From automata
to
RSK correspondence

«Words, Codes and
Algebraic Combinatorics»
Christophe Reutenauer Fest
Cetraro, 4 July 2013

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LaBRI, CNRS, Bordeaux
www.xavierviennot.org

7016 0 0201 4

SNCF

carte *Senior*

CARTE SENIOR

RAIL + PLUS

Nom REUTENAUER

Prénom CHRISTOPHE

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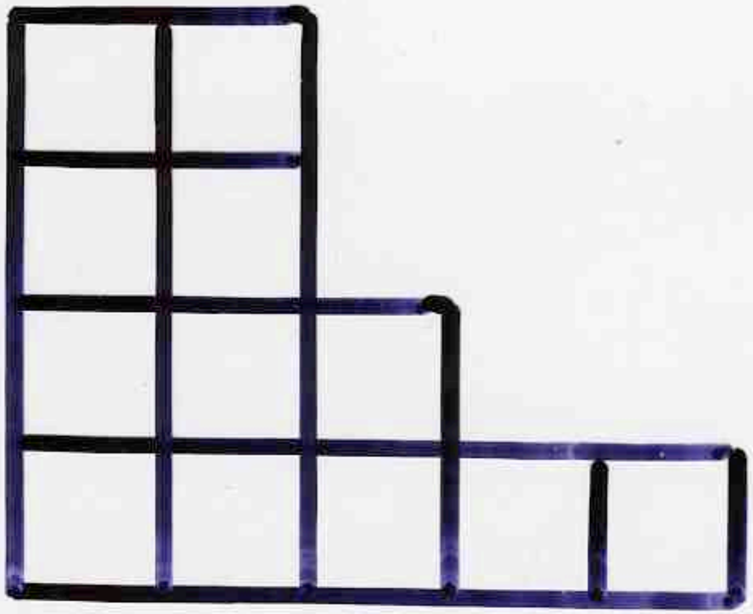
RSK

The Robinson-Shensted-Knuth correspondence

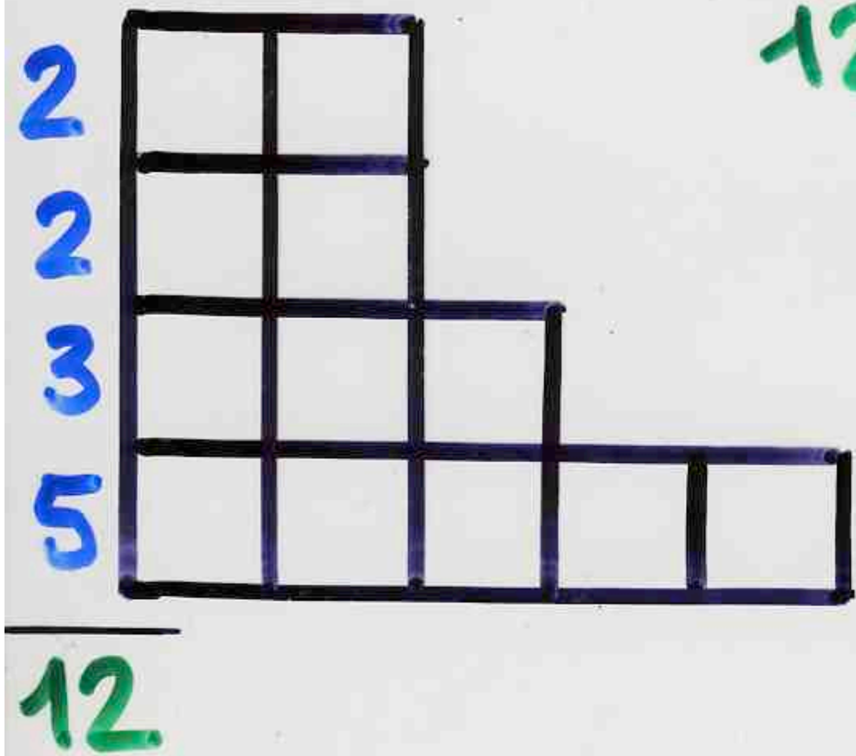
Algebraic combinatorics



Ferrers diagam

or

Young diagram

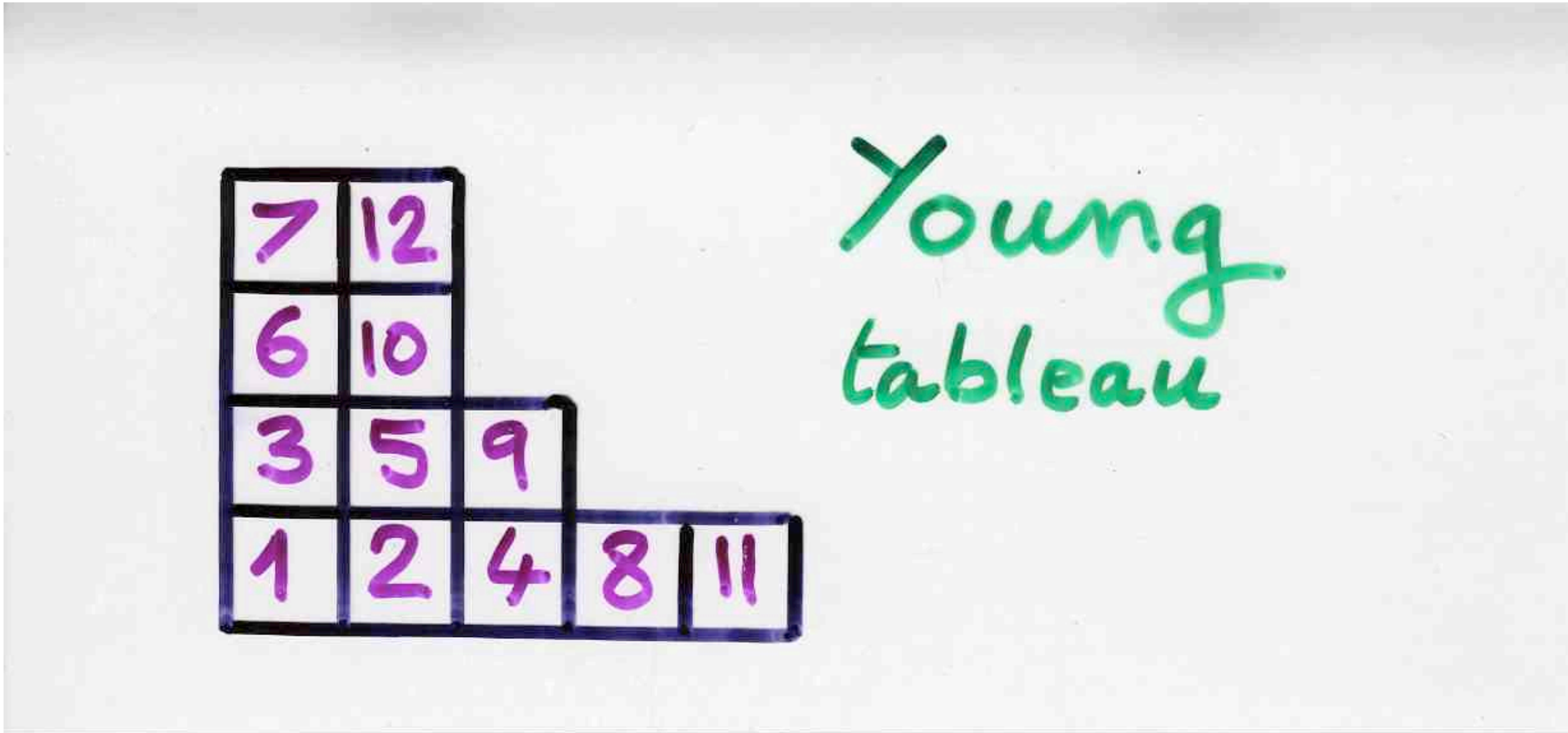


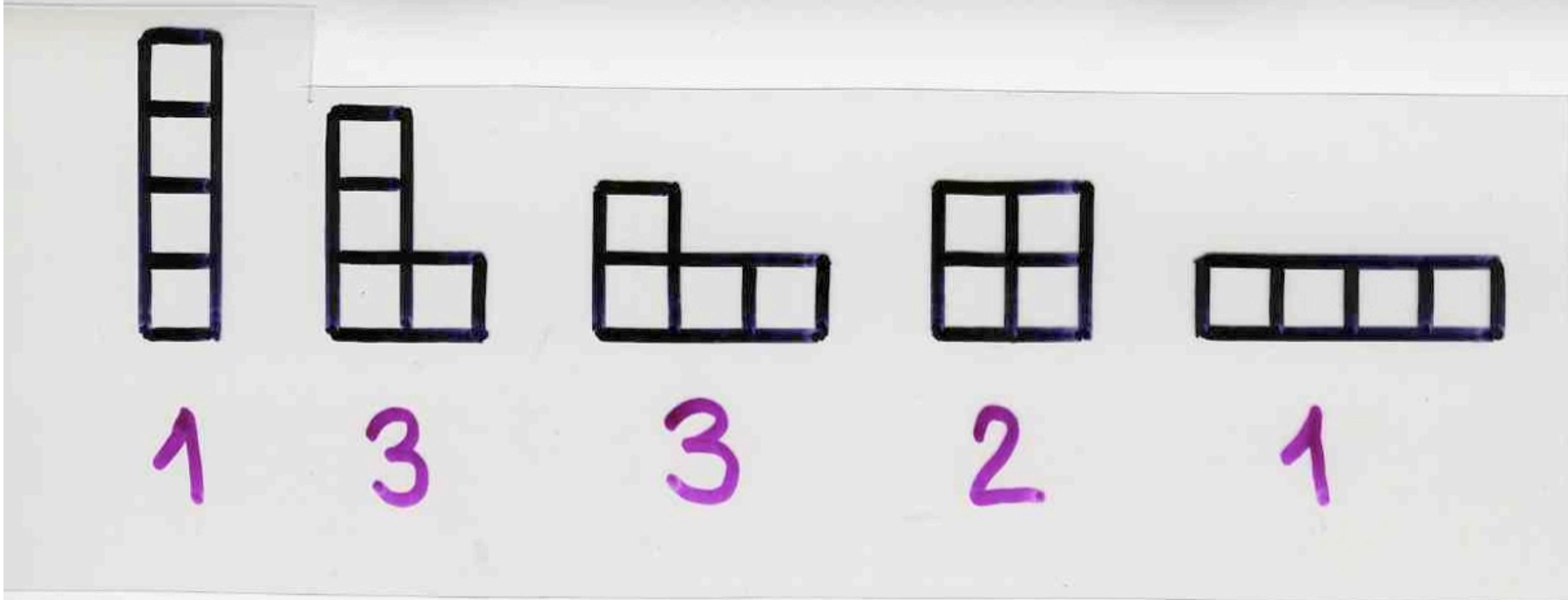
$12 = n = 5 + 3 + 2 + 2$

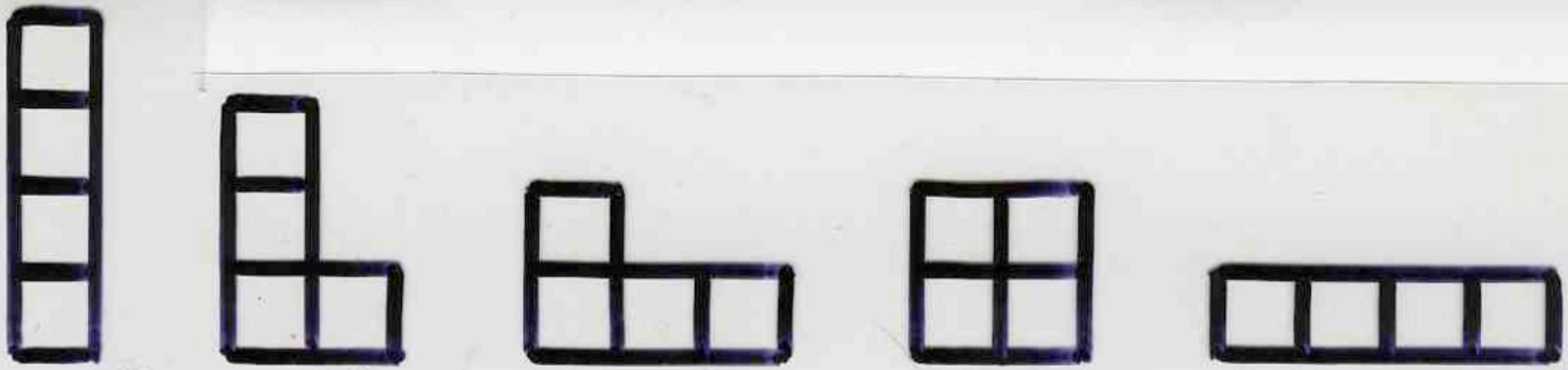
Ferrers

diagram

Partition of n


7 12
6 10
3 5 9
1 2 4 8 11
Young tableau

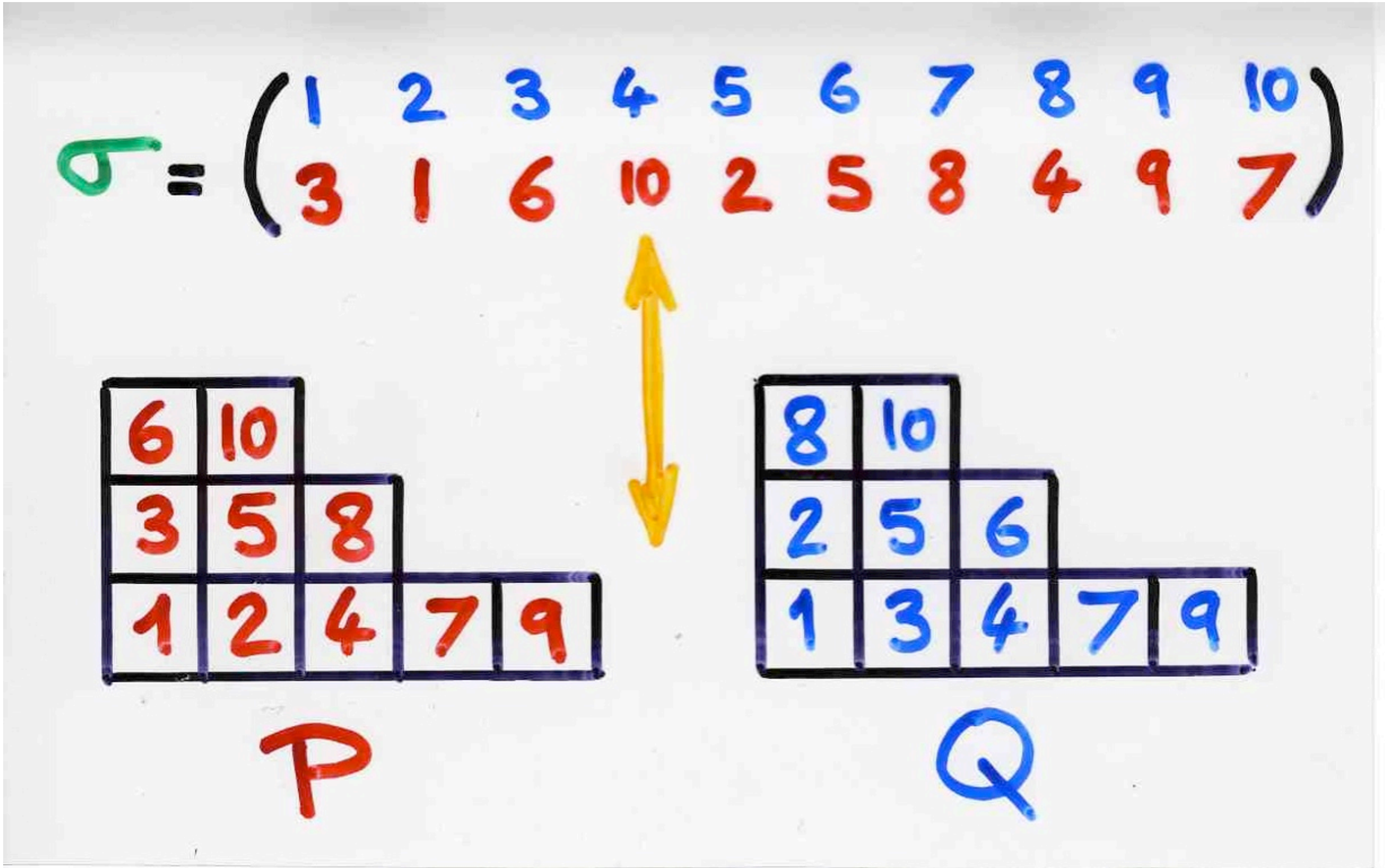

1
3
3
2
1



$$1^2 + 3^2 + 3^2 + 2^2 + 1^2$$

$$= 1 + 9 + 9 + 4 + 1$$

$$= 24 = 4!$$



The Robinson-Schensted correspondence between permutations and pair of (standard) Young tableaux with the same shape

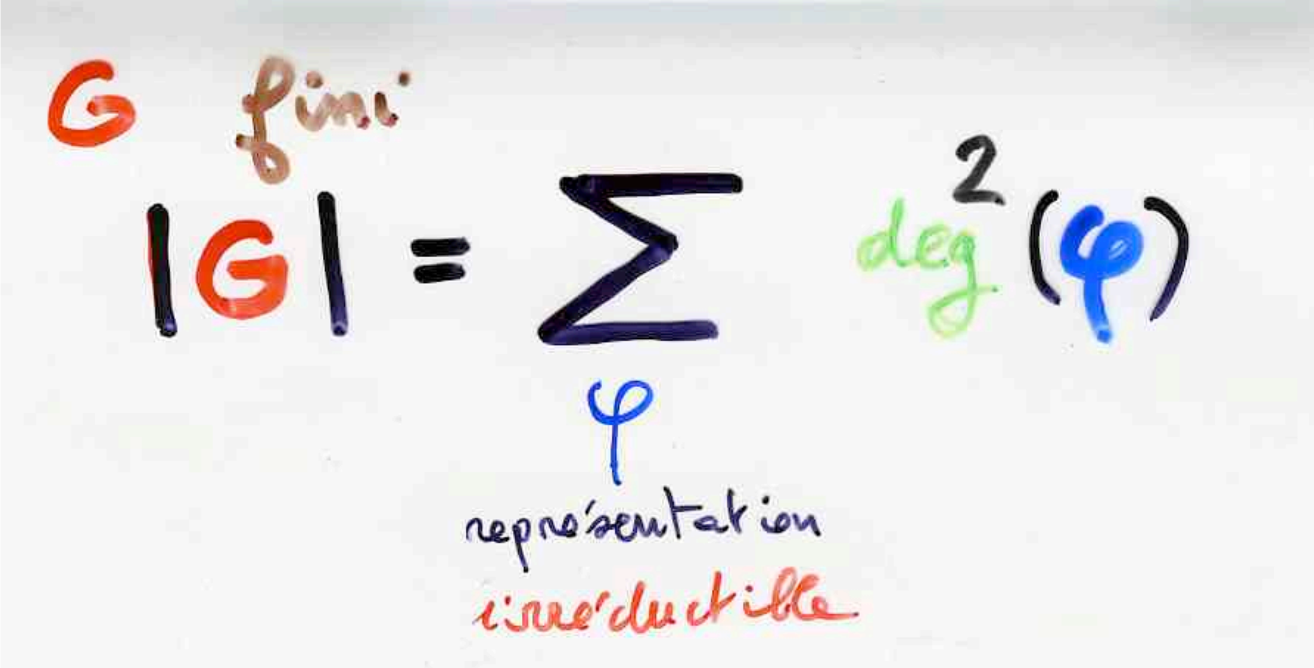
The Robinson-Shensted-Knuth correspondence

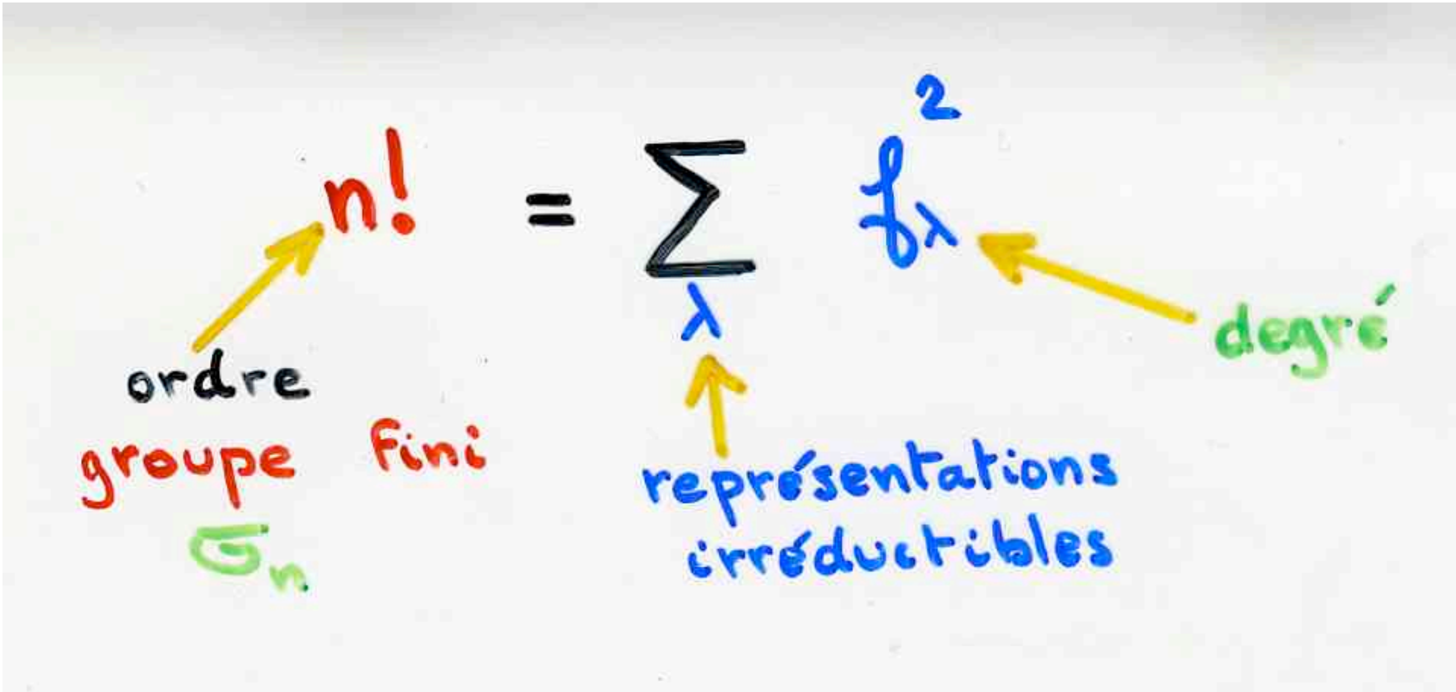
RSK

related to

the representation theory of finite groups

symmetric group of permutations


G fini
$|G| = \sum_{\varphi} \deg^2(\varphi)$
représentation
irréductible


$n! = \sum_{\lambda} f_\lambda^2$
ordre
groupe fini
$\mathfrak{S}_n$
représentations
irréductibles
degré

RSK with Schensted's insertions

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1					

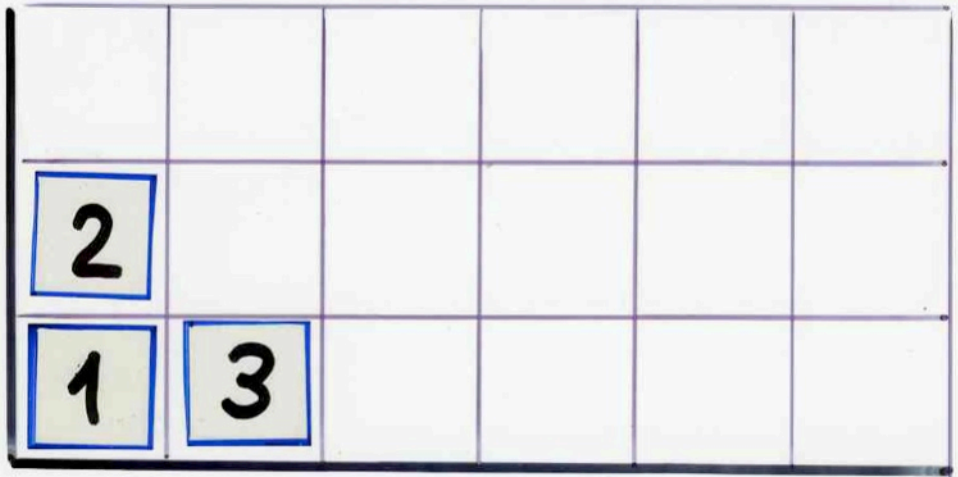
3					

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

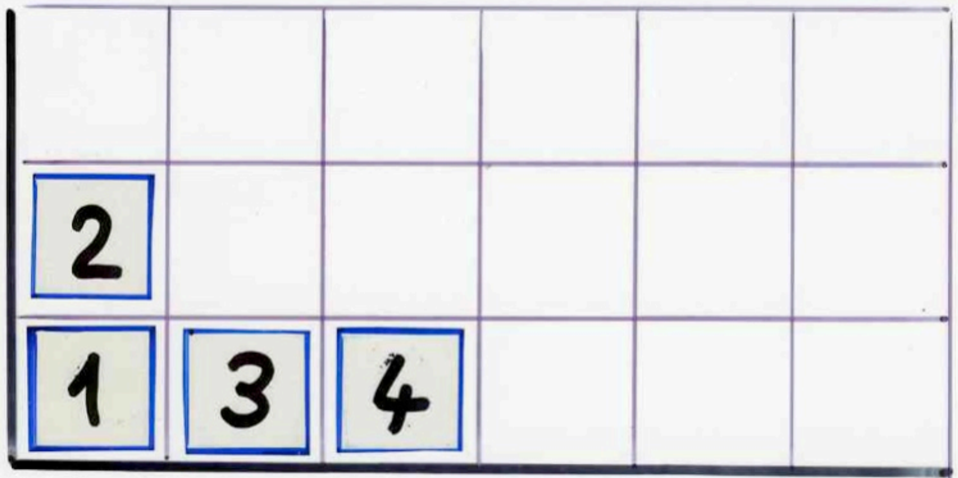
2					
1					

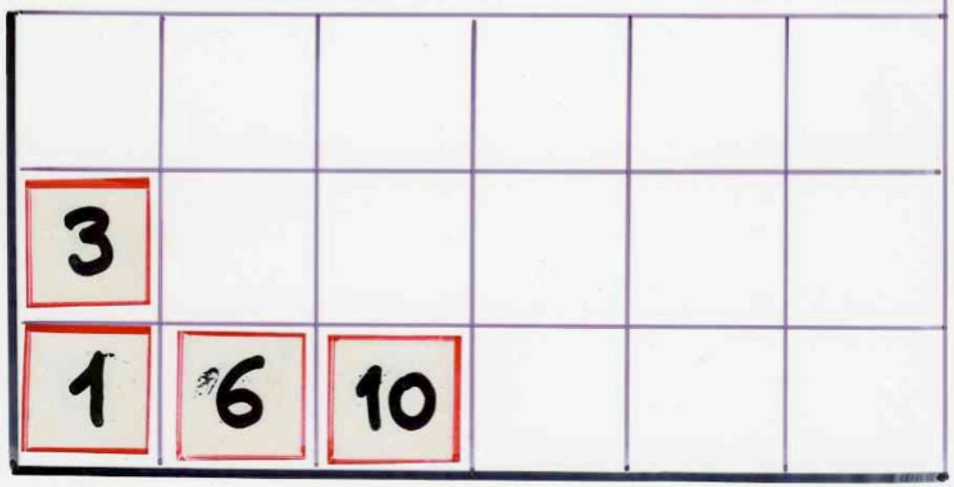
3					
1					

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

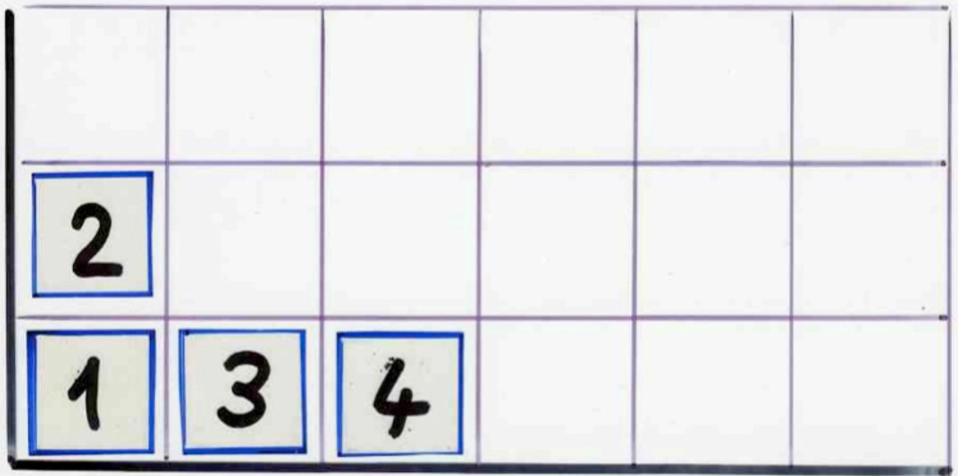


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



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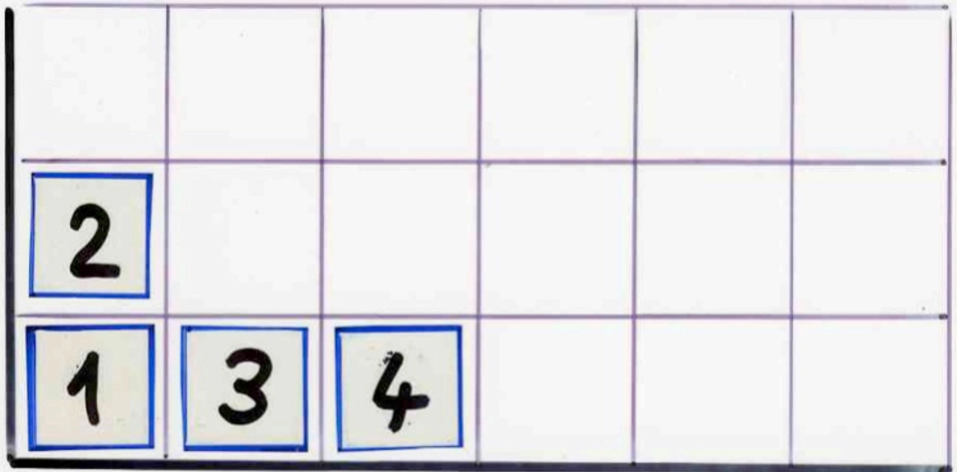

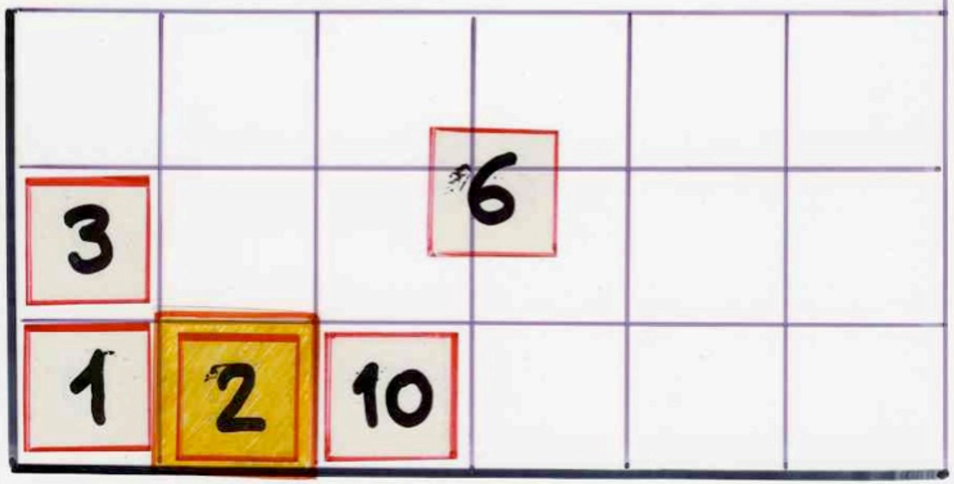

1	2	3	4	5	6	7	8	9	10
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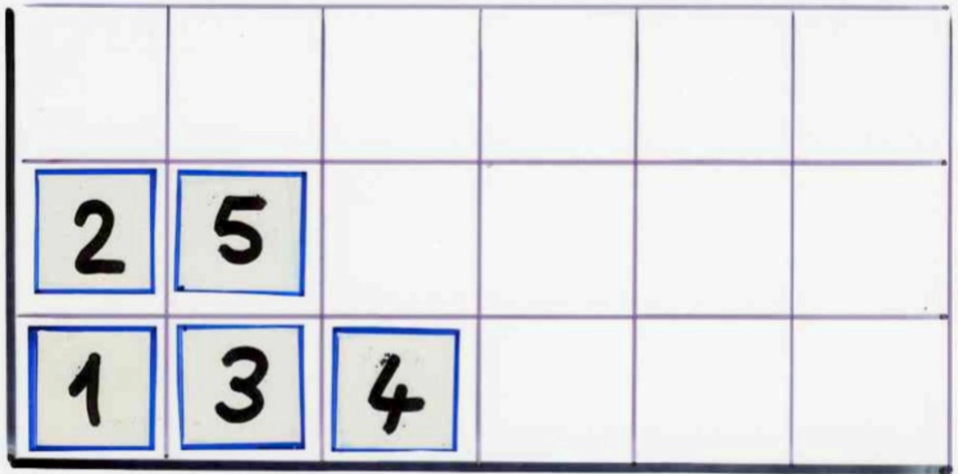

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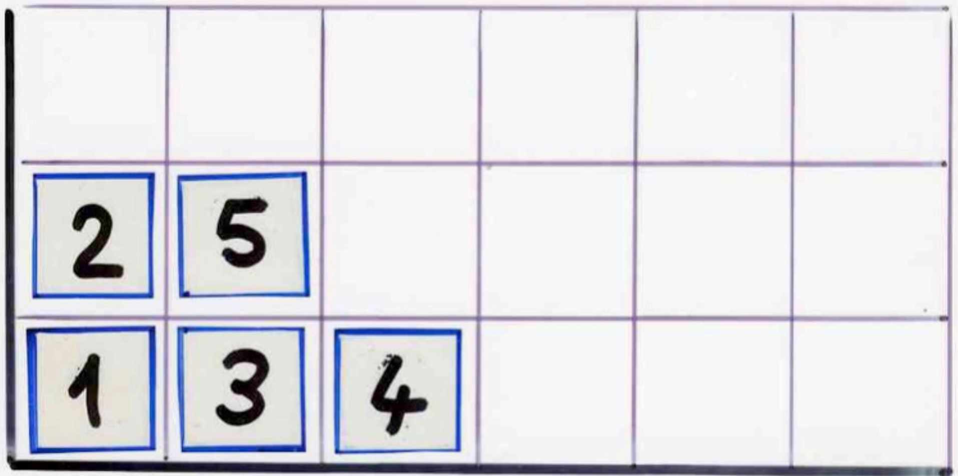

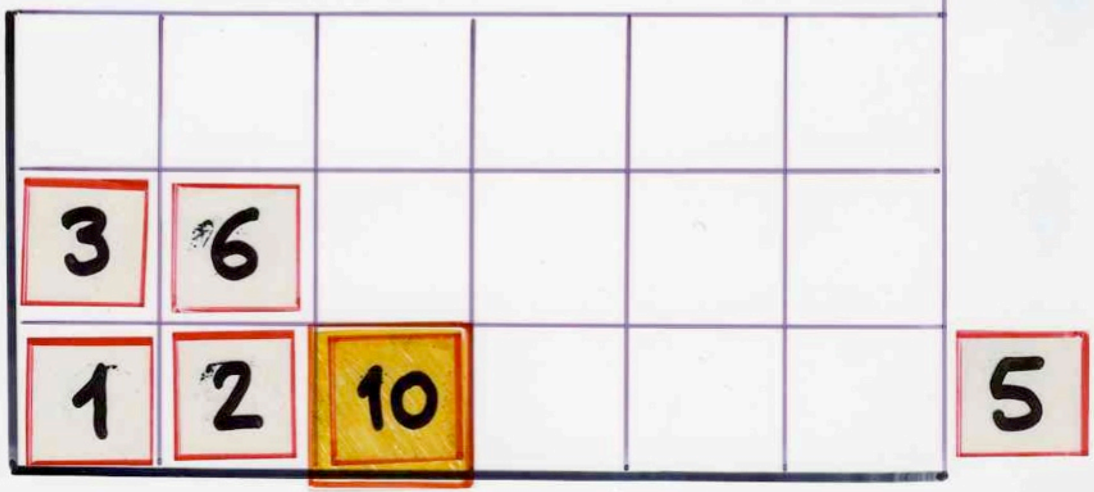

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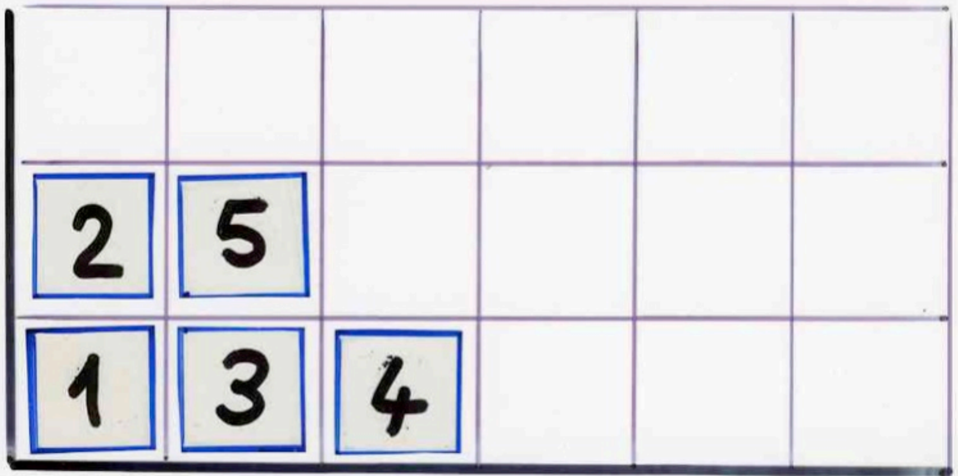
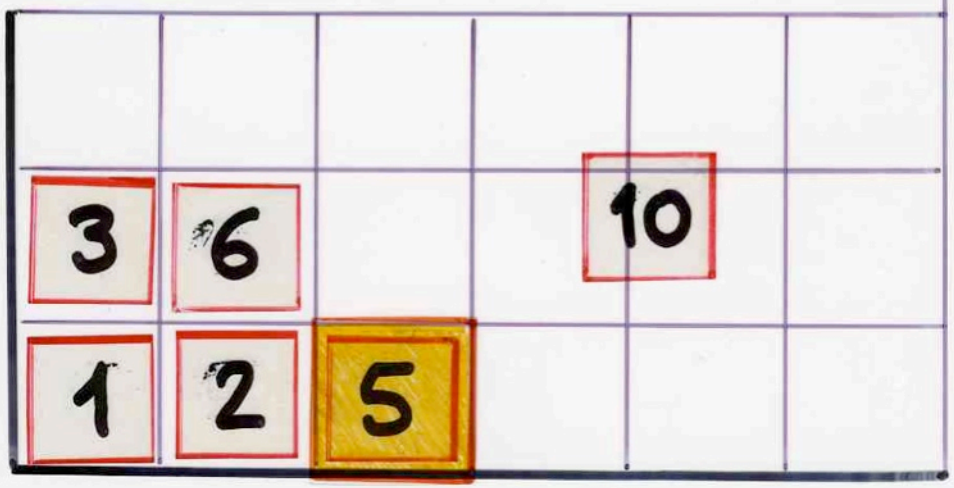




1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4			

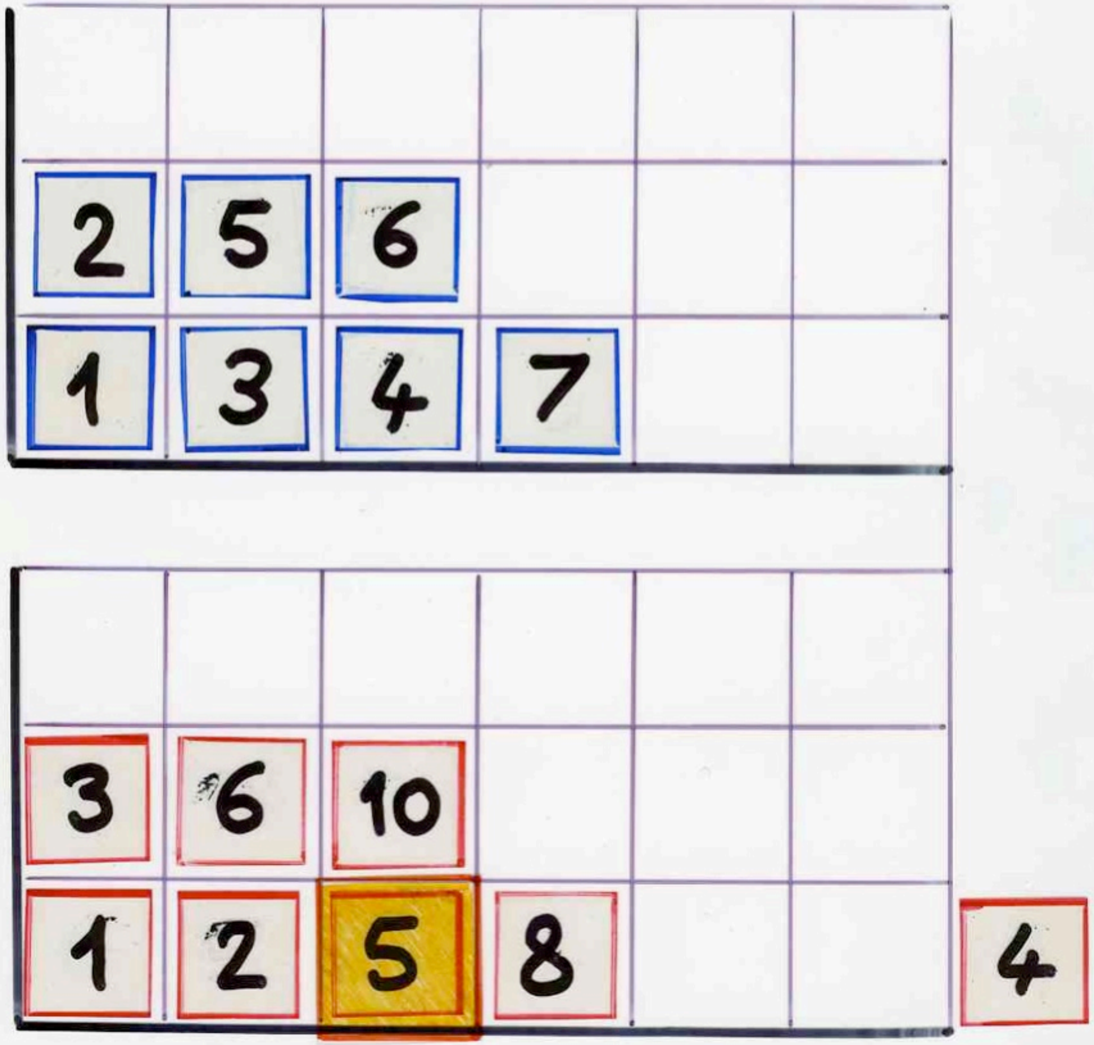
3	6	10			
1	2	5			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4	7		

3	6	10			
1	2	5	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



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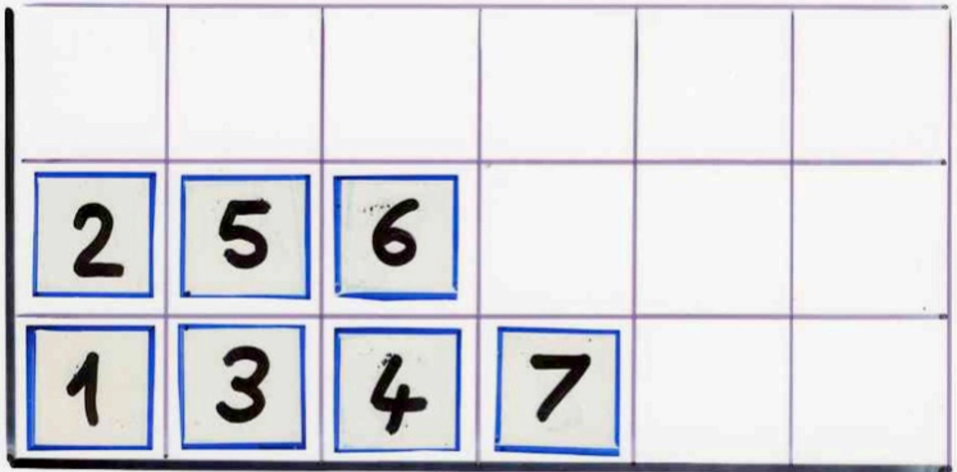

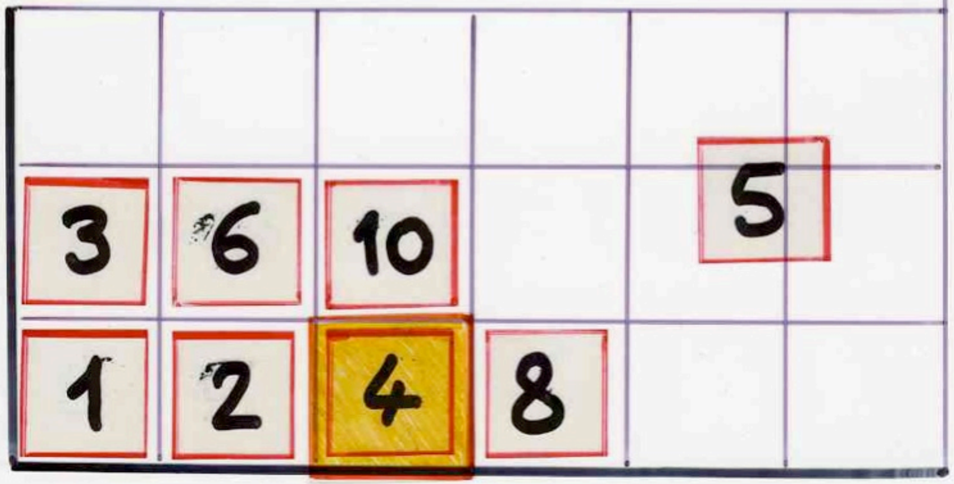

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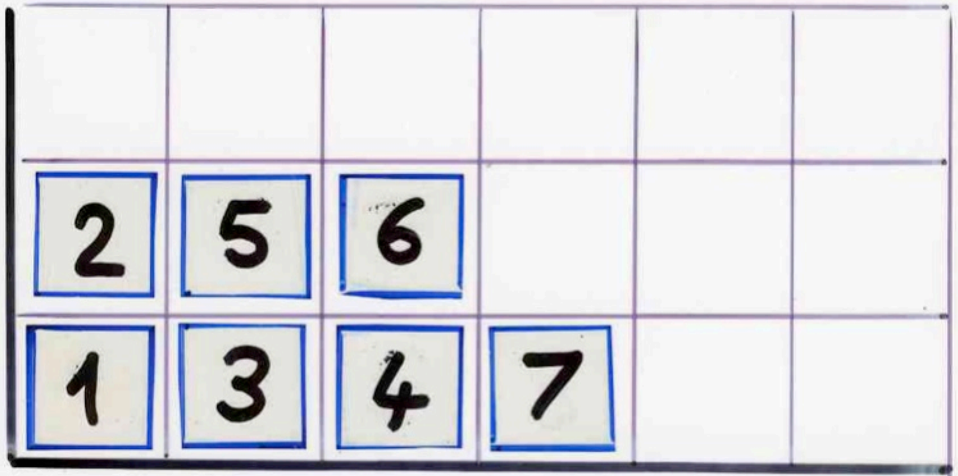

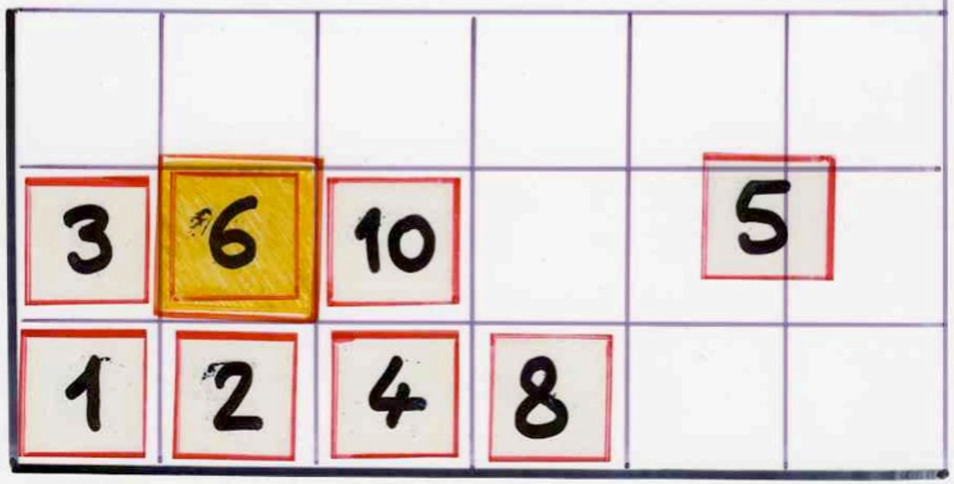

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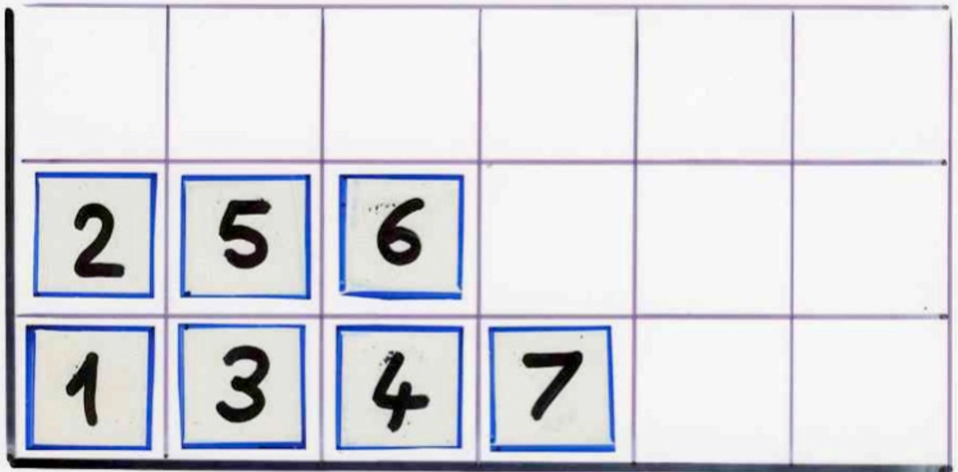


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7		

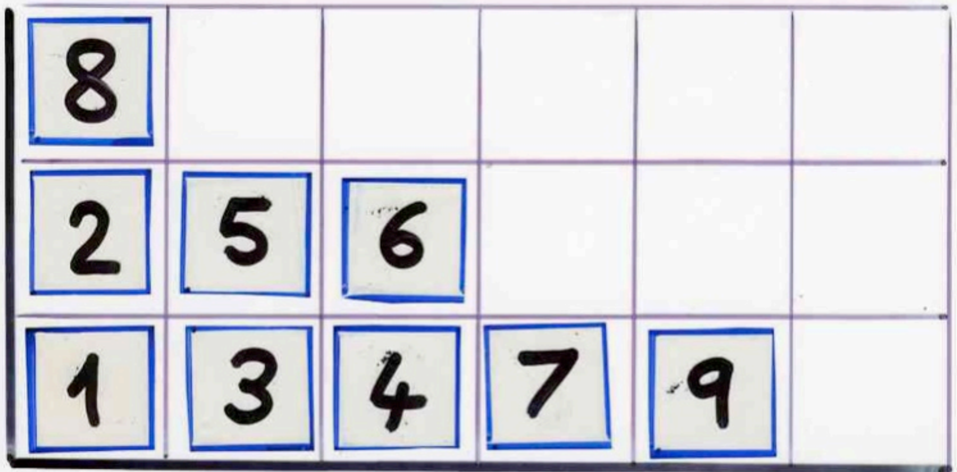
6					
3	5	10			
1	2	4	8		

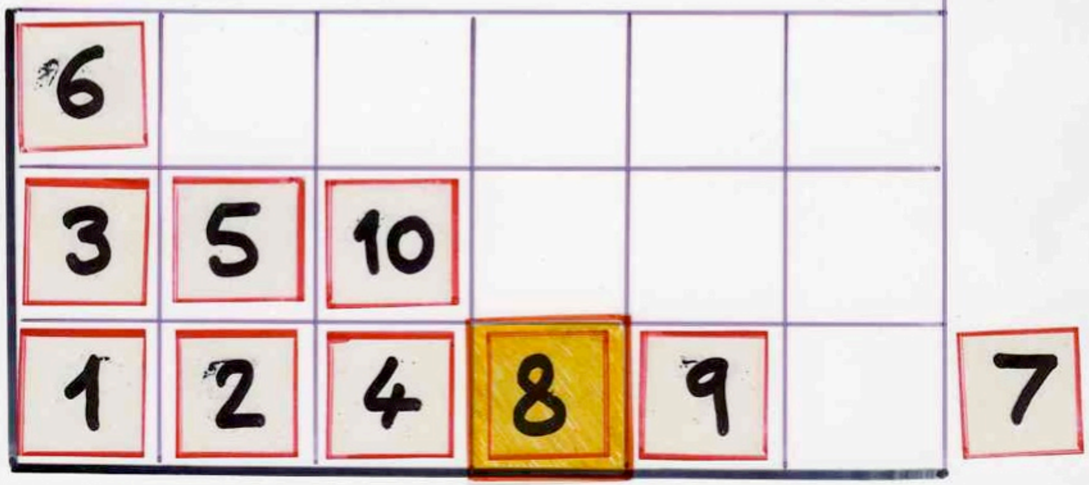
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6					
3	5	10			
1	2	4	8	9	

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





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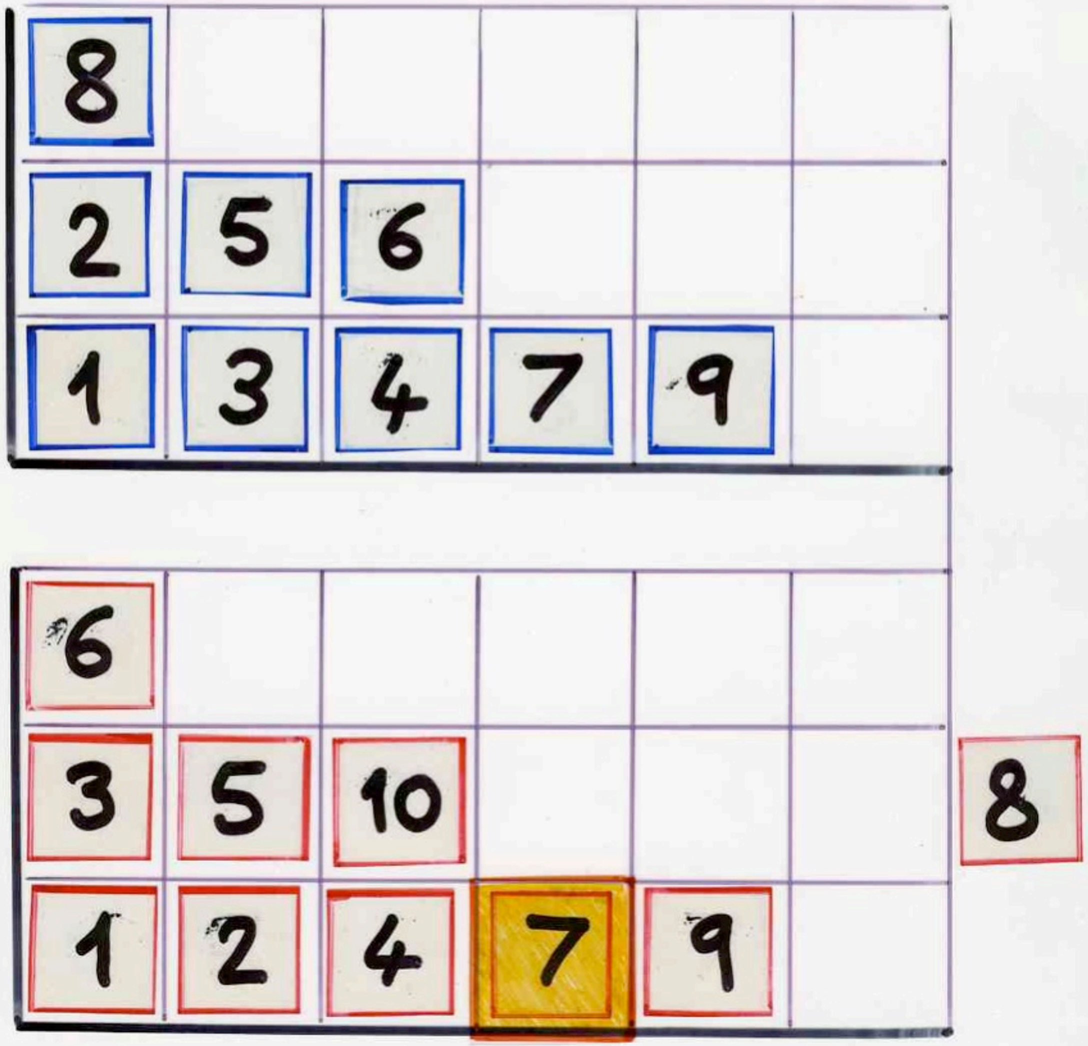

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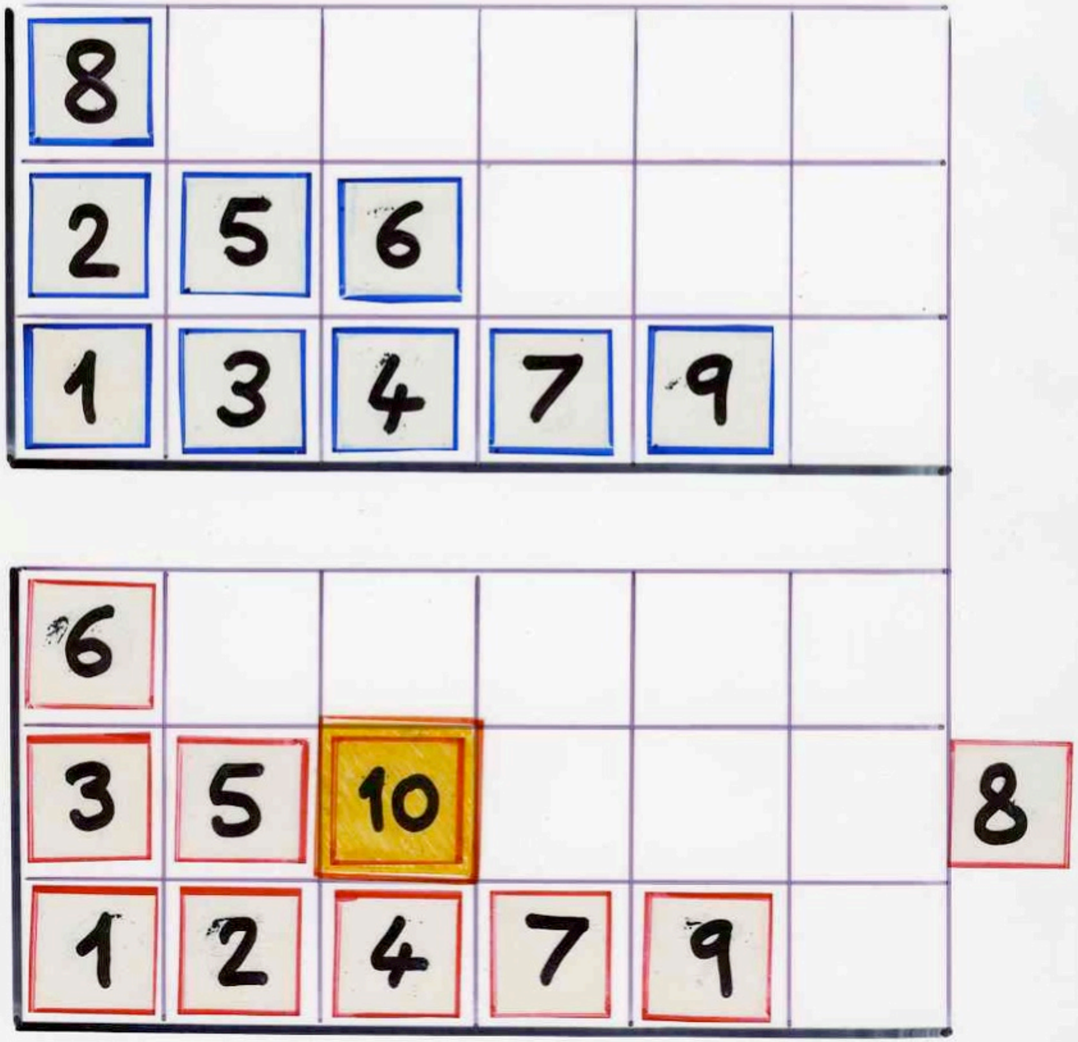

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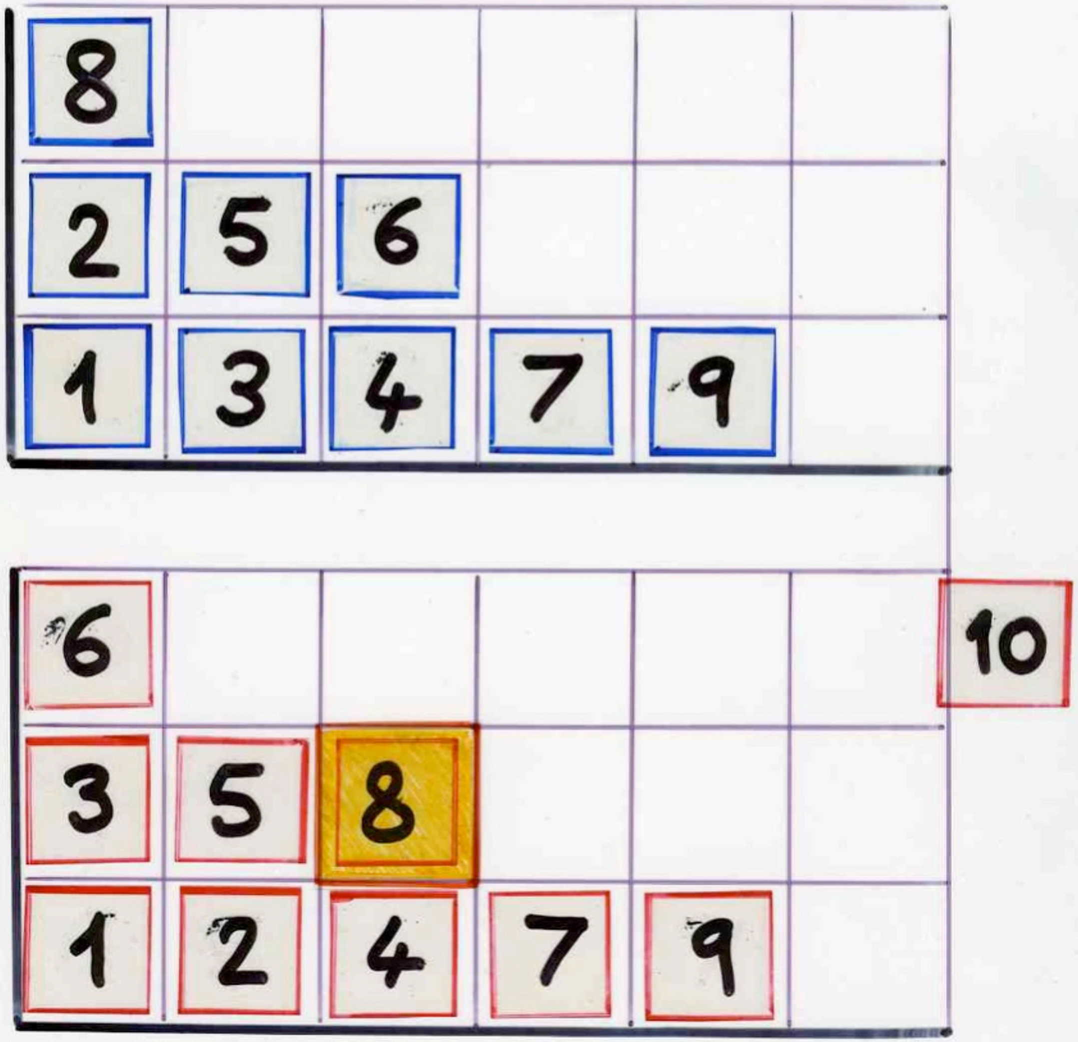


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

6	10				
3	5	8			
1	2	4	7	9	

$$\sigma \longleftrightarrow (P, Q)$$

$$\sigma^{-1} \longleftrightarrow (Q, P)$$

Donald Knuth (1972)

" The unusual nature of these coincidences might lead us to suspect that some sort of withcraft is operating behind the scenes "

Vol 3, *"The art of computer programming"*

Words, codes, languages, automata, ...

Theoretical computer science

finite automaton

L words recognized by a finite automaton

$$w = a_1 a_2 \ldots a_n$$

generating function for the number of words of length n

rational

$$\sum_{\substack{|w|=n \\ w \in L}} t^n = \frac{N(t)}{D(t)}$$

«pictures»
or geometric figures
or combinatorial objects
on a square lattice

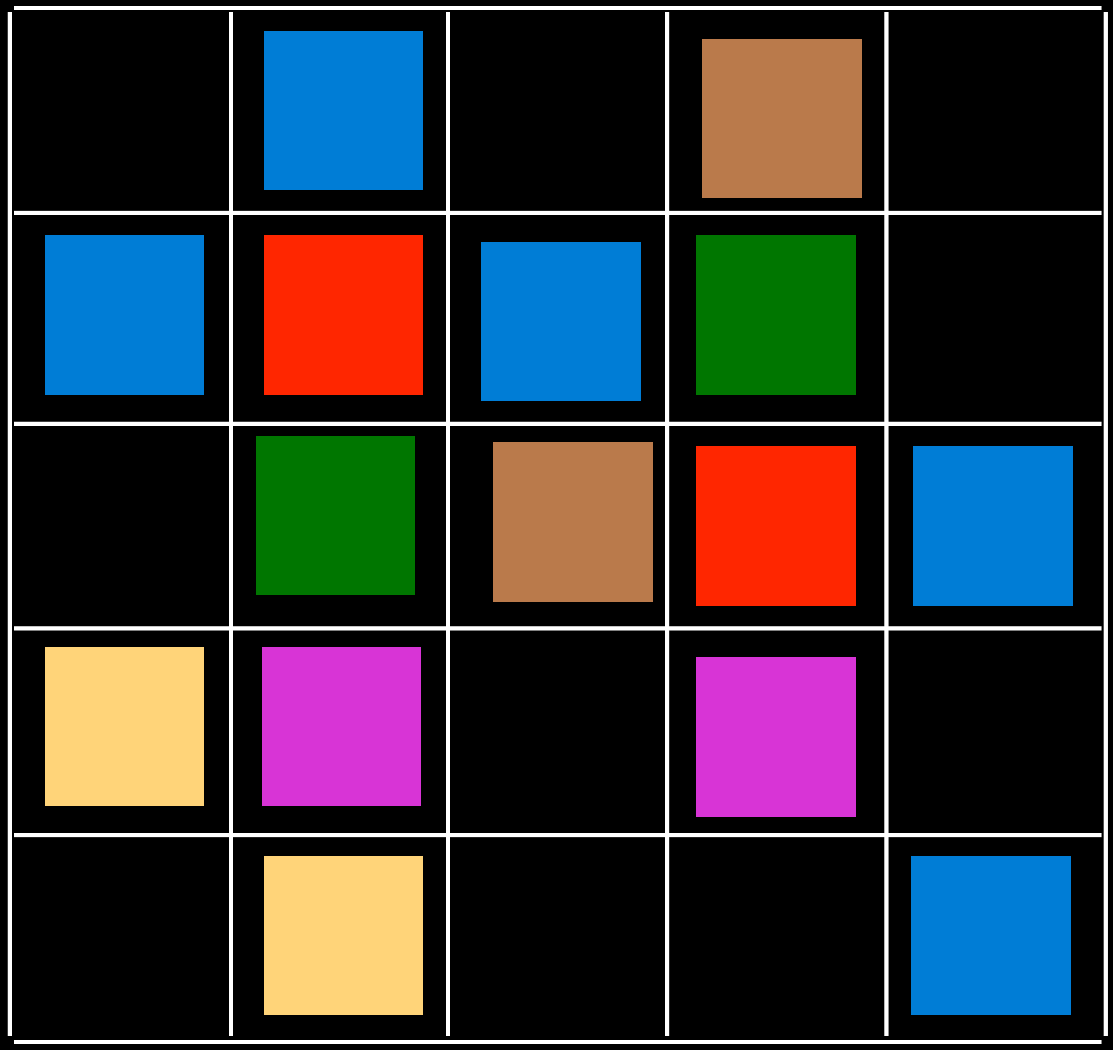
in relation
with physics

enumeration ?

«pictures» recognized by an automaton ?

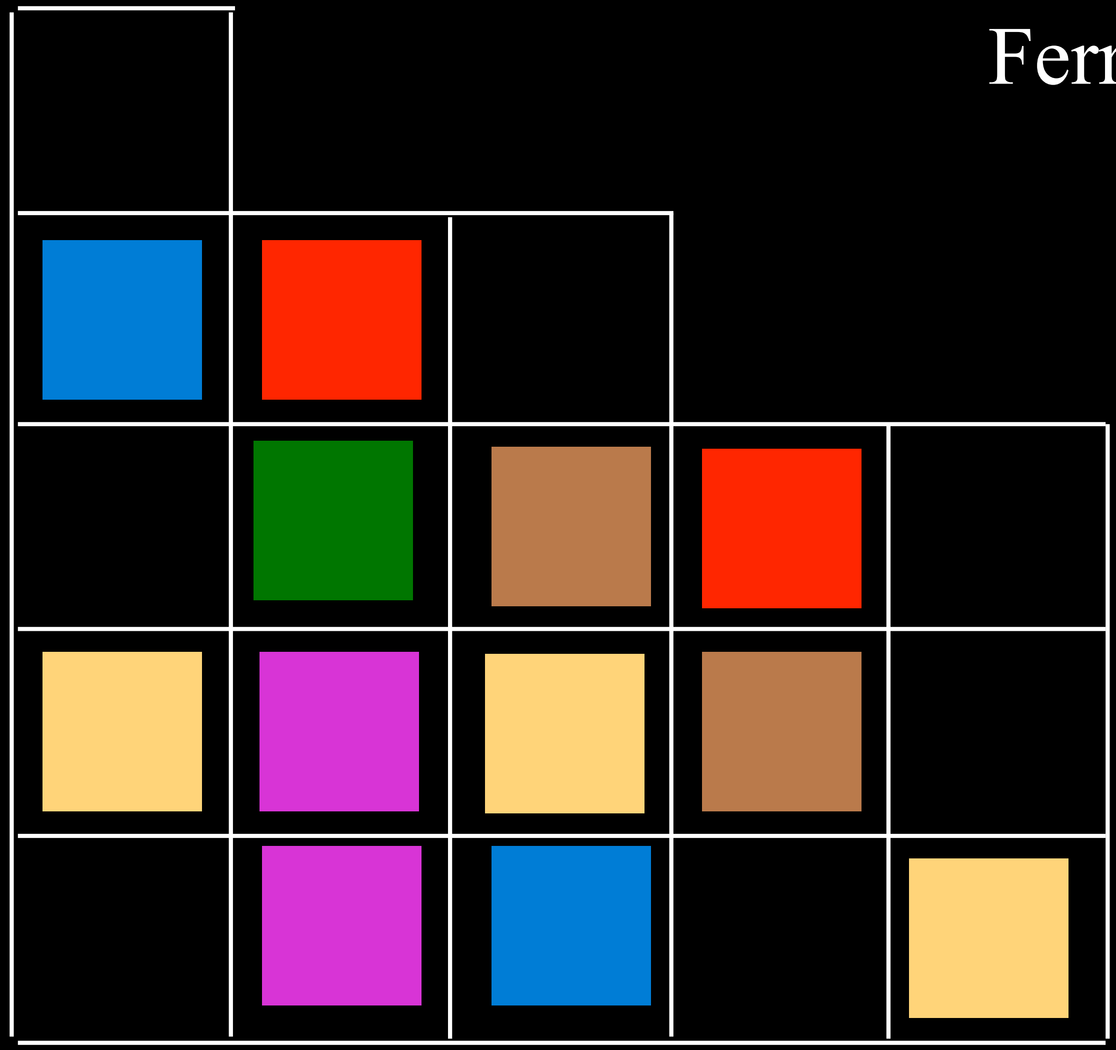
Planar automata

«picture»



«picture»

Young diagram
Ferrers diagam



Def. planar automaton $\mathcal{P}$

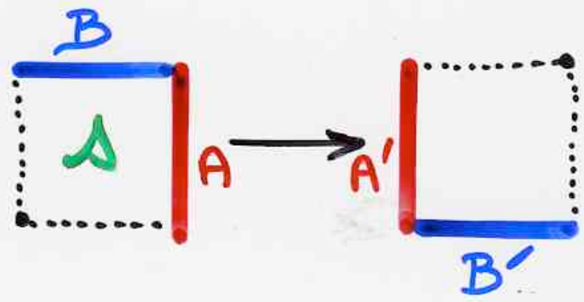
- 3 finite sets
  - $\mathcal{B}$ horizontal alphabet
  - $\mathcal{A}$ vertical alphabet (state)
  - S planar labels (state)

- θ (partial) transition function

$$(s, B, A) \xrightarrow{\theta} (B', A') \quad \text{or} \quad \emptyset$$

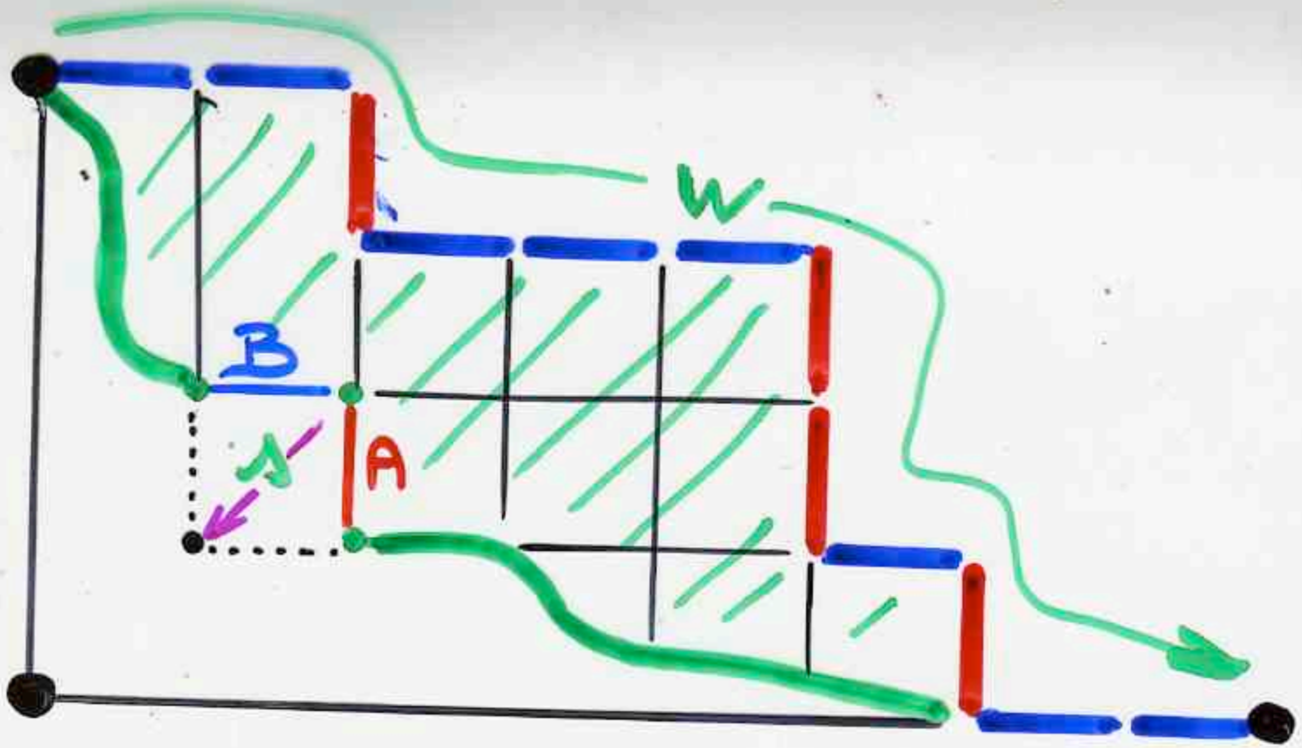
$s \in S$; $B, B' \in \mathcal{B}$; $A, A' \in \mathcal{A}$

planar rewriting

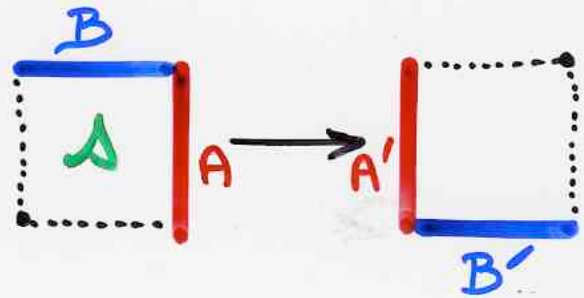


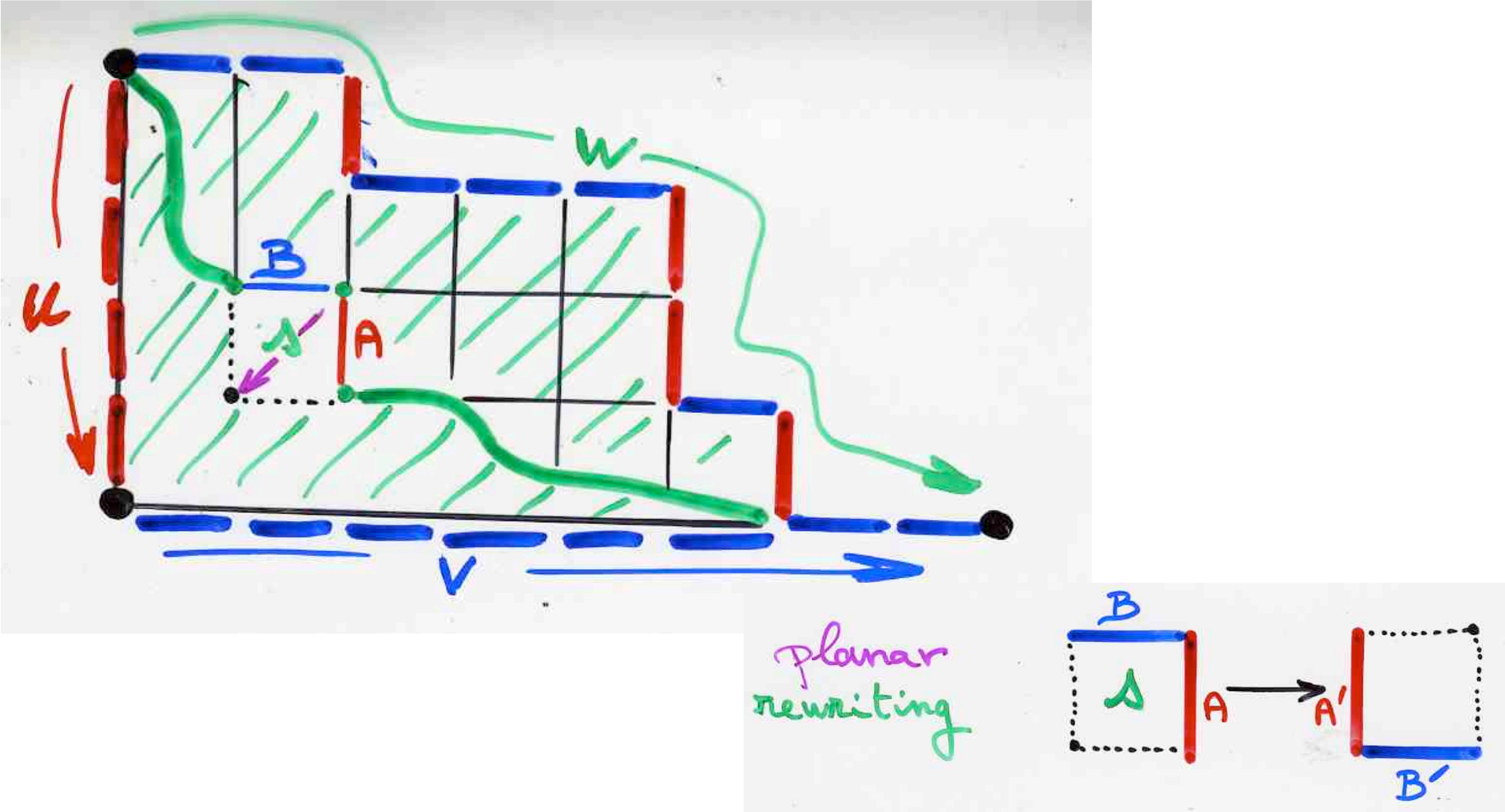
- $w \in (\mathcal{A} \cup \mathcal{B})^*$ initial word (state)
- uv, $u \in \mathcal{A}^*$, $v \in \mathcal{B}^*$ final word

Def. tableau T accepted by a planar automaton $P = (S, \mathcal{B}, \mathcal{A}, \theta, w, uv)$



planar rewriting





c(u,v;w) = number of tableaux T

accepted by the automata P

with initial state w and final state uv

Planar automata

example: ASM

alternating sign matrices

Def - **ASM** alternating sign matrix

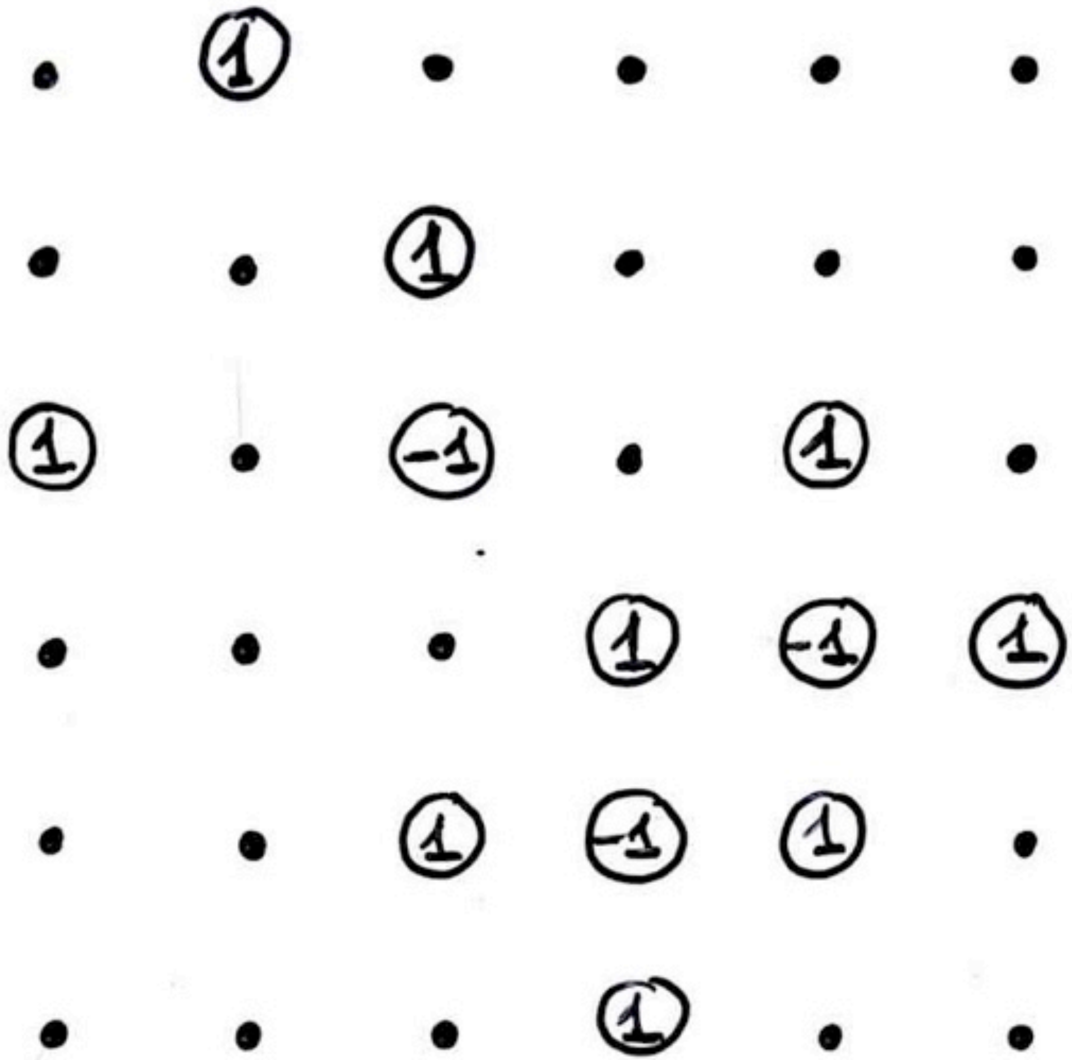
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(i) entries: 0, 1, −1

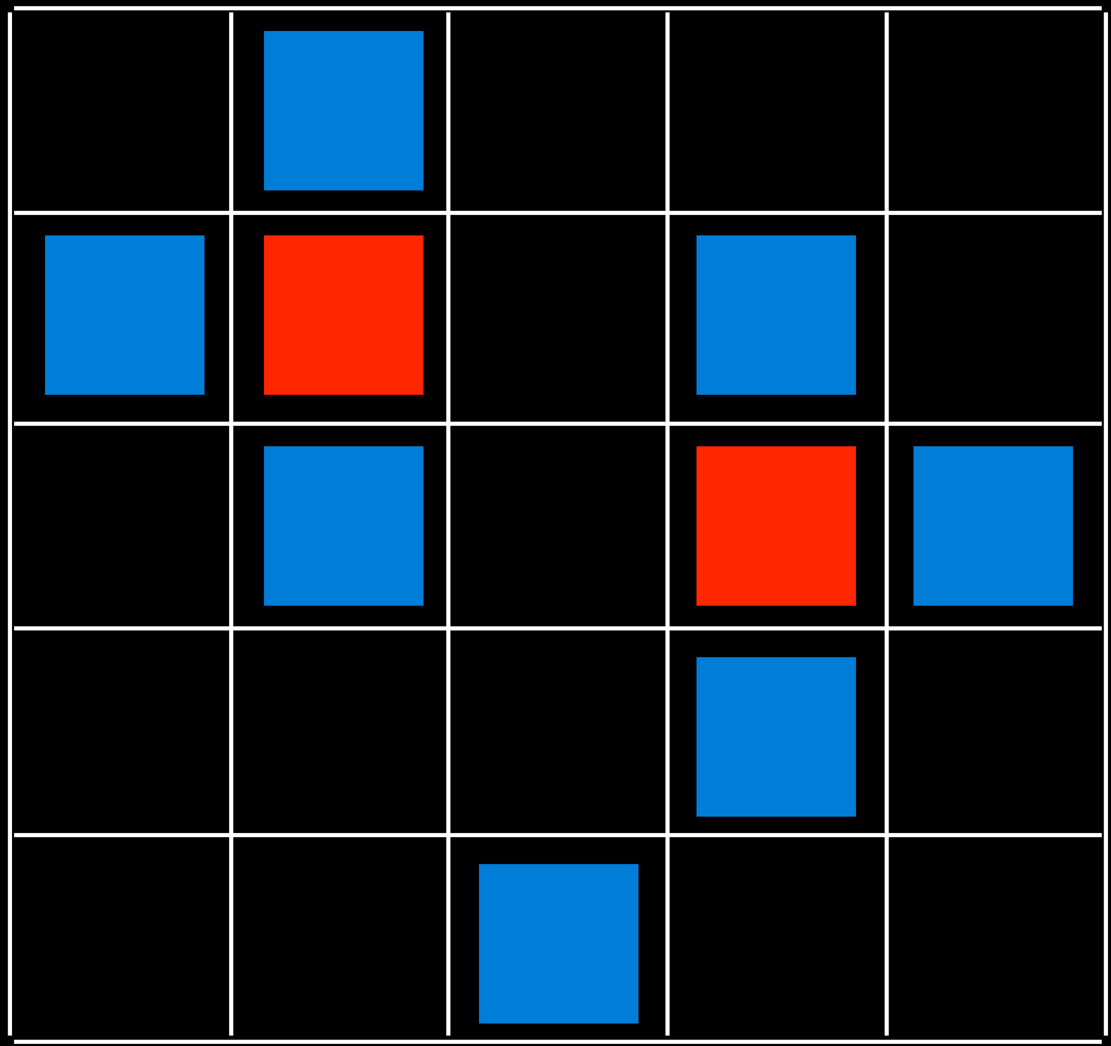
(ii) sum of entries in each {row, column} = 1

(iii) non-zero entries alternate in each {row, column}

ASM



Alternating sign matrices



Permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad + \; 6$$

permutations

$$1, \; 2, \; 7, \; 42, \; 429, \; \ldots$$

$$1, \ 2, \ 7, \ 42, \ 429, \ \ldots$$

$$\frac{1! \ 4! \ \cdots \ (3n-2)!}{n! \ (n+1)! \ \cdots \ (n+n-1)!}$$

?

alternating sign matrices (ex-) conjecture

Mills, Robbins, Rumsey (1982)

D. Zeilberger (1992– 1995)
(+ 90 checkers)

Proof of the **A. S. M.** conj.

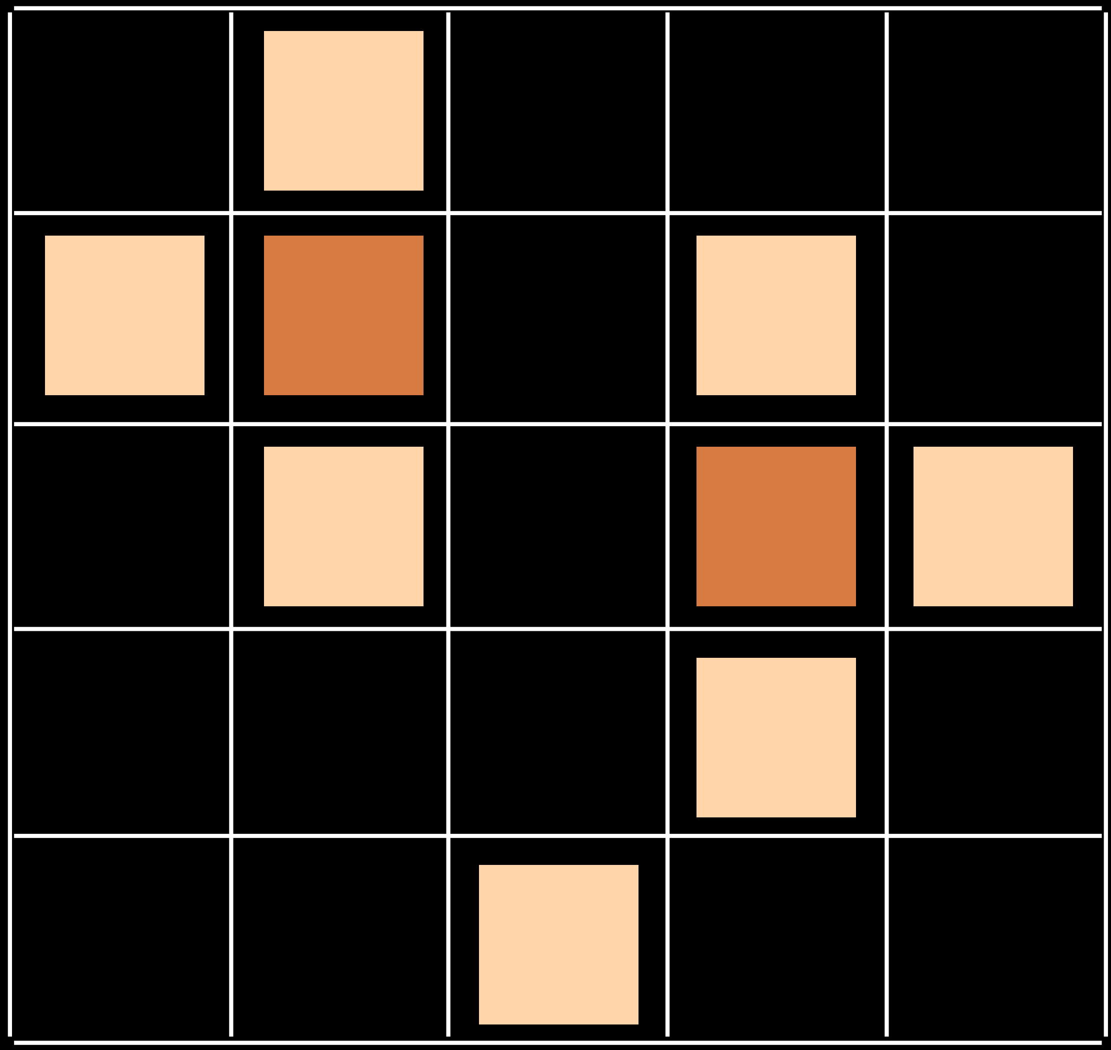
Kuperberg (1995)

6- vertex model

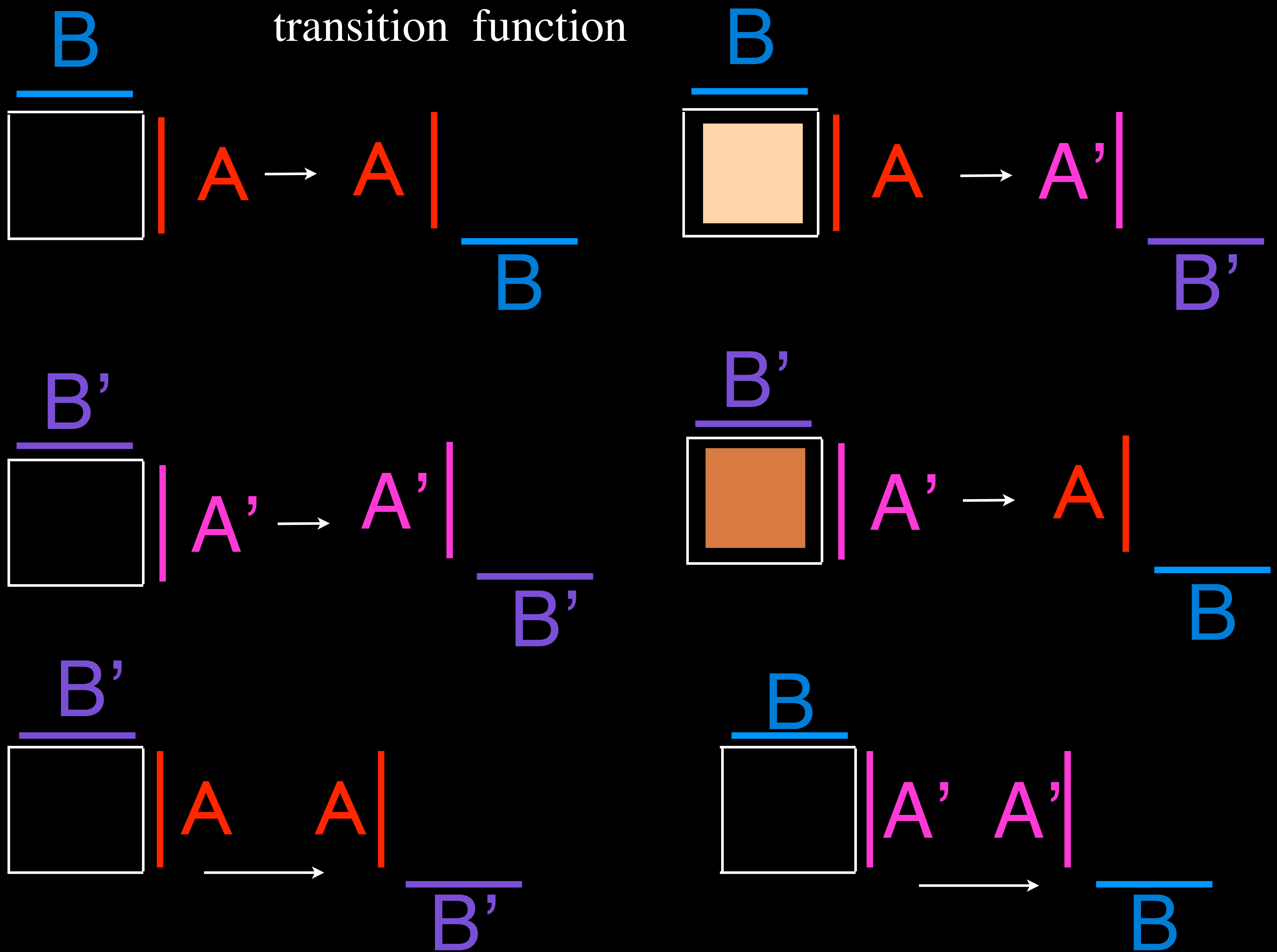
(ice model)

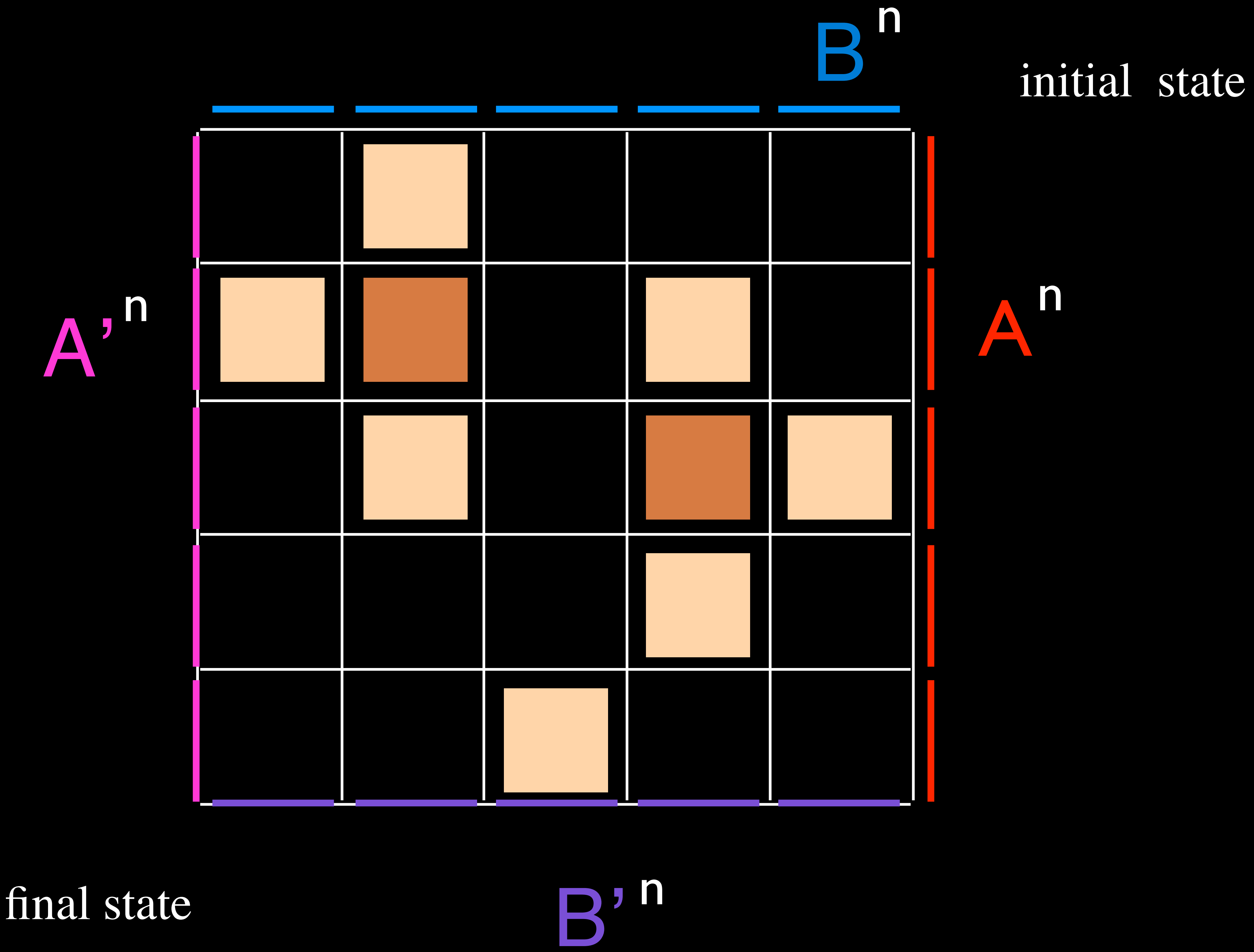
with domain wall boundary condition

alternating sign matrix

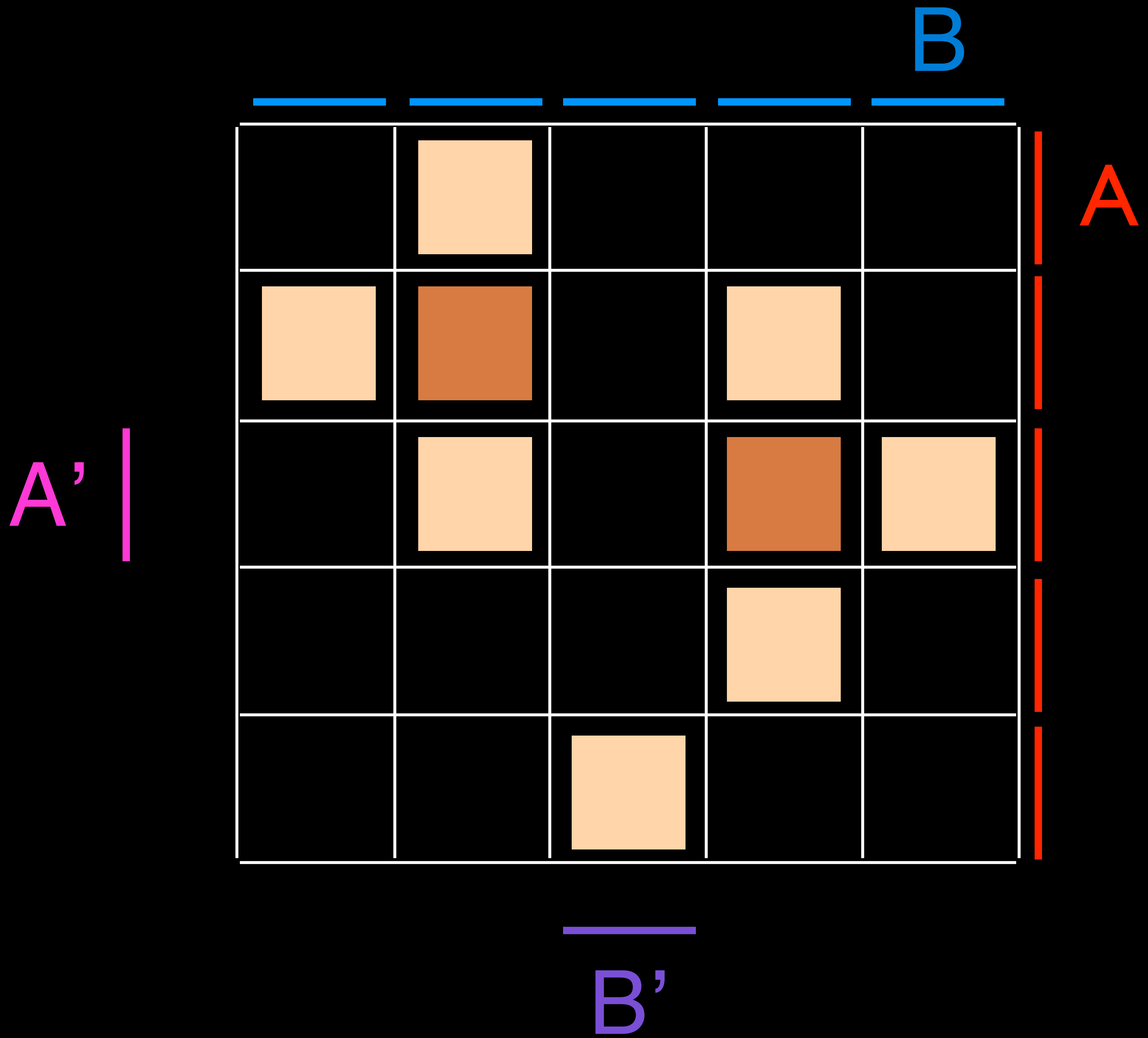


transition function

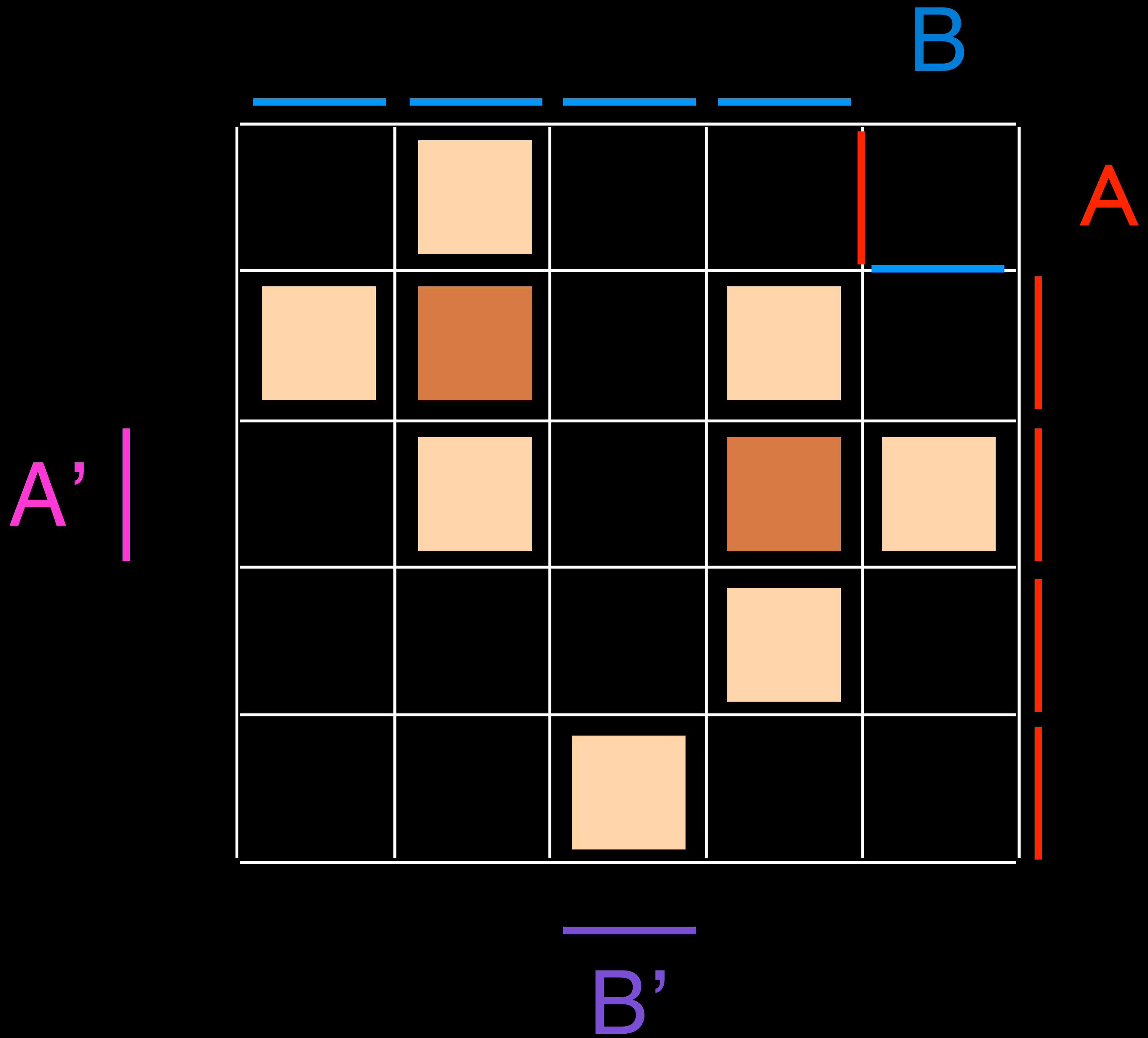


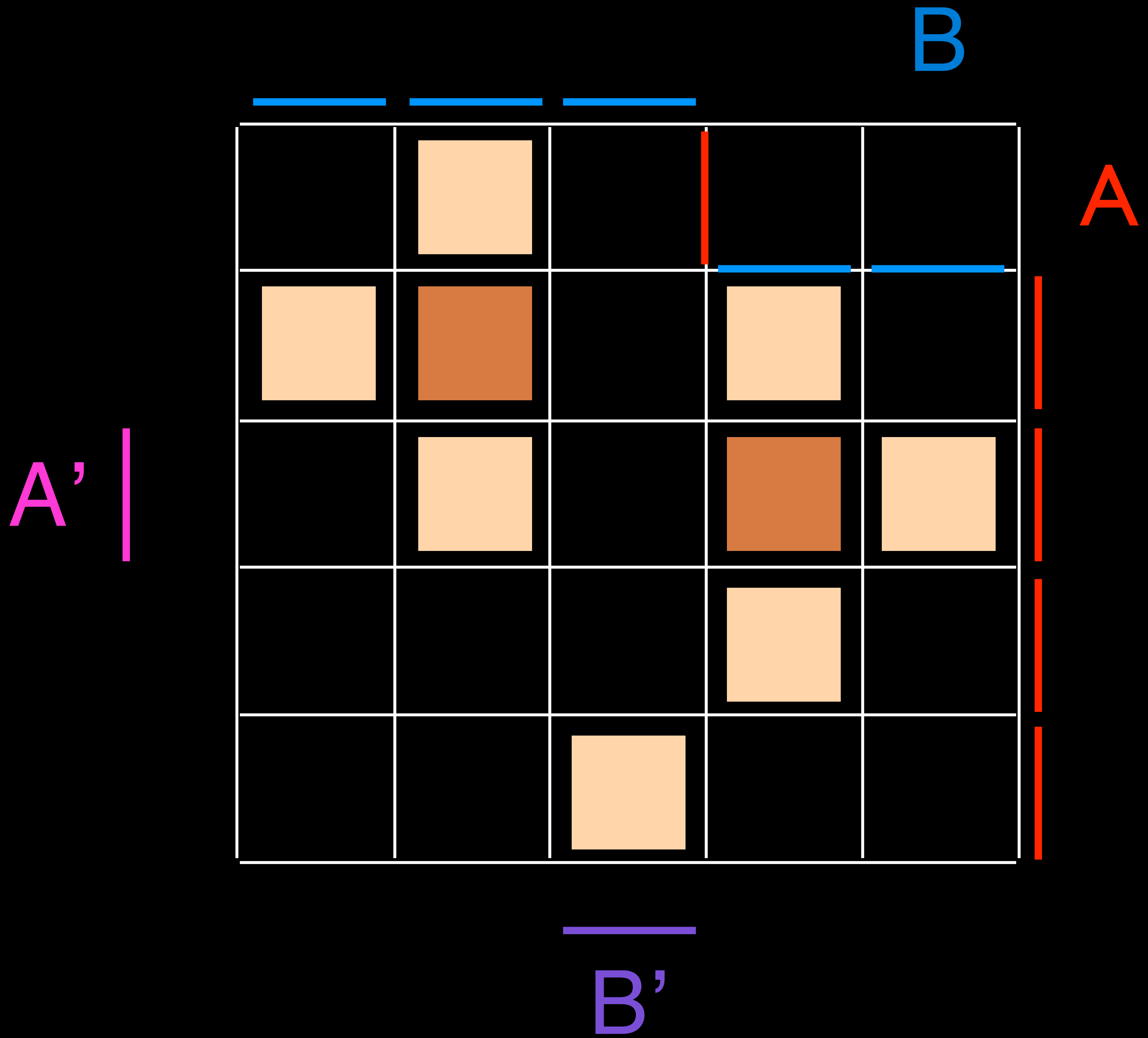


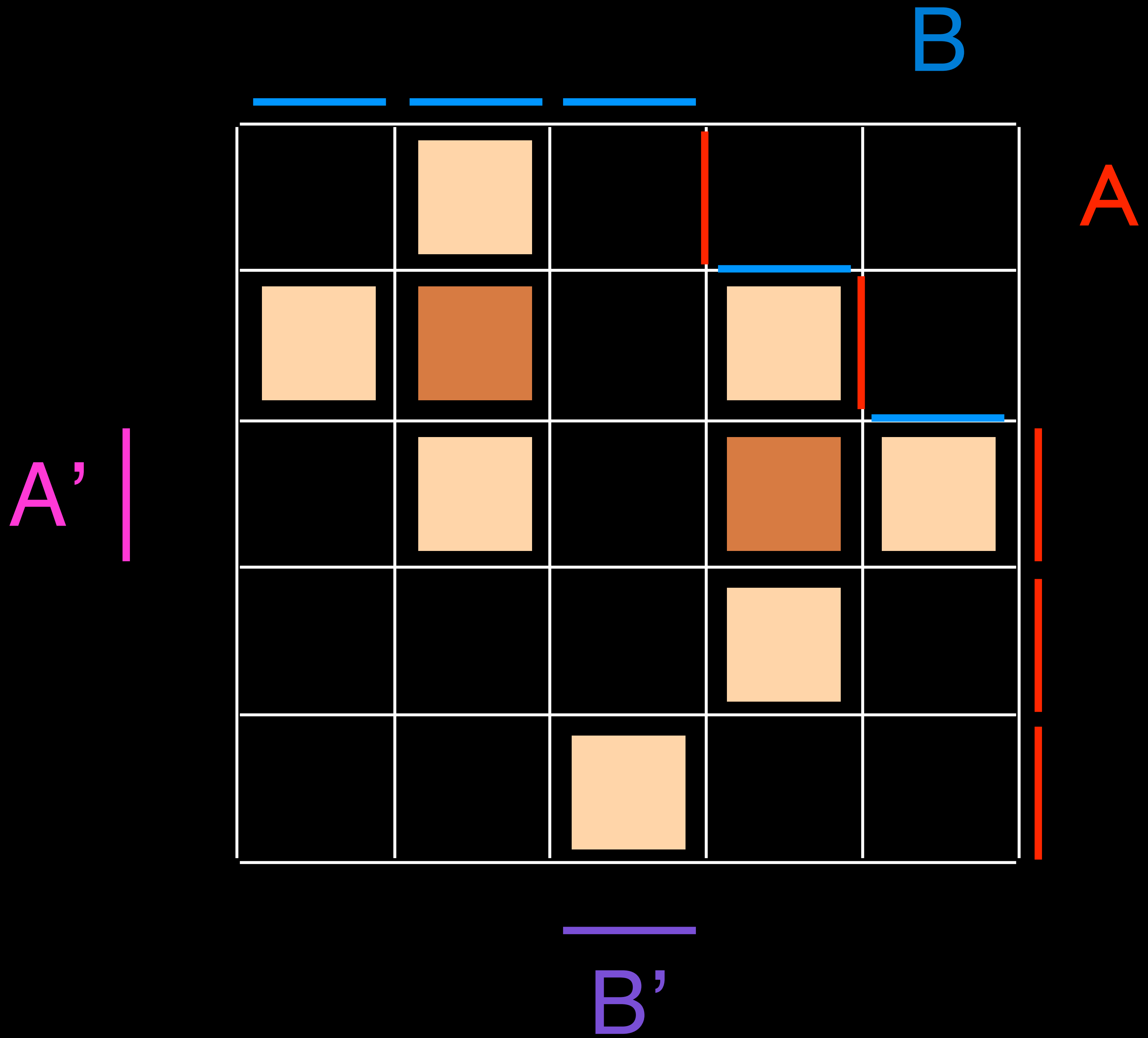
B^n
initial state
A'^n
A^n
final state
B'^n

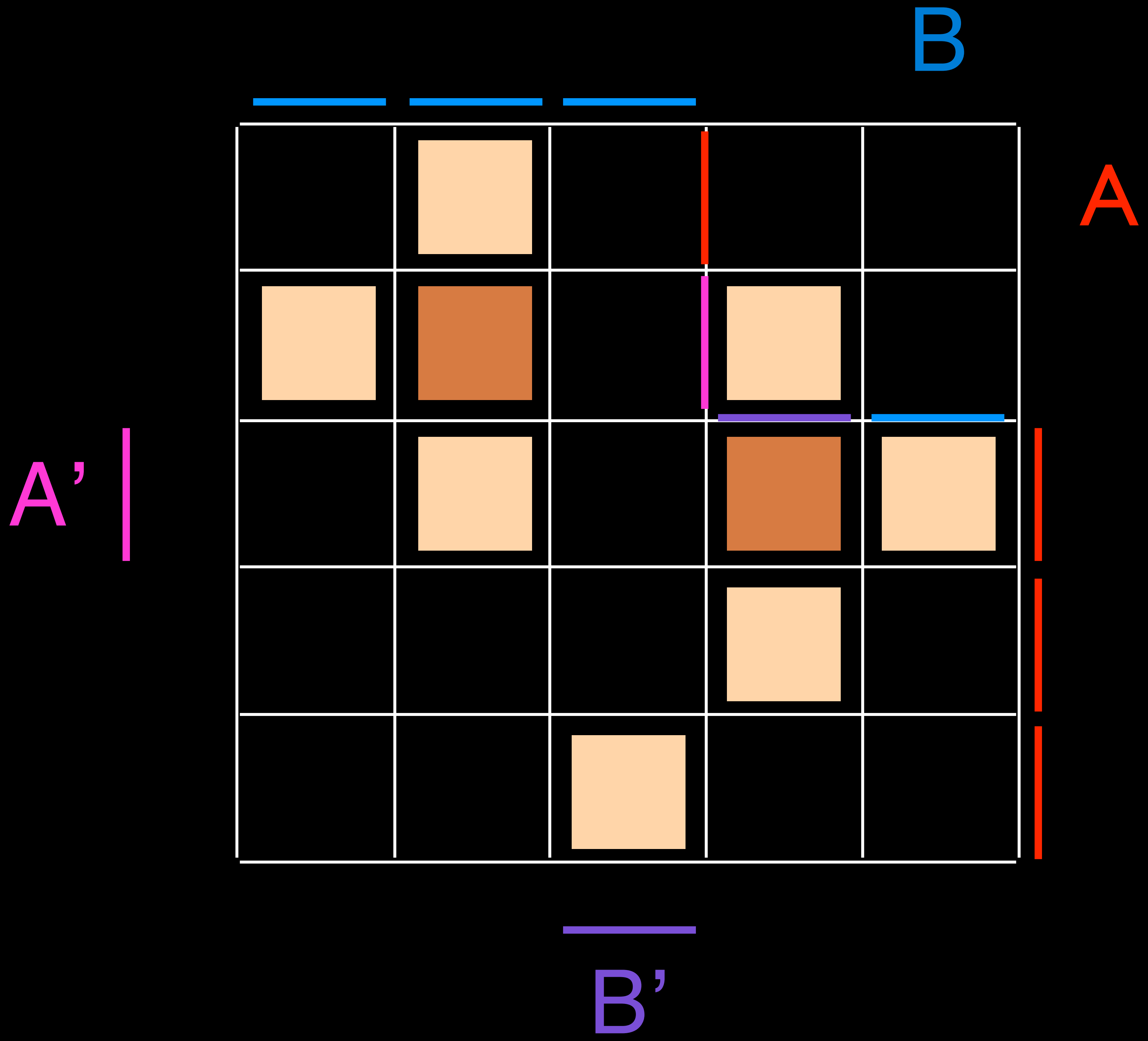


B
A
A’
B’


B
A
A'
B'


B
A
A'
B'


B
A
A'
B'


B
A
A'
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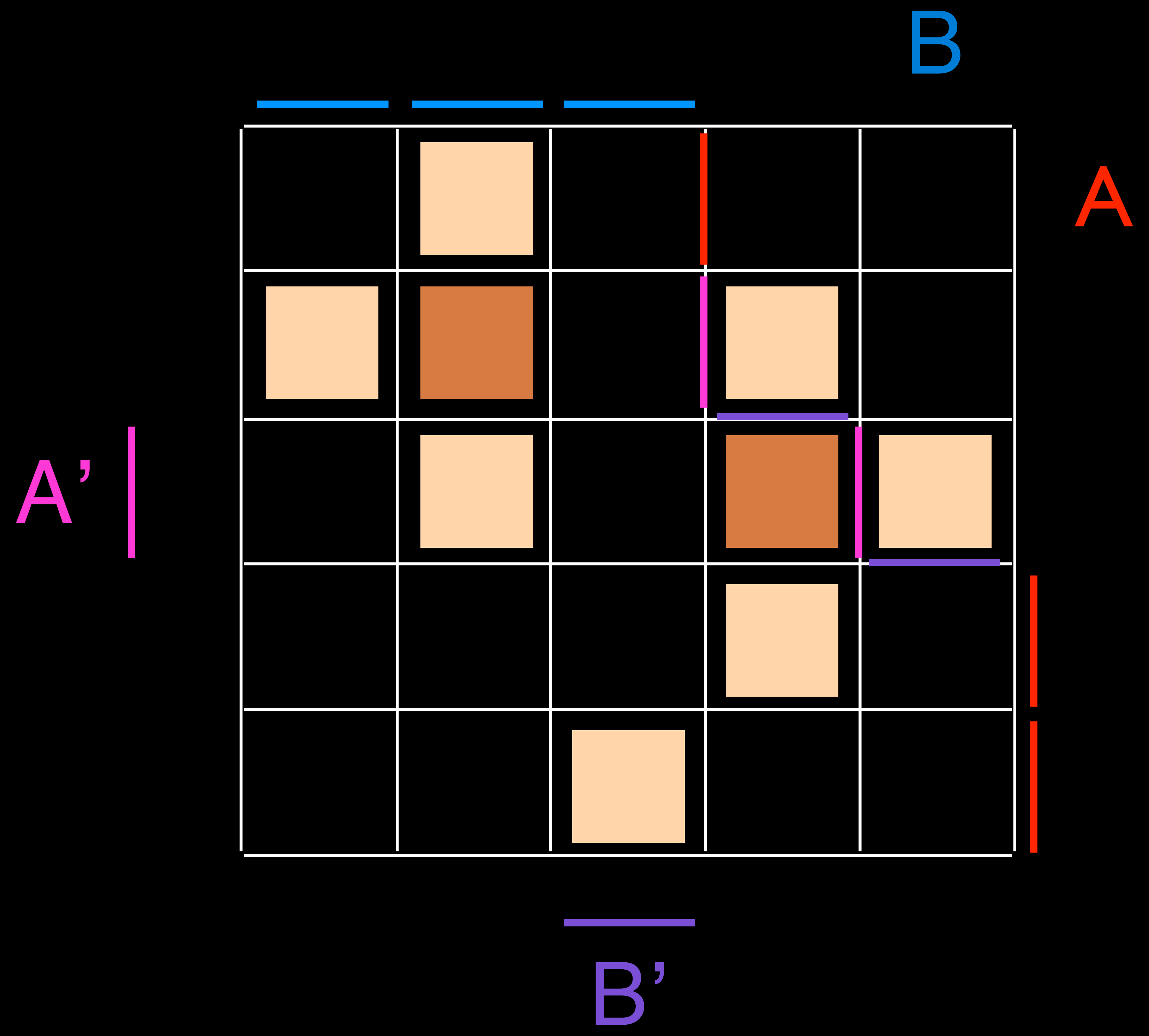
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A
A'
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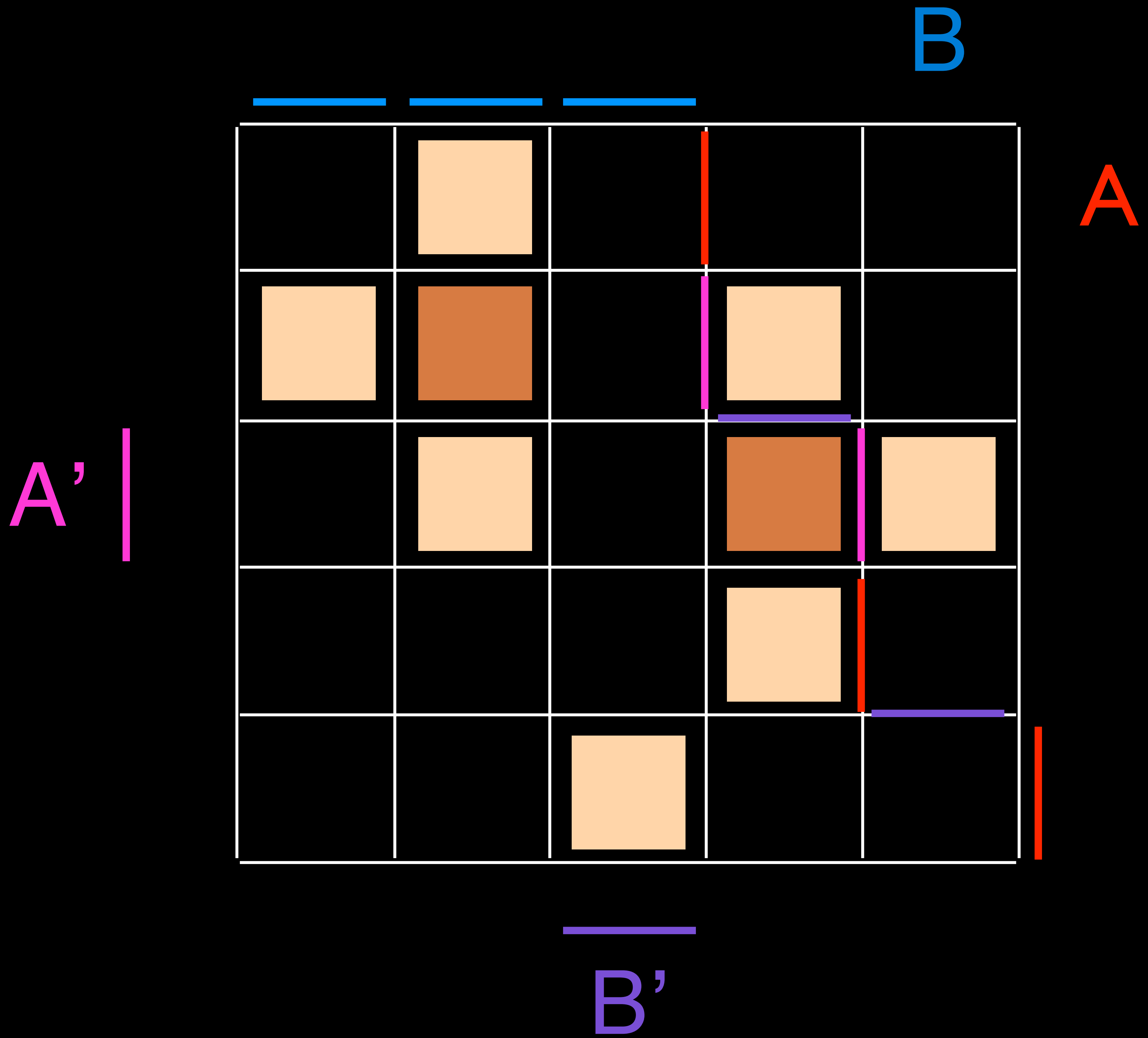
B
A
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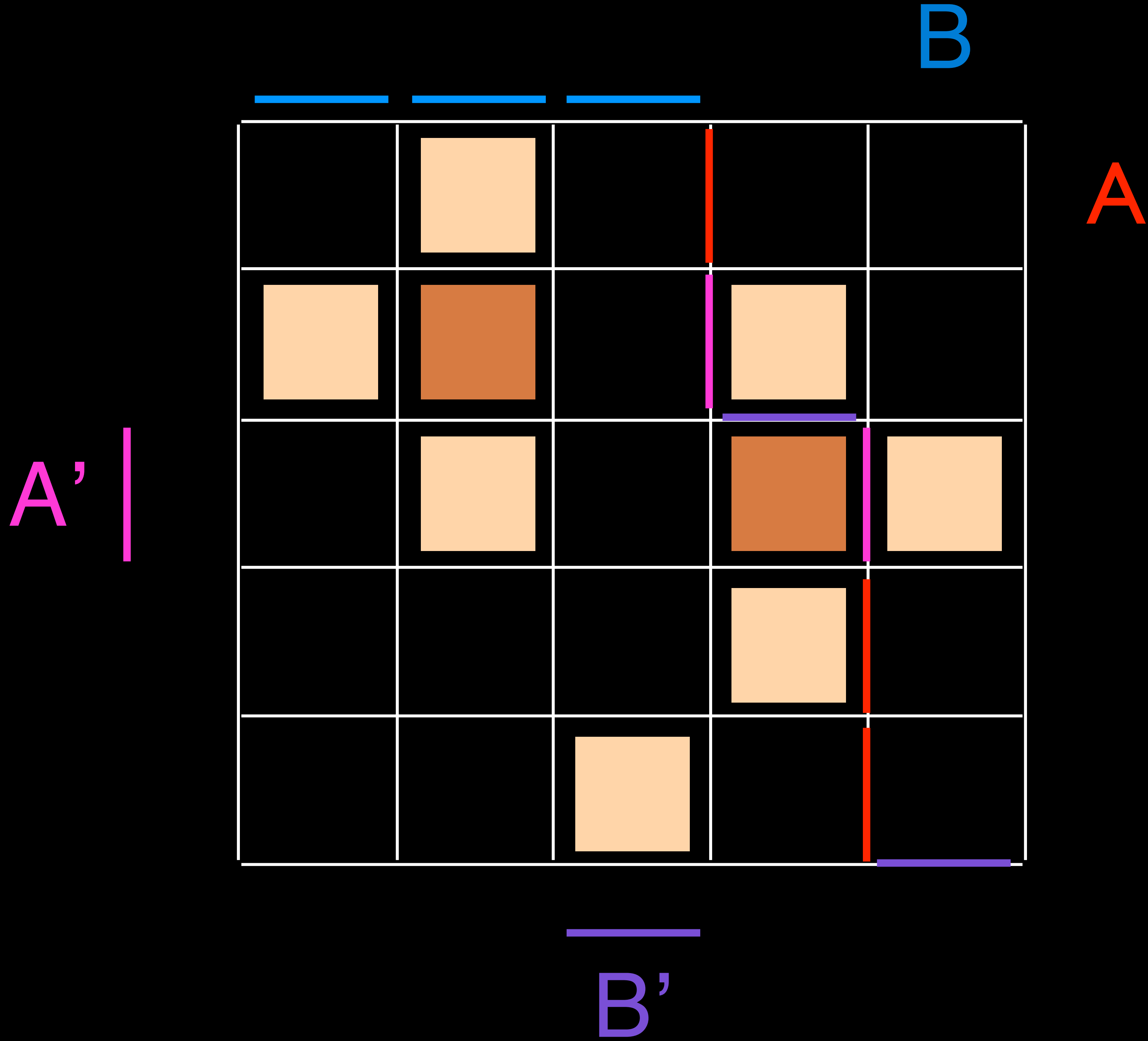
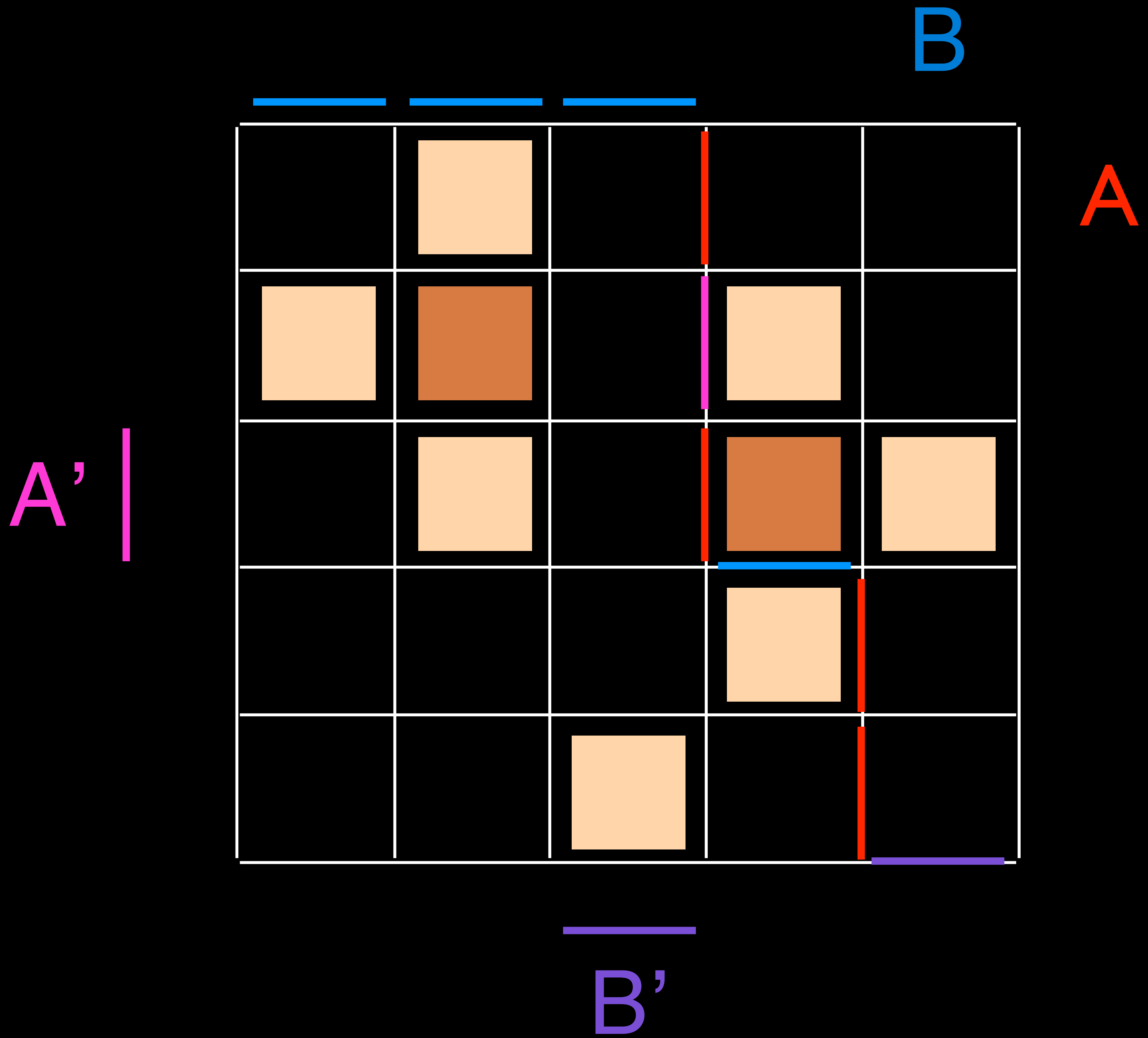
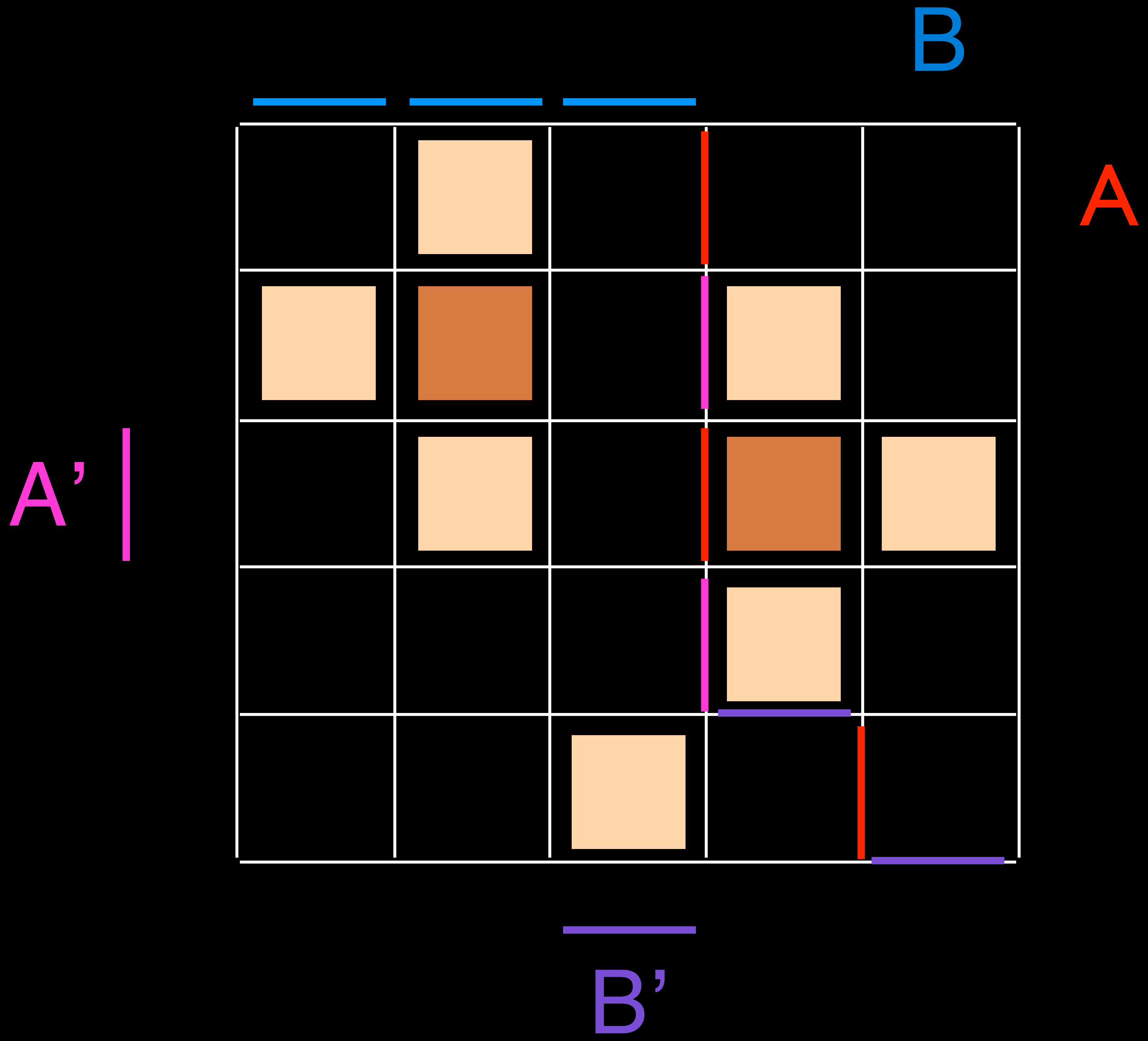
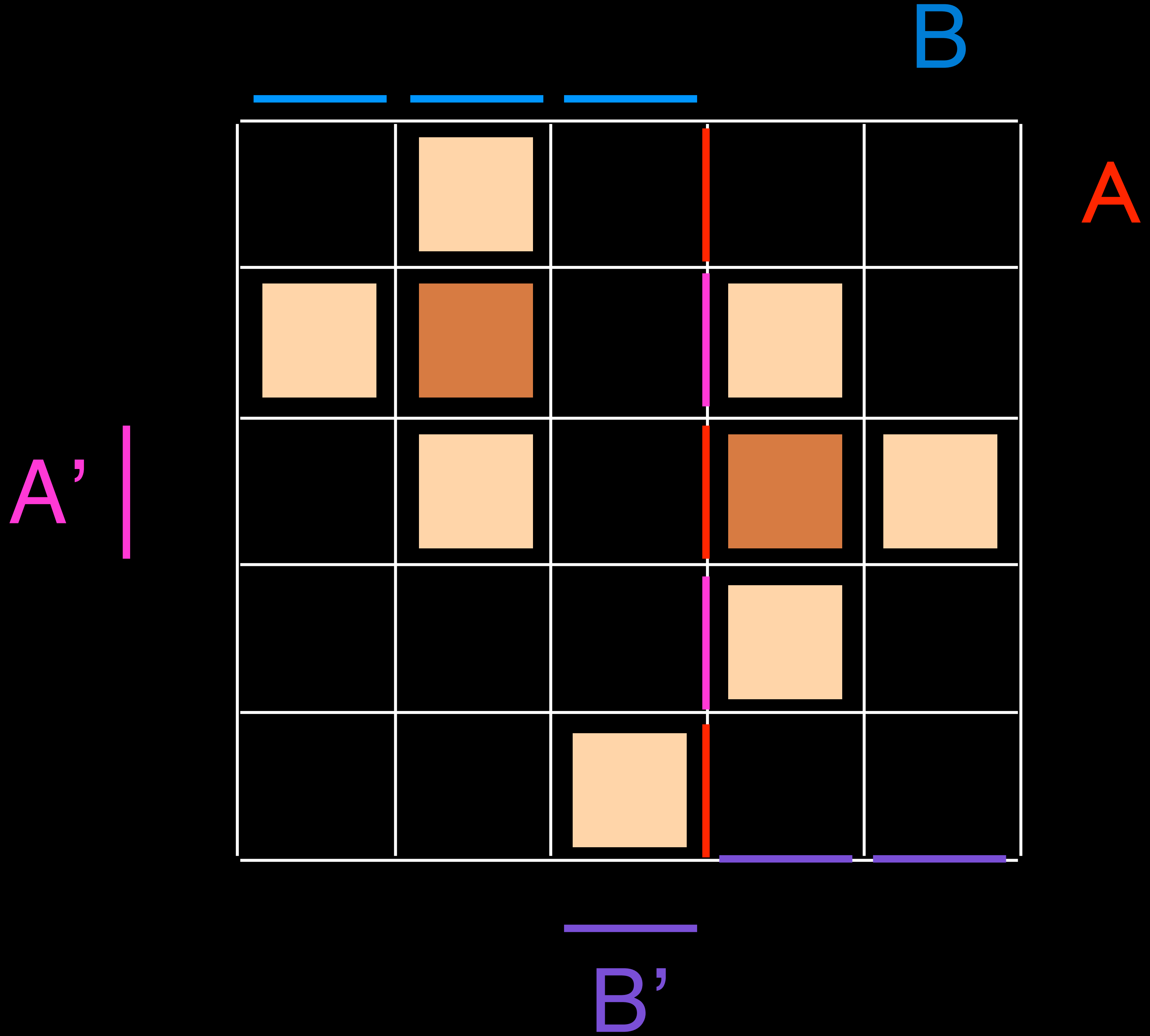
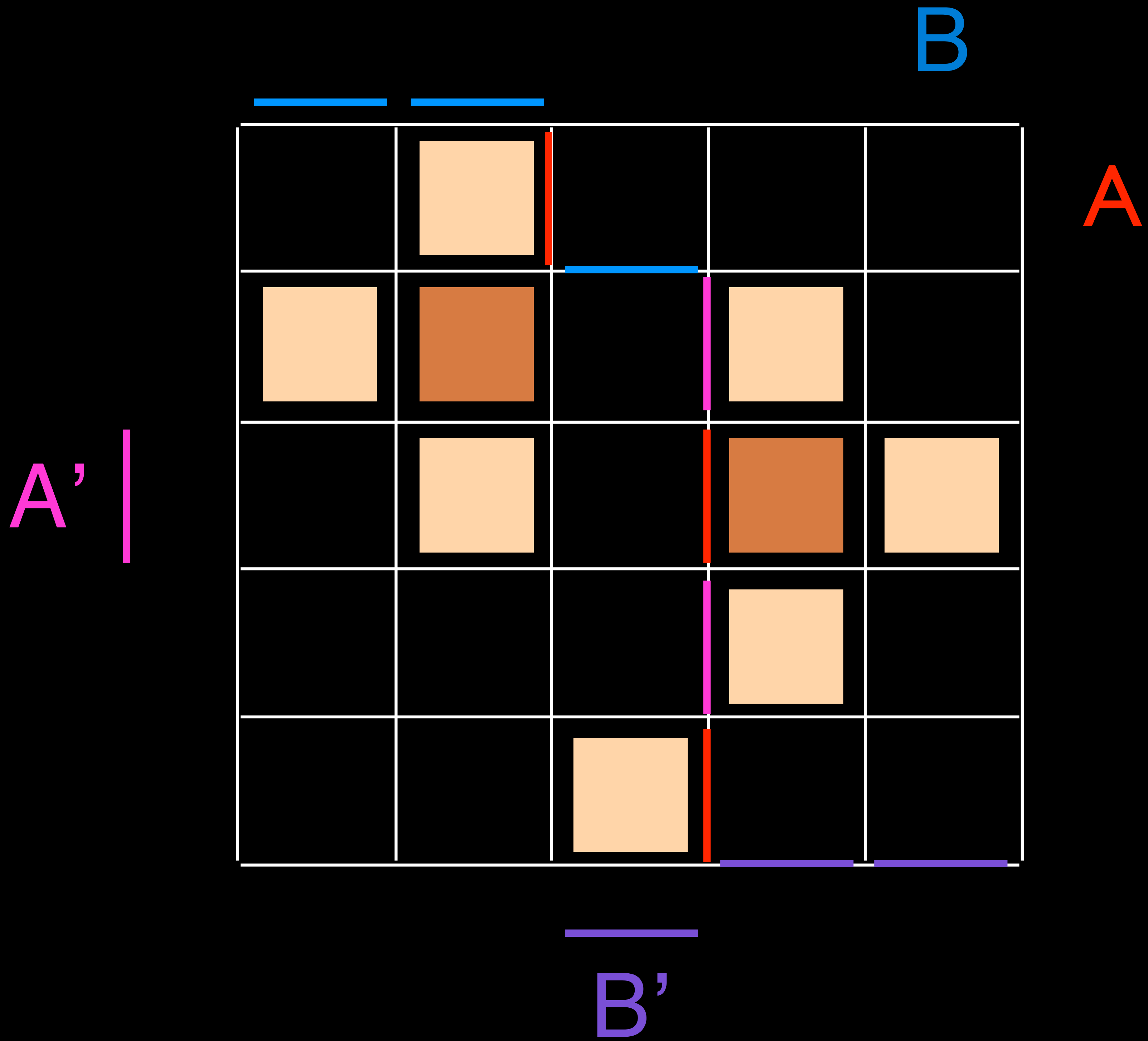


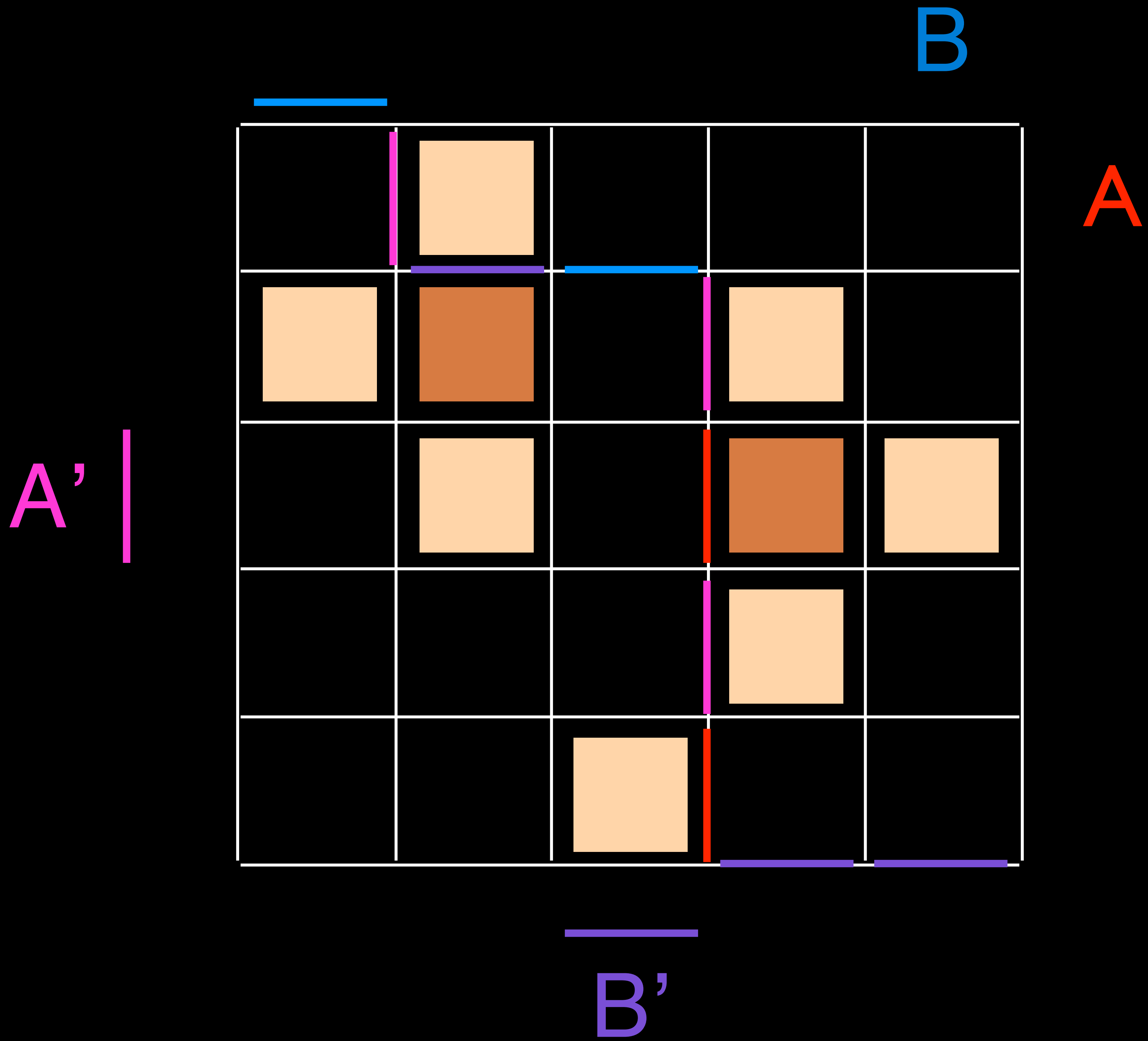
B
A
A’
B’

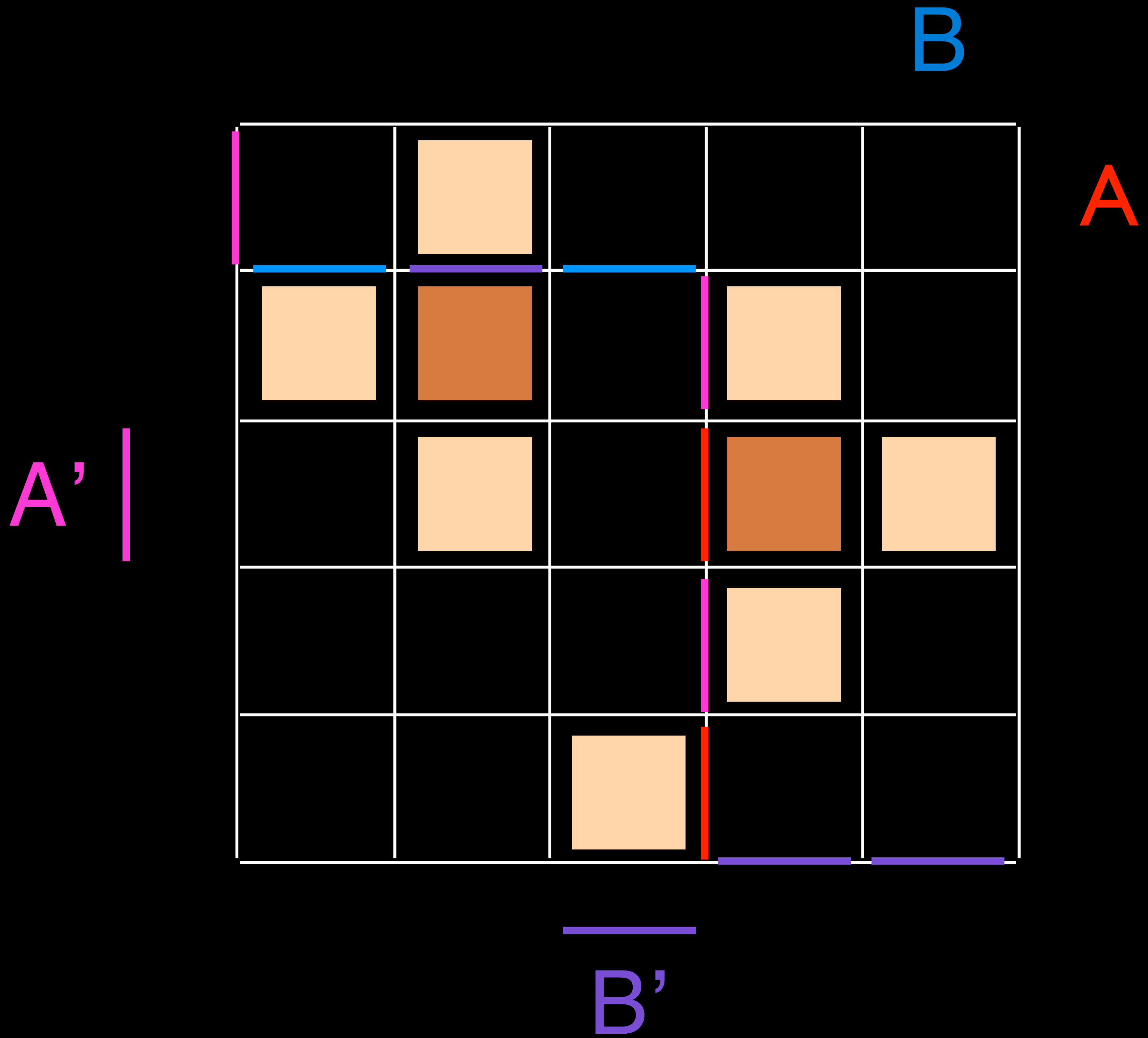

B
A
A'
B'

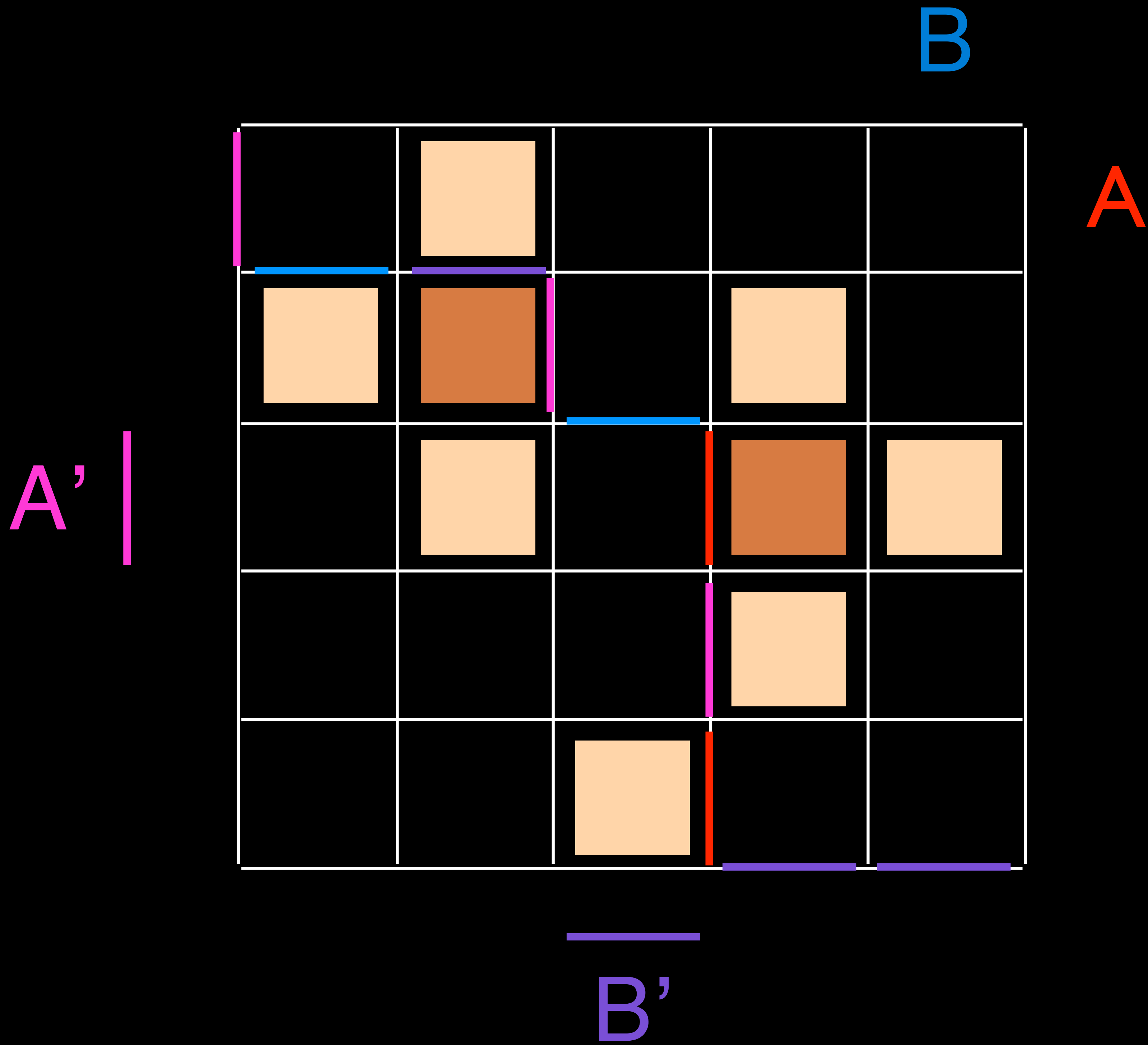

B
A
A'
B'

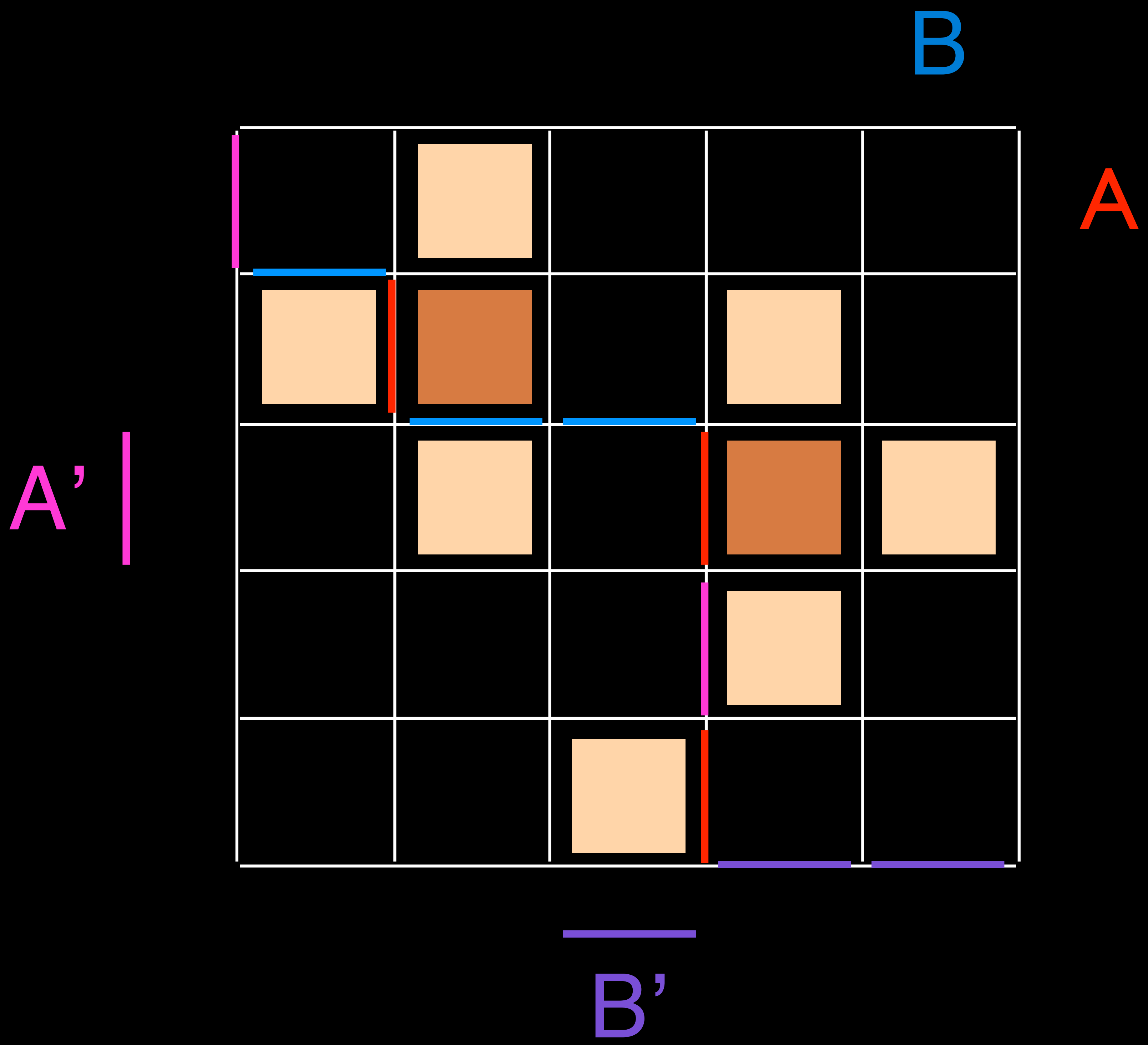

B
A
A'
B'

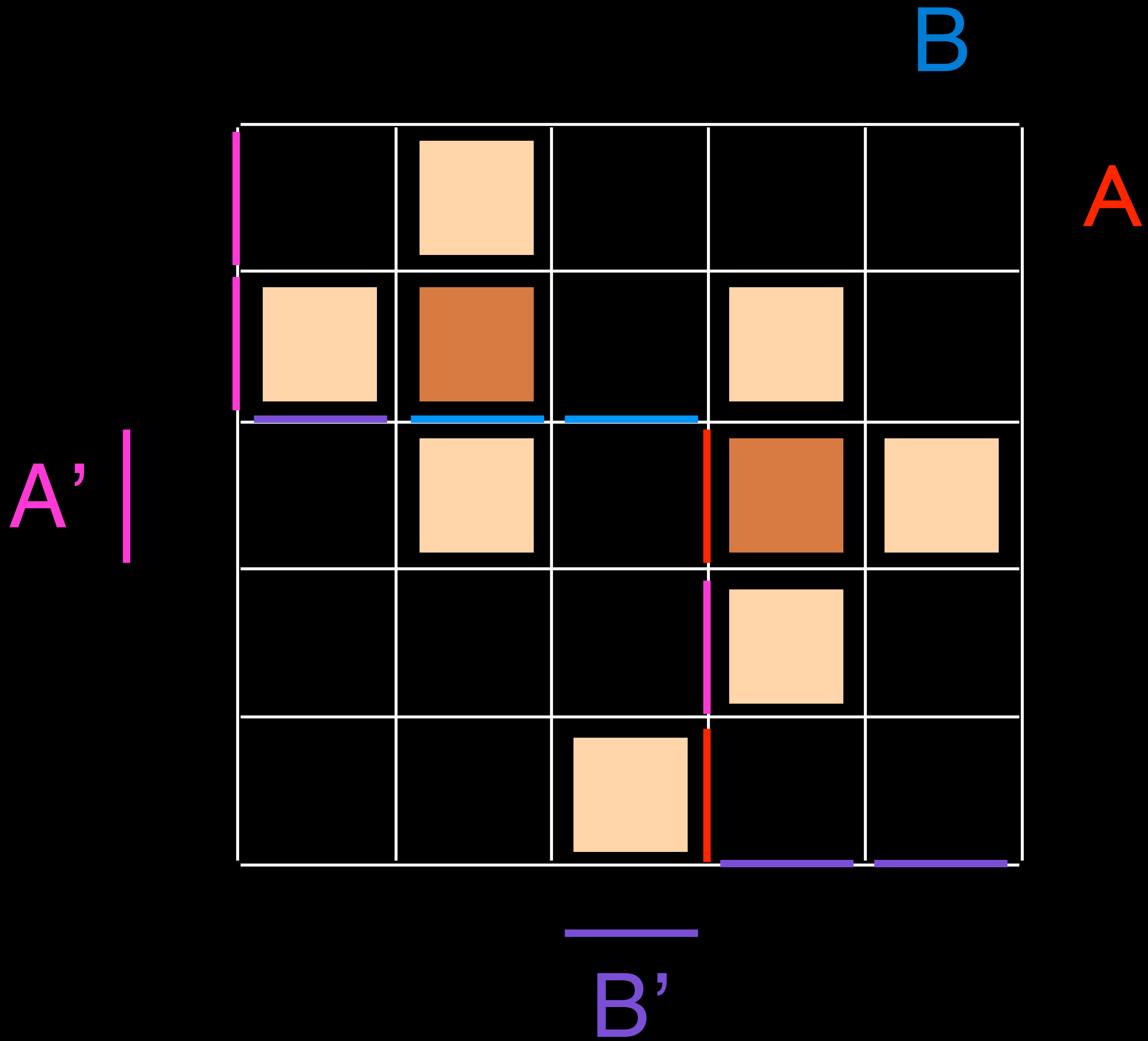

B
A
A’
B’


B
A
A’
B’

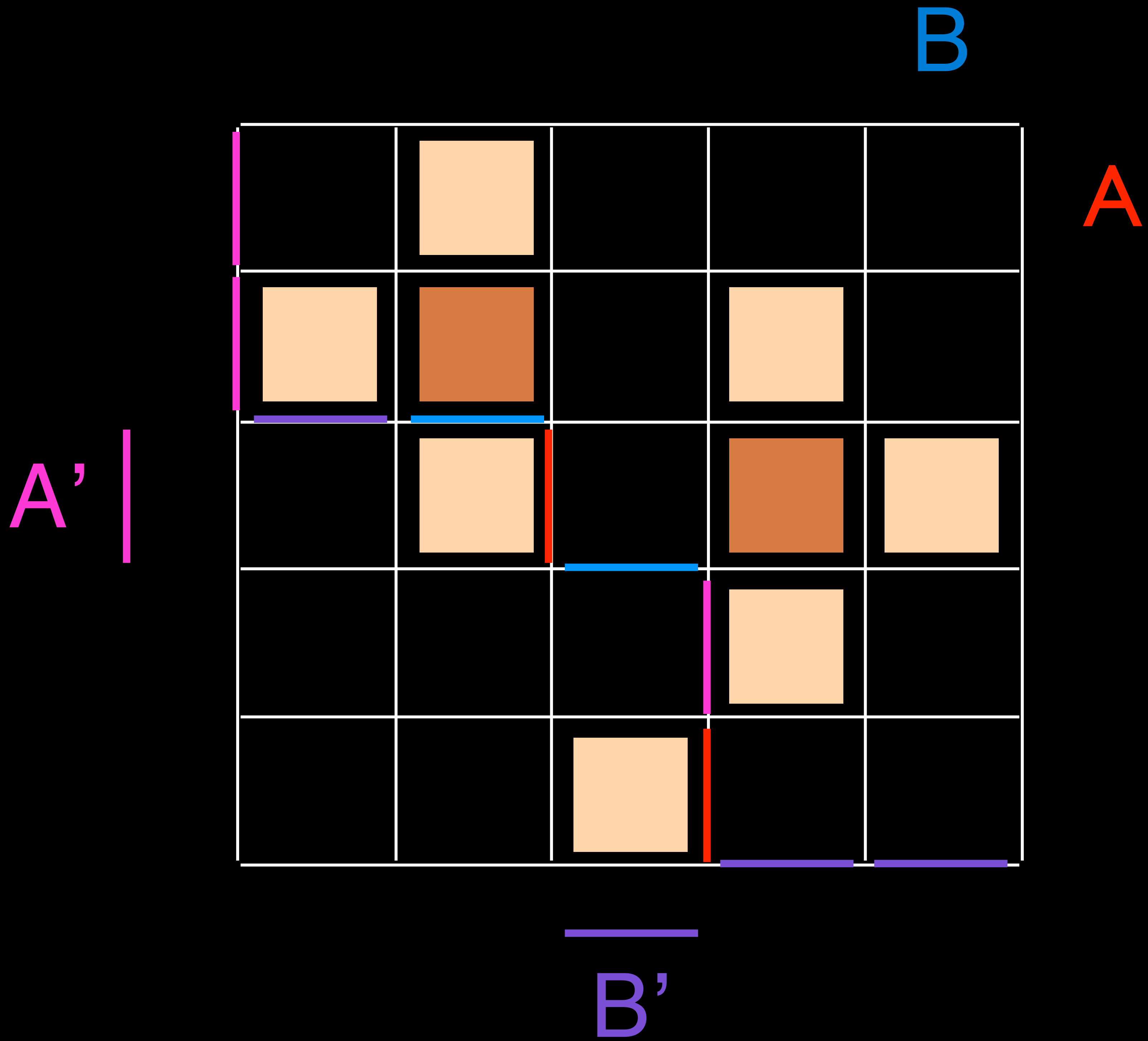

B
A
A'
B'

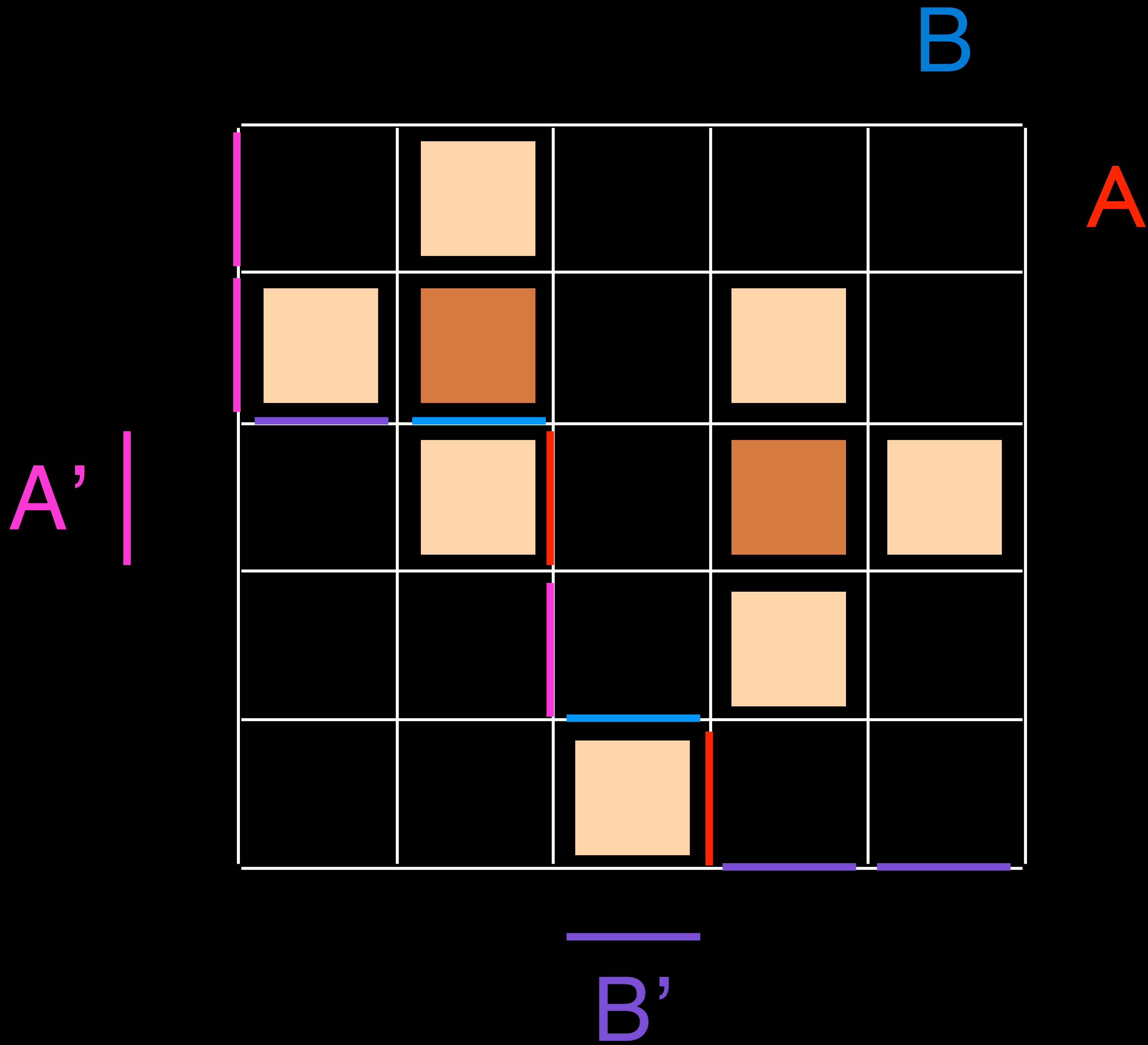

B
A
A’
B’


B
A
A'
B'

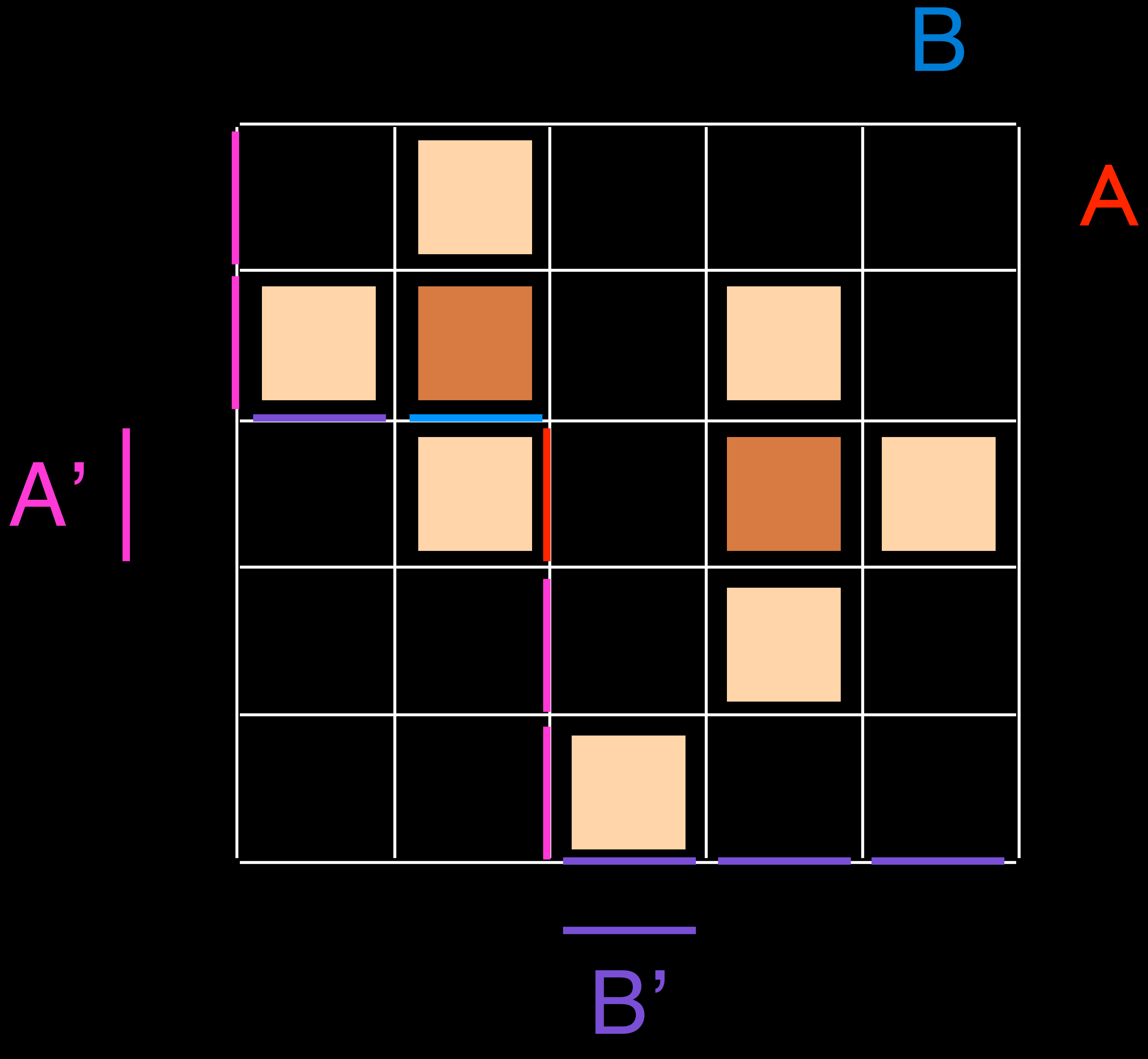


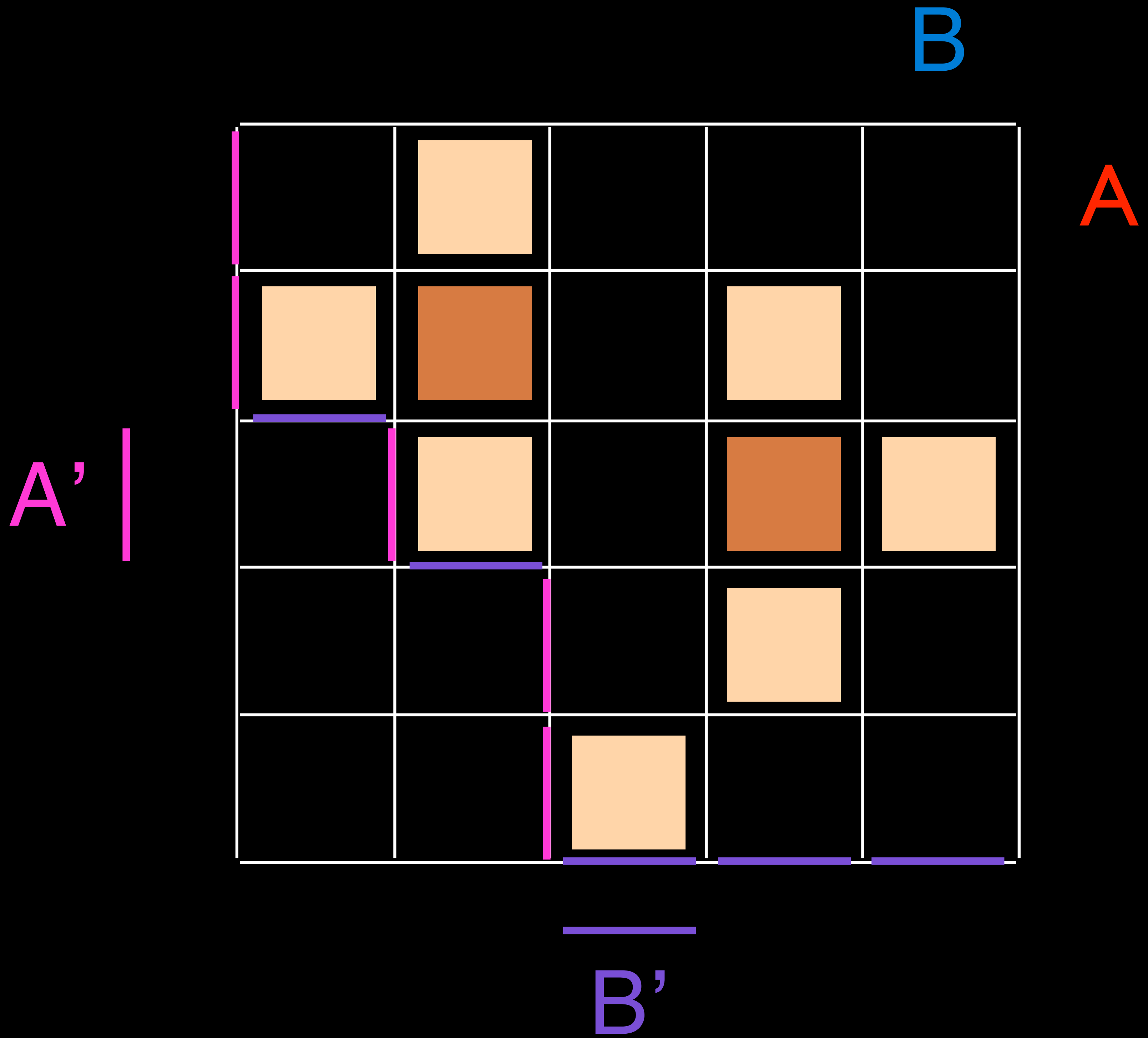
B
A
A'
B'


B
A
A'
B'

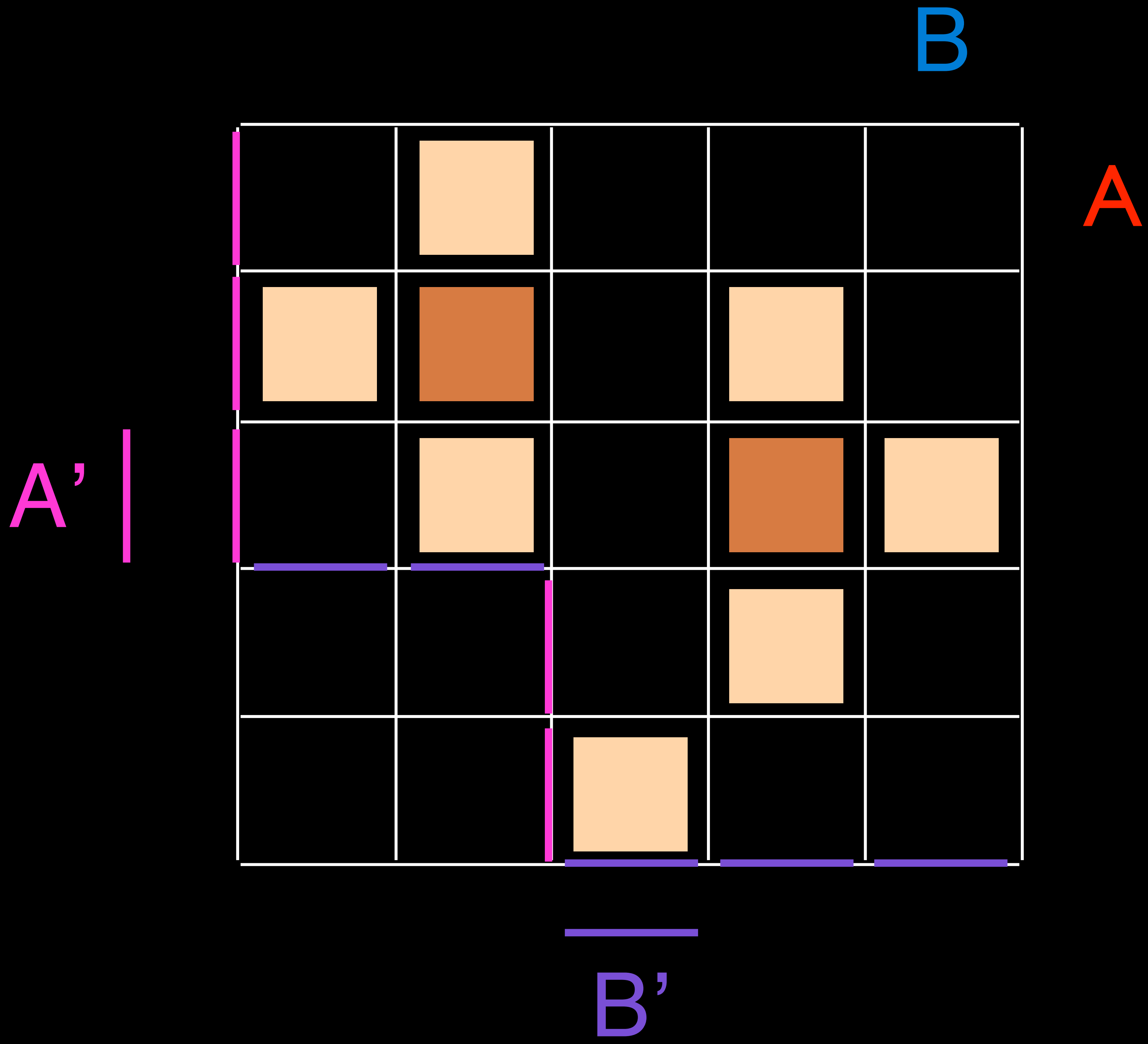


B
A
A'
B'

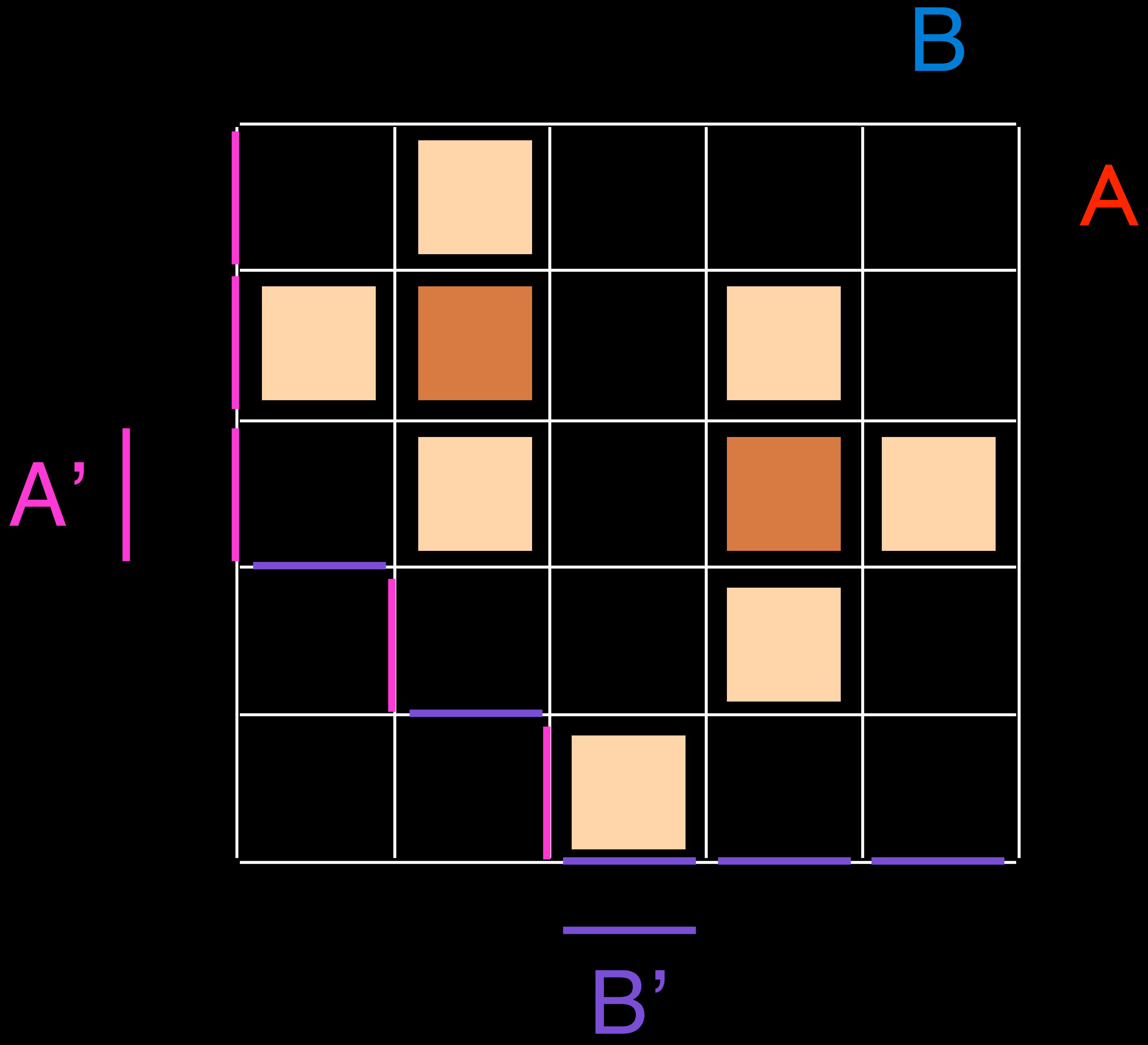

B
A
A'
B'

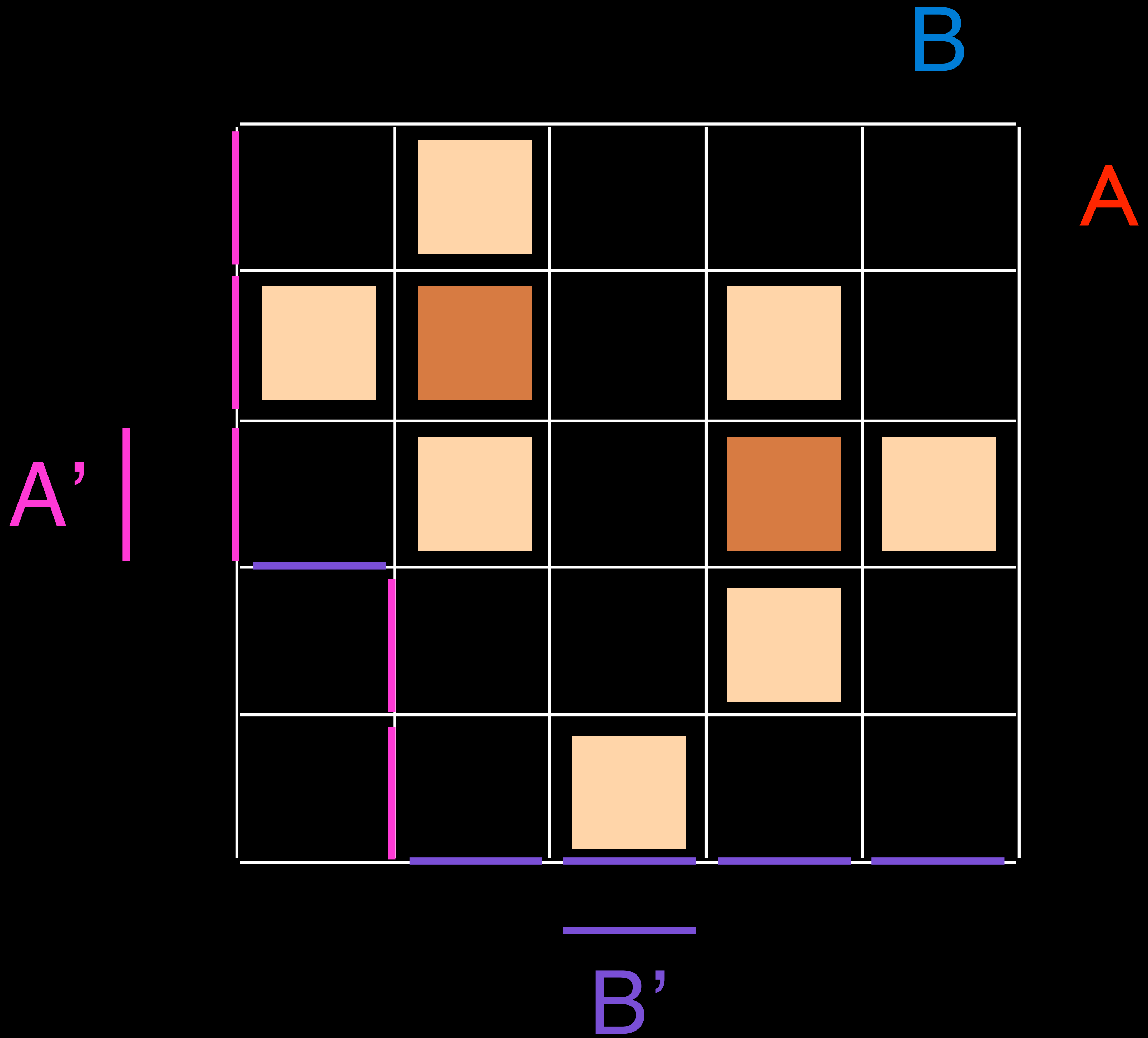


B
A
A'
B'

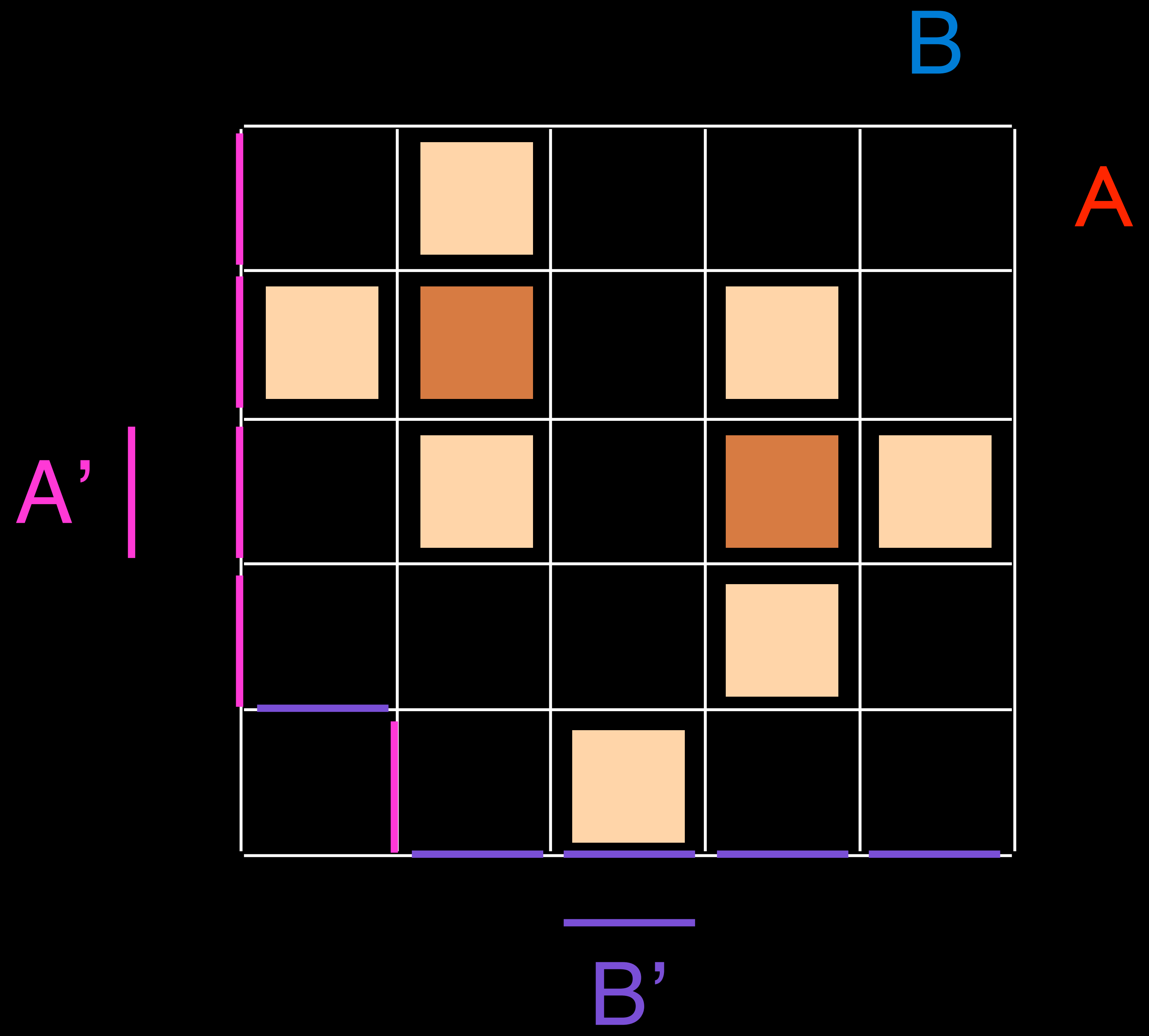


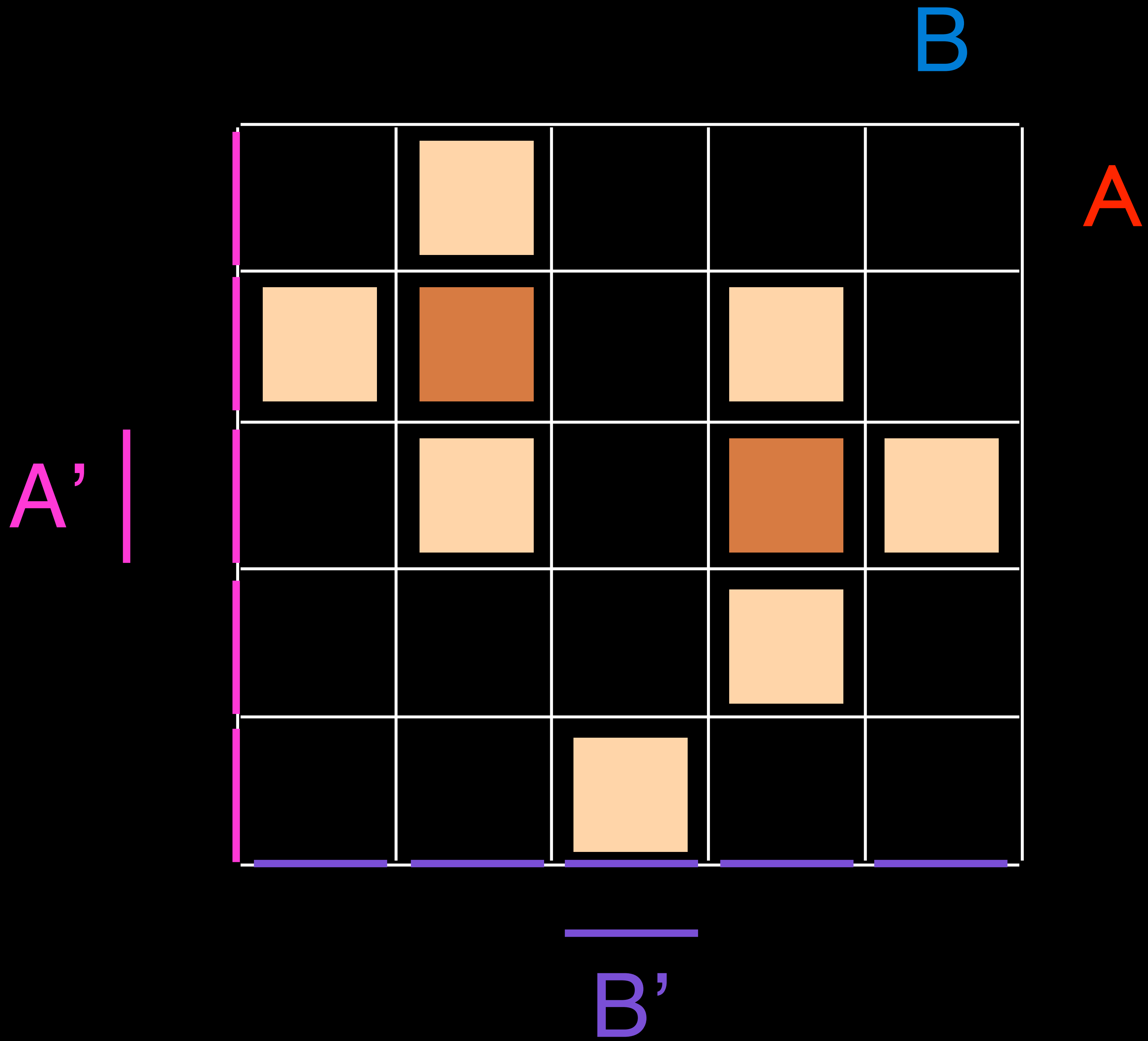
B
A
A'
B'

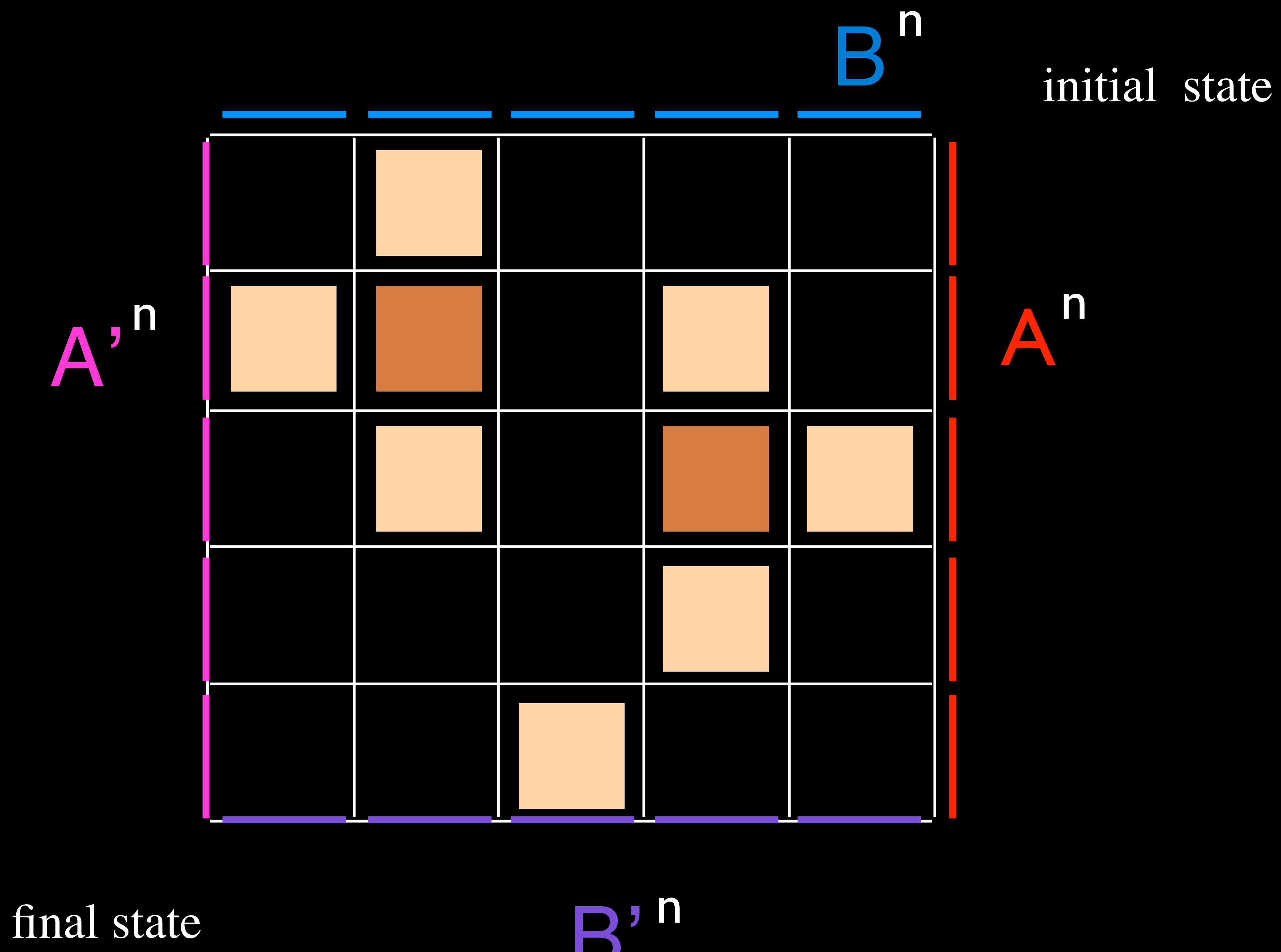

B
A
A'
B'



B
A
A'
B'


B
A
A’
B’


B
A
A'
B'



B ^{n}
initial state
A' ^{n}
A ^{n}
final state
B' ^{n}

The RSK planar automaton

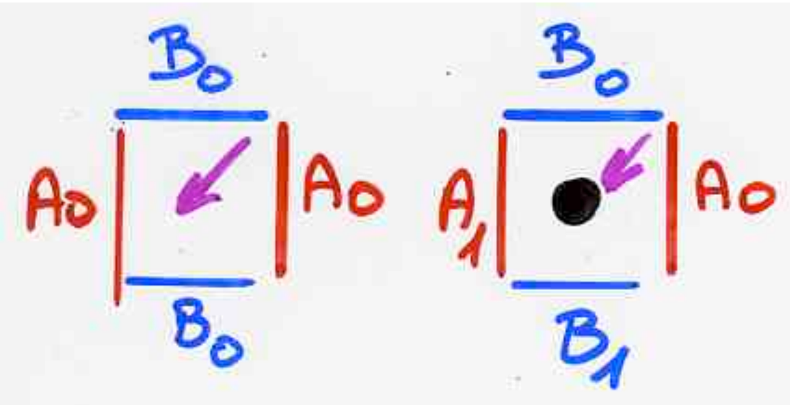
The "RSK planar automaton"

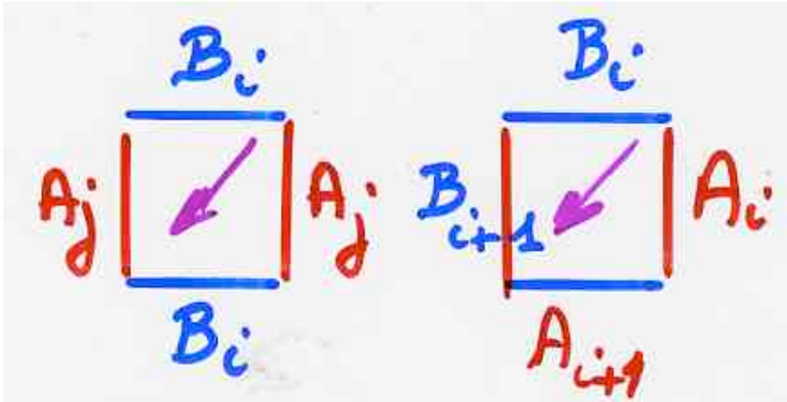
$\mathcal{B} = \{B_0, B_1, \ldots B_k\}$

$\mathcal{A} = \{A_0 \; A_1 \; A_k\}$

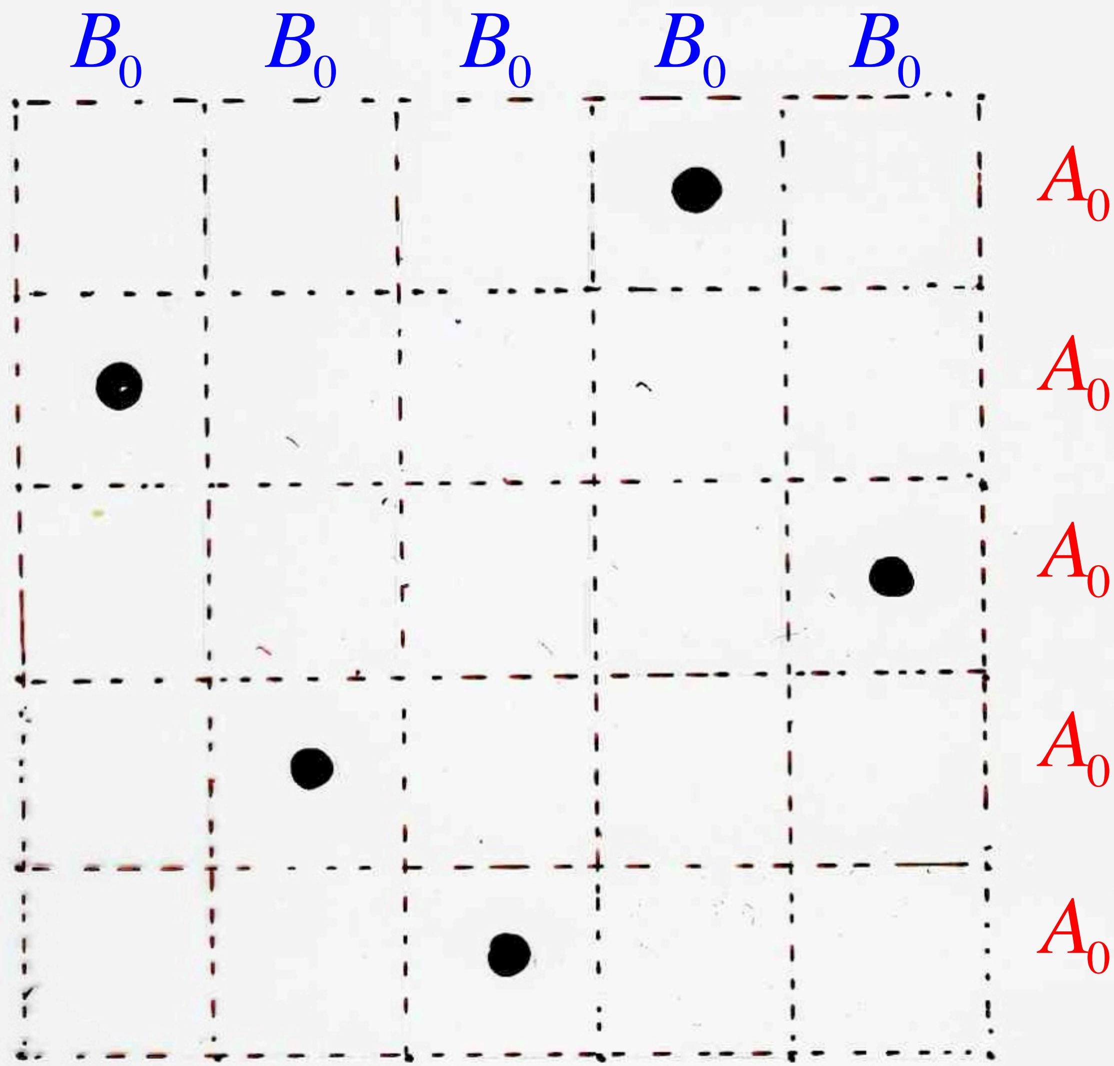
$w \in \{B_0, A_0\}^*$

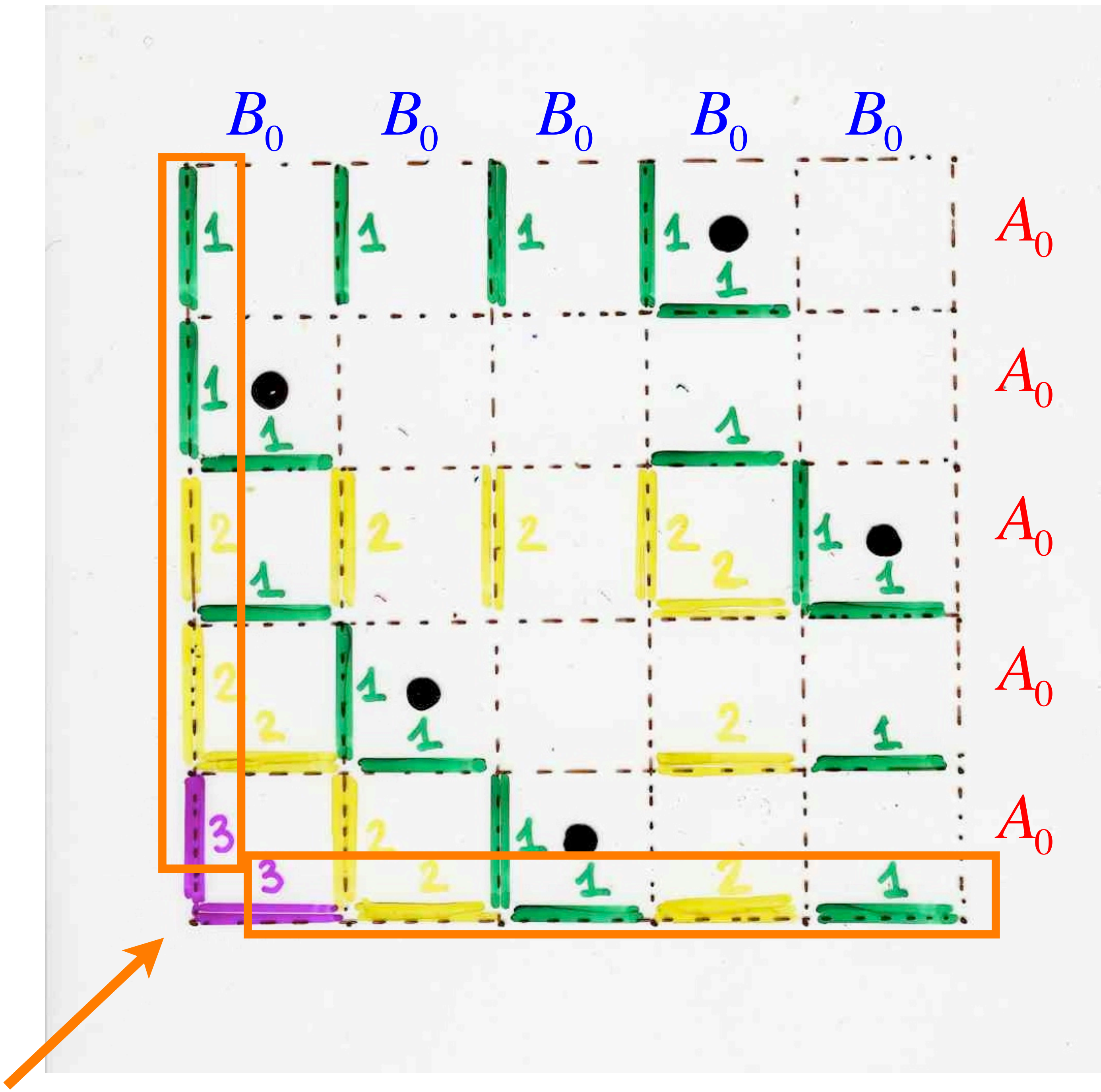
$S = \{\square, \boxdot\}$



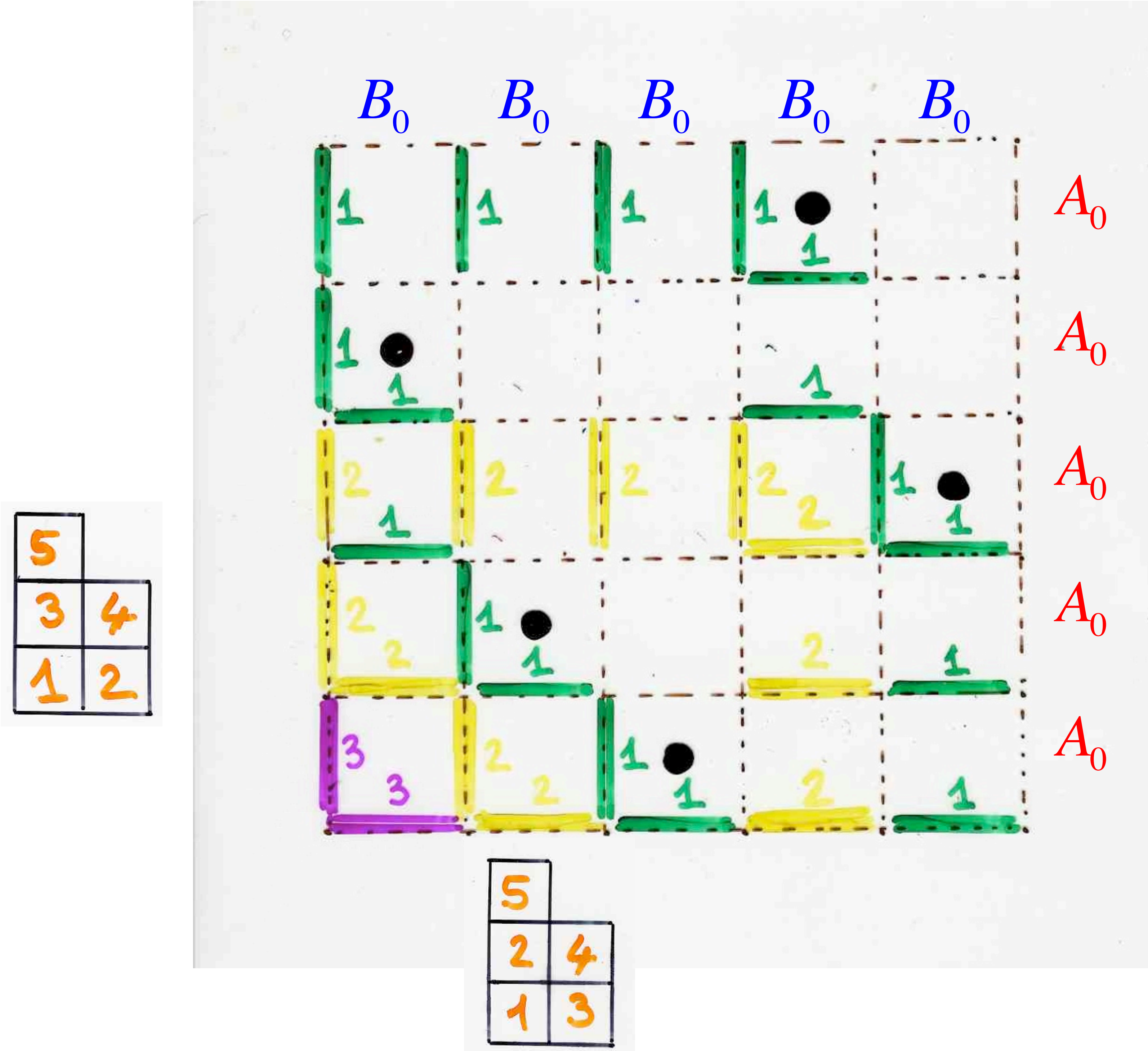


$i \neq j$


B_0
B_0
B_0
B_0
B_0
A_0
A_0
A_0
A_0
A_0

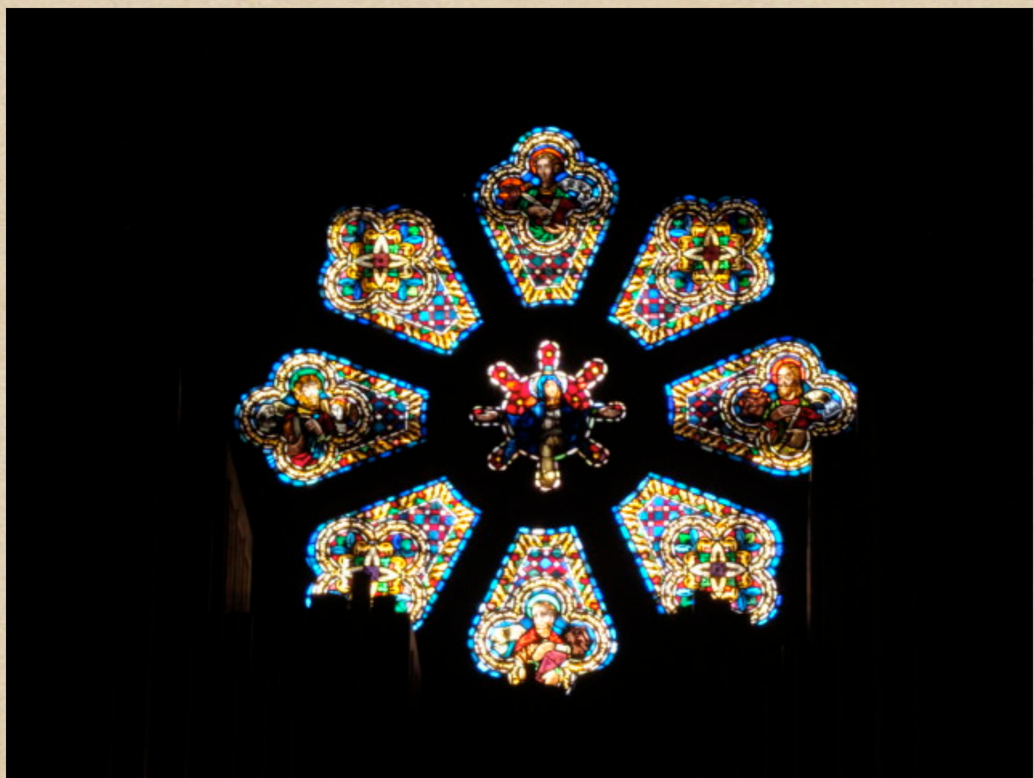


B_0 B_0 B_0 B_0 B_0
A_0
A_0
A_0
A_0
A_0

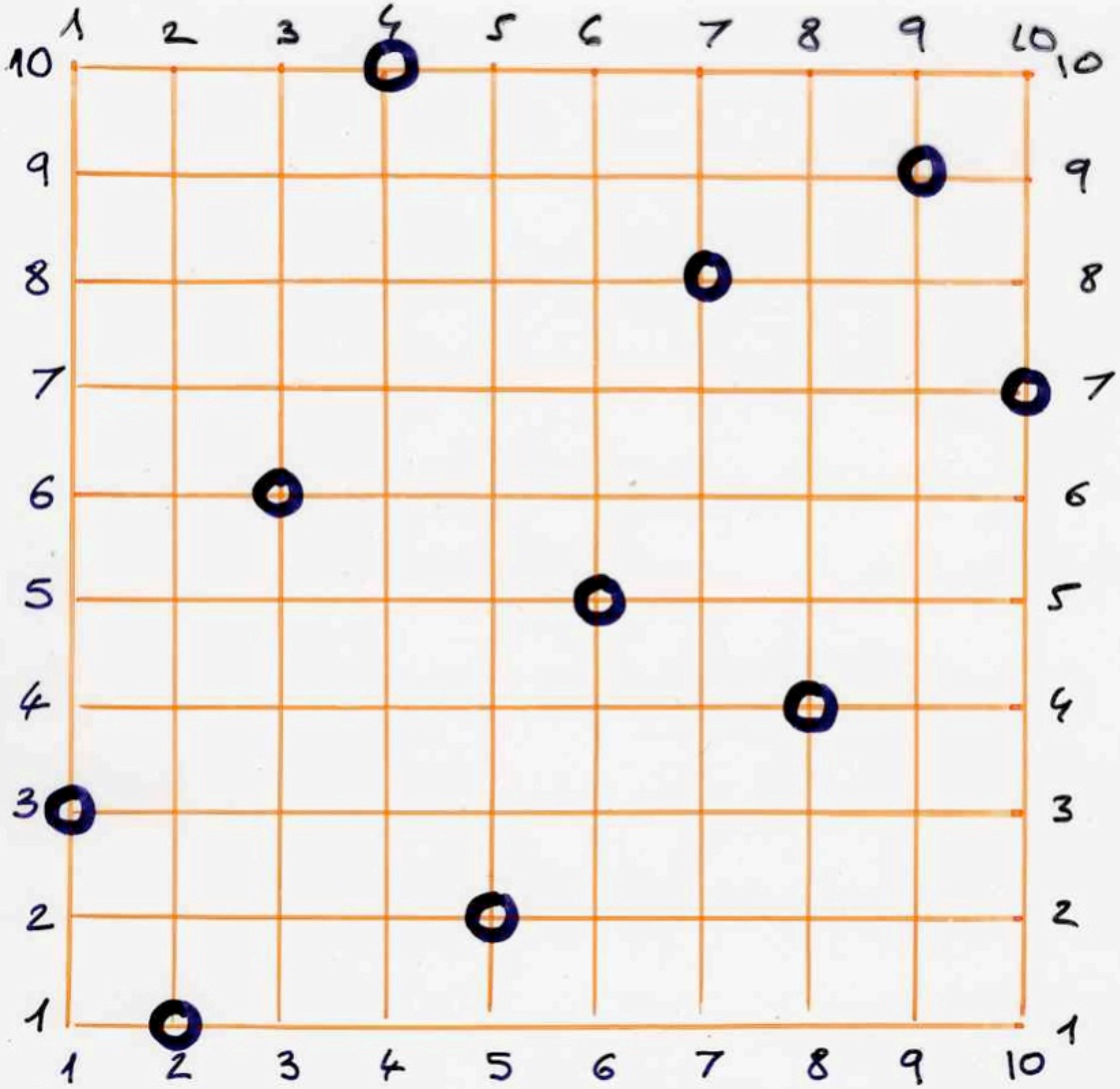


B_0 B_0 B_0 B_0 B_0
A_0
A_0
A_0
A_0
A_0

A geometric version of RSK with "light" and "shadow lines"

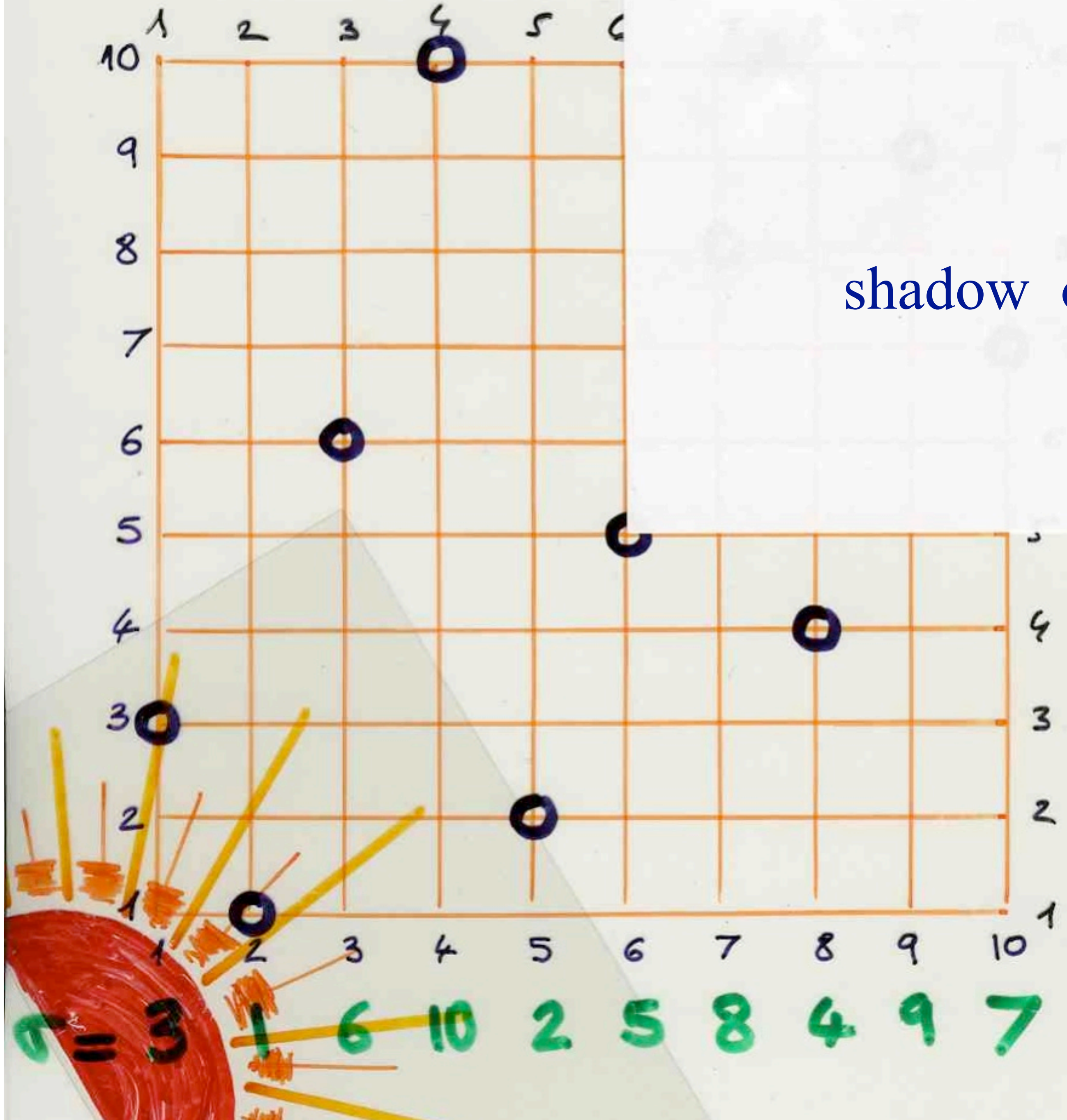


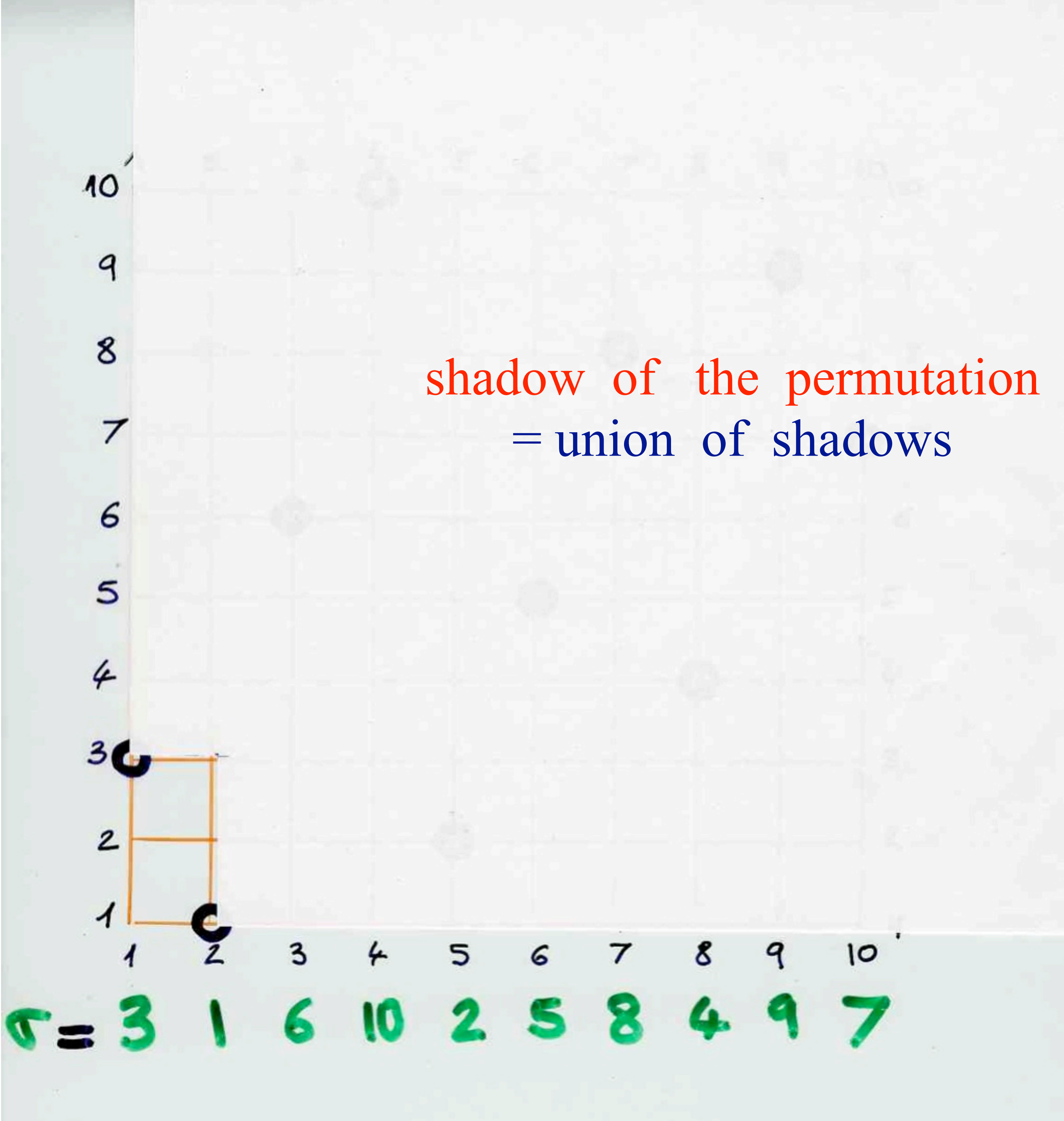
X.G.V., 1976



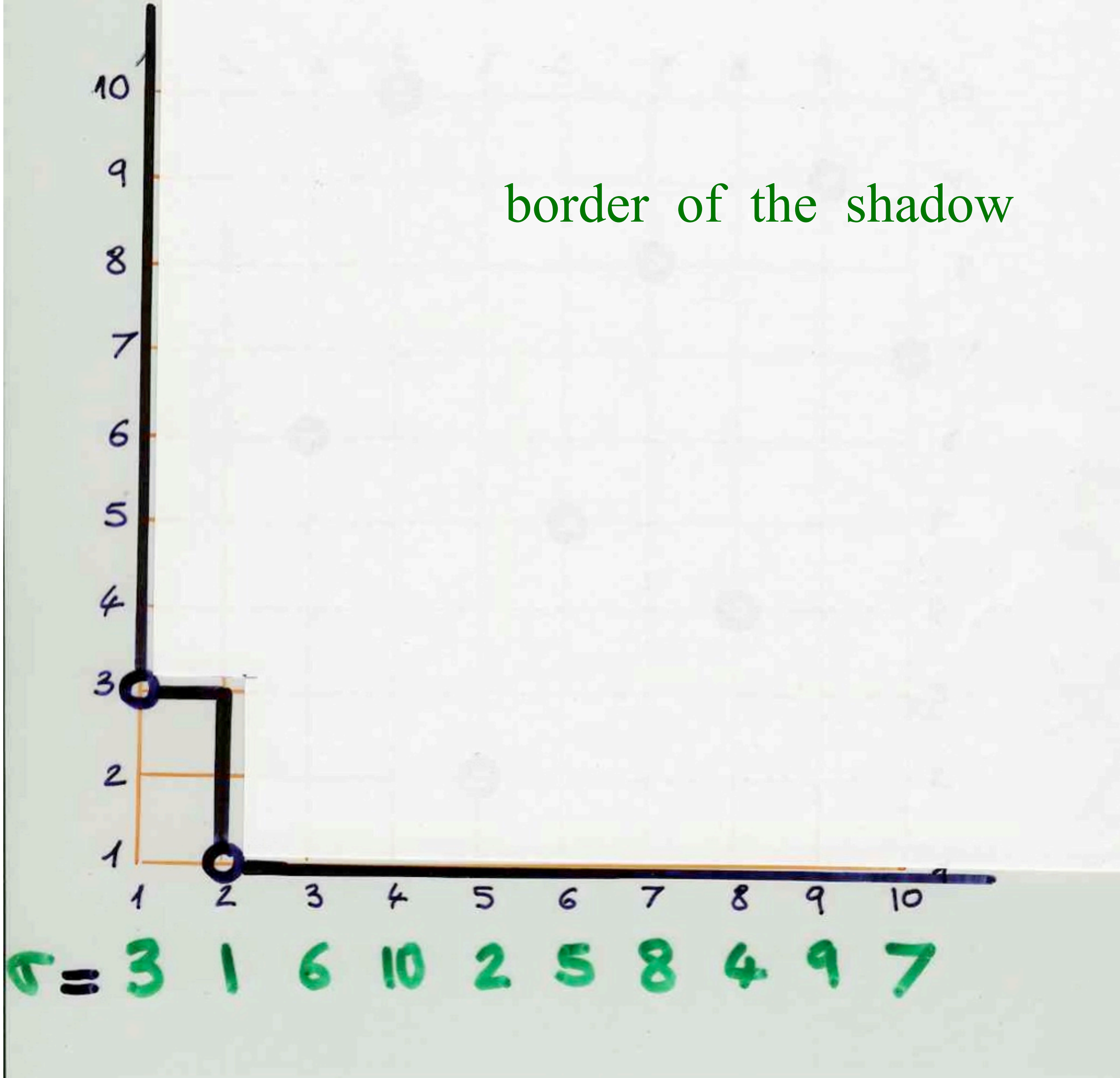
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

shadow of a point

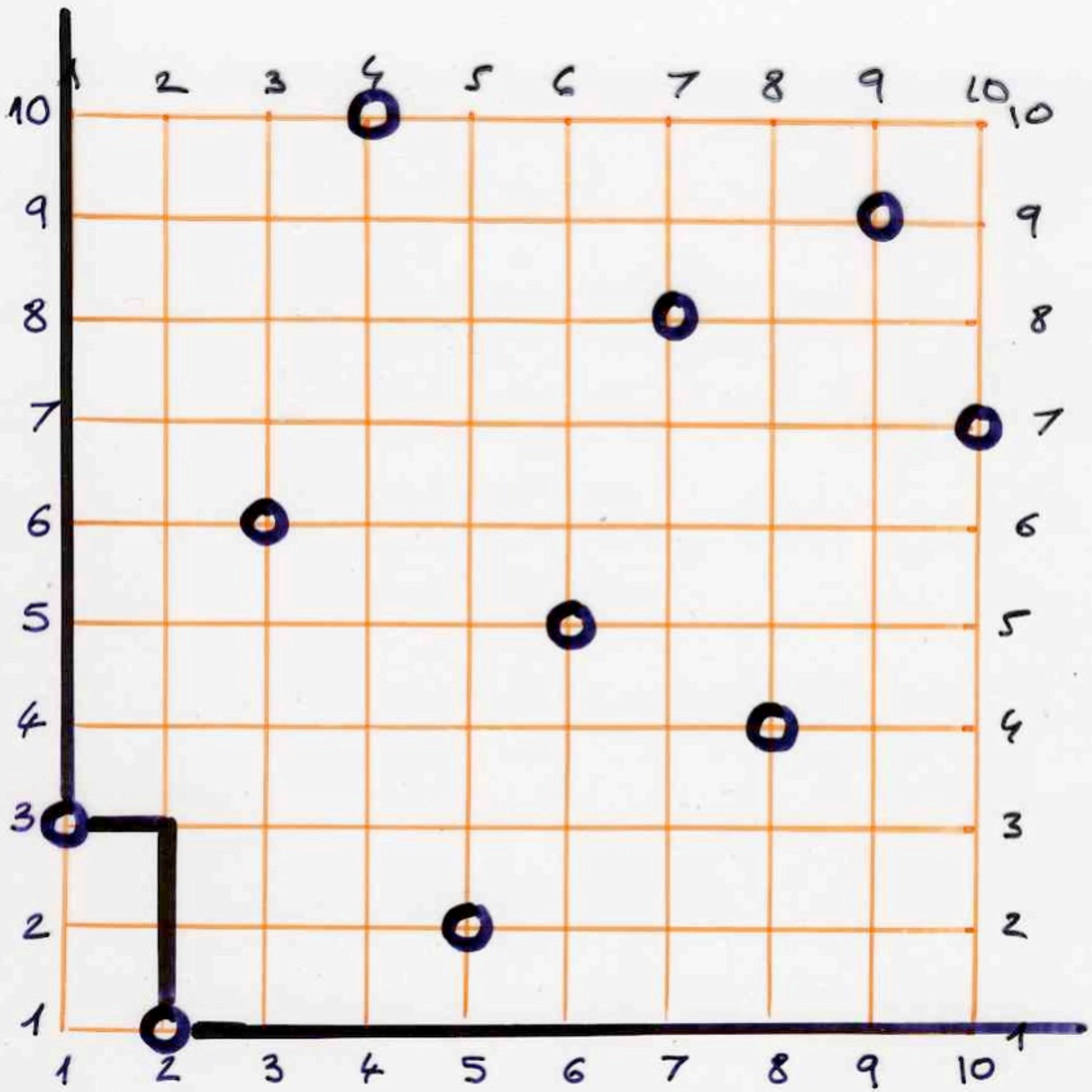




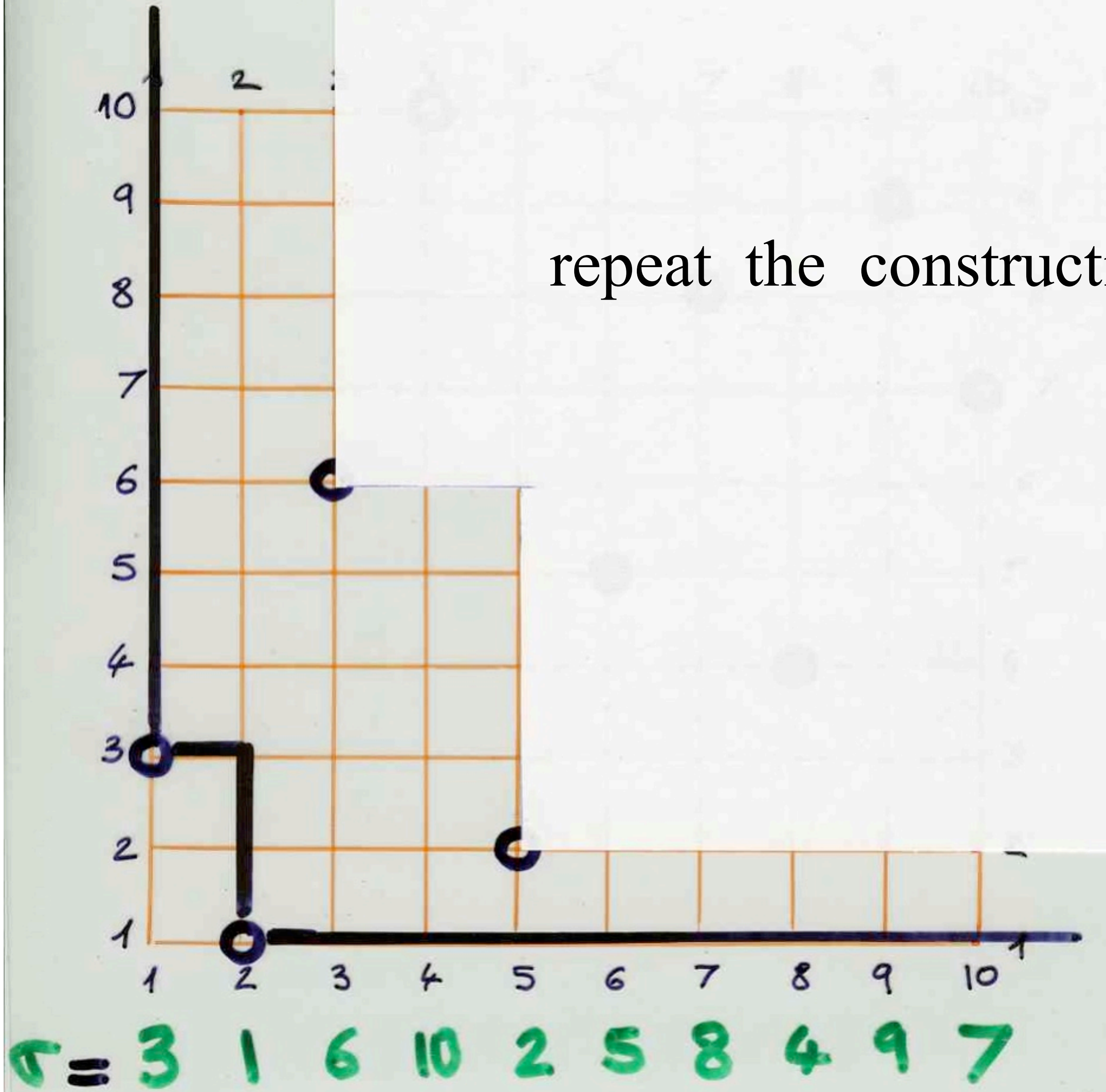
shadow of the permutation
= union of shadows
10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10
σ = 3 1 6 10 2 5 8 4 9 7



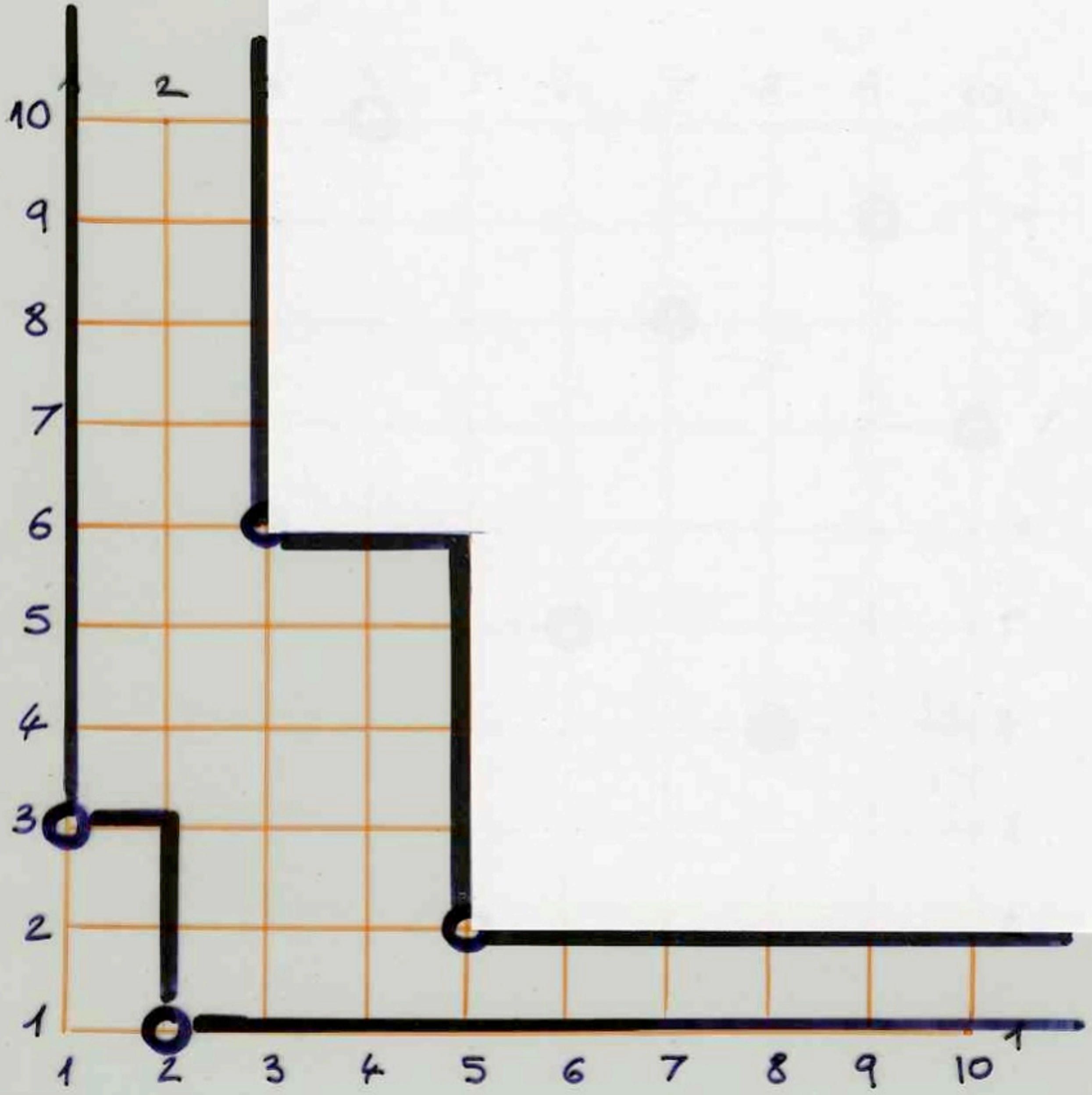
border of the shadow
10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10
σ = 3 1 6 10 2 5 8 4 9 7



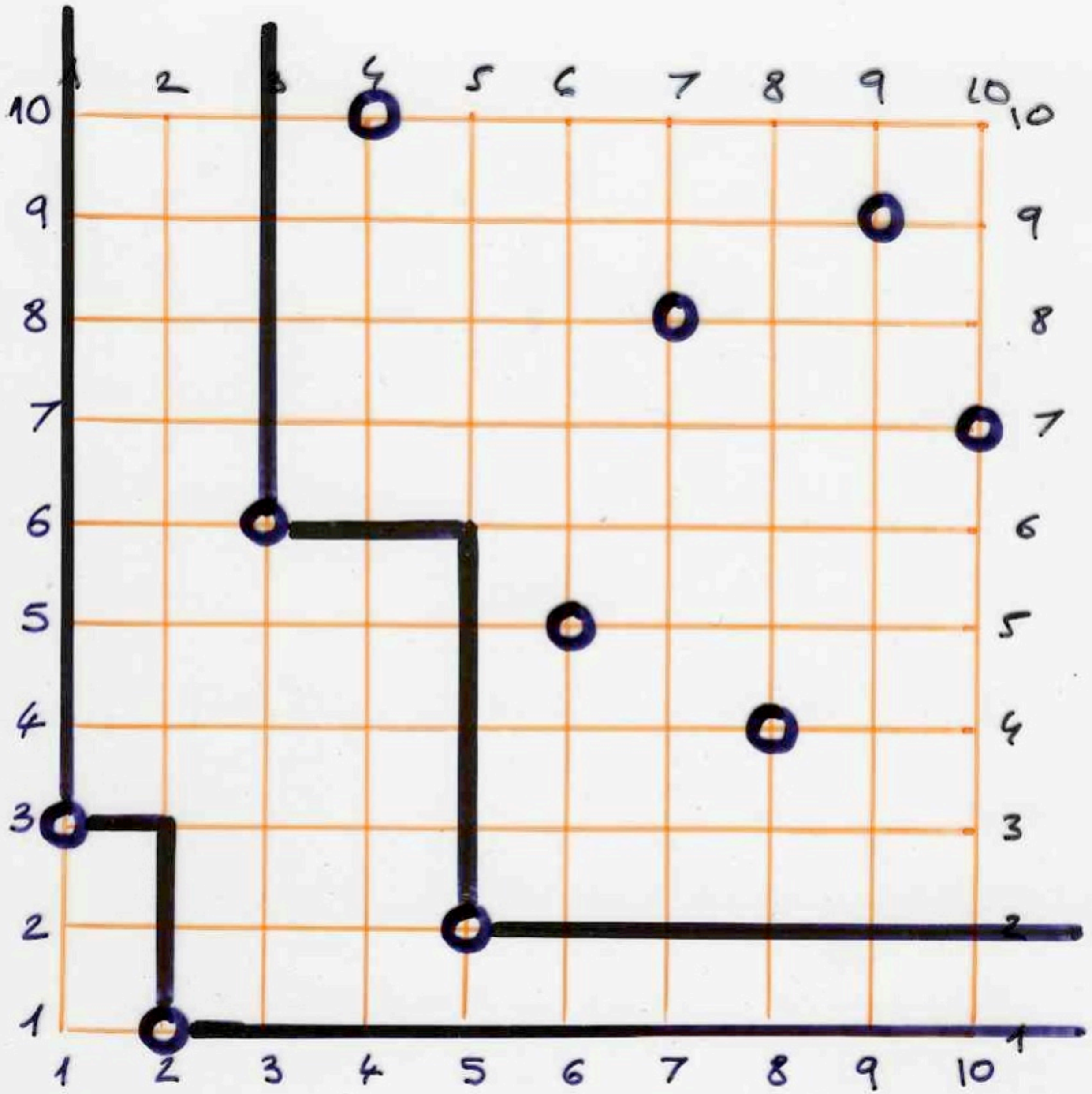
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



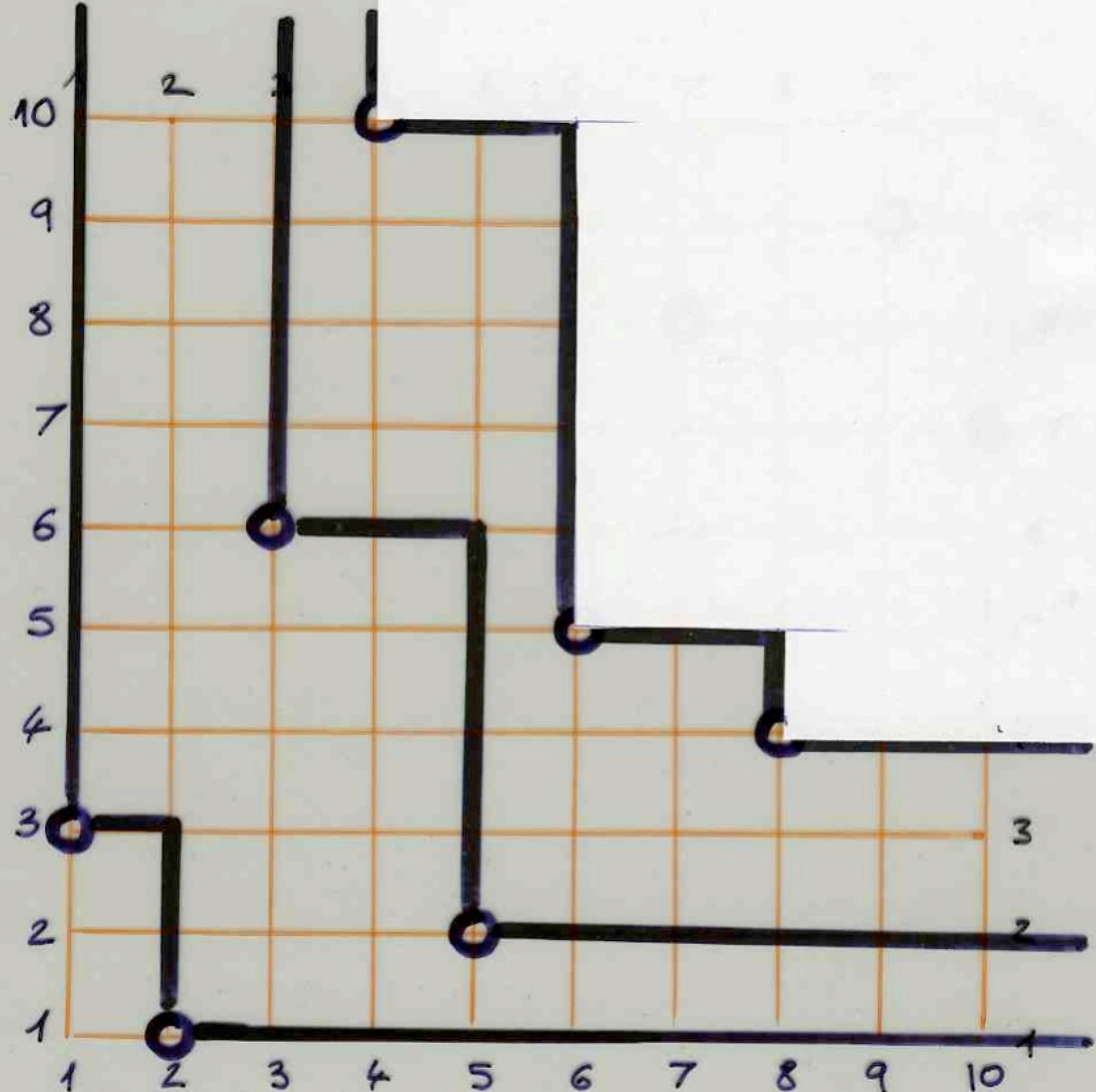
repeat the construction ...



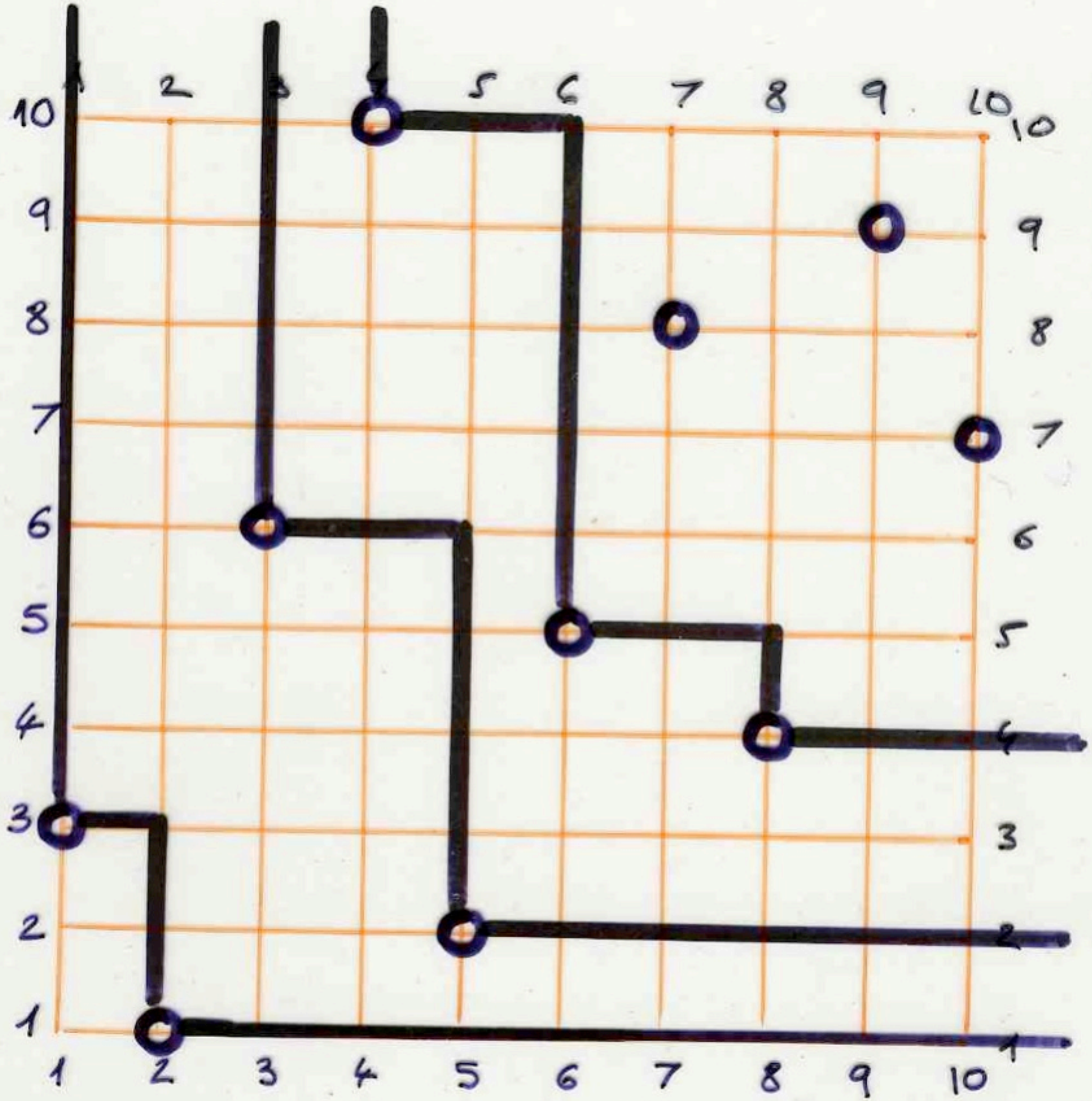
σ = 3 1 6 10 2 5 8 4 9 7



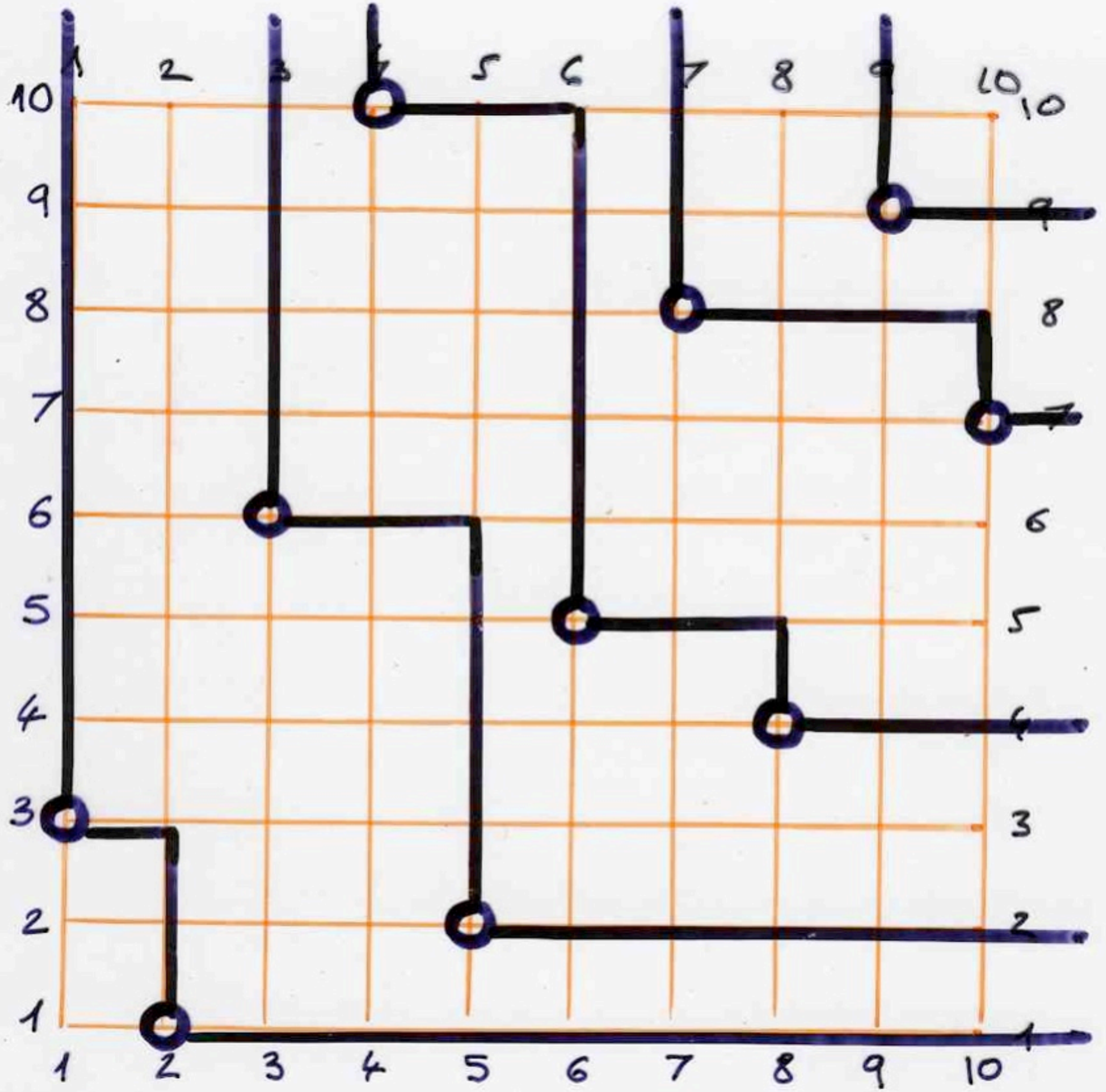
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



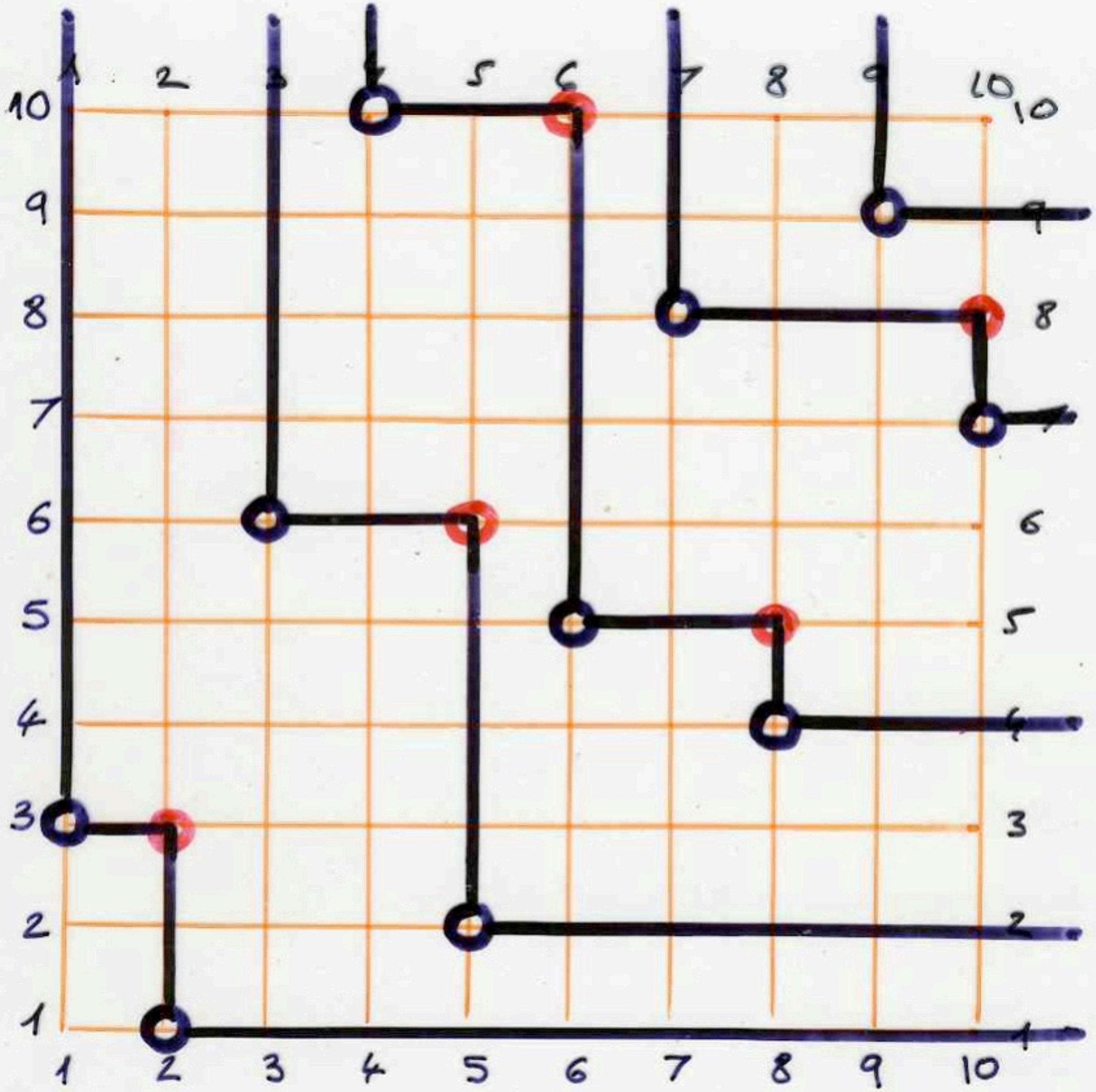
σ = 3 1 6 10 2 5 8 4 9 7



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

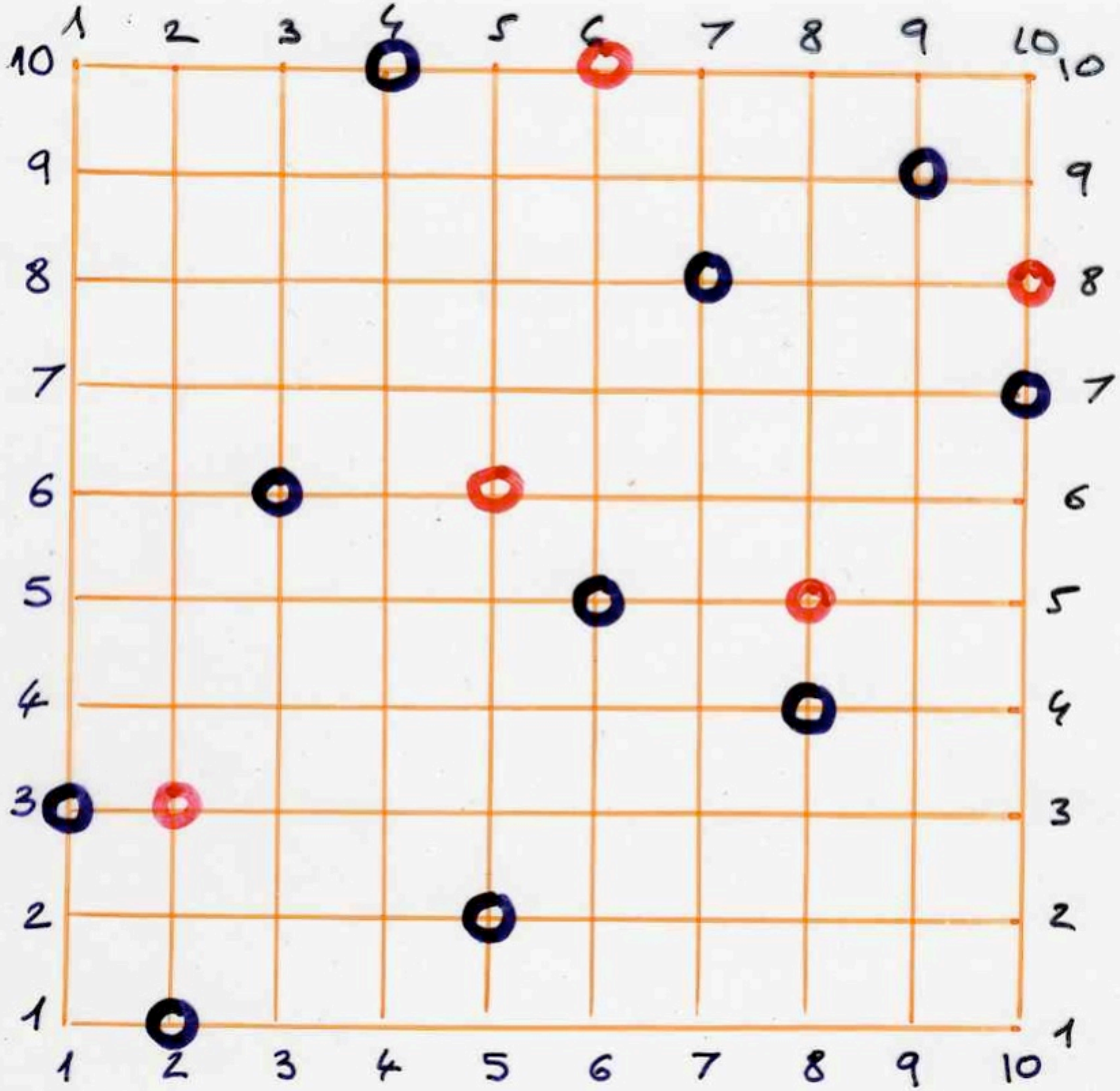


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



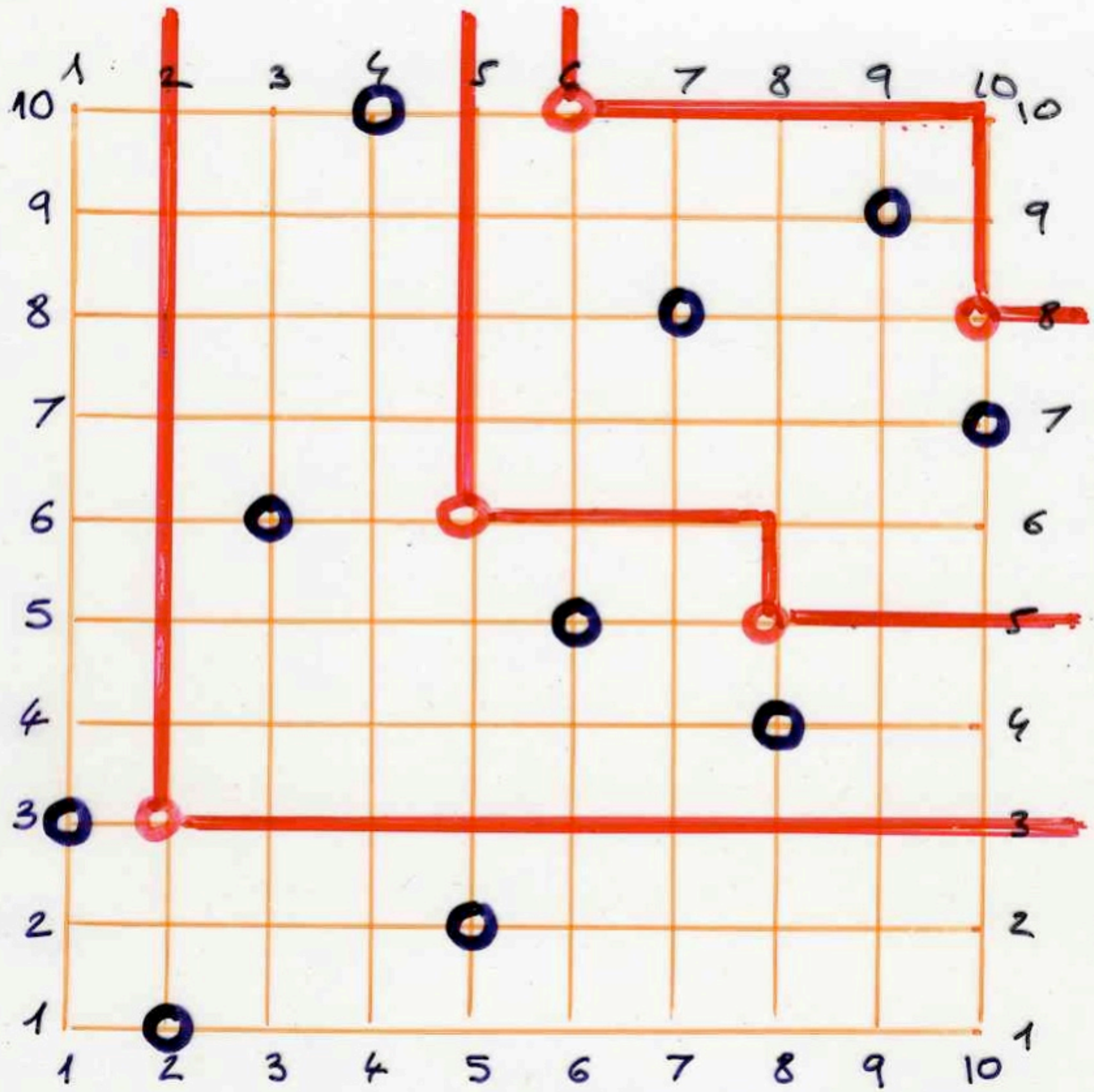
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

red points

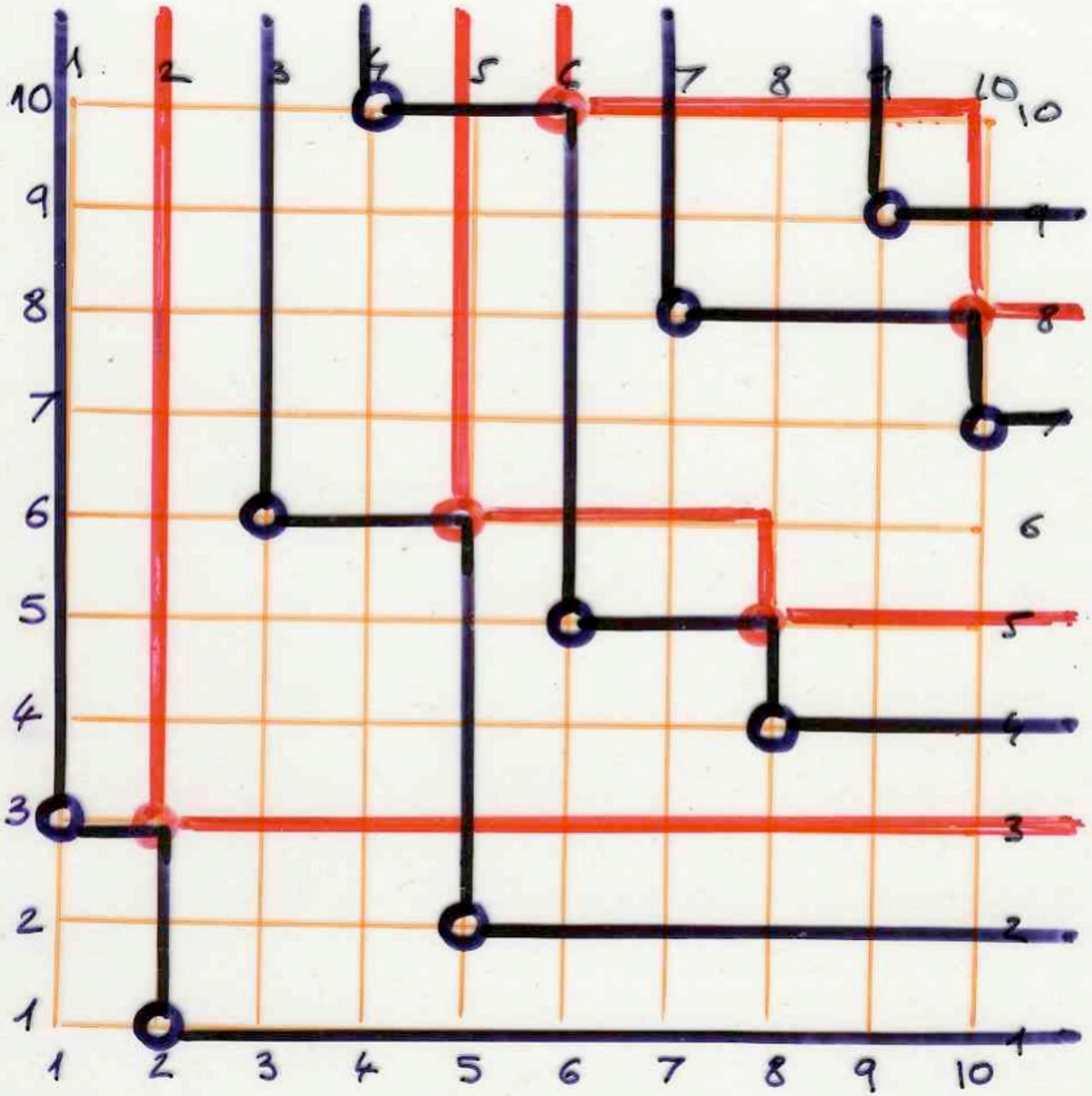


$\sigma = 3 \; 1 \; 6 \; 10 \; 2 \; 5 \; 8 \; 4 \; 9 \; 7$

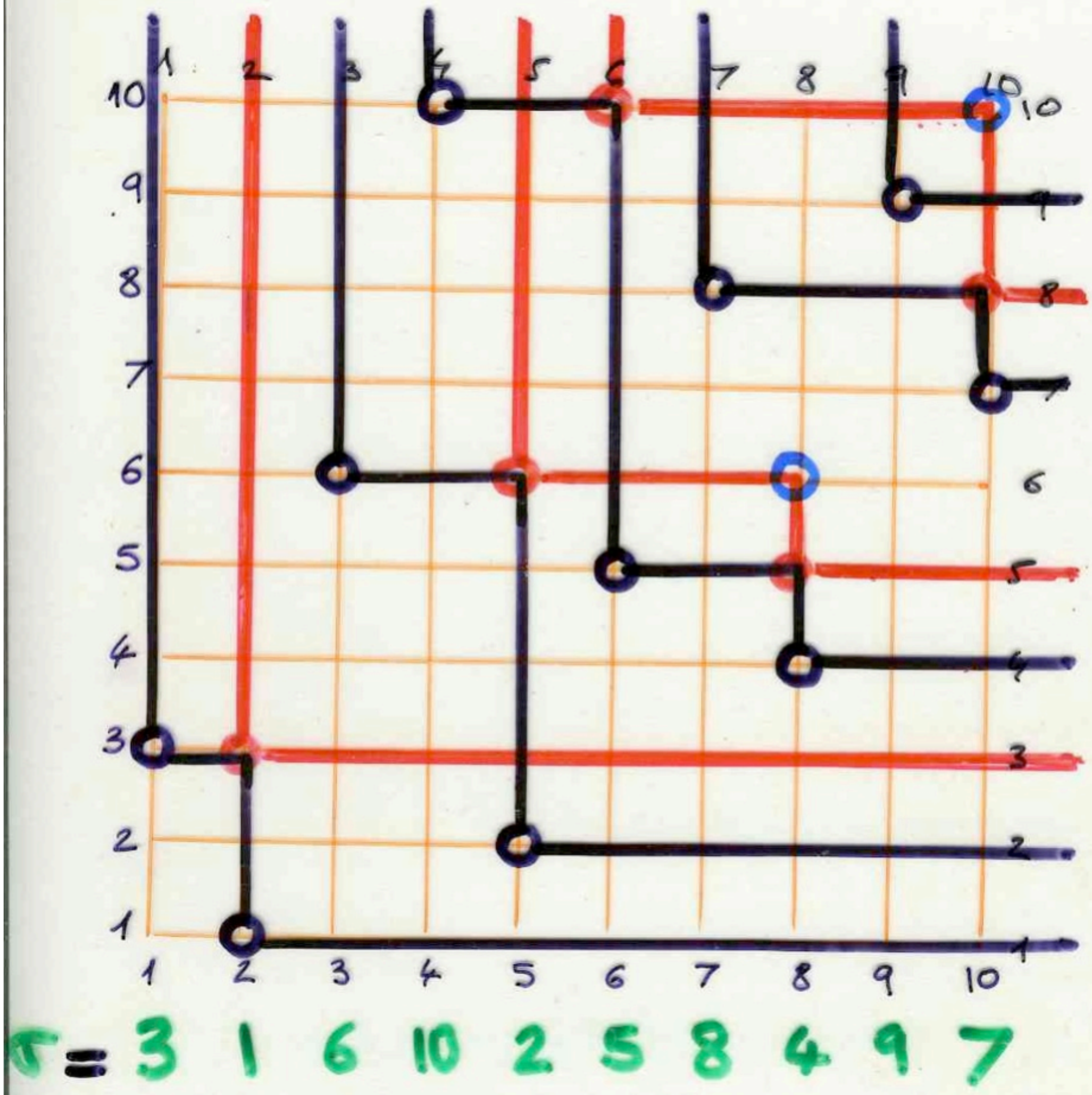
Repeat with the red points
the construction of sucessives shadows



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

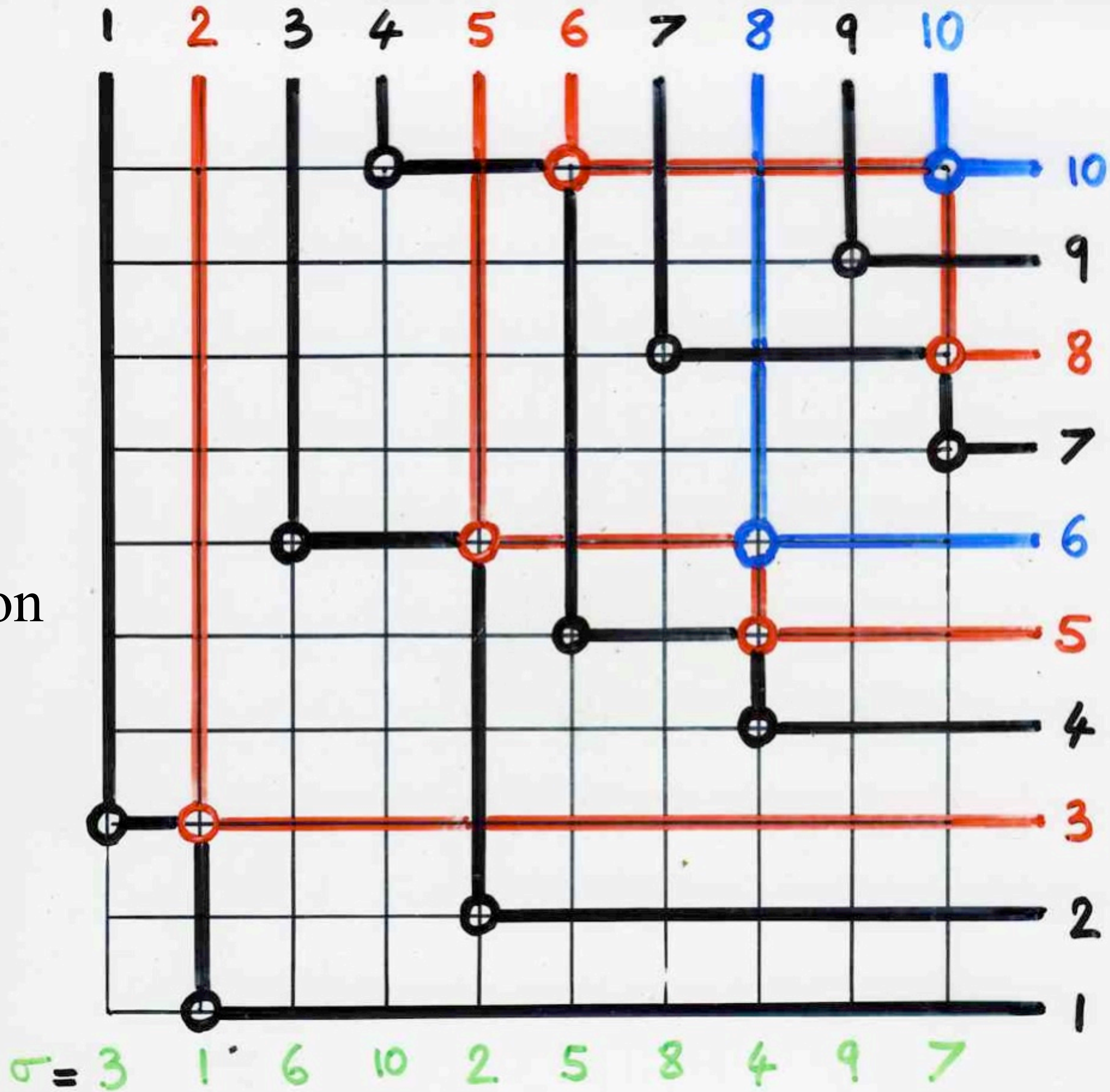


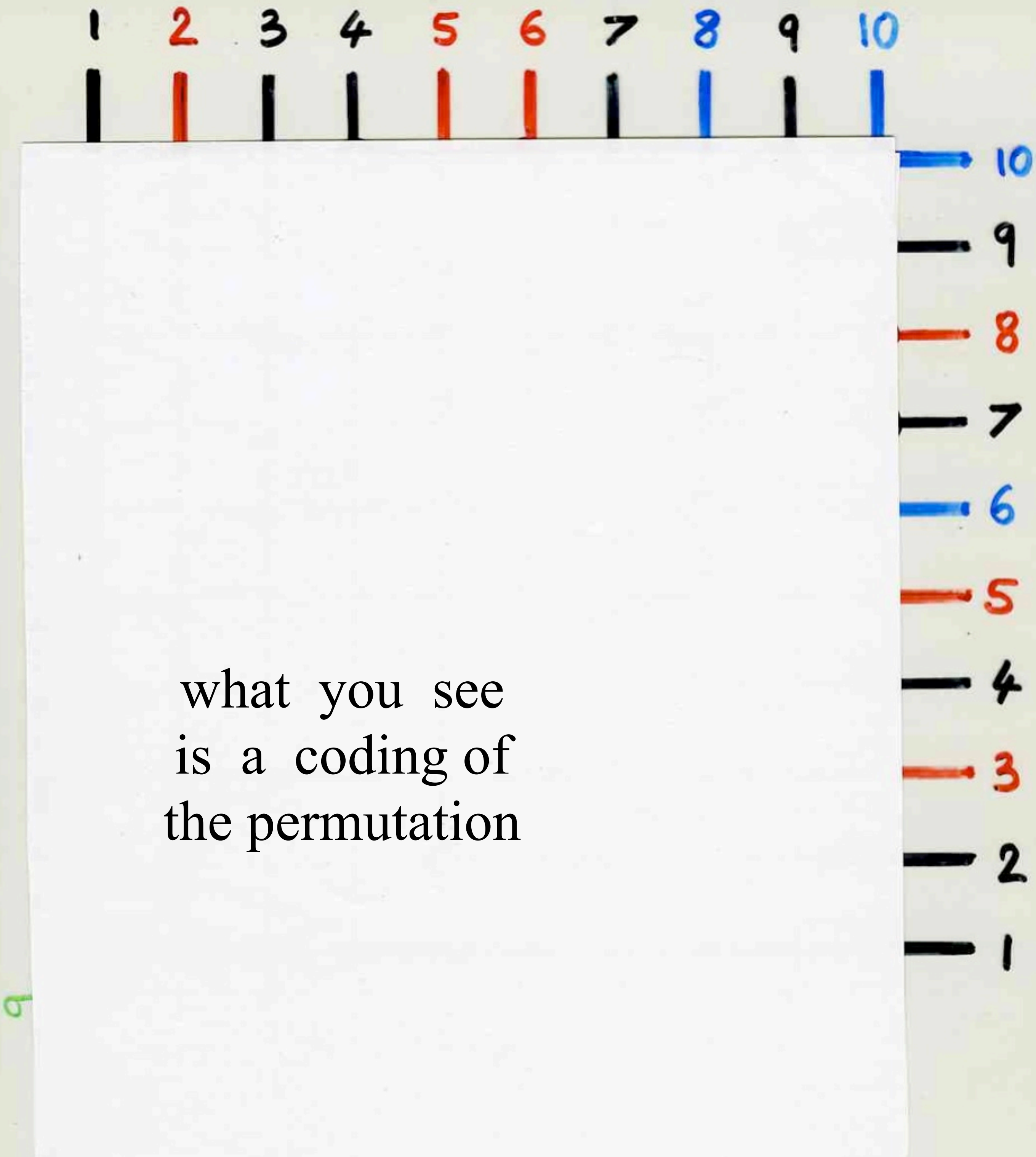
σ = 3 1 6 10 2 5 8 4 9 7



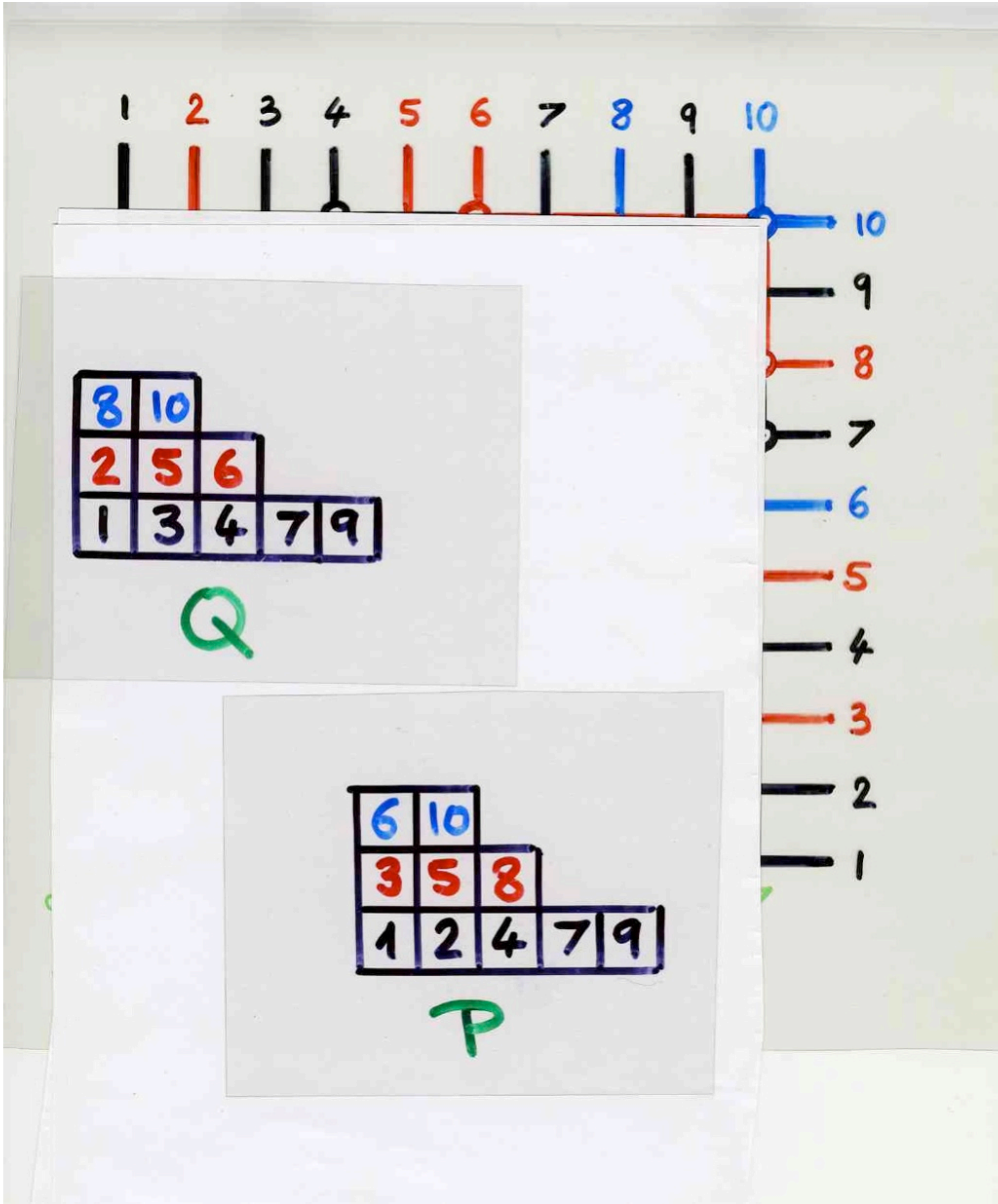
blue
points

no
green
points:
end
of the
construction

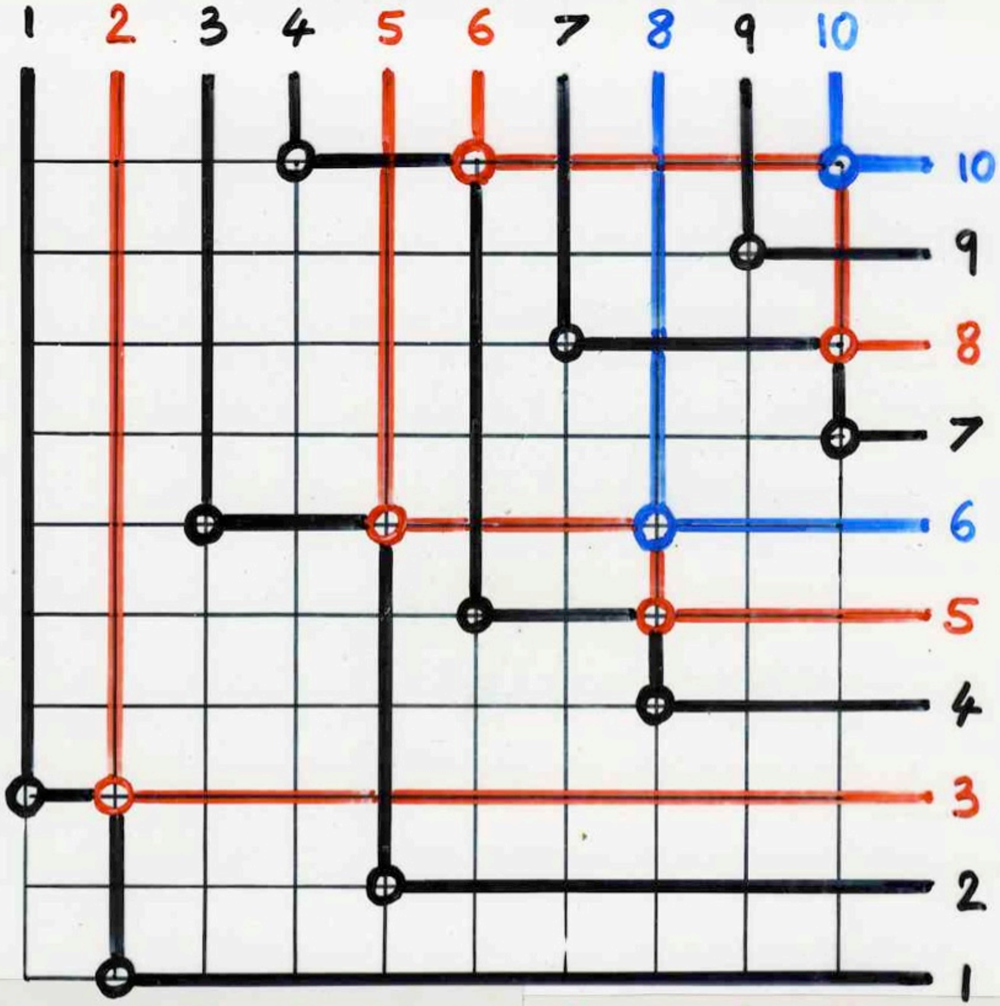




what you see
is a coding of
the permutation



1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
8 10
2 5 6
1 3 4 7 9
Q
6 10
3 5 8
1 2 4 7 9
P



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

6	10			
3	5	8		
1	2	4	7	9

P

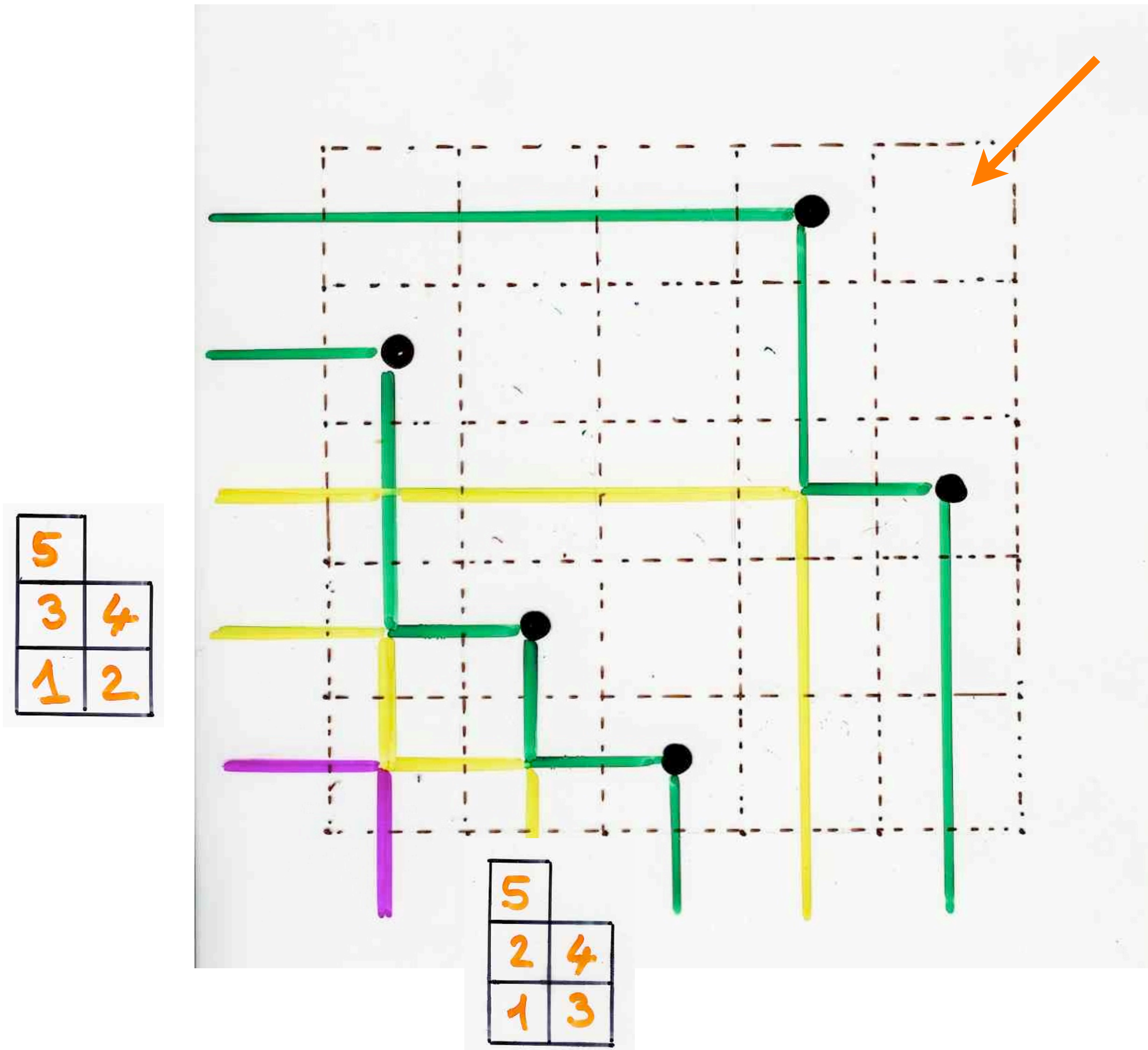
8	10			
2	5	6		
1	3	4	7	9

Q

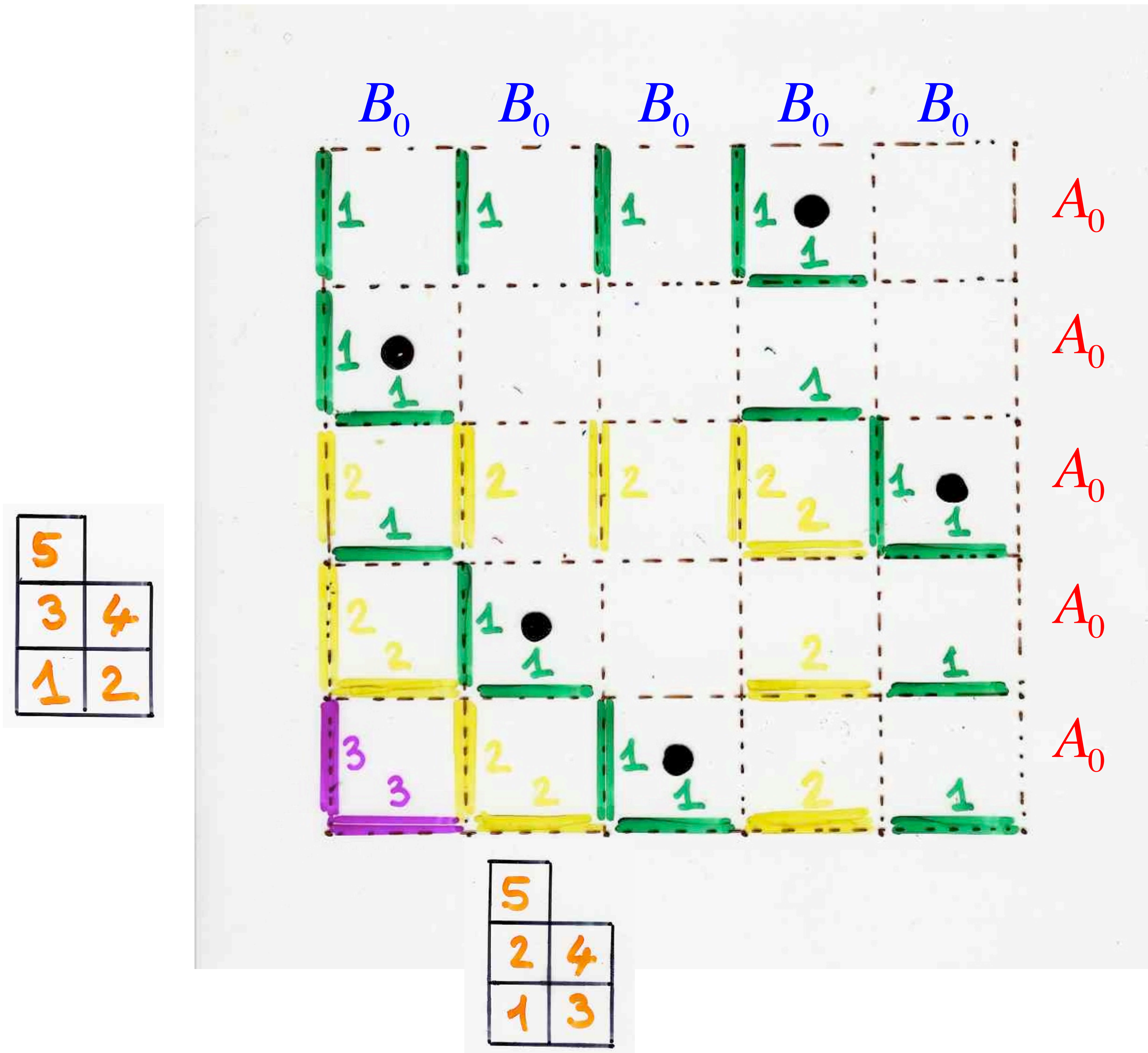
$$\sigma \longleftrightarrow (P, Q)$$

$$\sigma^{-1} \longleftrightarrow (Q, P)$$

The RSK planar automaton

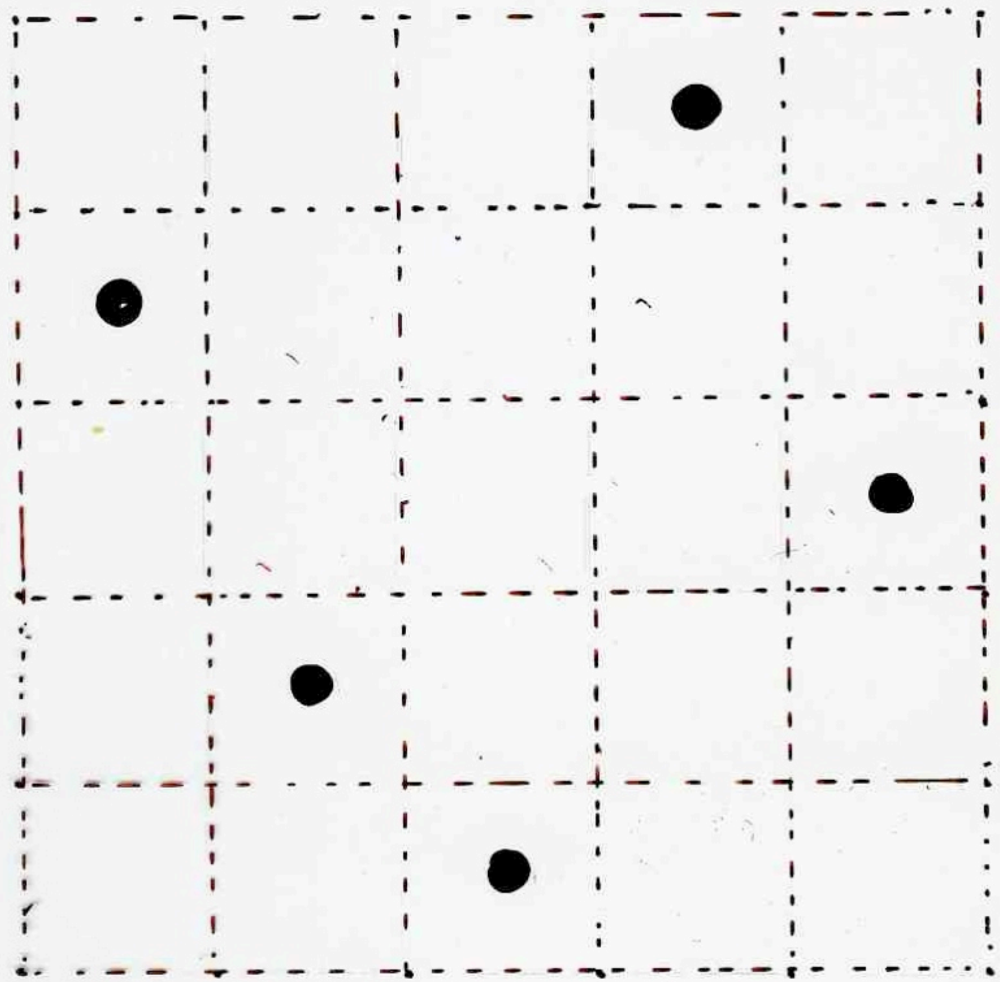


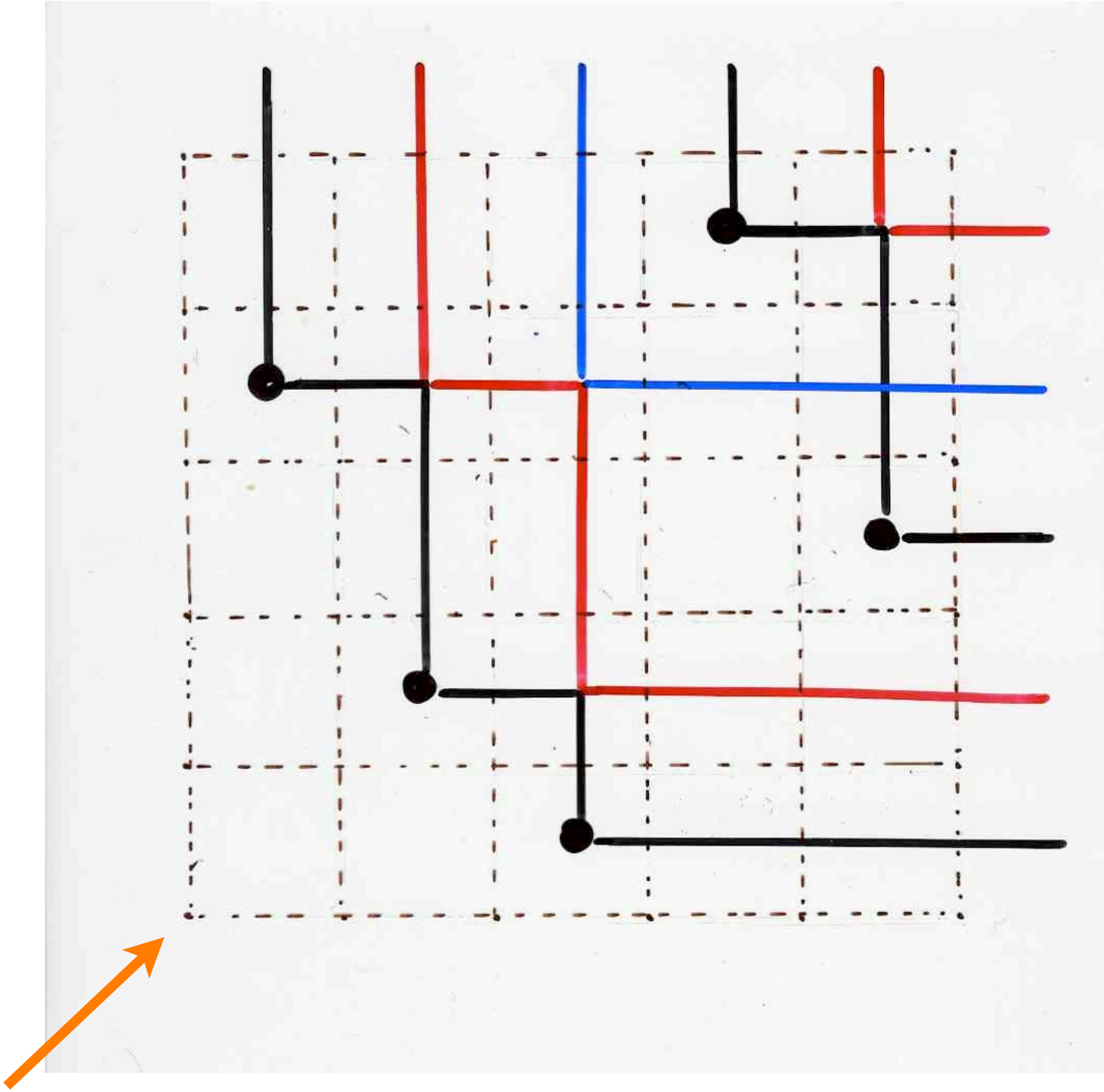
5
3 4
1 2
5
2 4
1 3

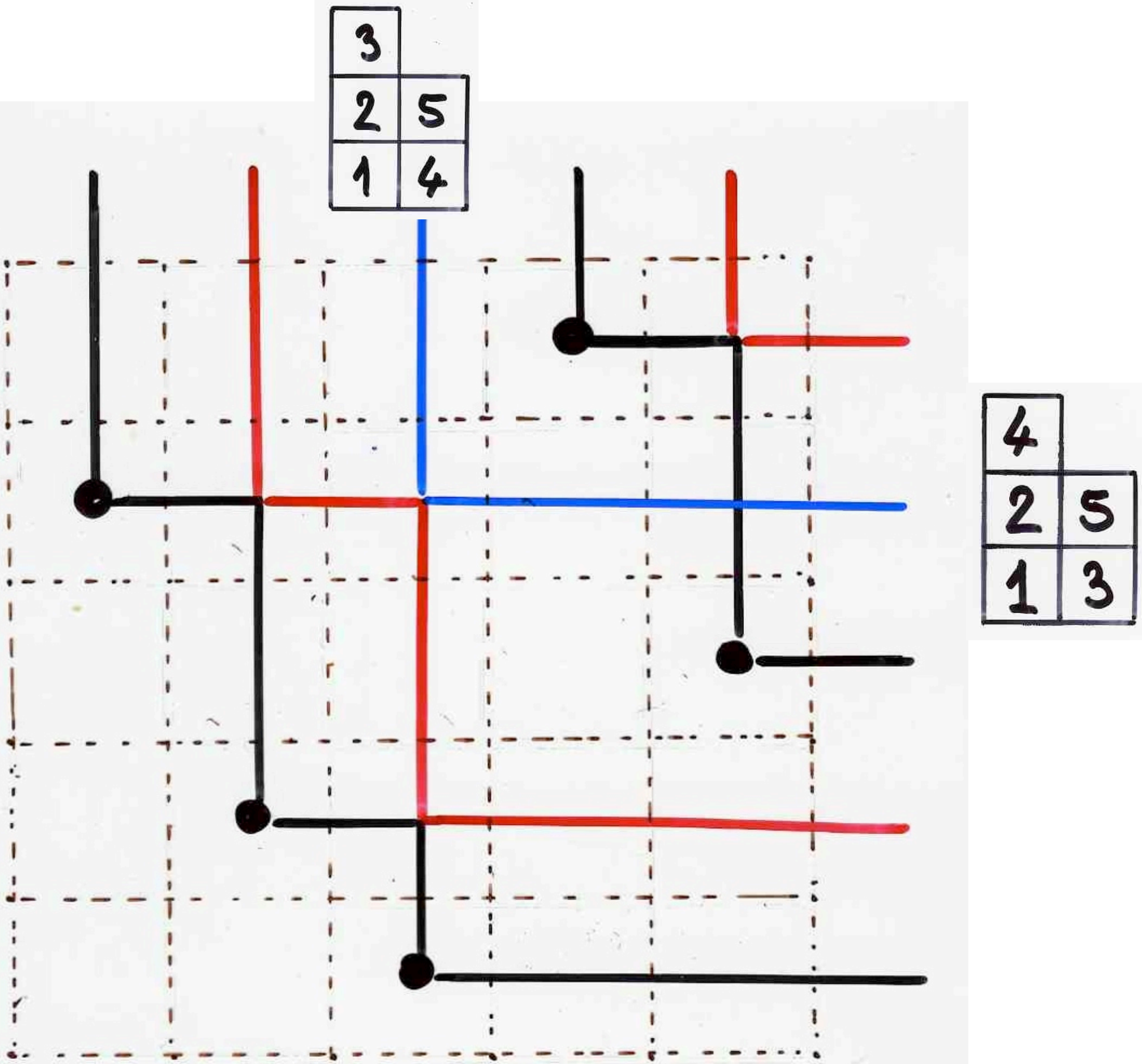


B_0 B_0 B_0 B_0 B_0
A_0
A_0
A_0
A_0
A_0

The reverse RSK planar automaton






3
2 5
1 4
4
2 5
1 3

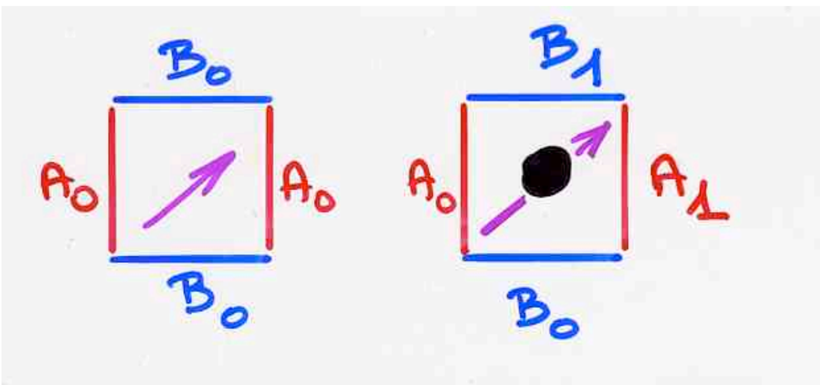
The RSK (reverse) planar automaton

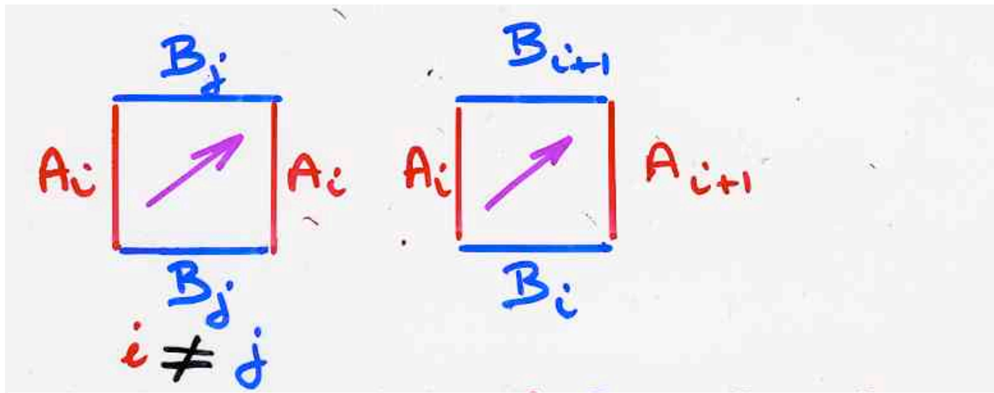
$\mathcal{B} = \{B_0, B_1, \dots B_k\}$

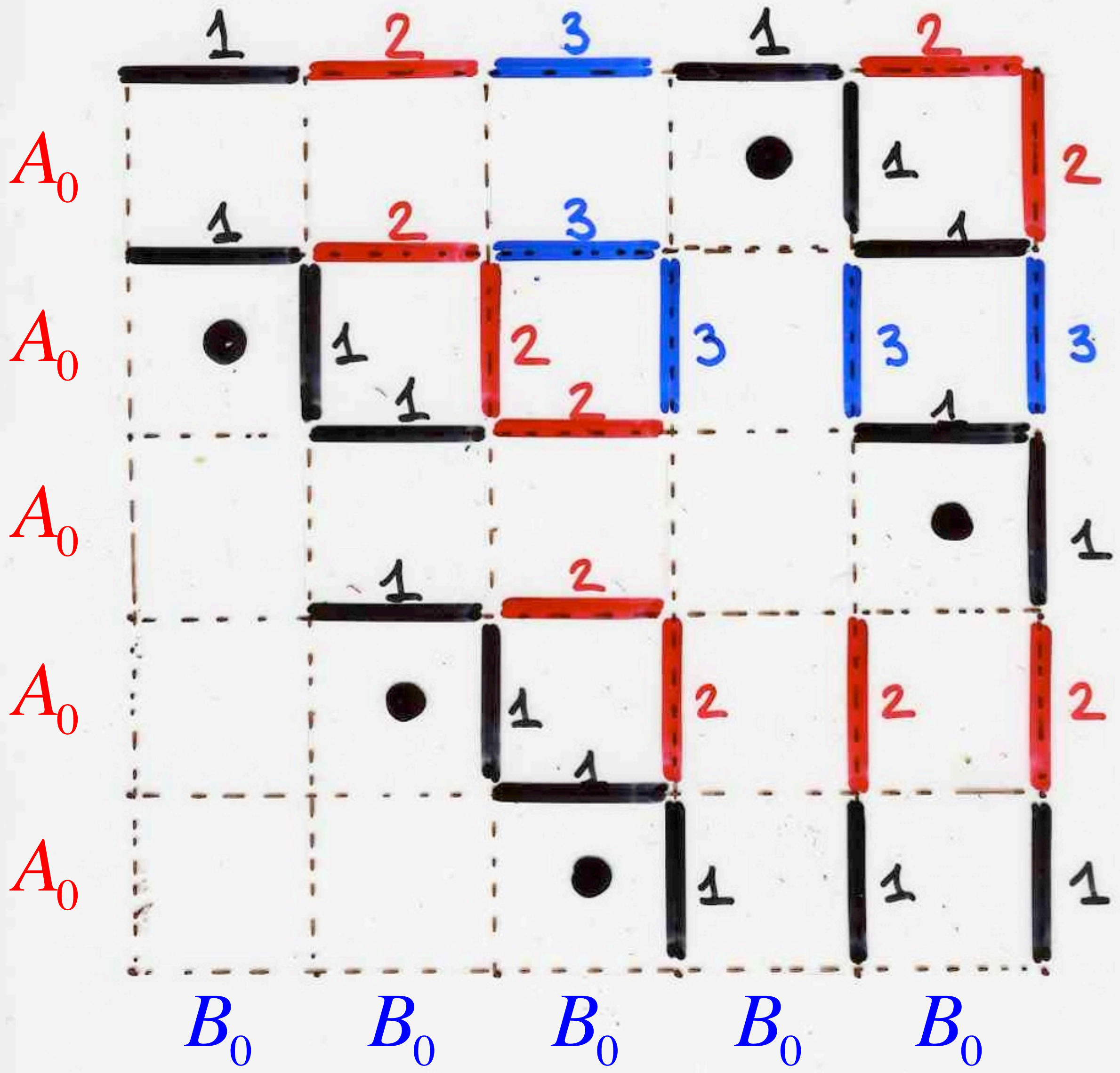
$\mathcal{A} = \{A_0 \quad A_1 \qquad A_k\}$

$w \in \{B_0, A_0\}^*$

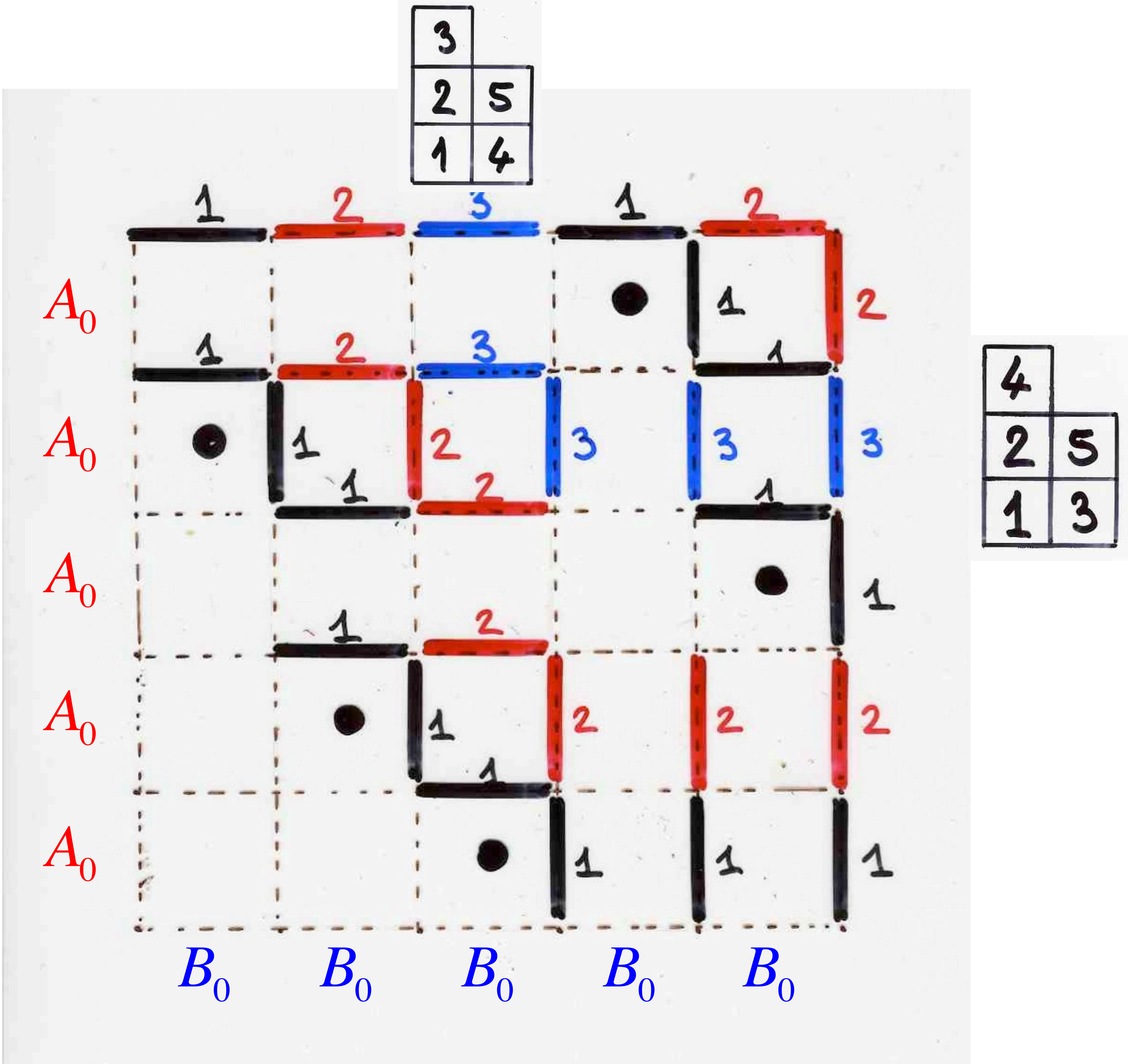
$S = \{\square, \boxdot\}$







1 2 3 1 2
A0
1 2
1 2 3
A0
1 2 3 3 3
1 2 1
A0
1
1 2
A0
1 2 2 2
1
A0
1 1 1
B0 B0 B0 B0 B0

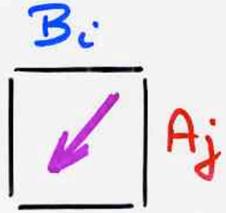

A0
A0
A0
A0
A0
B0
B0
B0
B0
B0

The bilateral RSK planar automaton

bilateral planar automaton RSK

$\mathcal{B} = \{B_i\}_{i \in \mathbb{Z} - \{0\}}$

$\mathcal{A} = \{A_j\}_{j \in \mathbb{Z} - \{0\}}$



$B_i A_j = A_j B_i$

$i \neq j$

$B_i A_i = A_{i-1} B_{i-1}$ $(i \neq 1)$

$B_1 A_1 = A_{-1} B_{-1}$

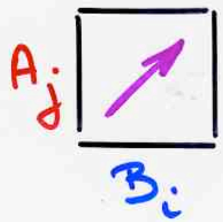
bilateral (reverse) planar automaton RSK

$A_j B_i = B_i A_j$

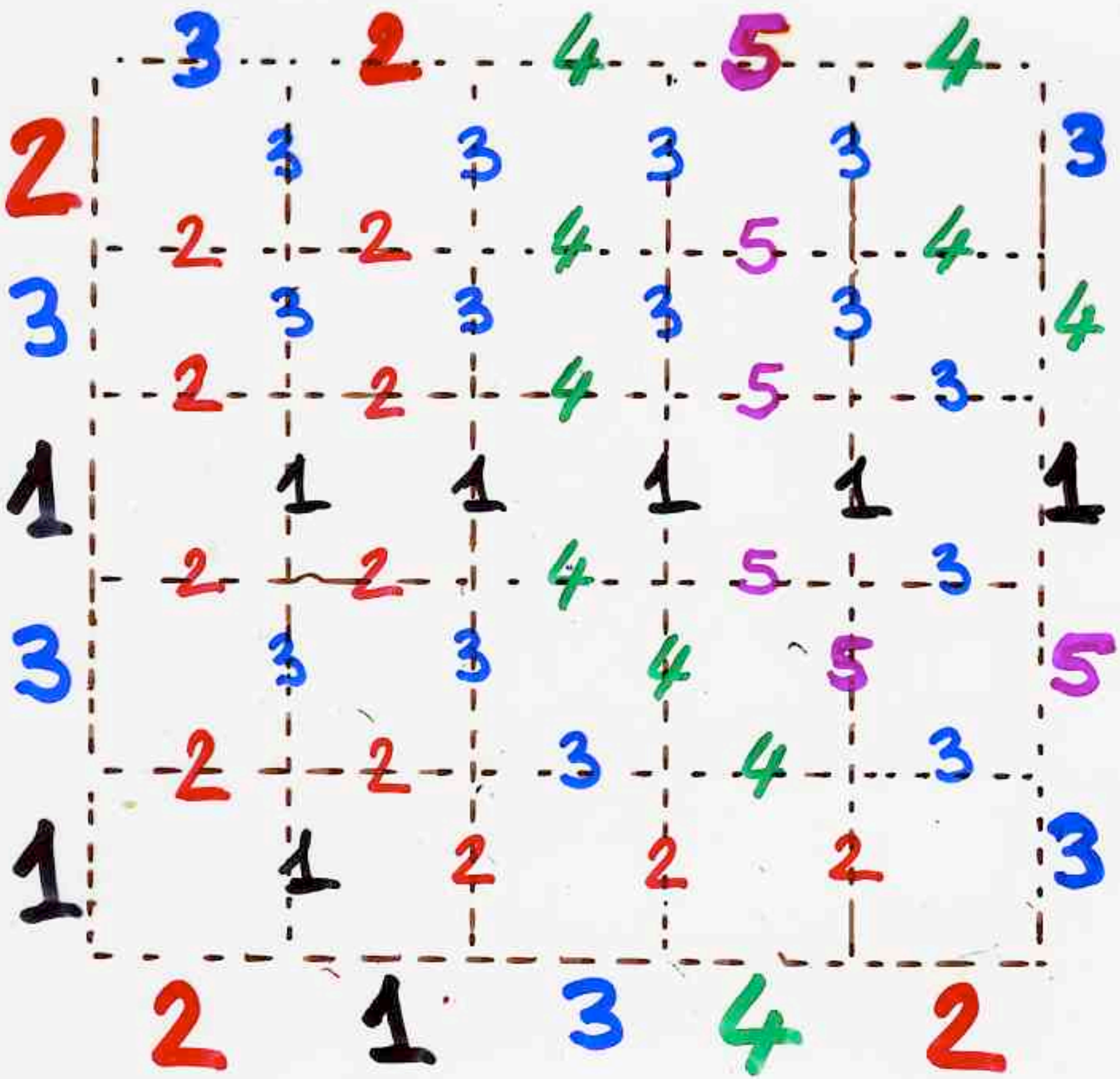
$i \neq j$

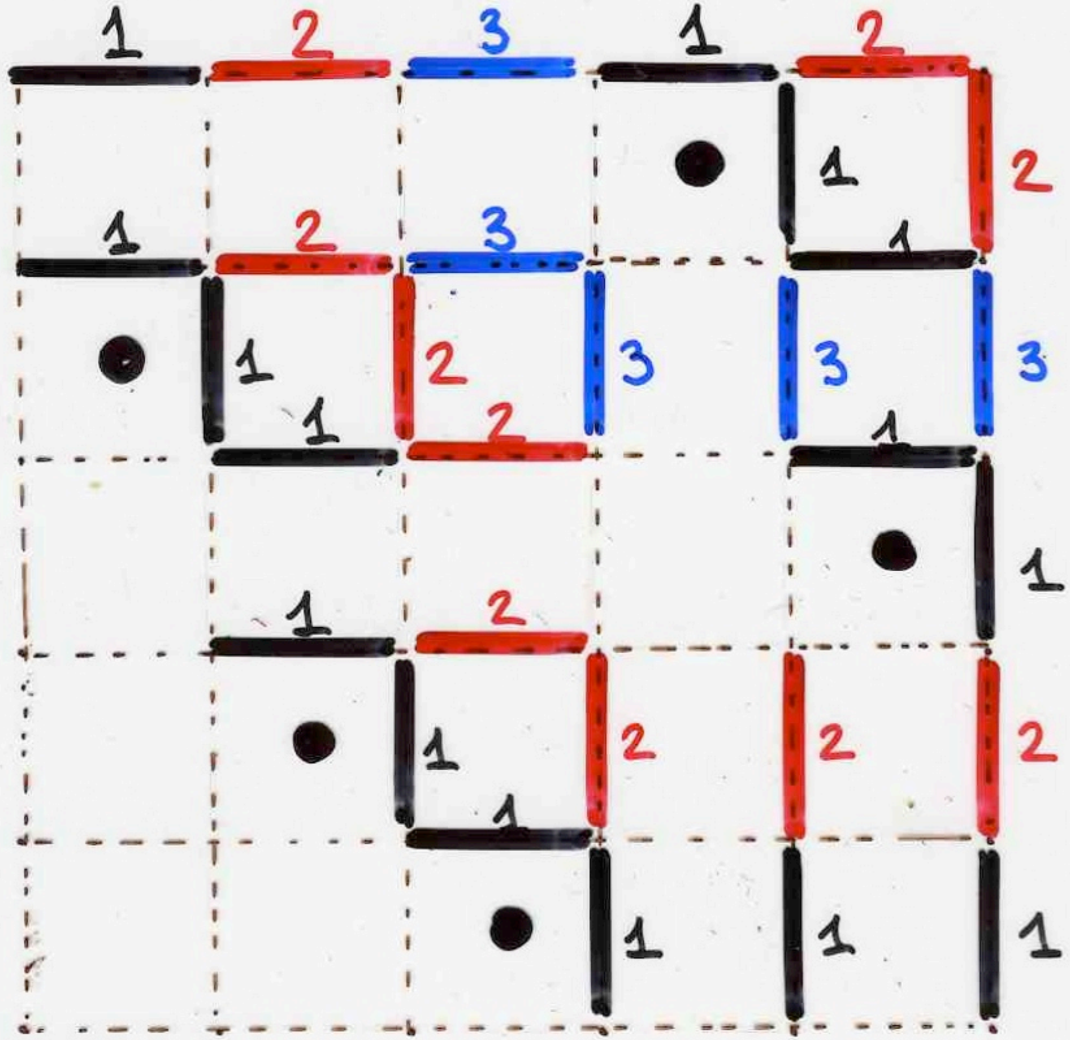
$A_i B_i = B_{i+1} A_{i+1}$ $(i \neq -1)$

$A_{-1} B_{-1} = B_1 A_1$

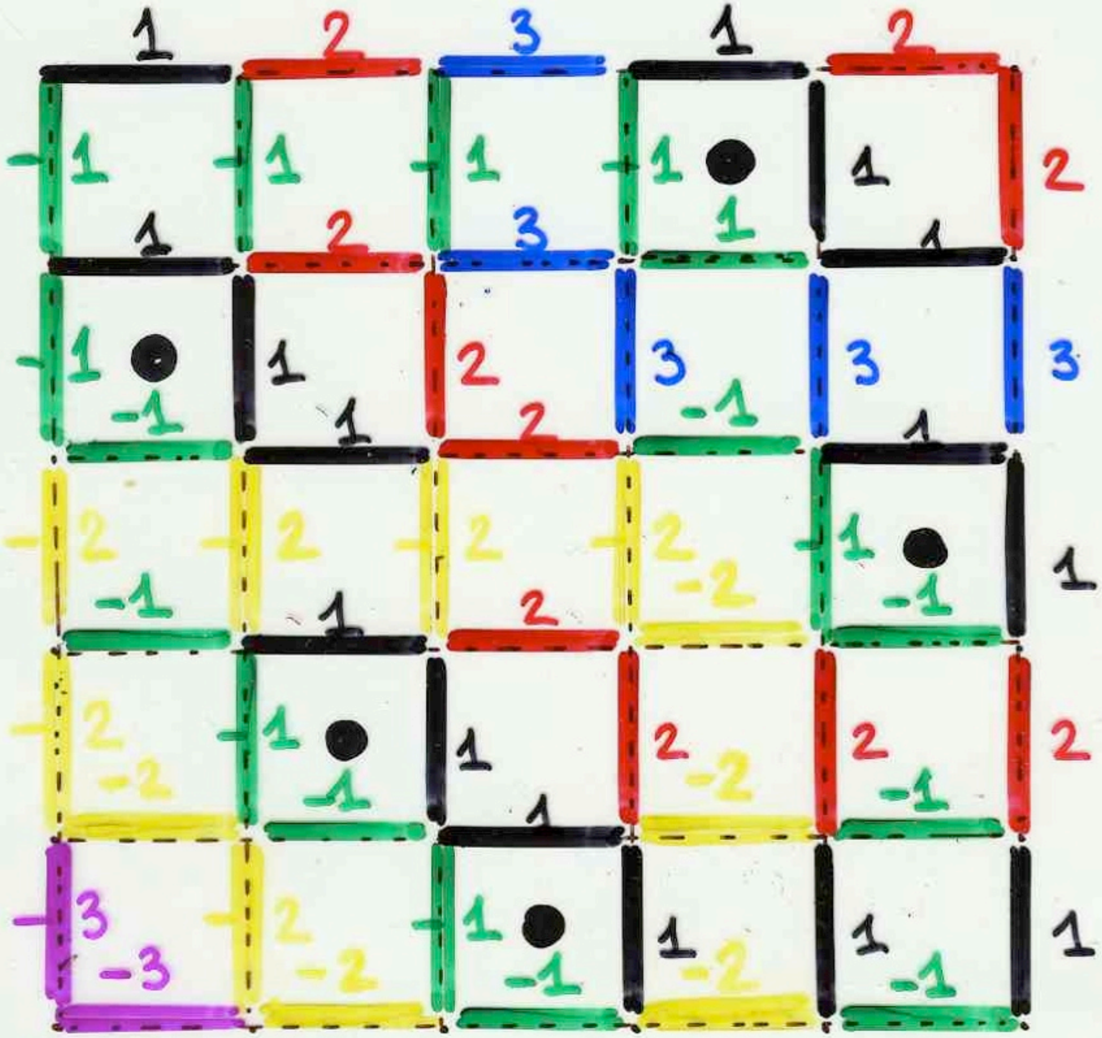


2					
3					
1					
3					
1					
	2	1	3	4	2


3 2 4 5 4
2 3 3 3 3 3
2 2 4 5 4
3 3 3 3 3 4
2 2 4 5 3
1 1 1 1 1 1
2 2 4 5 3
3 3 3 4 5 5
2 2 3 4 3
1 1 2 2 2 3
2 1 3 4 2

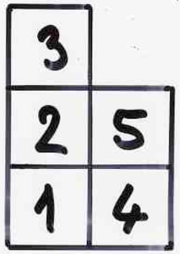


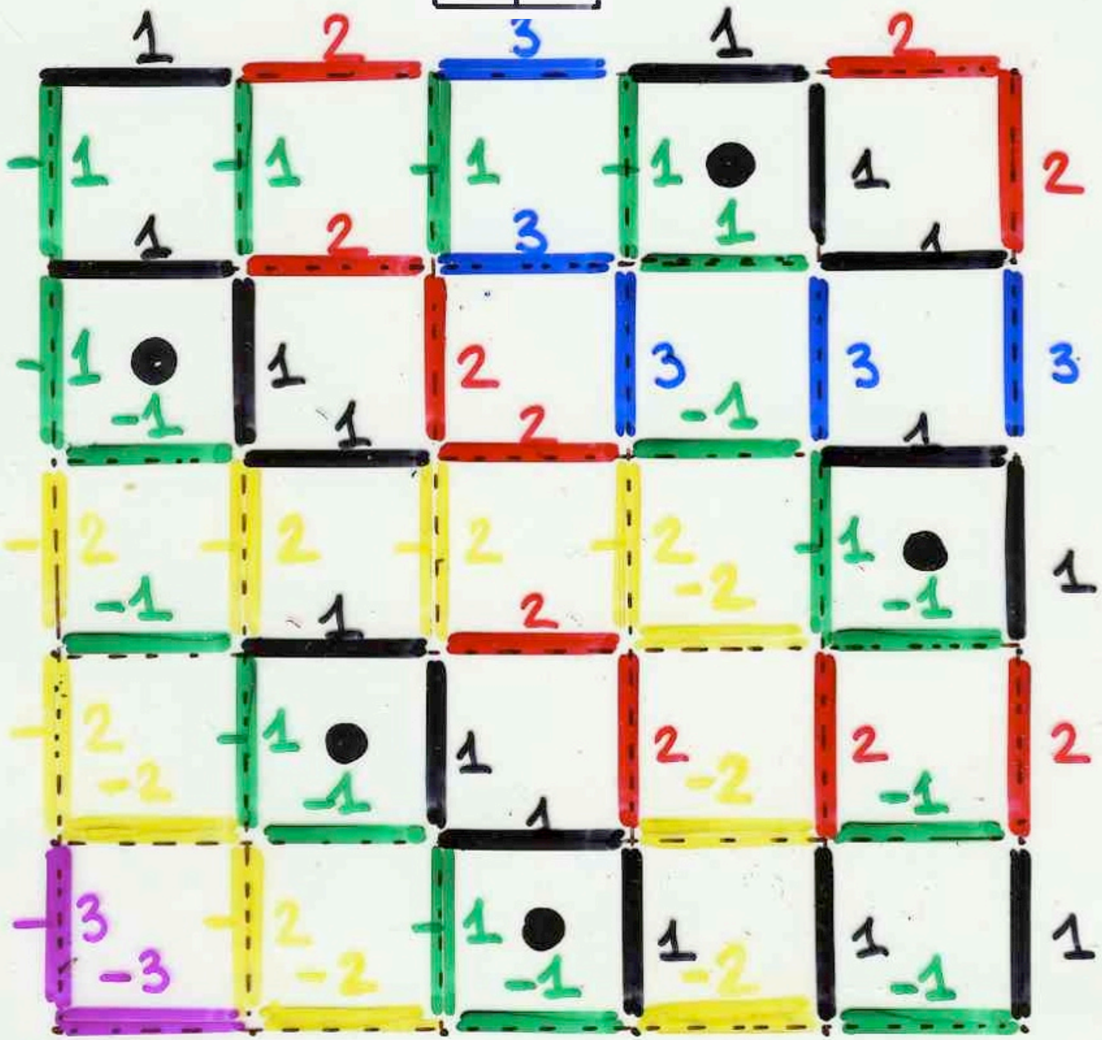
1 2 3 1 2
1 2
1
1 2 3
1 2 3 3 3
1 2 1
1
1 2
1 2 2 2
1
1 1 1

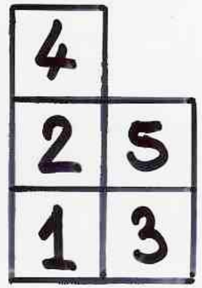


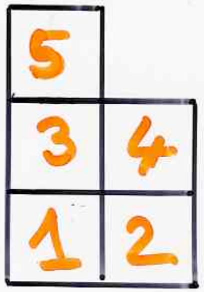
1 2 3 1 2
-1 1 -1 1 -1 1 -1 1 1 2
1 2 3 1 1
-1 1 1 2 3 3 3
-1 -1
1 2 1
-2 -2 -2 -2 -1 1
-1 -2 -1
1 2 1
-2 1 1 2 2 2
-2 -1 -2 -1
1
-3 -2 -1 1 1 1
-3 -2 -1 -2 -1

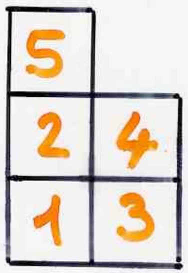
Schützenberger duality

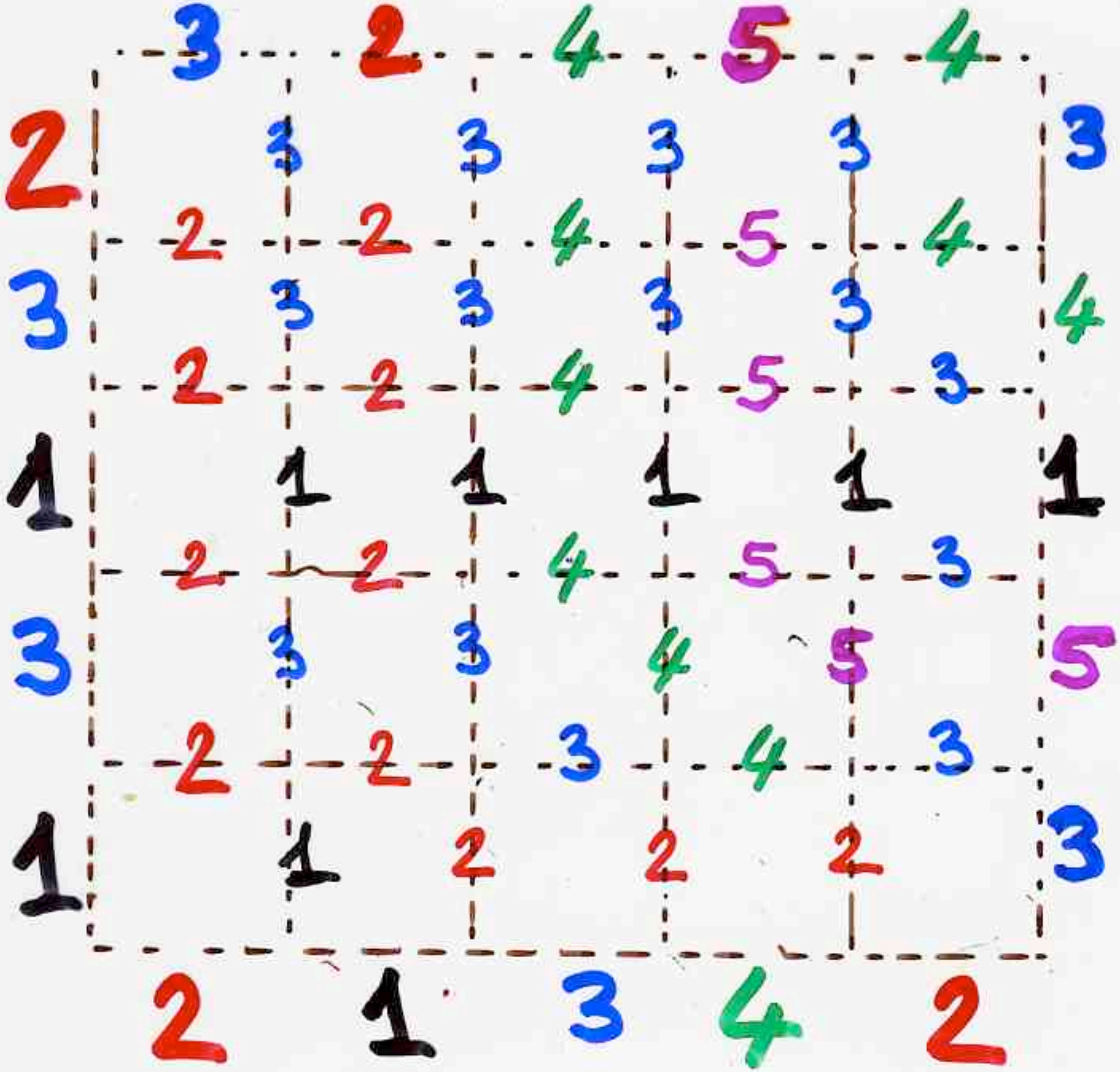











3 2 4 5 4
2 3 3 3 3 3
2 2 4 5 4
3 3 3 3 3 4
2 2 4 5 3
1 1 1 1 1 1
2 2 4 5 3
3 3 3 4 5 5
2 2 3 4 3
1 1 2 2 2 3
2 1 3 4 2

Relation
planar automata
and
quadratic algebras

the case of permutations

Heisenberg
operators
U, D

$$UD = DU + I$$

creation and annihilation opeators

quantum mechanics

normal ordering

$$UD = DU + I$$

Every word w with letters U and D
can be written in a unique way

$$w = \sum_{i,j \geq 0} c_{i,j}(w) D^i U^j$$

by applying a succession of substitutions

$$UD \rightarrow DU + I$$

independant of the order of the substitutions

normal ordering

$$UD = DU + I$$

$$
\begin{aligned}
UUDD &= UDUD + UD \\
&= DUUD + 2UD \\
&= DUDU + DU + 2(DU + I_d) \\
&= DDUU + 2DU \\
&= DDUU + 4DU + 2I_d
\end{aligned}
$$

$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} \, D^i U^i$$

normal ordering

$$c_{n,0} = n!$$

K. Penson, I. Solomon
P. Blasiak, A. Horzela
G. Duchamps

some quadratic algebra Q defined
by generators and relations

here $UD=DU+1$

↓ normal ordering

combinatorial objects
called Q-tableaux

here permutations, rooks placements

$$UD = DU + I$$

from commutation relations
to rewriting rules

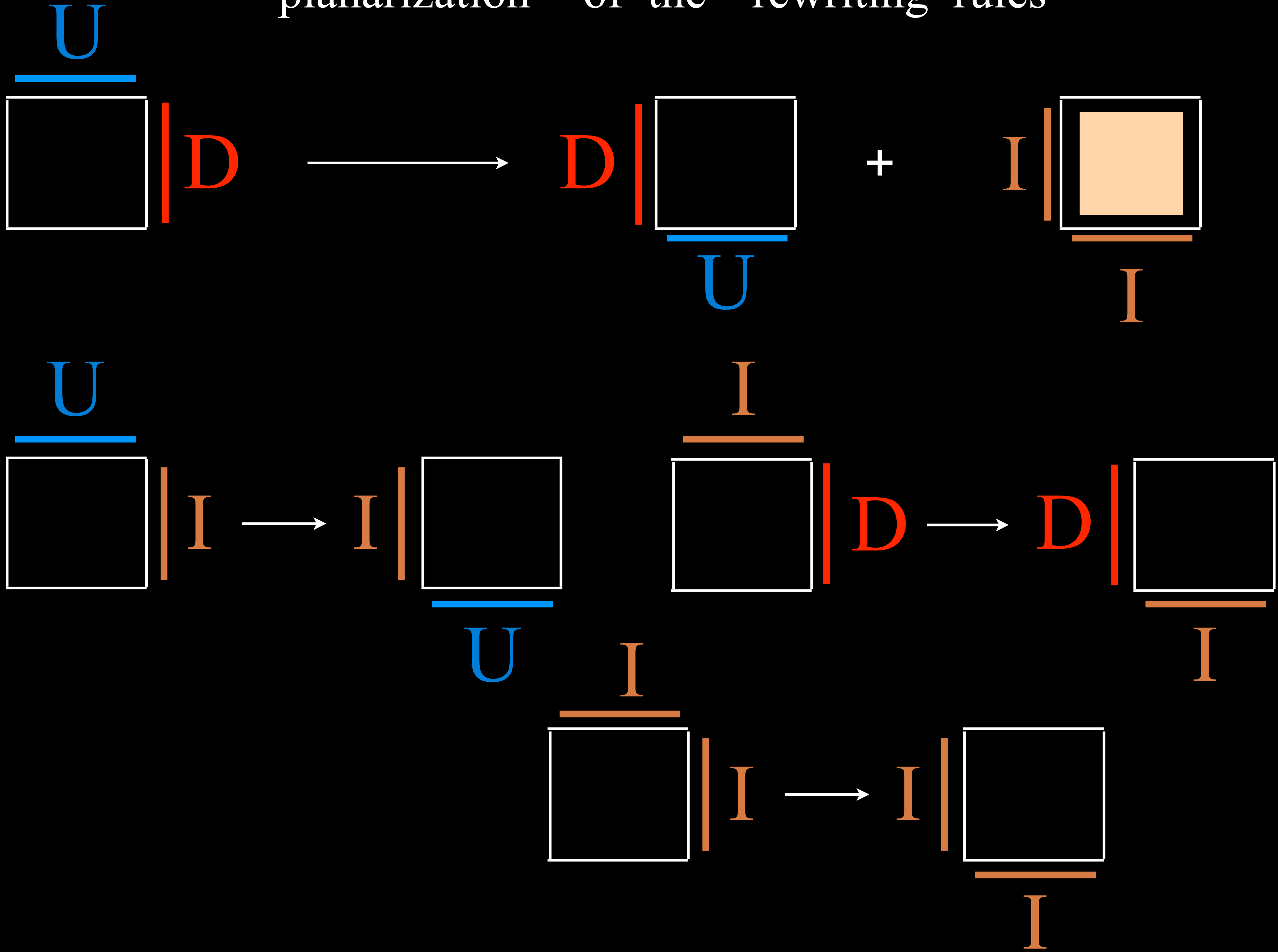
$$UD \rightarrow DU \qquad UD \rightarrow I$$

U U D U D D U D

U U D (U D) D U D

U U D (D U) D U D + U U D () D U D

“planarization” of the “rewriting rules”



$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

quadratic algebra

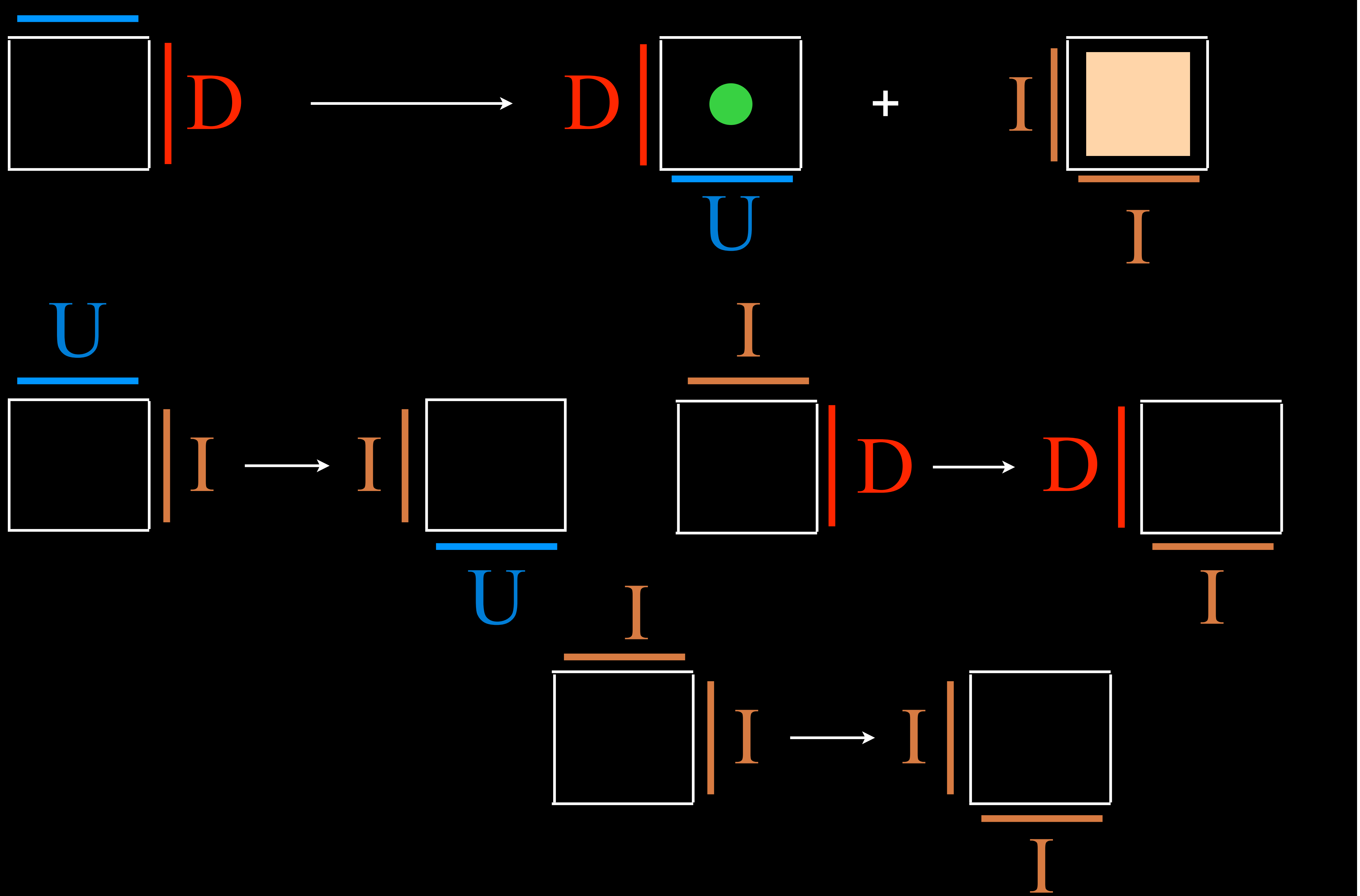
4 generators U, D, I_v, I_h

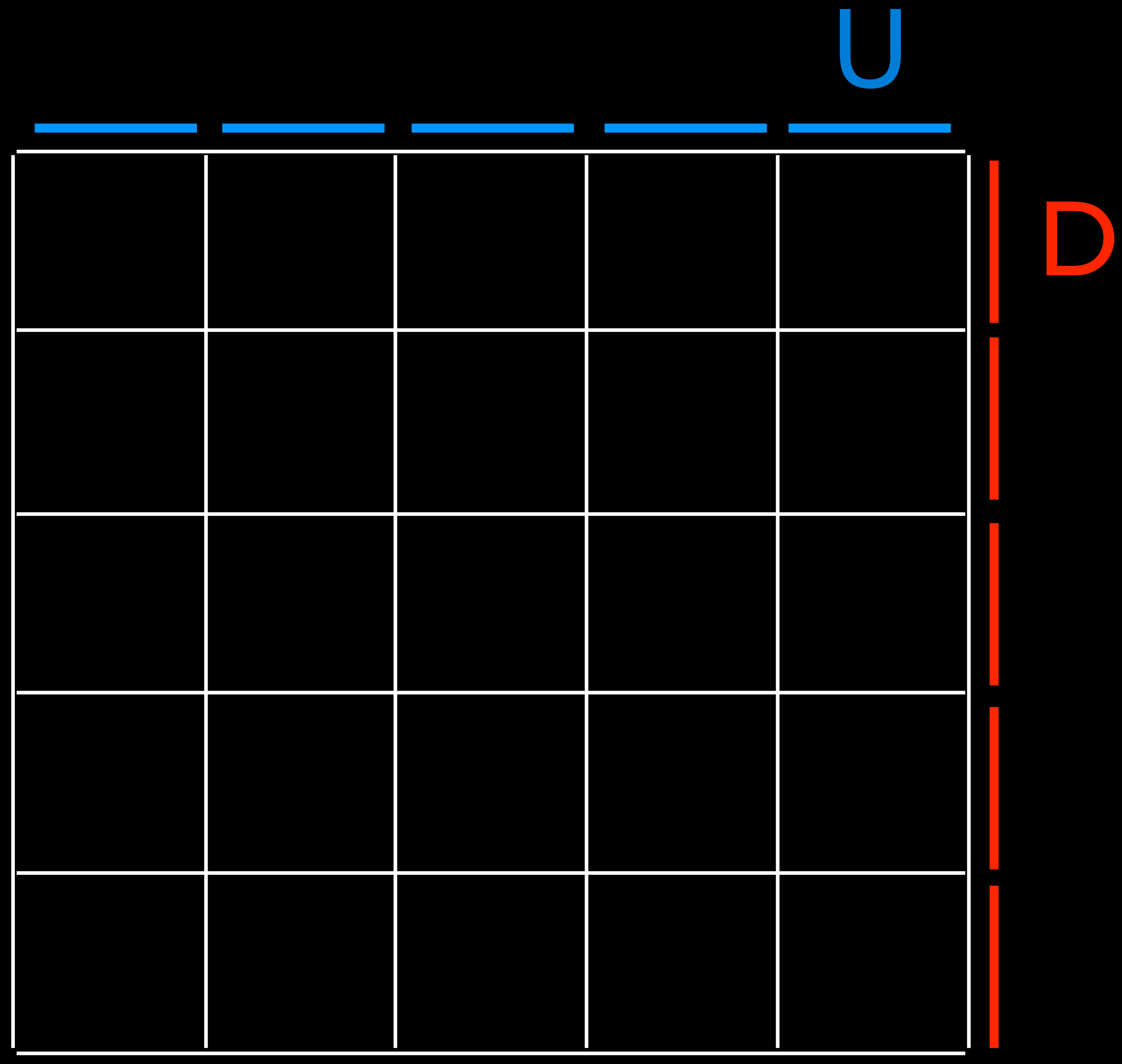
4 relations

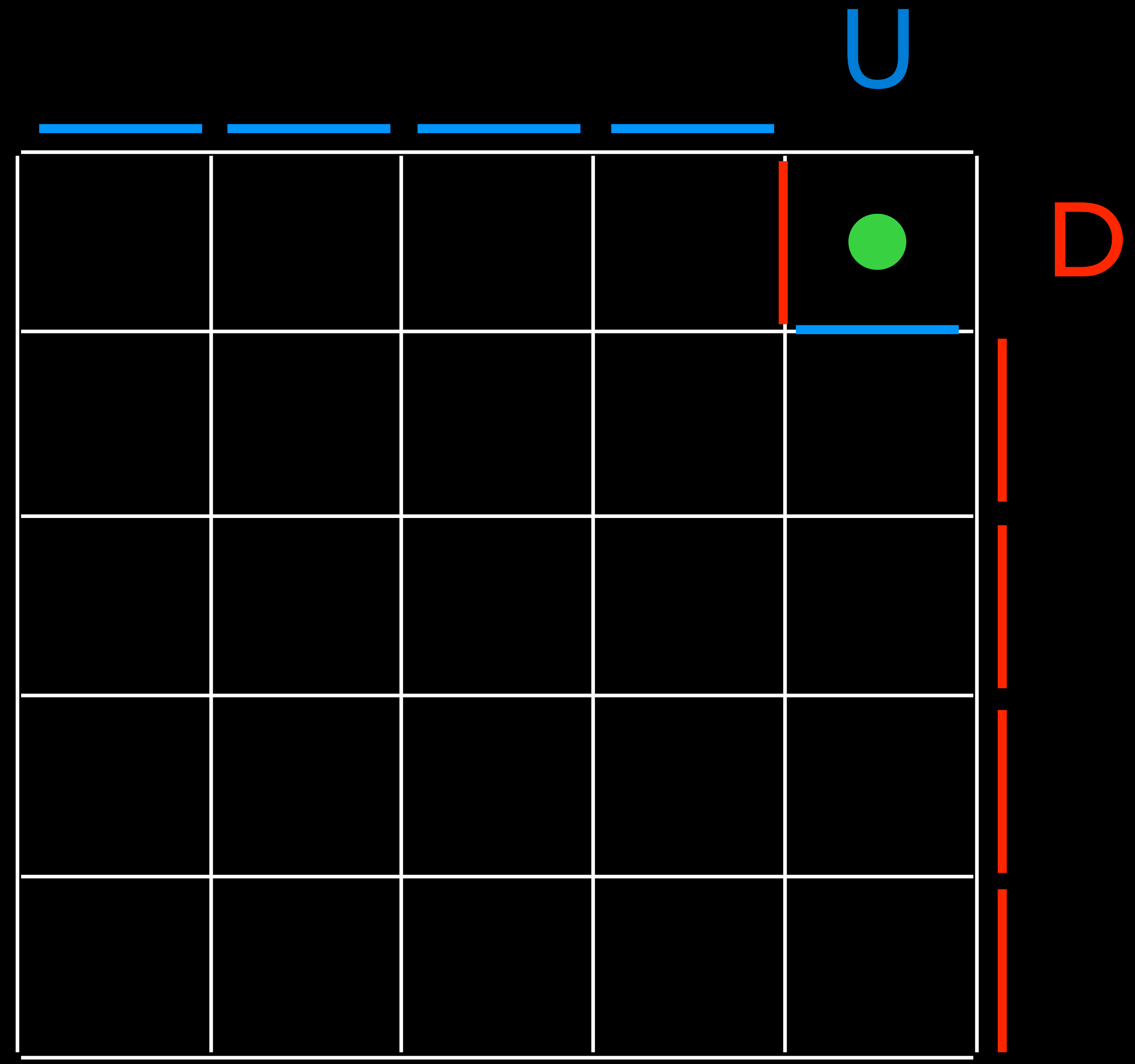
$$\begin{cases} UD \to DU \\ U I_v \to I_v U \\ I_h D \to D I_h \\ I_h I_v \to I_v I_h \end{cases} \qquad UD \to I_v I_h$$

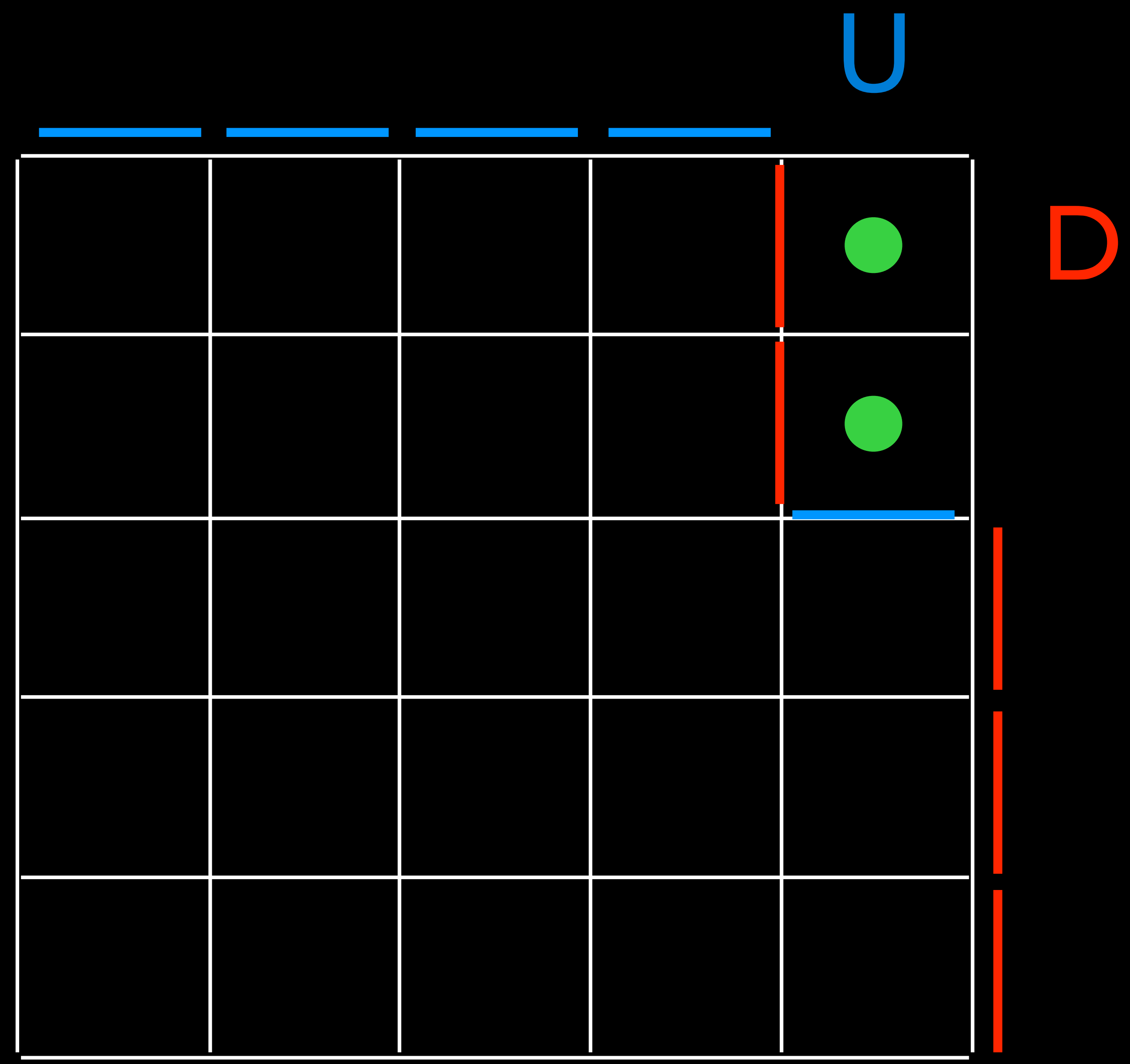
rewriting rules

$$UD = qDU + I$$




U
D


U
D


U
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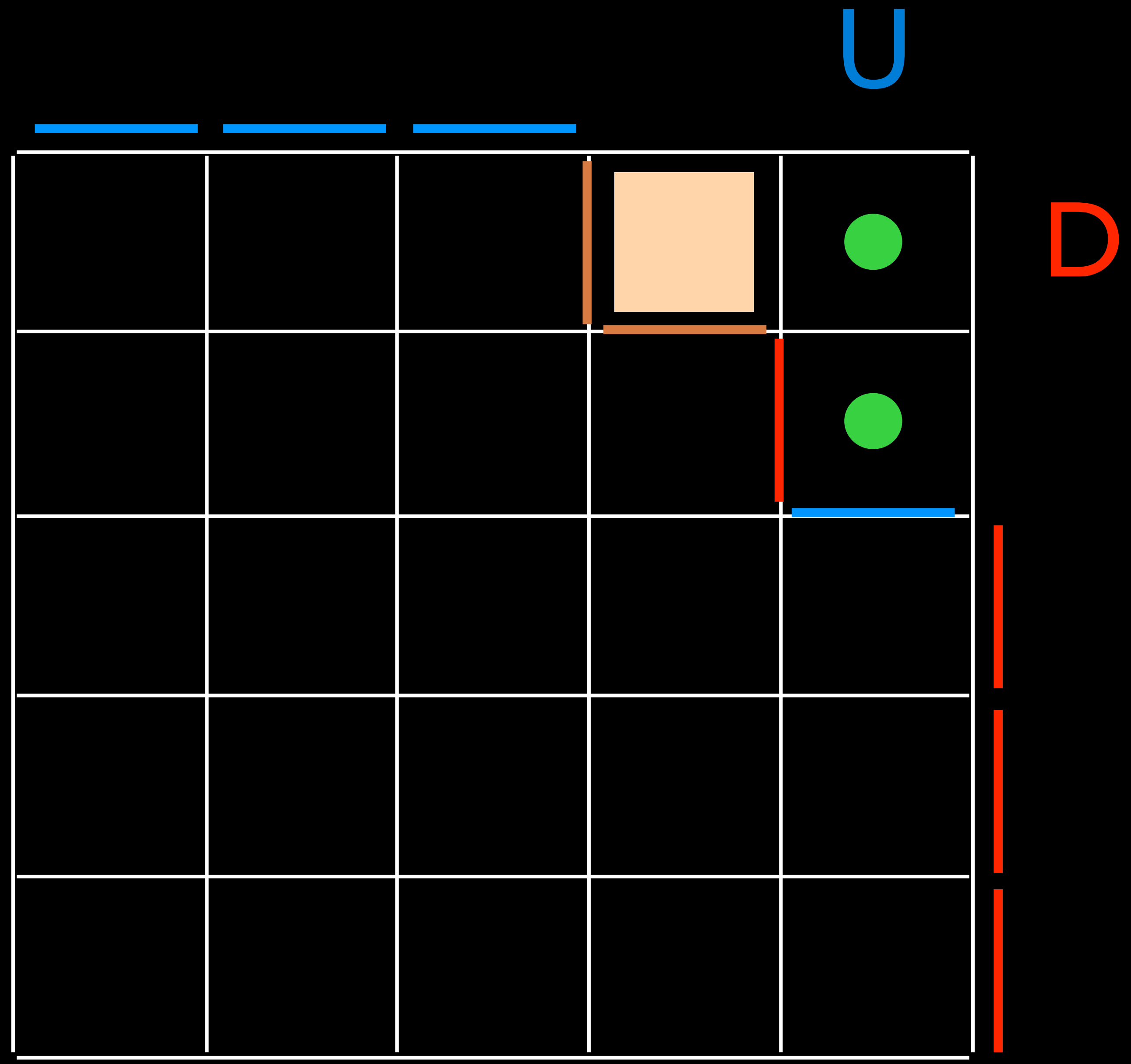
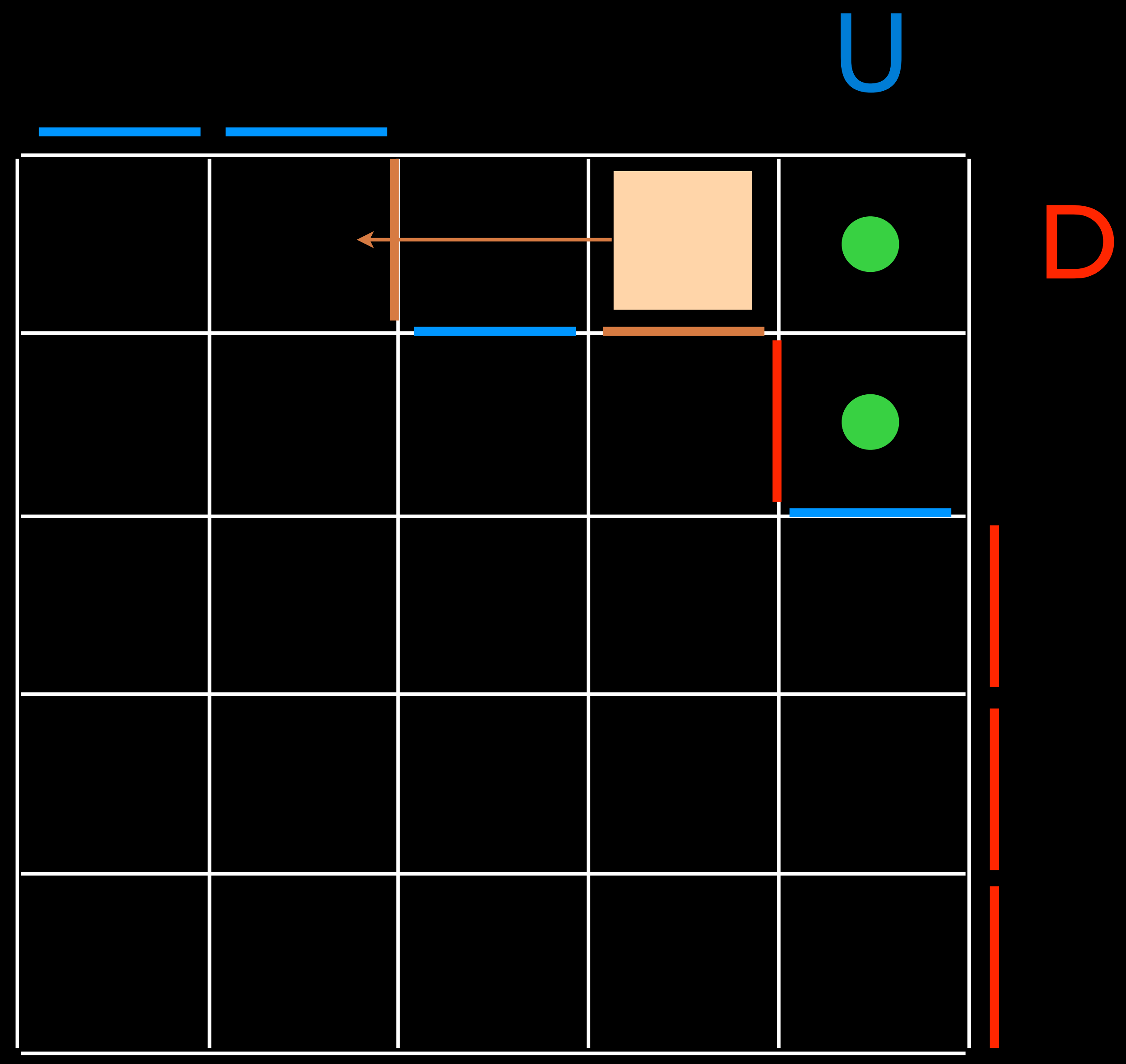
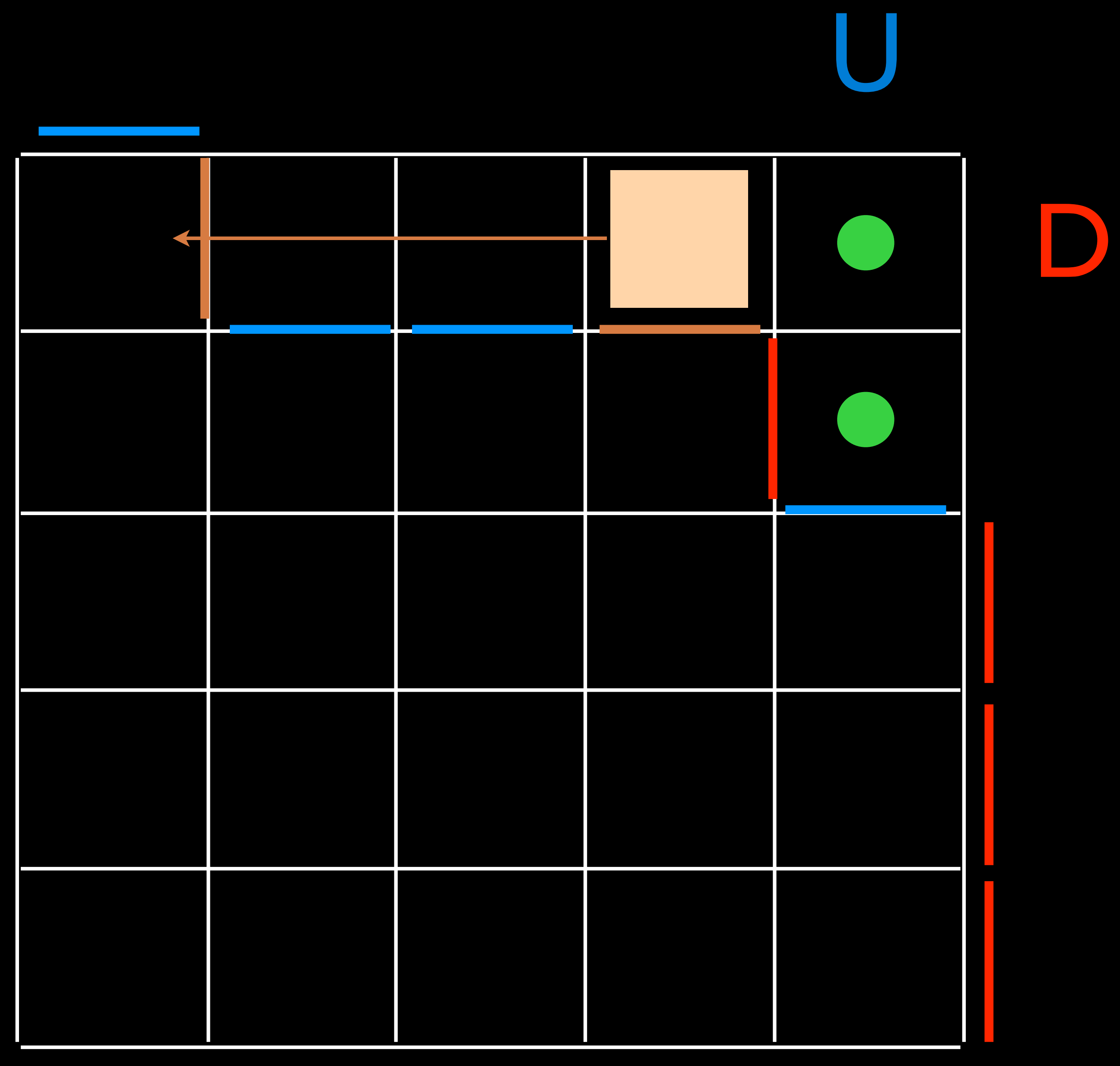
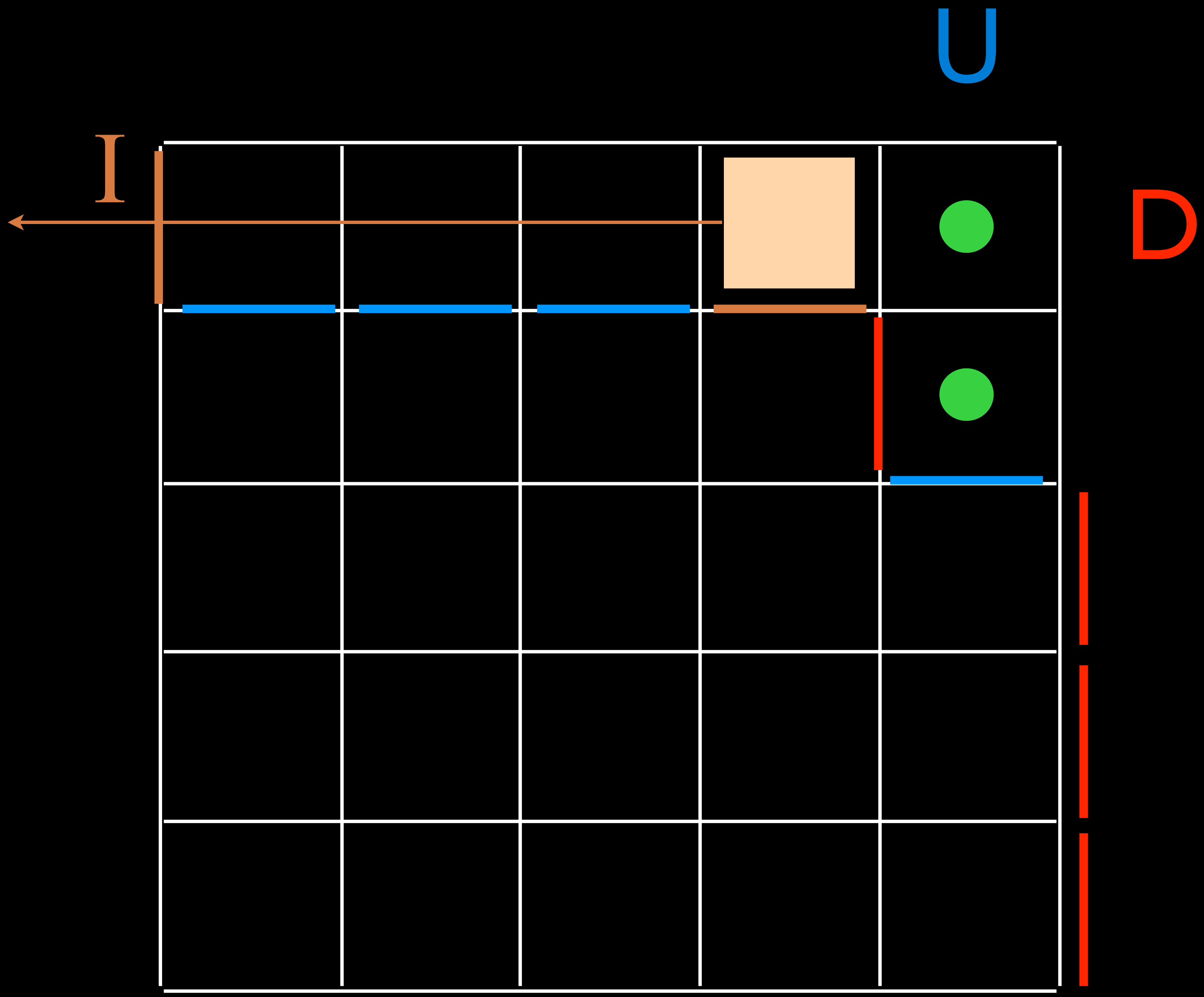

U
D


U
D


U
D


U
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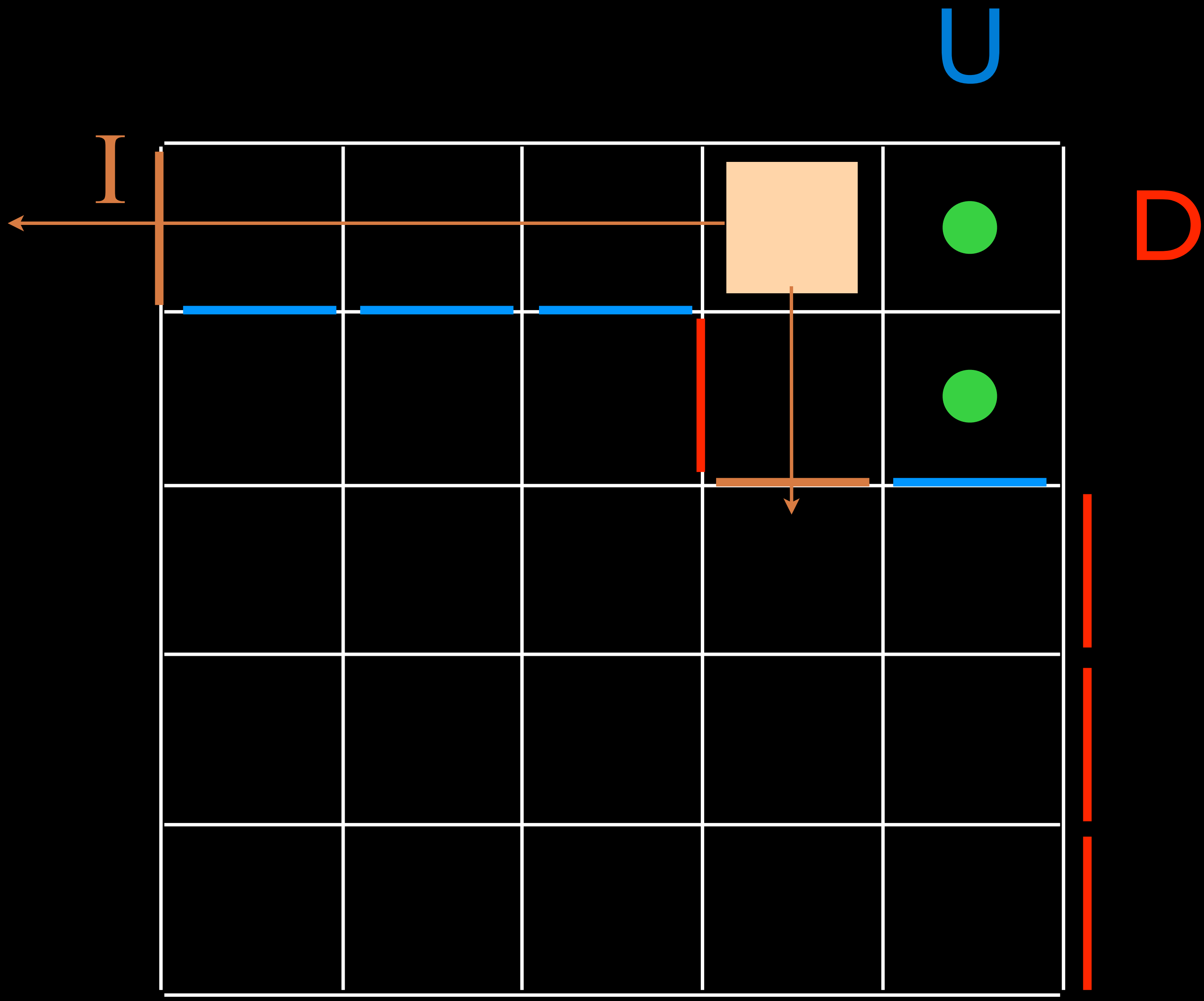
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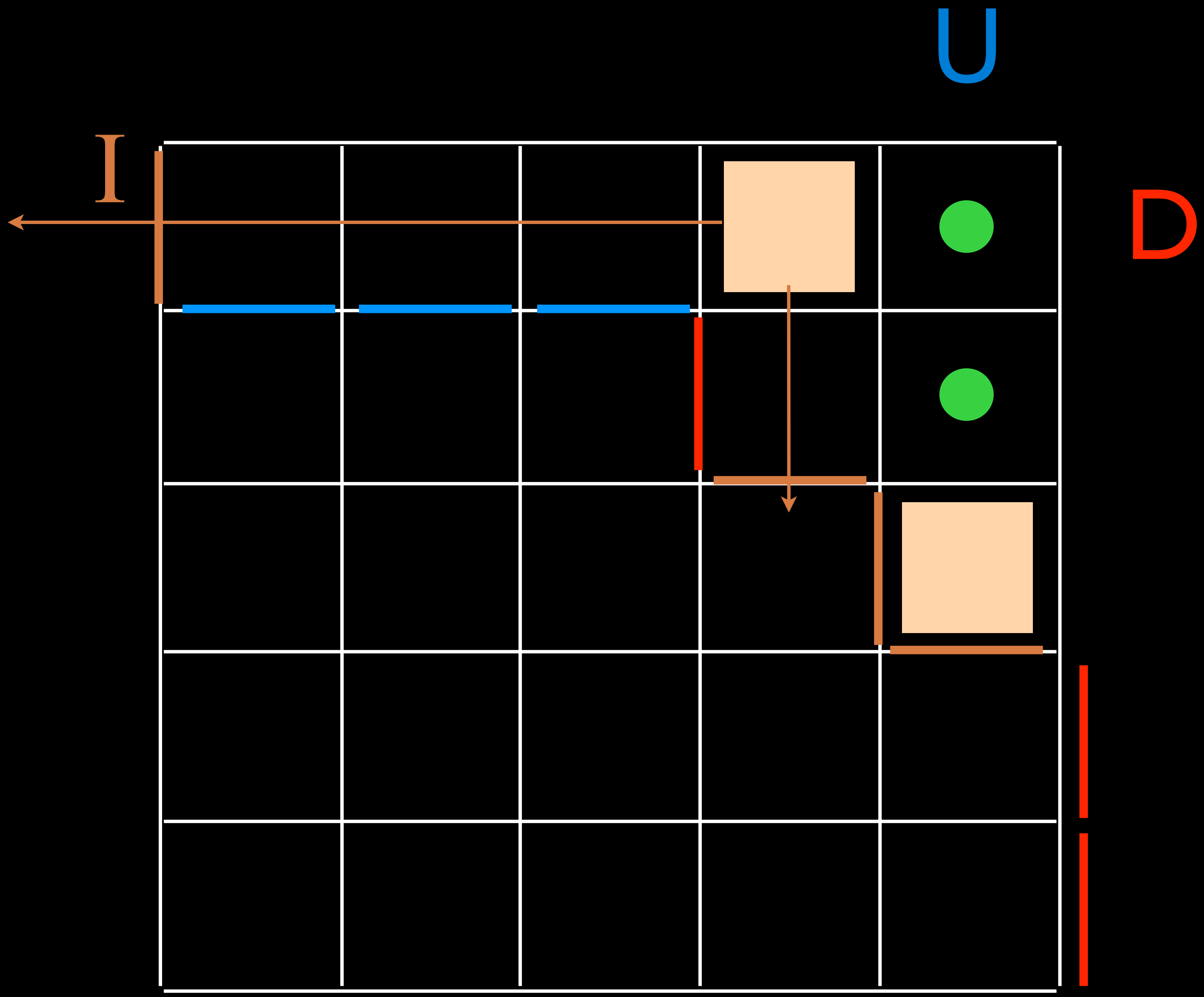
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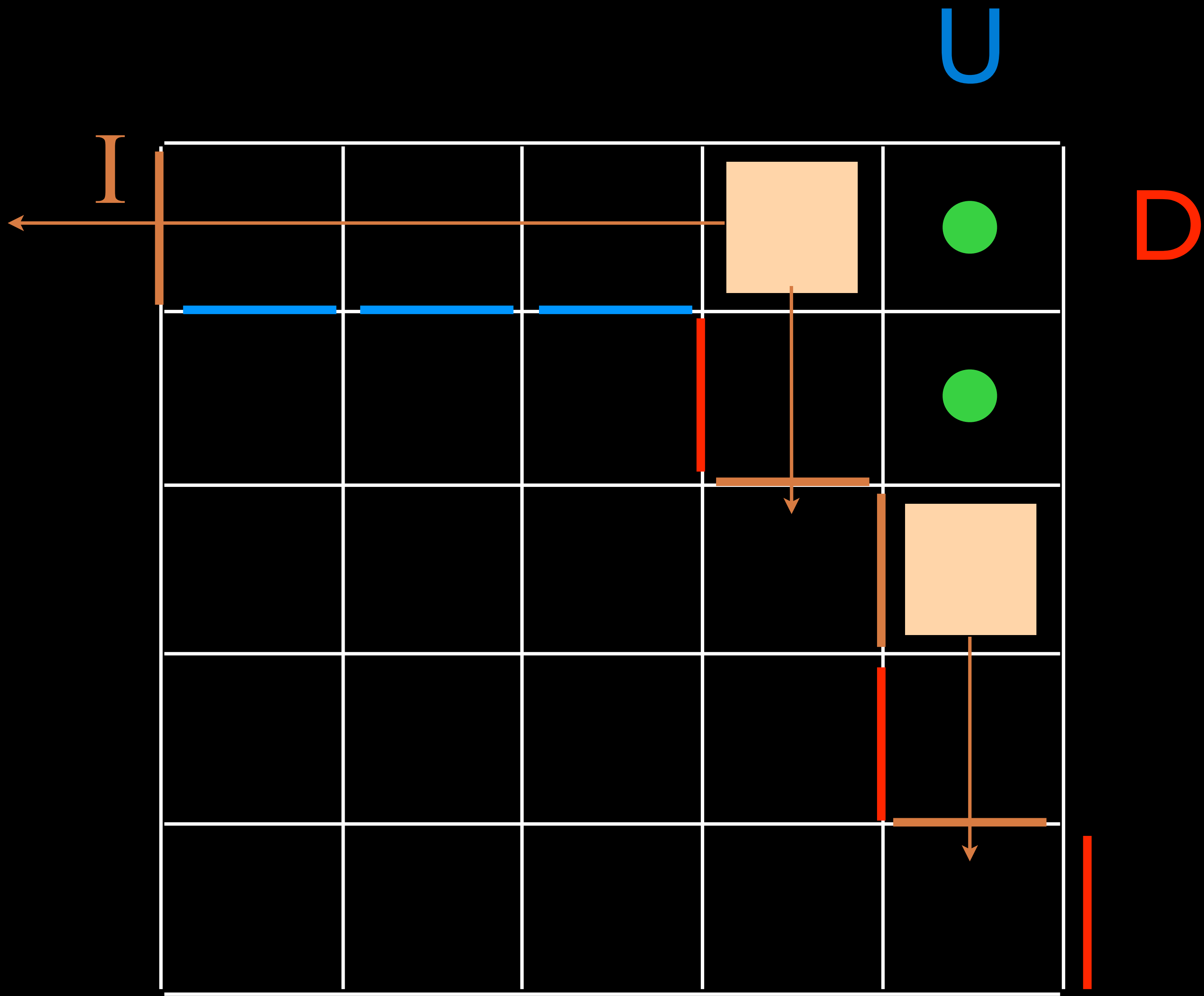
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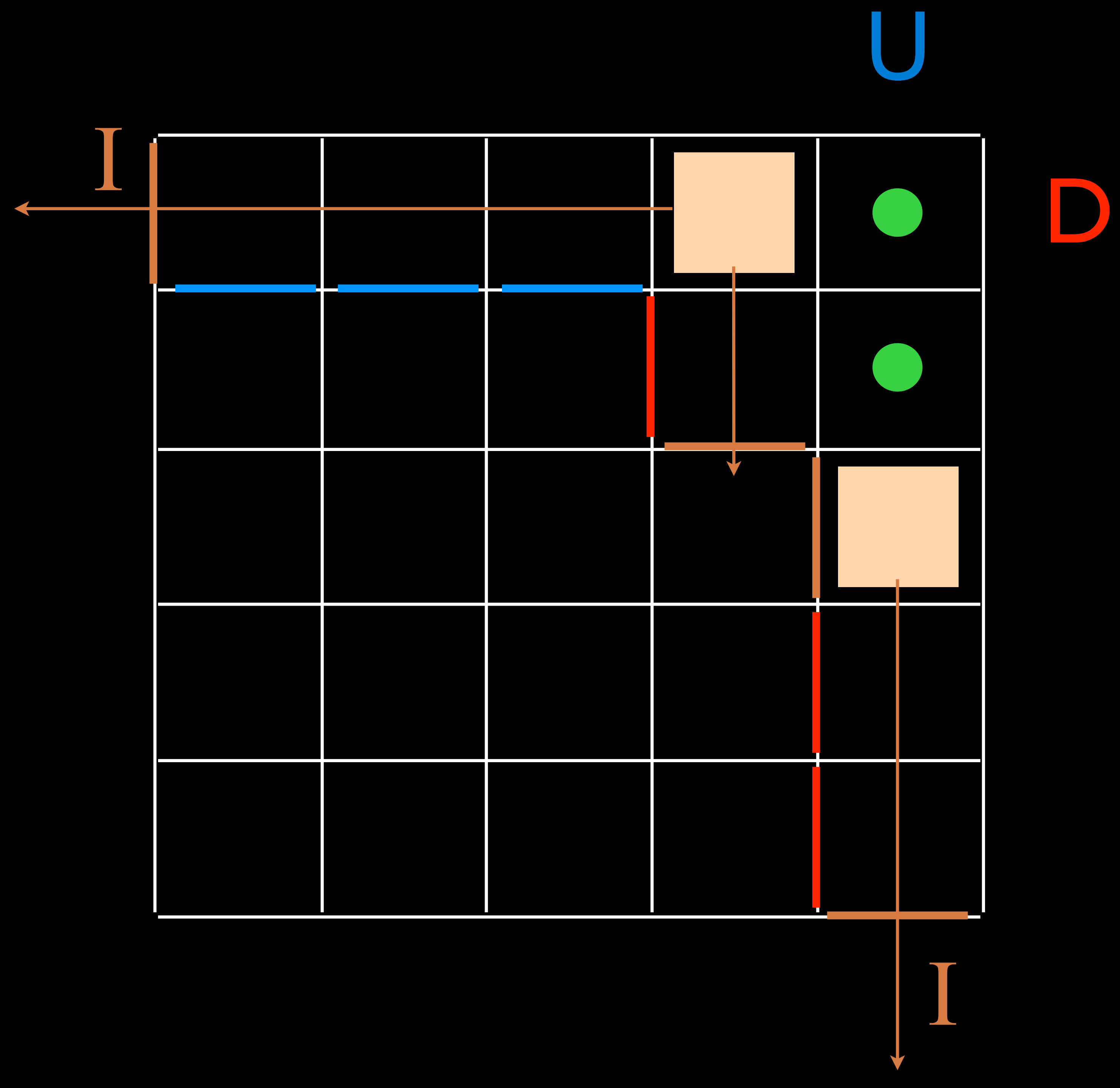
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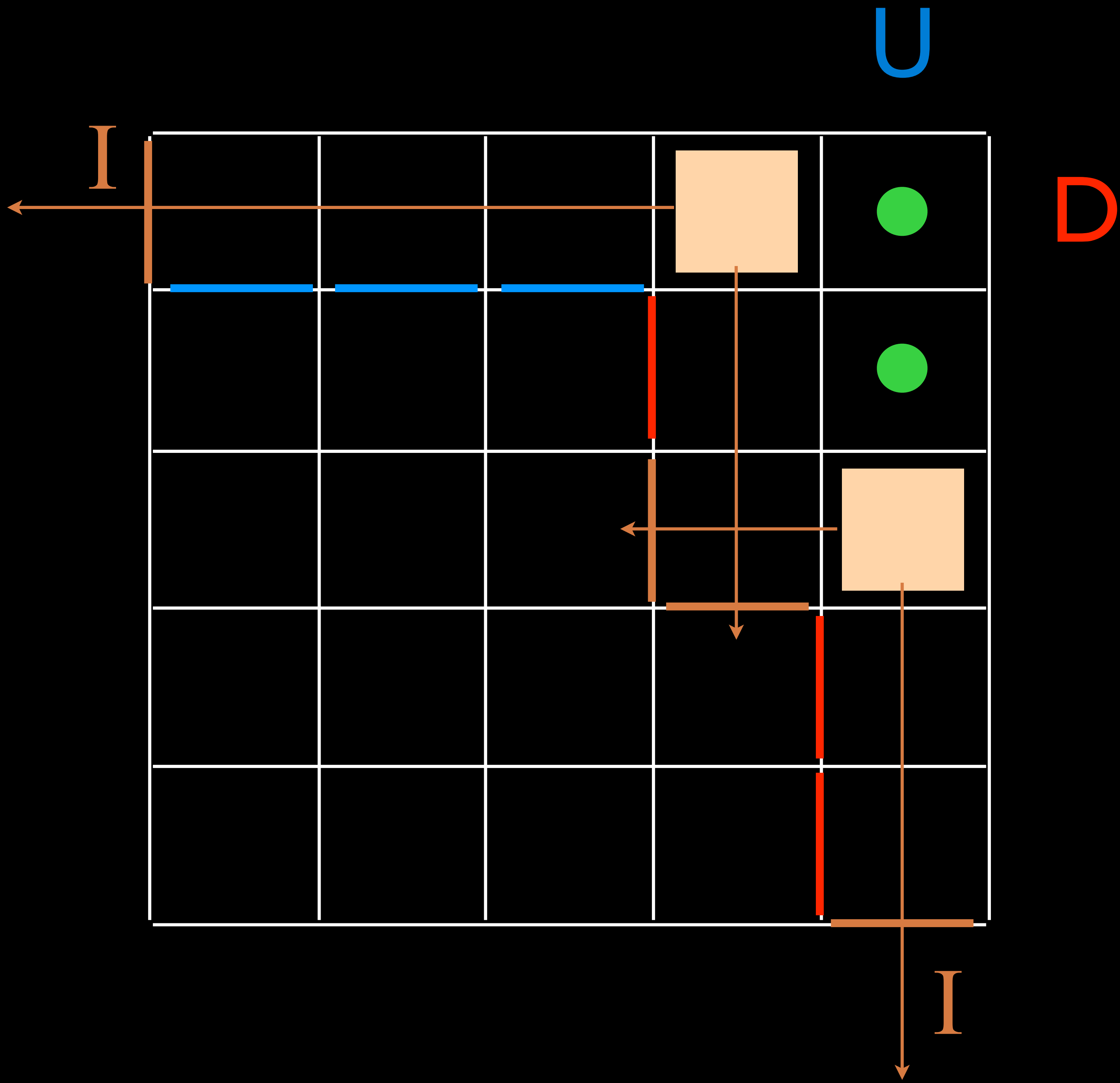
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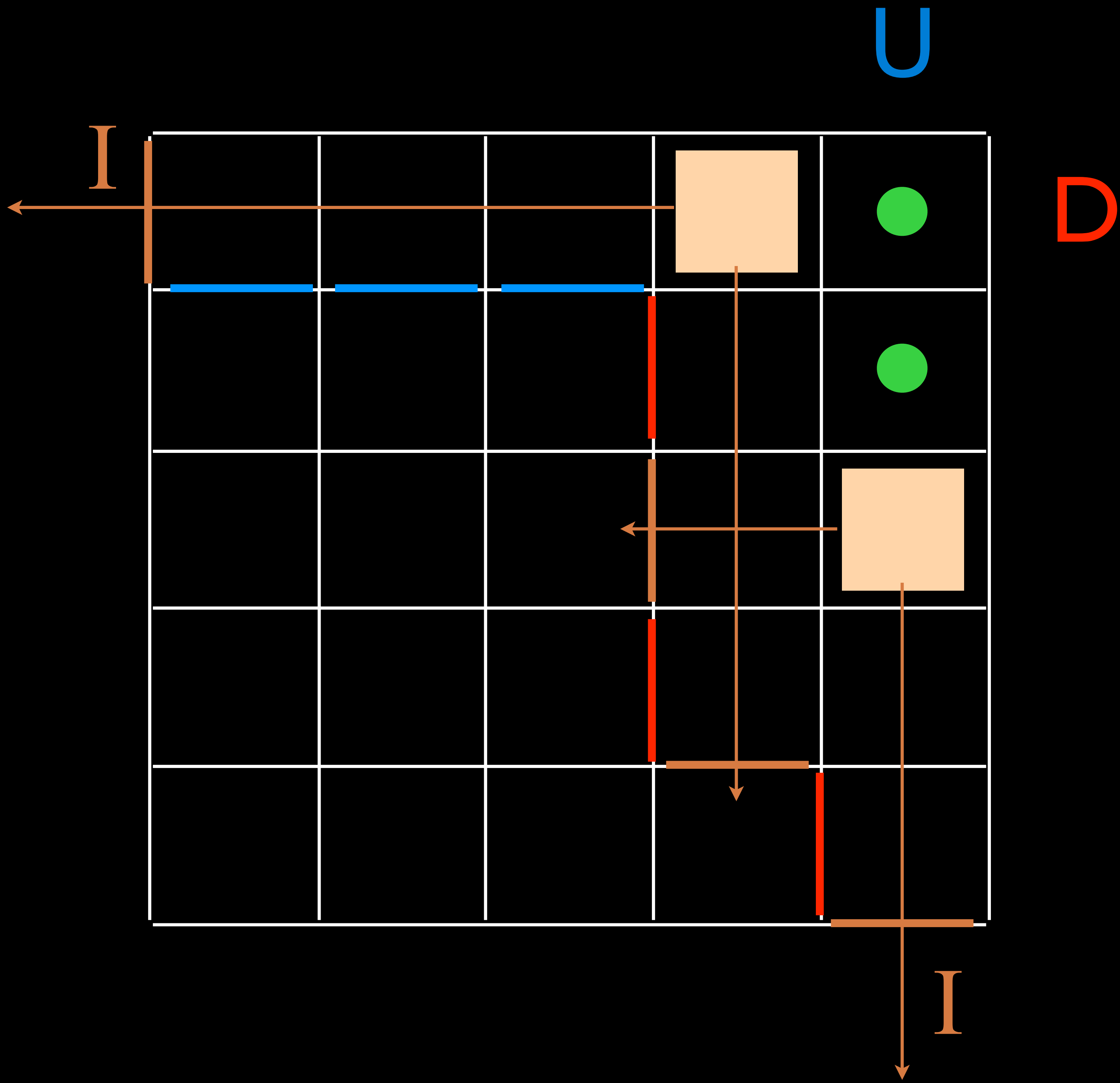
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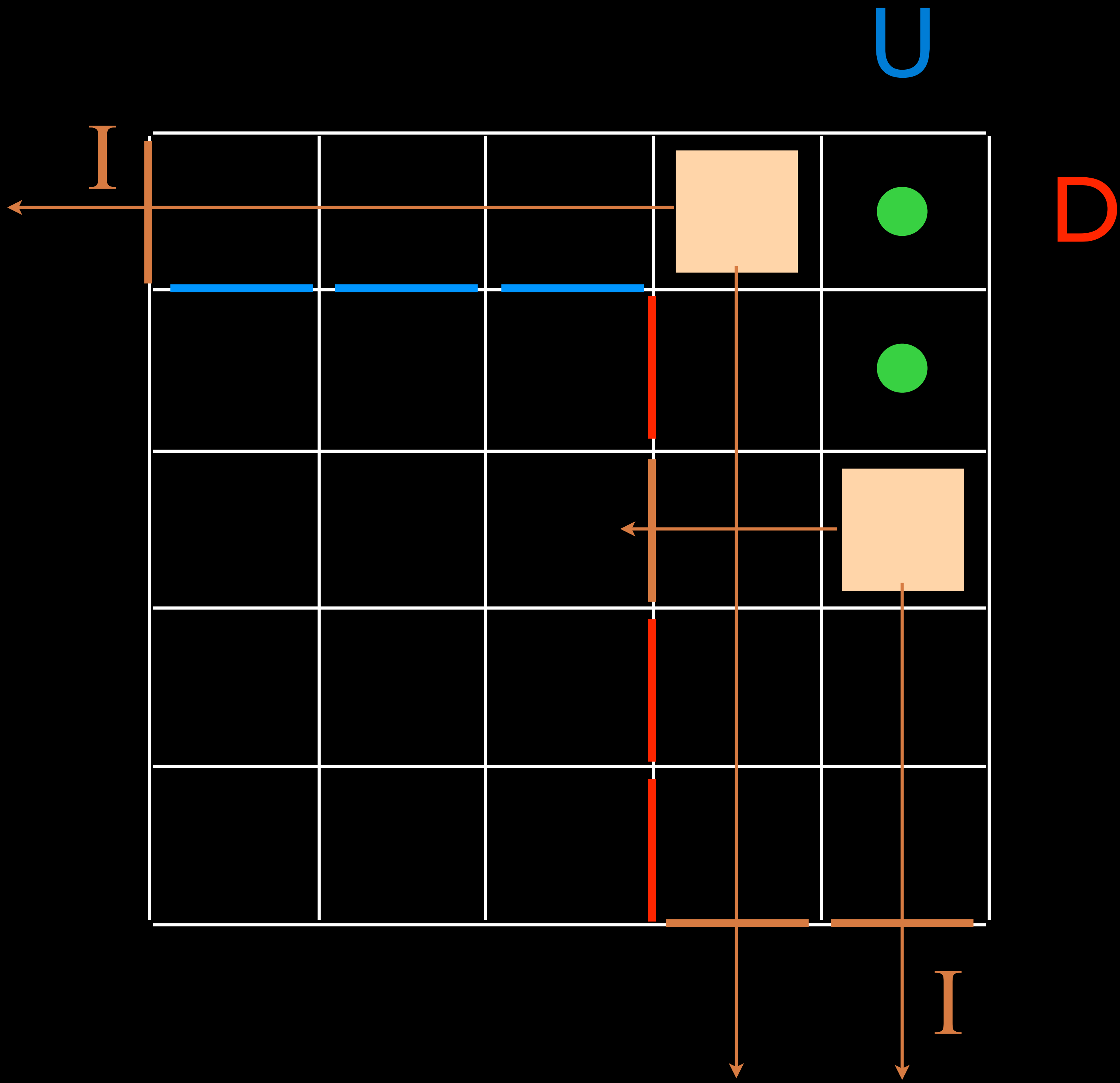

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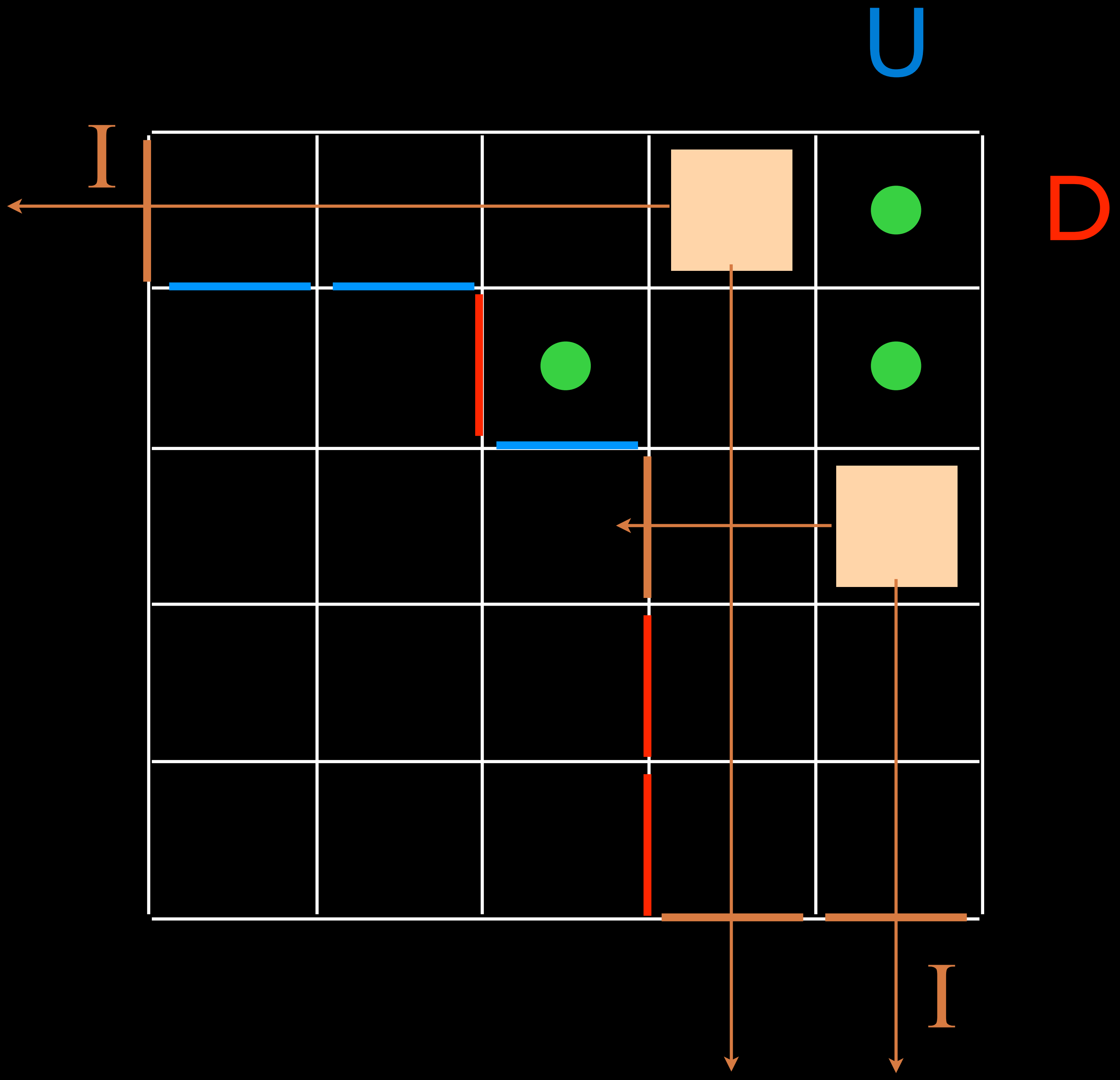

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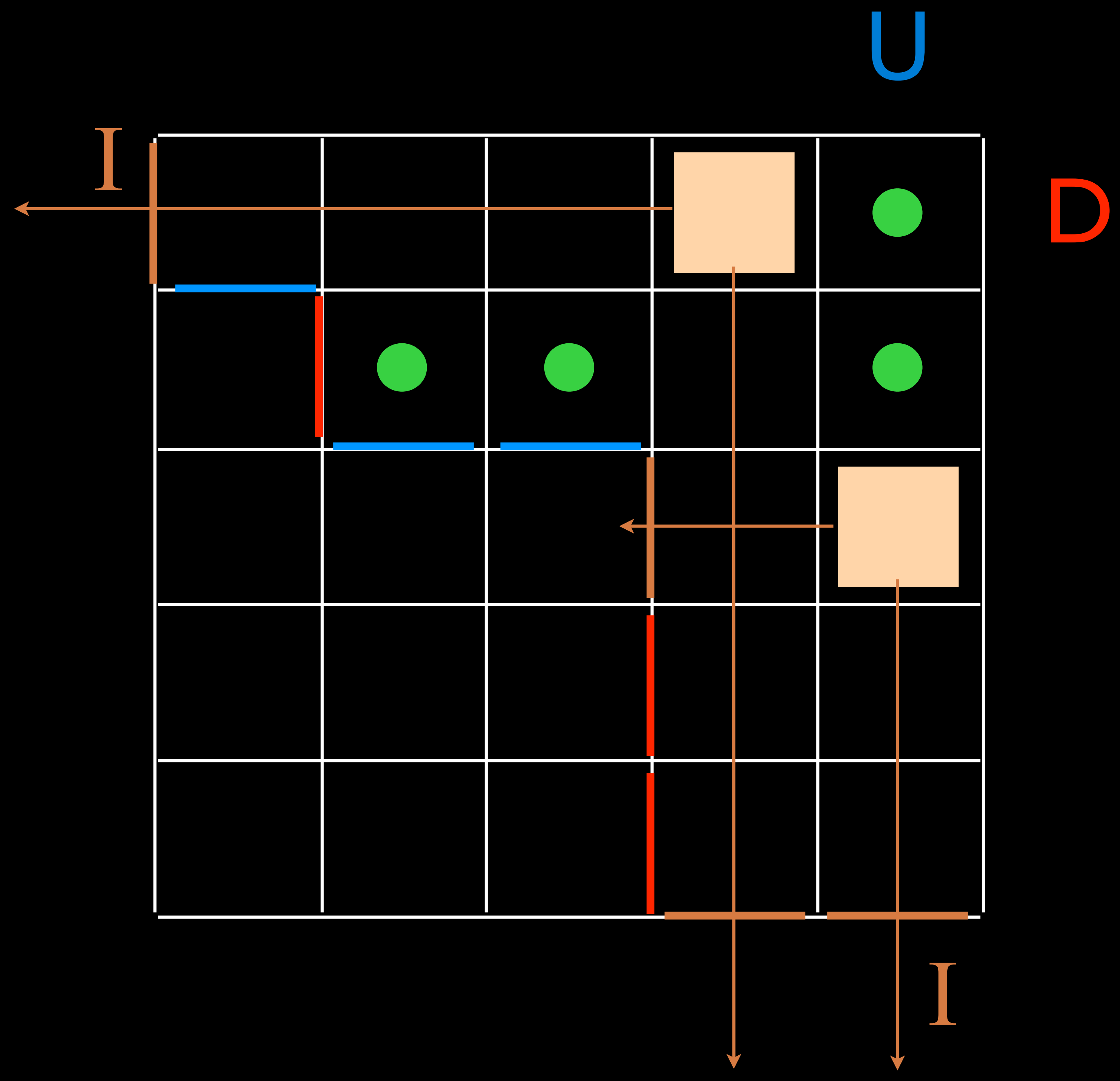
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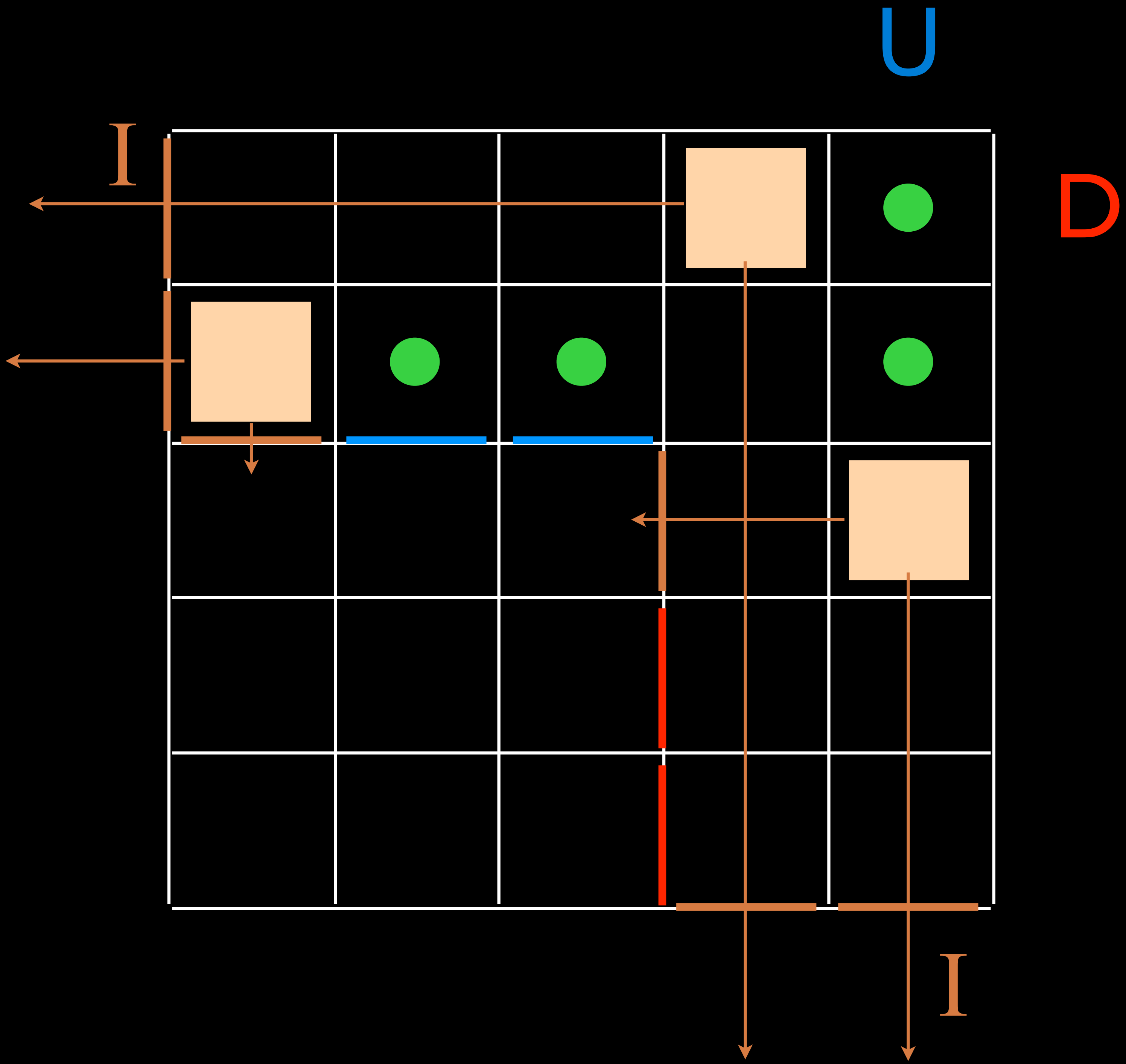
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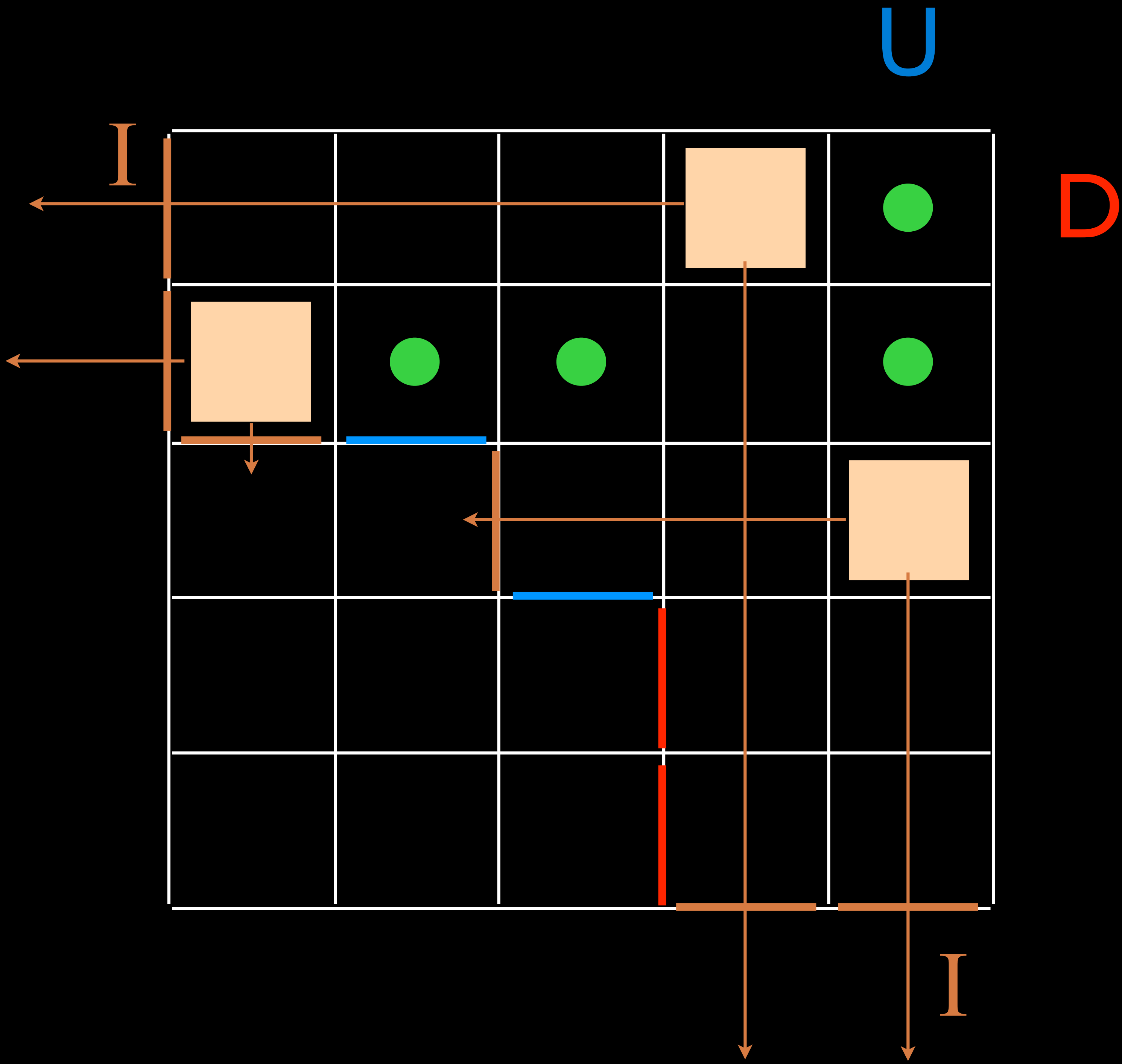
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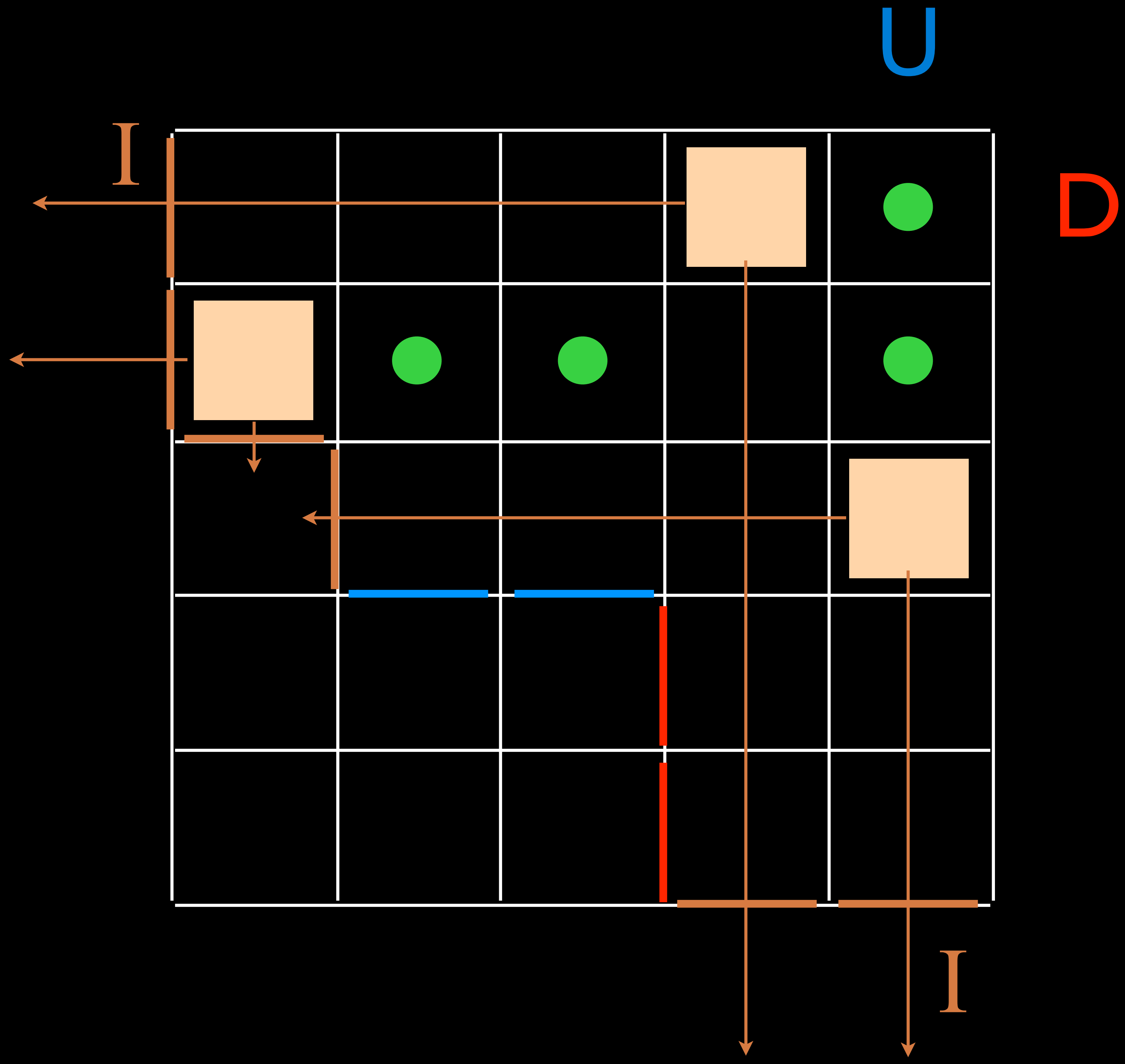
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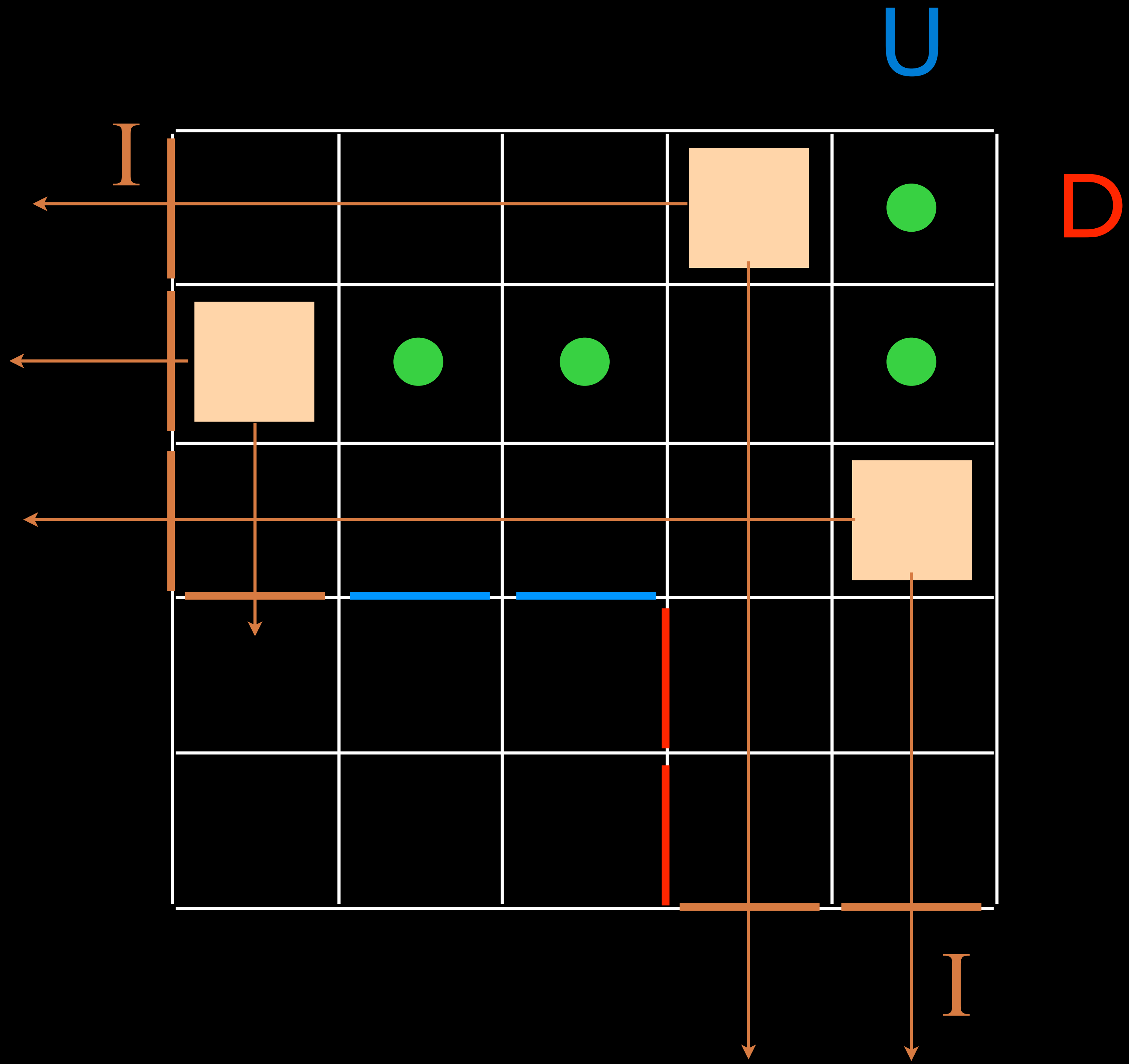
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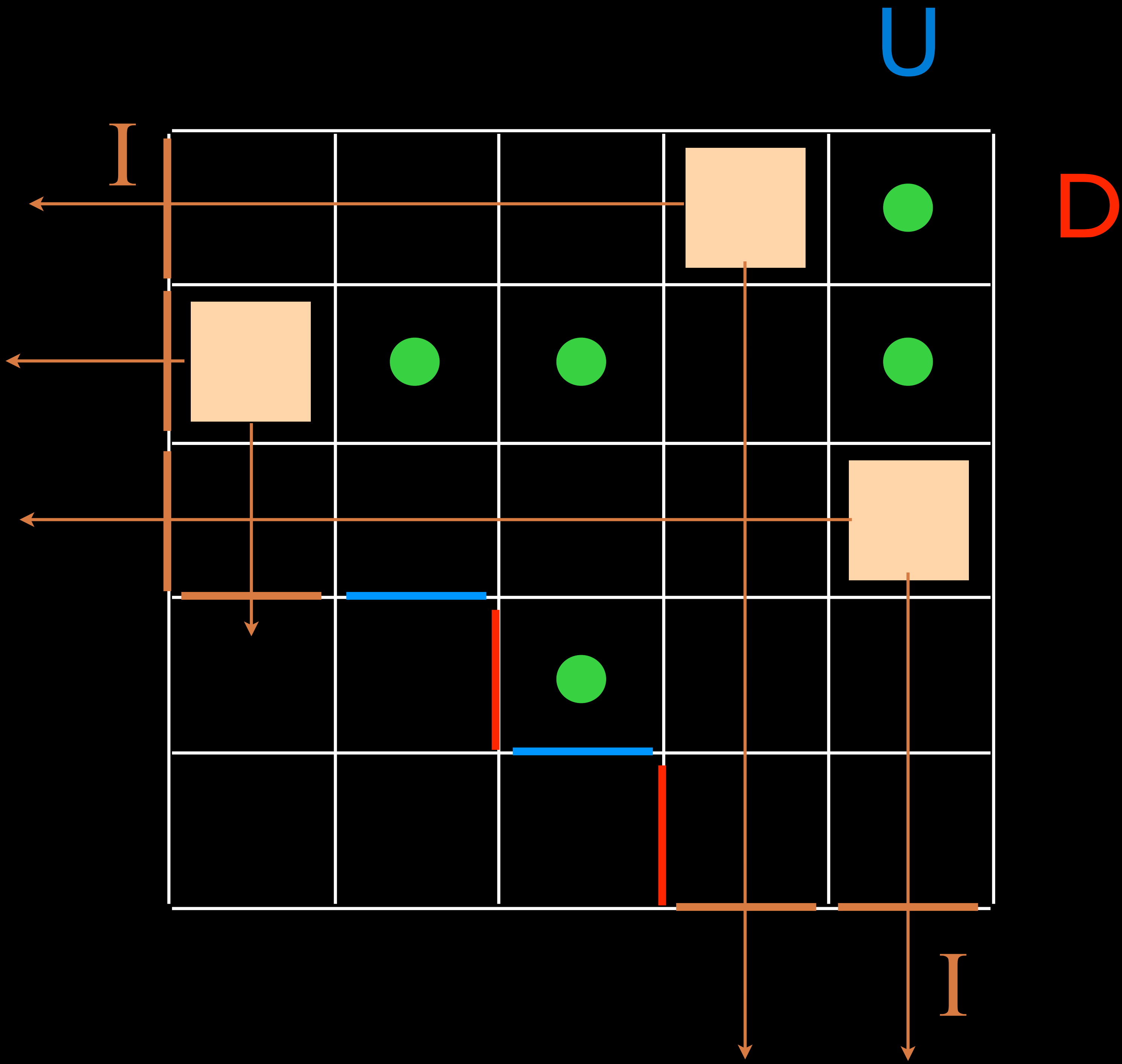

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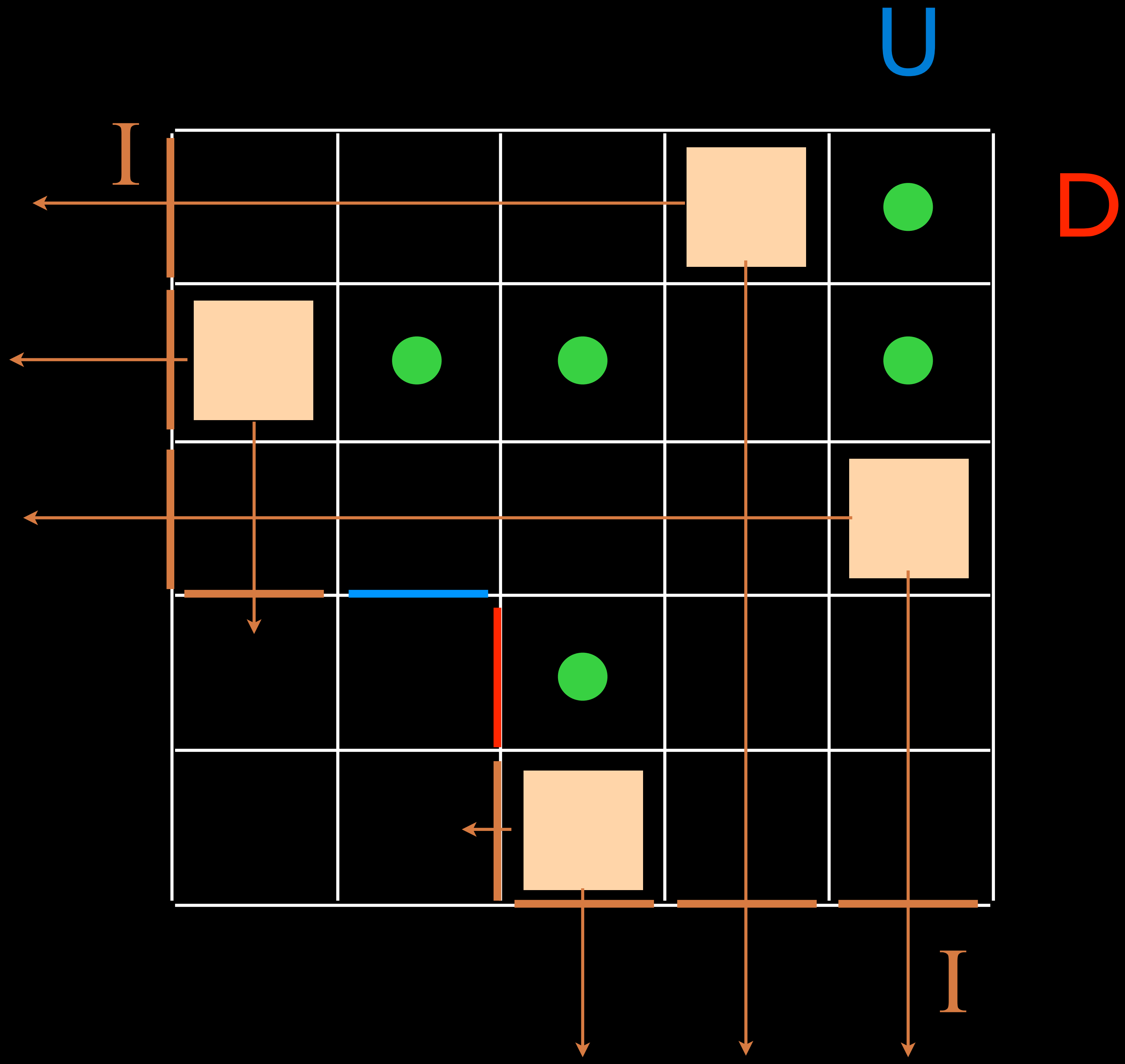
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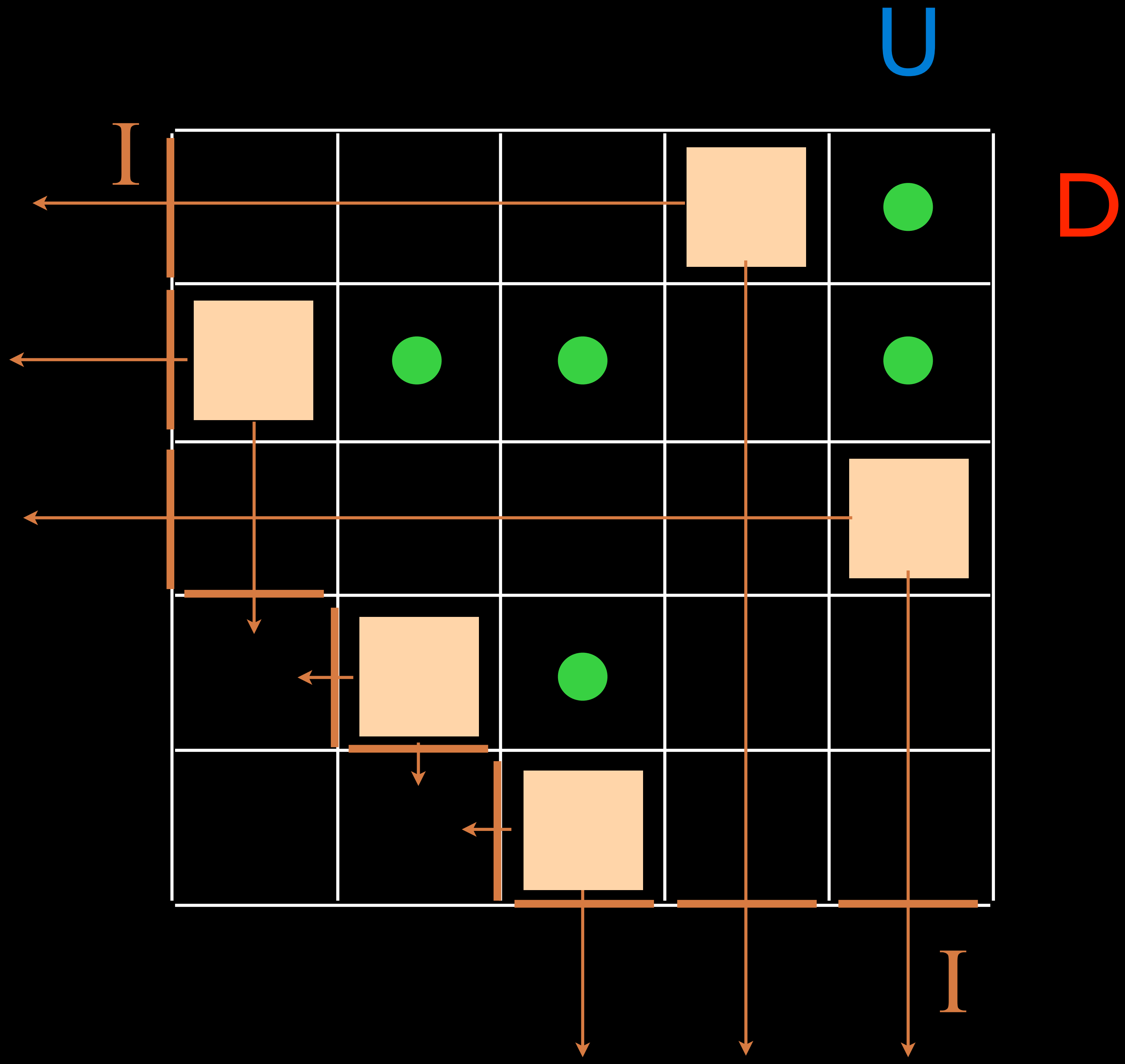

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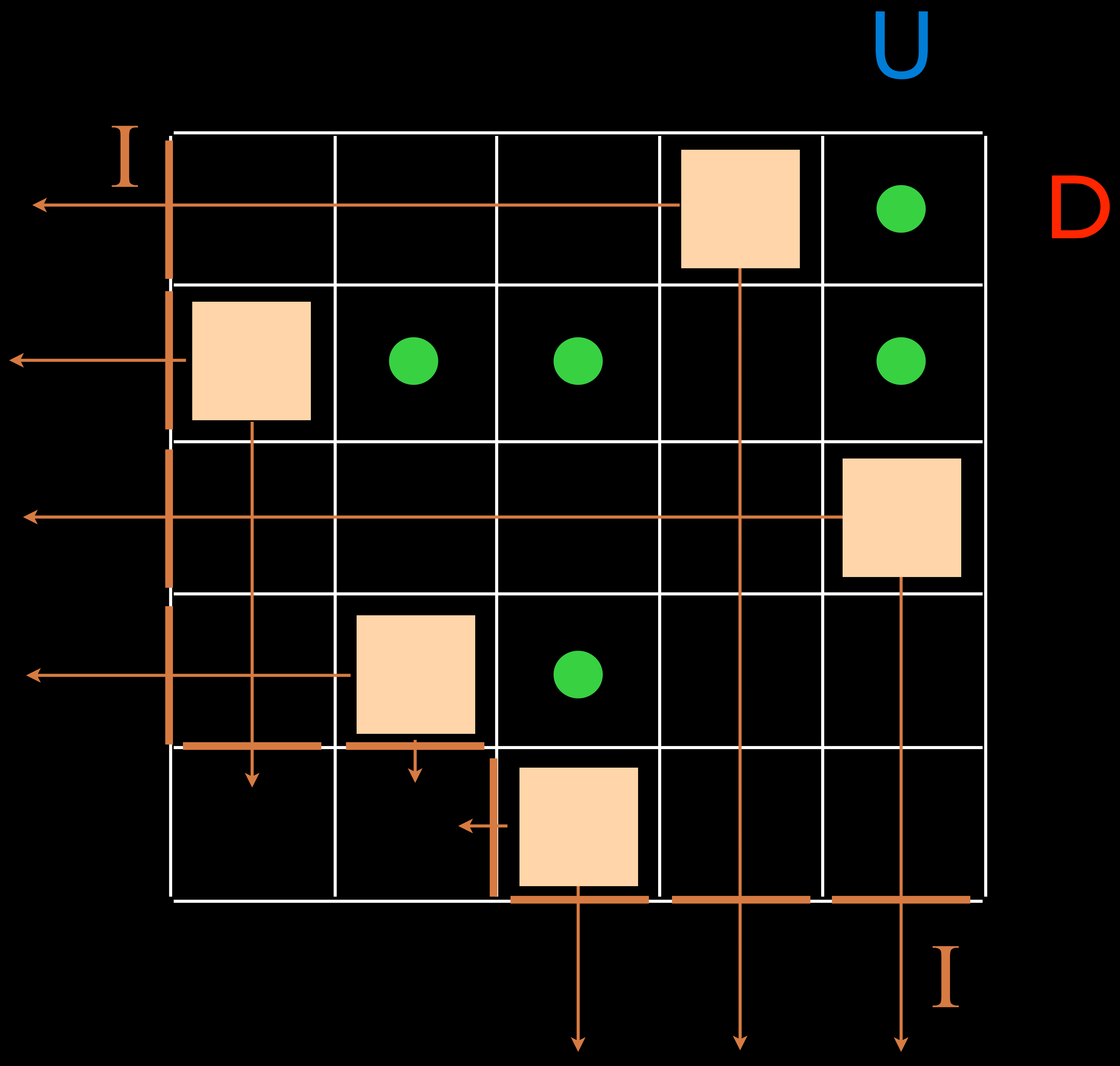
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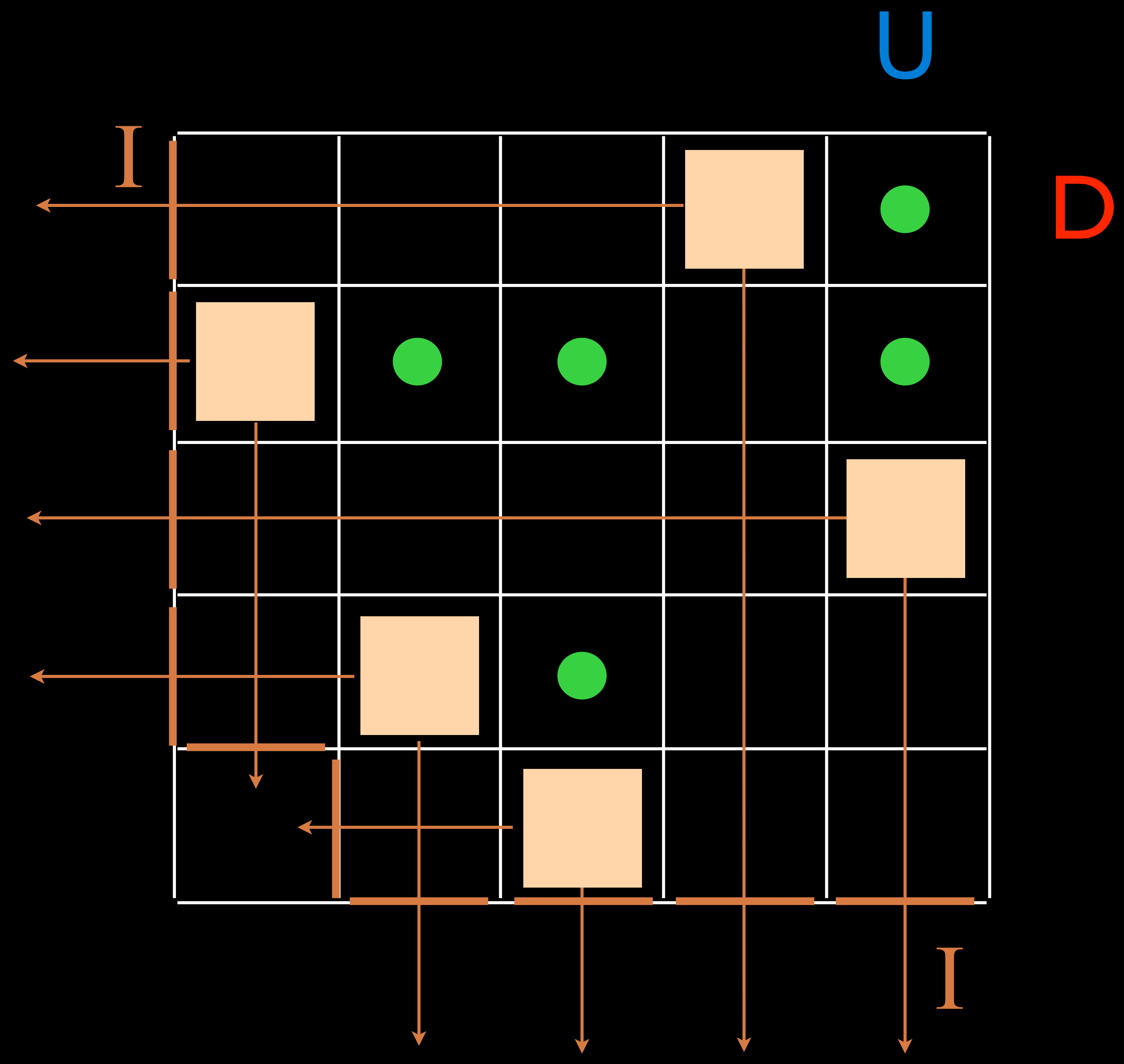
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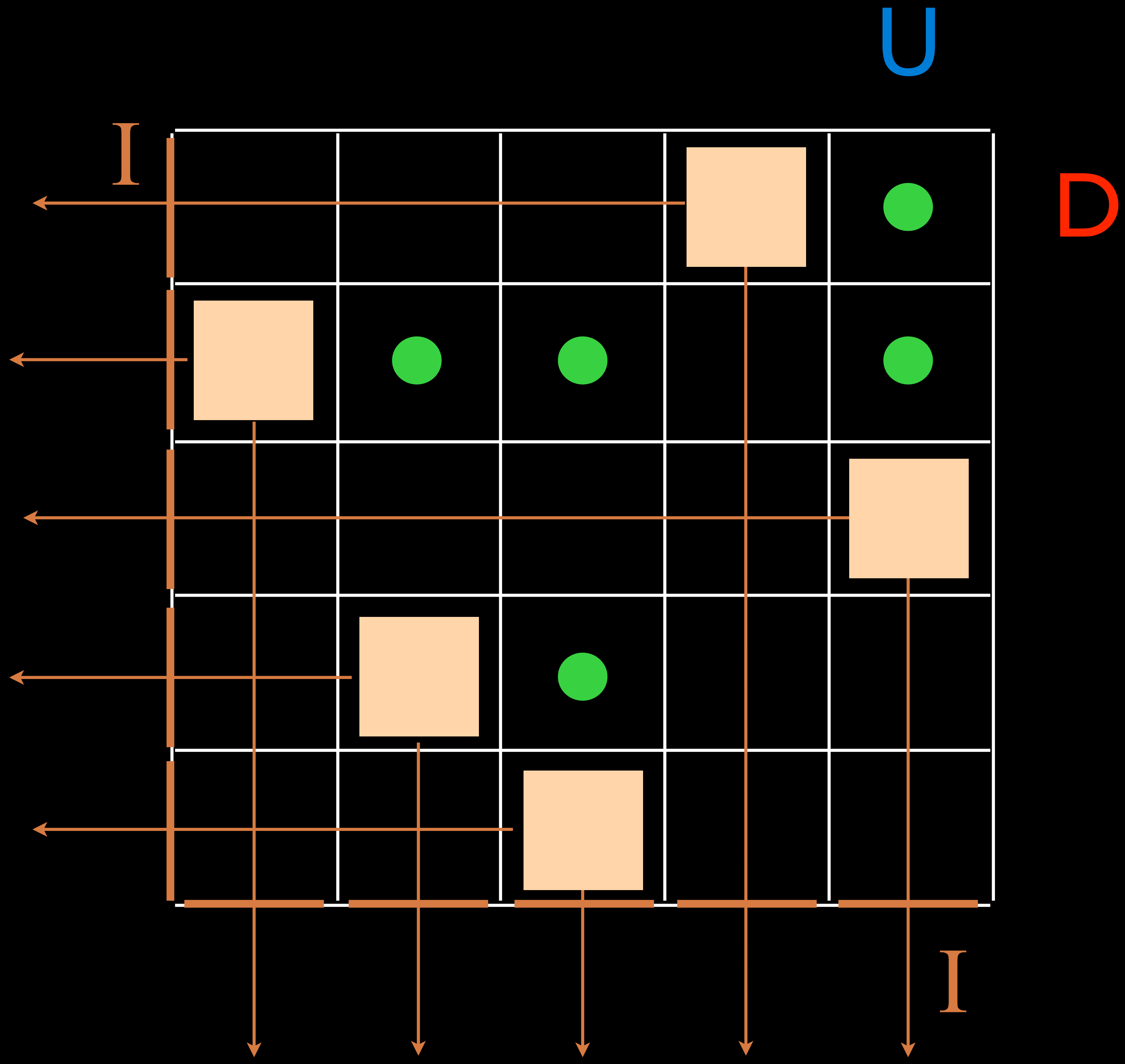
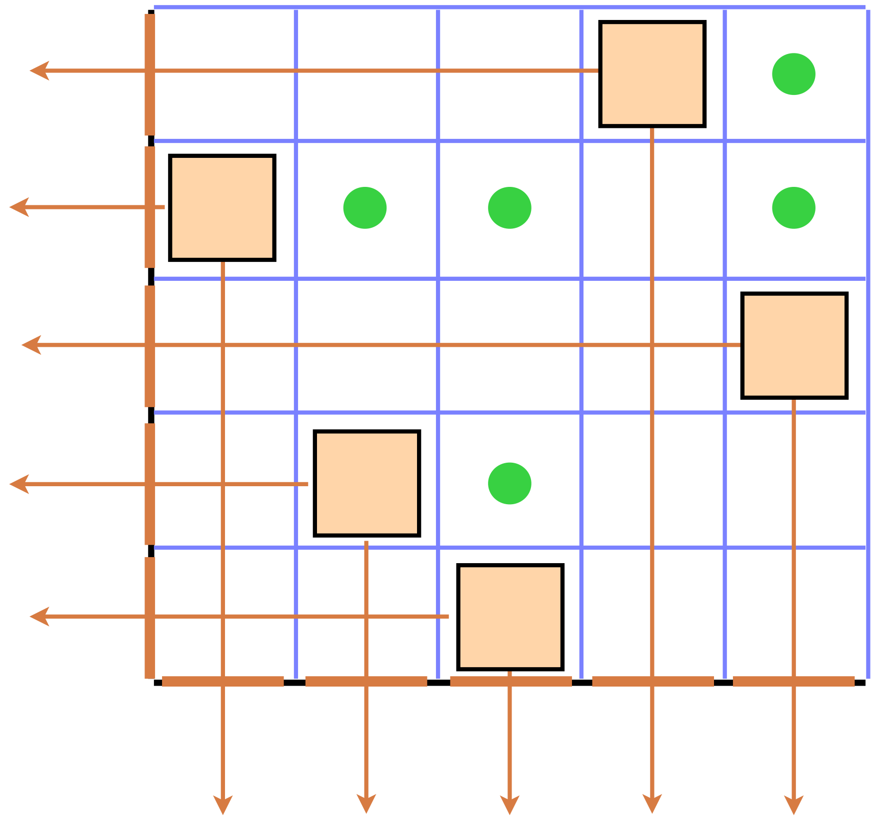


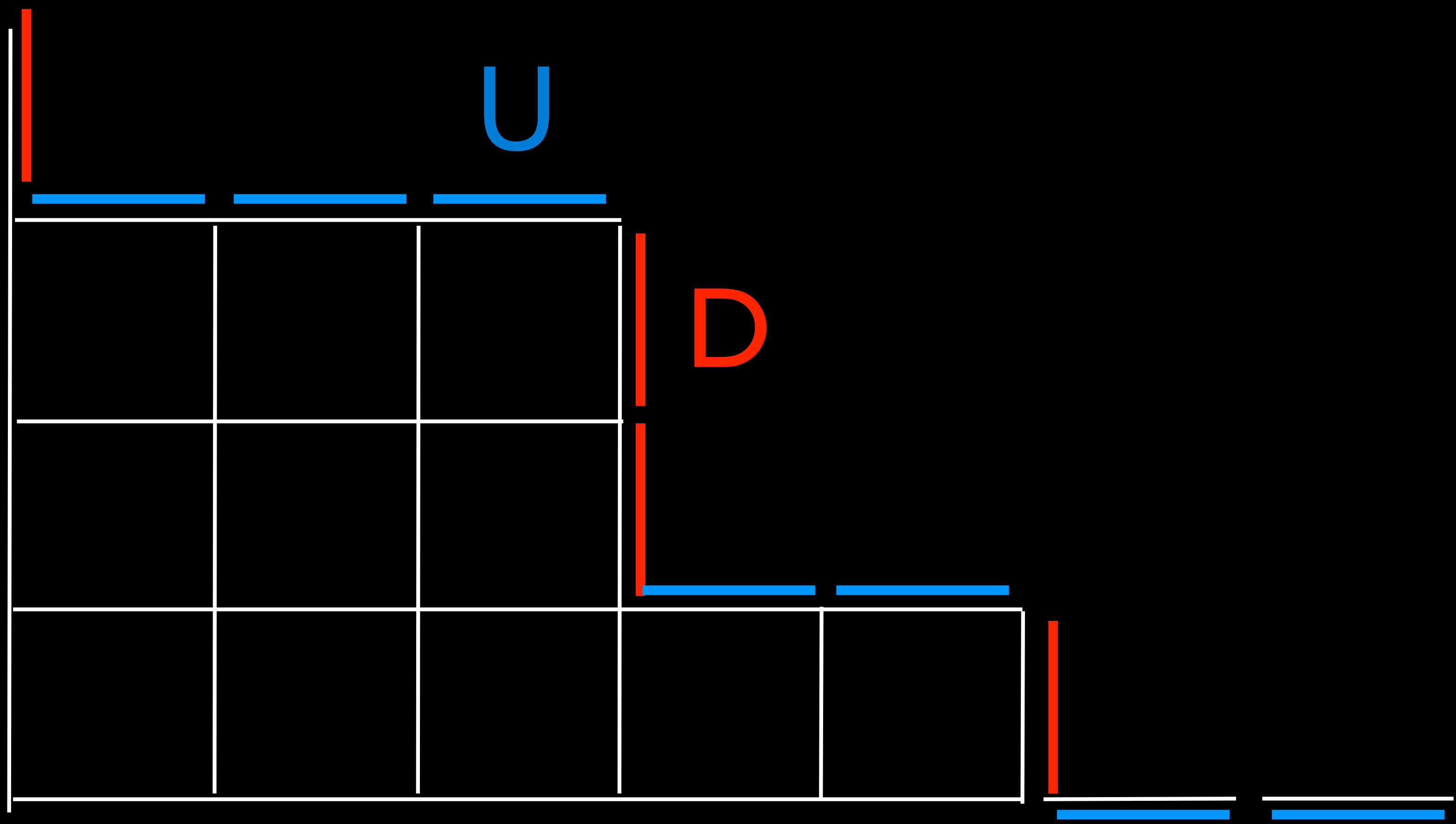
U
I
D
I

$$\begin{cases} UD = qDU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

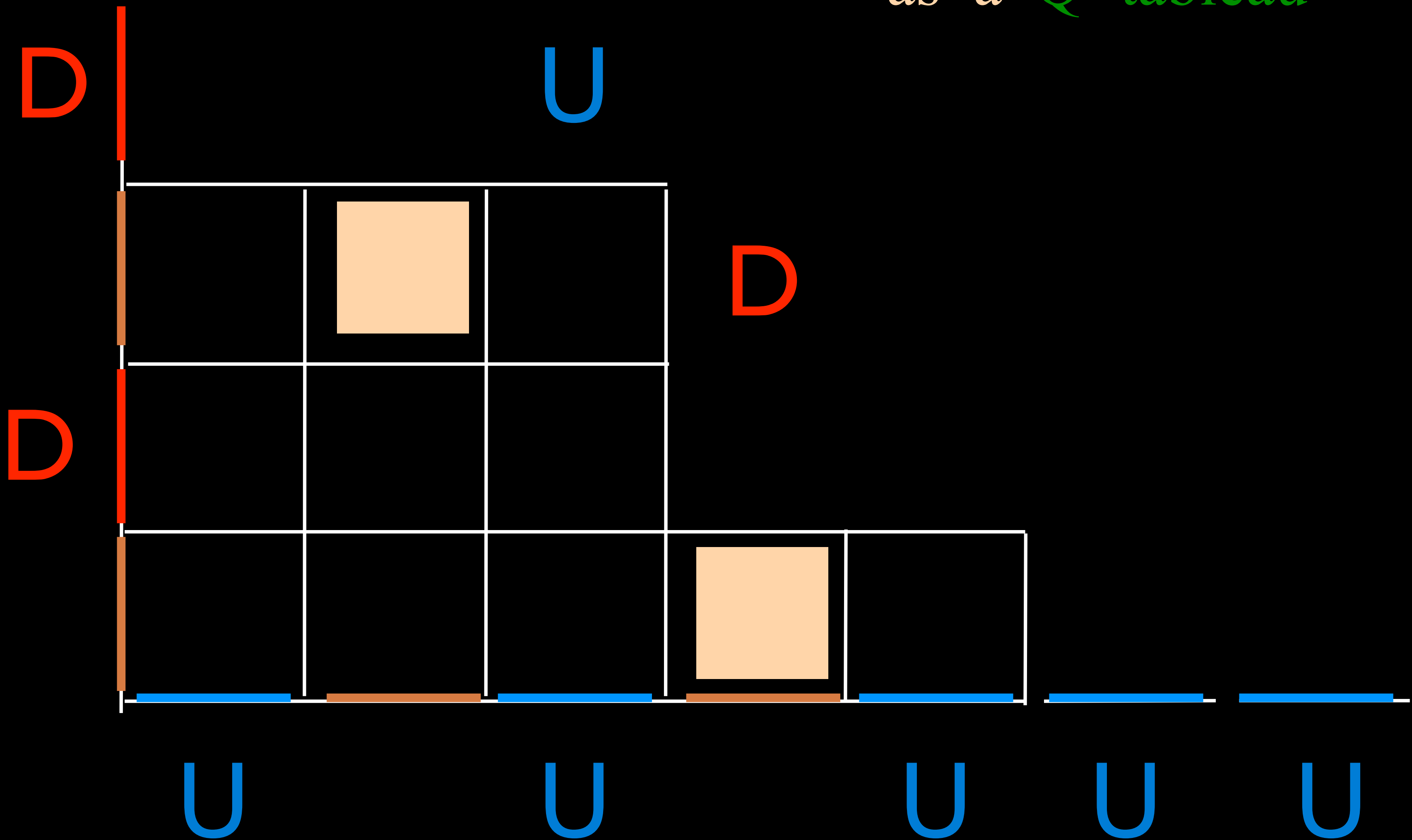
for a quadratic algebra Q
we will define
in the general theory
the notion of Q-tableaux
and complete Q-tableaux



rook placements


U
D

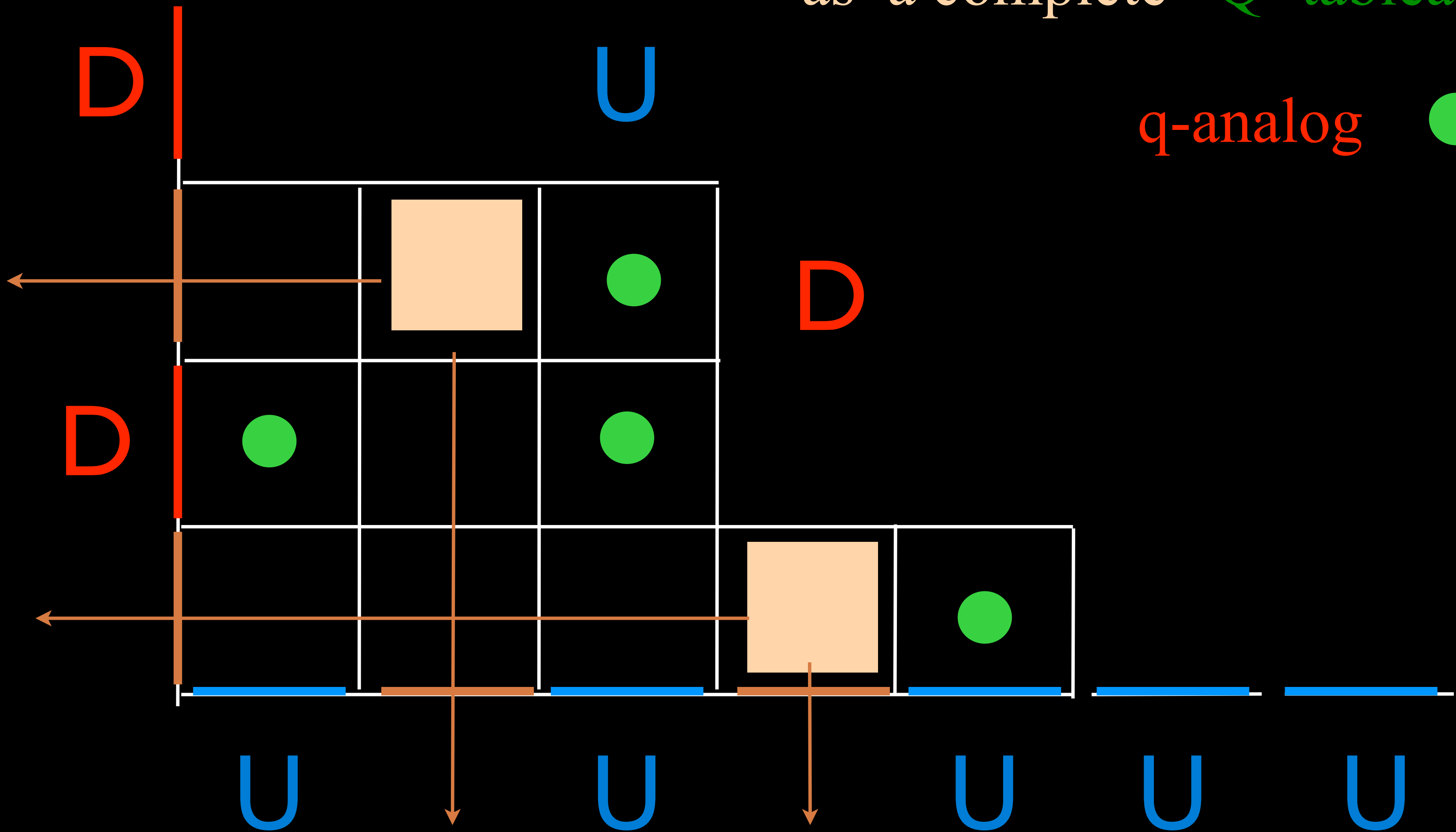
rook placement
as a Q- tableau



$$
\begin{cases}
UD = qDU + I_v I_h \\
U I_v = I_v U \\
I_h D = D I_h \\
I_h I_v = I_v I_h
\end{cases}
$$

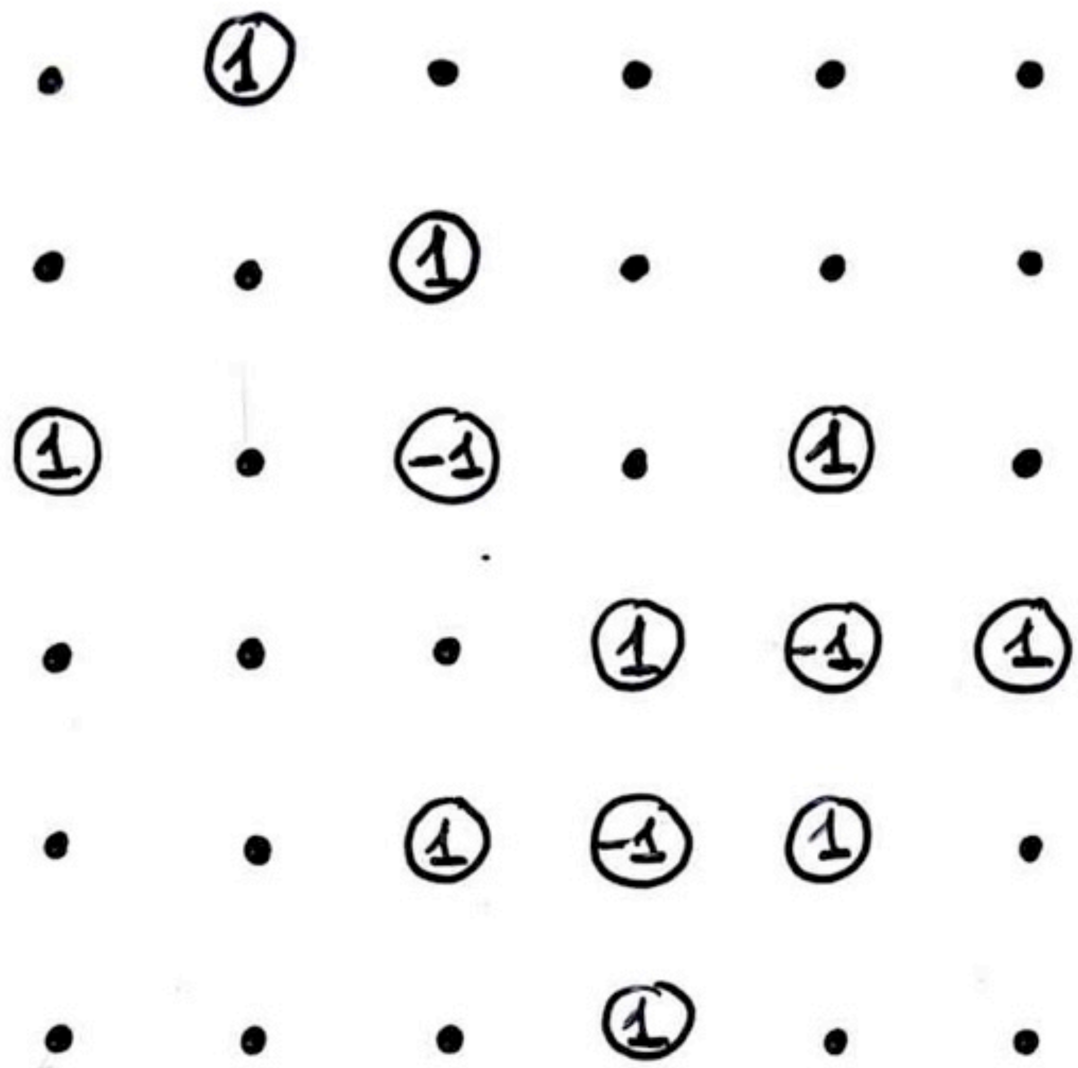
rook placement
as a complete Q- tableau

q-analog ●

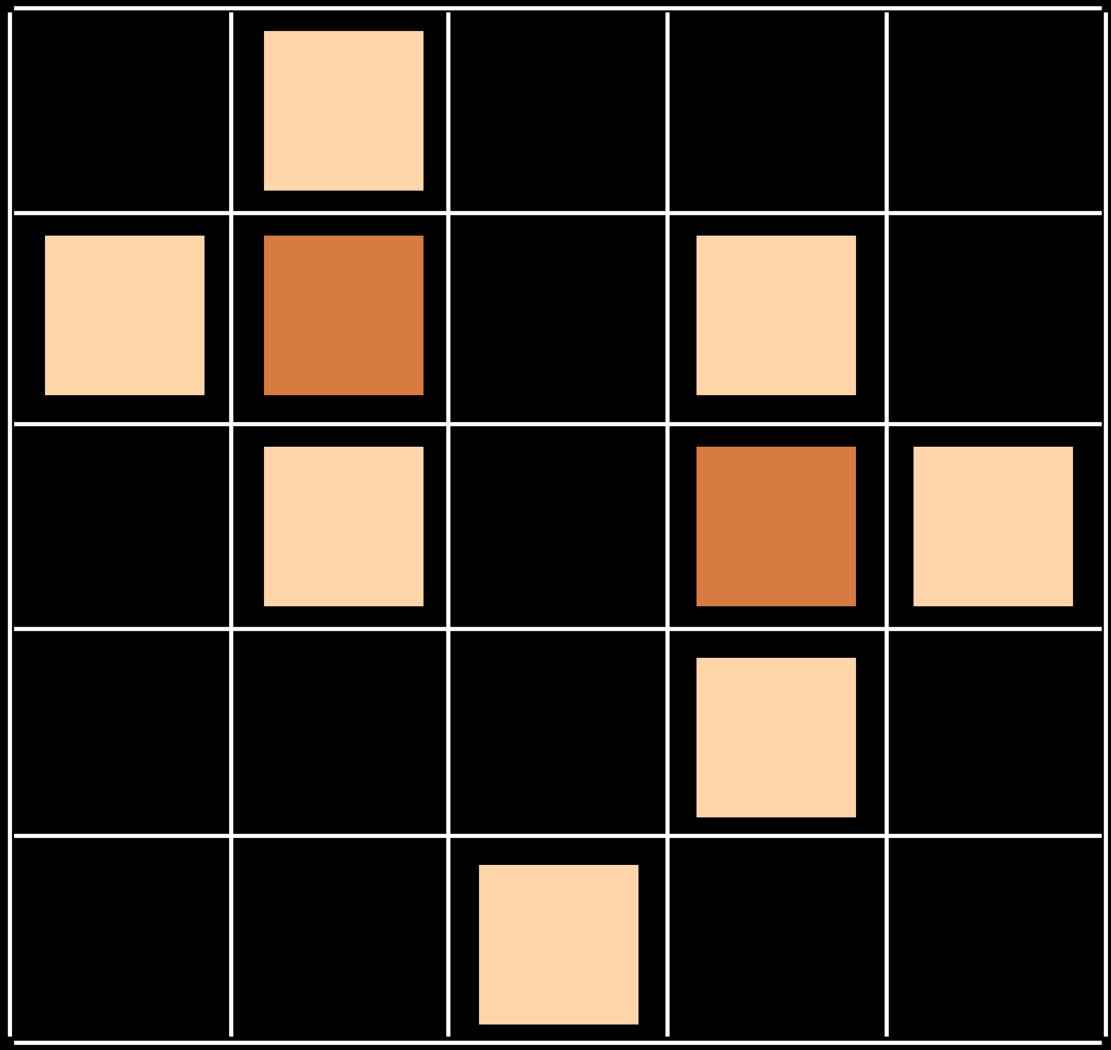


quadratic algebra for alternating sign matrices (ASM)

ASM



Alternating sign matrices



$A, A', B, B',$

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

Lemma. Any word $w(A, A', B, B')$ in letters A, A', B, B', can be uniquely written

$$\sum c(u,v;w)\ \underbrace{u(A,A')}_{\substack{\text{word}\\ \text{in } A,\,A'}}\ \underbrace{v(B,B')}_{\substack{\text{word}\\ \text{in } B,\,B'}}$$

Prop. For $w = B^n A^n$

$u = A'^n$, $v = B'^n$

$c(u,v;w)$ = the number of $n \times n$ ASM (alternating sign matrices)

The general theory

The cellular Ansatz

quadratic algebra Q (of a certain type)

(1) "planarisation" on a grid of the rewriting rules

Q- tableaux ------ planar automata

Quadratic algebra $\mathbb{Q}$

generators $\mathcal{B} = \{B_j\}_{j \in J}$

$\mathcal{A} = \{A_i\}_{i \in I}$

commutation relations

$$B_j A_i = \sum_{k,l} c_{ij}^{kl} A_k B_l \quad \text{for every } \begin{matrix} i \in I \\ j \in J \end{matrix}$$

<u>Lemma</u>. In $\mathbb{Q}$ every word $w \in (\mathcal{A} \cup \mathcal{B})^*$ can be written in a unique way

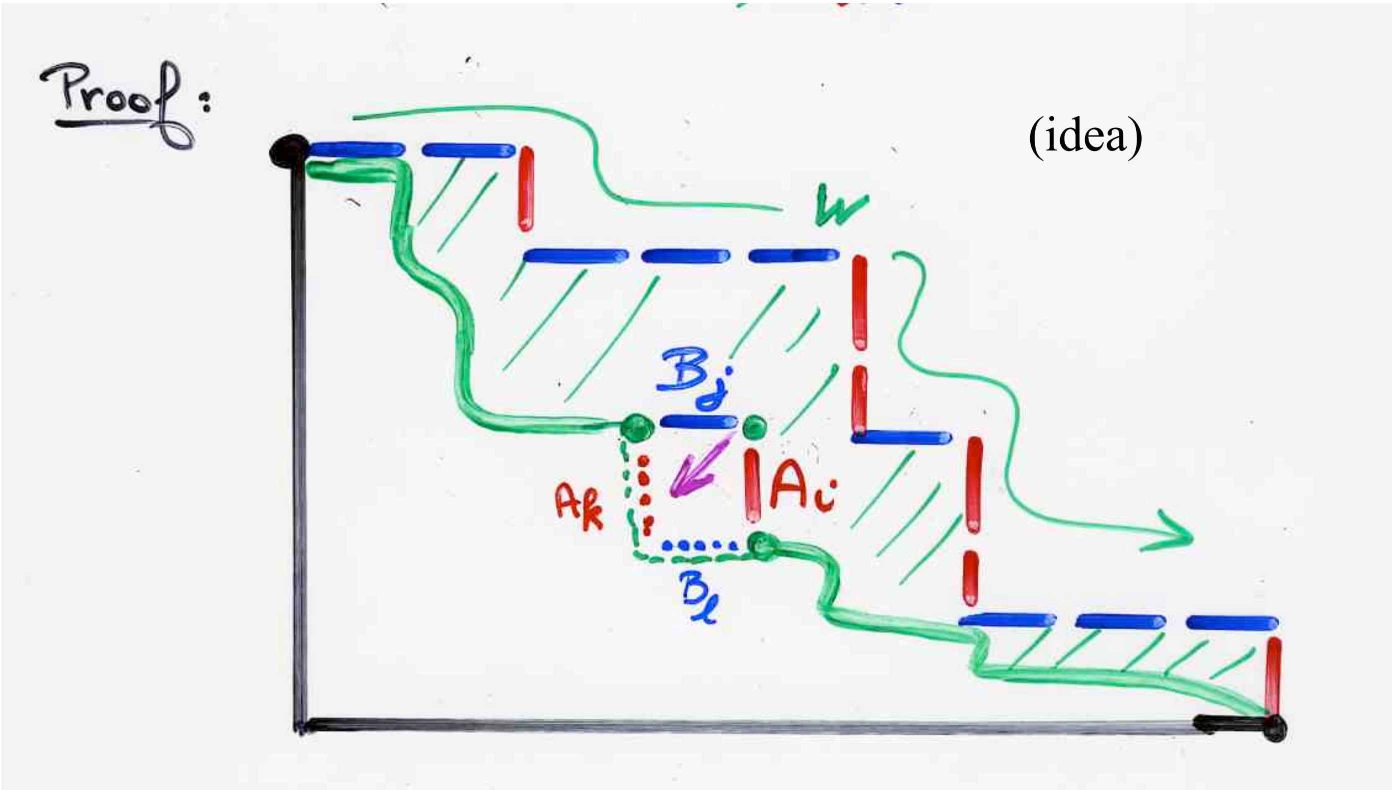
$$w = \sum_{\substack{u \in \mathcal{A}^* \\ v \in \mathcal{B}^*}} c(u, v; w)\, uv$$

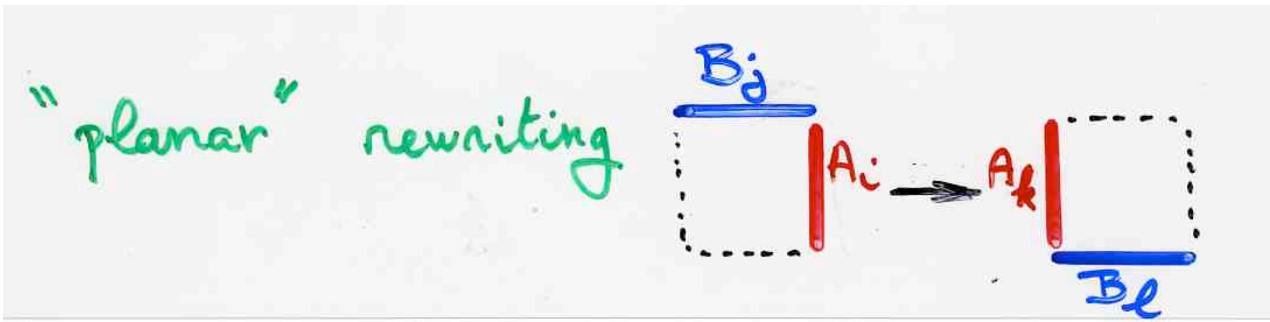
This polynomial can be obtained by successive rewriting rules :

any occurrence $B_j A_i \rightarrow \sum c_{ij}^{kl} A_k B_l$

until no more such occurrence.

(Lemma) independent of the order of rewriting


Proof:
(idea)
w
Bj
Ak
Ai
Bl

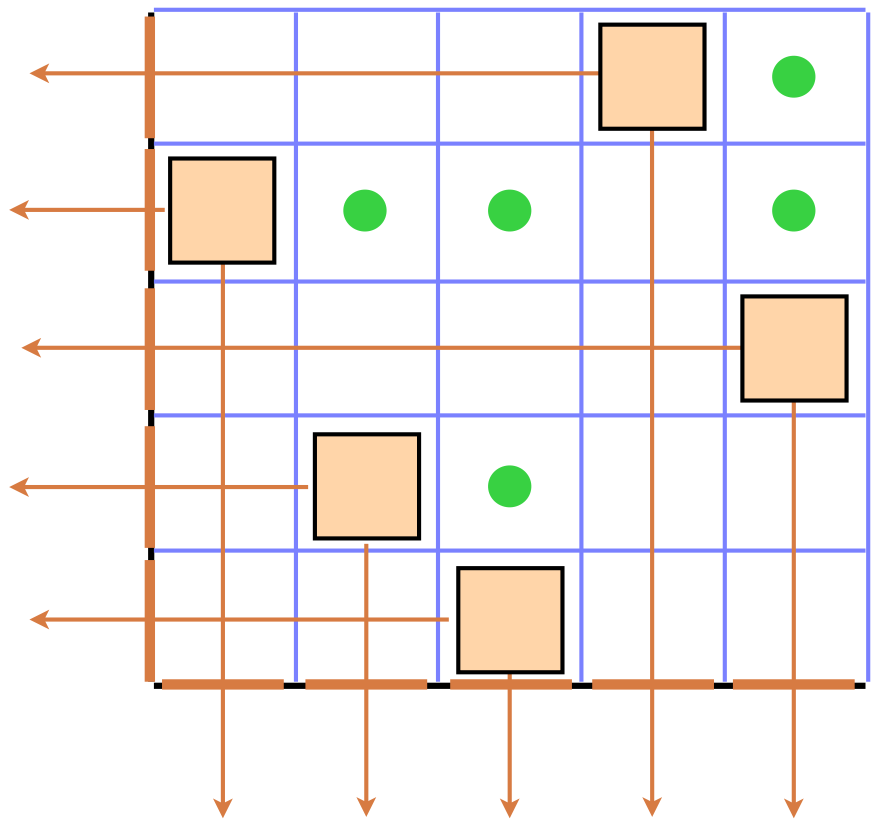

"planar" rewriting
Bj
Ai
Ak
Bl

Prop For any $w \in (\mathcal{A} \cup \mathcal{B})^*$, $u \in \mathcal{A}^*$, $v \in \mathcal{B}^*$

$$c(u, v; w) = \sum_{\substack{T \\ \text{complete } Q\text{-tableau} \\ \begin{cases} uwb(T) = w \\ lwb(T) = uv \end{cases}}} p(T)$$

example: permutations

$$\begin{cases} UD = qDU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

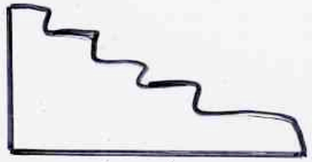


permutation as a
complete Q-tableau

complete

Def. Q-tableau

Ferrers diagram F

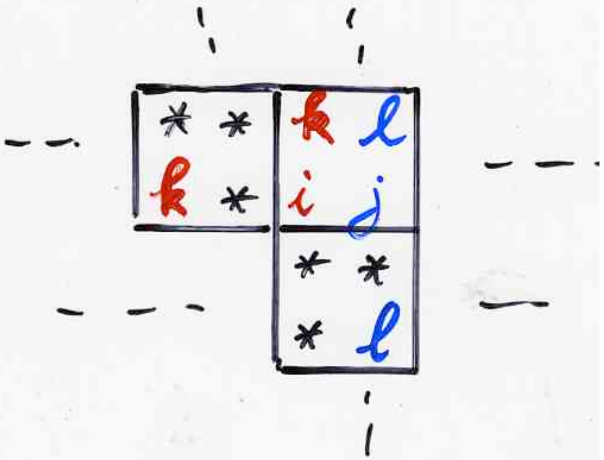


each cell $\alpha \in F$ labeled

$i, k \in I$

$j, l \in J$

with "compatibility" condition:



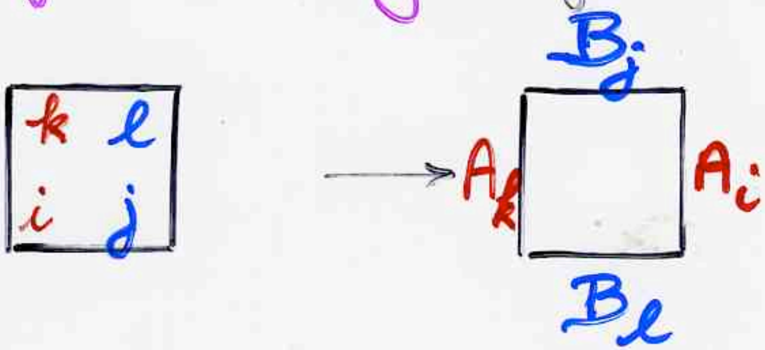
commutation relations

$$B_j A_i = \sum_{k,l} c_{ij}^{kl} A_k B_l \qquad i \in I, \quad j \in J$$

complete

Def. edge-labeling of a Q-tableau T

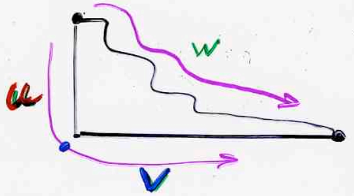
each cell α



Def. For T a complete Q-tableau

$uwb(T) \in (\mathcal{A} \cup \mathcal{B})^*$ upper word border

$lwb(T)$ lower word border



Def. weight of a complete Q-tableau T

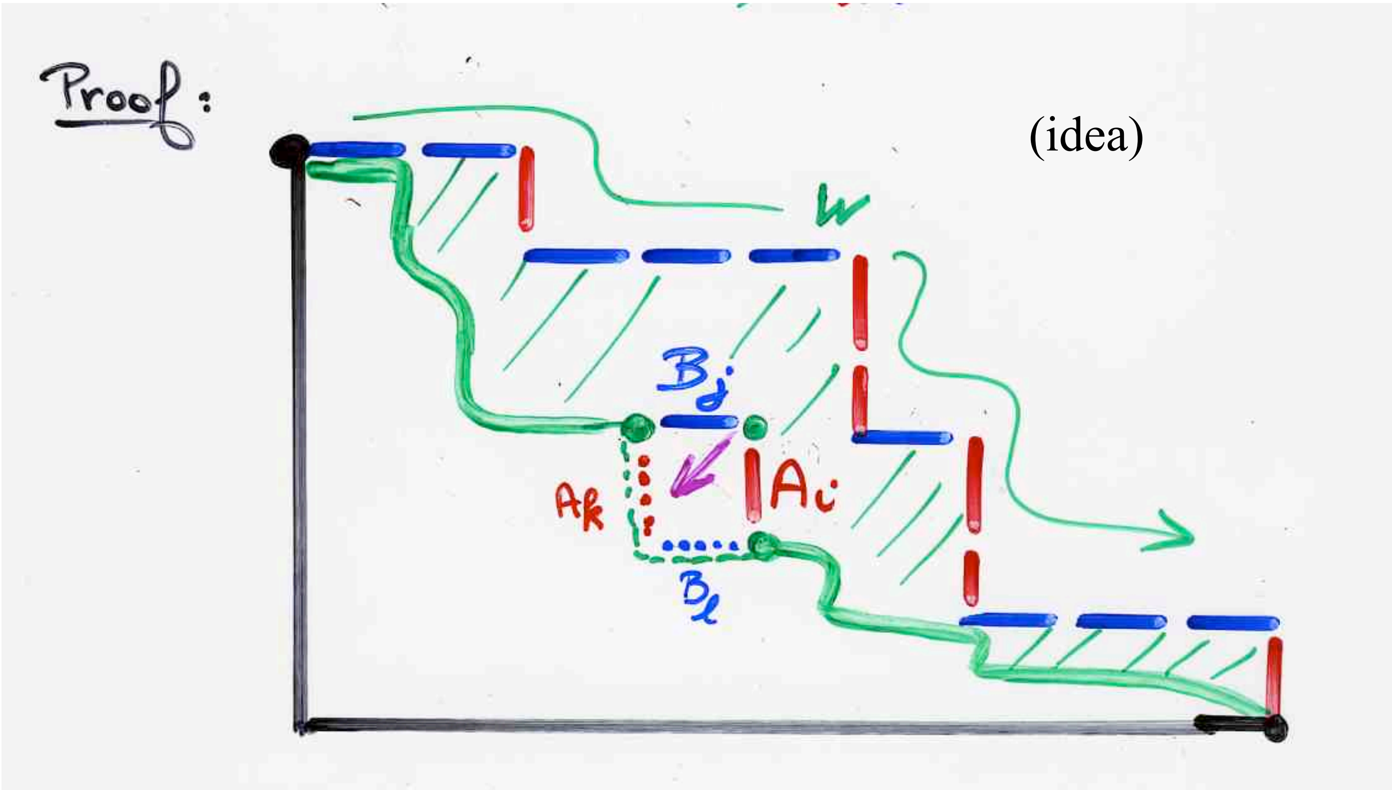
$$p(T) = \prod_{\substack{\text{cells} \\ \alpha \in F}} c_{i\,j}^{k\,l} \qquad \alpha = \boxed{\begin{matrix} k & l \\ i & j \end{matrix}}$$

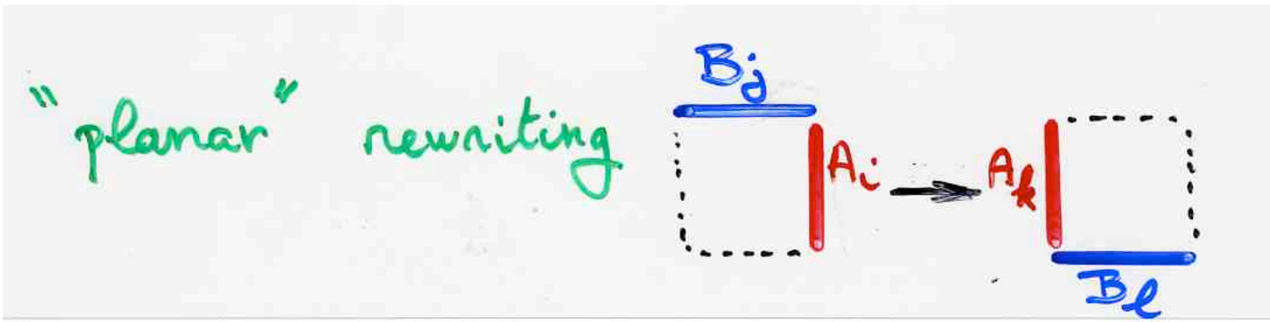
Prop For any $w \in (\mathcal{A} \cup \mathcal{B})^*$, $u \in \mathcal{A}^*$, $v \in \mathcal{B}^*$

$$c(u, v; w) = \sum_{T} p(T)$$

T complete Q-tableau

$$\begin{cases} uwb(T) = w \\ lwb(T) = uv \end{cases}$$


Proof:
(idea)
w
B_j
A_i
A_k
B_ℓ


"planar" rewriting
B_j
A_i
A_k
B_ℓ

complete Q-tableaux
and Q-tableaux
an example

$$UD = DU + I$$

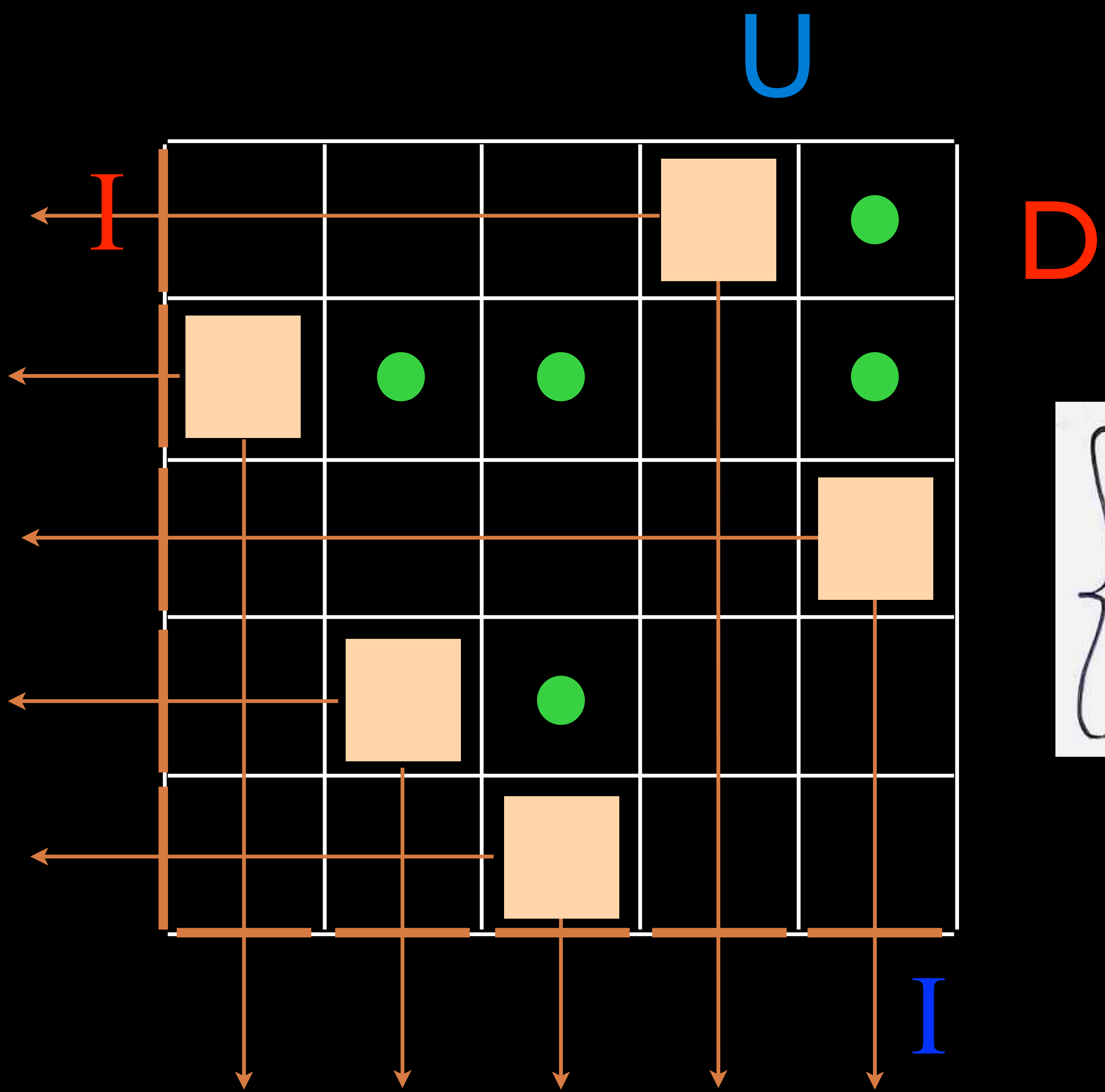
Weyl-Heisenberg algebra

$$\begin{cases} UD = qDU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

$$w = U^n D^n \qquad uv = I_v^n I_h^n \qquad c(u,v;w) = n!$$

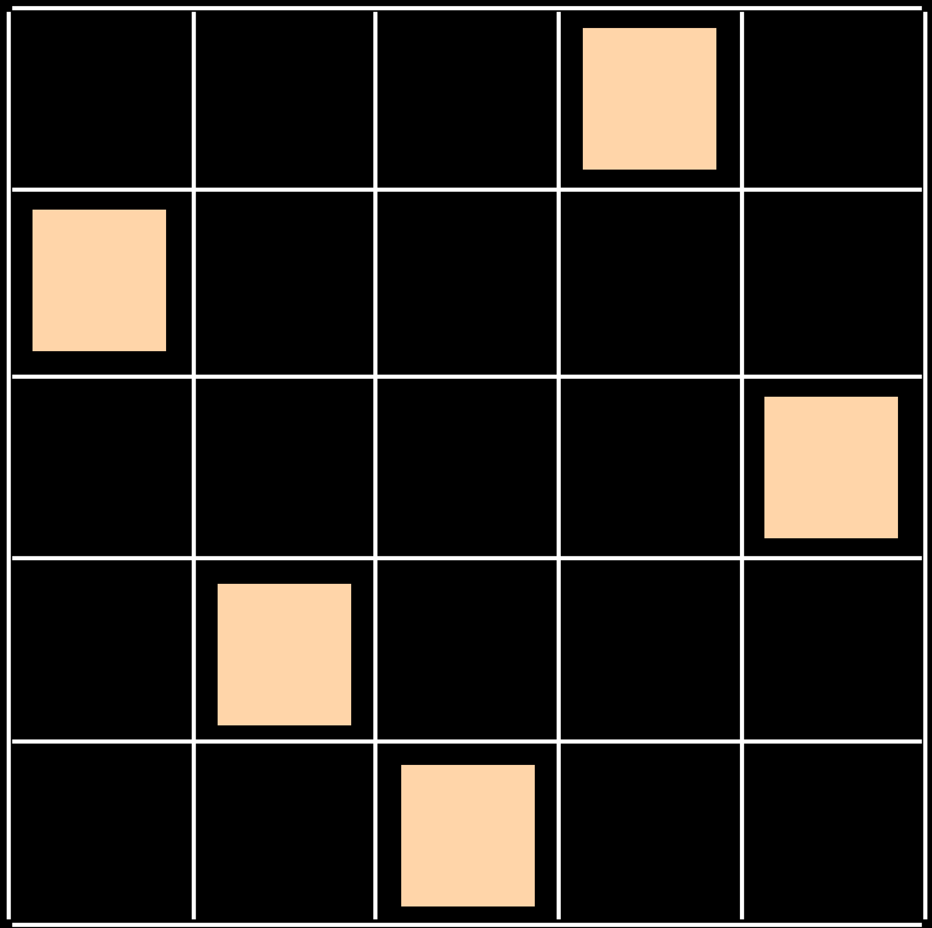
complete Q-tableau $\longleftrightarrow$ Permutations $\mathfrak{S}_n$

$$\begin{cases} uwb(T) = U^n D^n \\ lwb(T) = I_v^n I_h^n \end{cases}$$



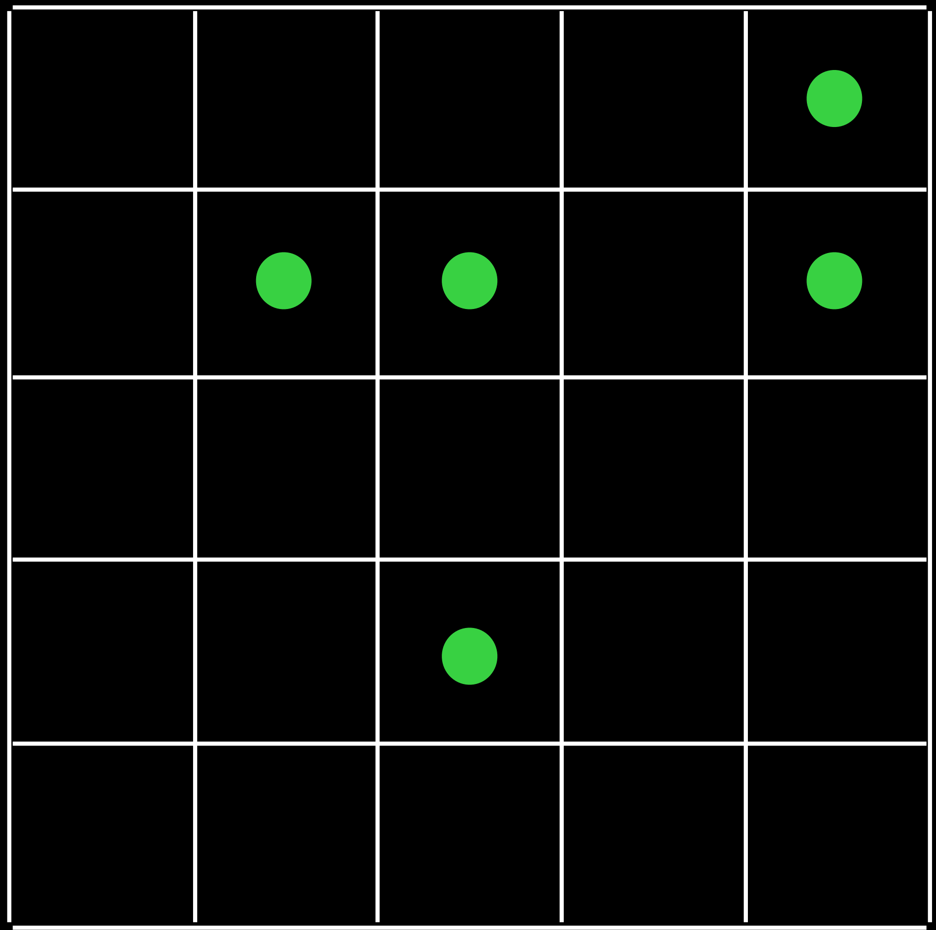
$$\begin{cases} UD = qDU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

permutation as a complete Q- tableau



$$
\begin{cases}
UD = qDU + I_v I_h \\
U I_v = I_v U \\
I_h D = D I_h \\
I_h I_v = I_v I_h
\end{cases}
$$

permutation as a Q- tableau



$$
\begin{cases}
UD = qDU + I_v I_h \\
U I_v = I_v U \\
I_h D = D I_h \\
I_h I_v = I_v I_h
\end{cases}
$$

another Q-tableau:
Rothe diagram of a permutation

definition Q-tableaux

S set of labels

$$\varphi : \left\{ \begin{bmatrix} k & l \\ i & j \end{bmatrix} \right\} = R \longrightarrow S$$

set of rewriting rules

$$B_j A_i \to C_{ij}^{kl} A_k B_l$$

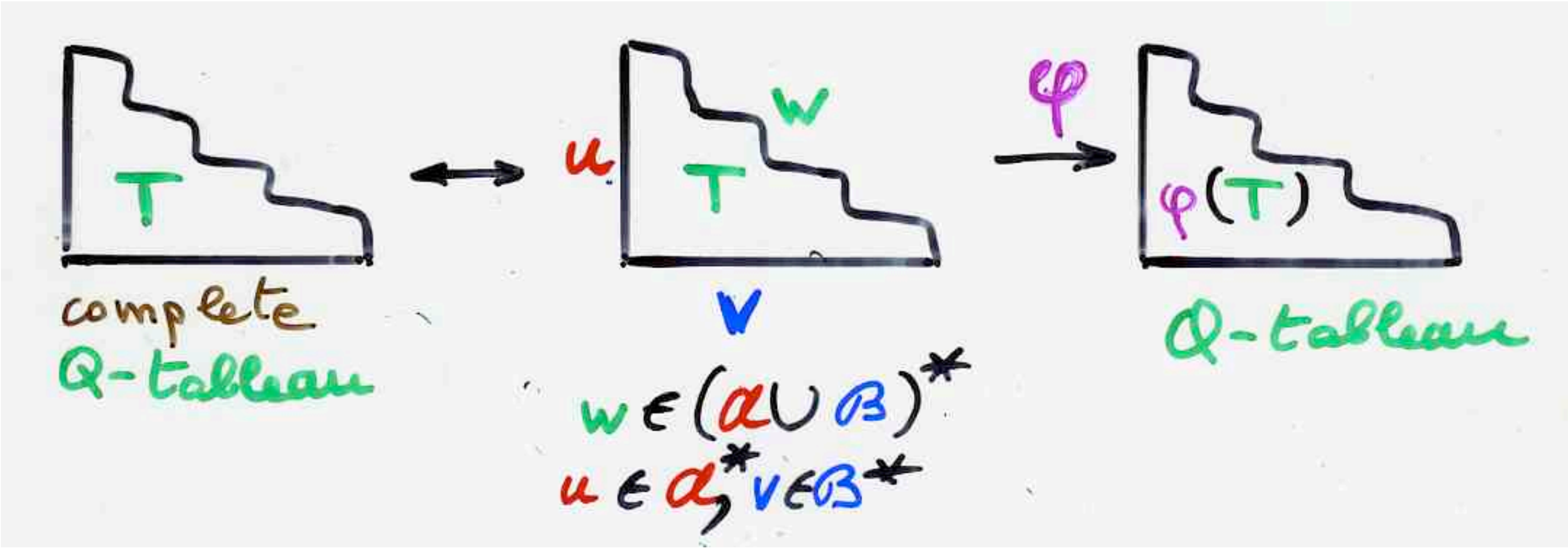
such that:

if $\begin{pmatrix} k & l \\ i & j \end{pmatrix} \neq \begin{pmatrix} k' & l' \\ i' & j' \end{pmatrix}$ and $\varphi\begin{pmatrix} k & l \\ i & j \end{pmatrix} = \varphi\begin{pmatrix} k' & l' \\ i' & j' \end{pmatrix}$

then $(i, j) \neq (i', j')$

<u>Def</u> Q-tableau

"image" by φ of a "complete Q-tableau"



w-compatible

w fixed

$\{$ set of Q-tableaux w-compatible $\}$

$\updownarrow$ bijection

$\{$ set of complete Q-tableaux T with uwb(T) = w $\}$

equivalence

Q-tableaux -- planar automaton

equivalence

Q-tableaux $\longleftrightarrow$ tableaux accepted by a planar automaton

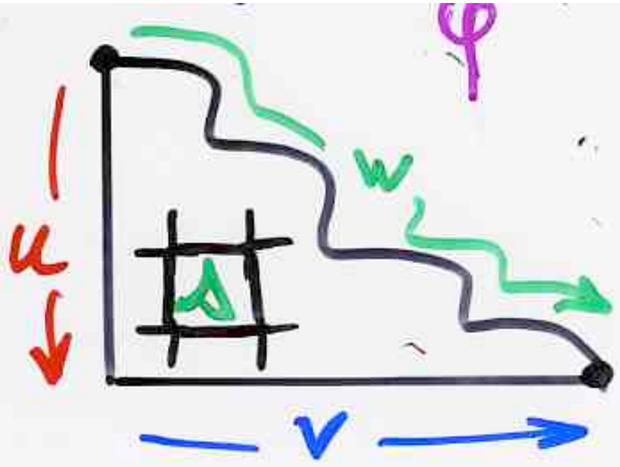
Q quadratic algebra φ

$\mathcal{P} = (S, \mathcal{B}, \alpha, \theta, w, uv)$

φ

with $\mathcal{P}$ satisfying

$$\theta(s, B, A) = \theta(t, B, A) \Rightarrow s = t$$



$$BA = \sum_{s \in S} A'B' \qquad (B', A') = \theta(s, B, A)$$

«Pictures»

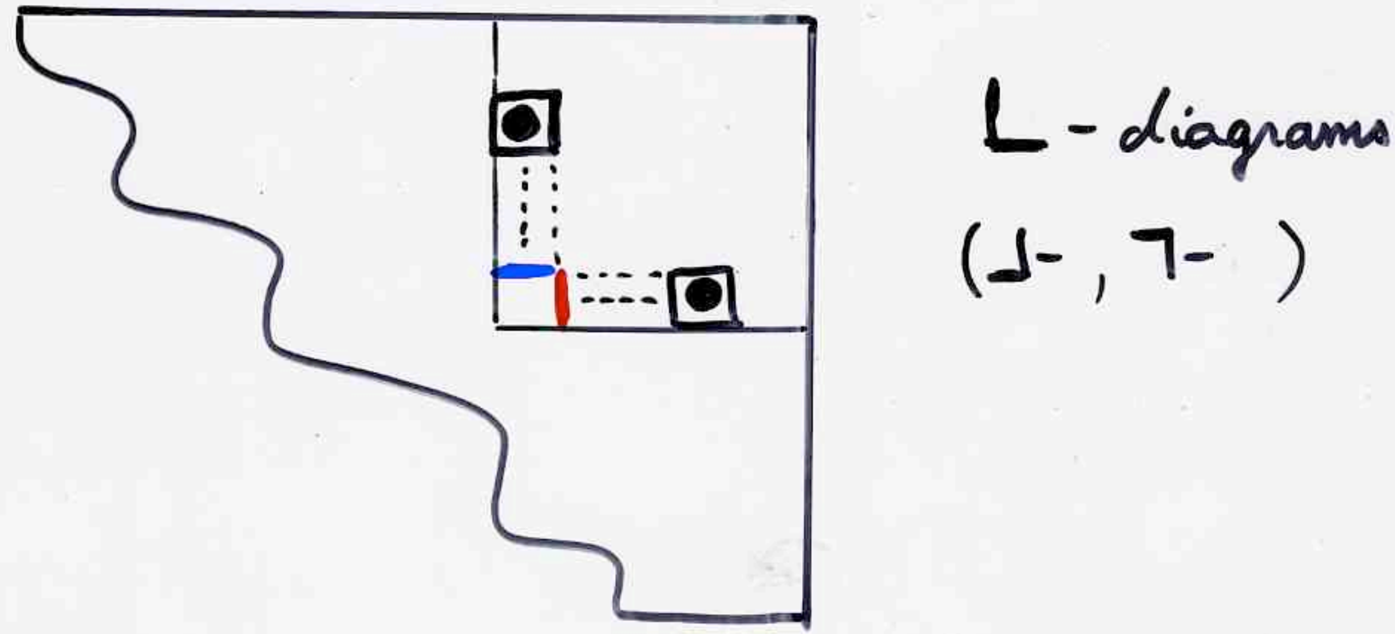
accepted by planar automata ?

permutation tableau

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

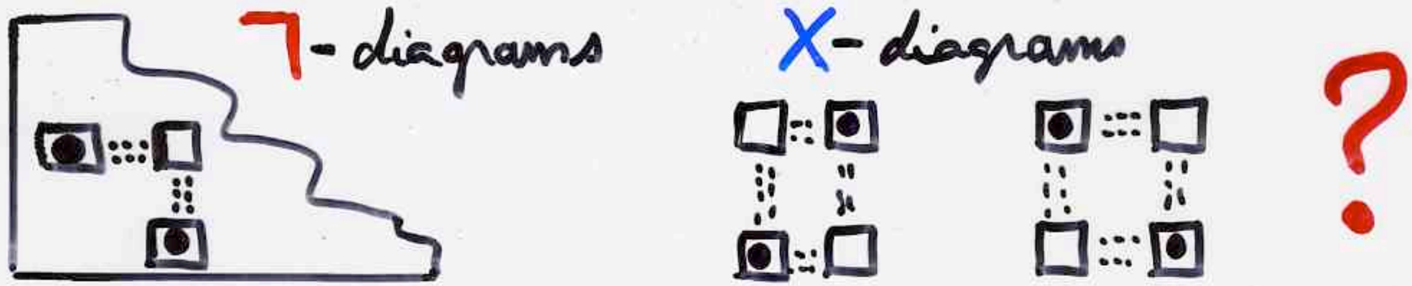
E. Steingrímsson, L. Williams (2005)


L - diagrams
(⅃- , ⅂-)

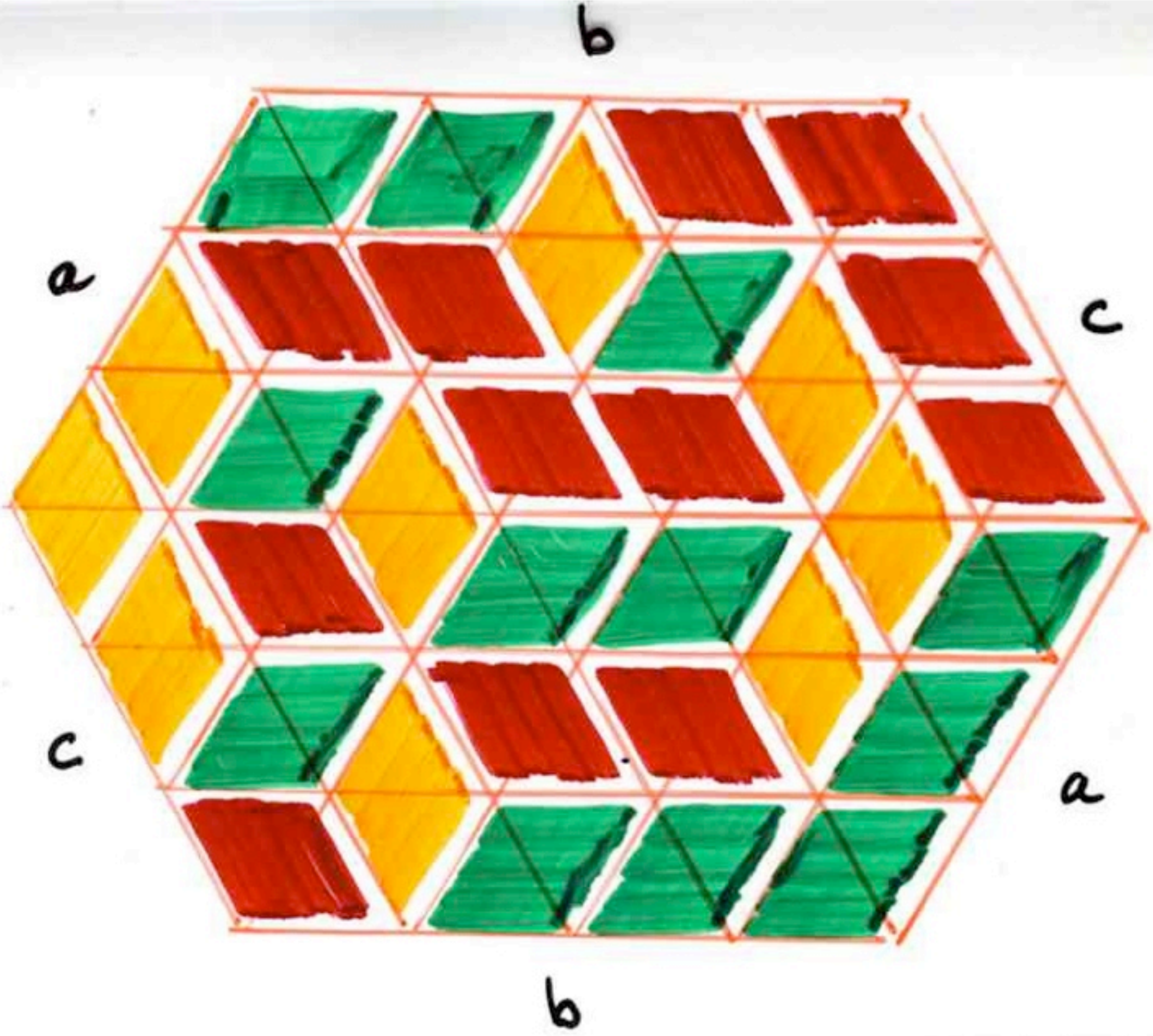

⅂ - diagrams

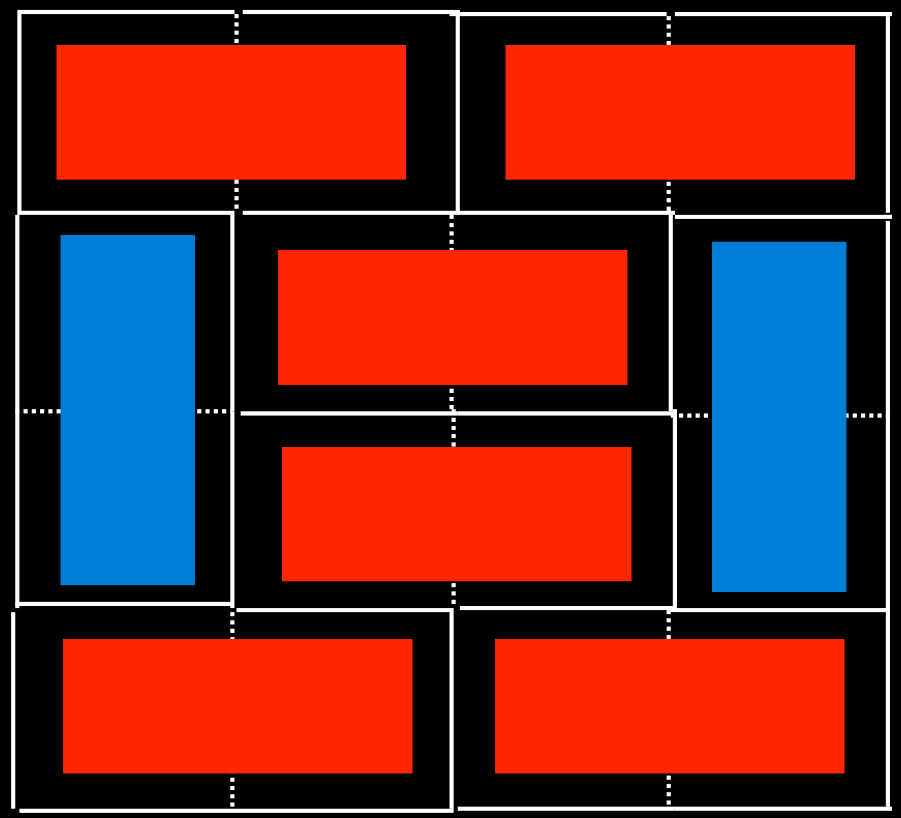
Bijections between pattern-avoiding fillings of Young diagrams

Josuat-Vergès (2008)



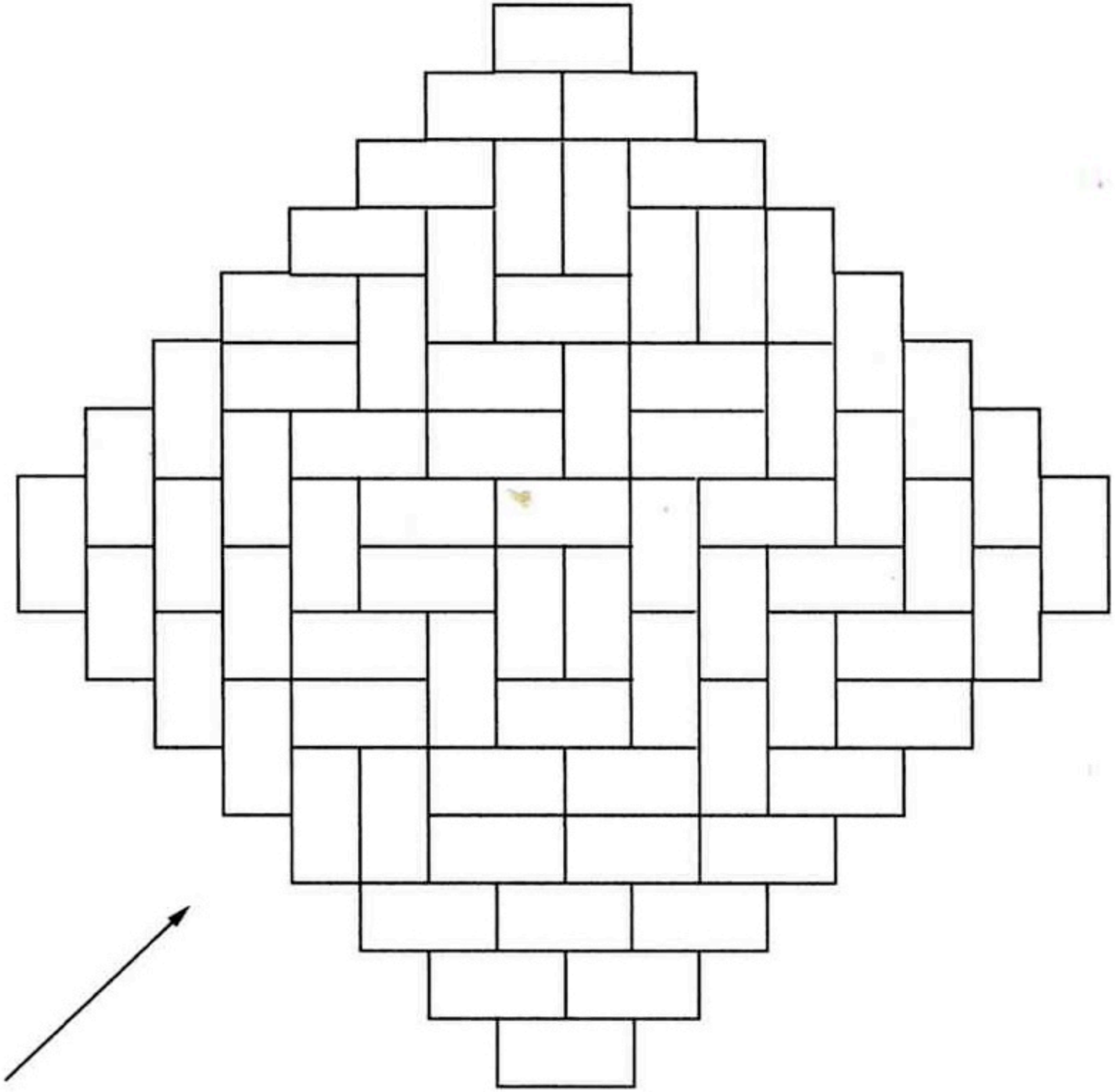
Tilings


b
a
c
c
a
b



a tiling
on the
square lattice

$2^{n(n-1)/2}$



Elkies,
Kuperberg,
Larsen,
Propp
(1992)

The 8-vertex algebra
(or XYZ - algebra)
(or Z - algebra)

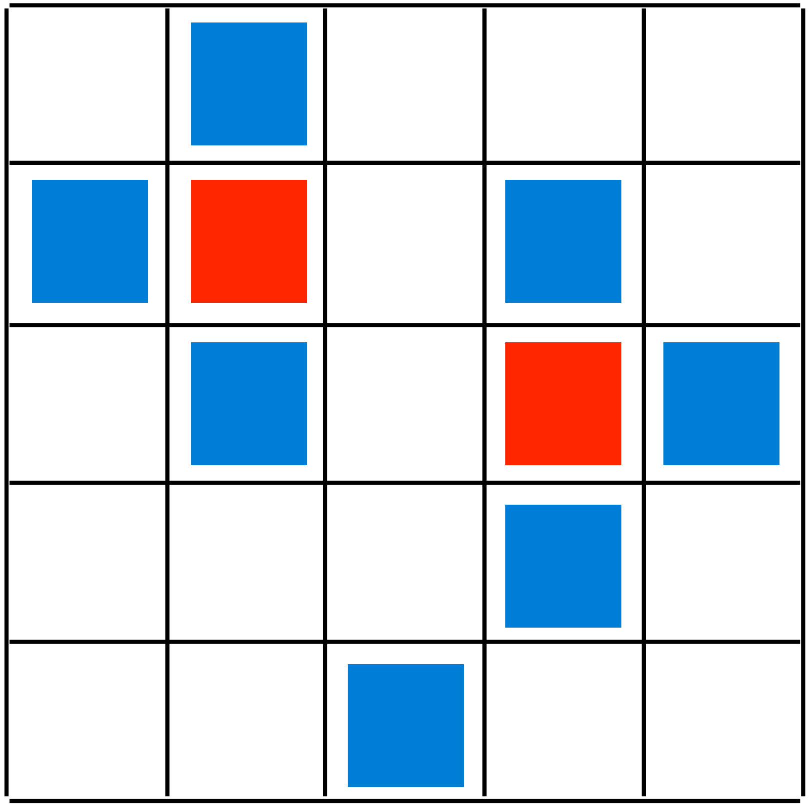
The quadratic algebra $\mathbf{Z}$

4 generators $B_\bullet \ A_\bullet \ B \ A$

8 parameters $q_{....}$, $t_{....}$

$$
\begin{cases}
B A = q_{\circ\circ} \, A B + t_{\circ\circ} \, A_\bullet B_\bullet \\
B_\bullet A_\bullet = q_{\bullet\bullet} \, A_\bullet B_\bullet + t_{\bullet\bullet} \, A B \\
B_\bullet A = q_{\bullet\circ} \, A B_\bullet + t_{\bullet\circ} \, A_\bullet B \\
B A_\bullet = q_{\circ\bullet} \, A_\bullet B + t_{\circ\bullet} \, A B_\bullet
\end{cases}
$$

$$t_{\bullet\circ} = t_{\circ\bullet} = 0$$



The quadratic algebra Z

4 generators $B_\bullet \; A_\bullet \; B \; A$

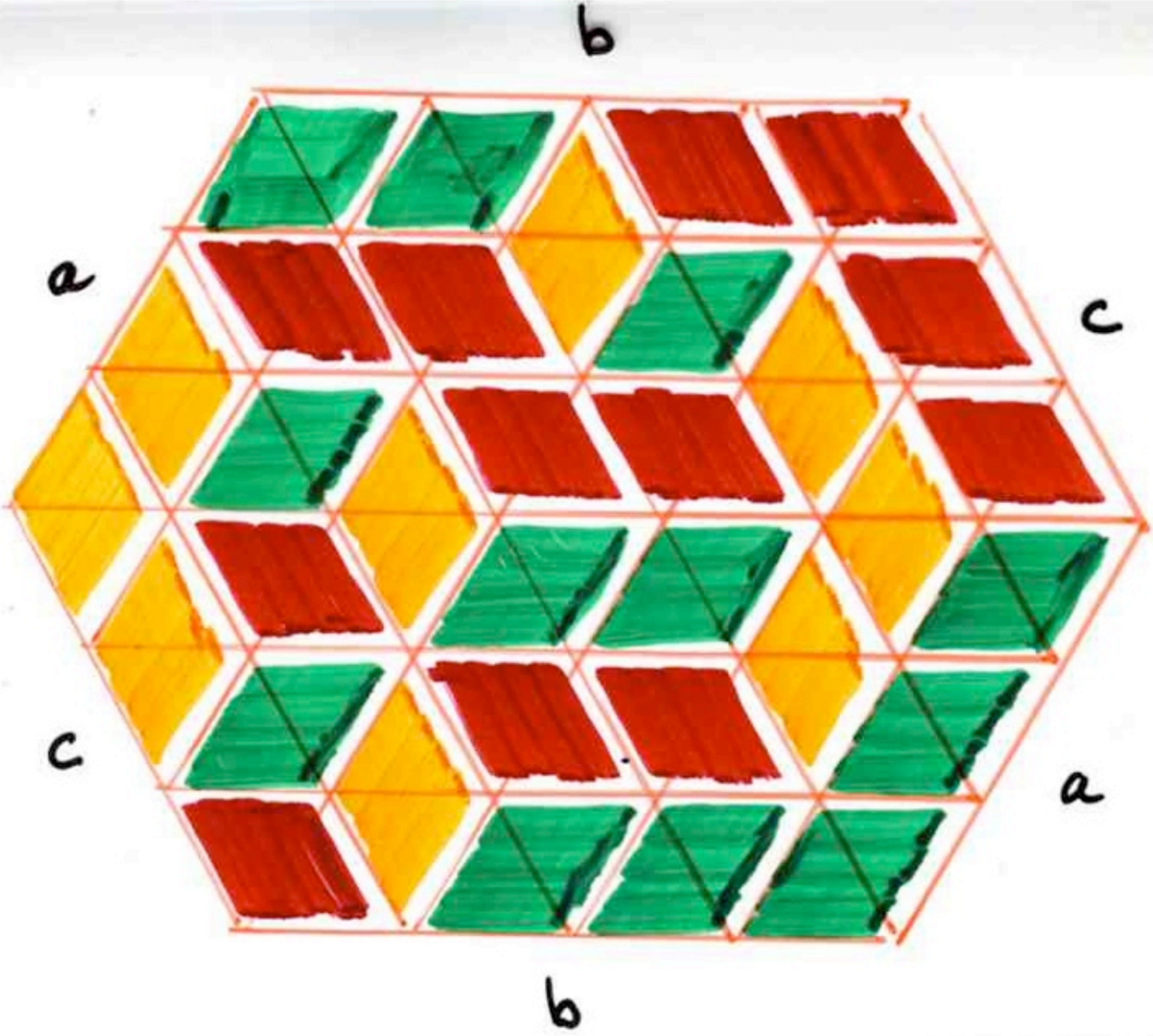
8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} BA = q_{\circ\circ} AB + t_{\circ\circ} A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet} A_\bullet B_\bullet + t_{\bullet\bullet} AB \\ B_\bullet A = q_{\bullet\circ} A B_\bullet + \bigcirc \, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} A_\bullet B + \bigcirc \, A B_\bullet \end{cases}$$

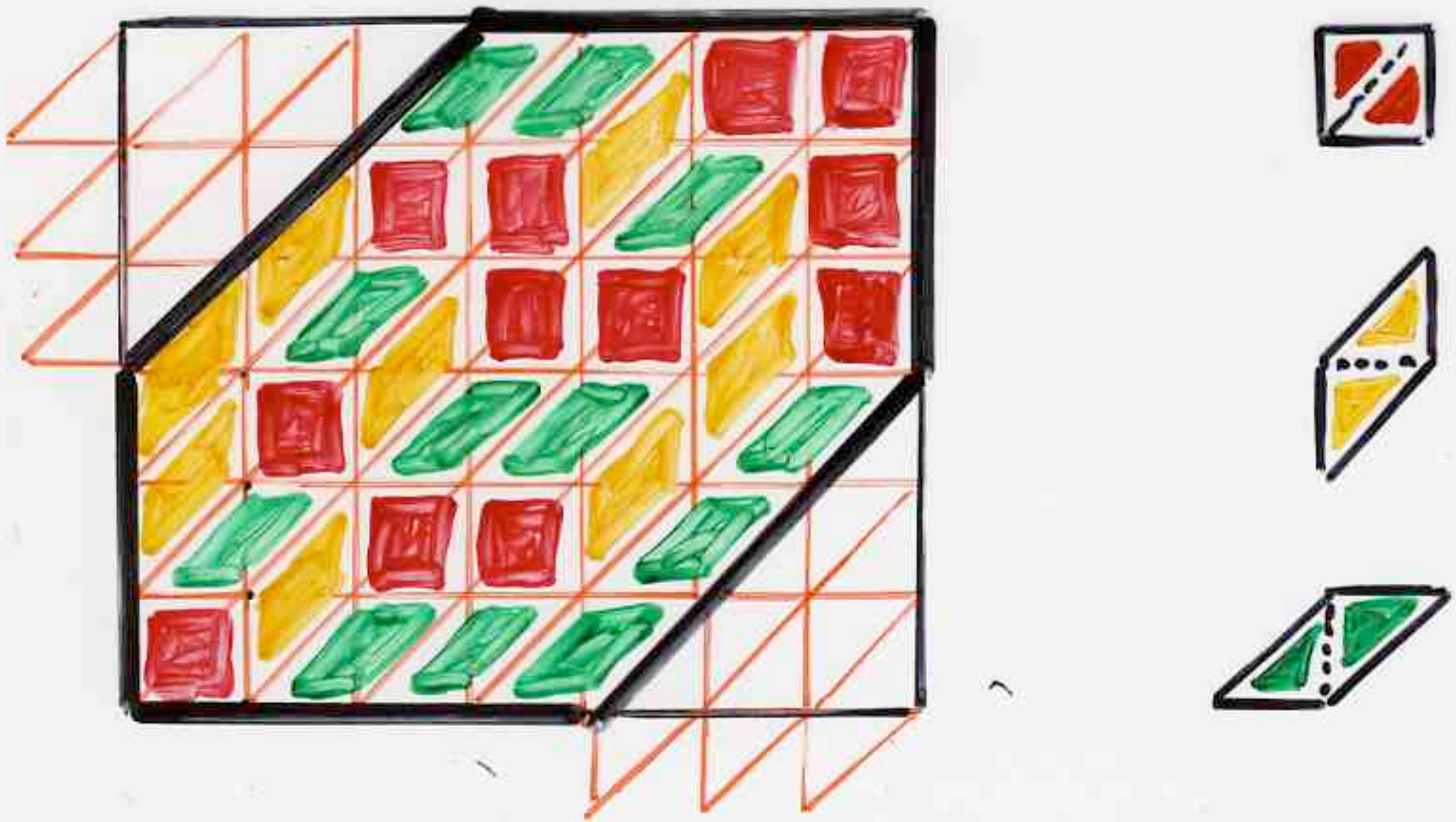
$w = B^n A^n \qquad uv = A_\bullet^n B_\bullet^n$

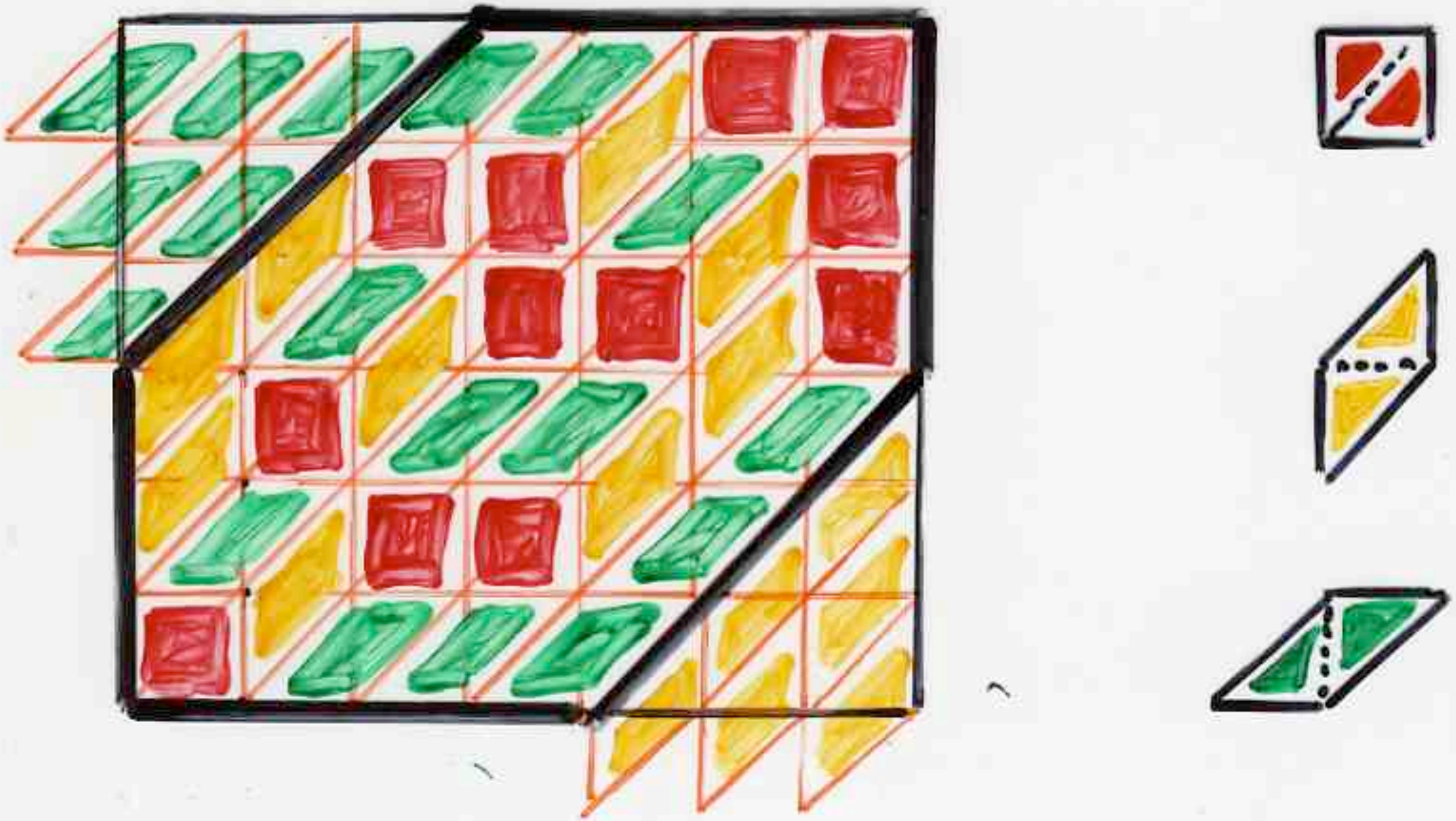
$c(u, v; w) =$ nb of ASM $n \times n$

rhombus tilings



b
a
c
c
a
b





$$\begin{cases} t_{\bullet\circ} = t_{\circ\bullet} = 0 \\ q_{\bullet\bullet} = 0 \end{cases} \qquad \text{(ASM)}$$

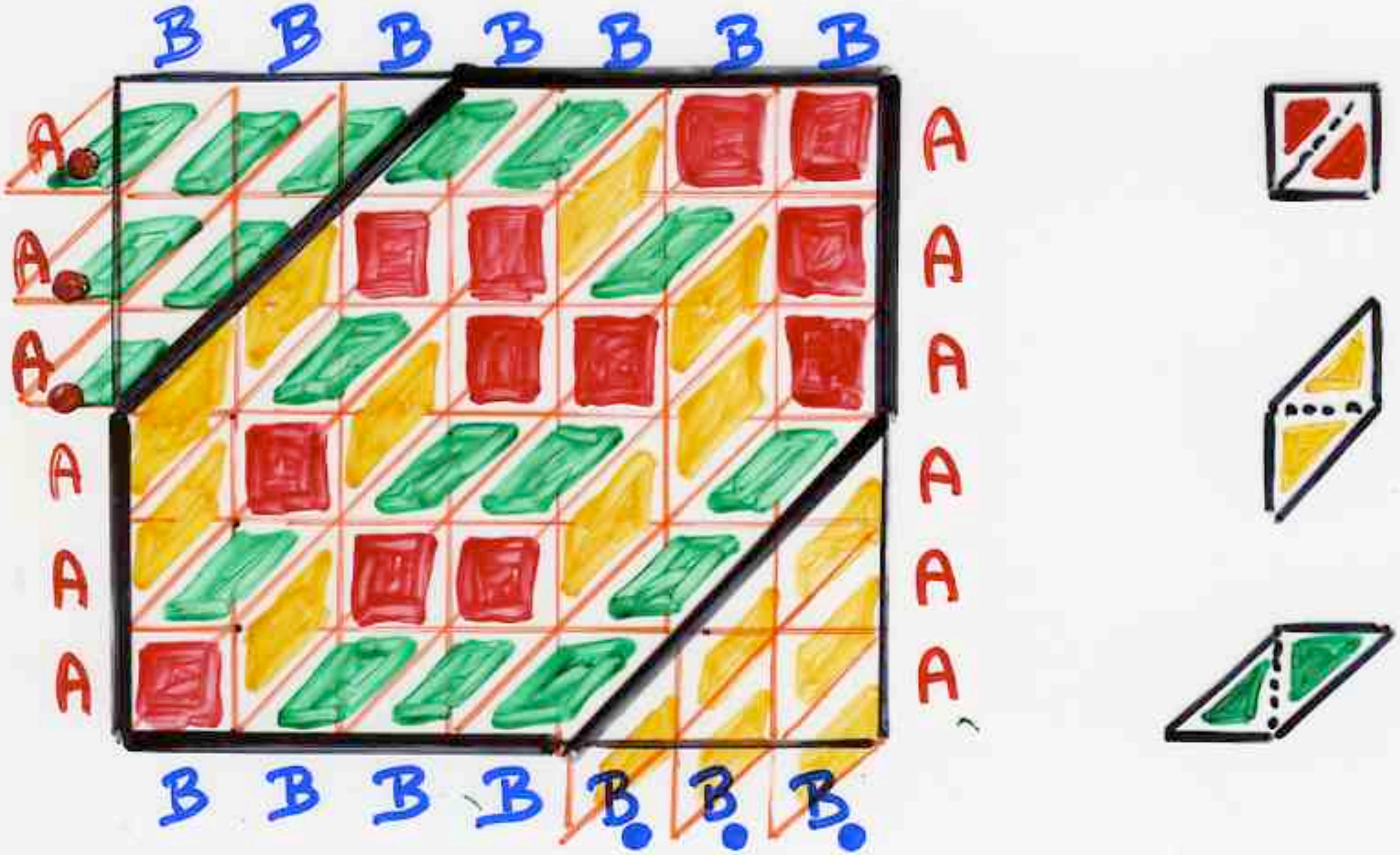
Rhombus tilings

The quadratic algebra Z

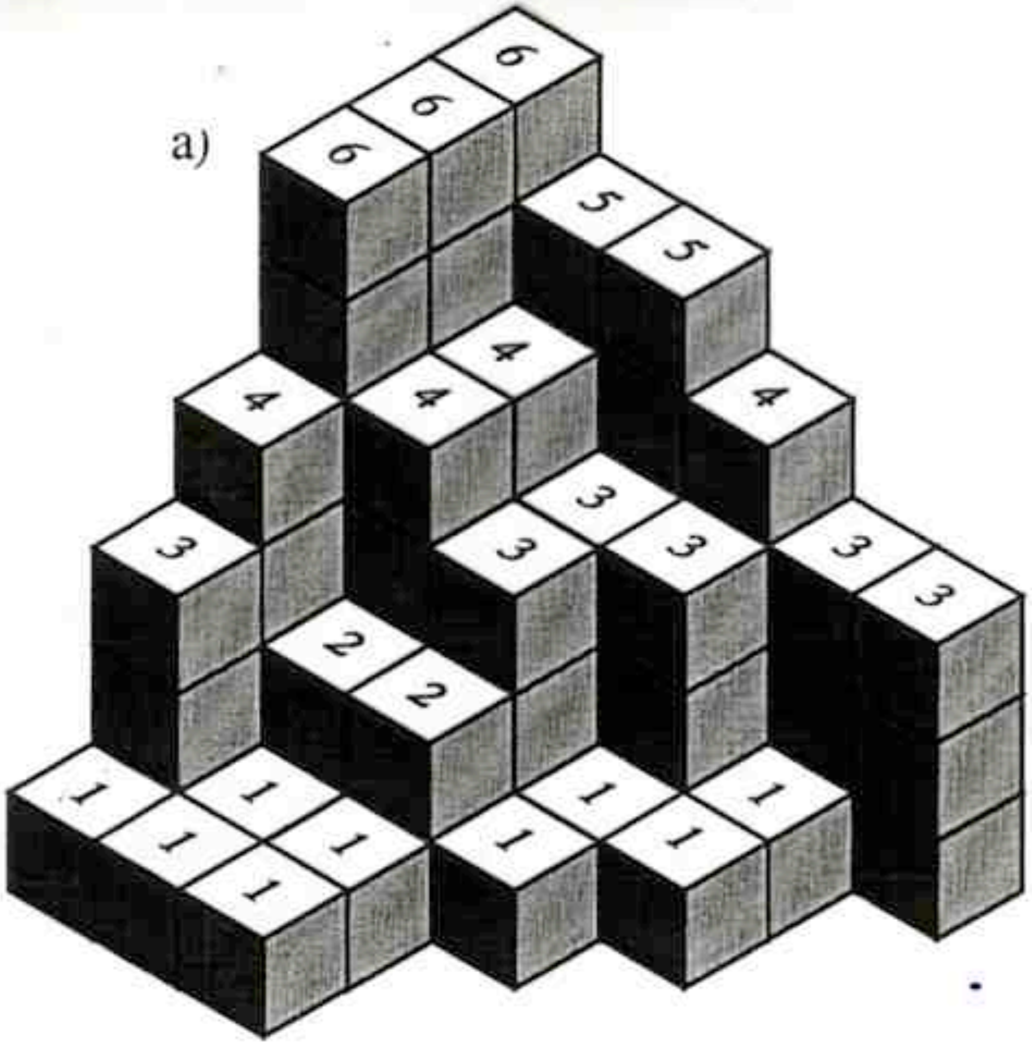
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} B A = q_{\circ\circ} A B + t_{\circ\circ} A_\bullet B_\bullet \\ B_\bullet A_\bullet = \bigcirc\, A_\bullet B_\bullet + t_{\bullet\bullet} A B \\ B_\bullet A = q_{\bullet\circ} A B_\bullet + \bigcirc\, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} A_\bullet B + \bigcirc\, A B_\bullet \end{cases}$$

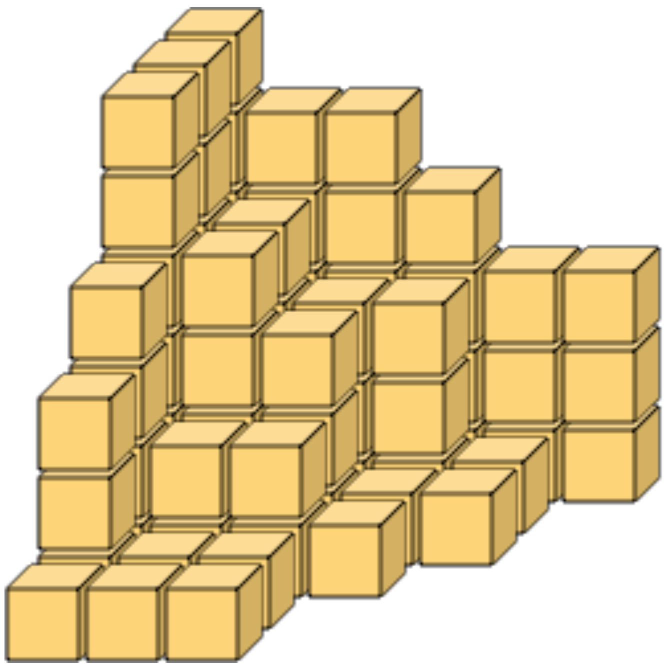

B B B B B B B
A A A A A A
A A A A A A
B B B B B B B

a)



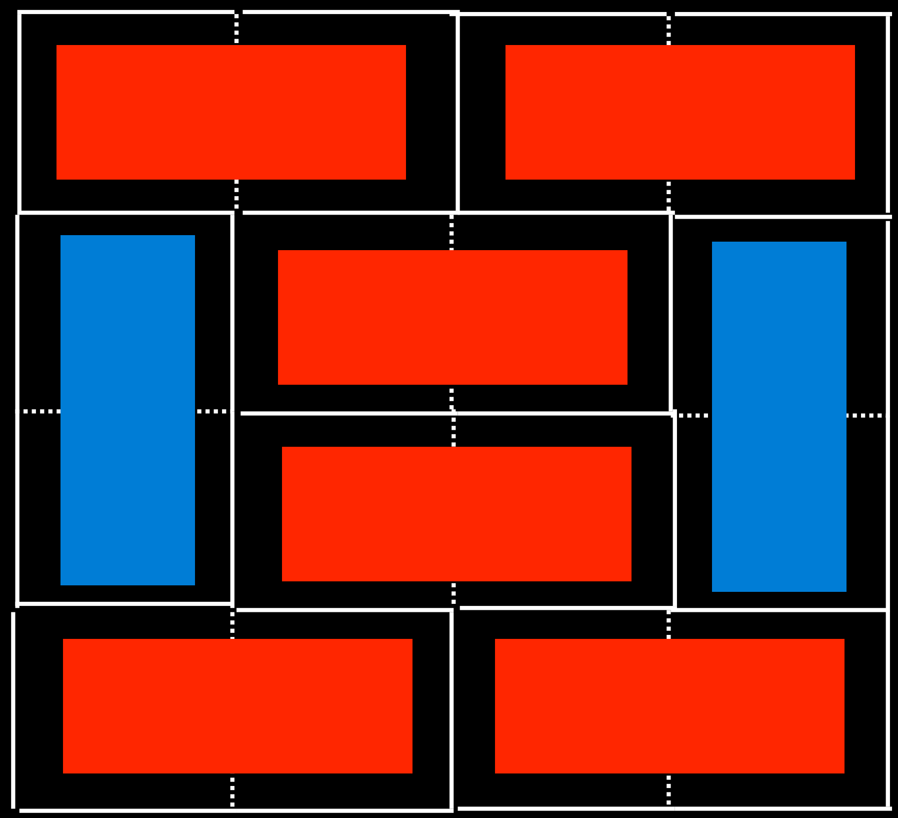
b)

6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			



$$\prod_{\substack{1 \leq i \leq a \\ 1 \leq j \leq b \\ 1 \leq k \leq c}} \frac{i+j+k-1}{i+j+k-2}$$

dimers tiling
on a square lattice



a tiling
on the
square lattice

The quadratic algebra $\mathbf{Z}$

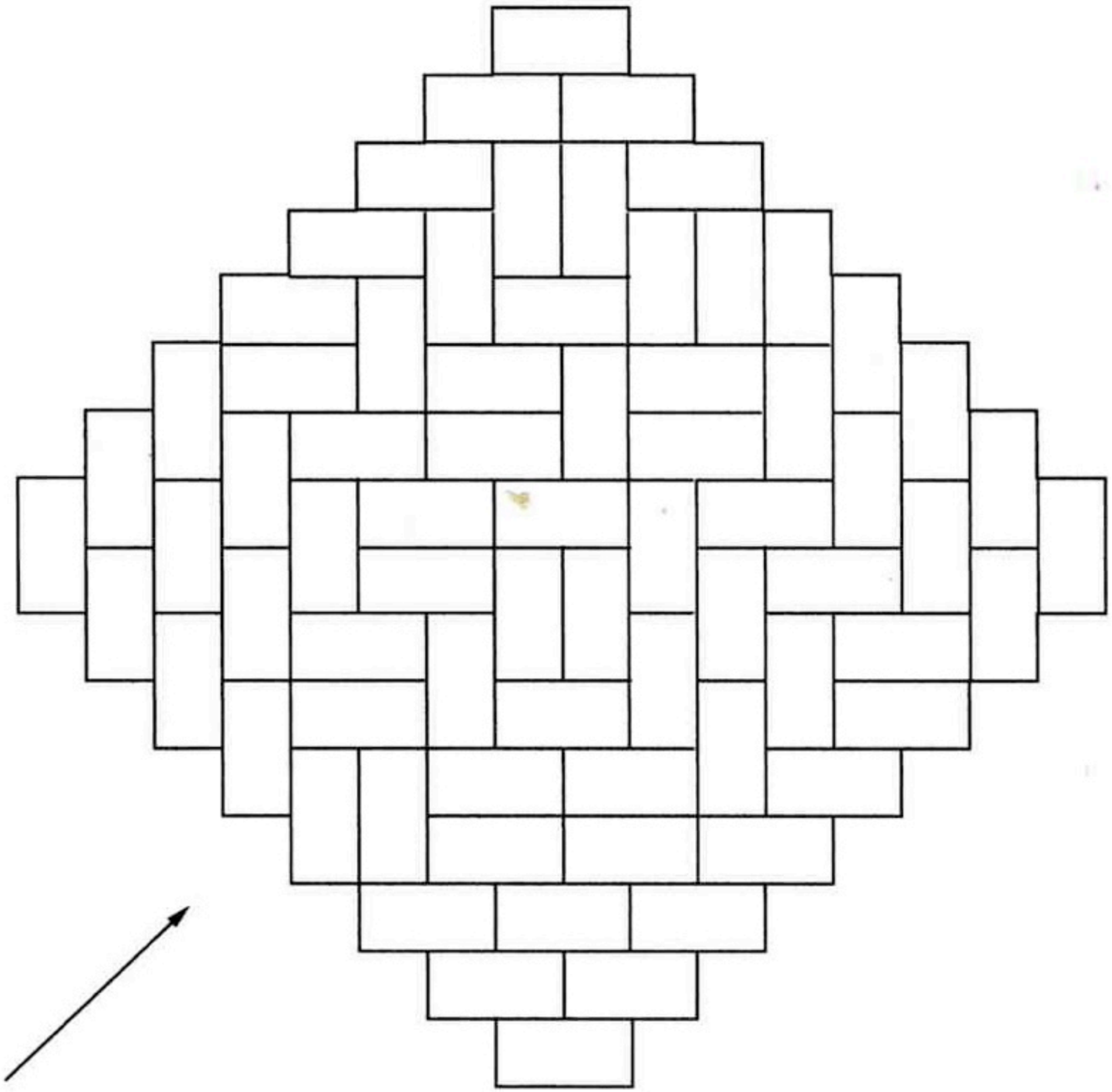
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\cdots}$, $t_{\cdots}$

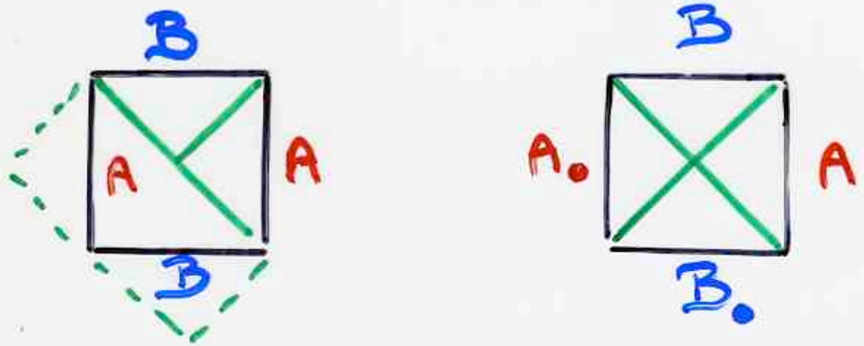
$$
\begin{cases}
BA = q_{00}\, AB + t_{00}\, A_\bullet B_\bullet \\
B_\bullet A_\bullet = \bigcirc\, A_\bullet B_\bullet + \bigcirc\, AB \\
B_\bullet A = q_{\bullet 0}\, A B_\bullet + \bigcirc\, A_\bullet B \\
B A_\bullet = q_{0 \bullet}\, A_\bullet B + \bigcirc\, A B_\bullet
\end{cases}
$$

Aztec tilings

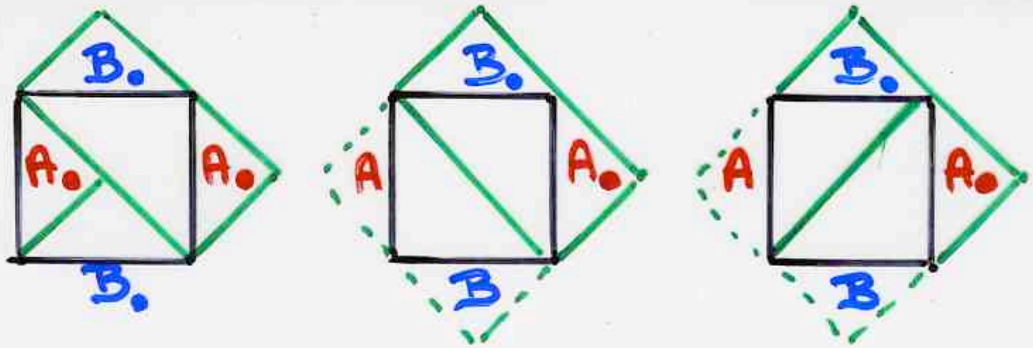
$2^{n(n-1)/2}$



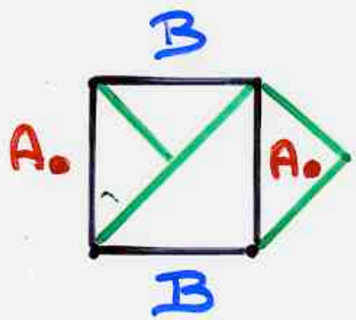
Elkies,
Kuperberg,
Larsen,
Propp
(1992)



$$BA = AB + A_{\bullet}B_{\bullet}$$



$$B_{\bullet}A_{\bullet} = A_{\bullet}B_{\bullet} + 2\,AB$$



$$B_{\bullet}A = AB_{\bullet} \qquad BA_{\bullet} = A_{\bullet}B$$

Aztec tilings

$$t_{\bullet\circ} = t_{\circ\bullet} = 0 \qquad \text{(ASM)}$$

$$t_{\bullet\bullet} = 2 \qquad \text{(nb of } -1 \text{ in ASM)}$$

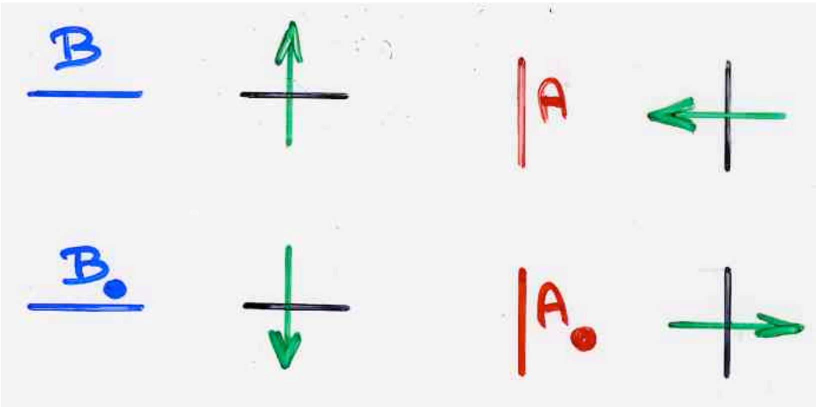
The quadratic algebra Z

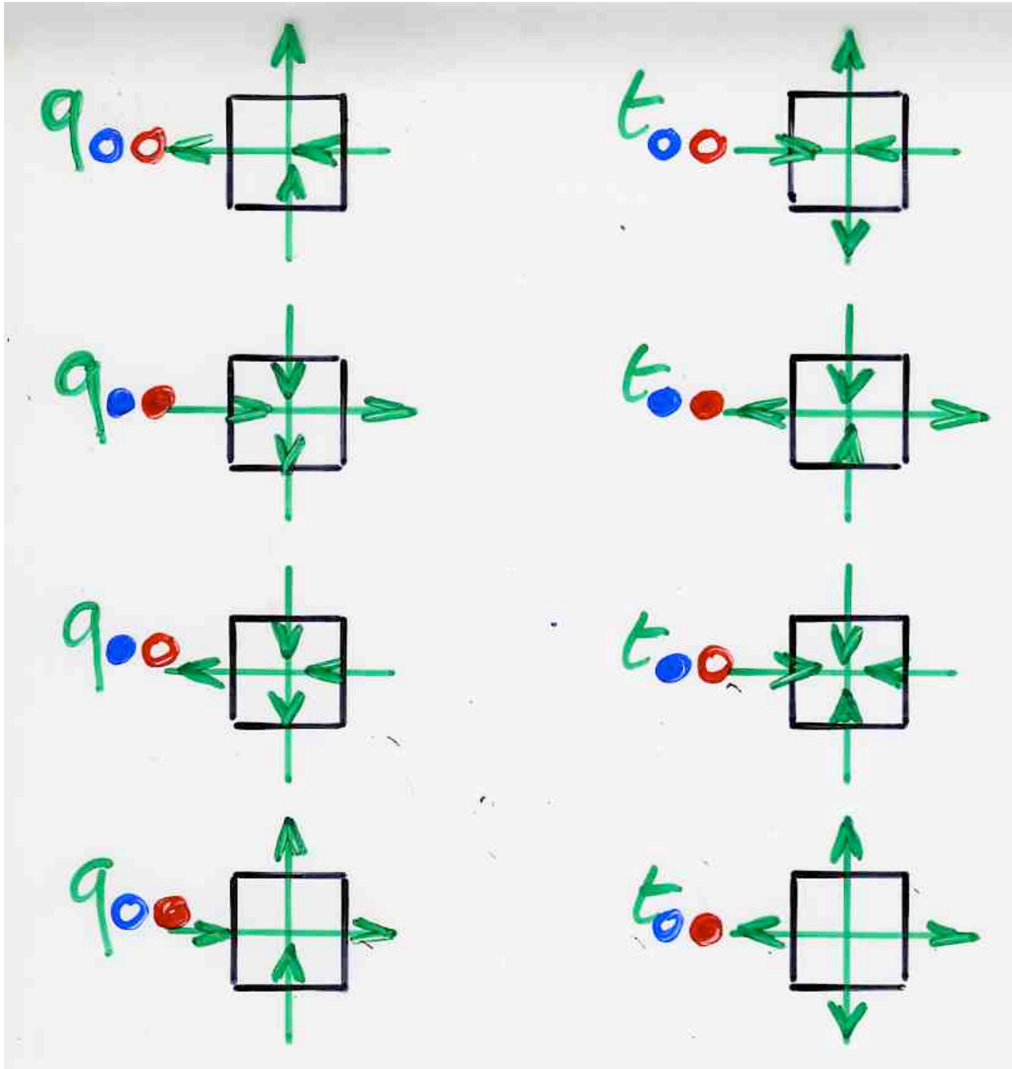
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} B A = q_{\circ\circ} \, A B + t_{\circ\circ} \, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet} \, A_\bullet B_\bullet + 2 \, A B \\ B_\bullet A = q_{\bullet\circ} \, A B_\bullet + 0 \, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} \, A_\bullet B + 0 \, A B_\bullet \end{cases}$$

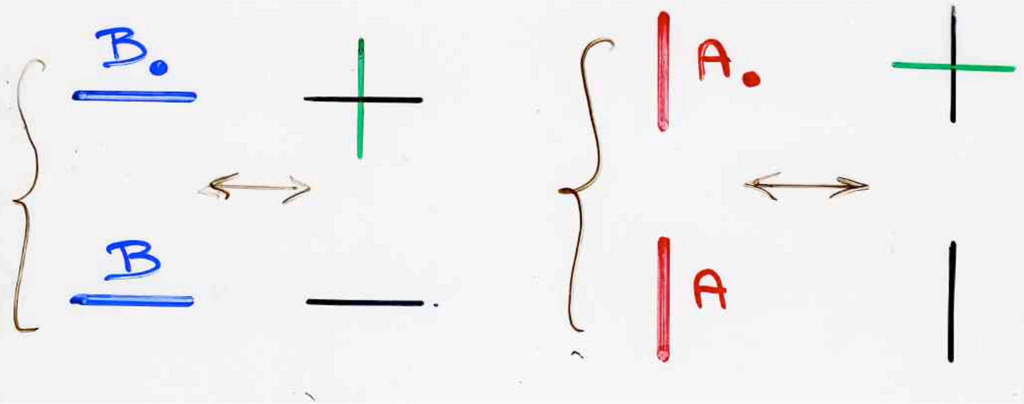
geometric interpretation of XYZ-tableaux

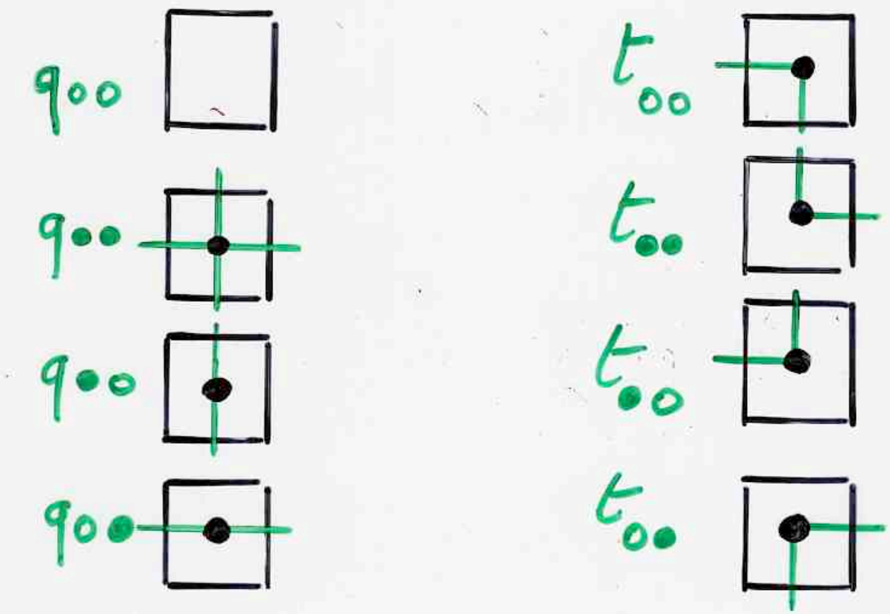




8 - vertex model

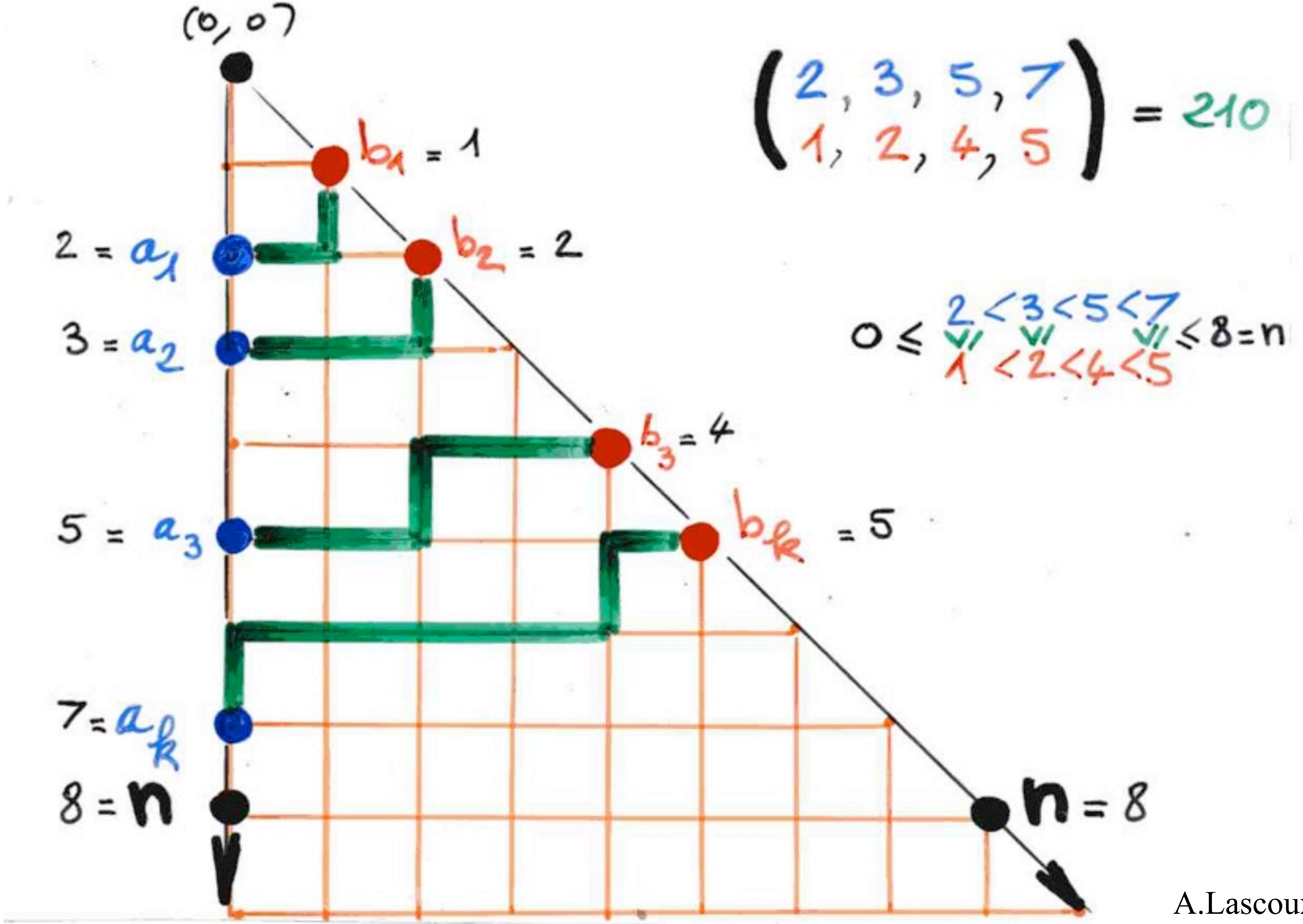
geometric interpretations of Z-tableaux





8 - vertex model

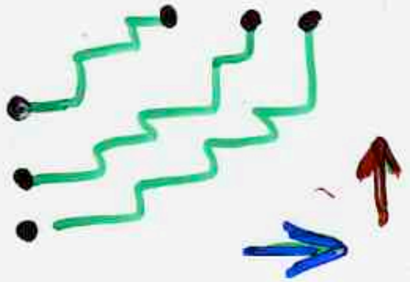
non-intersecting paths



A.Lascoux

example: binomial determinant

I.Gessel, X.G.V., 1985



$$\begin{cases} t_{\circ\circ} = 0 \\ q_{\bullet\bullet} = t_{\bullet\bullet} = 0 \end{cases}$$

The quadratic algebra $\mathbb{Z}$

4 generators $B_\bullet \; A_\bullet \; B \; A$

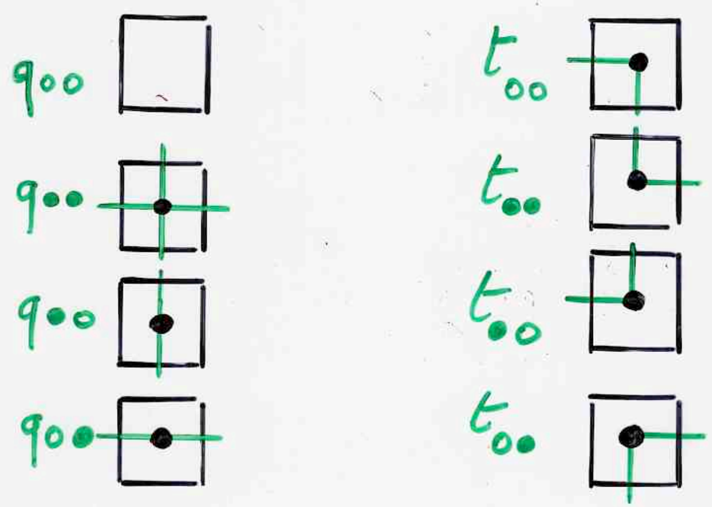
8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} B A = q_{\circ\circ} \, A B + \bigcirc \, A_\bullet B_\bullet \\ B_\bullet A_\bullet = \bigcirc \, A_\bullet B_\bullet + \bigcirc \, A B \\ B_\bullet A = q_{\bullet\circ} \, A B_\bullet + t_{\bullet\circ} \, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} \, A_\bullet B + t_{\circ\bullet} \, A B_\bullet \end{cases}$$

non intersecting paths

$$\begin{cases} q_{\bullet\bullet} = 0 \\ t_{\bullet\circ} = t_{\circ\bullet} = 0 \end{cases}$$

(ASM)
(osc. paths)



The quadratic algebra Z

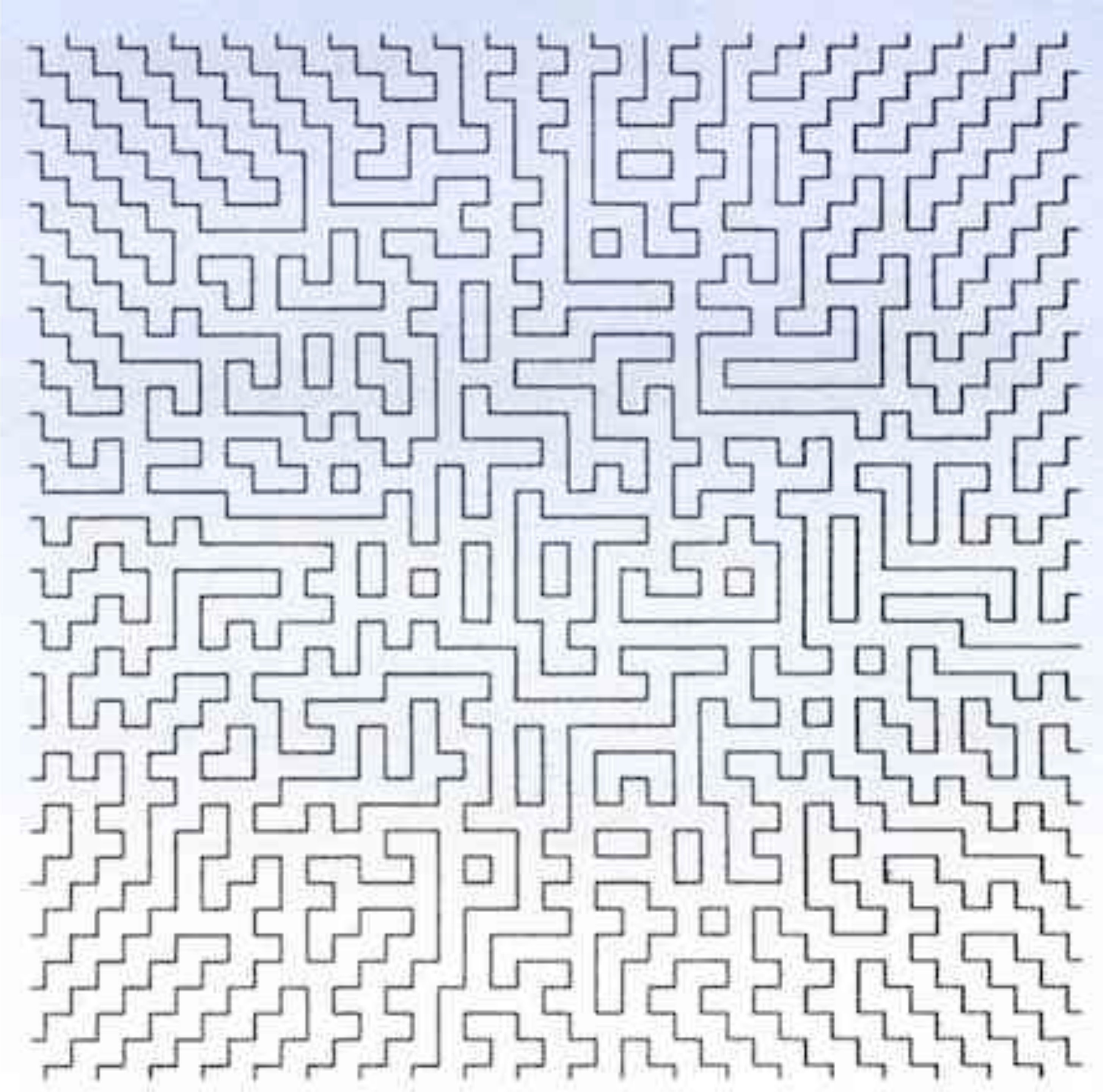
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} B\,A = q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = \bigcirc\, A_\bullet B_\bullet + t_{\bullet\bullet}\, A\,B \\ B_\bullet A = q_{\bullet\circ}\, A\,B_\bullet + \bigcirc\, A_\bullet B \\ B\,A_\bullet = q_{\circ\bullet}\, A_\bullet B + \bigcirc\, A\,B_\bullet \end{cases}$$

FPL
fully packed loops

random
FPL

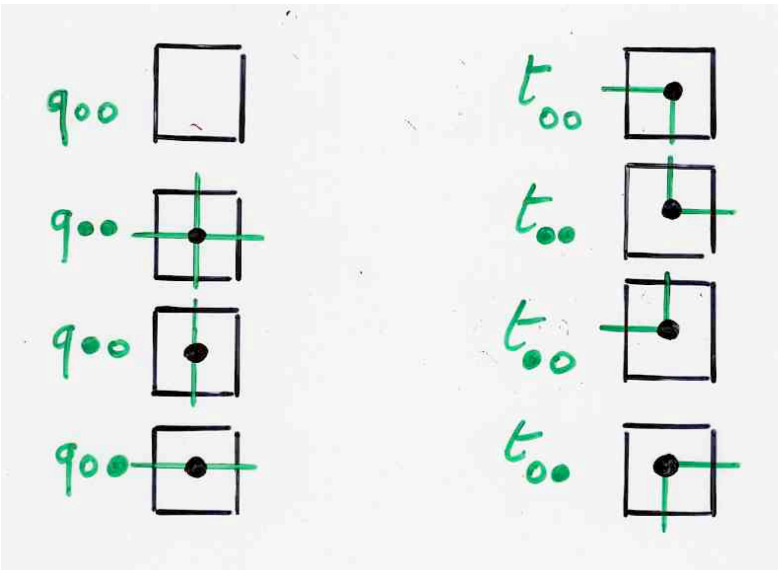


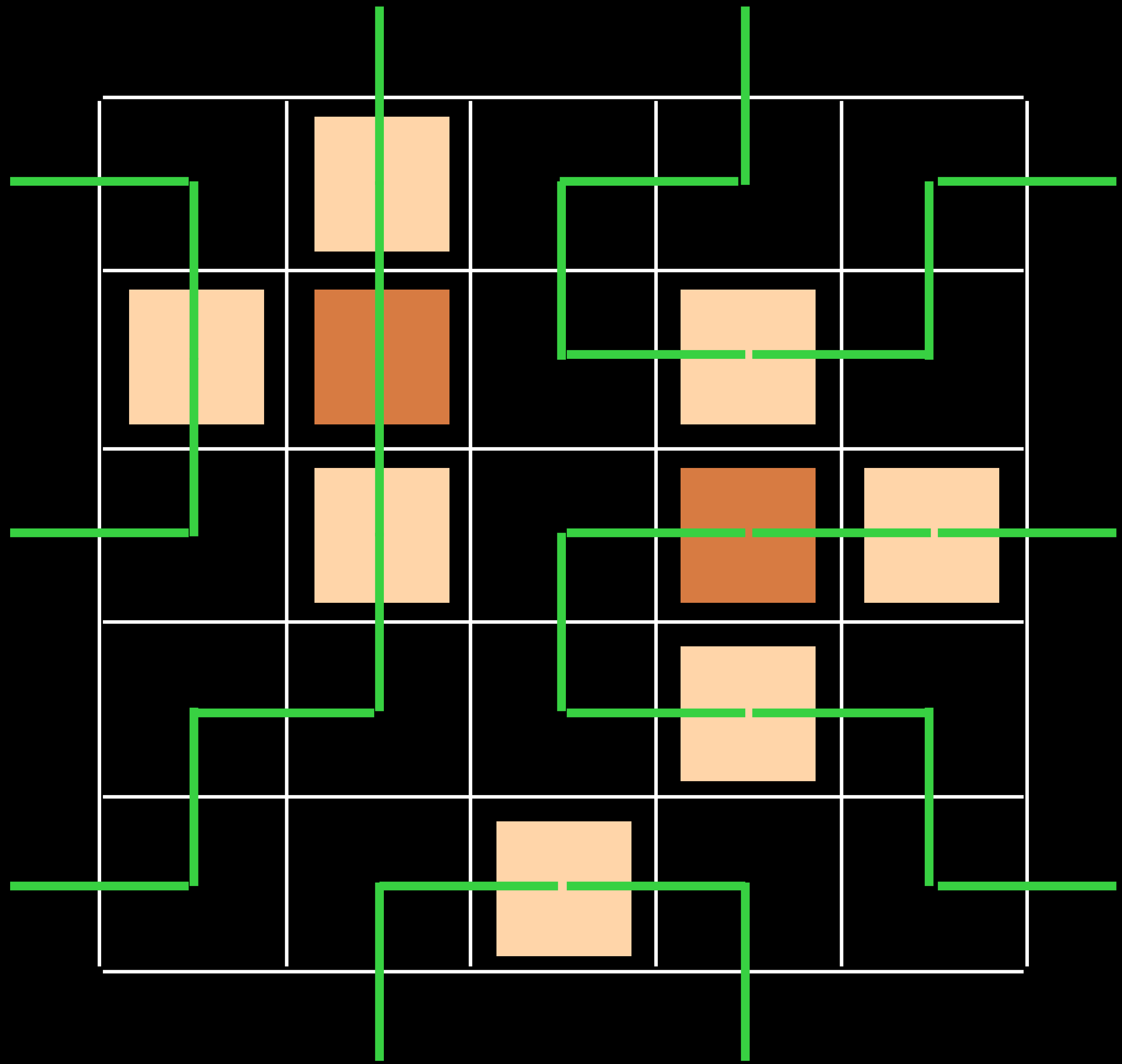
The quadratic algebra $\mathbb{Z}$

4 generators $B_\bullet\ A_\bullet\ B\ A$

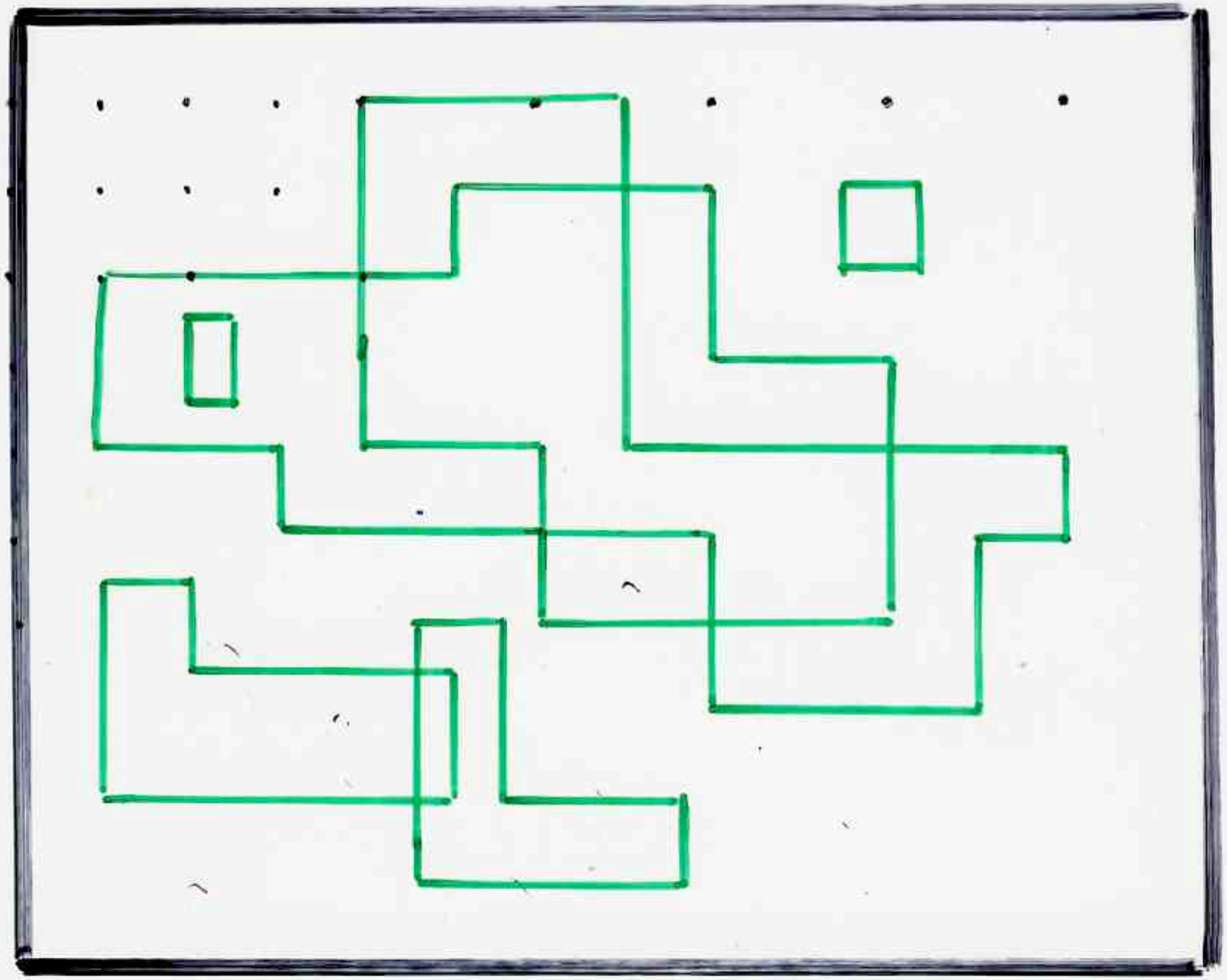
8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} BA = \bigcirc\, AB + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = \bigcirc\, A_\bullet B_\bullet + t_{\bullet\bullet}\, AB \\ B_\bullet A = q_{\bullet\circ}\, AB_\bullet + t_{\bullet\circ}\, A_\bullet B \\ BA_\bullet = q_{\circ\bullet}\, A_\bullet B + t_{\circ\bullet}\, AB_\bullet \end{cases}$$





Ising model



"closed" graph

Ising model

$w = B^m A^n$

$uv = A^n B^m$

XYZ-tableaux
or
B.A.BA configurations

The quadratic algebra Z

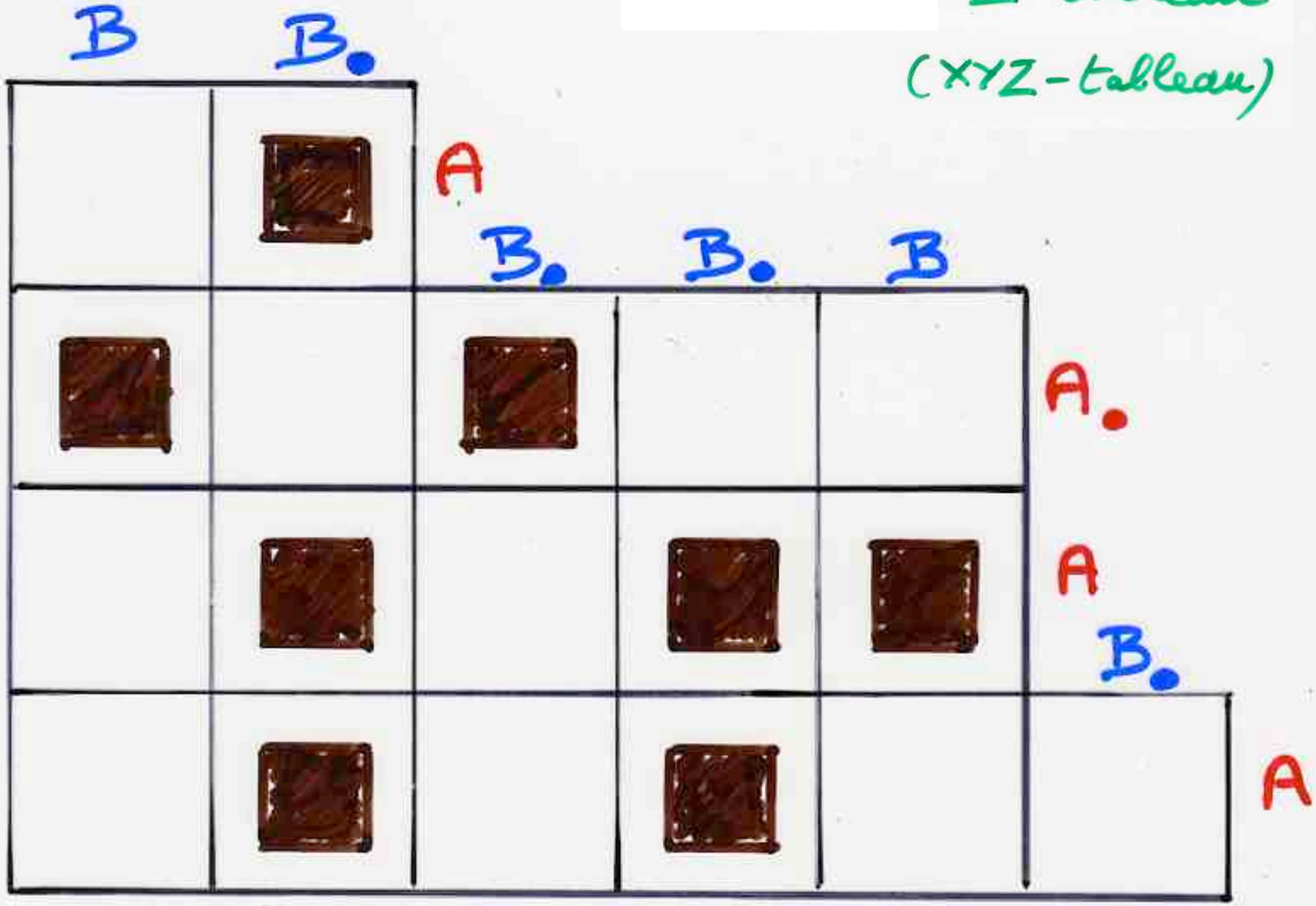
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$
\begin{cases}
B\,A = \square\, A\,B + \blacksquare\, A_\bullet B_\bullet \\
B_\bullet A_\bullet = \square\, A_\bullet B_\bullet + \blacksquare\, A\,B \\
B_\bullet A = \square\, A\,B_\bullet + \blacksquare\, A_\bullet B \\
B\,A_\bullet = \square\, A_\bullet B + \blacksquare\, A\,B_\bullet
\end{cases}
$$

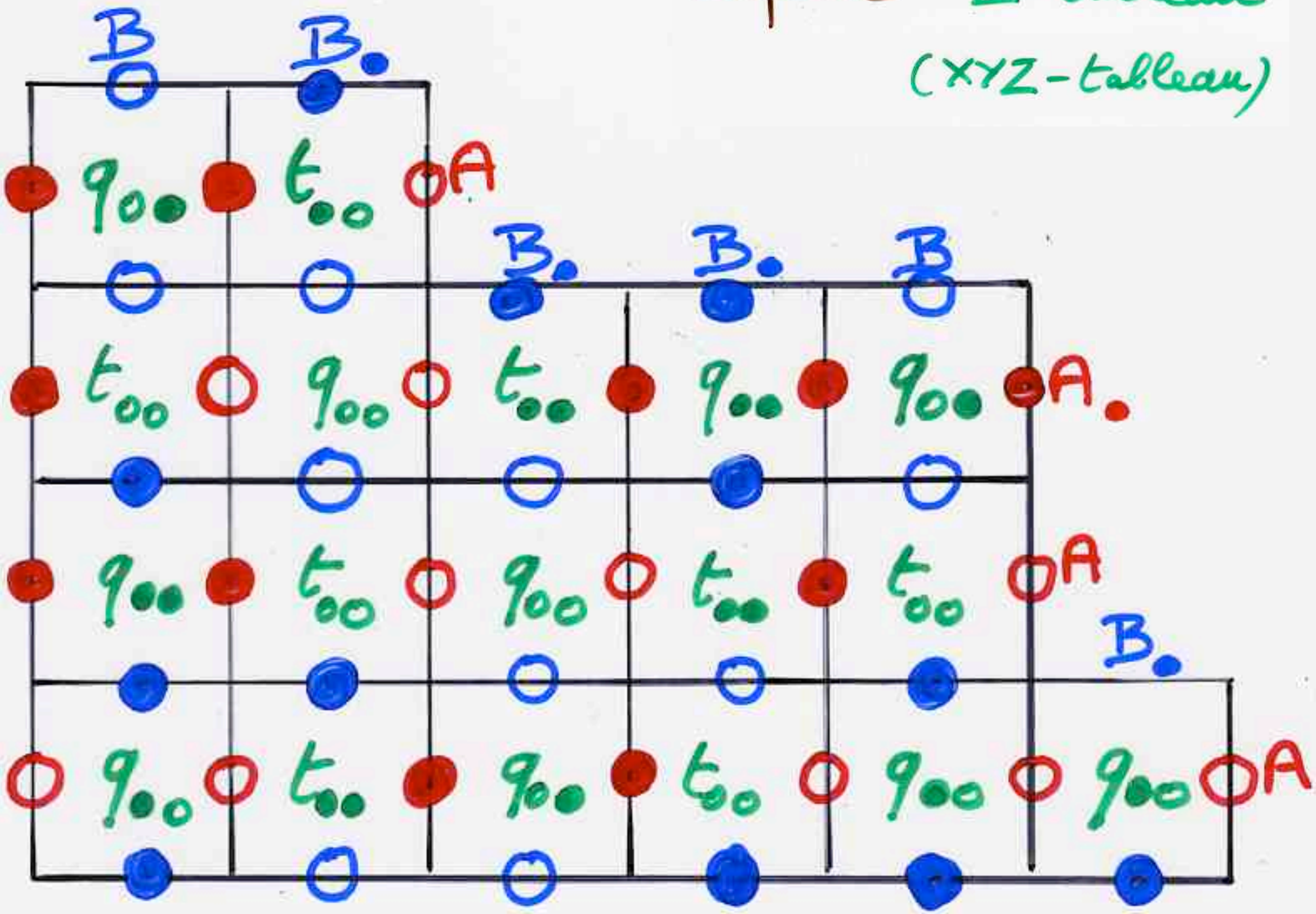
Z-tableau

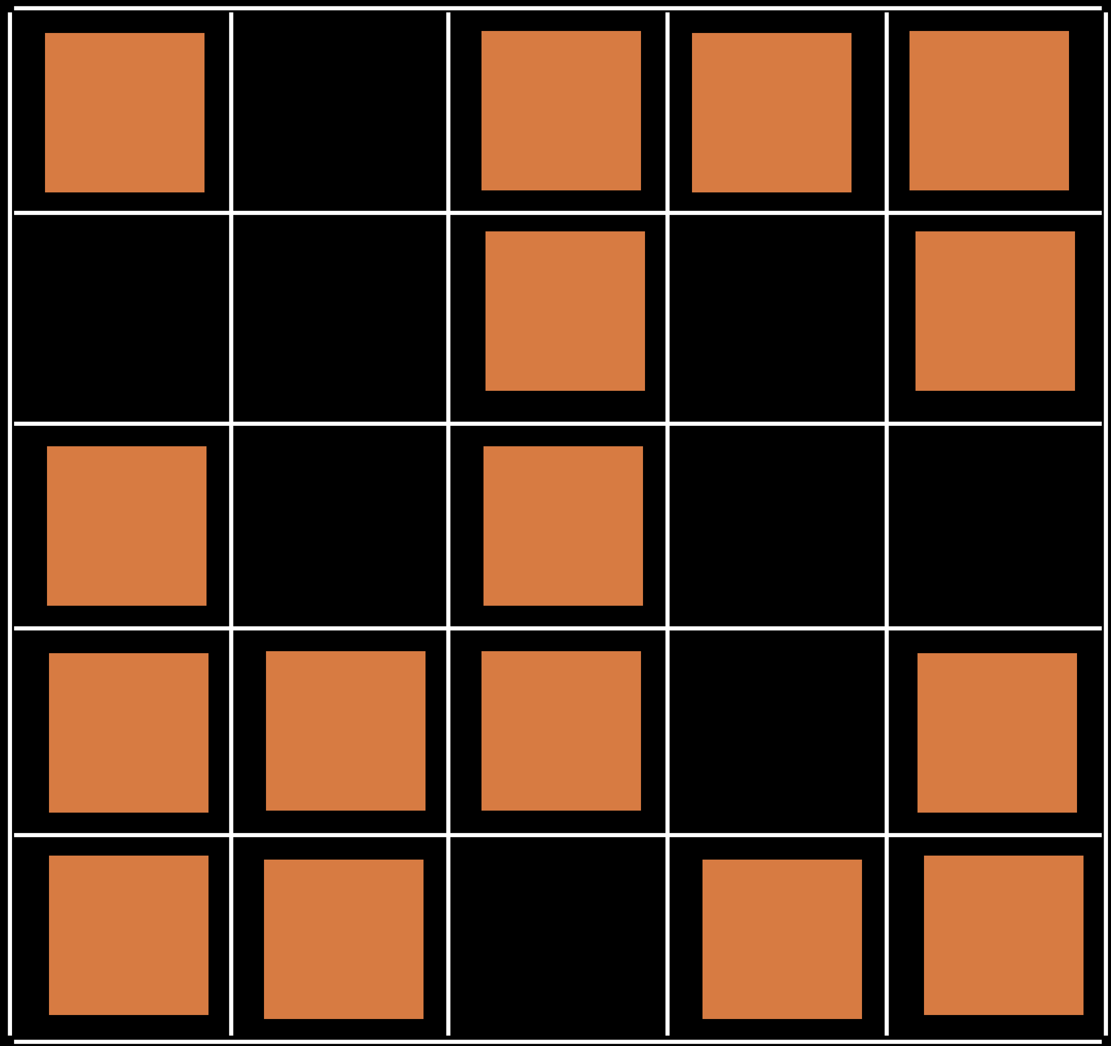
(XYZ-tableau)

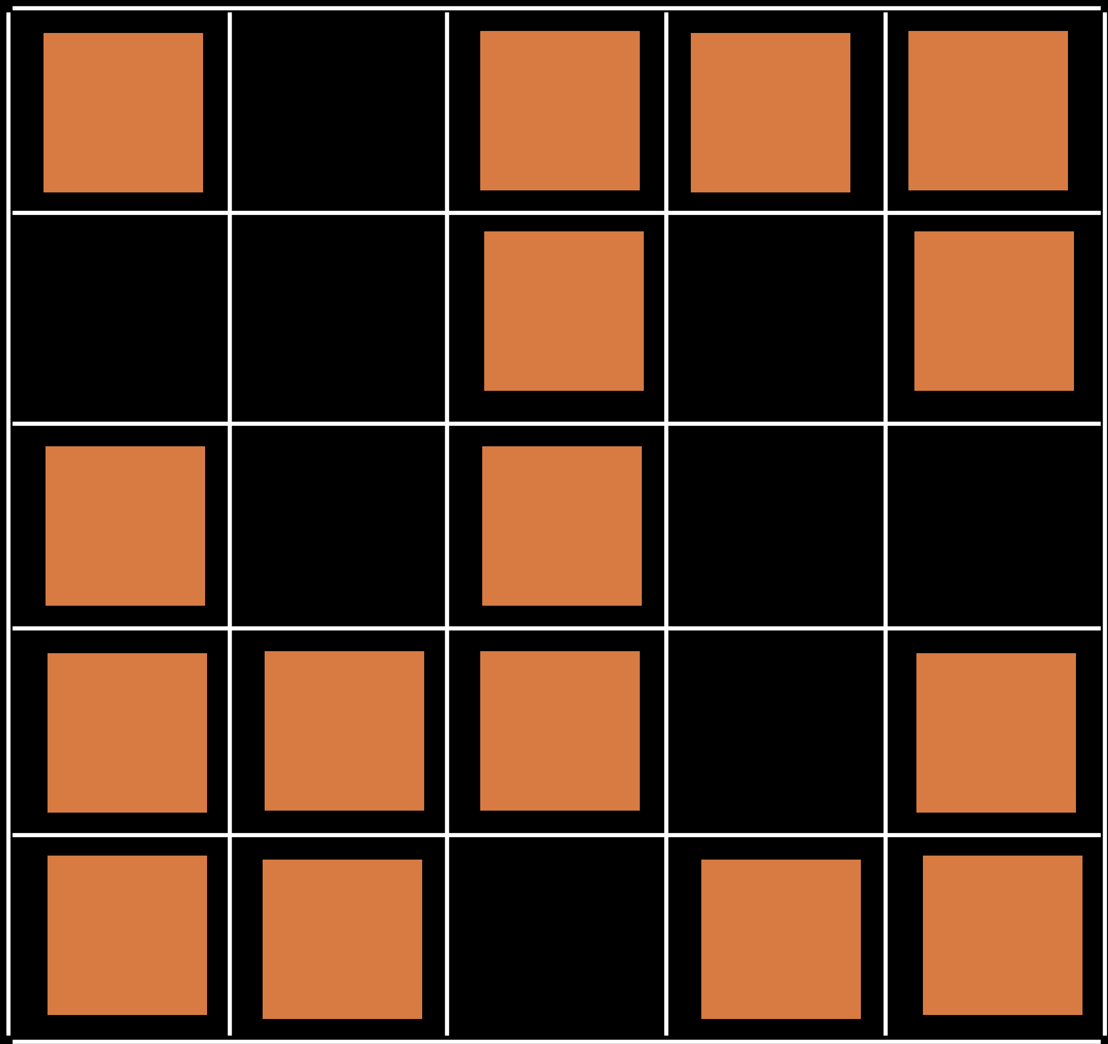


complete Z-tableau

(XYZ-tableau)

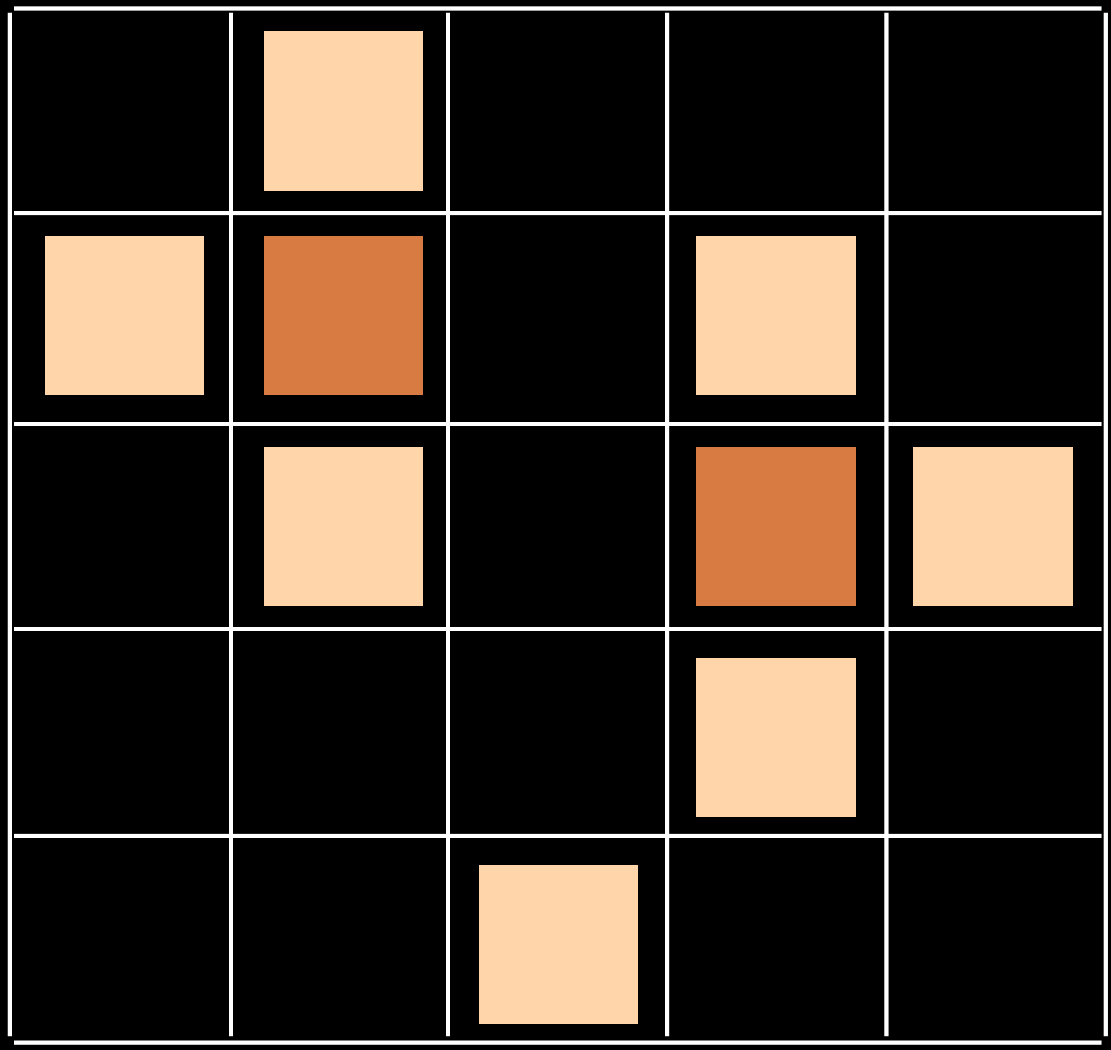




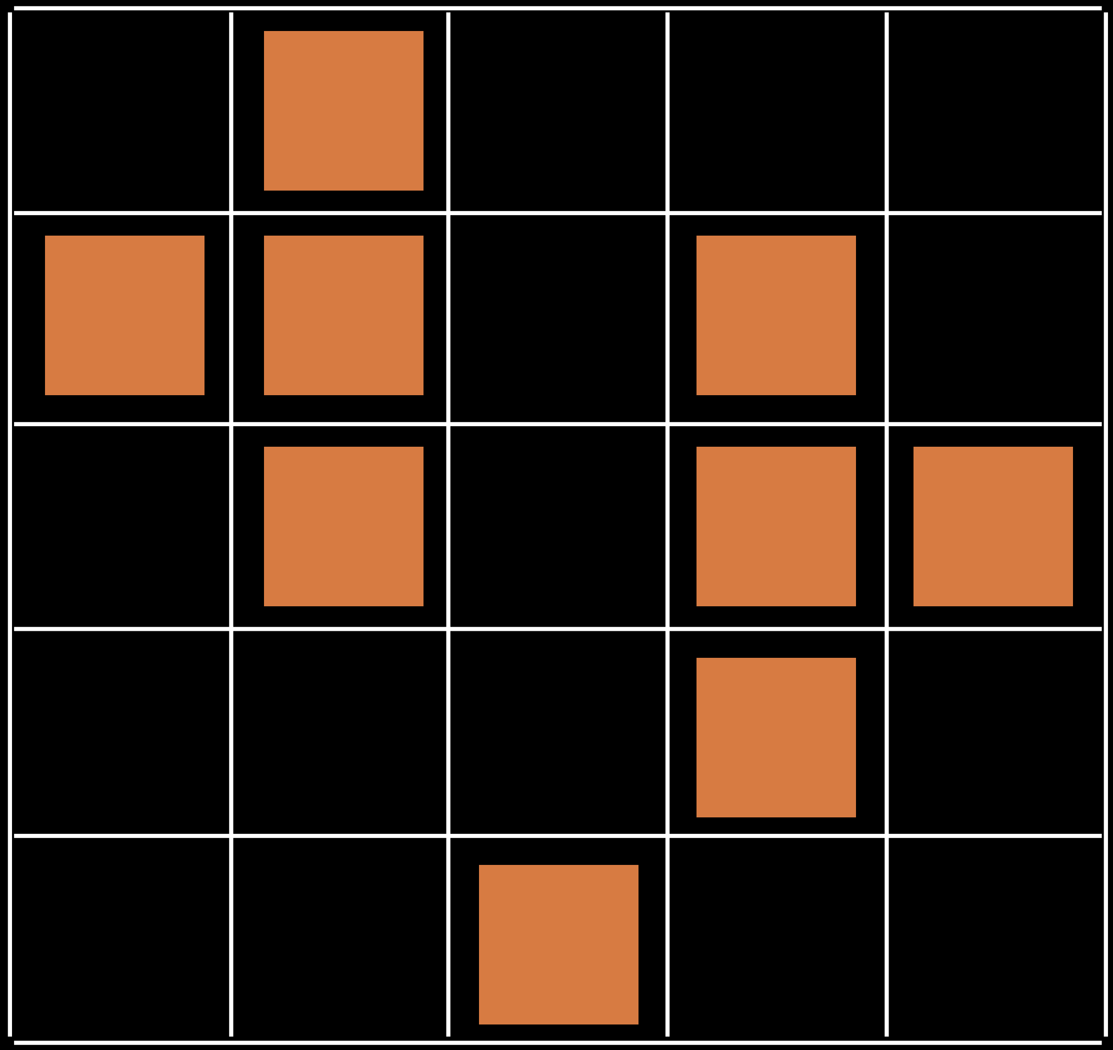


<u>Prop.</u> The number of configurations B.A. BA on $n \times n$ is $2^{(n^2)}$

alternating sign matrix



alternating sign matrix



Razumov - Stroganov (ex)- conjecture

ASM

alternating sign matrices

XXZ spin chains model

FPL

fully packed loops

proof by :

L. Cantini and A.Sportiello (March 2010)

arXiv: 1003.3376 [math.CO]

based on «Wieland rotation»

completely combinatorial proof

correlations functions
in XXZ spin chains

LPENSL-TH-05

Exact results for the σ^z two-point function of the XXZ chain at $\Delta = 1/2$

N. Kitanine[1], J. M. Maillet[2], N. A. Slavnov[3], V. Terras[4]

Abstract

We propose a new multiple integral representation for the correlation function $\langle \sigma_1^z \sigma_{m+1}^z \rangle$ of the XXZ spin-$\frac{1}{2}$ Heisenberg chain in the disordered regime. We show that for $\Delta = 1/2$ the integrals can be separated and computed exactly. As an example we give the explicit results up to the lattice distance $m = 8$. It turns out that the answer is given as integer numbers divided by $2^{(m+1)^2}$.

[1]LPTM, UMR 8089 du CNRS, Université de Cergy-Pontoise, France, kitanine@ptm.u-cergy.fr
[2]Laboratoire de Physique, UMR 5672 du CNRS, ENS Lyon, France, maillet@ens-lyon.fr
[3]Steklov Mathematical Institute, Moscow, Russia, nslavnov@mi.ras.ru
[4]LPTA, UMR 5207 du CNRS, Montpellier, France, terras@lpta.univ-montp2.fr

e^{2z_j}, it reduces to the derivatives of order $m-1$ with respect to each x_j at $x_1 = \cdots = x_n = e^{\frac{i\pi}{3}}$ and $x_{n+1} = \cdots = x_m = e^{-\frac{i\pi}{3}}$. If the lattice distance m is not too large, the representations (9), (11) can be successfully used to compute $\langle Q_\kappa(m)\rangle$ explicitely. As an example we give below the list of results for $P_m(\kappa) = 2^{m^2}\langle Q_\kappa(m)\rangle$ up to $m = 9$:

intergers ?

positivity ?

$$
\begin{aligned}
P_1(\kappa) &= 1+\kappa,\\
P_2(\kappa) &= 2+12\kappa+2\kappa^2,\\
P_3(\kappa) &= 7+249\kappa+249\kappa^2+7\kappa^3,\\
P_4(\kappa) &= 42+10004\kappa+45444\kappa^2+10004\kappa^3+42\kappa^4,\\
P_5(\kappa) &= 429+738174\kappa+16038613\kappa^2+16038613\kappa^3+738174\kappa^4+429\kappa^5,\\
P_6(\kappa) &= 7436+96289380\kappa+11424474588\kappa^2+45677933928\kappa^3+11424474588\kappa^4\\
&+96289380\kappa^5+7436\kappa^6,\\
P_7(\kappa) &= 218348+21798199390\kappa+15663567546585\kappa^2+265789610746333\kappa^3\\
&+265789610746333\kappa^4+15663567546585\kappa^5+21798199390\kappa^6+218348\kappa^7,\\
P_8(\kappa) &= 10850216+8485108350684\kappa+39461894378292782\kappa^2\\
&+3224112384882251896\kappa^3+11919578544950060460\kappa^4+3224112384882251896\kappa^5\\
&+39461894378292782\kappa^6+8485108350684\kappa^7+10850216\kappa^8\\
P_9(\kappa) &= 911835460+5649499685353257\kappa+177662495637443158524\kappa^2\\
&+77990624578576910368767\kappa^3+1130757526890914223990168\kappa^4
\end{aligned}
\qquad (12)
$$

e^{2z_j}, it reduces to the derivatives of order $m-1$ with respect to each x_j at $x_1 = \cdots = x_n = e^{\frac{i\pi}{3}}$ and $x_{n+1} = \cdots = x_m = e^{-\frac{i\pi}{3}}$. If the lattice distance m is not too large, the representations (9), (11) can be successfully used to compute $\langle Q_\kappa(m)\rangle$ explicitely. As an example we give below the list of results for $P_m(\kappa) = 2^{m^2}\langle Q_\kappa(m)\rangle$ up to $m = 9$:

intergers ?

positivity ?

combinatorial interpretation ?

ASM

FPL

$$
\begin{aligned}
P_1(\kappa) &= 1 + \kappa, \\
P_2(\kappa) &= 2 + 12\kappa + 2\kappa^2, \\
P_3(\kappa) &= 7 + 249\kappa + 249\kappa^2 + 7\kappa^3, \\
P_4(\kappa) &= 42 + 10004\kappa + 45444\kappa^2 + 10004\kappa^3 + 42\kappa^4, \\
P_5(\kappa) &= 429 + 738174\kappa + 16038613\kappa^2 + 16038613\kappa^3 + 738174\kappa^4 + 429\kappa^5, \\
P_6(\kappa) &= 7436 + 96289380\kappa + 11424474588\kappa^2 + 45677933928\kappa^3 + 11424474588\kappa^4 \\
&+ 96289380\kappa^5 + 7436\kappa^6, \\
P_7(\kappa) &= 218348 + 21798199390\kappa + 15663567546585\kappa^2 + 265789610746333\kappa^3 \\
&+ 265789610746333\kappa^4 + 15663567546585\kappa^5 + 21798199390\kappa^6 + 218348\kappa^7, \\
P_8(\kappa) &= 10850216 + 8485108350684\kappa + 3946189437829278 2\kappa^2 \\
&+ 32241123848822518 96\kappa^3 + 110195\text{[illegible]}8544950060460\kappa^4 + 3224112384882251896\kappa^5 \\
&+ 39461894378292782\kappa^6 + 8485108350684\kappa^7 + 10850216\kappa^8 \\
P_9(\kappa) &= 911835460 + 5649499685353257\kappa + 1776624956374431585 24\kappa^2 \\
&+ 7799062457857691036 8767\kappa^3 + 1130757526890914223990168\kappa^4
\end{aligned}
\tag{12}
$$

2^{n^2}

po

8 - vertex model

XYZ- spin chains model

analog of

Razumov - Stroganov conjecture ?

$$2^{n^2}$$

The cellular Ansatz

quadratic algebra Q (of a certain type)

(1) "planarisation" on a grid of the rewriting rules

Q- tableaux planar automata

"The cellular Ansatz"

combinatorial
objects
on a 2d lattice

Physics

bijections

"normal ordering"

UD = DU + Id

Weyl-Heisenberg

rooks placements

permutations

RSK

⟷

pairs of Tableaux Young

quadratic algebra Q

Q-tableaux

the XYZ algebra

ASM, (alternating sign matrices)

FPL (Fully packed loops)

tilings, non-crossing paths

commutations
rewriting rules

RSK automata

planarization

planar
automata

The cellular Ansatz

quadratic algebra Q (of a certain type)

(1) "planarisation" on a grid of the rewriting rules

Q- tableaux planar automata

(2) "planarization" on a grid of the bijection constructed from a representation of the algebra Q

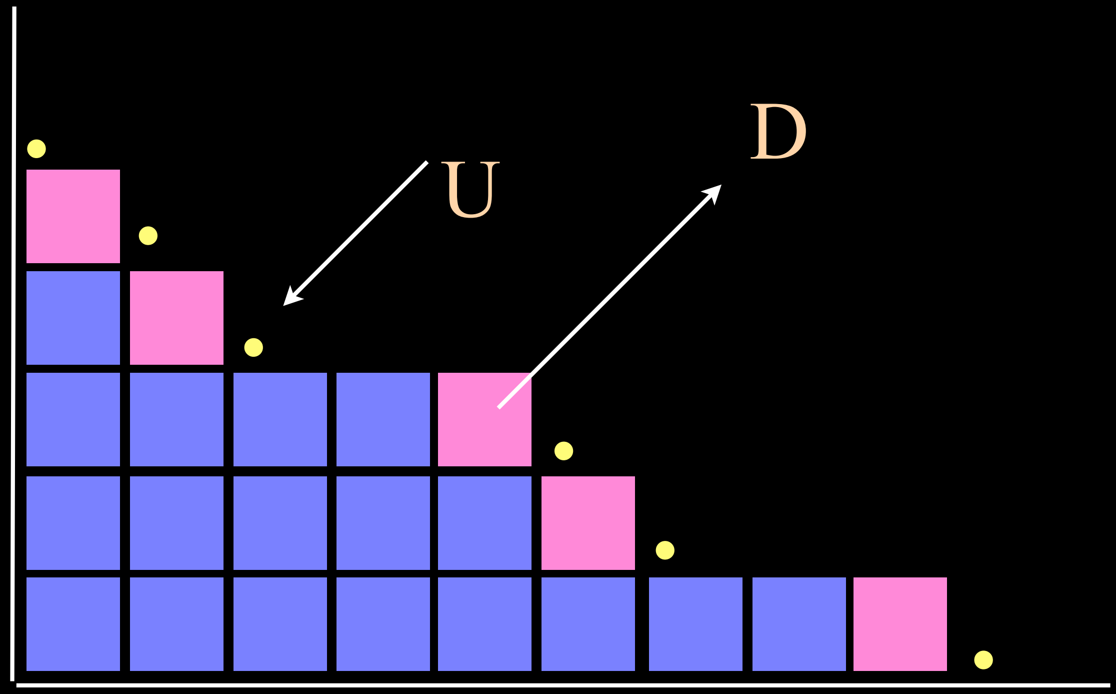
The cellular Ansatz

second part: $UD = DU + 1$

guided construction of a bijection from a representation of U and D acting on Ferrers diagrams

Sergey Fomin

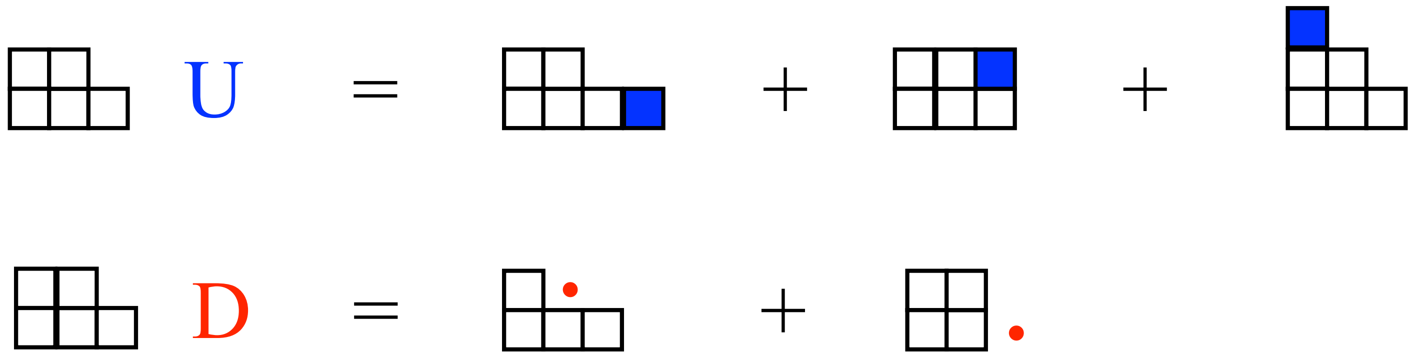
Operators U and D



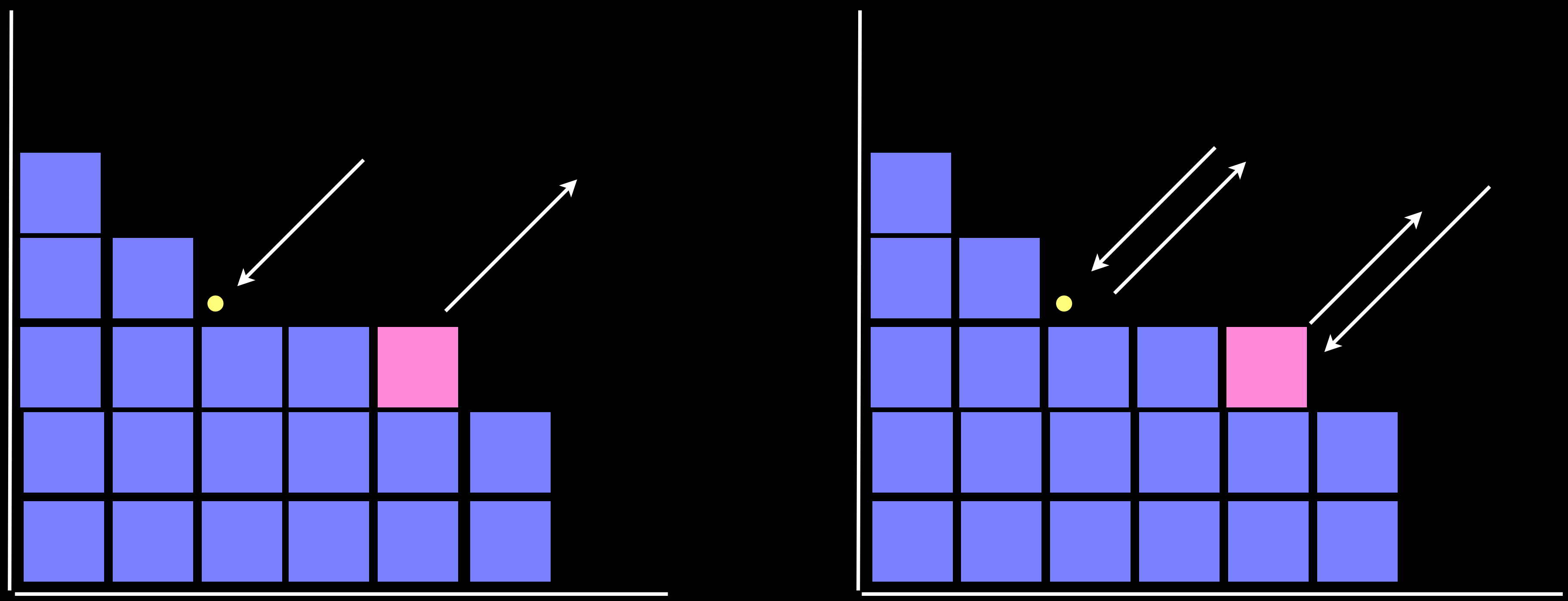
adding
or deleting
a cell in
a Ferrers
diagram

Young lattice

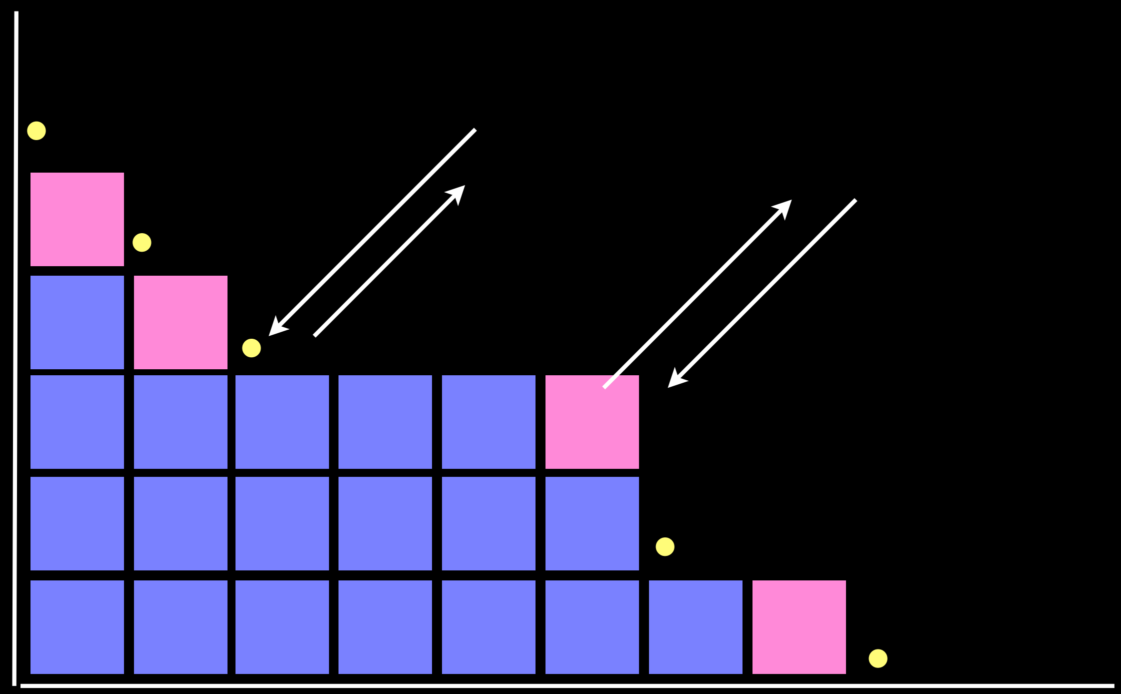
U and D are operators acting of the
vector space generated by Ferrers diagrams

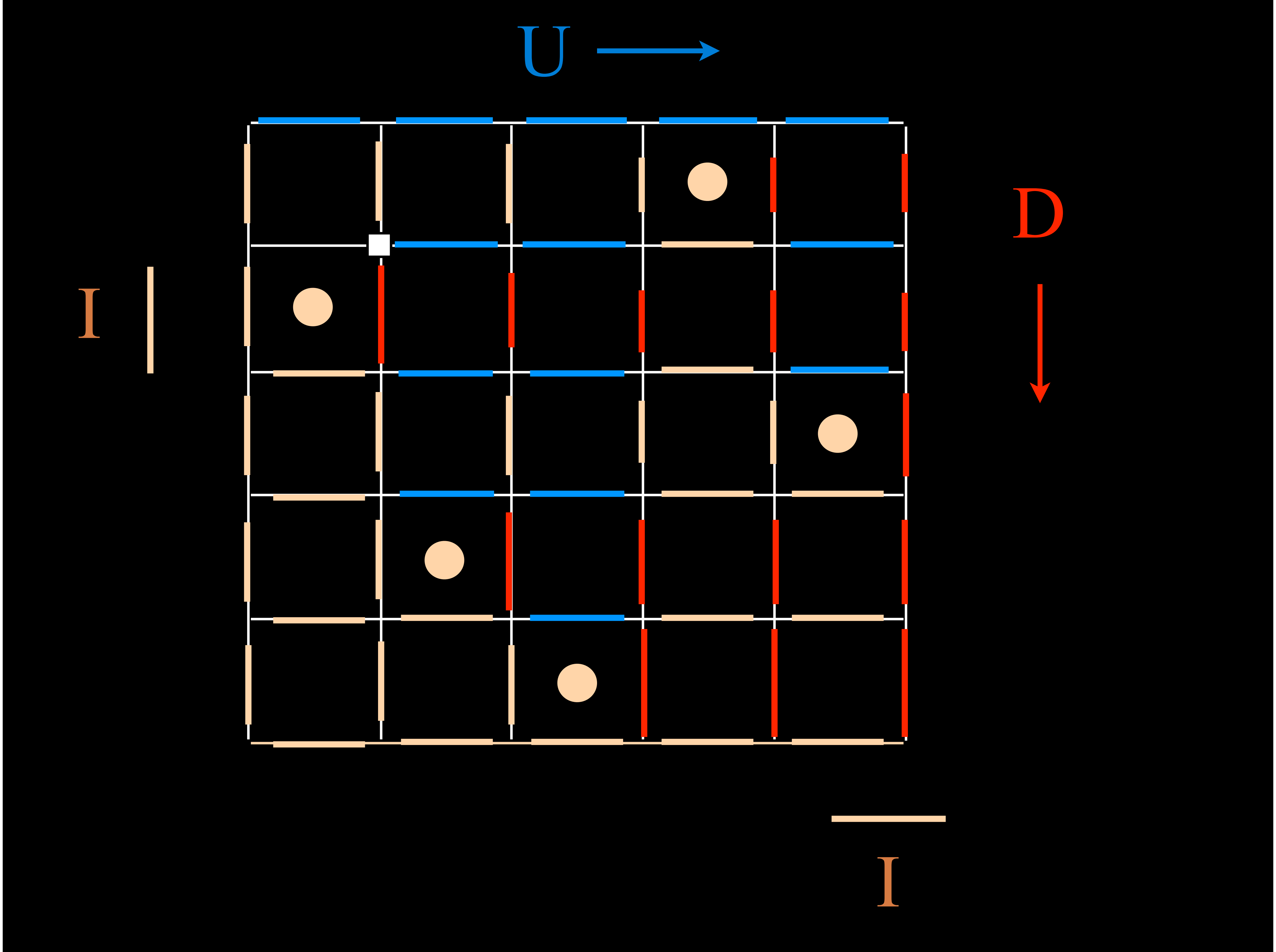


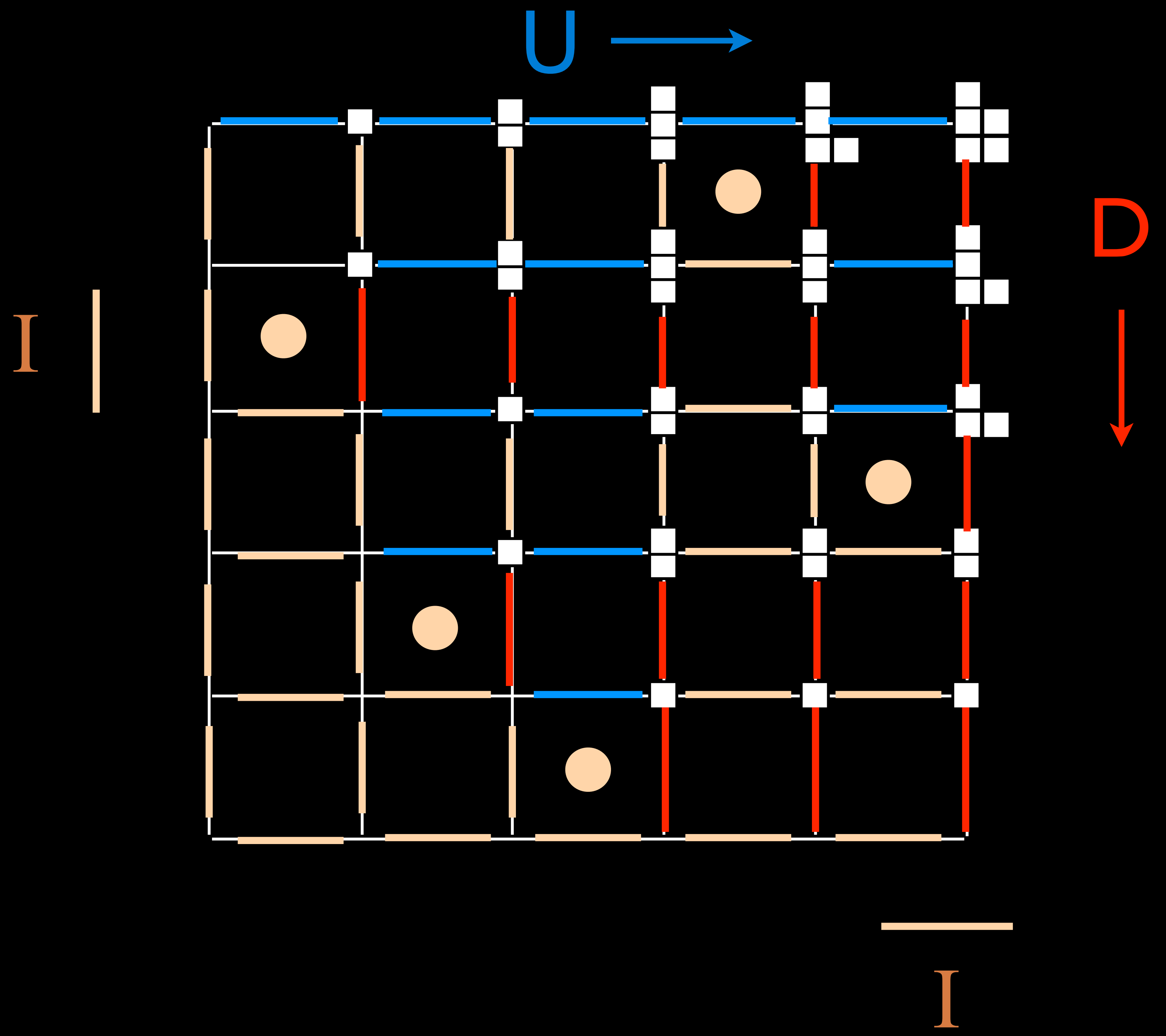
$$UD = DU + I$$

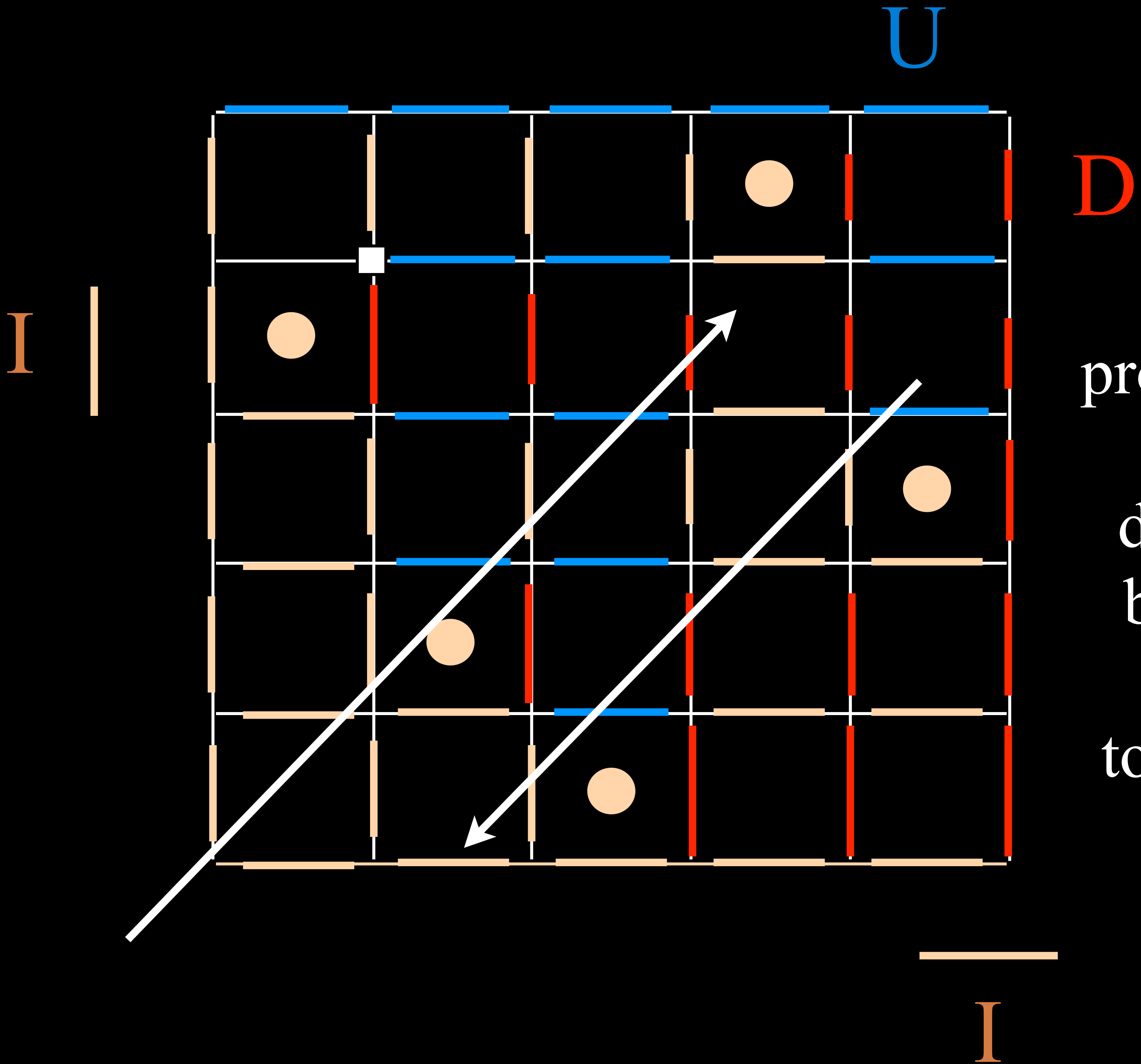


$$UD = DU + I$$

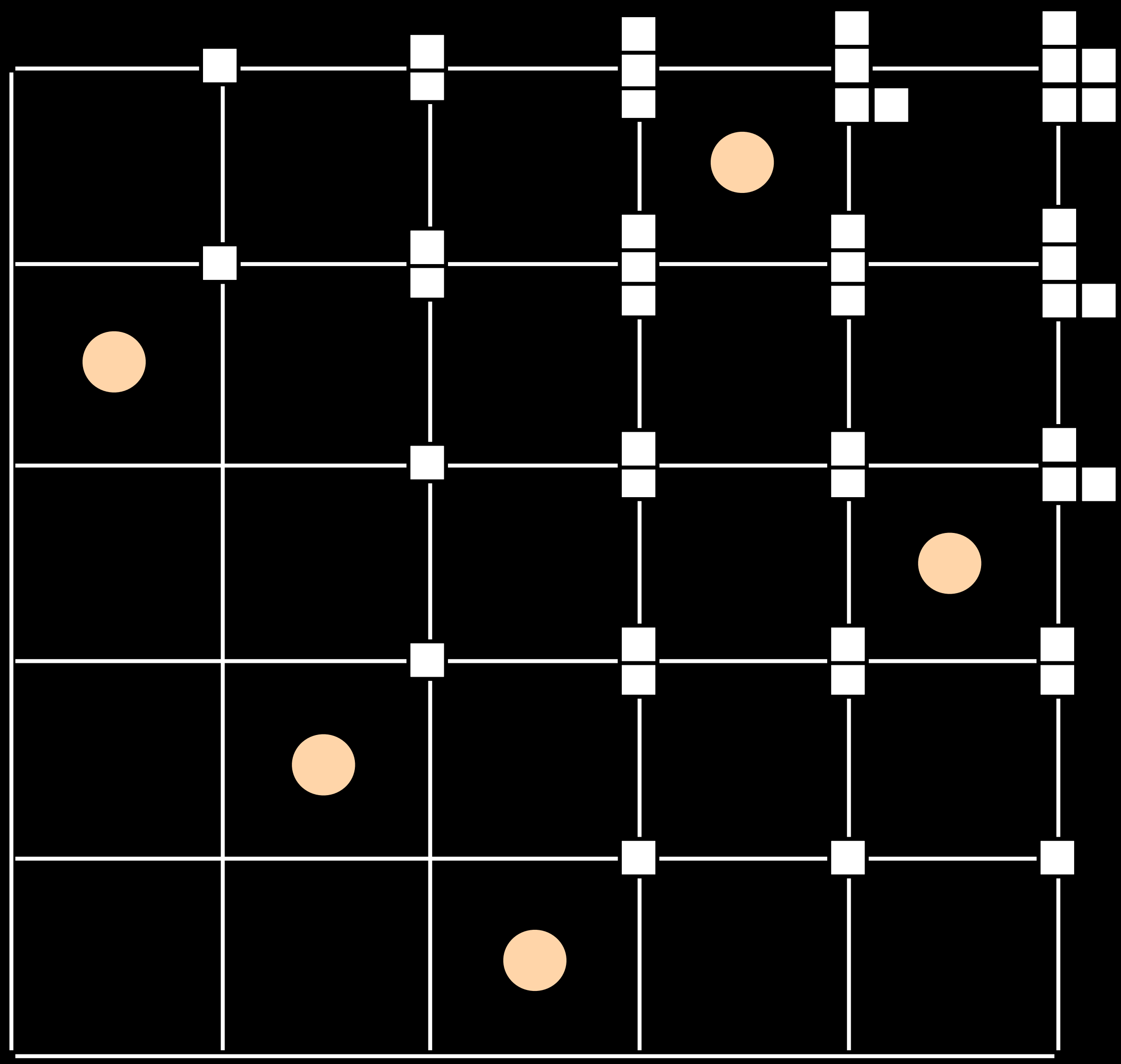



U
D
I
I


U
D
I
I

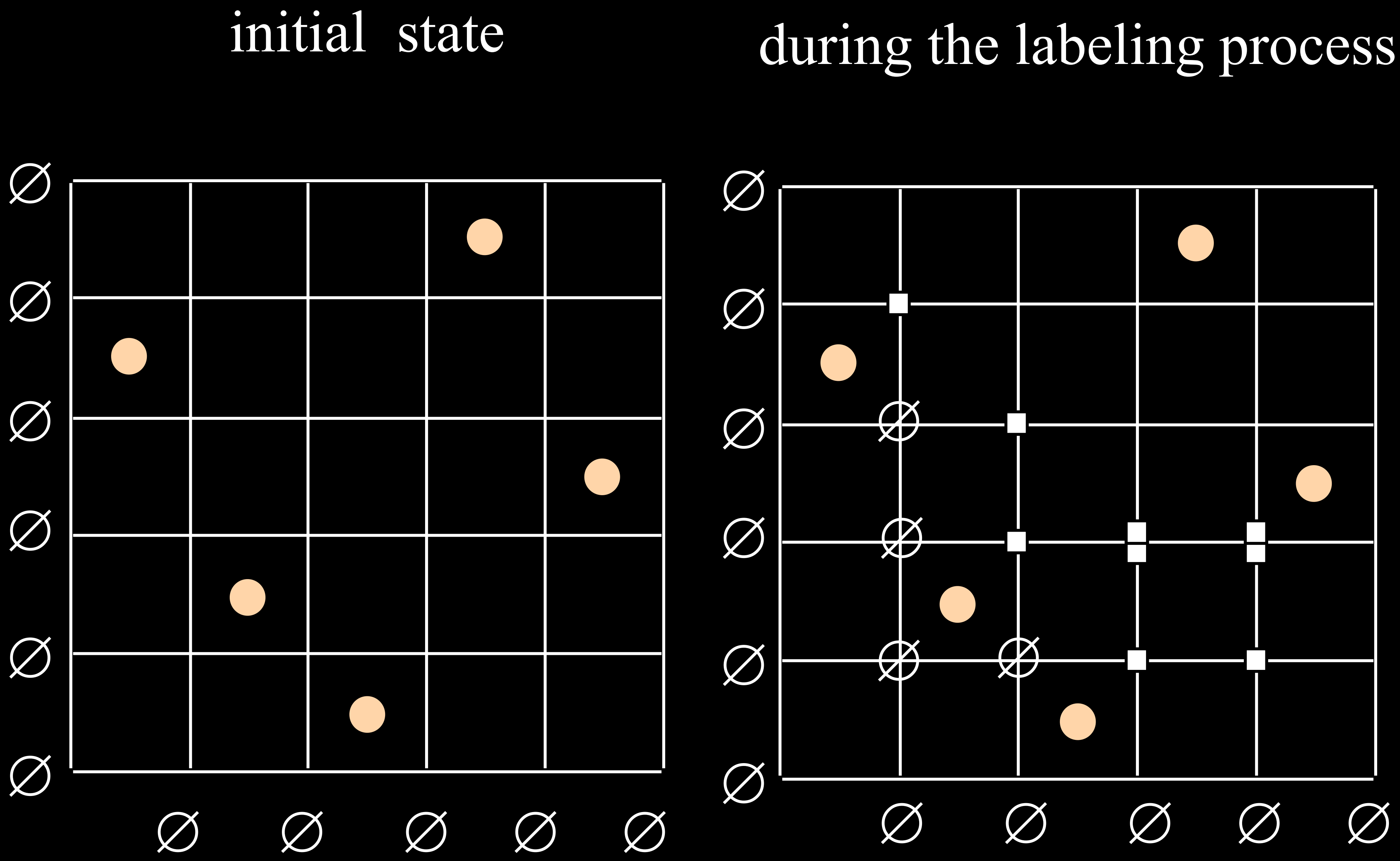


propagation
of the
diagrams
bijection
related
to one cell



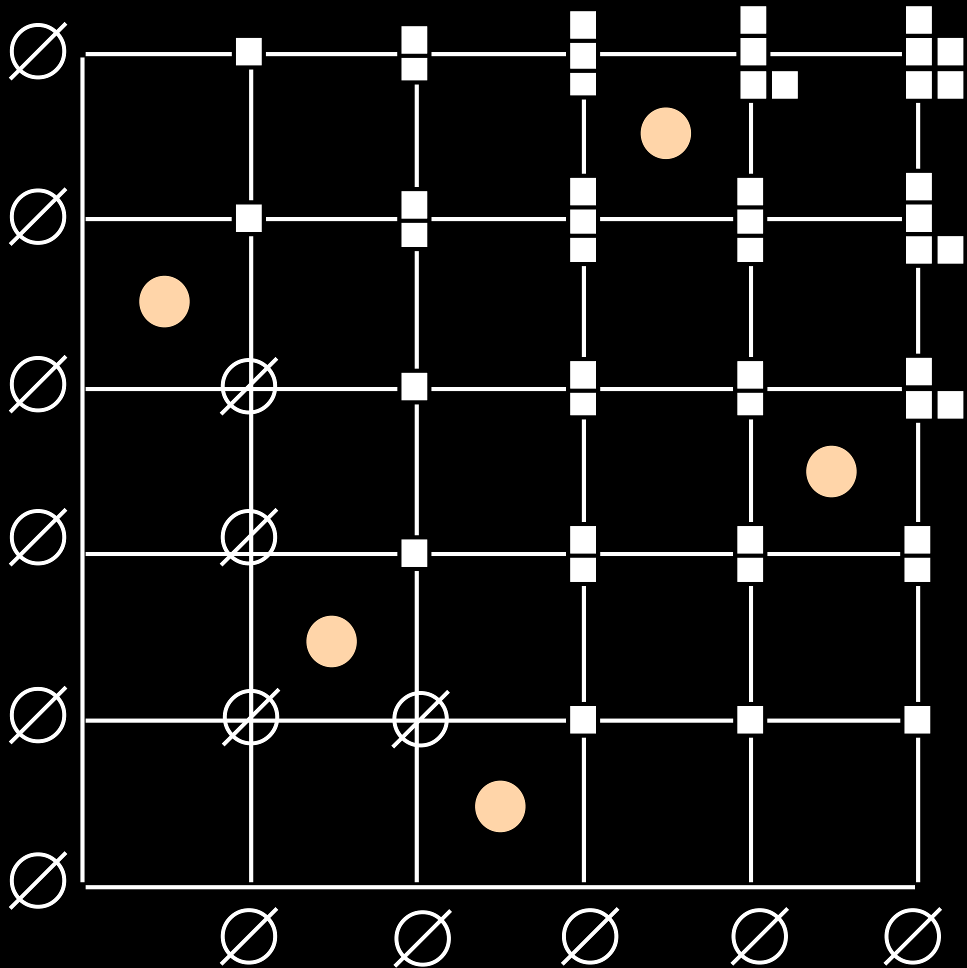
“local” algorithm on a grid or “growth diagrams”

Sergey Fomin

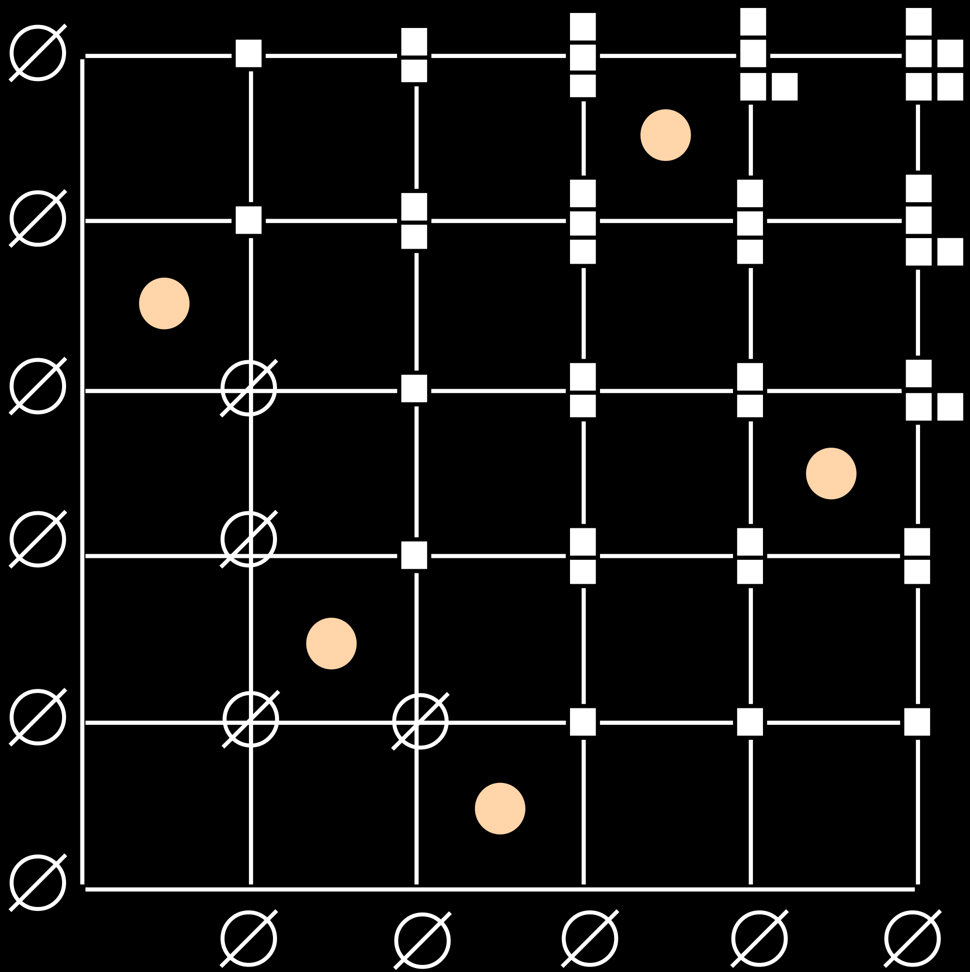


initial state
during the labeling process

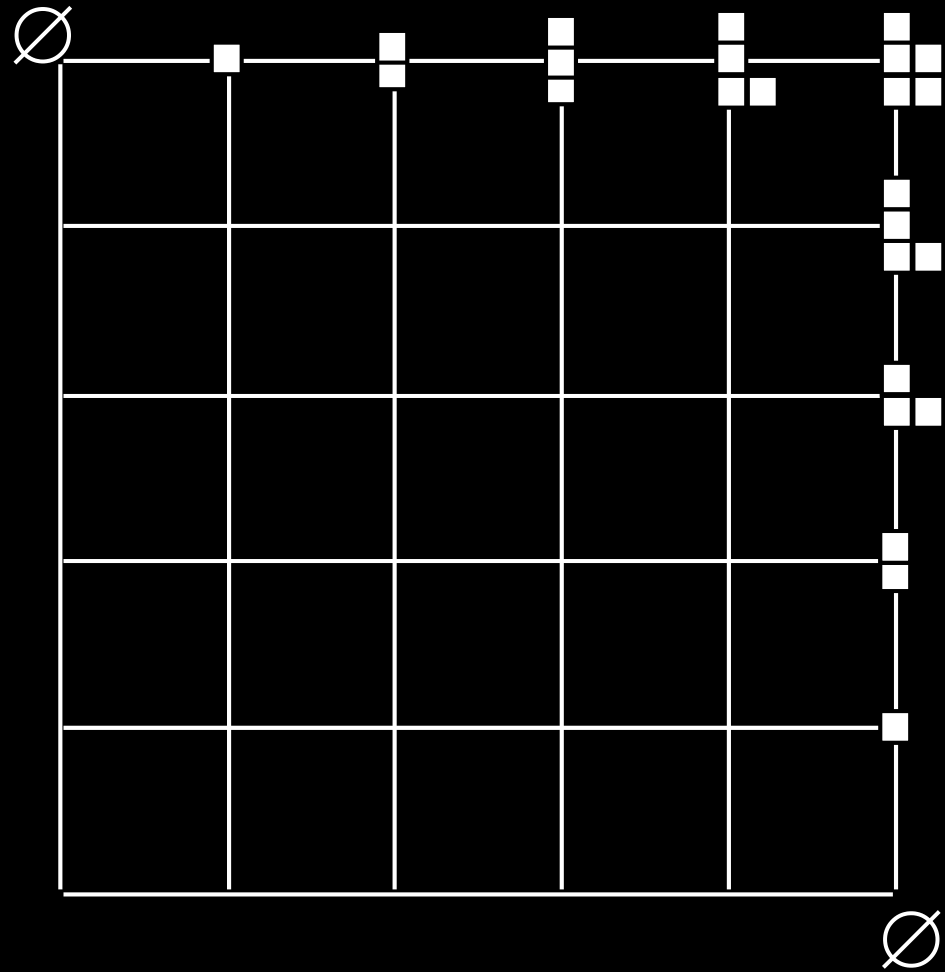
final state

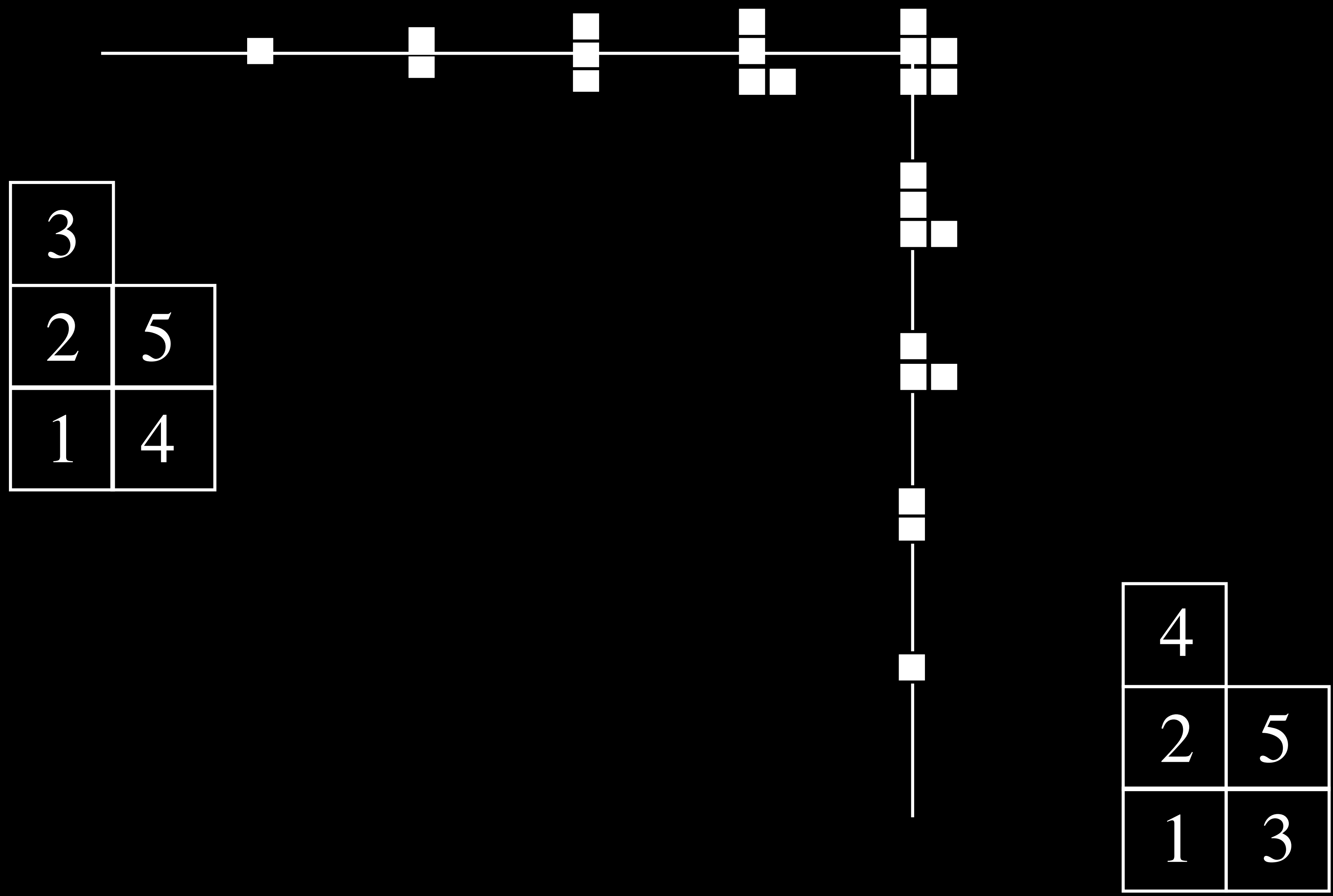


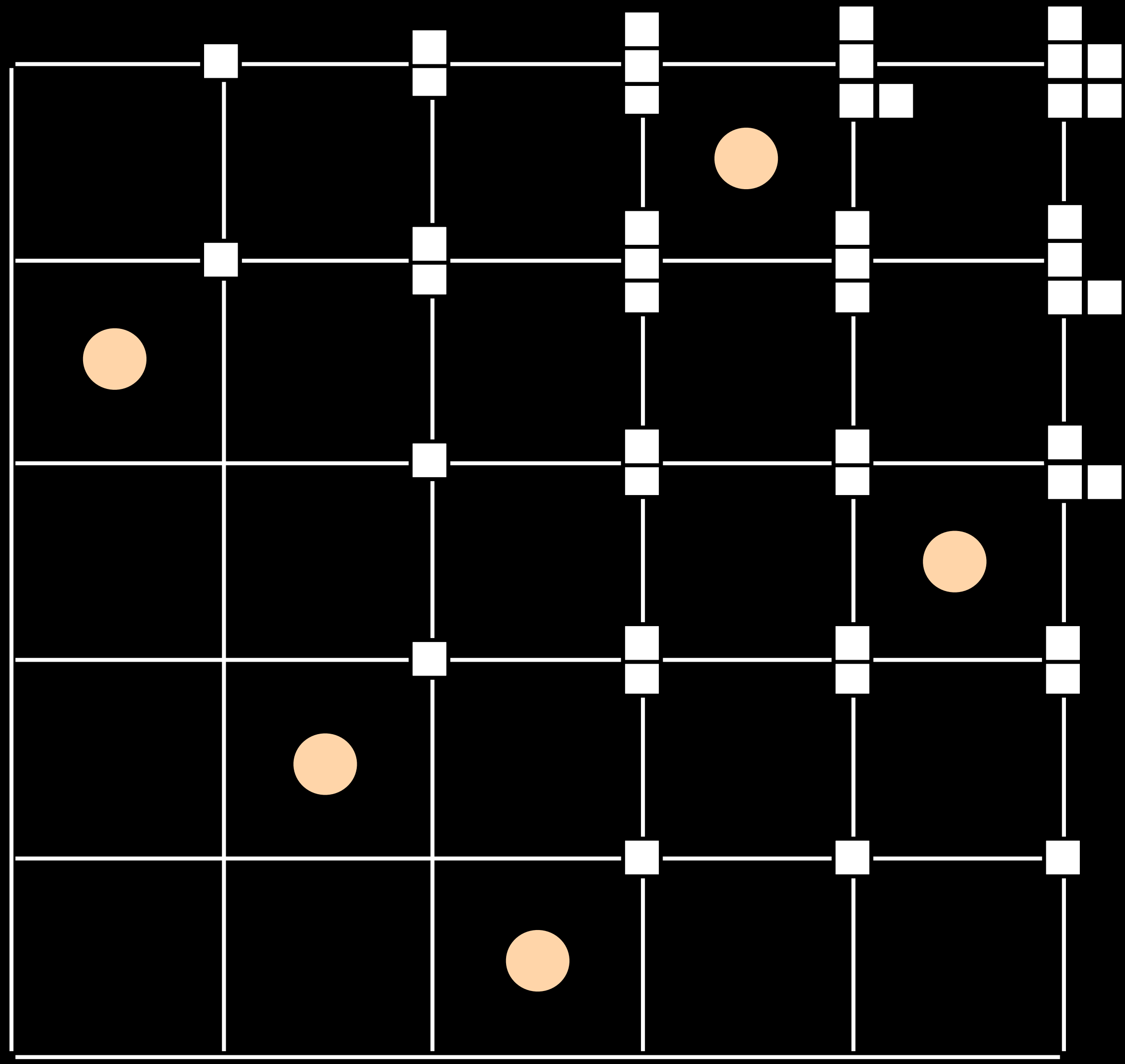
final state

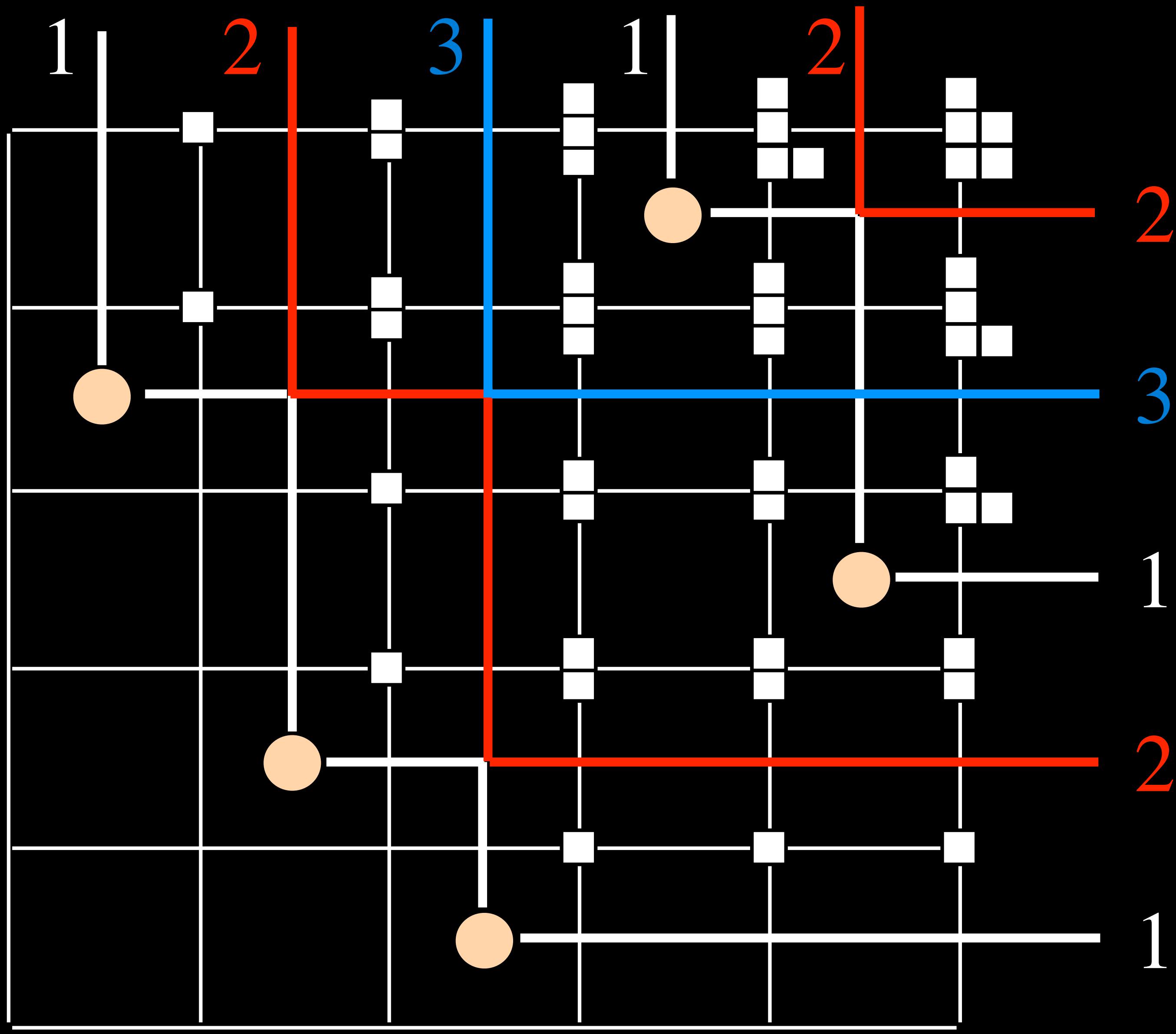


the pair (P,Q)

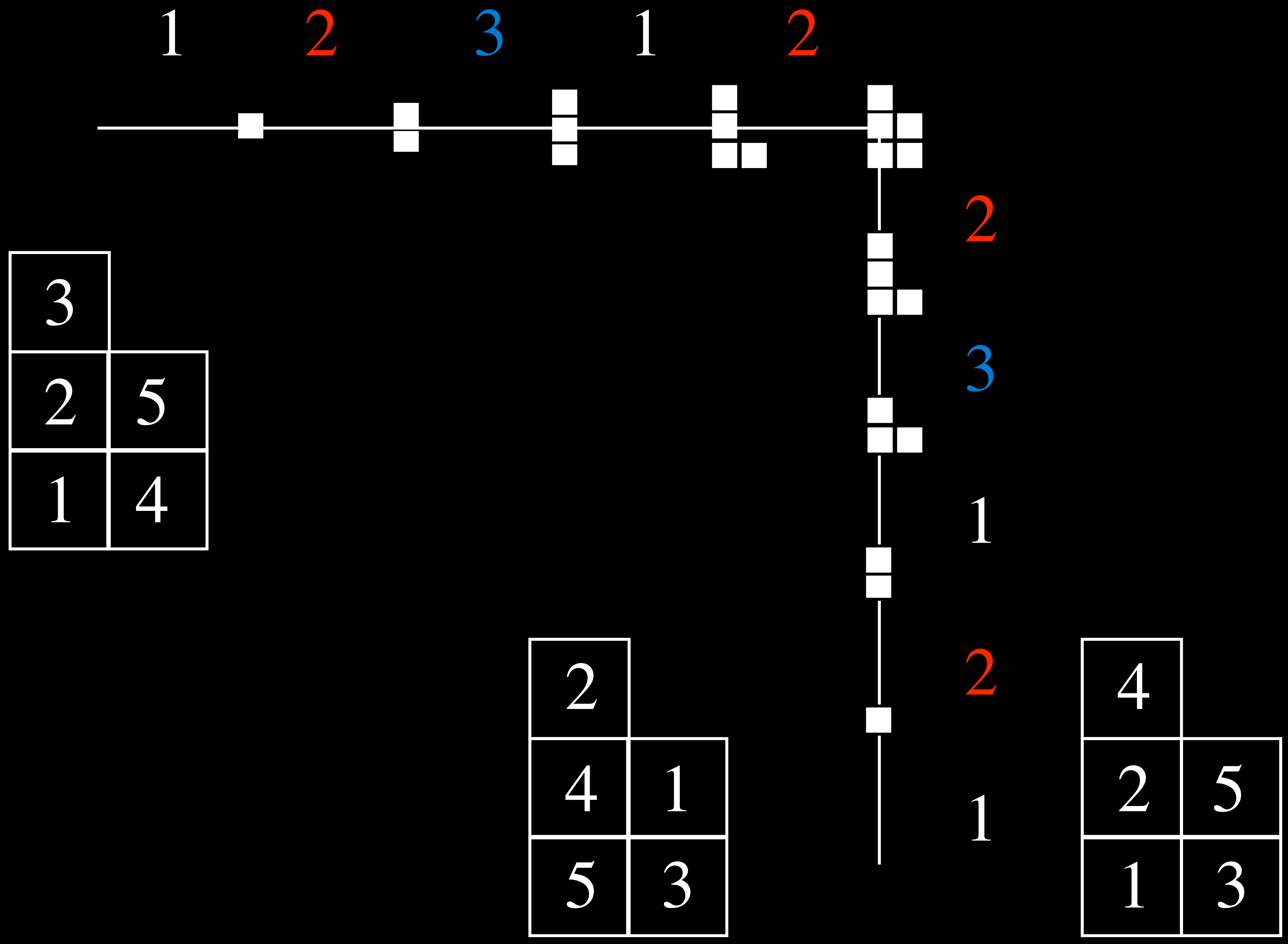



3
2 5
1 4
4
2 5
1 3

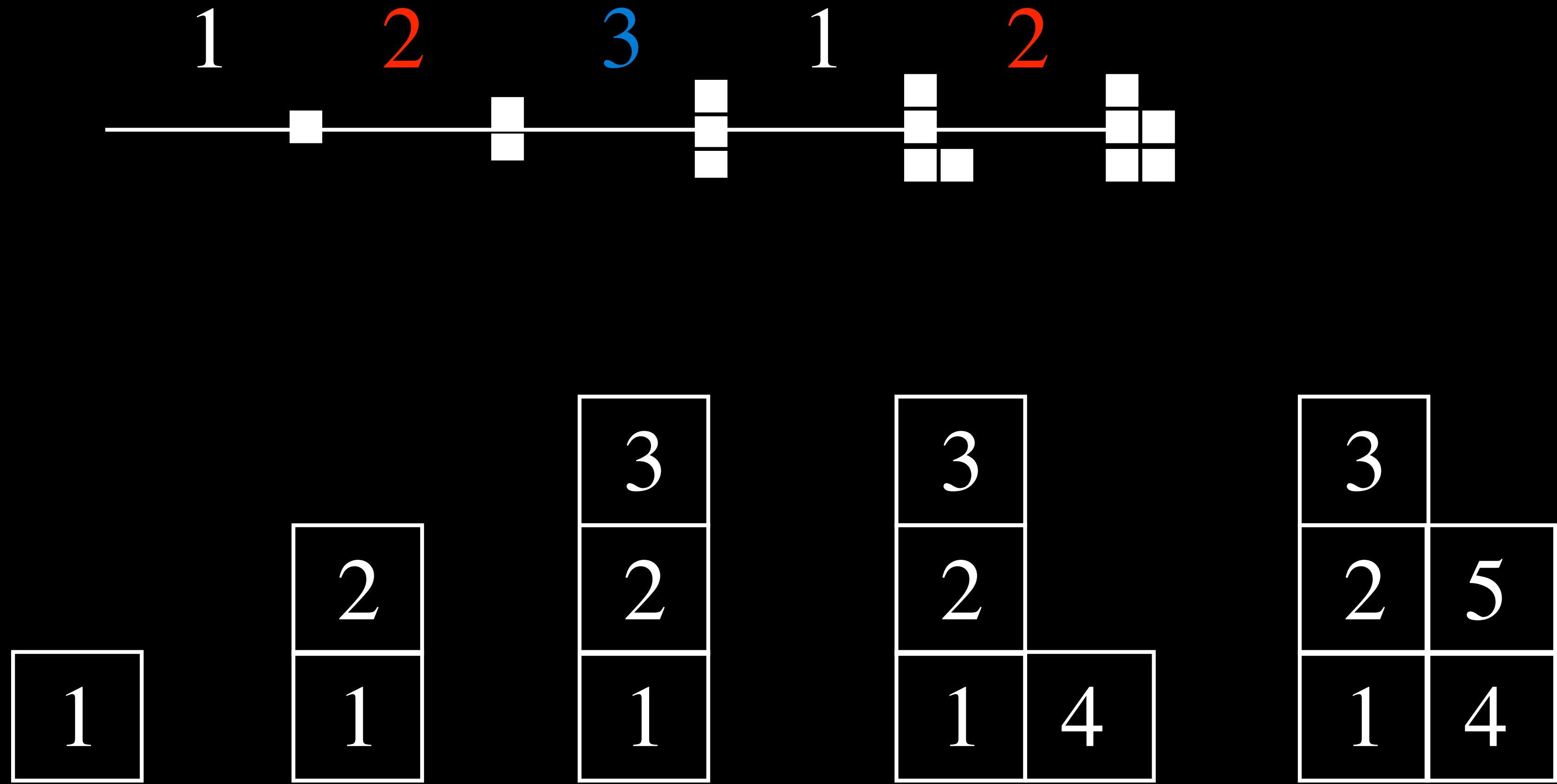




1
2
3
1
2
2
3
1
2
1



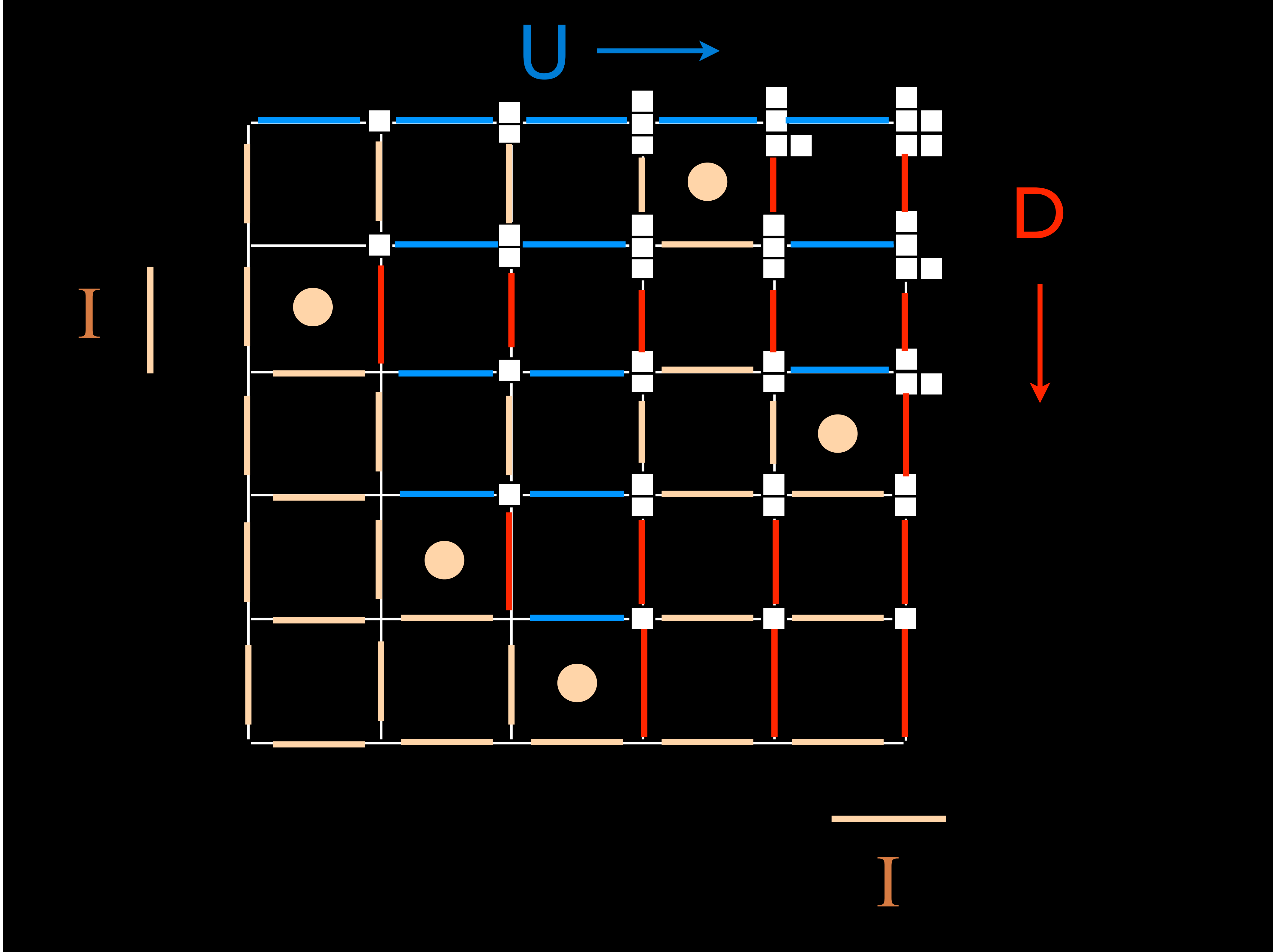
1 2 3 1 2
2 3 1 2 1
3
2 5
1 4
2
4 1
5 3
4
2 5
1 3



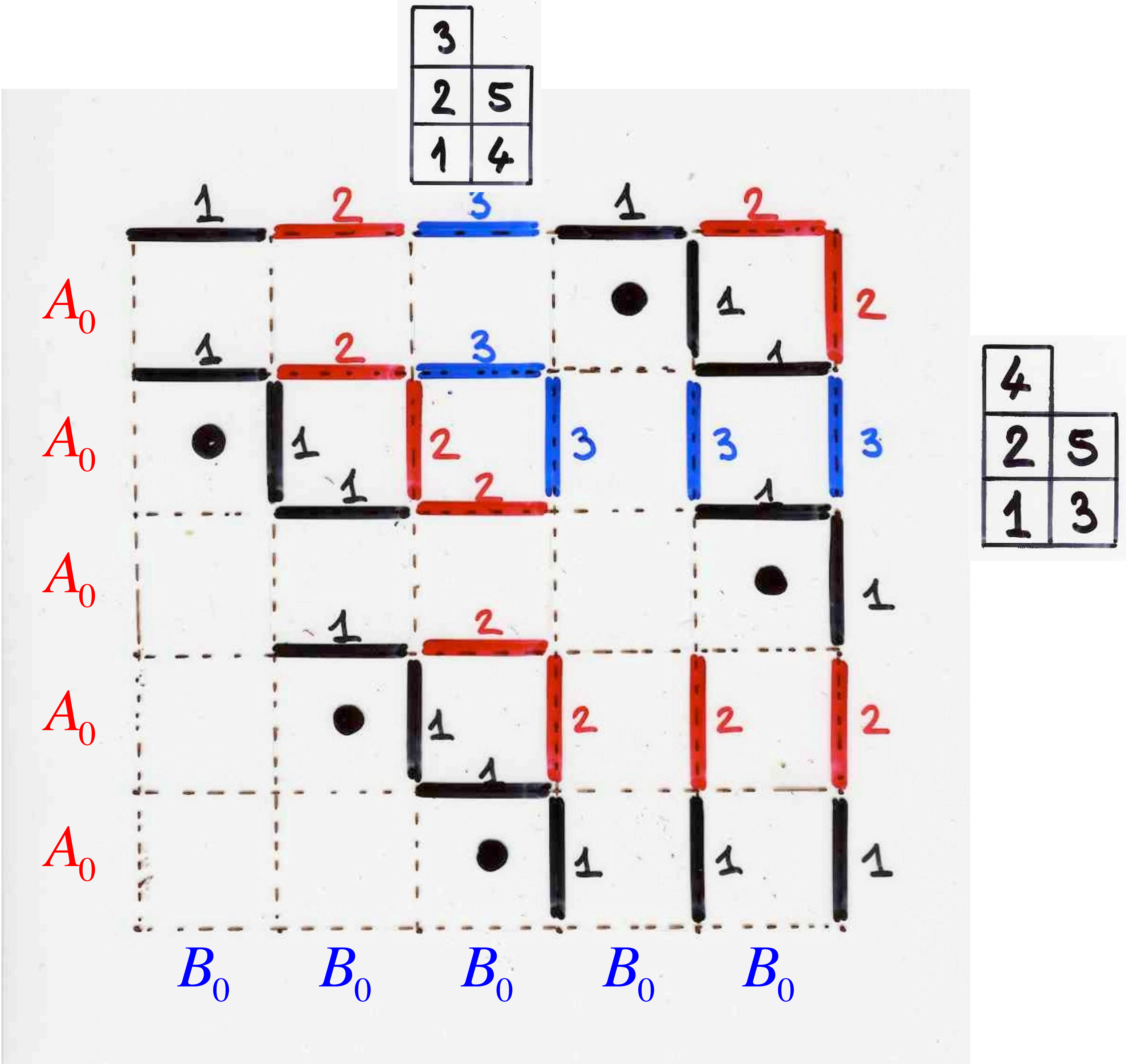
w = 1 2 3 1 2

Yamanuchi word

eqiuvalence with the RSK automaton



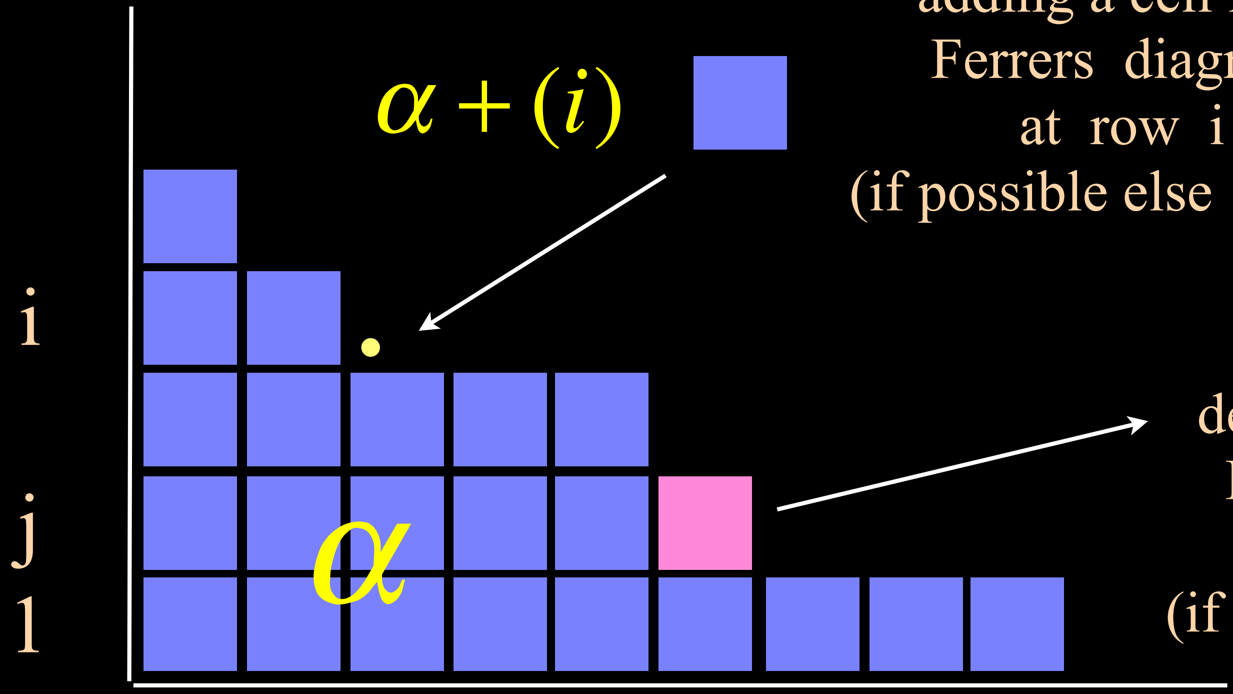
U
D
I
I



3
2 5
1 4
4
2 5
1 3
A_0
A_0
A_0
A_0
A_0
B_0 B_0 B_0 B_0 B_0

notations Operator U_i

$$U_i(\alpha) = \alpha + (i)$$



adding a cell in a
Ferrers diagram
at row i
(if possible else =0)

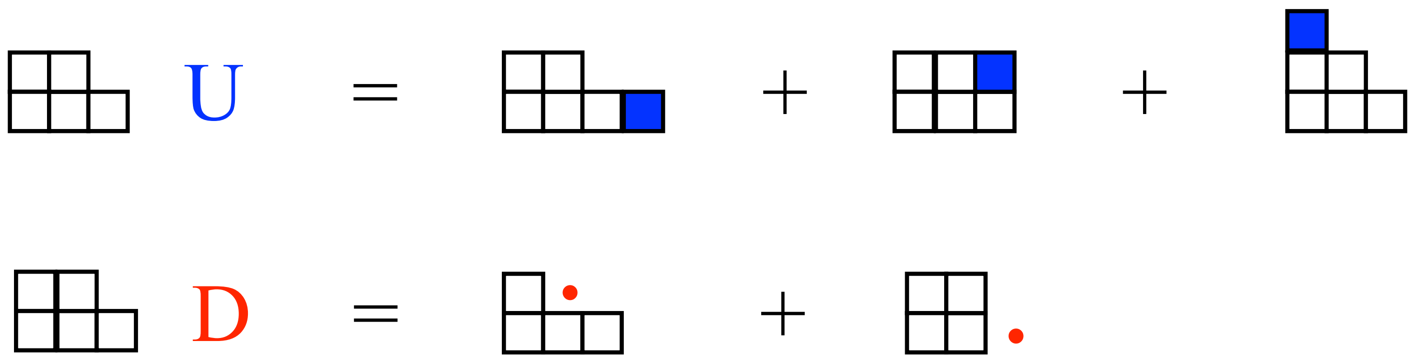
D_j

deleting a cell in a
Ferrers diagram
at row j
(if possible, else =0)

Operator D_j

$$U = \sum_{i \geq 0} U_i \qquad\qquad D = \sum_{i \geq 0} D_i$$

U and D are operators acting of the vector space generated by Ferrers diagrams



The cellular Ansatz

quadratic algebra Q (of a certain type)

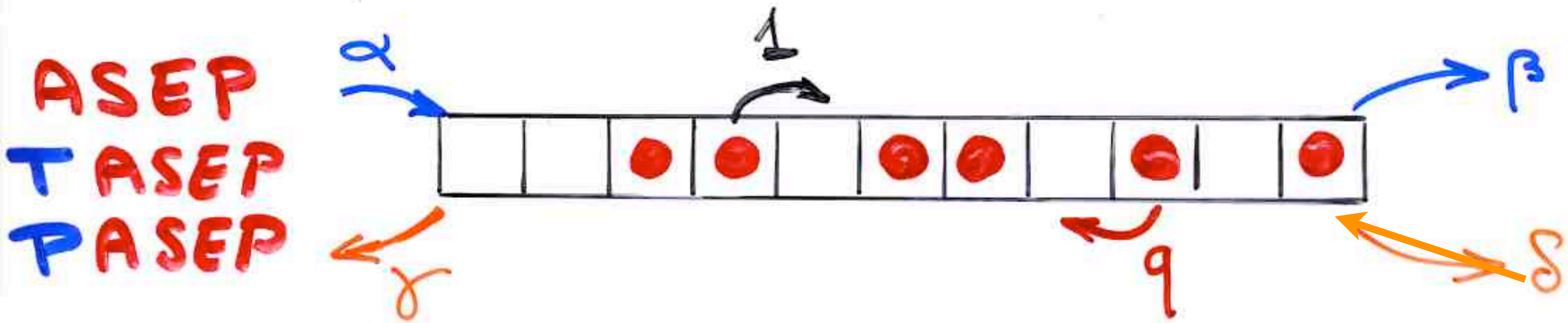
(1) "planarisation" on a grid of the rewriting rules

Q- tableaux planar automata

(2) from the representation of the algebra Q
construction of a bijection by «propagation»
on a grid of the commutation diagrams bijection
related to each cell

The PASEP algebra

$$DE = qED + E + D$$



$$DE = qED + E + D$$

The Matrix Ansatz Derrida, Evans, Hakim, Pasquier 1993

Combinatorics of the PASEP

TASEP

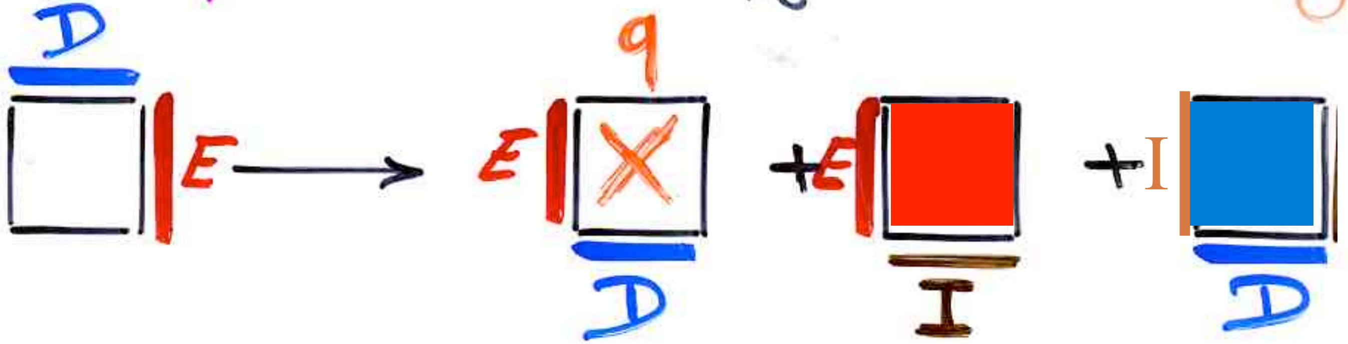
Brak, Essam (2003), Duchi, Schaeffer, (2004),
Angel (2005), XGV, (2007)

(P) ASEP

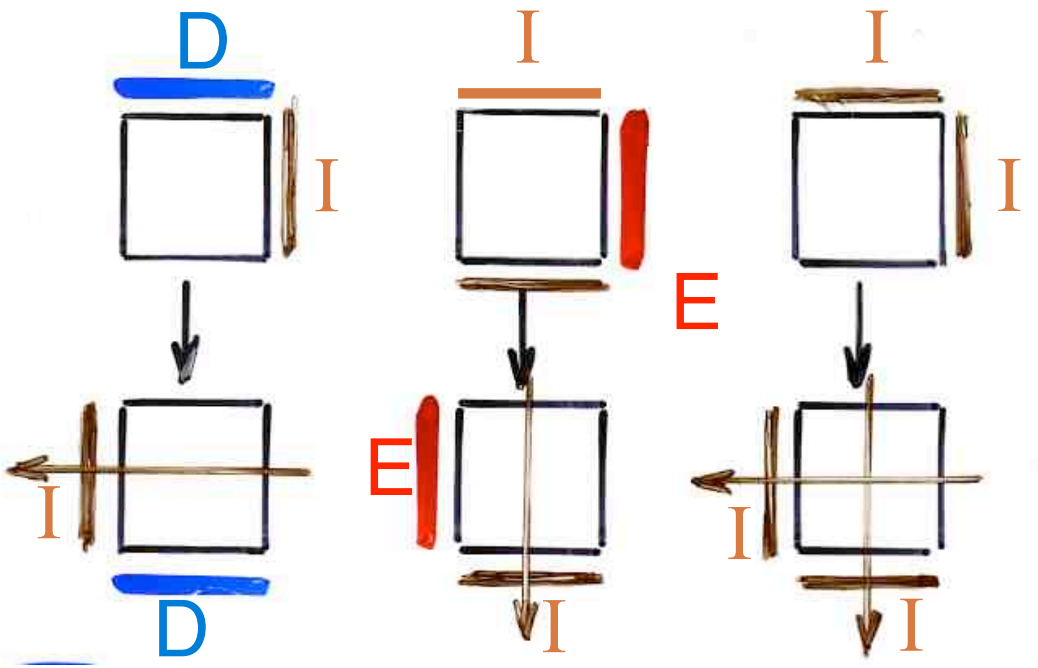
Brak, Corteel, Essam, Parviainen, Rechnitzer (2006)
Corteel, Williams (2006) (2008) (2009) XGV, (2008)
Corteel, Stanton, Stanley, Williams (2010)

Derrida, ...
Mallick, ... Golinelli, Mallick (2006)

Proof: "planarization" of the rewriting rules



I identity



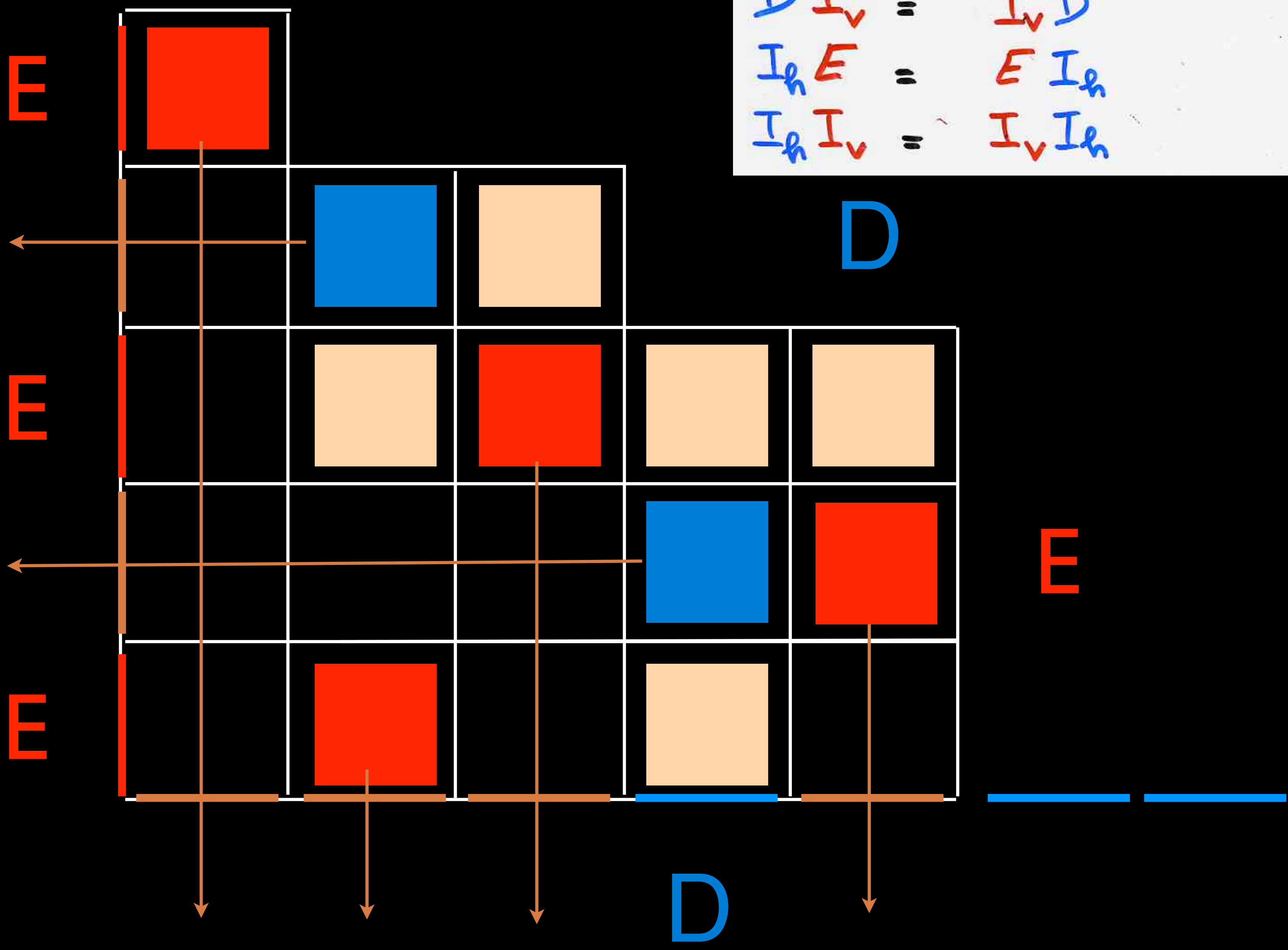
$$DE = qED + EI_h + I_v D$$

$$DI_v = I_v D$$

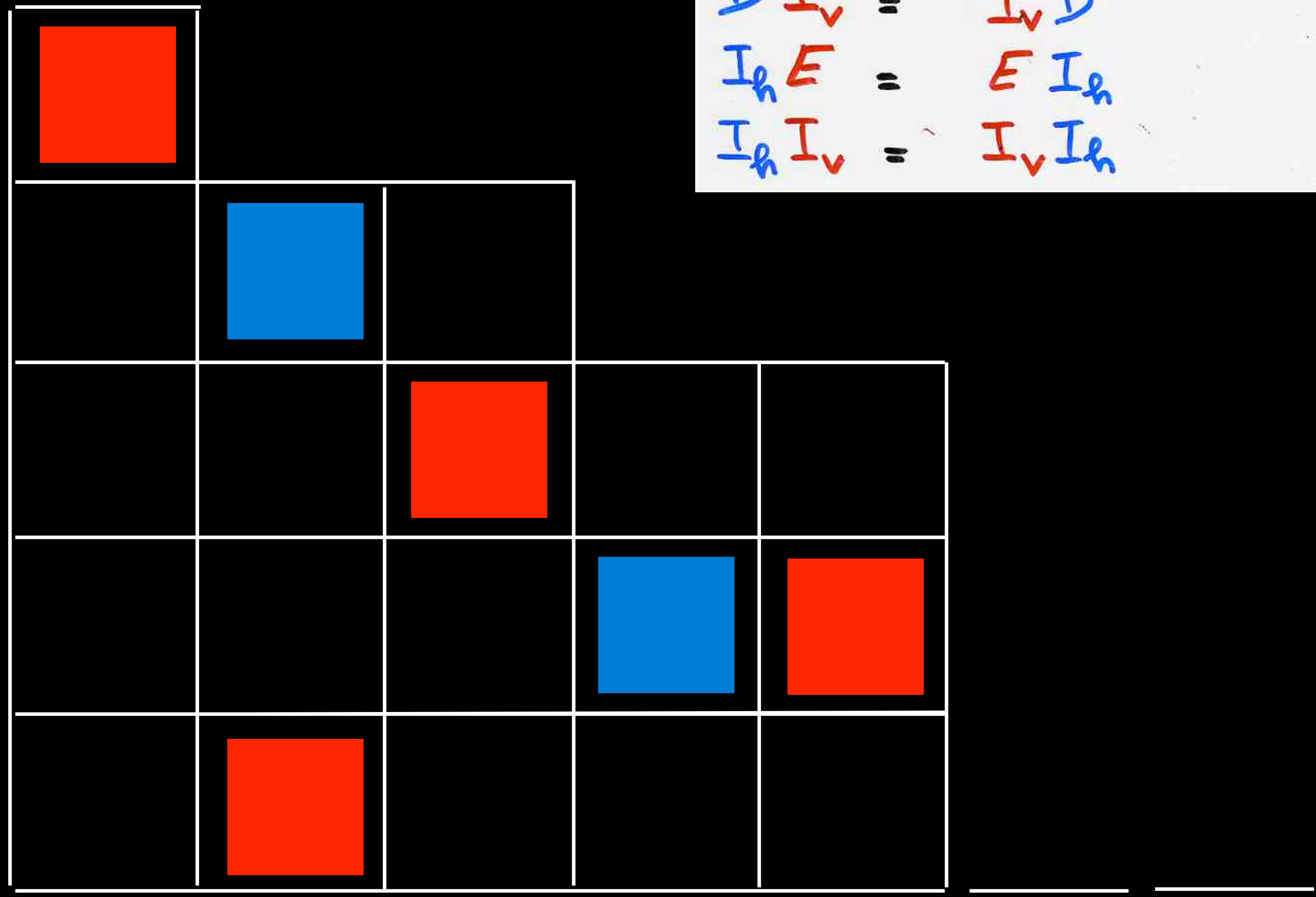
$$I_h E = E I_h$$

$$I_h I_v = I_v I_h$$

$$DE = q\,ED + EI_h + I_vD$$
$$DI_v = I_vD$$
$$I_hE = EI_h$$
$$I_hI_v = I_vI_h$$

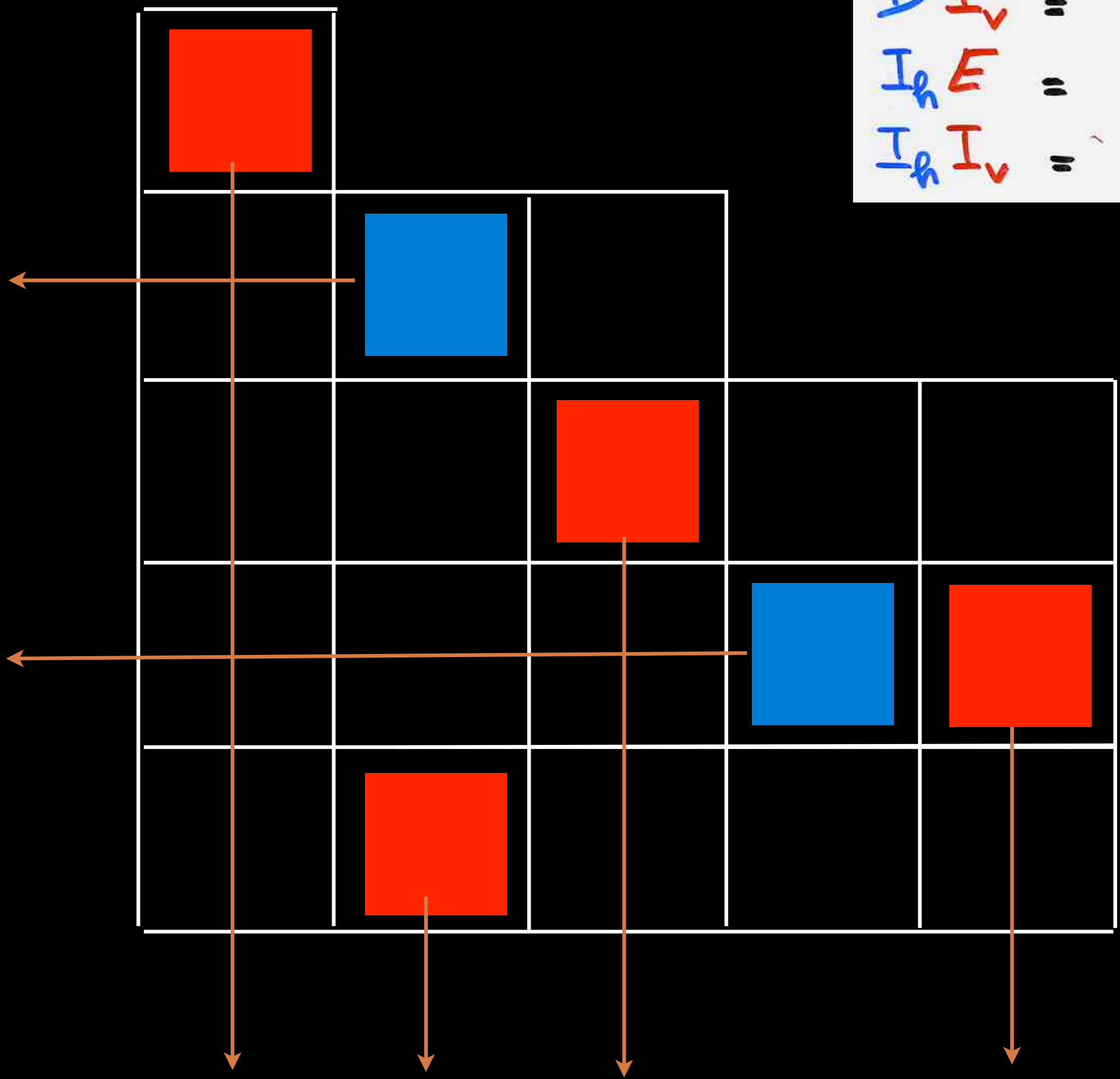


$$DE = qED + EI_h + I_vD$$
$$DI_v = I_vD$$
$$I_hE = EI_h$$
$$I_hI_v = I_vI_h$$



alternative tableau

$$DE = q\,ED + EI_h + I_vD$$
$$DI_v = I_vD$$
$$I_hE = EI_h$$
$$I_hI_v = I_vI_h$$

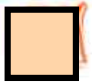


$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

unicity

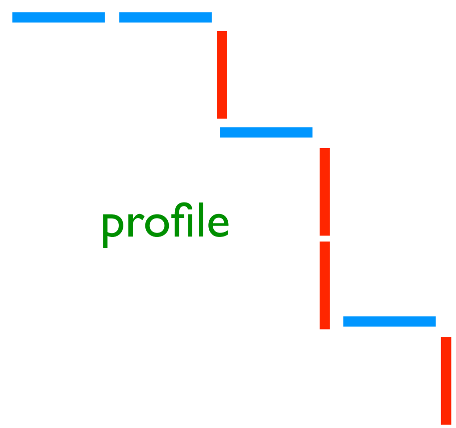
T alternative tableau with profile w

$k(T)$ = nb of 

$i(T)$ = nb of rows without blue cell

$j(T)$ = nb of columns without red cell

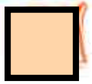
w = D D E D E E D E ⟶



$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

T alternative tableau with profile w

$k(T)$ = nb of 

$i(T)$ = nb of rows without blue cell

$j(T)$ = nb of columns without red cell

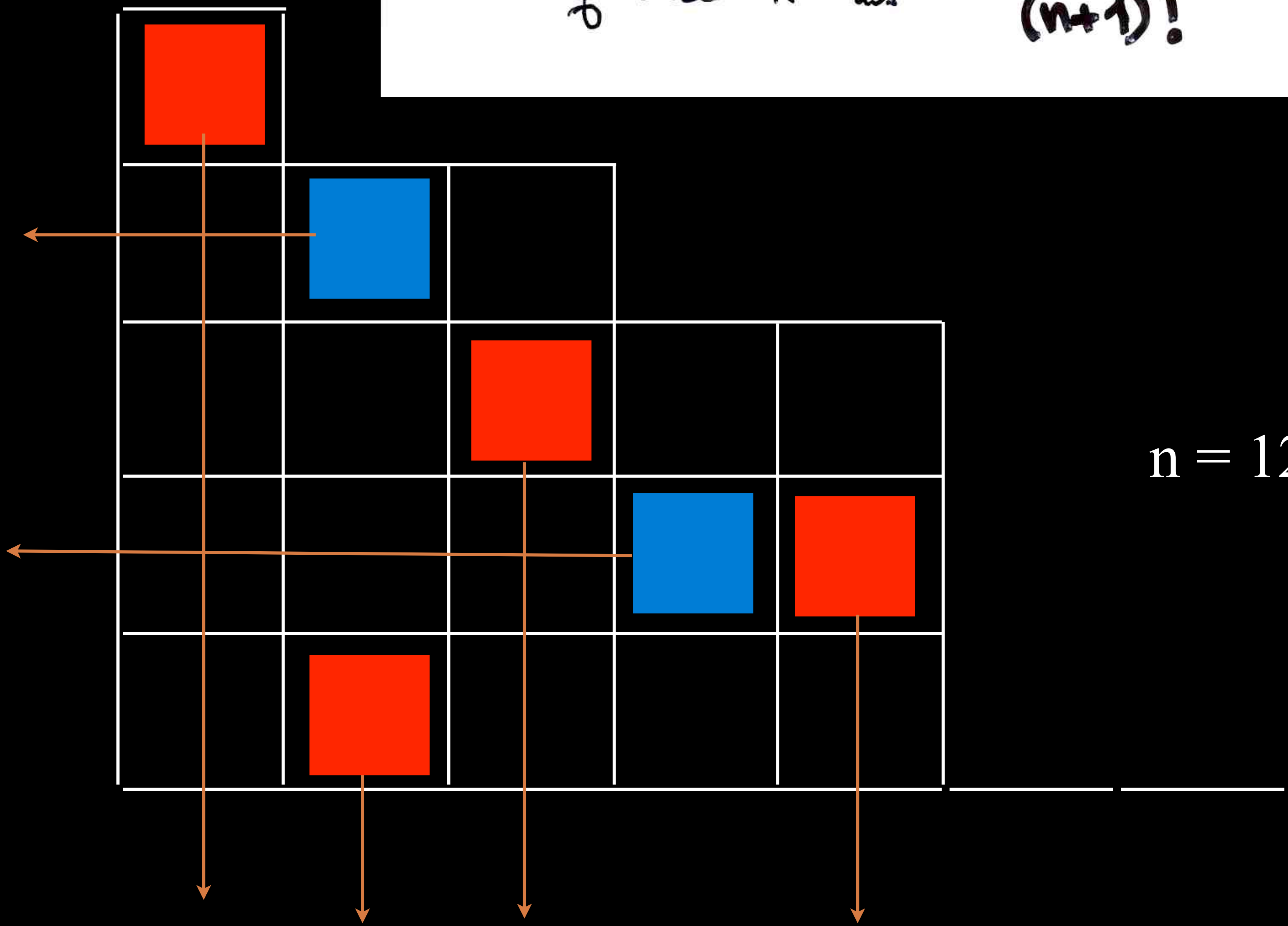
→ stationary probabilities

$E \to 1/\alpha$

$D \to 1/\beta$

The Matrix Ansatz   Derrida, Evans, Hakim, Pasquier 1993

Prop. The number of alternative tableaux of size n is (n+1)!



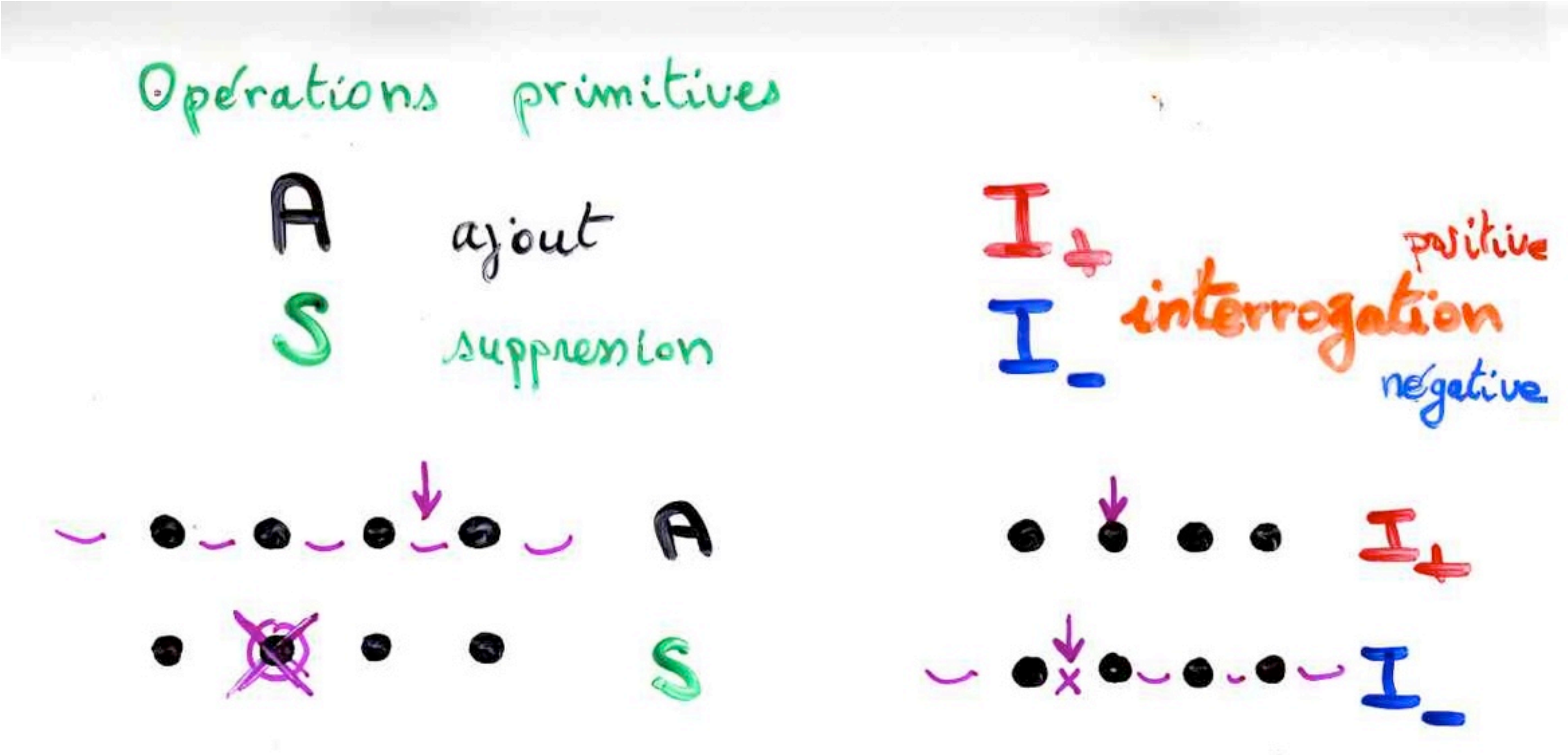
n = 12

ex: n=2

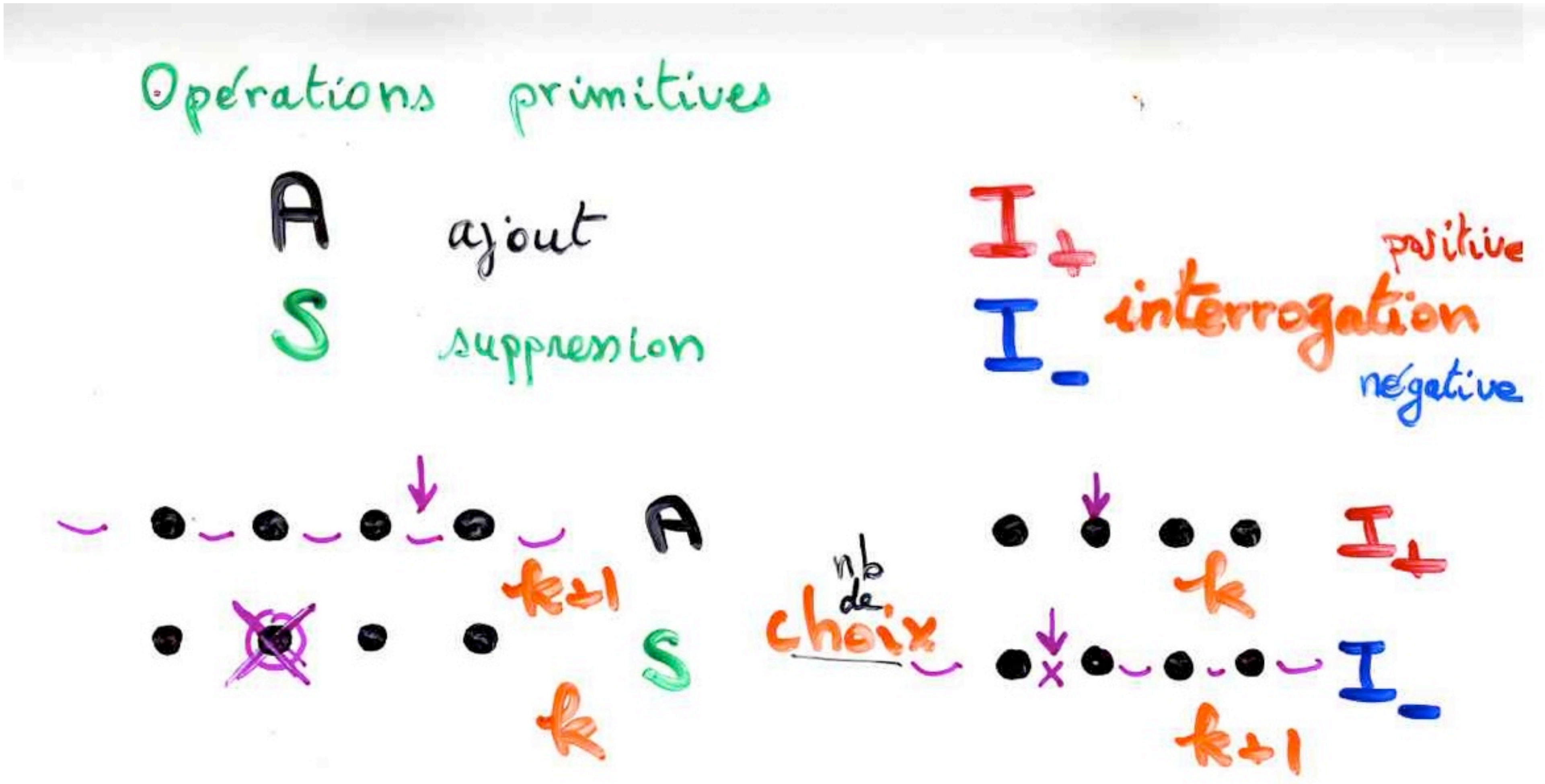
for the PASEP algebra

$$DE = qED + E + D$$

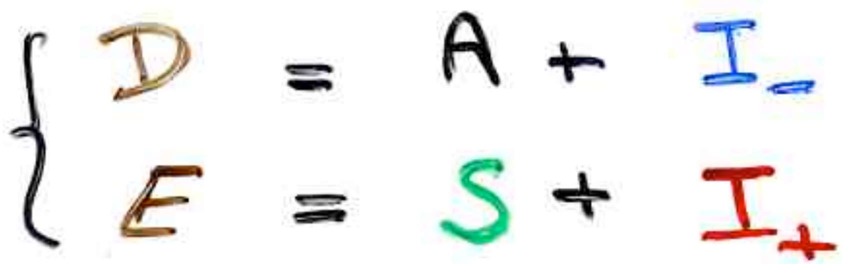
representation with operators
related to the combinatorial theory
of orthogonal polynomials
and data structures in computer science



Primitive operations
for “dictionnaries” data structure:
add or delete any elements, asking questions (with
positive or negative answer)



number of choices for each
primitive operations



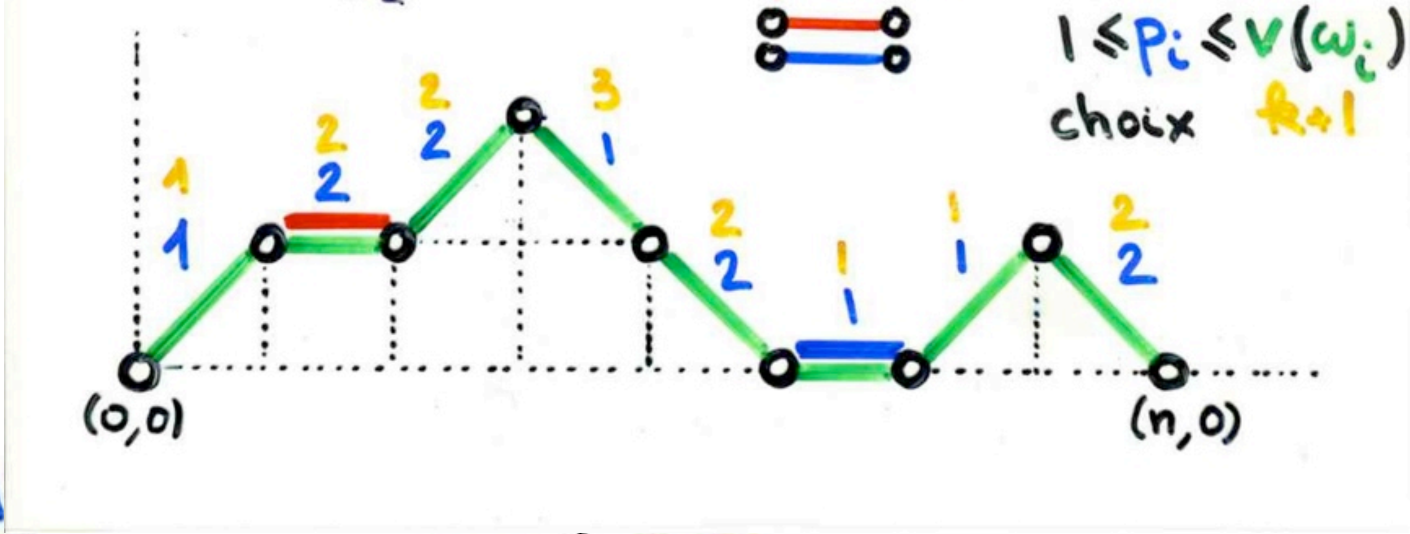
Combinatorial theory of

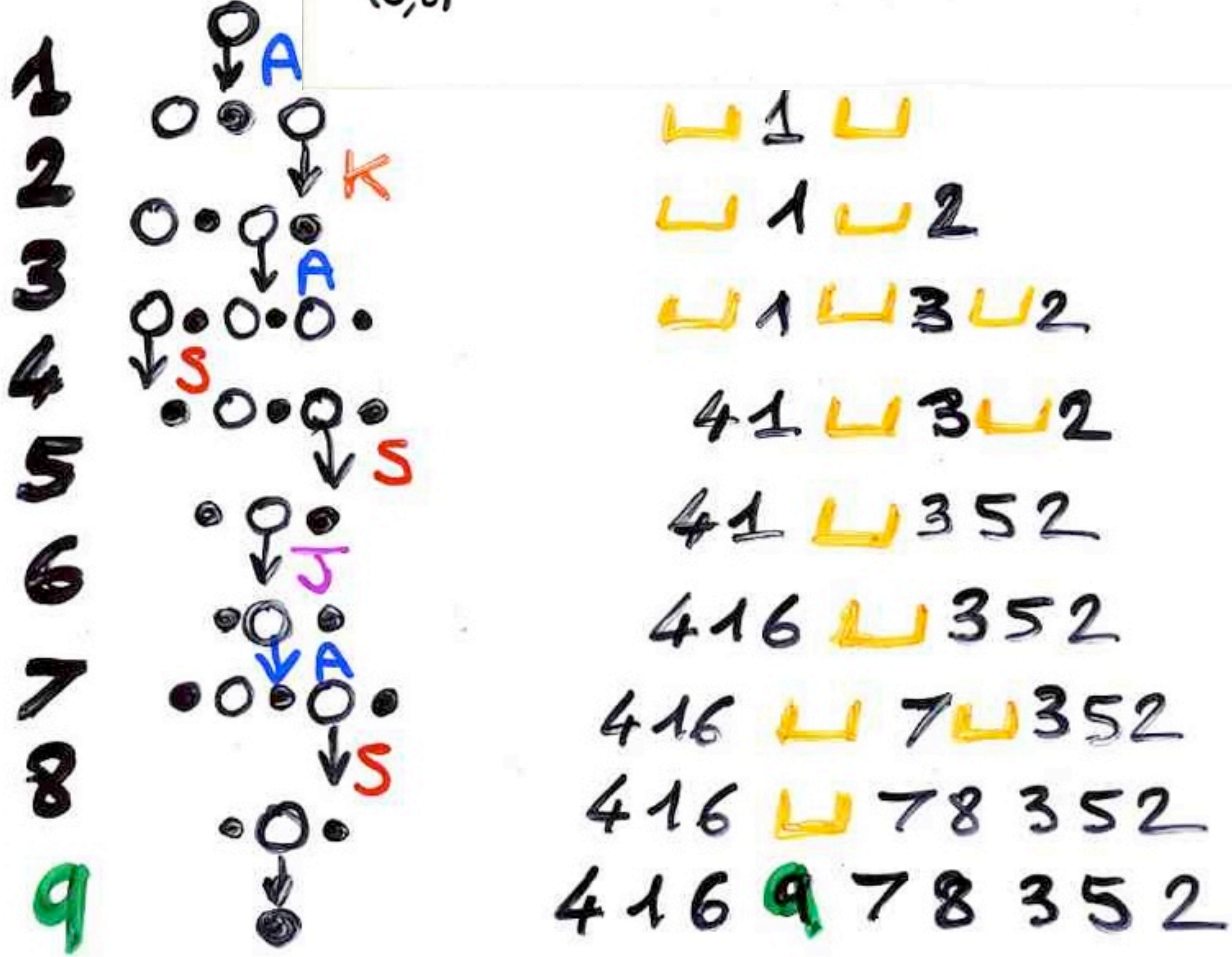
(formal) orthogonal polynomials

n! moments of laguerre polynomials

bijection permutations --- Laguerre histories

(certain weighted paths)


1 ≤ p_i ≤ v(ω_i)
choix k+1
(0,0)
(n,0)


1 2 3 4 5 6 7 8 9
A K A S S J A S
1
1 2
1 3 2
4 1 3 2
4 1 3 5 2
4 1 6 3 5 2
4 1 6 7 3 5 2
4 1 6 7 8 3 5 2
4 1 6 9 7 8 3 5 2

Combinatorial theory of

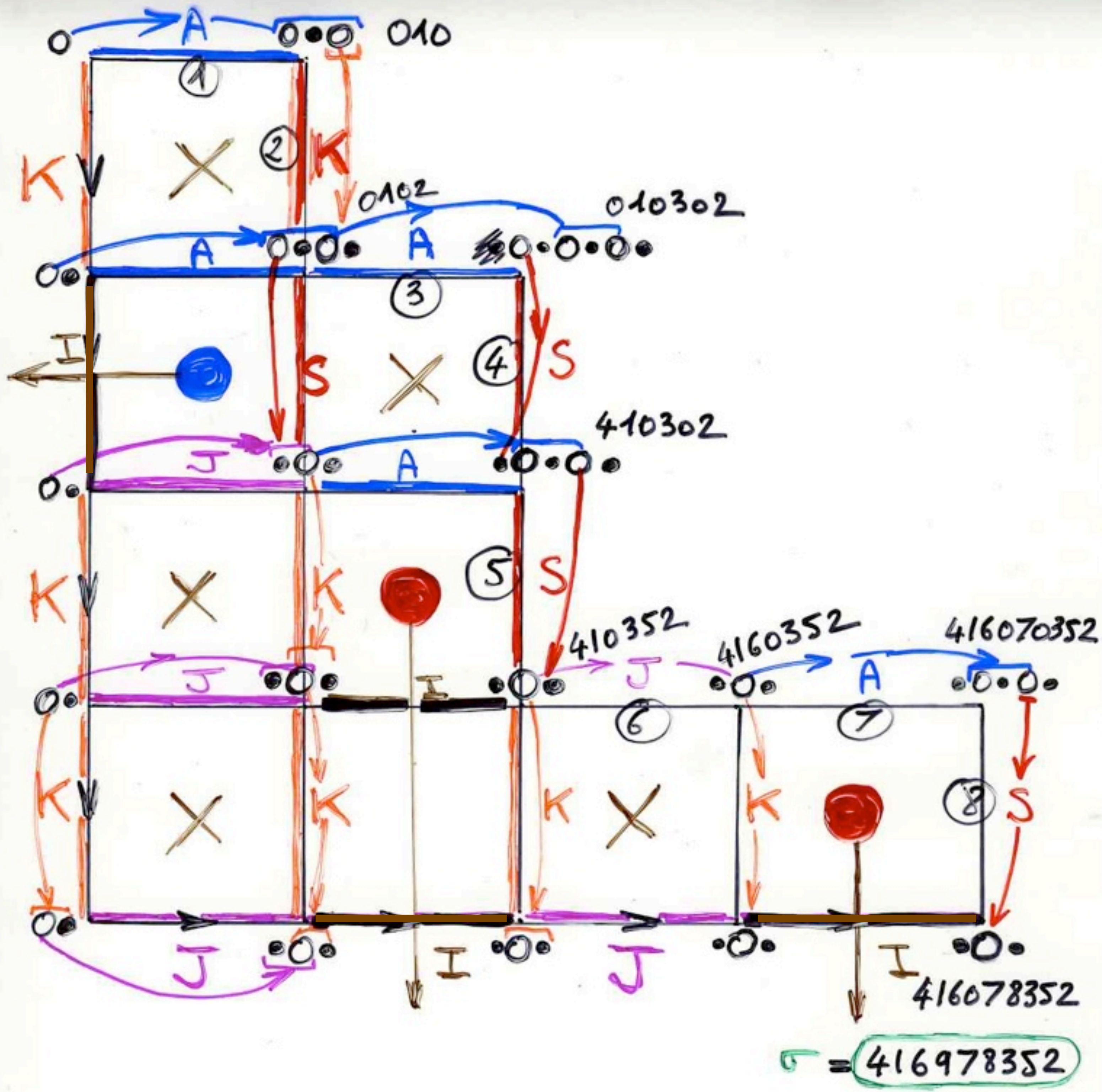
(formal) orthogonal polynomials

n! moments of laguerre polynomials

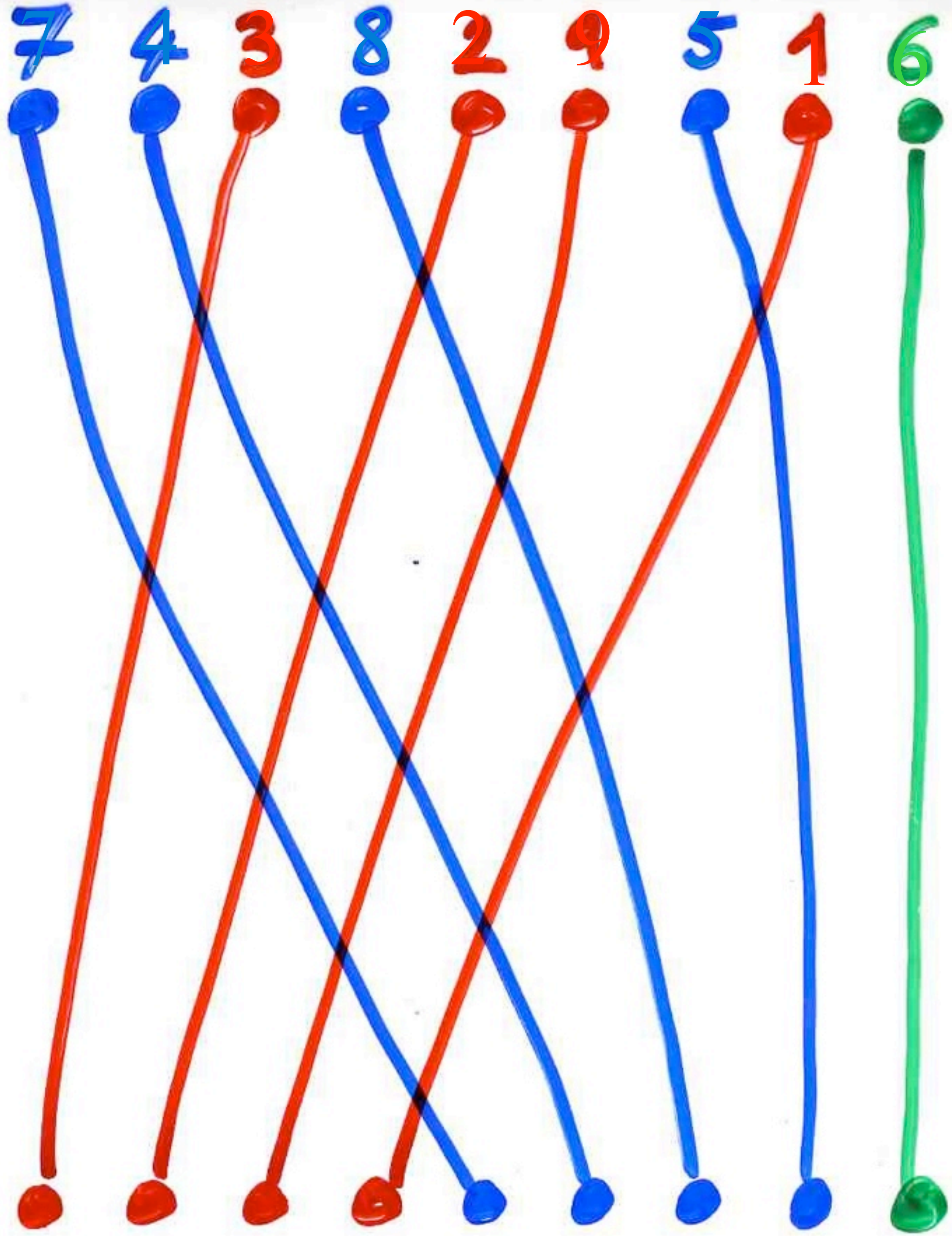
bijection permutations --- Laguerre histories

(certain weighted paths)

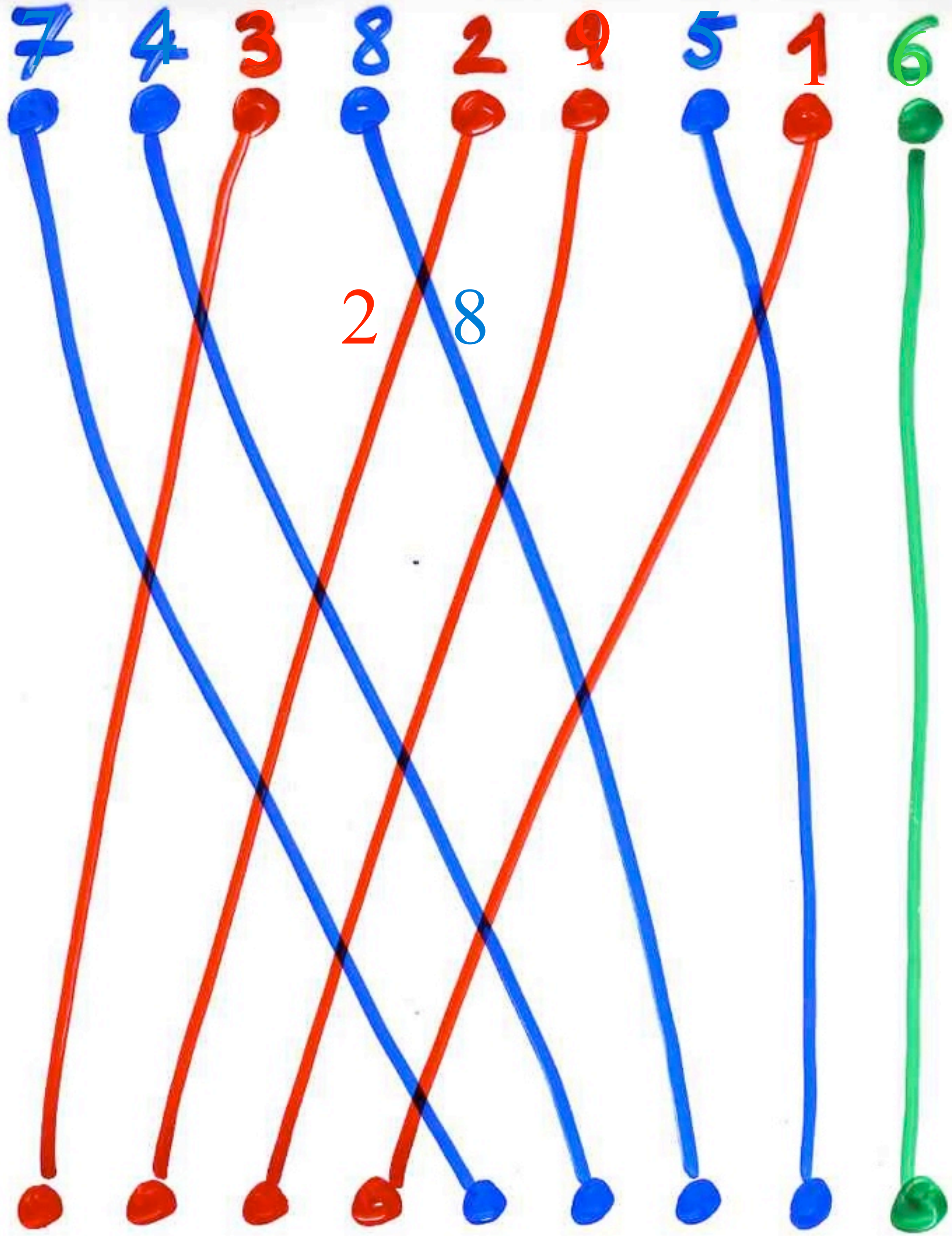
bijection alternative tableaux --- Laguerre histories



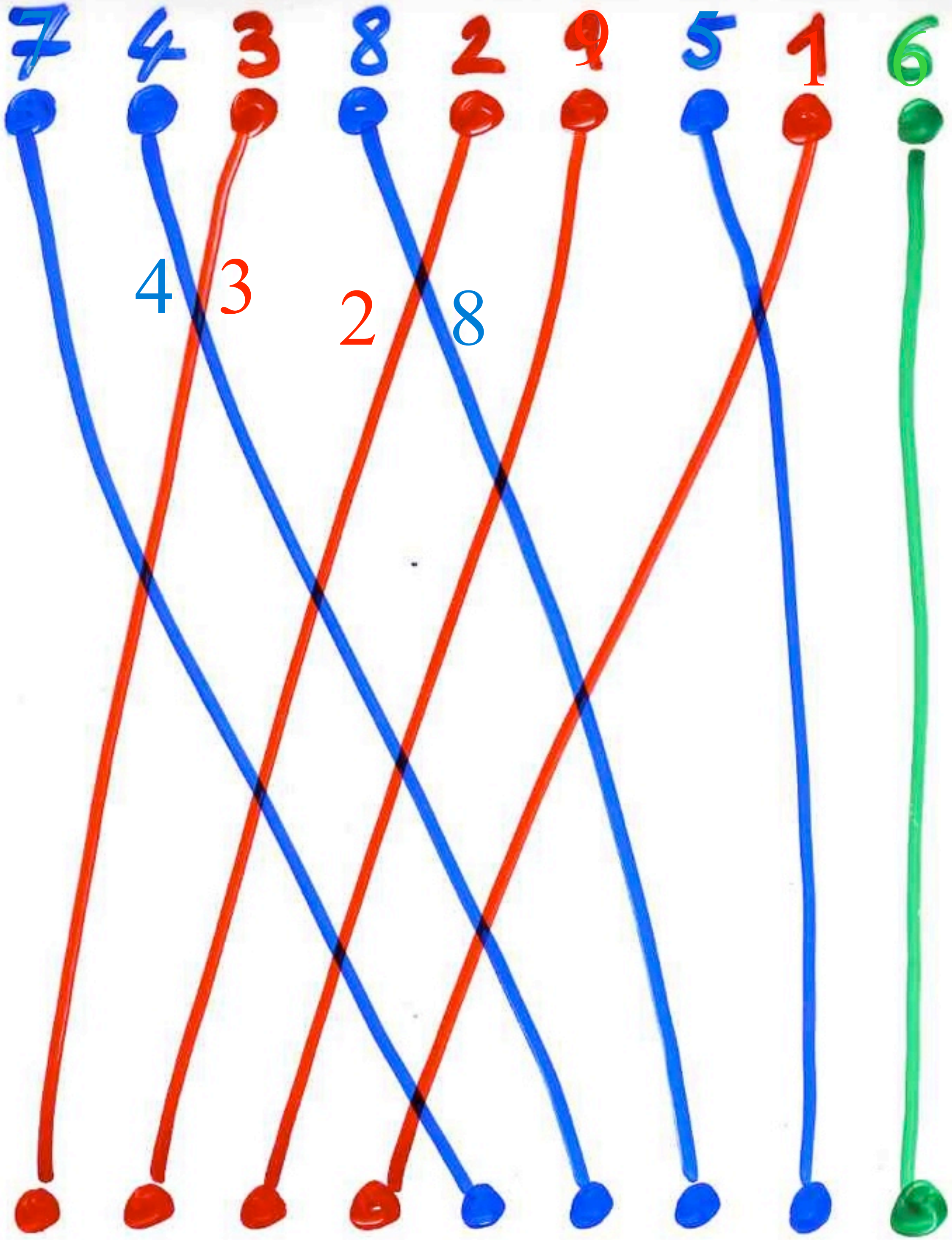
010
0102
010302
410302
410352
4160352
416070352
416078352
σ = 416978352

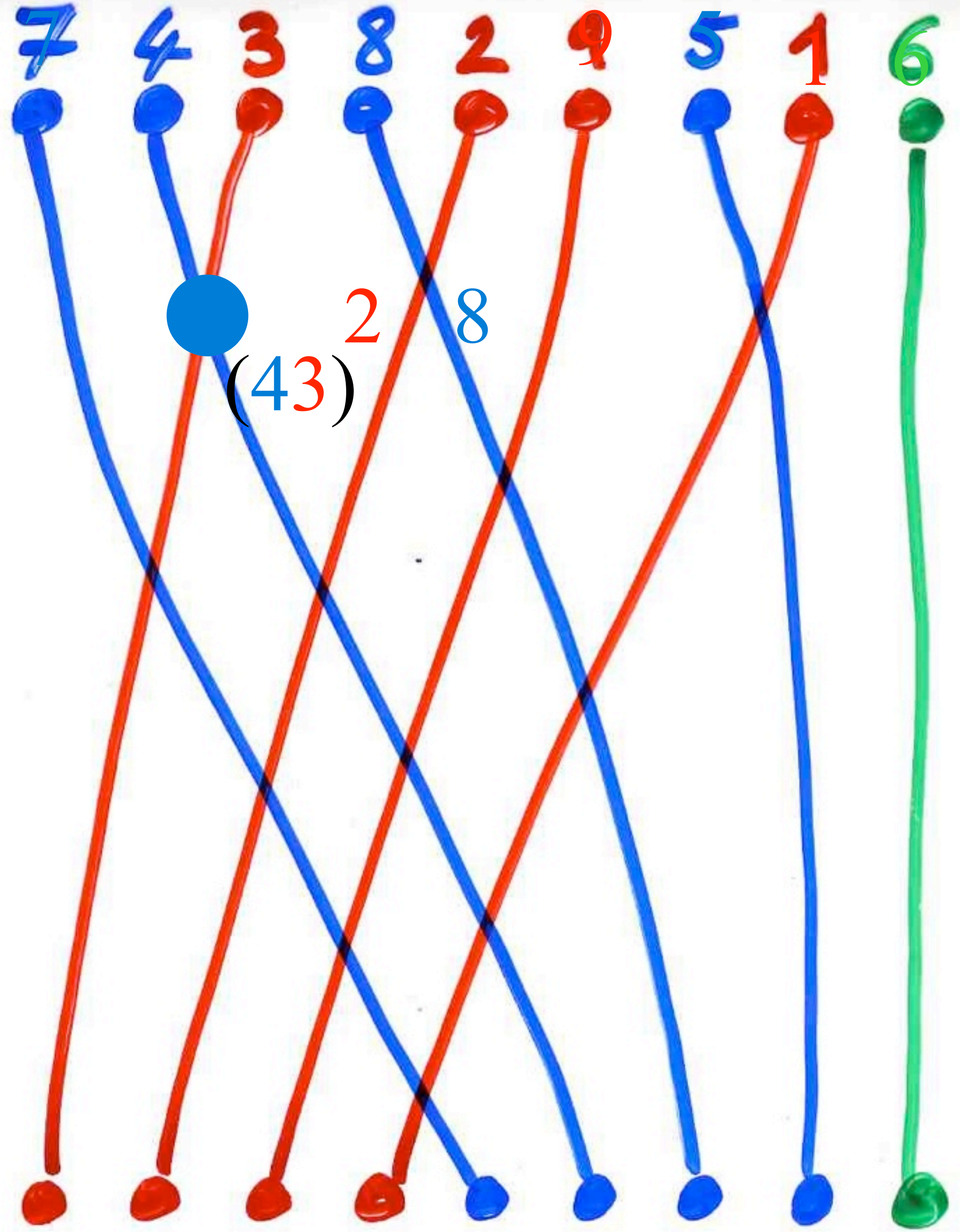


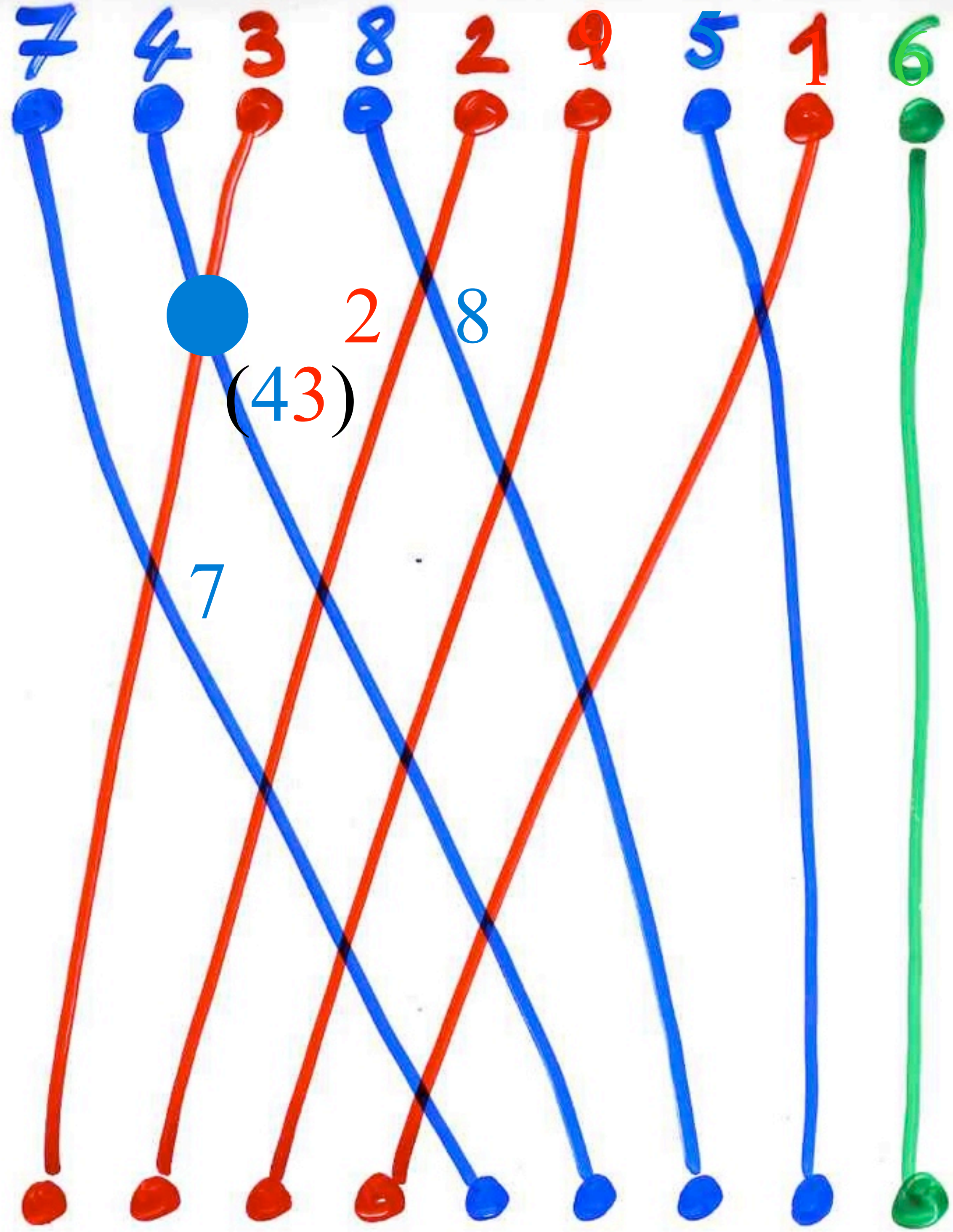
7 4 3 8 2 9 5 1 6



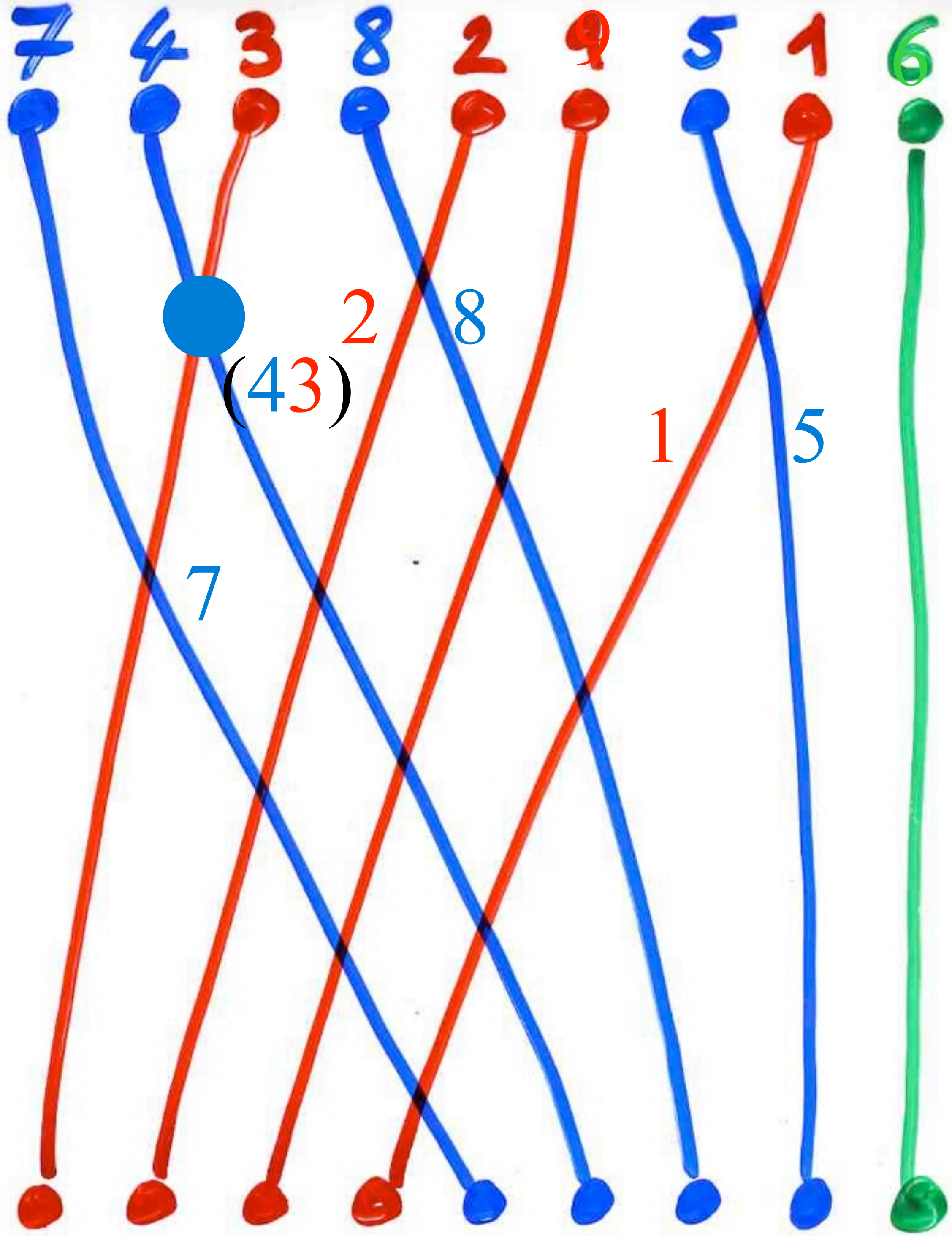
7 4 3 8 2 9 5 1 6
2 8


7
4
3
8
2
9
5
1
6
4
3
2
8

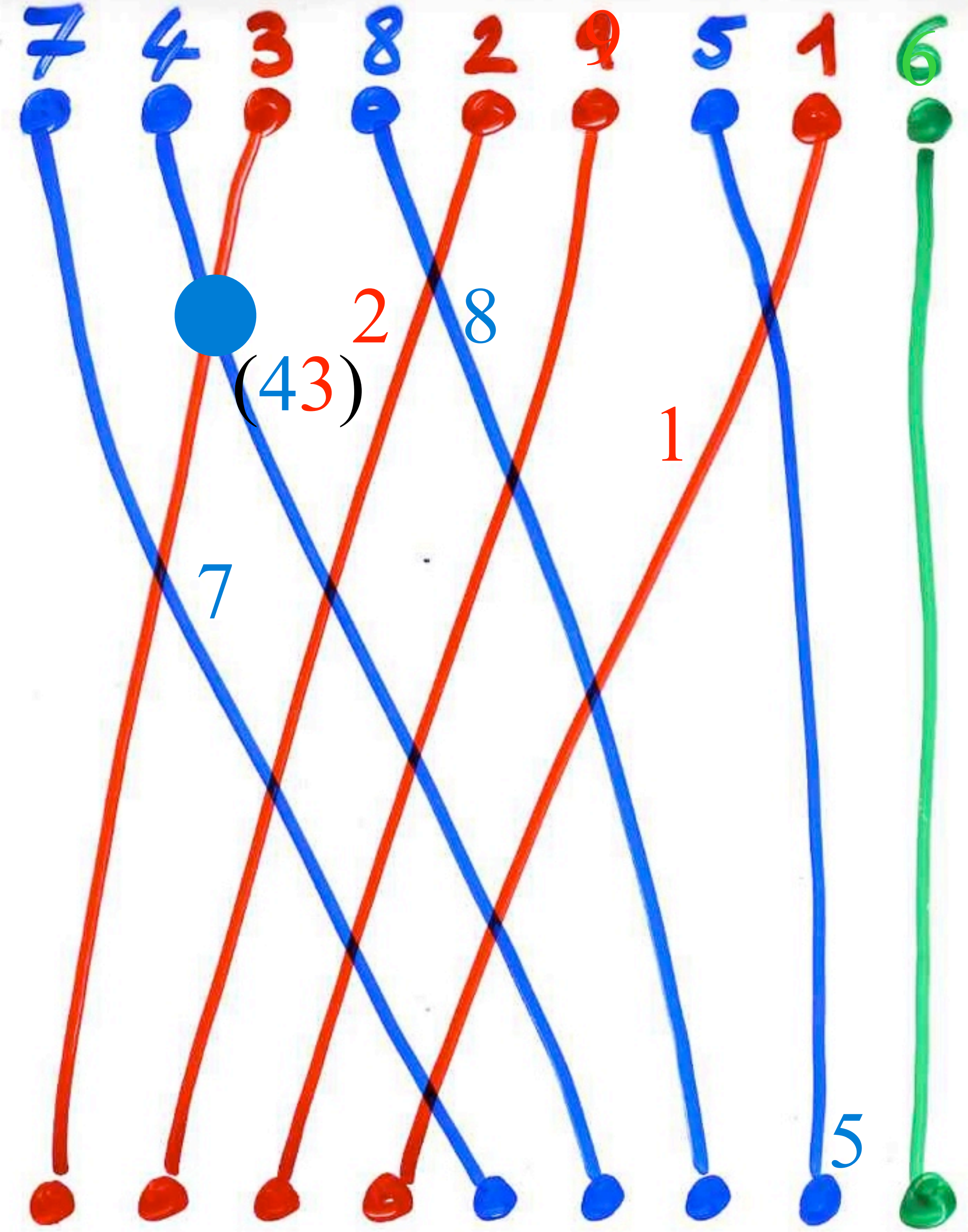

7 4 3 8 2 9 5 1 6
2
8
(43)



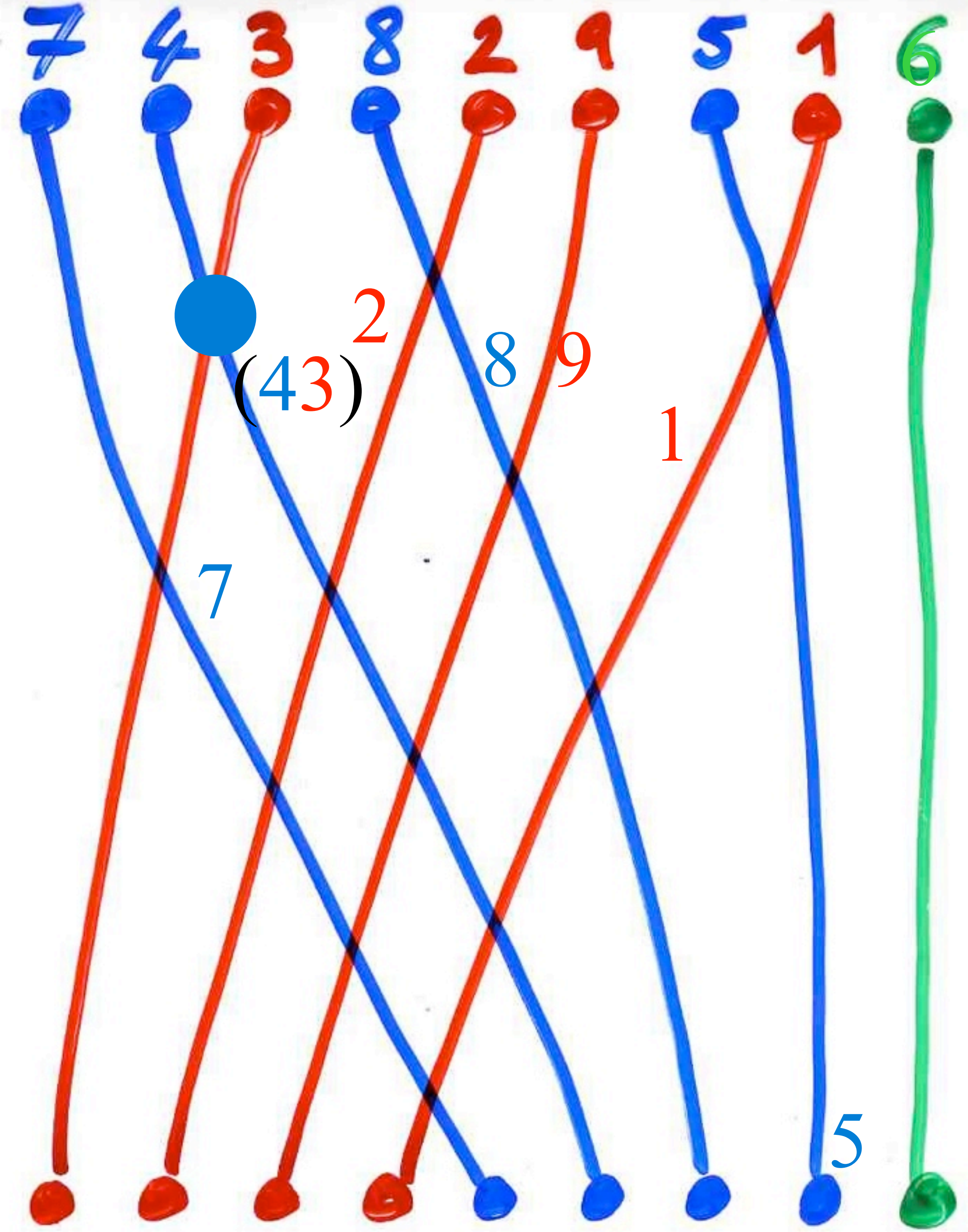
7 4 3 8 2 9 5 1 6
2
8
(43)
7

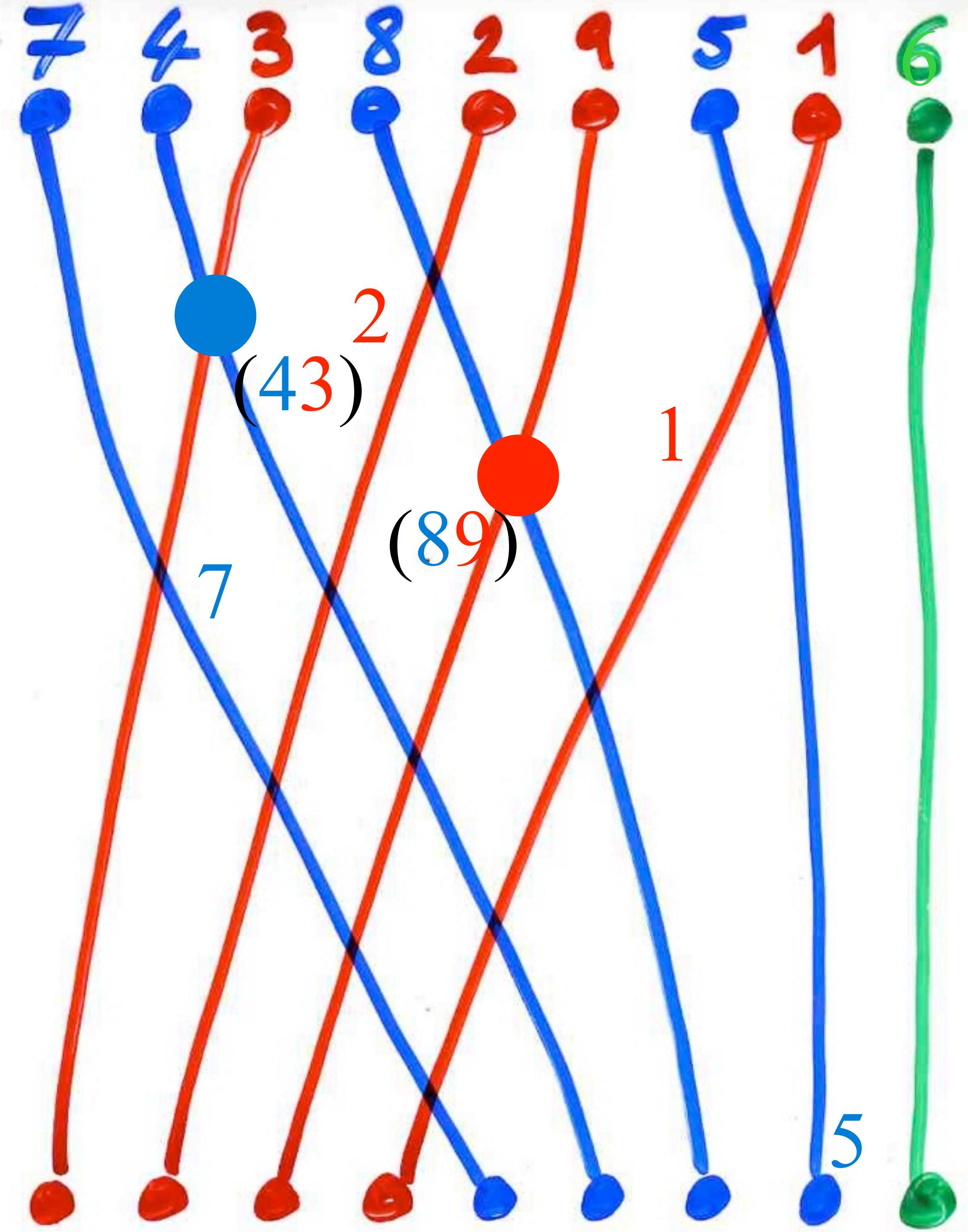


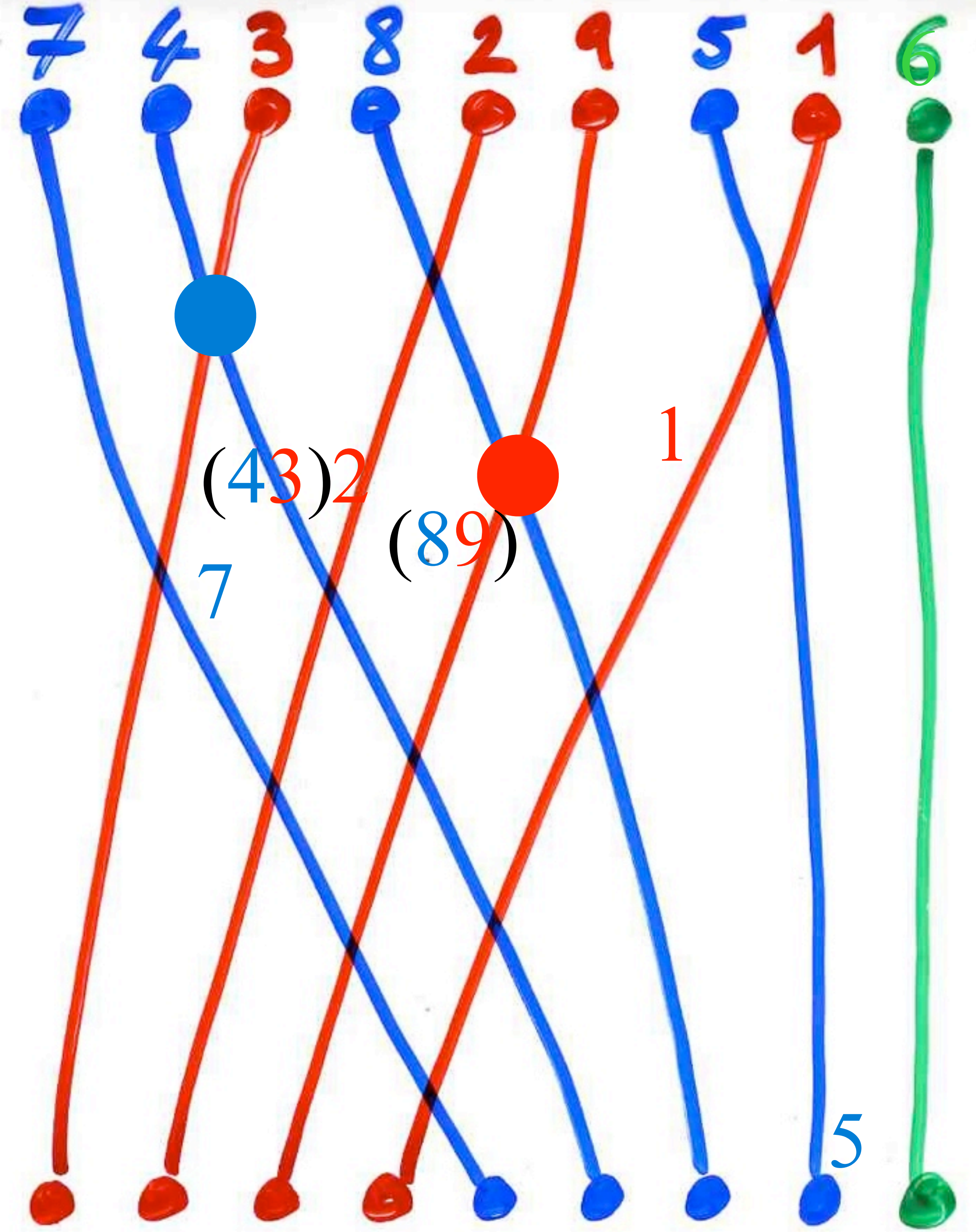
7 4 3 8 2 9 5 1 6
2
8
(43)
1
5
7

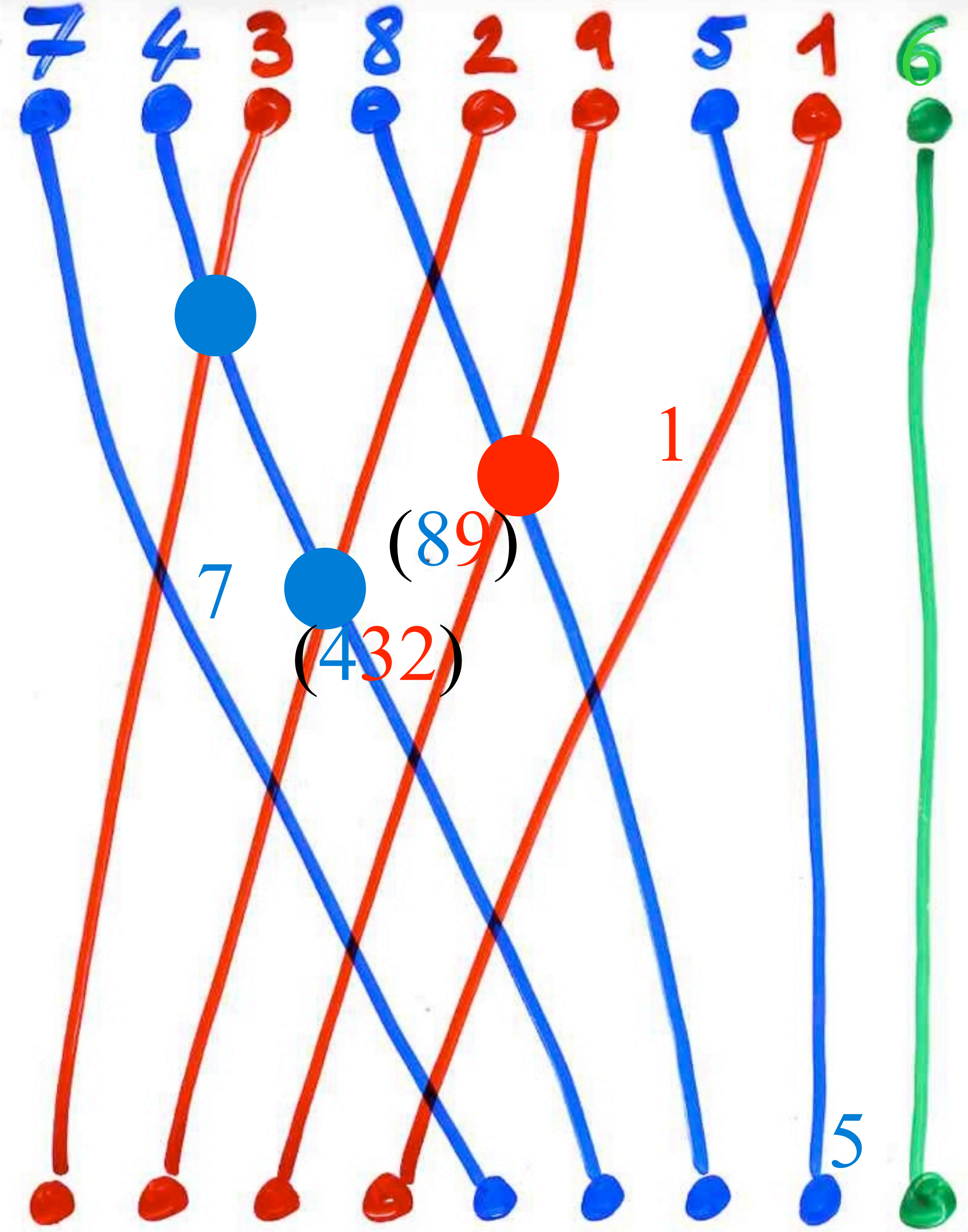


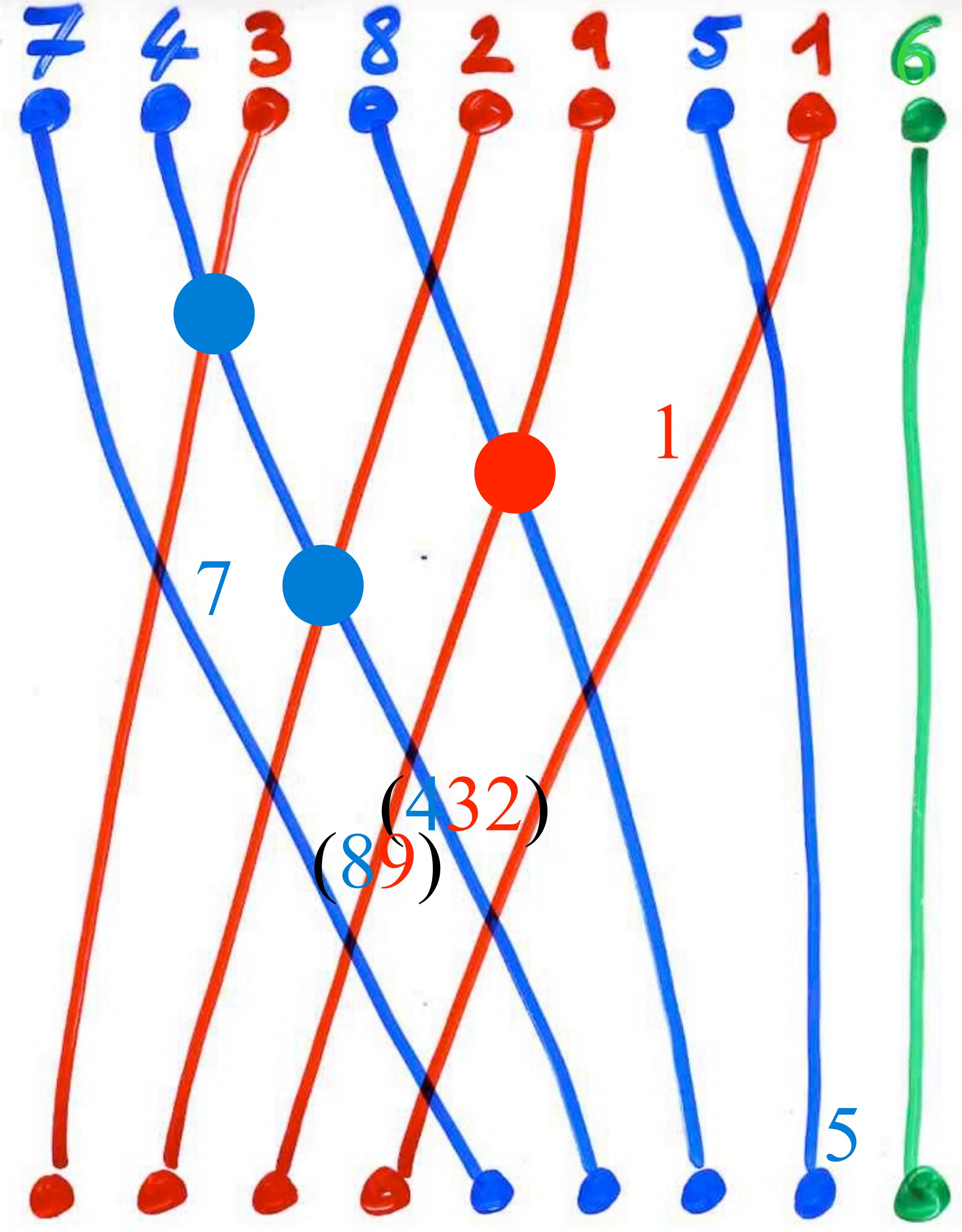
7 4 3 8 2 9 5 1 6
2
8
(43)
1
7
5


7 4 3 8 2 9 5 1 6
(43)
2
8
9
1
7
5

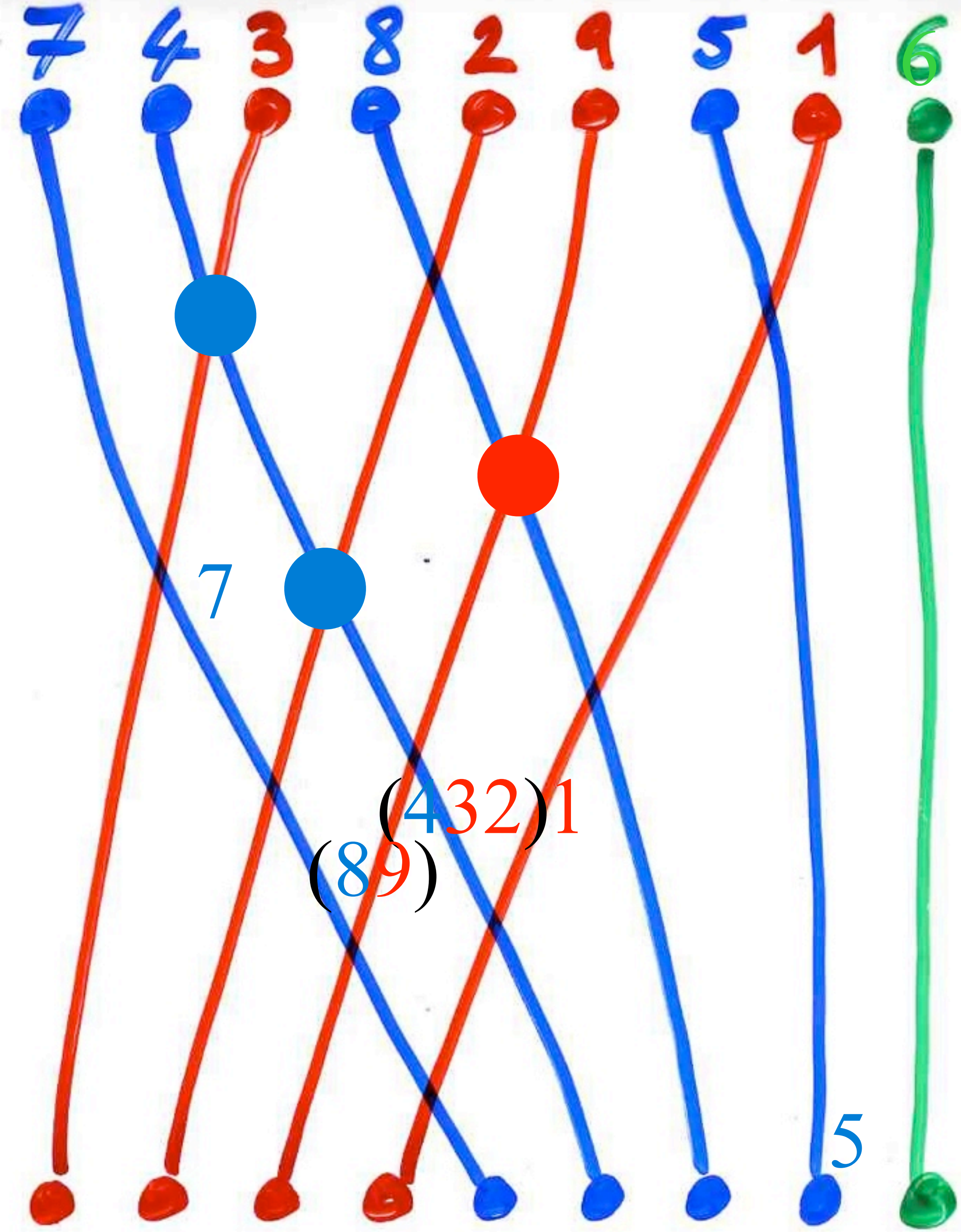

7 4 3 8 2 9 5 1 6
2
(43)
1
(89)
7
5


7 4 3 8 2 9 5 1 6
(43)2
(89)
1
7
5

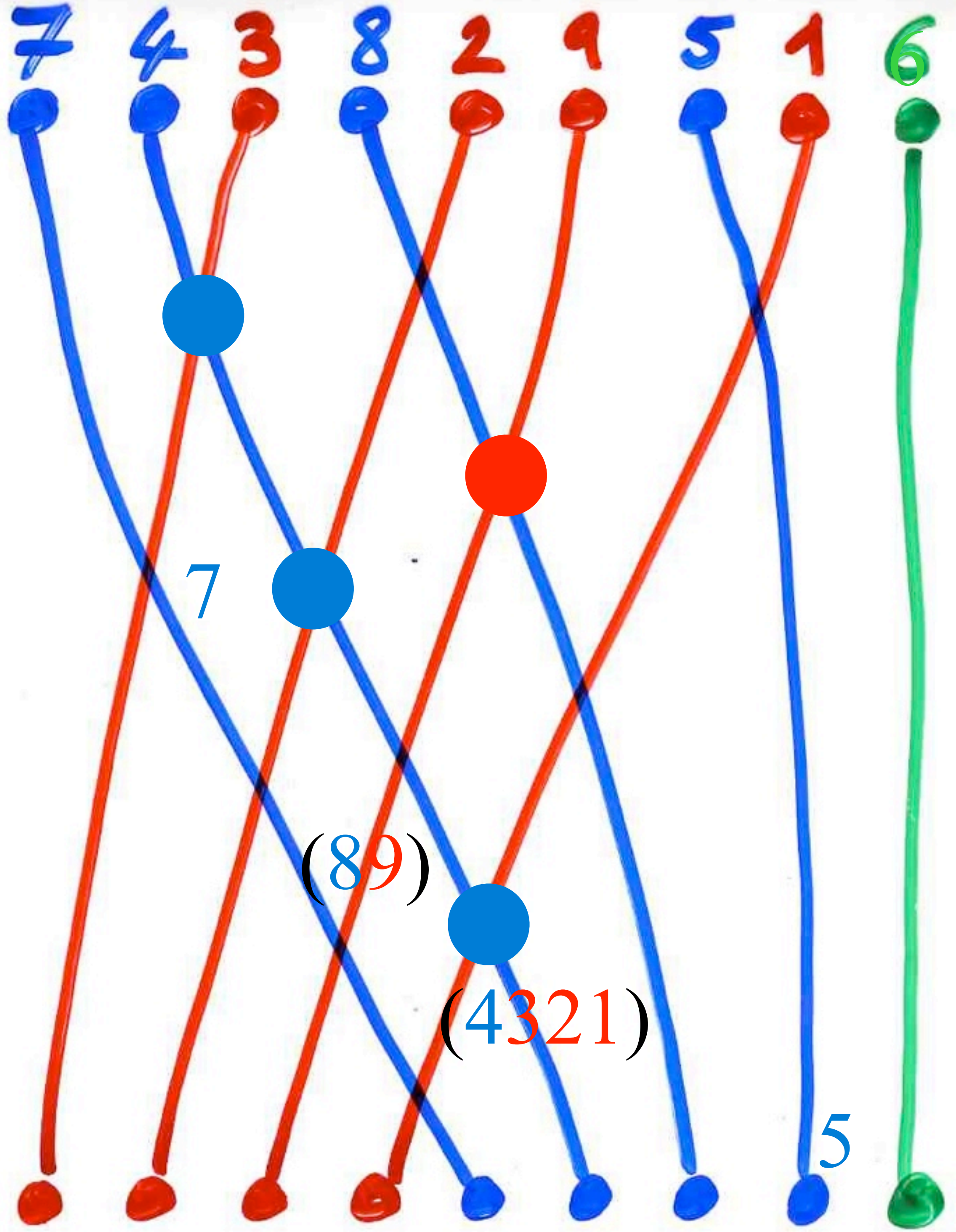

7 4 3 8 2 9 5 1 6
1
(89)
7
(432)
5

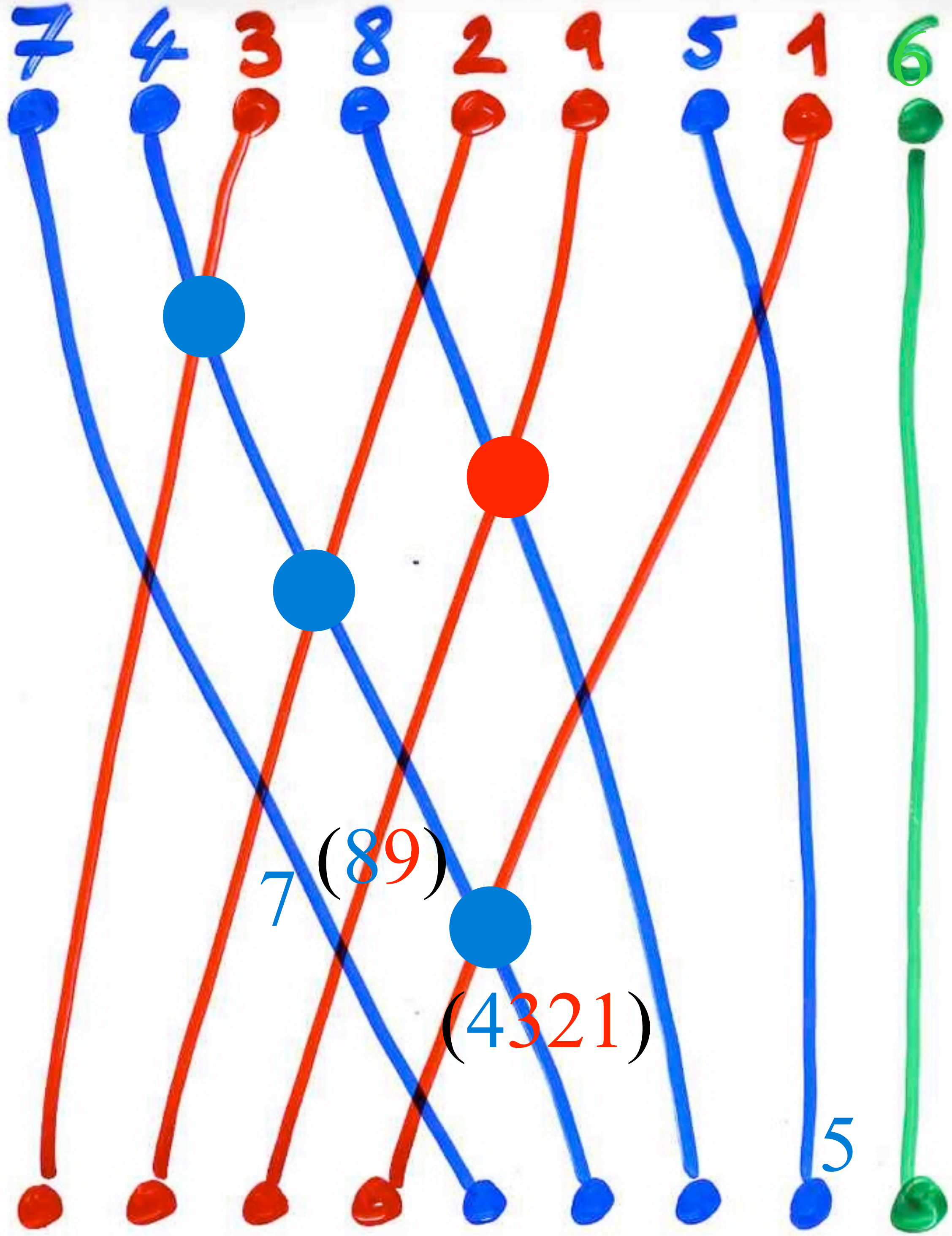


7 4 3 8 2 9 5 1 6
1
7
(432)
(89)
5

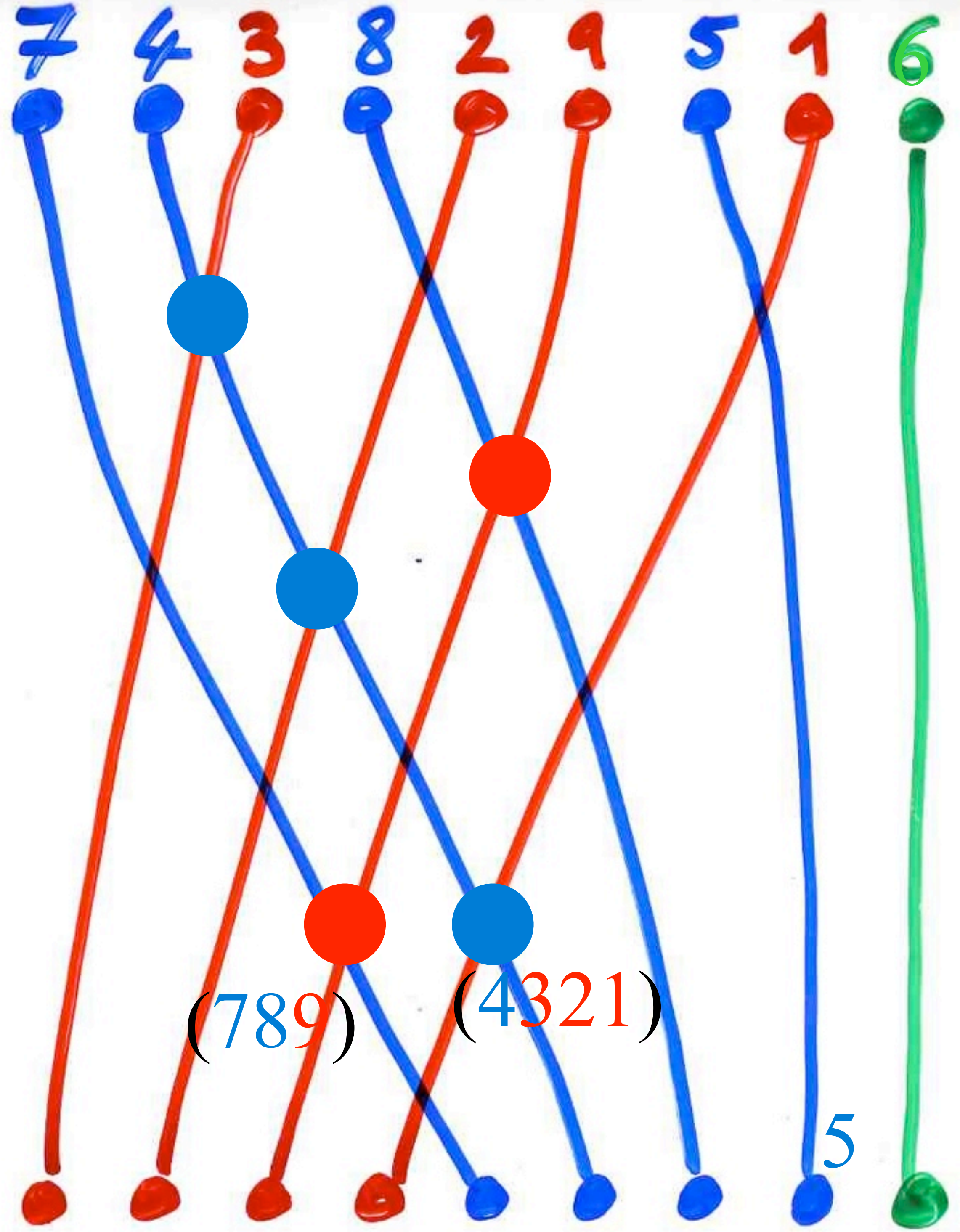


7 4 3 8 2 9 5 1 6
7
(432)1
(89)
5

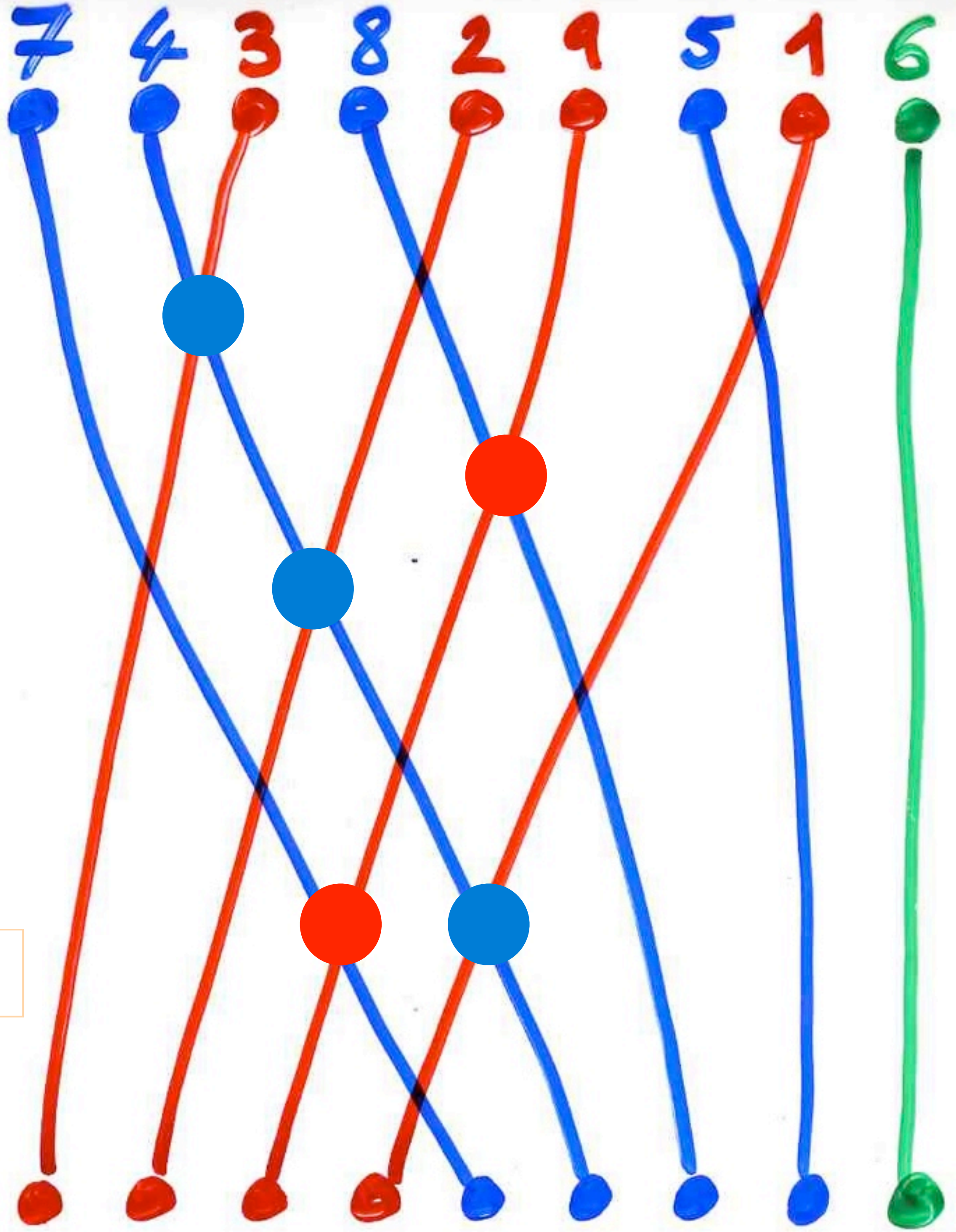

7 4 3 8 2 9 5 1 6
7
(89)
(4321)
5



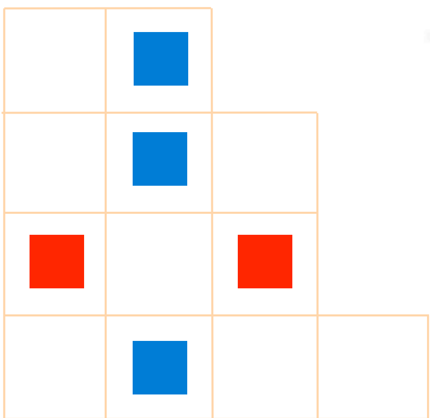
7 4 3 8 2 9 5 1 6
7
(89)
(4321)
5


7 4 3 8 2 9 5 1 6
(789)
(4321)
5

"exchange-fusion" algorithm



(789) (4321) 5 6



stationary probabilities for the PASEP, q-Laguerre

other Q-tableaux:

permutation tableaux
tree-like tableaux
staircase tableaux

TASEP $q=0$ $DE=E+D$

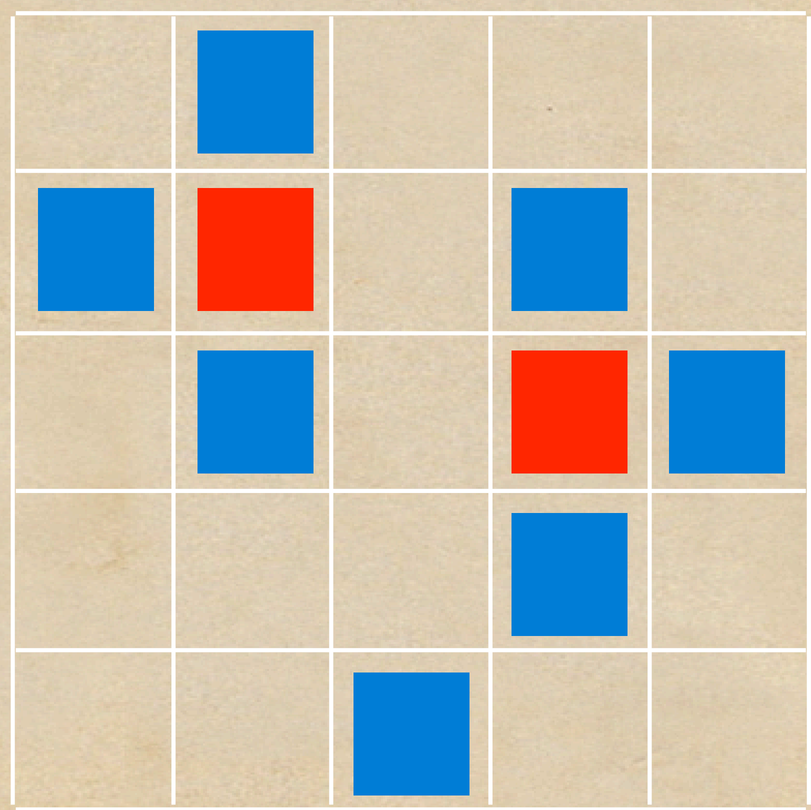
Catalan alternative tableaux

bijection with binary trees

relation with
the Loday-Ronco Hopf algebra on binary trees

Claudia-Christophe Hopf algebra on permutations

analog for ASM ?



"The cellular Ansatz"

combinatorial
objects
on a 2d lattice

representation
by operators

Physics

data structures
"histories"

orthogonal
polynomials

bijections

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

rooks placements

permutations

RSK

⟷ pairs of Tableaux Young

$DE = qED + E + D$

PASEP

alternative tableaux ⟷ permutations

tree-like tableaux

Laguerre histories

dynamical systems in physics
stationary probabilities

reverse Q-tableaux

quadratic algebra Q

Q-tableaux

the XYZ algebra
ASM, (alternating sign matrices)
FPL (Fully packed loops)
tilings, non-crossing paths

?

commutations
rewriting rules

RSK automata

planarization

planar
automata

reverse planar
automata

The cellular Ansatz

quadratic algebra Q (of a certain type)

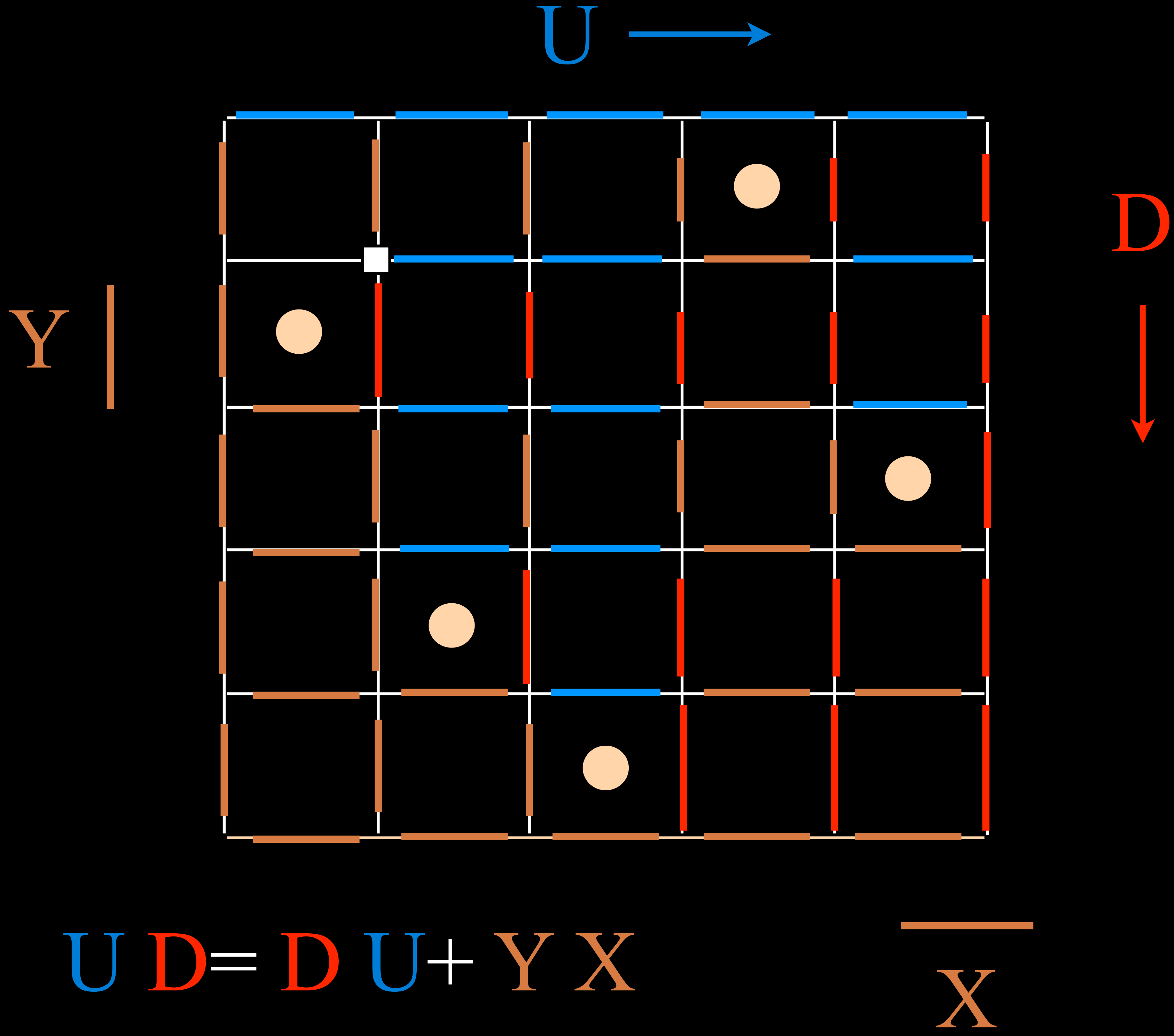
(1) "planarization" on a grid of the rewriting rules

Q- tableaux planar automata

(2) "planarization" on a grid of the bijection constructed from the representation of the algebra Q

(3) how to guess a representation: demultiplication of the commutation relations

«demultiplication»
of the commutation relations
in a quadratic algebra Q


U
D
Y
X
U D= D U+ Y X

$$\begin{cases} UD = DU + YX \\ UY = YU \\ XU = UX \\ XY = YX \end{cases}$$

"duplication" of the commutation relations defining the algebra Q

$$\begin{cases} UD = DU + YX \\ UY = YU \\ XU = UX \\ XY = YX \end{cases}$$

"duplication" of the commutation relations defining the algebra Q

$$UD = DU + Y_1 X_1$$

$$X_1 Y_1 = Y_2 X_2$$

$$\begin{cases} UD = DU + YX \\ UY = YU \\ XU = UX \\ XY = YX \end{cases}$$

"duplication" of the commutation relations defining the algebra Q

$$UD = DU + Y_1 X_1$$

$$X_1 Y_1 = Y_2 X_2$$

$$X_2 Y_2 = Y_3 X_3$$

$$\begin{cases} UD = DU + YX \\ UY = YU \\ XU = UX \\ XY = YX \end{cases}$$

"duplication" of the commutation relations defining the algebra Q

$$UD = DU + Y_1 X_1$$

$$X_1 Y_1 = Y_2 X_2$$

$$X_2 Y_2 = Y_3 X_3$$

$$\cdots\cdots\cdots$$

$$X_i Y_i = Y_{i+1} X_{i+1}$$

$$\cdots\cdots\cdots$$

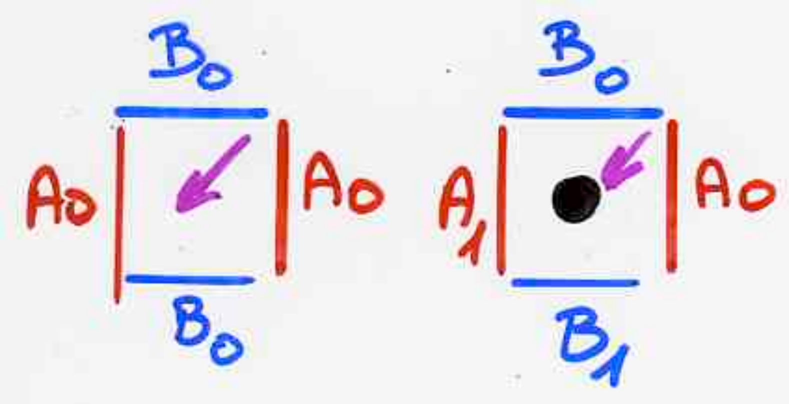
$$U Y_i = Y_i U$$

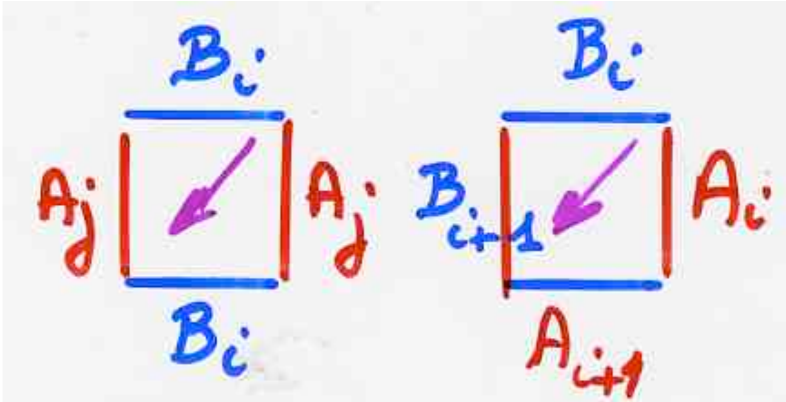
$$X_j U = U X_j$$

$$\cdots\cdots\cdots$$

we get back

the RSK planar automaton





$i \neq j$

$$\begin{cases} U = B_0 \\ X_i = B_i \end{cases} \quad i \geq 1$$

$$\begin{cases} D = A_0 \\ Y_i = A_i \end{cases} \quad i \geq 1$$

$$\begin{cases} UD = DU + YX \\ UY = YU \\ XU = UX \\ XY = YX \end{cases}$$

another demultiplication of the algebra $UD=DU+Id$

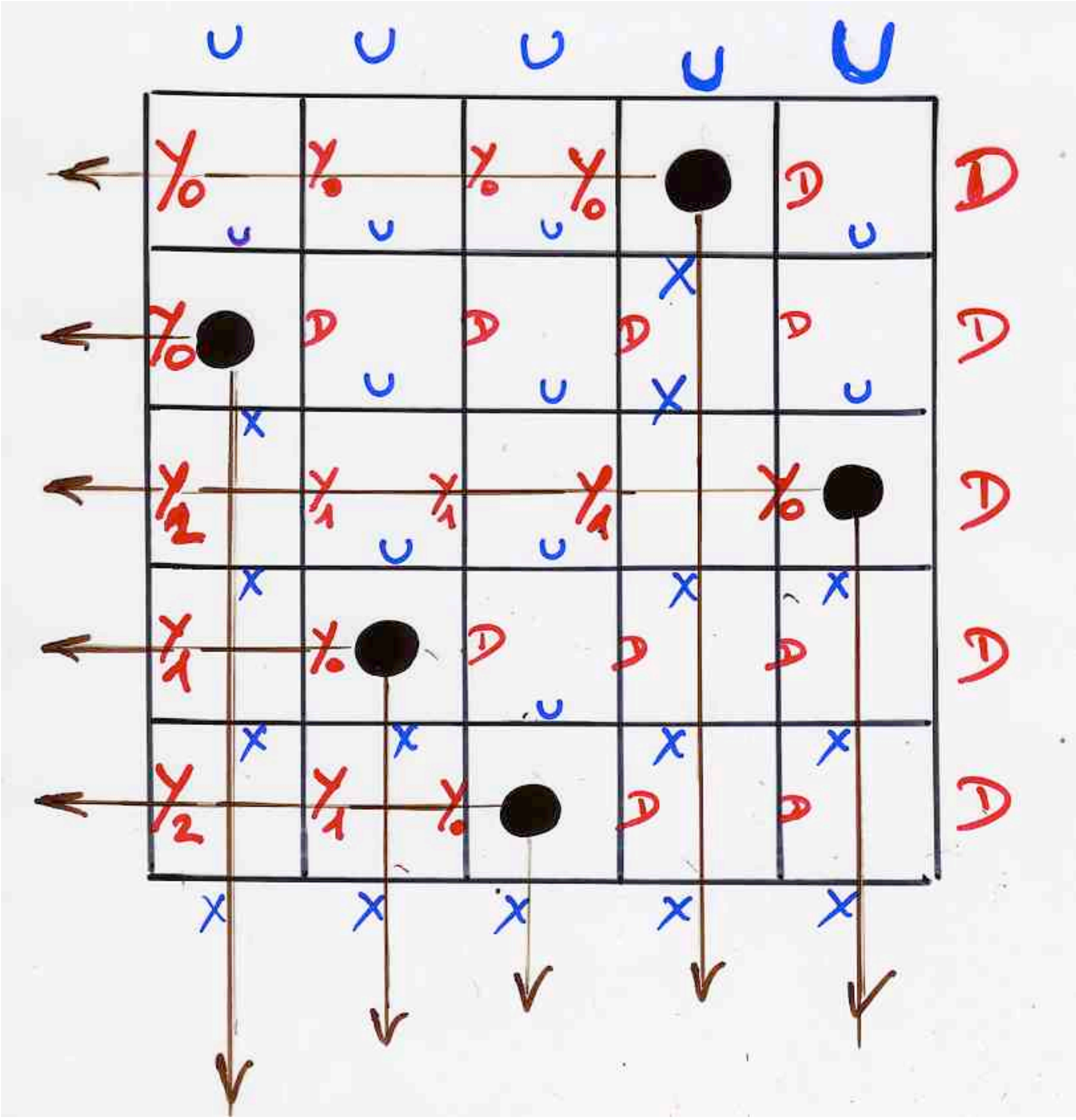
$$\begin{cases} UD = DU + YX \\ UY = YU \\ XU = UX \\ XY = YX \end{cases}$$

another "duplication" of the commutation relations of the algebra of the Q

$$UD = DU + Y_0 X$$

$$\begin{cases} X Y_0 = Y_1 X \\ X Y_1 = Y_2 X \\ X Y_2 = Y_3 X \\ \text{- - - - - - - -} \\ X Y_i \quad Y_{i+1} X \\ \text{- - - - - - - -} \end{cases}$$

$$XU = UX$$
$$U Y_i = Y_i U$$



$$UD = DU + Y_0 X$$

$$\begin{cases} X Y_0 = Y_1 X \\ X Y_1 = Y_2 X \\ X Y_2 = Y_3 X \\ \cdots \\ X Y_i = Y_{i+1} X \\ \cdots \end{cases}$$

$$XU = UX$$

$$U Y_i = Y_i U$$

→ bijections

- permutation ⟷ inversion table
- involution no fixed points ⟷ "Hermite histories"
- involution colored fixed points ⟷ towers placement

demultiplication in the XYZ algebra and the ASM algebra

$A, A', B, B',$

commutations

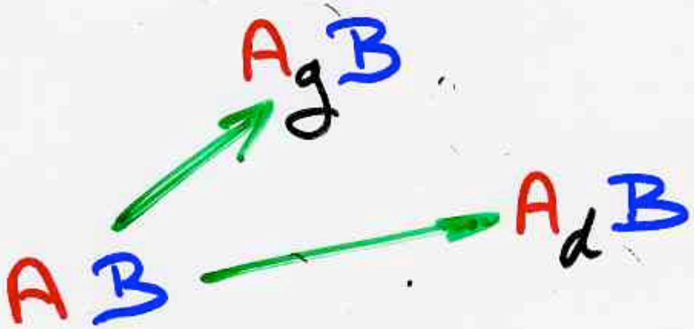
$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

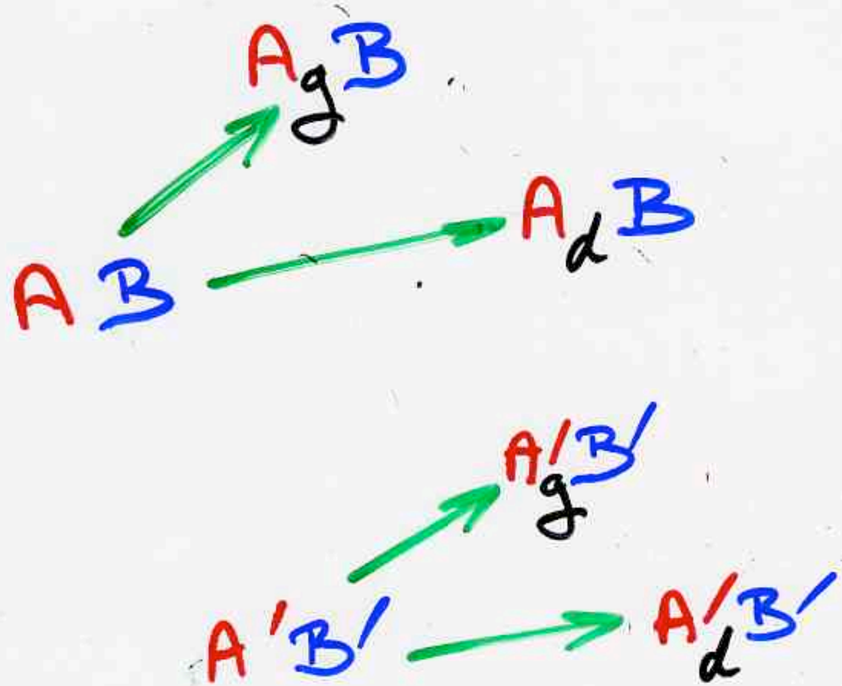
$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$



commutations

$$\begin{cases} B\,A = A\,B + A'\,B' \\ B'\,A' = A'\,B' + A\,B \end{cases}$$

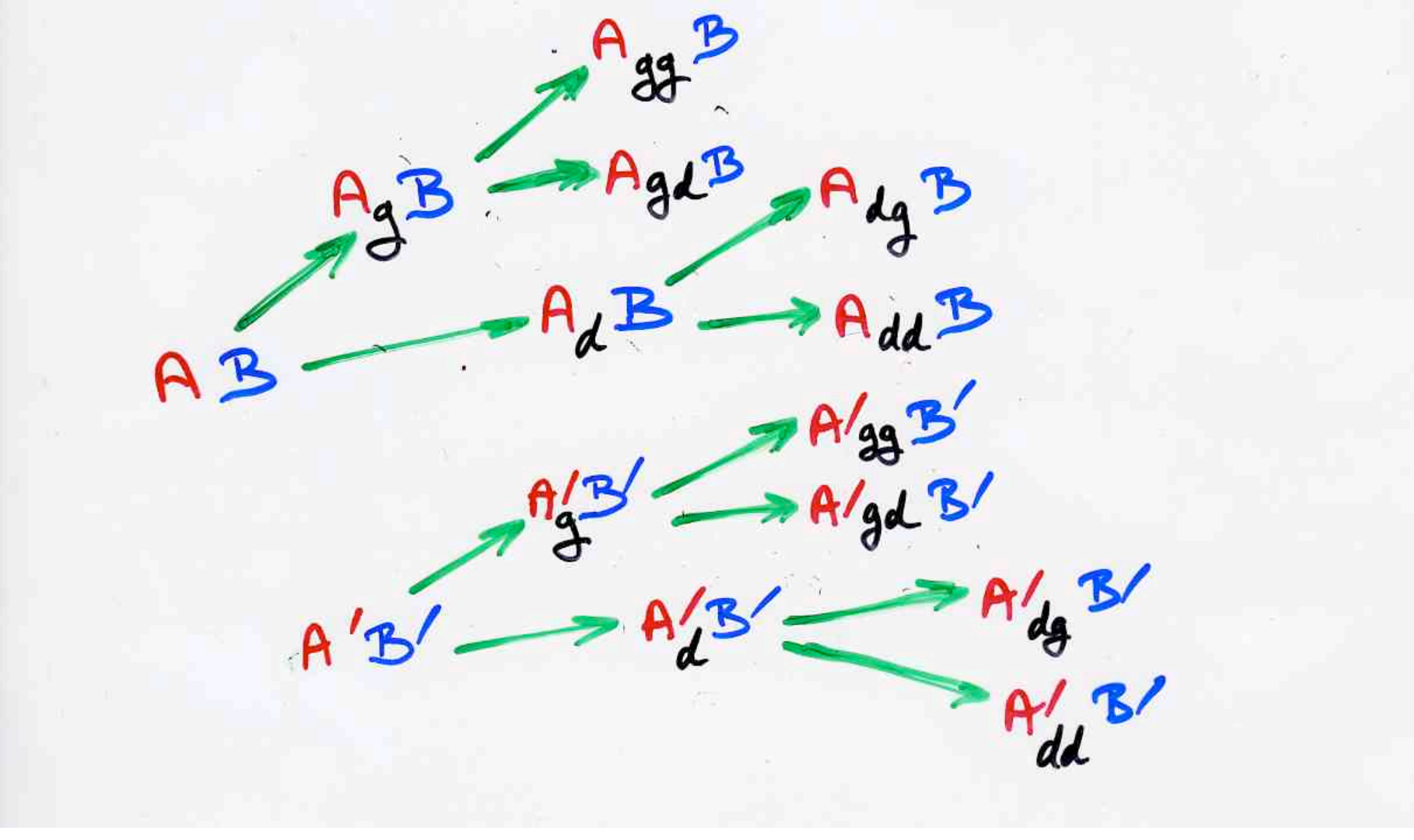
$$\begin{cases} B'\,A = A\,B' \\ B\,A' = A'\,B \end{cases}$$



commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

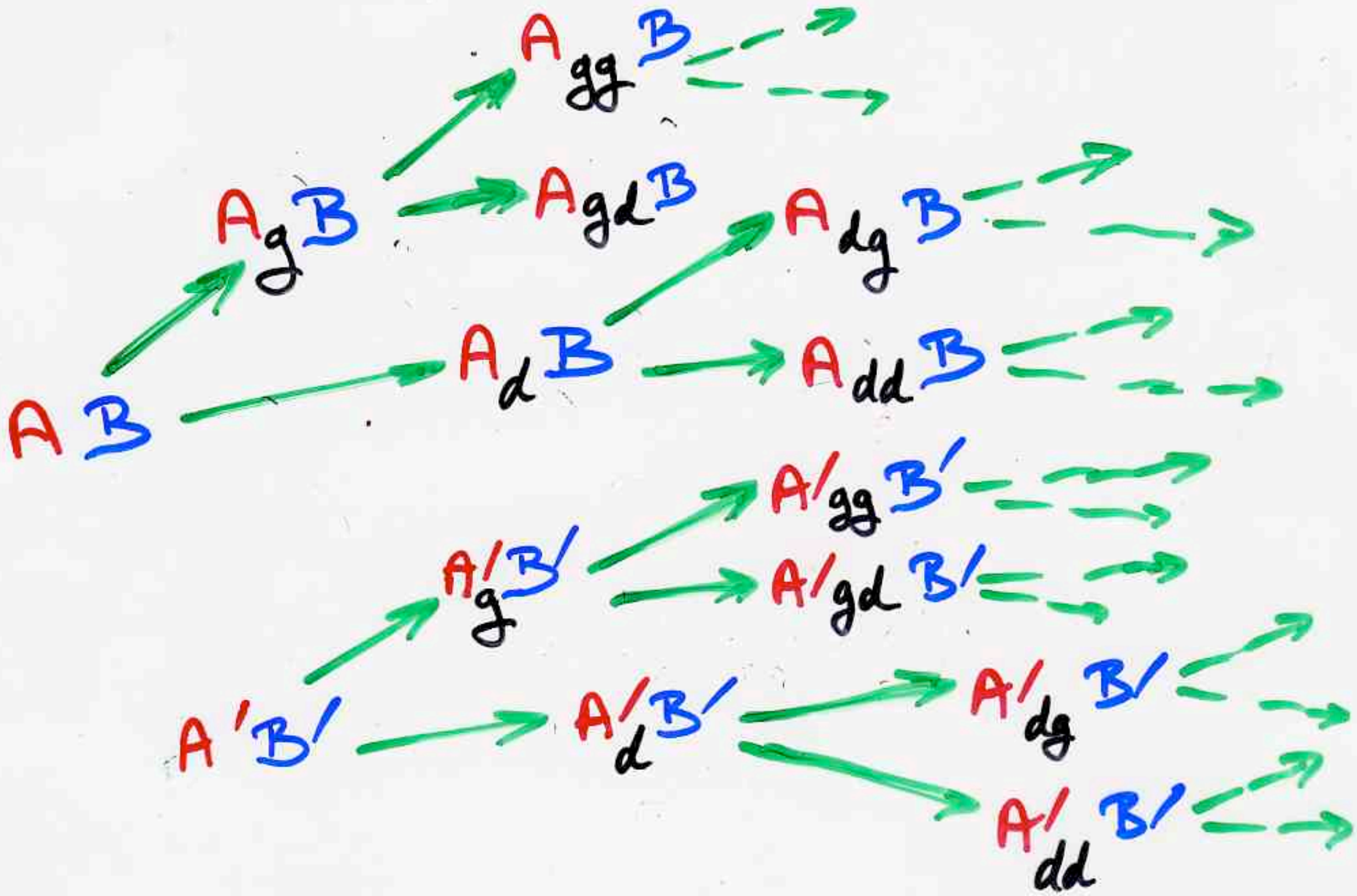
$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$



commutations

$$\begin{cases} B A = A B + A' B' \\ B' A' = A' B' + A B \end{cases}$$

$$\begin{cases} B' A = A B' \\ B A' = A' B \end{cases}$$

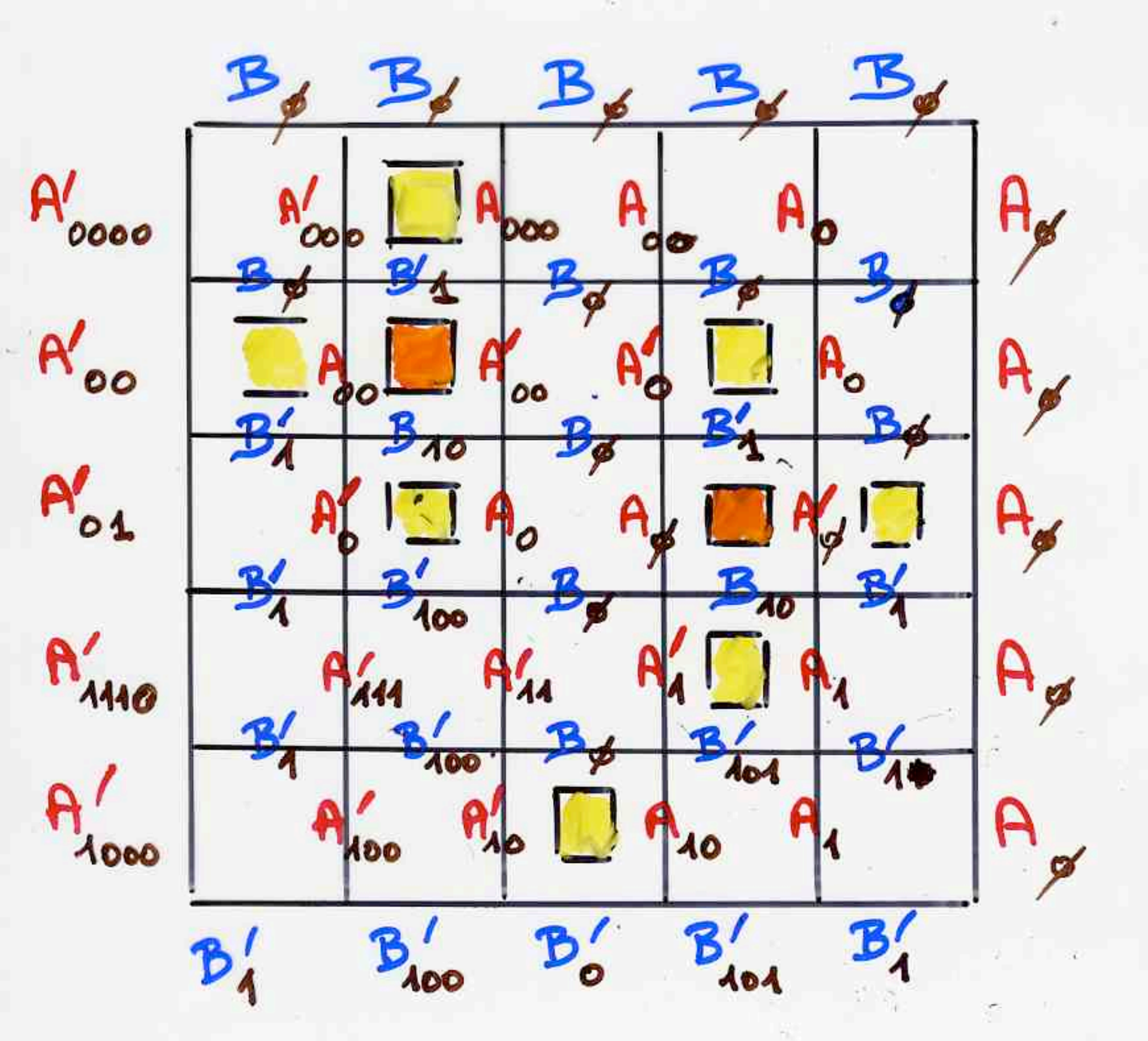


The quadratic algebra $\mathbb{Z}$

4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\cdots}$, $t_{\cdots}$

$$
\begin{cases}
B\,A = q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet B_\bullet \\
B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, A\,B \\
B_\bullet A = q_{\bullet\circ}\, A\,B_\bullet + t_{\bullet\circ}\, A_\bullet B \\
B\,A_\bullet = q_{\circ\bullet}\, A_\bullet B + t_{\circ\bullet}\, A\,B_\bullet
\end{cases}
$$



The bijection $T \to (P, Q)$

here T alternating sign matrix

·	1	·	·	·
1	-1	·	1	·
·	1	·	-1	1
·	·	·	1	·
·	·	1	·	·

$P = (0000, 00, 01, 1110, 1000)$

$Q = (1, 100, 0, 101, 1)$

more problems ...

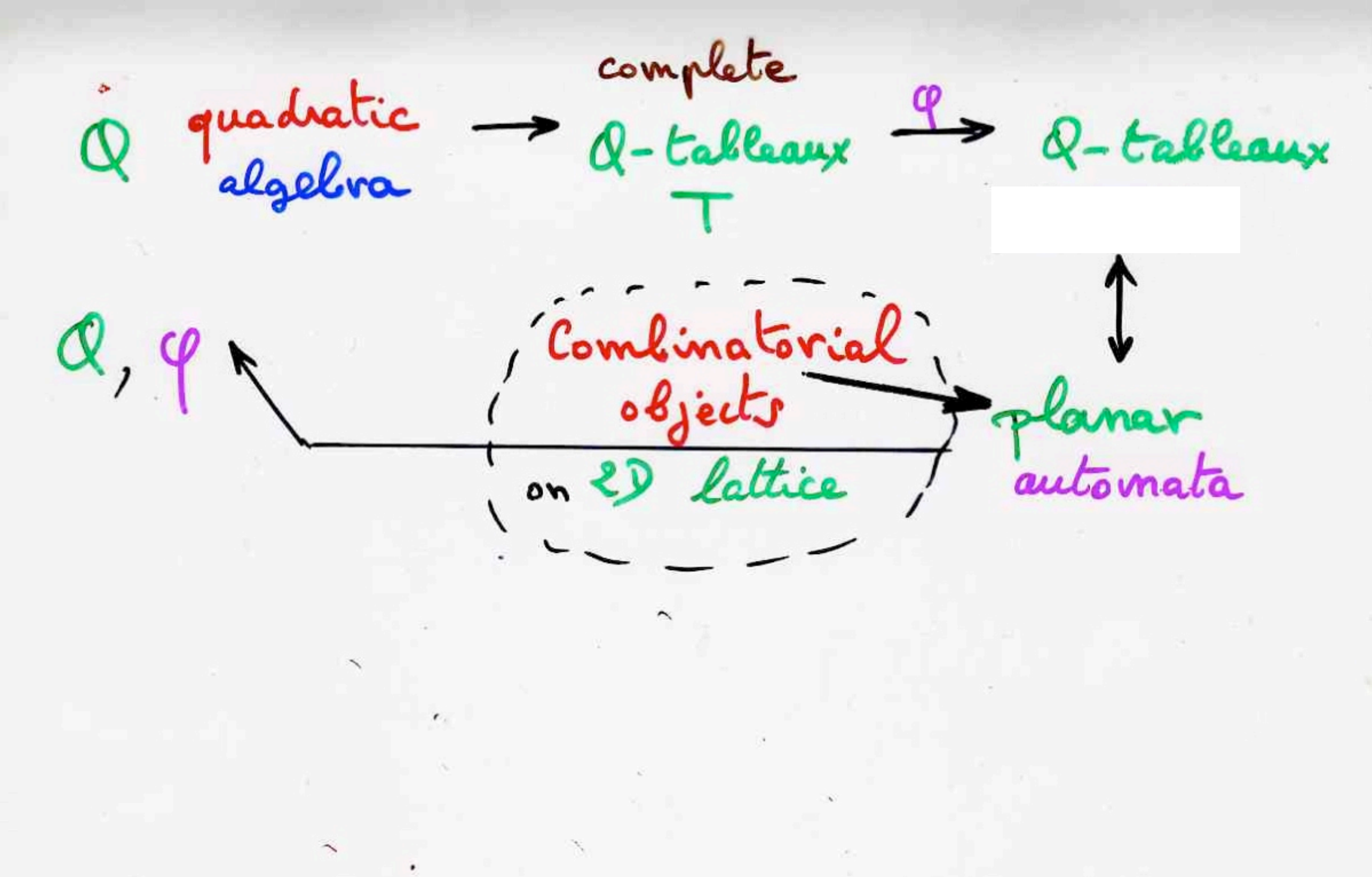
- formula for $c(u, v; w)$?

  Q determinant ?

- or at least efficient procedure for computing $c(u, v; w)$?

- generating function ?

conclusion



Q quadratic algebra
complete Q-tableaux T
φ
Q-tableaux
Q, φ
Combinatorial objects on 2D lattice
planar automata

"The cellular Ansatz"

combinatorial objects on a 2d lattice

Physics

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

rooks placements

permutations

$DE = qED + E + D$

PASEP

alternative tableaux

dynamical systems in physics stationary probabilities

quadratic algebra Q

Q-tableaux

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automata

representation
by operators

data structures "histories" orthogonal polynomials

bijections

RSK

permutations ⟷ pairs of Tableaux Young

alternative tableaux ⟷ permutations
Laguerre histories

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stationary probabilities

reverse Q-tableaux

quadratic algebra Q

Q-tableaux

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ASM, (alternating sign matrices)

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demultiplication of equations in algebra Q

commutations rewriting rules

RSK automata

planarization

planar automata

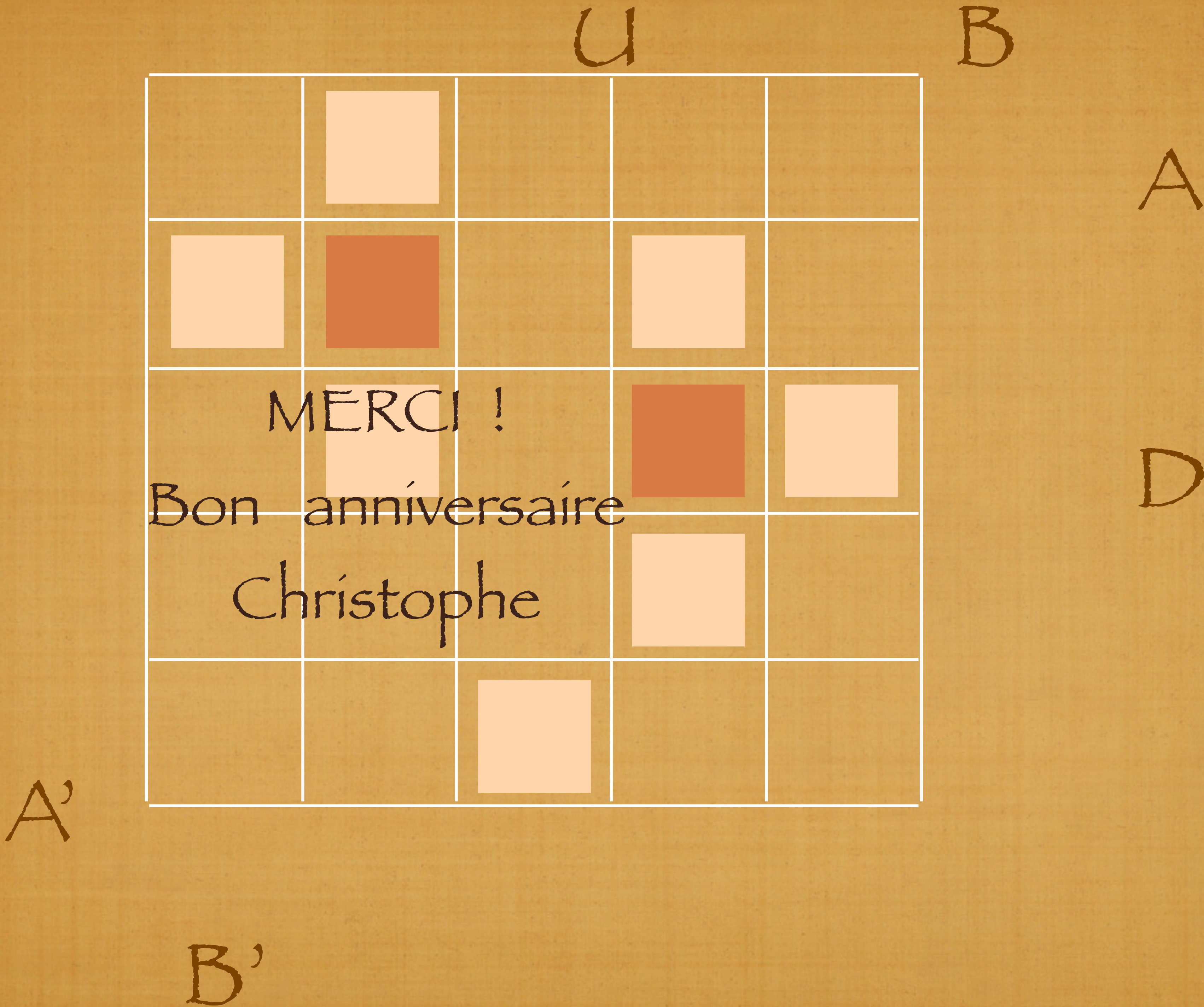
reverse planar automata

bijection BABA - pair (P,Q)

website Xavier Viennot

main website   www.xavierviennot.org

secondary website: Courses   cours.xavierviennot.org
- course IIT Bombay 2013 (20 hours)



MERCI !

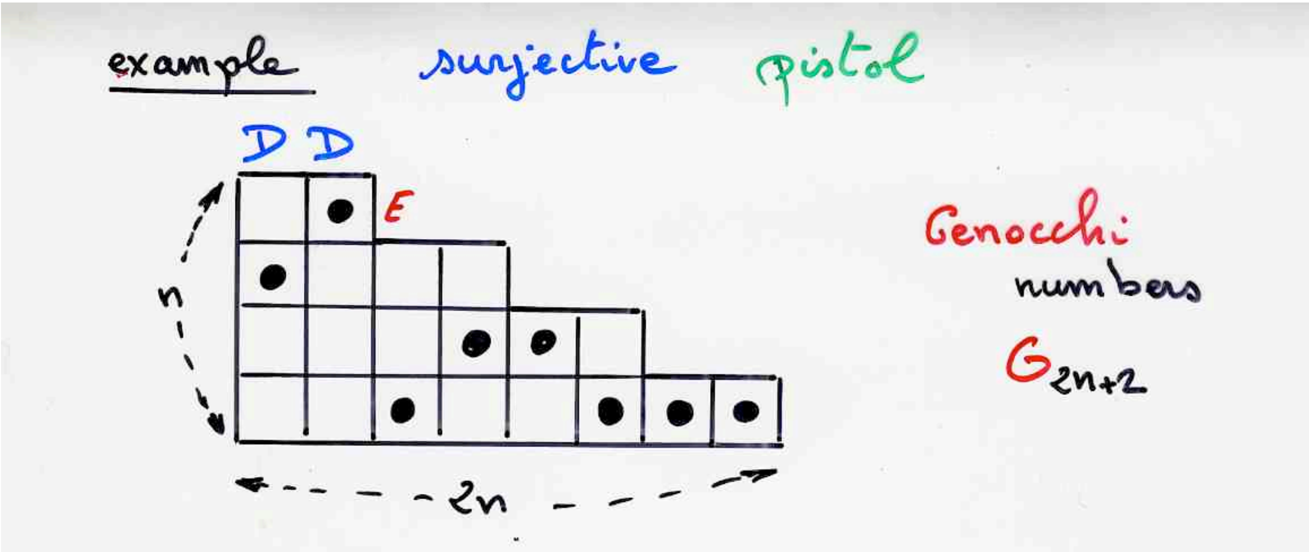
Bon anniversaire

Christophe

Planar automata

examples

example 1



one point in each column
at least one point in each row

nombres de Genocchi

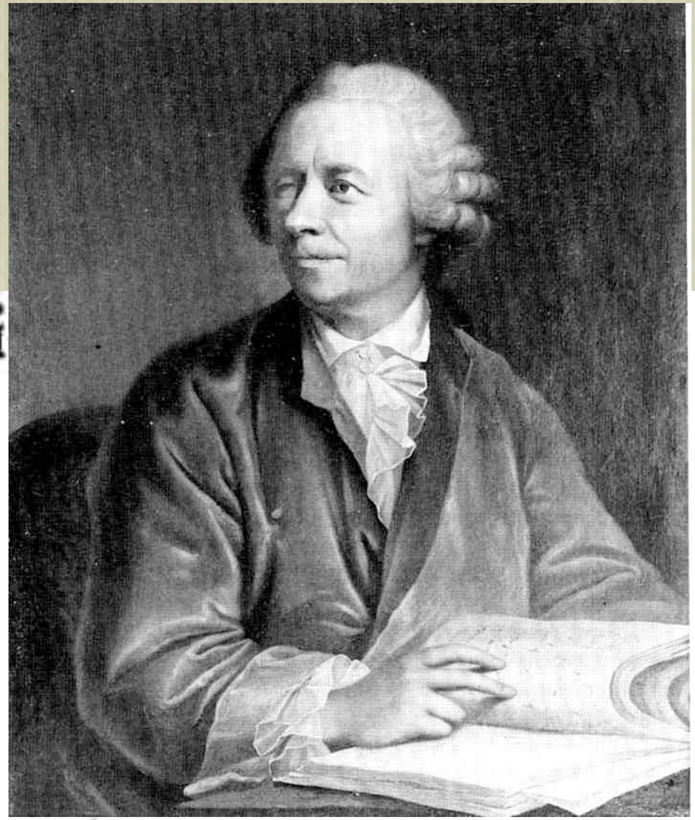
$$G_{2n} = 2(2^{2n} - 1) B_{2n}$$

Bernoulli

$$2^{2n} G_{2n+2} = (n+1) T_{2n+1}$$

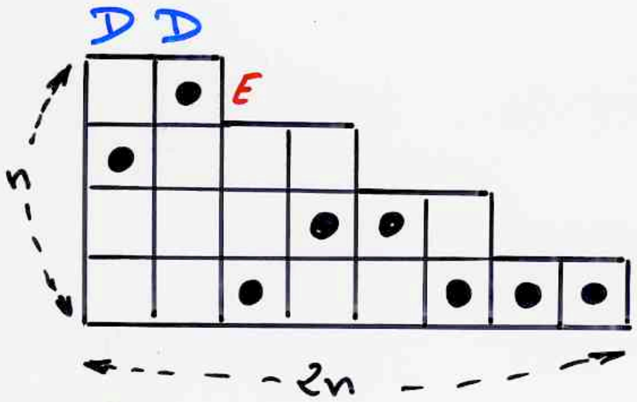
Angelo Genocchi
1817 - 1889

Hinc igitur calculo inſtituto reperi



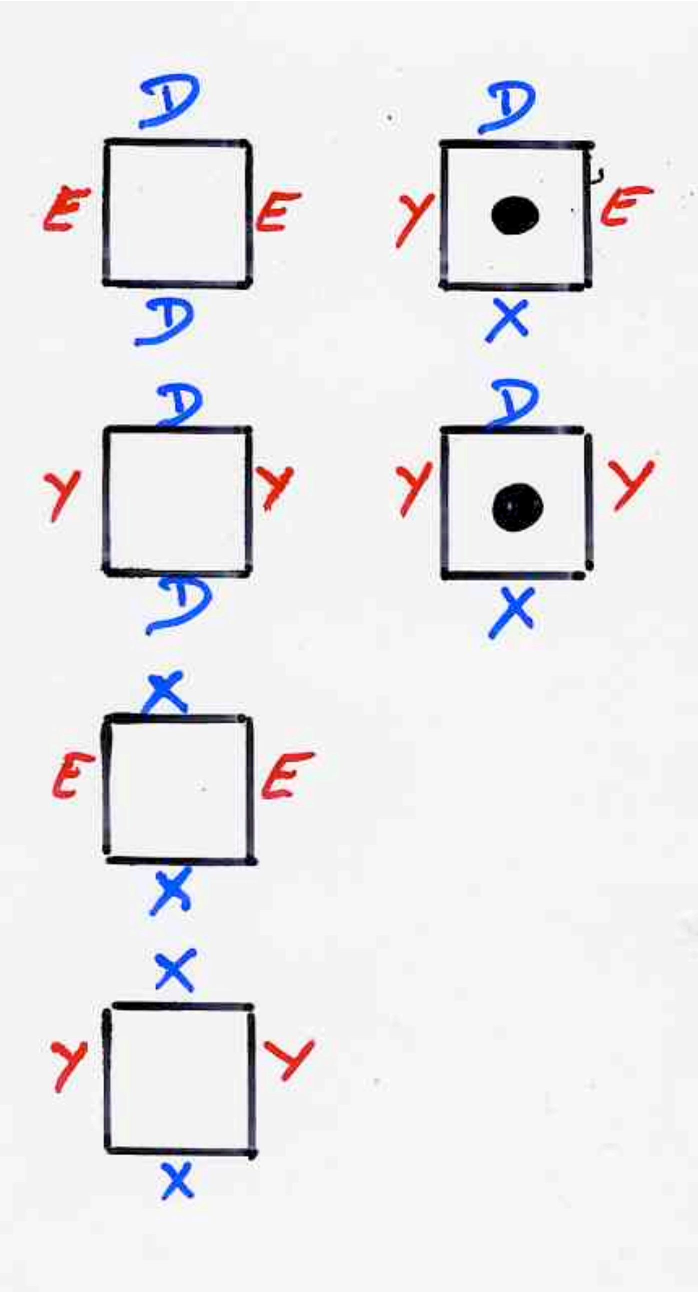
$$A = 1$$
$$B = 1$$
$$C = 3$$
$$D = 17$$
$$E = 155 = 5\cdot 31$$
$$F = 2073 = 691\cdot 3$$
$$G = 38227 = 7\cdot 5461 = 7\cdot\frac{127\cdot 129}{3}$$
$$H = 929569 = 3617\cdot 257$$
$$I = 28820619 = 43867\cdot 9\cdot 73 \quad \&c.$$

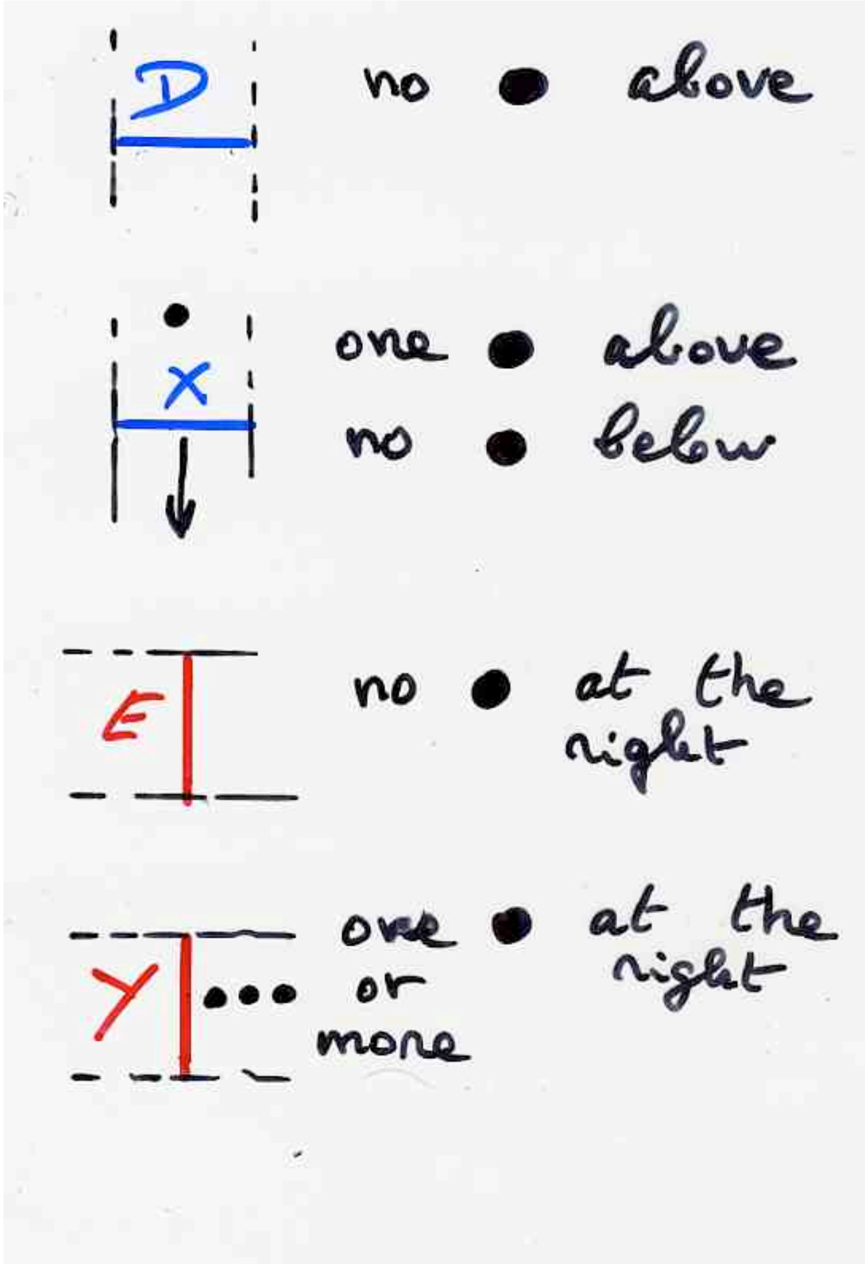
example surjective pistol



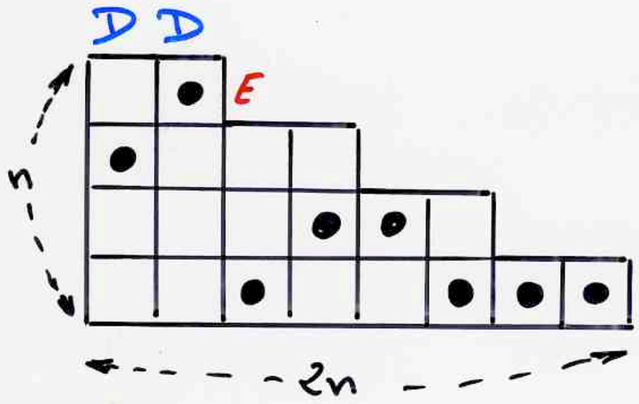
Genocchi numbers

G_{2n+2}


D
E E
D
Y E
X
D
Y Y
D
D
Y Y
X
X
E E
X
X
Y Y
X


D
no ● above
X
one ● above
no ● below
E
no ● at the right
Y
one ● or more
at the right

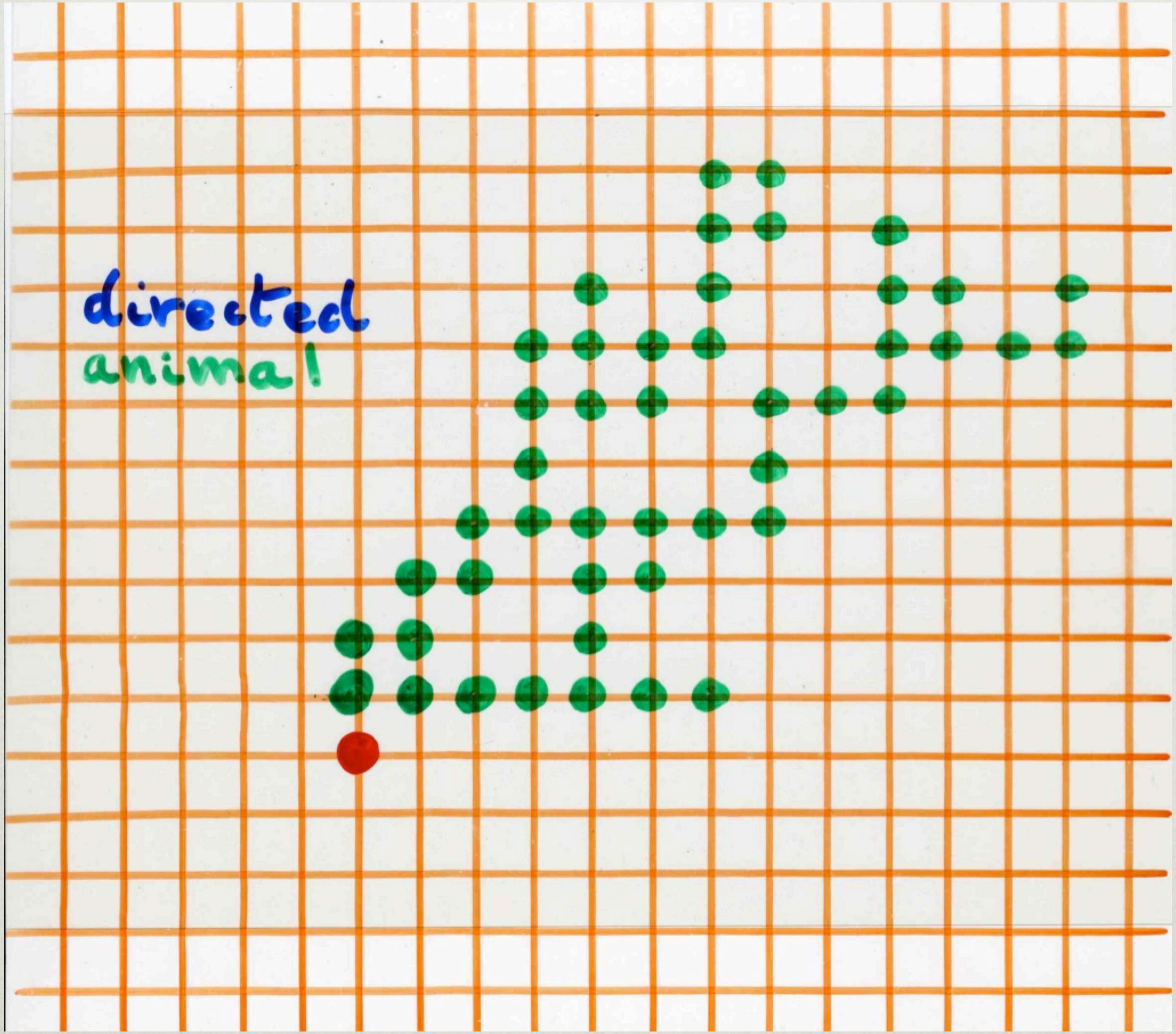
example surjective pistol



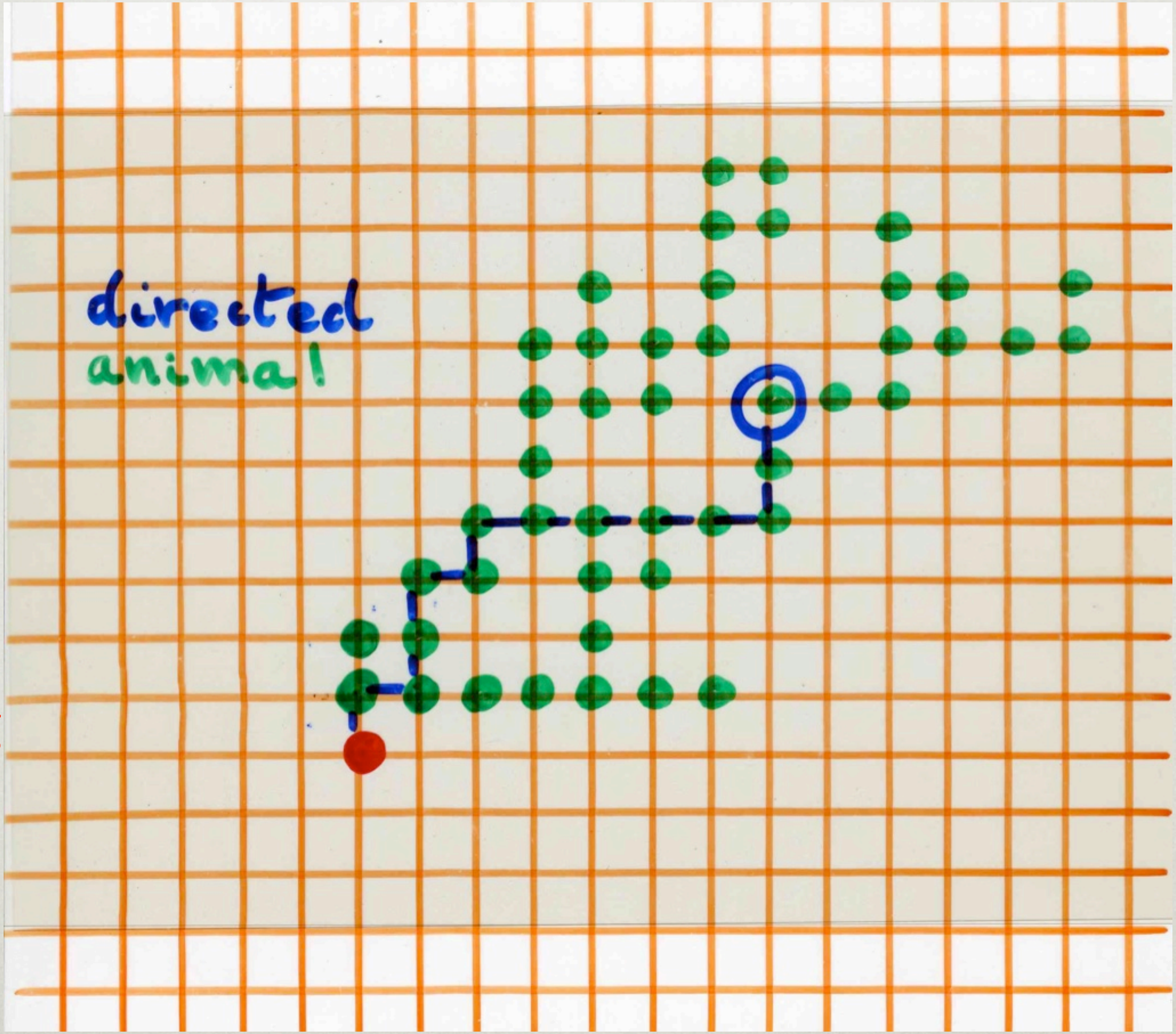
Genocchi numbers G_{2n+2}

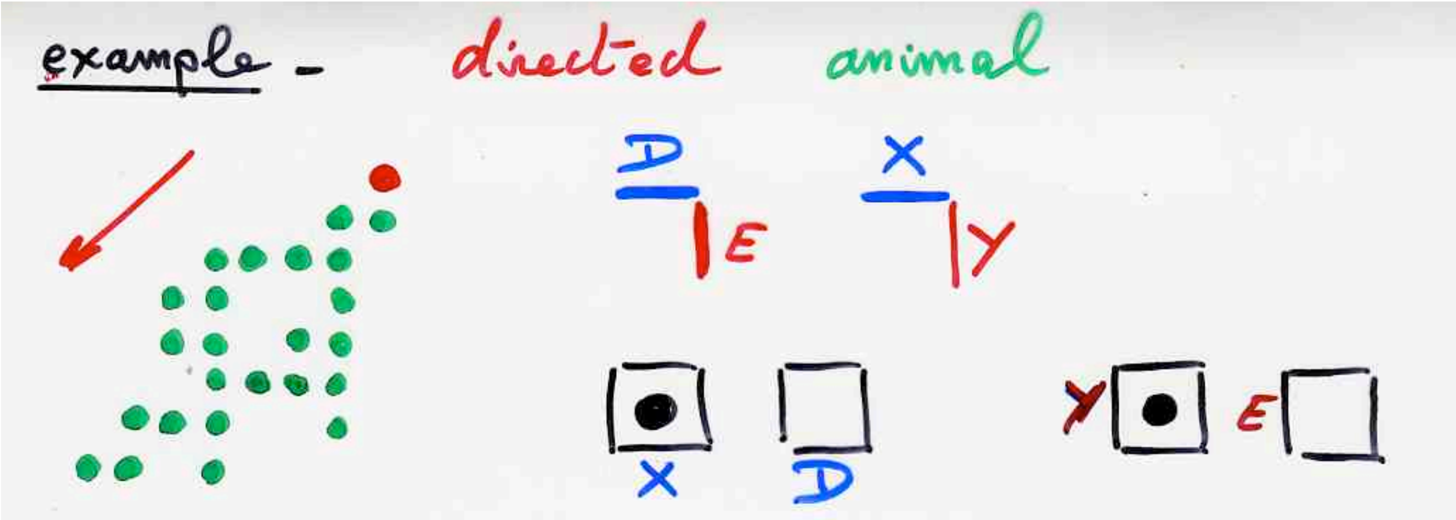
$$c\,(\underset{u}{Y^n},\ \underset{v}{X^{2n}}\ ;\ \underset{w}{(D^2E)^n}) = G_{2n+2}$$

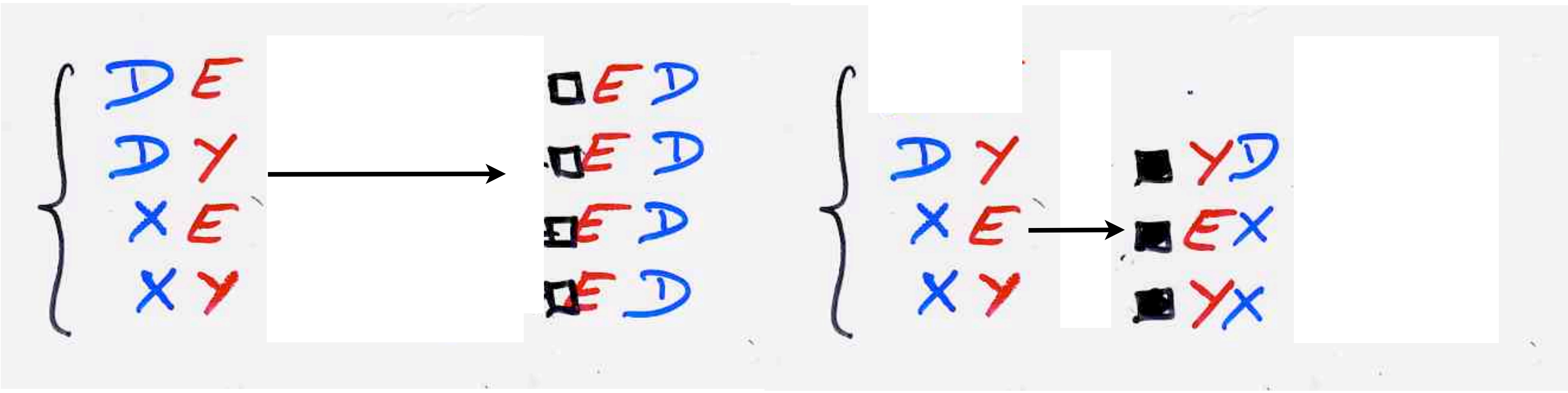
example 2



algebraic
generating
function

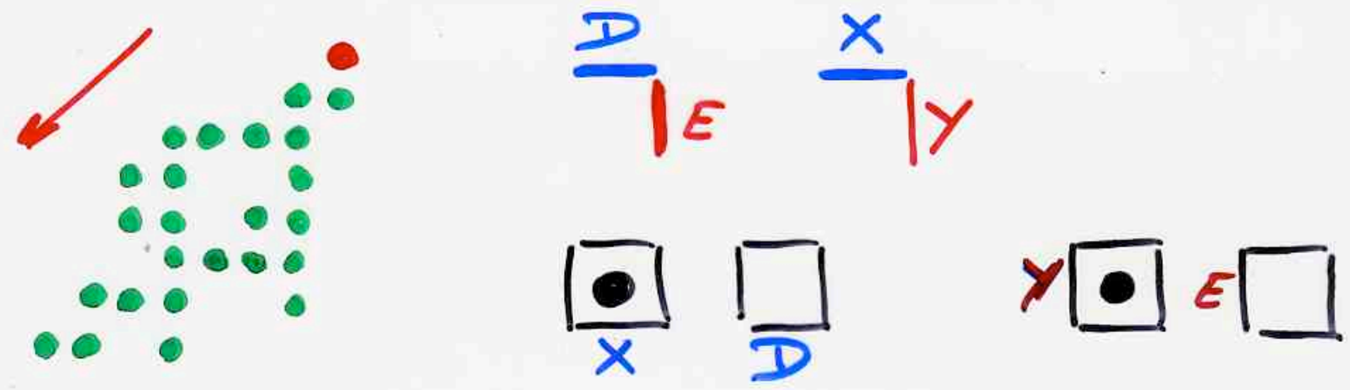



example - directed animal
D
E
X
Y
X
D
Y
E


D E
D Y
X E
X Y
E D
E D
E D
E D
D Y
X E
X Y
Y D
E X
Y X

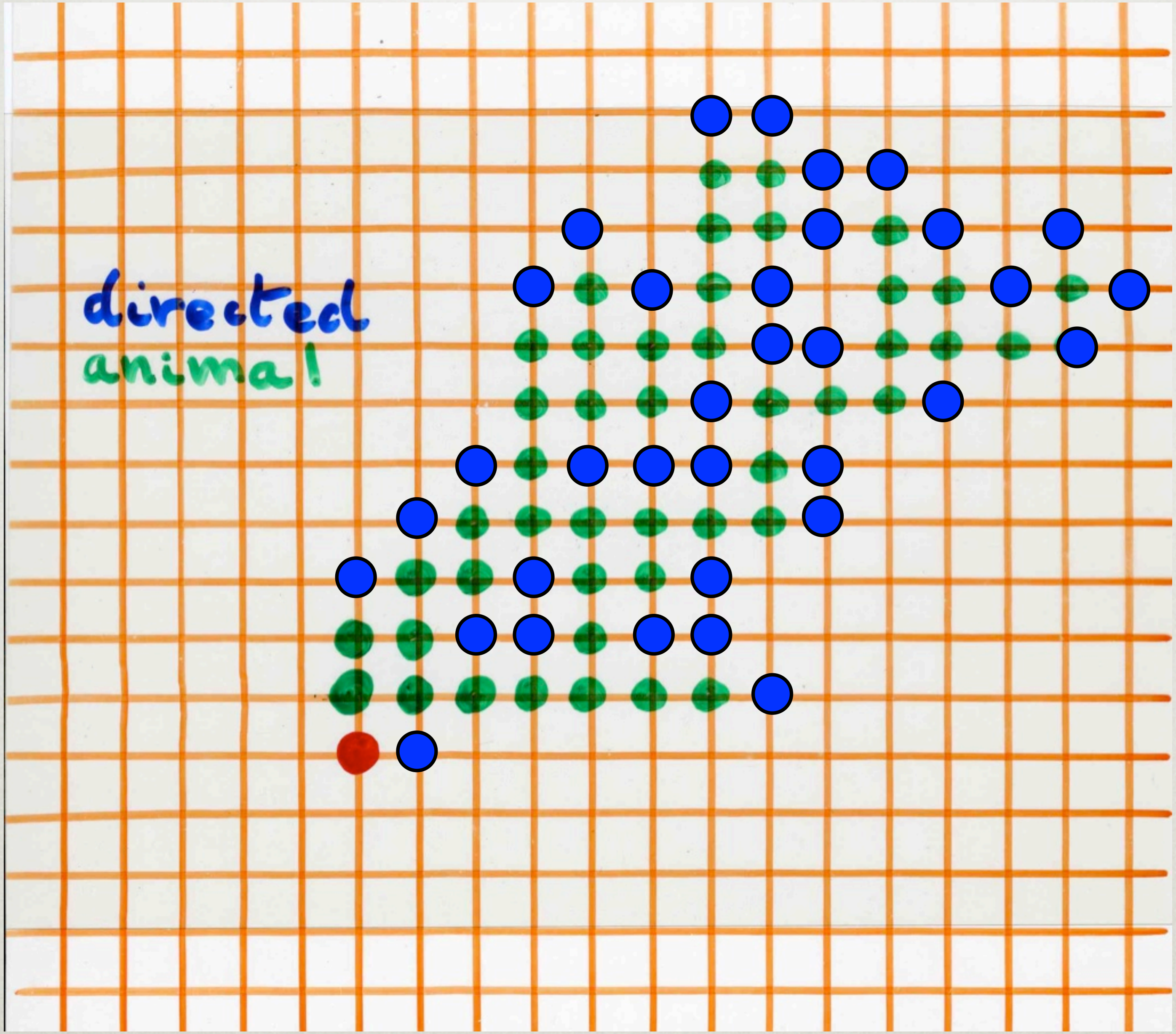
The directed animals algebra

example - directed animal



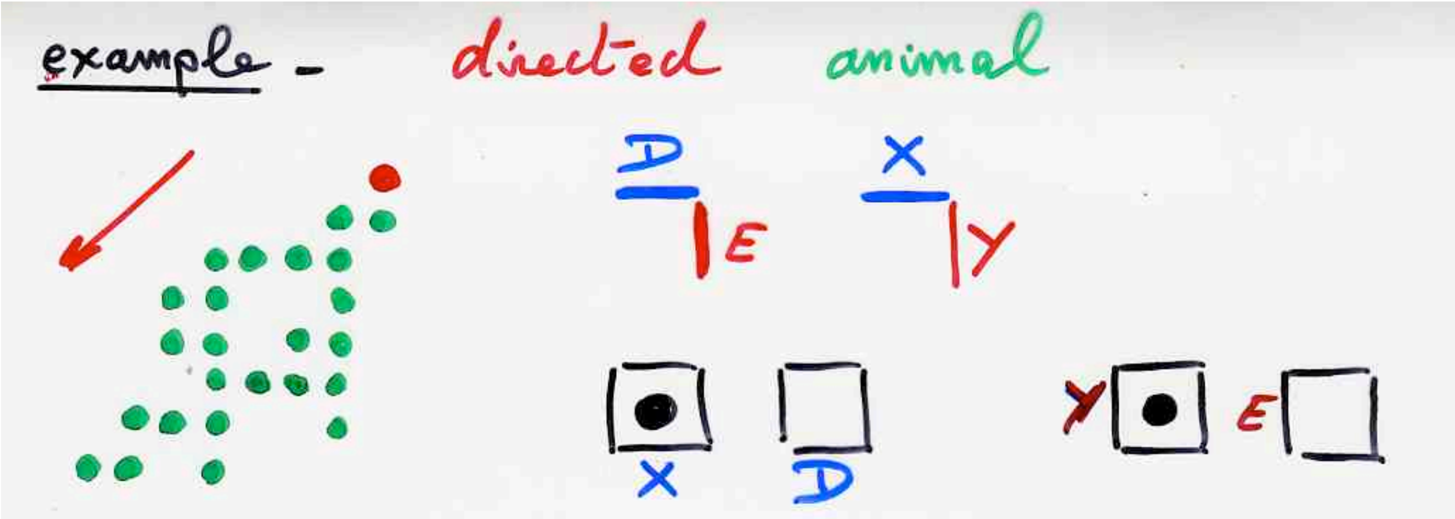
$$\begin{cases} DE = \square ED \\ DY = \blacksquare YD + \square ED \\ XE = \blacksquare EX + \square ED \\ XY = \blacksquare YX + \square ED \end{cases}$$

J.-F. Marckert 2011, 2012

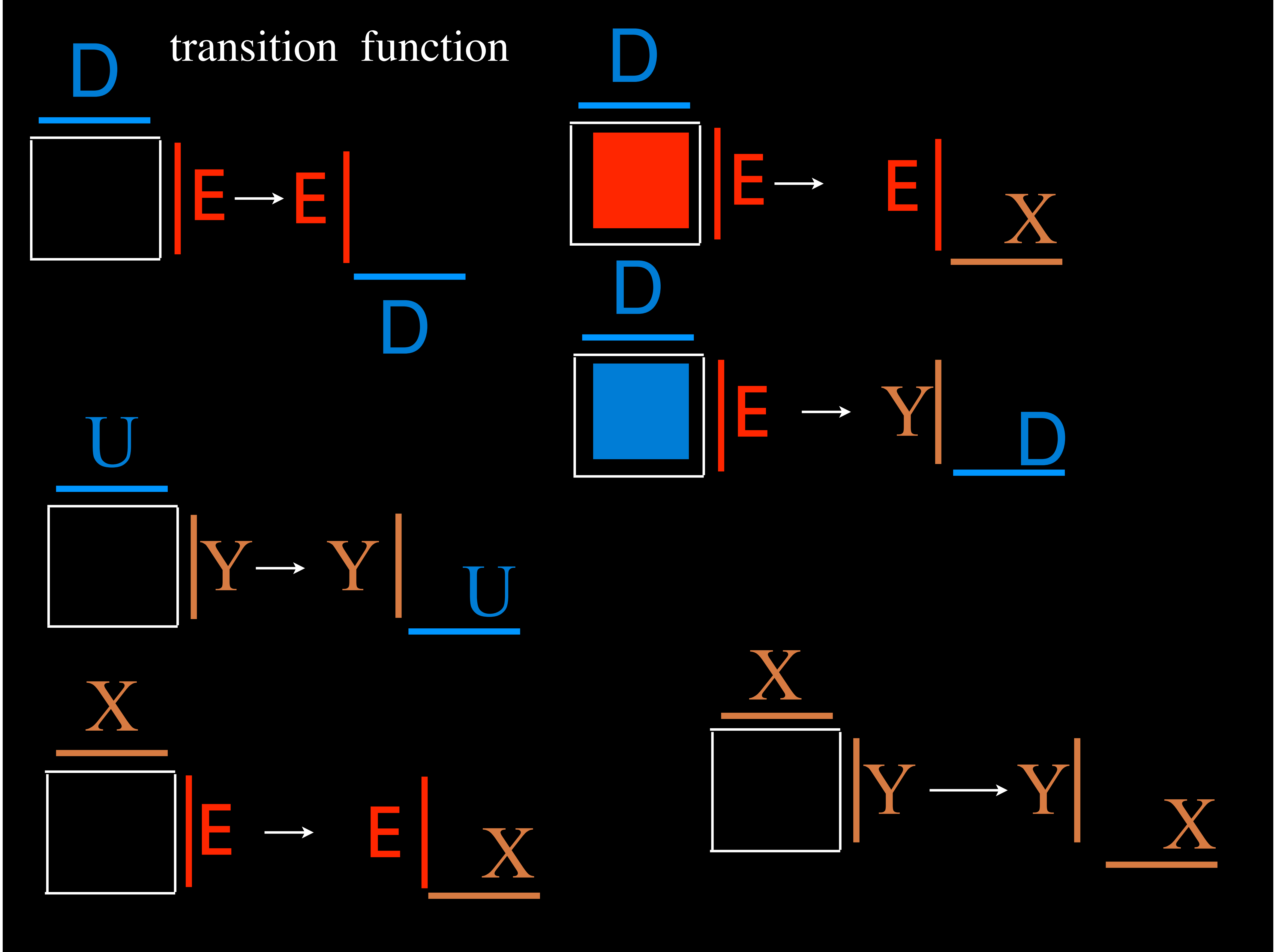


counting by number of points + directed perimeter

quadratic and rewriting systems



$$
\begin{cases}
DE = \square\, ED \\
DY = \blacksquare\, YD + \bullet\, ED \\
XE = \blacksquare\, EX + \bullet\, ED \\
XY = \blacksquare\, YX + \bullet\, ED
\end{cases}
$$


transition function
D
E → E
D
D
E → E
X
D
E → Y
D
U
Y → Y
U
X
E → E
X
X
Y → Y
X