The magic of Young tableaux

S.S. Pillai Endowment Lecture

Ramanujan Institute, Chennai
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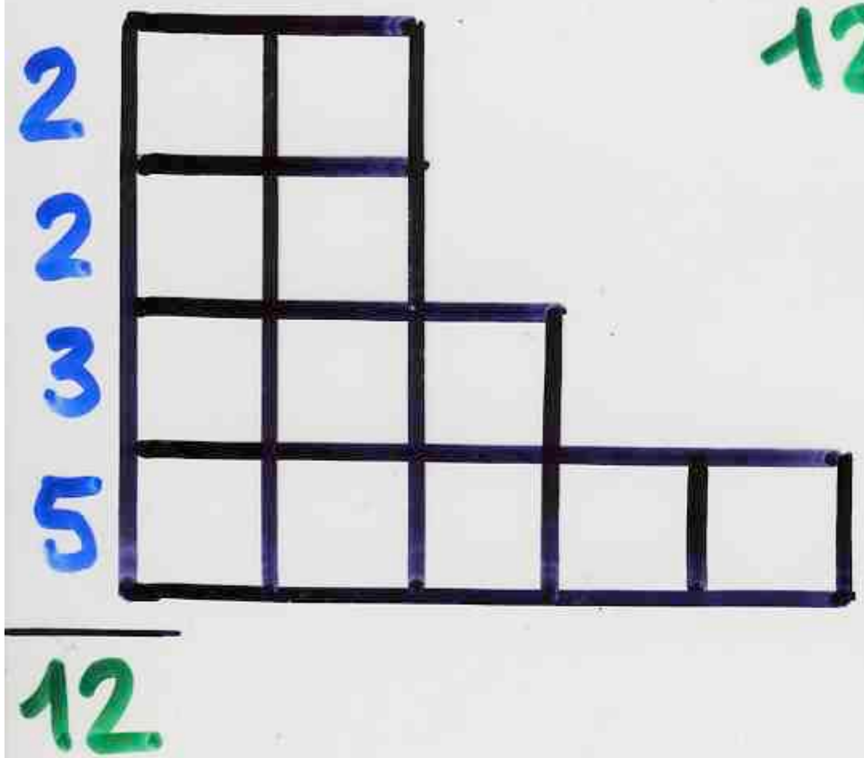
Young tableaux

$12 = n = 5 + 3 + 2 + 2$

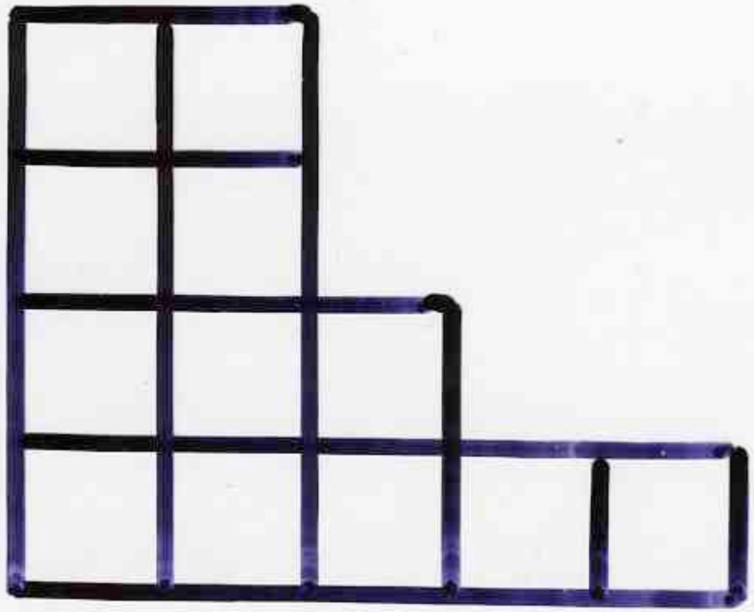


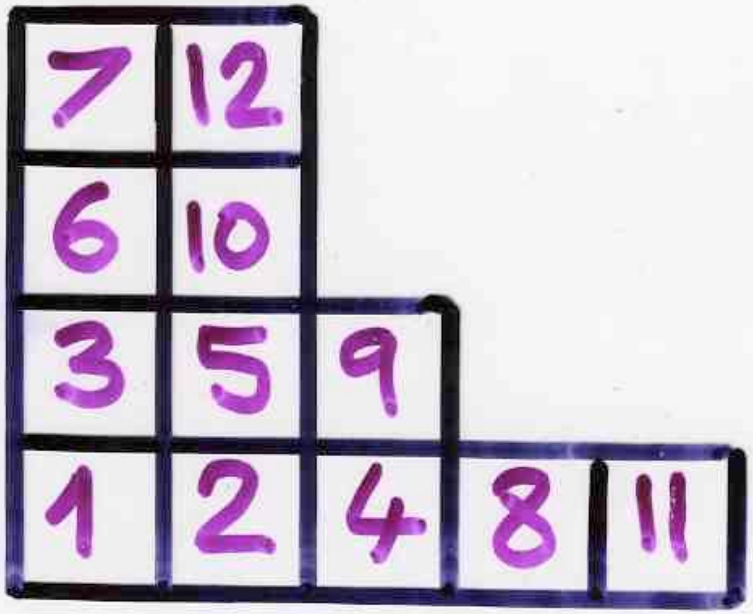
Ferrers

diagram

Partition of n

λ





Young tableau

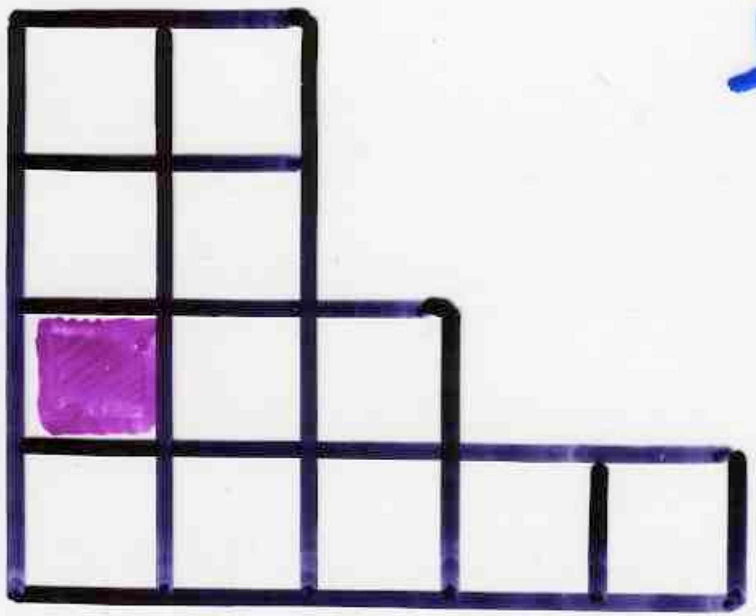
shape λ

f_λ = nb of Young tableaux shape λ

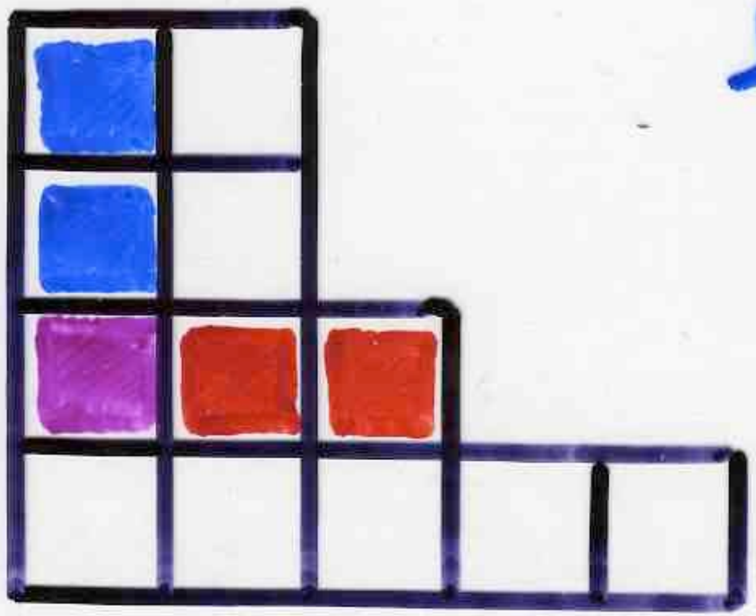
Hook length formula

J.S. Frame, G. de B. Robinson et R.M. Thrall, 1954

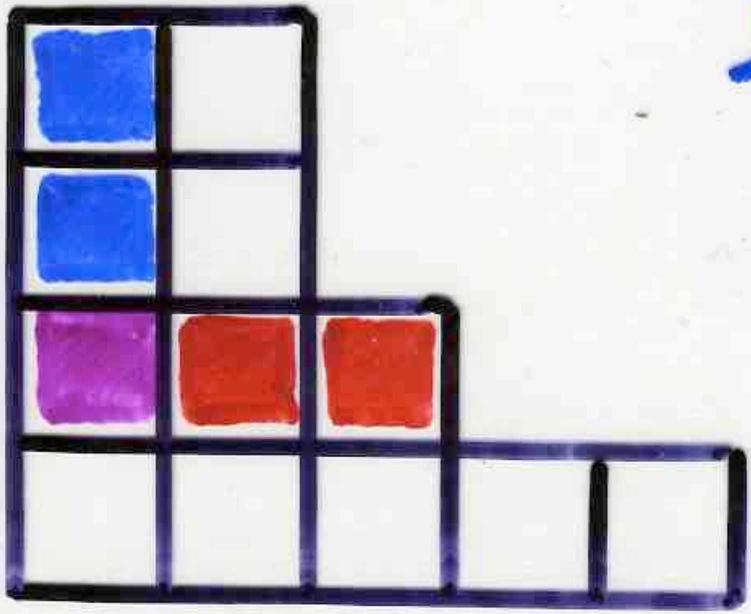
..... Franzblau-Zeilberger, Remmel, Greene-Wilf, Krattenthaler, Novelli- Pak-Stoyanovski, ...



hook



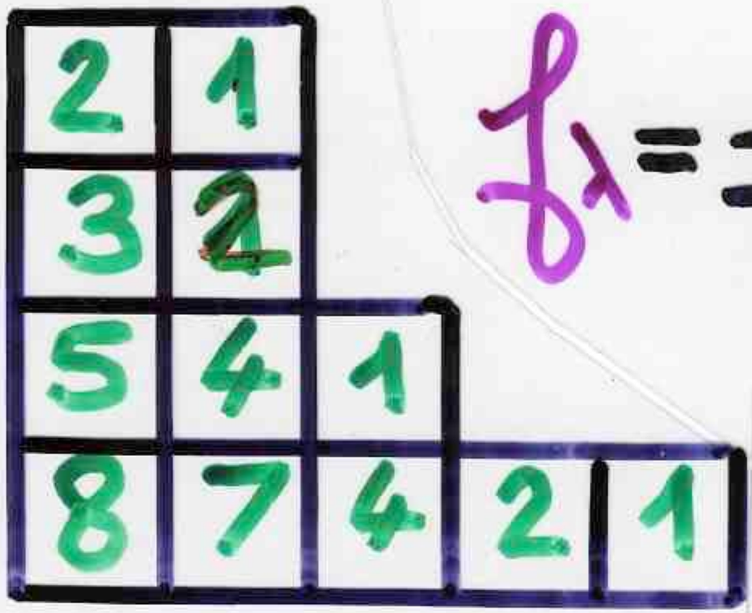
hook



hook length

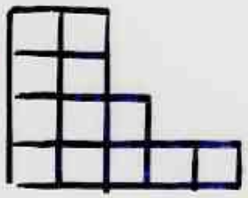
5

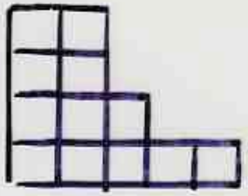
2	1			
3	2			
5	4	1		
8	7	4	2	1



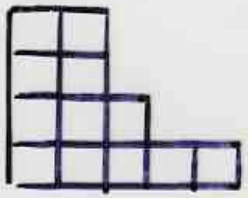
$$f_\lambda = \frac{n!}{\prod_x h_x}$$

hook length formula

f  $=$

f 

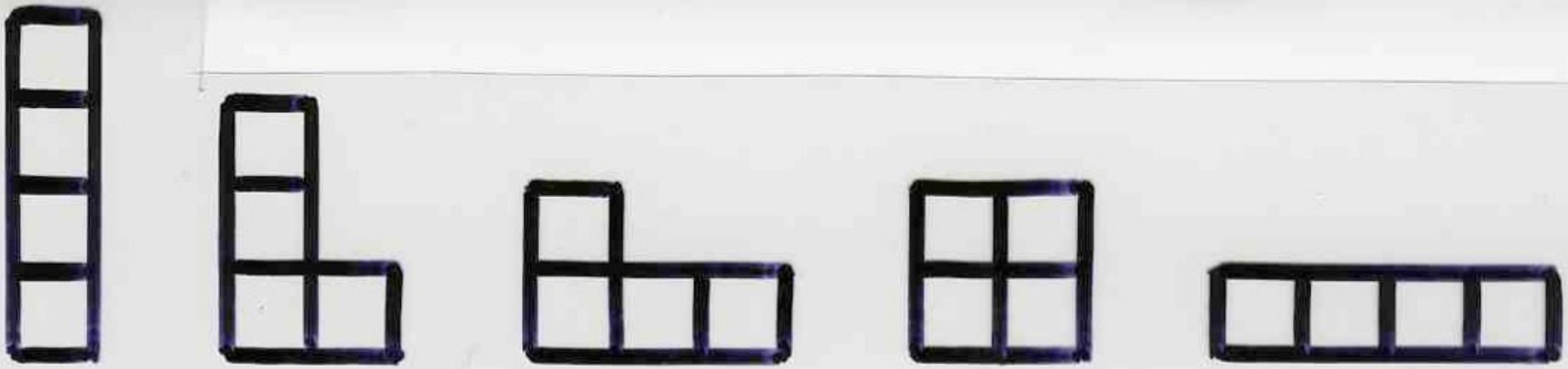
$$= \frac{1 \cdot 2 \times 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1^3 \times 2^3 \times 3 \times 4^2 \cdot 5 \cdot 7 \cdot 8}$$

f  $= \dfrac{1 \cdot 2 \times 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1^{3} \times 2^{3} \times 3 \times 4^{2} \cdot 5 \cdot 7 \cdot 8}$

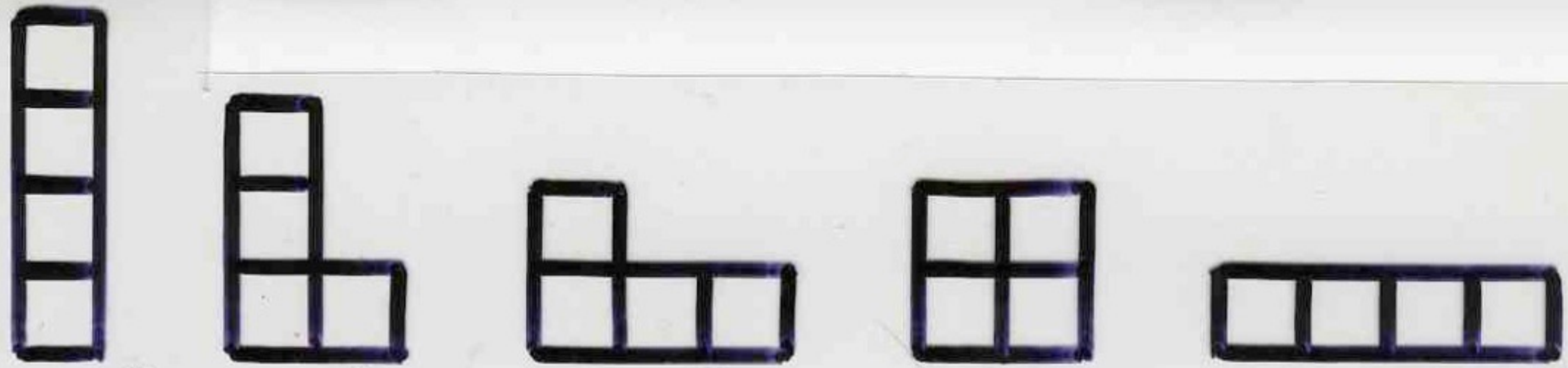
$= 3^{4} \times 5 \times 11 = 4455$

An introduction to RSK

G. de B. Robinson, 1938
C. Schensted, 1961
D. Knuth, 1970



1 3 3 2 1



$$1^2 + 3^2 + 3^2 + 2^2 + 1^2$$

$$= 1 + 9 + 9 + 4 + 1$$

$$= 24 = 4!$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

number of permutations on $\{1, 2, \ldots, n\}$ $= 1 \times 2 \times 3 \times \ldots \times n$

$= n!$

$$n! = \sum_{\substack{\lambda \\ \text{partitions} \\ \text{of } n}} \left(f_\lambda \right)^2$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

$$P = \begin{array}{|c|c|c|c|c|} \hline 6 & 10 \\ \cline{1-2} 3 & 5 & 8 \\ \cline{1-3} 1 & 2 & 4 & 7 & 9 \\ \hline \end{array} \qquad Q = \begin{array}{|c|c|c|c|c|} \hline 8 & 10 \\ \cline{1-2} 2 & 5 & 6 \\ \cline{1-3} 1 & 3 & 4 & 7 & 9 \\ \hline \end{array}$$

The Robinson-Schensted correspondence between permutations and pairs of (standard) Young tableaux with the same shape

RSK with Schensted's insertions

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1					

3					

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2					
1					

3					
1					

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2					
1	3				

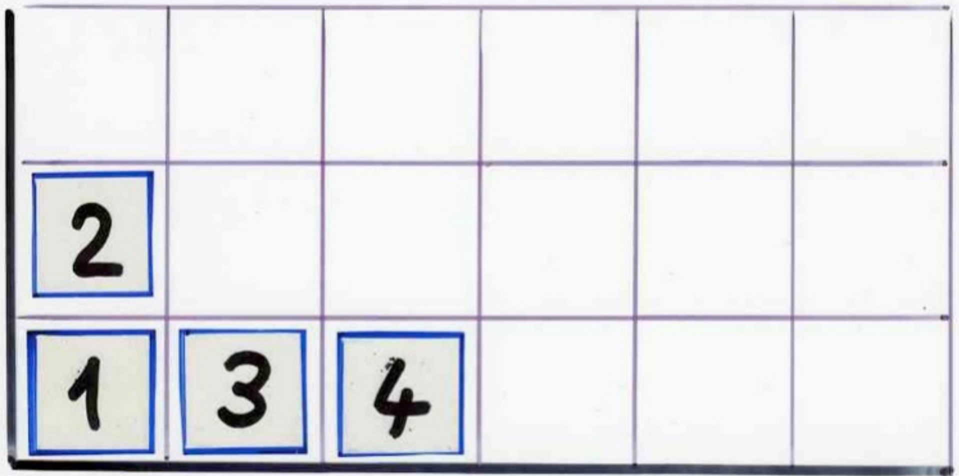
3					
1	6				

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

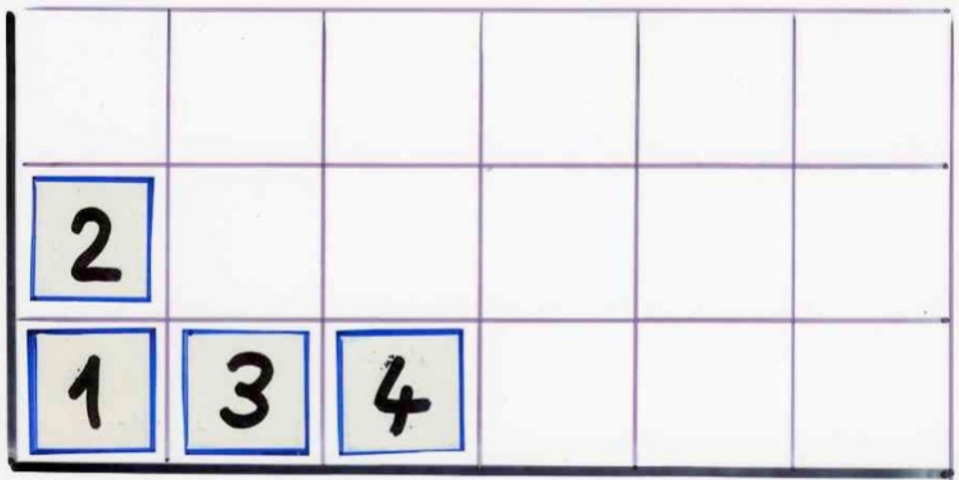
2					
1	3	4			

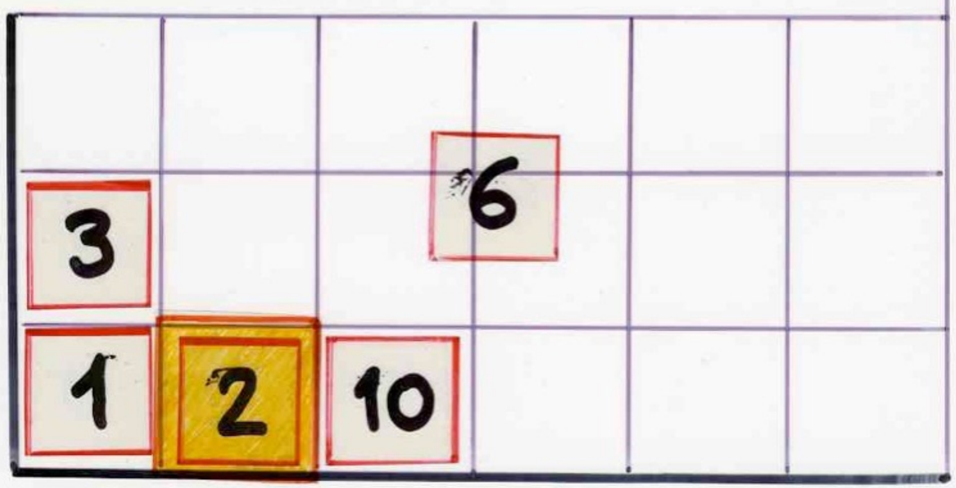
3					
1	6	10			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



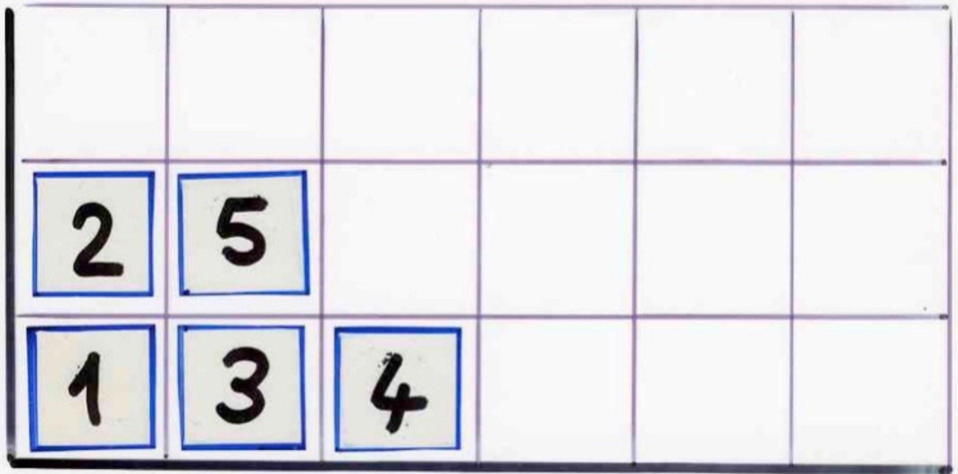


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

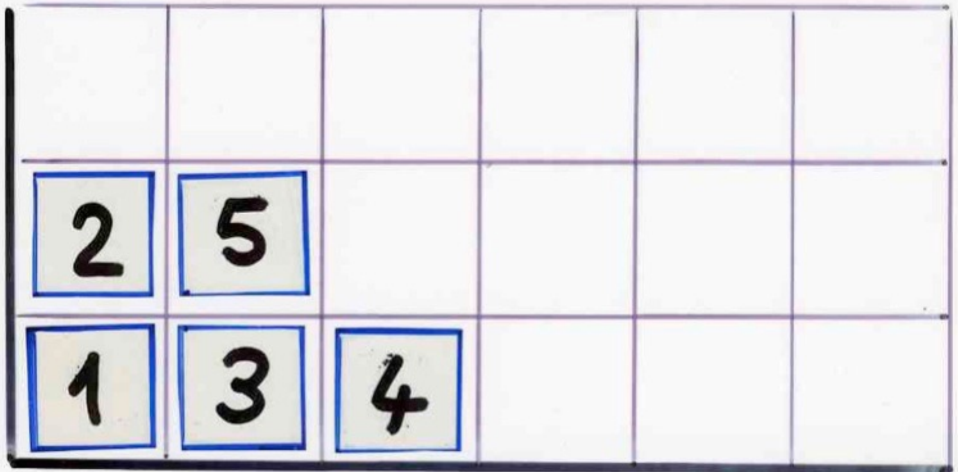
2	5				
1	3	4			

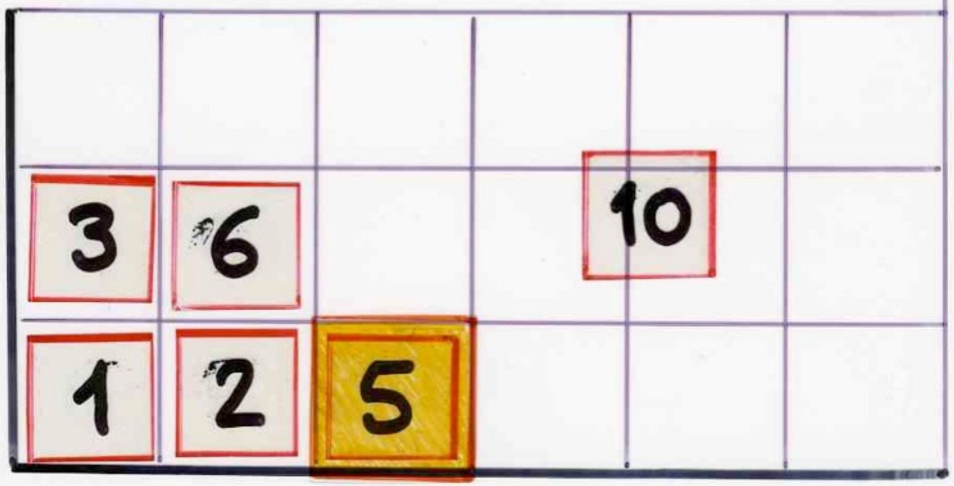
3	6				
1	2	10			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4			

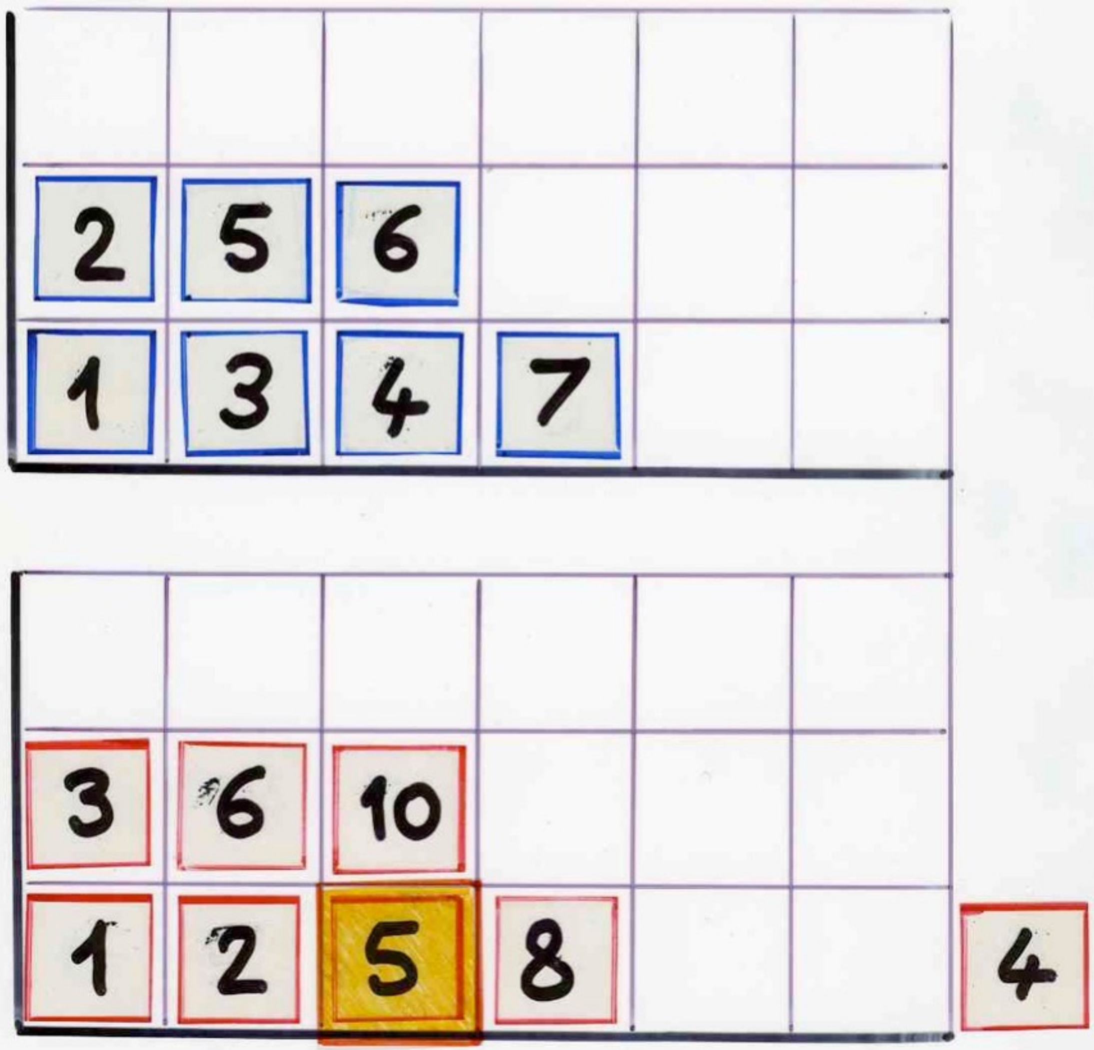
3	6	10			
1	2	5			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

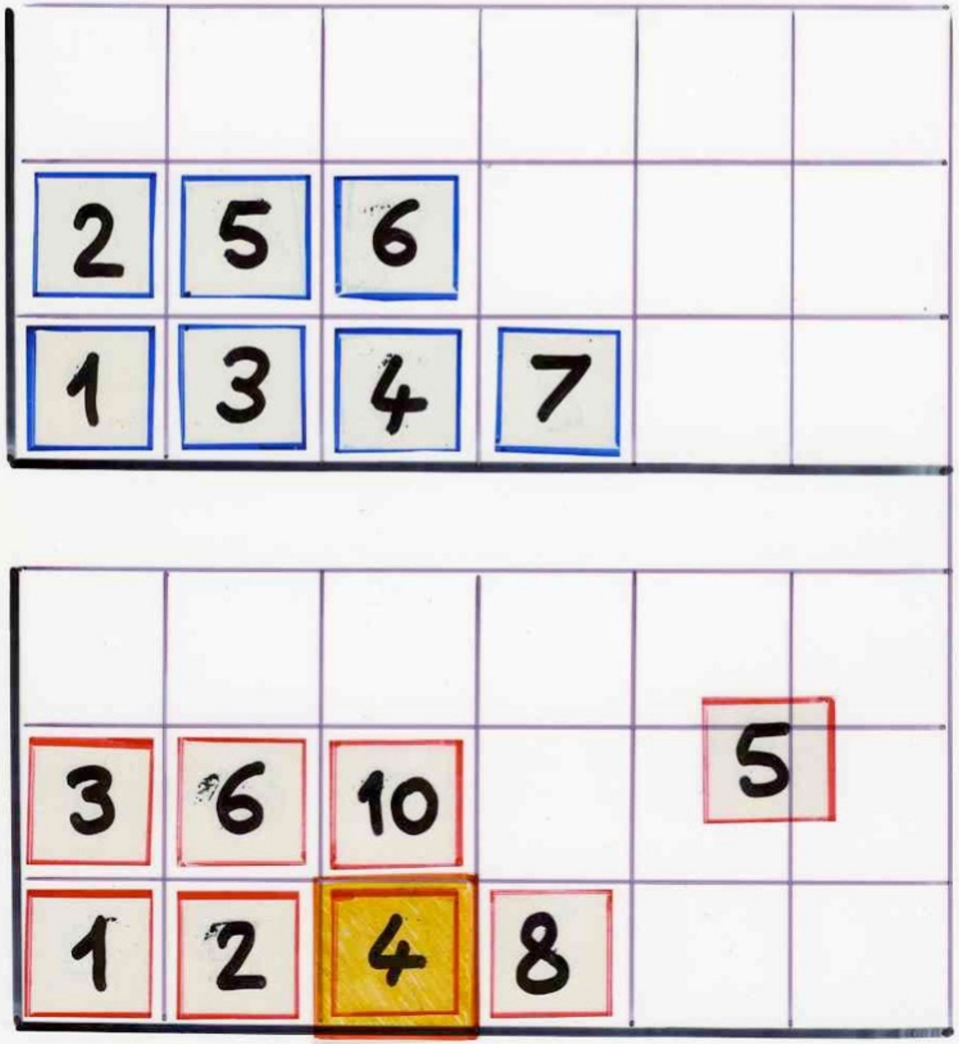
2	5	6			
1	3	4	7		

3	6	10			
1	2	5	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

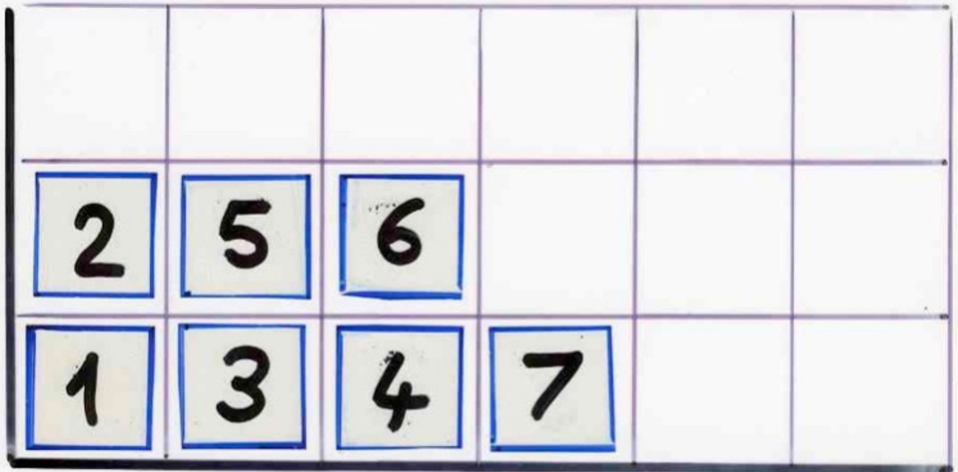


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4	7		

3	6	10		5	
1	2	4	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7		

6					
3	5	10			
1	2	4	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6					
3	5	10			
1	2	4	8	9	

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

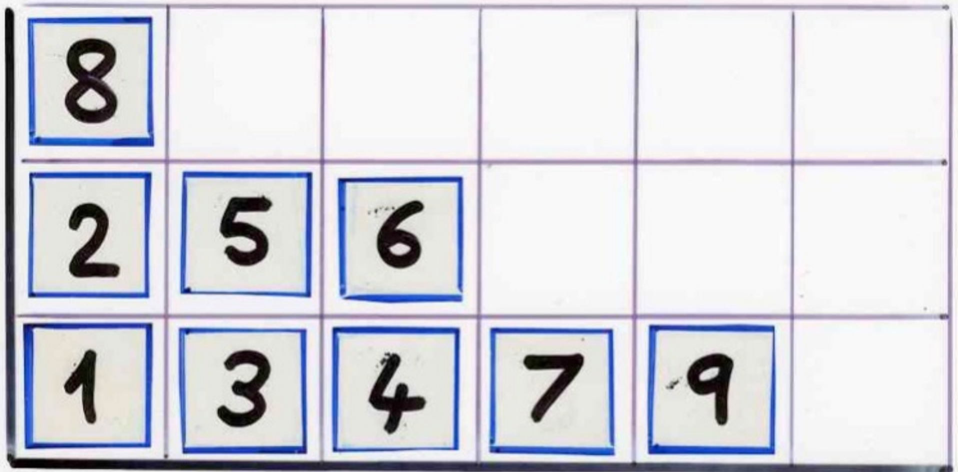
6						
3	5	10				
1	2	4	8	9		7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

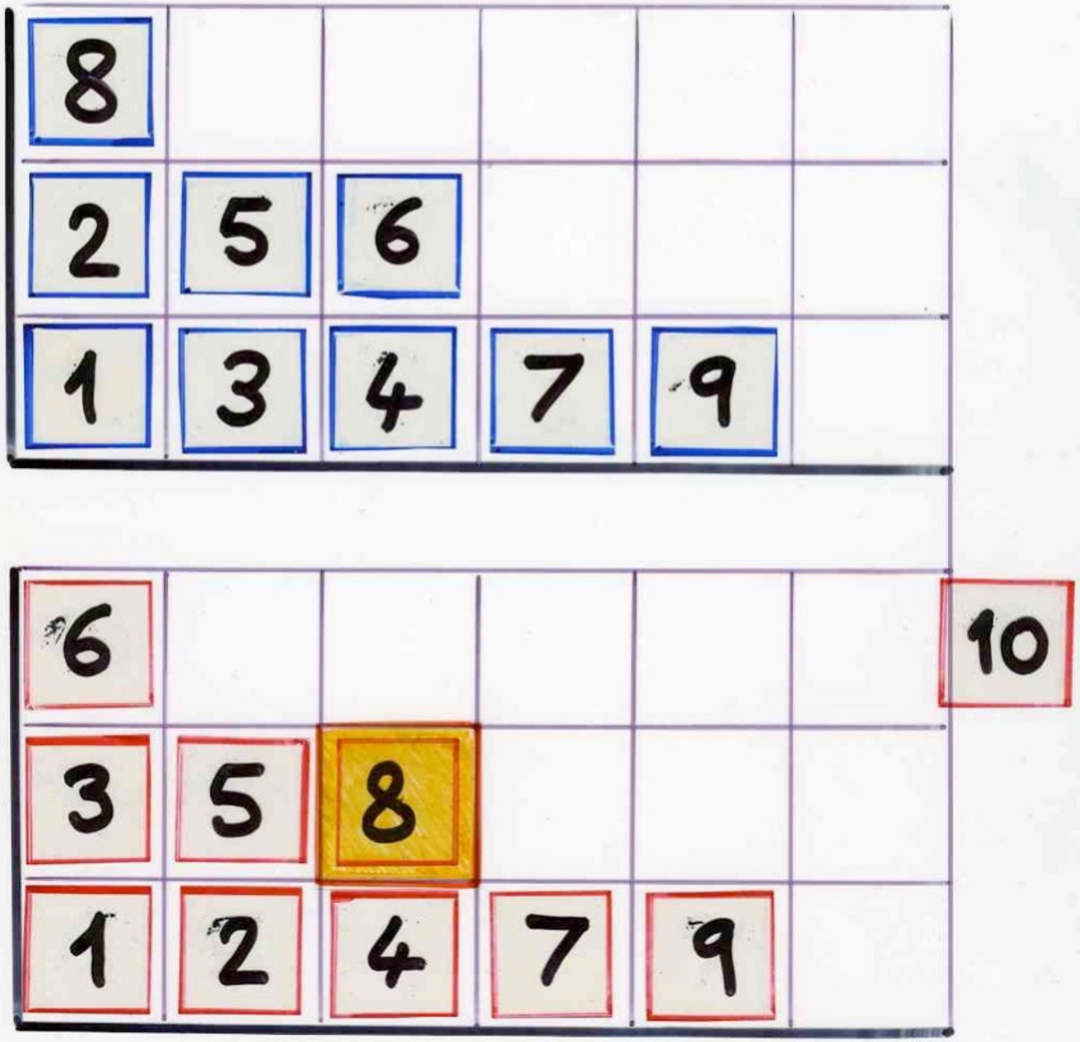
8					
2	5	6			
1	3	4	7	9	

6						
3	5	10				8
1	2	4	7	9		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

6	10				
3	5	8			
1	2	4	7	9	

$$\sigma \longleftrightarrow (P, Q)$$

$$? \longleftrightarrow (Q, P)$$

The group of permutations

Group theory

Definition Group G $(x, y) \to x * y$

(i) associativity $(x*y)*z = x*(y*z)$

(ii) neutral element $x*e = e*x = x$

(iii) inverse $(x*y = y*x = e$

y unique $y = x^{-1}$

examples

- integers $\mathbb{Z}$ for addition $+$
  inverse $-x$
- $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, for multiplication
  inverse $1/x$
- Permutations $\mathfrak{S}_n$ symmetric group

composition
of two permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$$

$$\sigma \circ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$$

e = identity permutation $= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

inverse $\quad \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix}$

$$\sigma \longleftrightarrow (P, Q)$$

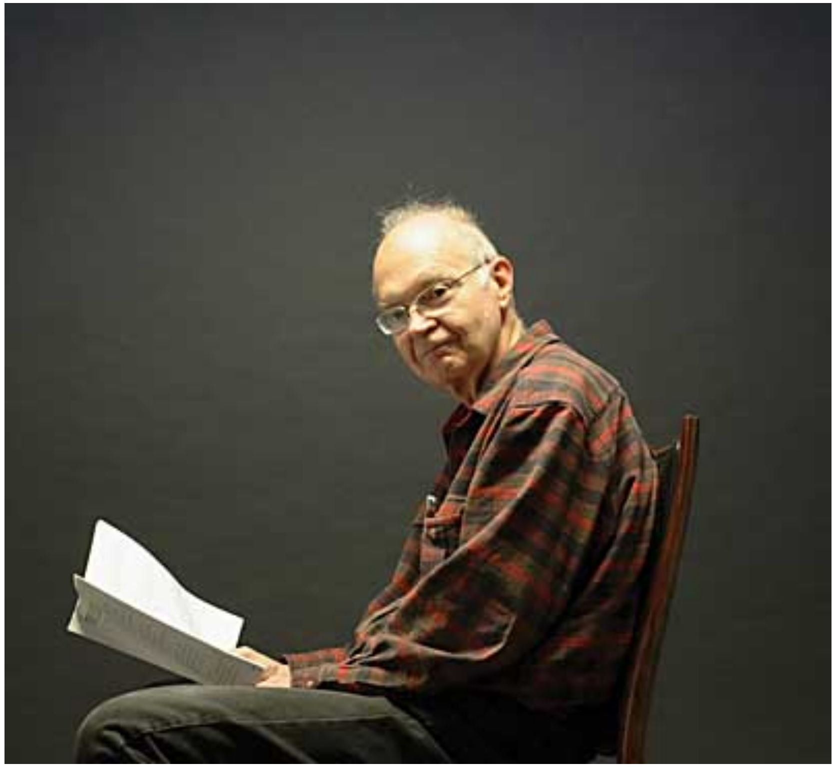
$$\sigma^{-1} \longleftrightarrow (Q, P)$$

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

6	10				
3	5	8			
1	2	4	7	9	

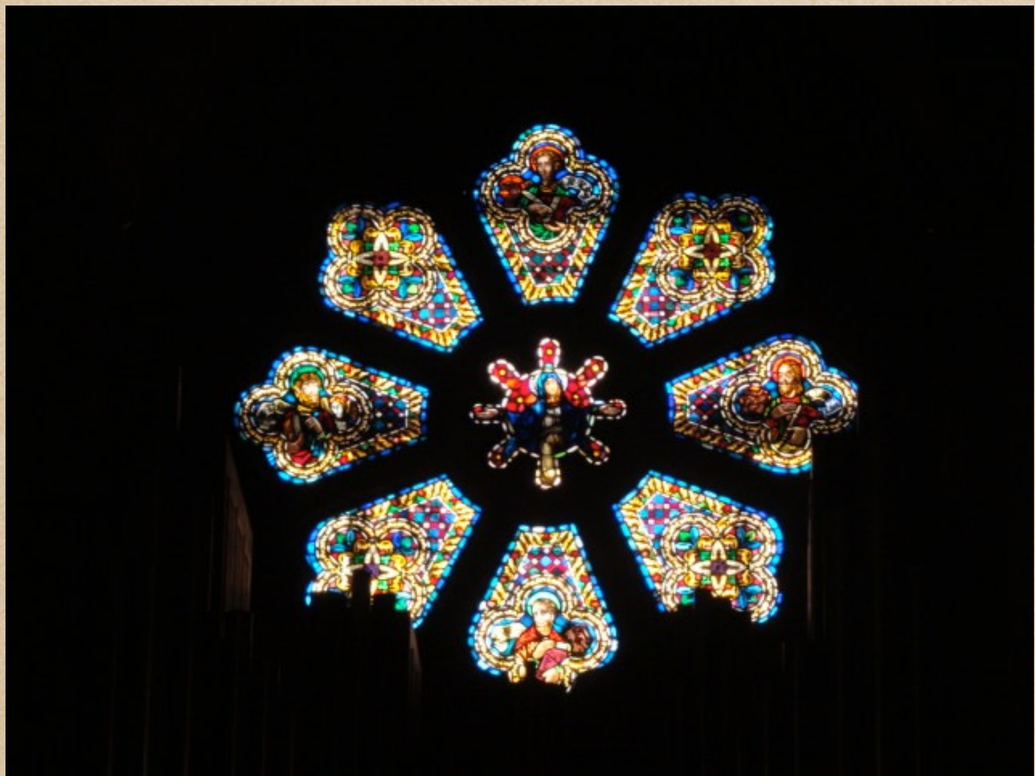
Vol 3, *"The art of computer programming"*



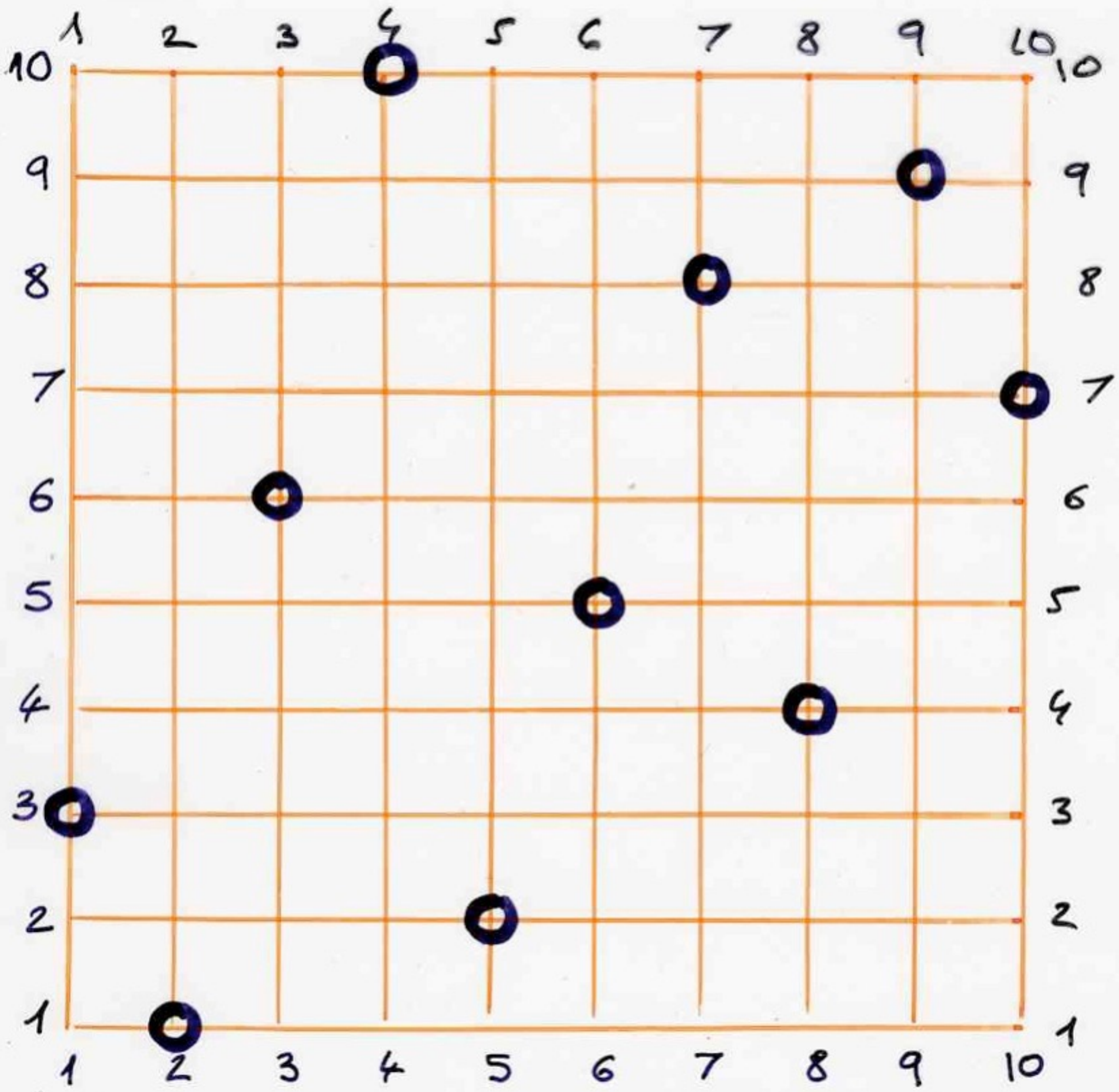
Donald Knuth (1972)

" The unusual nature of these coincidences might lead us to suspect that some sort of withcraft is operating behind the scenes "

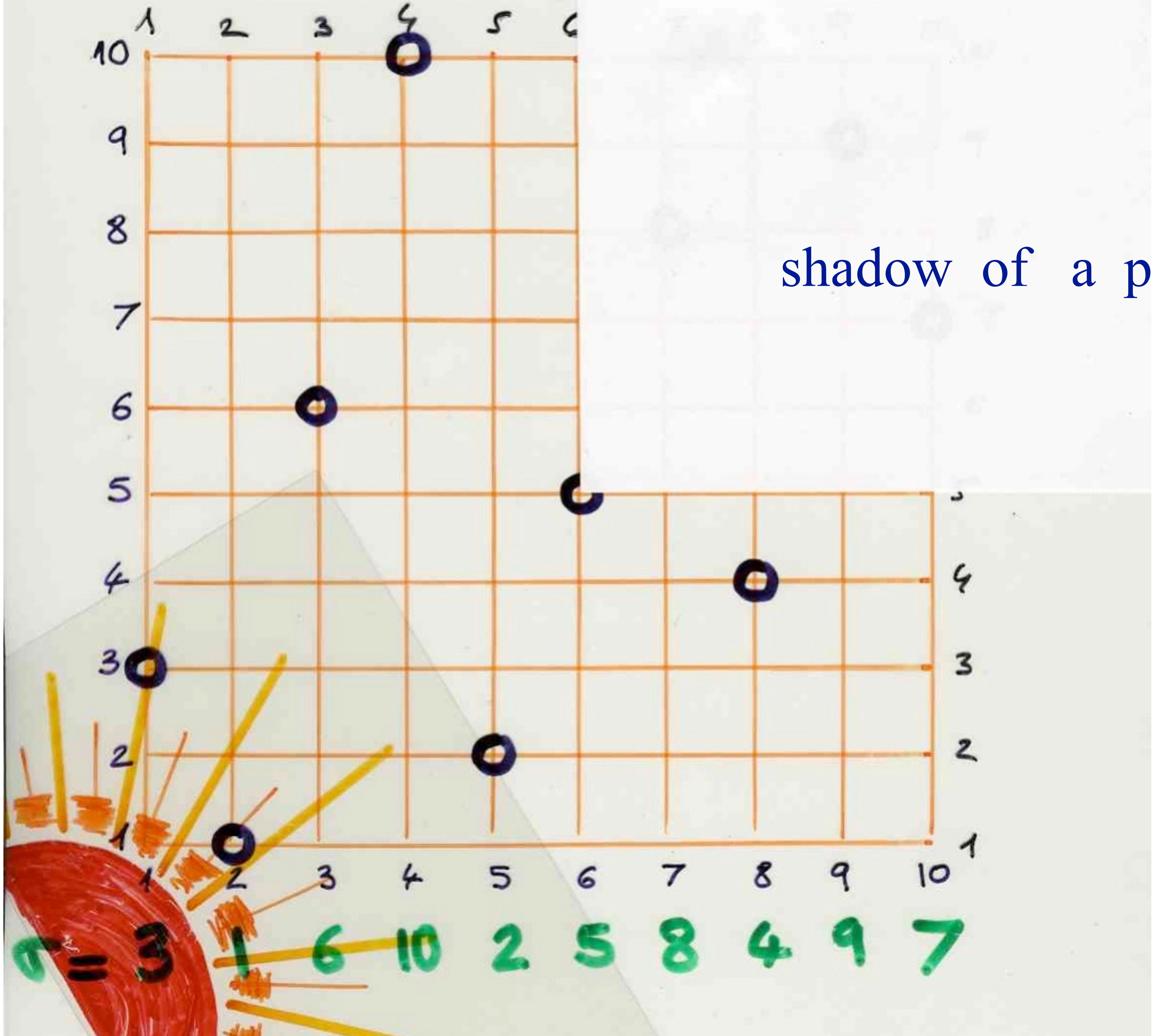
A geometric version of RSK with "light" and "shadow lines"



X.V., 1976

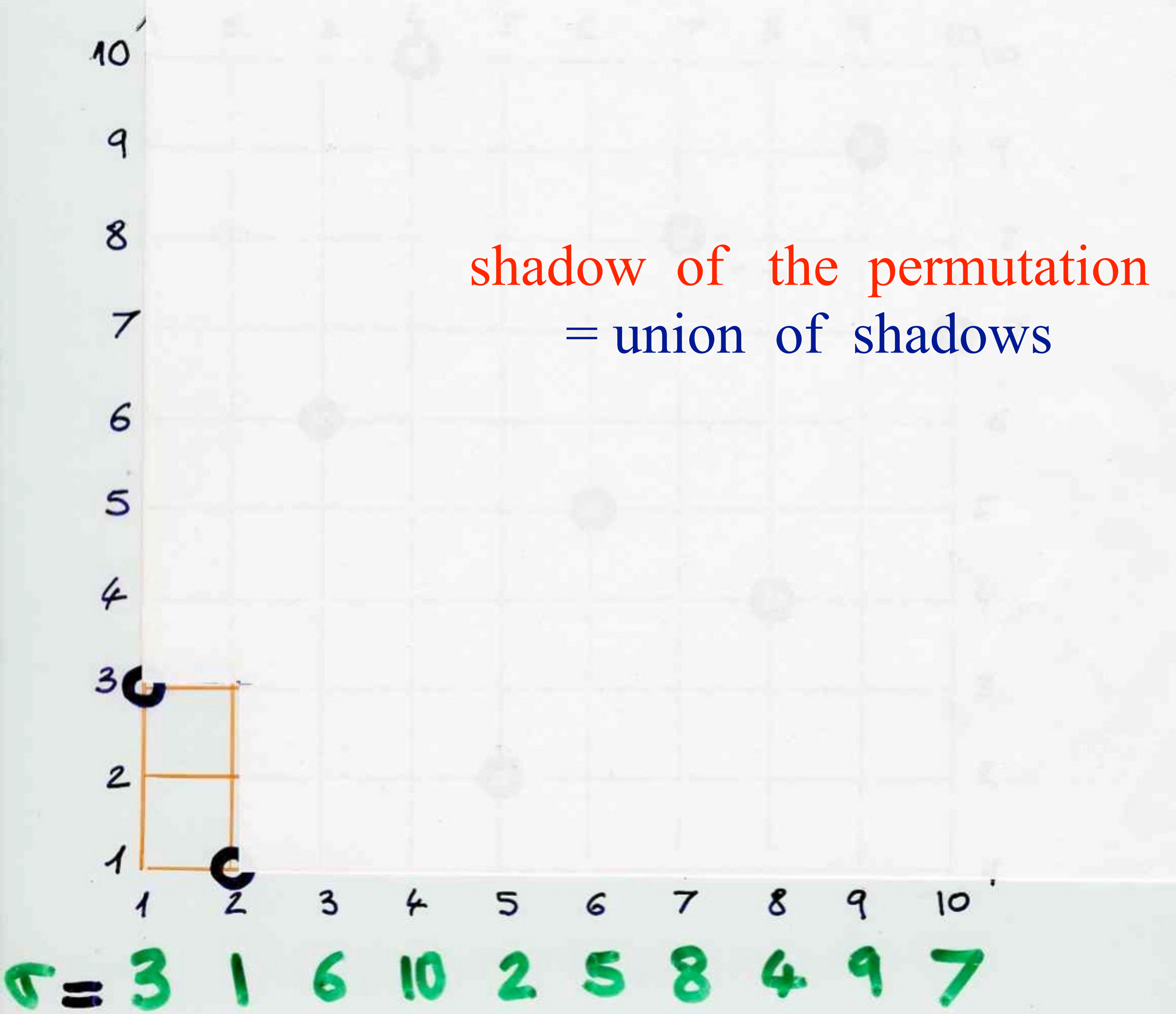


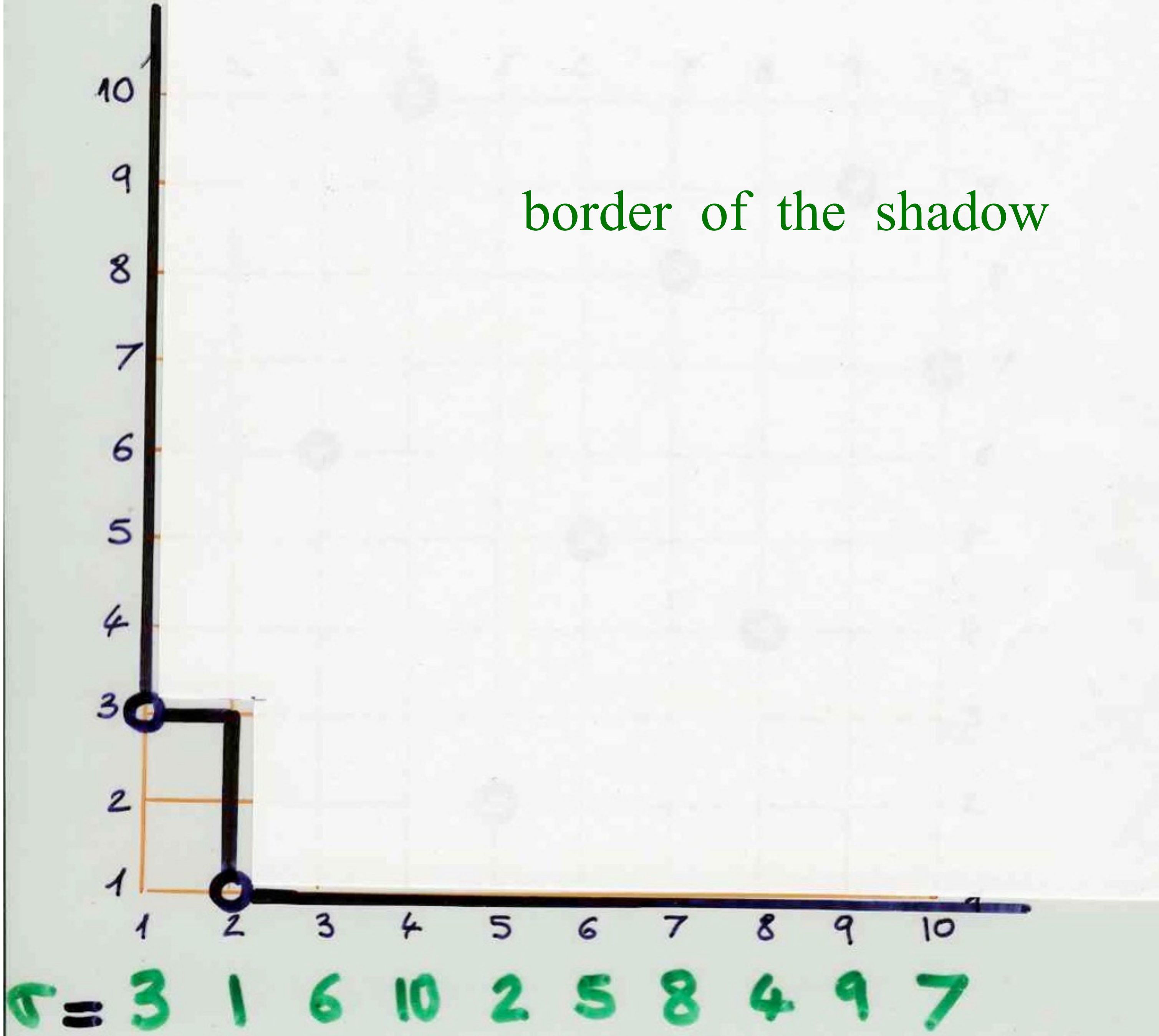
$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$



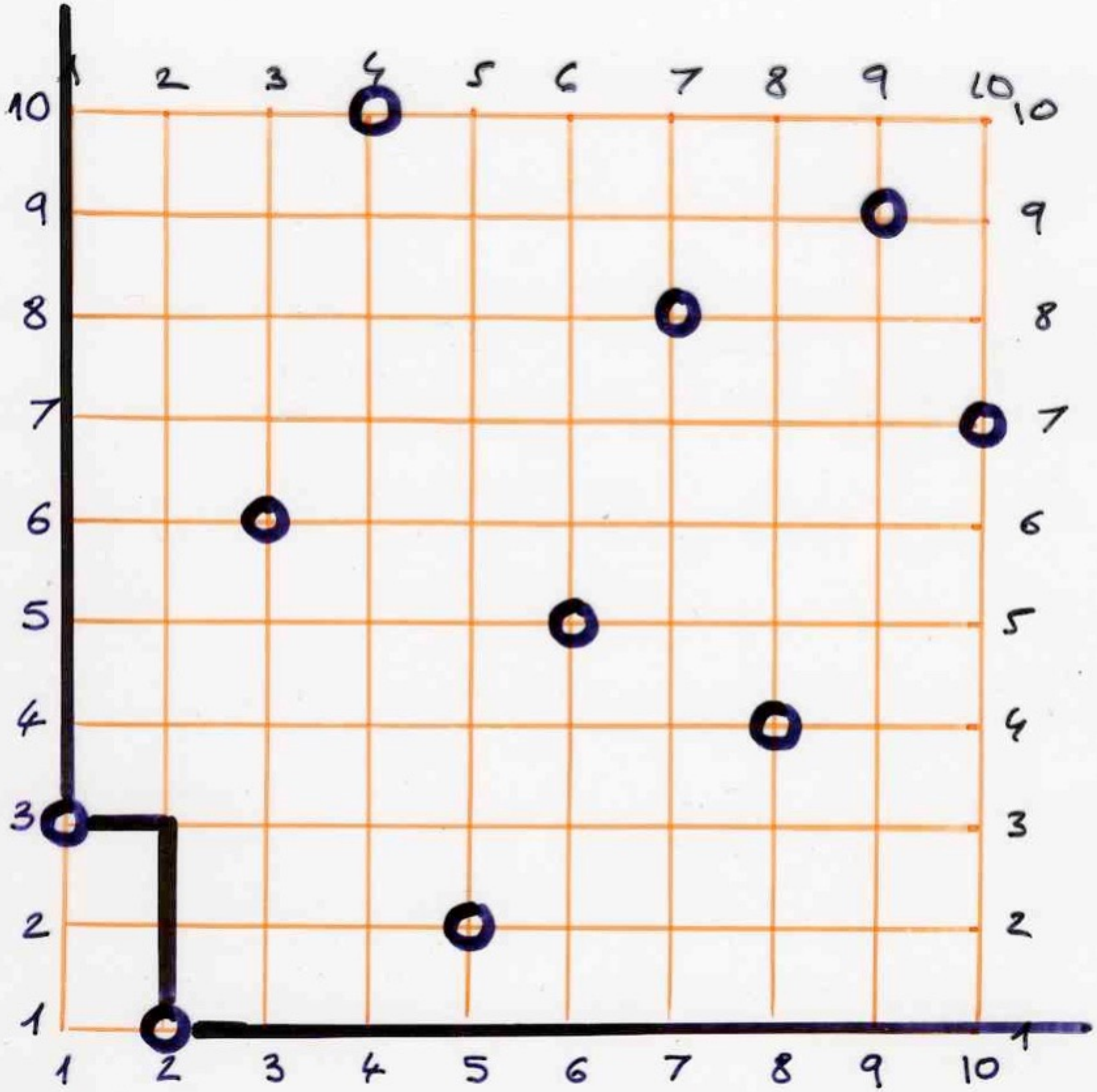
shadow of a point

shadow of the permutation
= union of shadows



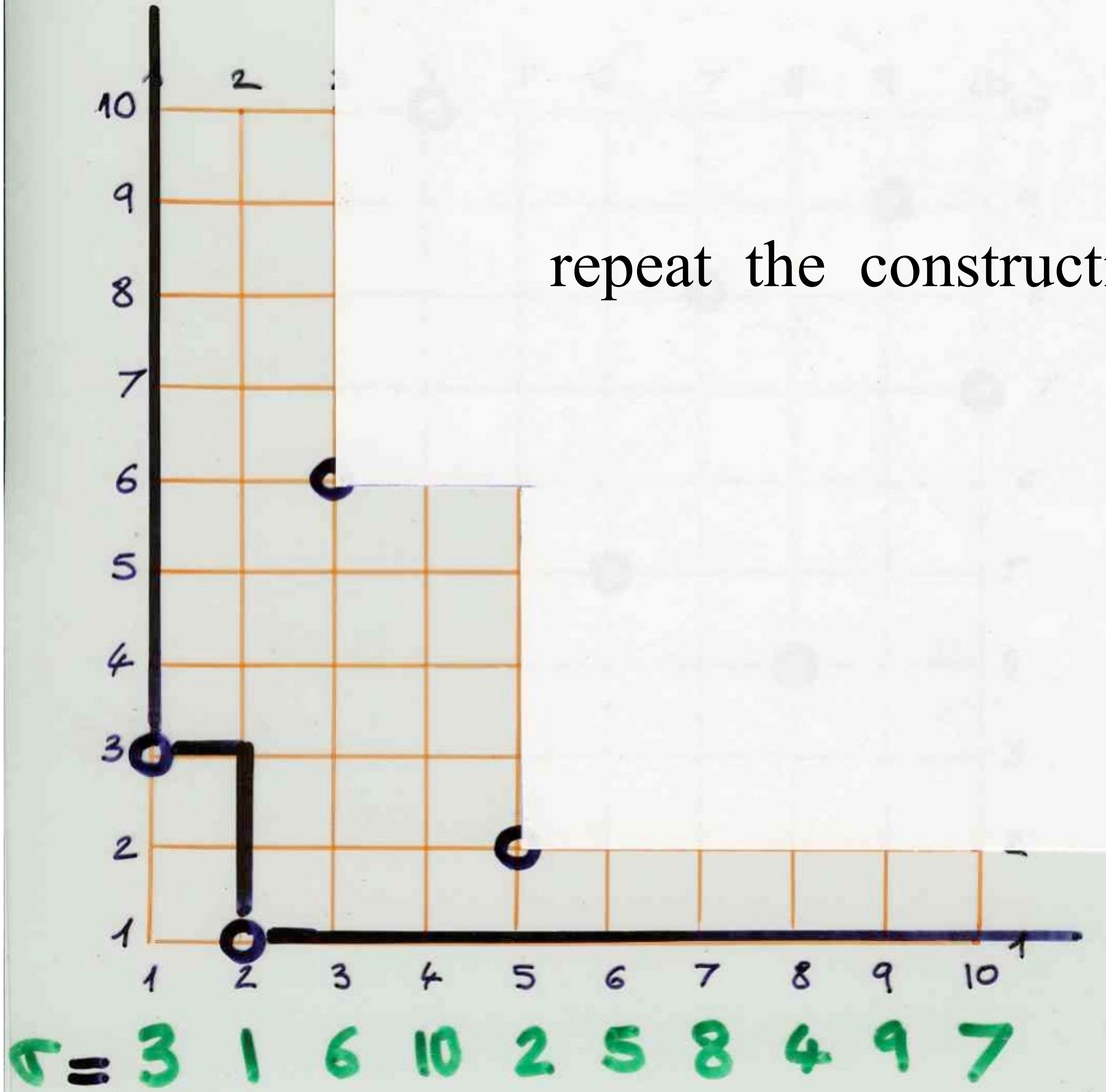


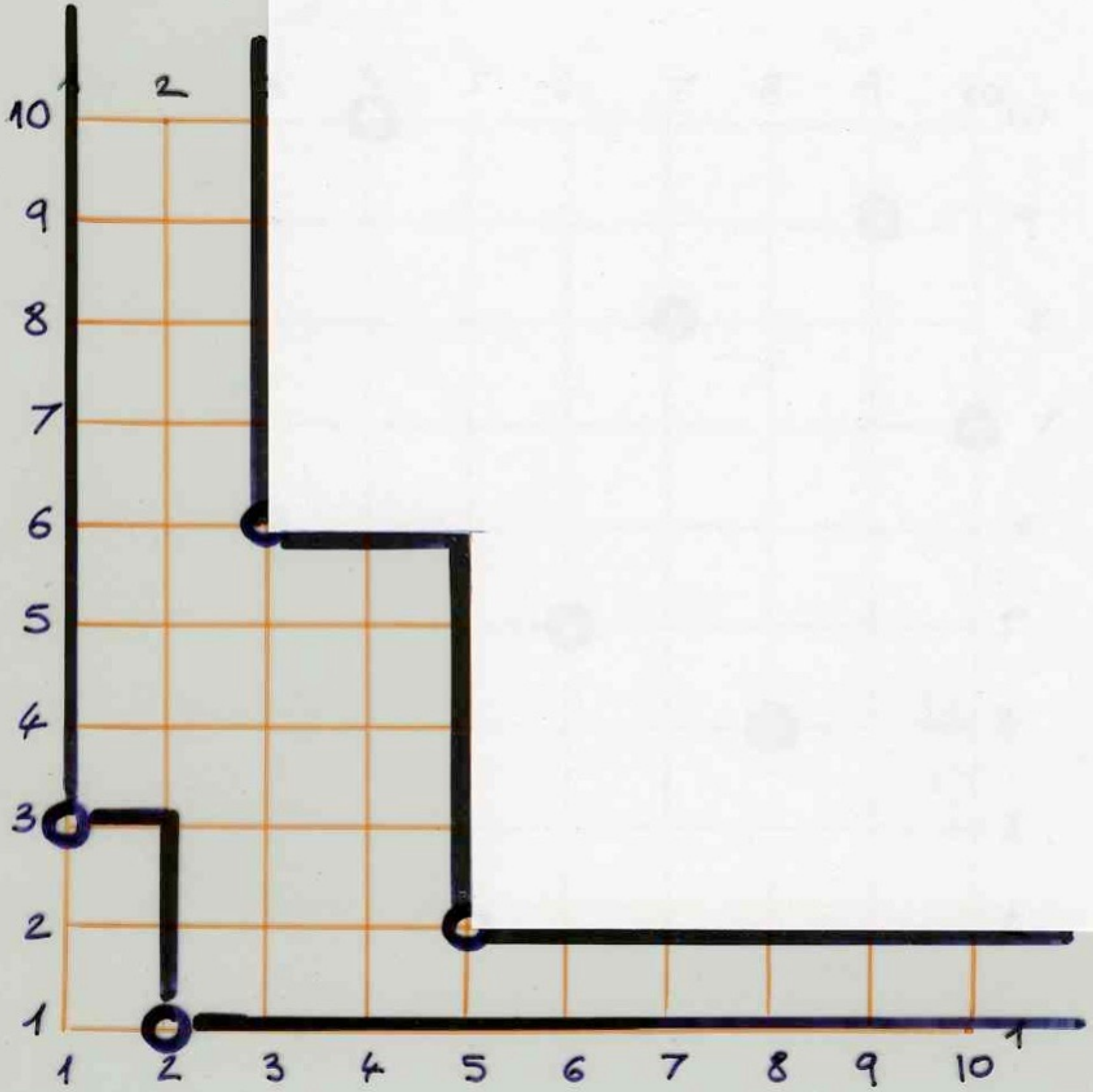
border of the shadow
10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10
σ = 3 1 6 10 2 5 8 4 9 7



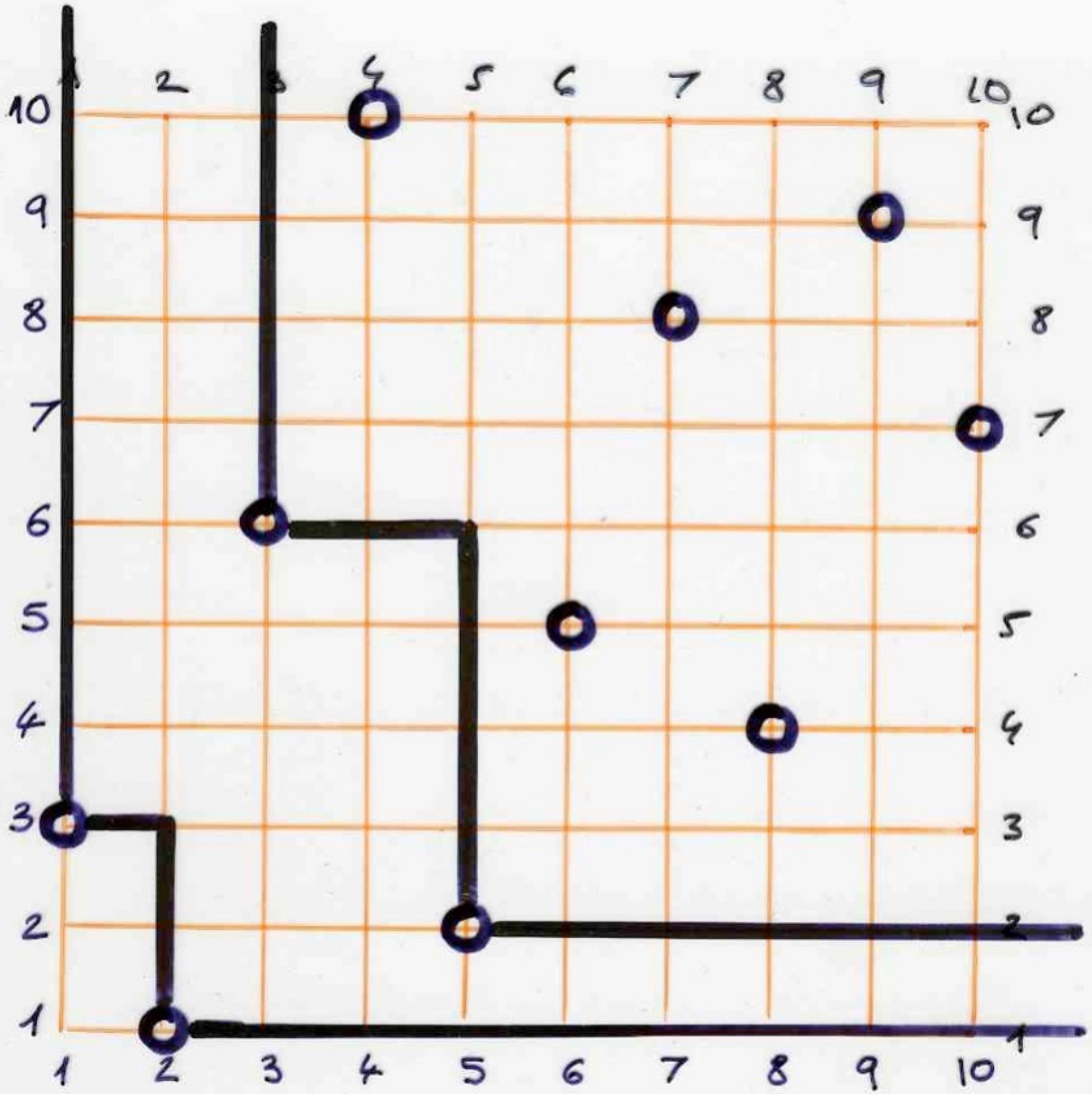
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

repeat the construction ...

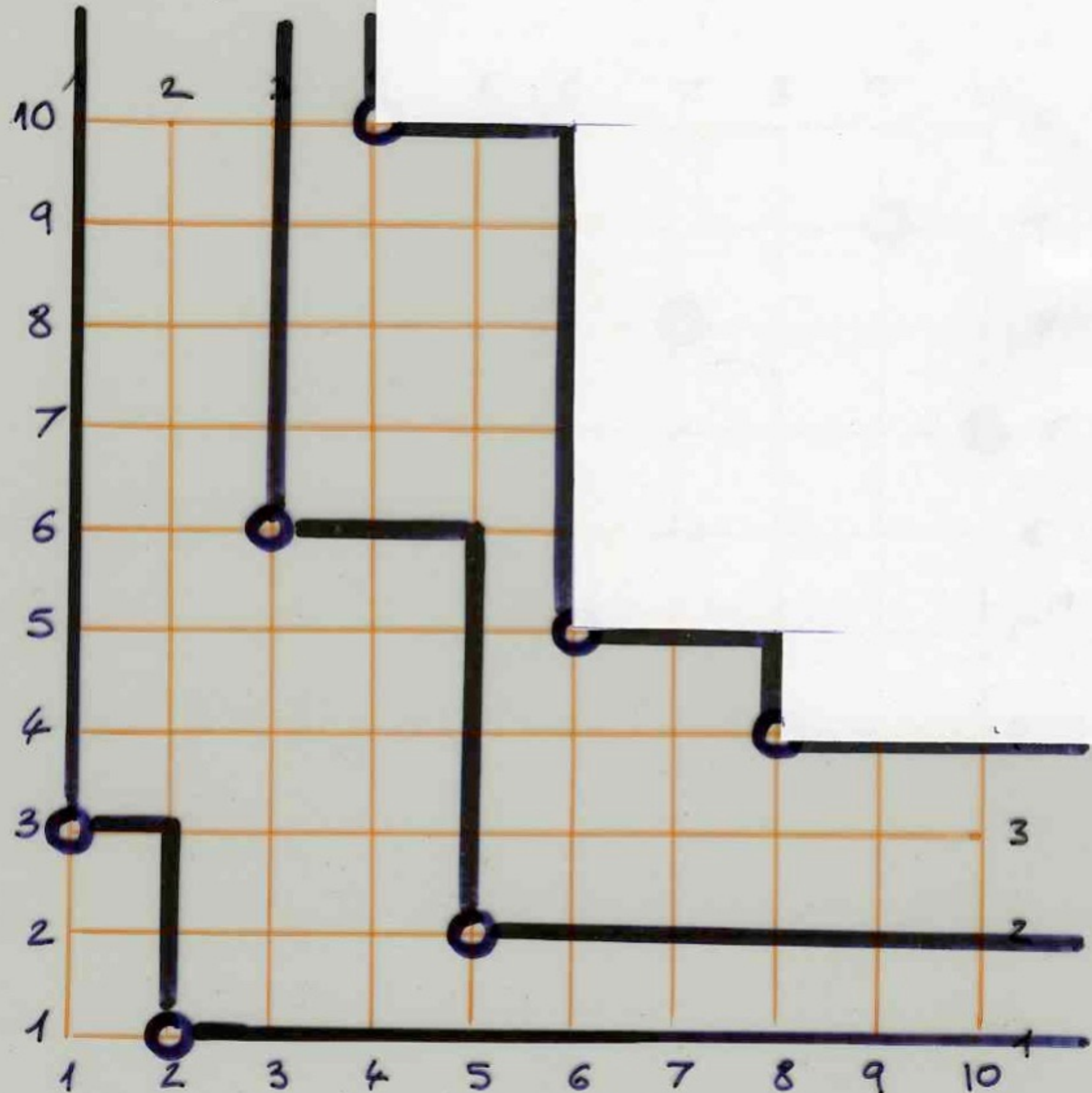




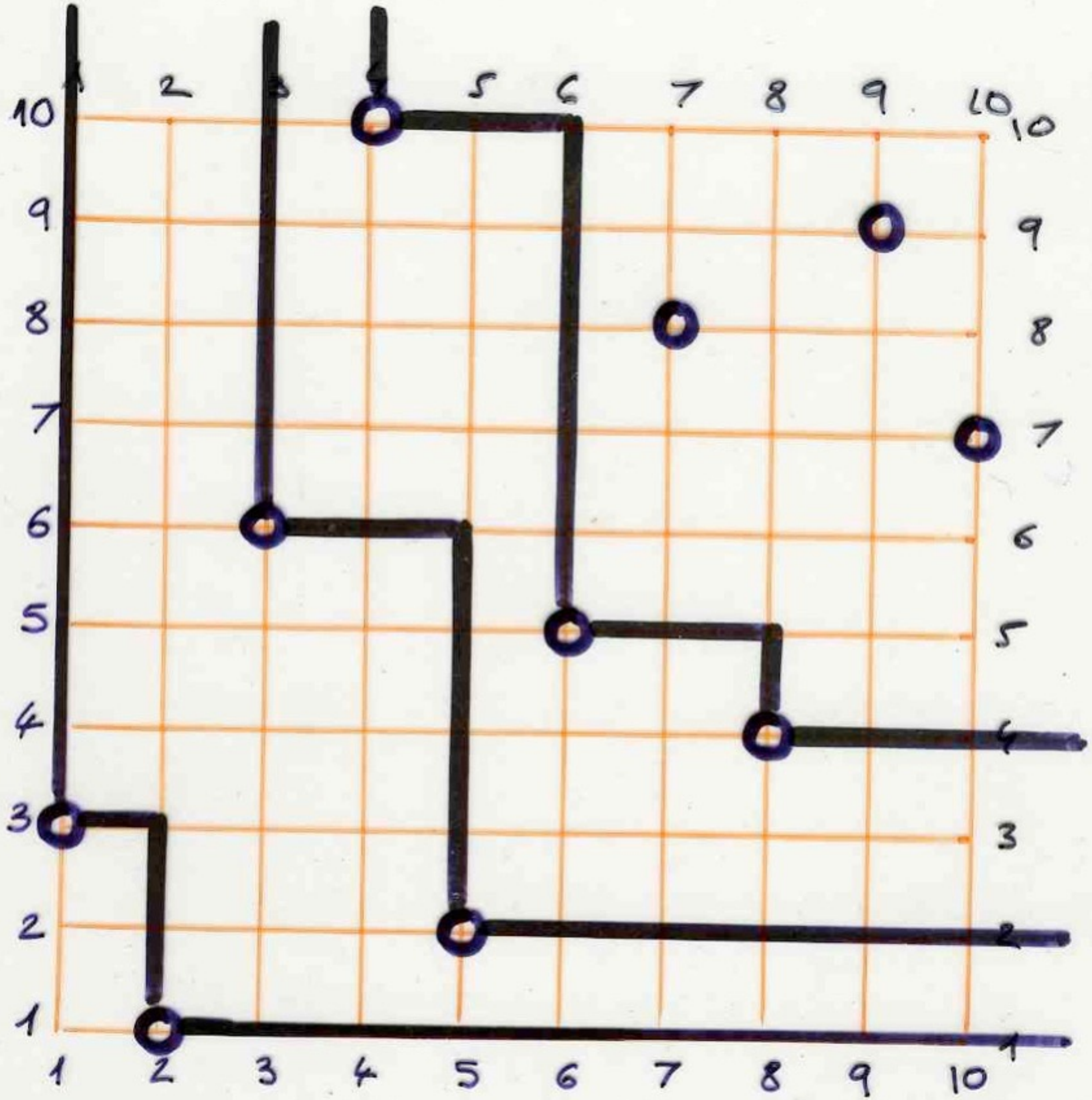
$$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$$



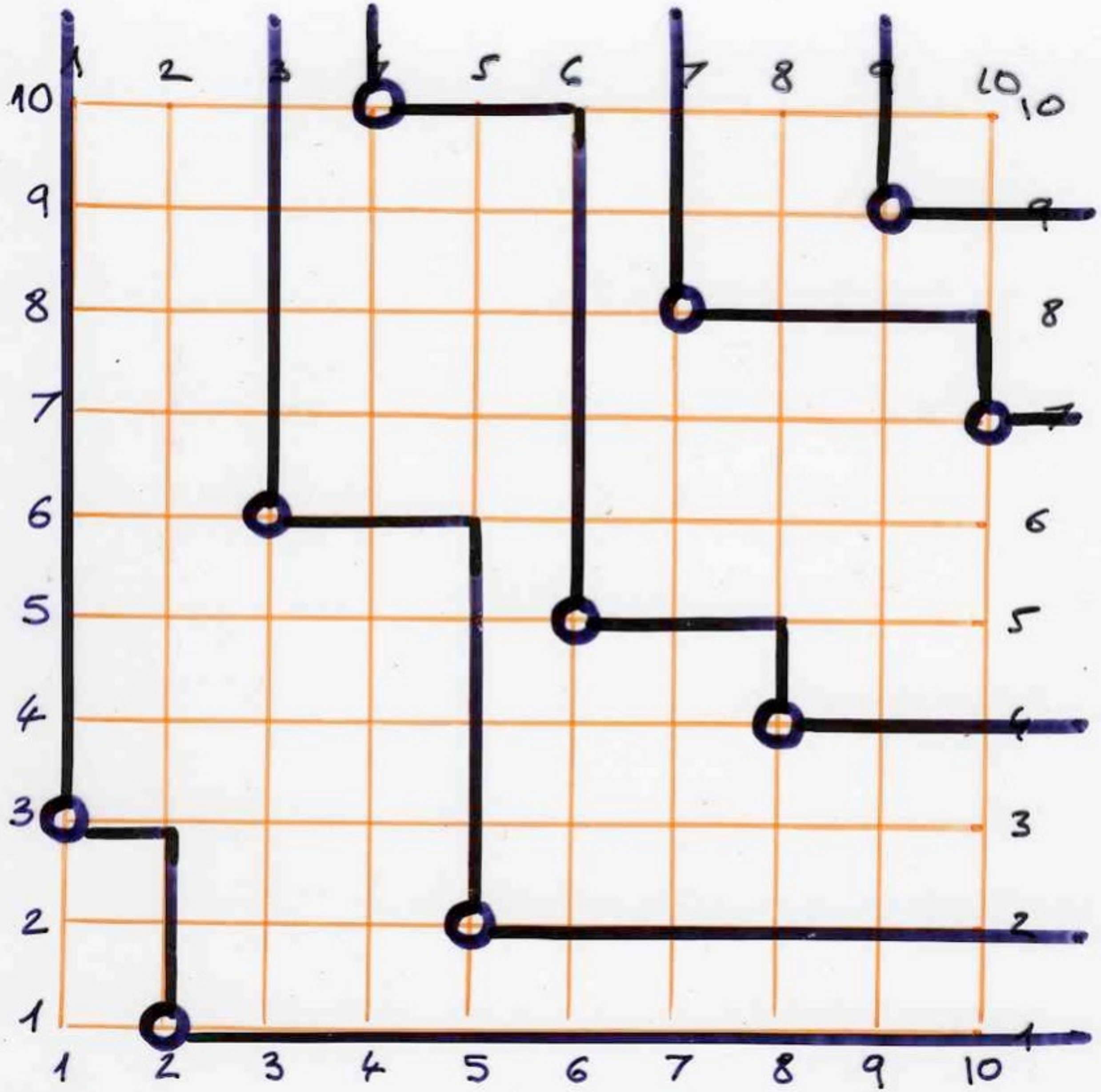
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



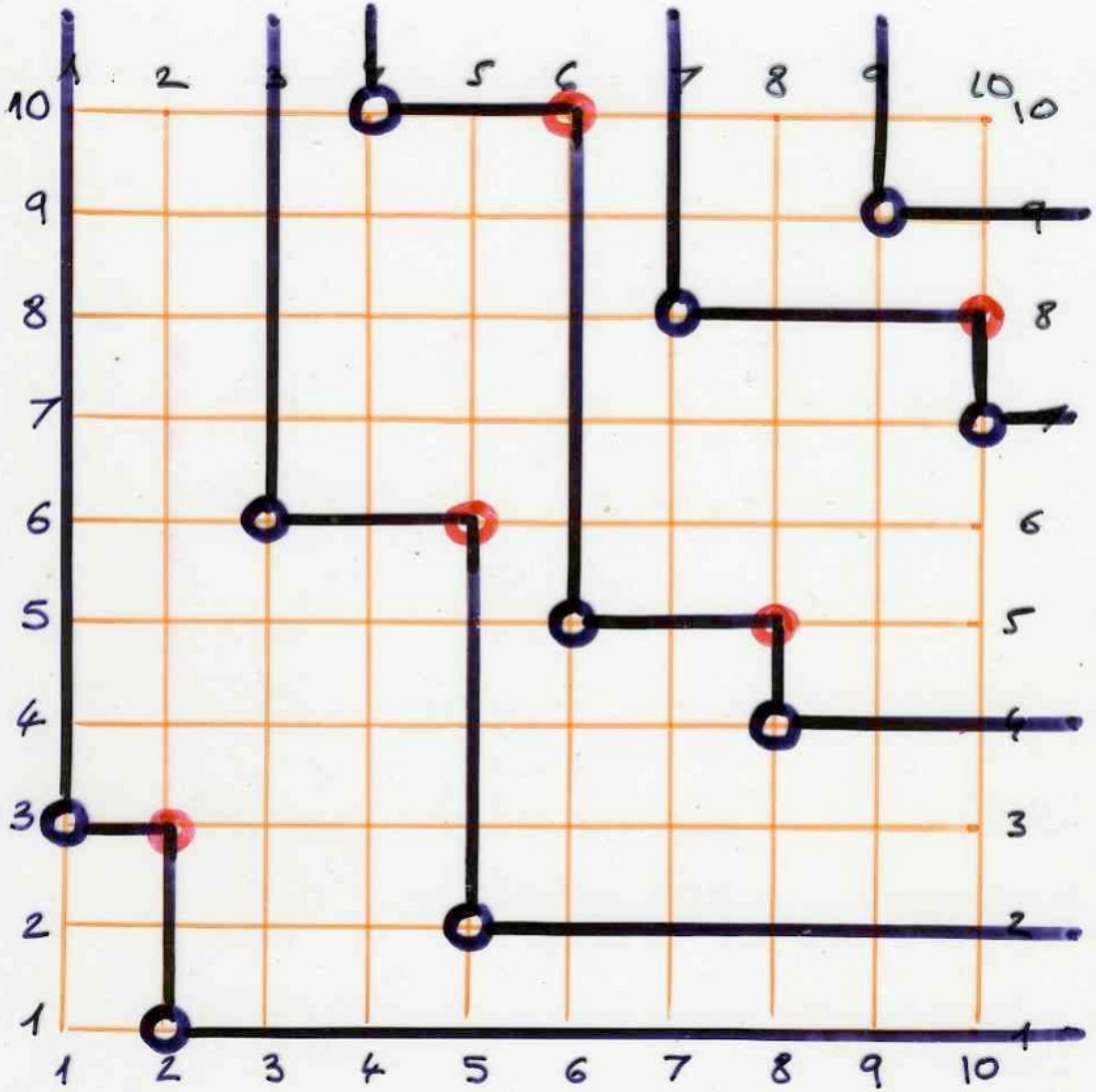
$$\sigma = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$$



$$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$$

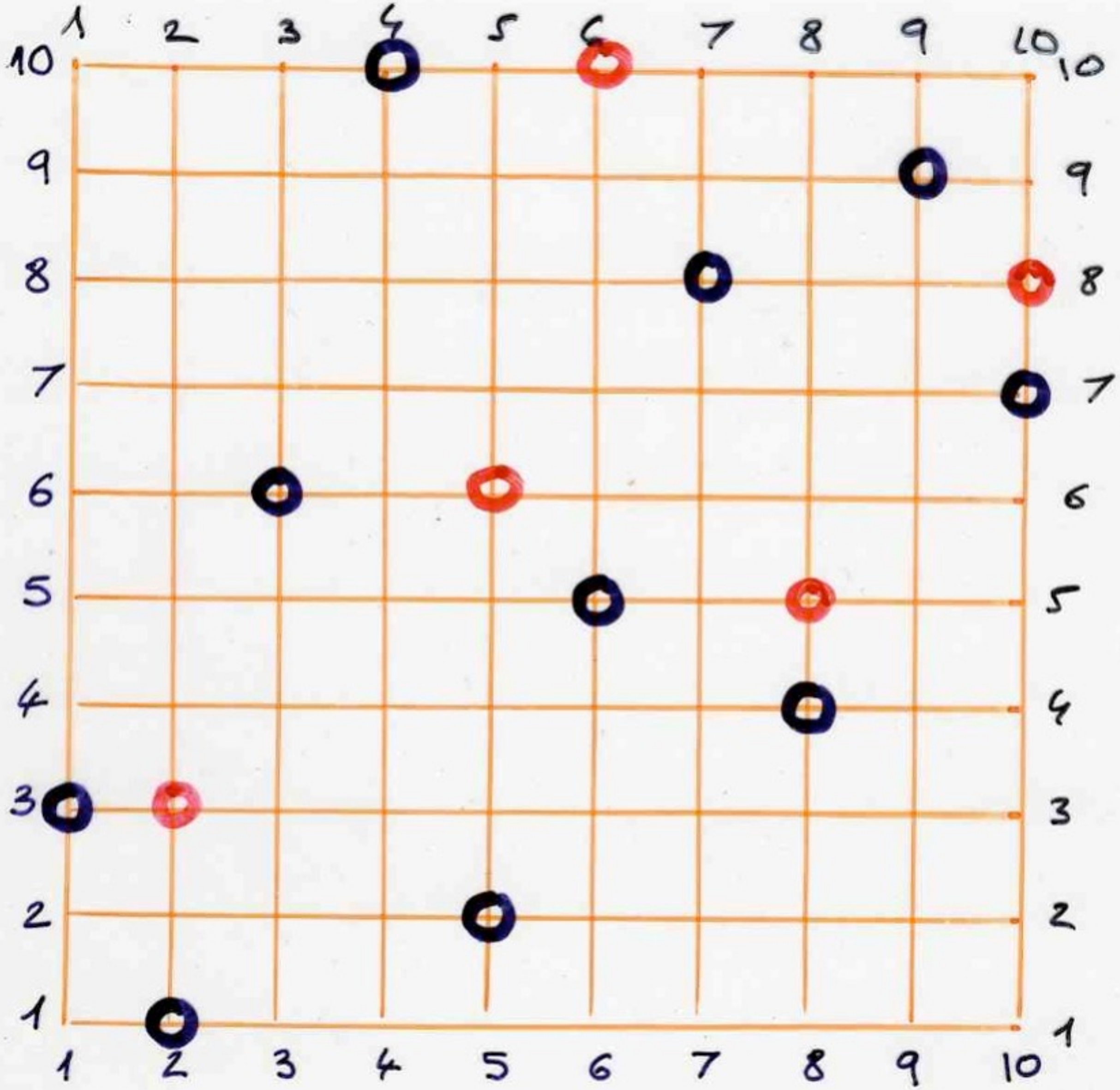


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



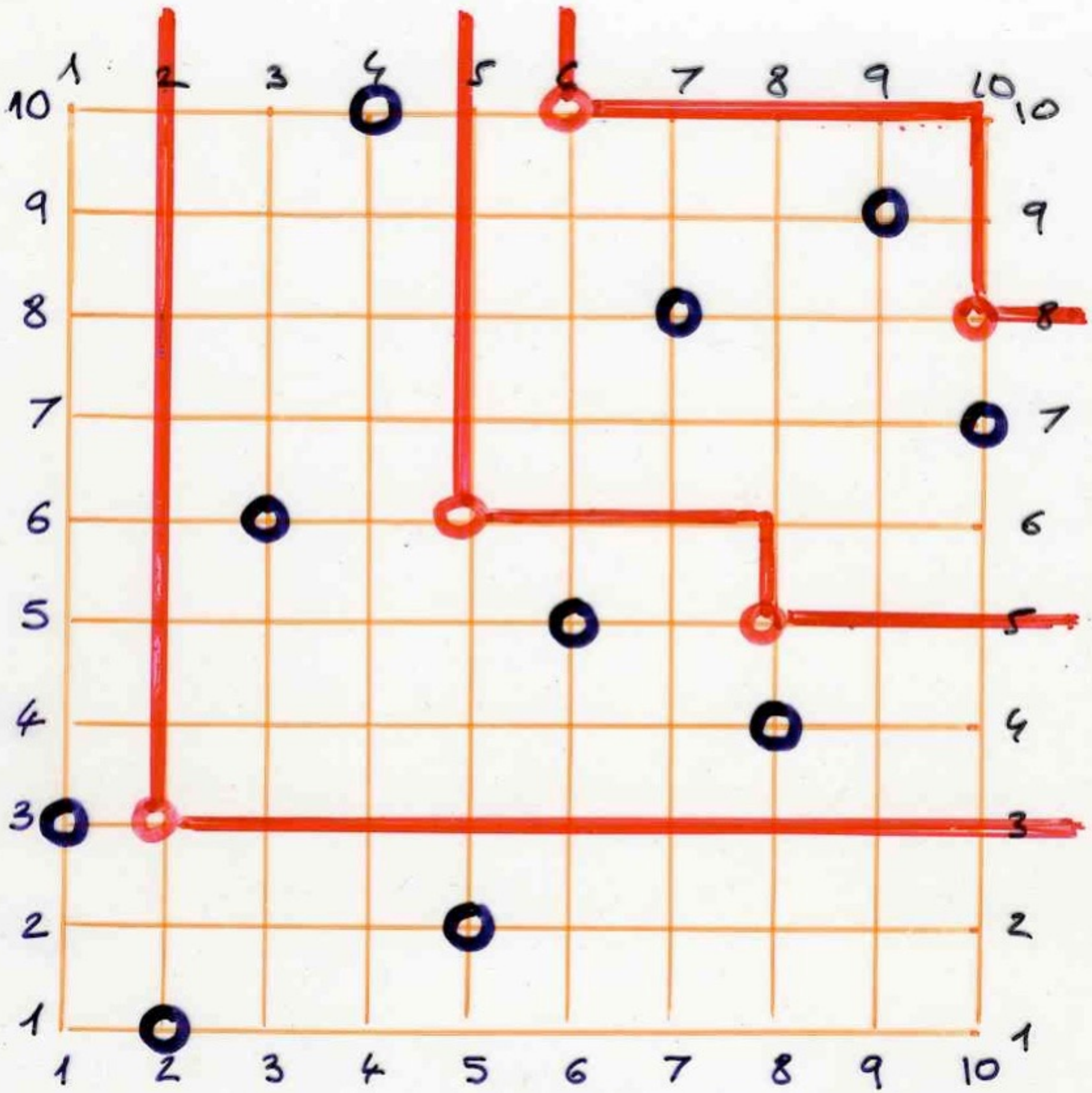
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

red points

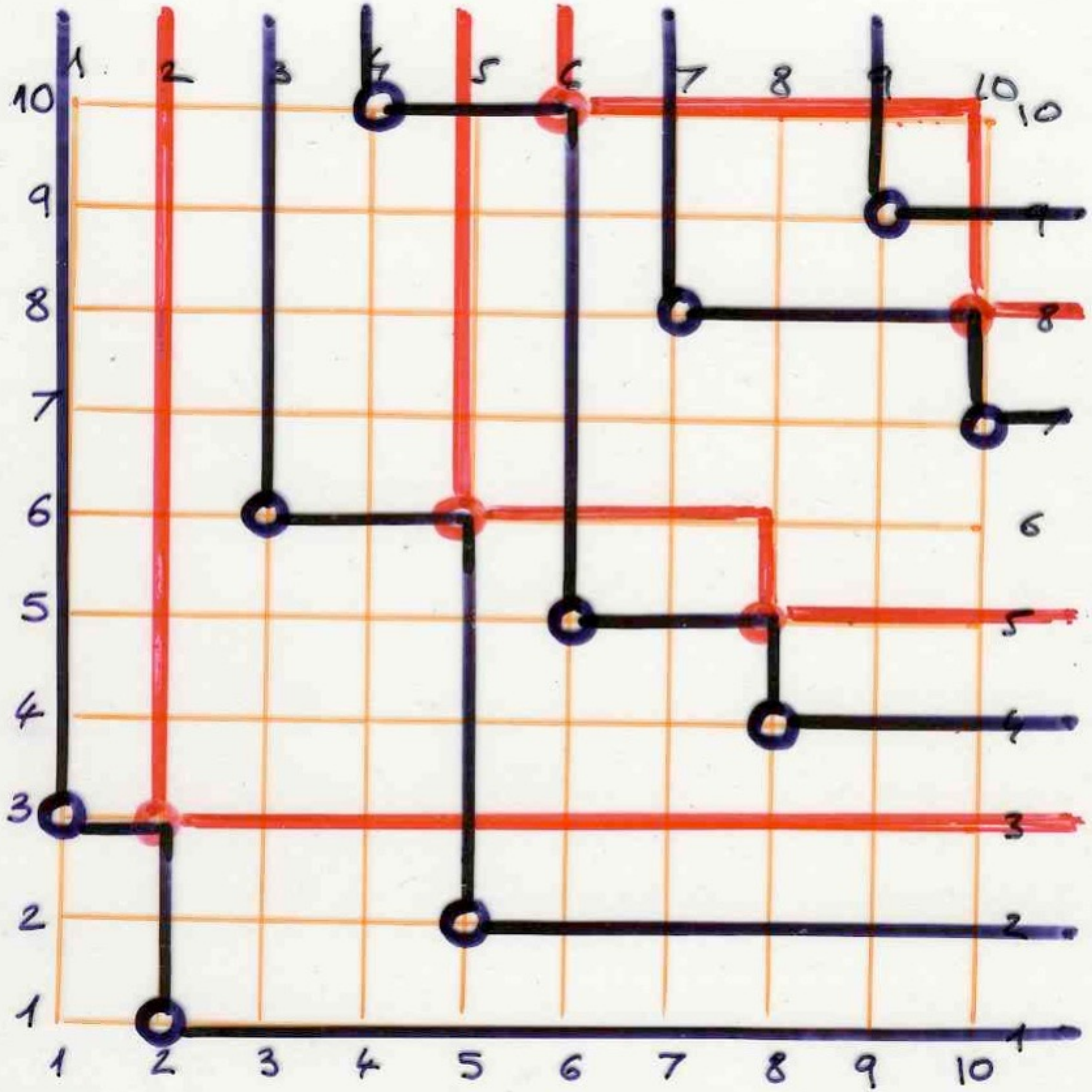


$$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$$

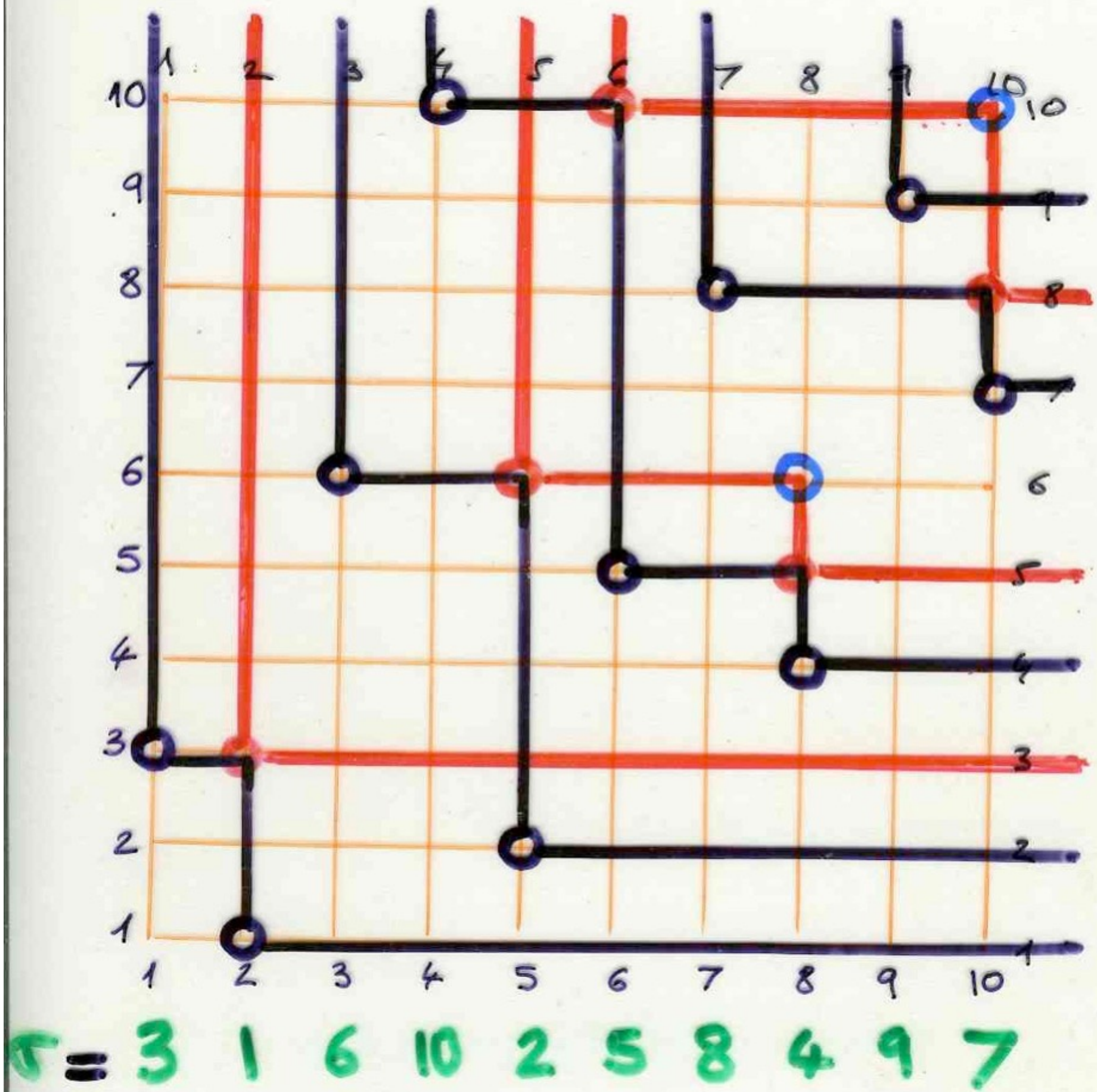
Repeat with the red points
the construction of sucessives shadows



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

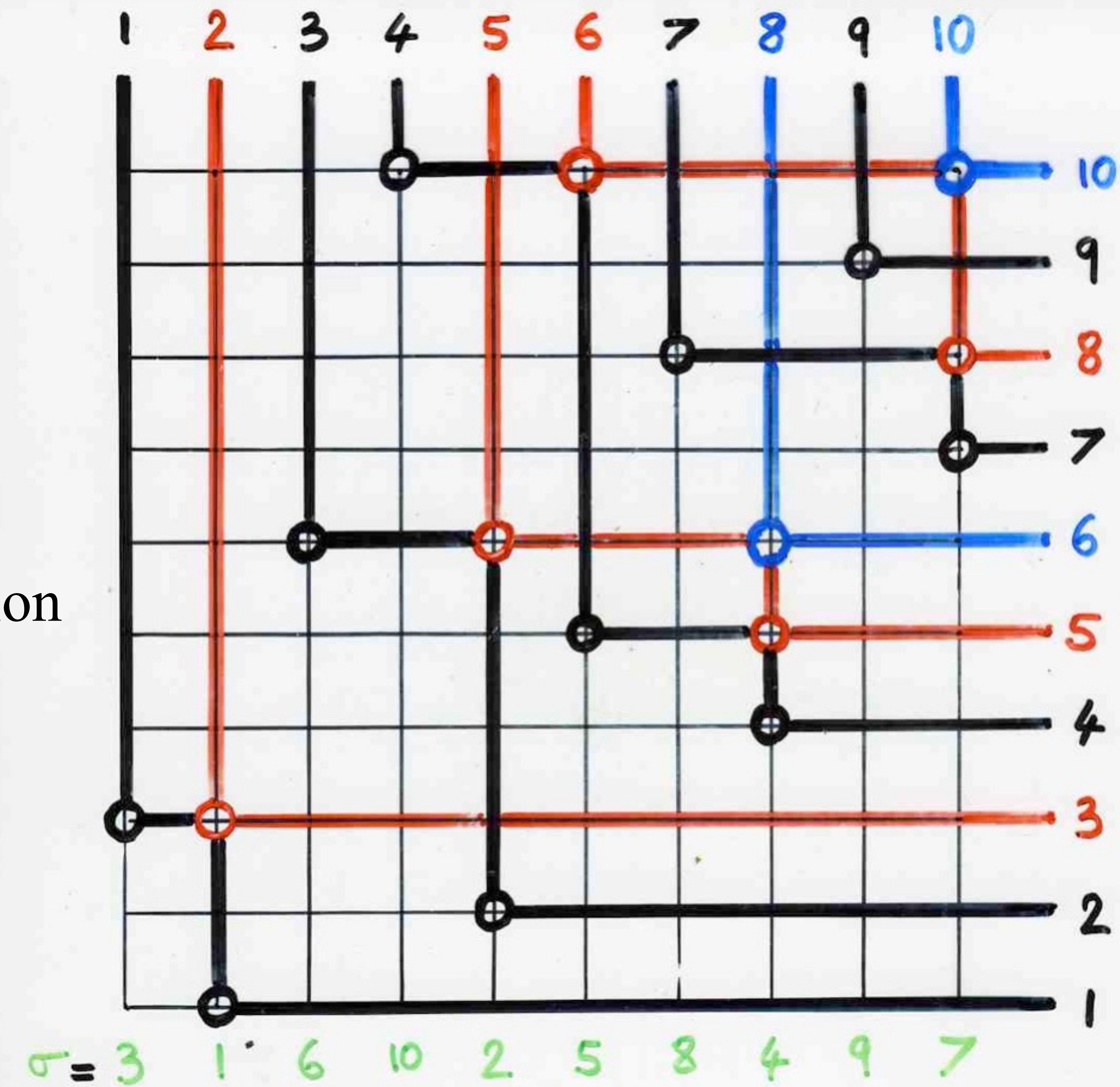


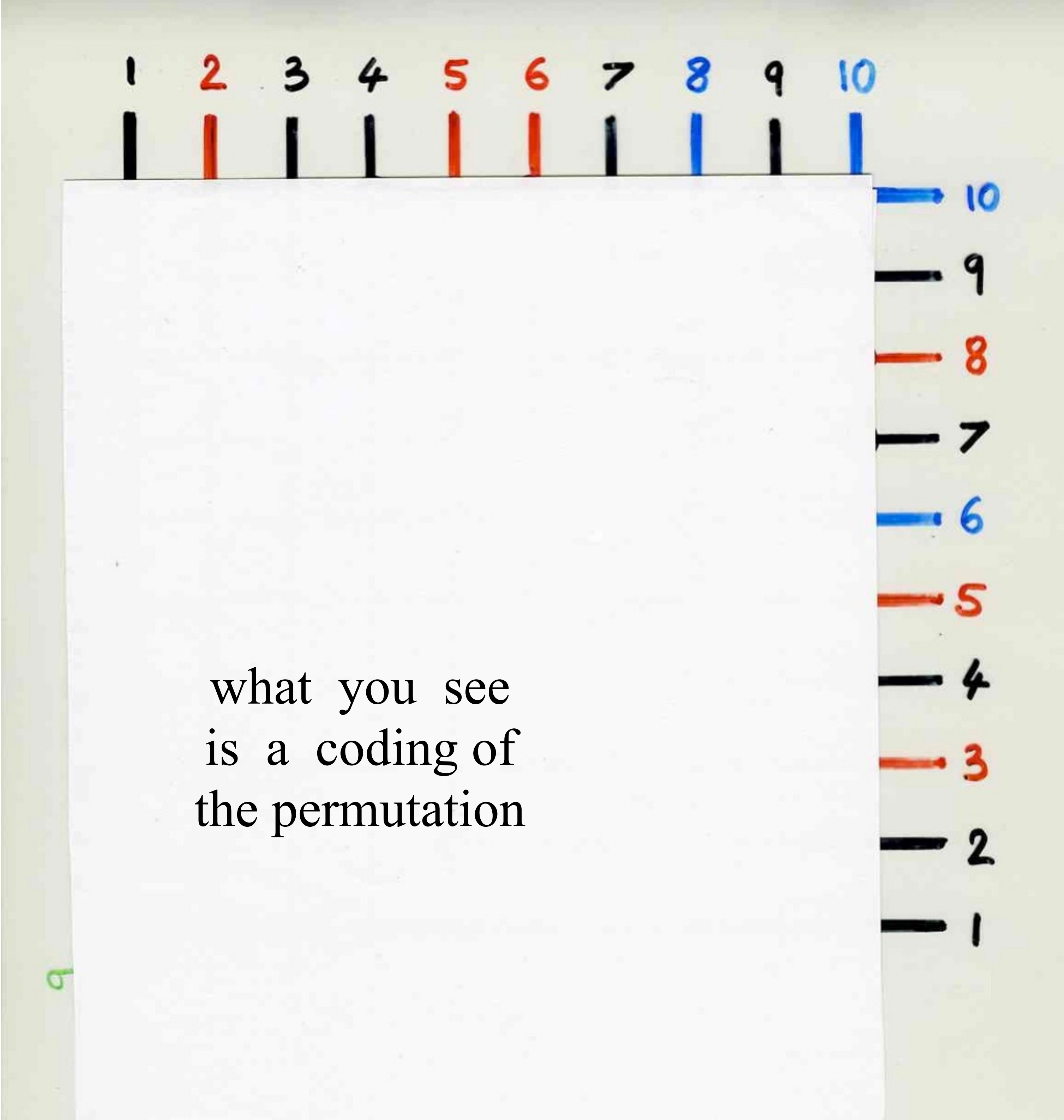
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

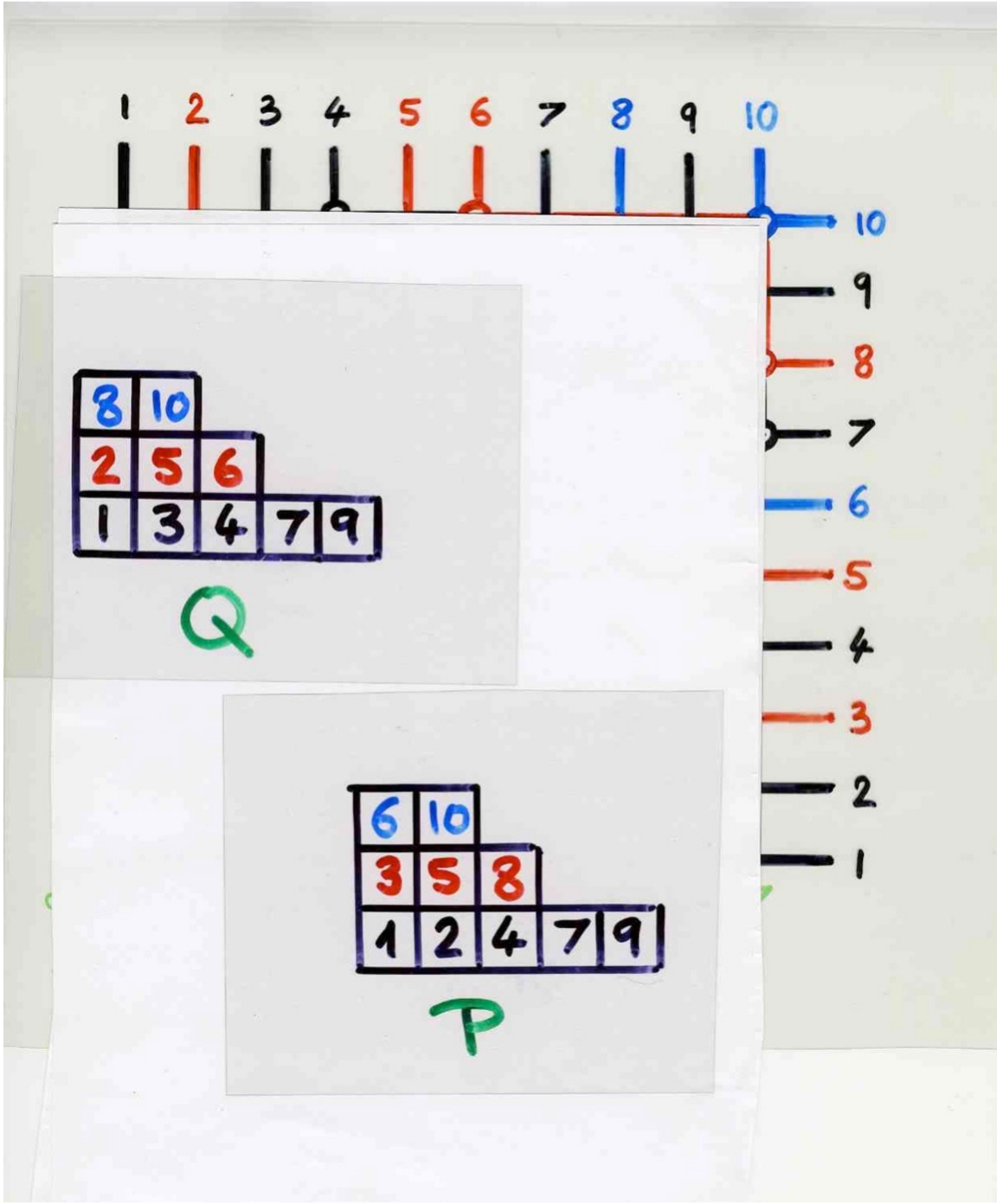


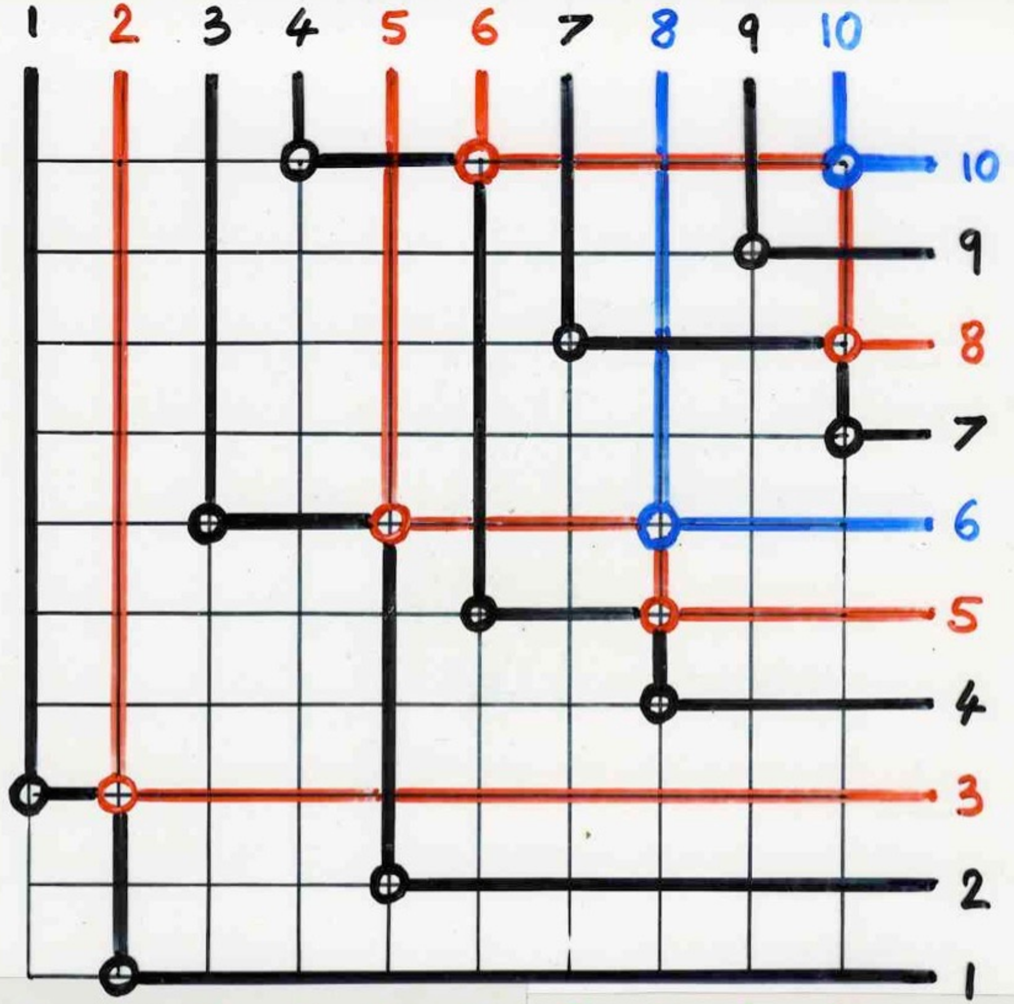
blue
points

no
green
points:
end
of the
construction




1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
what you see
is a coding of
the permutation


1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
8 10
2 5 6
1 3 4 7 9
Q
6 10
3 5 8
1 2 4 7 9
P



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

6	10			
3	5	8		
1	2	4	7	9

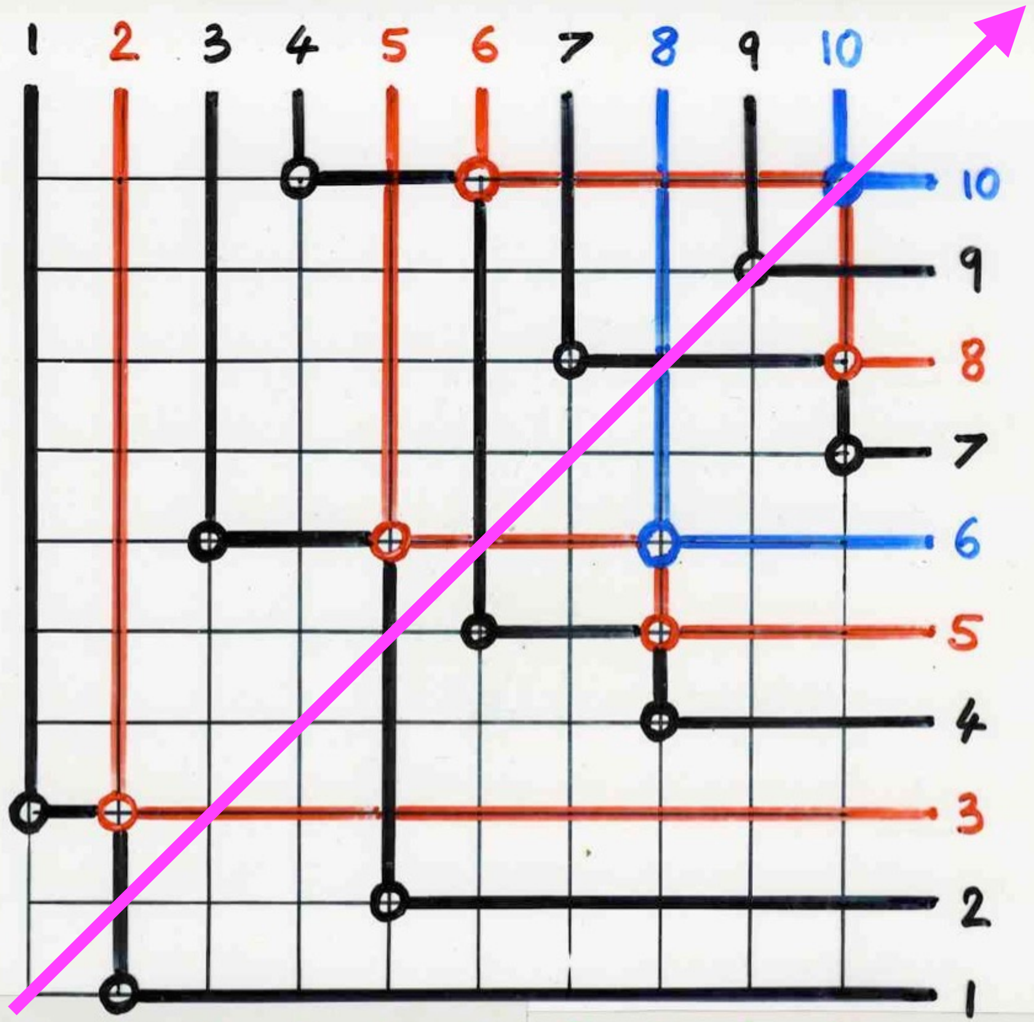
P

8	10			
2	5	6		
1	3	4	7	9

Q

$$\sigma \longleftrightarrow (P, Q)$$

$$\sigma^{-1} \longleftrightarrow (Q, P)$$



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

6	10			
3	5	8		
1	2	4	7	9

P

8	10			
2	5	6		
1	3	4	7	9

Q

more about groups theory

Group theory

<u>Definition</u> Group G $(x,y) \to x * y$

(i) associativity $(x*y)*z = x*(y*z)$

(ii) neutral element $x*e = e*x = x$

(iii) inverse $(x*y = y*x = e$

y unique $y = x^{-1}$

<u>examples</u>

- integers $\mathbb{Z}$ for addition $+$

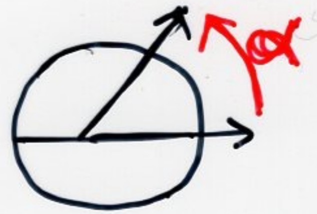
  inverse $-x$

- $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, for multiplication

  inverse $1/x$

- Permutations $\mathfrak{S}_n$ symmetric group other examples of groups

- Rotation in the plane



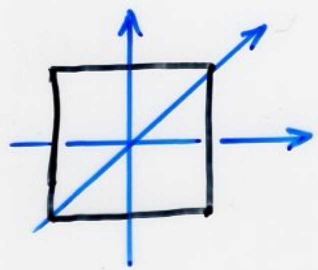
R_α

$$R_\alpha \circ R_\beta = R_{\alpha+\beta}$$

$$R_\alpha^{-1} = R_{-\alpha}$$

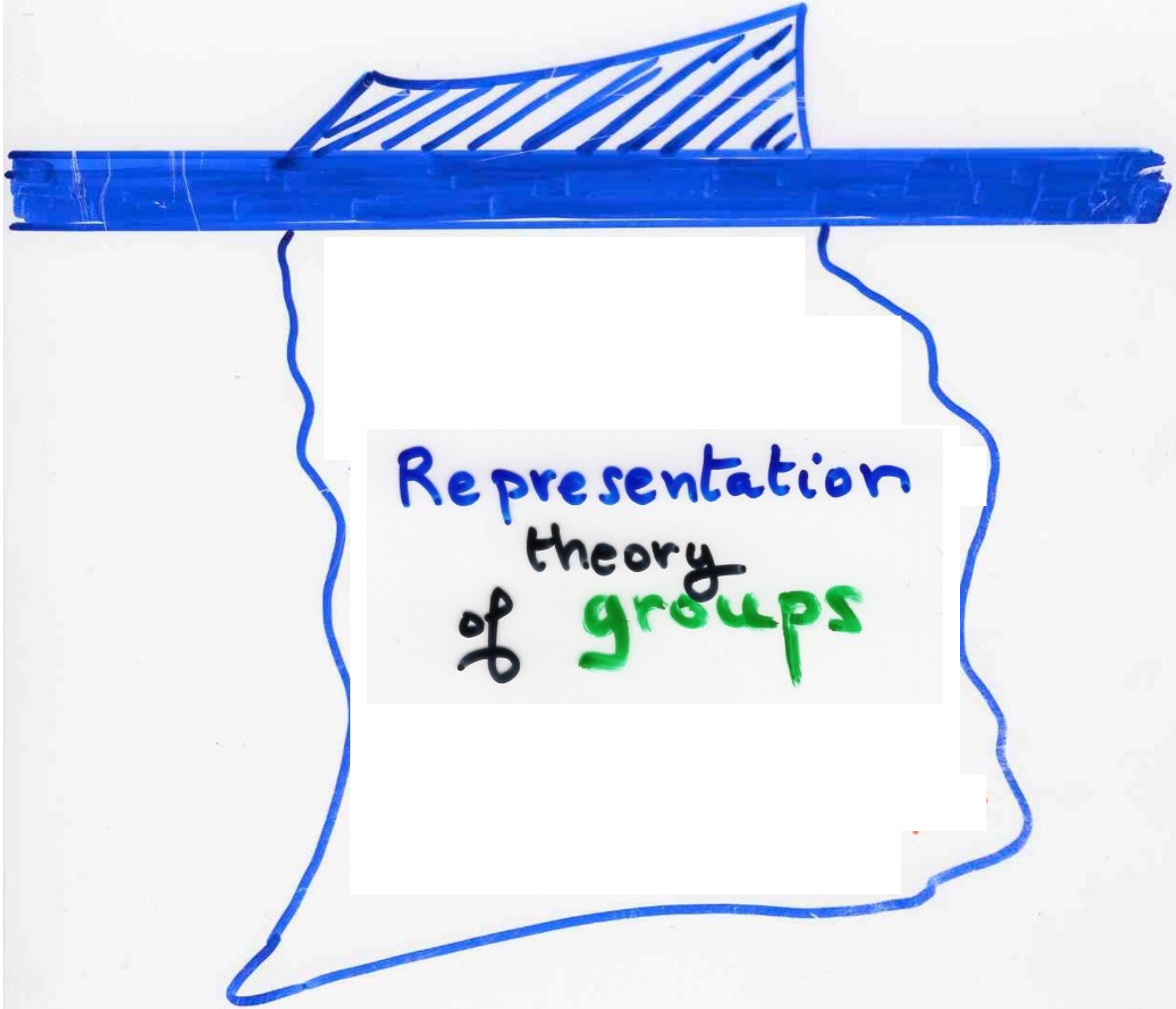
- $(n \times n)$ matrices with coefficients in $\mathbb{R}$

- symmetries of the square



The Robinson-Schensted correspondence

The Robinson-Schensted correspondence



Representation theory of groups

see a group G as a (sub)-group of matrices

$$G \to \text{Matrices}$$
$n \times n$, coeff. in $\mathbb{R}$

see G as a group of transformations

Important in Physics
standard model of particles

4 fundamental forces $\left\{\begin{array}{l}\text{electro-magnetism}\\ \text{strong}\\ \text{weak}\end{array}\right\}$ + gravity

for every group

representation $\xrightarrow[\text{decomposition}]{}$ into irreducible representations

analogy [every number n $= p_1^{\alpha_1} \cdots p_r^{\alpha_r}$

prime numbers decomposition

Case of the group $\mathfrak{S}_n$ permutations

irreducible representations $\longleftrightarrow$ partition λ of n

dimension of the irreducible representation (= order of the matrices) $=$ f_λ number of Young tableaux with shape λ

in (finite) group theory :

$$|G| = \sum_{R} (\deg R)^2$$

order of the group

irreducible representation

for the symmetric group $\mathfrak{S}_n$:

$$n! = \sum_{\lambda} (f_\lambda)^2$$

partitions of n

algebraic combinatorics

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

$$\updownarrow$$

6	10			
3	5	8		
1	2	4	7	9

P

8	10			
2	5	6		
1	3	4	7	9

Q

The Robinson-Schensted correspondence between permutations and pairs of (standard) Young tableaux with the same shape

“The bijective paradigm”

二

$=$

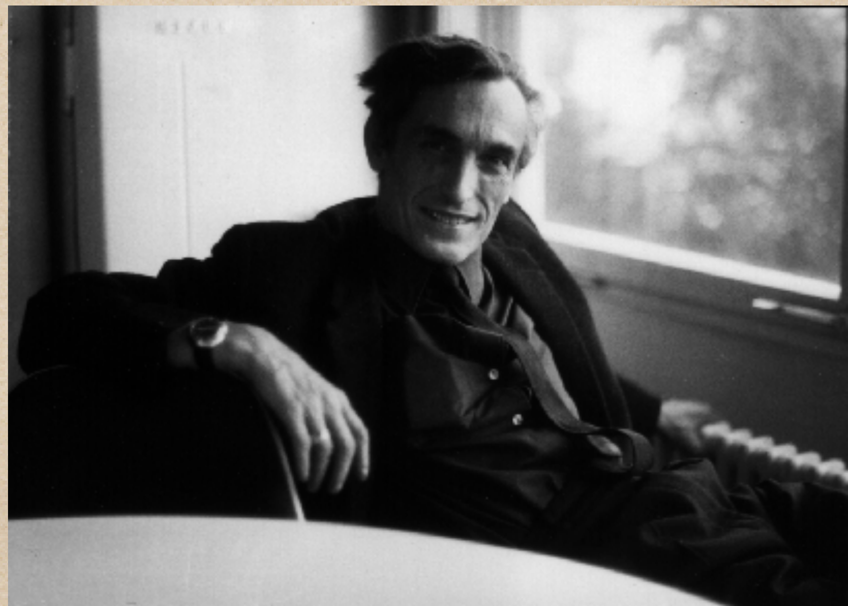
$\longleftrightarrow$

$=$

better understanding

$\longleftrightarrow$

Jeu de taquin



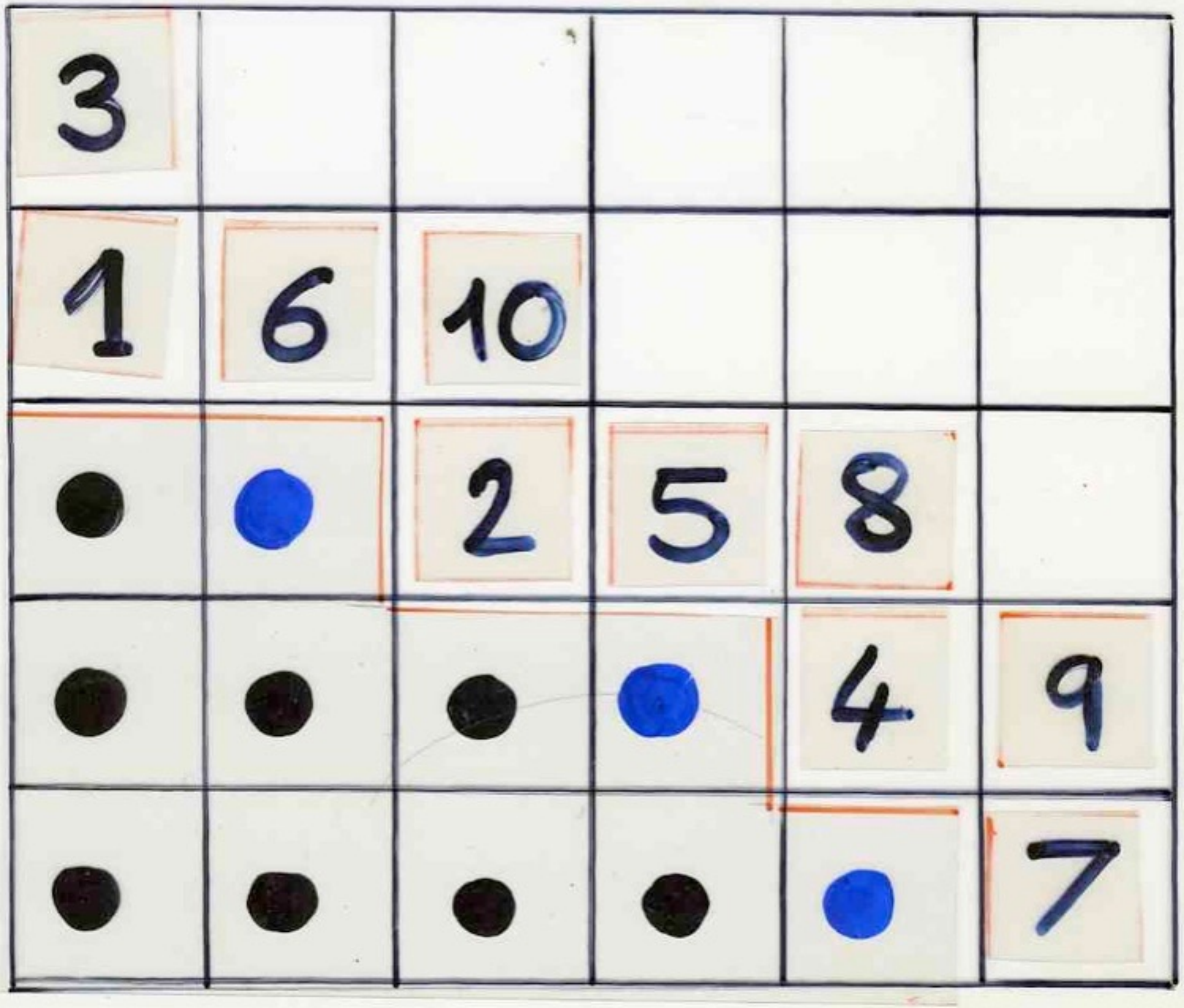
M.P. Schützenberger

$$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$$

$$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$$

3					
1	6	10			
		2	5	8	
				4	9
					7

$$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$$



3					
6	1	10			
		2	5	8	
				4	9
					7

$$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$$

6	10				
3	5	8			
1	2	4	7	9	

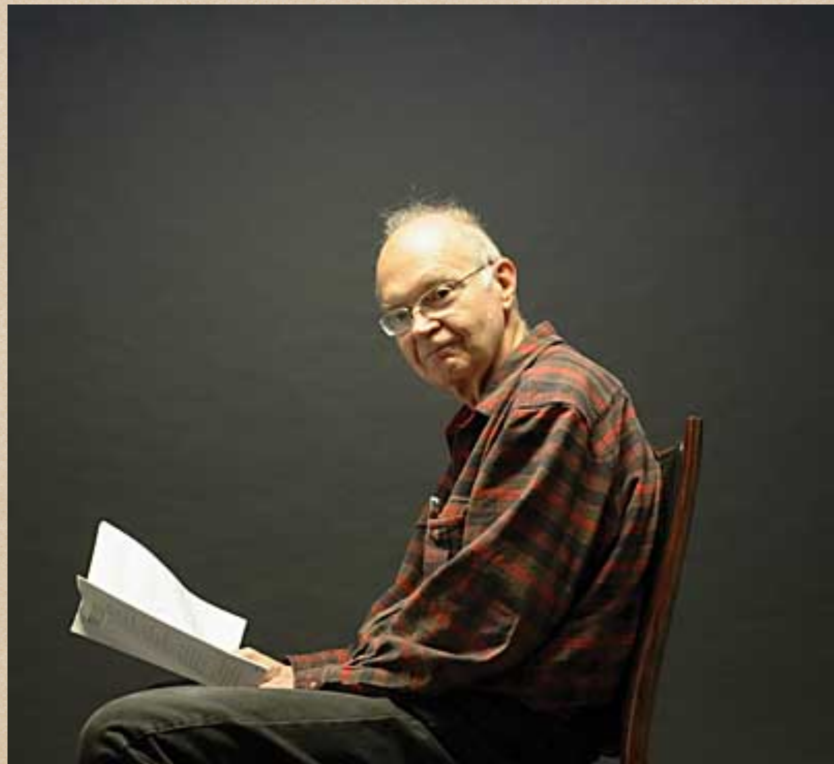
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

6	10				
3	5	8			
1	2	4	7	9	

Knuth's transpositions

D. Knuth, 1970



Knuth's transpositions (1970)

$$\sigma = \sigma(1) \cdots \underbrace{\sigma(i)}_{x} \underbrace{\sigma(i+1)}_{y} \cdots \sigma(n)$$

$z = \sigma(i-1)$ or $\sigma(i+1)$ is between x and y

$x < z < y$ or $y < z < x$

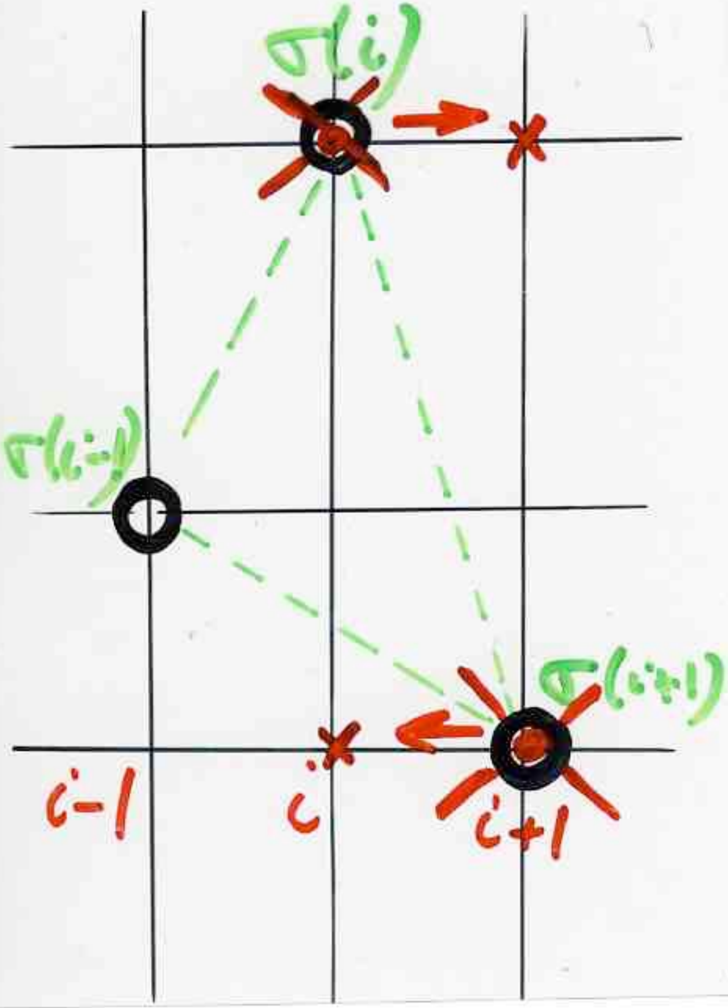
$$P(\sigma) = P(\tau)$$

$$\tau = \sigma(1) \cdots y\, x \cdots \sigma(n)$$

Knuth

Transpositions

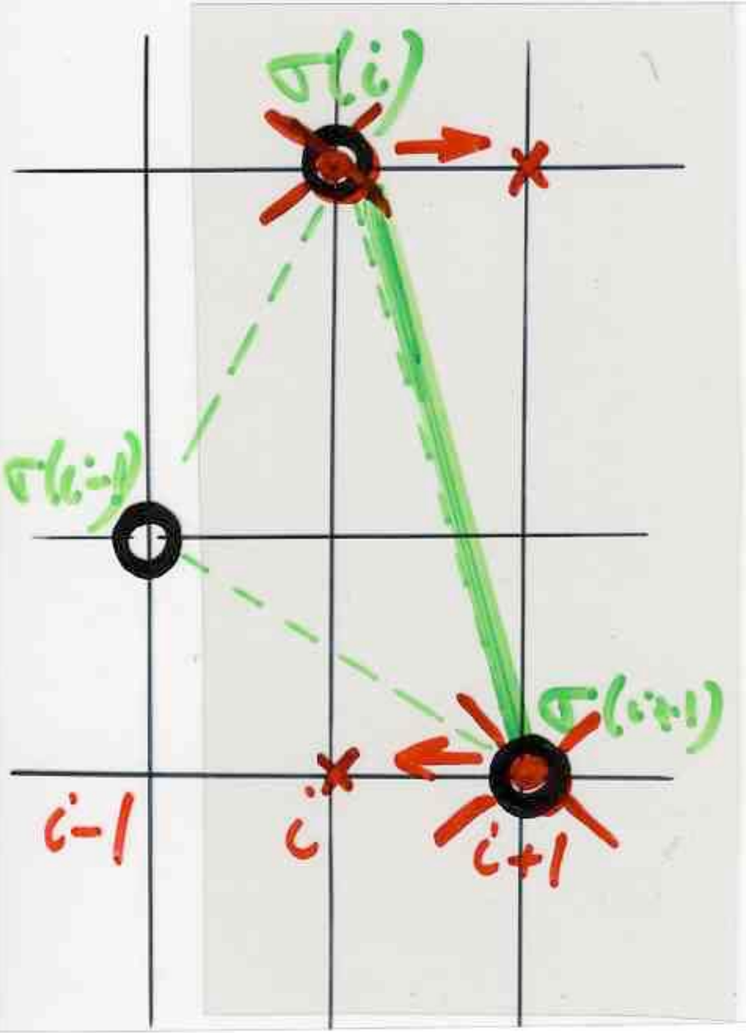
(1970)



Knuth

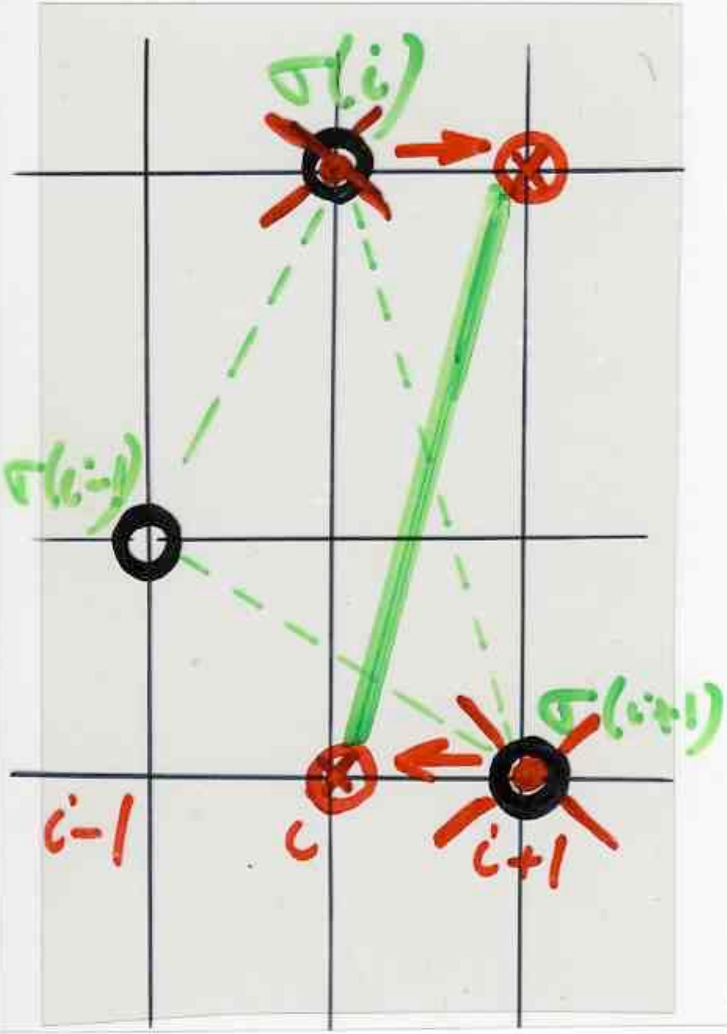
Transpositions

(1970)



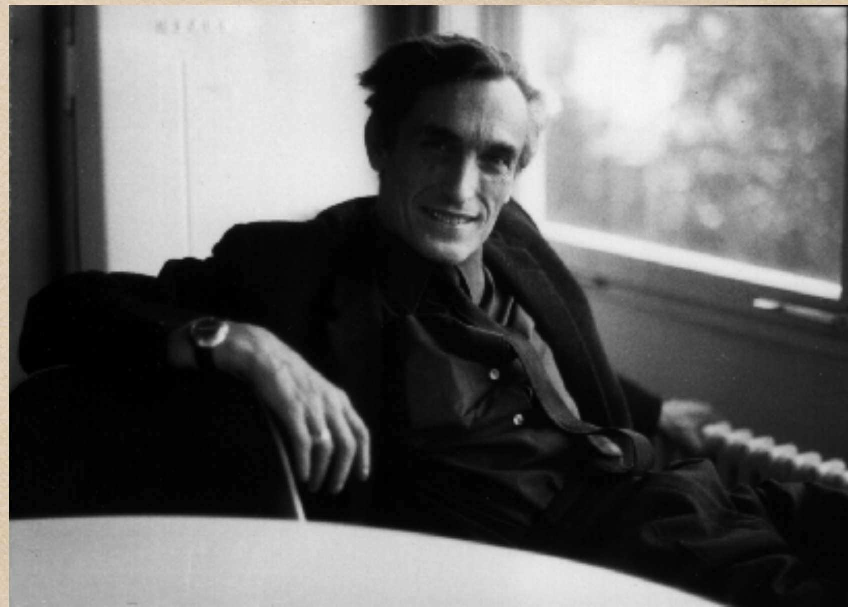
Knuth

Transpositions (1970)



plactic monoid

A. Lascoux



M.P. Schützenberger

Plactic monoid

X alphabet

$X^* / \equiv$

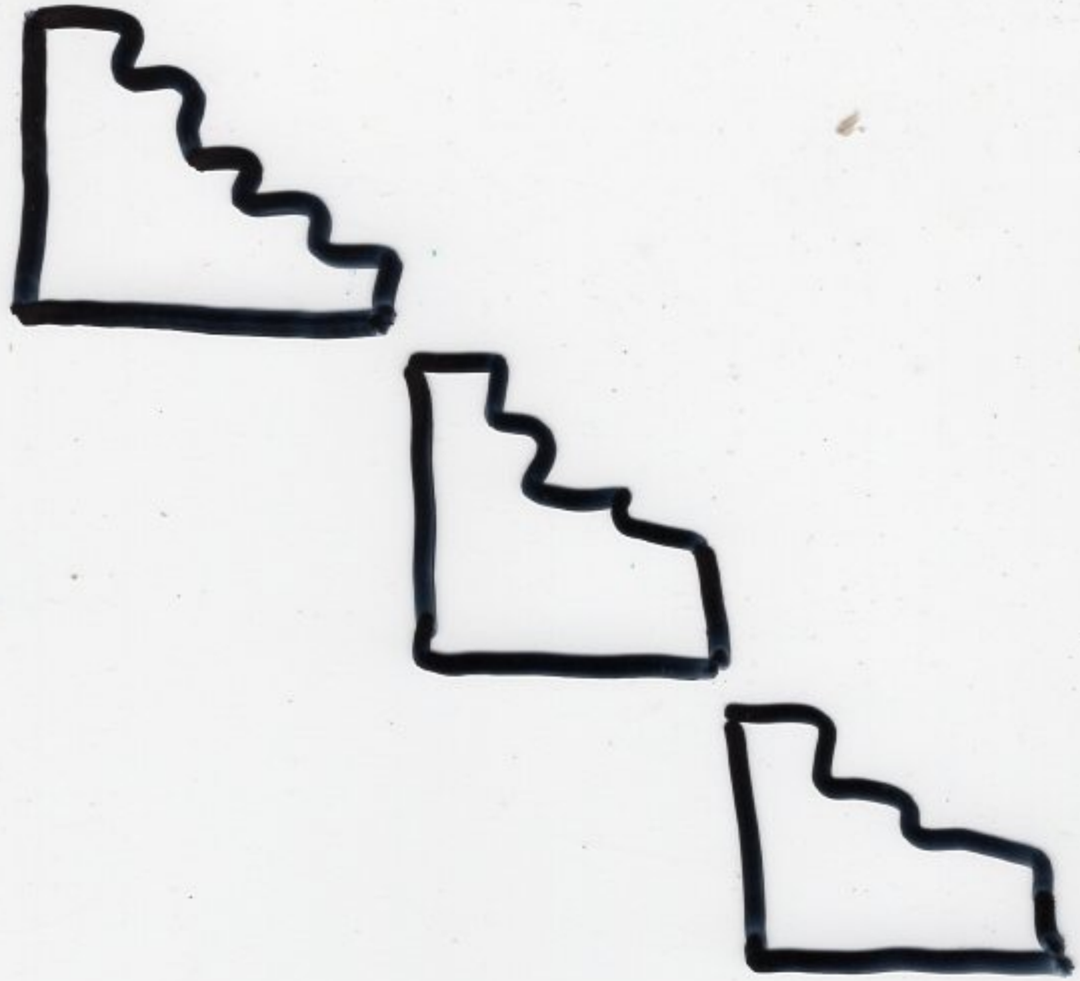
$$\begin{cases} yzx \equiv yxz & x < y \leq z \\ xzy \equiv zxy & x \leq y < z \end{cases}$$

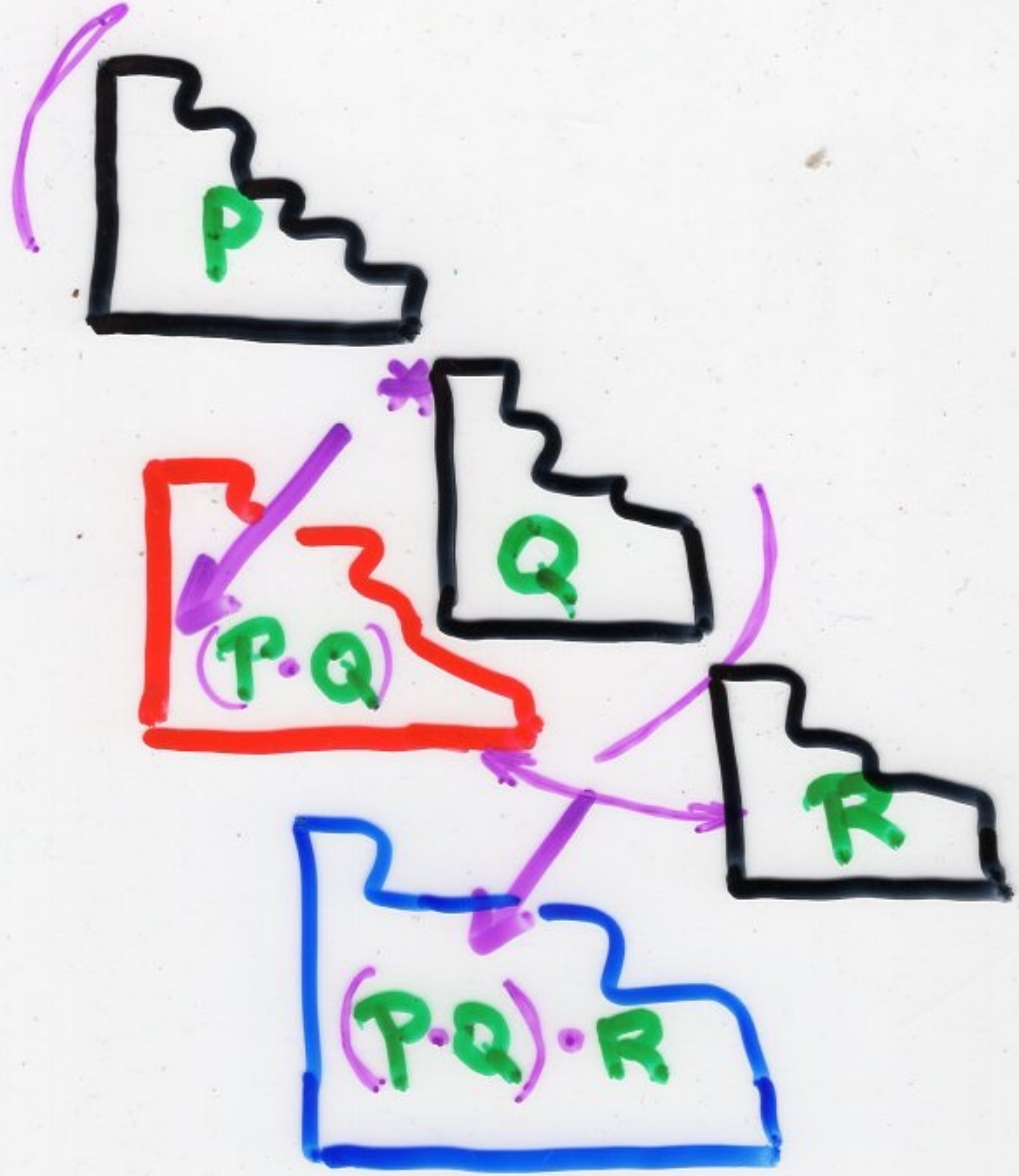
A. Lascoux, M.P. Schützenberger (1978)

Plactic monoid

8	8		
3	5		
2	2	3	
1	1	1	2

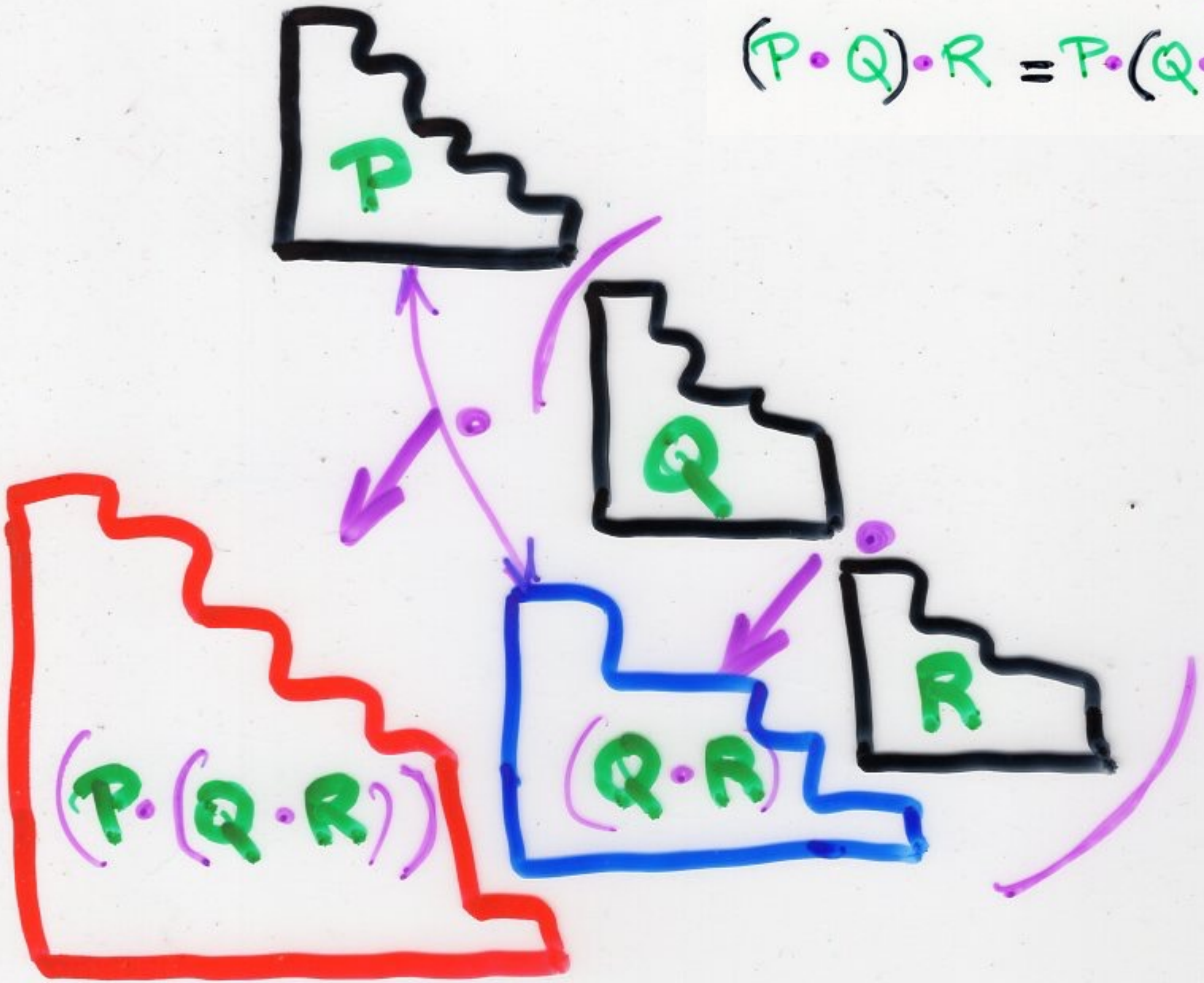
4	5	7		
2	4	4		
1	1	2	2	5

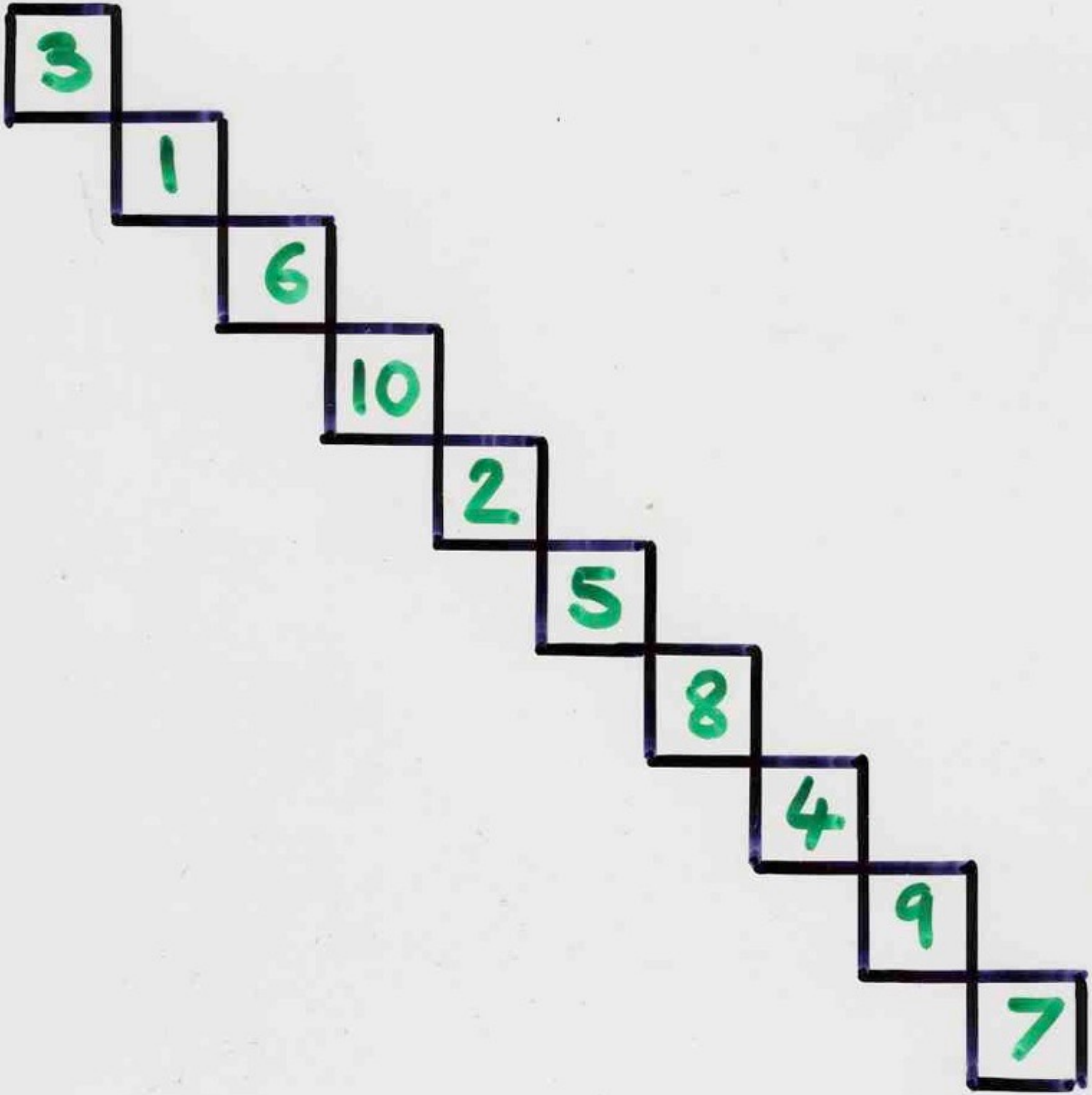


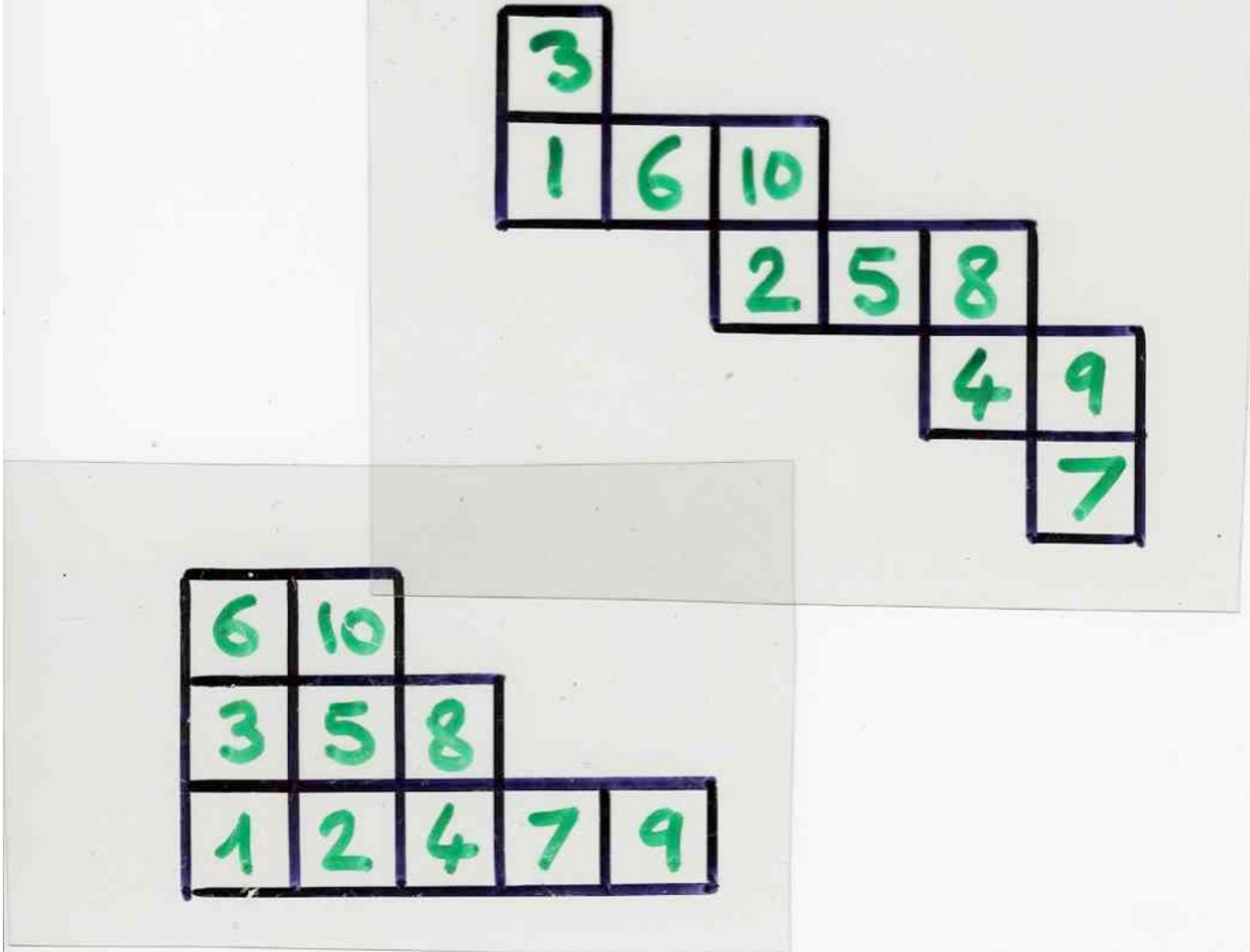

P
Q
(P·Q)
R
(P·Q)·R

associativity

$$(P \cdot Q) \cdot R = P \cdot (Q \cdot R)$$




3
1
6
10
2
5
8
4
9
7


3
1 6 10
2 5 8
4 9
7
6 10
3 5 8
1 2 4 7 9

Schur functions

fonction de Schur

$$s_\lambda(x_1, x_2, \dots, x_n) = \frac{\det\left(x_j^{n-i+\lambda_i}\right)_{1\le i,j\le n}}{\det\left(x_j^{n-i}\right)_{1\le i,j\le n}}$$

$\lambda = (\lambda_1, \dots, \lambda_n)$

Issai Schur 1875–1941

théorie des invariants

Cauchy 1812

Jacobi 1841 det (homogène)

N. Trudi 1864

Schur Functions

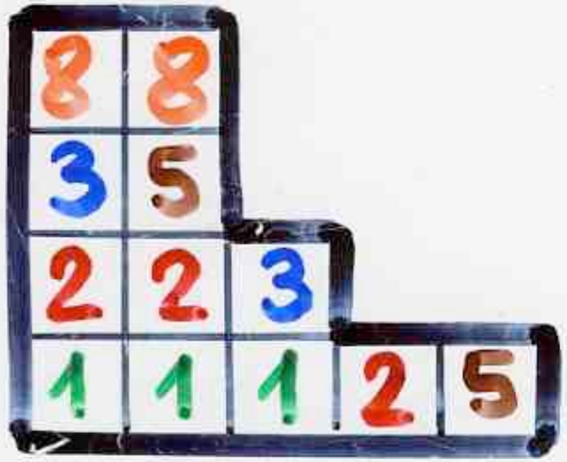
$$s_\lambda(x_1, x_2, \dots, x_m) = \sum_{T} v(T)$$

Young tableau
shape λ
entries $1, 2, \dots, m$

Jacobi (1841)
Schur (1901)

Littlewood-Richardson (1934)

basis of **symmetric** functions



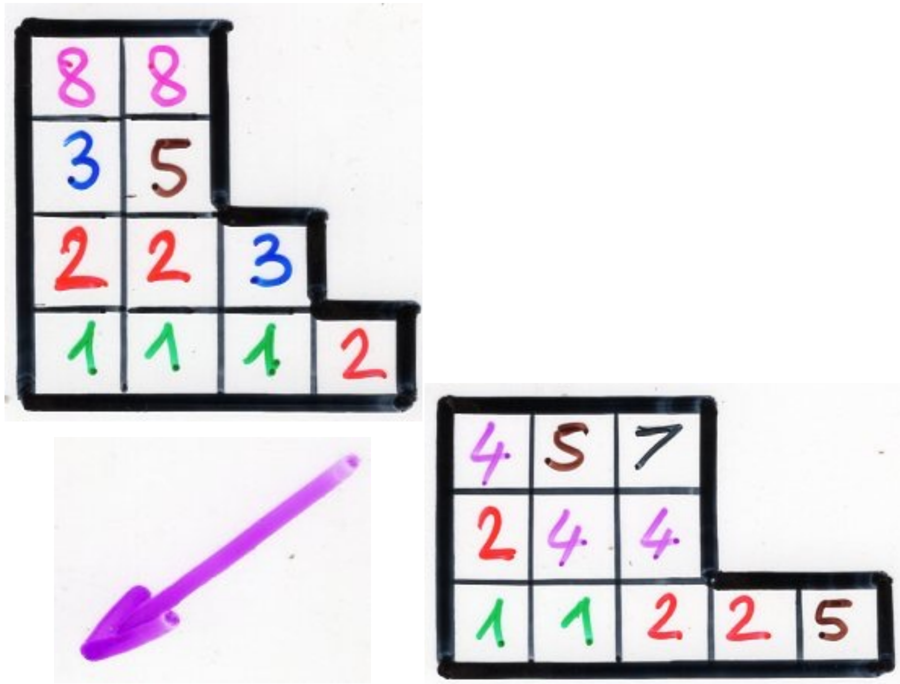
Schur functions

$$s_\lambda \, s_\mu = \sum_\nu g_{\lambda,\mu,\nu} \, s_\nu$$

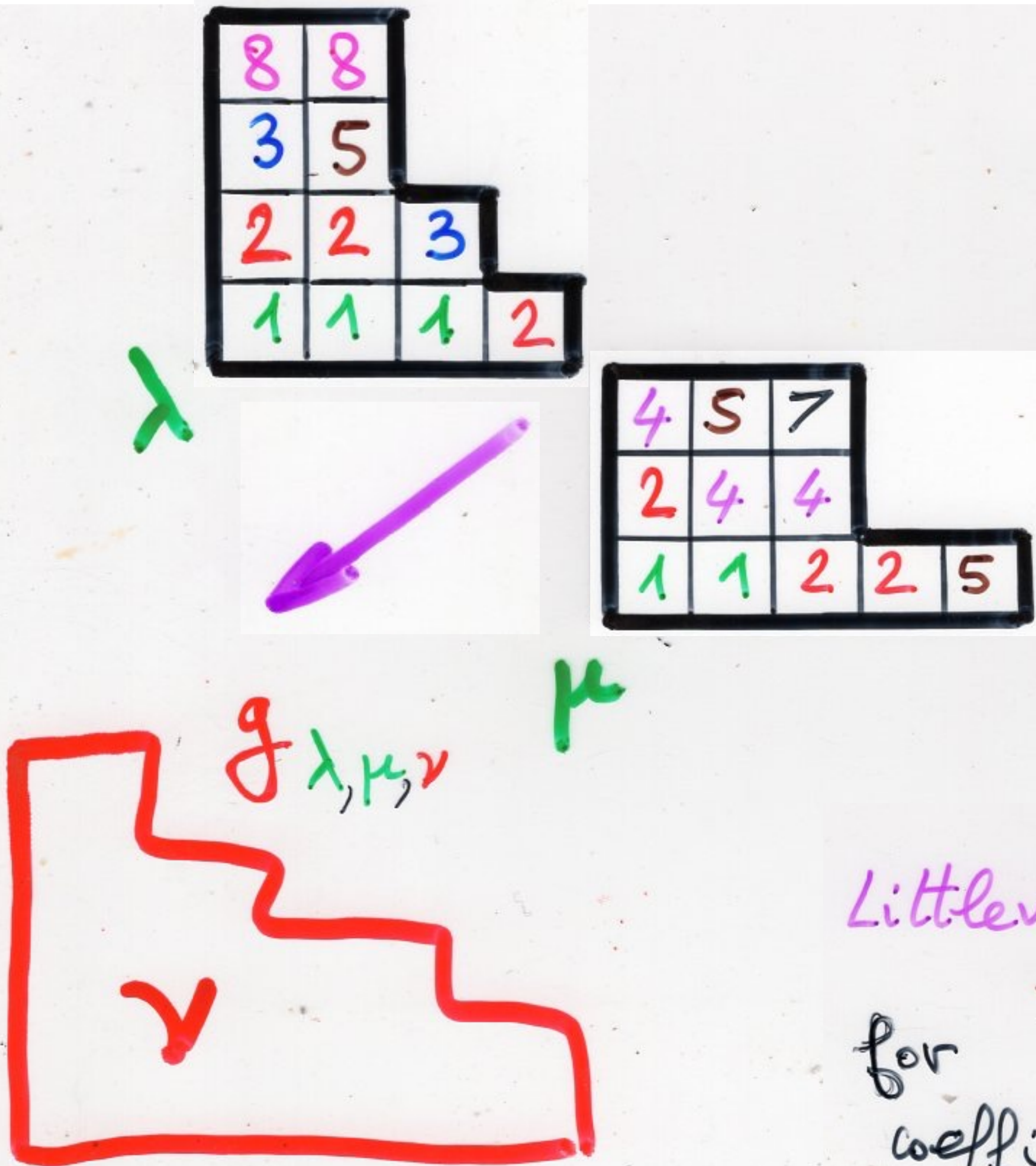
$s_\lambda(x_1, \dots, x_m)$

Littlewood–Richardson

Plactic monoid



Plactic monoid



Littlewood-Richardson rule (1934) for computing the coefficients $g_{\lambda,\mu,\nu}$

Thank you !

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ॐ सरस्वत्यै नमः।