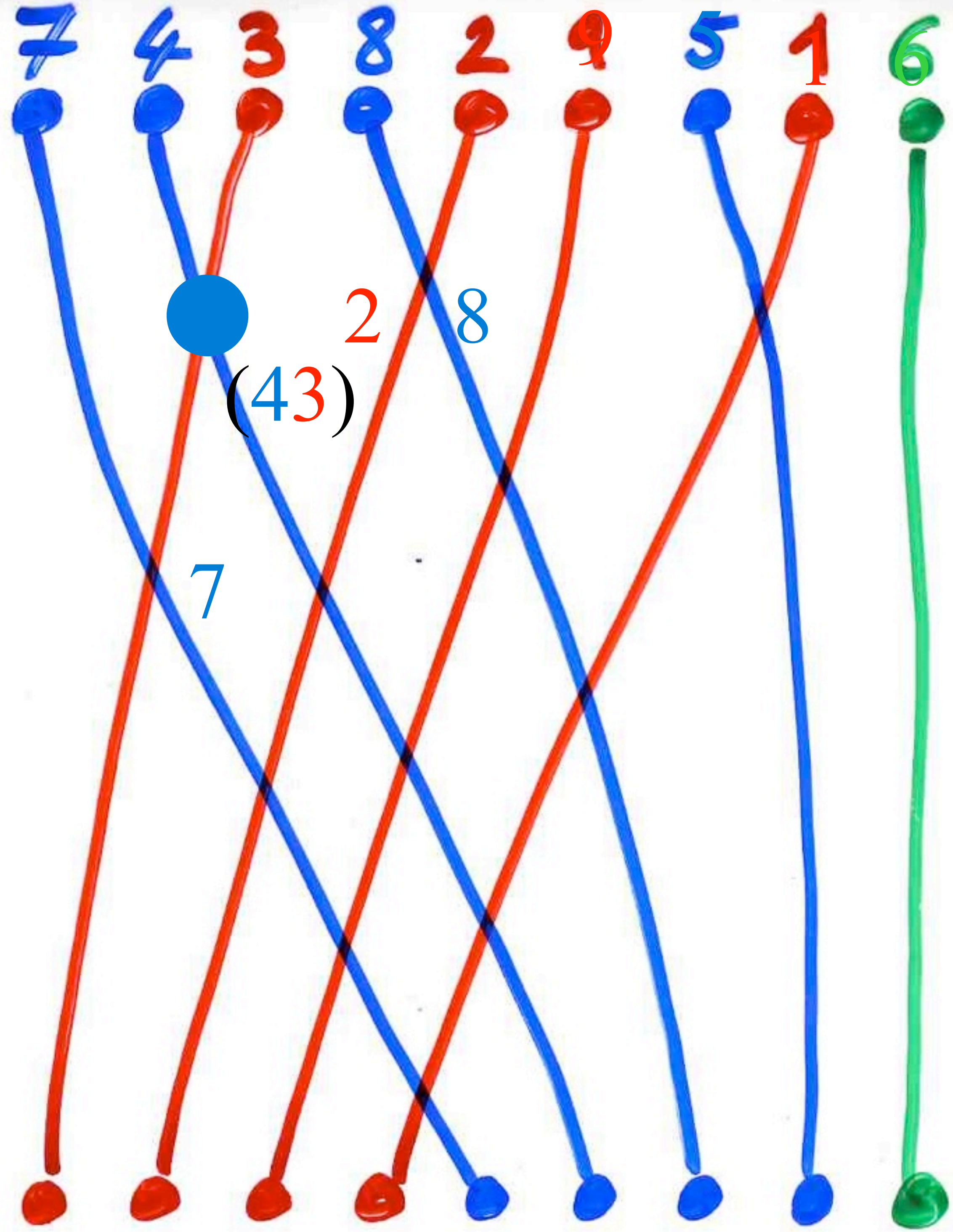
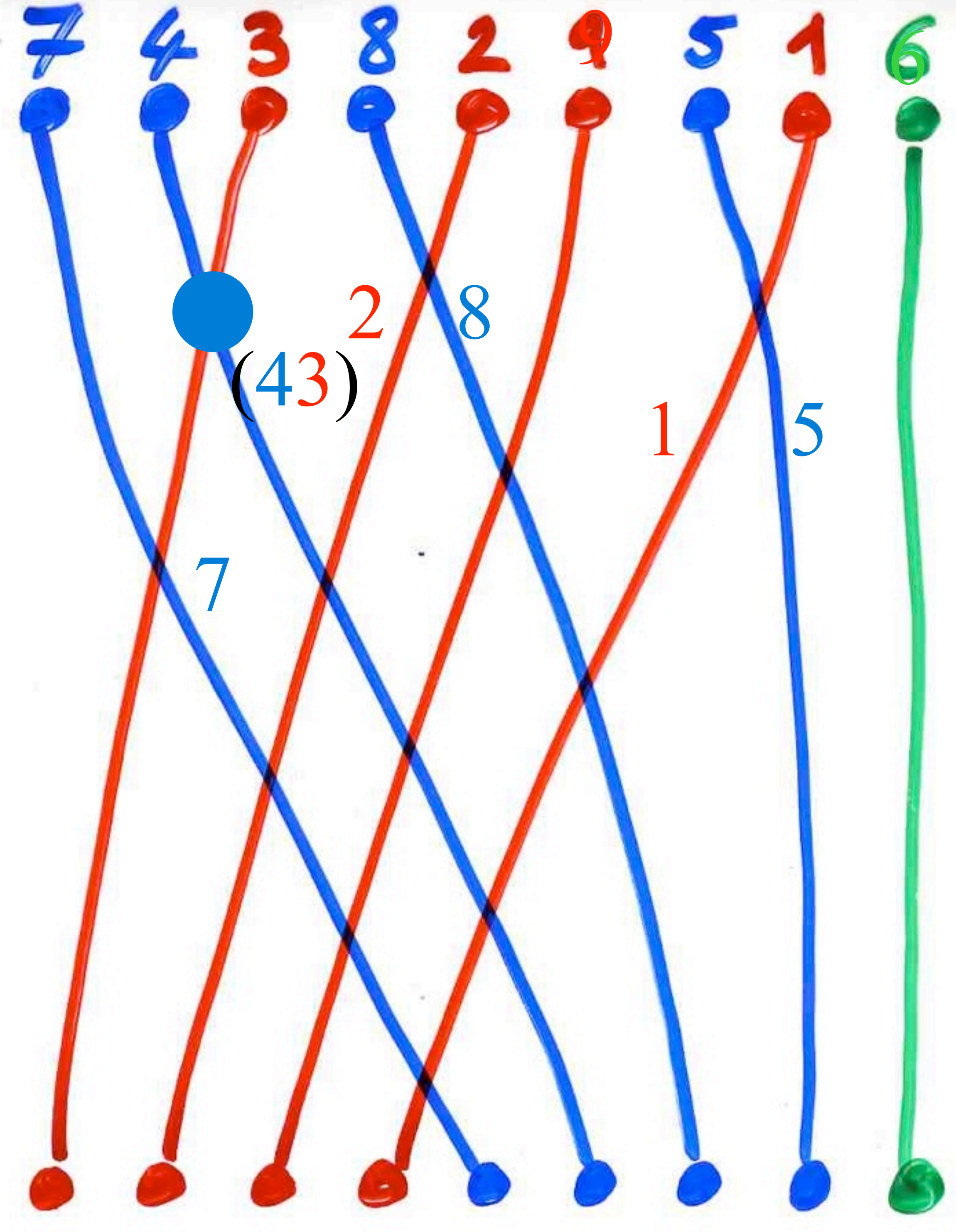
4 3
2 8



7 4 3 8 2 9 5 1 6
(43)
2
8



7 4 3 8 2 9 5 1 6
2
8
(43)
7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2
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(43)
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5
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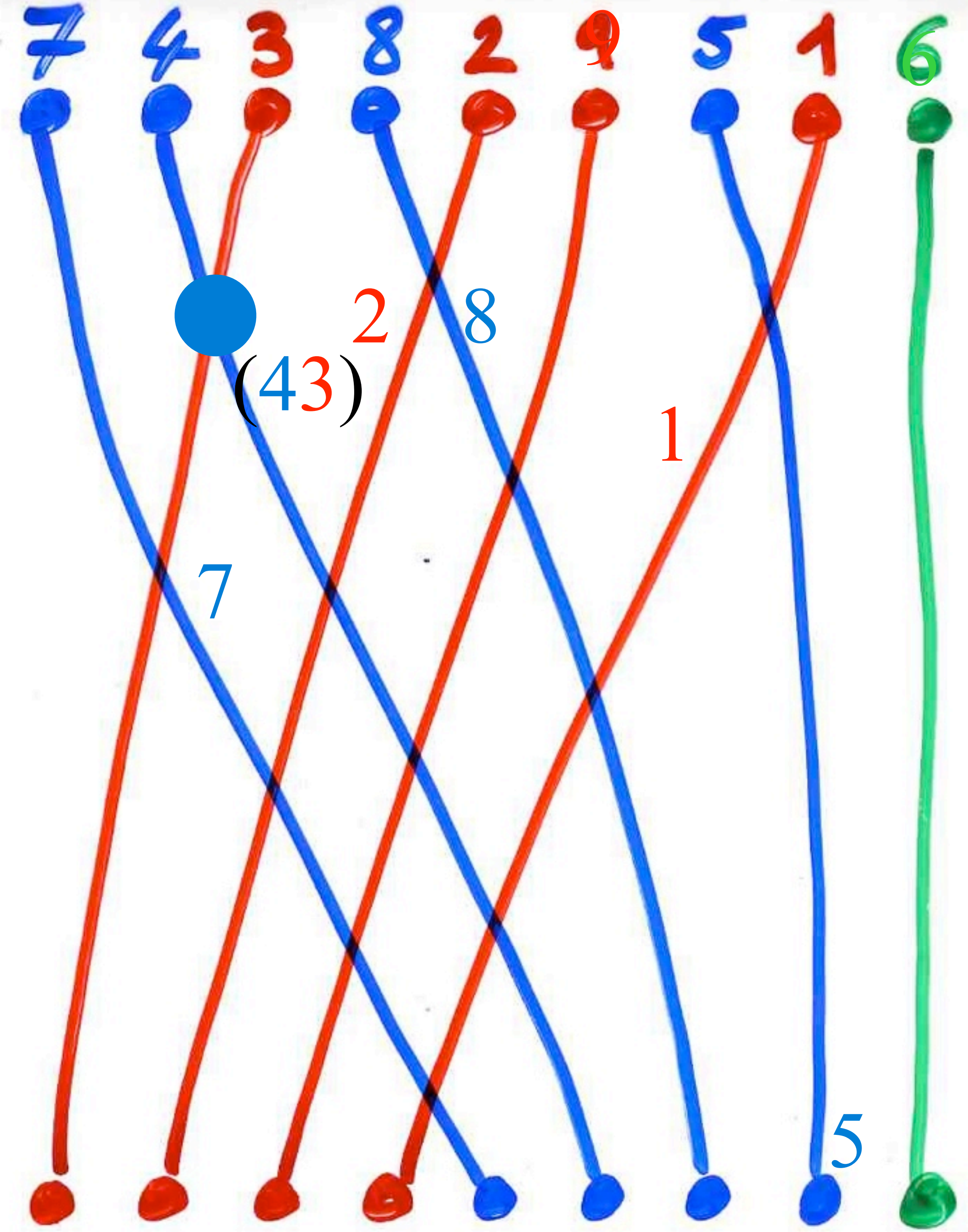

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2
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(43)
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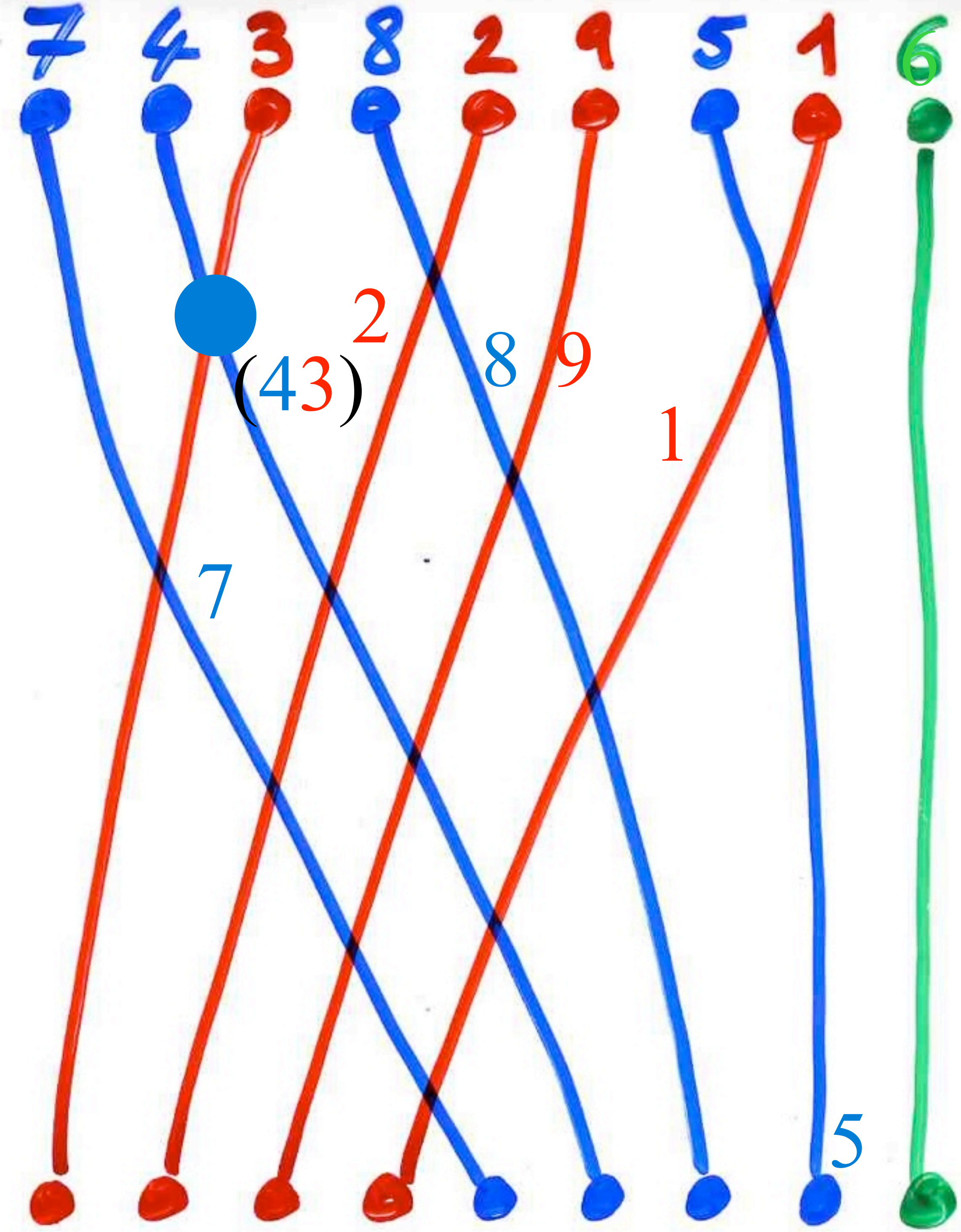
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(43)
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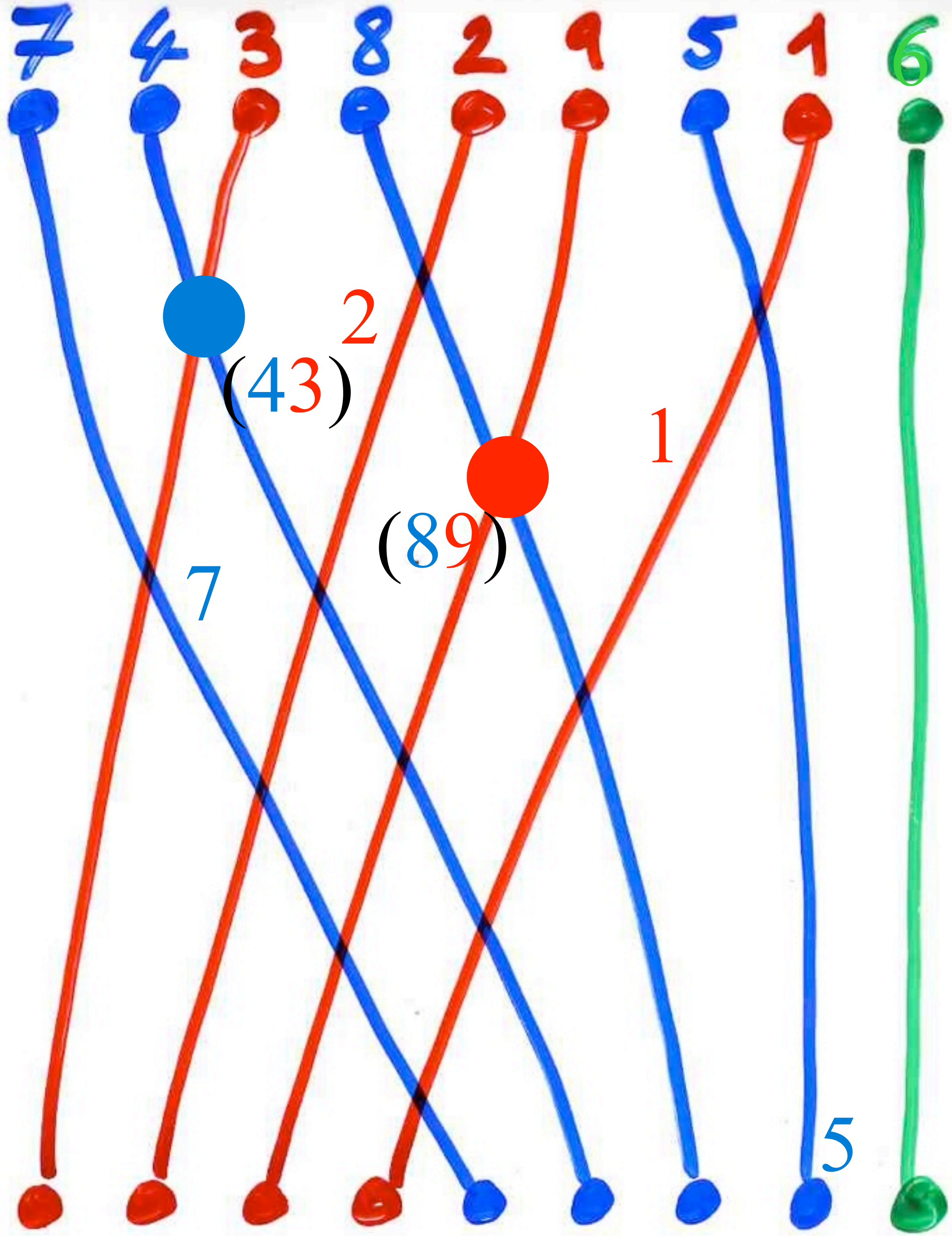
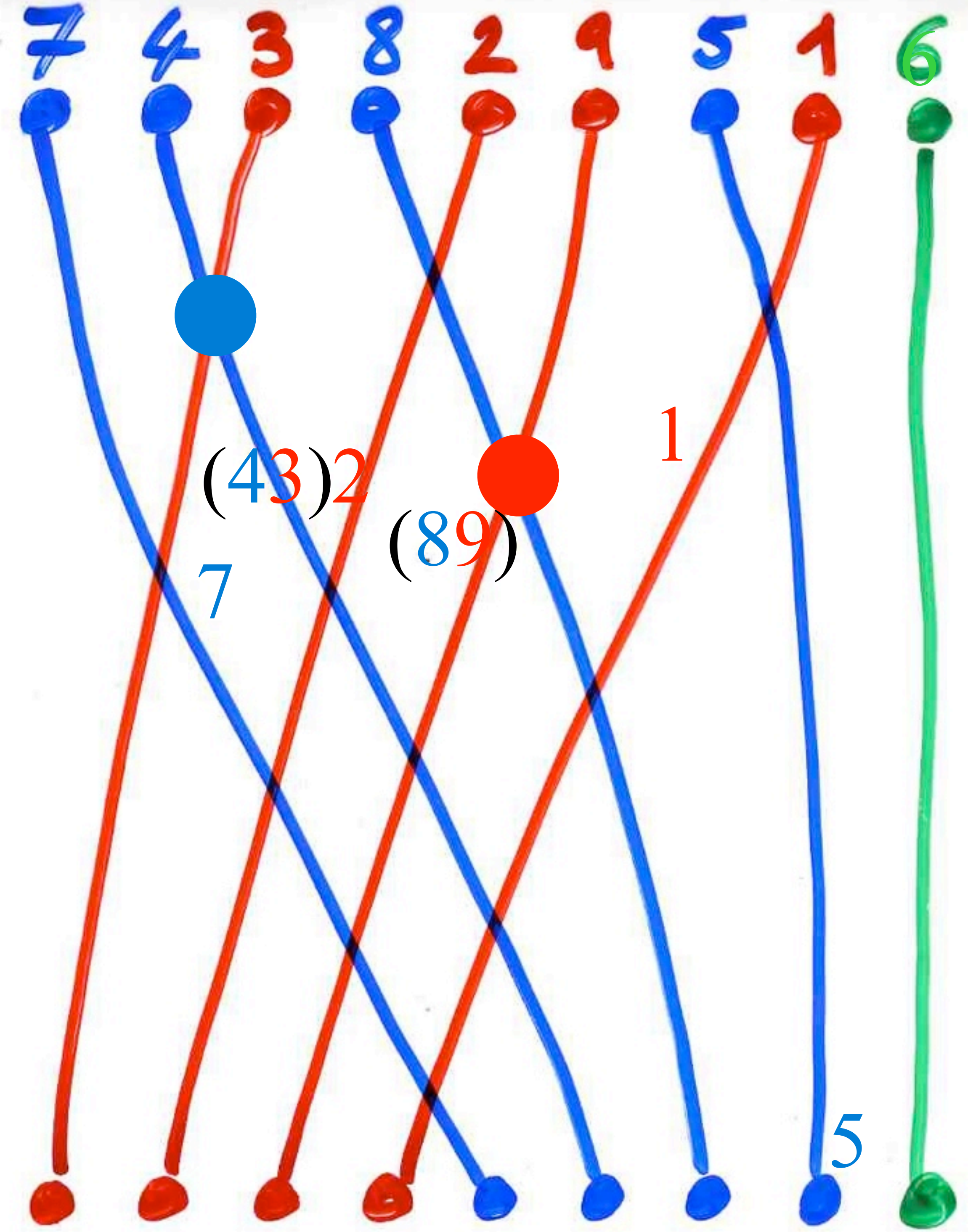

7 4 3 8 2 9 5 1 6
(43)
2
1
(89)
7
5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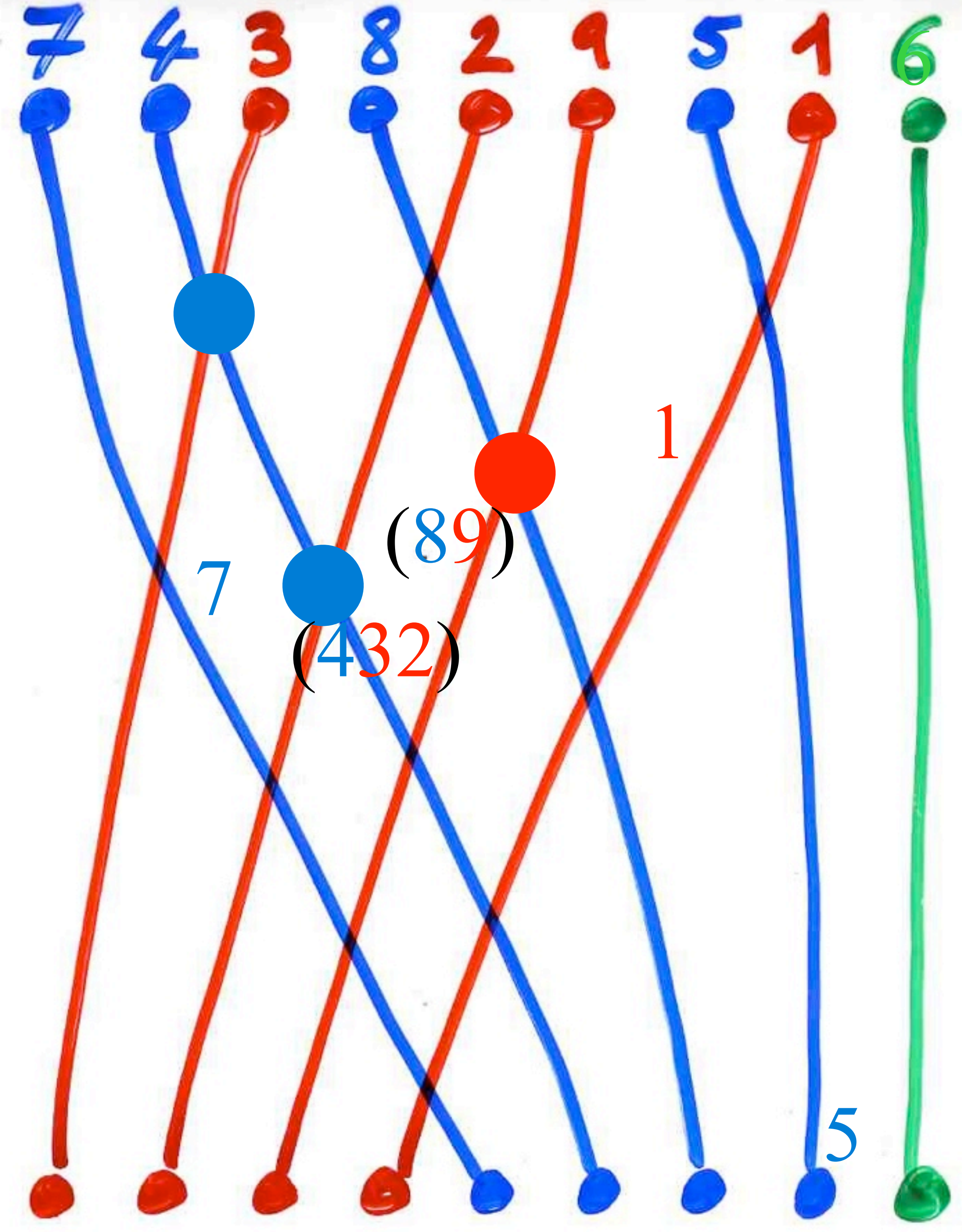
(43)2
(89)
1
7
5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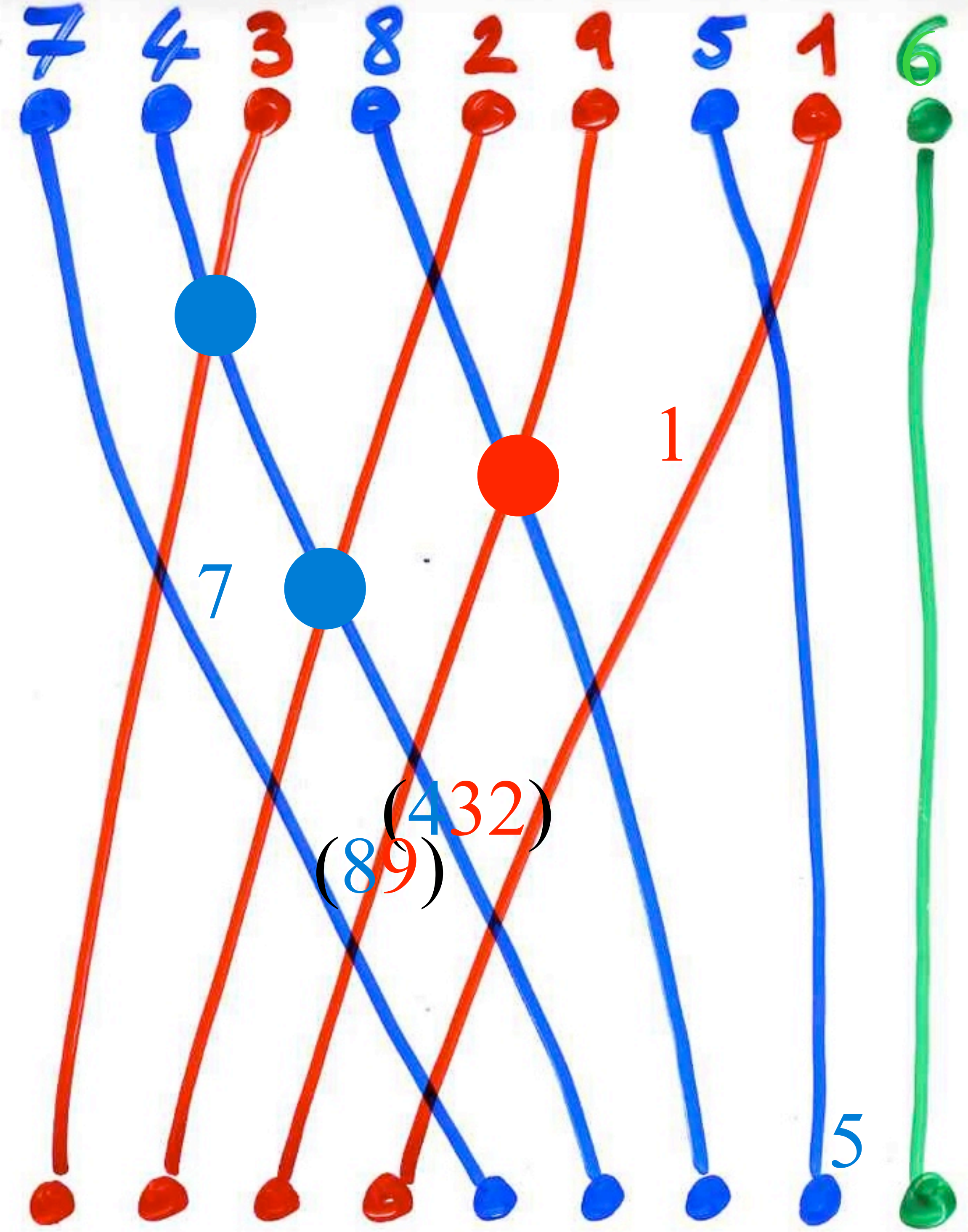
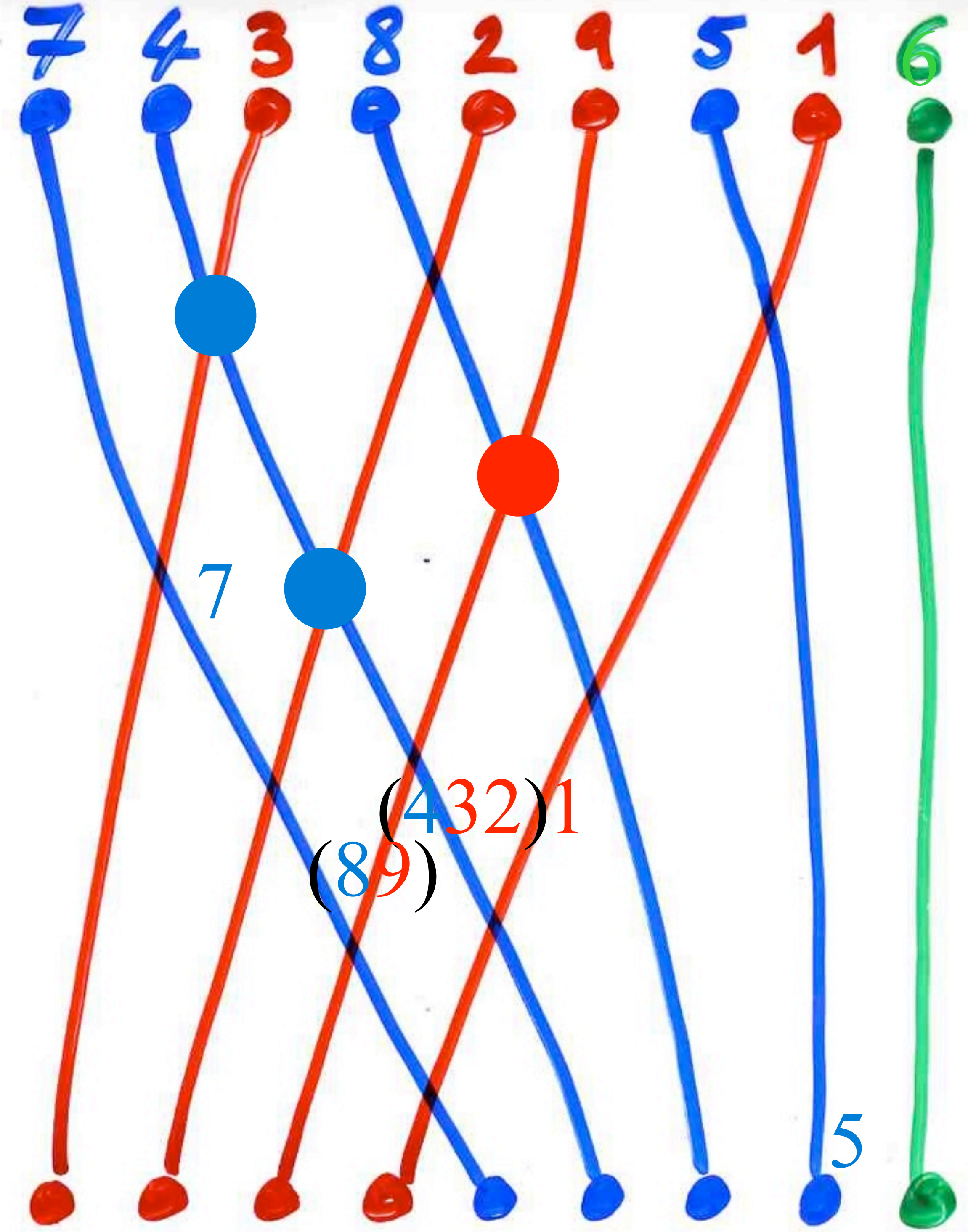
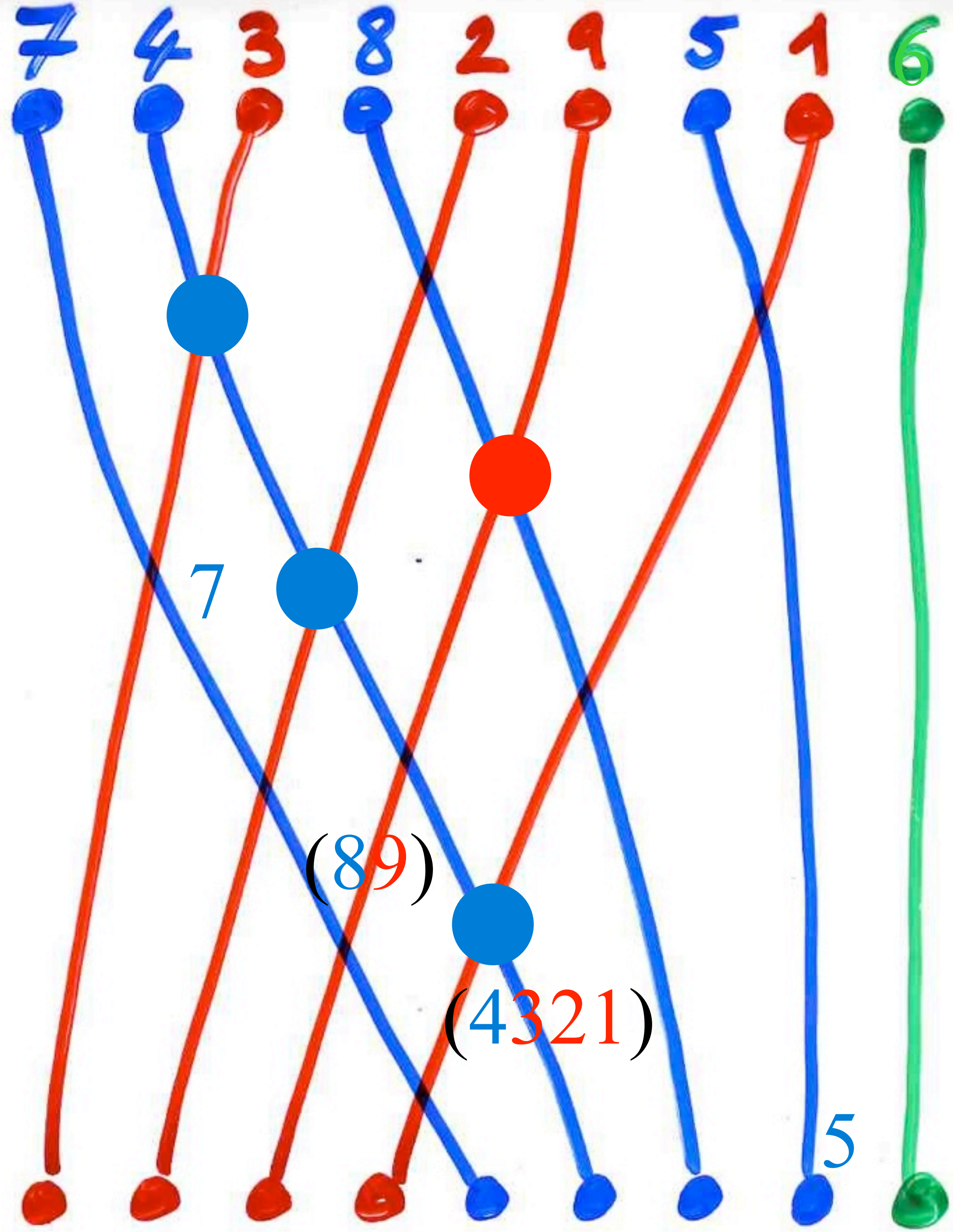
1
(89)
7
(432)
5



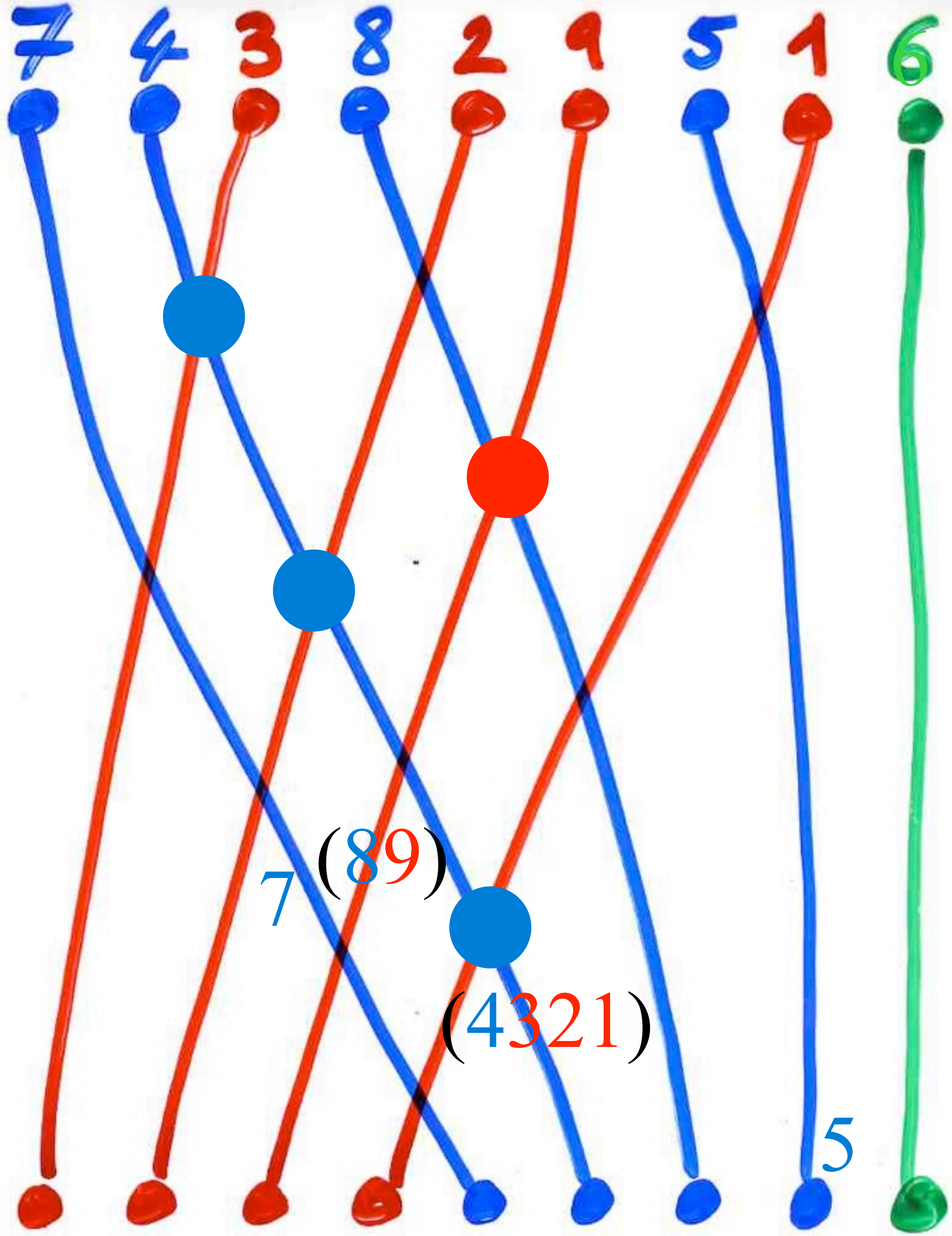
7 4 3 8 2 9 5 1 6
1
7
(432)
(89)
5



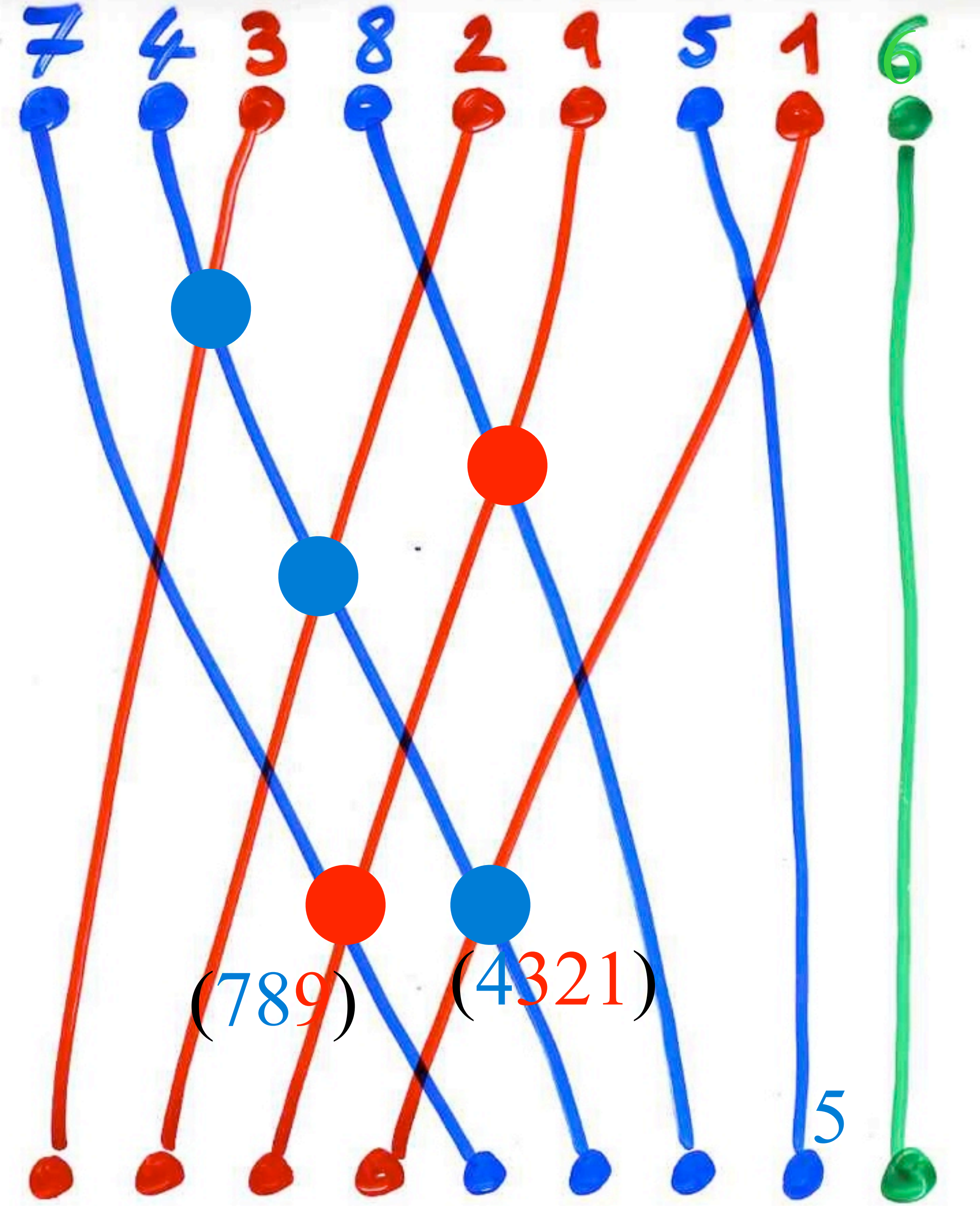
7 4 3 8 2 9 5 1 6
7
(432)1
(89)
5



7 4 3 8 2 9 5 1 6
7
(89)
(4321)
5

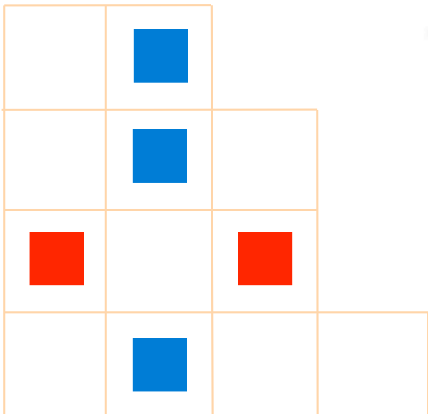


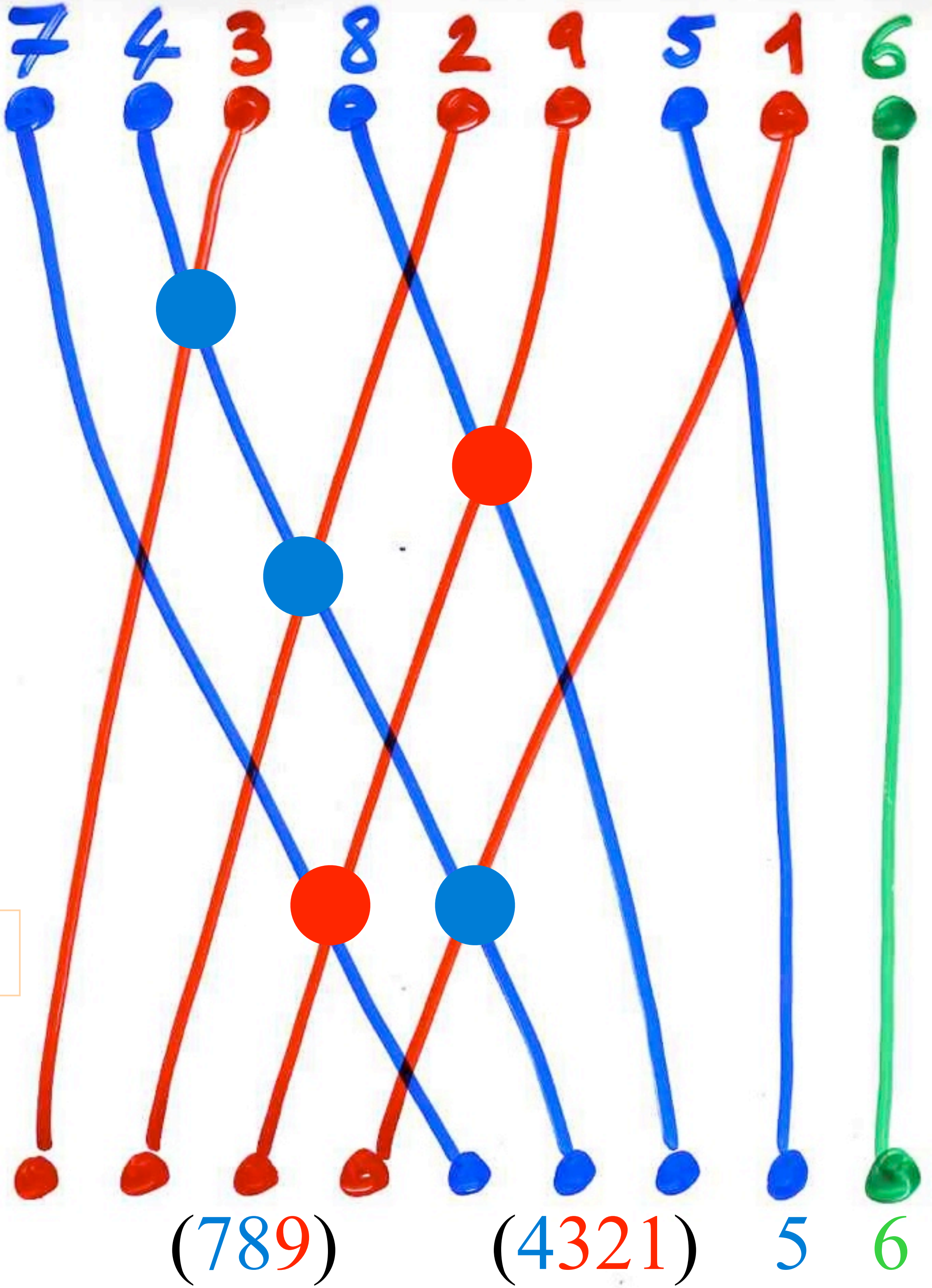
7 4 3 8 2 9 5 1 6
7
(89)
(4321)
5



7 4 3 8 2 9 5 1 6
(789)
(4321)
5

“exchange-fusion” algorithm





equivalent to a bijection

S. Corteel, P. Nadeau (2009)

with permutations tableaux

«canonical bijections»