Heaps of pieces
and
Lorentzian triangulations
in 2D quatum gravity

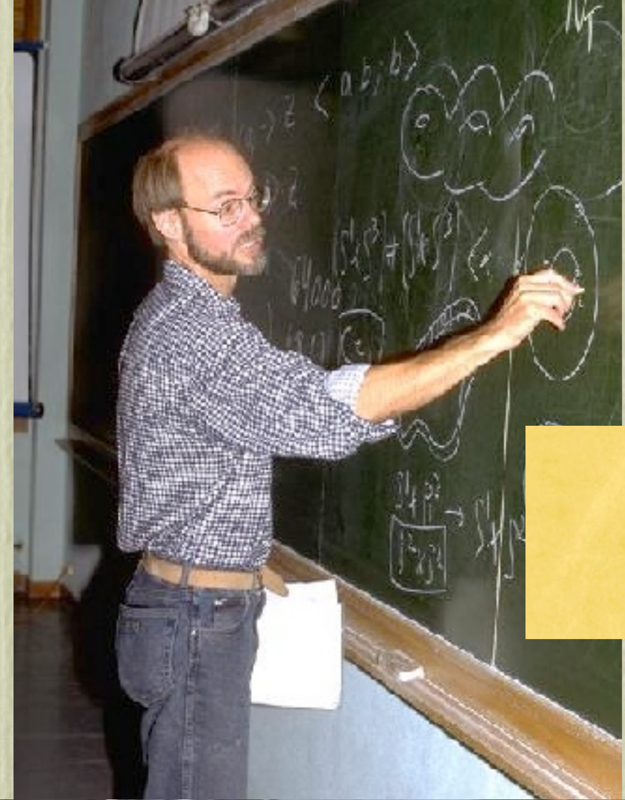
Erwin Schrödinger Institute
Vienna, 20 june 2014

Xavier Viennot
LaBRI, CNRS, Bordeaux

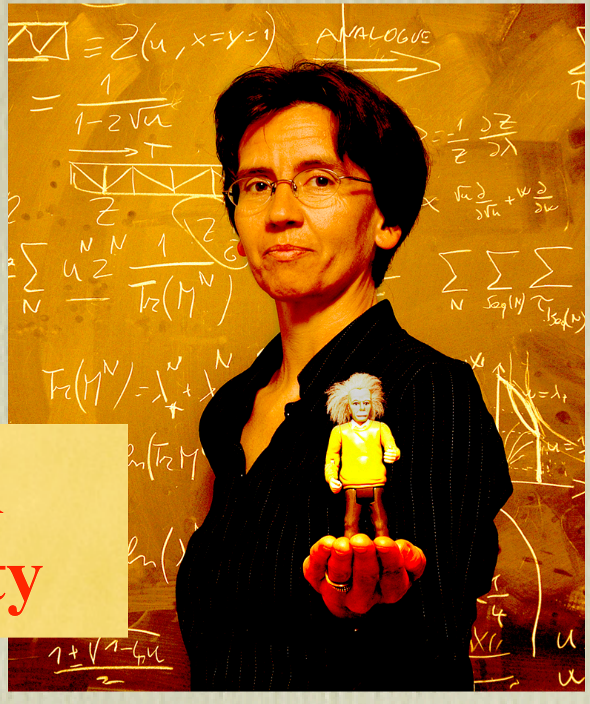
- J. Ambjørn, R. Loll, "Non-perturbative Lorentzian quantum gravity and topology change", Nucl. Phys. B536 (1998) 407-434
  arXiv: hep-th/9805108

- P. Di Francesco, E. Guitter, C. Kristjansen, Integrable 2D Lorentzian gravity and random walks", Nucl. Phys. B567 (2000) 515-553
  arXiv: hep-th/9907084

gravitation quantique



J.Ambjørn



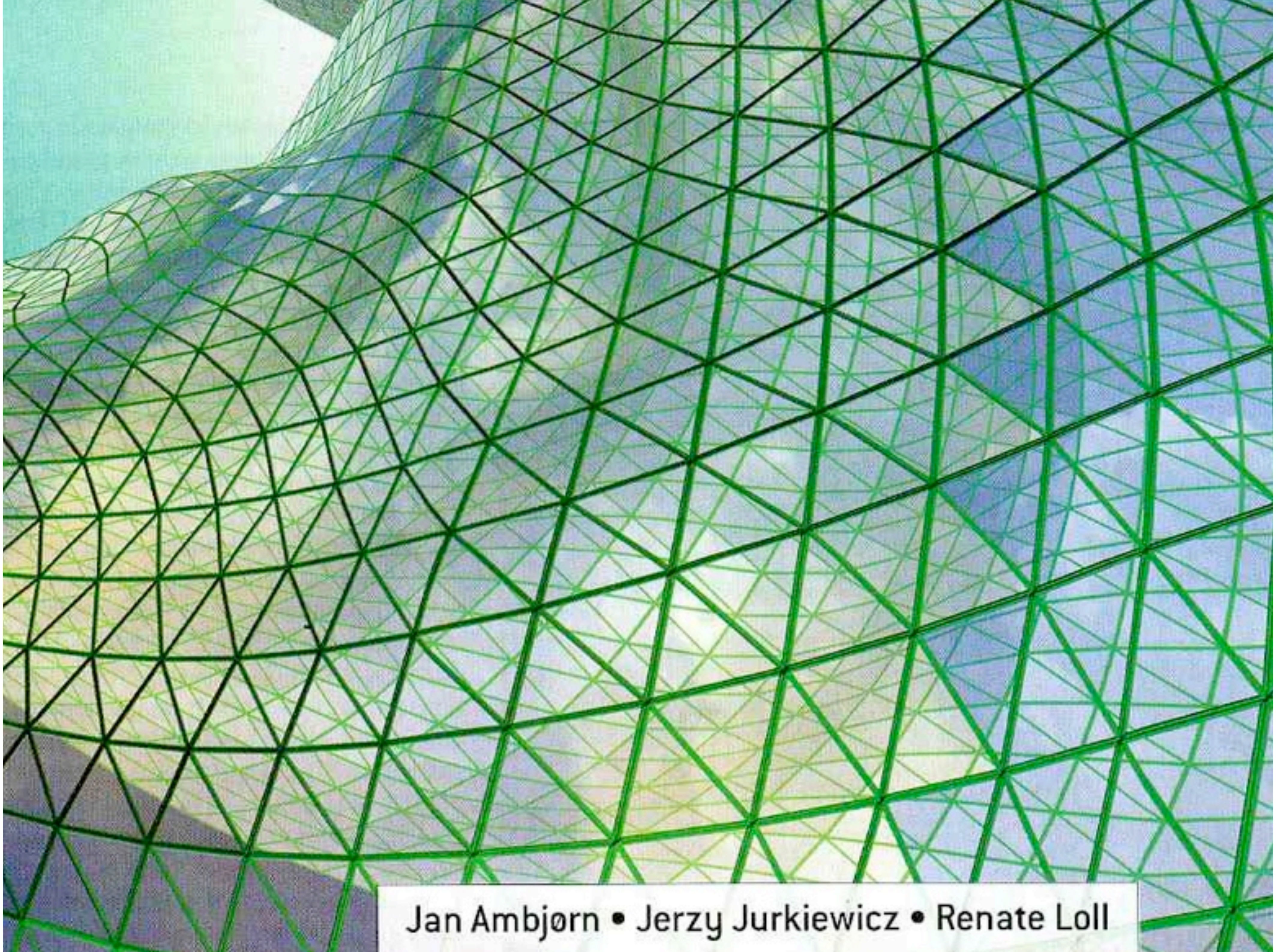
R. Loll

2D Lorentzian quantum gravity

P. Di Francesco

E.Guitter

C. Kristjansen



Jan Ambjørn • Jerzy Jurkiewicz • Renate Loll

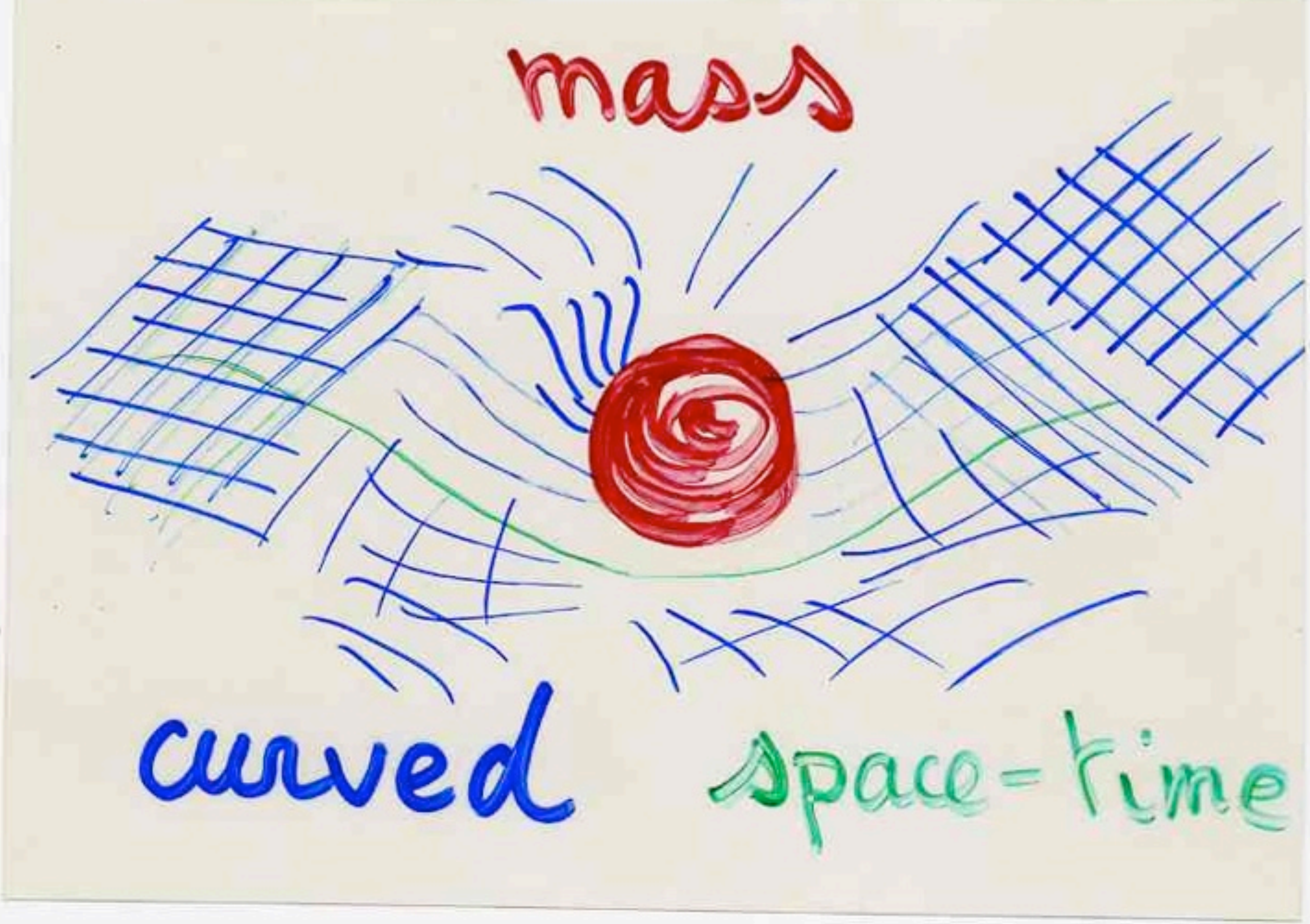

POUR LA
SCIENCE
pourlascience.com
Septembre 2008
Édition française de Scientific American
Le ve
Des algu
• L'Univers quantique auto-organisé
• Que s'est-il passé à Toungouska il y a 100 ans ?
• Comment détecter les images truquées
• D'où viennent les larves ?
M 02687 - 371 - F: 5,95 €
L'univers quantique auto-organisé

Lorentzian
quantum
gravity

$+++-$ (space: $+++$, time: $-$)

2D

$1 + 1$ (space + time)


mass
curved
space-time

.... In quantum gravity we are instructed to sum over all geometries connecting, say, two spatial boundaries of length ℓ_1 and ℓ_2, with the weight of each geometry g given by

(5) $$e^{iS[g]} \qquad S[g] = \Lambda \int \sqrt{-g} \,, \quad (\text{in } 2d)$$

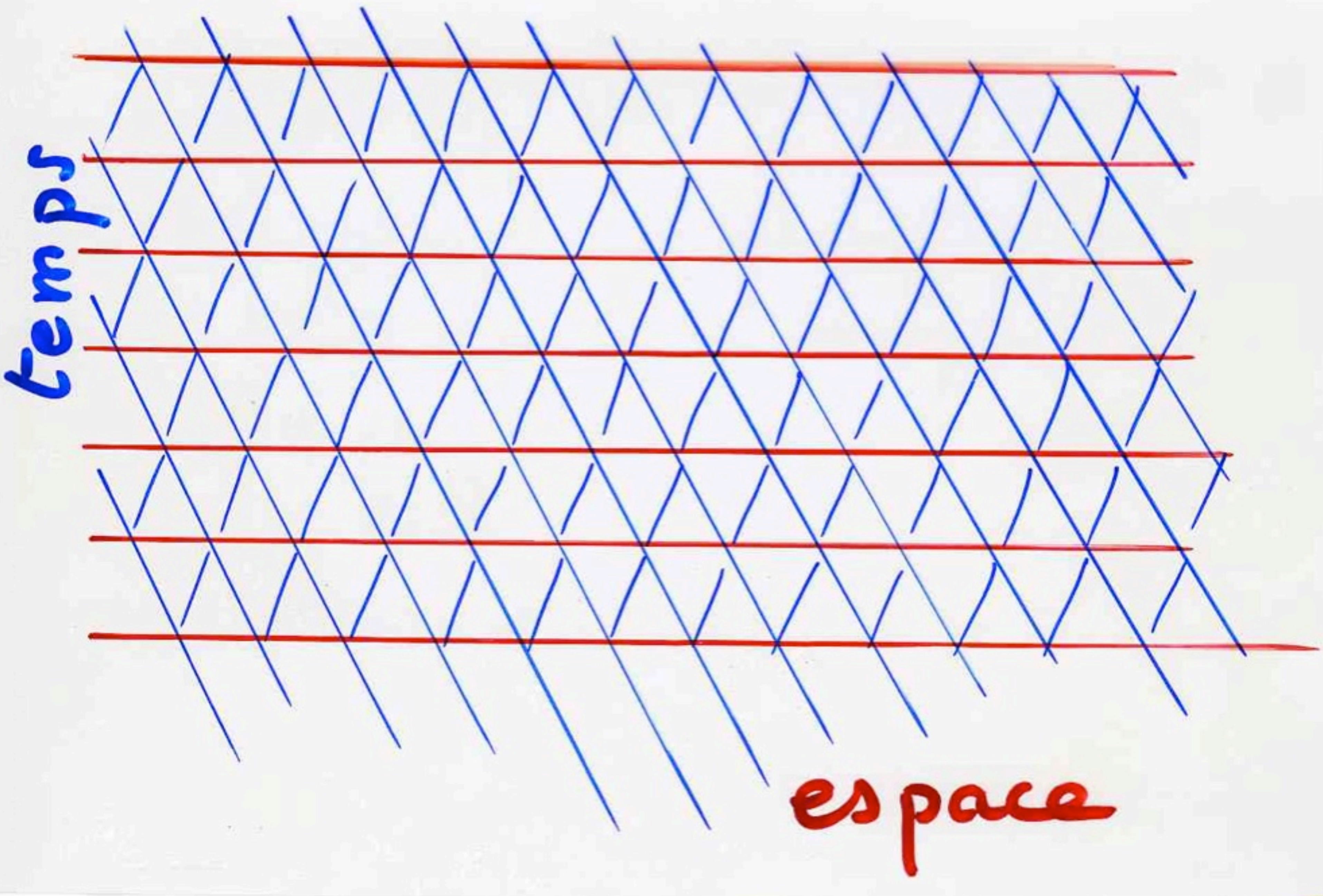
where Λ is the cosmological constant.

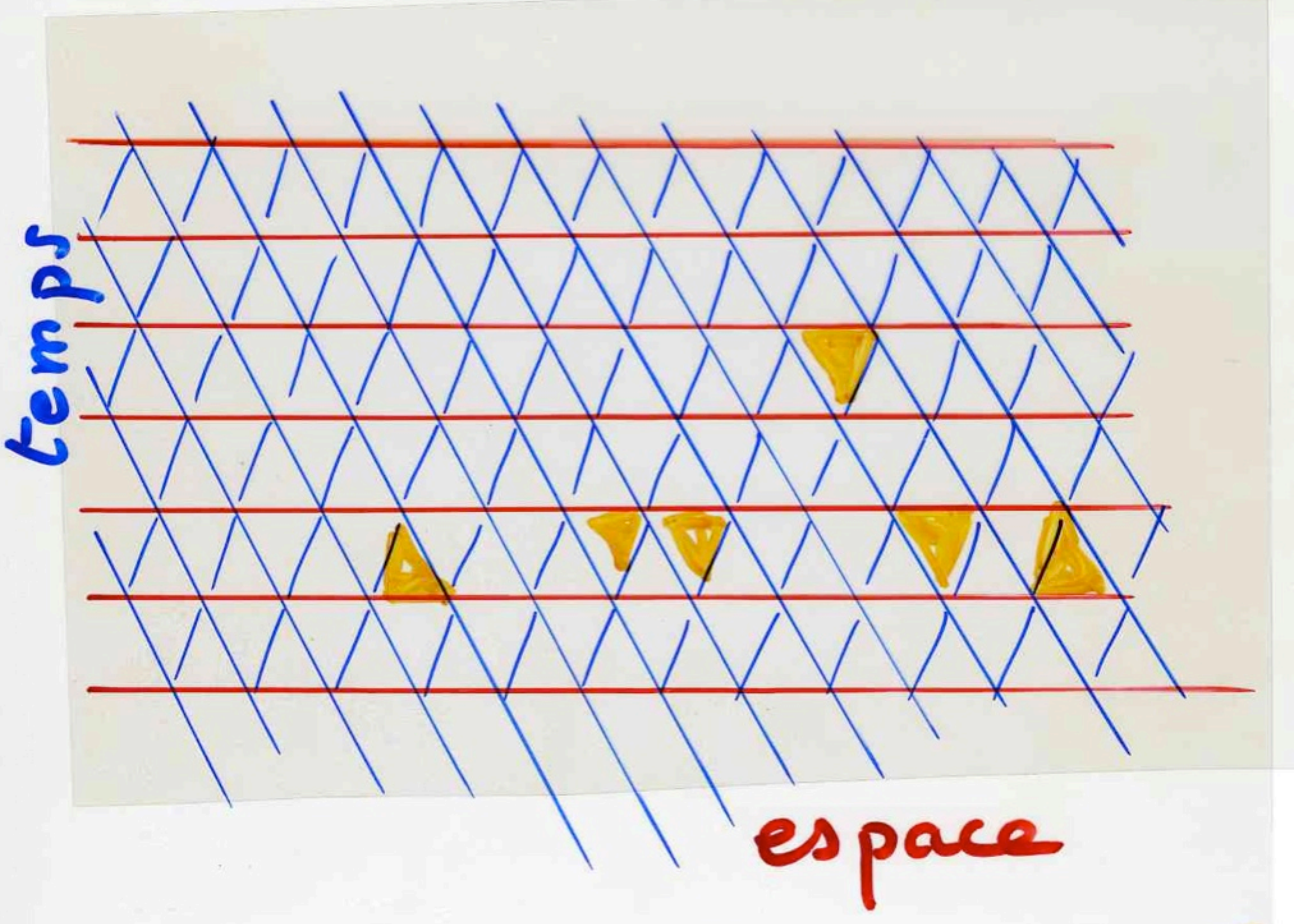
(21) $$F_t(x) = F \frac{1 - xF + F^{2t-1}(x - F)}{1 - xF + F^{2t-1}(x - F)}$$

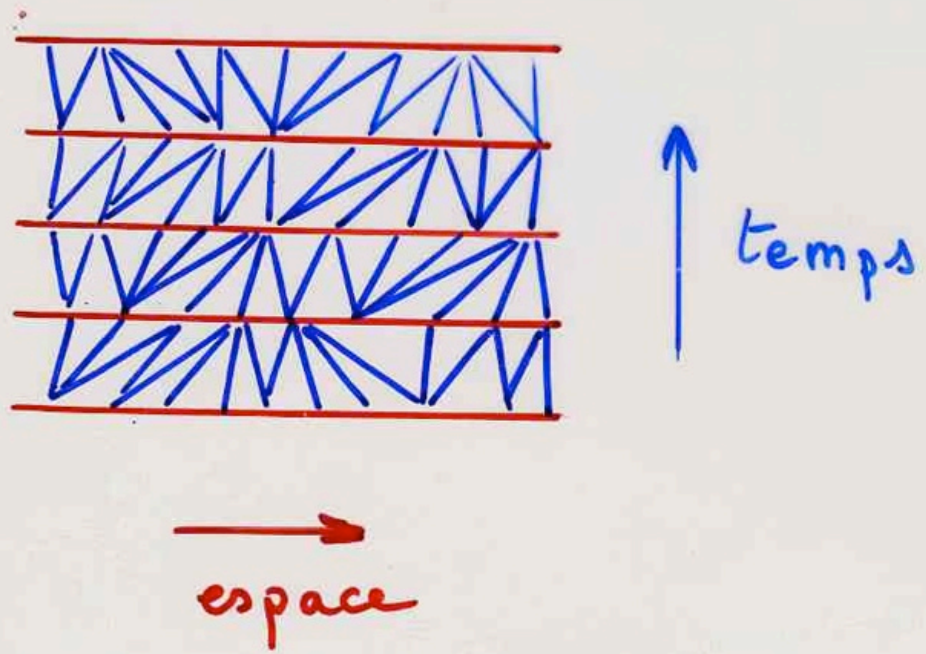
$$F = \frac{1 - \sqrt{1 - 4g^2}}{2g}$$

Catalan number !

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$


temps
espace


temps
espace



Lorentzian

triangulation

euclidian

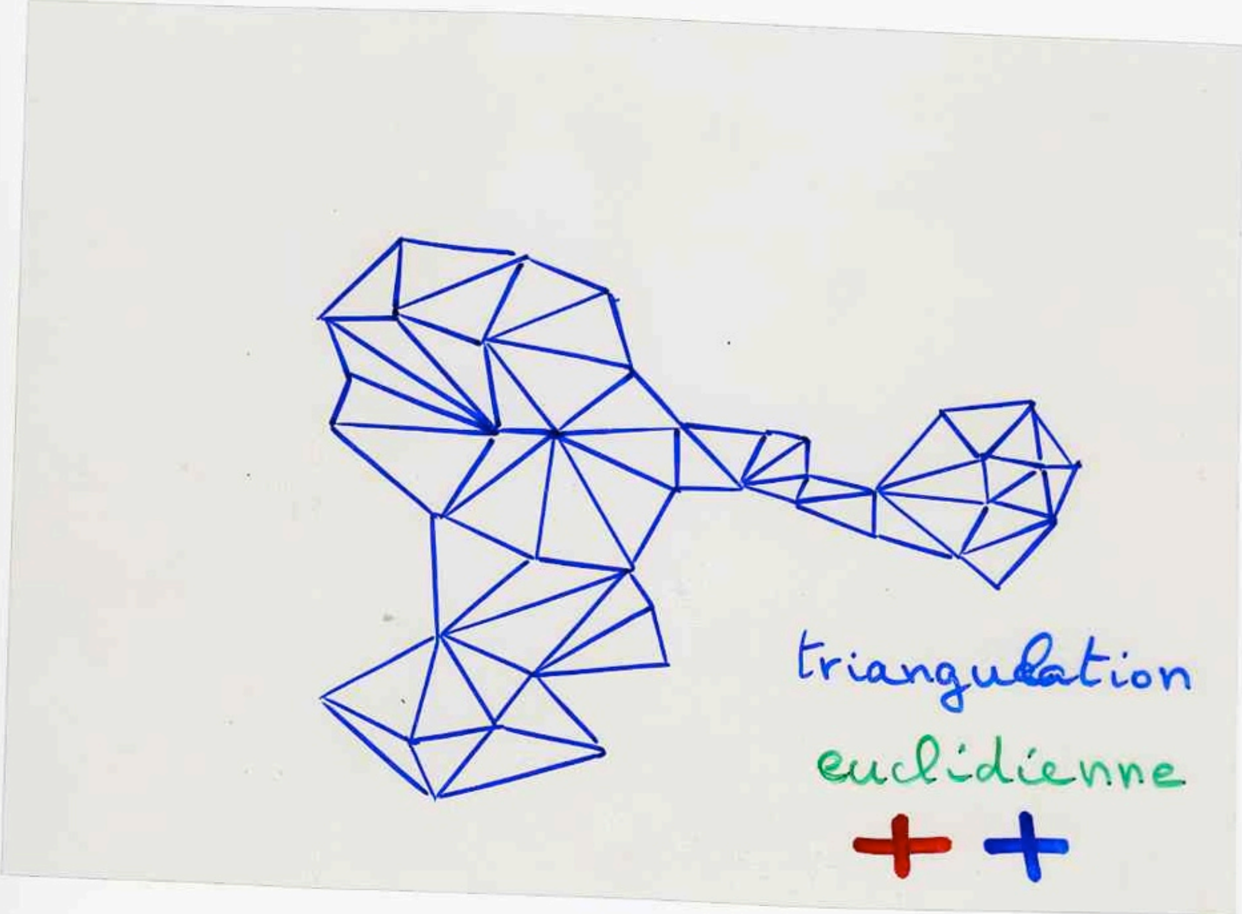
+ + + +

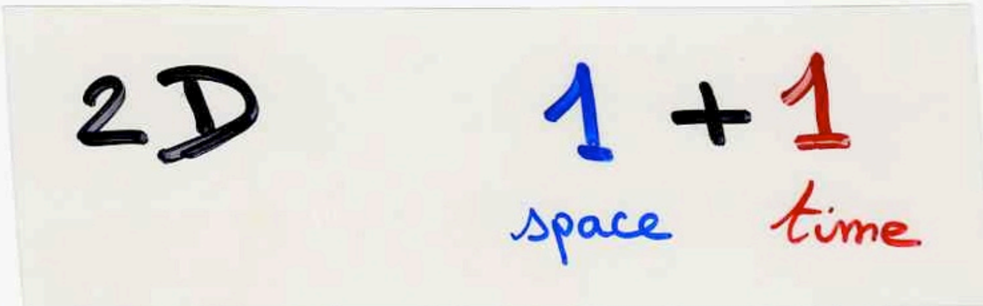
space time

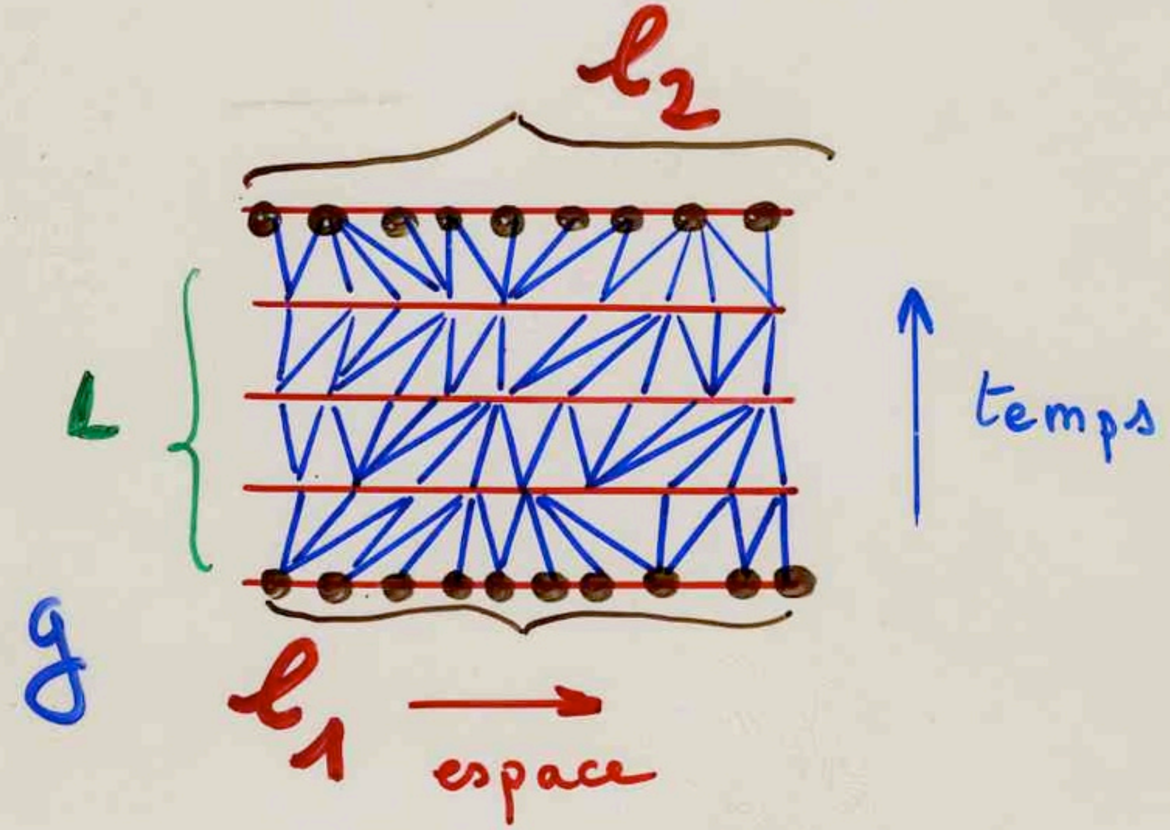
+ + + −

space time

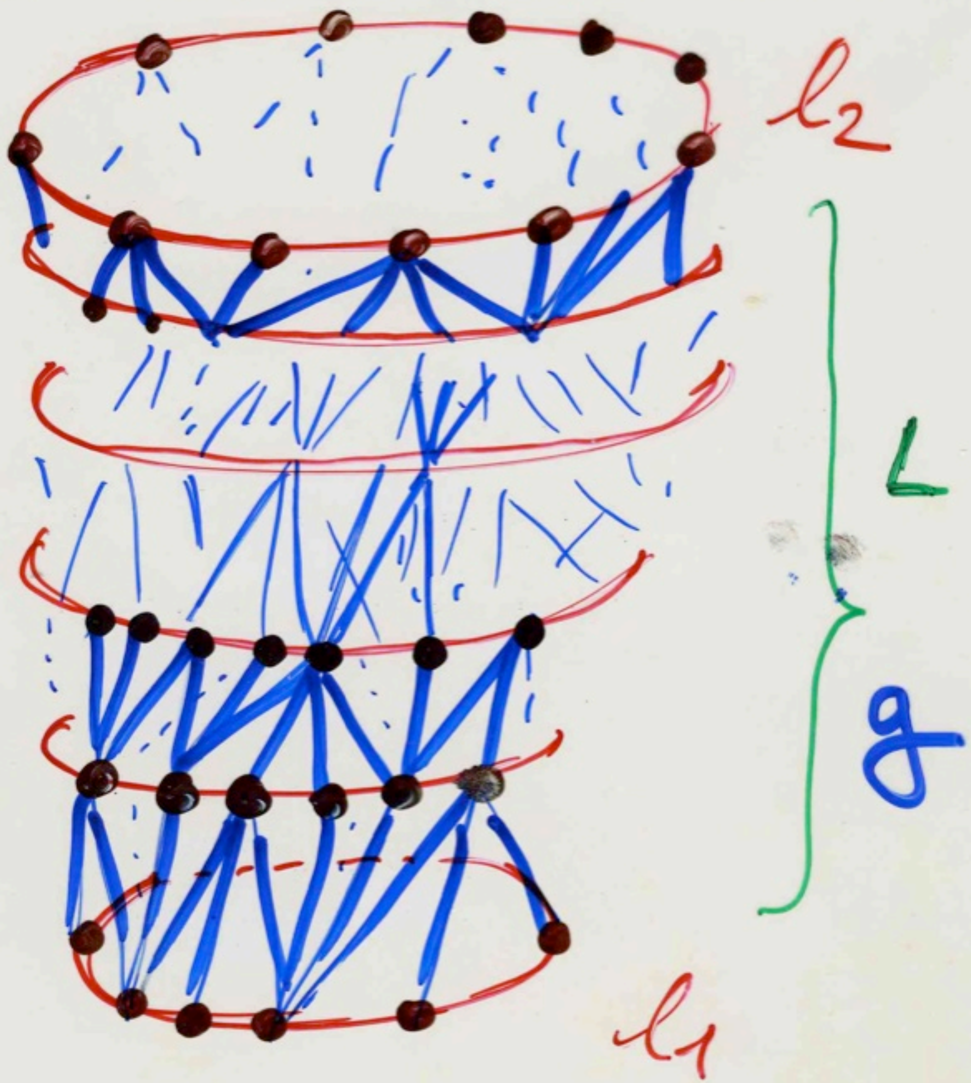
Lorentzian
quantum
gravity


triangulation
euclidienne
+ +

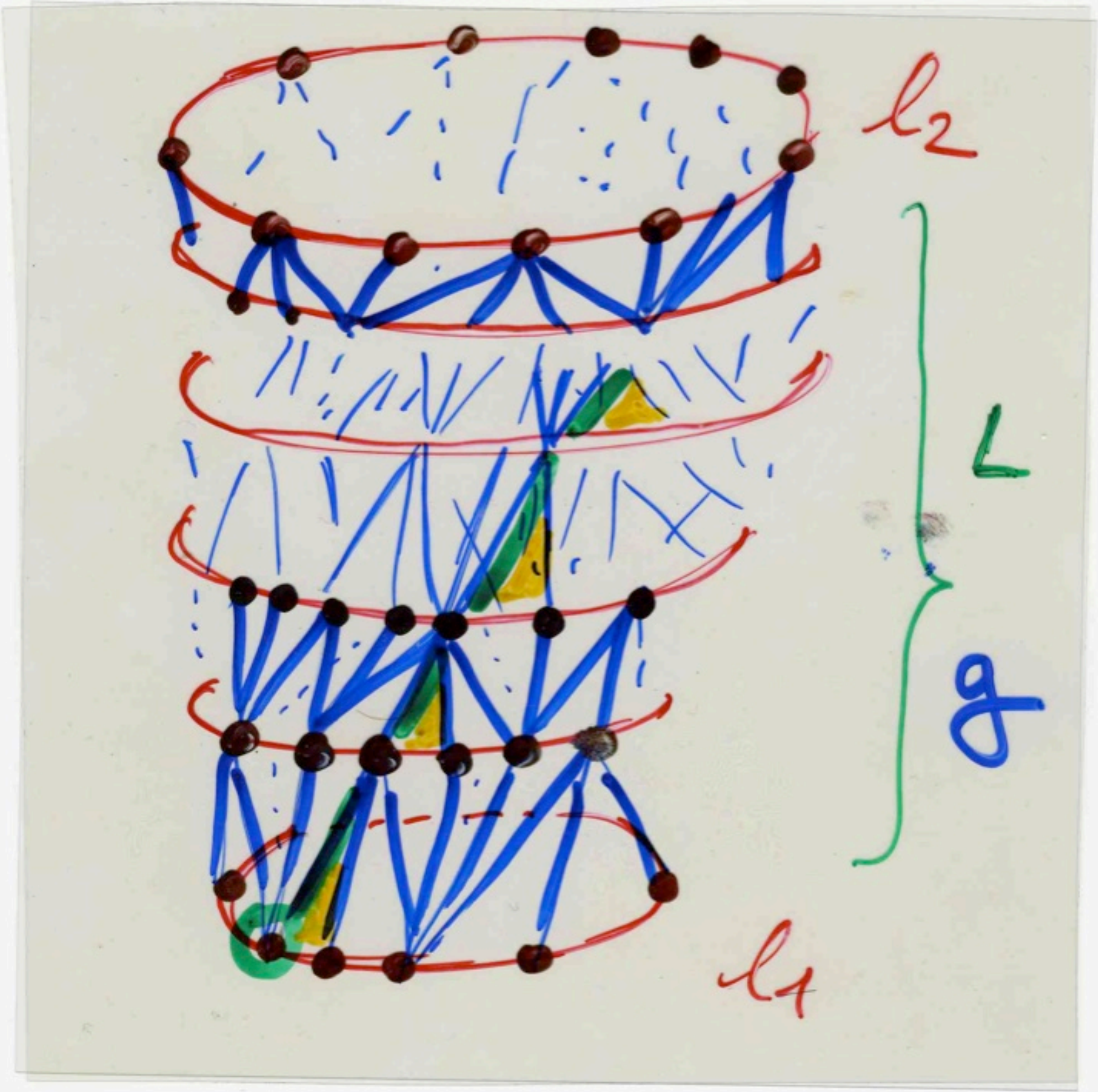

2D
1 + 1
space
time



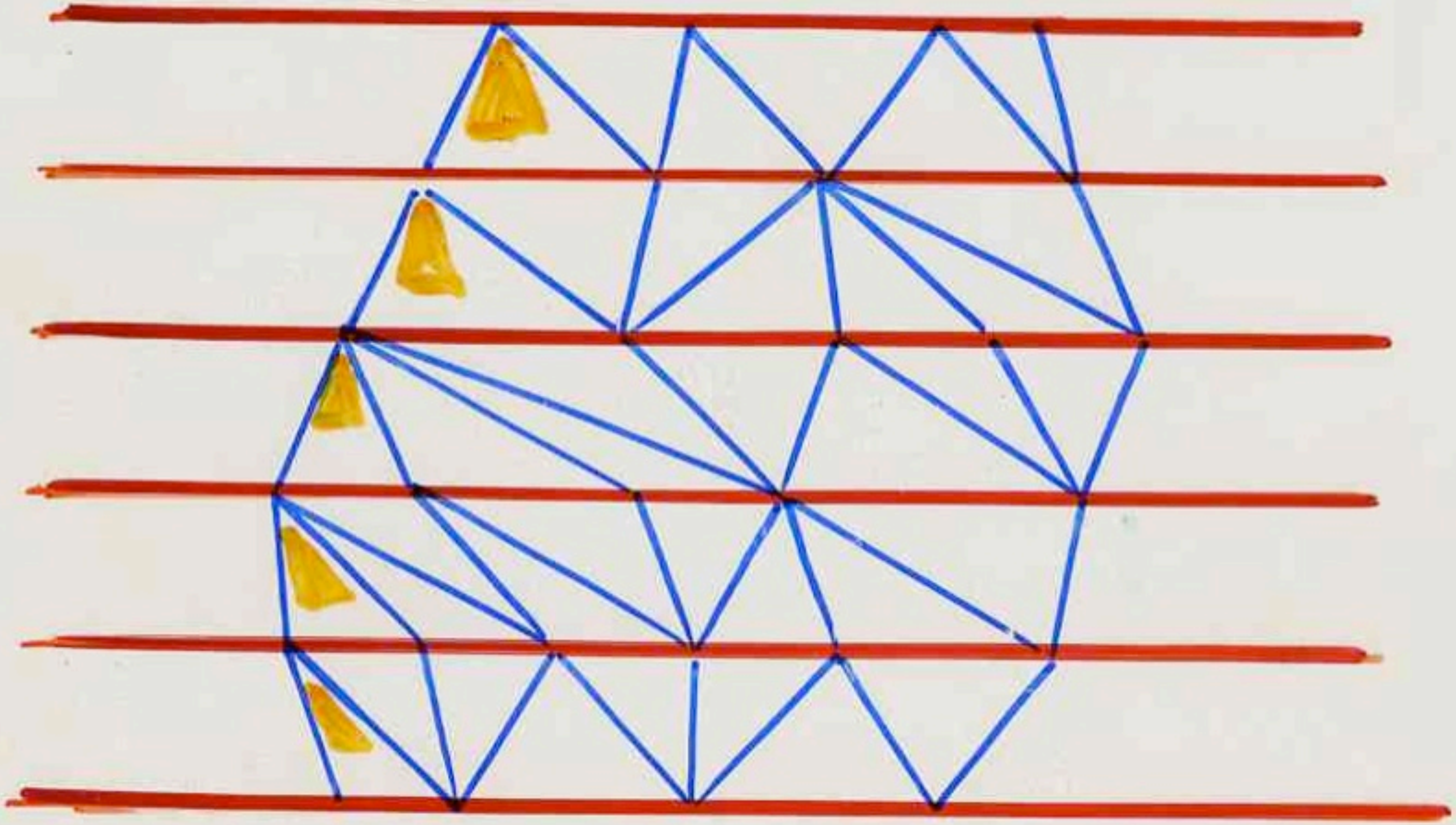
Path integral amplitude for the propagation from geometry ℓ_1 to ℓ_2

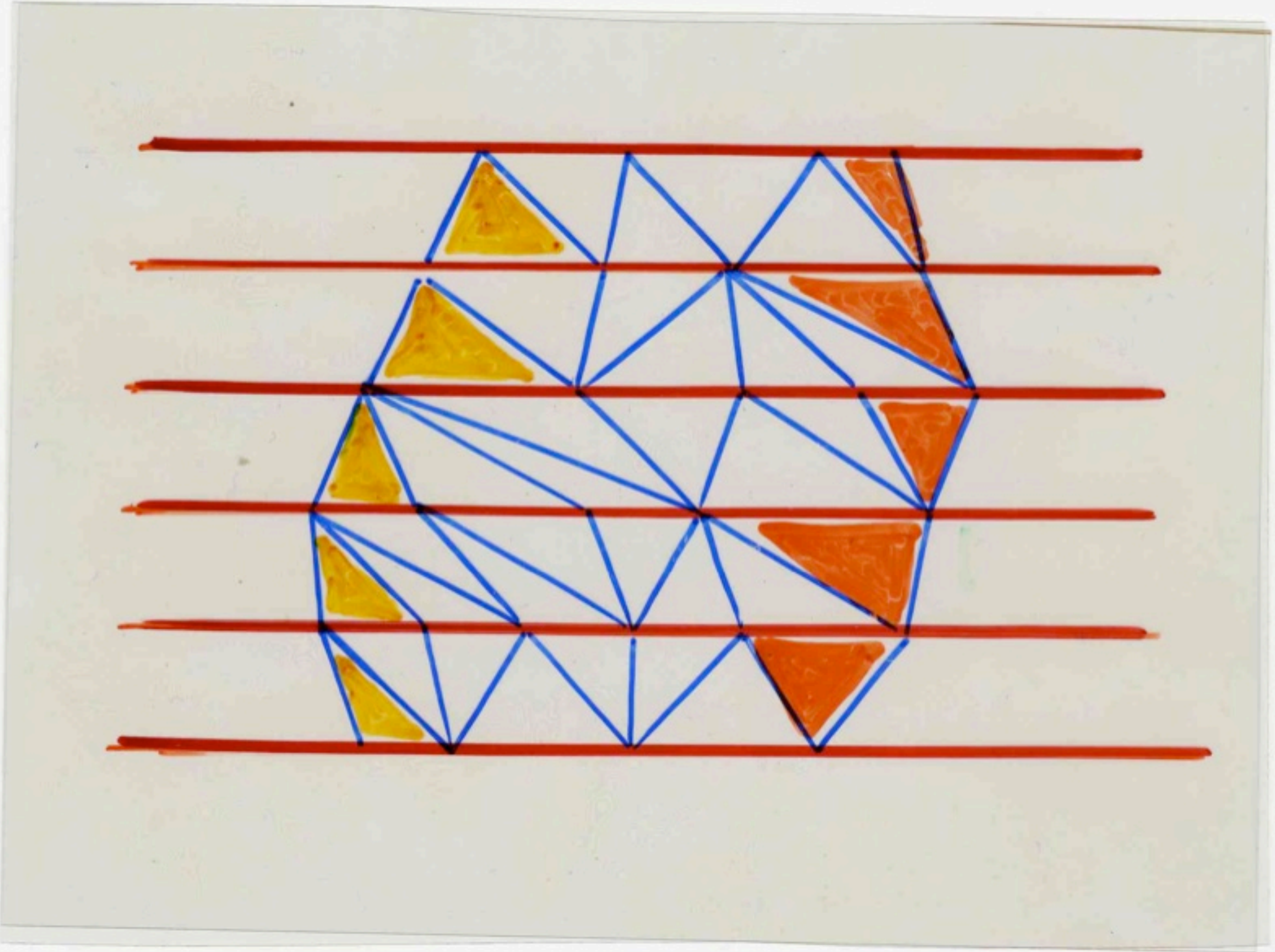


l2
L
g
l1



l2
L
g
l1





Dyck paths

↕

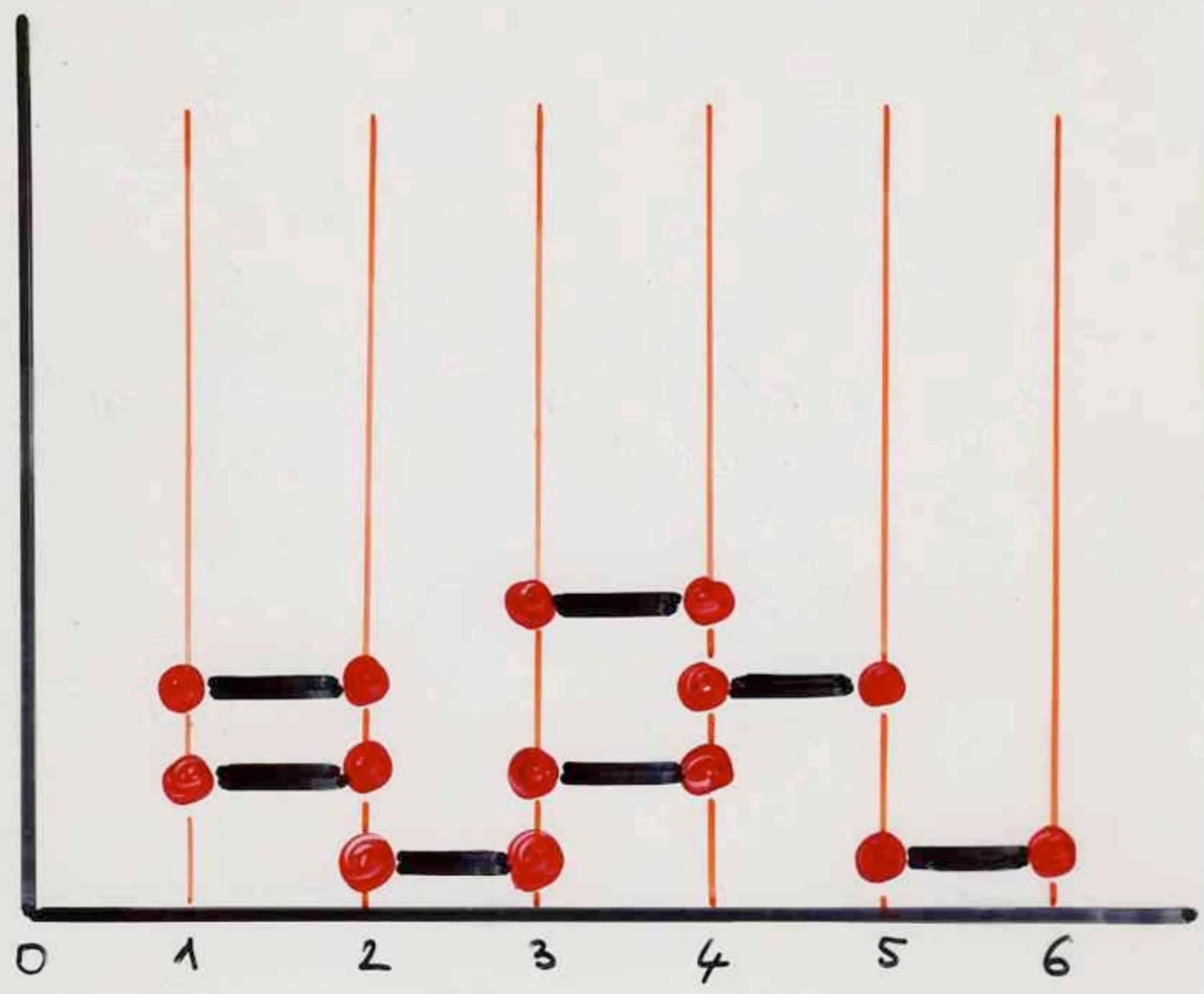
Heaps of dimers

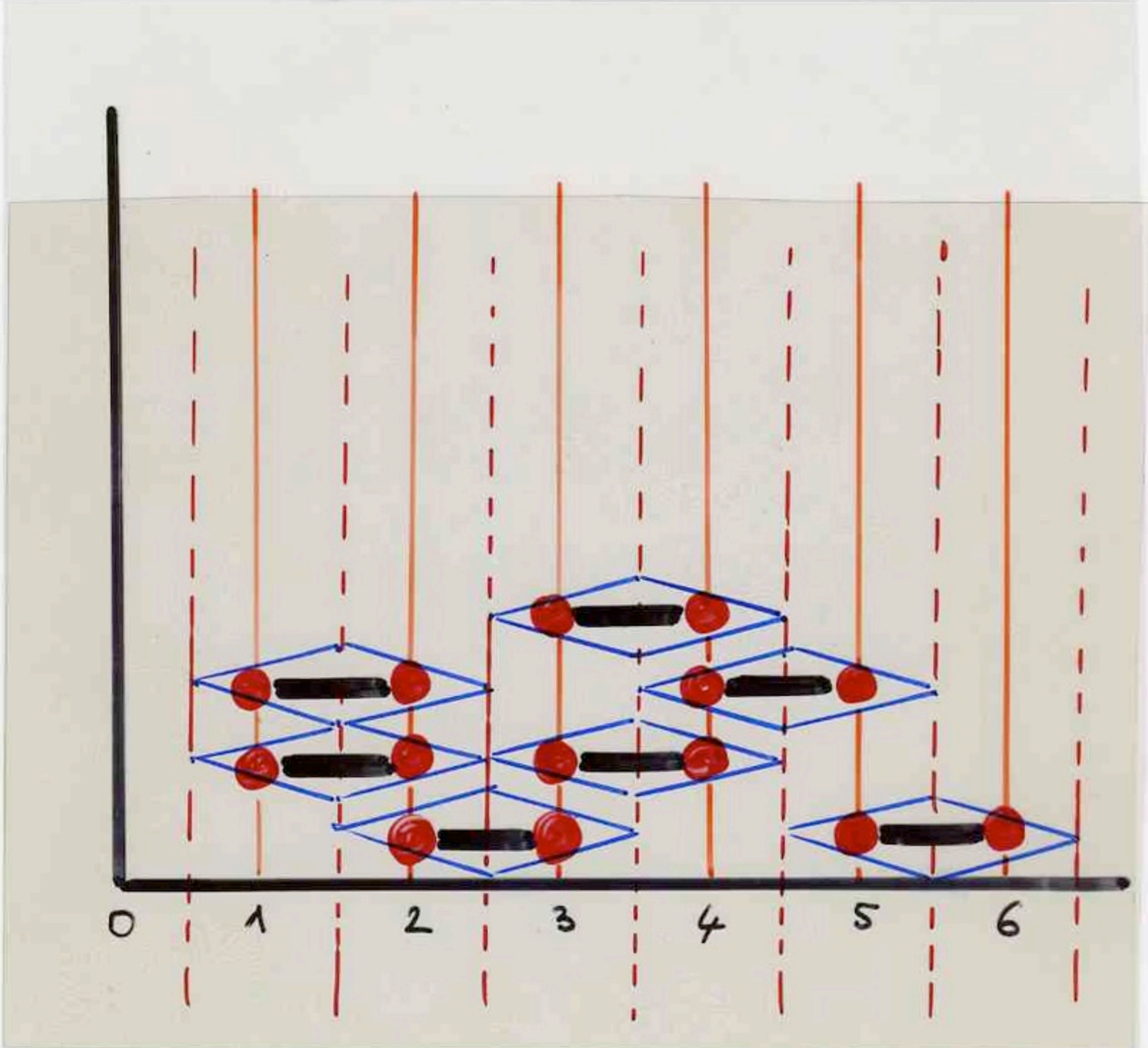
(Pyramids)

↕

Lorentzian triangulations

(*) border condition


0
1
2
3
4
5
6


0
1
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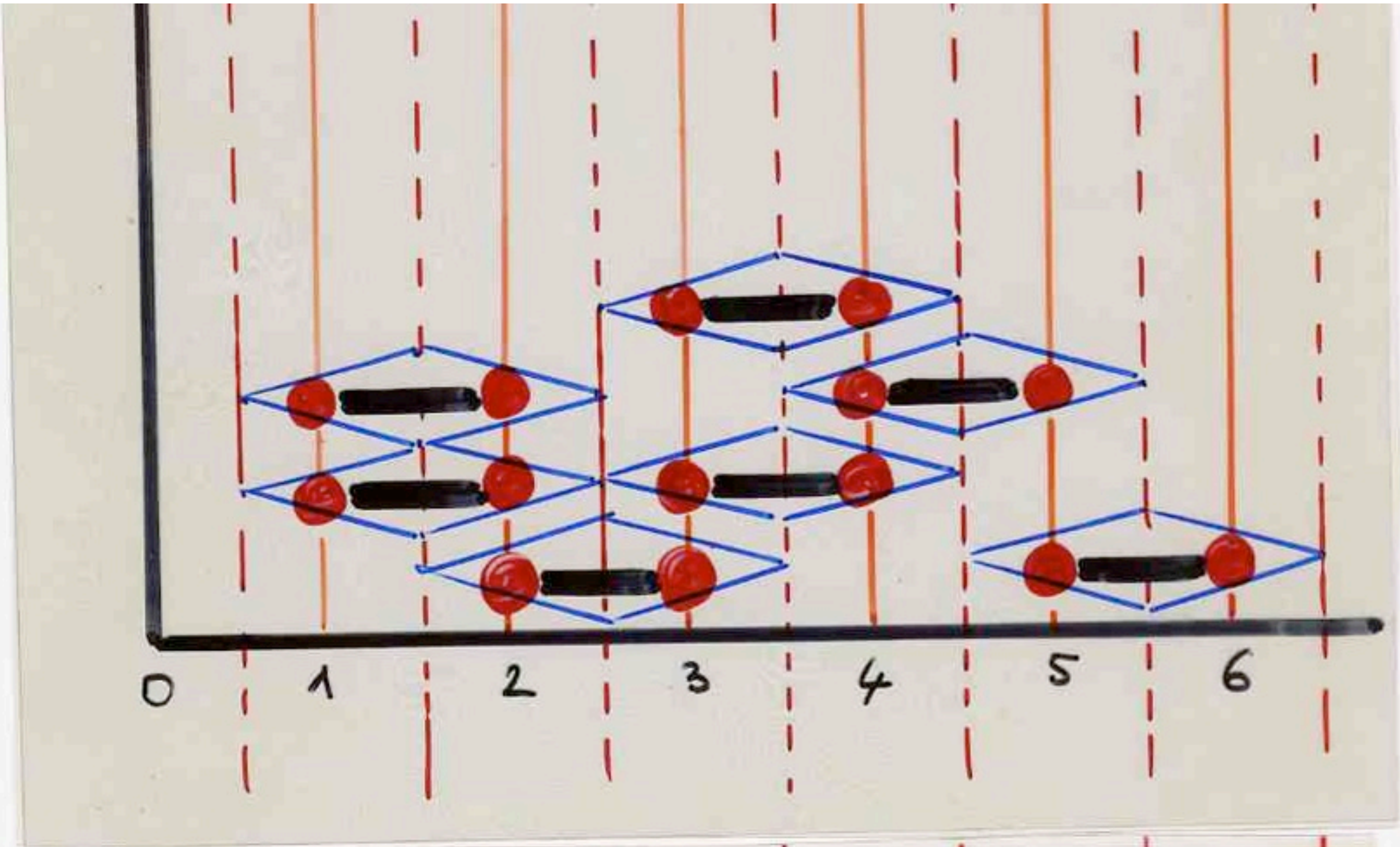
0
1
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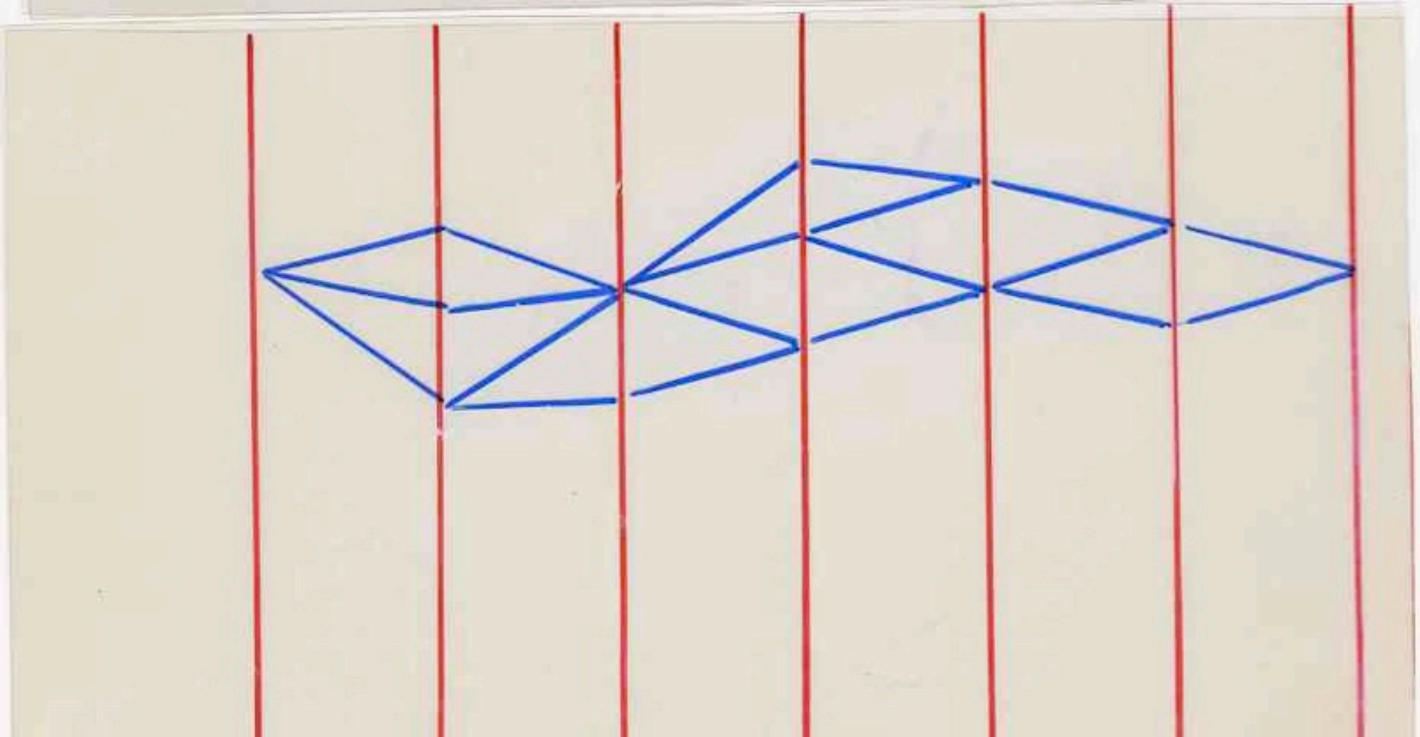

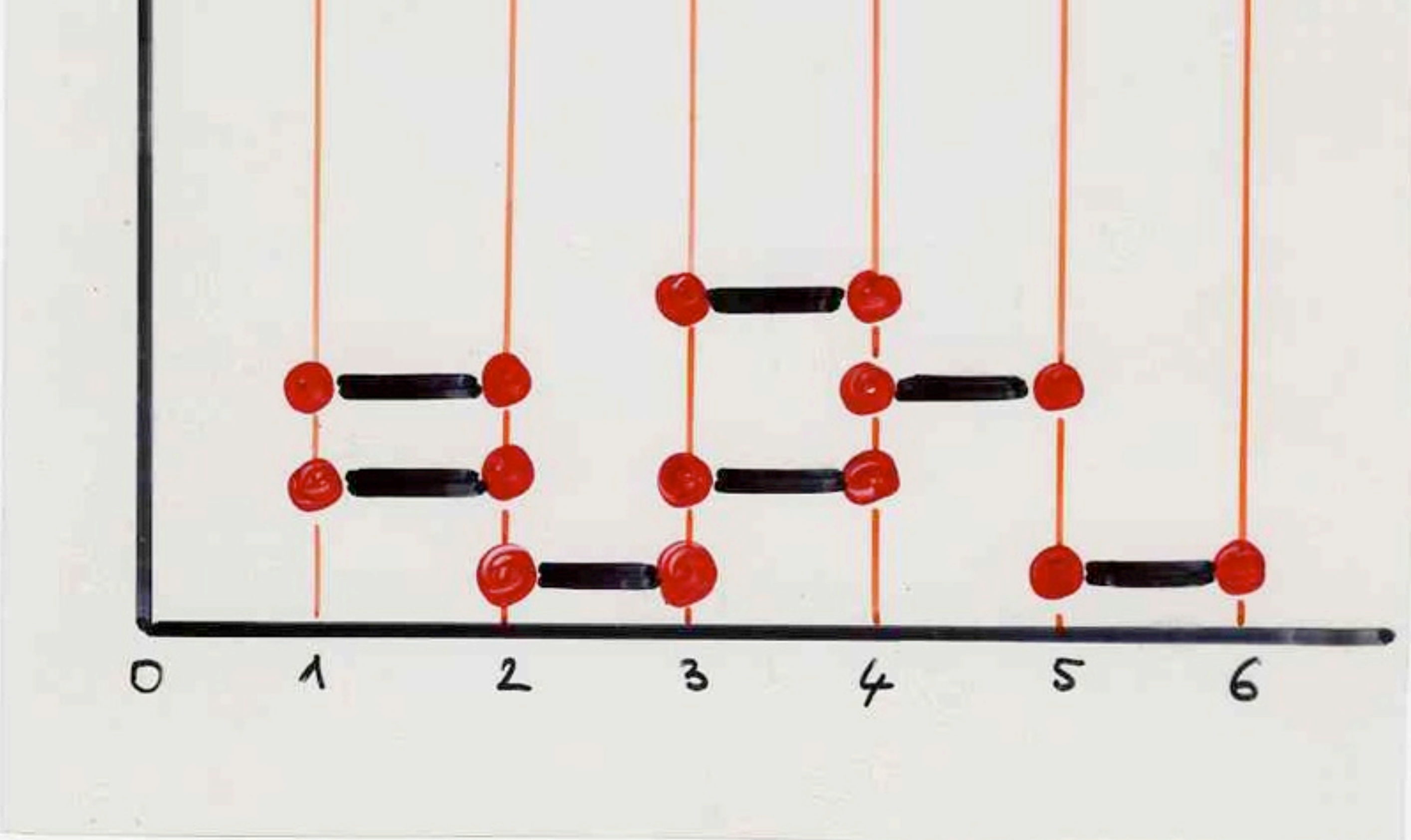
0
1
2
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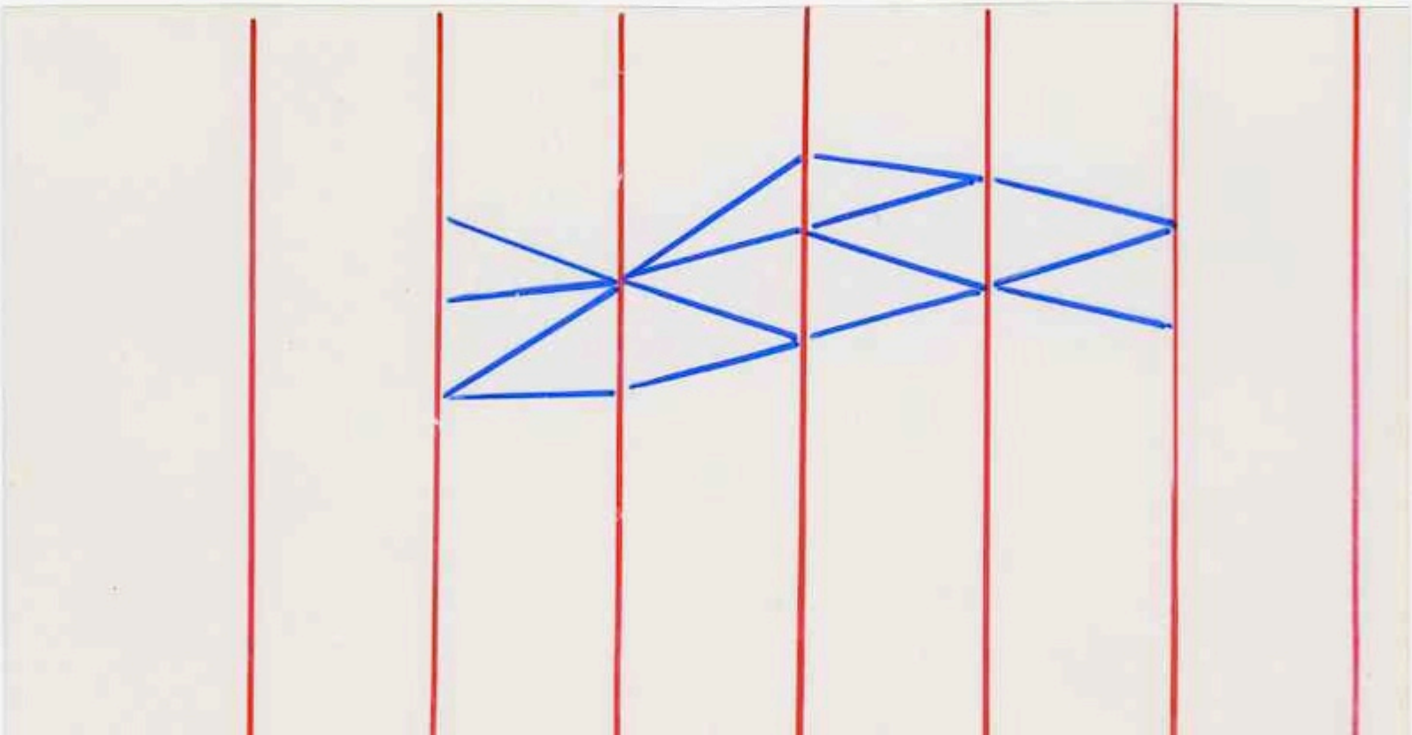
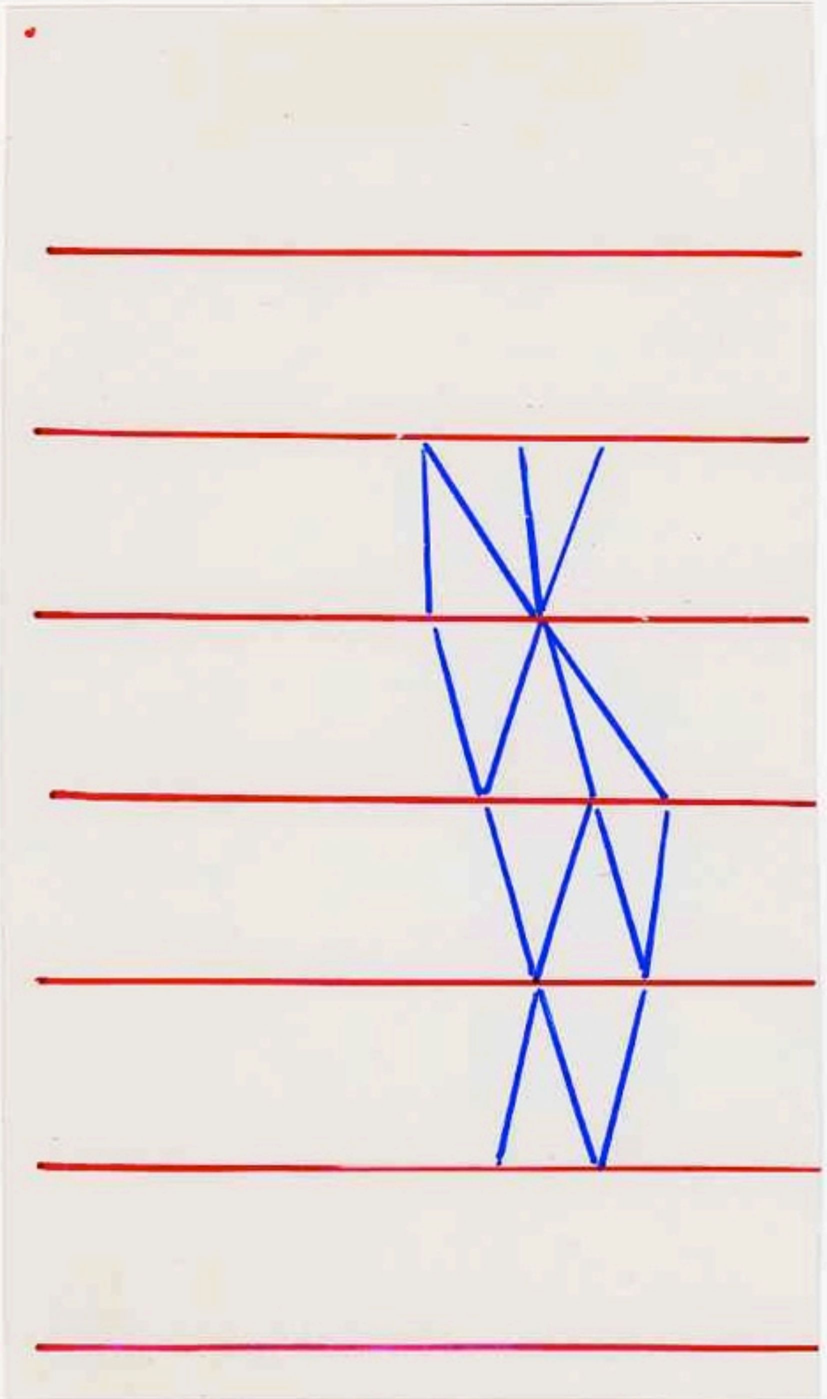
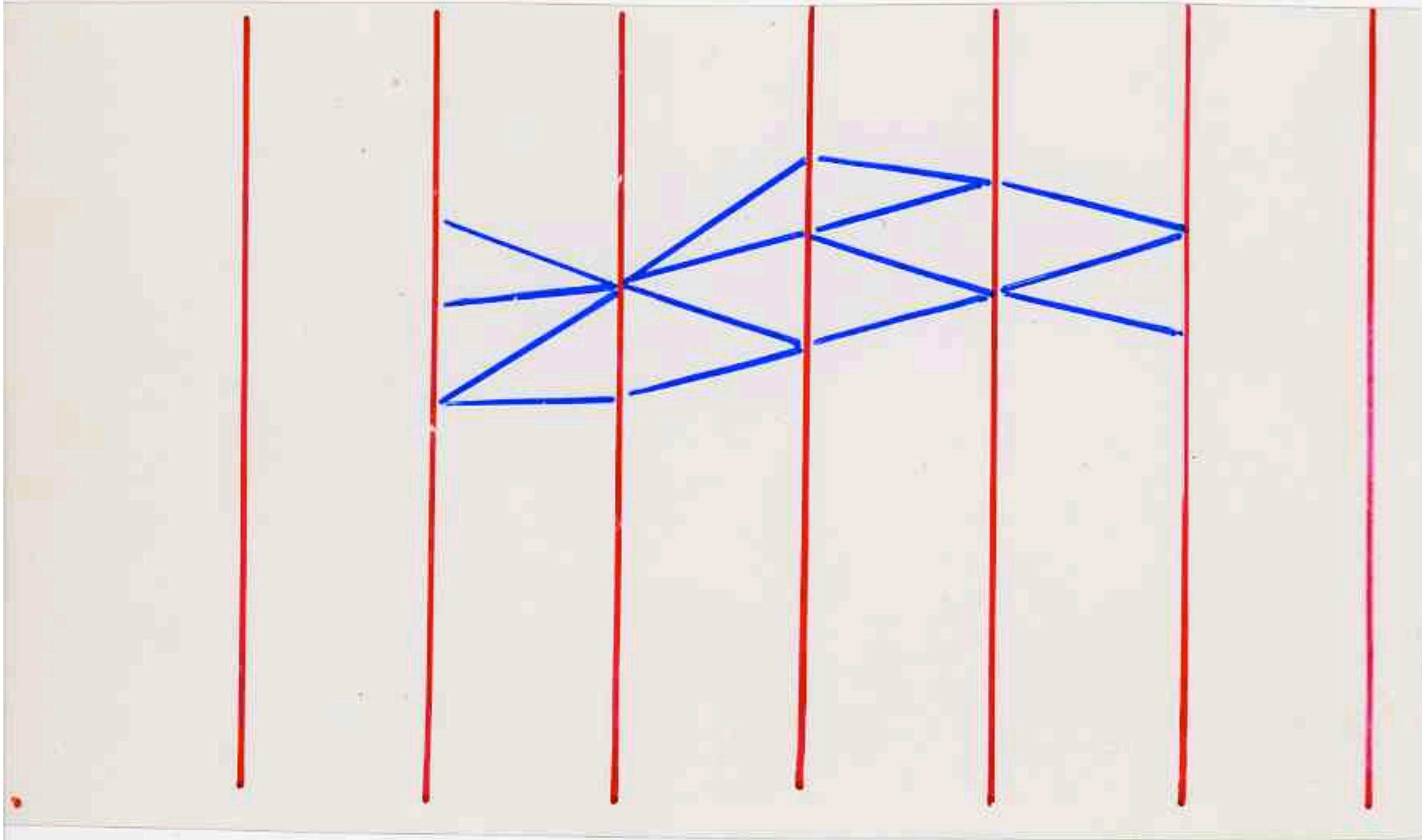
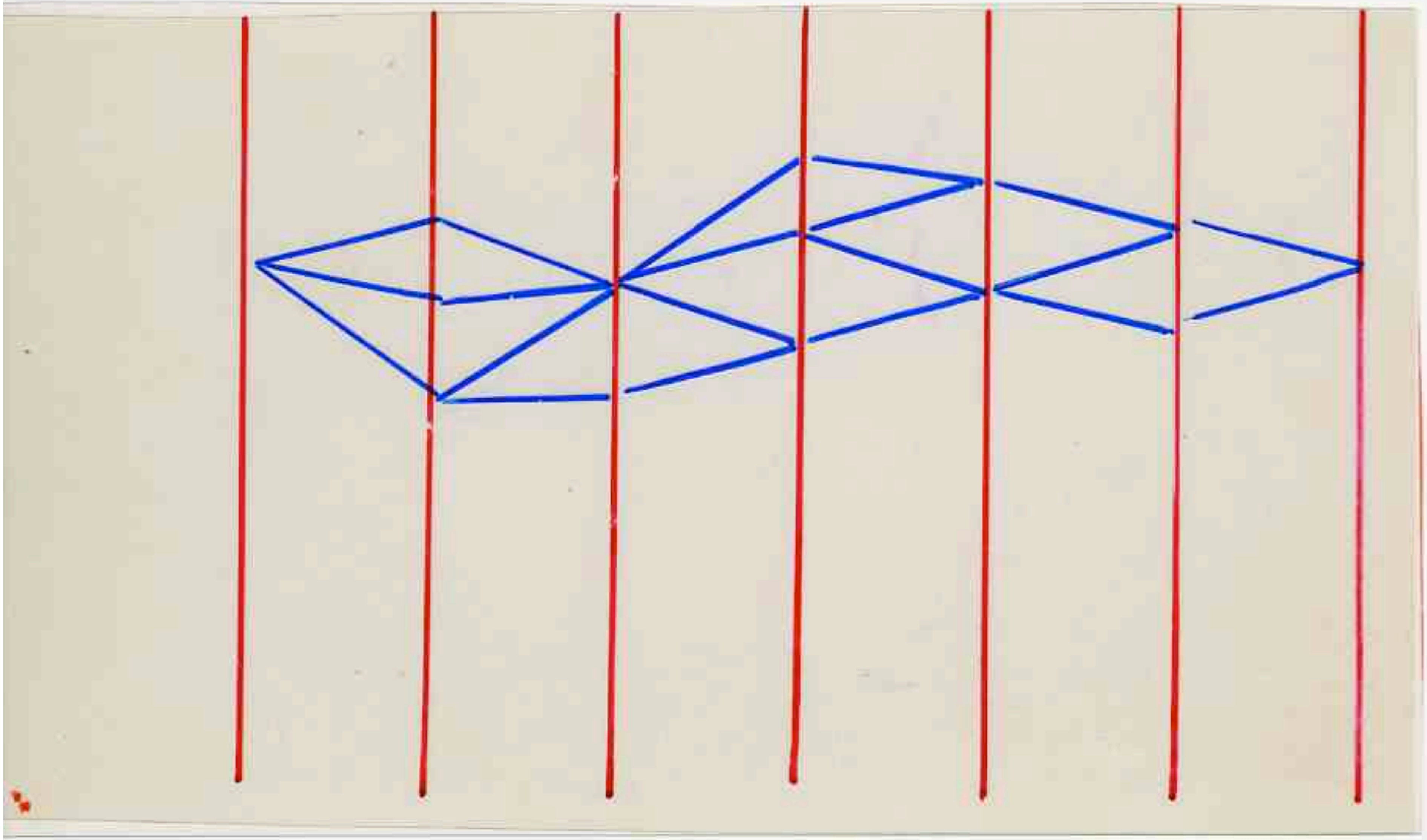
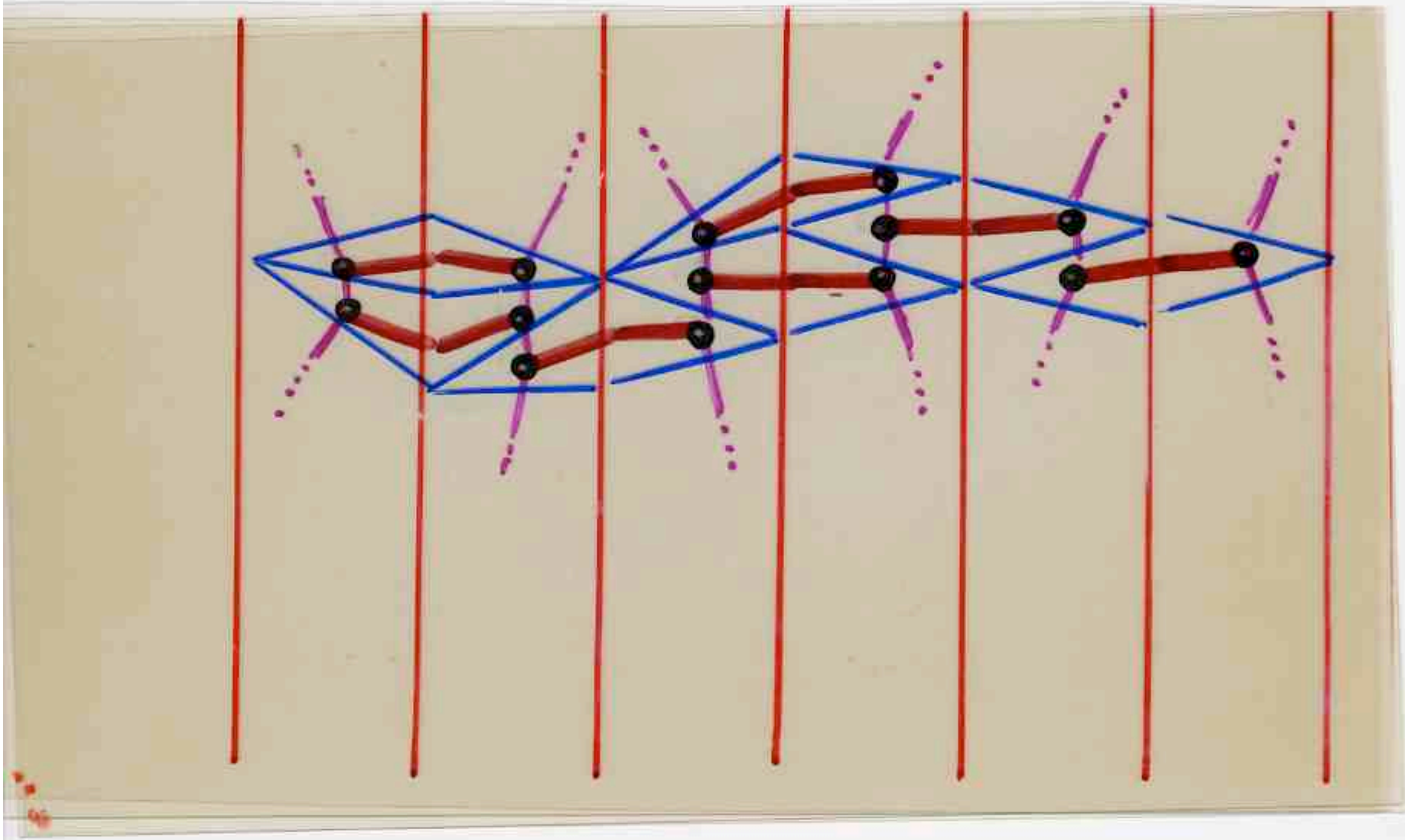
6

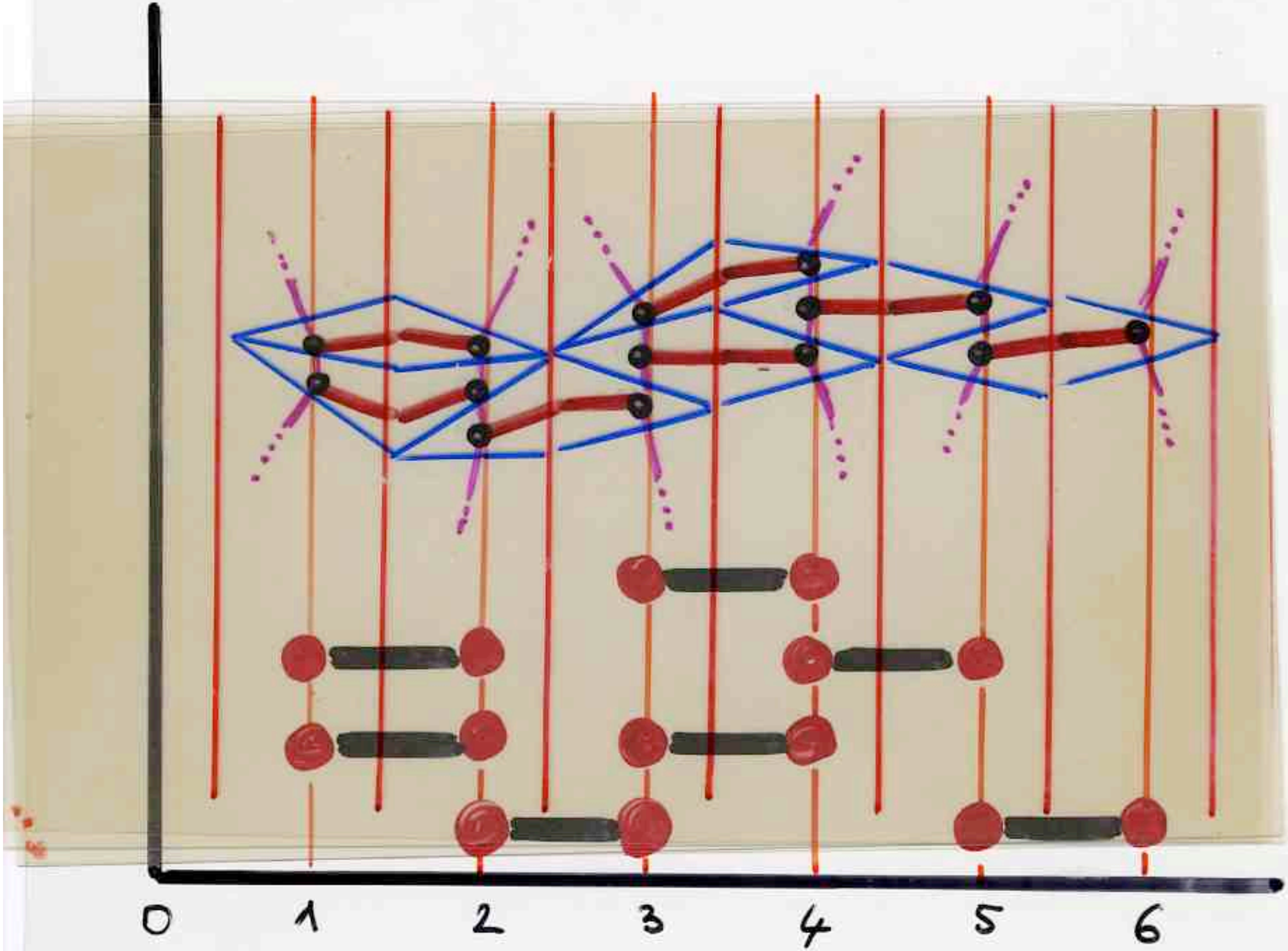











0
1
2
3
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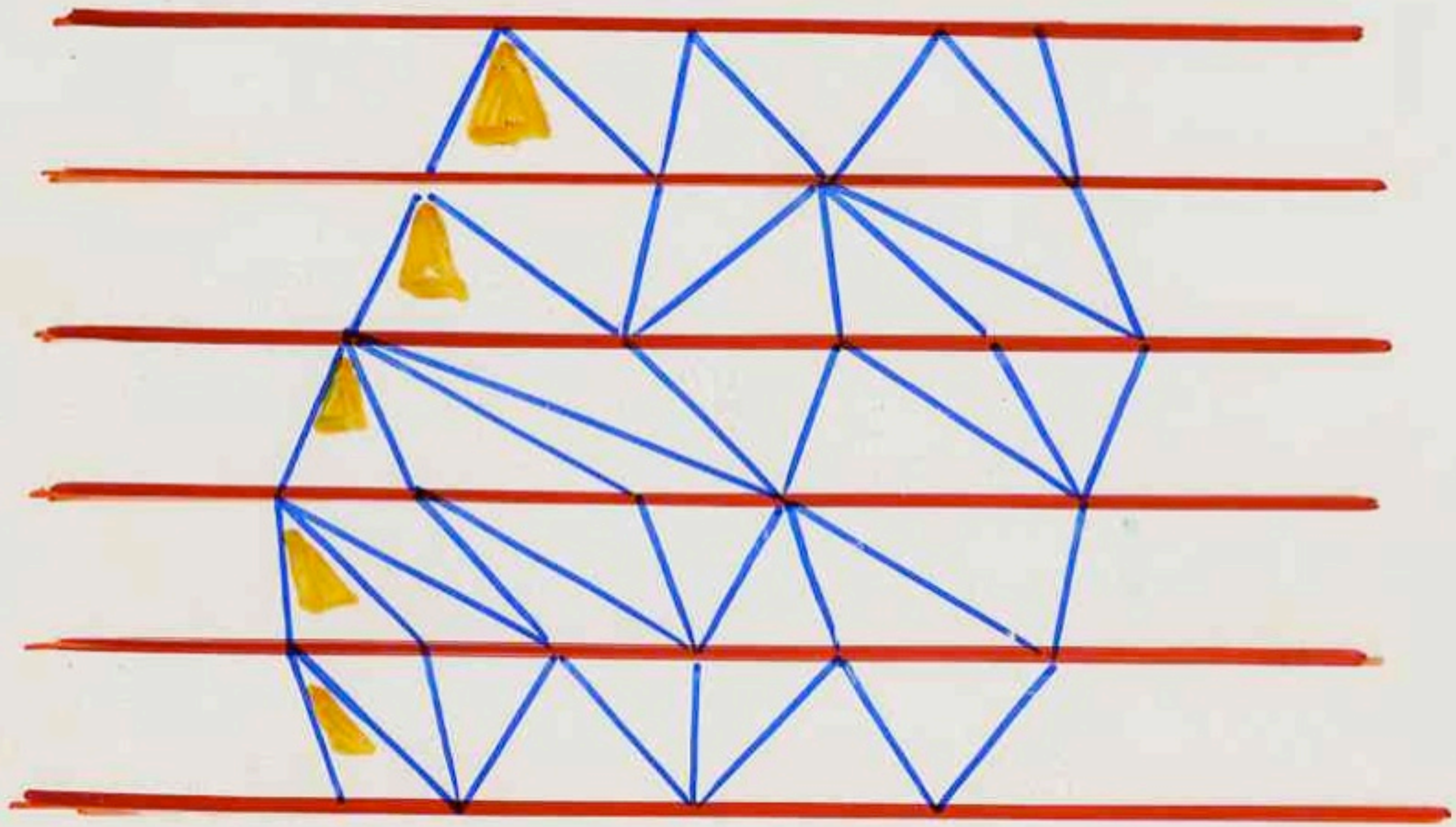
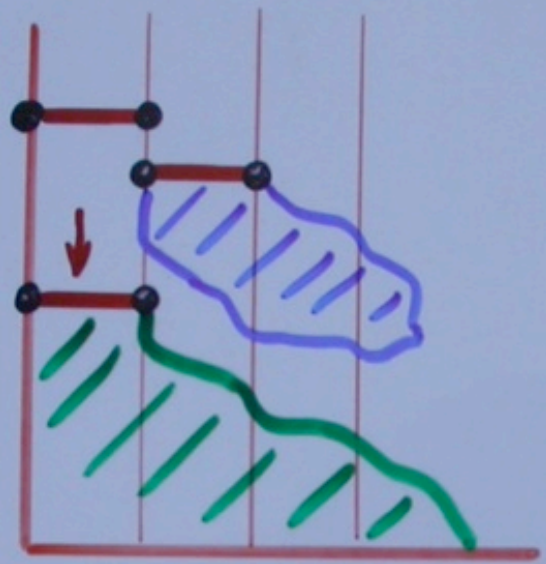
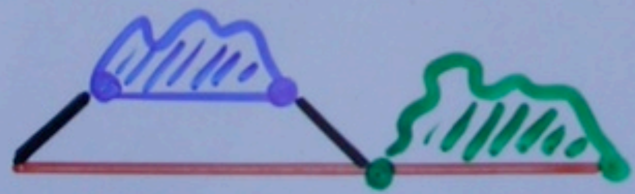
5
6





$$y = 1 + t\,y^2$$

Half-pyramid

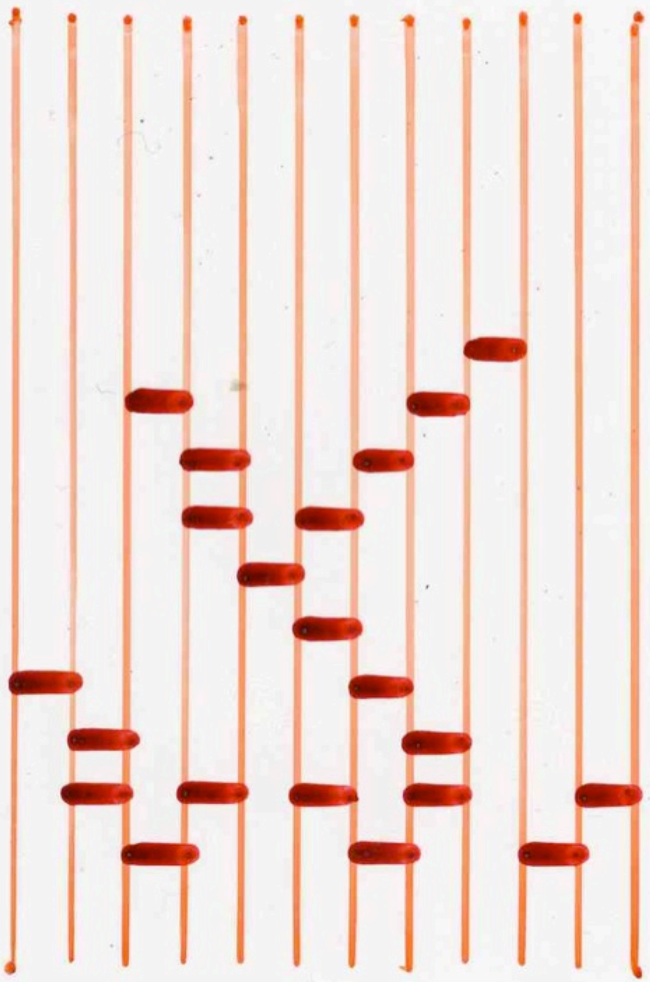


$$y = 1 + t\,y^2$$

Dyck path

Lorentzian triangulations
in 2D qantum gravity
without border conditions

the nordic decomposition
of a heap of dimers



connected heap of dimers

$$Q(t) = \frac{1-2t-\sqrt{1-4t}}{2t}$$

generating function for

half-pyramid $(\neq \emptyset)$

$$= \sum_{n \geqslant 1} C_n t^n$$

Catalan

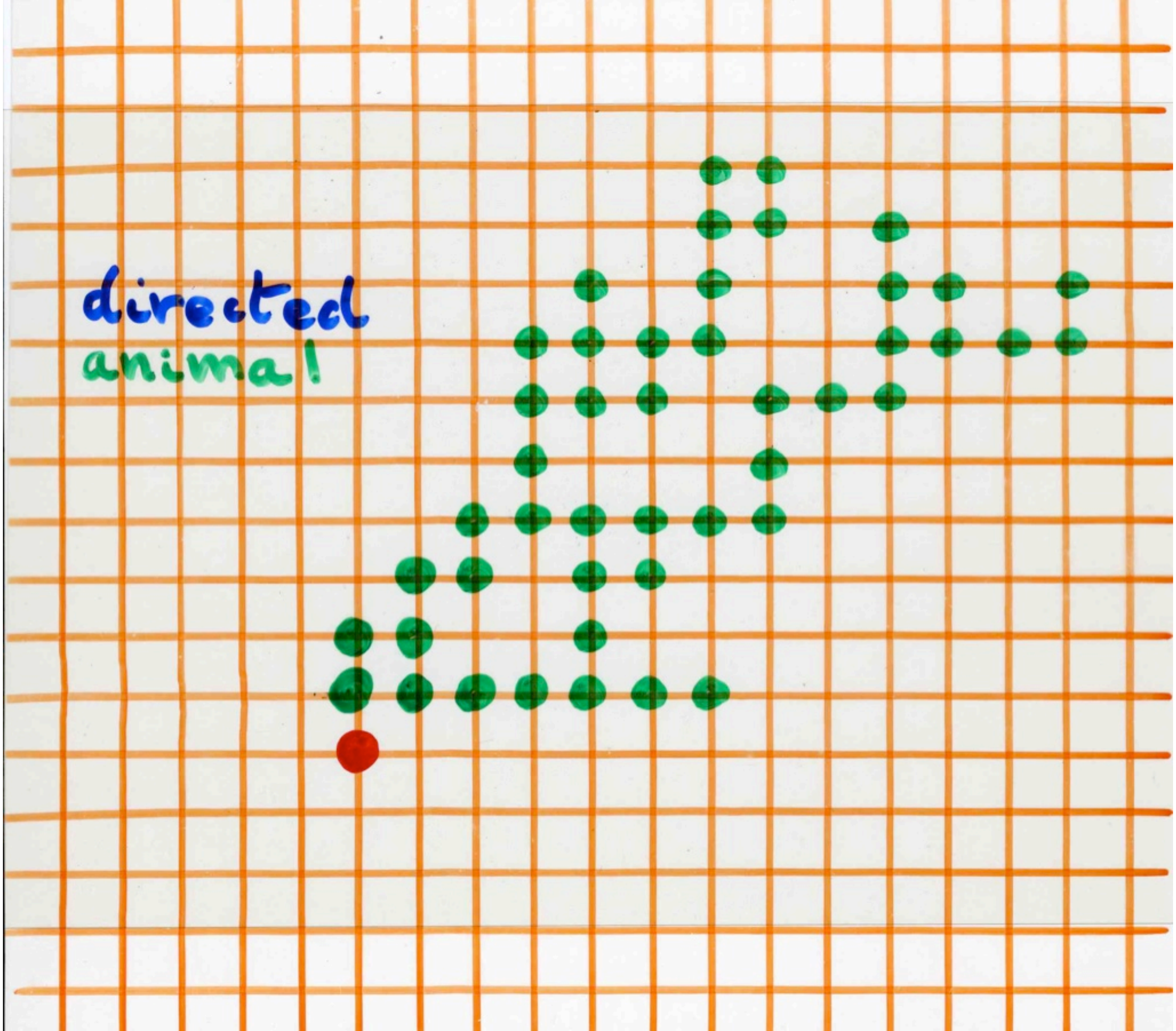
$$C = \frac{Q}{(1-Q)\left[1-\sum_{k\geq 1} \frac{Q^{k+1}(1+Q)^{-k-1}}{1-Q^k}\right]}$$

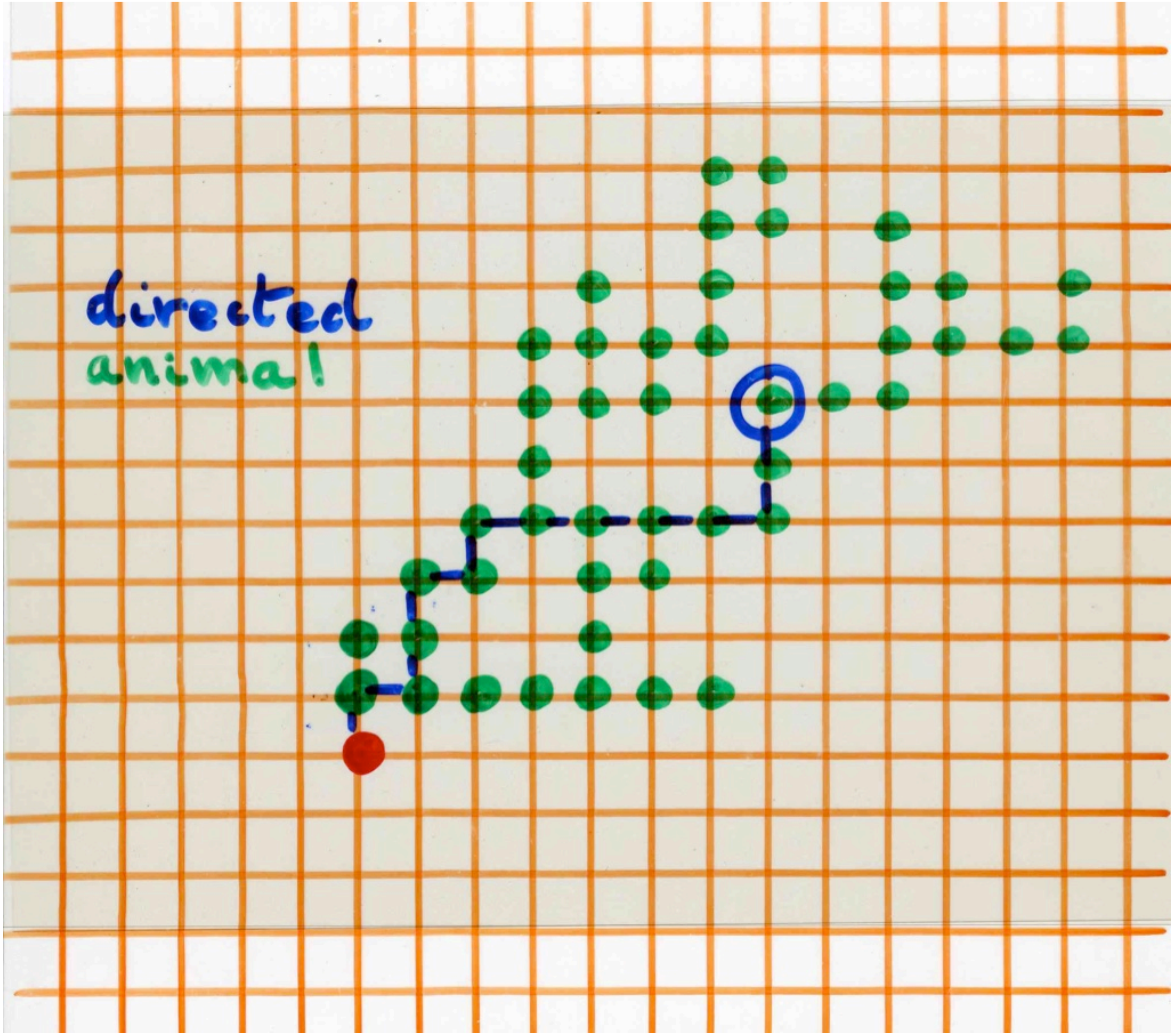
$$\sum_{k\geq 1} \frac{Q^{k+1}(1+Q)^{-k-1}}{1-Q^k} = \sum_{k\geq 1} \frac{Q^{k+1}}{1-Q^k(1+Q)}$$

Bousquet-Mélou, Rechnitzer (2002)

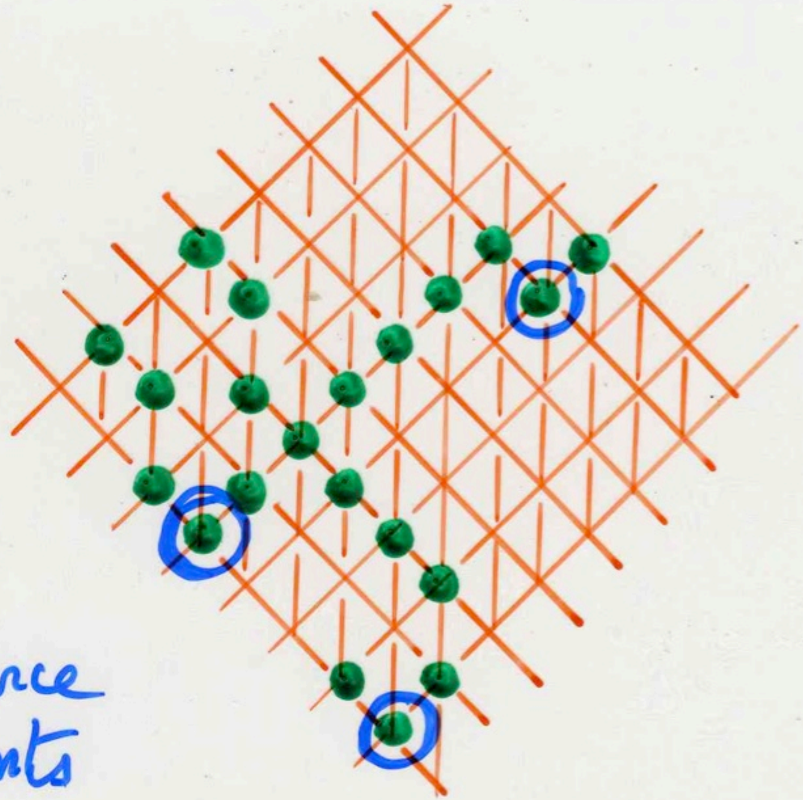
multidirected animals on square or triangular lattices

Mireille Bousquet-Mélou, A. Rechnitzer (2002)


directed
animal


directed
animal!

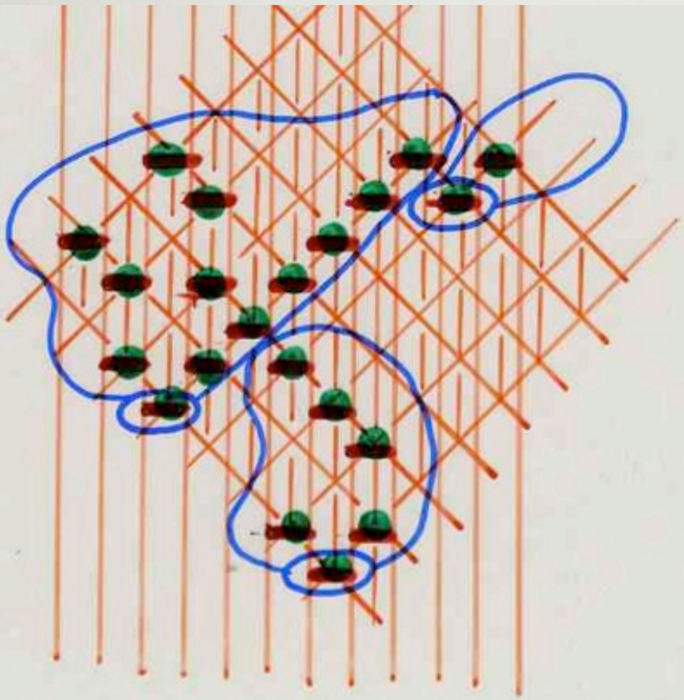
source points



multidirected animal

(triangular lattice)

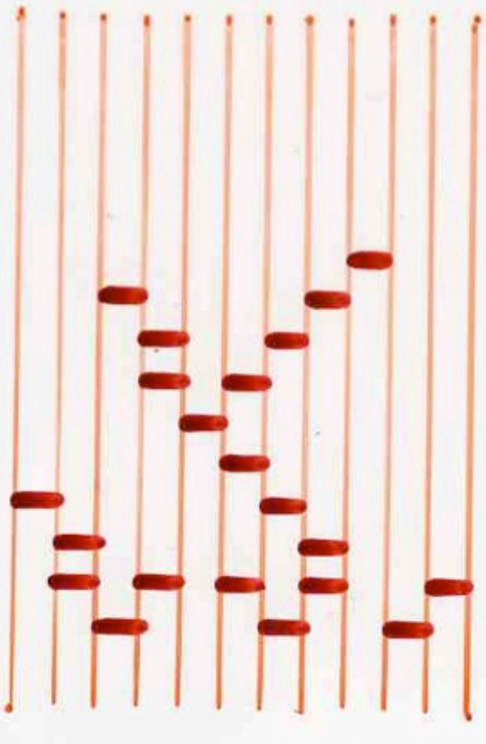
(Bousquet-Mélou, Rechnitzer, 2002)



multidirected animal

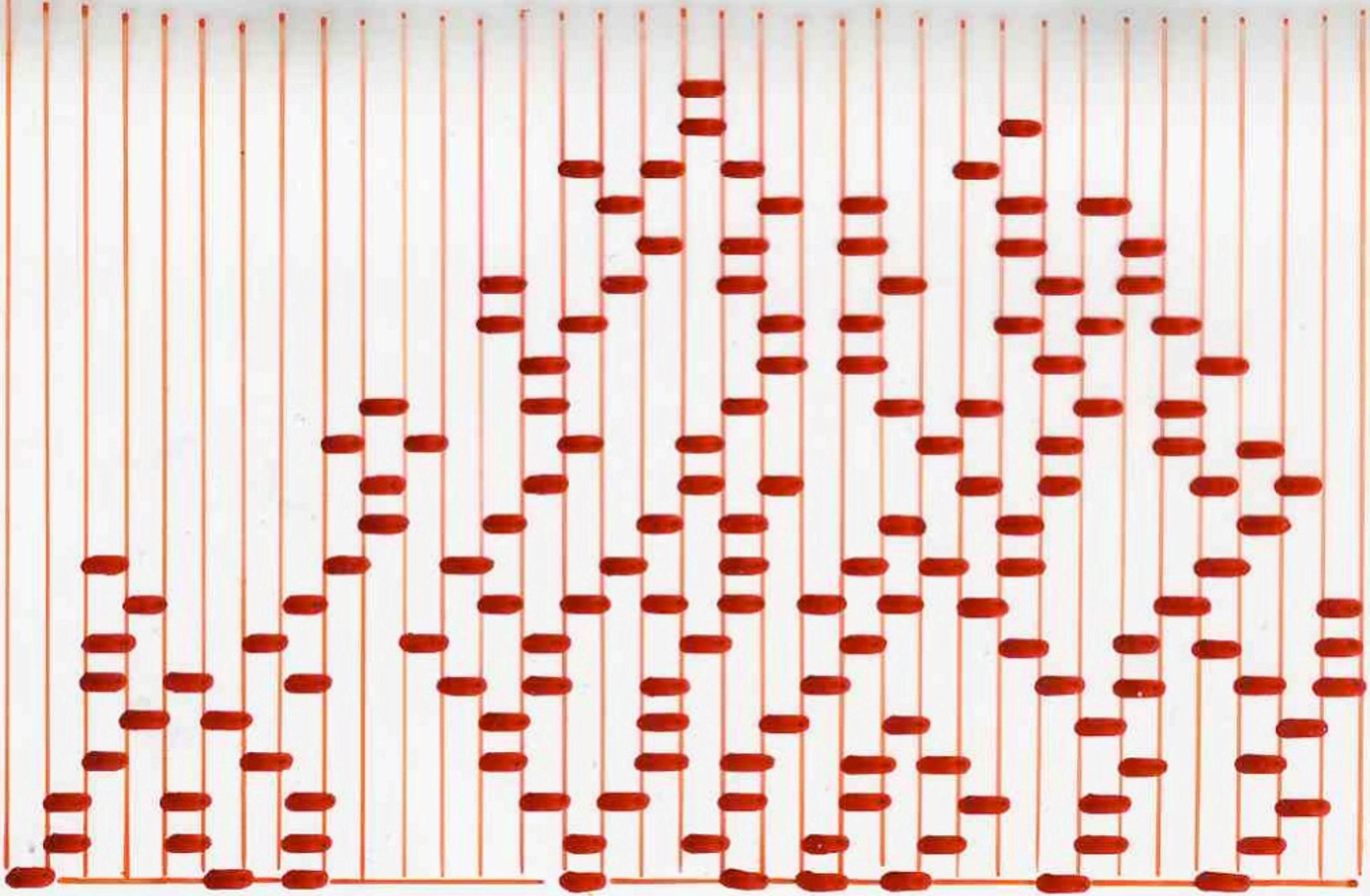
(triangular lattice)

(Bousquet-Mélou, Rechnitzer, 2002)

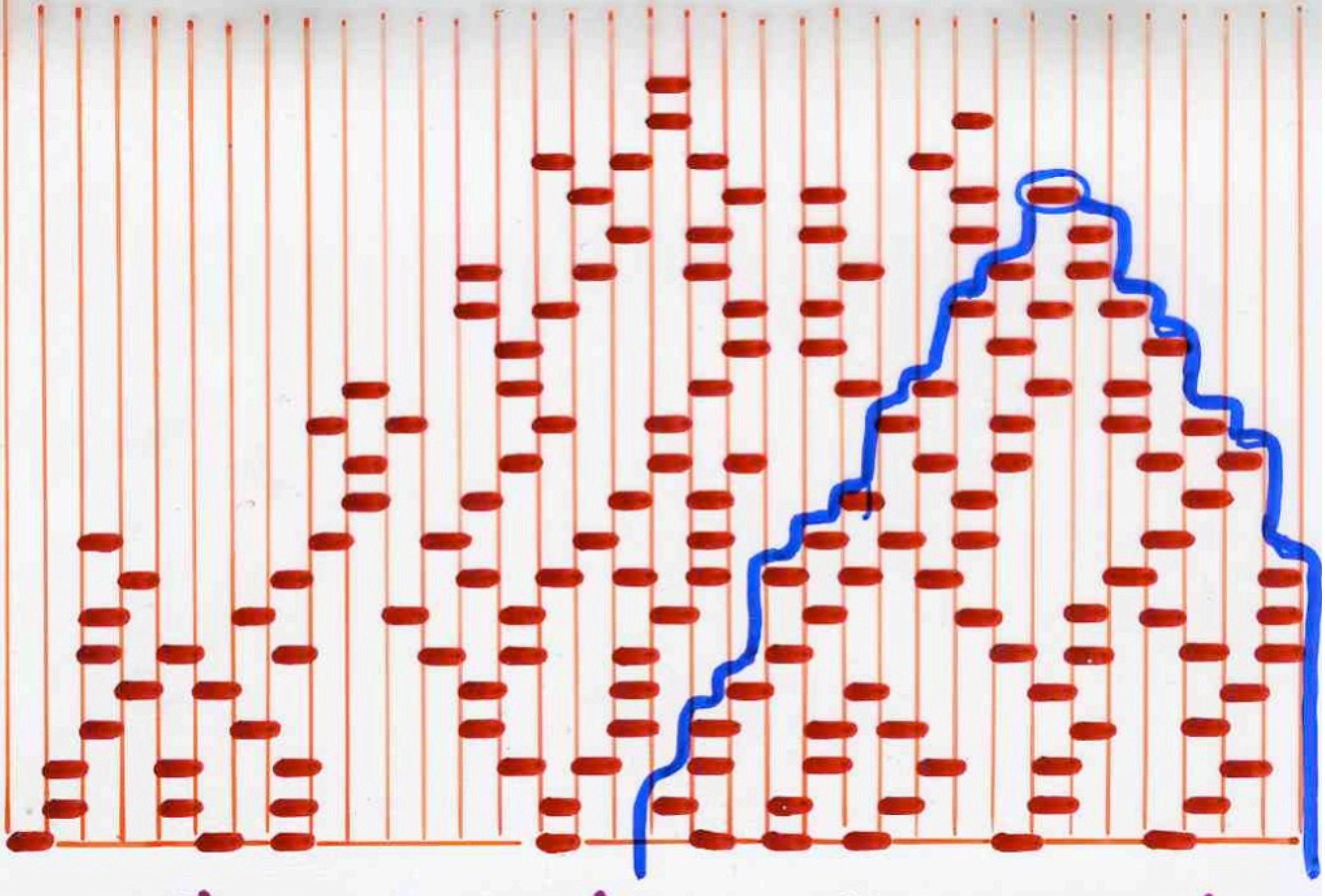


connected heap of dimers

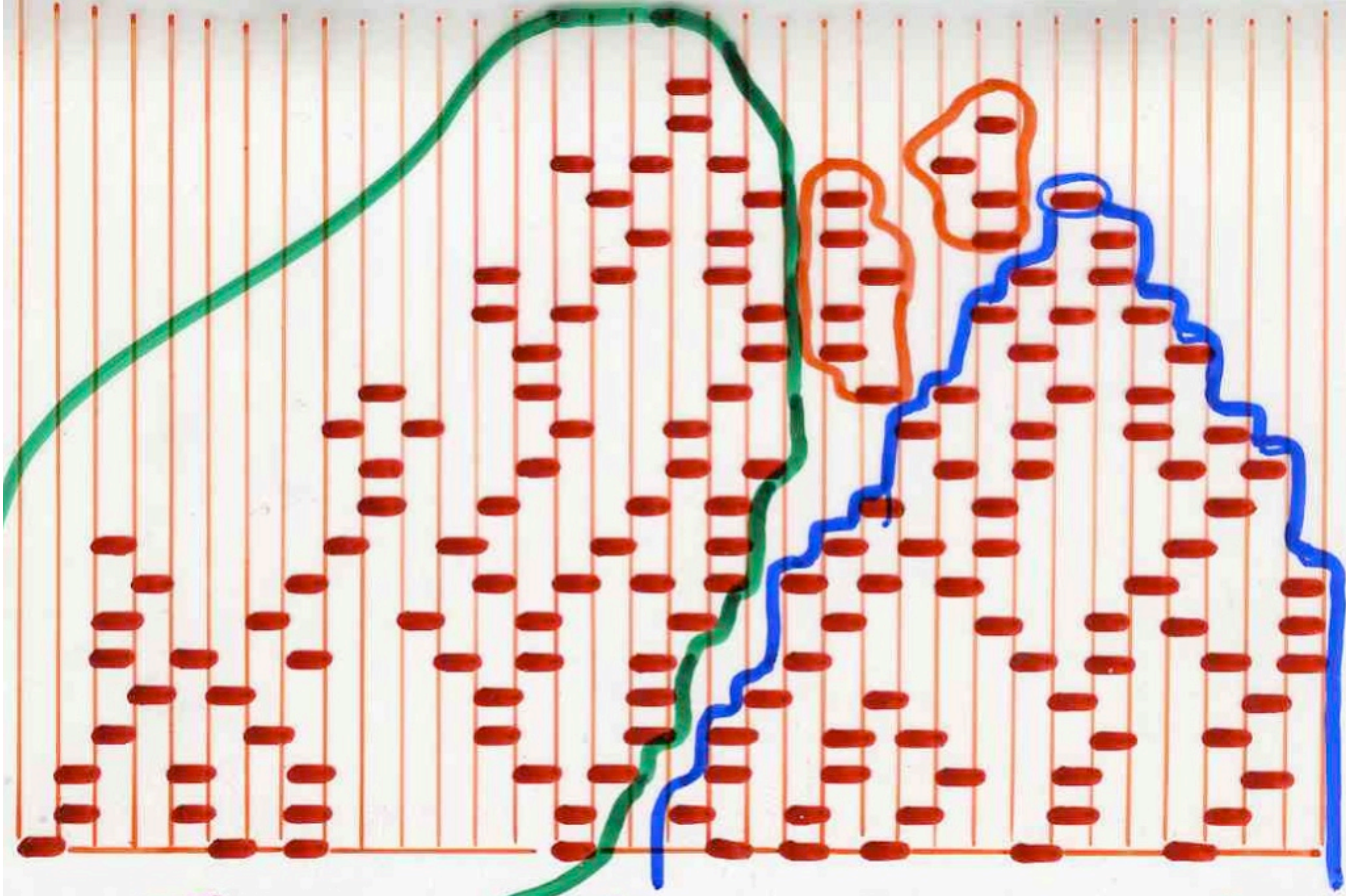
the nordic decomposition of a heap of dimers



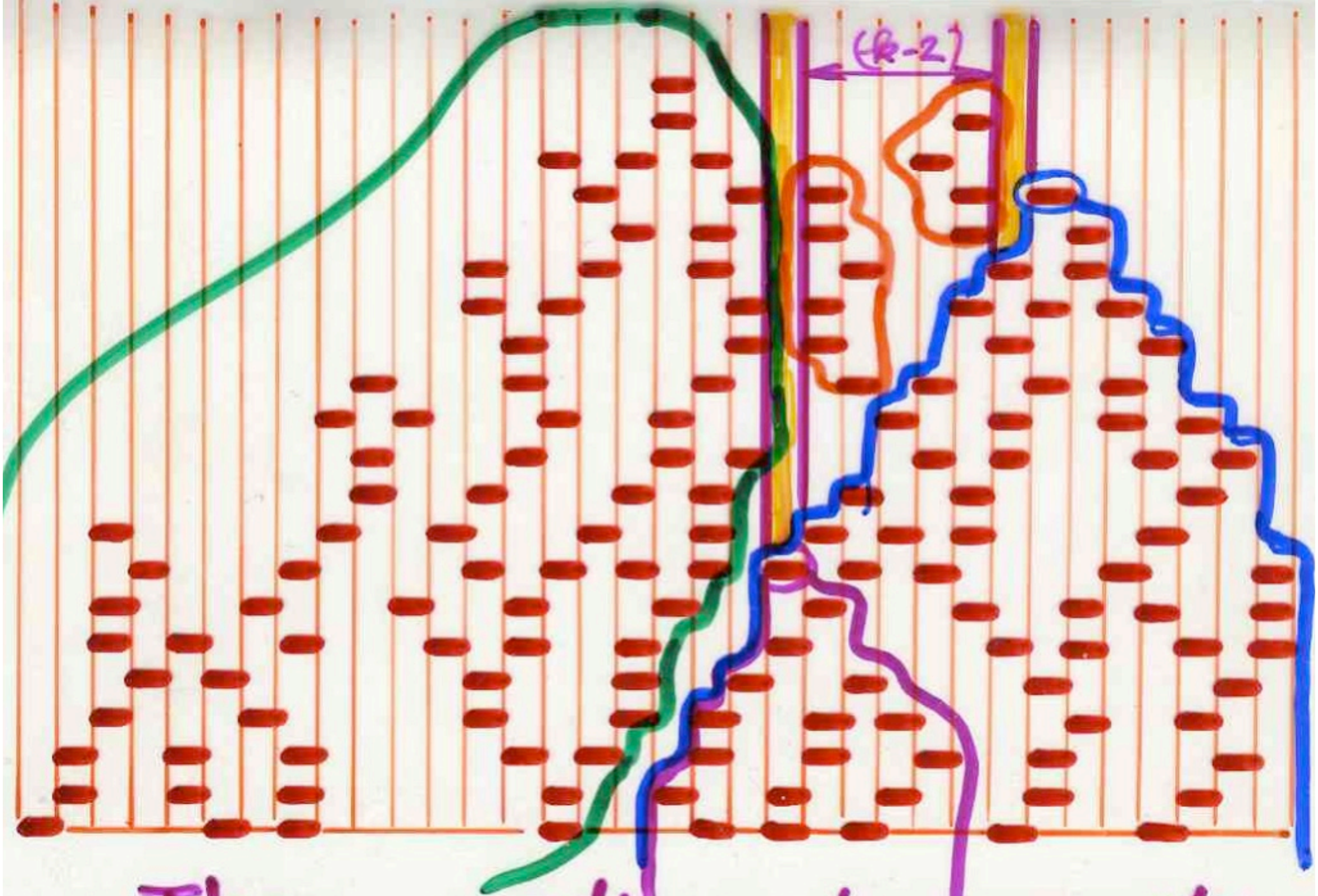
The nordic decomposition



The nordic decomposition



The nordic decomposition



The nordic decomposition

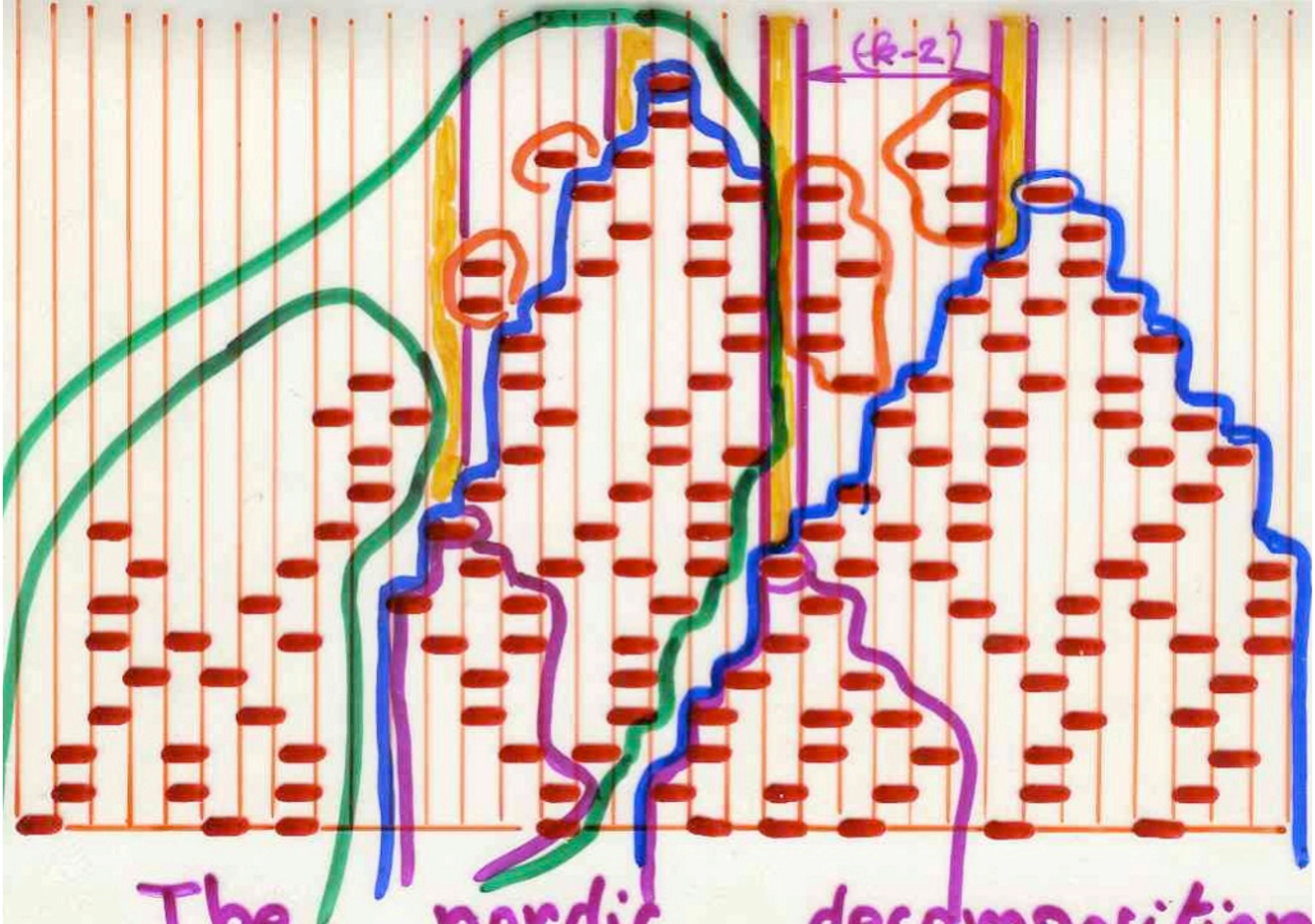
$$C = \frac{Q}{1-Q} + C \sum_{k \geq 1} \frac{Q}{1-Q} \times Q^k \times \frac{1}{F_{k-1}}$$

$$F_n = \frac{(1-Q^{n+1})}{(1-Q)(1+Q)^n}$$

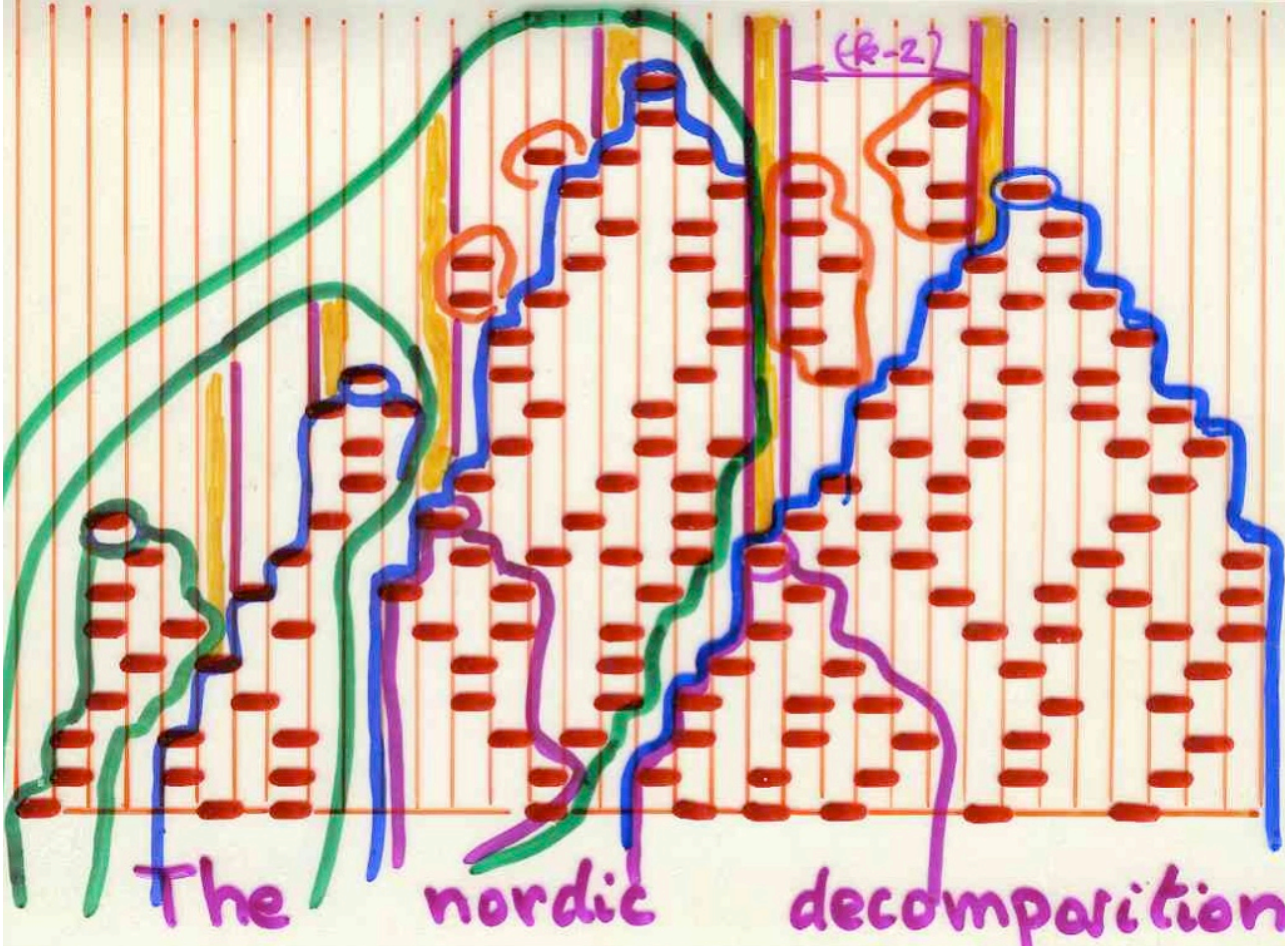
$$C = \frac{Q}{1-Q} \times \frac{1}{\left[1 - \left(\sum_{k \geq 1} \frac{Q}{1-Q} \times Q^{k} \times \frac{1}{F_{k-1}}\right)\right]}$$

connected heap

bijective proof X.V. (2002)

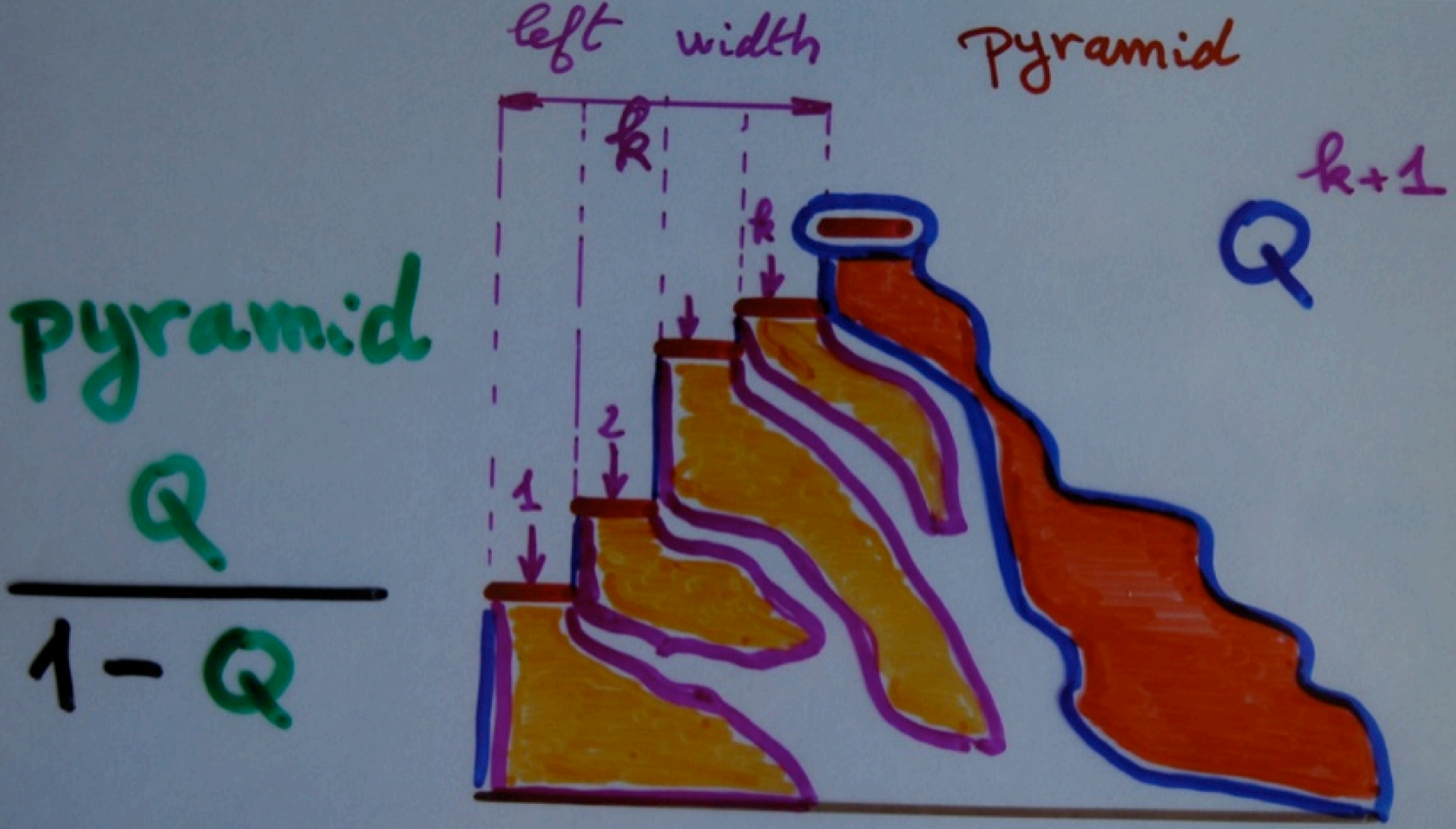


The nordic decomposition


(k-2)
The nordic decomposition

$$C = \frac{Q}{1-Q} + C \sum_{k \geq 1} \frac{Q}{1-Q} \times Q^{k} \times \frac{1}{F_{k-1}}$$


left width
Pyramid
k
k+1
Q
k
1
2


left width
pyramid
k
k
1
2
k+1
Q
pyramid
Q
1 - Q

$$F_n = \frac{(1 - Q^{n+1})}{(1-Q)(1+Q)^n}$$

$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n} \times (1 + Q + \ldots + Q^n)$$

$$C = \frac{Q}{(1-Q)\left[1 - \sum_{k \geq 1} \frac{Q^{k+1}(1+Q)^{-k-1}}{1-Q^{k}}\right]}$$

$$C = \frac{Q}{(1-Q)\left[1-\sum_{k\geq 1}\frac{Q^{k+1}(1+Q)^{-k-1}}{1-Q^{k}}\right]}$$

$$= \sum_{k\geq 1}\frac{Q^{k+1}}{1-Q^{k}(1+Q)}$$

$C(t)$ g.f. connected heap

(Bousquet-Mélou, Rechnitzer) 2002

$$C(t) = \frac{Q}{(1-Q)\left[1 - \sum_{k\geq 1} \frac{Q^{k+1}}{1-Q^{k}(1+Q)}\right]}$$

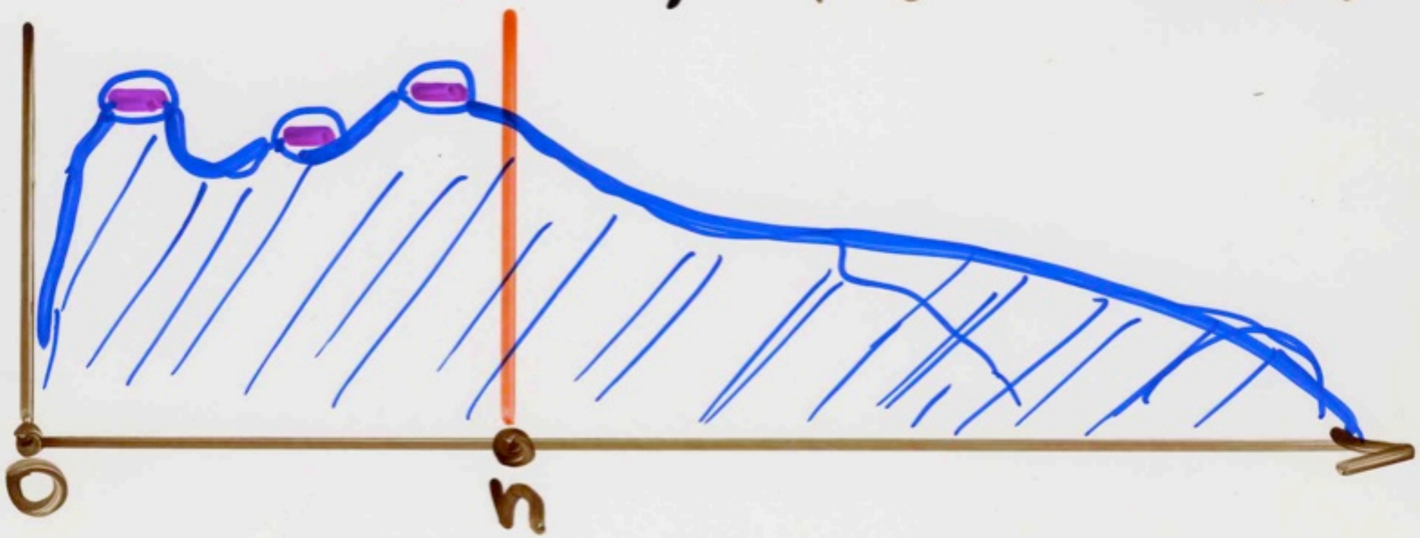
$$F_n = \frac{(1 - Q^{n+1})}{(1-Q)(1+Q)^n}$$

$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n} \times (1 + Q + \dots + Q^n)$$

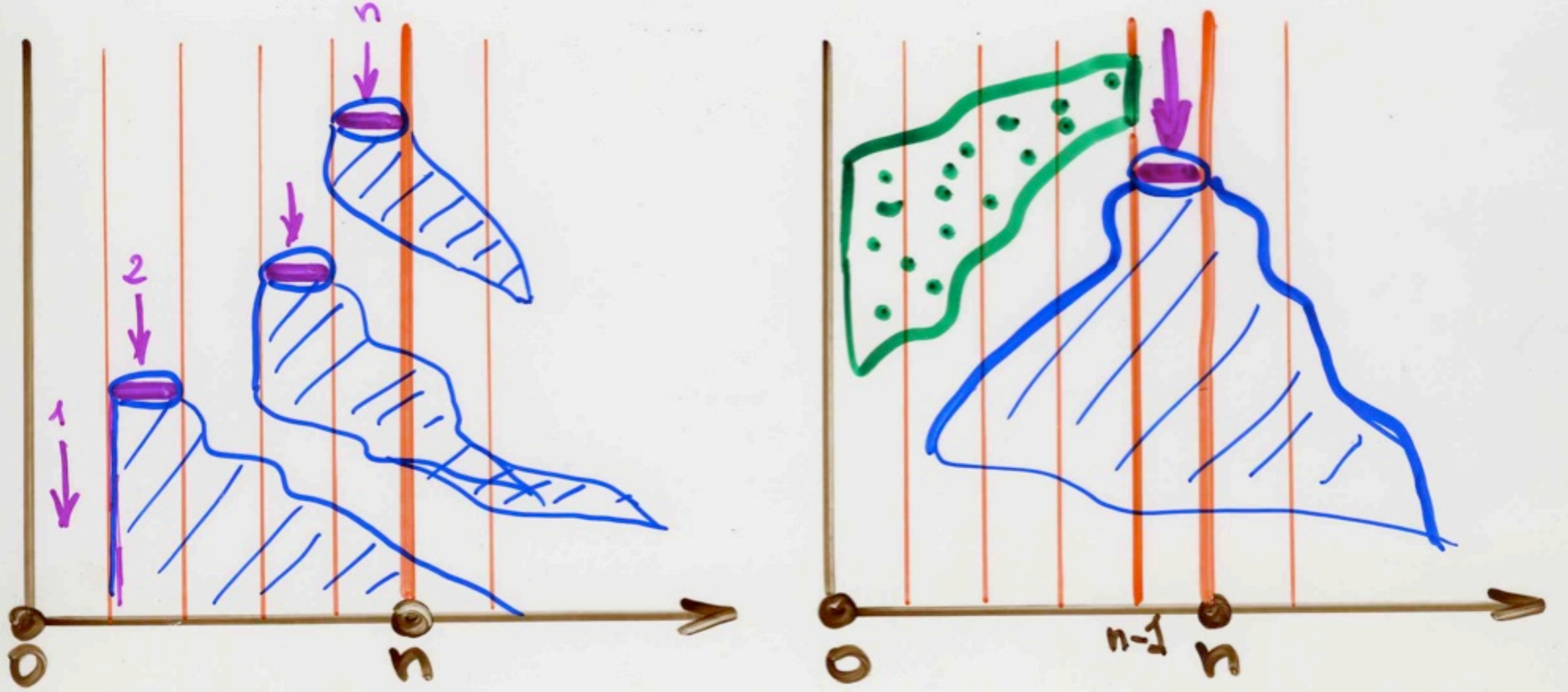
$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n} \times (1+Q+\ldots+Q^n)$$

heaps of dimers on $[0,\infty[$

maximal pieces, projection $\subseteq [0,n]$



$$\underbrace{(1+Q)}_{D}{}^{n} = \frac{1}{F_n} \times (1+Q+\dots+Q^n)$$

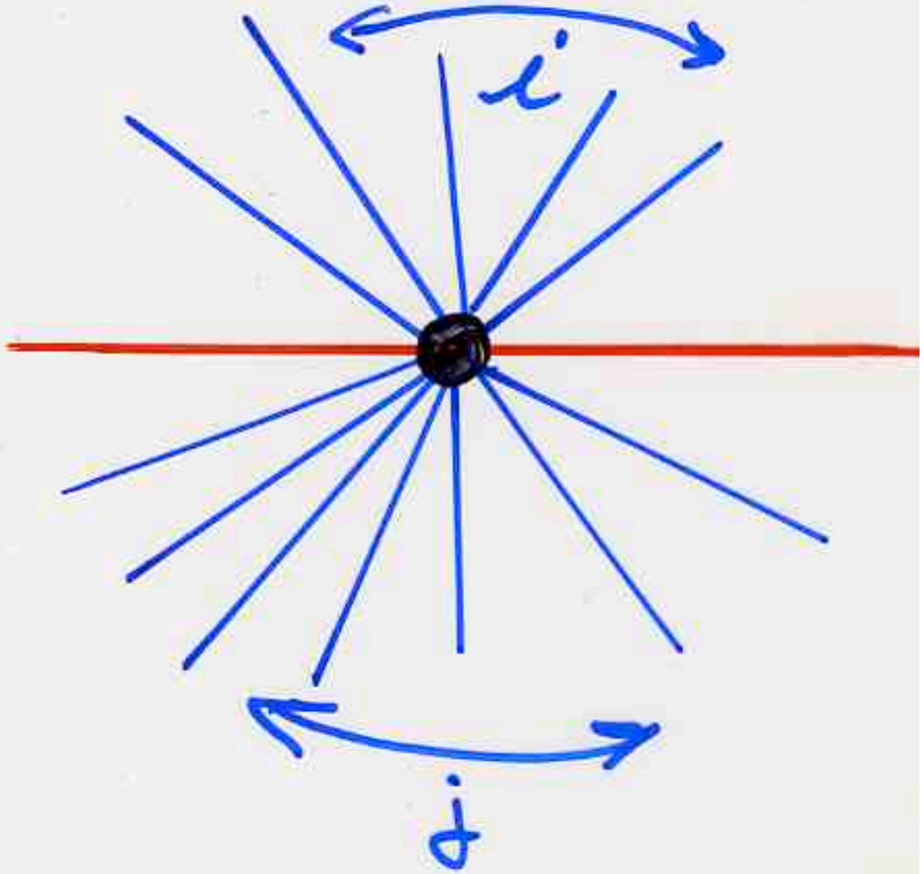


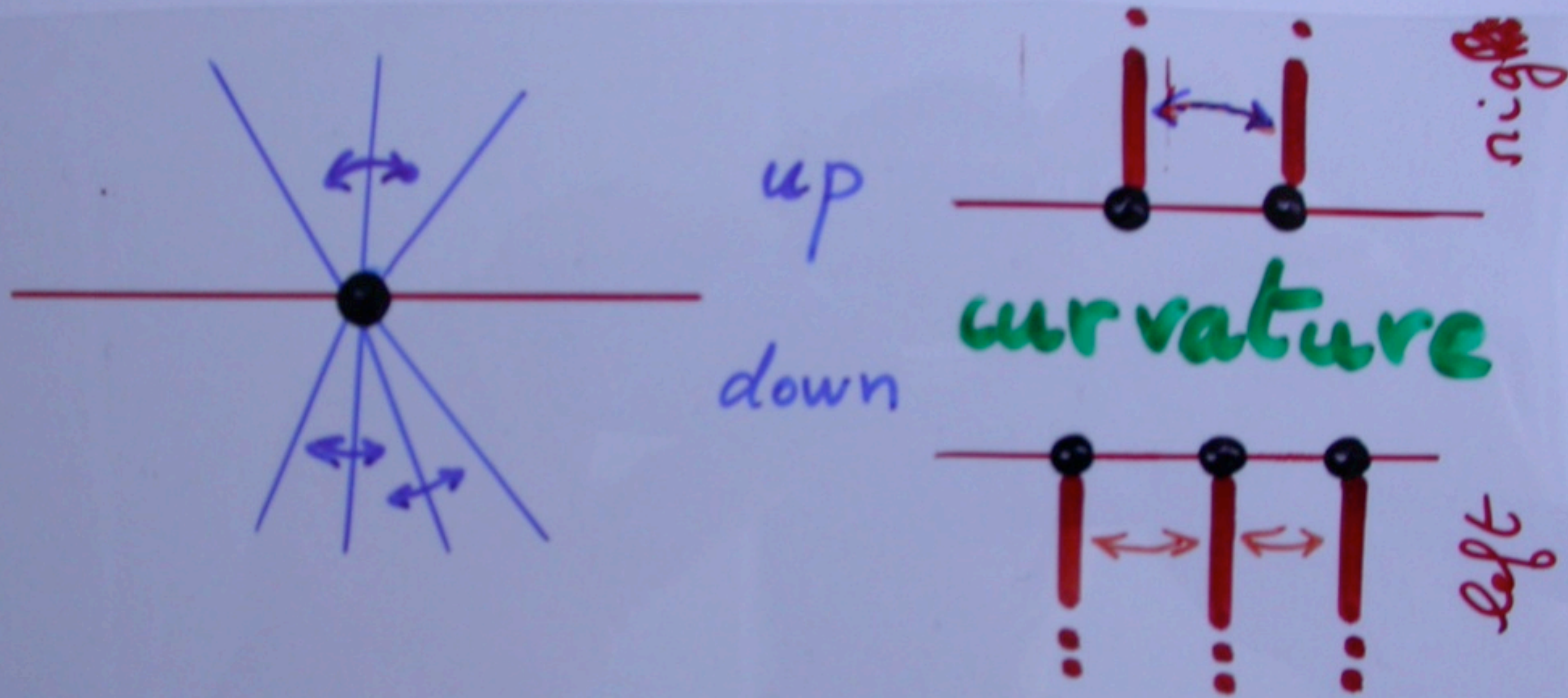
curvature

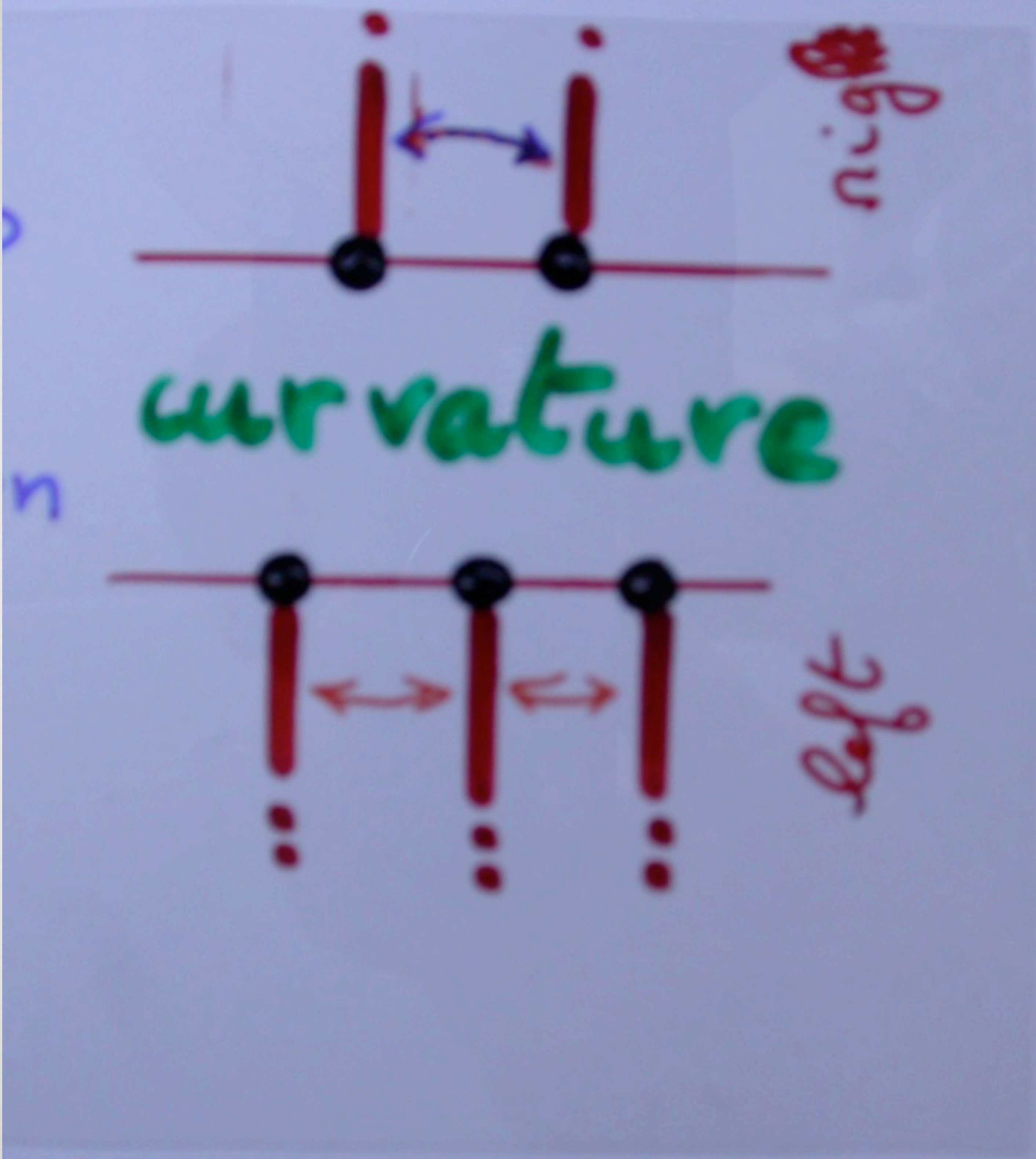
curvature

$$a^{|i-3|+|j-3|}$$

$$\prod_{\text{all points}} a^{(\dots)}$$



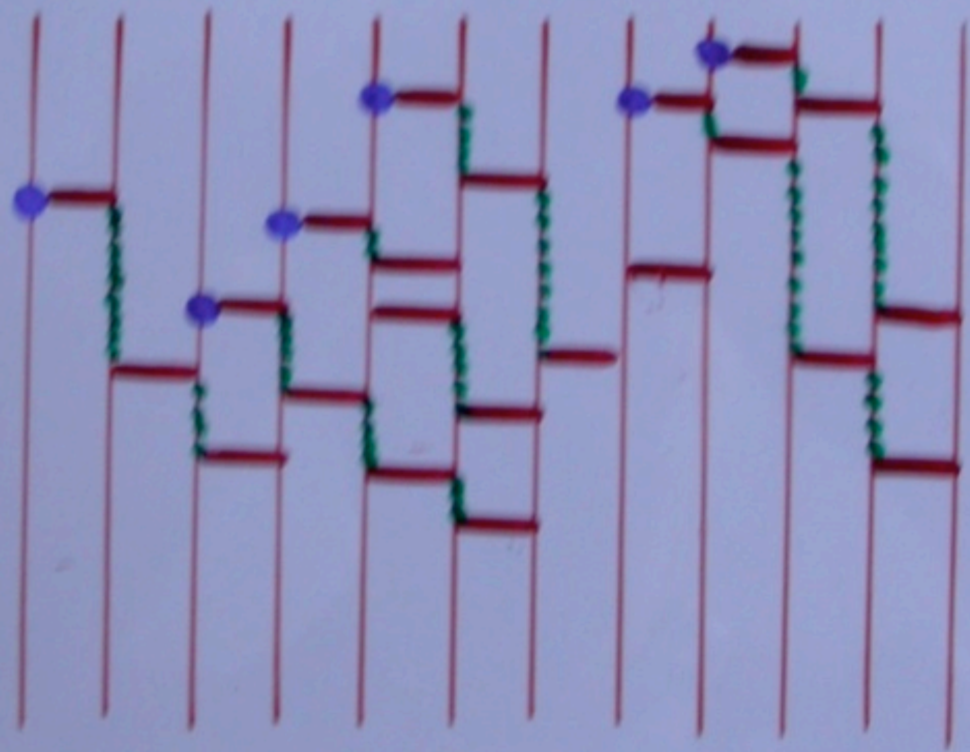

up
down
curvature
right
left


curvature
left

up - curvature

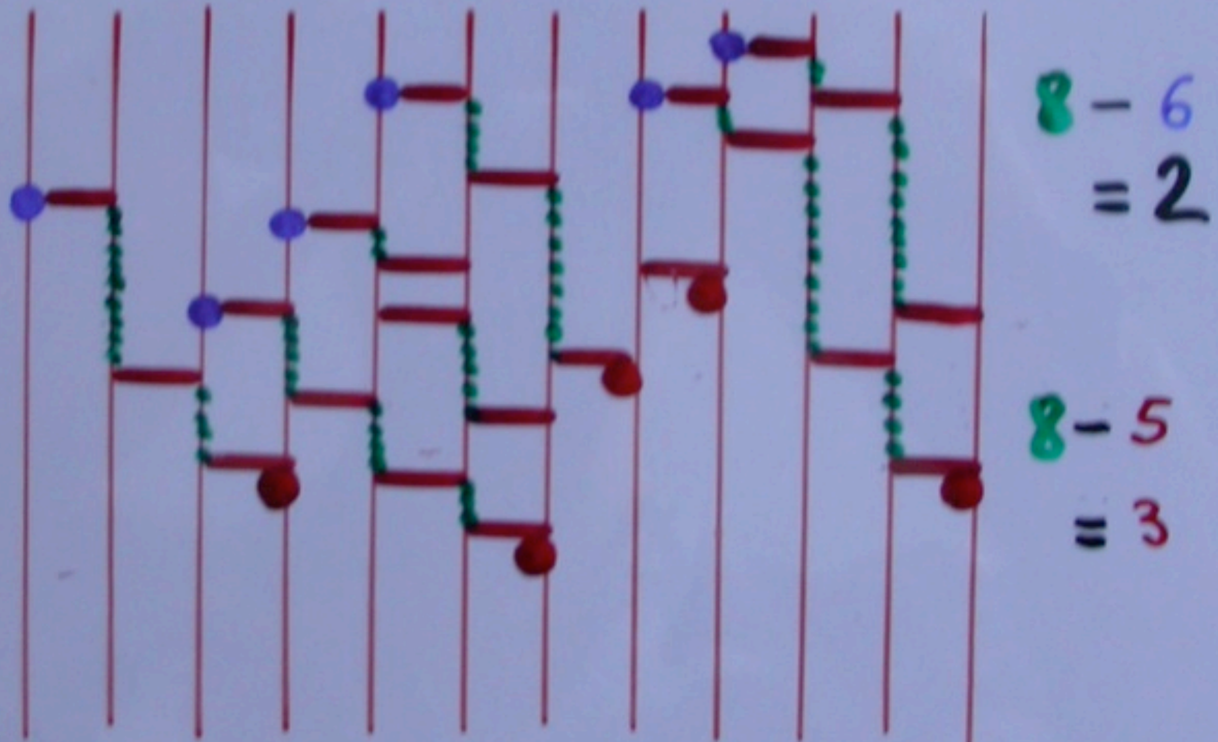
= nb of segments

− (nb of "left - nails" visible from above)



up- / down- **curvature** = nb of **segments**

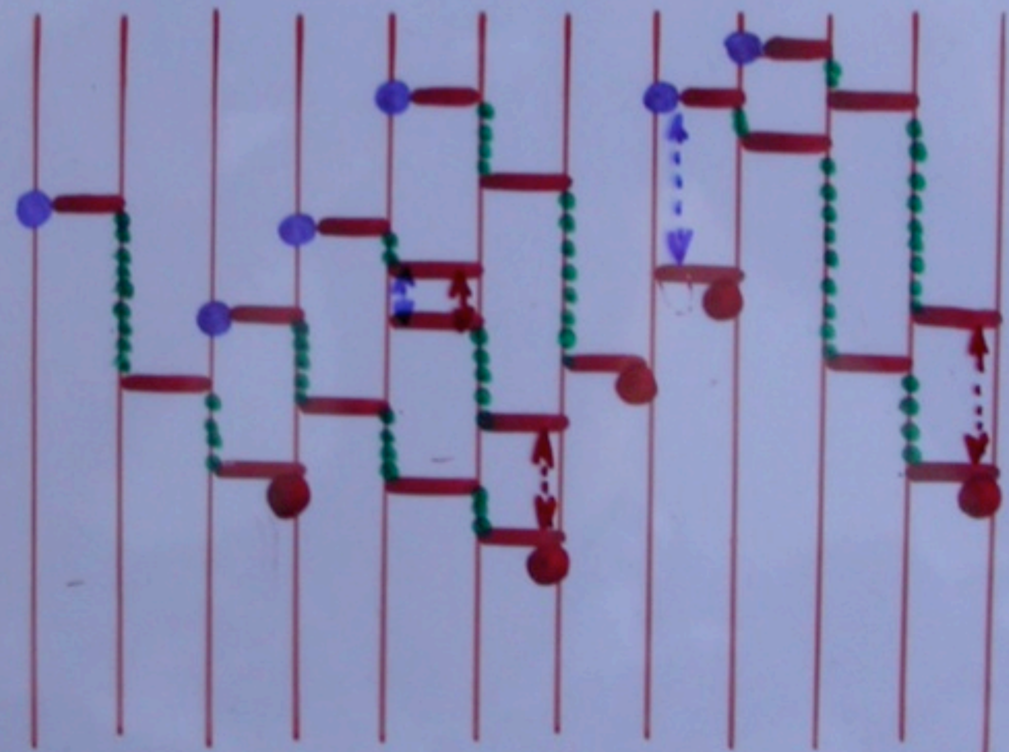
− (nb of "left- / right- nails" visible from above / below)



up - down **curvature**

= nb of **segments**

− (nb of "left - right nails" visible from above below)



8 − 6

= 2

8 − 5

= 3

heaps of dimers on [0, n] with

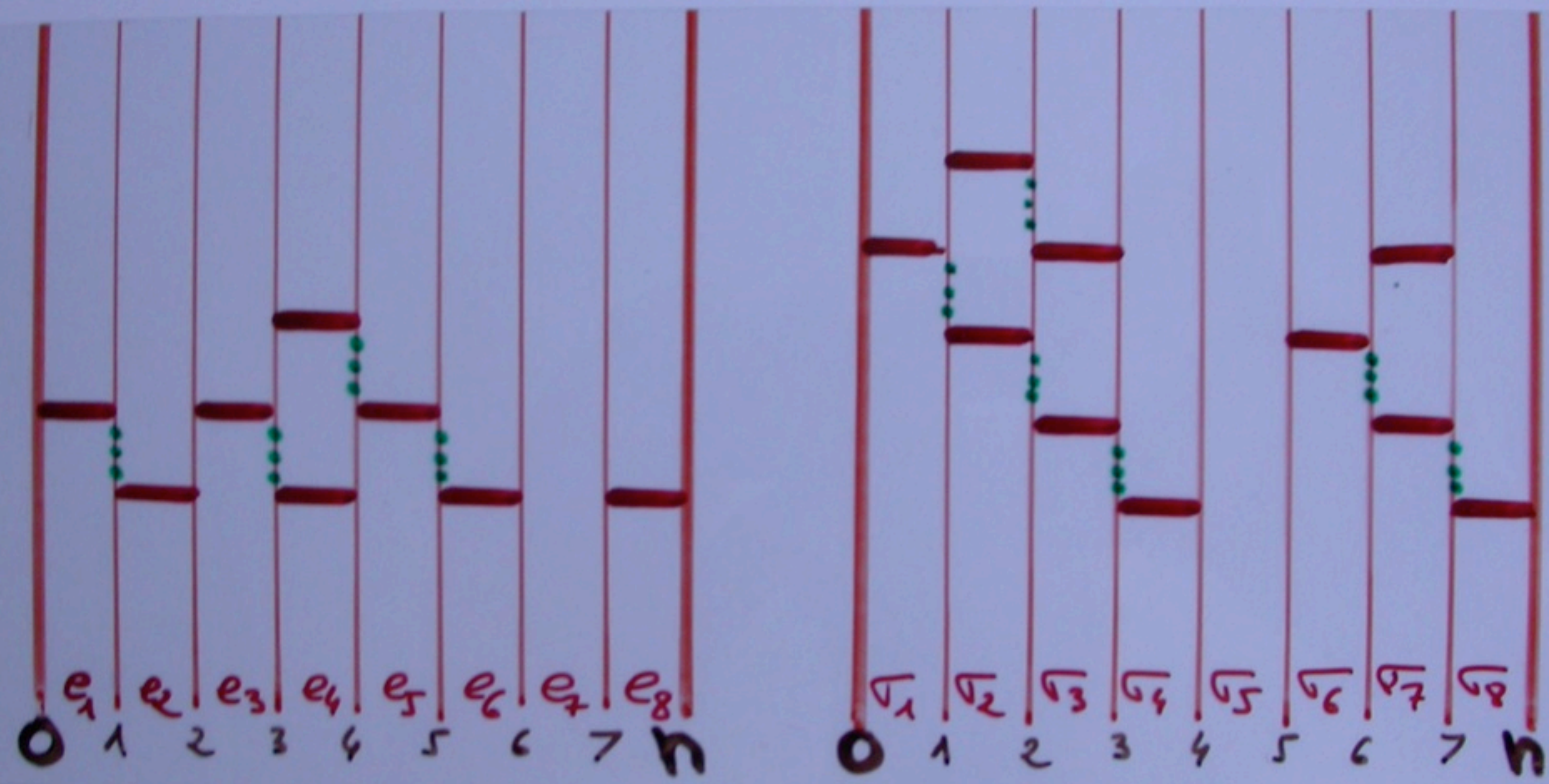
	number
(total) curvature = 0	?
up- curvature = 0	?

heaps of dimers

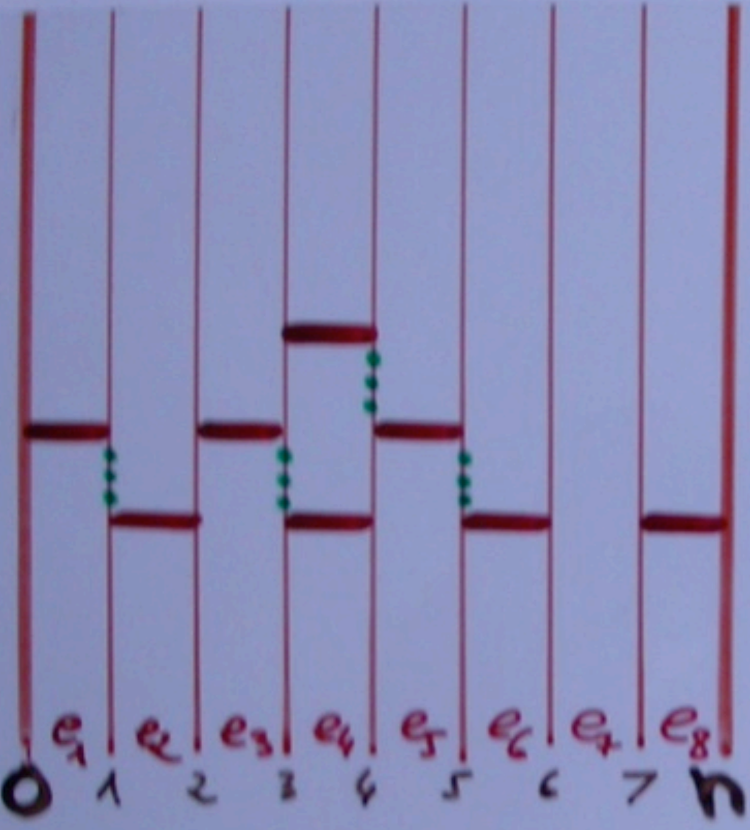
on $[0, n]$

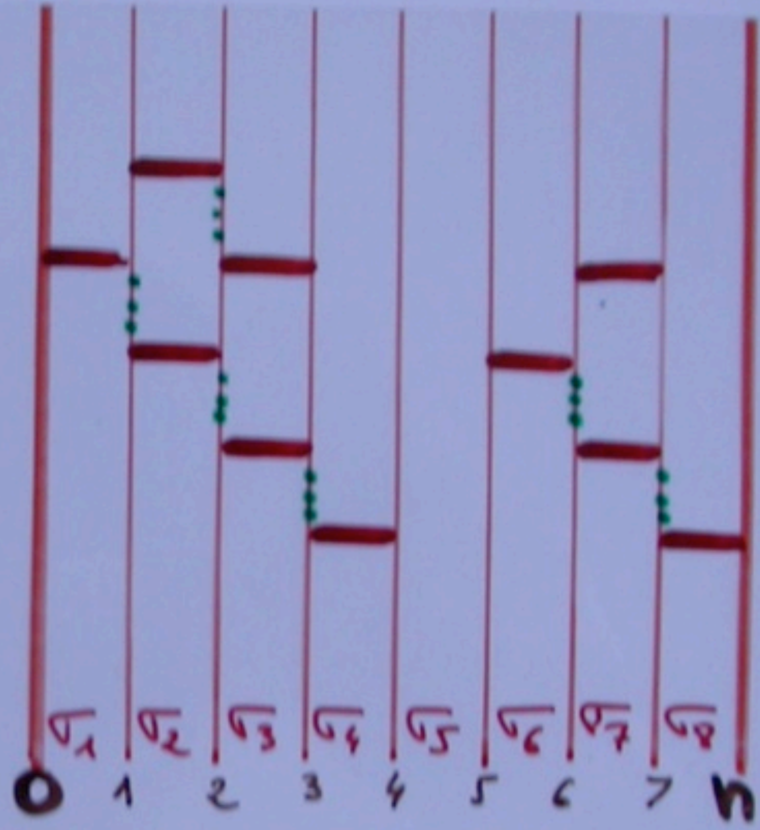
with

(total) curvature $= 0$	C_n
up-curvature $= 0$	$n!$



e1 e2 e3 e4 e5 e6 e7 e8
0 1 2 3 4 5 6 7 n
σ1 σ2 σ3 σ4 σ5 σ6 σ7 σ8
0 1 2 3 4 5 6 7 n



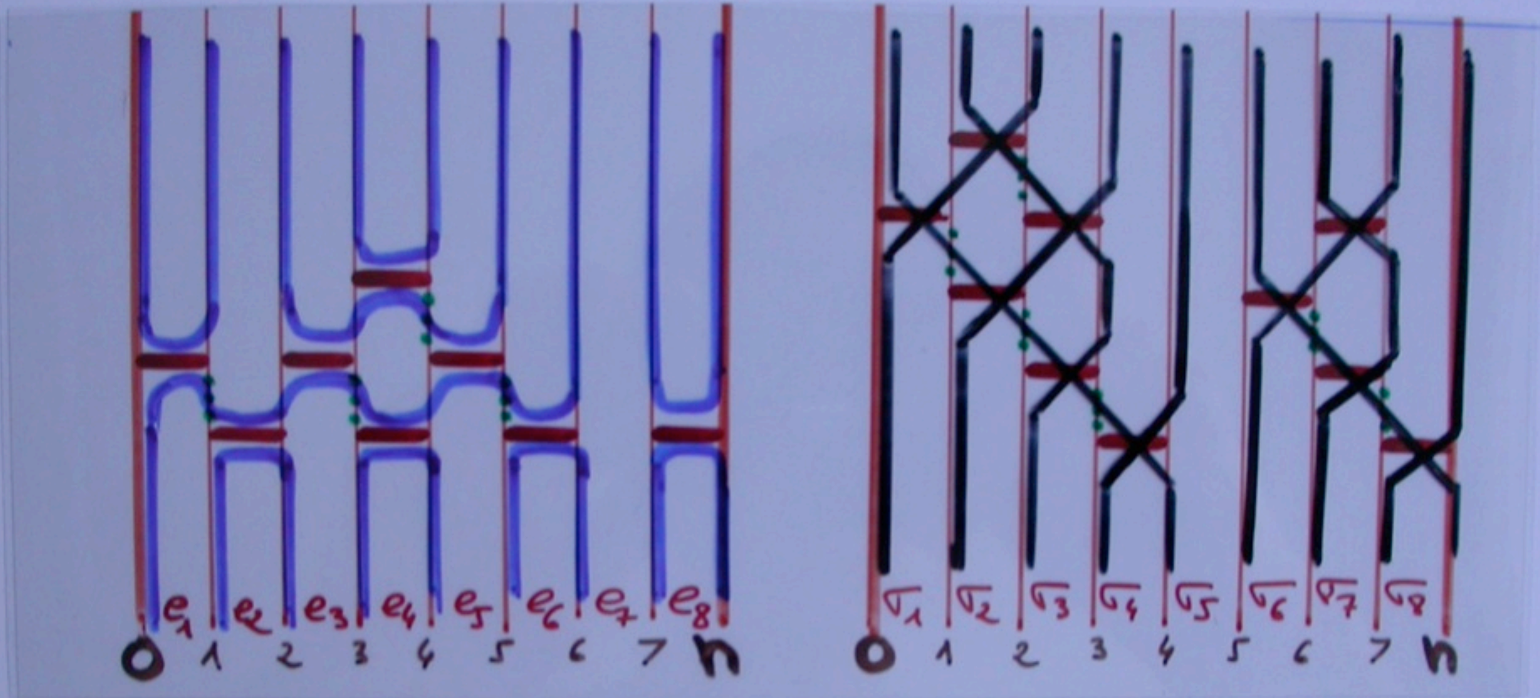


basis
Temperley-Lieb
algebra

$(e_2 e_1)(e_4 e_3)(e_6 e_5 e_4)(e_8)$

basis
symmetric group
algebra

$(\sigma_4 \sigma_3 \sigma_2 \sigma_1)(\sigma_3 \sigma_2)(\sigma_3 \sigma_7 \sigma_6)(\sigma_7)$



basis
Temperley-Lieb
algebra

$(e_2 e_1)(e_4 e_3)(e_6 e_5 e_4)(e_8)$

basis
symmetric group
algebra

$(\sigma_4 \sigma_3 \sigma_2 \sigma_1)(\sigma_3 \sigma_2)(\sigma_3 \sigma_7 \sigma_6)(\sigma_7)$

limite continue

$$G_\Lambda(L_1, L_2; T) = \frac{e^{-(\coth\sqrt{\Lambda}T)\sqrt{\Lambda}(L_1+L_2)}}{\operatorname{sh}\sqrt{\Lambda}\,T} \; \frac{\sqrt{\Lambda L_1 L_2}}{L_2} \, I_1\left(\frac{2\sqrt{\Lambda L_1 L_2}}{\operatorname{sh}\sqrt{\Lambda}\,T}\right)$$

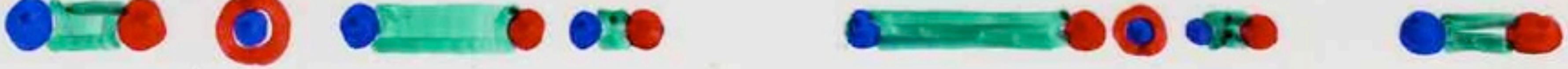
continuum limit

Bessel functions

q-Bessel


80
FRIEDRICH WILHELM BESSEL
1784 Astronom und Mathematiker 1846
DEUTSCHE BUNDESPOST
1984

$$D = \sum_{n \geq 0} \frac{(-1)^n \, t^n q^n \, q^{\binom{n}{2}}}{(1-q)\cdots(1-q^n)(1-uq)\cdots(1-uq^n)}$$



$$D = \sum_{C} (-1)^{|C|} v(C)$$

(q-Bessel)

C configuration of 2 by 2 disjoint segments

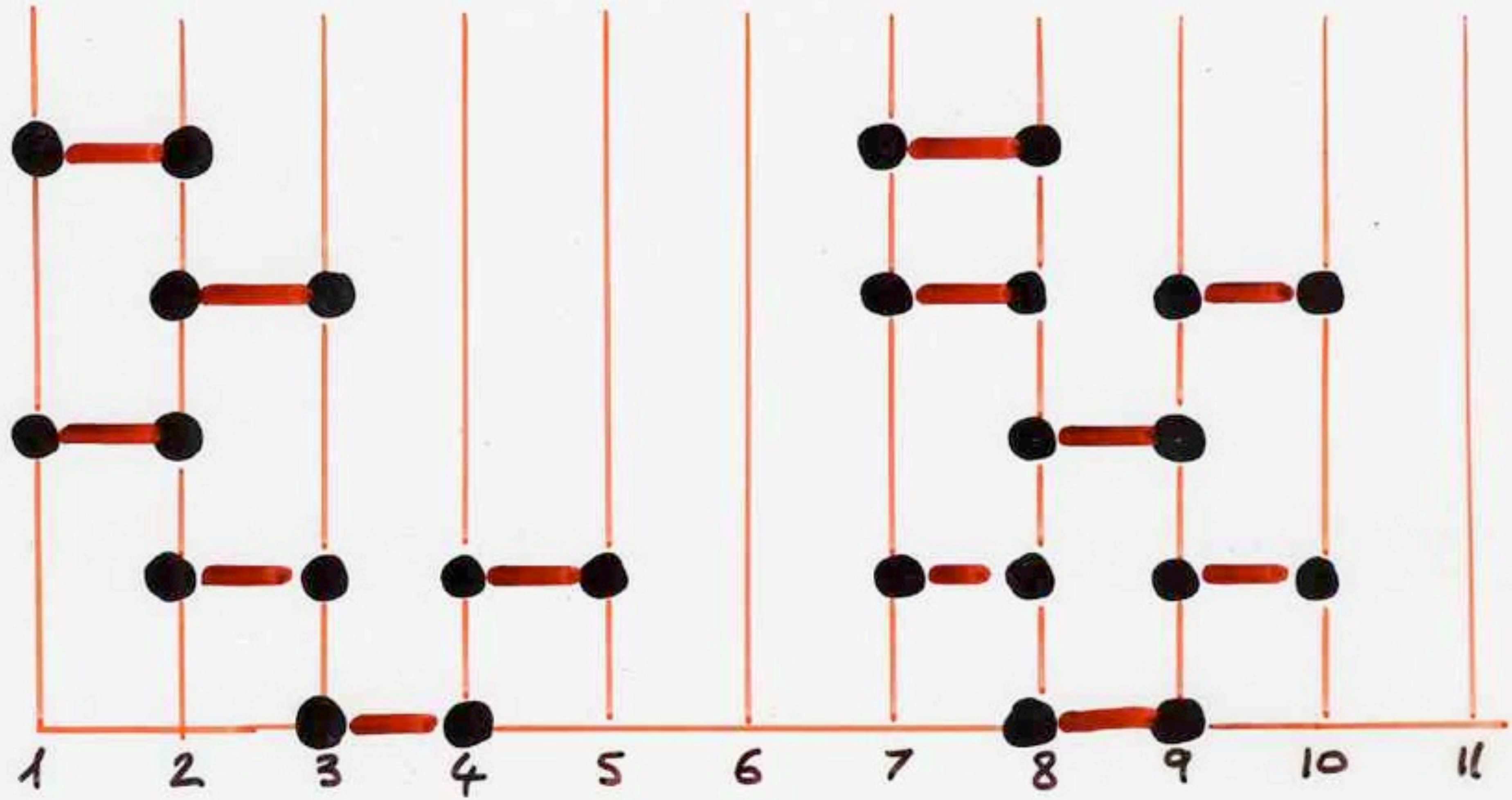
$$v(C) = \prod v(\text{each segment})$$

the stairs decomposition of a heap of dimers

(useful for:

-basis of Temperley-Lieb algebra

- curvature in 2D causal Lorentzian triangulations)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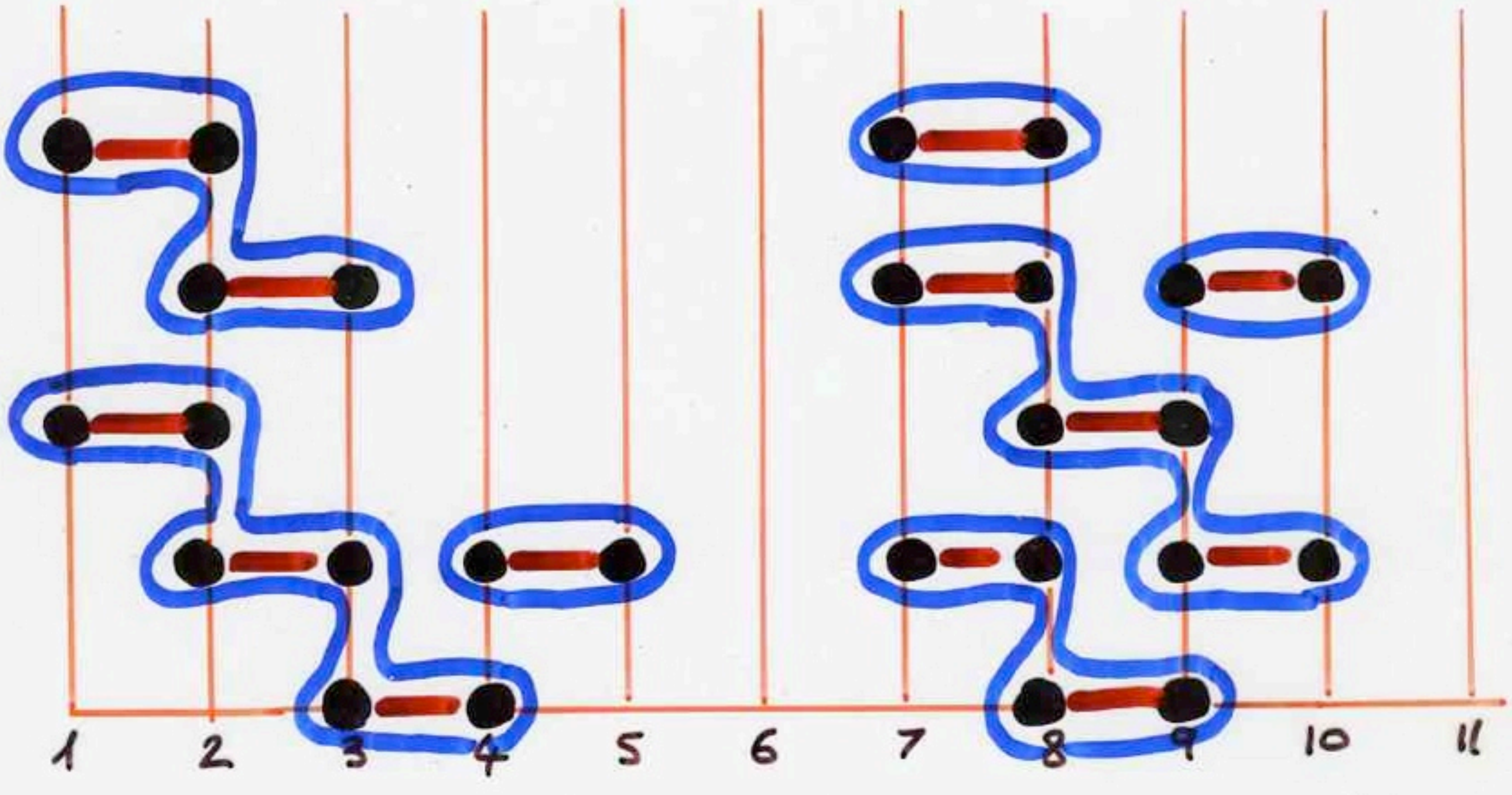
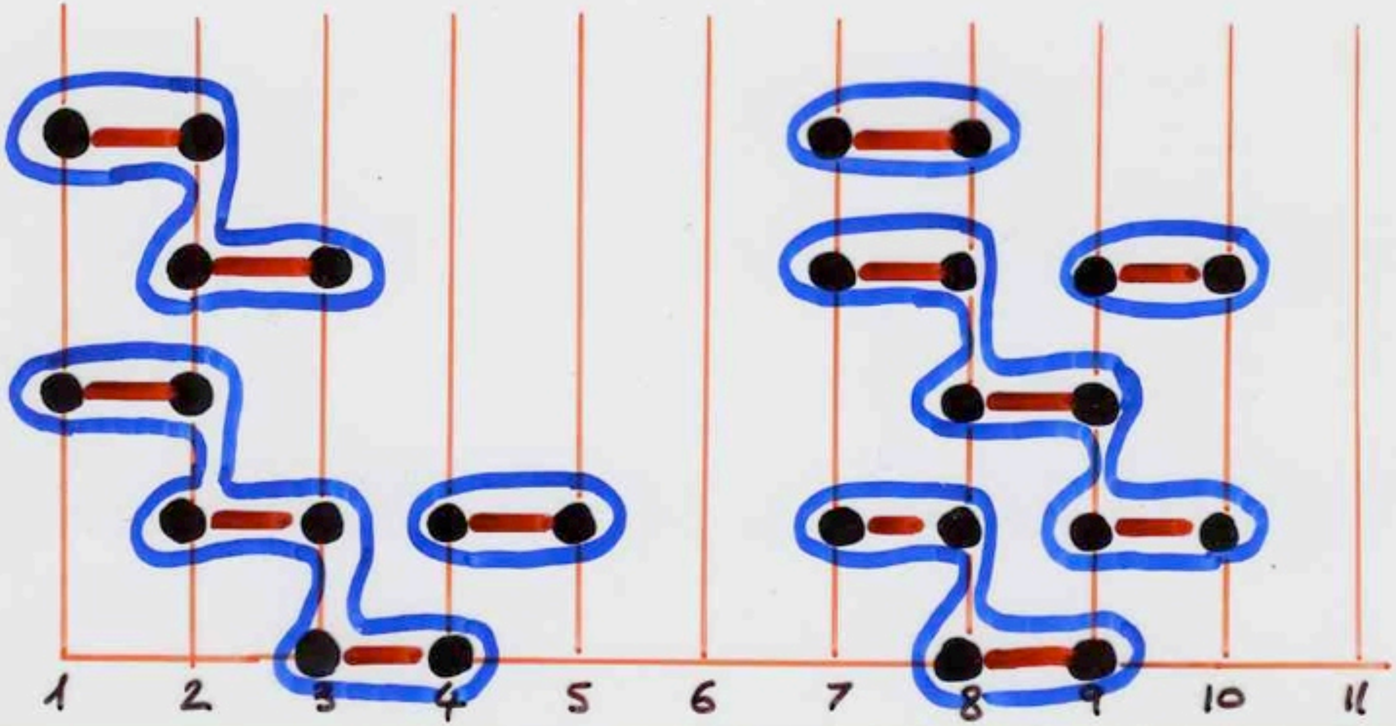
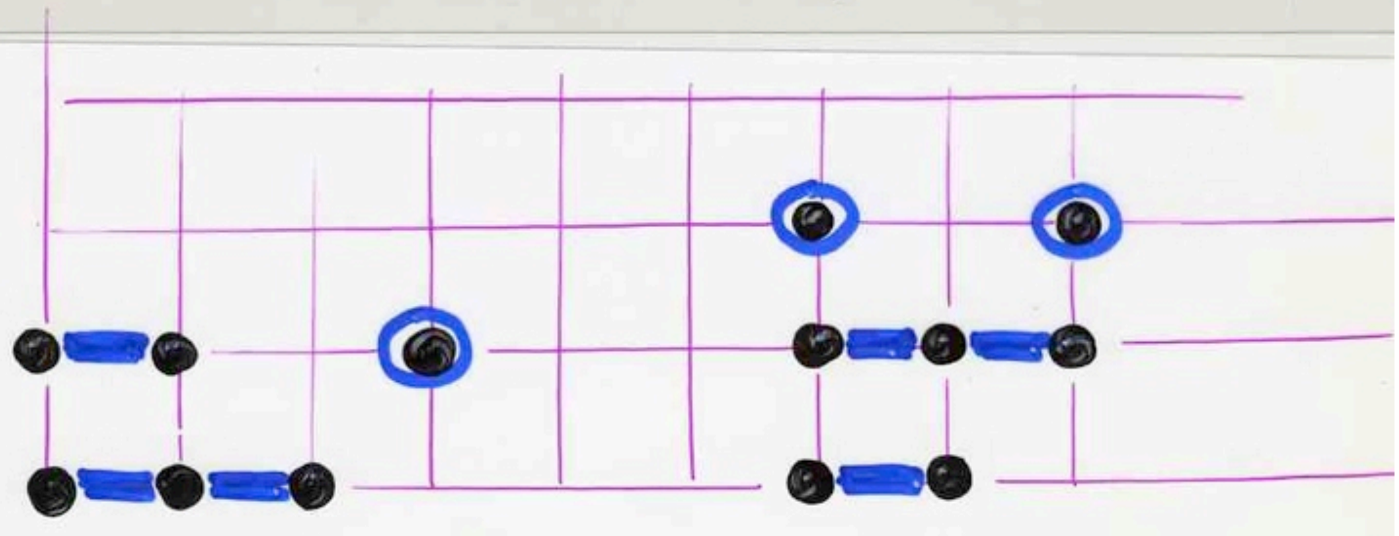
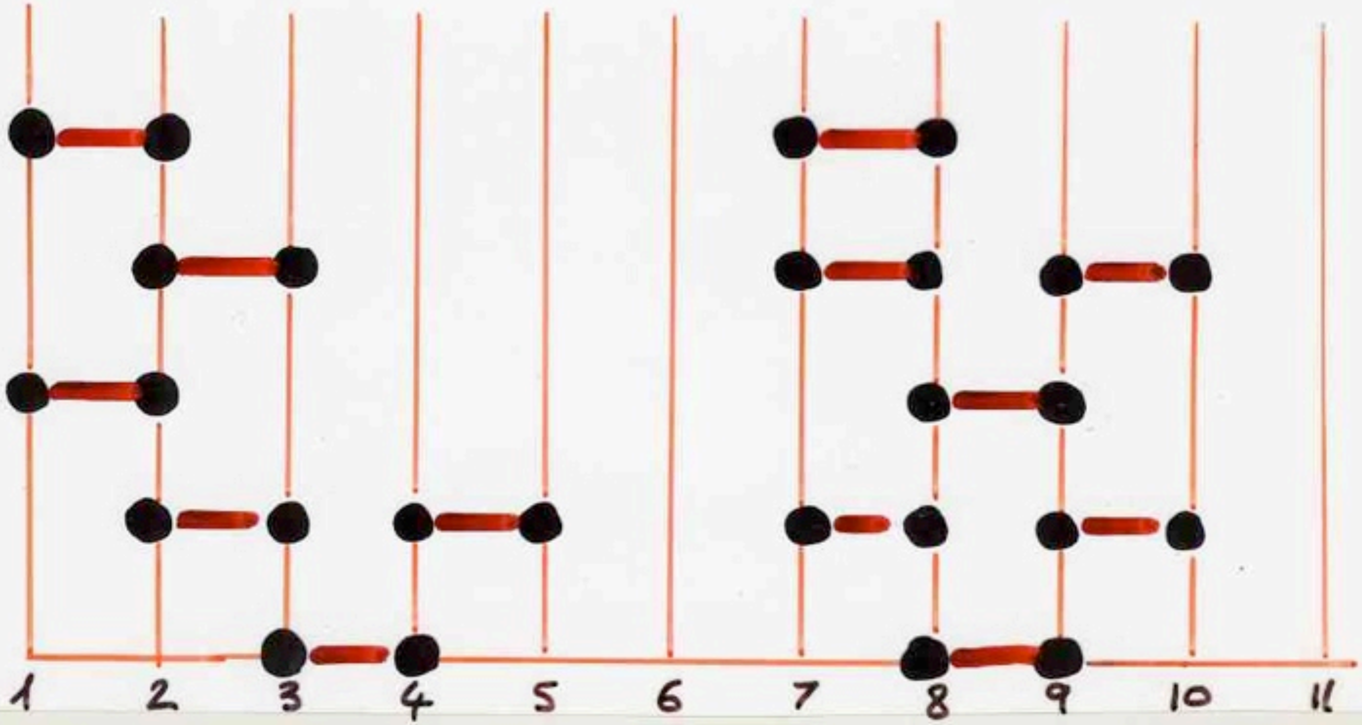


stairs decomposition



stairs decomposition

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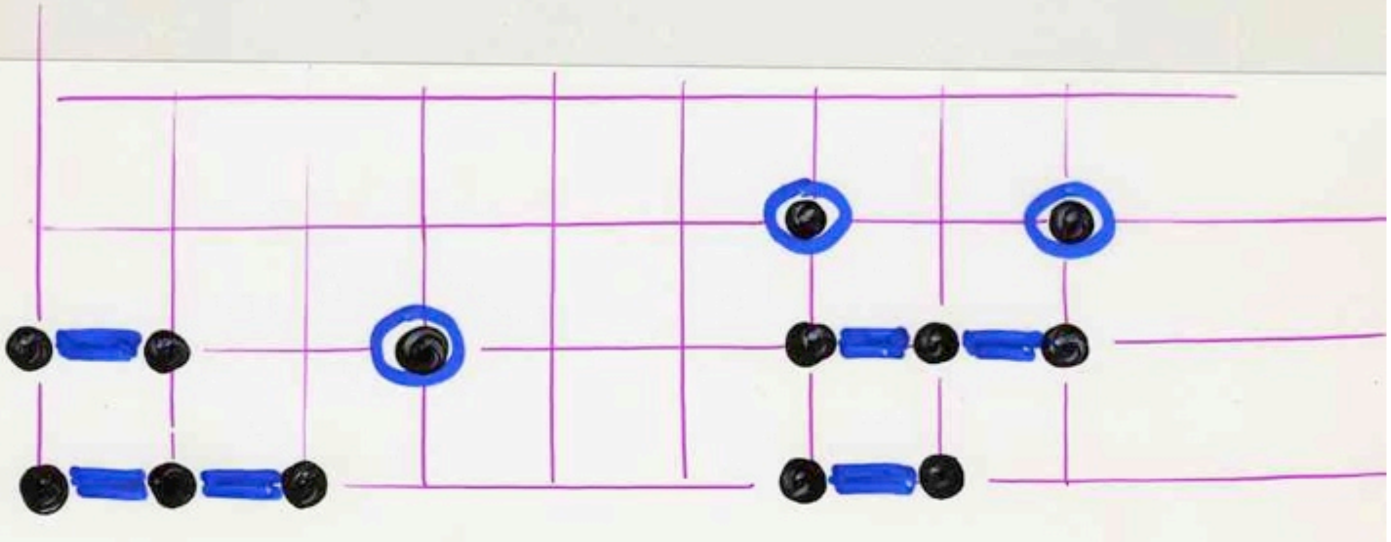
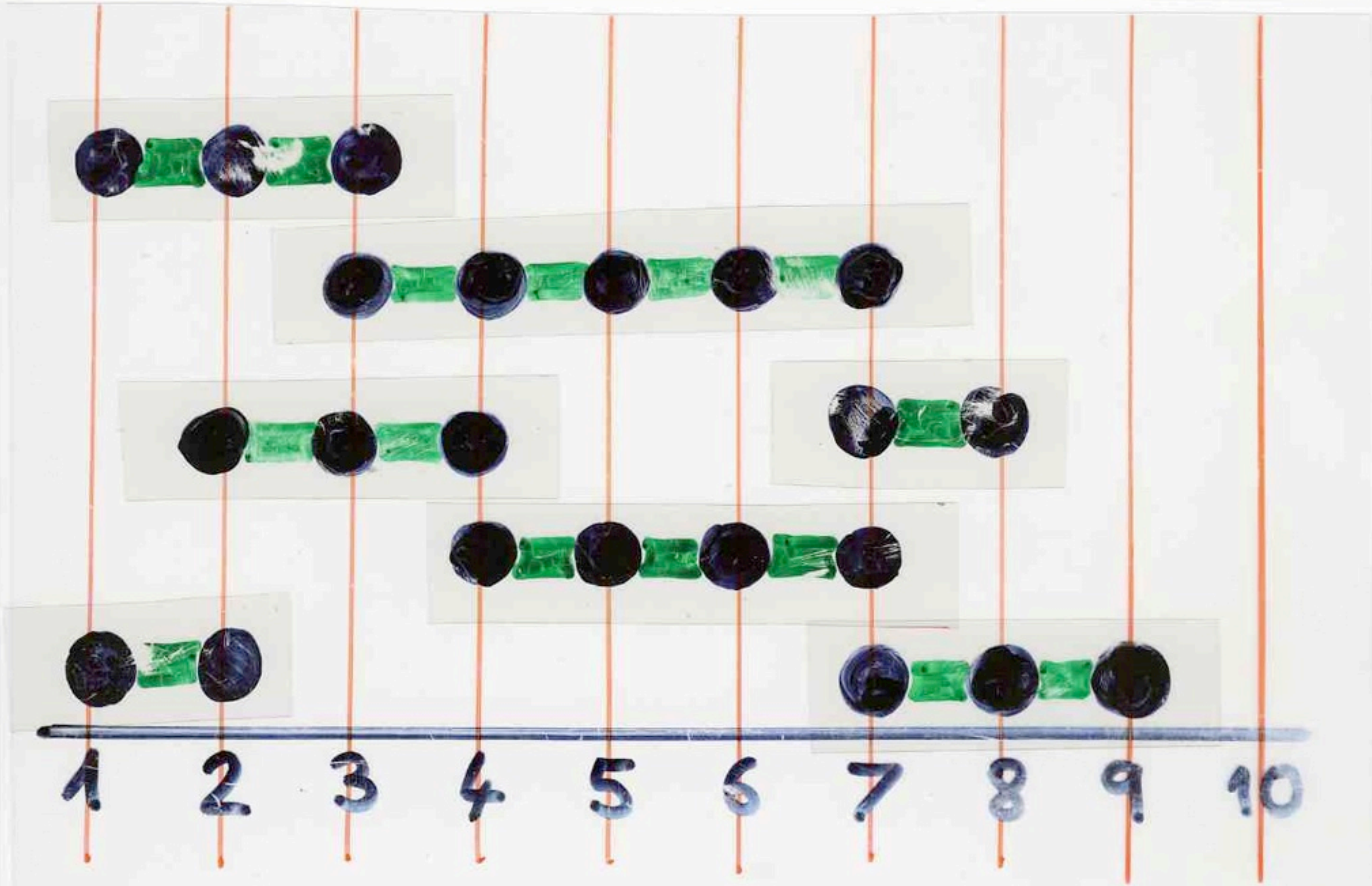


generating function

$$f(t,u;q) = \frac{N}{D}$$



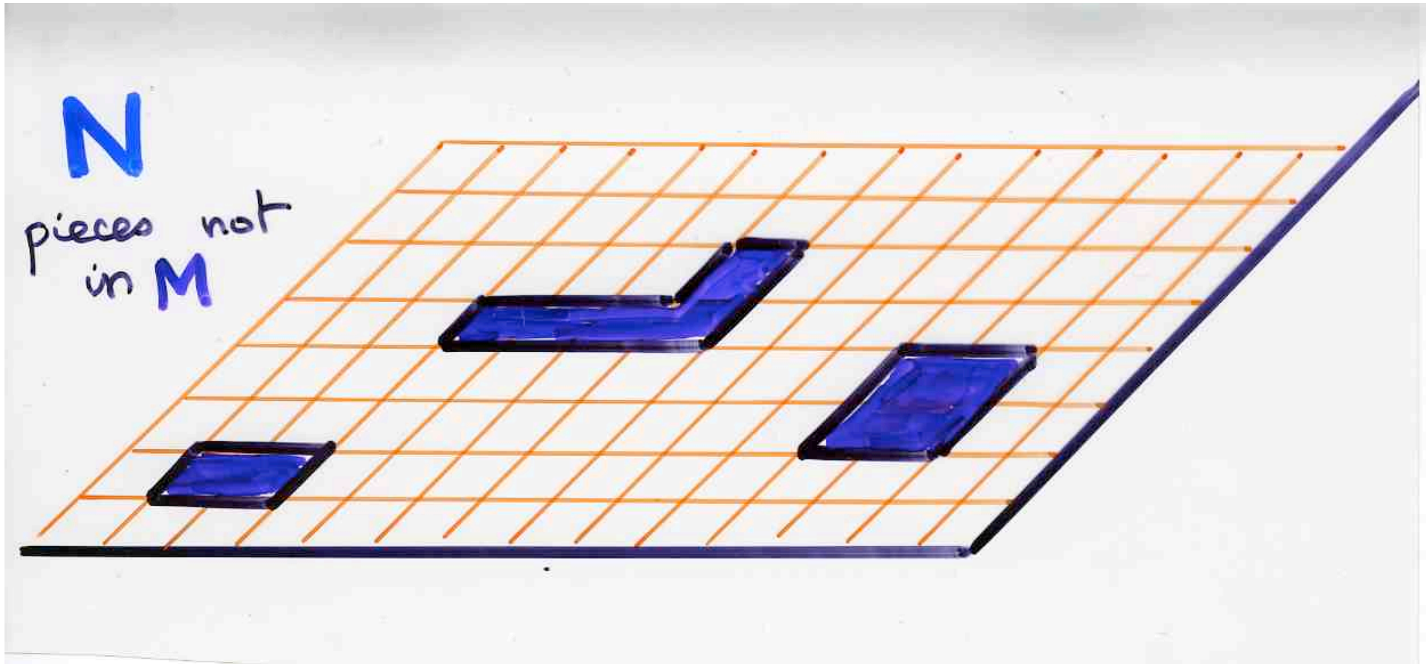
extension of the inversion lemma

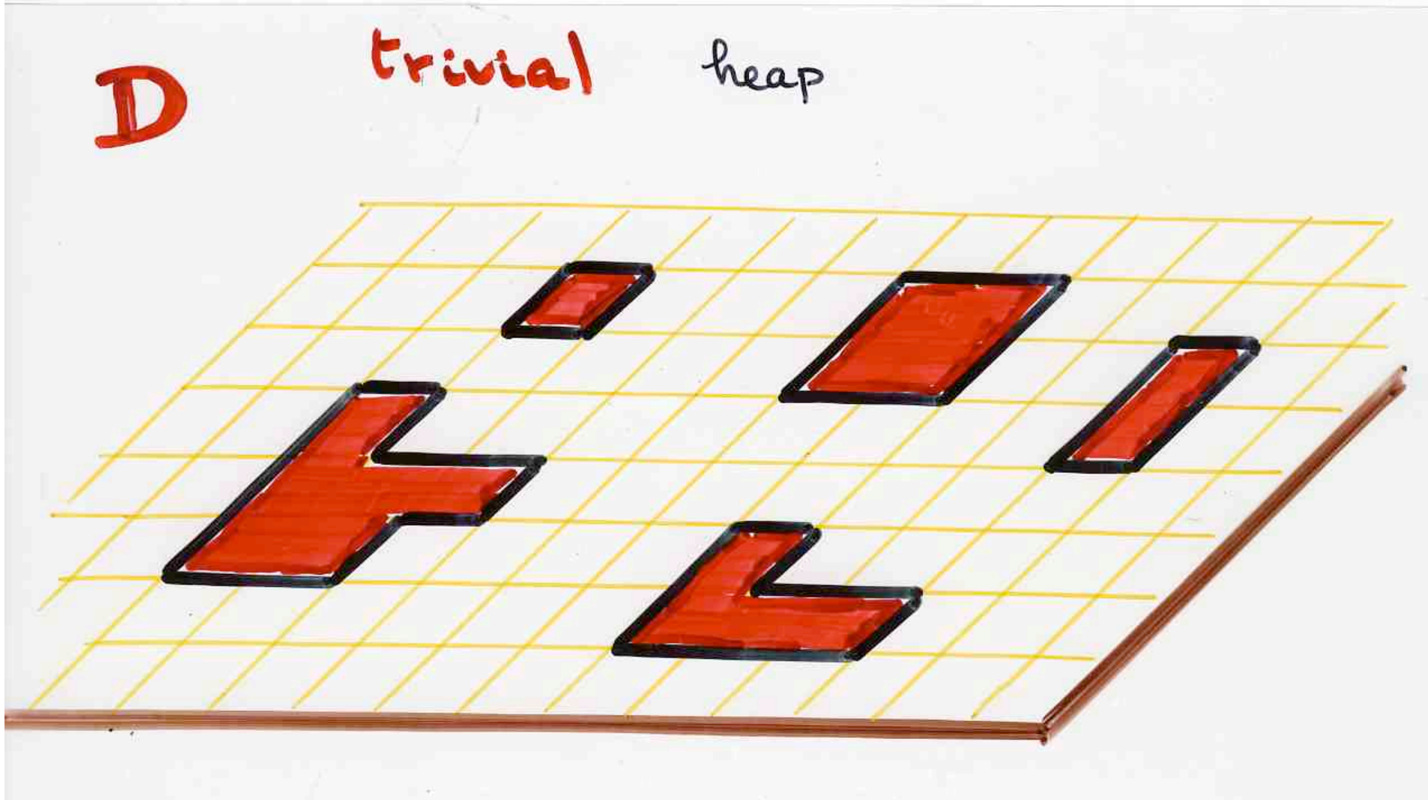
$M \subseteq P$

$$\sum_{\substack{E \\ \Pi(\text{maximal pieces}) \in M}} v(E) = \frac{N}{D}$$

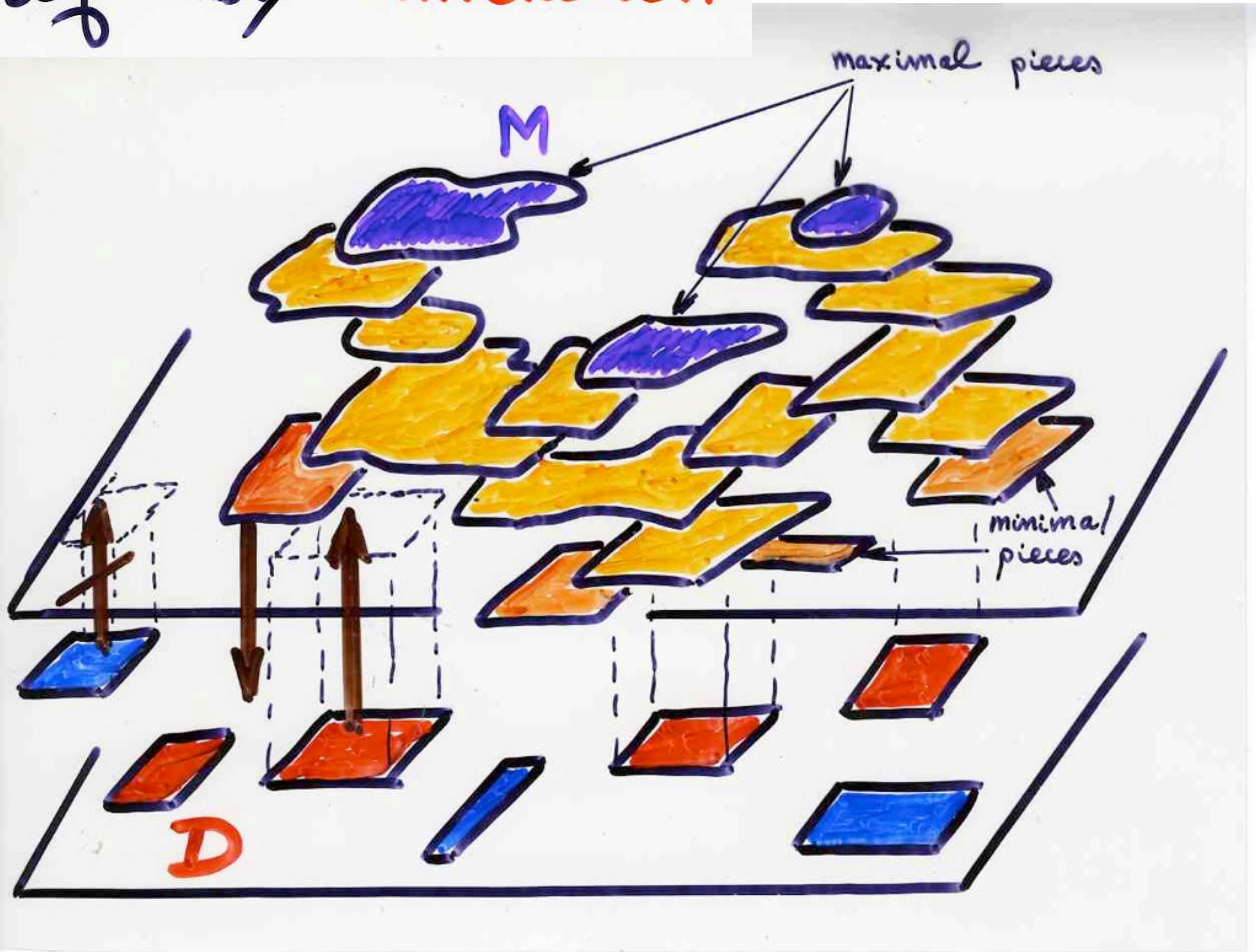
$$D = \sum_{\substack{F \\ \text{trivial heaps}}} (-1)^{|F|} v(F)$$

$$N = \sum_{\substack{F \\ \text{trivial heaps} \\ \text{pieces} \notin M}} (-1)^{|F|} v(F)$$

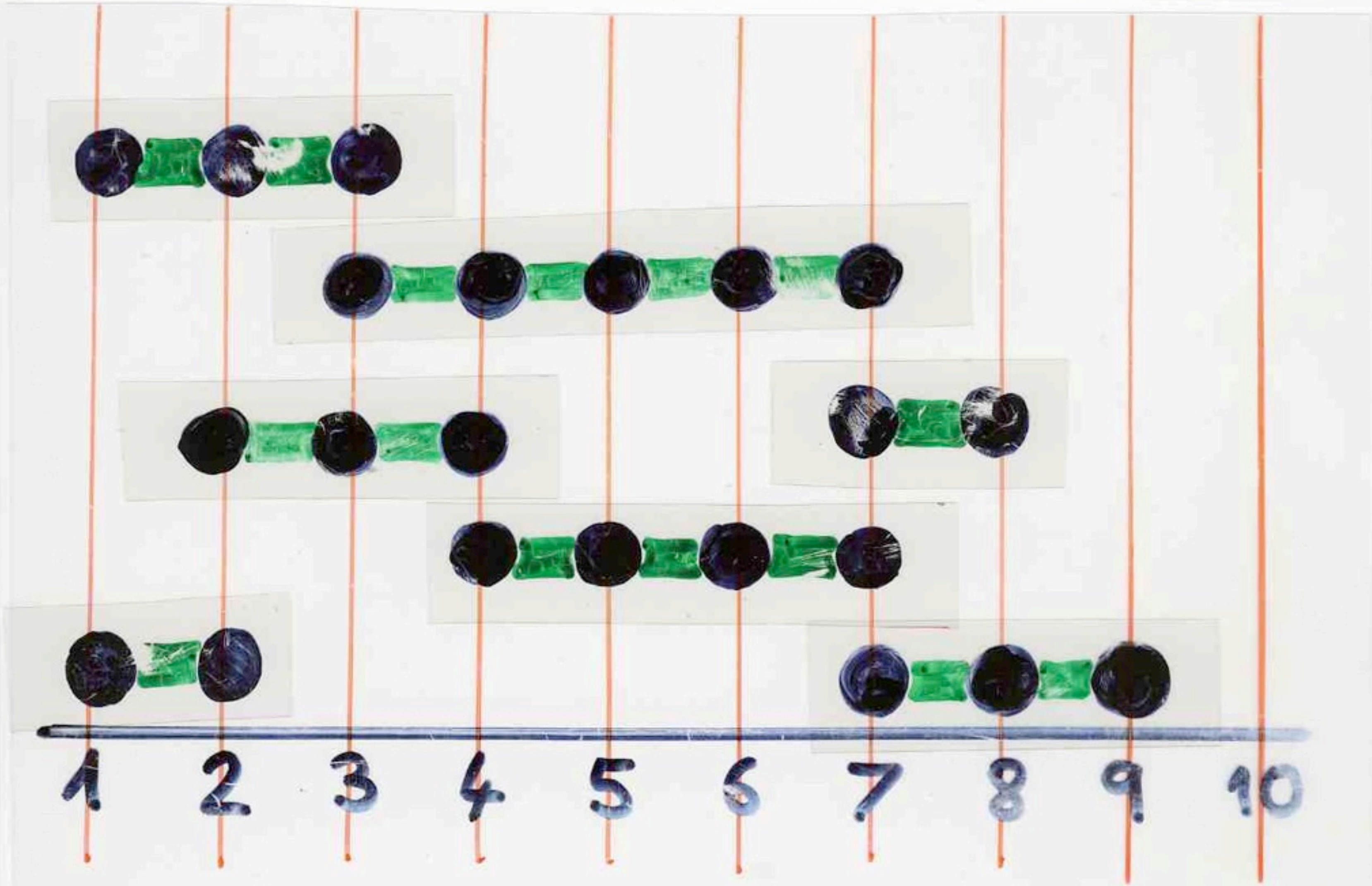

N
pieces not in M


D
trivial heap

Proof by involution



$$f(t,u;q) = \frac{N}{D}$$



Segments $v([i,j]) = q^{j} t u^{(j-i)}$

$$D = \sum_{n \geq 0} \frac{(-1)^n t^n q^n q^{\binom{n}{2}}}{(1-q)\cdots(1-q^n)(1-uq)\cdots(1-uq^n)}$$

$$D = \sum_{\mathcal{C}} (-1)^{|\mathcal{C}|} v(\mathcal{C})$$

(q-Bessel)

$\mathcal{C}$ configuration of 2 by 2 disjoint segments

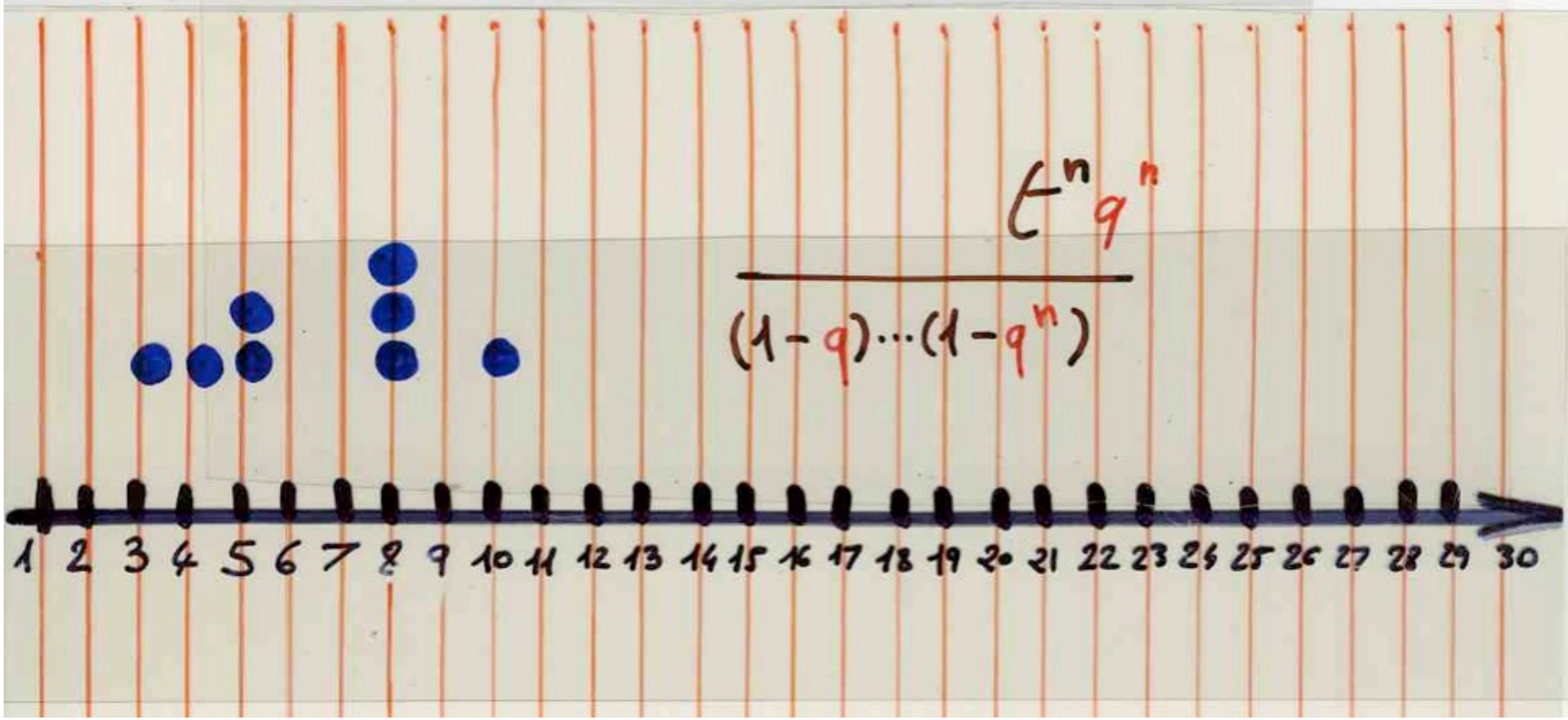
$$v(\mathcal{C}) = \prod v(\text{each segment})$$

$$D = \sum_{n \geq 0} \frac{(-1)^n \, t^n q^n \, q^{\binom{n}{2}}}{(1-q)\cdots(1-q^n)(1-uq)\cdots(1-uq^n)}$$

from integers partitions

to q-Bessel functions

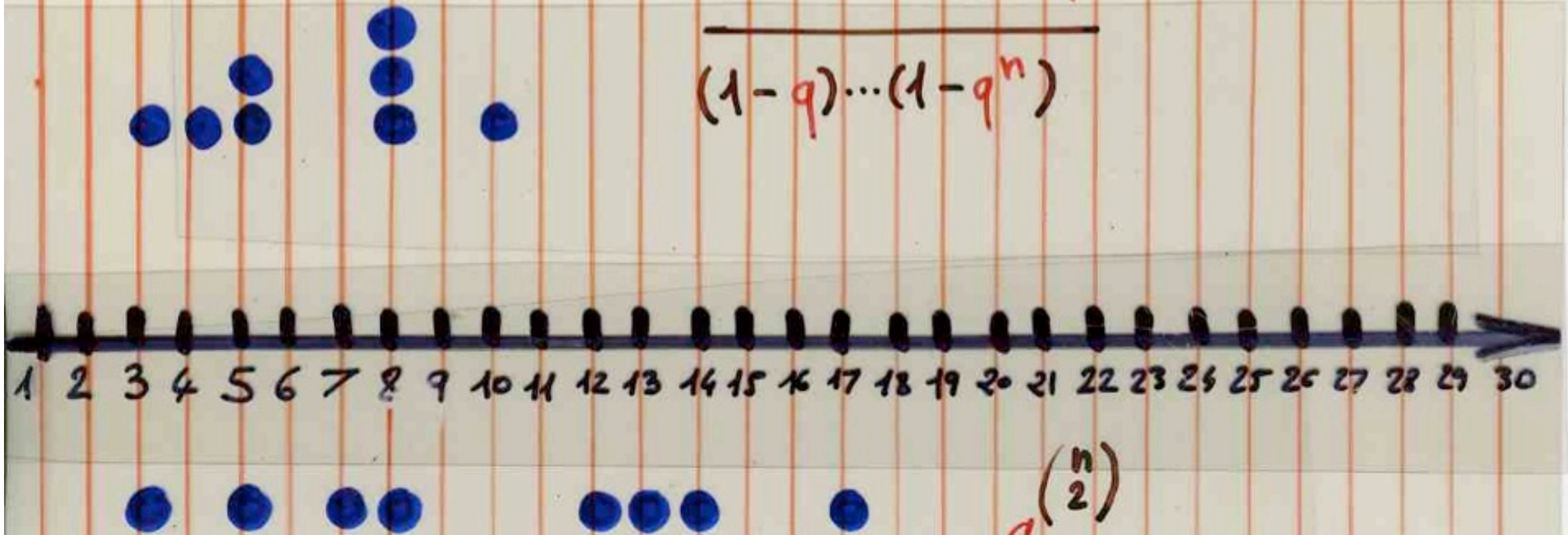
$$D = \sum_{n \geq 0} \frac{(-1)^n \, q^{\binom{n}{2}}}{(1-uq)\cdots(1-uq^n)}$$



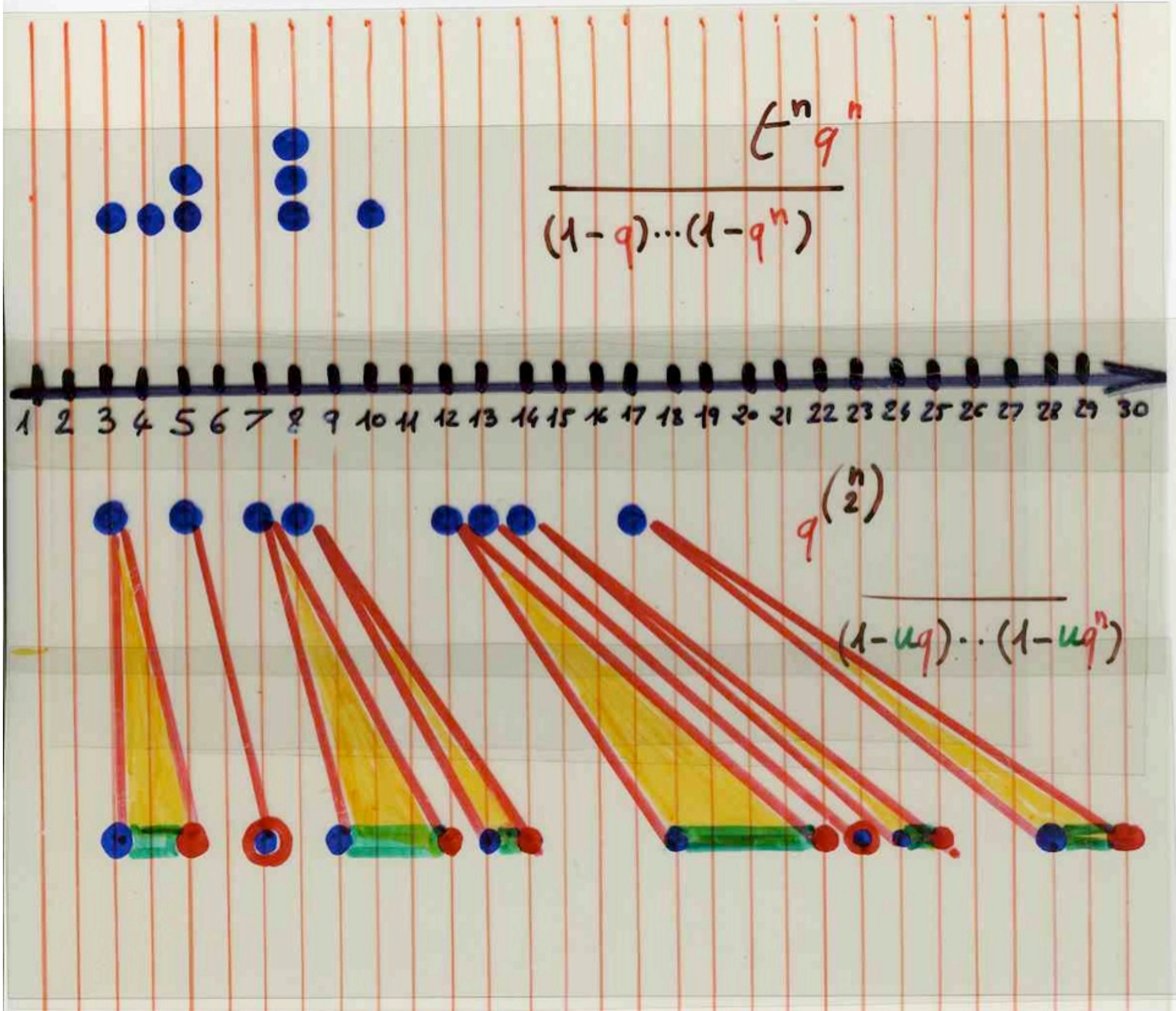
$$\frac{t^n q^n}{(1-q)\cdots(1-q^n)}$$

$$D = \sum_{n \geq 0} \frac{(-1)^n}{(1-uq)\cdots(1-uq^n)}$$

$$\frac{t^n q^n}{(1-q)\cdots(1-q^n)}$$

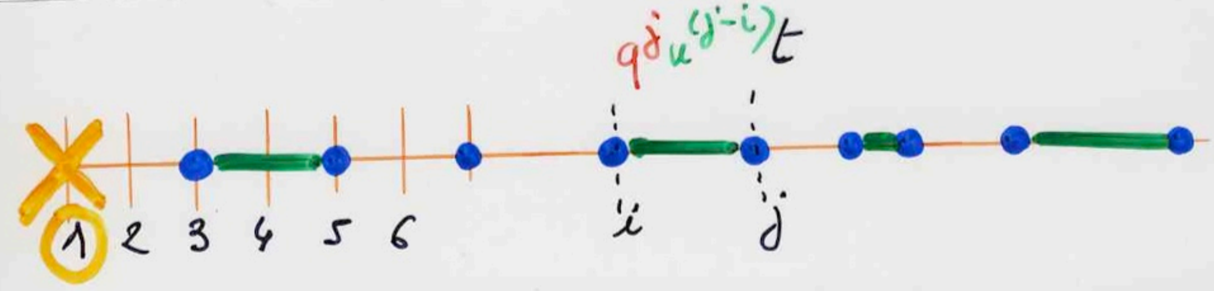


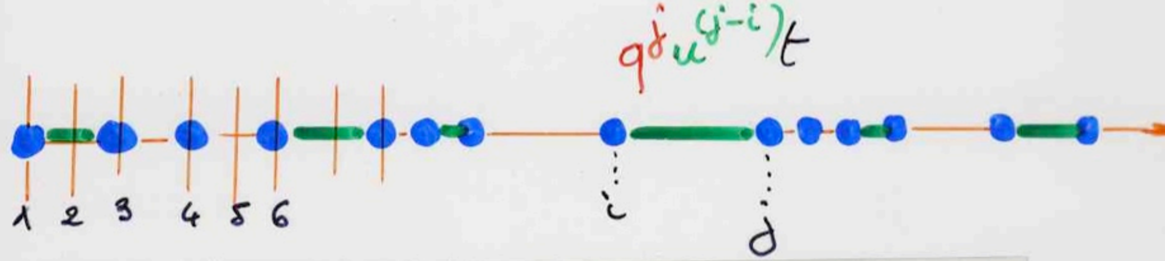
$$q^{\binom{n}{2}}$$



$\frac{t^n q^n}{(1-q)\cdots(1-q^n)}$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30
$\frac{q^{\binom{n}{2}}}{(1-uq)\cdots(1-uq^n)}$

$$N = u \sum_{n \geq 1} \frac{(-1)^{n-1}\, t^n\, q^n\, q^{\binom{n}{2}}}{(1-q)\cdots(1-q^{n-1})(1-uq)\cdots(1-uq^n)}$$





Segments $v([i,j]) = q^j\, t\, u^{(j-i)}$

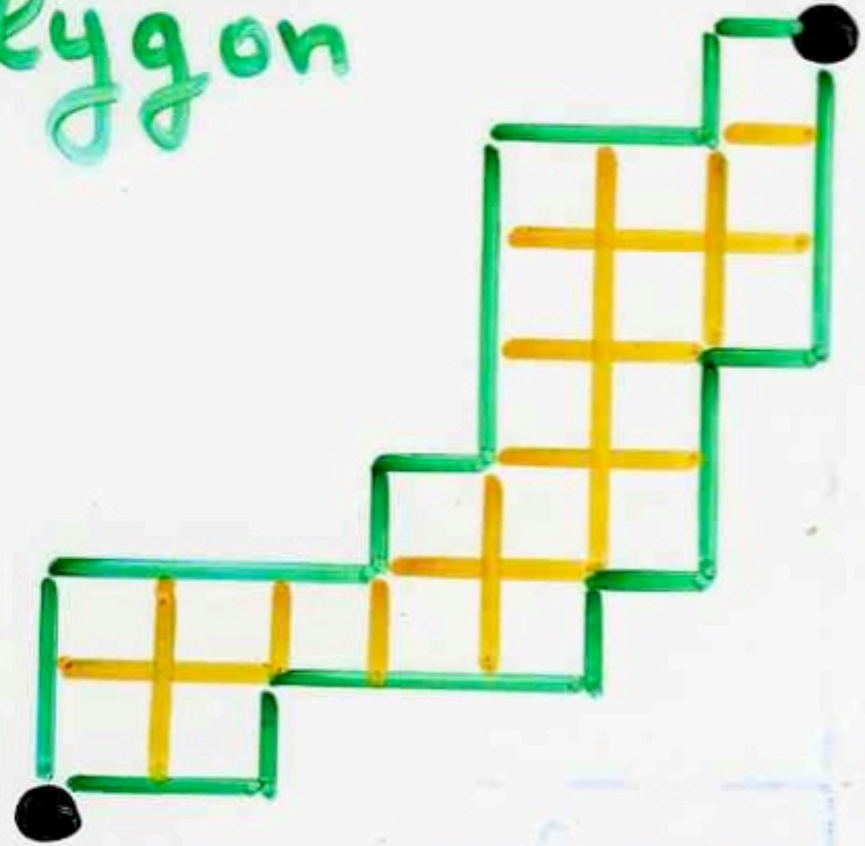
$$D = \sum_{n \geq 0} \frac{(-1)^n\, t^n\, q^n\, q^{\binom{n}{2}}}{(1-q)\cdots(1-q^n)(1-uq)\cdots(1-uq^n)}$$

Parallelogram polyominoes
(staircase polygons)

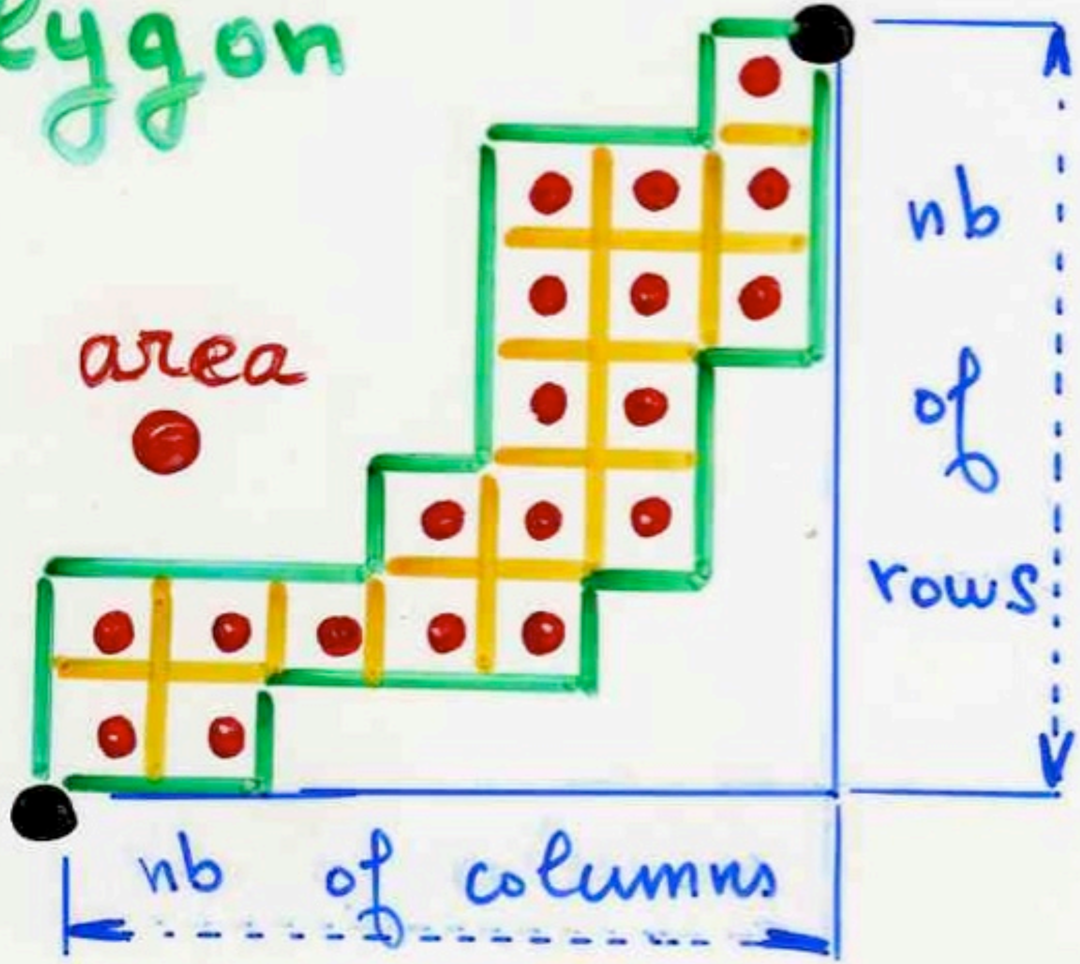
and q-Bessel functions

M.Bousquet-Mélou, X.V.
Adv. in Applied Maths. (1992)

staircase
polygon



staircase polygon



$a_{m,n,p}$

generating function

$$f(x, y; q) = \sum_{m,n,p} a_{m,n,p} \, x^m y^n q^p$$

$$= \sum_{P} x^{c(P)} y^{r(P)} q^{\alpha(P)}$$

P: staircase polygons

$c(P)$: nb of columns

$r(P)$: nb of rows

$\alpha(P)$: area

parallelogram polyominoes

$$\begin{cases} x & \text{length} \quad (\text{nb of columns}) \\ y & \text{height} \quad (\text{" rows "}) \\ q & \text{area} \end{cases}$$

Klarner, Rivest (1974)
Bender

Delest, Fedou (1989)

Brak, Guttmann (1990)

Bousquet-Mélou, Viennot (1990)

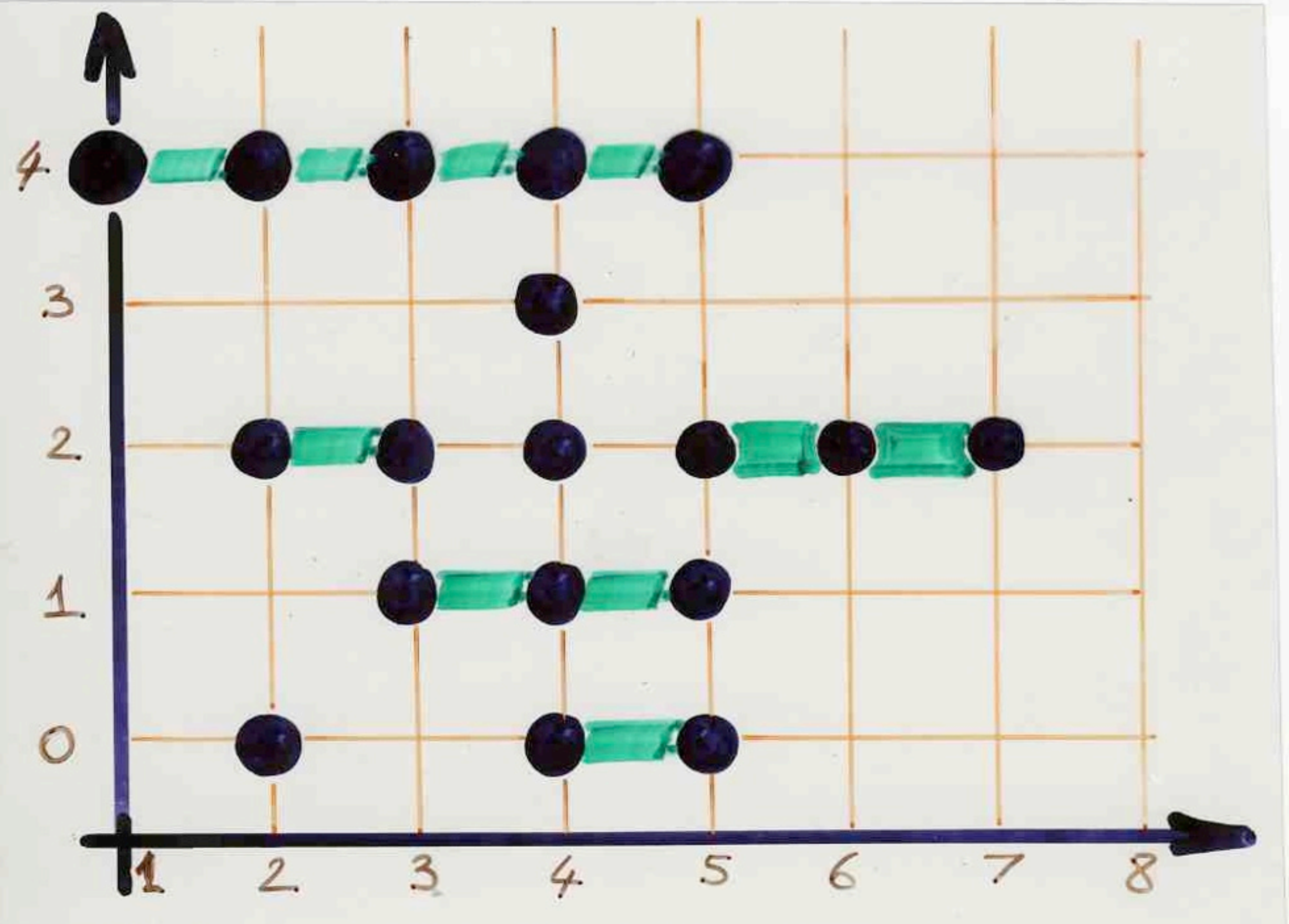
$$y \frac{J_1(x,y,q)}{J_0(x,y,q)}$$

$$J_0 = \sum_{n \geq 0} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n (yq)_n}$$

$$J_1 = \sum_{n \geq 1} \frac{(-1)^{n-1} x^n q^{\binom{n+1}{2}}}{(q)_{n-1} (yq)_n}$$

notation $(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$

bijetion parallelogram polyominoes
semi-pyramids of segments



poids $q^{j} t u^{(j-i)}$

bijection

- pyramids of segments E on $\mathbb{N}^+$

$$\pi\left(\begin{matrix}\text{unique}\\ \text{maximal}\\ \text{piece}\end{matrix}\right) = [1, k] \quad , k \geq 0$$

- parallelogram polyominoes Λ

$$q^{a(\Lambda)} \, t^{c(\Lambda)} \, u^{r(\Lambda)-1} = v(E)$$

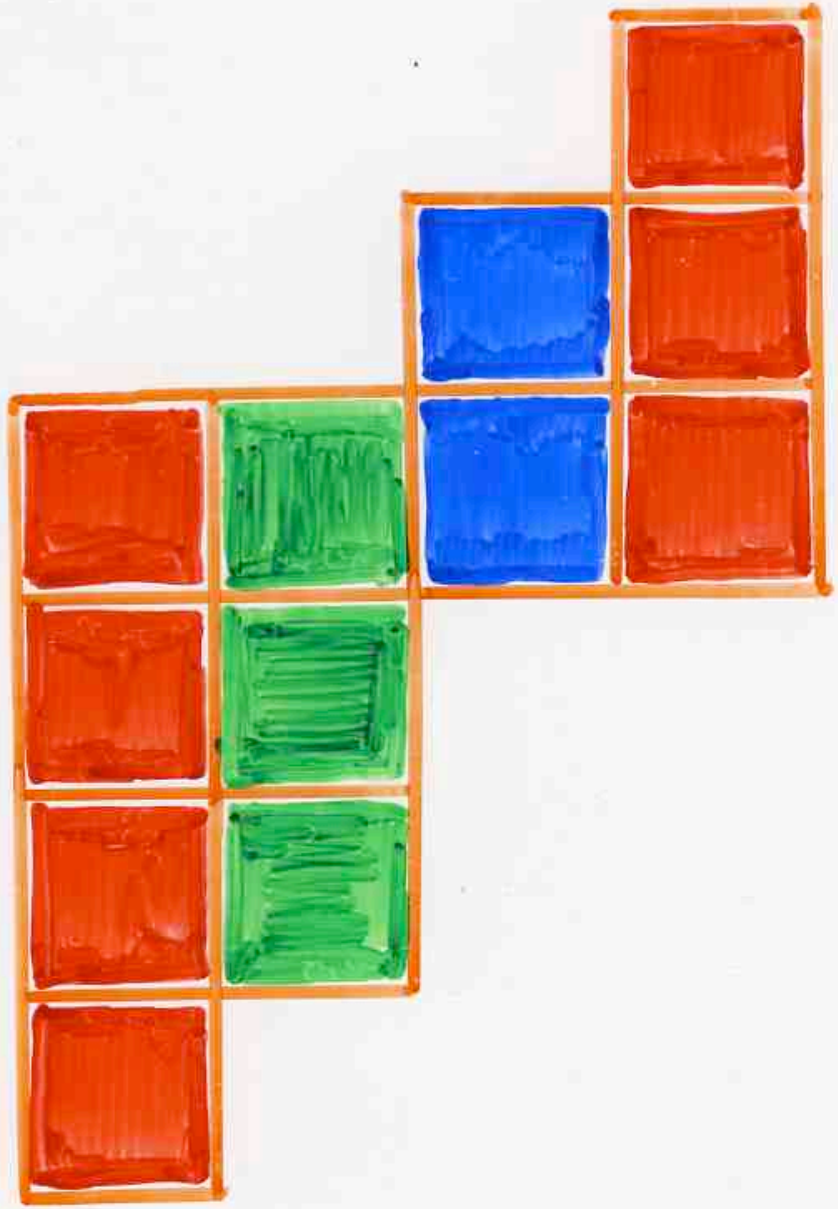
q ↑ area

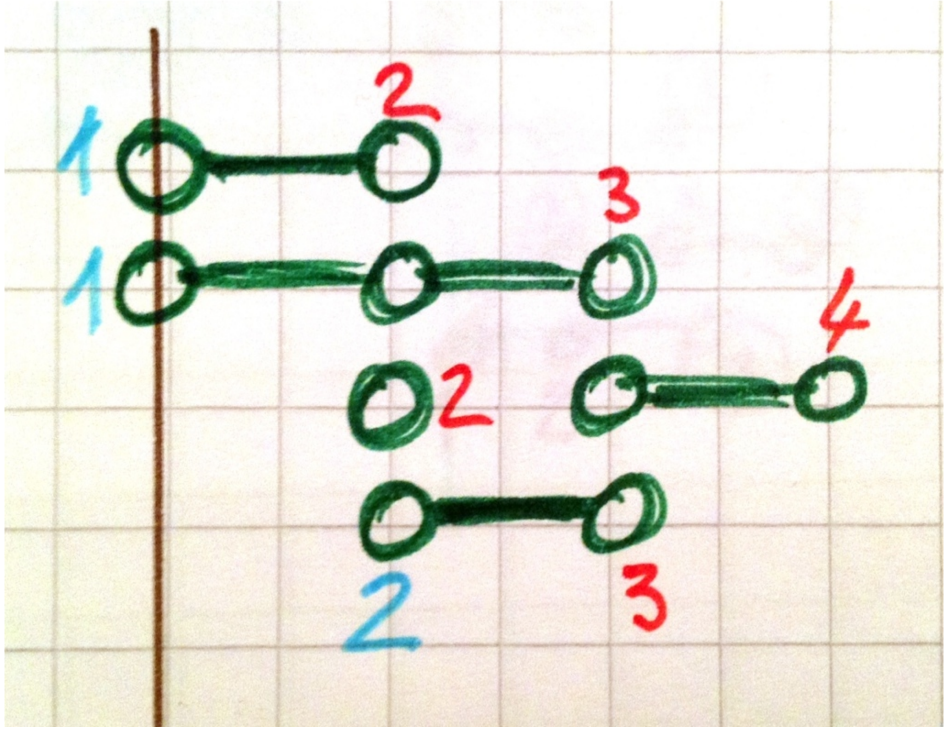
t ↑ number of columns

u ↑ number of rows

$$v([i, j]) = q^{j} t u^{(j-i)}$$

segment

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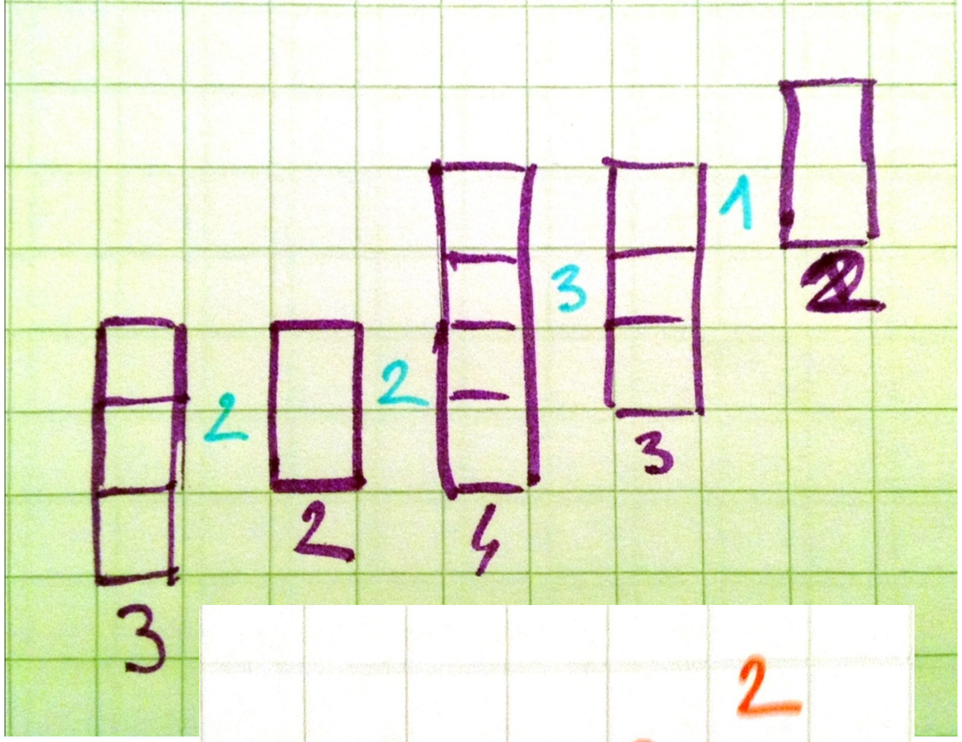
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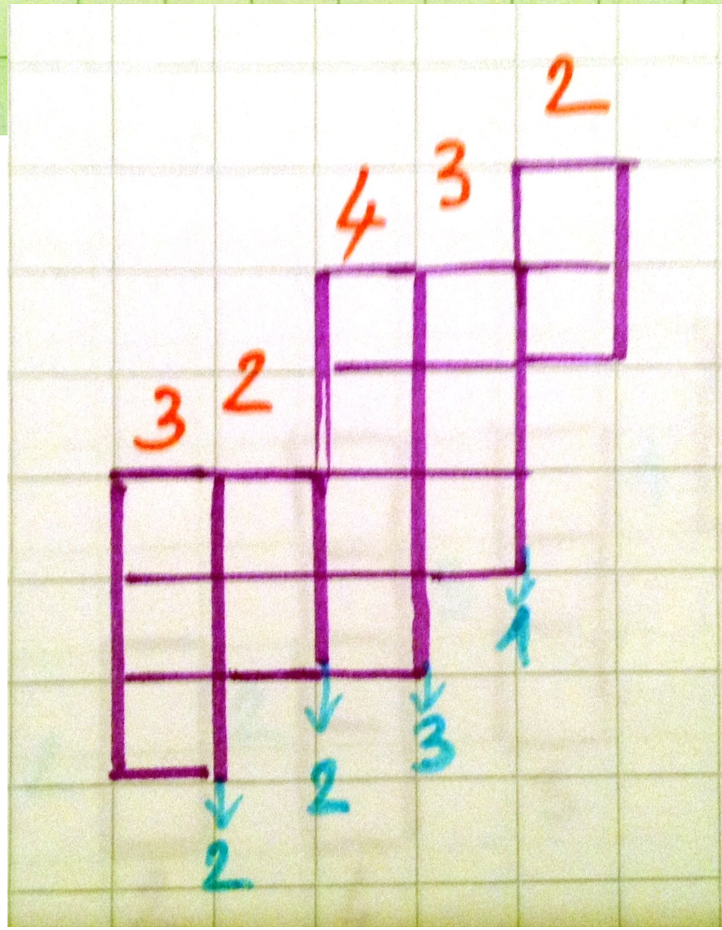
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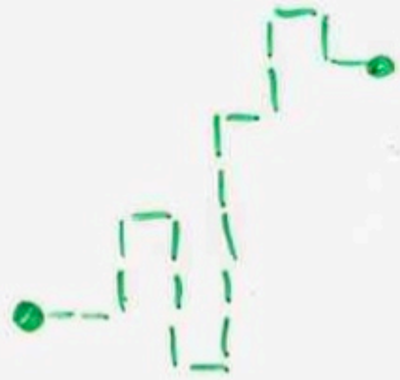
q-Bessel functions in statistical physics

9- Bessel

self-avoiding **walks**



self-avoiding **polygons**

(vesicles)

some subclasss :

exactly soldved

explicit formulae for **generating function**

partition function

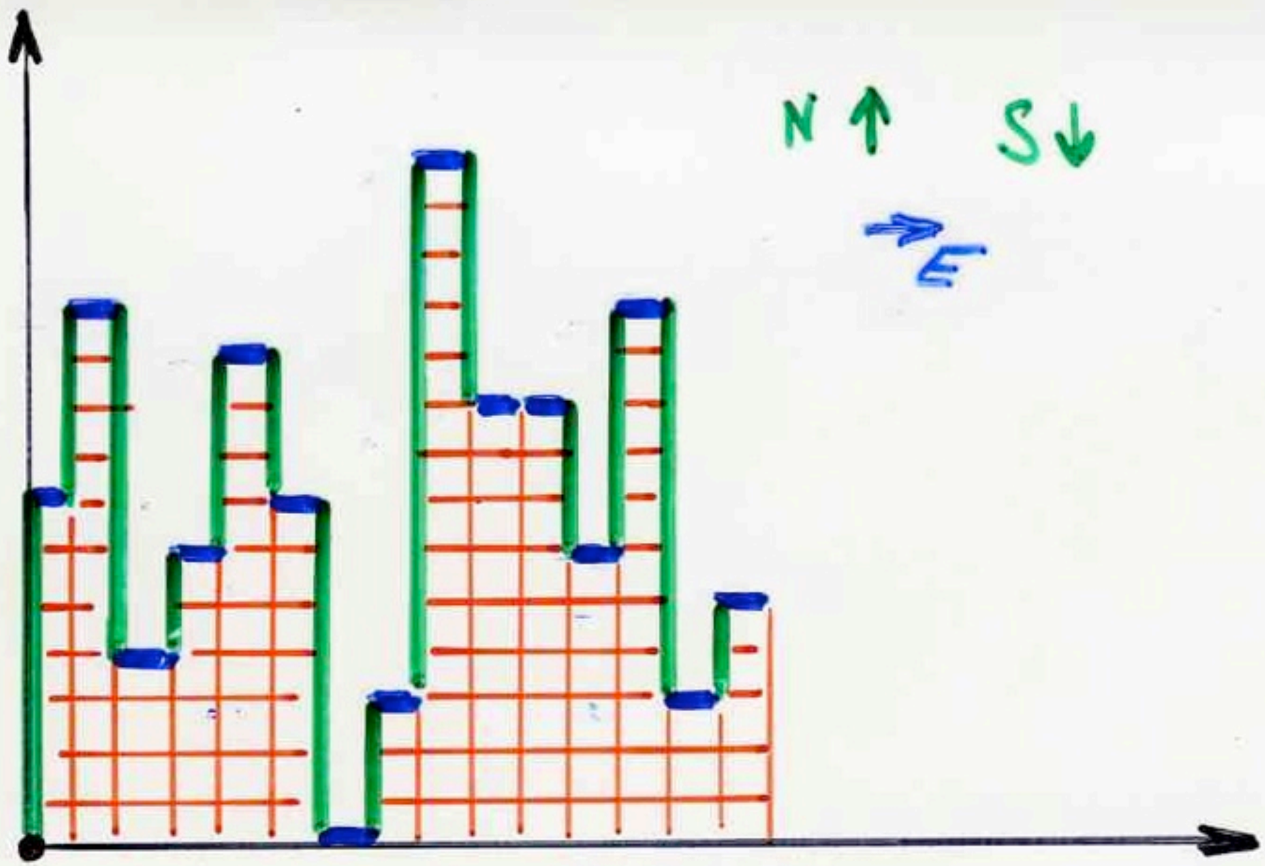
discrete (1+1)-dimensional

SOS model with

- magnetic field
- boundary potential
- surface interactions

exact solution

A. Owczarek, T. Prellberg (1993)



Partially directed self-avoiding walks (paths)

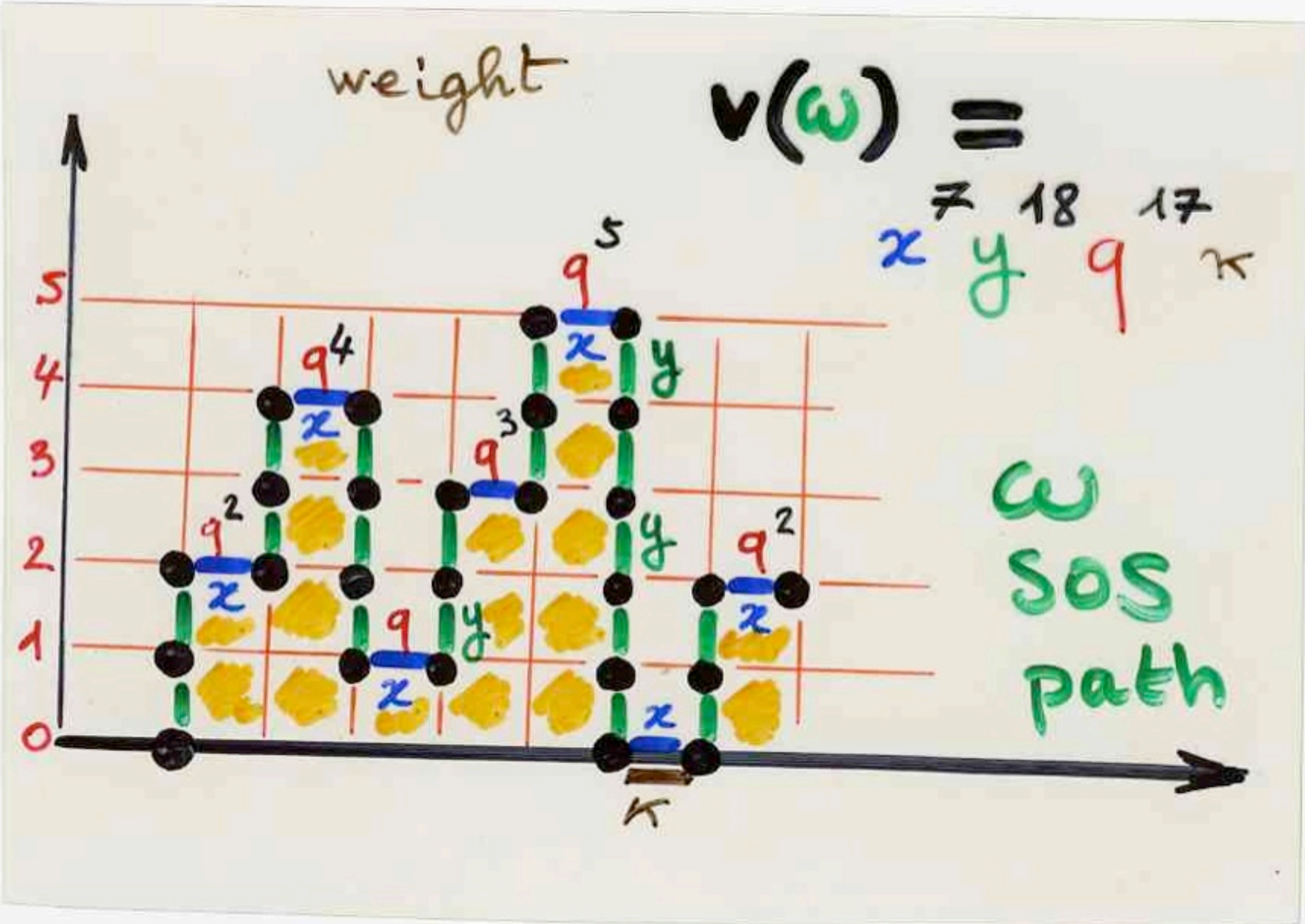
$$G(x, y, q, \kappa) = \sum_{\substack{\omega \\ \text{SOS path}}} v(\omega)$$

<u>weight</u>

↑ N ↓ S $\quad y$

$\underline{\text{level}}$ j $\quad$ •→• E $\quad \begin{cases} x\, q^{j} & \text{if } j > 0 \\ x\, \kappa & \text{if } j = 0 \end{cases}$

A. Owczarek, T. Prellberg (1993)


weight
v(ω) =
x^7 y^18 q^17 κ
5
4
3
2
1
0
q^5
q^4
q^3
q^2
q
q^2
x
y
κ
ω
SOS
path

ou encore :

$$\sum_{\omega} v(\omega) = x \, \frac{H\left(qy^2, q, x(1-y^2)q\right)}{H\left(y^2, q, x(1-y^2)\right)}$$

chemins SOS

arrivant au niveau 0

Owczarek, Prellberg (1993)

<u>notations :</u> $$H(u,q,t) = \sum_{n\geq 0} \frac{(-t)^n \, q^{\binom{n}{2}}}{(u,q)_n \, (q,q)_n}$$

avec $(u,q)_n = (1-u)(1-uq)\cdots(1-uq^{n-1})$

$(u)_n$

$$J_0 = H(uq, q, xq)$$

$$J_1 = H(uq, q, xq) - H(uq, q, xq^2)$$

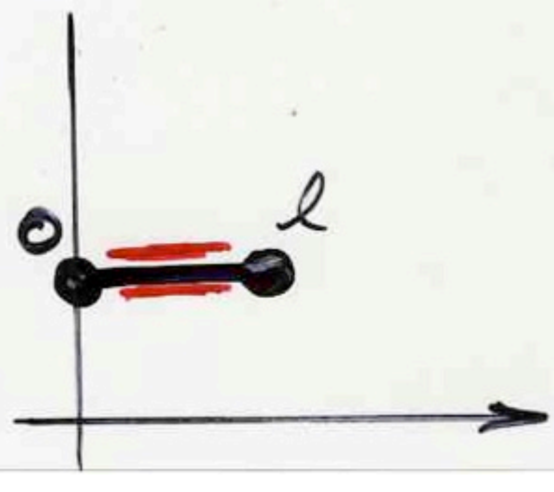
$$\frac{J_1}{J_0} \quad \text{or} \quad \frac{H(uq,\ q,\ xq^2)}{H(uq,\ q,\ xq)}$$

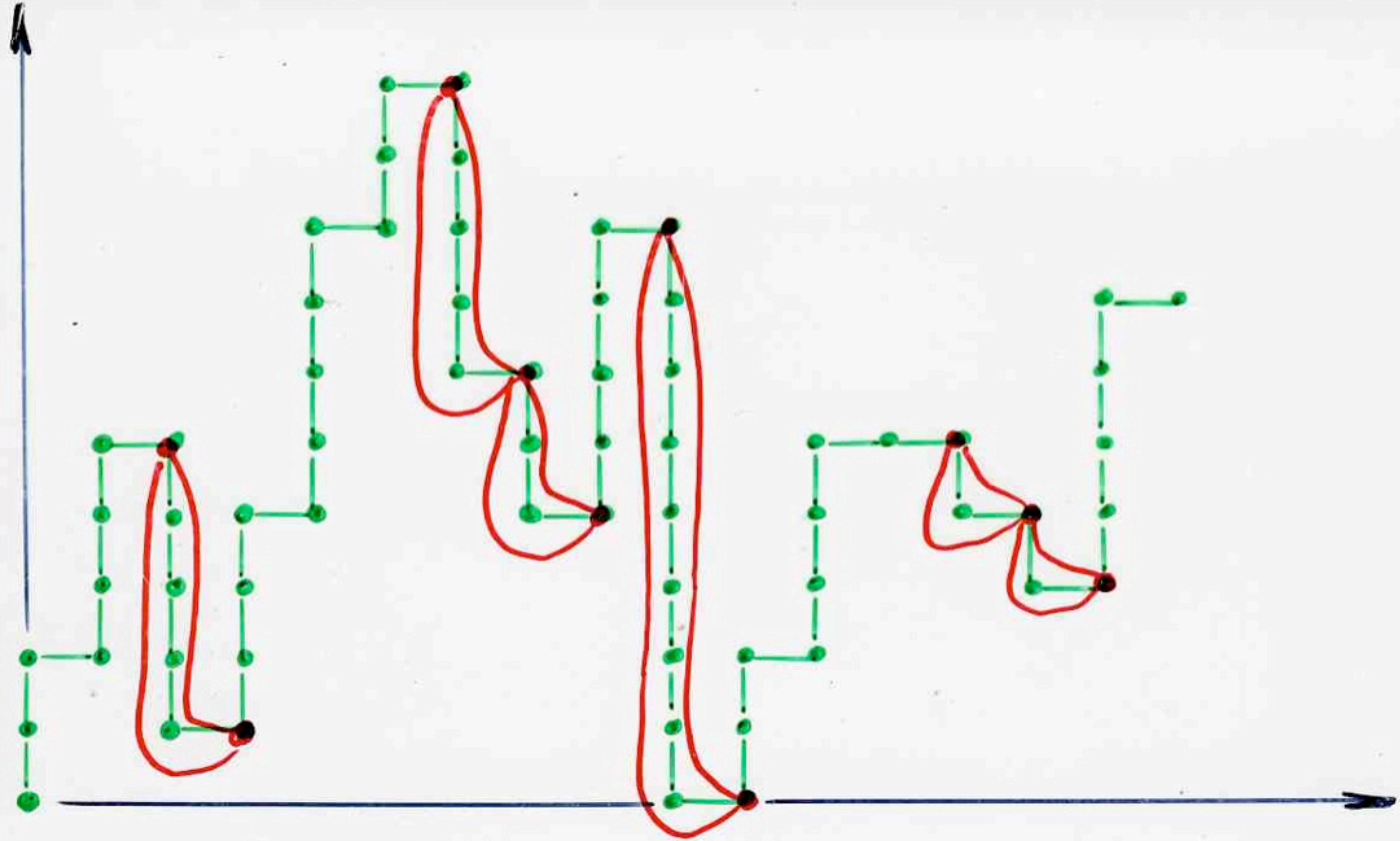
$$\frac{N}{D}$$

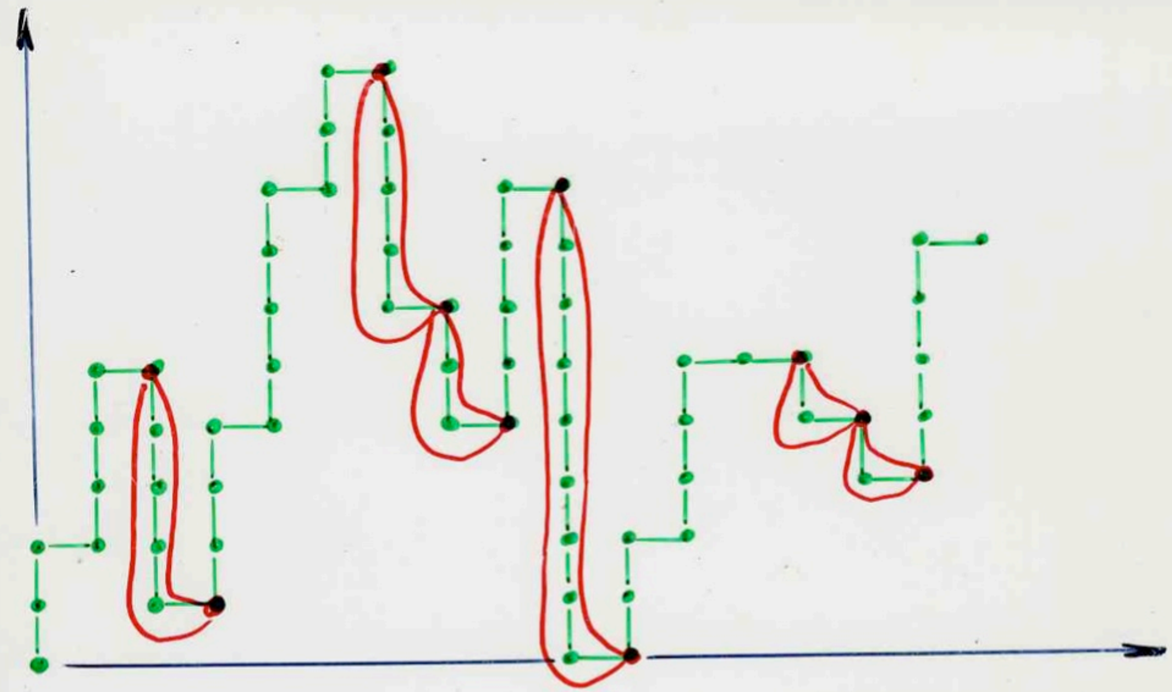
$$= \sum_{P} v(P)$$

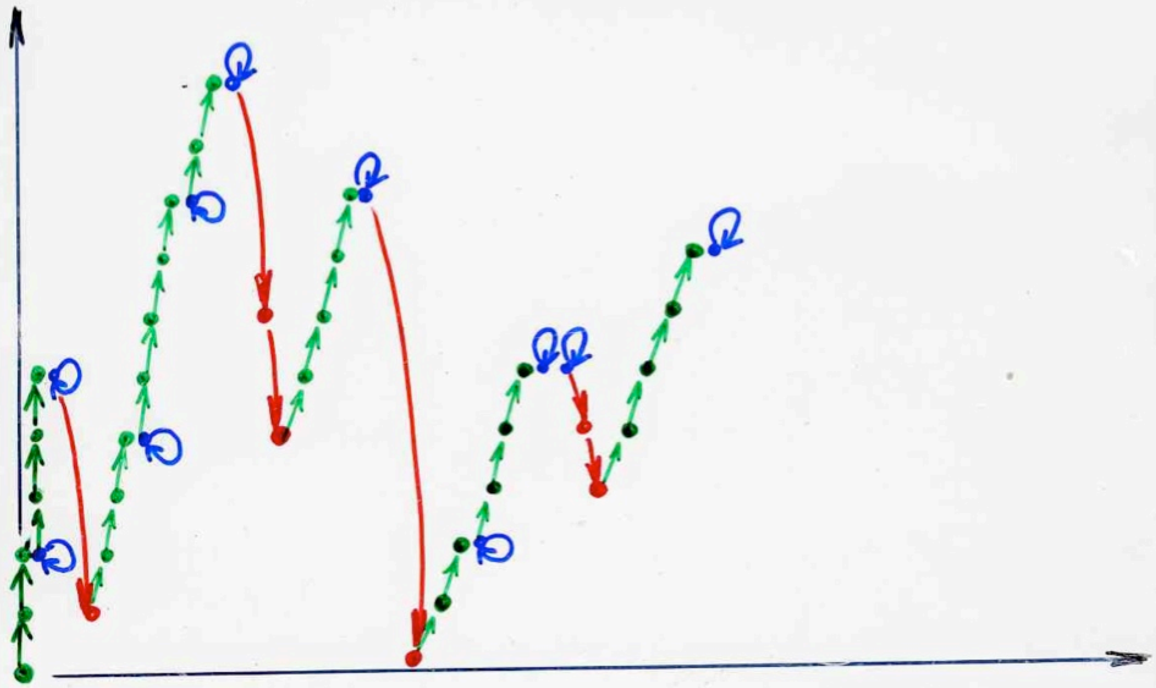
P pyramid

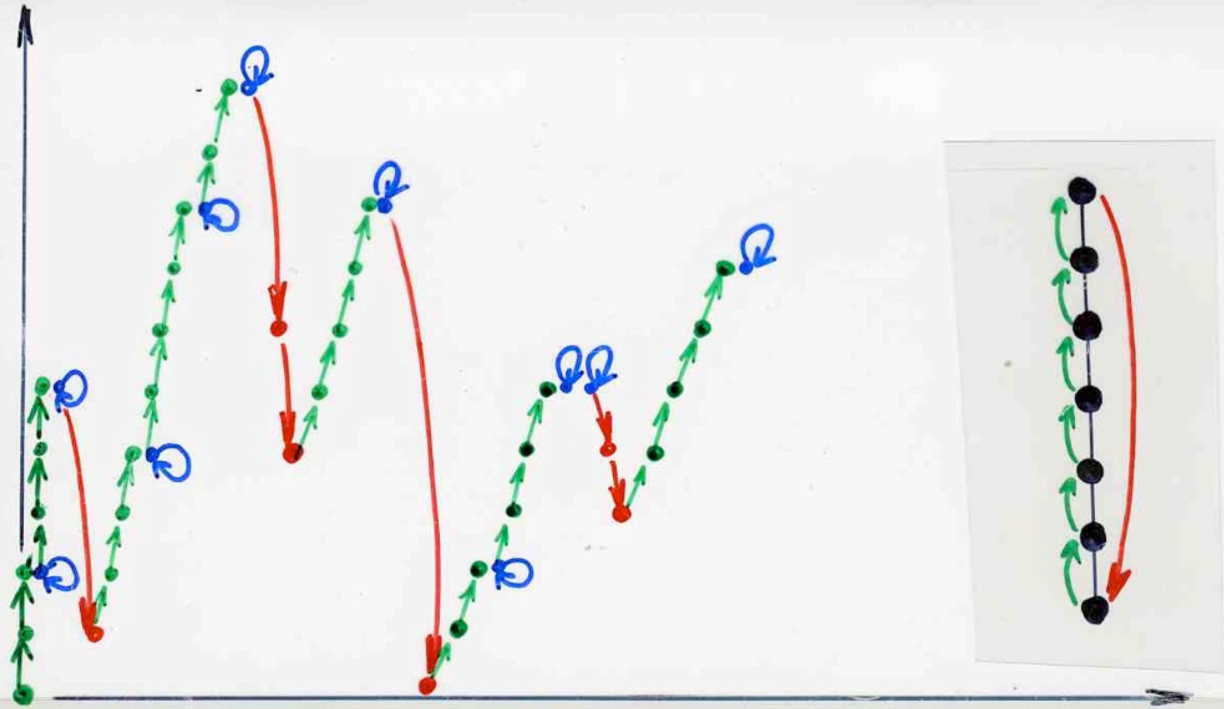
maximal piece is

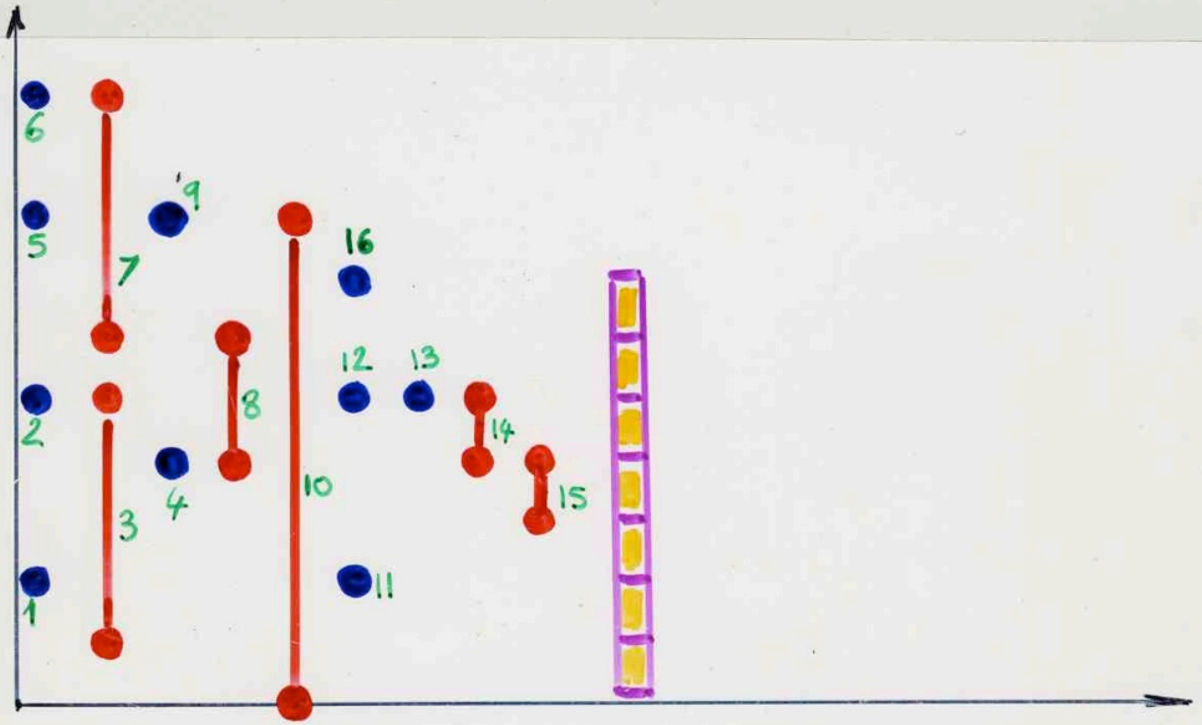








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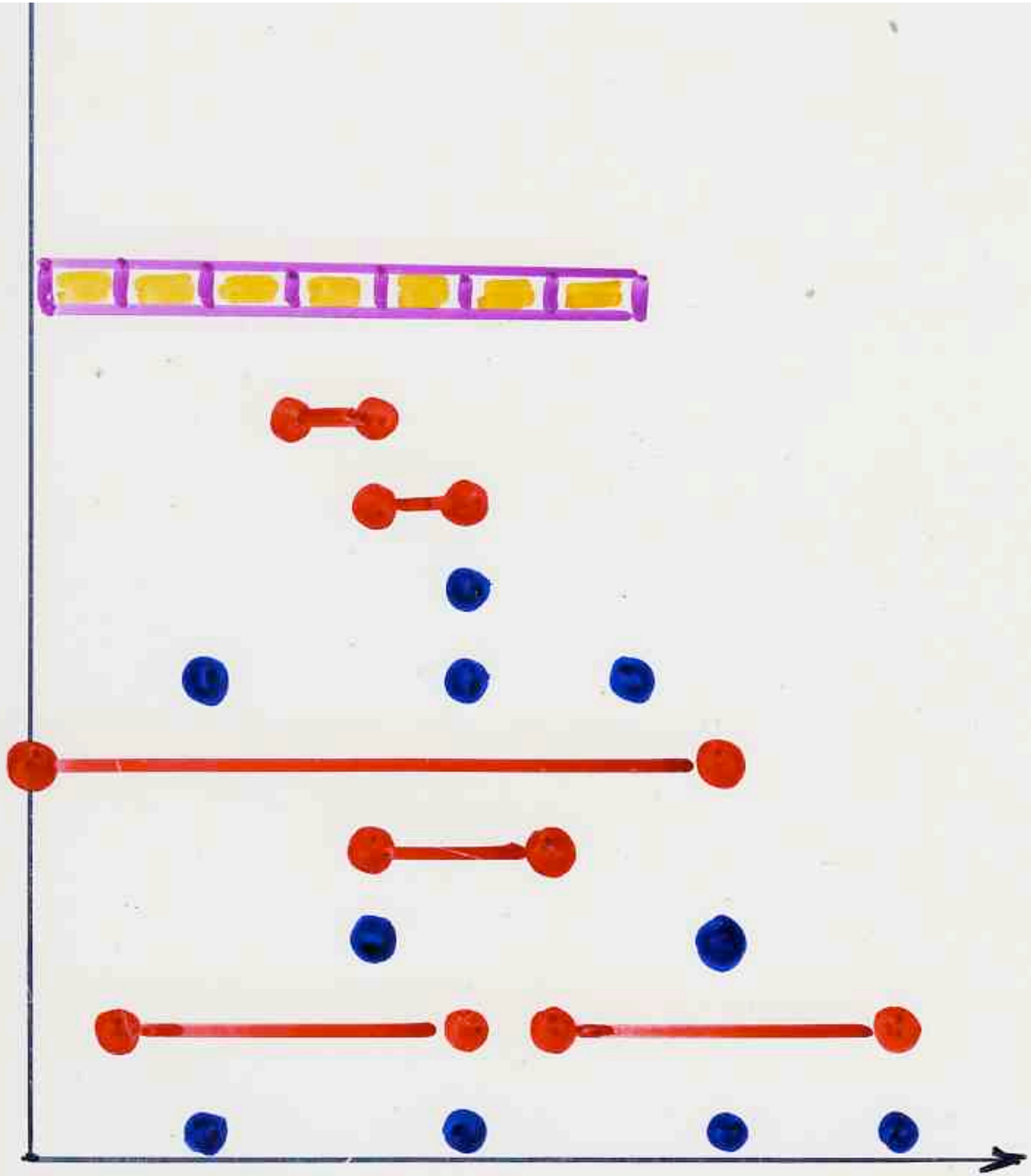
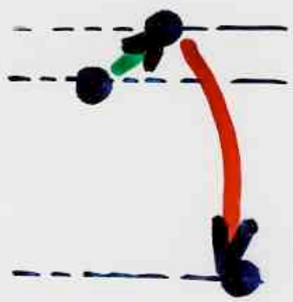


$$v_g([i,j]) = t\,u^{(j-i)}\,q^{i}$$

$$0 \leq i \leq j$$

Paths with no

peaks



$$t \leftarrow x(1-y^2)$$

- q-Bessel

chemins partiellement dirigés avec interaction

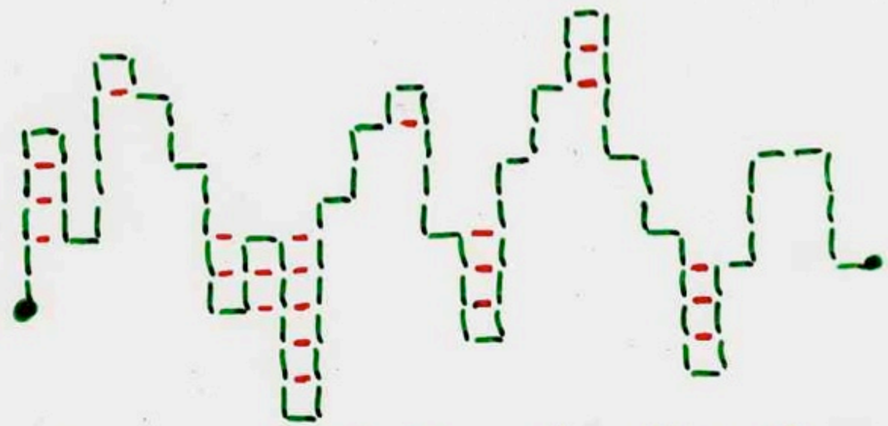
"effondrement" des polymères

Brak, Guttmann, Whittington 1992

Owczarek, Prellberg, Brak 1993

Zwanzig, Lauritzen 1968, 1970

autres familles de polyominos convexes



chemin partiellement dirigé avec interactions