Algèbres d'opérateurs
et
Physique combinatoire

(part 1)

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colloquium de l'IMJ
Institut mathématiques de Jussieu

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Physics

quantum mechanics:
spin chain model

Spin chains and combinatorics

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The XXZ quantum spin chain model with periodic boundary conditions is one of the most popular integrable models which has been investigating by the Bethe Ansatz method during the last 35 years [3]. It is described by the Hamiltonian

$$H_{XXZ} = -\sum_{j=1}^{N} \left\{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right\}, \qquad \vec{\sigma}_{N+1} = \vec{\sigma}_1. \tag{1}$$

The nonzero wave function components are

$$N=3: \quad \psi_{001}=1;$$
$$N=5: \quad \psi_{00011}=1, \quad \psi_{00101}=2;$$
$$N=7: \quad \psi_{0000111}=1, \quad \psi_{0001101}=\psi_{0001011}=3, \quad \psi_{0010011}=4 \quad \psi_{0010101}=7.$$

All components not included in the list can be obtained by shifting. Notice that the components of the ground state are positive in accordance with the Perron–Frobenius theorem.

Let us continue the list. For $N = 9$ the components of the eigenvector with the energy $-27/2$ and $S_z = -1/2$ are

$$\begin{array}{llll} \psi_{000001111}=1, & \psi_{000010111}=4, & \psi_{000011011}=6, & \psi_{000100111}=7, \\ \psi_{000101011}=17, & \psi_{000101101}=14, & \psi_{000110011}=12, & \psi_{001001011}=21, \\ & \psi_{001010011}=25, & \psi_{001010101}=42. & \end{array}$$

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We omit nonzero components which can be obtained by the reflection of the order of sites since this transformation is a symmetry of our state, as it is for the ground state. For example, we have

$$\psi_{000011101}=\psi_{000010111}=4.$$

1, 2, 7, 42, 429, ...

?

M1791 0, 1, 2, 7, 32, 181, 1214, 9403, 82508, 808393, 8743994, 103459471, 1328953592, 18414450877, 273749755382, 4345634192131, 73362643649444, 1312349454922513
$a(n)=n.a(n-1)+(n-2)a(n-2)$. Ref R1 188. [0,3; A0153, N0706]

$$\text{E.g.f.:}\quad (1-x)^{-3}\, e^{-x}.$$

M1792 1, 1, 2, 7, 32, 181, 1232, 9787, 88832, 907081, 10291712, 128445967, 1748805632, 25794366781, 409725396992, 6973071372547, 126585529106432
Expansion of $1/(1-\sinh x)$. Ref ARS 10 138 80. [0,3; A6154]

M1793 0, 1, 1, 2, 7, 32, 184, 1268, 10186, 93356, 960646, 10959452, 137221954, 1870087808, 27548231008, 436081302248, 7380628161076, 132975267434552
Stochastic matrices of integers. Ref DUMJ 35 659 68. [0,4; A0987, N0707]

M1794 1, 2, 7, 33, 192
Permutations of length n with n in second orbit. Ref C1 258. [2,2; A6595]

M1795 1, 2, 7, 34, 209, 1546, 13327, 130922, 1441729, 17572114, 234662231, 3405357682, 53334454417, 896324308634, 16083557845279, 306827170866106
$a(n)=2n.a(n-1)-(n-1)^2a(n-2)$. Ref SE33 78. [0,2; A2720, N0708]

M1796 1, 2, 7, 34, 257, 2606, 32300, 440564, 6384634
Polyhedra with n nodes. Ref GR67 424. UPG B15. Dil92. [4,2; A0944, N0709]

M1797 2, 7, 35, 219, 1594, 12935, 113945, 1070324, 10586856, 109259633, 1168384157, 12877168147, 145656436074, 1685157199175, 19886174611045
Two-rowed truncated monotone triangles. Ref JCT A42 277 86. Zei93. [1,1; A6947]

M1798 1, 1, 2, 7, 35, 228, 1834, 17382, 195866, 2487832, 35499576, 562356672, 9794156448, 186025364016, 3826961710272, 84775065603888, 2011929826983504
Coefficients of iterated exponentials. Ref SMA 11 353 45. [0,3; A0154, N0710]

M1799 1, 2, 7, 35, 228, 1834, 17582, 195866, 2487832, 35499576, 562356672, 9794156448, 186025364016, 3826961710272, 84775065603888, 2011929826983504
Expansion of ln $(1+\ln(1+x))$. [0,2; A3713]

M1800 1, 0, 1, 2, 7, 36, 300, 3218, 42335, 644808
Circular diagrams with n chords. Ref BarN94. [0,4; A7474]

M1801 1, 2, 7, 36, 317, 5624, 251610, 33642660, 14685630688
$n \times n$ binary matrices. Ref CPM 89 217 64. SLC 19 79 88. [0,2; A2724, N0711]

M1802 2, 7, 37, 216, 1780, 32652
Semigroups of order n with 2 idempotents. Ref MAL 2 2 67. SGF 14 71 77. [2,1; A2787, N0712]

M1803 1, 2, 7, 37, 266, 2431, 27007, 353522, 5329837, 90960751, 1733584106, 36496226977, 841146804577, 21065166341402, 569600638022431
$a(n)=(2n-1)a(n-1)+a(n-2)$. Ref RCI 77. [0,2; A1515, N0713]

M1804 1, 1, 2, 7, 38, 291, 2932, 36961, 561948, 10026505, 205608536, 4767440679, 123373203208, 3525630110107, 110284283006640, 3748357699560961
Forests of labeled trees with n nodes. Ref JCT 5 96 68. SIAD 3 574 90. [0,3; A1858, N0714]

M1805 1, 1, 2, 7, 40, 357, 4824, 96428, 2800472, 116473461
n-element partial orders contained in linear order. Ref nbh. [0,3; A6455]

M1806 1, 2, 7, 41, 346, 3797, 51157, 816356, 15050581, 314726117, 7359554632, 190283748371, 5389914888541, 165983936096162, 5521346346543307
Planted binary phylogenetic trees with n labels. Ref LNM 884 196 81. [1,2; A6677]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \, . \, . \, n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 64539836938720749739356
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

M1810 0, 1, 2, 7, 44, 361, 3654, 44207, 622552, 10005041, 180713290, 3624270839, 79914671748, 1921576392793, 50040900884366, 1403066801155039
Modified Bessel function $K_n(1)$. Ref AS1 429. [0,3; A0155, N0716]

M1811 0, 1, 2, 7, 44, 447, 6749, 142176, 3987677, 143698548, 6470422337, 356016927083, 23503587609815, 1833635850492653, 166884365982441238
$a(n)=n(n-1)a(n-1)/2+a(n-2)$. [0,3; A1046, N0717]

M1812 1, 2, 7, 44, 529, 12278, 565723, 51409856, 9371059621, 3387887032202, 2463333456292207, 3557380311703796564, 10339081666350180289849
Sum of Gaussian binomial coefficients $[n,k]$ for $q=4$. Ref TU69 76. GJ83 99. ARS A17 328 84. [0,2; A6118]

M1813 2, 7, 52, 2133, 2590407, 3374951541062, 5695183504479116640376509, 16217557574922386301420514191523784895639577710480
Free binary trees of height n. Ref JCIS 17 180 92. [1,1; A5588]

M1814 1, 1, 2, 7, 56, 2212, 2595782, 3374959180831, 5695183504489239067484387, 16217557574922386301420531277071365103168734284282
Planted 3-trees of height n. Ref RSE 59(2) 159 39. CMB 11 87 68. JCIS 17 180 92. [0,3; A2658, N0718]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
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M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 64539836938720749739356
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Combinatorics

ASM

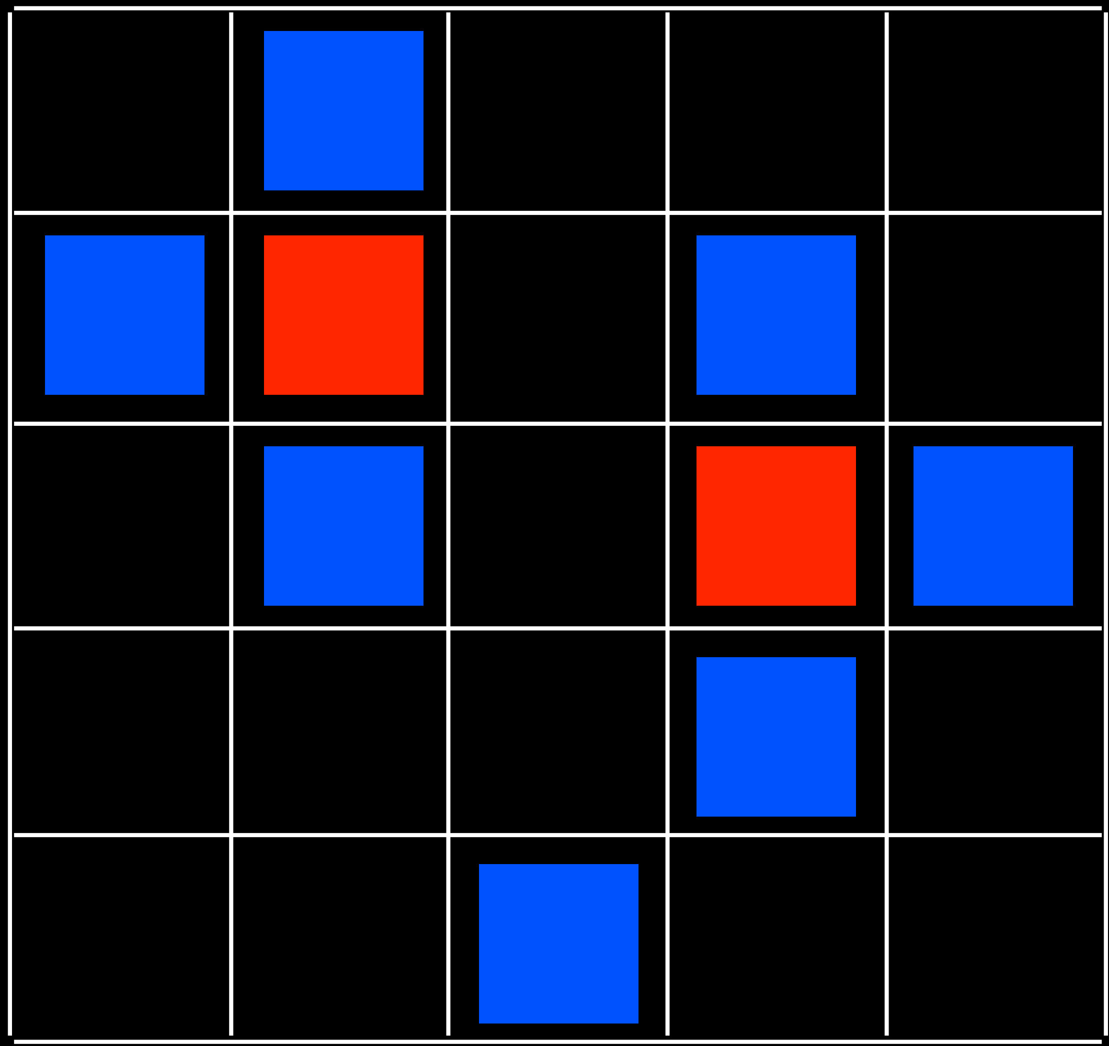
Alternating sign matrices

Alternating sign matrices

- entries: 0, 1, -1
- sum in rows and columns = 1
- non 0 entries alternate in sign
  in each row and column

ex:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



Permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad + \underset{\text{permutations}}{6}$$

$$1,\ 2,\ 7,\ 42,\ 429,\ \ldots$$

" What else have you got in your pocket ?" he
went on, turning to A''

" Only a thimble,"

" Hand it over her

Then they all crow

while the Dodo solem

Lewis Carroll

"Alice aux pays des merveilles"

C. I. Dodgson (1866)

Condensation of determinants

$$det(M) = \frac{M_{NO} M_{SE} - M_{NE} M_{SO}}{M_C}$$

enumeration of ASM

$$1,\ 2,\ 7,\ 42,\ 429,\ \ldots.$$

$$\frac{1!\ 4!\ \ ?\ \ (3n-2)!}{n!\ (n+1\ \ ?\ \ (n+n-1)!}$$

alternating sign matrices conjecture

Mills, Robbins, Rumsey (1982)

Robbins

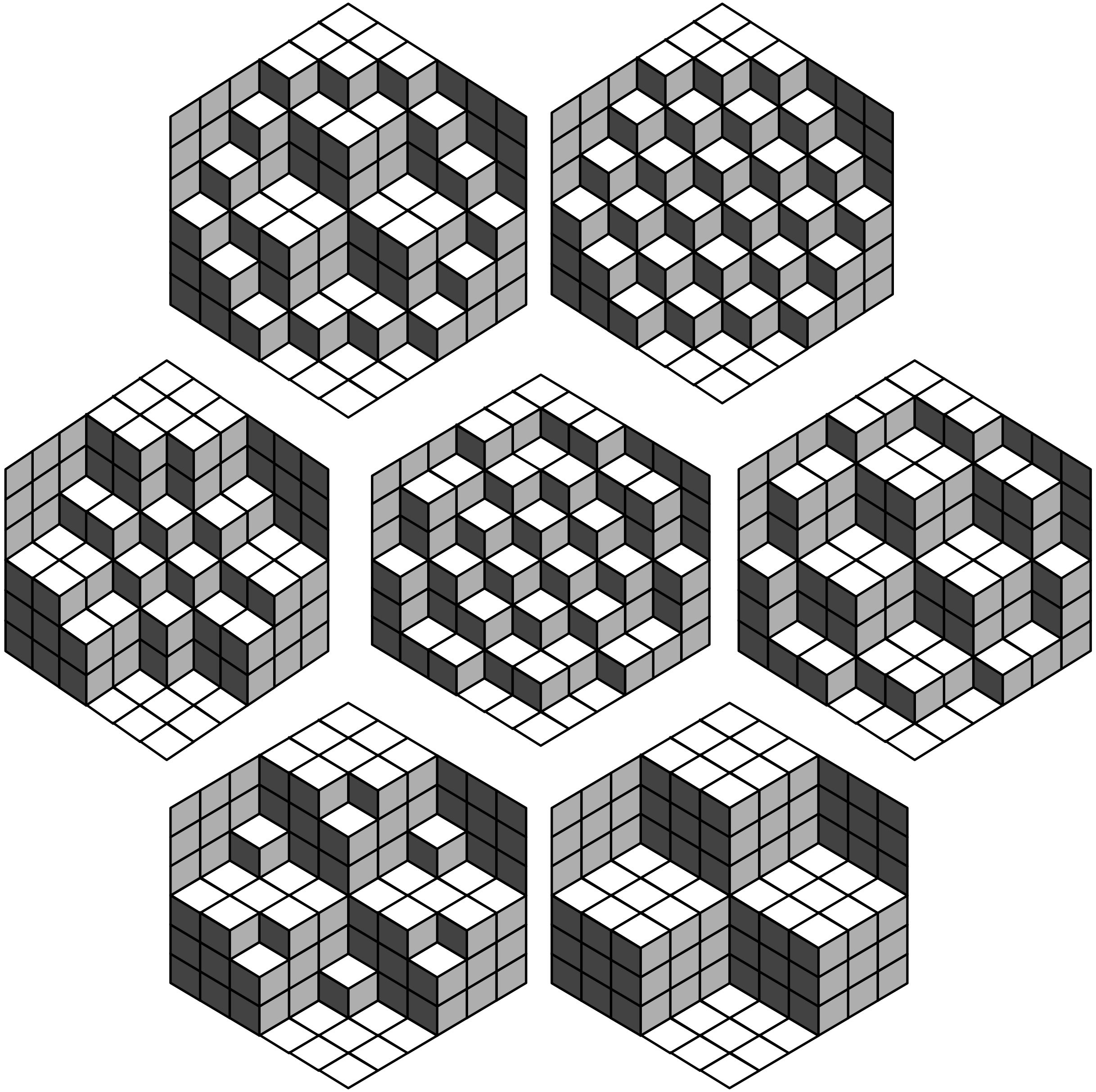
The Mathematical Intelligencer (1991)

“These conjectures are of such compelling simplicity that it is hard to understand how any mathematician can bear the pain of living without understanding why they are true”

D. Zeilberger (1992- 1995)

(+ 90 checkers)

Proof of the A. S. M. conj.



the last of the ten

the T.S.S.C.P.P. conjecture

"tour de force" G. Andrews (1994)

$$\frac{1!\,4!\,7!\cdots(3n-2)!}{n!\,(n+1)!\,(n+2)!\cdots(n+n-1)!}$$

Kuperberg (1995)

6- vertex model

(ice model)

with domain wall boundary condition

Proofs and Confirmations
The story of the alternating sign matrix conjecture

David M. Bressoud
Macalester College
Saint Paul, MN

July 28, 1997

Razumov - Stroganov
(ex)- conjecture 2000-2001

proof by : L. Cantini and A.Sportiello (March 2010)

The nonzero wave function components are

$$
\begin{aligned}
N = 3: &\quad \psi_{001} = 1; \\
N = 5: &\quad \psi_{00011} = 1, \quad \psi_{00101} = 2; \\
N = 7: &\quad \psi_{0000111} = 1, \quad \psi_{0001101} = \psi_{0001011} = 3, \quad \psi_{0010011} = 4 \quad \psi_{0010101} = 7.
\end{aligned}
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All components not included in the list can be obtained by shifting. Notice that the components of the ground state are positive in accordance with the Perron–Frobenius theorem.

Let us continue the list. For $N = 9$ the components of the eigenvector with the energy $-27/2$ and $S_z = -1/2$ are

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$$

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$$\psi_{000011101} = \psi_{000010111} = 4.$$

Combinatorial physics

Physique combinatoire

Quadratic algebras
and
physics

Heisenberg
operators
U, D

$$UD = DU + 1$$

"normal ordering"

$$UD = DU + I$$

Lemma

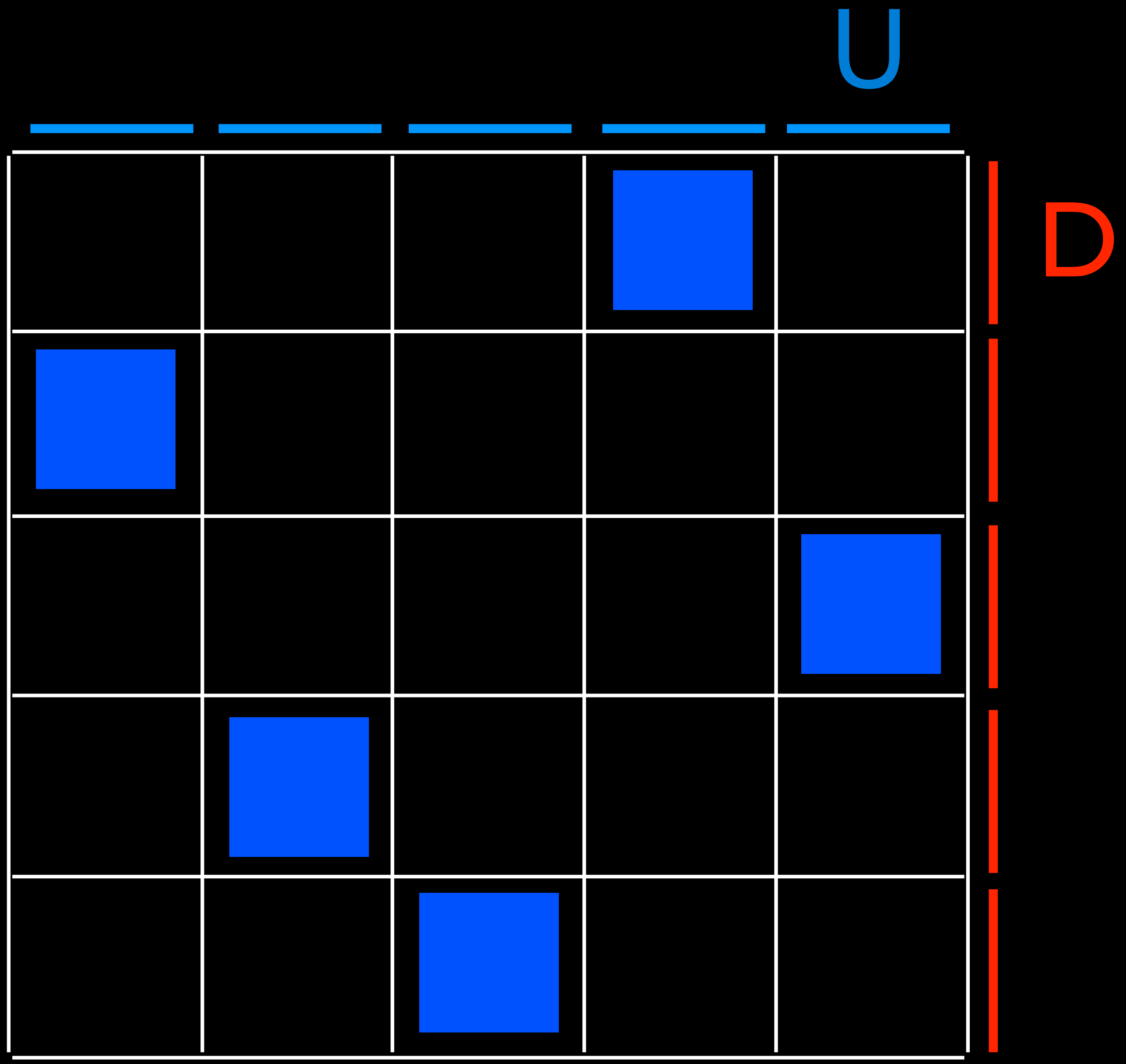
Every word w with letters U and D can be written in a unique way

$$w = \sum_{i,j \geq 0} c_{i,j}(w) D^i U^j$$

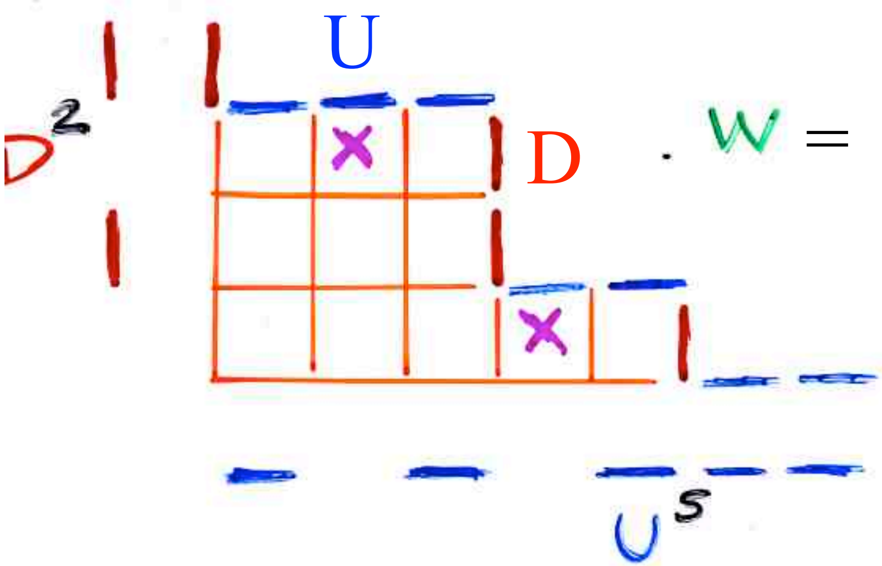
$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} D^i U^i$$

normal ordering

$$c_{n,0} = n!$$



permutations n!



$w = D\,U\,U\,U\,D\,D\,U\,U\,D\,U\,U$

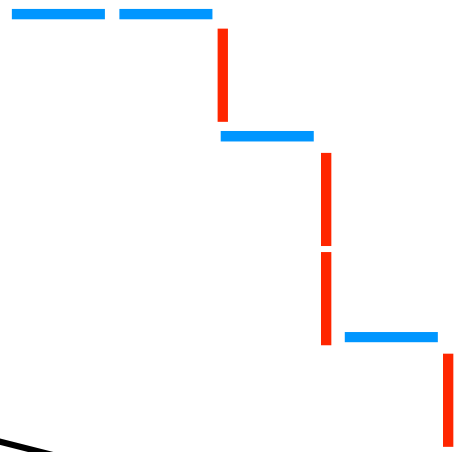
Towers
placements on a
Ferrers diagram

The PASEP algebra

$$DE = qED + E + D$$

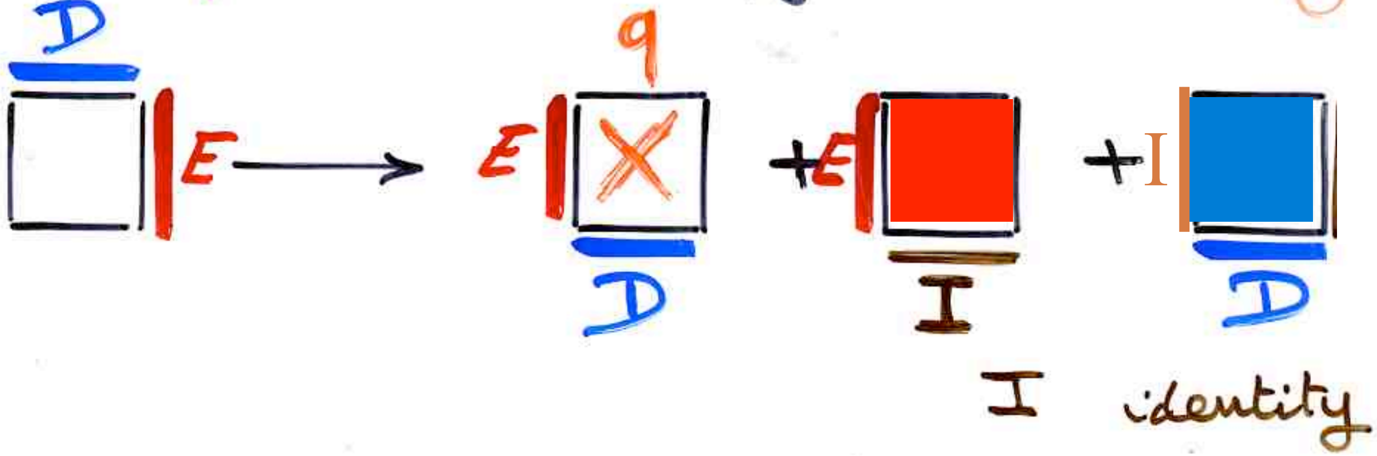
D D E D E E D E

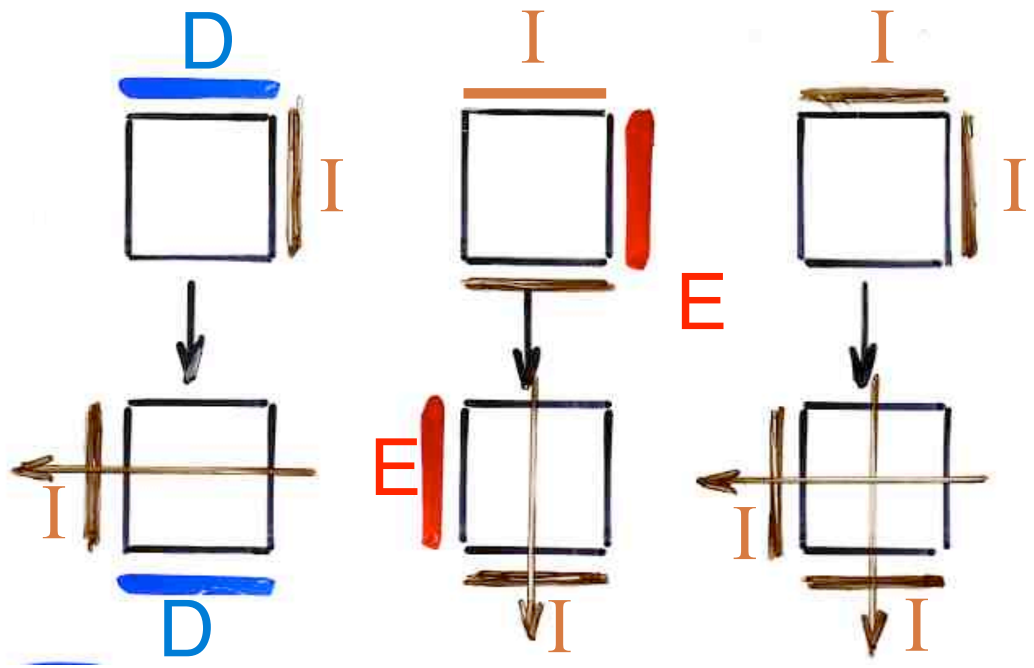
D DE (D E) E D E

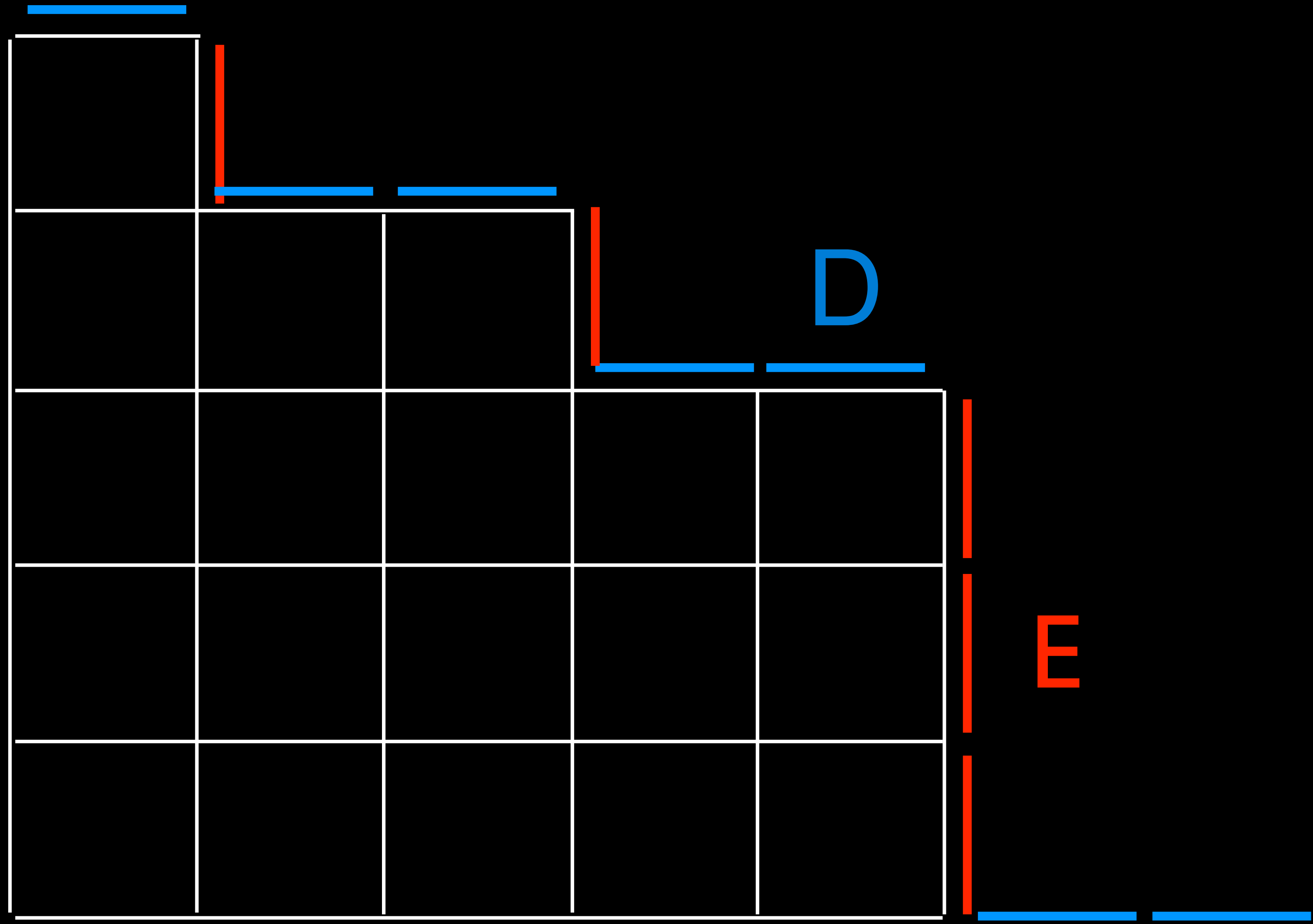


DDE(E)EDE + DDE(ED)EDE + DDE(D)EDE

Proof: "planarization" of the rewriting rules



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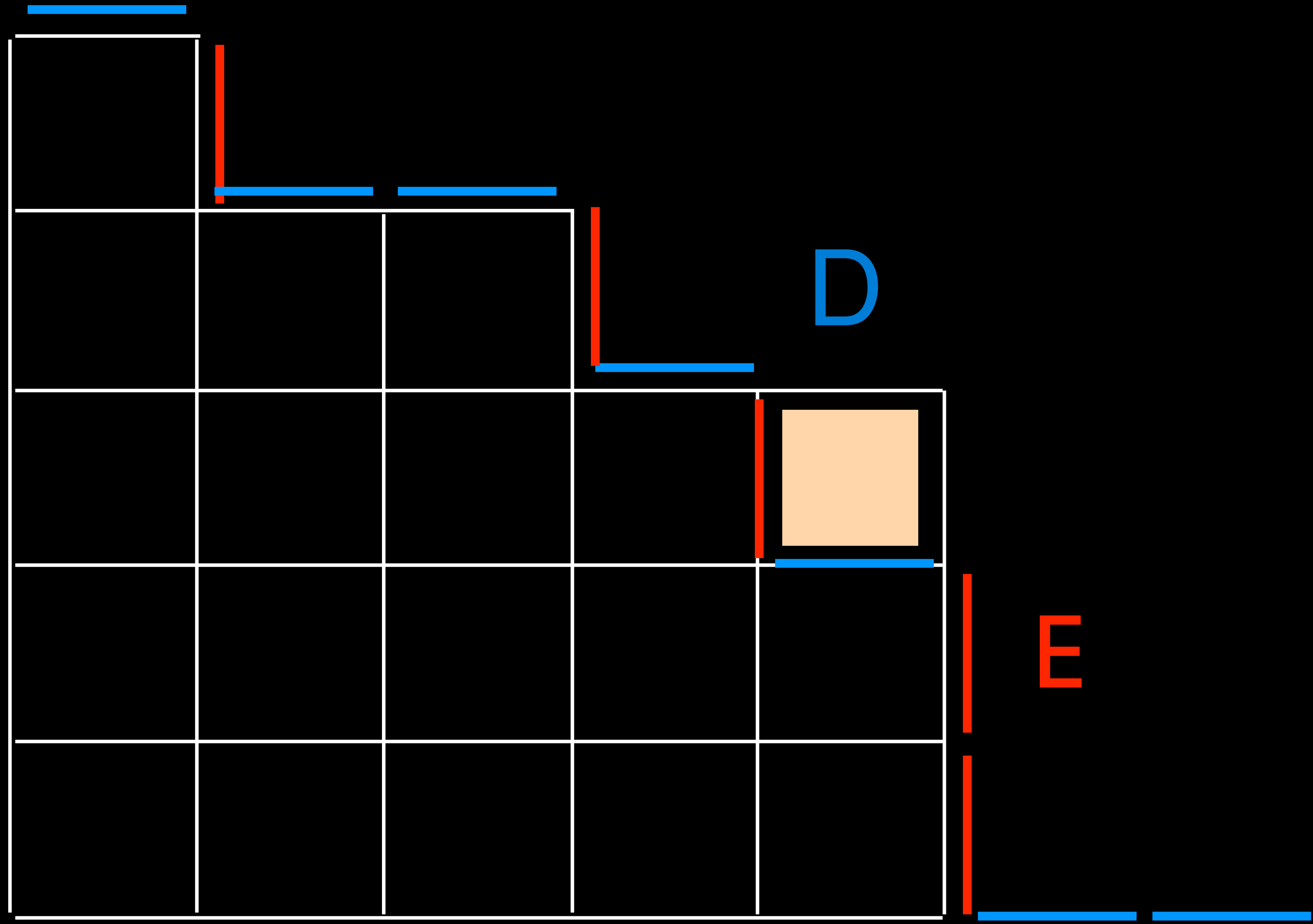
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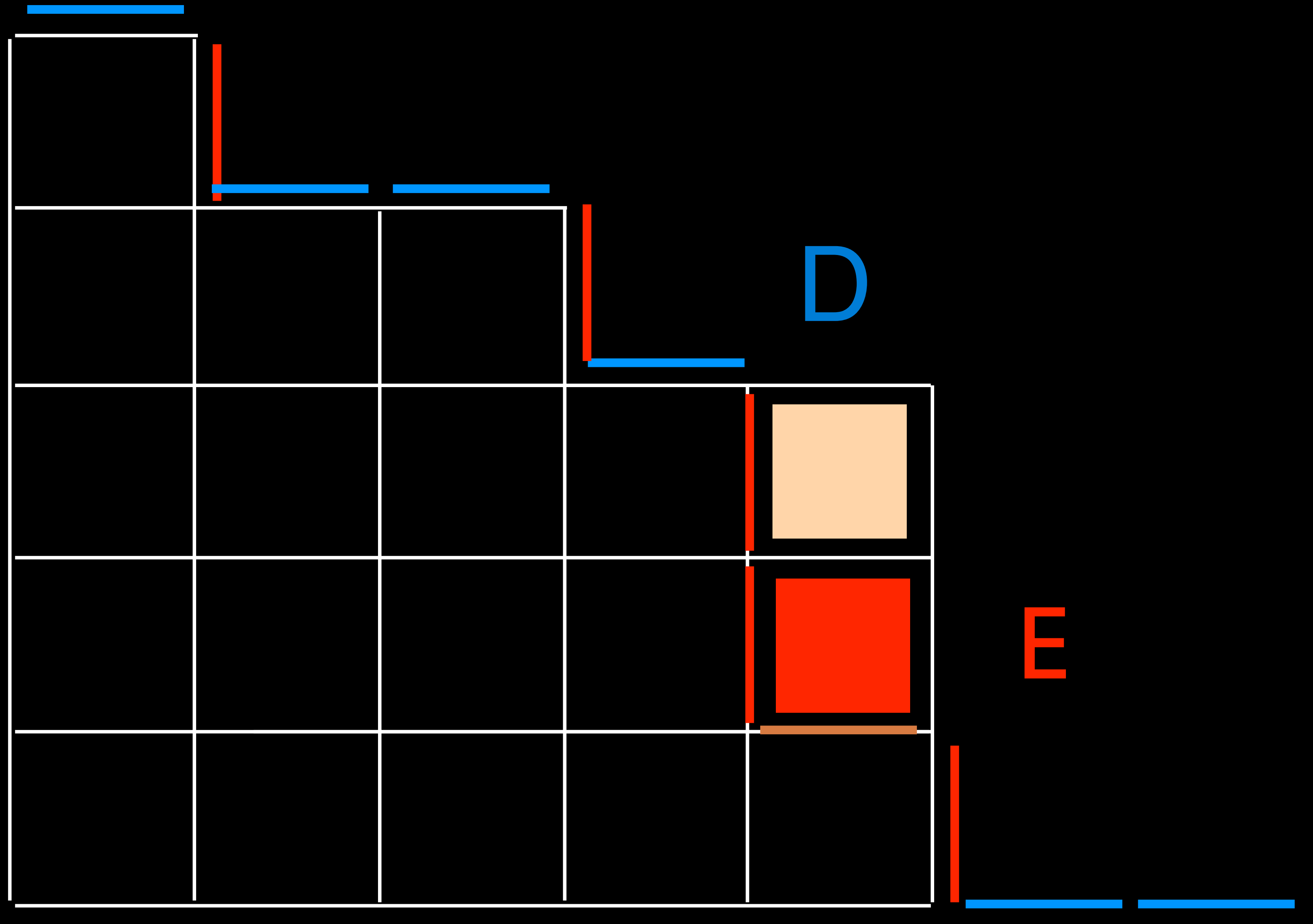
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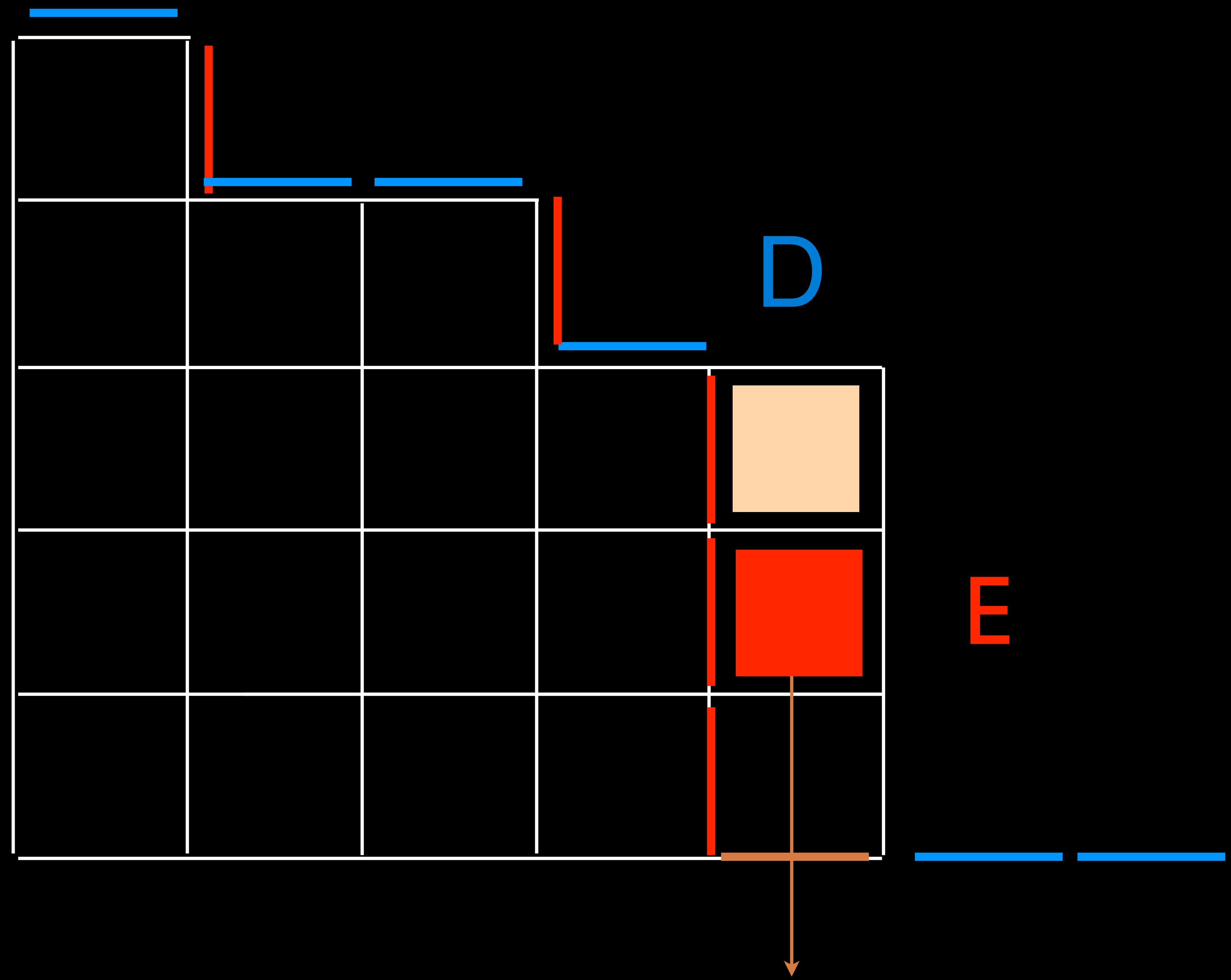
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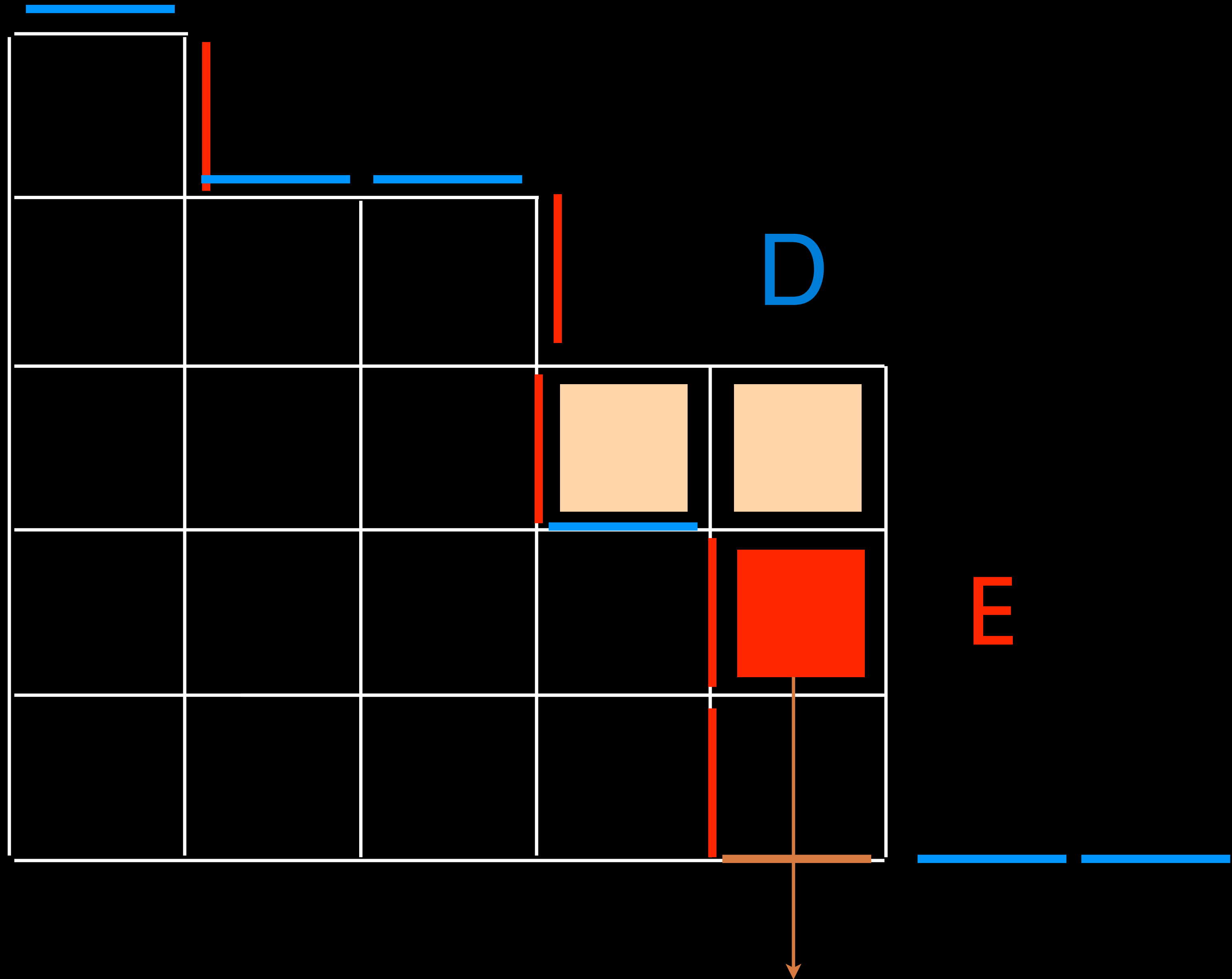
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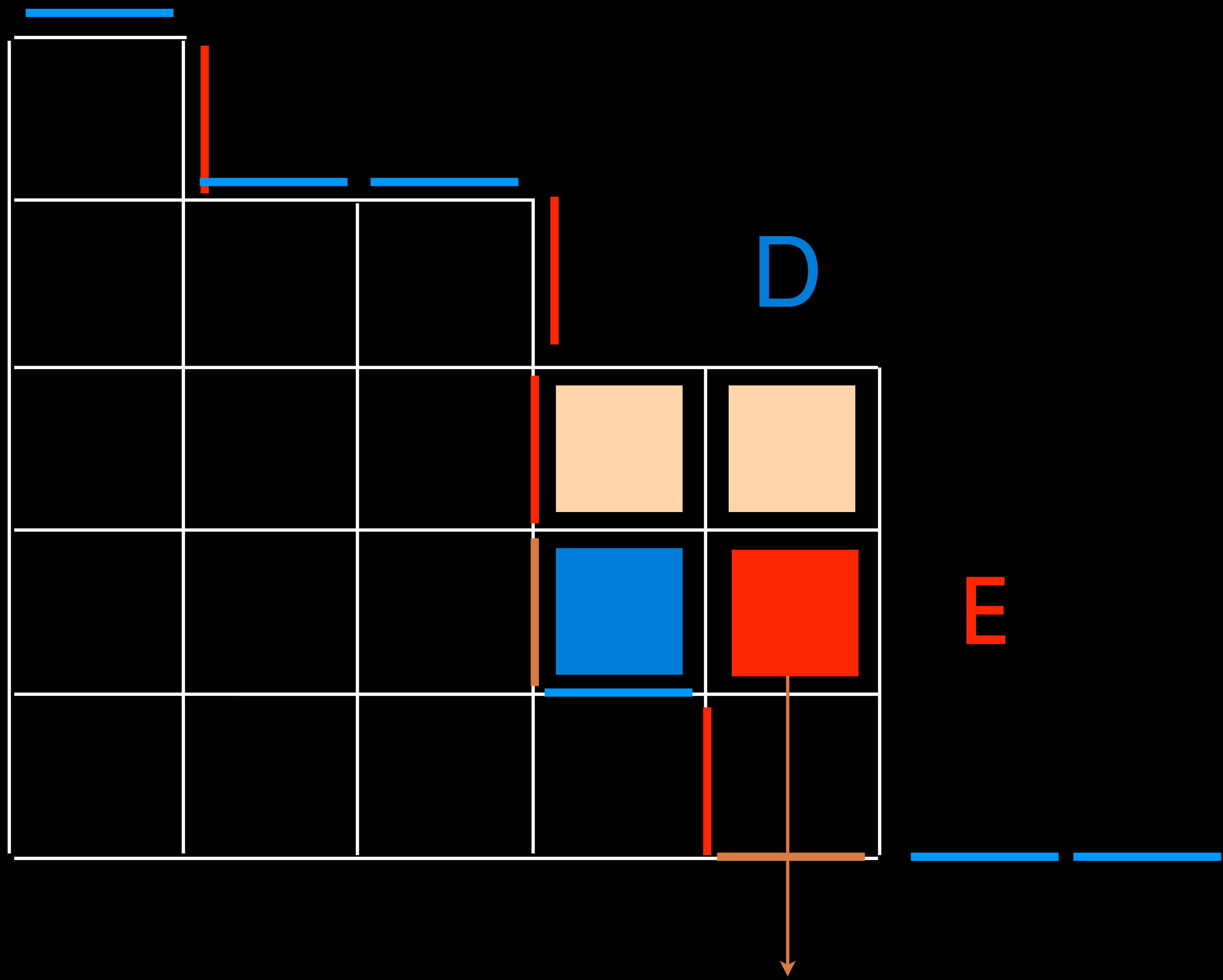
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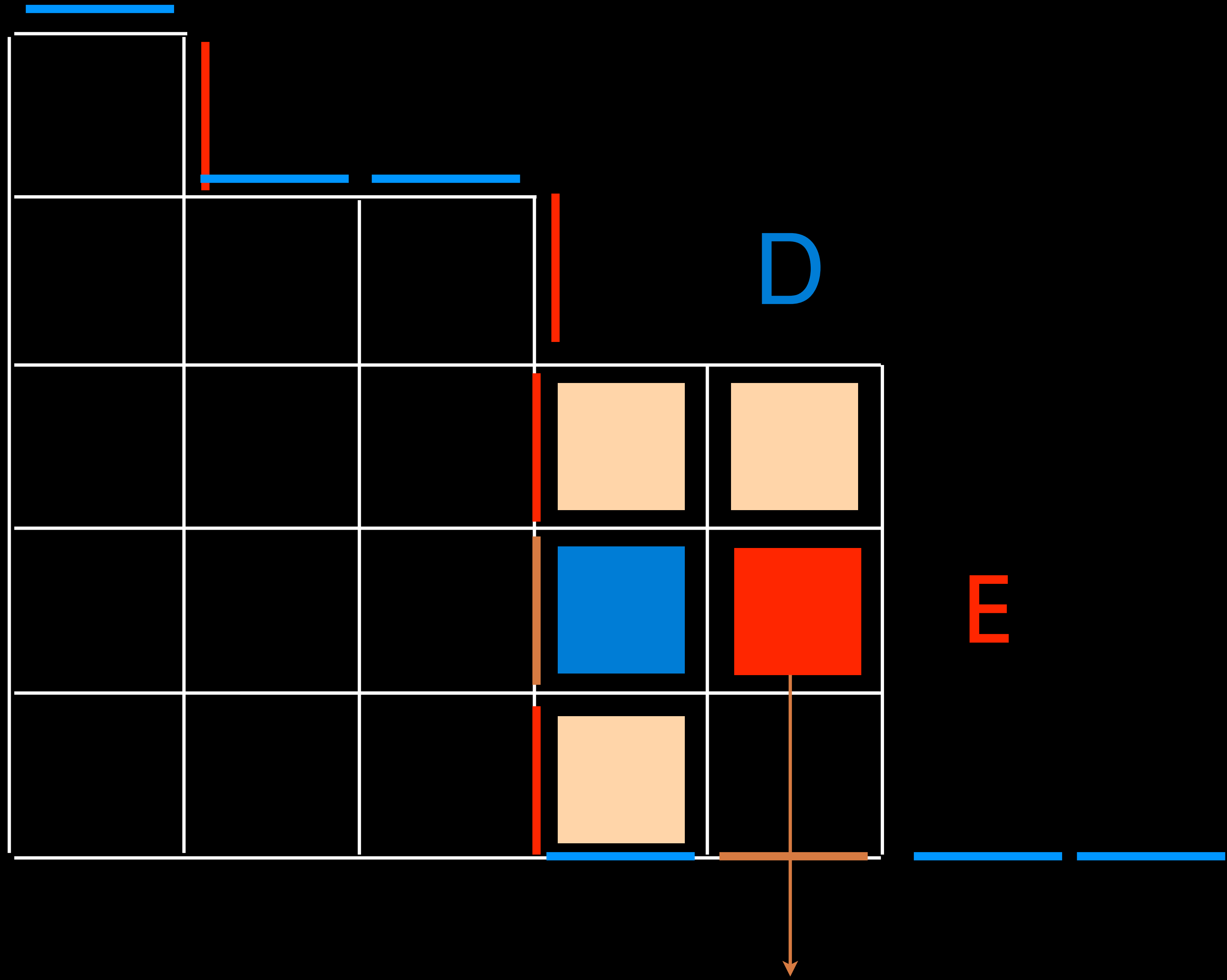
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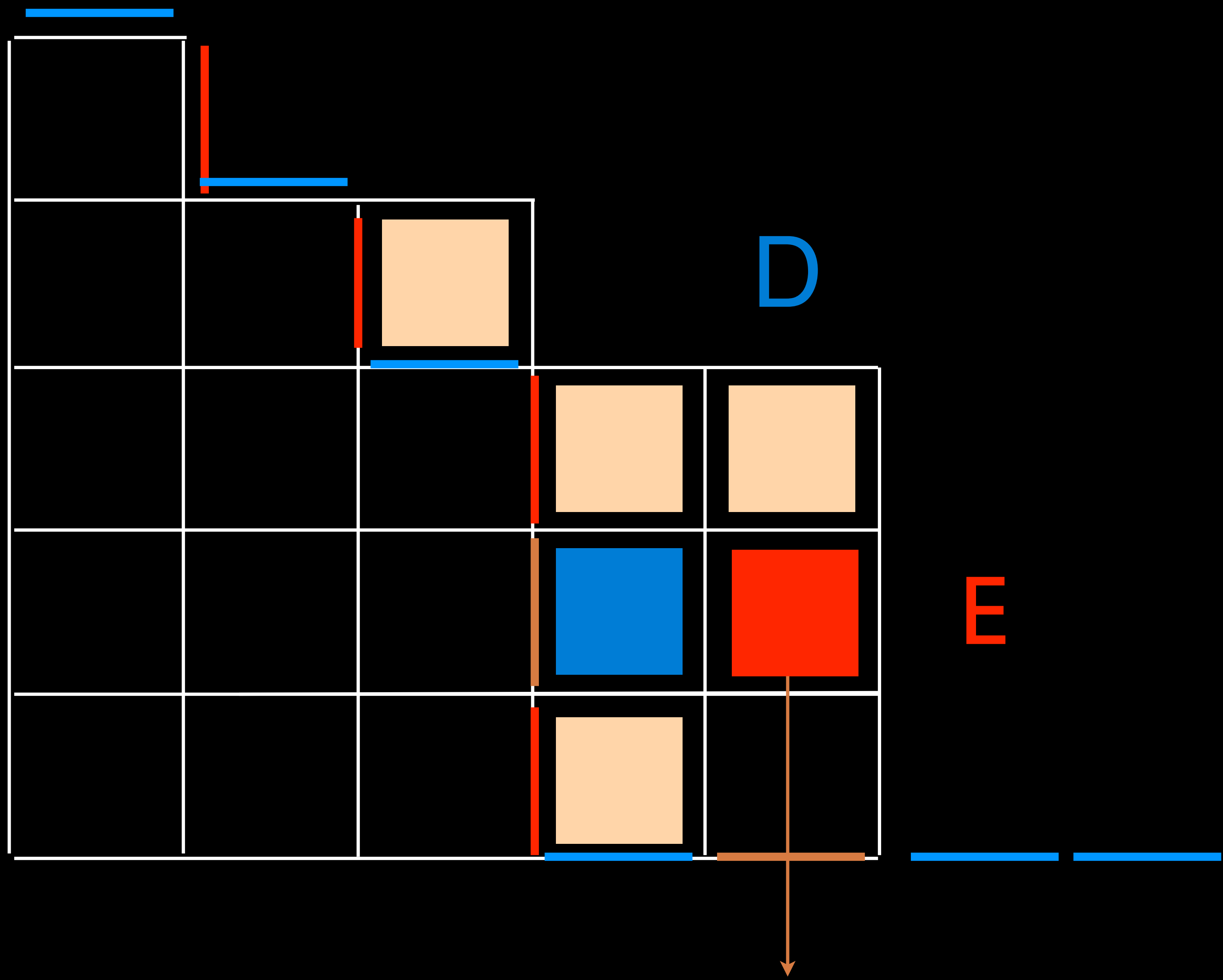
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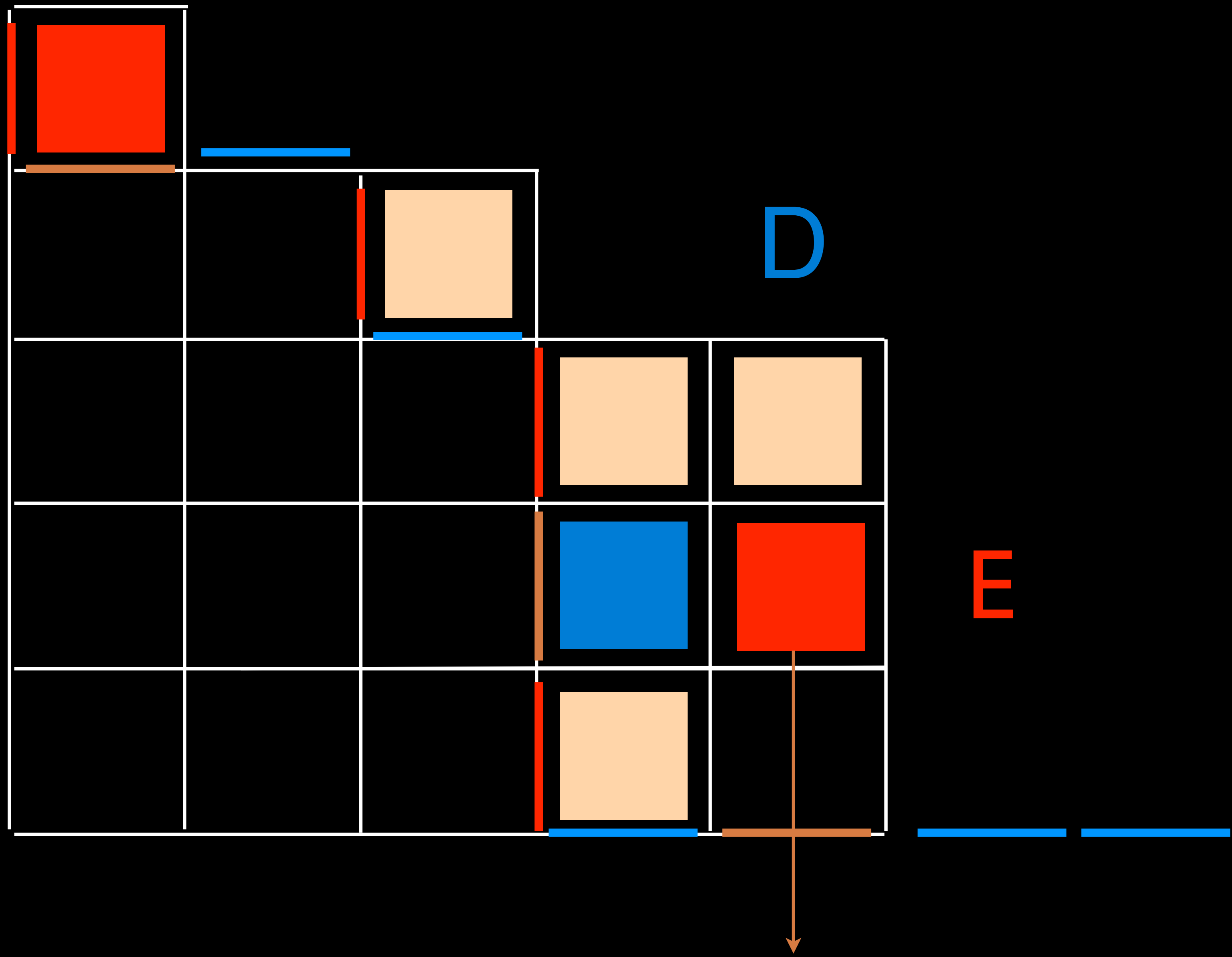
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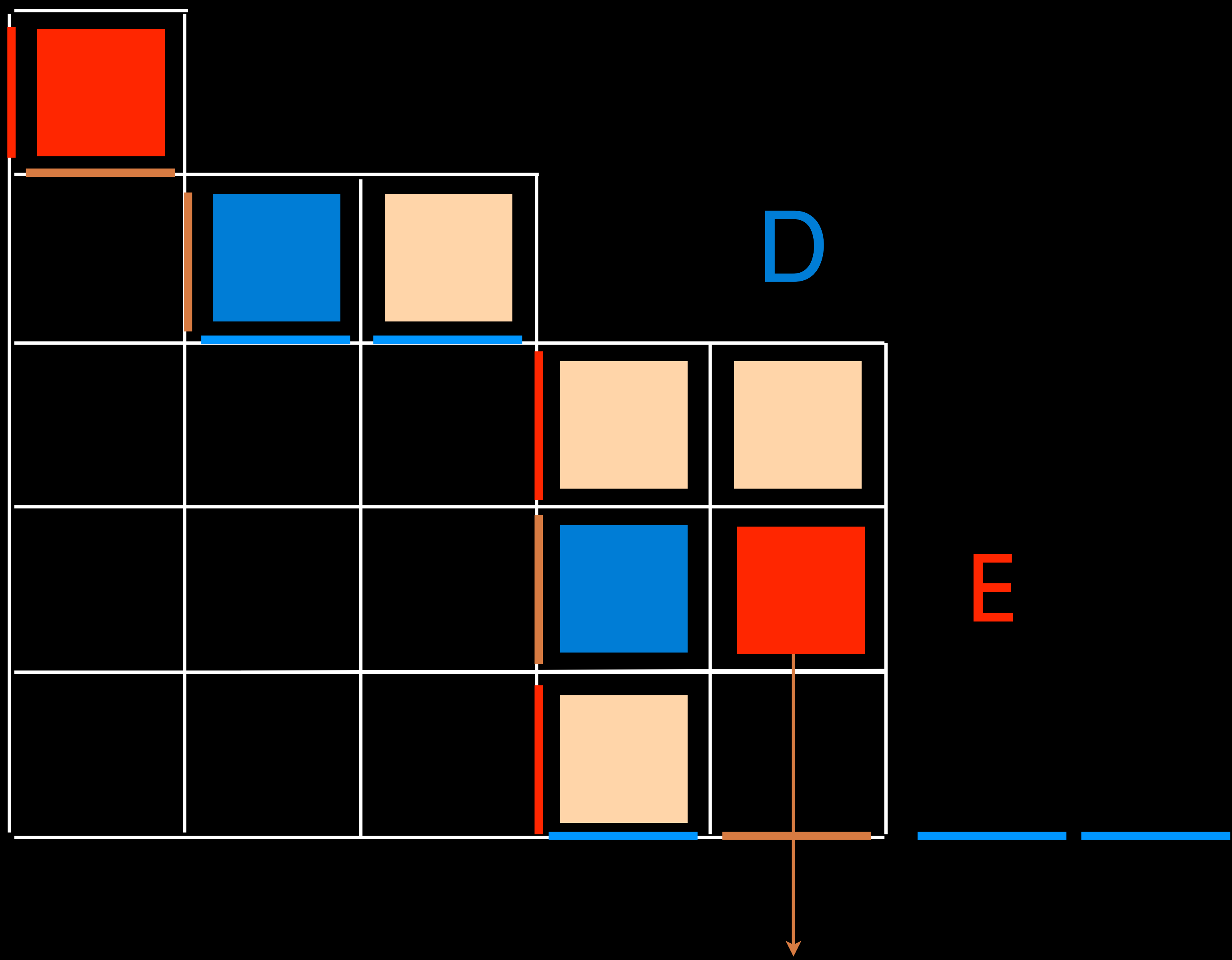
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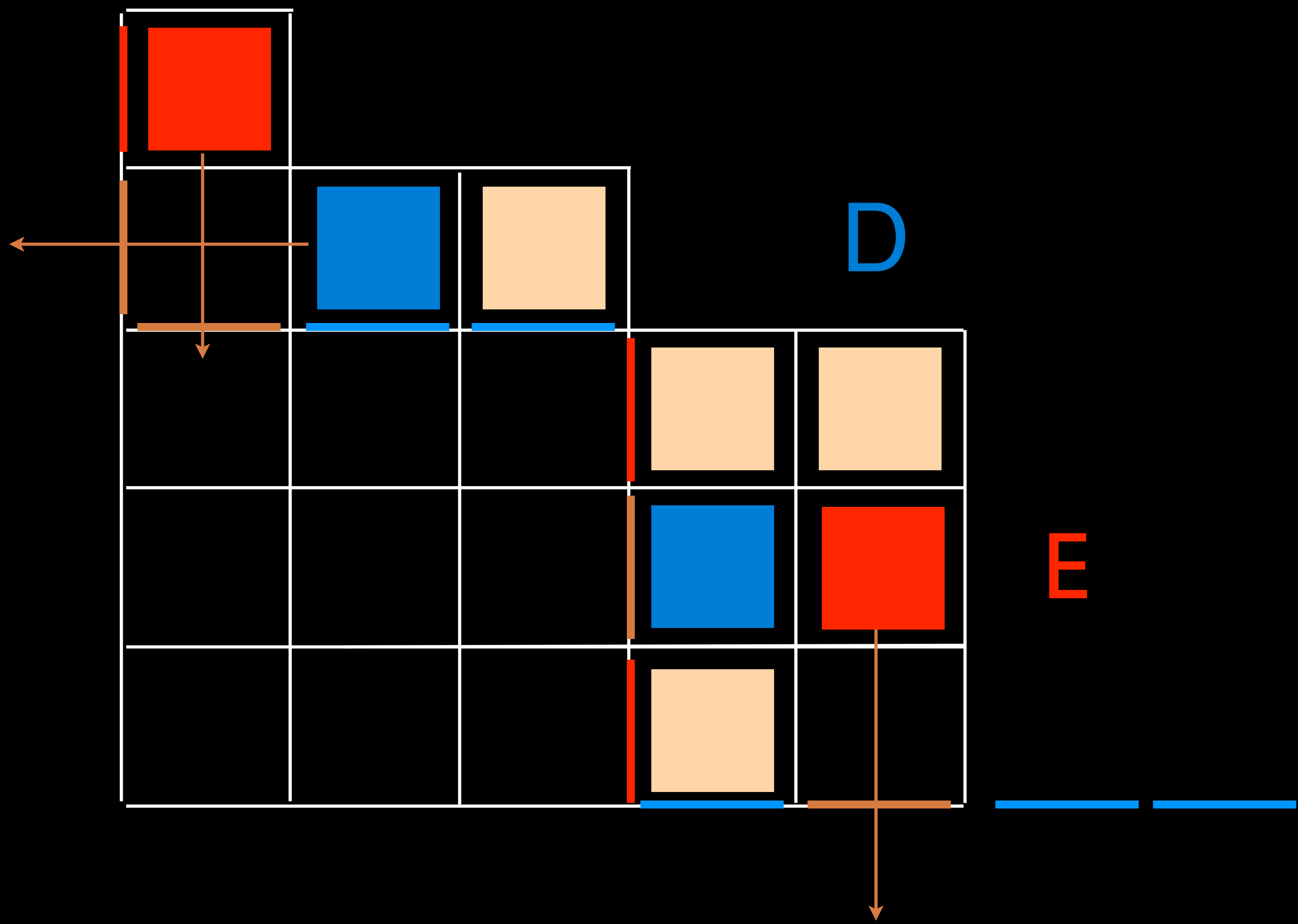
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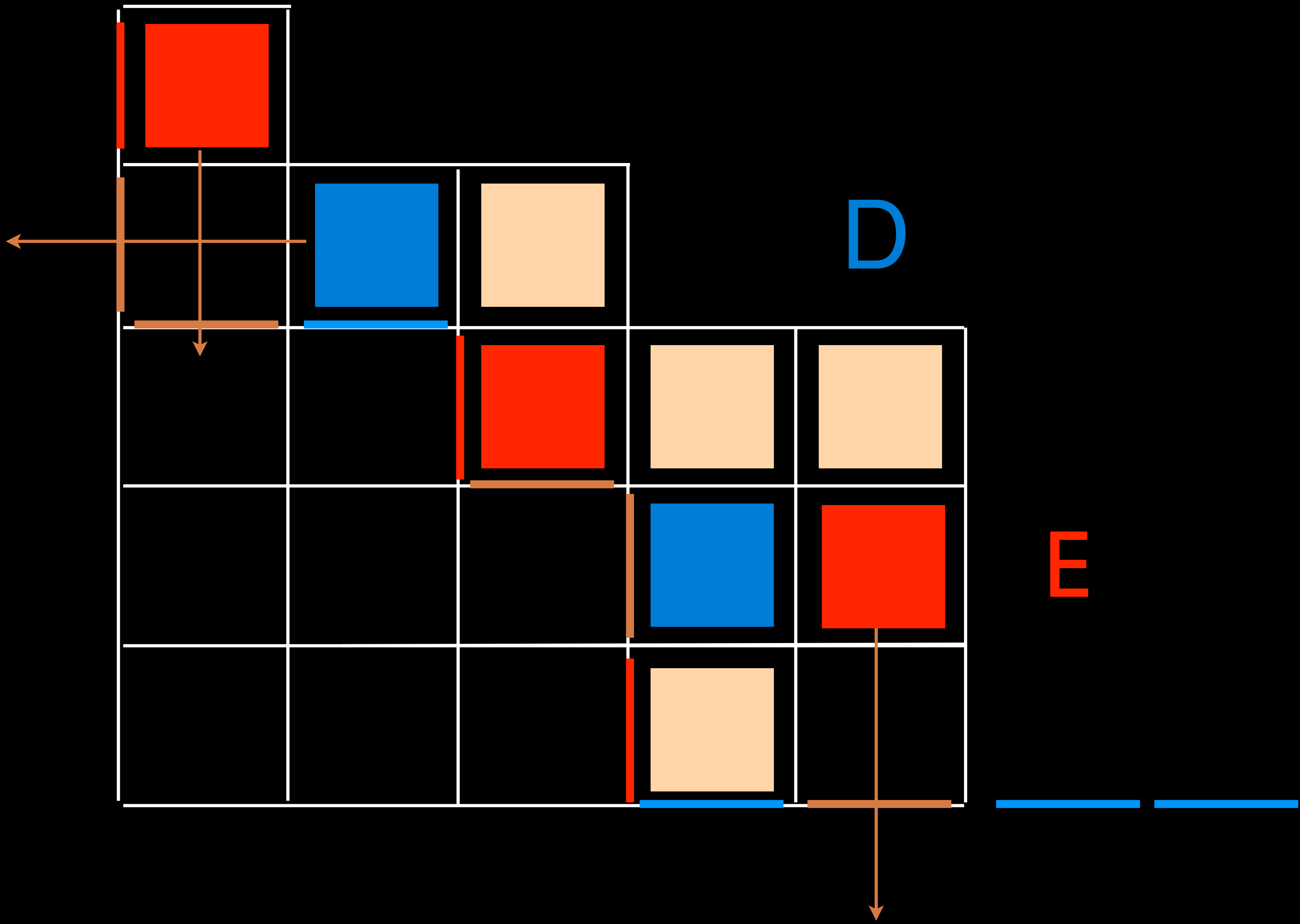
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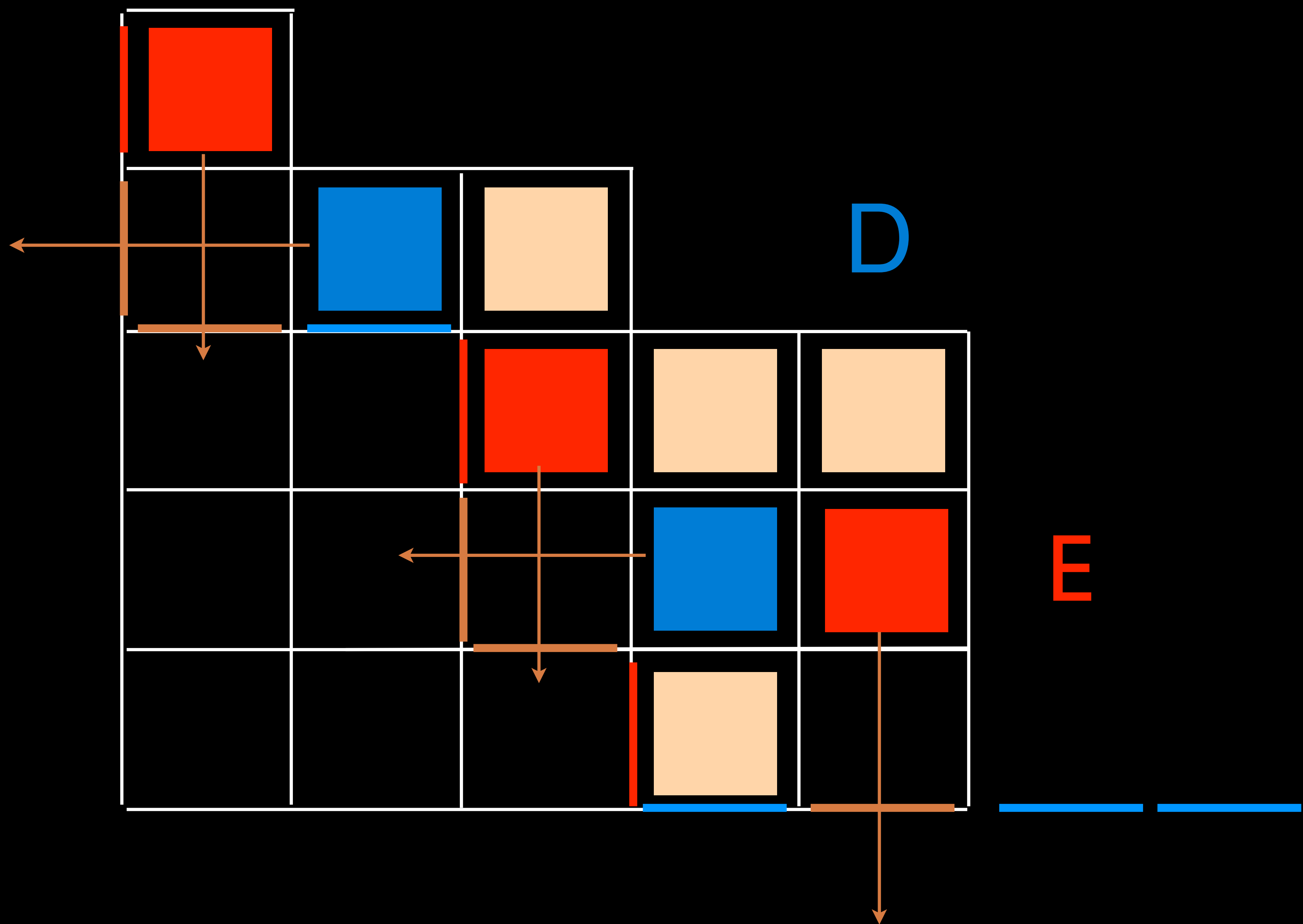
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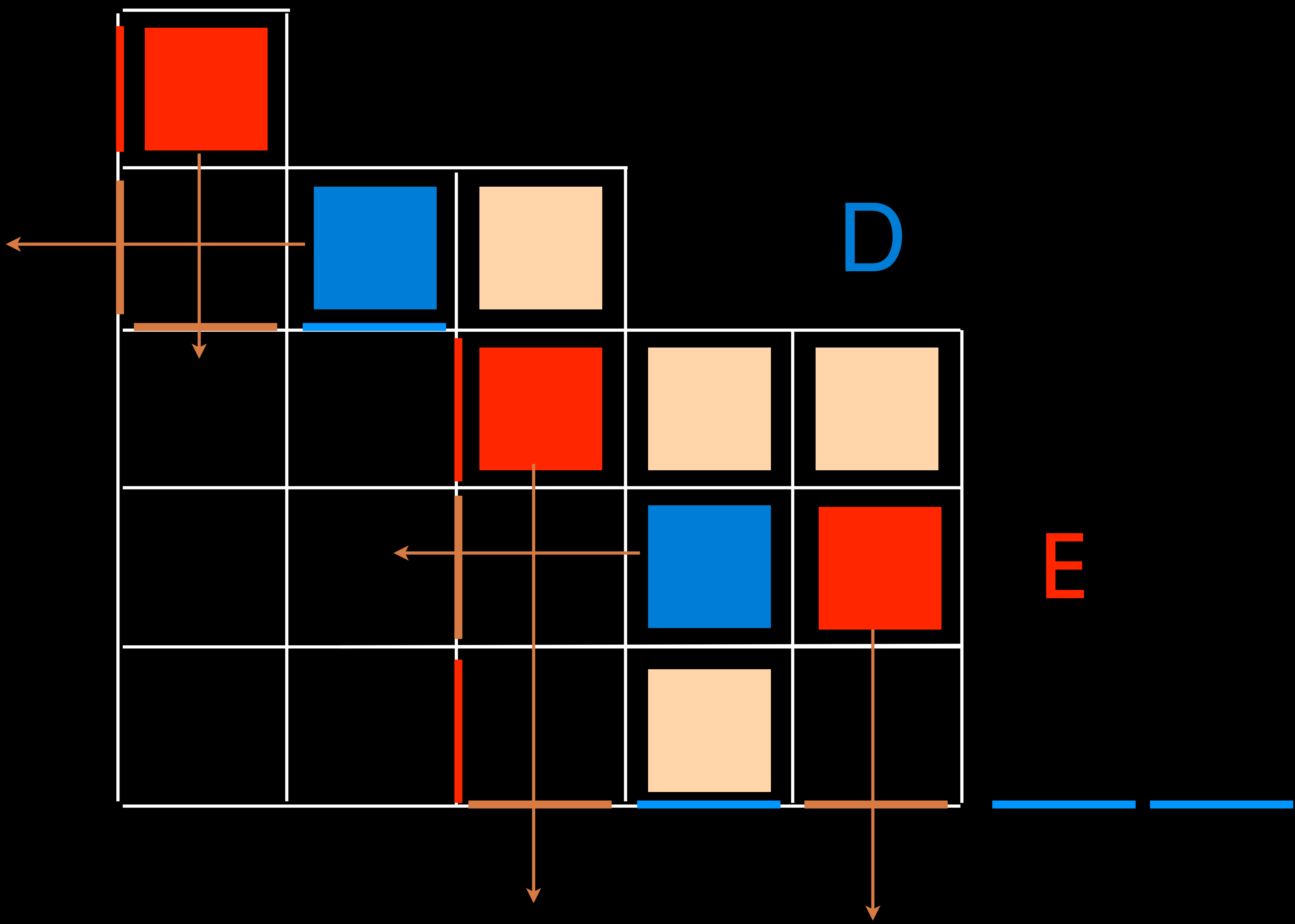
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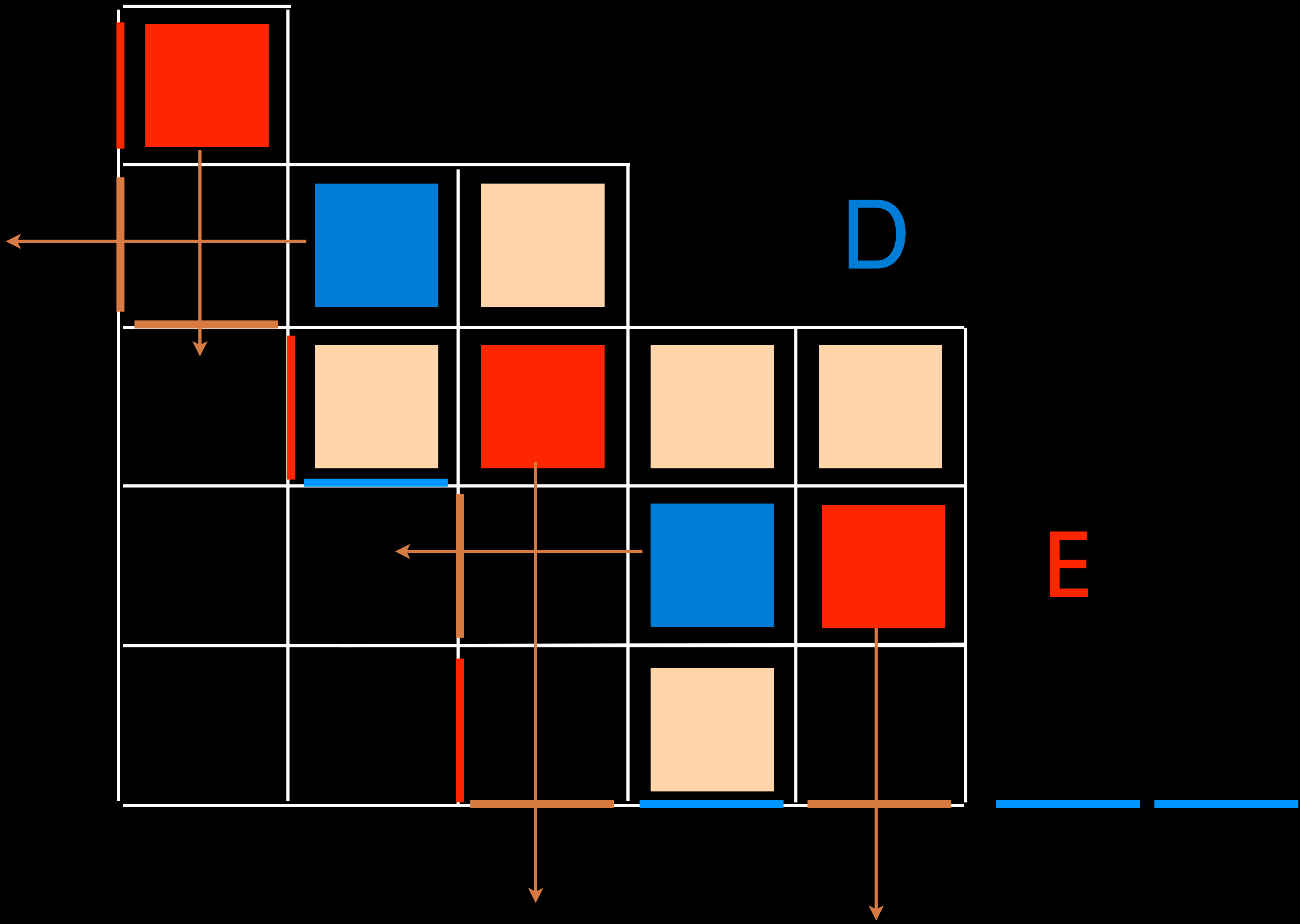
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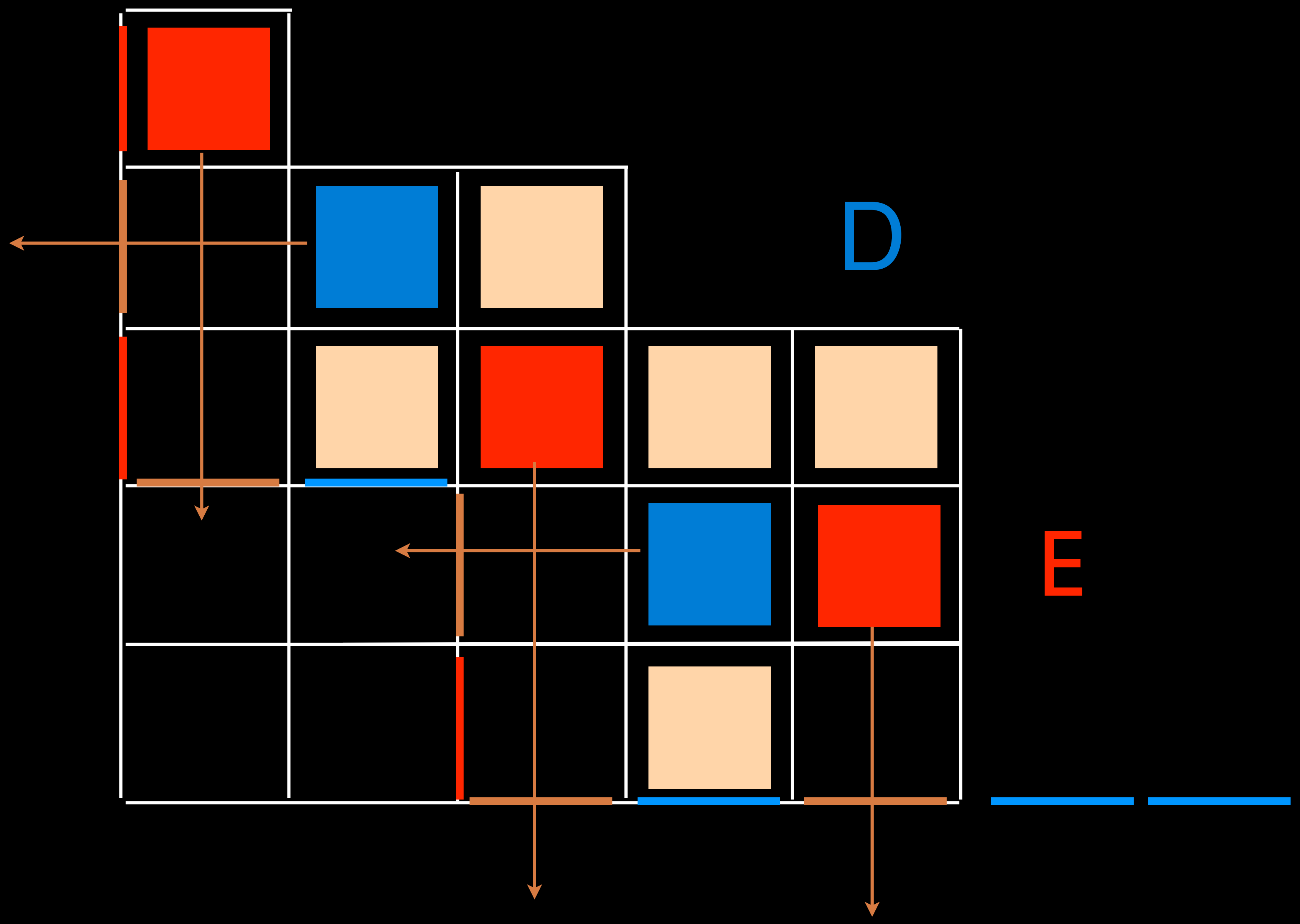
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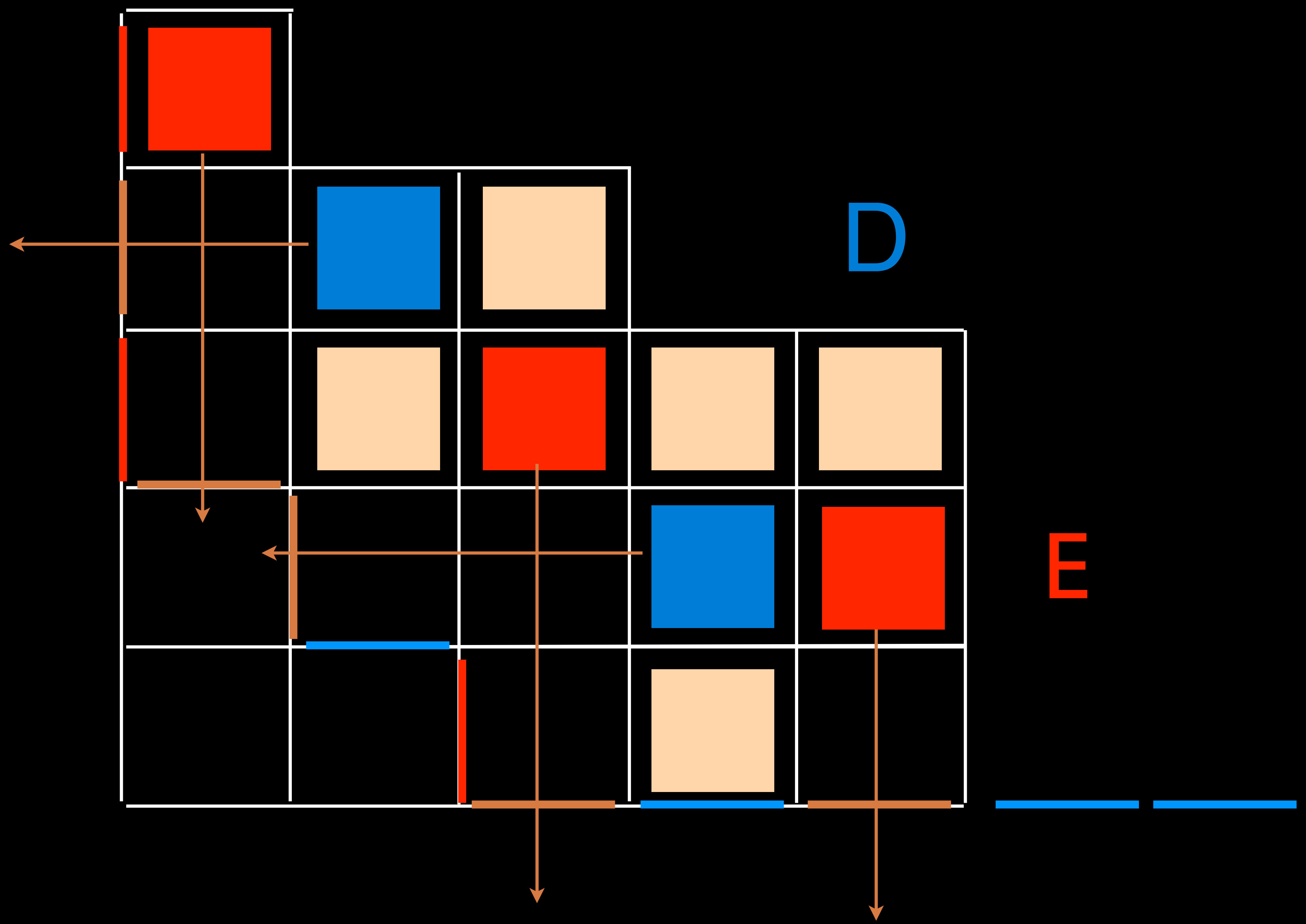
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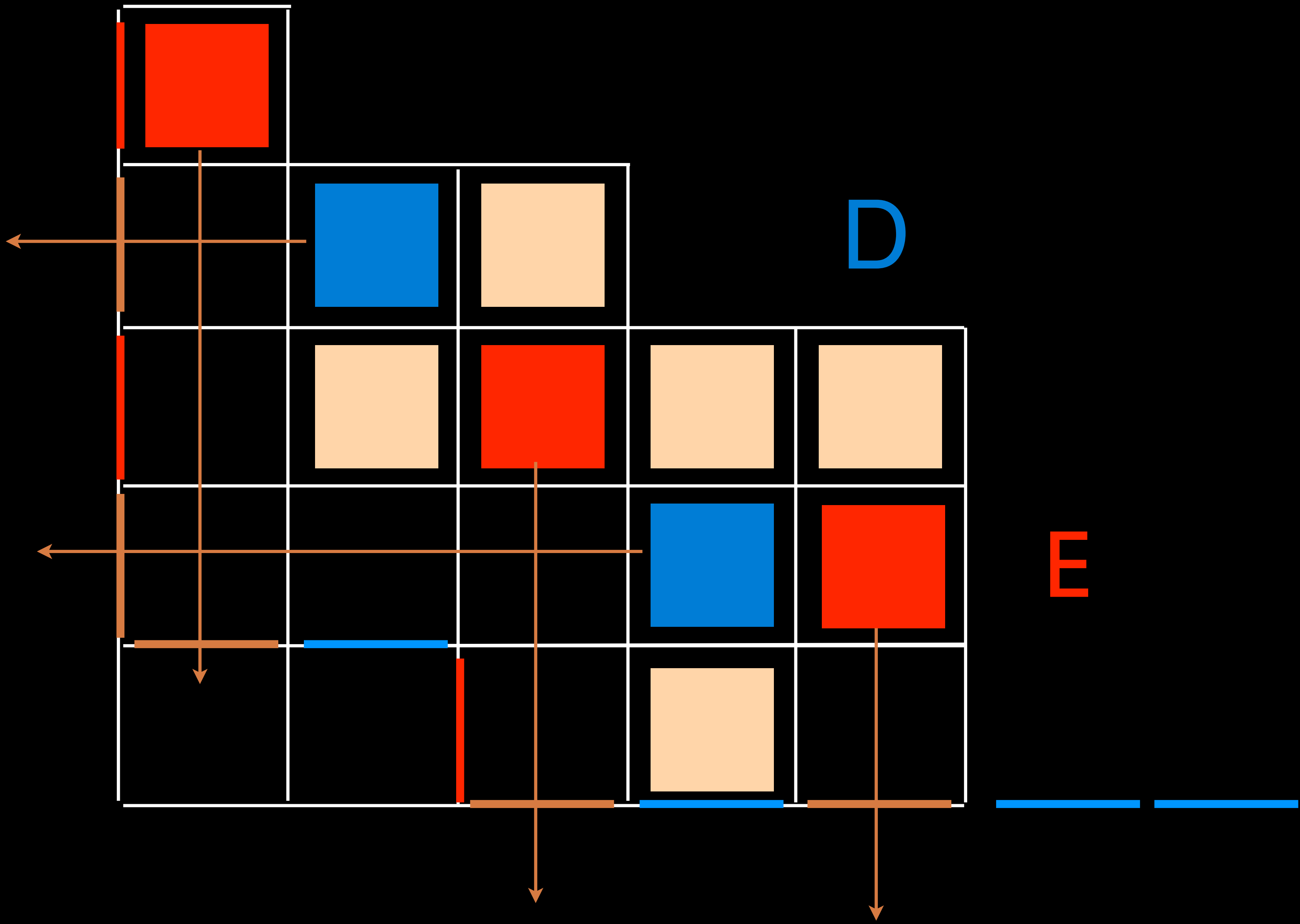
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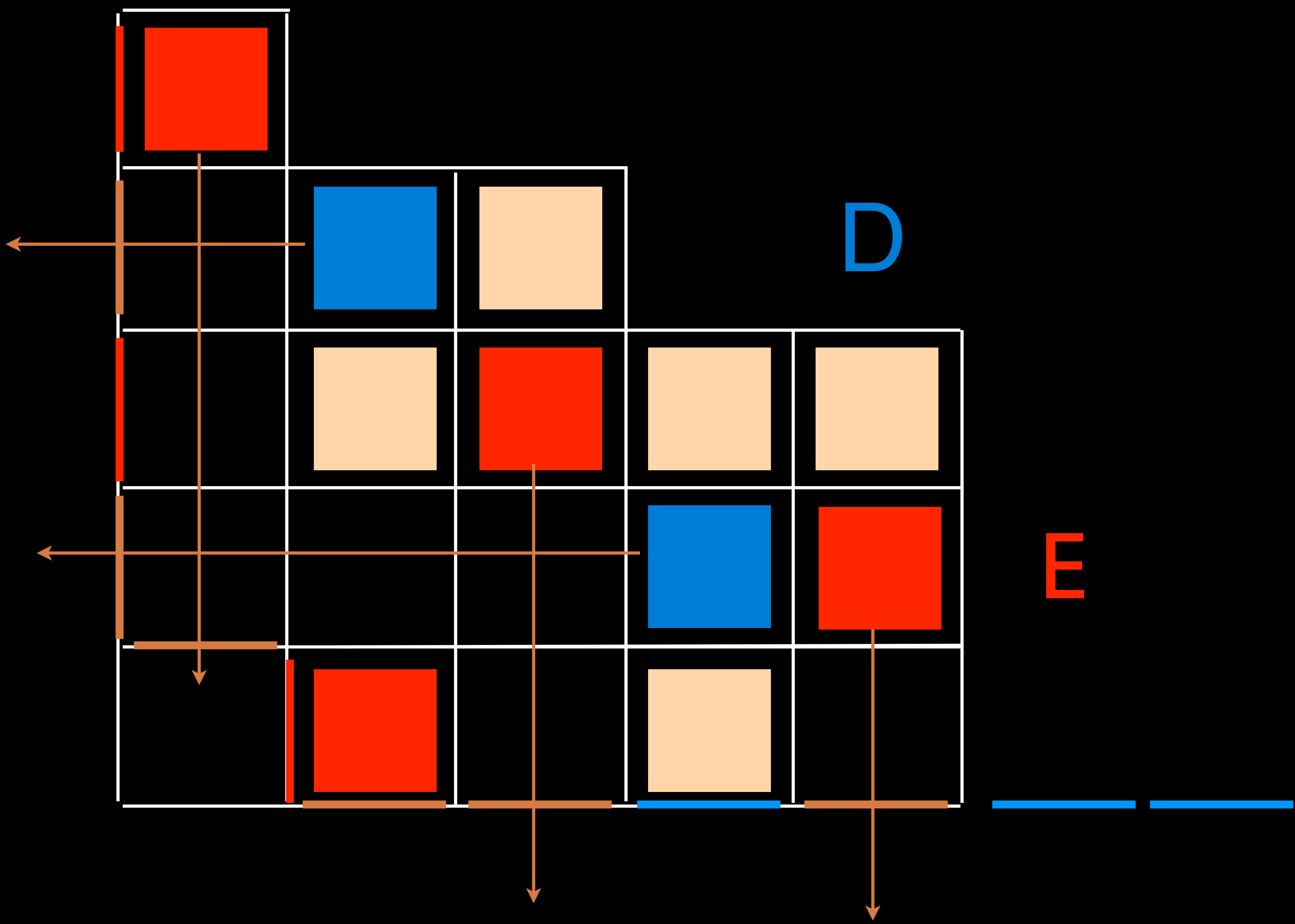
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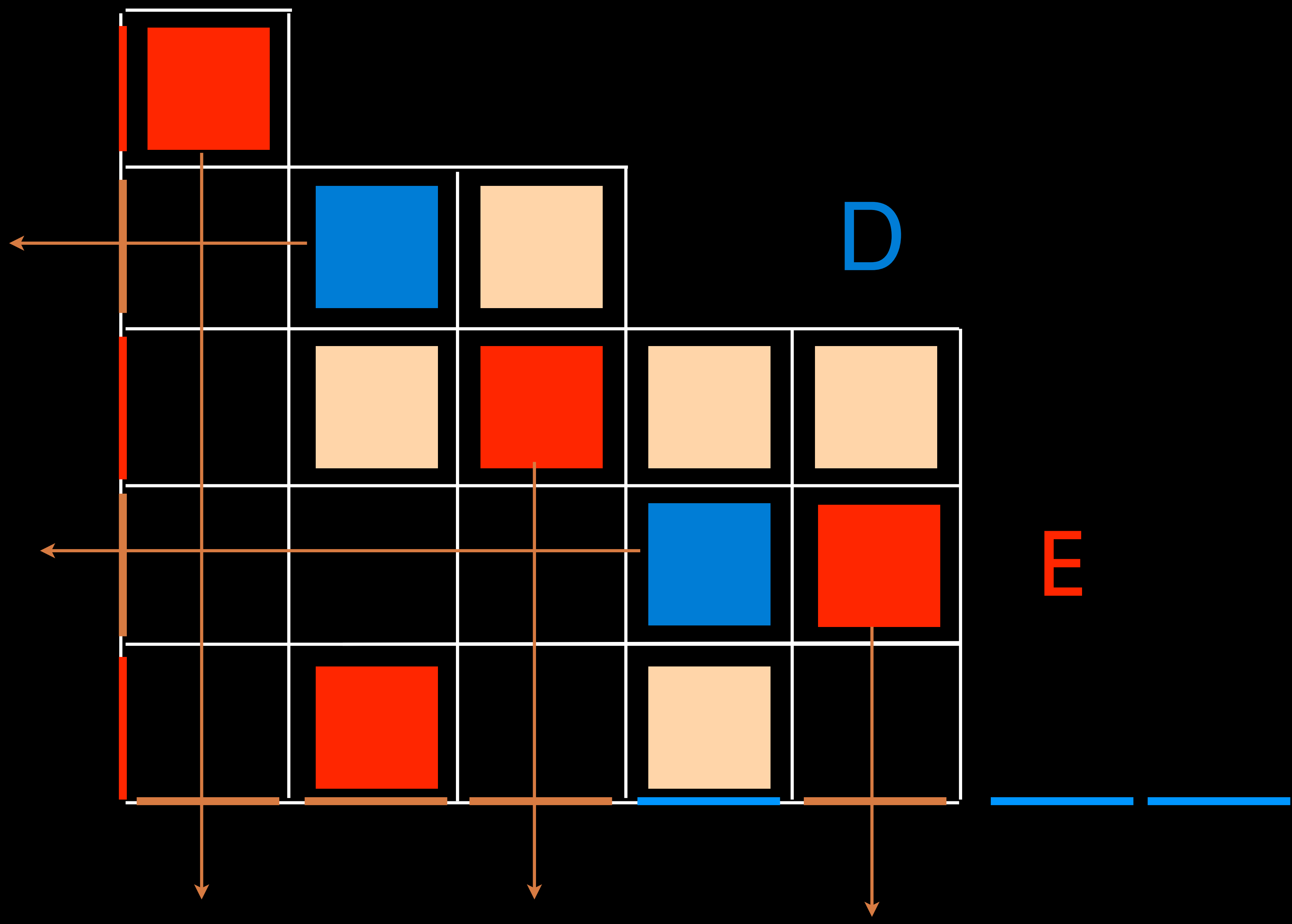
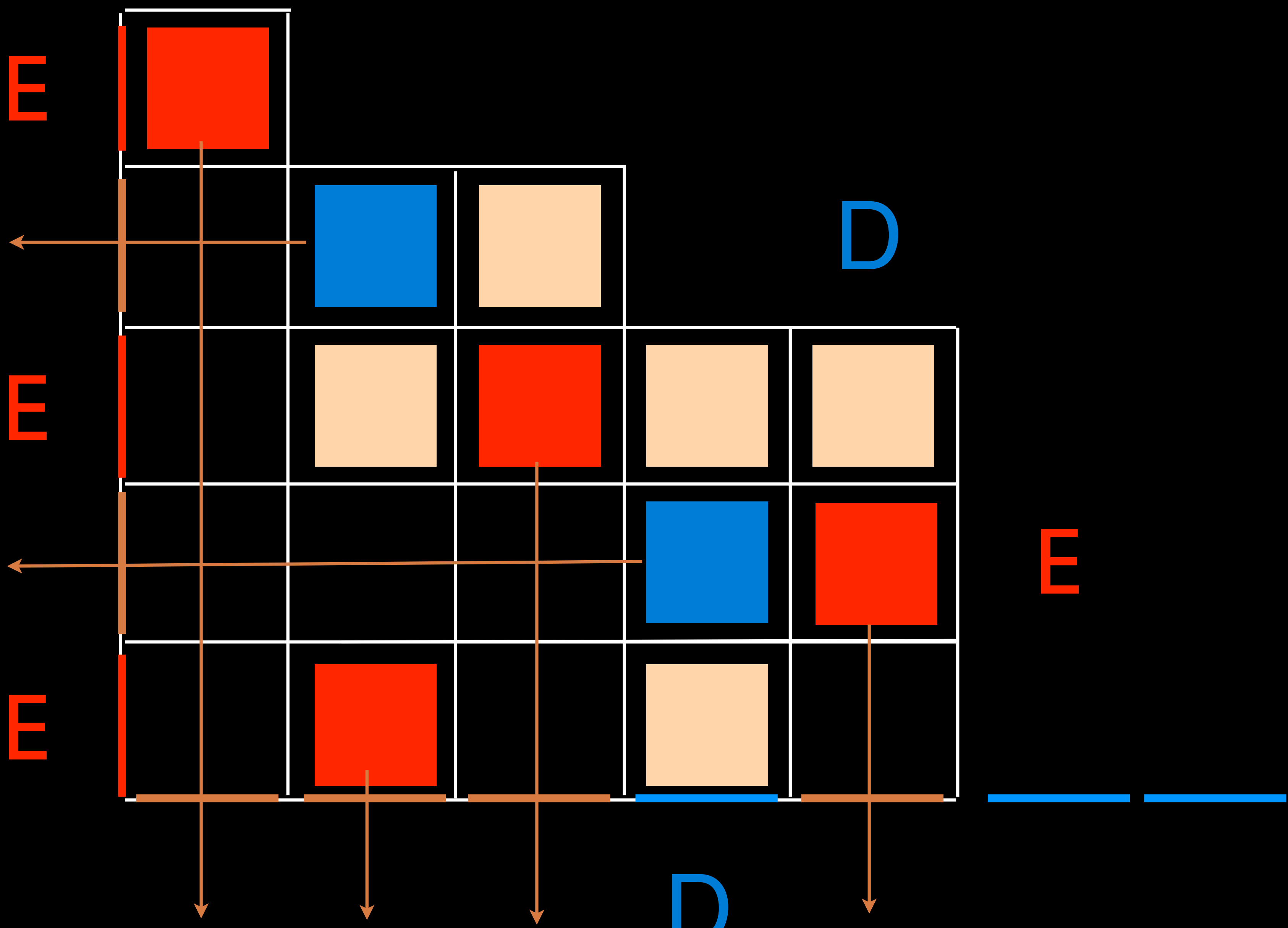
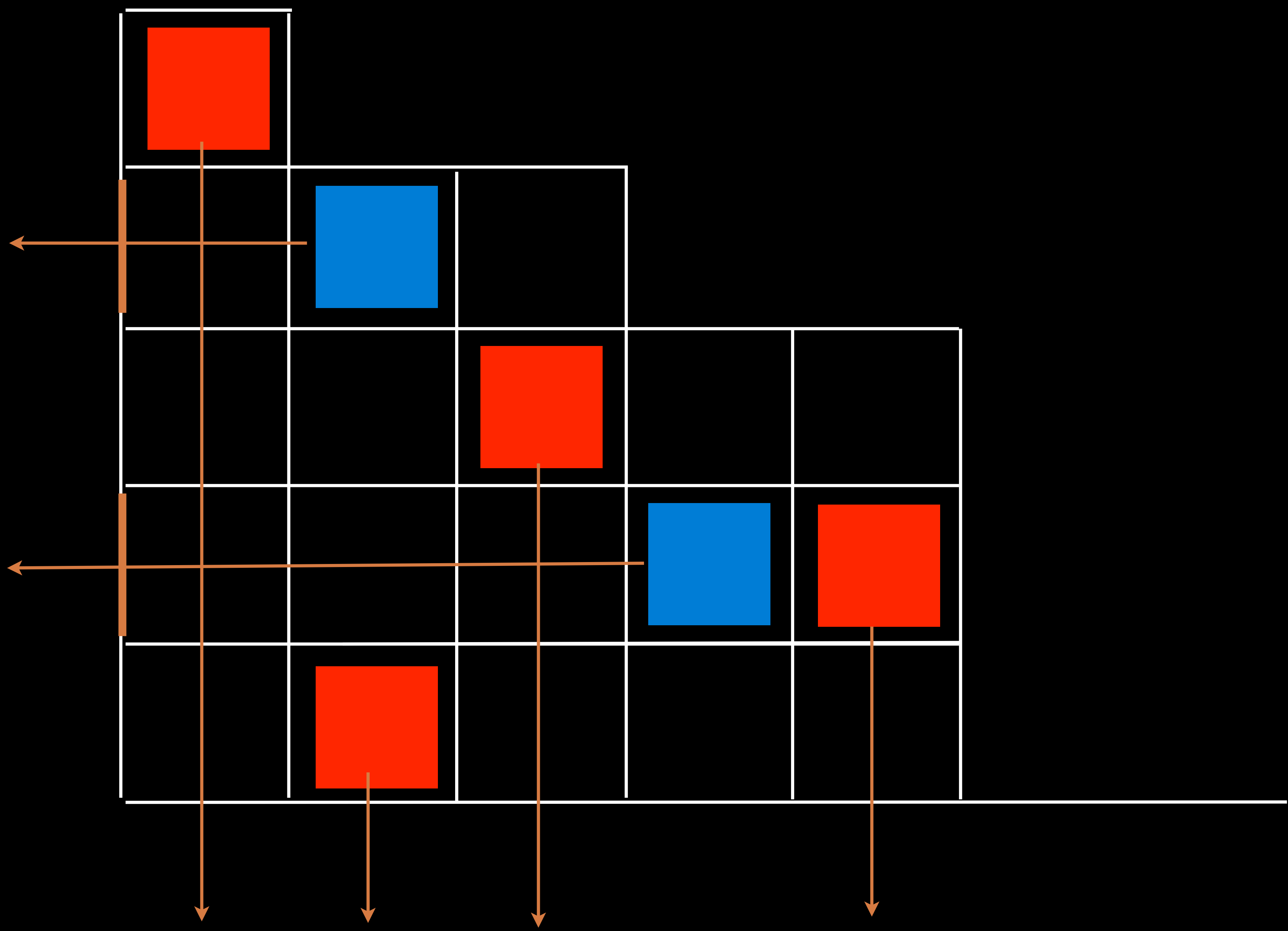


D
E

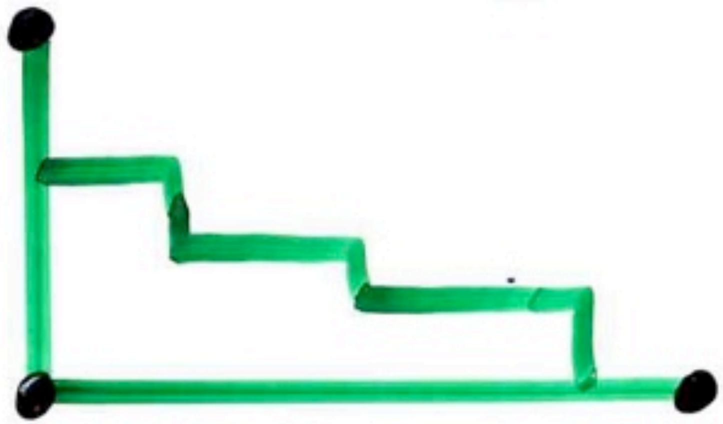


E
D
E
E
E
D



alternative tableau

- Ferrers diagram F

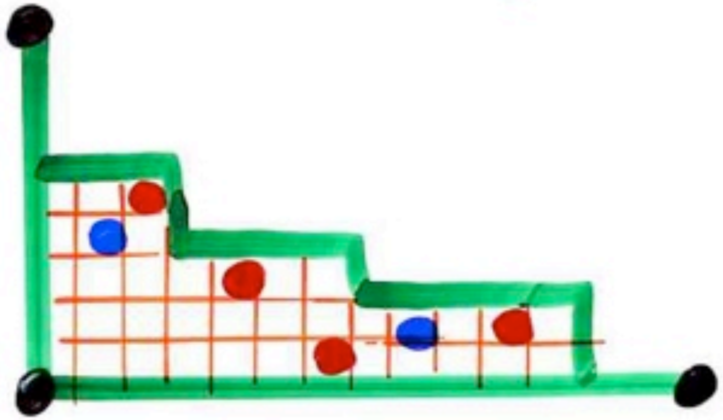


(possibly empty rows or column)

(nb of rows) + (nb of columns) $= n$

alternative tableau

- Ferrers diagram **F** (possibly empty rows or column)

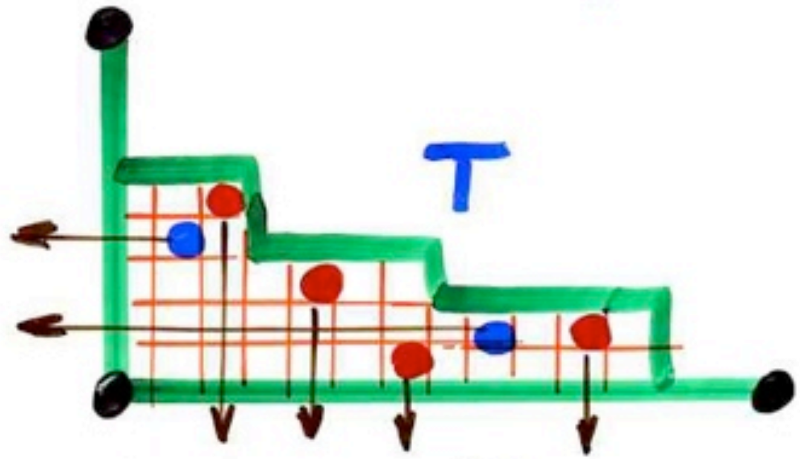


(nb of rows) + (nb of columns) = n

- some cells are coloured **red** or **blue**

alternative tableau T

- Ferrers diagram F (possibly empty rows or column)



(nb of rows) + (nb of columns) = n

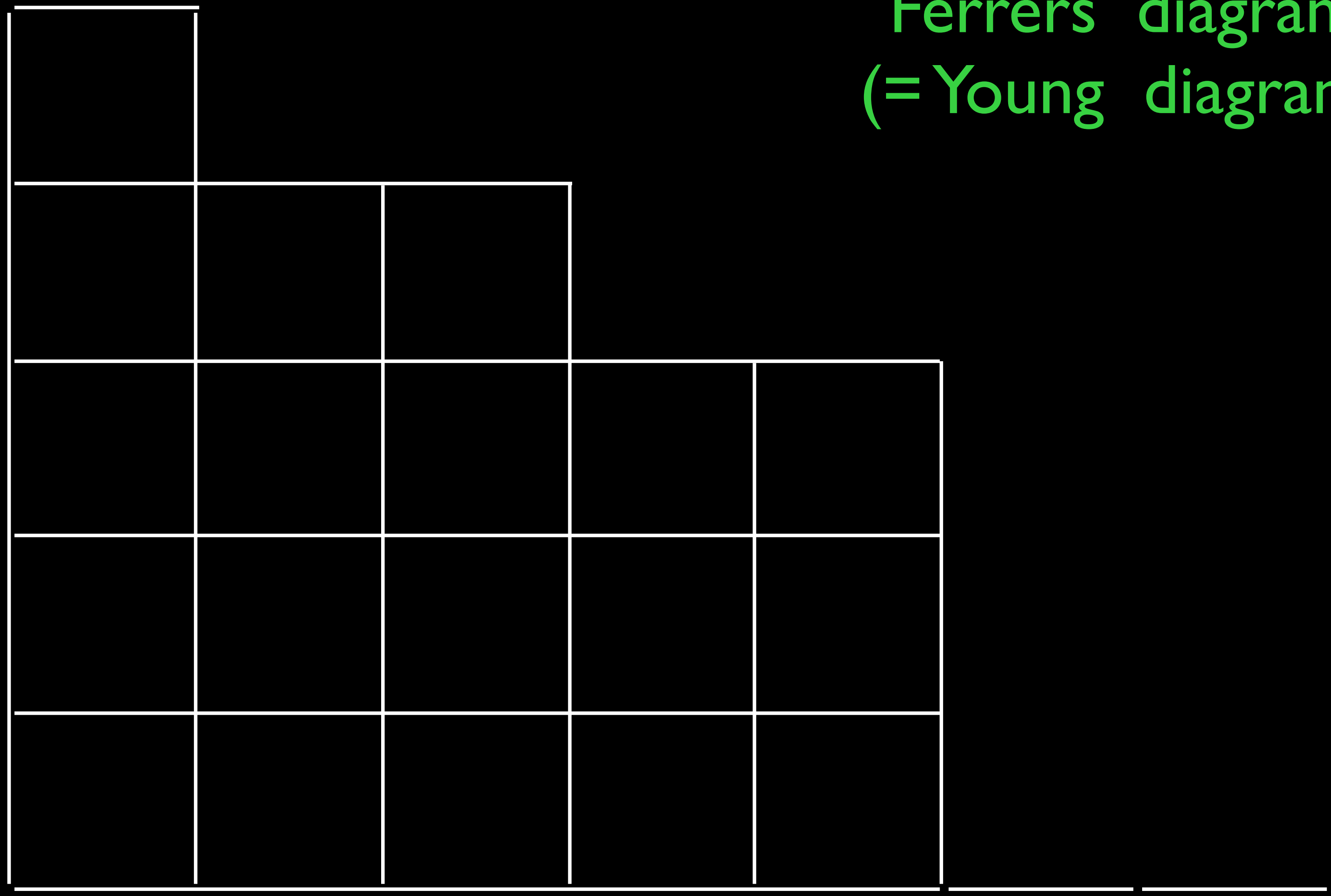
- some cells are coloured **red** or **blue**

- $\begin{cases} \text{no coloured cell at the left of } \boxed{\color{blue}\bullet} \\ \text{no coloured cell below } \boxed{\color{red}\bullet} \end{cases}$

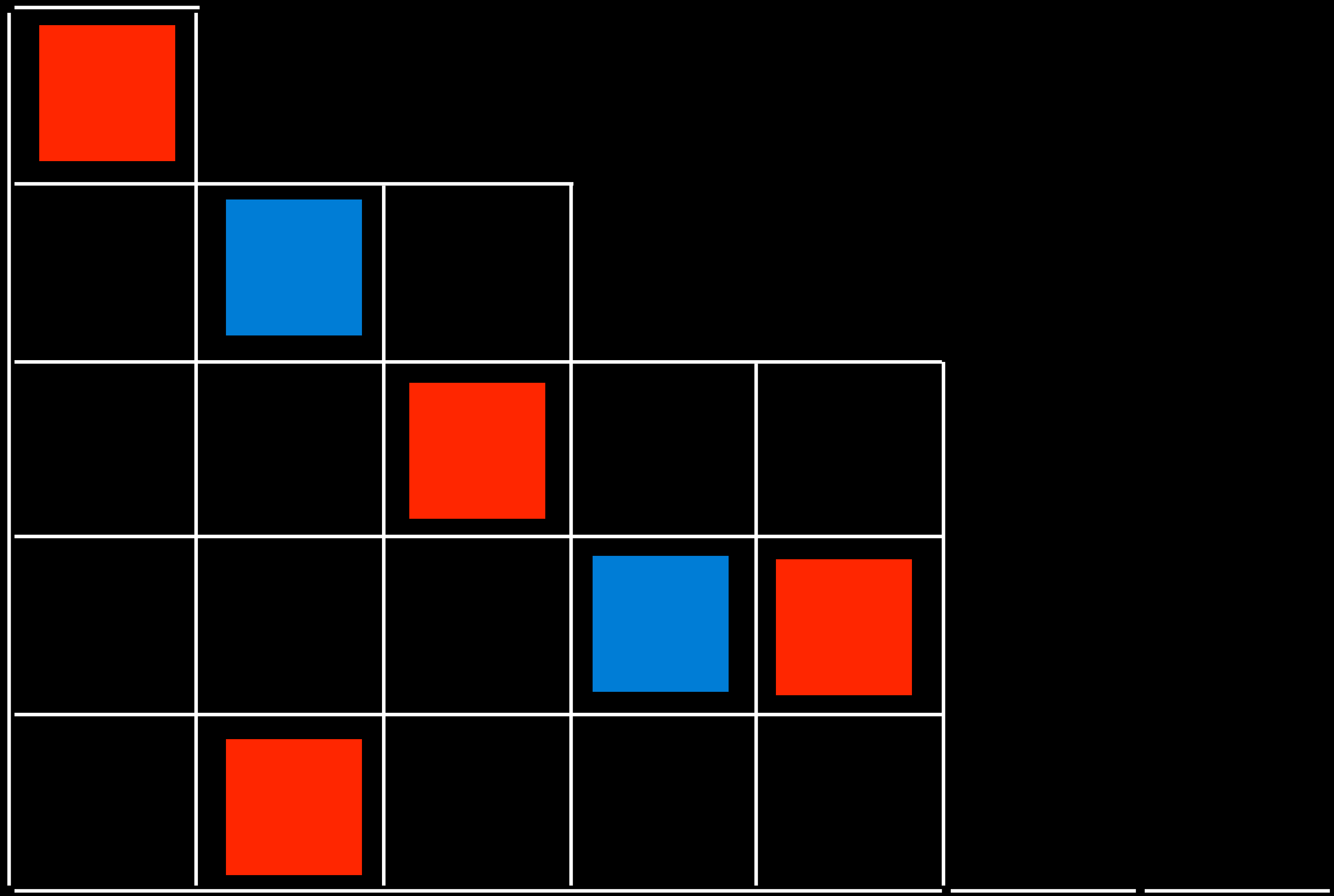
n size of T

alternative tableau

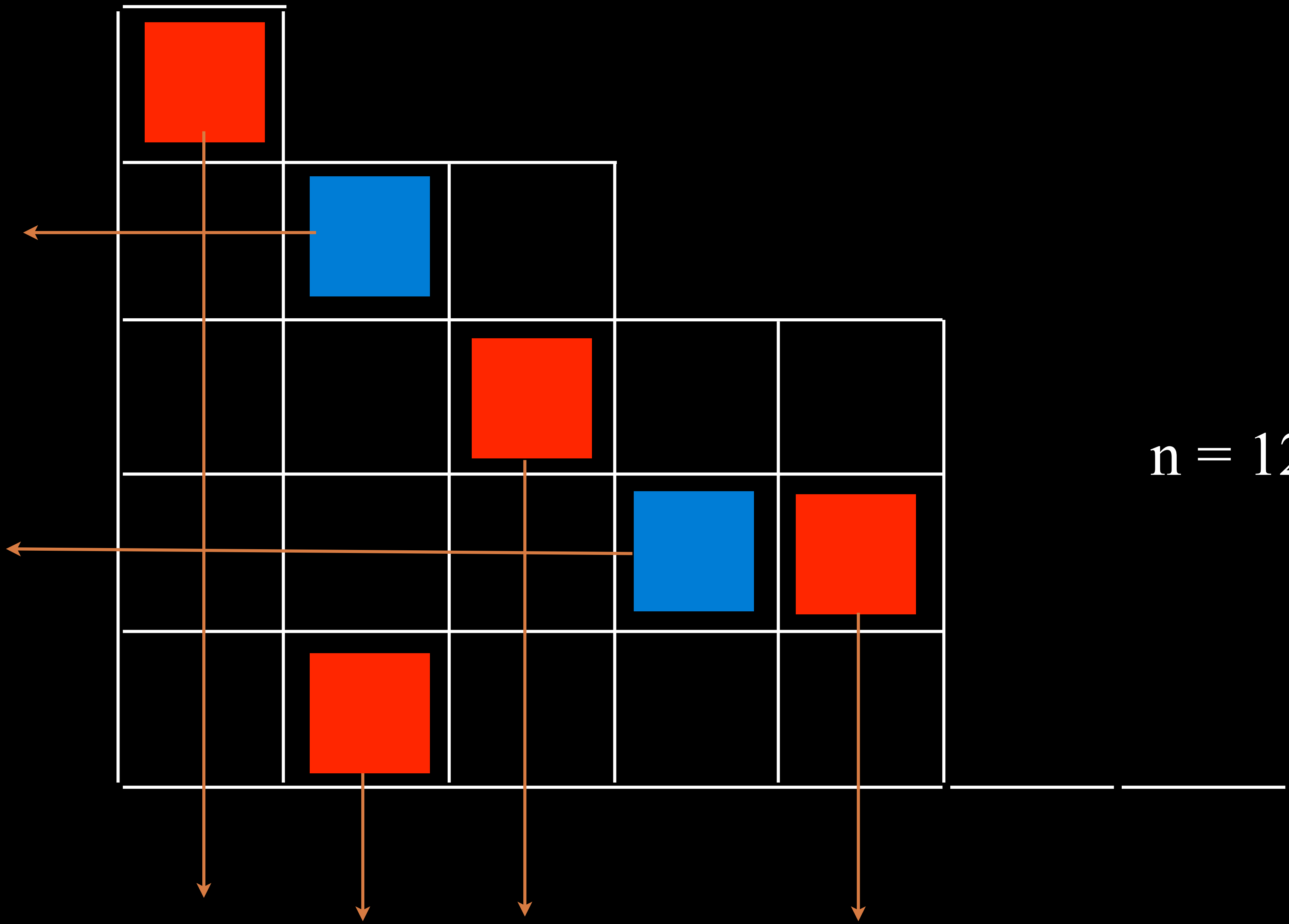
Ferrers diagram
(= Young diagram)



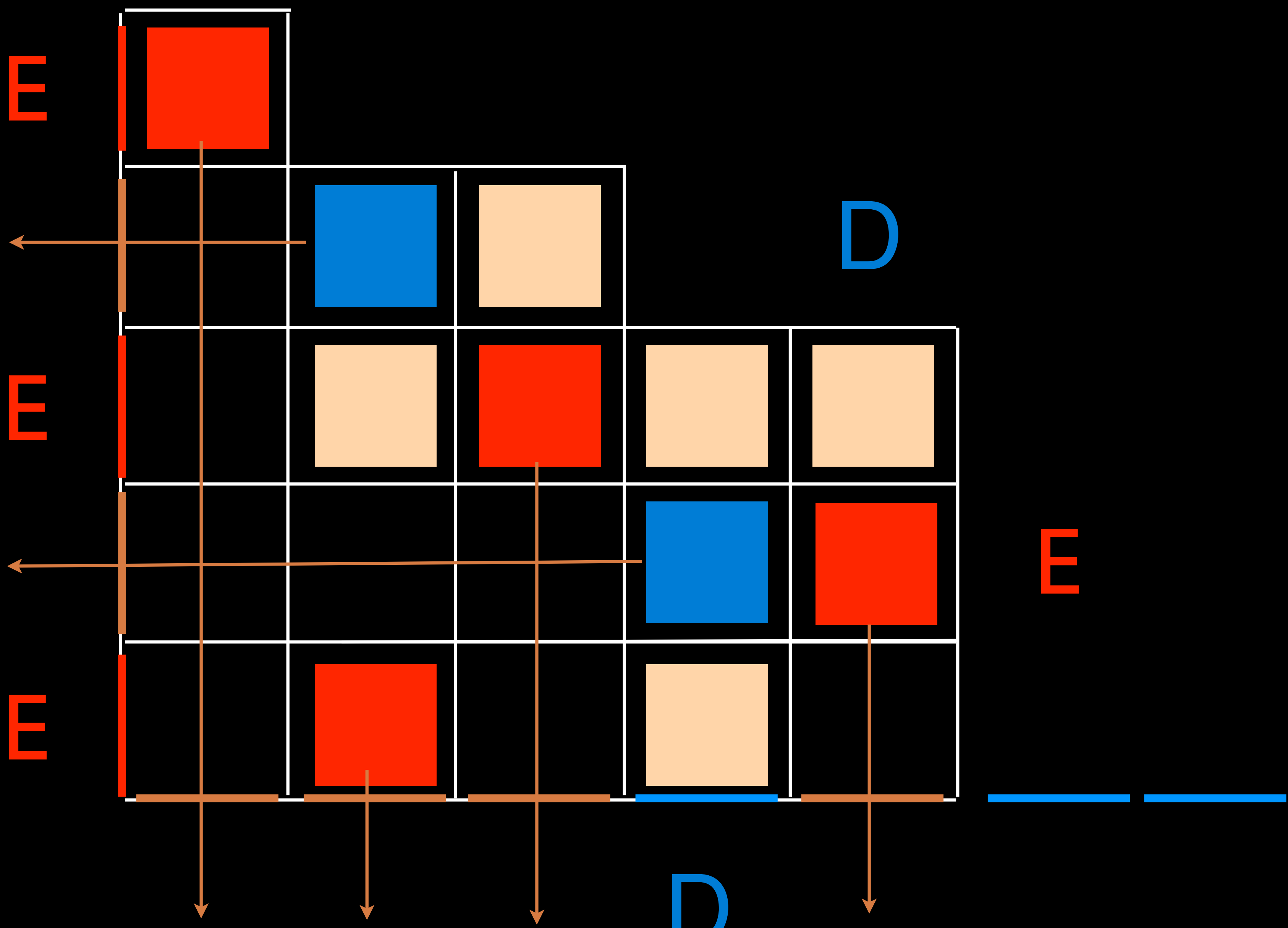
alternative tableau



alternative tableau

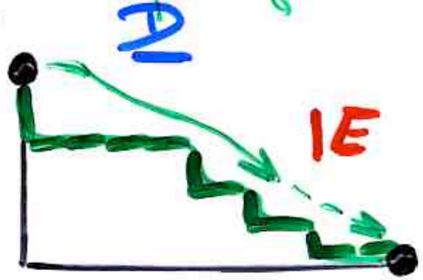


n = 12



E
D
E
E
E
D

Def - profile of an alternative tableau



word $w \in \{E, D\}^*$

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T \text{ alternative tableau with profile } w} q^{k(T)} E^{i(T)} D^{j(T)}$$

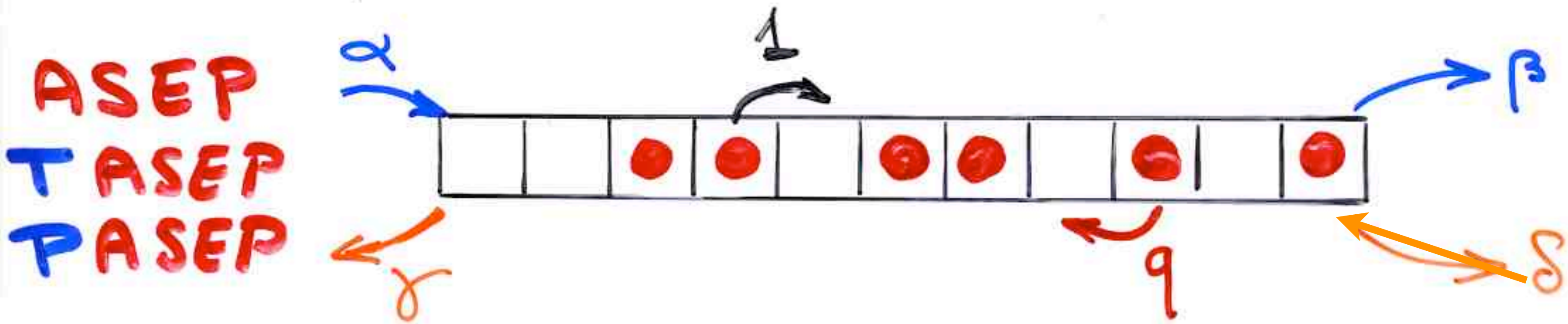
$k(T)$ = nb of $\square$

$i(T)$ = nb of rows without blue cell

$j(T)$ = nb of columns without red cell

The PASEP

Partialy asymmetric exclusion process


ASEP
TASEP
PASEP
α
1
β
γ
q
δ

$$\mathcal{P}_n(\tau_1, \ldots, \tau_n) = f_n(\tau_1, \ldots, \tau_n) / Z_n$$

$$Z_n = \sum_{\tau} f_n(\tau_1, \ldots, \tau_n)$$ partition function

The Matrix Ansatz

Derrida, Evans, Hakim, Pasquier (1993)

stationary probabilities for the PASEP

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

T alternative tableau with profile w

$k(T)$ = nb of $\square$

$i(T)$ = nb of rows without blue cell

$j(T)$ = nb of columns without red cell

Cor. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ (PASEP) is

$$\mathrm{proba}_{\tau}(q\,; \alpha, \beta) = \frac{1}{Z_n} \sum_{T} q^{L(T)} \alpha^{-f(T)} \beta^{-u(T)}$$

T: alternative tableaux, profile τ

$\begin{cases} f(T) \\ u(T) \end{cases}$ nb of $\begin{cases} \text{rows} \\ \text{columns} \end{cases}$ without cell

$L(T)$ nb of cells

permutation tableau

S. Corteel, L. Williams (2007) (2008) (2009) permutation tableau

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)

TASEP

Shapiro, Zeilberger (1982)
Brak, Essam (2003), Duchi, Schaeffer, (2004),
Angel (2005), XGV (2007)

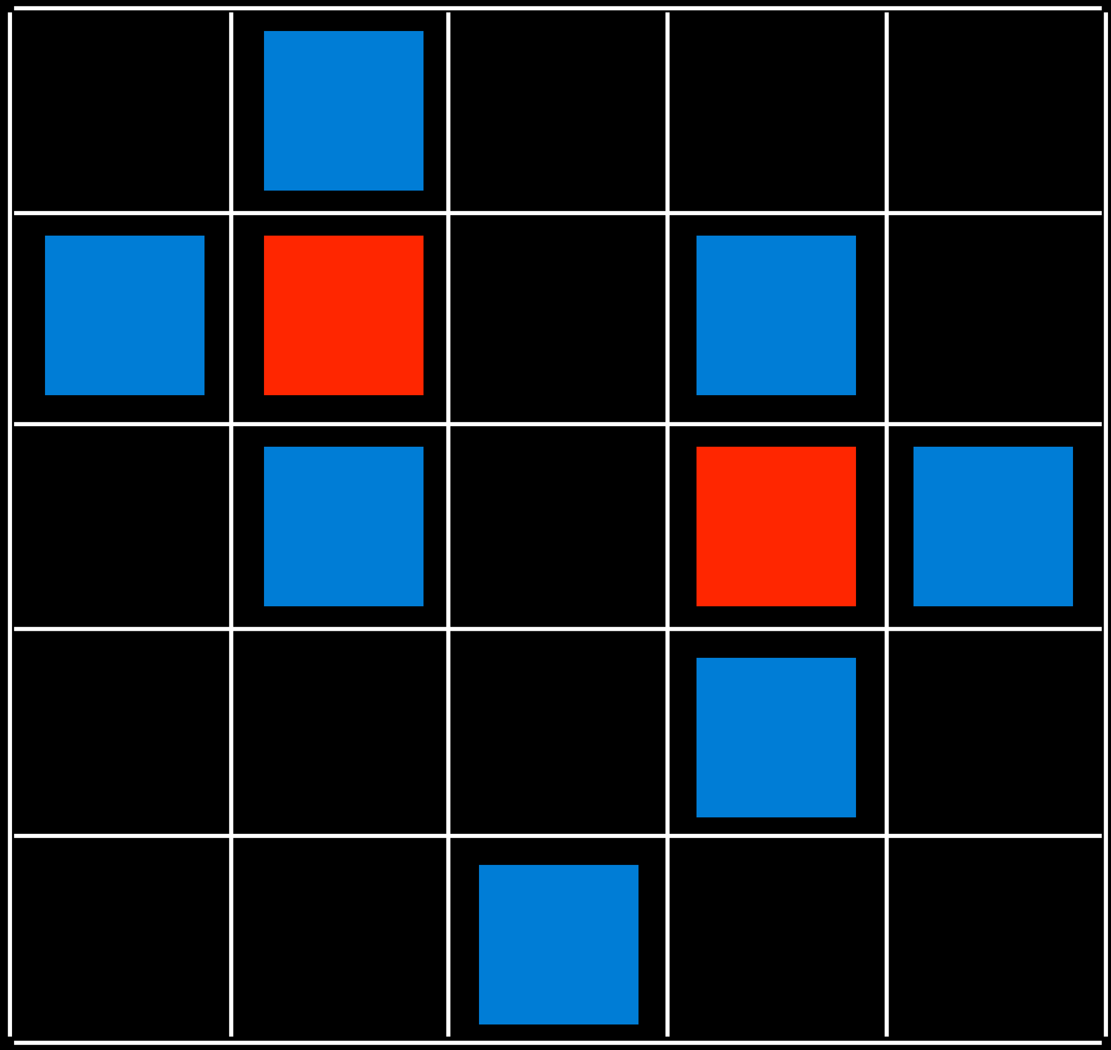
(P) ASEP

Brak, Corteel, Essam, Parviainen, Rechnitzer (2006)
Corteel, Williams (2006,..., 2010)
Corteel, Stanton, Stanley, Williams (2011)
Josuat-Vergès (2008,..., 2010)

Derrida, ...
Mallick, ... Golinelli, Mallick (2006)

....

alternating sign matrices (ASM)
and a quadratic algebra



A, A', B, B',

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

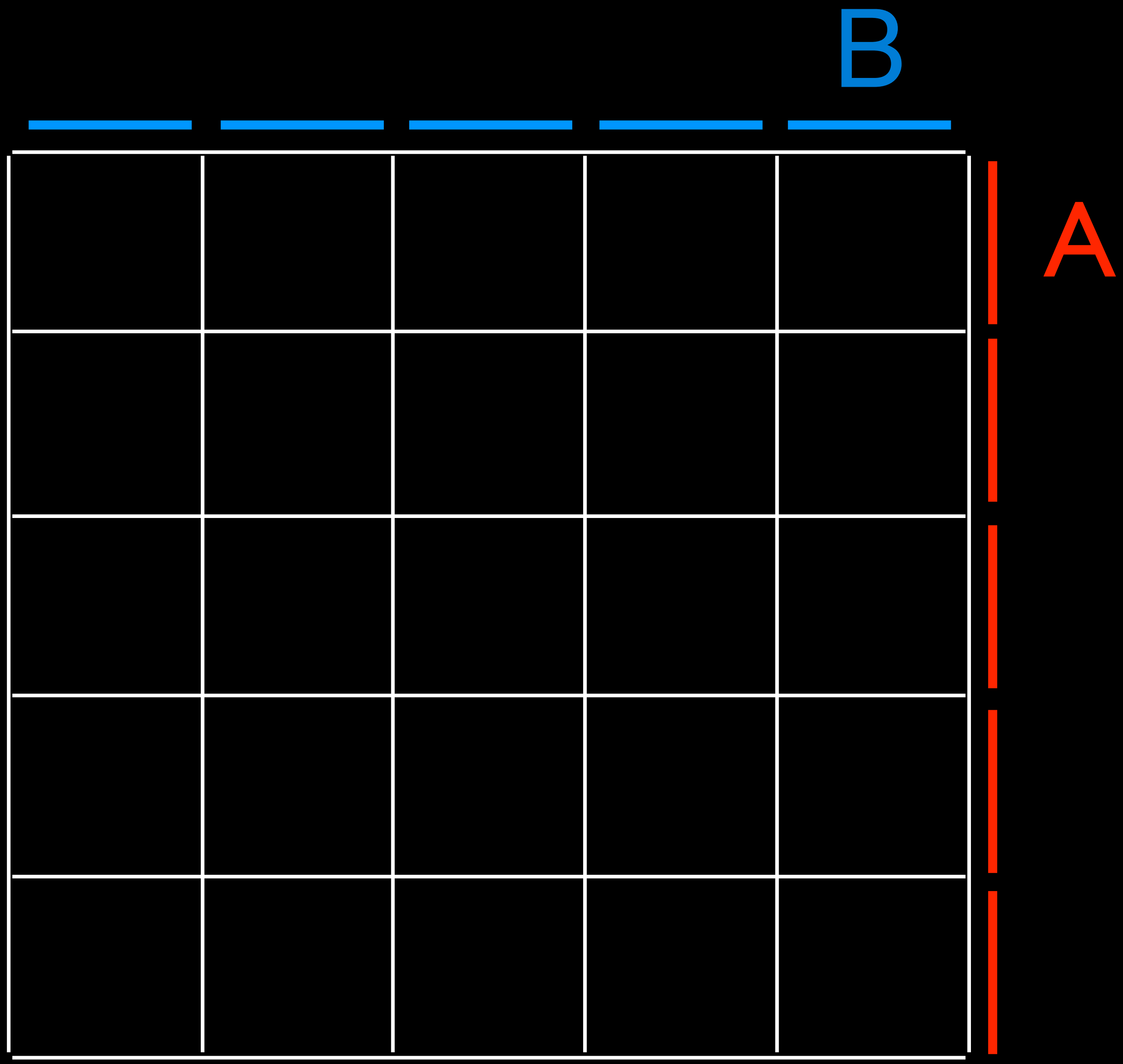
Lemma. Any word $w(A, A', B, B')$ in letters A, A', B, B', can be uniquely written

$$\sum c(u,v;w)\ \underbrace{u(A,A')}_{\substack{\text{word}\\ \text{in } A,\, A'}}\ \underbrace{v(B,B')}_{\substack{\text{word}\\ \text{in } B,\, B'}}$$

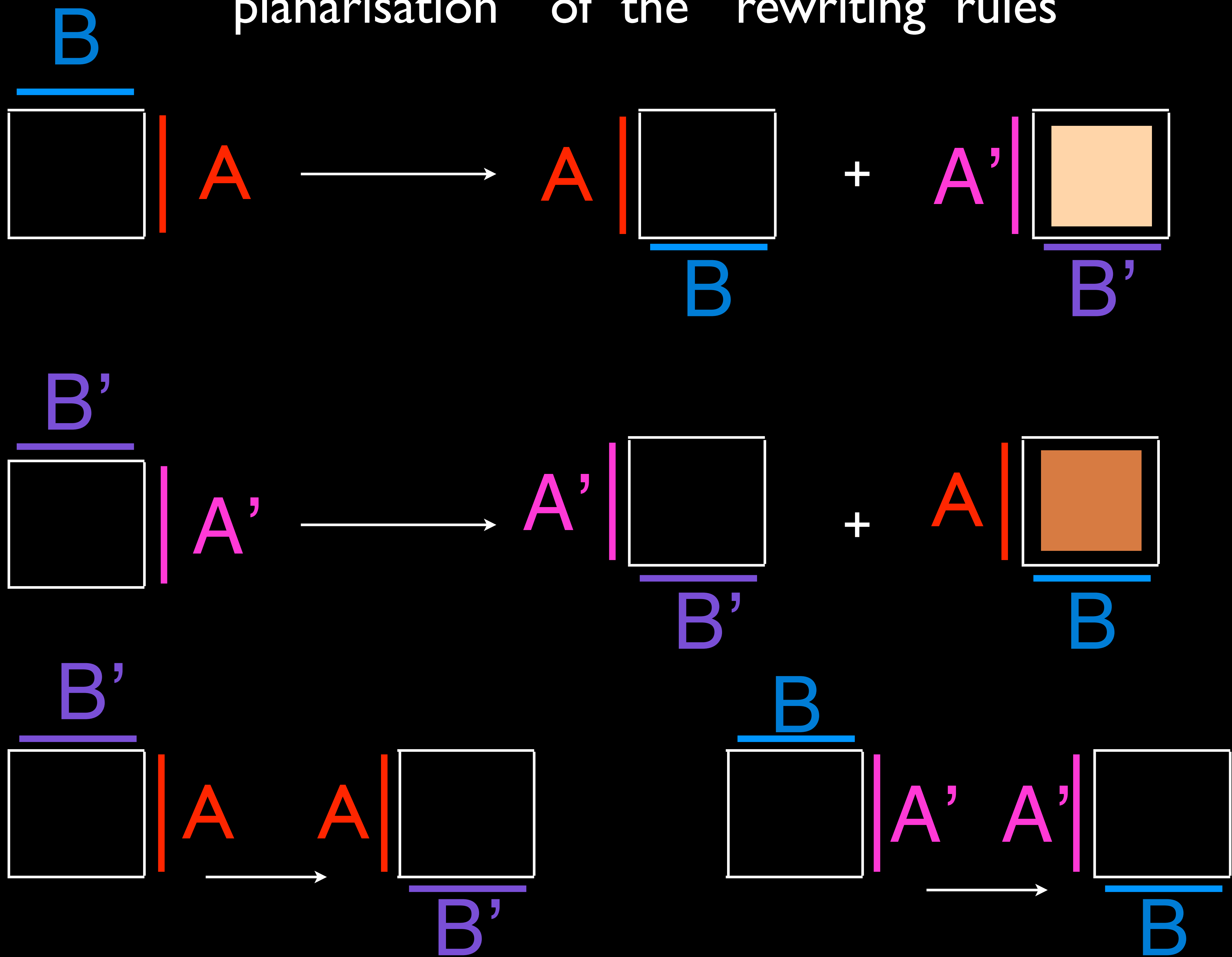
Prop. For $w = B^n A^n$

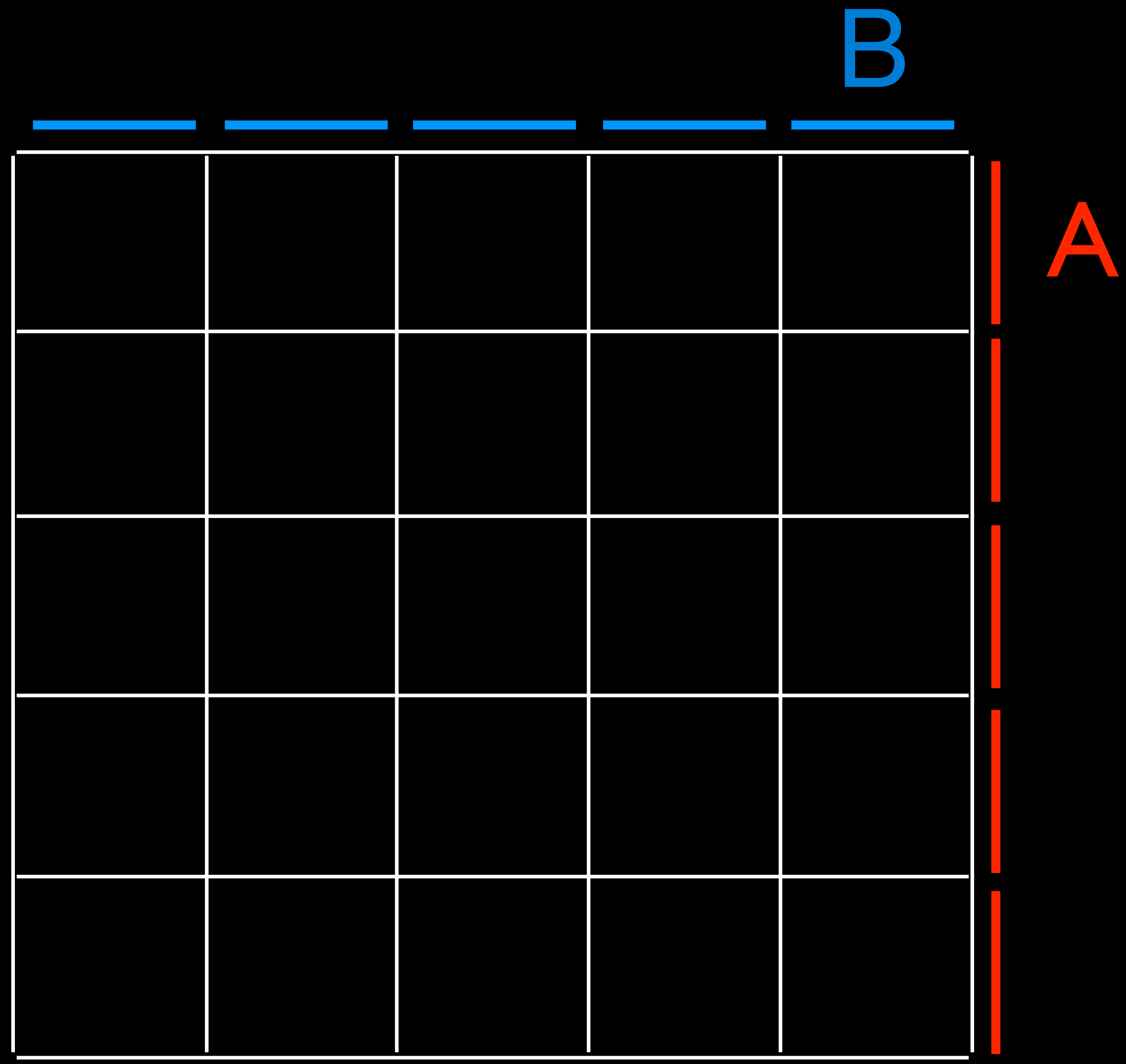
$u = A'^n$, $v = B'^n$

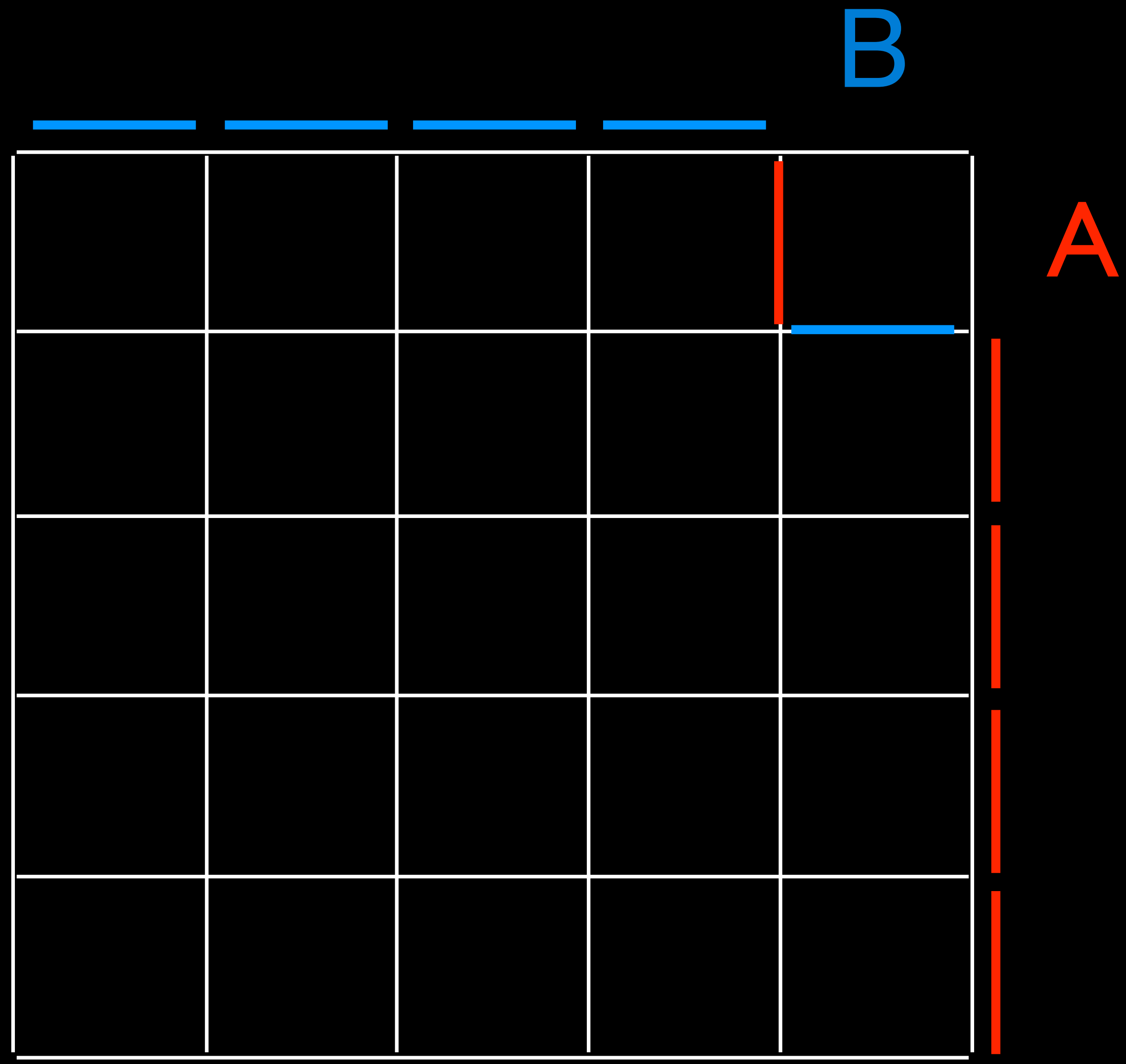
$c(u,v;w) =$ the number of $n \times n$ ASM (alternating sign matrices) “planar” proof:

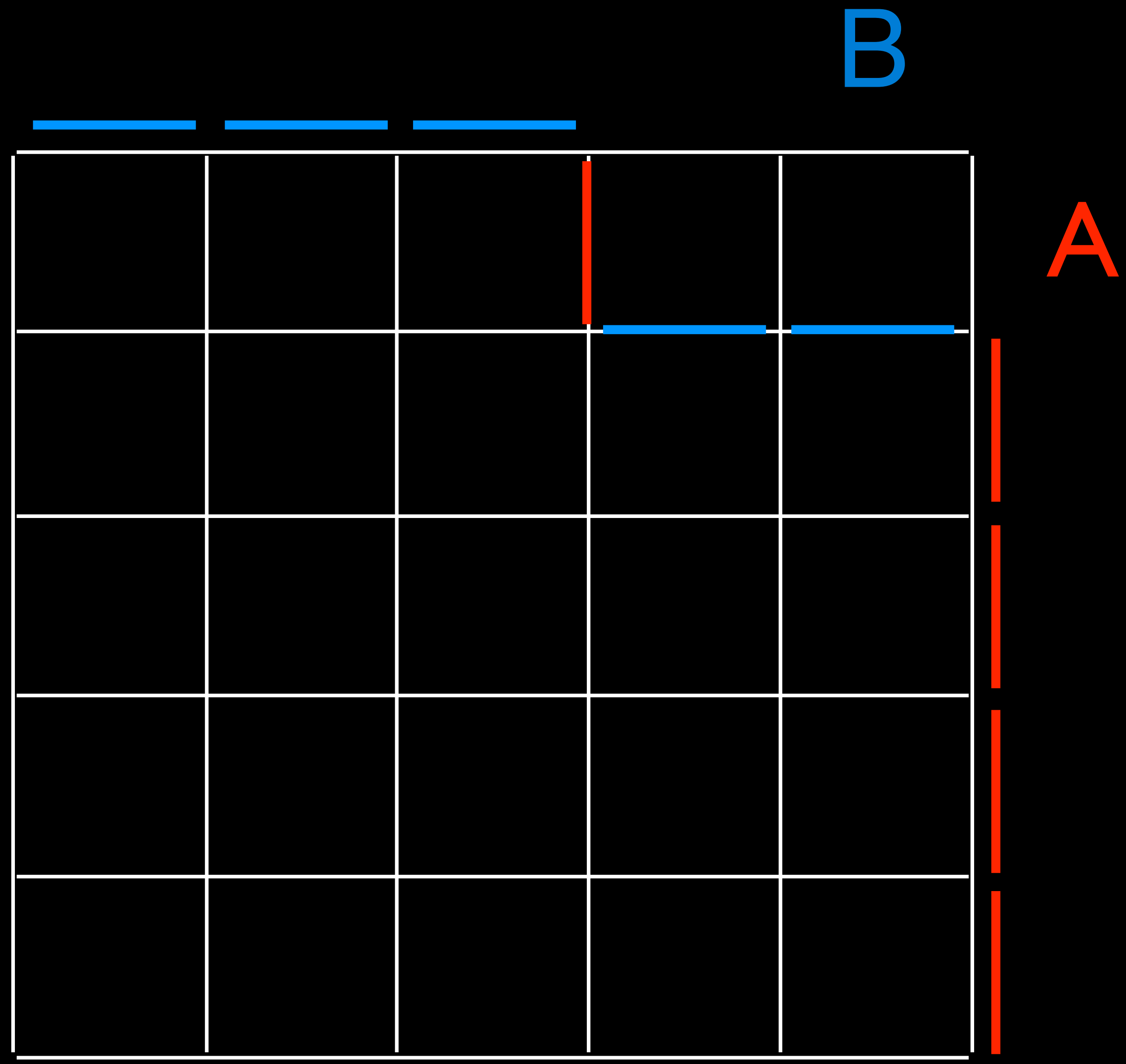


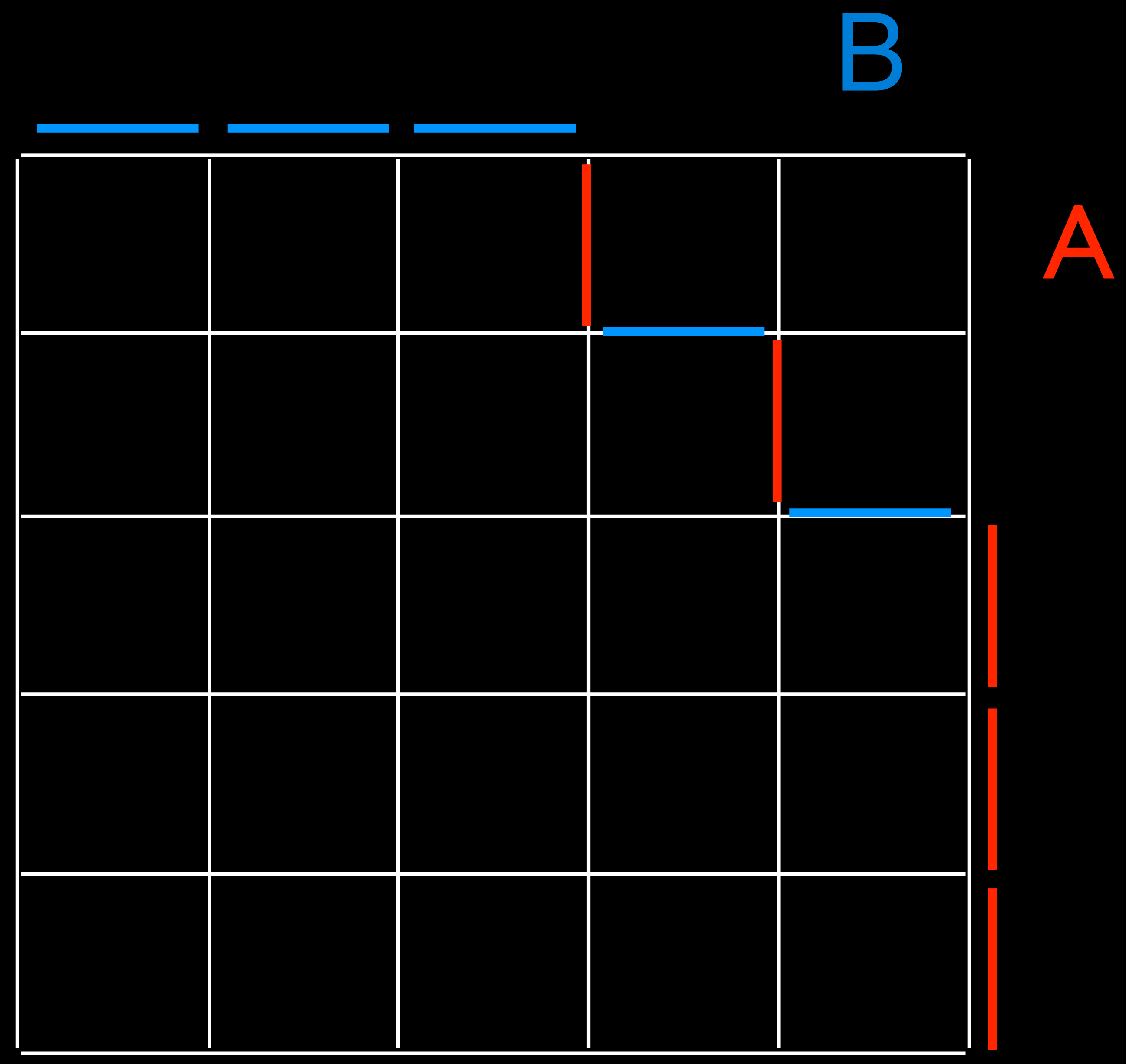
“planarisation” of the “rewriting rules”

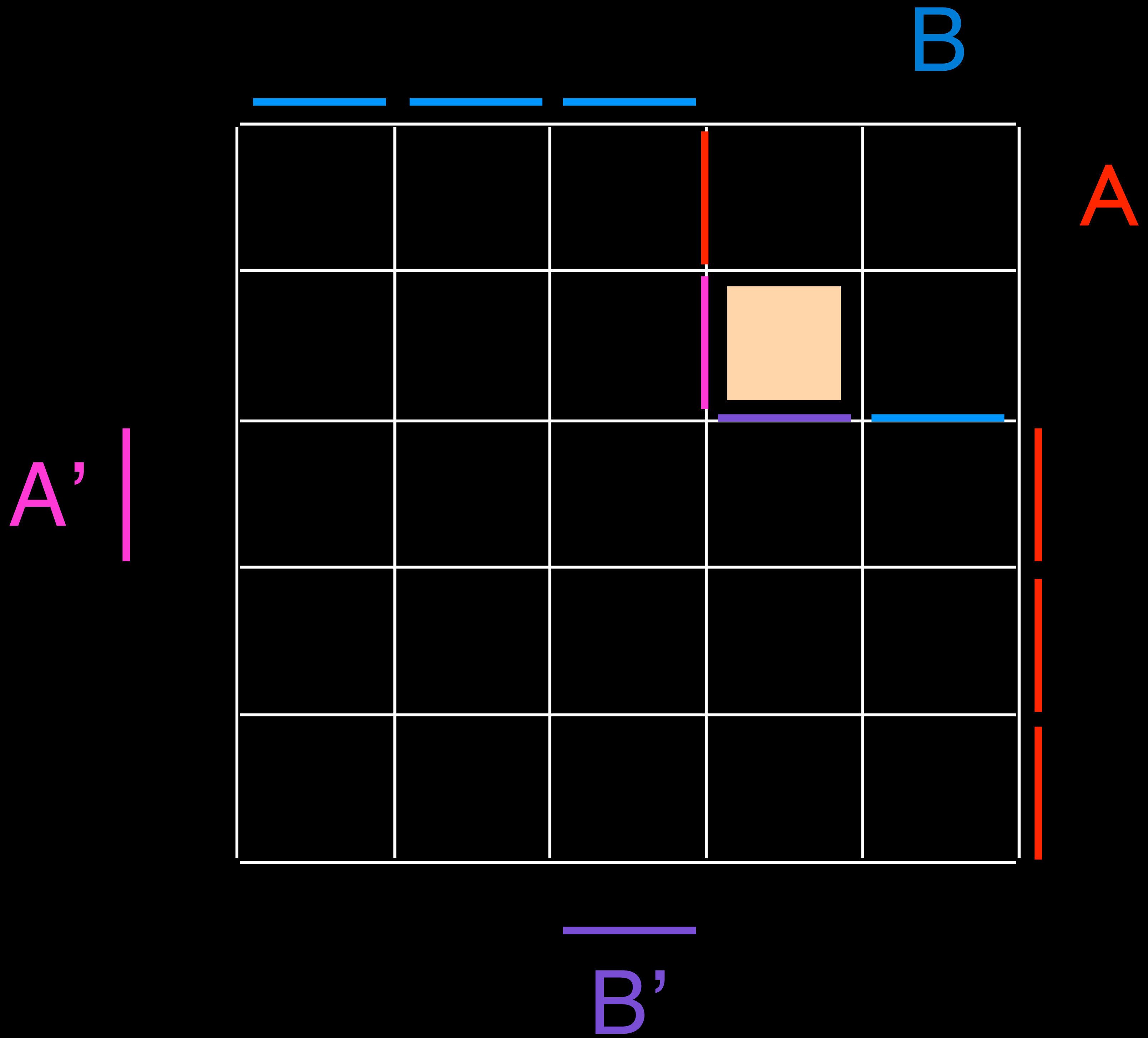


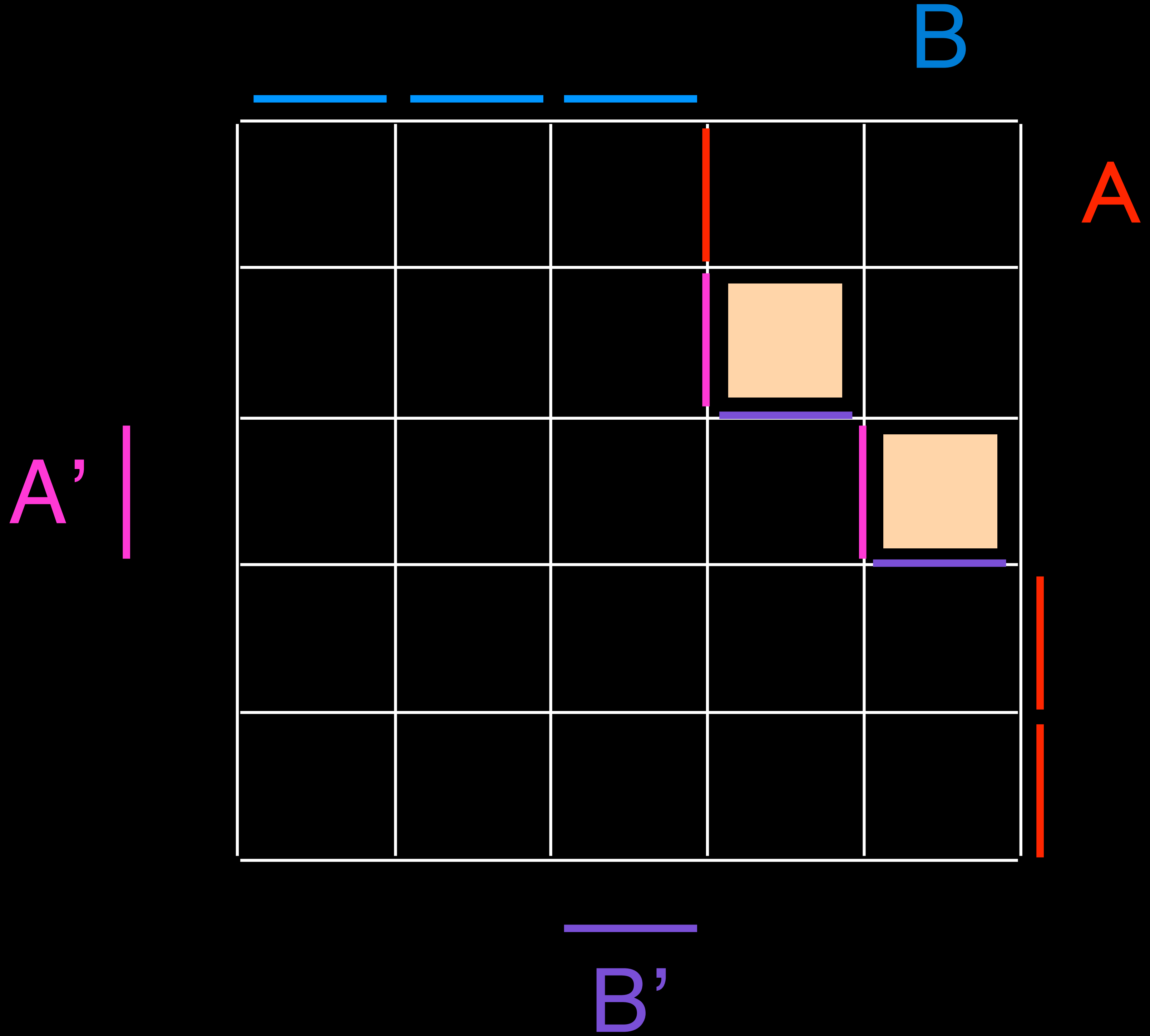

B
A


B
A

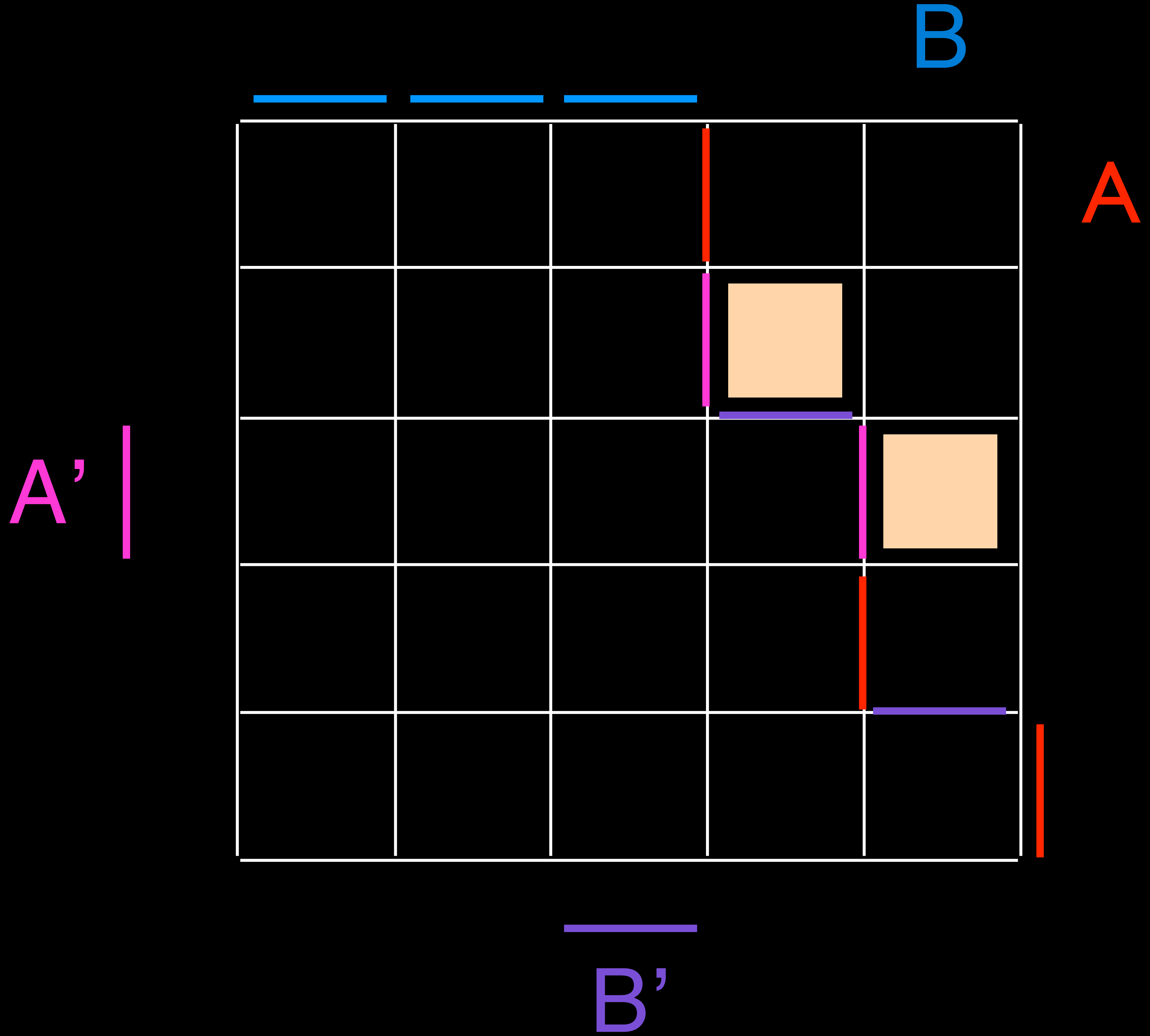

B
A

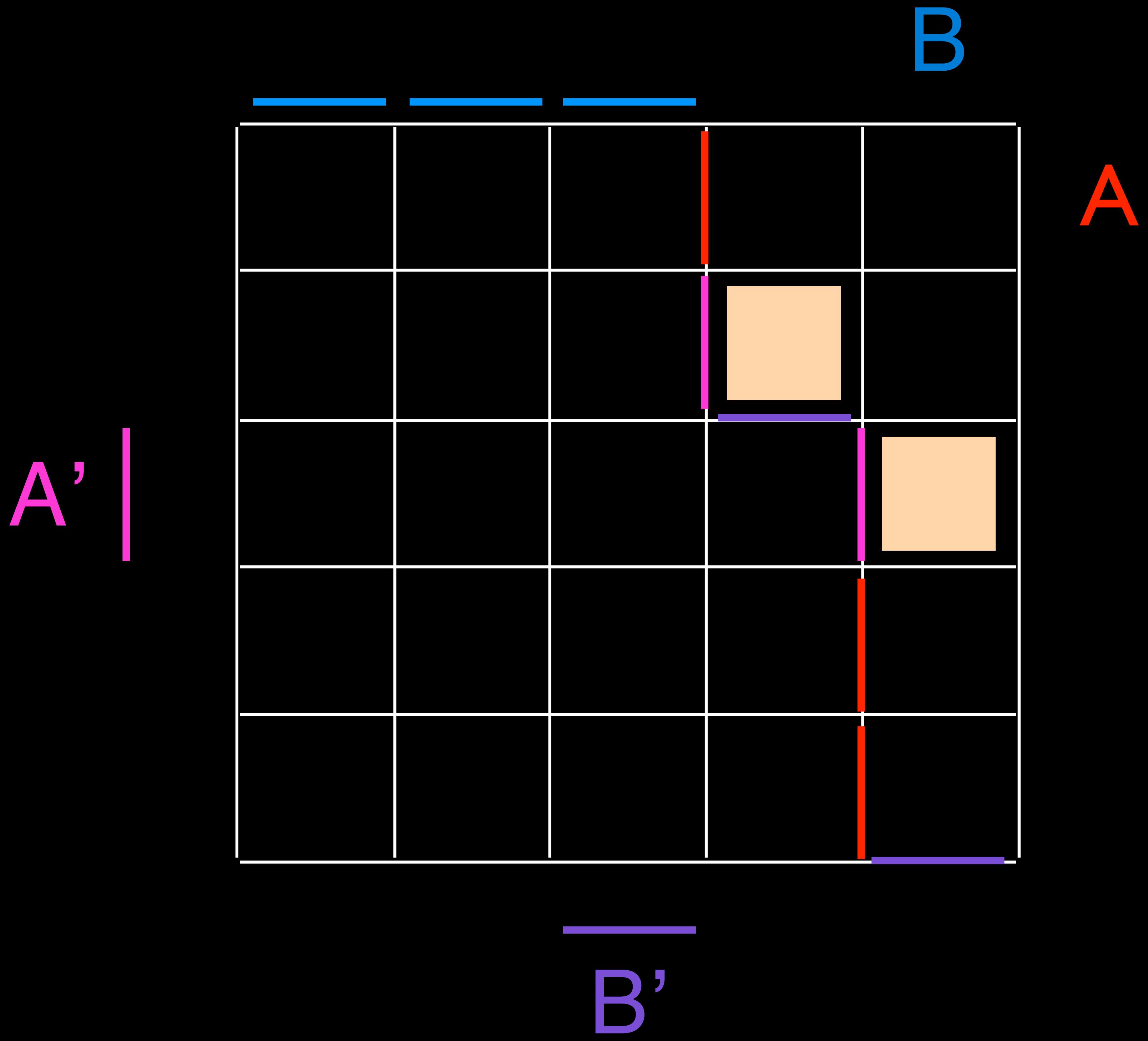

B
A

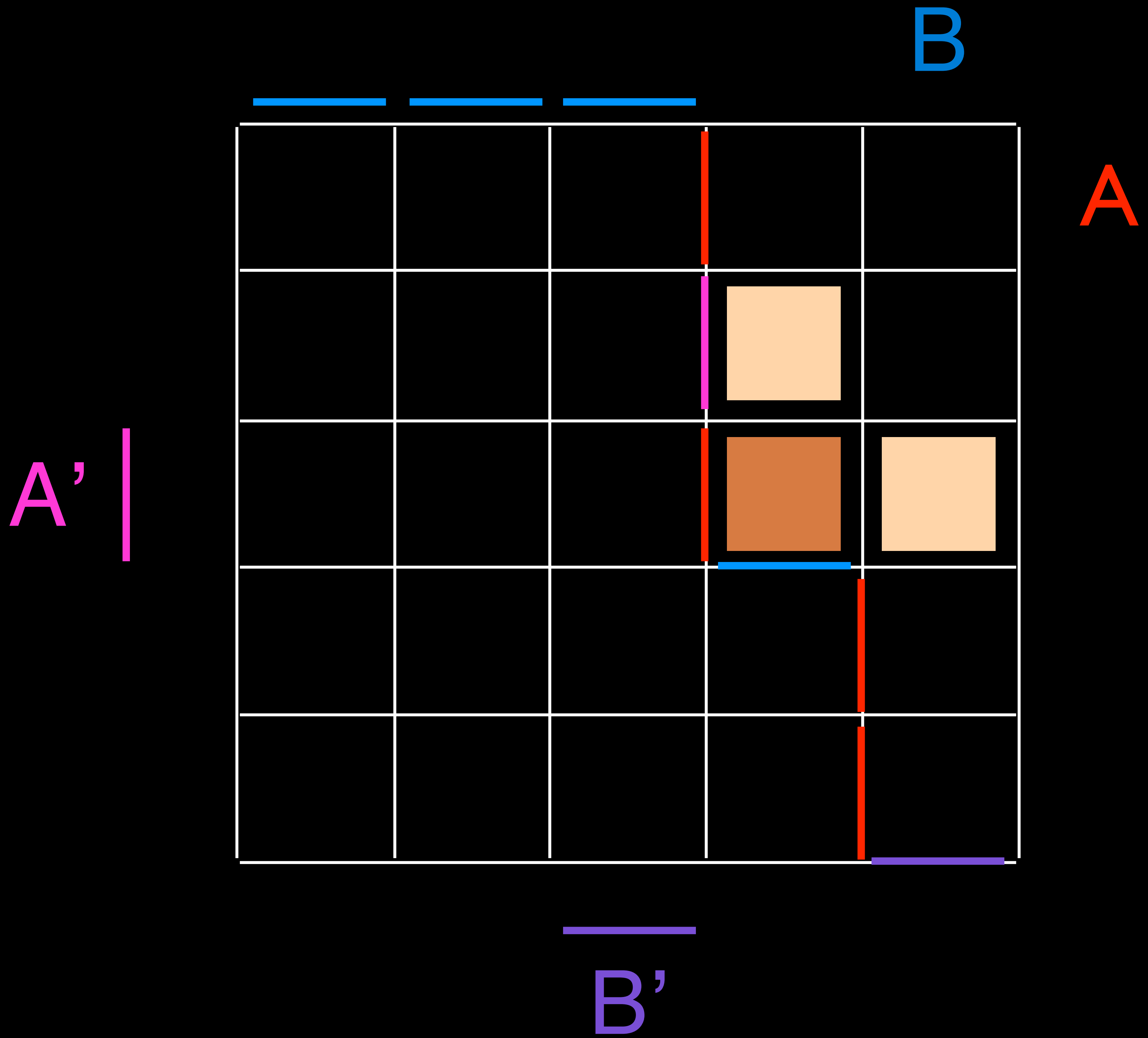

B
A
A’
B’

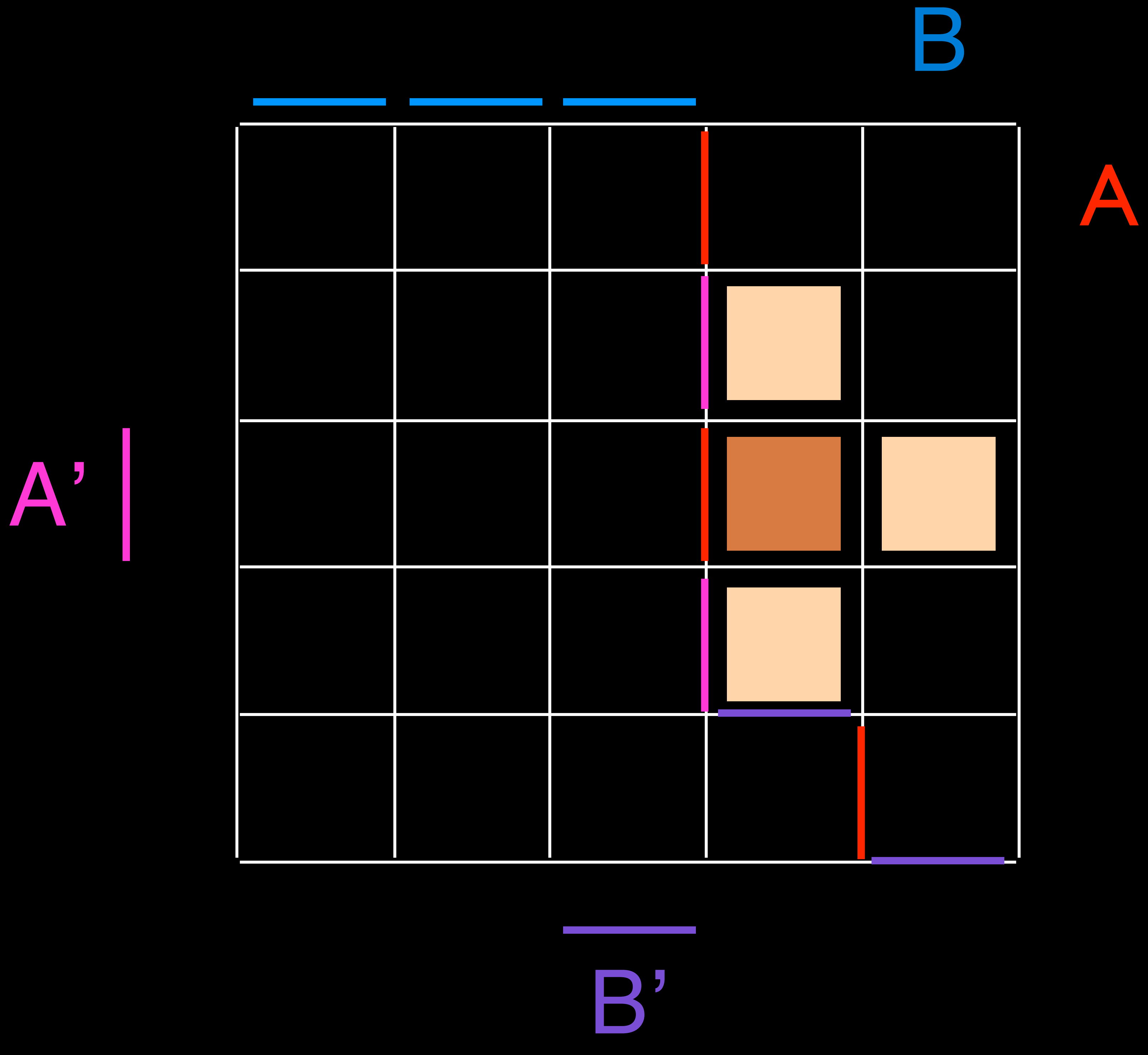


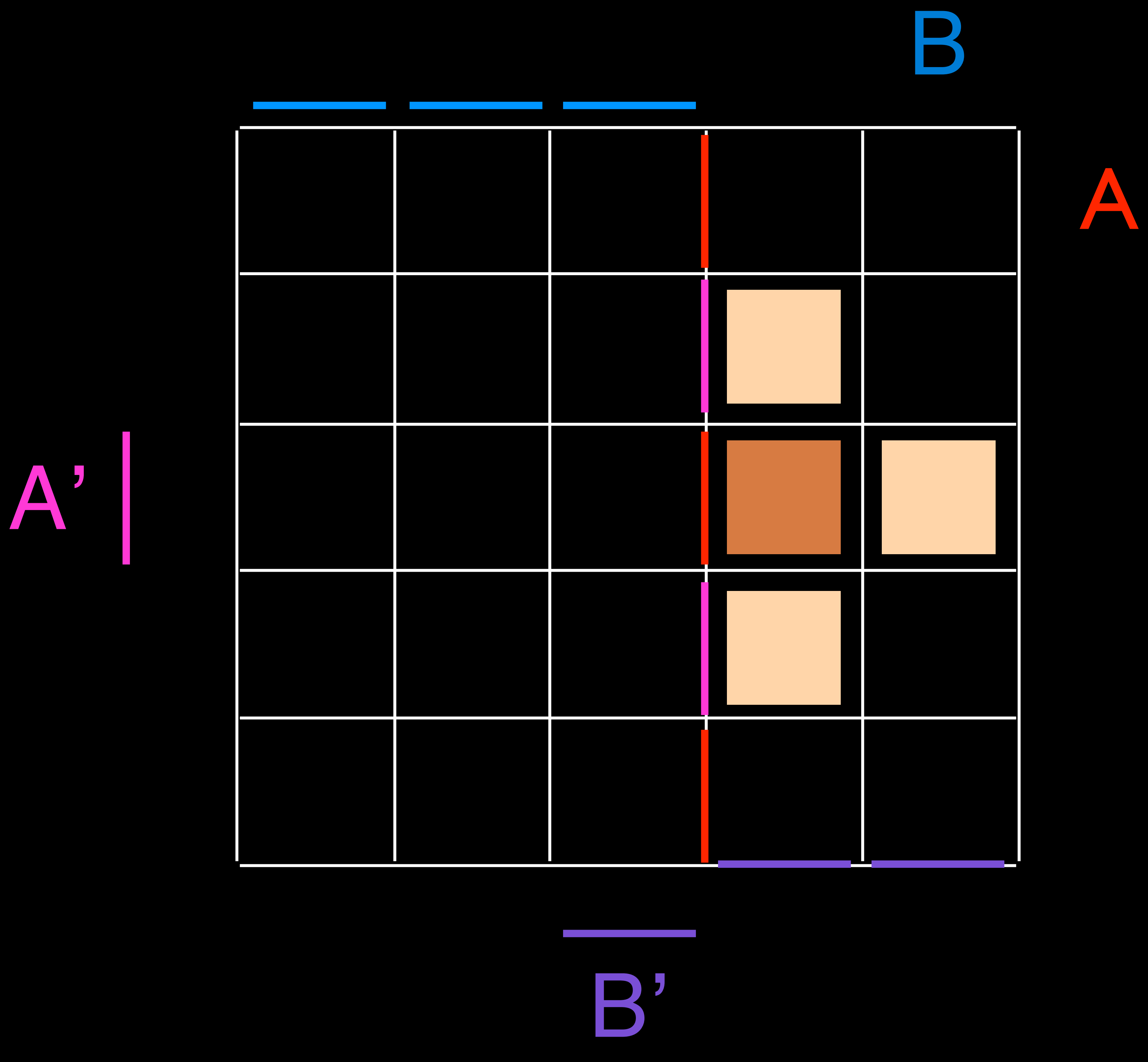
B
A
A'
B'

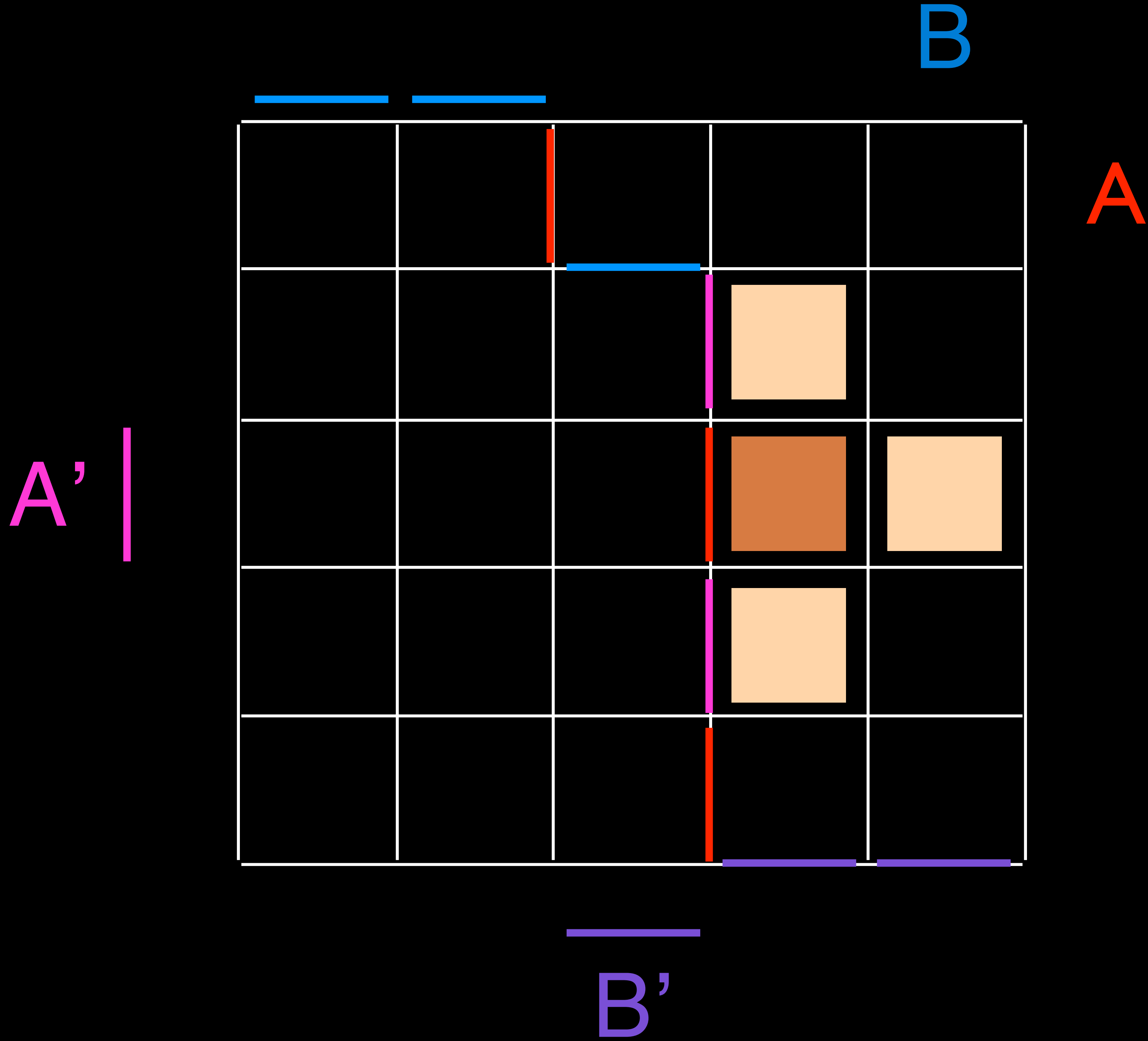

B
A
A'
B'

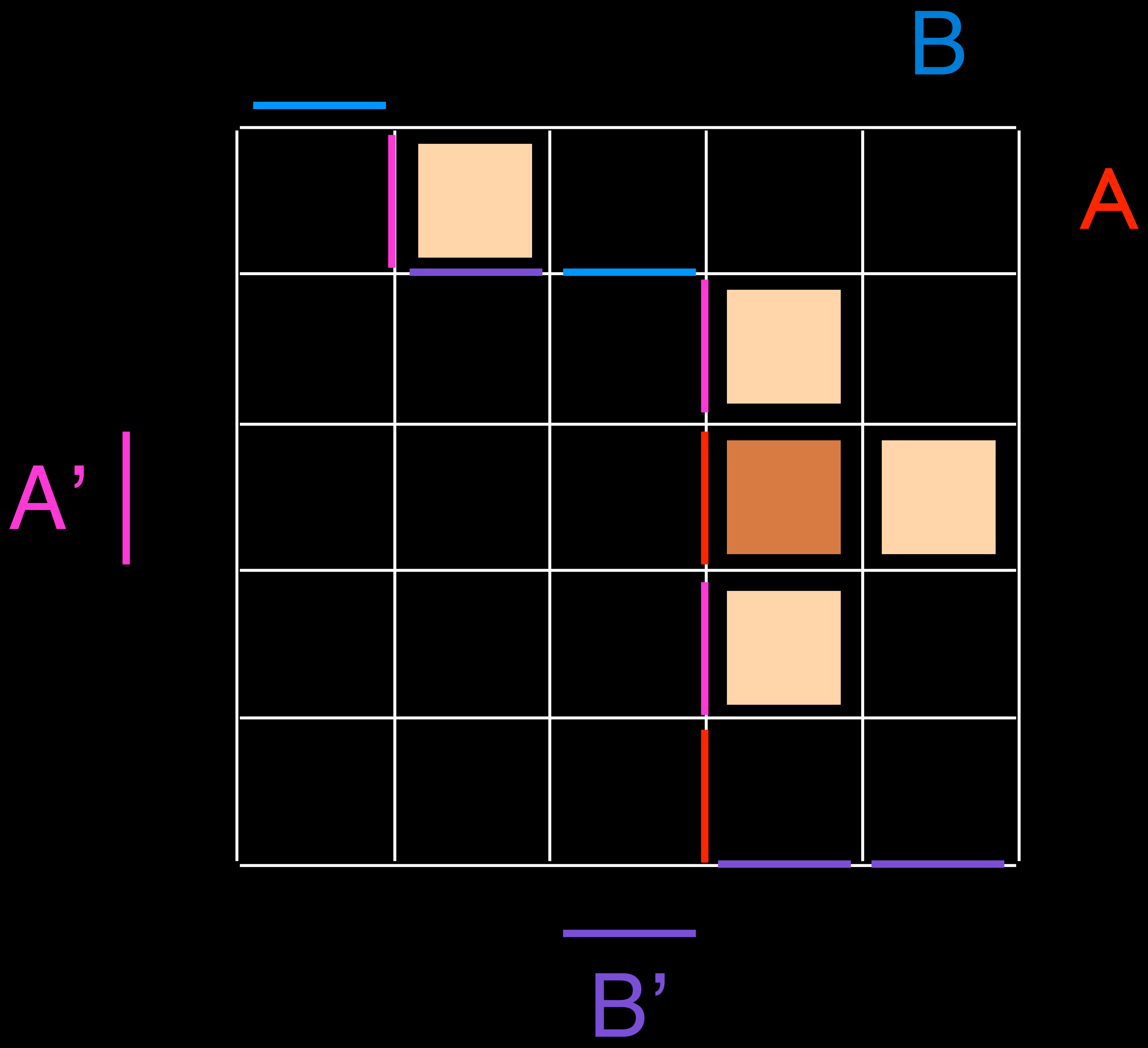

B
A
A’
B’


B
A
A'
B'

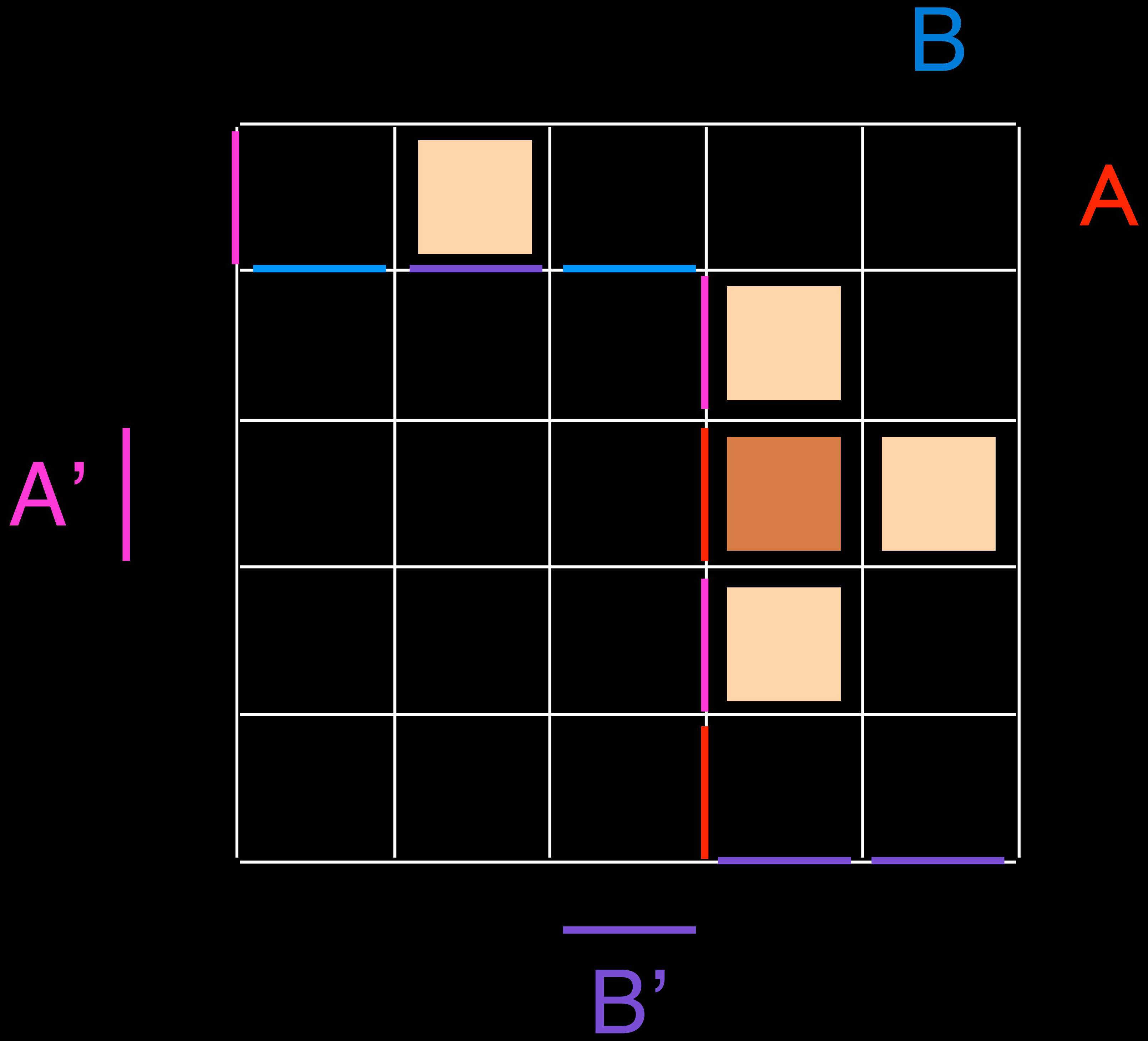

B
A
A'
B'

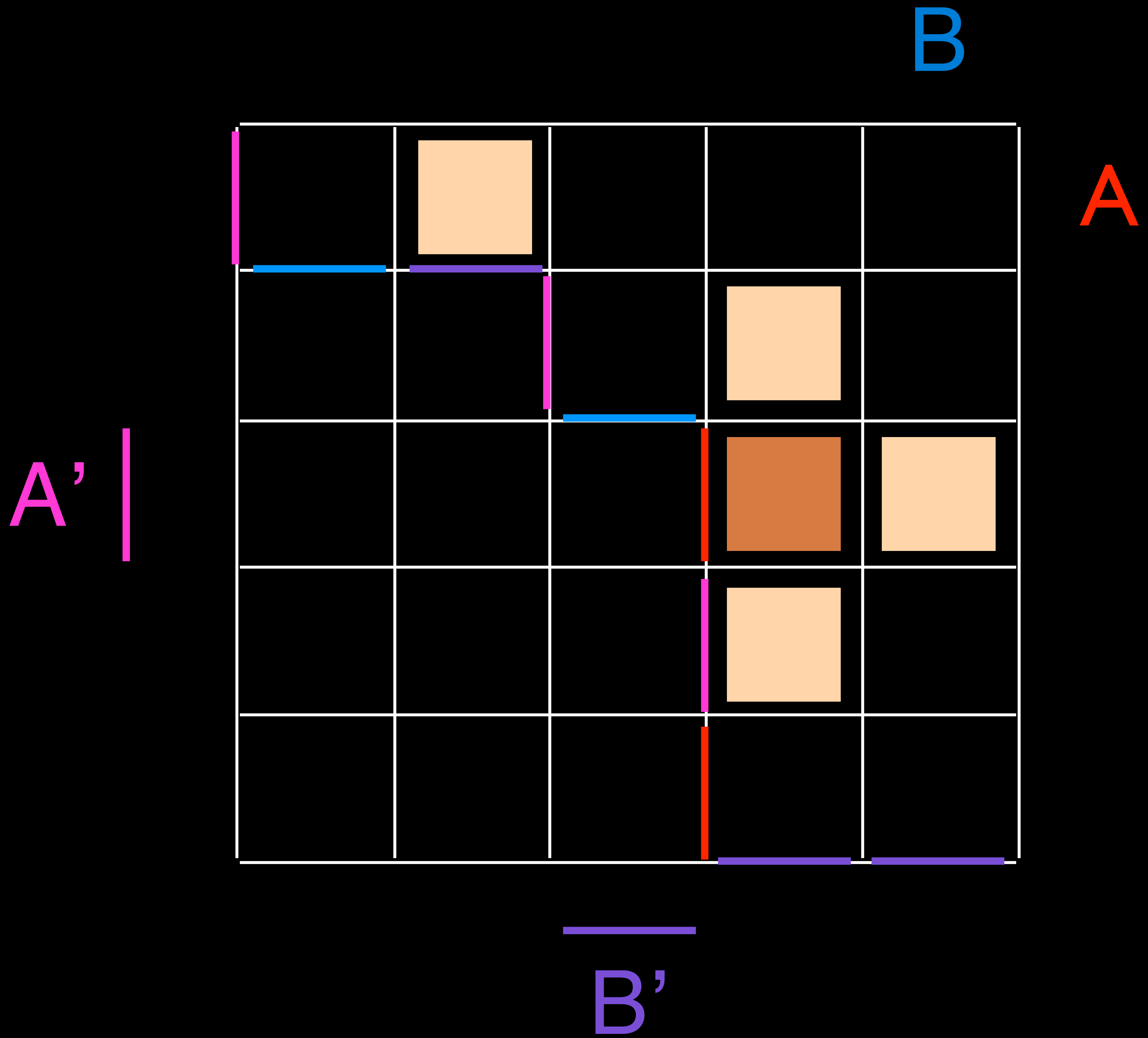

B
A
A'
B'

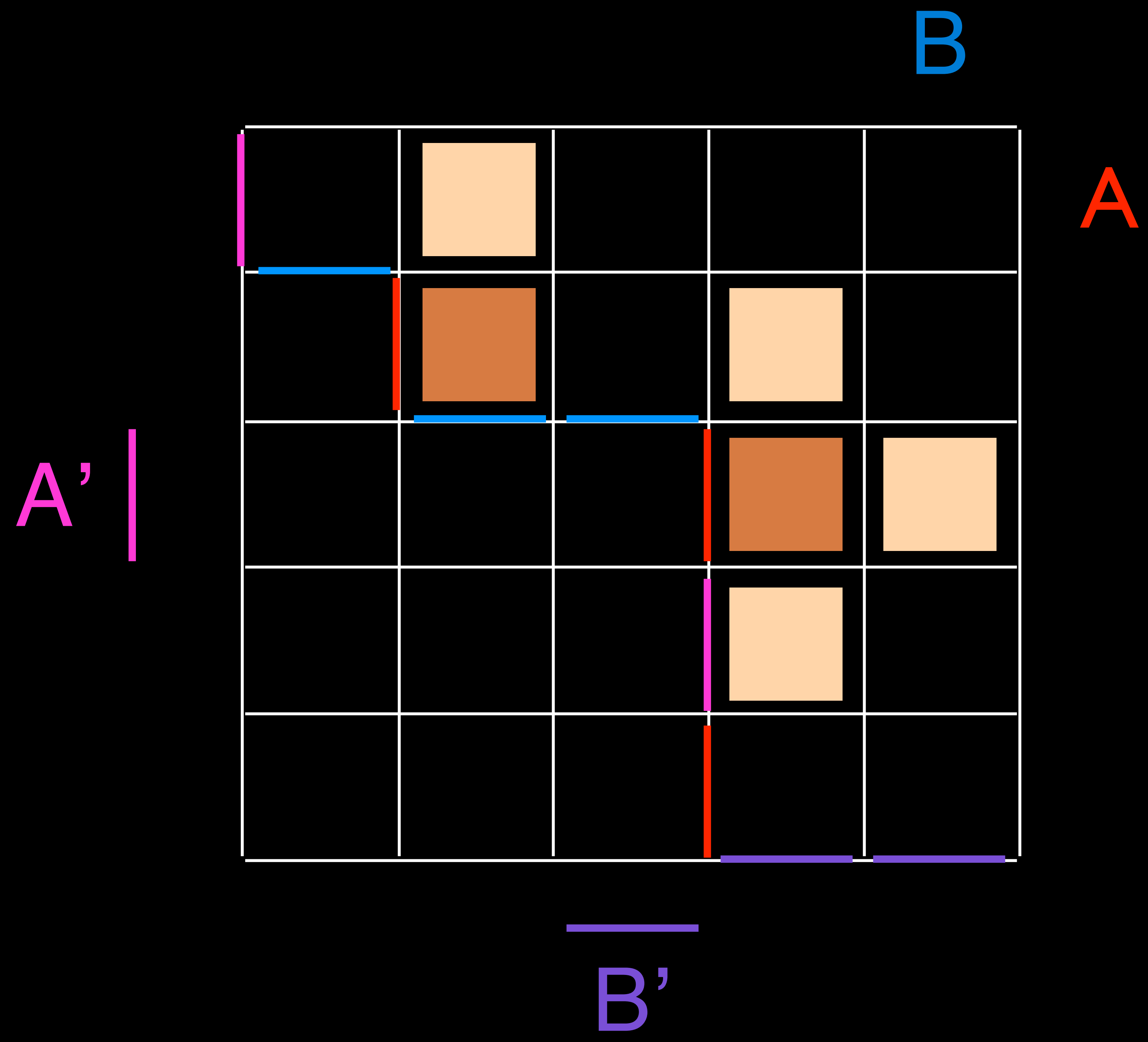

B
A
A'
B'

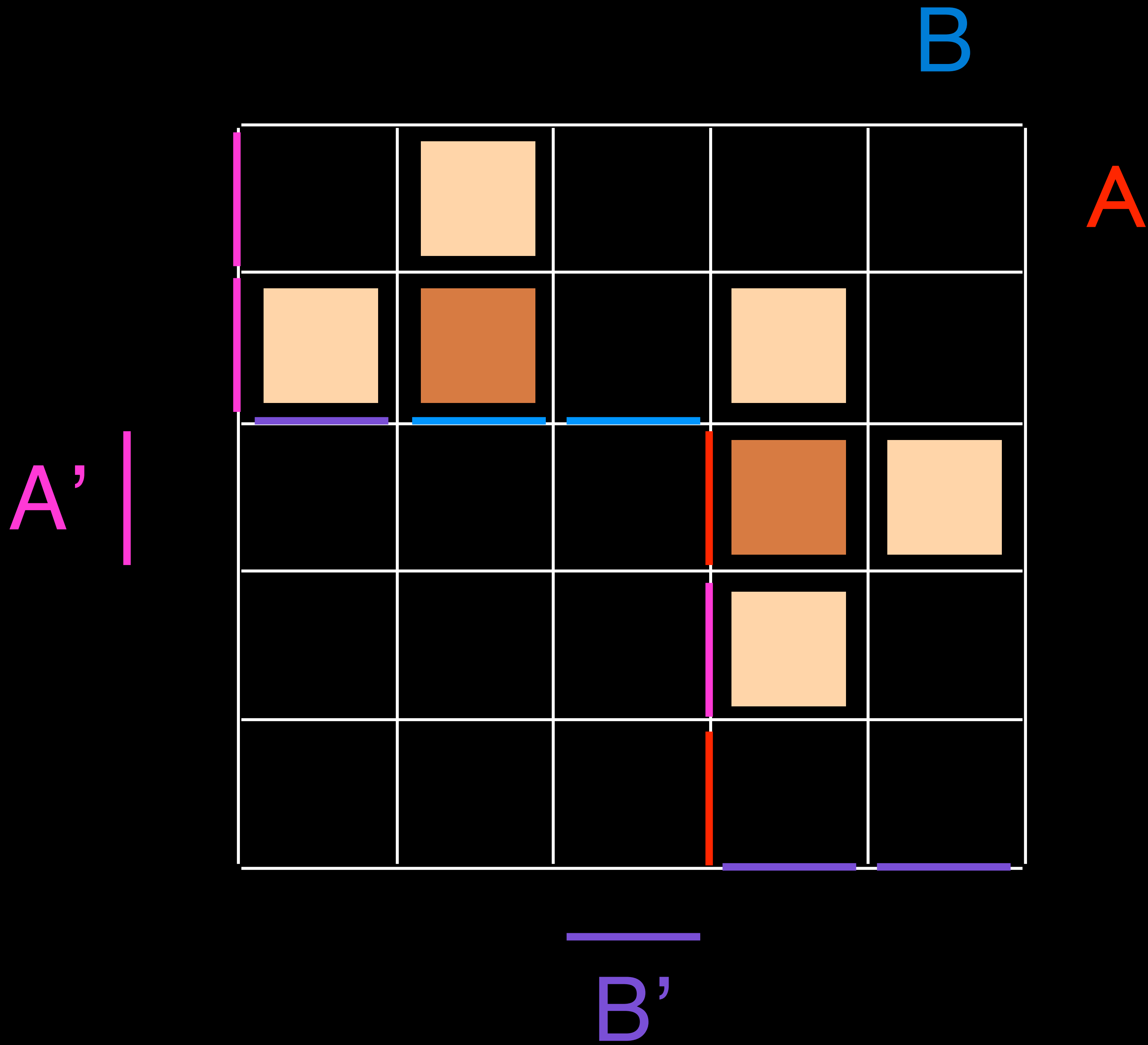


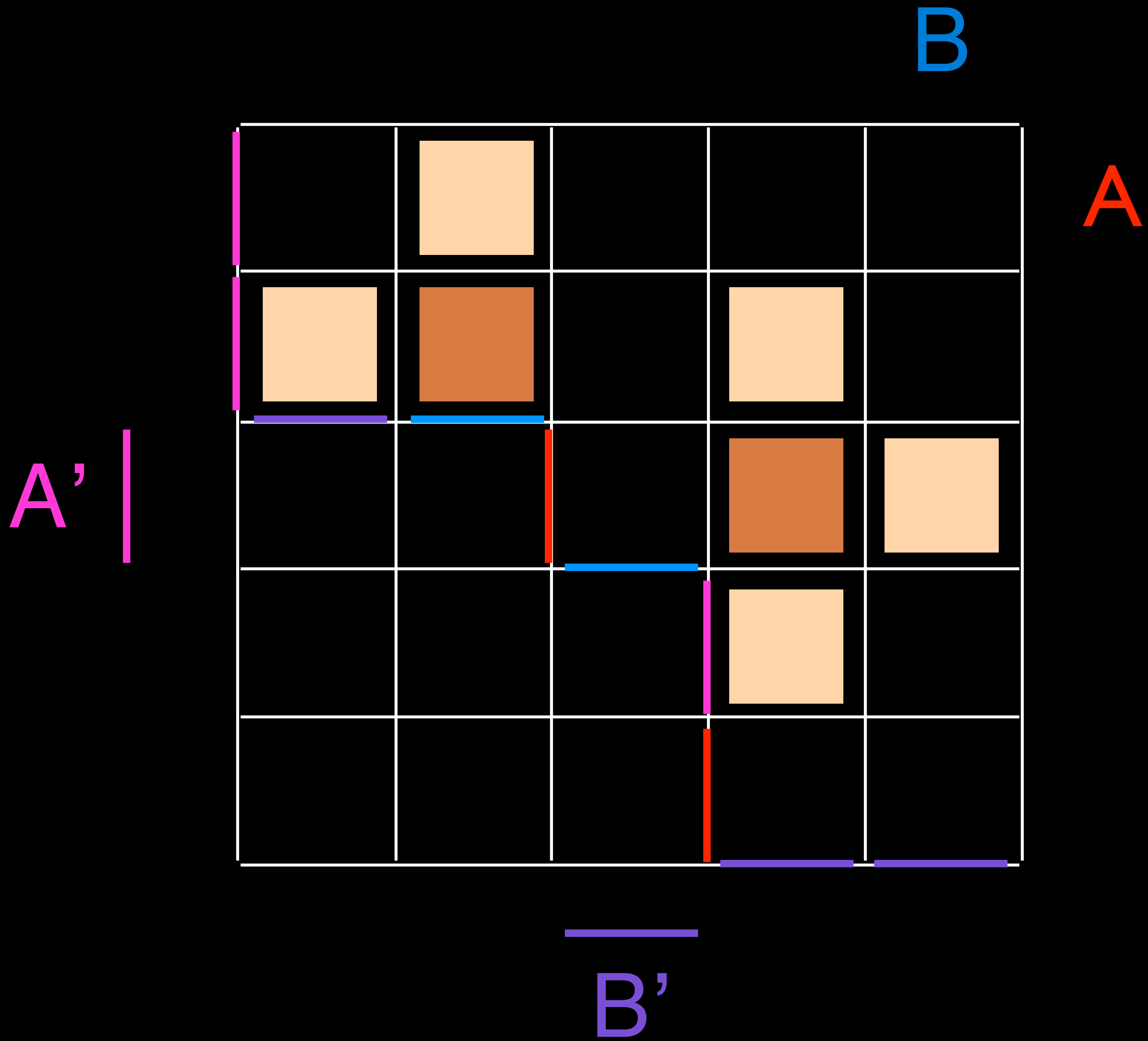
B
A
A'
B'


B
A
A'
B'

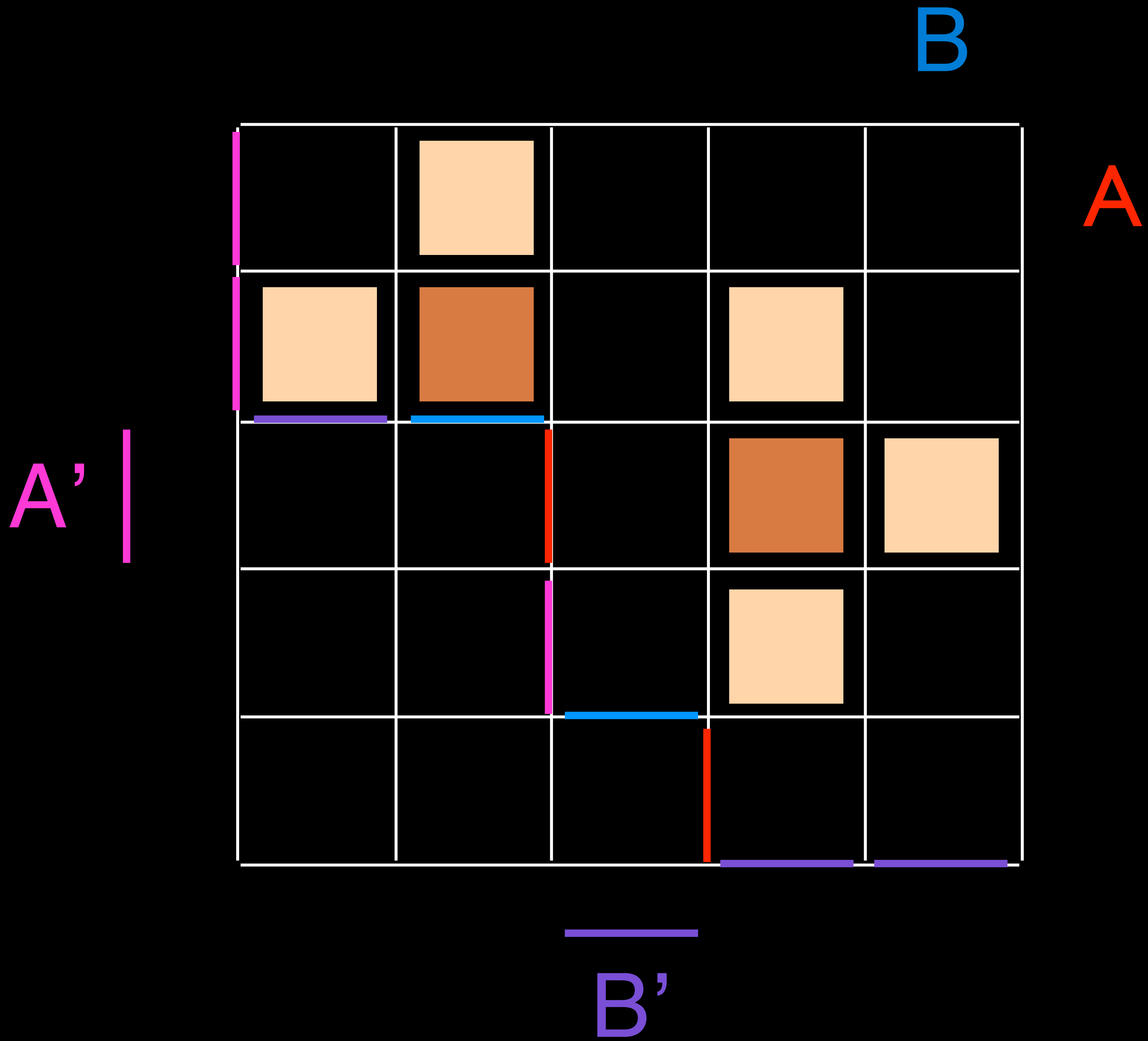

B
A
A'
B'

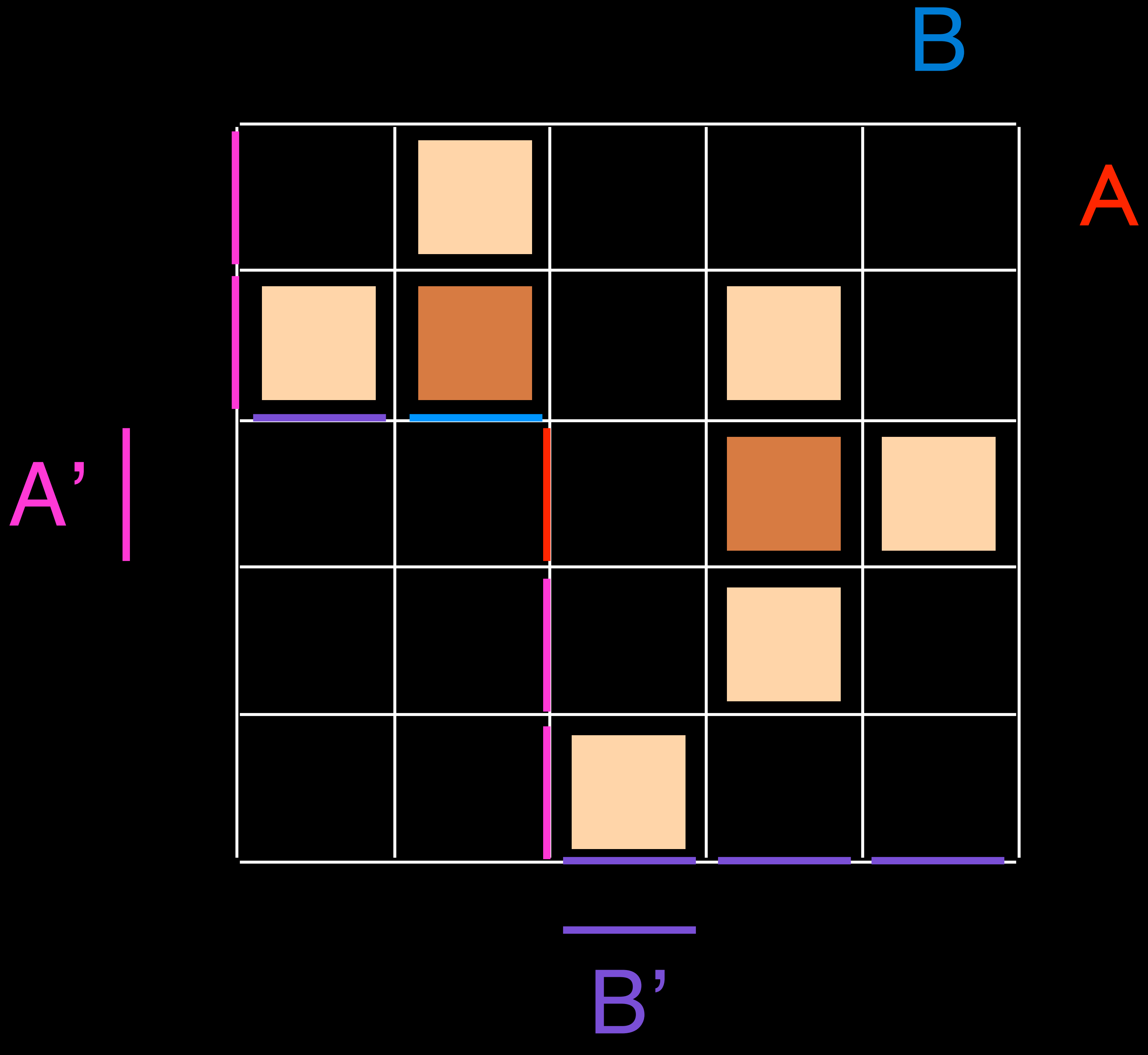

B
A
A'
B'

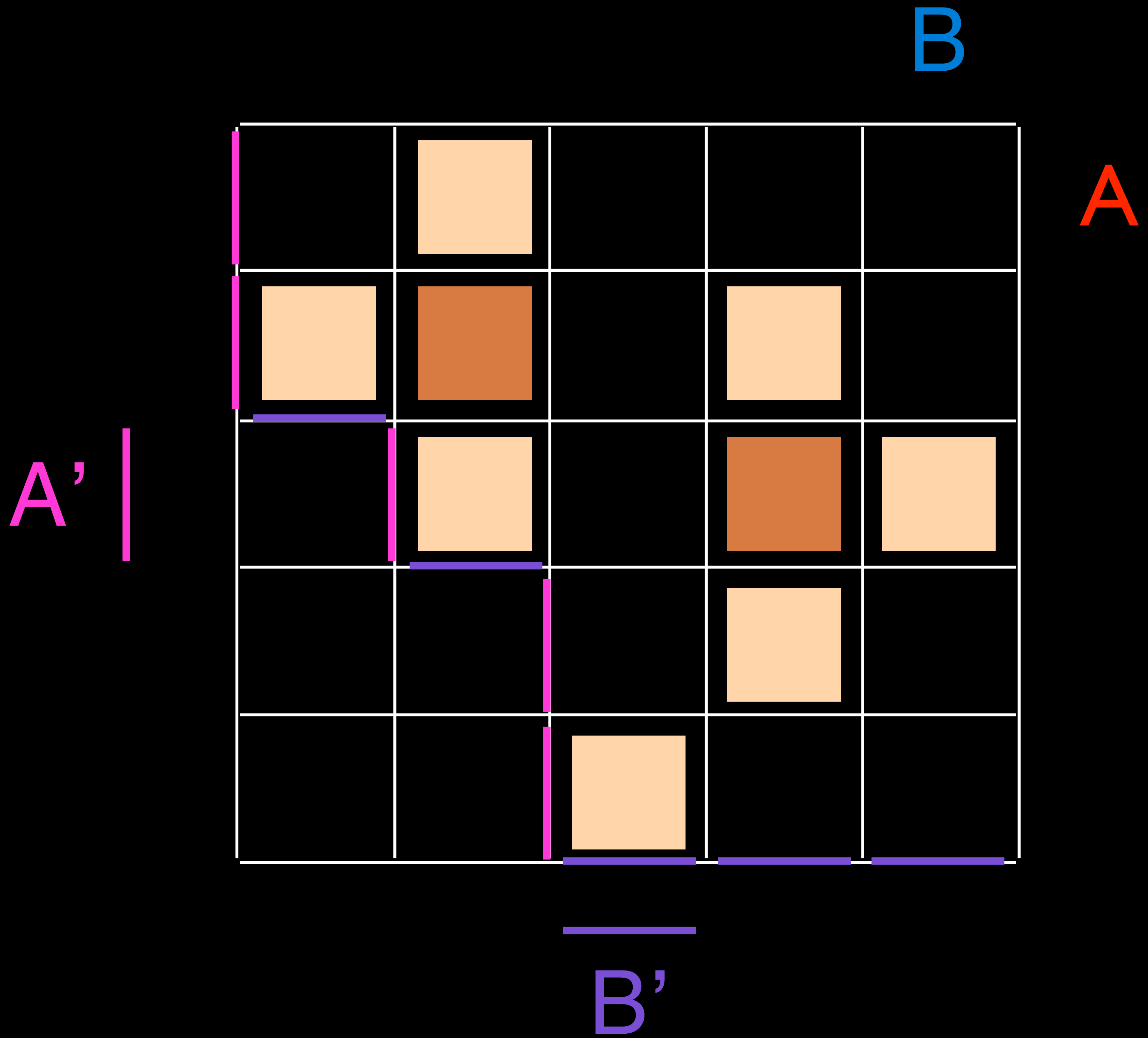

B
A
A'
B'



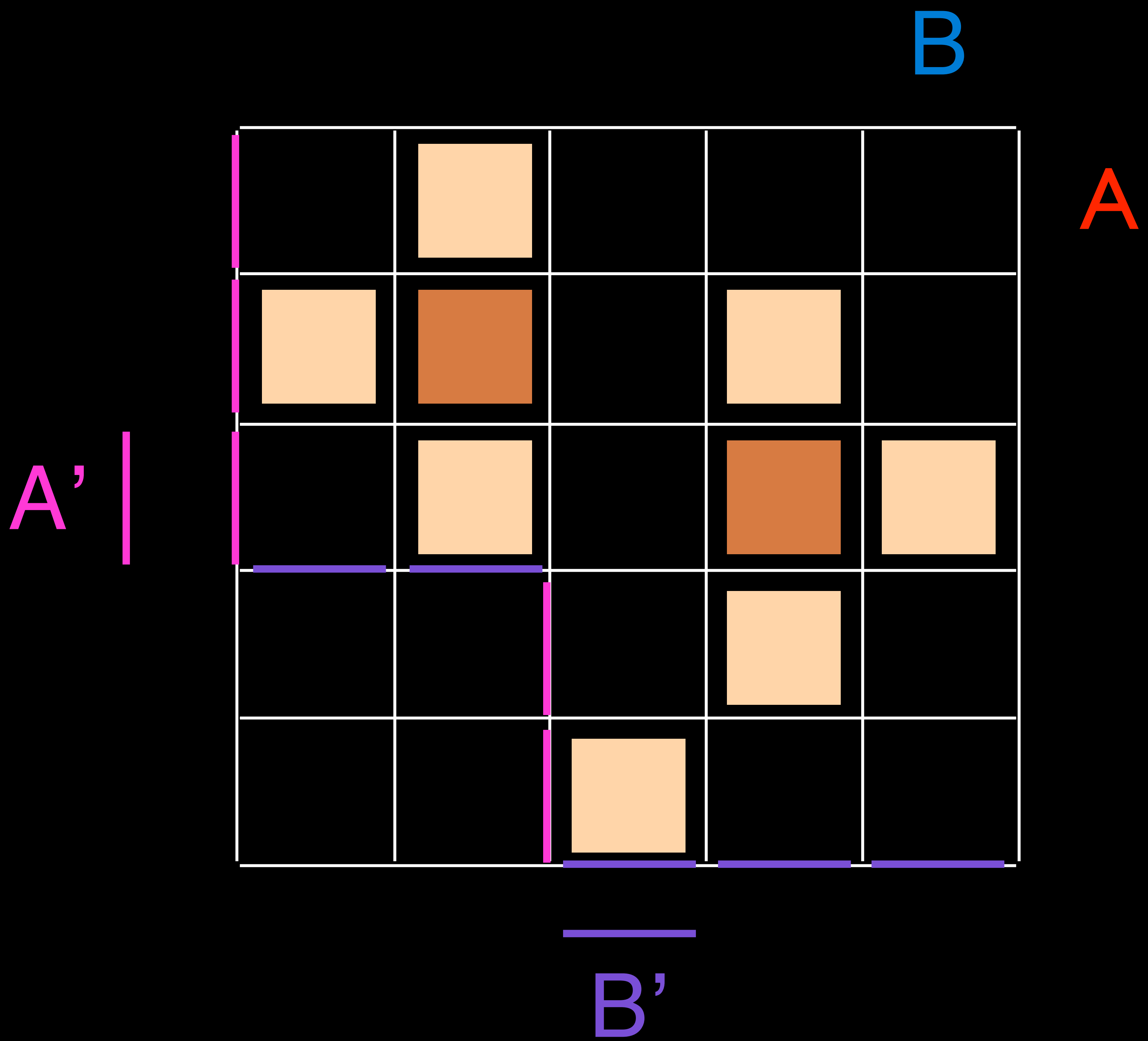
B
A
A'
B'


B
A
A'
B'

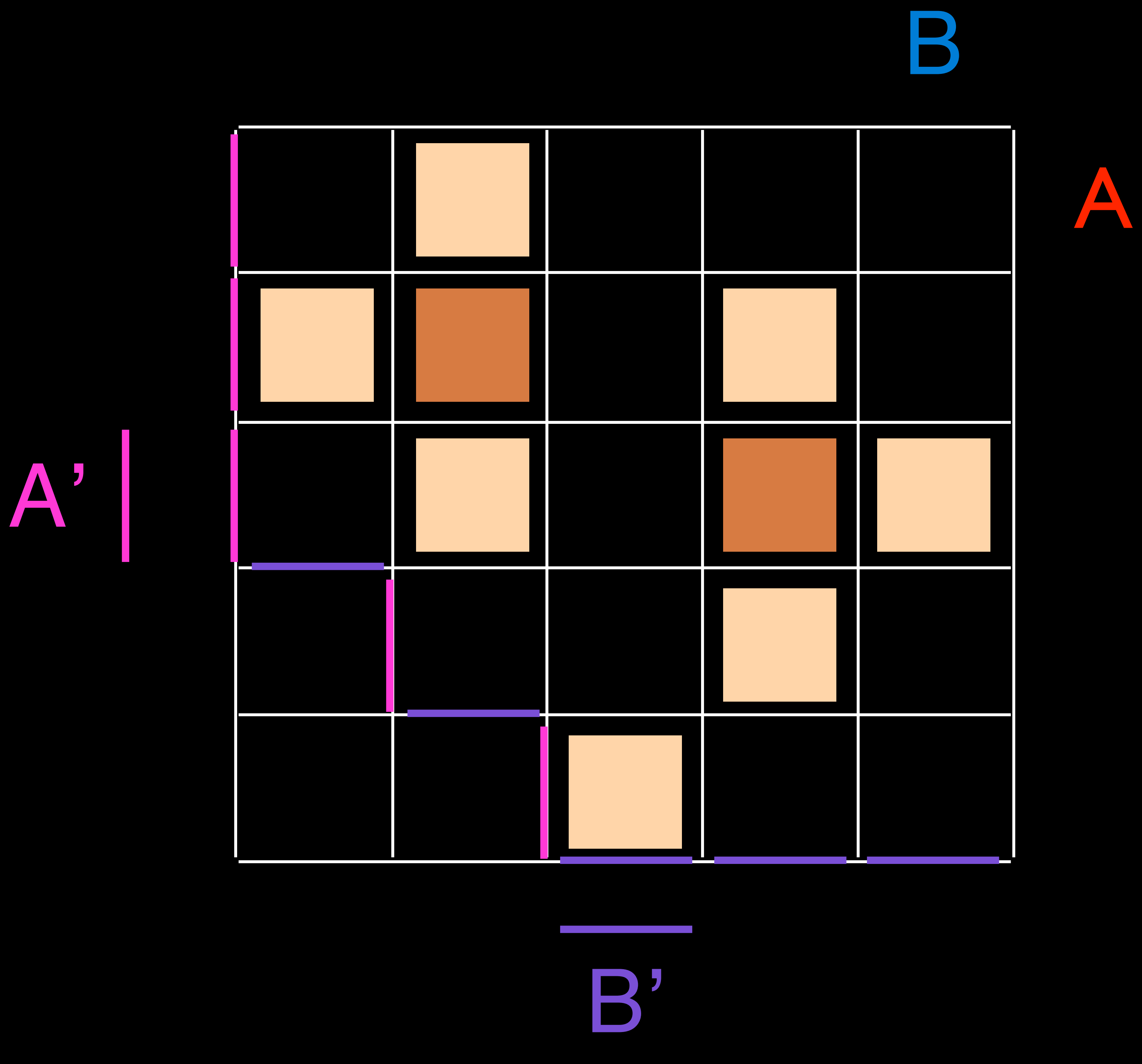

B
A
A'
B'

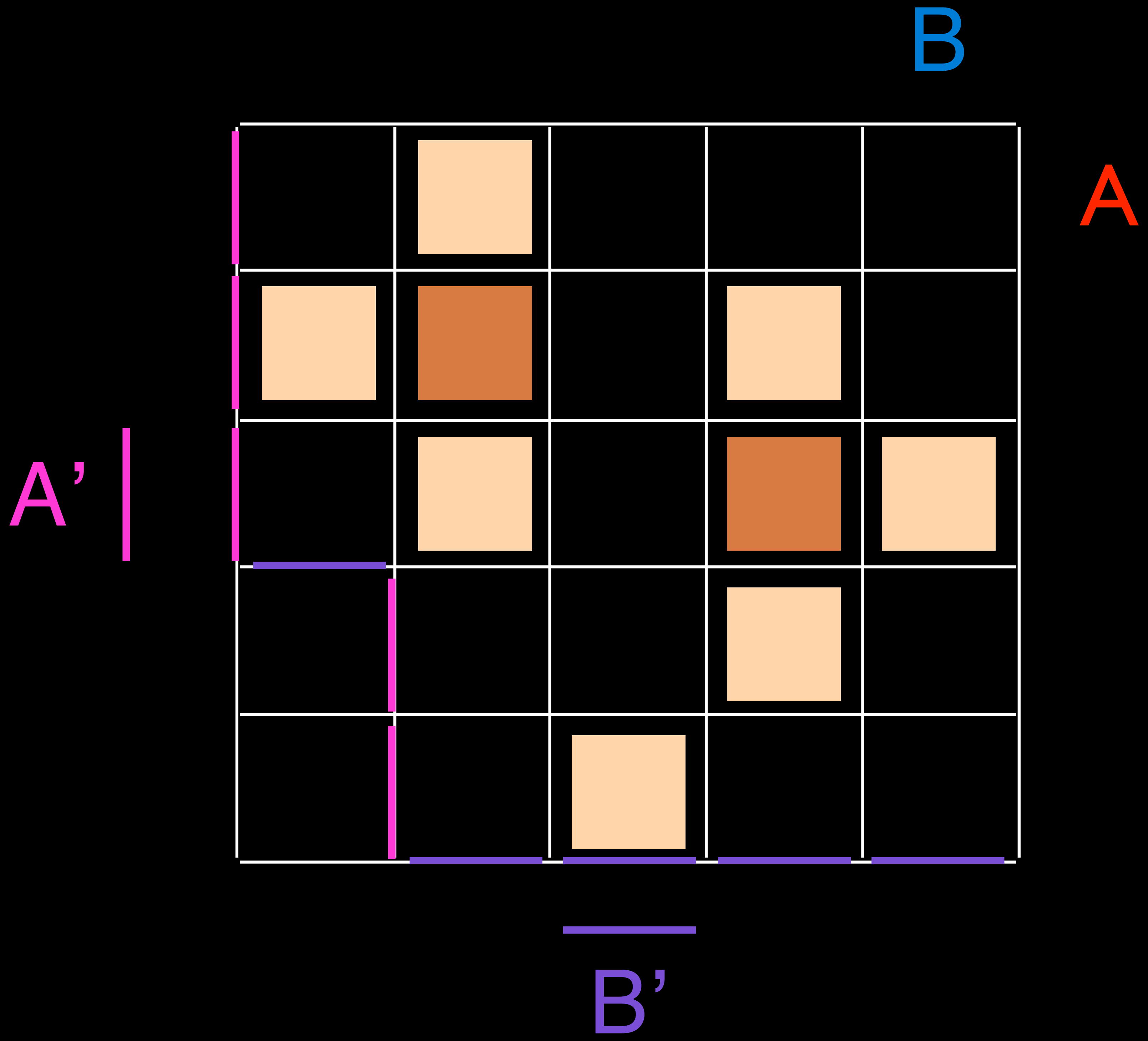


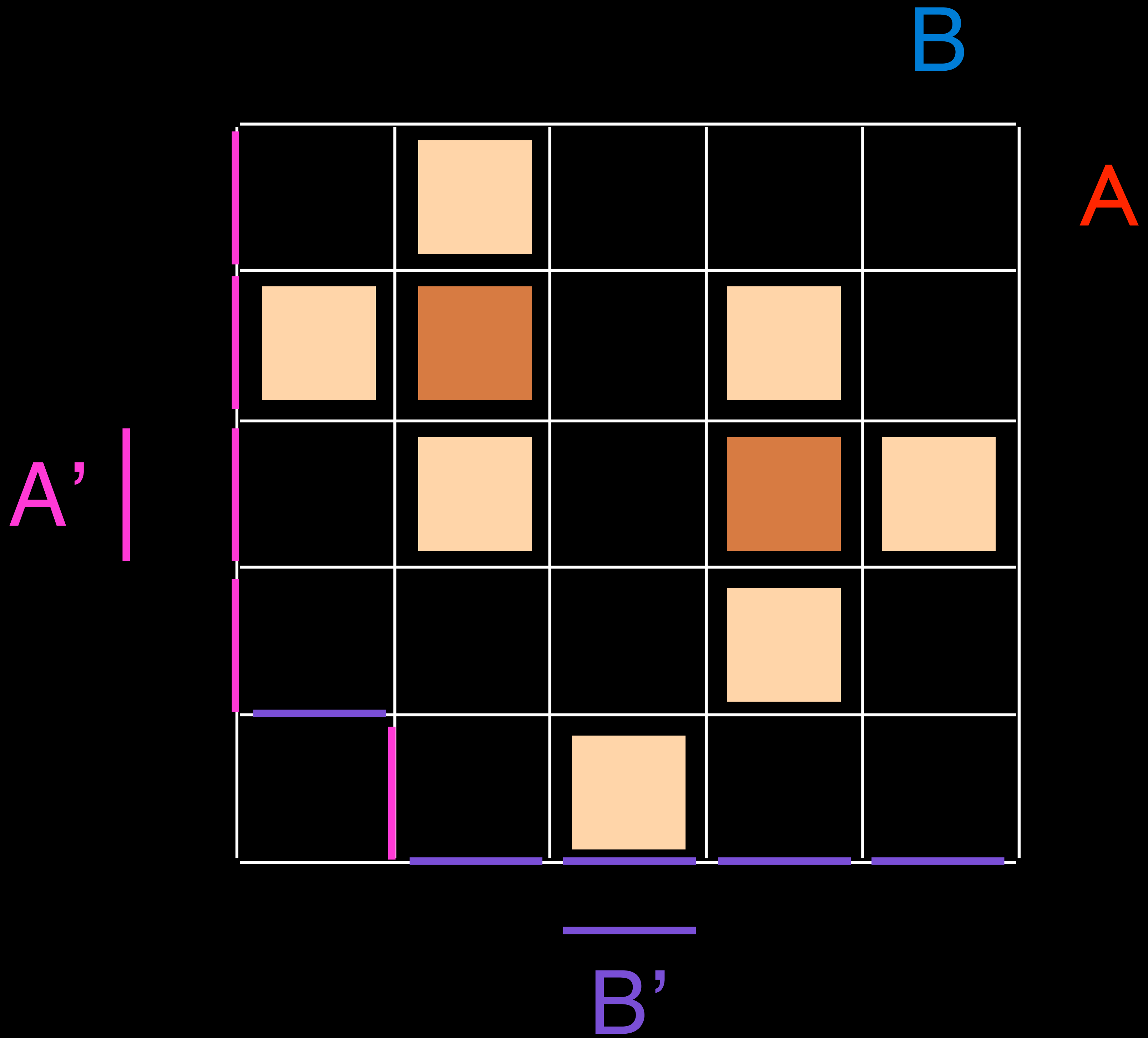
B
A
A'
B'

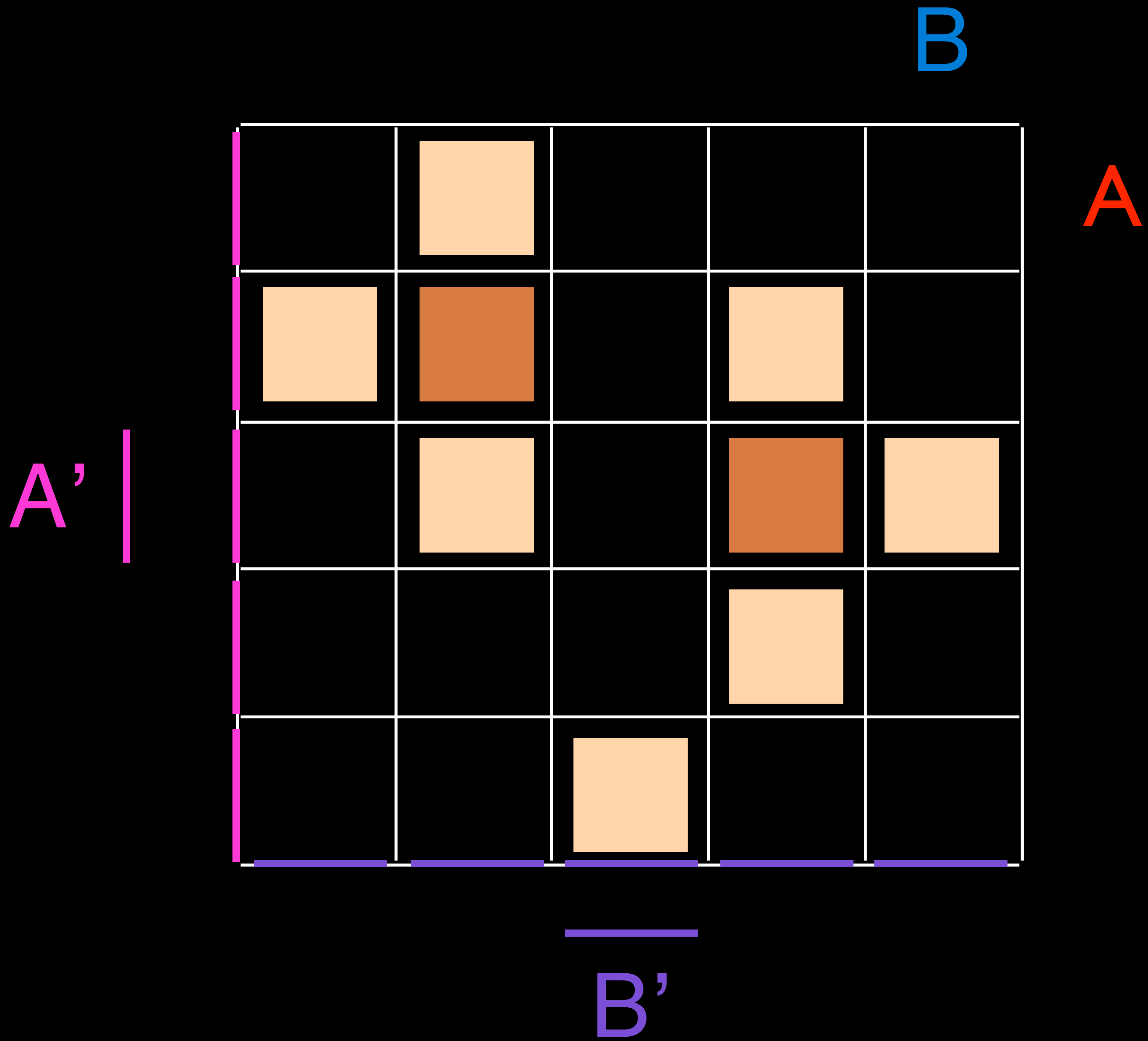


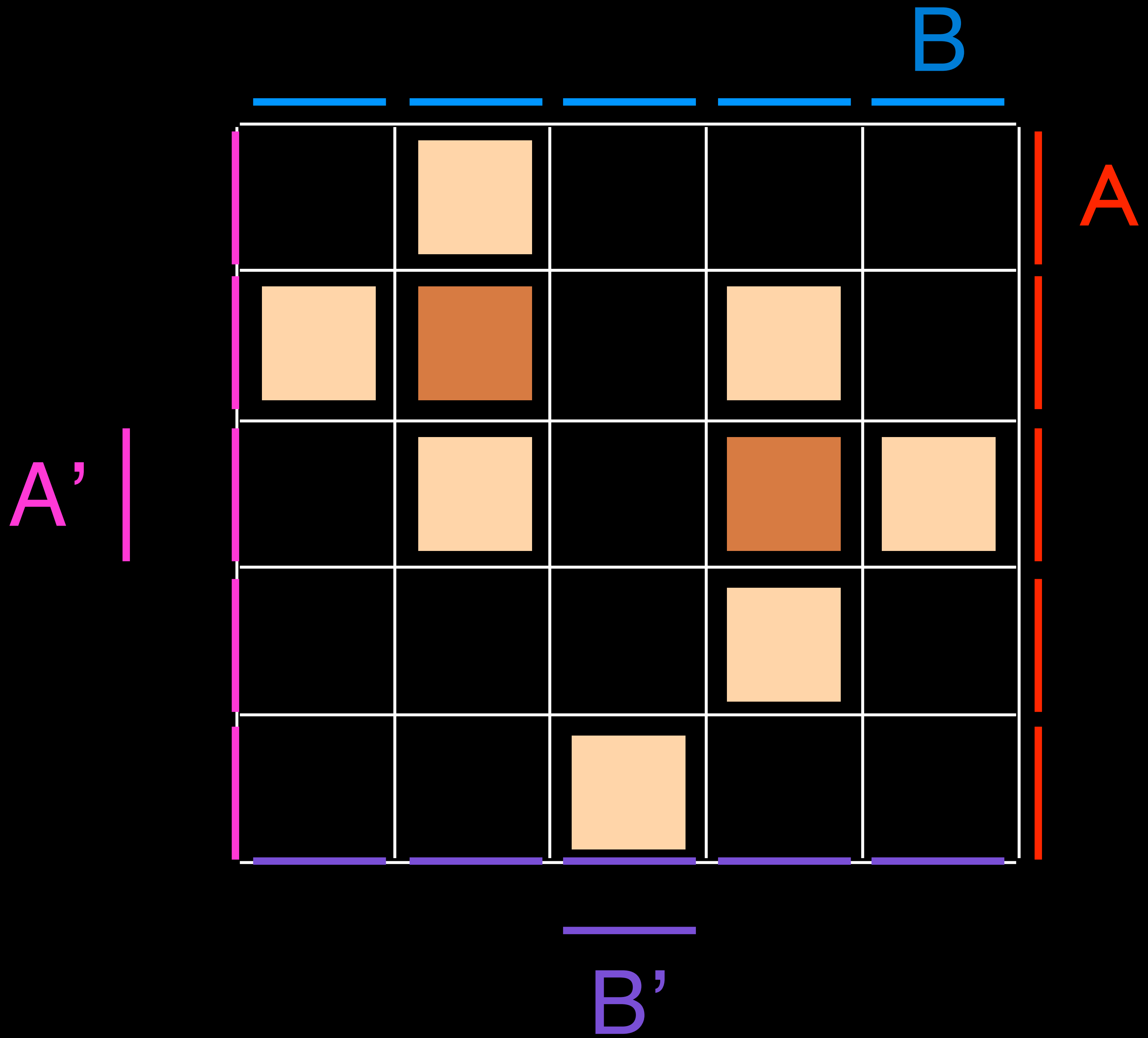
B
A
A'
B'

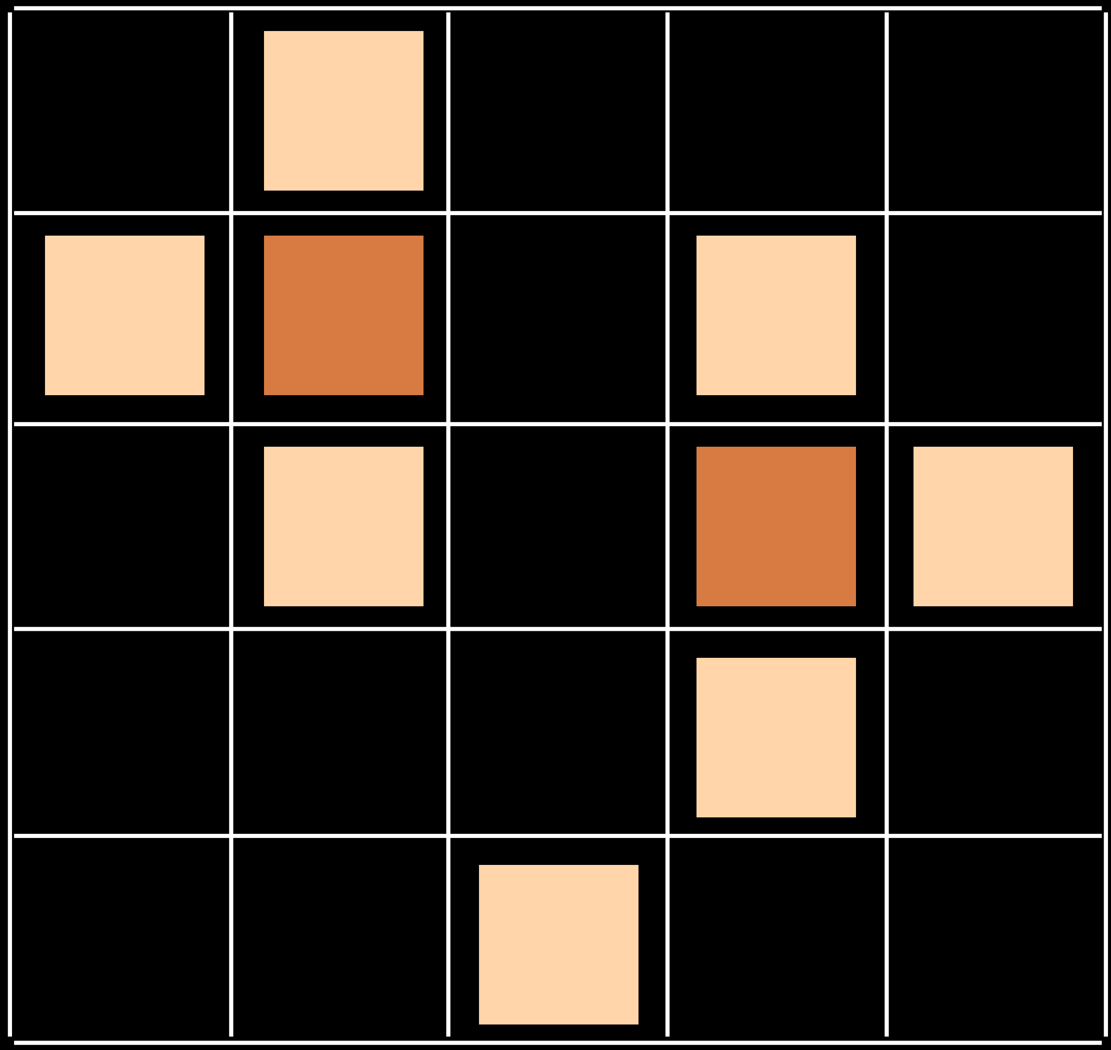

B
A
A'
B'


B
A
A'
B'


B
A
A'
B'


B
A
A'
B'


B
A
A'
B'



The «cellular Ansatz»
first part:

combinatorial objects
(Q-tableaux)
associated to a quadratic algebra Q

Q-tableaux

Quadratic algebra $\mathcal{Q}$

generators $\mathcal{B} = \{B_j\}_{j \in J}$

$\mathcal{A} = \{A_i\}_{i \in I}$

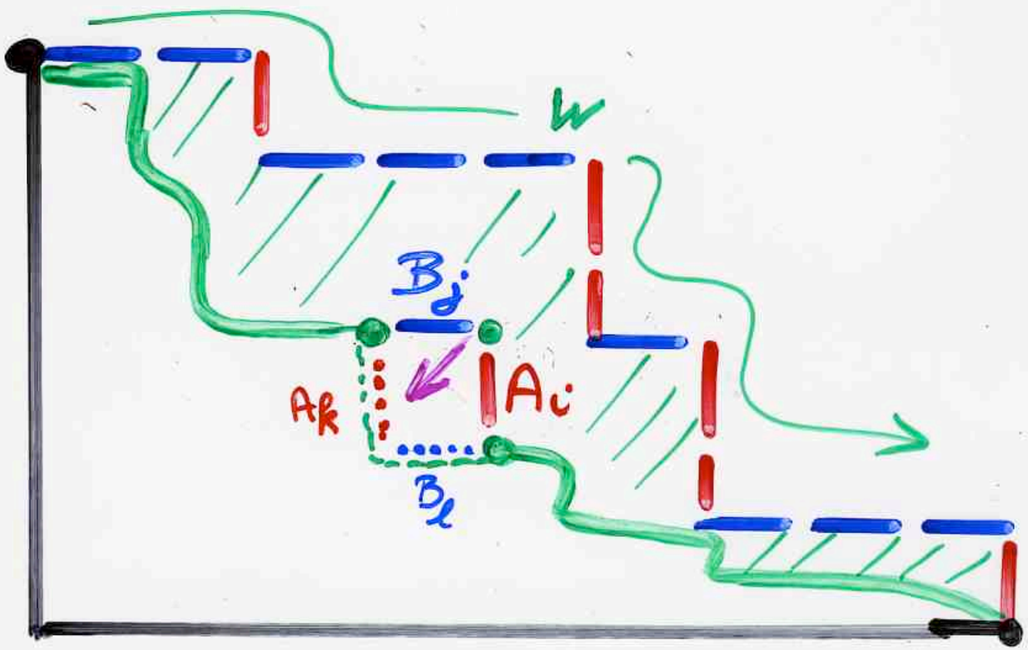
commutation relations

$$B_j A_i = \sum_{k,l} c_{ij}^{kl} A_k B_l \qquad \begin{array}{l} i \in I \\ j \in J \end{array}$$

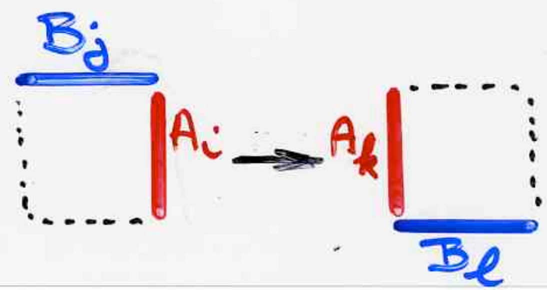
Lemma. In $\mathcal{Q}$ every word $w \in (\mathcal{A} \cup \mathcal{B})^*$ can be written in a unique way

$$w = \sum_{\substack{u \in \mathcal{A}^* \\ v \in \mathcal{B}^*}} c(u, v; w)\, uv$$

Proof:



"planar" rewriting



S set of labels

$$\varphi : \underbrace{\left\{ \begin{bmatrix} k & l \\ i & j \end{bmatrix} \right\}}_{\text{set of rewriting rules}} = R \longrightarrow S$$

$$B_j A_i \rightarrow c_{ij}^{kl} A_k B_l$$

Def

Q-tableau

"image" by φ of a "complete Q-tableau"