At the crossroad of algebra, combinatorics, physics and probabilities: tableaux and exclusion model

Mathematisches
Kolloquium
Universität Wien
18 june 2014

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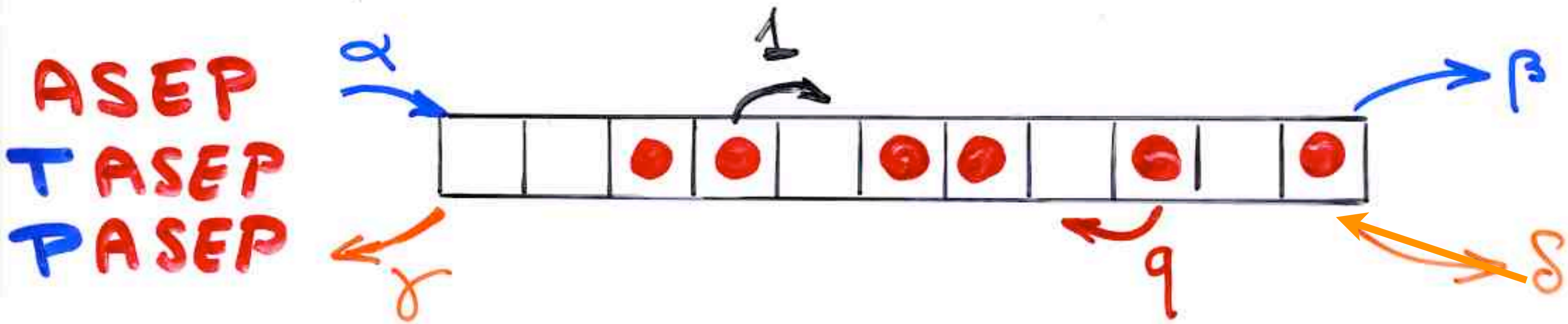
CNRS, Bordeaux, France

The PASEP
(ASEP)

(Partially) ASymmetric Exclusion Process


ASEP
TASEP
PASEP
α
1
β
γ
q
δ

boundary induced phase transitions

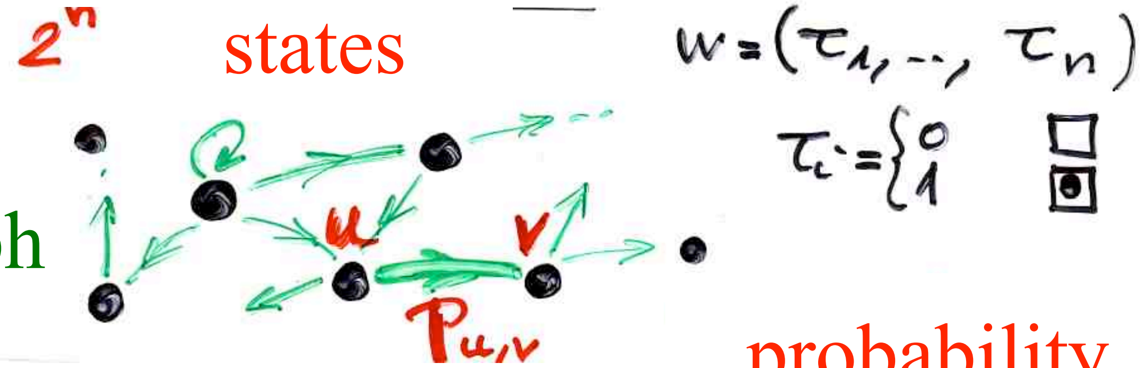
molecular diffusion
linear array of enzymes
biopolymers
traffic flow

------ formation of shocks

Markov chains

states

graph



probability

S : states

$M = (p_{u,v})_{u,v \in S}$ probabilities matrix (stochastic)

$\pi = (p_u, \ldots)$ vector (time t)

$\pi \cdot M$ vector (time t+1)

$$p_v^{(t+1)} = \sum_u p_u^{(t)} \, p_{u,v}$$

$t+1$ time t

stationary probabilities

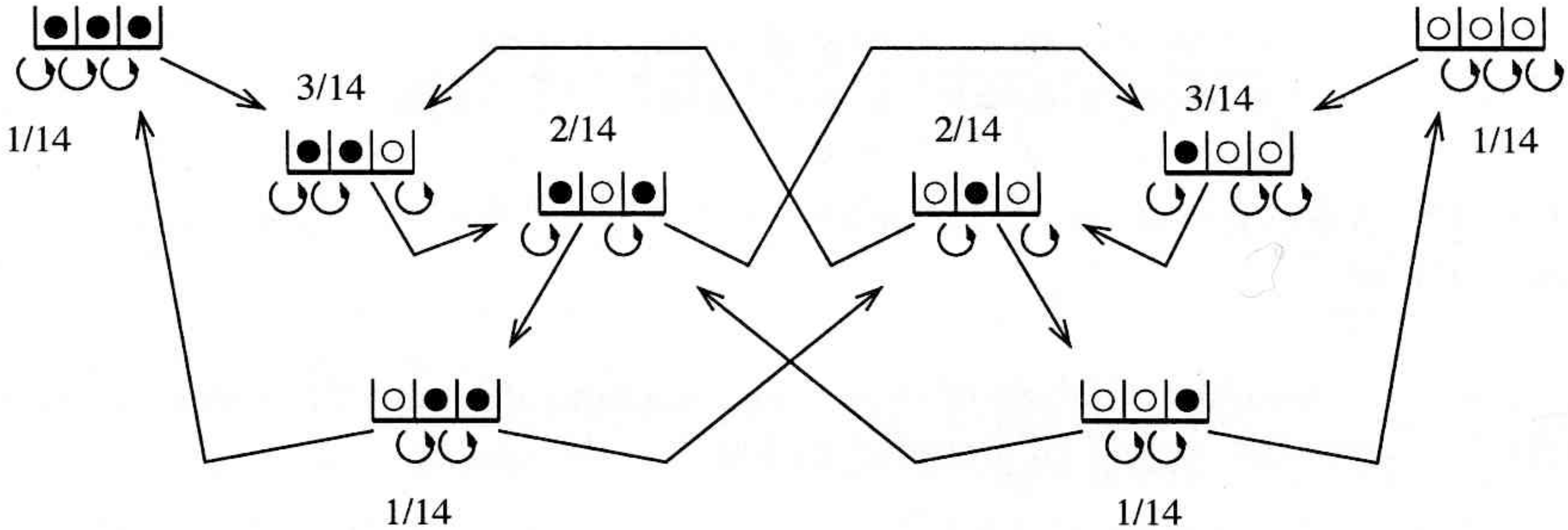
$\pi \cdot M = \pi$

unicity

eigenvector $\qquad M^T \qquad$ eigenvalue 1

time $\rightarrow \infty$

$$p_v = \sum_u p_u \, p_{u,v}$$


3/14
1/14
2/14
2/14
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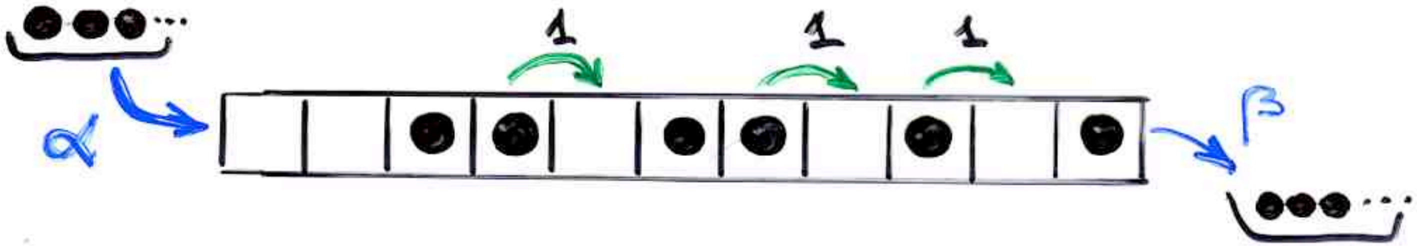
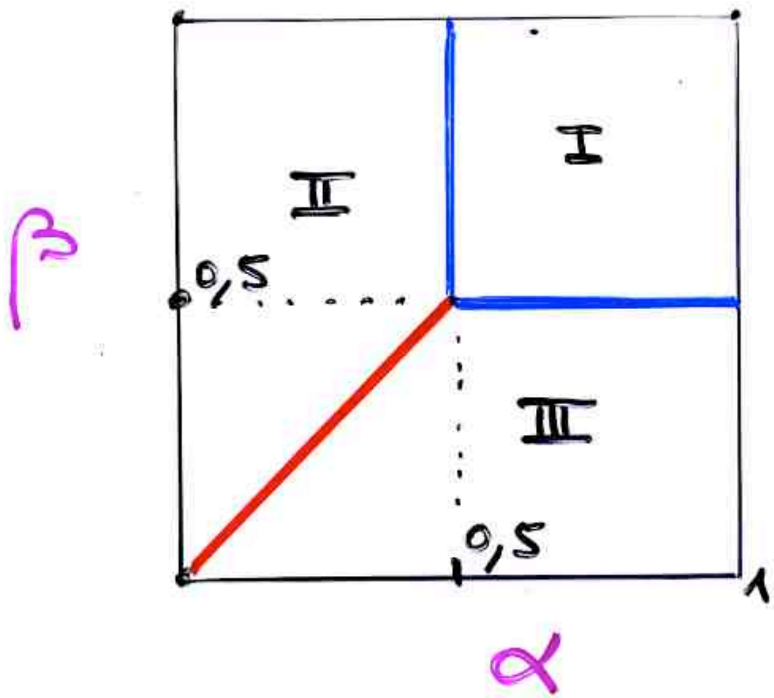
1/14
1/14

TASEP

"totally asymmetric exclusion process"





$n \to \infty$

$\rho = \langle \tau_i \rangle$ = taux moyen d'occupation

i loin des bords

(I) $\rho = 1/2$

(II) $\rho = \alpha$

(III) $\rho = 1-\beta$

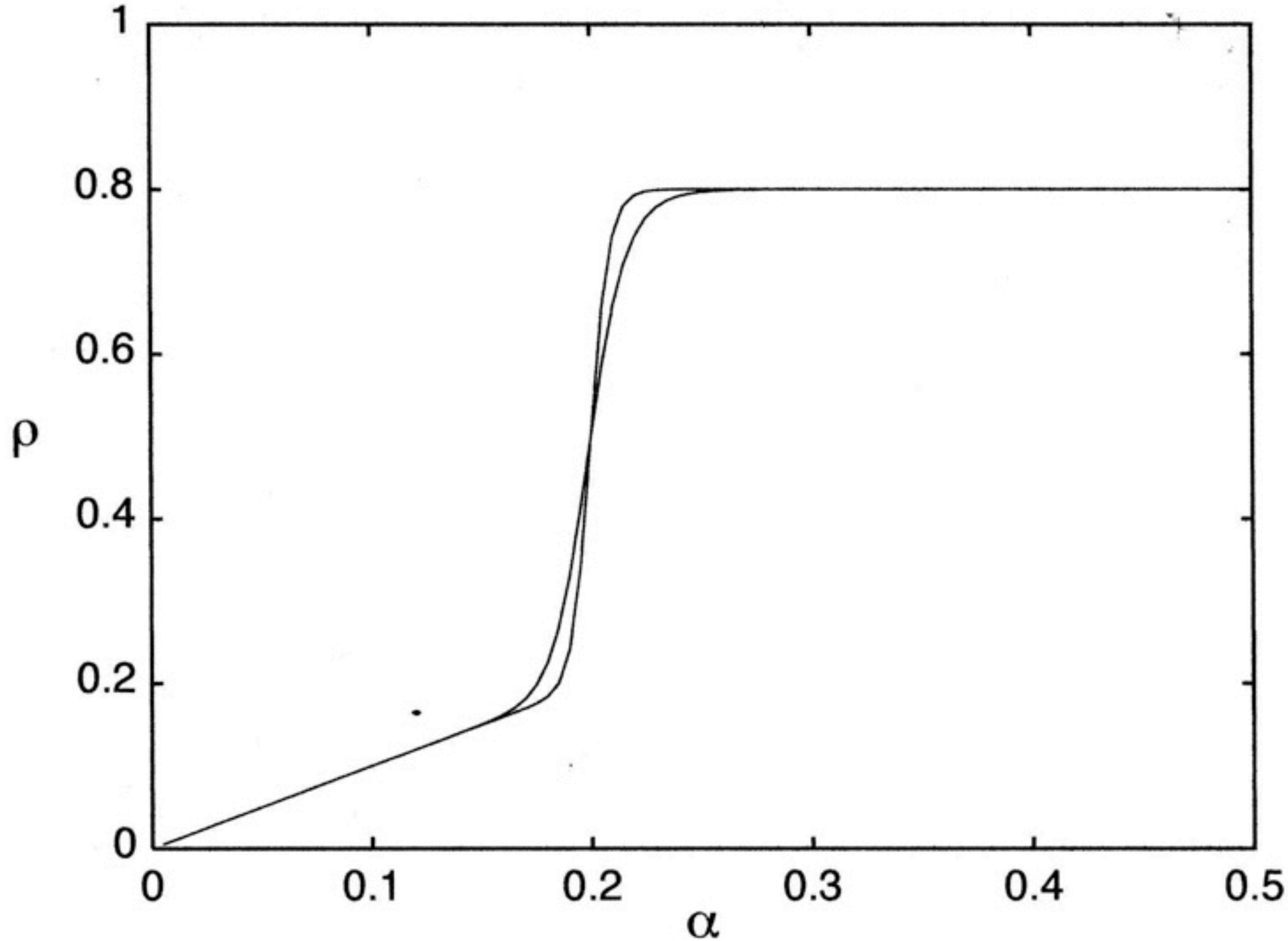
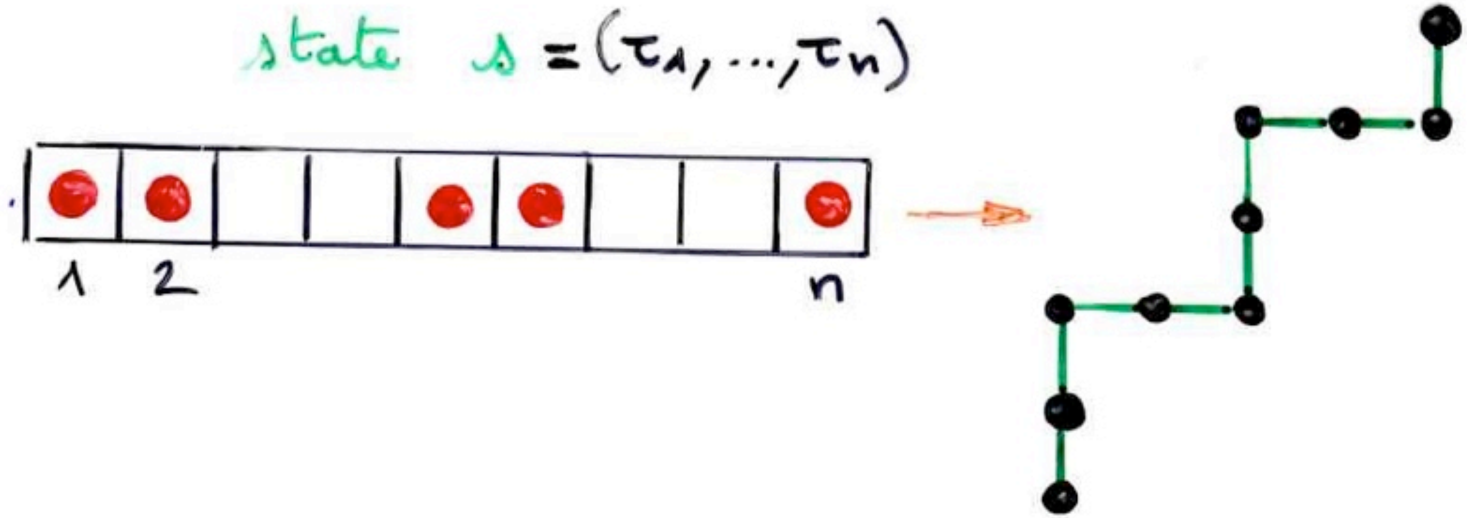


Figure 2: The average occupation $\rho = \langle \tau_{(N+1)/2} \rangle$ of the central site versus α for $N = 61$ and $N = 121$ when $\beta = .2$.

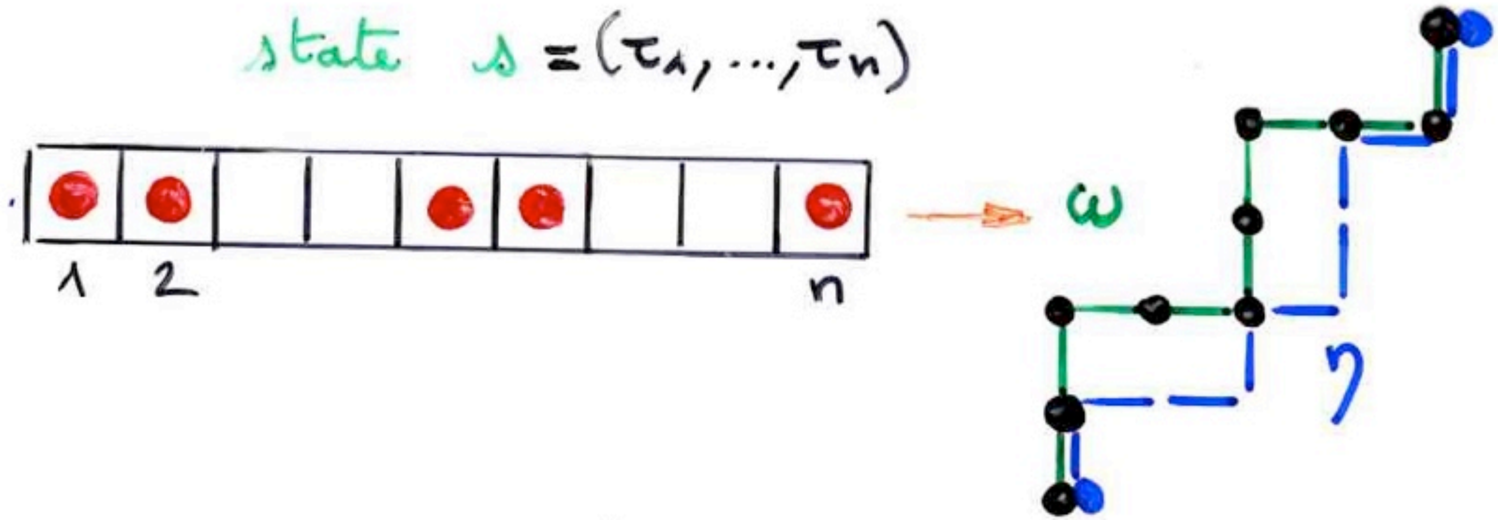
state $s = (\tau_1, \dots, \tau_n)$



$\mathcal{P}_n(s) =$

Shapiro, Zeilberger, 1982

state $s = (\tau_1, \dots, \tau_n)$



$$\mathcal{P}_n(s) = \frac{1}{C_{n+1}} \left(\text{number of paths } \eta \text{ below the path } \omega \text{ associated to } s \right)$$

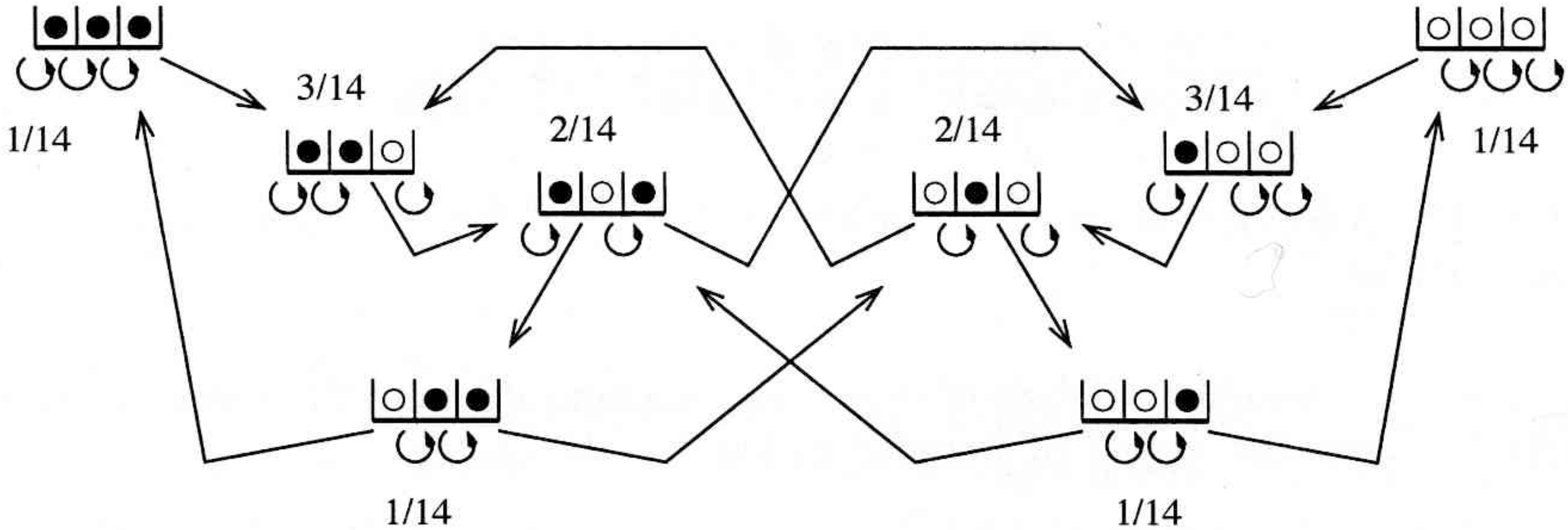
Shapiro, Zeilberger, 1982

Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\; n!}$$

$$n! = 1 \times 2 \times \dots \times n$$



1/14
3/14
2/14
2/14
3/14
1/14
1/14
1/14

Combinatorics of the PASEP

TASEP

Brak, Essam (2003), Duchi, Schaeffer, (2004),
Angel (2005), XGV, (2007)

(P) ASEP

Brak, Corteel, Essam, Parviainen, Rechnitzer (2006)
Corteel, Williams (2006) (2008) (2009) XGV, (2008)
Corteel, Stanton, Stanley, Williams (2010)

Derrida, ...
Mallick, ... Golinelli, Mallick (2006)

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

Askey-Wilson polynomials

The Matrix Ansatz

Derrida, Evans, Hakim, Pasquier 1993

$$\mathbb{P}_n(\tau_1, \dots, \tau_n) = f_n(\tau_1, \dots, \tau_n) / Z_n$$

$$Z_n = \sum_{\tau} f_n(\tau_1, \dots, \tau_n)$$

partition function

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

$$\begin{cases} DE = q\,ED + D + E \\ (\beta D - \delta E)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \gamma D) = \langle W| \end{cases}$$

Then

$f_n(\tau_1, \dots, \tau_n)$

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

TASEP

q=0

$$
\begin{cases}
DE = \square + D + E \\
(\beta D - \square)|V\rangle = |V\rangle \\
\langle W|(\alpha E - \square) = \langle W|
\end{cases}
$$

Then

$$
f_n(\tau_1, \dots, \tau_n) = \langle W| \prod_{i=1}^{n} (\tau_i D + (1-\tau_i) E) |V\rangle
$$

examples:

TASEP

$$\begin{cases} DE = D + E \\ D|V\rangle = \bar{\beta}\,|V\rangle \\ \langle W|E = \bar{\alpha}\,\langle W| \end{cases}$$

examples:

TASEP

$$D = \begin{bmatrix} 0 & \bar\beta & 0 & \cdots \\ & 0 & \bar\beta & \ddots \\ & & 0 & \ddots \\ 0 & & & \ddots \end{bmatrix} \qquad E = \begin{bmatrix} \bar\alpha & 0 & \cdots & & \\ \bar\alpha\beta & 1 & 0 & & \\ \bar\alpha\beta^2 & \beta & 1 & 0 & \\ \bar\alpha\beta^3 & \beta^2 & \beta & 1 & \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}$$

(infinite matrices)

$$\bar\beta = \frac{1}{\beta}, \quad \bar\alpha = \frac{1}{\alpha}$$

$$\langle W| = (1, 0, \ldots, 0, \ldots) \qquad |V\rangle = (1, 1, \ldots, 1, \ldots)^T$$

$$D = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ & 0 & 1 & \ddots \\ & & 0 & \ddots \\ 0 & & & \ddots \end{bmatrix} \qquad E = \begin{bmatrix} \bar\beta & & & \\ \bar\beta & 1 & & \\ \vdots & \vdots & \ddots & \\ \bar\beta & 1 & \cdots & 1 \end{bmatrix}$$

(infinite matrices)

$$\bar\beta = \frac{1}{\beta}, \quad \bar\alpha = \frac{1}{\alpha}$$

$$\langle W| = (1, 0, \ldots) \qquad |V\rangle = (1, \bar\alpha, \bar\alpha^2, \ldots)^T$$

examples:

TASEP

$$D = \begin{bmatrix} \bar\beta & \kappa & & & 0 \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & 0 & & 1 & 1 \\ 0 & & & & 1 \end{bmatrix} \qquad E = \begin{bmatrix} \bar\alpha & & & & 0 \\ \kappa & 1 & & & \\ & 1 & \ddots & & \\ & & \ddots & 1 & \\ 0 & & & 1 & 1 \end{bmatrix}$$

(infinite matrices)

$$\langle W| = (1, 0, \cdots, 0) \qquad |V\rangle = (1, 0, \cdots, 0)$$

$$\bar\alpha = \frac{1}{\alpha} \qquad \bar\beta = \frac{1}{\beta} \qquad \kappa^2 = \bar\alpha + \bar\beta - \bar\alpha\bar\beta$$

The PASEP algebra

$$DE = qED + E + D$$

$$w = \sum_{i,j \geq 0} c_{i,j} E^i D^j$$

D D E D E E D E

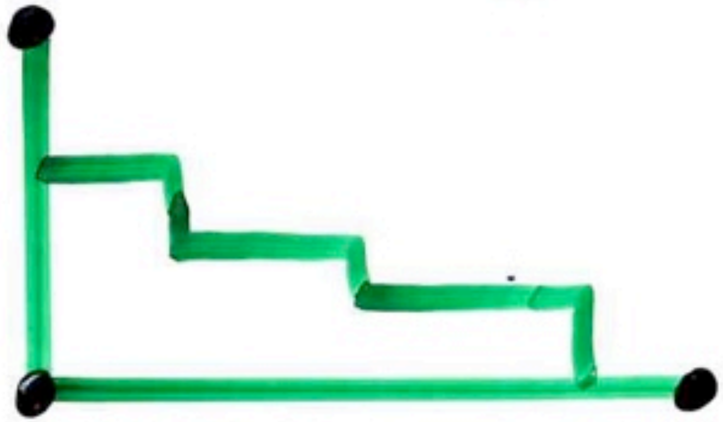
D DE (D E) E D E

DDE(E)EDE $+_q$ DDE(ED)EDE + DDE(D)EDE

alternative tableaux

alternative tableau

- Ferrers diagram F

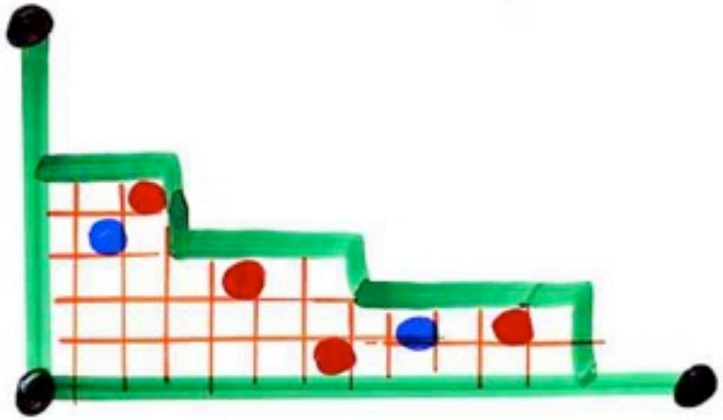


(possibly empty rows or column)

(nb of rows) + (nb of columns) $= n$

alternative tableau

- Ferrers diagram **F** (possibly empty rows or column)

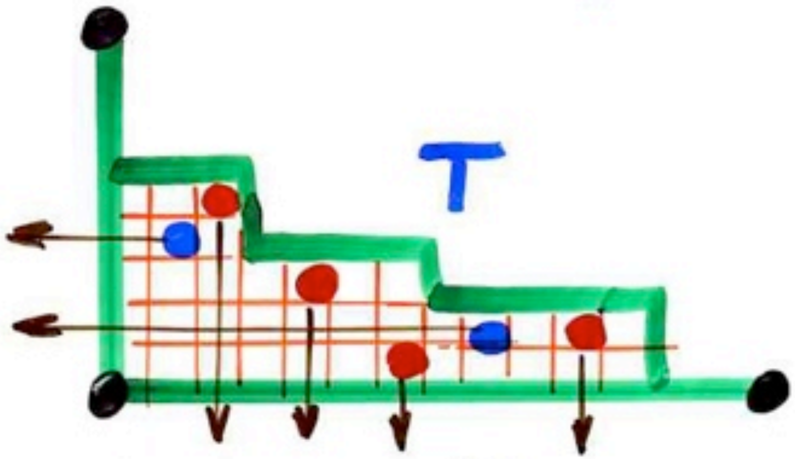


(nb of rows) + (nb of columns) $= n$

- some cells are coloured **red** or **blue**

alternative tableau T

- Ferrers diagram F (possibly empty rows or column)



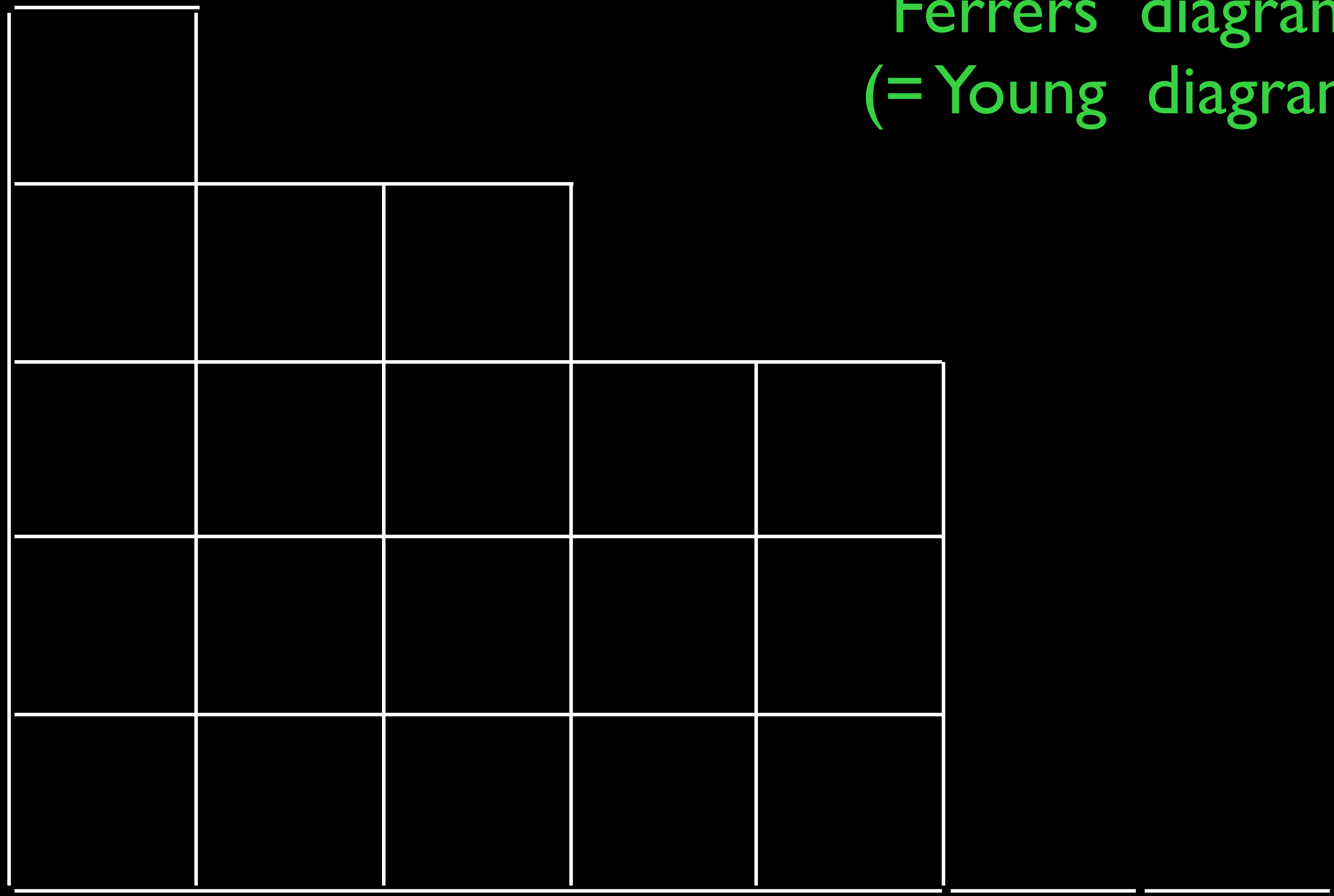
(nb of rows) + (nb of columns) = n

- some cells are coloured **red** or **blue**
- { no coloured cell at the left of 
  { no coloured cell below 

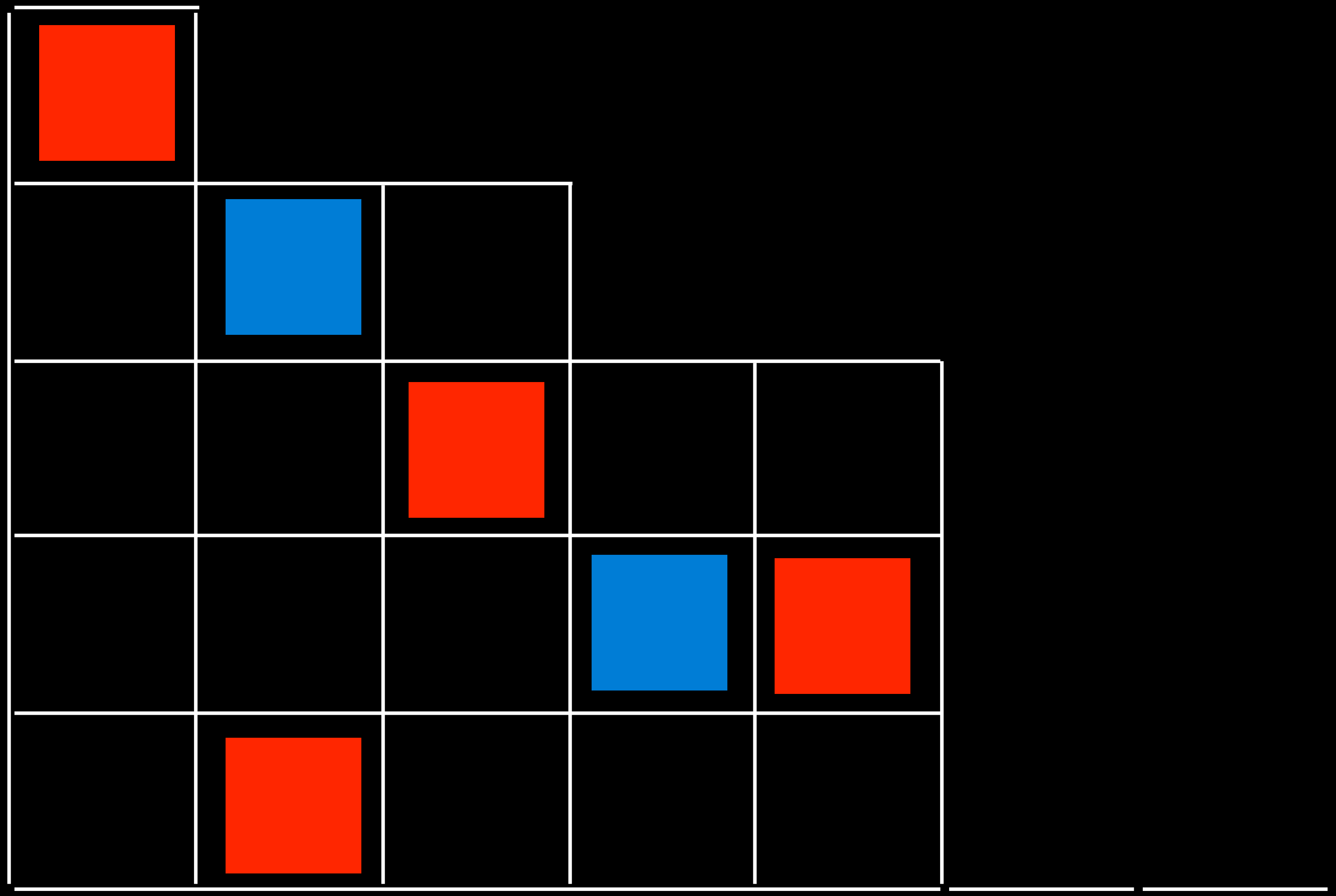
n size of T

alternative tableau

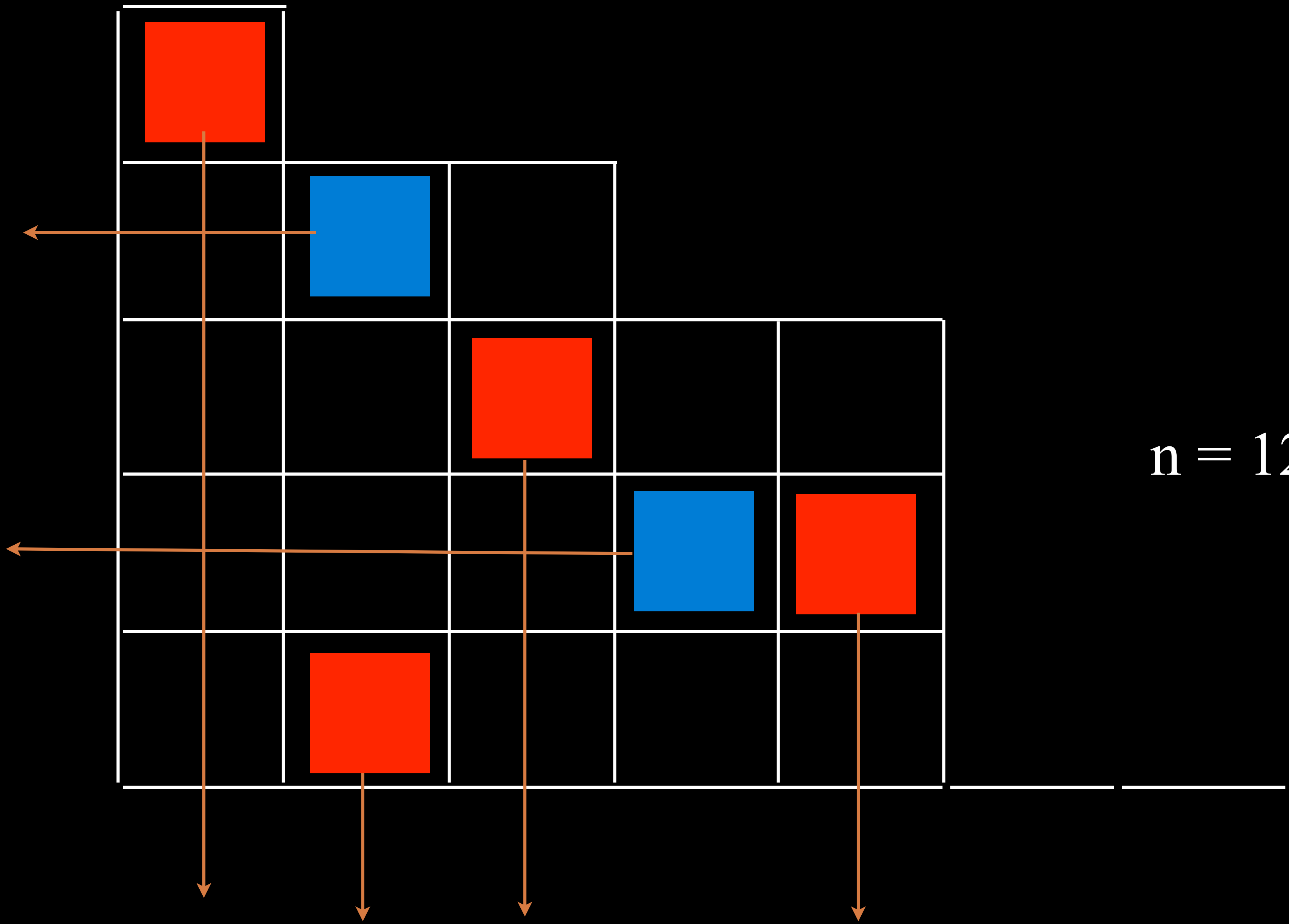
Ferrers diagram
(= Young diagram)



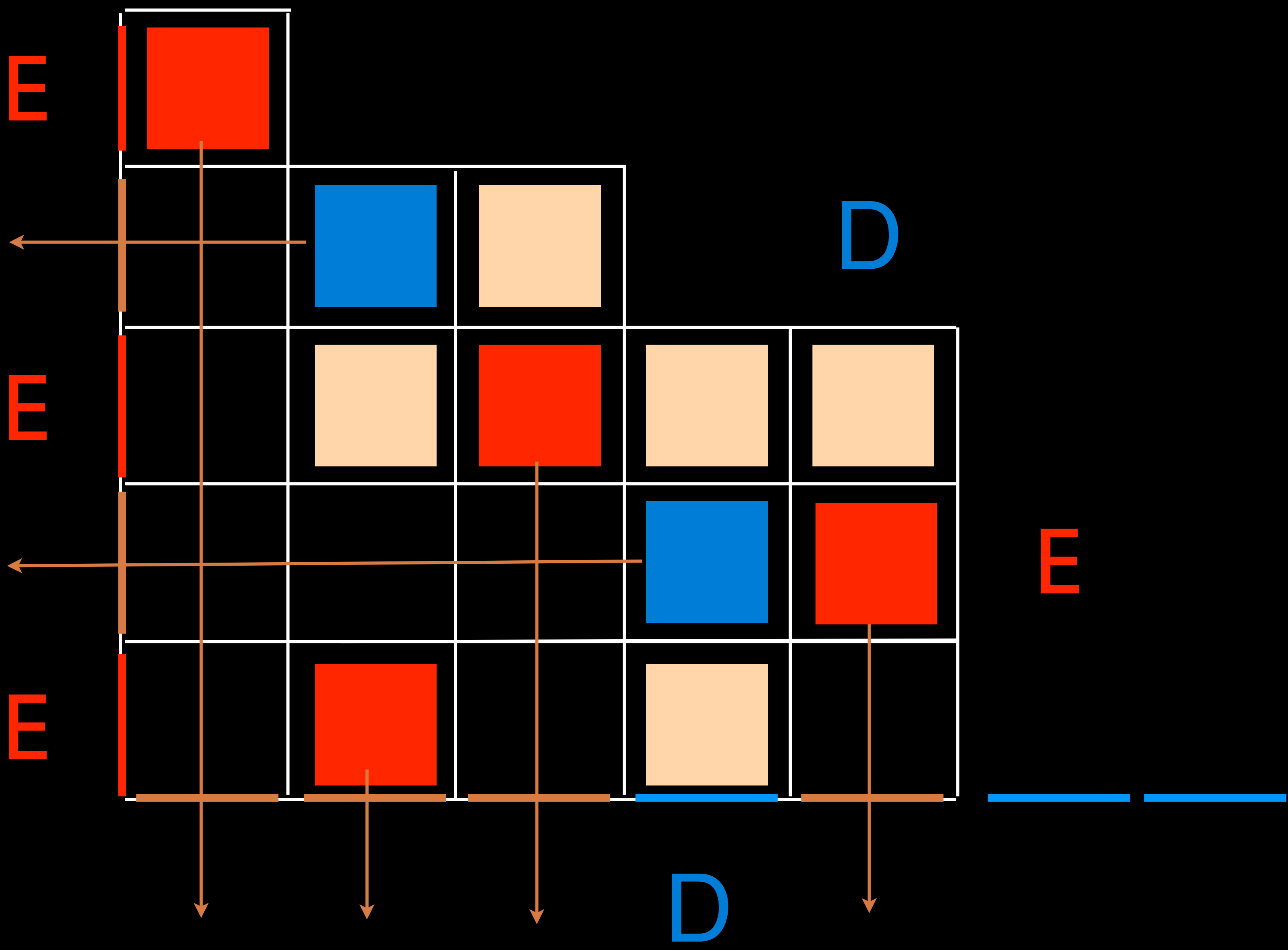
alternative tableau



alternative tableau



n = 12



E
E
E
D
E
D

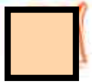
Def - profile of an alternative tableau

word $w \in \{E, D\}^*$

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

T alternative tableau with profile w

$k(T)$ = nb of 

$i(T)$ = nb of rows without blue cell

$j(T)$ = nb of columns without red cell

stationary probabilities for the PASEP

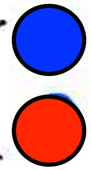
$$\begin{cases} DE = qED + D + E \\ DV = \bar{\beta} V & \bar{\beta} = 1/\beta \\ WE = \bar{\alpha} W & \bar{\alpha} = 1/\alpha \end{cases}$$

$$W E^{i} D^{j} V = \bar{\alpha}^{i} \bar{\beta}^{j} \underbrace{WV}_{1}$$

Cor. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ (PASEP) is

$$\mathrm{proba}_{\tau}(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{T} q^{L(T)} \alpha^{-f(T)} \beta^{-u(T)}$$

T alternative tableaux, profile τ

$f(T)$ / $u(T)$: nb of rows / columns without  cell

$L(T)$: nb of cells 

permutation tableau

S. Corteel, L. Williams
(2007) (2008) (2009)

permutation tableau

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)

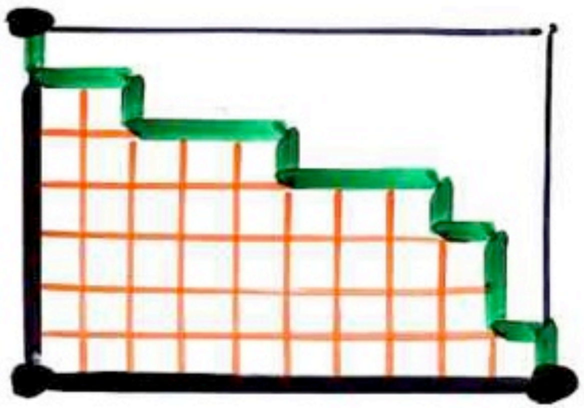
Corteel, Williams (2006) PASEP

Partially Asymmetric Exclusion Process

M. Josuat-Vergès (2007)

Permutation Tableau

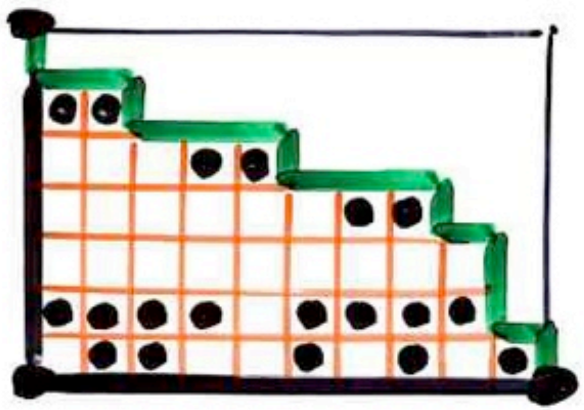
Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$

rectangle



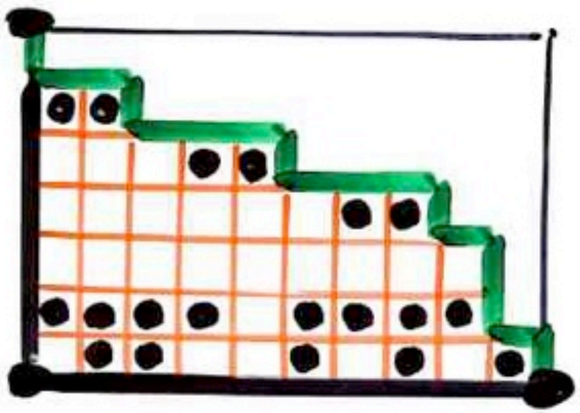
filling of the cells with 0 and 1

(i)

(ii)

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

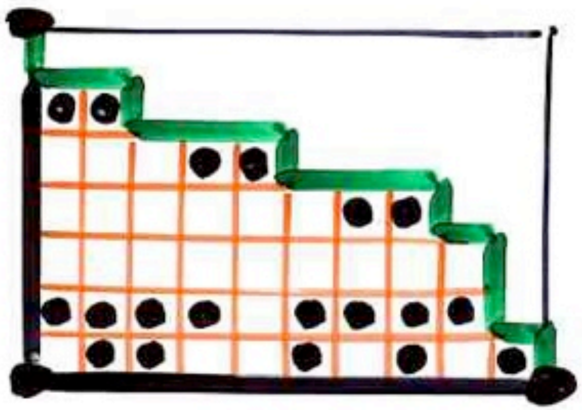
(i) in each column : at least one 1

(ii)

□ = 0 ⊡ = 1

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



□ = 0 ▣ = 1

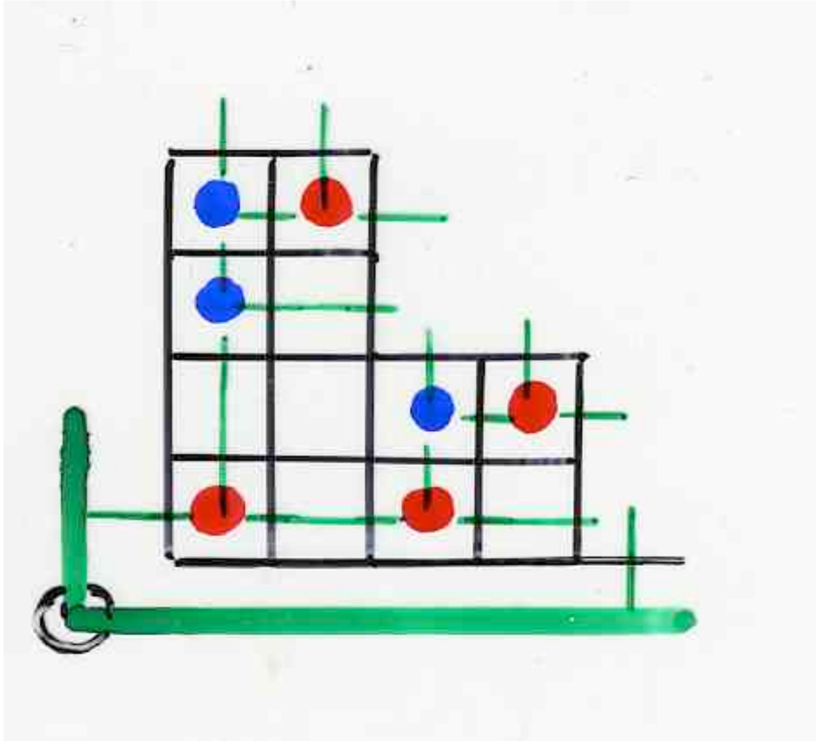
filling of the cells with 0 and 1

(i) in each column: at least one 1

(ii) 

forbidden

tree-like tableaux



J.-C. Aval, A. Boussicault,
P. Nadeau, ... (2011)

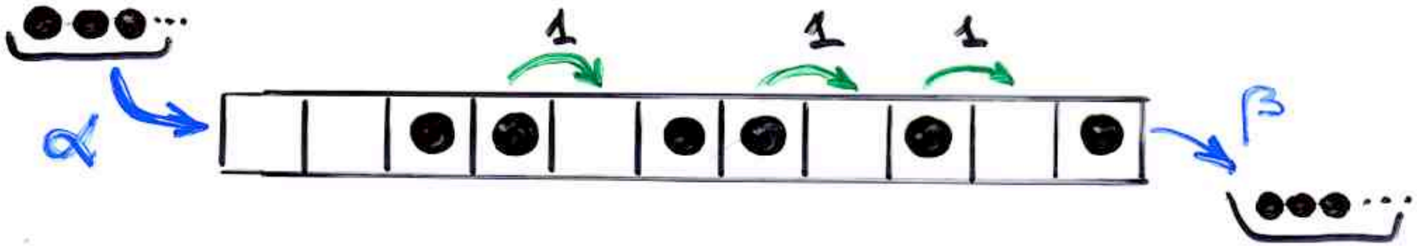
J.-C. Aval, A. Boussicault,
M. Bouvel, M. Silimbani (2013)

TASEP

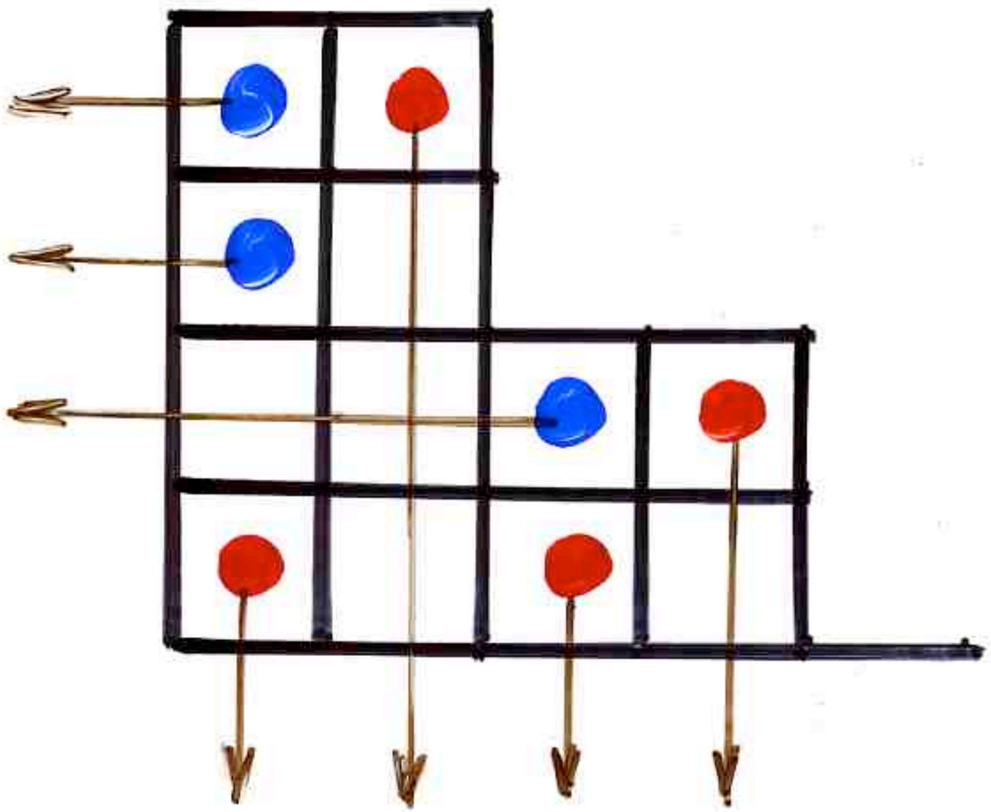
Totally asymmetric exclusion process

TASEP

"totally asymmetric exclusion process"



Def Catalan alternative tableau T
alt. tab. without cells ☒
i.e: every empty cell is below a red cell or
on the left of a blue cell

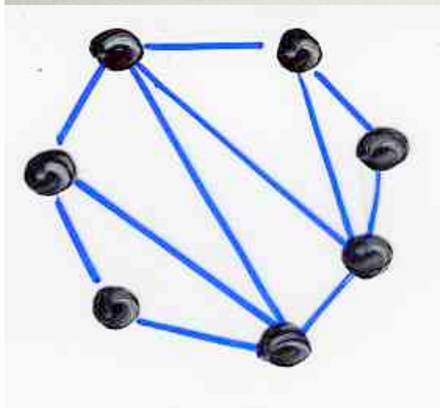


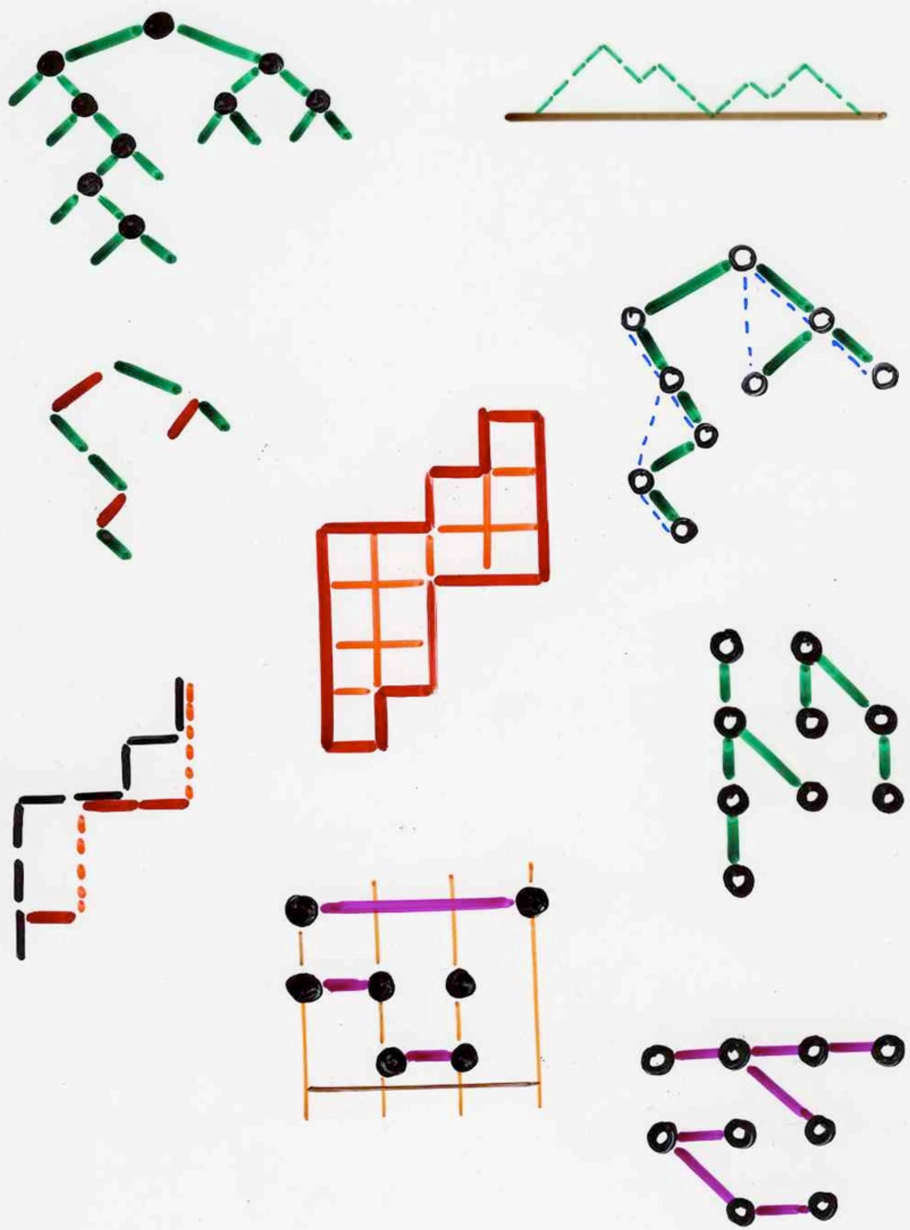
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\ n!}$$

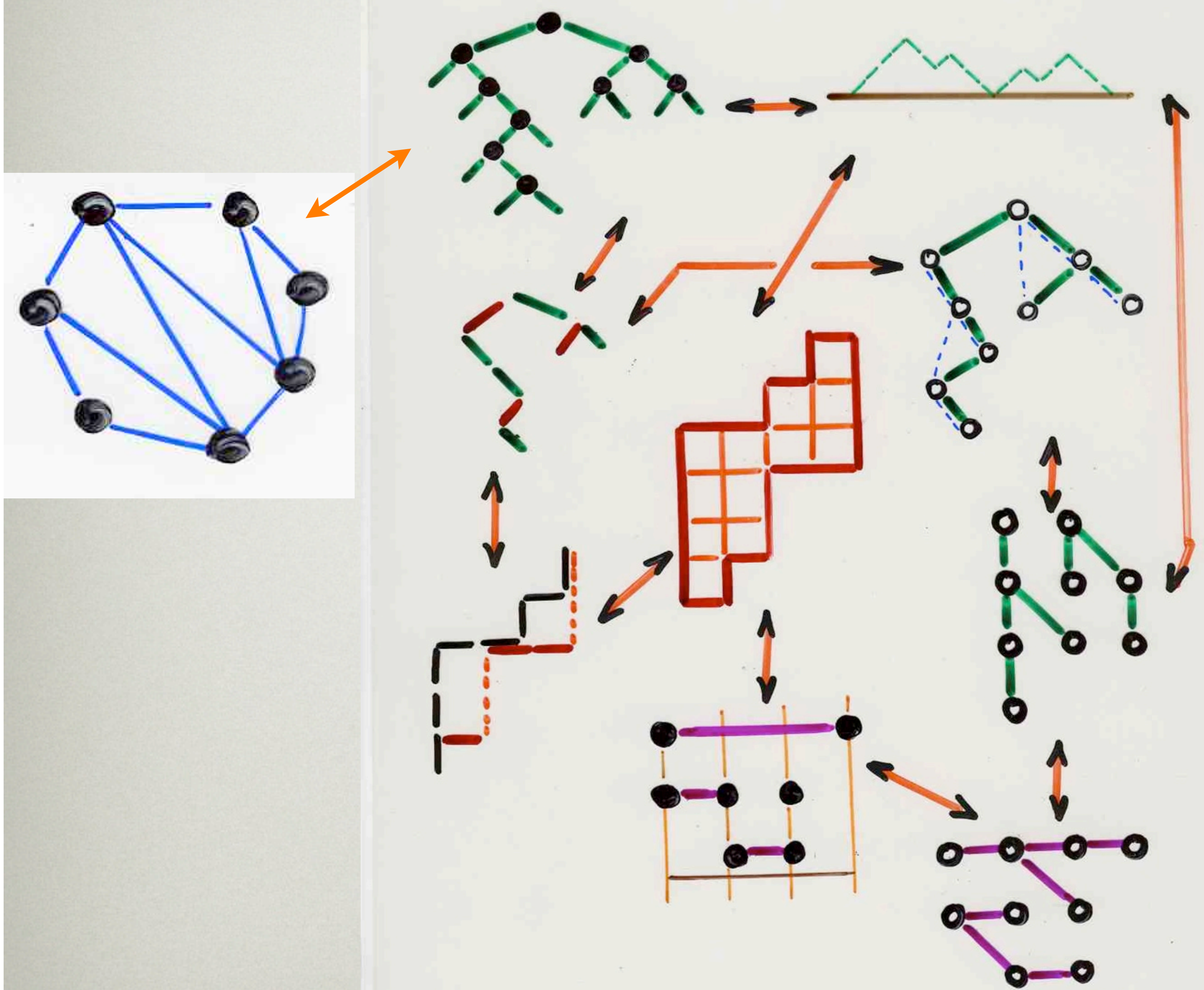
$$n! = 1 \times 2 \times \dots \times n$$

the Catalan garden



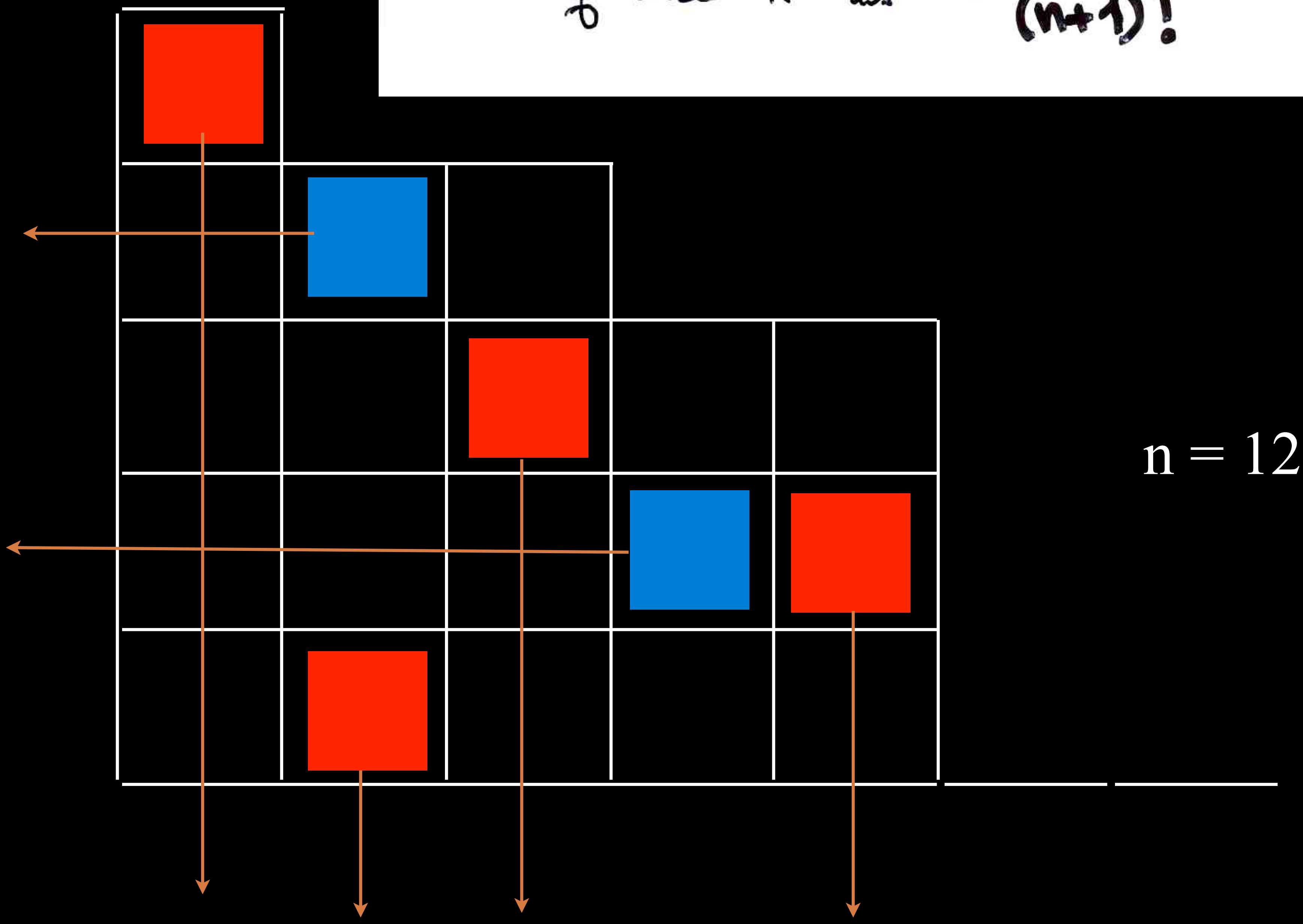


the Catalan garden



number of alternative tableaux

Prop. The number of alternative tableaux of size n is (n+1)!



ex: n=2

number of
alternative tableaux

with alternating shape

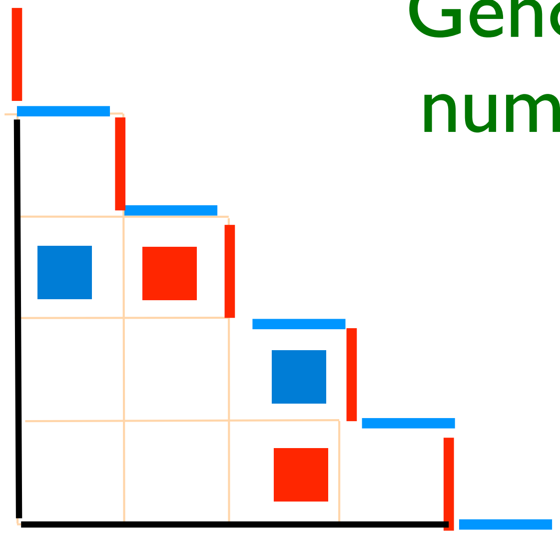
nombres de Genocchi

$$G_{2n} = 2(2^{2n} - 1) B_{2n}$$

Bernoulli

$$2^{2n} G_{2n+2} = (n+1) T_{2n+1}$$

Genocchi numbers

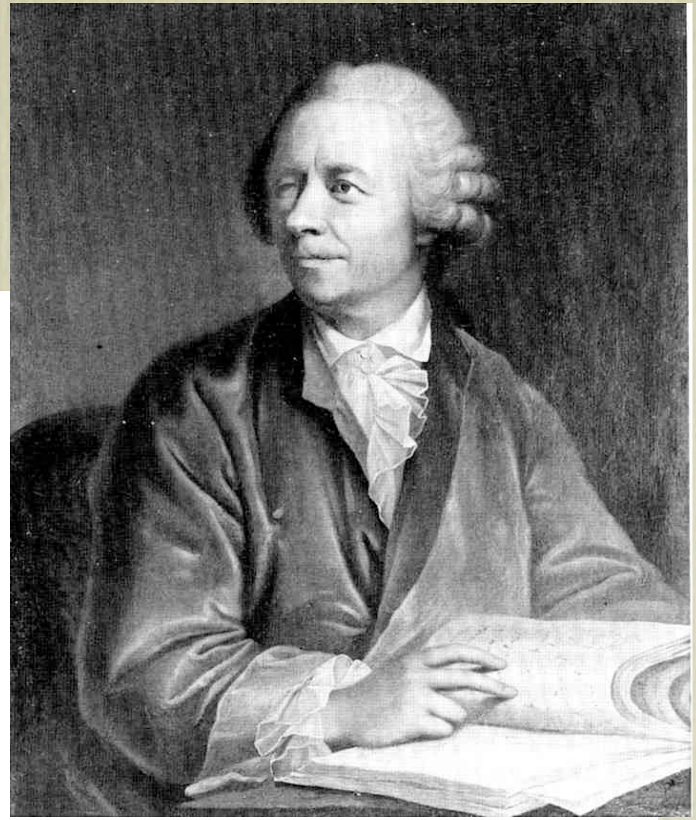


alternating shape

Angelo Genocchi
1817 - 1889

Hinc igitur calculo inſtituto reperietur:

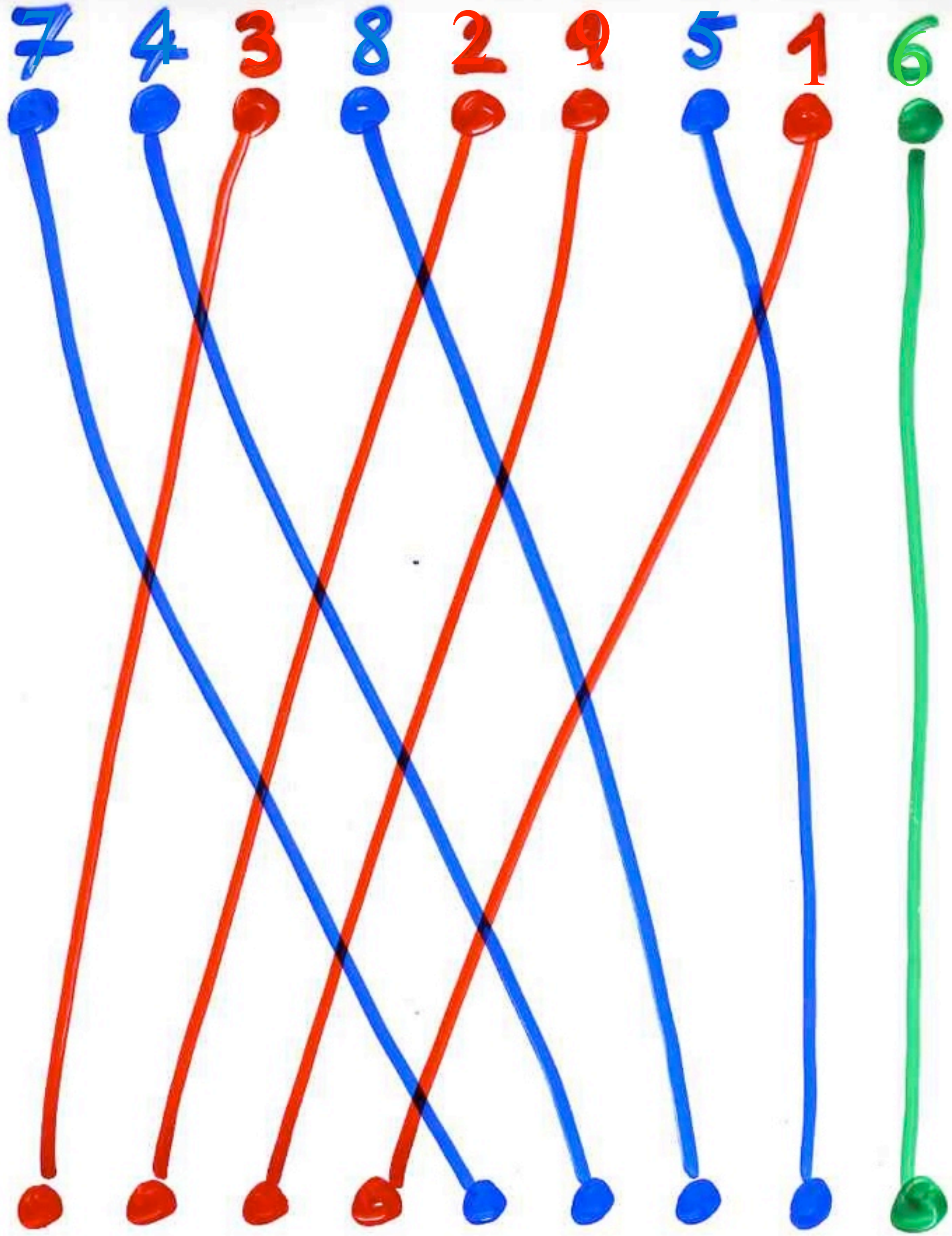
$$A = 1$$
$$B = 1$$
$$C = 3$$
$$D = 17$$
$$E = 155 = 5.31$$
$$F = 2073 = 691.3$$
$$G = 38227 = 7.5461 = 7.\frac{127.129}{3}.$$
$$H = 929569 = 3617.257$$
$$I = 28820619 = 43867.9.73 \quad \&c.$$

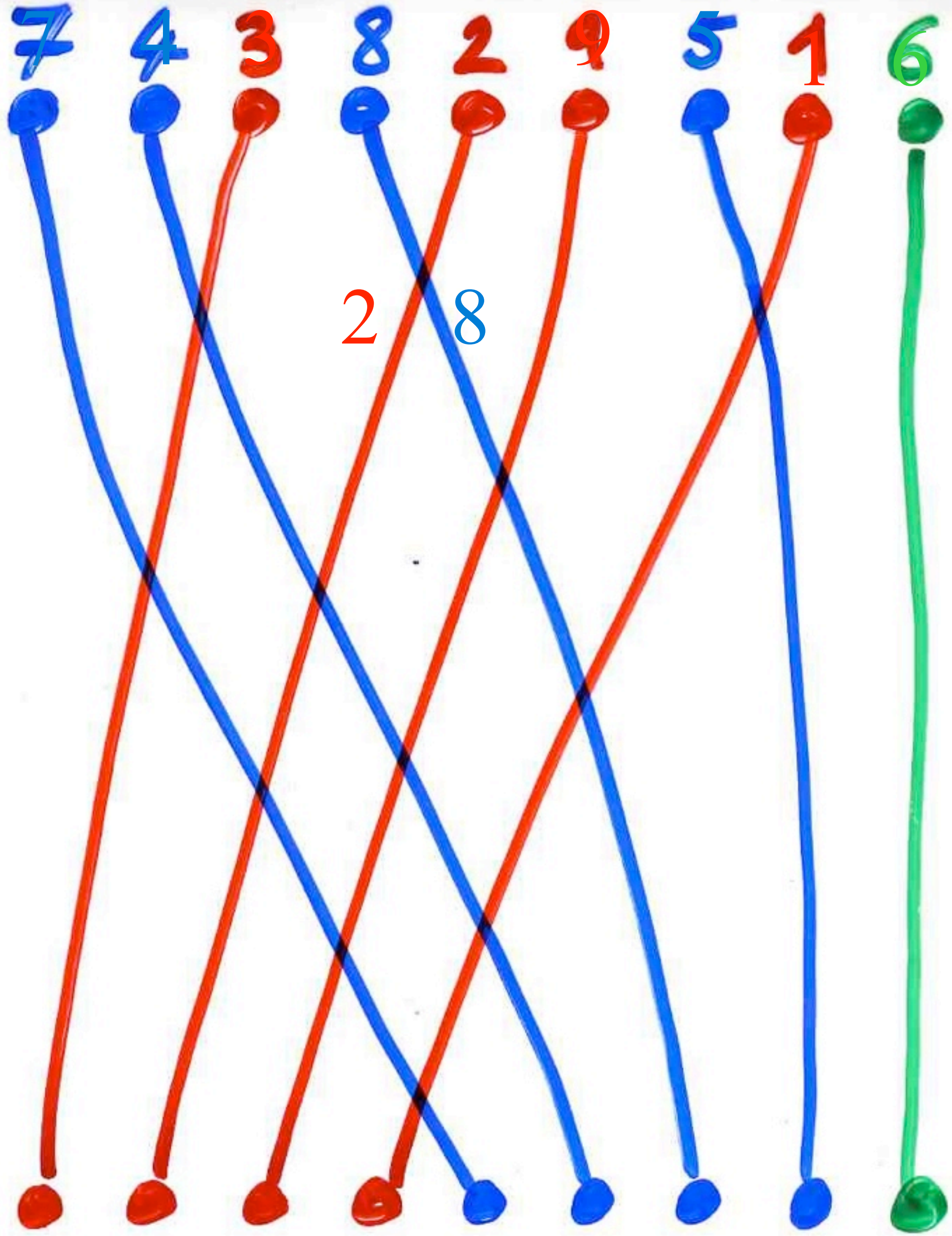


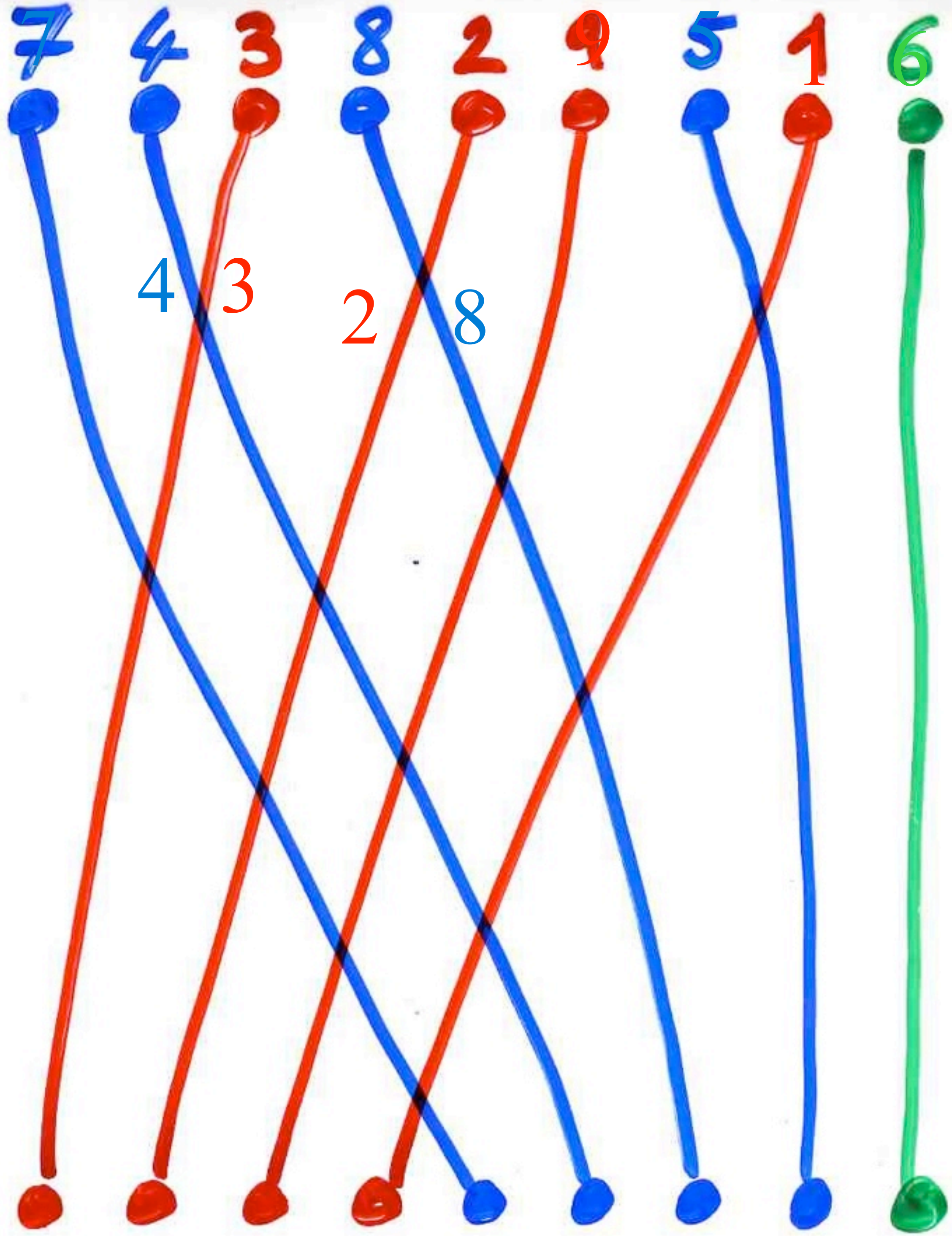
bijection

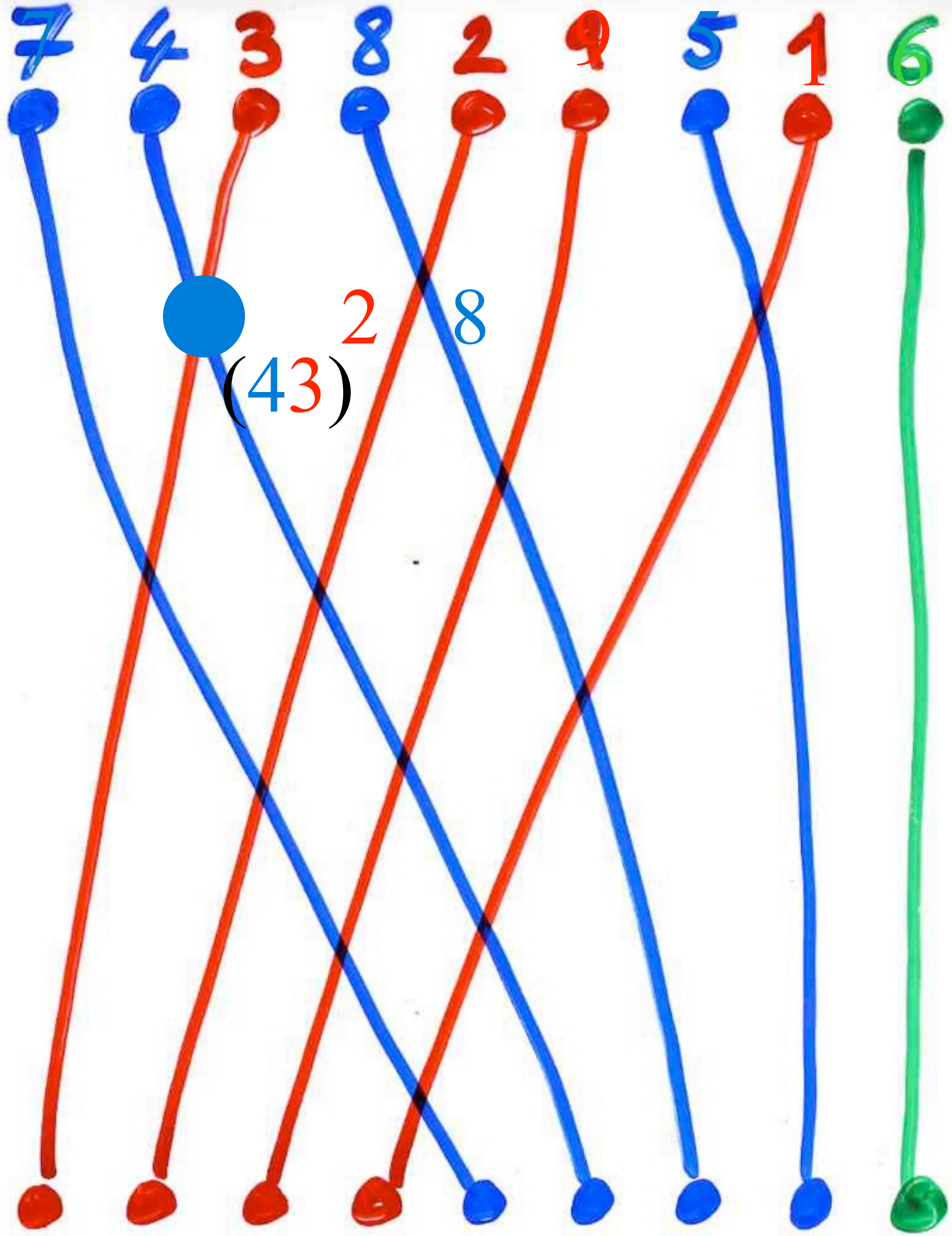
permutations --- alternative tableaux

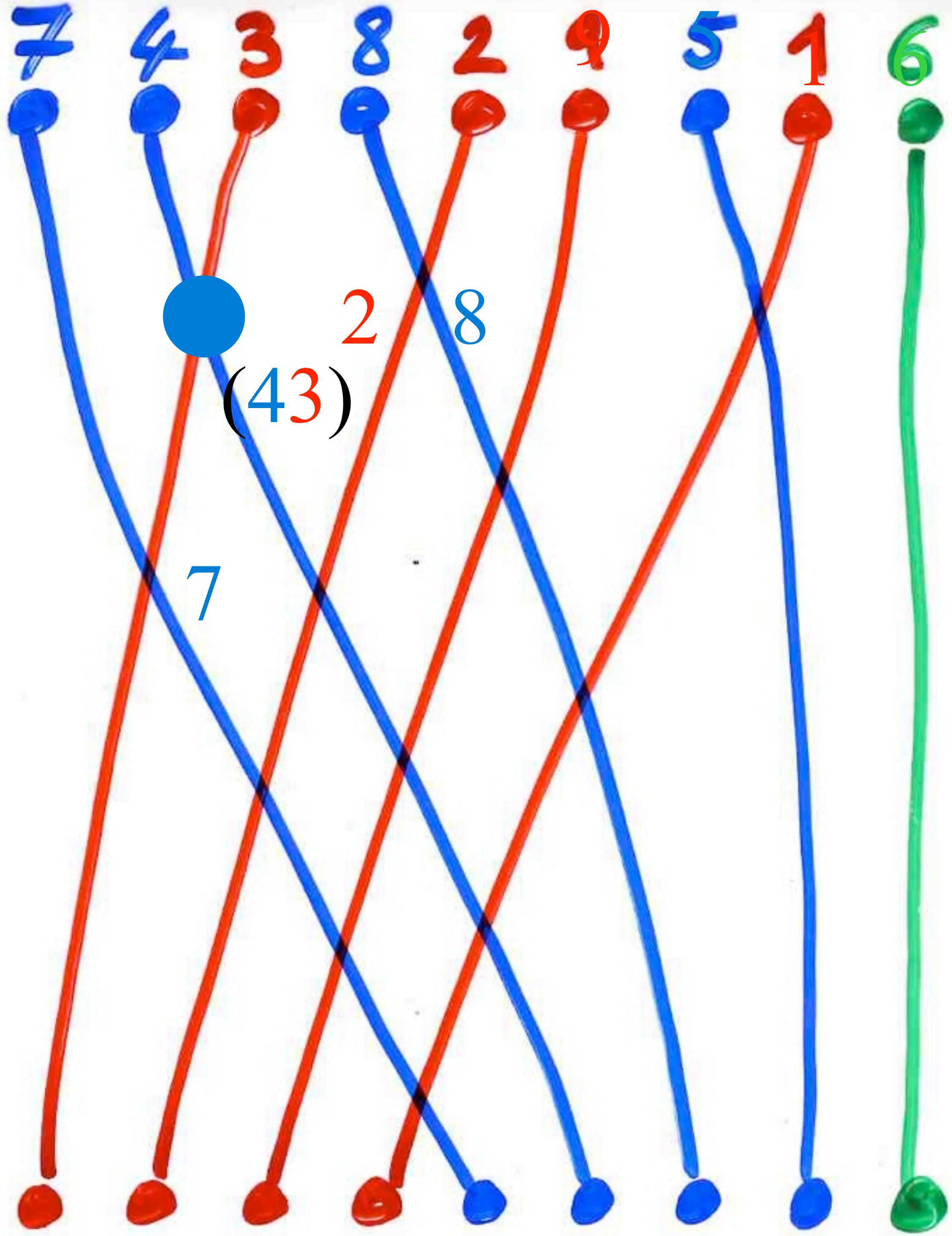
The "exchange-fusion" algorithm


7 4 3 8 2 9 5 1 6

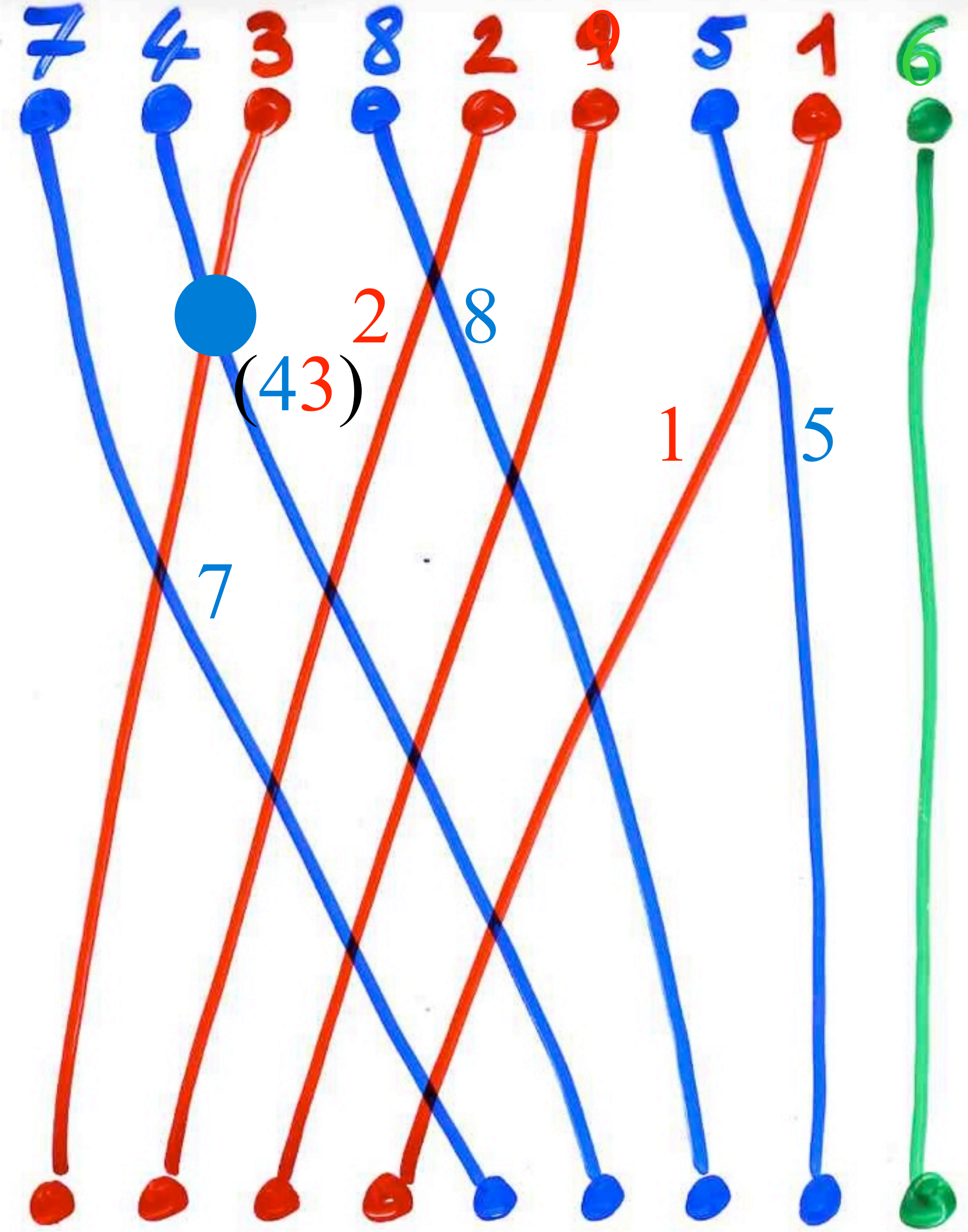

7 4 3 8 2 9 5 1 6
2
8


7 4 3 8 2 9 5 1 6
4 3 2 8


7 4 3 8 2 9 5 1 6
2 8
(43)



7 4 3 8 2 9 5 1 6
2
8
(43)
7



7 4 3 8 2 9 5 1 6
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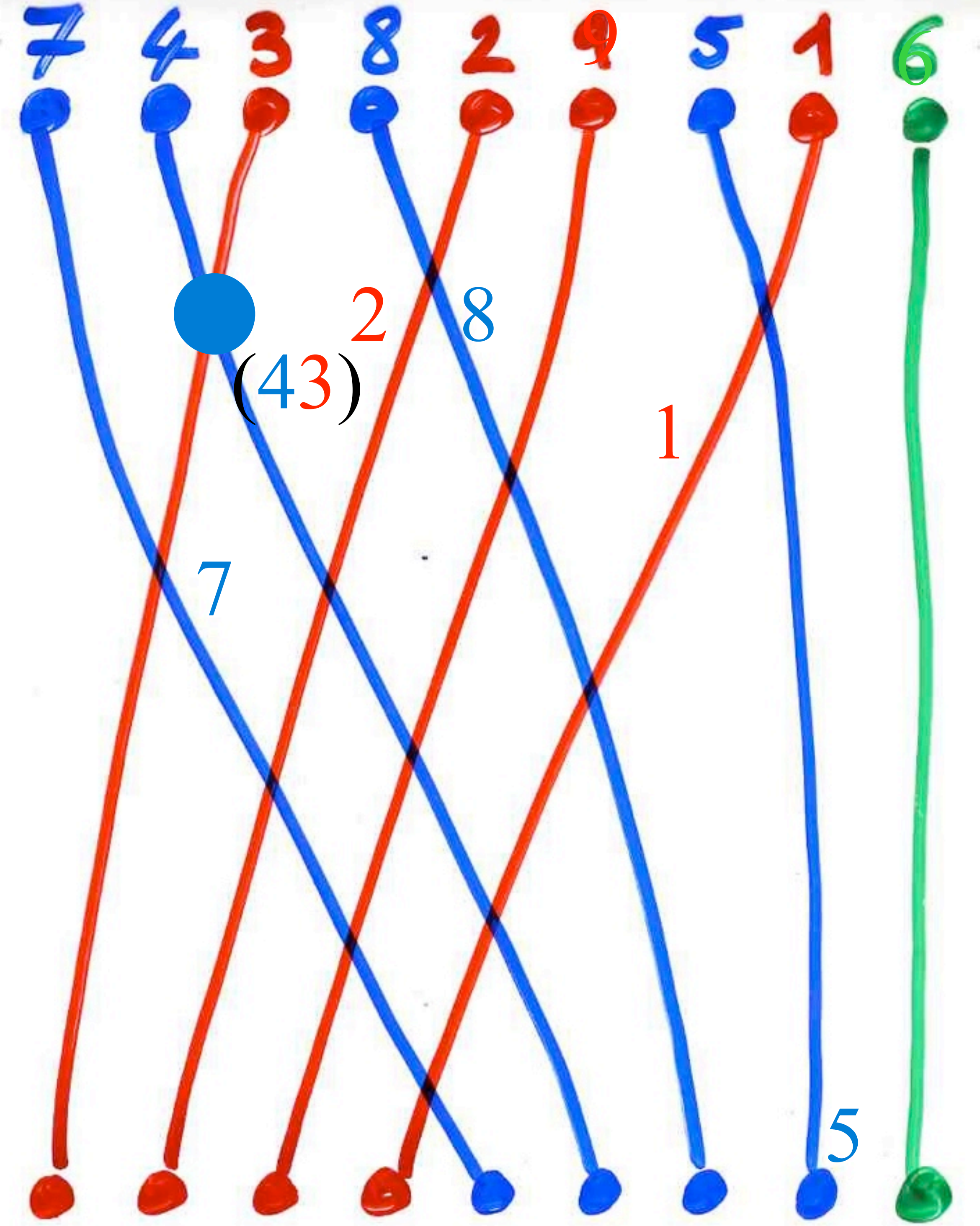

7 4 3 8 2 9 5 1 6
(43)
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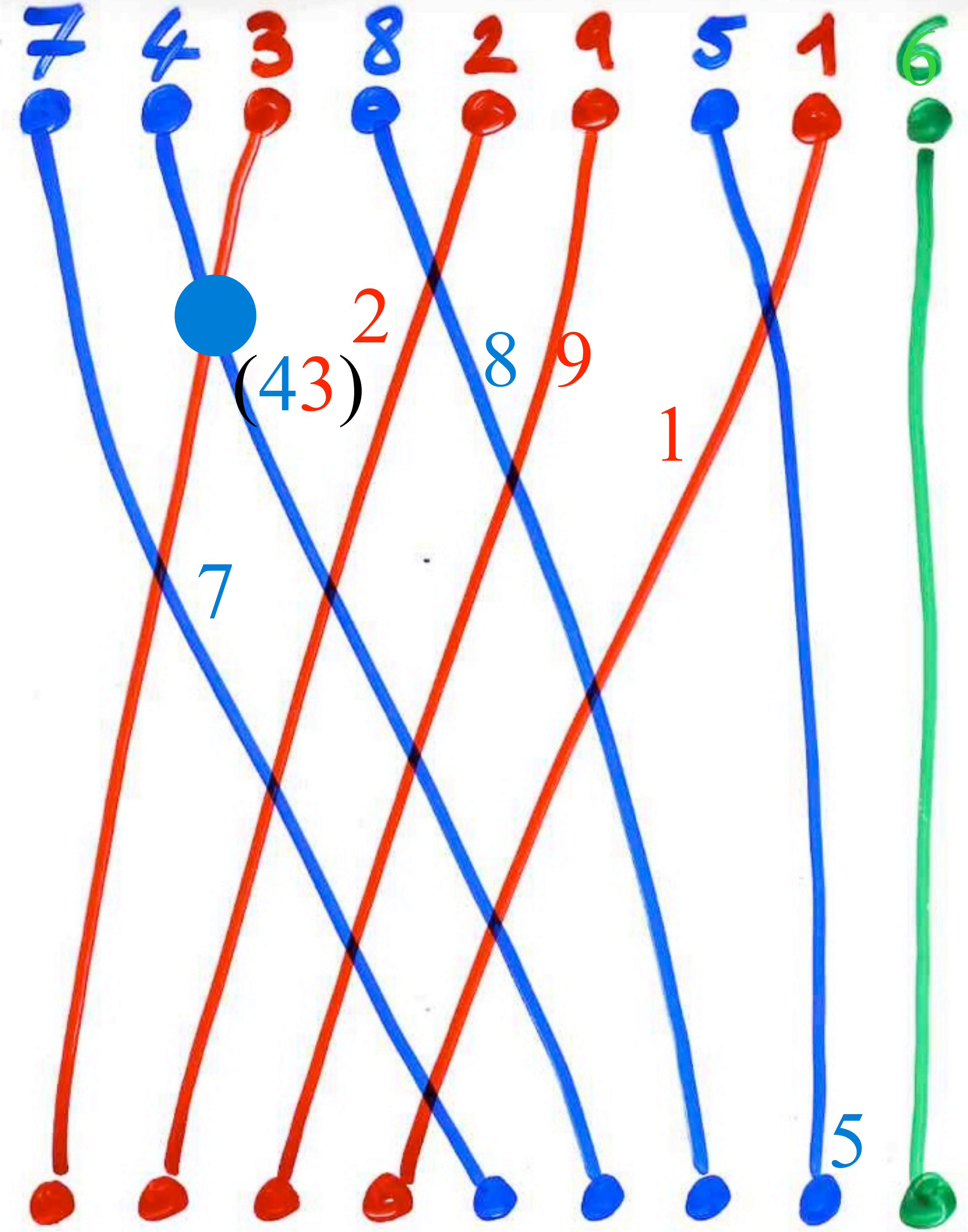

7 4 3 8 2 9 5 1 6
(43)
2
8
9
1
7
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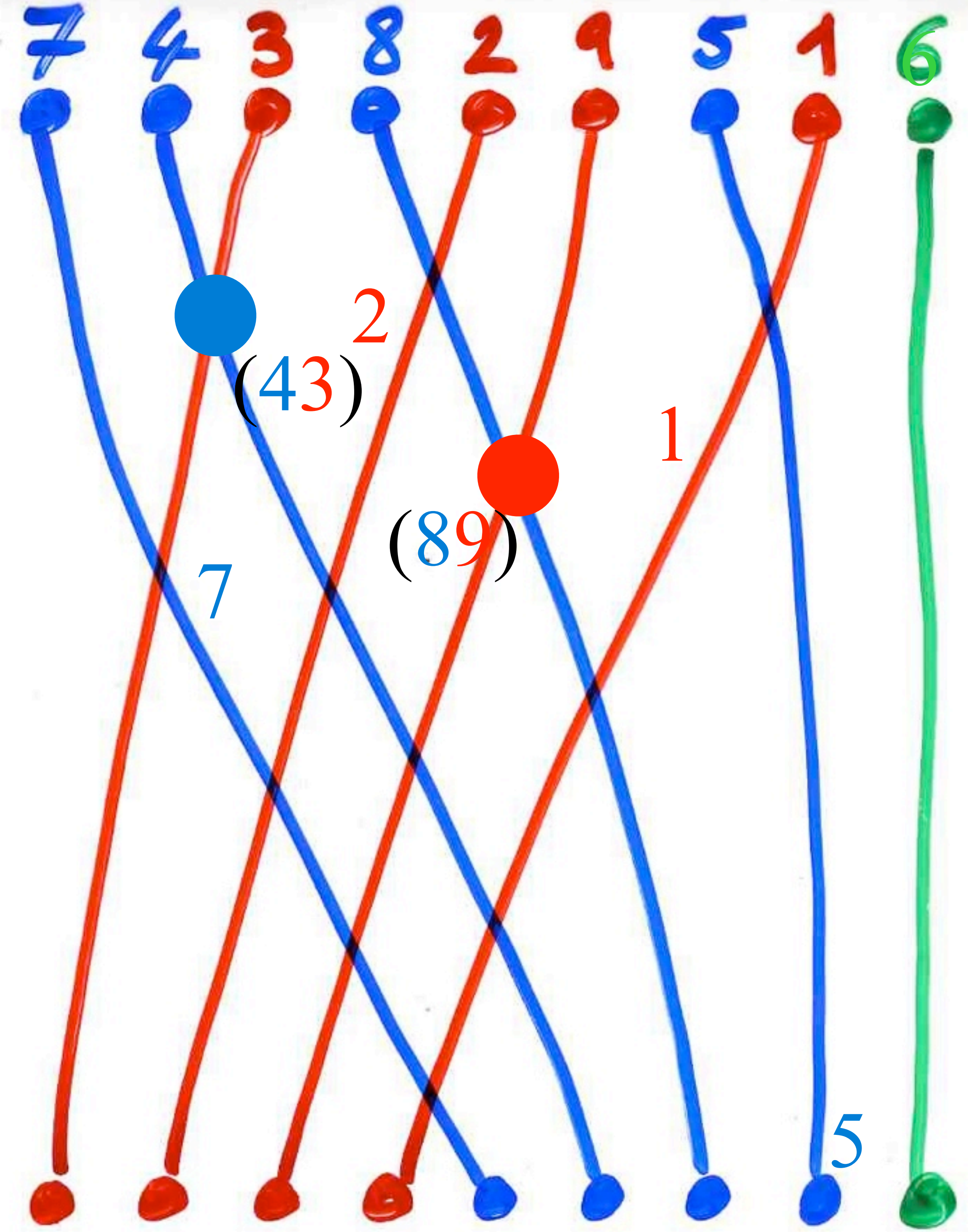
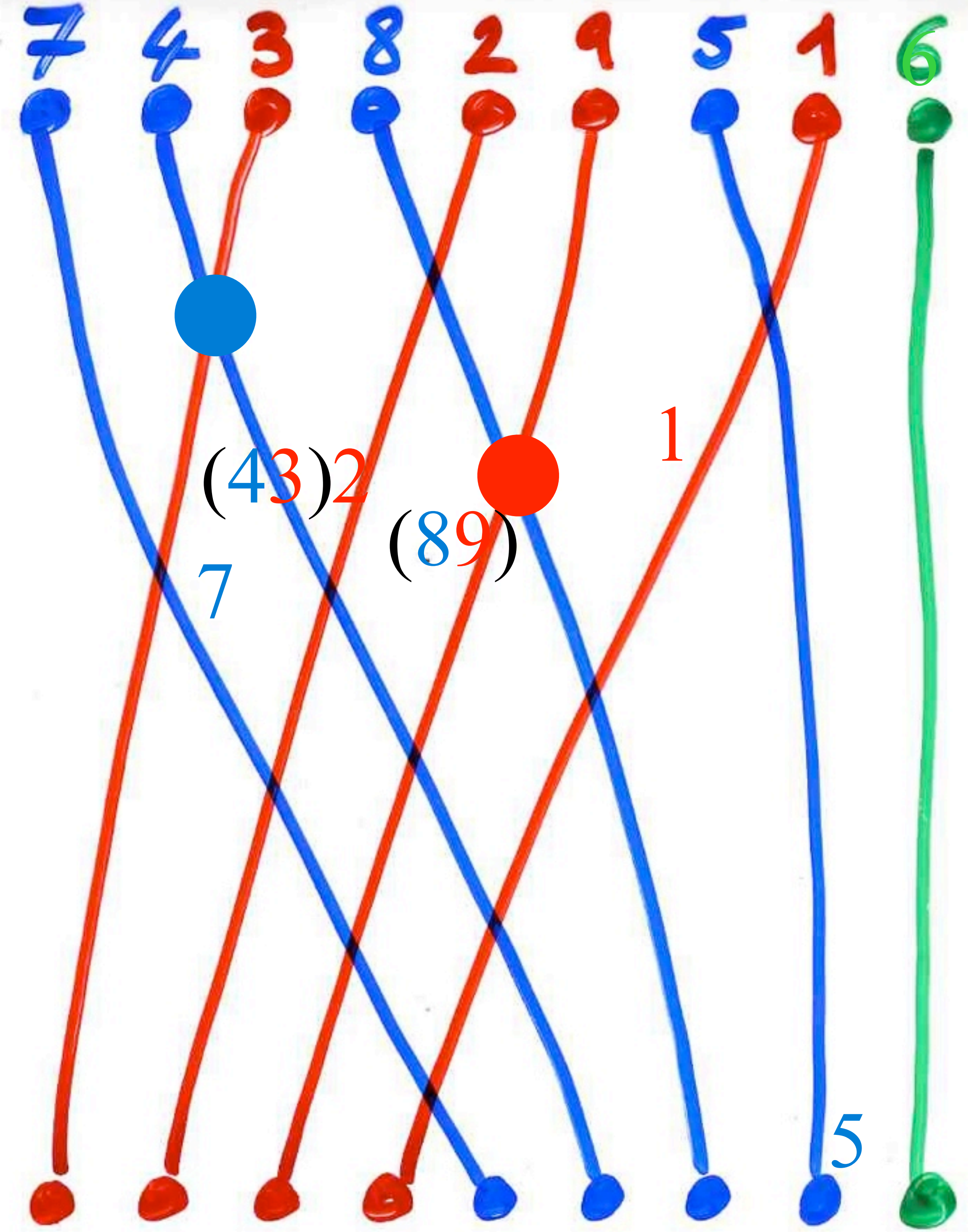
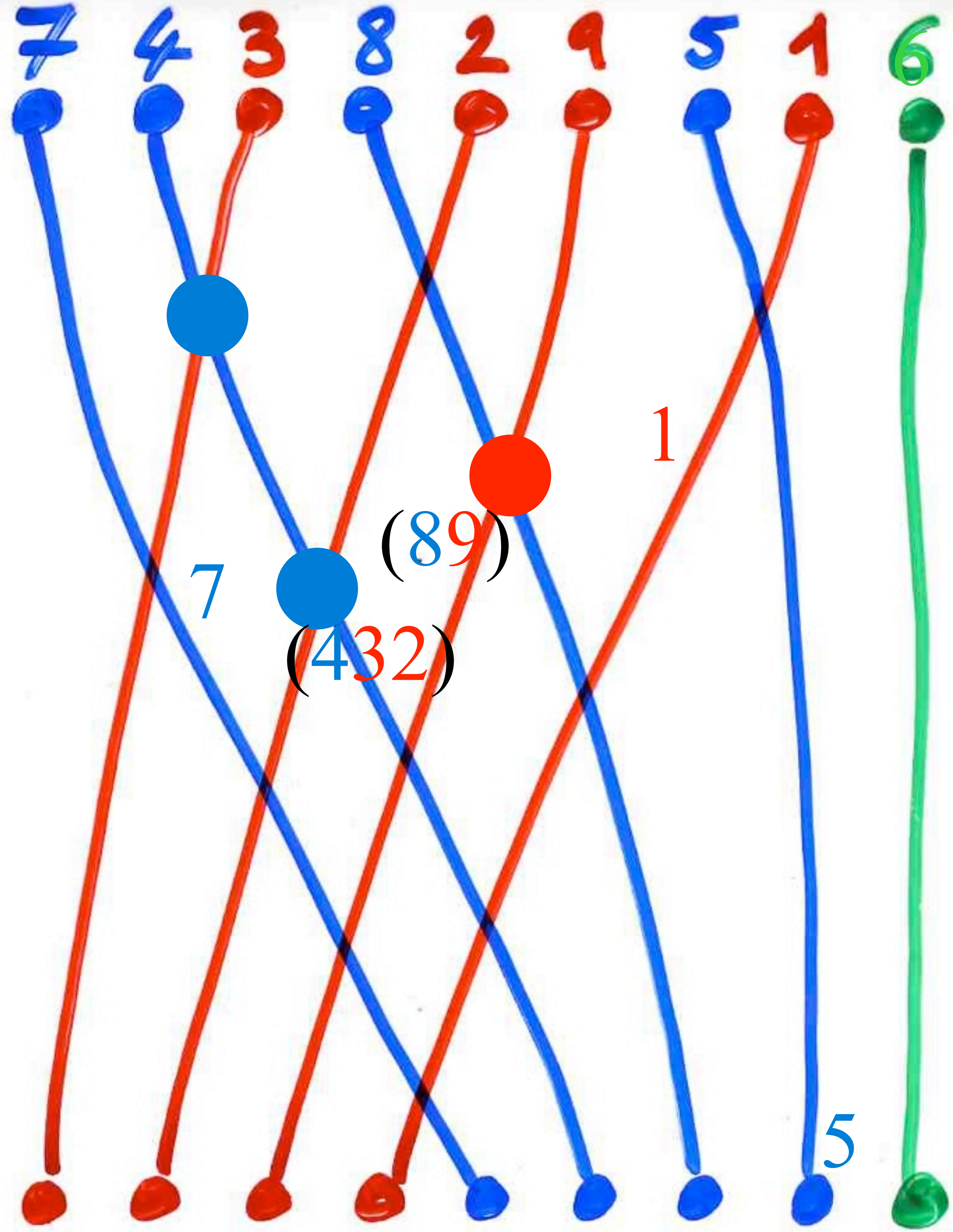


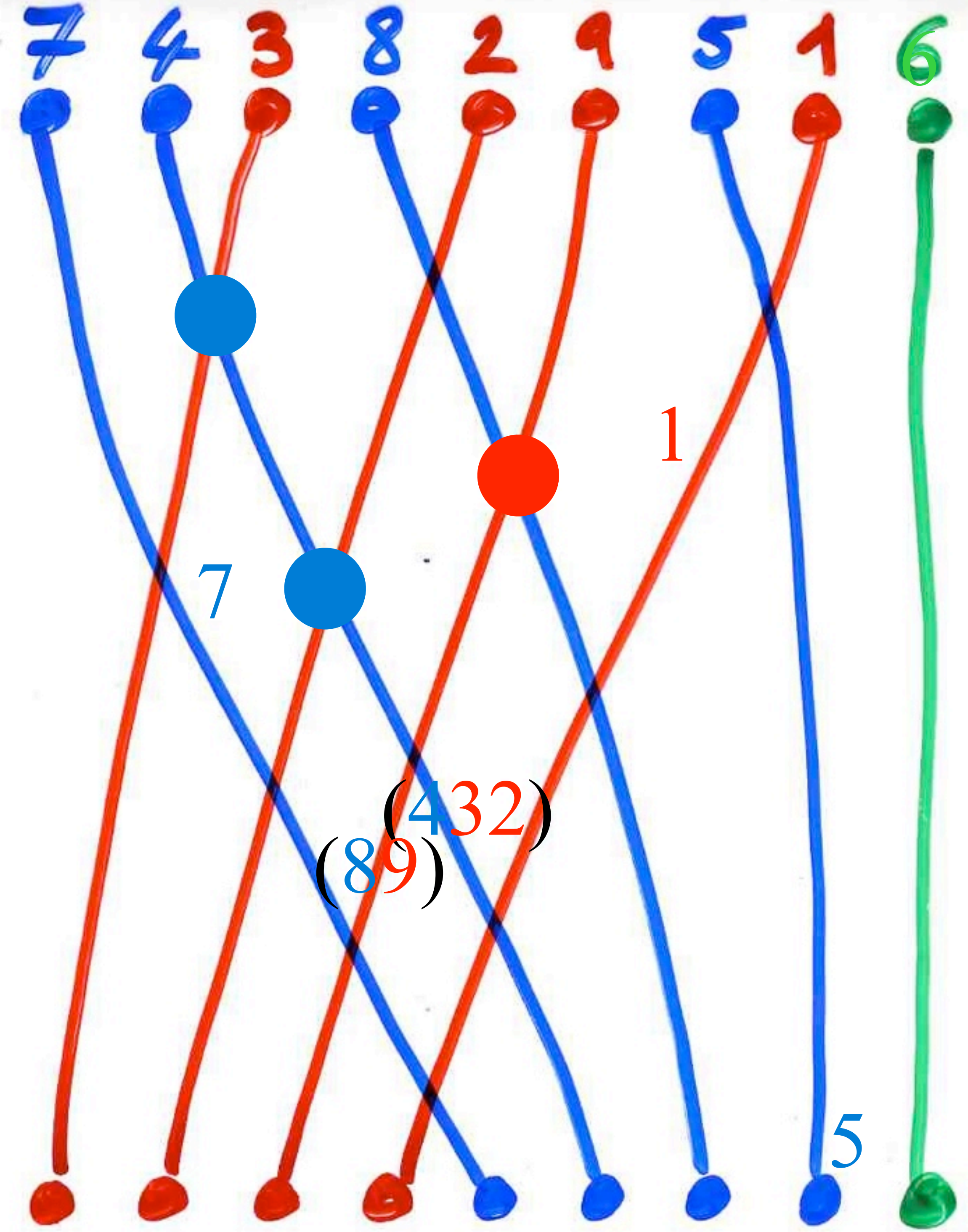
7 4 3 8 2 9 5 1 6
2
(43)
1
(89)
7
5



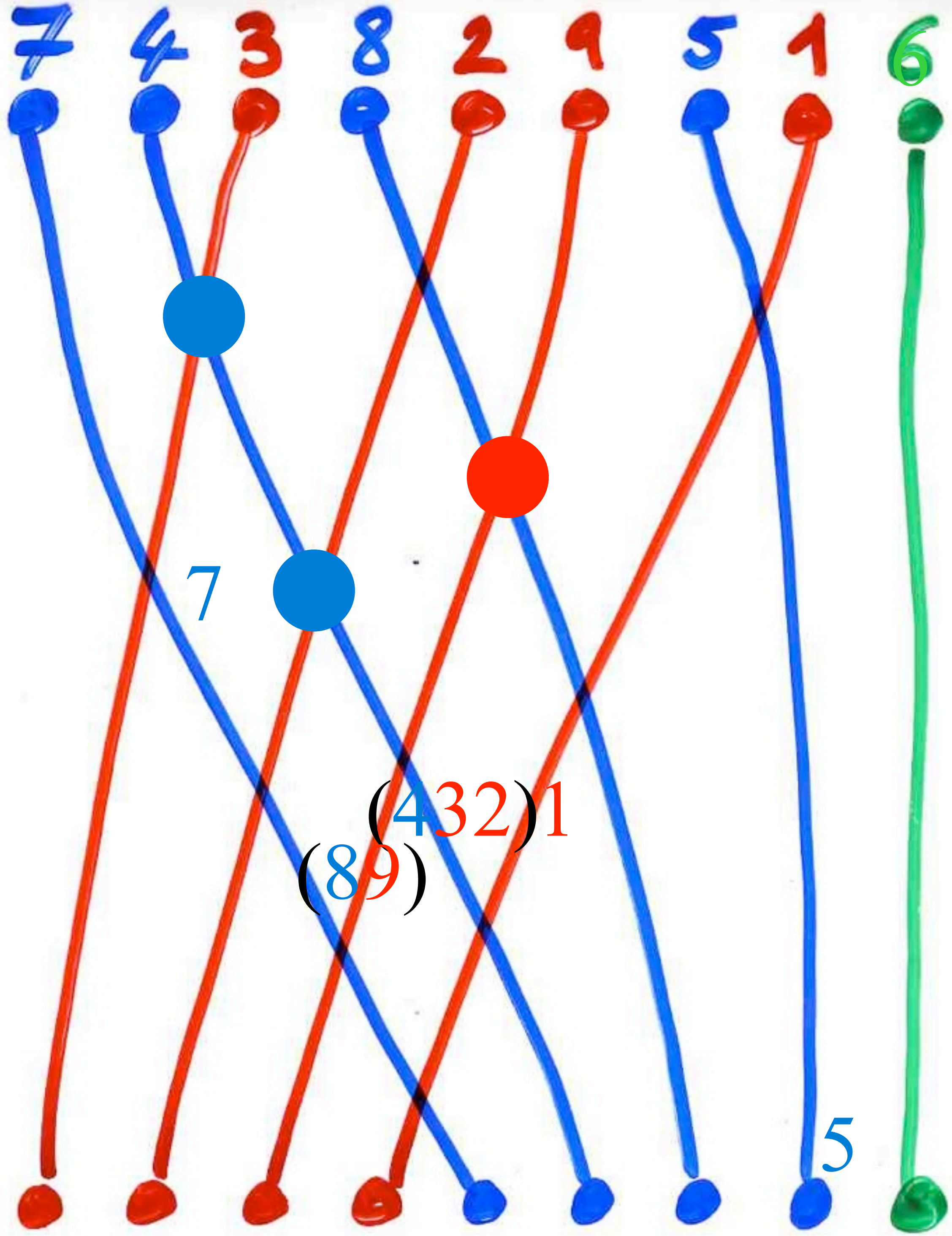
7 4 3 8 2 9 5 1 6
(43)2
(89)
1
7
5



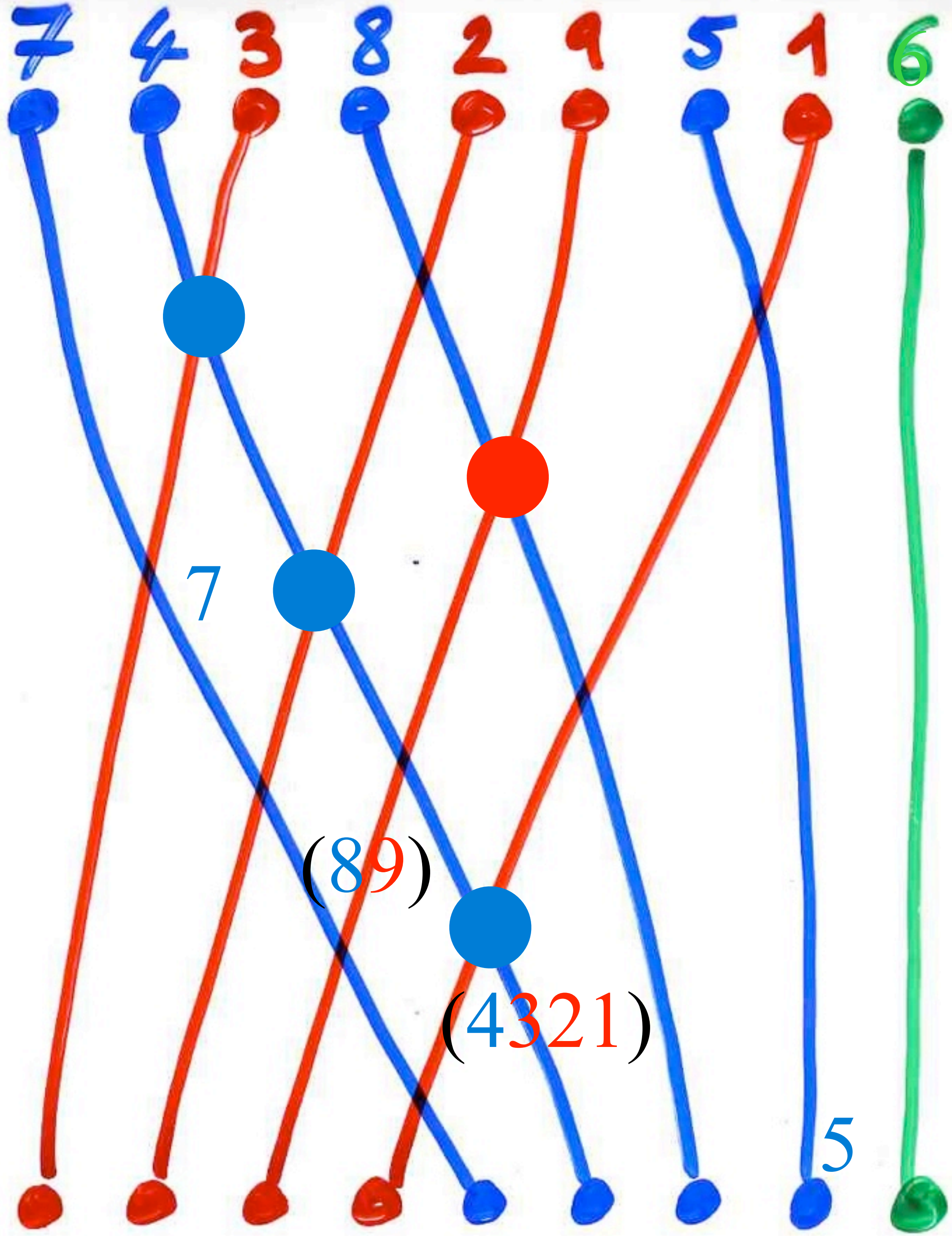
7 4 3 8 2 9 5 1 6
1
(89)
7
(432)
5



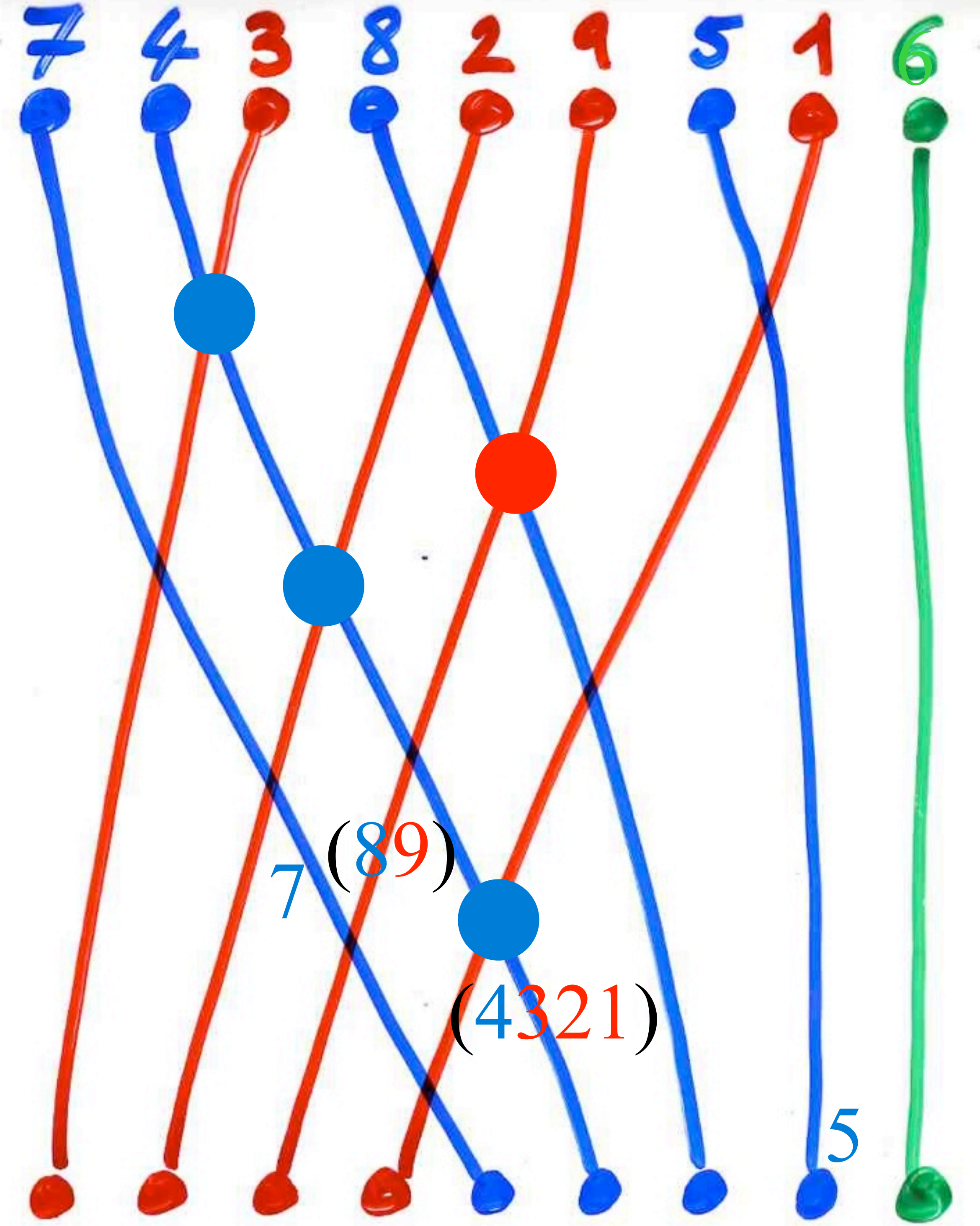
7 4 3 8 2 9 5 1 6
1
7
(432)
(89)
5

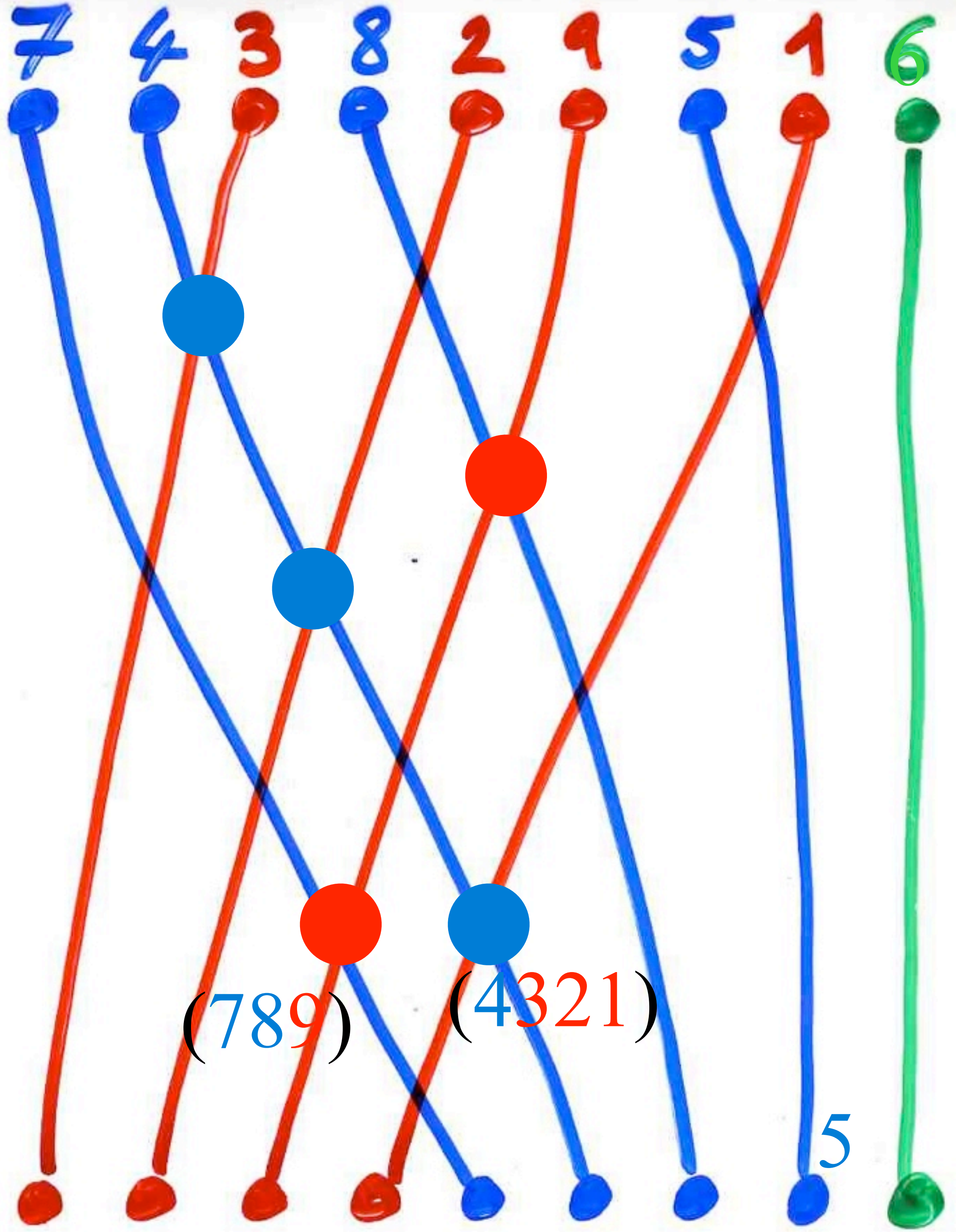


7 4 3 8 2 9 5 1 6
7
(432)1
(89)
5



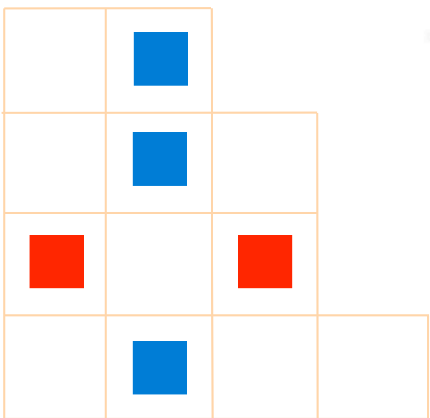
7 4 3 8 2 9 5 1 6
7
(89)
(4321)
5

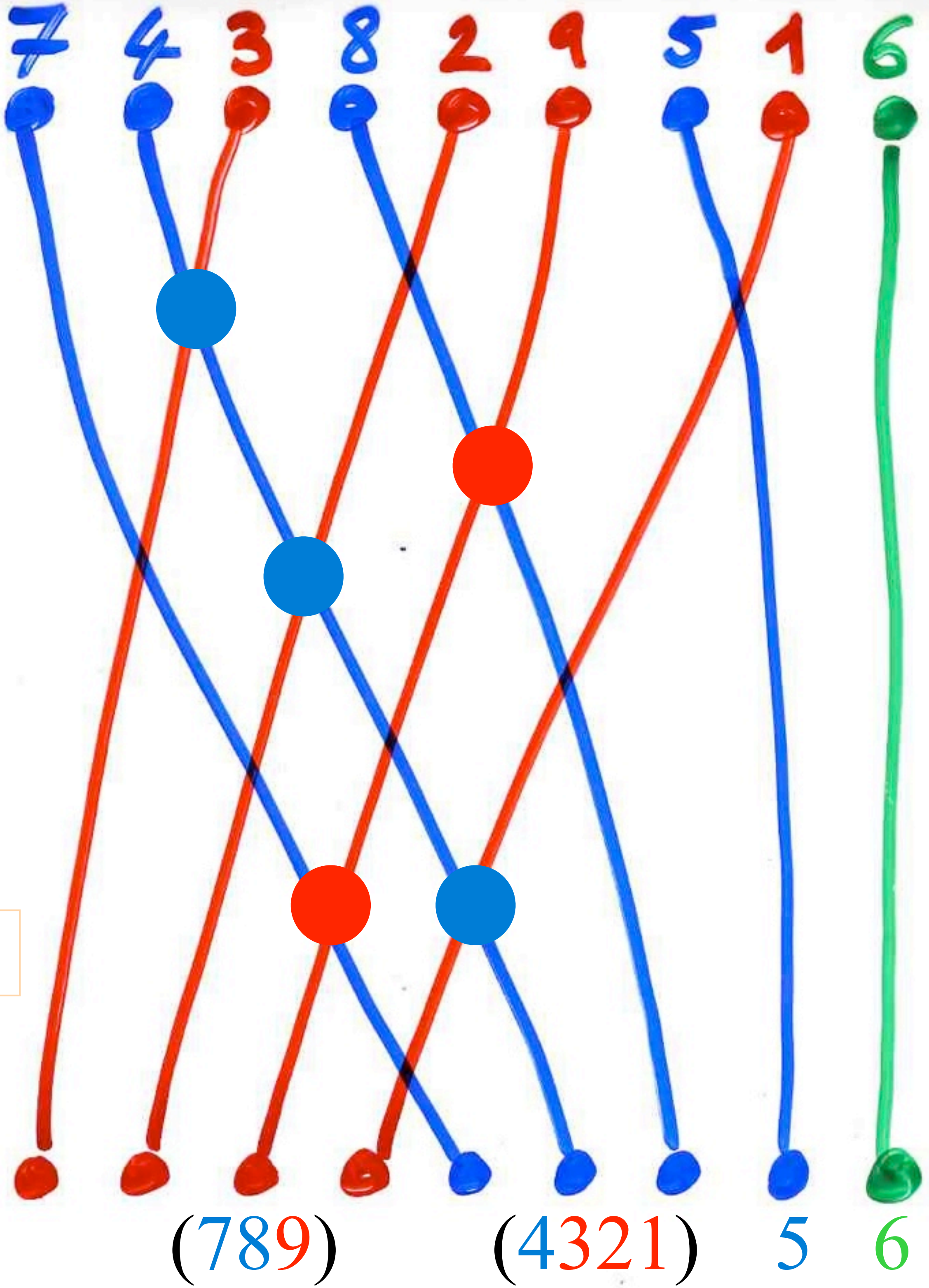

7 4 3 8 2 9 5 1 6
(89)
7
(4321)
5



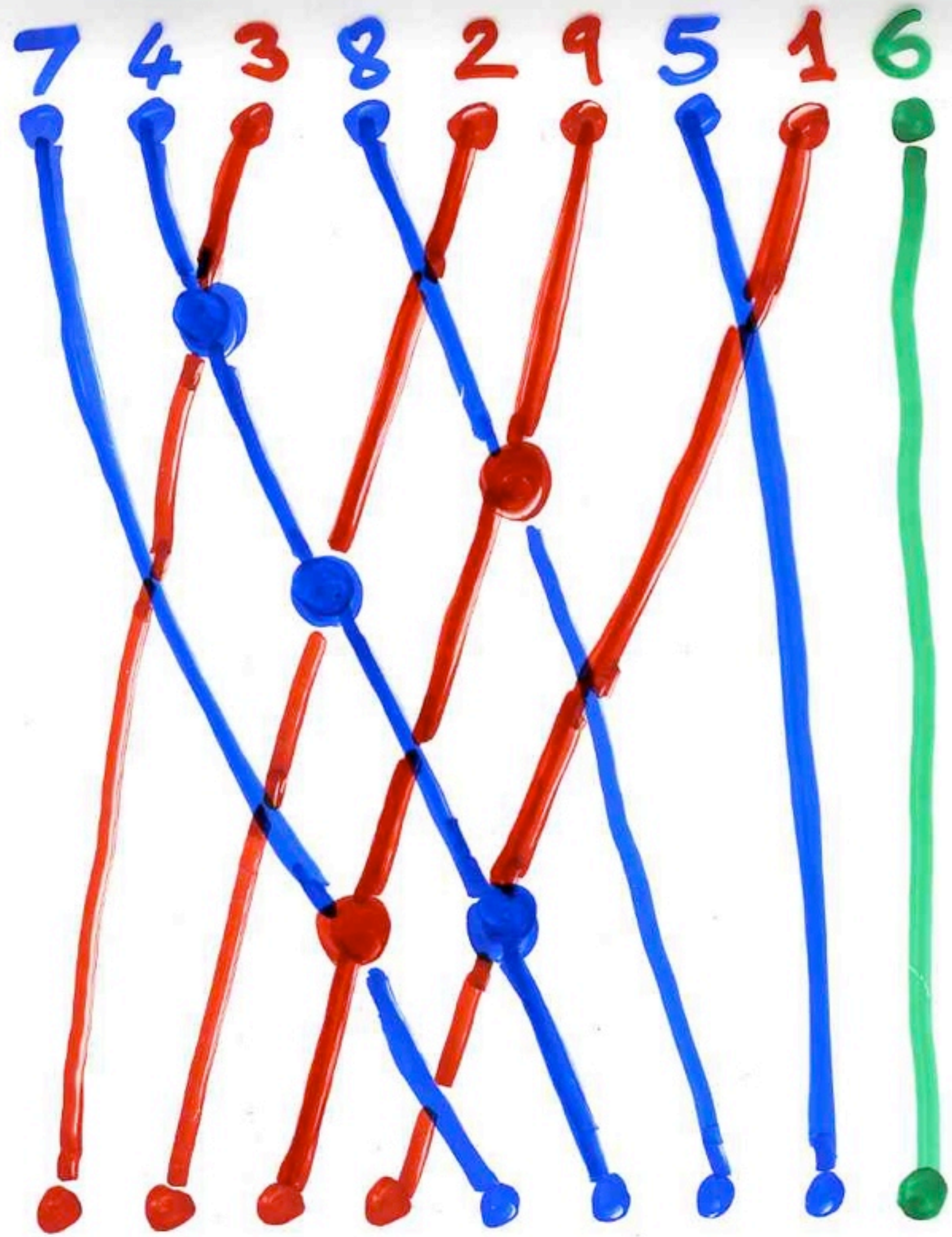
7 4 3 8 2 9 5 1 6
(789)
(4321)
5

“exchange-fusion” algorithm





The inverse “exchange-fusion” algorithm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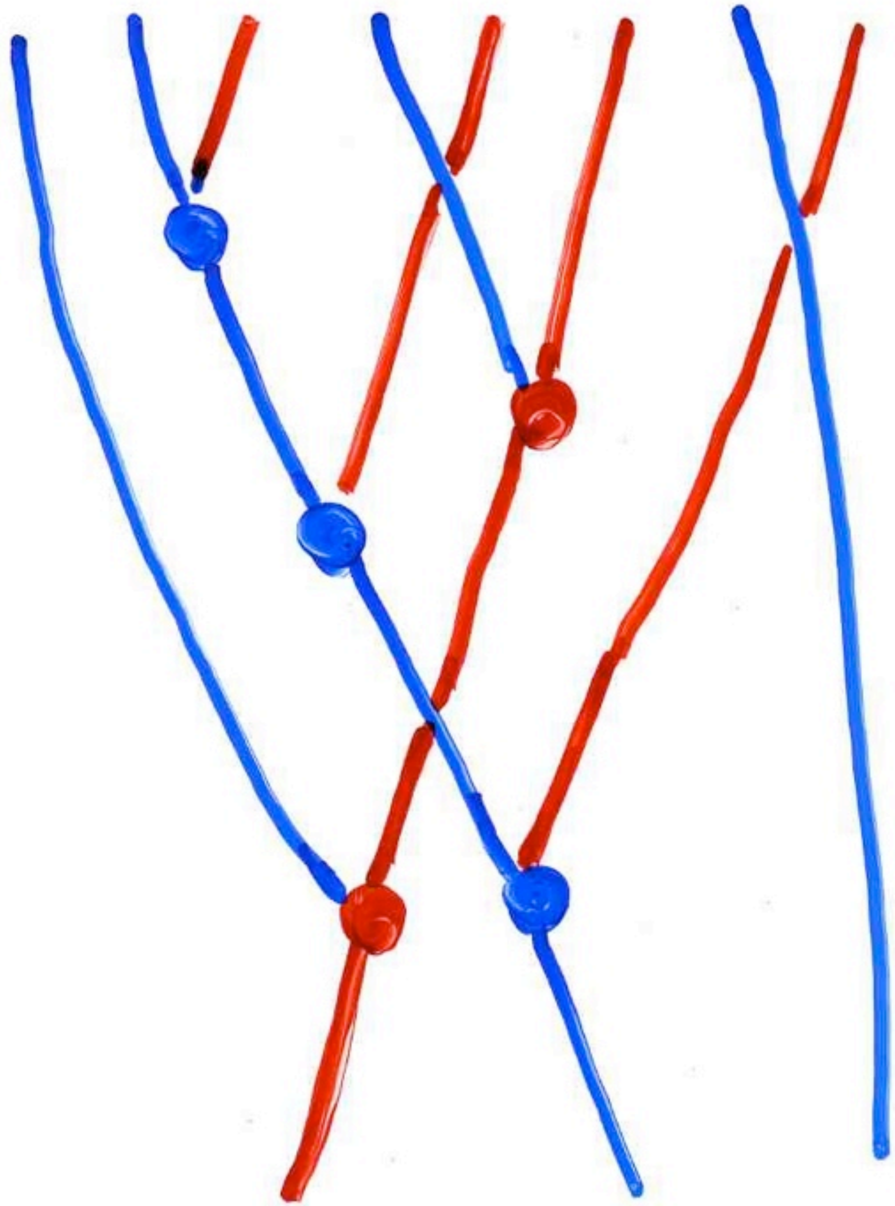
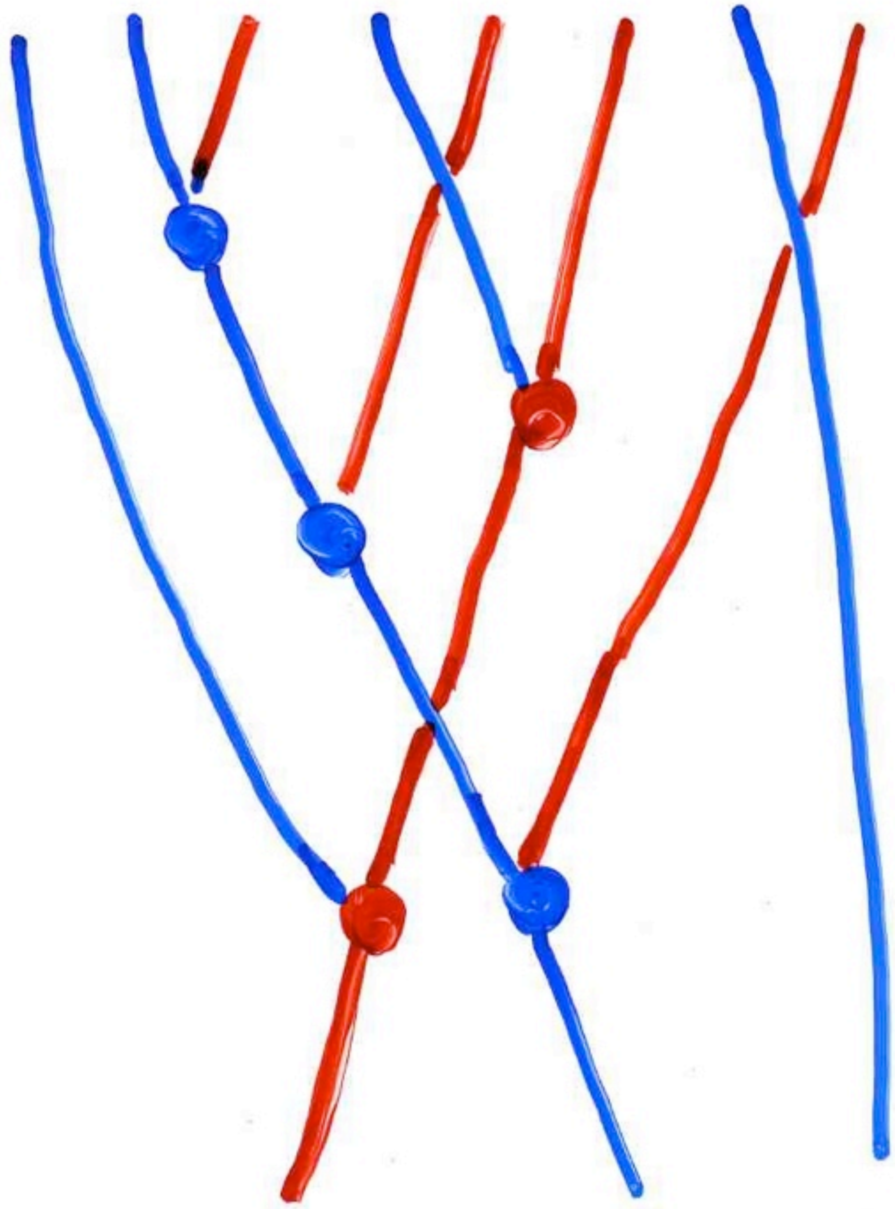
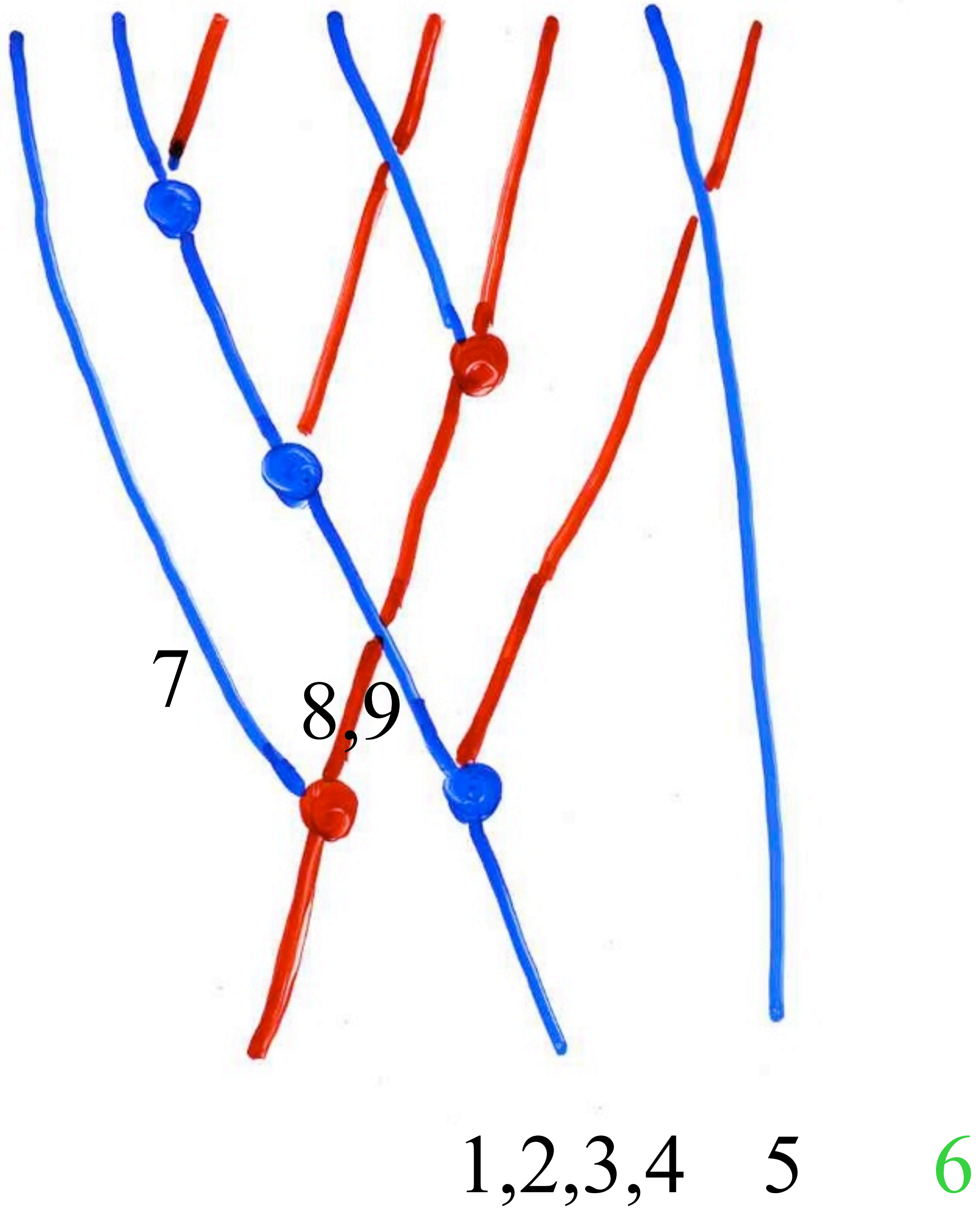
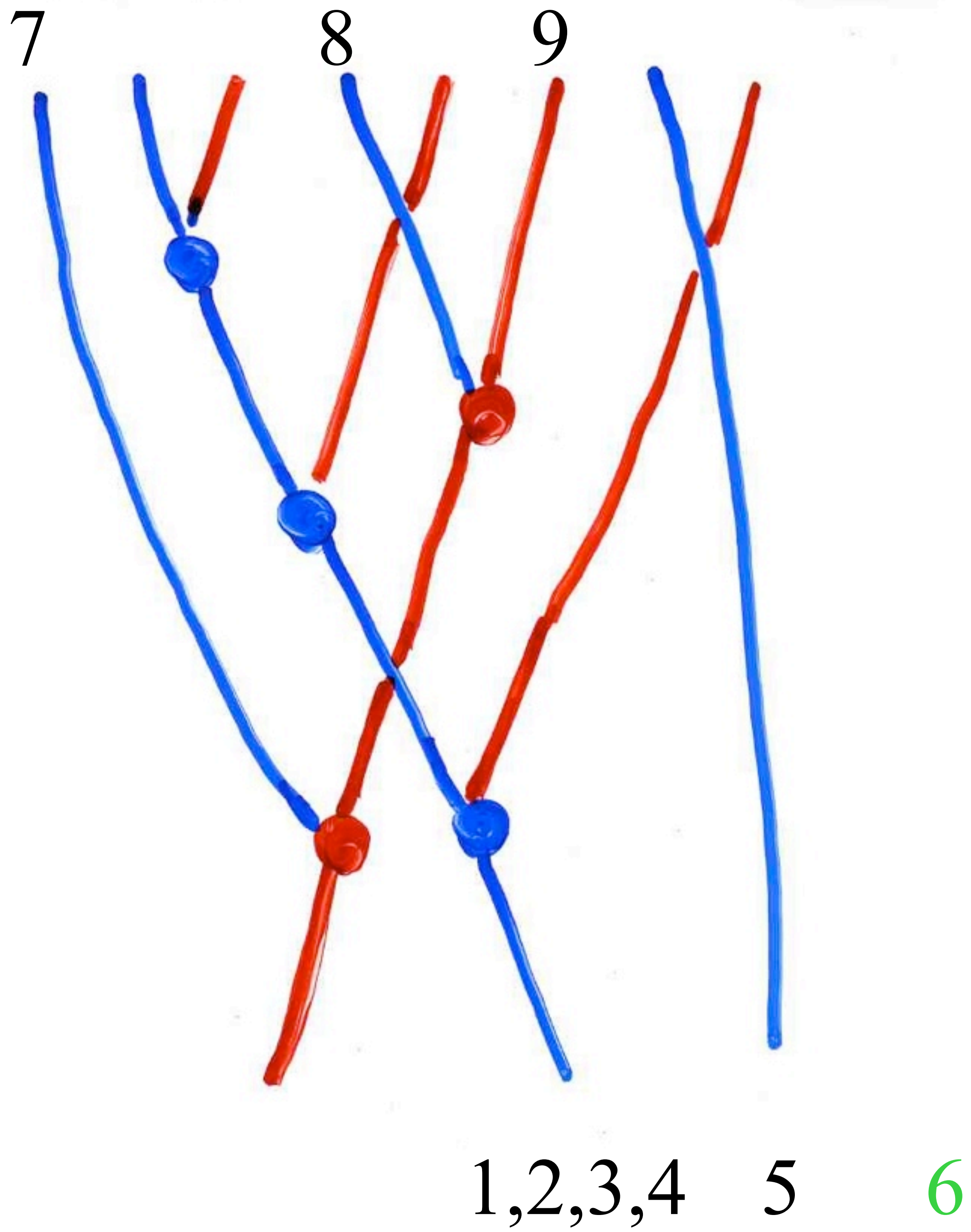


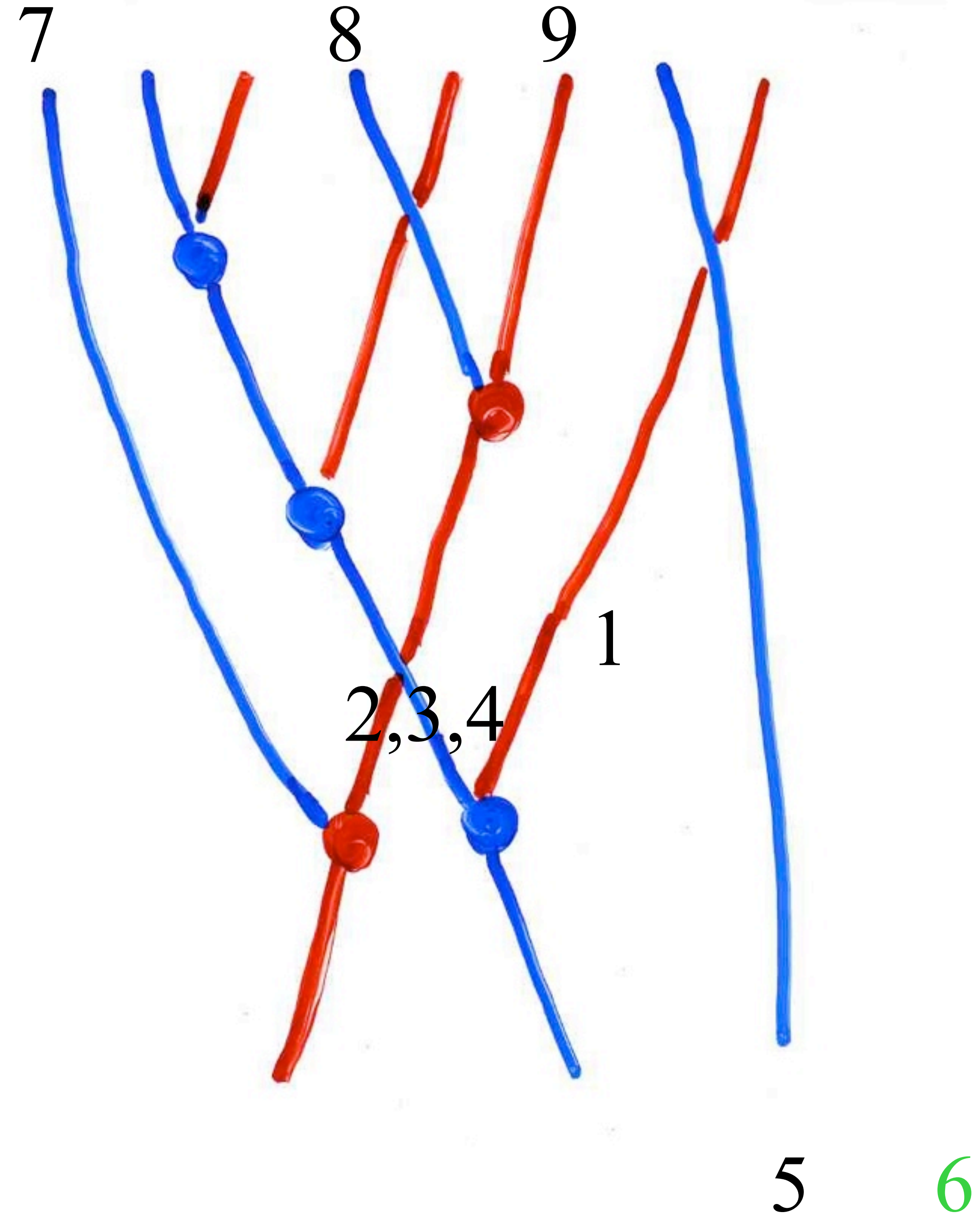


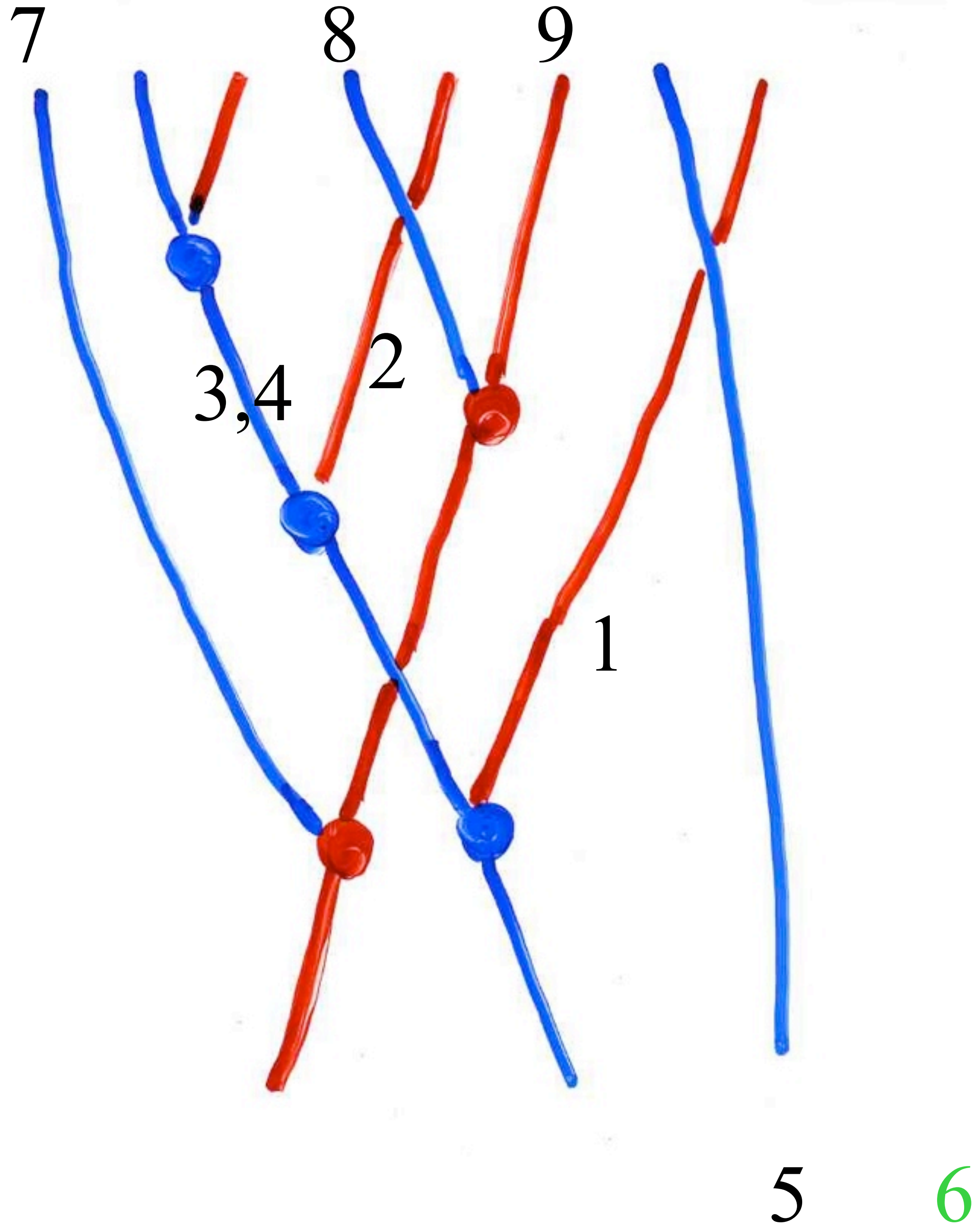
7,8,9 1,2,3,4 5 6

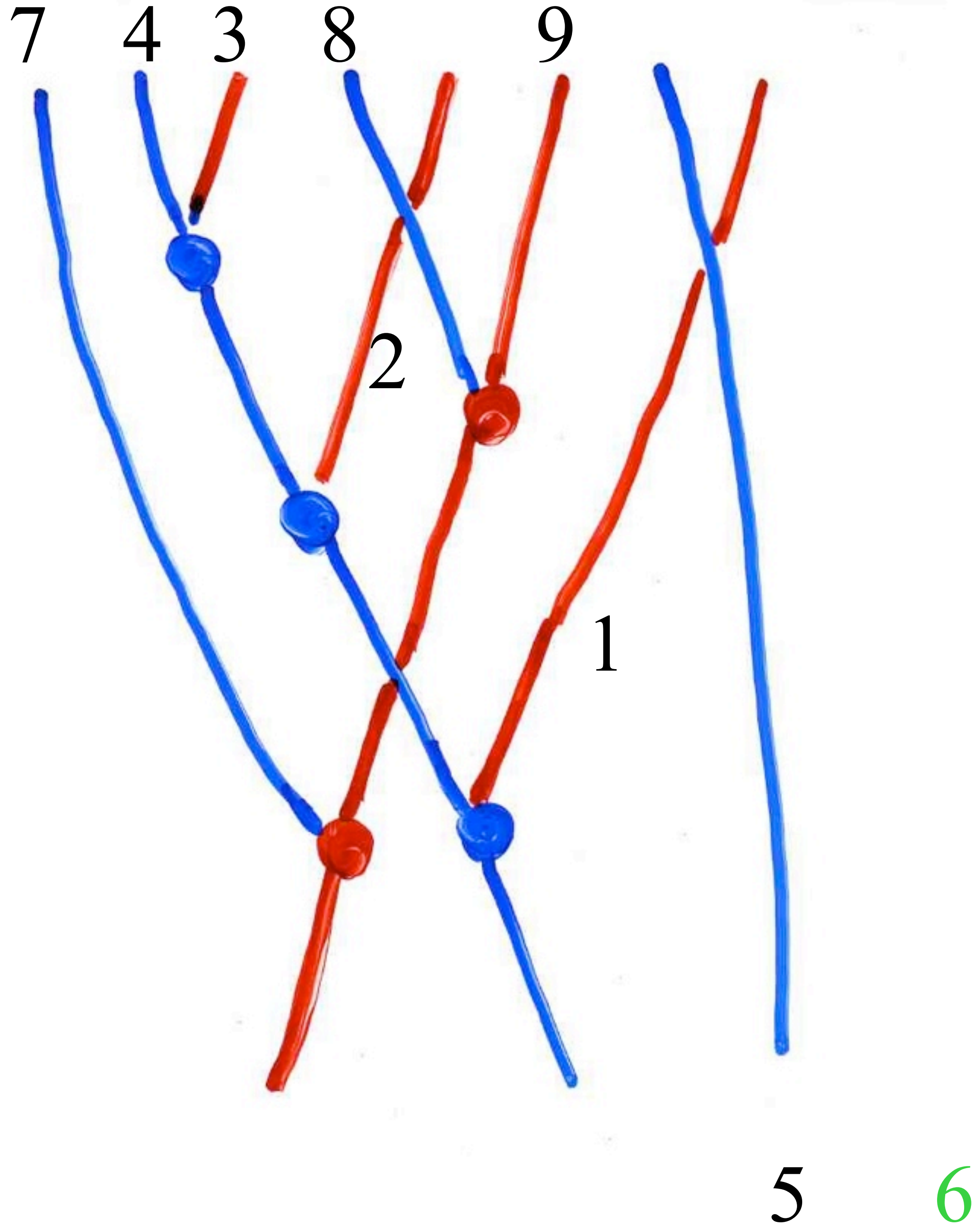


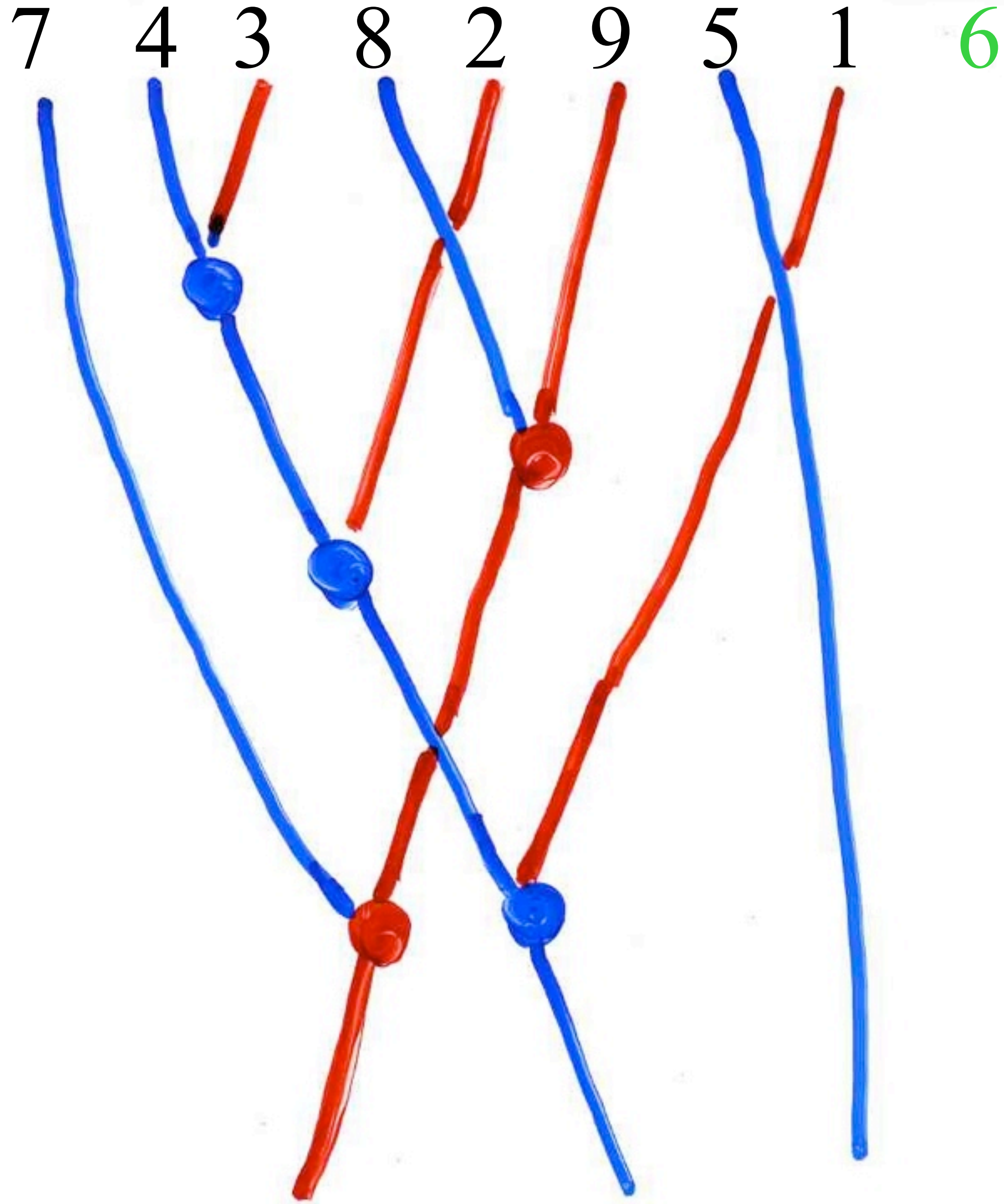
7
8,9
1,2,3,4
5
6


7
8
9
1,2,3,4
5
6


7
8
9
1
2,3,4
5
6


7
8
9
2
3,4
1
5
6


7
4
3
8
9
2
1
5
6


7 4 3 8 2 9 5 1 6

this bijection was constructed
from a combinatorial representation
of the PASEP algebra

and using the methodology
of the «Cellular Ansatz»

"The cellular Ansatz"

combinatorial
objects
on a 2d lattice

representation
by operators

Physics

bijections

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

RSK

permutations ⟷ pairs of Tableaux Young

alternative tableaux ⟷ permutations

$DE = qED + E + D$

PASEP

dynamical systems in physics
stationary probabilities

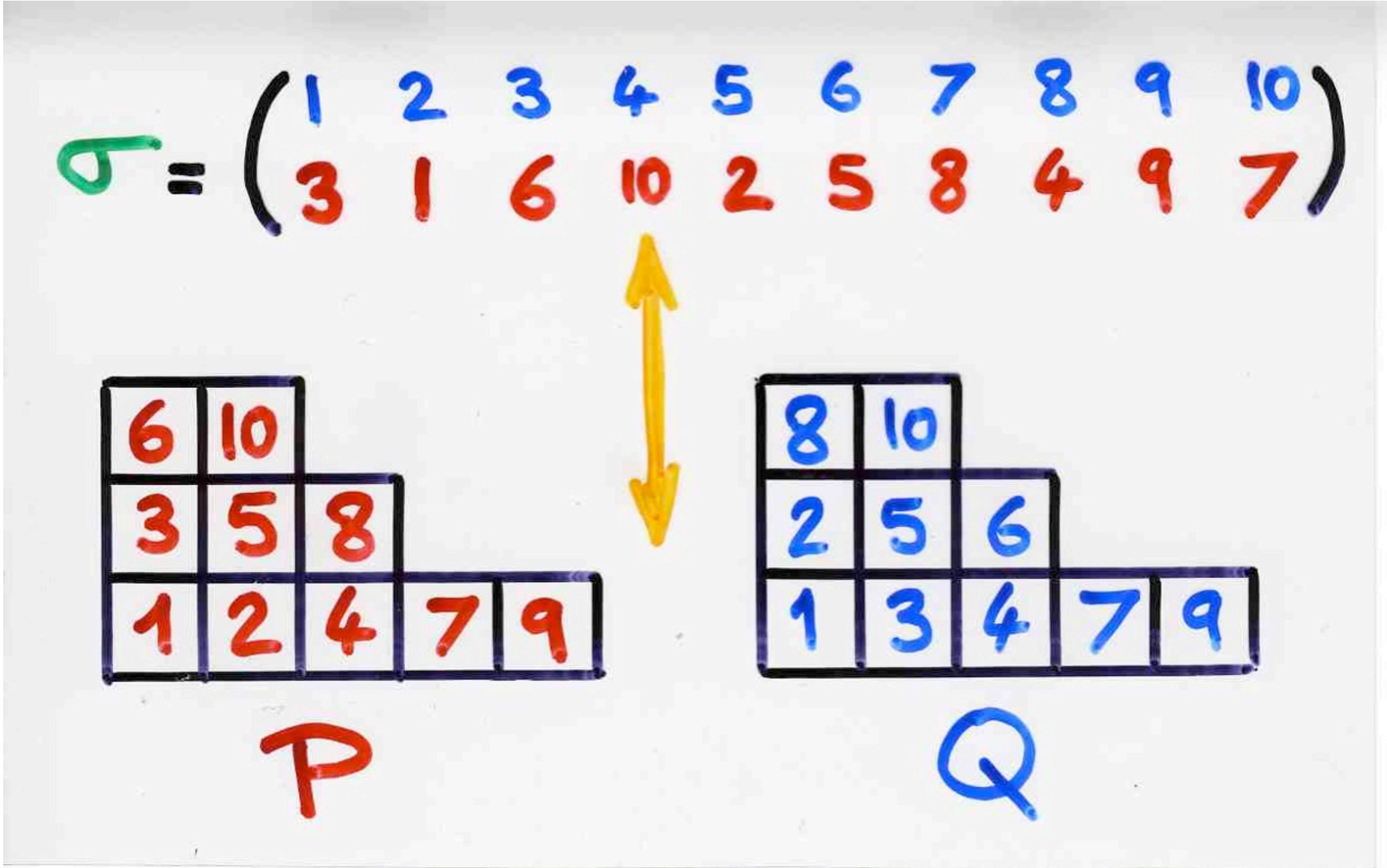
quadratic algebra Q

Q-tableaux

commutations
rewriting rules

planarization

The RSK correspondence
(Robinson-Schensted-Knuth)



The Robinson-Schensted correspondence (RSK) between permutations and pair of (standard) Young tableaux with the same shape

Heisenberg operators
U, D

$$UD = DU + I$$

$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} \, D^i U^i$$

normal ordering

$$c_{n,0} = n!$$

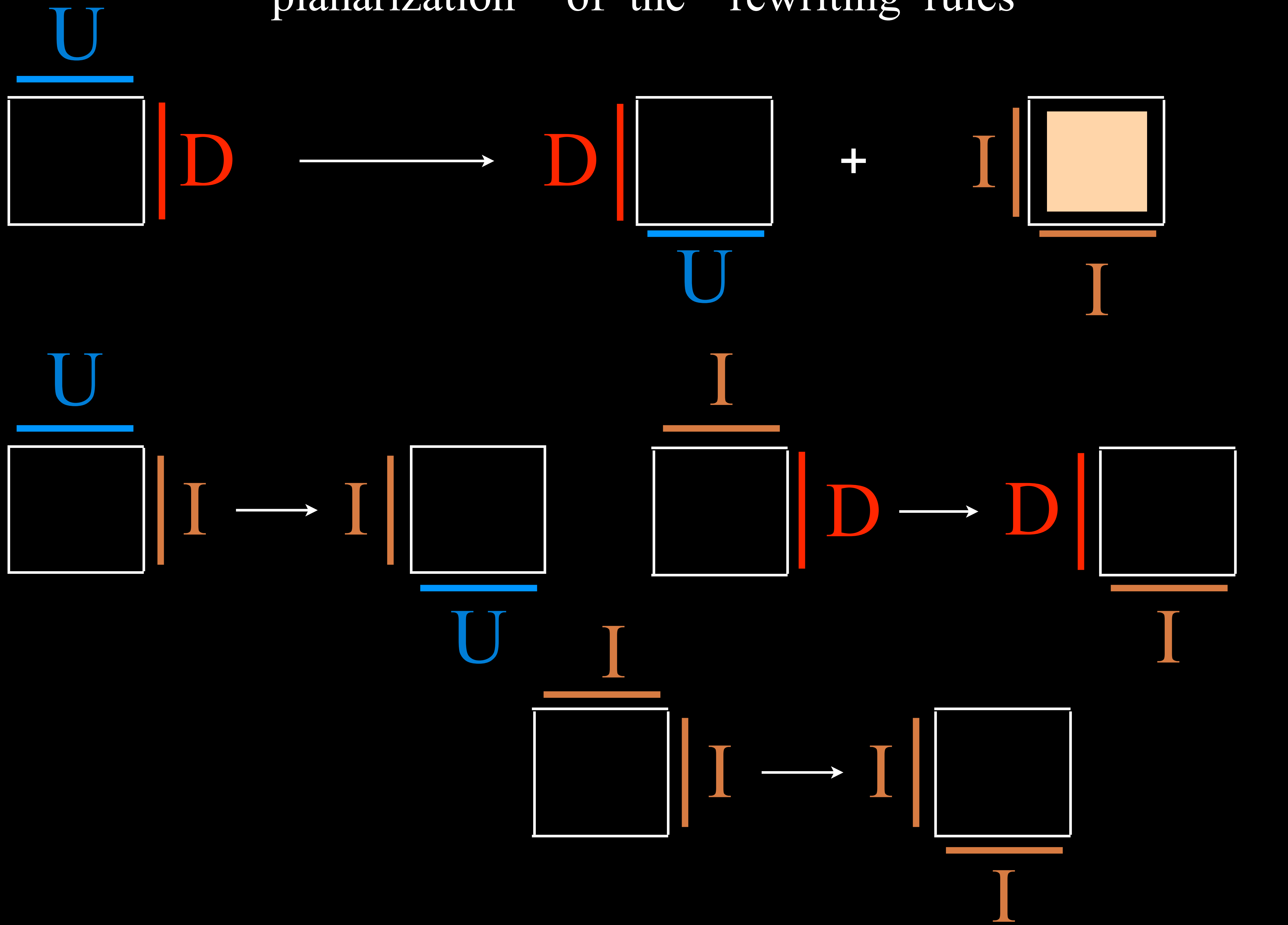
The cellular Ansatz

quadratic algebra Q (of a certain type)

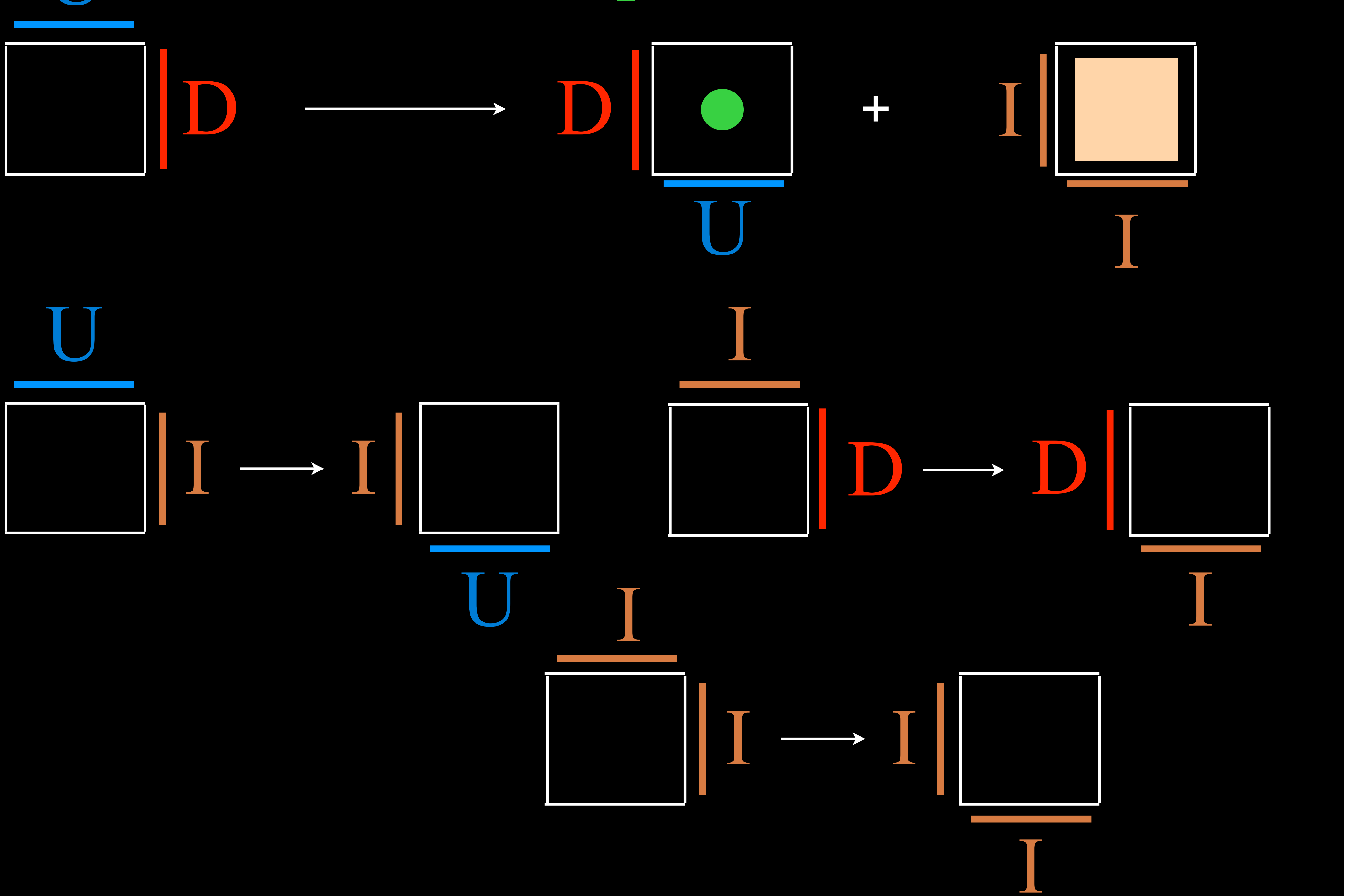
(I) "planarisation" on a grid of the rewriting rules

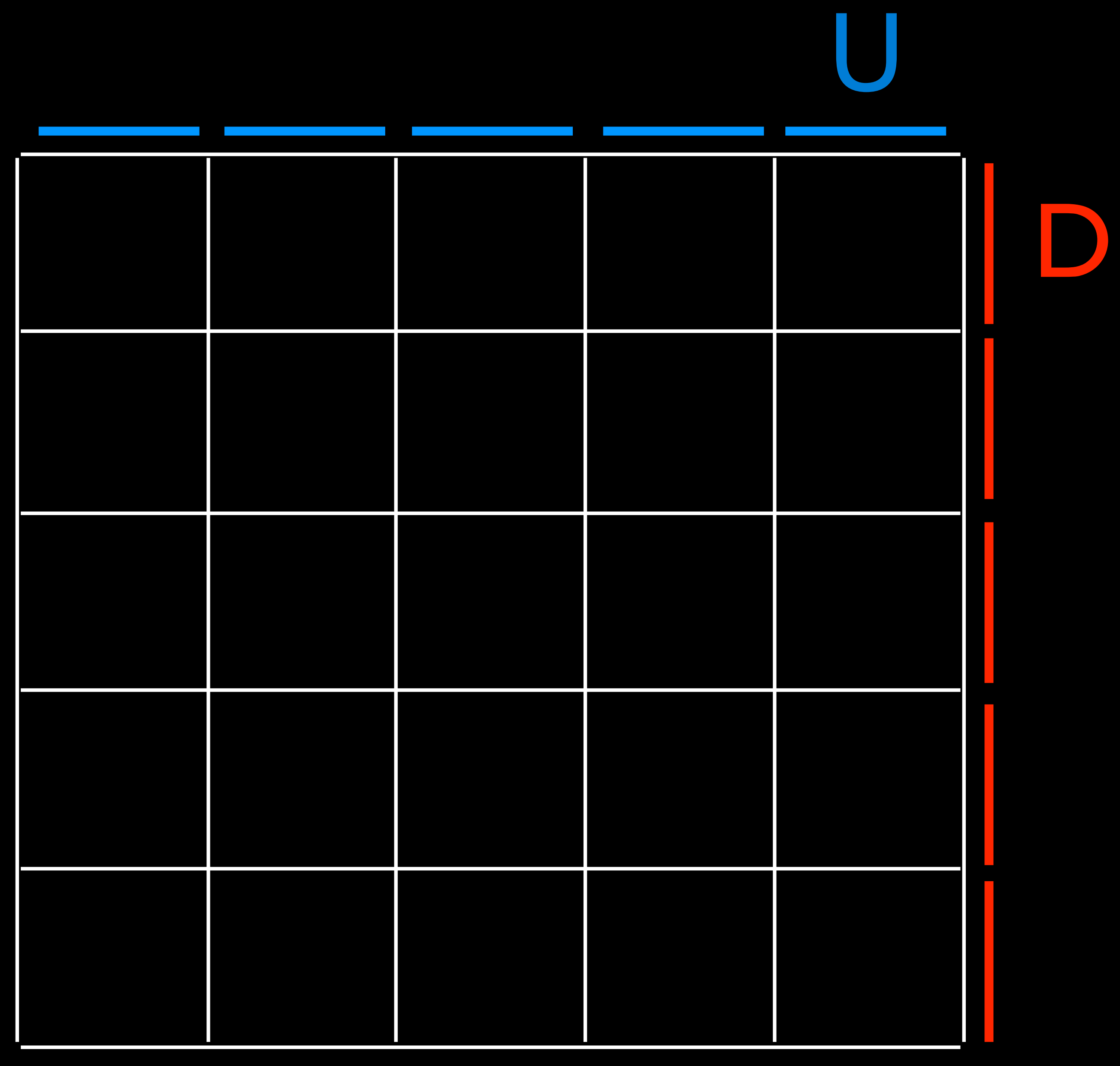
Q- tableaux

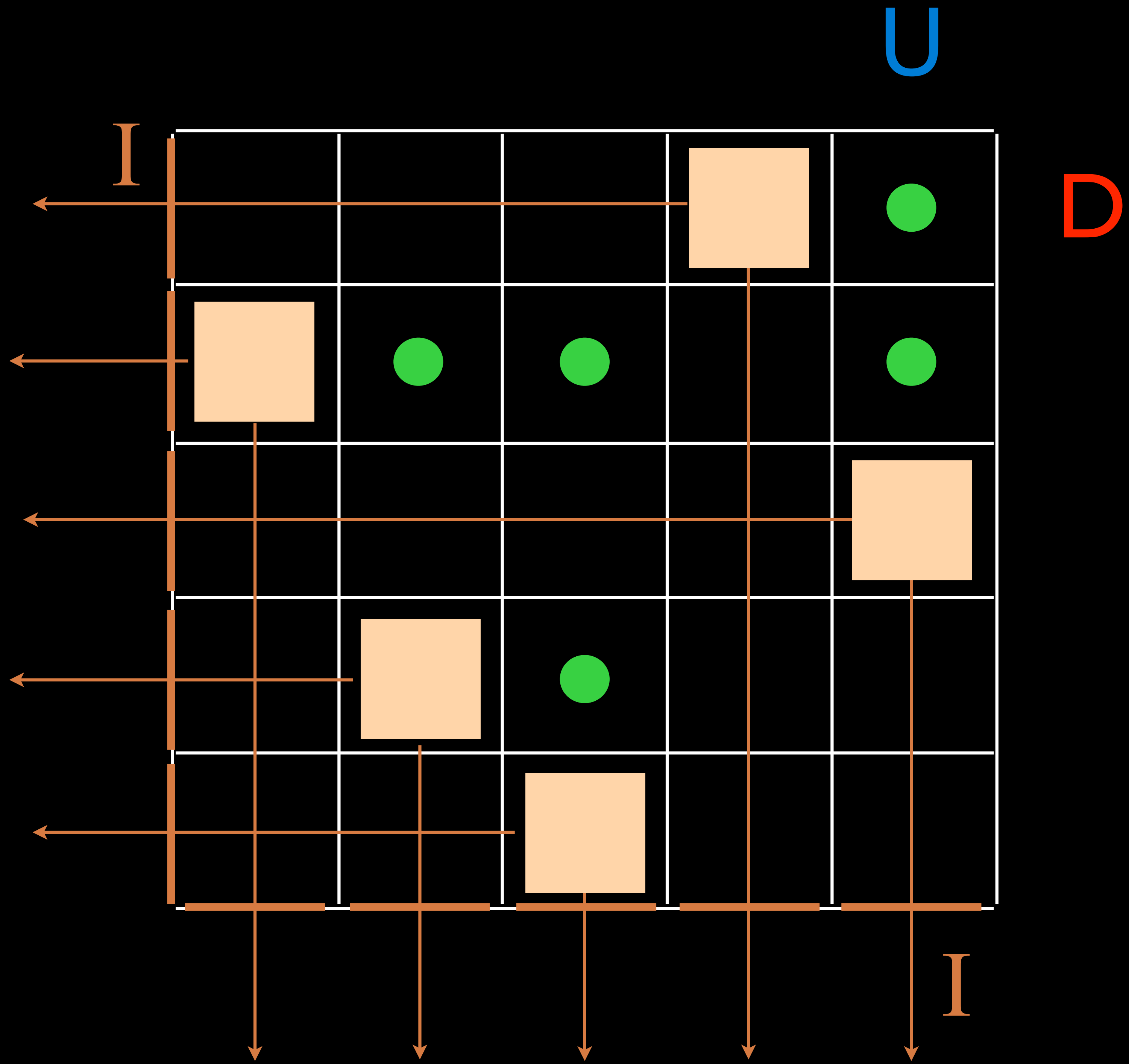
“planarization” of the “rewriting rules”



$$UD = qDU + I$$




U
D


U
D
I
I

$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} \, D^i U^i$$

normal ordering

$$c_{n,0} = n!$$

The cellular Ansatz

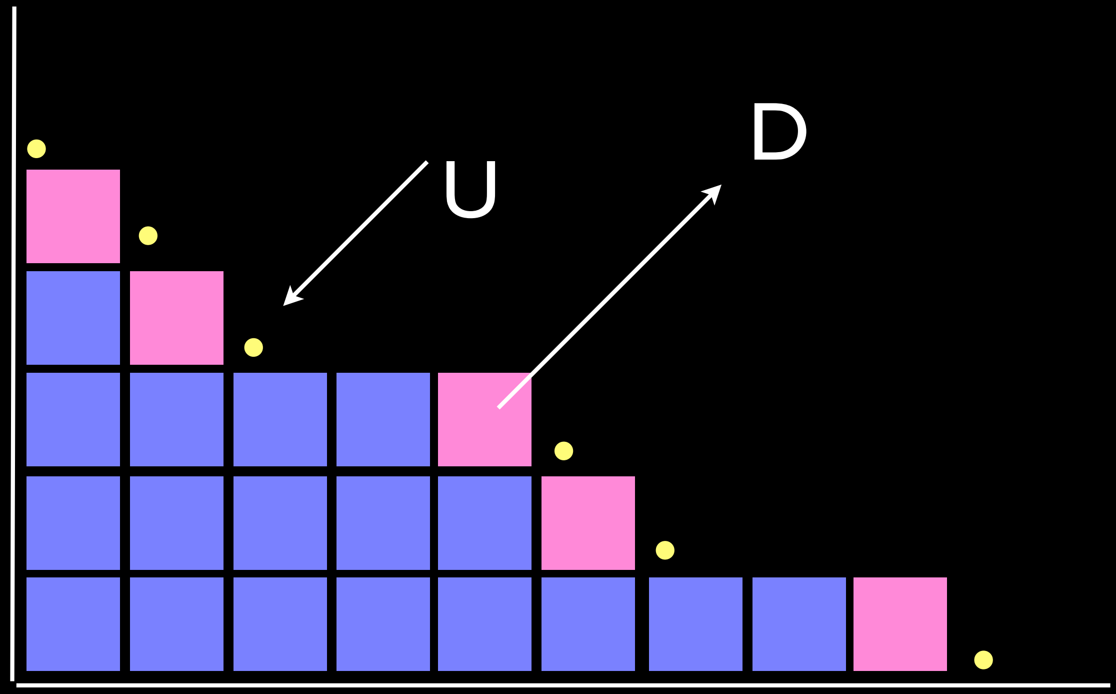
(2)

guided construction
of a bijection
(from a representation of the quadratic
algebra Q with "combinatorial operators")

representation of the operators U, D

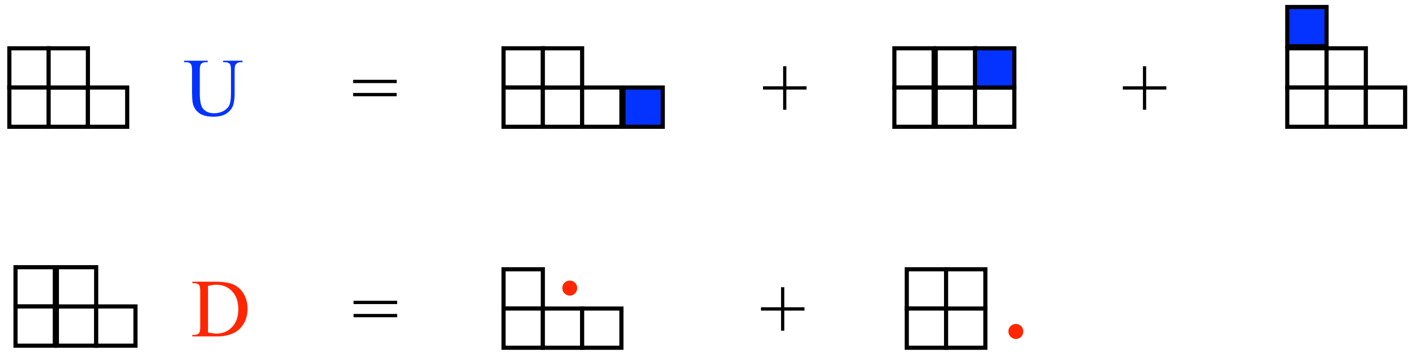
Sergey Fomin
(with C. K.)

Operators U and D



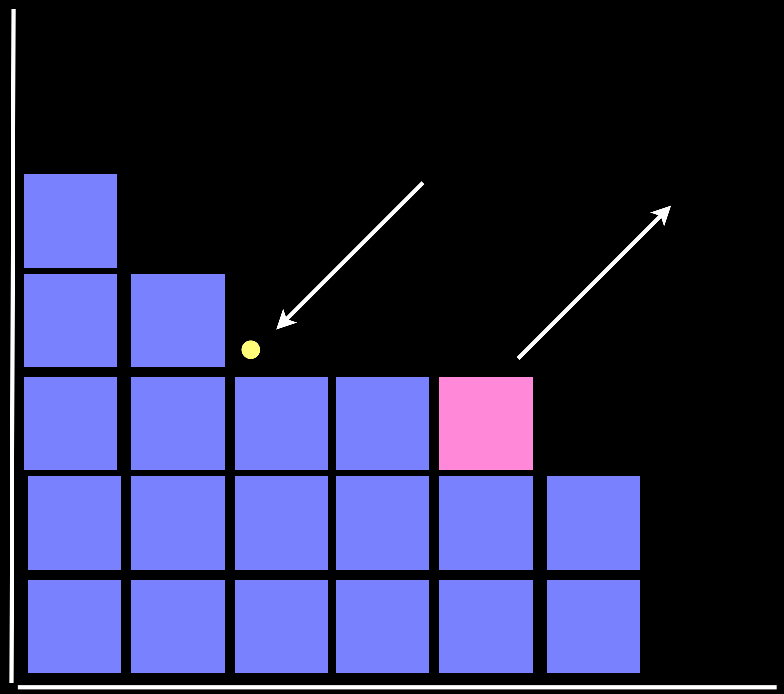
adding
or deleting
a cell in
a Ferrers
diagram

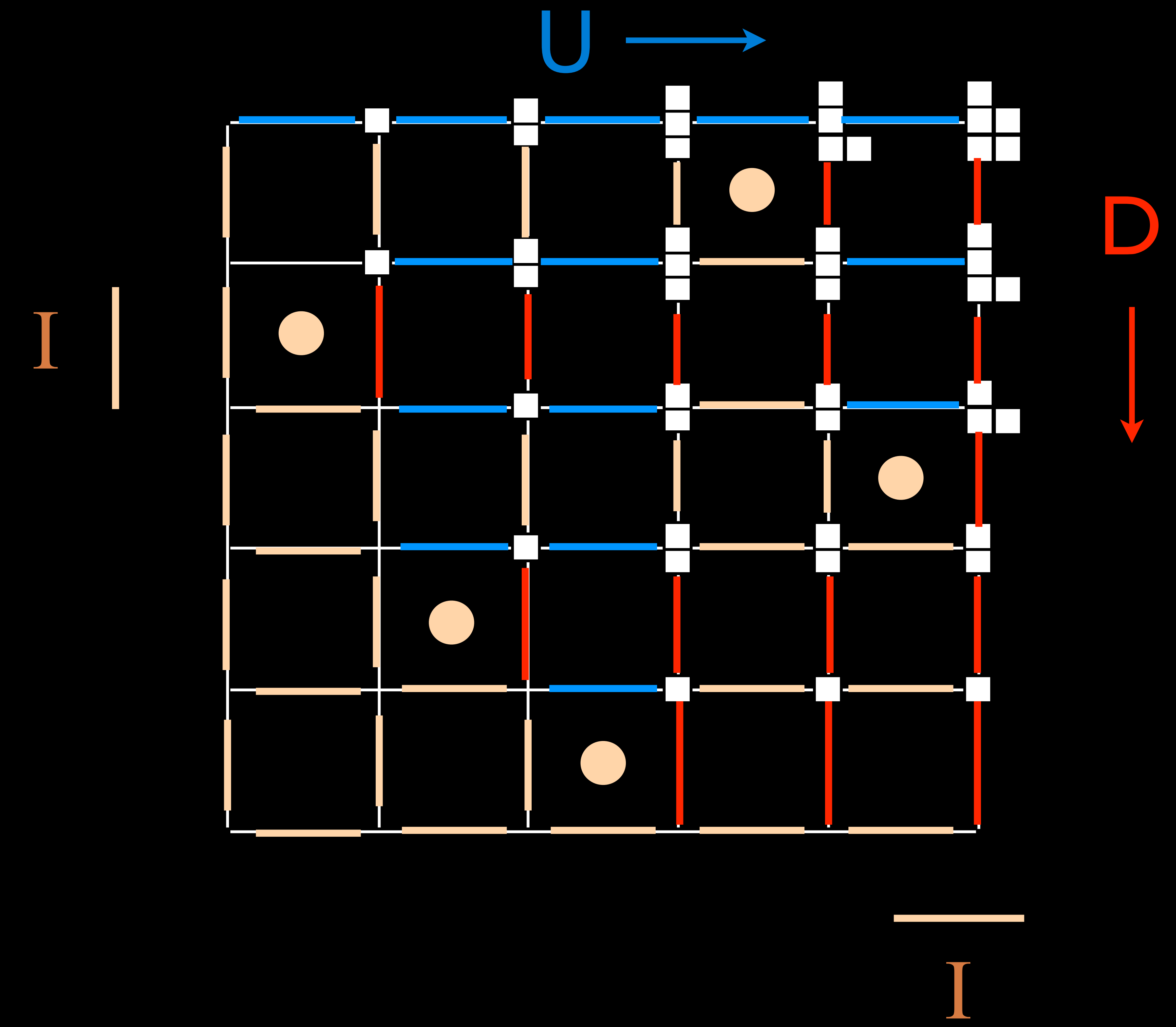
Young lattice

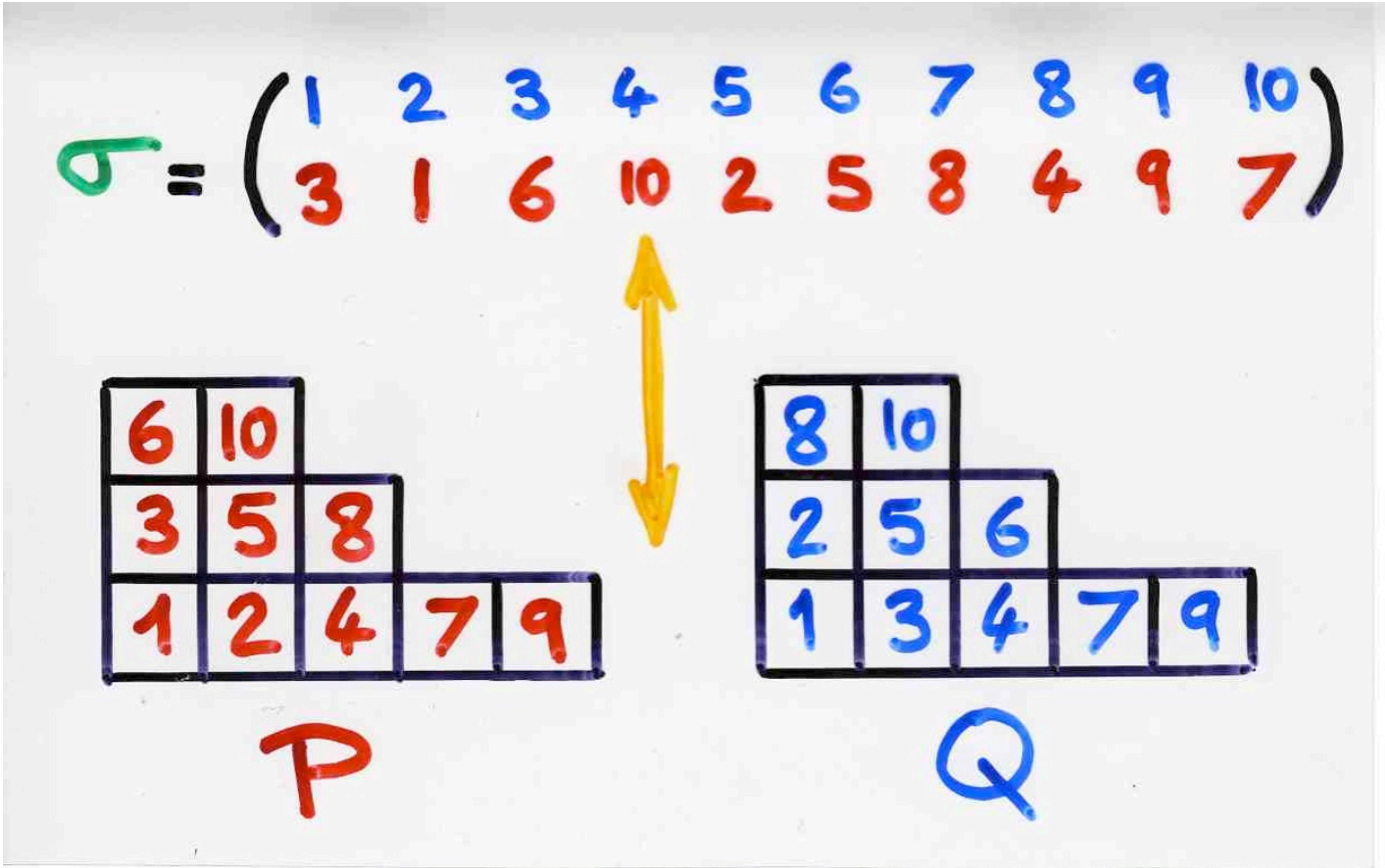

U
D

Heisenberg commutation relation

$$UD = DU + I$$




U
D
I
I


σ =
1 2 3 4 5 6 7 8 9 10
3 1 6 10 2 5 8 4 9 7
6 10
3 5 8
1 2 4 7 9
P
8 10
2 5 6
1 3 4 7 9
Q

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alternative tableaux ⟷ permutations

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quadratic algebra Q

Q-tableaux

commutations
rewriting rules

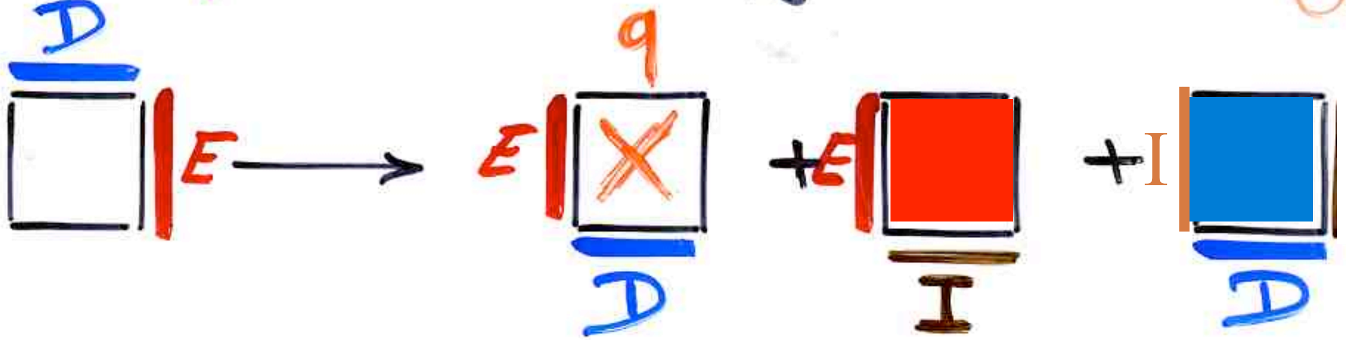
planarization

for the PASEP algebra

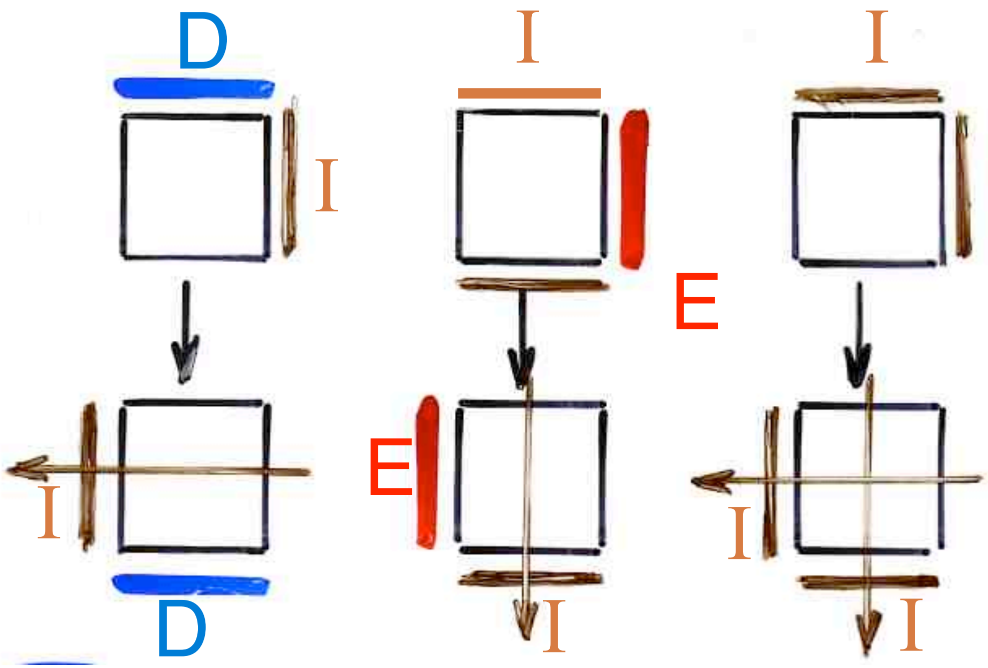
$$DE = qED + E + D$$

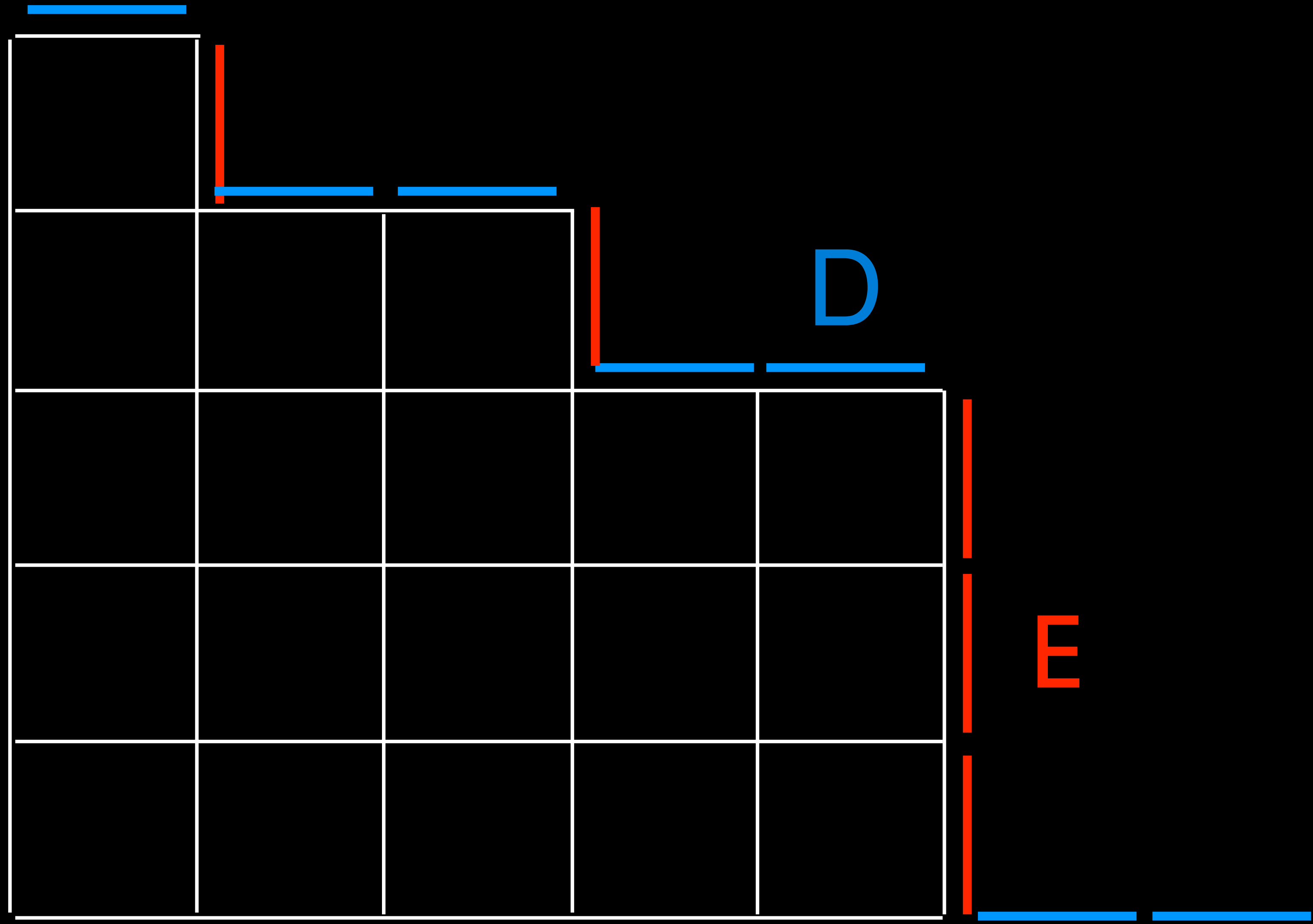
(1) planarization

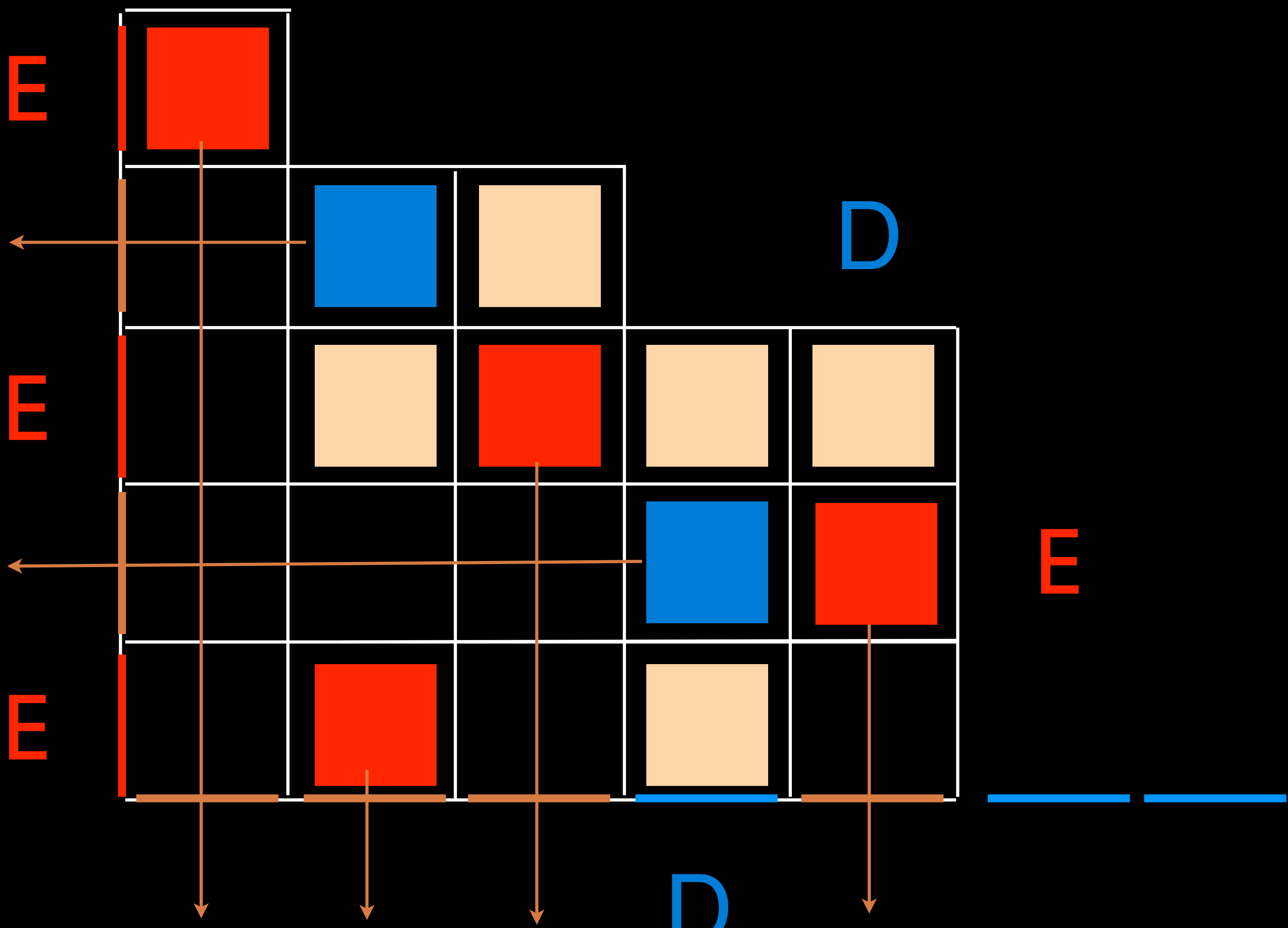
Proof: "planarization" of the rewriting rules



I identity




D
E


E
D
E
E
E
D

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

T alternative tableau with profile w

$k(T)$ = nb of □

$i(T)$ = nb of rows without blue cell

$j(T)$ = nb of columns without red cell

for the PASEP algebra

$$DE = qED + E + D$$

(2) representation with operators
related to the combinatorial theory
of orthogonal polynomials
and data structures in computer science

representation
of the
operators
E and D

$$DE = ED + E + D$$

V vector space generated by $\mathcal{B}$ basis

$\mathcal{B}$ alternating words two letters $\{○, ●\}$
(no occurrence of ○○ or ●●)

4 operators A, S, J, K

4 operators A, S, J, K, $u \in B$

$\langle u | A = \sum_{\text{letter } \circ \text{ of } u} v$, v obtained by:
○ → ○●○

$\langle u | S = \sum_{\circ \text{ of } u} v$ v obtained by:
○ → ●
(and ●● → ● ●●● → ●)

$\langle u | J = \sum_{\circ \text{ of } u} v$ v, ○ → ●○
(and ●● → ●)

$\langle u | K = \sum_{\circ \text{ of } u} v$ v, ○ → ○●
(and ●● → ●)

○●○●|S = ●○● + ○●

Lemma.

$$AS = SA + J + K$$

$$AK = KA + A$$

$$JS = SJ + S$$

$$JK = KJ$$

$$D = A + J$$

$$E = S + K$$

$$D = A + J$$

$$E = S + K$$

$$DE = (A+J)(S+K)$$

$$= AS + AK + JS + JK$$

$$= \underbrace{\underbrace{(SA + KA + SJ + KJ)}_{(S+K)(A+J)}}_{ED} + \underbrace{J + K + A + S}_{E + D}$$

J. Françon 1976

data structure histories

"histoires de fichiers"

24

17

10

8

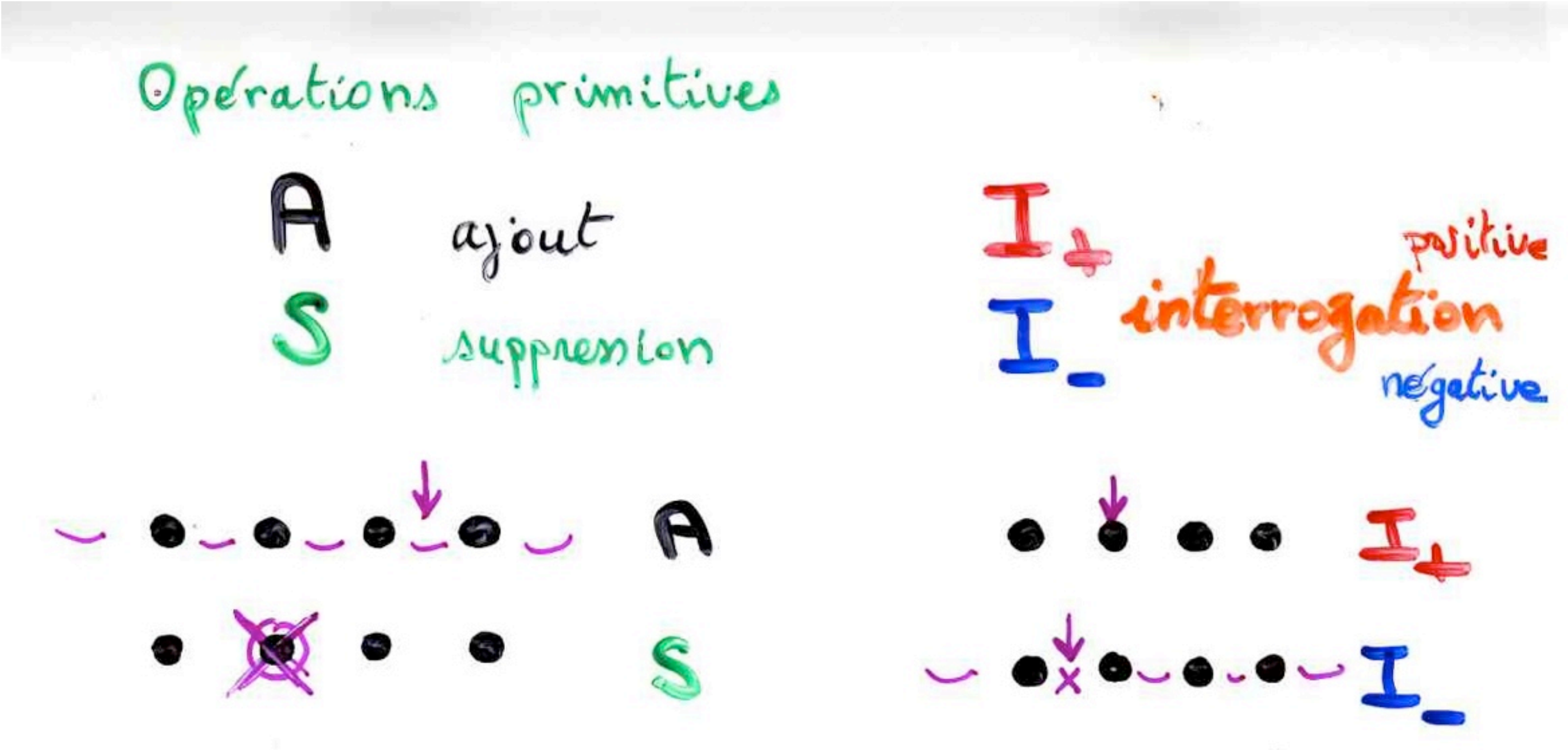
24

17

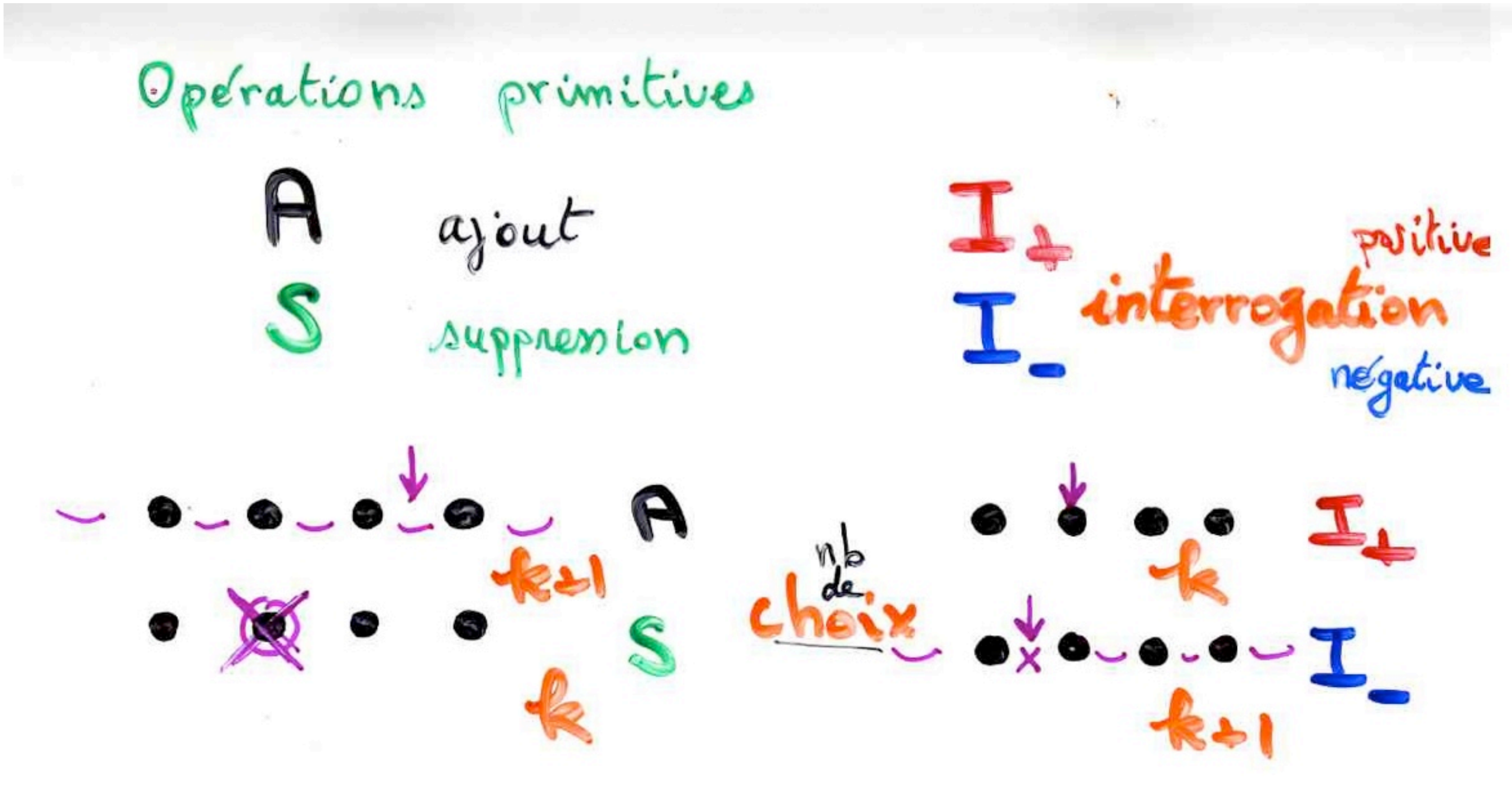
← 12

10

8



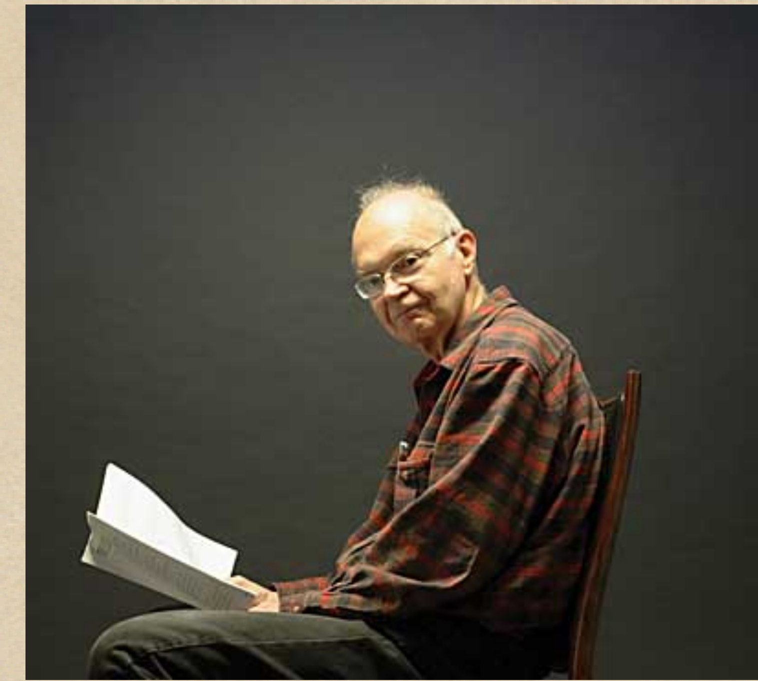
Primitive operations
for “dictionnaries” data structure:
add or delete any elements, asking questions (with positive or negative answer)



number of choices for each
primitive operations

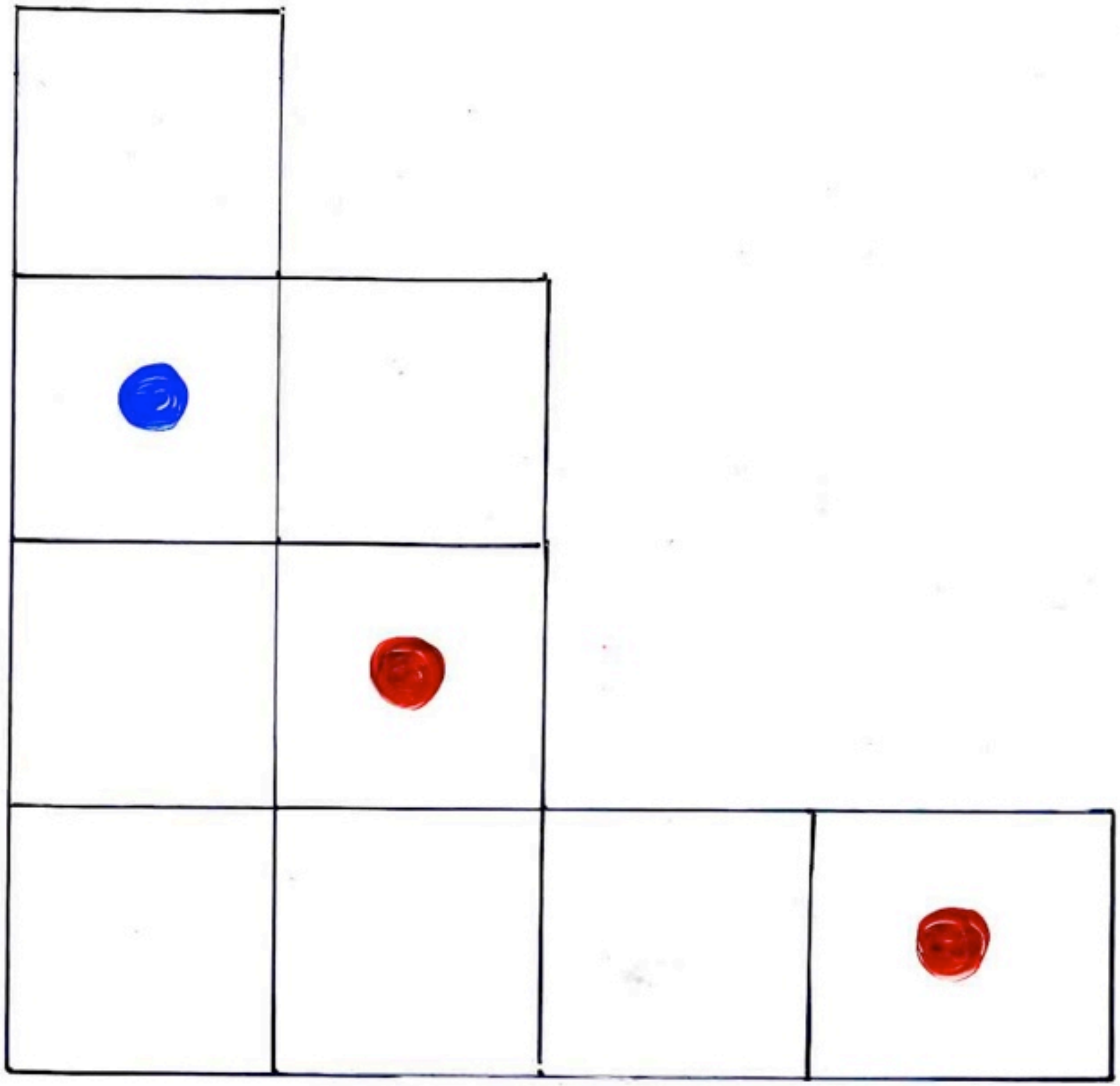
$$\begin{cases} D = A + I_- \\ E = S + I_+ \end{cases}$$

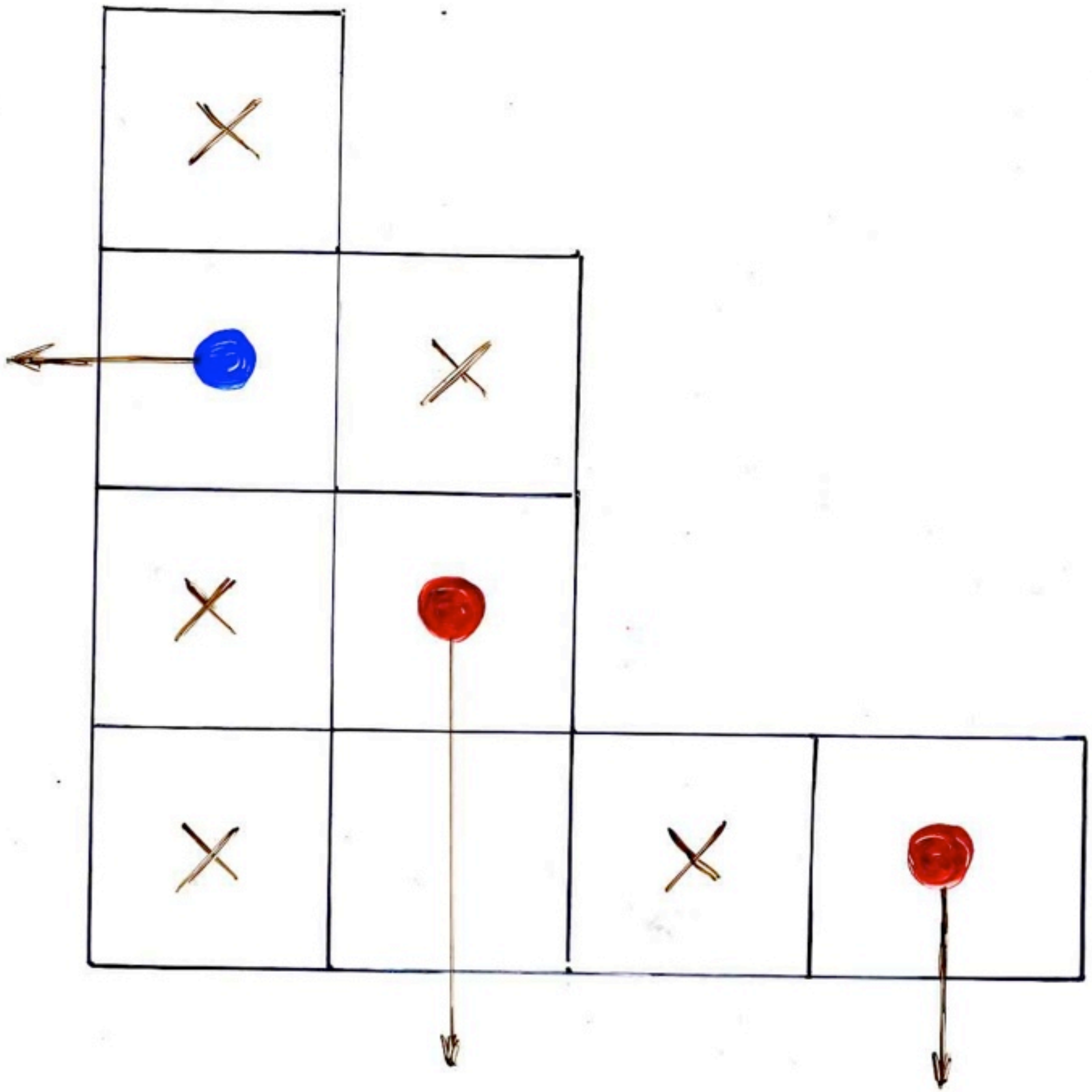
data structure
integrated cost

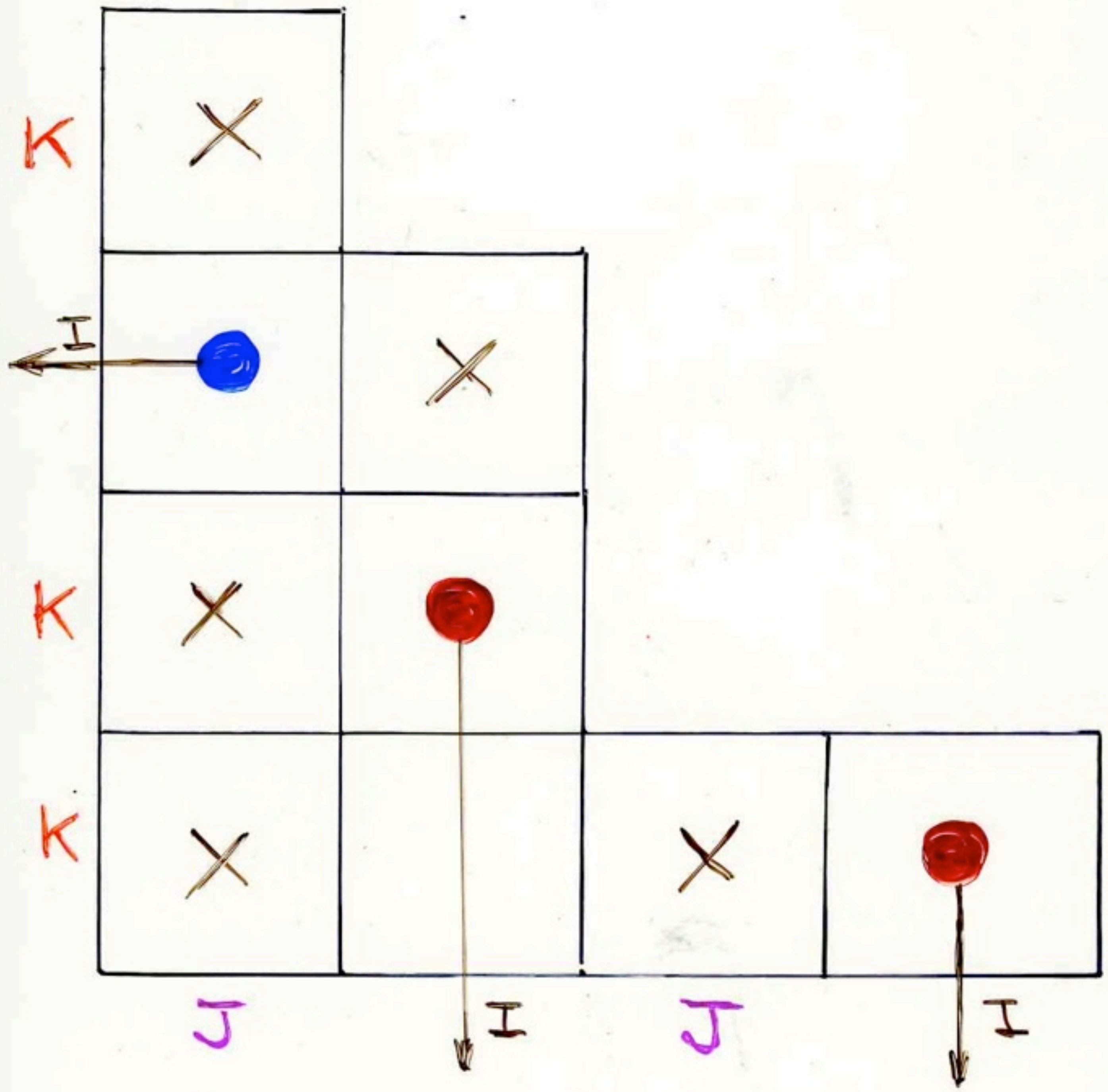


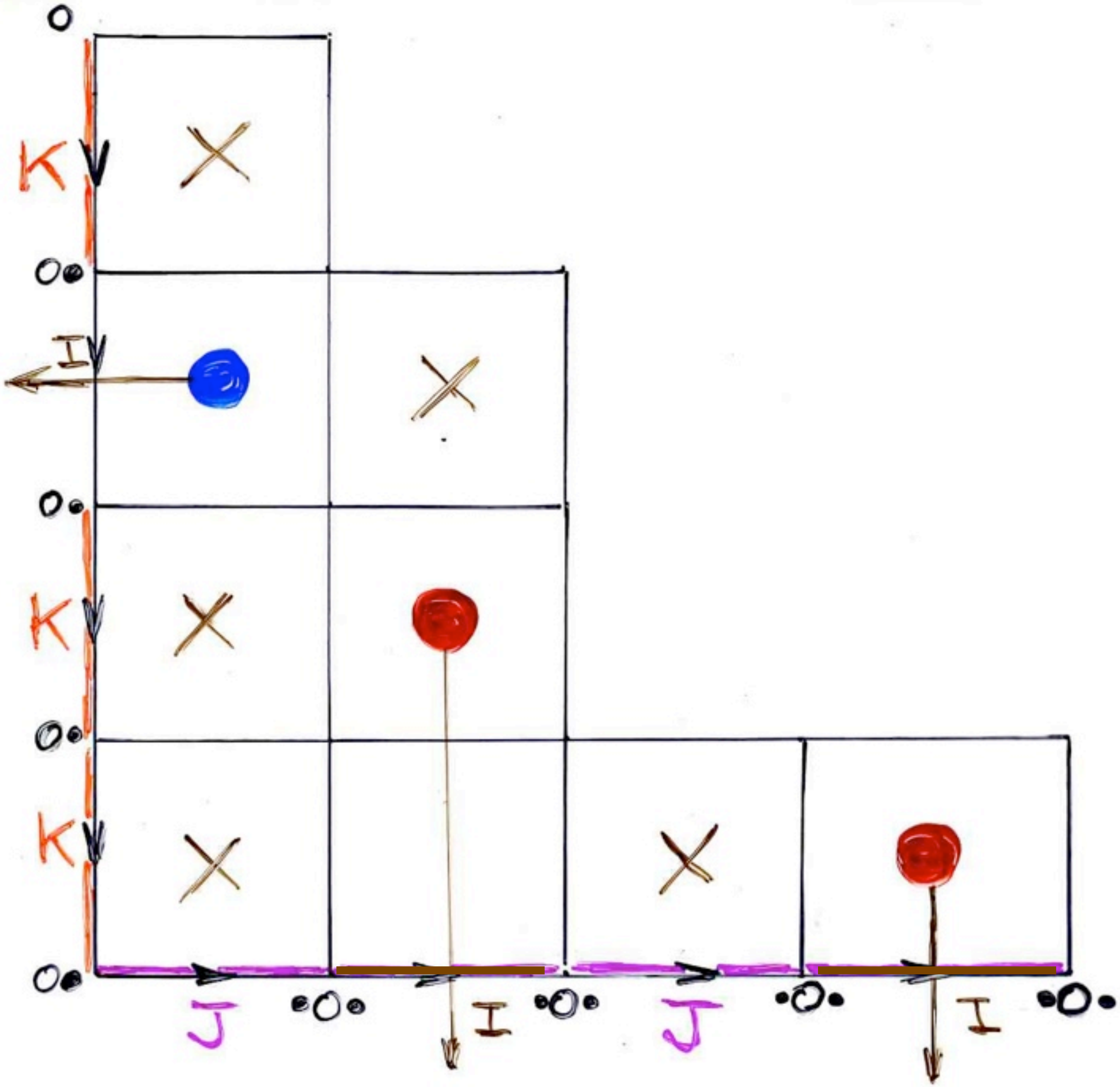
D. Knuth

P.Flajolet

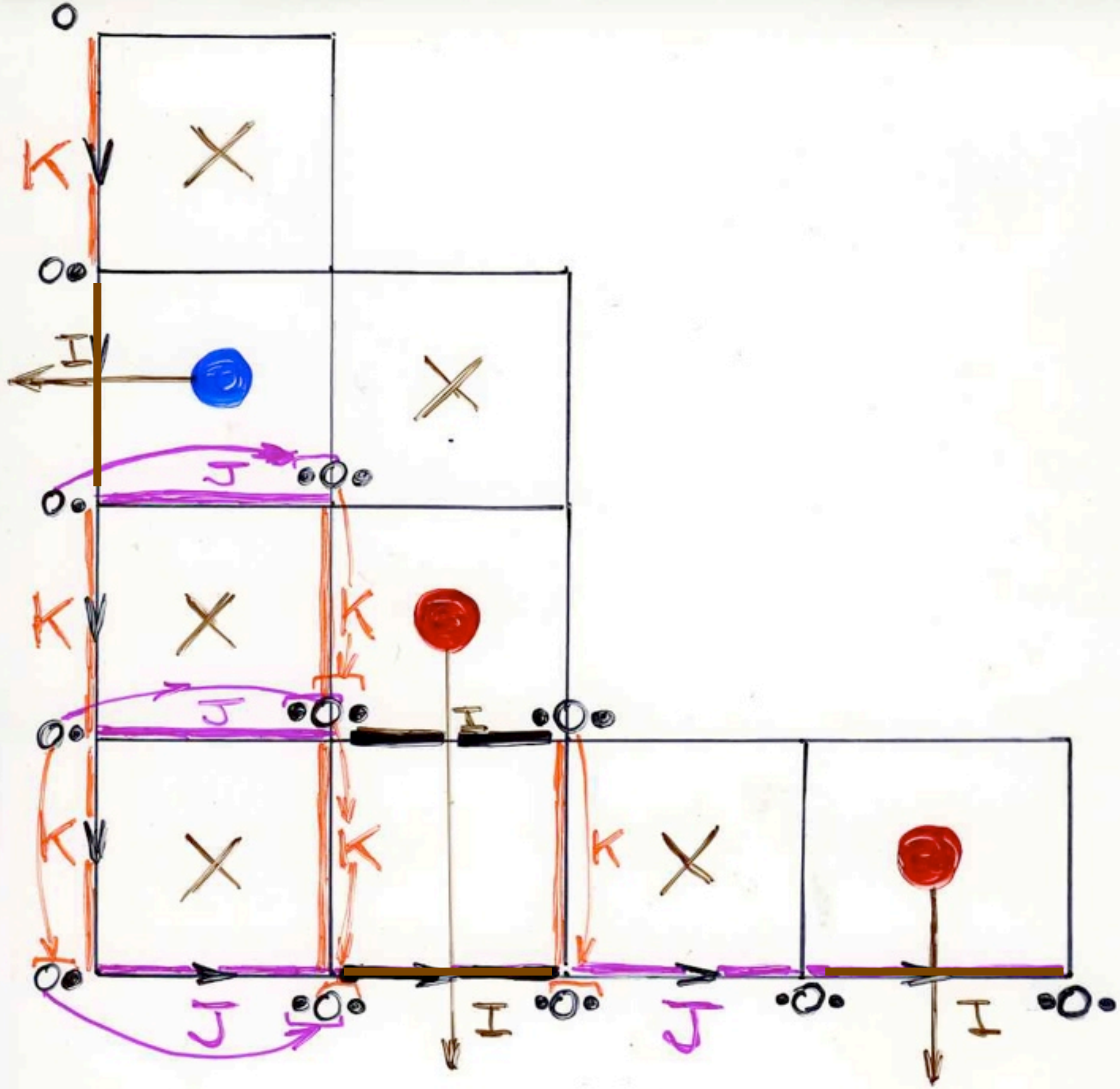





K
I
K
K
J
I
J
I



K
I
K
K
J
I
J
I



K
I
J
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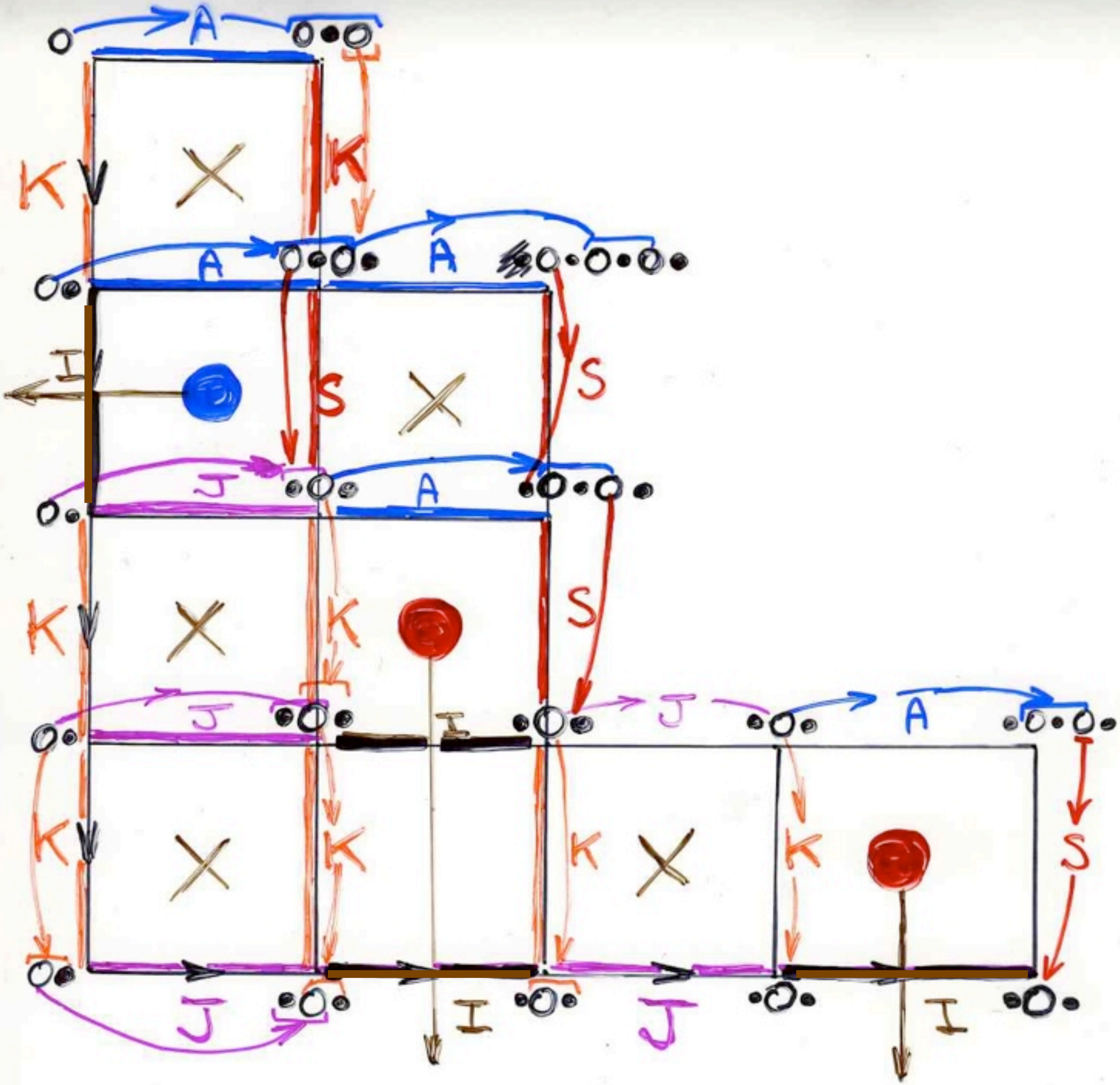
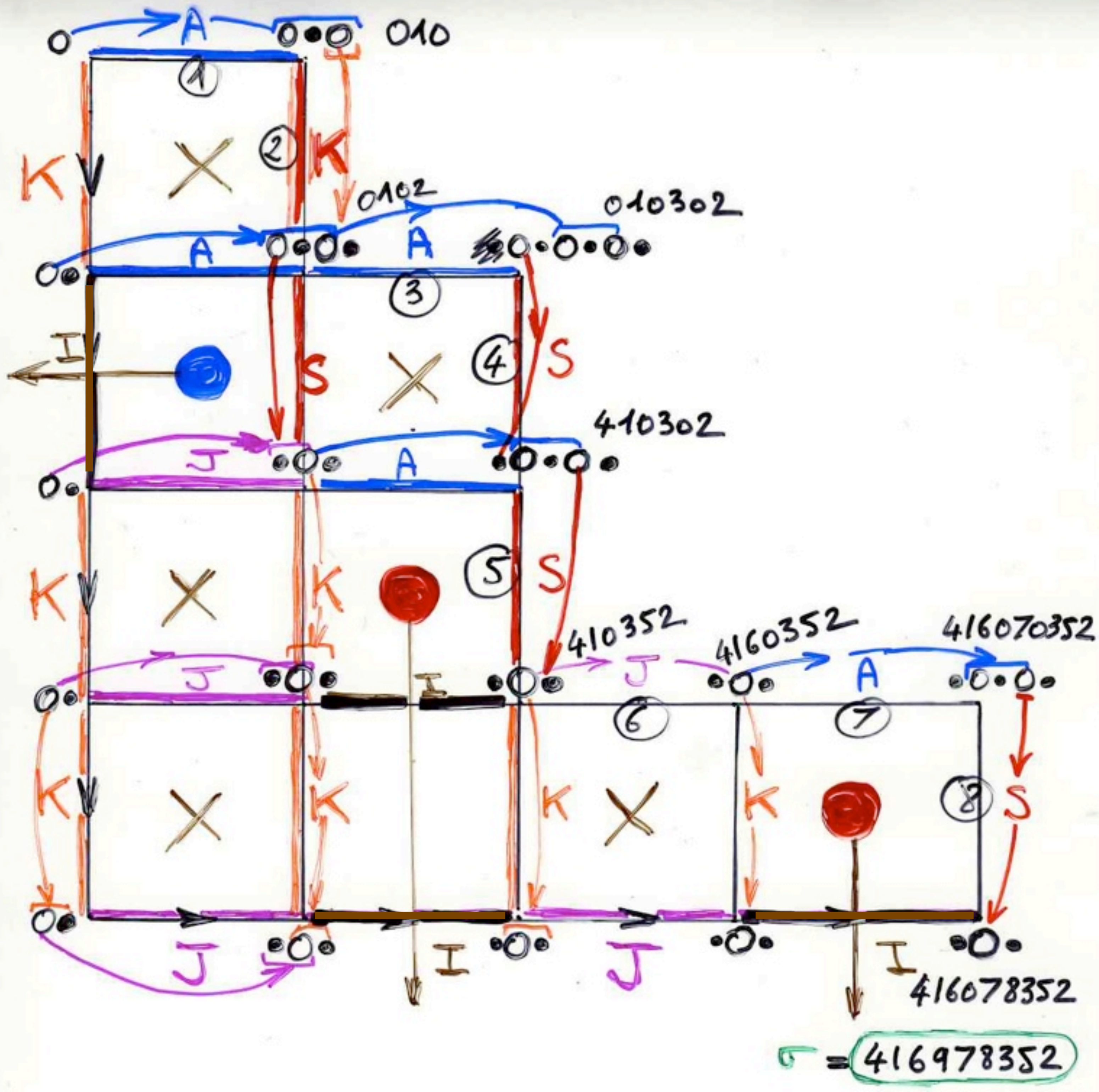
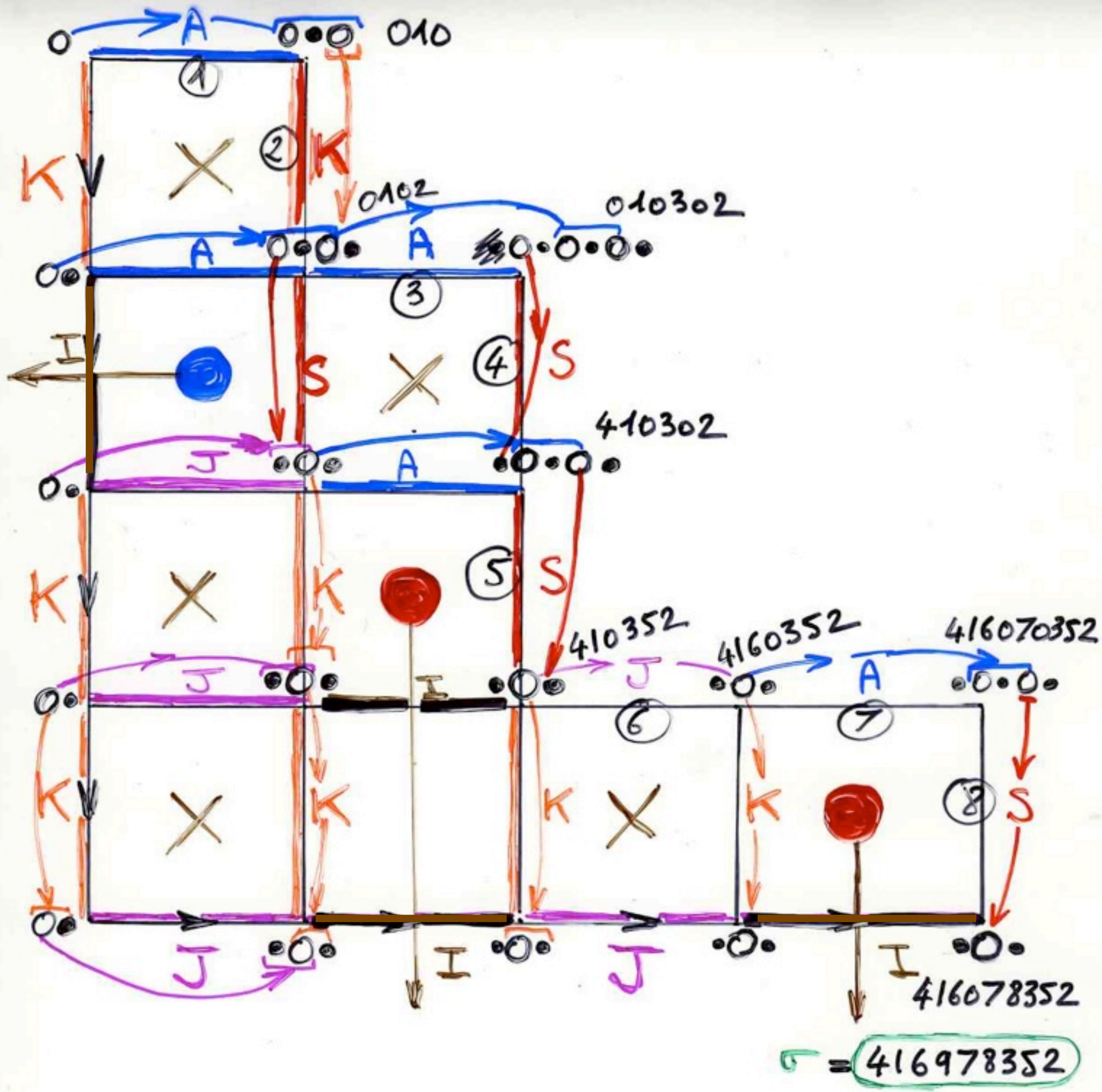
I


A
K
A
A
S
S
I
J
A
K
K
S
J
J
A
K
K
K
K
S
J
I
J
I


A
010
K
0102
010302
A
A
I
S
S
410302
J
A
K
K
S
410352
4160352
416070352
J
J
A
I
S
K
K
K
K
K
J
I
J
I
416078352
σ = 416978352



010
0102
010302
410302
410352
4160352
416070352
416078352
σ = 416978352

"The cellular Ansatz"

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Physics

data structures
"histories"

orthogonal
polynomials

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planarization

(formal) orthogonal polynomials

Orthogonal polynomials

Def. $\{P_n(x)\}_{n\geq 0}$ $\quad P_n(x) \in \mathbb{K}[x]$

orthogonal iff $\exists\ f : \mathbb{K}[x] \longrightarrow \mathbb{K}$

linear functional

$$\begin{cases} \text{(i)} & \deg\left(P_n(x)\right) = n & (\forall n \geq 0) \\ \text{(ii)} & f\left(P_k P_\ell\right) = 0 & \text{for } k \neq \ell \geq 0 \\ \text{(iii)} & f\left(P_k^2\right) \neq 0 & \text{for } k \geq 0 \end{cases}$$

$$\mathcal{f}(x^n) = \mu_n \qquad (n \geqslant 0)$$

moments

$$\mathcal{f}(PQ) = \int_a^b P(x)\,Q(x)\,d\mu$$

mesure

Thm. (Favard)

- $\{P_n(x)\}_{n\geq 0}$ sequence of monic polynomials, $\deg(P_n)=n$
- $\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ coeff. in $\mathbb{K}$

orthogonality $\Longleftrightarrow$

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$(\forall k \geq 1)$

3 terms linear recurrence relation

combinatorial interpretation of the moments

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

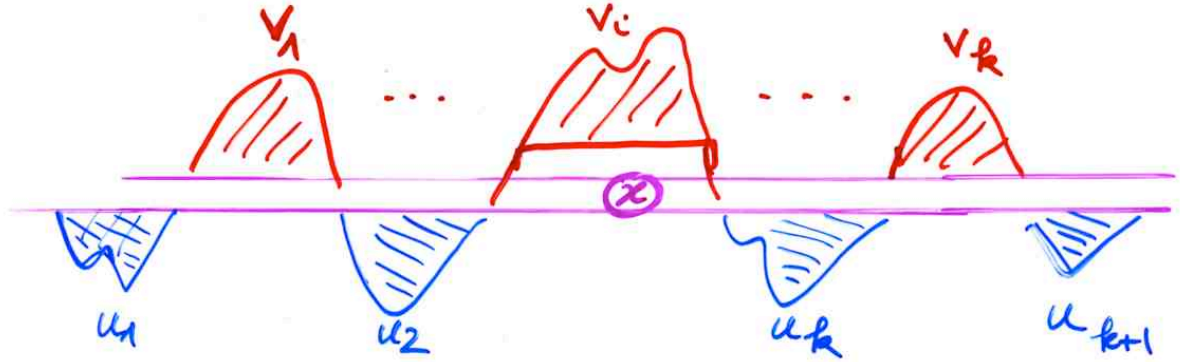
$$P_0 = 1 \qquad P_1 = x - b_0$$

$$\mu_n = (n+1)!$$

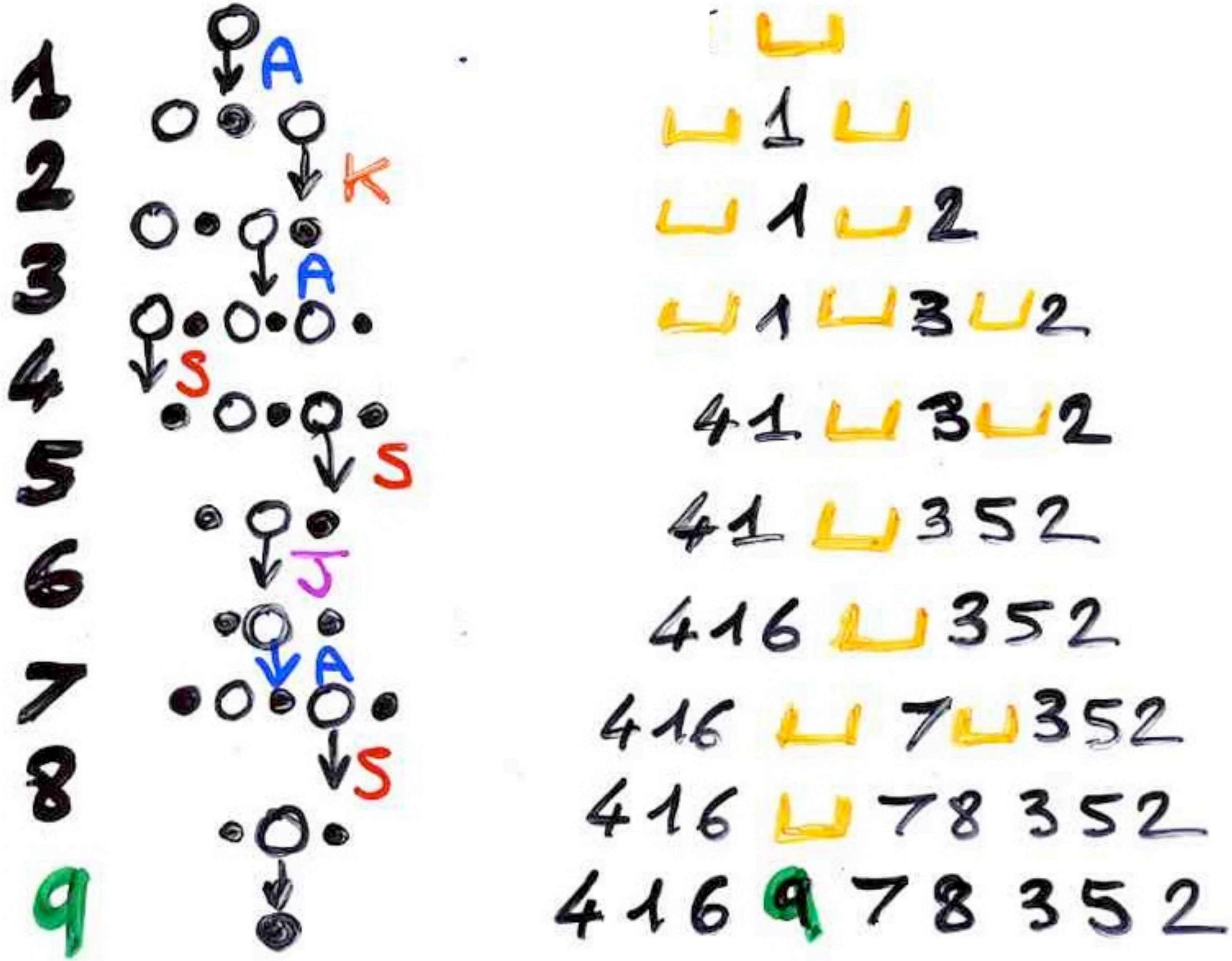
$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

Laguerre polynomial

$$J(t) = \cfrac{1}{1 - 2t - \cfrac{1 \cdot 2t^2}{1 - 4t - \cfrac{2 \cdot 3t^2}{\dots}}}$$



v1
vi
vk
x
u1
u2
uk
uk+1



1
2
3
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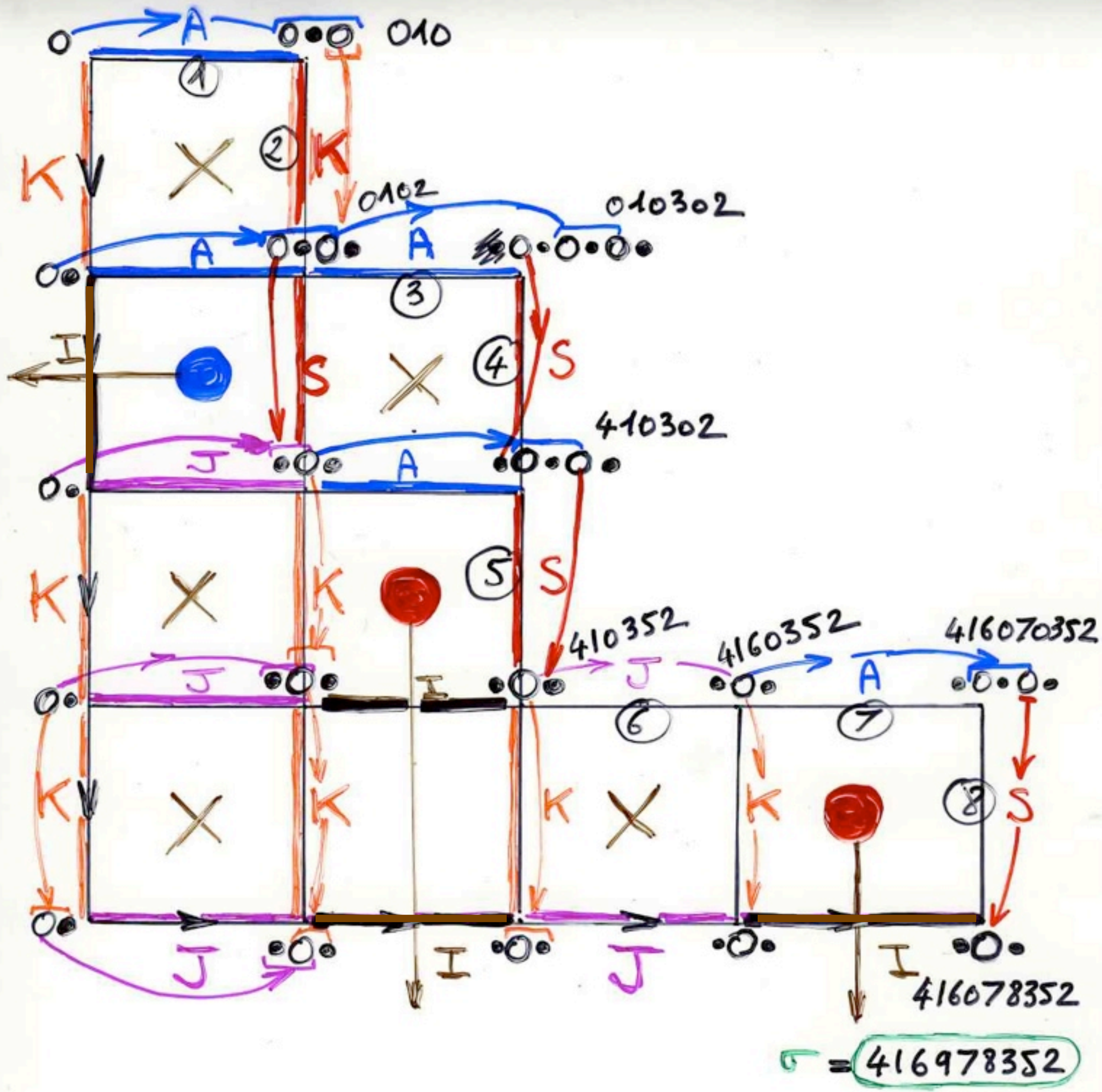
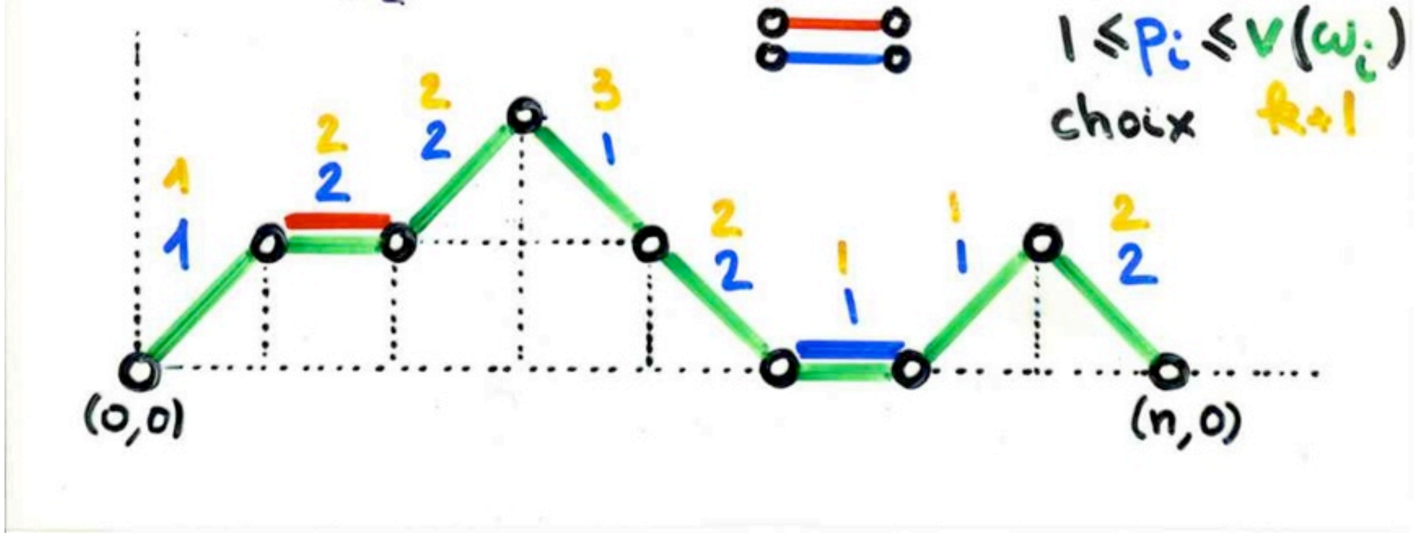
5
6
7
8
9
A
K
A
S
S
J
A
S
1
1 2
1 3 2
4 1 3 2
4 1 3 5 2
4 1 6 3 5 2
4 1 6 7 3 5 2
4 1 6 7 8 3 5 2
4 1 6 9 7 8 3 5 2



010
0102
010302
410302
410352
4160352
416070352
416078352
σ = 416978352

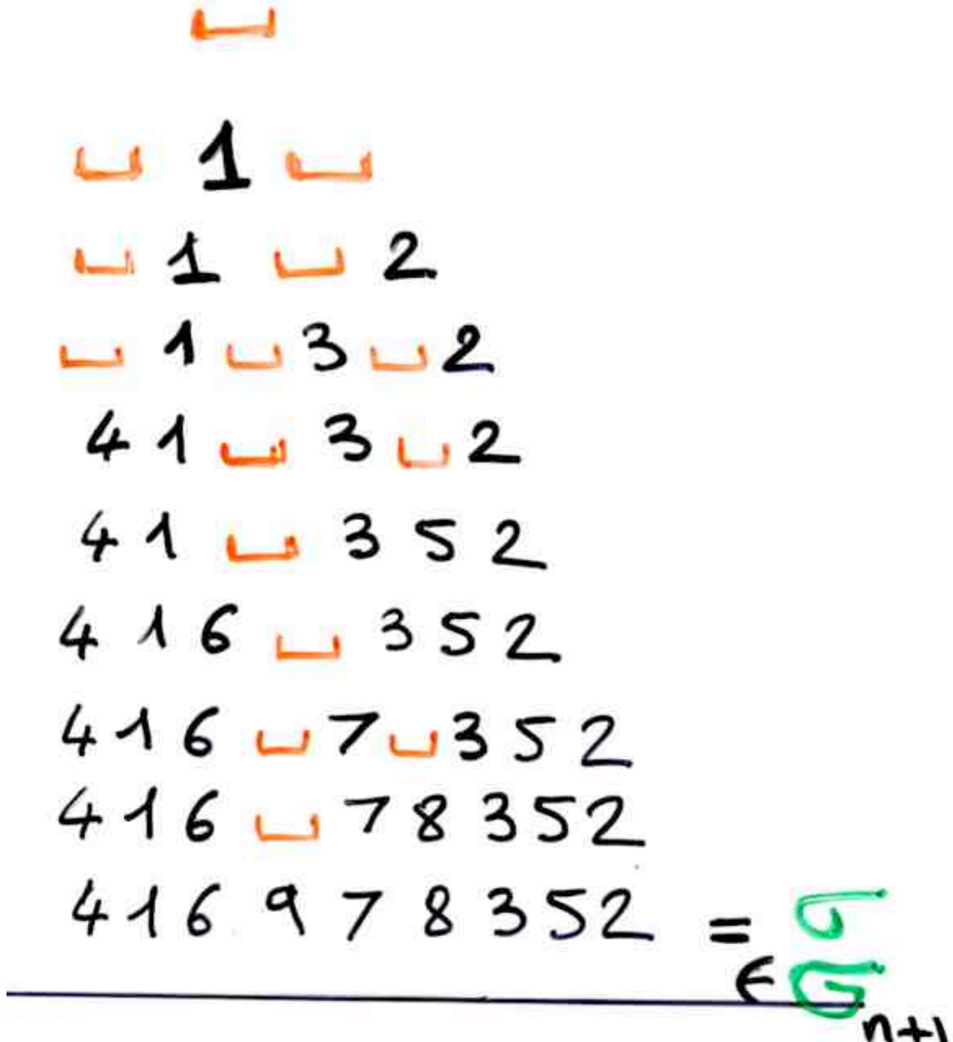
"q-analogue" of Laguerre histories



choices function

1	2	3	4	5	6	7	8
1	2	2	1	2	1	1	2
0	1	1	0	1	0	0	1

q-Laguerre : q^4



q-Laguerre

$L_n^{I\,\beta}(x;q)$

$\beta = \alpha + 1$

$$\begin{cases} b_{k,q}^{(\beta)} = [k]_q + [k+1;\beta]_q \\ \lambda_{k,q}^{(\beta)} = [k]_q \cdot [k;\beta]_q \end{cases}$$

$$[k;\beta]_q = \beta + q + q^2 + \cdots + q^{k-1}$$

$$\mu_n = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\binom{2n}{n-k} - \binom{2n}{n-k-2} \right) \left(\sum_{i=0}^{k} q^{i(k+1-i)} \right) y$$

Corteel, Josuat-Vergès, Prellberg, Rubey (2008)

general PASEP

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\,\hat{a} = 1$$

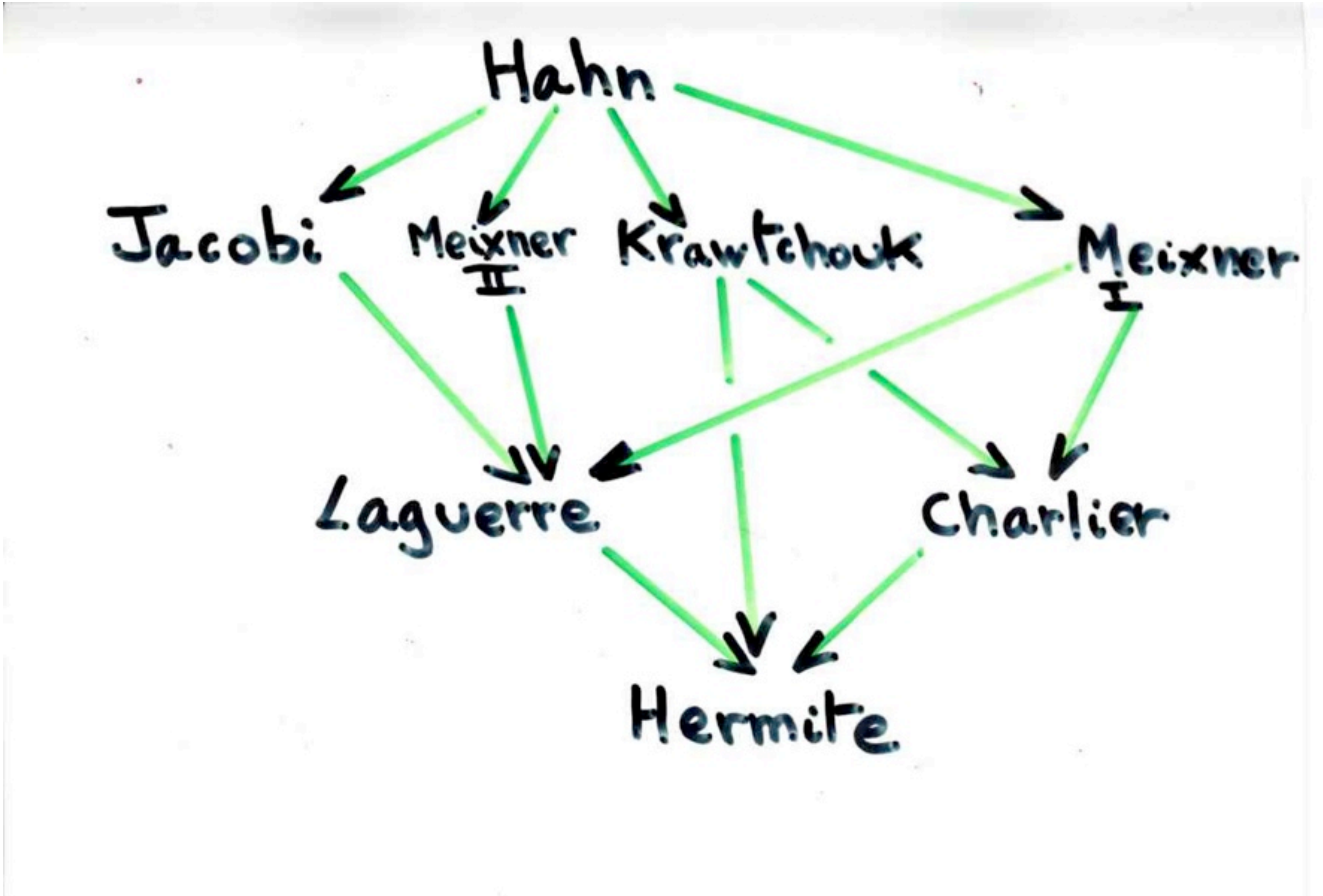
→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

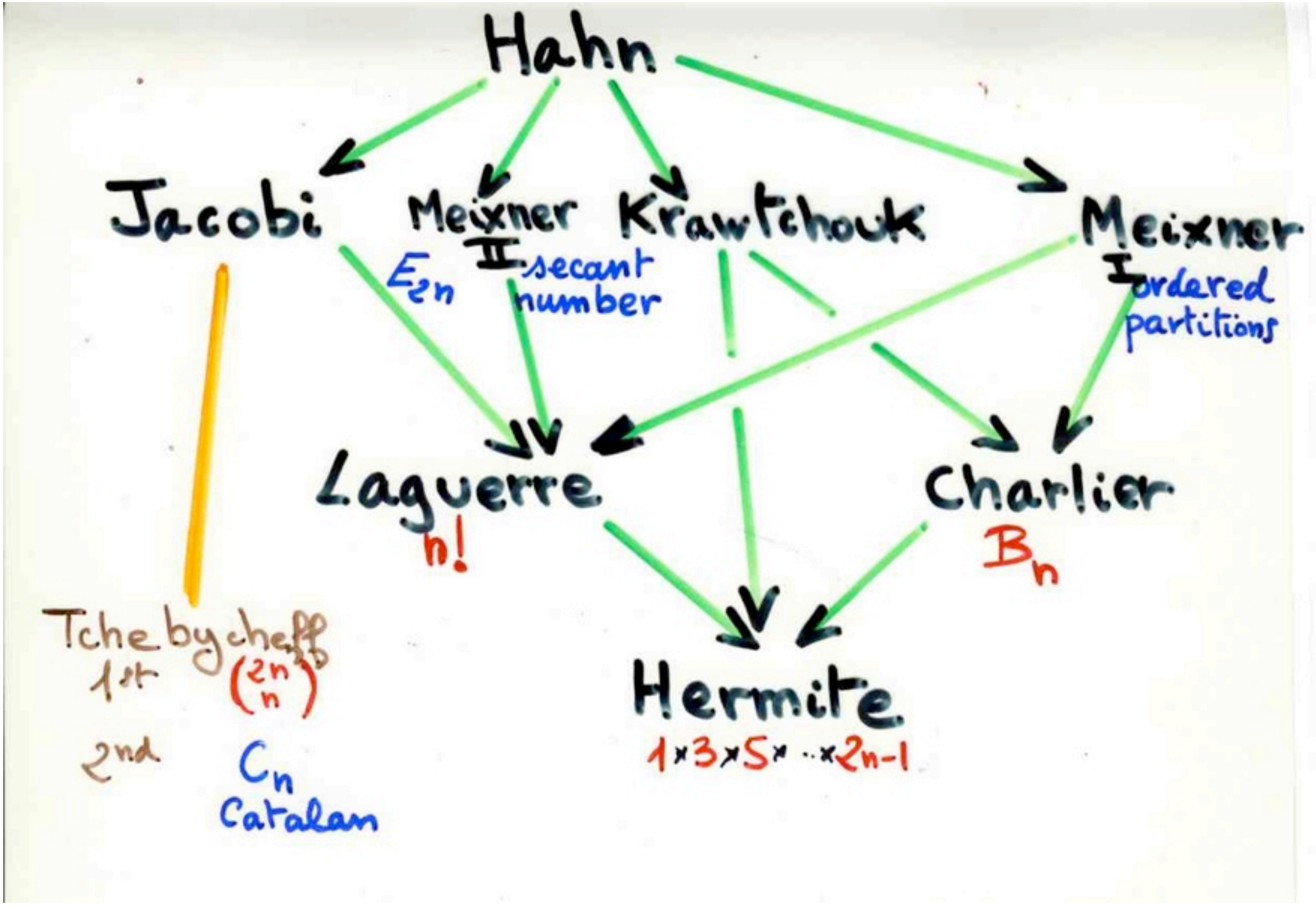
Askey-Wilson polynomials

Askey tableau

Askey-Wilson



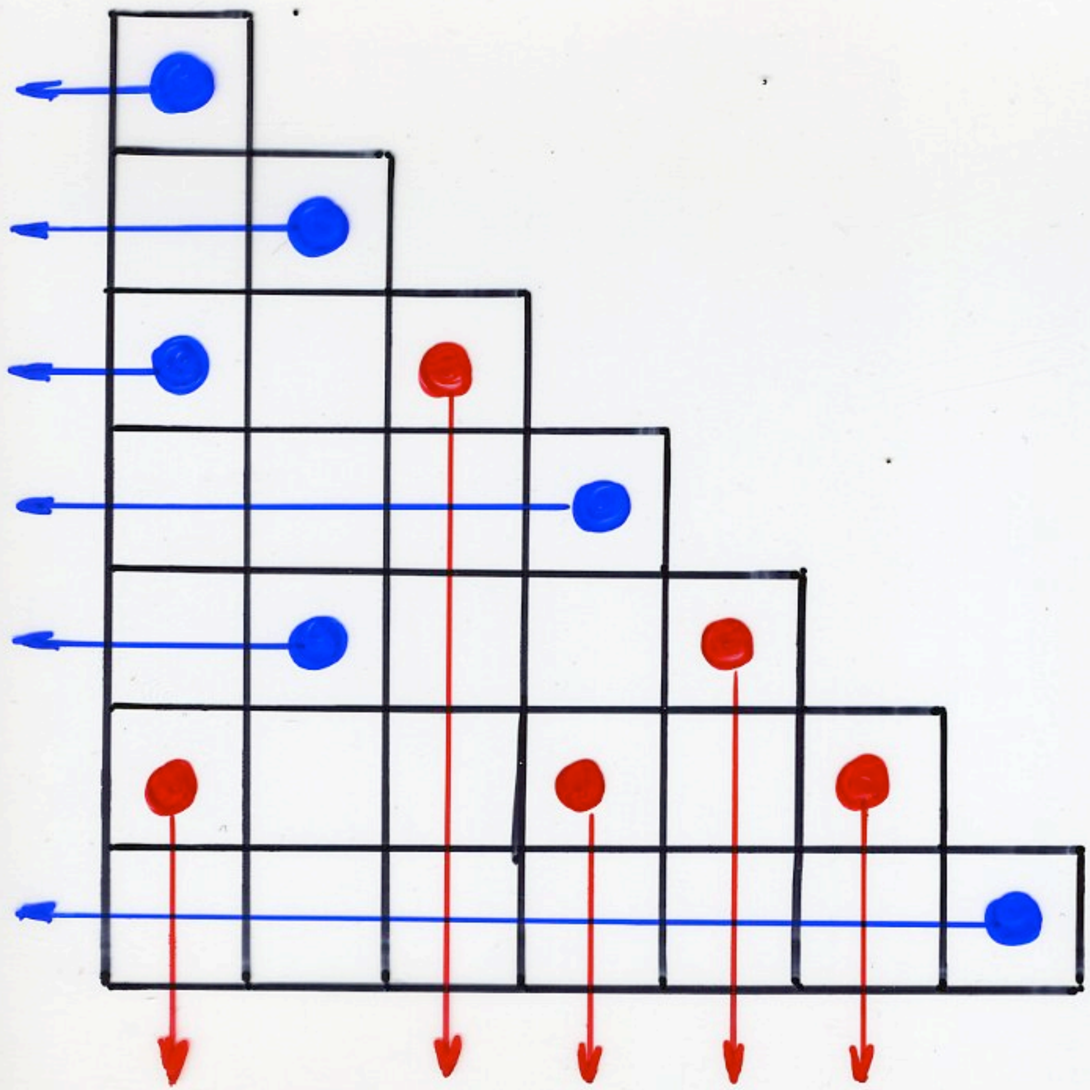
Askey-Wilson

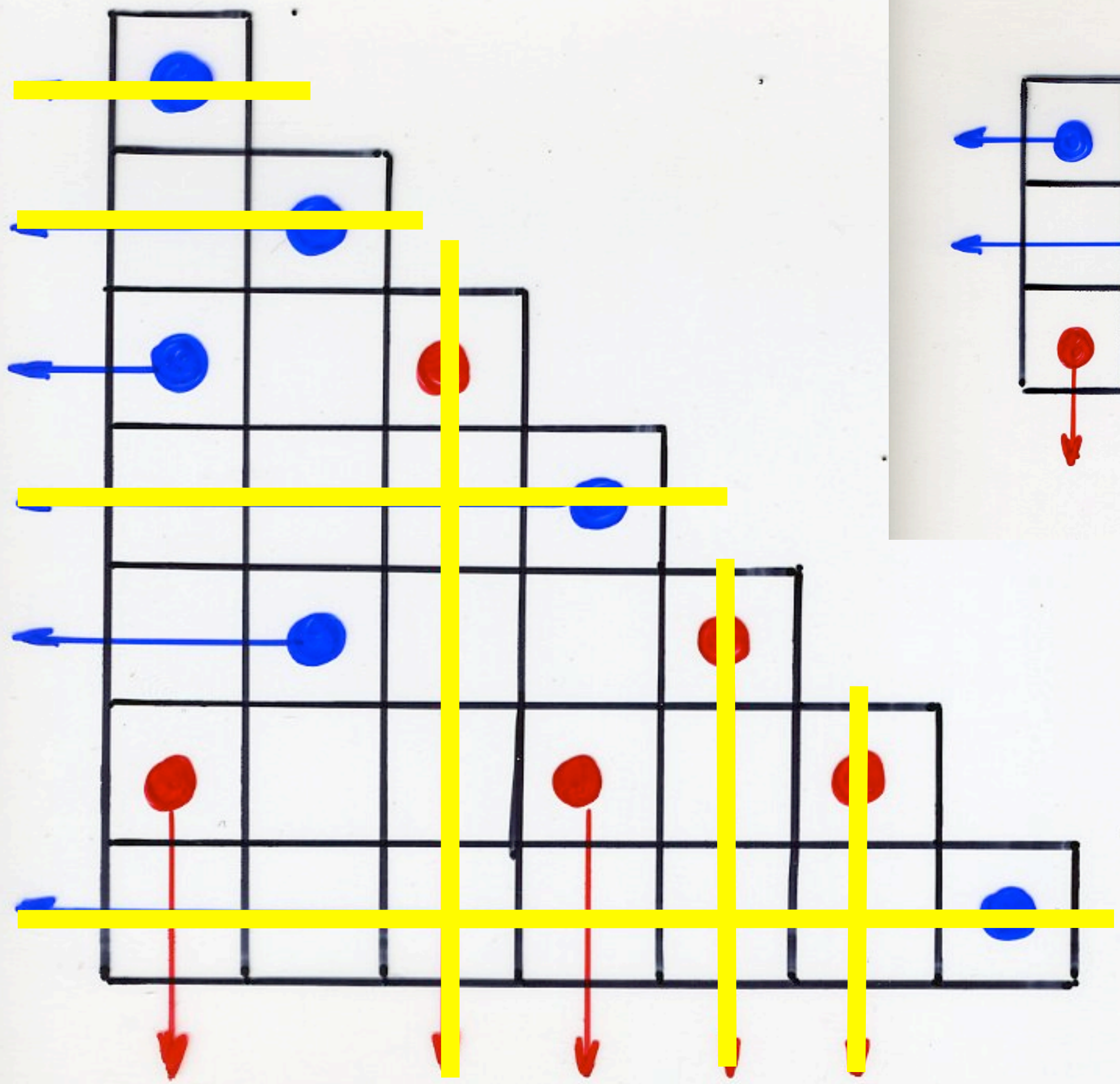


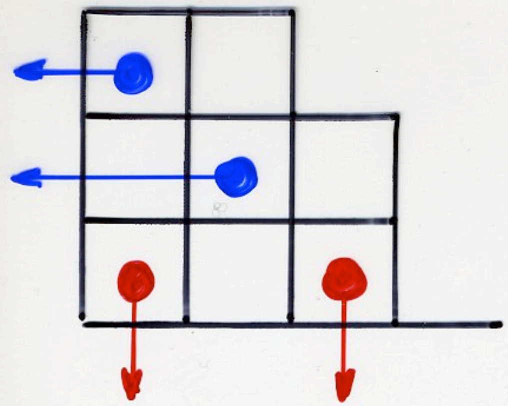
staircase tableaux

Corteel, Williams, 2009

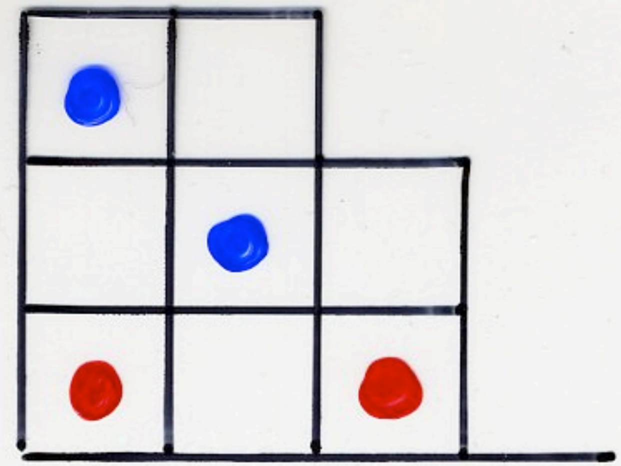
Corteel, Stanley, Stanton, Williams, 2010



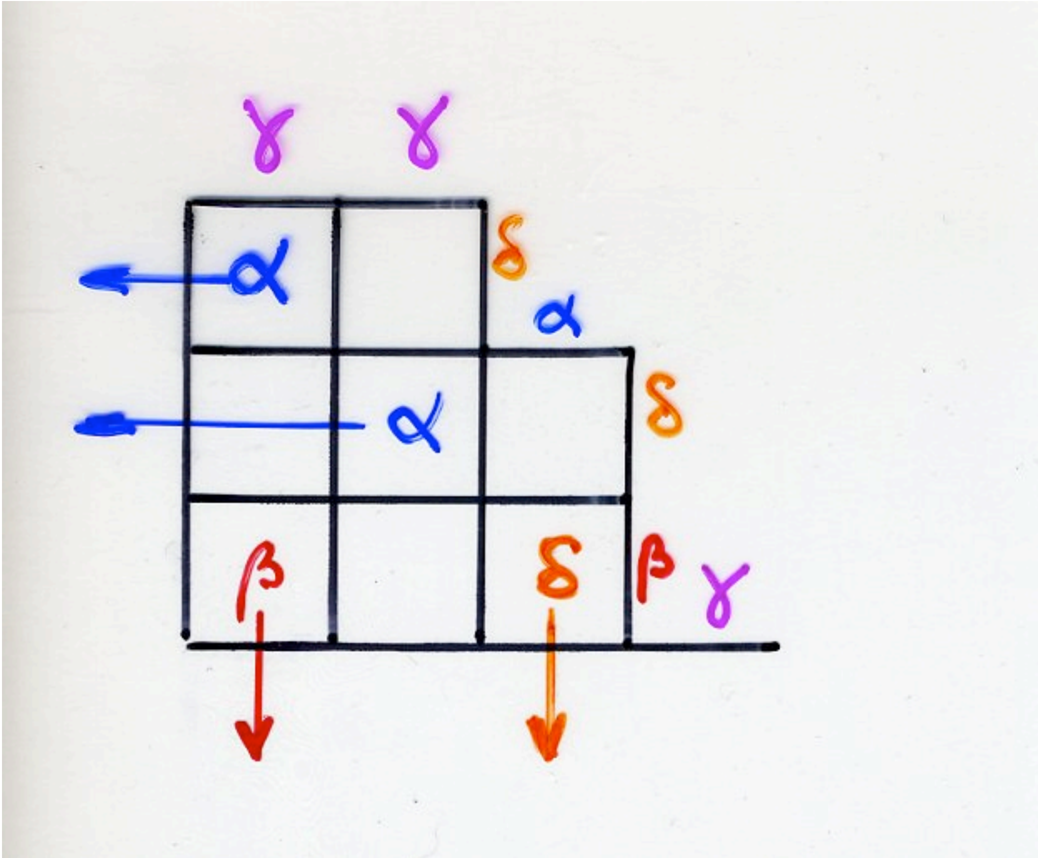




nb of 2-colored alternative tableaux $= 2^n \cdot n!$

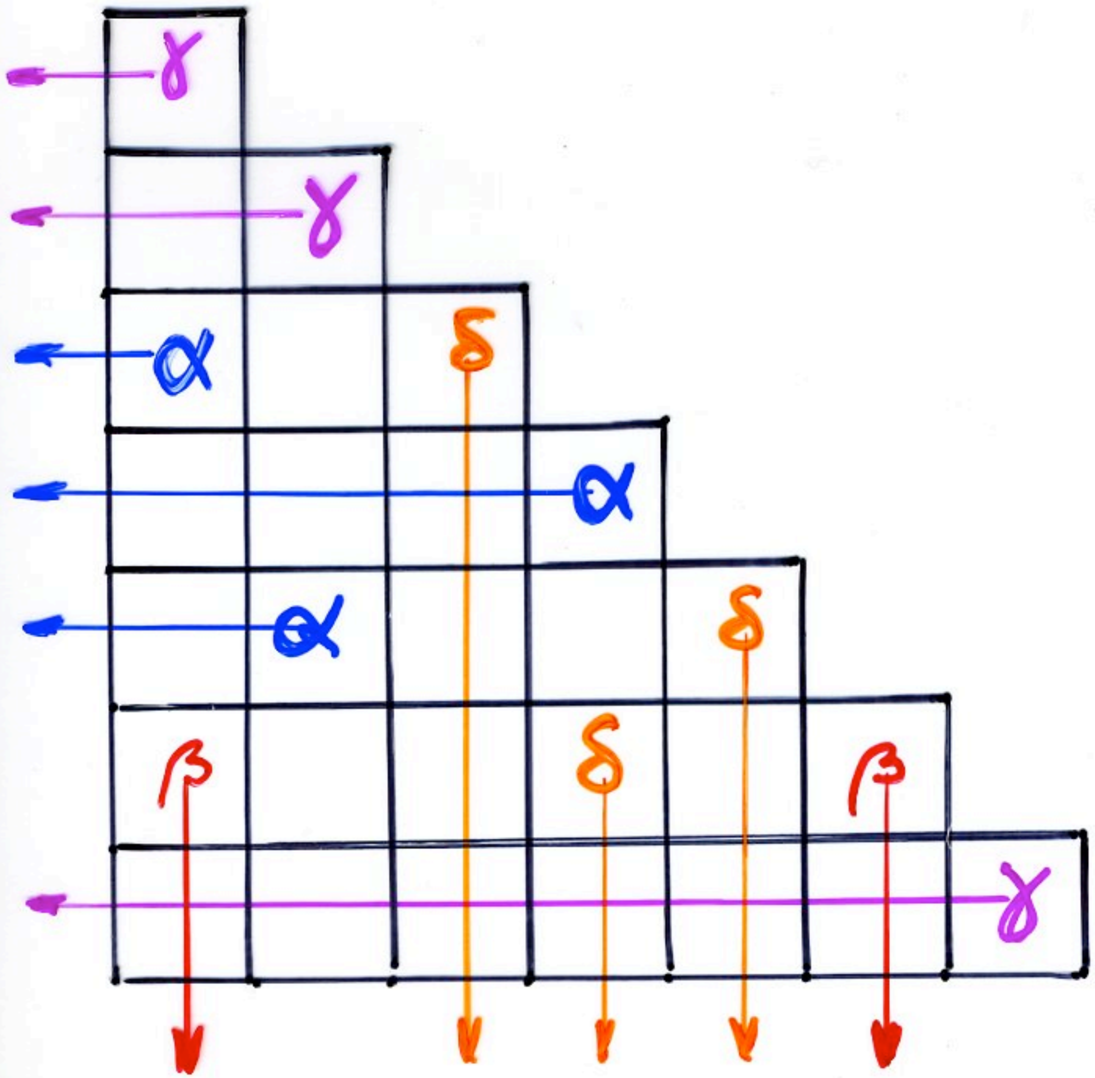


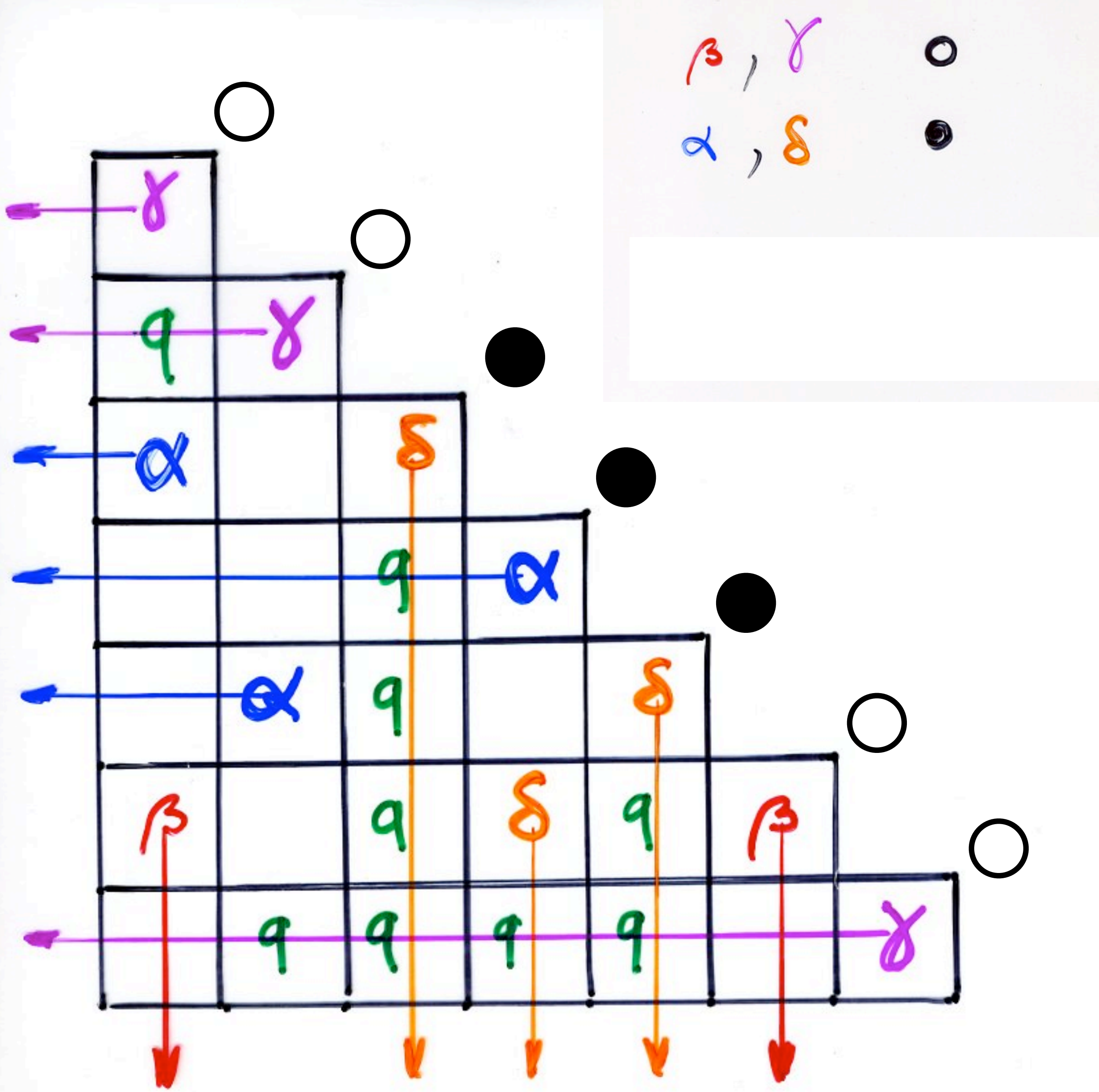
nb of 2-colored alternative tableaux $= 2^n \cdot n!$

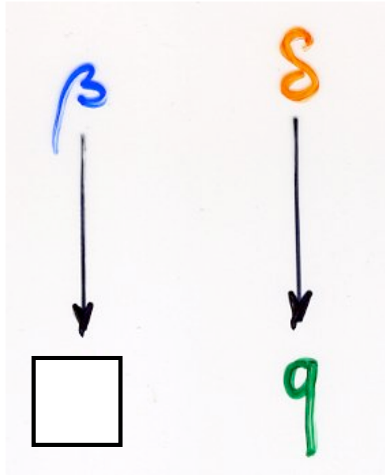


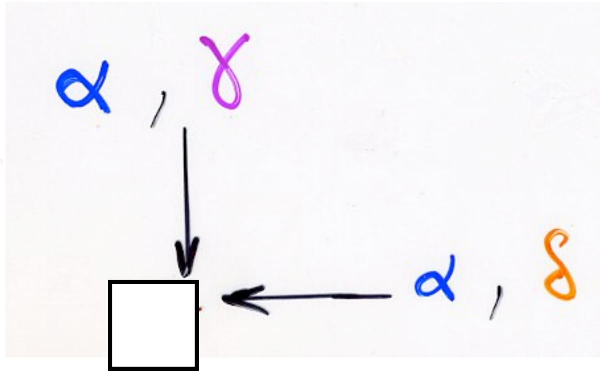
nb of staircase tableaux $= 4^n \cdot n!$

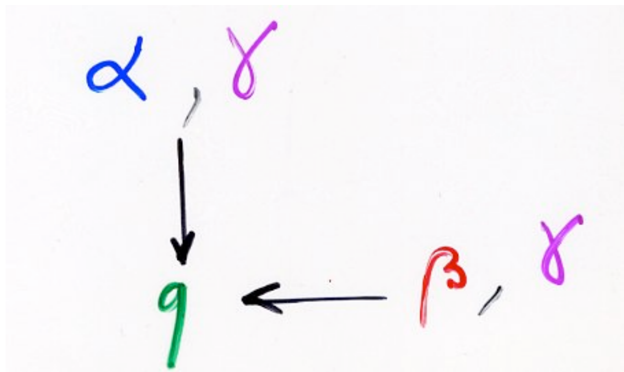
staircase tableaux




γ
q γ
α δ
q α
α q δ
β q δ q β
q q q q γ
β , γ
α , δ


β
δ
q


α , γ
α , δ


α , γ
q
β , γ

steady state probability PASEP

$$\frac{1}{Z_n} Z_\tau(\alpha, \beta, \gamma, \delta ; q)$$

$$Z_n = \sum_\tau Z_\tau$$

$\tau = (\tau_1, \dots, \tau_n)$ state

relation with moments of Askey-Wilson polynomials

Corteel, Williams, 2009

Corteel, Stanley, Stanton, Williams, 2010

The cellular Ansatz

From quadratic algebra Q
to combinatorial objects (Q-tableaux)
and bijections

"The cellular Ansatz"

combinatorial objects on a 2d lattice

representation by operators

Physics

data structures "histories"

bijections

orthogonal polynomials

"normal ordering"

UD = DU + Id

Weyl-Heisenberg

DE = qED + E + D

PASEP

towers placements

permutations ⟷ pairs of Tableaux Young (RSK)

alternative tableaux ⟷ permutations

Laguerre histories

Q-tableaux

ex: ASM, (alternating sign matrices)

FPL (fully packed loops)

tilings, 8-vertex

quadratic algebra Q

?

commutations

rewriting rules

planarisation

planar automata

Koszul algebras

duality

website Xavier Viennot

main website   www.xavierviennot.org

page «exposés»:

secondary website: Courses   cours.xavierviennot.org

- course IIT Bombay 2013 (20 hours)

Thank you !