The combinatorics of some exclusion model in physics

Roma

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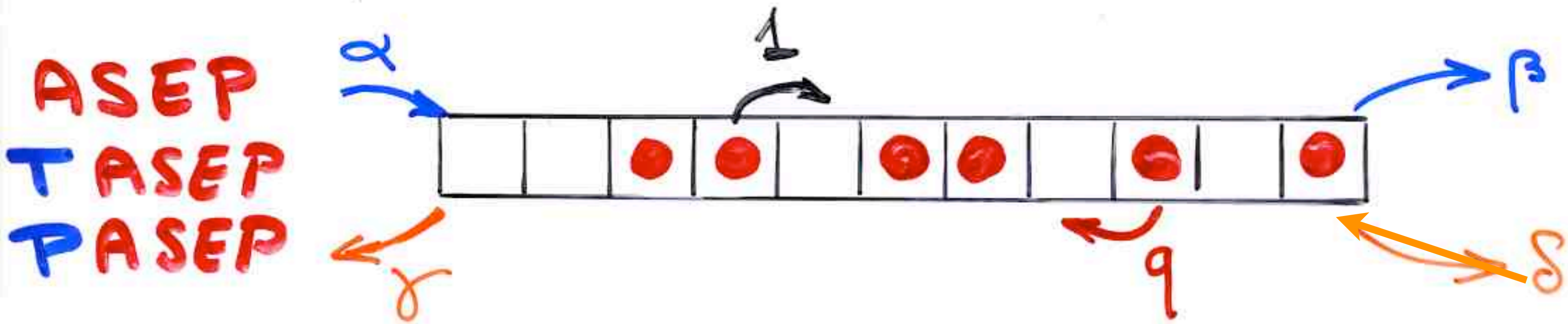
The PASEP algebra

$$DE = qED + E + D$$

The PASEP

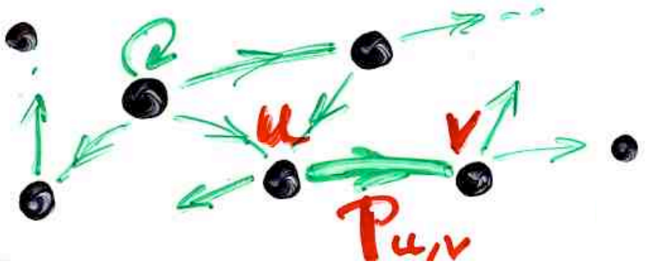

ASEP
TASEP
PASEP
α
1
β
γ
q
δ

Markov chains

2^n states — $w = (\tau_1, \ldots, \tau_n)$

$\tau_i = \begin{cases} 0 \\ 1 \end{cases}$

graph



probability

S : states

$M = (p_{u,v})_{u,v \in S}$ probabilities matrix (stochastic)

$\pi = (p_u, \dots)$ vector (time t)

$\pi \cdot M$ vector (time t+1)

$$p_v^{(t+1)} = \sum_u p_u^{(t)} \, p_{u,v}$$

$t+1$ time t

stationary probabilities

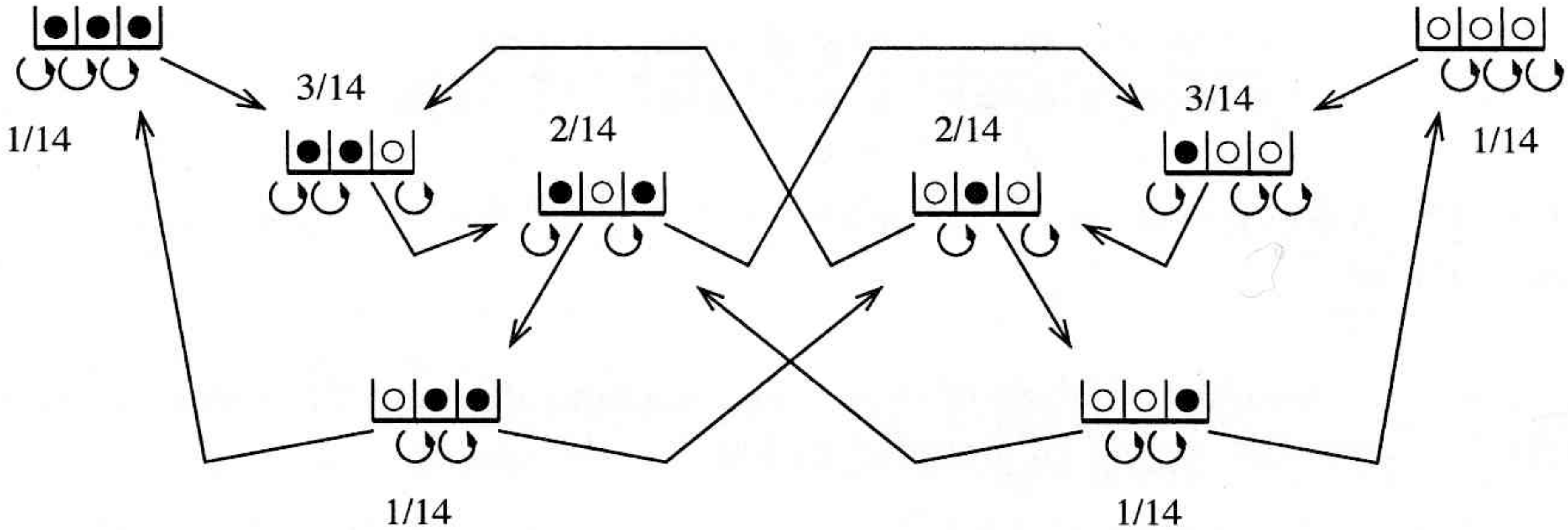
$$\pi \cdot M = \pi$$

unicity

eigenvector $\qquad M^T$ eigenvalue 1

time $\longrightarrow \infty$

$$p_v = \sum_u p_u \, p_{u,v}$$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non-equilibrium statistical mechanics

... relaxation → stationary state

states $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$

$$\tau_i = \begin{cases} 1 & \text{site } i \text{ occupied} \\ 0 & \text{site } i \text{ empty} \end{cases}$$

unique stationary state $\quad \dfrac{d}{dt} P_n(\tau_1, \ldots, \tau_n) = 0$

Derrida, Evans, Hakim, Pasquier (1993)

boundary induced phase transitions

molecular diffusion

linear array of enzymes

biopolymers

traffic flow

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formation of shocks

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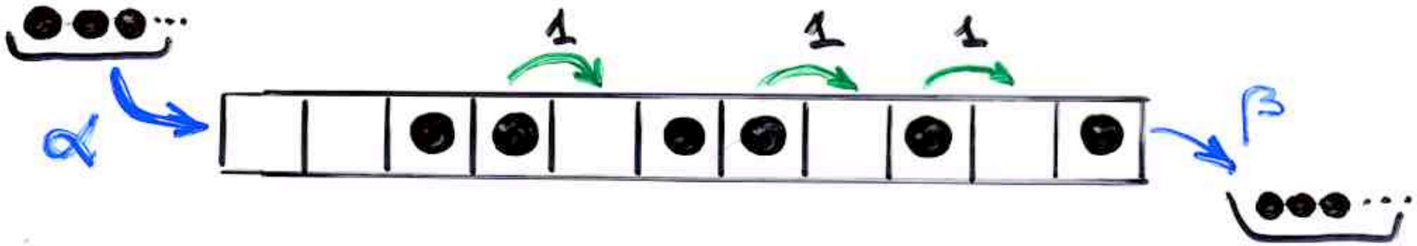
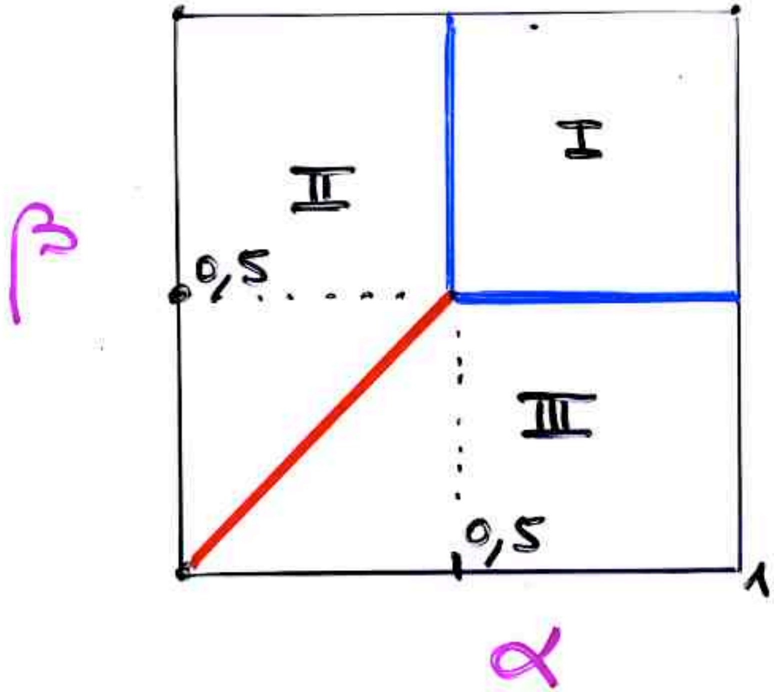
TASEP

"totally asymmetric exclusion process"





$n \to \infty$

$\rho = \langle \tau_i \rangle$ = taux moyen d'occupation

i loin des bords

(I) $\rho = 1/2$

(II) $\rho = \alpha$

(III) $\rho = 1-\beta$

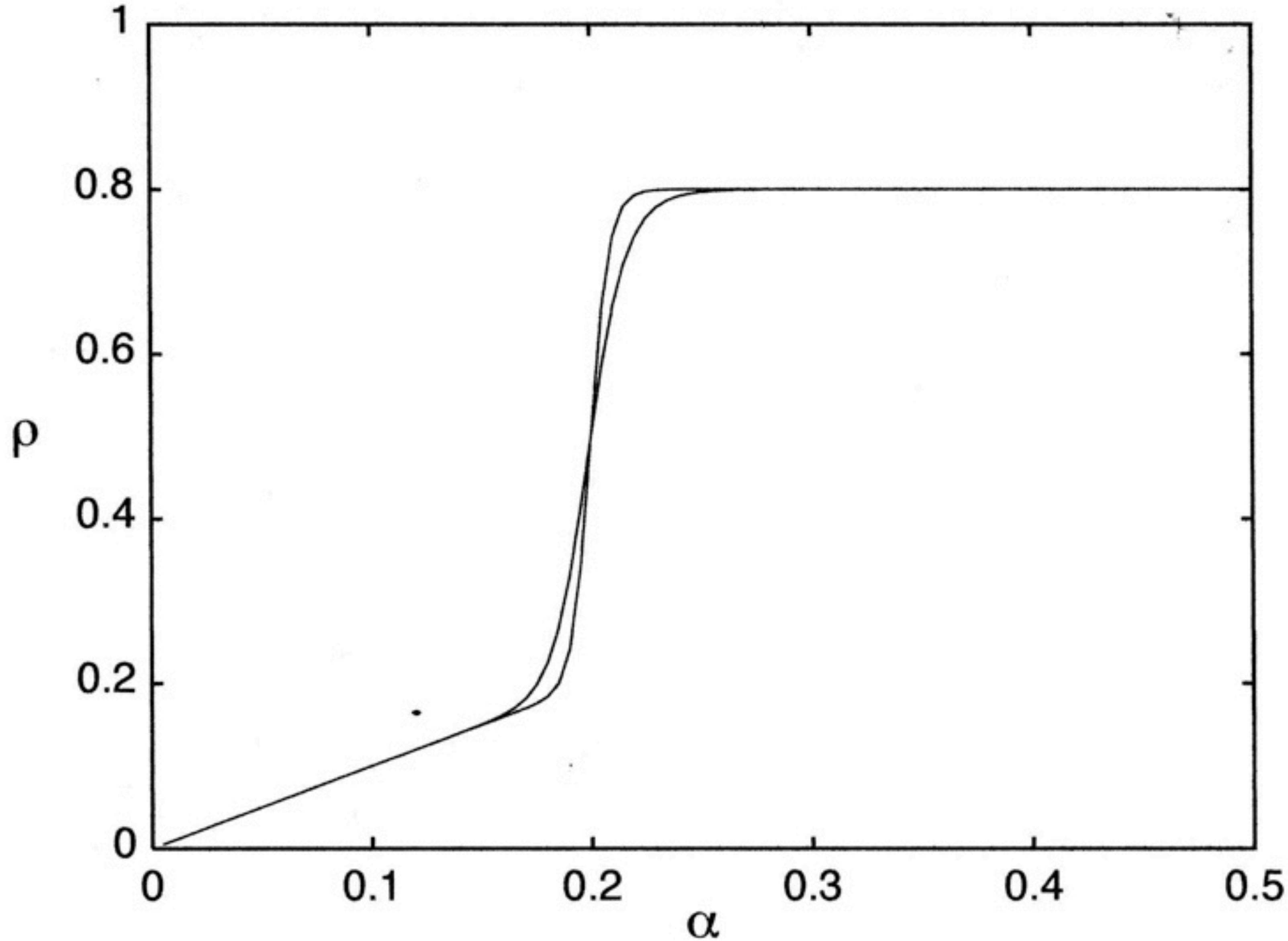
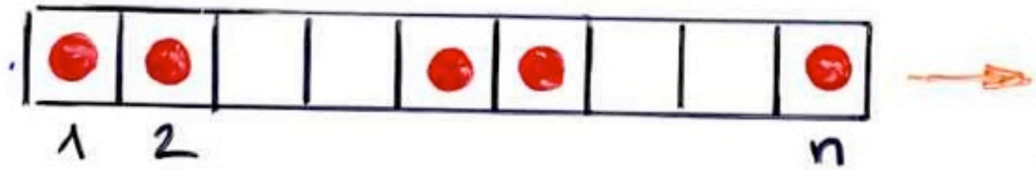
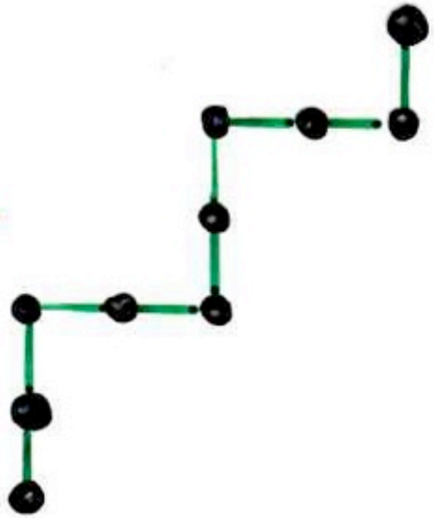


Figure 2: The average occupation $\rho = \langle \tau_{(N+1)/2} \rangle$ of the central site versus α for $N = 61$ and $N = 121$ when $\beta = .2$.

state $s = (\tau_1, \dots, \tau_n)$

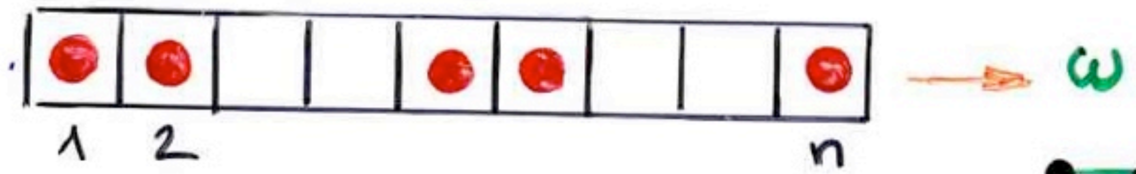


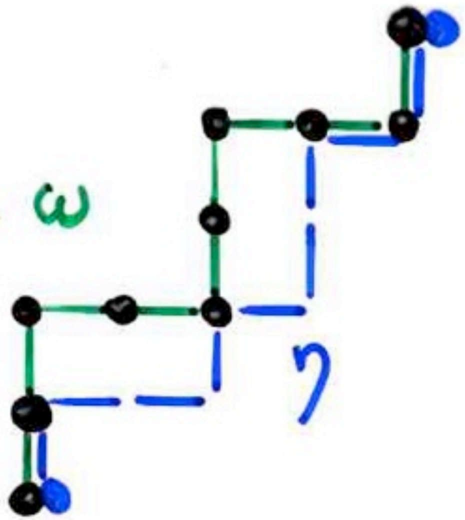


$\mathcal{P}_n(s) =$

Shapiro, Zeilberger, 1982

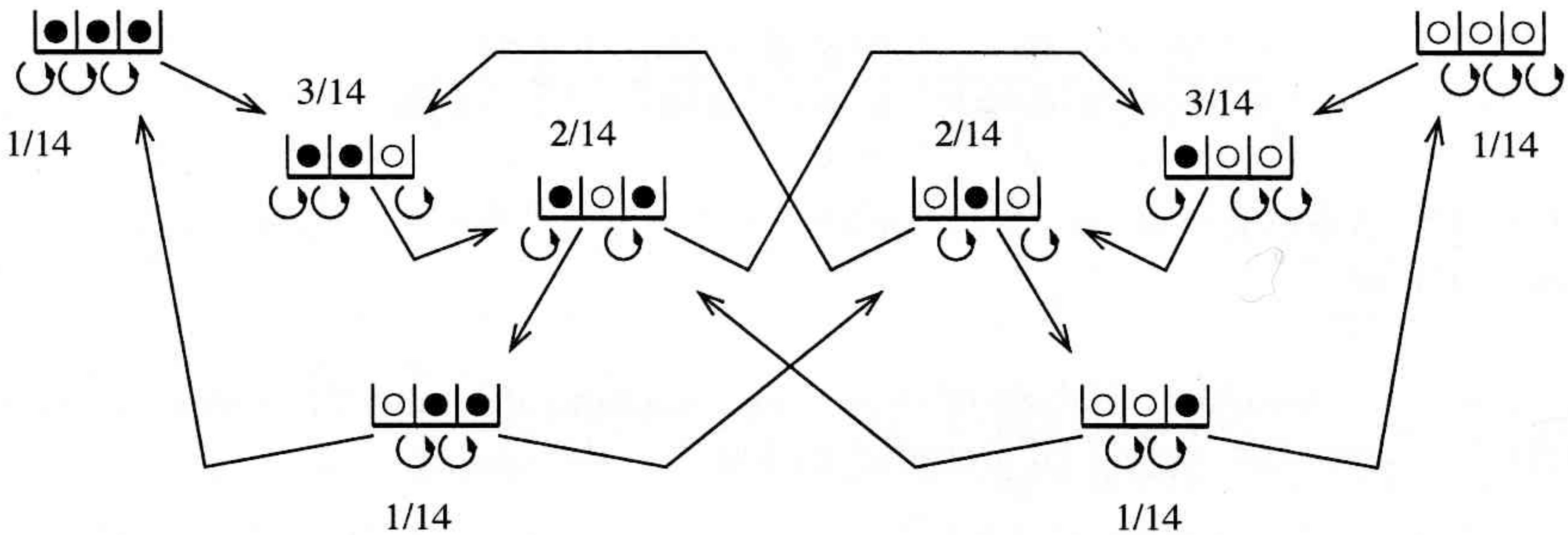
state $s = (\tau_1, \dots, \tau_n)$





$$\mathcal{P}_n(s) = \frac{1}{C_{n+1}} \left(\begin{array}{c} \text{number of paths } \eta \\ \text{below the path } \omega \\ \text{associated to } s \end{array} \right)$$

Shapiro, Zeilberger, 1982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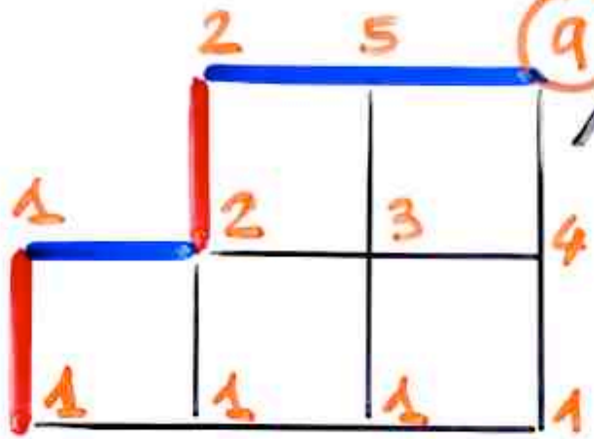
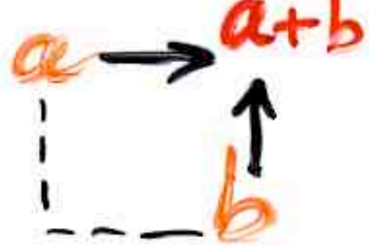
$\delta = (1,0,1,0,0) \quad \lambda = (1,2,2)$



$$\frac{9}{C_6} = P(1,0,1,0,0)$$



Combinatorics of the PASEP

TASEP

Brak, Essam (2003), Duchi, Schaeffer, (2004),
Angel (2005), XGV, (2007)

(P) ASEP

Brak, Corteel, Essam, Parviainen, Rechnitzer (2006)
Corteel, Williams (2006) (2008) (2009) XGV, (2008)
Corteel, Stanton, Stanley, Williams (2010)

Derrida, ...
Mallick, ... Golinelli, Mallick (2006)

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q- Hermite polynomial

$\alpha, \beta, q \qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$
$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^+$$

$$\hat{a}\,\hat{a}^+ - q\,\hat{a}^+\,\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \qquad q$

Askey-Wilson polynomials

The Matrix Ansatz

Derrida, Evans, Hakim, Pasquier 1993

$$\mathbb{P}_n(\tau_1, \ldots, \tau_n) = f_n(\tau_1, \ldots, \tau_n) / Z_n$$

$$Z_n = \sum_{\tau} f_n(\tau_1, \ldots, \tau_n)$$

partition function

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

$$\begin{cases} DE = q\,ED + D + E \\ (\beta D - \delta E)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \gamma D) = \langle W| \end{cases}$$

Then

$f_n(\tau_1, \dots, \tau_n)$

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

TASEP

q=0

$$\begin{cases} DE = \boxed{} + D + E \\ (\beta D - \blacksquare)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \blacksquare) = \langle W| \end{cases}$$

Then

$$f_n(\tau_1, \dots, \tau_n) = \langle W| \prod_{i=1}^{n} (\tau_i D + (1-\tau_i) E) |V\rangle$$

examples:

TASEP

$$\begin{cases} DE = D + E \\ D|V\rangle = \bar{\beta}\,|V\rangle \\ \langle W|E = \bar{\alpha}\,\langle W| \end{cases}$$

examples:

TASEP

$$D = \begin{bmatrix} 0 & \bar{\beta} & 0 & \cdots \\ & 0 & \bar{\beta} & \ddots \\ & & \ddots & \ddots \\ 0 & & & 0 \end{bmatrix} \qquad E = \begin{bmatrix} \bar{\alpha} & 0 & \cdots & 0 \\ \bar{\alpha}\beta & 1 & \ddots & \vdots \\ \bar{\alpha}\beta^2 & \beta & 1 & 0 \\ \bar{\alpha}\beta^3 & \beta^2 & \beta & 1 \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}$$

(infinite matrices) $\bar{\beta} = \frac{1}{\beta}$, $\bar{\alpha} = \frac{1}{\alpha}$

$$\langle W| = (1, 0, \dots, 0, \dots) \qquad |V\rangle = (1, 1, \dots, 1, \dots)^T$$

$$D = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ & 0 & 1 & \ddots \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} \qquad E = \begin{bmatrix} \bar{\beta} & & & \\ \bar{\beta} & 1 & & \\ \vdots & & \ddots & \\ \bar{\beta} & 1 & \cdots & 1 \end{bmatrix}$$

(infinite matrices) $\bar{\beta} = \frac{1}{\beta}$, $\bar{\alpha} = \frac{1}{\alpha}$

$$\langle W| = (1, 0, \dots) \qquad |V\rangle = (1, \bar{\alpha}, \bar{\alpha}^2, \dots)^T$$

examples:

TASEP

$$D = \begin{bmatrix} \bar{\beta} & \kappa & & & 0 \\ 0 & 1 & 1 & & \\ & & \ddots & \ddots & \\ & 0 & & 1 & 1 \\ 0 & & & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} \bar{\alpha} & 0 & & & 0 \\ \kappa & 1 & & 0 & \\ 0 & 1 & \ddots & & \\ & 0 & \ddots & \ddots & 0 \\ 0 & & 0 & 1 & 1 \end{bmatrix}$$

(infinite matrices)

$$\langle W| = (1, 0, \cdots, 0) \qquad |V\rangle = (1, 0, \cdots, 0)$$

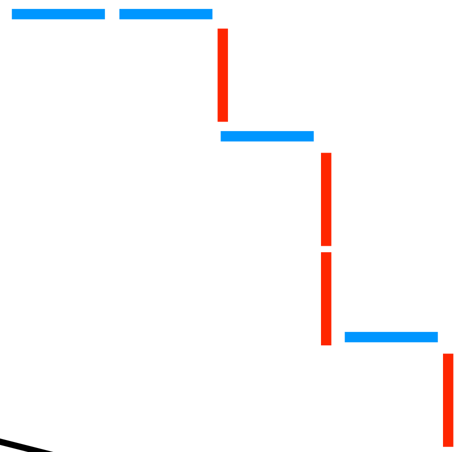
$$\bar{\alpha} = \frac{1}{\alpha} \qquad \bar{\beta} = \frac{1}{\beta} \qquad \kappa^2 = \bar{\alpha} + \bar{\beta} - \bar{\alpha}\bar{\beta}$$

The PASEP algebra

$$DE = qED + E + D$$

$$w = \sum_{i,j\geq 0} c_{i,j} E^i D^j$$

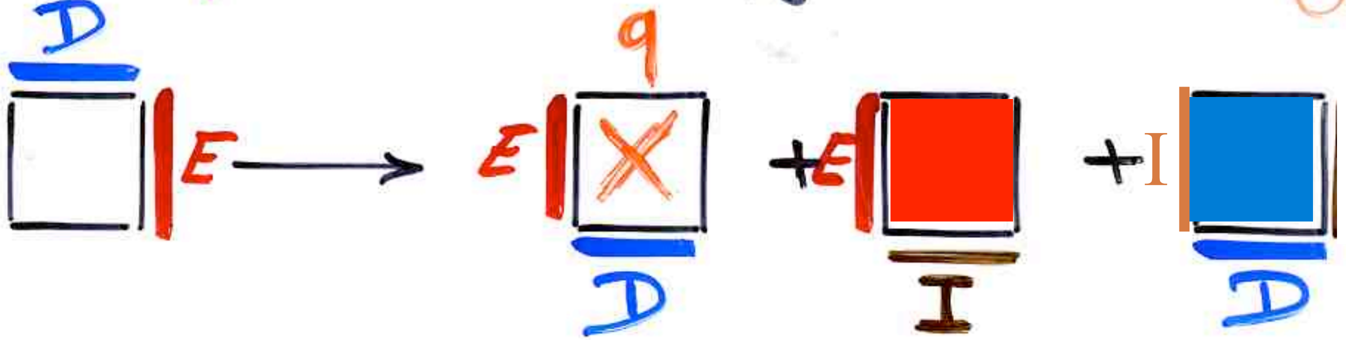
D D E D E E D E



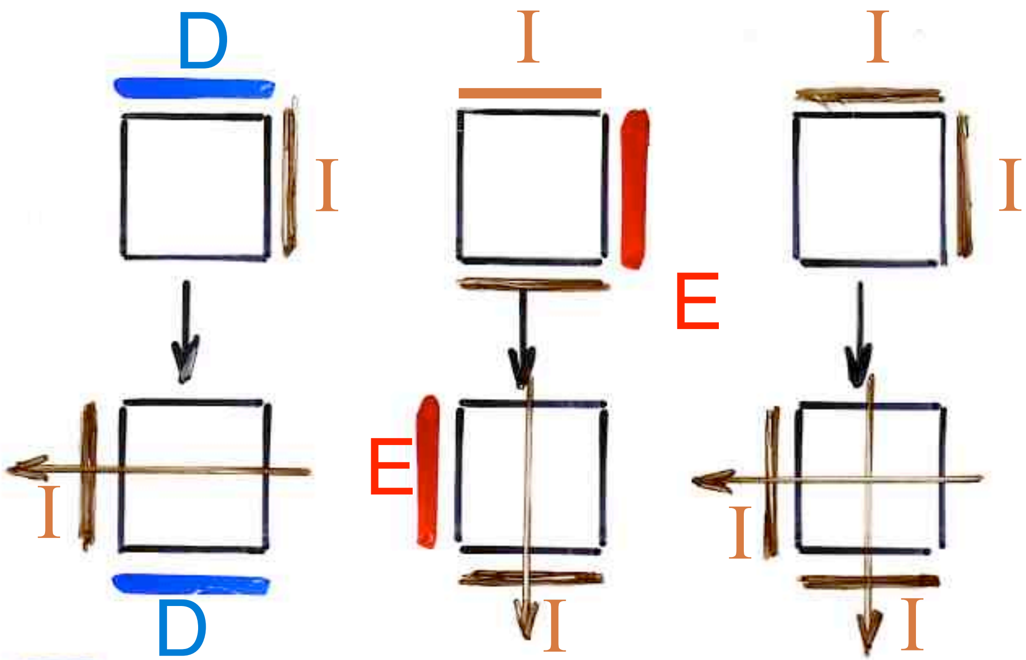
D DE (D E) E D E

DDE(E)EDE $+_q$ DDE(ED)EDE + DDE(D)EDE

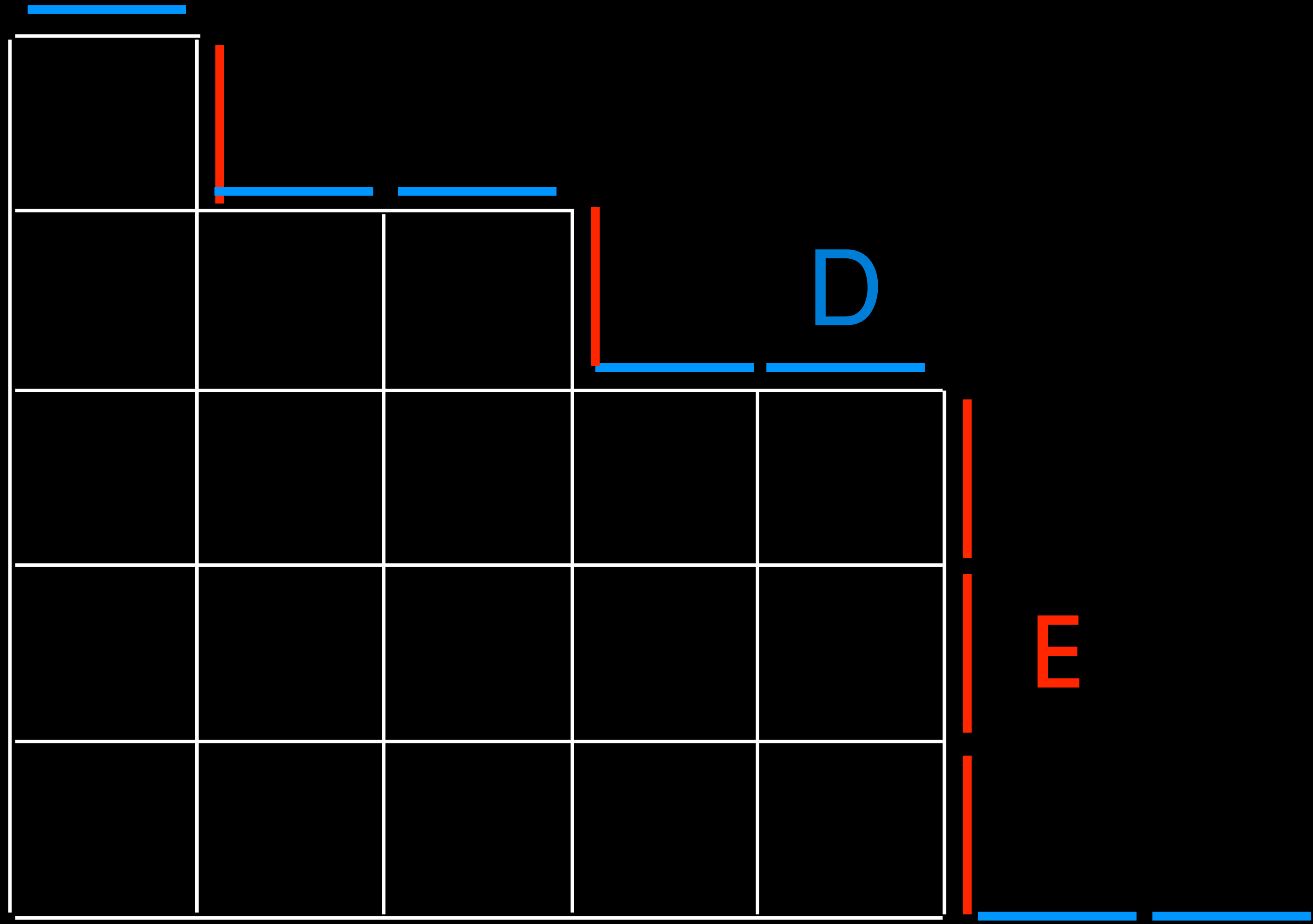
Proof: "planarization" of the rewriting rules



I identity



$$DE = qED + EI_h + I_vD$$
$$DI_v = I_vD$$
$$I_hE = EI_h$$
$$I_hI_v = I_vI_h$$



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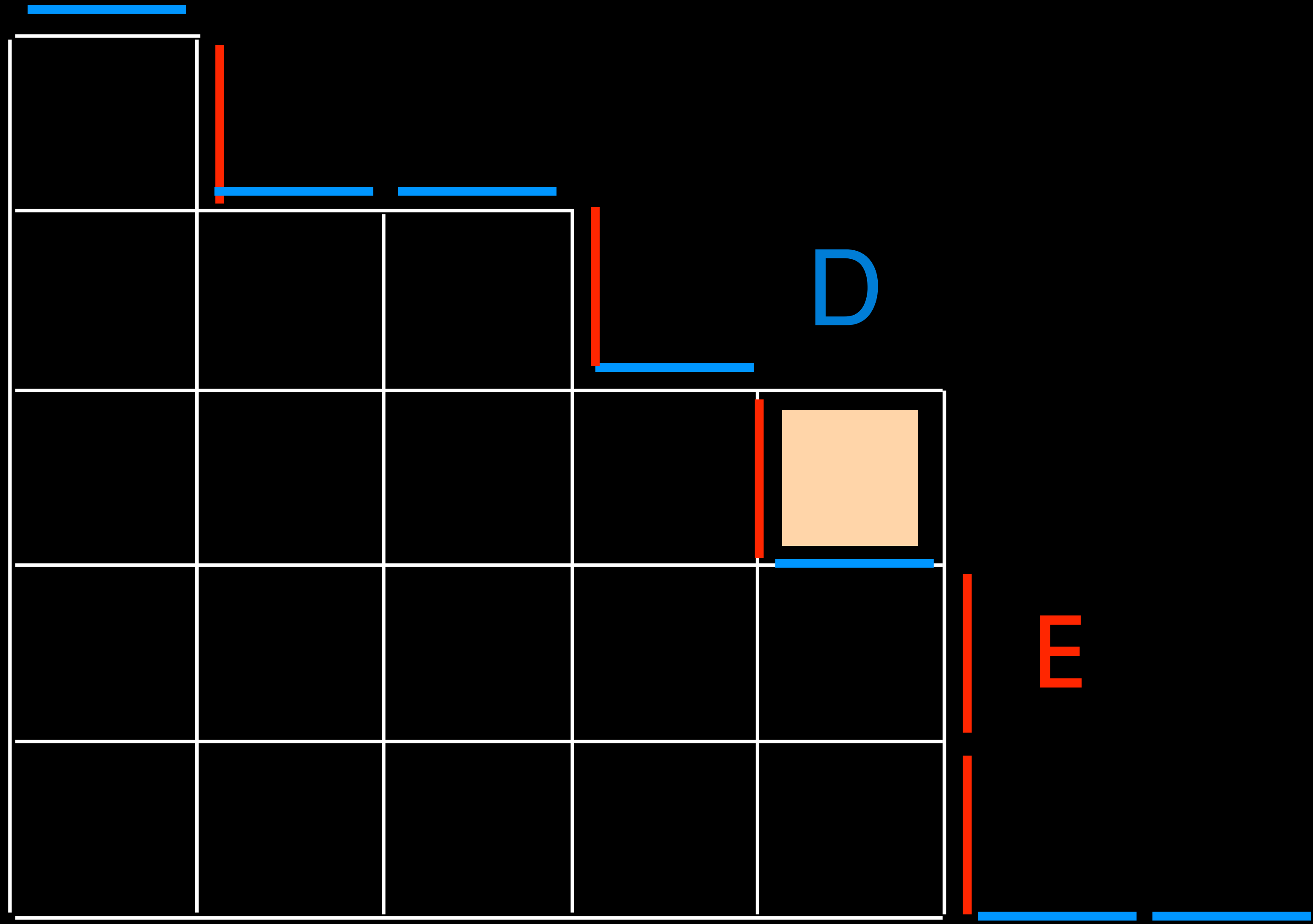
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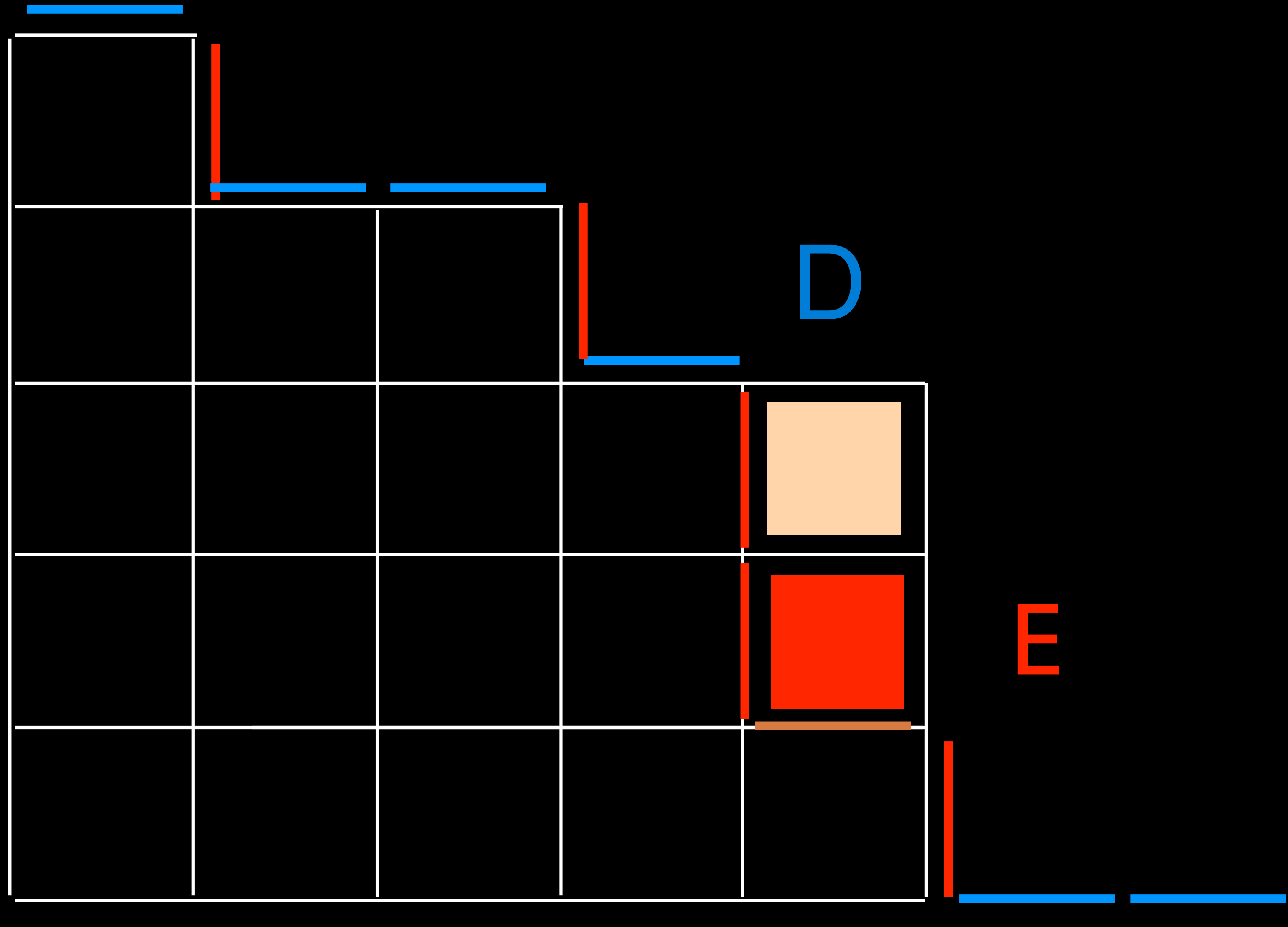
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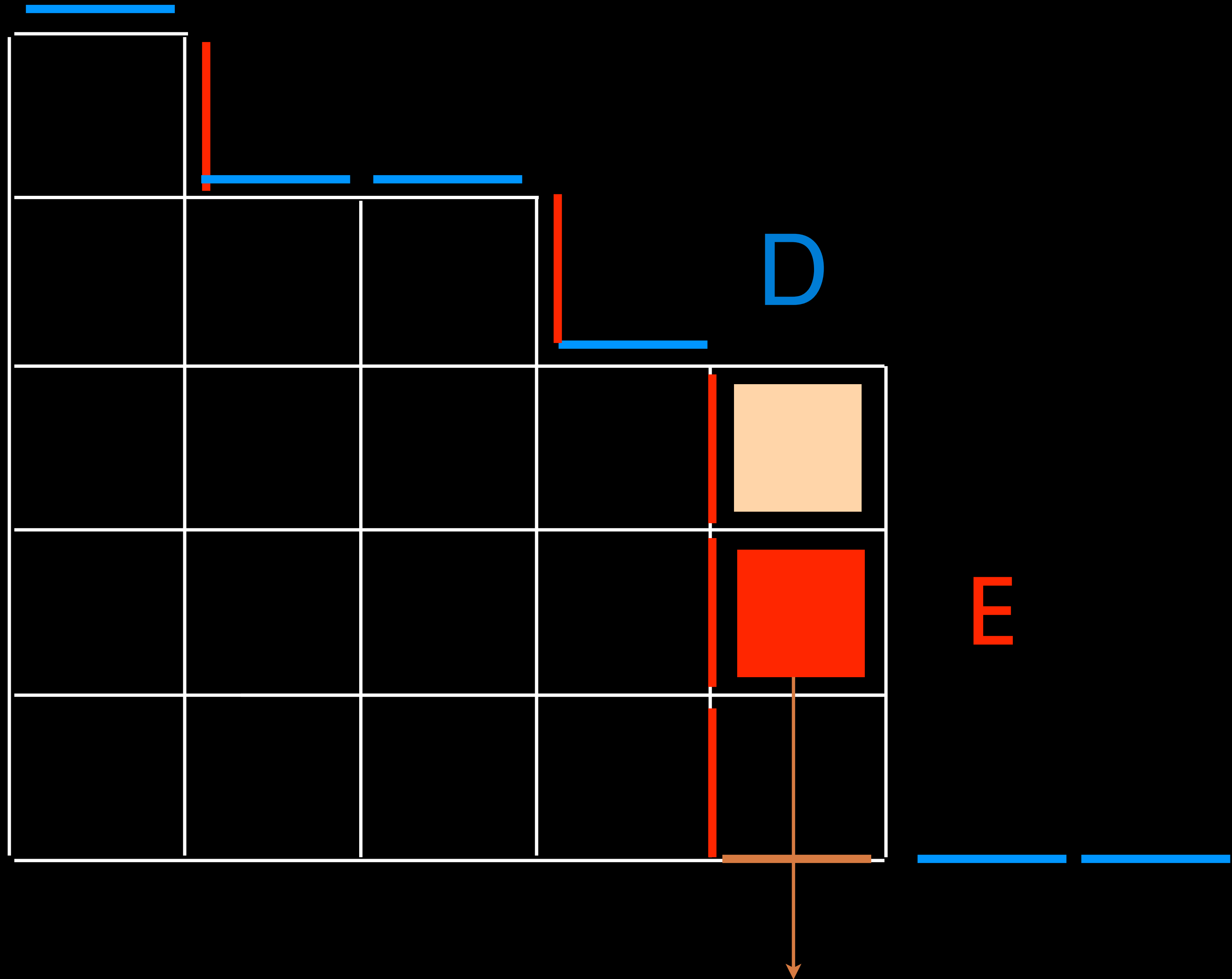
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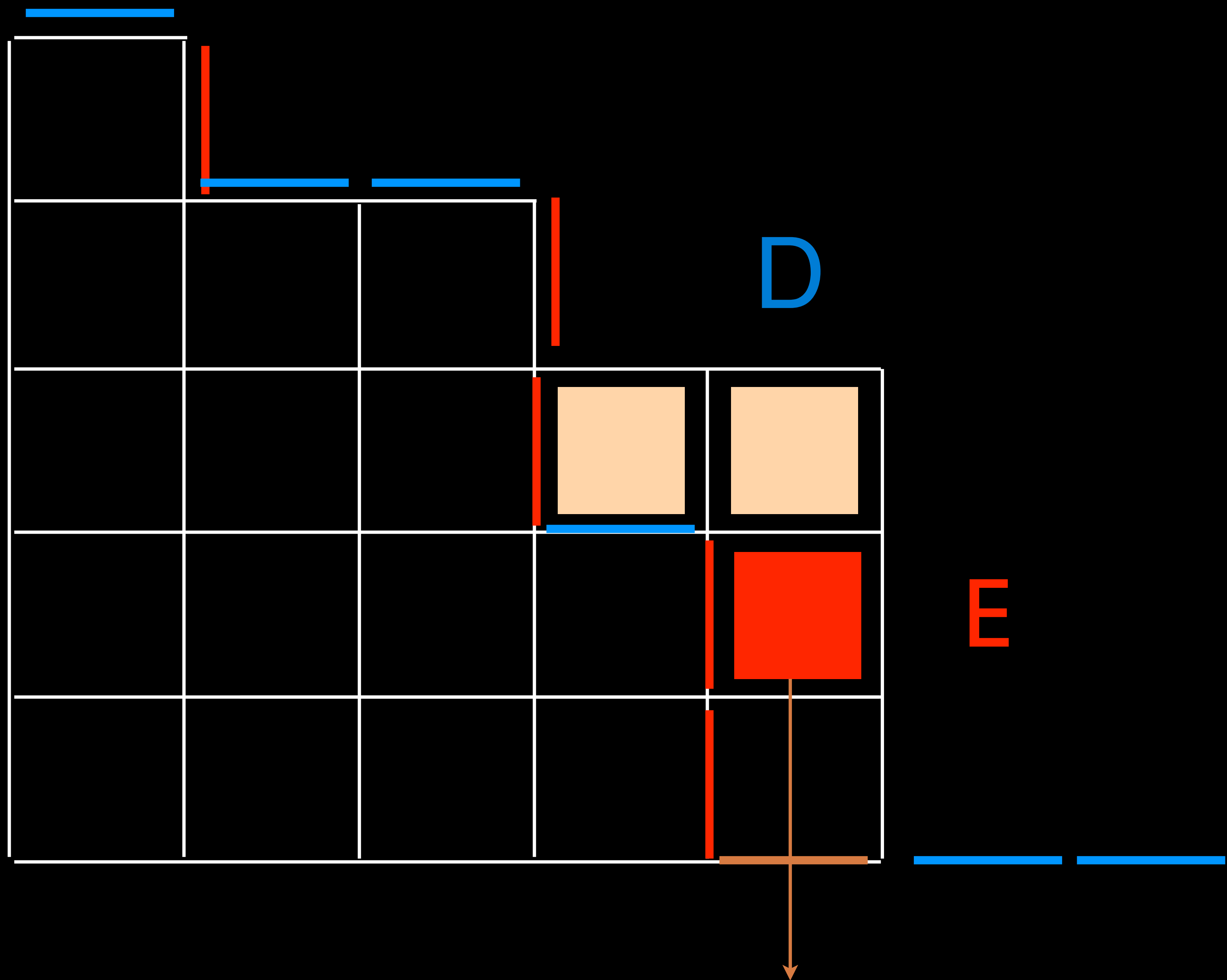
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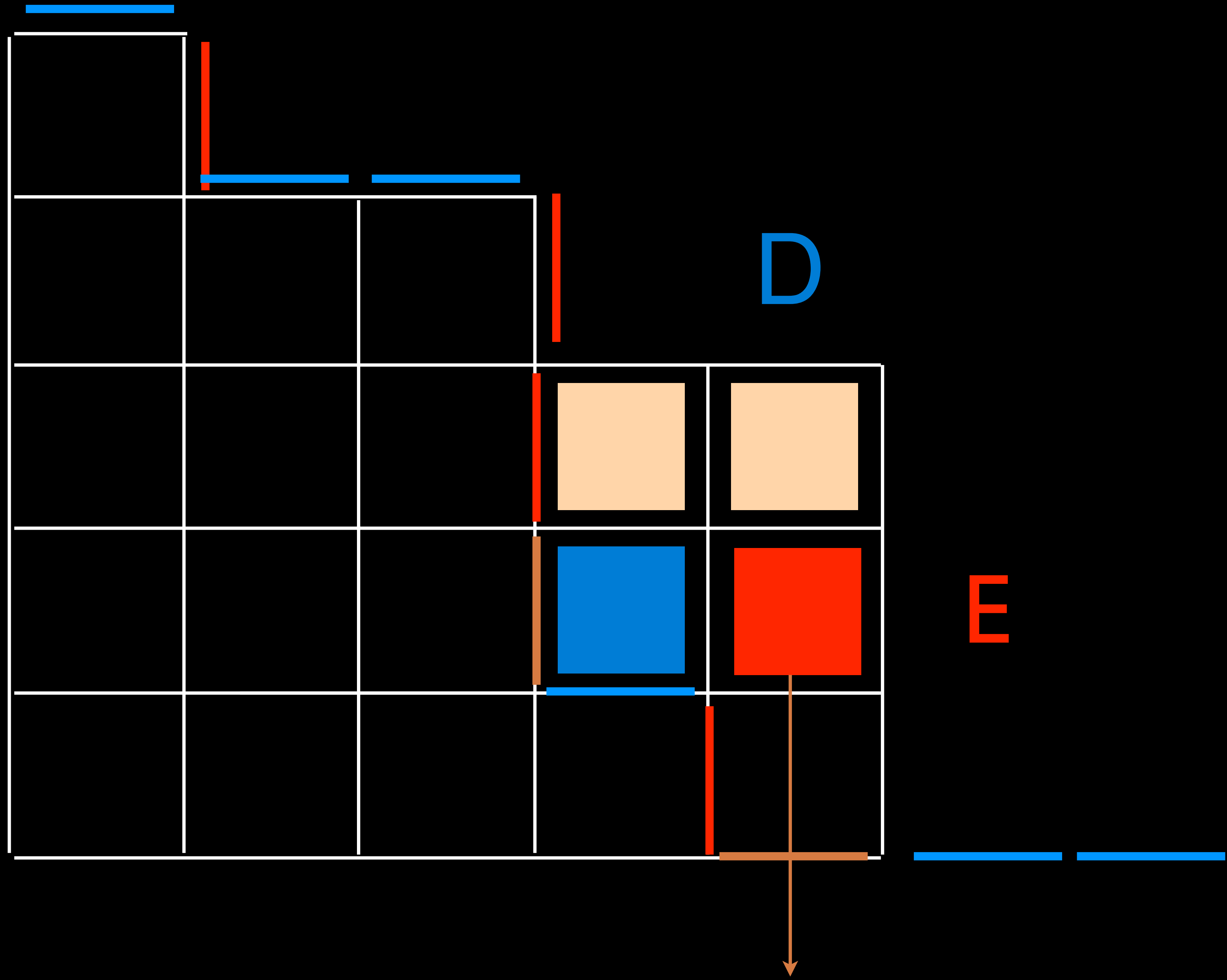
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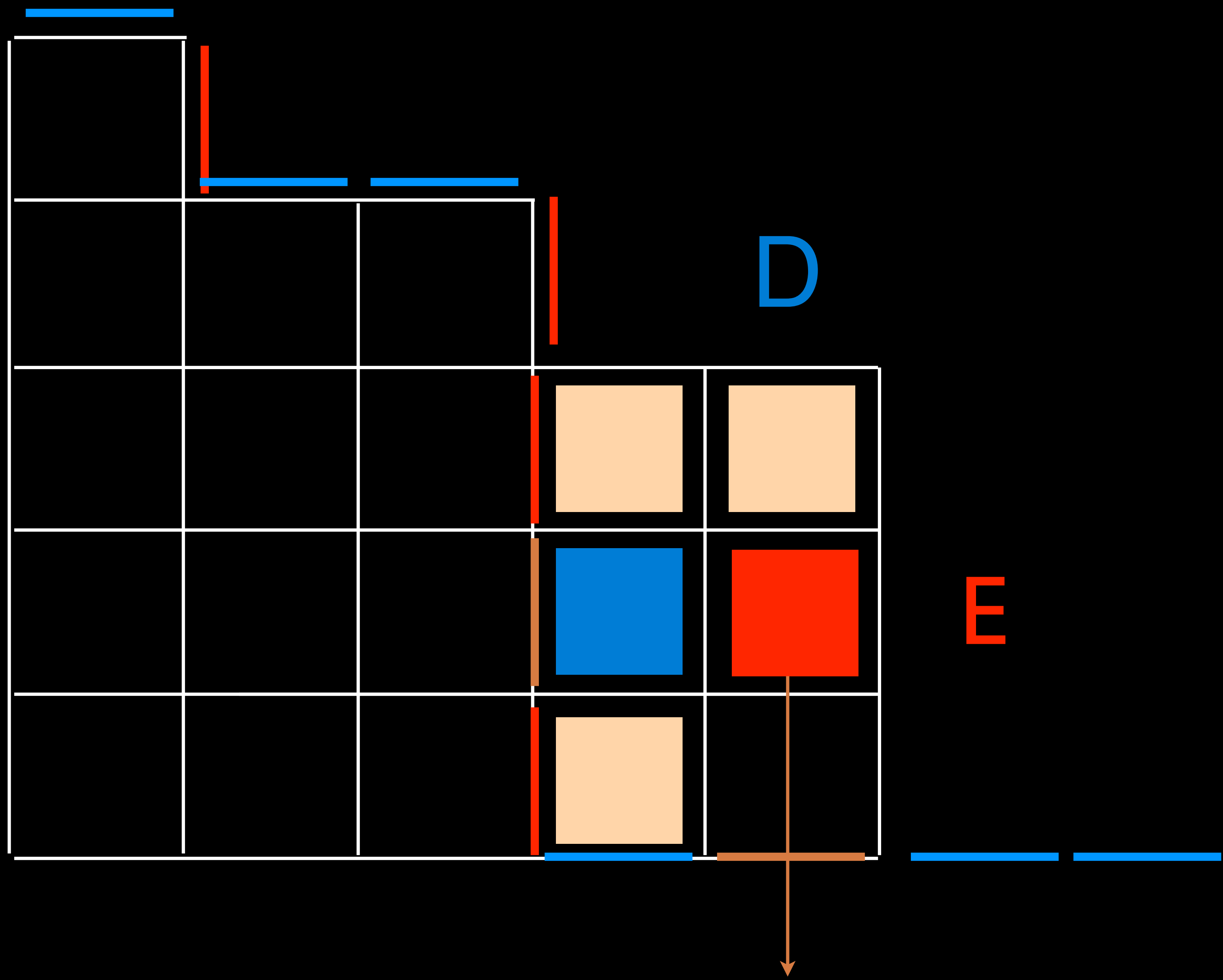
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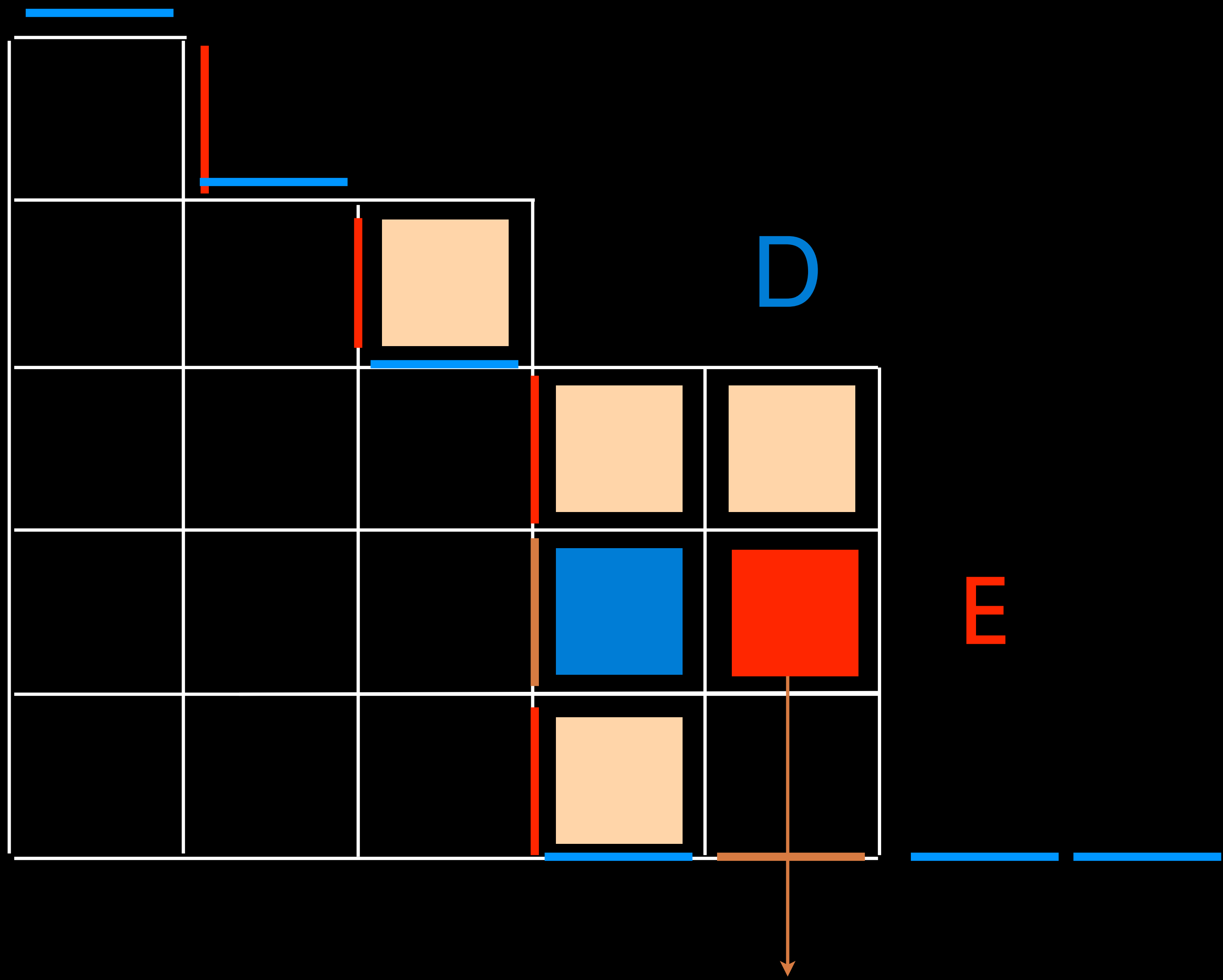

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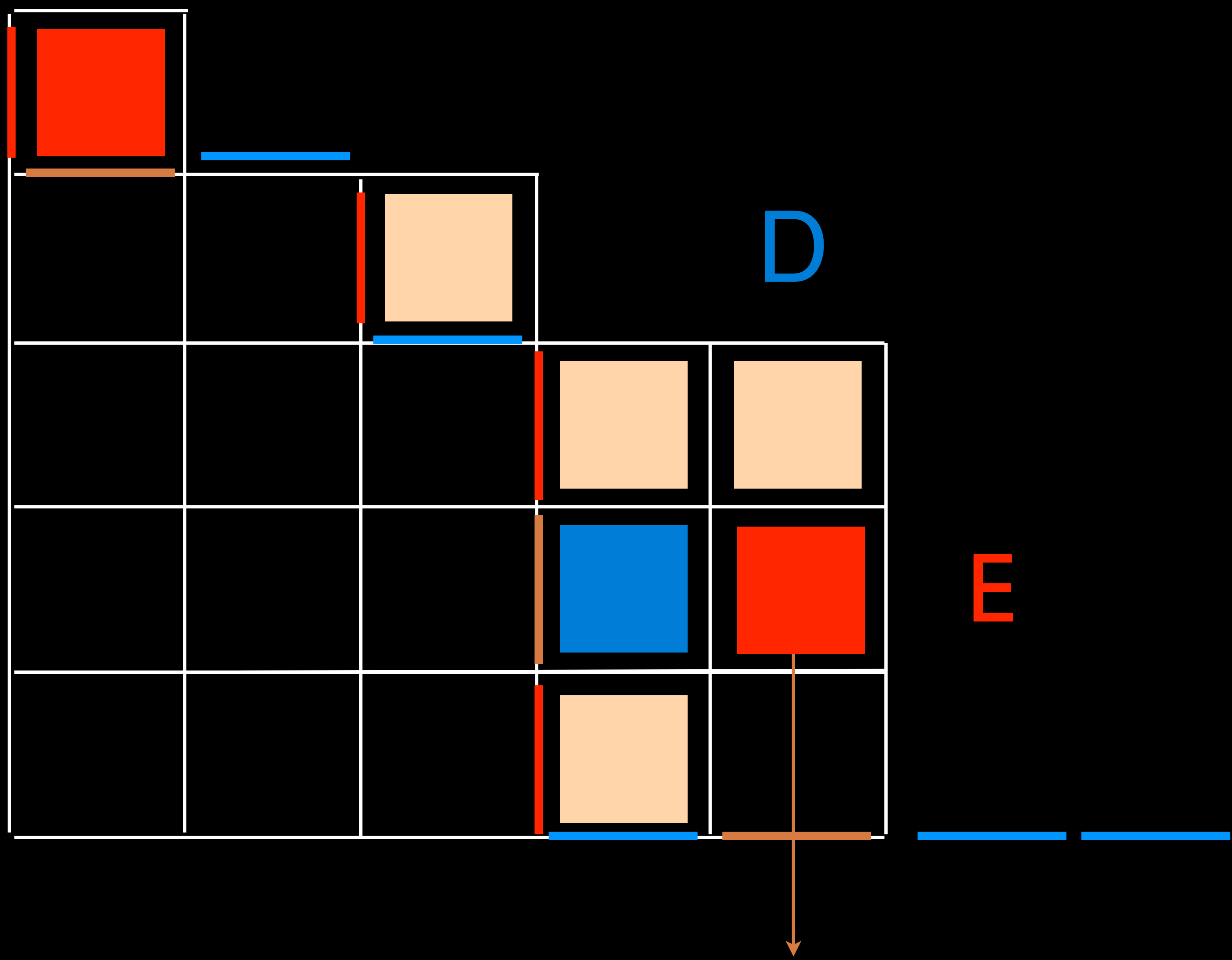
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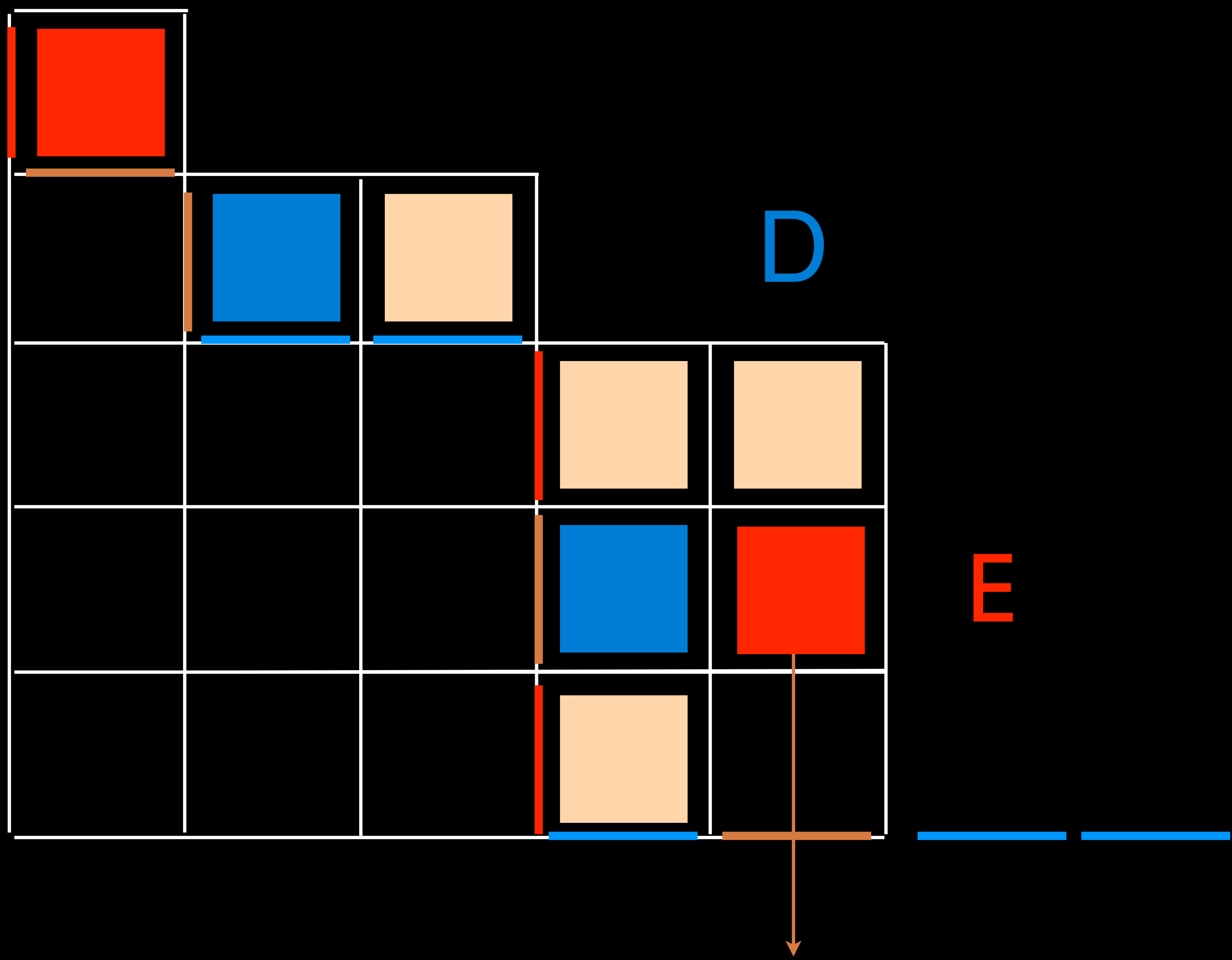
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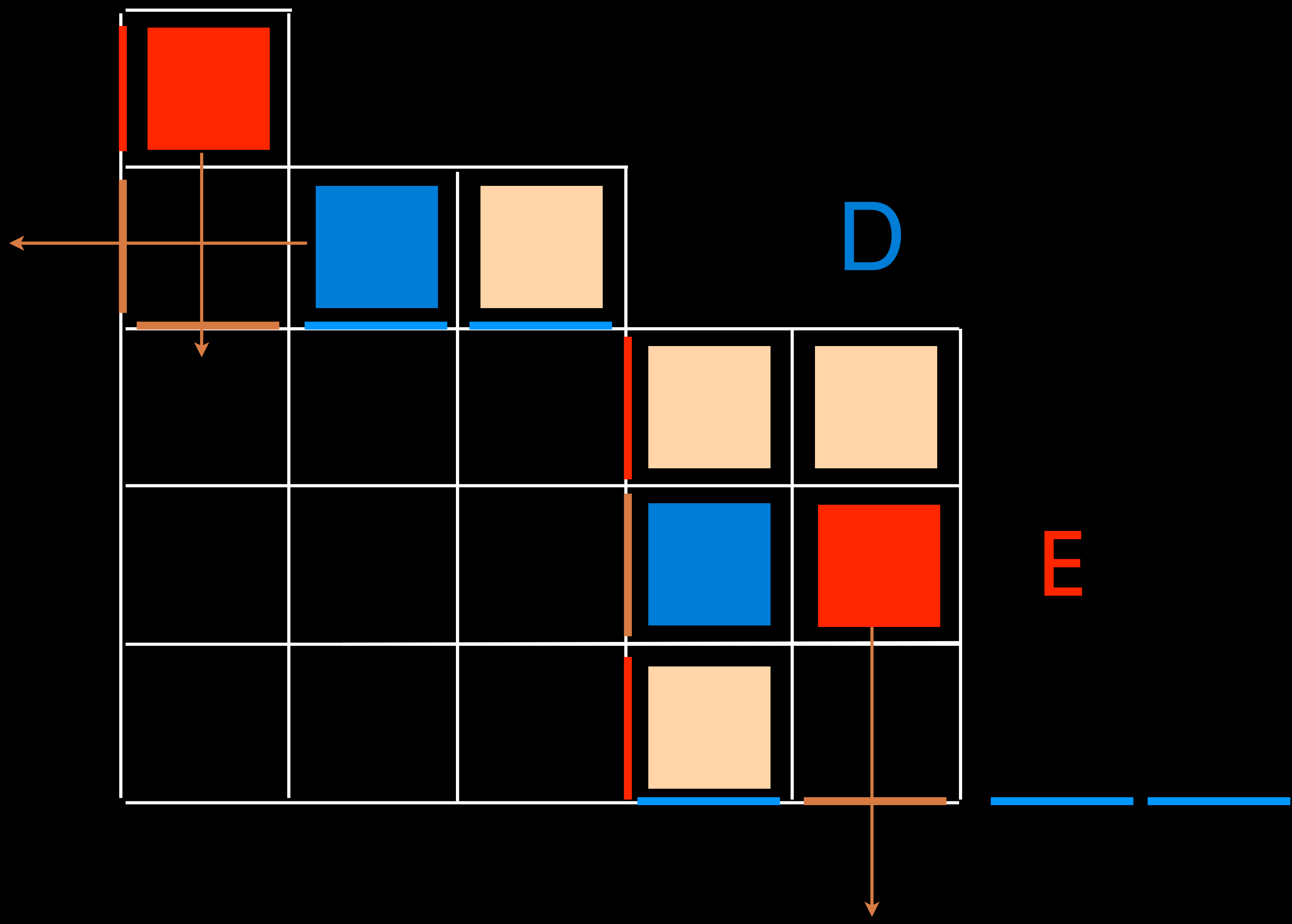
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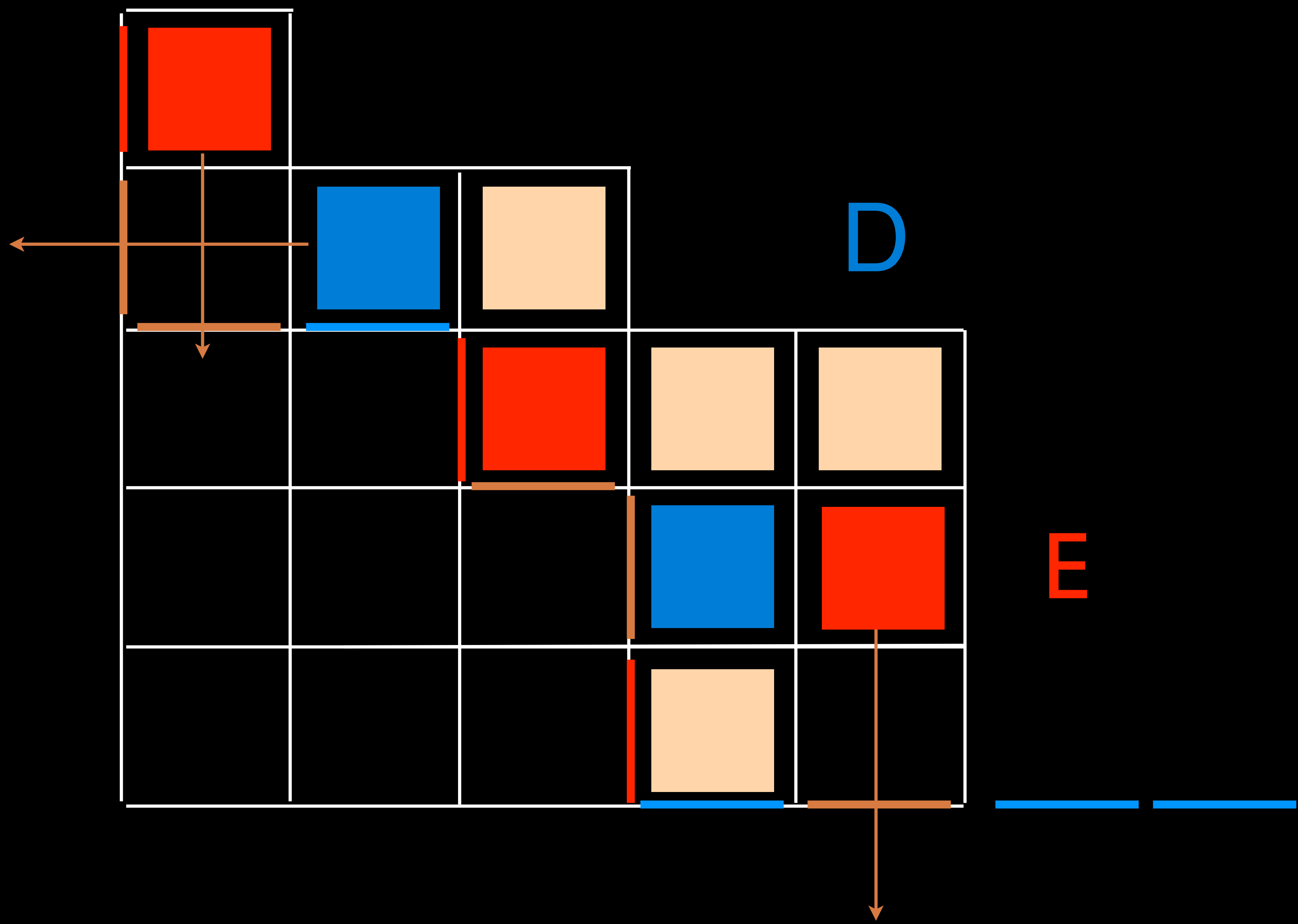
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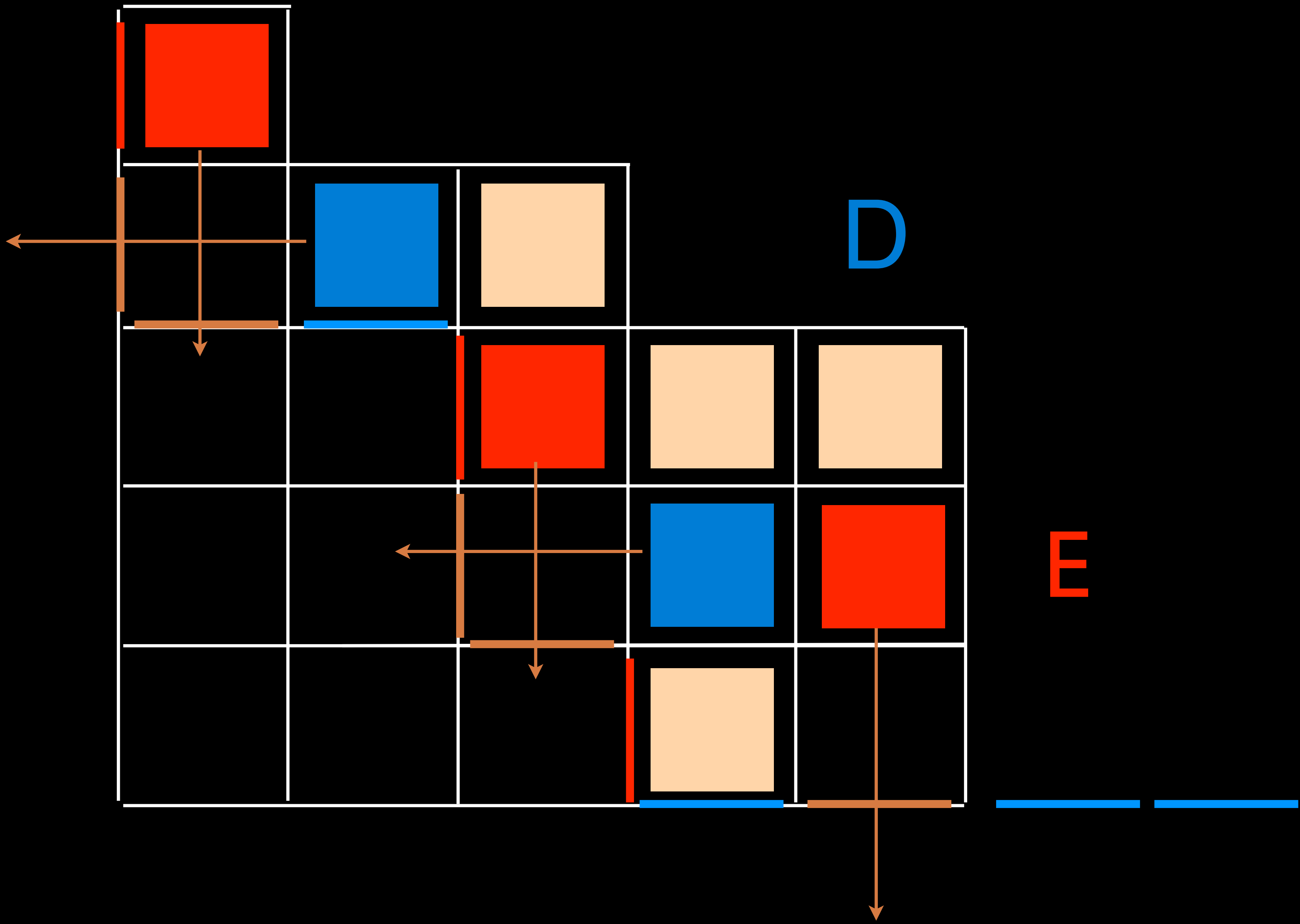
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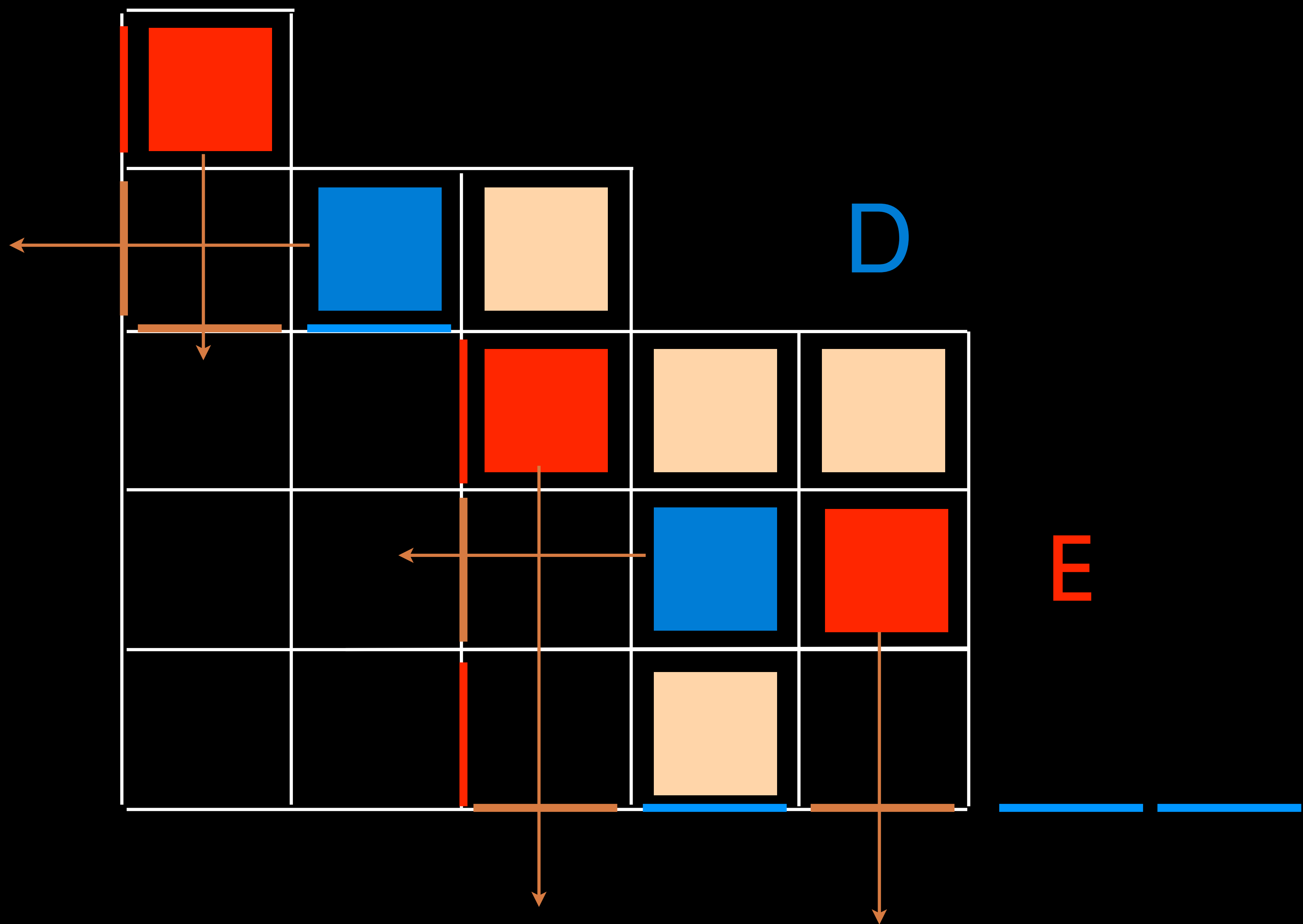
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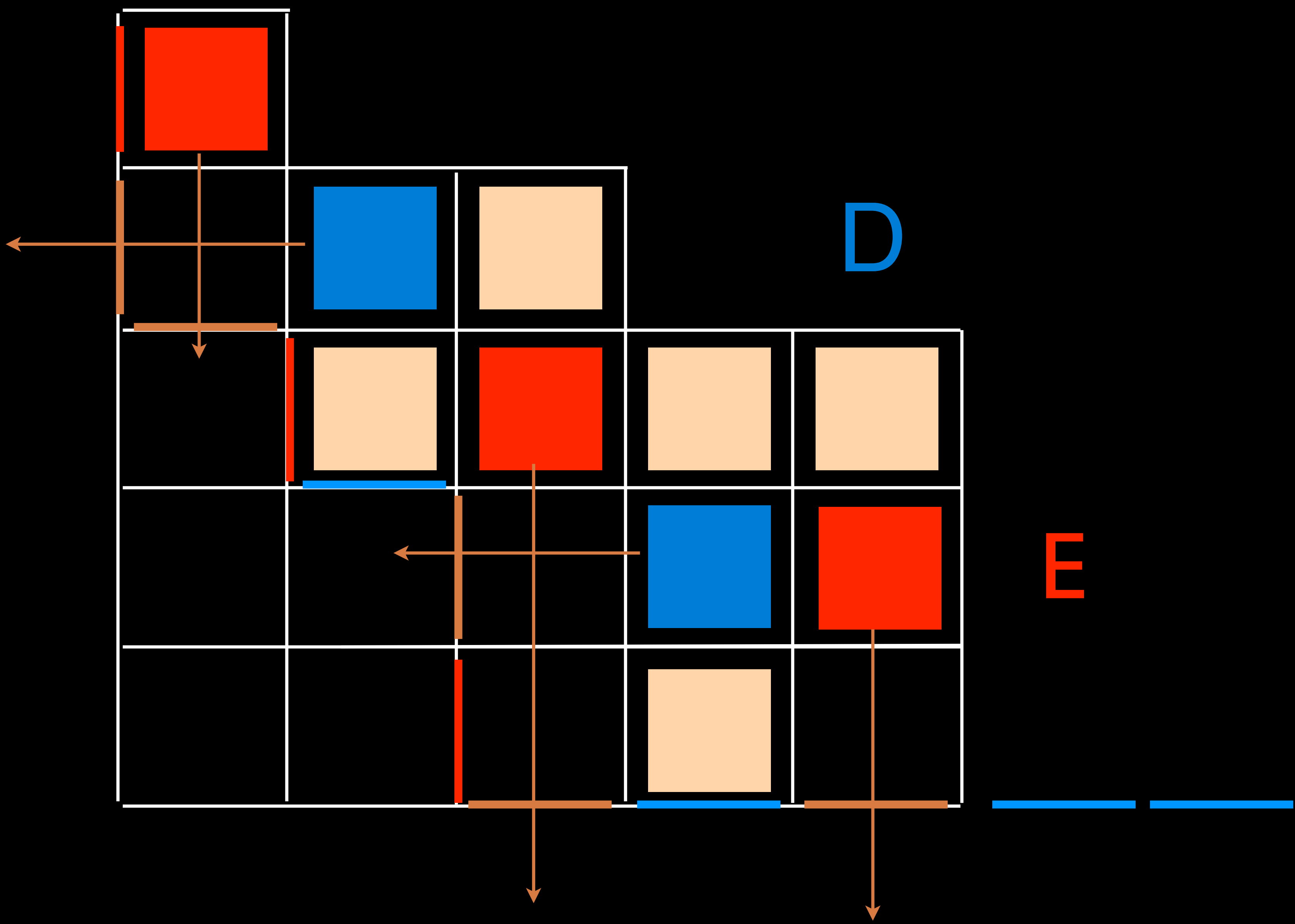
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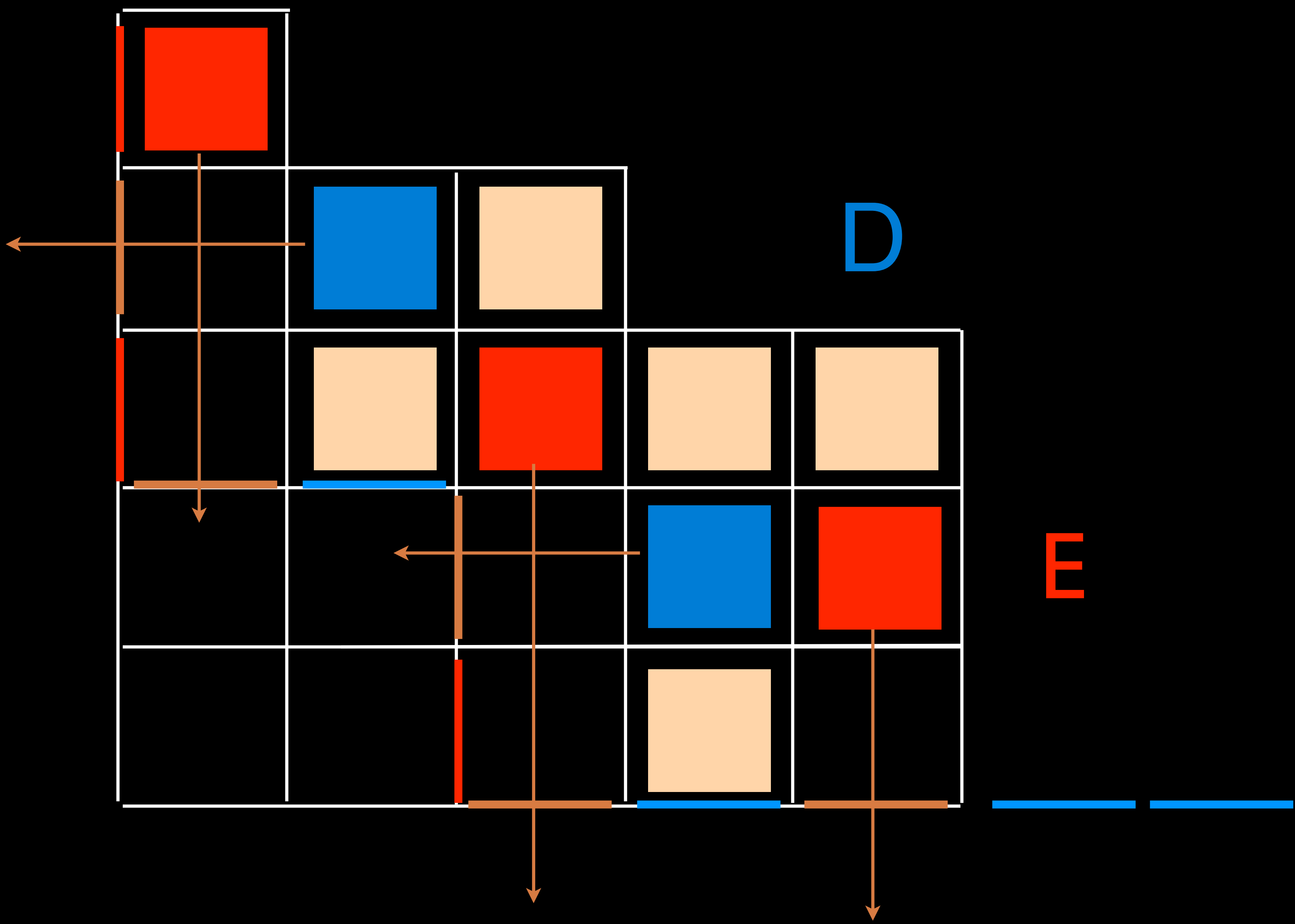
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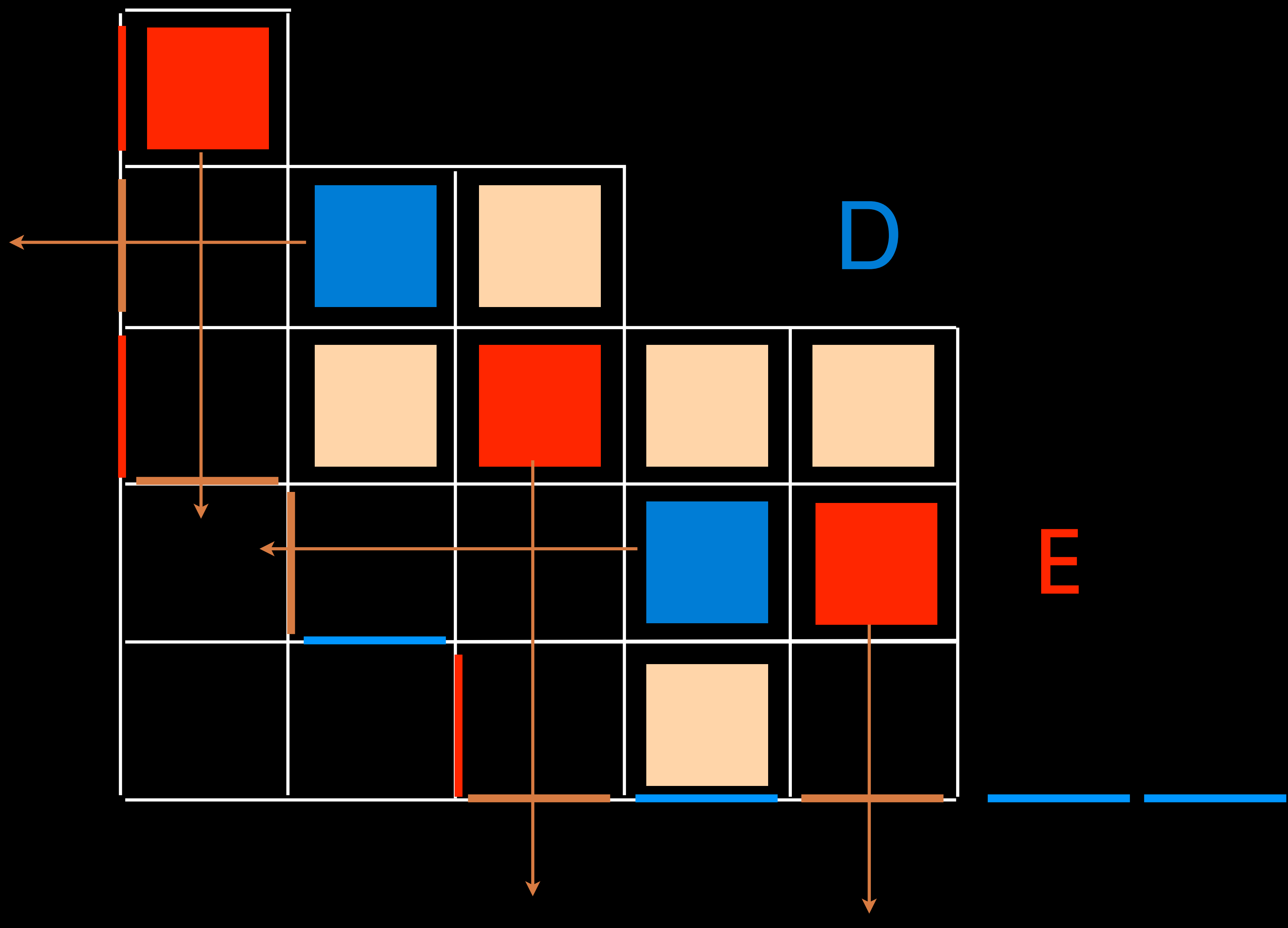
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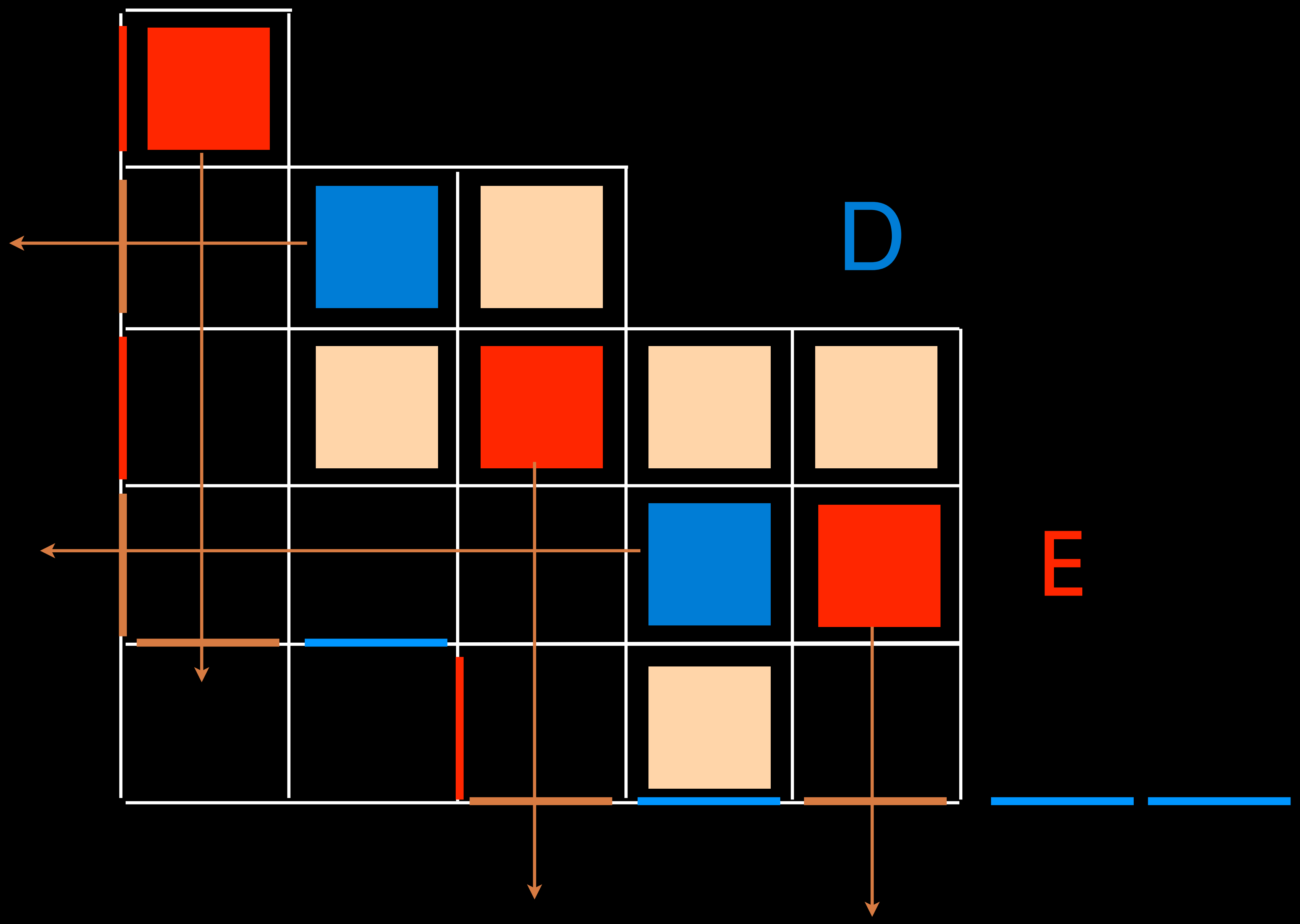
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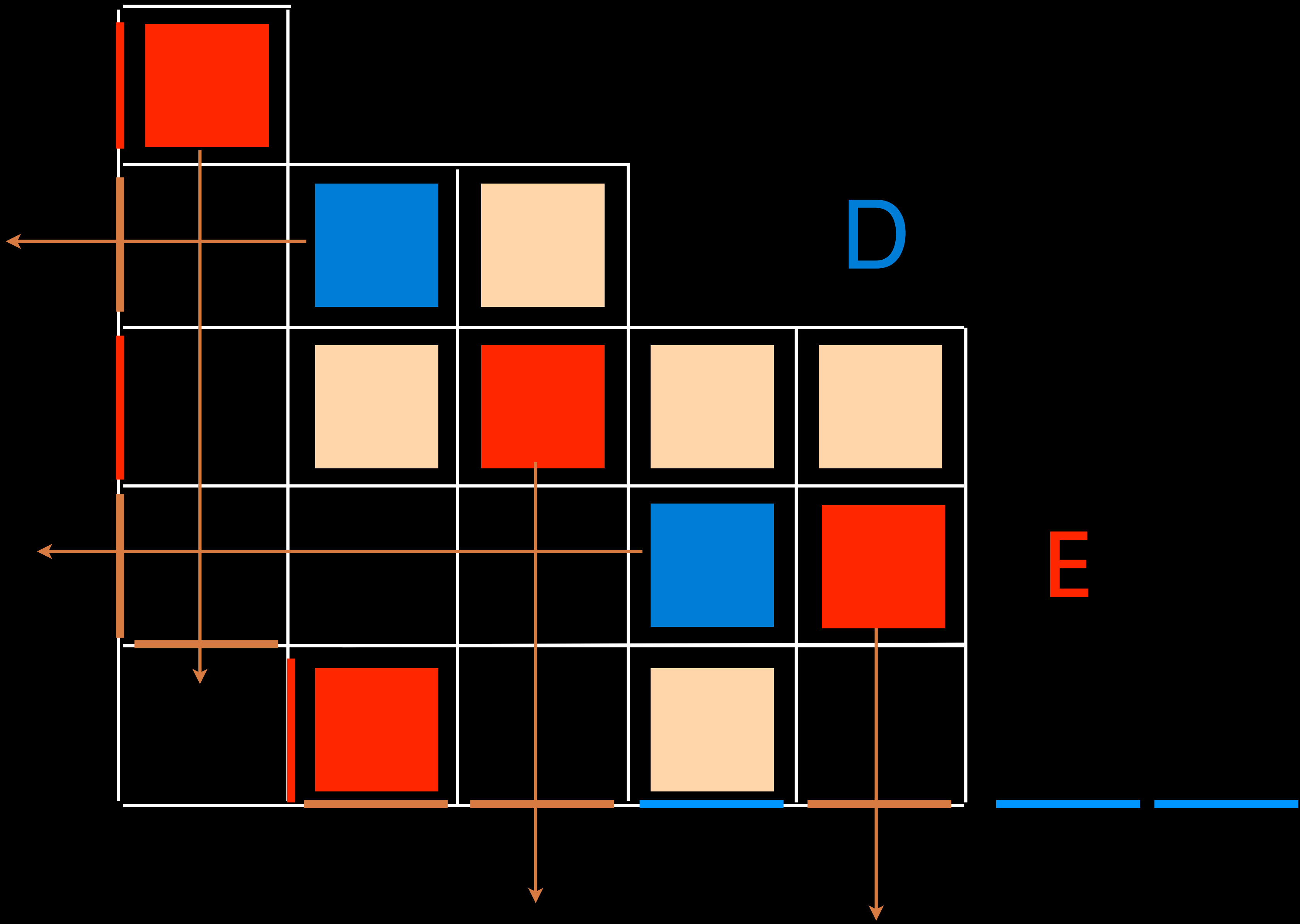
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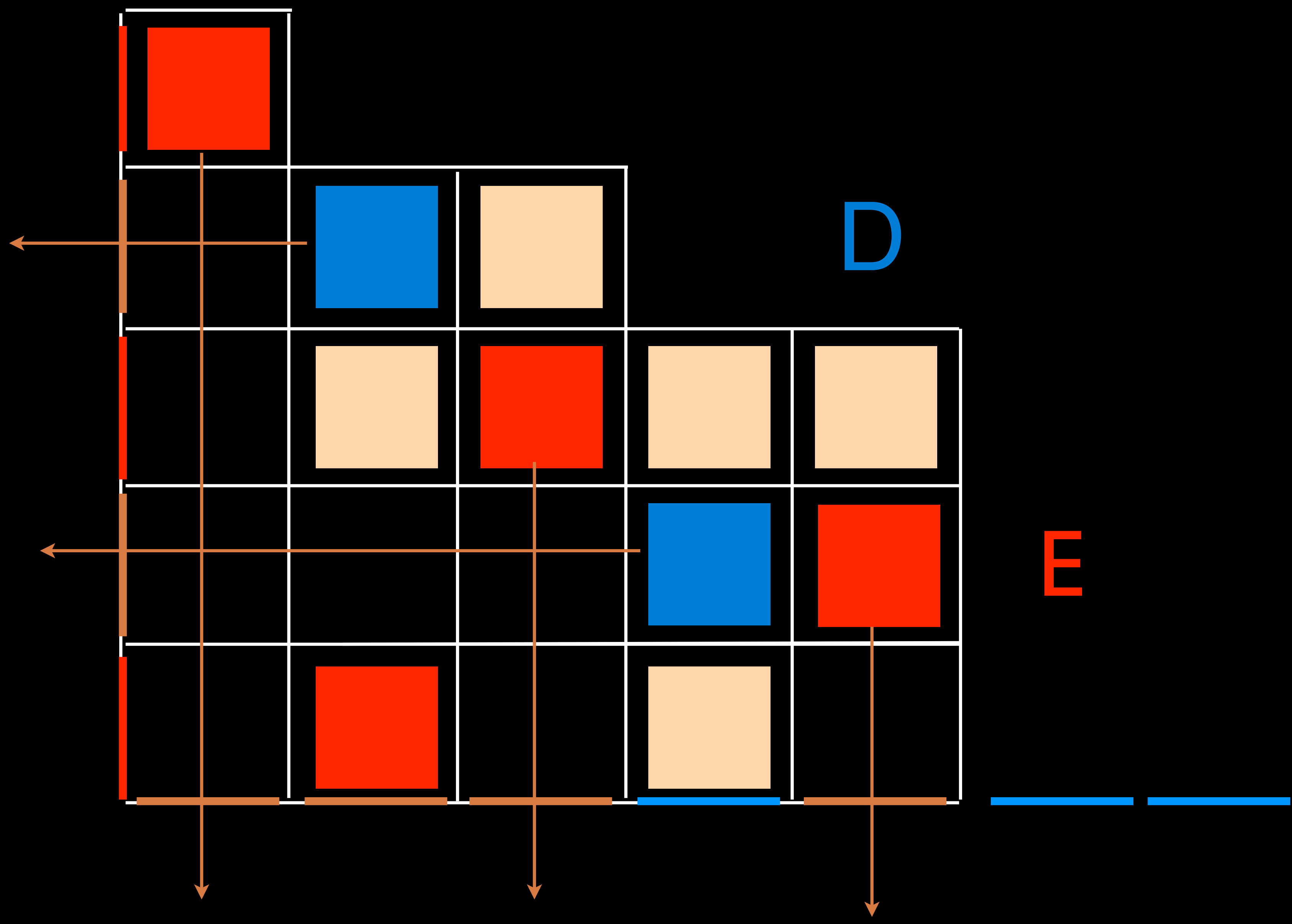
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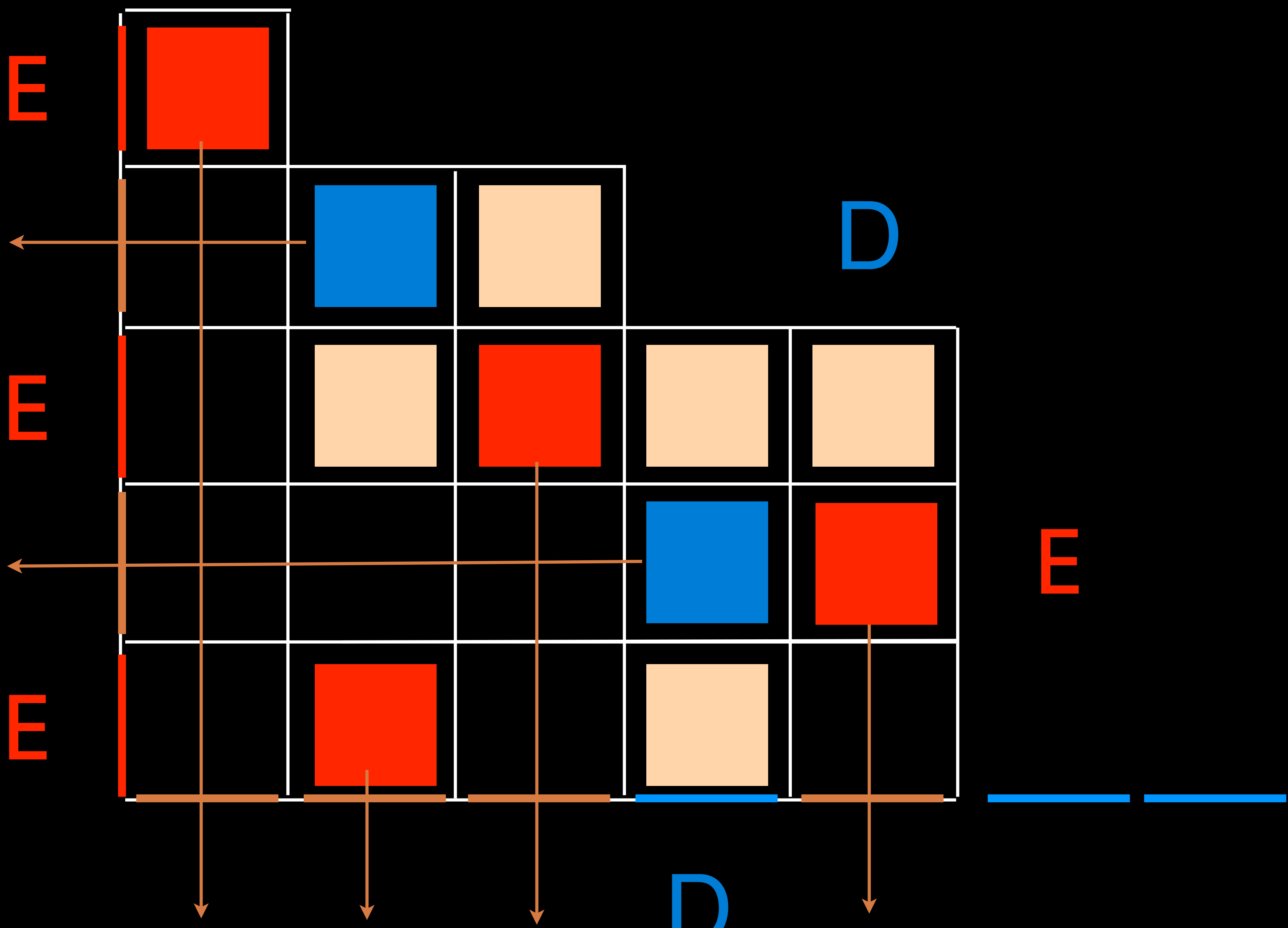
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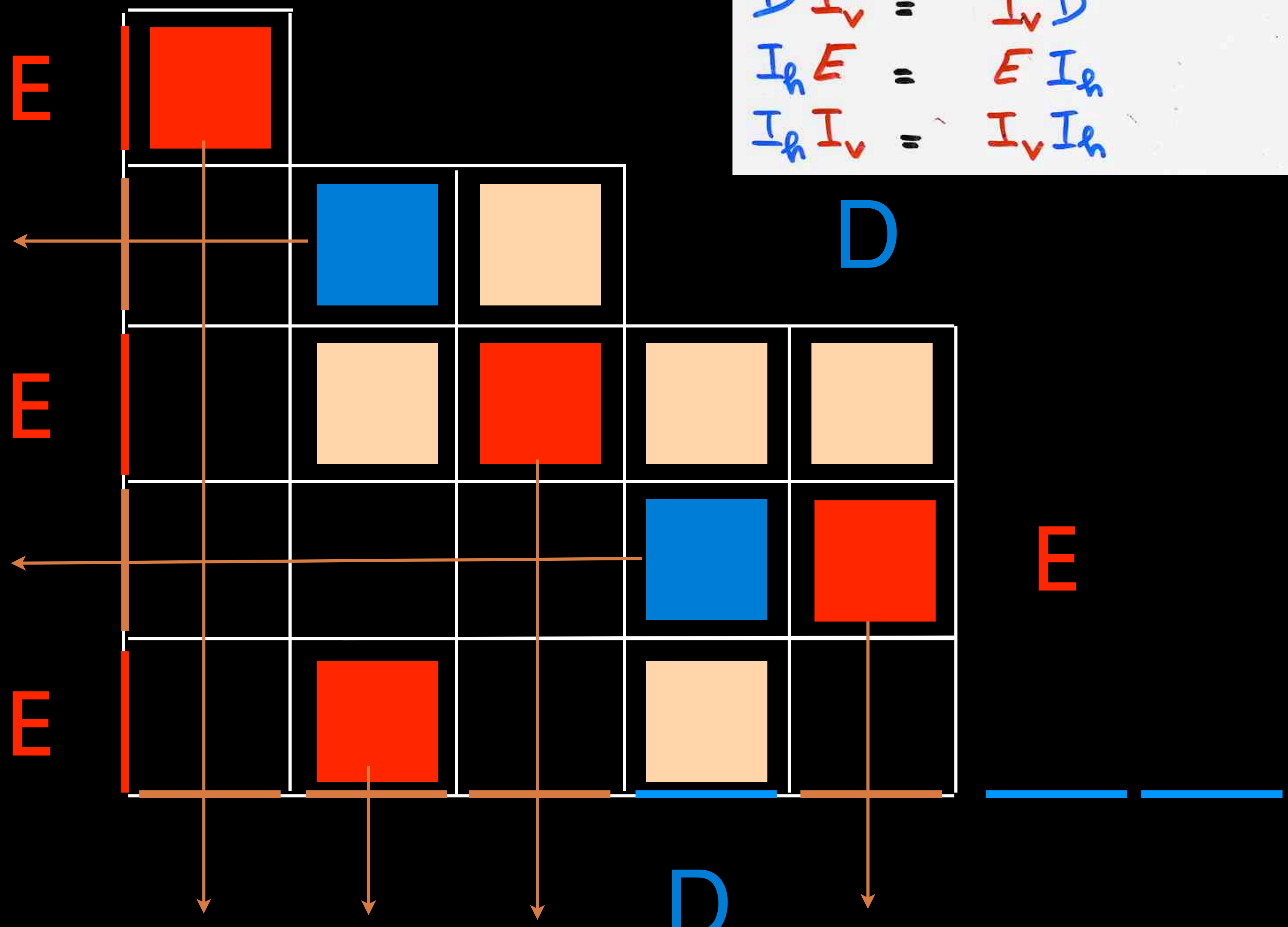
E
D

$$DE = q\,ED + EI_h + I_v D$$
$$DI_v = I_v D$$
$$I_h E = E I_h$$
$$I_h I_v = I_v I_h$$



$$DE = qED + EI_h + I_vD$$
$$DI_v = I_vD$$
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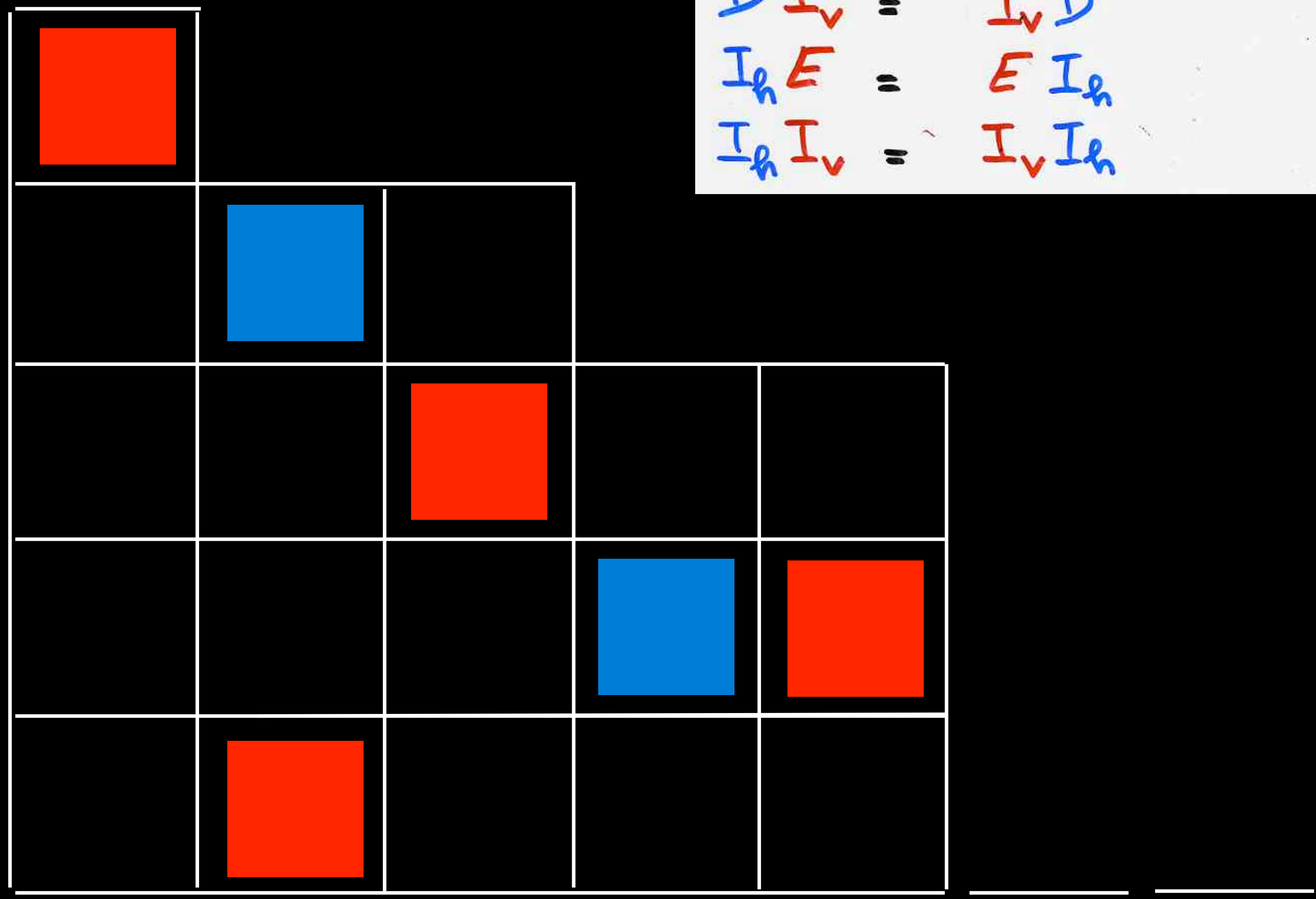
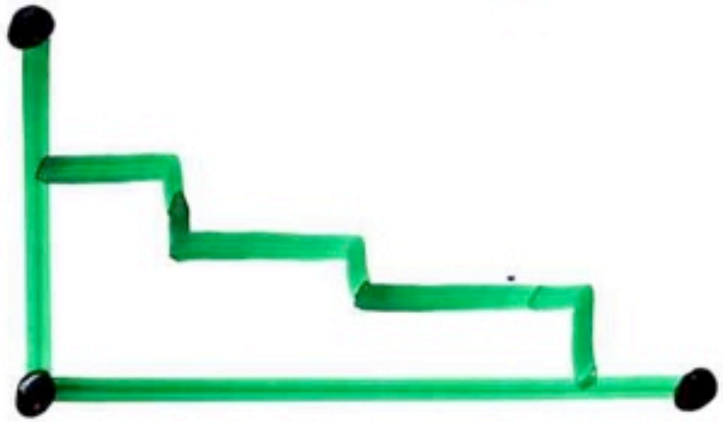
$$I_hI_v = I_vI_h$$



alternative tableaux

alternative tableau

- Ferrers diagram F



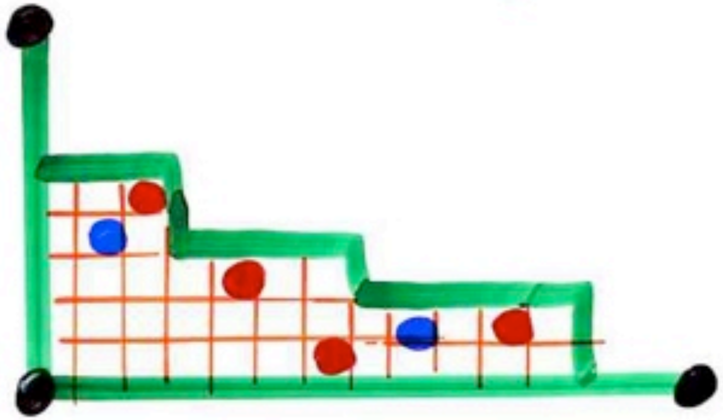
(possibly empty rows or column)

(nb of rows) + (nb of columns) $= n$

alternative tableau

- Ferrers diagram F

(possibly empty rows or column)

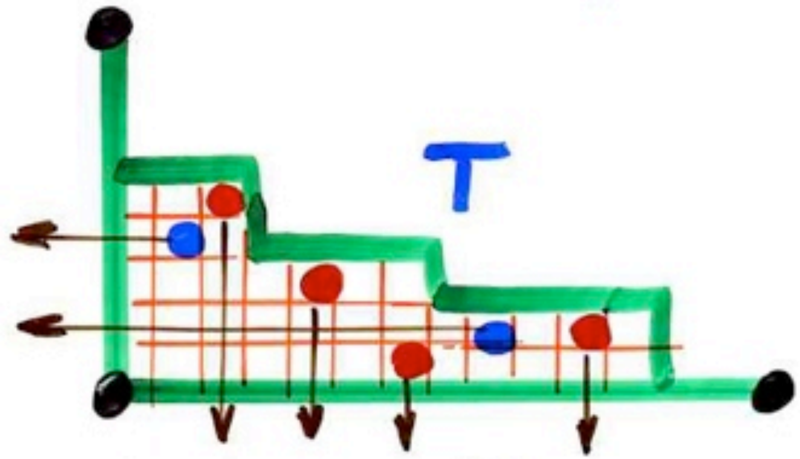


(nb of rows) + (nb of columns) $= n$

- some cells are coloured **red** or **blue**

alternative tableau T

- Ferrers diagram F (possibly empty rows or column)



(nb of rows) + (nb of columns) = n

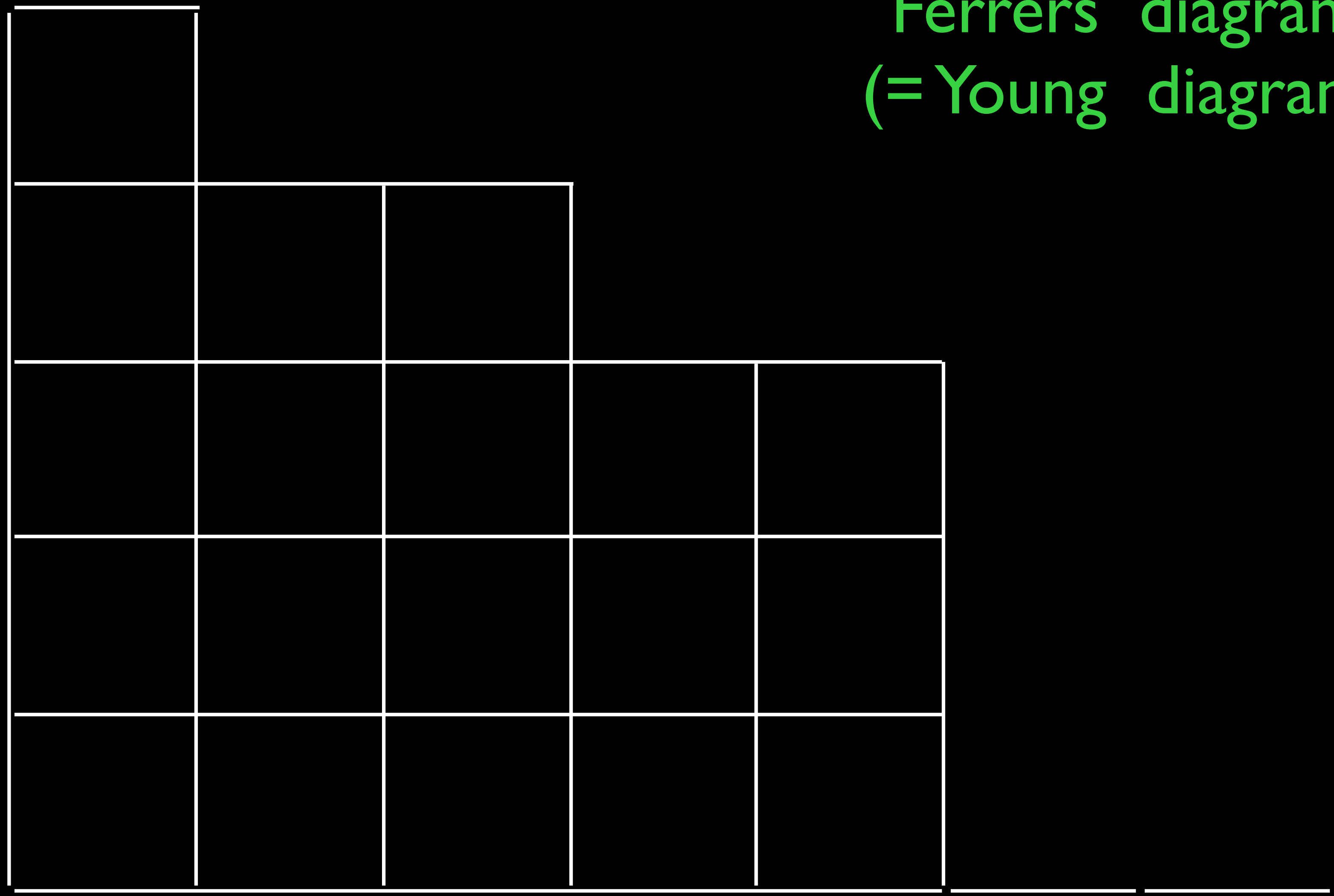
- some cells are coloured **red** or **blue**

- $\begin{cases} \text{no coloured cell at the left of } \boxed{\color{blue}\bullet} \\ \text{no coloured cell below } \boxed{\color{red}\bullet} \end{cases}$

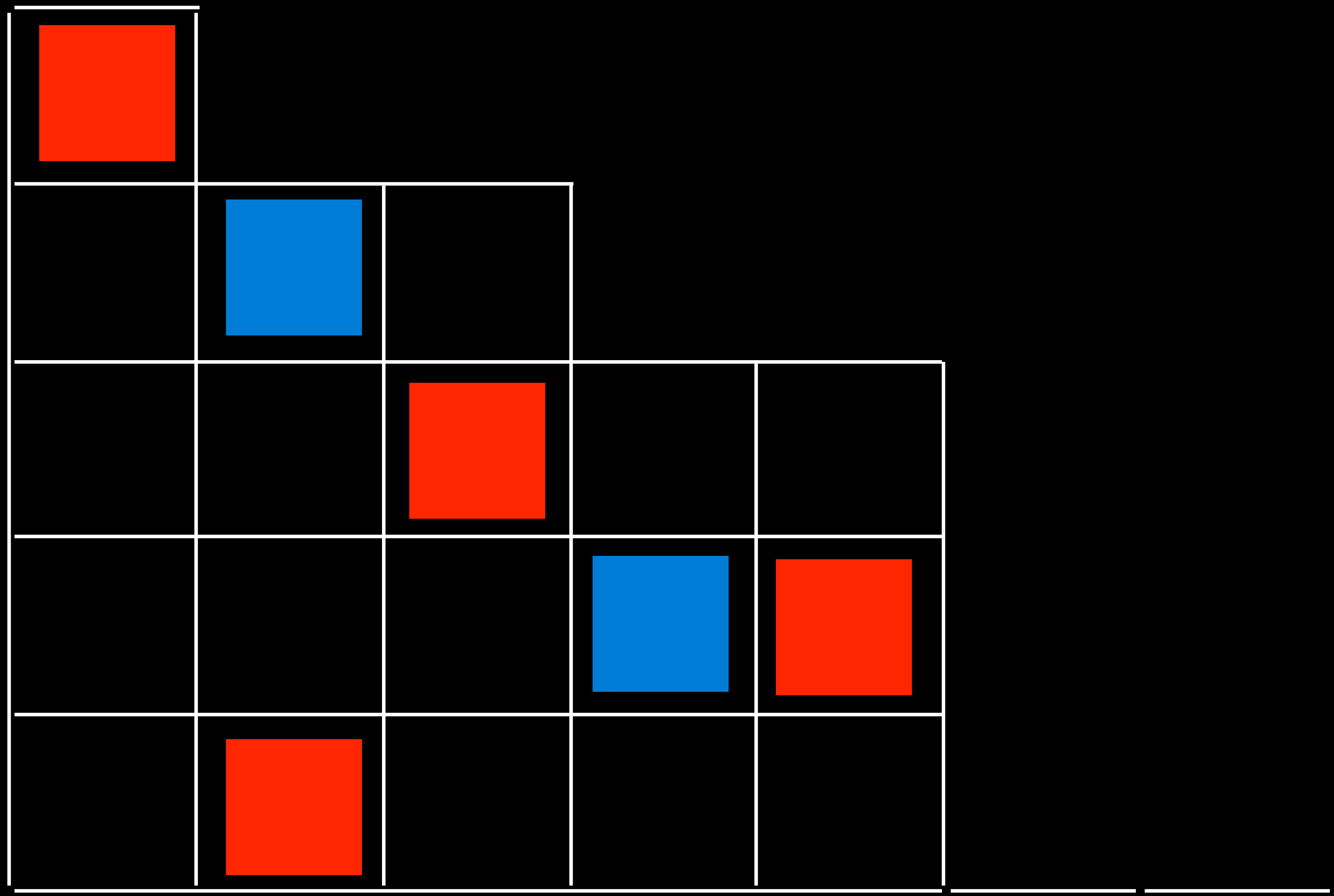
n size of T

alternative tableau

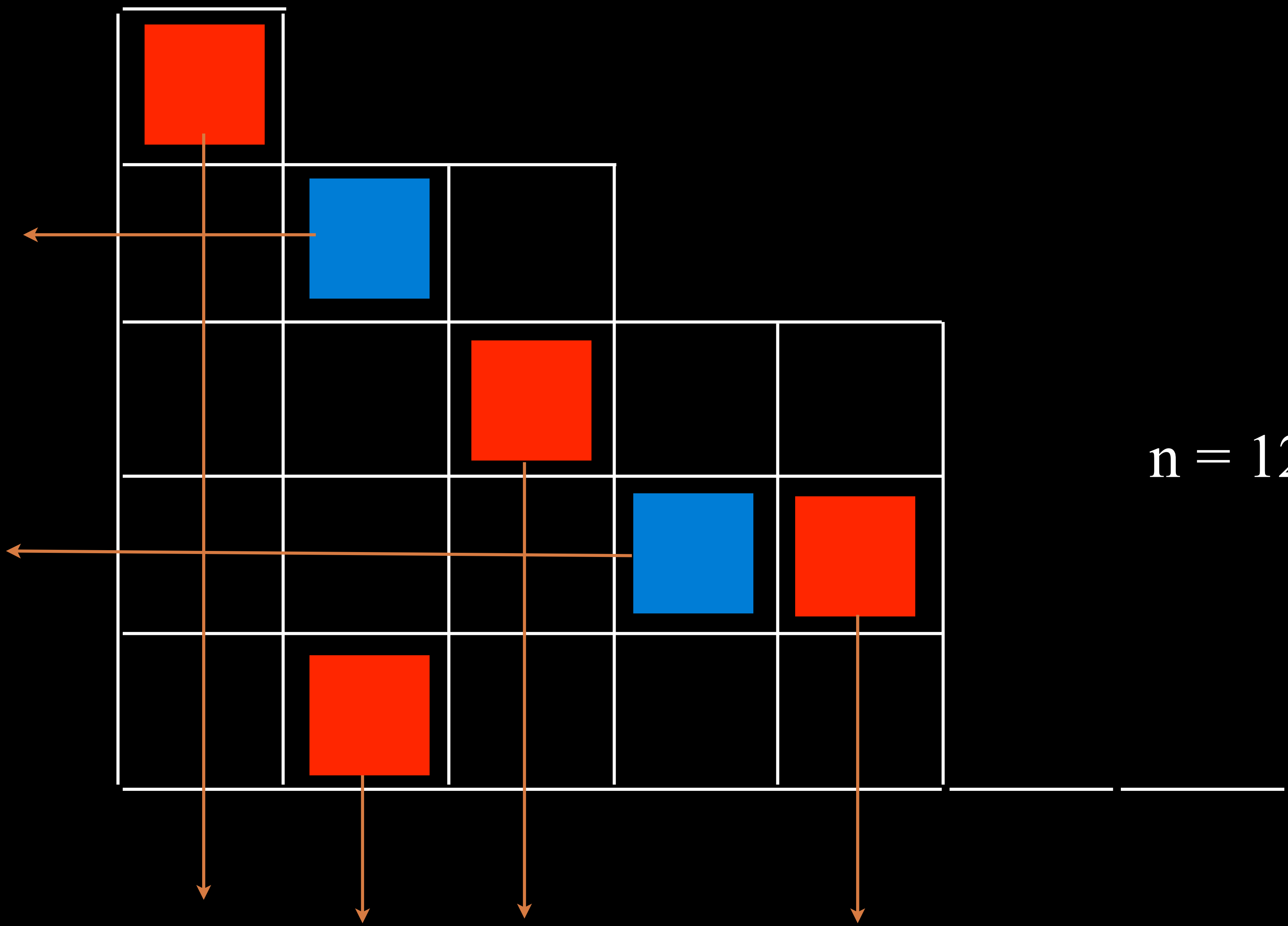
Ferrers diagram
(= Young diagram)



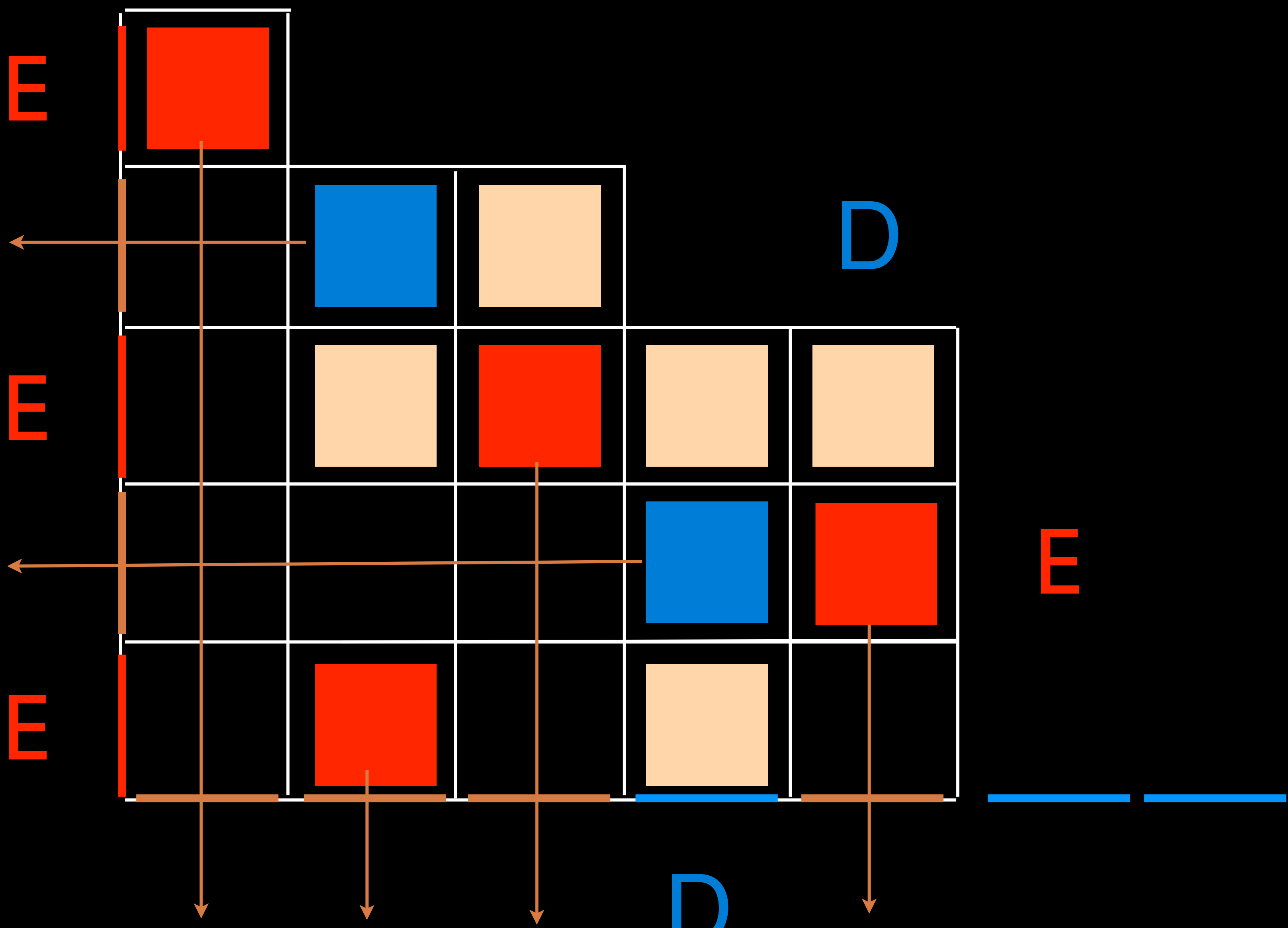
alternative tableau



alternative tableau



$n = 12$


E
D
E
E
E
D

Def - profile of an alternative tableau

word $w \in \{E, D\}^*$

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

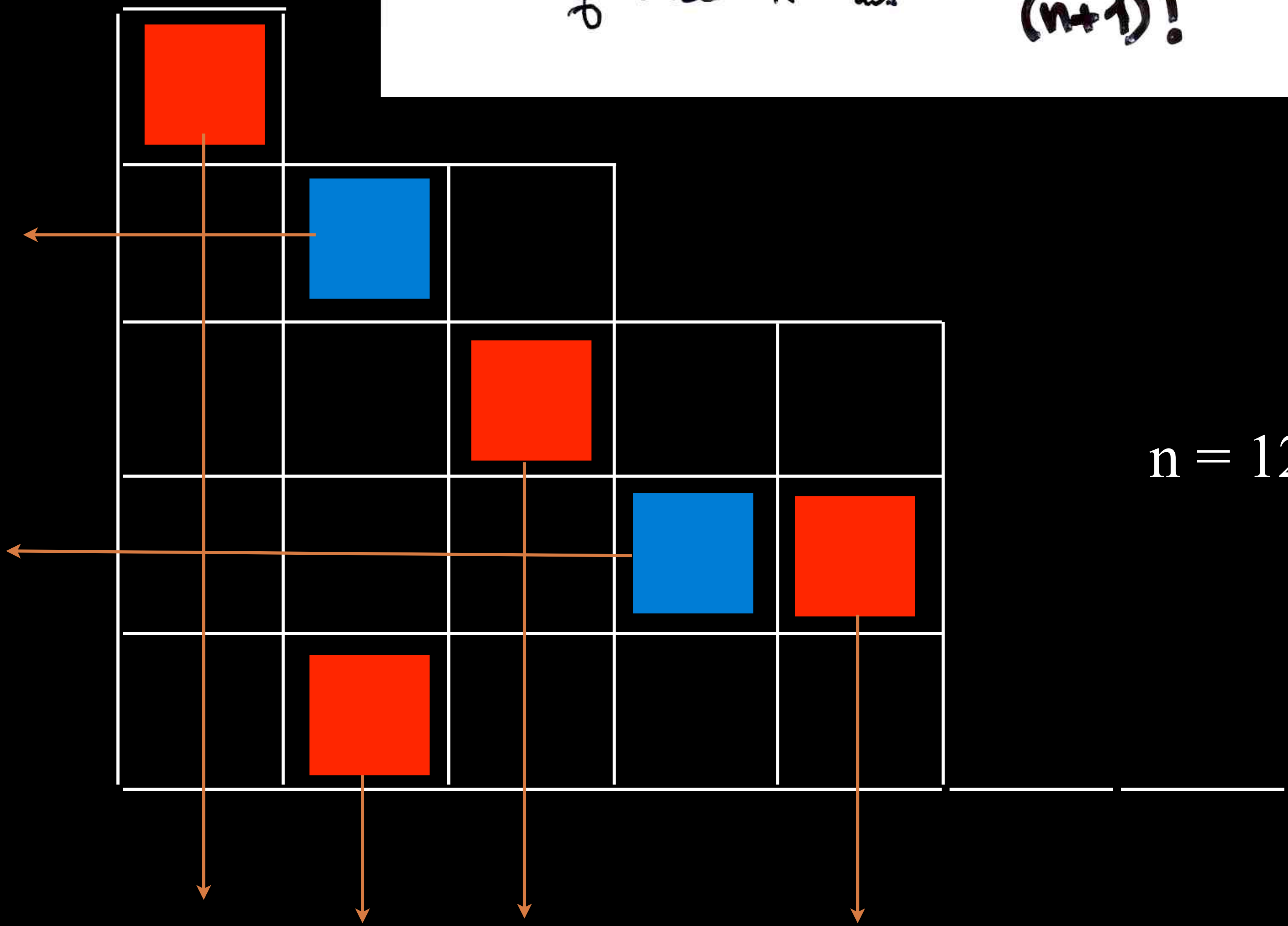
T alternative tableau with profile w

$k(T)$ = nb of $\boxed{}$

$i(T)$ = nb of rows without blue cell

$j(T)$ = nb of columns without red cell

Prop. The number of alternative tableaux of size n is $(n+1)!$



$n = 12$

ex: $n=2$

stationary probabilities for the PASEP

$$\begin{cases} DE = qED + D + E \\ DV = \bar{\beta} V & \bar{\beta} = 1/\beta \\ WE = \bar{\alpha} W & \bar{\alpha} = 1/\alpha \end{cases}$$

$$W E^{i} D^{j} V = \bar{\alpha}^{i} \bar{\beta}^{j} \underbrace{WV}_{1}$$

Cor. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ (PASEP) is

$$\mathrm{proba}_{\tau}(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{T} q^{L(T)} \alpha^{-f(T)} \beta^{-u(T)}$$

alternative tableaux profile τ

$$\begin{cases} f(T) & \text{nb of } \begin{cases}\text{rows} \\ \text{columns}\end{cases} \text{ without } \begin{cases} \bullet \\ \bullet \end{cases} \text{ cell} \\ u(T) & \\ L(T) & \text{nb of cells } \square \end{cases}$$

permutation tableau

S. Corteel, L. Williams
(2007) (2008) (2009)

permutation tableau

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)

Corteel, Williams (2006) PASEP

Partially Asymmetric Exclusion Process

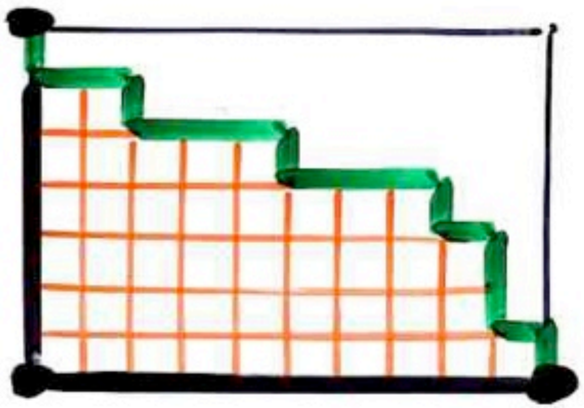
M. Josuat-Vergès (2007)

permutation tableaux

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$

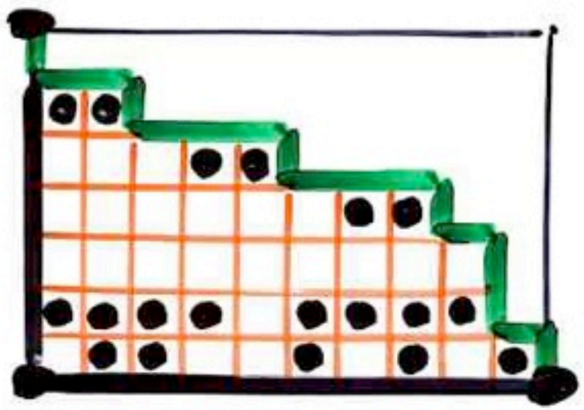
rectangle



Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$

rectangle



filling of the cells with 0 and 1

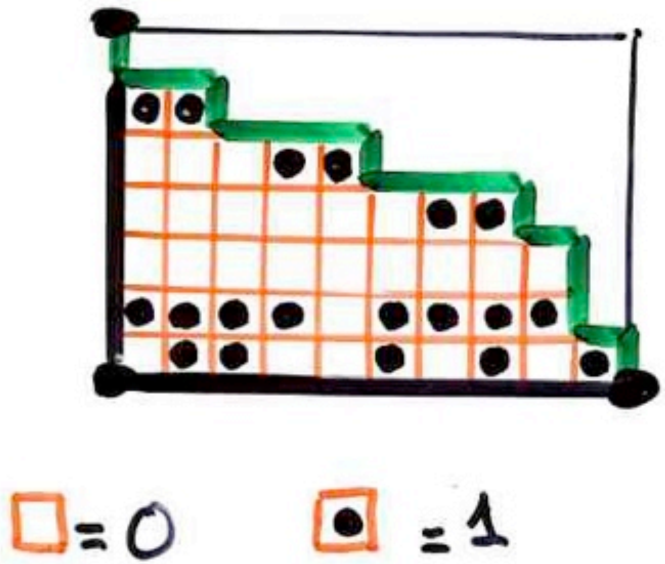
(i)

(ii)

$\square = 0$ $\boxed{\bullet} = 1$

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



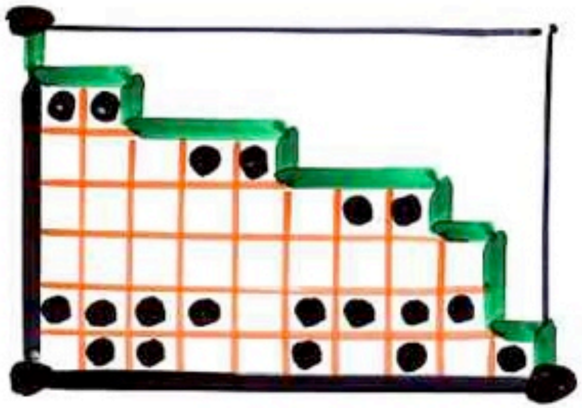
filling of the cells with 0 and 1

(i) in each column : at least one 1

(ii)

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



filling of the cells
with 0 and 1

(i) in each column :
at least one 1

□ = 0 ⊡ = 1

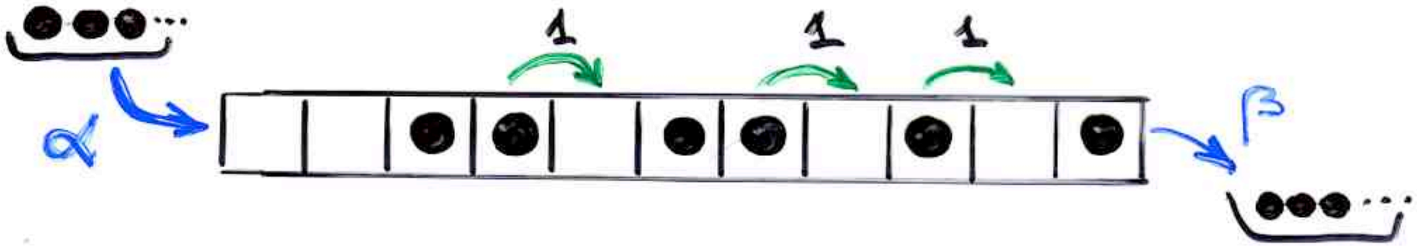
(ii)
$$\begin{array}{ccc} 1 & \cdots\cdots & 0 \\ & & \vdots \\ & & 1 \end{array}$$
forbidden

TASEP

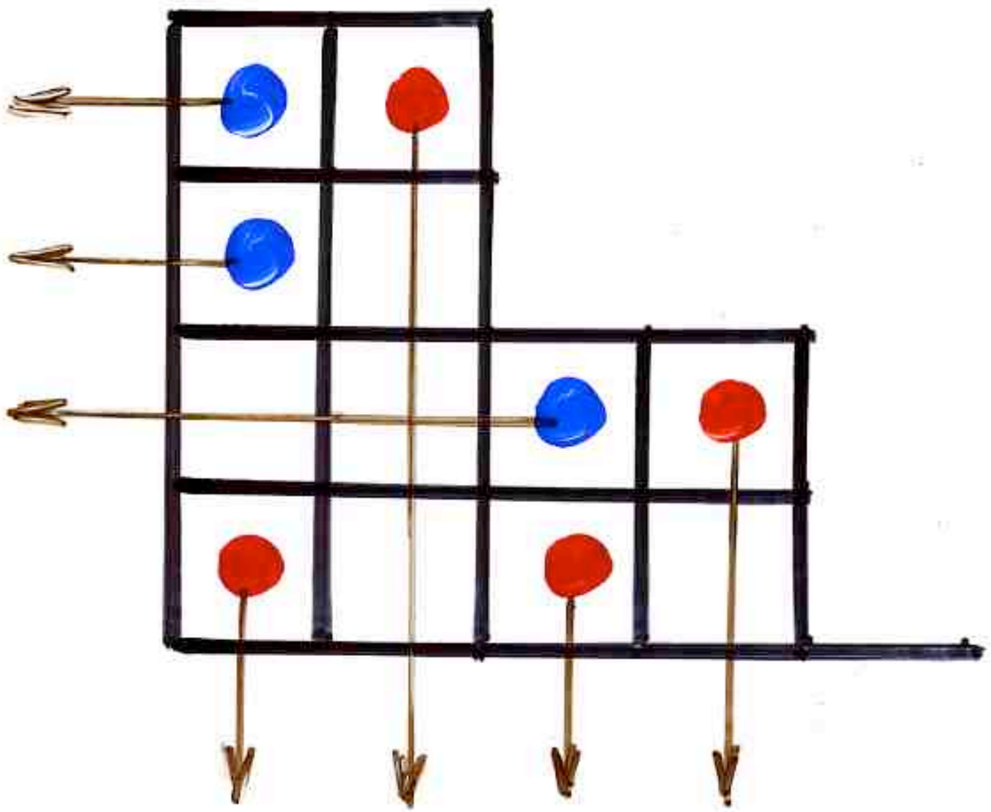
Totally asymmetric exclusion process

TASEP

"totally asymmetric exclusion process"



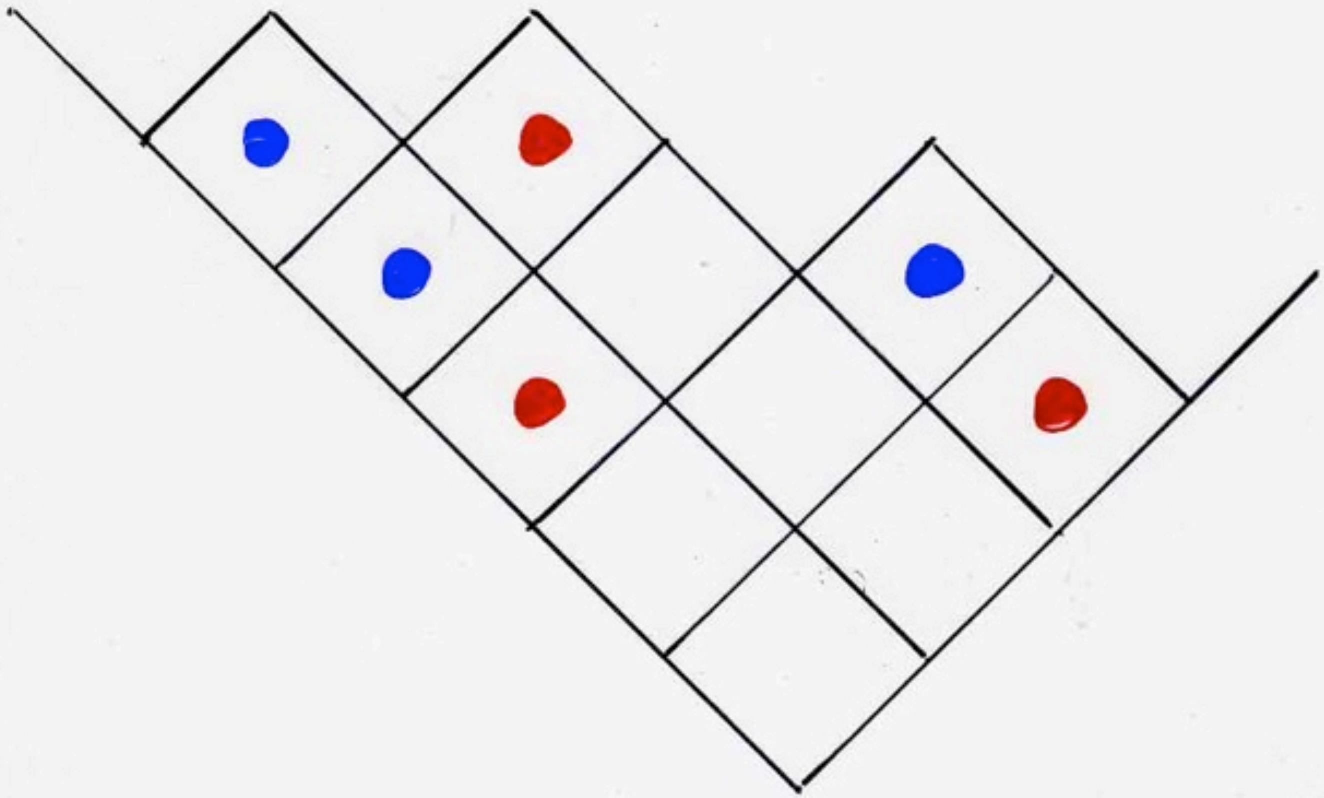
Def Catalan alternative tableau T
alt. tab. without cells $\boxed{\times}$
i.e: every empty cell is below a red cell or
on the left of a blue cell

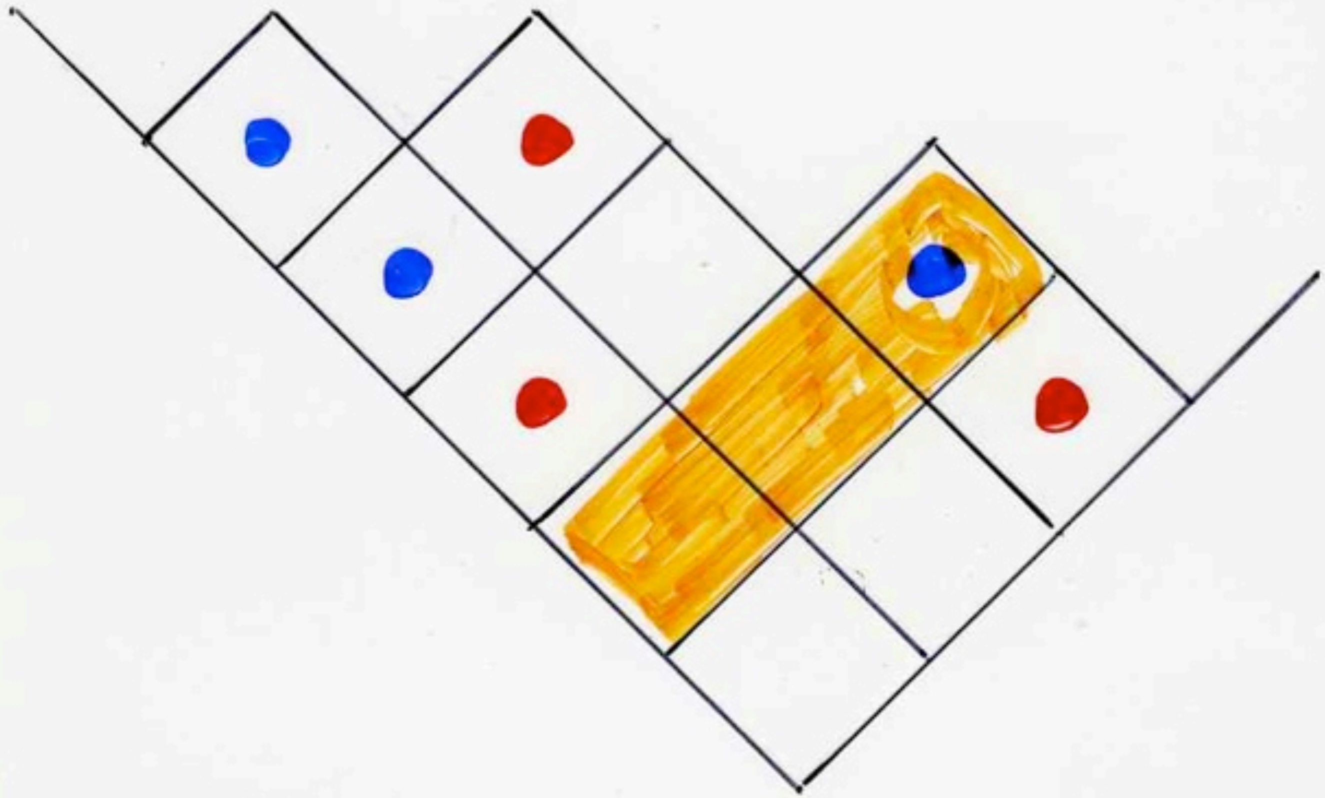


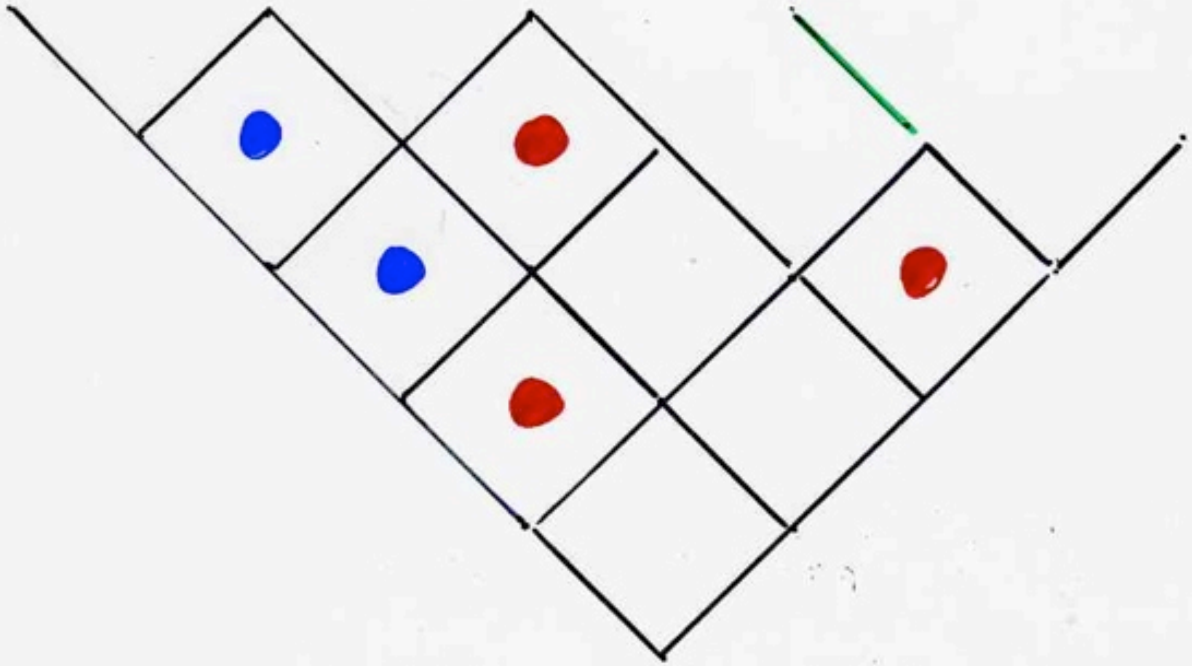
Bijection

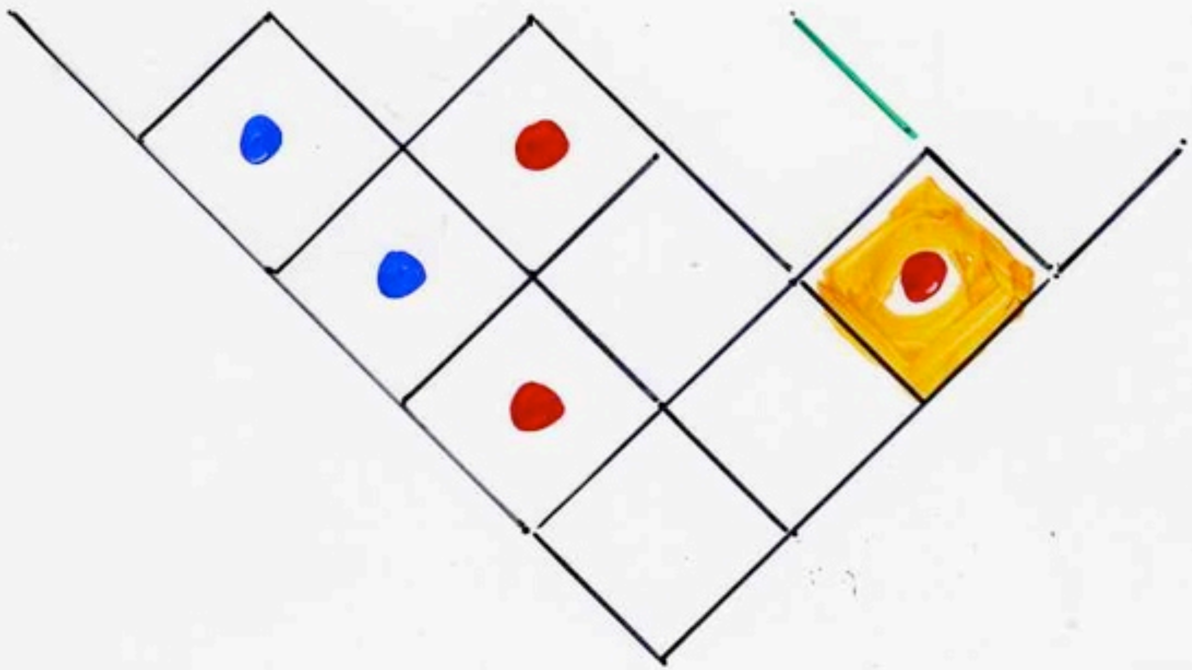
alternative Catalan tableaux

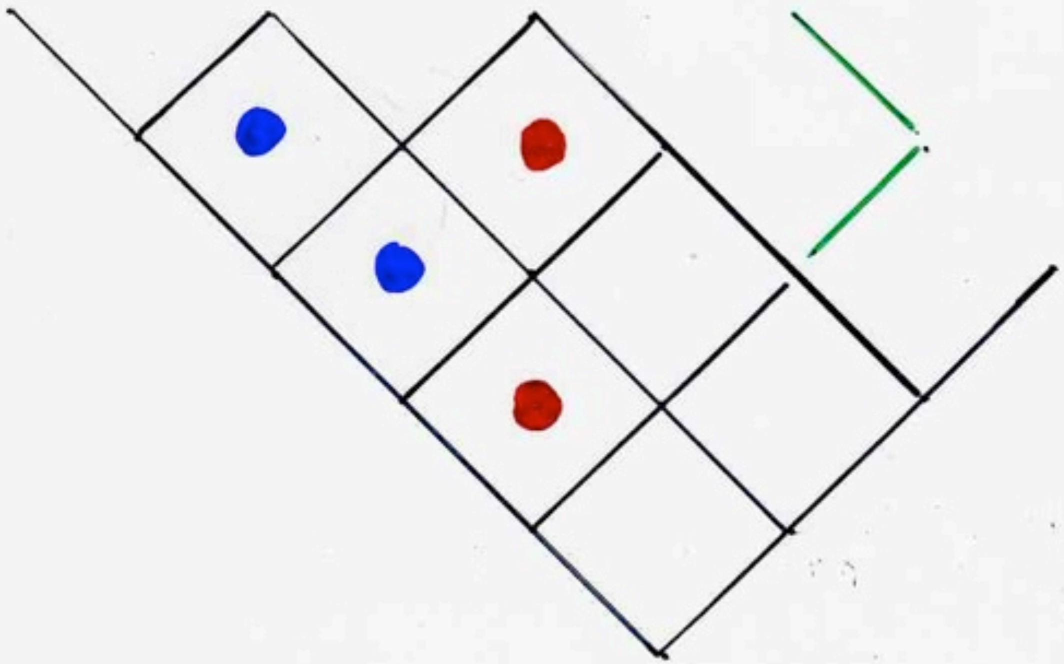
binary trees

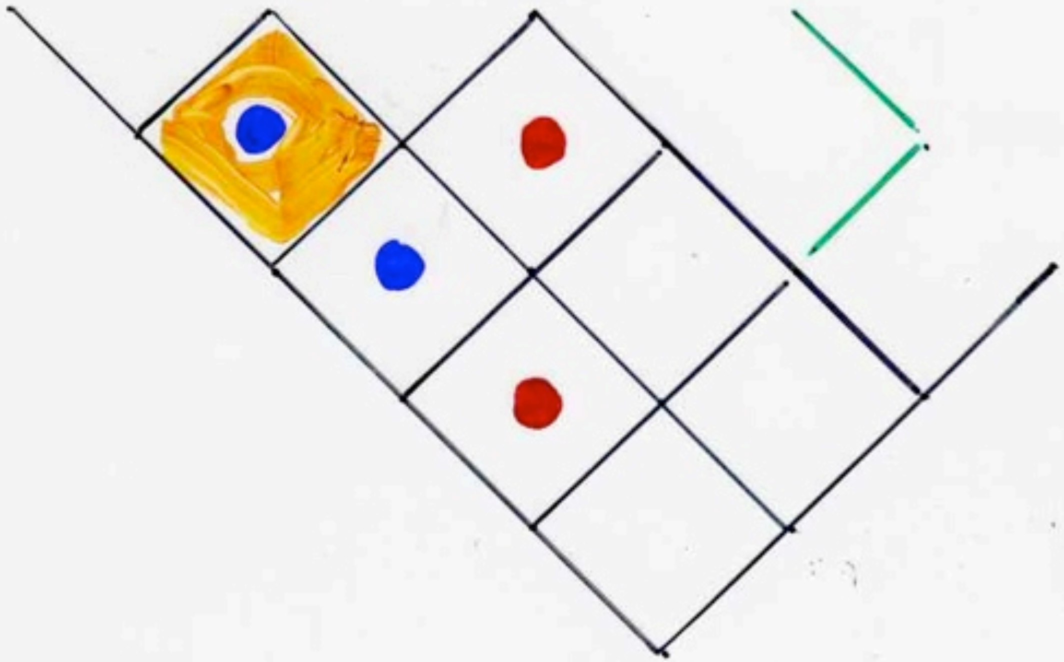


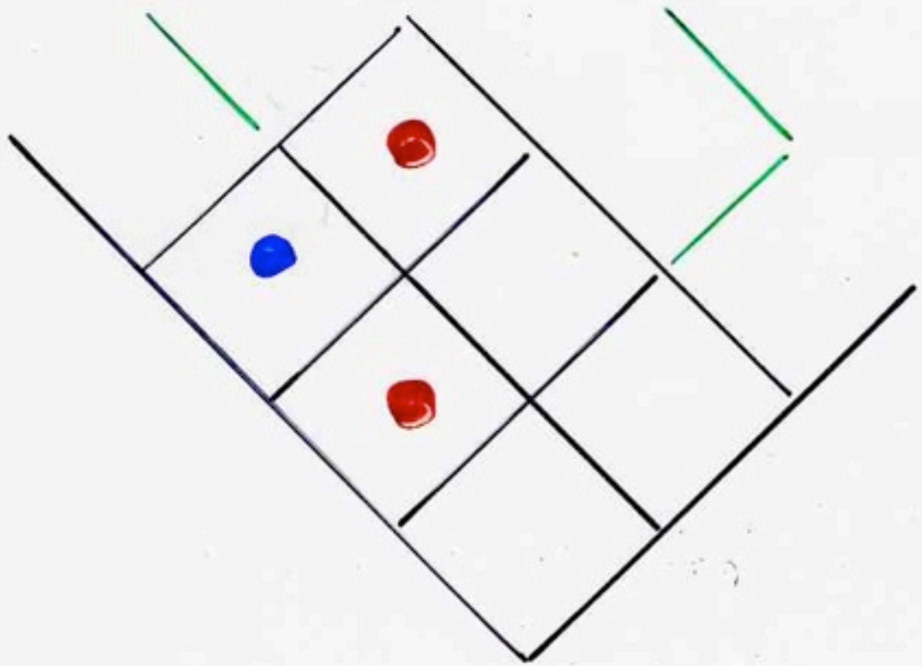


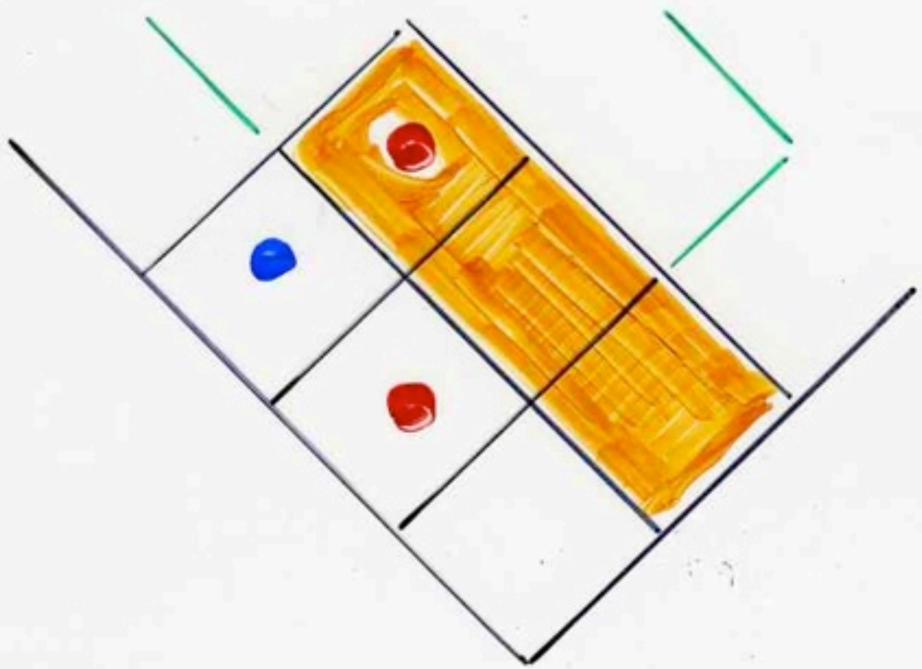


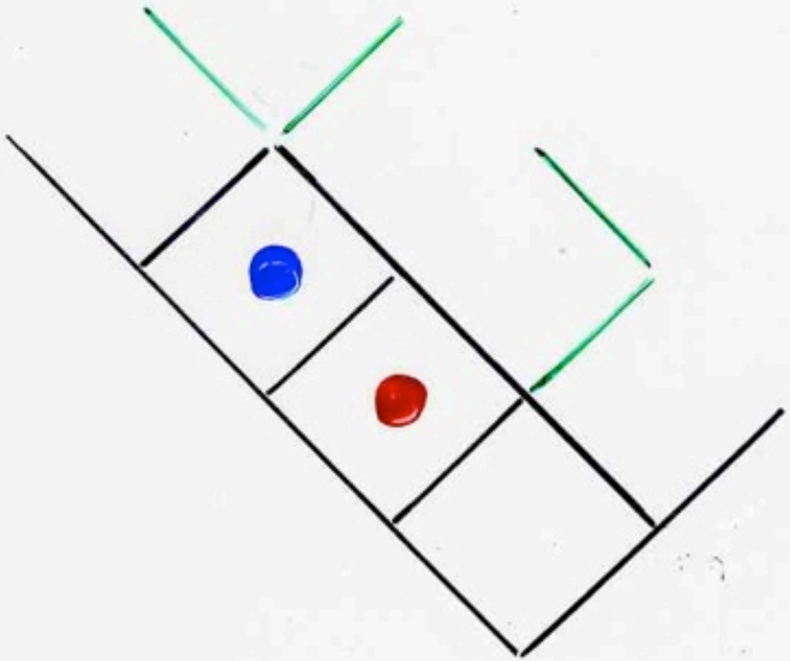


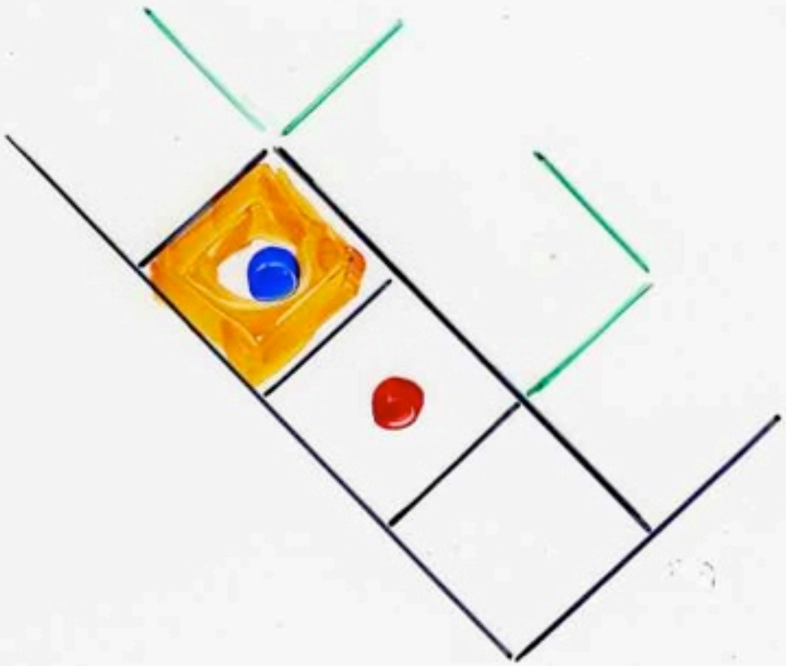


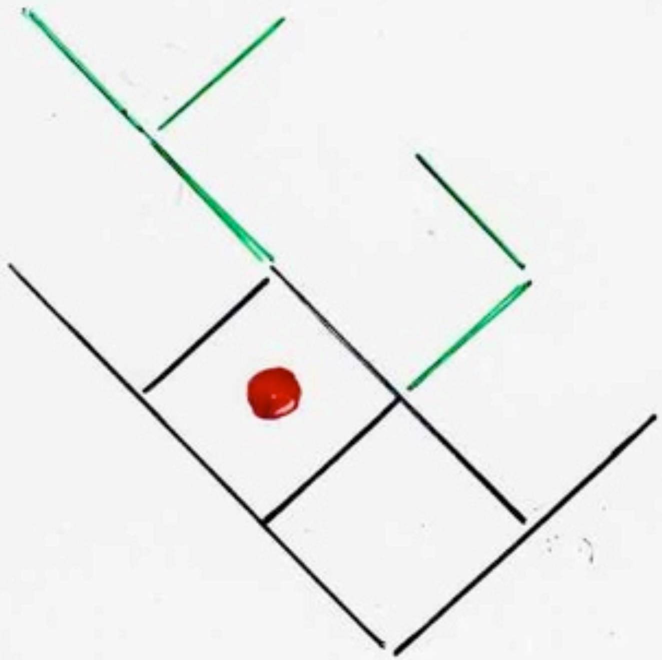


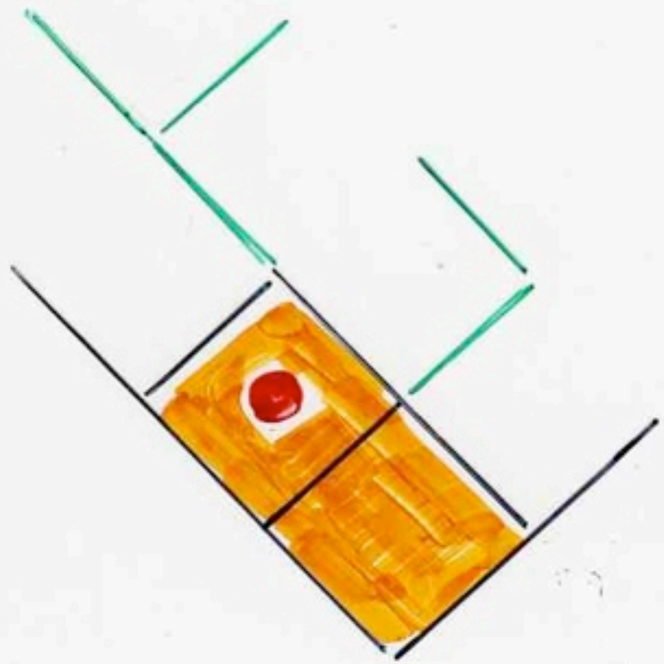


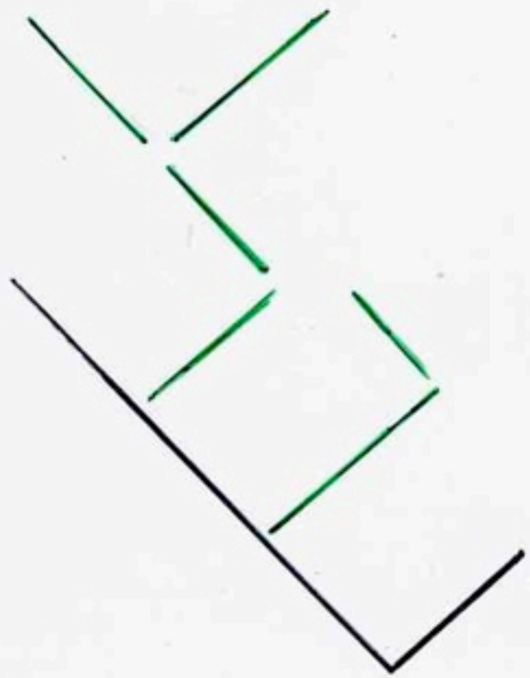




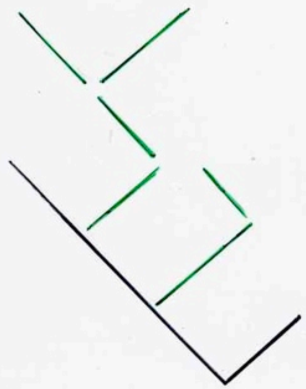


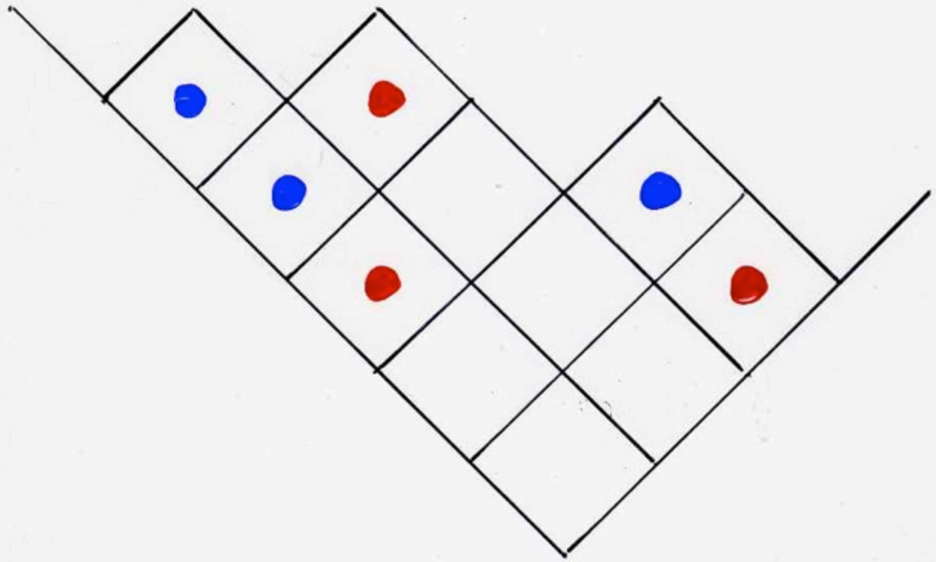




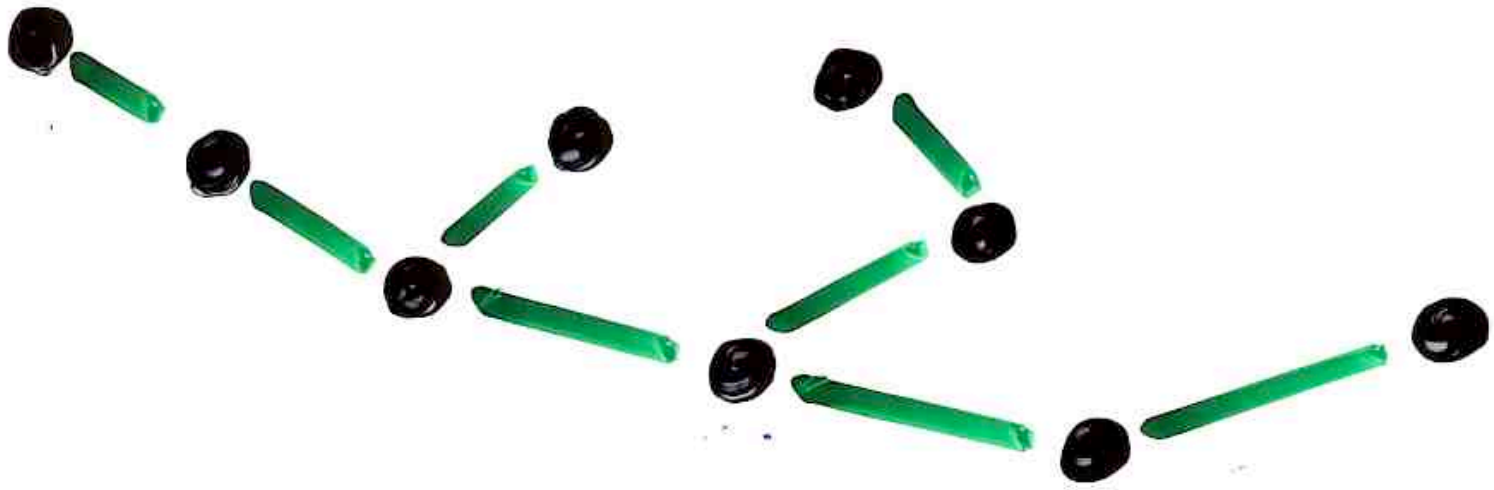


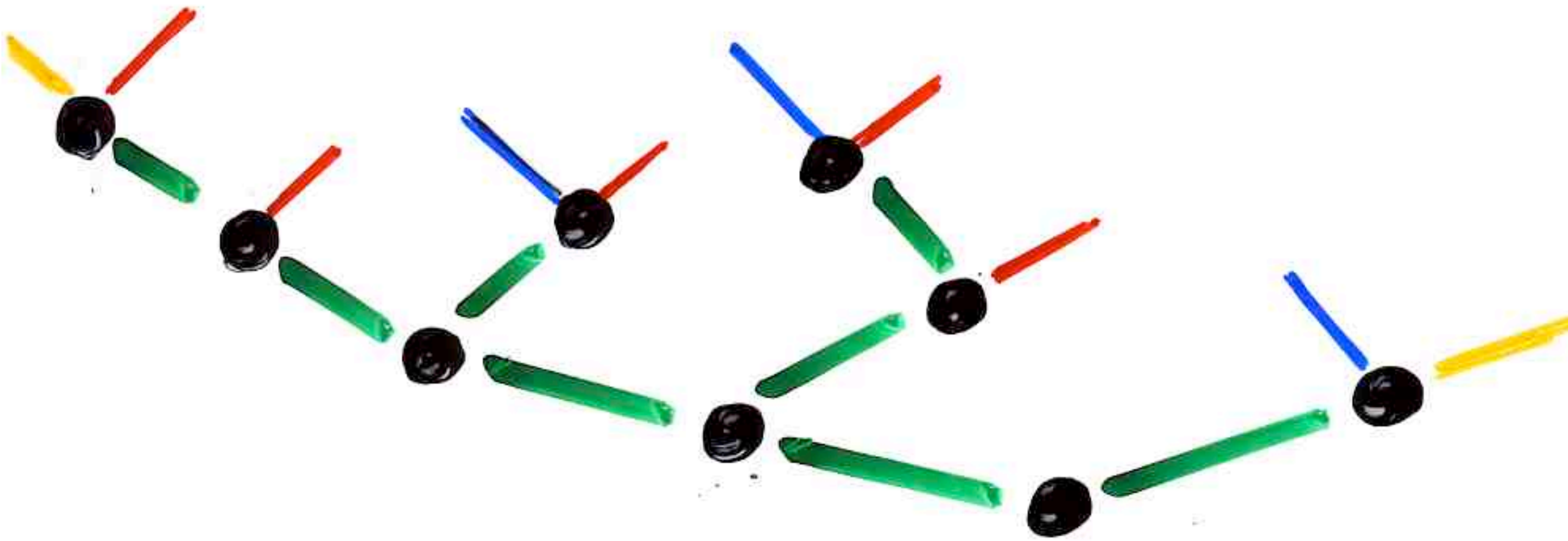
Bijection tableaux alternatifs de Catalan taille n $\longleftrightarrow$ arbres binaires n arêtes





profil (bord) du diagramme de Ferrers $\longleftarrow\text{---}\longrightarrow$ canopée





canopy of a binary tree

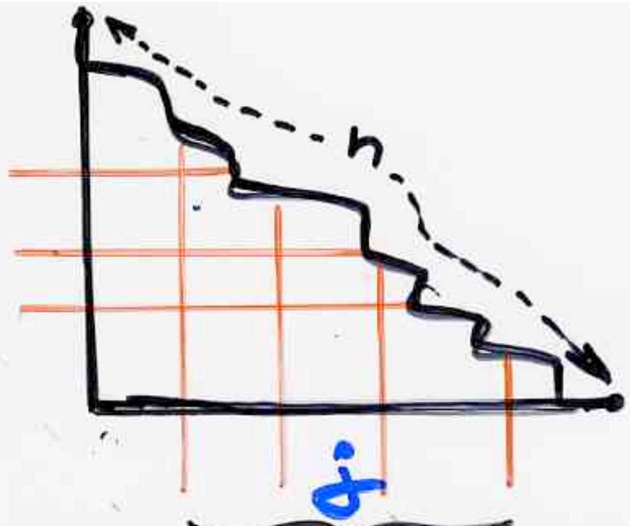
$$C(B) = - - + - + - - +$$

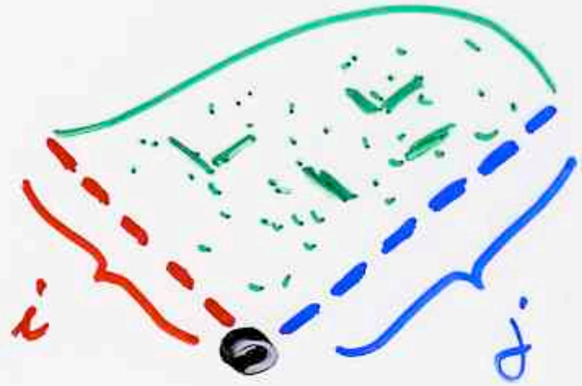
Bijection

tableaux **alternatifs** de Catalan taille n $\longleftrightarrow$ arbres binaires n arêtes

profil (bord) du diagramme de Ferrers $\longleftrightarrow$ canopée

nb de lignes libres (sans ←●) i





nb de colonnes libres (sans ●↓)

$$s = (\tau_1, \dots, \tau_n)$$

$$P_n(s ; \alpha, \beta) = \frac{1}{Z_n} \sum_{B} \bar{\alpha}^{\,lb(B)} \, \bar{\beta}^{\,rb(B)}$$

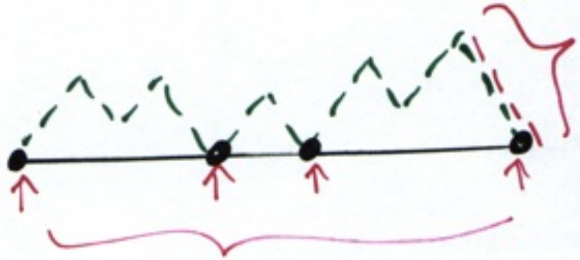
binary trees

canopy s

$$Z_n = \sum_{i=1}^{n} \frac{i}{2n-i} \binom{2n-i}{n} \frac{\bar{\alpha}^{(i+1)} - \bar{\beta}^{(i+1)}}{\bar{\alpha} - \bar{\beta}}$$

partition function

"ballot" numbers



TASEP $q=0$ $DE=E+D$

Catalan alternative tableaux

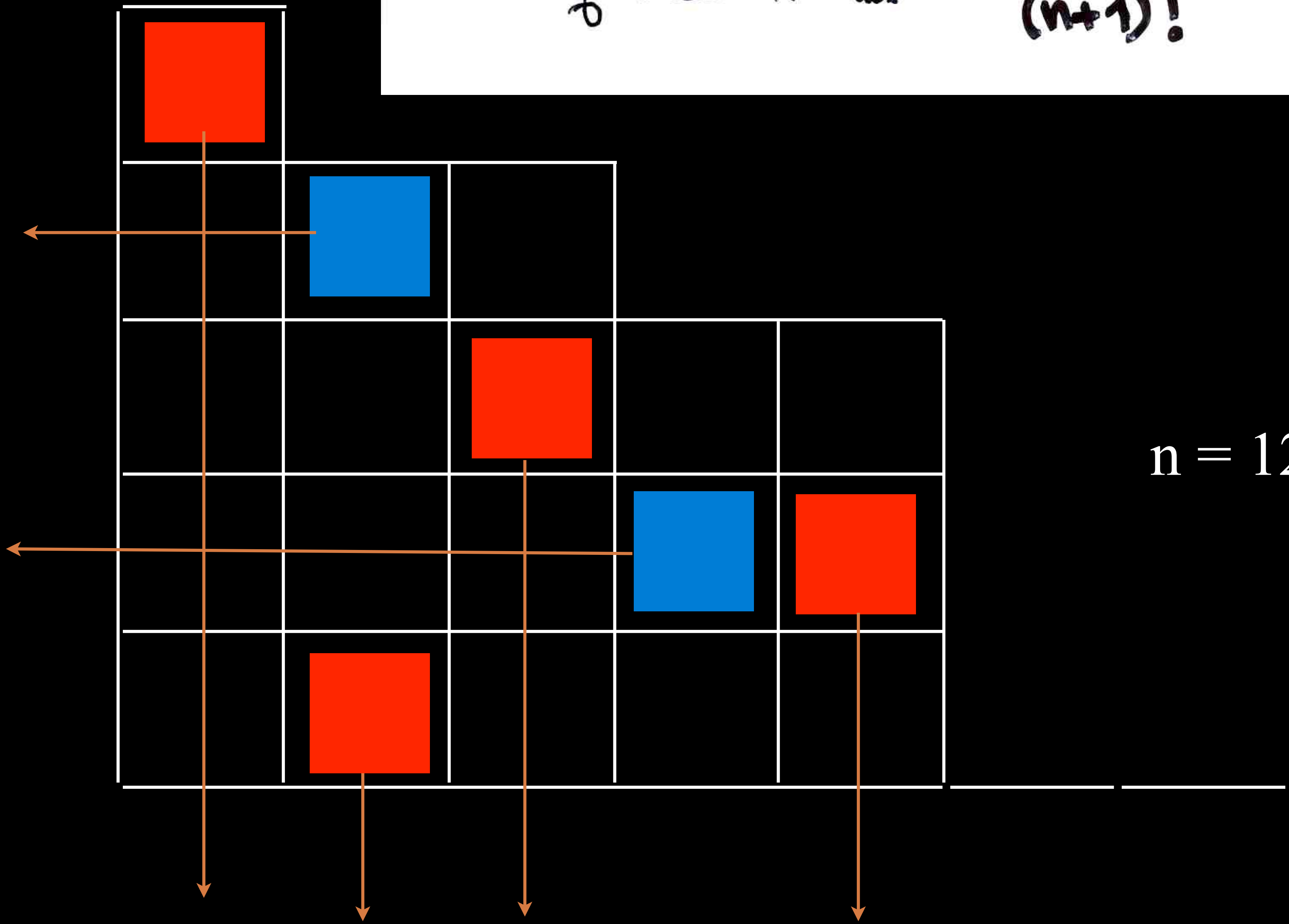
bijection with binary trees

relation with
the Loday-Ronco Hopf algebra on binary trees

Claudia-Christophe Hopf algebra on permutations

number of
alternative tableaux

Prop. The number of alternative tableaux of size n is $(n+1)!$



$n = 12$

ex: $n=2$

bijection

permutations --- alternative tableaux

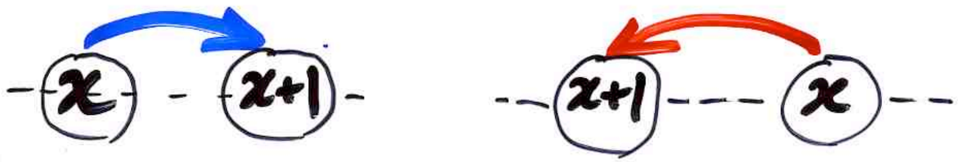
The "exchange-fusion" algorithm

Def- Permutation $\sigma = \sigma(1) \cdots \sigma(n)$

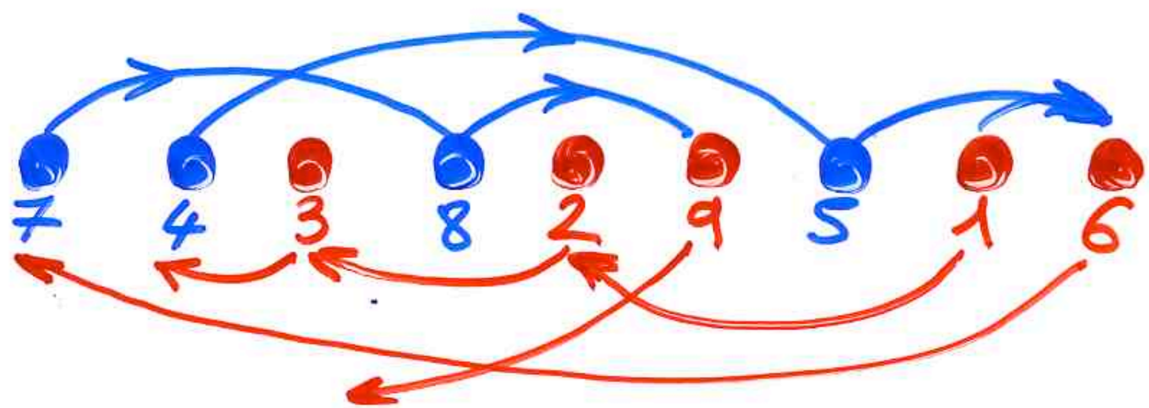
$x = \sigma(i)$, $1 \leq x < n$

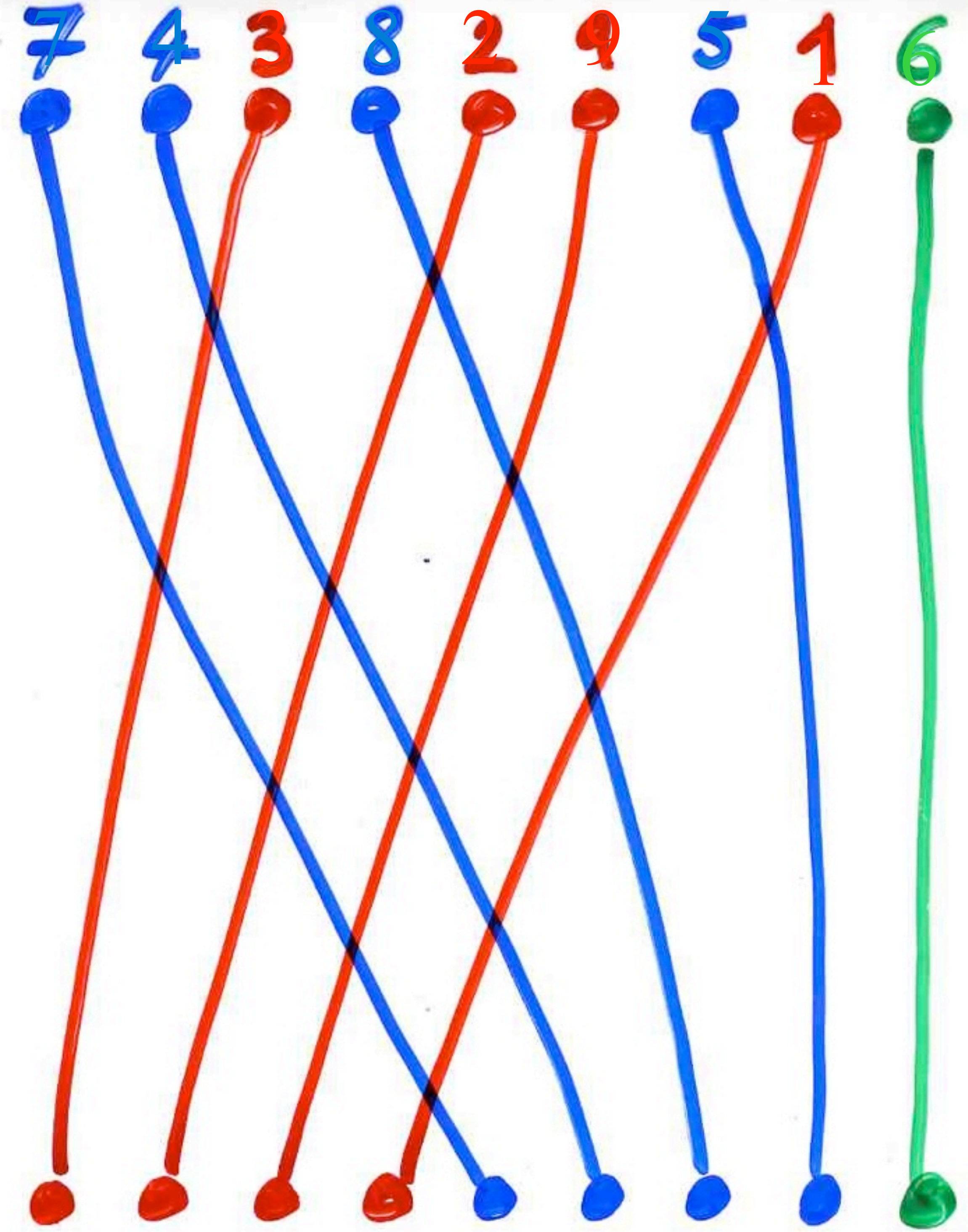
$$\text{(valeur)}\ x \begin{cases} \text{avance} \\ \text{recul} \end{cases} \quad x+1 = \sigma(j), \quad \begin{cases} i<j \\ j<i \end{cases}$$

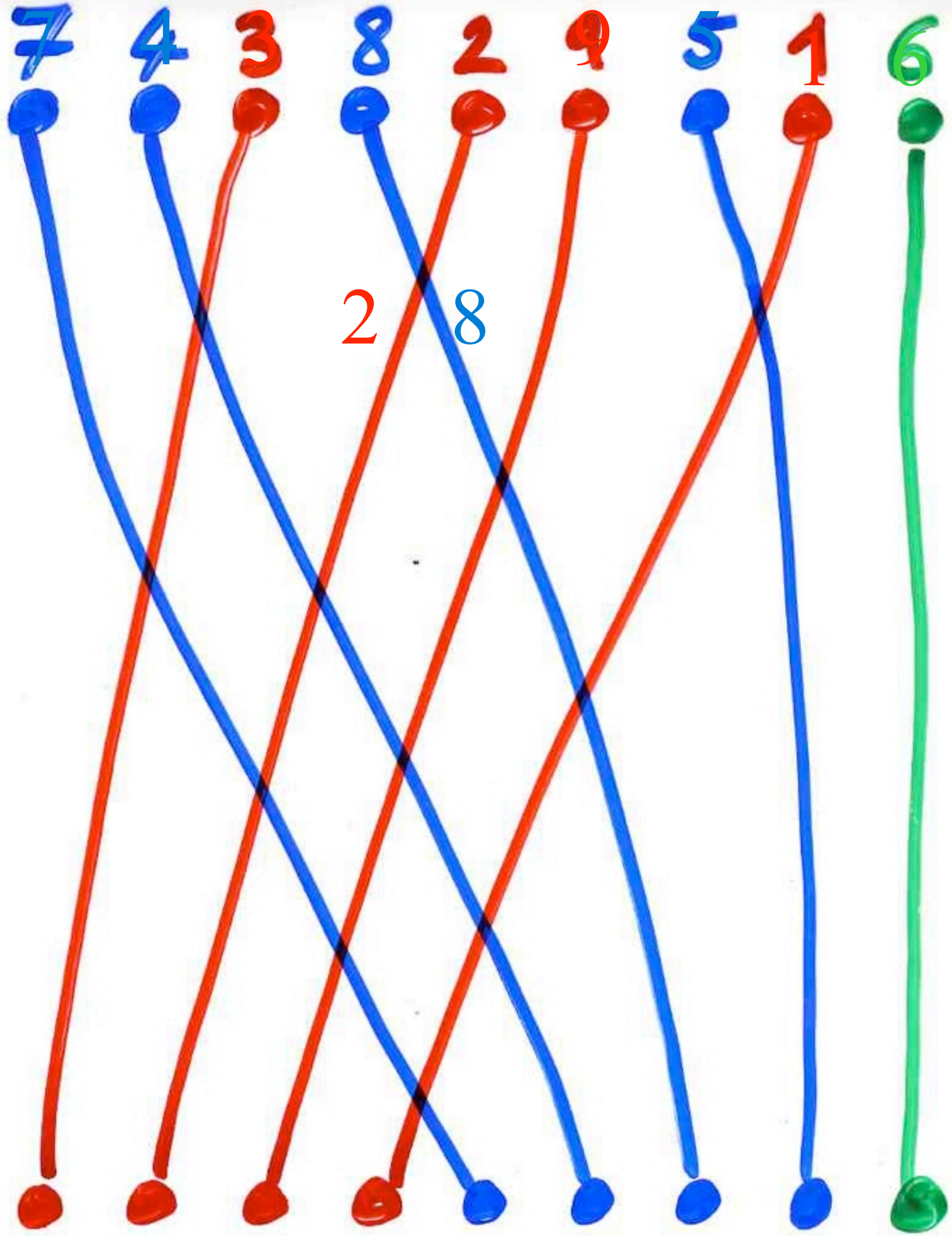
- convention $x = n$ est un recul

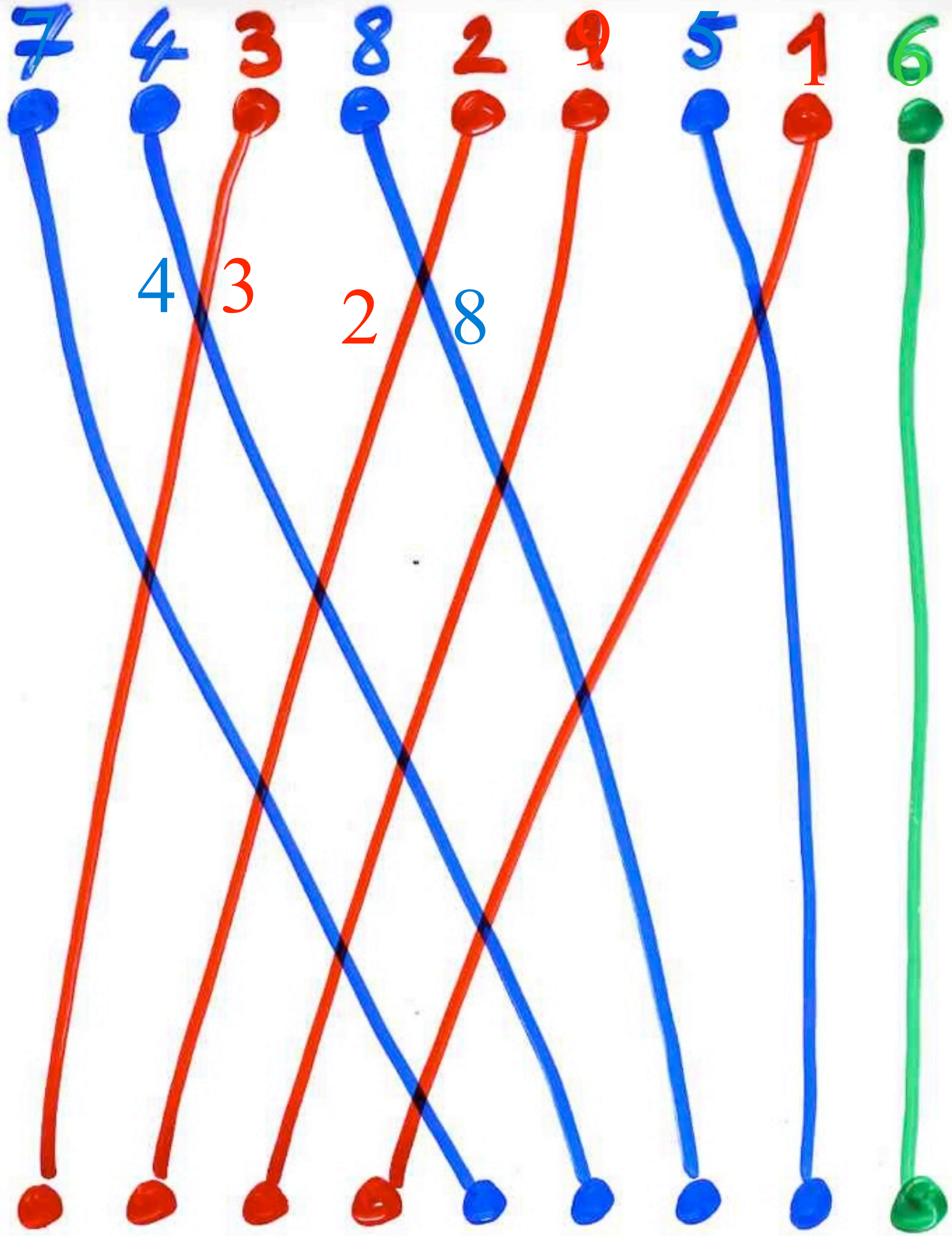


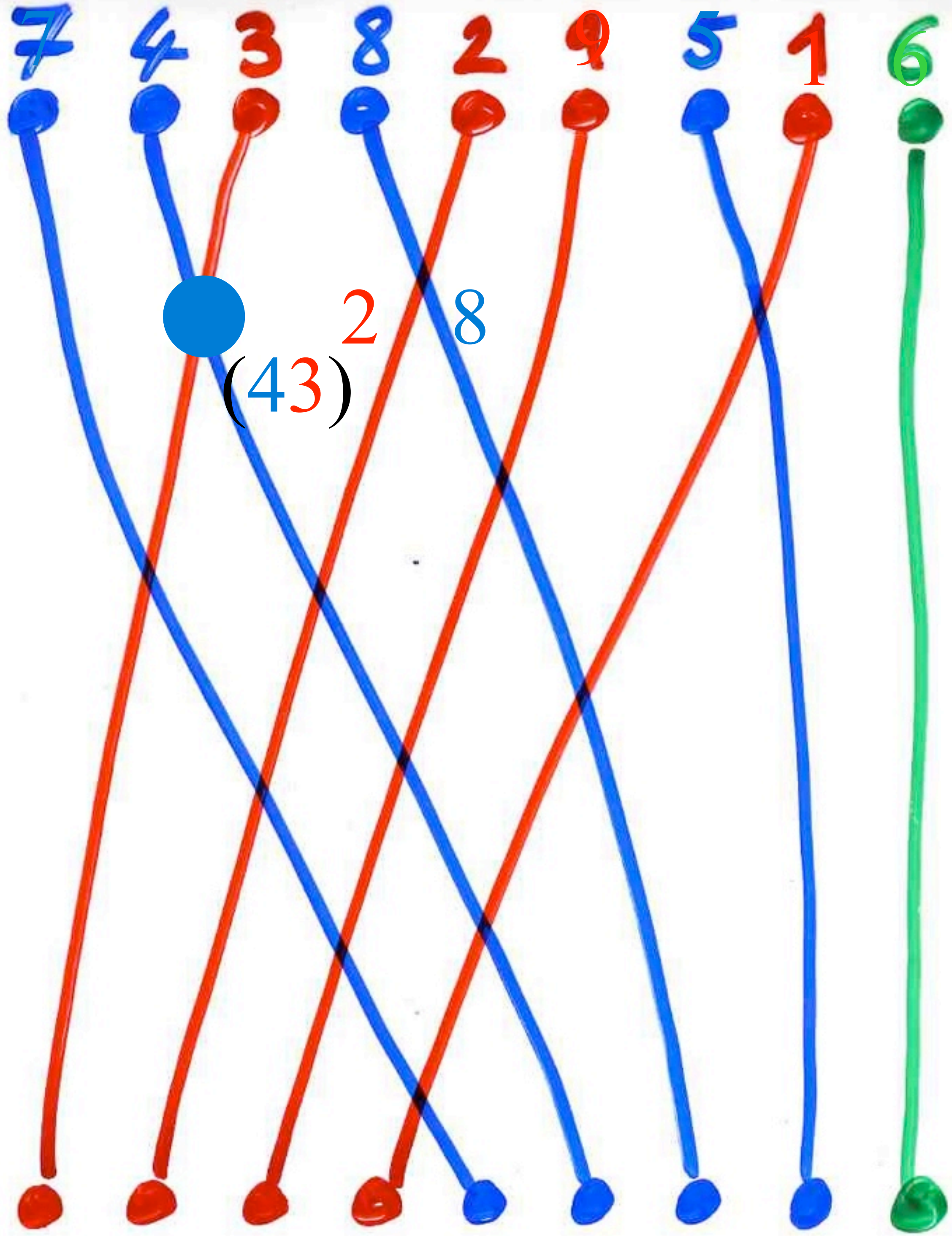
$\sigma =$ 7 4 3 8 2 9 5 1 6



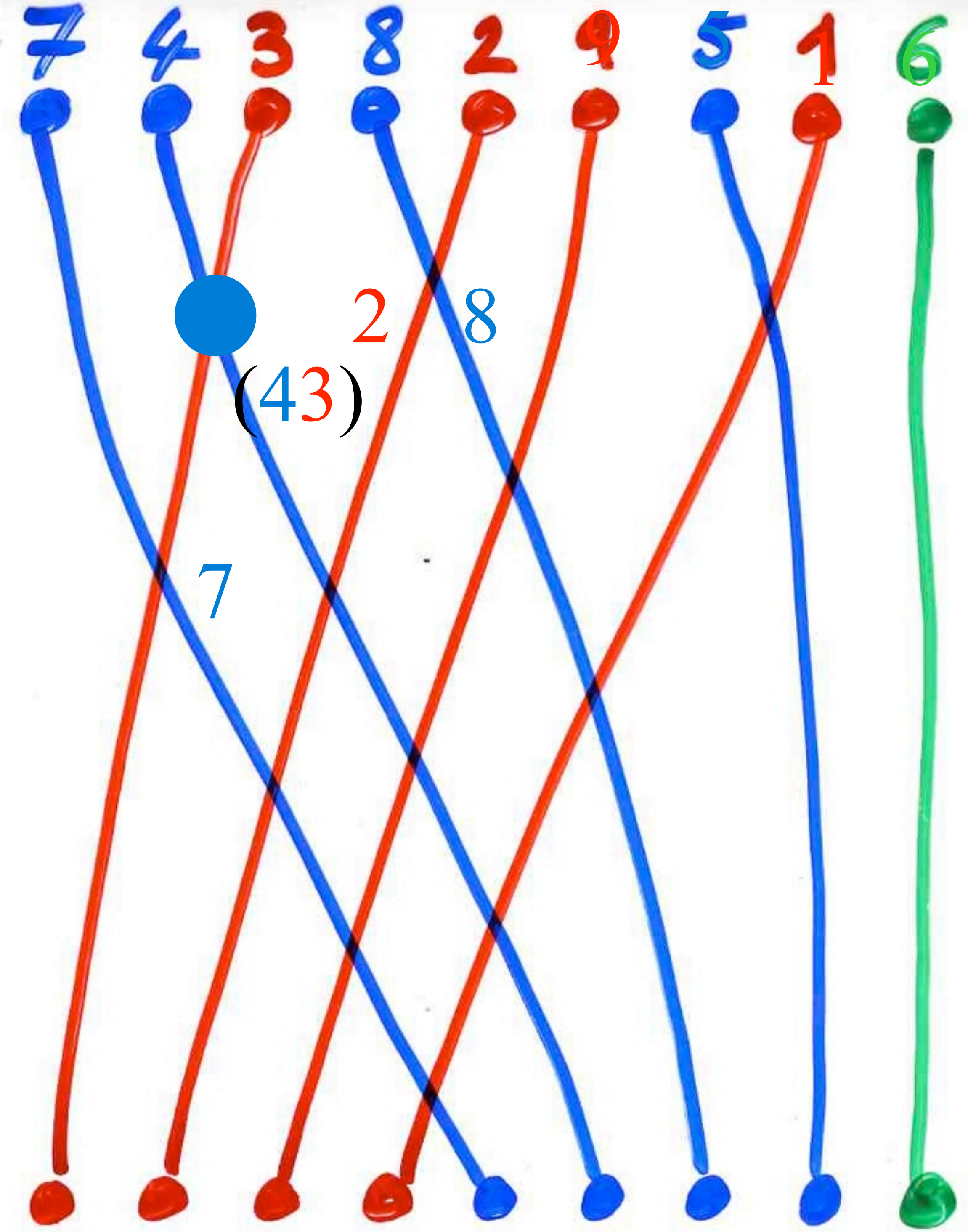

7 4 3 8 2 9 5 1 6

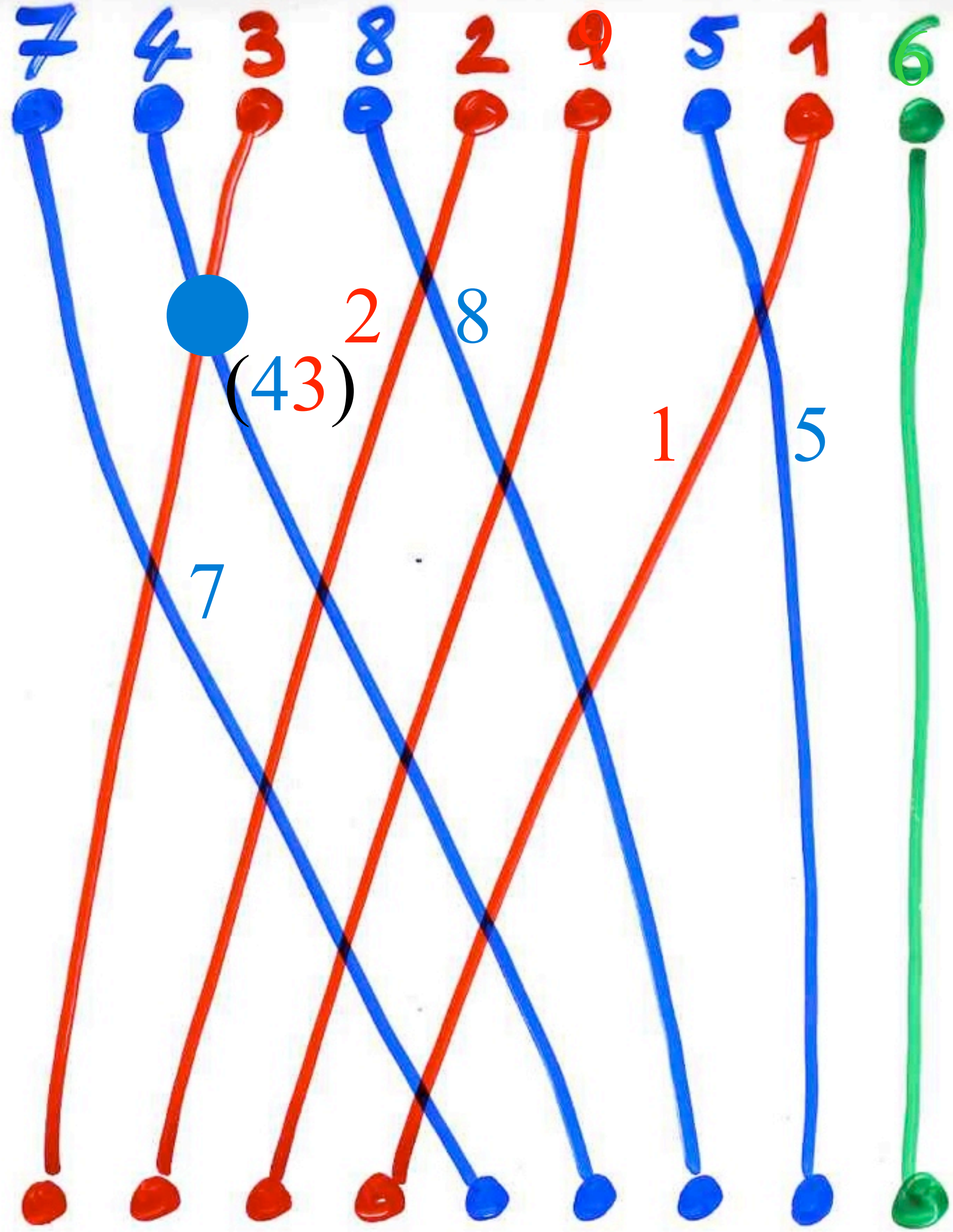

7 4 3 8 2 9 5 1 6
2
8


7 4 3 8 2 9 5 1 6
4 3
2 8

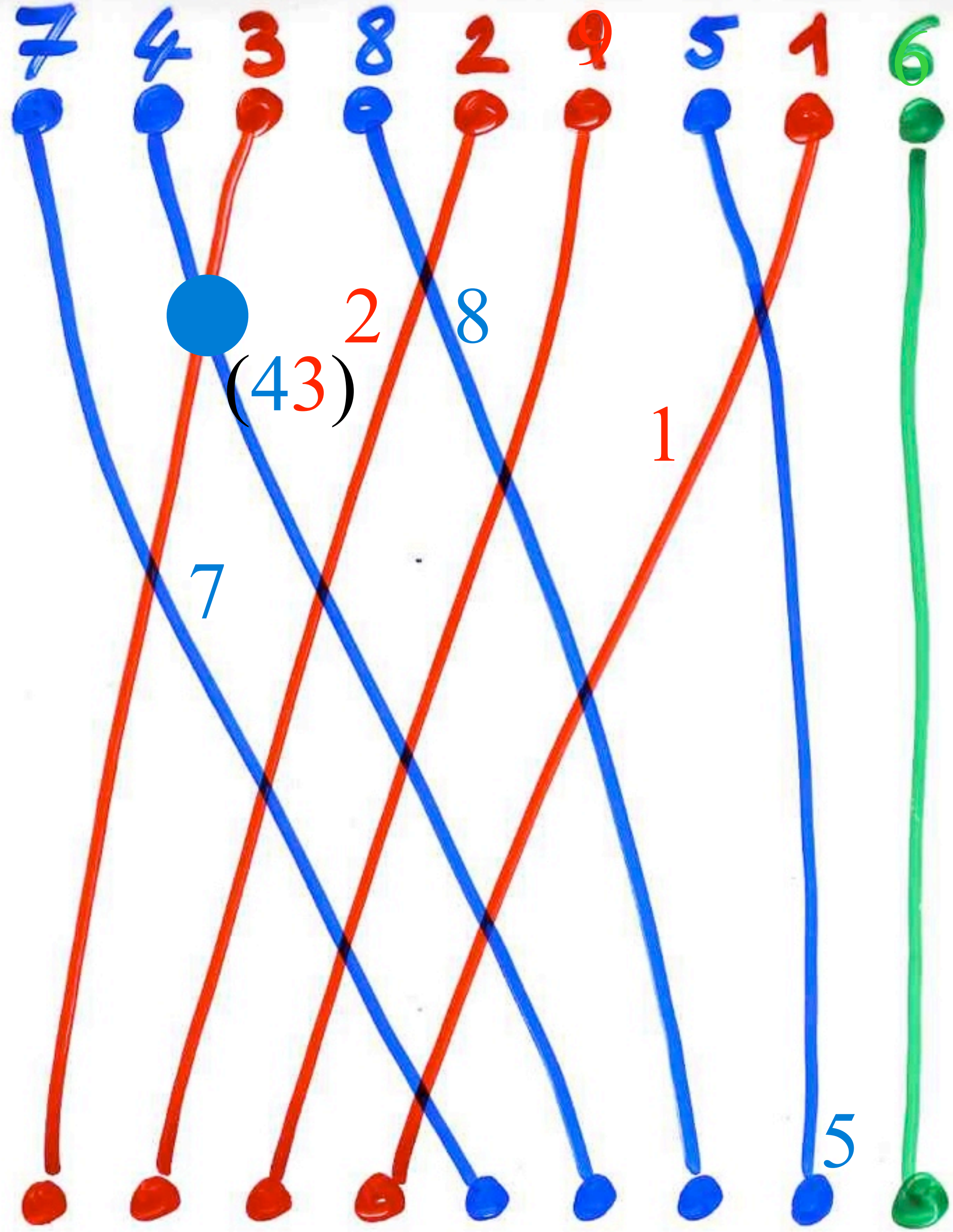


7 4 3 8 2 9 5 1 6
2
8
(43)

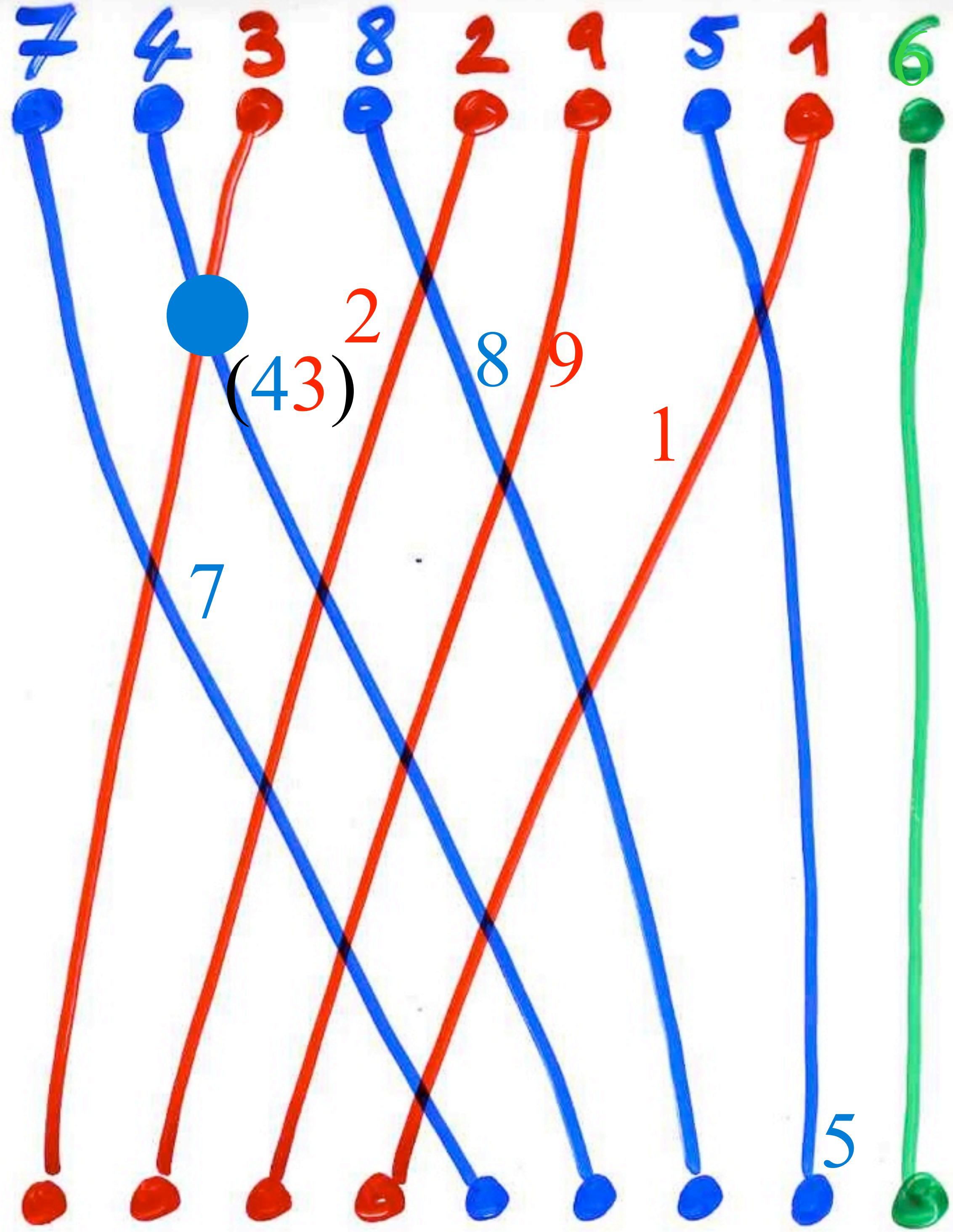

7 4 3 8 2 9 5 1 6
2
8
(43)
7



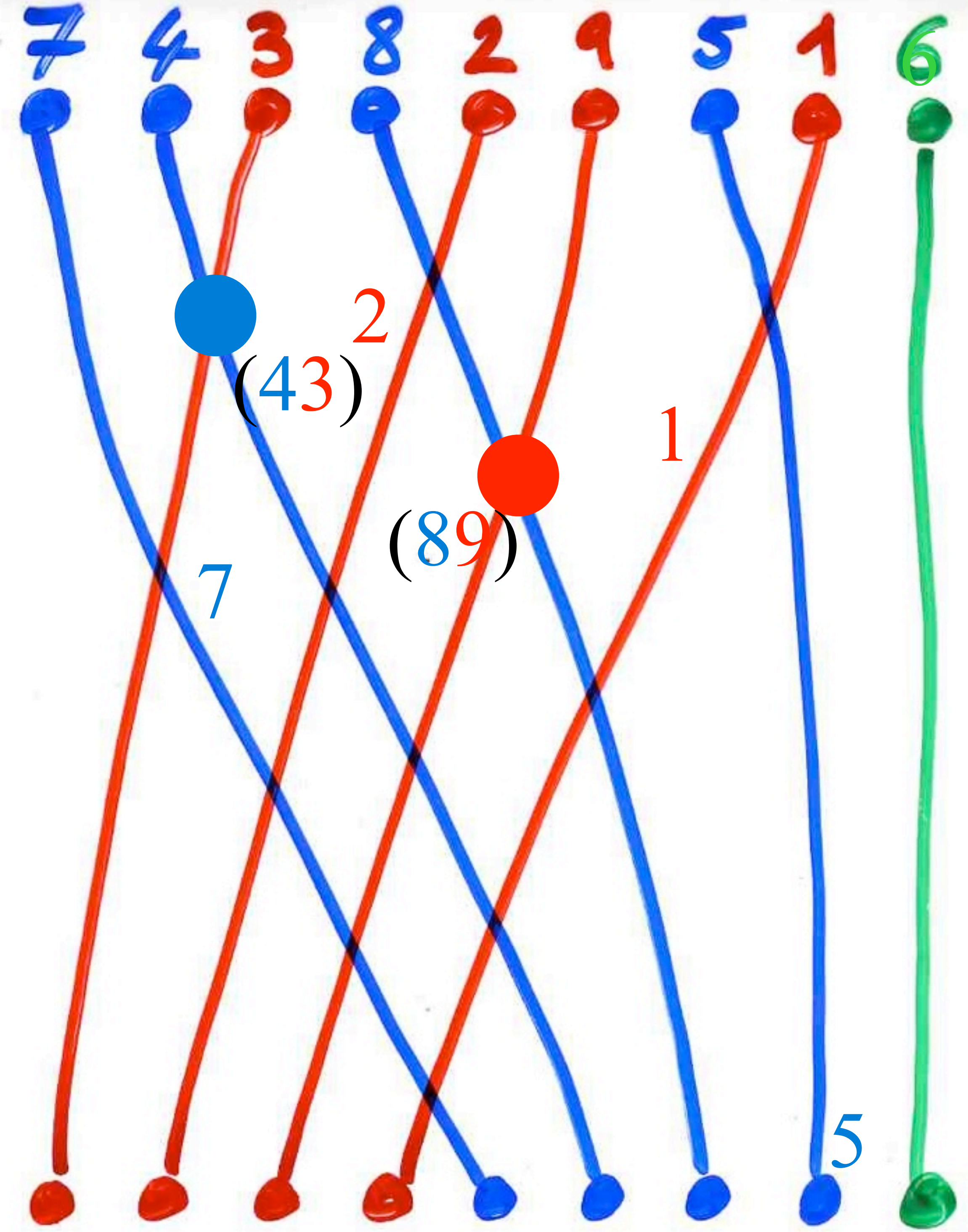
7 4 3 8 2 9 5 1 6
(43)
2
8
1
5
7

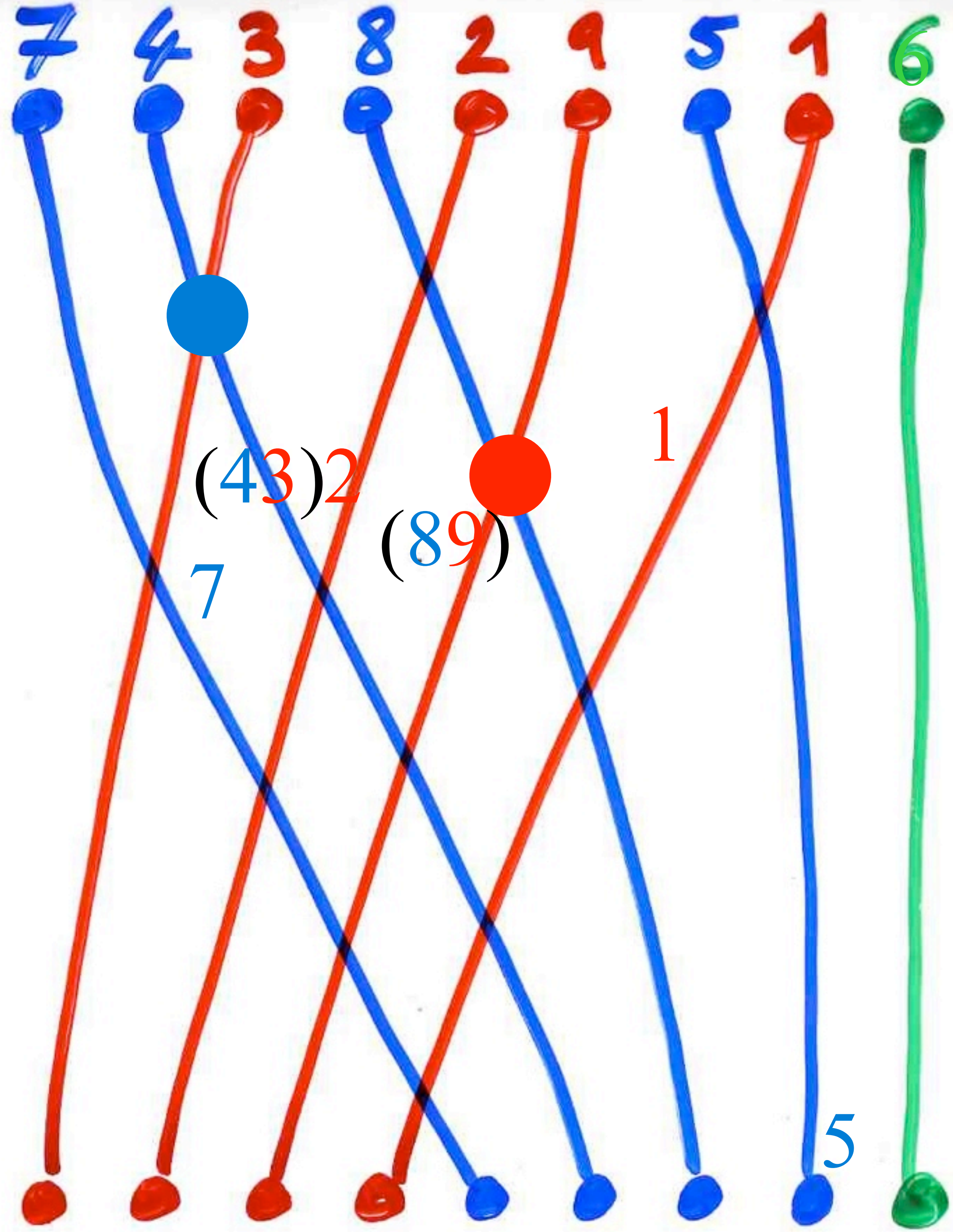


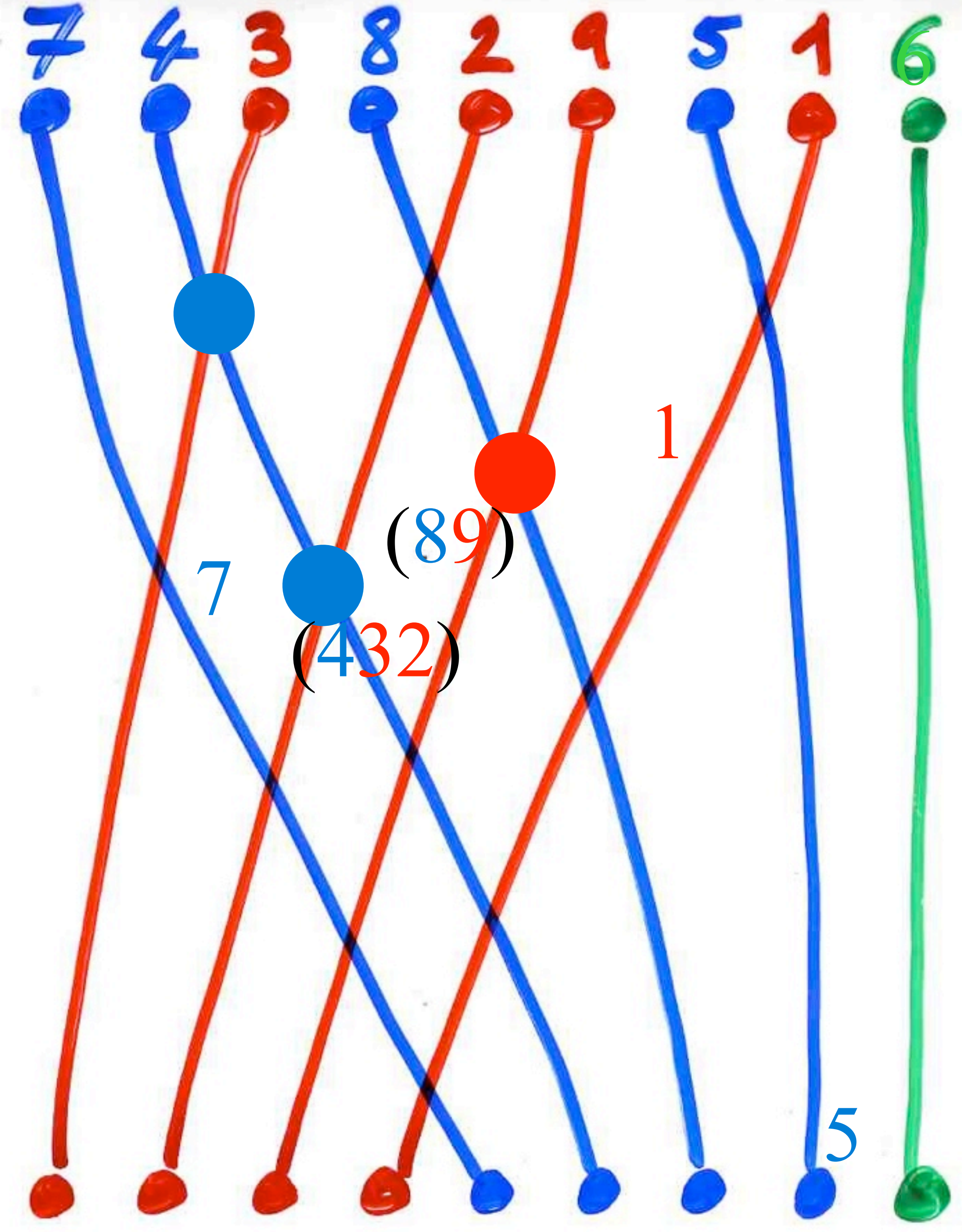
7 4 3 8 2 9 5 1 6
(43)
2
8
1
7
5



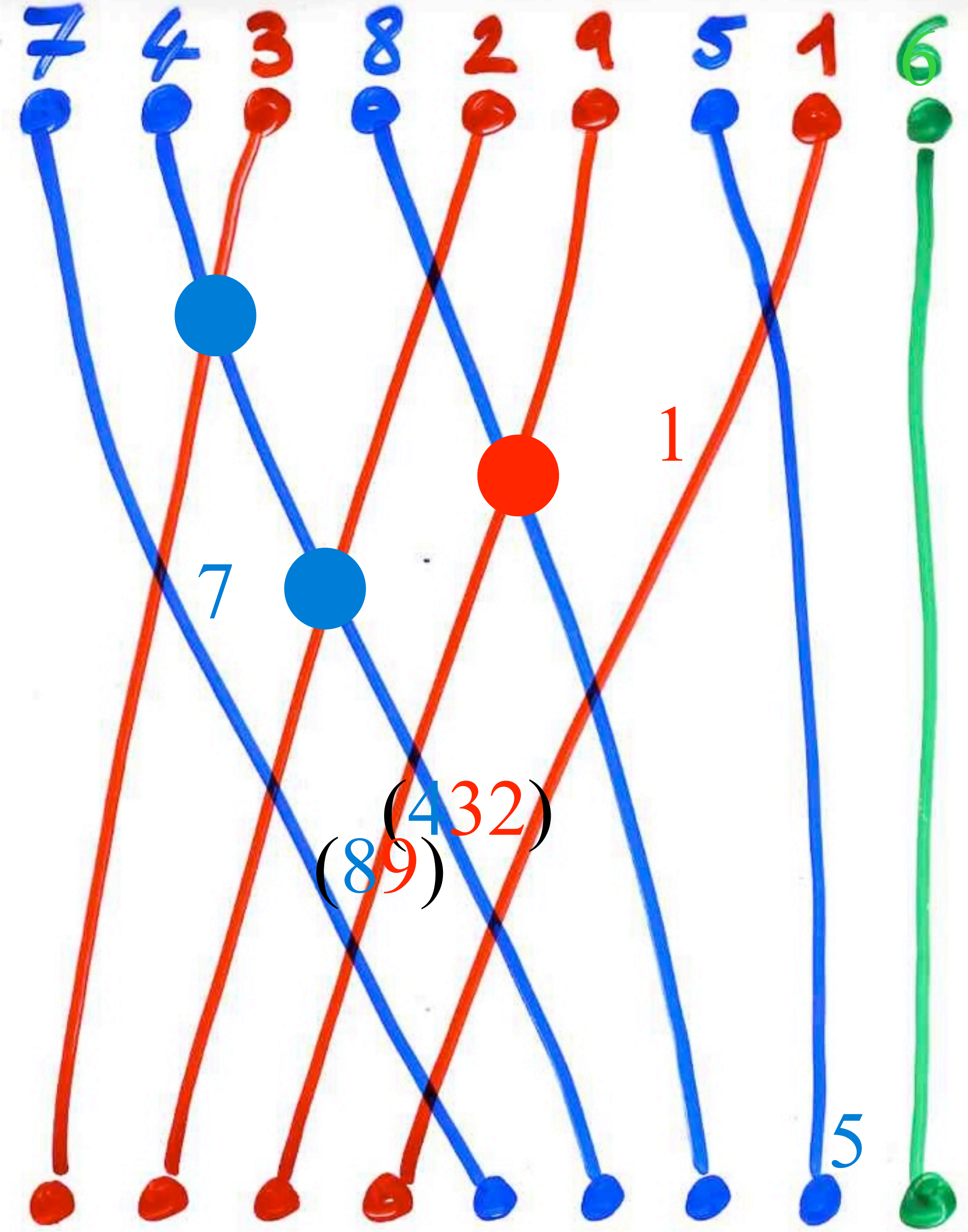
7 4 3 8 2 9 5 1 6
(43)
2
8
9
1
7
5


7 4 3 8 2 9 5 1 6
(43)
2
1
(89)
7
5

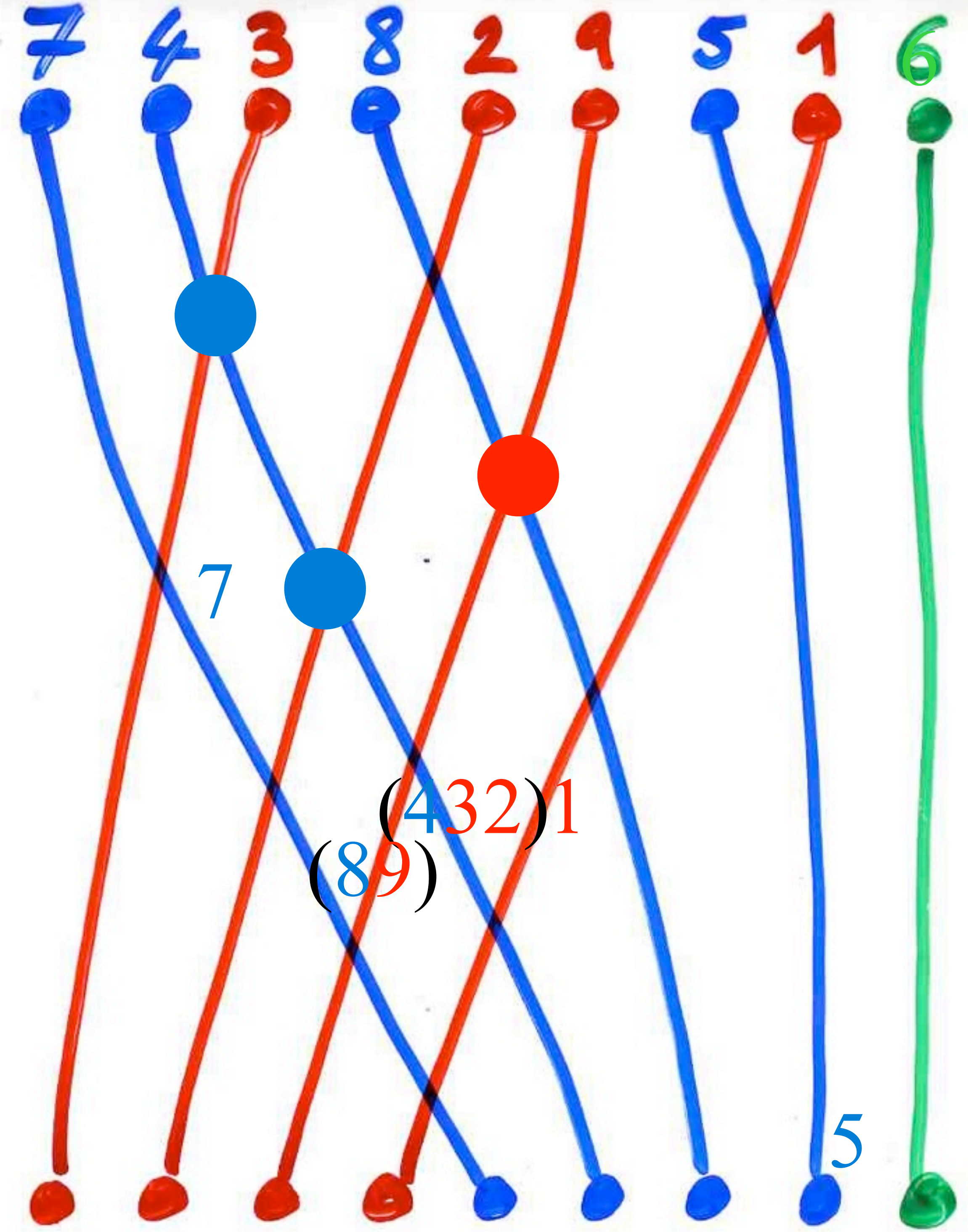

7 4 3 8 2 9 5 1 6
(43)2
(89)
1
7
5

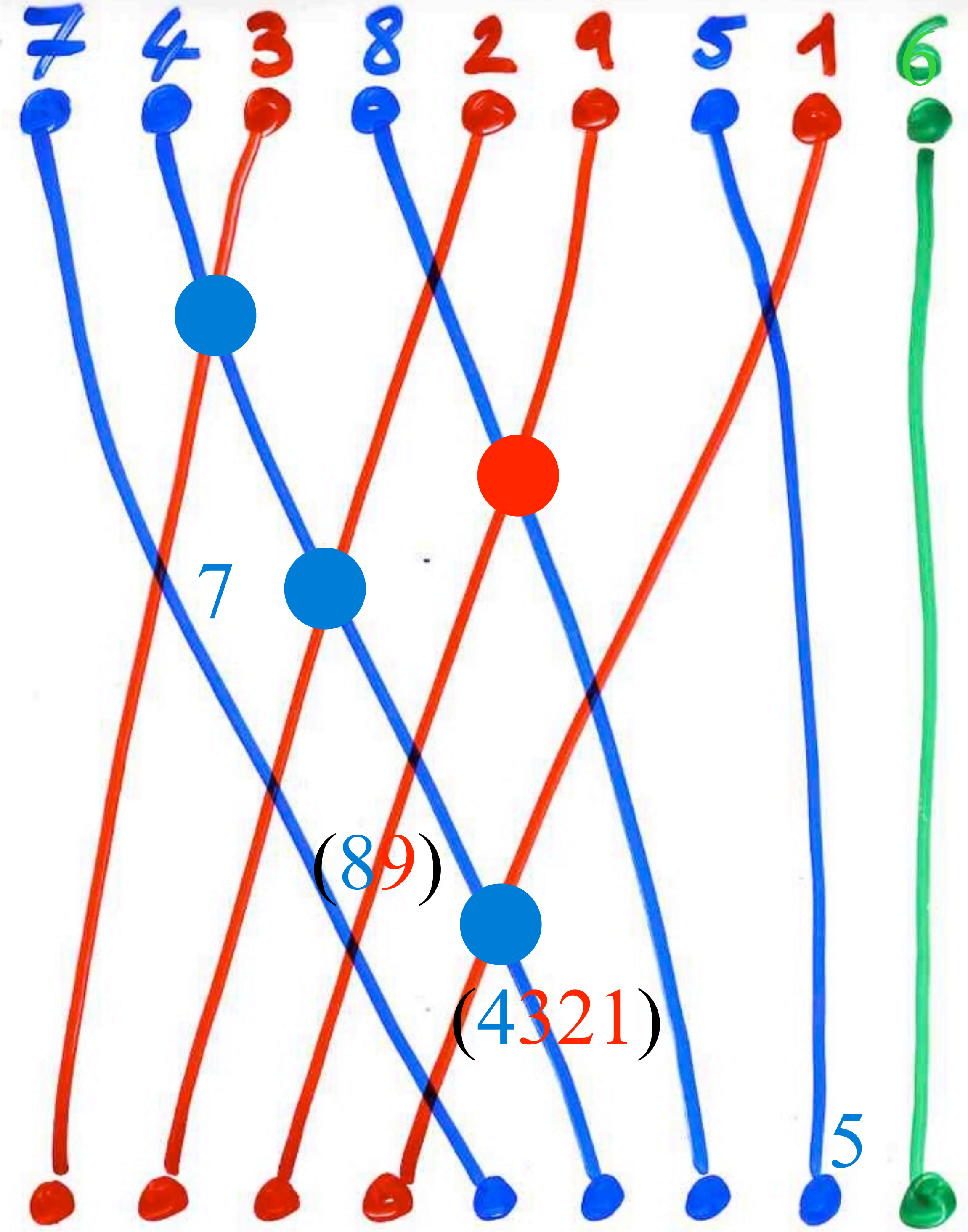


7 4 3 8 2 9 5 1 6
1
(89)
7
(432)
5

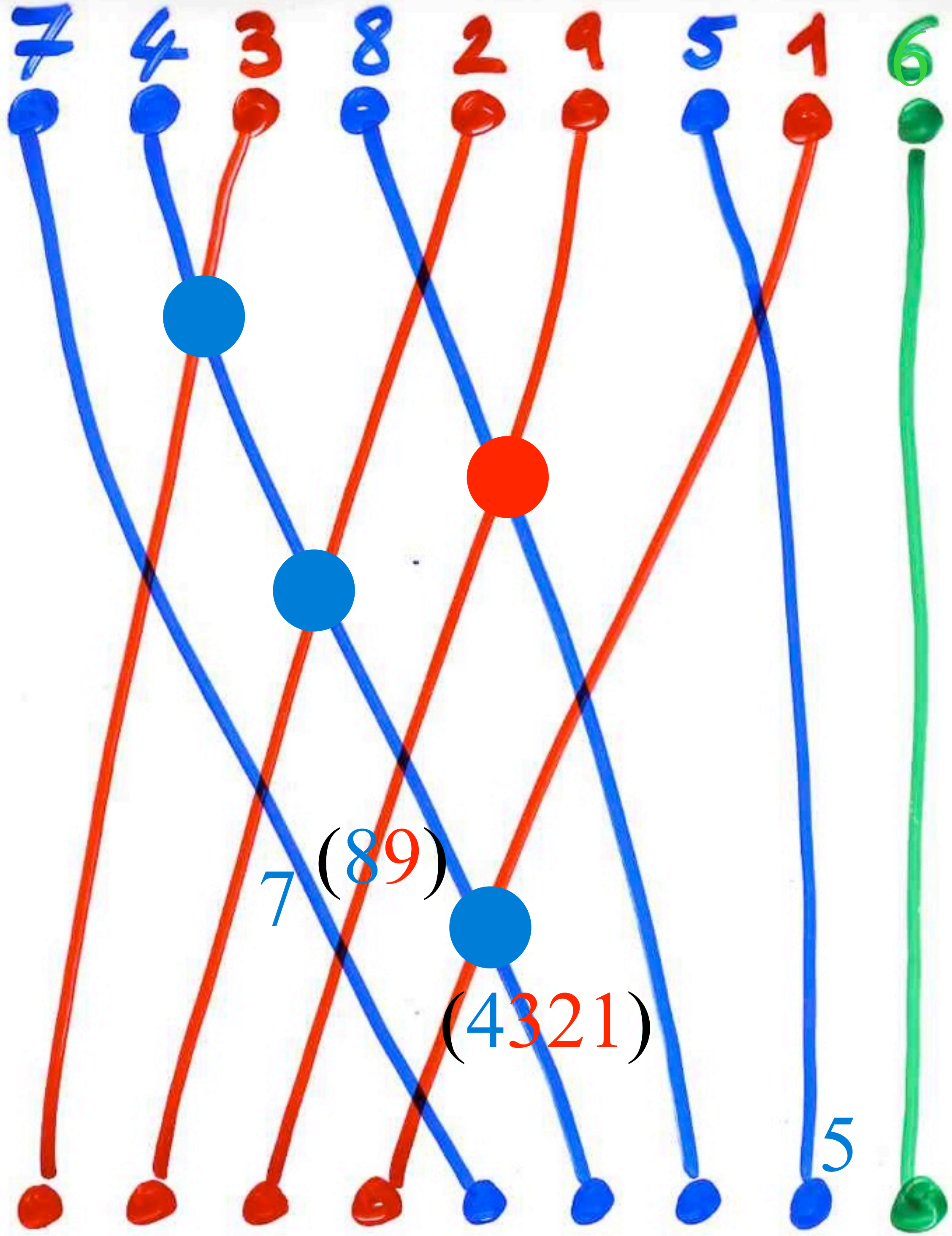


7 4 3 8 2 9 5 1 6
1
7
(432)
(89)
5

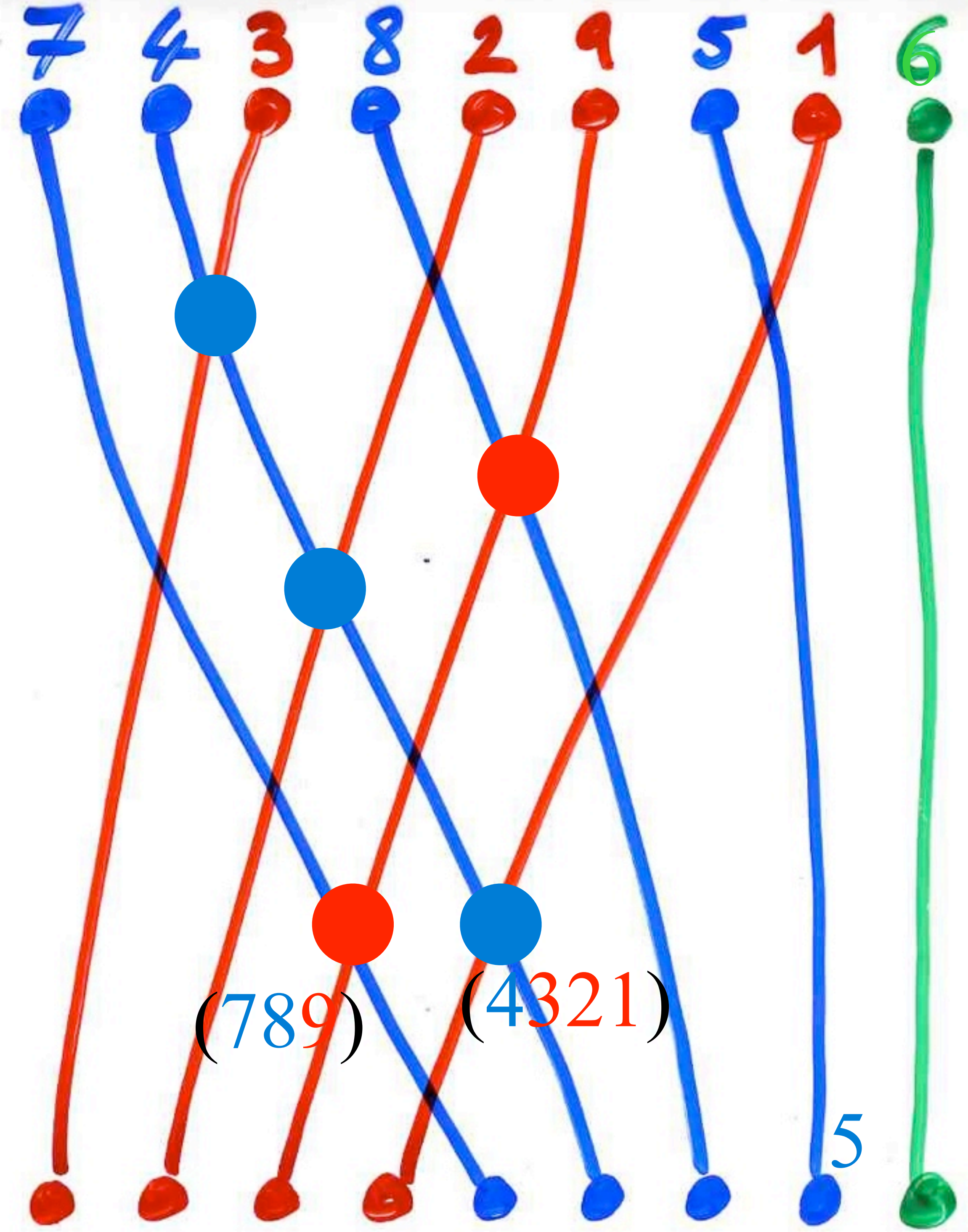

7 4 3 8 2 9 5 1 6
7
(432)1
(89)
5



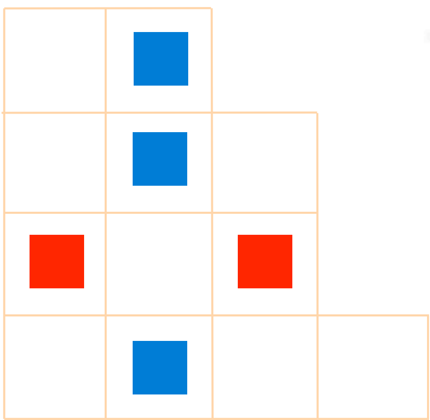
7 4 3 8 2 9 5 1 6
7
(89)
(4321)
5

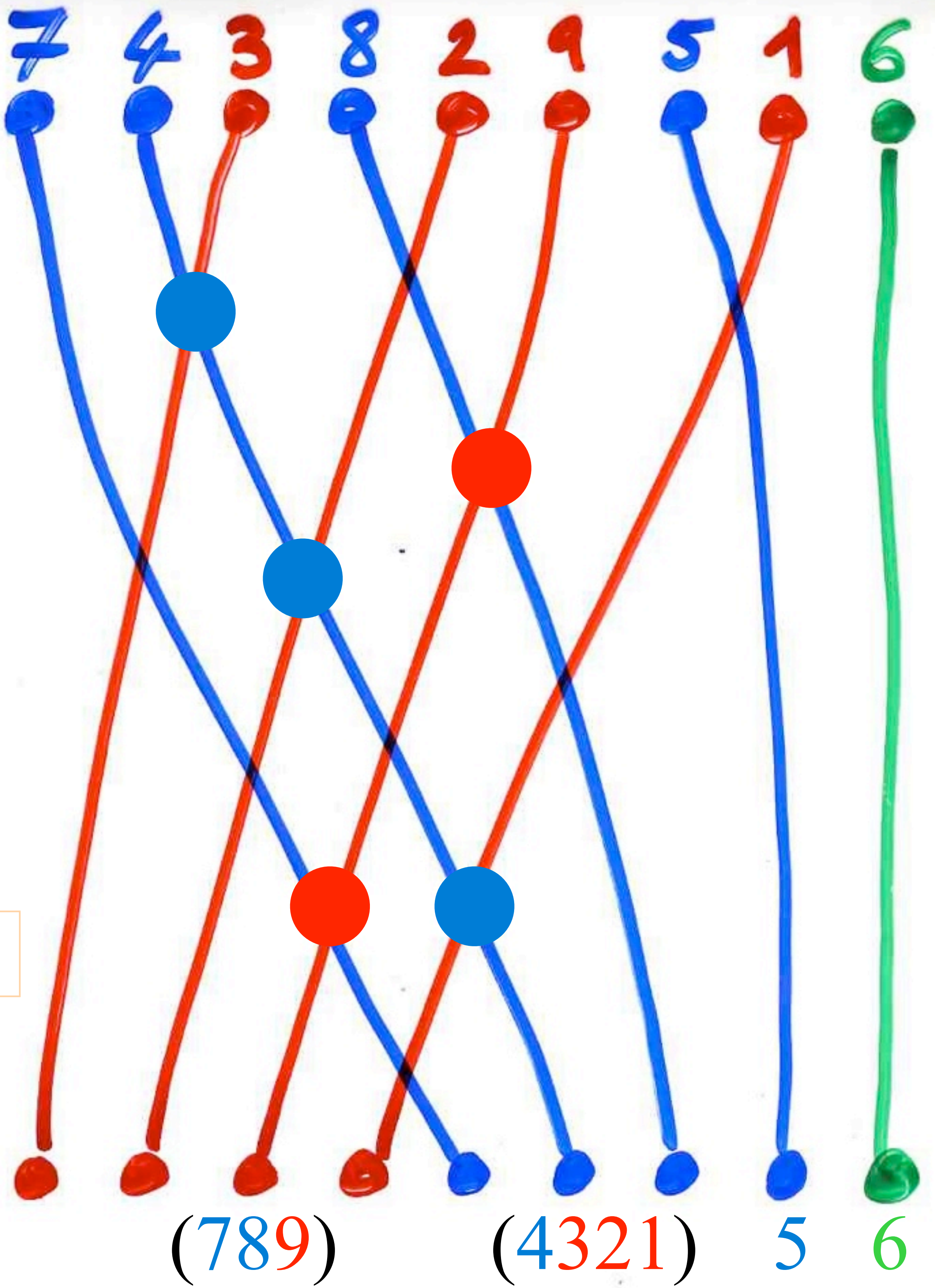


7 4 3 8 2 9 5 1 6
(89)
7
(4321)
5

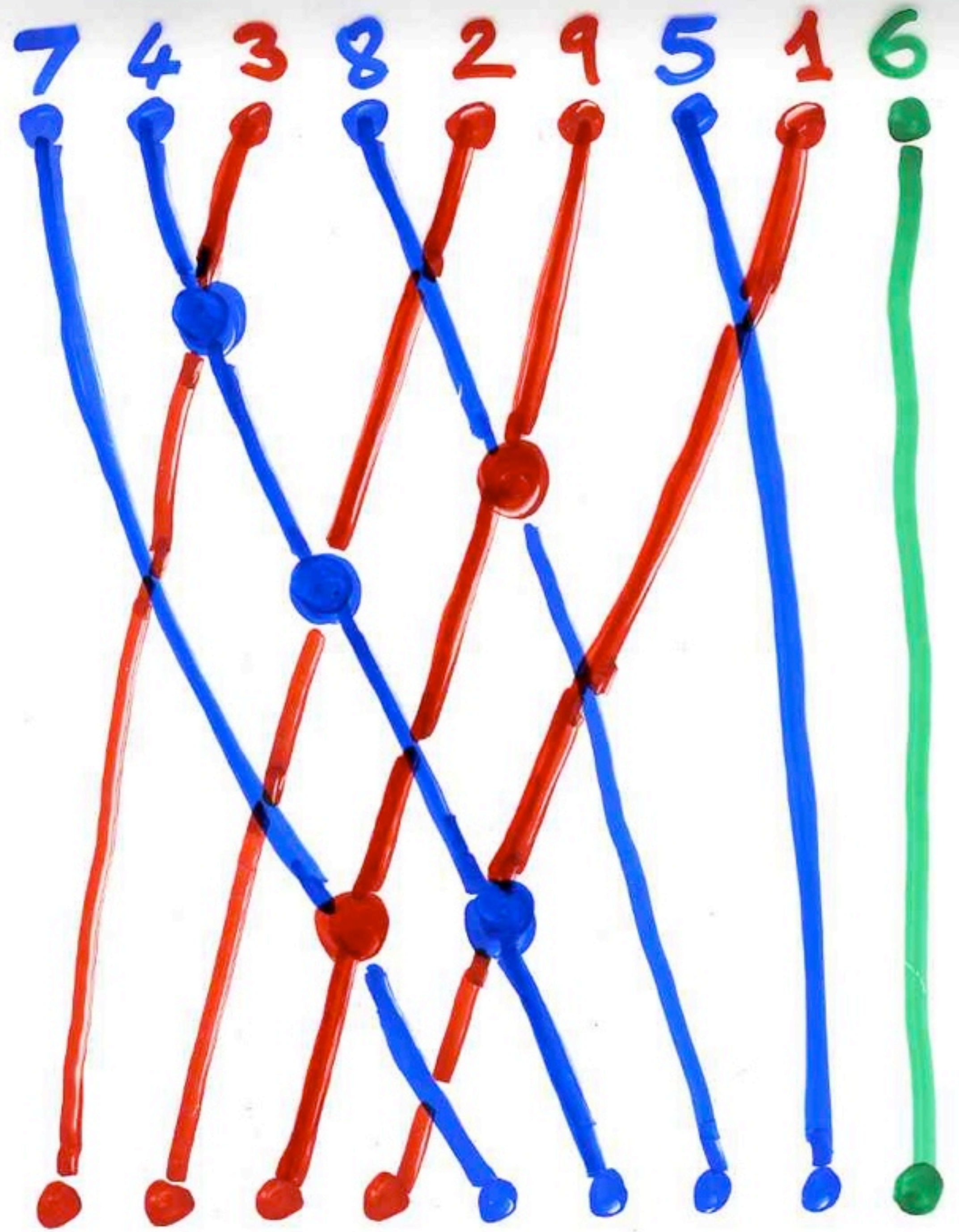

7 4 3 8 2 9 5 1 6
(789)
(4321)
5

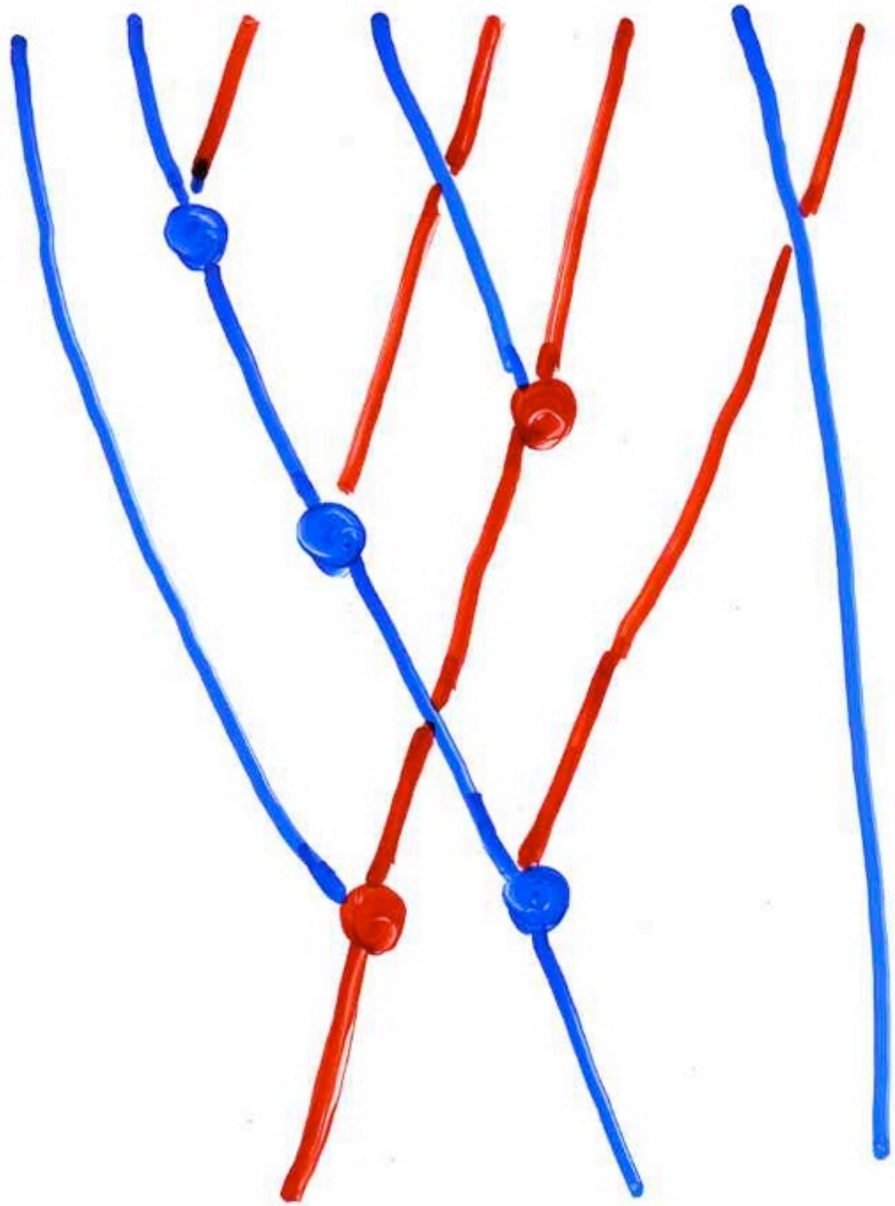
“exchange-fusion” algorithm

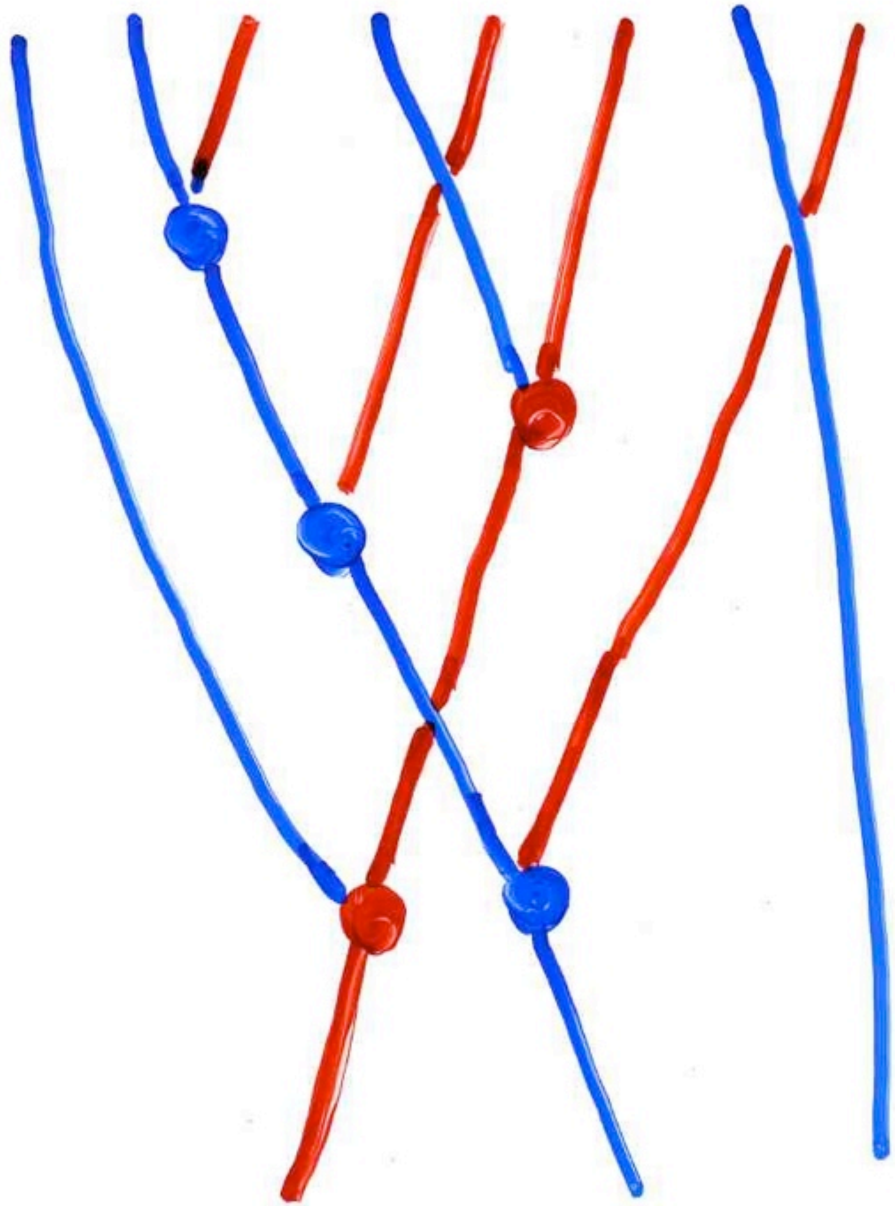




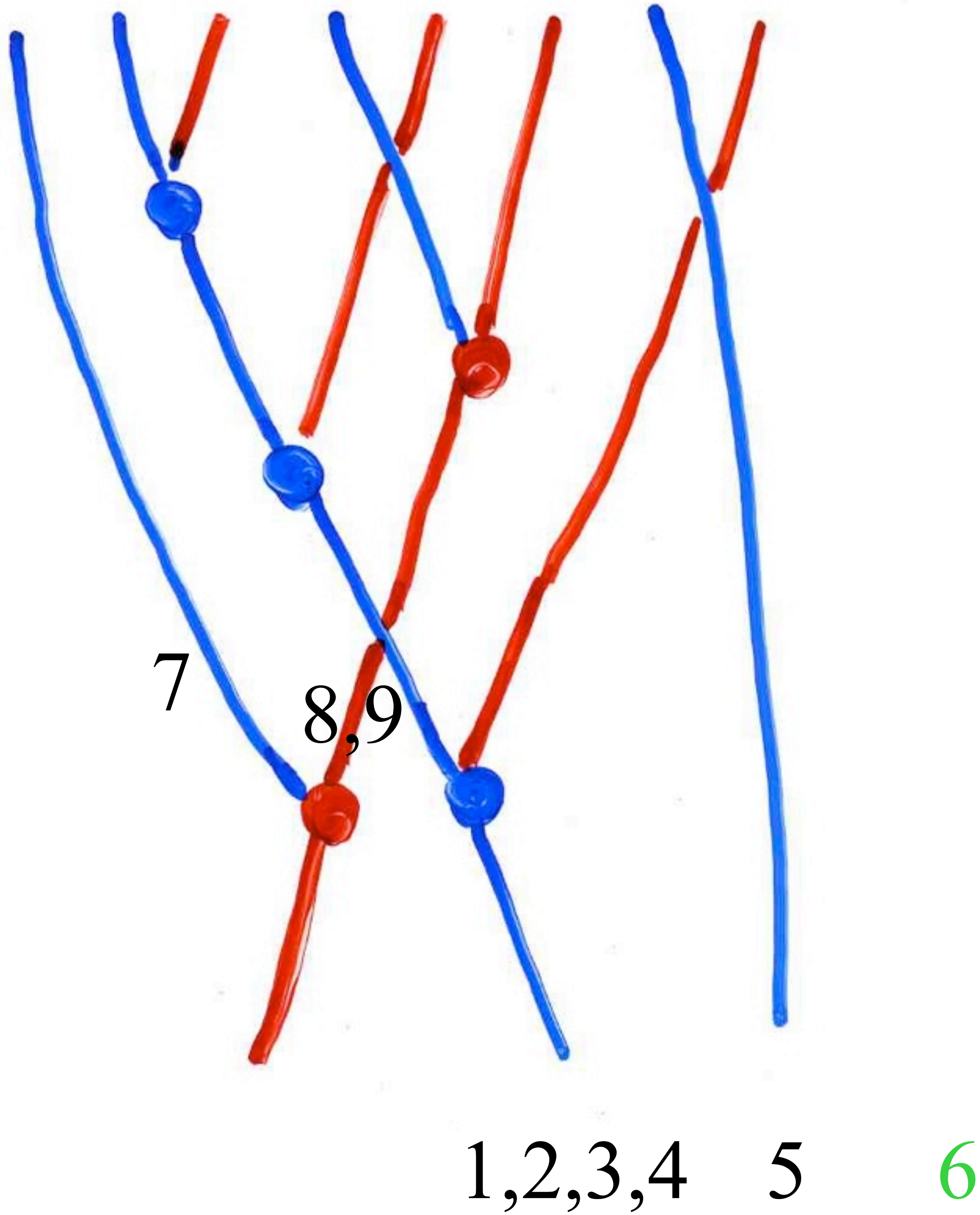
The inverse "exchange-fusion" algorithm

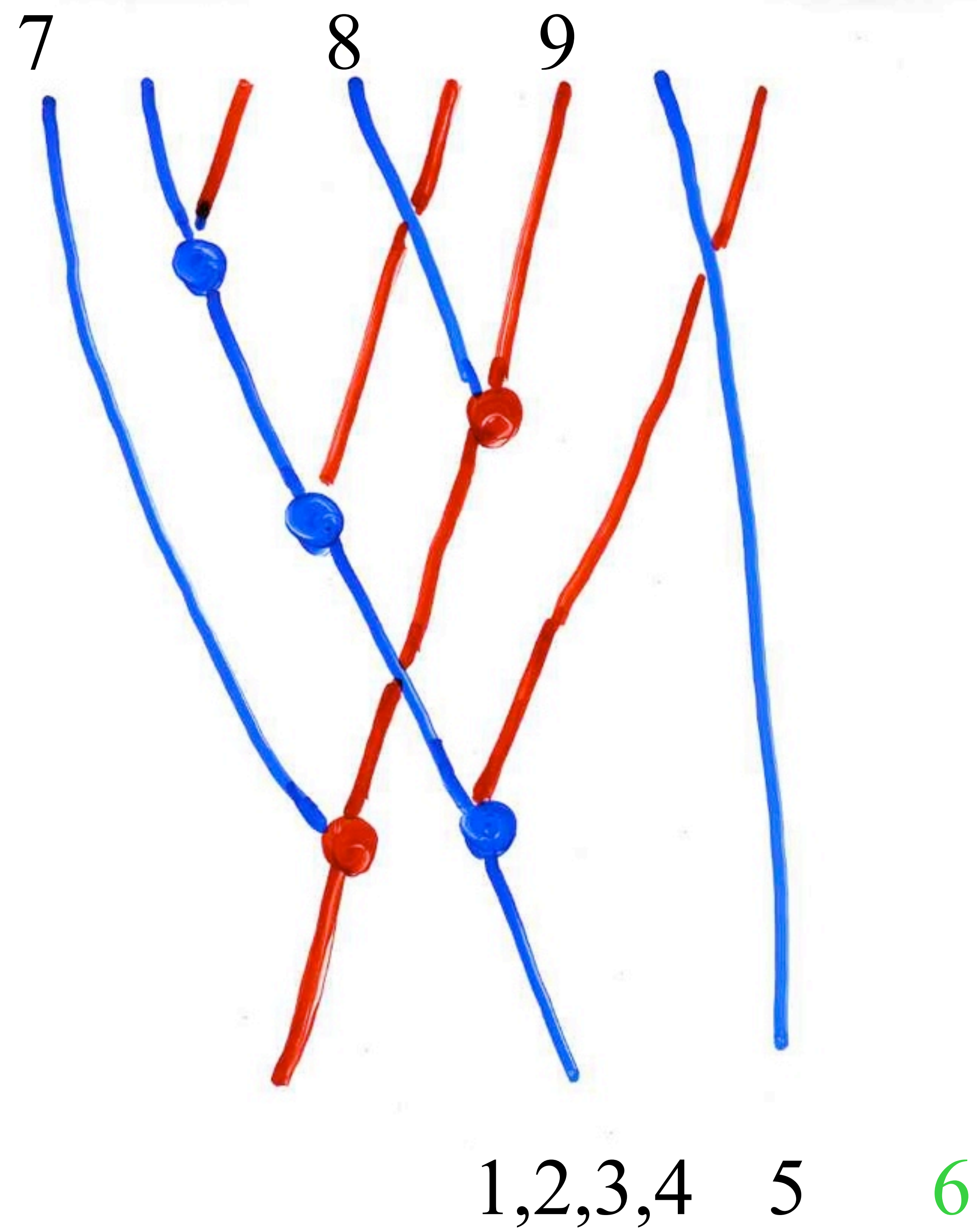

7 4 3 8 2 9 5 1 6

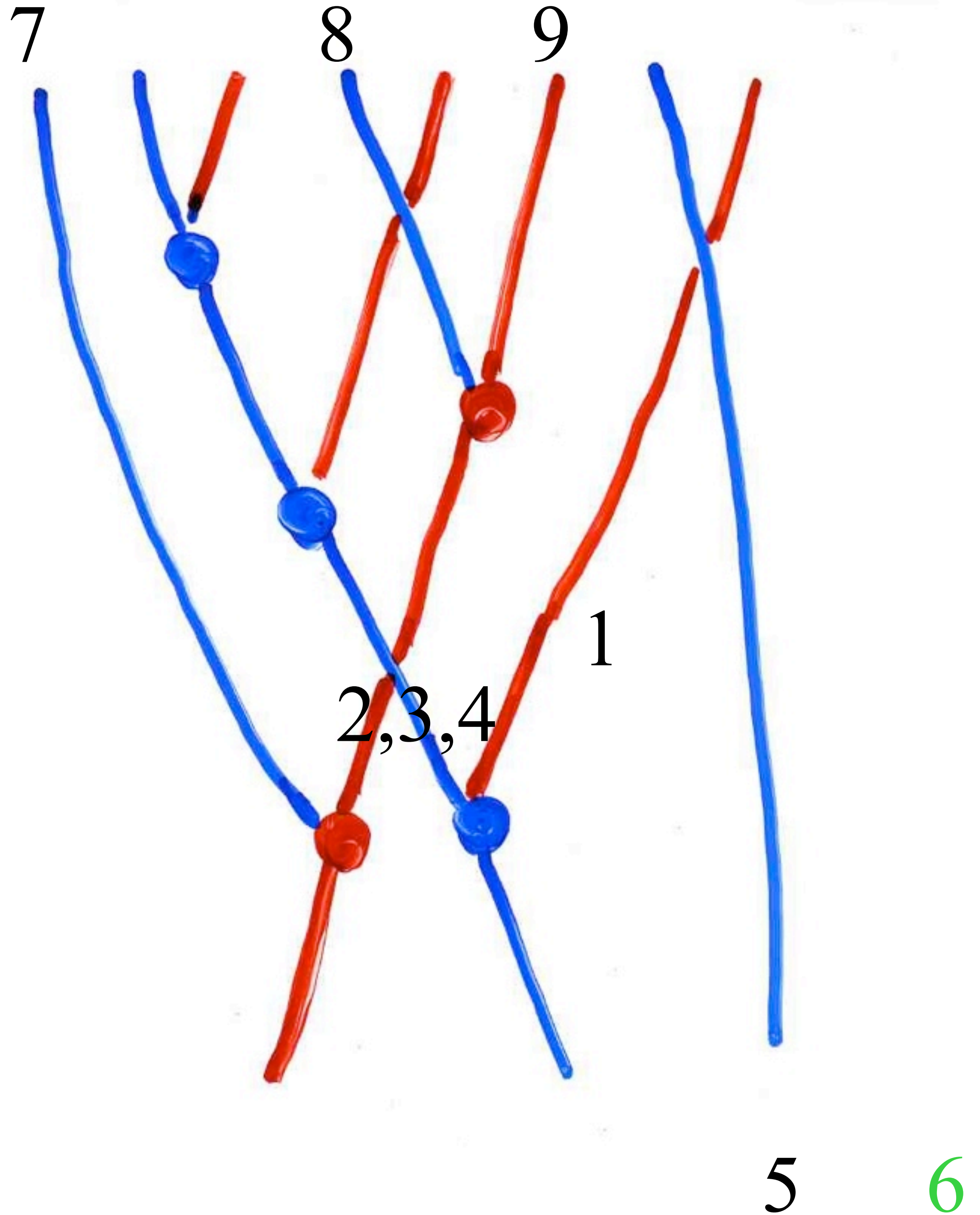


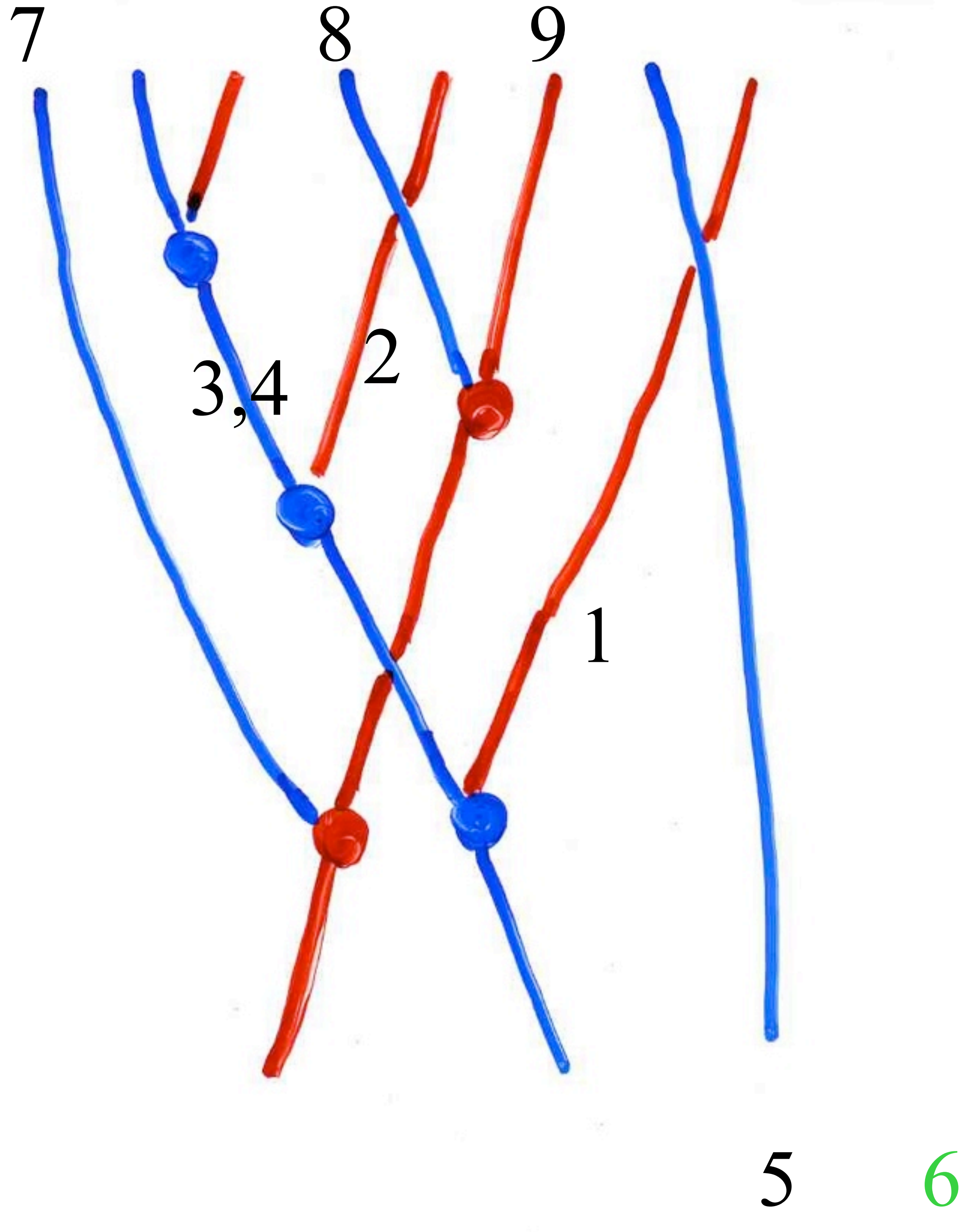


7,8,9 1,2,3,4 5 6

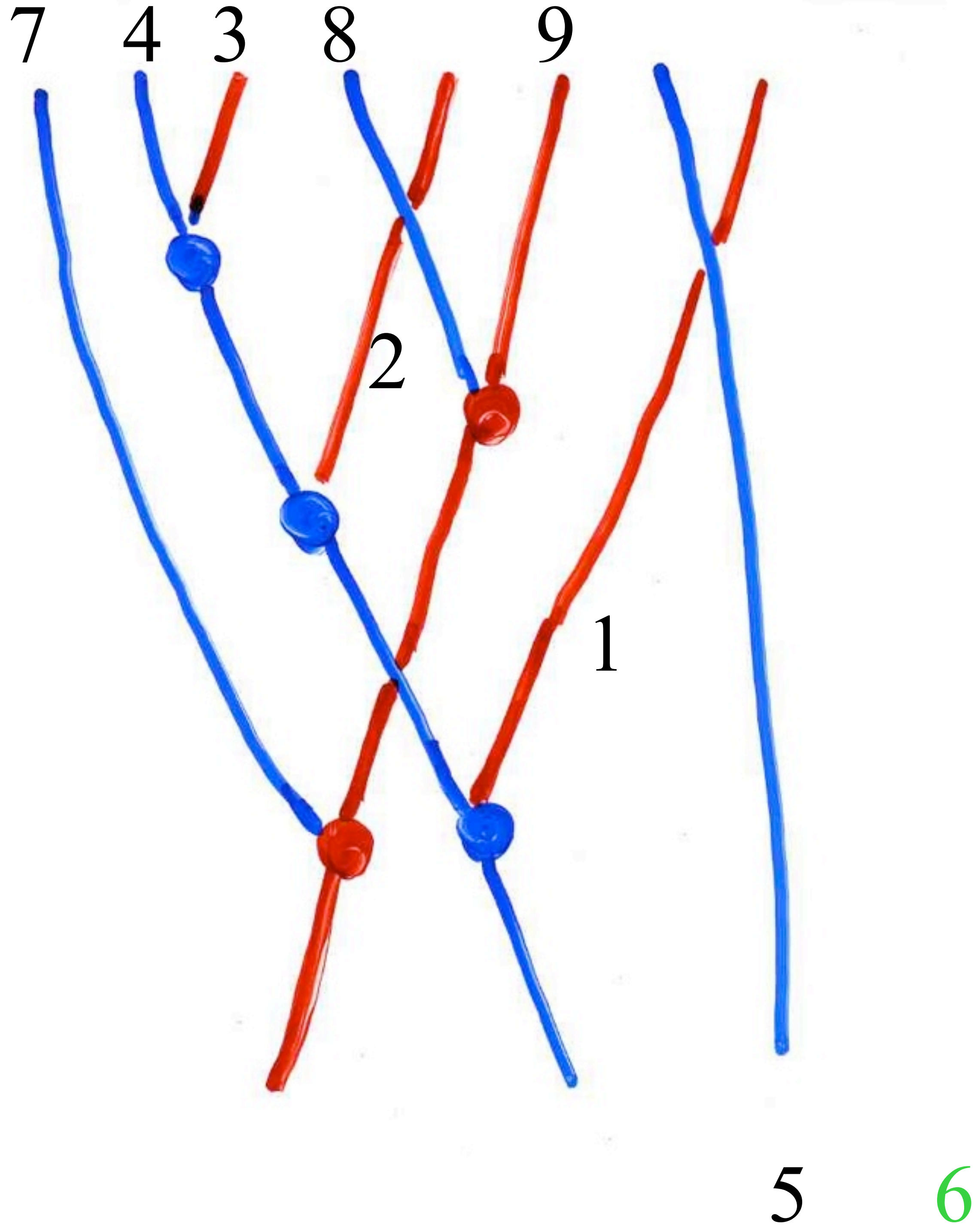

7
8,9
1,2,3,4
5
6

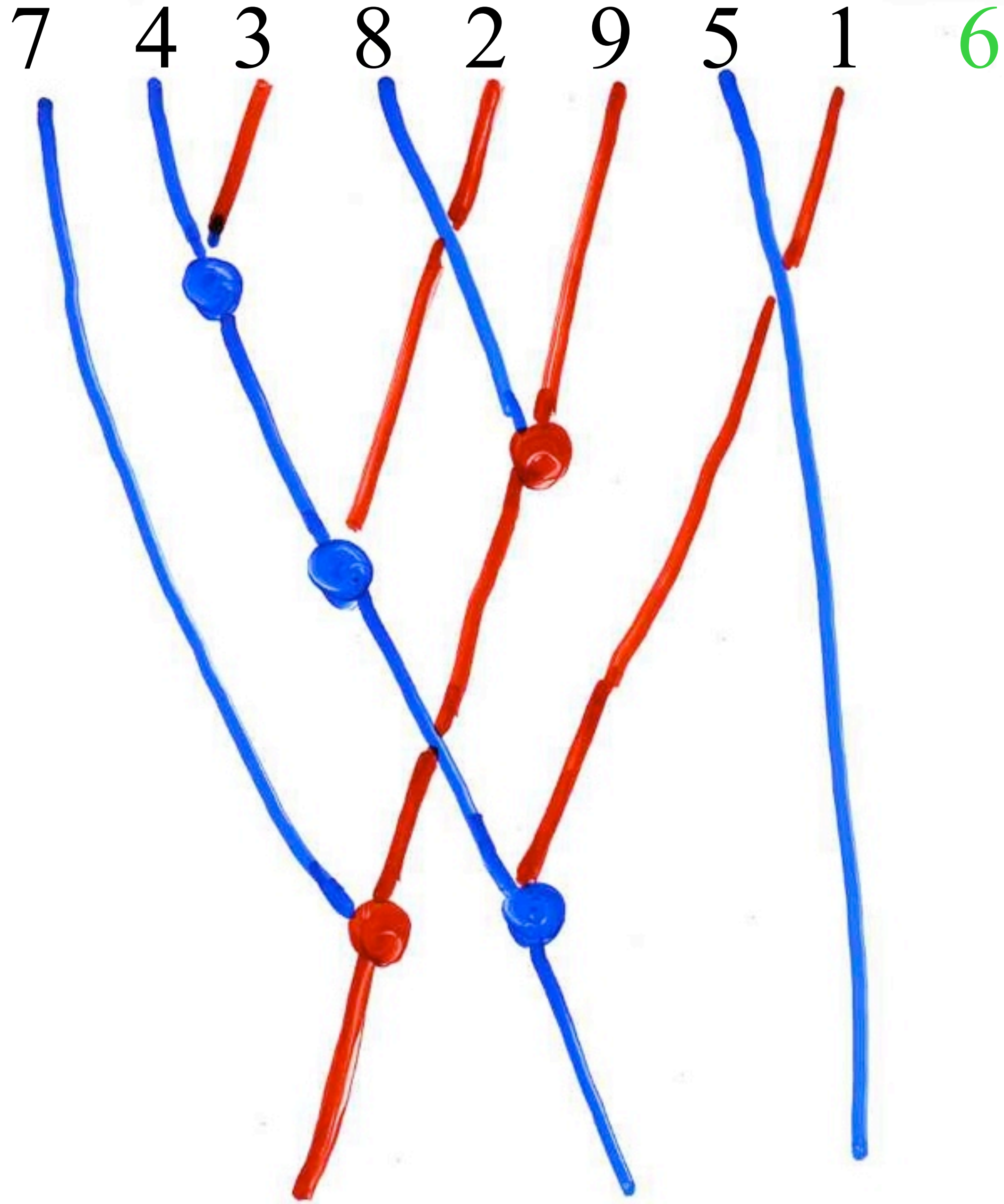

7
8
9
1,2,3,4
5
6


7
8
9
1
2,3,4
5
6



7
8
9
3,4
2
1
5
6


7 4 3 8 9
2
1
5 6


7 4 3 8 2 9 5 1 6

this bijection was constructed
from a combinatorial representation
of the PASEP algebra

and using the methodology
of the «Cellular Ansatz»

"The cellular Ansatz"

combinatorial
objects
on a 2d lattice

representation
by operators

Physics

data structures
"histories"

orthogonal
polynomials

bijections

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

rooks placements

permutations

RSK

$\longleftrightarrow$ pairs of Tableaux Young

$DE = qED + E + D$

PASEP

alternative tableaux $\longleftrightarrow$ permutations

Laguerre histories

dynamical systems in physics
stationary probabilities

quadratic algebra Q

Q-tableaux

commutations
rewriting rules

planarization

for the PASEP algebra

$$DE = qED + E + D$$

representation with operators
related to the combinatorial theory
of orthogonal polynomials
and data structures in computer science

J. Françon 1976

data structure histories

"histoires de fichiers"

24

17

10

8

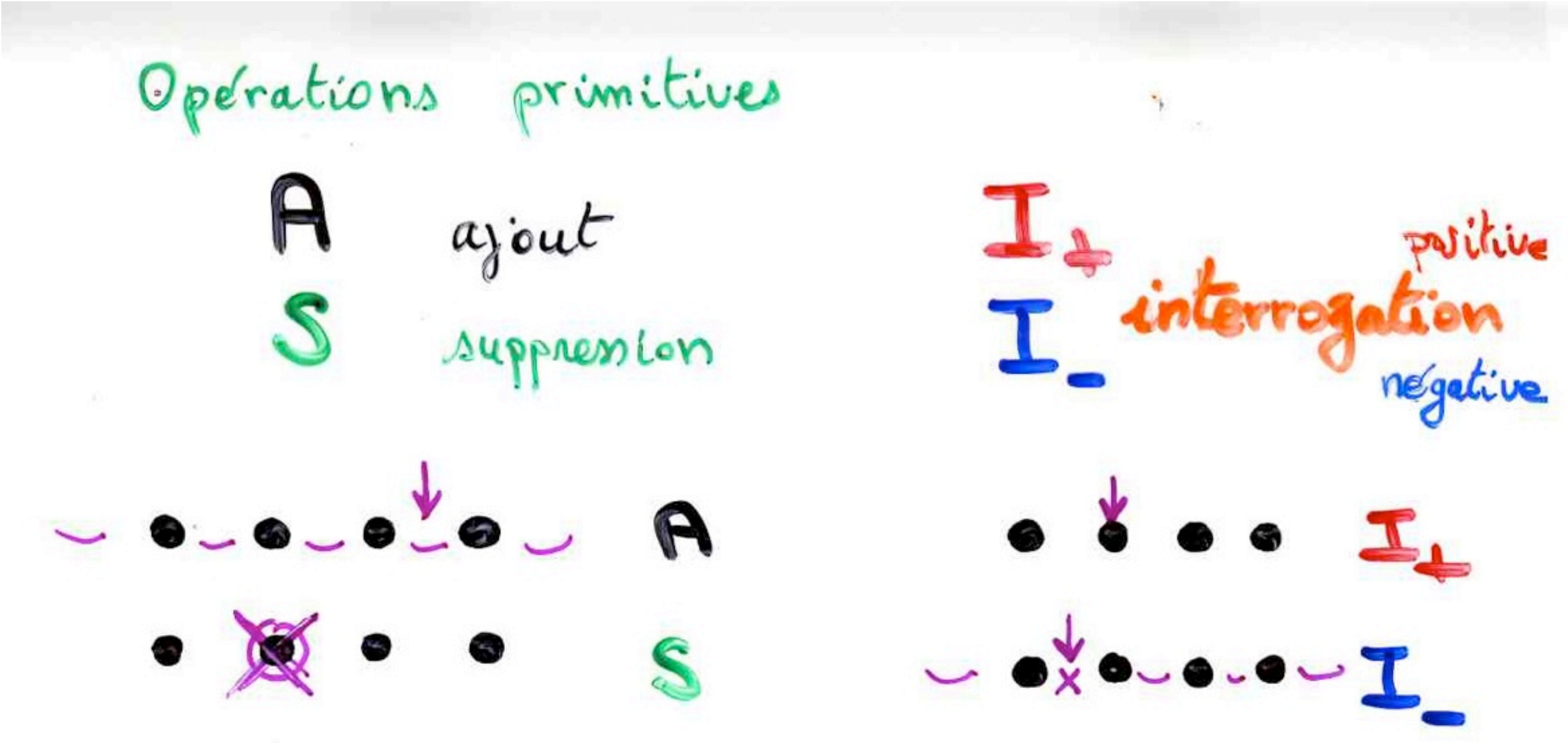
24

17

12

10

8



Primitive operations
for “dictionnaries” data structure:
add or delete any elements, asking questions (with
positive or negative answer)

Opérations primitives

A ajout

S suppression

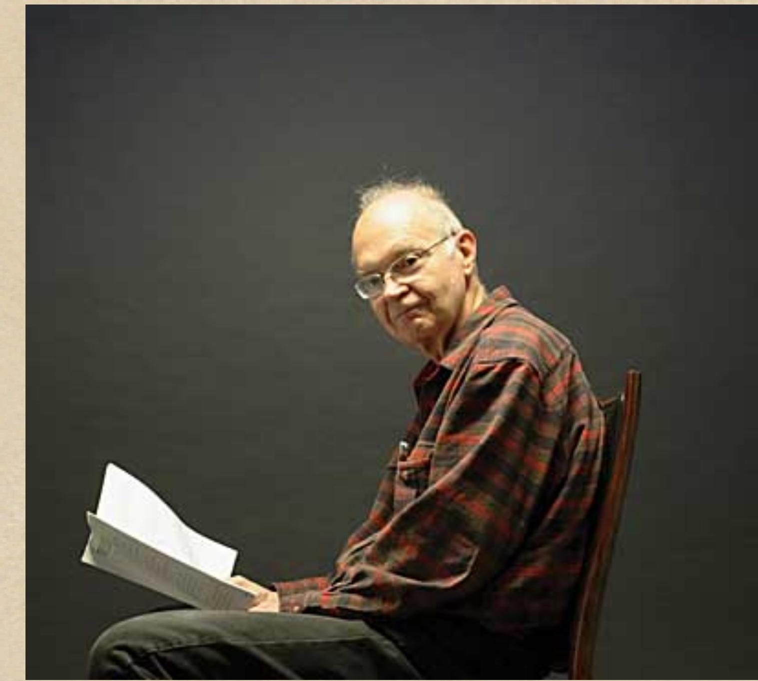
I_+ interrogation positive

I_- interrogation négative

number of choices for each
primitive operations

$$\begin{cases} D = A + I_- \\ E = S + I_+ \end{cases}$$

data structure
integrated cost



D. Knuth

P.Flajolet

Calcul du coût intégré

d'une structure de donnée

pour une séquence aléatoire

d'opérations primitives

Françon, Flajolet, Vuillemin (1980, ...)

connaissant le coût moyen

d'une opération primitive.

representation
of the
operators
E and D

$$DE = ED + E + D$$

V vector space generated by $\mathcal{B}$ basis

$\mathcal{B}$ alternating words two letters $\{\circ, \bullet\}$
(no occurrence of $\circ\circ$ or $\bullet\bullet$)

4 operators A, S, J, K

4 operators A, S, J, K, $\quad u \in B$

$$\langle u | A = \sum_{\substack{\text{letter } \circ \\ \text{of } u}} v \quad , \quad v \text{ obtained by: } \circ \to \circ\bullet\circ$$

$$\langle u | S = \sum_{\circ \text{ of } u} v \qquad v \text{ obtained by: } \circ \to \bullet$$

(and $\bullet\bullet \to \bullet \quad \bullet\bullet\bullet \to \bullet$)

$$\langle u | J = \sum_{\circ \text{ of } u} v \qquad v, \quad \circ \to \bullet\circ$$

(and $\bullet\bullet \to \bullet$)

$$\langle u | K = \sum_{\substack{\circ \\ \text{of } u}} v \qquad v, \quad \circ \to \circ\bullet$$

(and $\bullet\bullet \to \bullet$)

$$\circ\bullet\circ\bullet \,|\, S = \bullet\circ\bullet + \circ\bullet$$

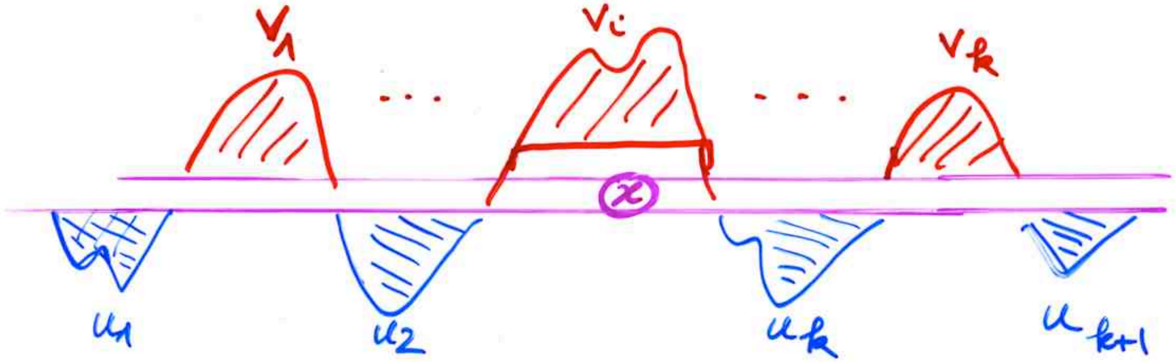
Lemma.

$$AS = SA + J + K$$
$$AK = KA + A$$
$$JS = SJ + S$$
$$JK = KJ$$

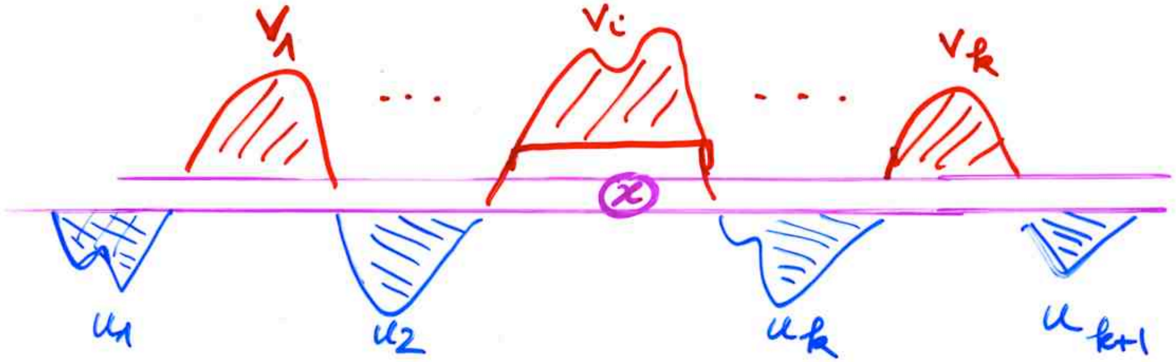
$$D = A + J$$
$$E = S + K$$

$$D = A + J$$
$$E = S + K$$

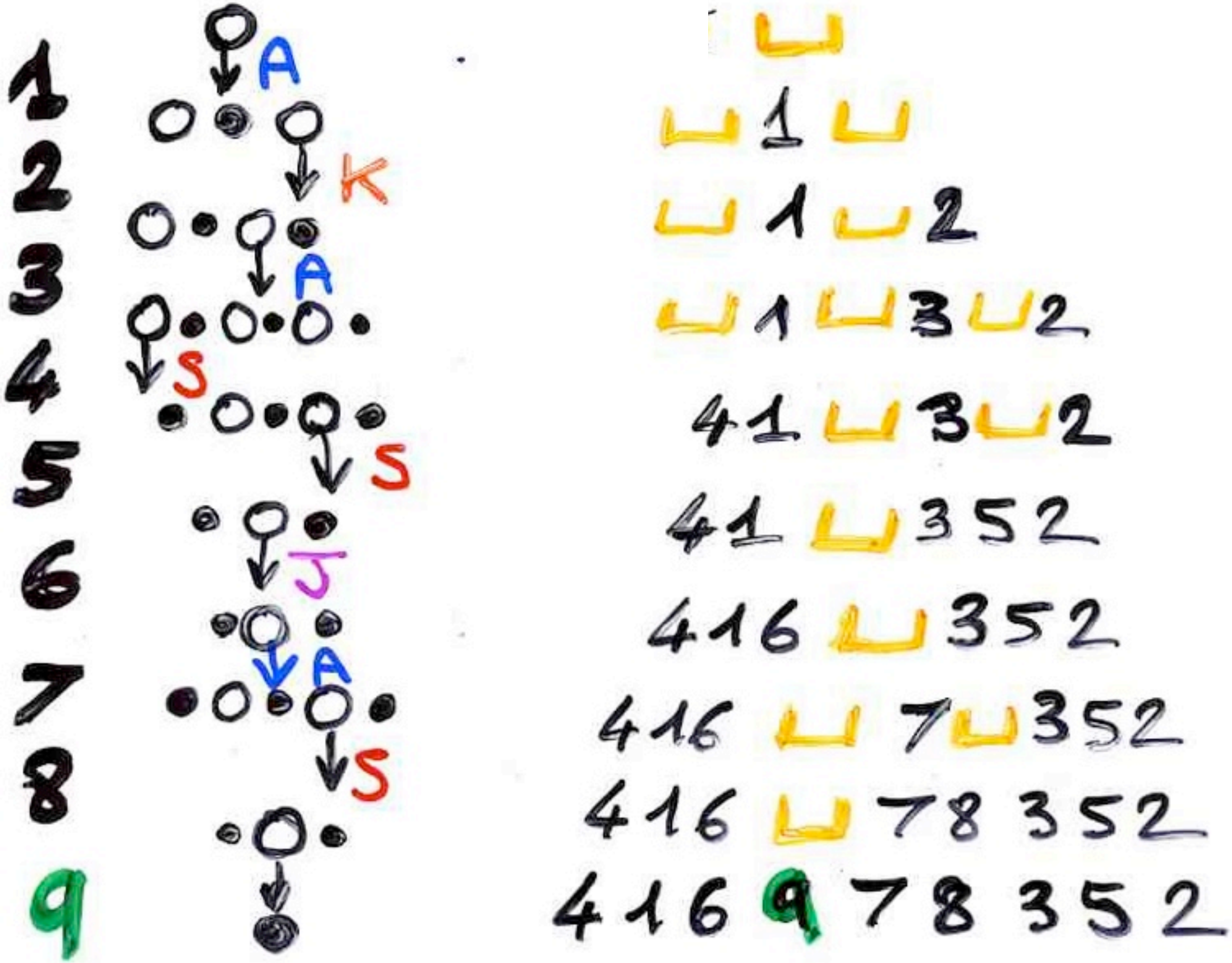
$$
\begin{aligned}
DE &= (A+J)(S+K) \\
&= AS + AK + JS + JK \\
&= \underbrace{\underbrace{(SA + KA + SJ + KJ)}_{(S+K)(A+J)}}_{ED} + \underbrace{J + K + A + S}_{E+D}
\end{aligned}
$$



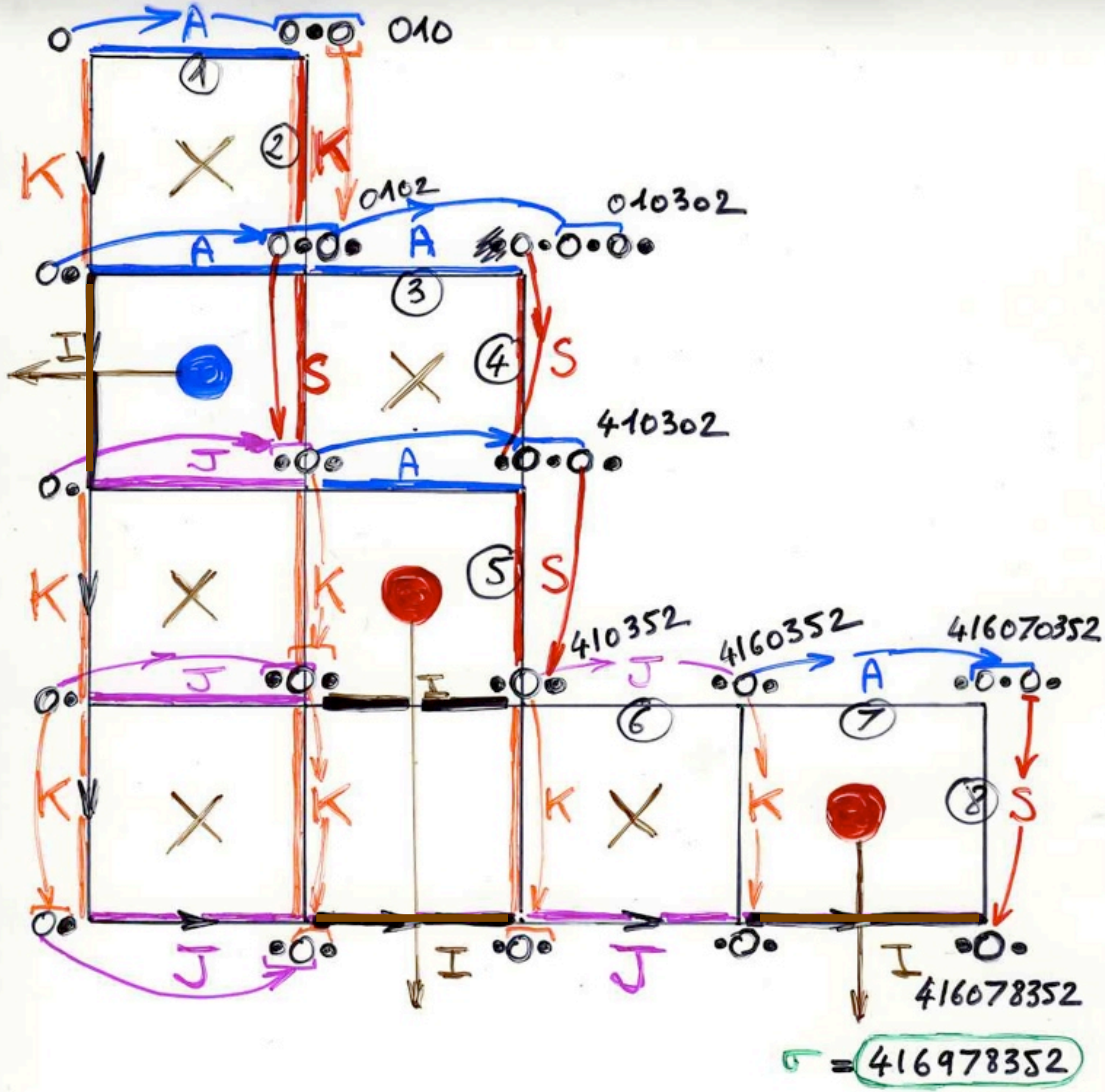
	$\sqcup$
1	$\sqcup\ 1\ \sqcup$
2	$\sqcup\ 1\ \sqcup\ 2$
3	$\sqcup\ 1\ \sqcup\ 3\ \sqcup\ 2$
4	$4\ 1\ \sqcup\ 3\ \sqcup\ 2$
5	$4\ 1\ \sqcup\ 3\ 5\ 2$
6	$4\ 1\ 6\ \sqcup\ 3\ 5\ 2$
7	$4\ 1\ 6\ \sqcup\ 7\ \sqcup\ 3\ 5\ 2$
8	$4\ 1\ 6\ \sqcup\ 7\ 8\ 3\ 5\ 2$
9	$4\ 1\ 6\ 9\ 7\ 8\ 3\ 5\ 2$



v_1
v_i
v_k
x
u_1
u_2
u_k
u_{k+1}



1
2
3
4
5
6
7
8
9
A
K
A
S
S
J
A
S
1
1 2
1 3 2
4 1 3 2
4 1 3 5 2
4 1 6 3 5 2
4 1 6 7 3 5 2
4 1 6 7 8 3 5 2
4 1 6 9 7 8 3 5 2



A
010
K
K
0102
010302
A
A
I
S
S
410302
J
A
K
K
S
410352
4160352
416070352
J
J
A
I
K
K
K
K
S
J
I
J
I
416078352
σ = 416978352

"The cellular Ansatz"

combinatorial
objects
on a 2d lattice

representation
by operators

Physics

data structures
"histories"

orthogonal
polynomials

bijections

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

rooks placements

permutations

RSK

pairs of Tableaux Young

$DE = qED + E + D$

PASEP

alternative tableaux

permutations

Laguerre histories

dynamical systems in physics
stationary probabilities

quadratic algebra Q

Q-tableaux

commutations
rewriting rules

planarization

"The cellular Ansatz"

combinatorial objects on a 2d lattice

representation by operators

Physics

data structures "histories"

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$UD = DU + Id$

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permutations $\longleftrightarrow$ RSK pairs of Tableaux Young

$DE = qED + E + D$

PASEP

alternative tableaux $\longleftrightarrow$ permutations

tree-like tableaux

Laguerre histories

dynamical systems in physics
stationary probabilities

reverse Q-tableaux

quadratic algebra Q

Q-tableaux

the XYZ algebra
ASM, (alternating sign matrices)
FPL (Fully packed loops)
tilings, non-crossing paths

?

commutations
rewriting rules

RSK automata

planarization

planar automata

reverse planar automata

Laguerre histories

The FV bijection

Françon-XV 1978

Bijection Permutations $n+1$

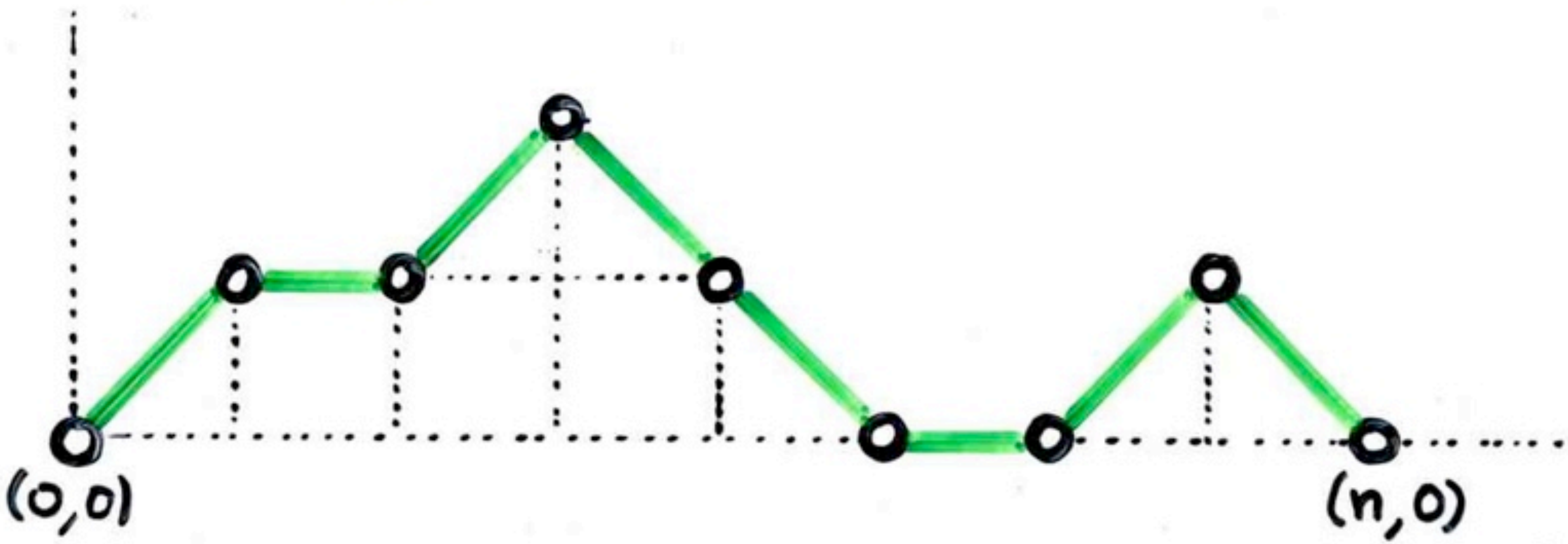
Histoires de Laguerre $(\gamma_c \, ; \, f)$ n

Bijection **Permutations** $n+1$

Histoires de Laguerre $(\gamma_c \,;\, f)$ n

Chemin de Motzkin

n



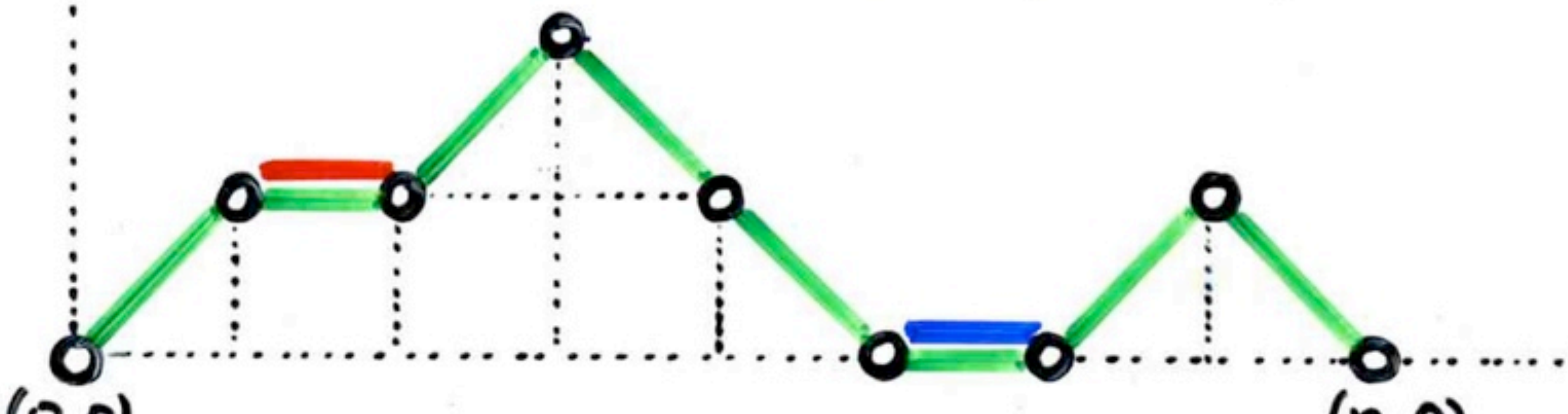
Bijection Permutations $n+1$

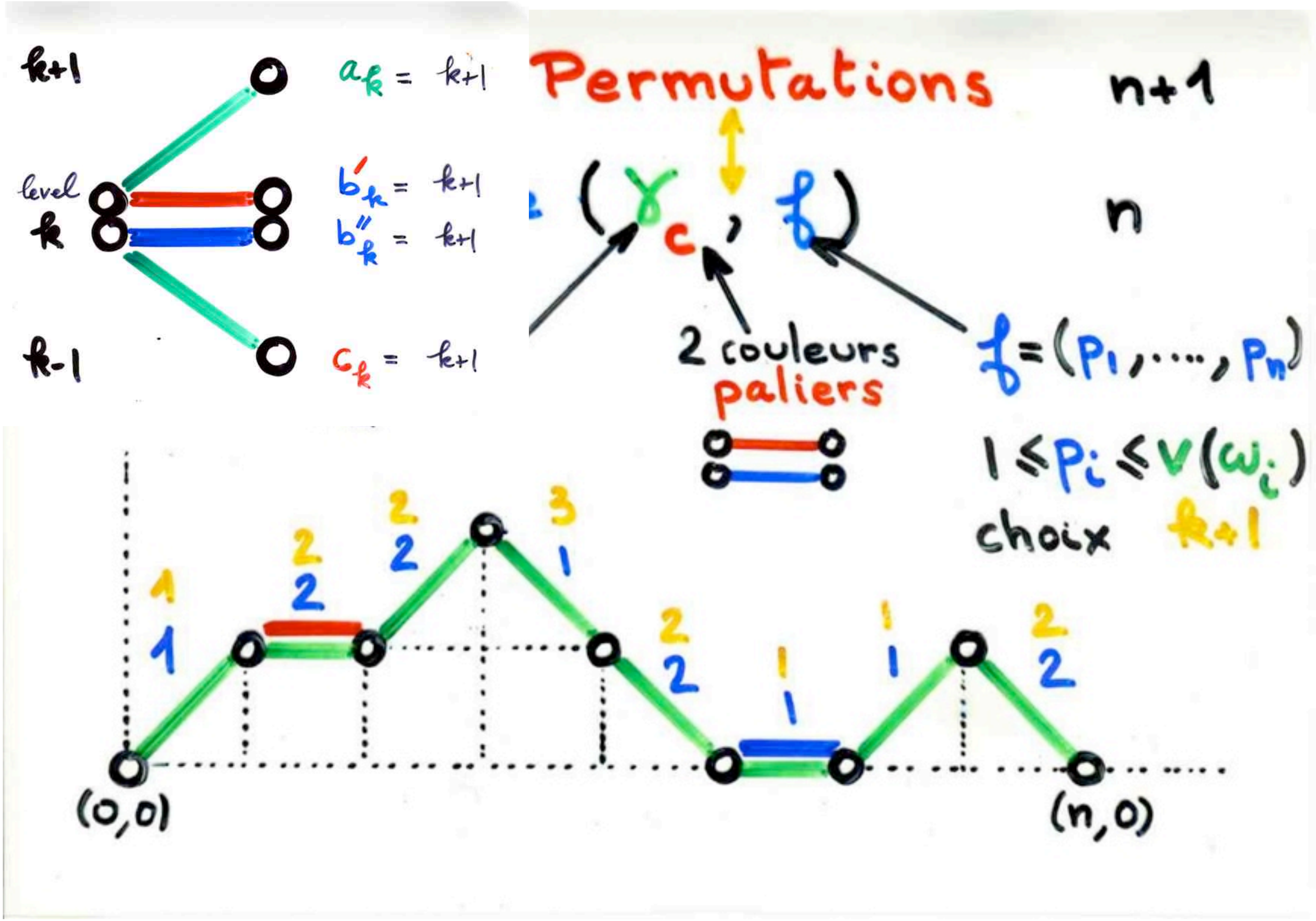
Histoires de Laguerre (γ_c, f) n

Chemin de Motzkin

$n\cancel{-1}$

2 couleurs paliers



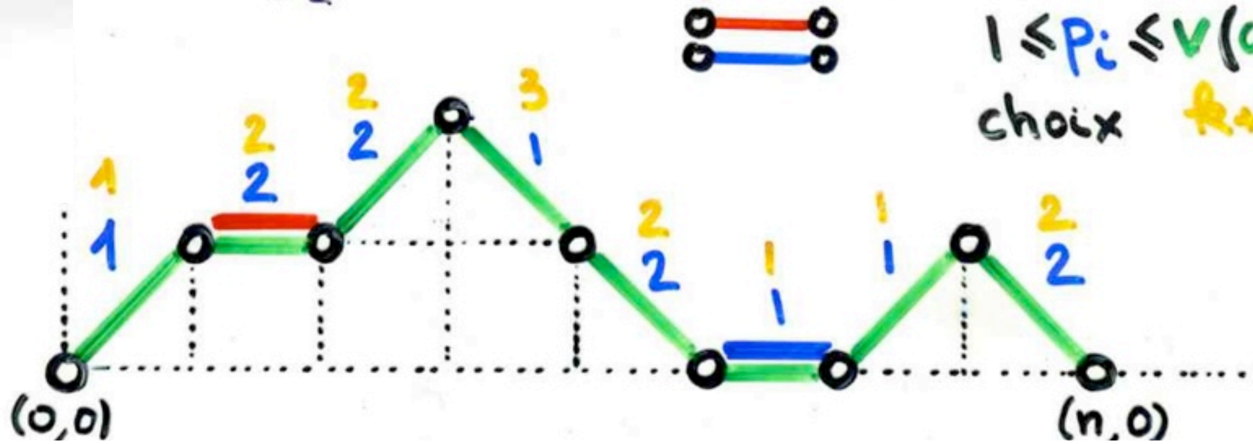


k+1
level
k
k-1
a_k = k+1
b'_k = k+1
b''_k = k+1
c_k = k+1
Permutations
n+1
(γ_c , f)
n
2 couleurs
paliers
f=(p_1, ..., p_n)
1 ≤ p_i ≤ v(ω_i)
choix k+1
(0,0)
(n,0)

Bijection

Histoires de Laguerre $(\omega\,;(p_1,\dots,p_n)) \longleftrightarrow$ permutations $(n+1)!$

$h = (\omega_c \, ; (p_1, \dots, p_n))$



$1 \leq p_i \leq v(\omega_i)$

choix $k+1$

x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9	•		

␣

␣ 1 ␣

␣ 1 ␣ 2

␣ 1 ␣ 3 ␣ 2

4 1 ␣ 3 ␣ 2

4 1 ␣ 3 5 2

4 1 6 ␣ 3 5 2

4 1 6 ␣ 7 ␣ 3 5 2

4 1 6 ␣ 7 8 3 5 2

4 1 6 9 7 8 3 5 2 $= \sigma \in \mathfrak{S}_{n+1}$

(formal) orthogonal polynomials

Orthogonal polynomials

Def. $\{P_n(x)\}_{n\geq 0}$ $\quad P_n(x) \in \mathbb{K}[x]$

orthogonal iff $\exists\ f : \mathbb{K}[x] \longrightarrow \mathbb{K}$

linear functional

$$\begin{cases} \text{(i)} \quad \deg(P_n(x)) = n & (\forall n \geq 0) \\ \text{(ii)} \quad f(P_k P_l) = 0 & \text{for } k \neq l \geq 0 \\ \text{(iii)} \quad f(P_k^2) \neq 0 & \text{for } k \geq 0 \end{cases}$$

$$f(x^n) = \mu_n \qquad (n \geq 0)$$

moments

$$f(PQ) = \int_a^b P(x)\,Q(x)\,d\mu$$

mesure

combinatorial interpretation of the moments

Thm. (Favard)

- $\{P_n(x)\}_{n\geqslant 0}$ sequence of monic polynomials, $\deg(P_n)=n$
- $\{b_k\}_{k\geqslant 0}$, $\{\lambda_k\}_{k\geqslant 1}$ coeff. in $\mathbb{K}$

orthogonality $\Longleftrightarrow$

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

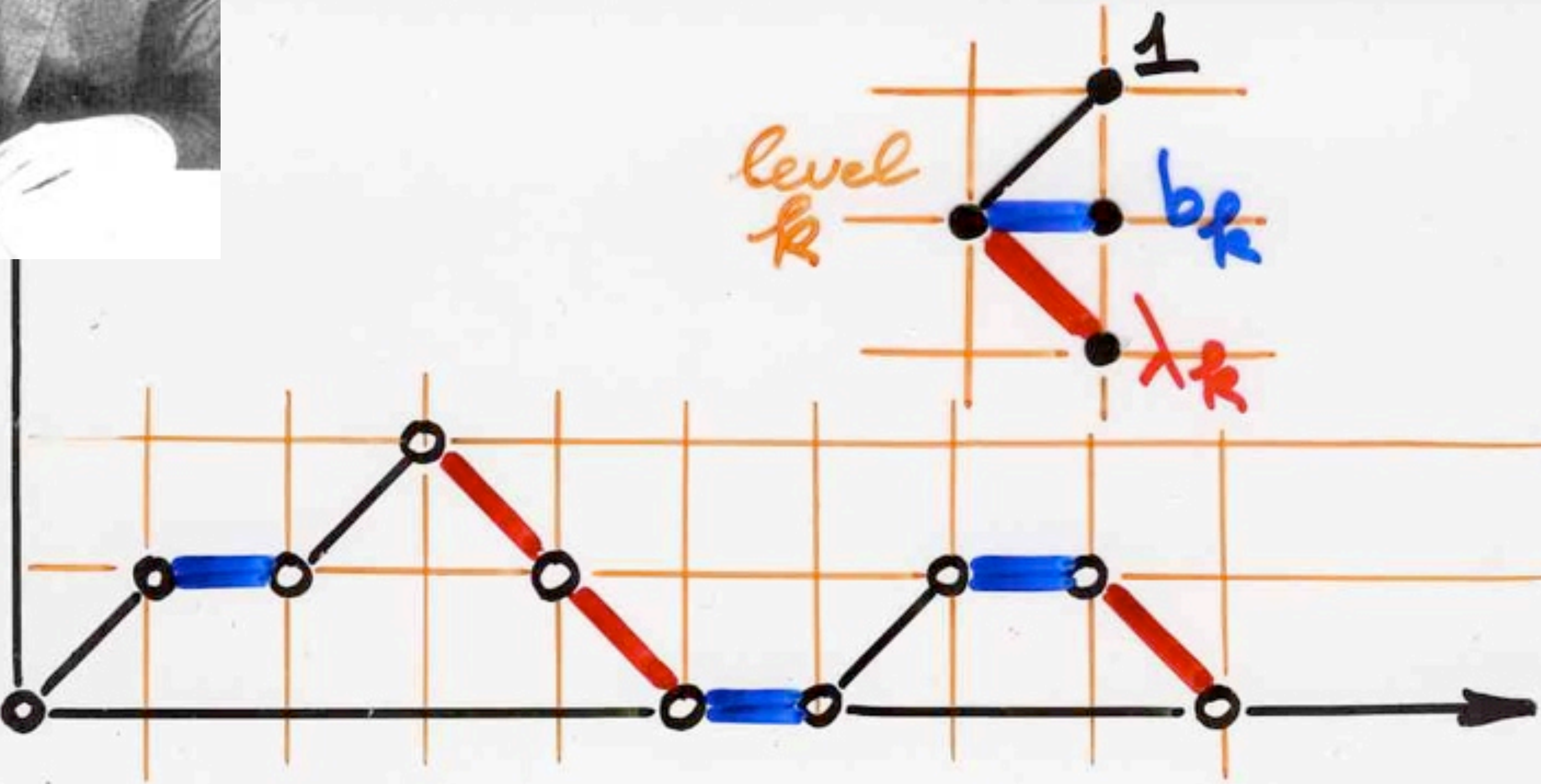
$(\forall k\geqslant 1)$

3 terms linear recurrence relation

$\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$

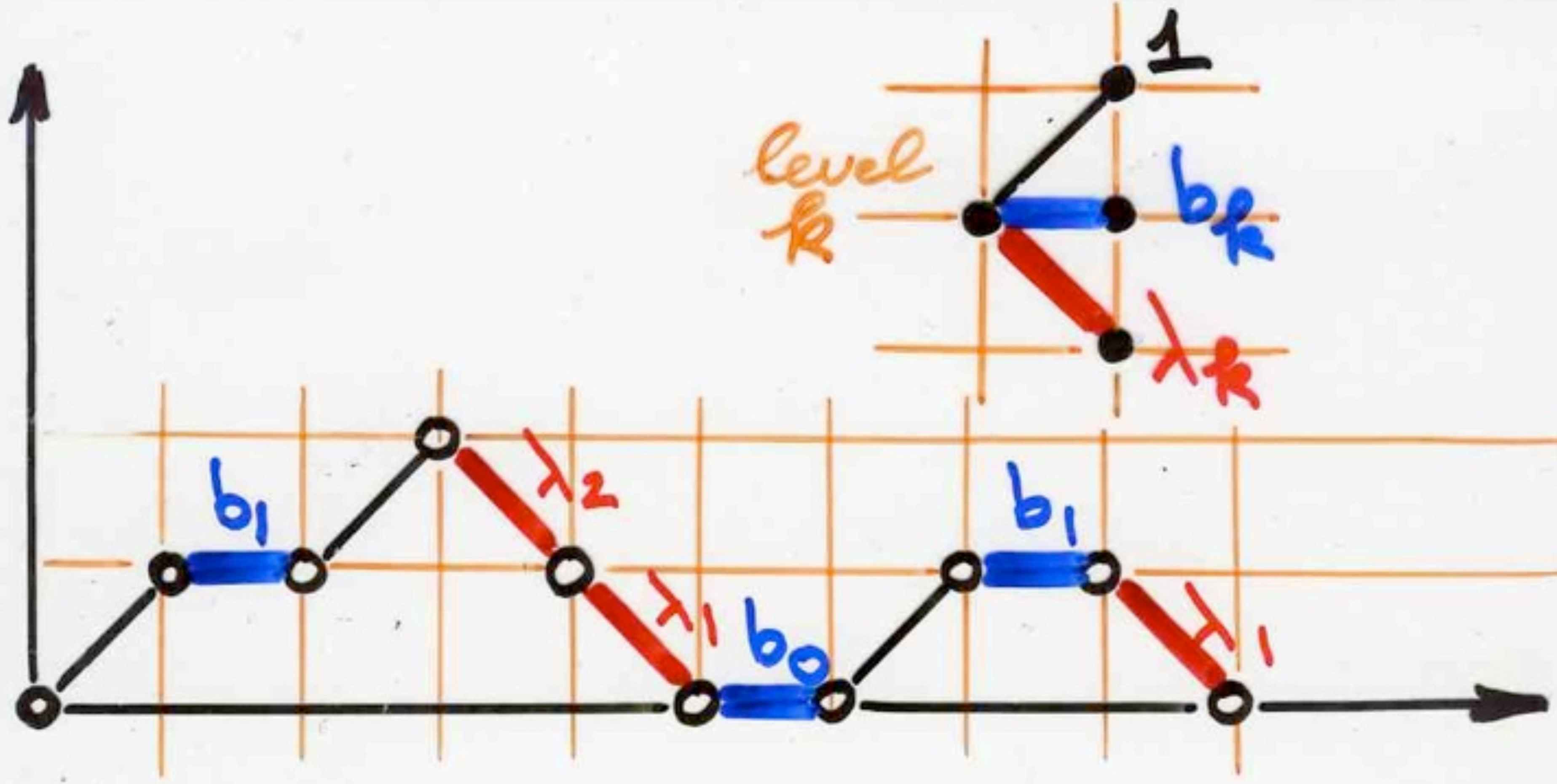
$b_k , \lambda_k \in \mathbb{K}$ ring

valuation v



ω Motzkin path

valuation v



ω Motzkin path

$$v(\omega) = b_0\, b_1^2\, \lambda_1^2\, \lambda_2$$

$$f(x^n) = \mu_n \qquad (n \geqslant 0)$$

moments

$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path

$|\omega| = n$

Laguerre histories
and
Laguerre polynomials

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$P_0 = 1 \qquad P_1 = x - b_0$$

$$\mu_n = (n+1)!$$

$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

Laguerre

polynomial

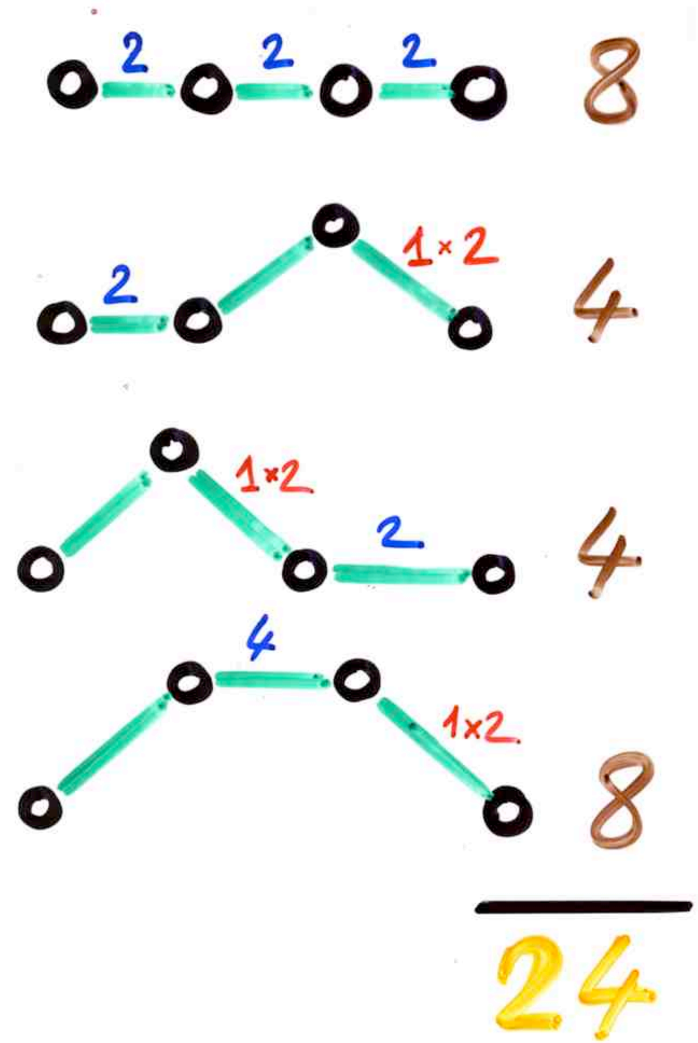
$$J(t) = \cfrac{1}{1 - 2t - \cfrac{1\cdot 2t^2}{1 - 4t - \cfrac{2\cdot 3t^2}{\cdots}}}$$

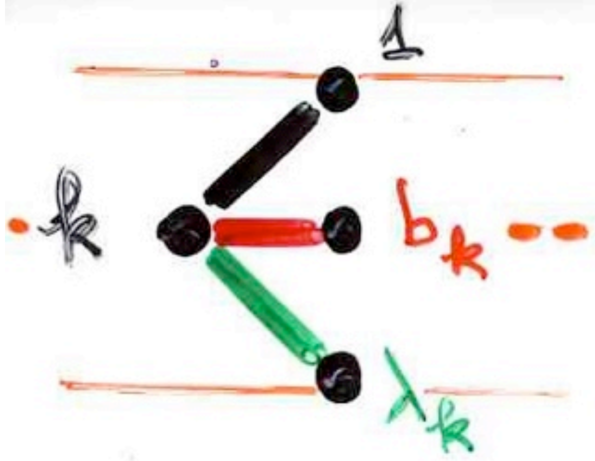
Laguerre $L_n^{(1)}(x)$

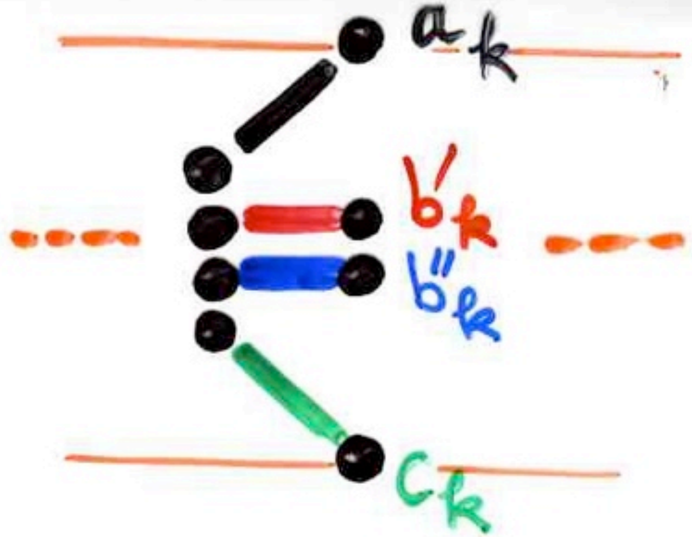
moment $\mu_n = (n+1)!$

$b_k = 2k+2$

$\lambda_k = k(k+1)$







$$b_k = b'_k + b''_k$$

$$a_{k-1} c_k = \lambda_k$$

Laguerre $L_n^{(1)}(x)$

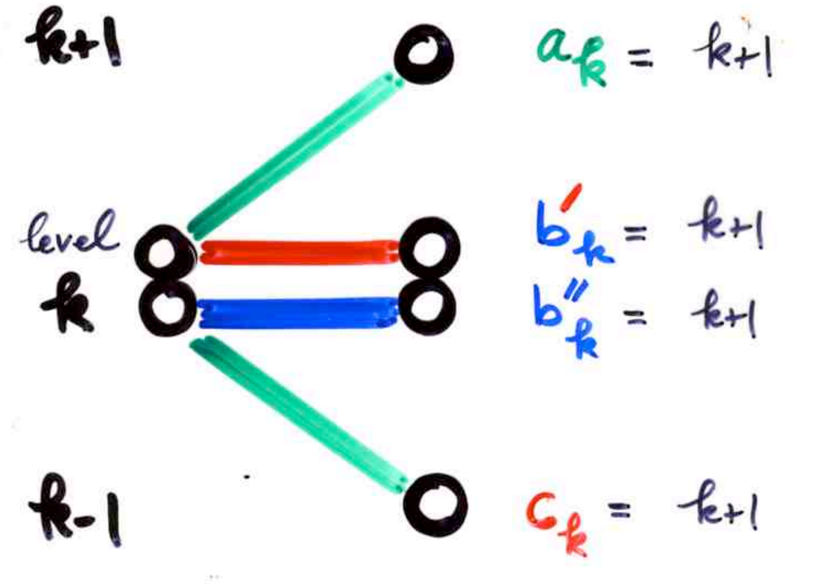
$\mu_n = (n+1)!$

$a_k = k+1$

$b'_k = k+1$

$b''_k = k+1$

$c_k = k+1$



Permutations n+1

$\updownarrow$

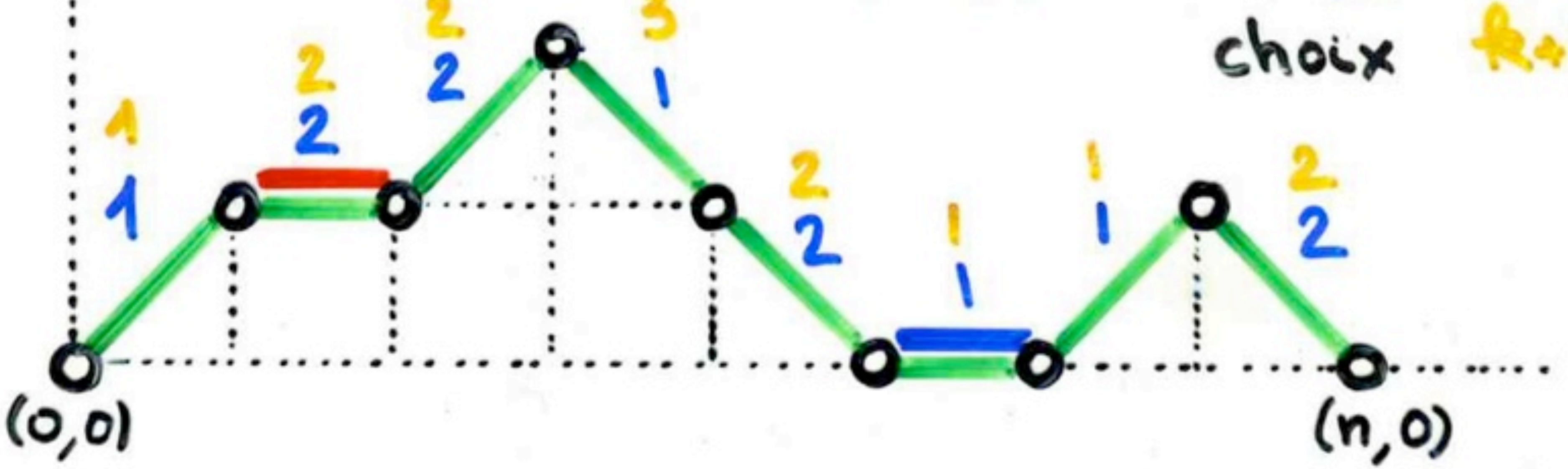
(γ_c, f) n

2 couleurs
paliers

$f=(p_1,\dots,p_n)$

$1 \leq p_i \leq v(\omega_i)$

choix $k+1$



3 parameters

Cor. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ (PASEP) is

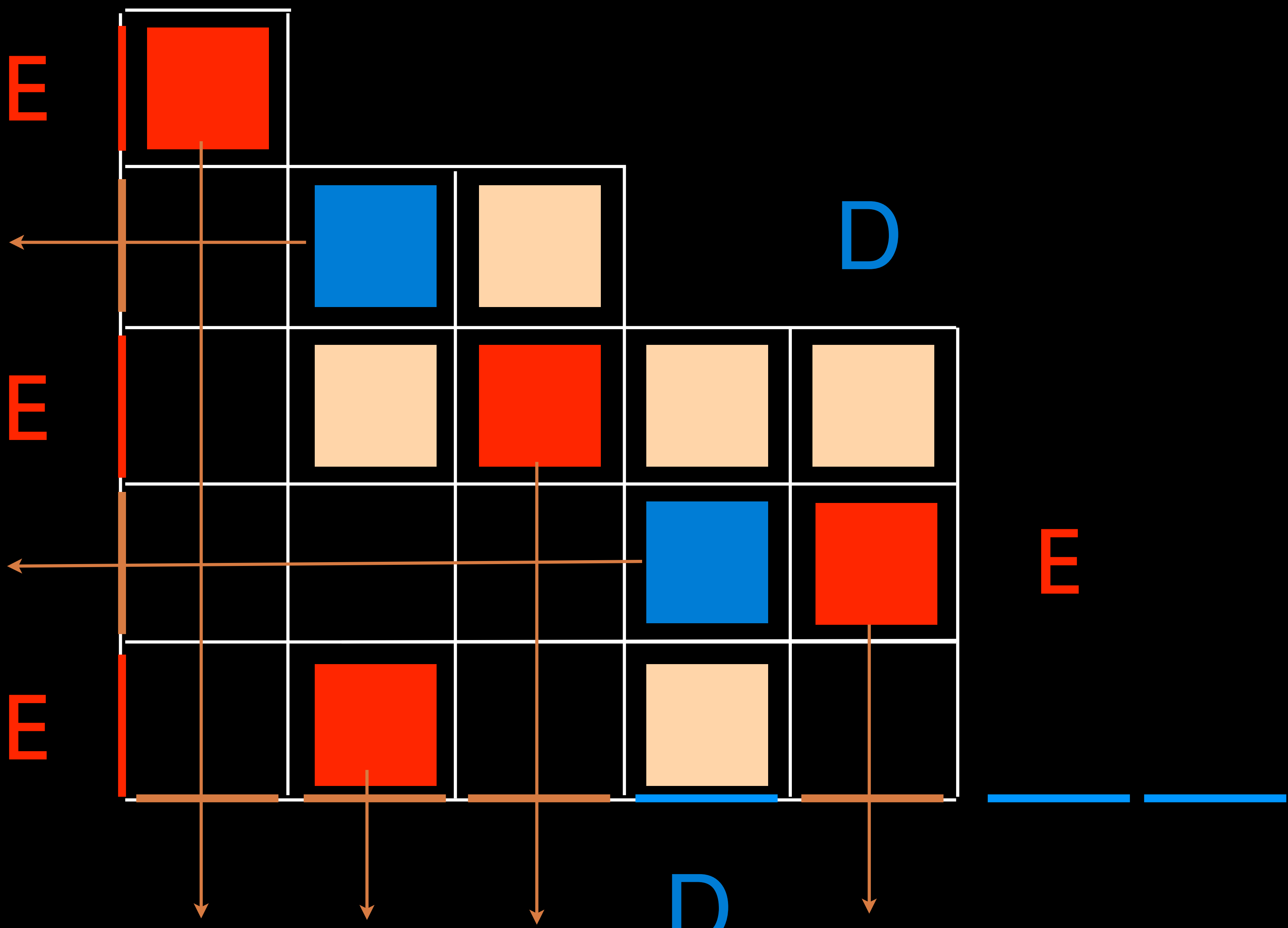
$$\mathrm{proba}_{\tau}(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{T} q^{L(T)} \alpha^{-f(T)} \beta^{-u(T)}$$

alternative tableaux
profile τ

$$\begin{cases} f(T) & \text{nb of rows without cell} \\ u(T) & \text{nb of columns without cell} \\ L(T) & \text{nb of cells} \end{cases}$$

permutation tableau

S. Corteel, L. Williams
(2007) (2008) (2009)



E
D
E
E
E
D

total order

$\{1, 2, \ldots, n\}$

$$\sigma = 7\ 2\ 3\ 9\ 6\ 8\ 5\ 1\ 4$$

word

left-to-right

right-to-left

minimum elements

$$\sigma = 7\ 2\ 3\ 9\ 6\ 8\ 5\ \boxed{1}\ \boxed{4}$$

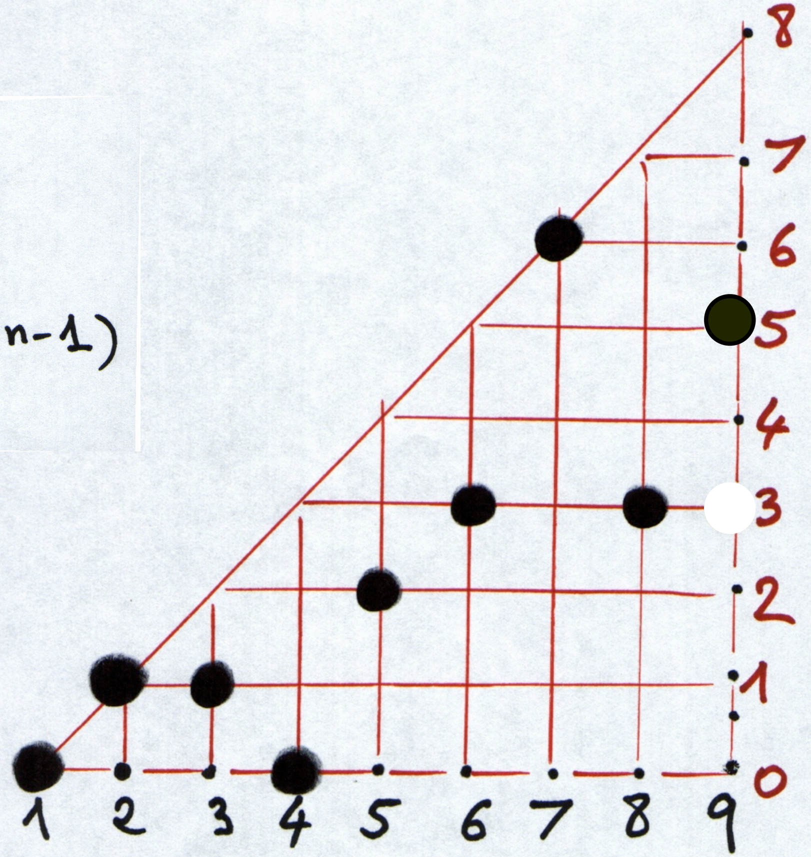
left-to-right
right-to-left minimum elements

$\sigma = 7\ 2\ 3\ 9\ 6\ 8\ 5\ 1\ 4$

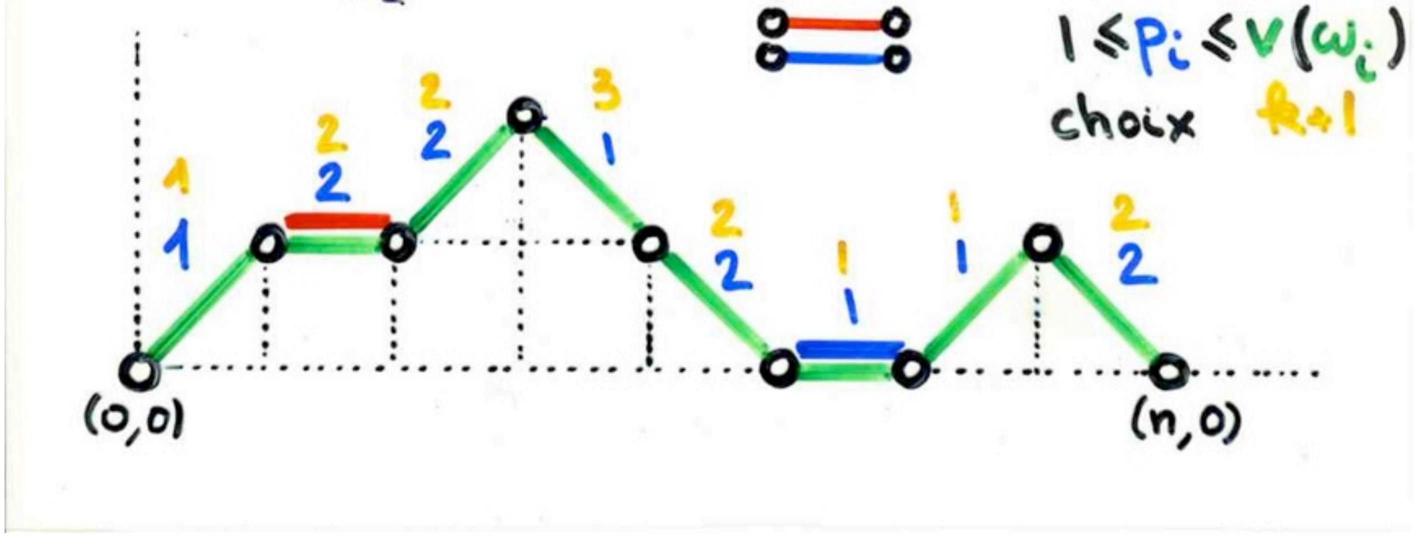
Stirling numbers $s_{n,k}$

$$\sum_{k\geq 1} s_{n,k}\, x^k = x(x+1)\dots(x+n-1)$$

$xy(x+y)(x+1+y)\dots(x+n-2+y)$



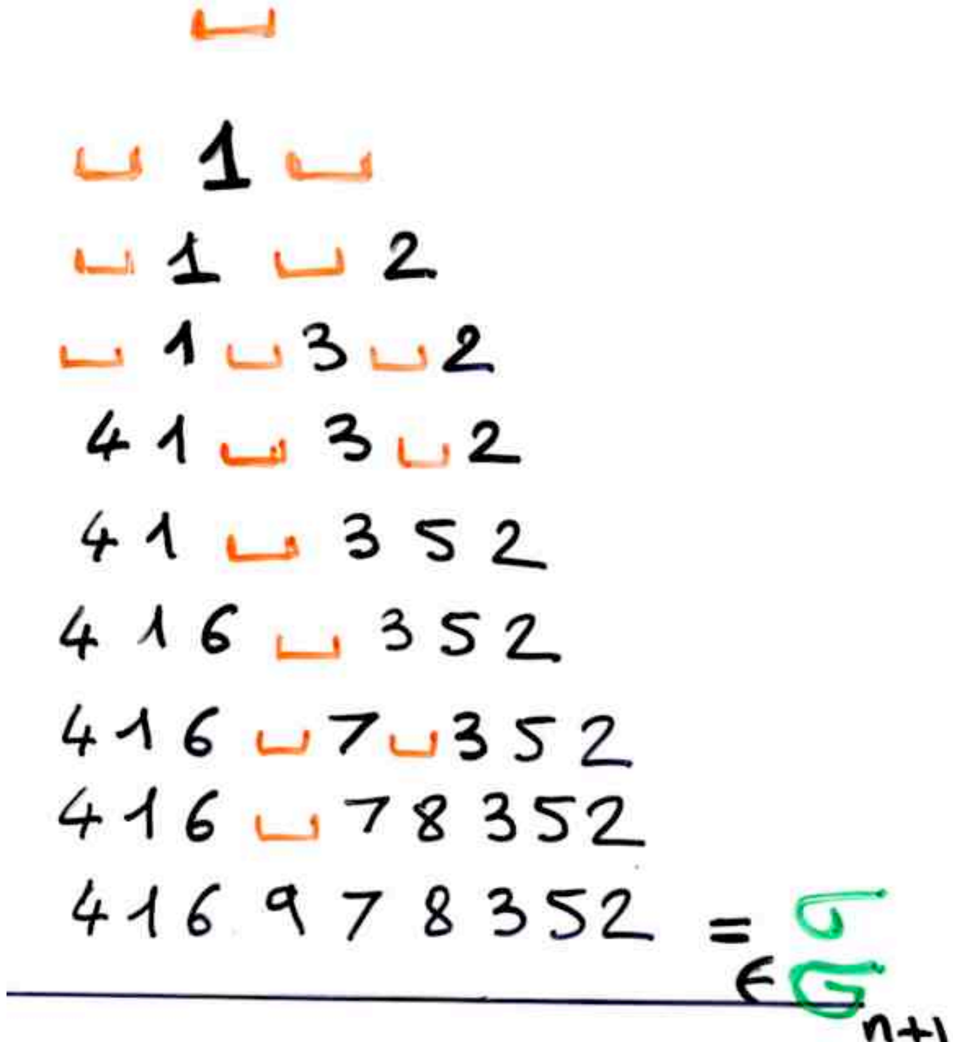
“q-analogue” of Laguerre histories



choices function

1	2	3	4	5	6	7	8
1	2	2	1	2	1	1	2
0	1	1	0	1	0	0	1

q-Laguerre : q^4



q-Laguerre

$$L_n^{\beta}(x;q)$$

$\beta = \alpha + 1$

$$\begin{cases} b_{k,q}^{(\beta)} = [k]_q + [k+1;\beta]_q \\ \lambda_{k,q}^{(\beta)} = [k]_q \cdot [k;\beta]_q \end{cases}$$

$$[k;\beta]_q = \beta + q + q^2 + \cdots + q^{k-1}$$

$$\mu_n = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\binom{2n}{n-k} - \binom{2n}{n-k-2} \right) \left(\sum_{i=0}^{k} q^{i(k+1-i)} \right)$$

Corteel, Josuat-Vergès, Prellberg, Rubey (2008)

general PASEP

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$
$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\,\hat{a} = 1$$

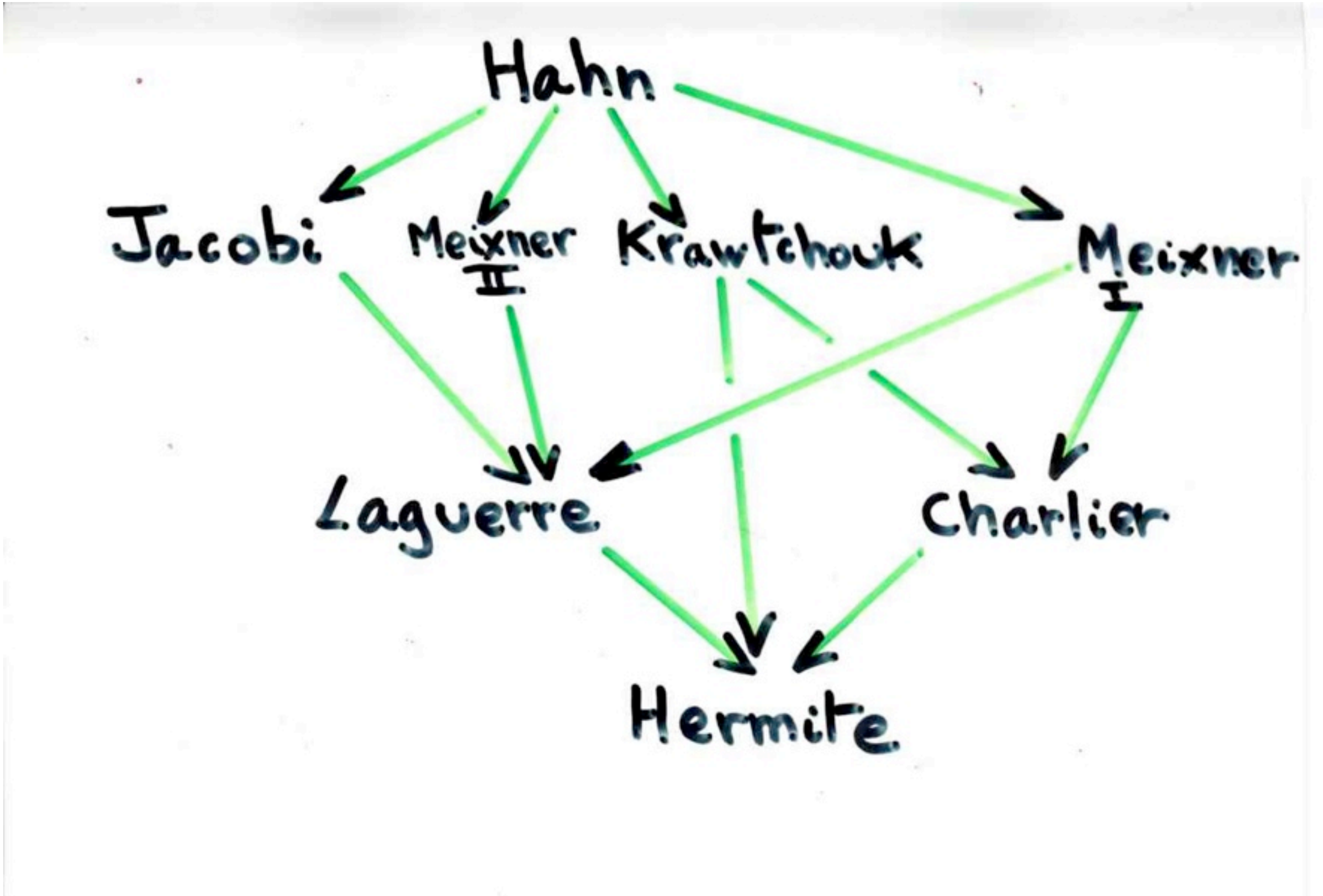
→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta,$ q

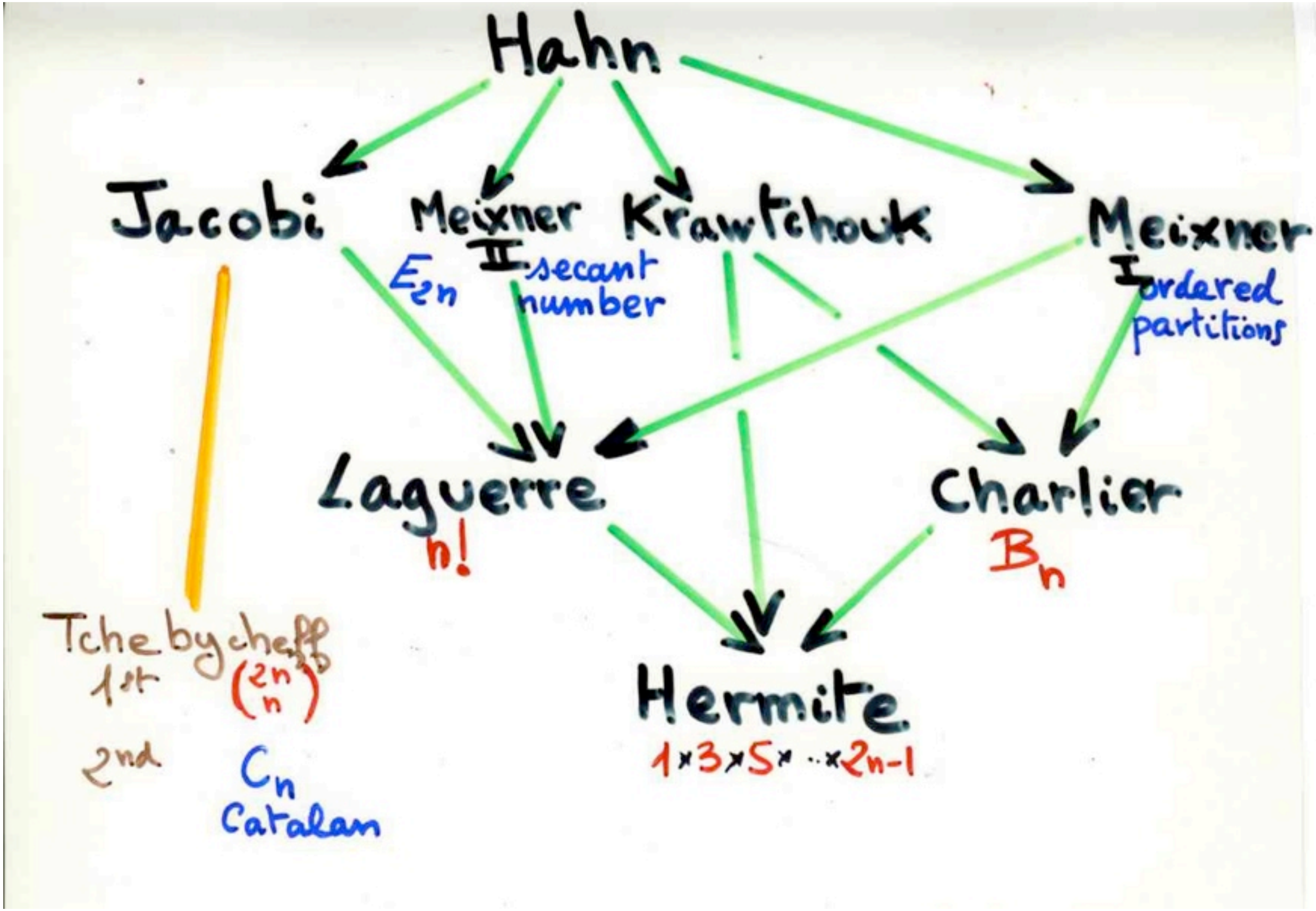
Askey-Wilson polynomials

Askey tableau

Askey-Wilson



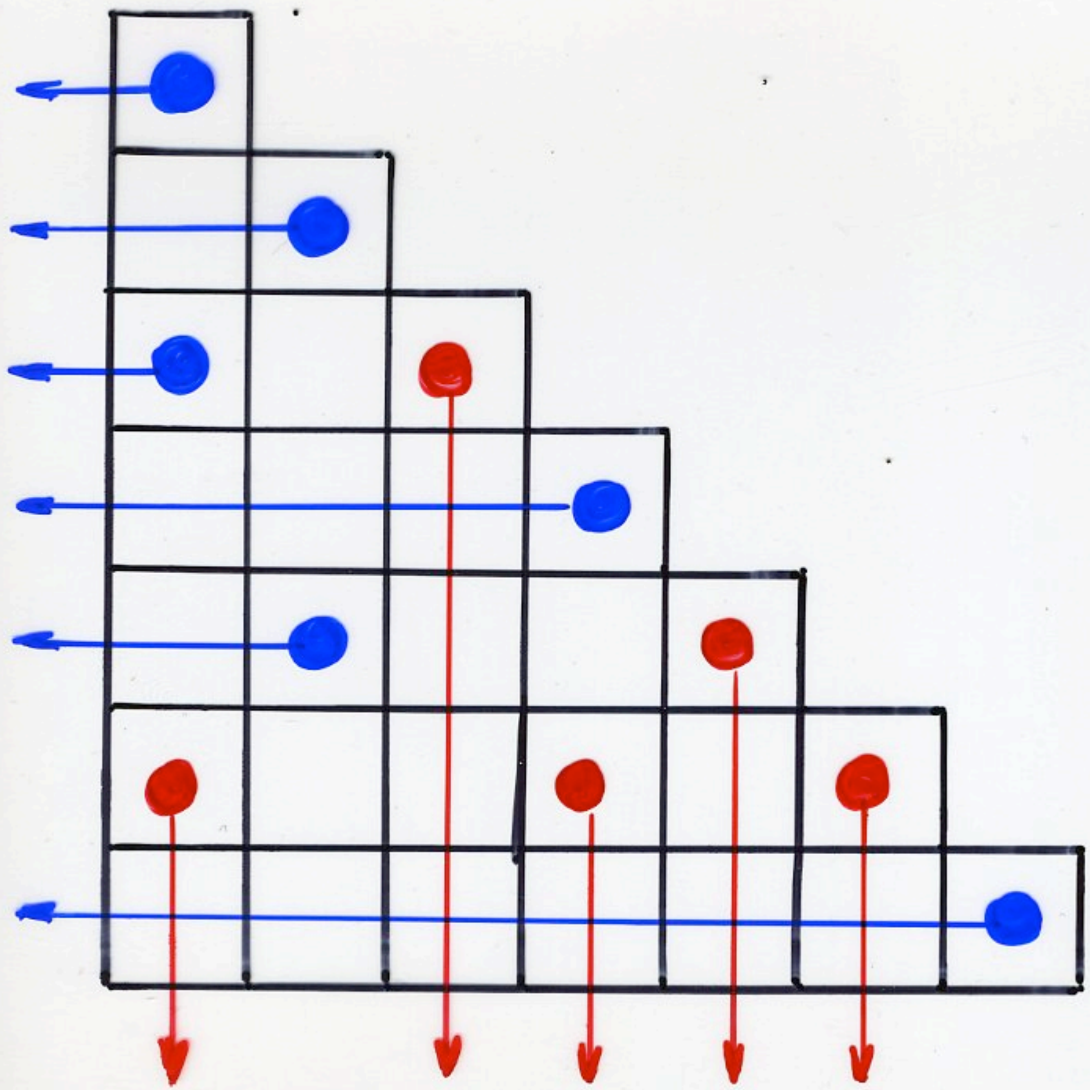
Askey-Wilson

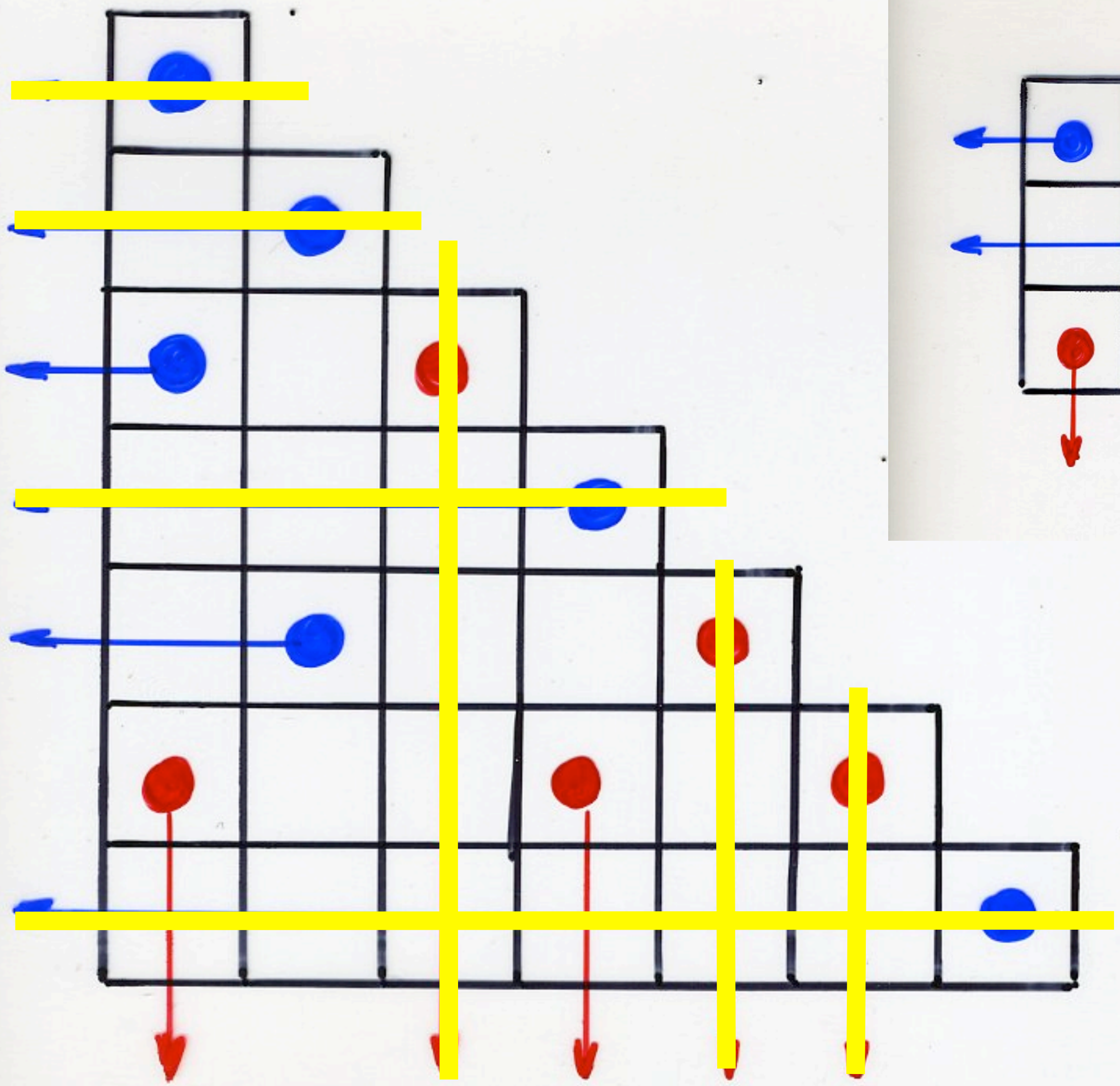


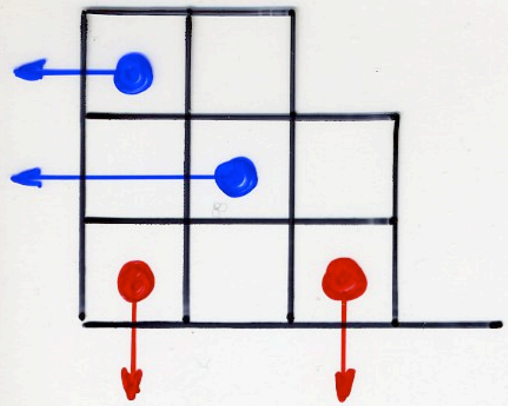
staircase tableaux

Corteel, Williams, 2009

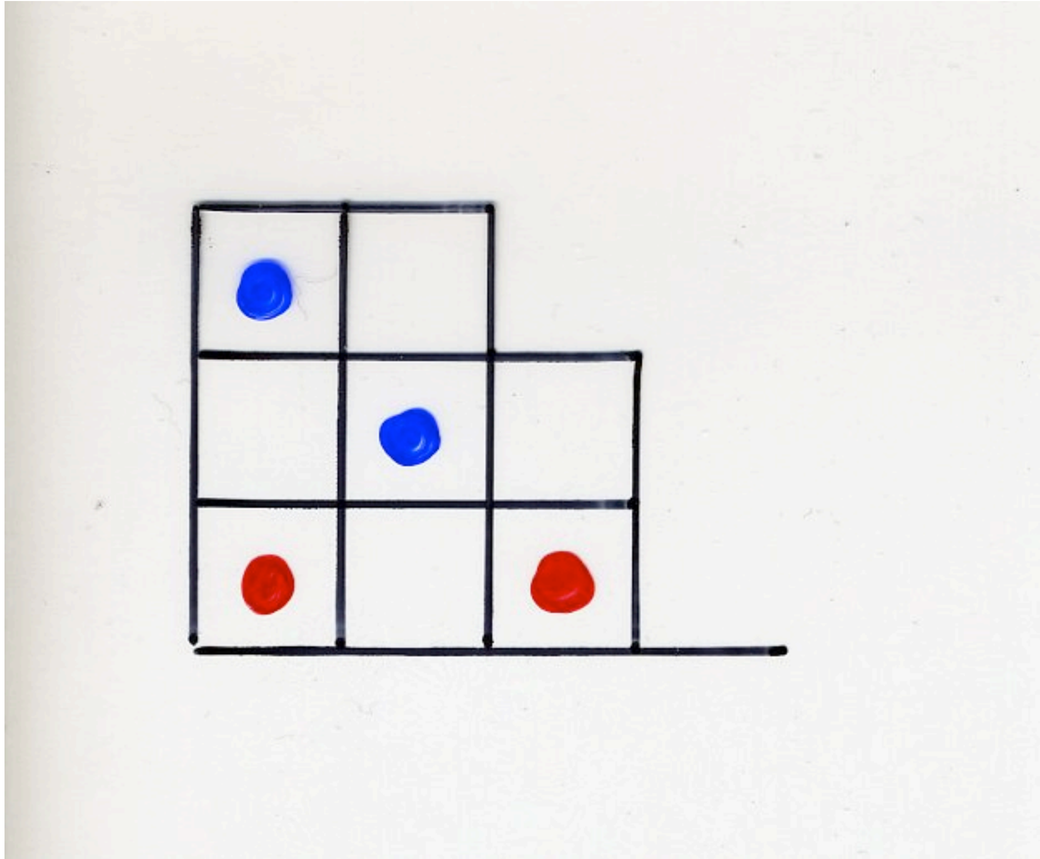
Corteel, Stanley, Stanton, Williams, 2010



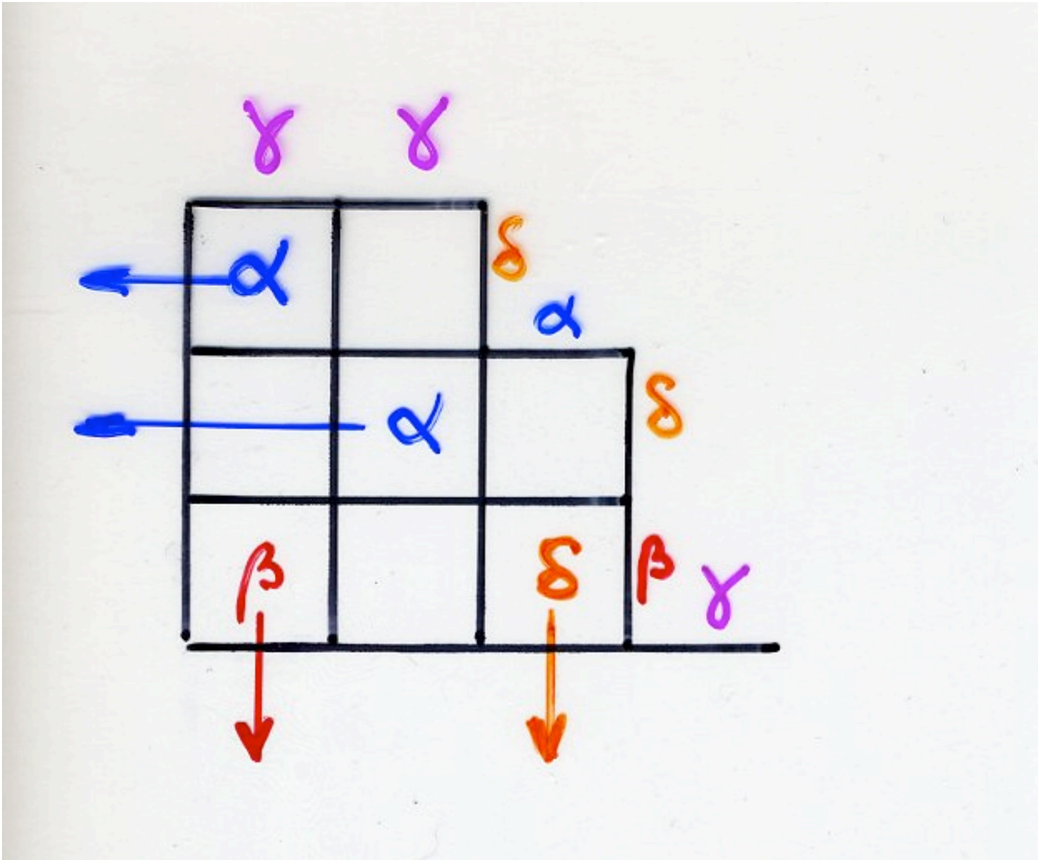




nb of 2-colored alternative tableaux $= 2^n \cdot n!$

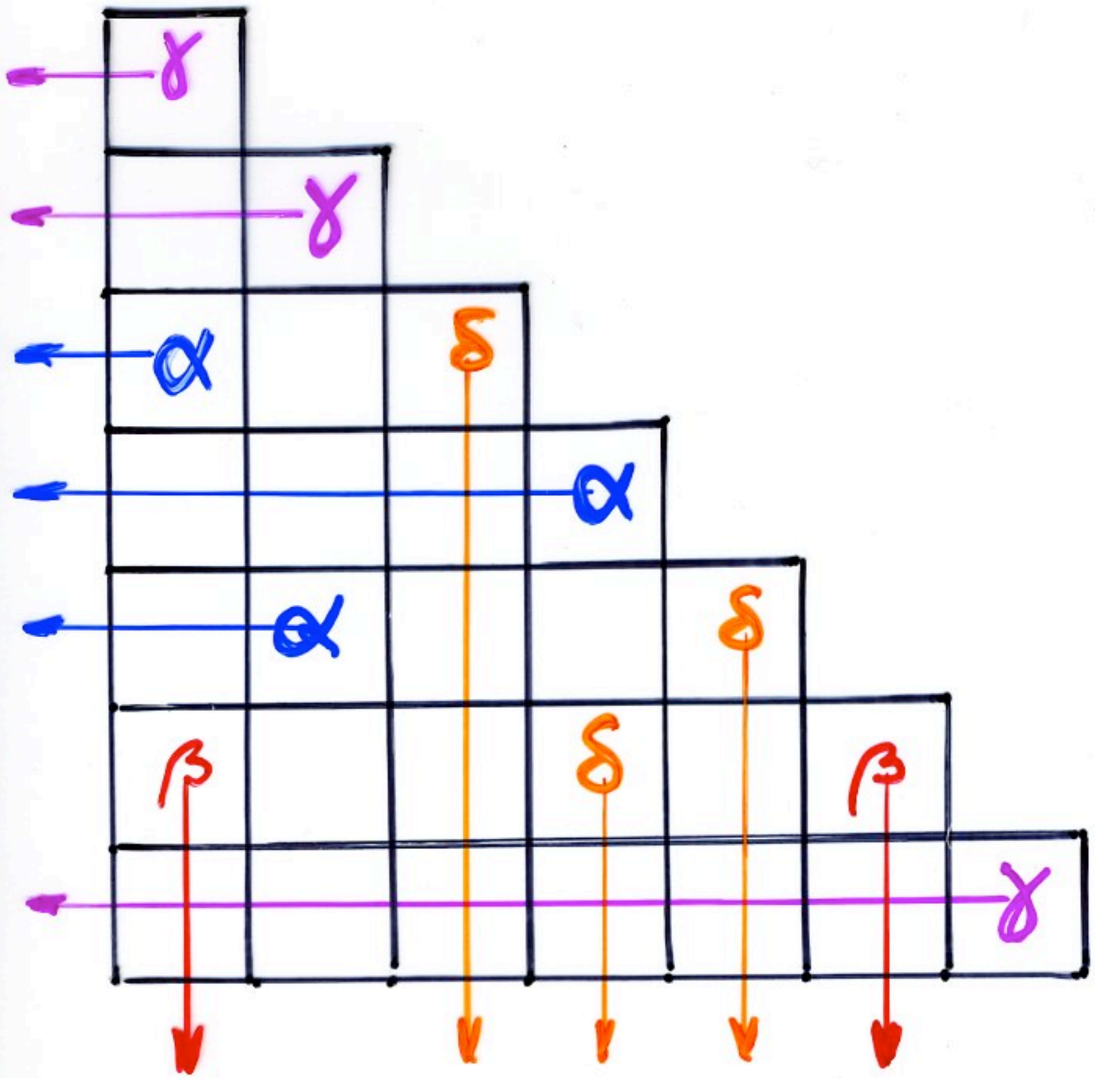


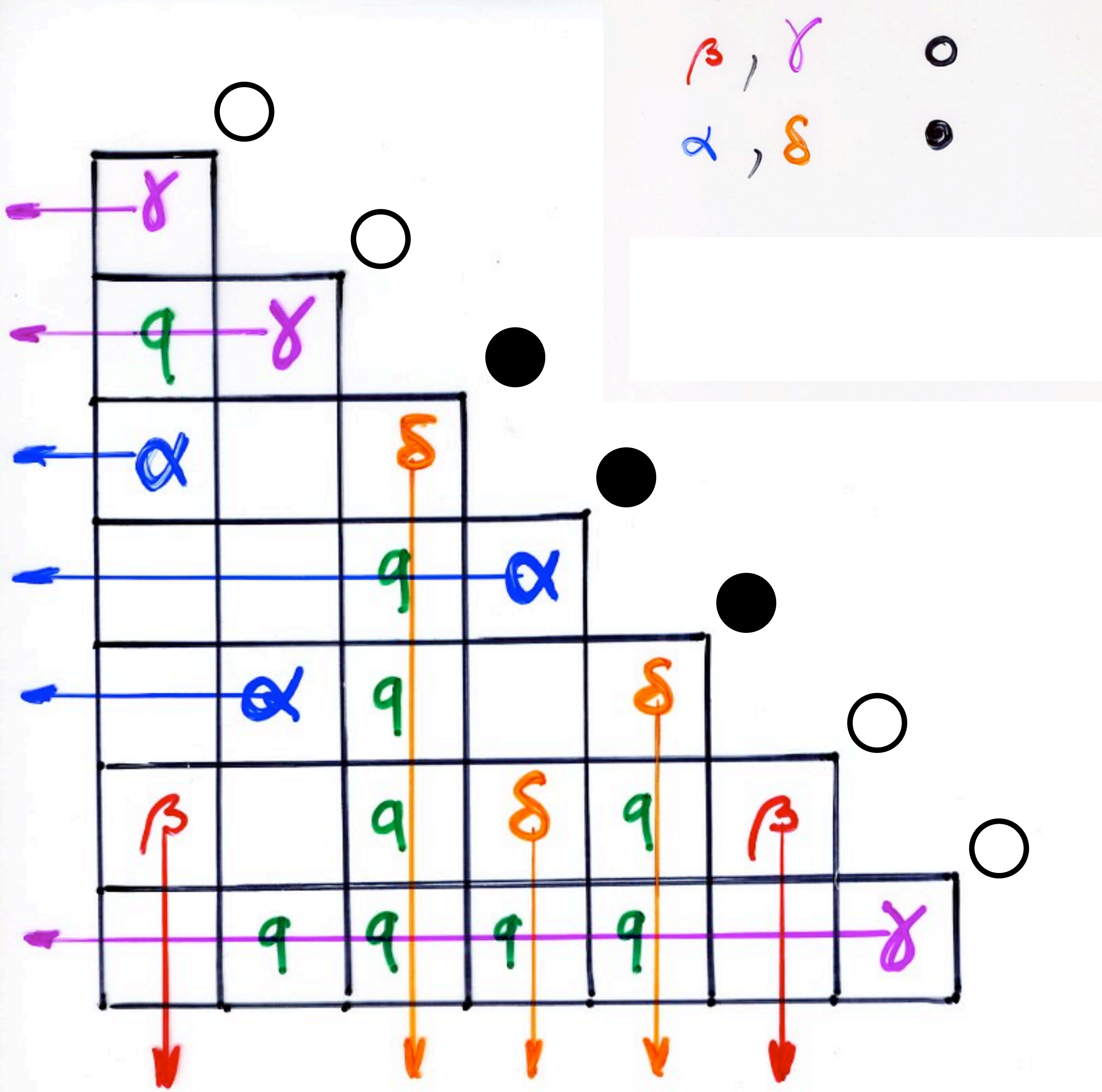
nb of 2-colored alternative tableaux $= 2^n \cdot n!$

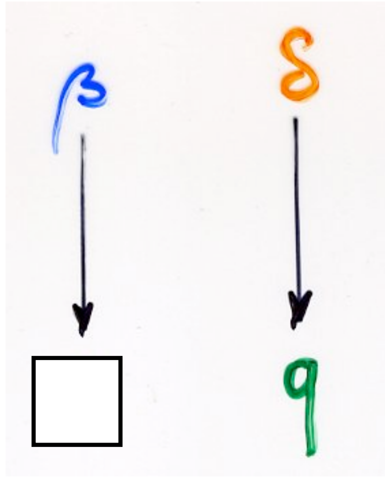


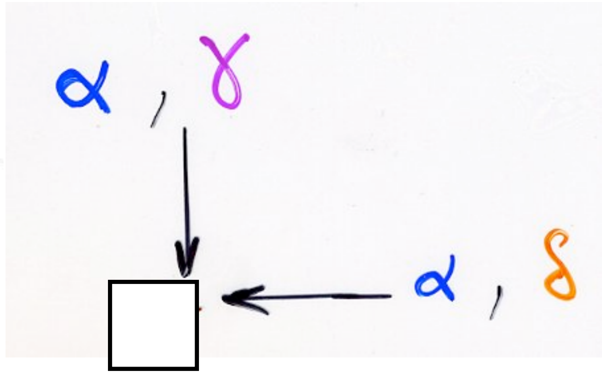
nb of staircase tableaux $= 4^n \cdot n!$

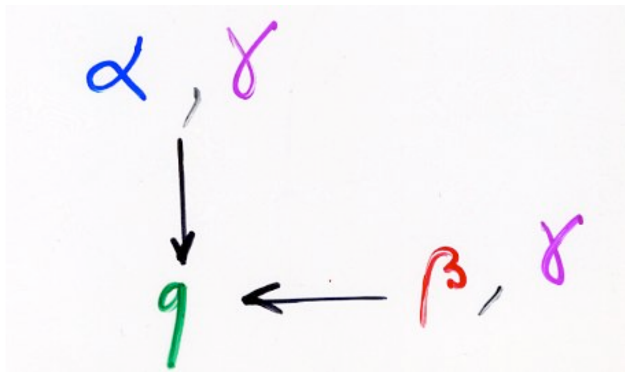
staircase tableaux




γ
q γ
α δ
q α
α q δ
β q δ q β
q q q q γ
β , γ
α , δ


β
δ
q


α , γ
α , δ


α , γ
q
β , γ

steady state probability PASEP

$$\frac{1}{Z_n} Z_\tau(\alpha, \beta, \gamma, \delta ; q \quad)$$

$$Z_n = \sum_{\tau} Z_\tau$$

$\tau = (\tau_1, \dots, \tau_n)$
state

relation with moments of Askey-Wilson polynomials

Corteel, Williams, 2009
Corteel, Stanley, Stanton, Williams, 2010

The cellular Ansatz

From quadratic algebra Q
to combinatorial objects (Q-tableaux)
and bijections

"The cellular Ansatz"

combinatorial
objects
on a 2d lattice

representation
by operators

Physics

data structures
"histories"

bijections

orthogonal
polynomials

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

$DE = qED + E + D$

PASEP

towers placements

permutations $\xleftrightarrow{\text{RSK}}$ pairs of Tableaux Young

alternative tableaux $\longleftrightarrow$ permutations

Laguerre histories

Q-tableaux

ex: ASM,
(alternating sign matrices)
FPL (fully packed loops)
tilings, 8-vertex

quadratic algebra Q

?

commutations
rewriting rules

planarisation

planar
automata

Koszul algebras
duality

website Xavier Viennot

main website www.xavierviennot.org

page «exposés»:

The combinatorics of some exclusion model in physics

IIT Madras (Indian Institute of Technology), March 2, 2012 slides (14,7 Mo)

secondary website: Courses cours.xavierviennot.org

- course IIT Bombay 2013 (20 hours)

Thank you !

Genocchi sequence of a permutation

Def. Genocchi sequence of a permutation

$$\sigma = \sigma(1) \qquad \sigma(n)$$

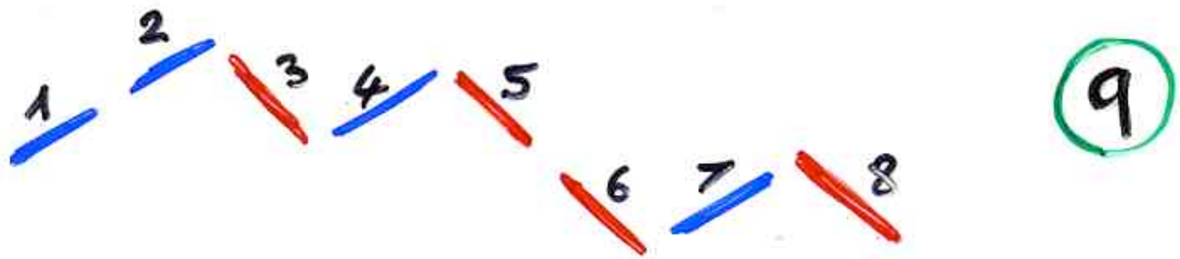
$$G(\sigma) = z_1 \ \ldots \ z_{n-1}$$

$$z_x = \begin{cases} a & (\text{ascent}) \\ d & (\text{descent}) \end{cases} \qquad \underset{\text{"value"}}{x} = \sigma(\underset{\text{"index"}}{i}) \begin{array}{l} < \sigma(i+1) \\ > \sigma(i+1) \end{array}$$

$1 \leq x \leq n-1$

convention: $\sigma(n+1) = 0$ ($\sigma(n)$ is a descent)

ex: $\sigma = (8 \ 5 \ 3 \ 2 \ 7 \ 9 \ 1 \ 4 \ 6)$



σ permutation

(valeur) x $\begin{cases} \text{avance} \\ \text{recul} \end{cases}$ ssi (indice) x $\begin{cases} \text{montée} \\ \text{descente} \end{cases}$

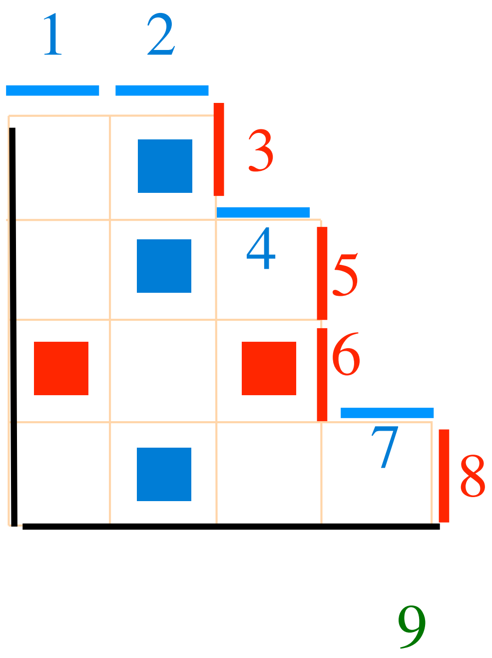
$\sigma^{-1}(x) < \sigma^{-1}(x+1)$

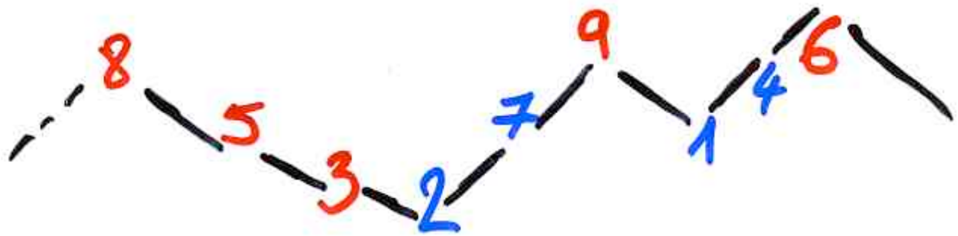
$\sigma^{-1}(x) > \sigma^{-1}(x+1)$

convention : $\sigma(n)$ descente

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 3 & 8 & 2 & 9 & 5 & 1 & 6 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \end{pmatrix}$$





“Genocchi shape” of a permutation

alternating sequence $d\,a\,d\,a\,d \ldots a\,d\,a$

Prop. (Dumont, 1974)
The nb of permutations on $\{1, 2, \ldots, 2n\}$ having an alternating Genocchi sequence is the Genocchi numbers G_{2n+2}

nombres de Genocchi

$$G_{2n} = 2(2^{2n} - 1) B_{2n}$$

Bernoulli

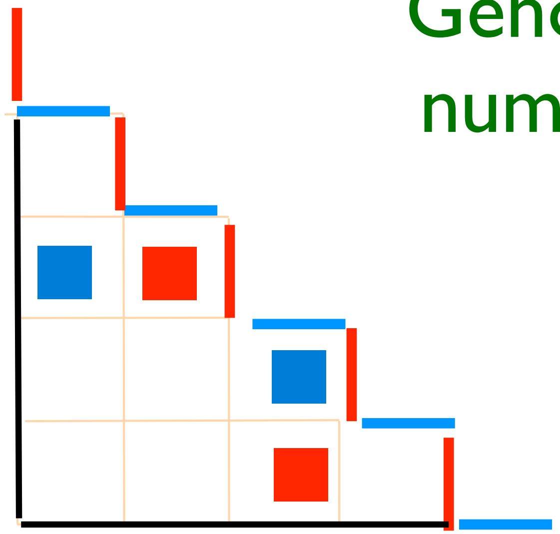
$$2^{2n} G_{2n+2} = (n+1) T_{2n+1}$$

nombres de Genocchi

$$G_{2n} = 2(2^{2n} - 1) B_{2n}$$

Bernoulli

$$2^{2n} G_{2n+2} = (n+1) T_{2n+1}$$



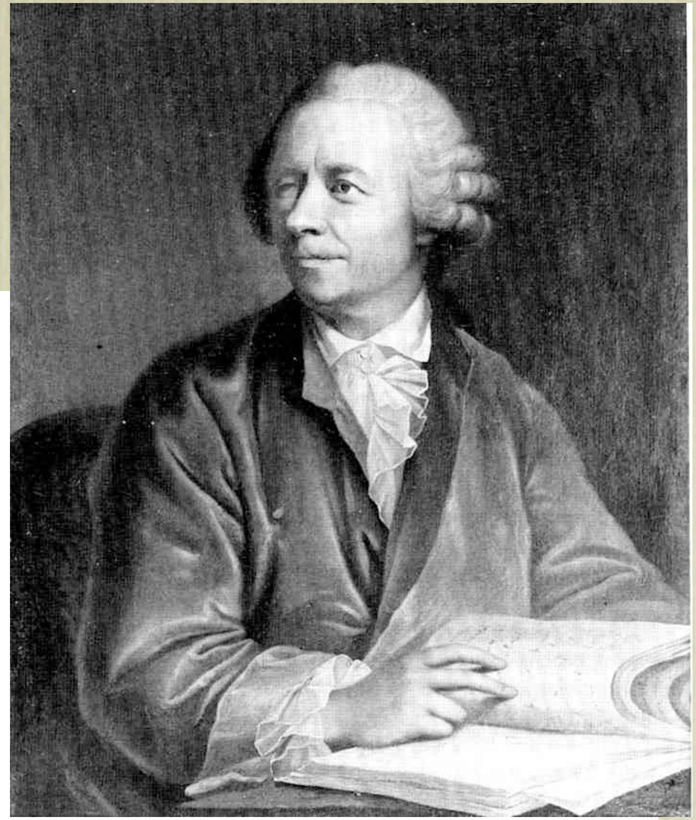
Genocchi numbers

alternating shape

Angelo Genocchi
1817 - 1889

Hinc igitur calculo instituto reperietur:

$$
\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= 3 \\
D &= 17 \\
E &= 155 = 5.31 \\
F &= 2073 = 691.3 \\
G &= 38227 = 7.5461 = 7.\frac{127.129}{3}. \\
H &= 929569 = 3617.257 \\
I &= 28820619 = 43867.9.73 \quad \&c.
\end{aligned}
$$



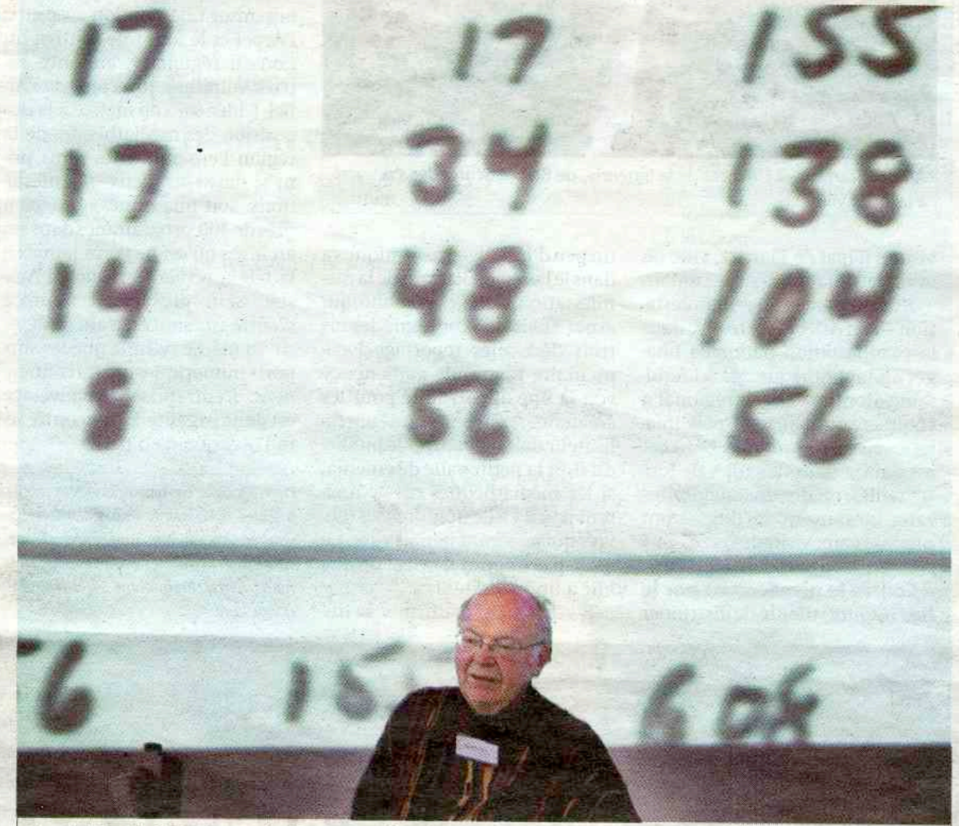
BORDEAUX 1. Le professeur Donald Knuth consacre sa vie à la programmation informatique, considérée comme un art. Il vient d'être sacré docteur honoris causa à Bordeaux

L'ermite de l'informatique

: Bernard Broustet

Une sommité de l'informatique mondiale a séjourné en Gironde ces derniers jours. Donald Knuth, 69 ans, a été sacré mardi docteur honoris causa de l'université Bordeaux 1, après avoir été lundi au centre d'une journée d'échanges qui réunissait une bonne partie du gratin français et européen de la recherche en informatique (1).

Depuis son premier contact, il y a un demi-siècle, avec un monumental et dinosaurien IBM 650, Donald Knuth n'a cessé d'être habité par la passion de l'informatique. Physicien, puis mathématicien de formation, ce géant affable et modeste a voué sa vie à ce qu'il appelle « l'art de la programmation informatique ». Car, à ses yeux, plus qu'une technique, c'est une forme d'activité qui requiert à la fois rigueur, intuition et sens esthétique. Les programmes informatiques réussis ont une sorte de beauté à laquelle même les non-spécialistes peuvent être sensibles.



Donald Knuth, à Bordeaux, le 29 octobre. À 69 ans, il animait une journée d'échanges avec le gratin européen de la recherche en informatique PHOTO LAURENT THEILLET

Une encyclopédie. Au long de sa carrière académique (pour l'essentiel à l'université californienne de Stanford), Donald Knuth a fait preuve d'une grande fécondité, en jouant notamment un rôle essentiel dans le développement de langages toujours utilisés par la communauté des mathématiciens. Mais, à 55 ans, le professeur Knuth a décidé de prendre sa retraite de Stanford. Il trouve que les fonctions administratives sont trop absorbantes pour lui permettre de mener à bien l'œuvre entamée à la fin des années 60 sous le titre de « Art of computer programming », sorte d'encyclopédie de l'algorithmique et de la programmation informatique. Donald Knuth a publié, il y a quelque temps déjà, les trois premiers volumes de cette gigantesque somme, traduite en russe, en japonais, en polonais, etc. mais pas en français. Le quatrième tome est pour bientôt. Et Donald Knuth se dit décidé à poursuivre sa tâche tant qu'il en aura la force. Ses ouvrages, dont les ventes cumulées au fil des ans approchent le million d'exemplaires, visent essentiellement les informaticiens et créateurs de programmes. Une communauté certes minoritaire à travers le monde, mais qui se trouve investie d'une mission considérable. En quelques décennies, l'écriture informatique a aidé à résoudre d'innombrables problèmes. « Mais il y en a tant d'autres qui attendent des solutions, notamment dans le domaine médical », affirme le professeur émérite de Stanford.

Un chèque de 2,56 dollars. Pour mener à bien sa tâche, Donald Knuth s'est imposé une vie d'ermite. D'ordinaire, sa journée débute par la bibliothèque ou la piscine. Après quoi, il passe tout le reste de son temps à sa table de travail, dimanche compris. Il n'a plus d'e-mail depuis le début des années 90, considérant que le courrier électronique représente une perte de temps, dès lors qu'on veut aller au fond des choses et non pas rester à leur surface. Une secrétaire lui fait passer les messages considérés comme les plus urgents. Pour le reste, Donald Knuth demande qu'on lui écrive par courrier ordinaire ou par fax, dont il prend parfois connaissance avec des mois de retard. Il s'oblige, en revanche, à tenir aussi scrupuleusement que possible sa promesse d'envoyer un chèque de 2,56 dollars à tout lecteur ayant détecté une erreur dans un de ses livres. Par ailleurs, pour se détendre, il pratique l'orgue, appris dans sa prime jeunesse auprès de son père qui partagea sa vie entre la musique et l'enseignement.

L'orgue de Sainte-Croix. Donald Knuth n'est pas fermé aux choses de ce monde. Sur son site Internet, à la rubrique « Questions qui ne me sont pas fréquemment posées », il demande entre autres : « Pourquoi mon pays a-t-il le droit d'occuper l'Irak ? », « Pourquoi mon pays ne soutient-il pas une Cour internationale de justice ? » Mais cet homme de conscience ne se veut pas militant, pas plus qu'il n'aspire au vedettariat et à la richesse. « Beaucoup de gens, dit-il, ont tendance à considérer que l'informatique, c'est surtout des histoires de business, d'entreprise. Ce n'est pas mon cas. » Sortant de sa semi-réclusion, Donald Knuth s'est donc laissé convaincre d'accepter les hommages de l'université de Bordeaux, après celles de Harvard, d'Oxford, de Tübingen. Il a eu le coup de foudre pour la beauté et l'agrément de la ville. Et il n'oubliera sans doute pas de sitôt l'orgue illustre de l'église Sainte-Croix (2), sur lequel il a eu le bonheur d'exercer son talent.

(1) Ces journées étaient organisées par le Laboratoire bordelais de recherche en informatique (Labri).

(2) Thierry Semenoux, professeur d'orgue au conservatoire de Bordeaux, a joué dans ce domaine un rôle de cicérone auprès de Donald Knuth.