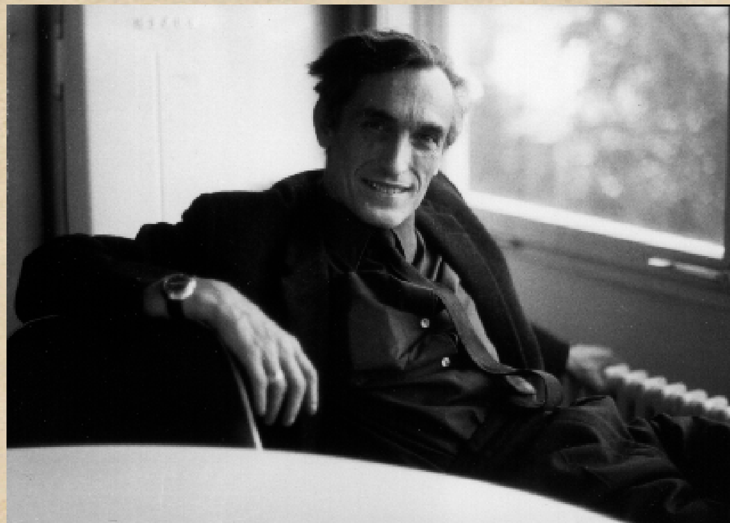
“Jeu de taquin” pour les arbres binaires

GT, LaBRI

19 Décembre 2008

xgv

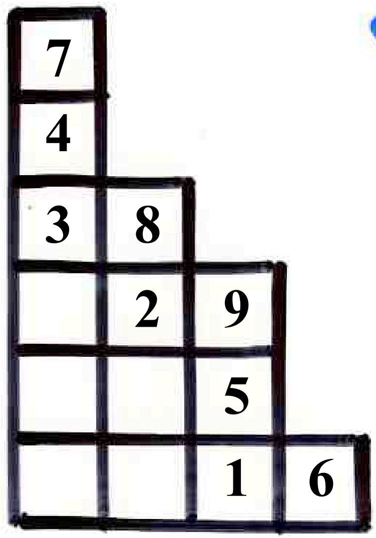
§1 Schützenberger
"jeu de taquin"



$$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$$

"jeu de taquin"

M.P. Schützenberger

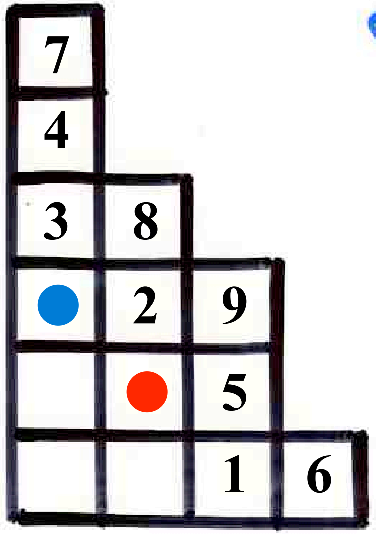


$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6 \ldots$

up-down sequence − − + − + − − +

"jeu de taquin"

M.P. Schützenberger



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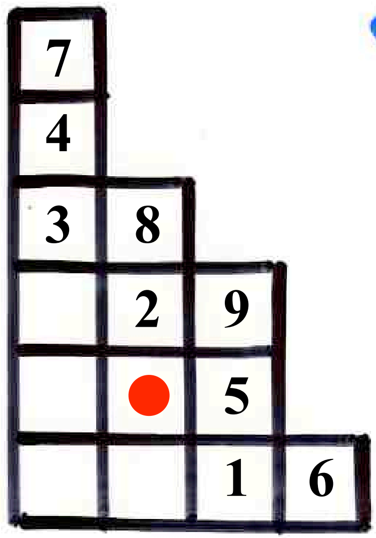

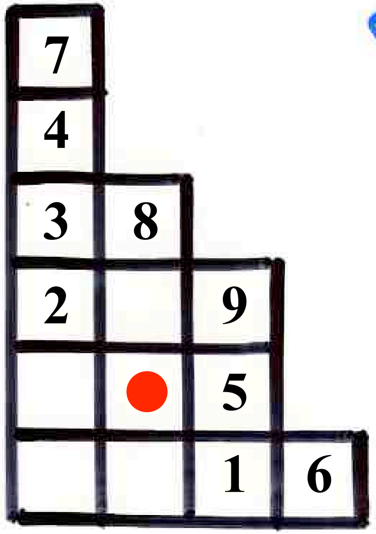

"jeu de taquin"

M.P. Schützenberger

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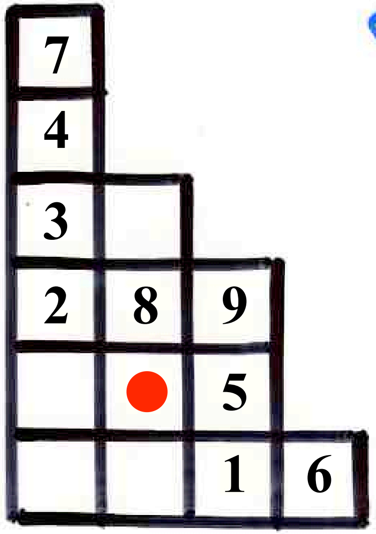

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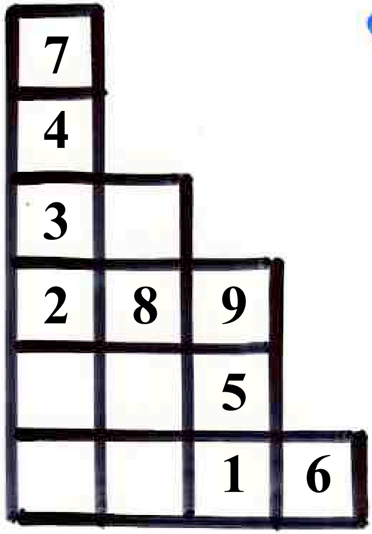

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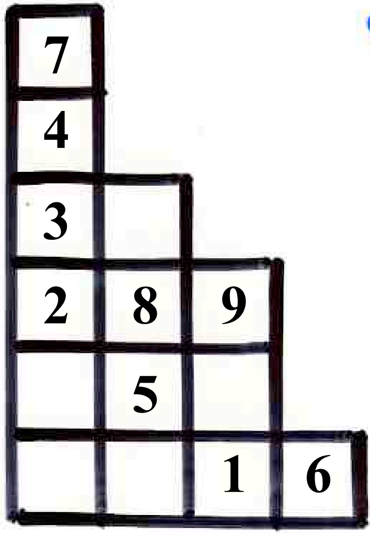

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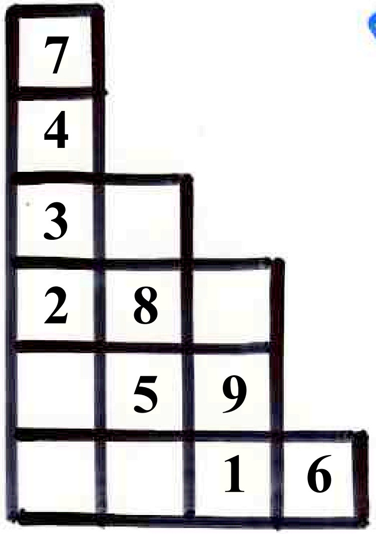

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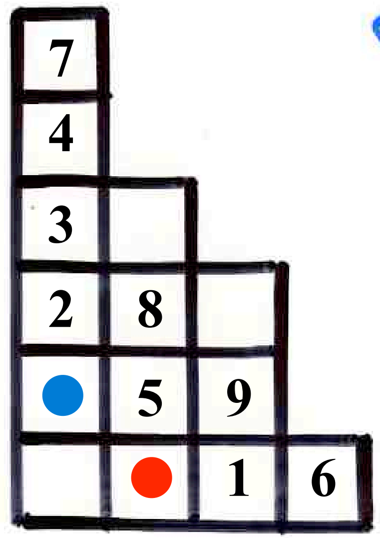

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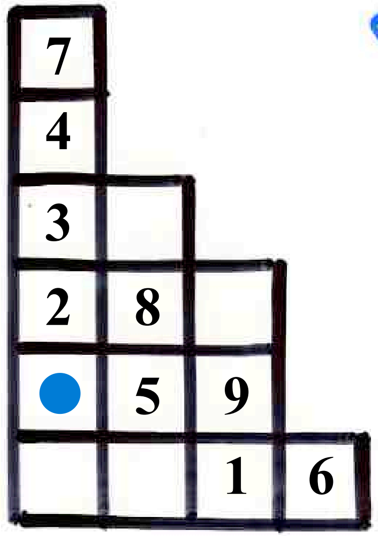

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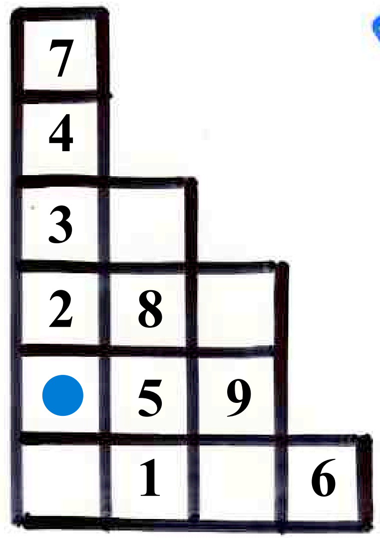

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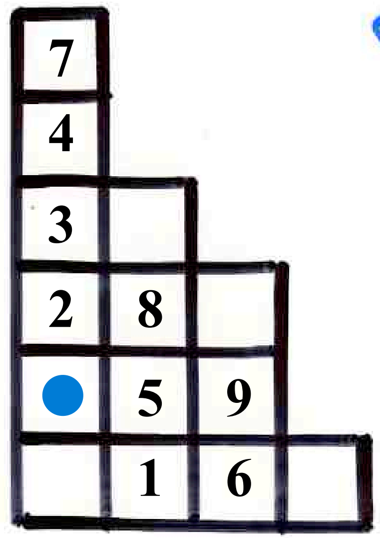

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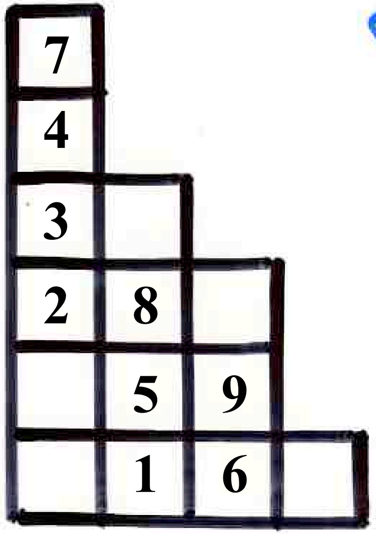

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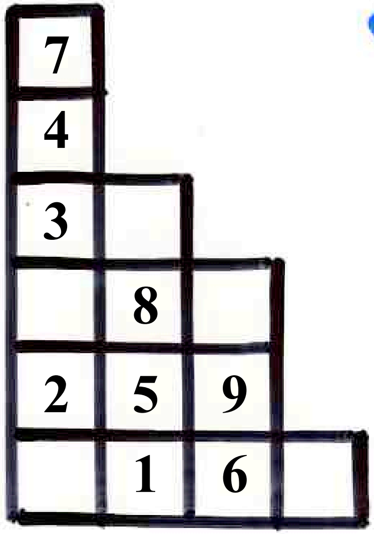

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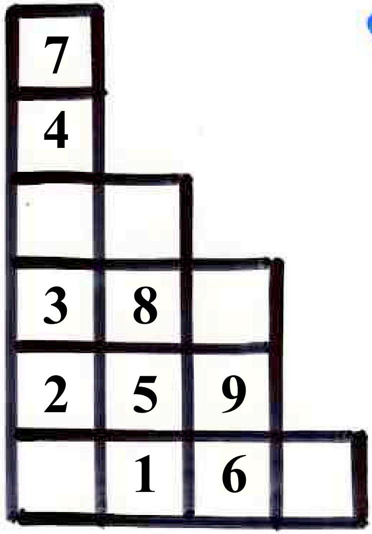

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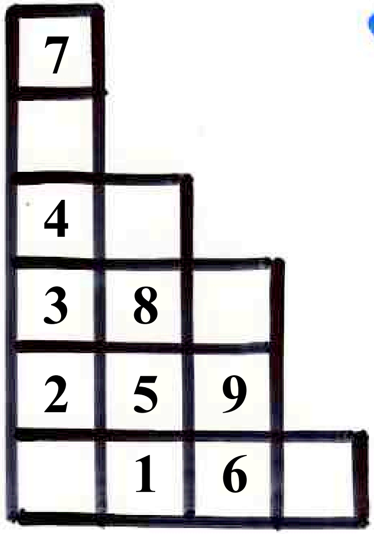

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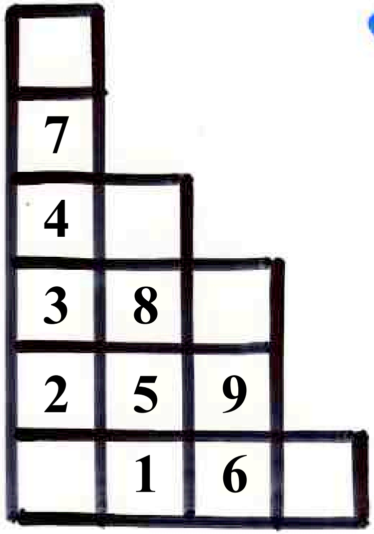

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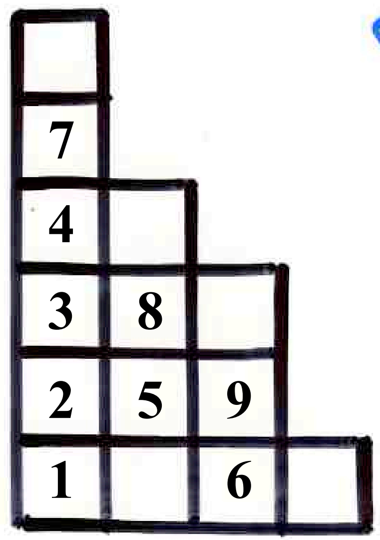

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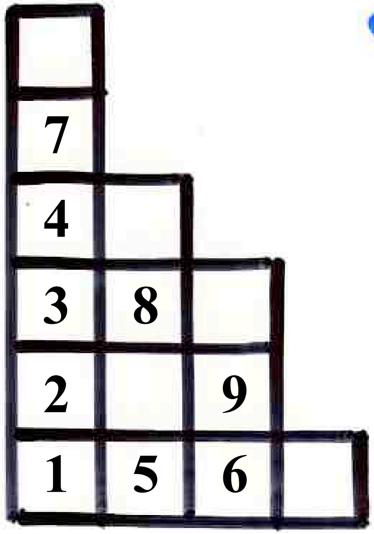

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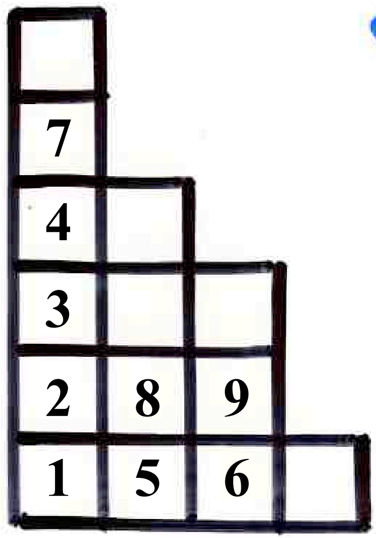
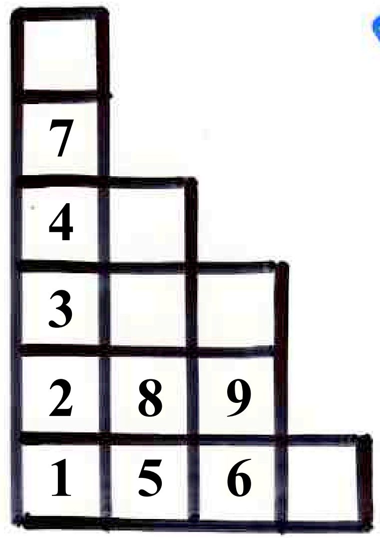


"jeu de taquin"

M.P. Schützenberger

$P(\sigma)$



Young tableau

$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$

$\sigma \longrightarrow (P(\sigma), Q(\sigma))$

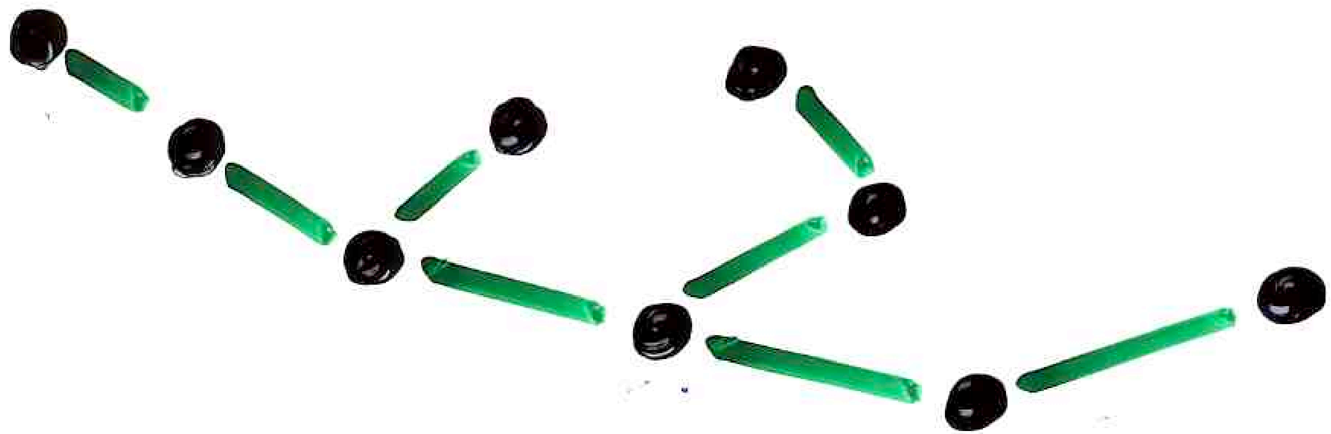
$Q(\sigma) = P(\sigma^{-1})$

RSK

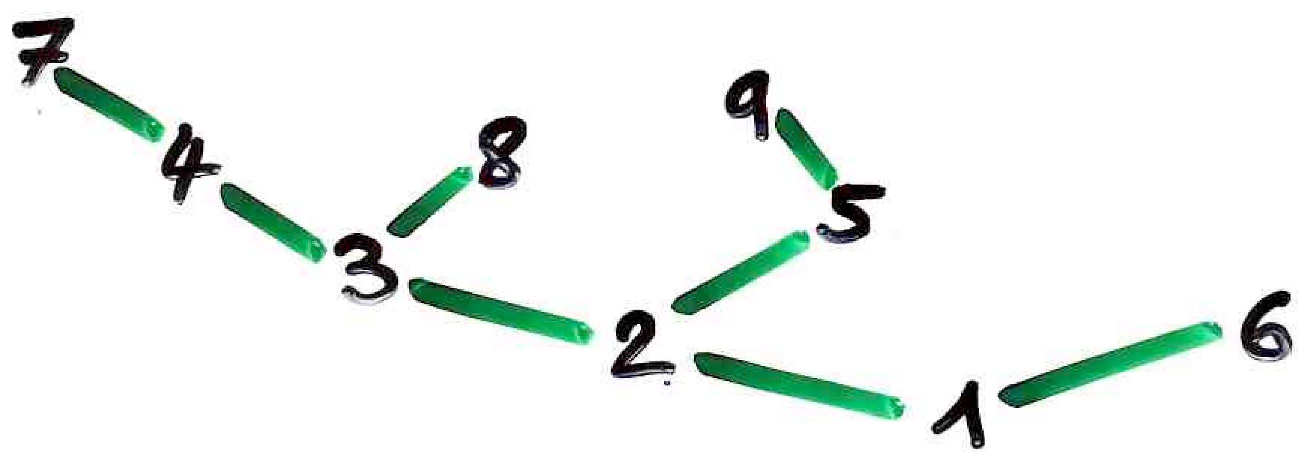
Robinson-Schensted-Knuth

§2 increasing binary trees

Def- Increasing binary tree

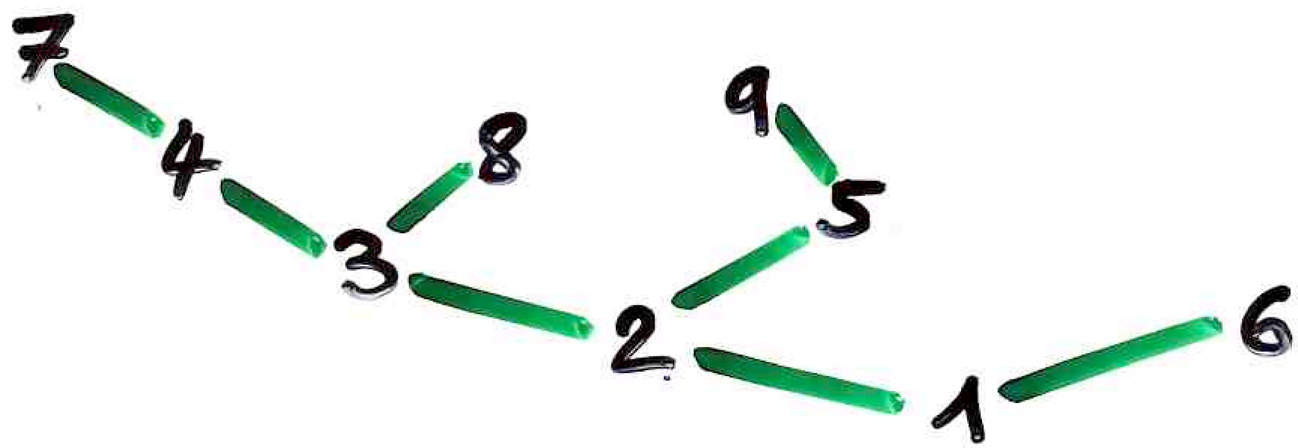


Def- Increasing binary tree



Bijection

increasing binary tree T ⟷ permutation σ



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$

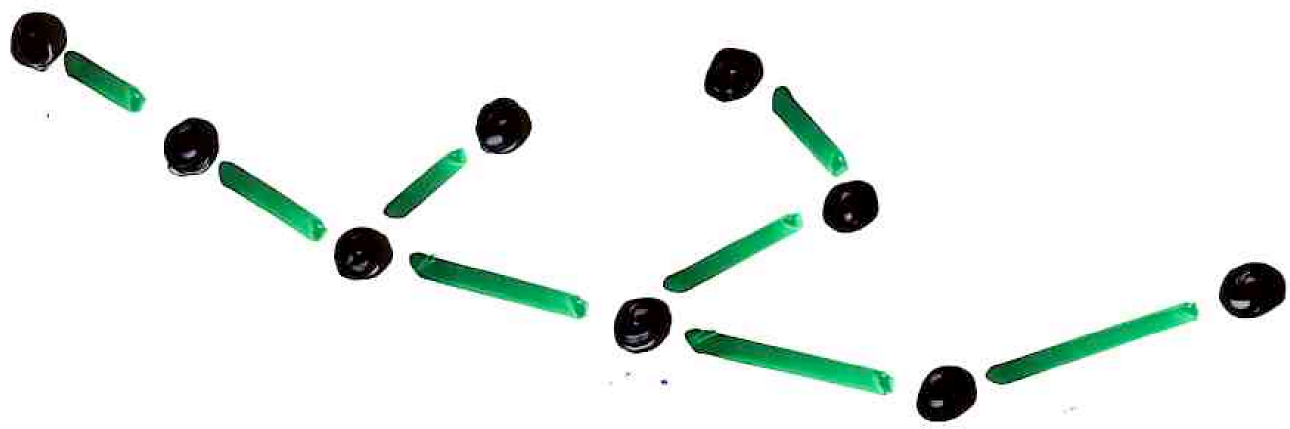
Bijection   increasing binary tree T $\longleftrightarrow$ permutation σ

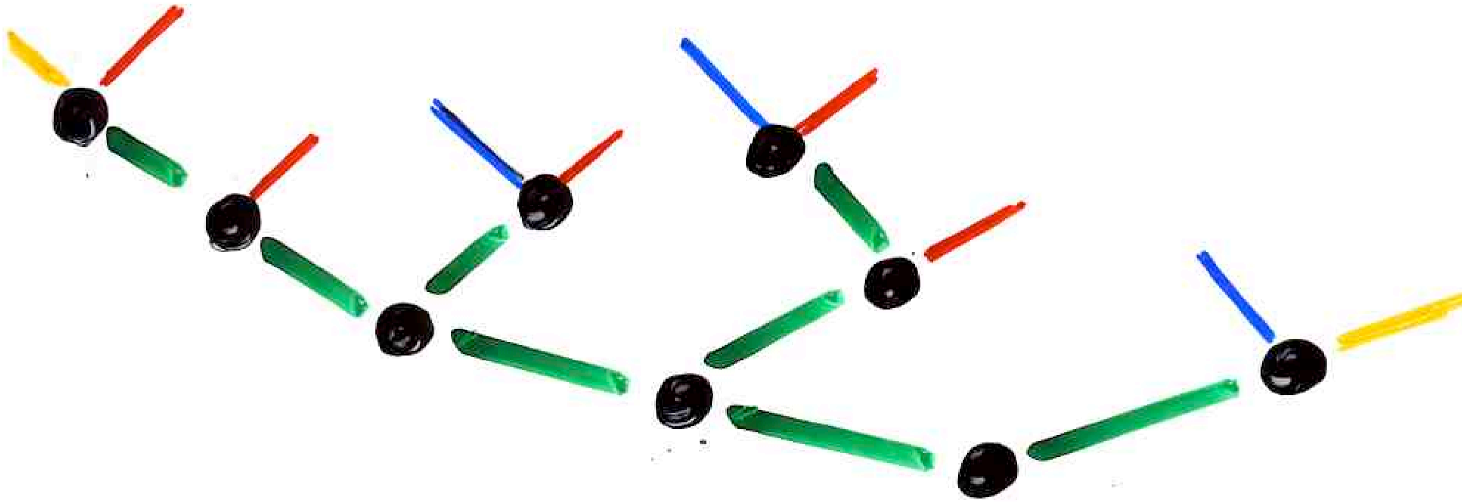
$T \xrightarrow{\pi} \sigma$ symmetric order of vertices (or "projection")

$\sigma \xrightarrow{\delta} T$ "déployé"

word $w = u\,m\,v$ m (unique) minimum letter

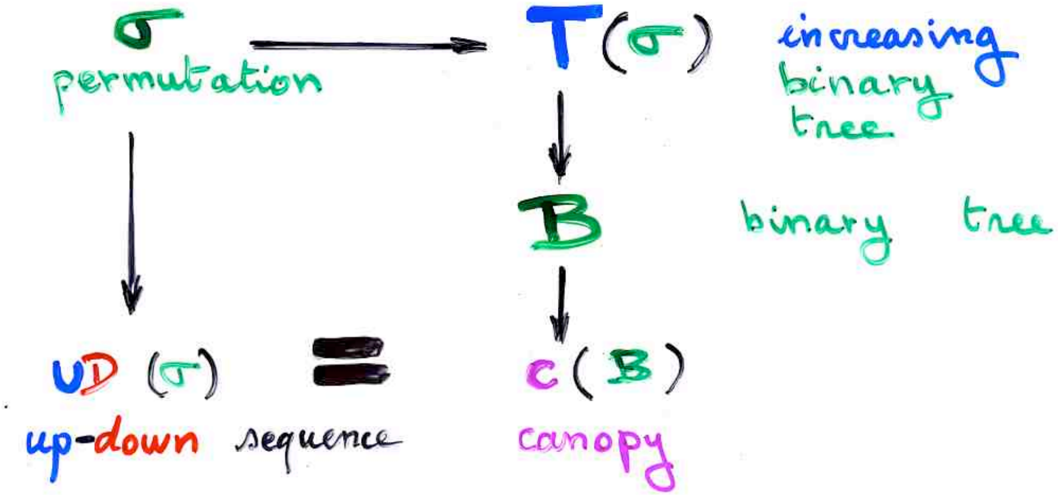
$\delta(w) =$ $\delta(u)$ \ m / $\delta(v)$

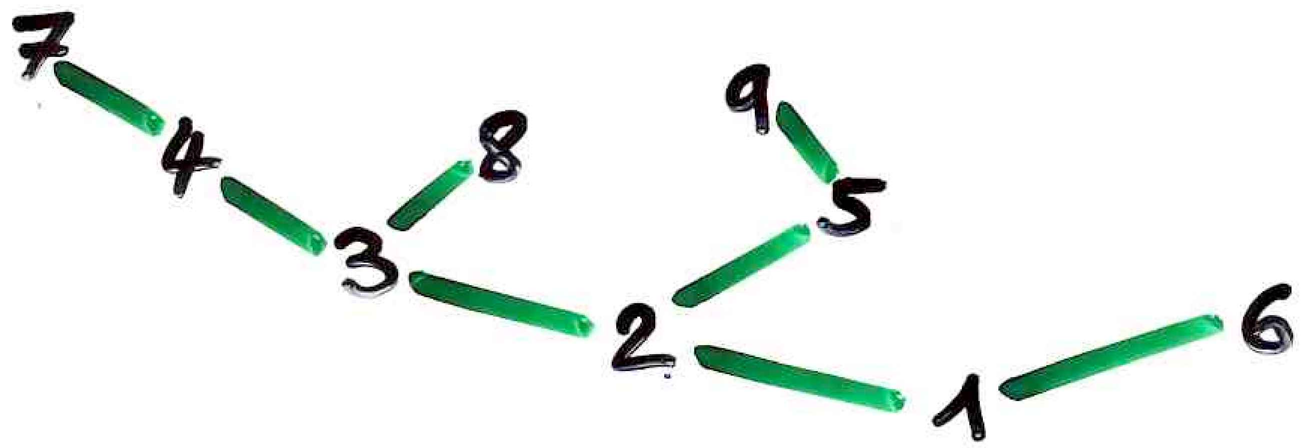




canopy of a binary tree

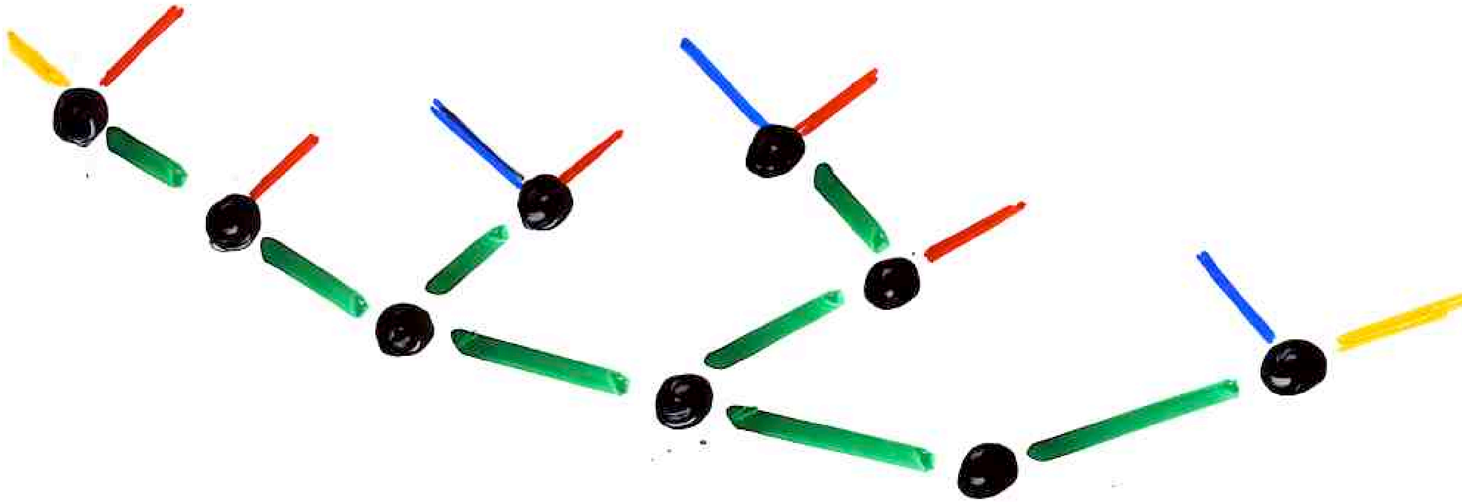
$C(B) = - - + - + - - +$


σ
permutation
T(σ)
increasing
binary
tree
B
binary tree
UD (σ)
up-down sequence
=
c(B)
canopy



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6 \ldots$

up-down sequence $-\ -\ +\ -\ +\ -\ -\ +$



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6 \ldots$

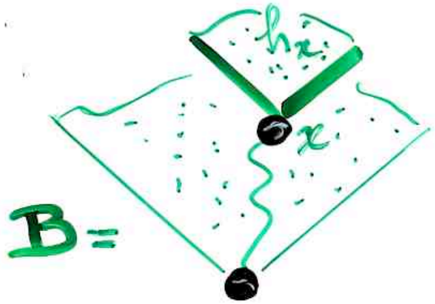
up-down sequence: $-\ -\ +\ -\ +\ -\ -\ +$

"hook-length formula"

$$\frac{n!}{\prod_x h_x}$$

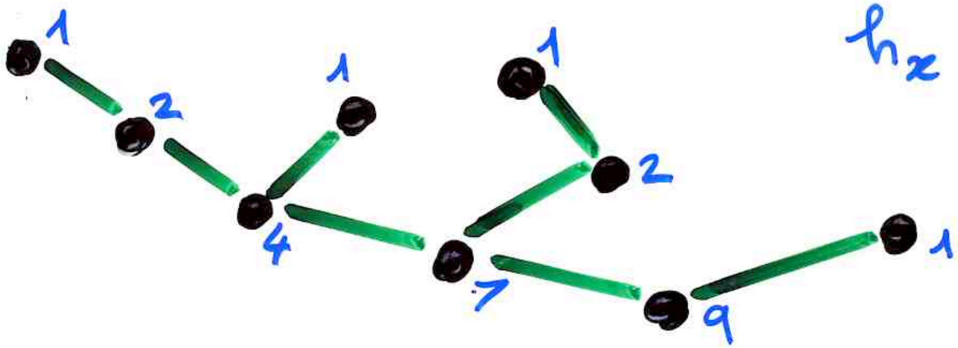
n nb of vertices

product of size of sub-trees



nb of increasing binary tree for a binary tree B

ex:

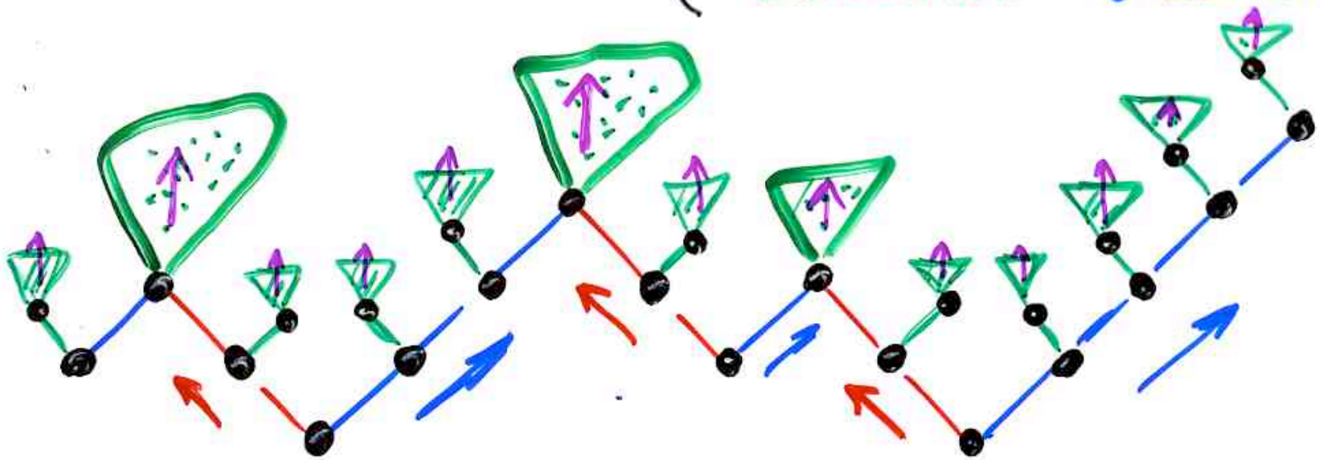


"hook-length" h_x

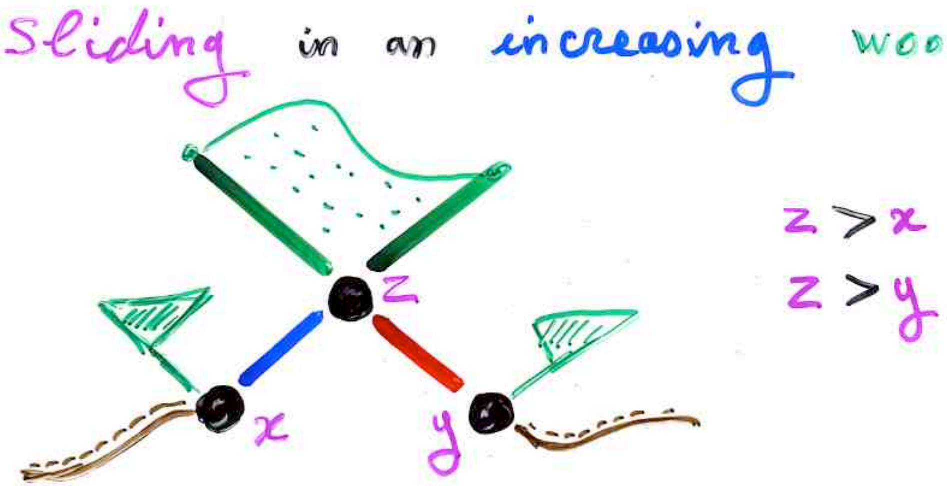
$$\frac{9!}{2^2 . 4 . 7 . 9} = 360$$

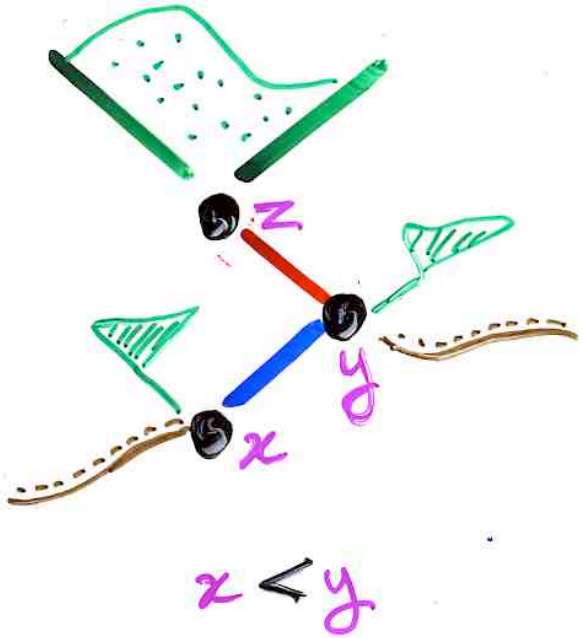
§ 3 "jeu de taquin" for increasing binary trees

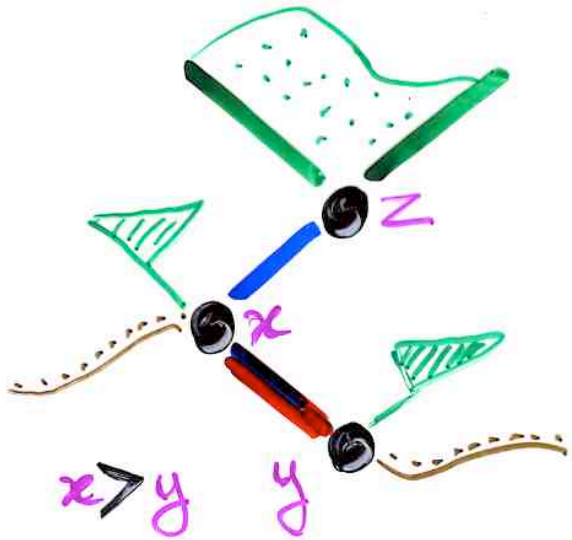
Def- Increasing woods ("buissons" croissants)



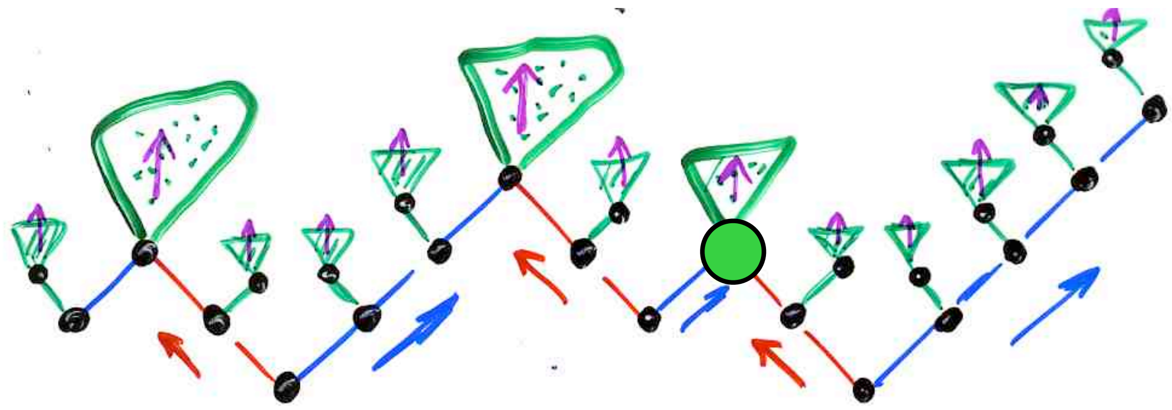
Def - Sliding in an increasing woods

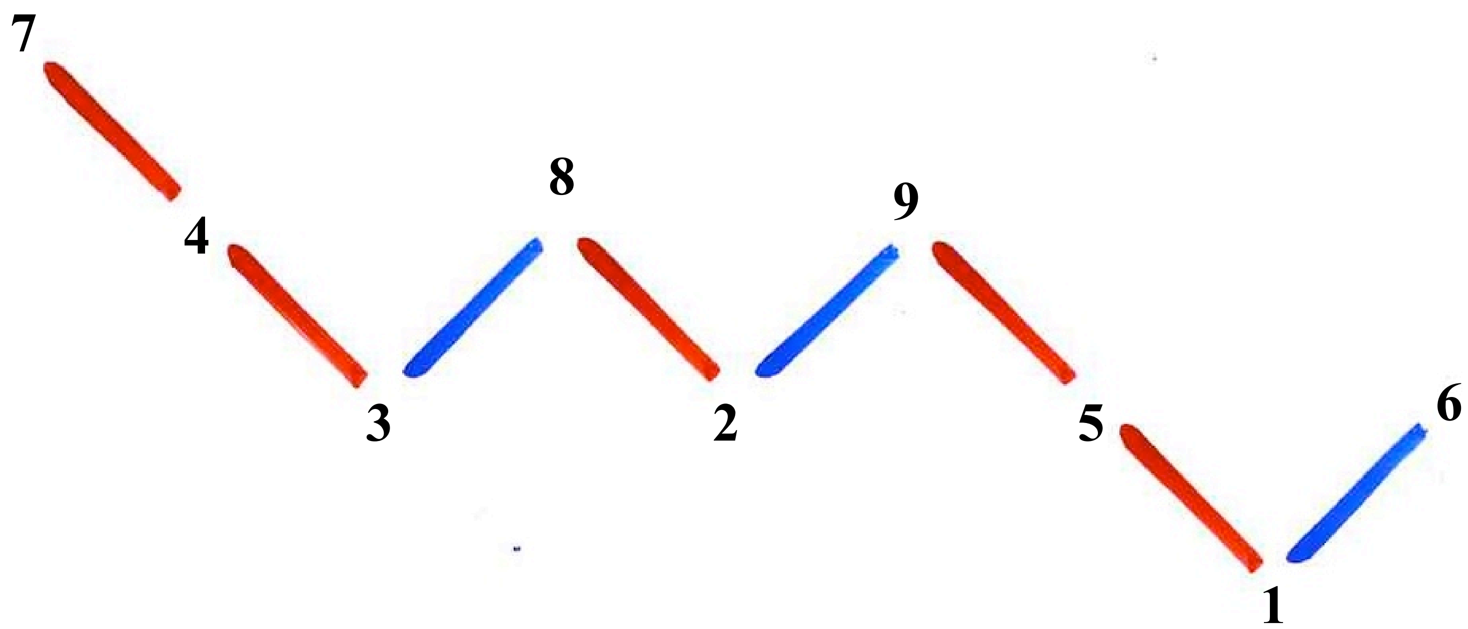






"jeu de taquin" for increasing woods

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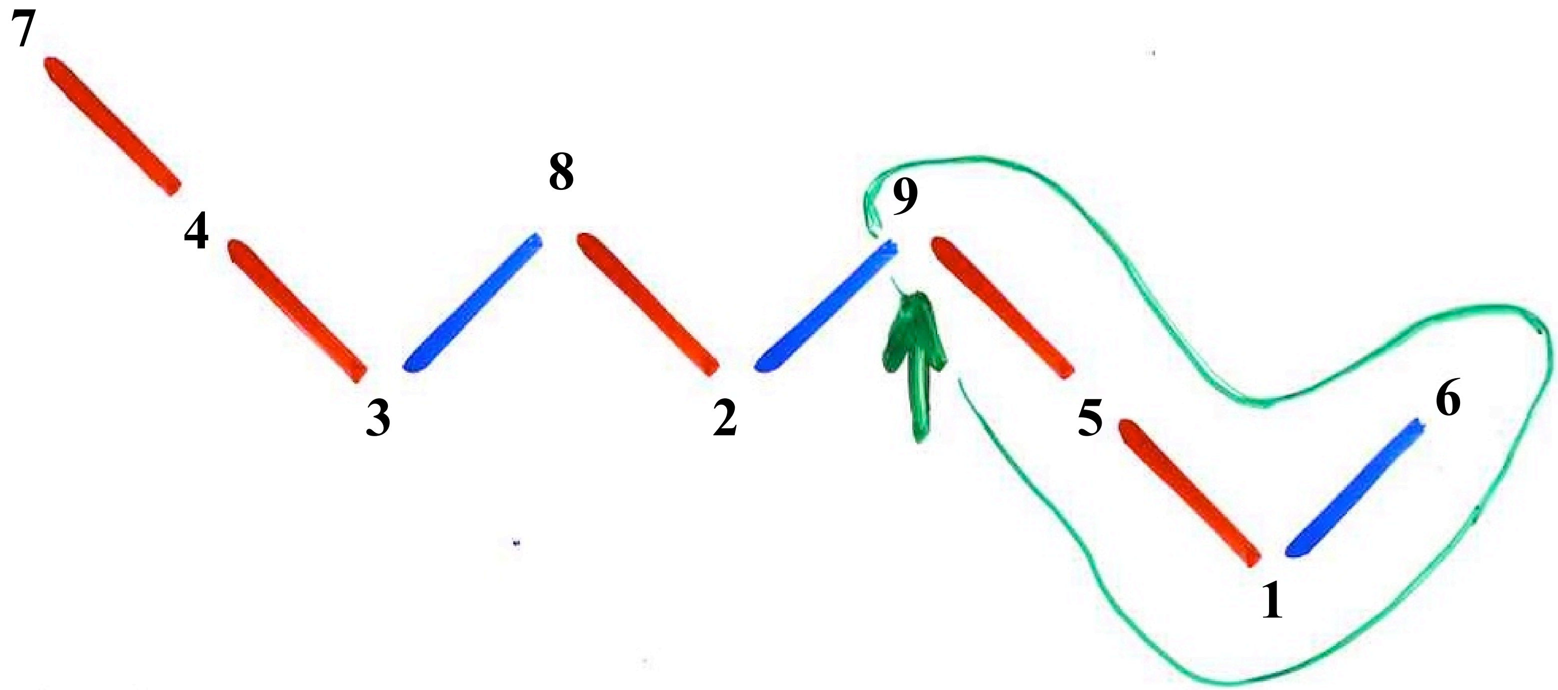
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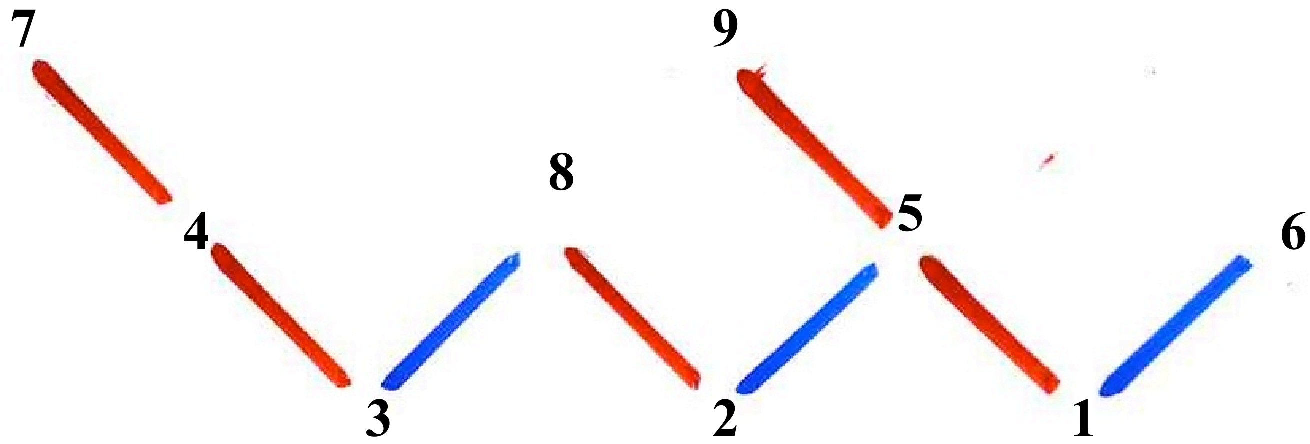
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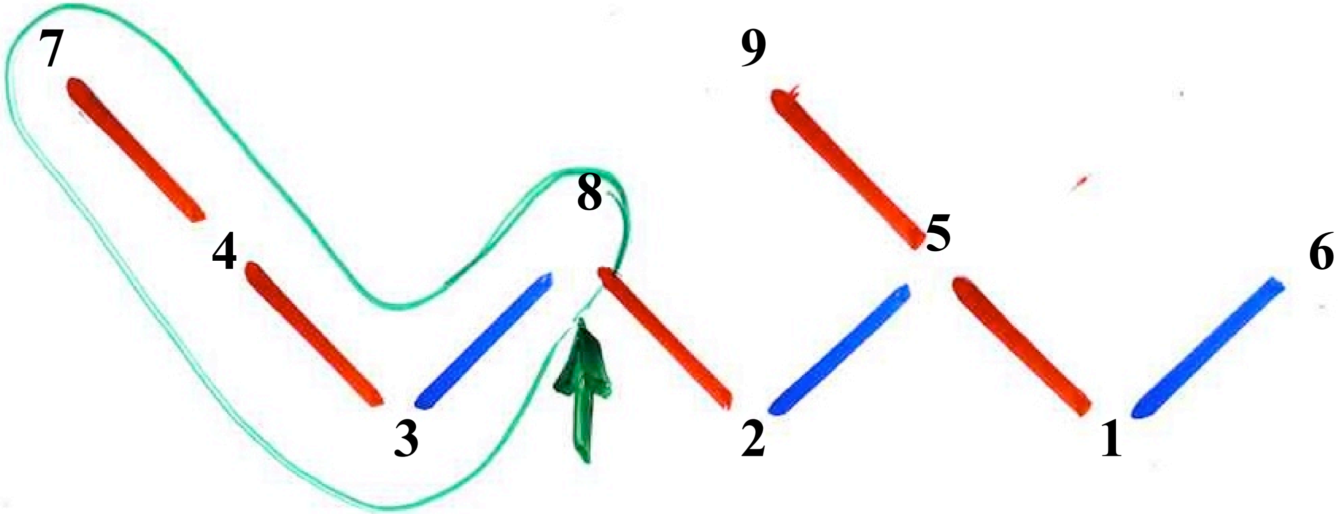
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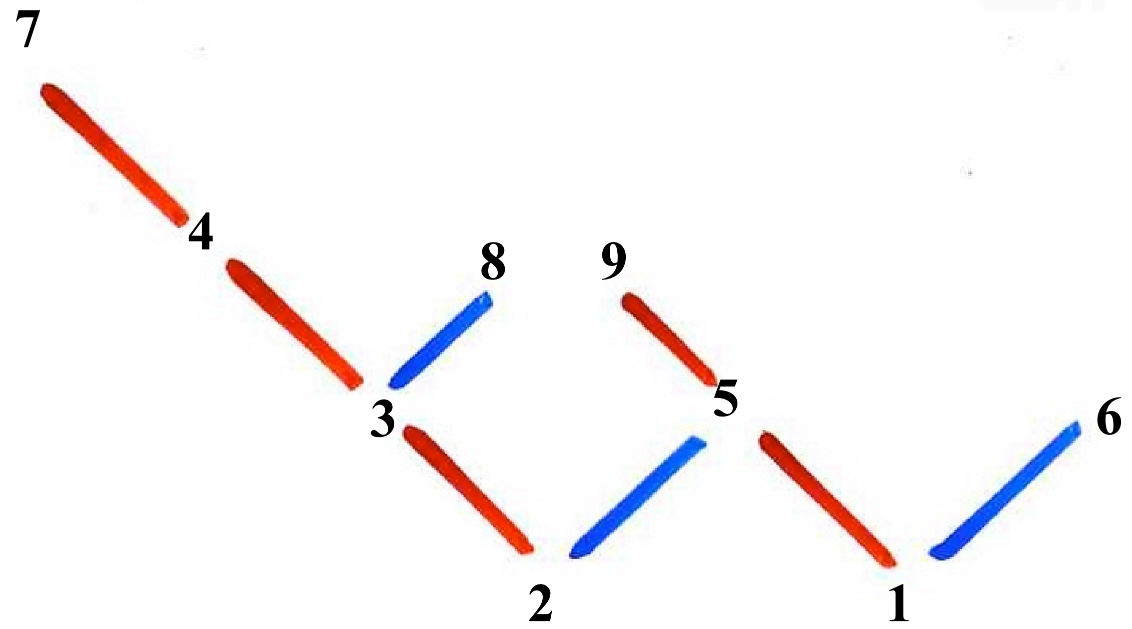
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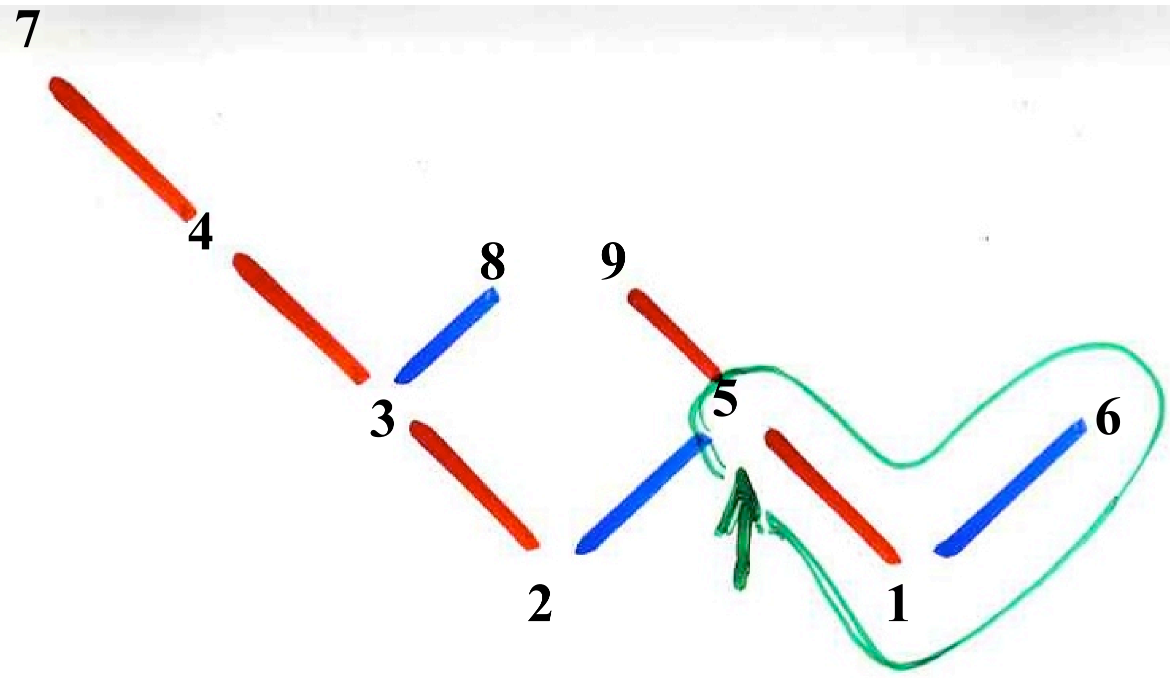

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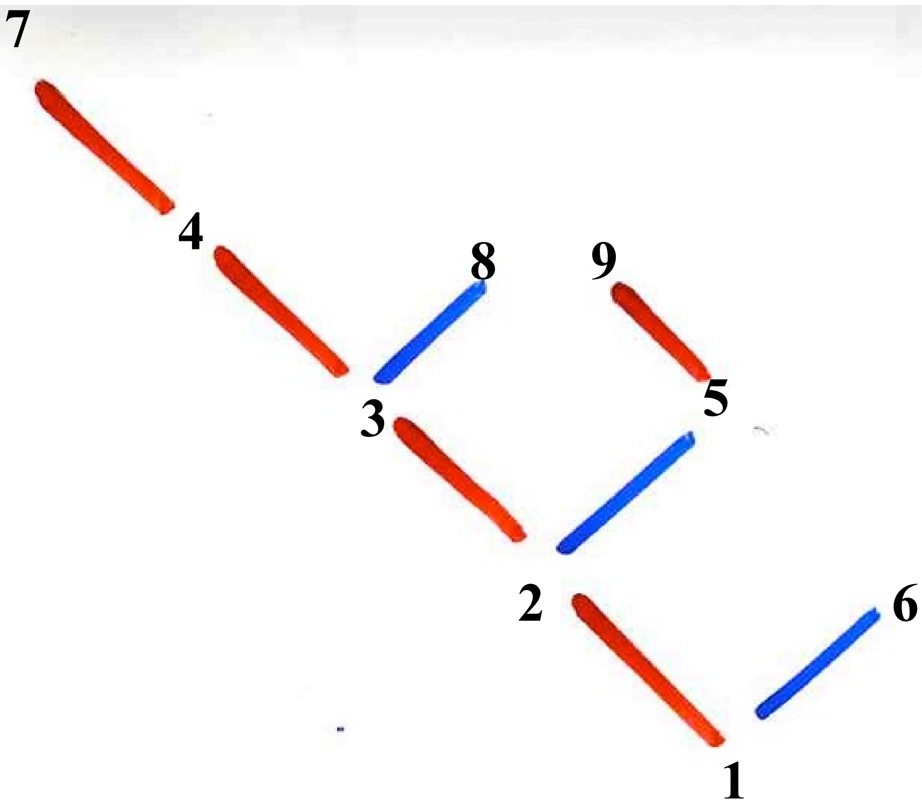
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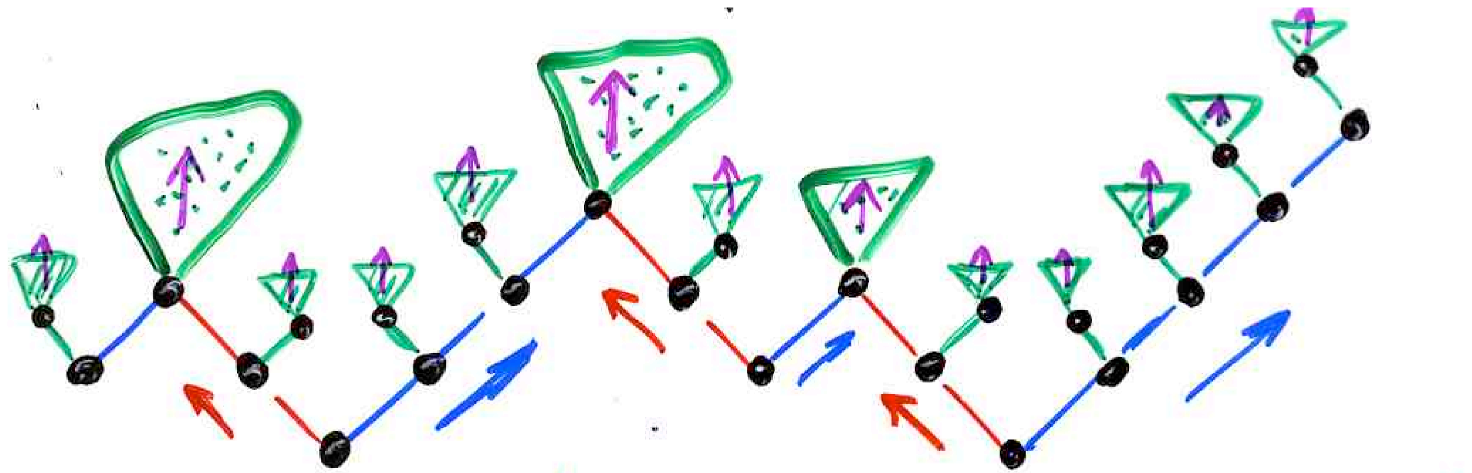


symmetric order for woods

Lemma. invariance of the symmetric order through slidings of an increasing woods

Cor. $\sigma \rightarrow$ UD-wood 

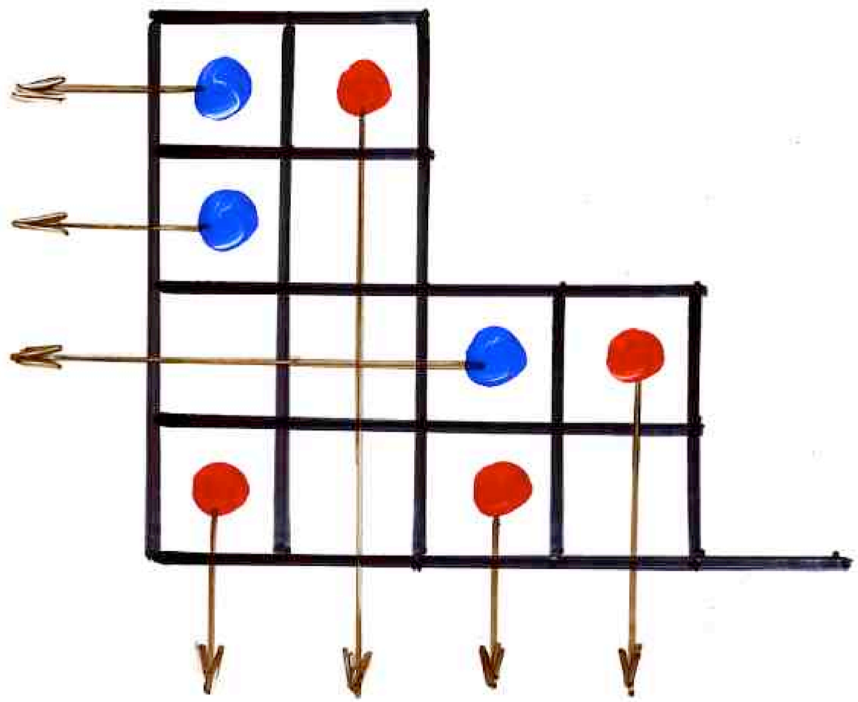
$\downarrow$ "jeu de taquin"

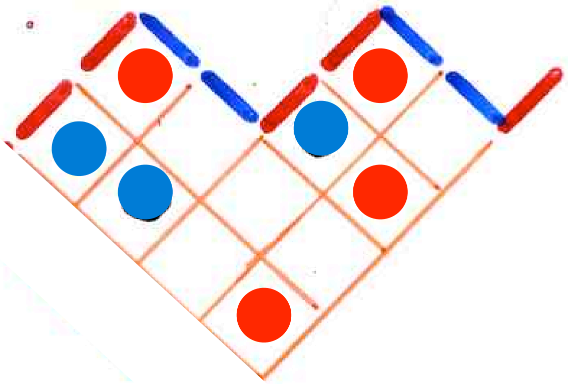
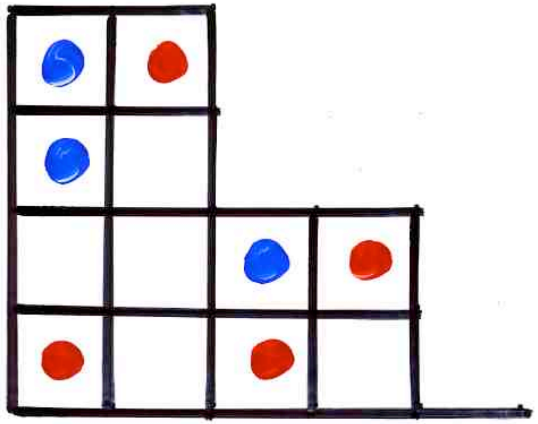
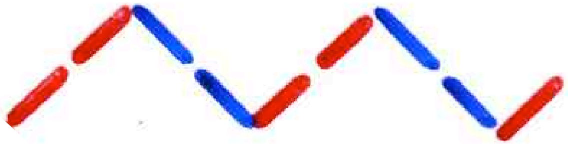
$T(\sigma) = \delta(\sigma)$ déployé

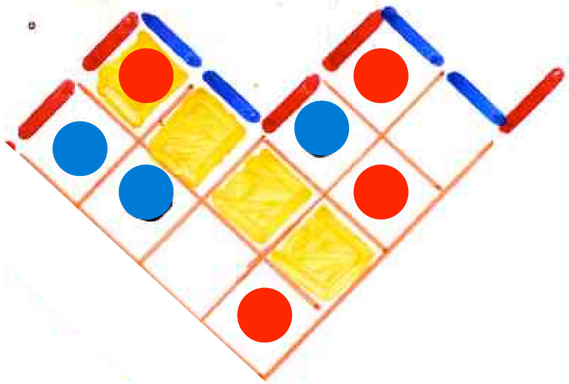
increasing binary tree

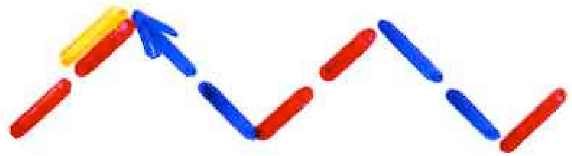
§ 4
jeu de taquin
for
binary trees

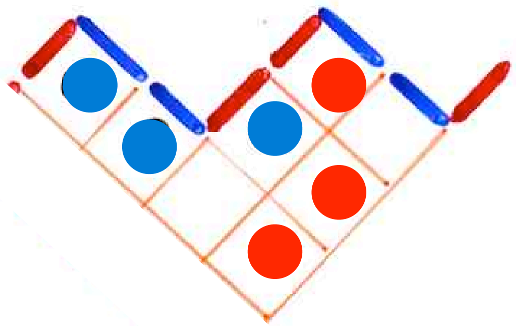
Def Catalan alternative tableau T
alt. tab. without cells ☒
i.e: every empty cell is below a red cell or on the left of a blue cell

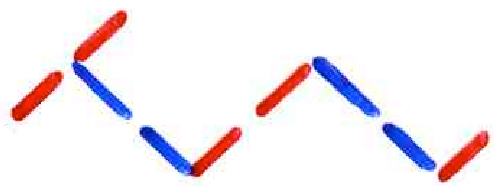


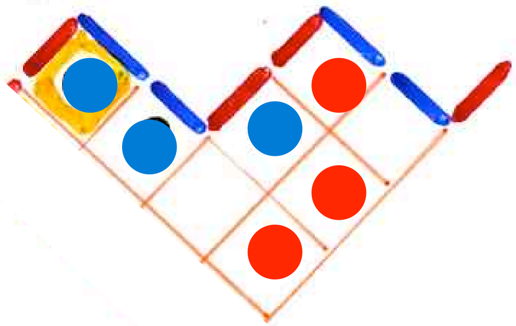




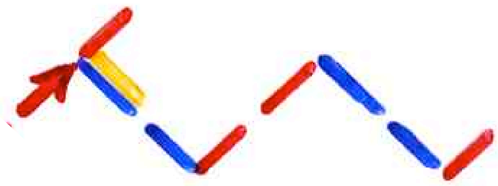


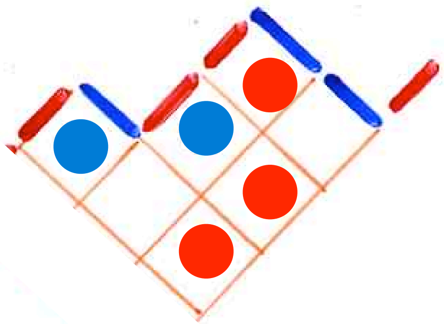


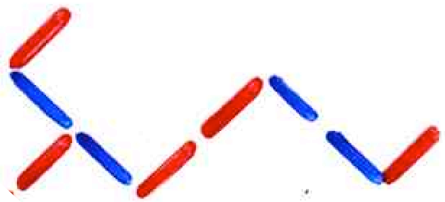


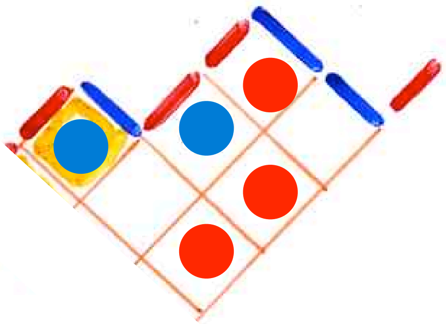


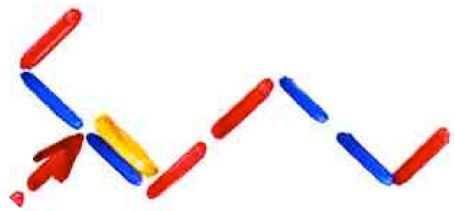


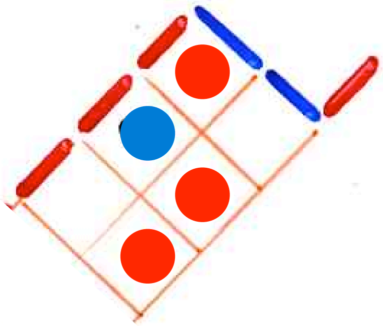


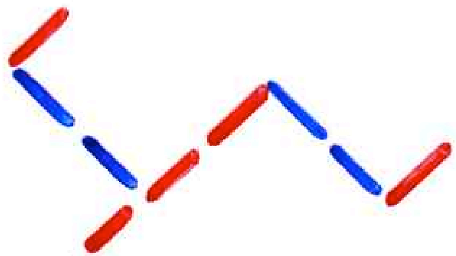


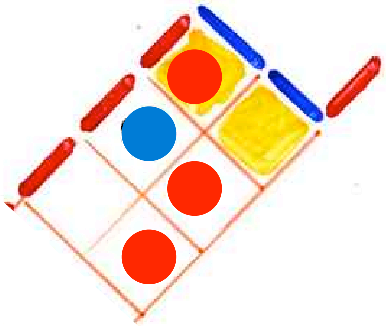


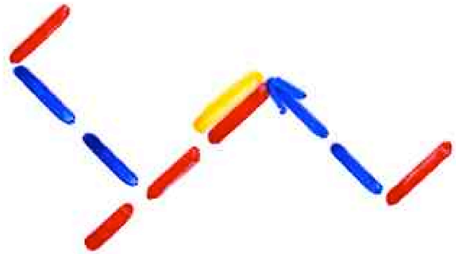


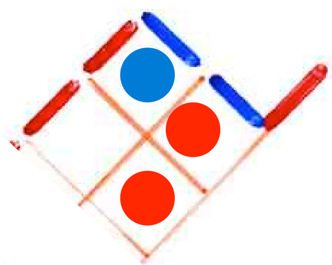


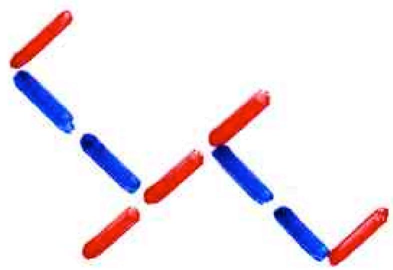


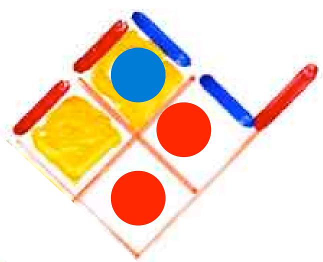


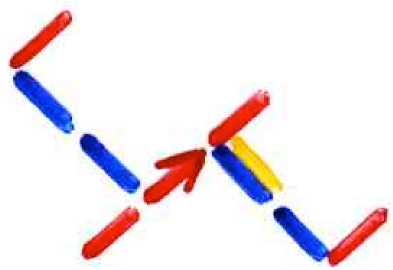


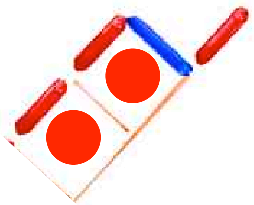


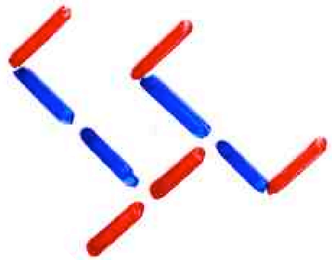


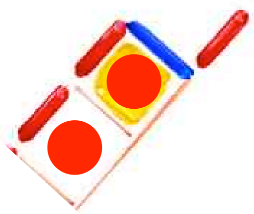


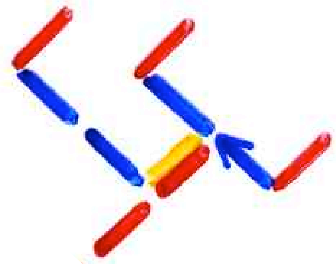


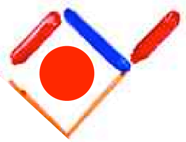


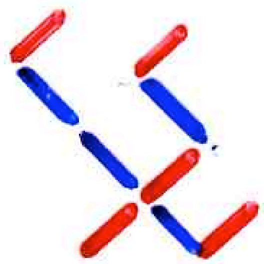


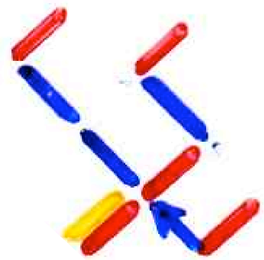


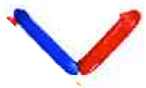


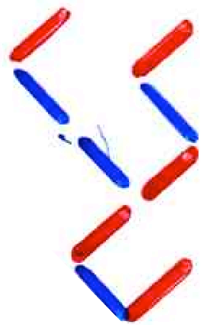












§ 5 Up-down and Genocchi sequences

Def. Genocchi sequence of a permutation

$$\sigma = \sigma(1) \quad \sigma(n)$$

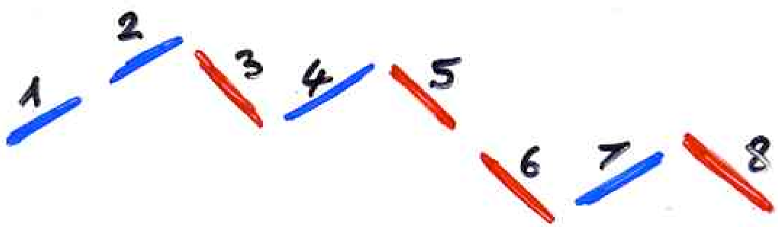
$$G(\sigma) = z_1 \dots z_{n-1}$$

$$z_x = \begin{cases} a & \text{(ascent)} \\ d & \text{(descent)} \end{cases} \qquad x = \sigma(i) \begin{matrix} < \sigma(i+1) \\ > \sigma(i+1) \end{matrix}$$

$1 \leq x \leq n-1$ "value" "index"

convention: $\sigma(n+1) = 0$ ($\sigma(n)$ is a descent)

ex: $\sigma = (8\ 5\ 3\ 2\ 7\ 9\ 1\ 4\ 6)$



alternating sequence dadad...ada

Prop. (Dumont, 1974)
The nb of permutations on $\{1,2,..,2n\}$ having an alternating Genocchi sequence is the Genocchi numbers G_{2n+2}

nombres de Genocchi

$$G_{2n} = 2(2^{2n}-1)\,B_{2n}$$

Bernoulli

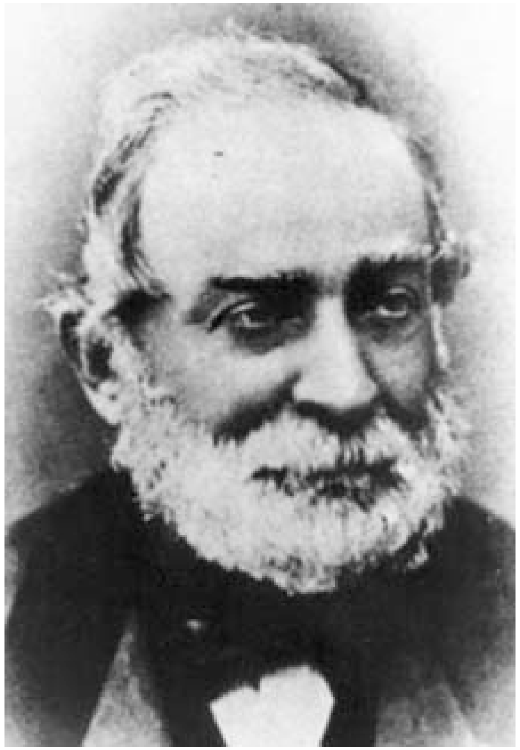
$$2^{2n}G_{2n+2} = (n+1)\,T_{2n+1}$$

nombres de Genocchi

$$G_{2n} = 2(2^{2n} - 1) B_{2n}$$

Bernoulli

$$2^{2n} G_{2n+2} = (n+1) T_{2n+1}$$



Angelo Genocchi
1817 - 1889

$$\operatorname{tg}(t) = \sum_{n \geq 0} T_{2n+1} \frac{t^{2n+1}}{(2n+1)!}$$

$$\frac{1}{\cos(t)} = \sum_{n \geq 0} E_{2n} \frac{t^{2n}}{(2n)!}$$

E_{2n} $\{1, 5, 61, 1385, \ldots\}$

nombres sécant (d'Euler)

T_{2n+1} $\{1, 2, 16, 272, 7936, \ldots\}$

nombres tangents

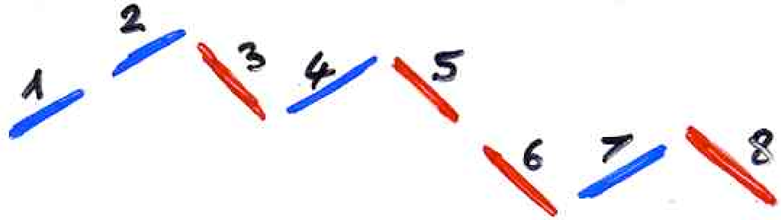
Permutations alternantes

D. André (1880)

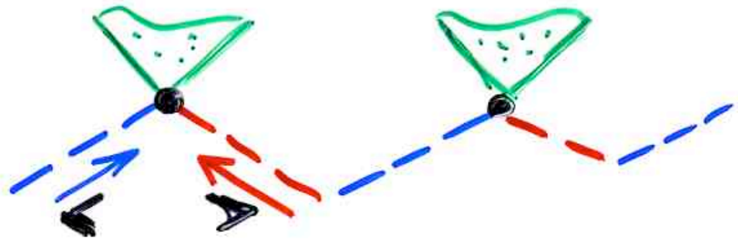
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 2 & 9 & 7 & 8 & 4 & 5 & 1 & 3 \end{pmatrix}$$

$$6 \searrow 2 \nearrow 9 \searrow 7 \nearrow 8 \searrow 4 \nearrow 5 \searrow 1 \nearrow 3$$

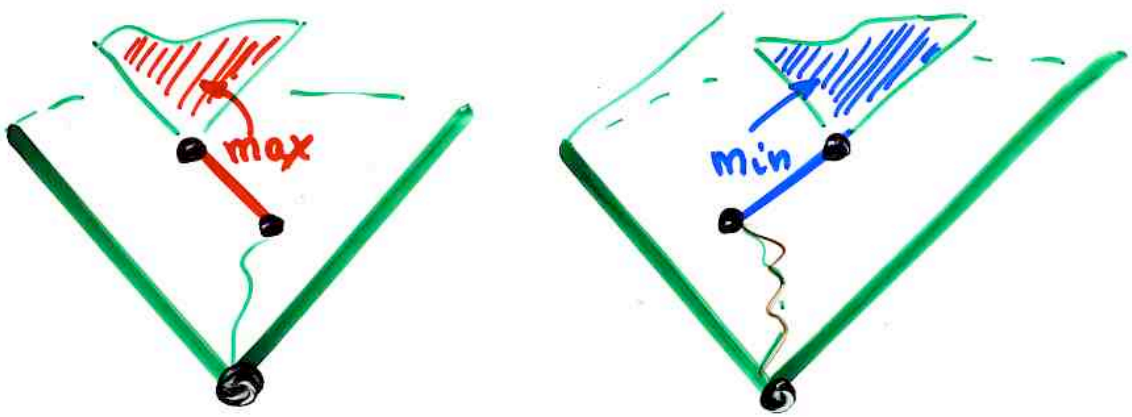
§ 6 alternative binary trees

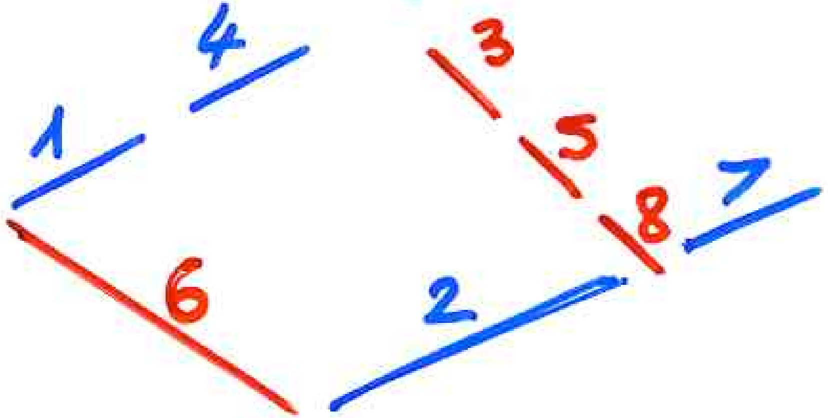


Def. edge-alternative woods

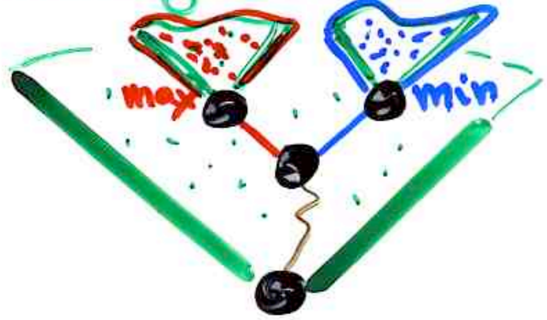


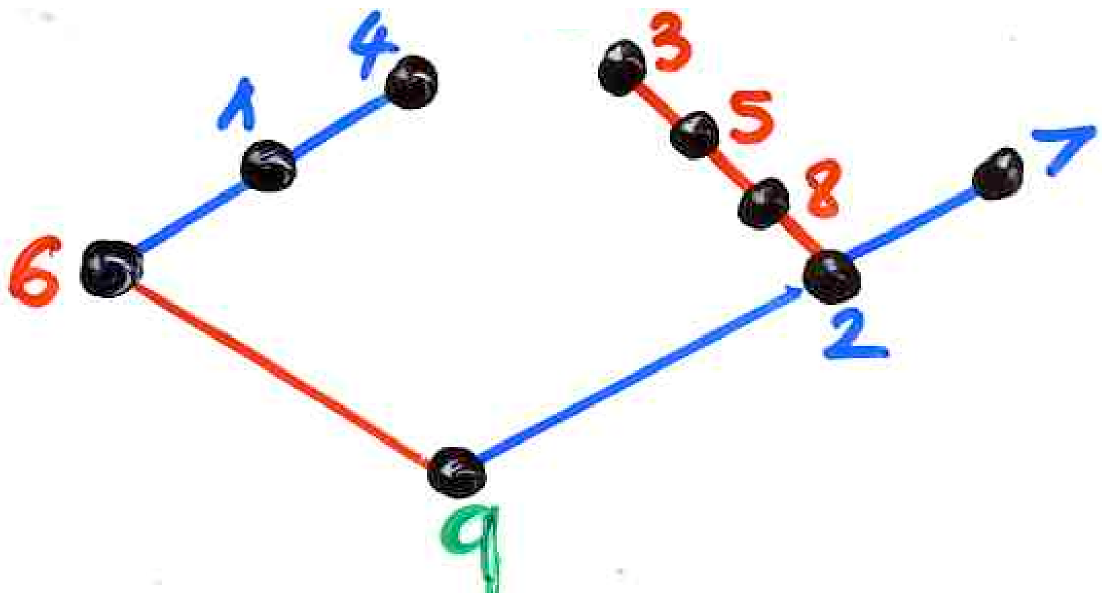
Def edge-alternative binary trees





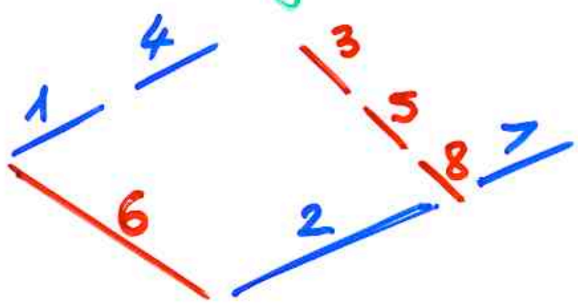
Def alternative binary tree



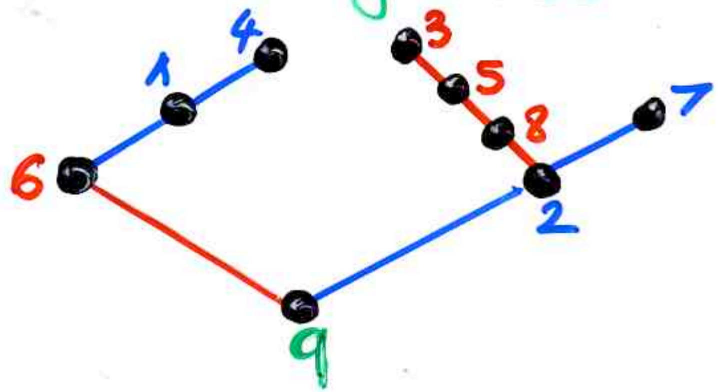


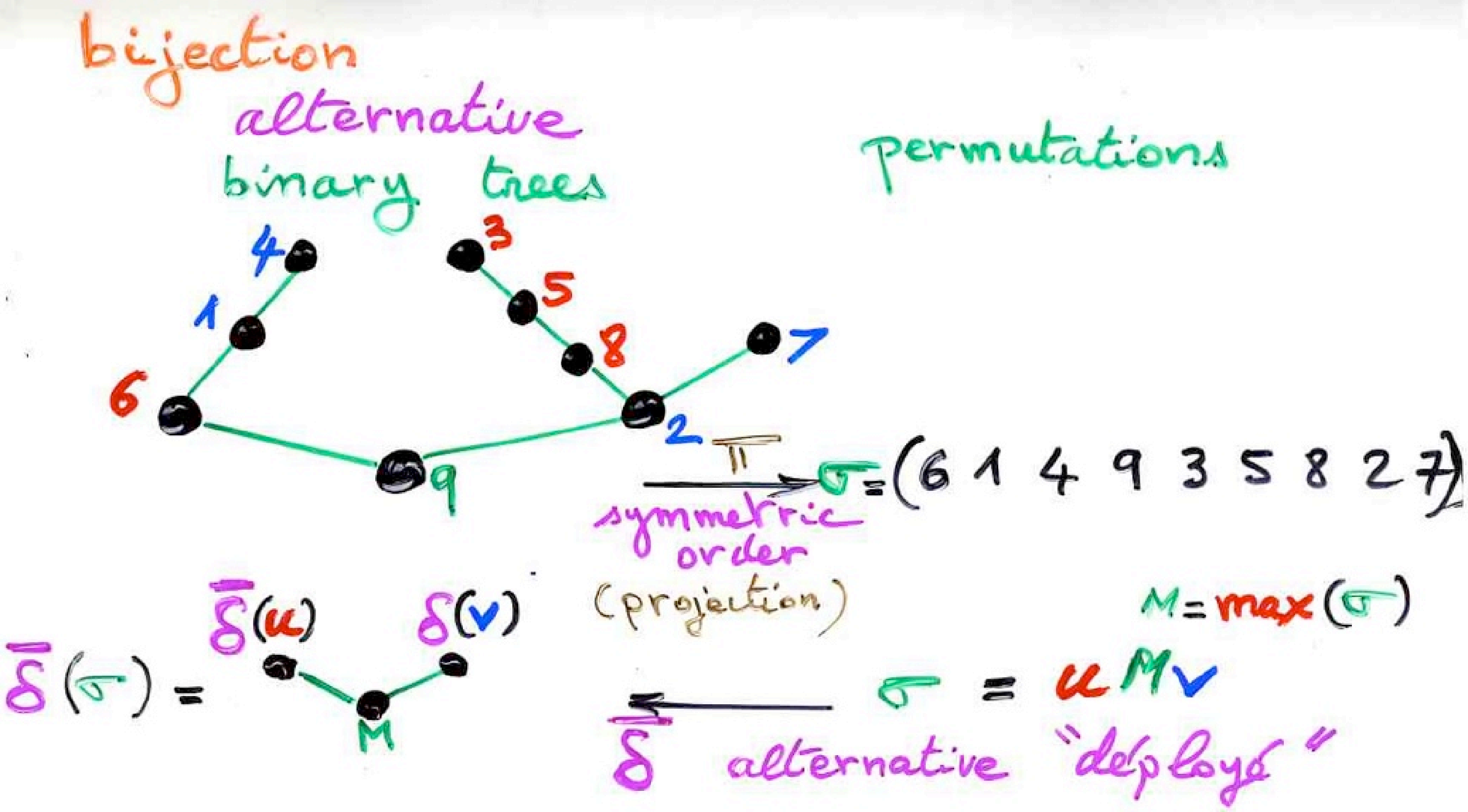
bijection

edge-alternative binary tree



alternative binary tree





bijection
alternative binary trees
permutations
4
1
6
3
5
8
7
2
9
π
σ = (6 1 4 9 3 5 8 2 7)
symmetric order
(projection)
M = max(σ)
δ̄(σ) =
δ̄(u)
δ(v)
M
σ = u M v
δ̄
alternative "déployé"

"hook length" formula

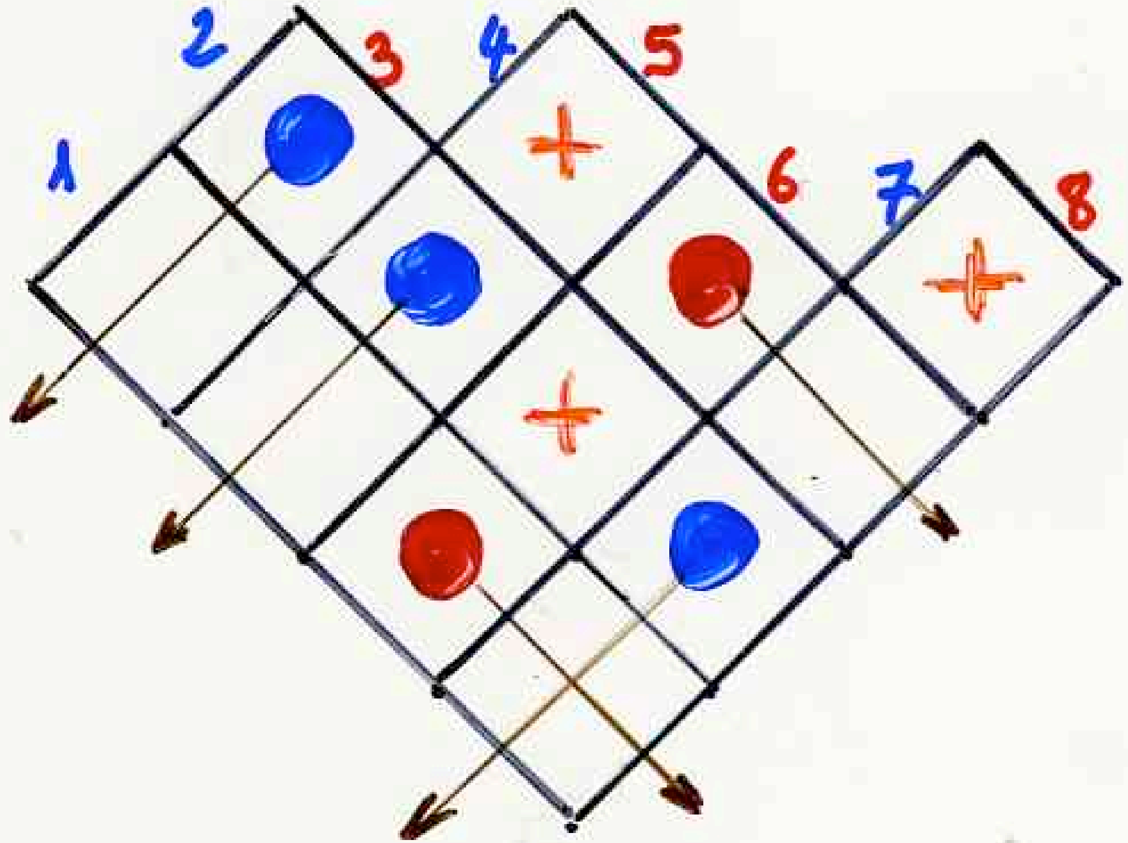
same as for

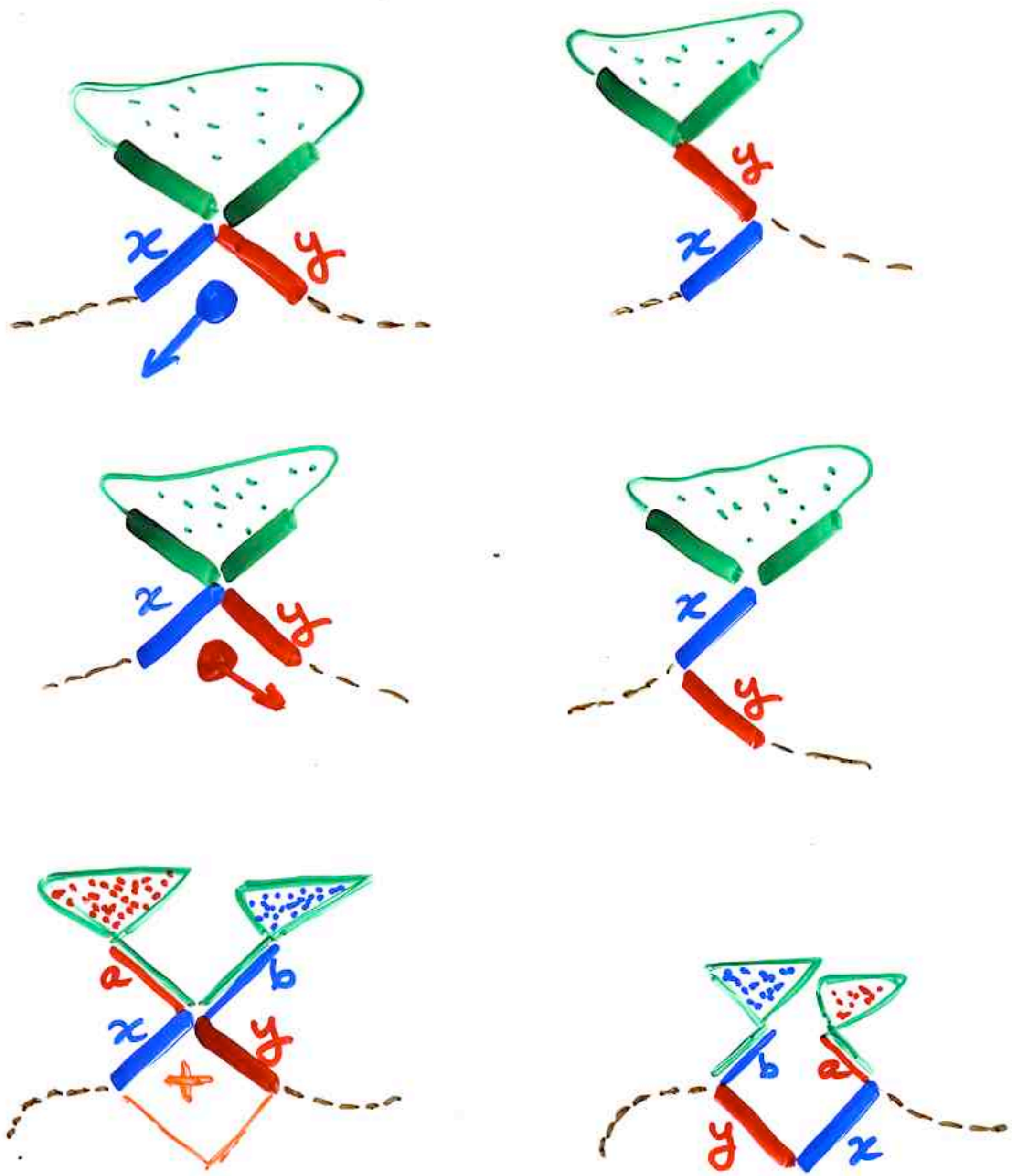
increasing

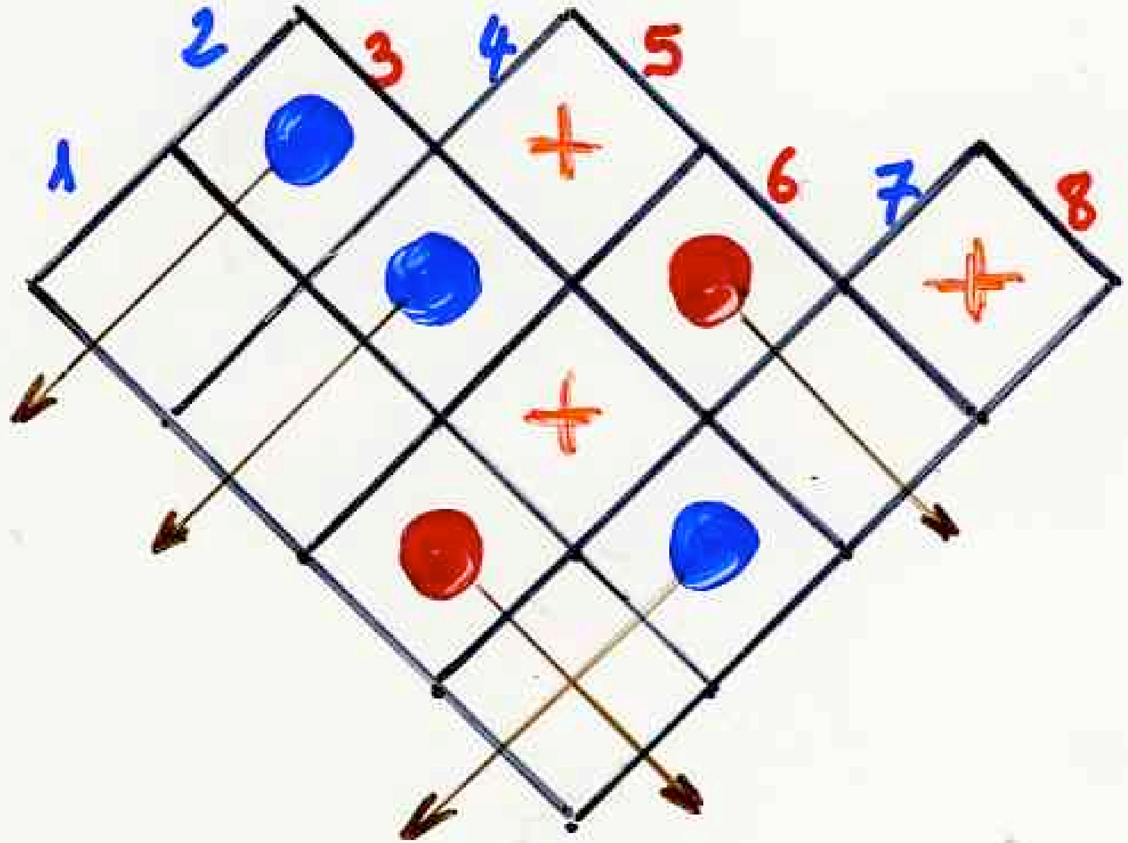
binary tree

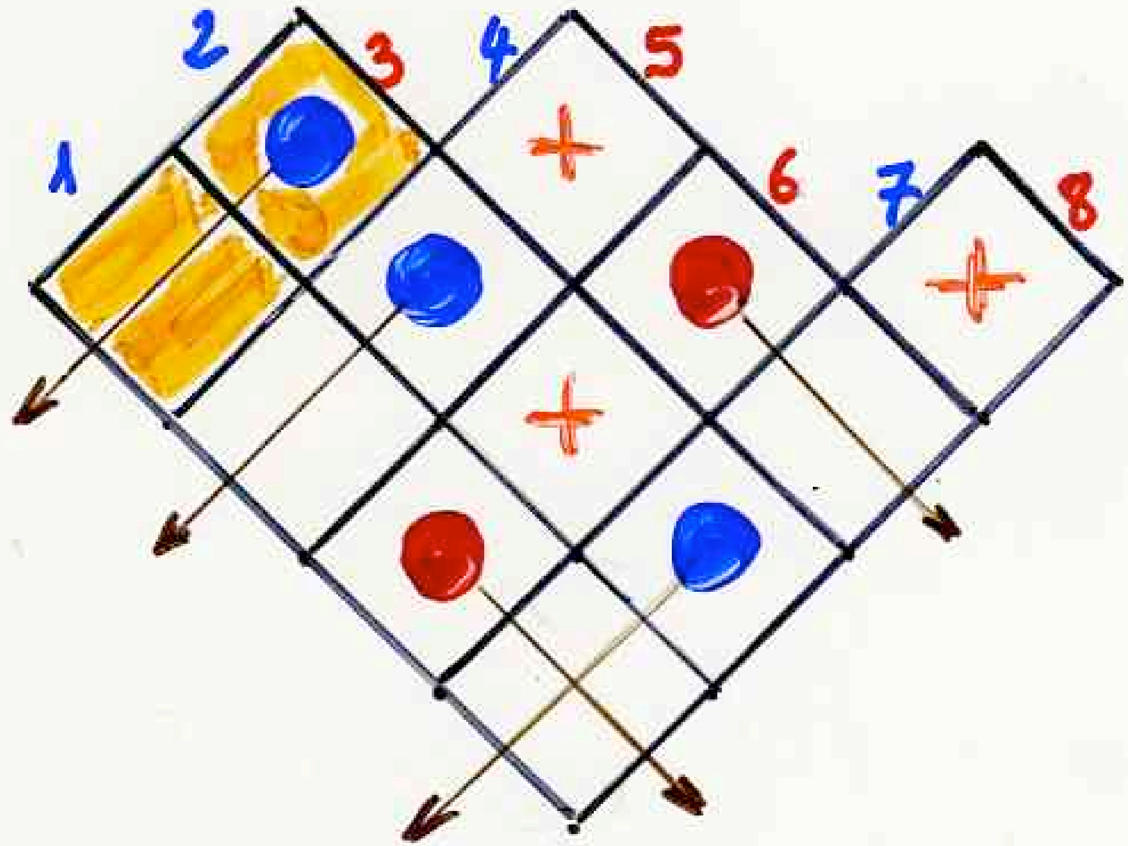
$$\frac{n!}{\prod_{x} h_x}$$

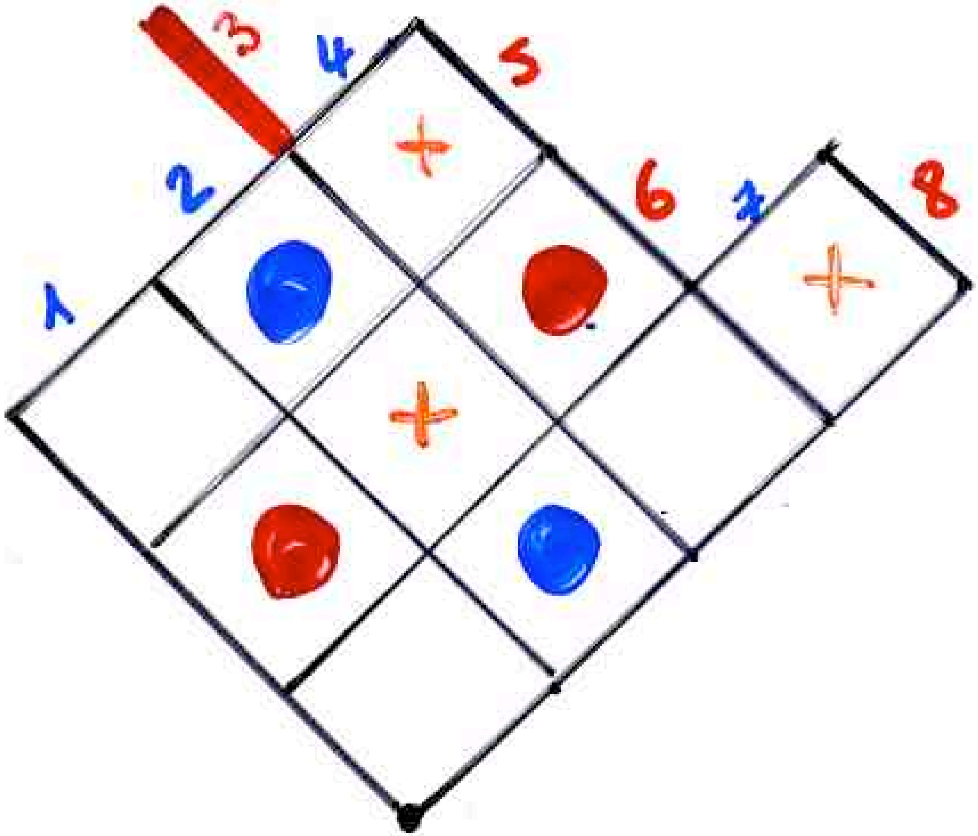
§ 7 jeu de taquin for alternative binary trees

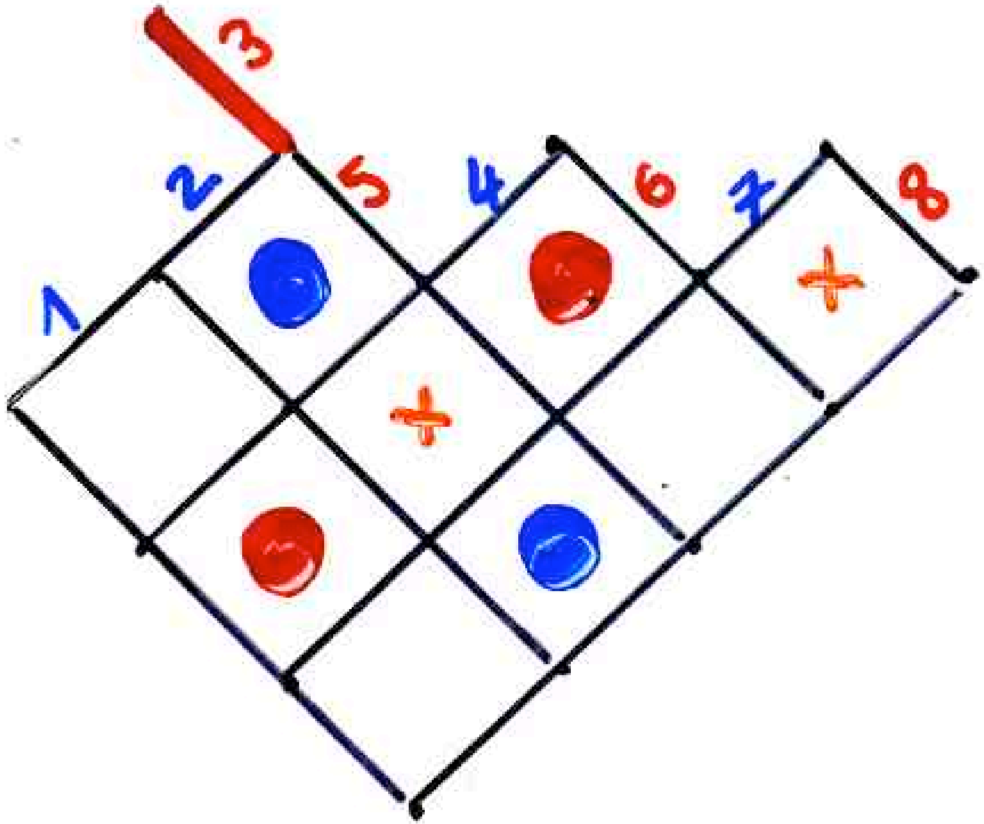

1
2
3
4
5
6
7
8

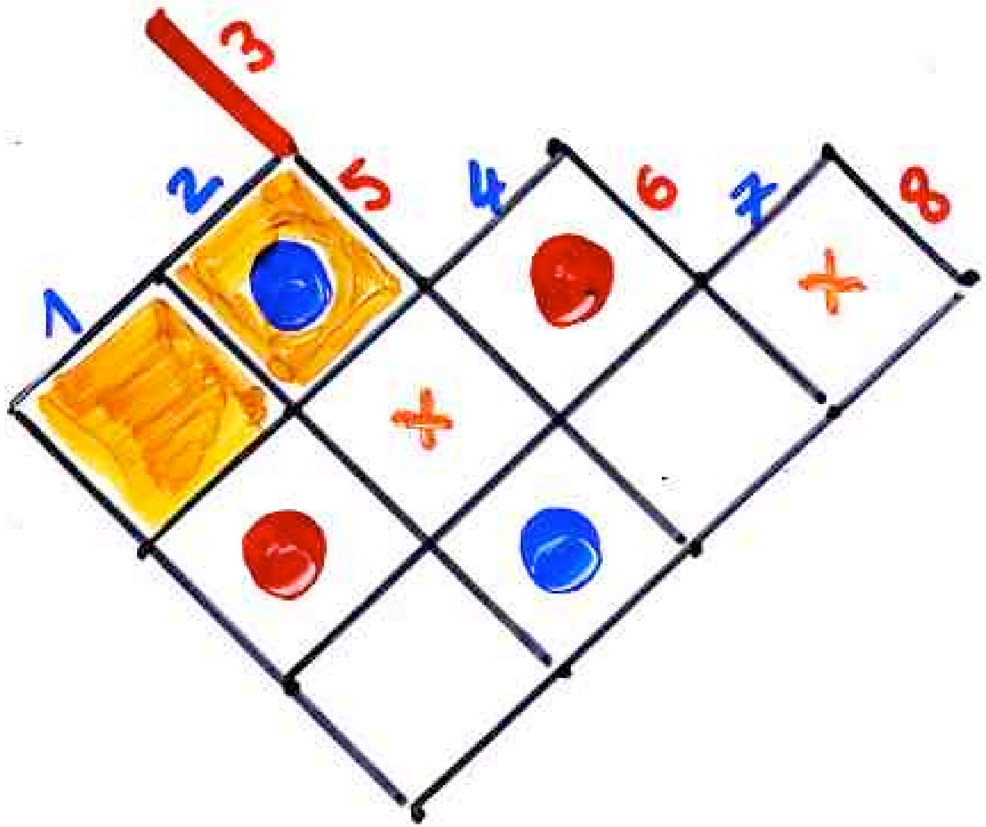

x
y
y
x
x
y
x
y
a
b
x
y
b
a
y
x

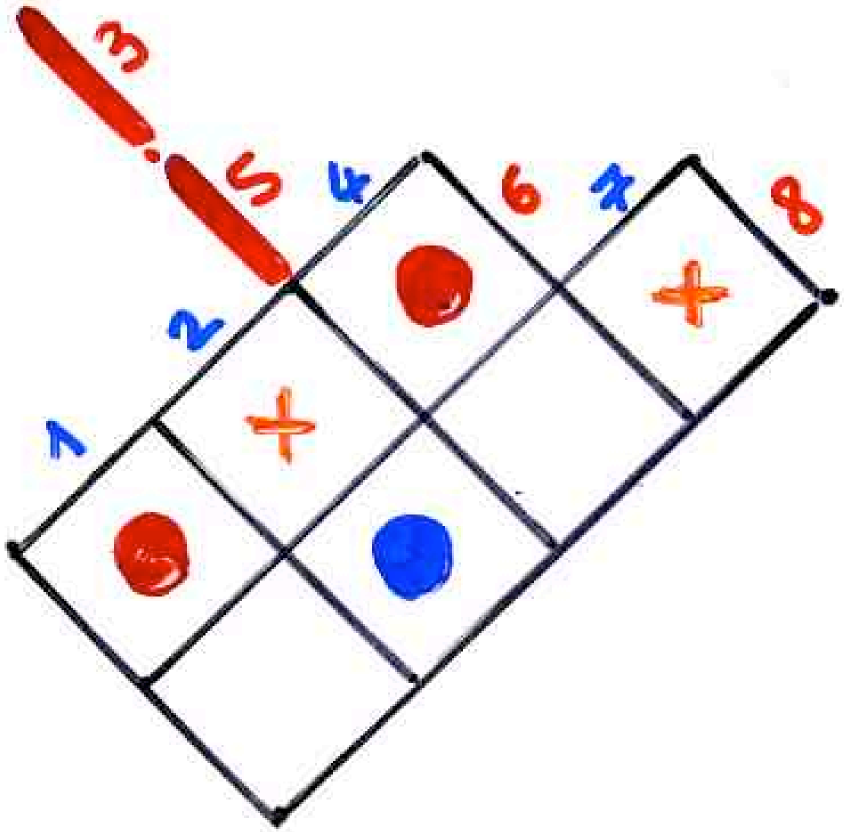

1
2
3
4
5
6
7
8

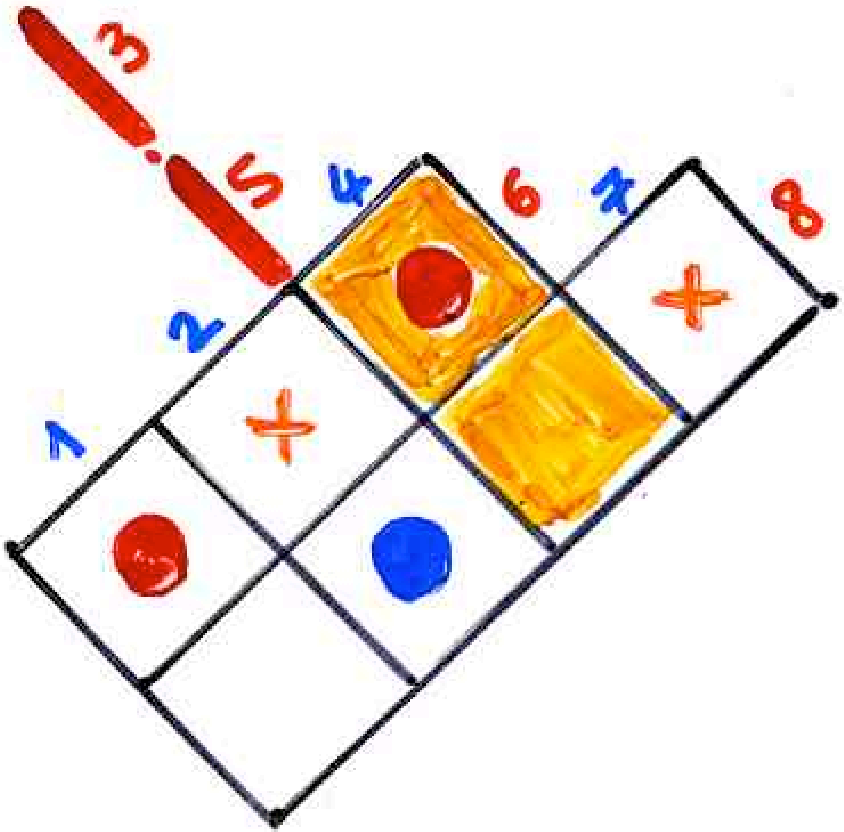

1
2
3
4
5
6
7
8

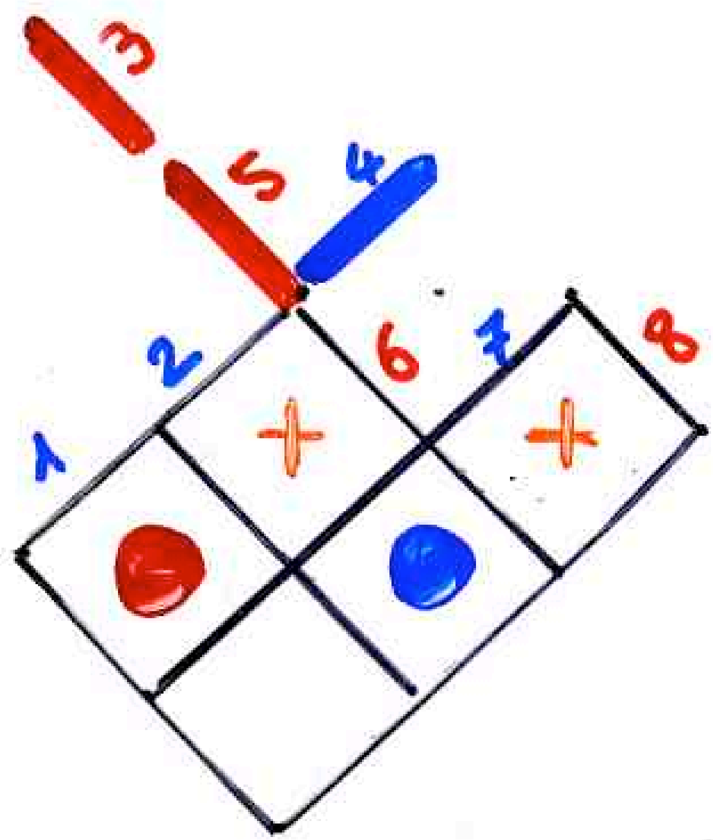

1
2
3
4
5
6
7
8

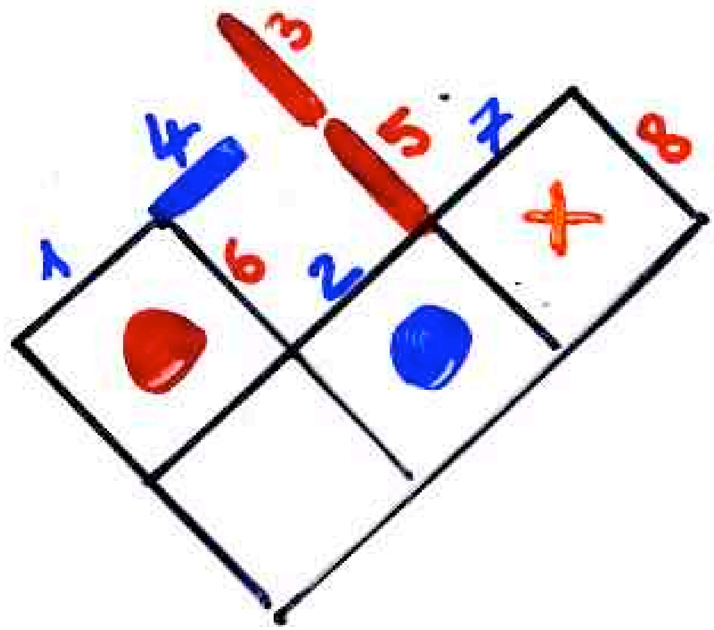

1
2
3
5
4
6
7
8

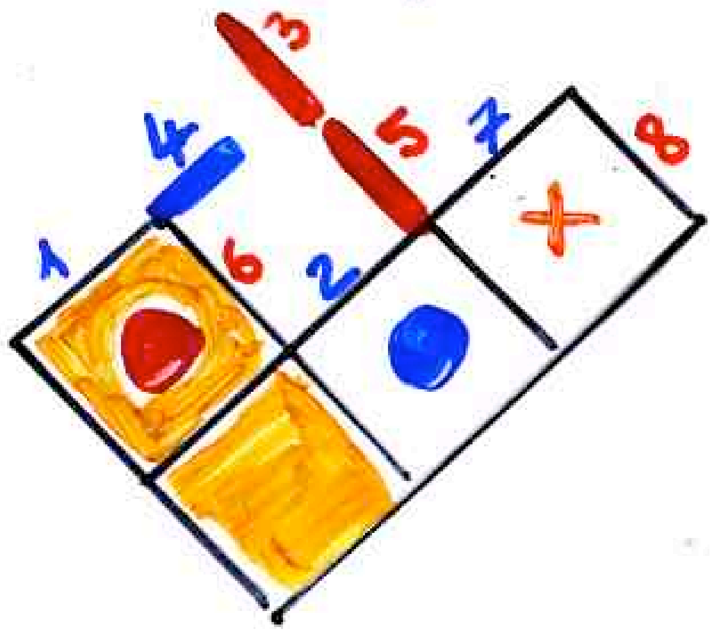

1
2
3
5
4
6
7
8


3
5
4
6
7
8
2
1

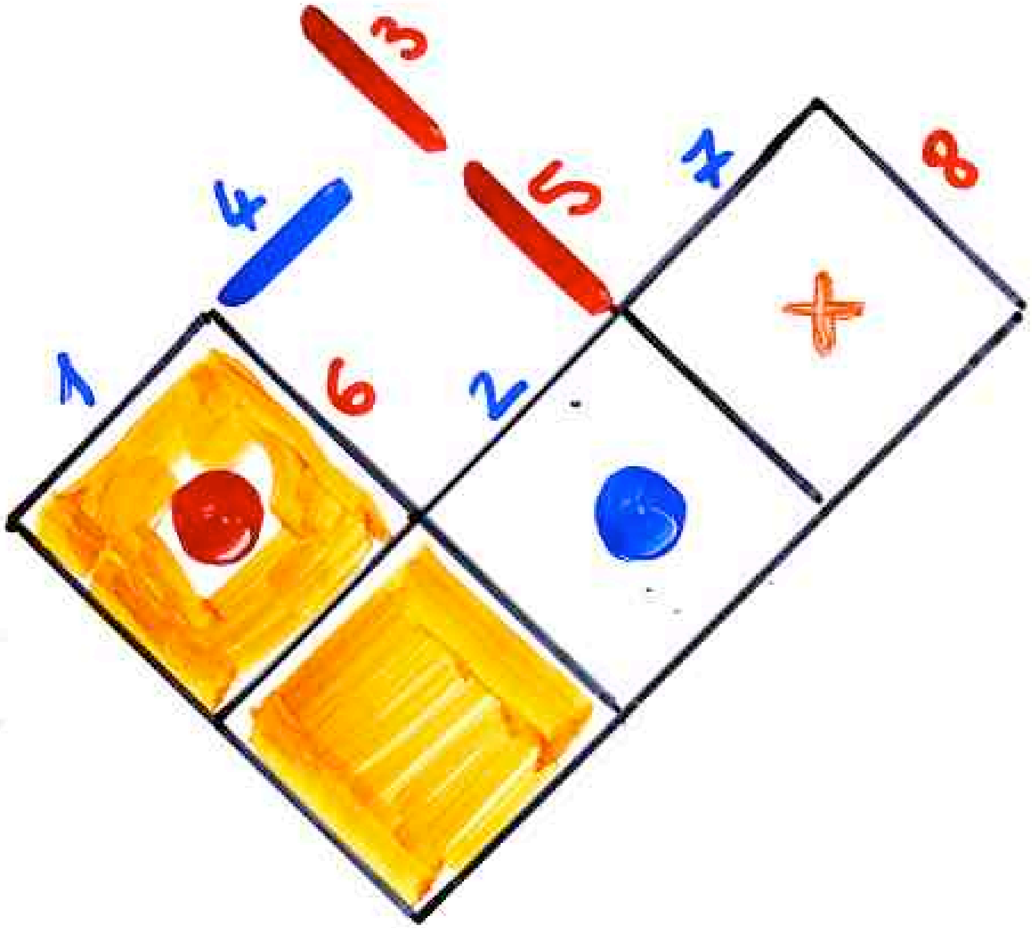

3
5
4
6
7
8
2
1

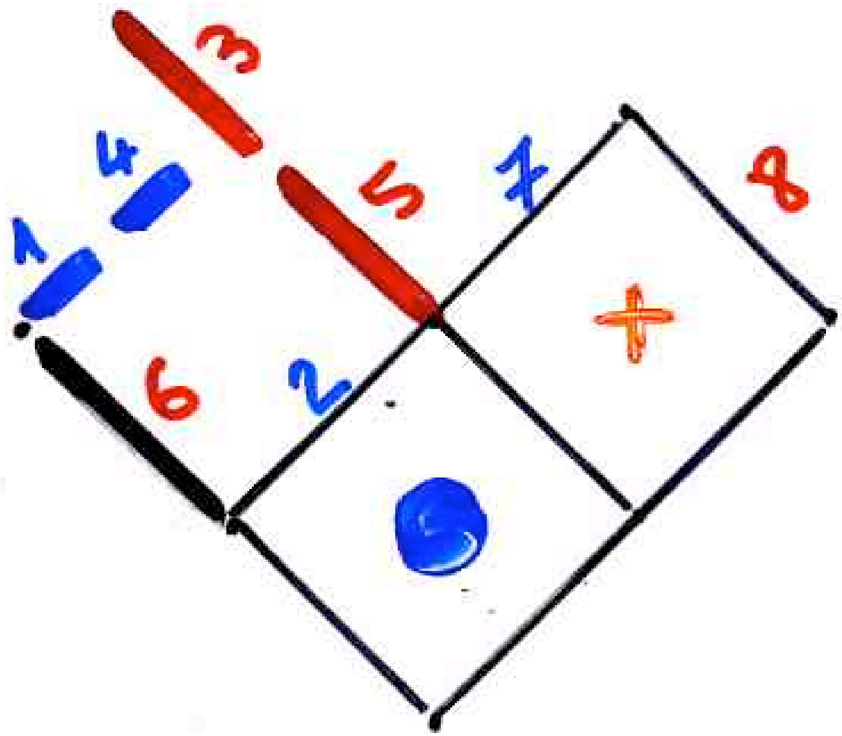

3
5
4
2
6
7
8
1


1
4
3
6
2
5
7
8

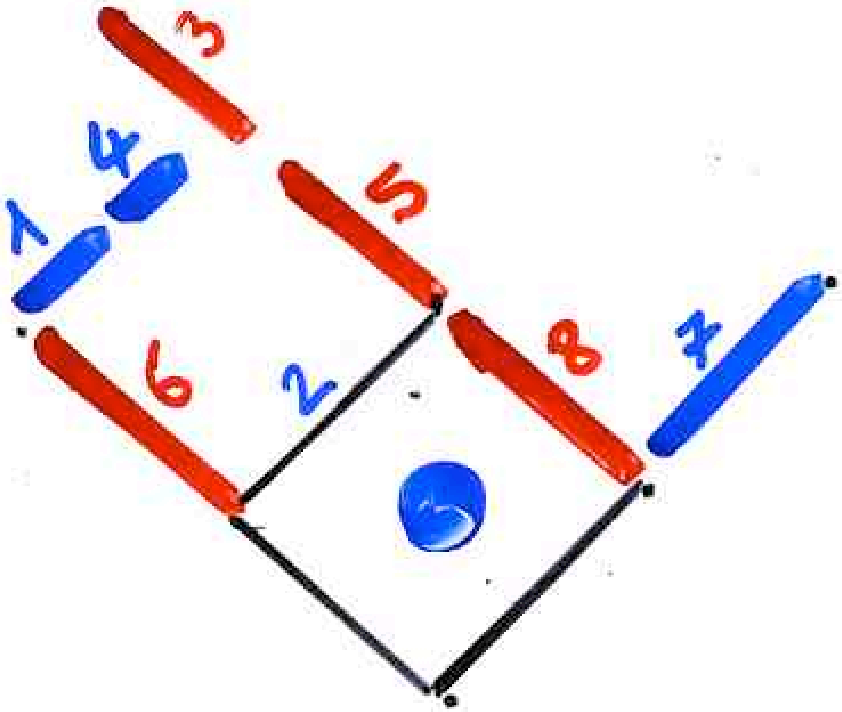


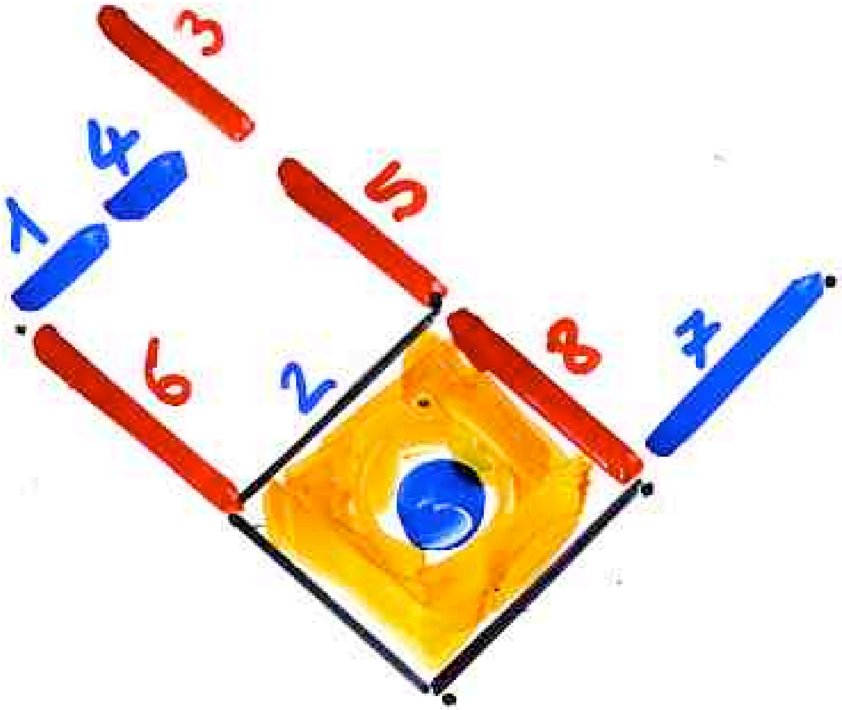
3
4
5
7
8
1
6
2

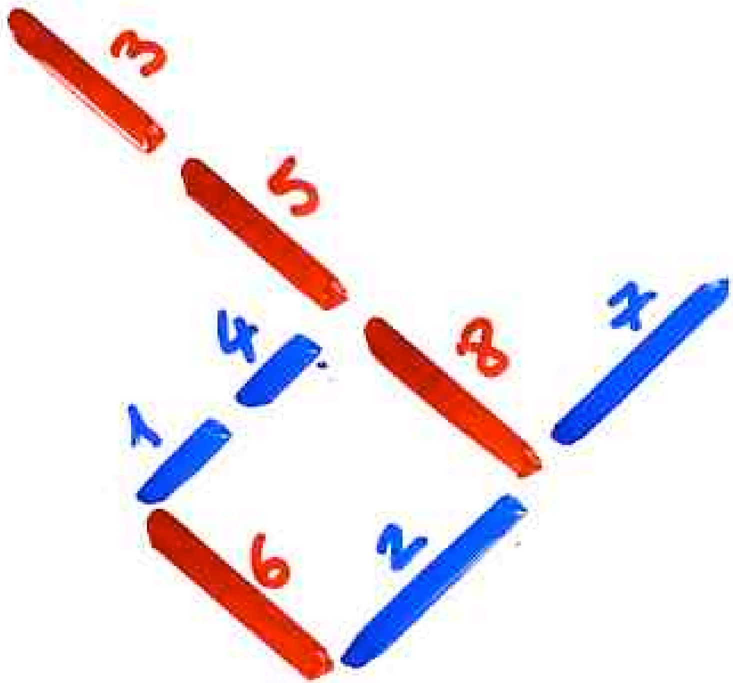

3
4
5
7
8
1
6
2

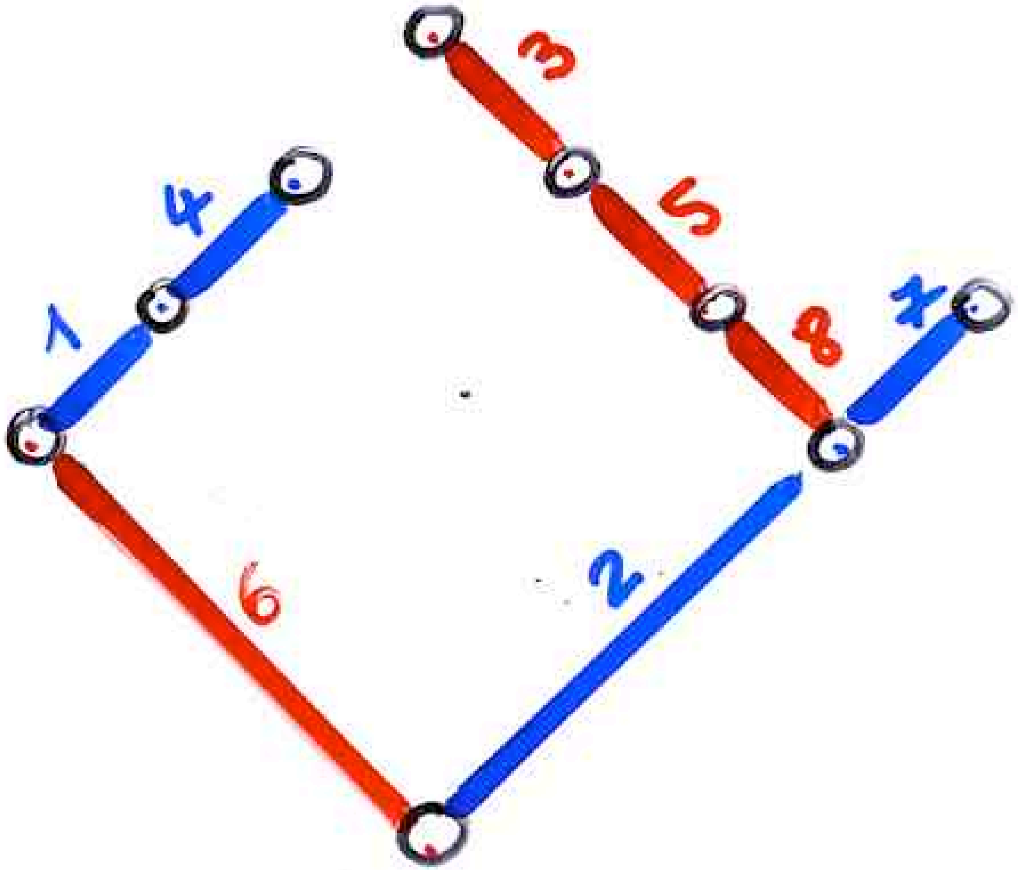


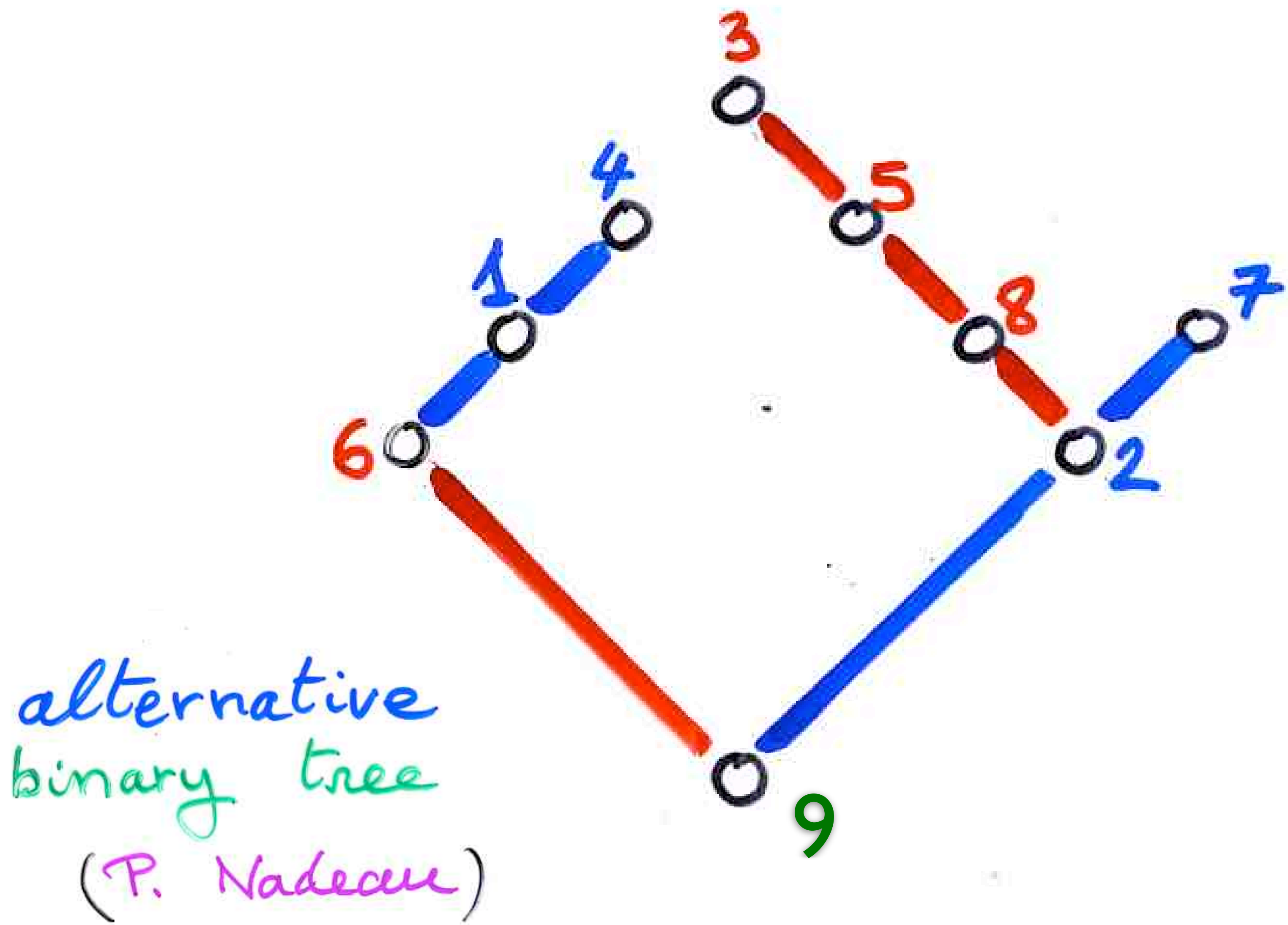
1
2
3
4
5
6
7
8


3
4
1
5
6
2
8
7


3
4
1
5
6
2
8
7

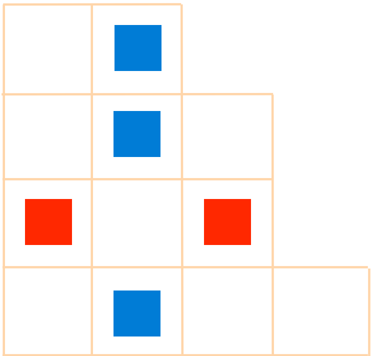
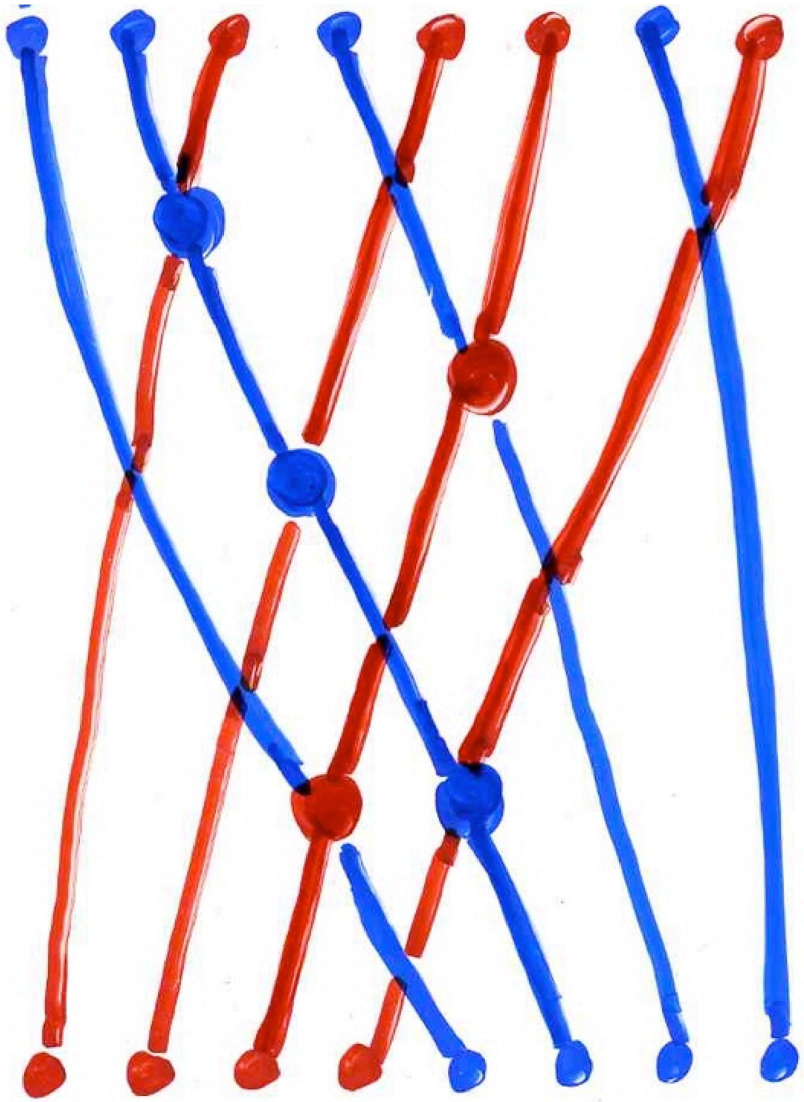

3
5
4
8
7
1
6
2

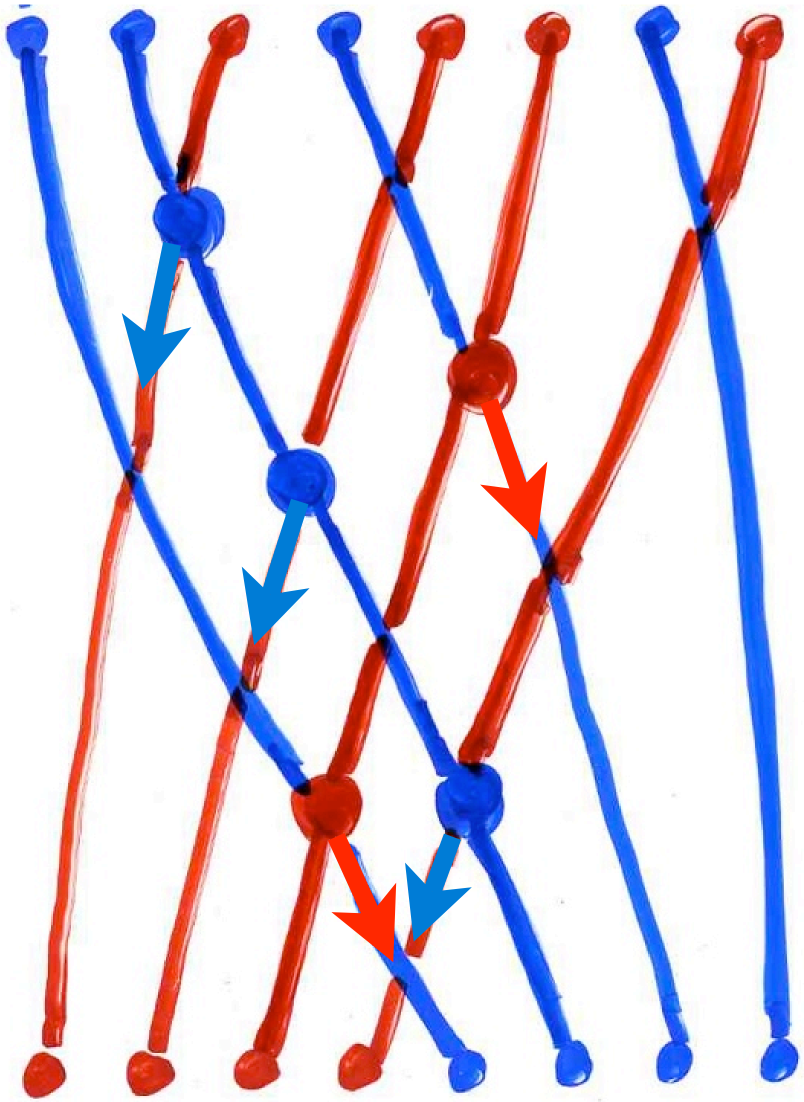

3
4
5
1
8
7
6
2

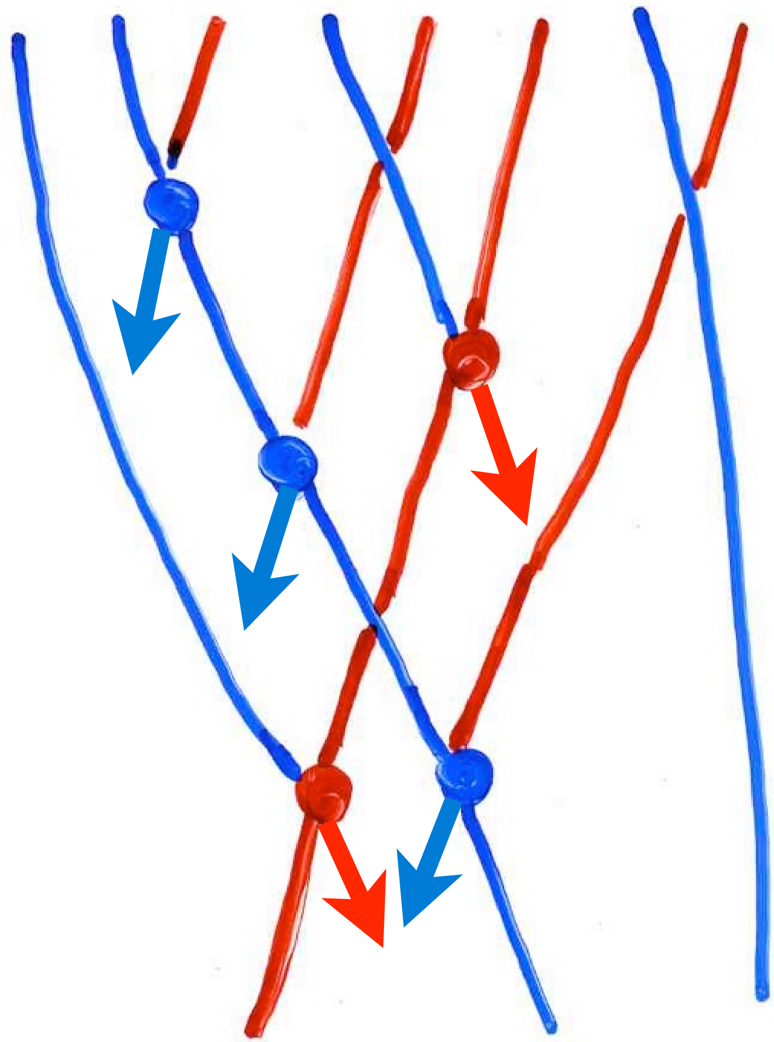

3
4
5
1
8
7
6
2
9
alternative
binary tree
(P. Nadeau)

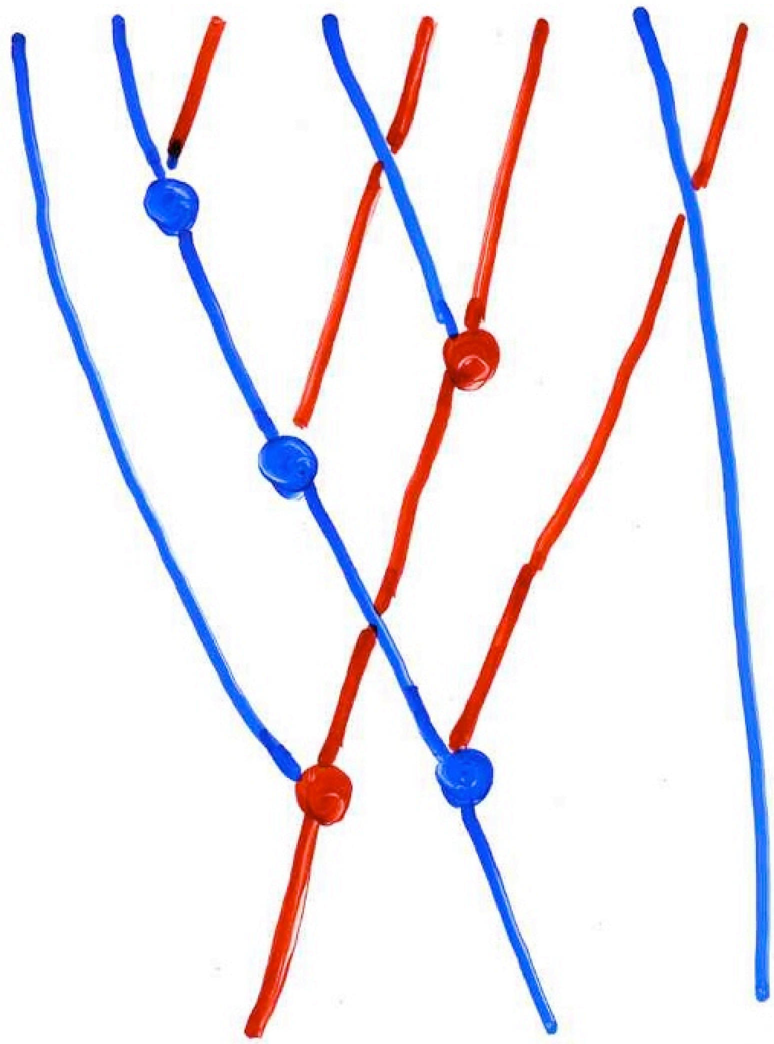
§ 8

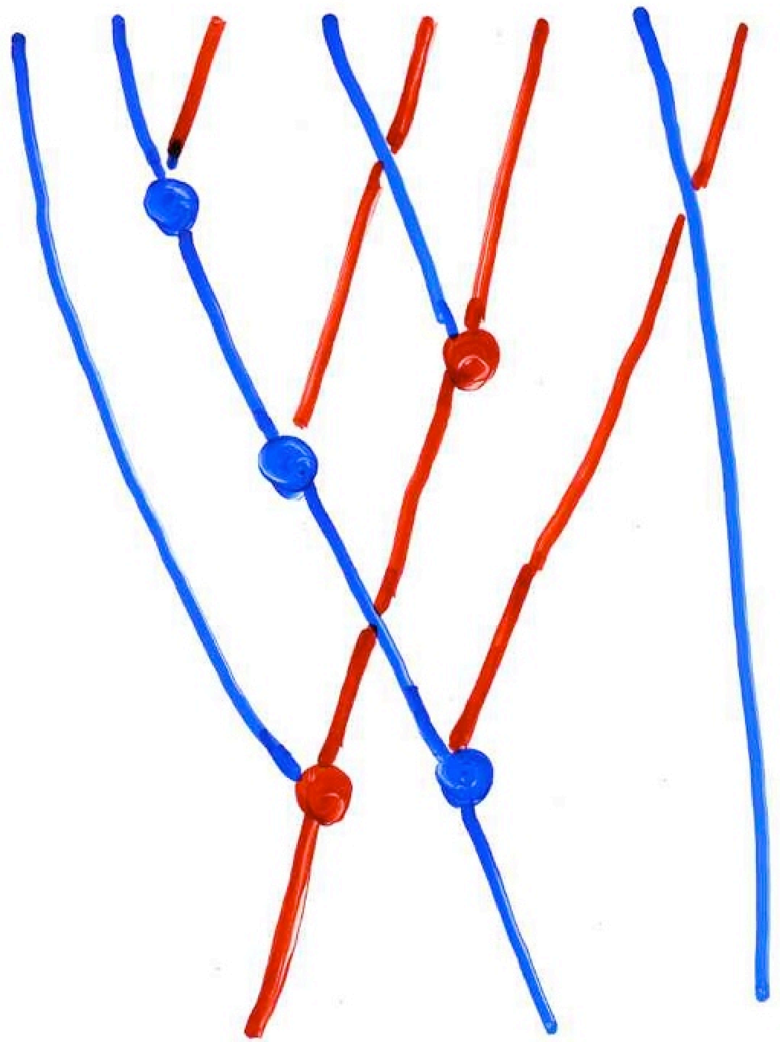
The inverse fusion algorithm

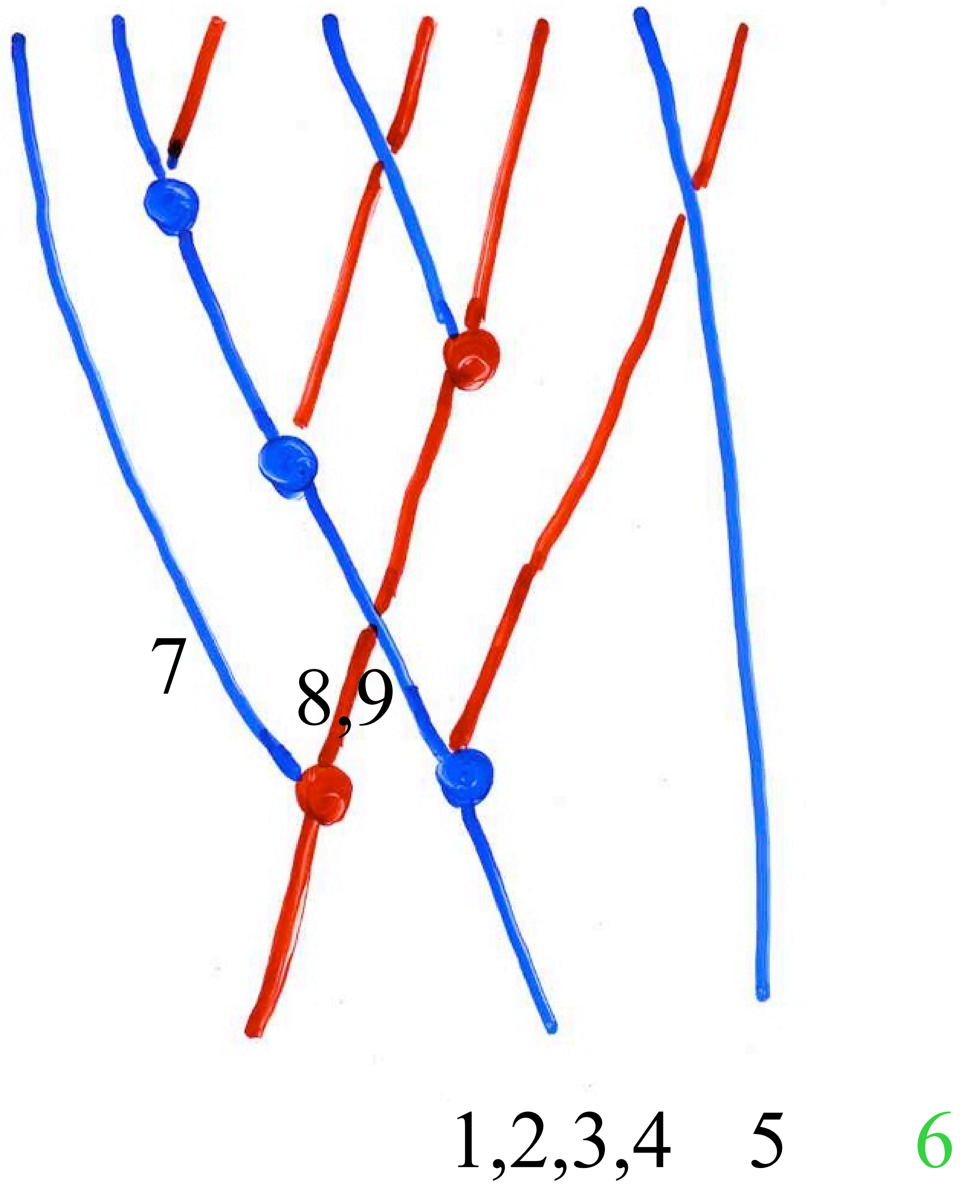


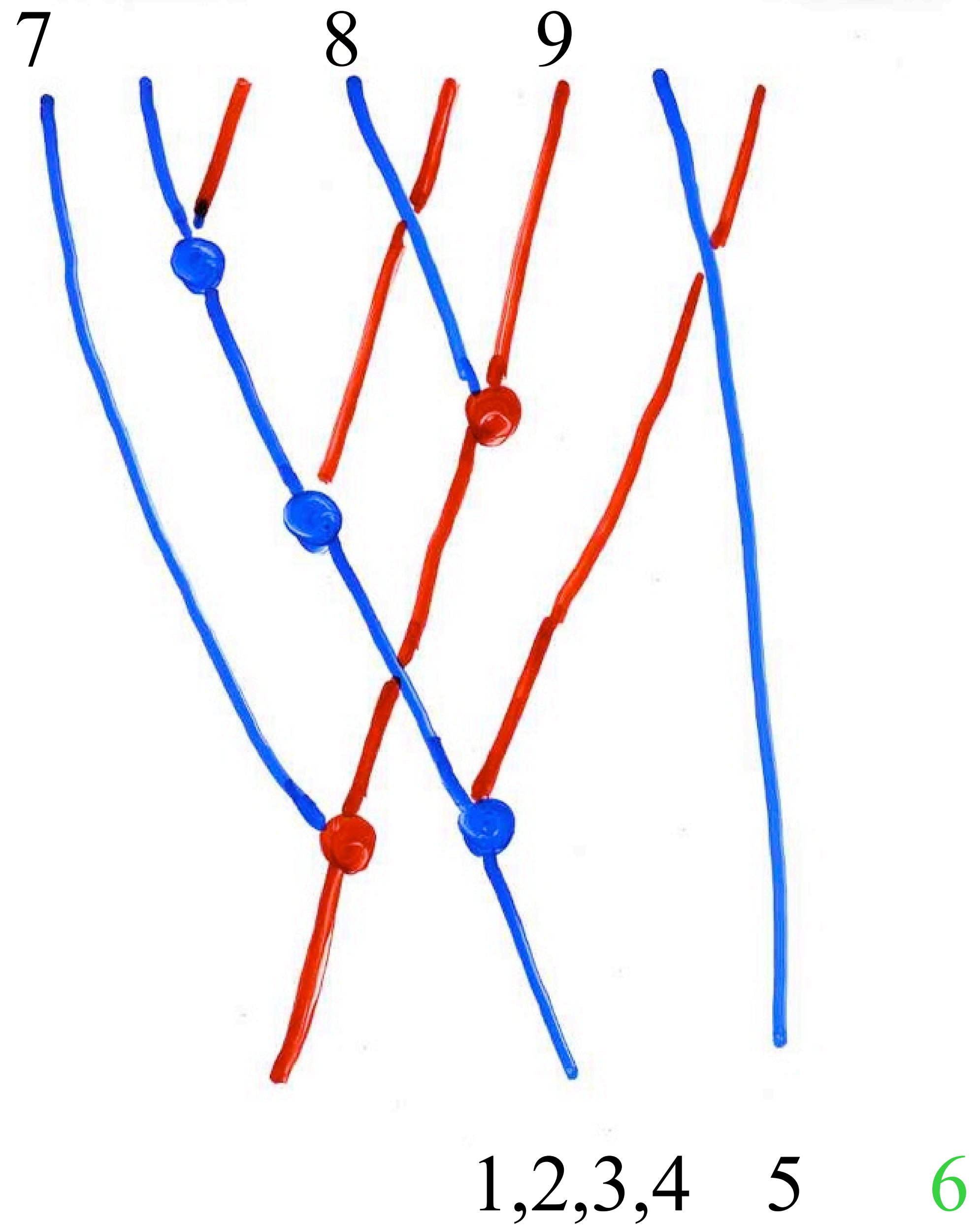


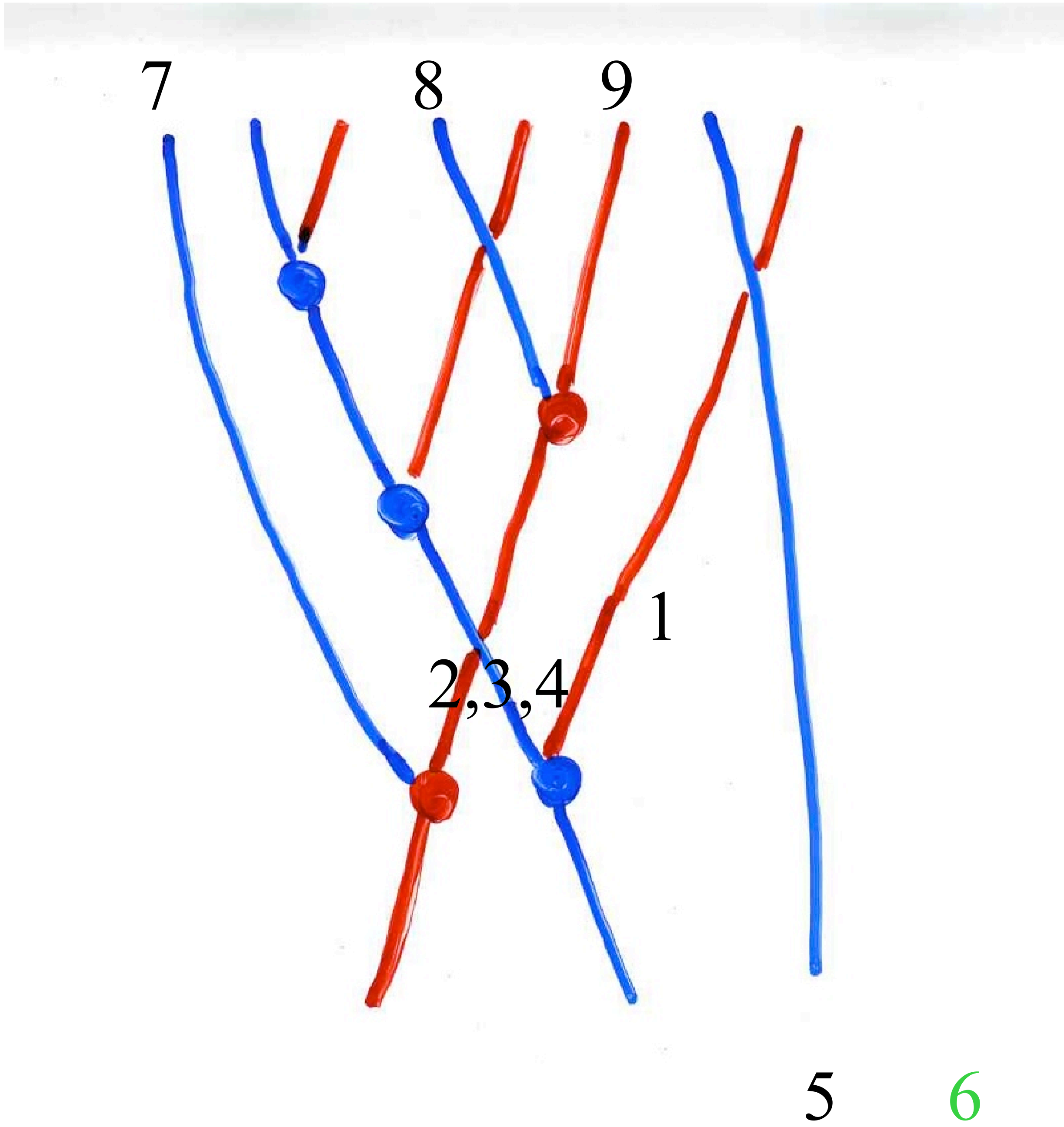


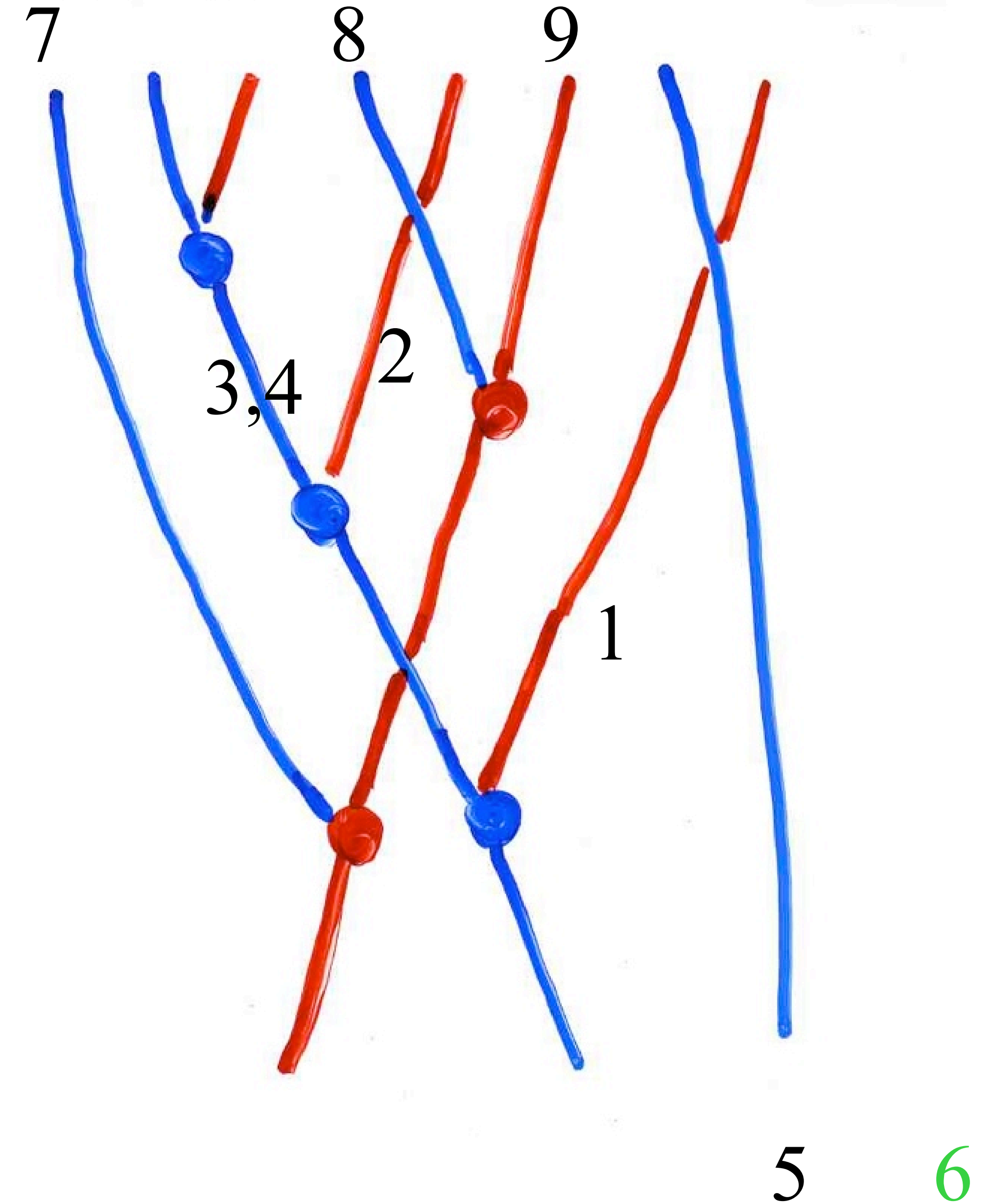


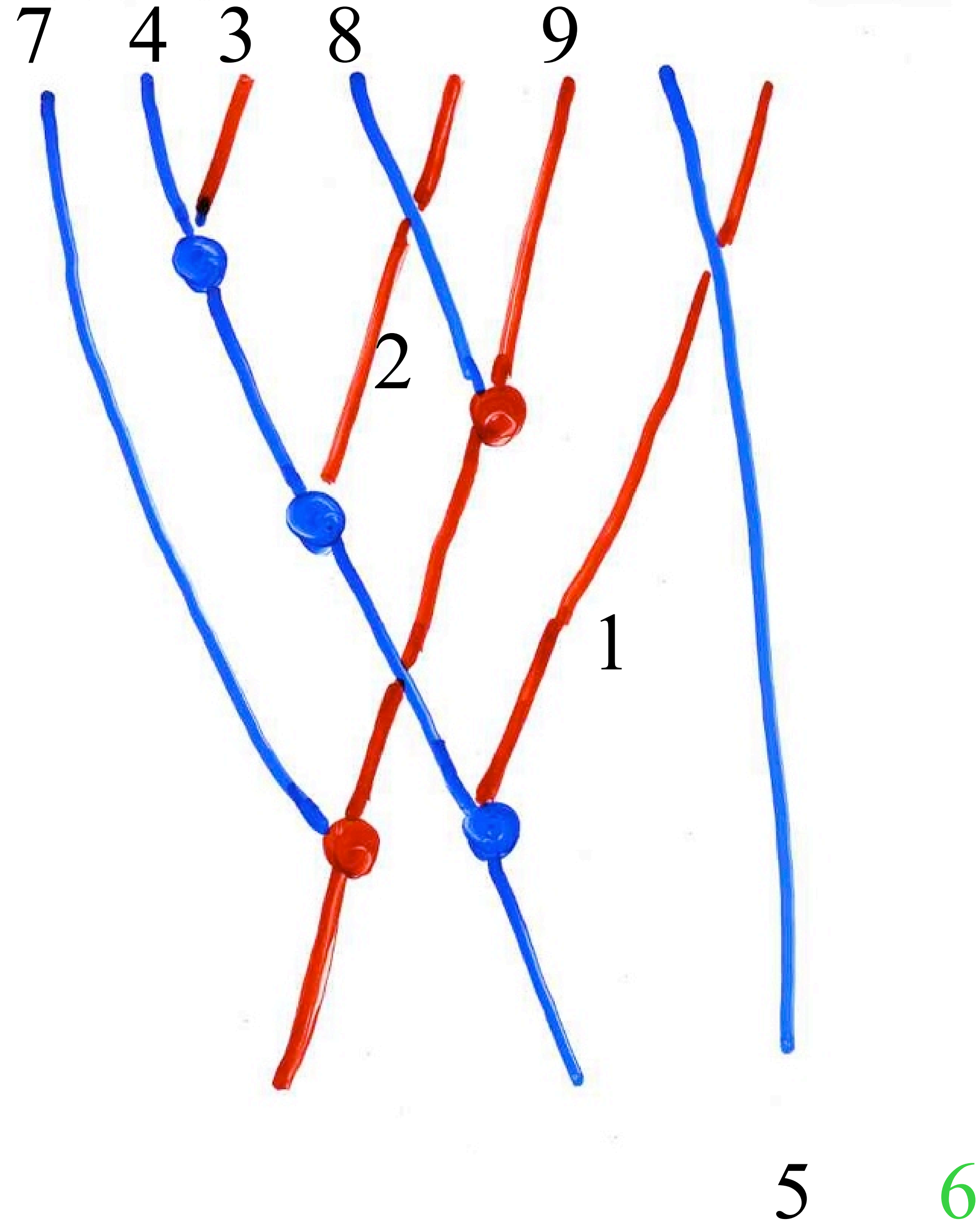
7,8,9 1,2,3,4 5 6


7
8,9
1,2,3,4
5
6

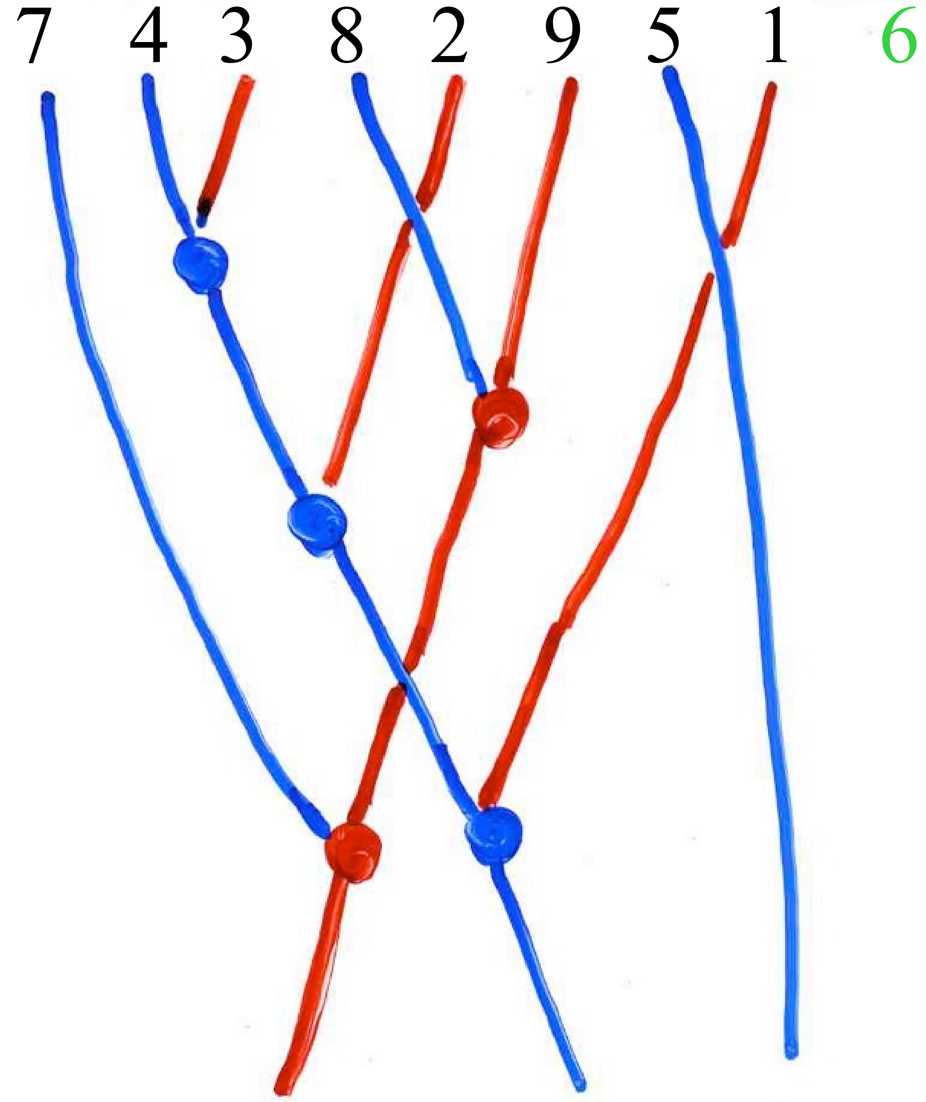

7
8
9
1,2,3,4
5
6

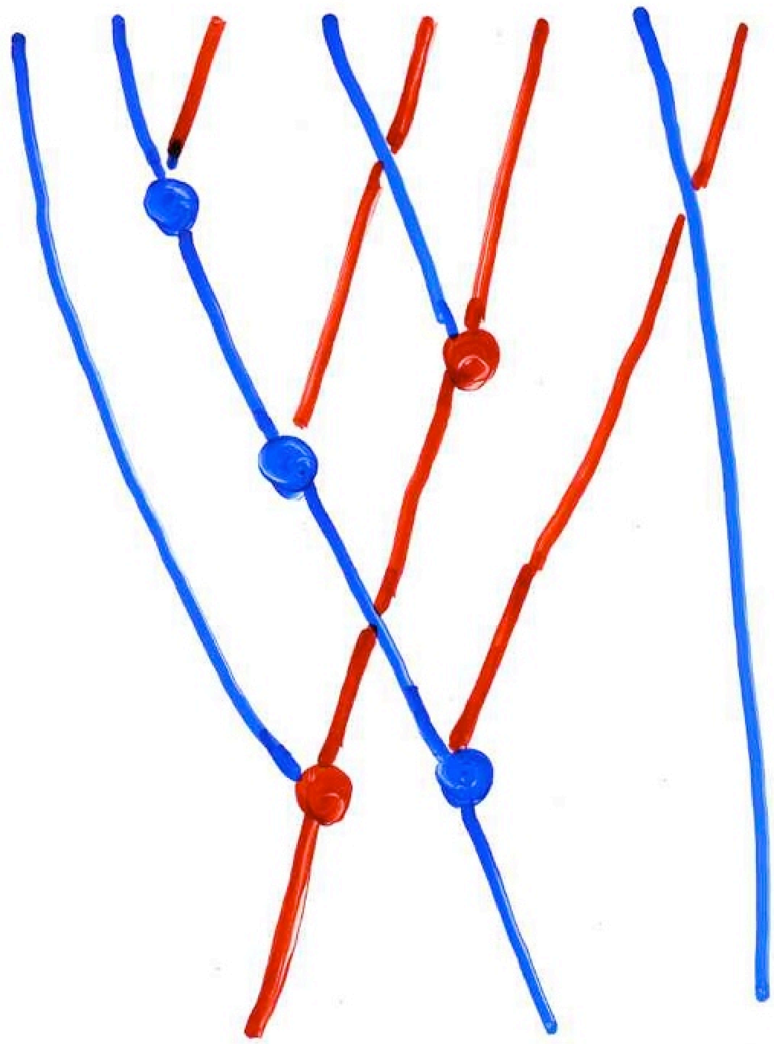

7
8
9
1
2,3,4
5
6

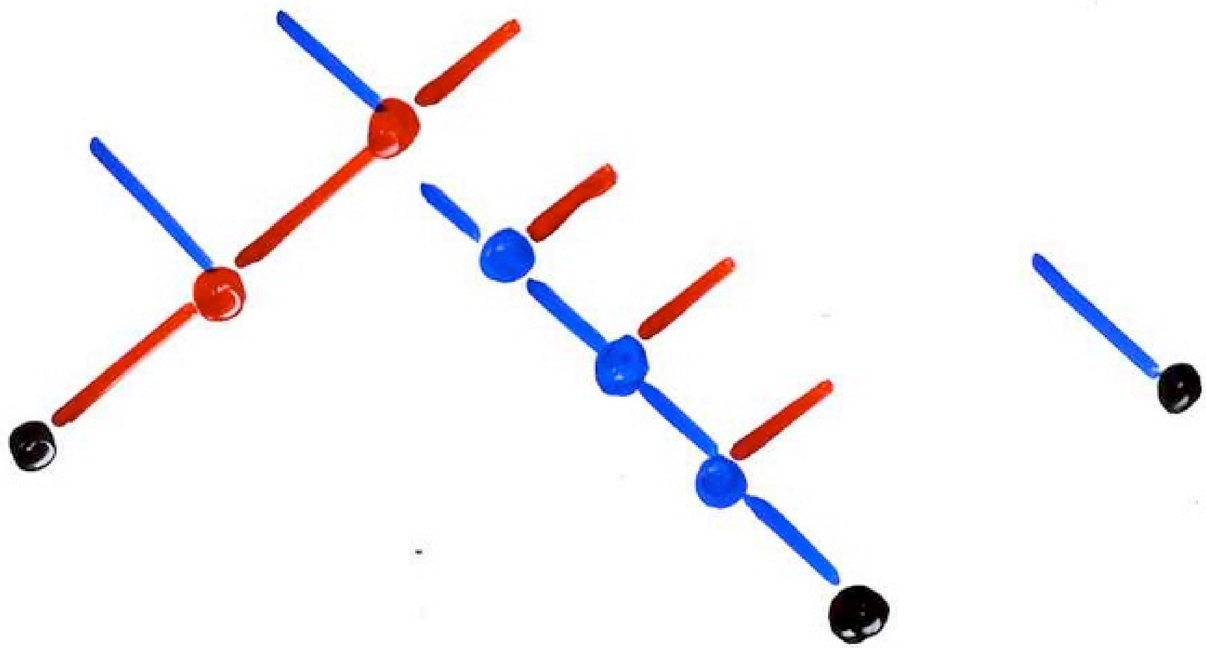

7
8
9
3,4
2
1
5
6

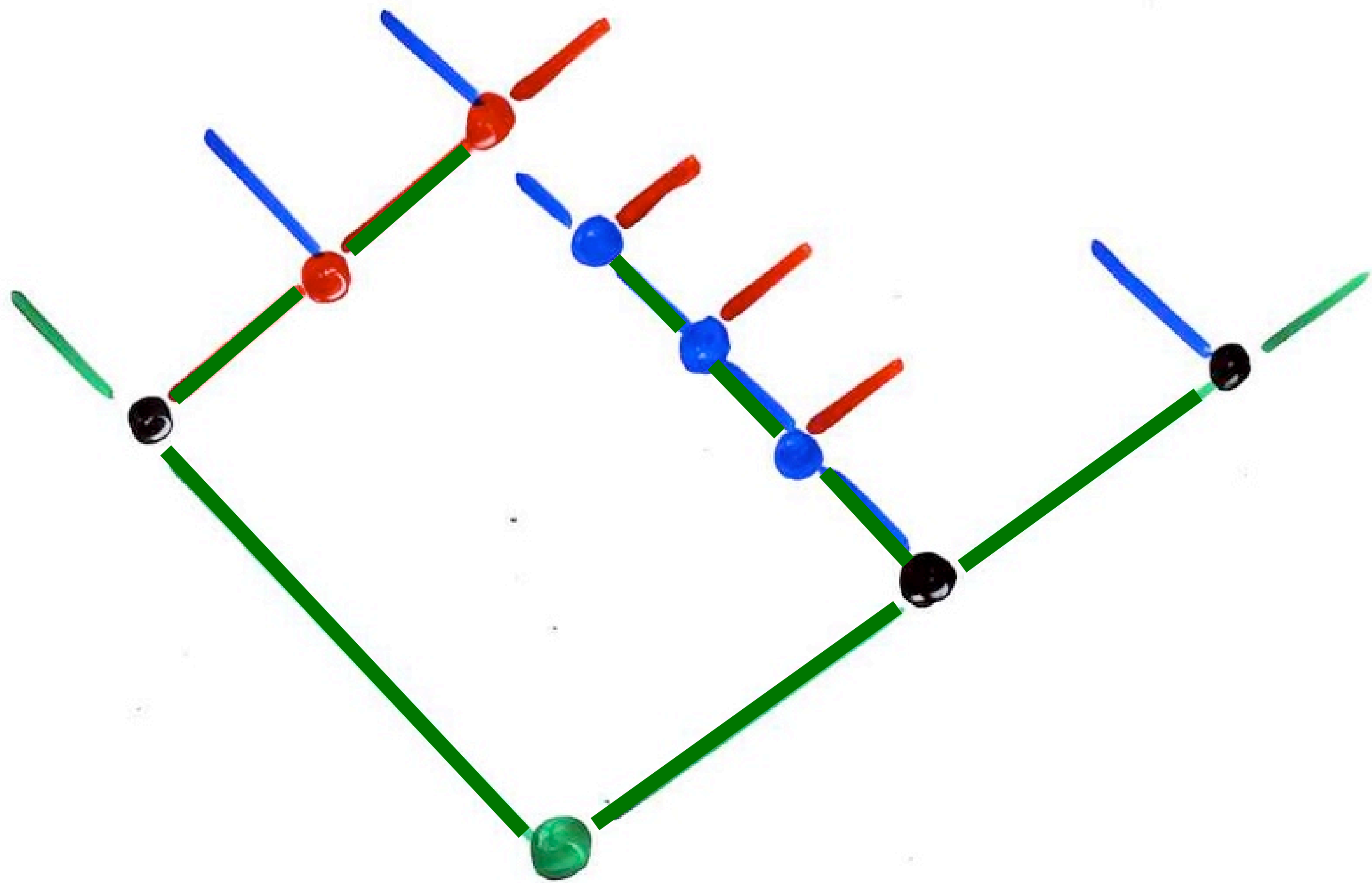


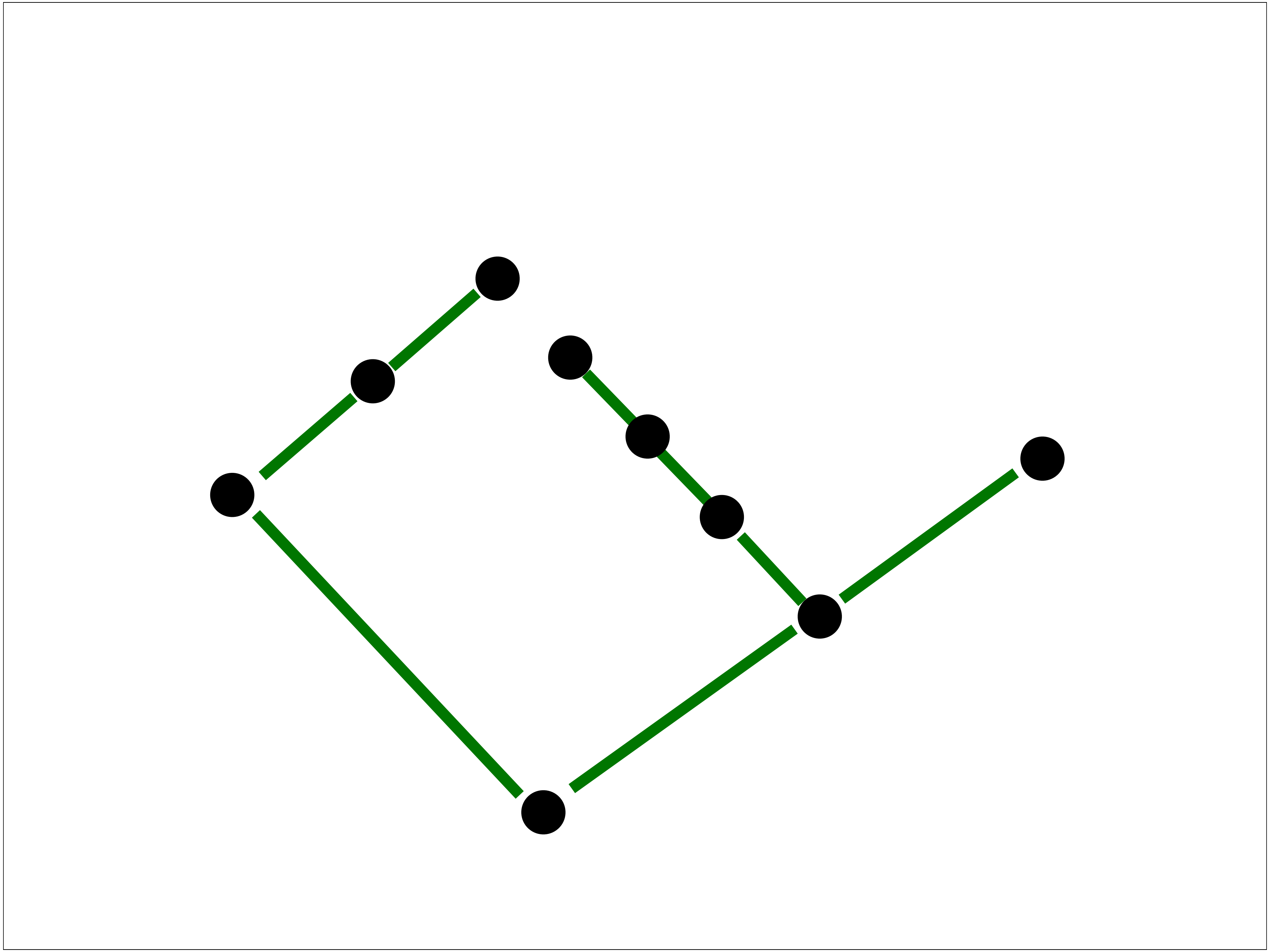
7
4
3
8
9
2
1
5
6


7 4 3 8 2 9 5 1 6

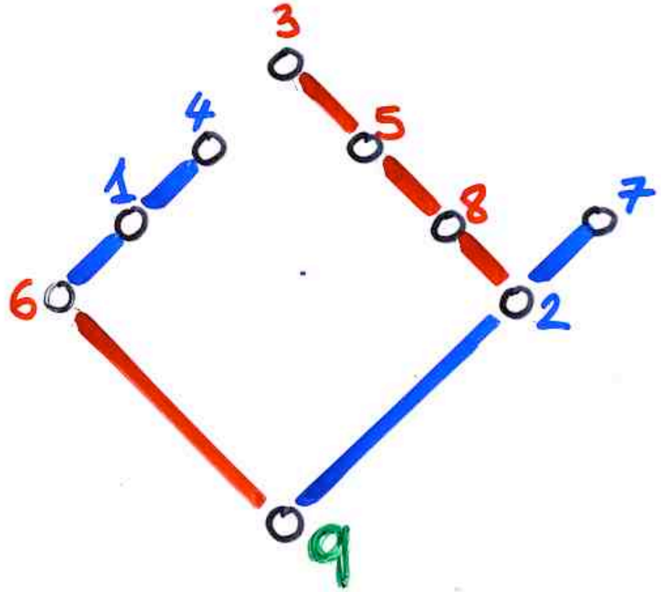


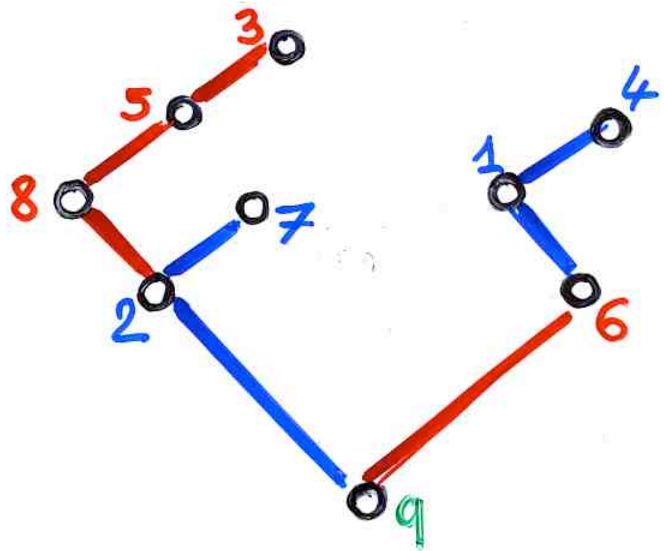




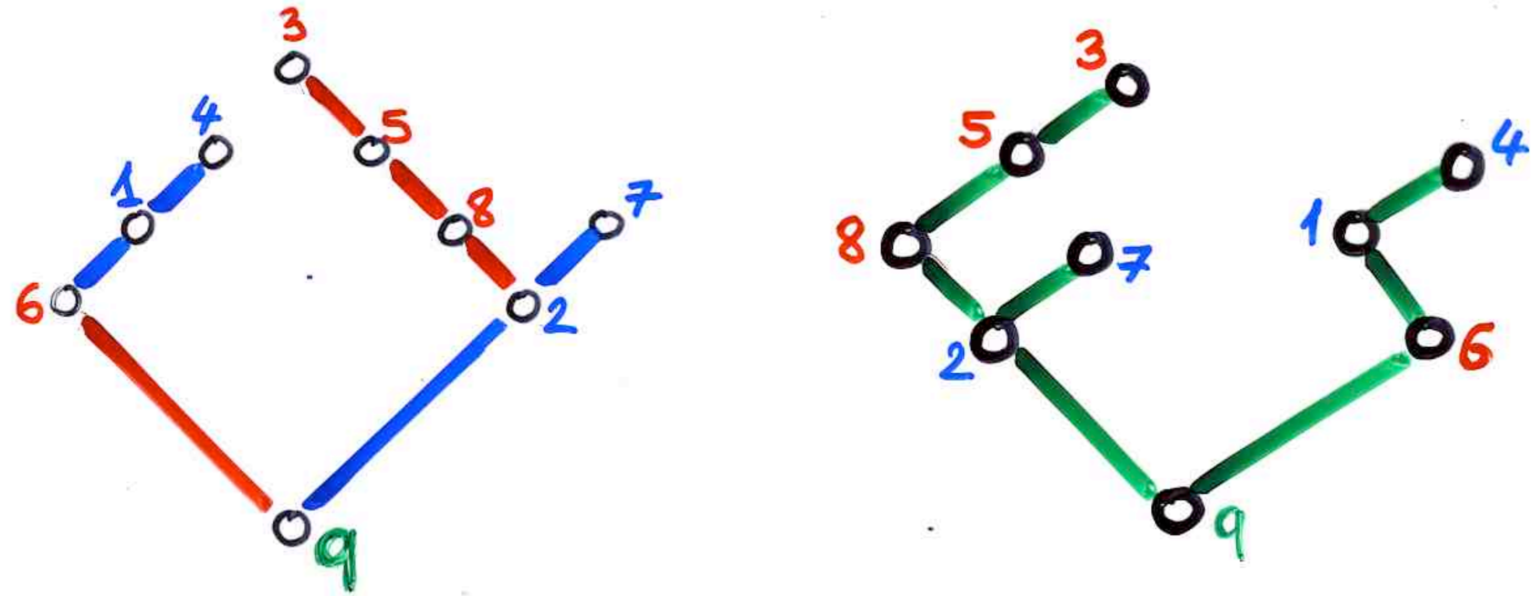


§ 9
The twisted symmetric order





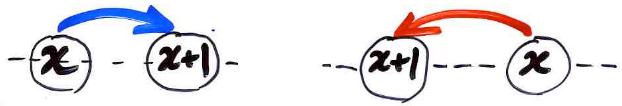
"twisted"
symmetric
order



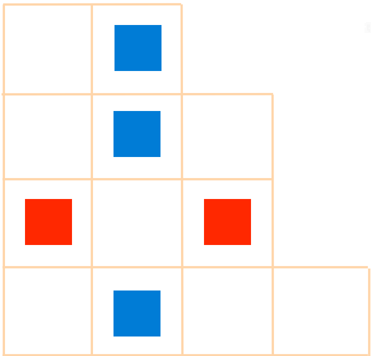
"twisted" symmetric order

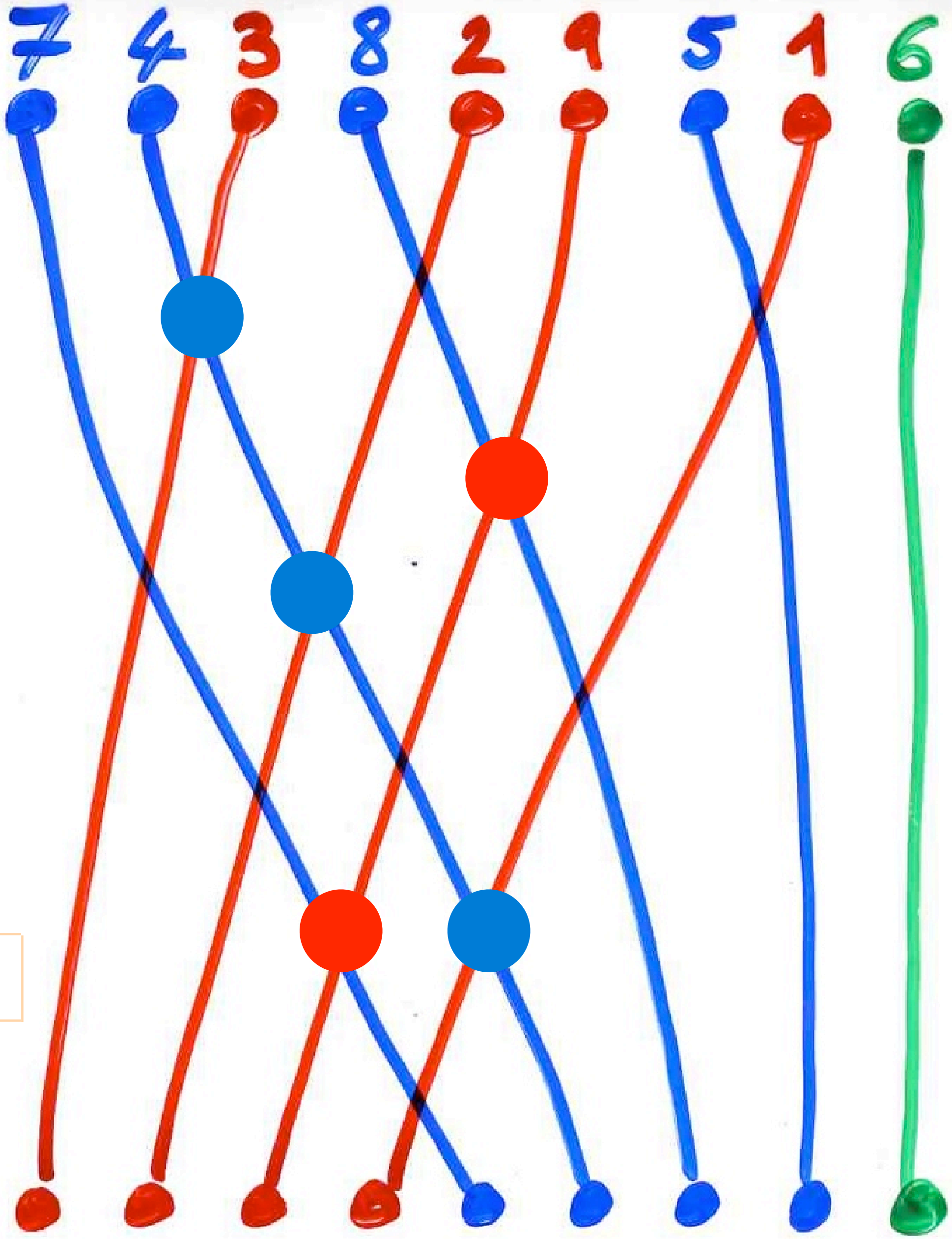
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 3 & 8 & 2 & 9 & 5 & 1 & 6 \end{pmatrix}$$



“exchange-fusion” algorithm





Novelli, Thibon, Williams (April 2008)

Hall-Littlewood functions, Tevlin' bases (2007)

conjectures

§ 10

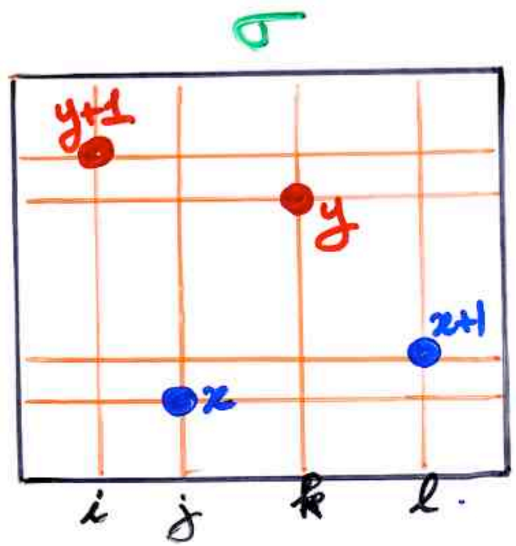
permutations with no $y+1, x, y, x+1$

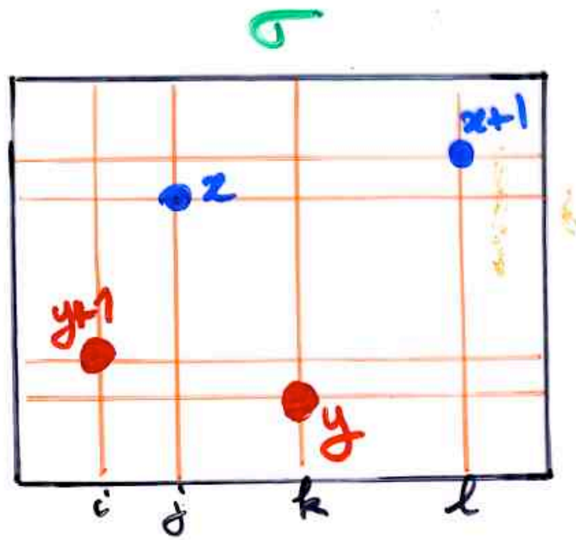
Permutations

with no subsequence of the type

$$\cdots (y+1) \cdots x \cdots y \cdots (x+1) \cdots$$

ex: $\sigma = 6\ 4\ 5\ 3\ 9\ 7\ 8\ (10)\ 1\ 2$





Permutations

with no subsequence of the type

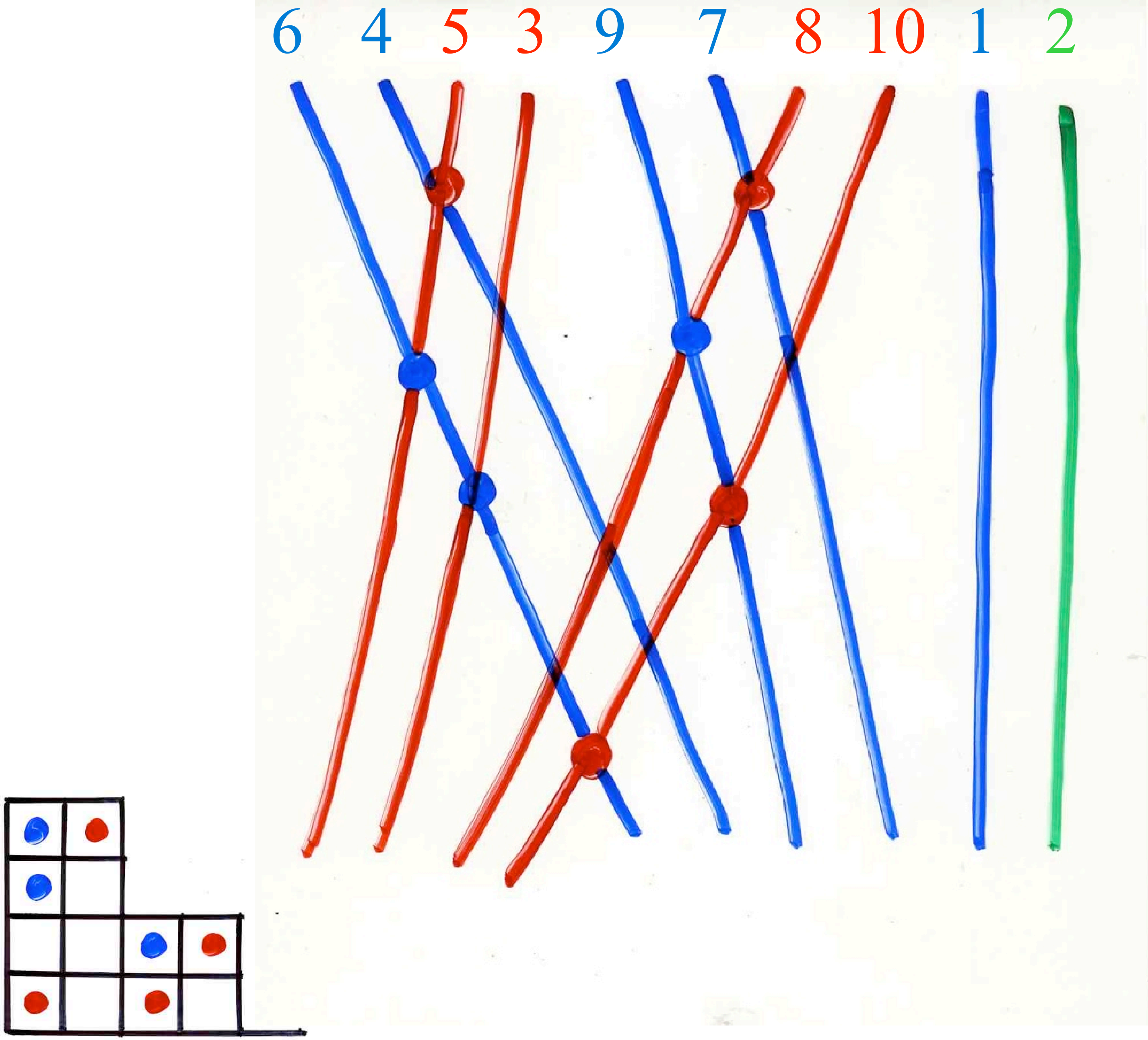
$$\cdots (y+1) \cdots x \cdots y \cdots (x+1) \cdots$$

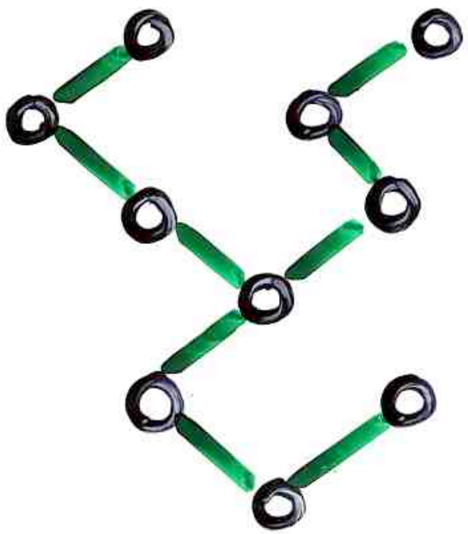
ex: $\sigma = 6\ 4\ 5\ 3\ 9\ 7\ 8\ (10)\ 1\ 2$

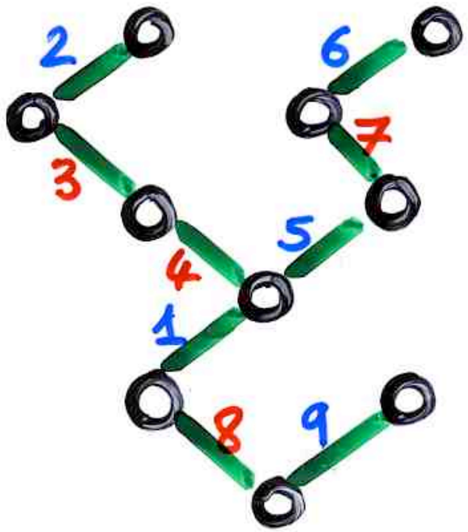
Prop. (O. Bernardi, 2008)
The number of such permutations on n elements is C_n Catalan number

Lemma. σ (permutation) $\Leftrightarrow$ T alternating tableau

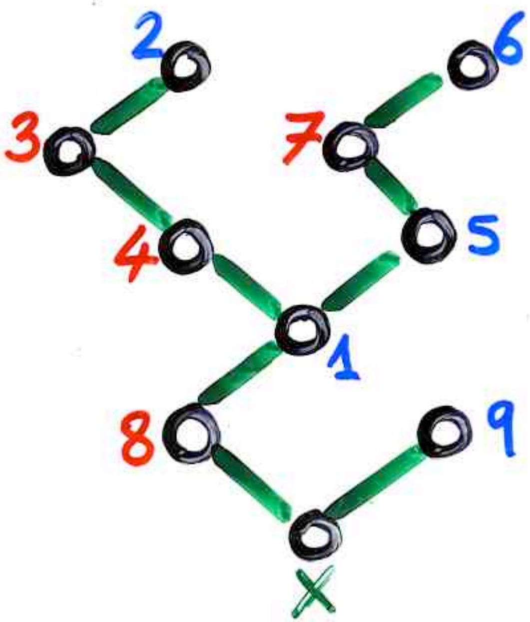
T has no crossing
$\Leftrightarrow$ σ has no subsequence of type
$(y+1) \cdots x \cdots y \cdots (x+1)$

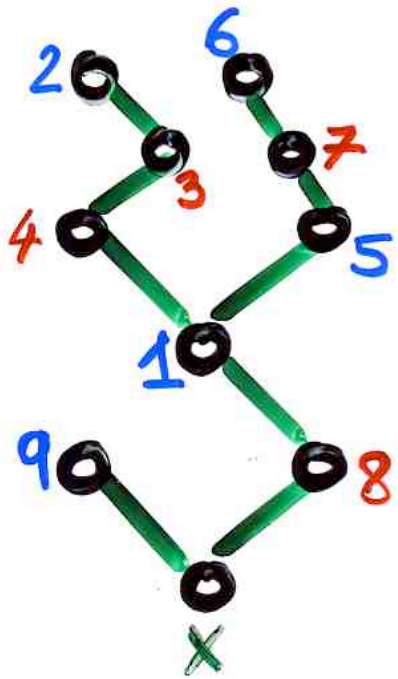

6 4 5 3 9 7 8 10 1 2



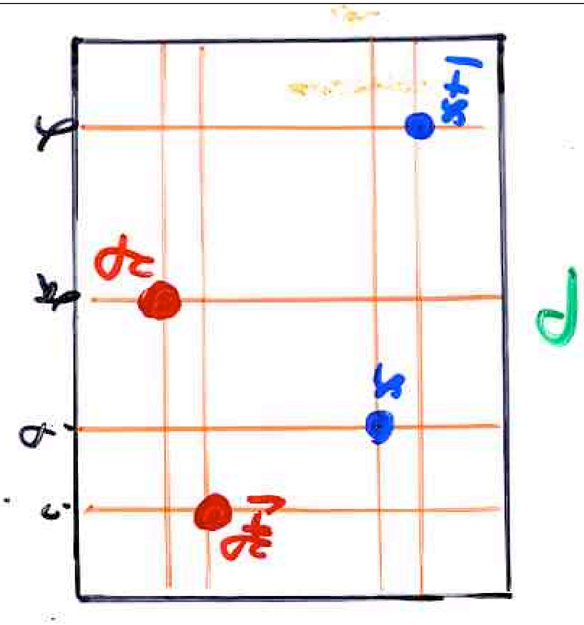

2
6
7
3
5
4
1
8
9

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & X \\ 6 & 4 & 5 & 3 & 9 & 7 & 8 & X & 1 & 2 \end{pmatrix}$$

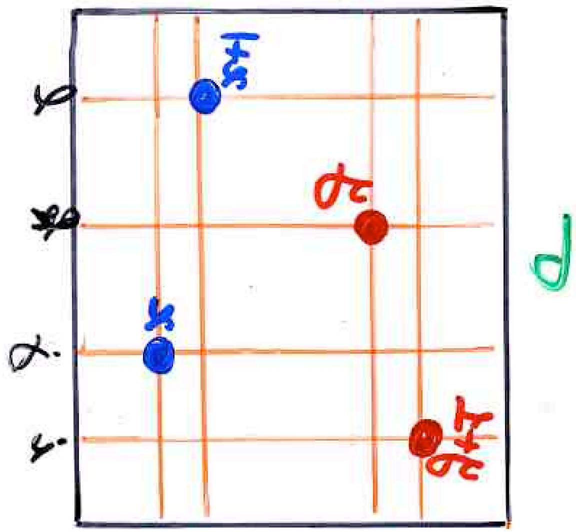




$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & X \\ 9 & X & 4 & 2 & 3 & 1 & 6 & 7 & 5 & 8 \end{pmatrix}$$



31- 24



24 - 31

Bonnes

fêtes

à

tous !

§1 Schützenberger "jeu de taquin"

§2 increasing binary trees

§3 "jeu de taquin" for increasing binary trees

§4 "jeu de taquin" for binary trees

§5 Up-down and Genocchi sequences

§6 alternative binary trees

§7 "jeu de taquin" for alternative binary trees

§8 the inverse fusion algrithm

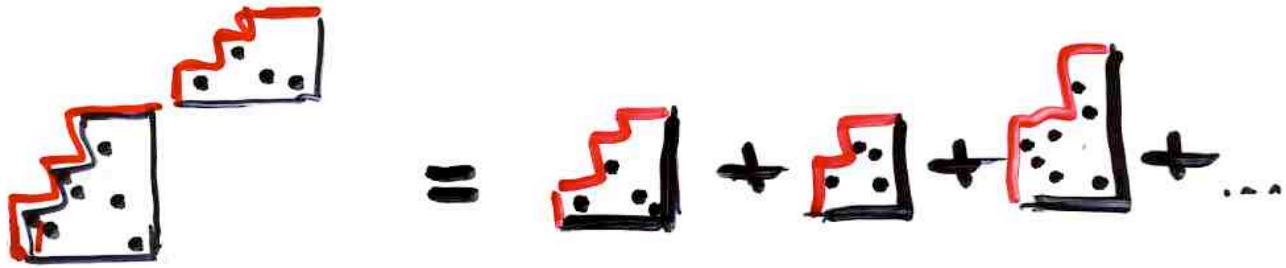
§9 the twisted symmetric order

§10 permutations with no $y+1, x, y, x+1$ Hopf algebra

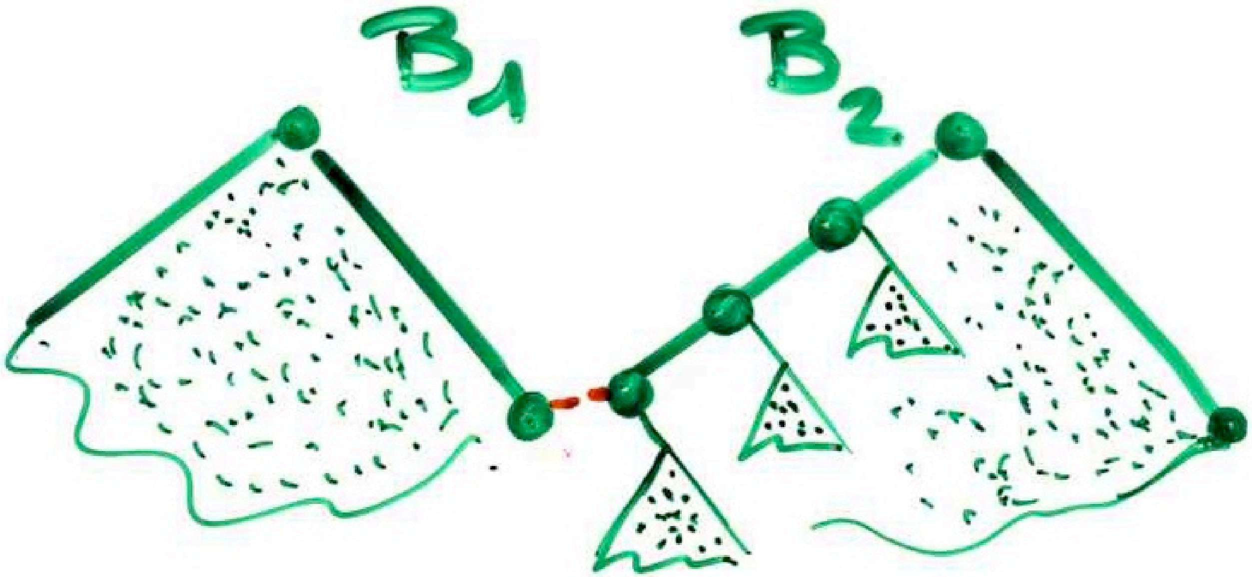
"exchange-fusion" algo "ex-fus" algo Catalan

Hopf algebra

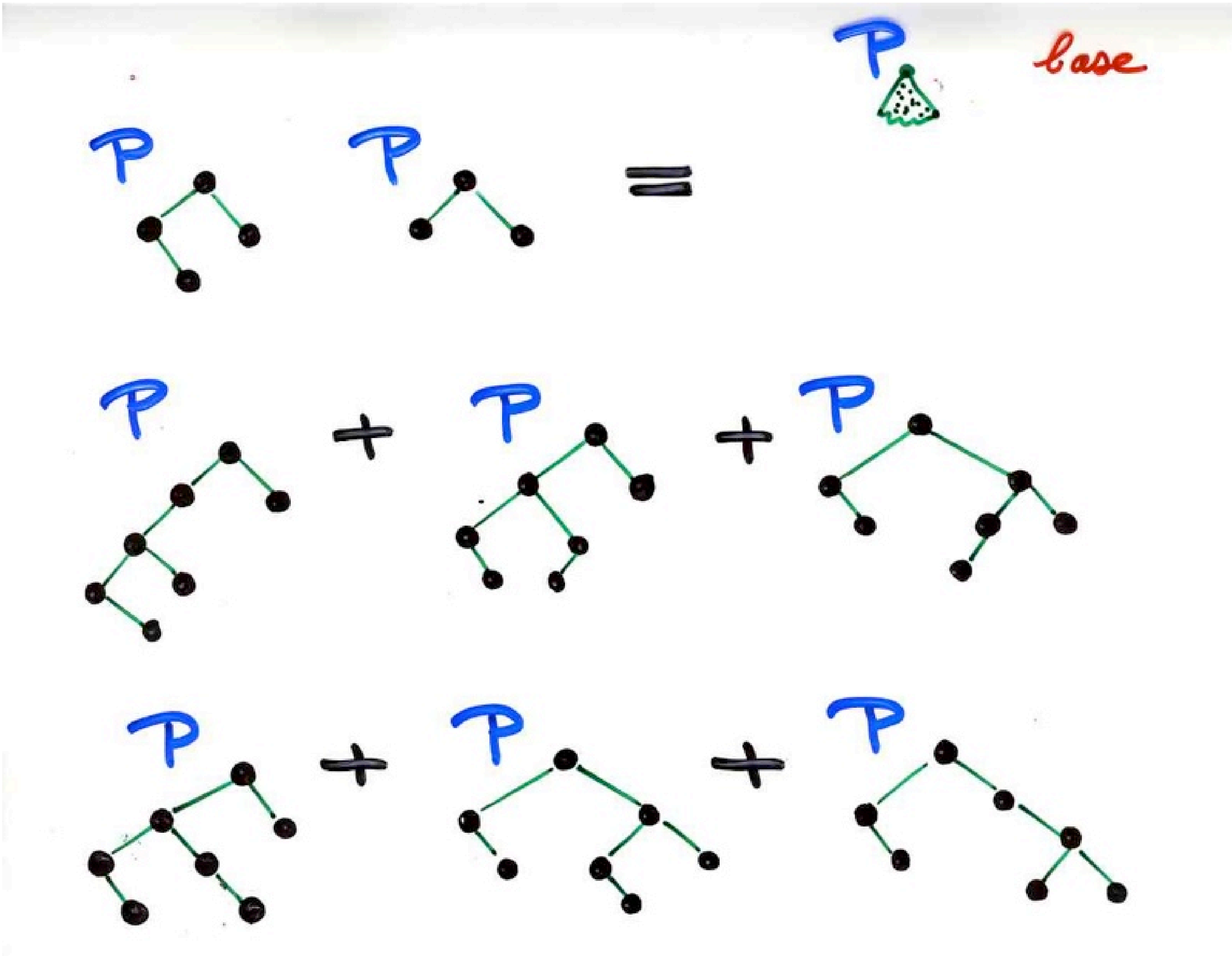
• Loday-Ronco Hopf algebra

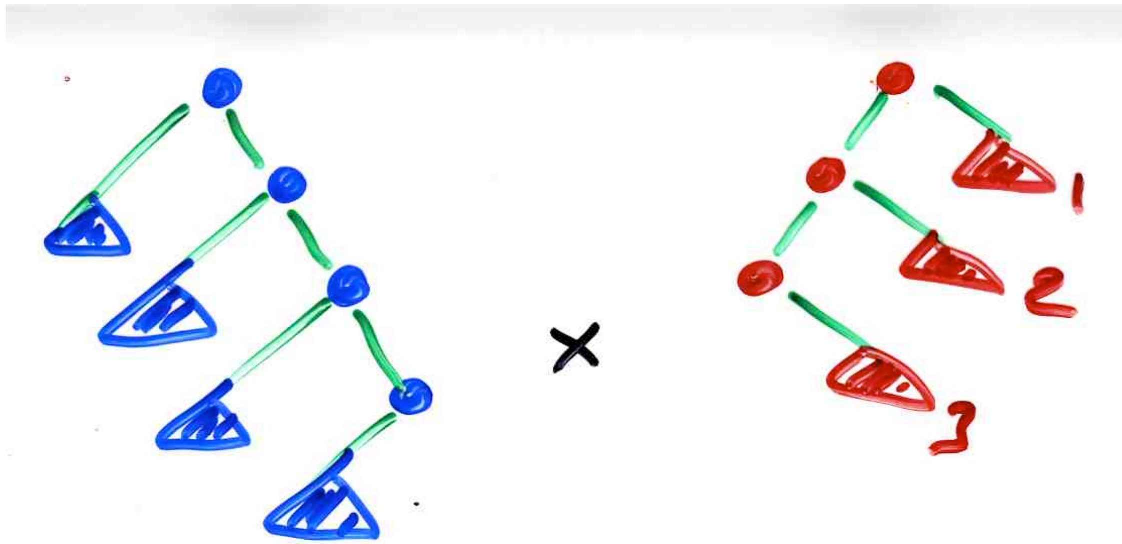


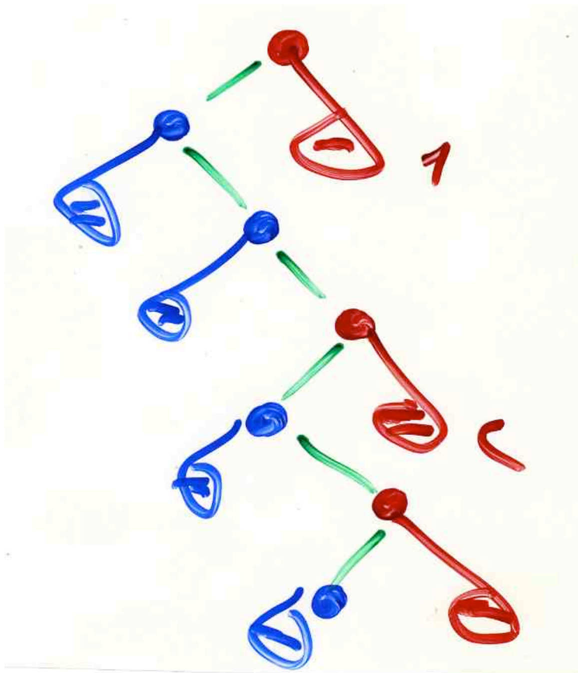
algèbre Loday, Ronco

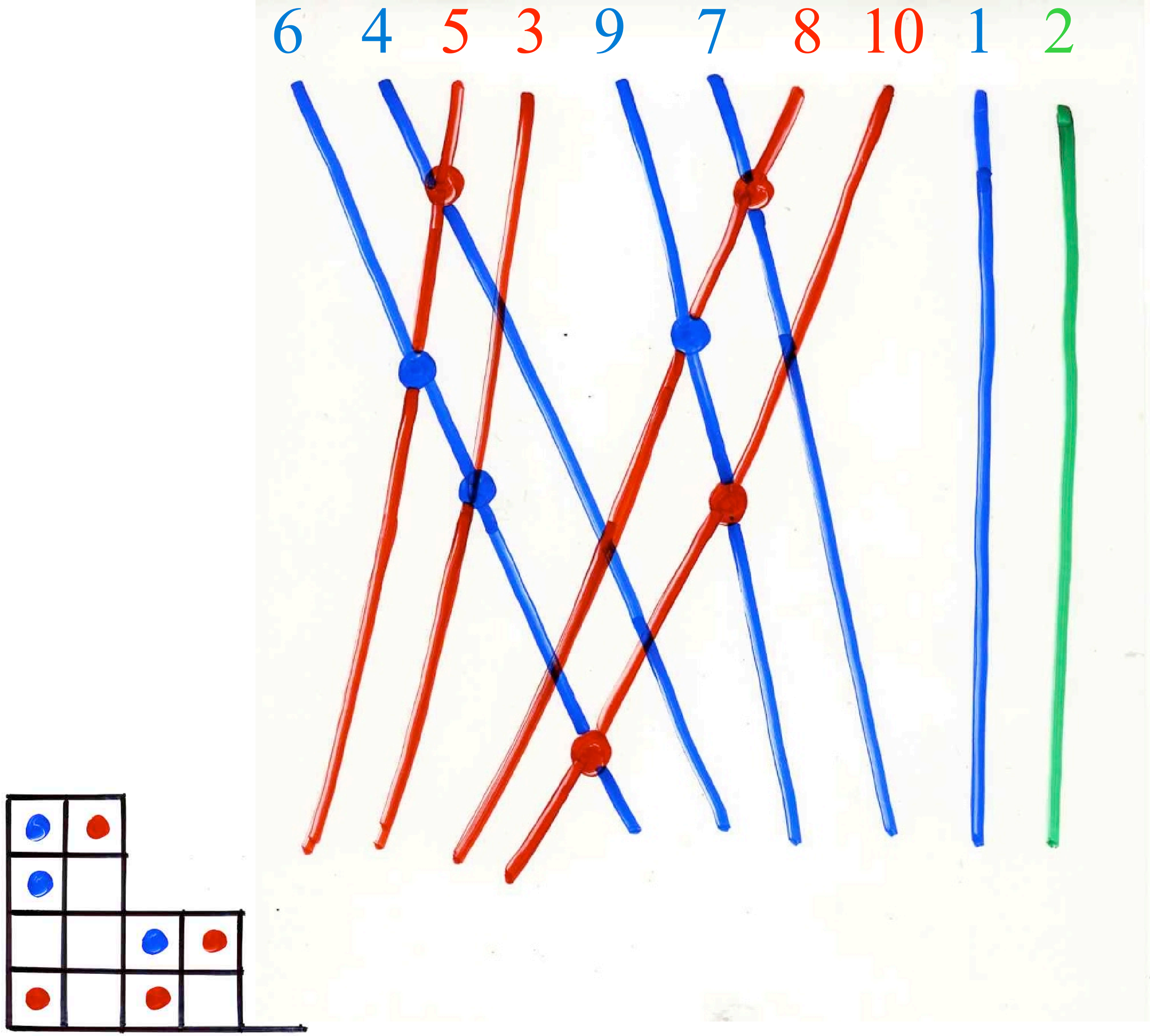


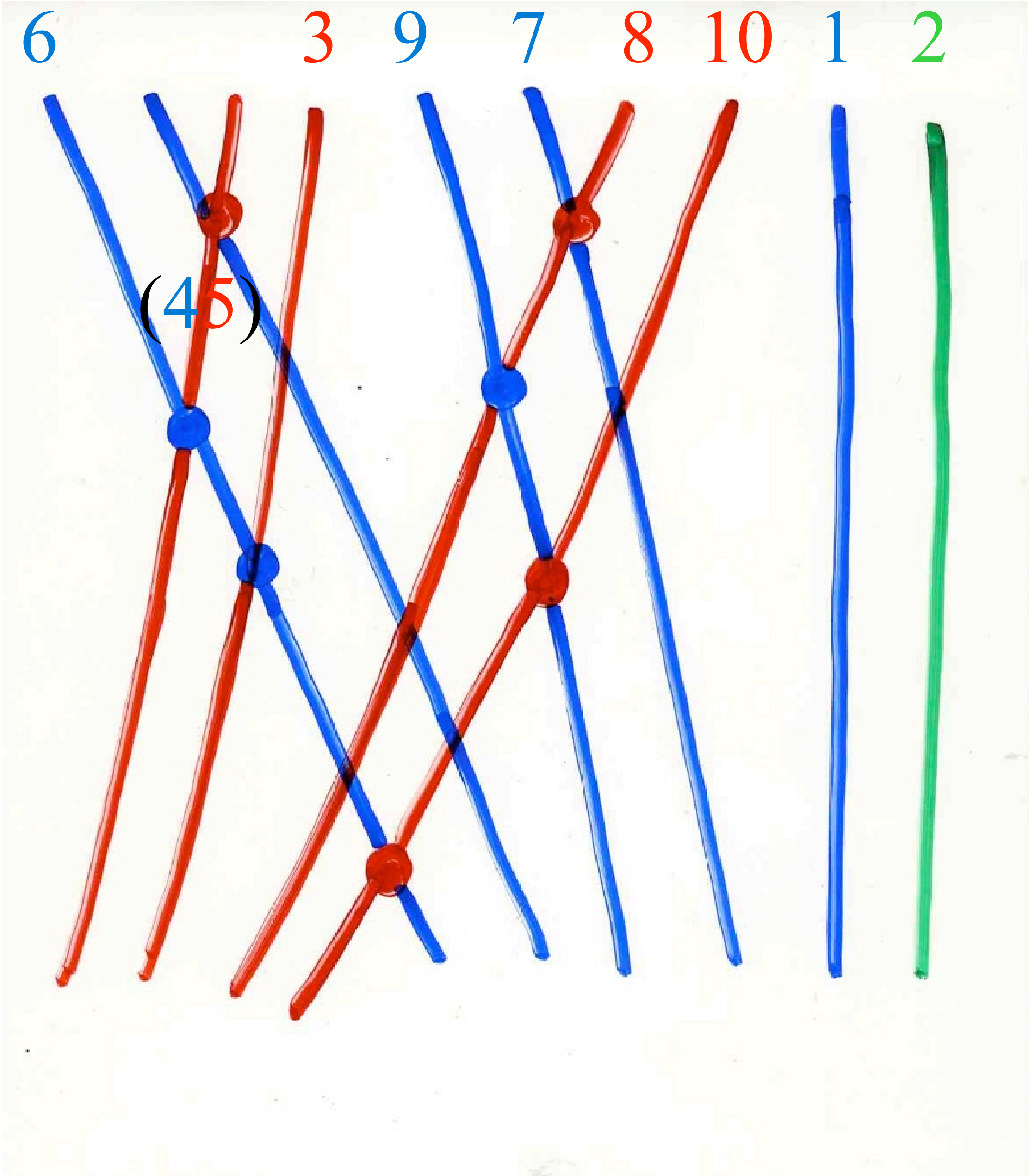
algorithme "Glisser - Pousser"

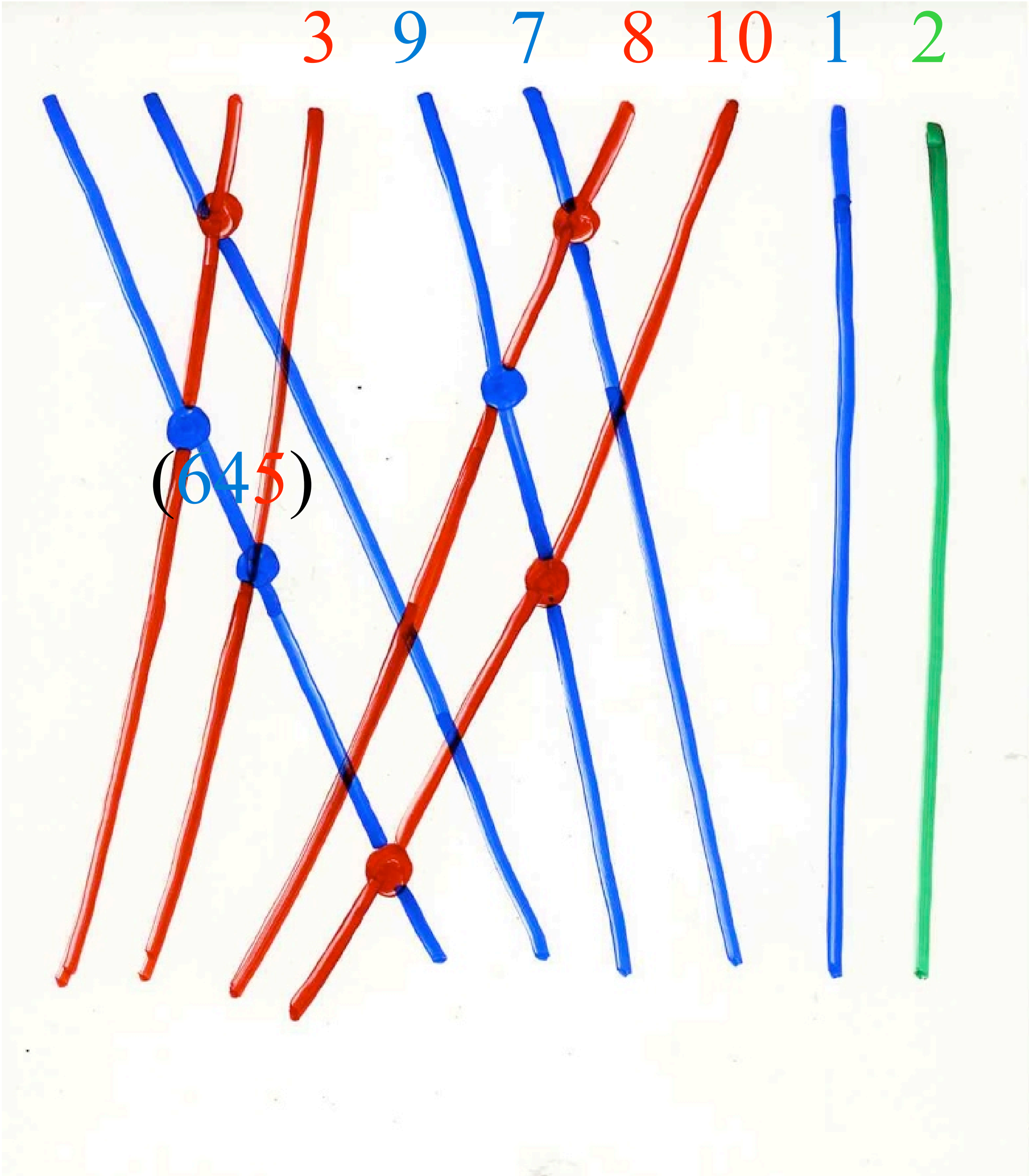

base

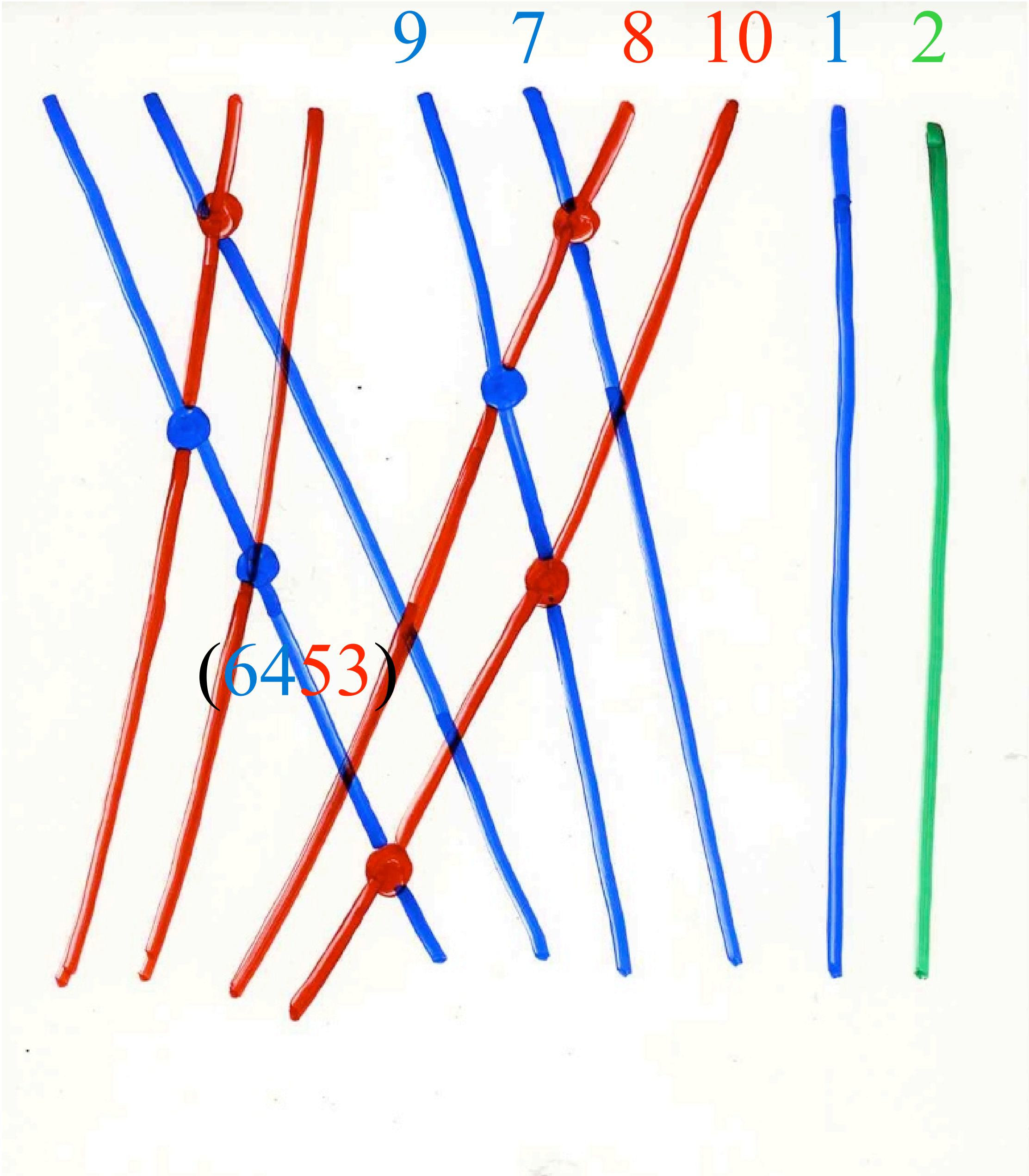

×
1
2
3

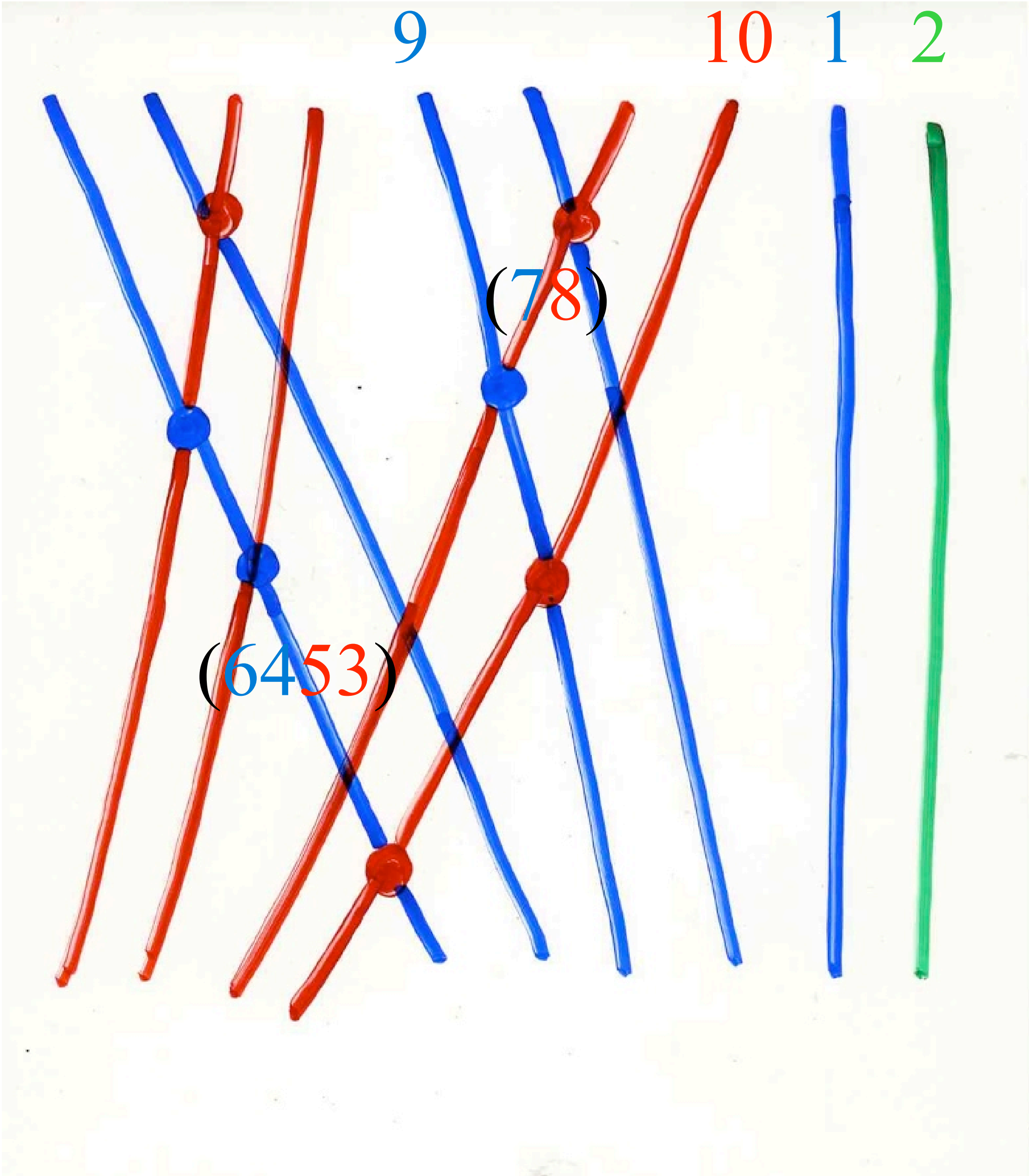

1

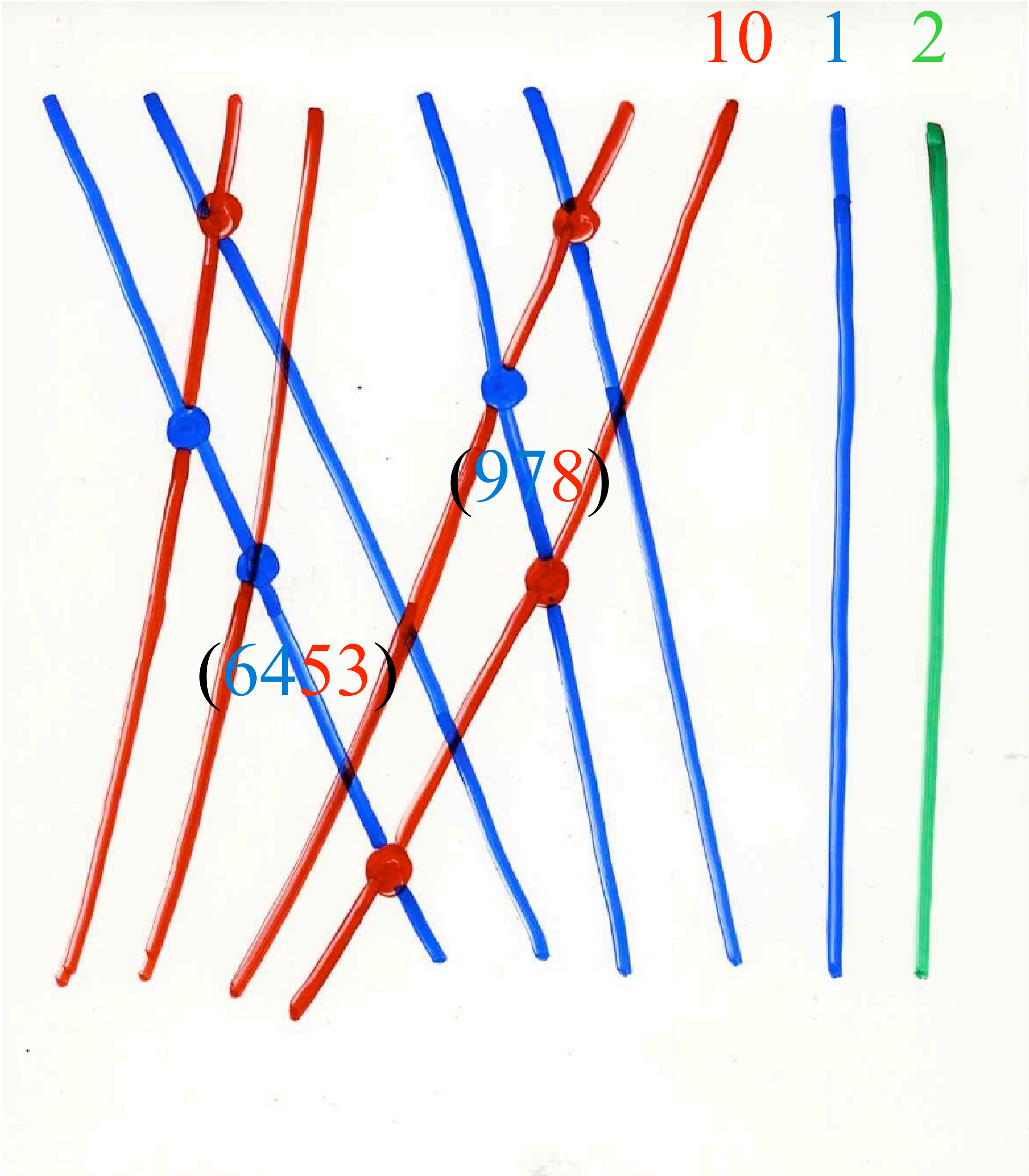

6 4 5 3 9 7 8 10 1 2

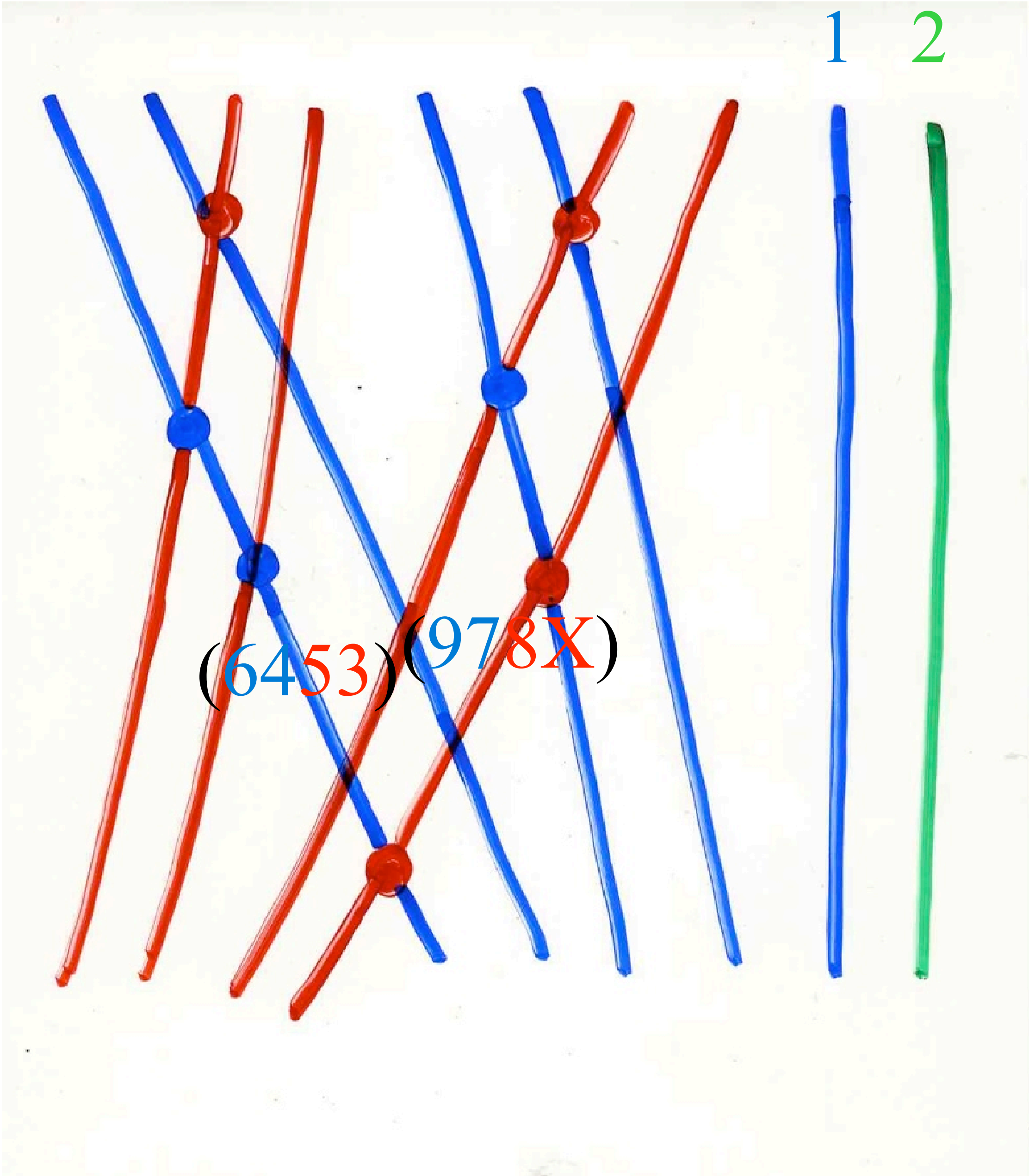

6
3
9
7
8
10
1
2
(45)

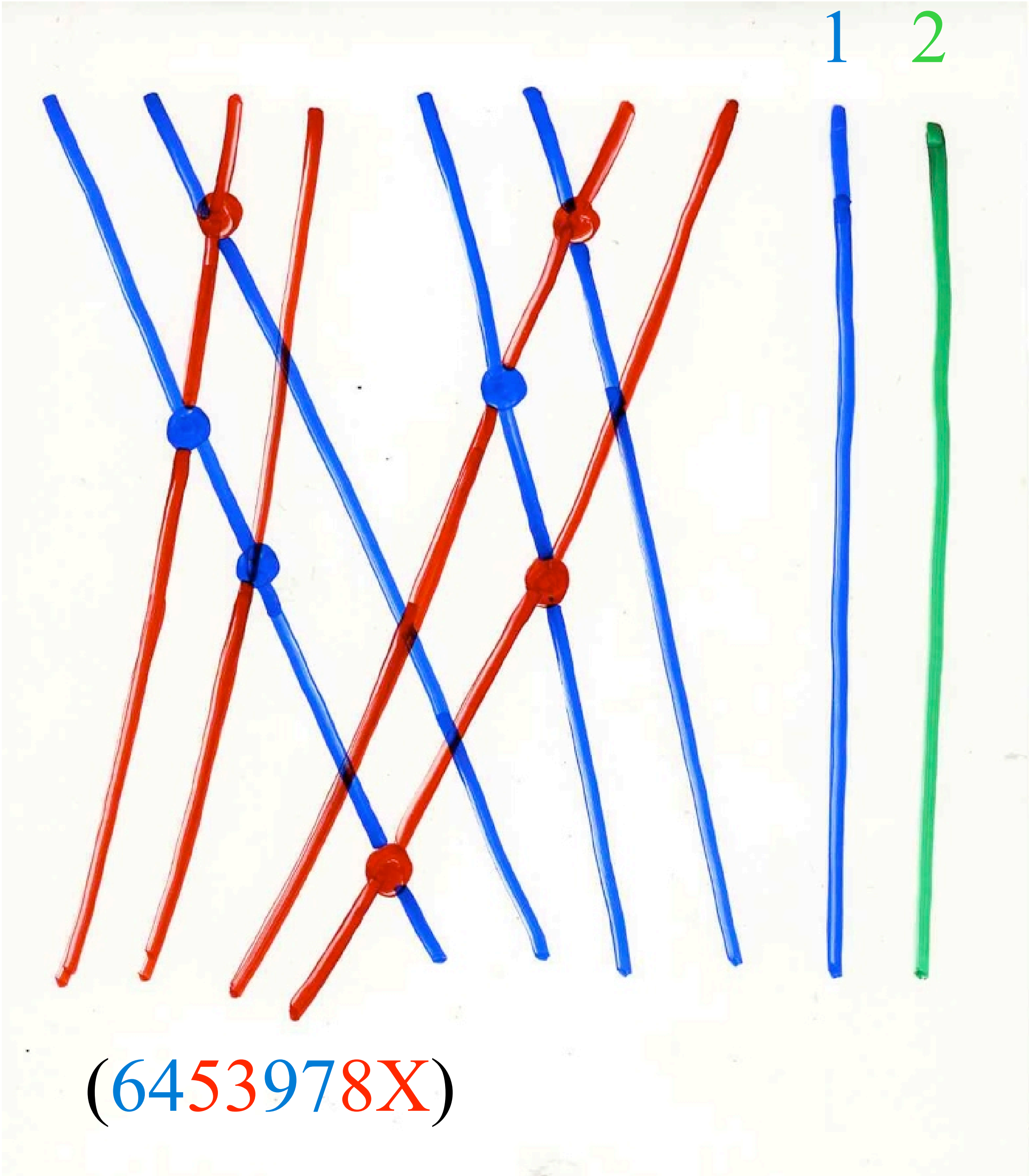

3 9 7 8 10 1 2
(645)

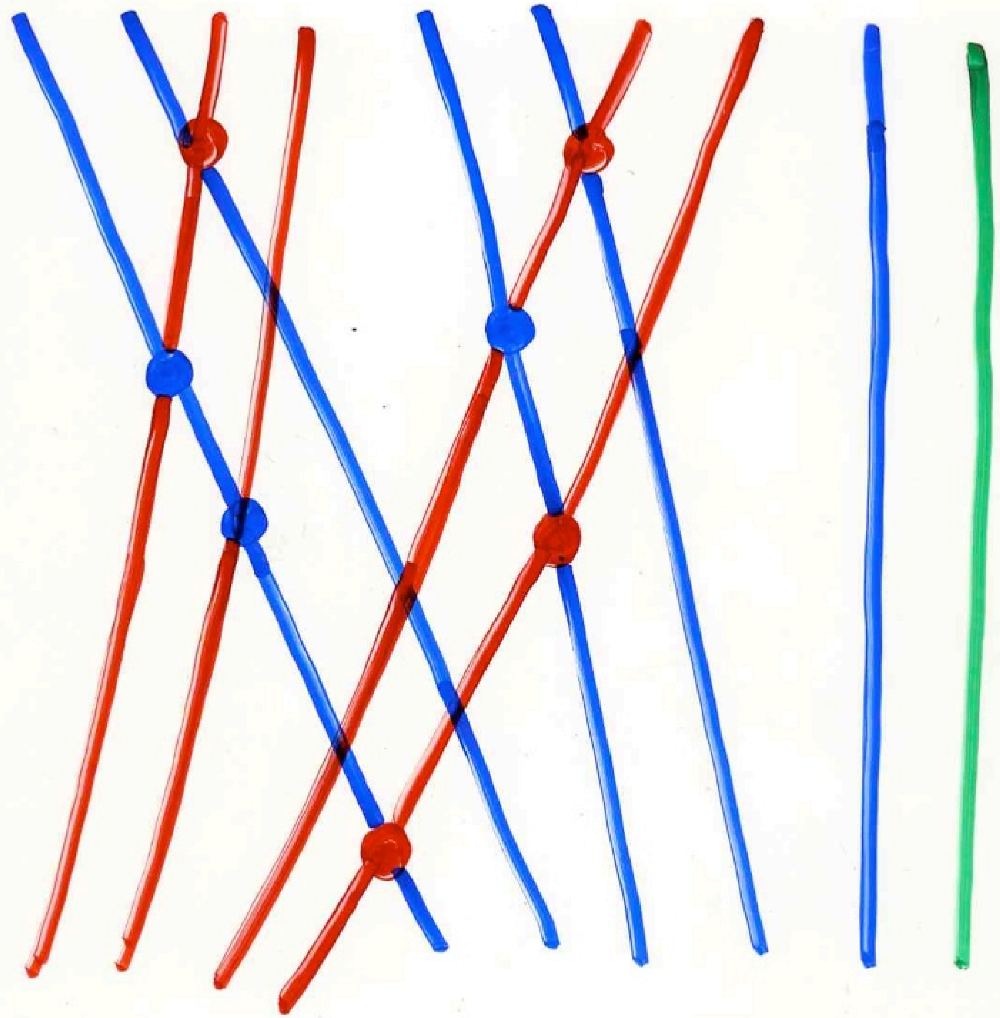

9 7 8 10 1 2
(6453)


9
10
1
2
(78)
(6453)

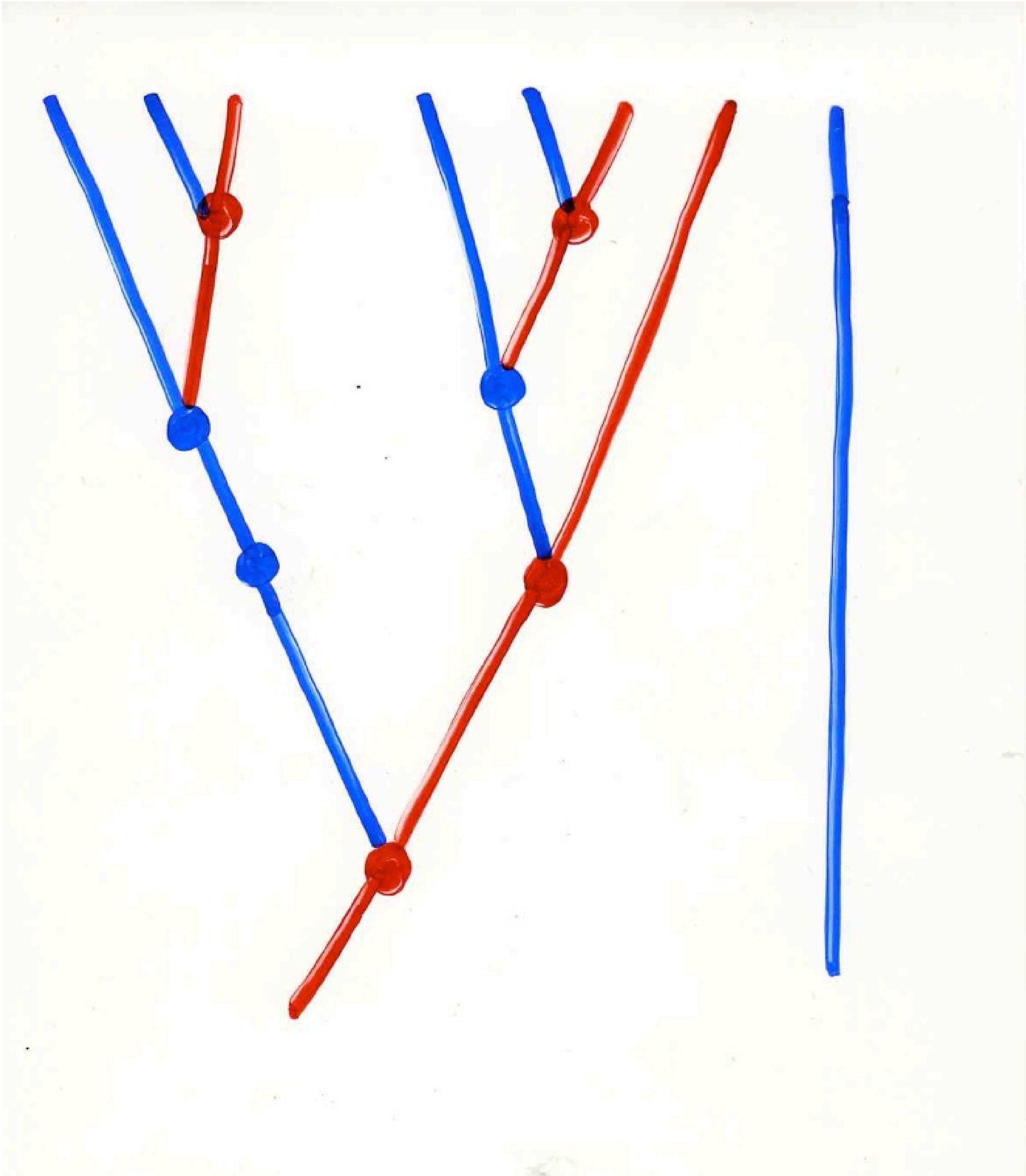

10 1 2
(978)
(6453)

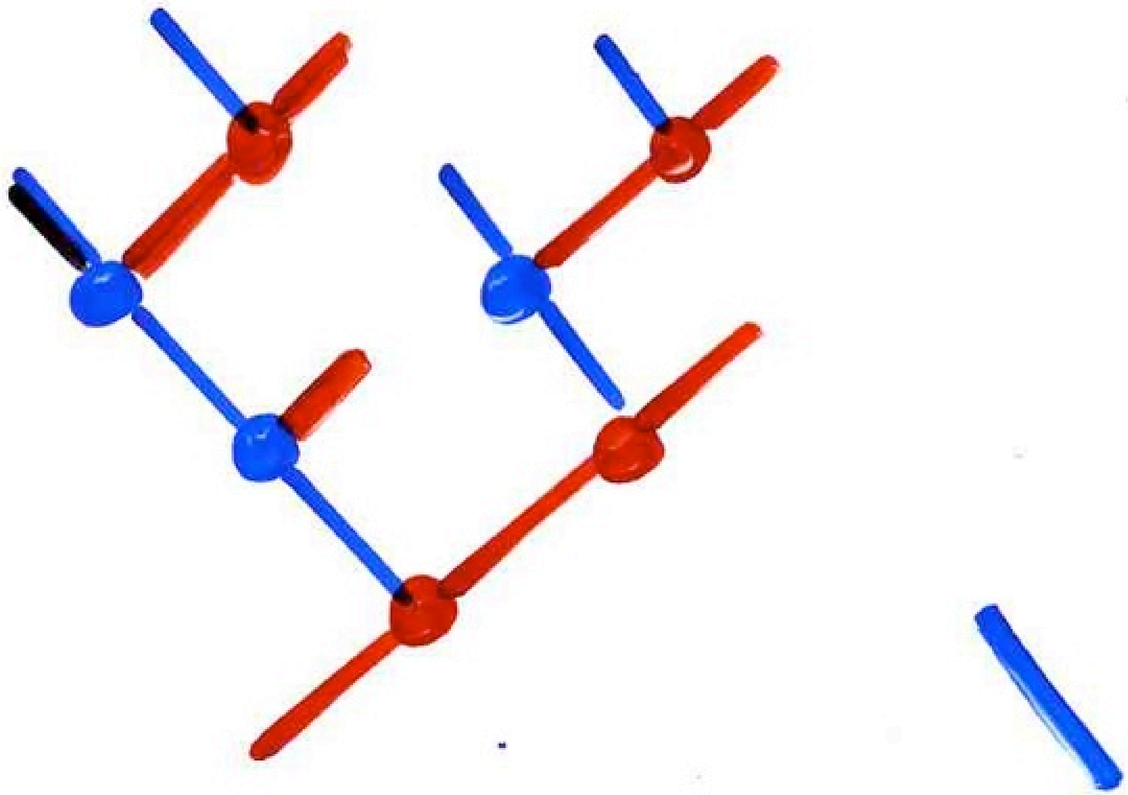

1
2
(6453)
(978X)

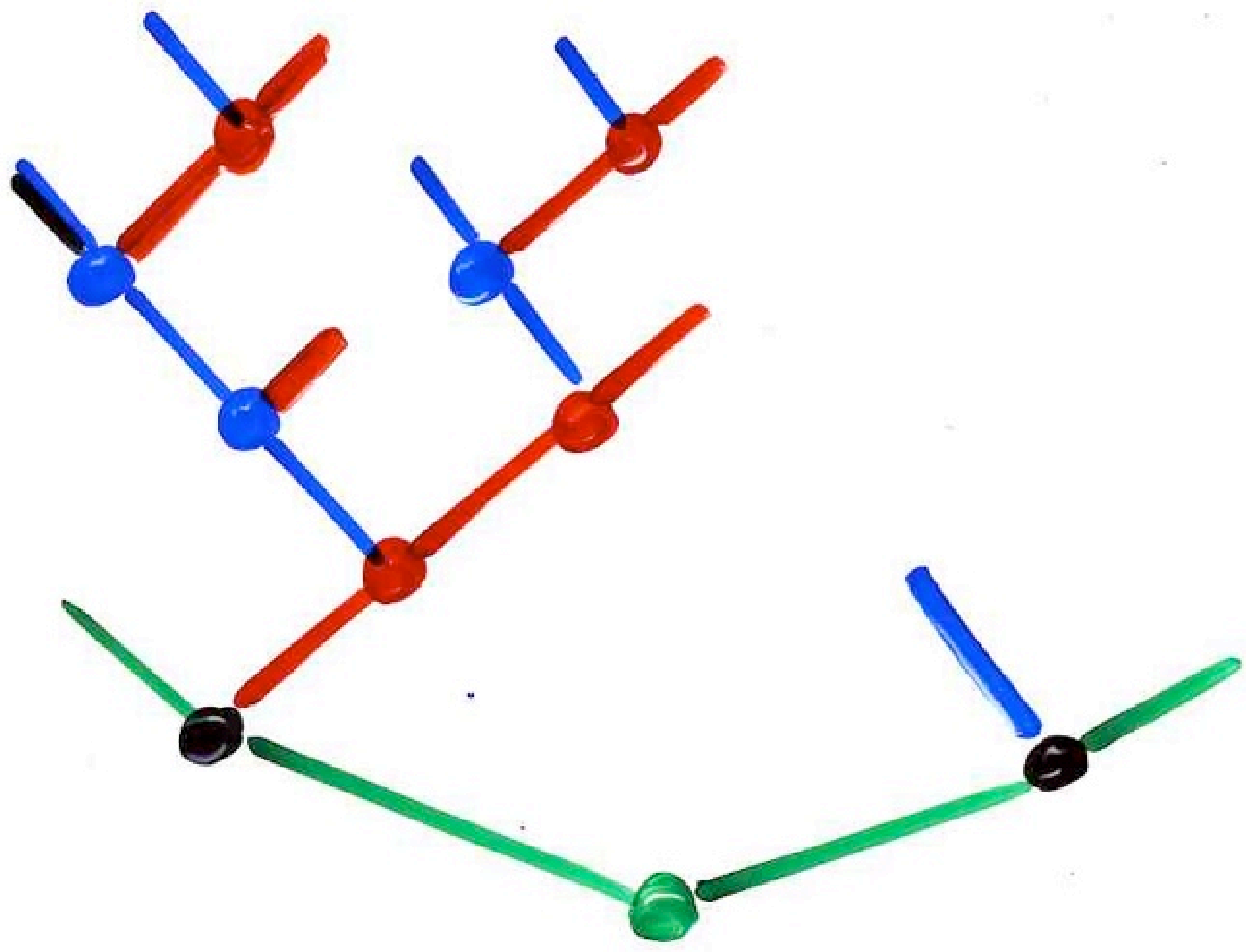

1
2
(645397 8X)

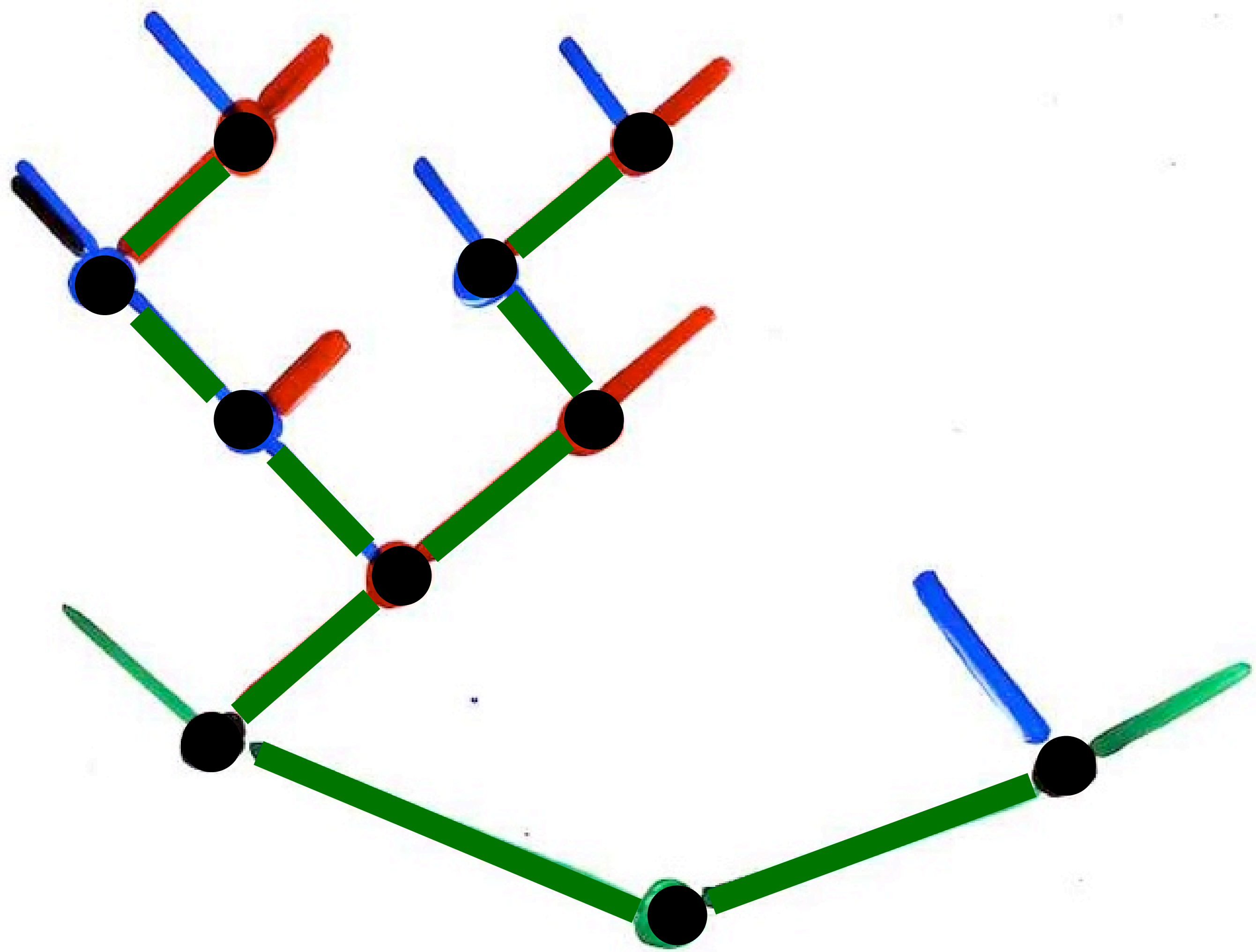


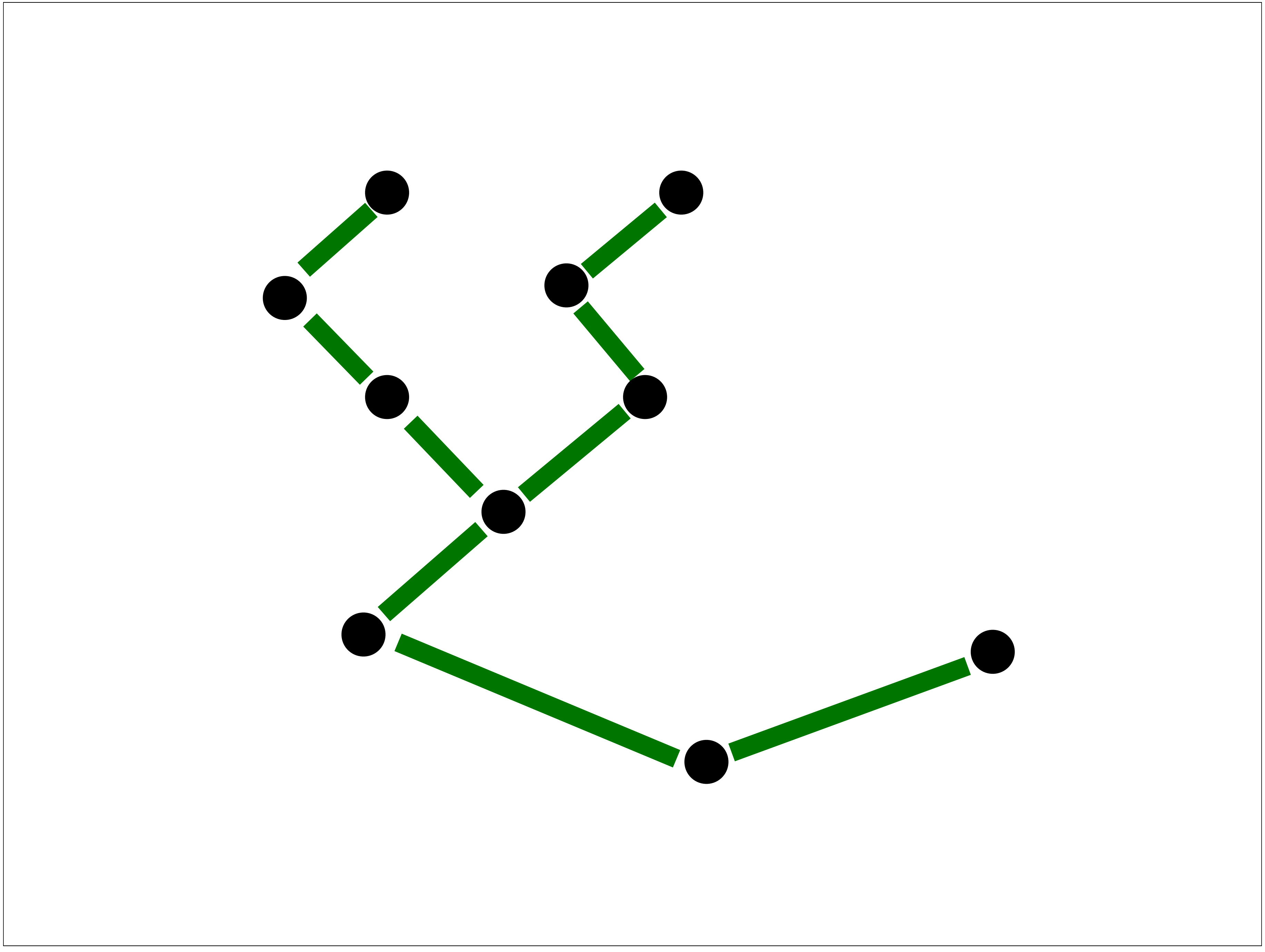
(6453978X) 1 2

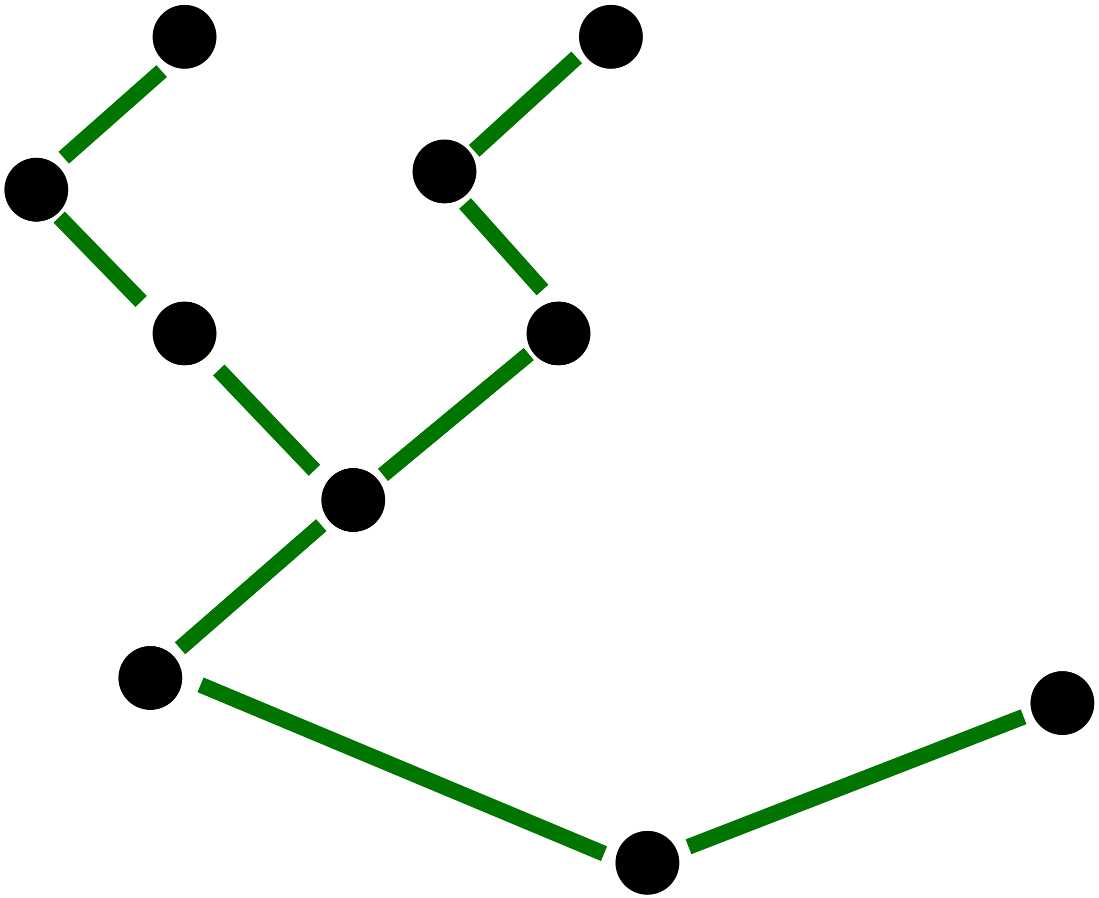












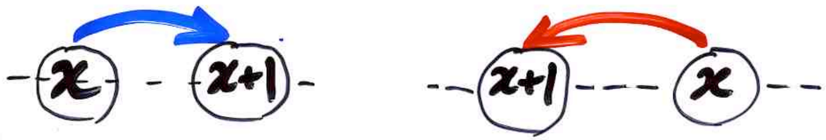
The “exchange-fusion” algorithm

Def- Permutation $\sigma = \sigma(1) \cdots \sigma(n)$

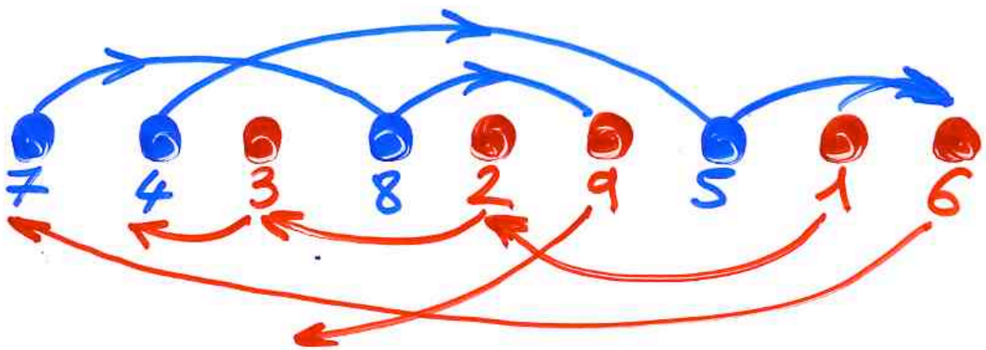
$x = \sigma(i)$, $1 \leq x < n$

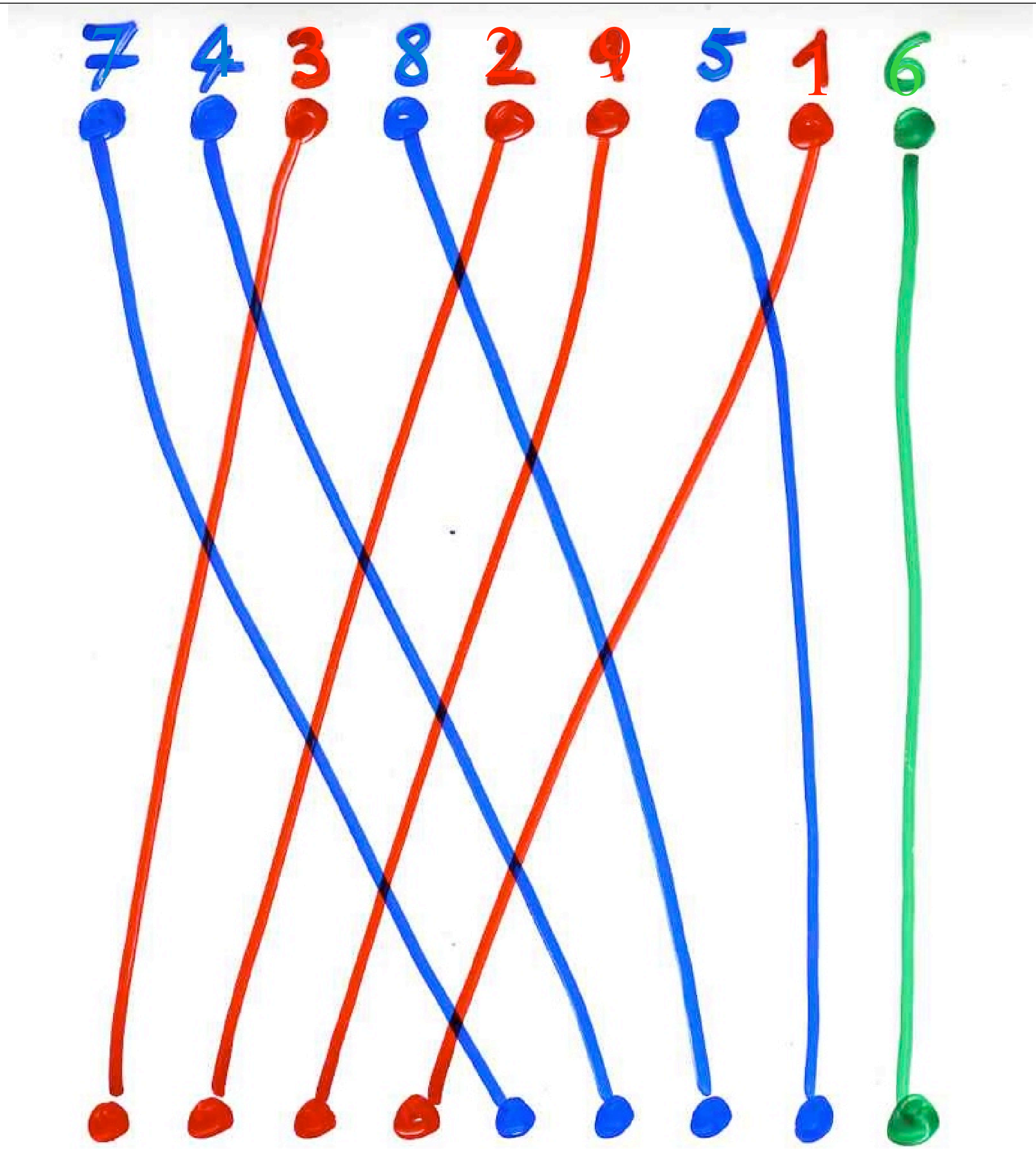
(valeur) x $\begin{cases} \text{avance} \\ \text{recul} \end{cases}$ $x+1 = \sigma(j)$, $\begin{cases} i < j \\ j < i \end{cases}$

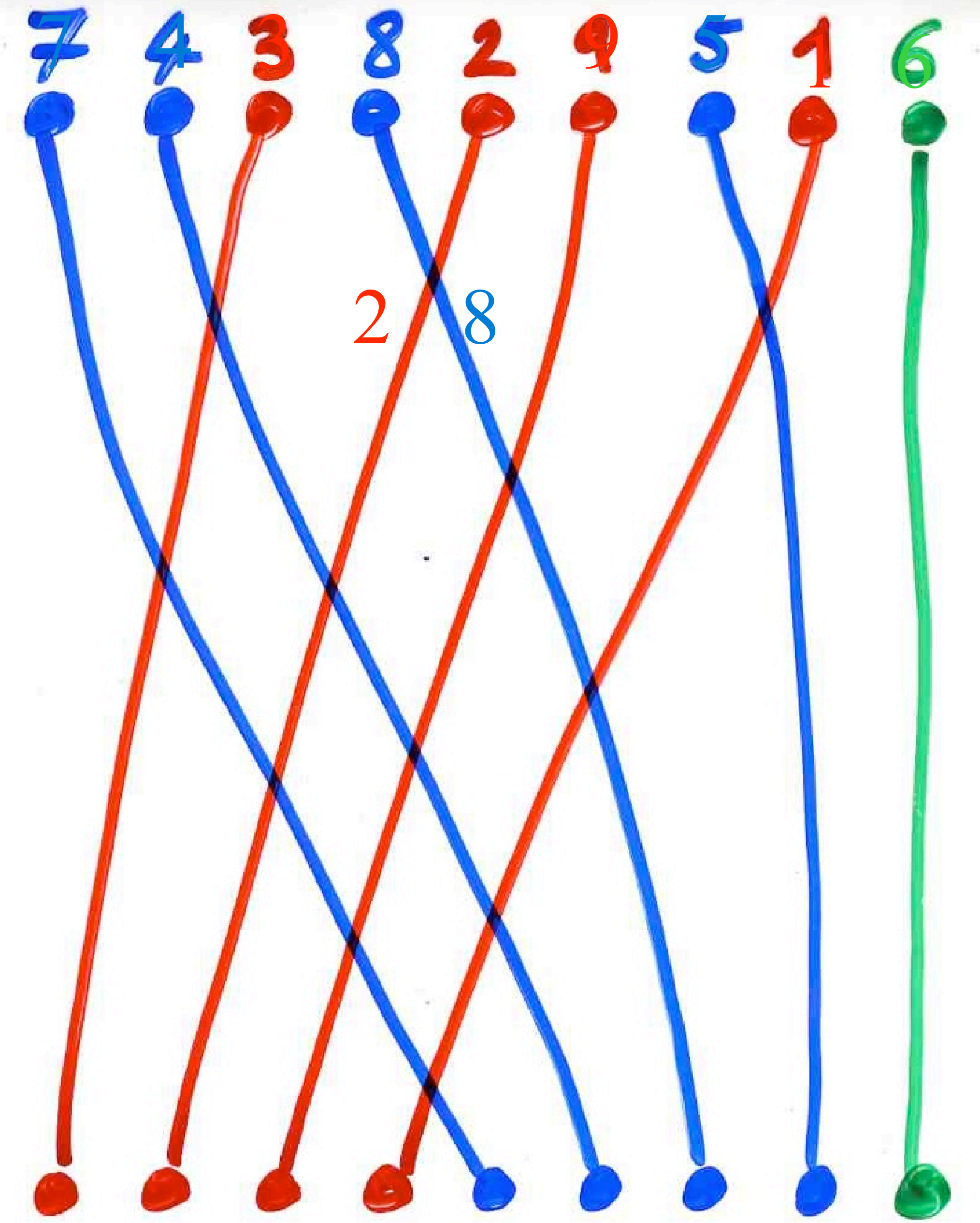
- convention $x = n$ est un recul

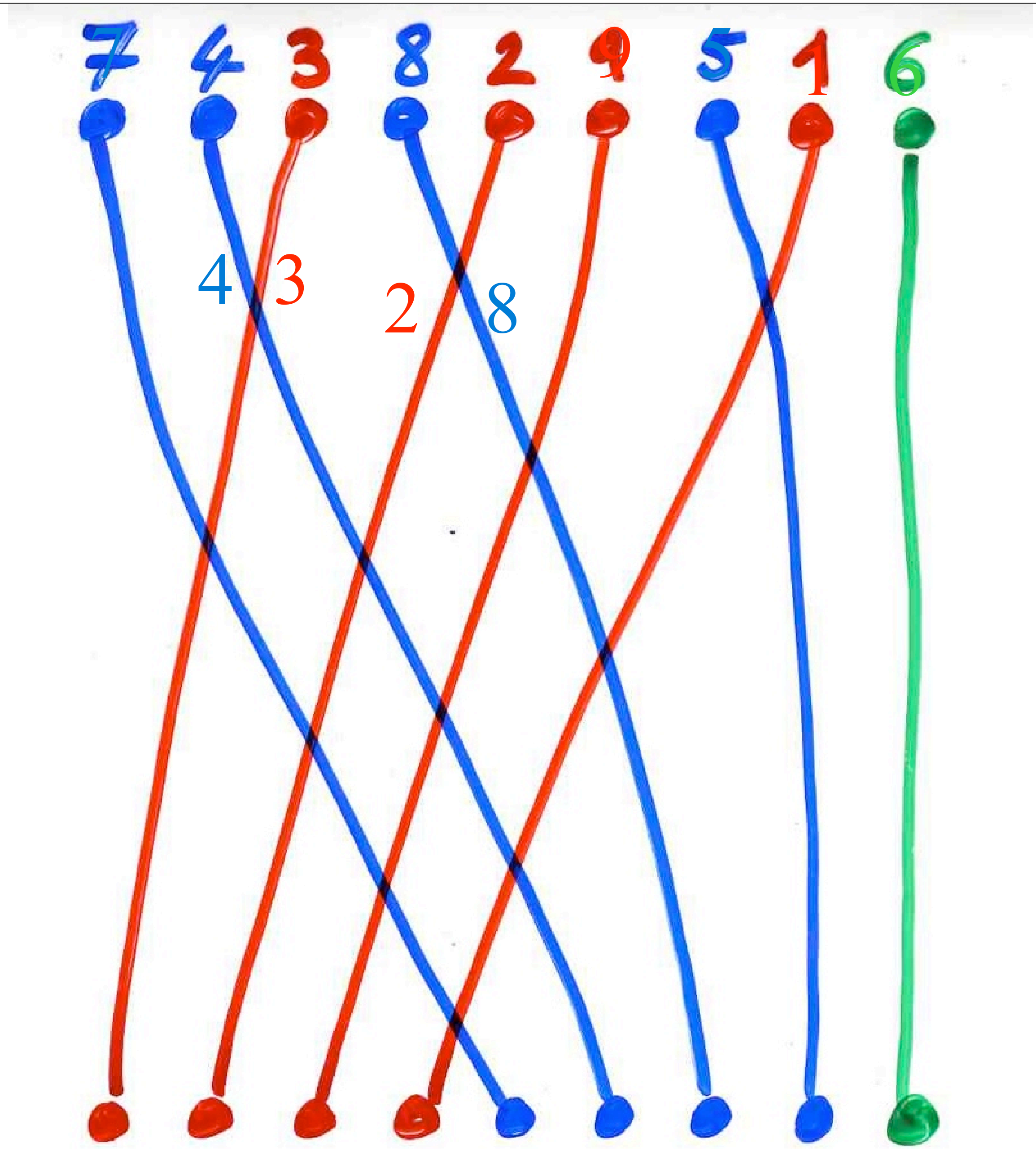


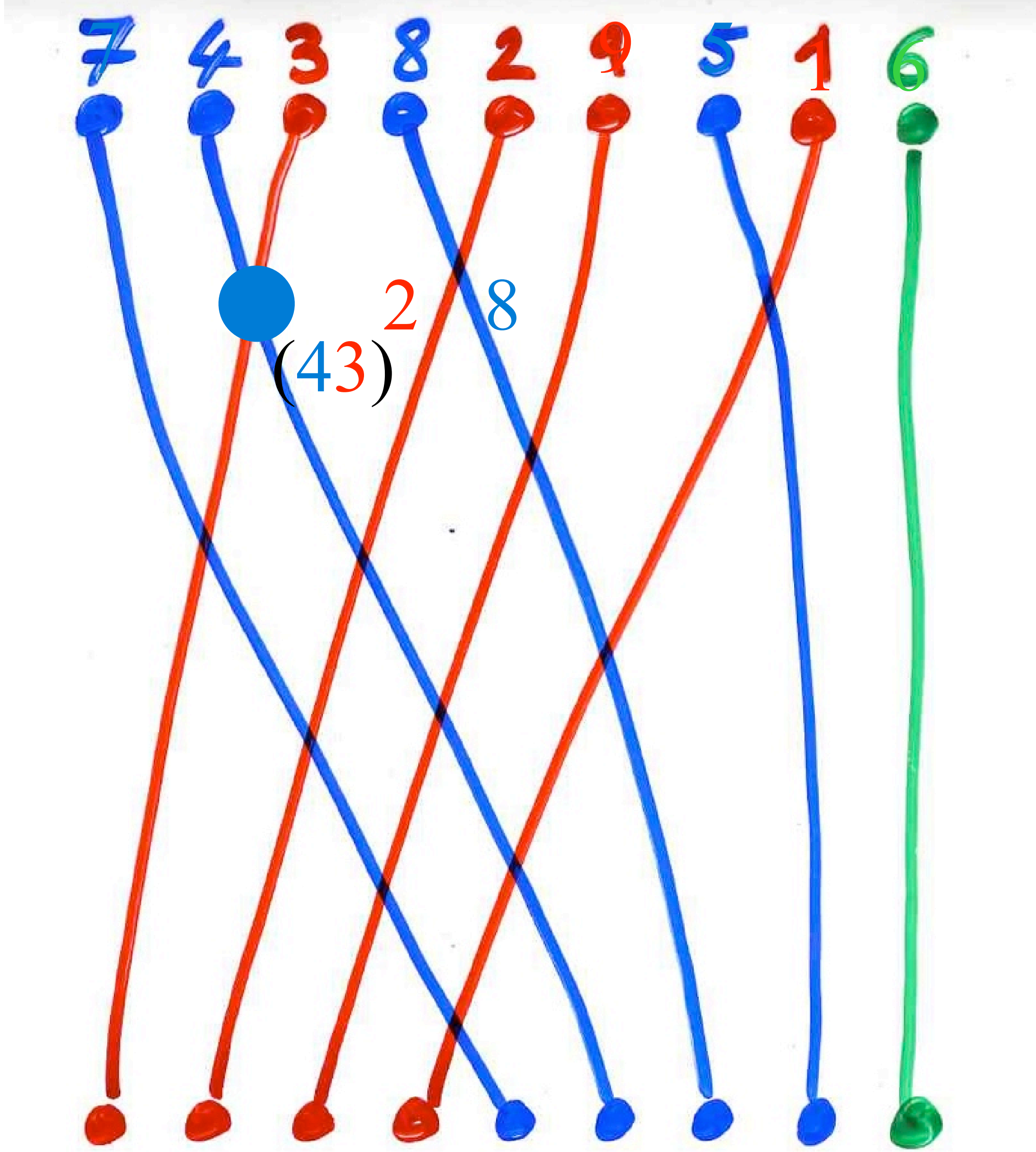
$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$



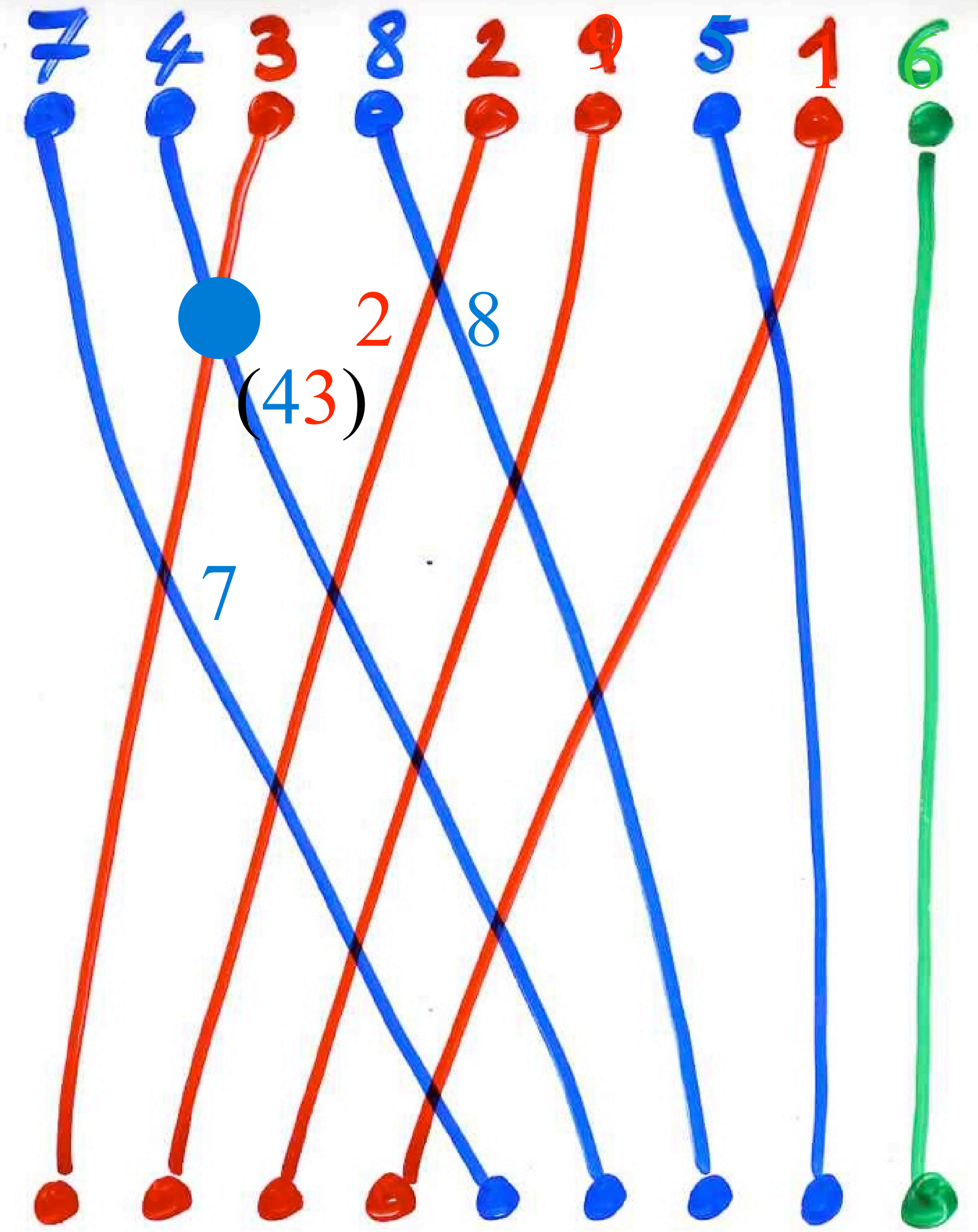

7 4 3 8 2 9 5 1 6

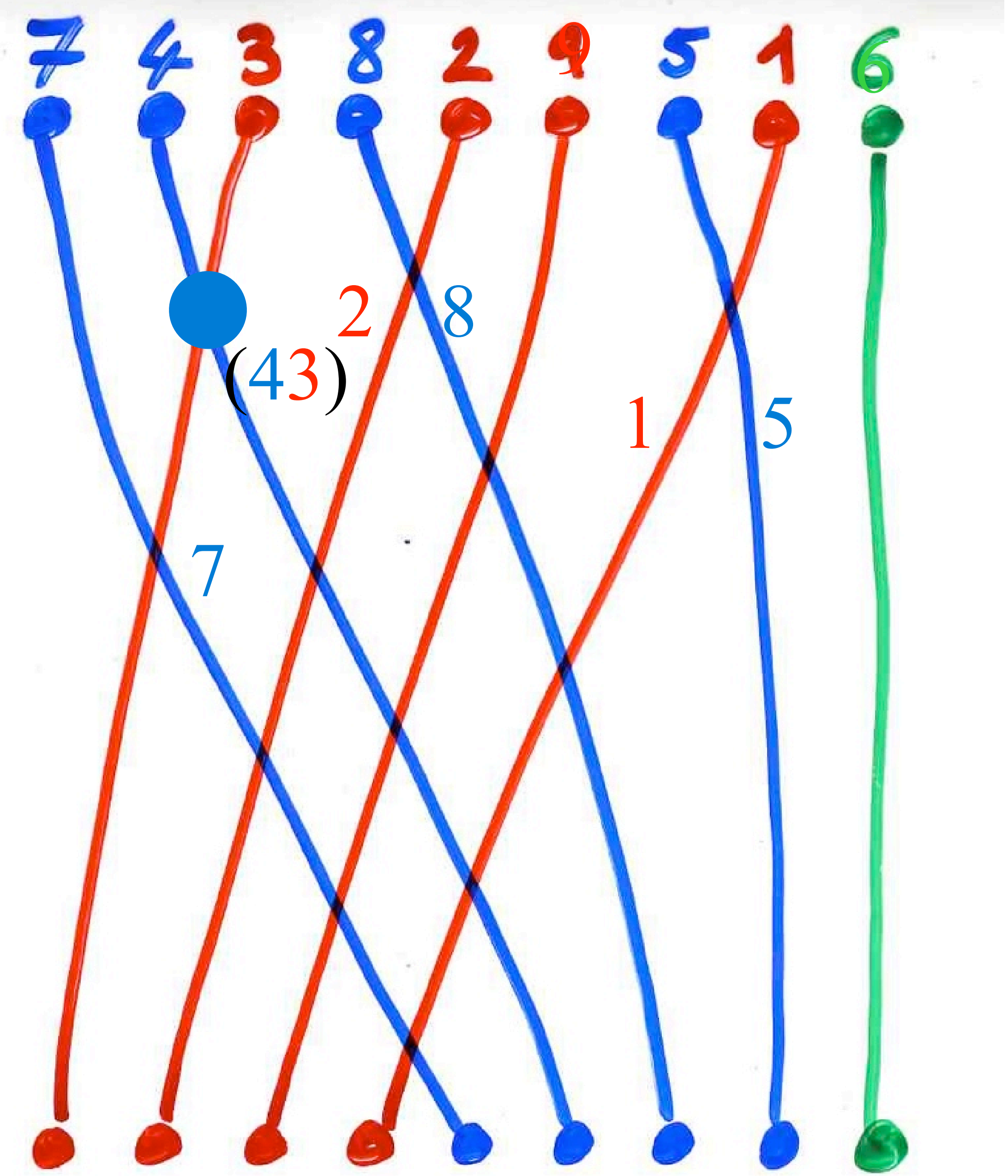

7 4 3 8 2 9 5 1 6
2 8

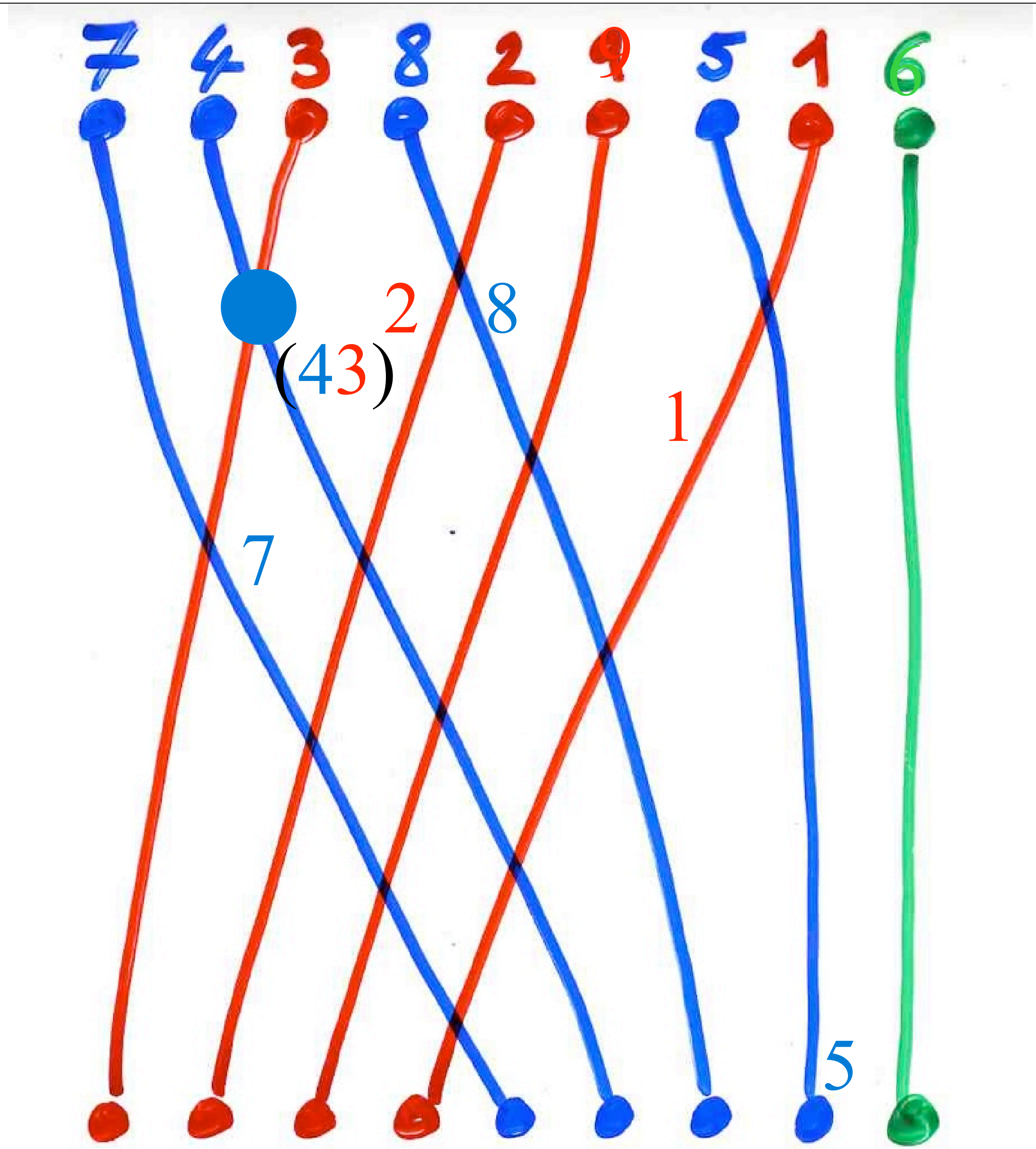

7 4 3 8 2 9 5 1 6
4 3 2 8

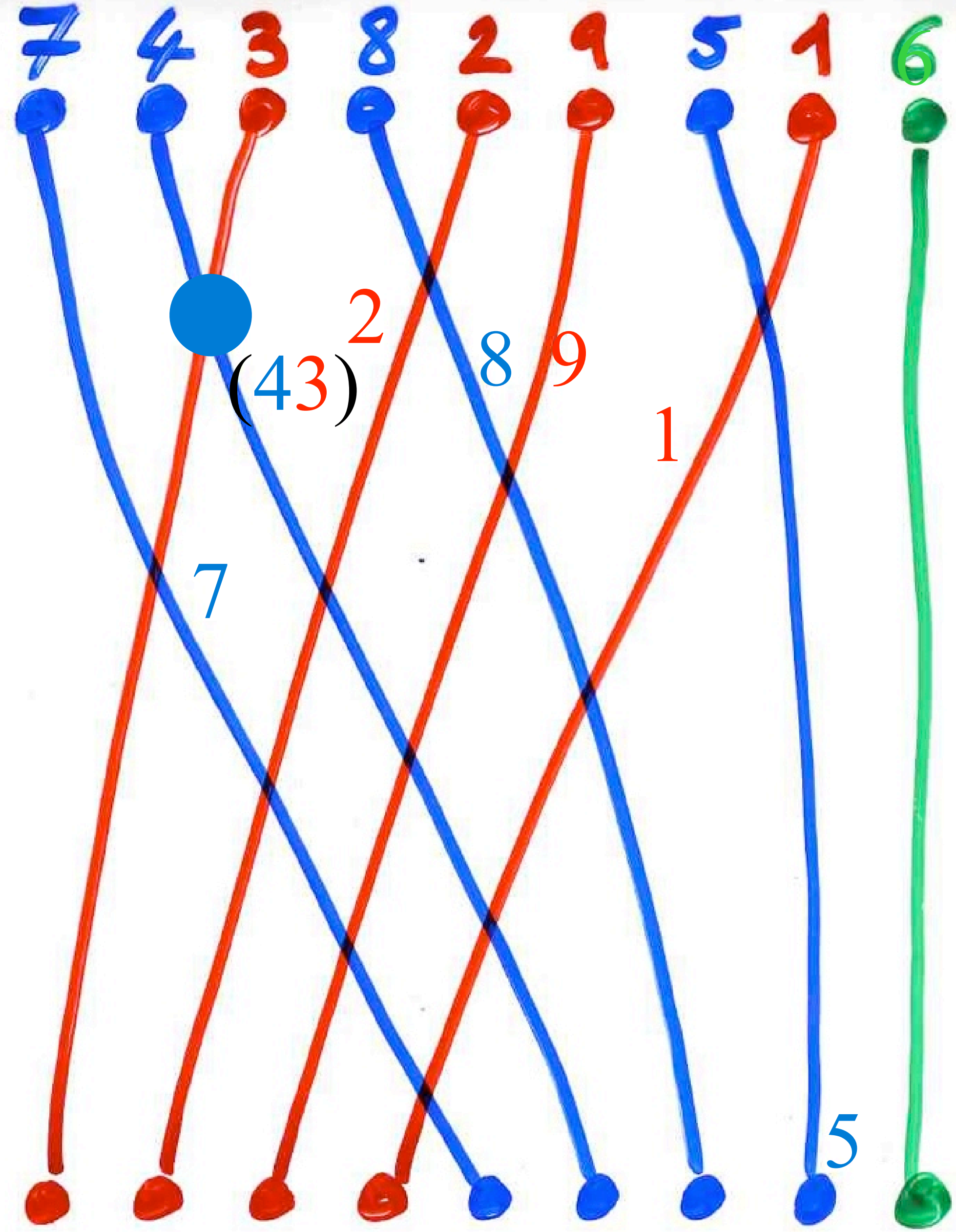


7 4 3 8 2 9 5 1 6
2 8
(43)

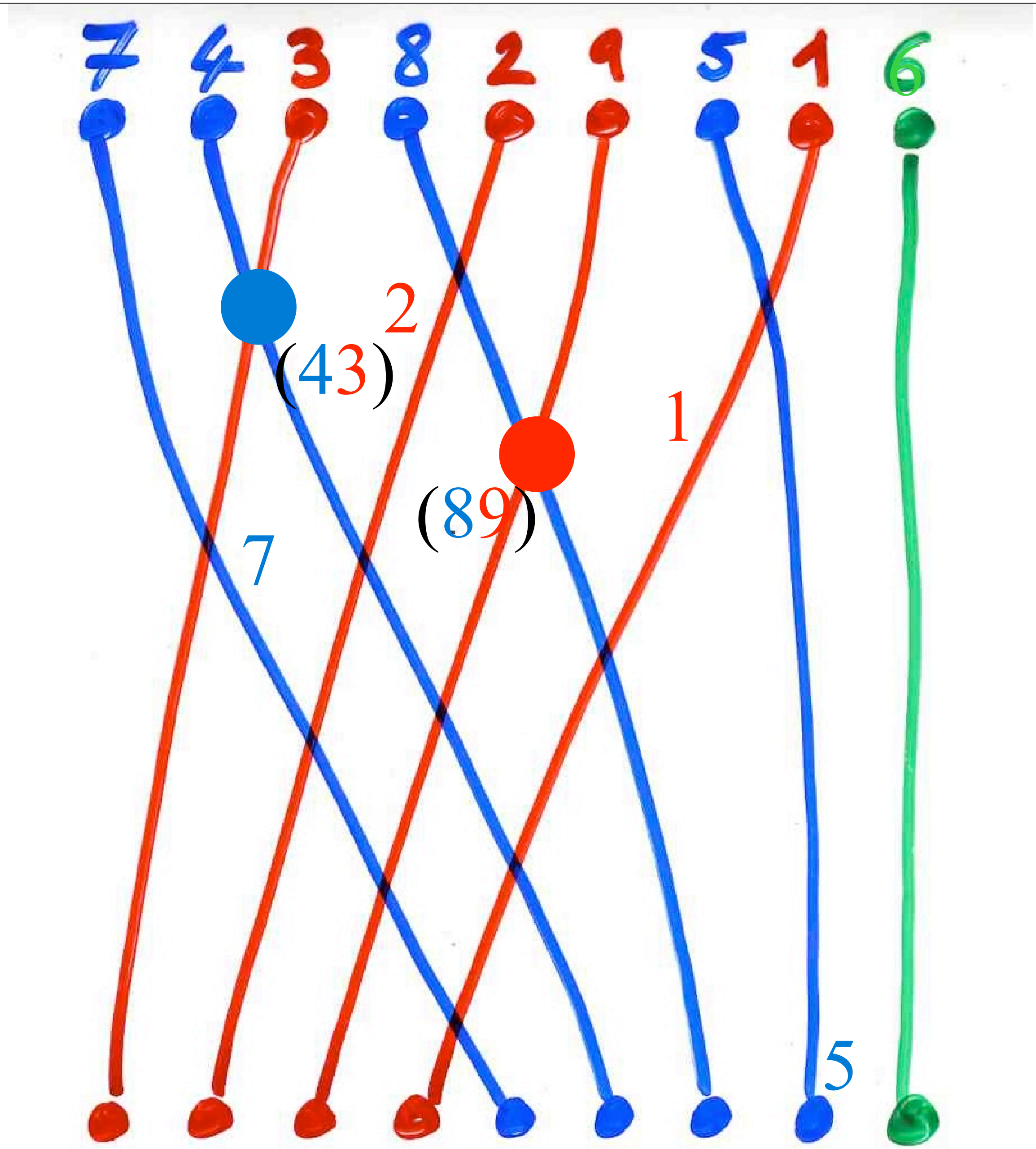

7 4 3 8 2 9 5 1 6
2
8
(43)
7

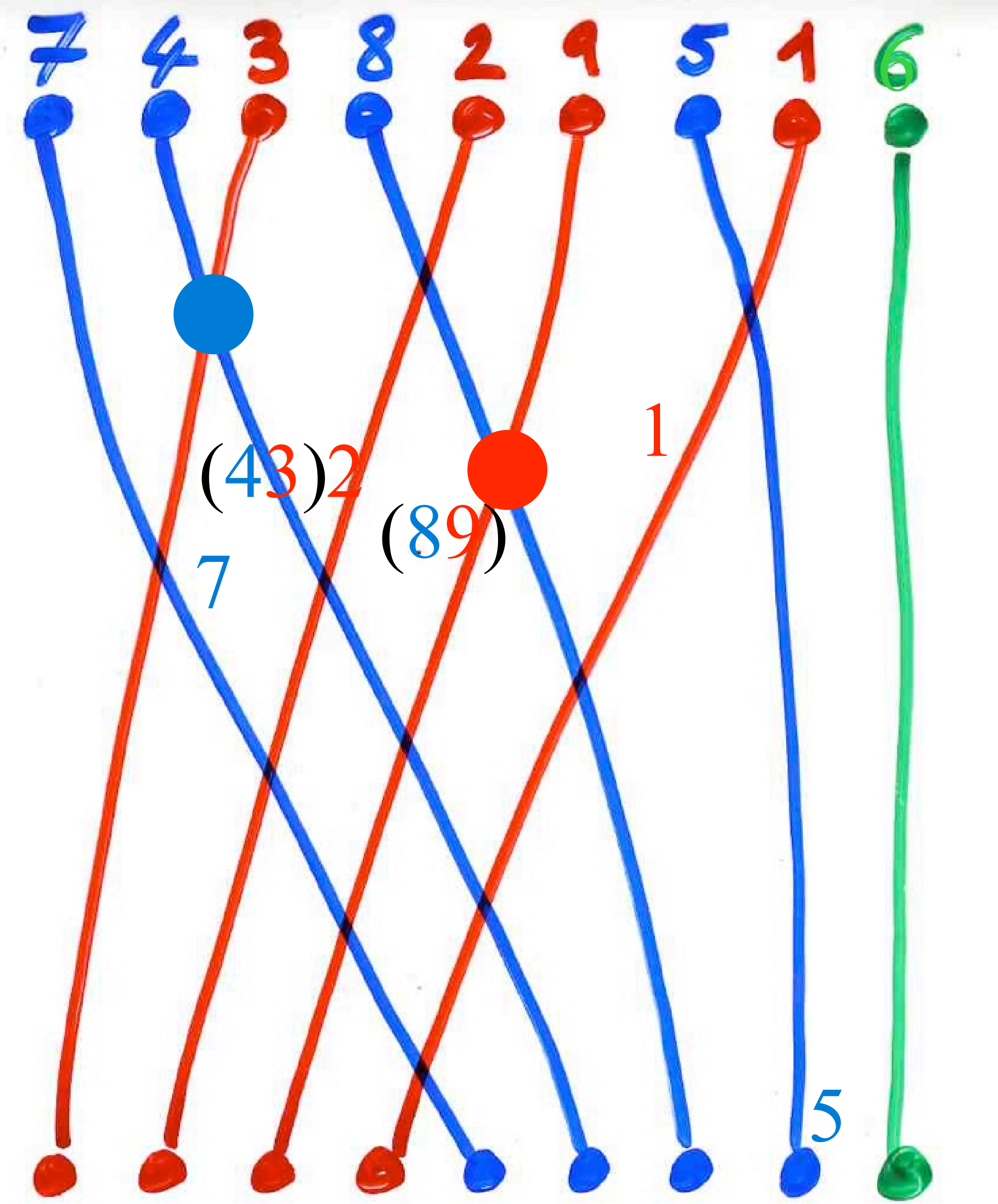

7 4 3 8 2 9 5 1 6
2
8
(43)
1
5
7

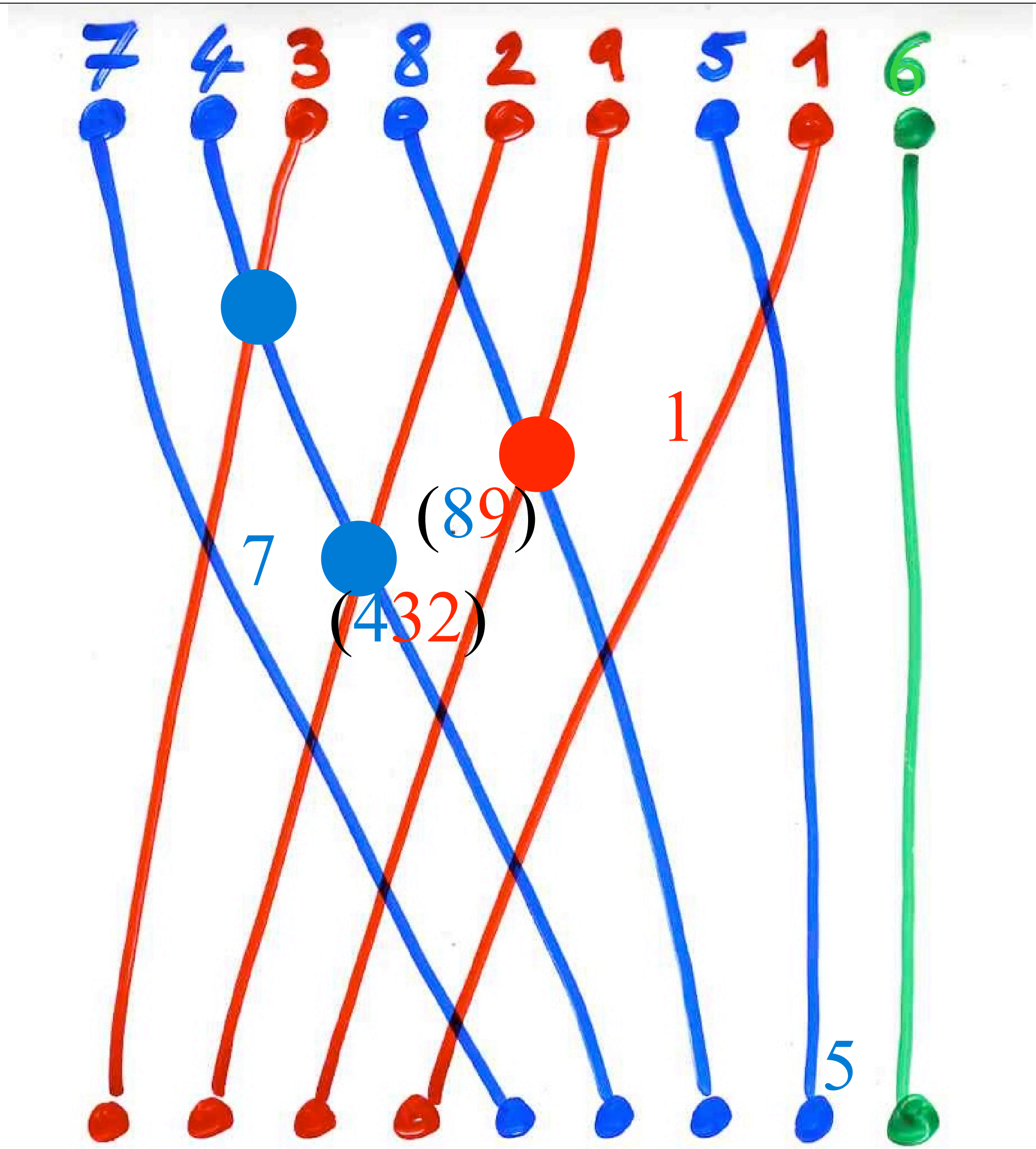

7 4 3 8 2 9 5 1 6
2
8
(43)
1
7
5

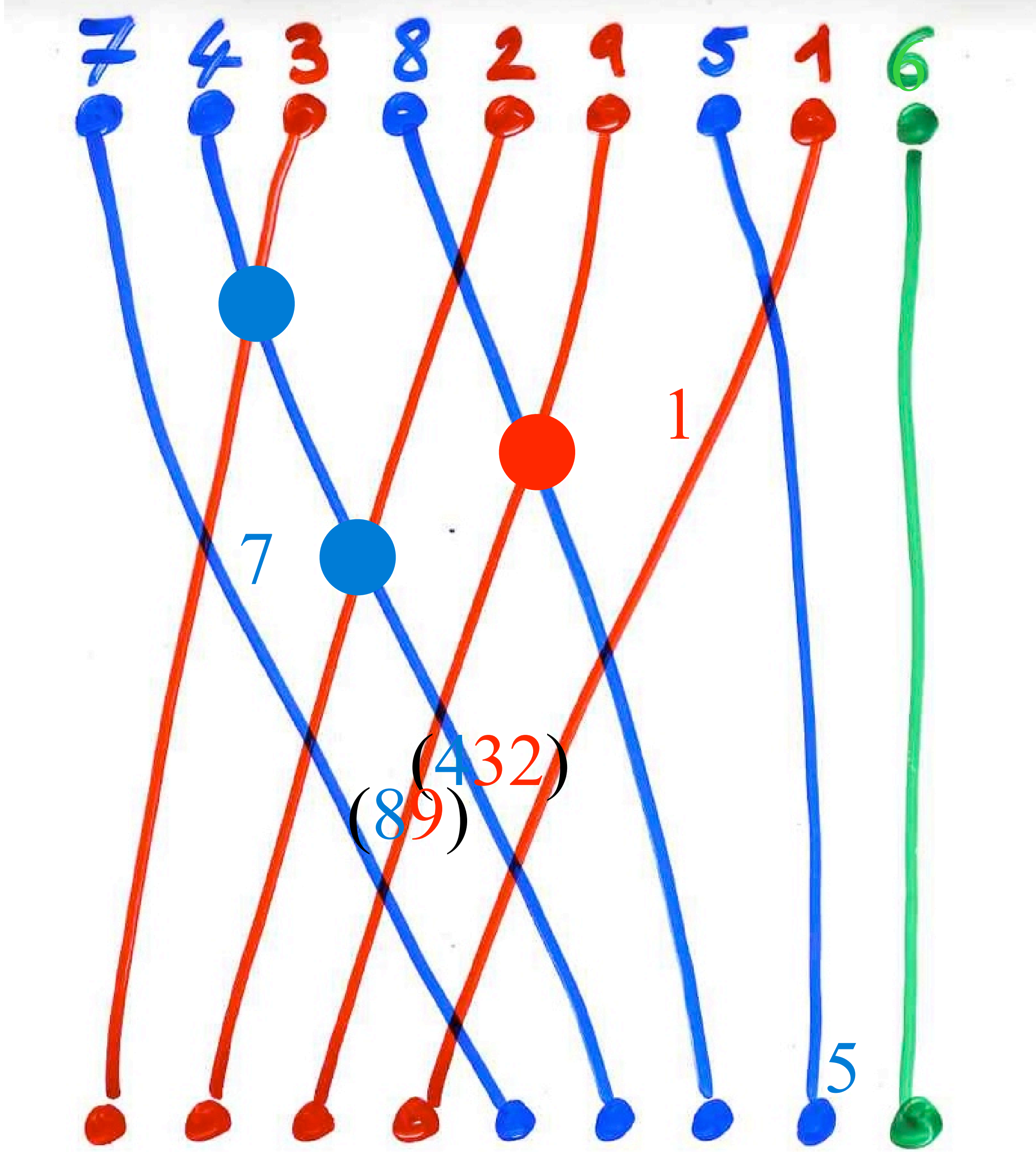


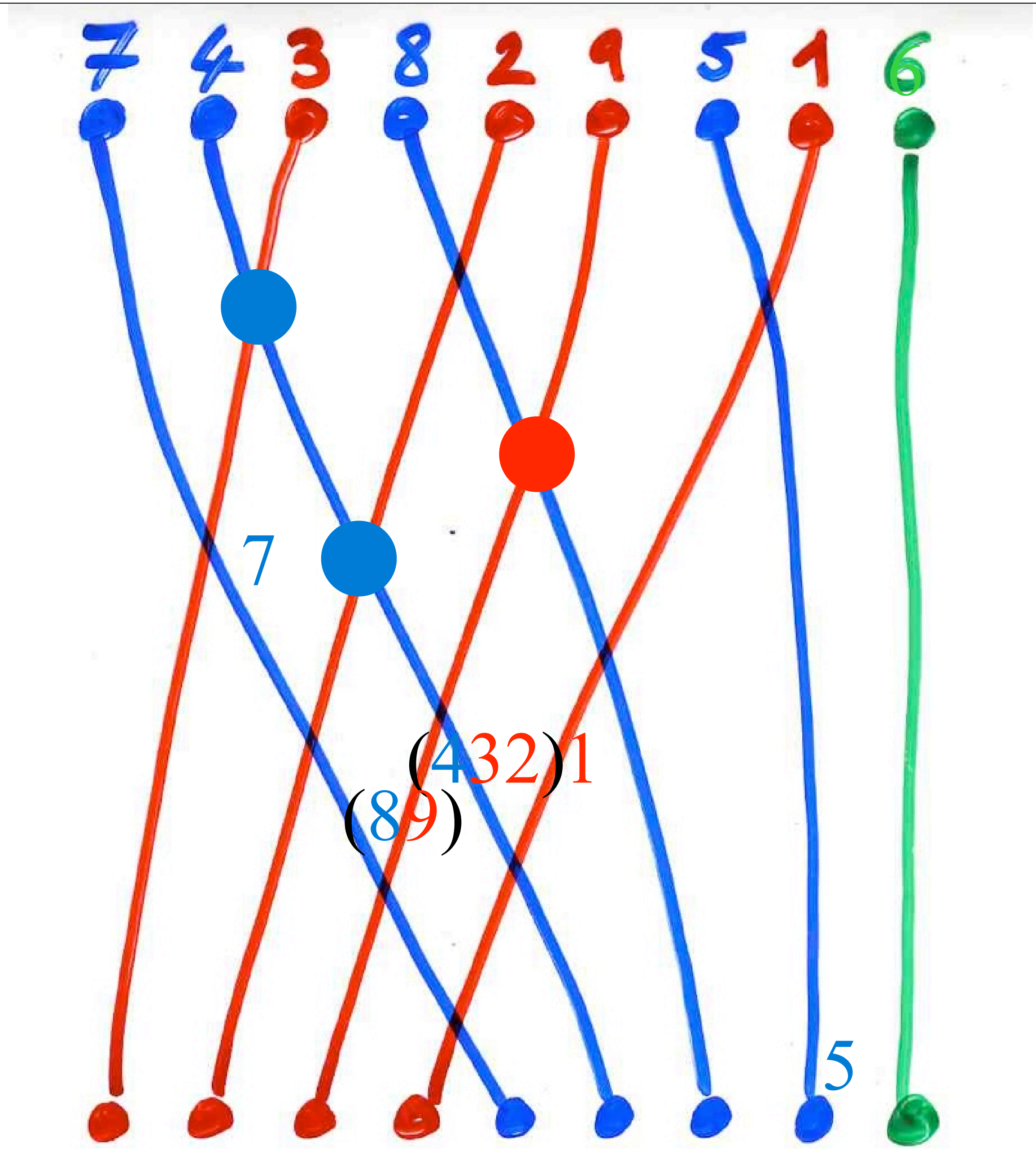
7 4 3 8 2 9 5 1 6
(43)
2
8
9
1
7
5

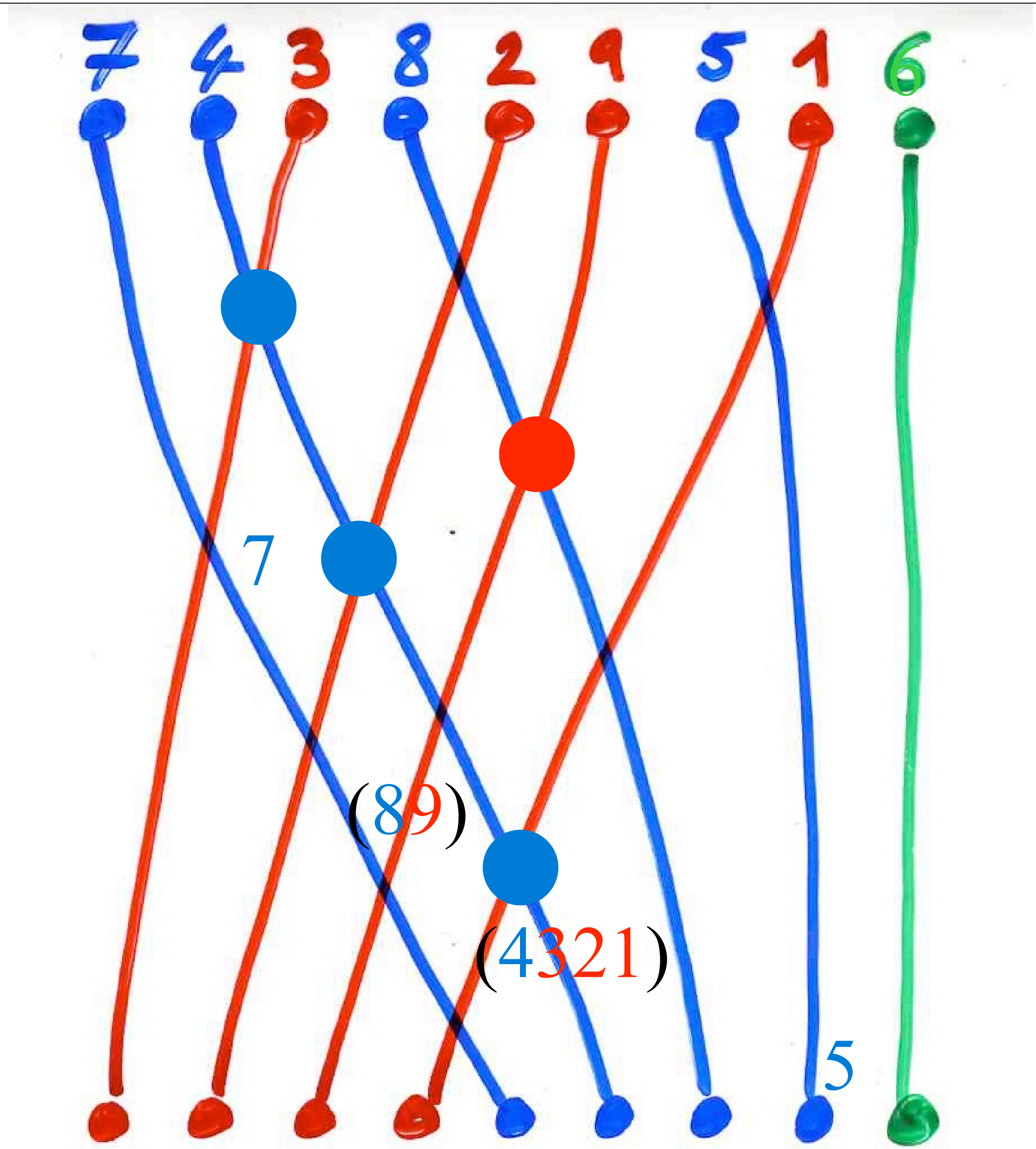

7 4 3 8 2 9 5 1 6
2
(43)
1
(89)
7
5


7 4 3 8 2 9 5 1 6
(43)2
1
(89)
7
5

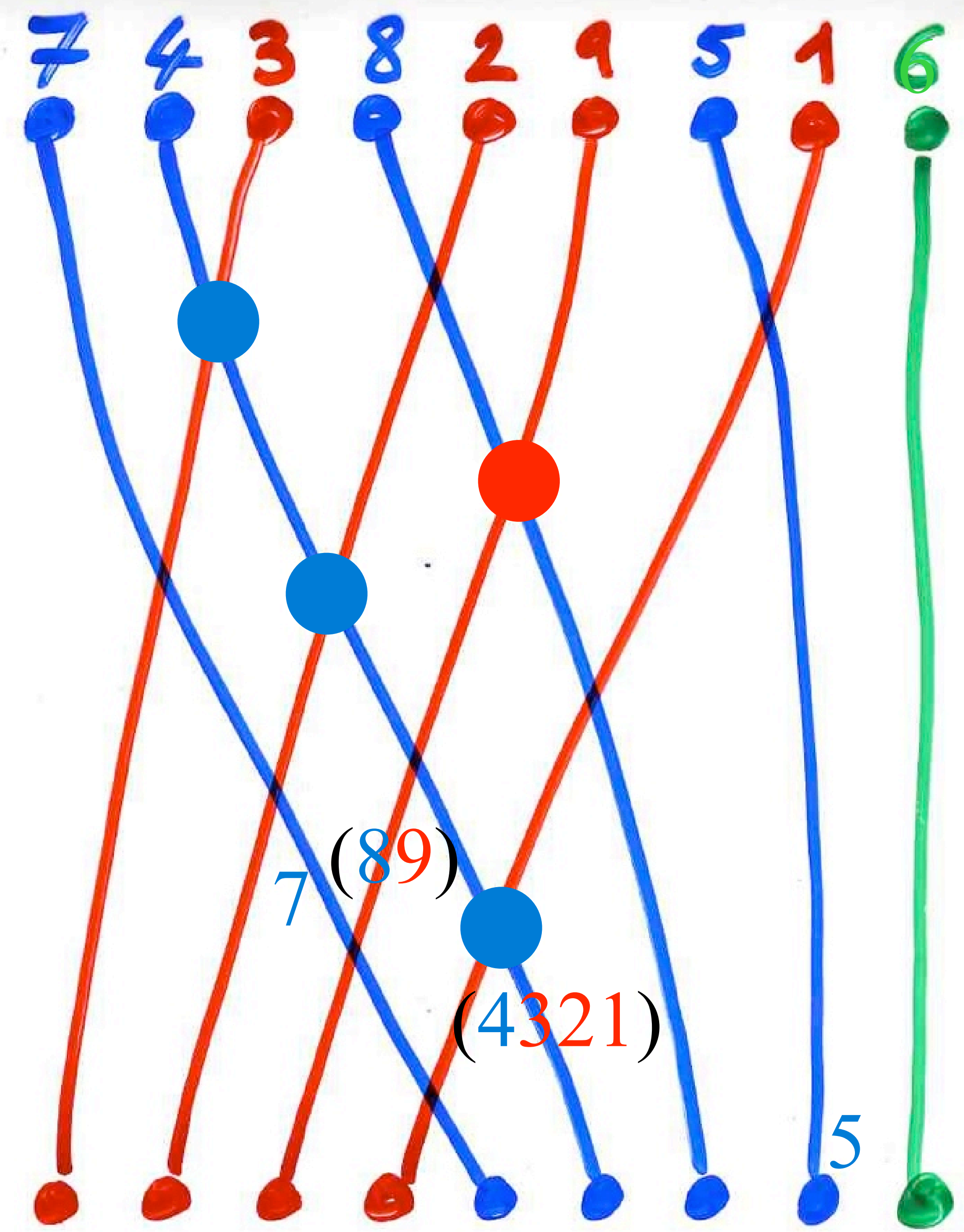

7 4 3 8 2 9 5 1 6
1
(89)
7
(432)
5


7 4 3 8 2 9 5 1 6
1
7
(432)
(89)
5

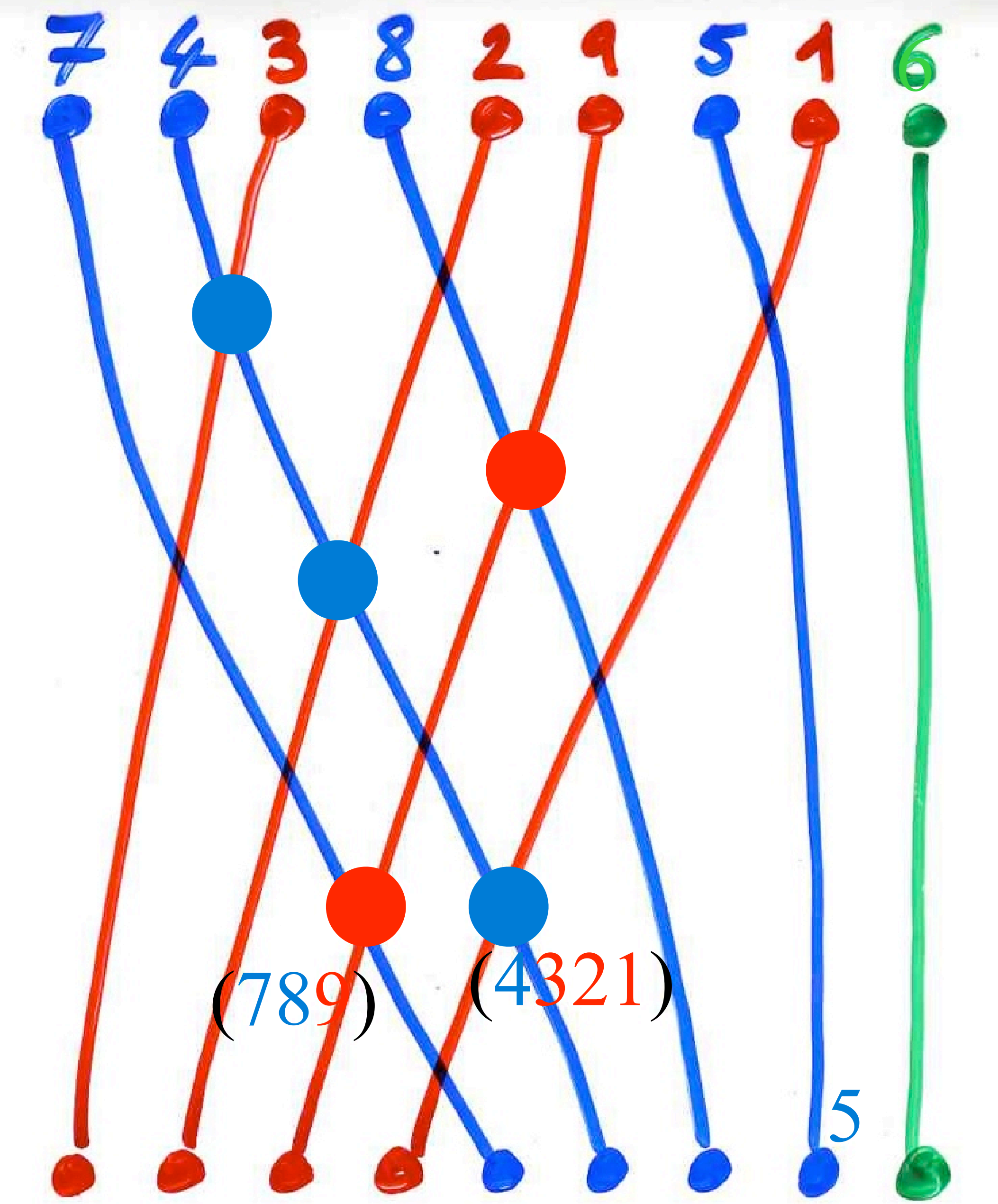

7 4 3 8 2 9 5 1 6
7
(432)1
(89)
5



7 4 3 8 2 9 5 1 6
7
(89)
(4321)
5

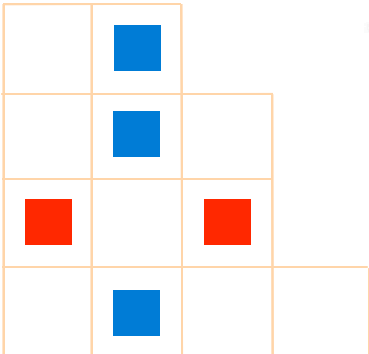


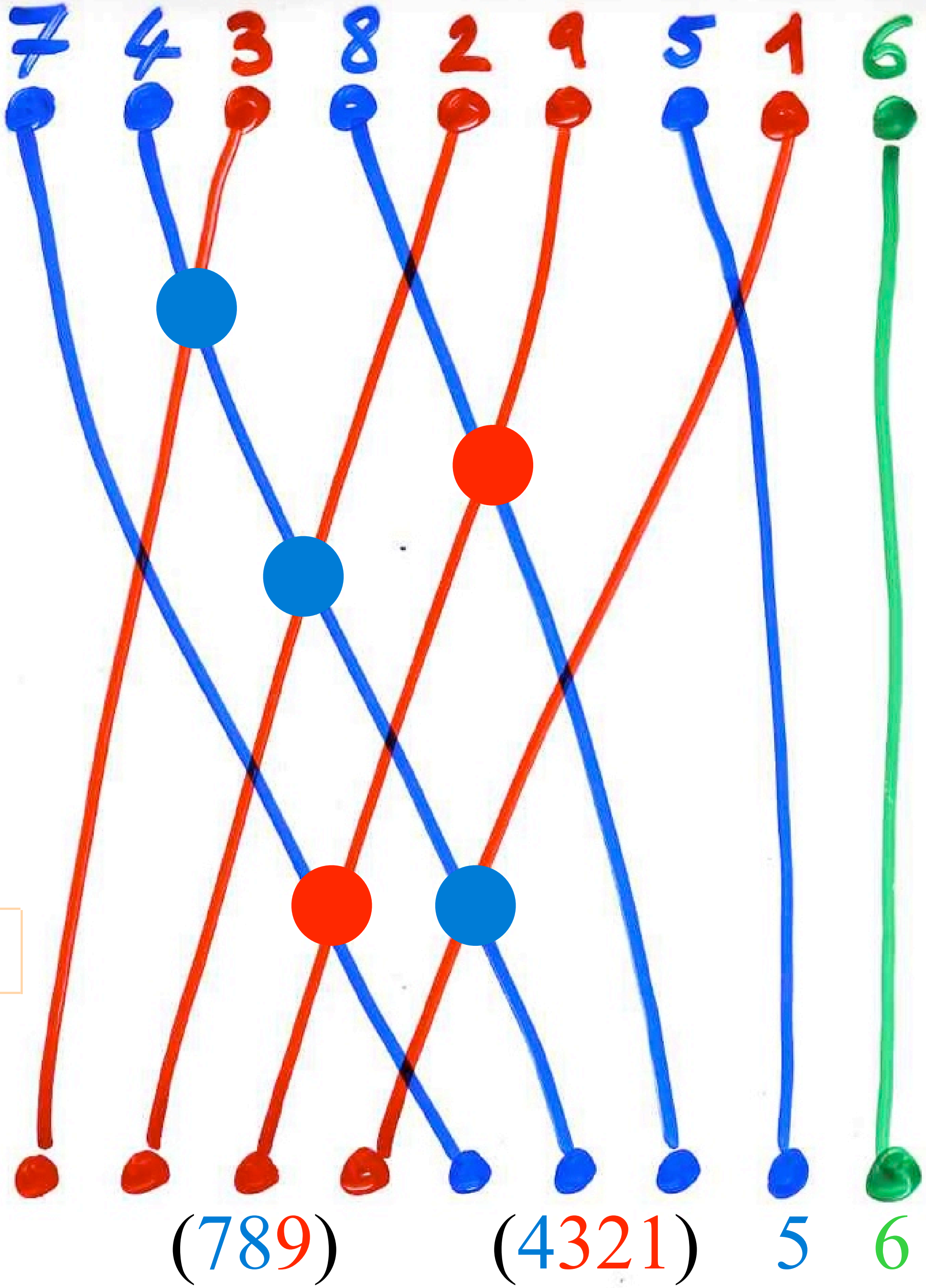
7 4 3 8 2 9 5 1 6
7
(89)
(4321)
5



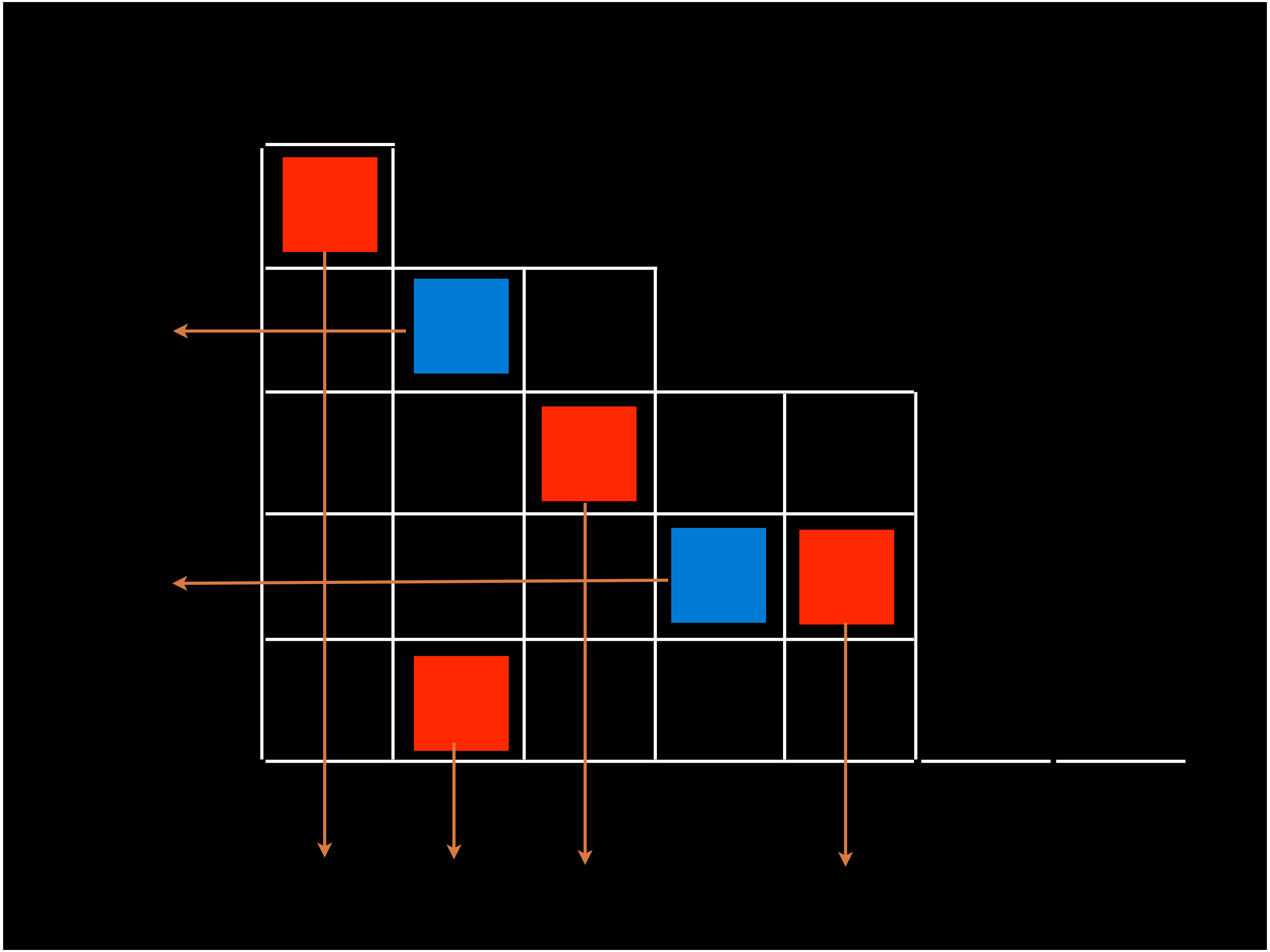
7 4 3 8 2 9 5 1 6
(789)
(4321)
5

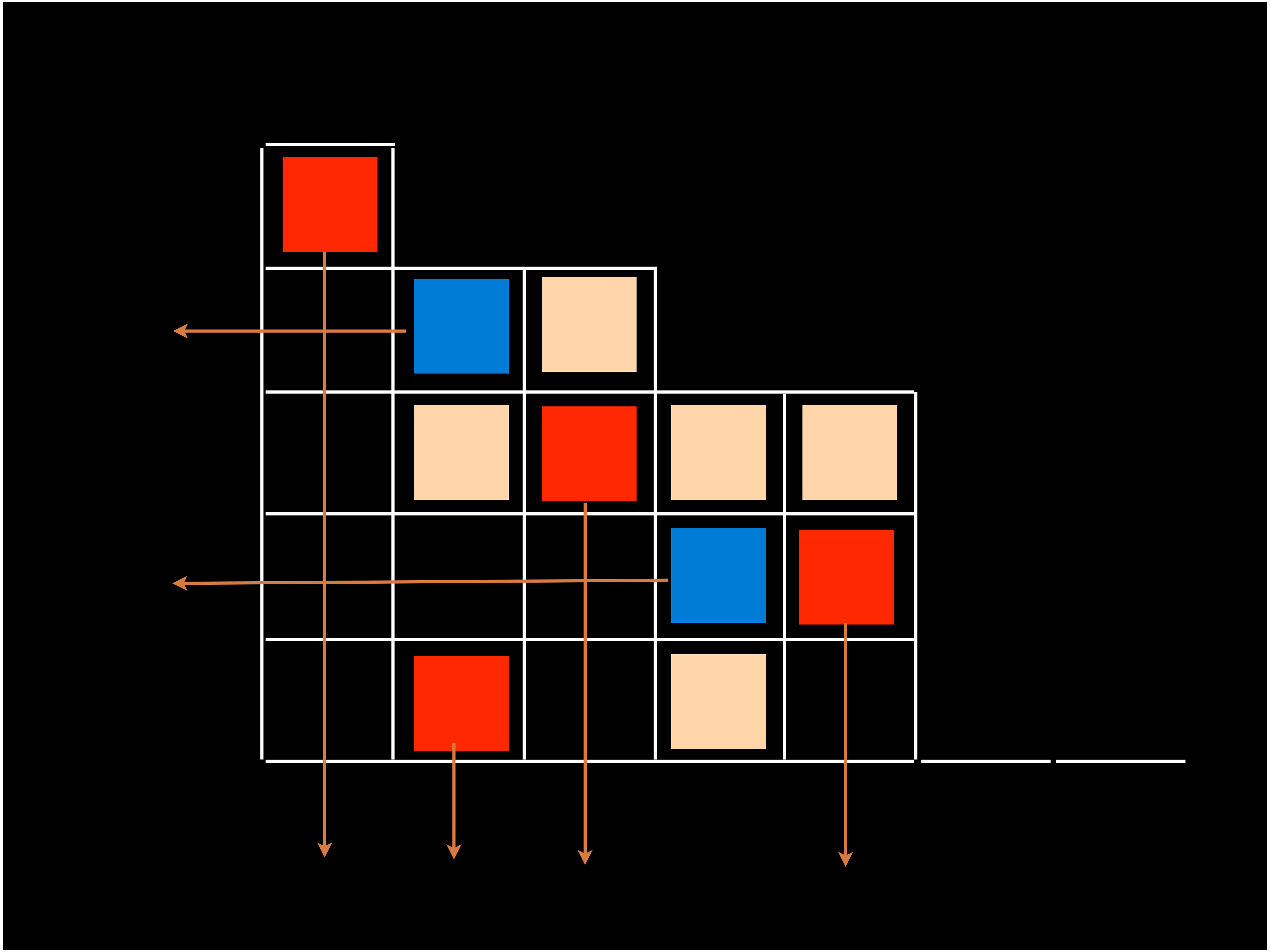
“exchange-fusion” algorithm





§ 5 Catalan alternative tableaux

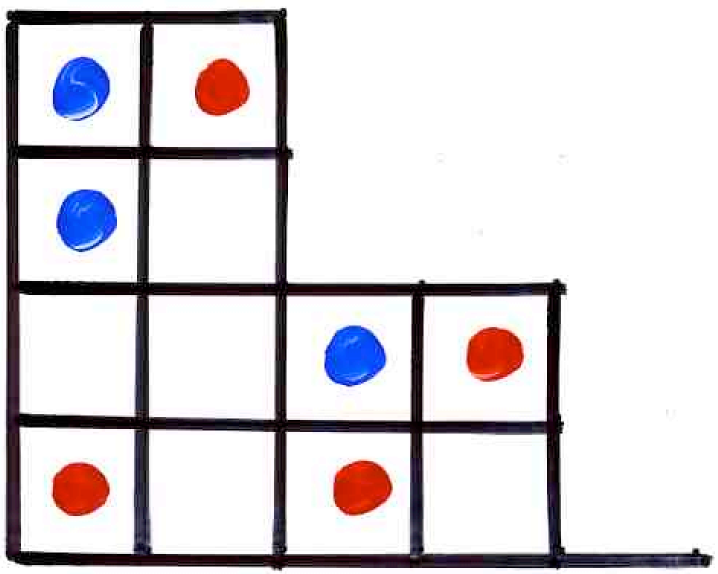




Def Catalan alternative tableau T

alt. tab. without cells ☒

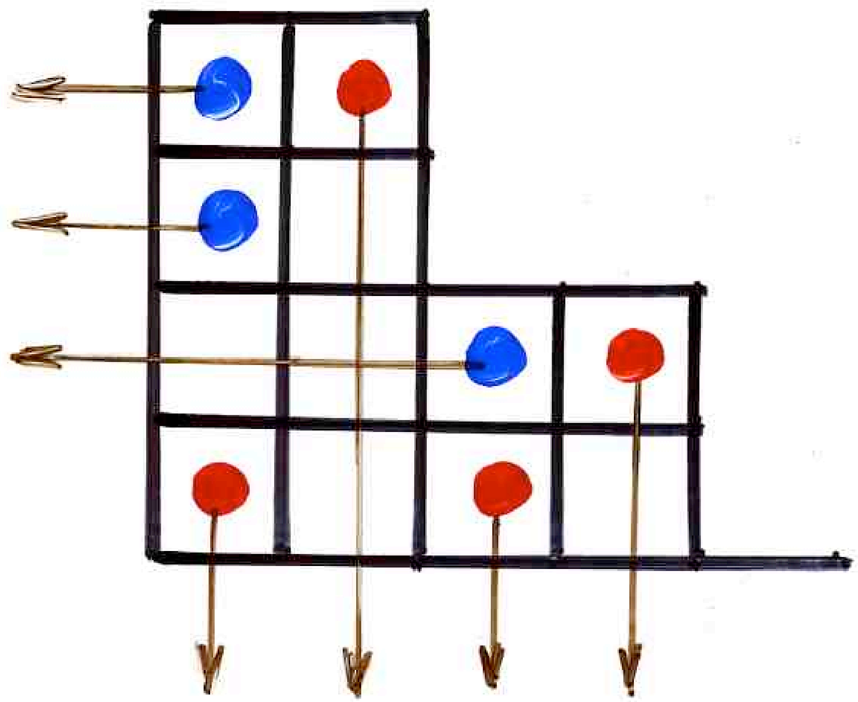
i.e: every empty cell is below a red cell or on the left of a blue cell



Def Catalan alternative tableau T

alt. tab. without cells ☒

i.e: every empty cell is below a red cell or on the left of a blue cell



Lemma. σ (permutation) $\Leftrightarrow$ T alternating tableau

T has no crossing

$\Leftrightarrow$ σ has no subsequence of type

$(y+1) \cdots x \cdots y \cdots (x+1)$

Prop. (O. Bernardi, 2008)

The number of such permutations on n elements is C_n Catalan number

ex: $\sigma = 6\ 4\ 5\ 3\ 9\ 7\ 8\ (10)\ 1\ 2$