An alternative approach to alternating sign matrices

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Vienna, 20 May 2008

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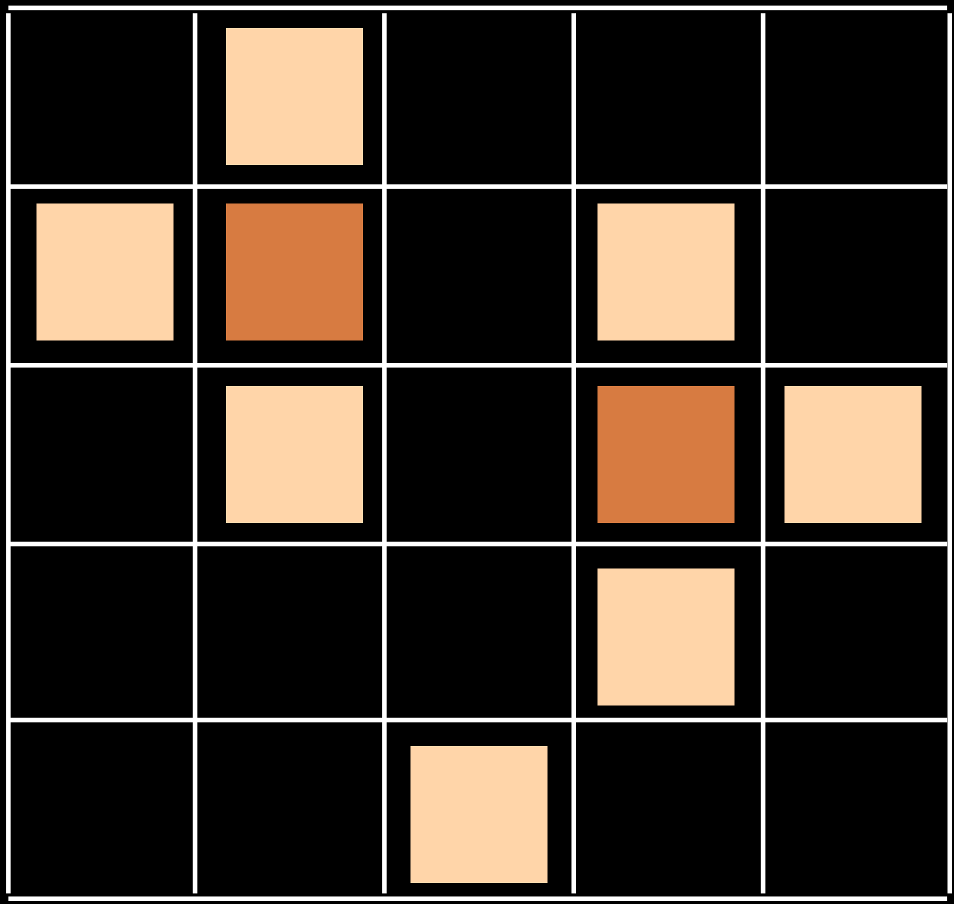
Def - ASM alternating sign matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(i) entries: 0, 1, −1

(ii) sum of entries in each $\left\{\begin{matrix}\text{row}\\ \text{column}\end{matrix}\right.$ = 1

(iii) non-zero entries alternate in each $\left\{\begin{matrix}\text{row}\\ \text{column}\end{matrix}\right.$



§1 Four operators A, A', B, B'

§2 Heisenberg operators U, D

§3 representation of operators U, D and "local rules" for RSK

§4 FPL and operators A, A, B, B

§5 FPL, Temperley-Lieb algebra, heaps of dimers

inspiration from some papers ?

§6 operators for the PASEP

Bijection Laguerre histories permutations

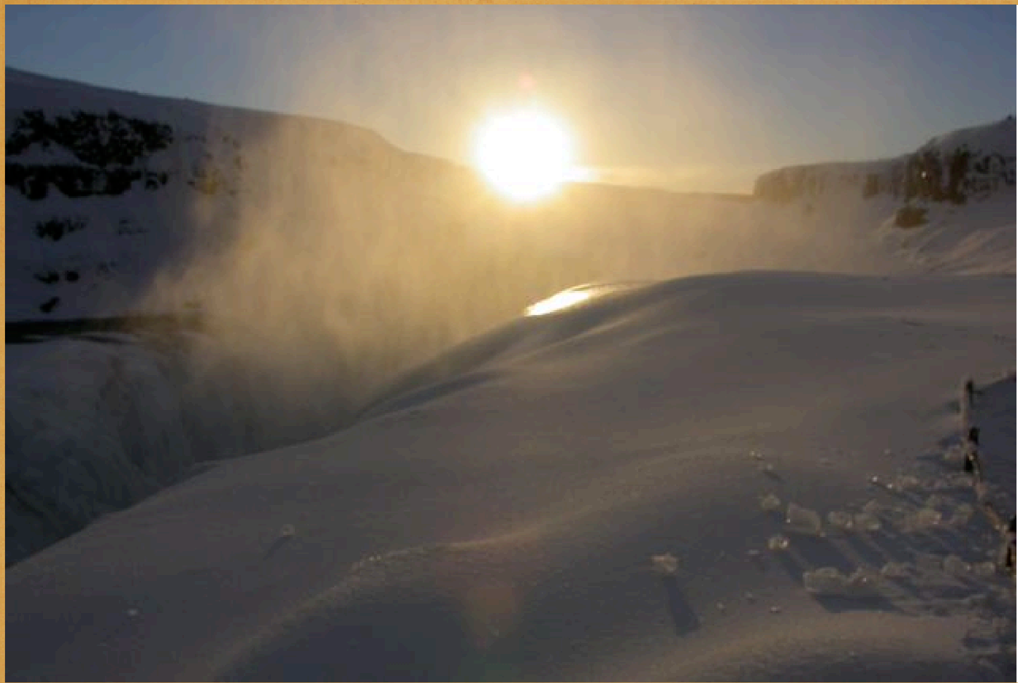
Conclusion: the "cellular ansatz"

xgv website

Supplement: §7 Other examples of the "cellular ansatz"

Complements: local RSK and geometric RSK

Temperley-Lieb algebra "contraction" of heaps of dimers



Islande hiver 02 xgv

§1 Four operators A, A', B, B'

$A, A', B, B',$

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

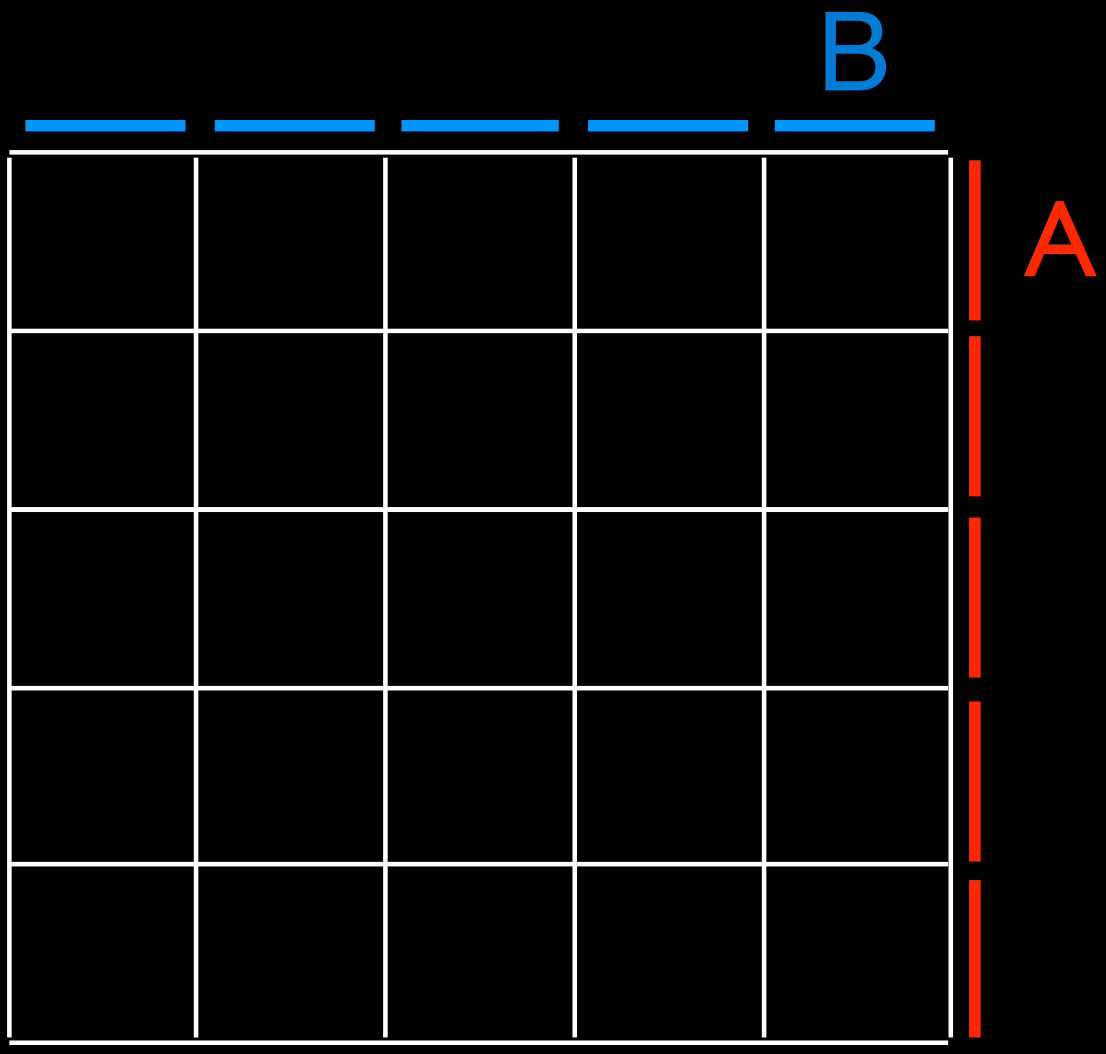
Lemma. Any word $w(A, A', B, B')$ in letters A, A', B, B', can be uniquely written

$$\sum C(u,v;w)\; \underbrace{u(A,A')}_{\text{word in } A,\,A'}\; \underbrace{v(B,B')}_{\text{word in } B,\,B'}$$

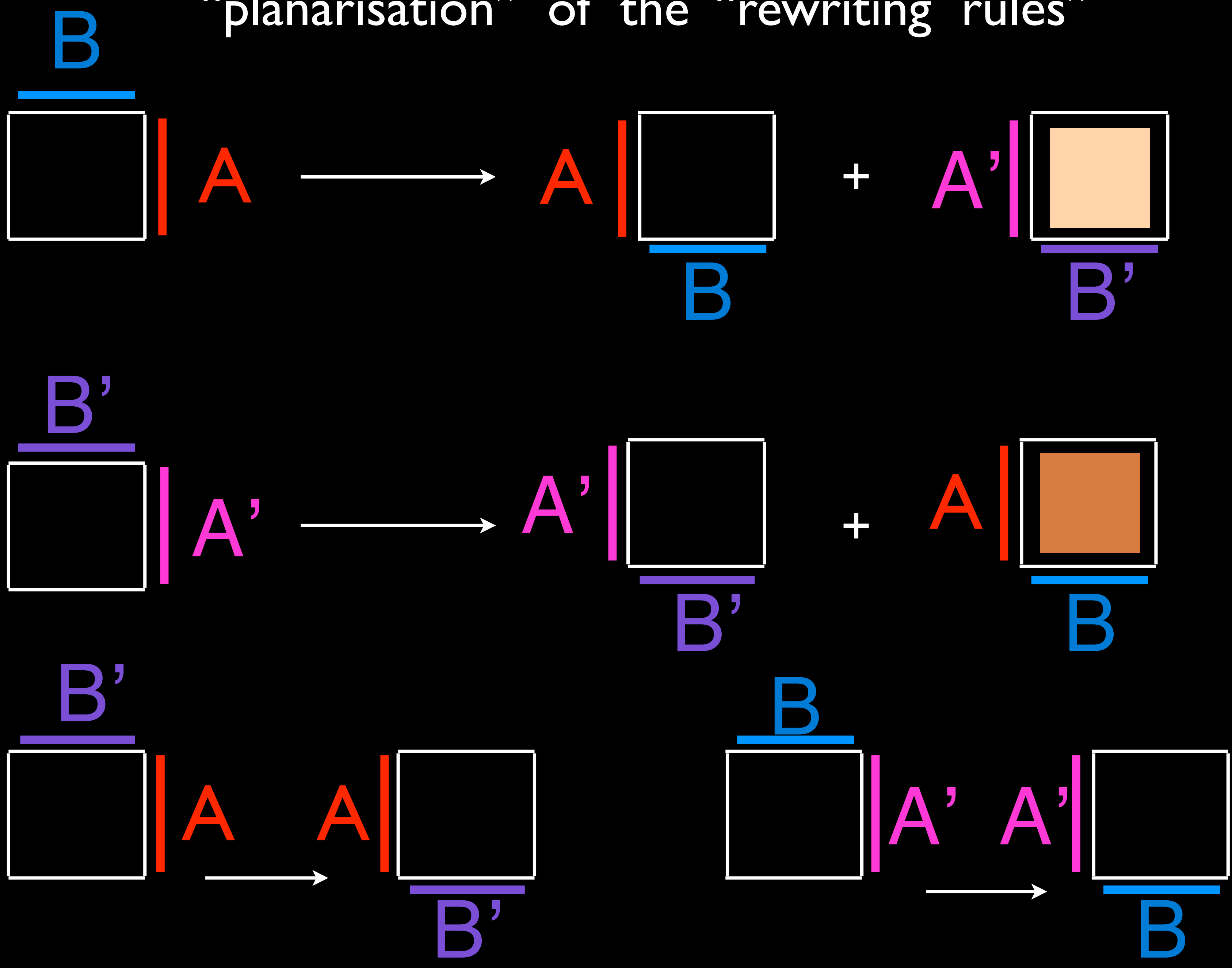
Prop. For $w = B^n A^m$

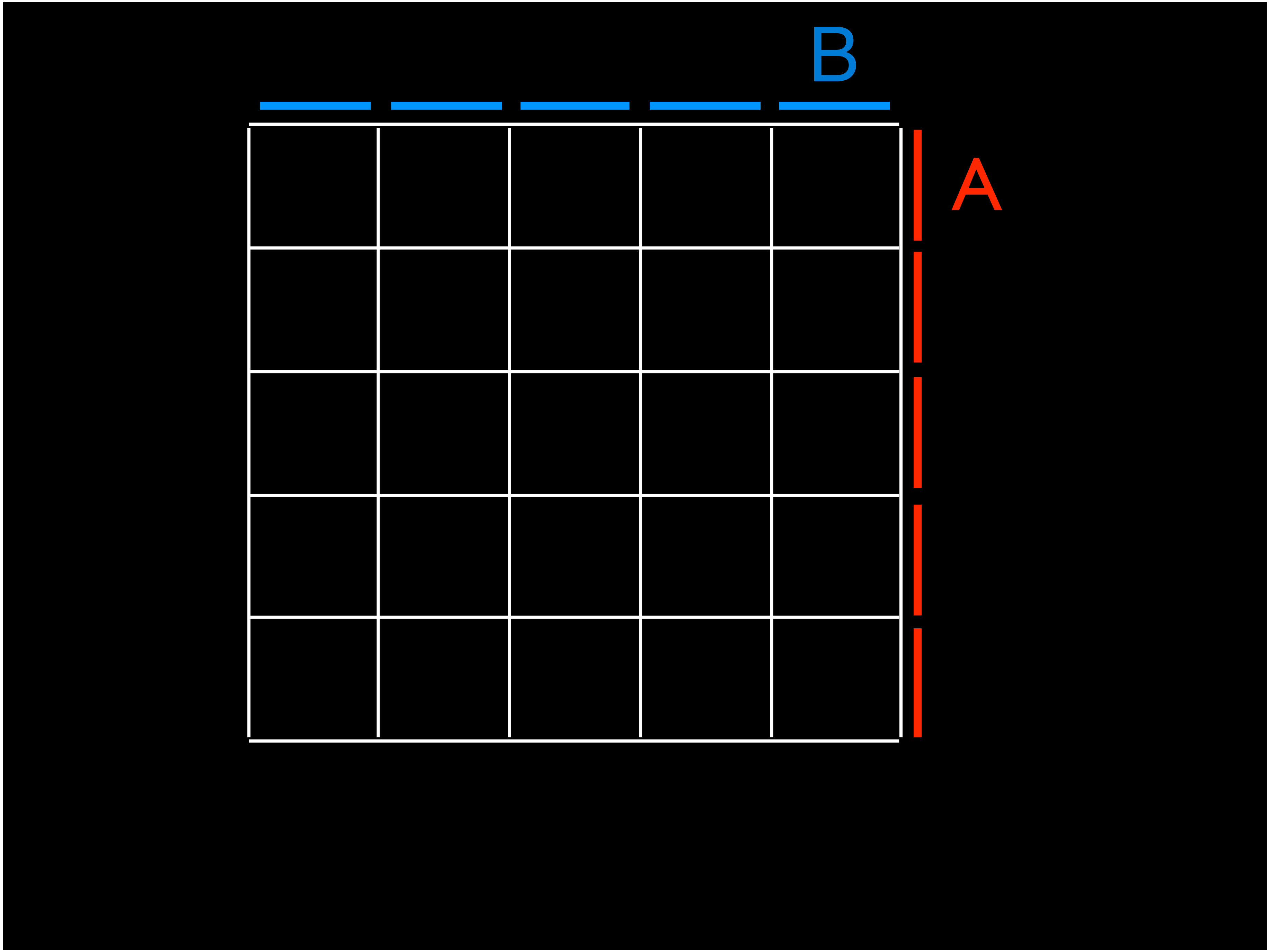
$u = A'^n$, $v = B'^n$

$C(u,v;w)$ = the number of $n \times n$ ASM (alternating sign matrices) “planar” proof:



“planarisation” of the “rewriting rules”

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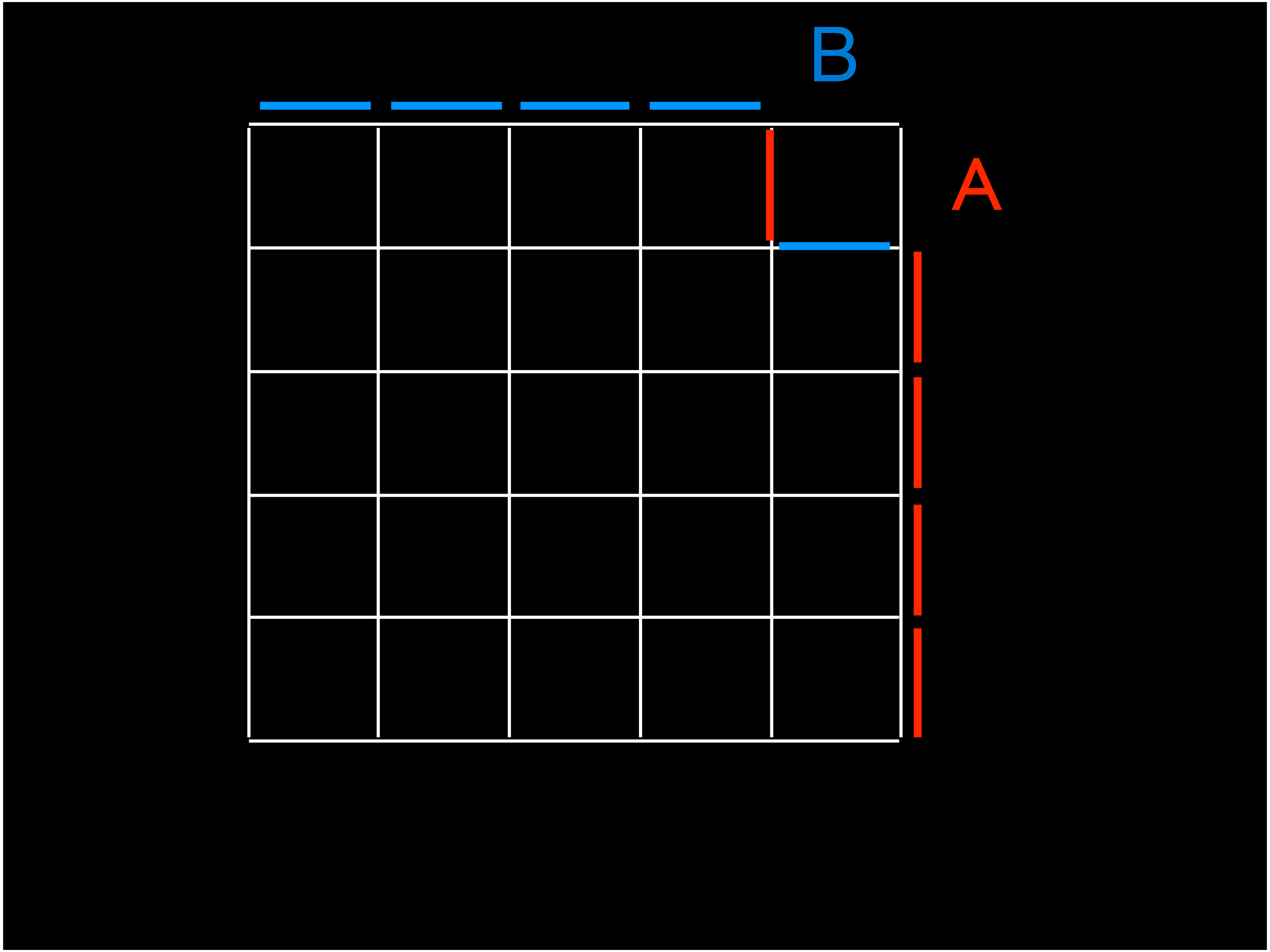
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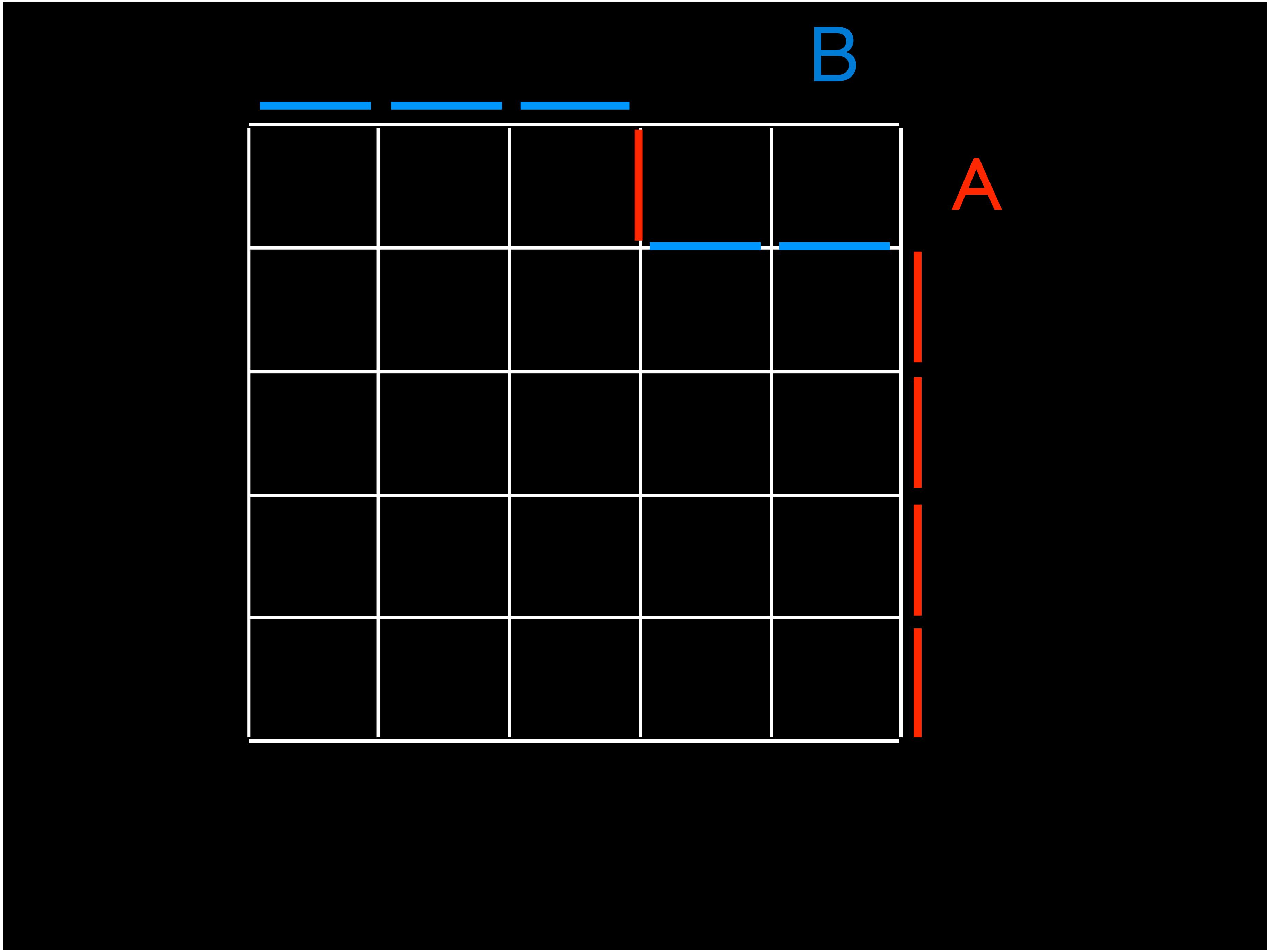
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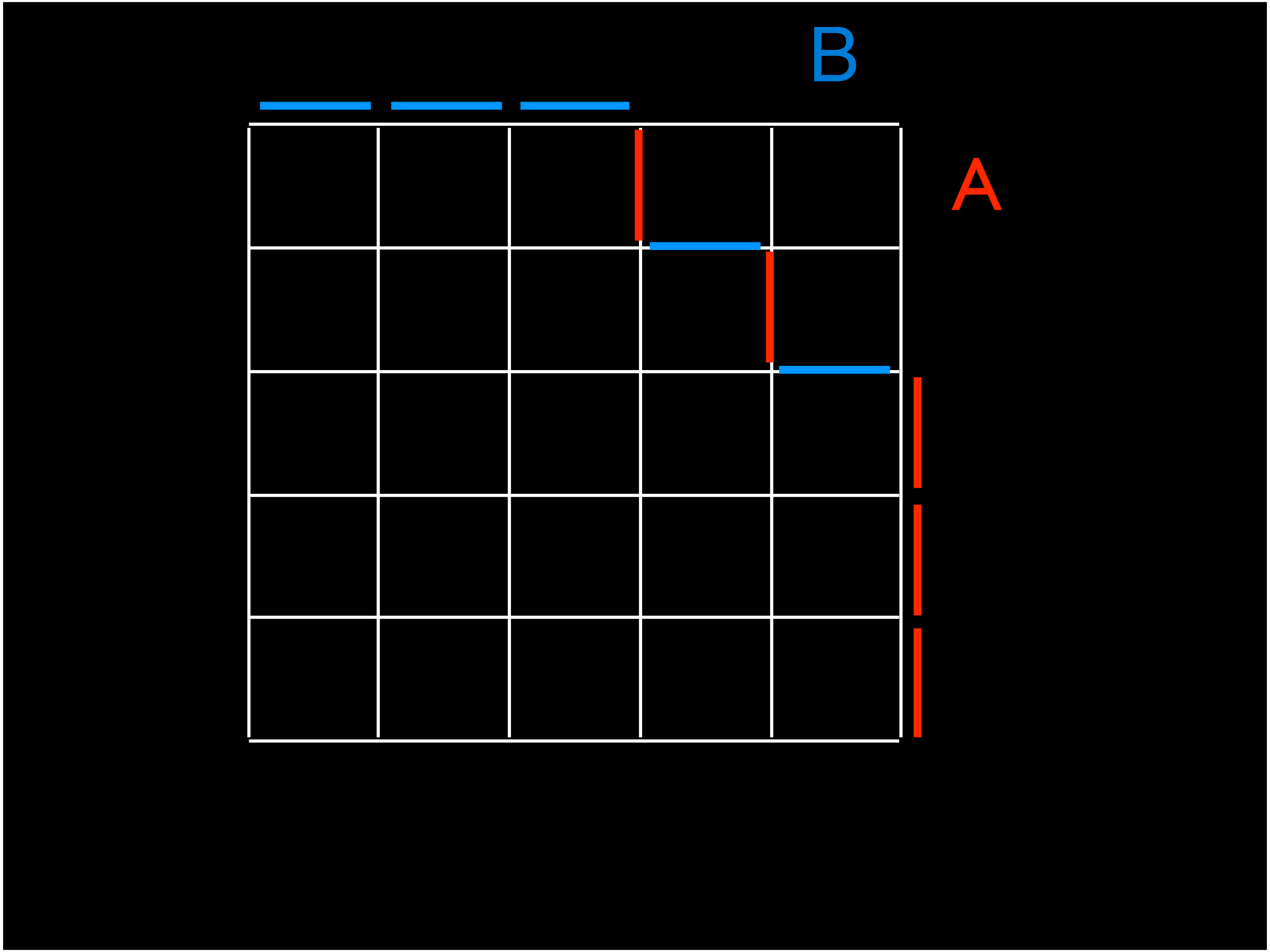
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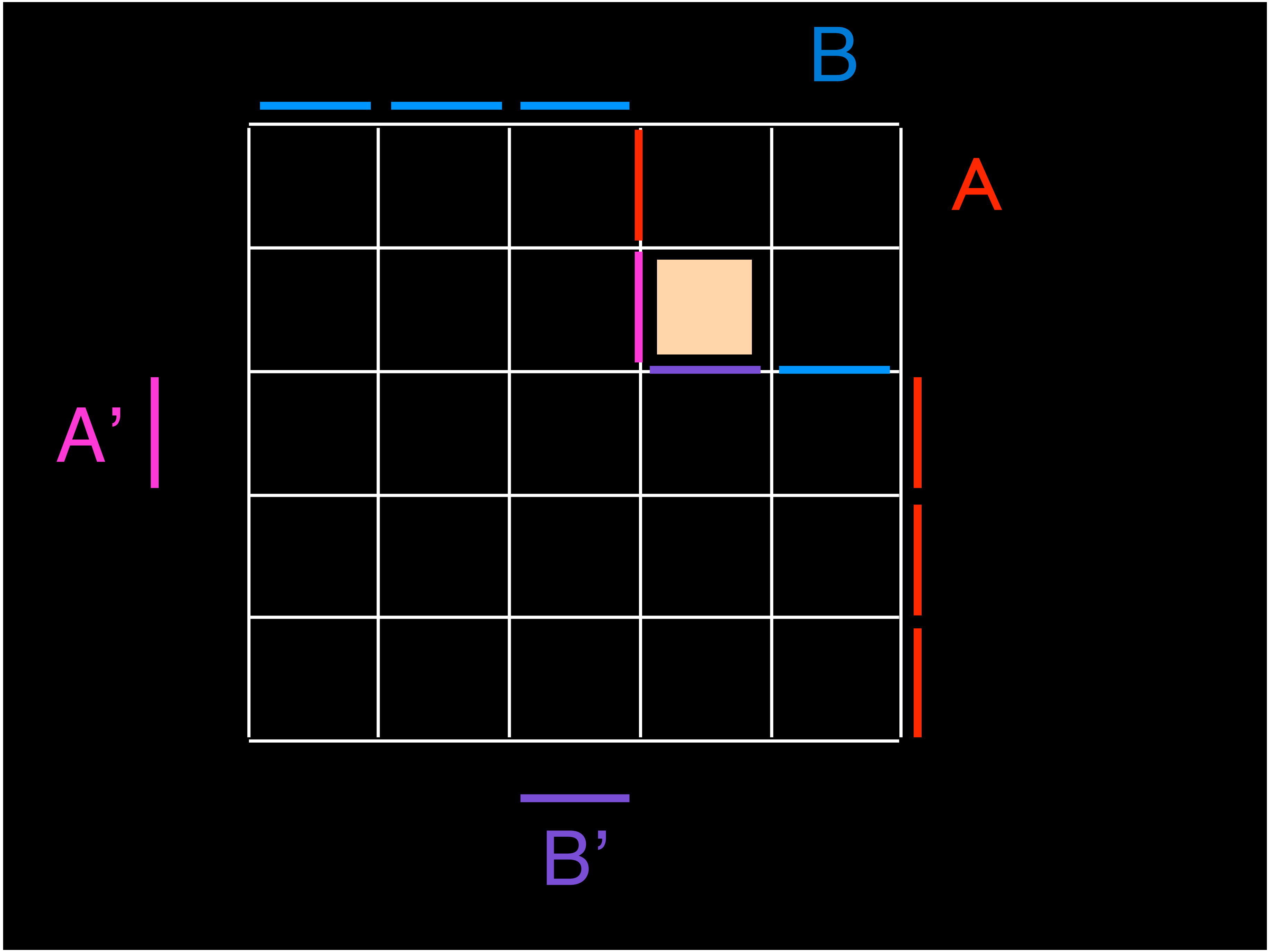
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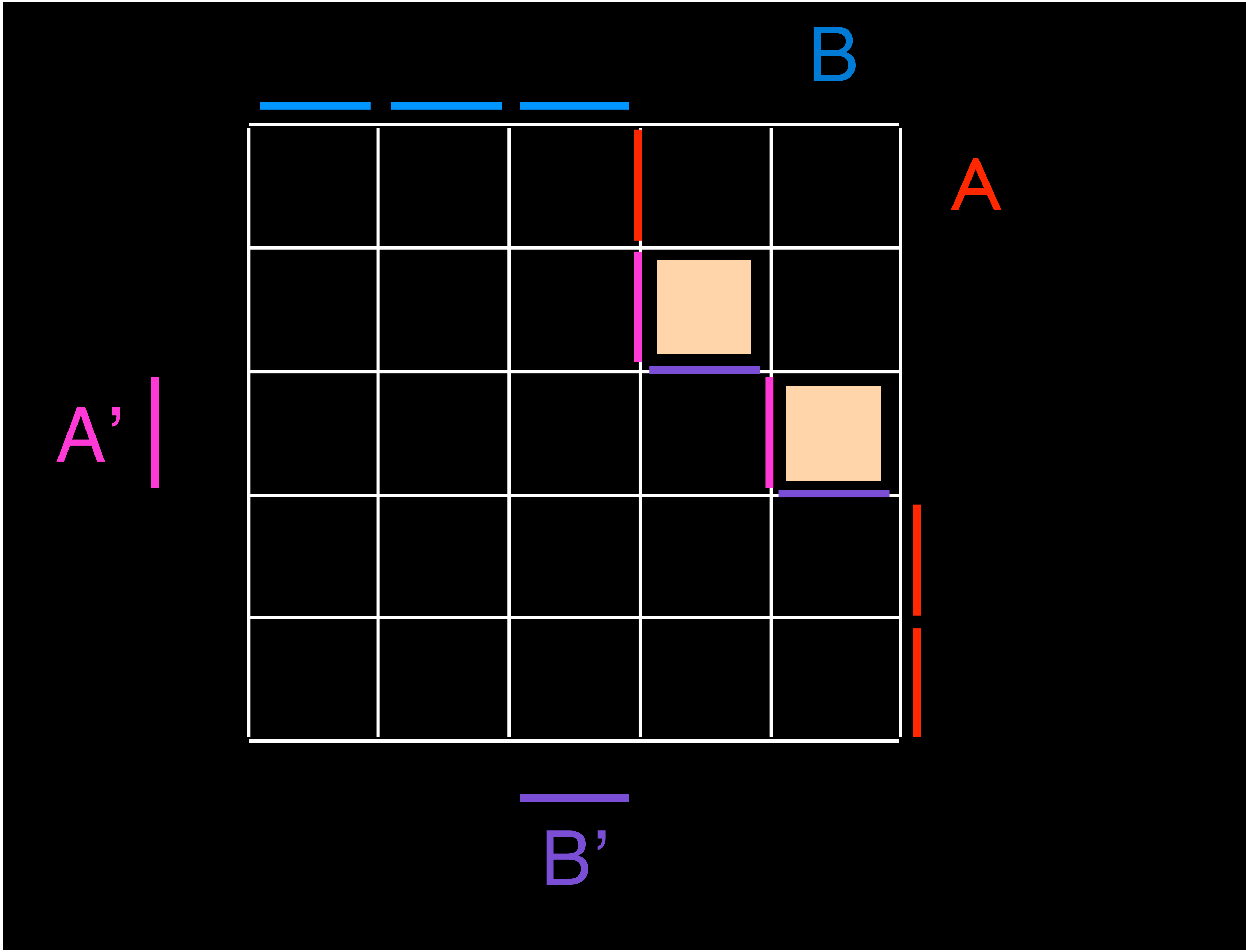

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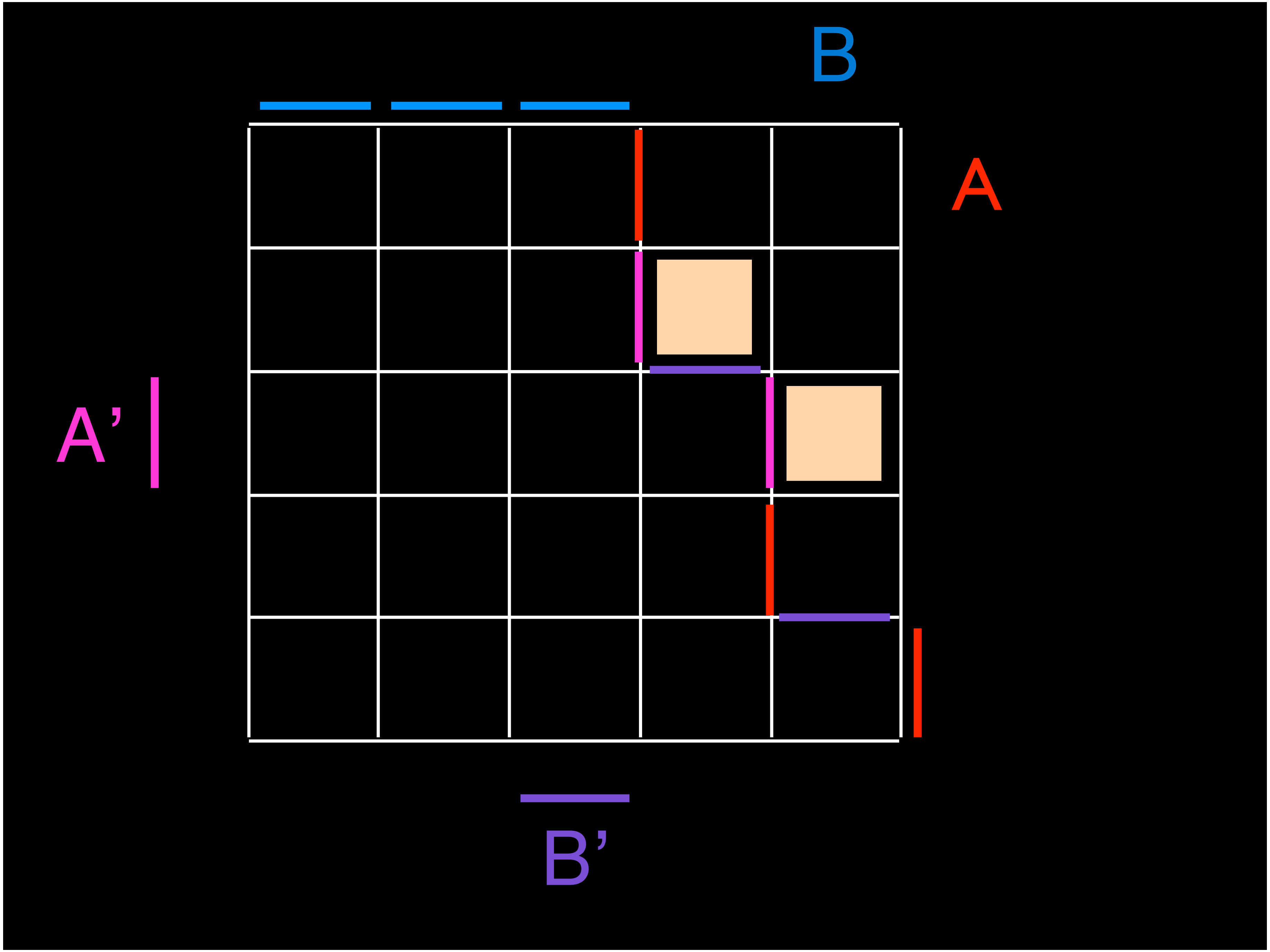
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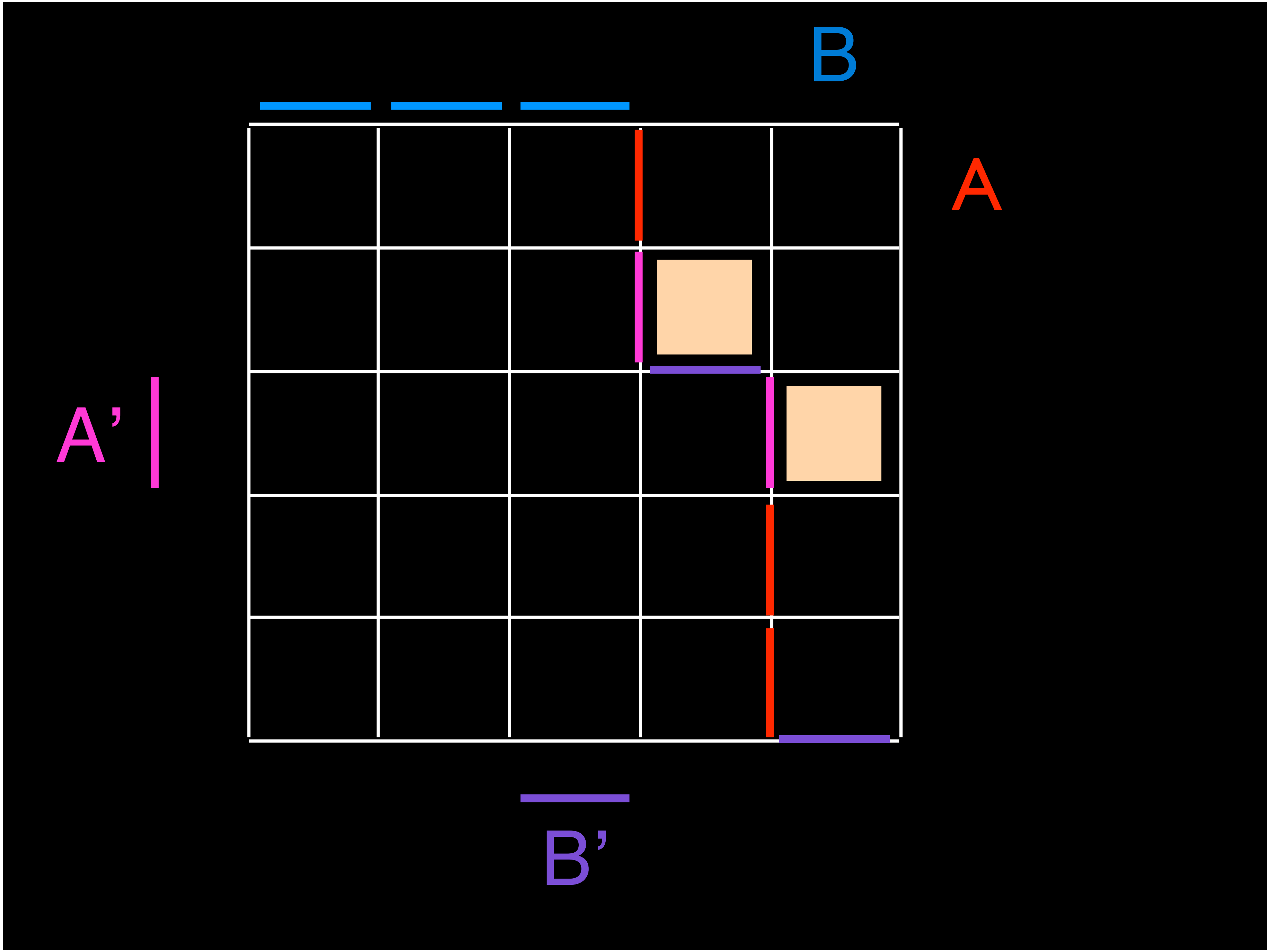
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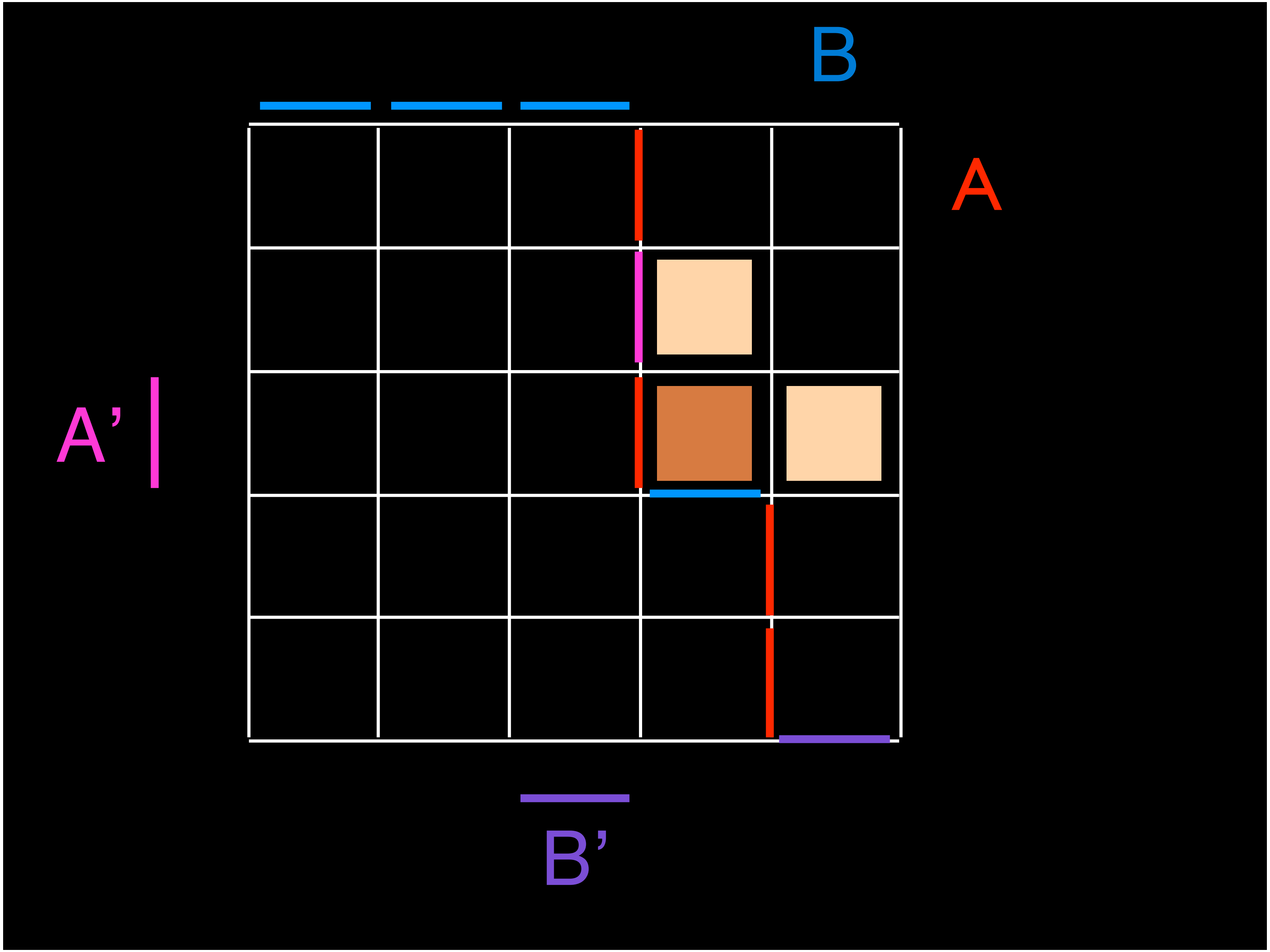
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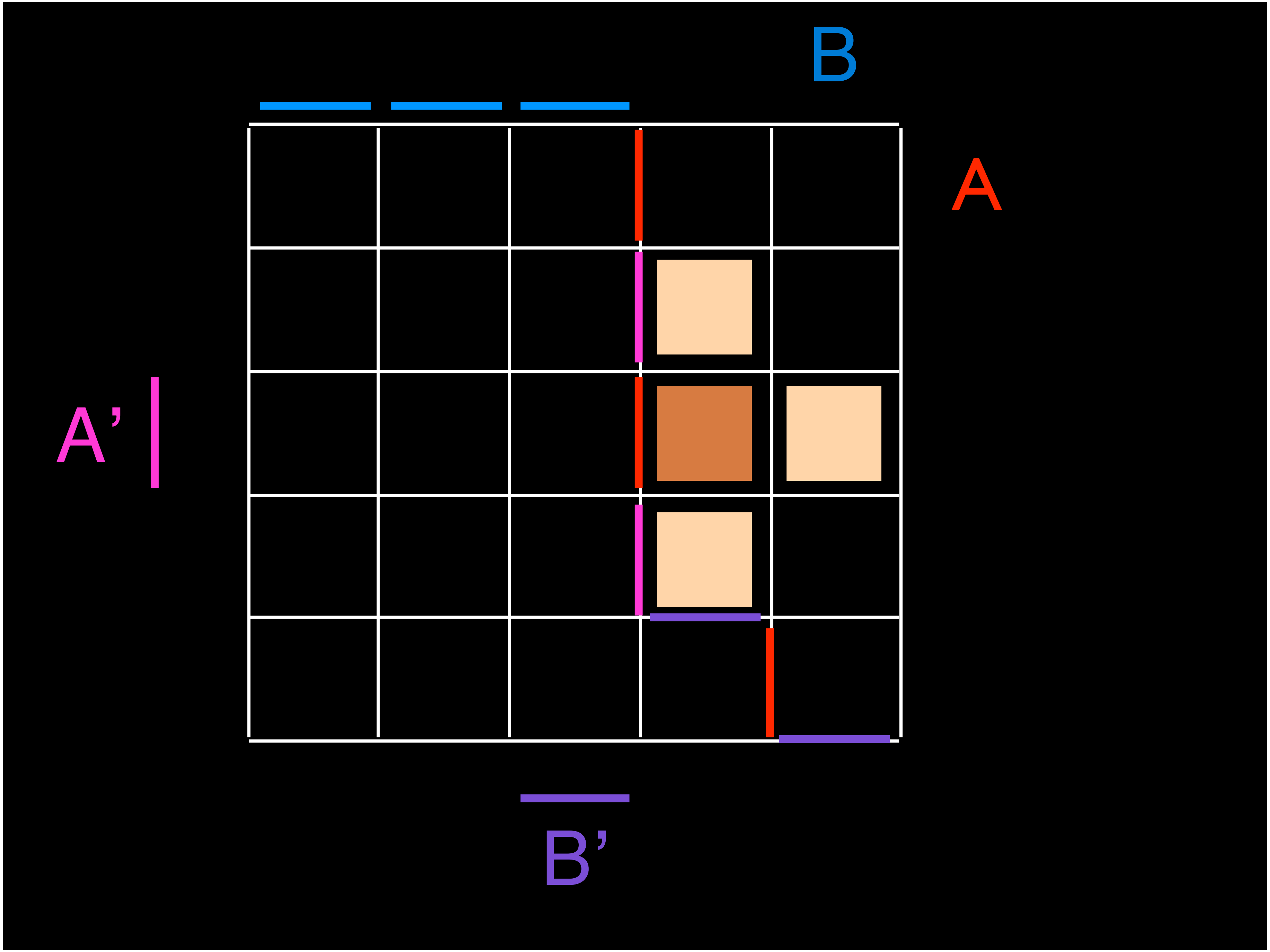
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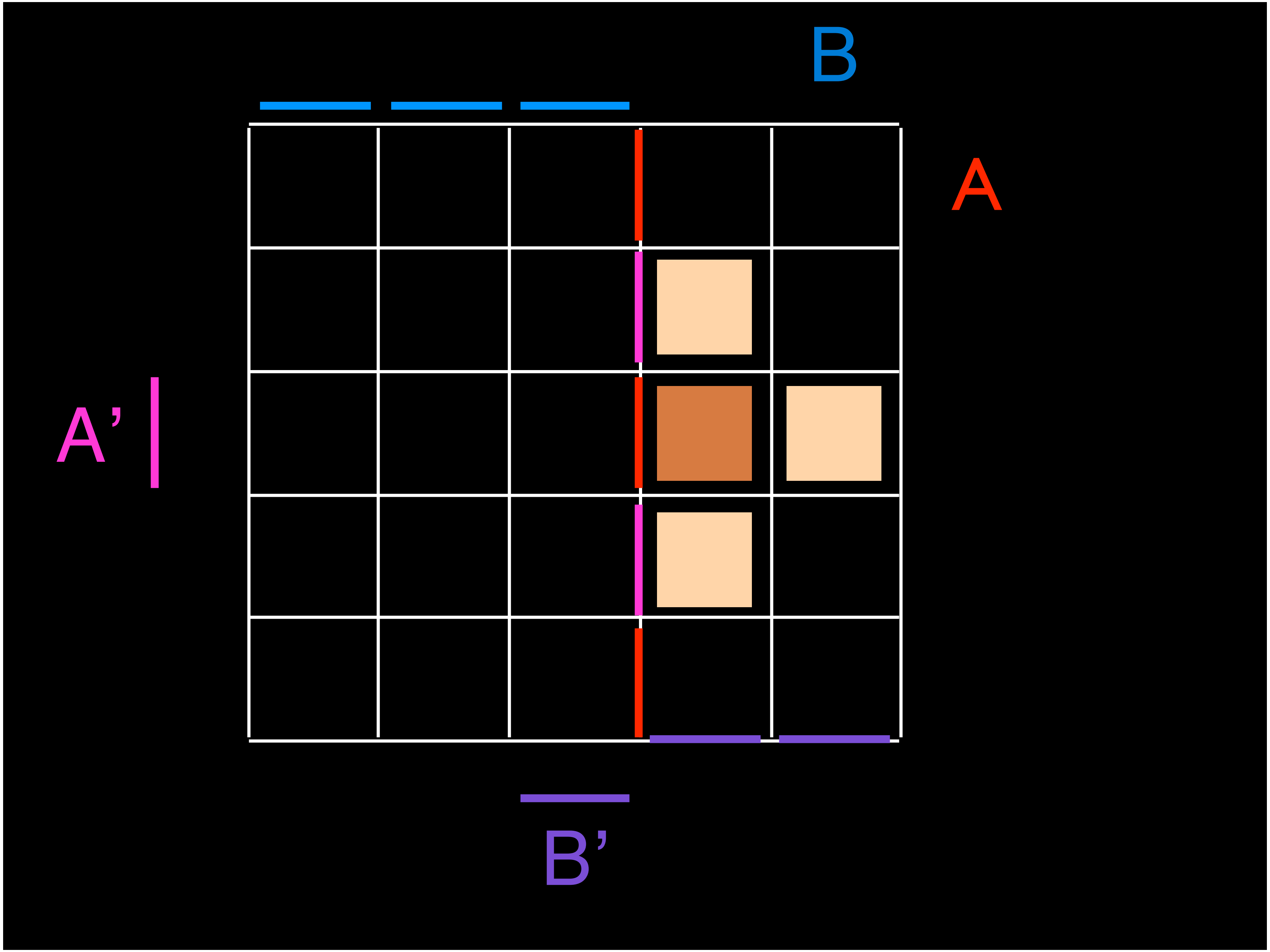
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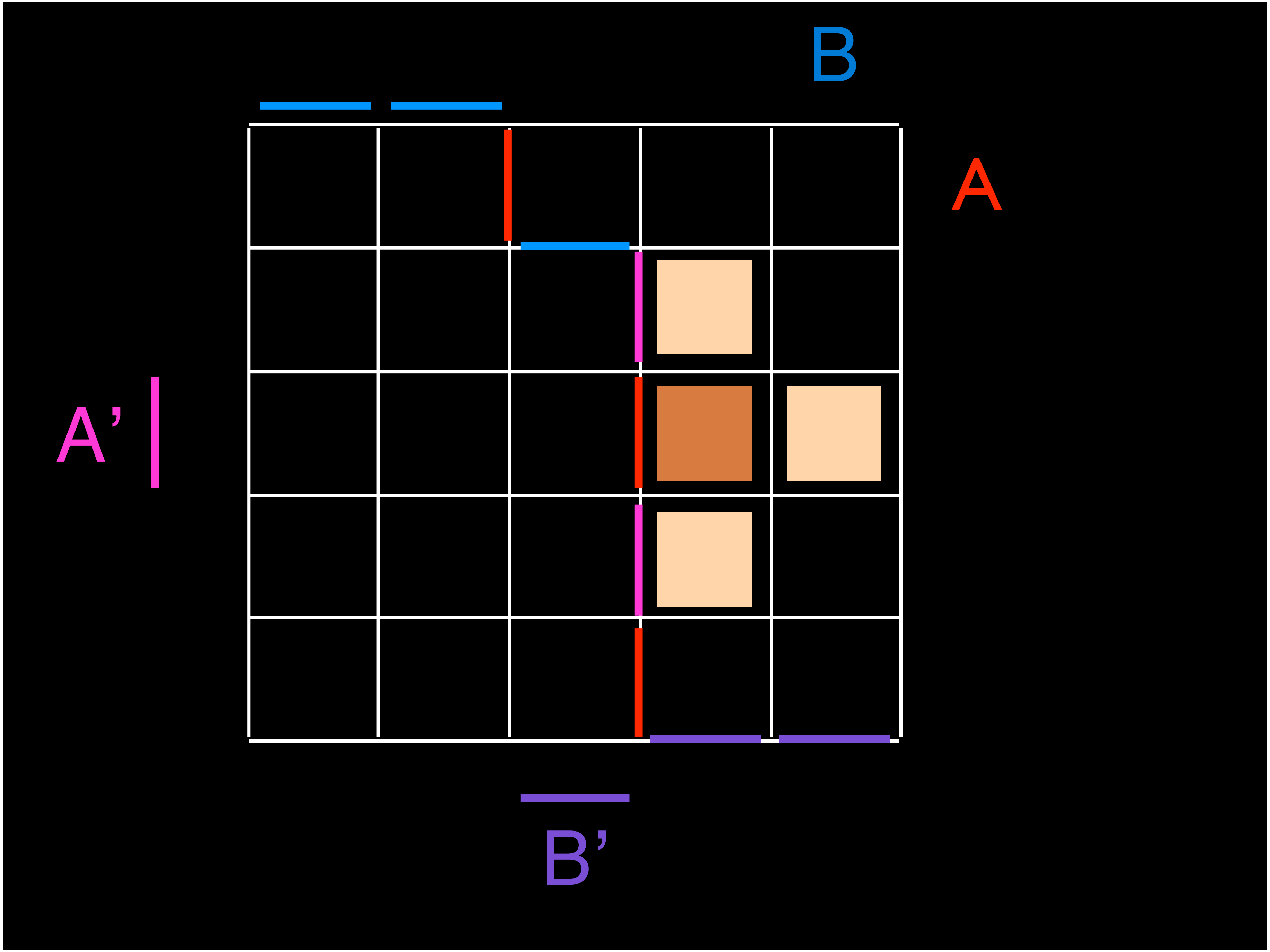

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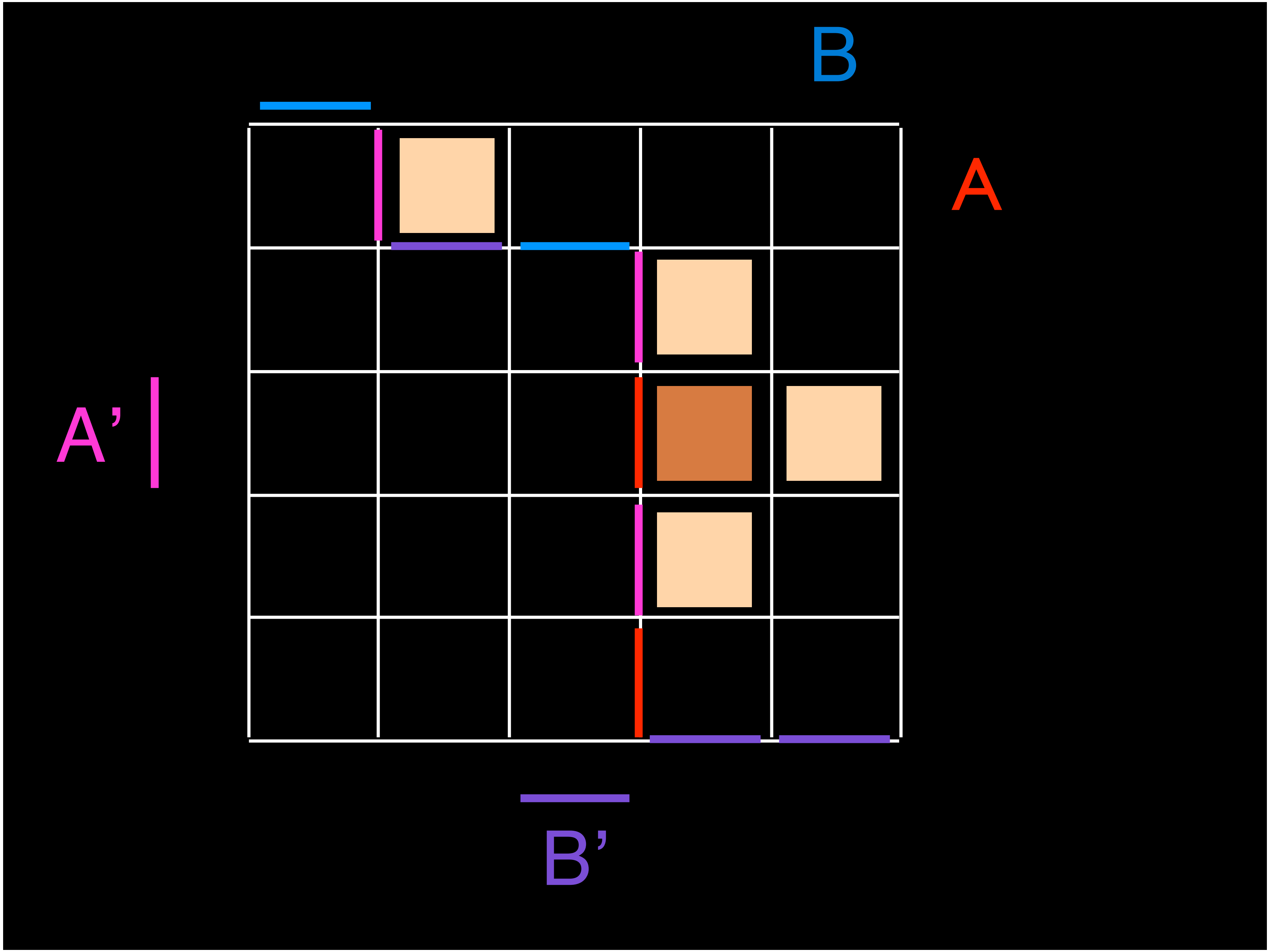

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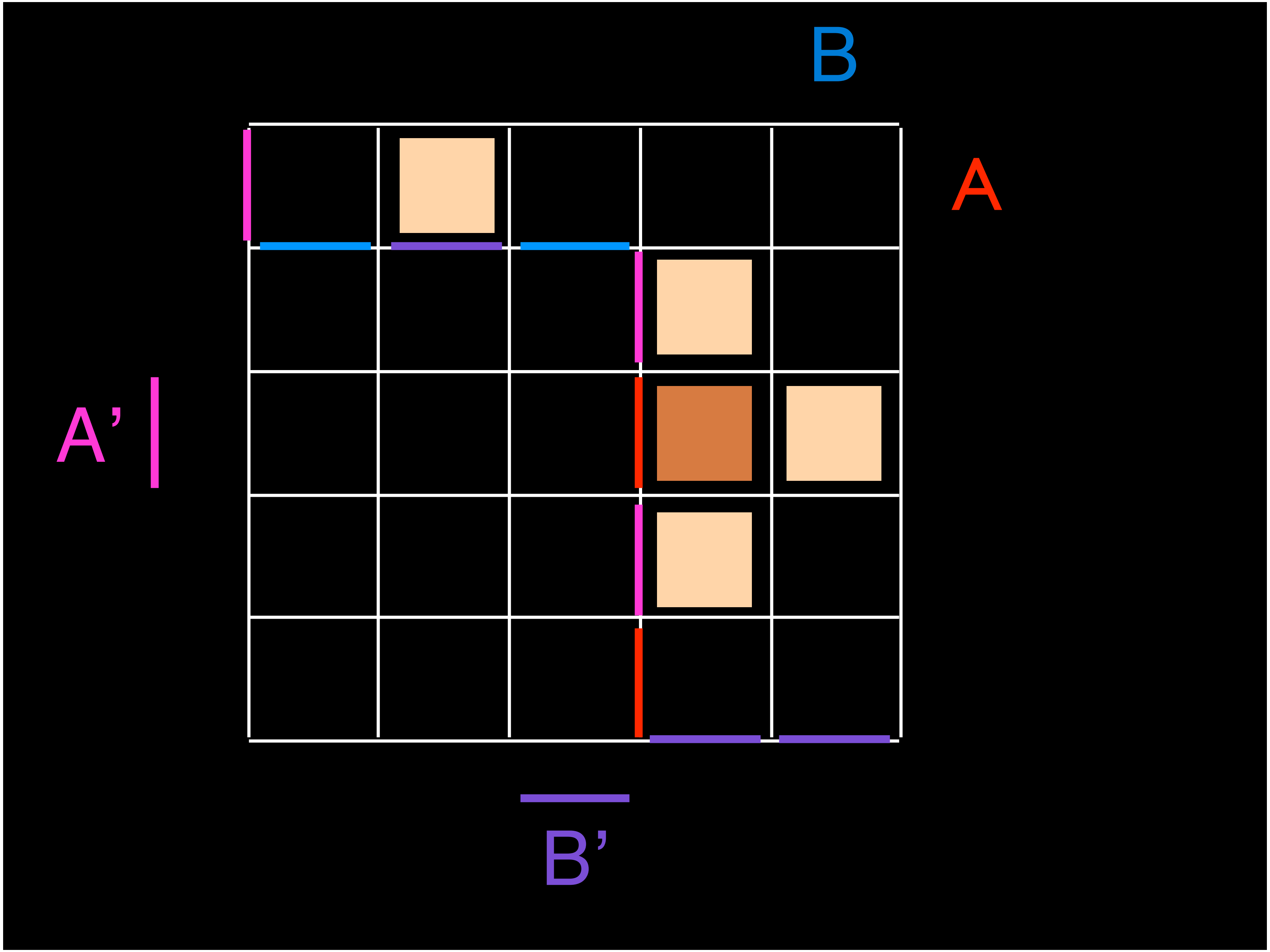
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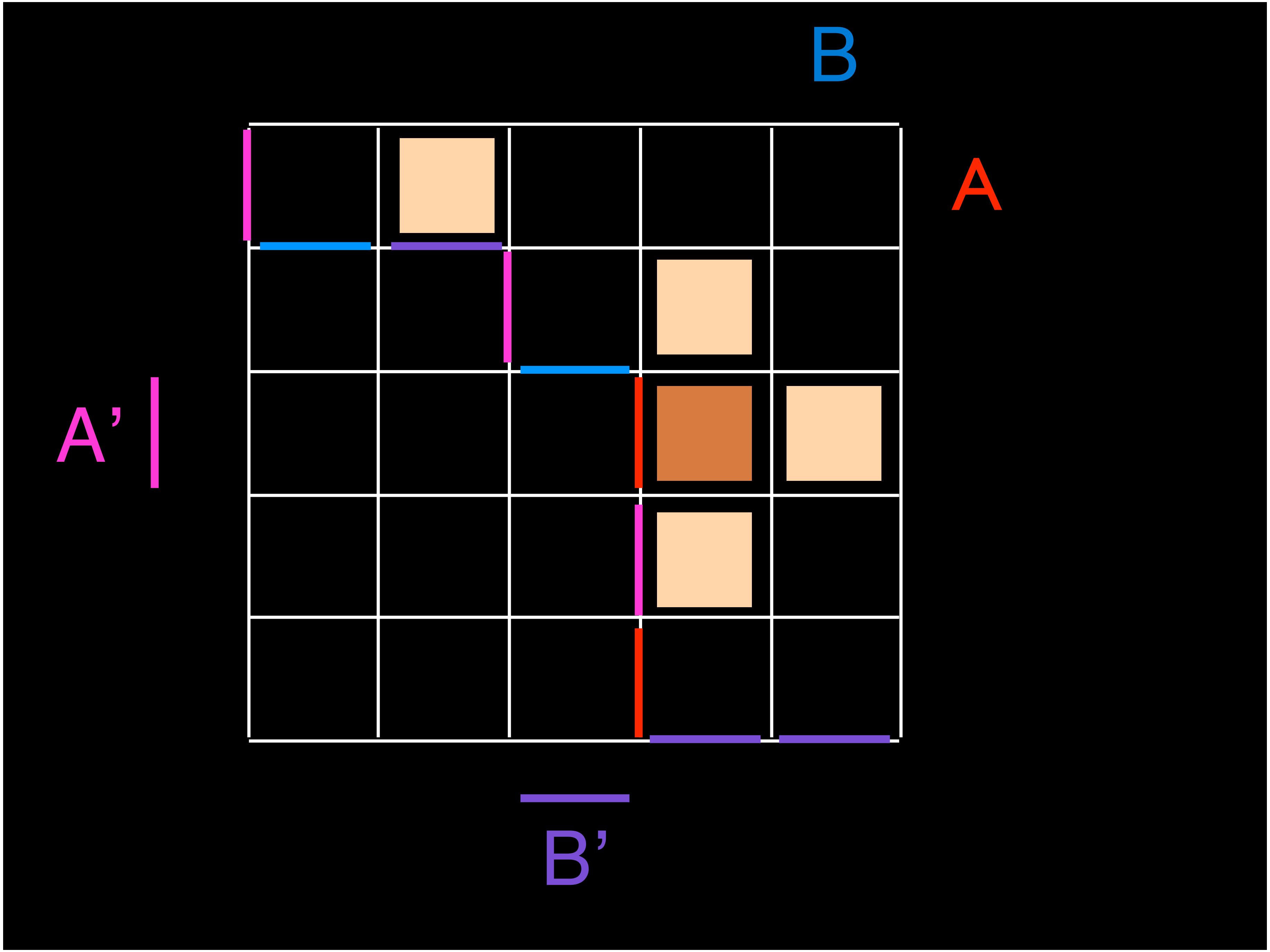
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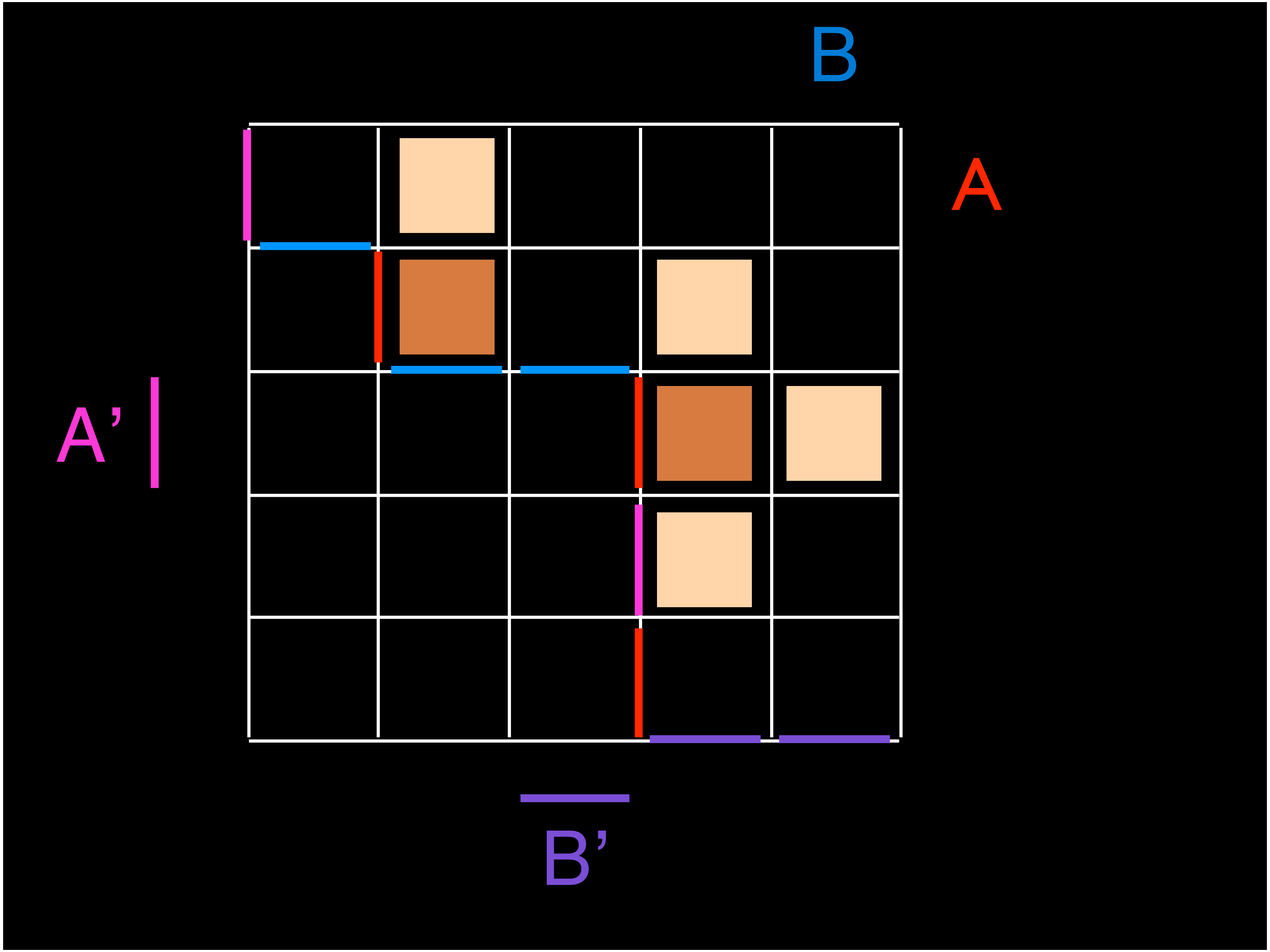
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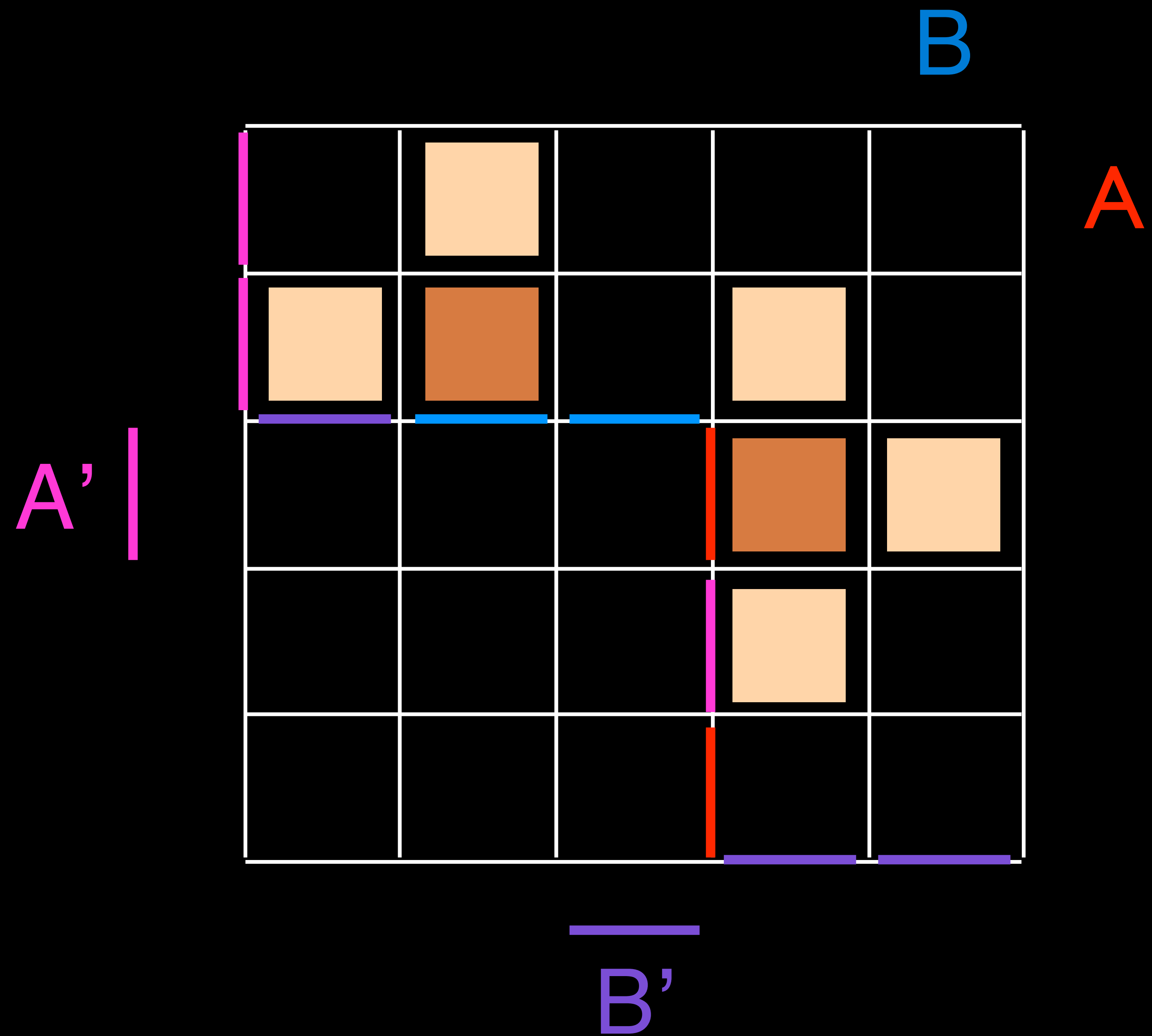
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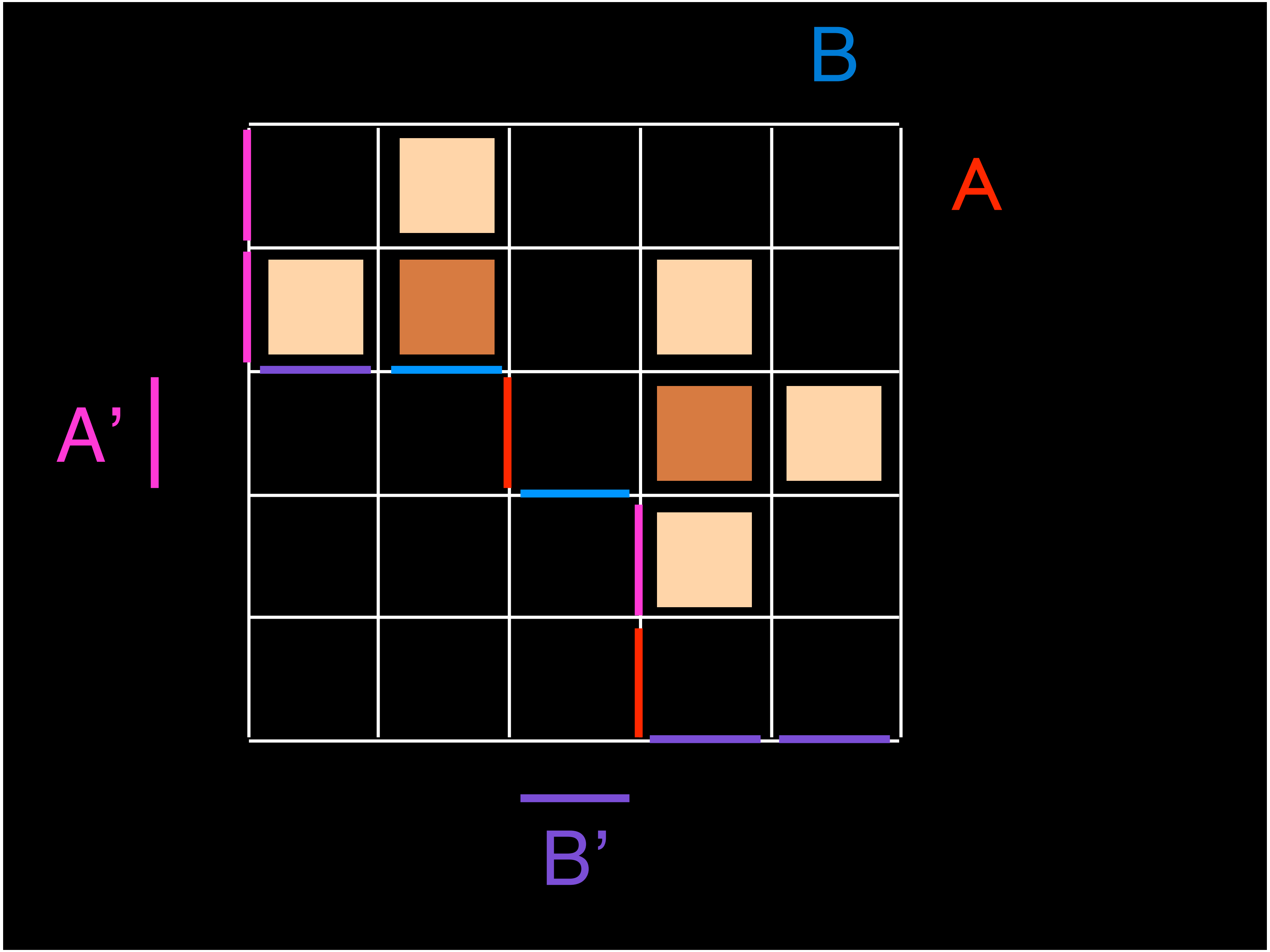

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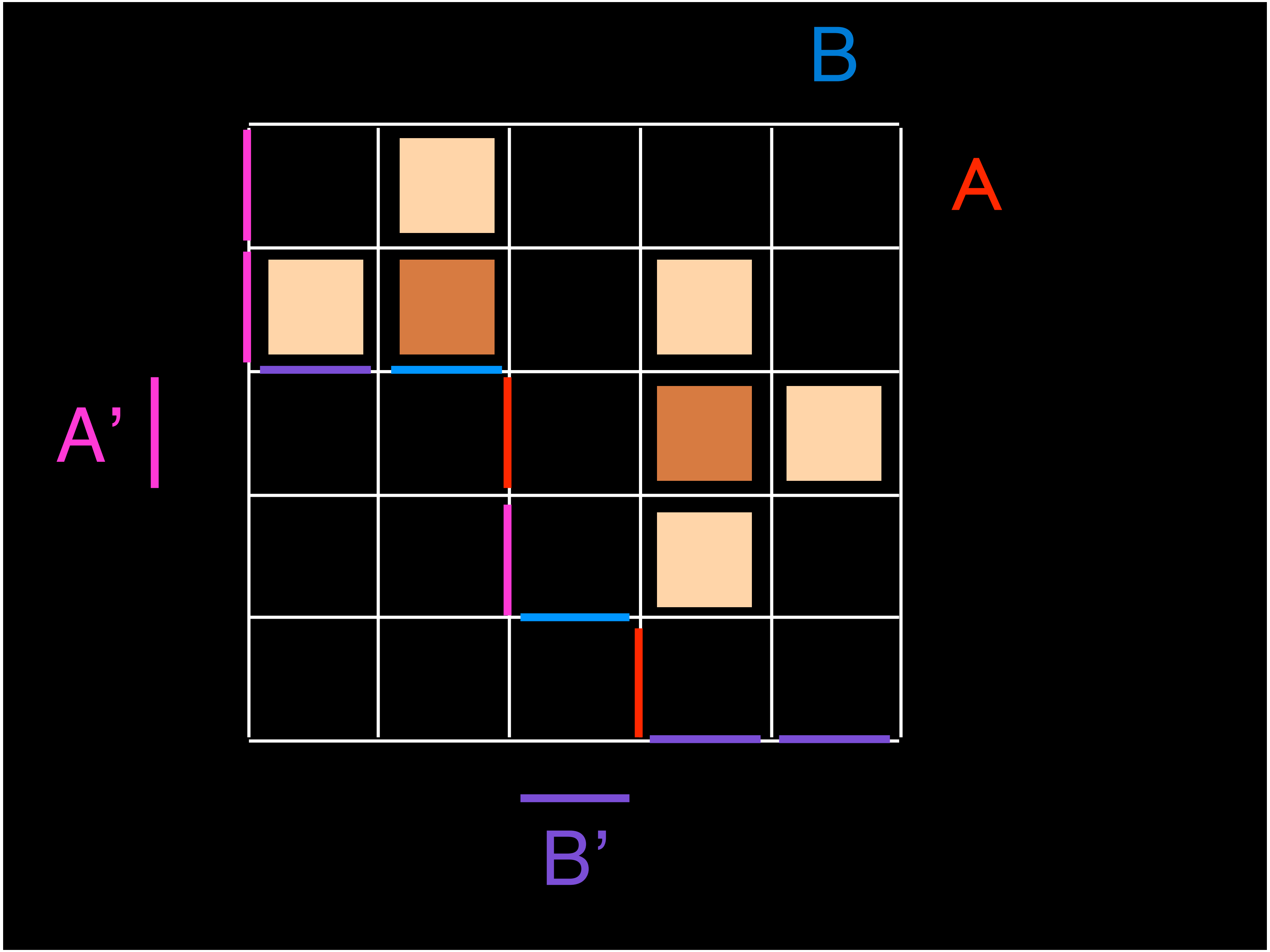

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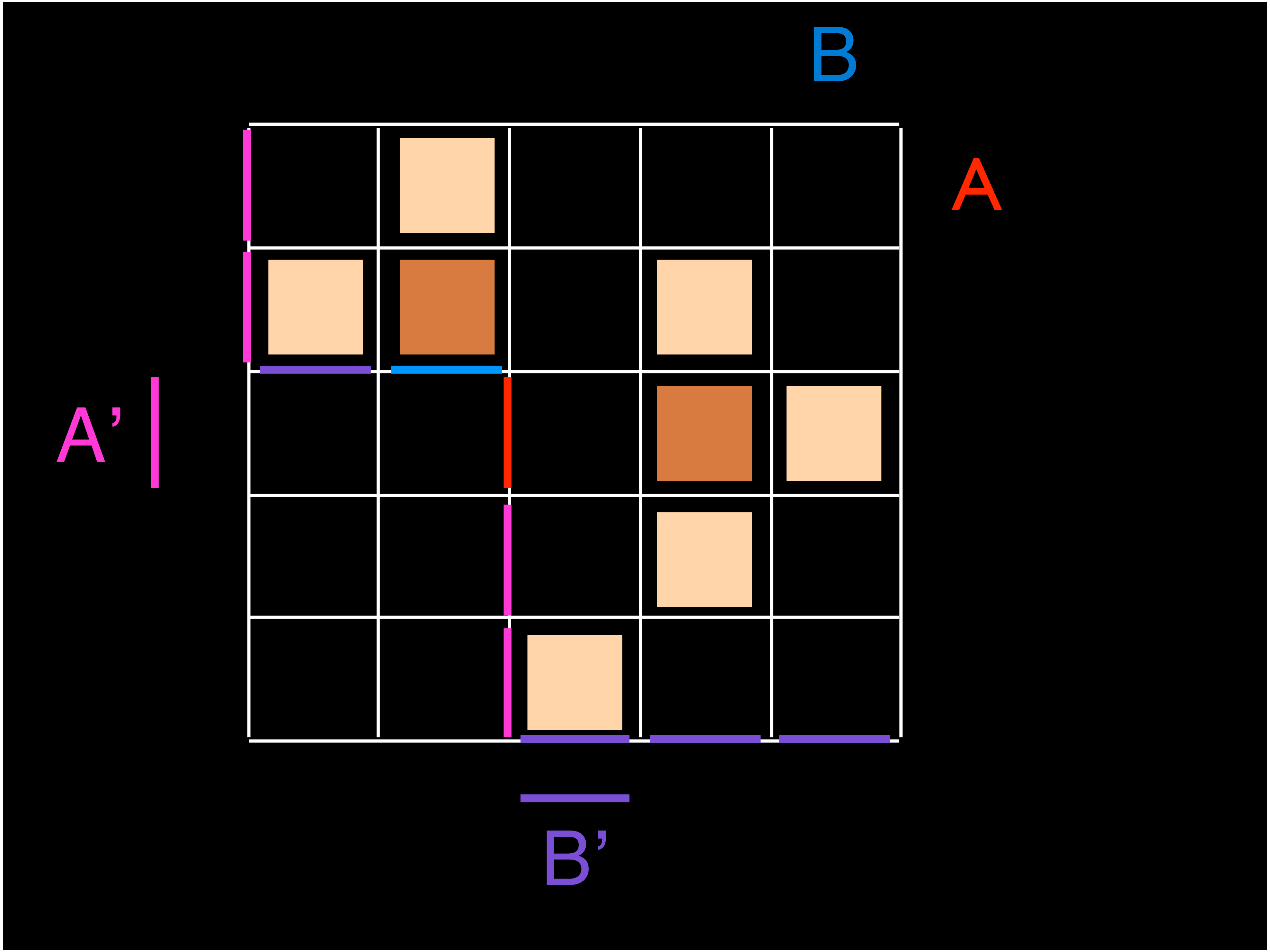
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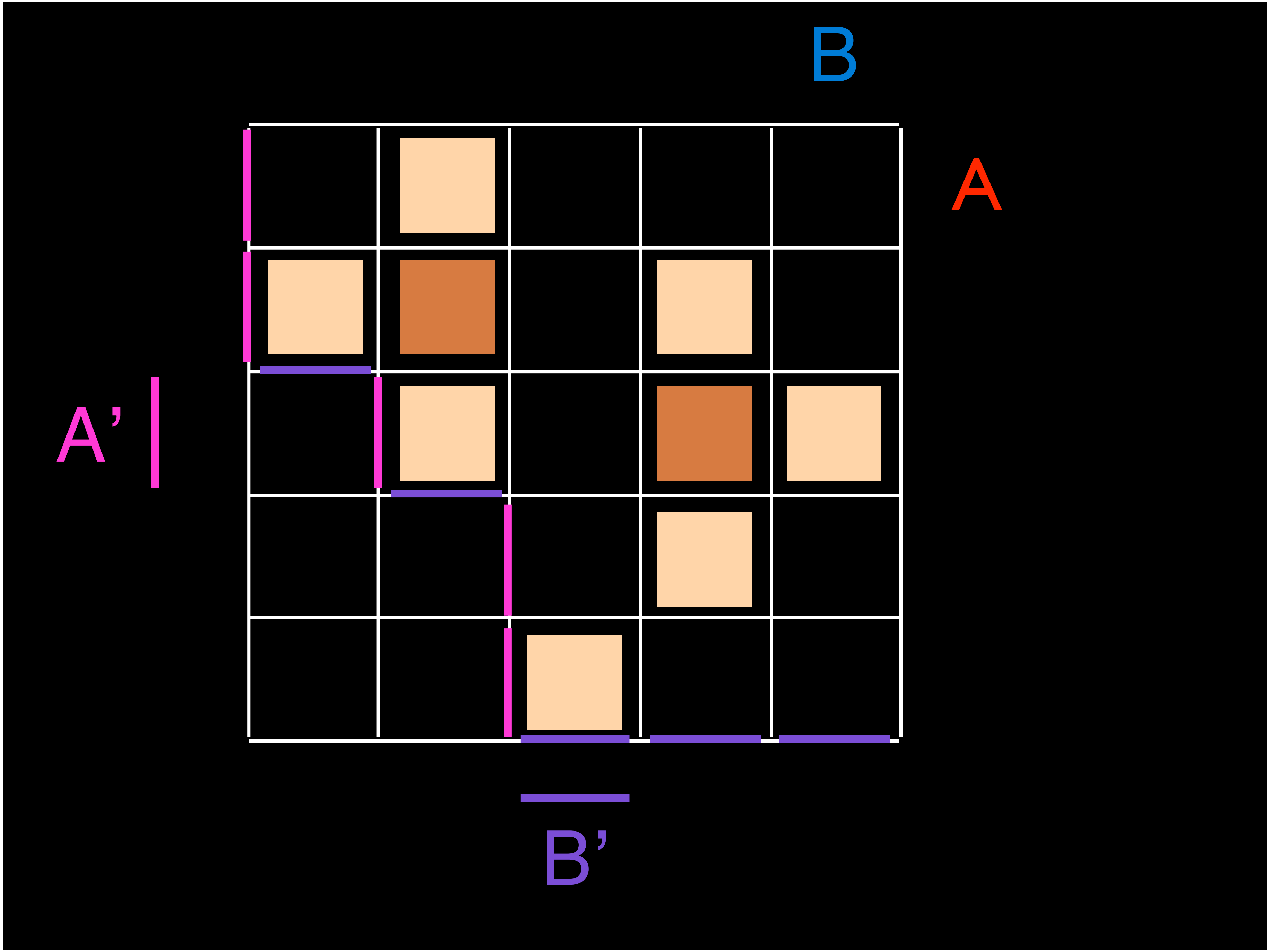
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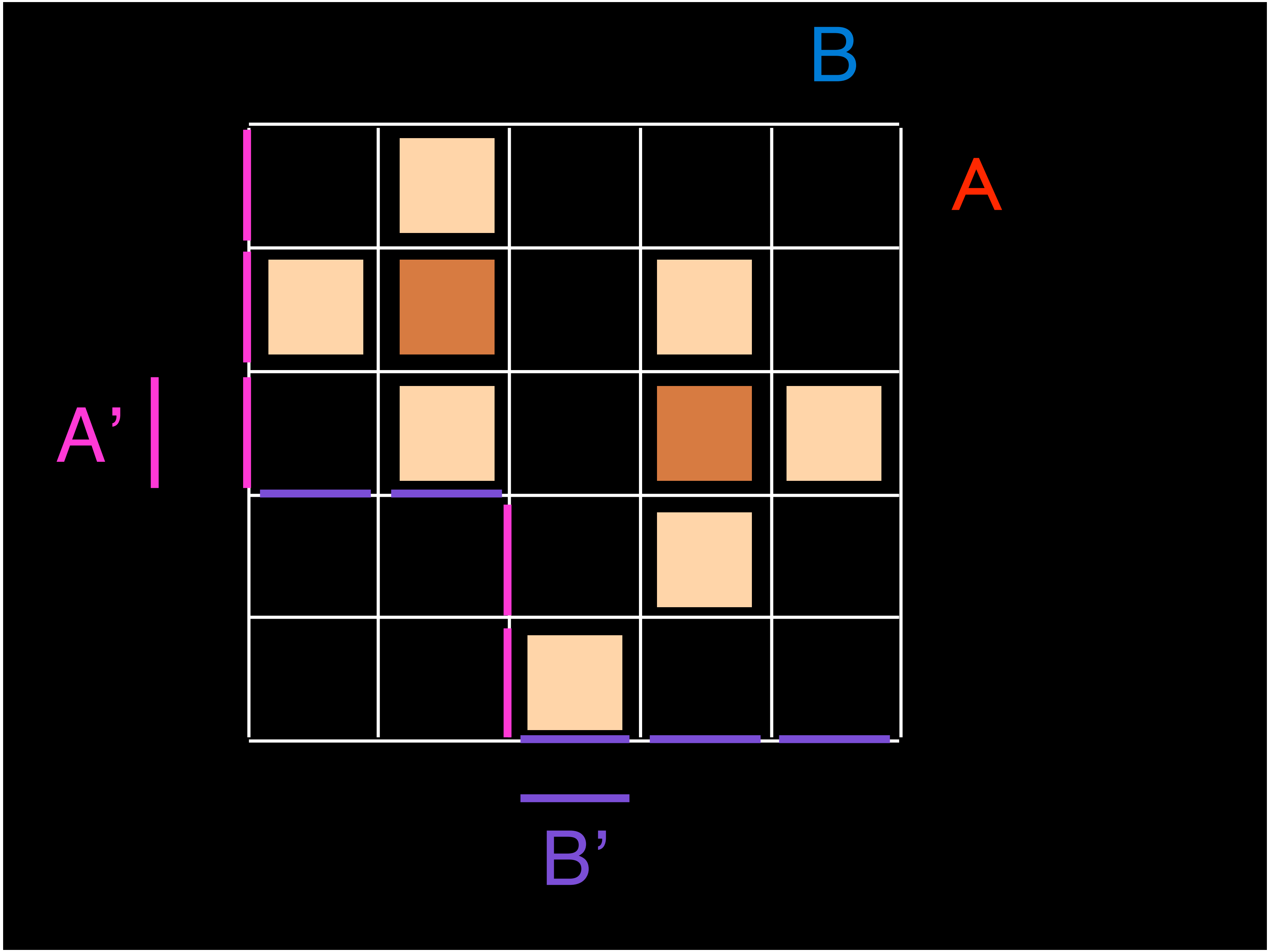
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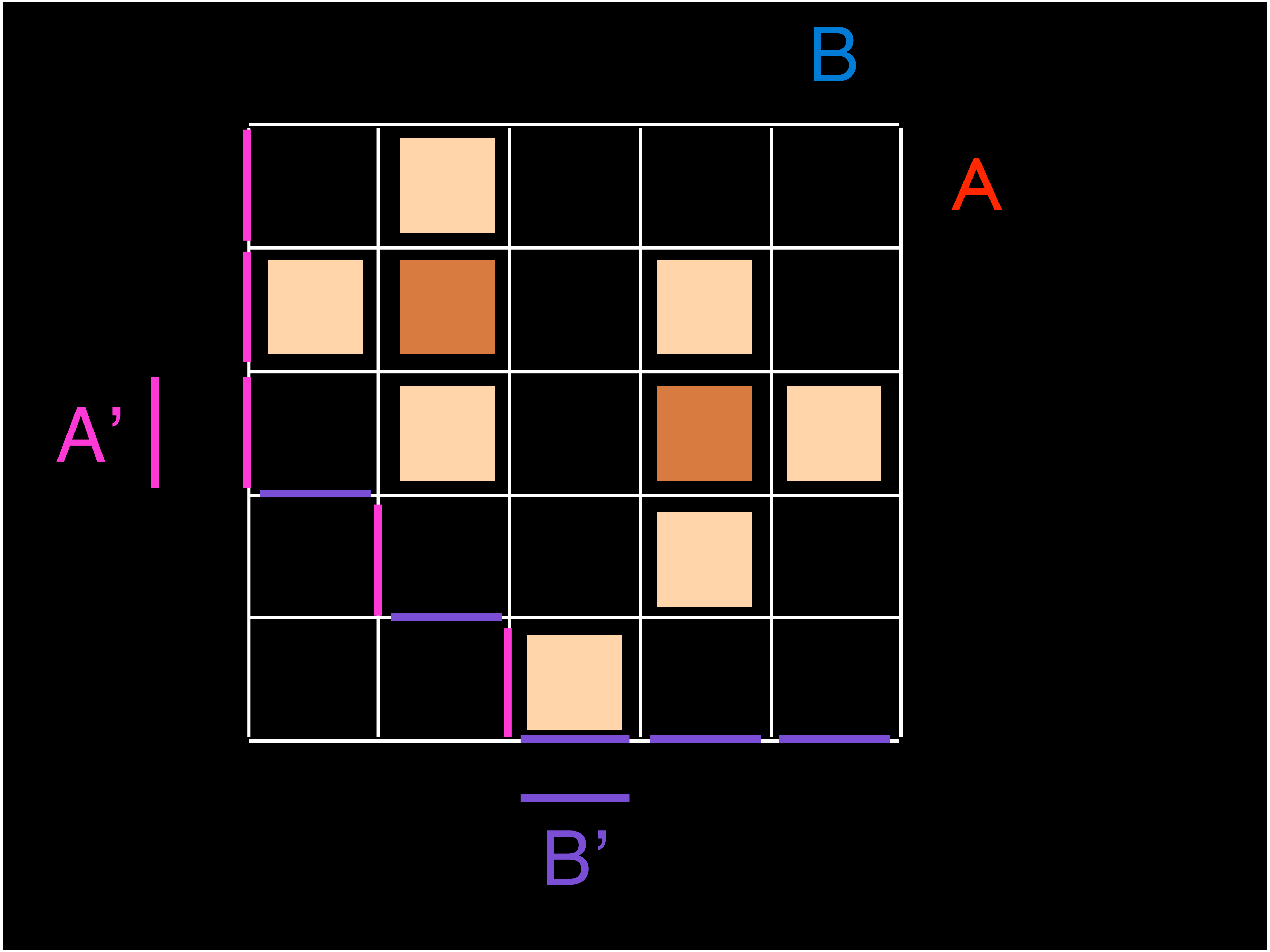

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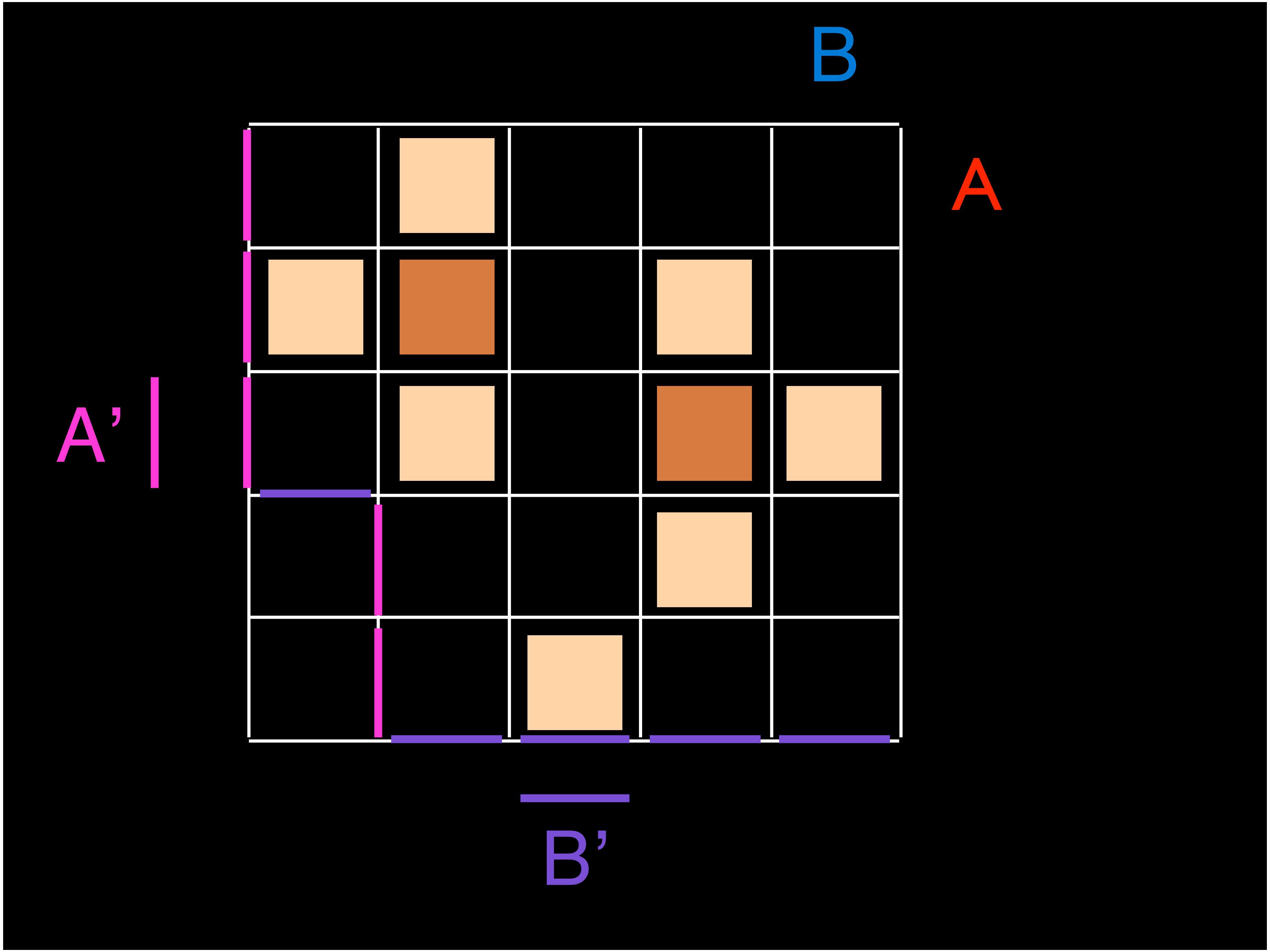
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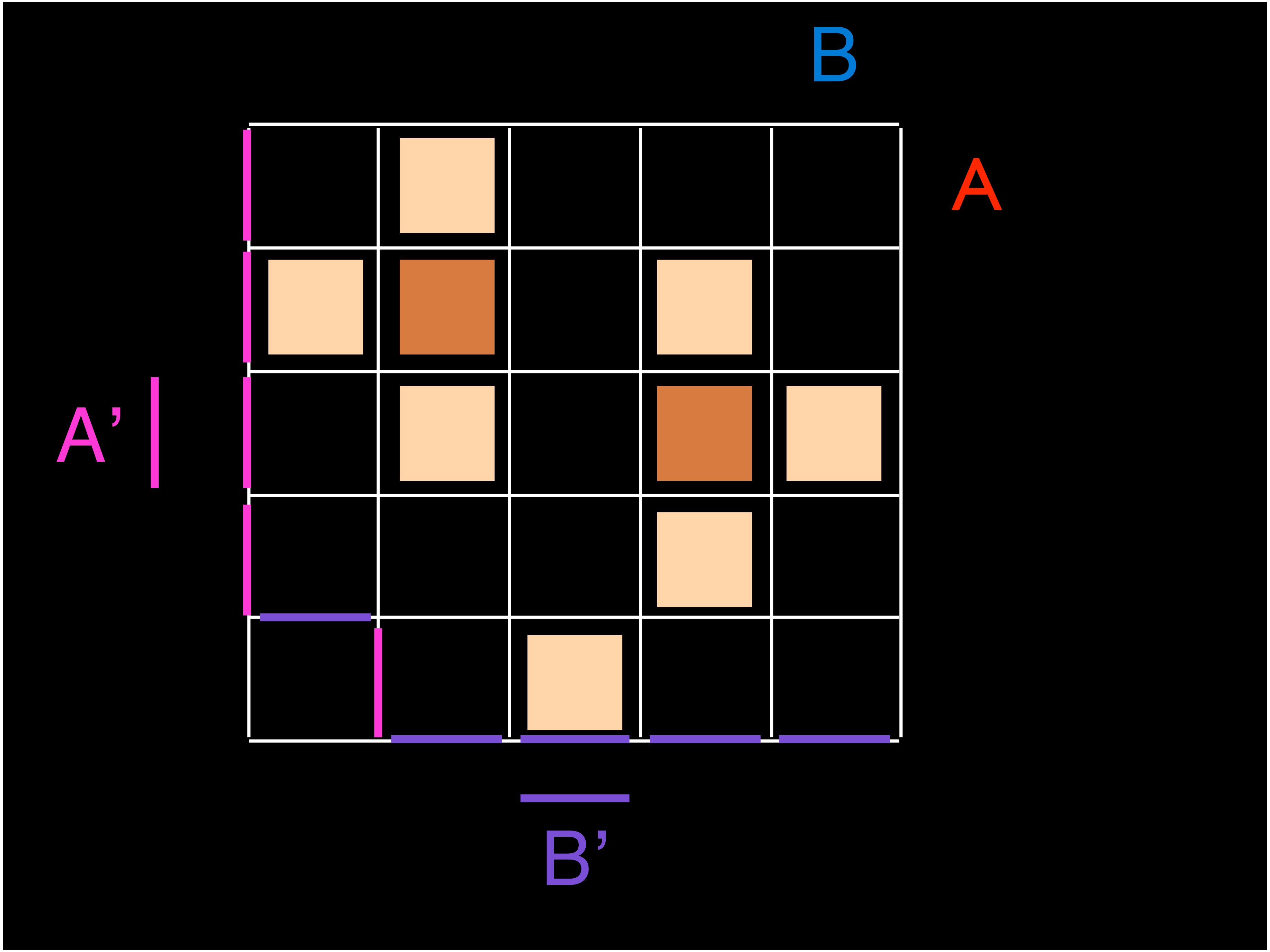
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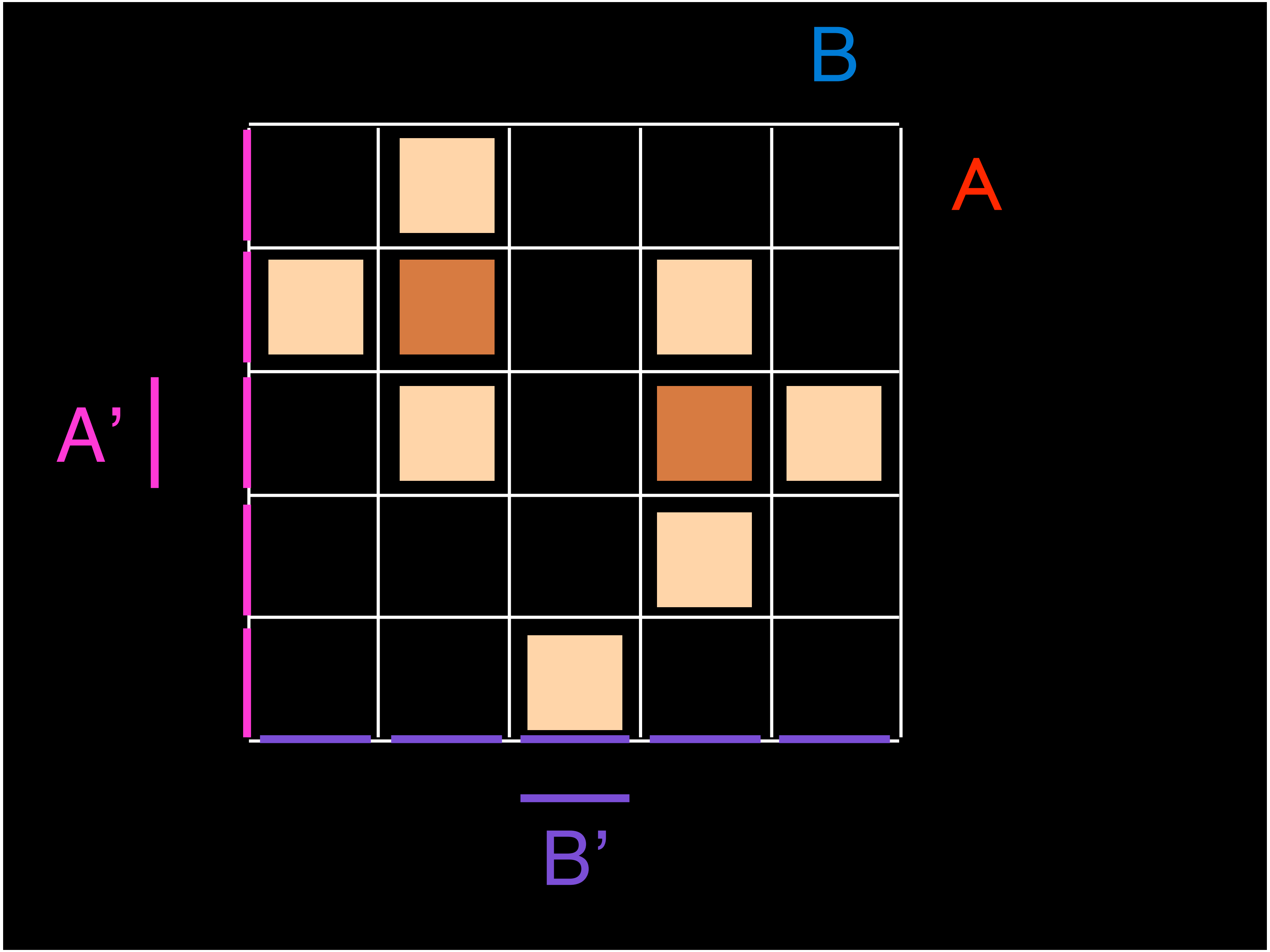

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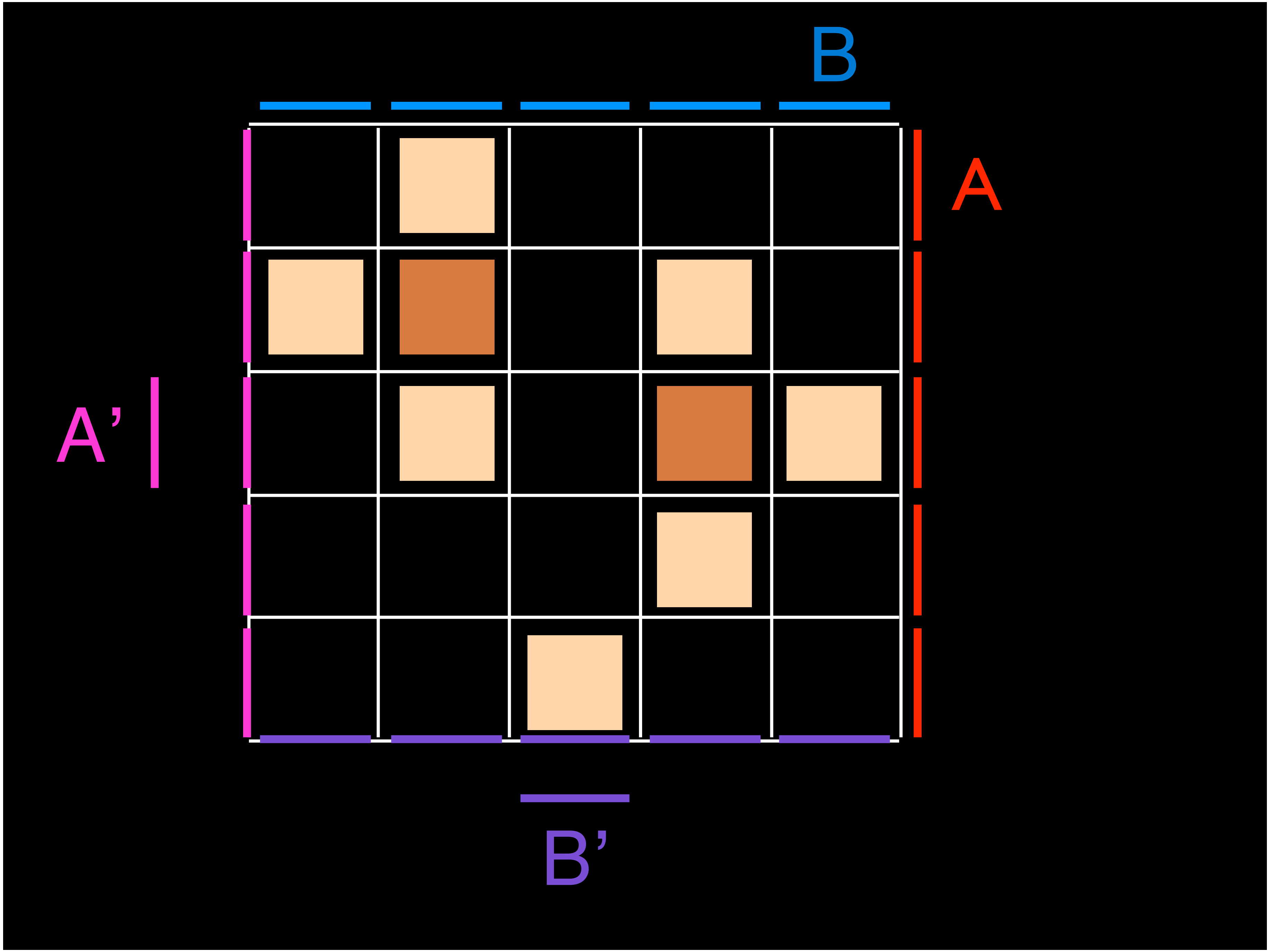

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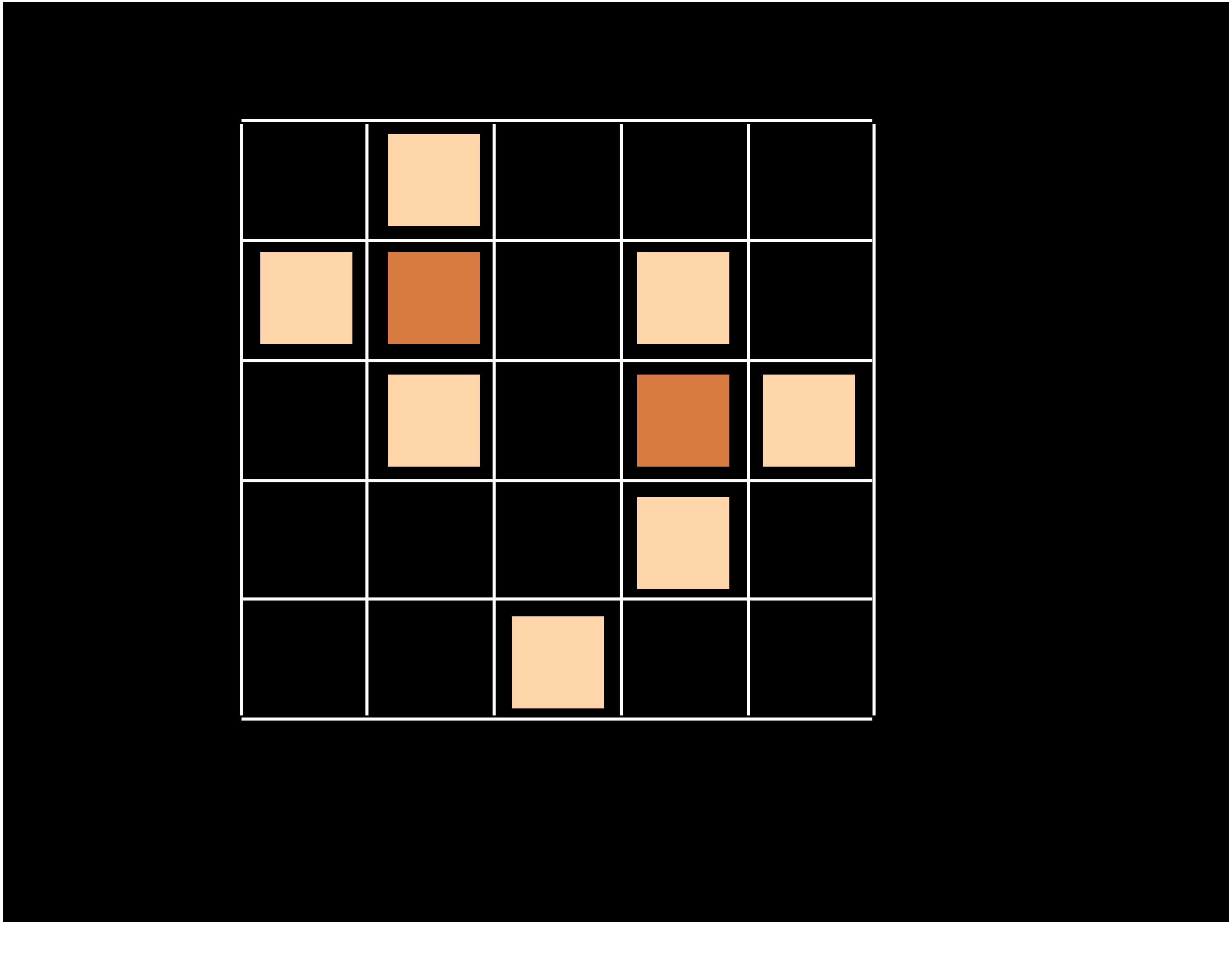

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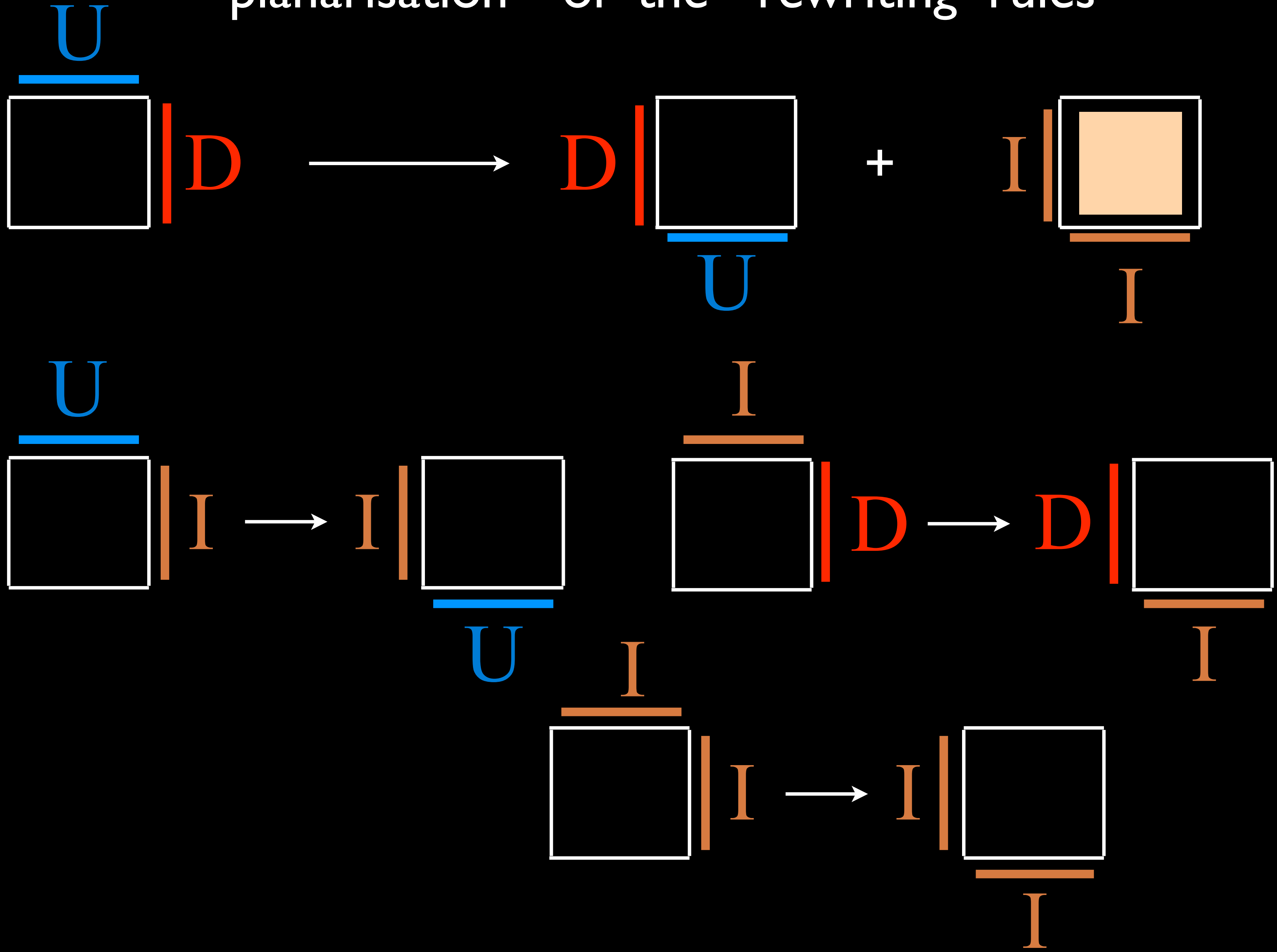
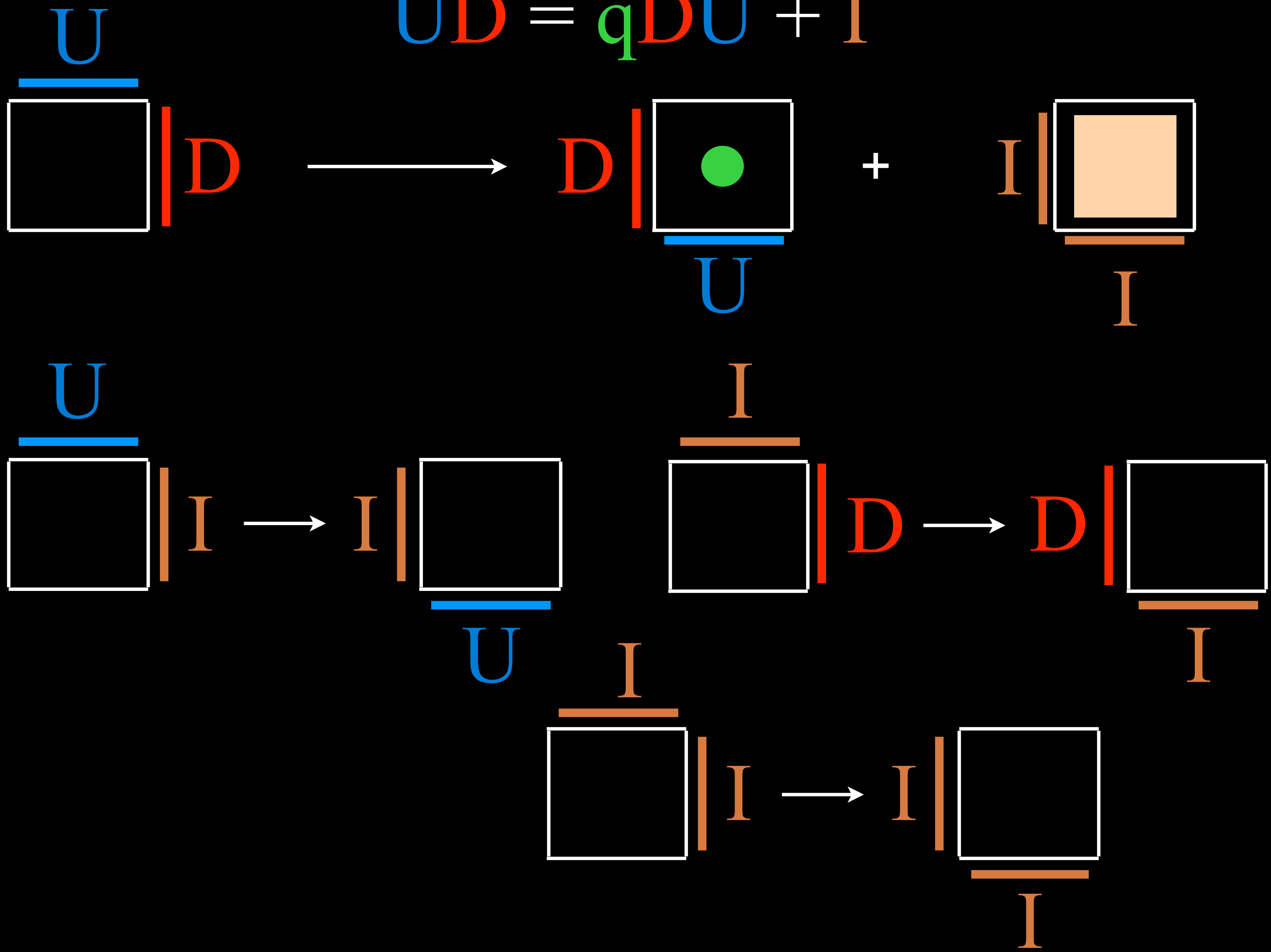
§2 Heisenberg operators U, D

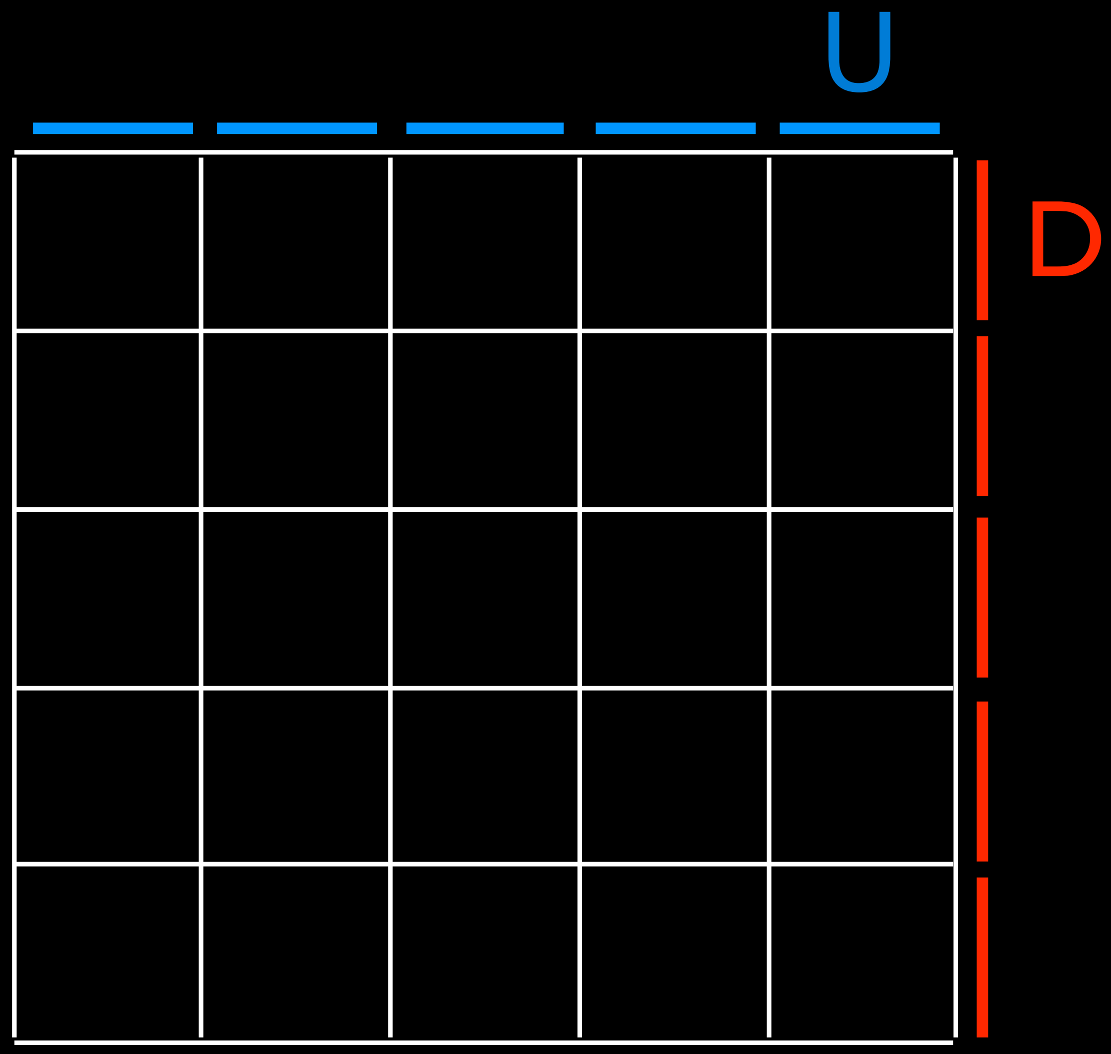
$$UD = q\,DU + 1$$

“planarisation” of the “rewriting rules”



$$UD = qDU + I$$

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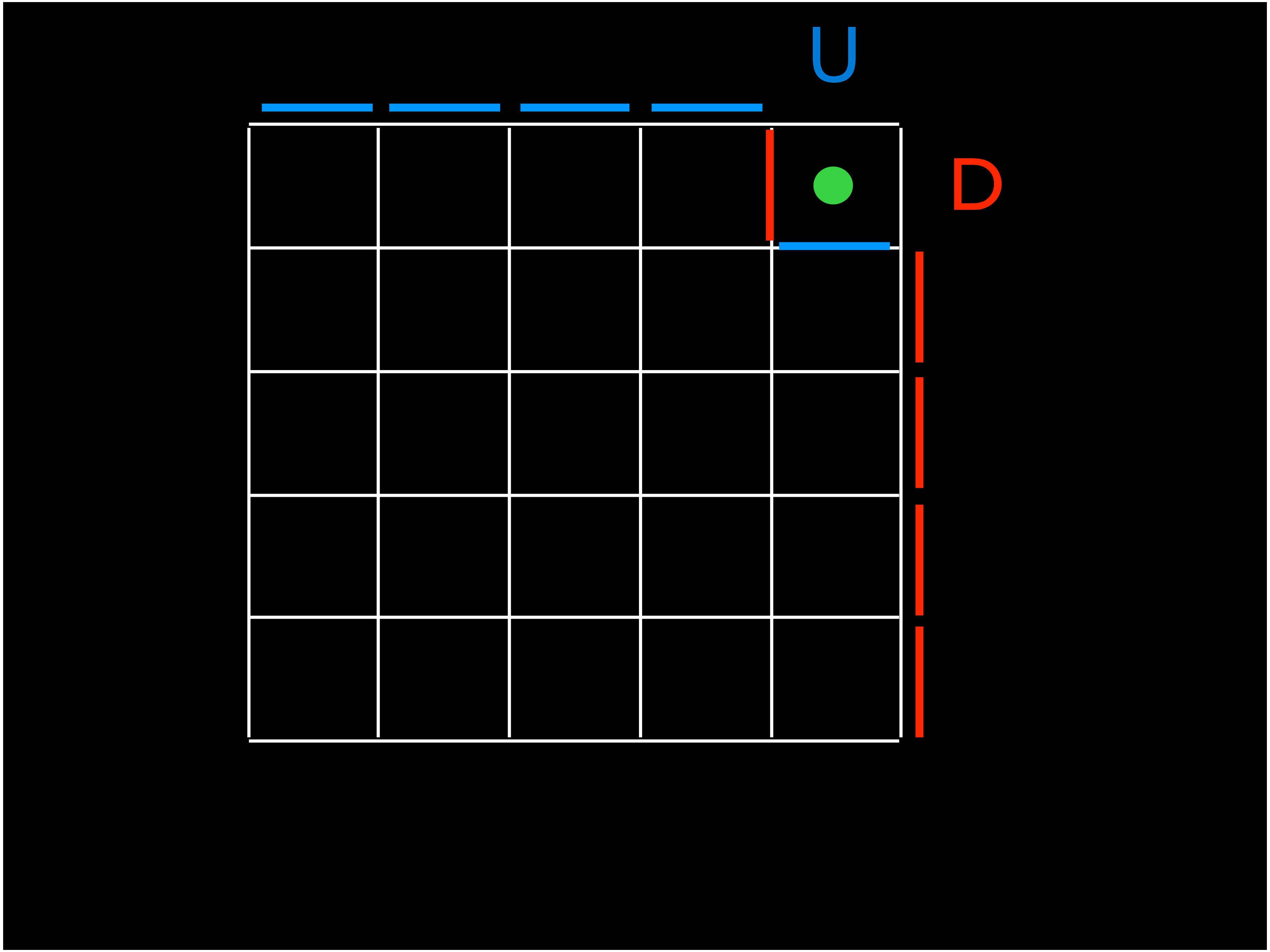
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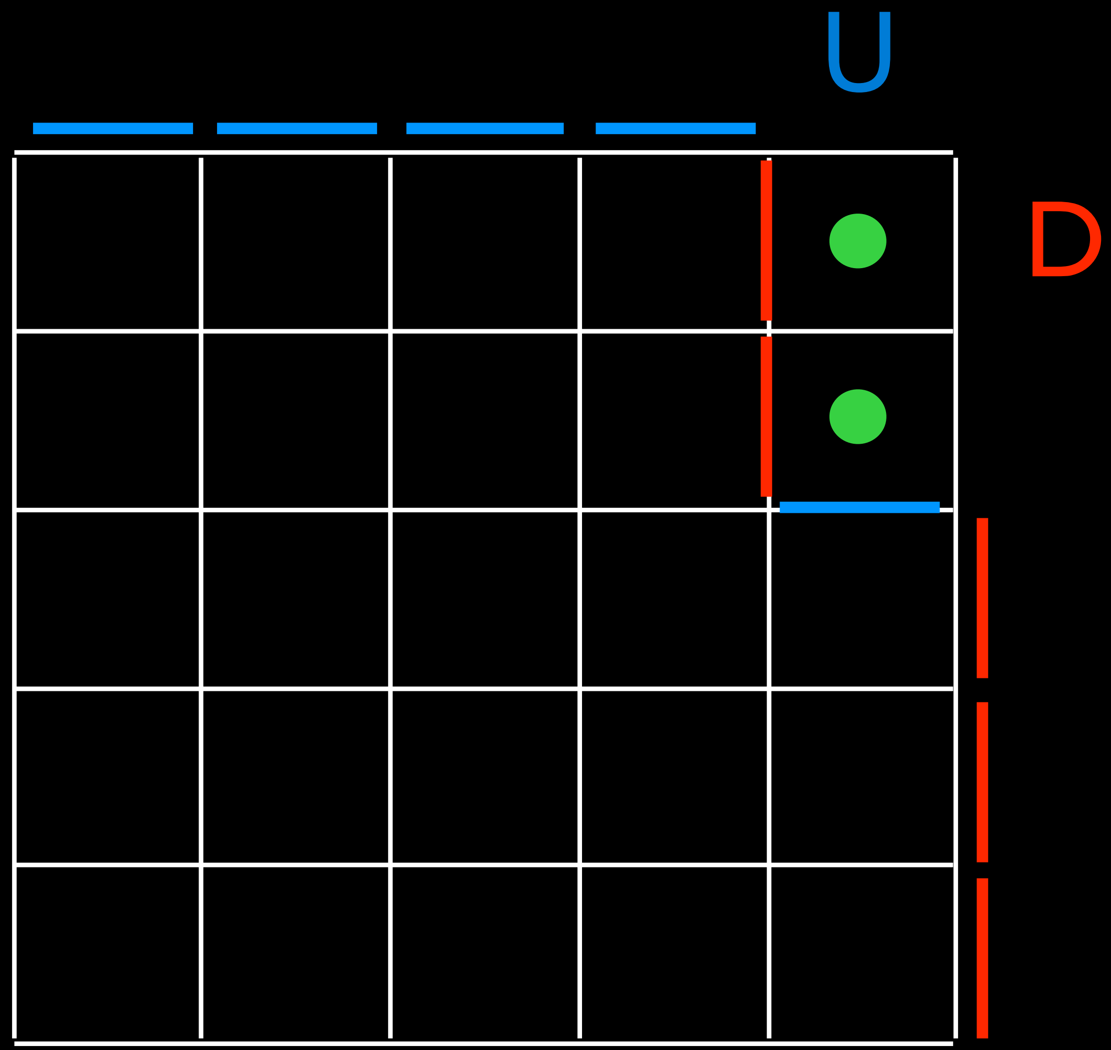
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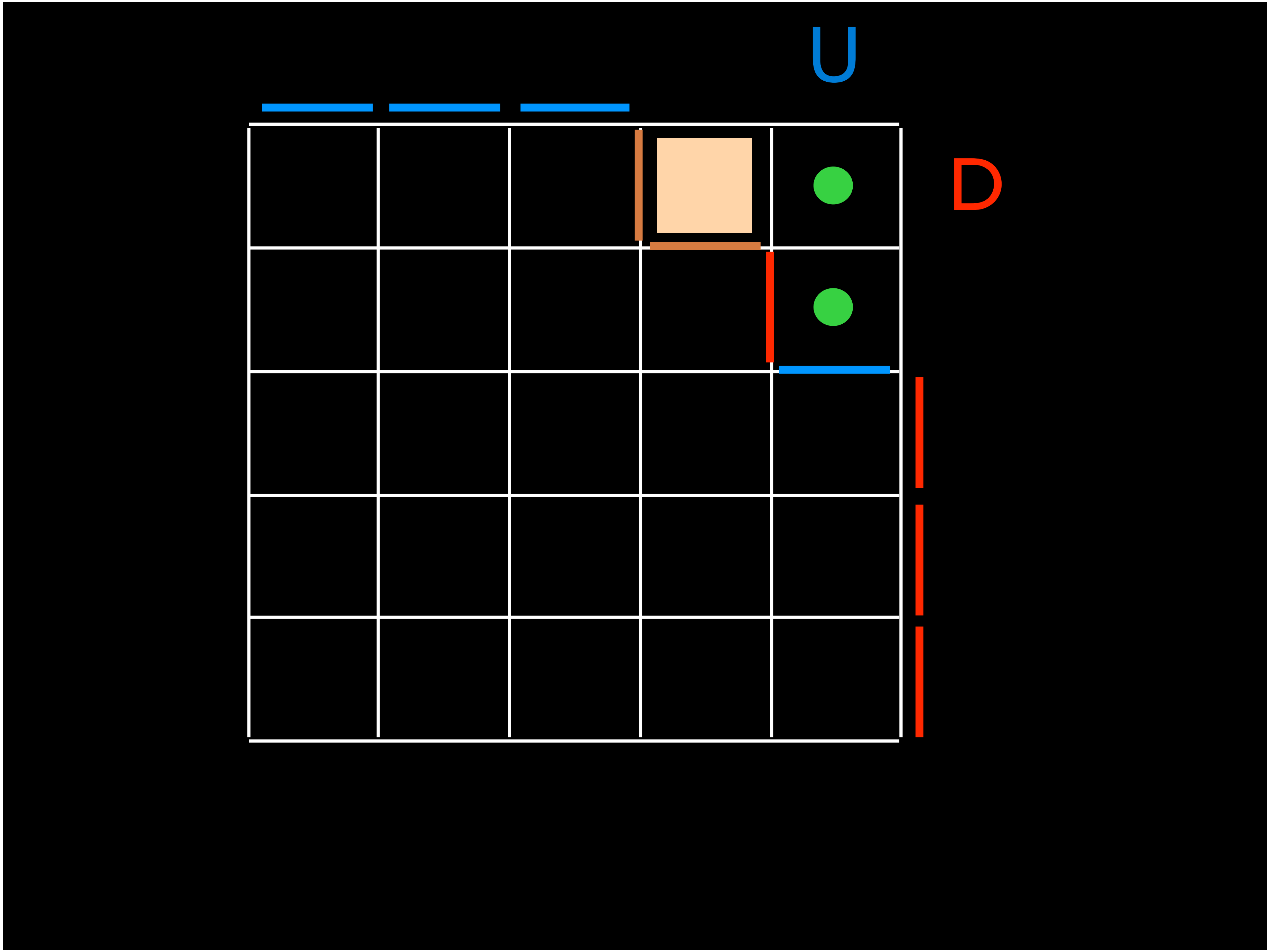
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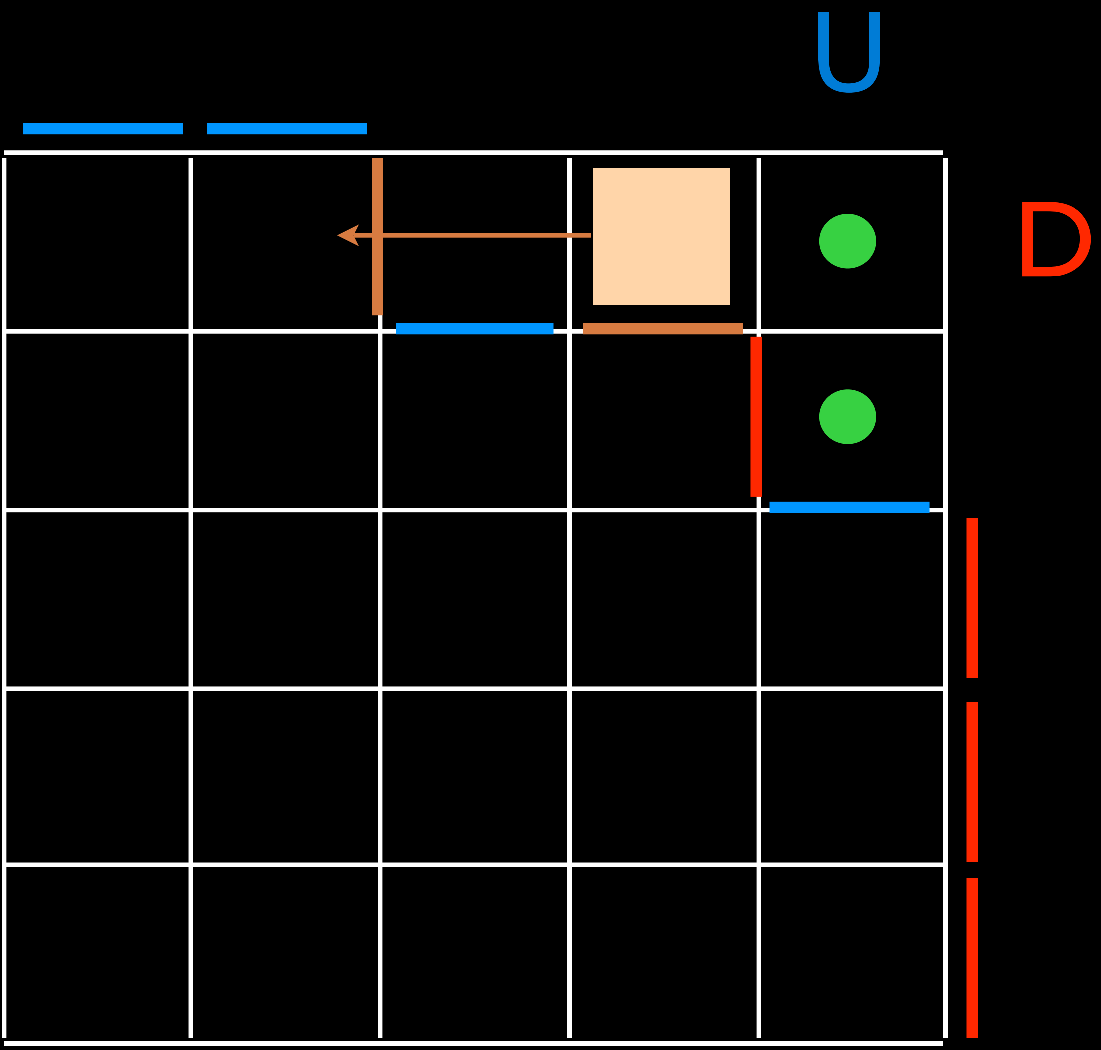
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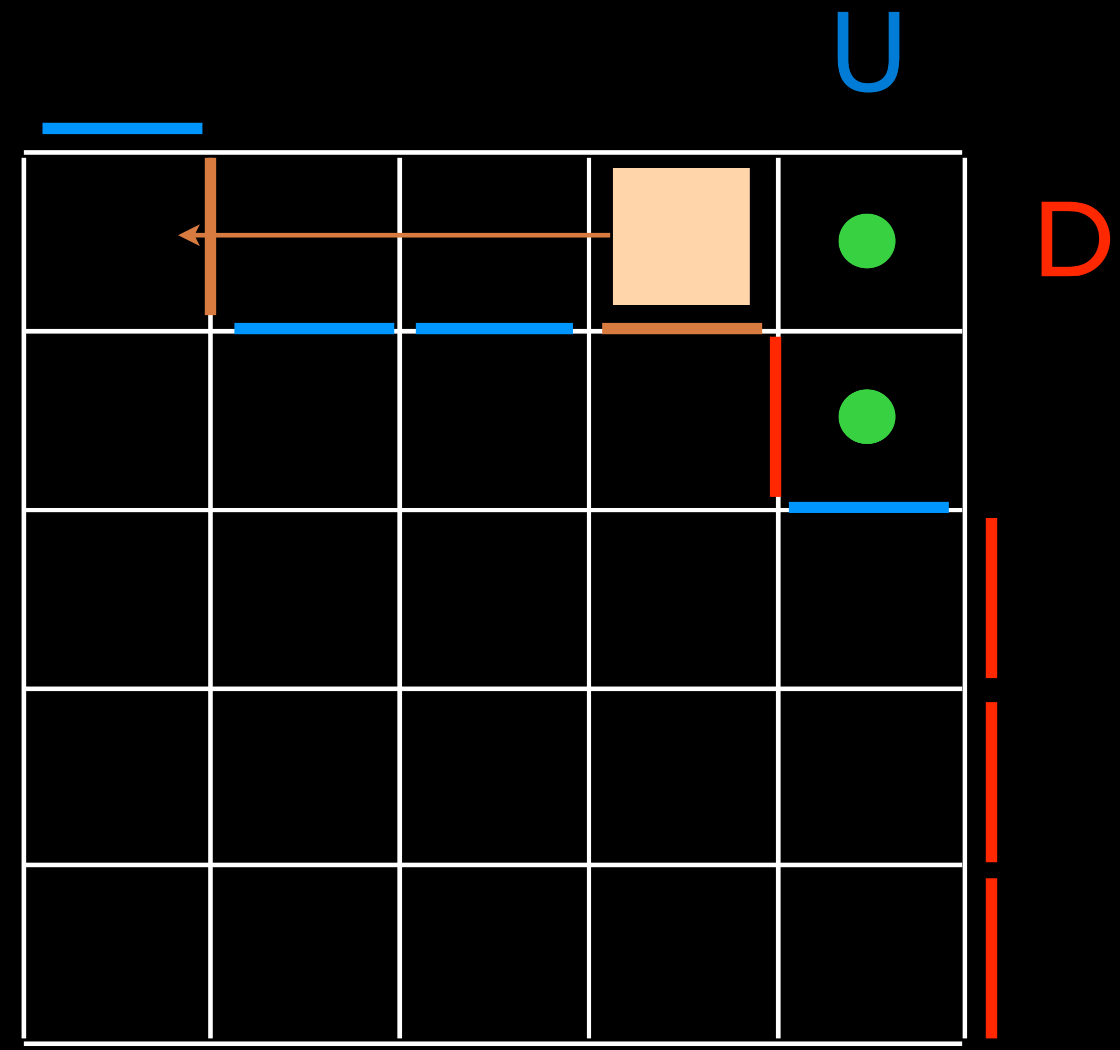
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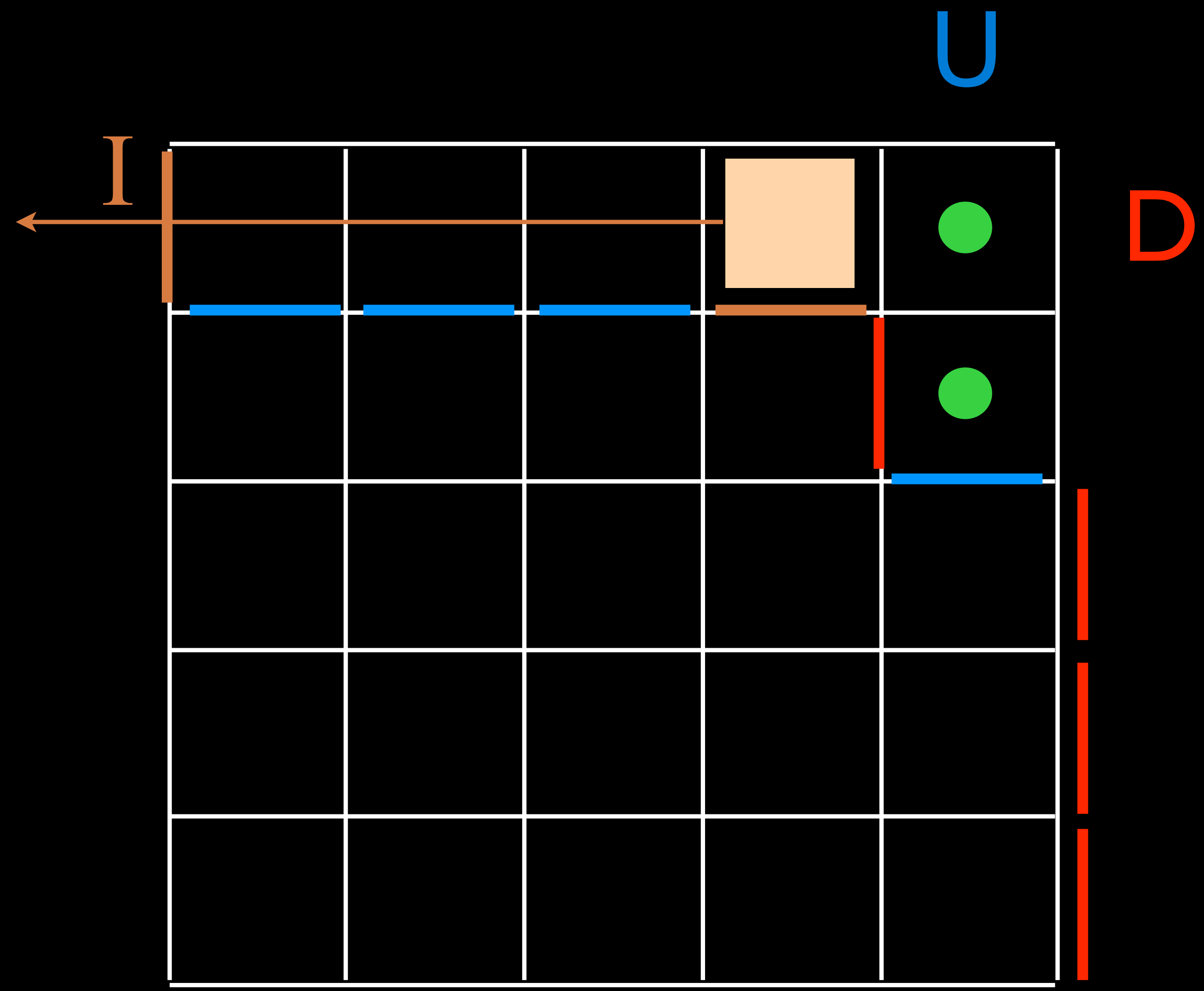

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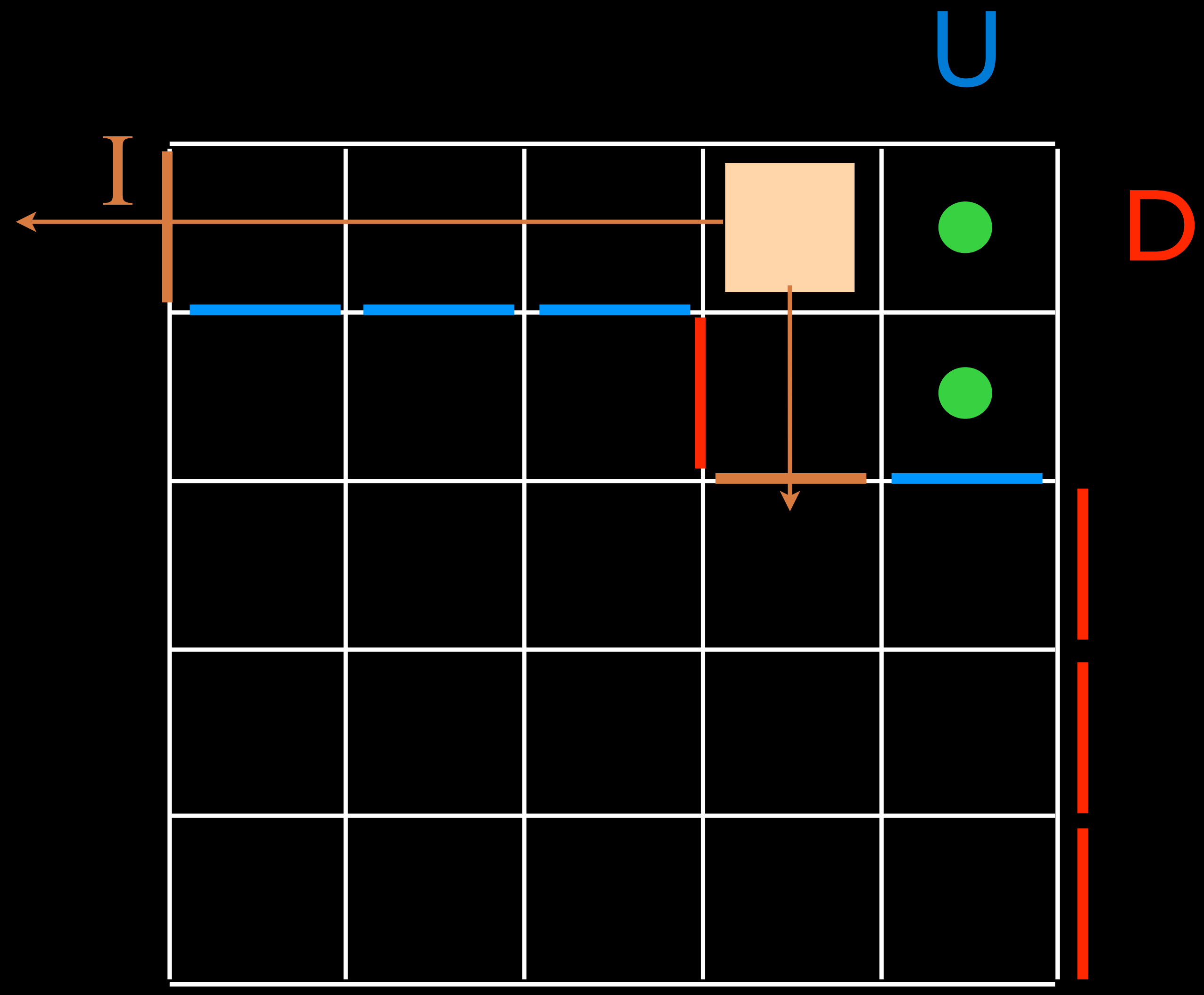
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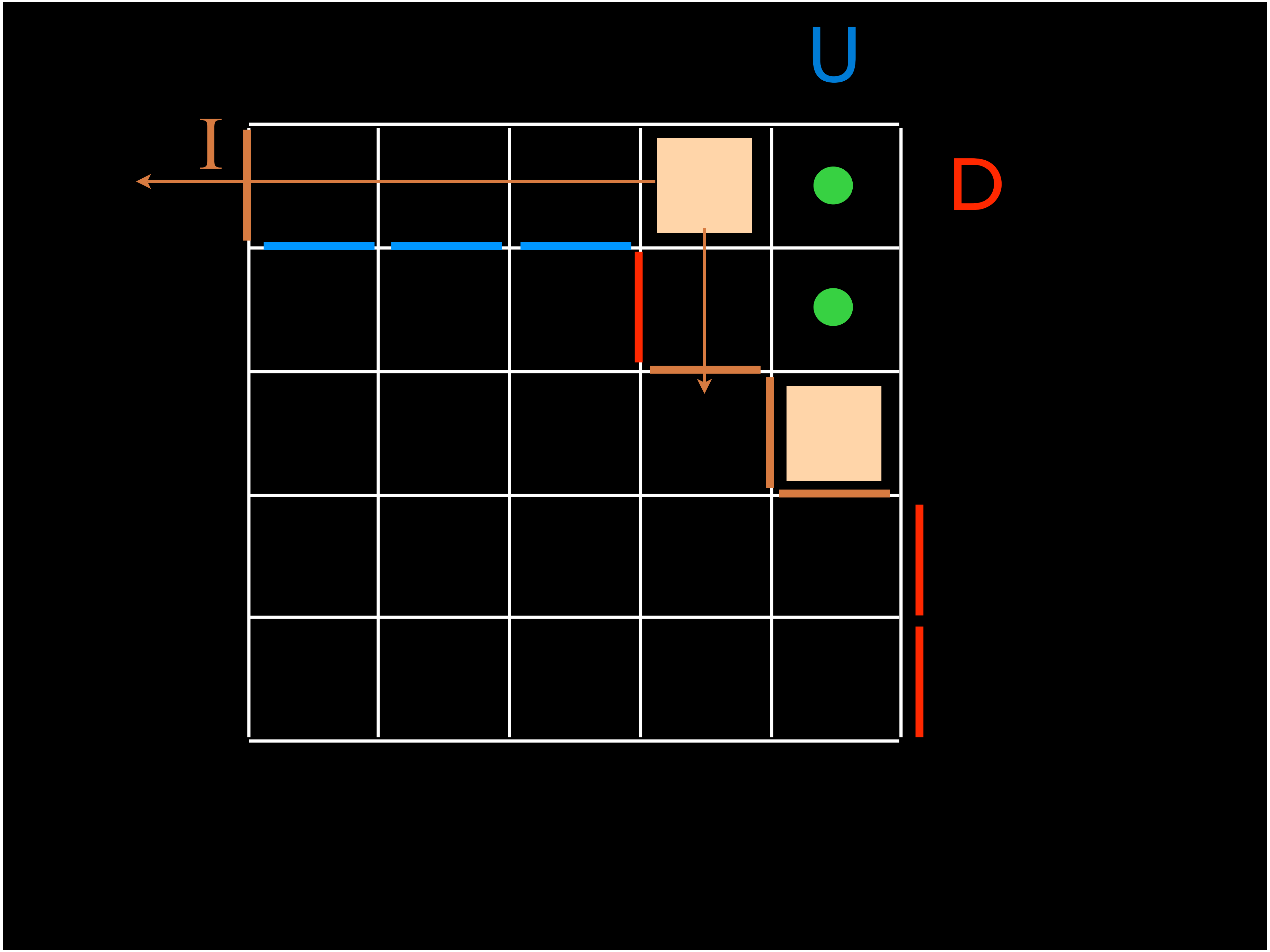
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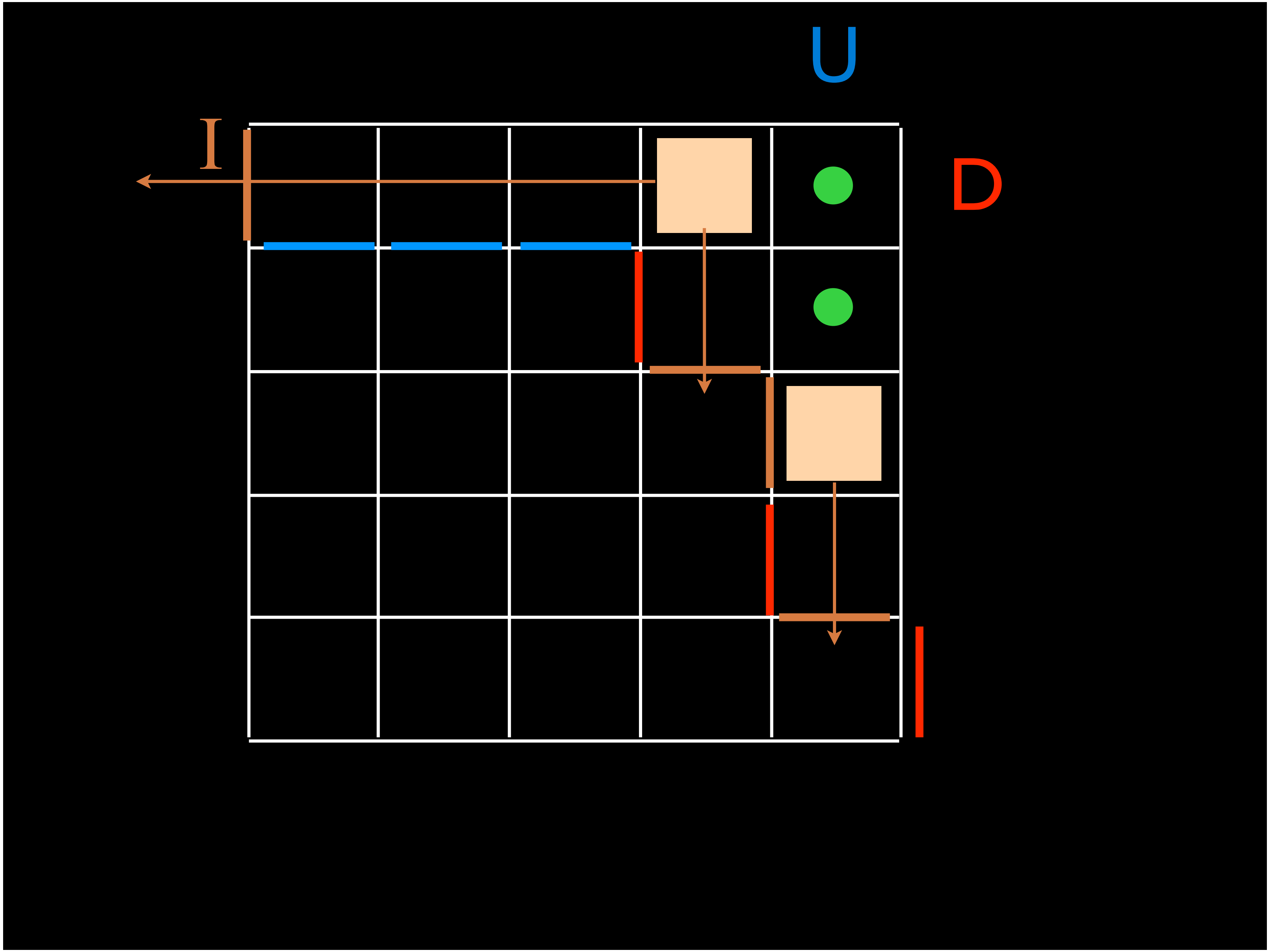

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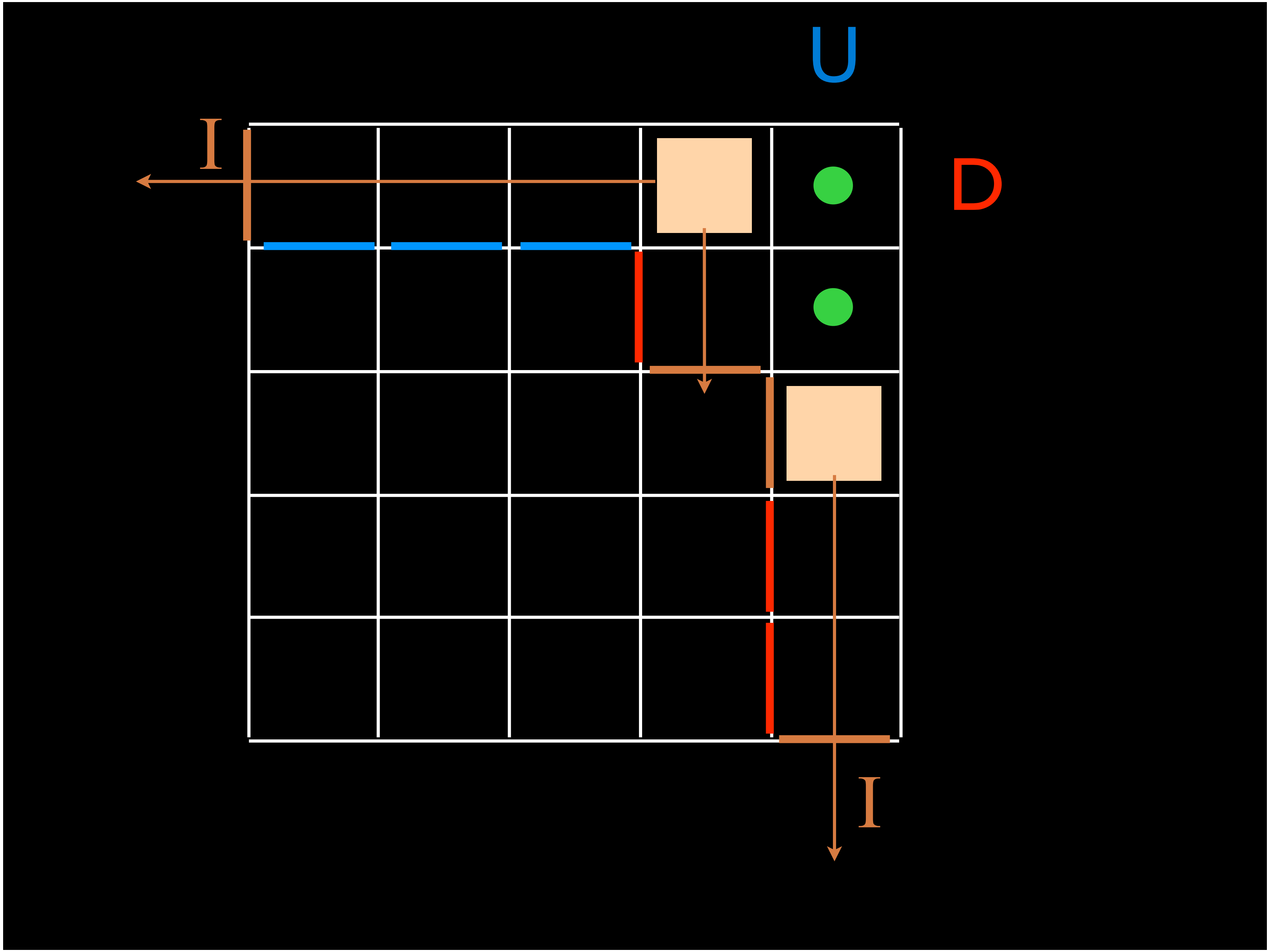
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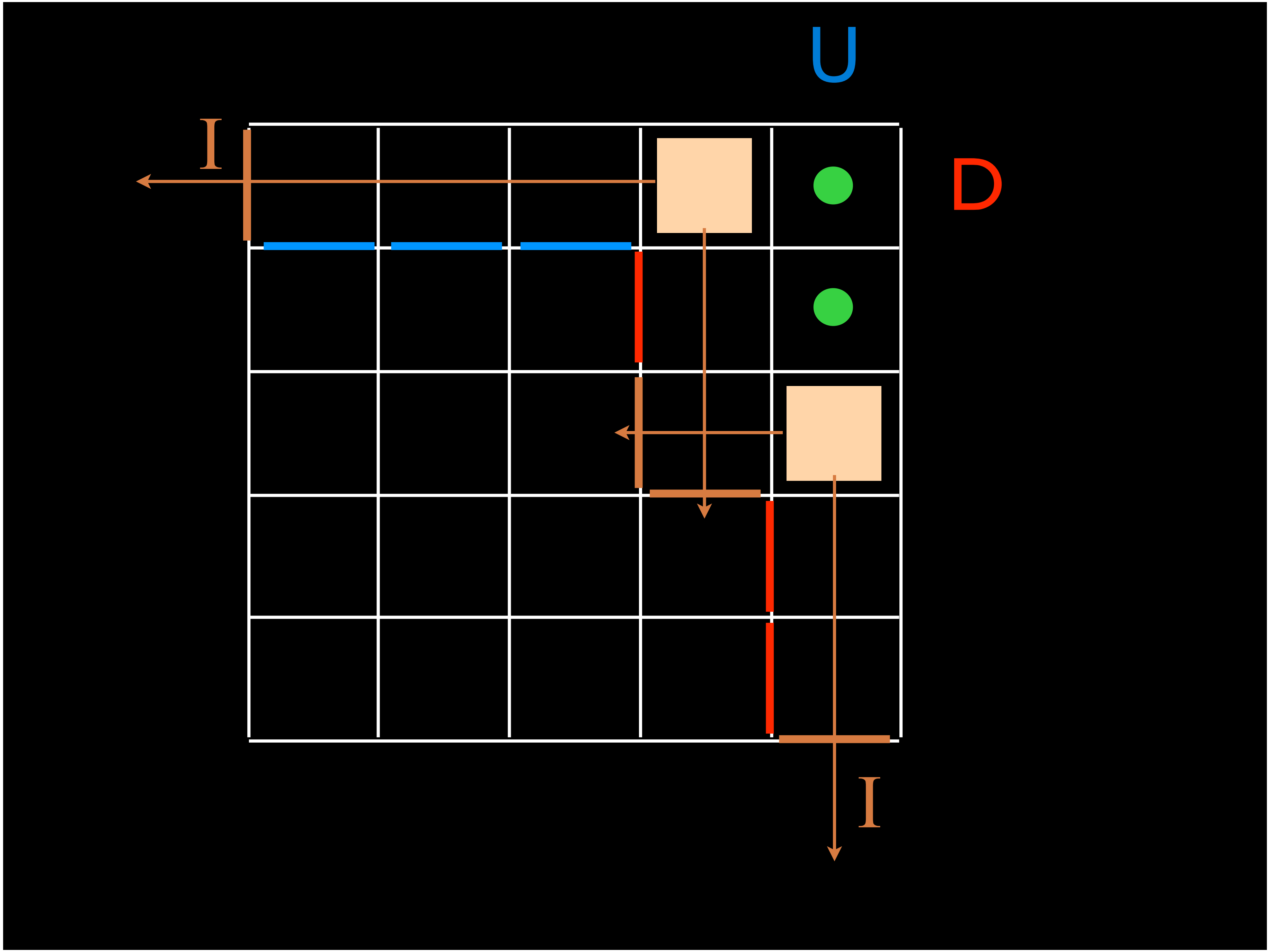
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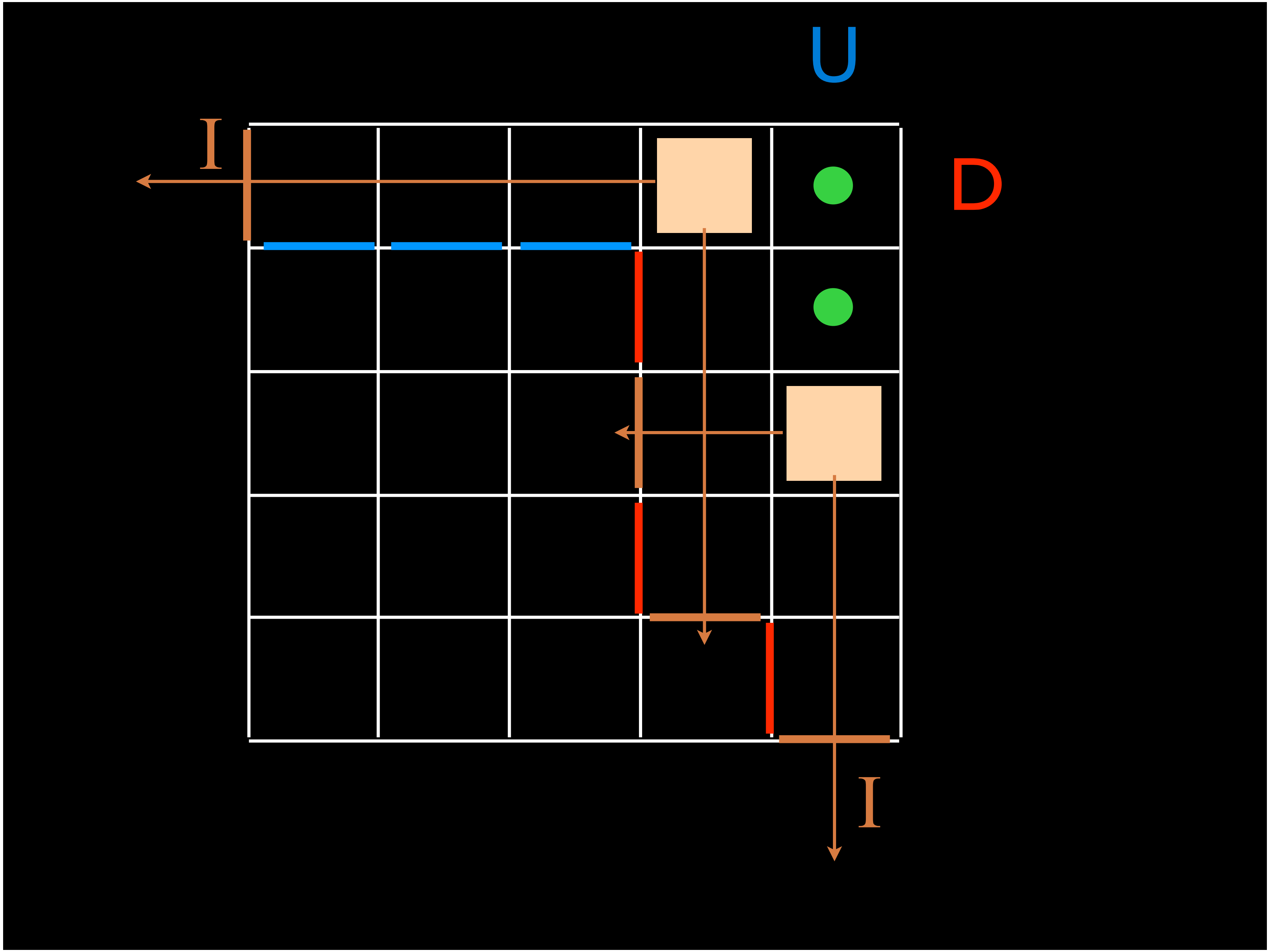

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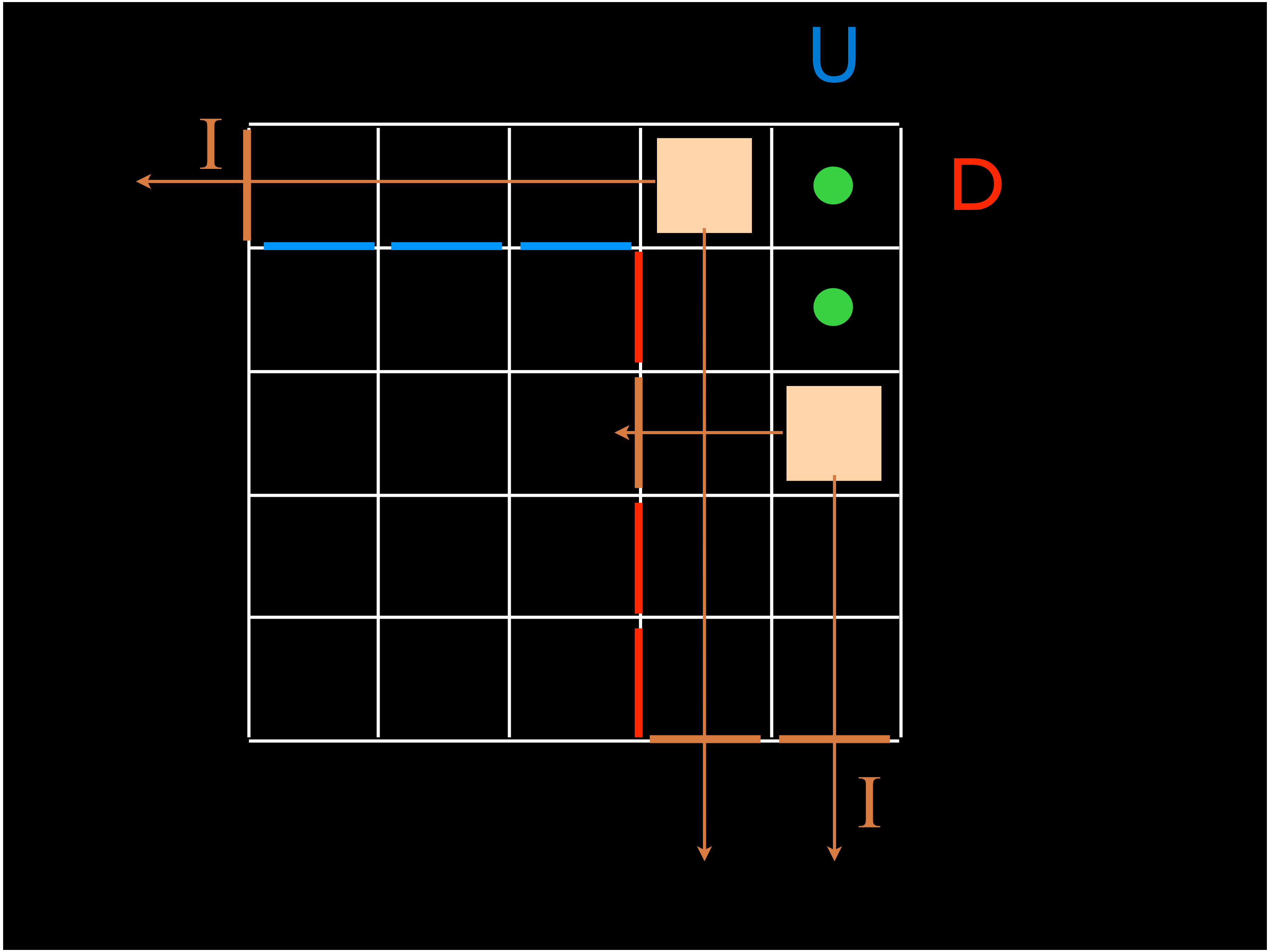
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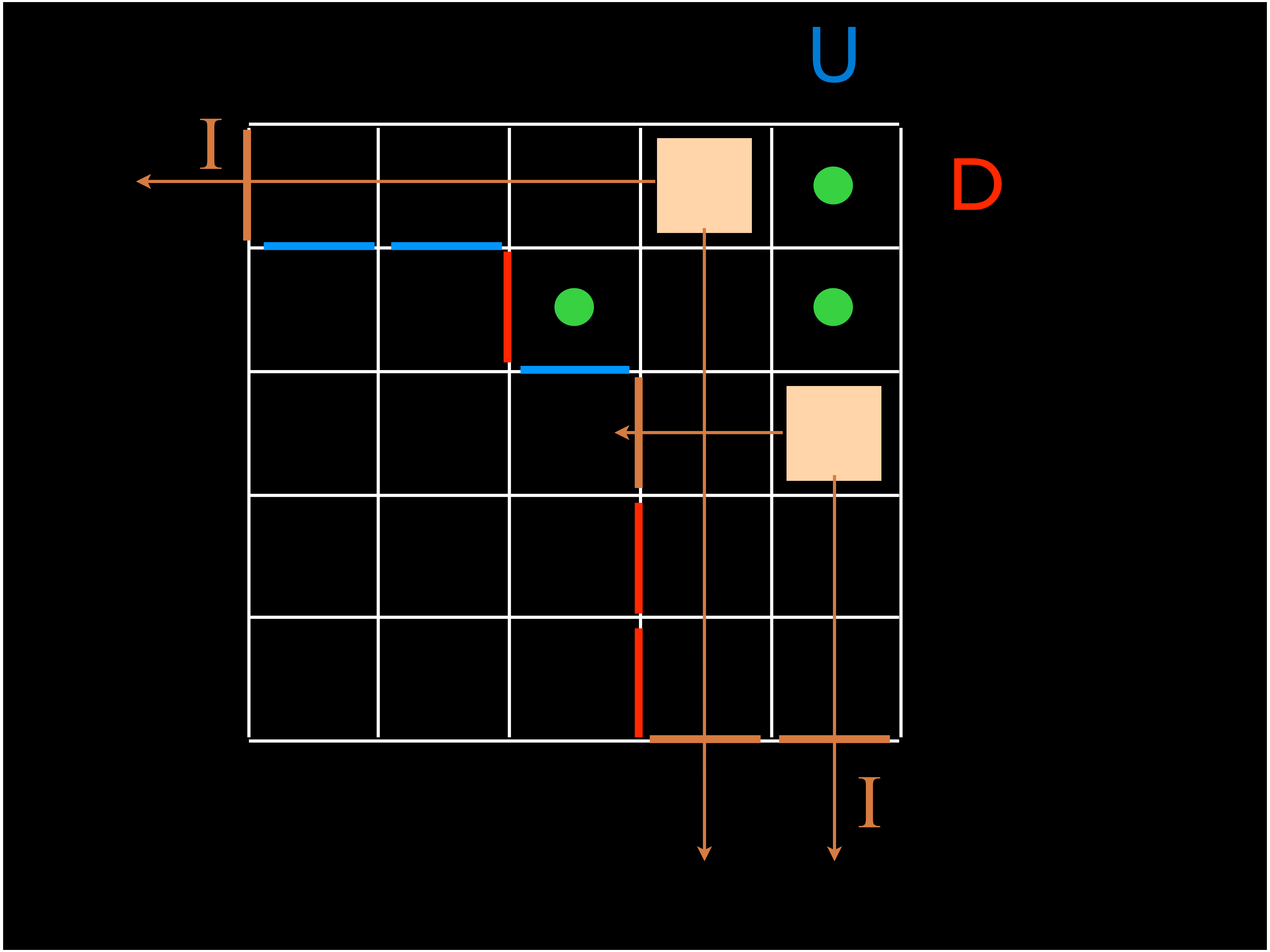

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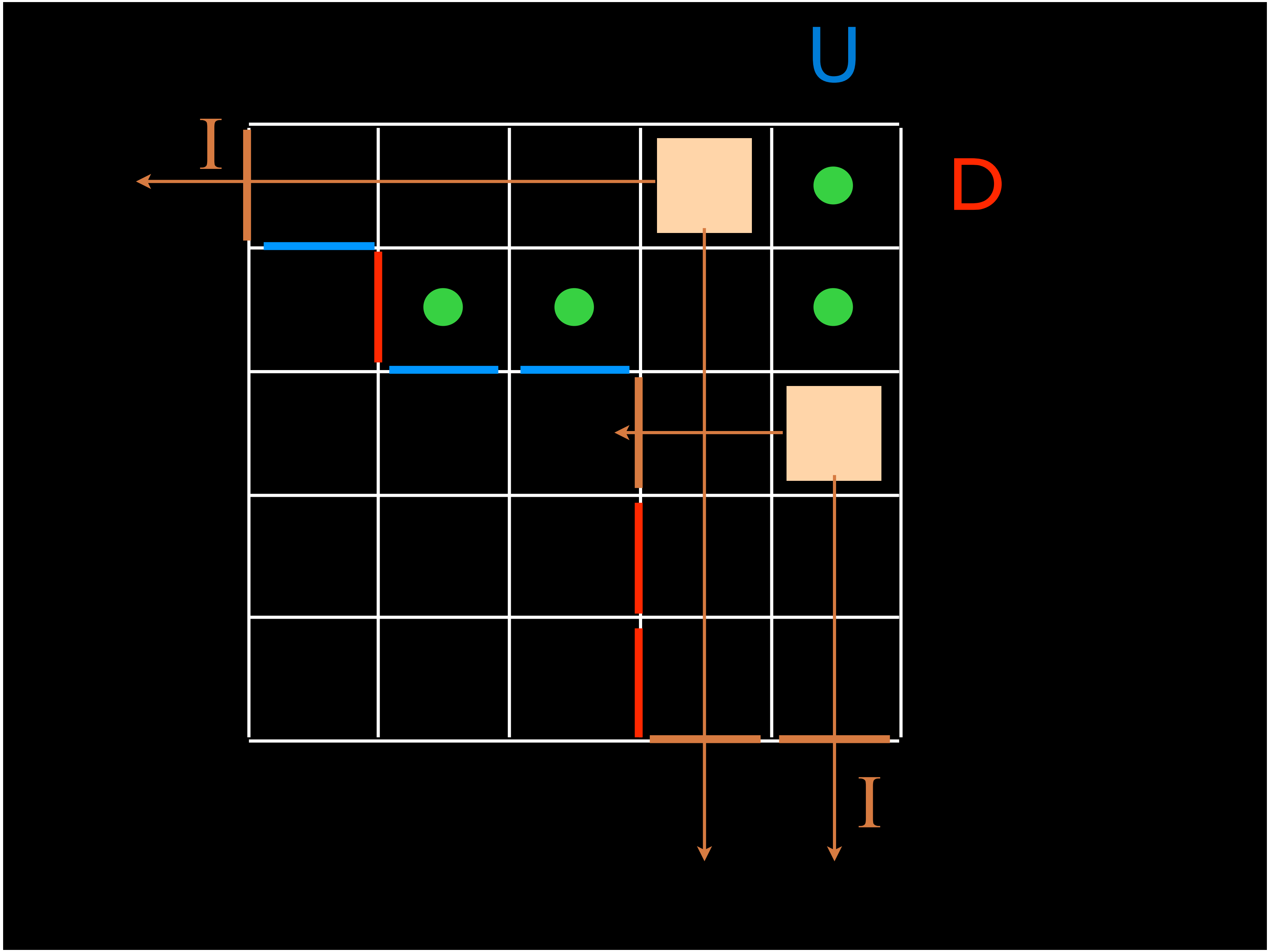
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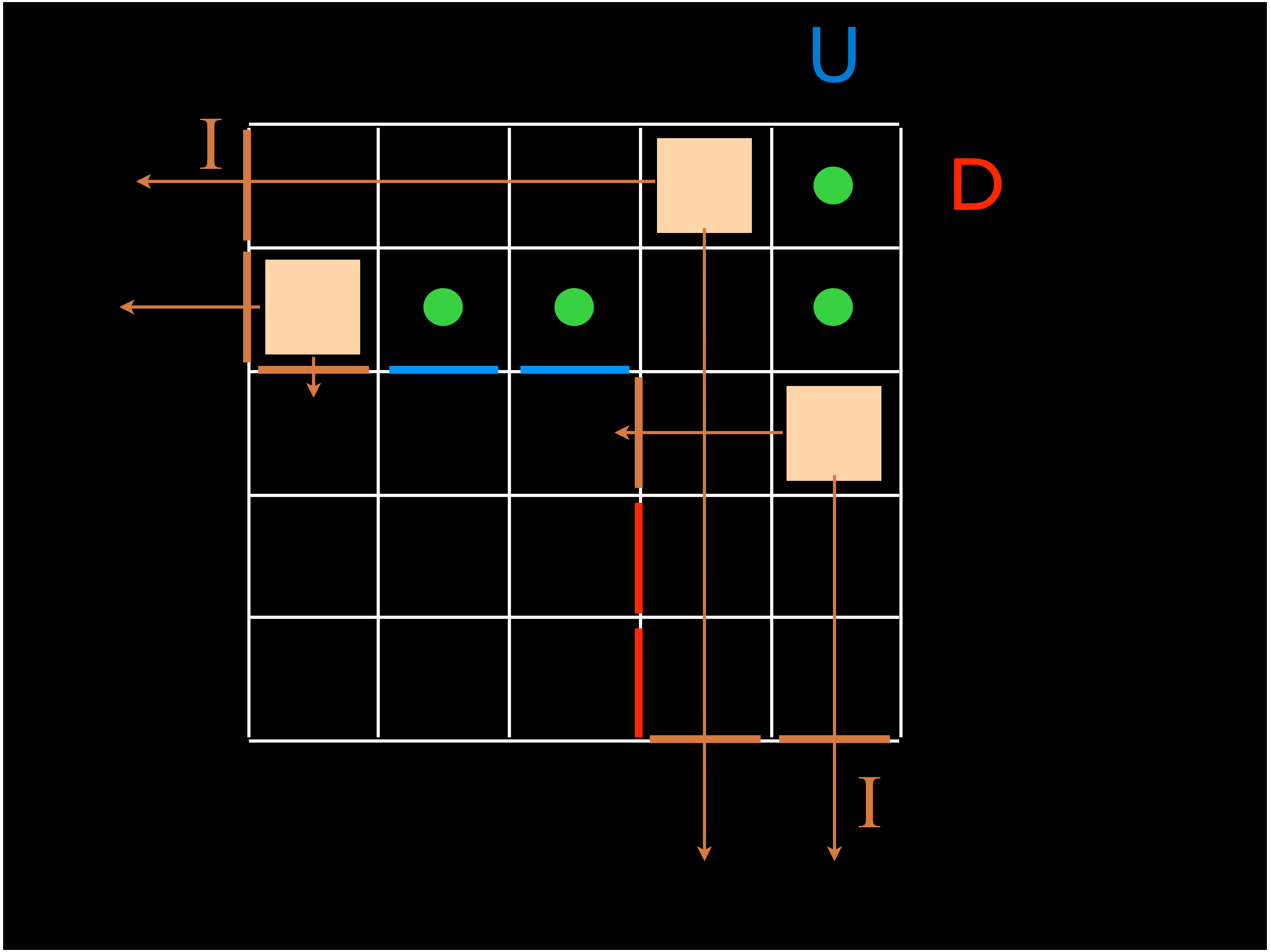
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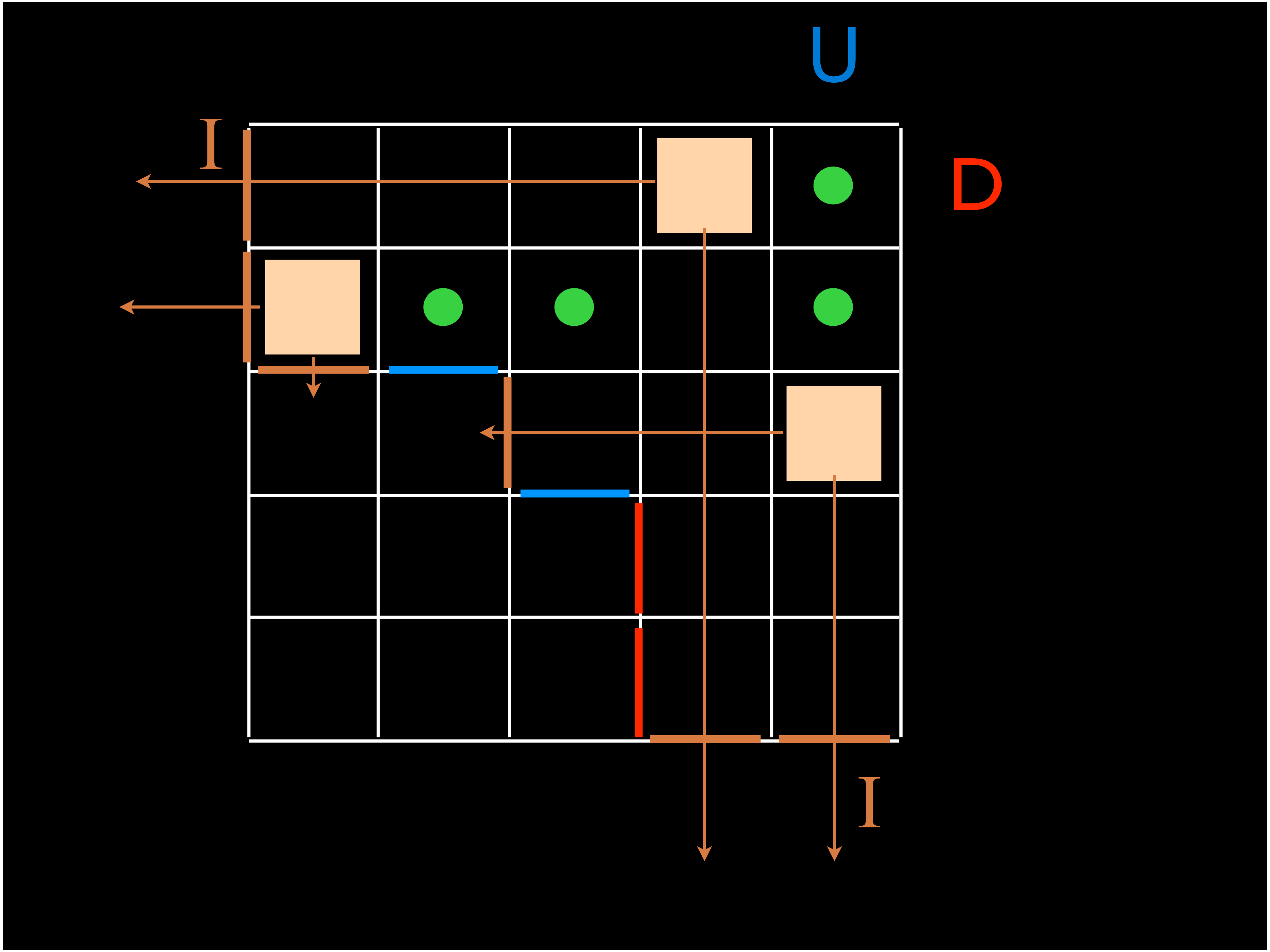

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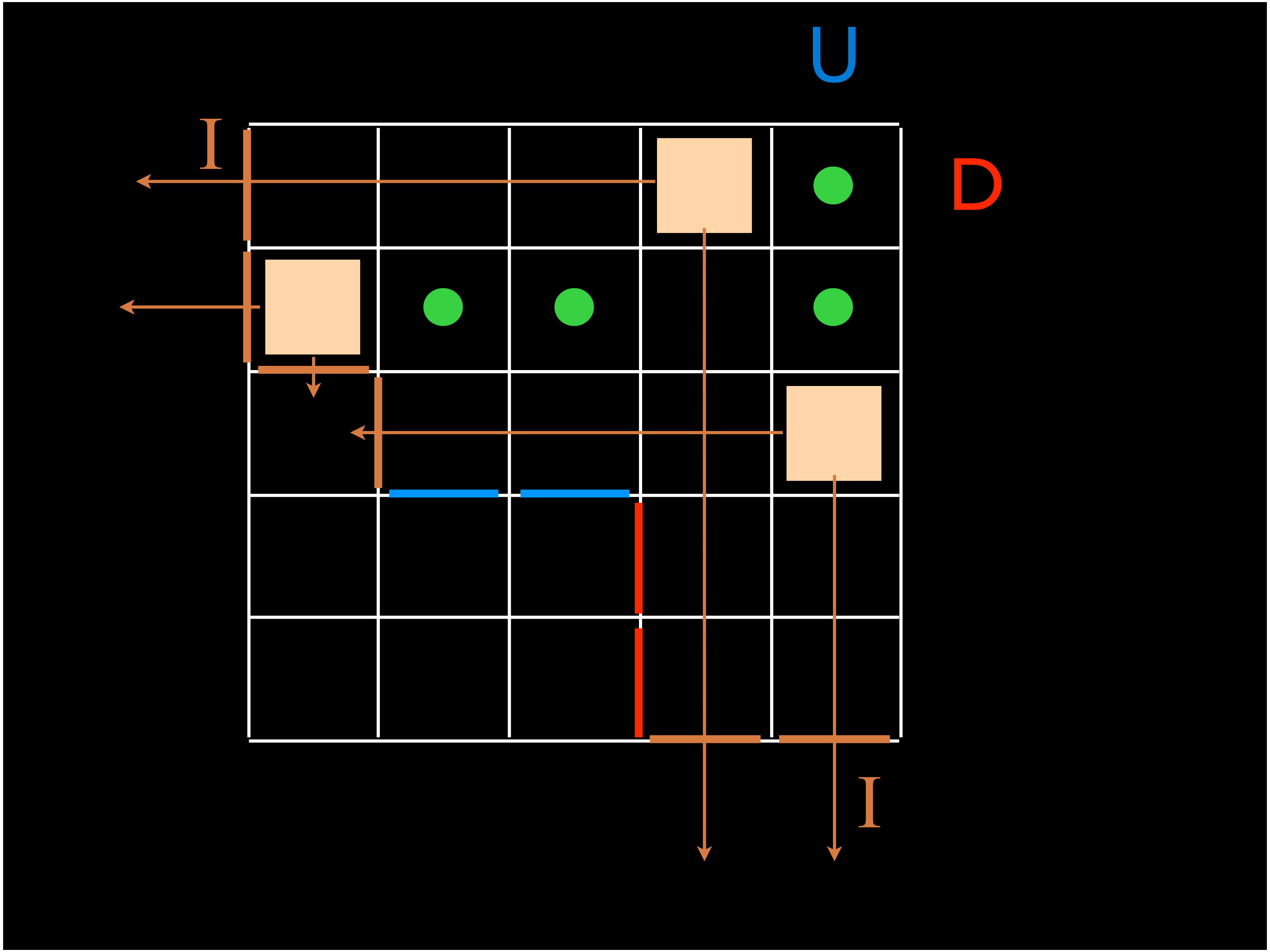
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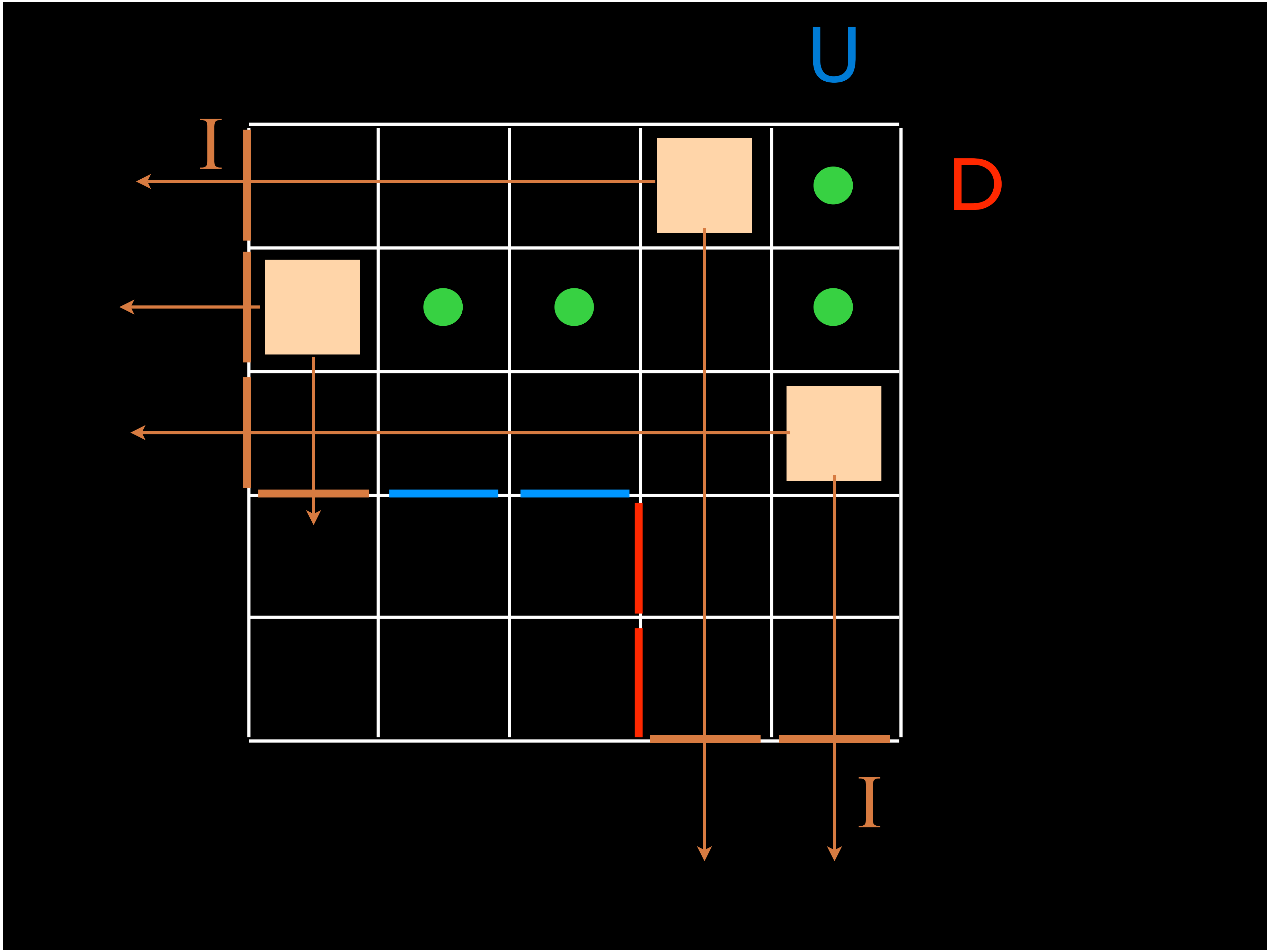

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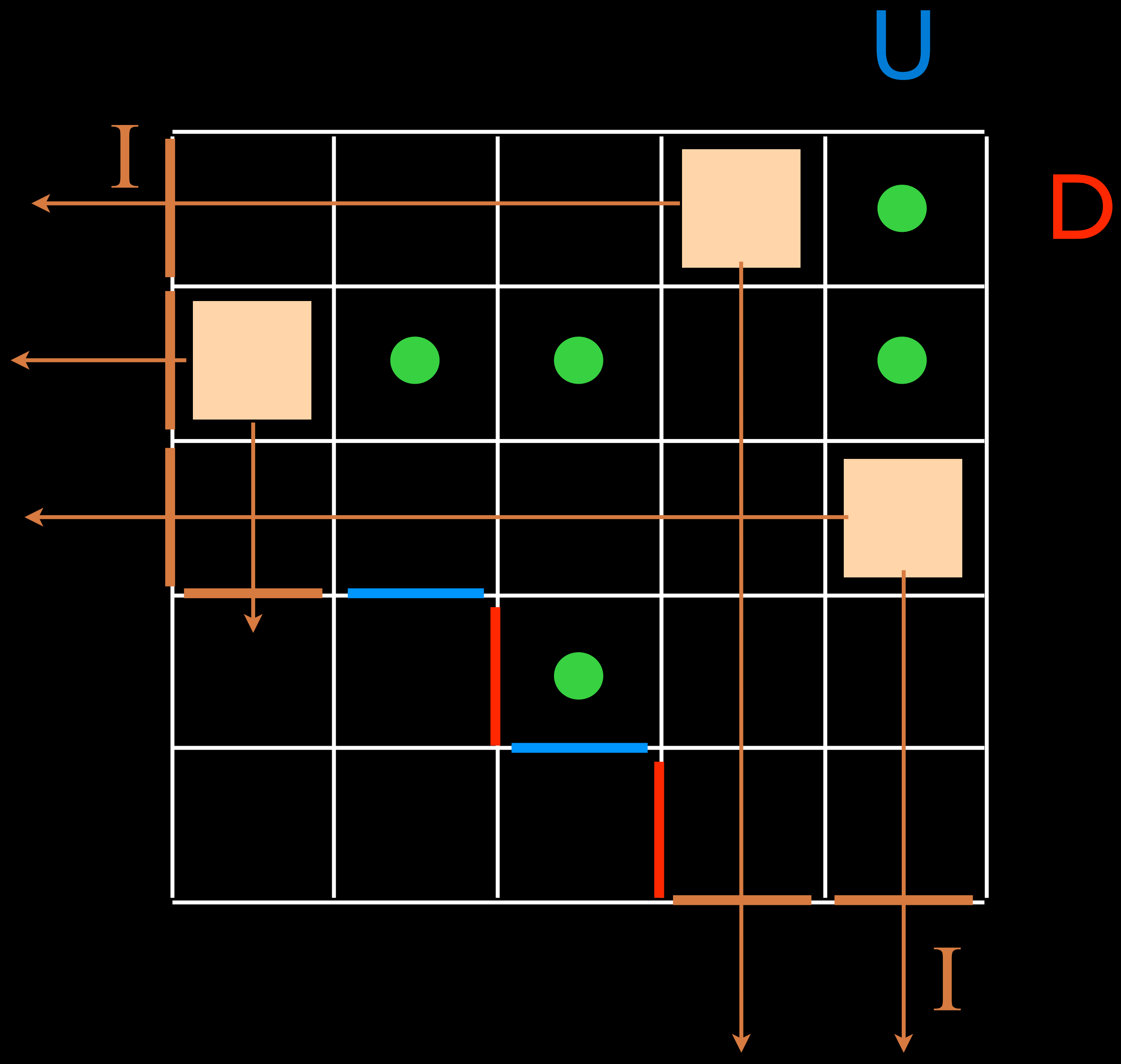
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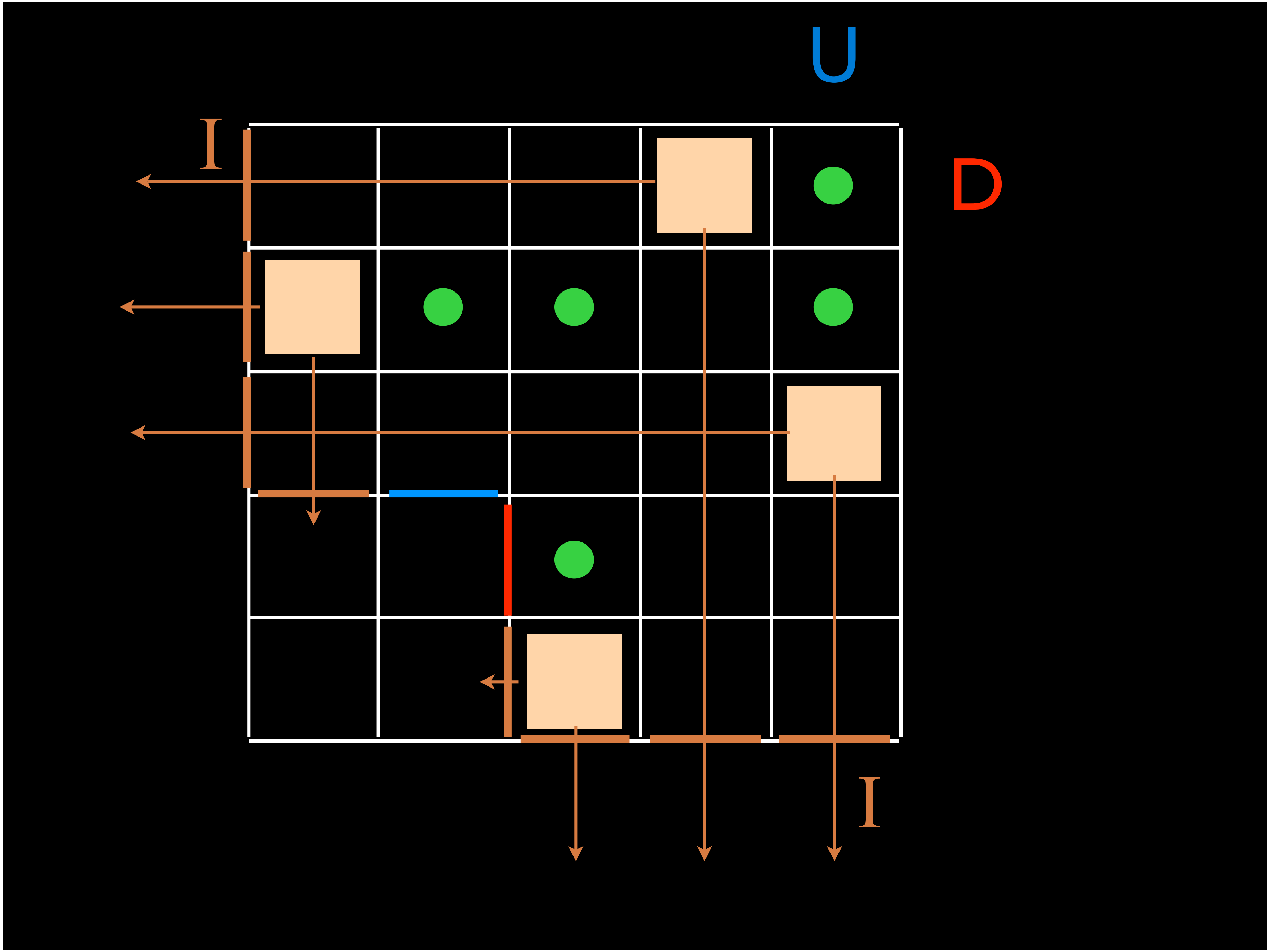

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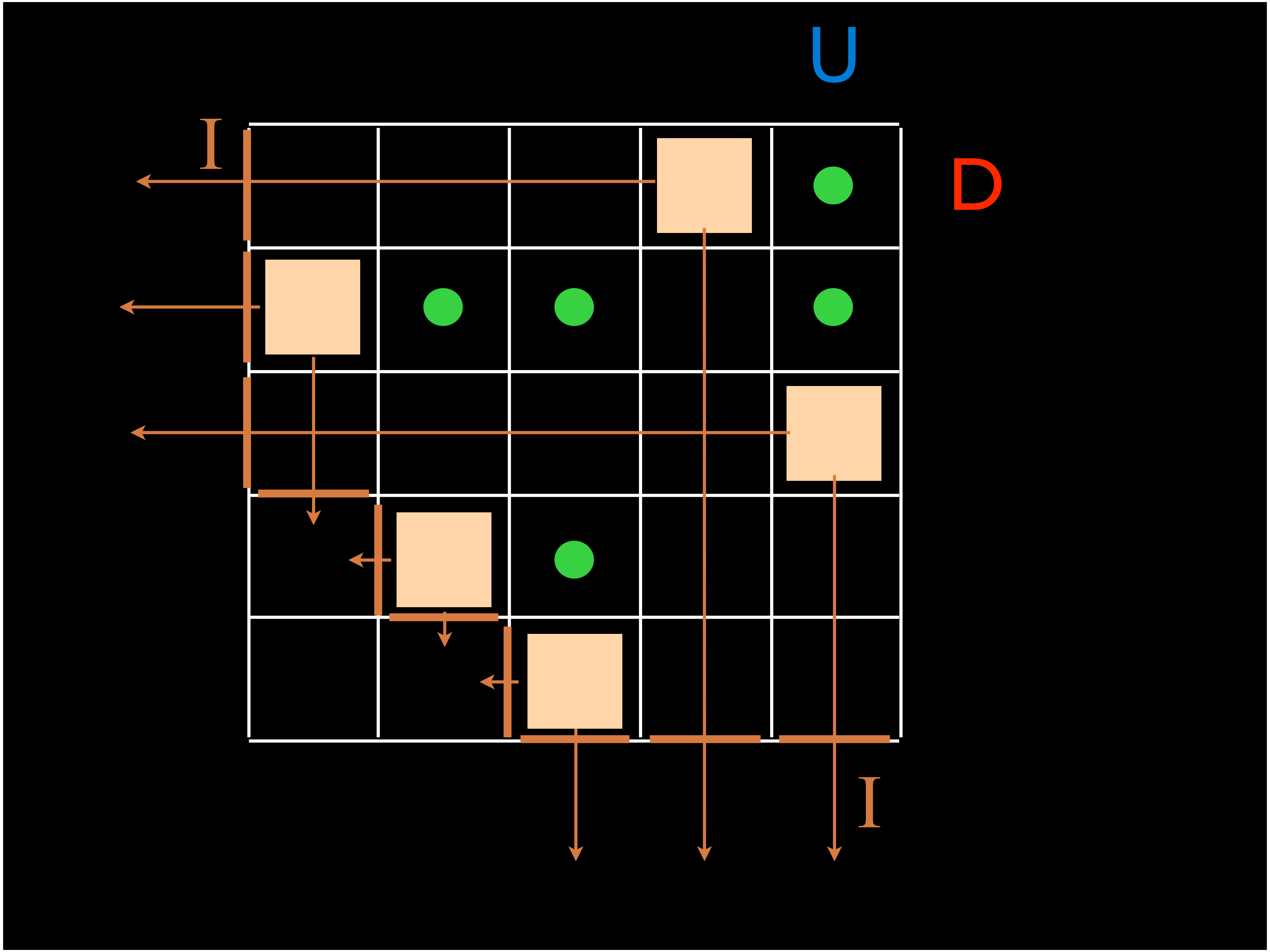
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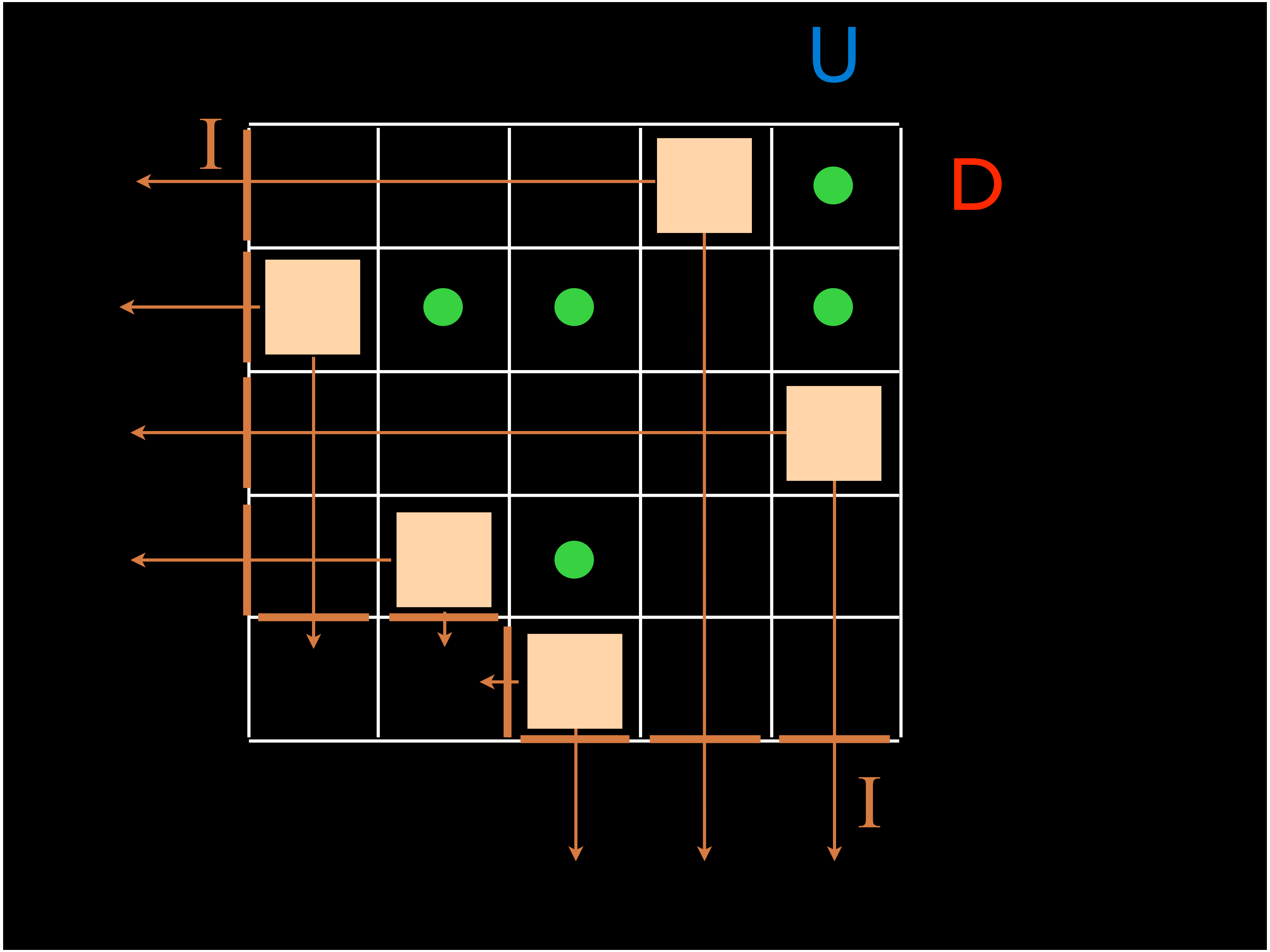
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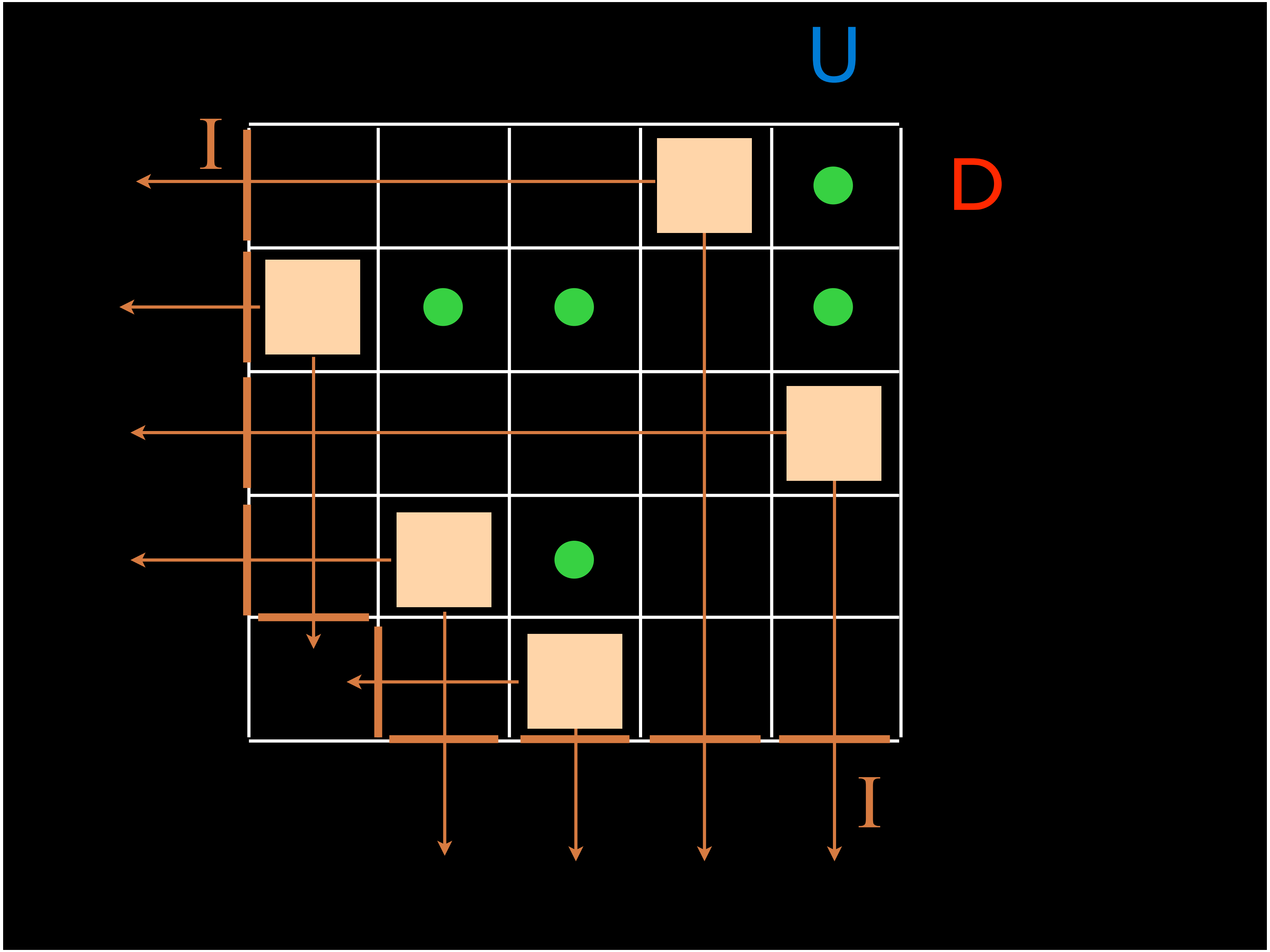
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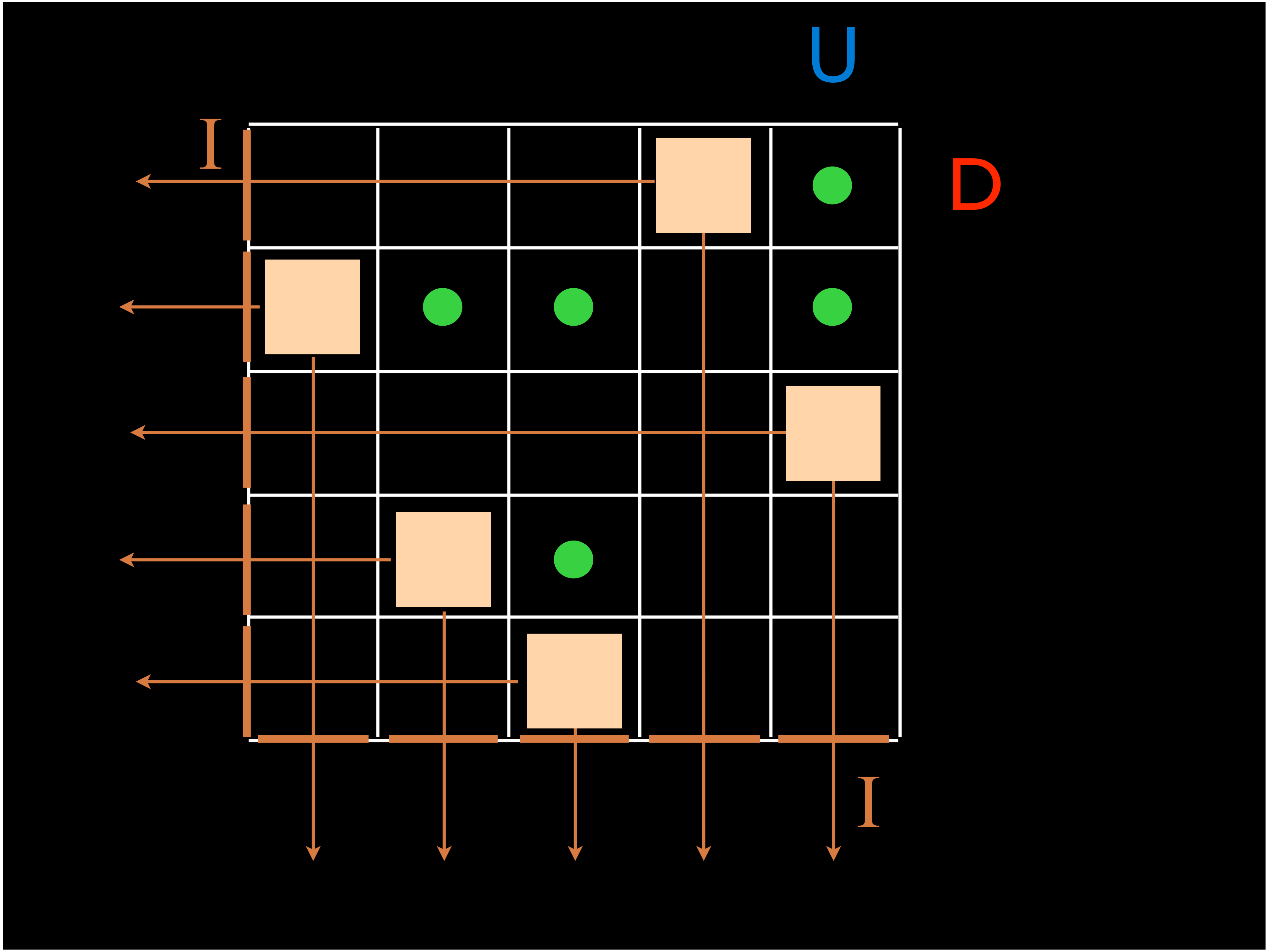
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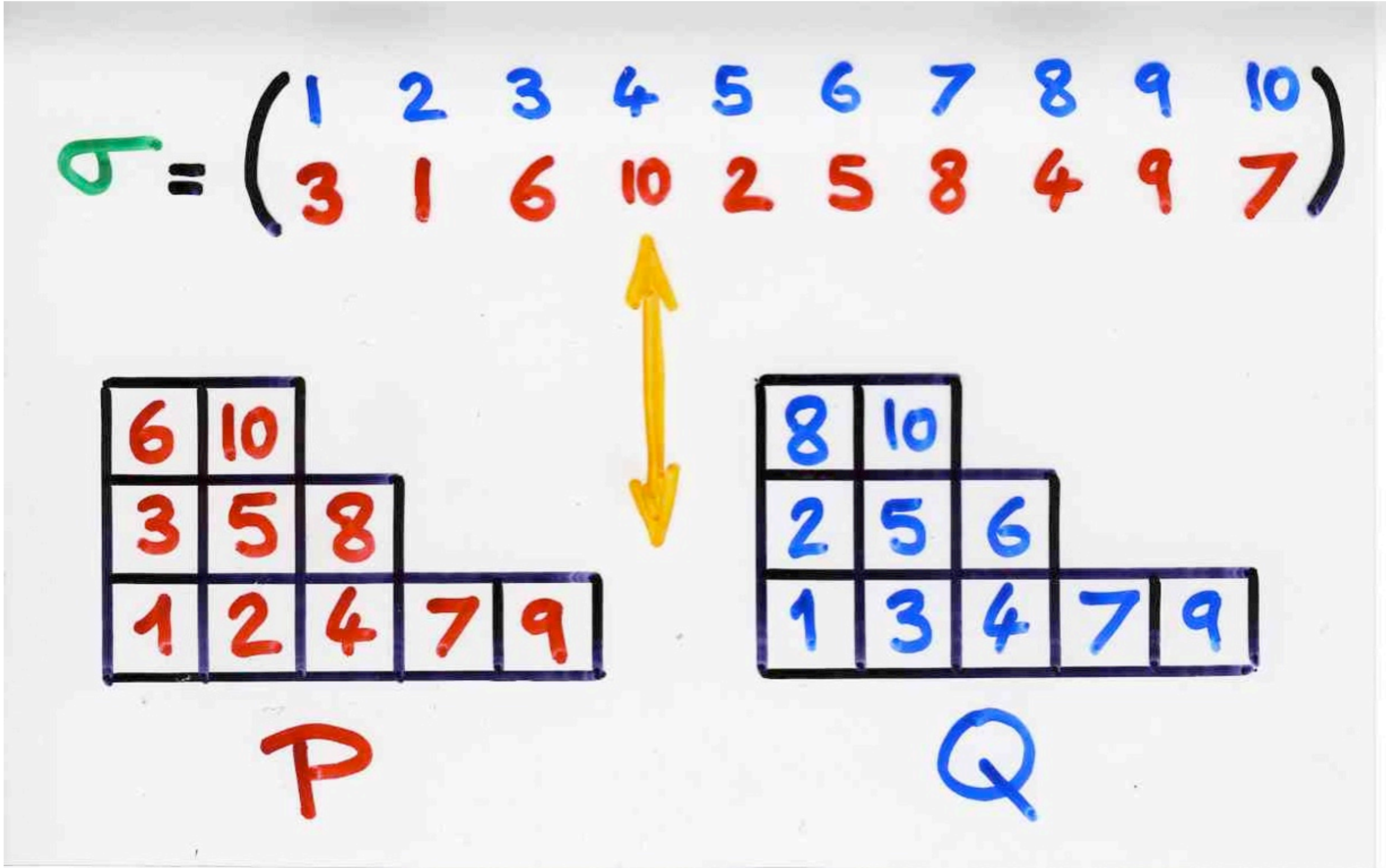
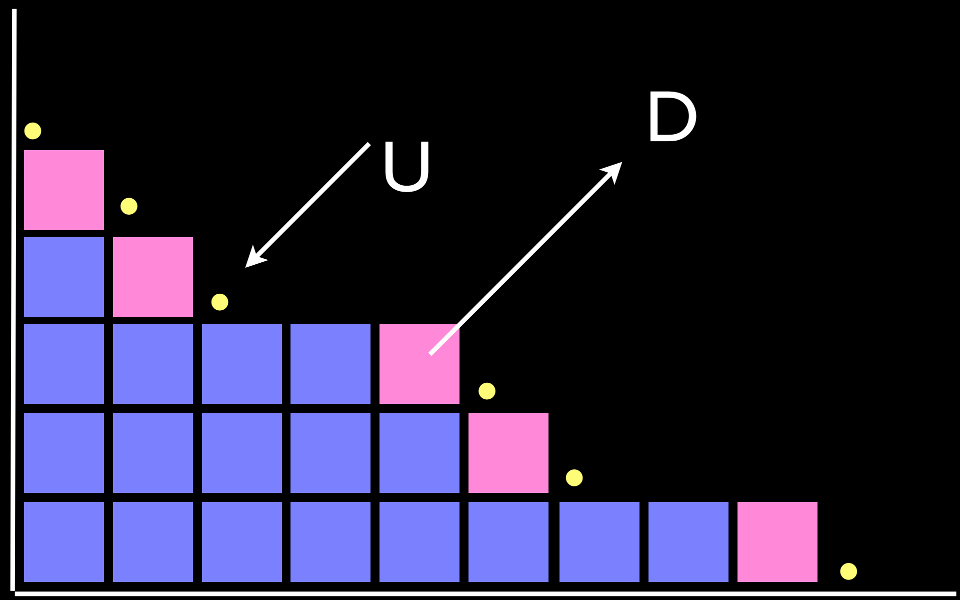
§3
representation
of
operators U, D
and
"local rules"
for RSK

Robinson-Schensted-Knuth correspondence



The Robinson-Schensted correspondence between permutations and pair of (standard) Young tableaux with the same shape

Operators U and D

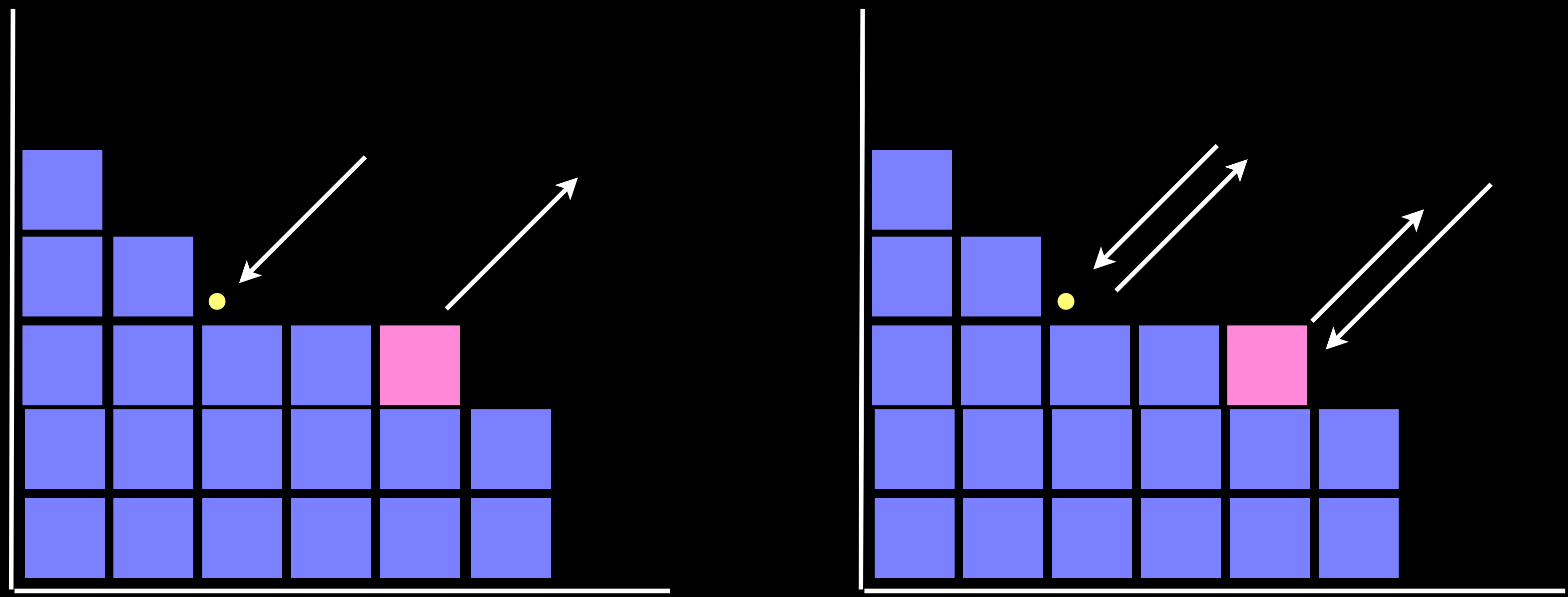


adding
or deleting
a cell in
a Ferrers
diagram

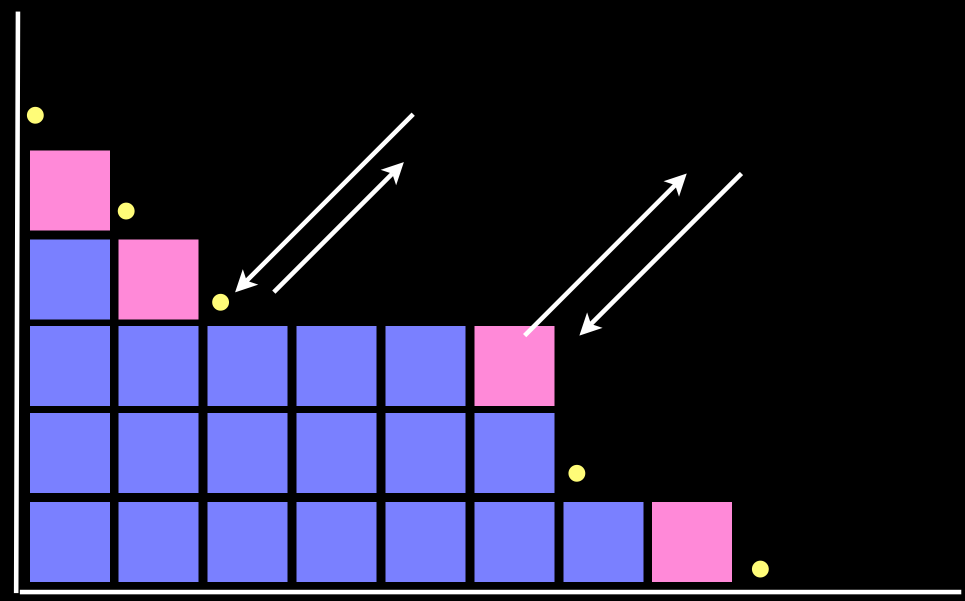
Young lattice

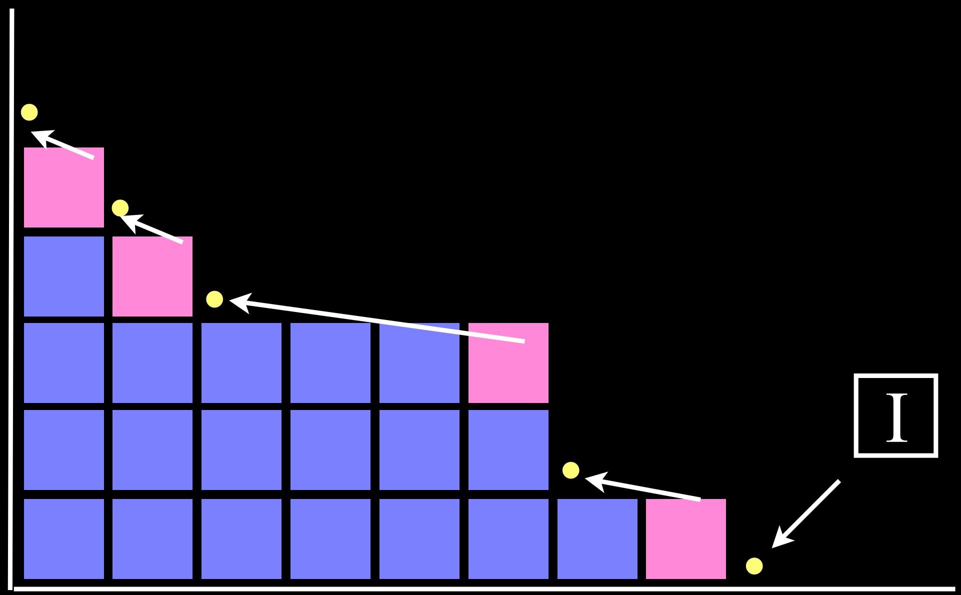
Heisenberg commutation relation

$$UD = DU + I$$



$$UD = DU + I$$



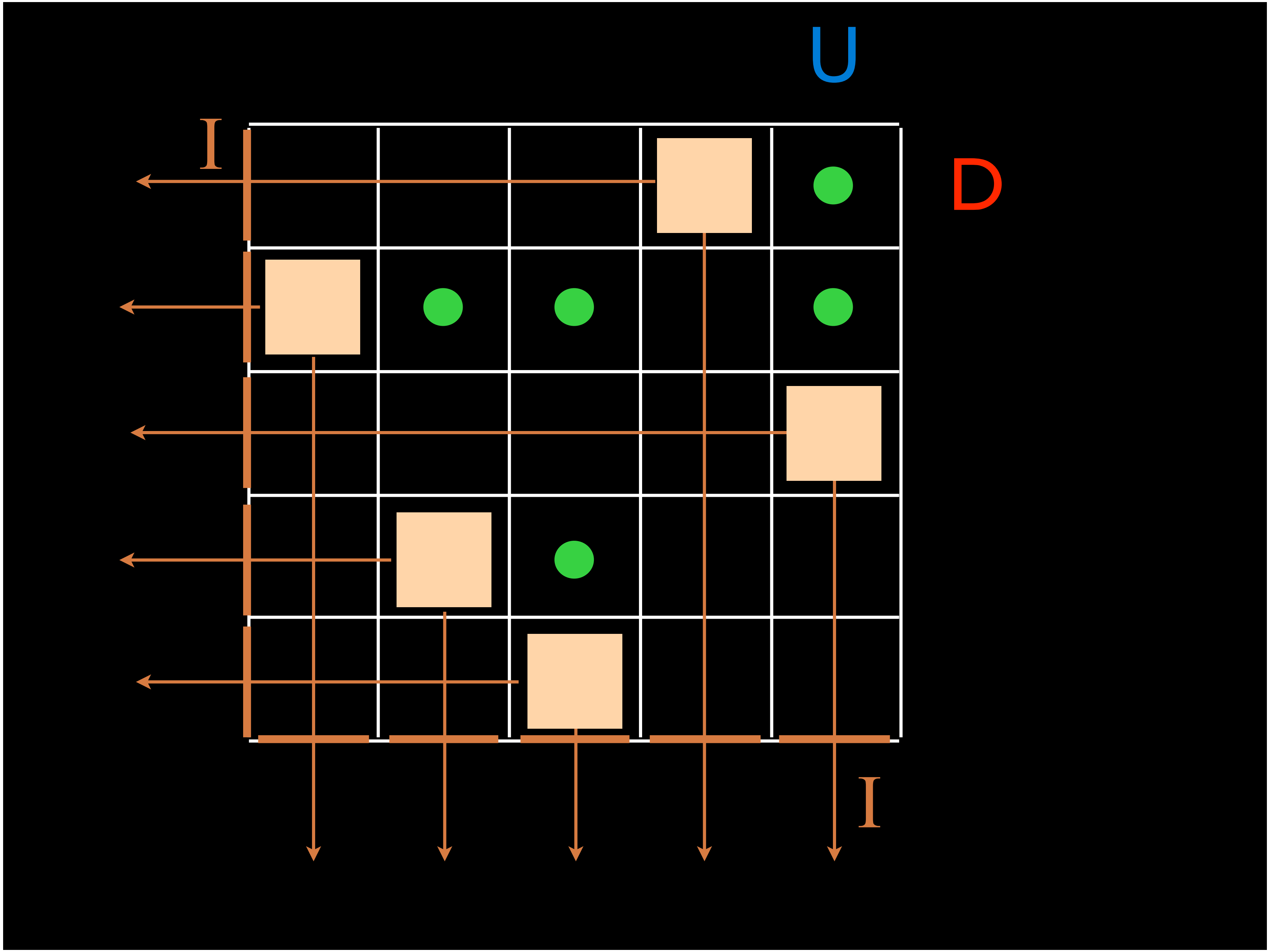


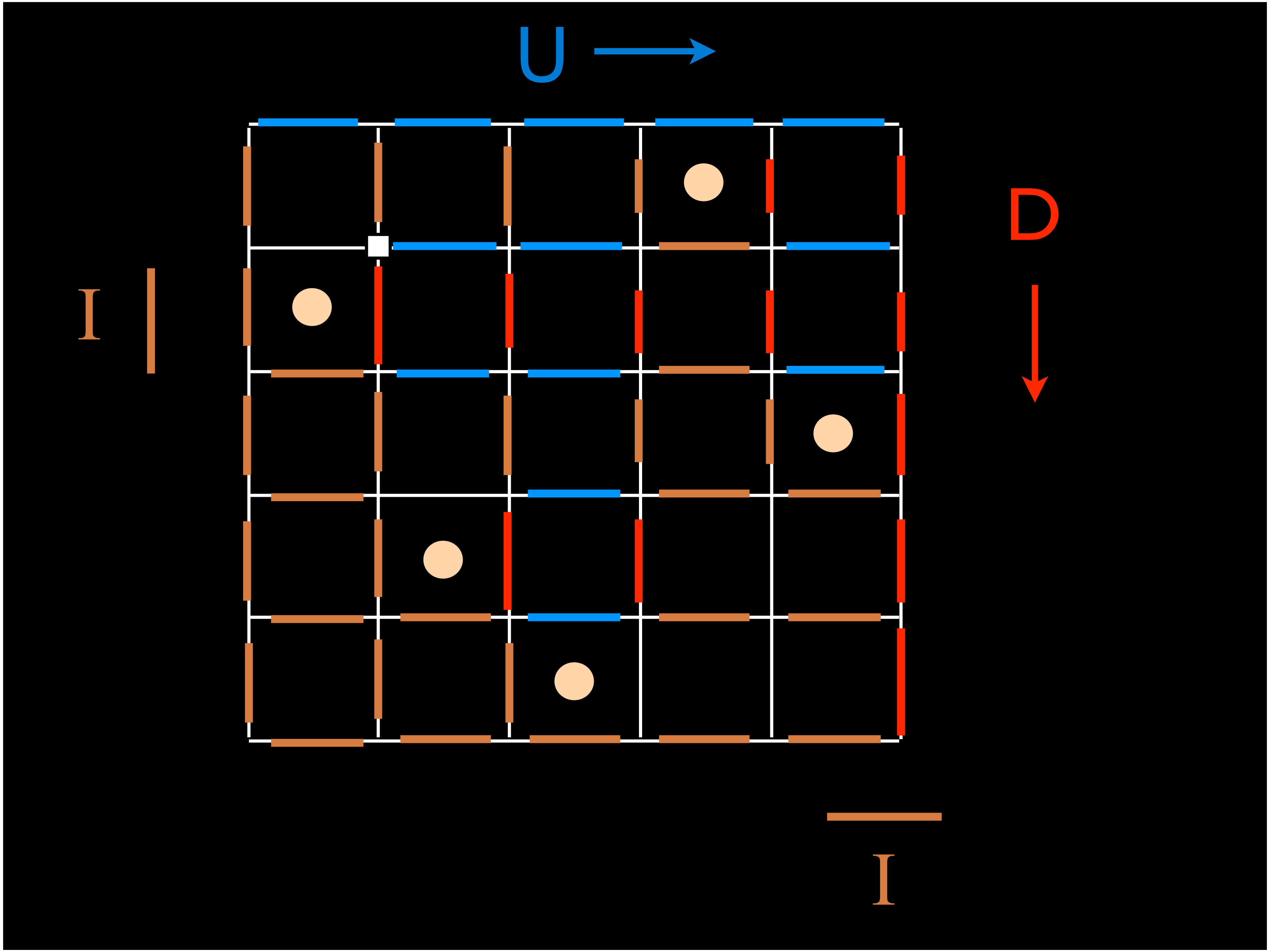
combinatorial “representation” of the commutation relation $UD = DU + I$

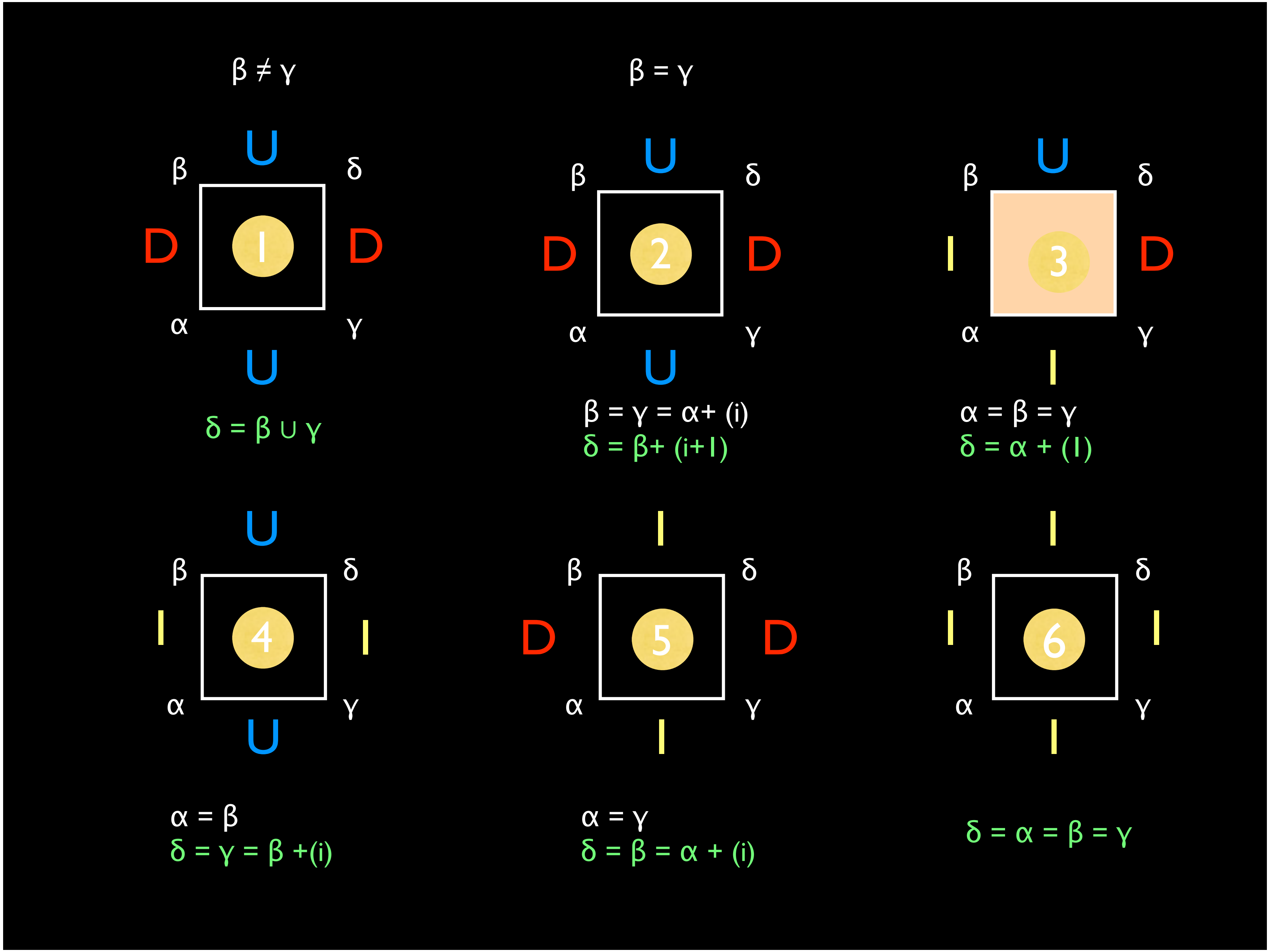
RSK with
Fomin's
"local rules"

$UD = q\,DU + 1$

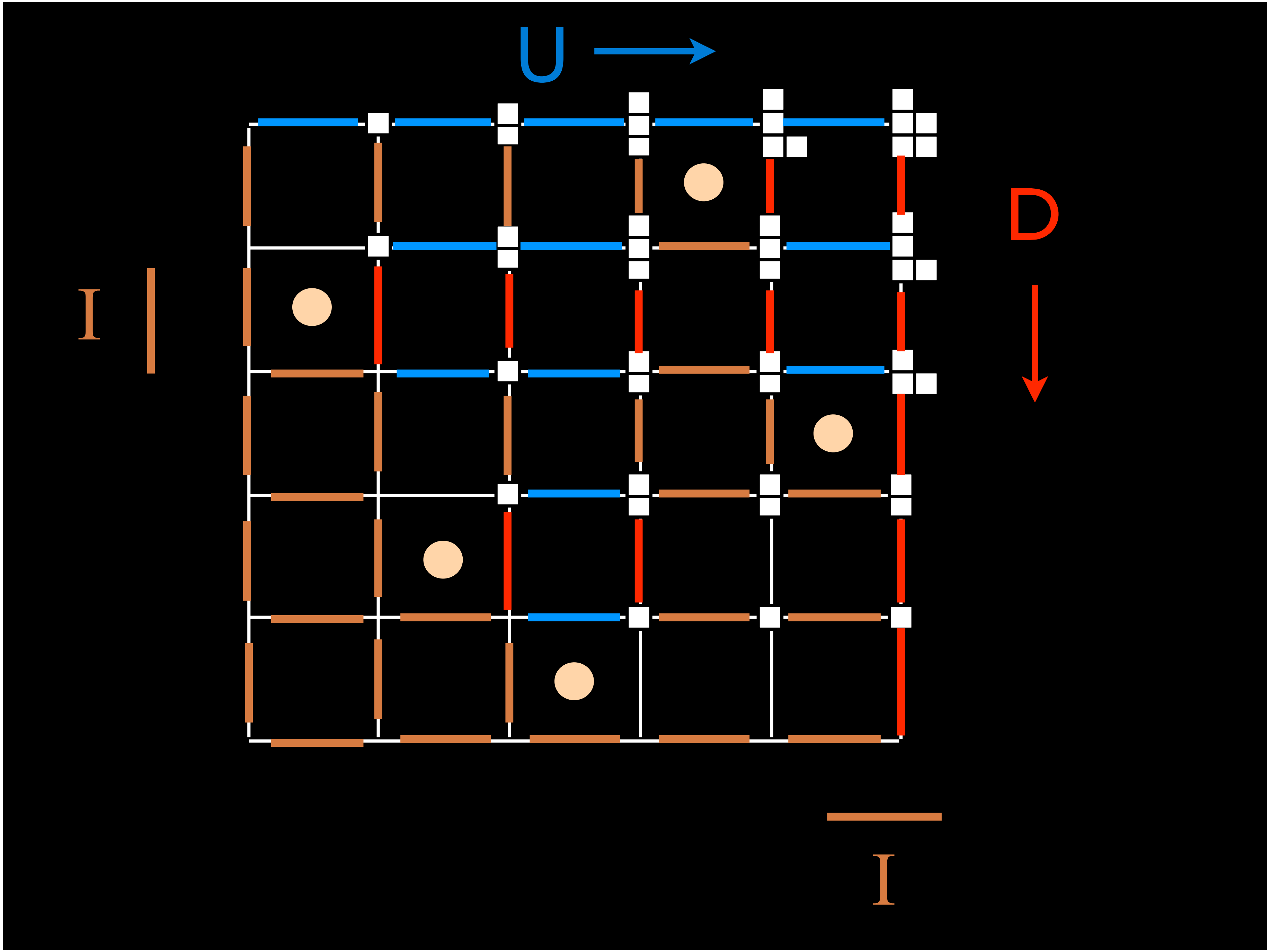
Sergey Fomin
(with C. K.)

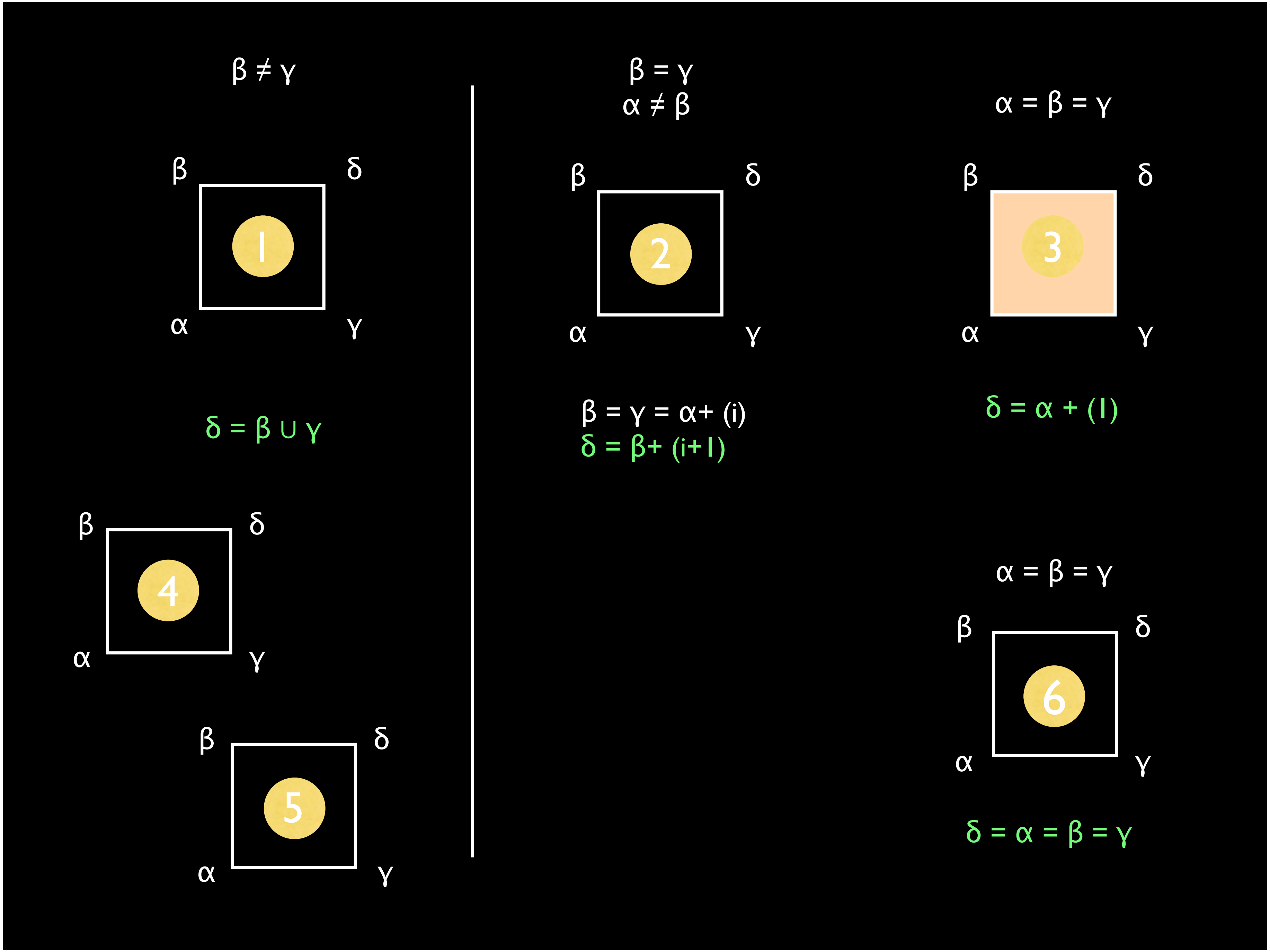

U
D
I
I


U
D
I
I

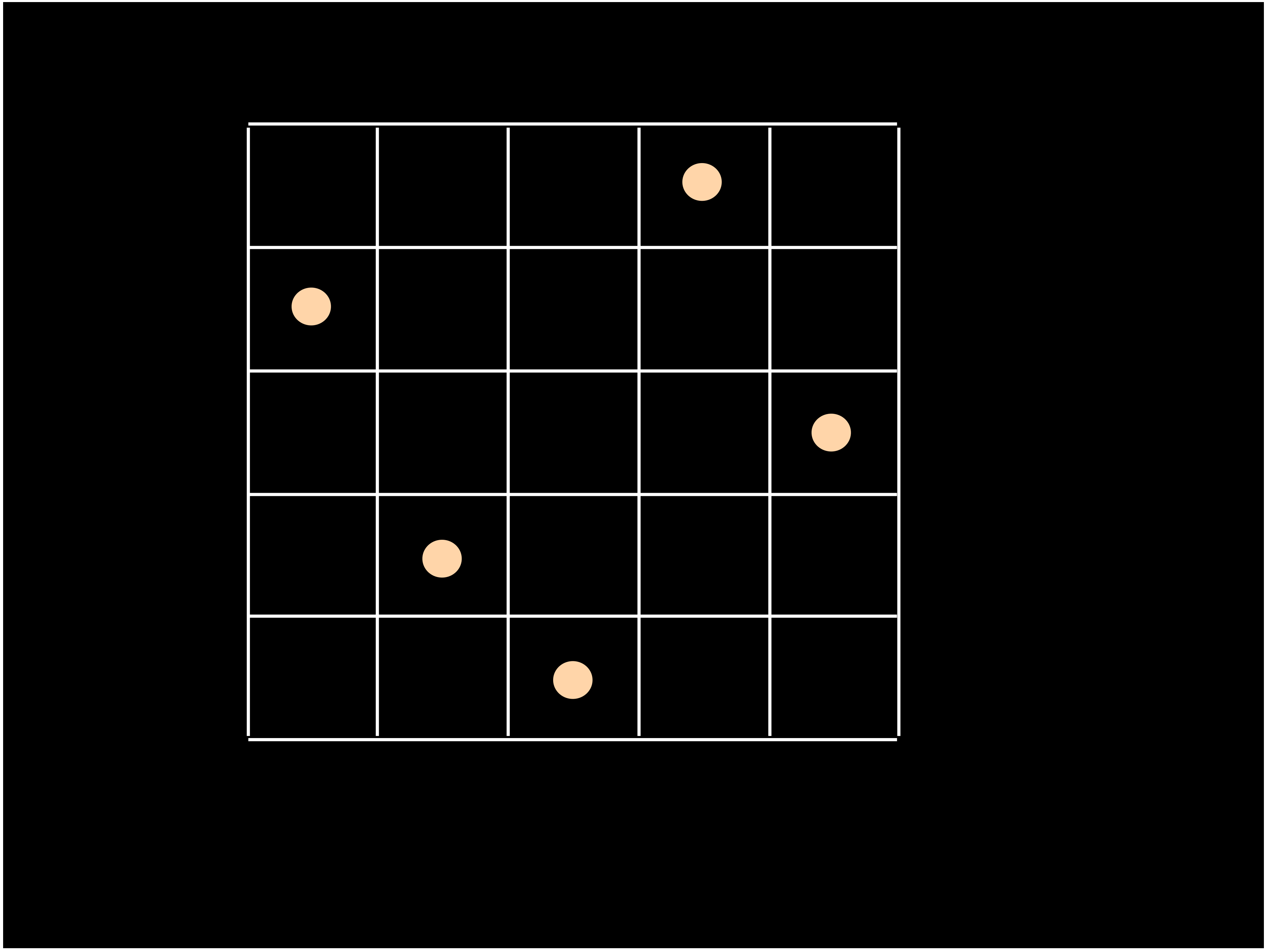


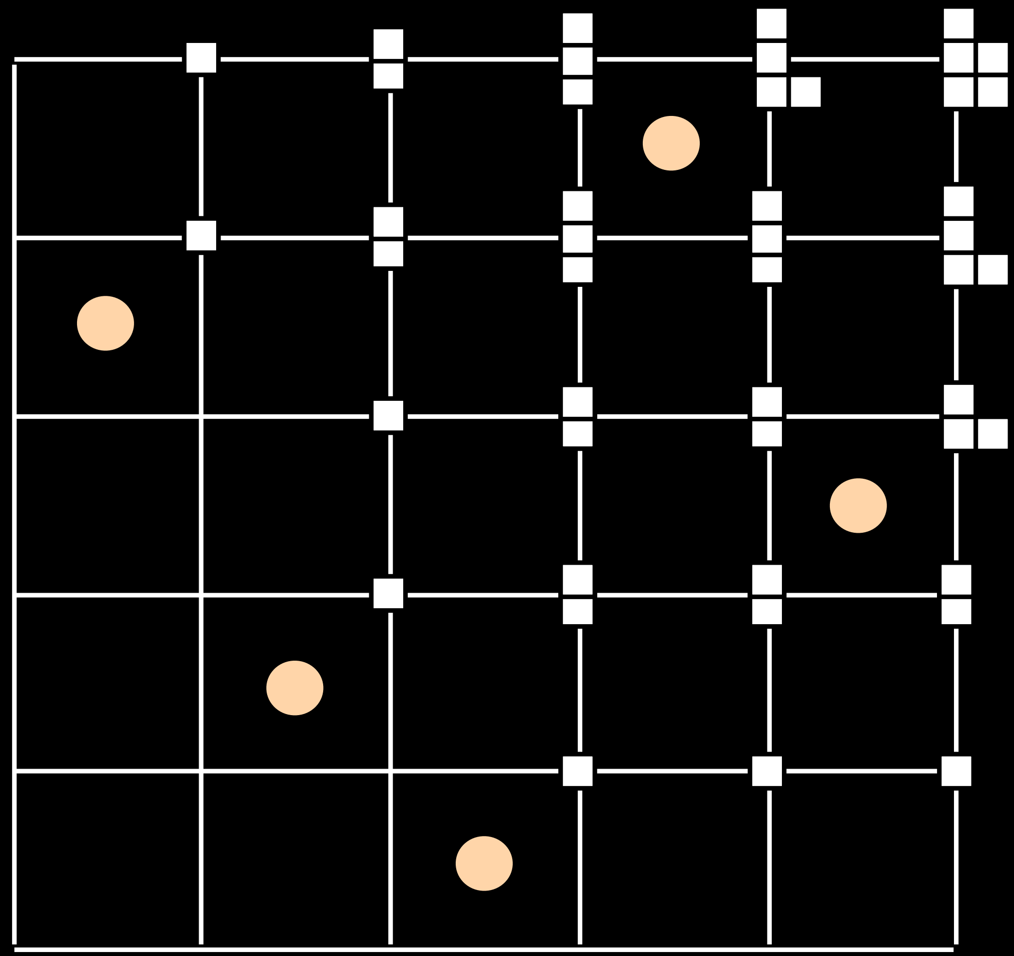
β ≠ γ
β = γ
U
β
δ
D
1
D
α
γ
U
δ = β ∪ γ
U
β
δ
D
2
D
α
γ
U
β = γ = α+ (i)
δ = β+ (i+1)
U
β
δ
I
3
D
α
γ
I
α = β = γ
δ = α + (1)
U
β
δ
I
4
I
α
γ
U
α = β
δ = γ = β +(i)
I
β
δ
D
5
D
α
γ
I
α = γ
δ = β = α + (i)
I
β
δ
I
6
I
α
γ
I
δ = α = β = γ


U
D
I
I



β ≠ γ
β
δ
1
α
γ
δ = β ∪ γ
β
δ
4
α
γ
β
δ
5
α
γ
β = γ
α ≠ β
β
δ
2
α
γ
β = γ = α+ (i)
δ = β+ (i+1)
α = β = γ
β
δ
3
α
γ
δ = α + (1)
α = β = γ
β
δ
6
α
γ
δ = α = β = γ





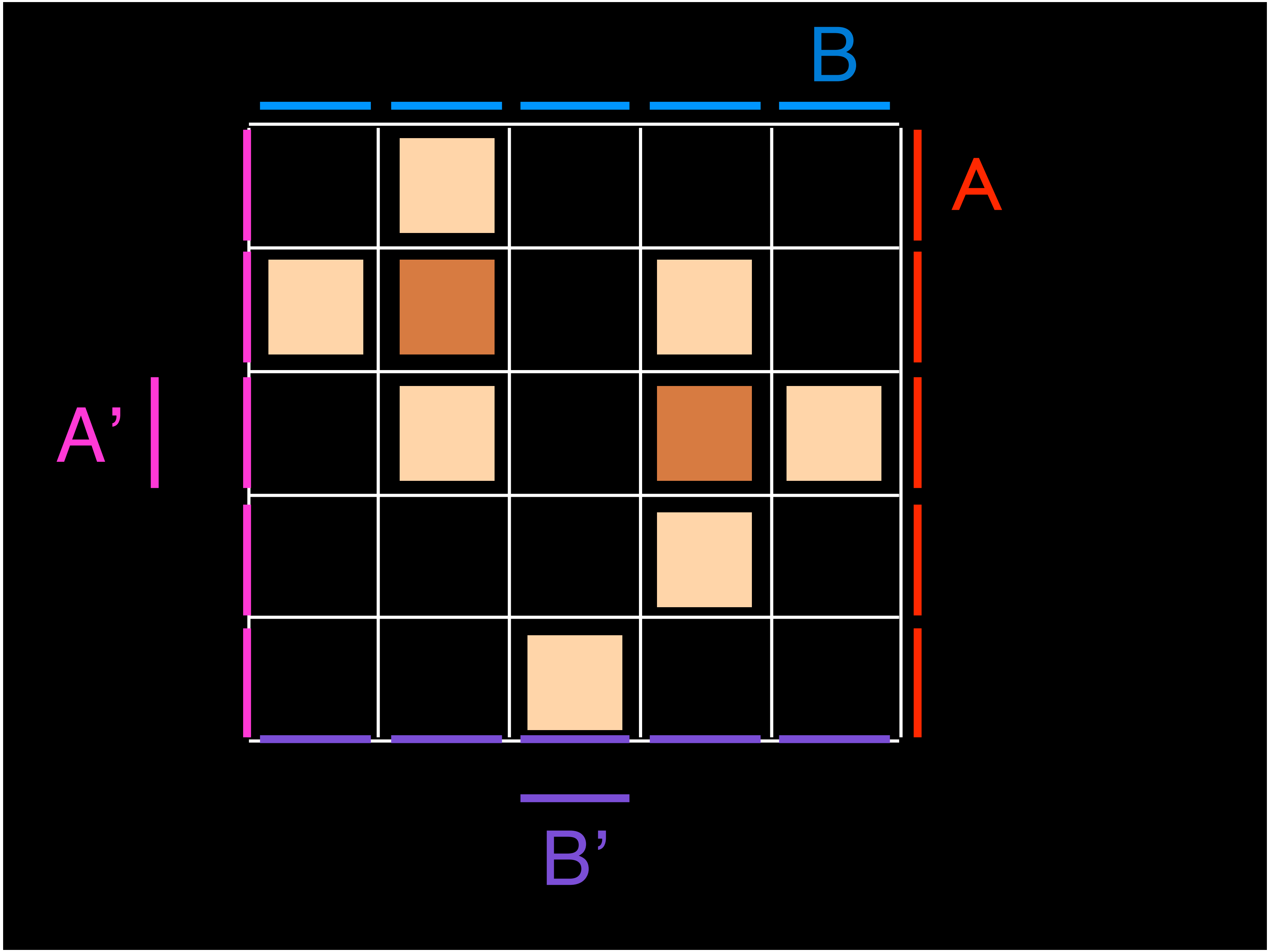
Questions.

- find a "combinatorial representation" for operators A, A', B, B'. ?
- analogue of RSK (Robinson-Schensted-Knuth) for ASM ?
- analogue of "local rules" (Fomin) ?
- direct proof of the formula

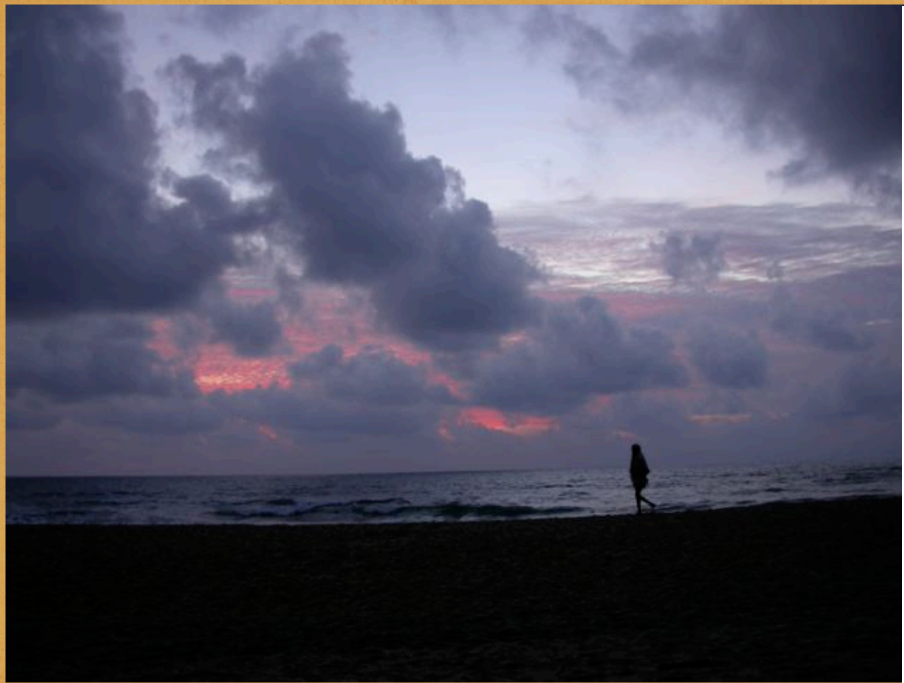
$$A_n = \prod_{j=1}^{n} \frac{(3j-2)!}{(n+j-1)!}$$?

(nb of ASM of size n)

= 1, 2, 7, 42, ...



B
A
A'
B'

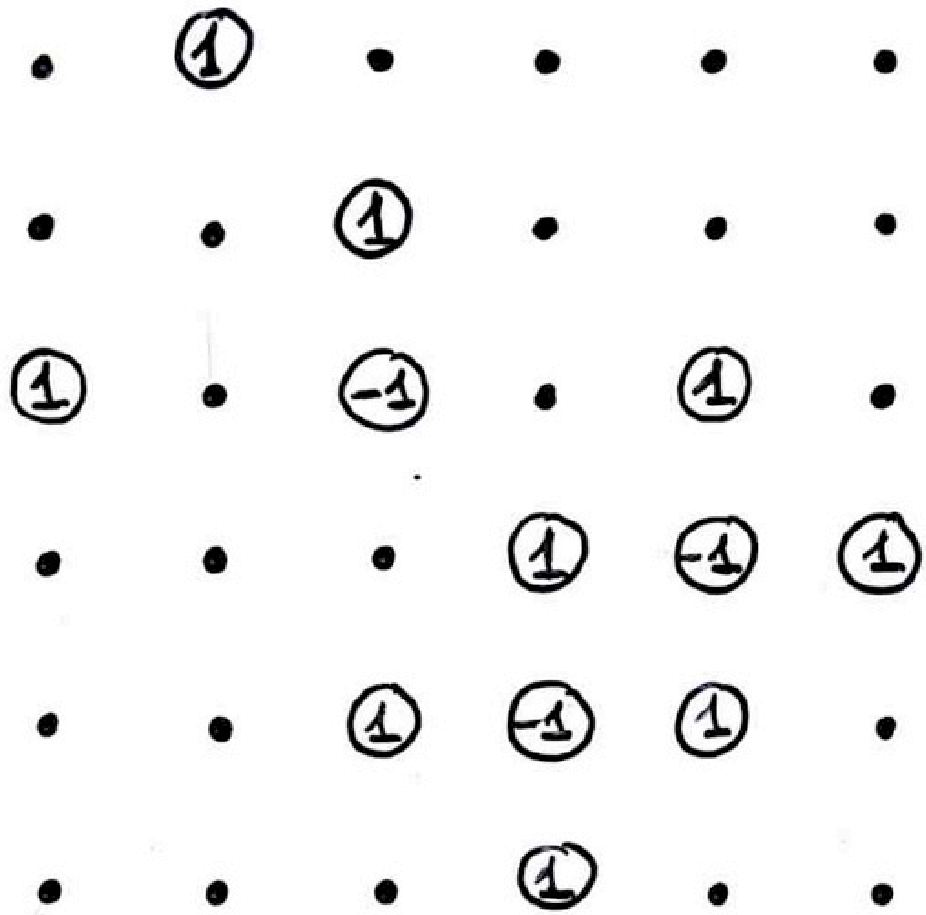


§4 FPL and operators A, $\underline{A}$, B, $\underline{B}$

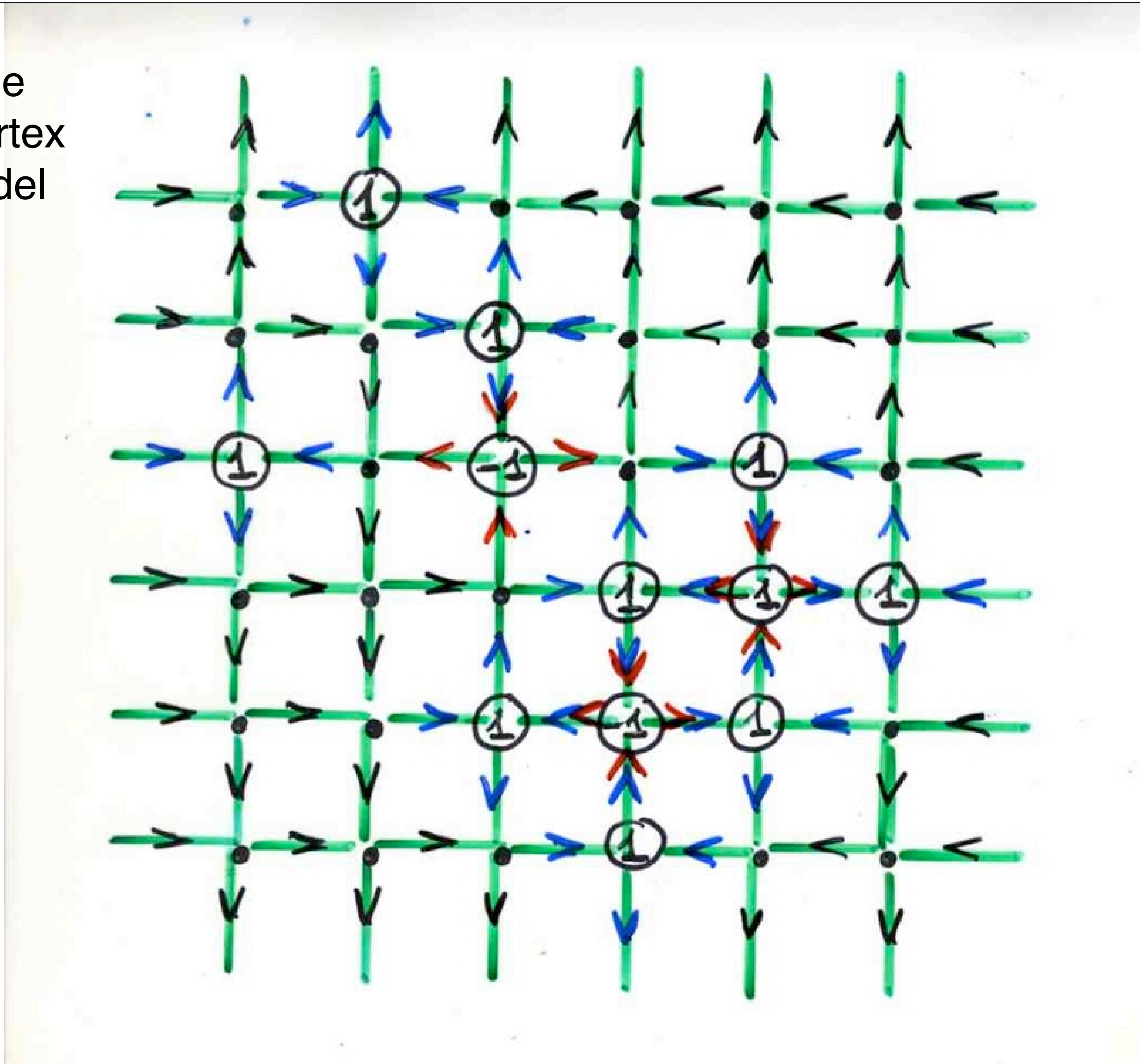
Tamil Nadu, Inde 02 xgv

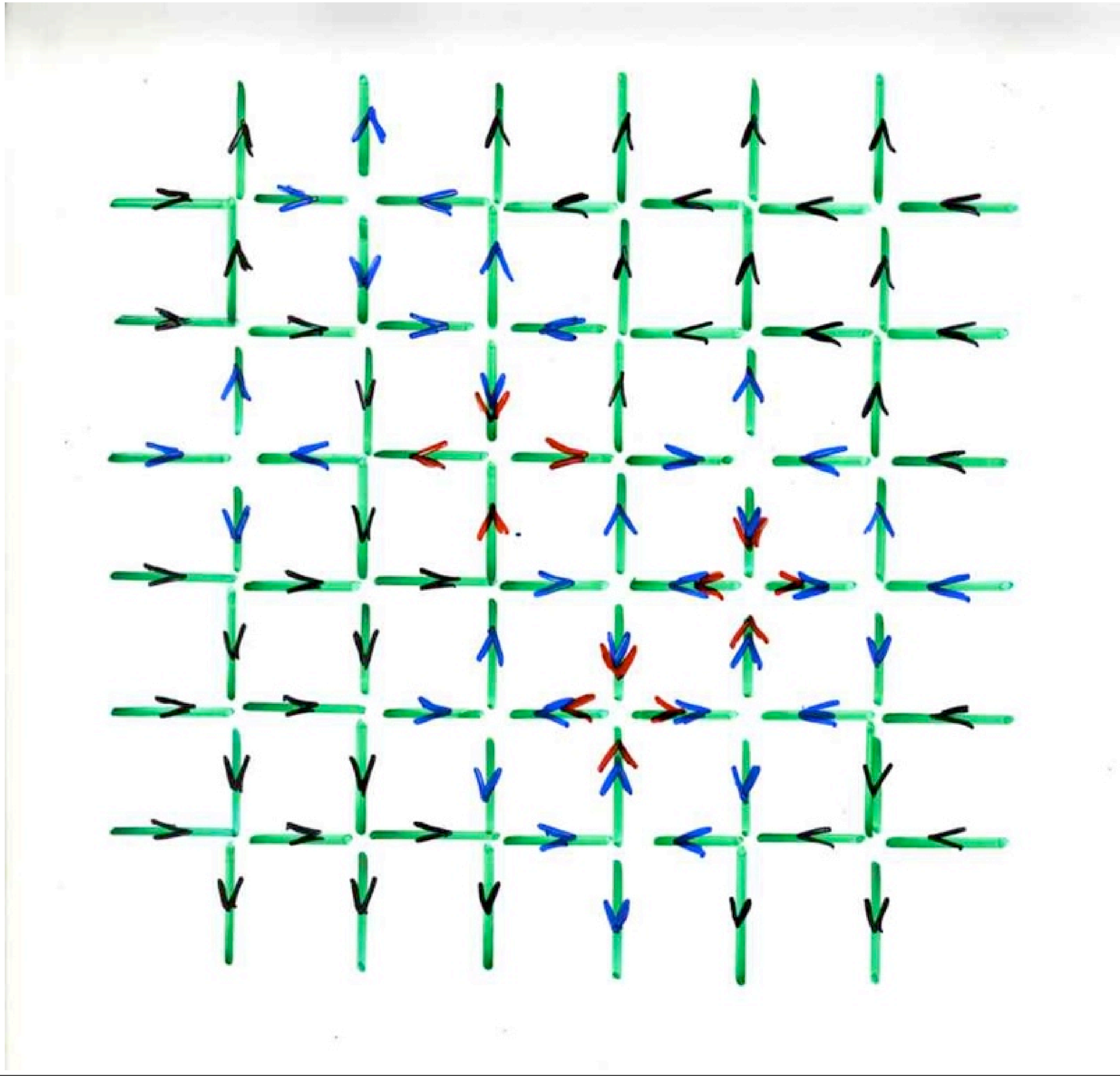
"Fully packed loop configurations"

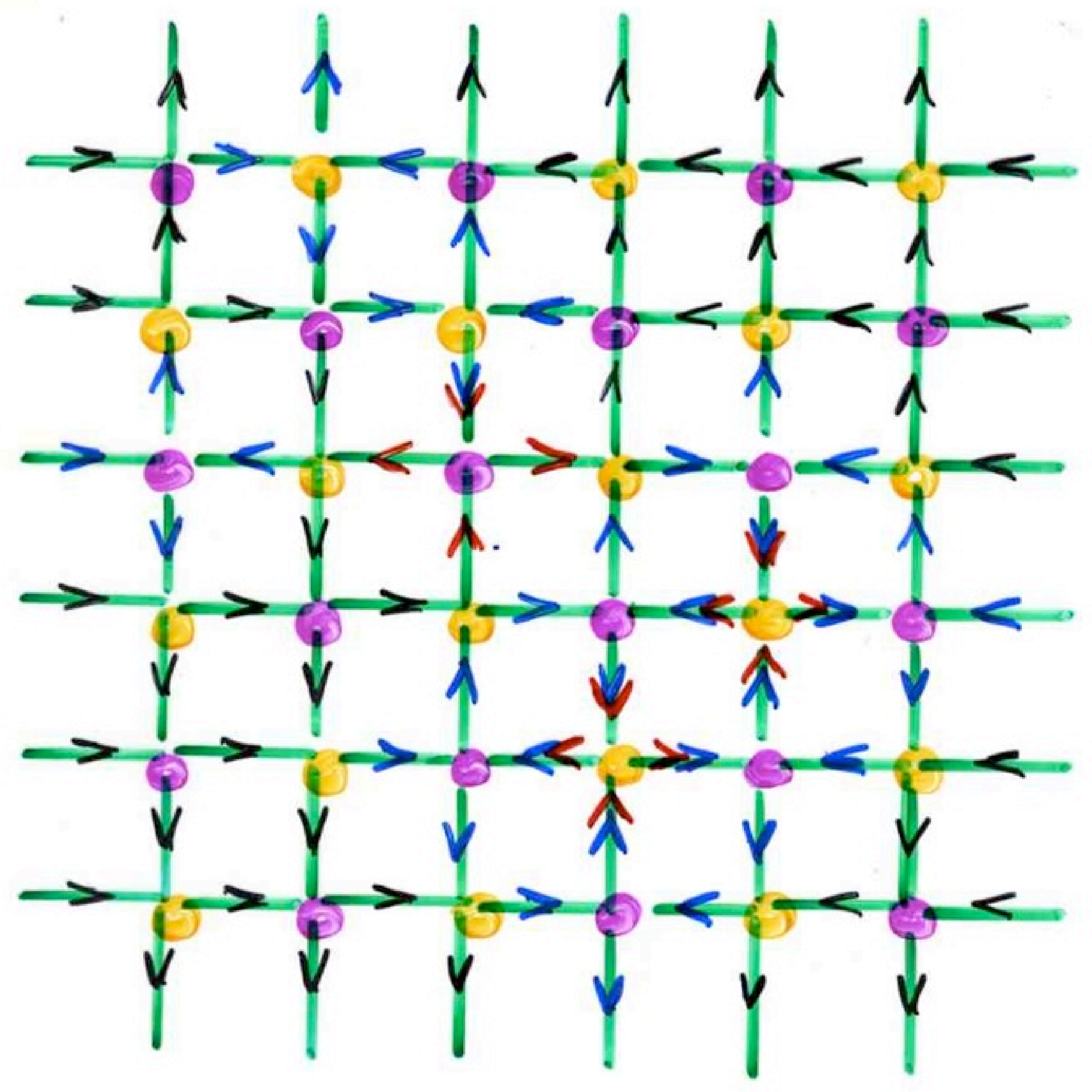
The
bijection
AMS
FPL

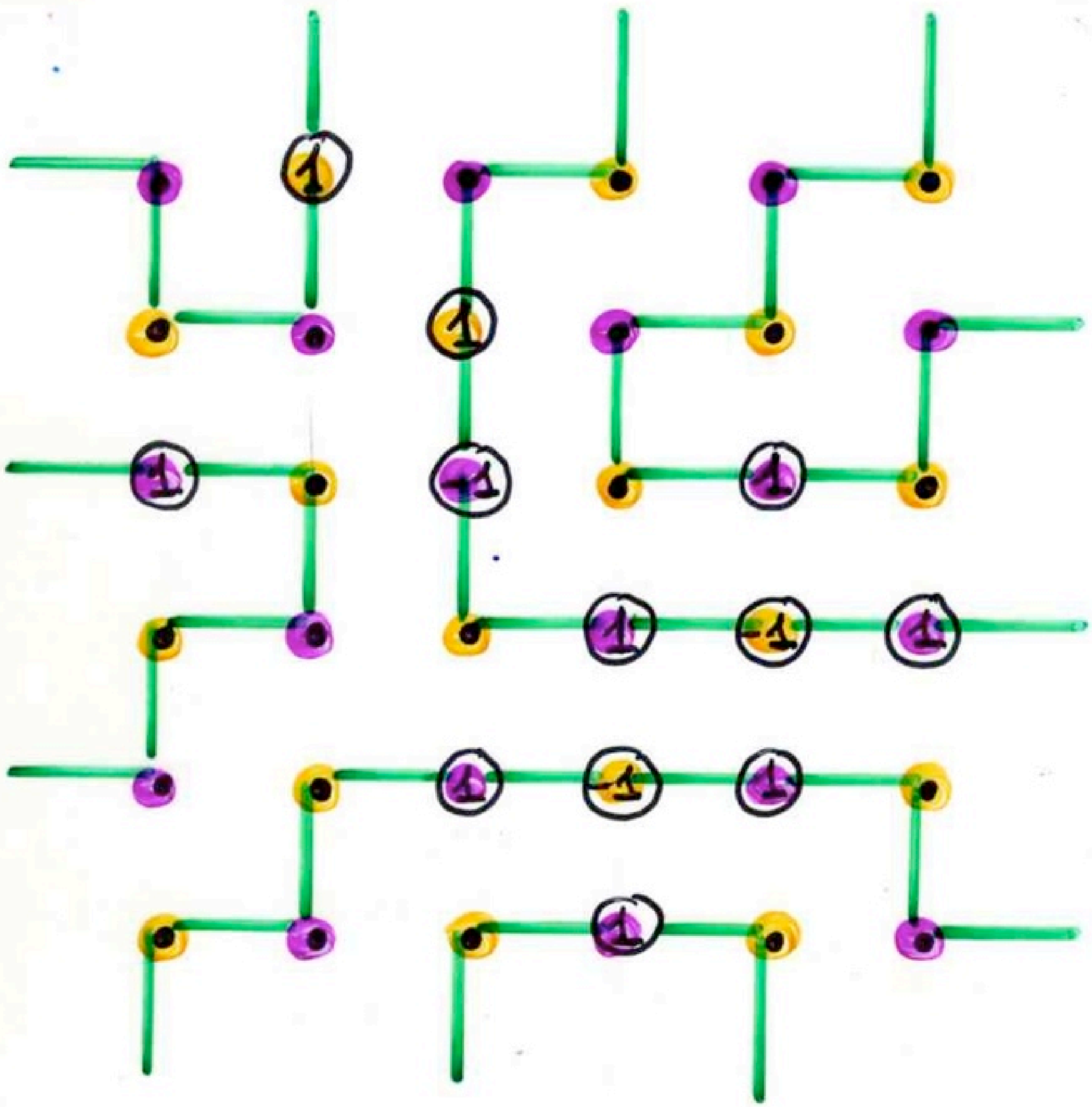


The
6-vertex
model



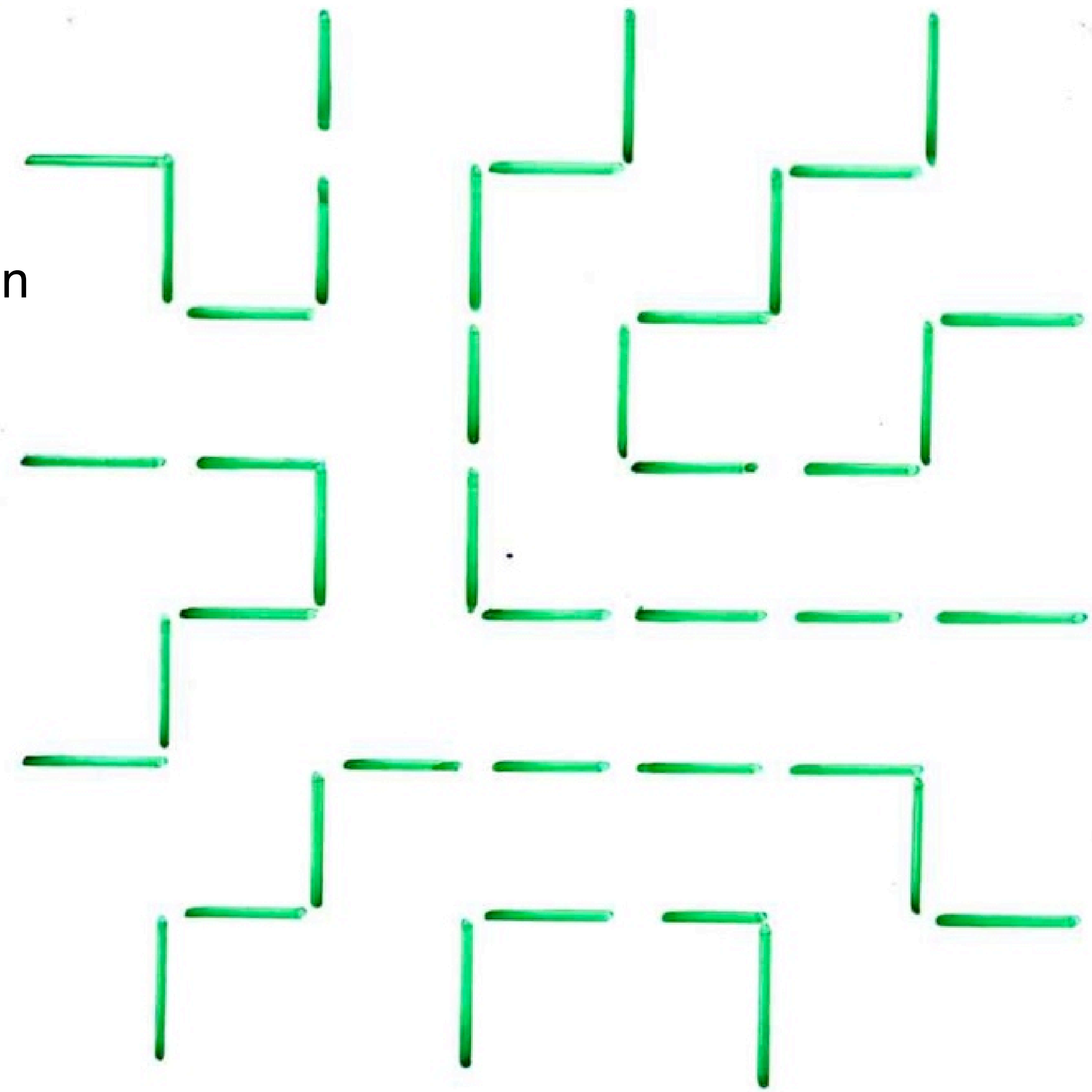




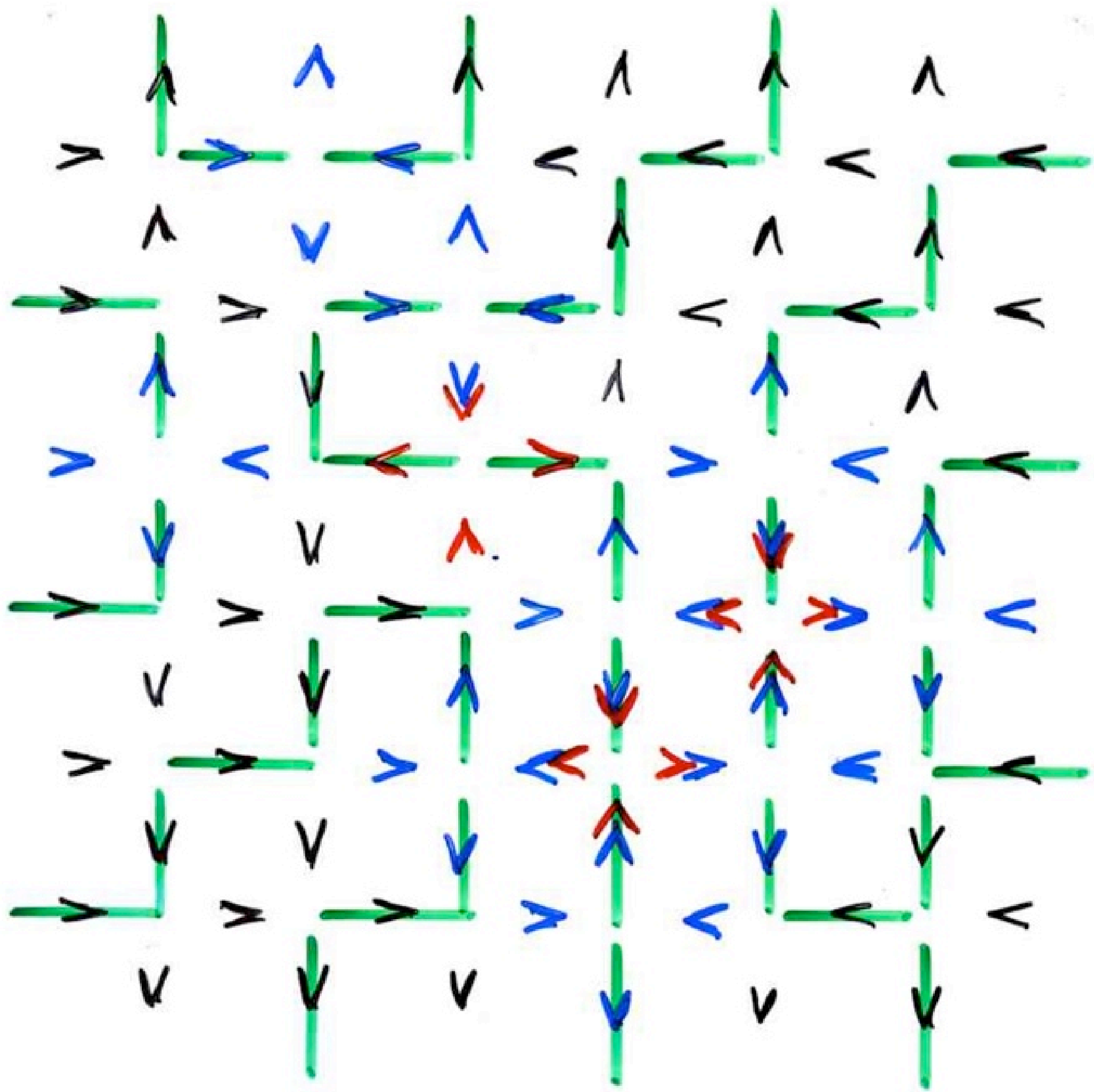


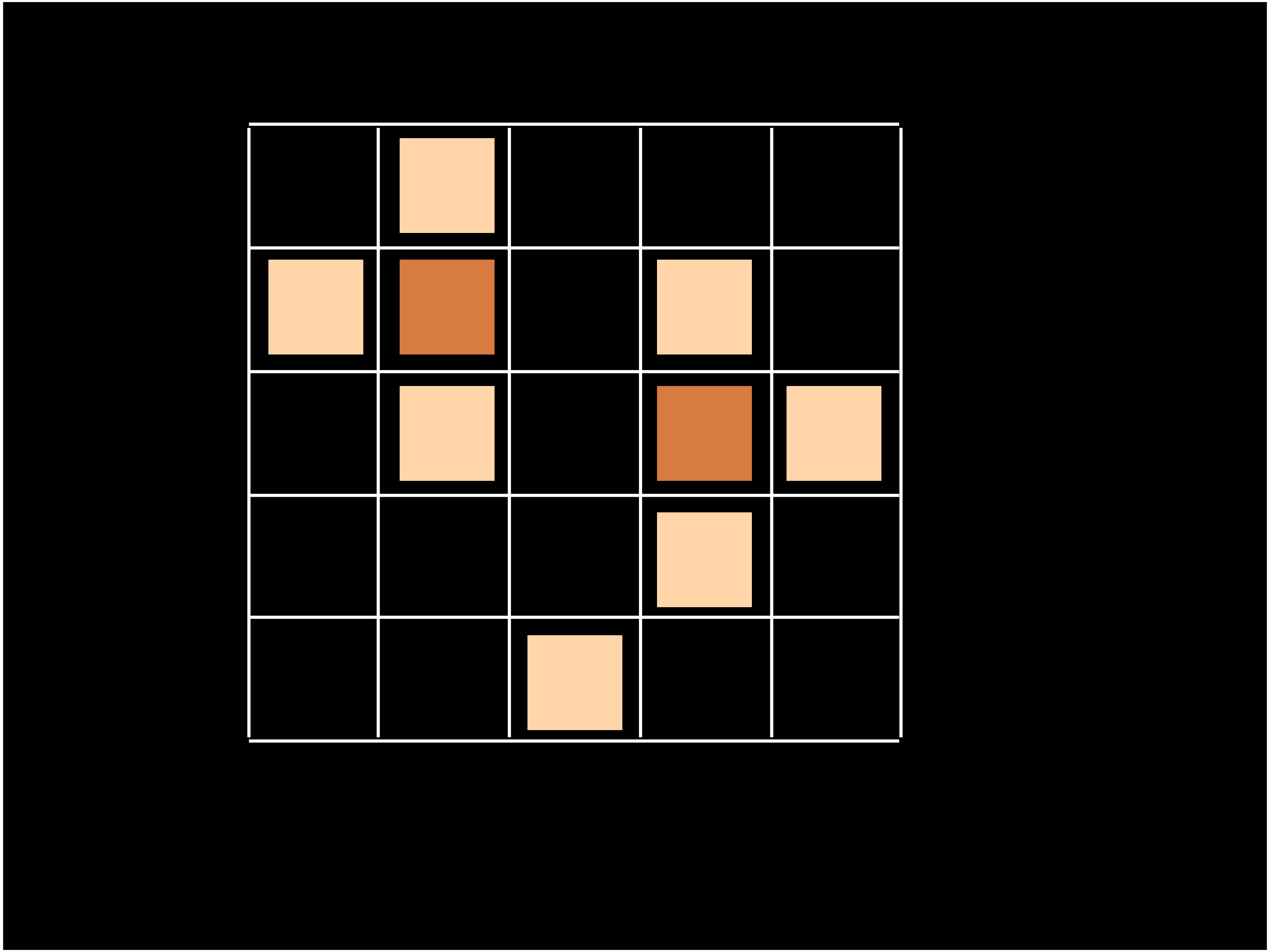
1
1
1
-1
1
1
-1
1
1
-1
1
1

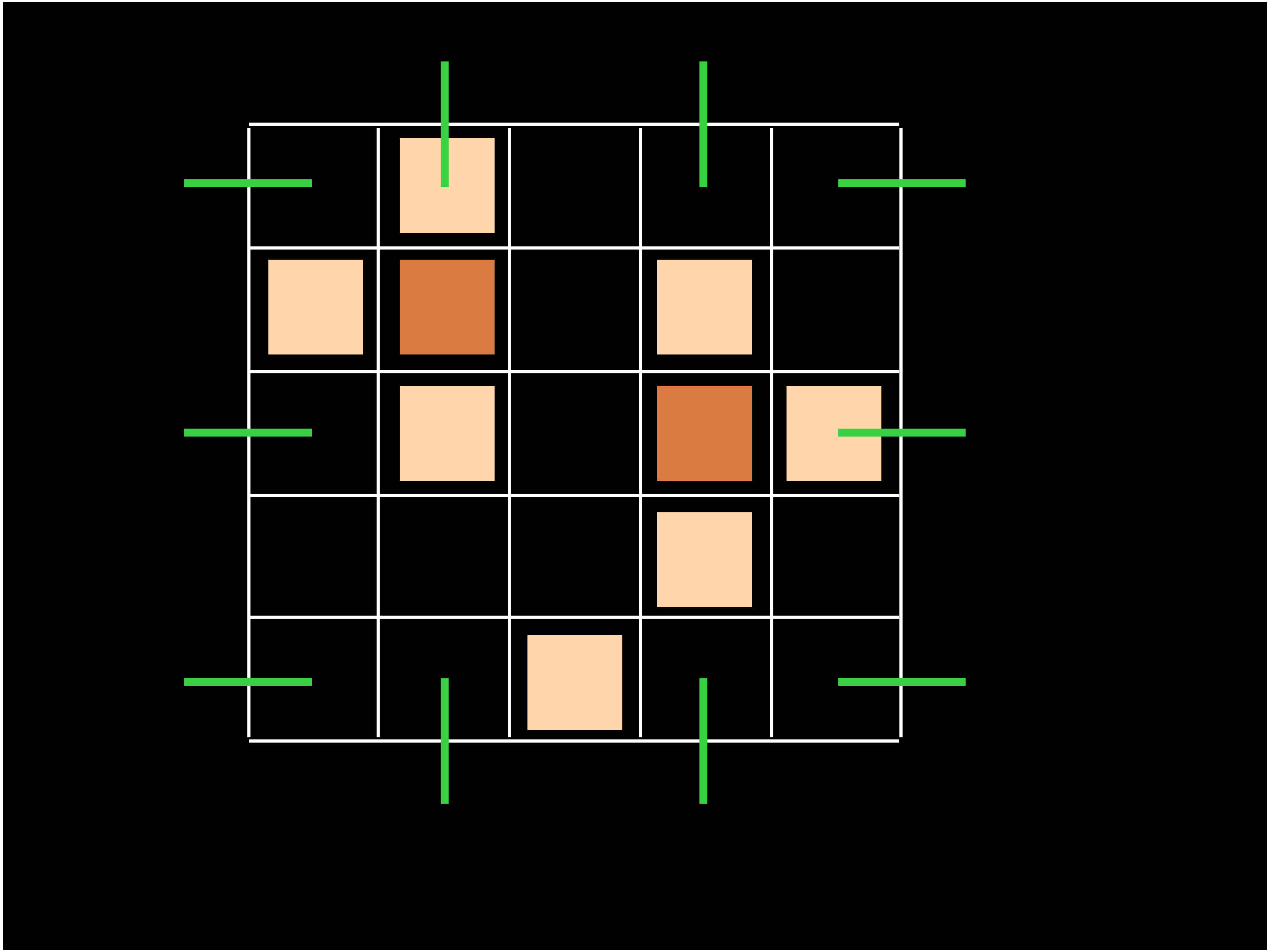
FPL
“Fully
Packed
Loop”
configuration

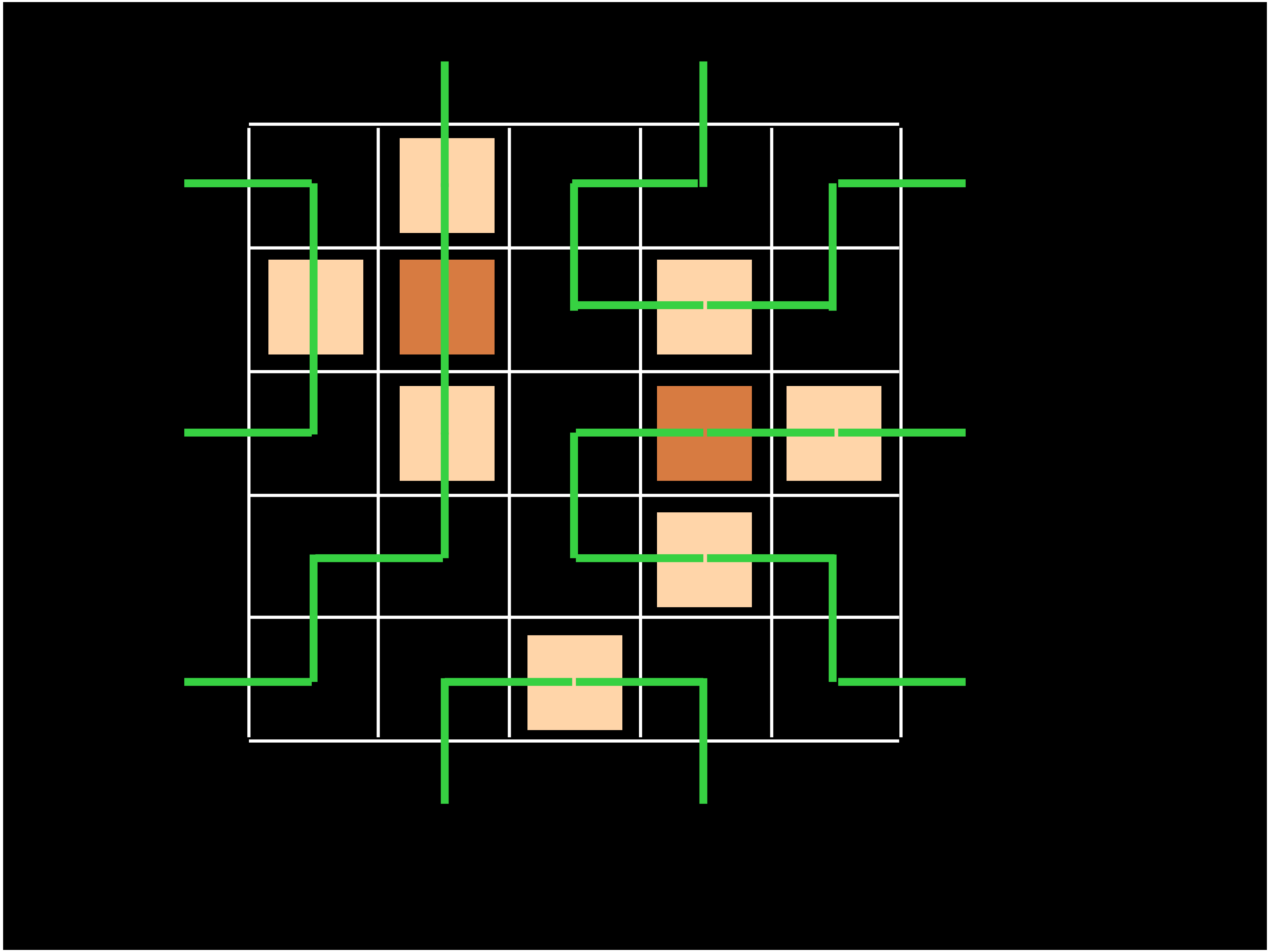


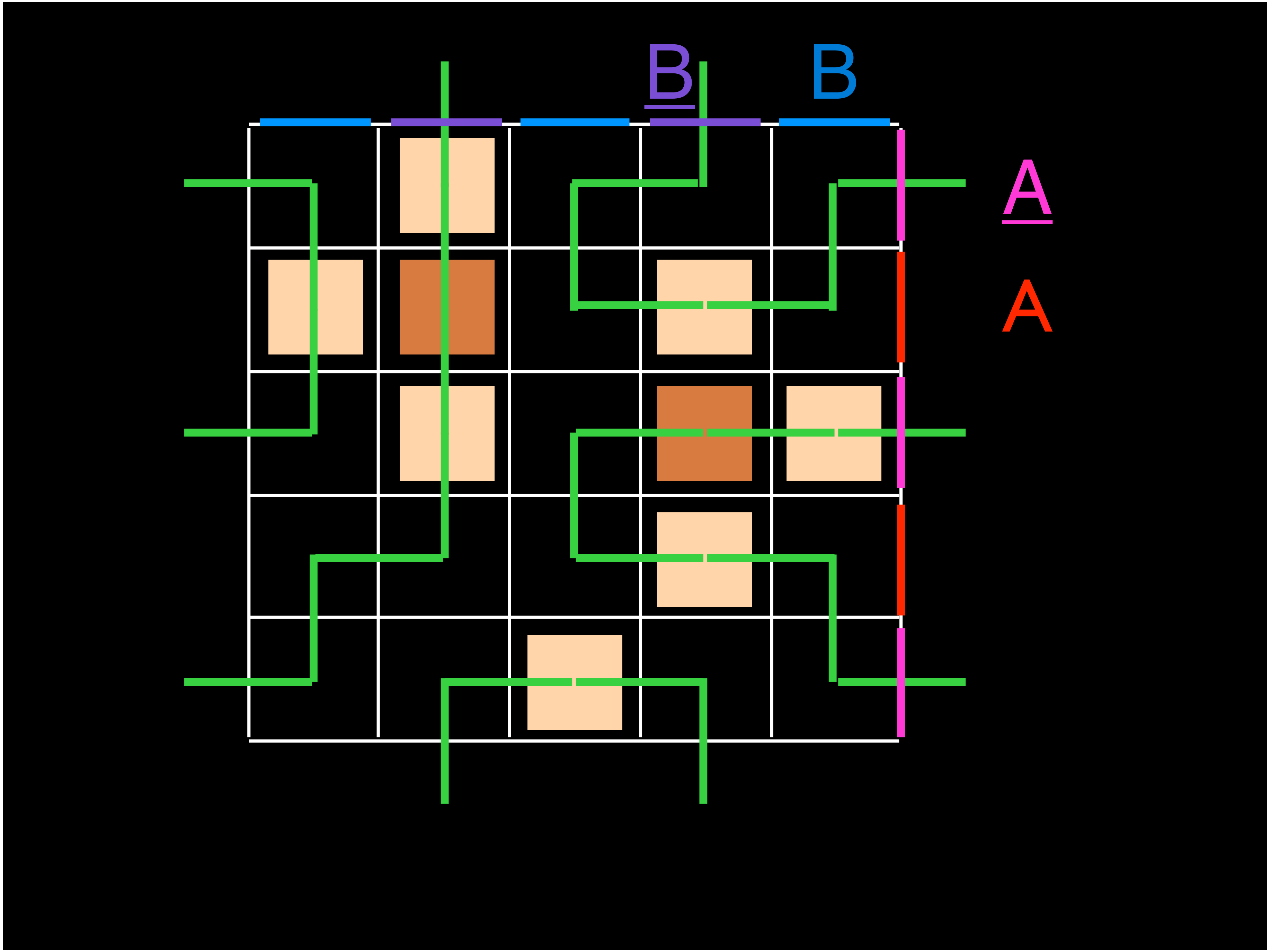
dual
FPL

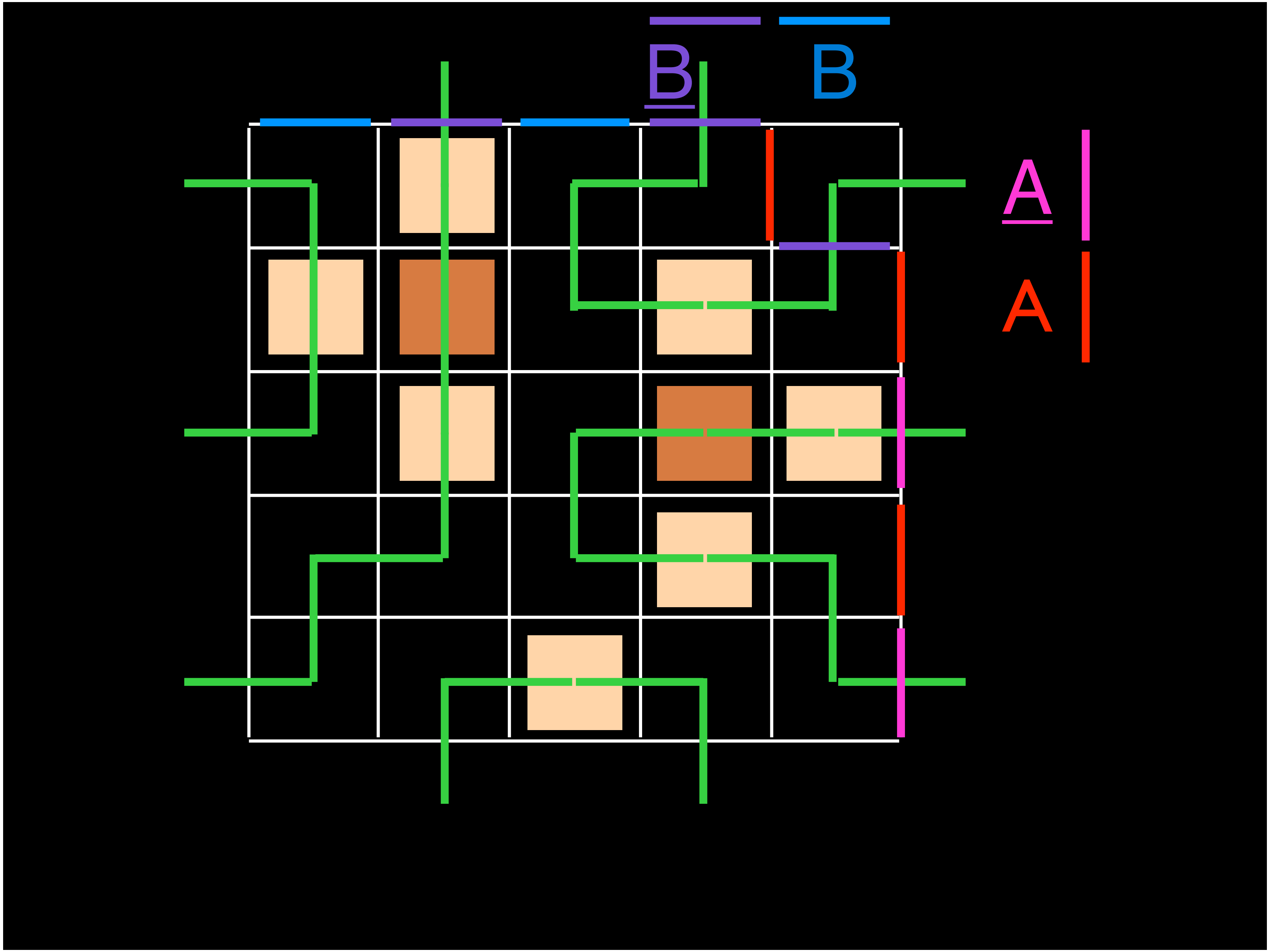




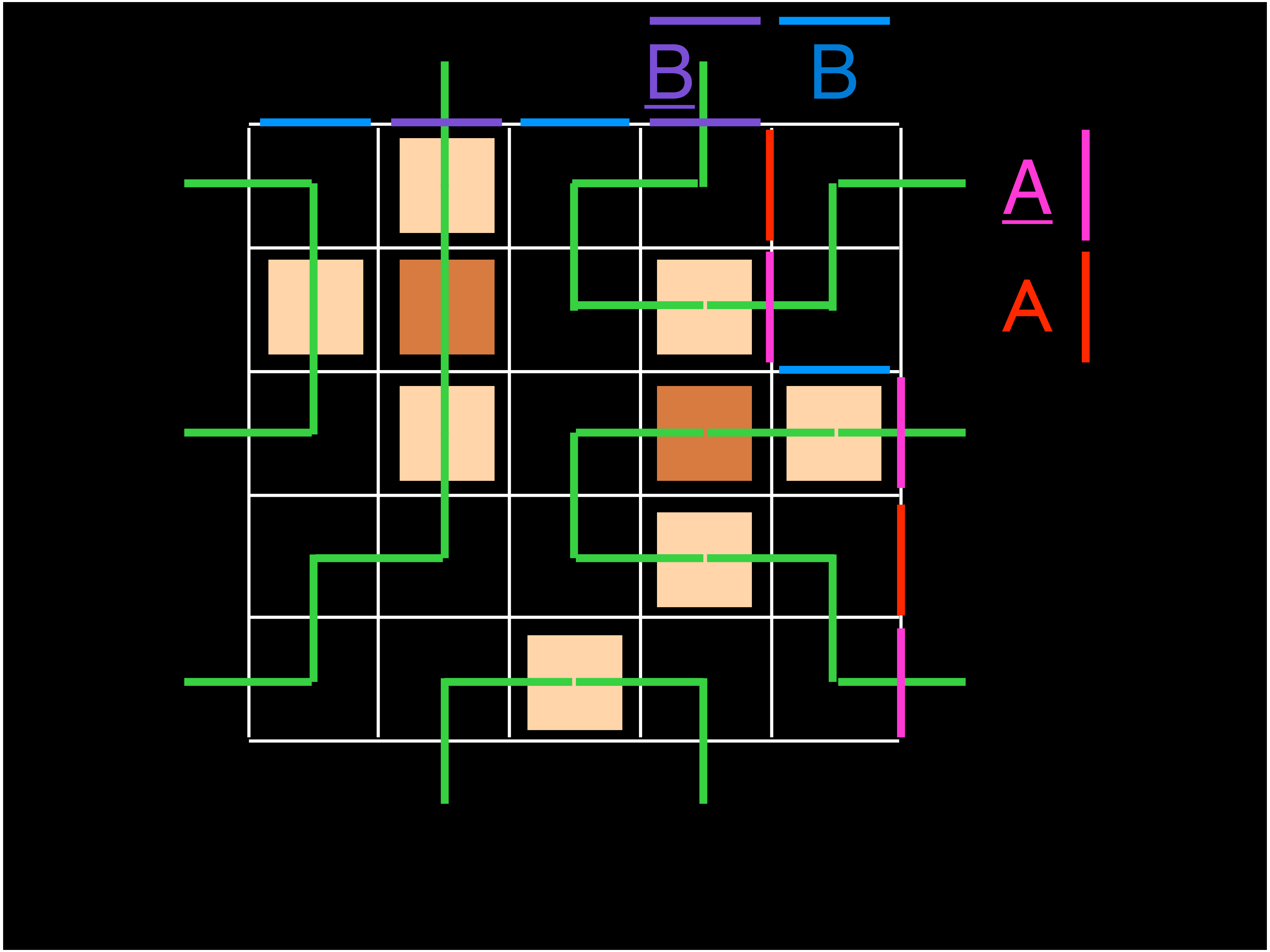


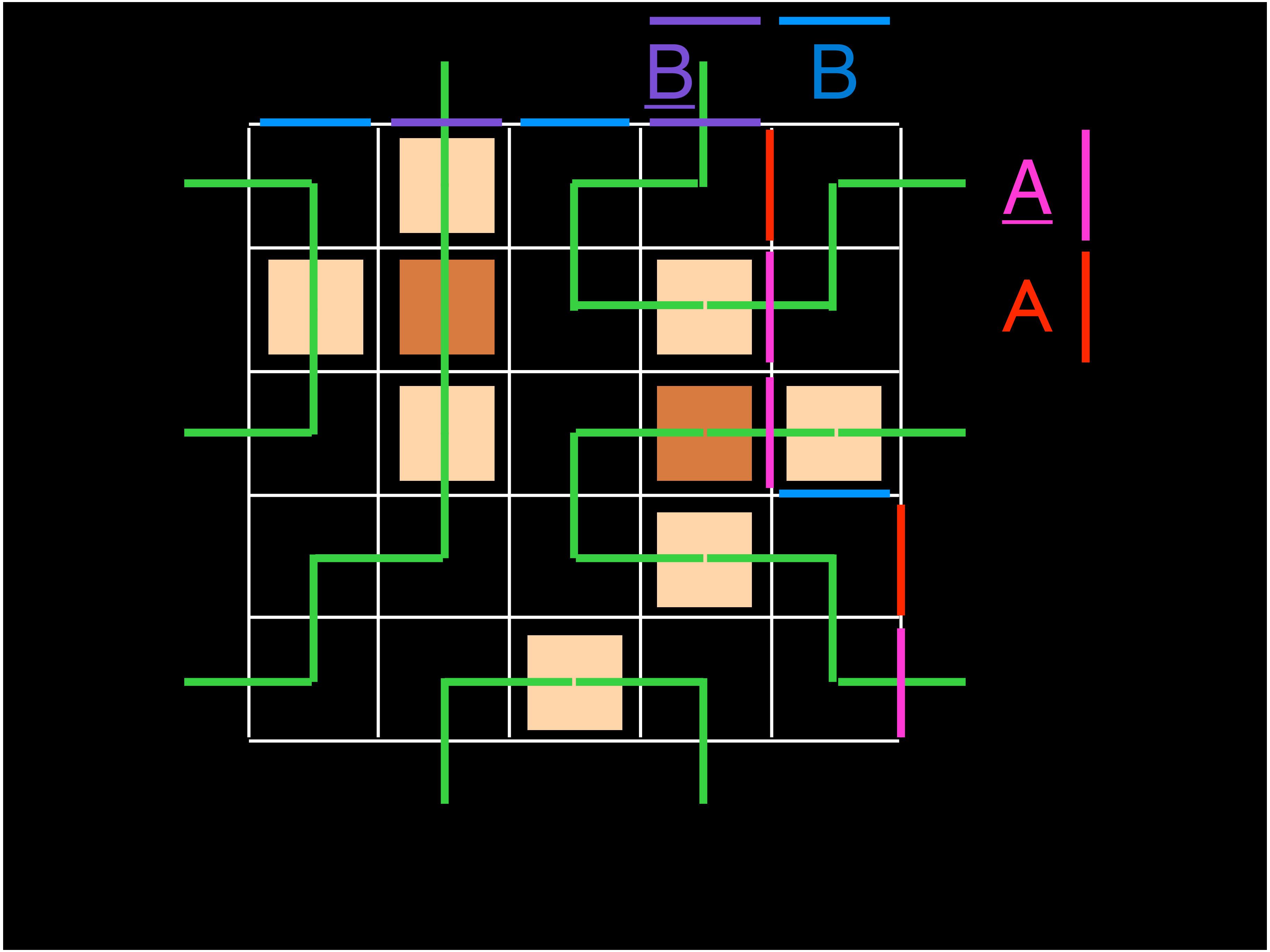



B
B
A
A

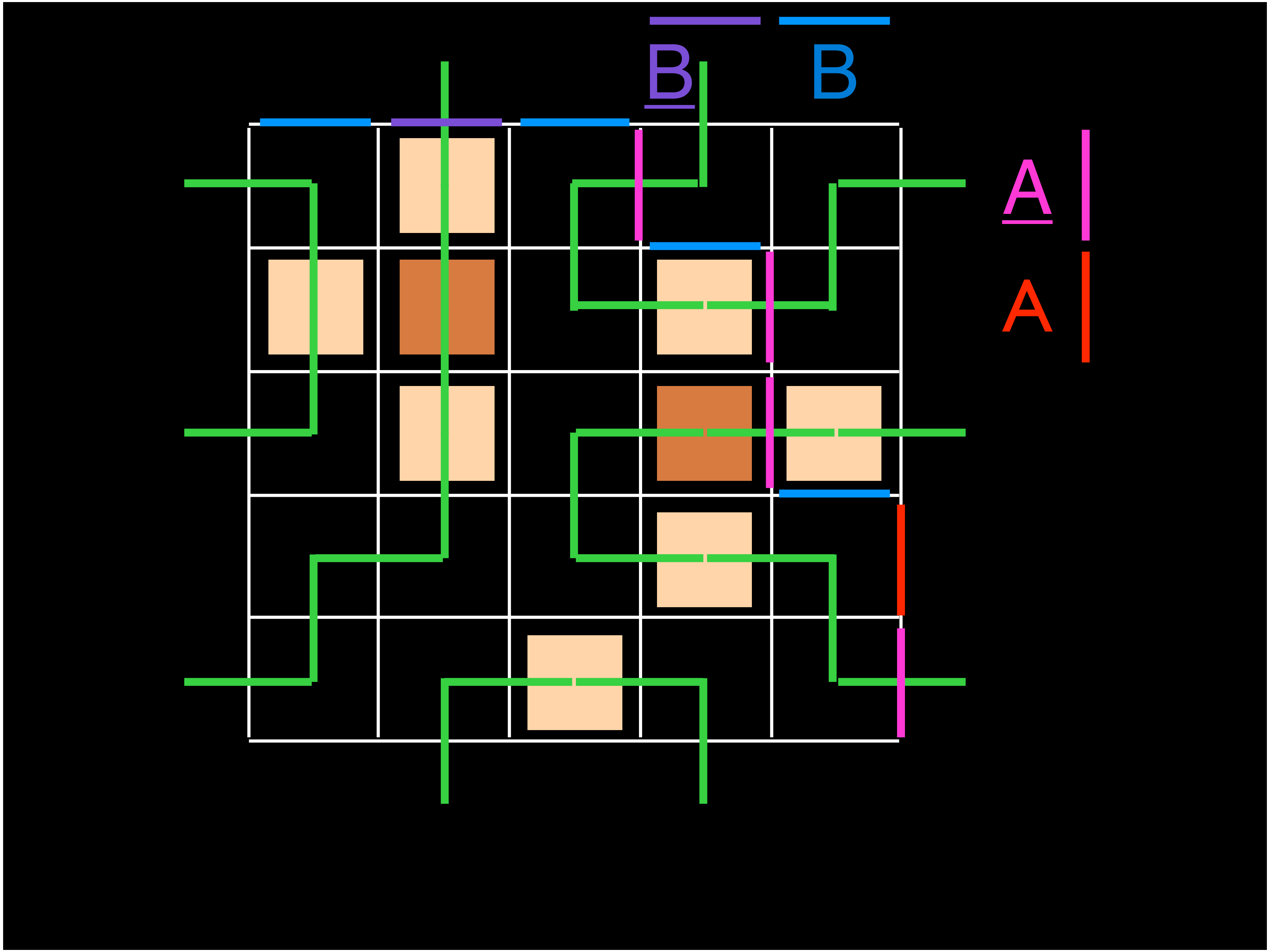


B
B
A
A


B
B
A
A



B̲
B
A̲
A



B
B
A
A

$$A,\ \bar{A},\ B,\ \bar{B}$$

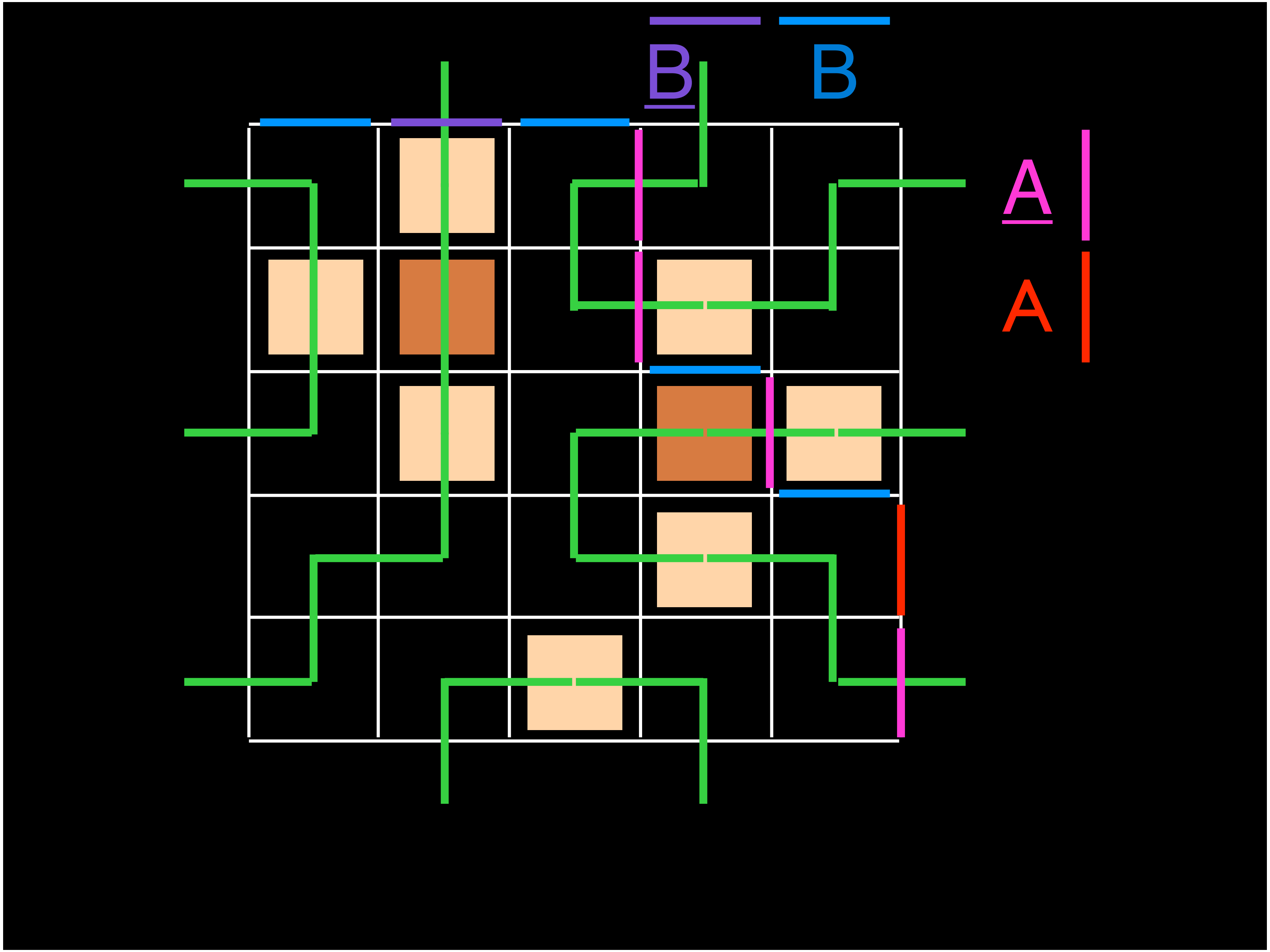
$$\begin{cases} B\bar{A} = \bar{A}B + A\bar{B} \\ \bar{B}A = A\bar{B} + \bar{A}B \end{cases}$$

$$\begin{cases} BA = \bar{A}\bar{B} \\ \bar{B}\bar{A} = AB \end{cases}$$

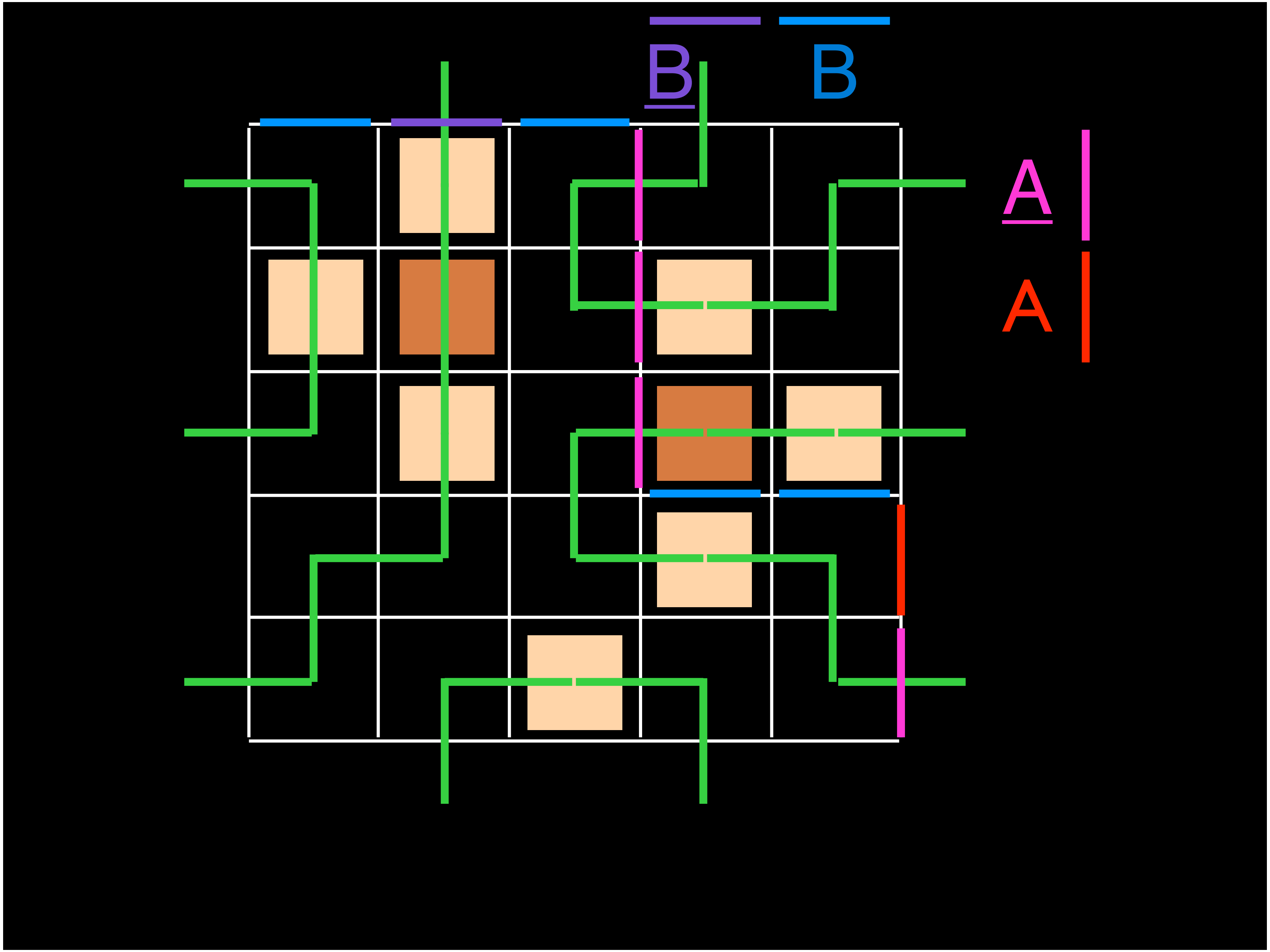
$$B \to b \qquad \bar{A} \to a$$

$$\bar{B} \to b' \qquad A \to a'$$

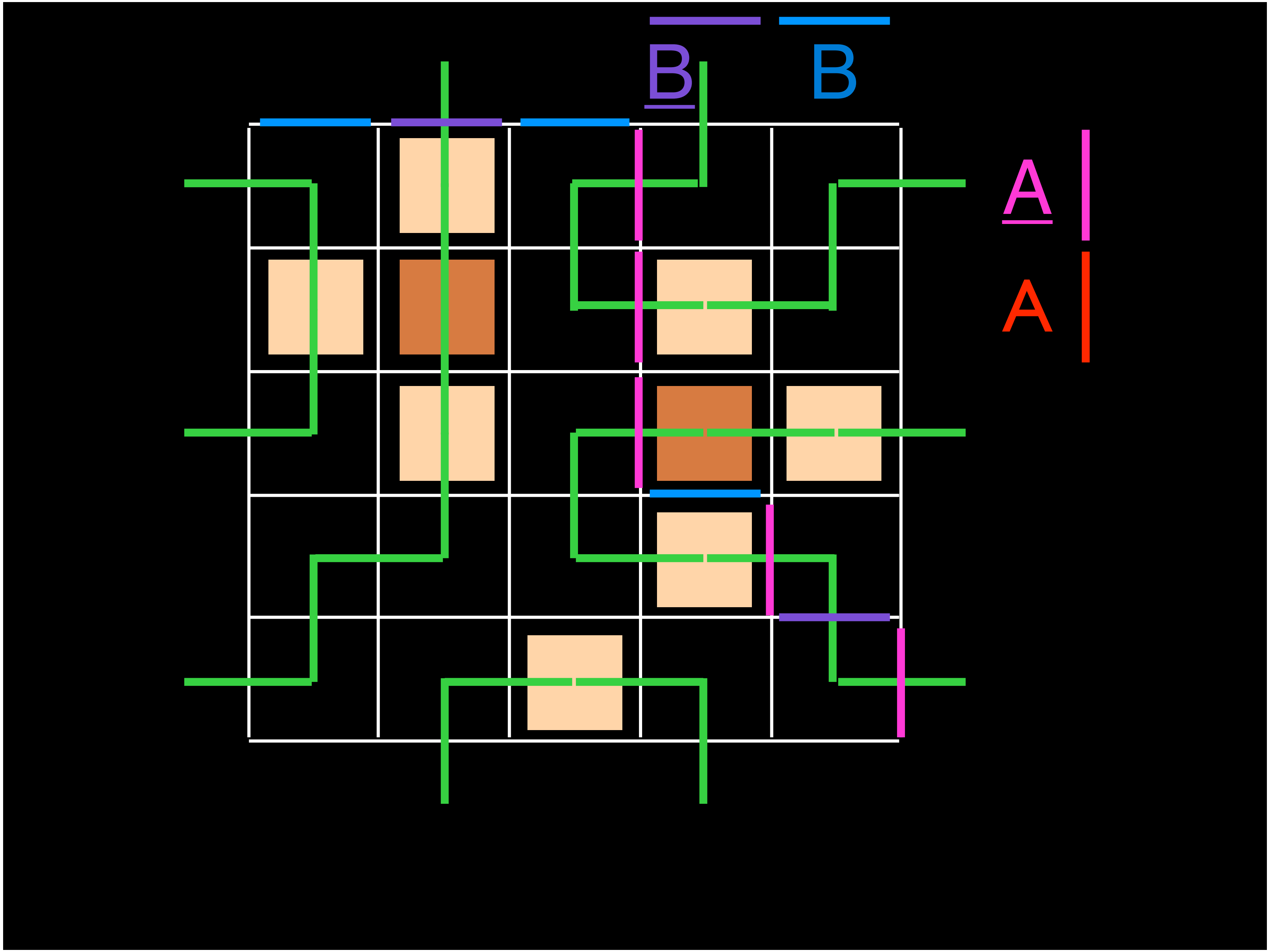
$$\begin{cases} ba = ab + a'b' \\ b'a' = a'b' + ab \end{cases} \qquad \begin{cases} ba' = ab' \\ b'a = a'b \end{cases}$$



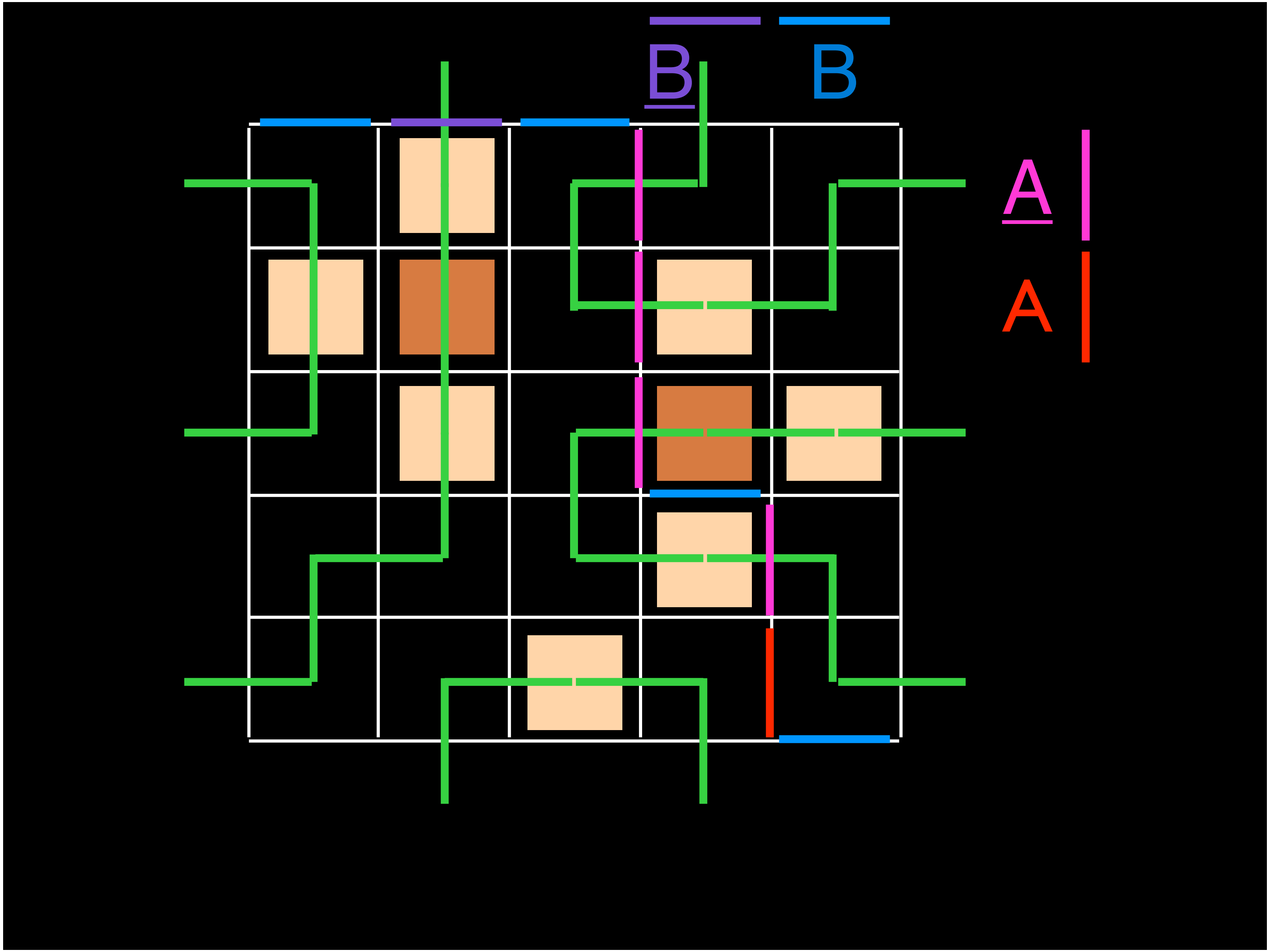
B
B
A
A



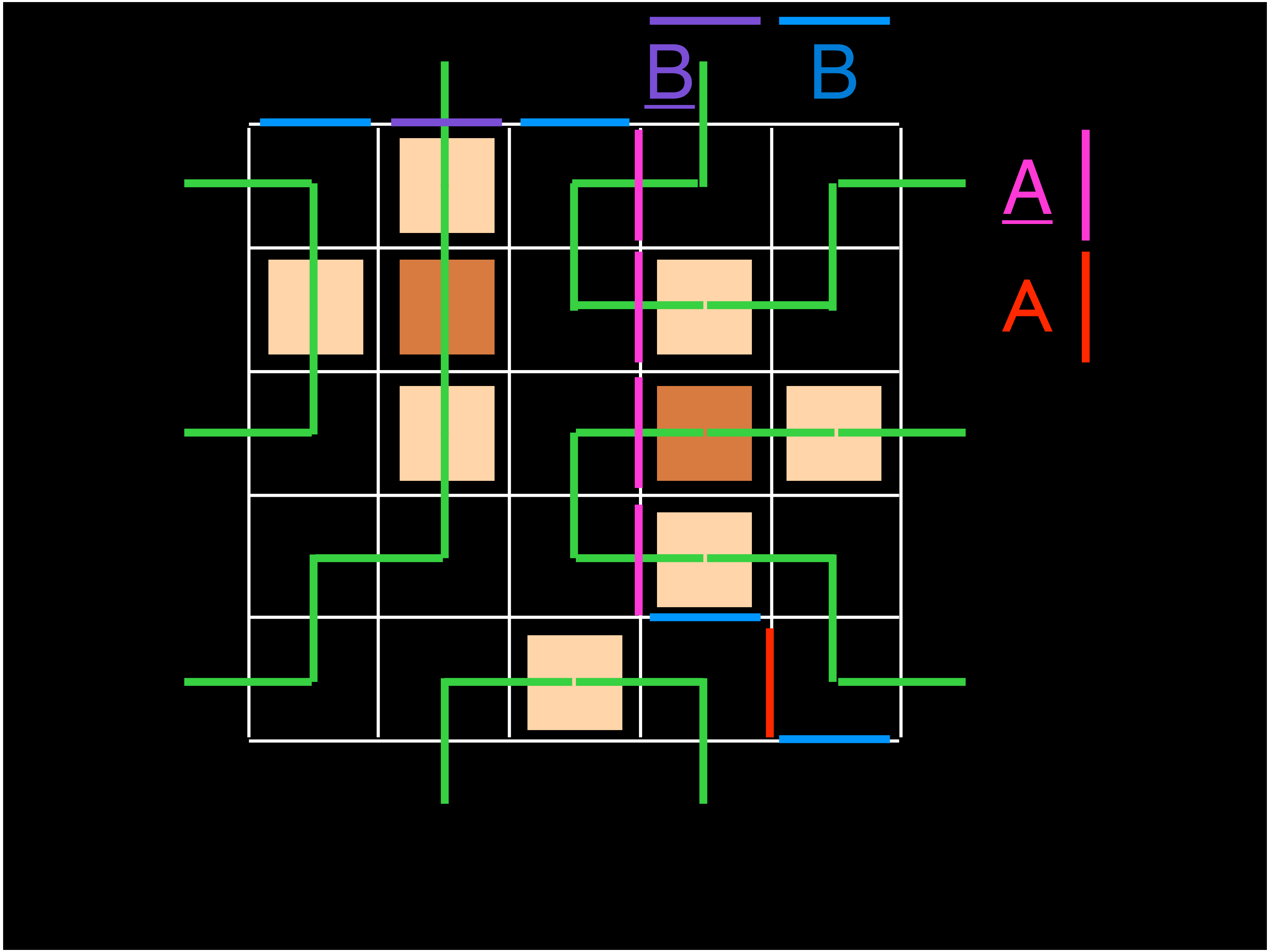
B
B
A
A

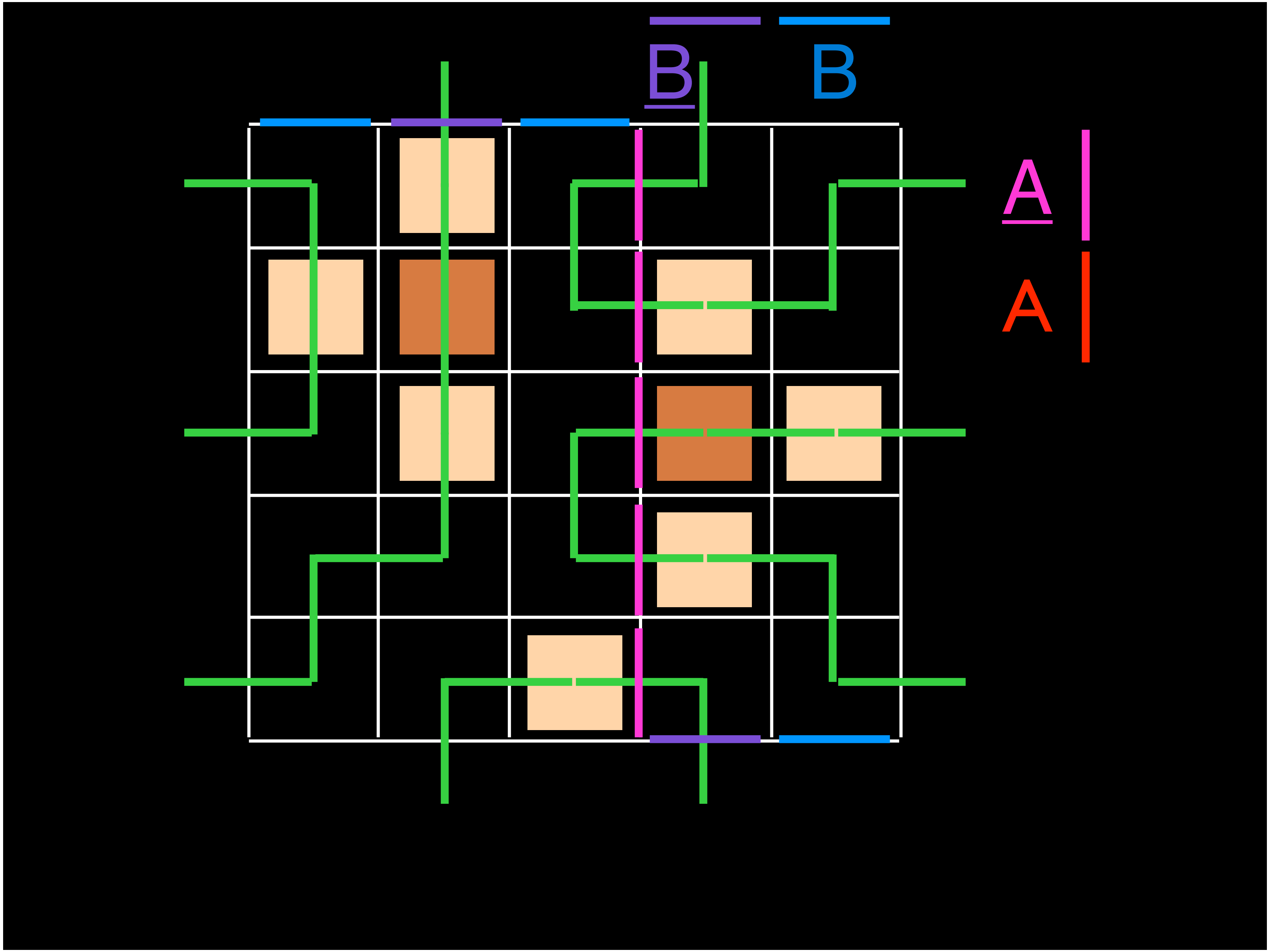


B
B
A
A

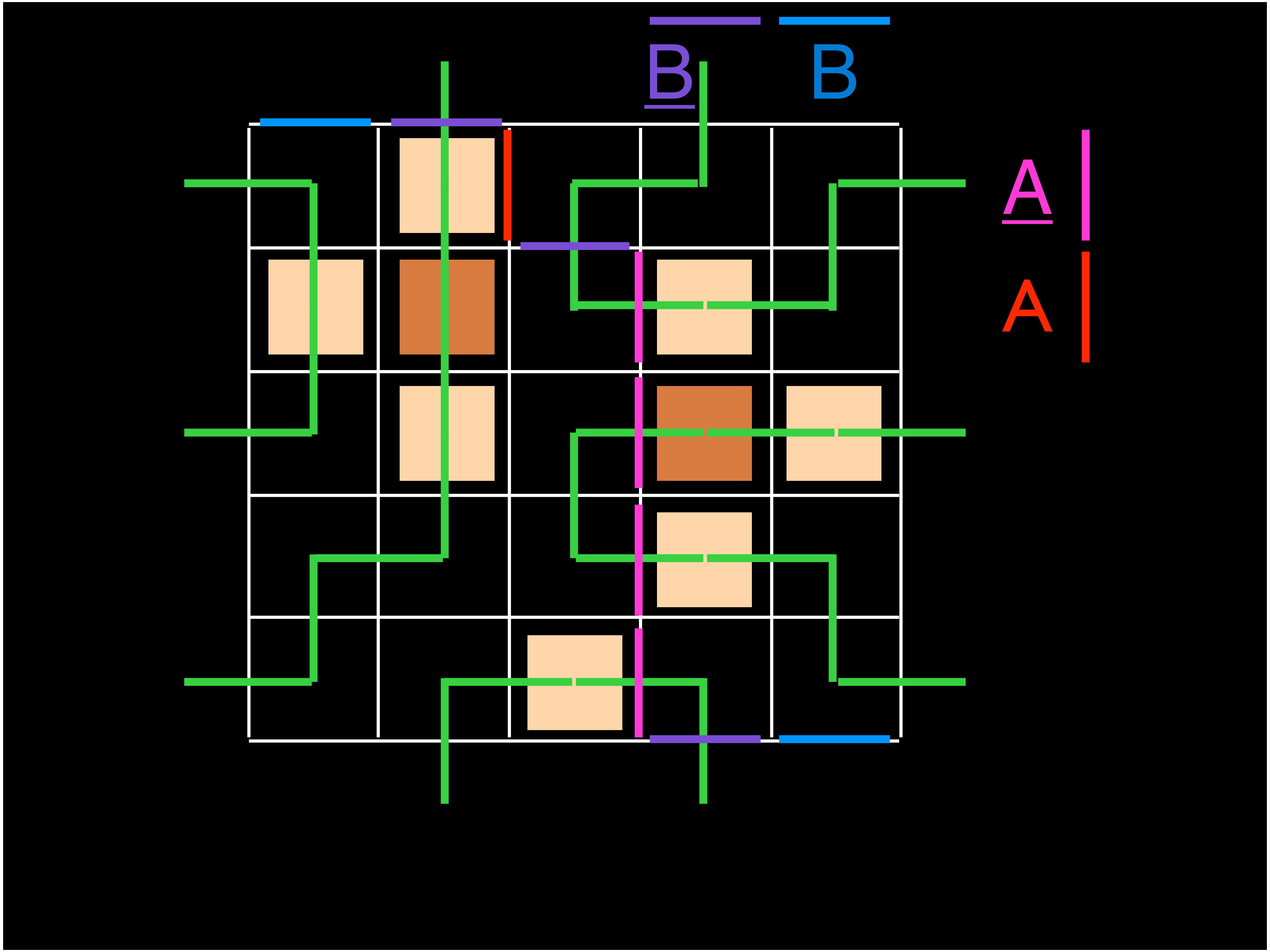


B
B
A
A

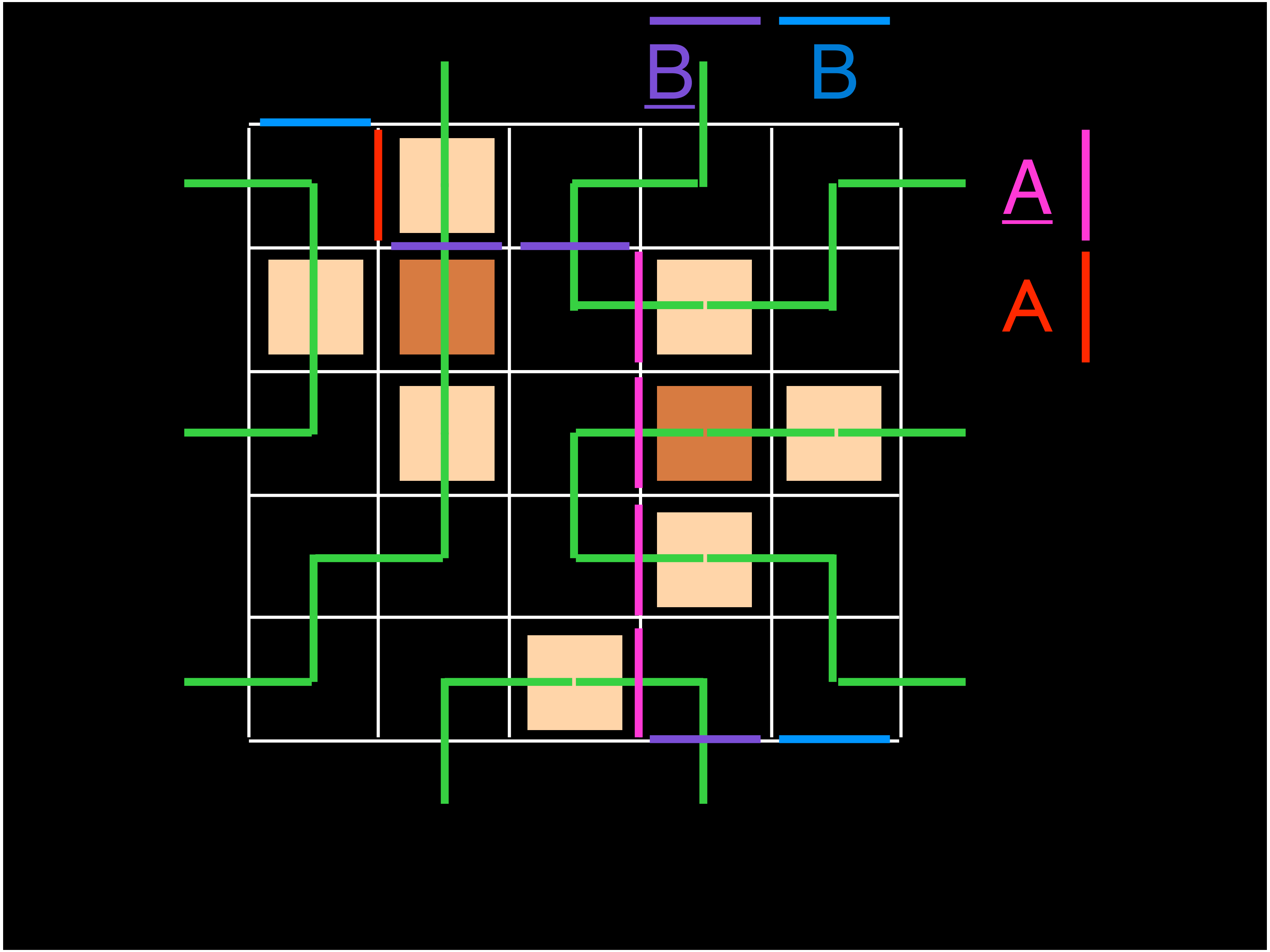

B
B
A
A



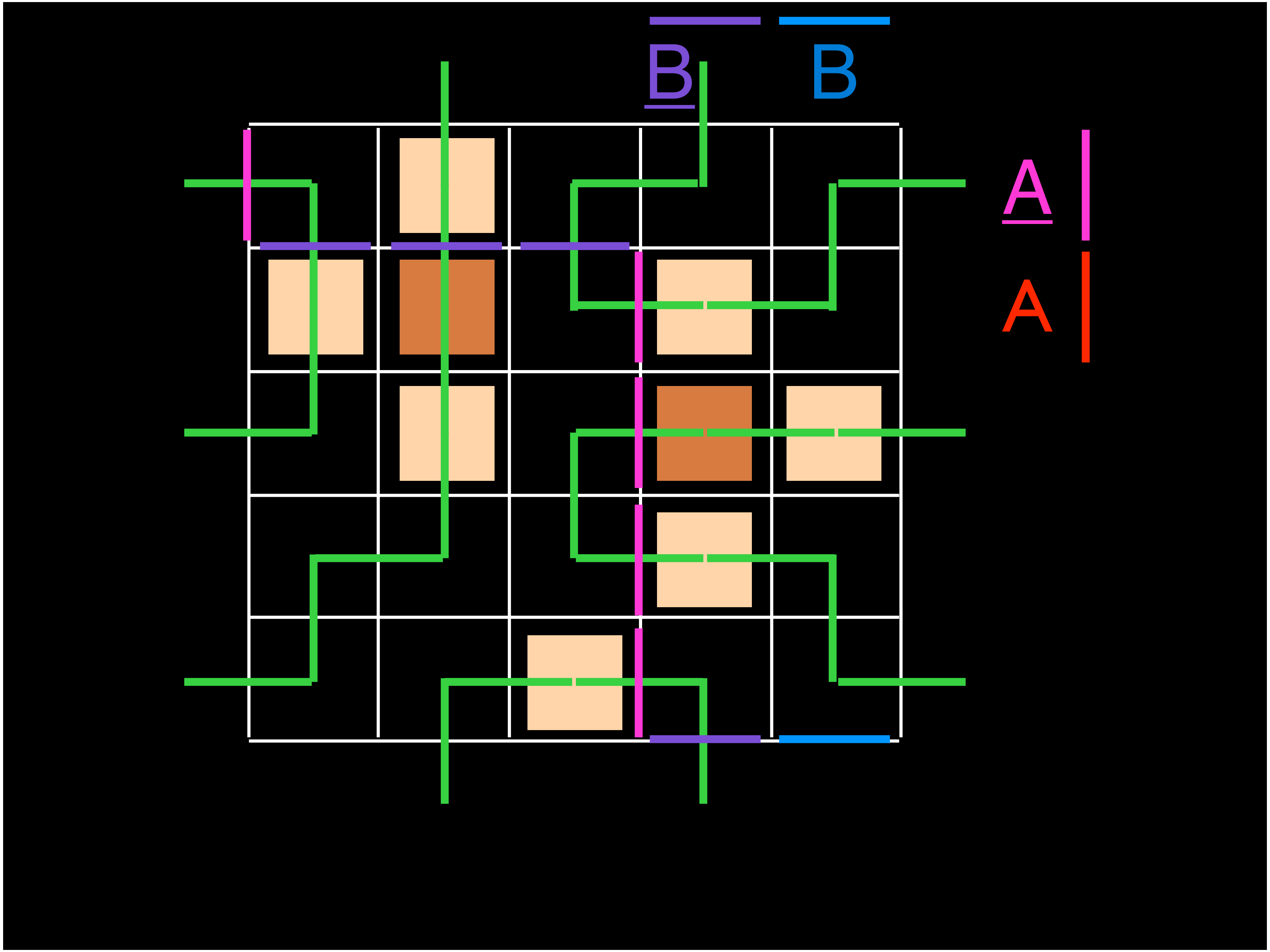
B
B
A
A

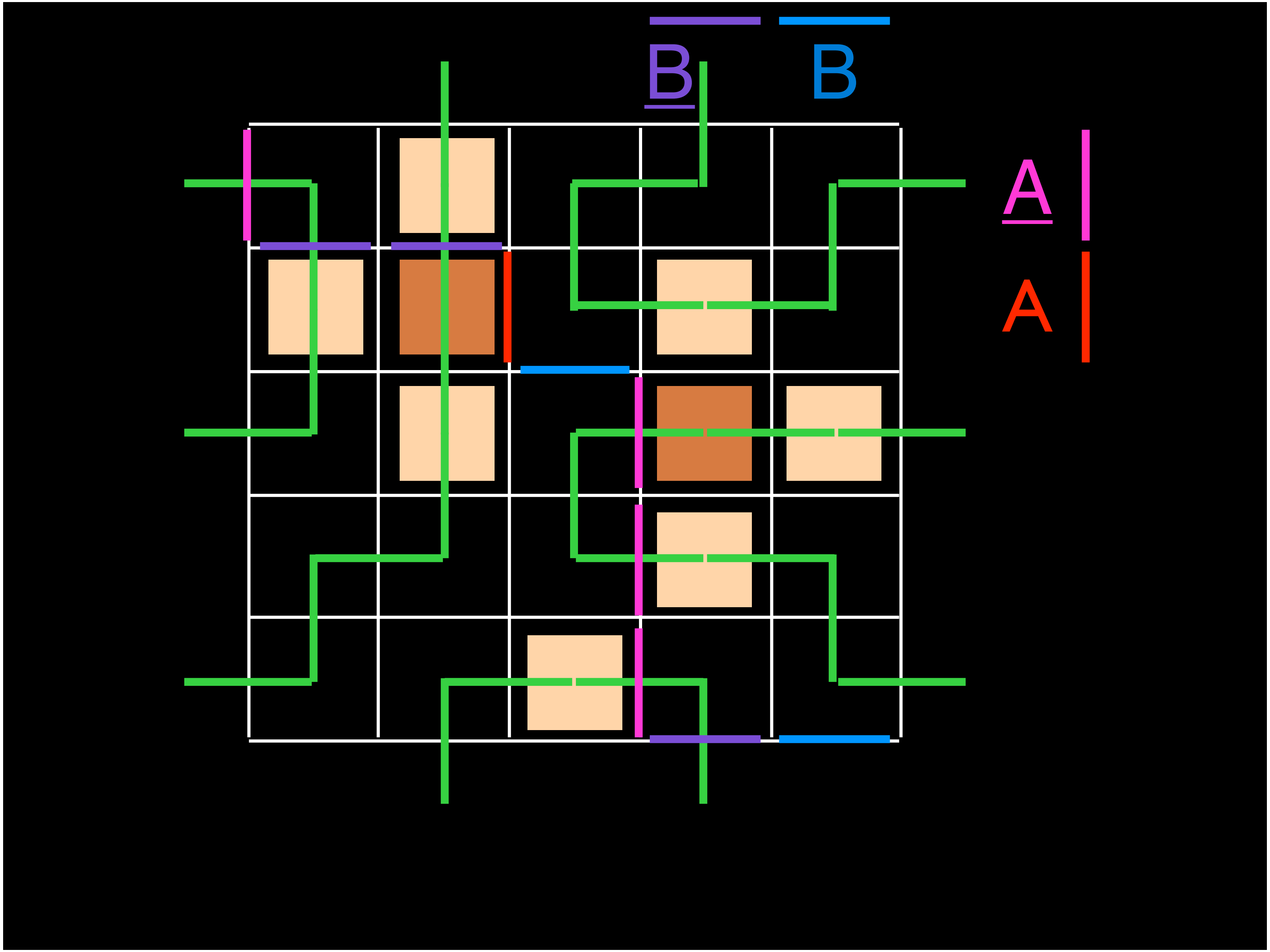


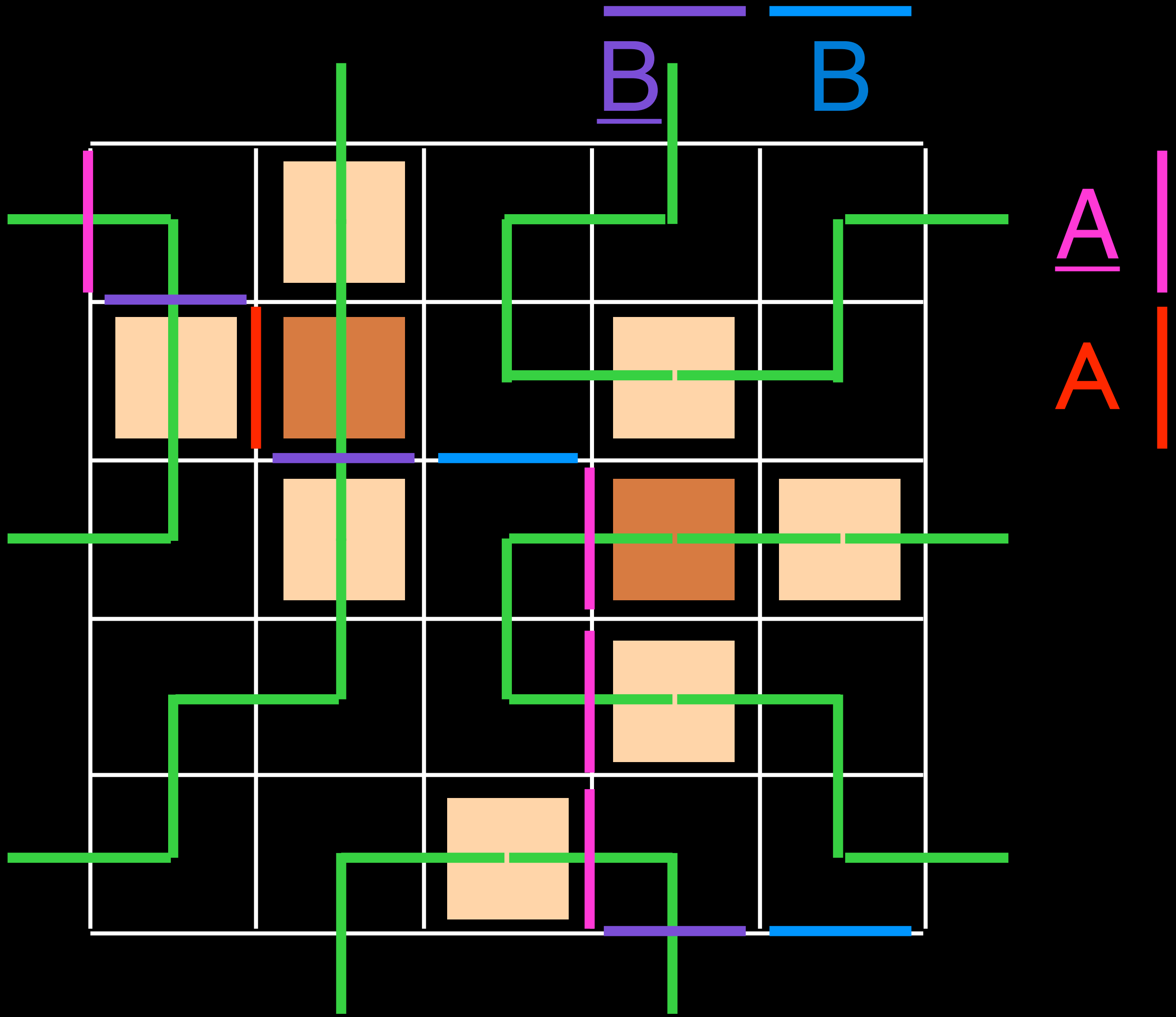
B
B
A
A



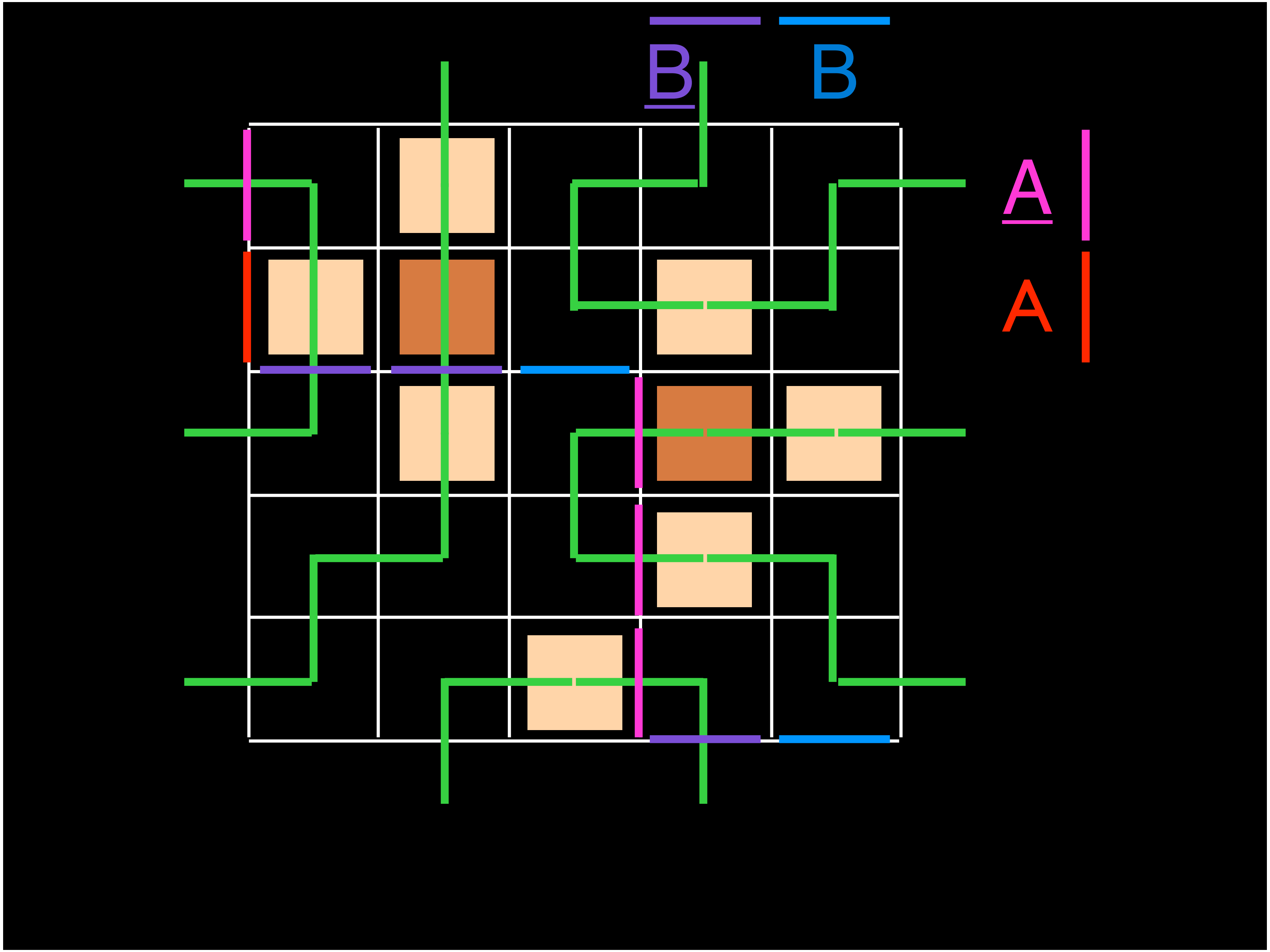
B
B
A
A


B
B
A
A

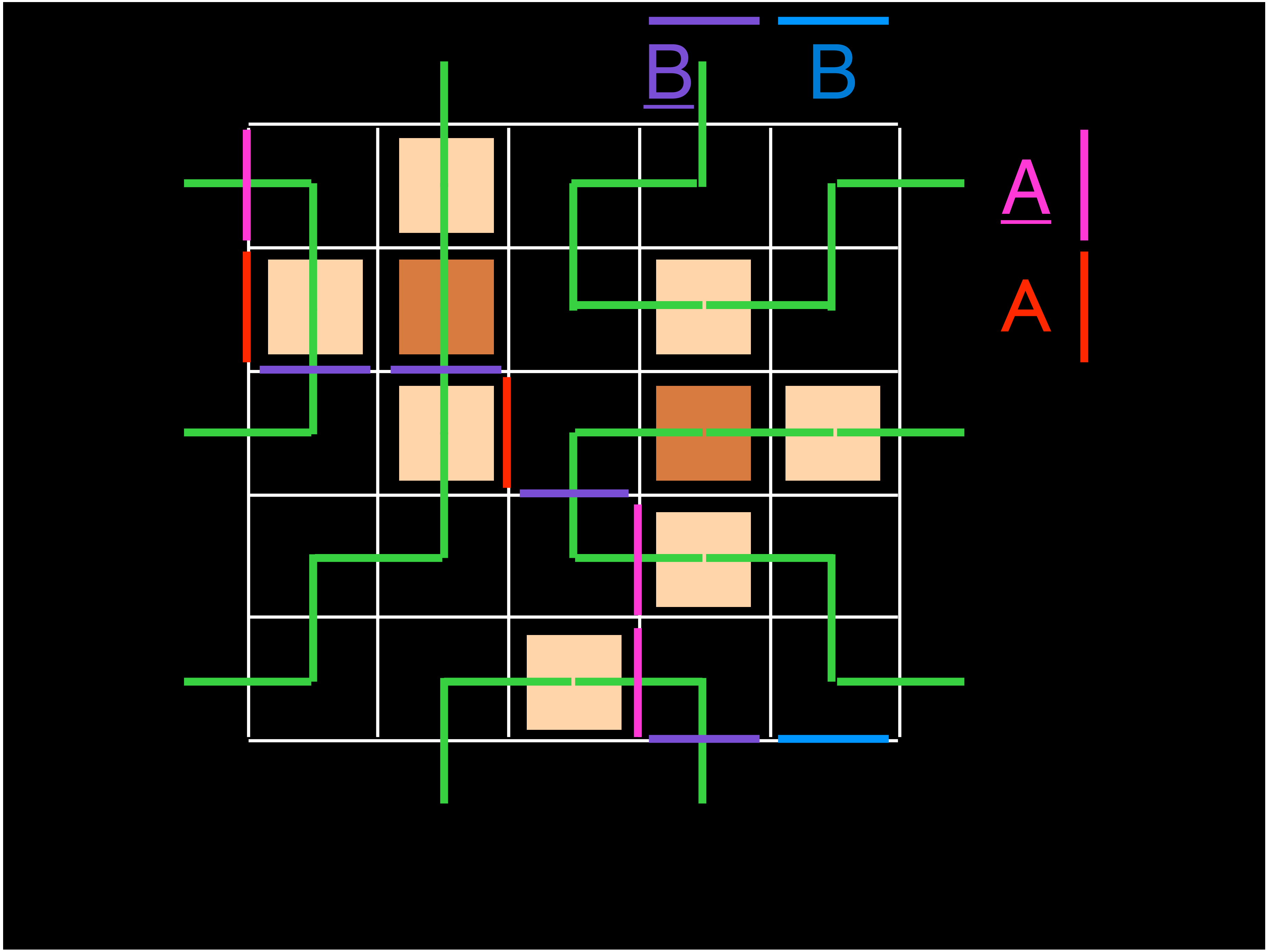

B
B
A
A



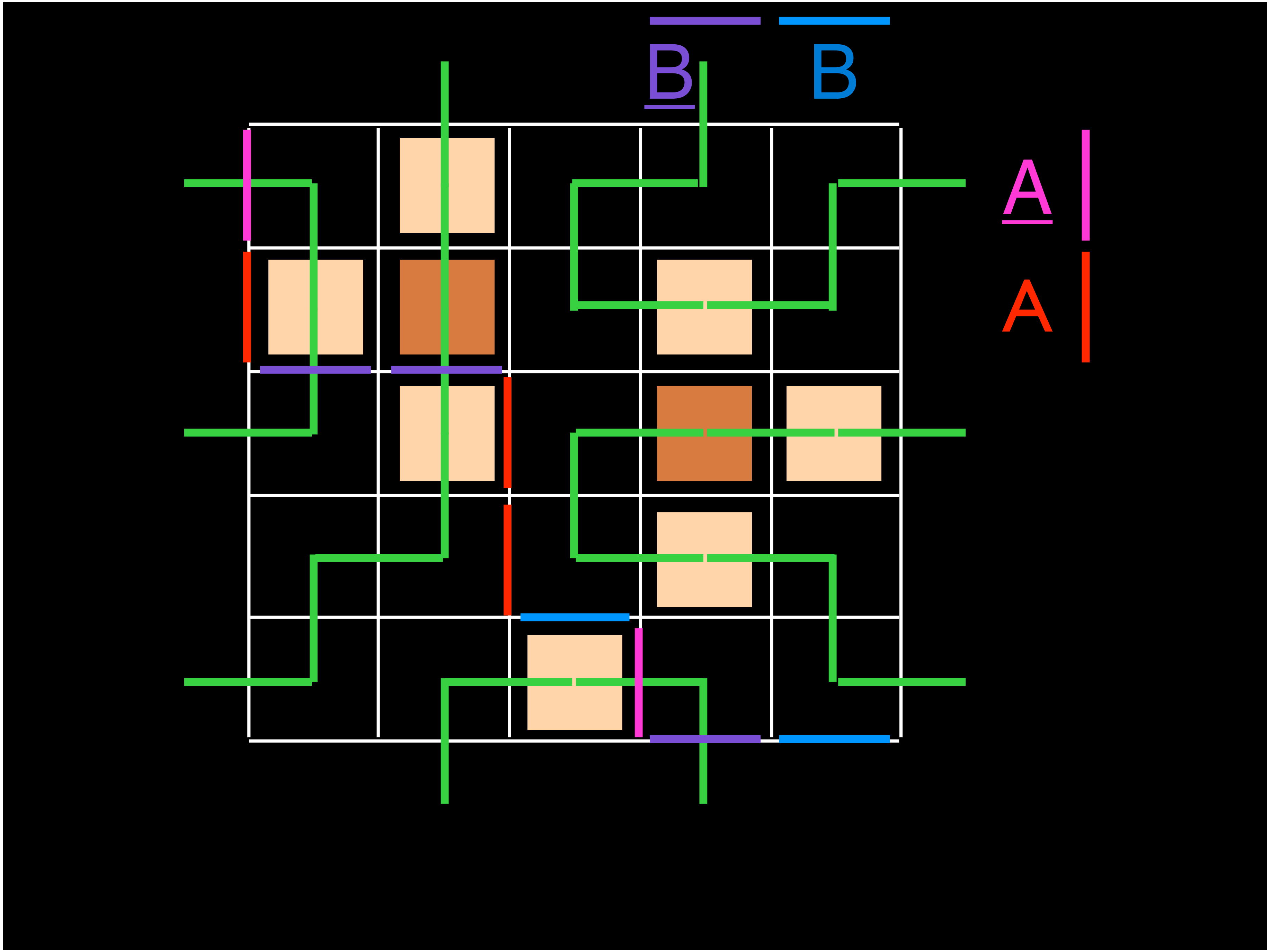
B
B
A
A



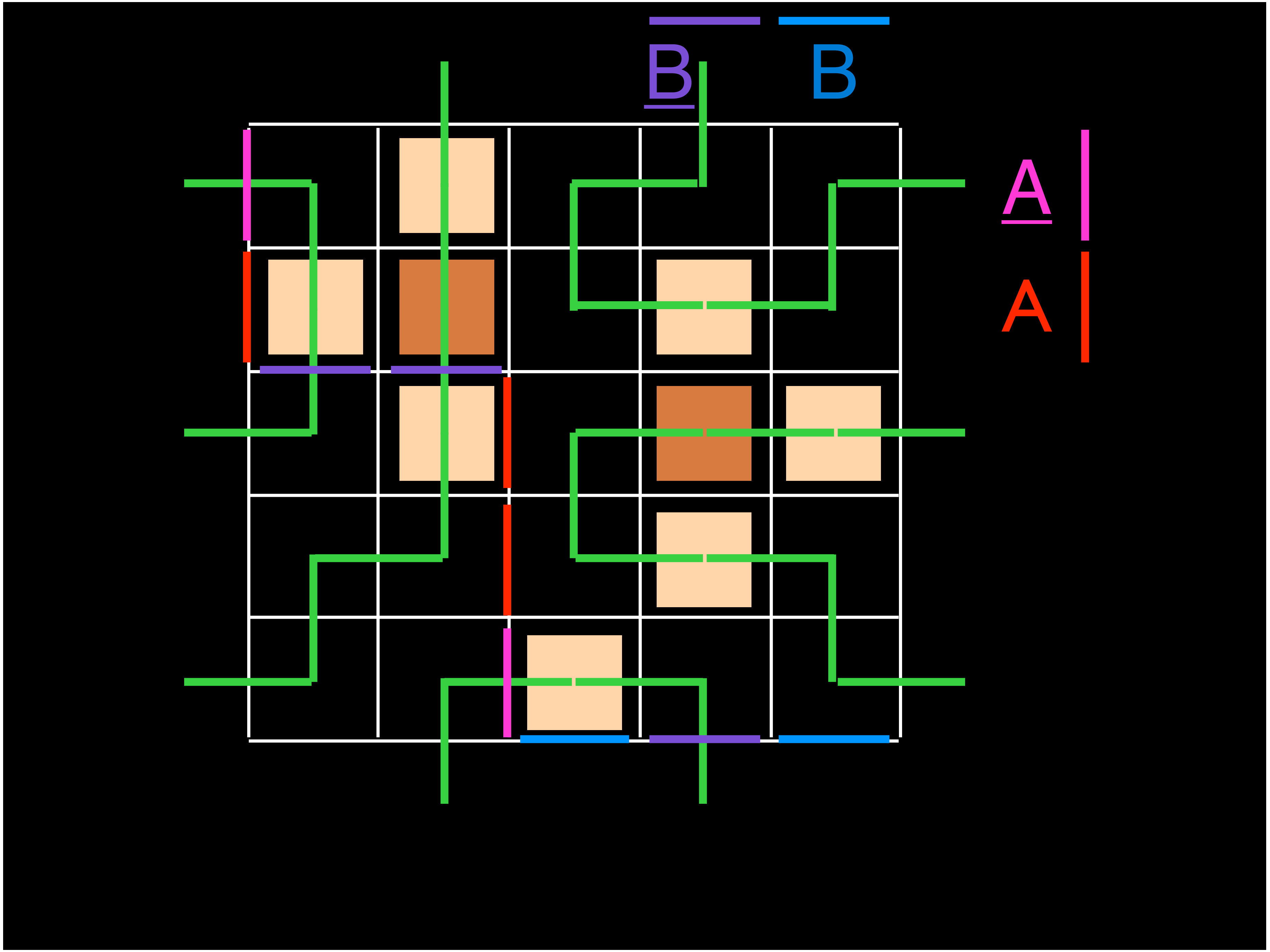
B
B
A
A



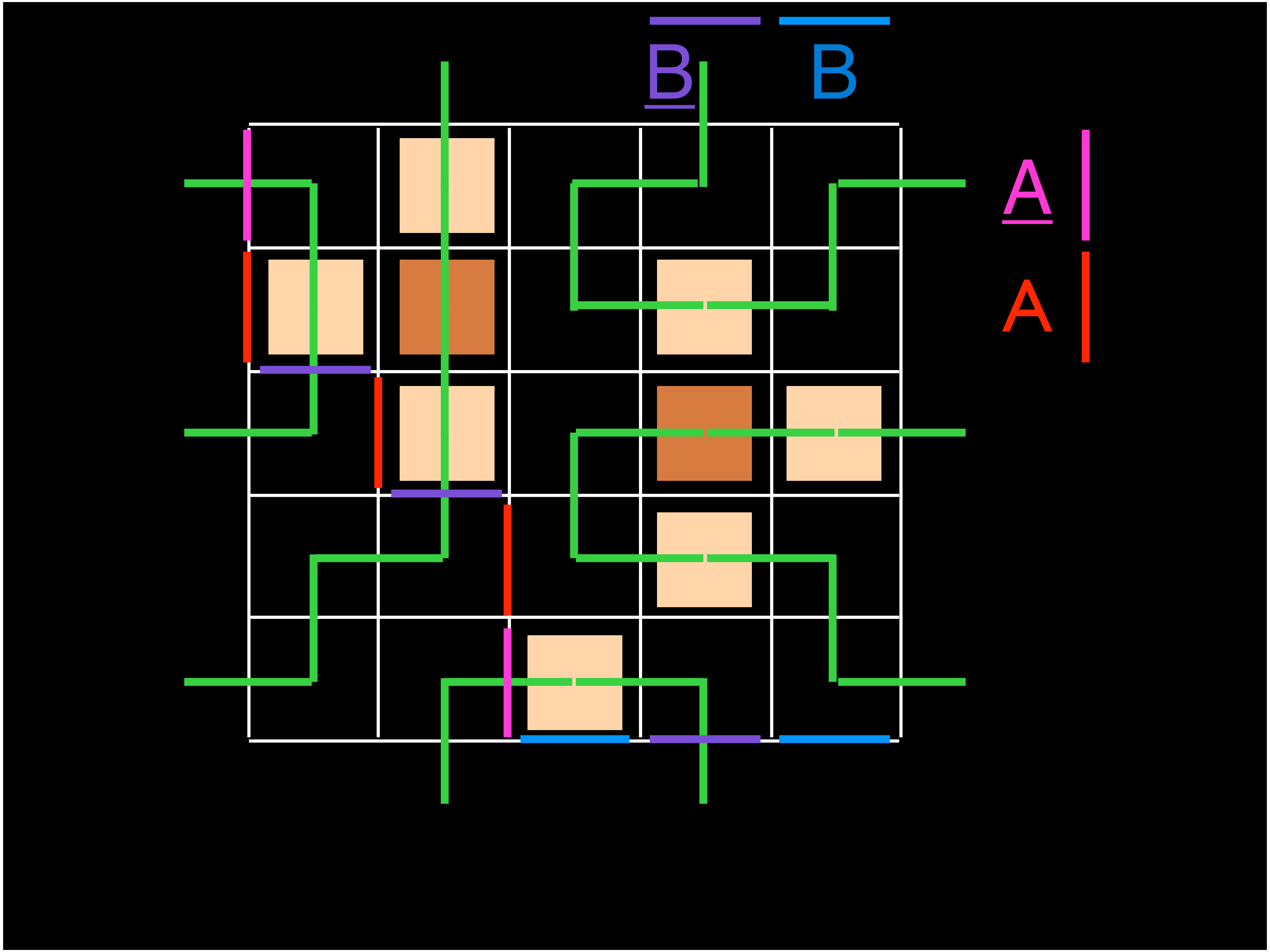
B̲
B
A̲
A

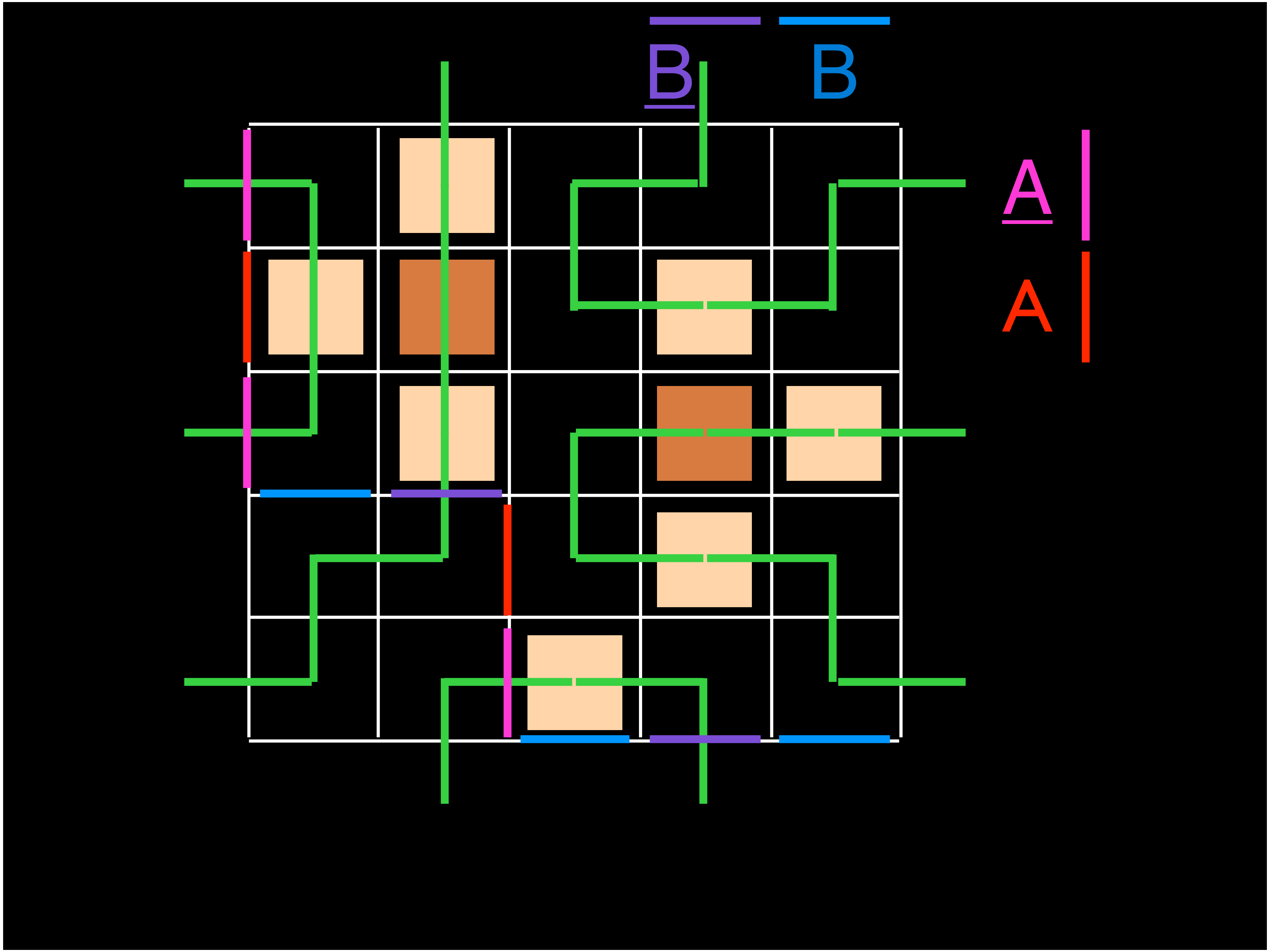


B
B
A
A

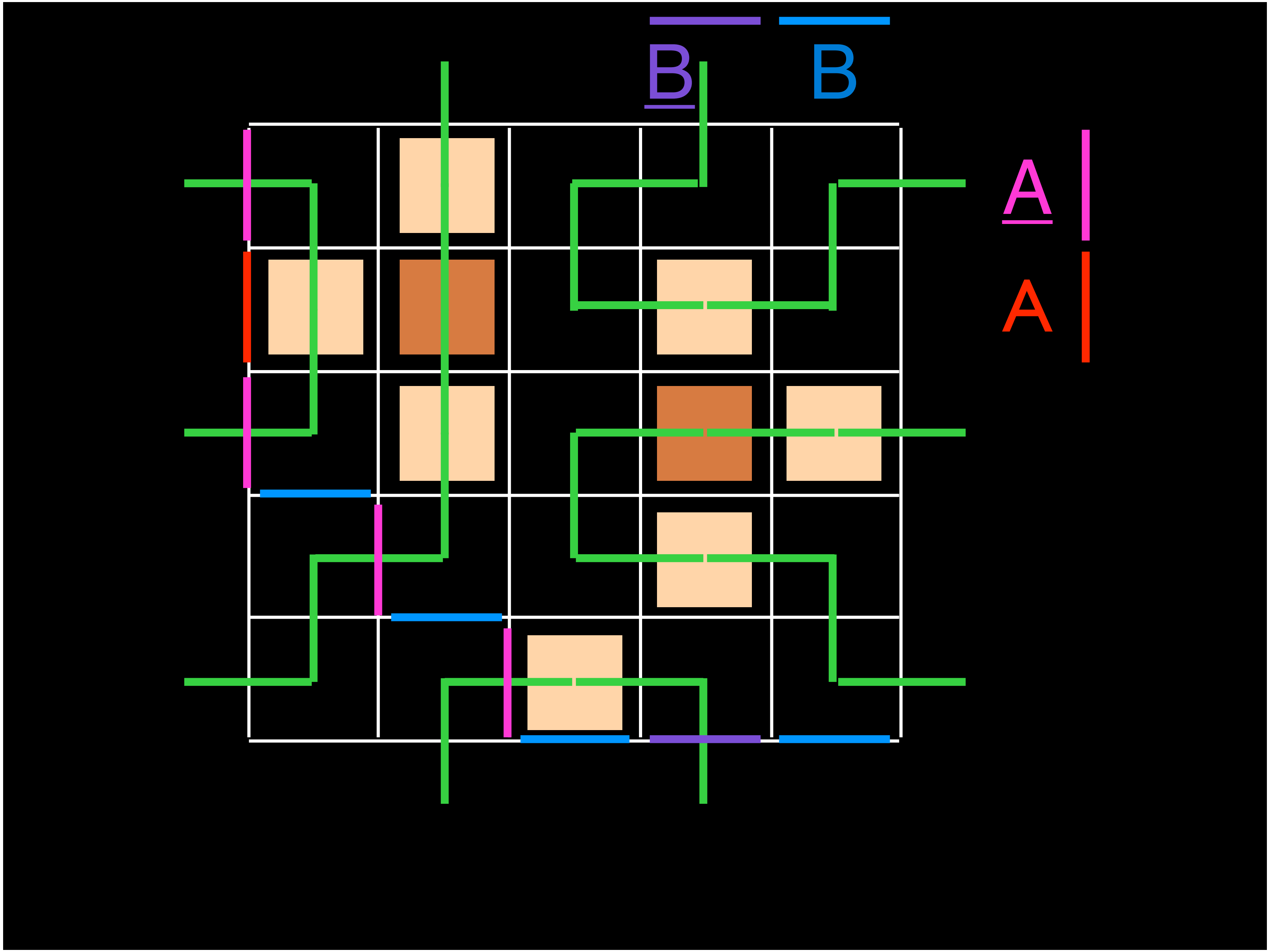


B̲
B
A̲
A

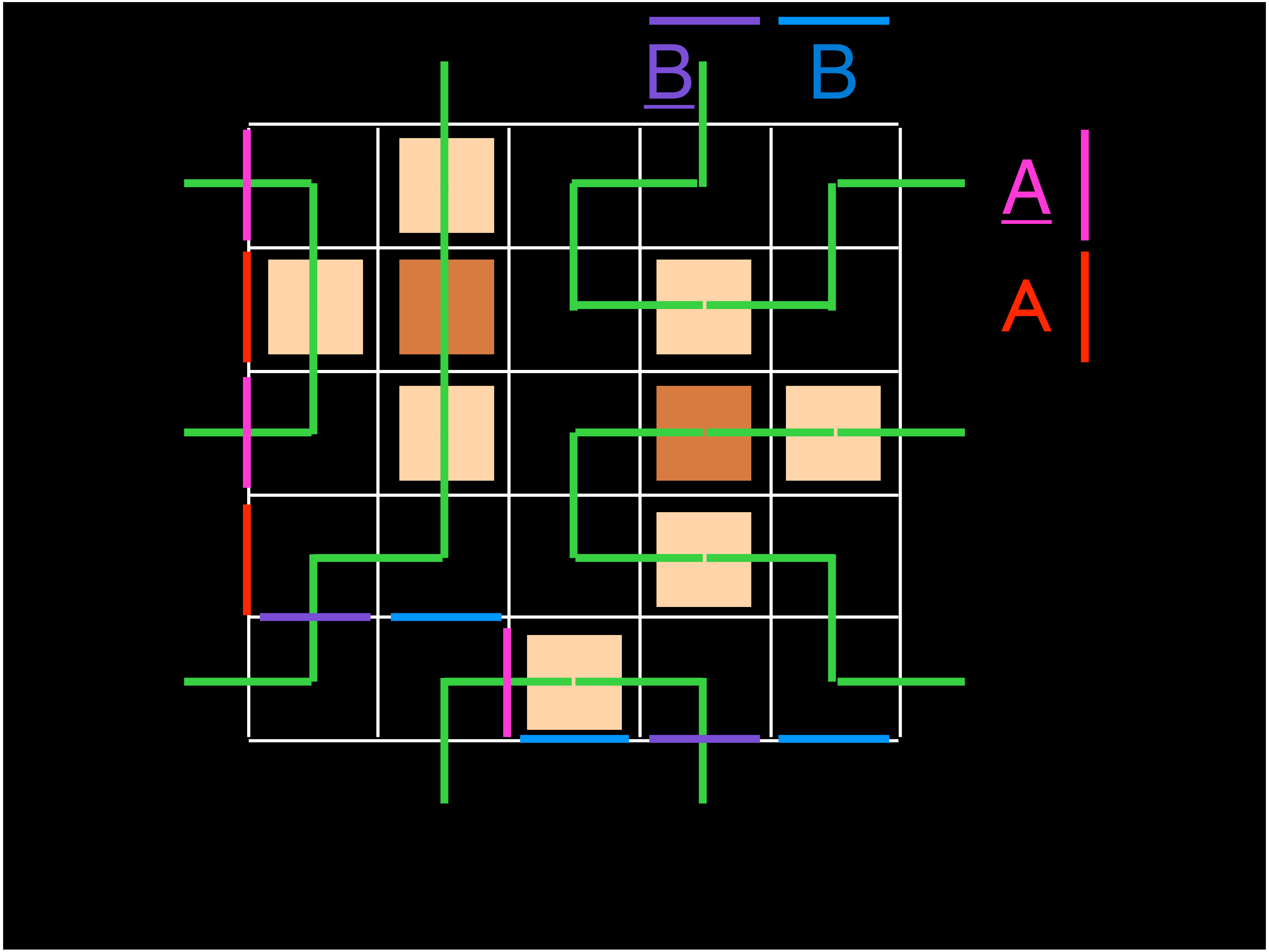

B
B
A
A

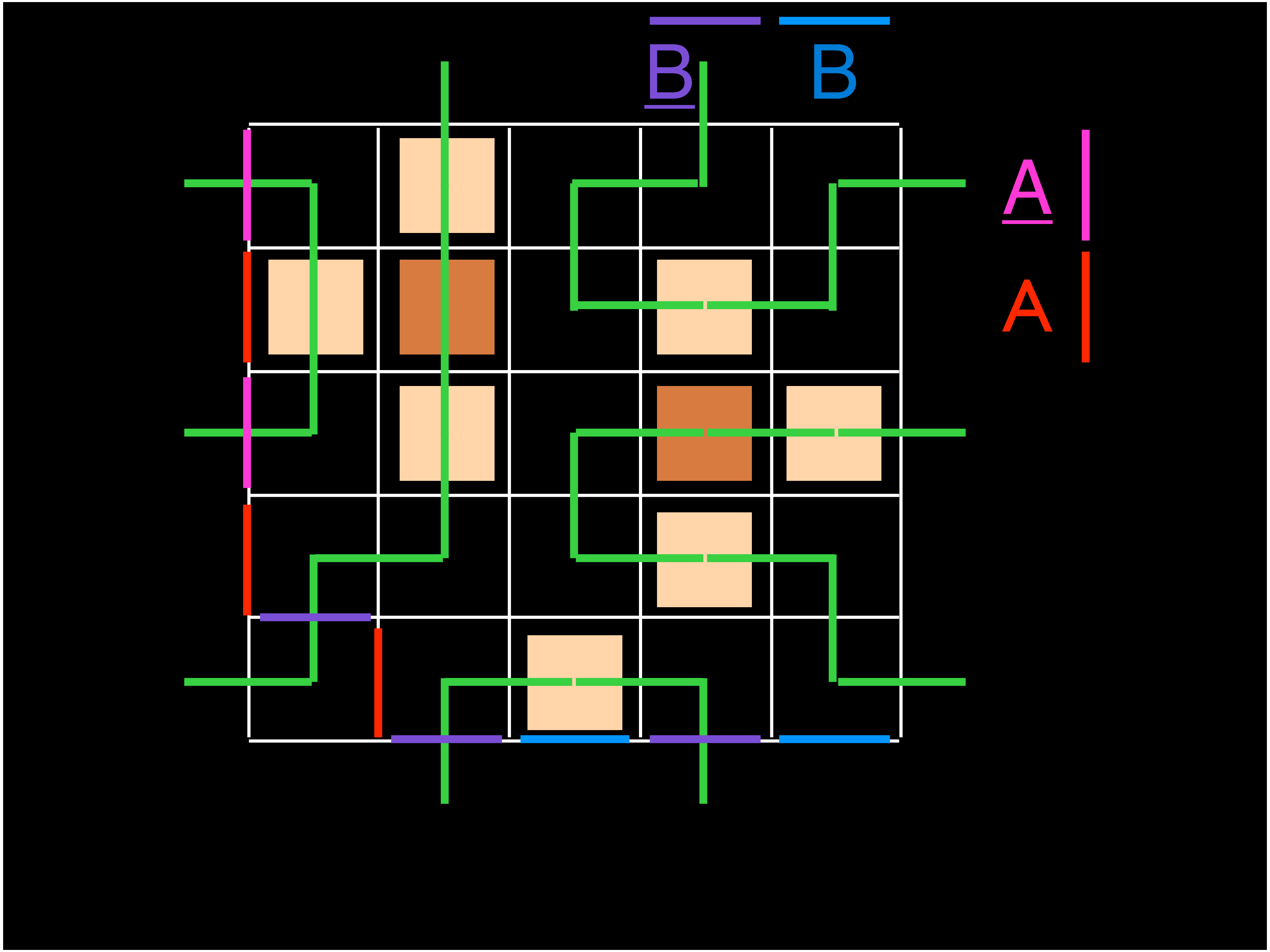


B
B
A
A

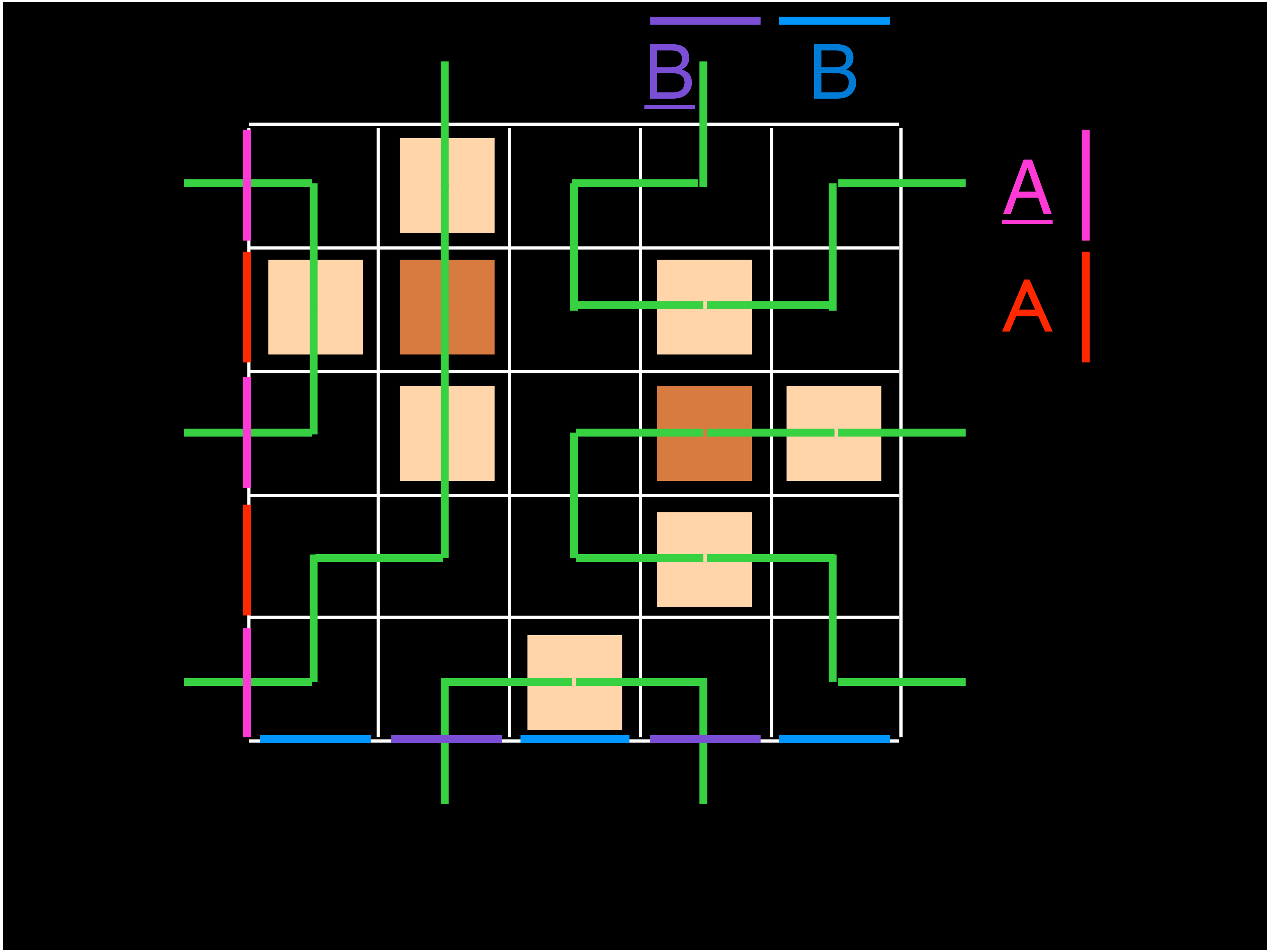


B
B
A
A

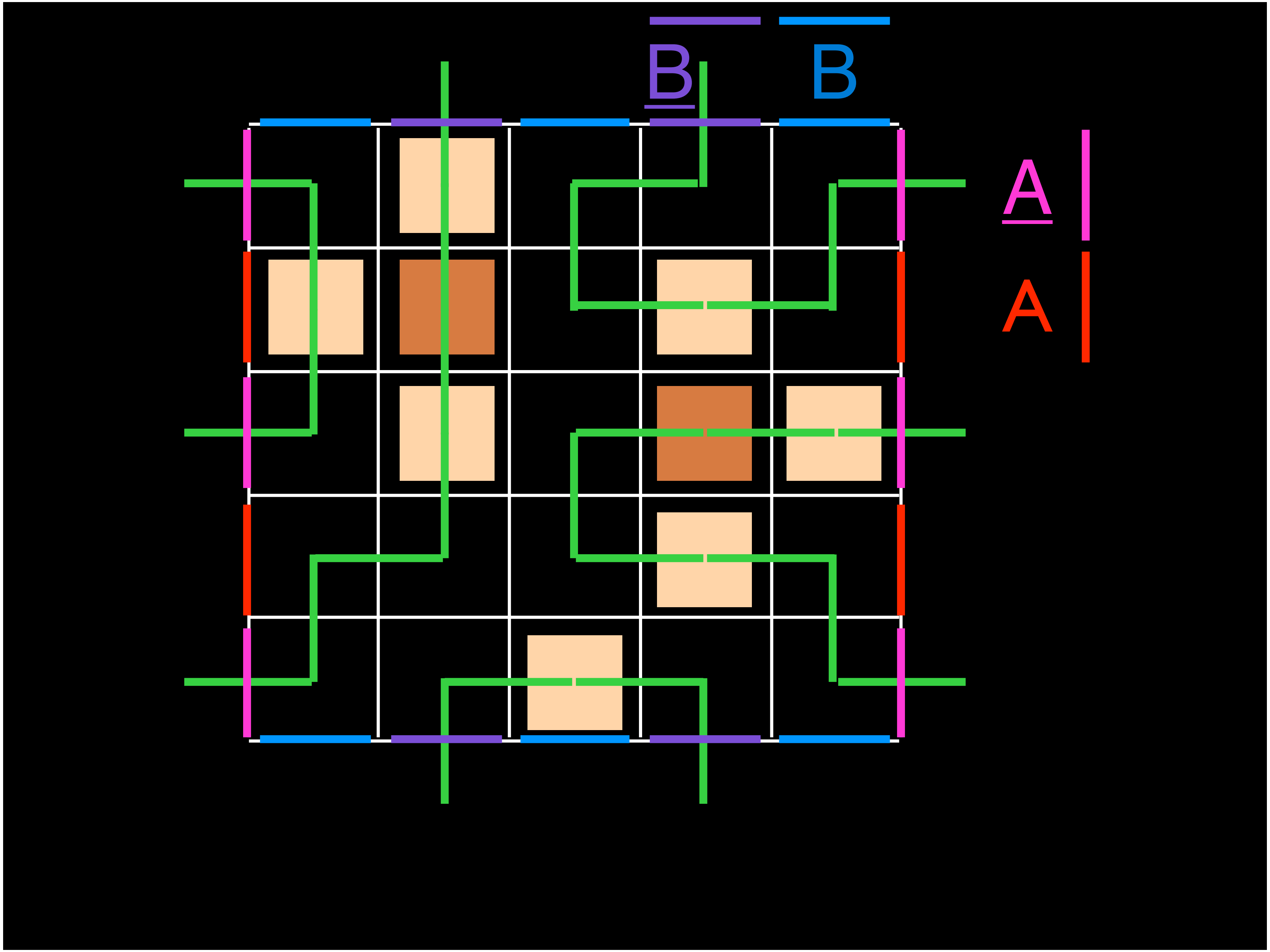

B
B
A
A



B
B
A
A



B
B
A
A


B̄
B
Ā
A

$$A,\ \bar{A},\ B,\ \bar{B}$$

$$\begin{cases} B\bar{A} = \bar{A}B + A\bar{B} \\ \bar{B}A = A\bar{B} + \bar{A}B \end{cases}$$

$$\begin{cases} BA = \bar{A}\bar{B} \\ \bar{B}\bar{A} = AB \end{cases}$$

$$B \rightarrow b \qquad \bar{A} \rightarrow a$$

$$\bar{B} \rightarrow b' \qquad A \rightarrow a'$$

$$\begin{cases} ba = ab + a'b' \\ b'a' = a'b' + ab \end{cases} \qquad \begin{cases} ba' = ab' \\ b'a = a'b \end{cases}$$

Lemma. Any word $w(A, \bar{A}, B, \bar{B})$ in letters $A, \bar{A}, B, \bar{B}$ can be uniquely written

$$\sum_{u,v} c(u,v;w)\, \underbrace{u(A,\bar{A})}_{\substack{\text{word}\\ \text{in } A,\bar{A}}}\ \underbrace{v(B,\bar{B})}_{\substack{\text{word}\\ \text{in } B,\bar{B}}}$$

Prop • For n even,

let $$w = \underbrace{B\bar{B} \dots B\bar{B}}_{n} \underbrace{A\bar{A} \dots A\bar{A}}_{n}$$

$$u = \underbrace{\bar{A}A \dots \bar{A}A}_{n} \qquad v = \underbrace{\bar{B}B \dots \bar{B}B}_{n}$$

• For n odd,

let $$w = \underbrace{B\bar{B} \dots B\bar{B}B}_{n} \underbrace{\bar{A}A \dots \bar{A}A\bar{A}}_{n}$$

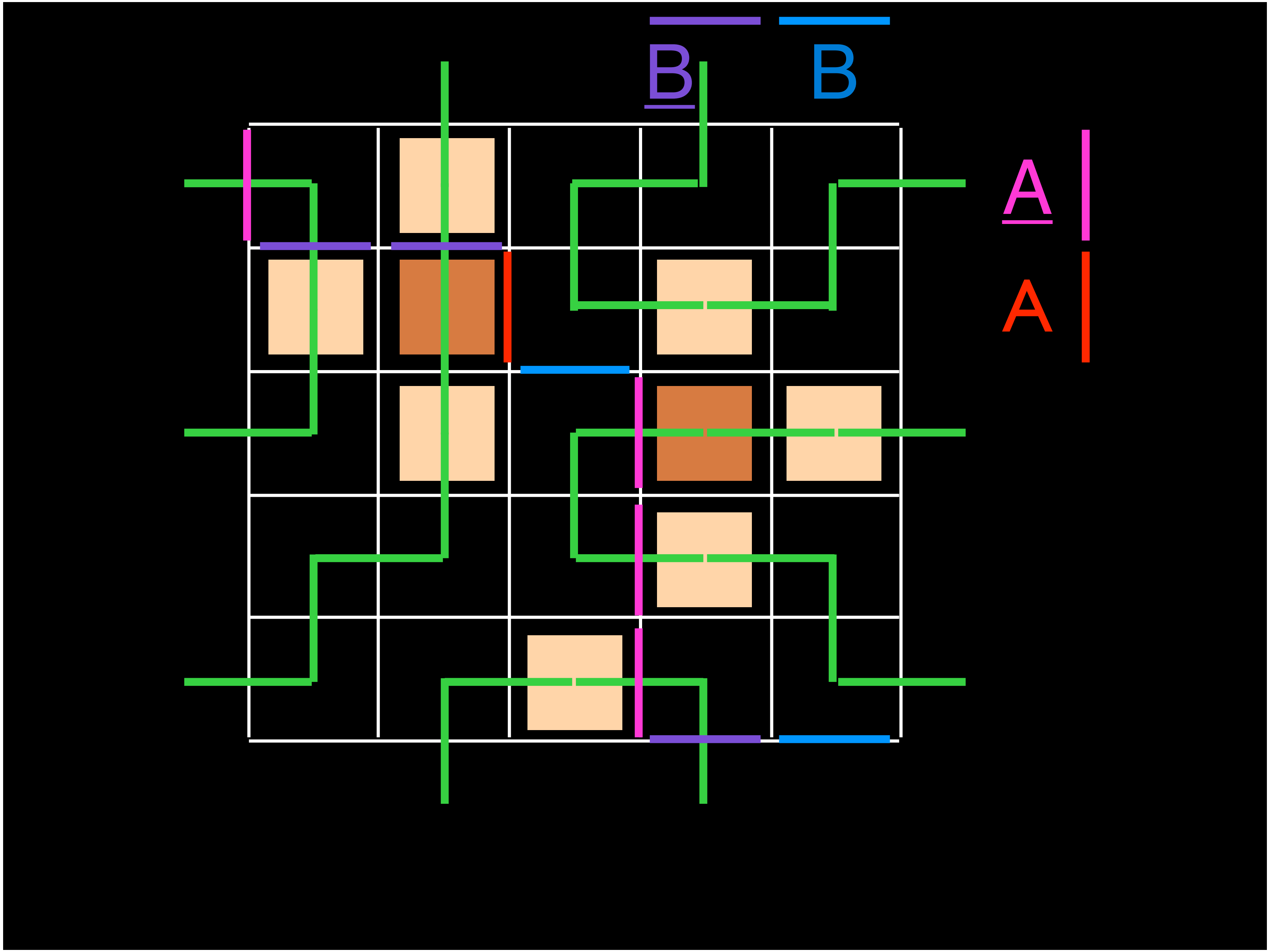
$$u = \underbrace{\bar{A}A \dots \bar{A}A\bar{A}}_{n} \qquad v = \underbrace{B\bar{B} \dots B\bar{B}B}_{n}$$

• then the coefficient $c(u, v; w)$ is equal to the number of **FPL** (fully packed loops) of size n.

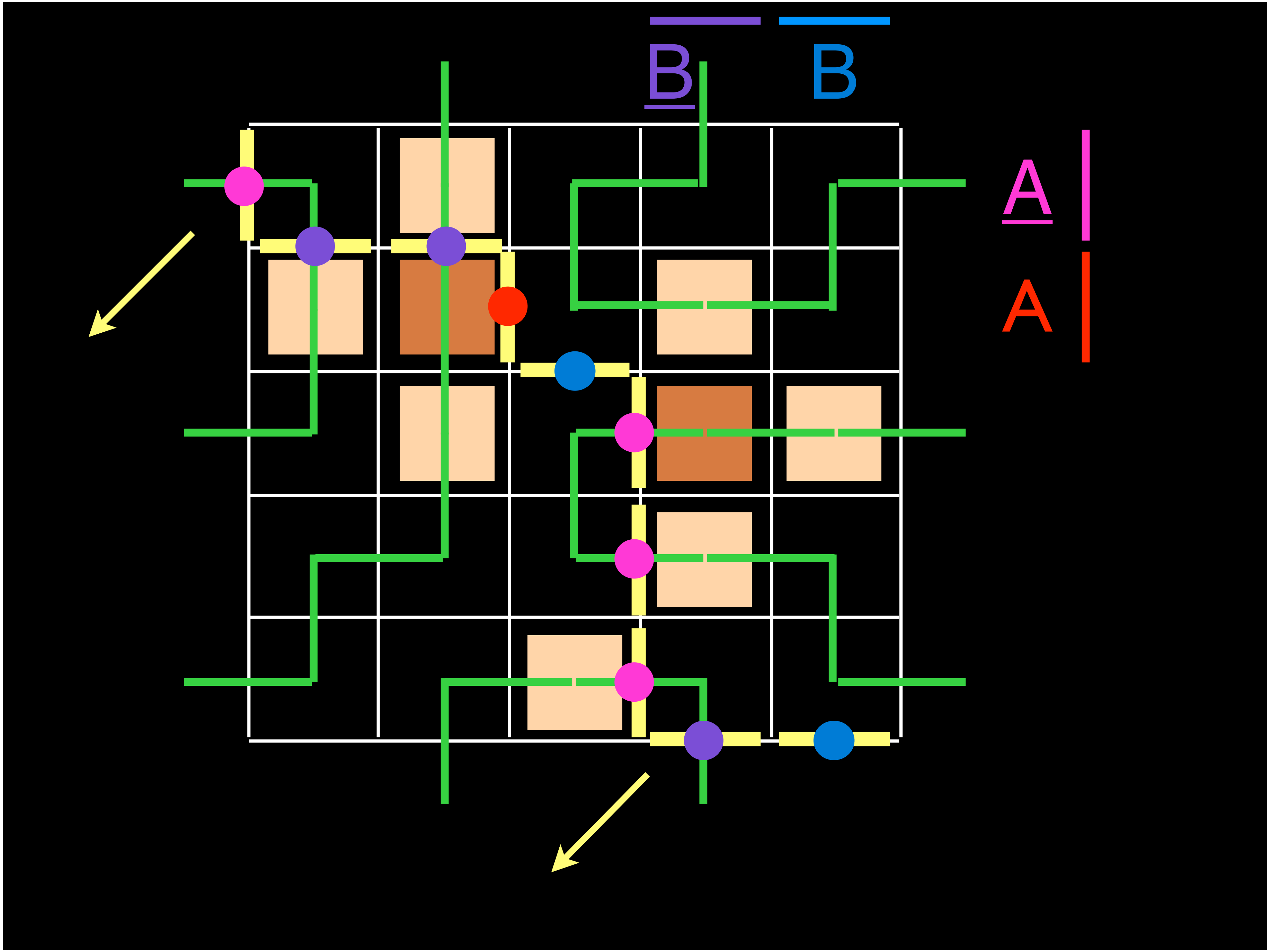
§5 FPL, Temperley-Lieb algebra, heaps of dimers

Kerala, Inde 02 xgv

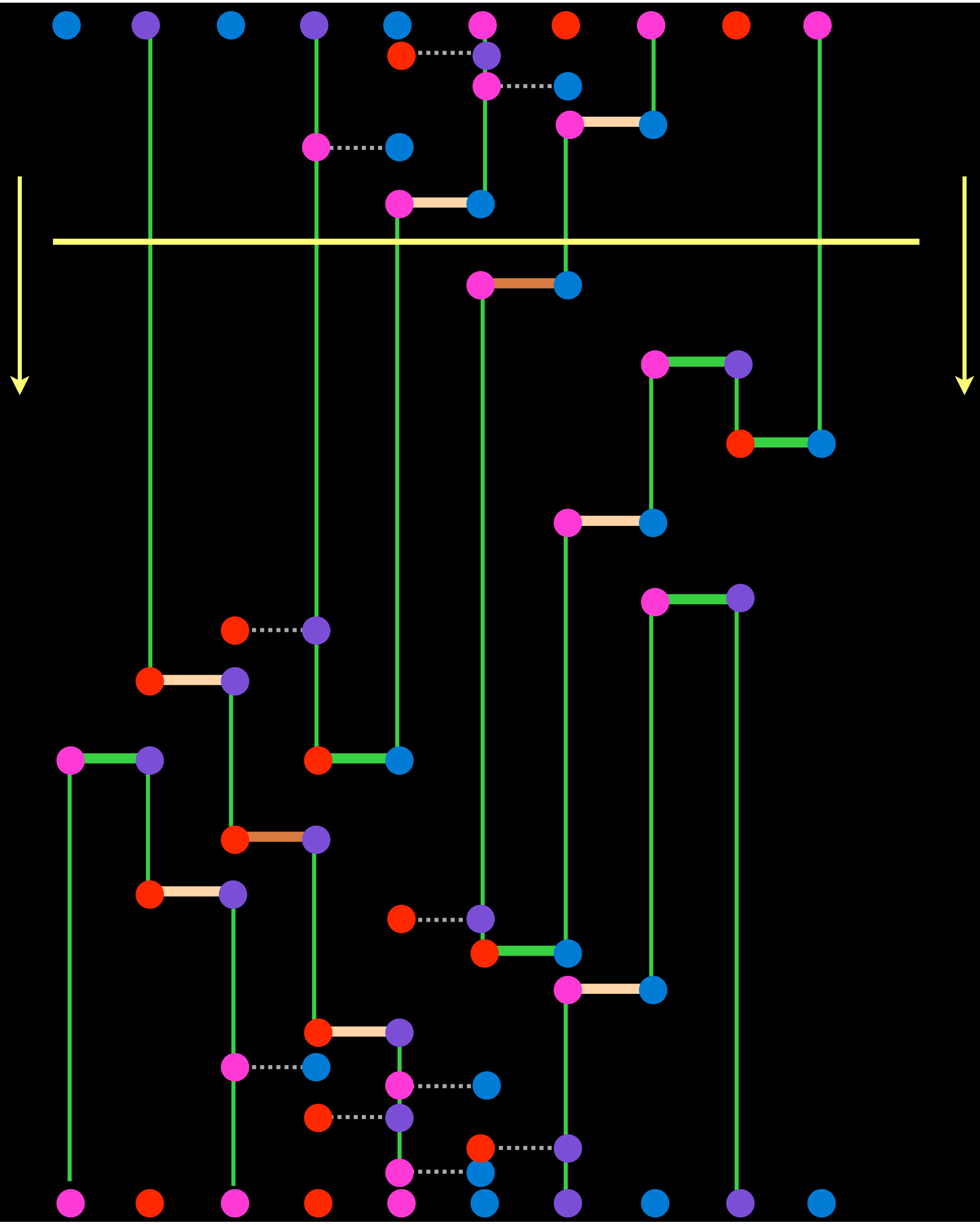
From FPL to "decorated" heaps of dimers with threads



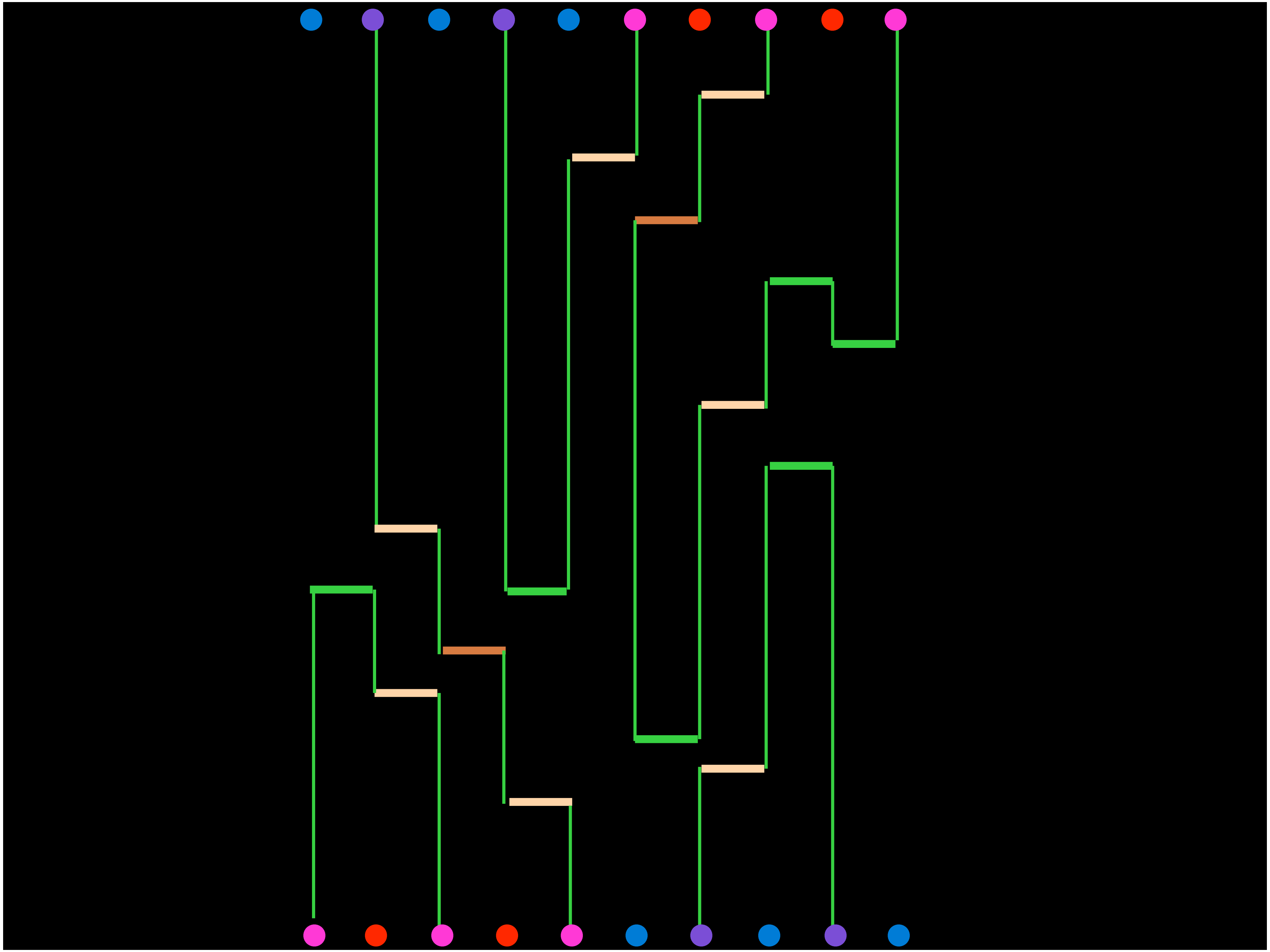
B
B
A
A

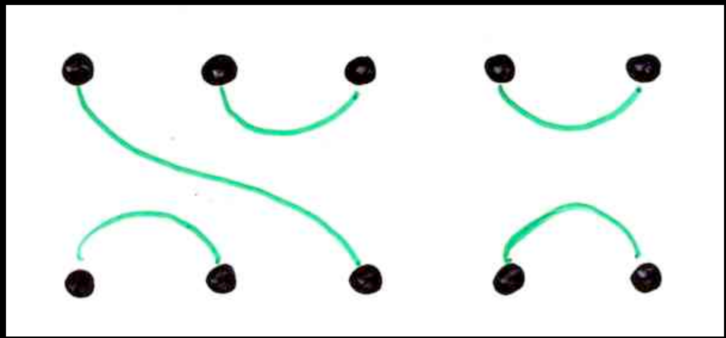


B
B
A
A

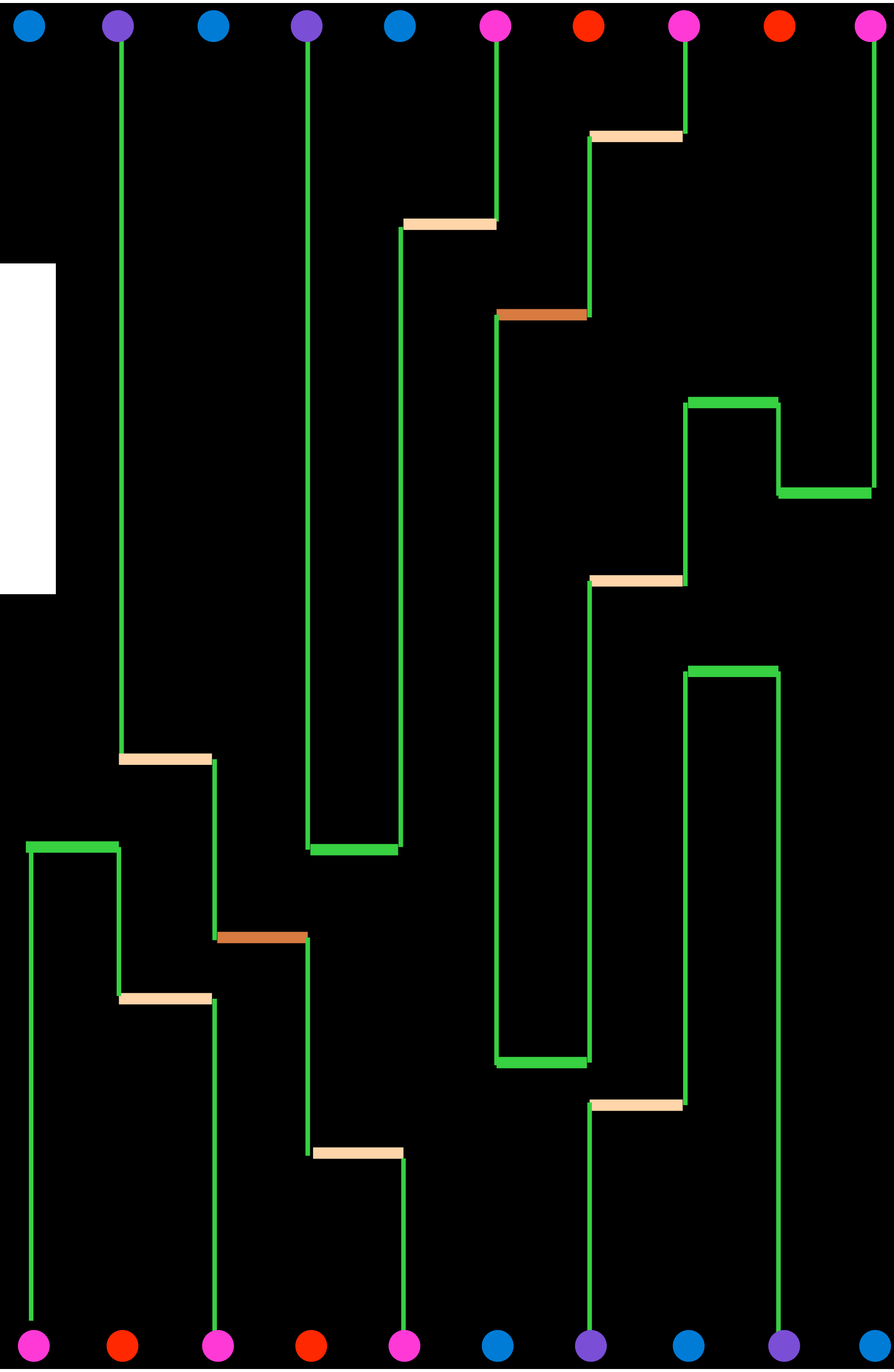


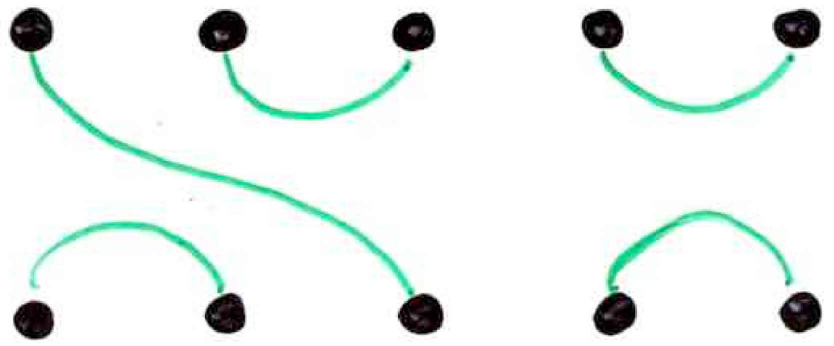
“decorated heaps of dimers” in bijection with FPL



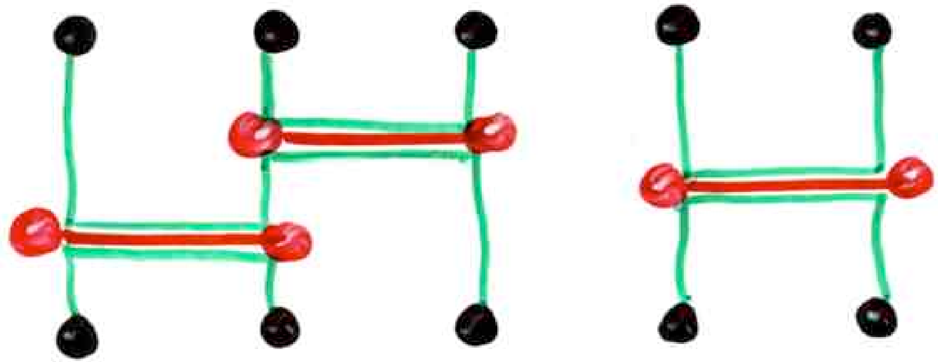


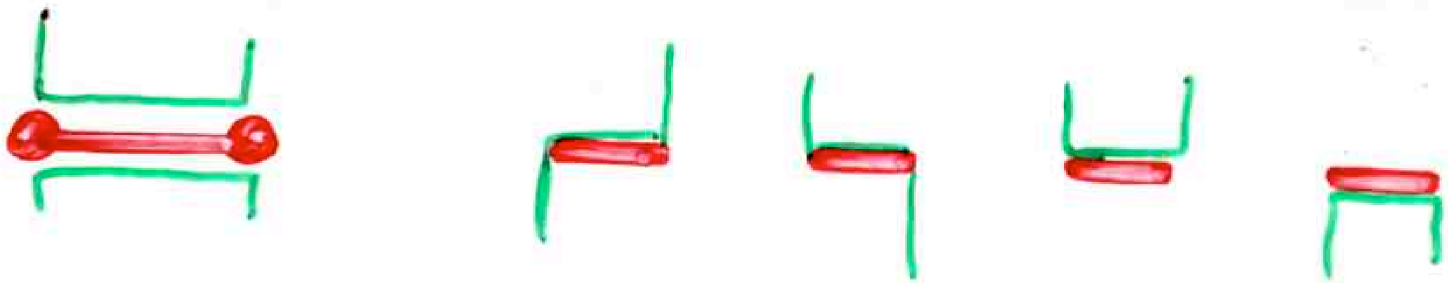
From “decorated heaps of dimers” to element of the Temperley-Lieb algebra





an element of the Temperley-Lieb algebra





“big dimers” and small “dimers”

Tamil Nadu, Inde 02 xgv

inspiration
from
some papers ?

Combinatorial representation

of operators A, A', B, B' ?

or operators A, $\underline{A}$, B, $\underline{B}$?

Shigechi, Uchiyama (2005)
cond-mat/0508090
"Boxed skew plane partition and integrable phase model"

. (7)

he monodromy operators.
obtained from

$$R(u,v)(T(u) \otimes T(v)) = (T(v) \otimes T(u))R(u,v). \tag{8}$$

Several explicit relations are listed in the following.

$$[A(u), A(v)] = [B(u), B(v)] = [C(u), C(v)] = [D(u), D(v)] = 0, \tag{9}$$

$$g(v,u)C(u)A(v) = f(v,u)C(v)A(u), \tag{10}$$

$$g(v,u)D(u)B(v) = f(v,u)D(v)B(u), \tag{11}$$

$$C(u)B(v) = g(u,v)\{A(u)D(v) - A(v)D(u)\} = g(u,v)\{D(u)A(v) - D(v)A(u)\}, \tag{12}$$

$$C(u)A(v) = f(v,u)A(v)C(u) + g(u,v)A(u)C(v), \tag{13}$$

$$D(u)B(v) = f(v,u)B(v)D(u) + g(u,v)B(u)D(v), \tag{14}$$

Note that these commutation relations can be reduced from those of the q-boson model.

2.2 Graphical representation of operators

The monodromy operators A, B, C and D are expressed as the linear combination of products of elements of L-operators. For later convenience, we introduce a graphical representation of the L-operator and the monodromy matrix in this subsection.

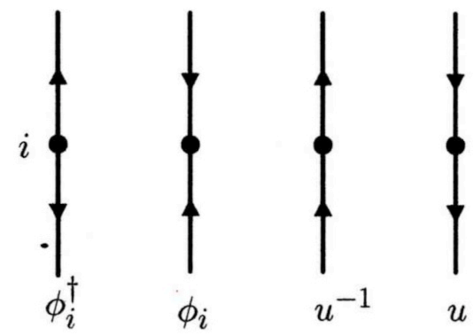


Figure 1: Vertex representation for the elements of L-operator.

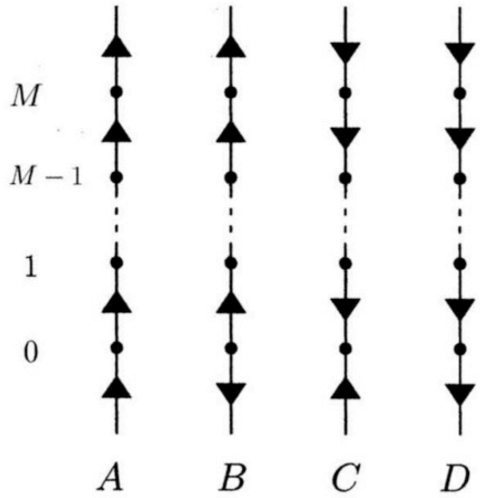


Figure 2: Basic arrow configurations of the operators A, B, C and D.

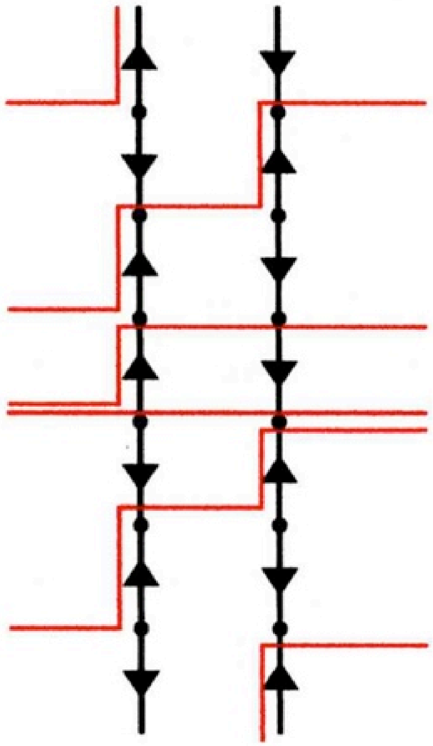


Figure 3: An example of a lattice path configuration corresponding to an arrow configuration.

$$\pi = \begin{pmatrix} & 3 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

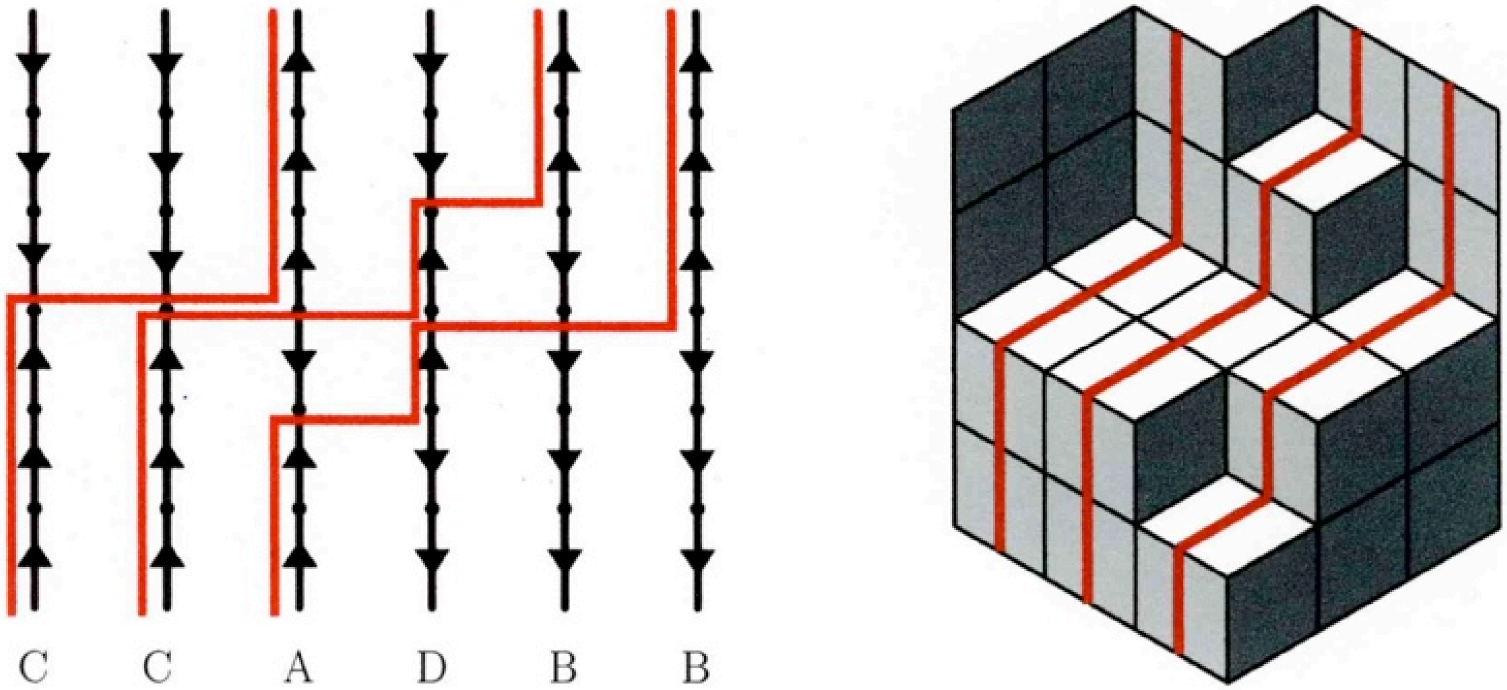


Figure 6: An example of identification of a skew plane partition with a lattice path configuration and a lozenge tiling.

Bogoliubov (2005) cond-mat/0503748

"Boxed plane partitions as an exactly solvable Boson model"

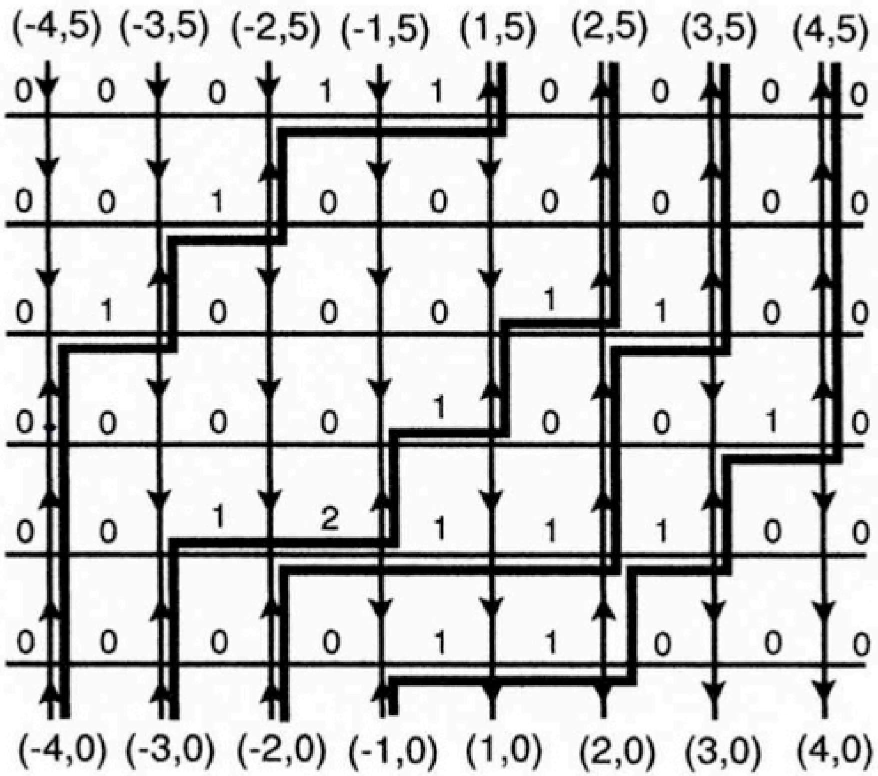


FIG. 6: A typical configuration of admissible lattice paths.

Taking the trace of the monodromy matrix over the auxiliary space $\mathcal{A}$, we obtain a one-parameter family of *transfer* matrices acting on $\mathcal{S}$

$$t(\lambda) = \mathrm{Tr}_{\mathcal{A}}(T(\lambda)) = A(\lambda) + D(\lambda)\,. \tag{59}$$

By taking the trace of equation (58) over $\mathcal{A} \otimes \mathcal{A}'$ and using the fact that $\mathcal{R}(\nu)$ is generically an invertible matrix, we deduce that the operators $t(\lambda)$ form a family of commuting operators (see e.g., Nepomechie, 1999). In particular, this family contains the translation operator $T = t(0)$ (that shifts all the particles simultaneously one site forward) and the Markov matrix $M = t'(0)/t(0)$.

The equation (58), when written explicitly, leads to 16 quadratic relations between the operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$ and the operators $A(\mu), B(\mu), C(\mu), D(\mu)$. In particular, we have

$$C(\lambda)C(\mu) = C(\mu)C(\lambda)\,, \tag{60}$$

$$A(\lambda)C(\mu) = (1 - q\nu)A(\mu)C(\lambda) + p\nu C(\mu)A(\lambda)\,, \tag{61}$$

$$D(\lambda)C(\mu) = p\nu C(\mu)D(\lambda) + (1 - p\nu)D(\mu)C(\lambda)\,. \tag{62}$$

These relations are used in the next subsection to construct the eigenvectors of the family of transfer matrices $t(\lambda)$.

Golinelli, Mallick (2006)

cond-mat/0611701

"The asymmetric simple exclusion process: an integrable model for non-equilibrium statistical mechanics"

Essler, Rittenberg (1995)

cond-mat/9506131

"Representation of the quadratic algebra and partially asymmetric diffusion with open boundaries"

Fock representation

of the quadratic algebra

$$x_1 A^2 + x_2 AB + x_3 BA + x_4 B^2 = x_5 A + x_6 B + x_7$$

de Gier, Essler (2005)

$$\frac{\mathrm{d}P_t}{\mathrm{d}t} = MP_t. \tag{1}$$

Here M is the PASEP transition matrix whose eigenvalues have non-positive real parts. The large time behaviour of the PASEP is dominated by the eigenstates of M with the largest real parts of the corresponding eigenvalues.

Bethe's Ansatz: It is well known that the transition matrix M is related to the Hamiltonian H of the open spin-1/2 XXZ quantum spin chain through a similarity transformation $M = -\sqrt{pq}\,U_\lambda H U_\lambda^{-1}$ where H is given by [10]

$$H = -\frac{1}{2}\sum_{j=1}^{L-1}\Big[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y - \Delta\sigma_j^z\sigma_{j+1}^z + h(\sigma_{j+1}^z - \sigma_j^z) + \Delta\Big] + B_1 + B_L. \tag{2}$$

The parameters Δ and h, and the boundary terms $B_{1,L}$ are related to the PASEP transition rates by

$$\Delta = -\frac{1}{2}(Q+Q^{-1}), \quad h = \frac{1}{2}(Q-Q^{-1}), \quad Q = \sqrt{\frac{q}{p}},$$

$$B_L = \frac{\beta+\delta-(\beta-\delta)\sigma_L^z - \frac{2\beta}{\lambda Q^{L-1}}\sigma_L^+ - 2\delta\lambda Q^{L-1}\sigma_L^-}{2\sqrt{pq}},$$

$$B_1 = \frac{\alpha+\gamma+(\alpha-\gamma)\sigma_1^z - 2\alpha\lambda\sigma_1^- - \frac{2\gamma}{\lambda}\sigma_1^+}{2\sqrt{pq}}. \tag{3}$$

Here λ is a free parameter on which the spectrum does not depend and $\sigma_j^\pm = (\sigma_j^x \pm \mathrm{i}\sigma_j^y)/2$.

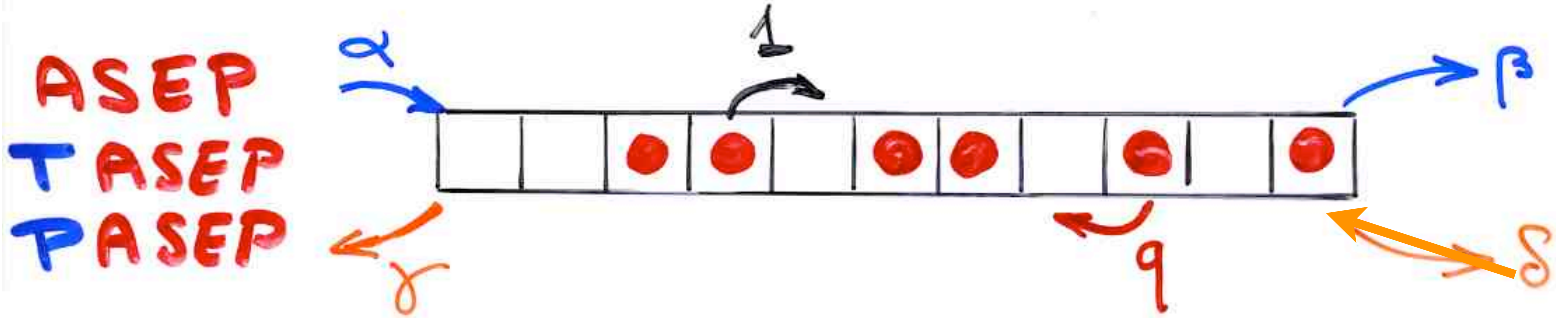
§6 operators for the PASEP

IIT Mumbai, Inde 02 xgv

$$DE = q\,ED + E + D$$

this section §6 “Operators for the PASEP” is a summary of parts of the talk given at the Isaac Newton Institute on 23 April 2008 on
“Alternative tableaux, permutations and asymmetric exclusion process”

for more details see the slides or the video
http://www.newton.cam.ac.uk/ (page “web seminar”



ASEP
TASEP
PASEP
α
1
β
γ
q
δ

$$P_n(\tau_1, \dots, \tau_n) = f_n(\tau_1, \dots, \tau_n) / Z_n$$

$$Z_n = \sum_{\tau} f_n(\tau_1, \dots, \tau_n)$$

partition function

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

$$\begin{cases} DE = q\,ED + D + E \\ (\beta D - \delta E)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \gamma D) = \langle W| \end{cases}$$

Then

$f_n(\tau_1, \dots, \tau_n)$

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

· V column vector, W row vector

$$\begin{cases} DE = q\,ED + D + E \\ (\beta D - \text{[redacted]})|V\rangle = |V\rangle \\ \langle W|(\alpha E - \text{[redacted]}) = \langle W| \end{cases}$$

Then

$$f_n(\tau_1, \dots, \tau_n) = \langle W| \prod_{i=1}^{n} (\tau_i D + (1-\tau_i) E) |V\rangle$$

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

TASEP

q=0

$$\begin{cases} DE = \square + D + E \\ (\beta D - [redacted])|V\rangle = |V\rangle \\ \langle W|(\alpha E - [redacted]) = \langle W| \end{cases}$$

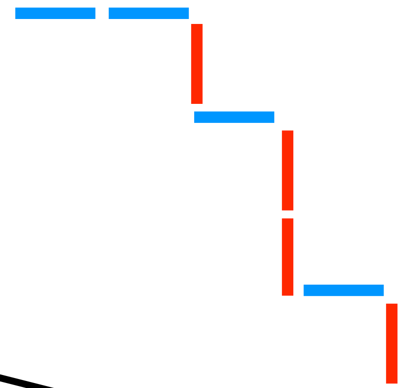
Then

$$f_n(\tau_1, \ldots, \tau_n) = \langle W| \prod_{i=1}^{n} (\tau_i D + (1-\tau_i) E) |V\rangle$$

$$DE = qED + E + D$$

commutation relation

"profile" w(E,D)



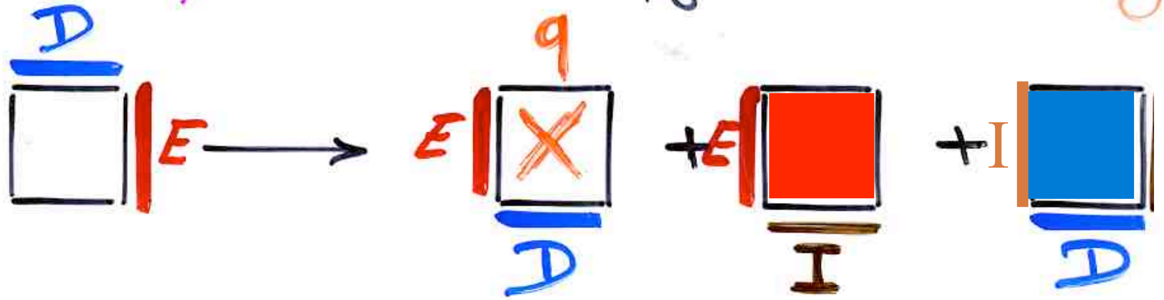
w(E,D) = D D E D E E D E

D DE (D E) E D E

DDE(E)EDE + DDE(ED)EDE + DDE(D)EDE

rewriting rules

Proof: "planarization" of the rewriting rules



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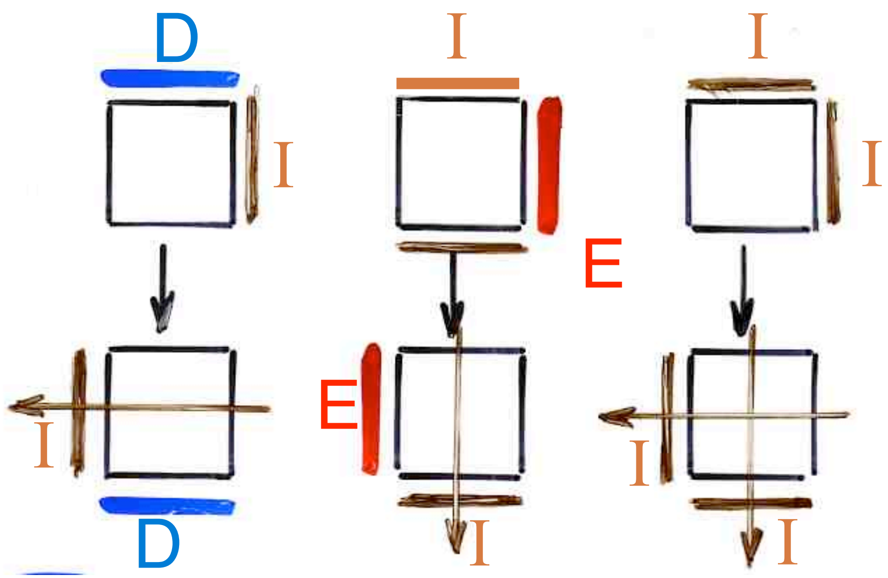

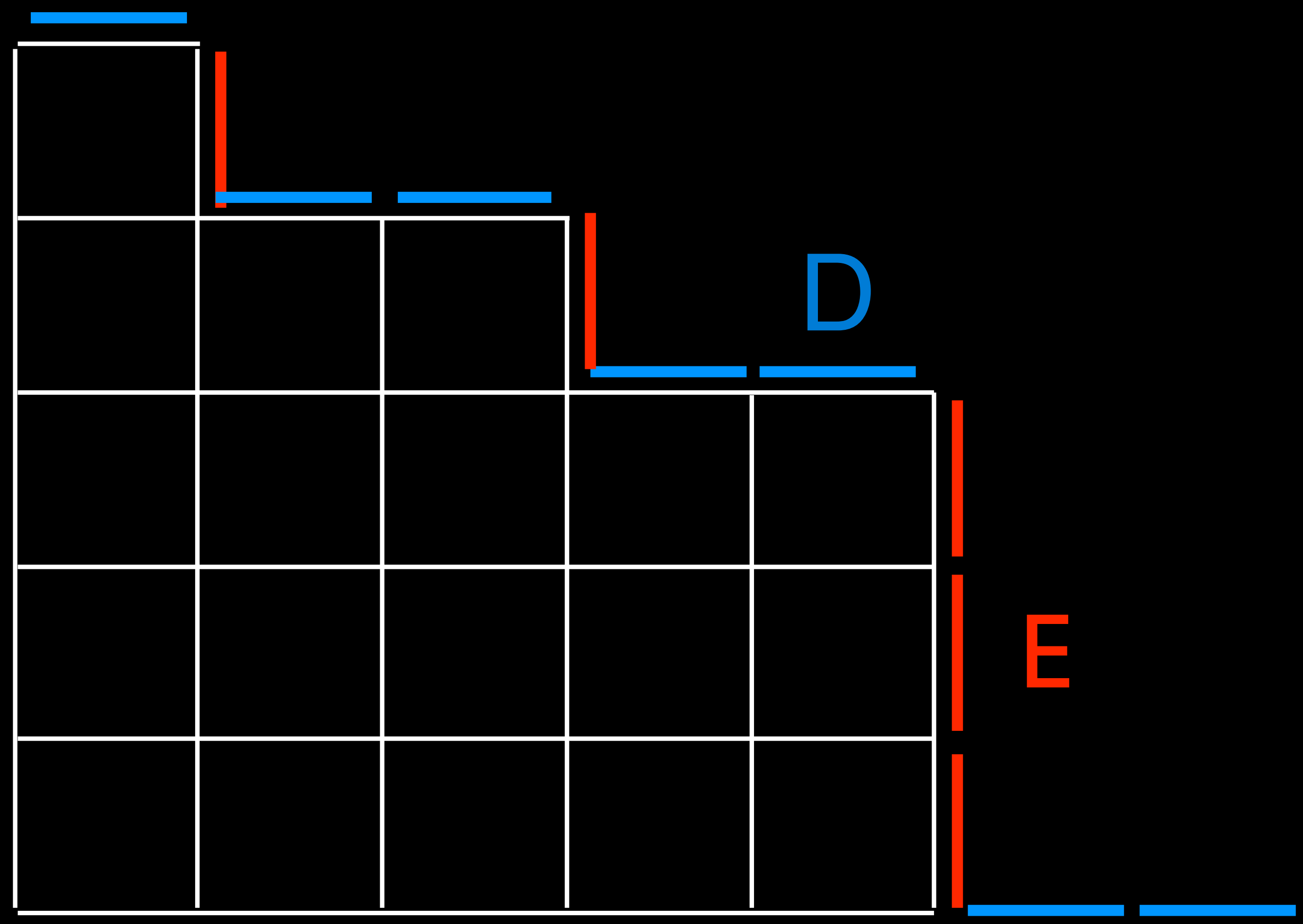
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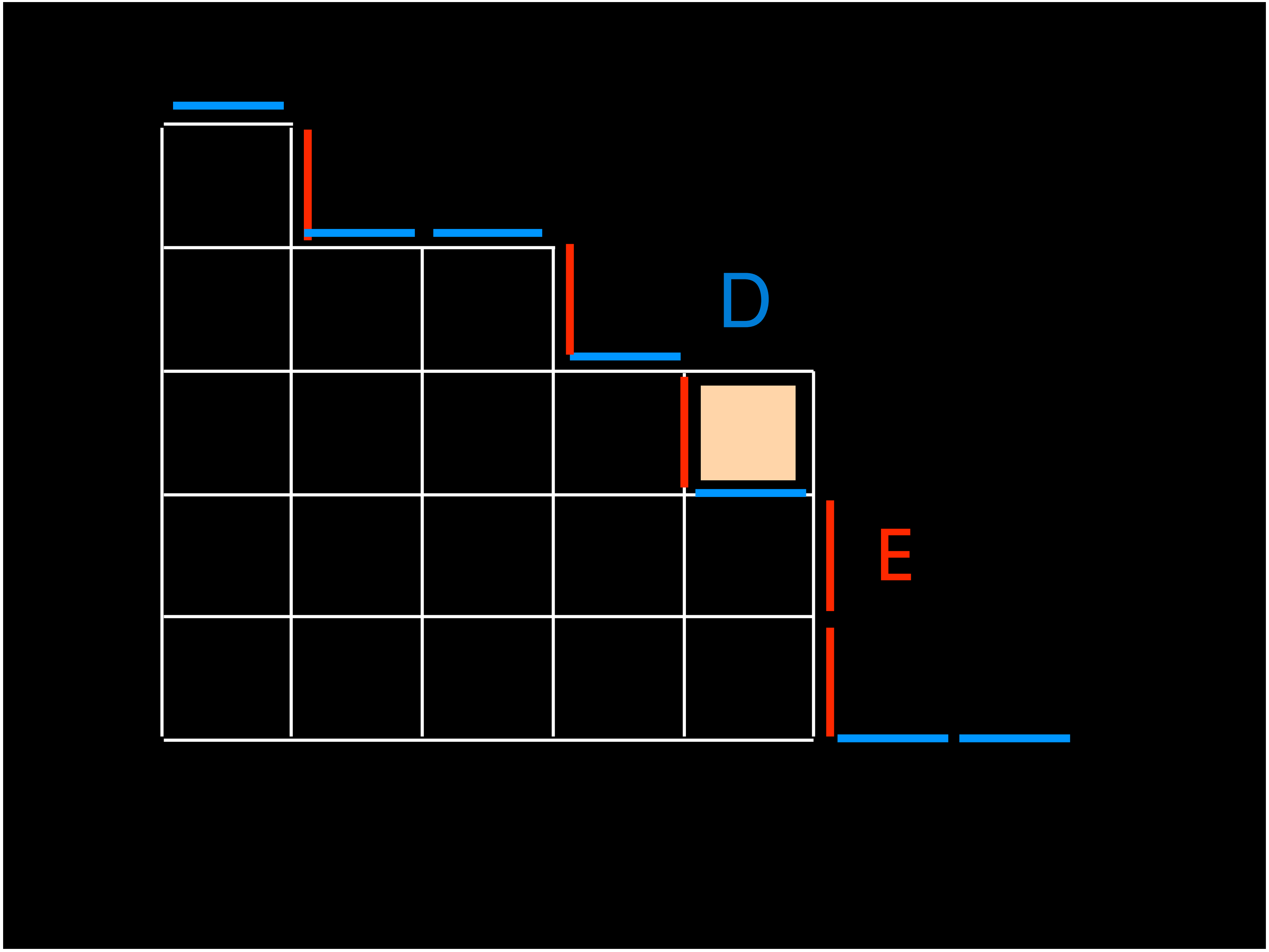
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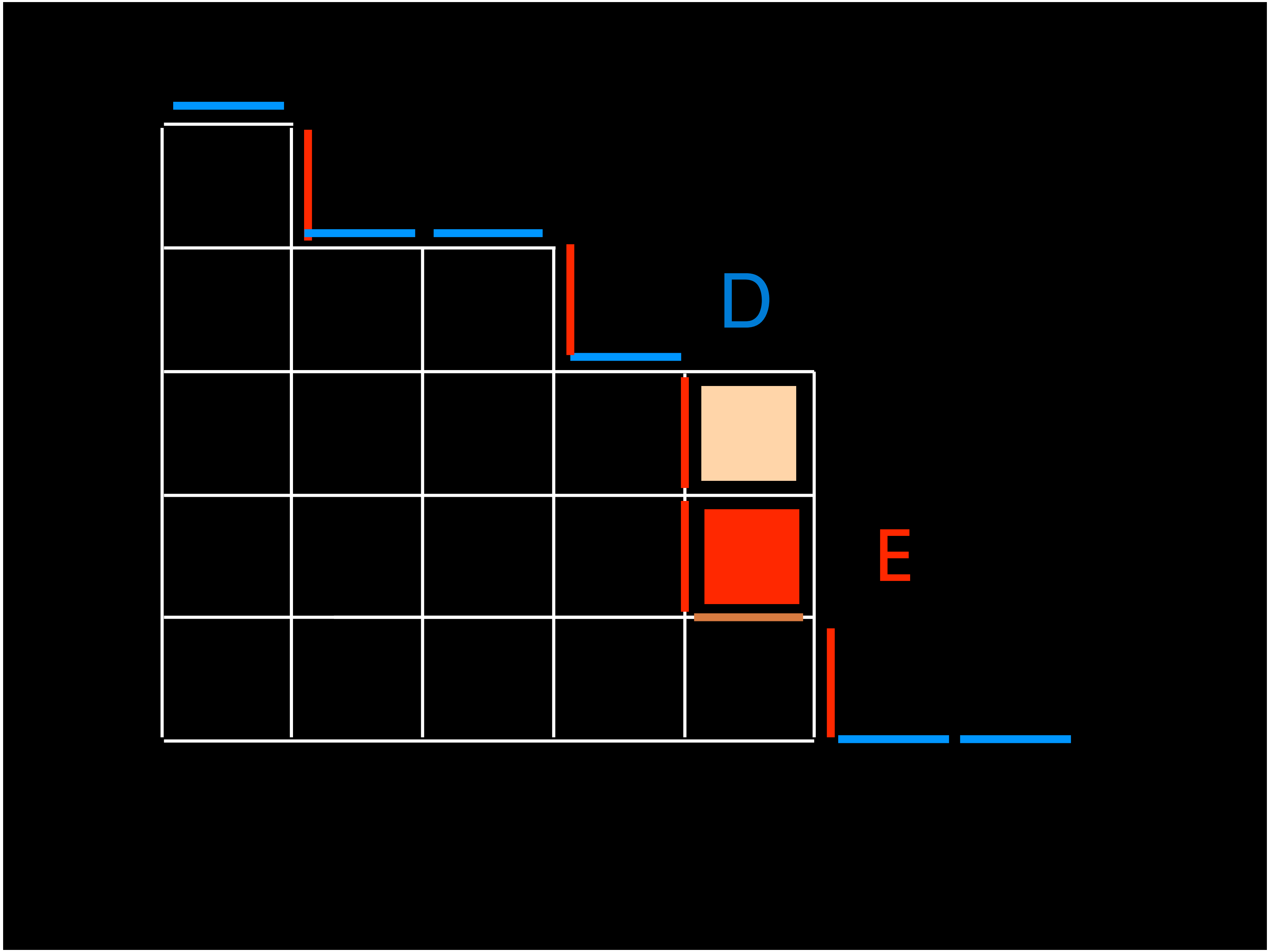
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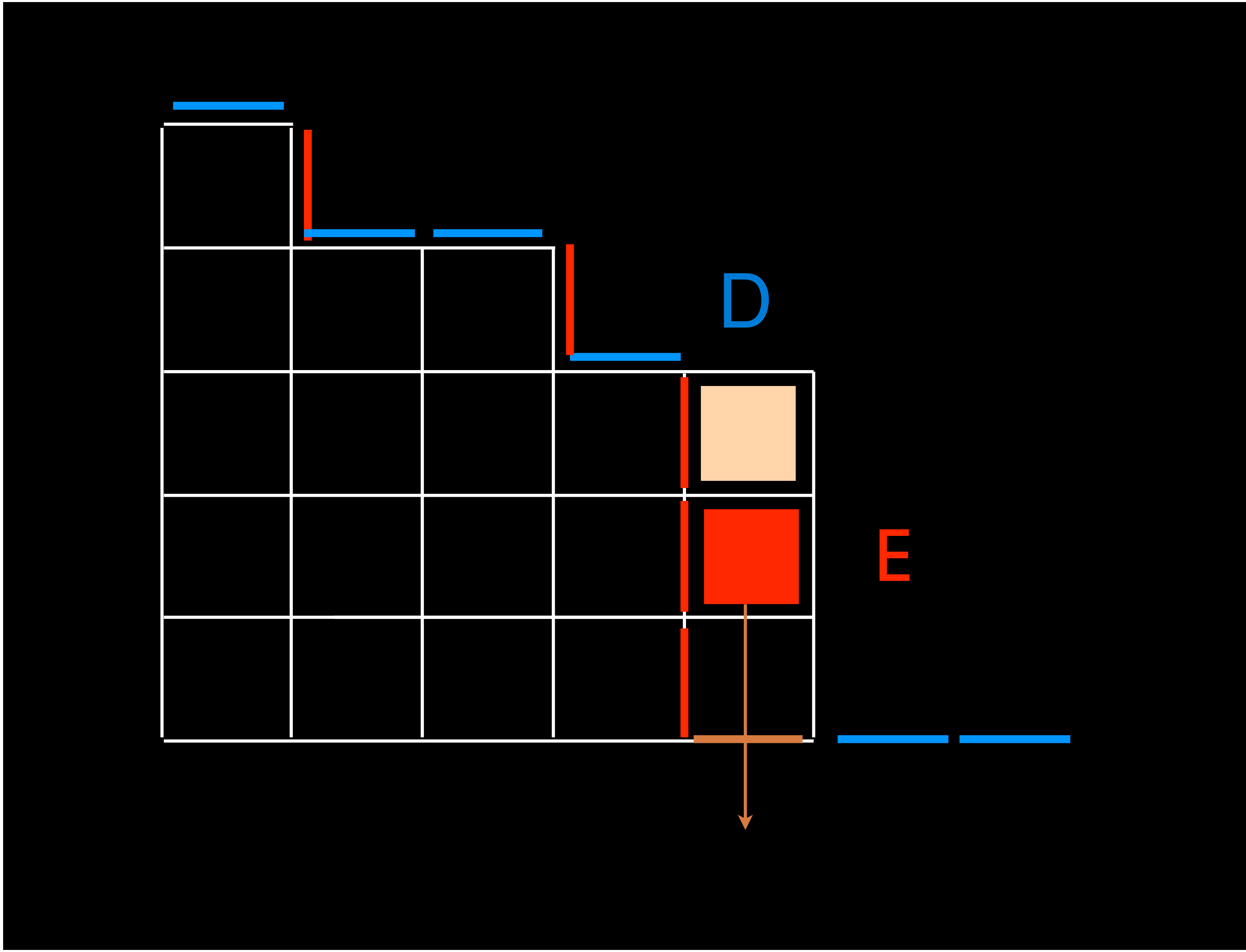
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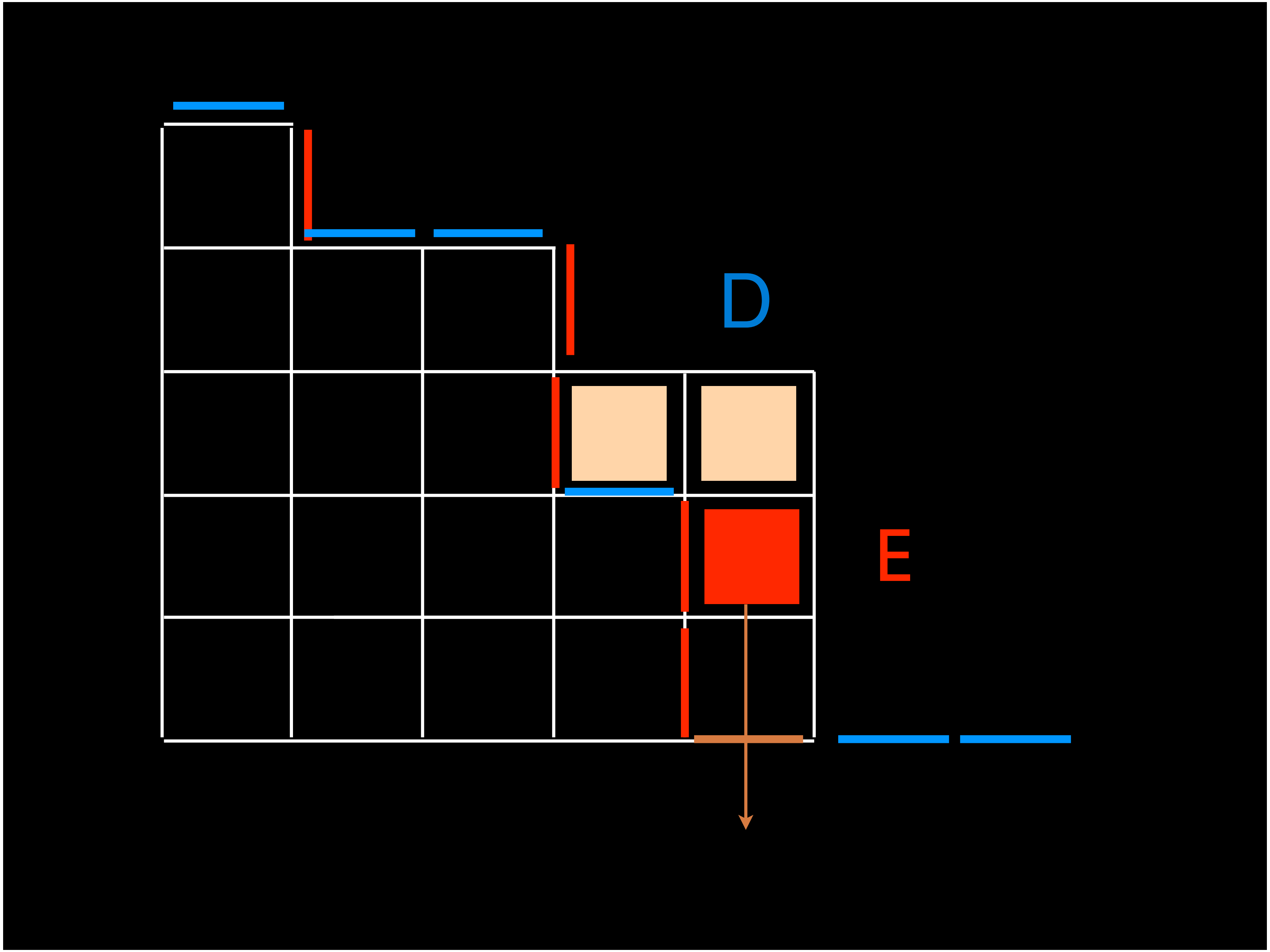
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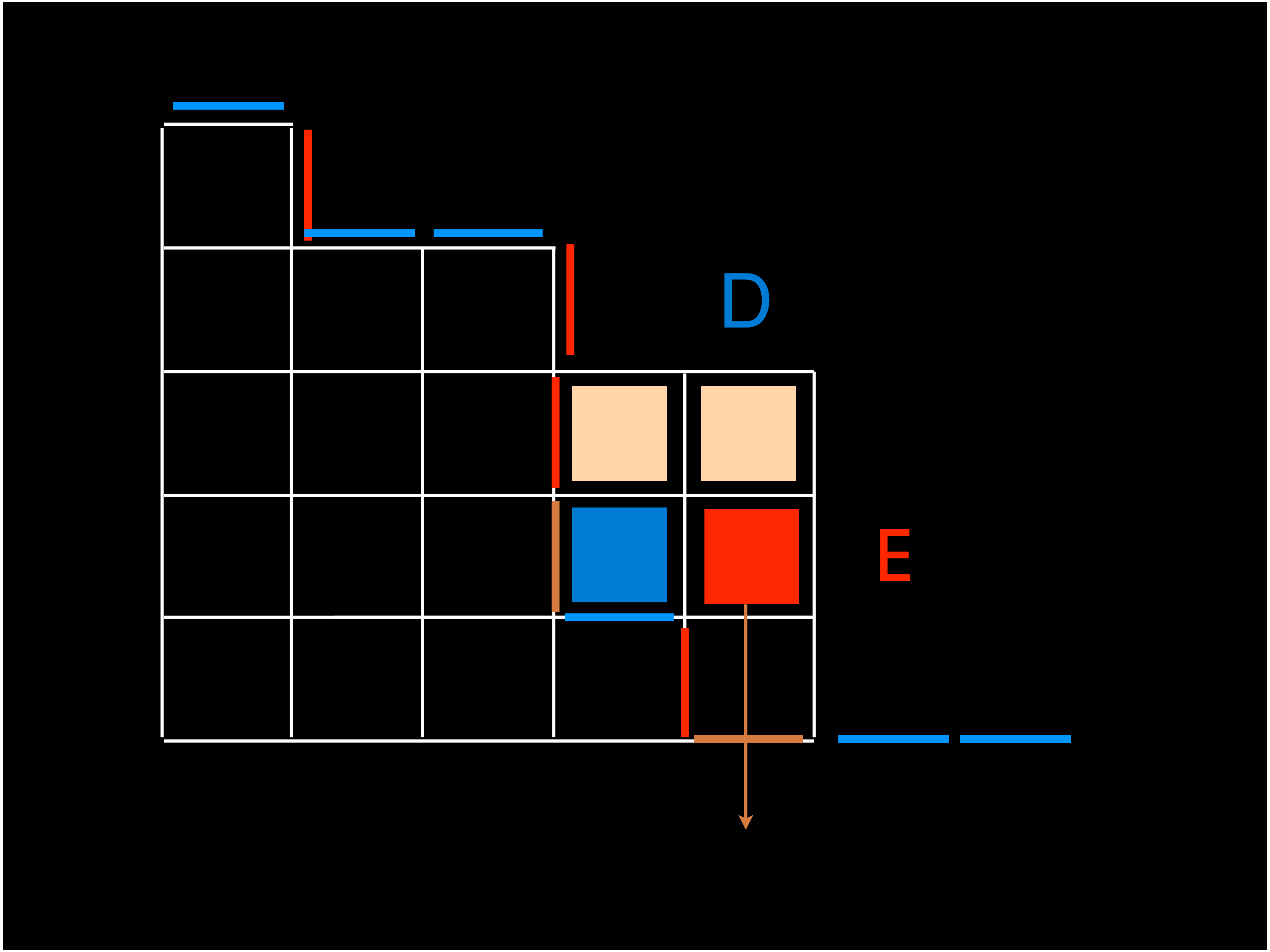
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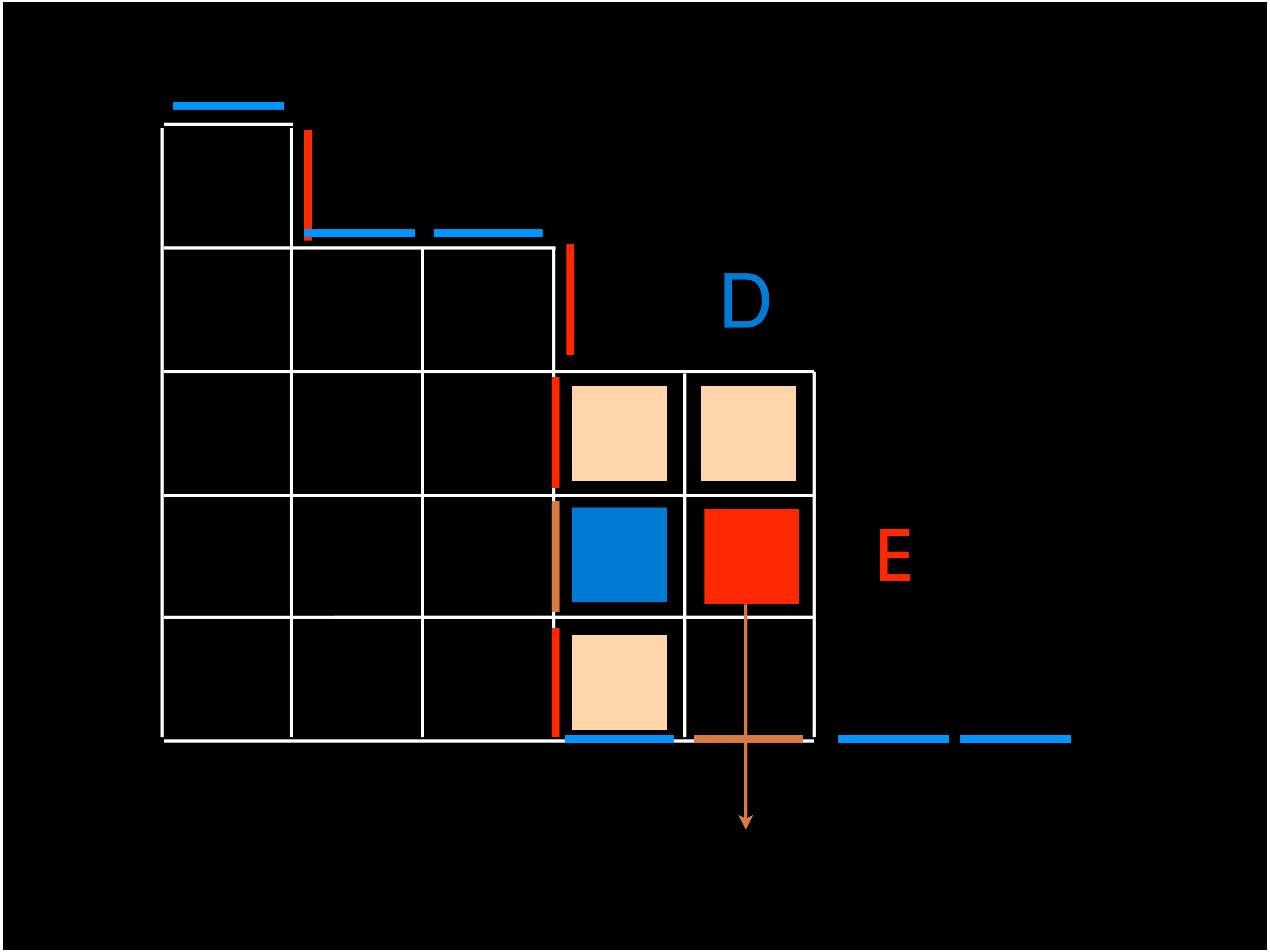
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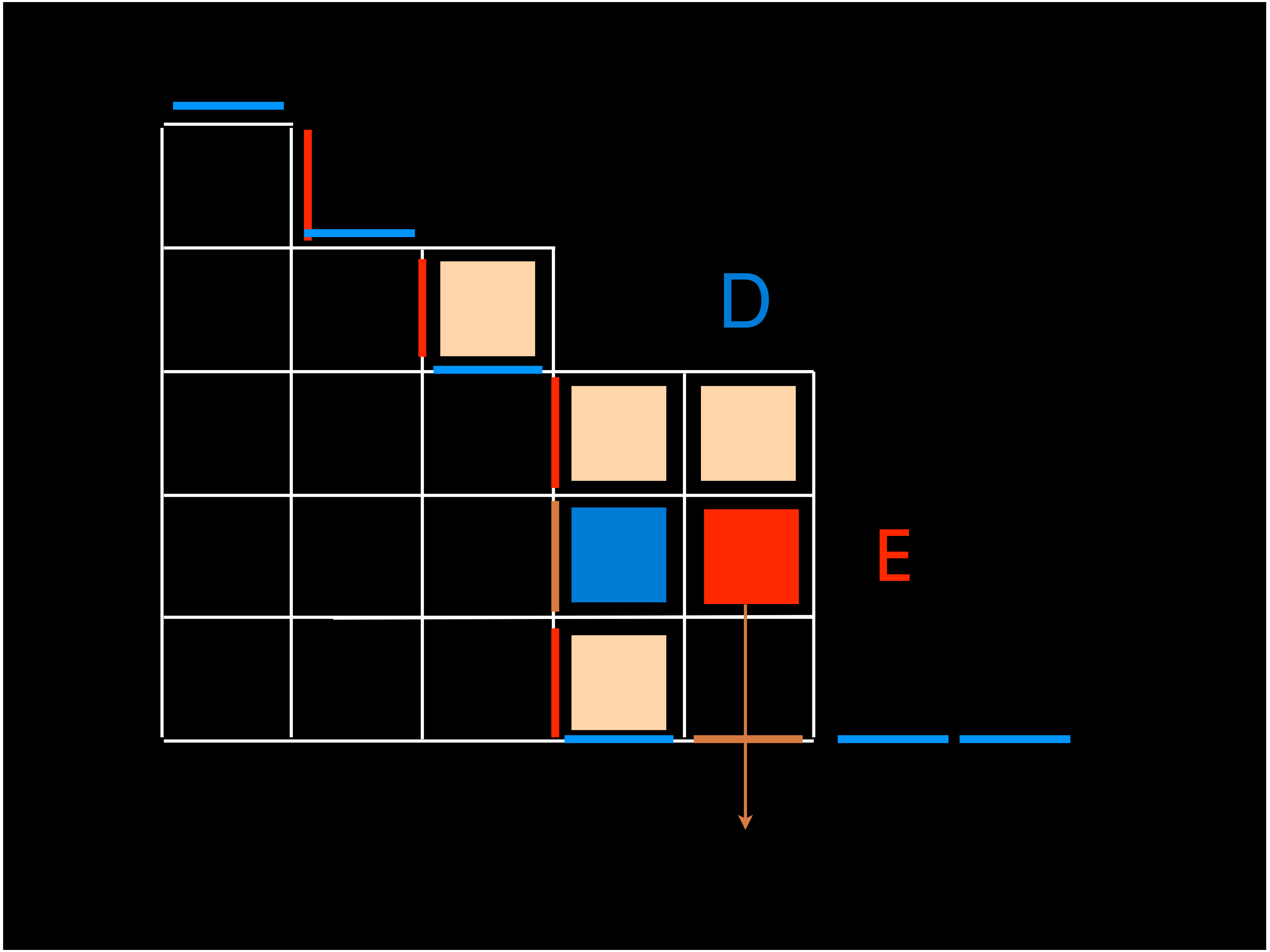
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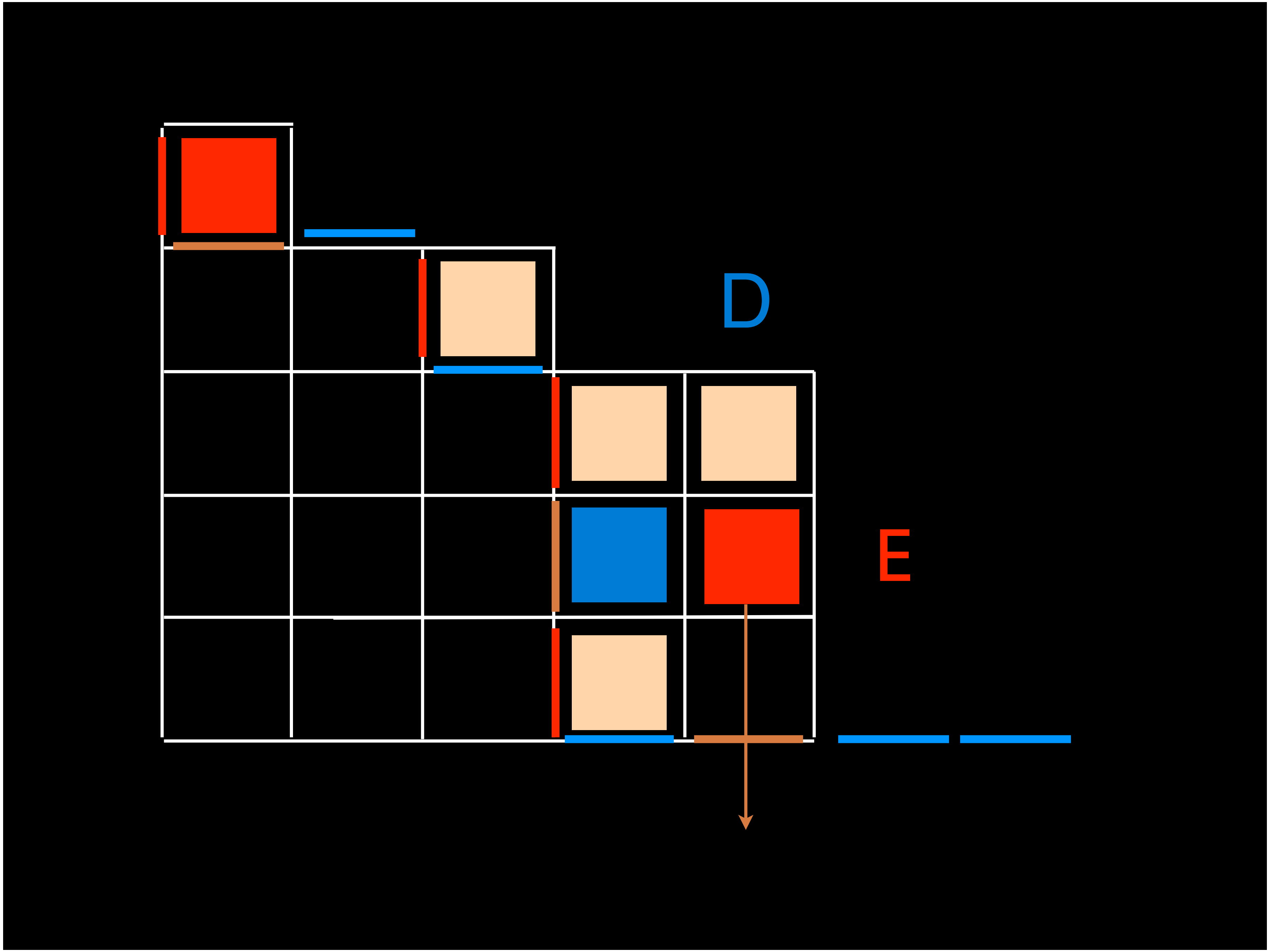

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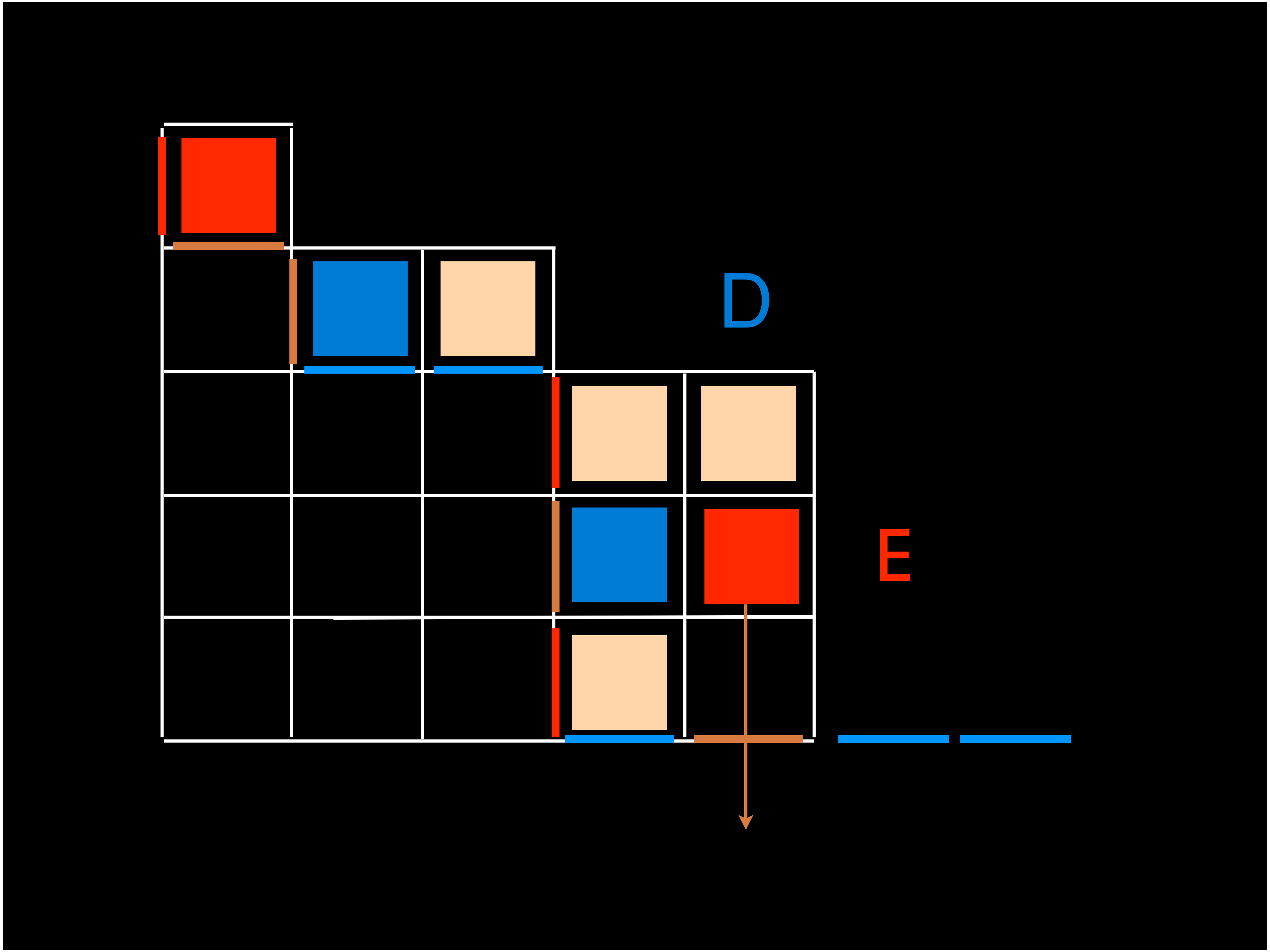

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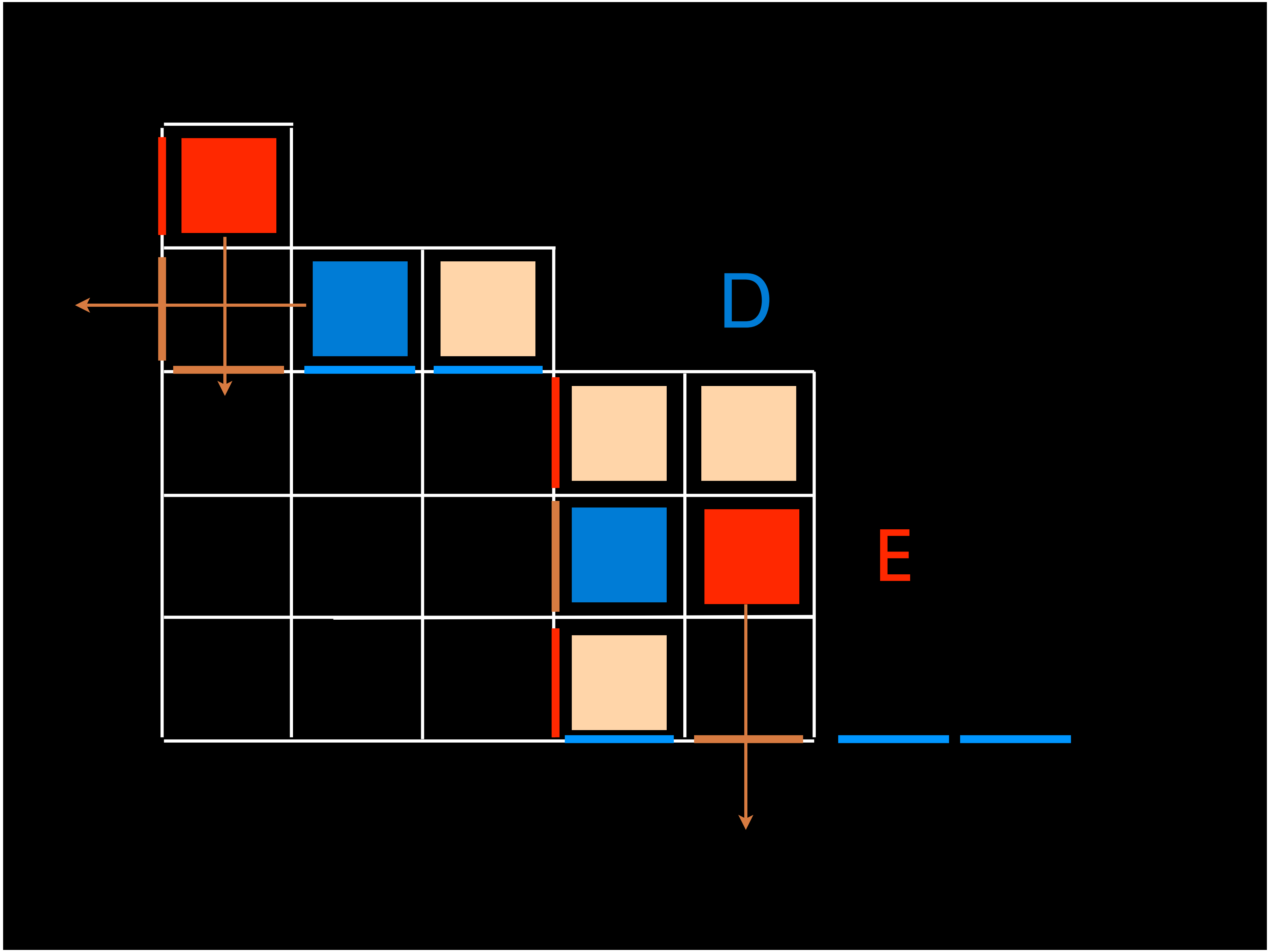
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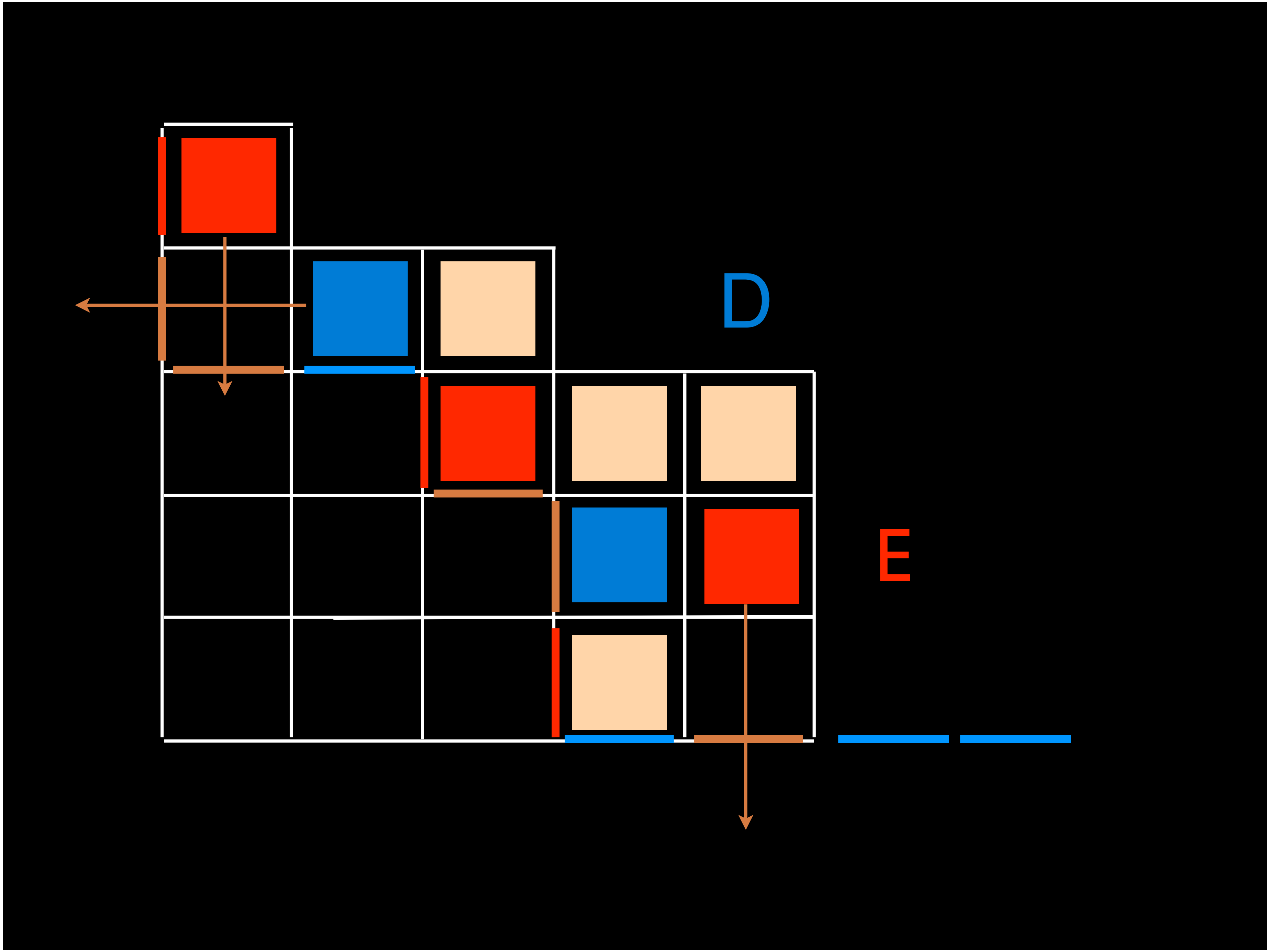

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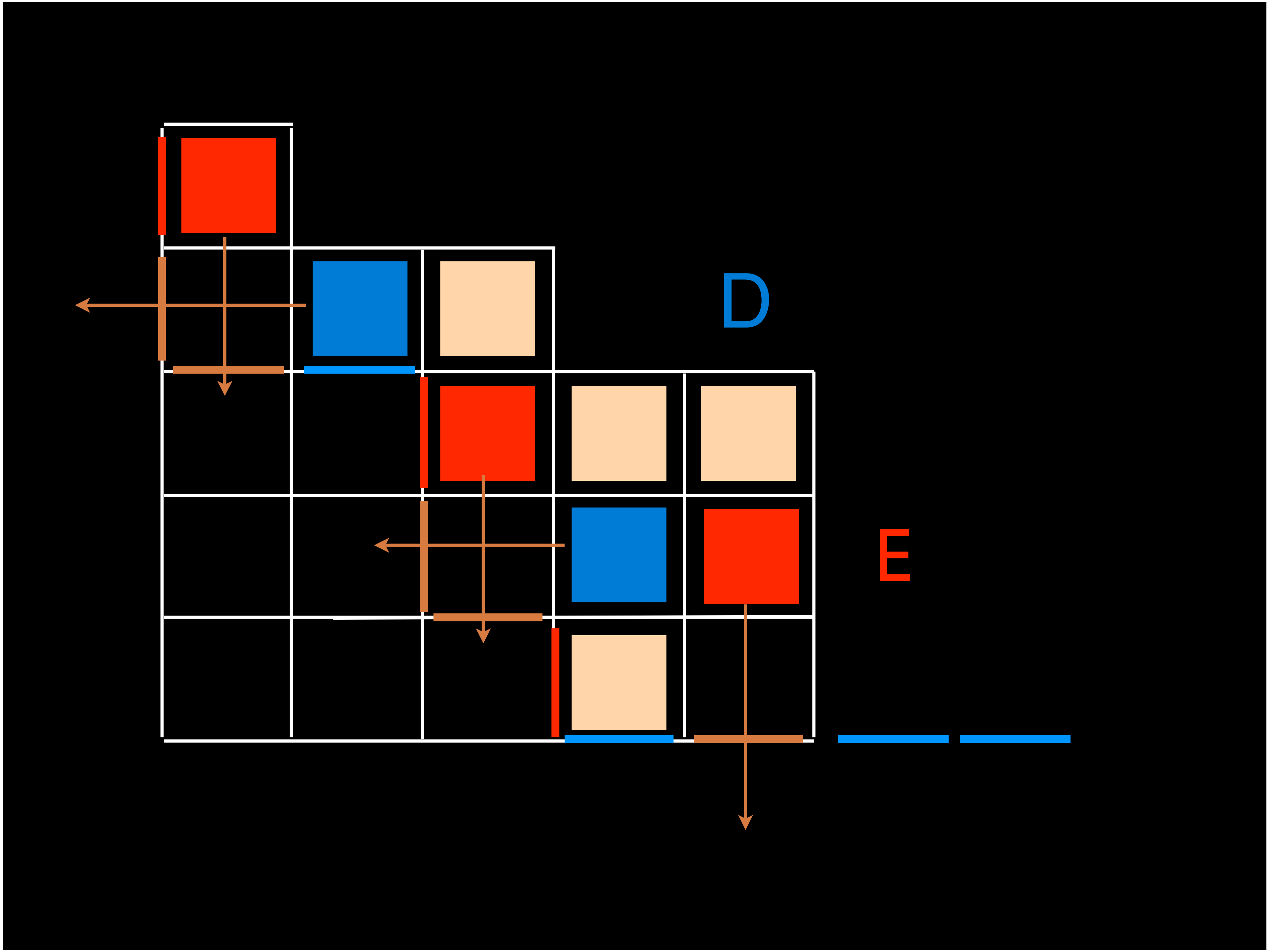
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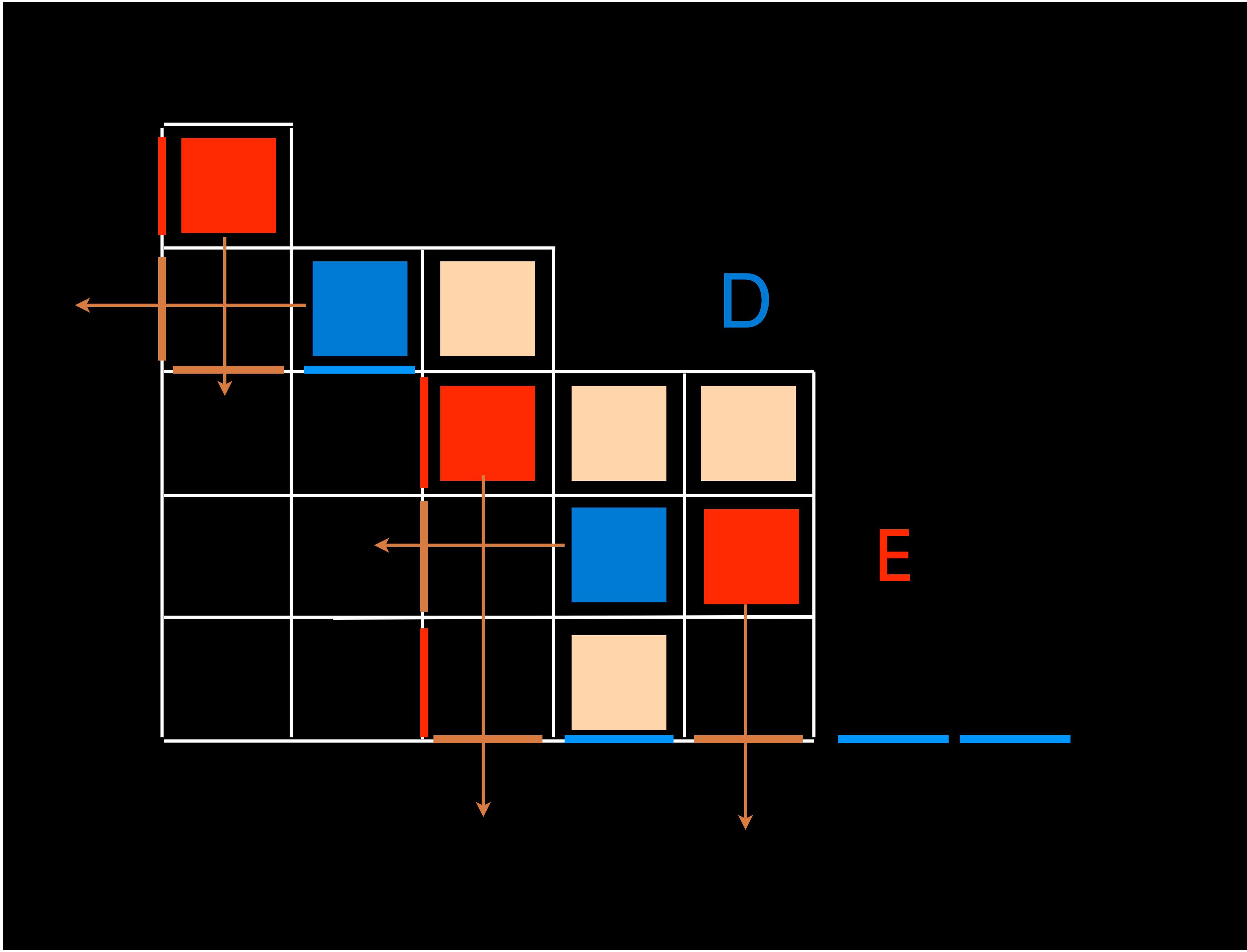
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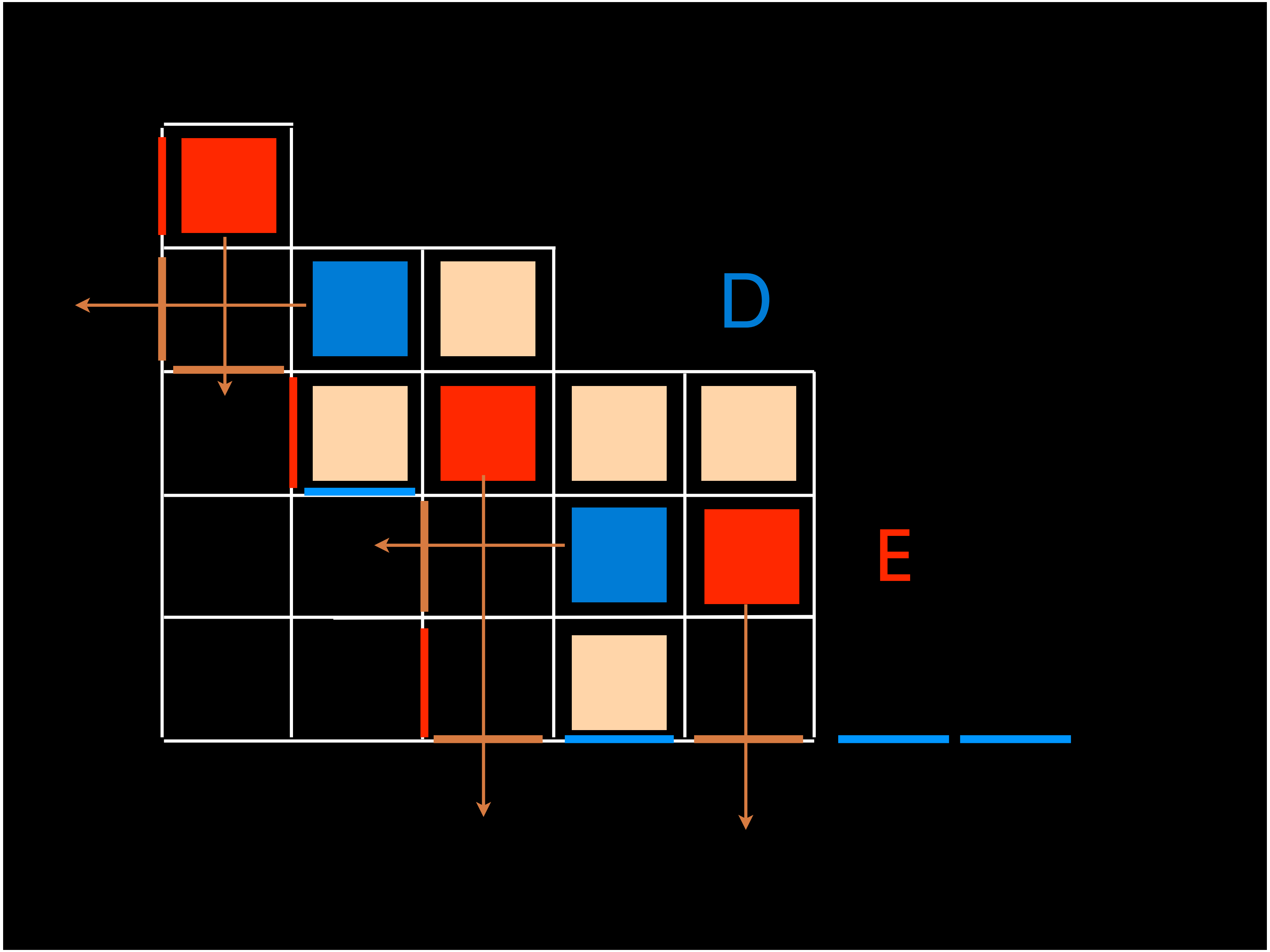
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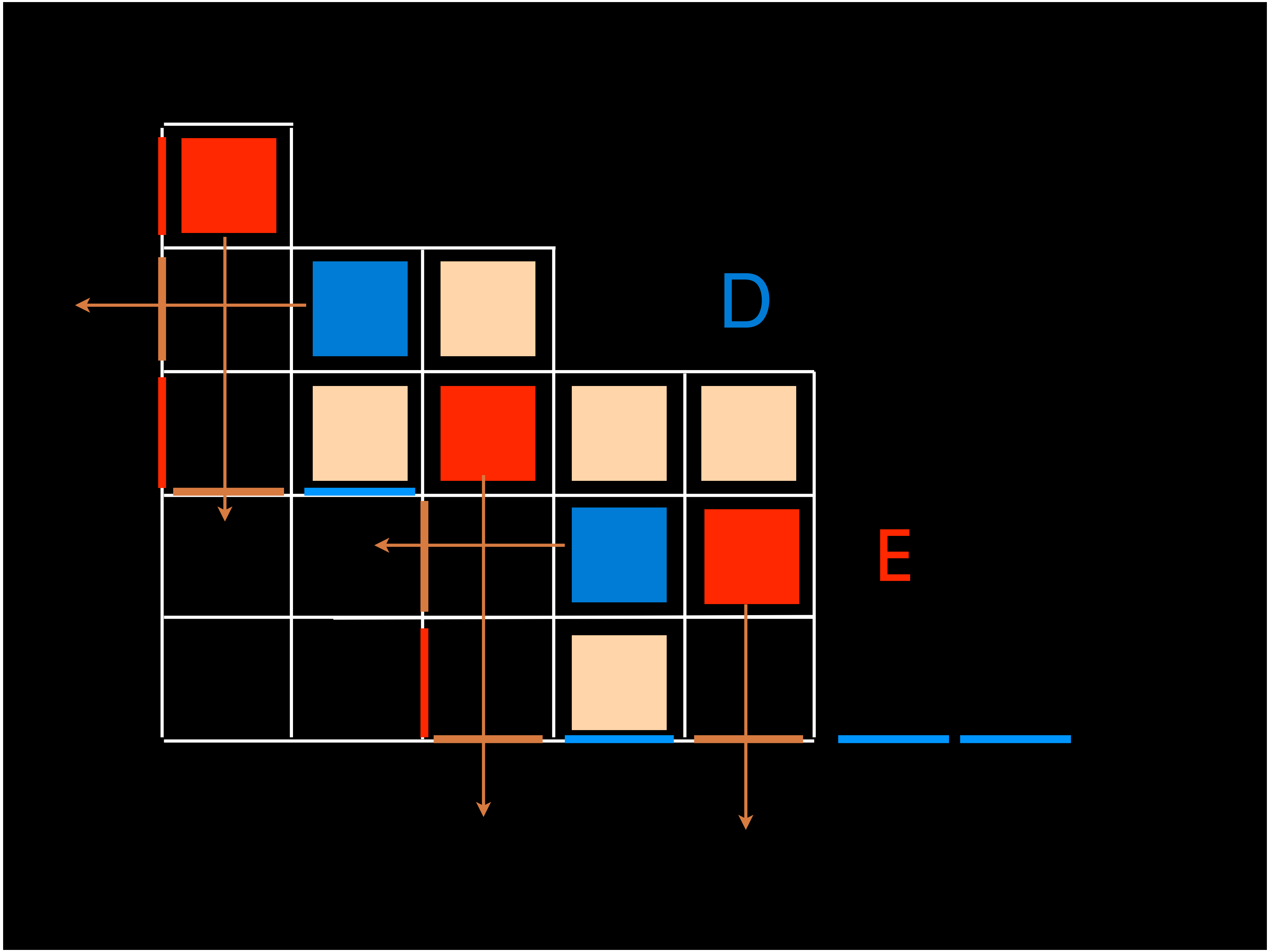
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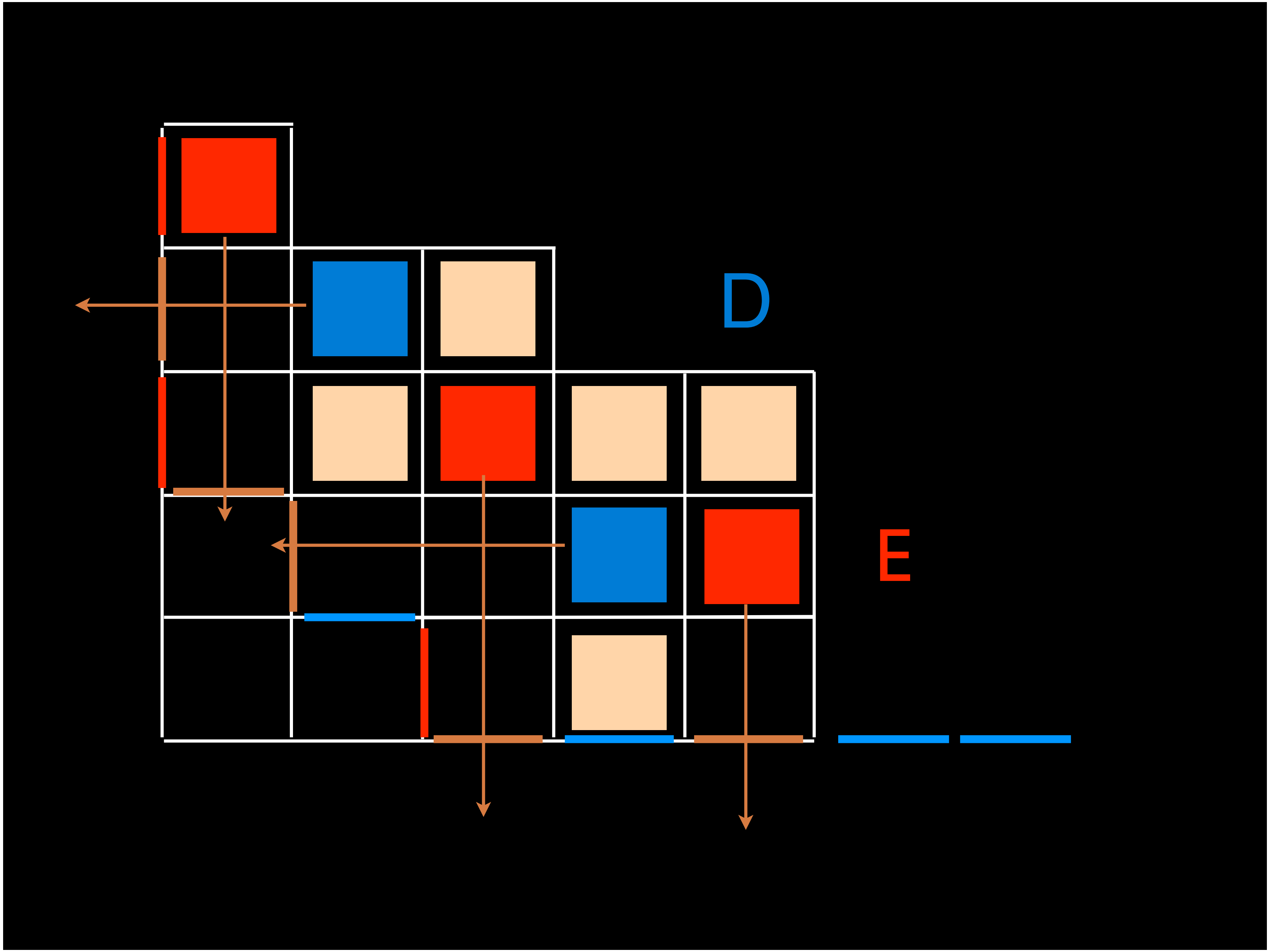
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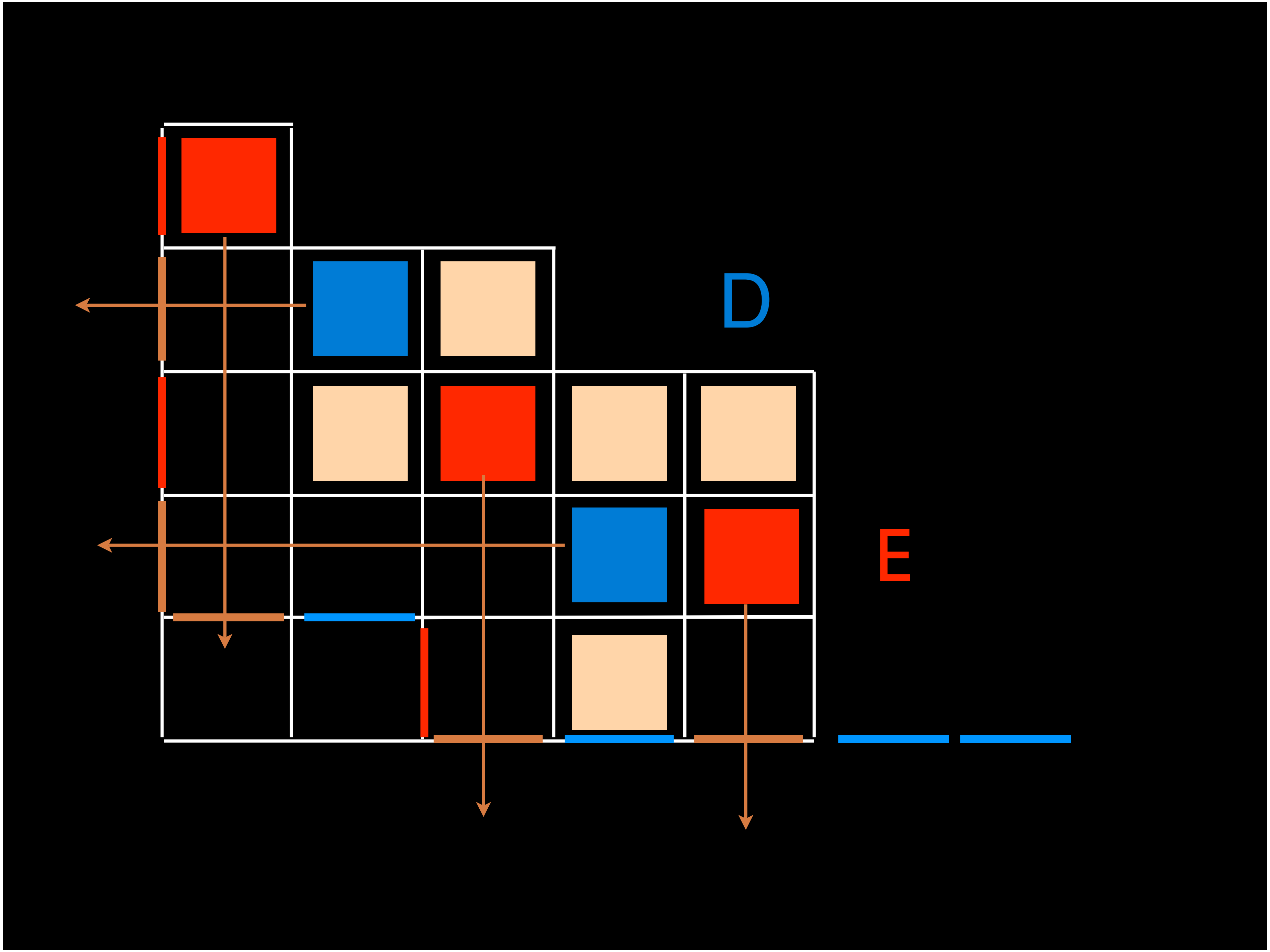
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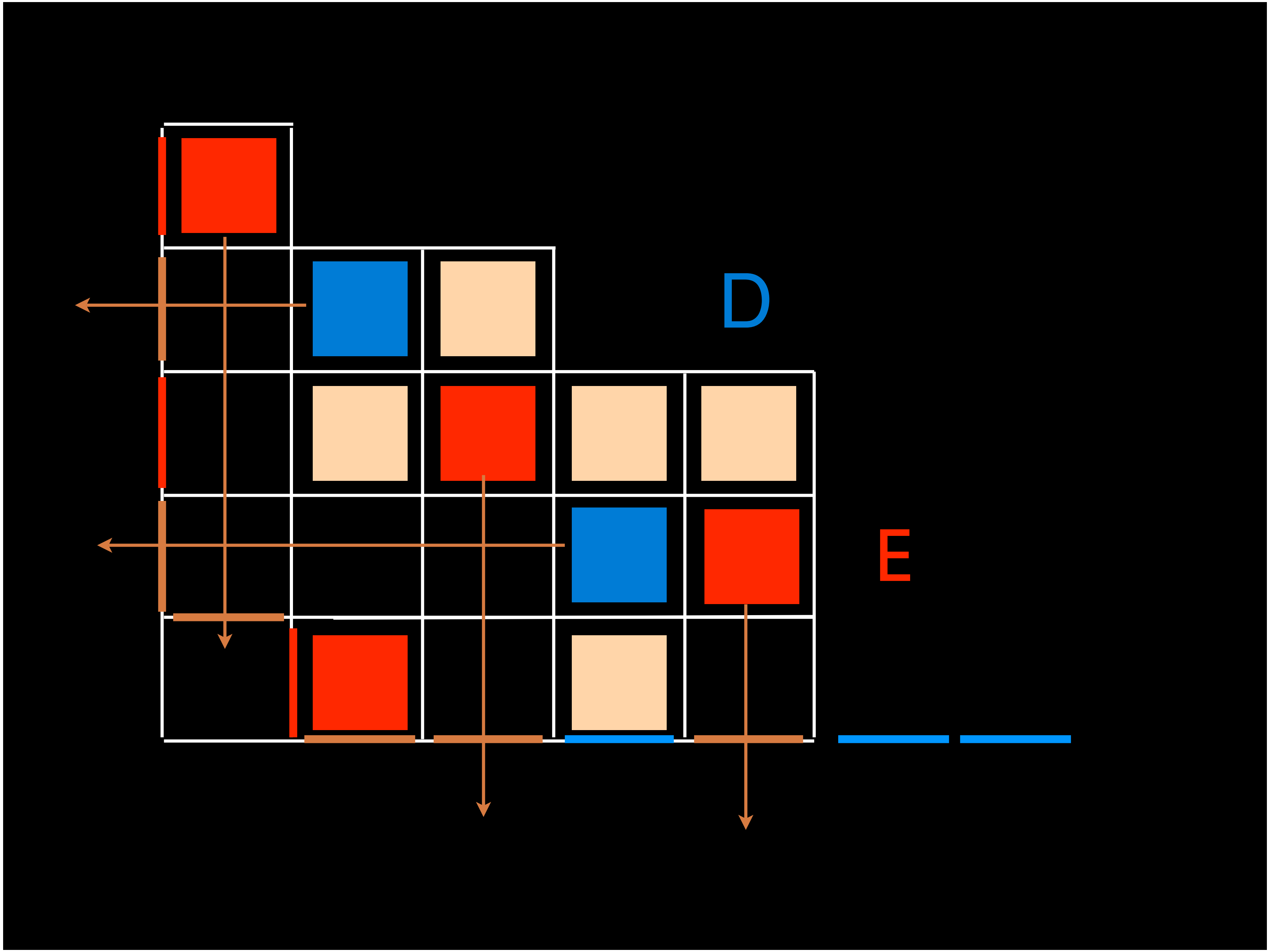
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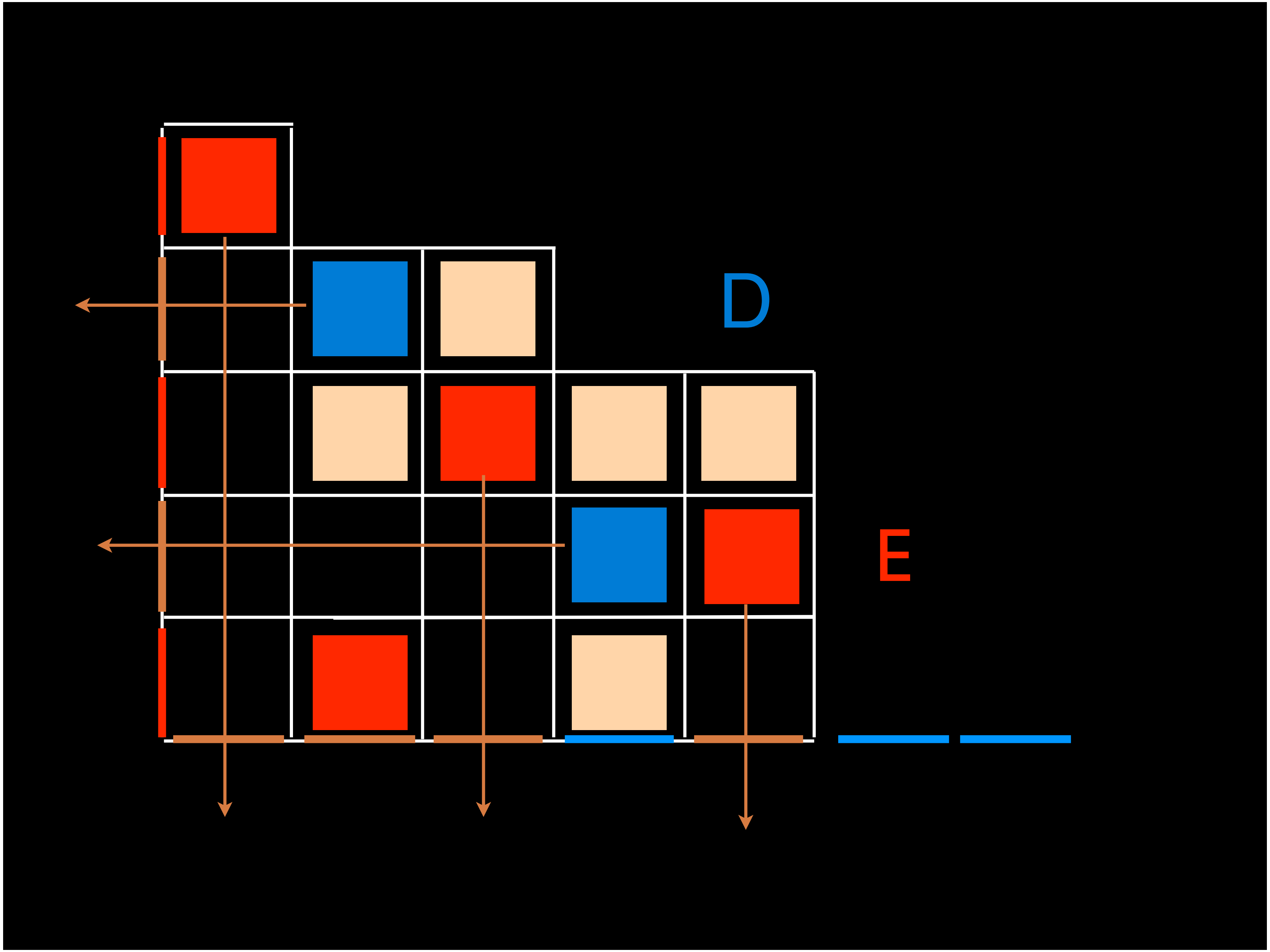
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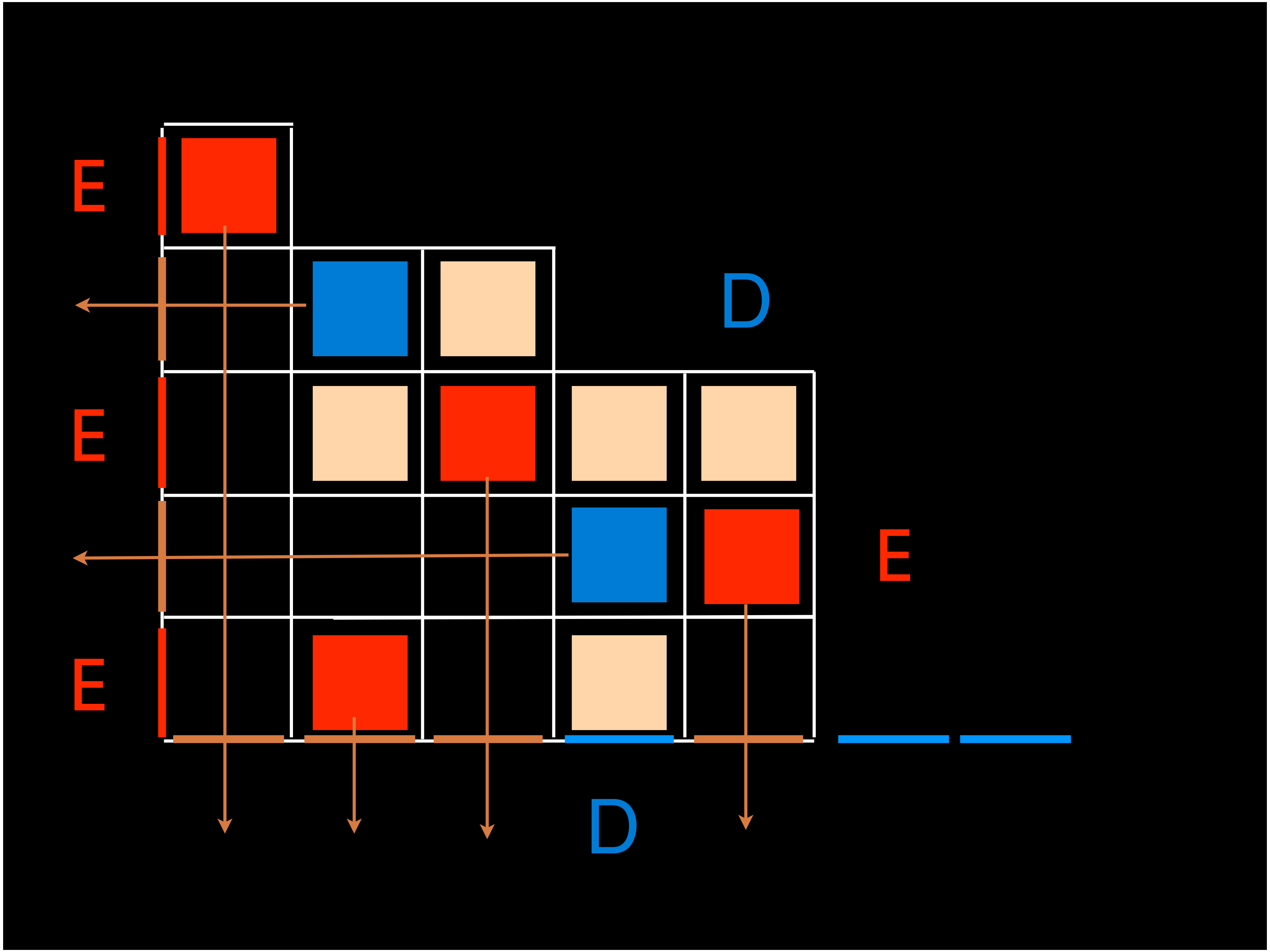
E
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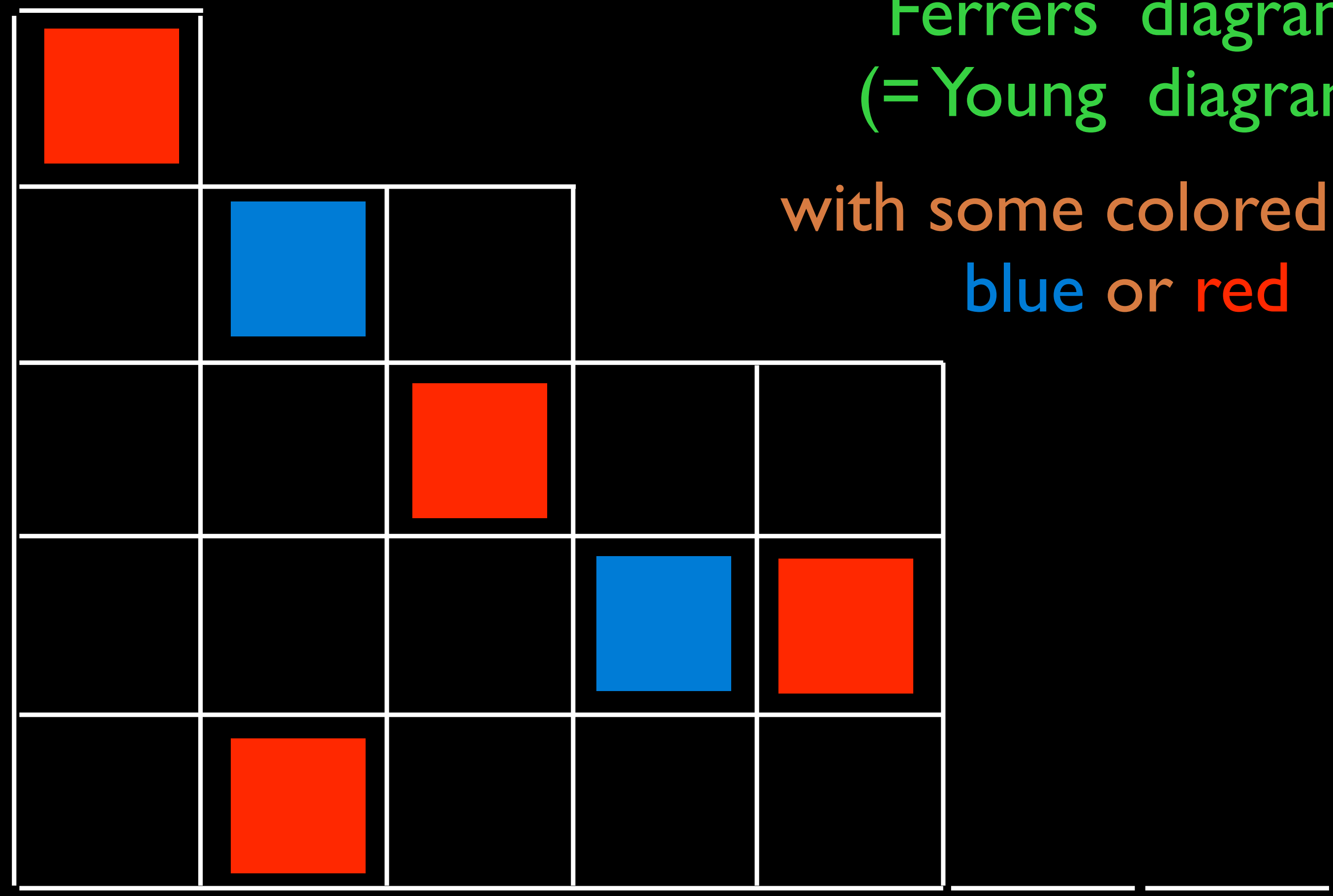
D

alternative tableau

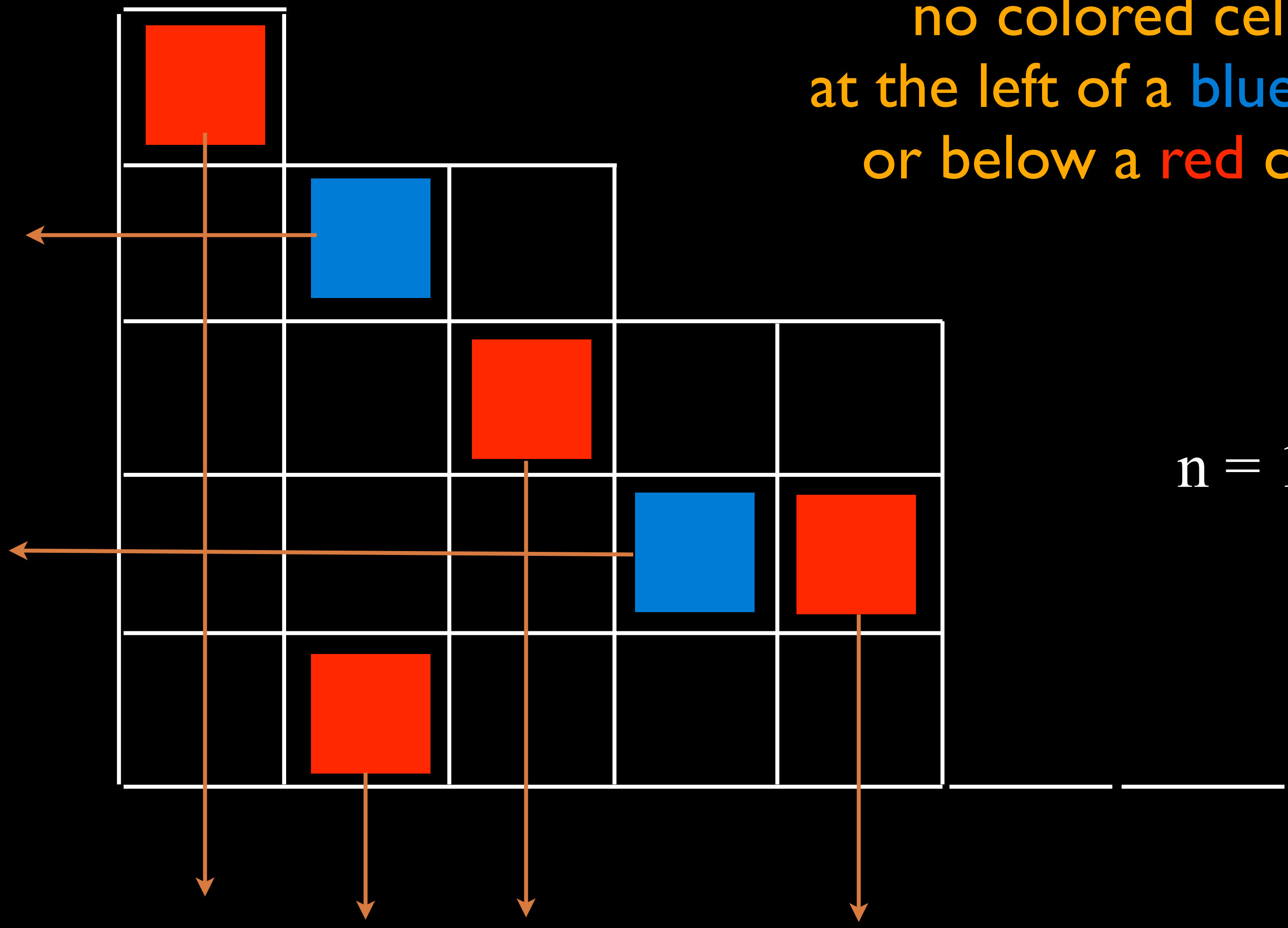
Ferrers diagram
(= Young diagram)

with some colored cells
blue or red



alternative tableau

no colored cell
at the left of a blue cell
or below a red cell



n = 12

from: $DE = qED + E + D$

we get: $$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

T alternative tableau with profile w

$k(T)$ = nb of $\boxtimes$

$i(T)$ = nb of columns without red cell

$j(T)$ = nb of rows without blue cell

suppose there exist “formal symbols” D, E, V, W, such that

$$\begin{cases} DE = qED + D + E \\ DV = \bar{\beta} V & \bar{\beta} = 1/\beta \\ WE = \bar{\alpha} W & \bar{\alpha} = 1/\alpha \end{cases}$$

“matrix ansatz”

$$W E^{i} D^{j} V = \bar{\alpha}^{i} \bar{\beta}^{j} \underbrace{WV}_{1}$$

we get immediately:

Cor. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ (PASEP) is

$$\text{proba}_\tau(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{T} q^{L(T)} \alpha^{-f(T)} \beta^{-u(T)}$$

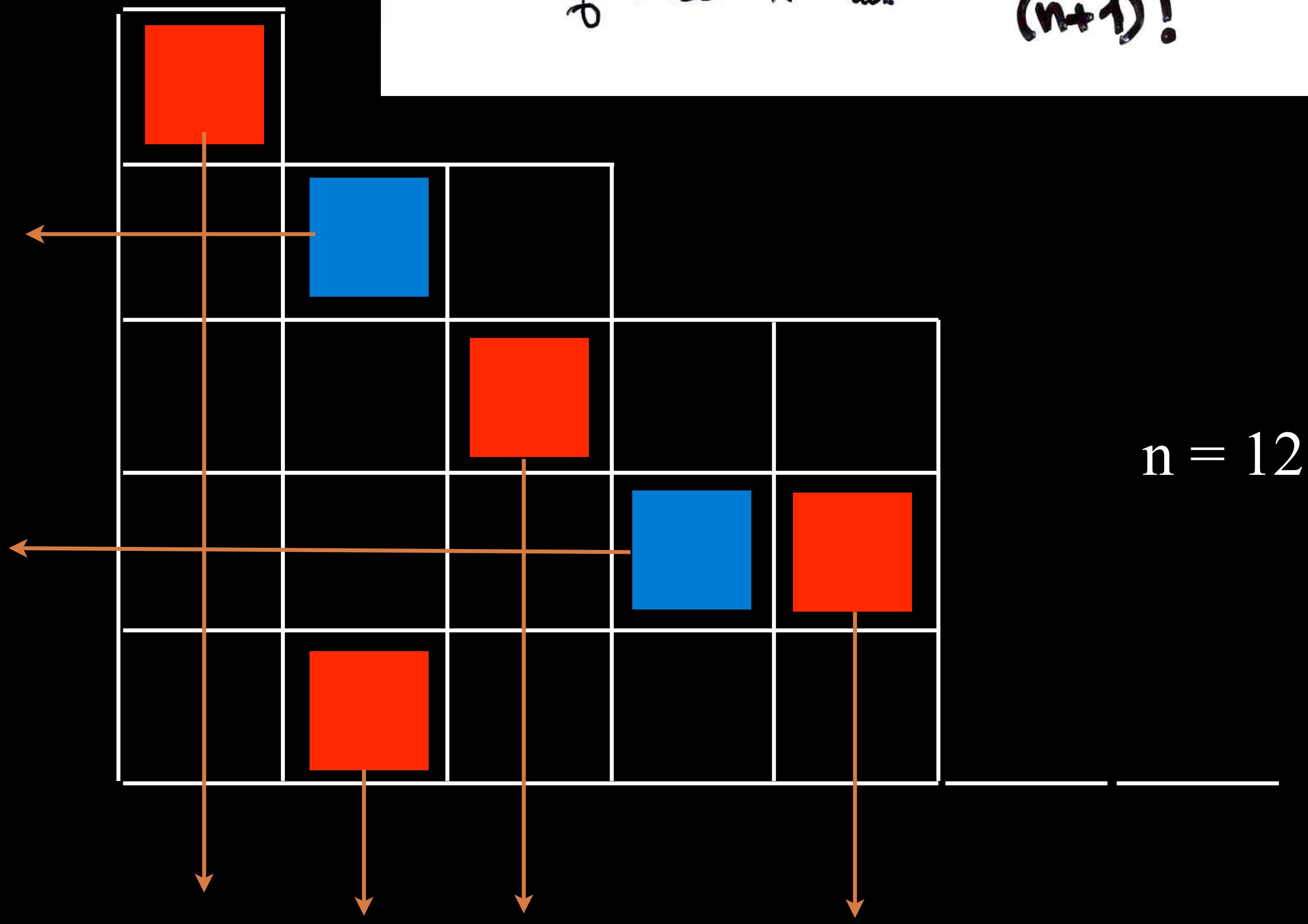
T alternative tableaux, profile τ

- $f(T)$ nb of rows without (red) cell
- $u(T)$ nb of columns without (blue) cell
- $L(T)$ nb of cells ⊠

called "free" row or column

Another equivalent interpretation has been given by S.Corteel and L.Williams in term of "permutations tableaux"(2006). For more details see §5 of the slides ot the talk given 23 April 2008 at the Isaac Newton Institute and references therein.

Prop. The number of alternative tableaux of size n is $(n+1)!$



V vector space generated by B basis

B alternating words two letters {○, ●}

(no occurrence of ○○ or ●●)

4 operators A, S, J, K

4 operators A, S, J, K, $u \in B$

$$\langle u | A = \sum_{\text{letter } \circ \text{ of } u} v$$, v obtained by: $\circ \rightarrow \circ \bullet \circ$

$$\langle u | S = \sum_{\circ \text{ of } u} v$$ v obtained by: $\circ \rightarrow \bullet$ (and $\bullet\bullet \rightarrow \bullet$ $\quad \bullet\bullet\bullet \rightarrow \bullet$)

$$\langle u | J = \sum_{\circ \text{ of } u} v$$ v, $\circ \rightarrow \bullet \circ$ (and $\bullet\bullet \rightarrow \bullet$)

$$\langle u | K = \sum_{\circ \text{ of } u} v$$ v, $\circ \rightarrow \circ \bullet$ (and $\bullet\bullet \rightarrow \bullet$)

$$\circ \bullet \circ \bullet | S = \bullet \circ \bullet + \circ \bullet$$

Lemma.

$$AS = SA + J + K$$

$$AK = KA + A$$

$$JS = SJ + S$$

$$JK = KJ$$

Lemma.

$$AS = SA + J + K$$

$$AK = KA + A$$

$$JS = SJ + S$$

$$JK = KJ$$

$$D = A + J$$

$$E = S + K$$

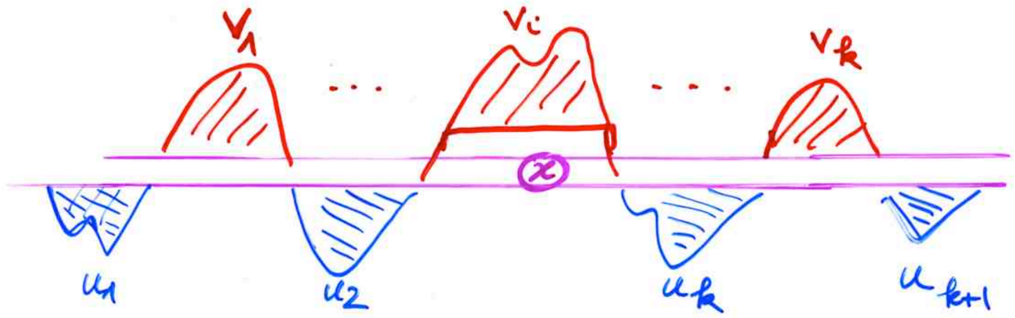
$$D = A + J$$

$$E = S + K$$

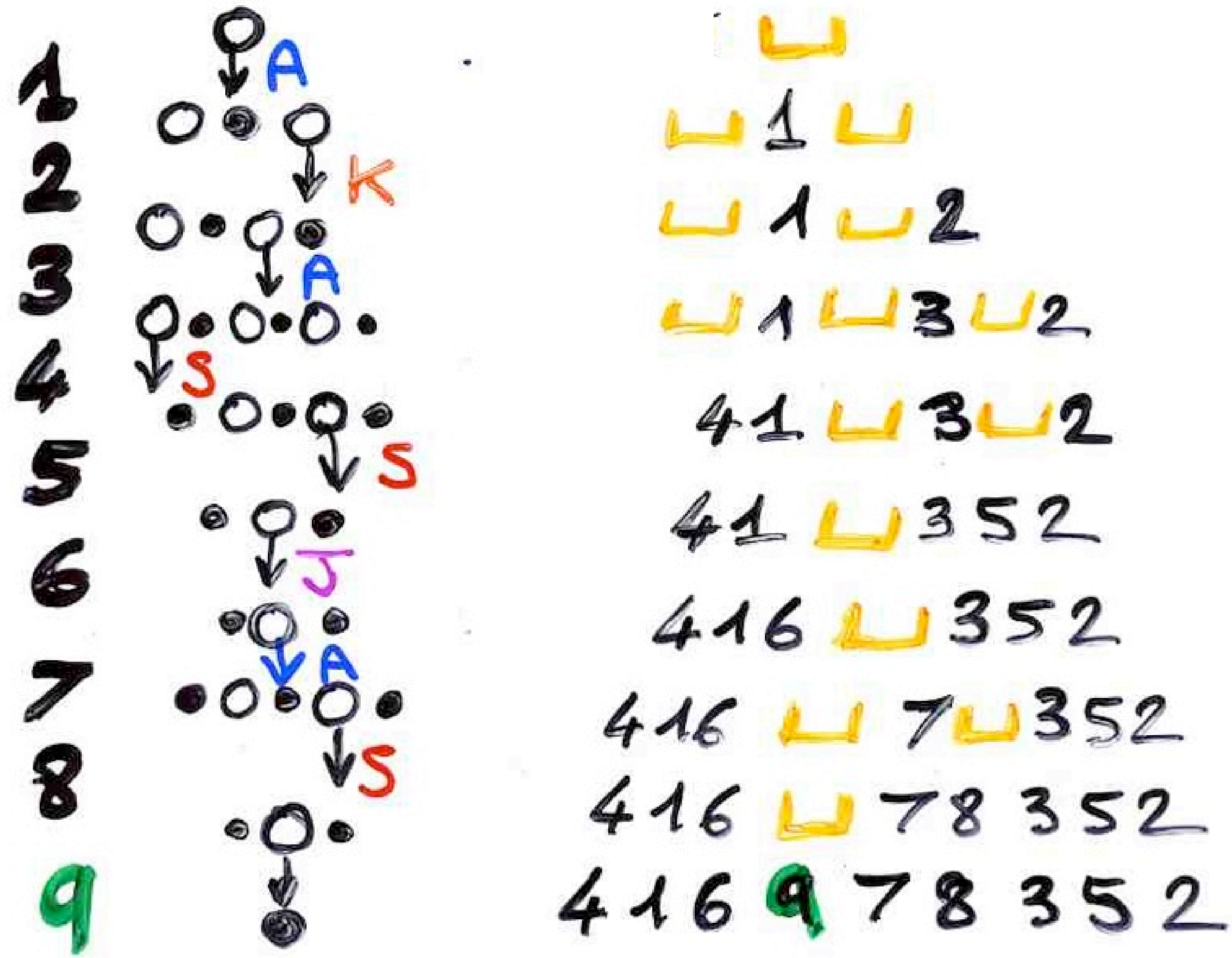
$$DE = (A+J)(S+K)$$

$$= AS + AK + JS + JK$$

$$= \underbrace{(SA + KA + SJ + KJ)}_{\underbrace{(S+K)(A+J)}_{ED}} + \underbrace{J + K + A + S}_{E + D}$$



v1
vi
vk
x
u1
u2
uk
uk+1



1
2
3
4
5
6
7
8
9
A
K
A
S
S
J
A
S
1
1 2
1 3 2
41 3 2
41 352
416 352
416 7 352
416 78352
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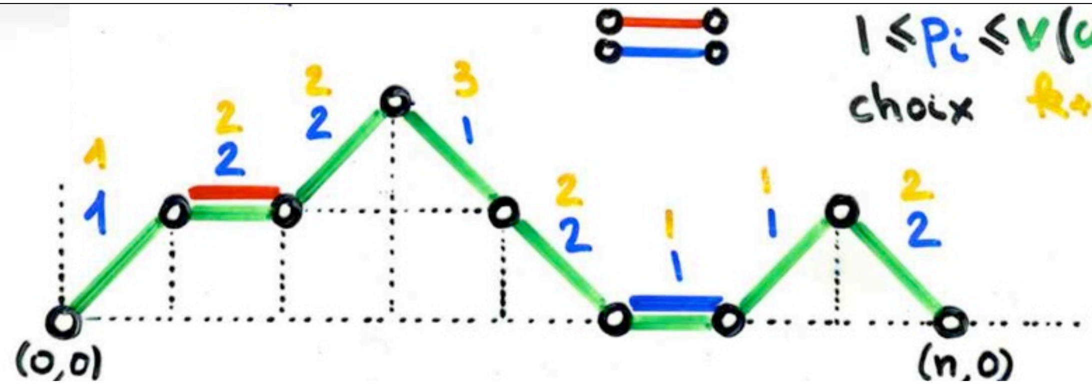
Bijection
Laguerre histories
permutations

Françon-xgv., 1978

$h = (\omega_c ; (p_1, \dots, p_n))$



x	ω_c	pos	v
1	↗	1	1
2	—	2	2
3	↗	2	2
4	↘	1	3
5	↘	2	2
6	—	1	1
7	↗	1	1
$n=$ 8	↘	2	2
9	•		

␣

␣ 1 ␣

␣ 1 ␣ 2

␣ 1 ␣ 3 ␣ 2

4 1 ␣ 3 ␣ 2

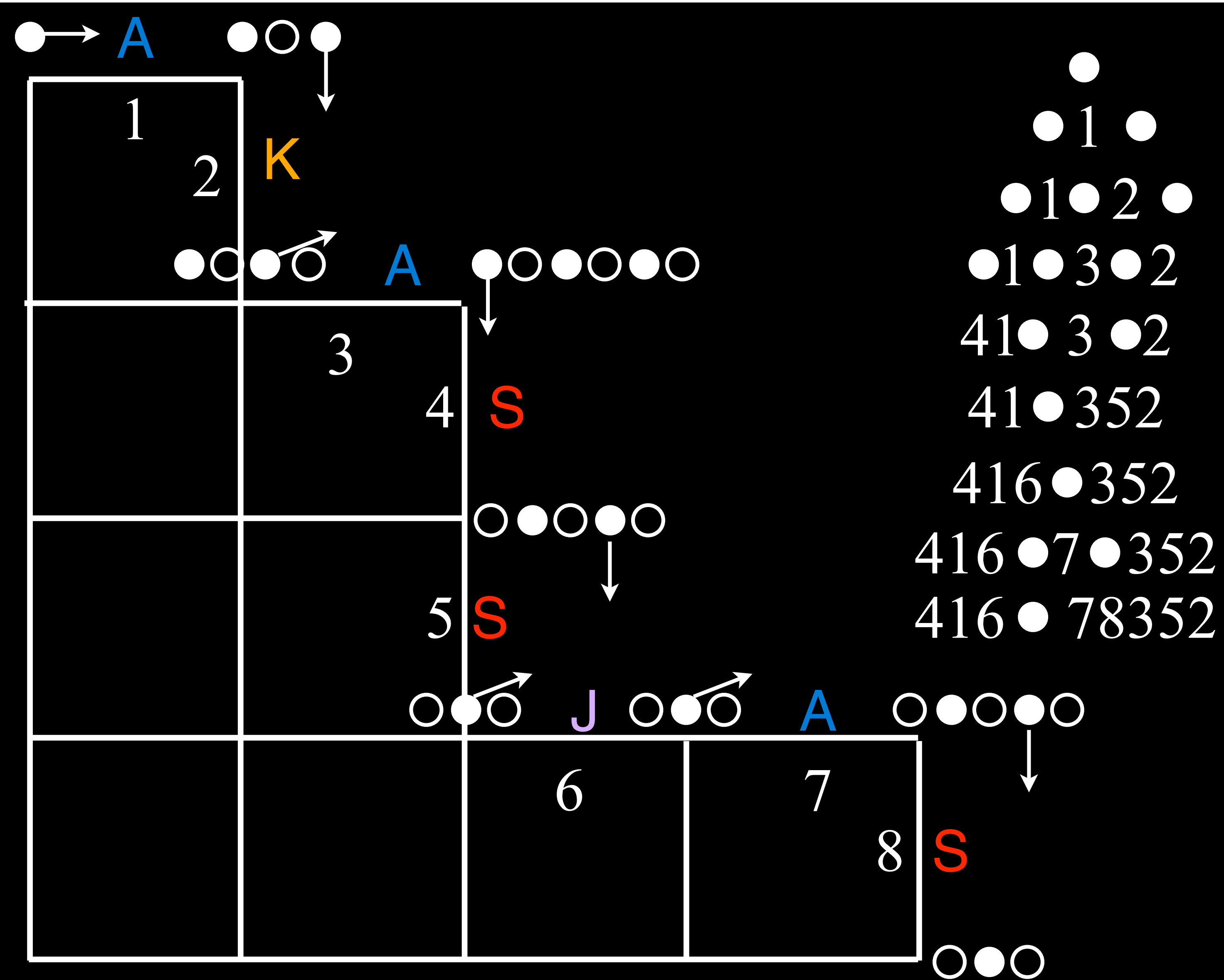
4 1 ␣ 3 5 2

4 1 6 ␣ 3 5 2

4 1 6 ␣ 7 ␣ 3 5 2

4 1 6 ␣ 7 8 3 5 2

4 1 6 9 7 8 3 5 2 $= \sigma \in \mathfrak{S}_{n+1}$



A permutation with its corresponding
“Laguerre history”

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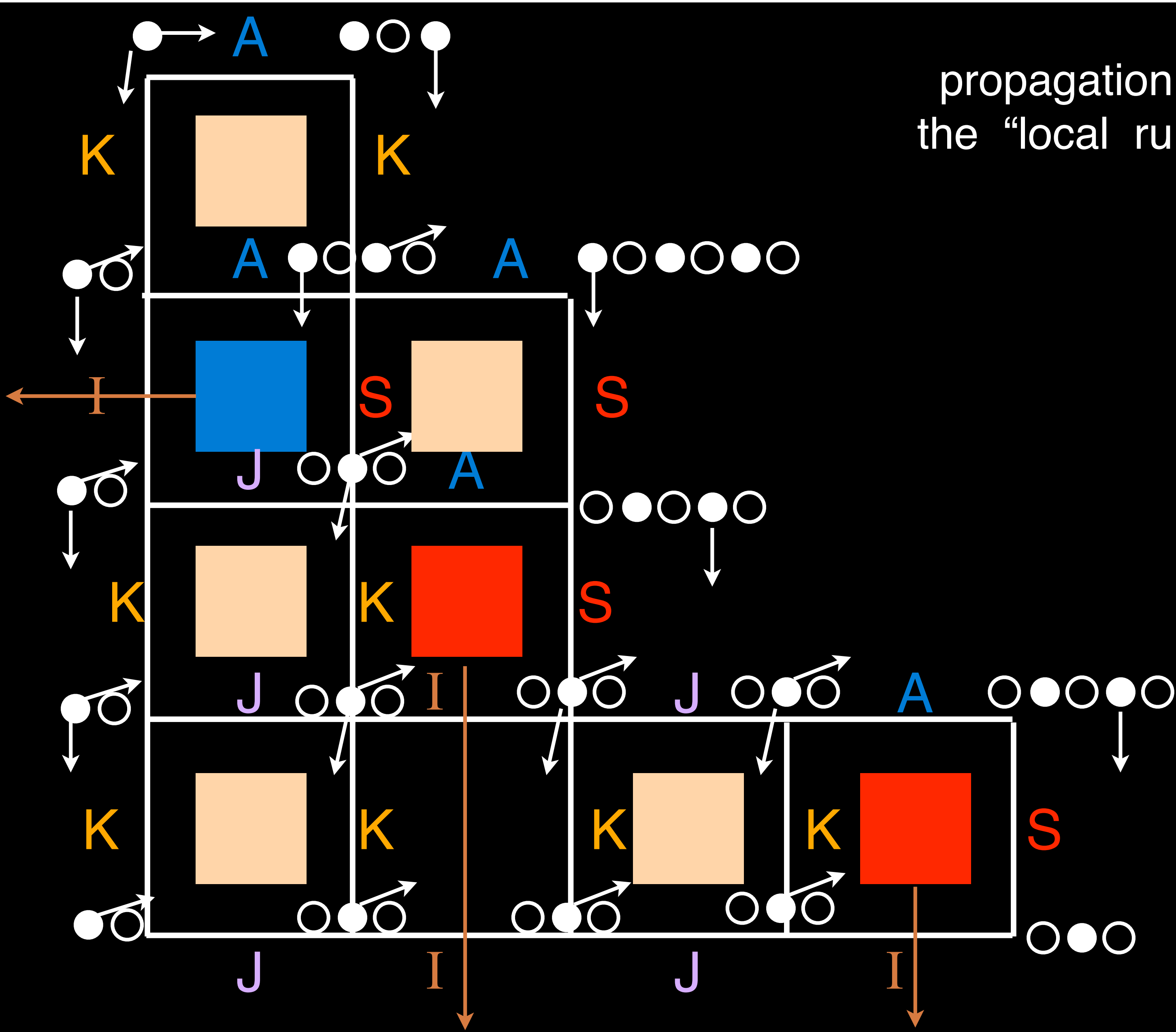
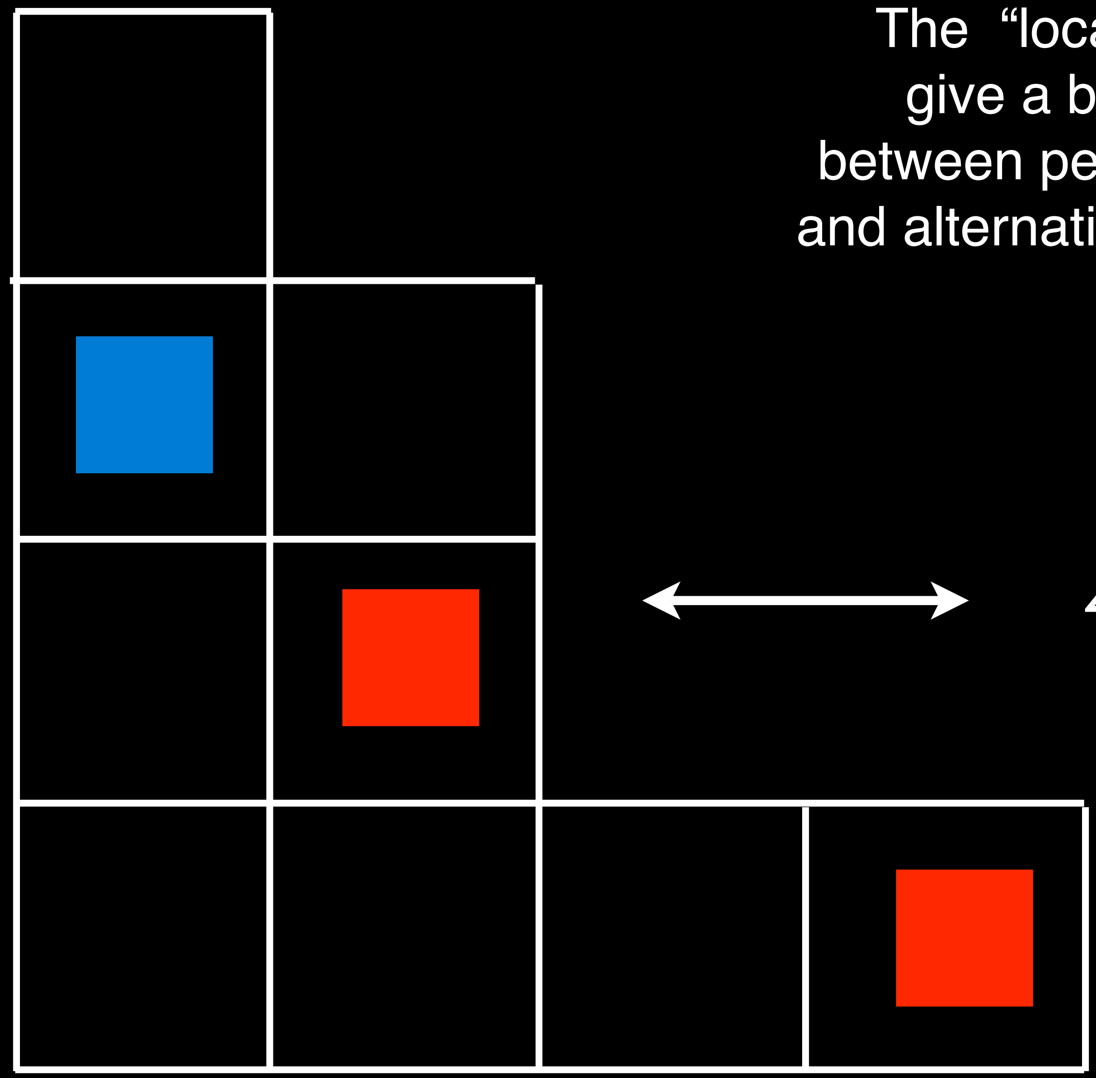
propagation of
the "local rules"



The "local rules"
give a bijection
between permutations
and alternative tableaux



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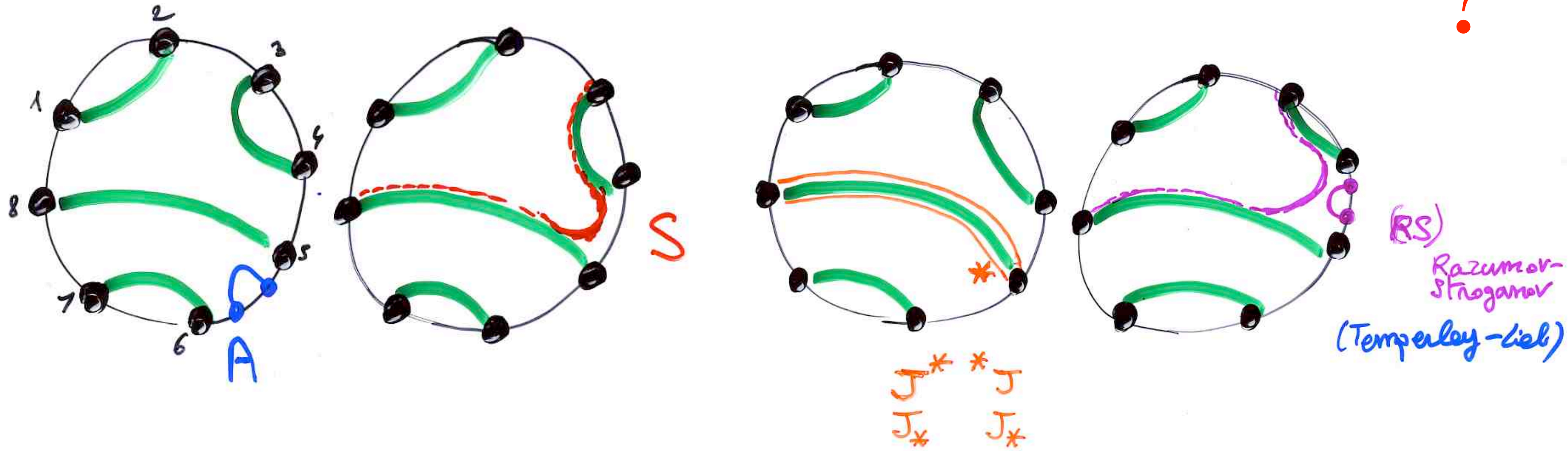
some other operators on chord diagrams

?



$$AS = SA + {}^{*}J + J^{*} + {}_{*}J + J_{*} + (RS)$$

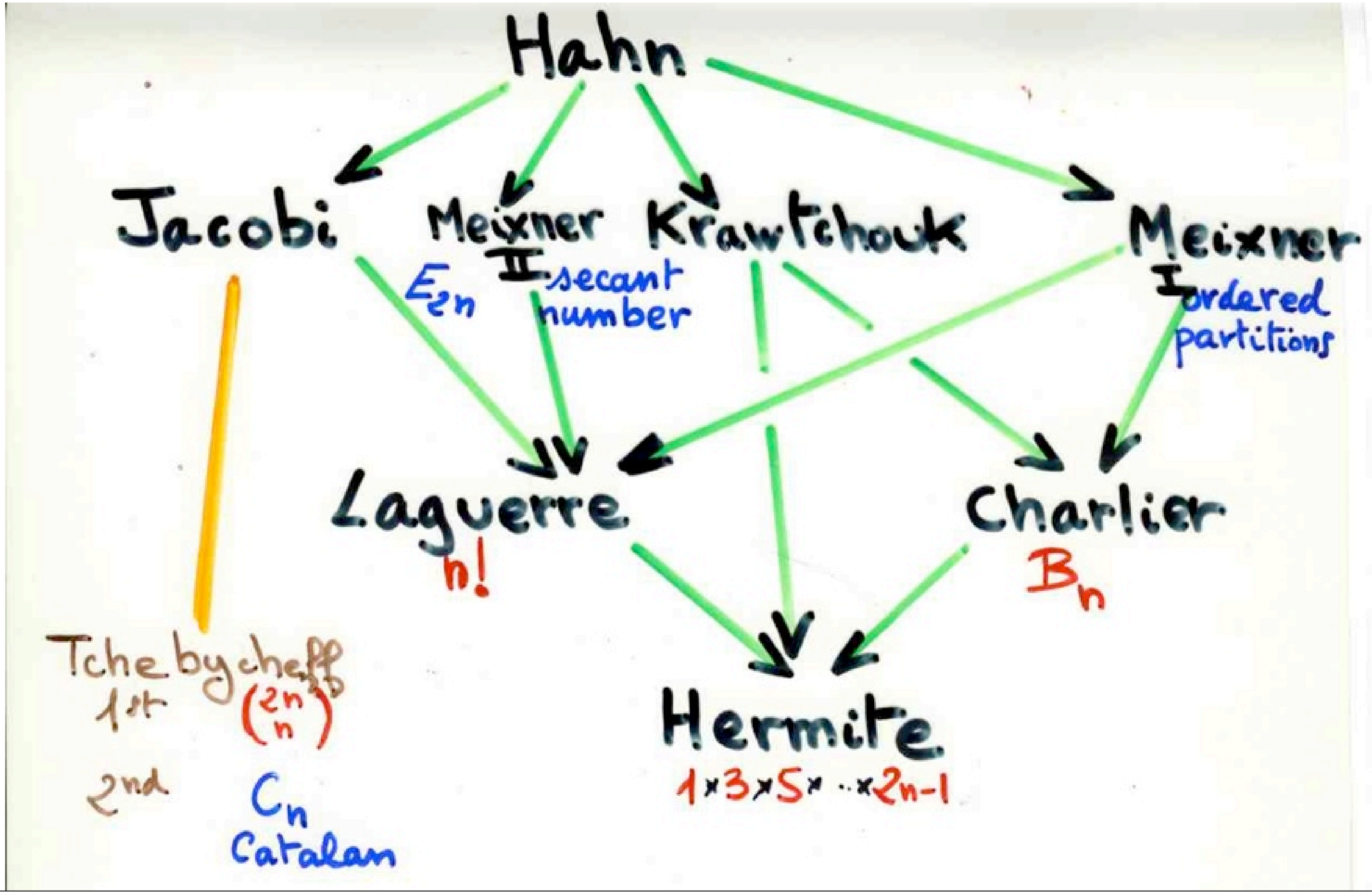
PASEP : binary words → Catalan → permutations

RS : Catalan ⇒ " ⇒ ASM

Hypercube → associahedron → permutohedron → alternah.

Orthogonal polynomials are behind operators and the story of PASEP, ASM and Alternating Tableaux

Askey-Wilson polynomials



- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\,\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta,$ q

Askey-Wilson polynomials

ASM

1-, 2-, 3- enumeration $A_n(x)$

Colomo, Pronco, (2004)

Hankel determinants

(continuous) Hahn, Meixner-Pollaczek,
(continuous) dual Hahn orthogonal polynomials

Ismail, Lin, Roan (2004)
XXZ spin chains and Askey-Wilson operator

Conclusion

from the “matrix ansatz” to the :

“The cellular ansatz”

Conclusion: In this talk I have presented a sort of

"cellular ansatz"

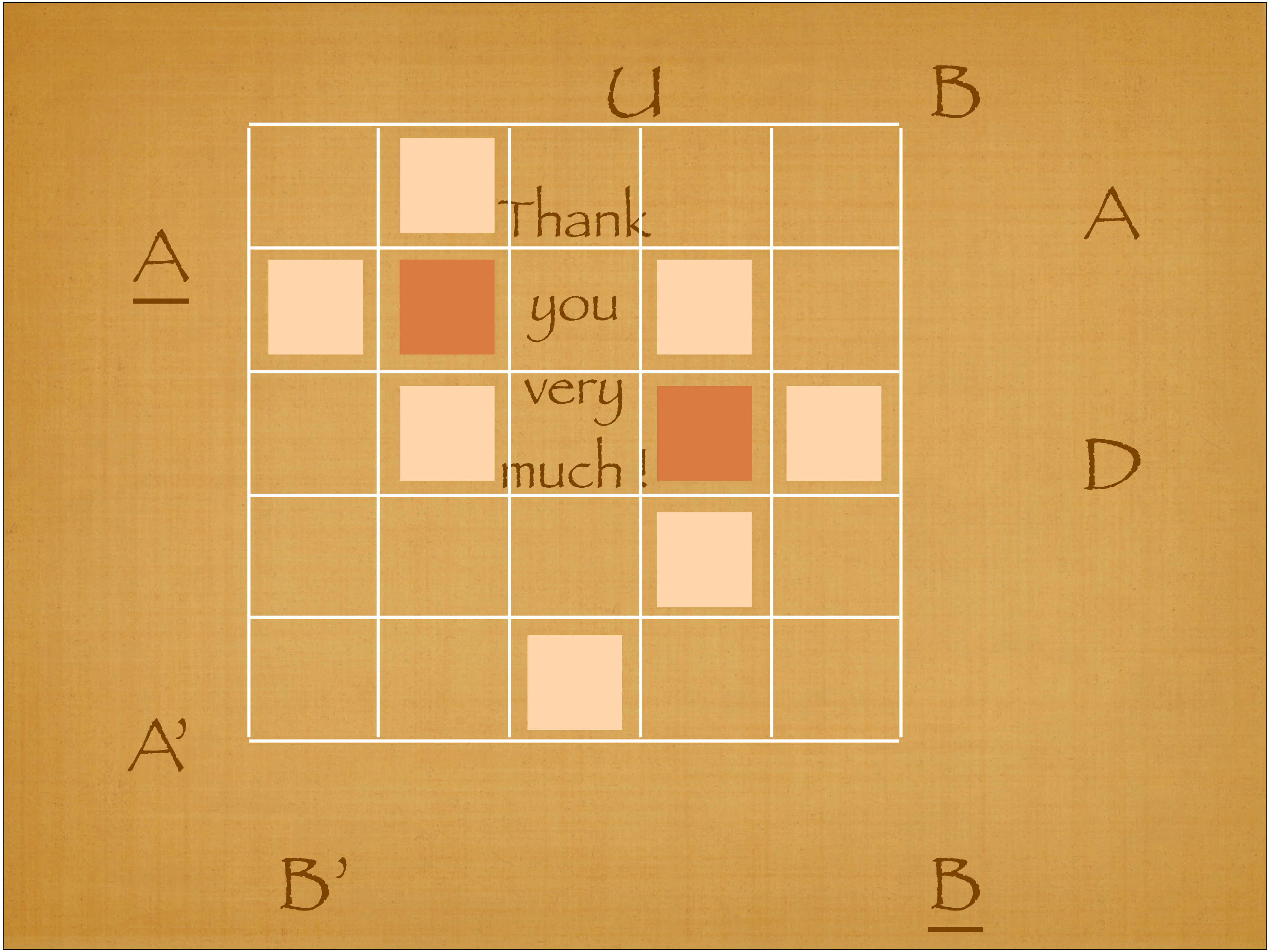
- Some (formal) operators satisfying some commutation relations are given and generate a certain quadratic algebra.

- The computations in this algebra are made by some (oriented) rewriting rules which are visualized in a planar way on a (square) elementary cell of a grid. May be the operator identity I has to be introduced as another formal operator.

- The rewriting of a word of the algebra is visualized by a kind of a 2D cellular oriented expansion. The edges of the grid are labeled by the operators, the cells are labeled by each of the possible rewriting rules.

- The grid with the final labeling of the cells is in bijection with a class P of combinatorial objects (Permutations, Alernative tableaux, ASM, FPL, Tilings, etc ...).

- If the operators can be represented as combinatorial operators acting on a certain class F of combinatorial objects, then a simple combinatorial explanation of the commutation rules can be "attached" to each labeled cell of the grid. The vertices of the grid becomes labeled by the objects of F and "local rules" should be defined. In the case (as in the two examples of RSK and Alternating tableaux) when only the labels of the cells, and not those of the edges, are needed for defining the local rules, then from the cellular propagation of these local rules across the grid, one obtain a bijection between the objects of P and some other objects coded by the sequence of the F-labels on the border of the grid.


U
B
Thank
you
very
much !
A
A
D
A'
B'
B

xgv website :

http://www.labri.fr/perso/viennot/

Recherche, cv, publications, exposés, diaporamas, livres, petite école, photos: voir ma page personnelle ici

Vulgarisation scientifique voir la page de l'association Cont'Science

downloadable papers, slides and lecture notes, etc ... here

(the summary on page "recherches" and most slides are in english)

→ **page "video"**

"Alternative tableaux, permutations and asymmetric exclusion process"

conference 23 April 2008,

Isaac Newton Institute for Mathematical science

or http://www.newton.cam.ac.uk/ (page "web seminar")

on *xgv website :* http://www.labri.fr/perso/viennot/

other slides of talks available at the **page "exposés"** ←

- For a more complete introduction to RSK and Fomin's local rules, see:

Robinson-Schensted-Knuth: RSK1 (pdf, 9,1 Mo)
groupe de travail de combinatoire, Bordeaux, LaBRI, Février 2005

Robinson-Schensted-Knuth: RSK2 (pdf,10,8Mo)
groupe de travail de combinatoire, Bordeaux, LaBRI, Février 2005

- For a more detailed description of the relationship between Temperley-Lieb algebra and heaps of dimers, see:

Jacobi-Trudi, Linström-Gessel-Viennot, Fomin-Kirillov, Lascoux-Schützenberger, Yang-Baxter, Temperley-Lieb, deux exposés au groupe de travail de Combinatoire énumérative, LaBRI, Bordeaux:

Première partie 28 Avril 2006 (pdf, 9,4 Mo)
Deuxième partie 12 Mai 2006 (pdf, 11,7 Mo)

- Also on this page "exposés" you can find the slides of the talk, with an extended abstract and a complementary set of slides:

- Alternative tableaux, permutations and partially asymmetric exclusion process, (pdf, 9,9 Mo) workshop "Statistical Mechanics & Quantum-Field Theory Methods in Combinatorial Enumeration", Isaac Newton Institute for Mathematical Science, Cambridge, 23 April 2008.

Papers corresponding to this talk and the talk on alternative tableaux (Isaac Newton Institute, April 2008), are in preparation:

X.G.Viennot, Alternative tableaux and permutations,

X.G.Viennot, Alternative tableaux and partially asymmetric exclusion process,

X.G.Viennot, A Robinson-Schensted like bijection for alternative tableaux,

X.G.Viennot, The “cellular ansatz”: an alternative approach to alternating sign matrices.

supported by project ANR BLAN06-2-134516 MARS

MARS: “Physique combinatoire: Matrices à Signes Alternant et la conjecture de Razumov-Stroganov

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Supplements

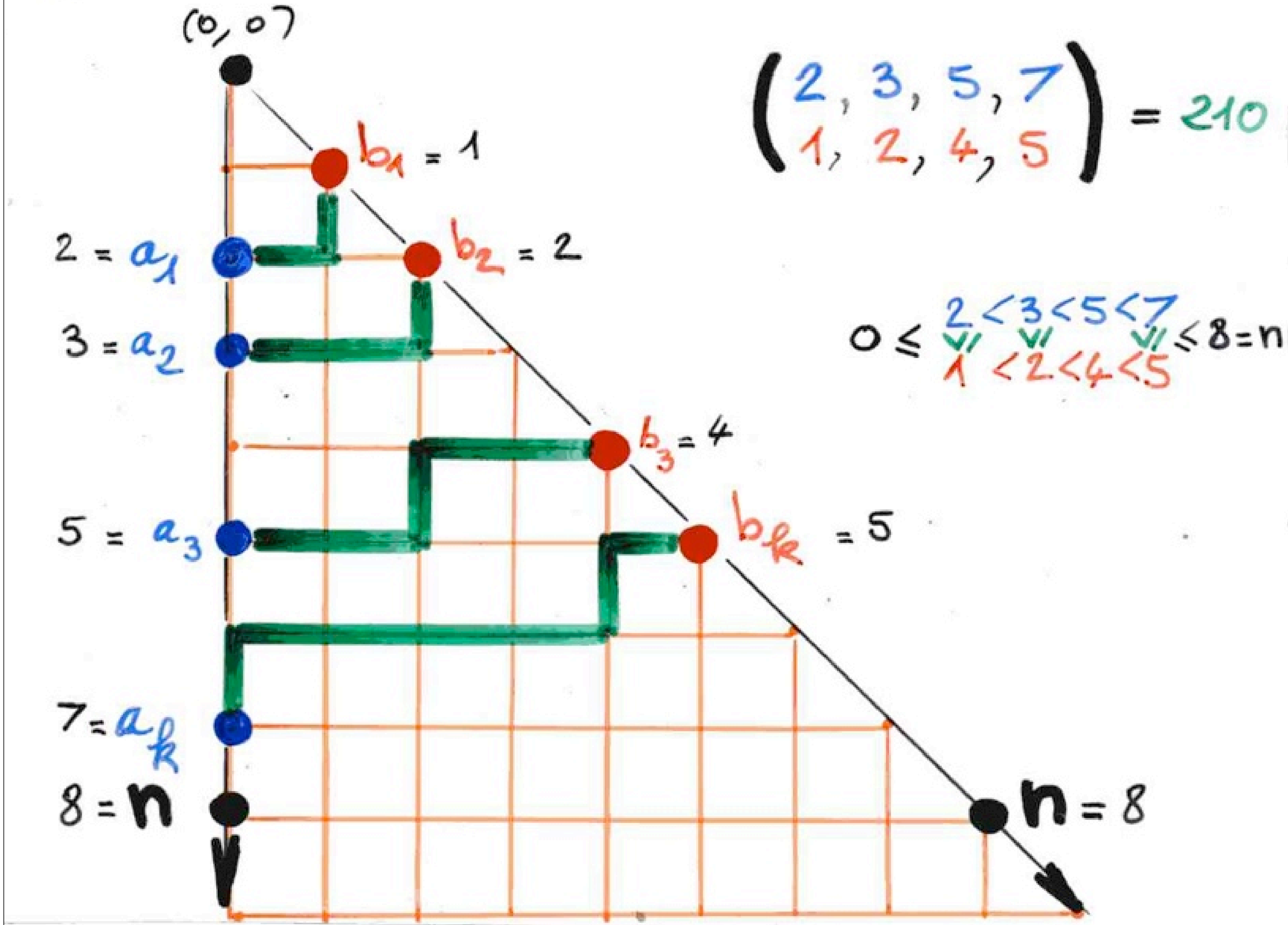
Islande hiver 02 xgv

§7 Other examples of the cellular ansatz

This section has been added after the talk. Many thanks to conversations with P. Di Francesco and P. Nadeau.

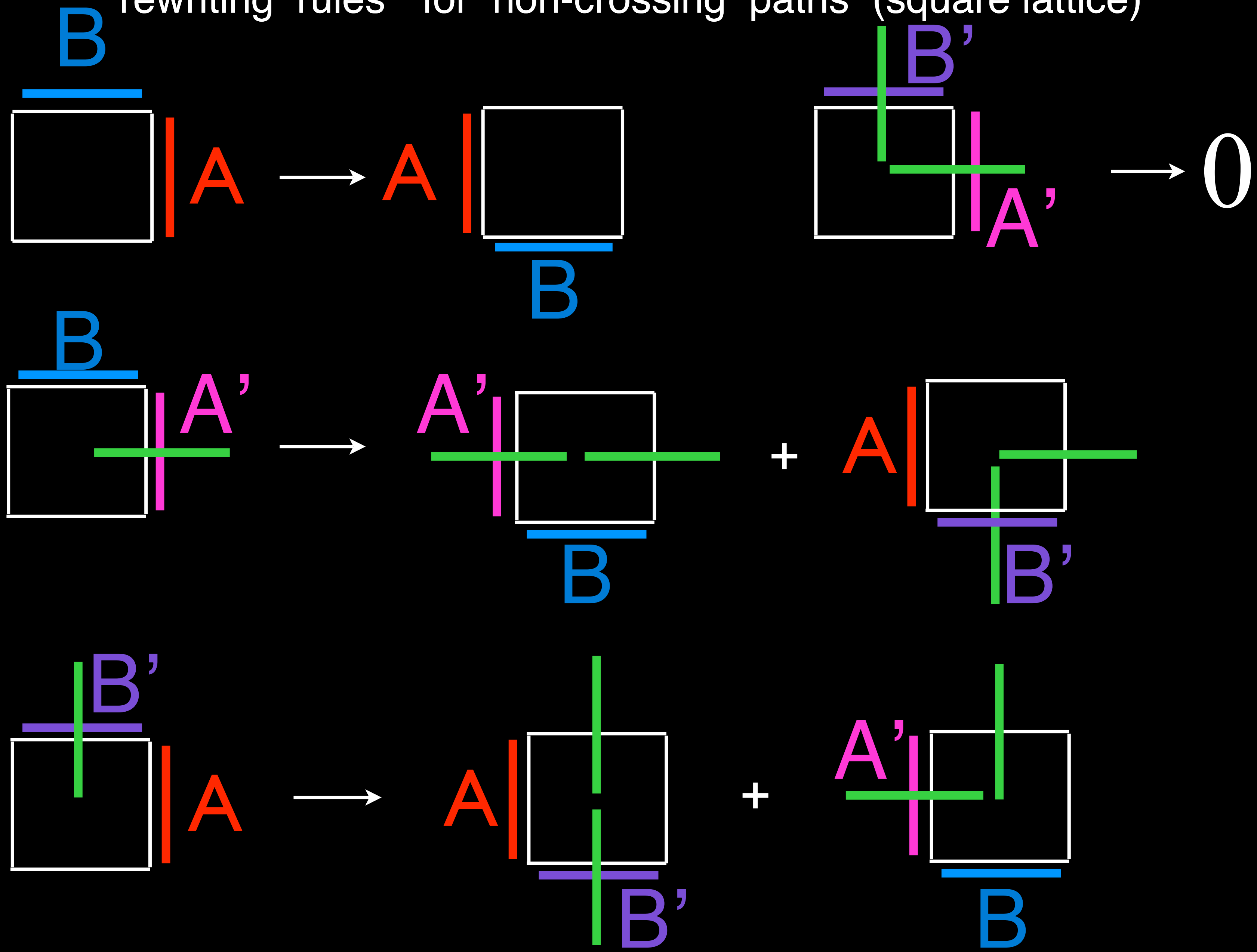
ESI, Vienna, 24 May 2008

Operators for non-crossing configuration of paths (square lattice)

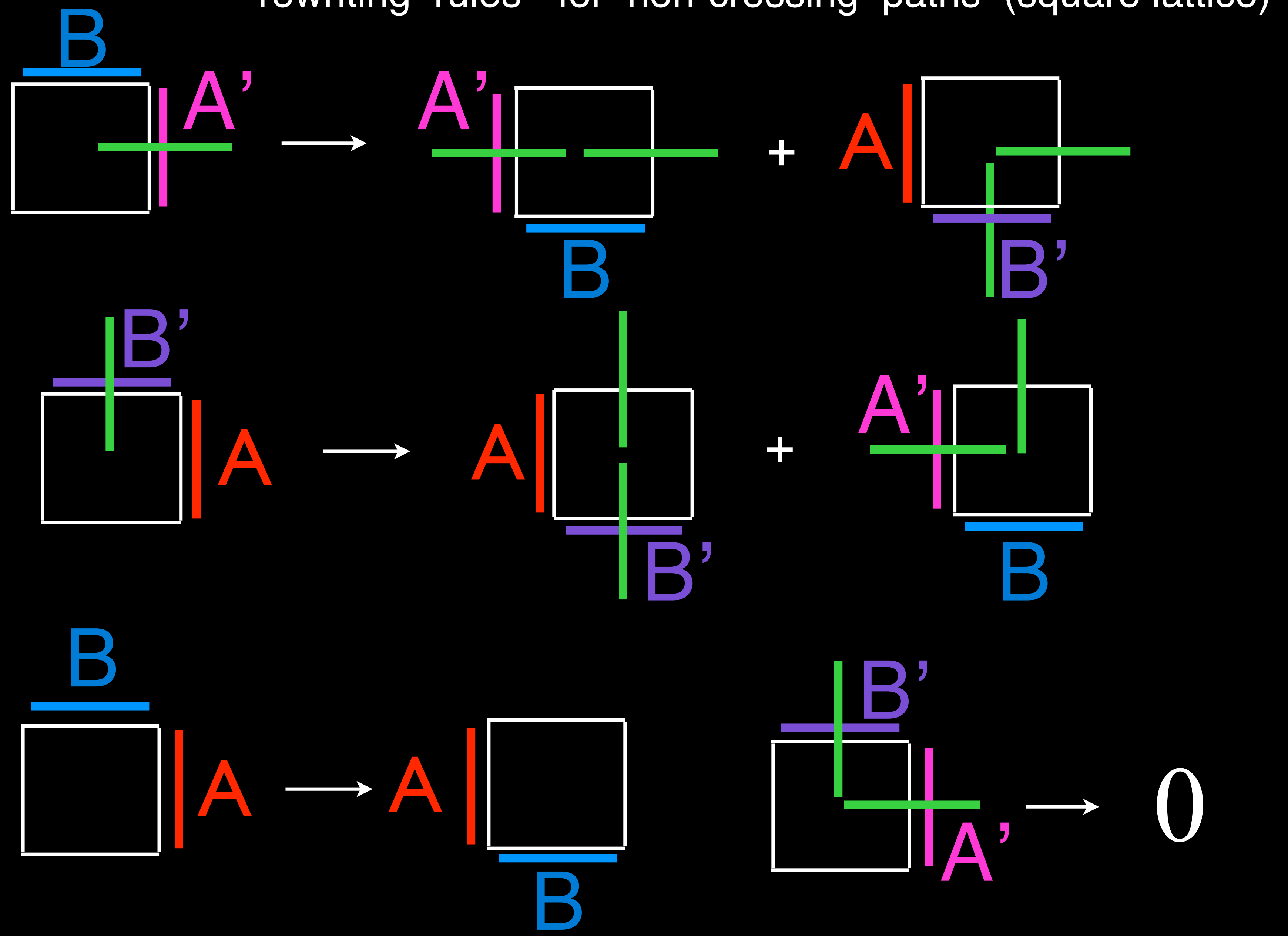


example: binomial determinant

“rewriting rules” for non-crossing paths (square lattice)



“rewriting rules” for non-crossing paths (square lattice)



operators and commutations for non-crossing paths
(square lattice)

$$B\,A' = A'\,B + A\,B'$$

$$B'\,A = A\,B' + A'\,B$$

$$BA = AB \qquad B'\,A' = 0$$

equivalently, exchanging A and A', it can be rewrittten as :

$$B\,A = A\,B + A'\,B'$$

$$B'\,A' = A'\,B' + A\,B$$

$$B\,A' = A'\,B \qquad B'\,A = 0$$

compare with the commutations with ASM or FPL !

example: binomial determinant

The binomial determinant $\binom{2, 3, 5, 7}{1, 2, 4, 5}$

is equal to the coefficient of the word

A A A' A' A A' A A' A B B B B B B B B B

0 1 2 3 4 5 6 7 8

when rewriting the word

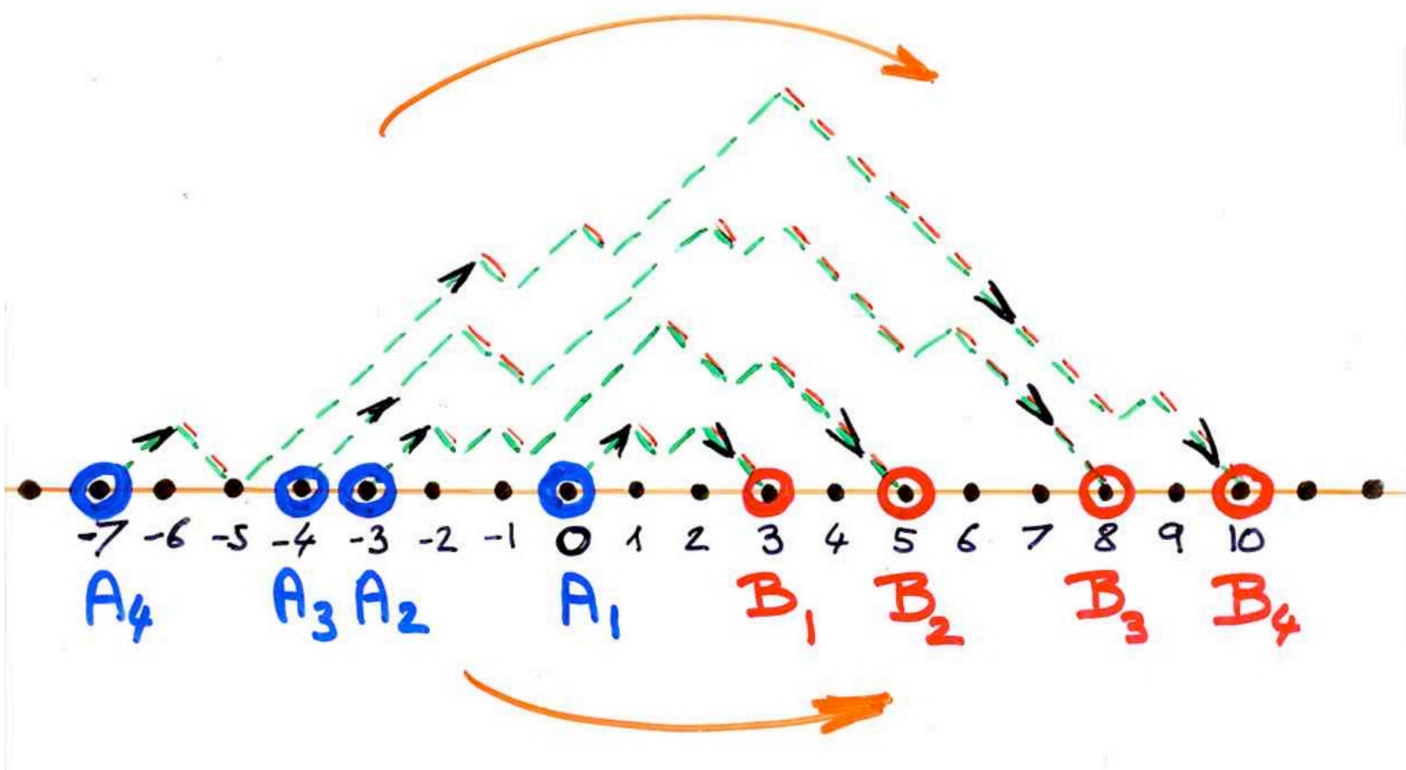
B A B' A B' A B A B' A B' A B A B A B A

0 1 2 3 4 5 6 7 8

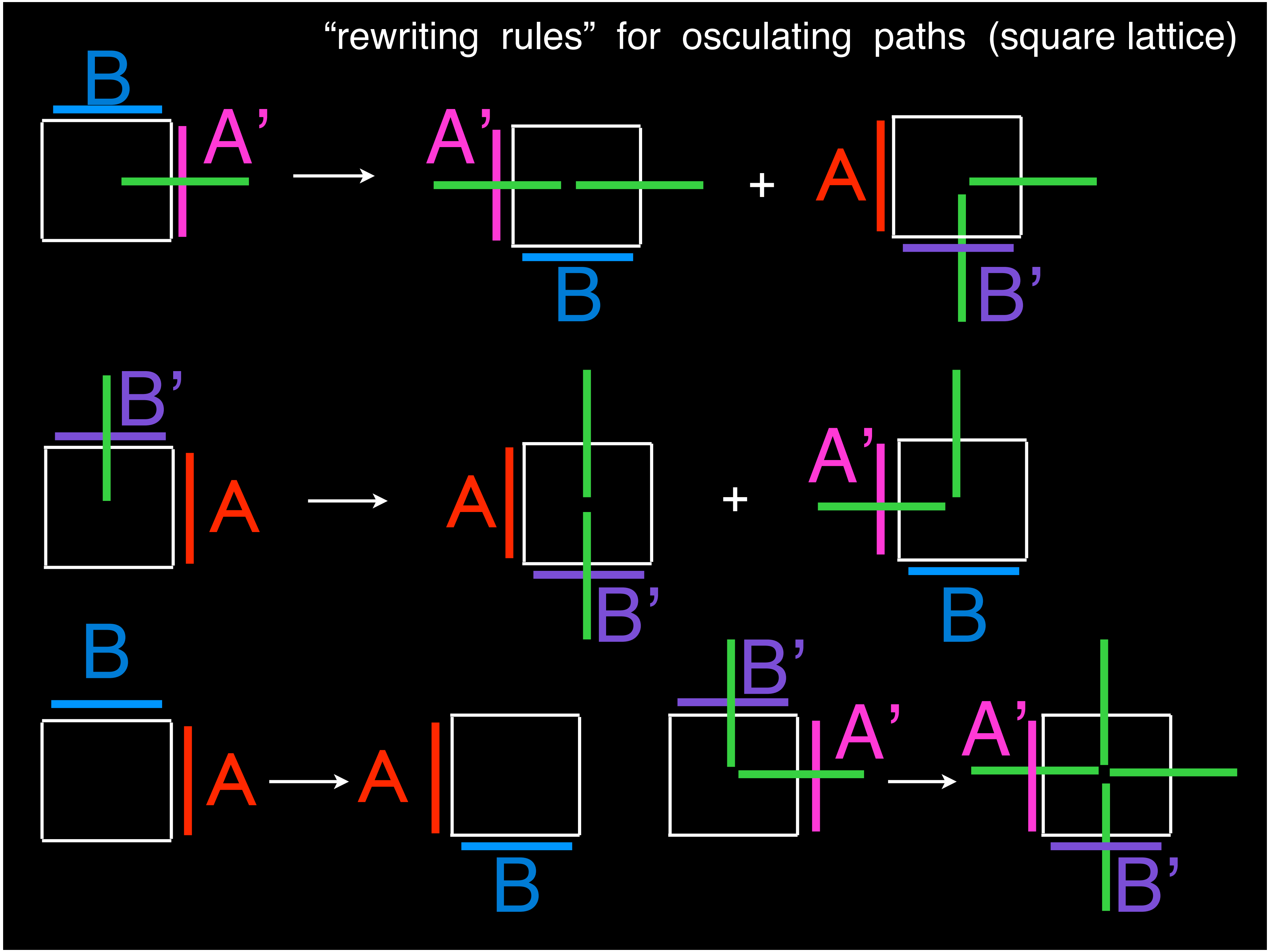
acording to the rewriting rules of the non-crossing configuration of paths (square lattice)

$$\begin{vmatrix} \mu_3 & \mu_5 & \mu_8 & \mu_{10} \\ \mu_6 & \mu_8 & \mu_{11} & \mu_{13} \\ \mu_7 & \mu_9 & \mu_{12} & \mu_{14} \\ \mu_{10} & \mu_{12} & \mu_{15} & \mu_{17} \end{vmatrix}$$

another example:
Hankel determinant
of moments
of orthogonal polynomials



Operators for configurations of osculating paths (square lattice)



“rewriting rules” for osculating paths (square lattice)
B
A’
A’
B
+
A
B’
B’
A
A
B’
+
A’
B
B
A
A
B
B’
A’
A’
B’

operators and commutations for osculating paths (square lattice)

$$B\,A' = A'\,B + A\,B'$$

$$B'\,A = A\,B' + A'\,B$$

$$BA = AB \qquad B'\,A' = A'\,B'$$

equivalently, exchanging A and A', it can be rewrittten as :

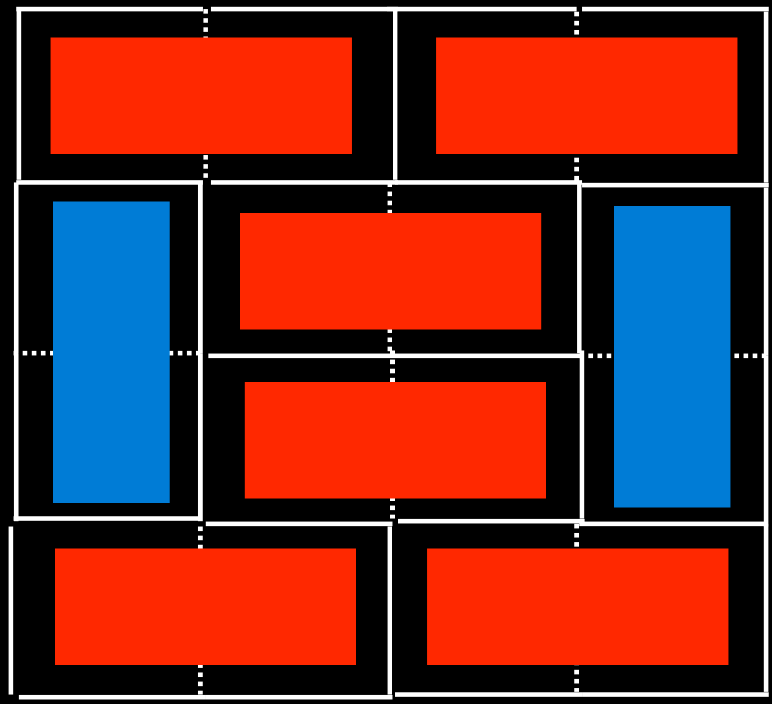
$$B\,A = A\,B + A'\,B'$$

$$B'\,A' = A'\,B' + A\,B$$

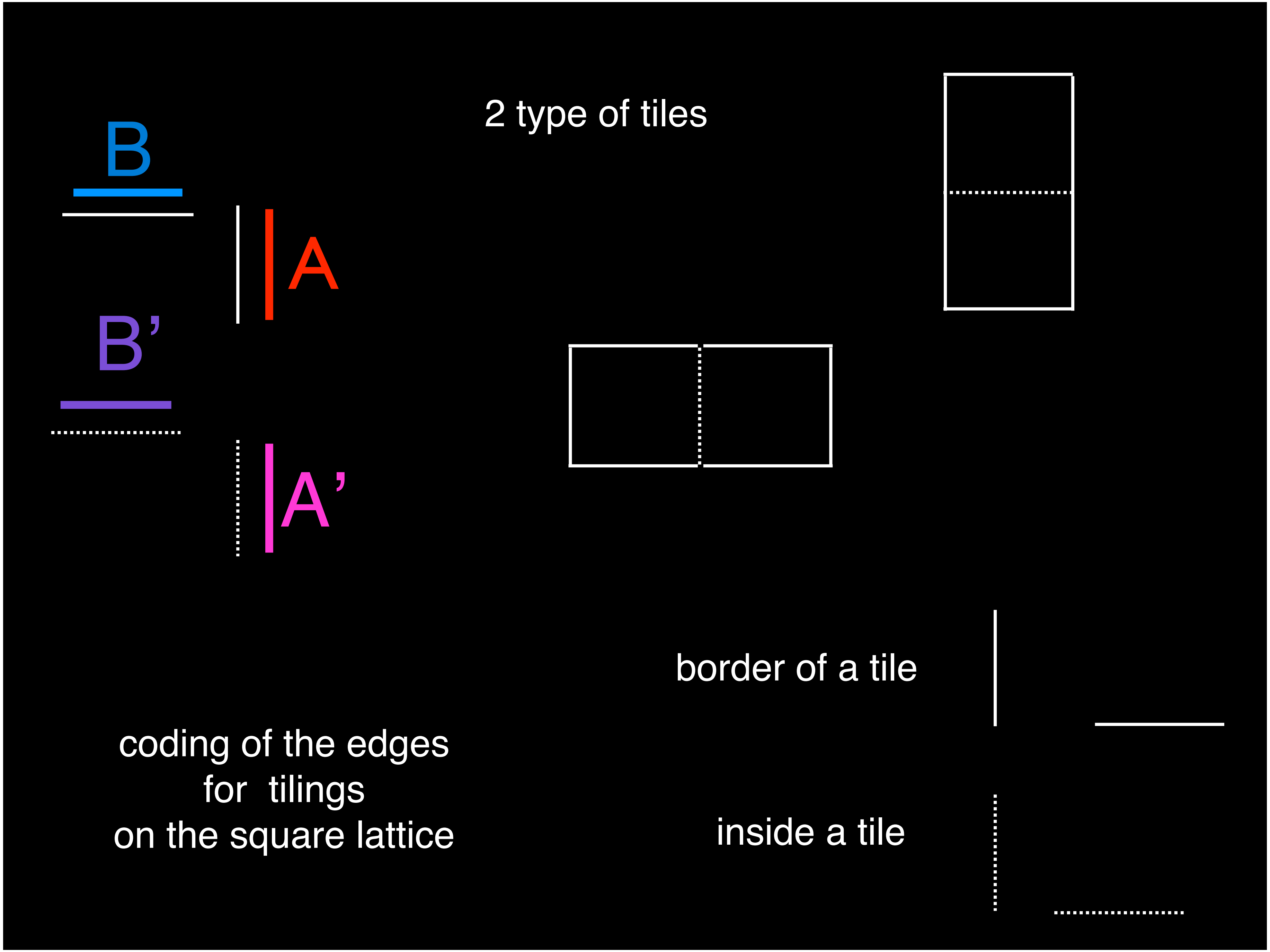
$$B\,A' = A'\,B \qquad B'\,A = A\,B'$$

same as for ASM (which are in bijection with configurations of osculating paths on a square grid)

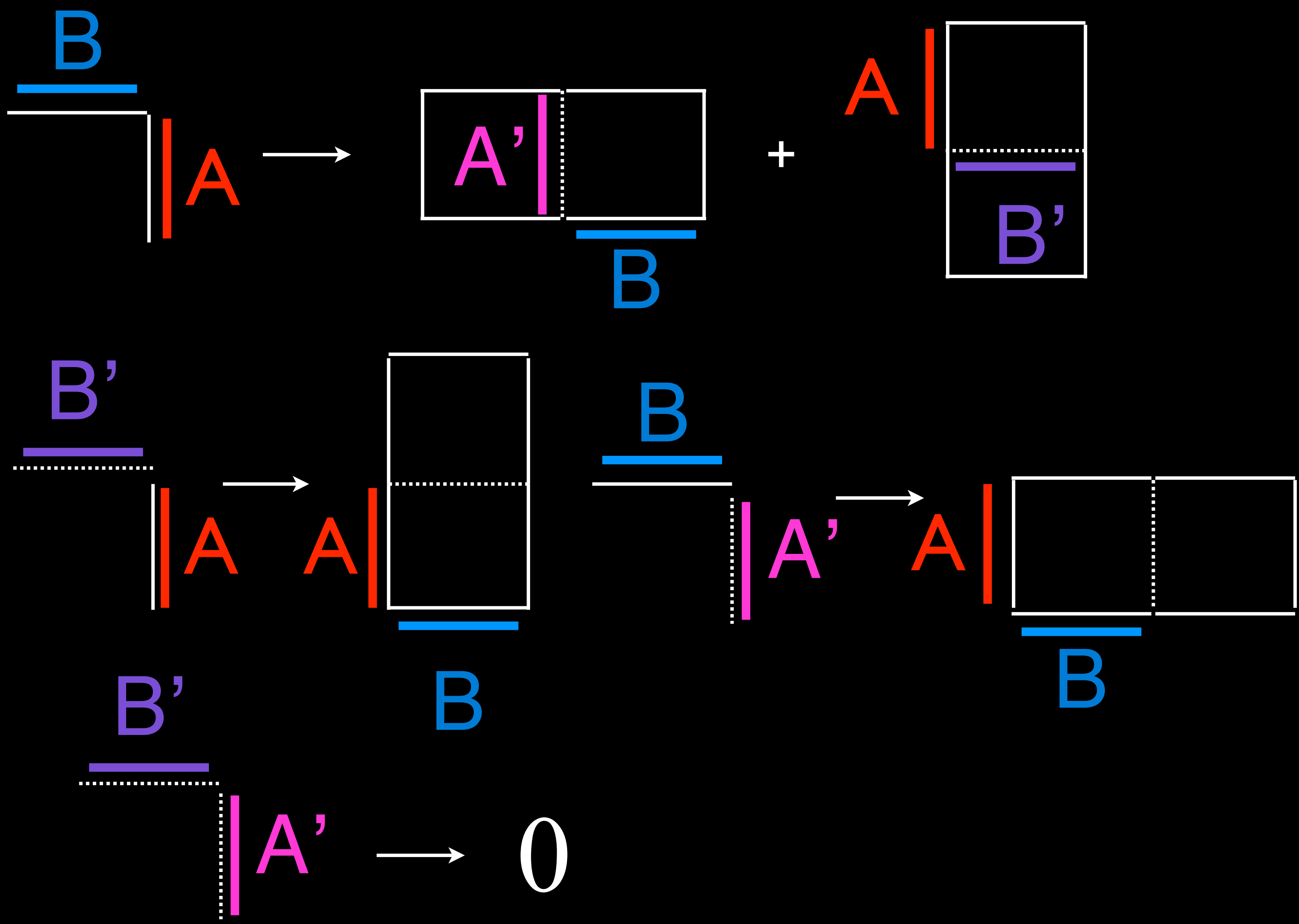
Opertors for dimers tiling
(square lattice)



a tiling
on the
square lattice


2 type of tiles
B
A
B’
A’
border of a tile
inside a tile
coding of the edges
for tilings
on the square lattice

“rewriting rules” for tilings (square lattice)



operators and commutations for tilings (square lattice)

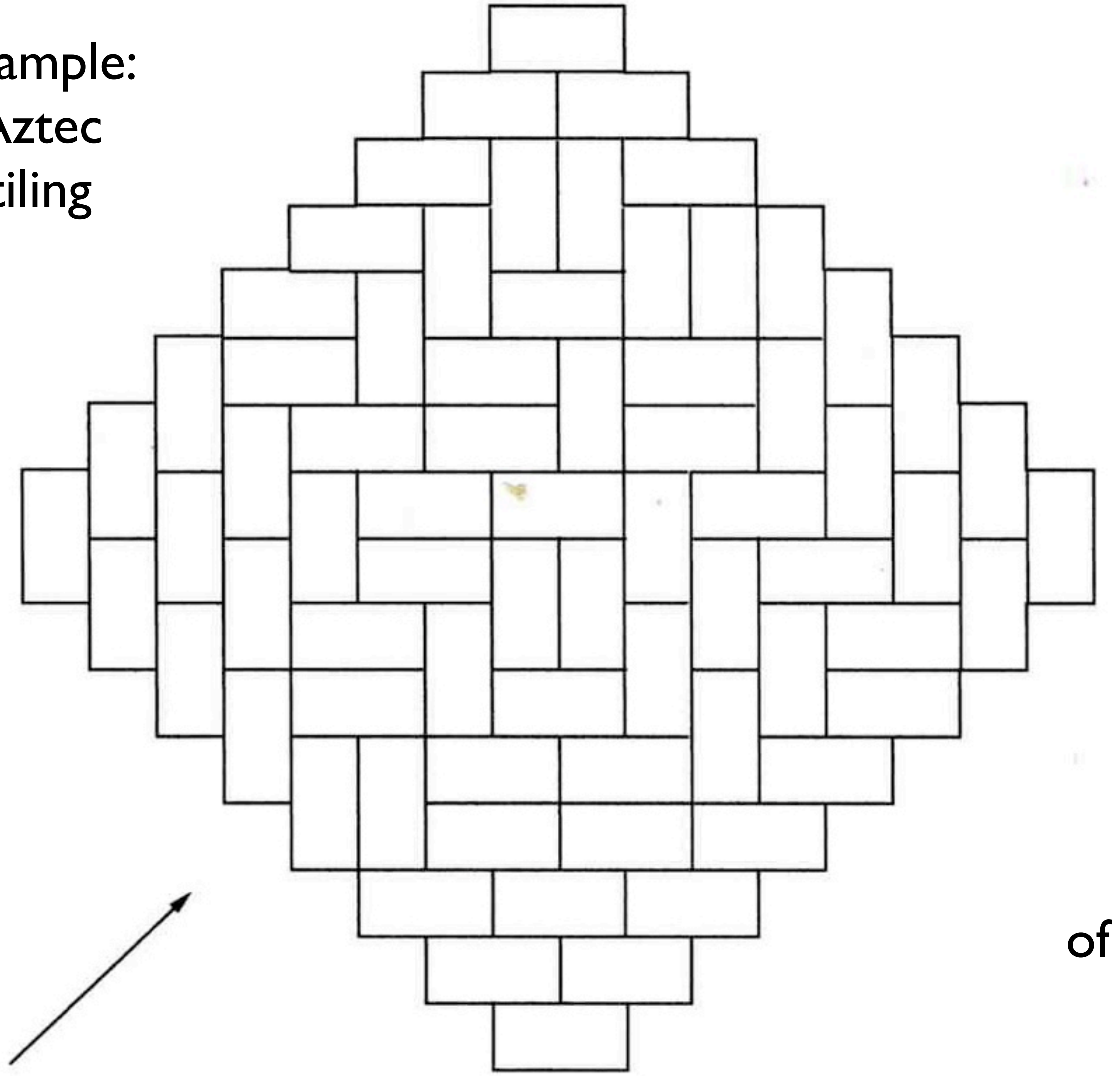
$$B A = A' B + A B'$$

$$B' A' = 0$$

$$B' A = A B$$

$$B A' = A B$$

example:
Aztec
tiling



exercise:
express in term
of words in operators
the well known
formula for the
number of tilings
of the Aztec diamond

Operators for tilings of the triangular lattice

After the talk, P. Di Francesco's gave a construction of four operators for the classical tilings of the triangular lattice coding the TSSCPP and DPP (totally symmetric self-complementary plane partitions and descending plane partitions).
Here is a presentation of these operators.

Operators and commutations for tilings of the triangular lattice

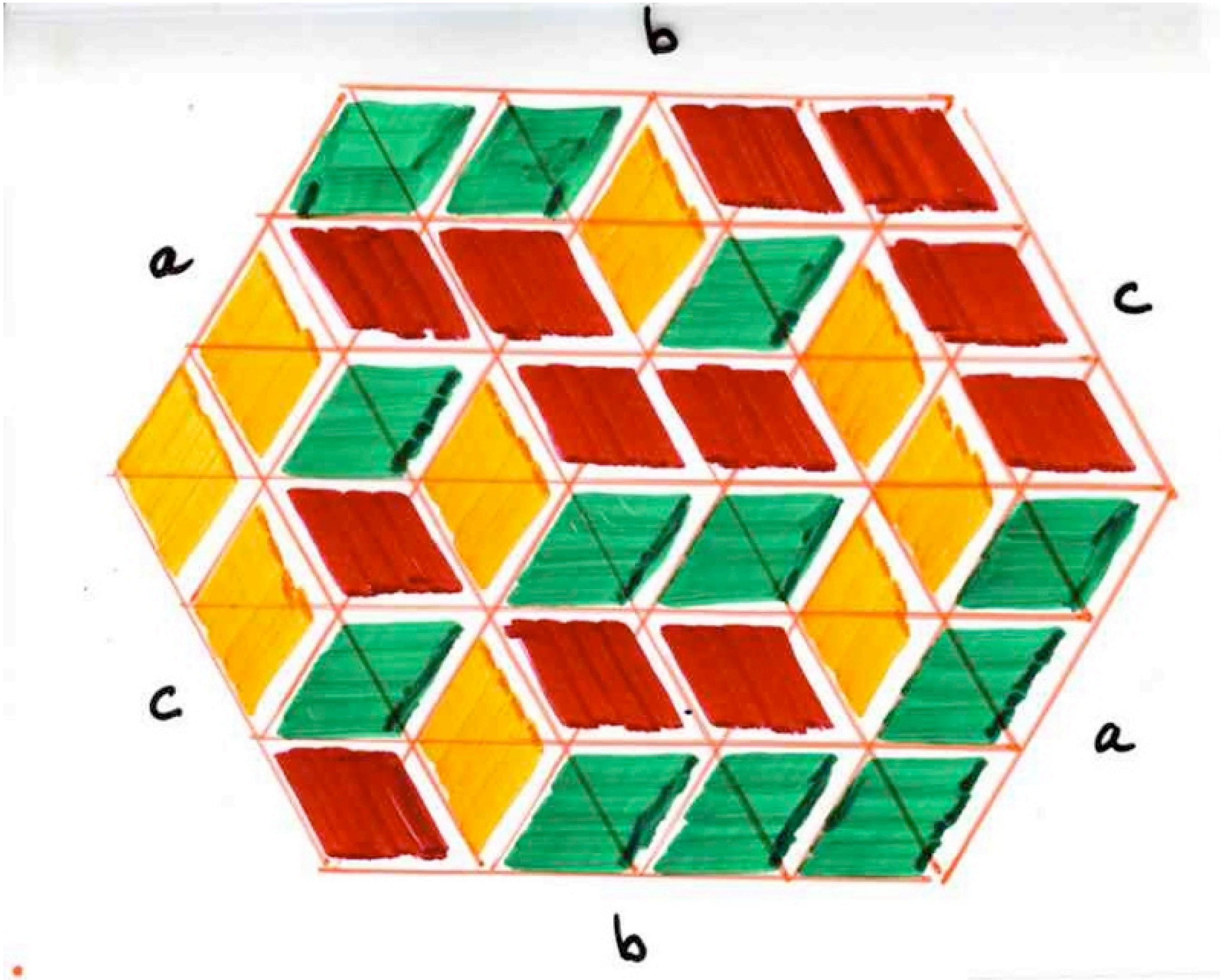
A, A', B, B',

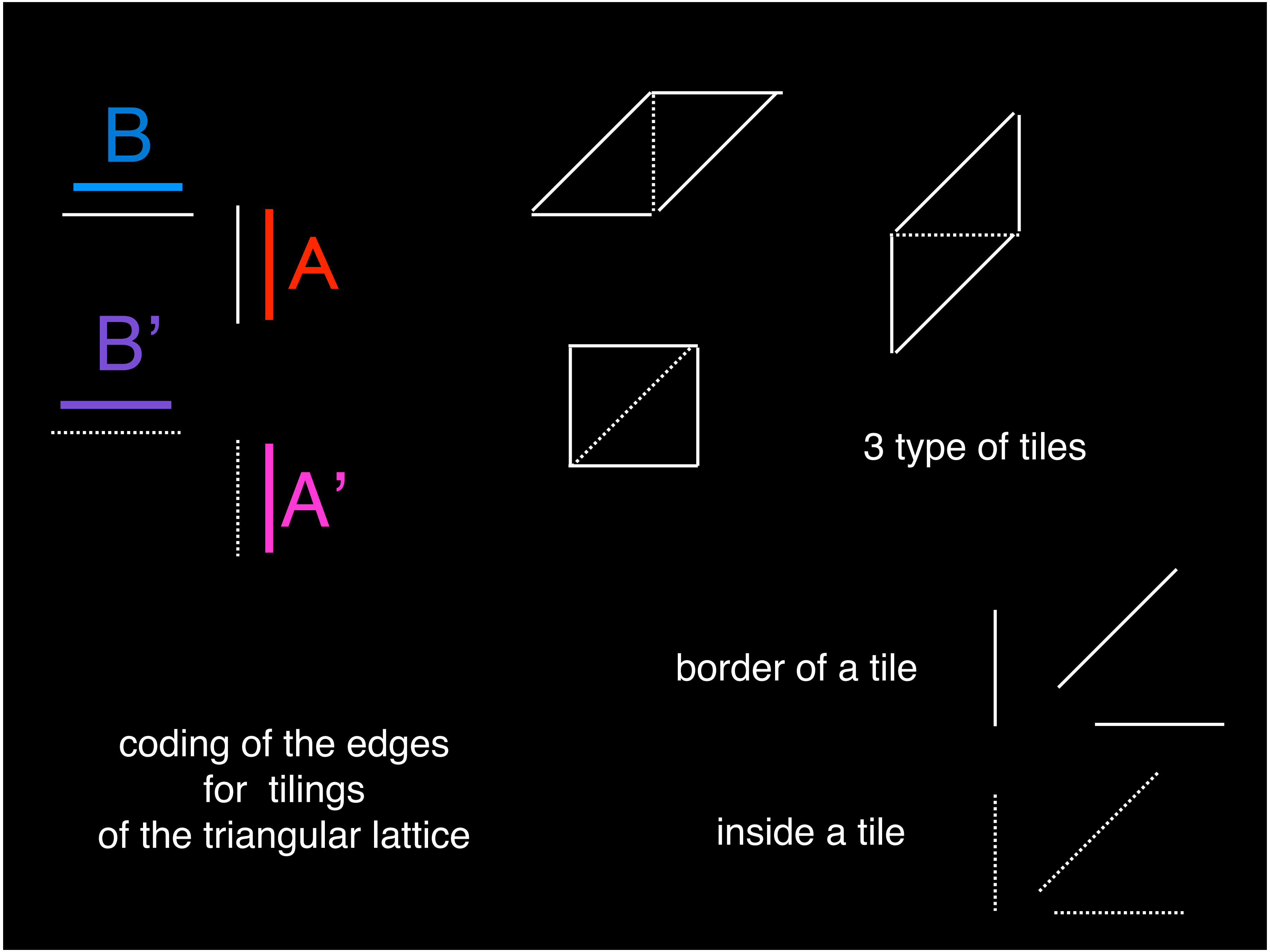
commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = AB \end{cases}$$

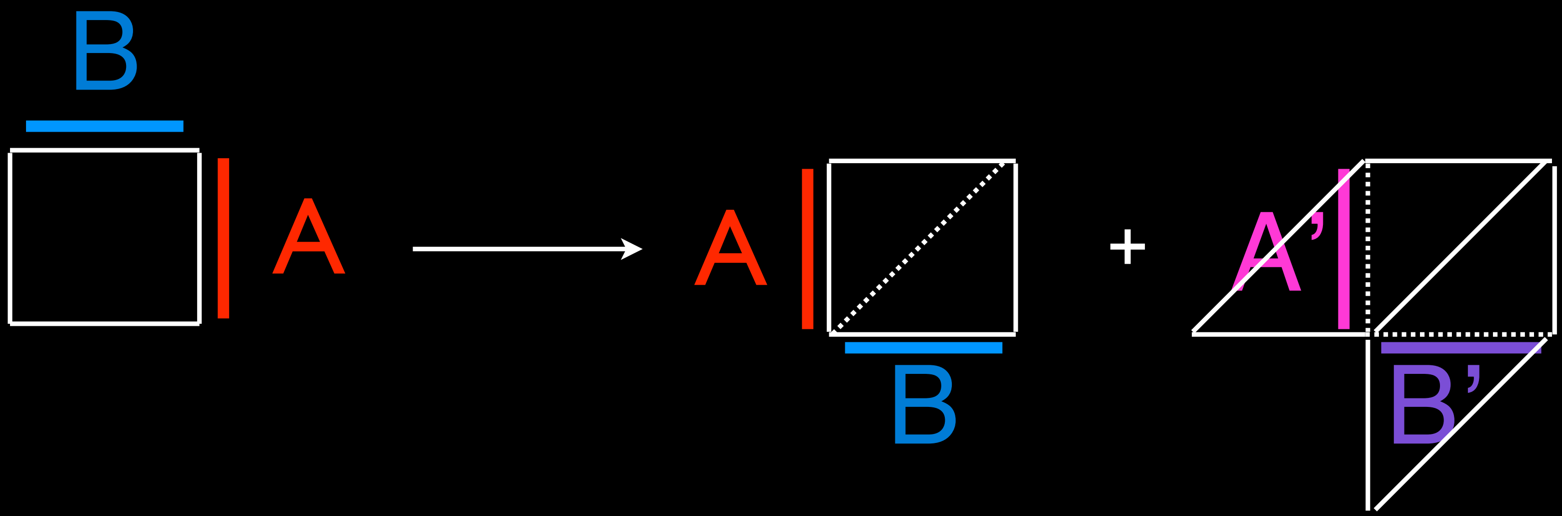
$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

(same as for ASM but with B’A’ ⟶ A’B’ missing !):

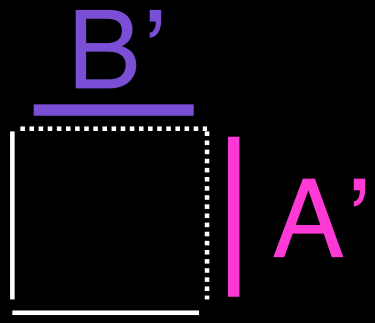

b
a
c
c
a
b

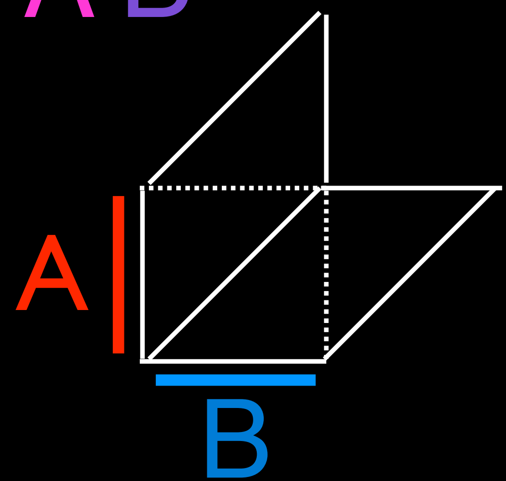

B
A
B'
A'
coding of the edges
for tilings
of the triangular lattice
3 type of tiles
border of a tile
inside a tile

“rewriting rules” for tilings of the triangular lattice



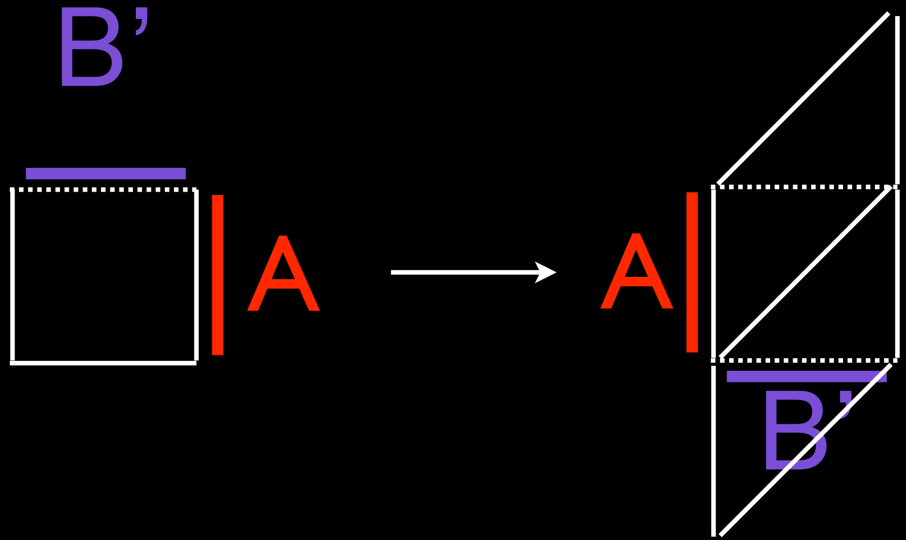
$$BA = AB + A'B'$$



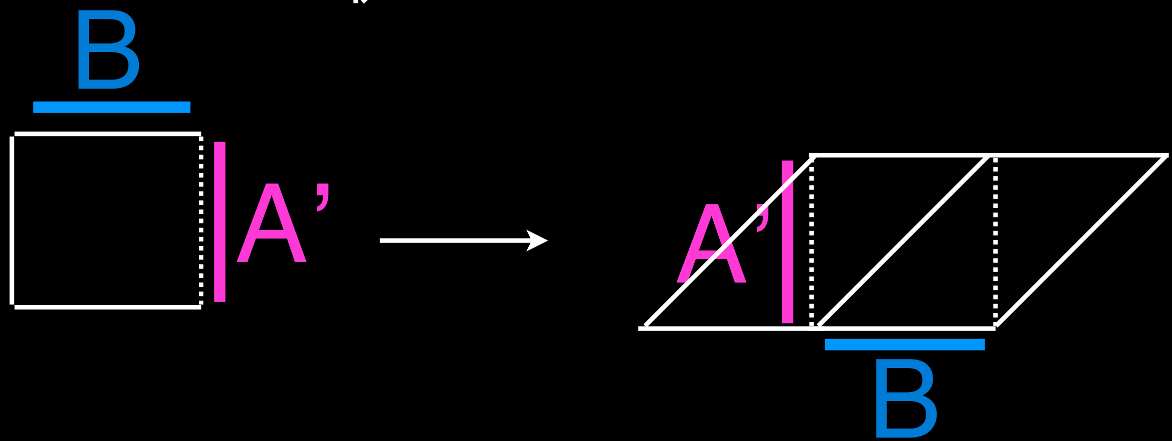


$$B'A' = AB$$

“rewriting rules” for tilings of the triangular lattice



$$B' A = A B'$$



$$B A' = A' B$$

“rewriting rules” for tilings of the triangular lattice

$$BA = AB + A'B'$$

$$B'A' = AB$$

$$B'A = AB'$$

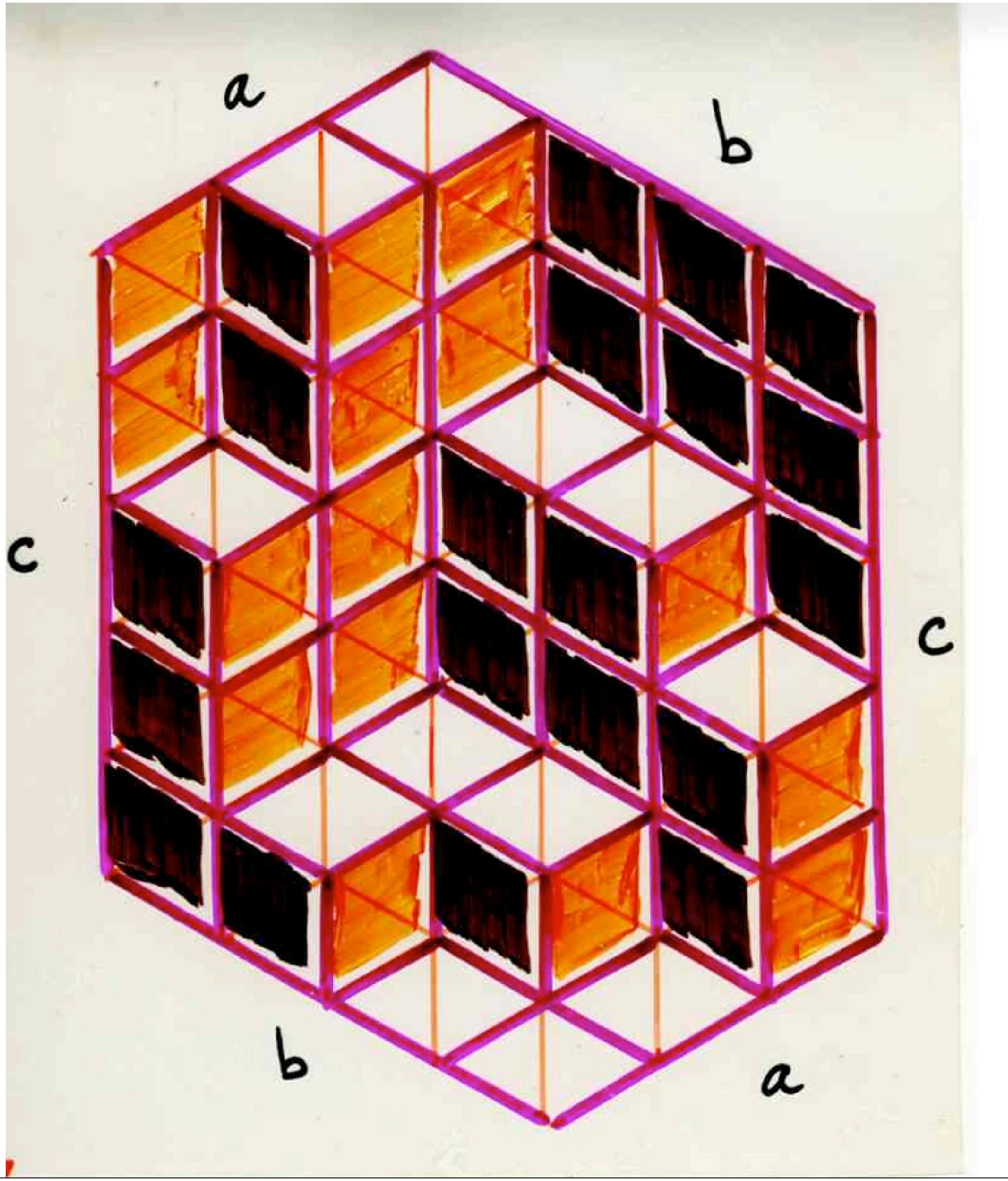
$$BA' = A'B$$

same as for ASM , except the rewriting rule

$B'A' \longrightarrow A'B'$ is forbidden

example:
plane
partitions
in a box

(MacMahon
formula)



Many examples of enumeration of tilings in a triangular lattice can be rewritten using these four operators A, B, A', B'

a simple example:

Prop. The number of plane partitions in a box k x l x m is the coefficient of the word

$$(A')^{k} A^{m} B^{l} (B')^{k}$$

when rewriting the word $B^{k+l} A^{m+k}$

according to the rewriting rules

$$BA \longrightarrow AB \quad \text{or} \quad A'B'$$

$$B'A' \longrightarrow AB$$

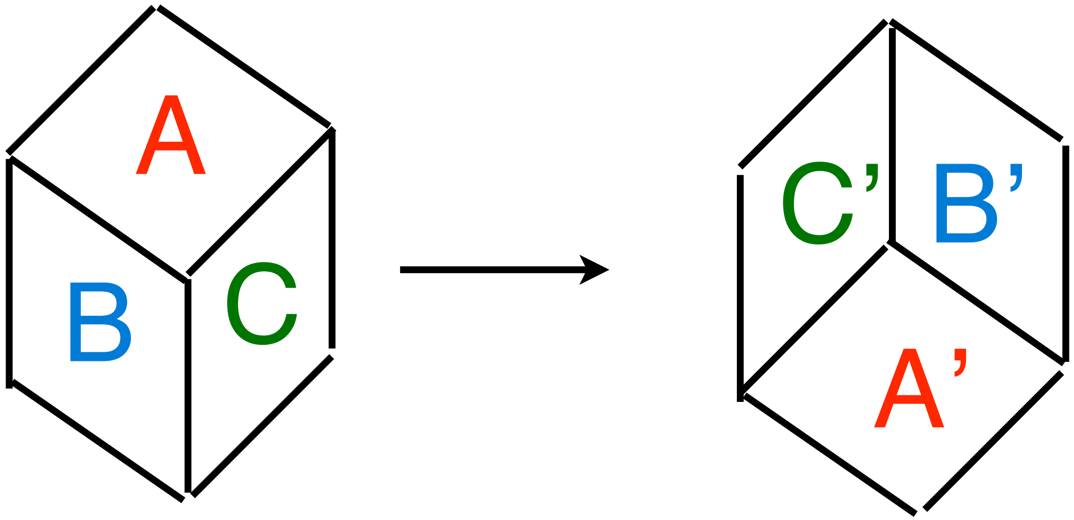
$$B A' \longrightarrow A' B \qquad\qquad B' A \longrightarrow A B'$$

question:

proof of the MacMahon formula (for q=1) with this algebraic approach ?

Remark:

In order to get the parameter q (number of boxes of the 3D diagram), one should consider "3D rewriting rules", ot the type:



this would lead to a 3D "cellular ansatz"

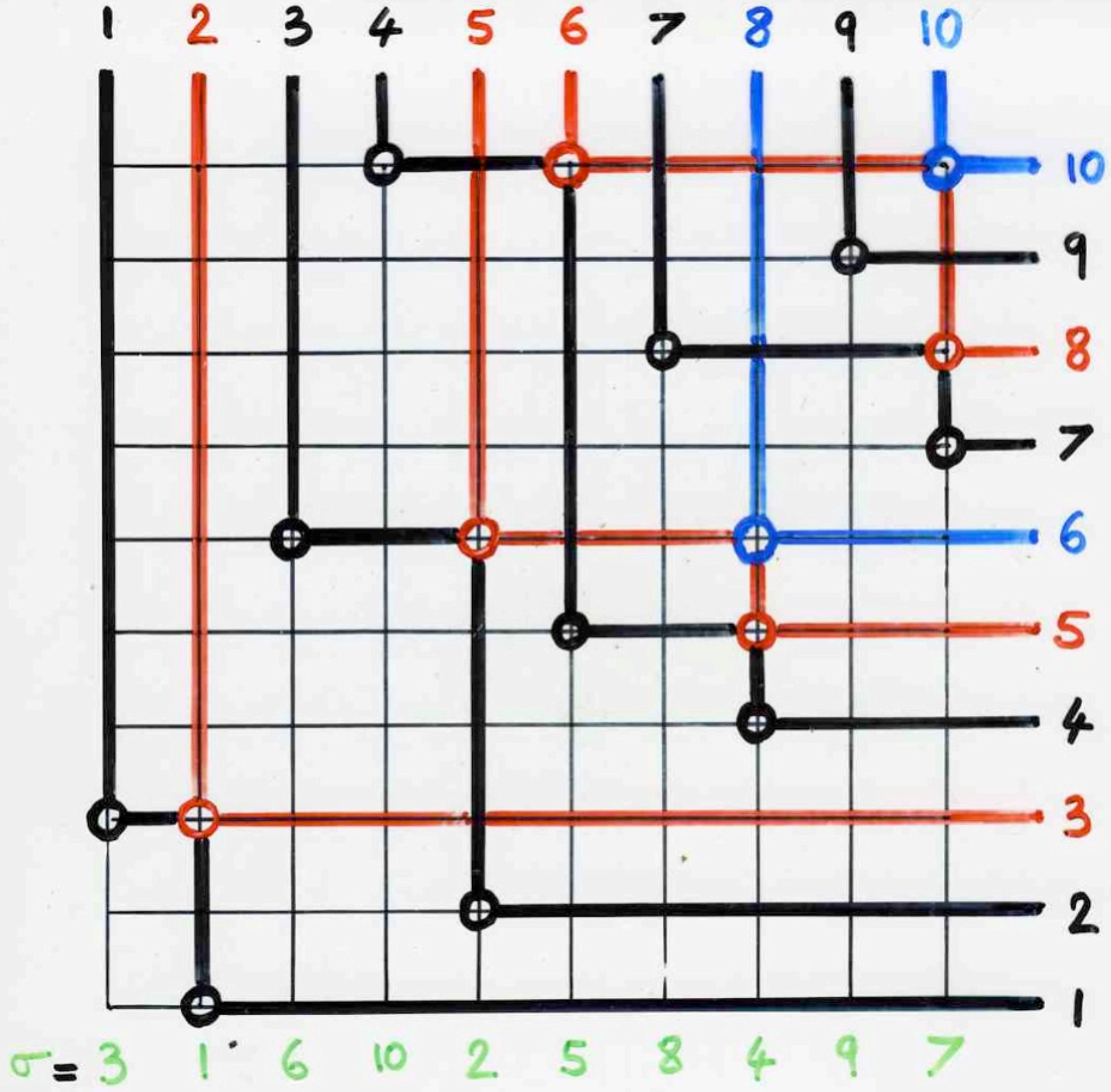
Yang-Baxter operators or equation is an example

Complements

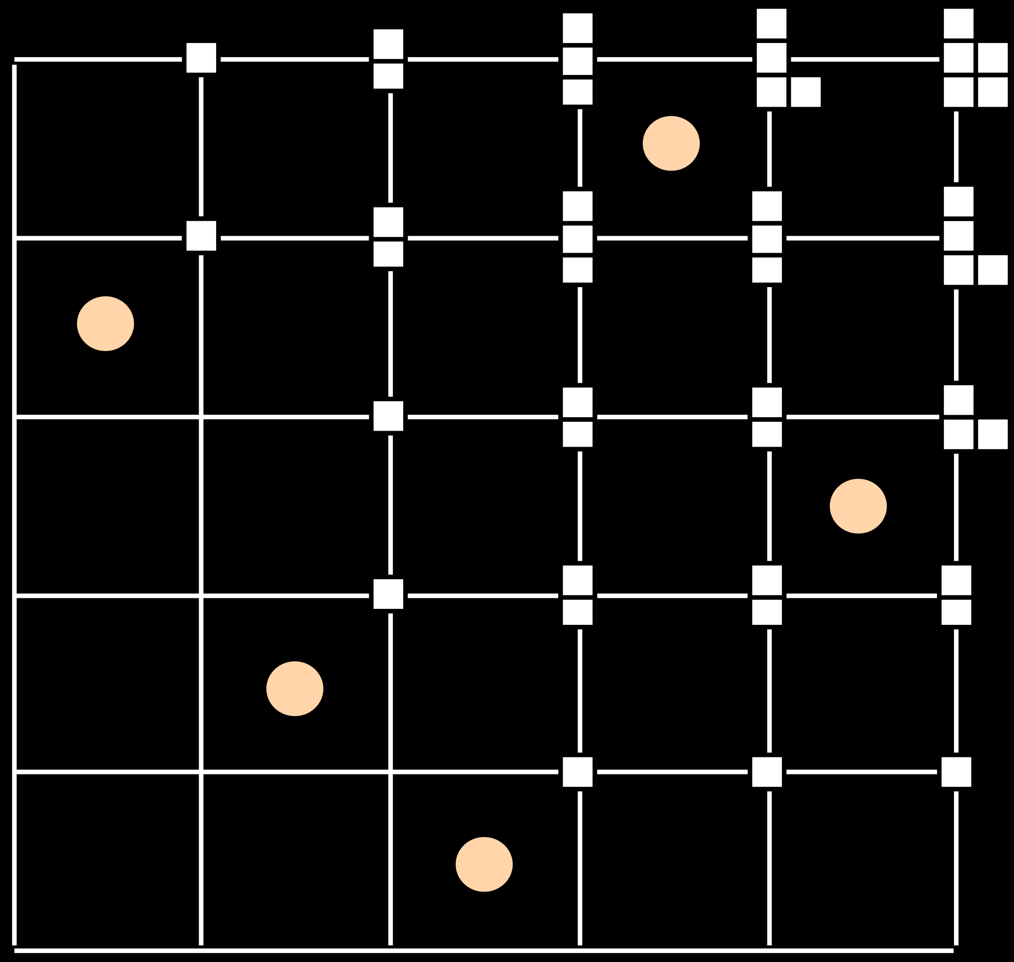
local RSK
and geometric RSK

(the geometric construction with "light" and "shadow" for RSK leads to a simple proof of the fact that RSK and the "local rules" gives the same bijection)

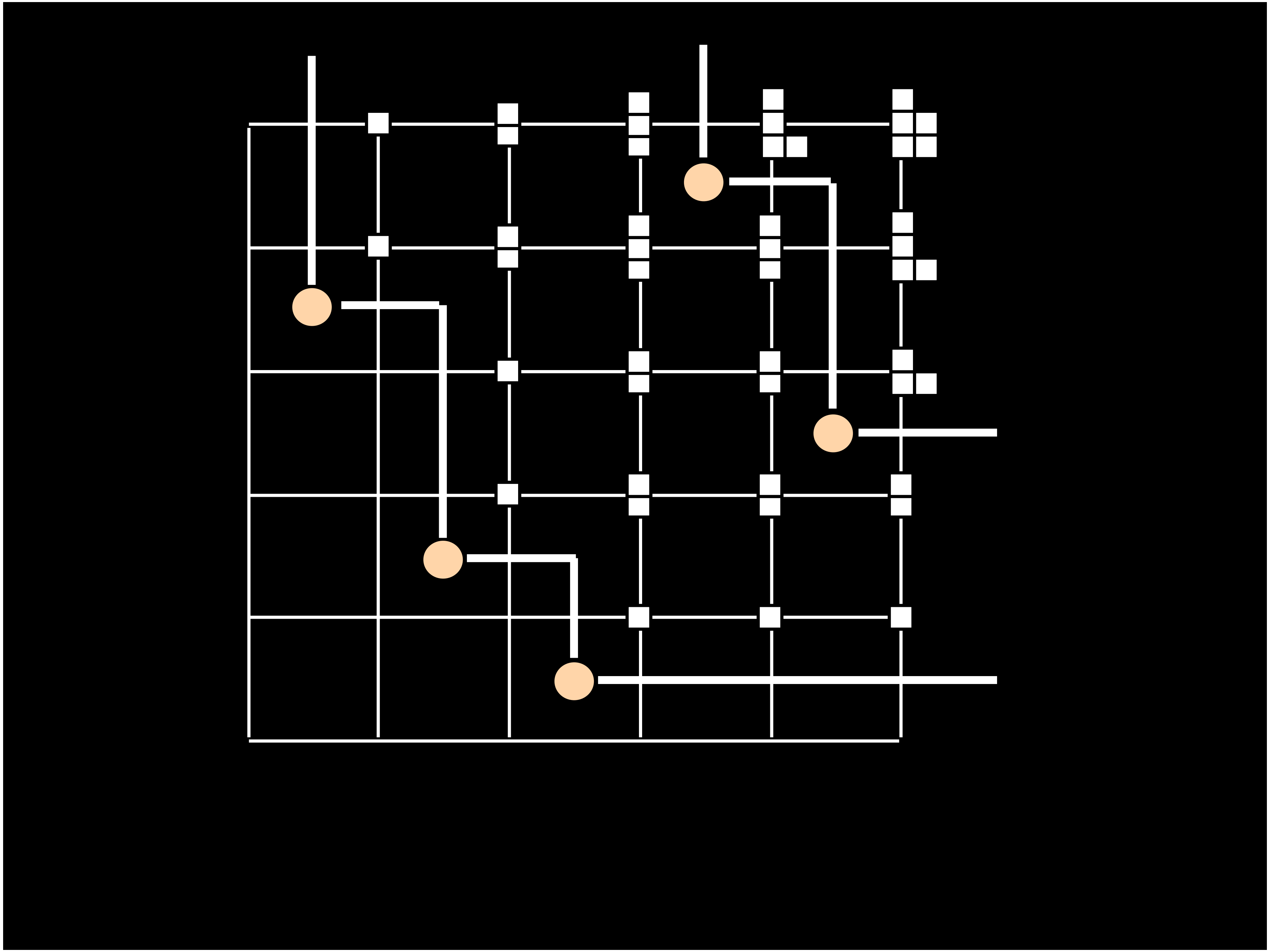
For a more complete introduction to RSK and Fomin's local rules,
see on my web site (page "exposés"):
Robinson-Schensted-Knuth: RSK1 (pdf, 9,1 Mo)
groupe de travail de combinatoire, Bordeaux, LaBRI, Février 2005
Robinson-Schensted-Knuth: RSK2 (pdf,10,8Mo)
groupe de travail de combinatoire, Bordeaux, LaBRI, Février 2005

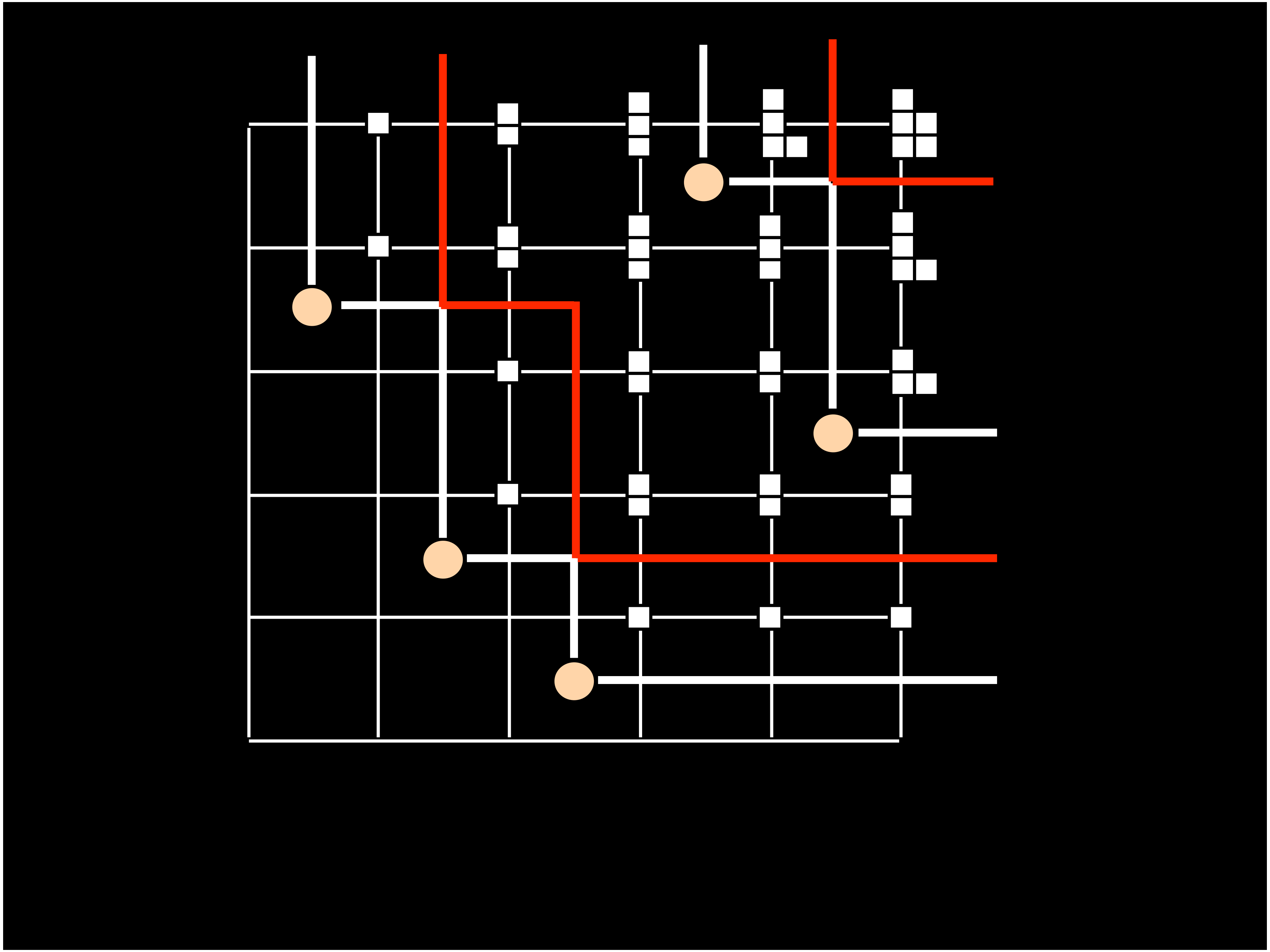


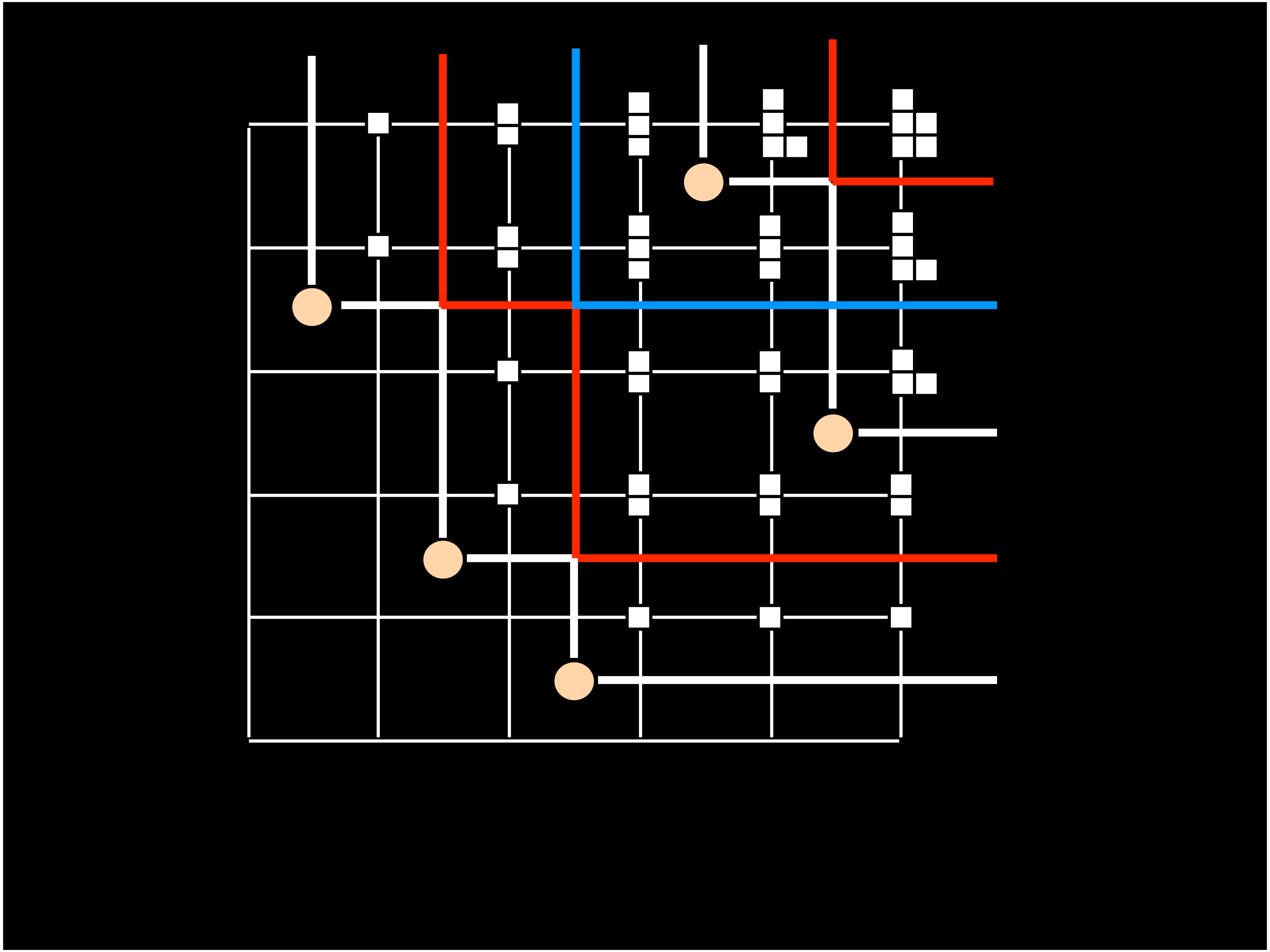
1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
σ = 3 1 6 10 2 5 8 4 9 7

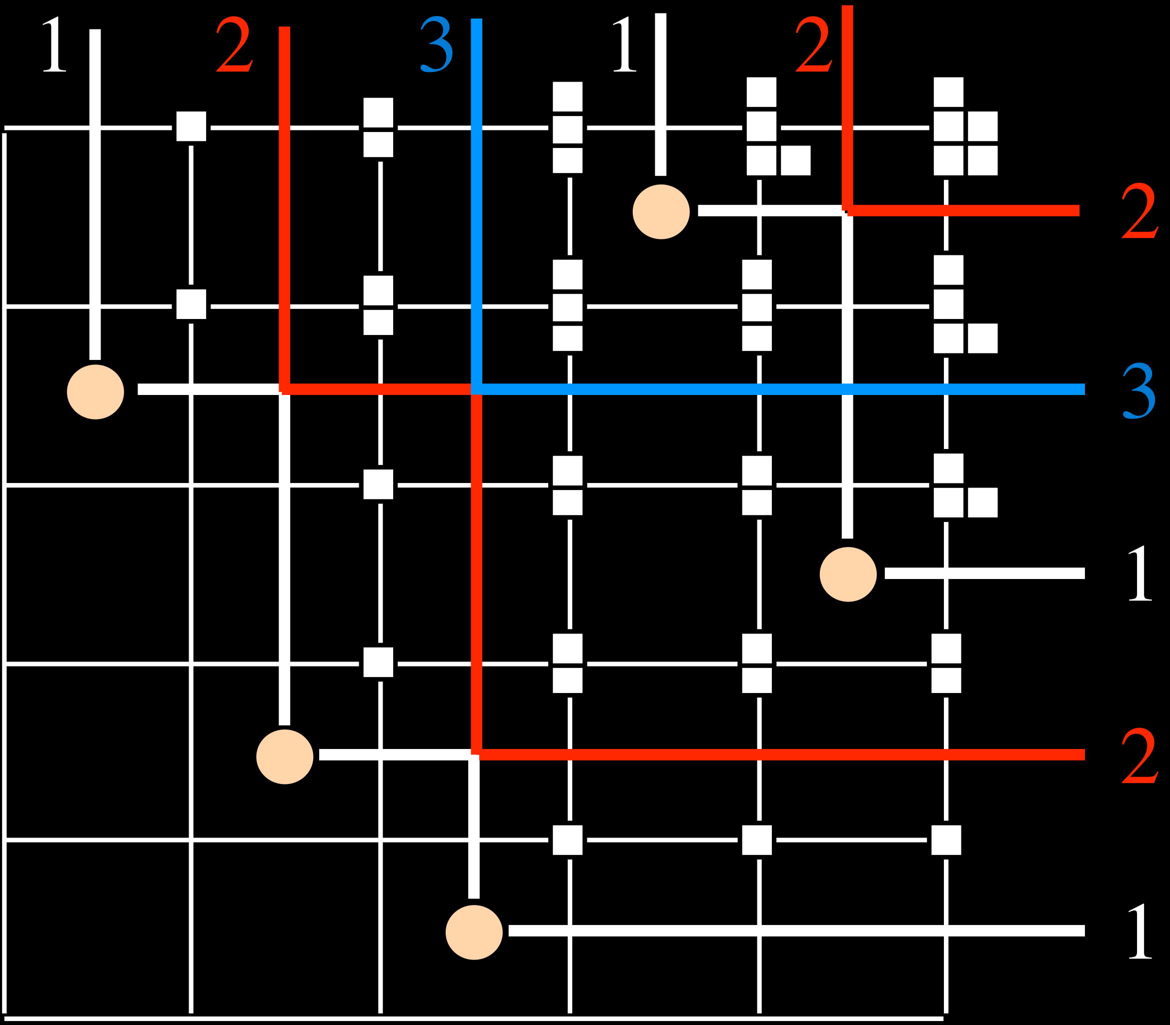


4 2 1 5 3

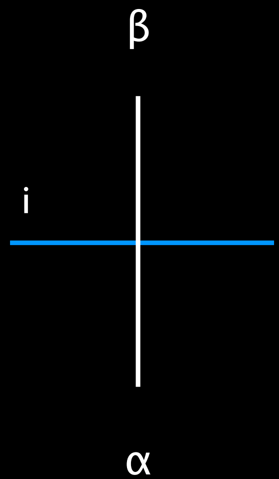




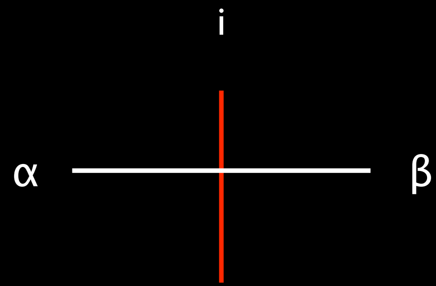


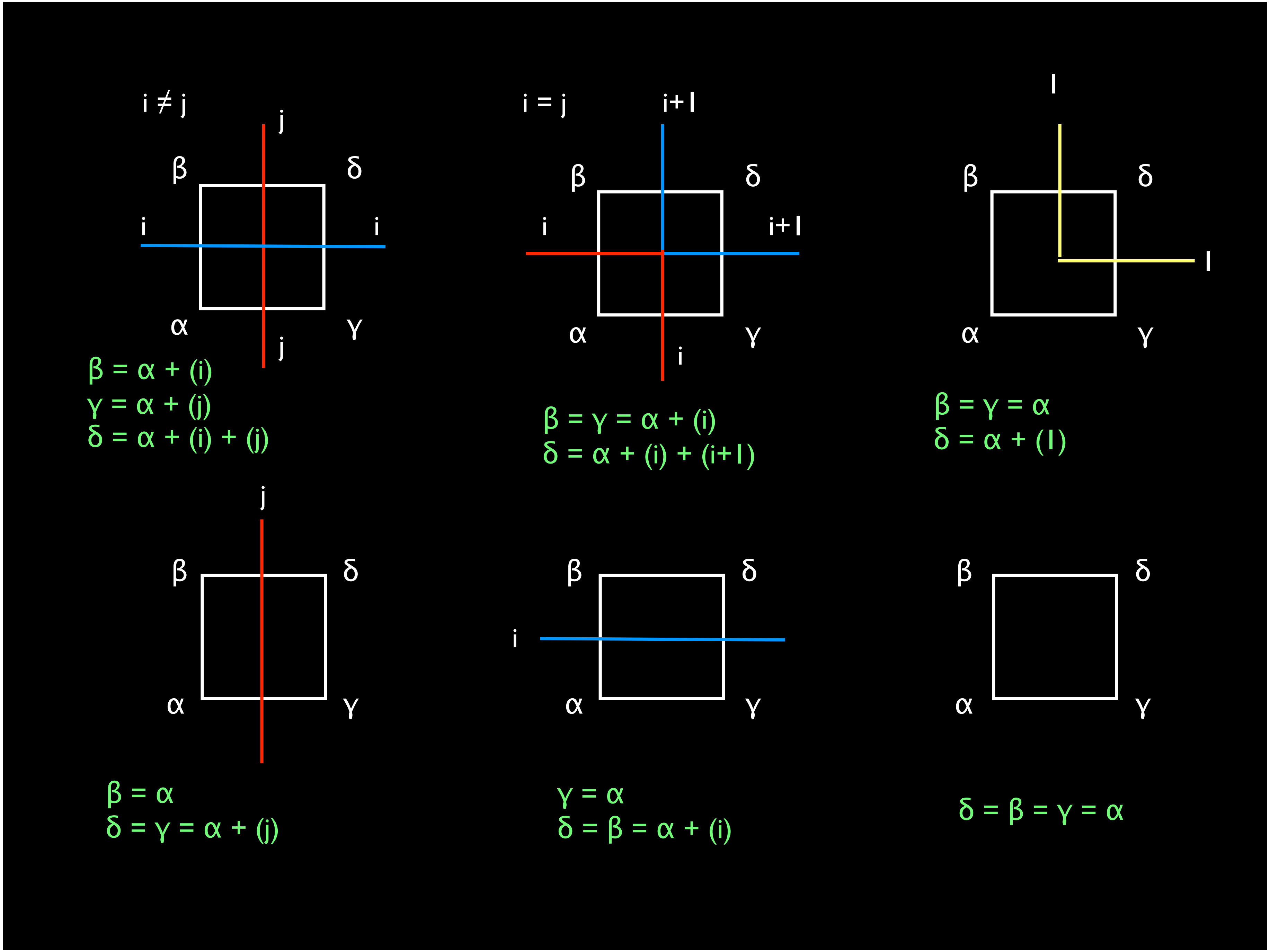


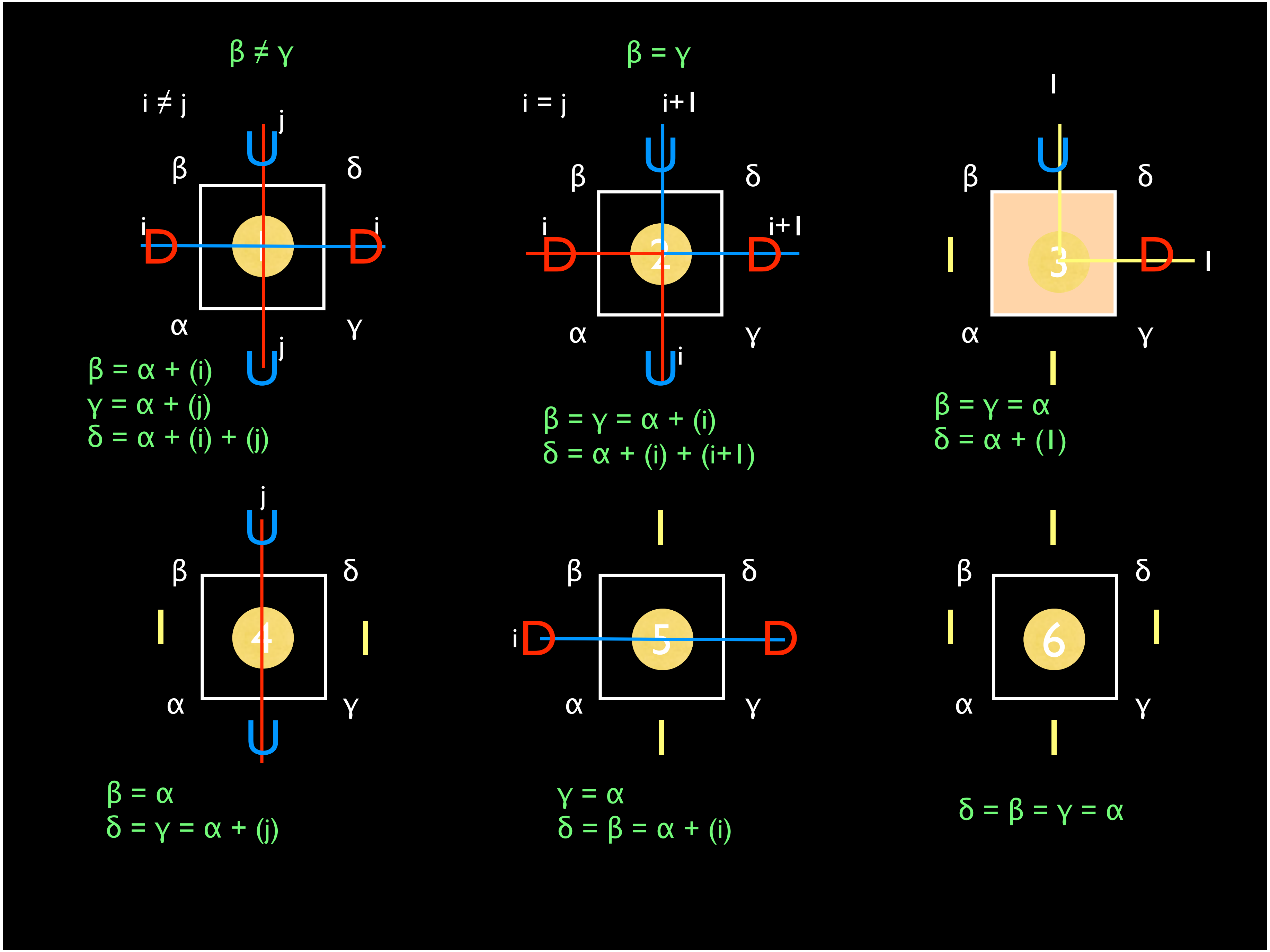
1
2
3
1
2
2
3
1
2
1

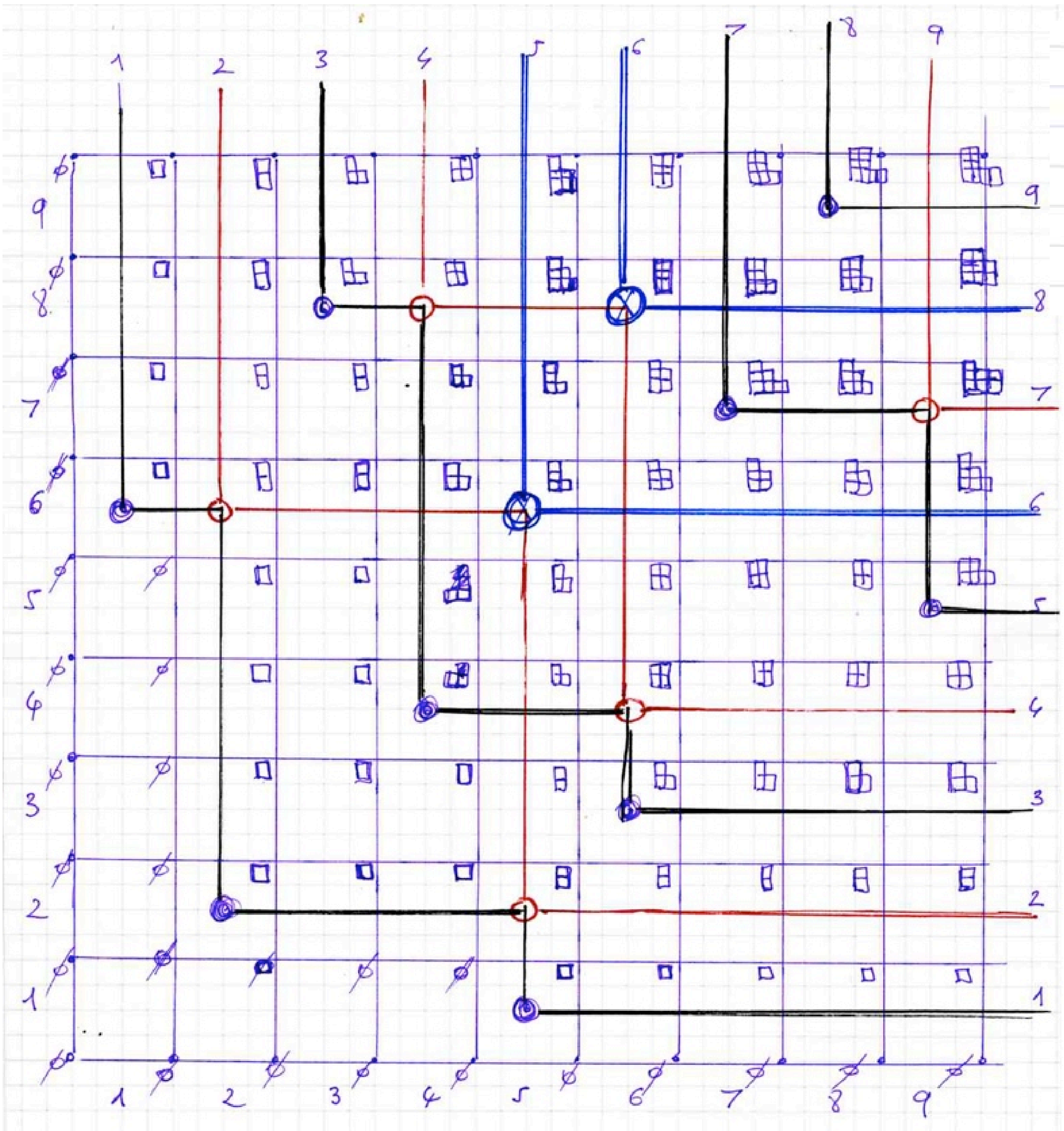


$$\beta = \alpha + (i)$$




i ≠ j
j
β
δ
i
i
α
γ
j
β = α + (i)
γ = α + (j)
δ = α + (i) + (j)
i = j
i+1
β
δ
i
i+1
α
γ
i
β = γ = α + (i)
δ = α + (i) + (i+1)
1
β
δ
1
α
γ
β = γ = α
δ = α + (1)
j
β
δ
α
γ
β = α
δ = γ = α + (j)
β
δ
i
α
γ
γ = α
δ = β = α + (i)
β
δ
α
γ
δ = β = γ = α


β ≠ γ
i ≠ j
β = α + (i)
γ = α + (j)
δ = α + (i) + (j)
β = γ
i = j
i+1
β = γ = α + (i)
δ = α + (i) + (i+1)
β = γ = α
δ = α + (1)
β = α
δ = γ = α + (j)
γ = α
δ = β = α + (i)
δ = β = γ = α



5	6		
2	4	9	
1	3	7	8

Q

another example with

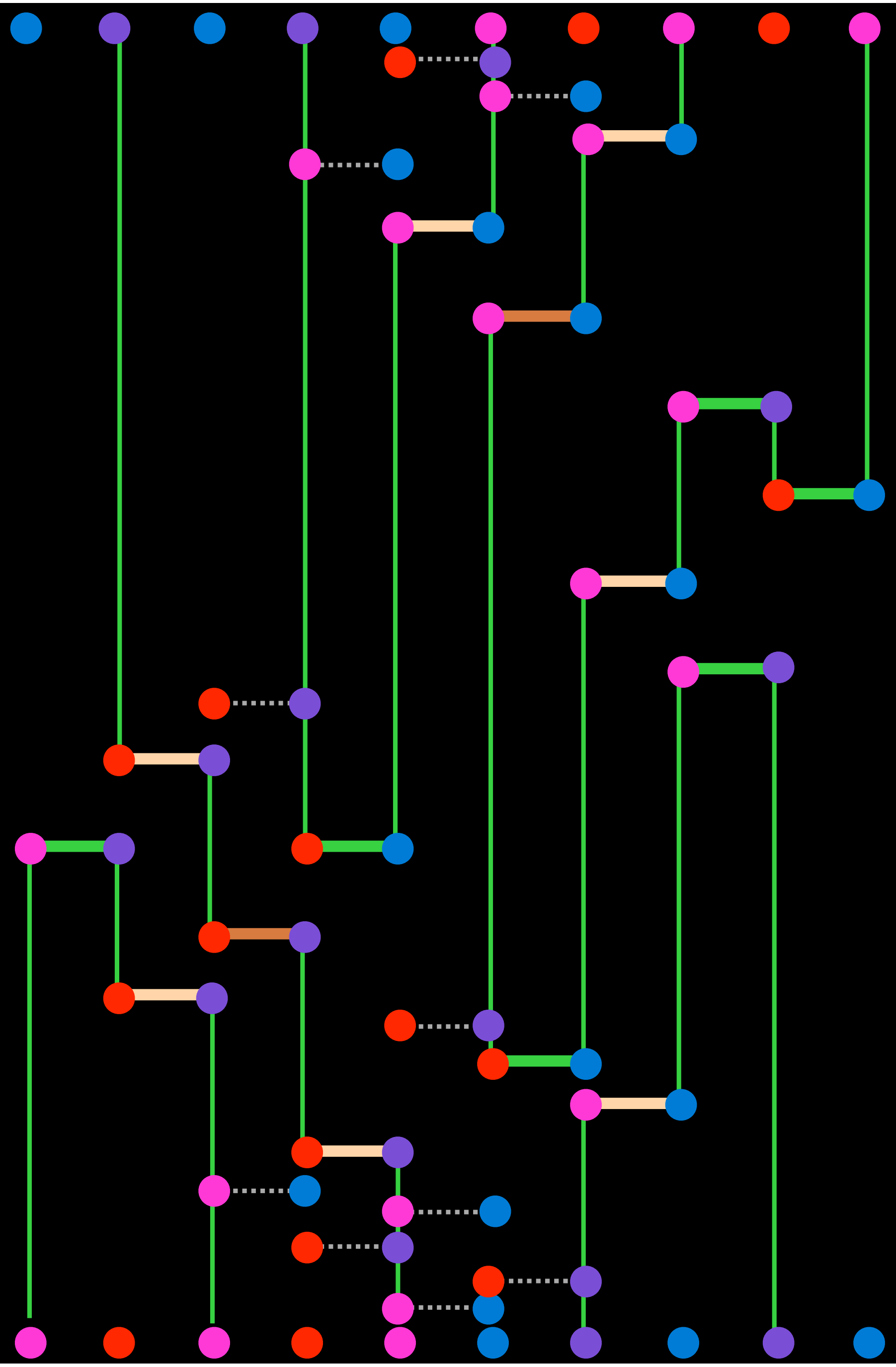
6 2 8 4 1 3 7 9 5

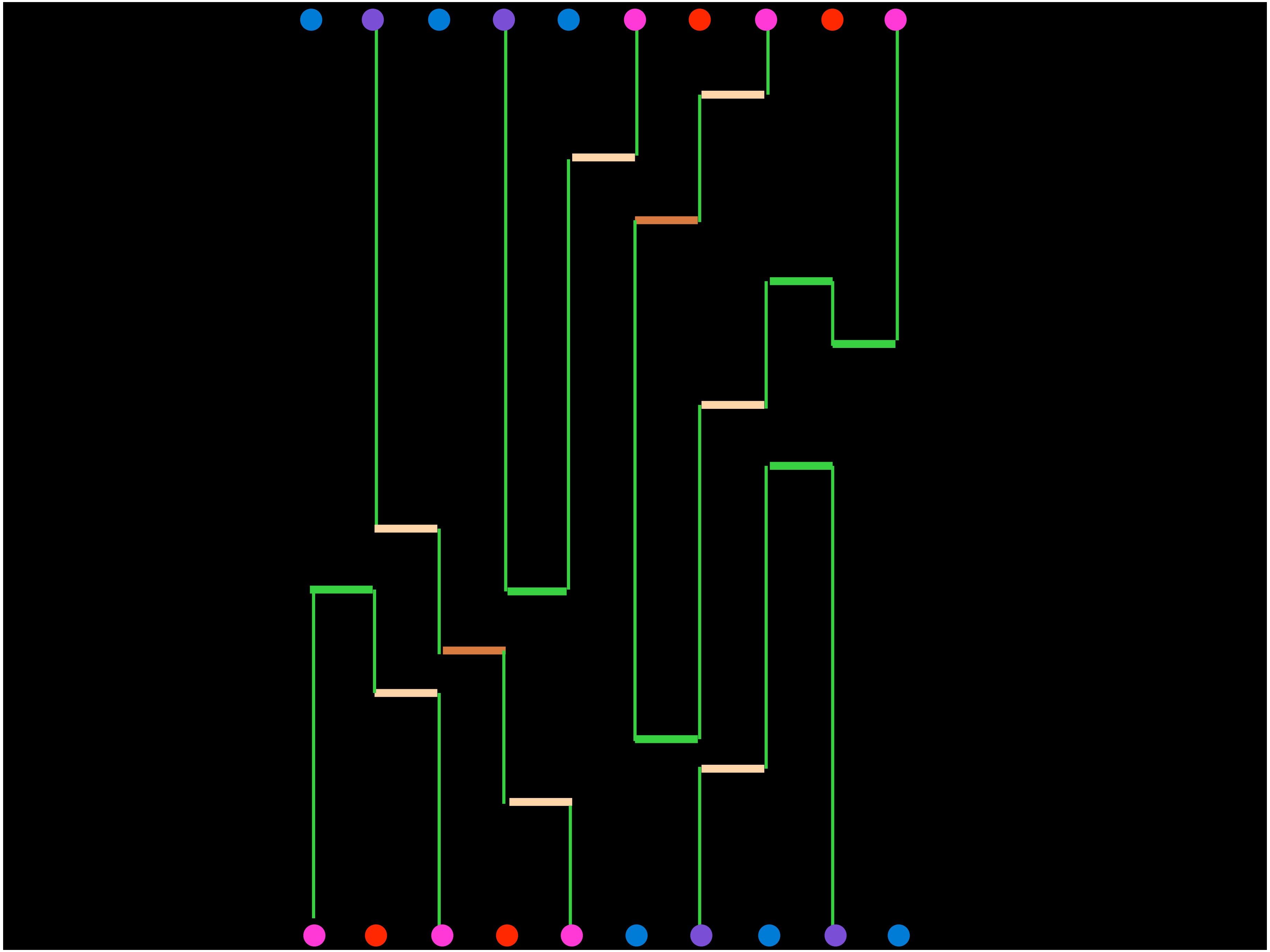
6	8		
2	4	7	
1	3	5	9

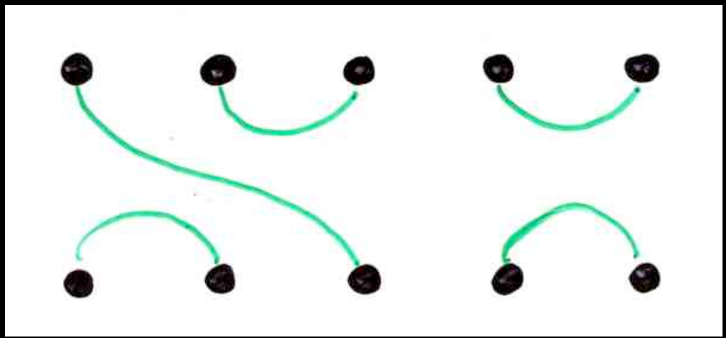
P

Temperley-Lieb algebra
and
decorated heaps of dimers

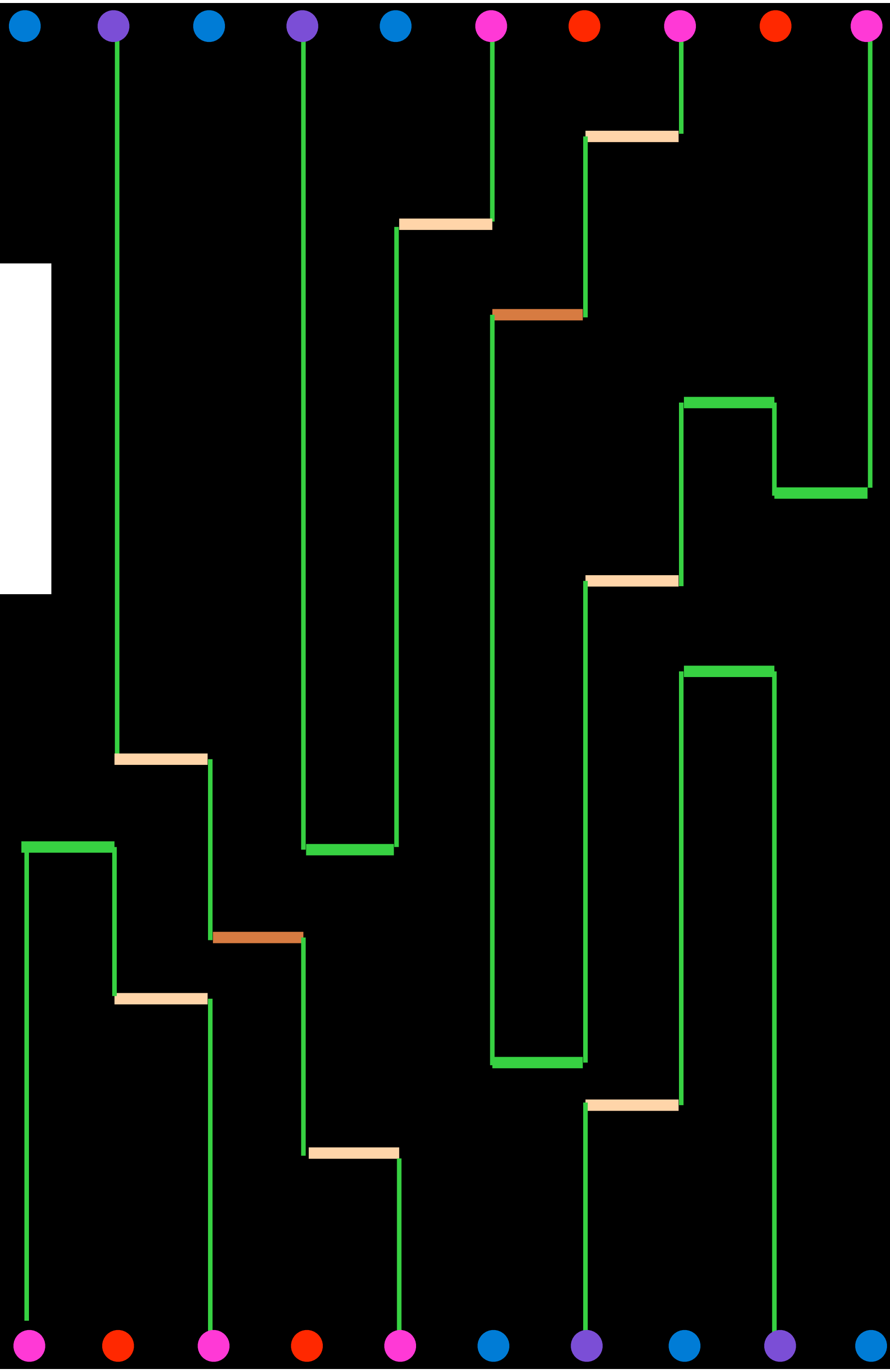
"decorated heaps of dimers" in bijection with FPL

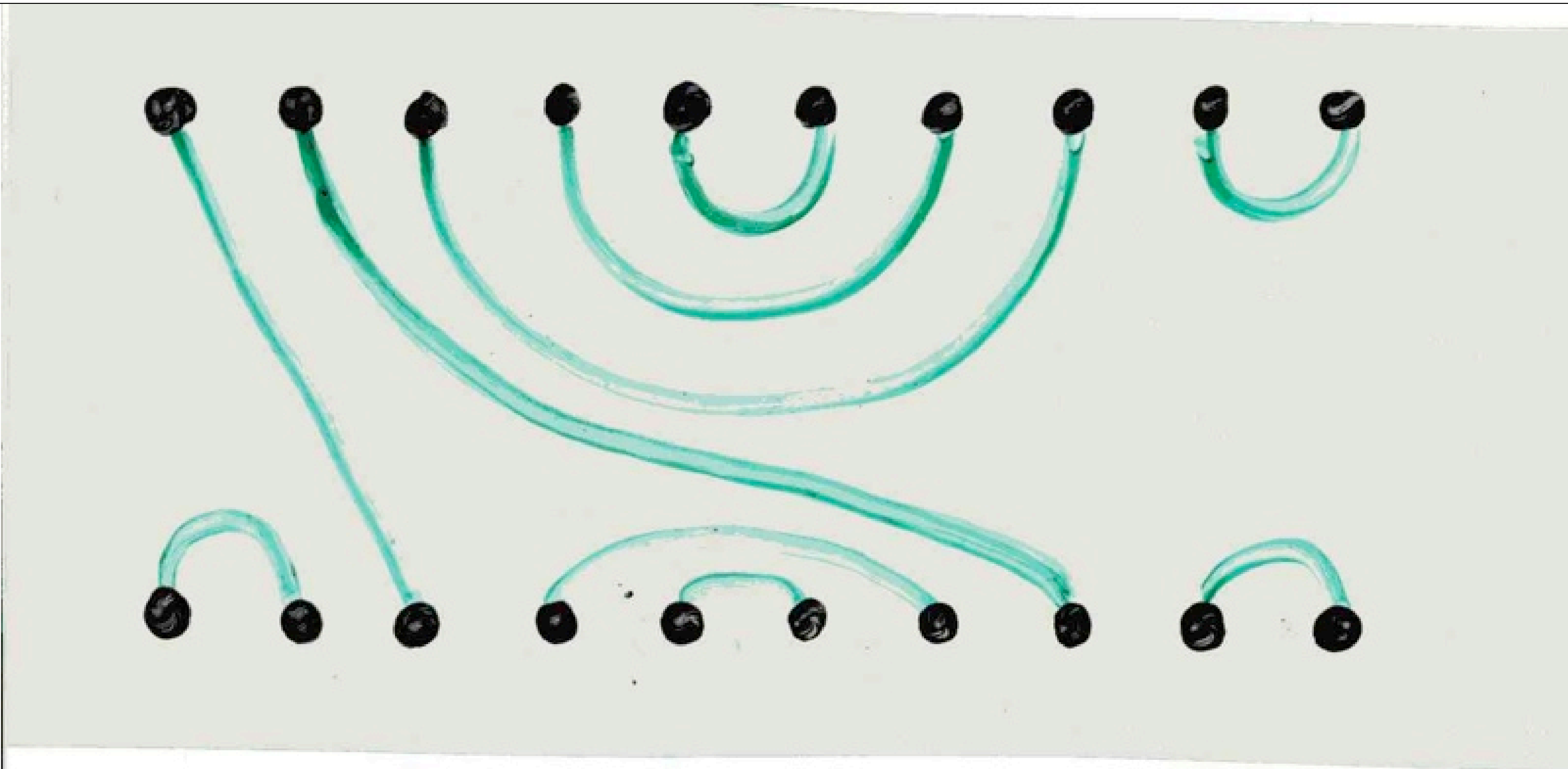




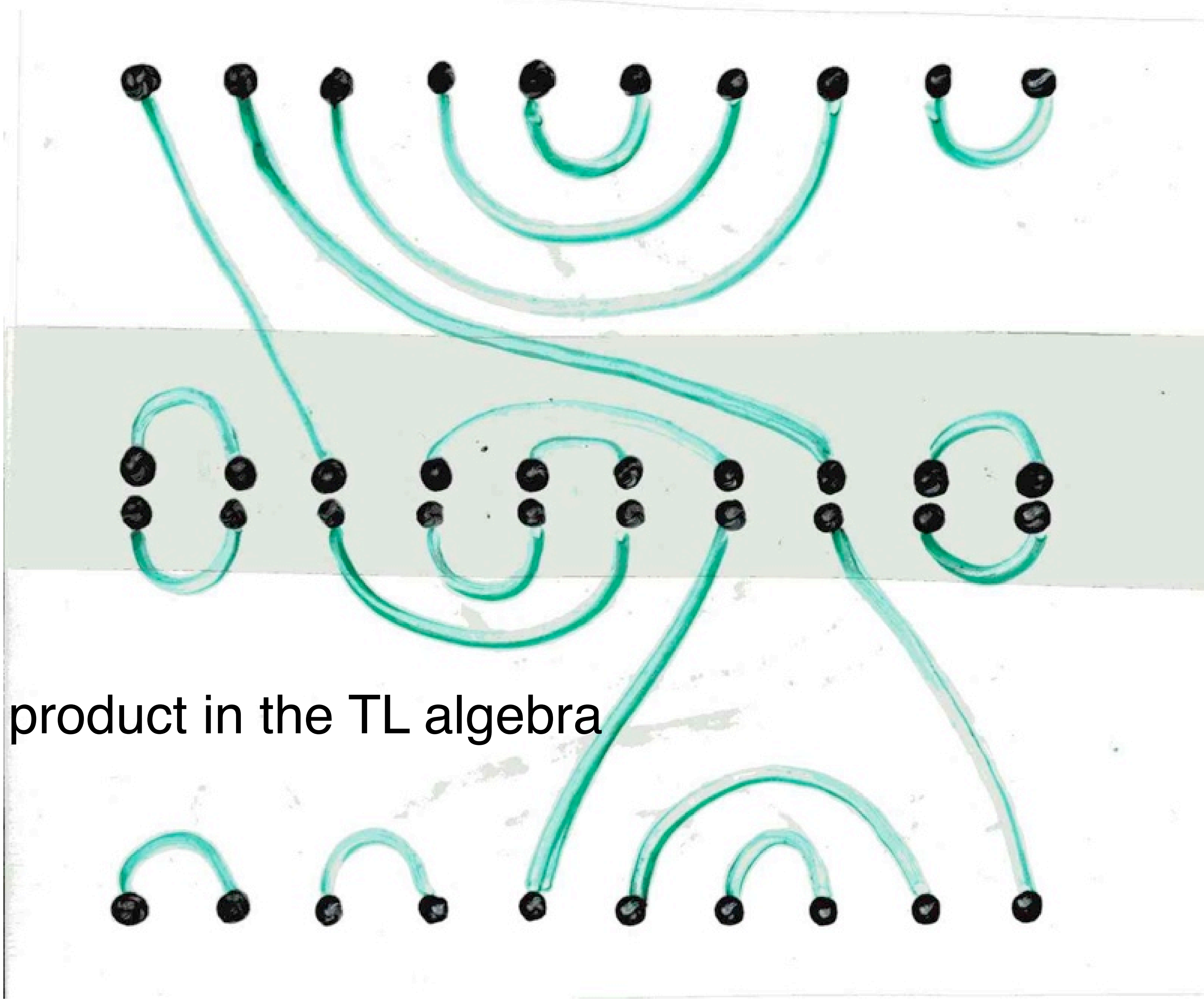


From
“decorated
heaps of
dimers” to
element of
the
Temperley-
Lieb algebra





An element of the TL algebra



product in the TL algebra

Temperley-Lieb algebra

$TL_n(\beta)$ generators $\{e_1, e_2, \cdots, e_{n-1}\}$

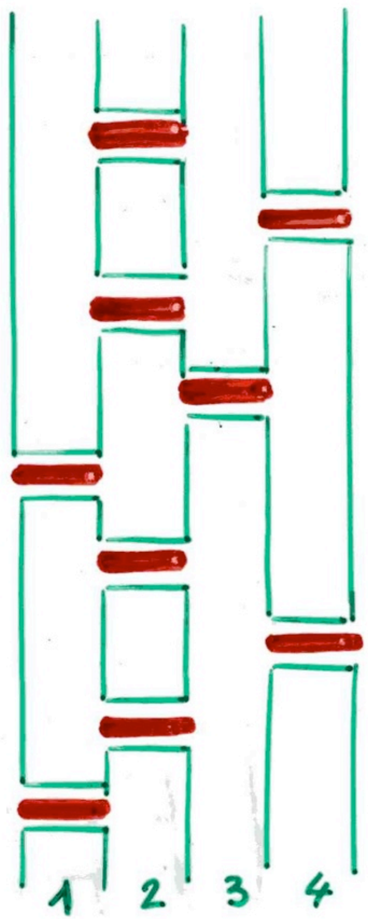
(i) $e_i \, e_j = e_j \, e_i$ $|i-j| \geq 2$

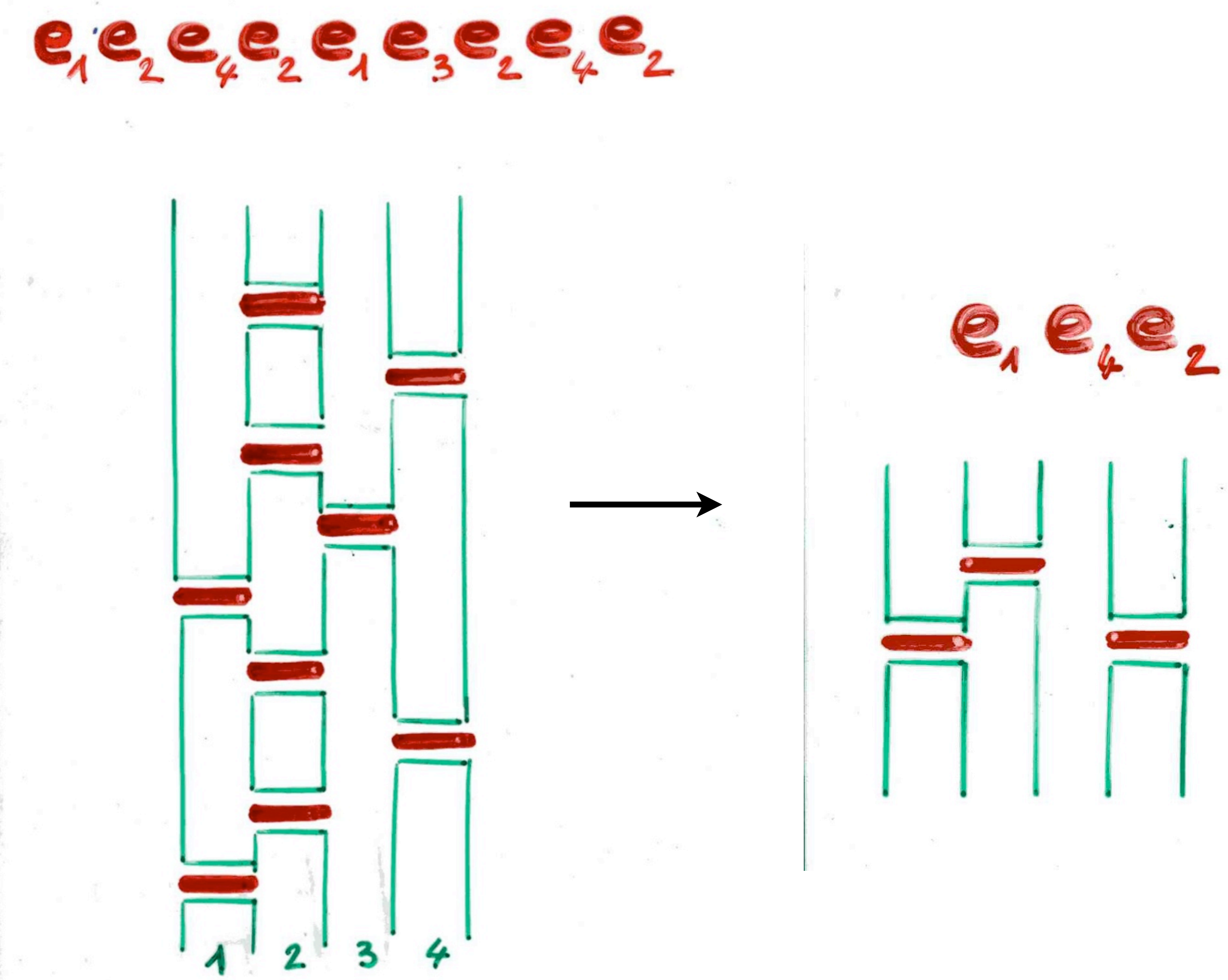
(ii) $e_i^2 = \beta e_i$

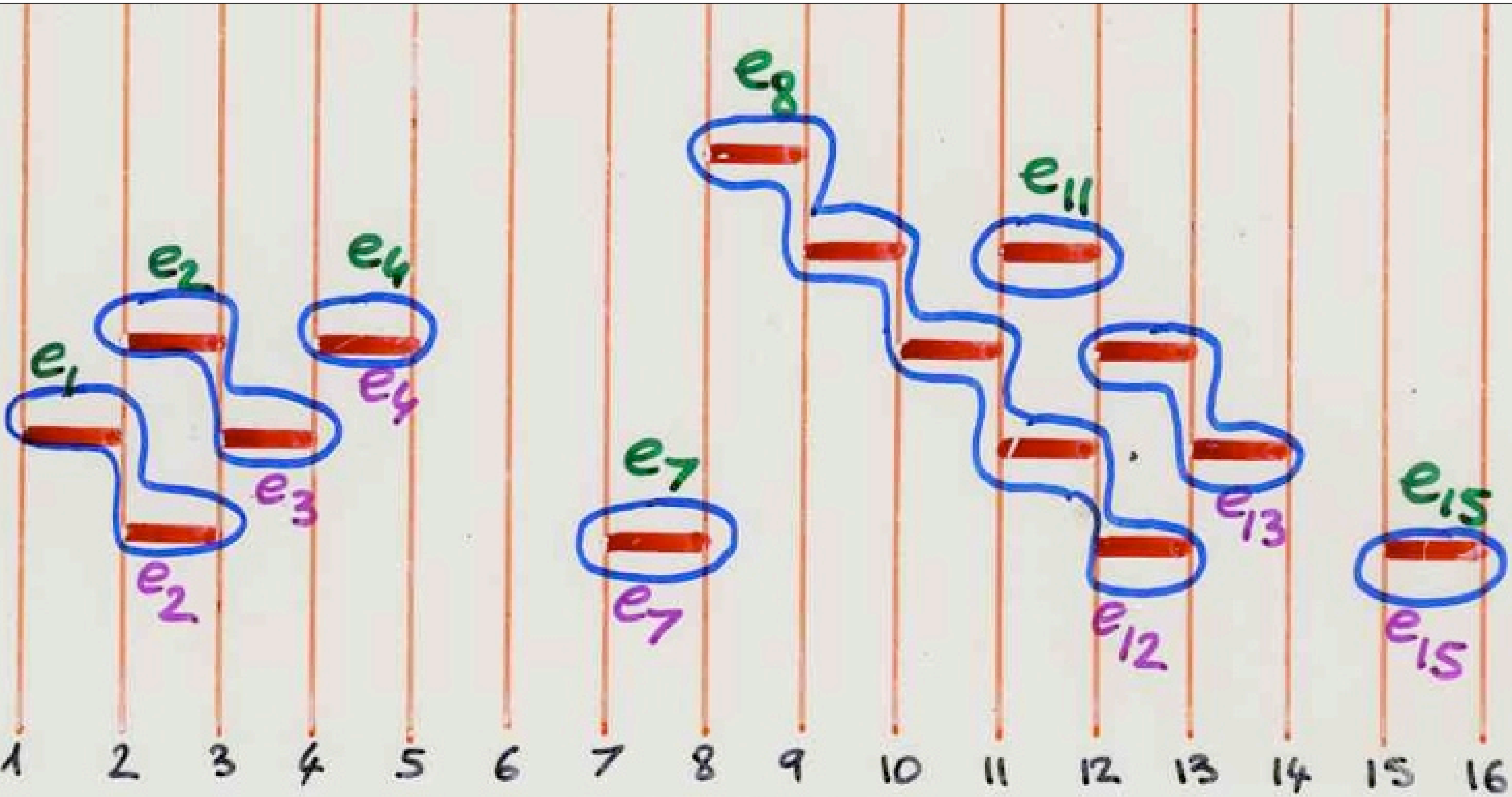
(iii) $\begin{cases} e_i \, e_{i+1} \, e_i = e_i \\ e_{i+1} \, e_i \, e_{i+1} = e_{i+1} \end{cases}$

β scalar

$e_1 e_2 e_4 e_2 e_1 e_3 e_2 e_4 e_2$

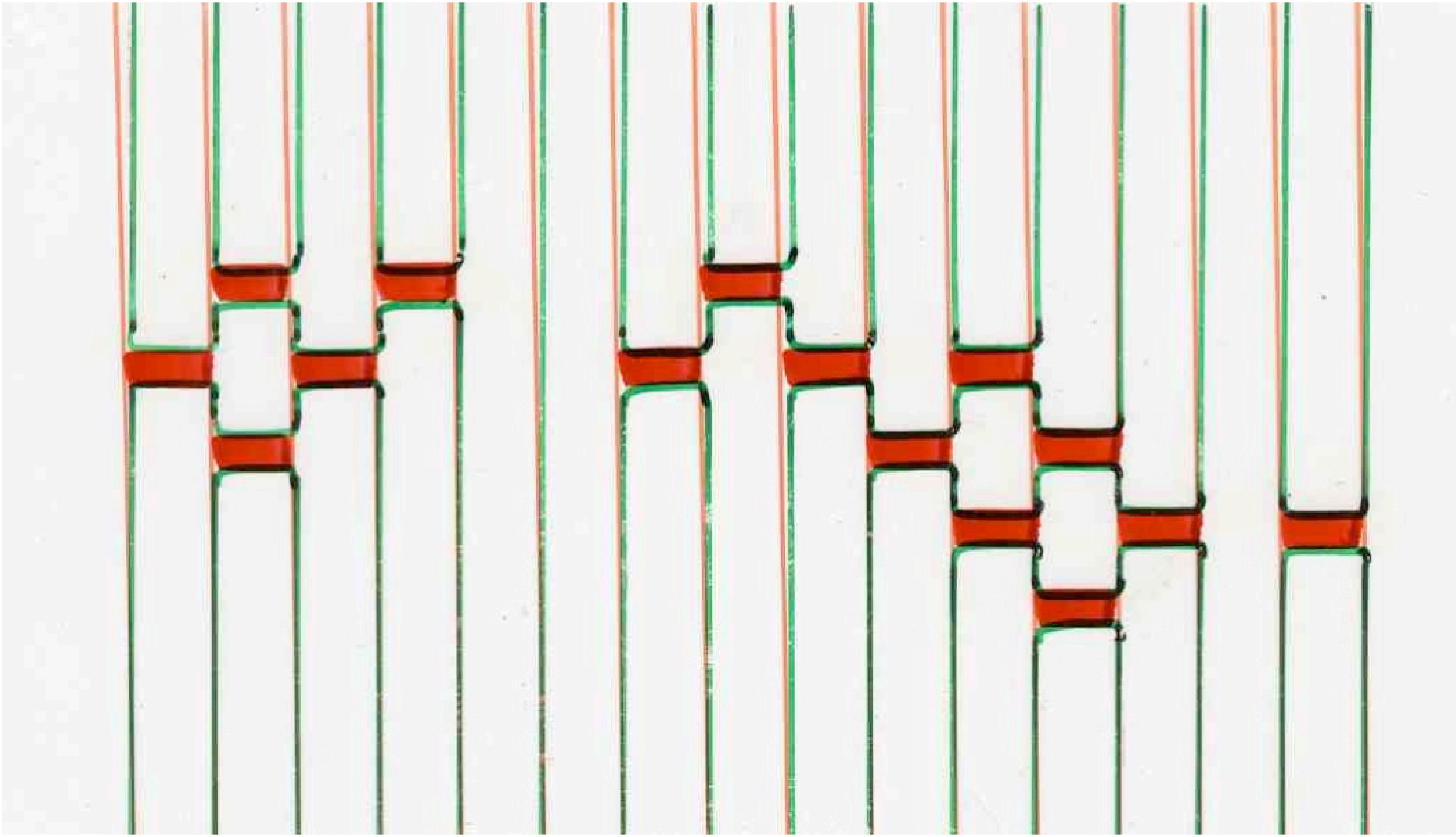



$e_1 e_2 e_4 e_2 e_1 e_3 e_2 e_4 e_2$
1 2 3 4
$e_1 e_4 e_2$

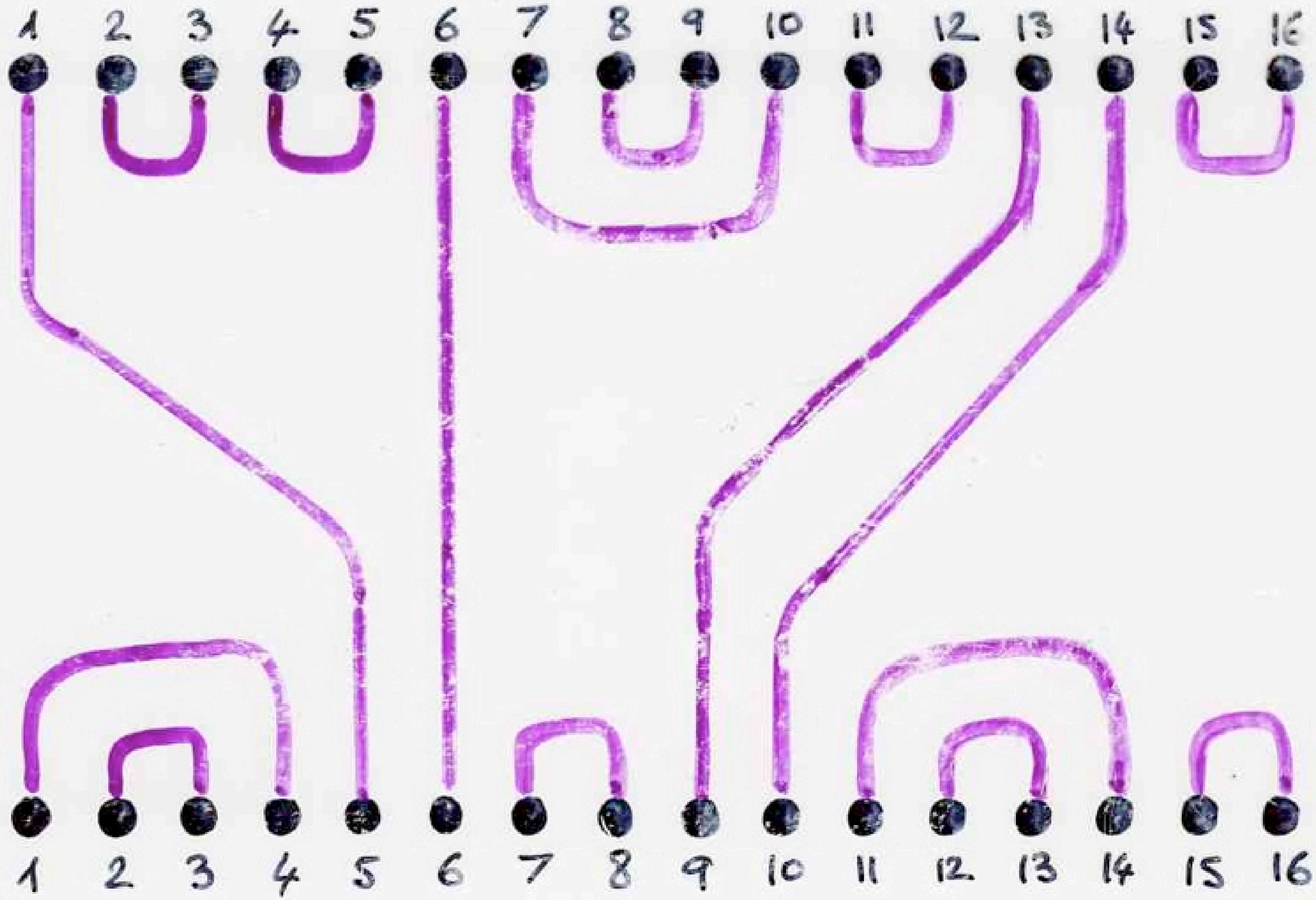


An element of the basis of the Temperley-Lieb algebra

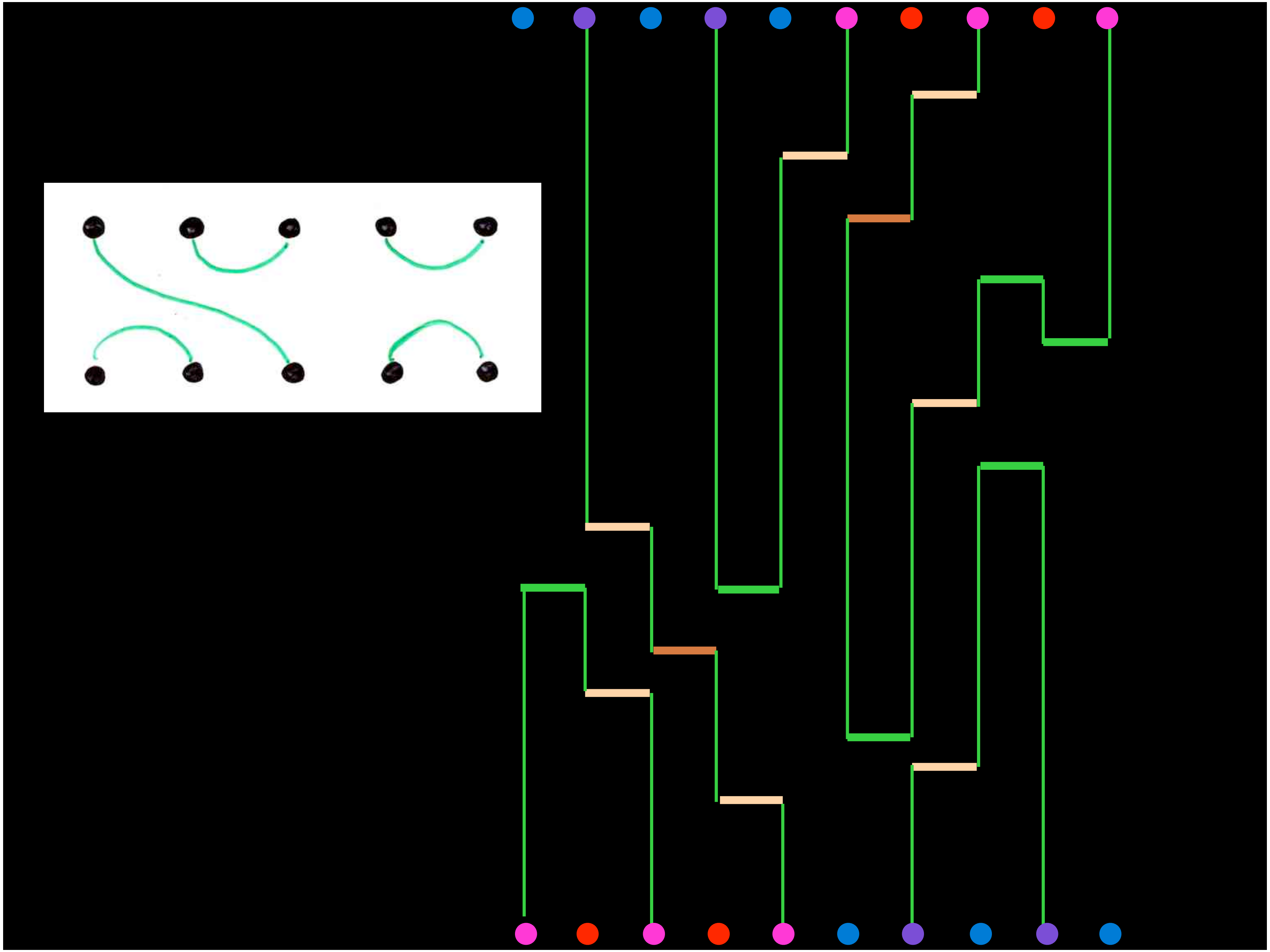
$$
\begin{array}{ccccccccccccccccc}
 & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & \\
 & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & \\
1 \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq n
\end{array}
$$

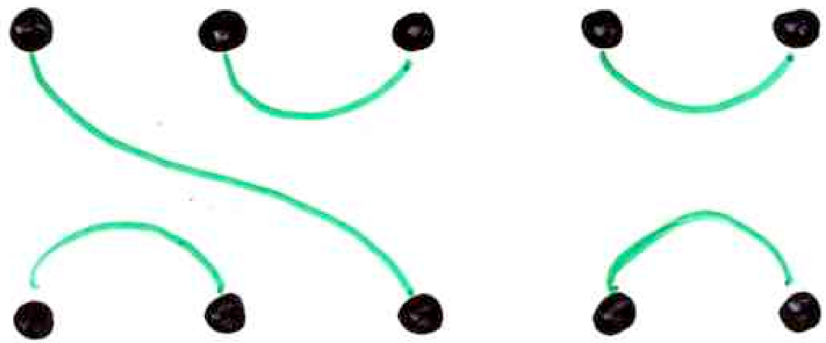


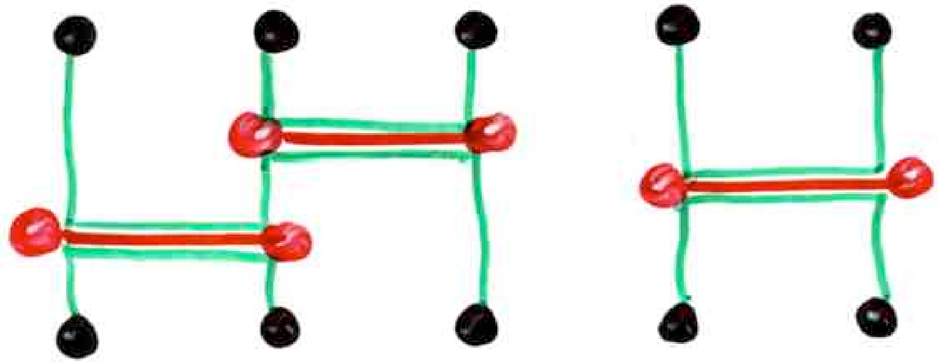
An element of the basis of the Temperley-Lieb algebra

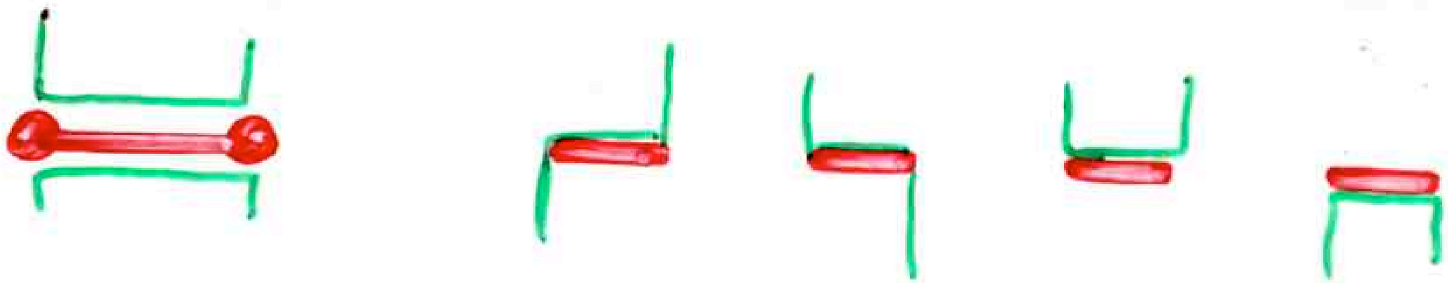


The same element of the basis of the Temperley-Lieb algebra









“big dimers” and small “dimers”

“contraction”
of
heap of dimers

May be it can be useful to study the map sending FPL on the “decorated” heaps of (small) dimers.

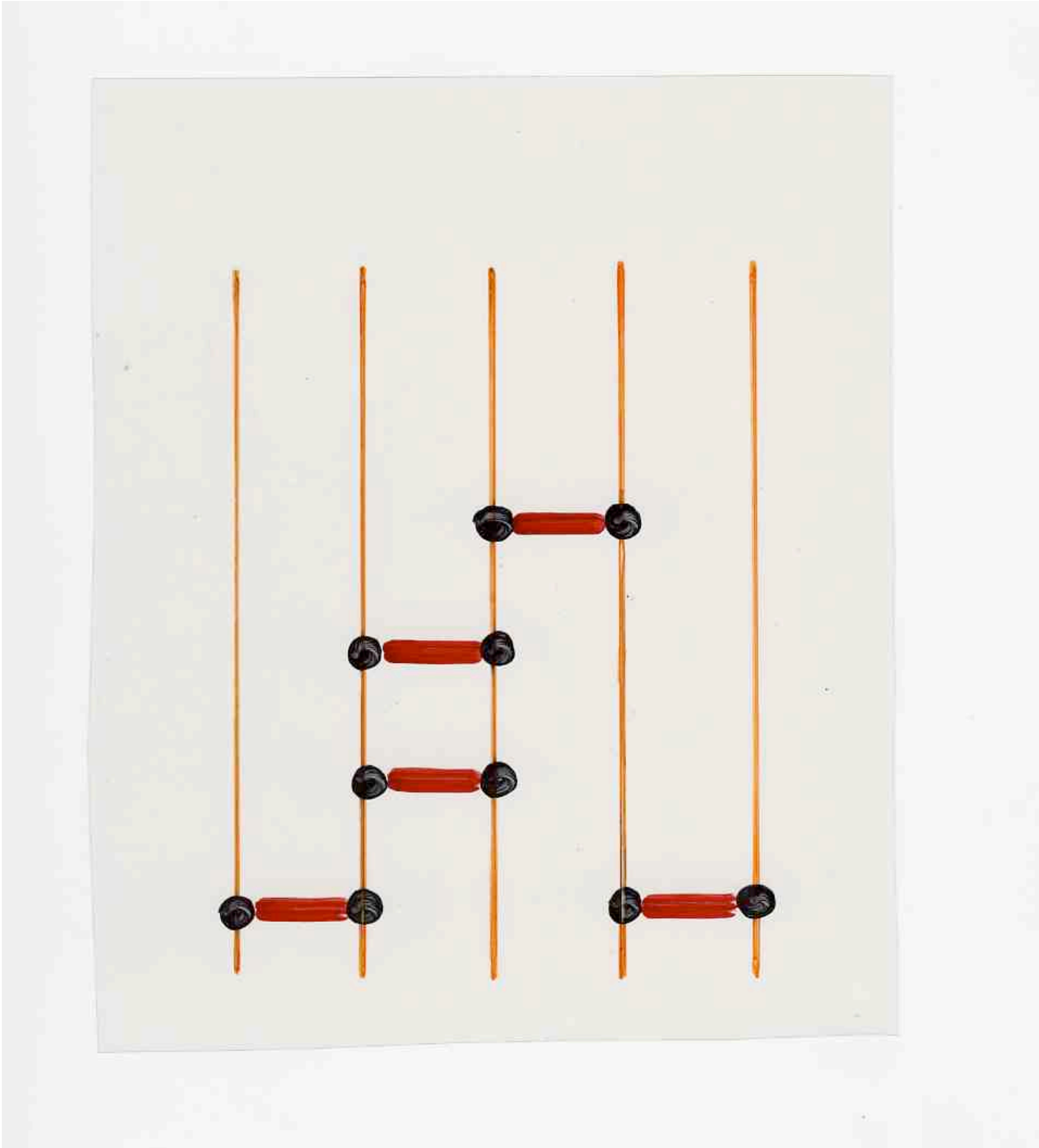
Here are example of situtions with some bijections between heaps of “big” dimers and heaps of “small “dimers

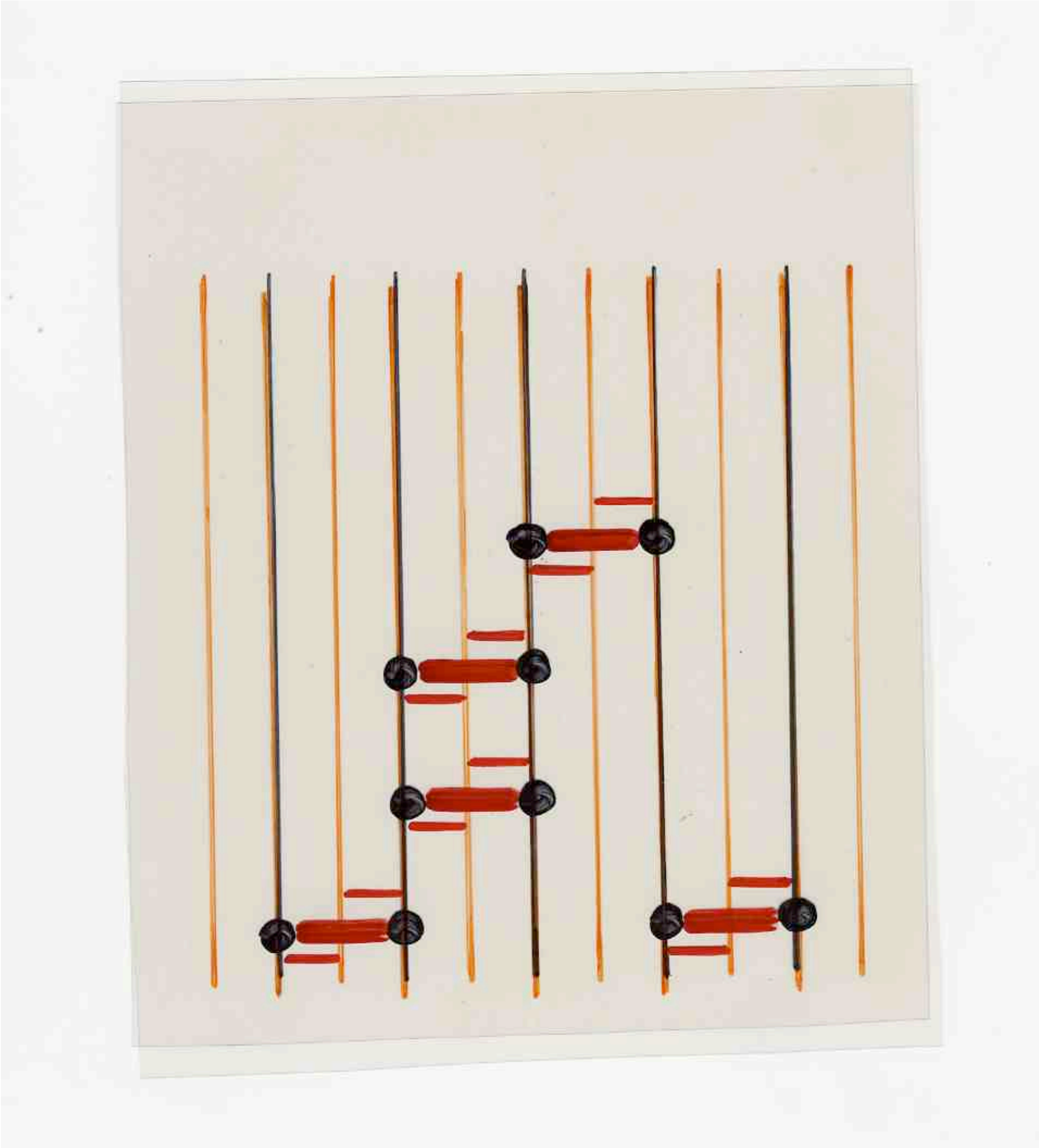
(1 big dimer= 2 small dimers)

Contraction in heaps of dimers

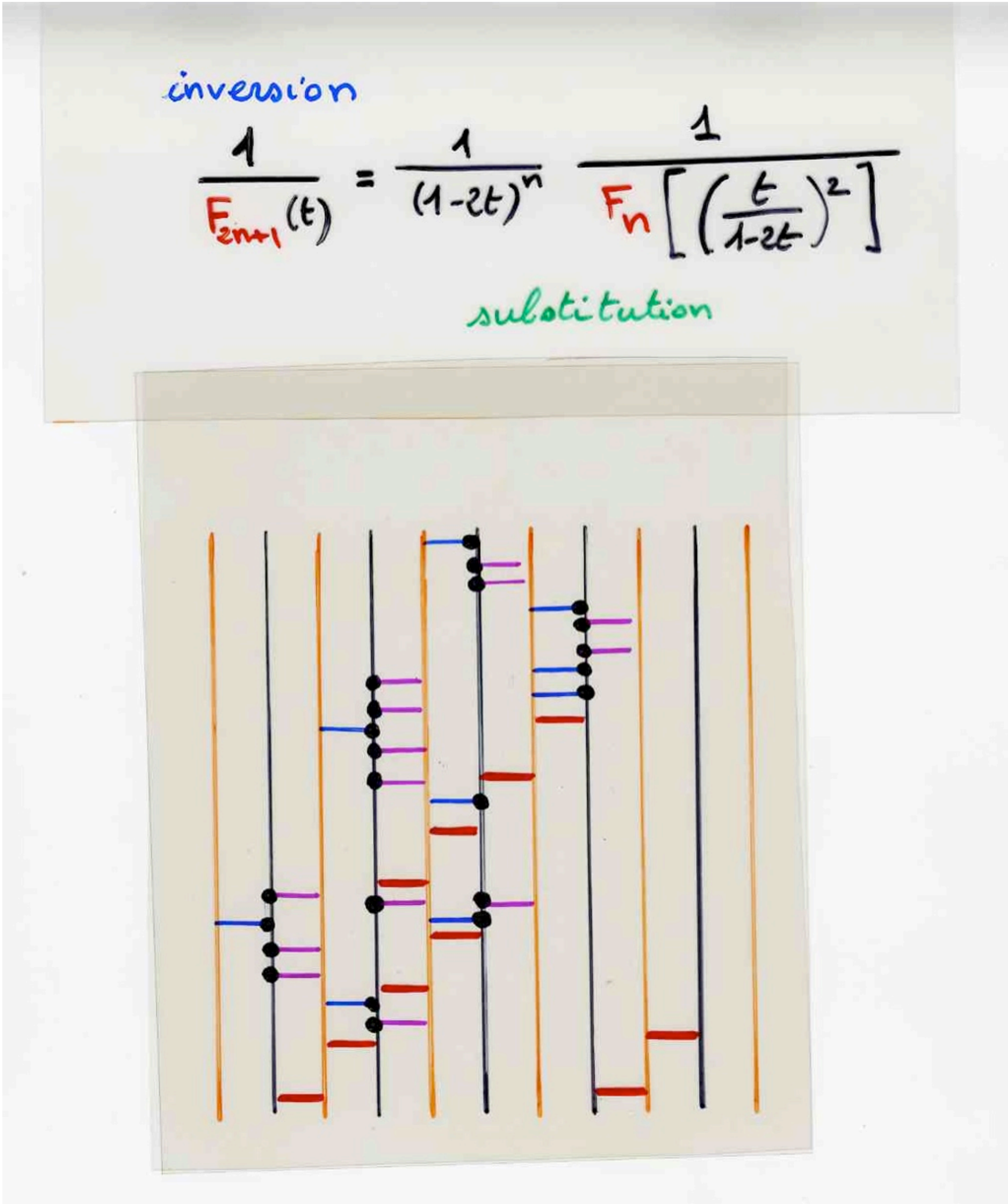
from "big" dimers
to "small" dimers

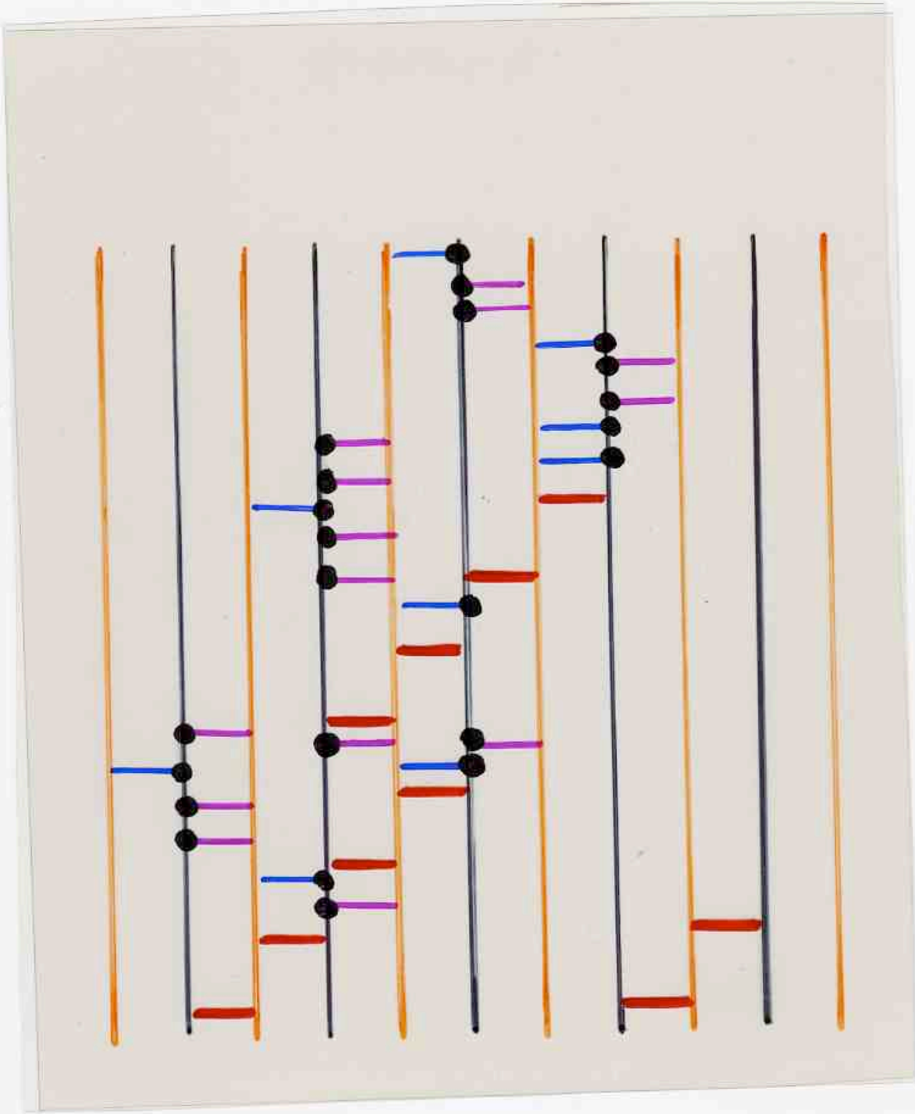
more details in the talk
A walk in the garden of bijections (pdf, 23,3 Mo)
(52th SLC, ViennotFest, Lucelle, Avril 2005
slides on my web site, page "exposés".

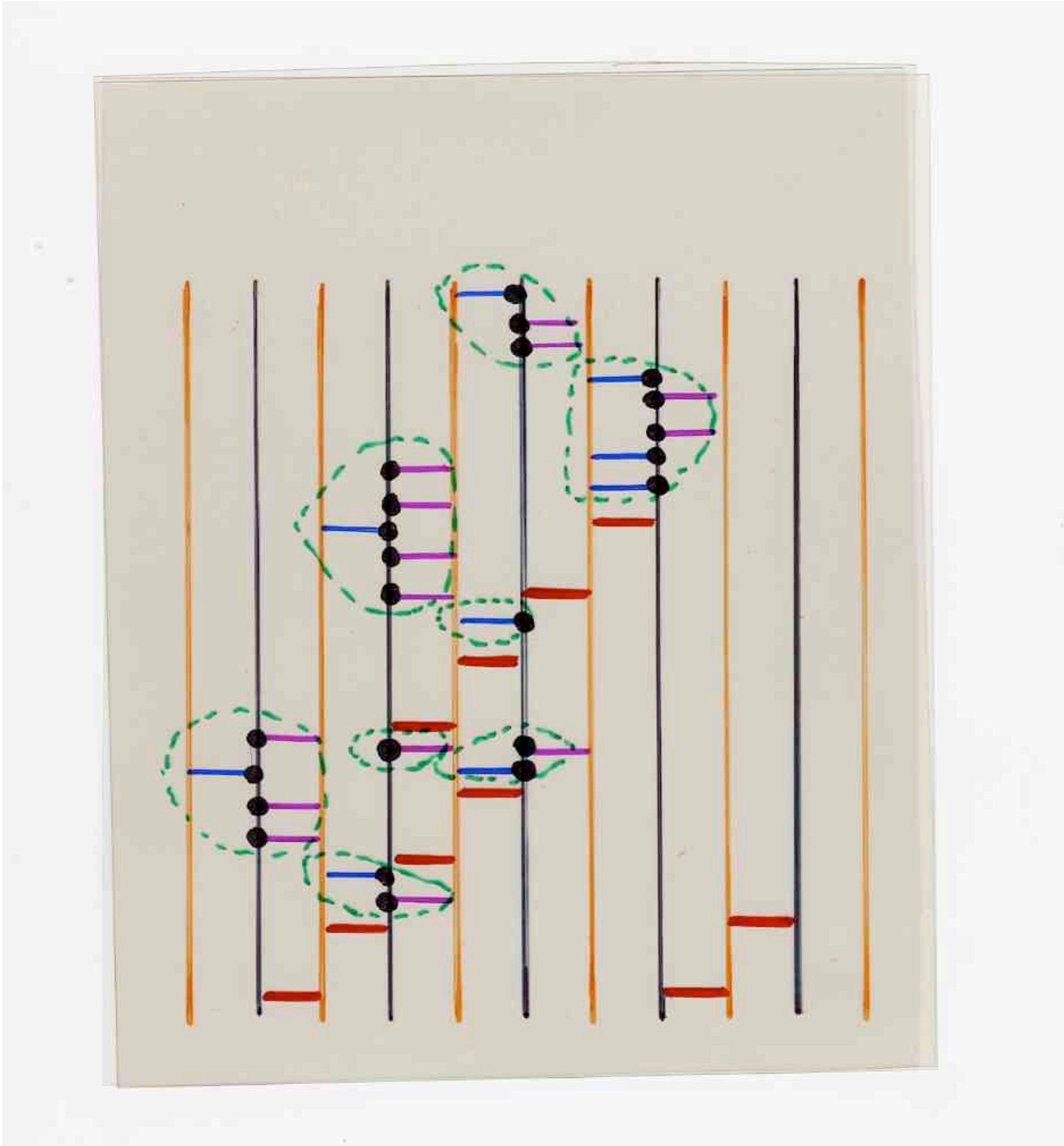




Fibonacci polynomials







bijective
proof !

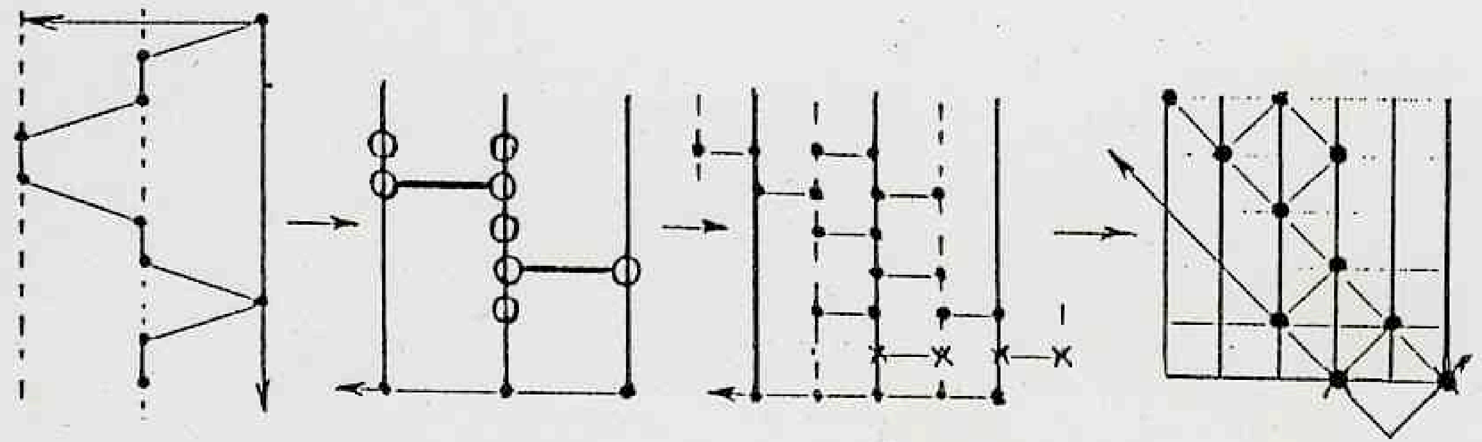


Fig. 37. Path, dimers-monomers heap and directed animal.

another example with directed animals on the square lattice
bijection: Motzkin path, heap of big dimers,
heap of small dimers, directed animal.

aurore boréale Nunavik xgv