Rhombic alternative tableaux for the 2-species PASEP

(work wirh Olya Mandelshtam, Berkeley)

GT LaBRI
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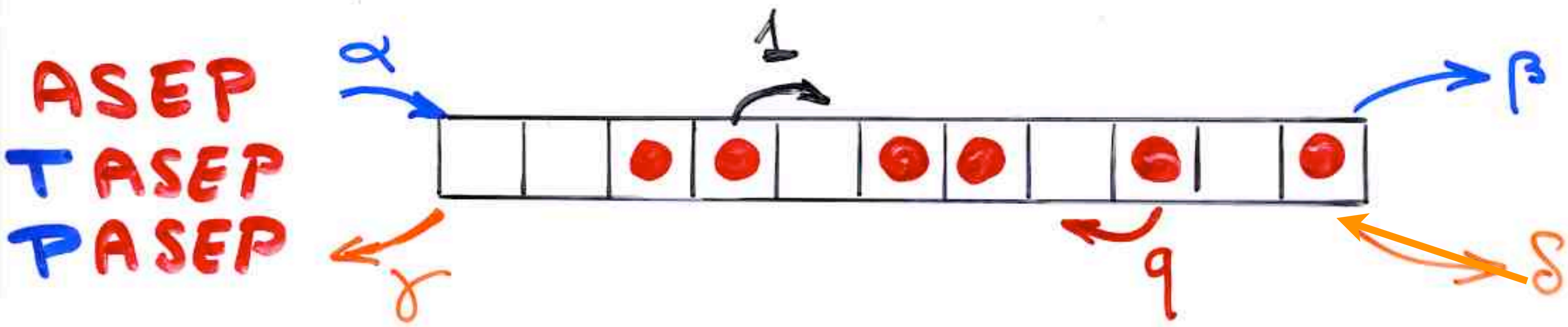
The PASEP


ASEP
TASEP
PASEP
α
1
β
γ
q
δ

boundary induced phase transitions

molecular diffusion

linear array of enzymes

biopolymers

traffic flow

------ formation of shocks

---

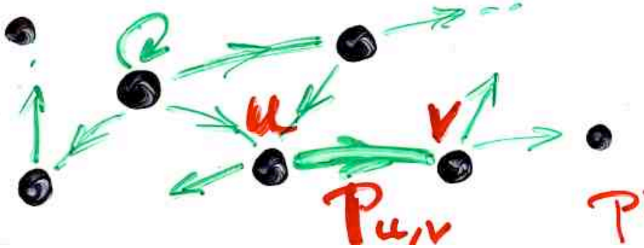
chaînes de Markov

2^n états

$w = (\tau_1, \dots, \tau_n)$

$\tau_i = \begin{cases} 0 \\ 1 \end{cases}$

graphe



probabilité

S : états

$M = (p_{u,v})_{u,v \in S}$ matrice de probabilités (stochastique)

$\pi = (p_u, \ldots)$ vecteur (temps t)

$\pi \cdot M$ vecteur (temps $t+1$)

$$p_v^{(t+1)} = \sum_u p_u^{(t)} \, p_{u,v}$$

$t+1$ temps t

probabilités stationnaires

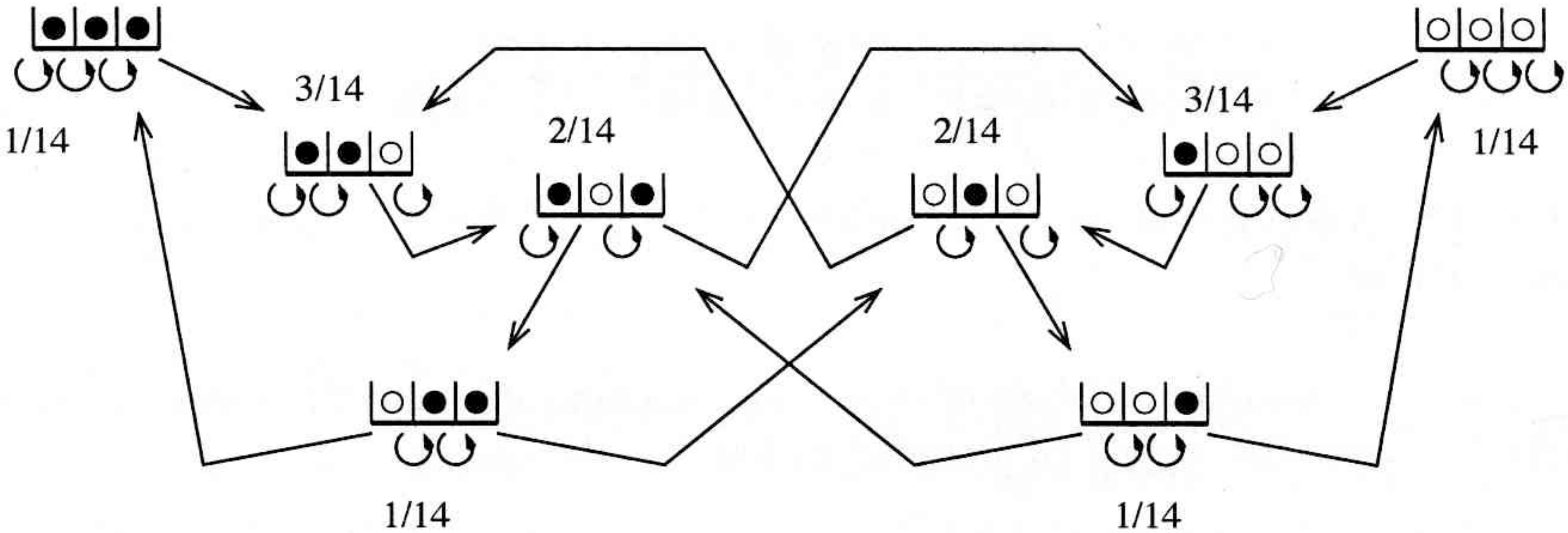
$$\pi \cdot M = \pi$$

unicité

vecteur propre M^T valeur propre 1

temps $\rightarrow \infty$

$$p_v = \sum_u p_u \, p_{u,v}$$


1/14
3/14
2/14
2/14
3/14
1/14
1/14
1/14

The Matrix Ansatz

Derrida, Evans, Hakim, Pasquier 1993

$$\mathbb{P}_n(\tau_1, \dots, \tau_n) = f_n(\tau_1, \dots, \tau_n) / Z_n$$

$$Z_n = \sum_{\tau} f_n(\tau_1, \dots, \tau_n)$$

partition function

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

$$\begin{cases} DE = q\,ED + D + E \\ (\beta D - \delta E)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \gamma D) = \langle W| \end{cases}$$

Then

$f_n(\tau_1, \dots, \tau_n)$

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

TASEP

D E matrices,

V column vector, W row vector

q=0

$$\begin{cases} DE = \square + D + E \\ (\beta D - \square)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \square) = \langle W| \end{cases}$$

Then

$$f_n(\tau_1, \dots, \tau_n) = \langle W| \prod_{i=1}^{n} (\tau_i D + (1-\tau_i) E) |V\rangle$$

orthogonal polynomials

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\quad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

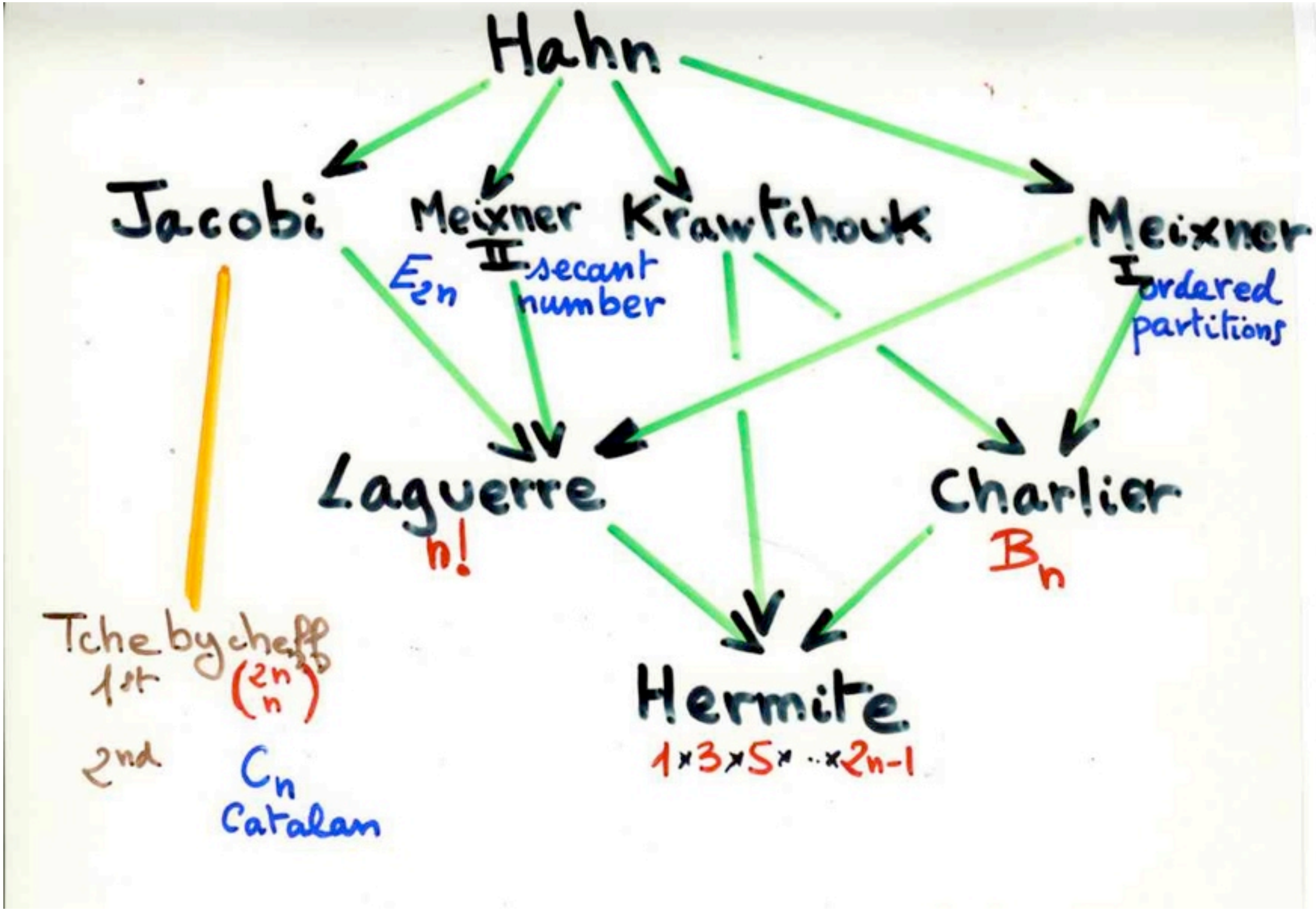
$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

Askey-Wilson polynomials

Askey-Wilson



combinatorics

permutations tableaux — A. Postnikov (2001), E. Steingrímsson (2005) + L. W.

S. Corteel, L. Williams (2007)

alternative tableaux

X.V. (2008)

tree-like tableaux

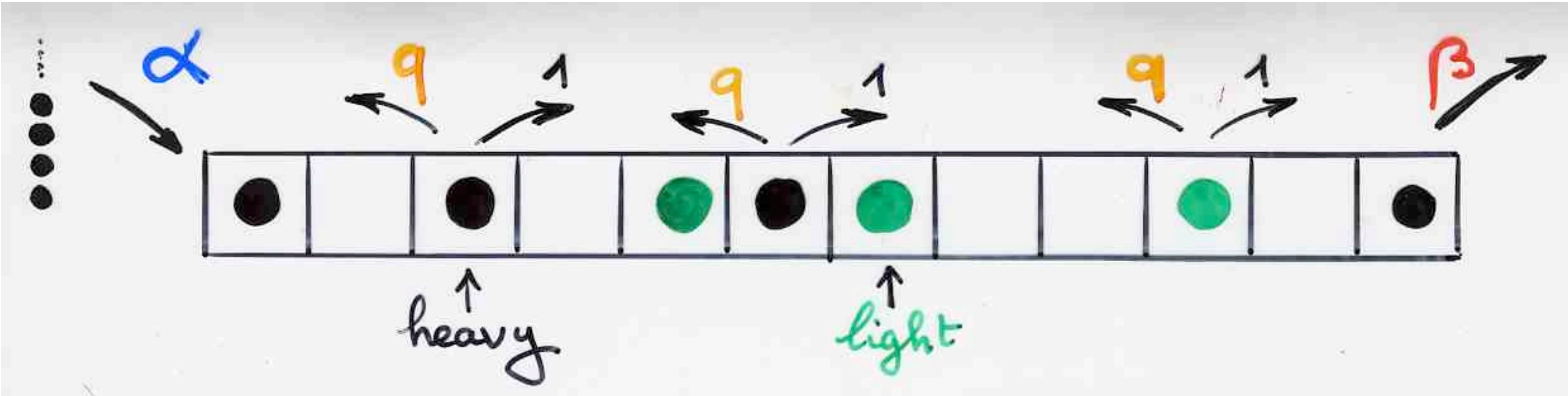
J.-C. Aval, A. Boussicault, P. Nadeau (2013)

staircase tableaux

S. Corteel, L. Williams (2011)

The 2-species PASEP

The 2-species PASEP



Matrix Ansatz

for the 2-species PASEP

Matrix Ansatz (Uchiyama, 2008)

$X = X_1 \dots X_n \qquad X_i \in \{ \bullet, \bullet, \circ \}$

D, E, A matrices

W row vector $\qquad V$ column vector

$$\begin{cases} DE = qED + D + E \\ DA = qAD + A \\ AE = qEA + A \end{cases}$$

$$\langle W | E = \frac{1}{\alpha} \langle W |$$

$$D | V \rangle = \frac{1}{\beta} | V \rangle$$

Matrix Ansatz (Uchiyama, 2008)

$X = X_1 \dots X_n$ $X_i \in \{\bullet, \textcolor{green}{\bullet}, \circ\}$

D, E, A matrices

W row vector V column vector

$$\mathrm{Prob}(X) = \frac{1}{Z_{n,r}} \langle W | \prod_{i=1}^{n} D\,\mathbb{1}_{(X_i=\bullet)} + A\,\mathbb{1}_{(X_i=\textcolor{green}{\bullet})} + E\,\mathbb{1}_{(X_i=\circ)} | V \rangle$$

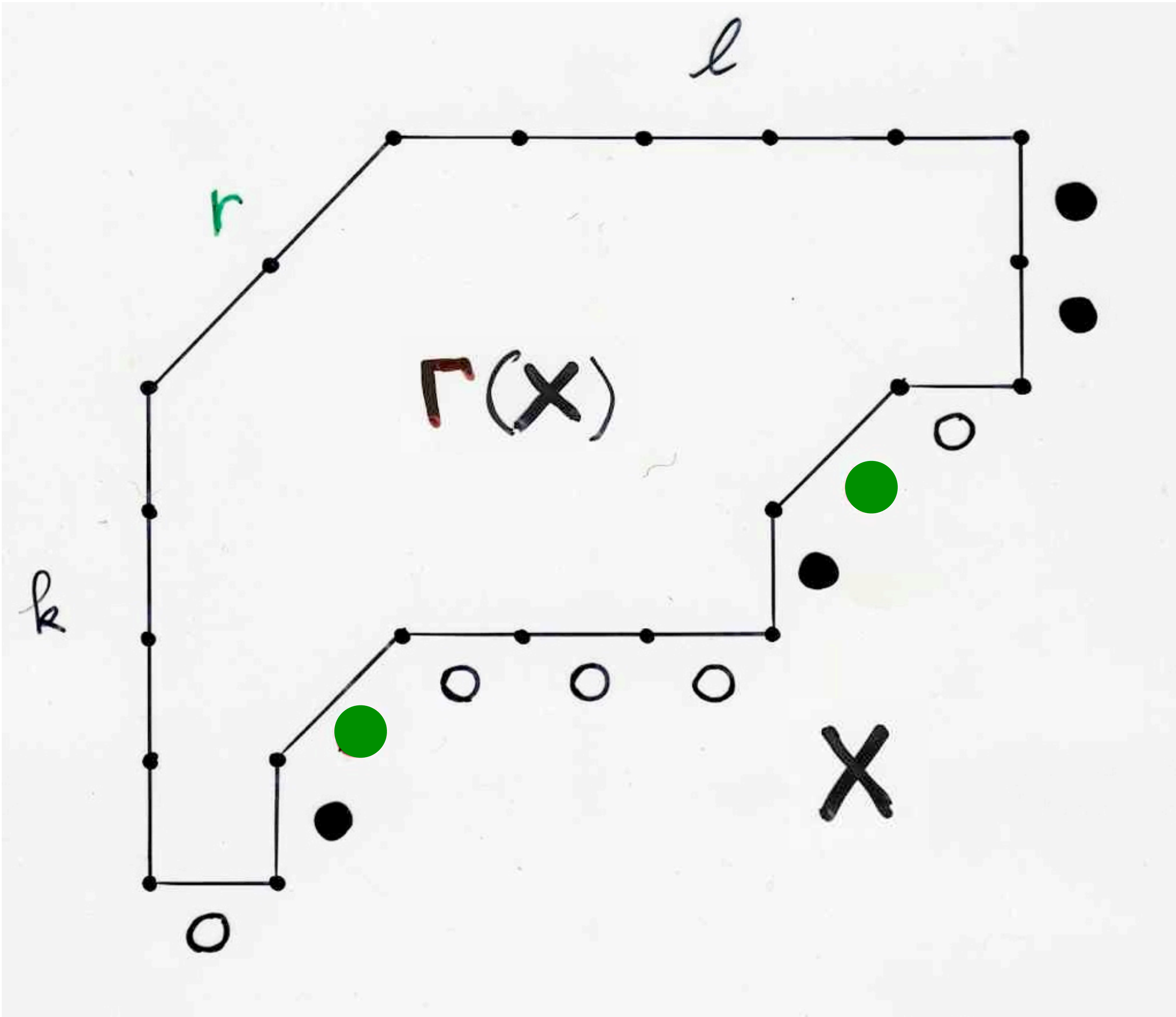
$Z_{n,r}$ = coeff. of y^r in $\langle W | (D + yA + E)^n | V \rangle$

Rhombic alternative tableaux

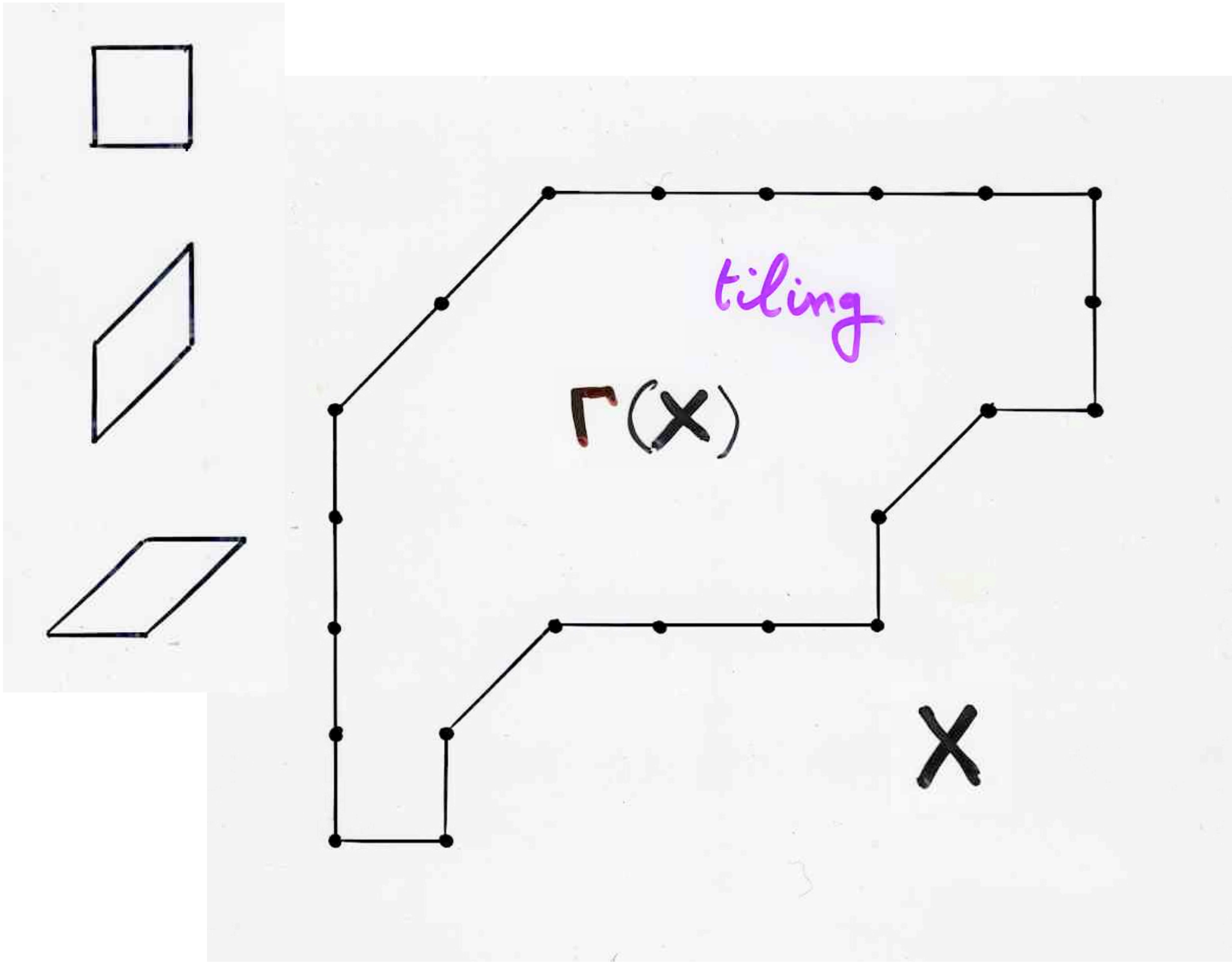
(RAT)

Rhombic alternative tableaux

(tableaux alternatifs rhomboïdaux)

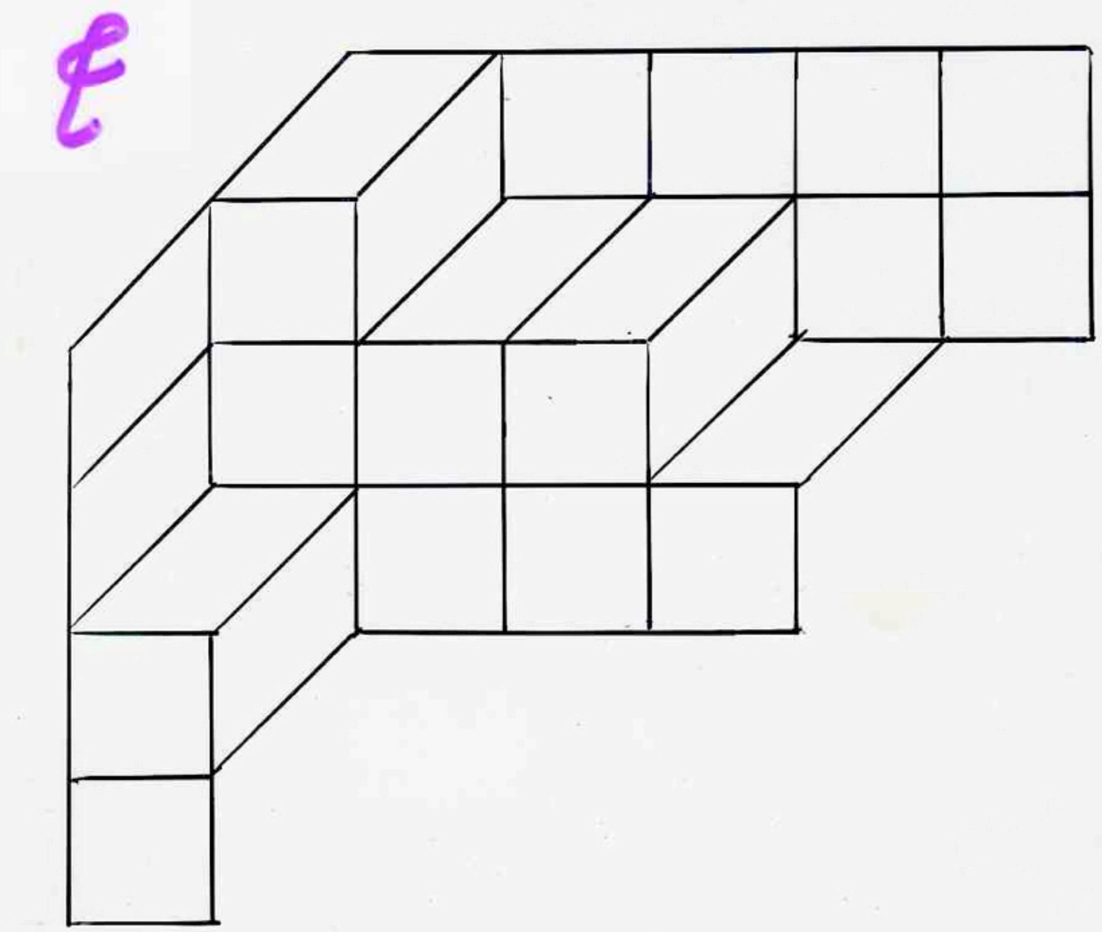


ℓ
r
Γ(X)
k
X



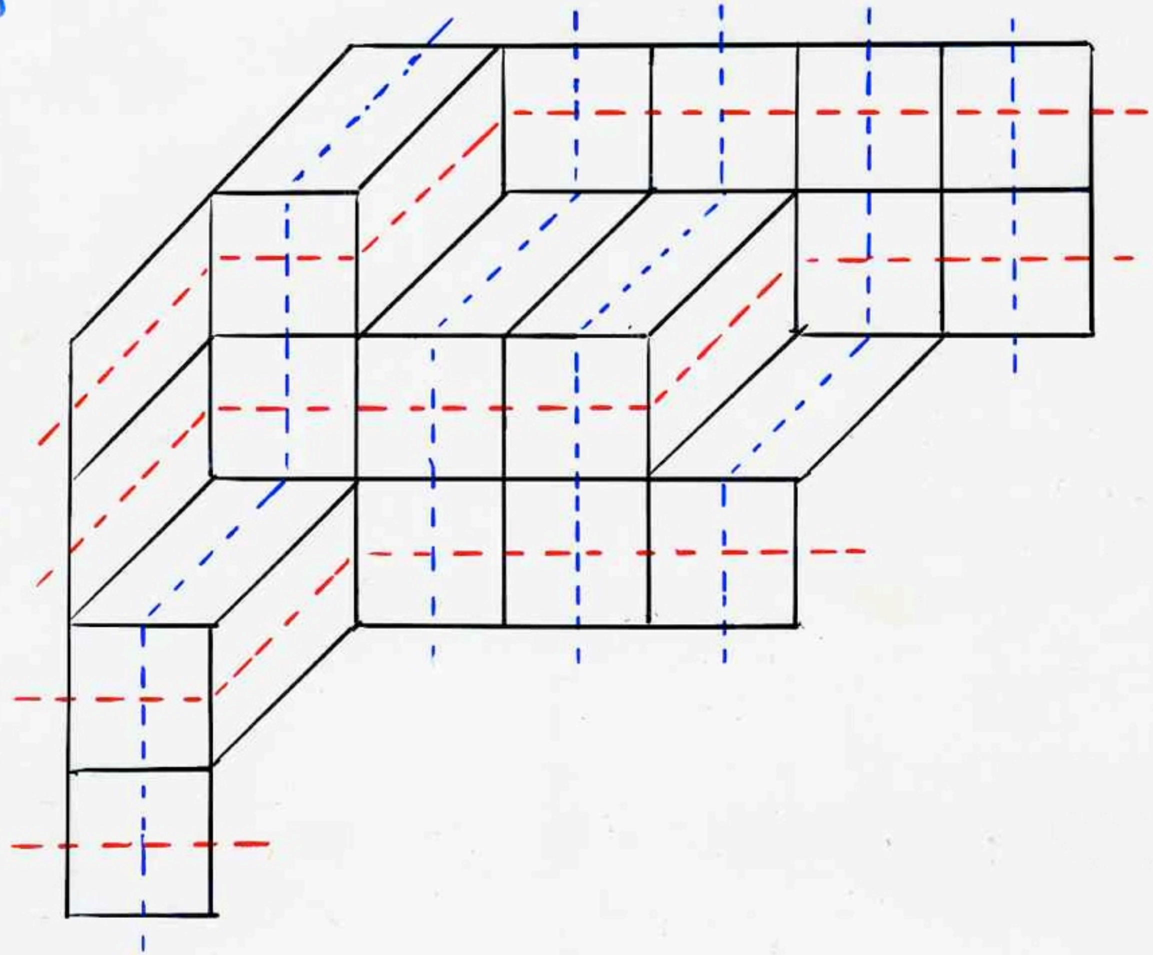
tiling
Γ(X)
X

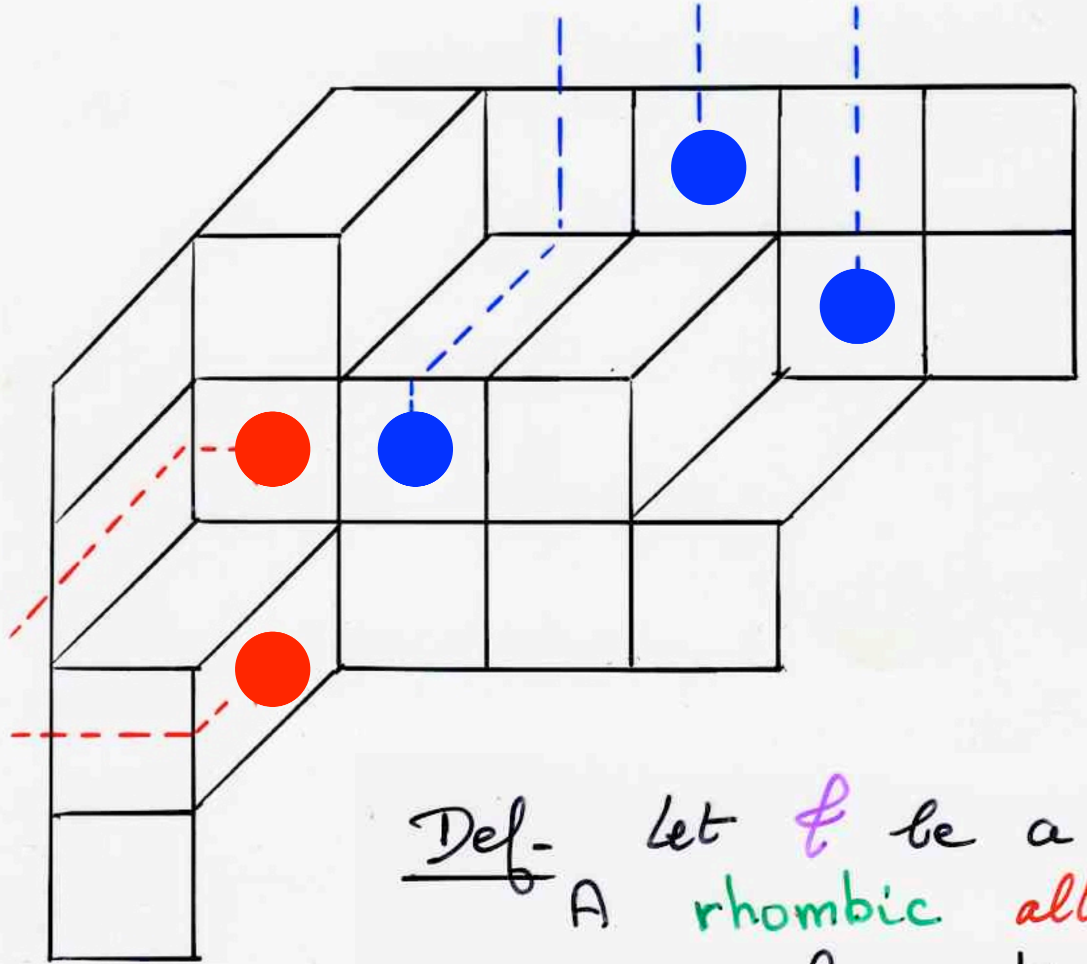
tiling



west-strips

north-strips





west-strips

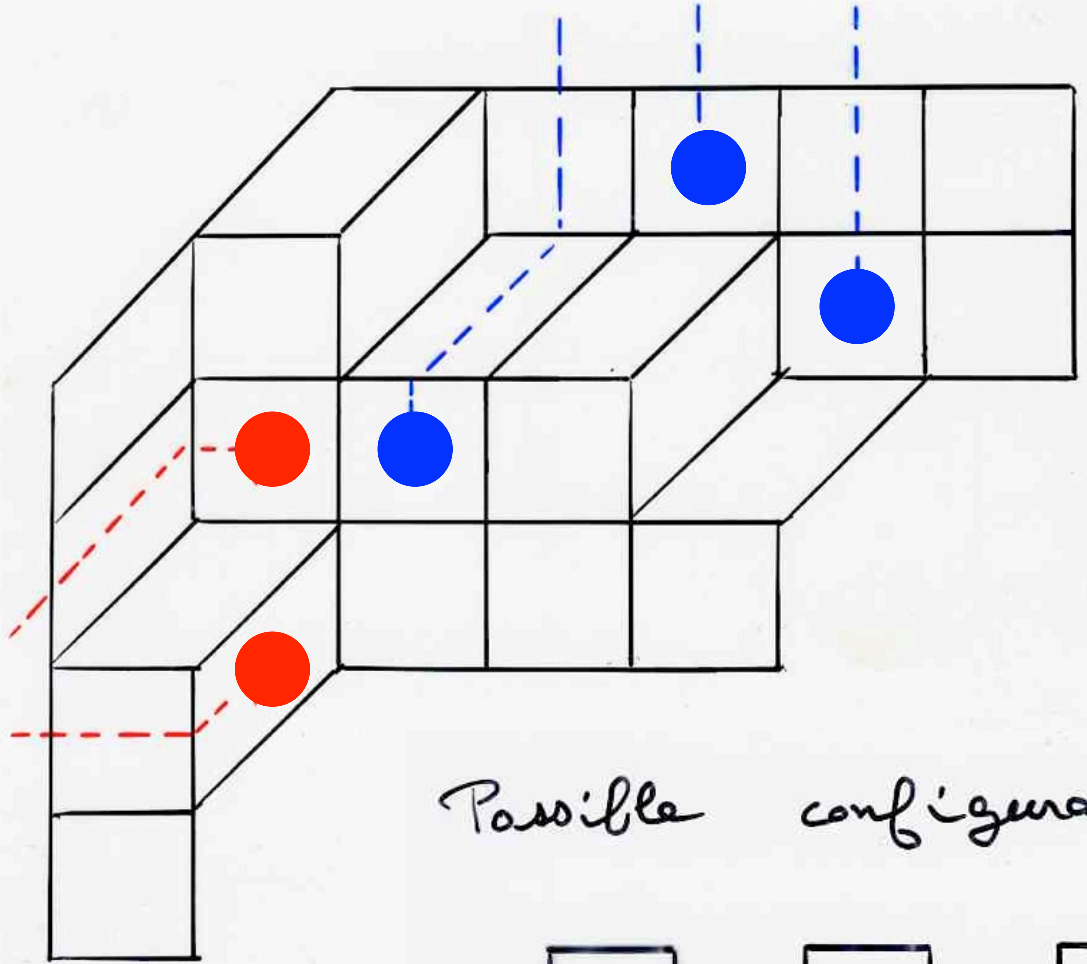
north-strips

Def- Let f be a tiling of $\Gamma(X)$

A rhombic alternative tableau T

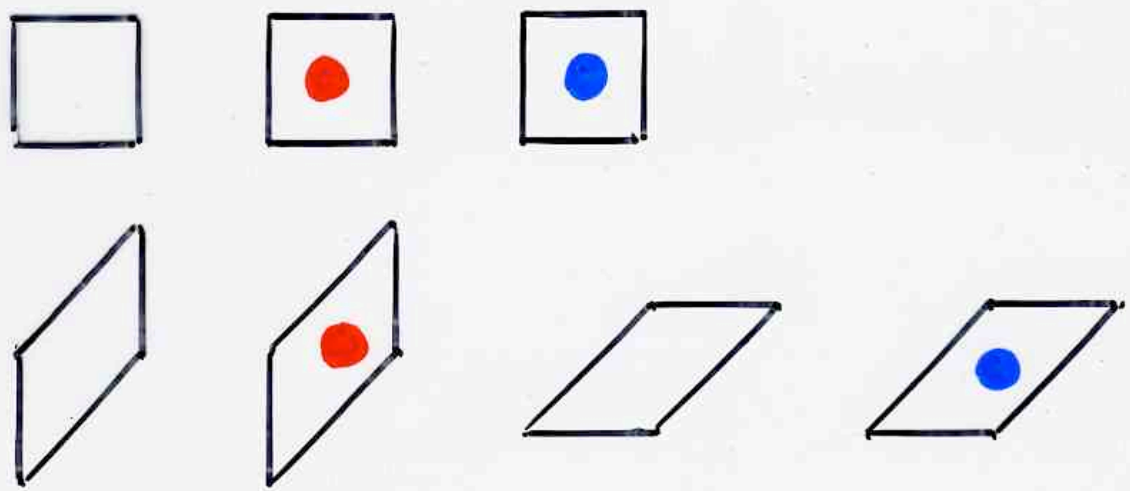
is a placement of ●, ● in the tiles such that:

- a ● is on a west-strip and any tile left of this ● is empty
- a ● is on a north-strip and any tile north of this ● is empty



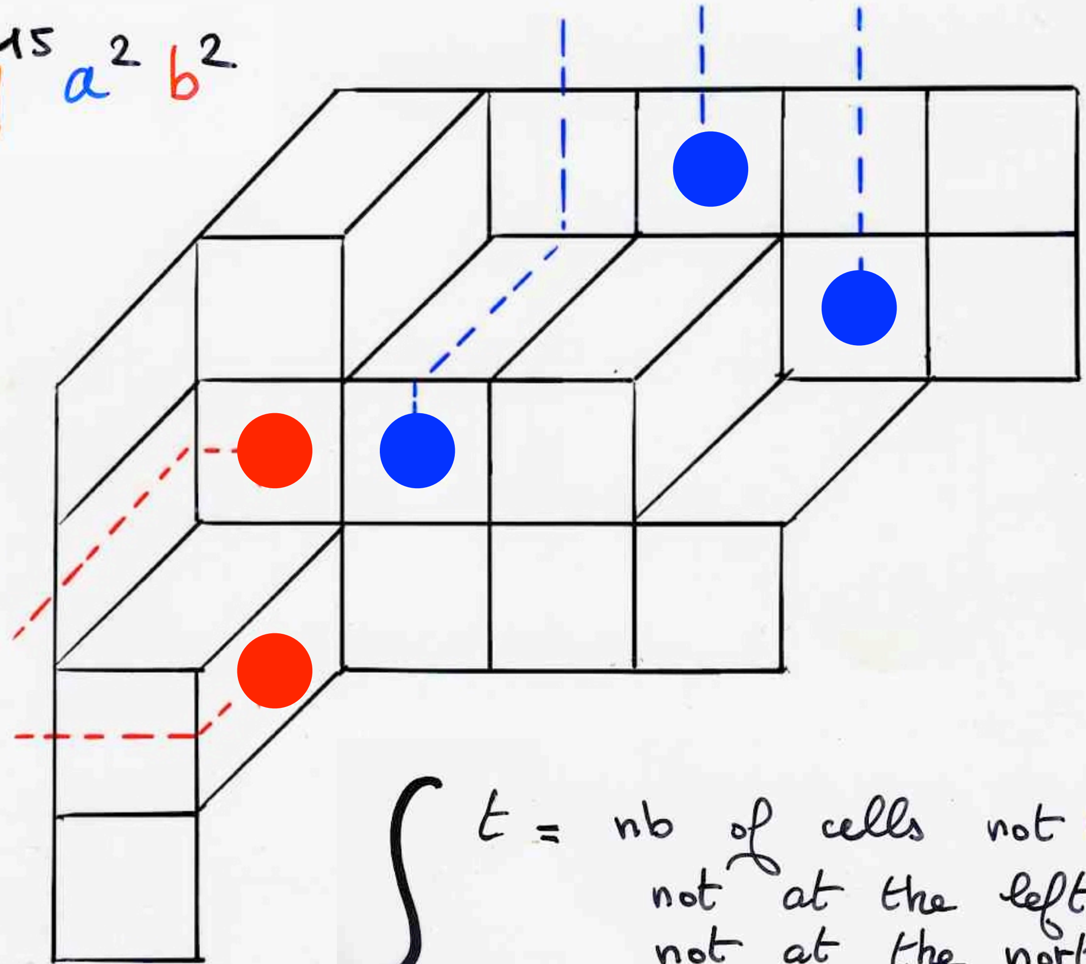
west-strips
north-strips

Possible configurations for a tile :



weight $\mathbf{wt}(T) = q^{t} a^{i} b^{j}$

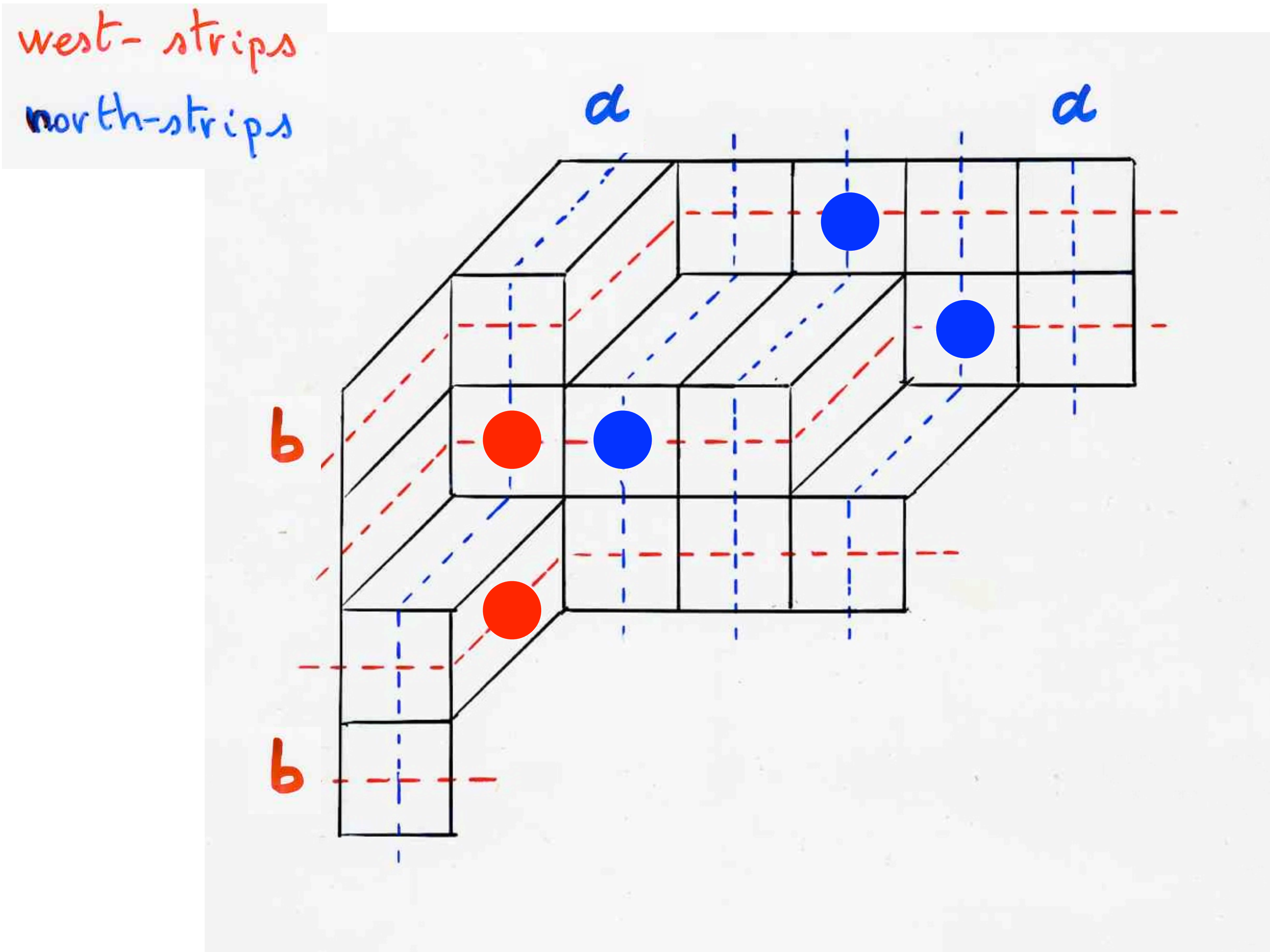
$wt(T) = q^{15} a^{2} b^{2}$

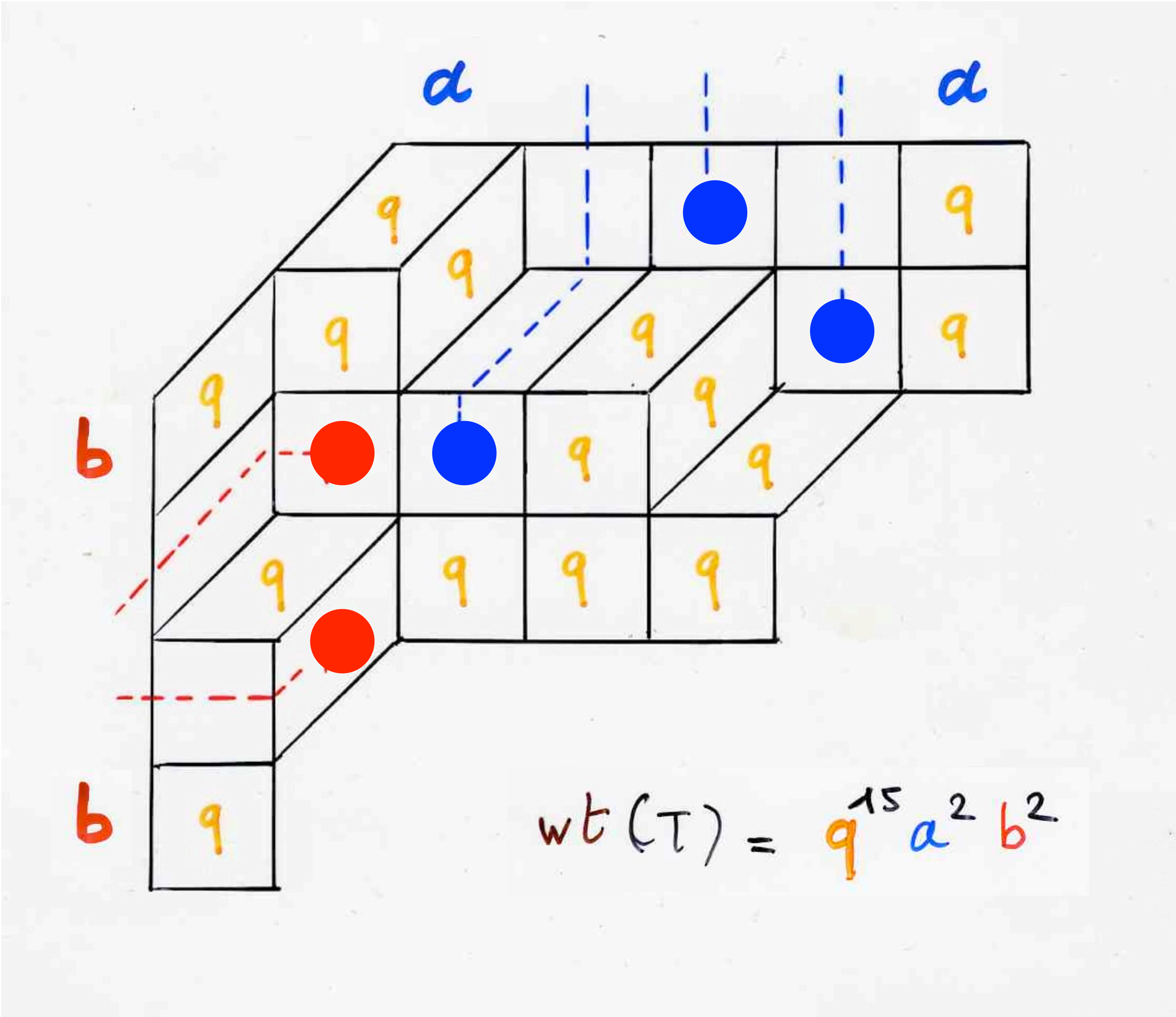


t = nb of cells not ●, not ●, not at the left of a ●, not at the north of a ●

i = nb of north-strips without a ●

j = nb of west-strips without a ●


west- strips
north-strips
a
a
b
b

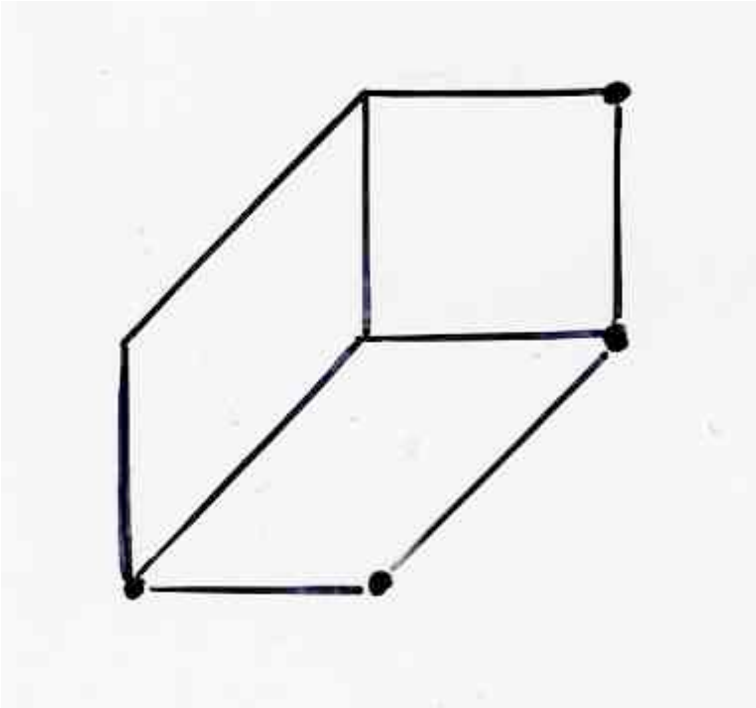
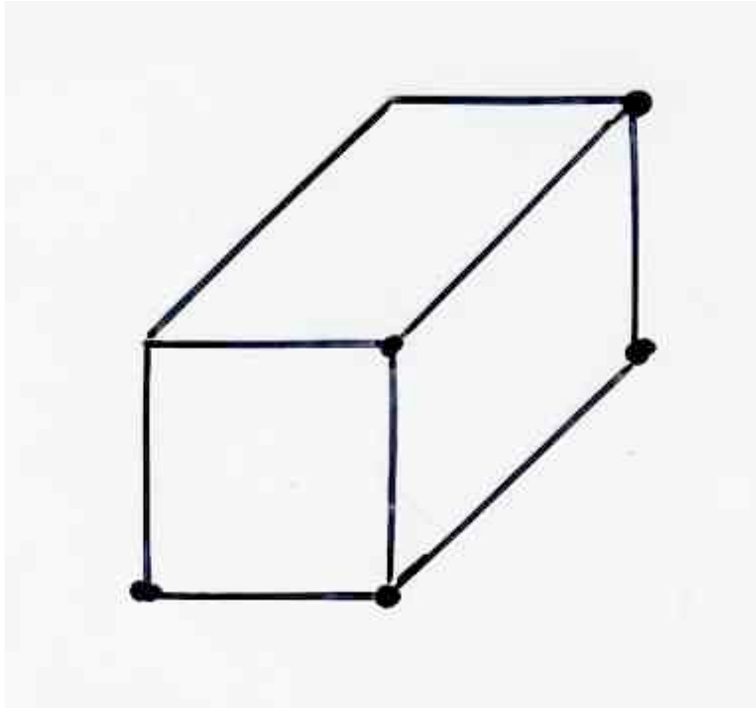

a
a
b
b
q
wt(T) = q^15 a^2 b^2

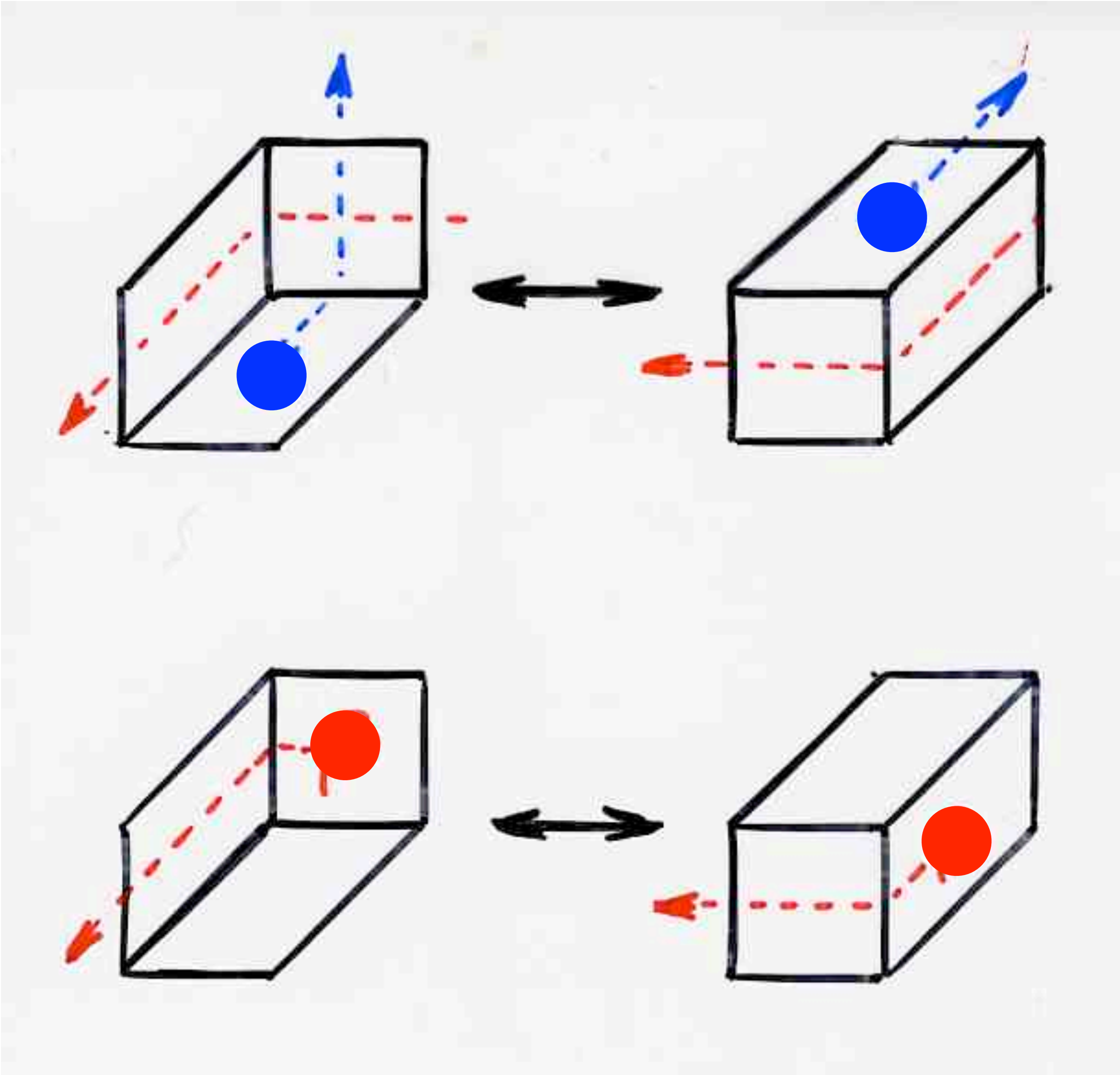
$R(X, \mathcal{T})$ set of rhombic alternative tableaux related to X, with the tiling $\mathcal{T}$ of $\Gamma(X)$

<u>Prop</u> X, $\Gamma(X)$ diagram

$\mathcal{T}$, $\mathcal{T}'$ tiling of $\Gamma(X)$

$$\sum_{T \in R(X, \mathcal{T})} wt(T) = \sum_{T \in R(X, \mathcal{T}')} wt(T)$$



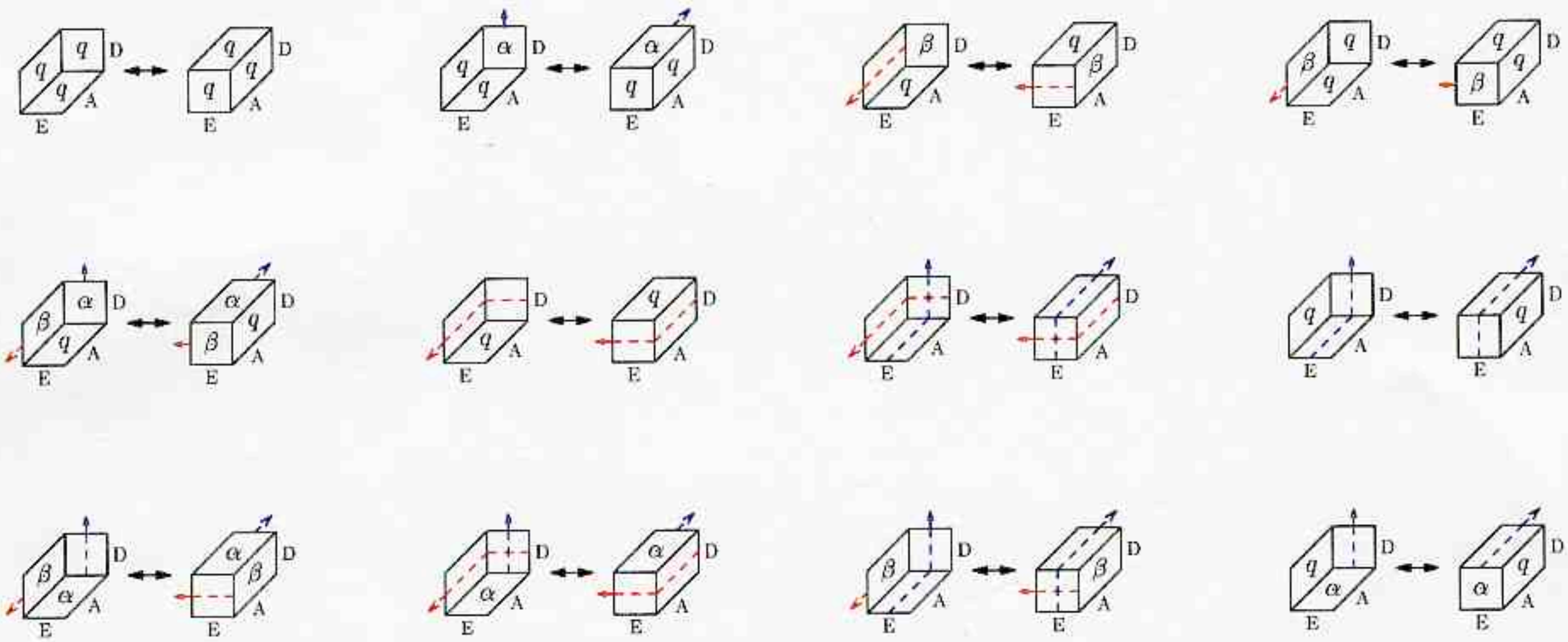


Figure 11: The involution ϕ from each possible filling of a minimal hexagon (left) to a maximal hexagon (right). The arrows imply compatibility requirements.

combinatorial interpretation of stationary probabilities

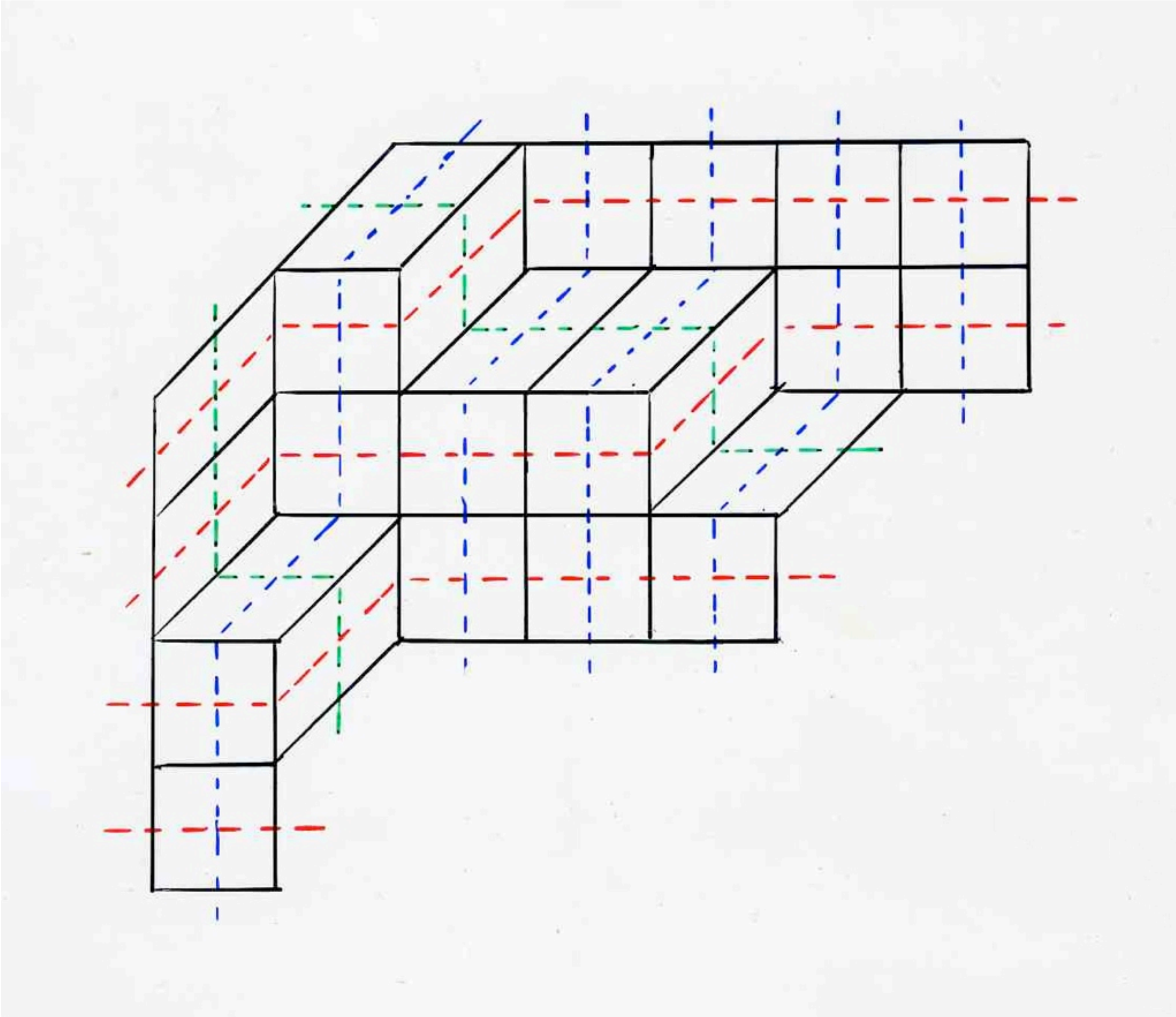
$$\mathrm{Prob}(X) = \frac{1}{Z^*_{n,r}} \sum_{T \in R(X, \tau_X)} \underbrace{q^t \left(\frac{1}{\alpha}\right)^i \left(\frac{1}{\beta}\right)^j}_{wt(T)}$$

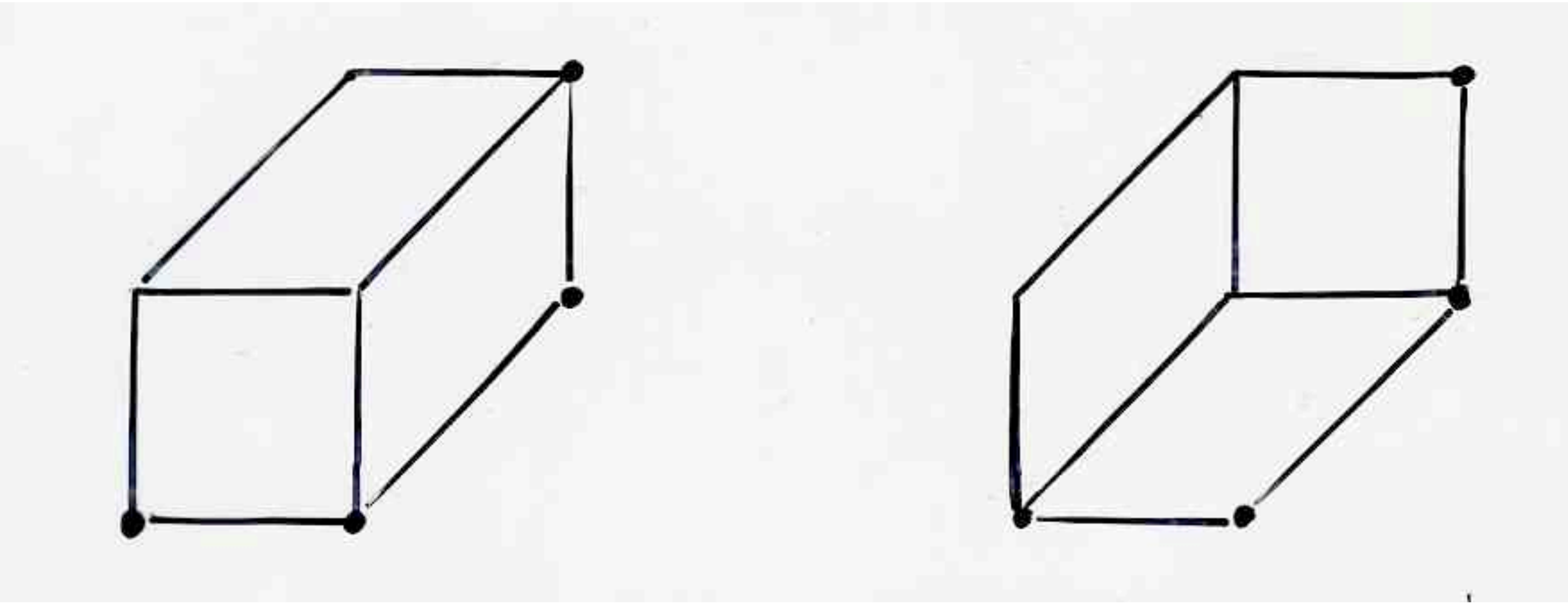
$$a = \frac{1}{\alpha}$$

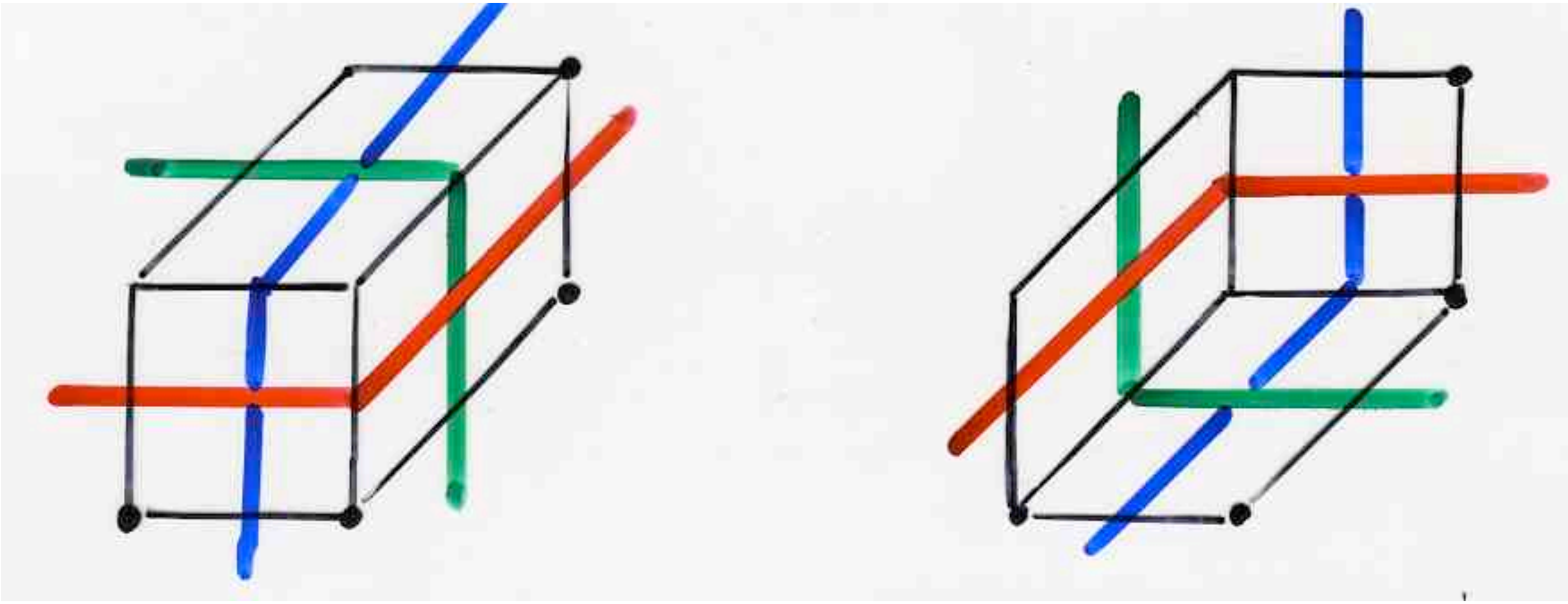
$$b = \frac{1}{\beta}$$

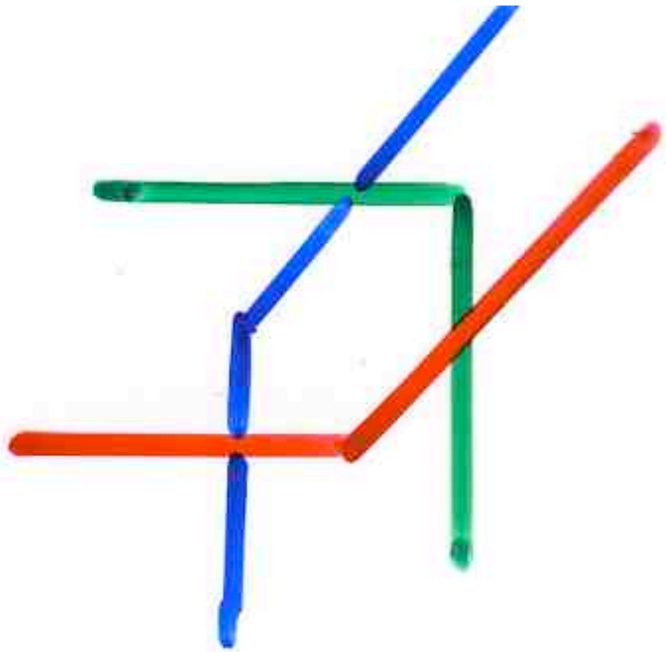
$$Z^*_{n,r} = \sum_{X} \sum_{T \in R(X, T_X)} q^t \left(\frac{1}{\alpha}\right)^i \left(\frac{1}{\beta}\right)^j$$

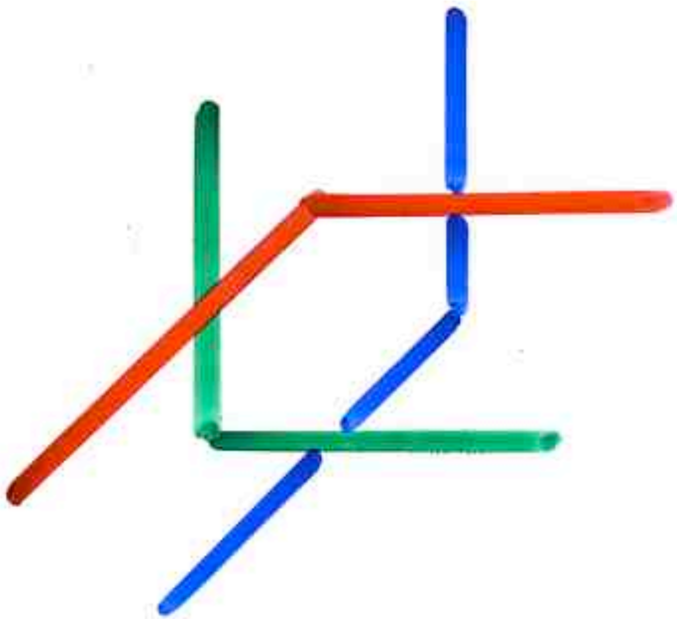
quelques remarques







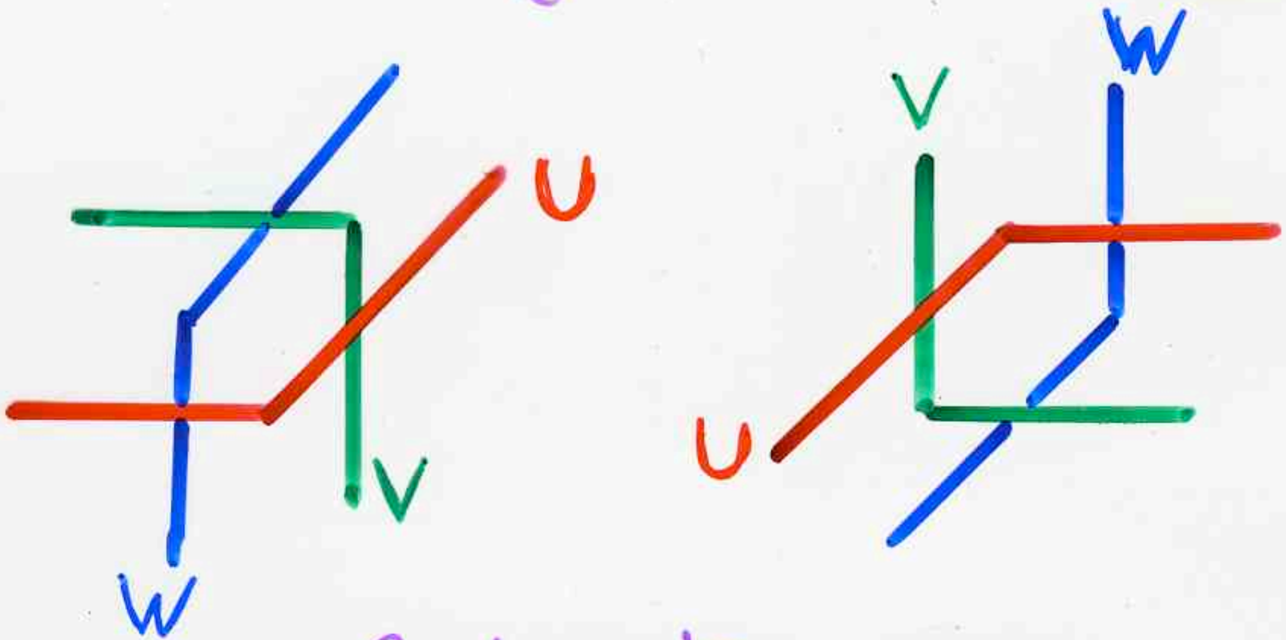




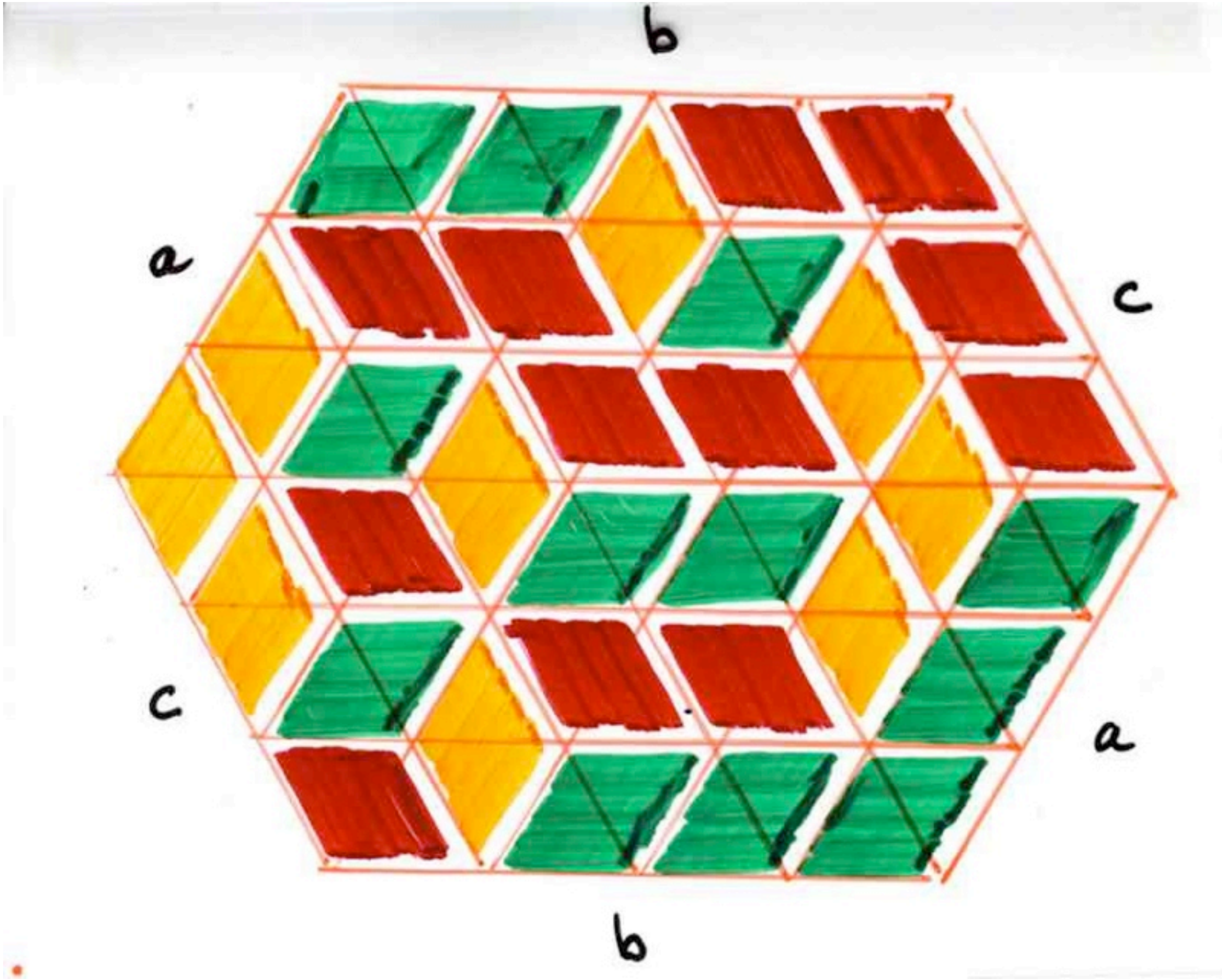
Yang-Baxter equation

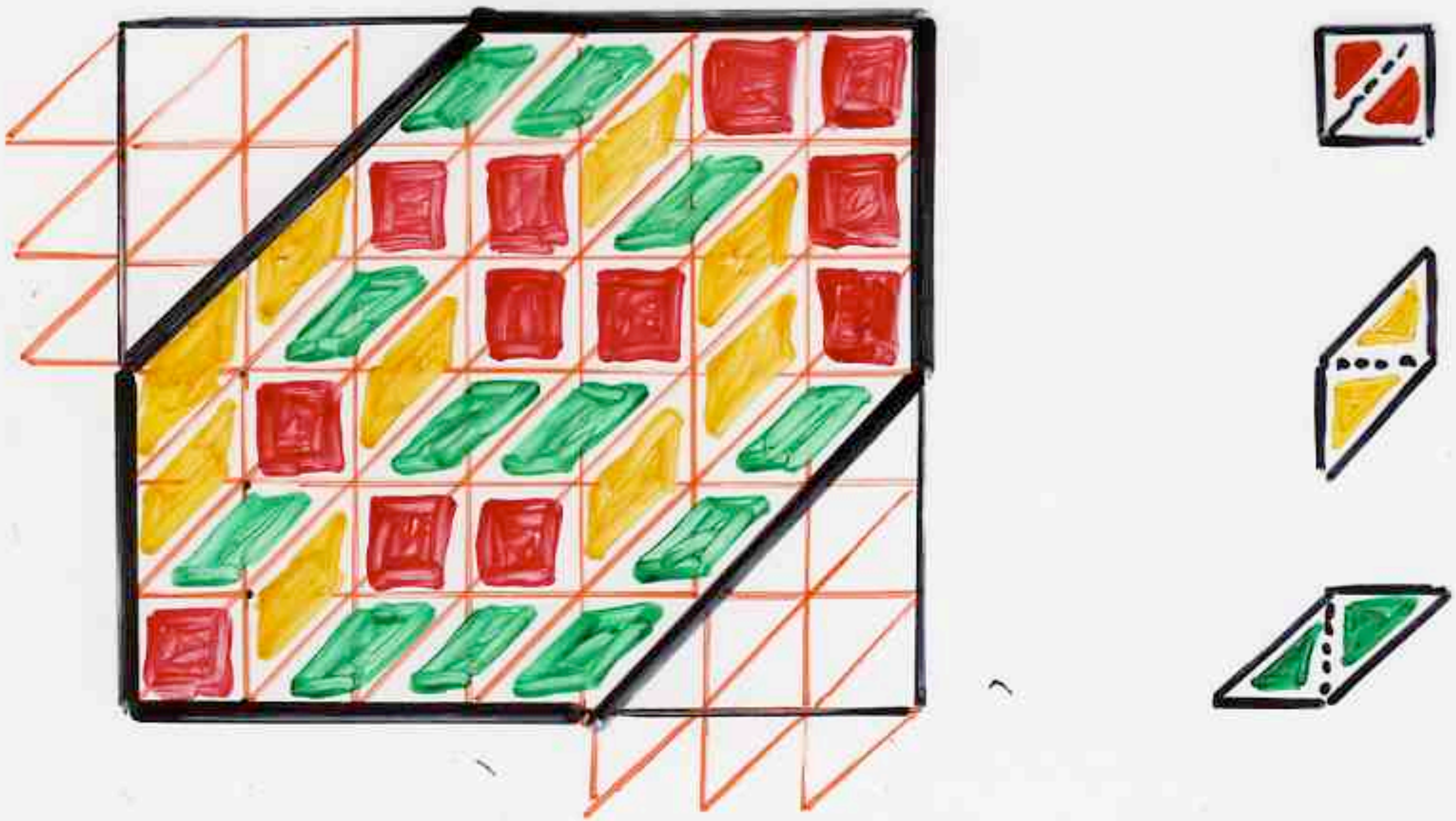
$$UVW = WVU$$

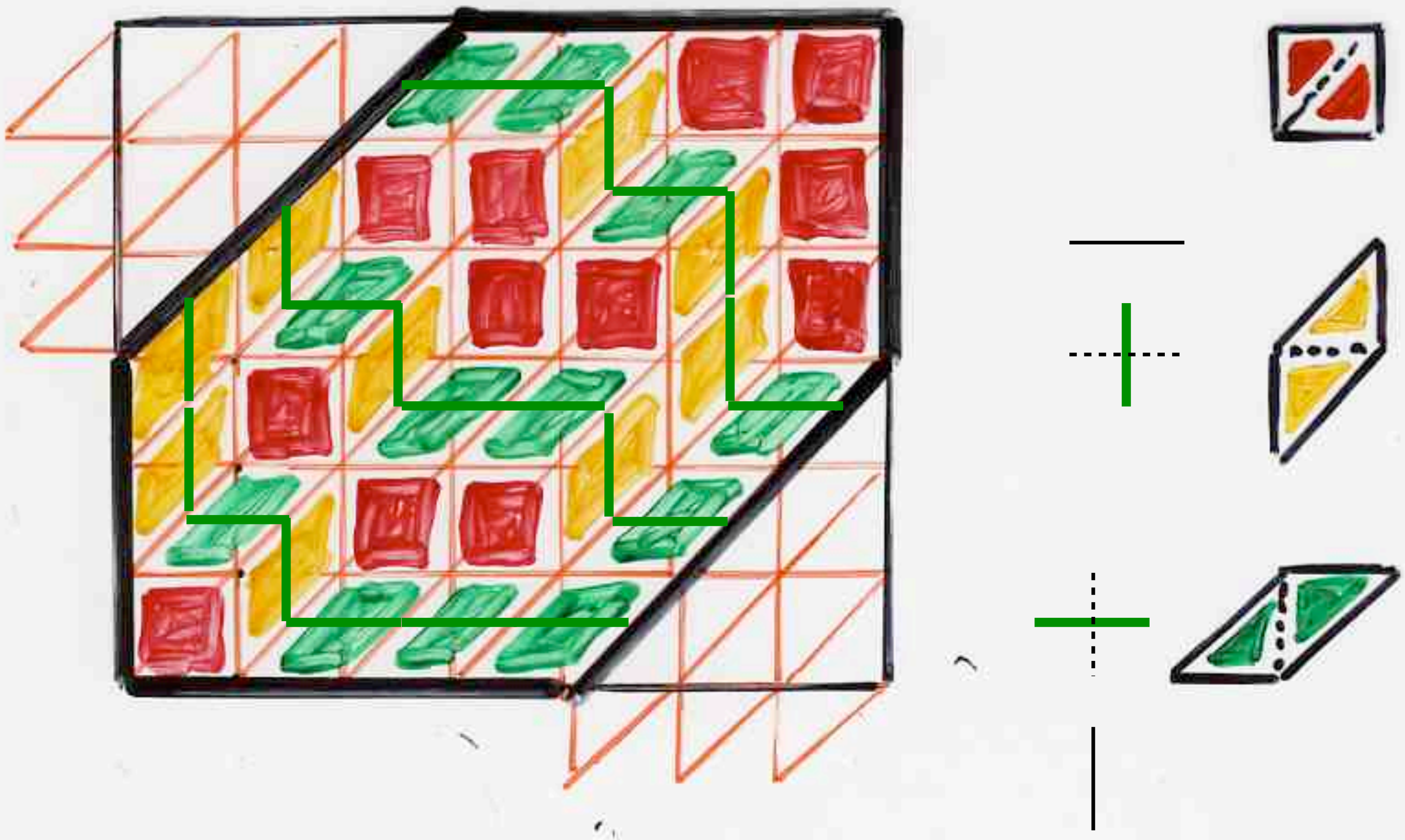
Yang-Baxter moves

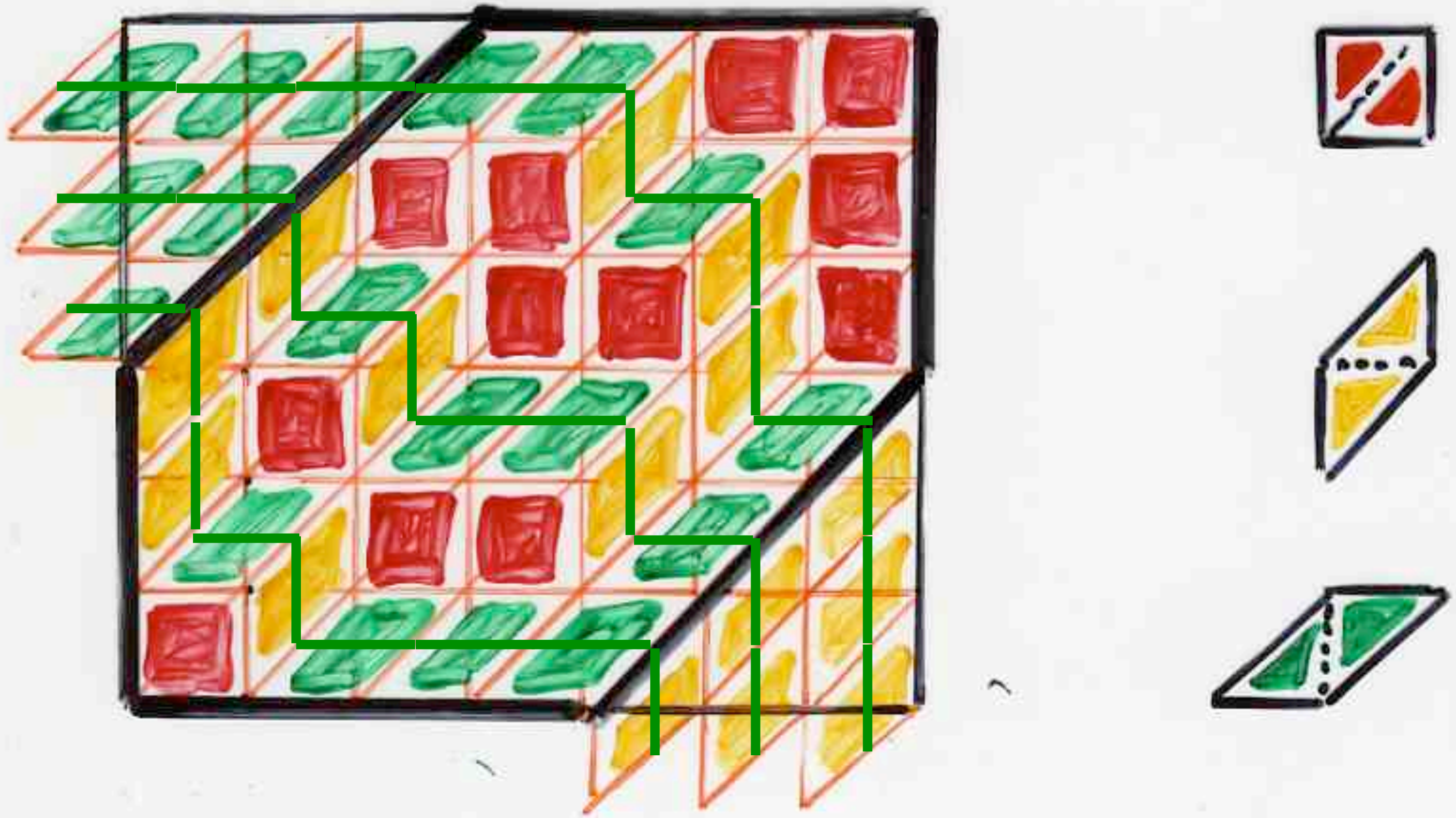


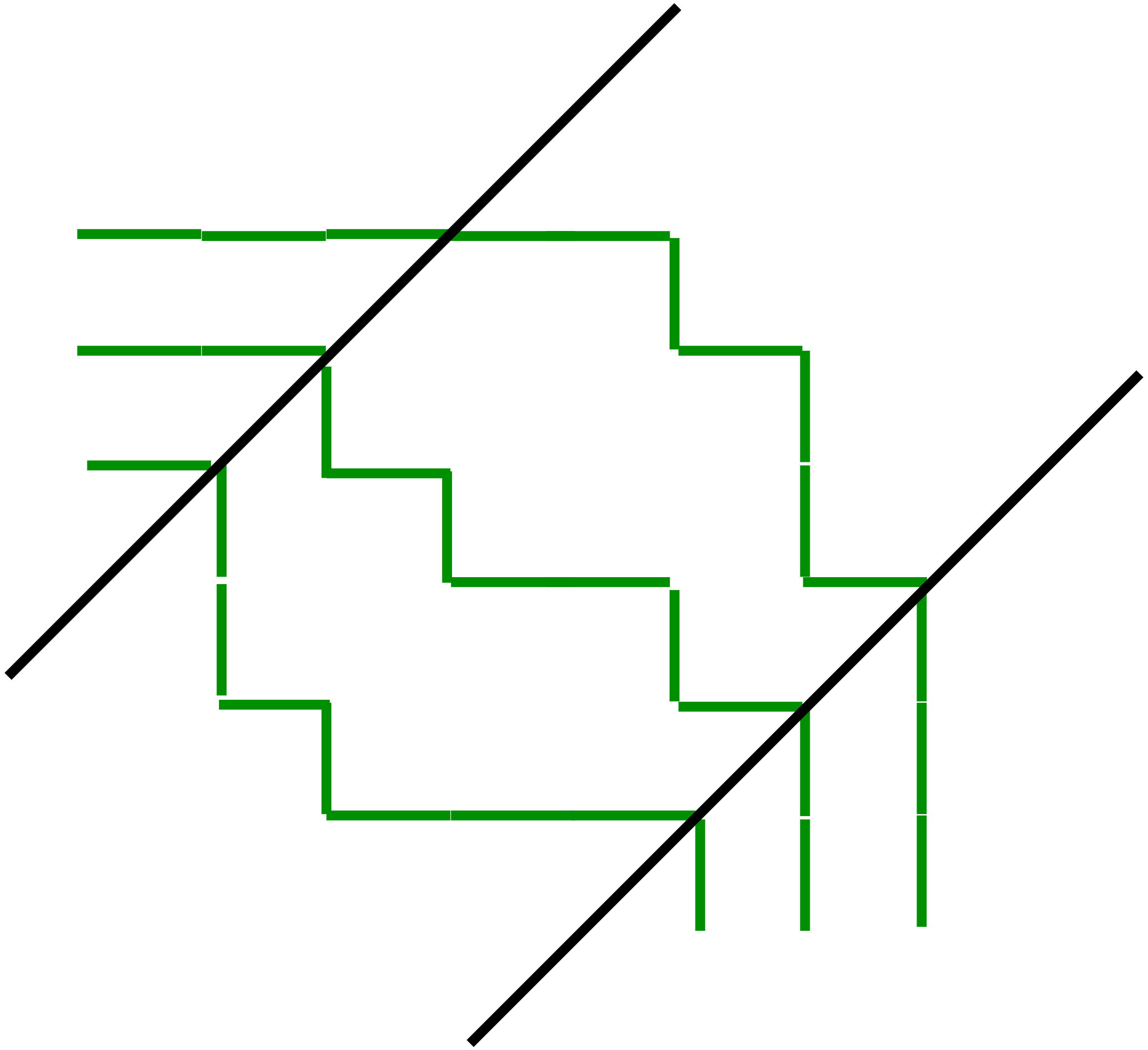
Reidemeister moves
(knot theory)


b
a
c
c
a
b

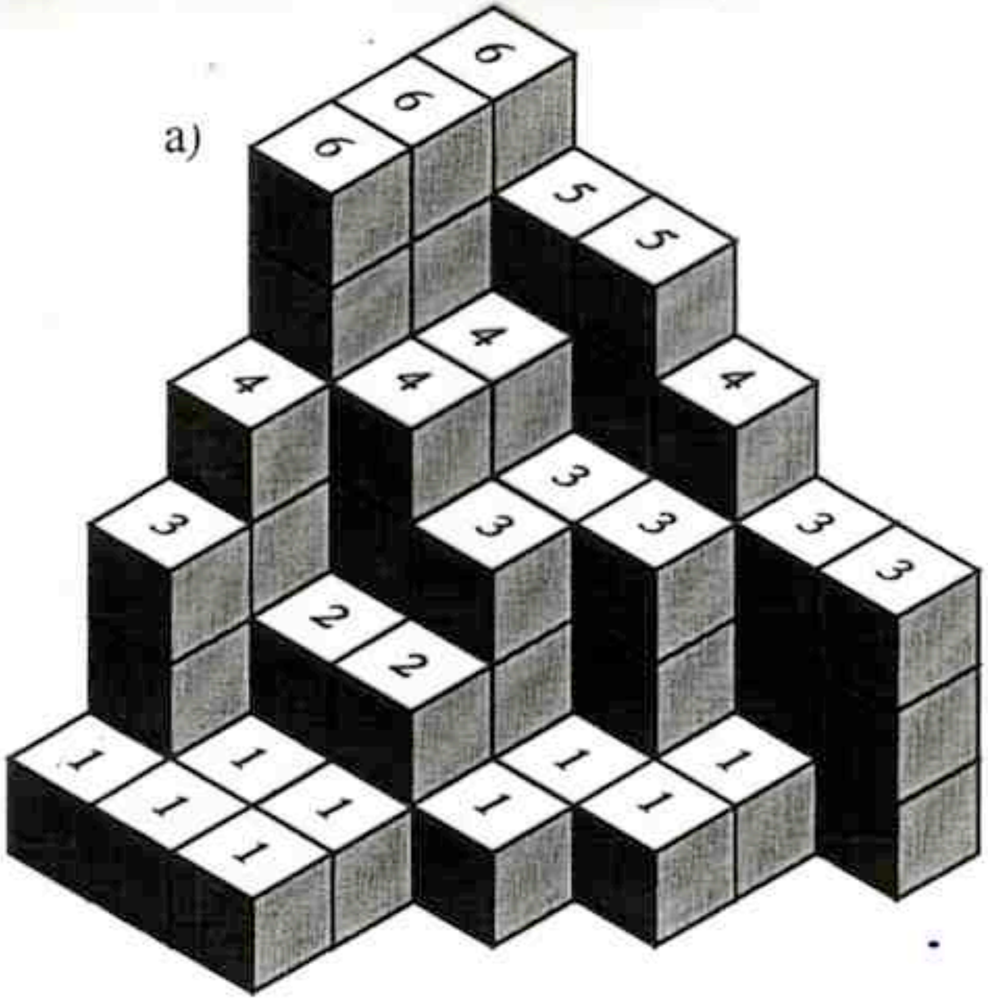






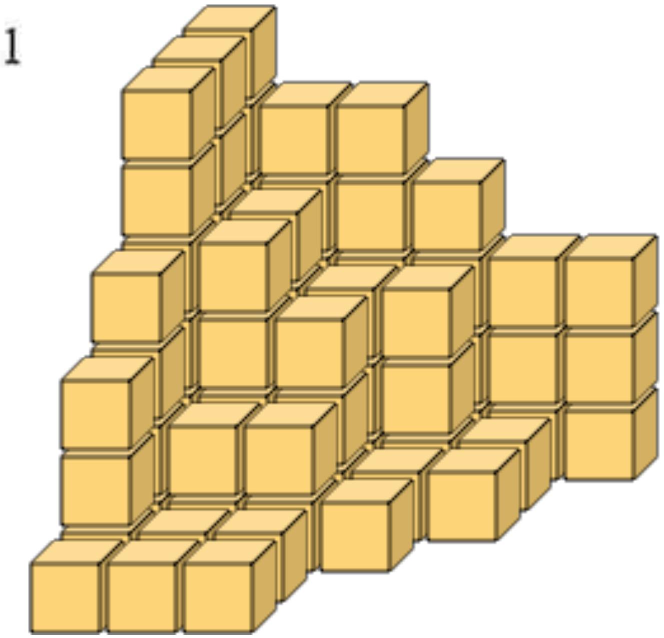


a)



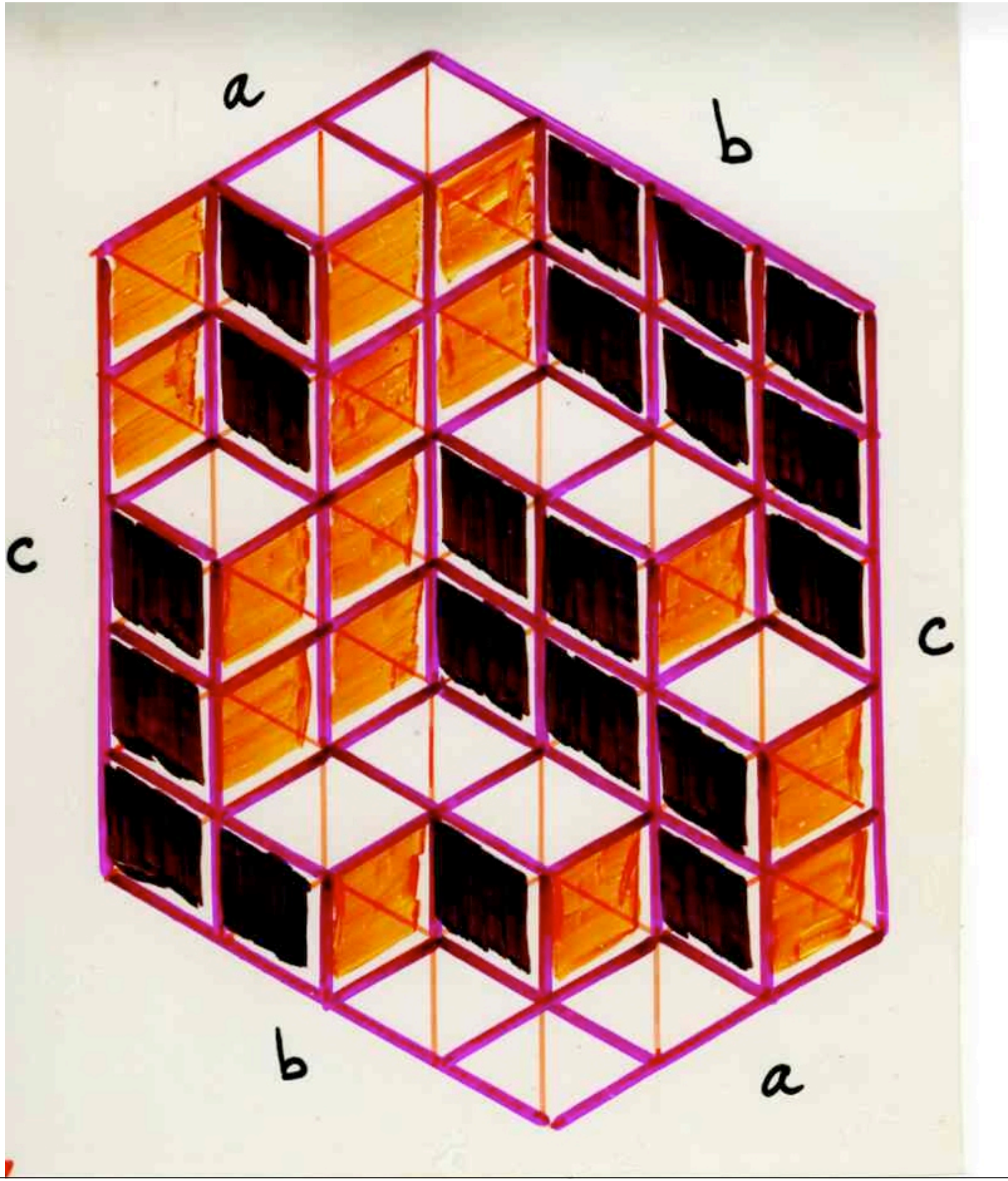
b)

6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			



example: plane partitions in a box

(MacMahon formula)

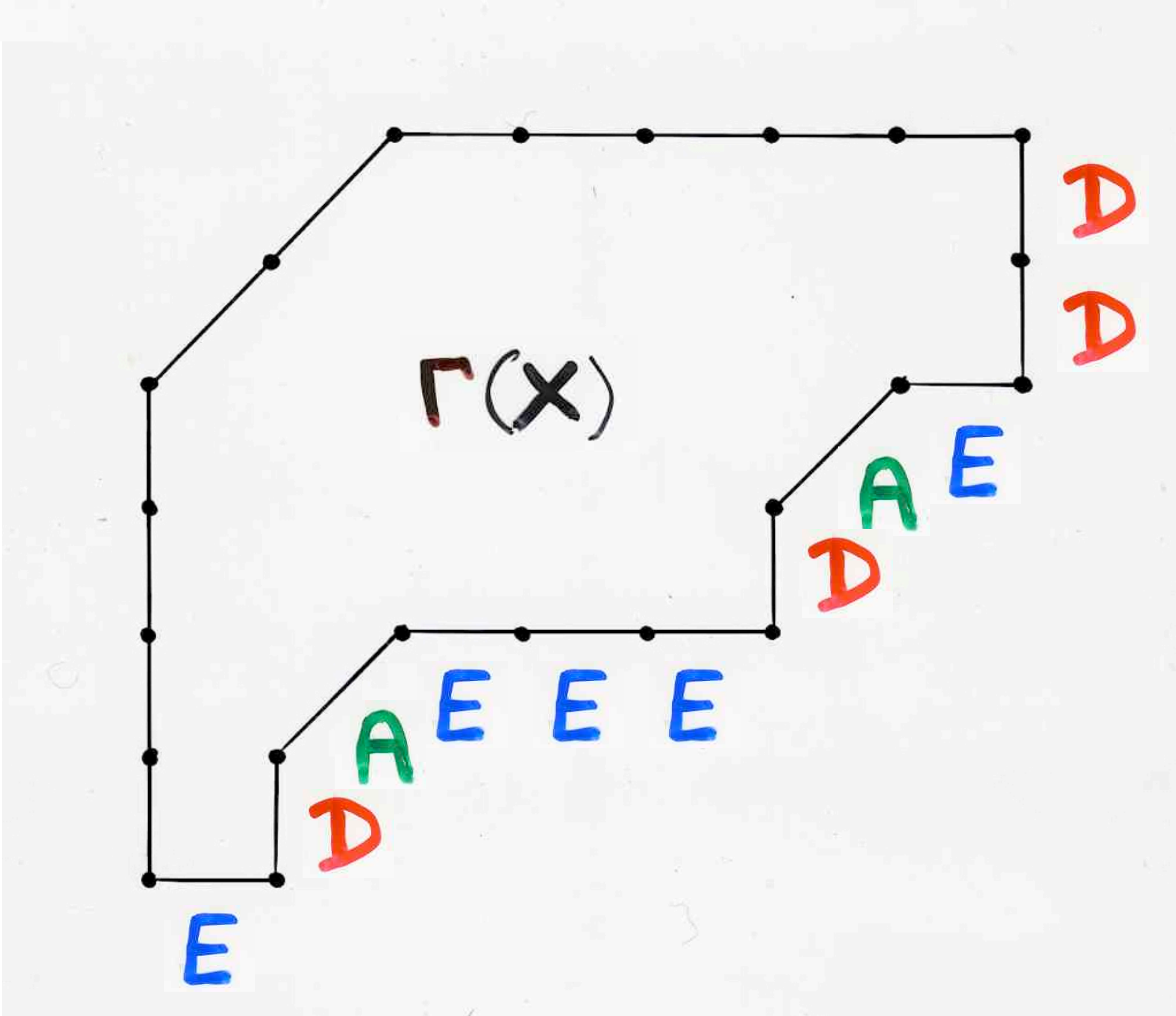


$$\prod_{\substack{1 \le i \le a \\ 1 \le j \le b \\ 1 \le k \le c}} \frac{i+j+k-1}{i+j+k-2}$$

2-PASEP algebra

$$\begin{cases} DE = qED + D + E \\ DA = qAD + A \\ AE = qEA + A \end{cases}$$

X = D D E A D E E E A D E



In the 2-PASEP algebra

Every word $X \in \{D, E, A\}^*$ can be expressed in a unique way

$$X = \sum_{T \in R(X, \mathcal{T})} q^t E^i A^r D^j$$

where:

$r = |X|_A$ (nb of A in the word X)

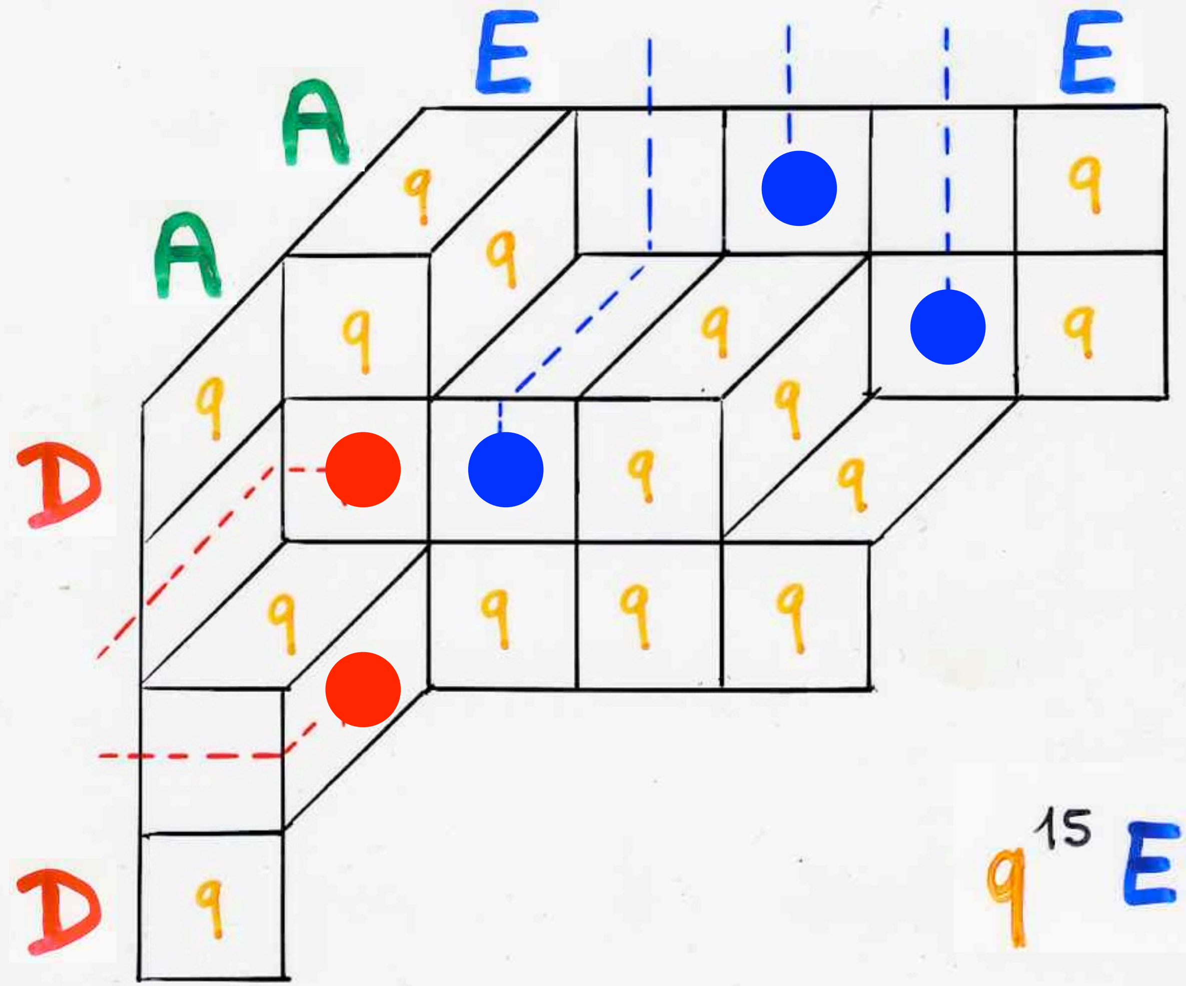
$\mathcal{T}$ a fixed tiling of $\Gamma(X)$

i = nb of free north-strips in T (not containing an ●)

j = nb of free south-strips in T (not containing a ●)

t = nb of cells labeled q in T

D D E A D E E E A D E



$$q^{15}E^{2}A^{2}D^{2}$$

$$
\begin{cases}
DE = qED + D + E \\
DA = qAD + A \\
AE = qEA + A
\end{cases}
$$

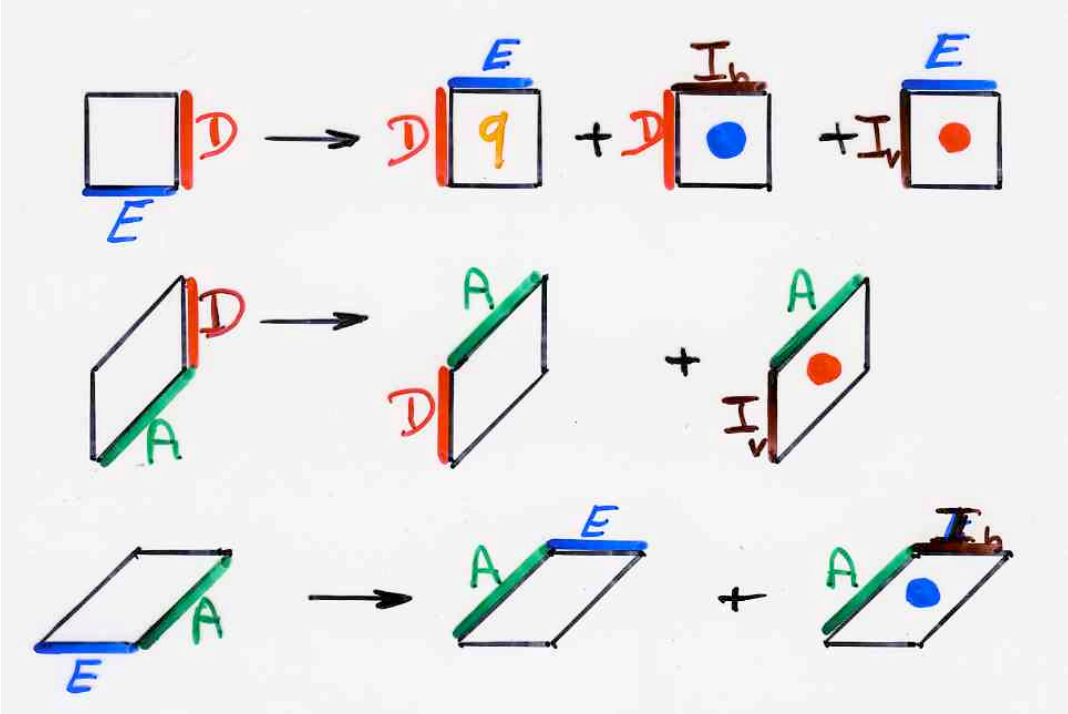
rewriting rules

$DE \longrightarrow qED$ or E or D

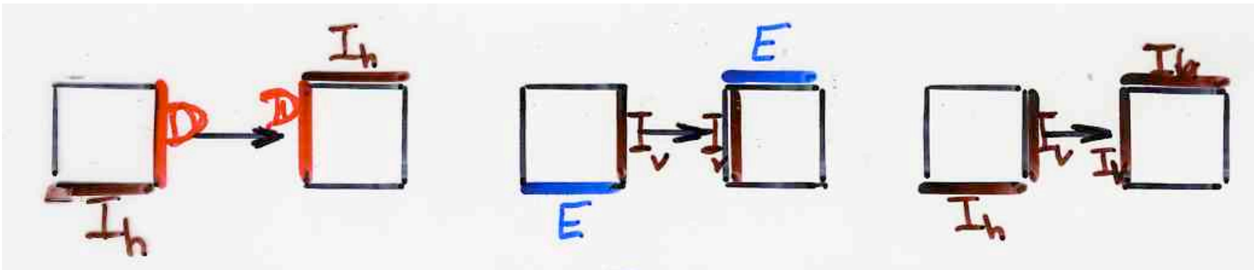
$DA \longrightarrow AD$ or A

$AE \longrightarrow EA$ or A

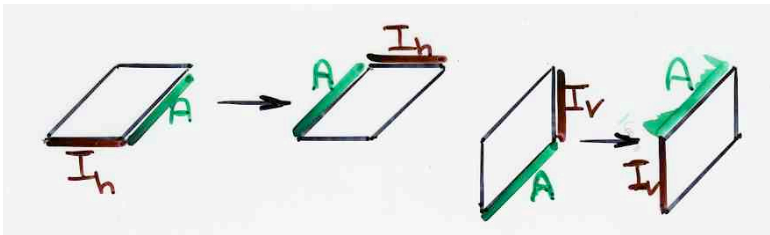
Planarisation of the rewriting rules



E
D
D
E
I_h
D
E
I_v
D
A
A
D
A
I_v
E
A
E
A
I_h
A



D
I_h
D
I_h
I_v
E
I_v
E
I_v
I_h
I_v
I_h

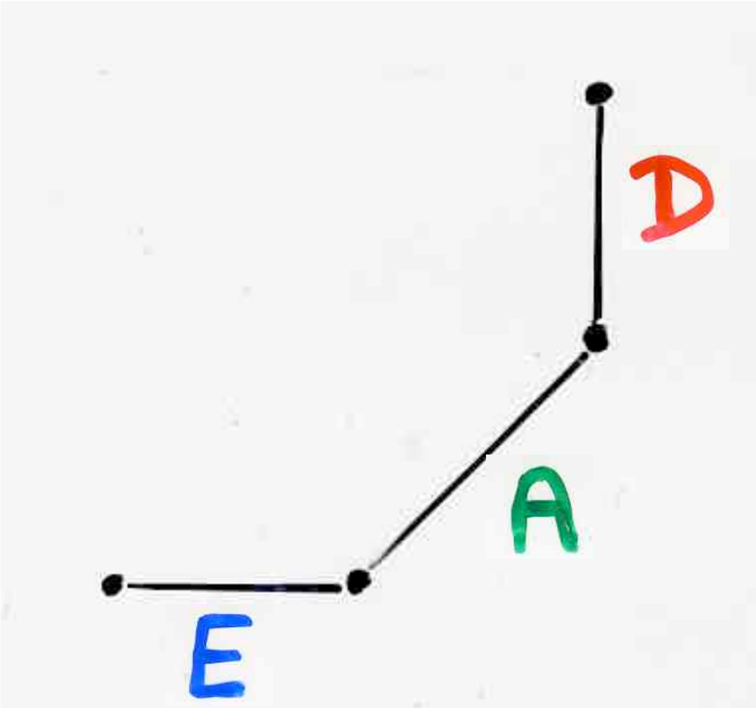


A
I_h
A
I_h
I_v
A
A
I_v

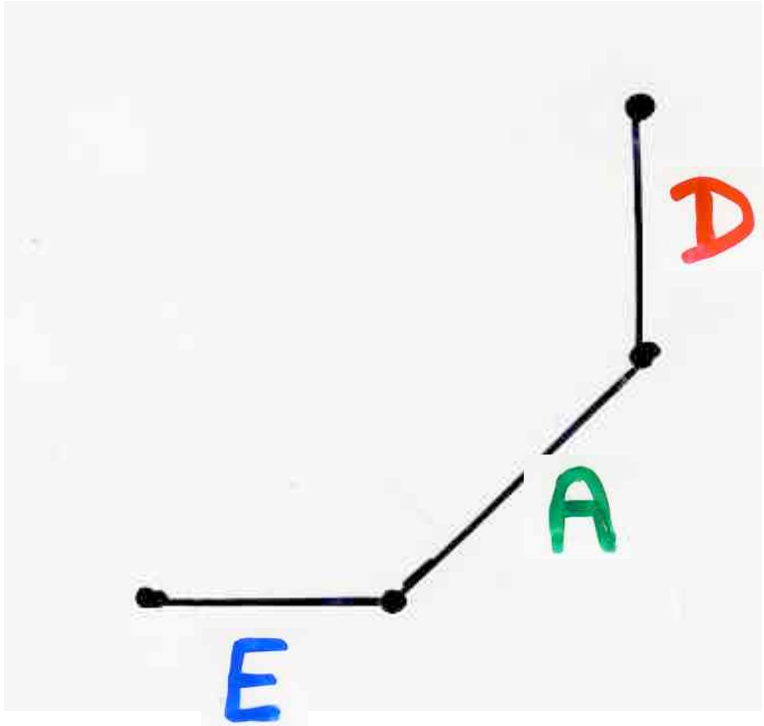
$$\begin{cases} DE = qED + I_v D + E I_v \\ DA = qAD + A I_v \\ AE = qEA + I_h A \end{cases}$$

$$\begin{cases} D I_h = I_h D \\ I_v E = E I_v \\ I_v I_h = I_h I_v \\ A I_h = I_h A \\ I_v A = A I_v \end{cases}$$

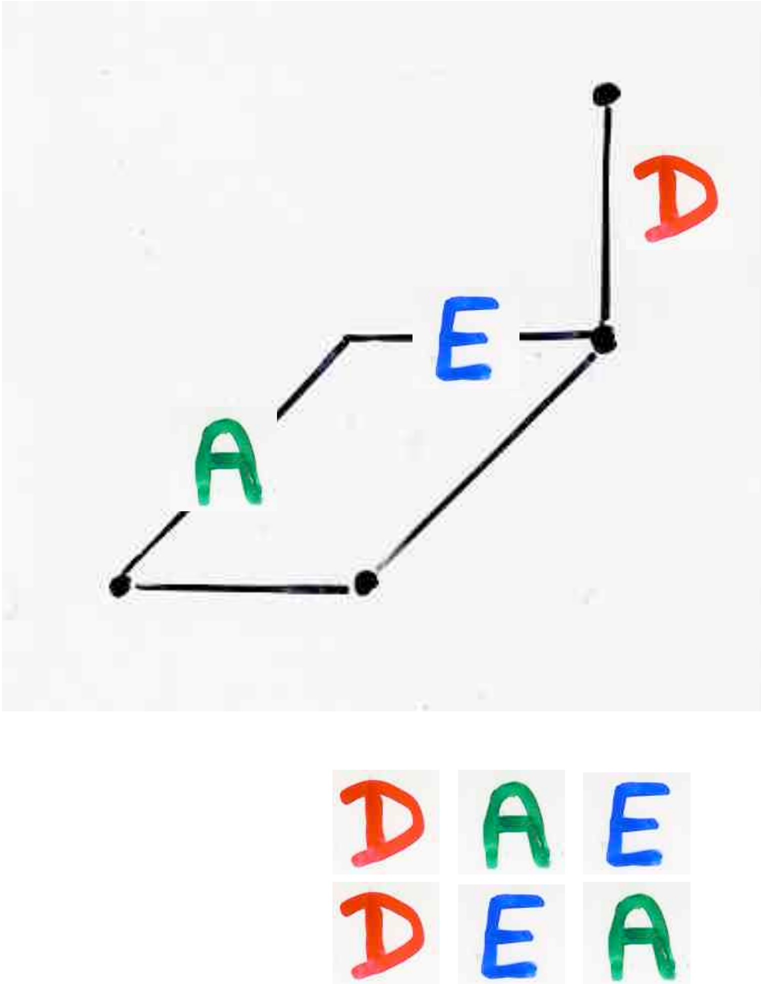
an example

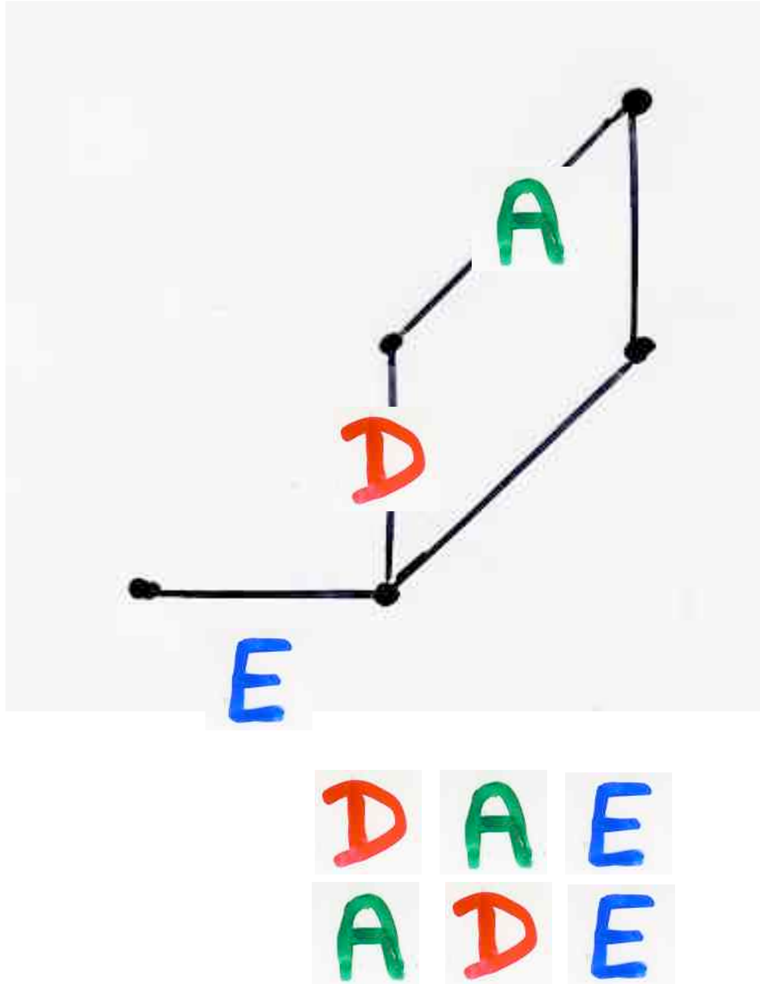


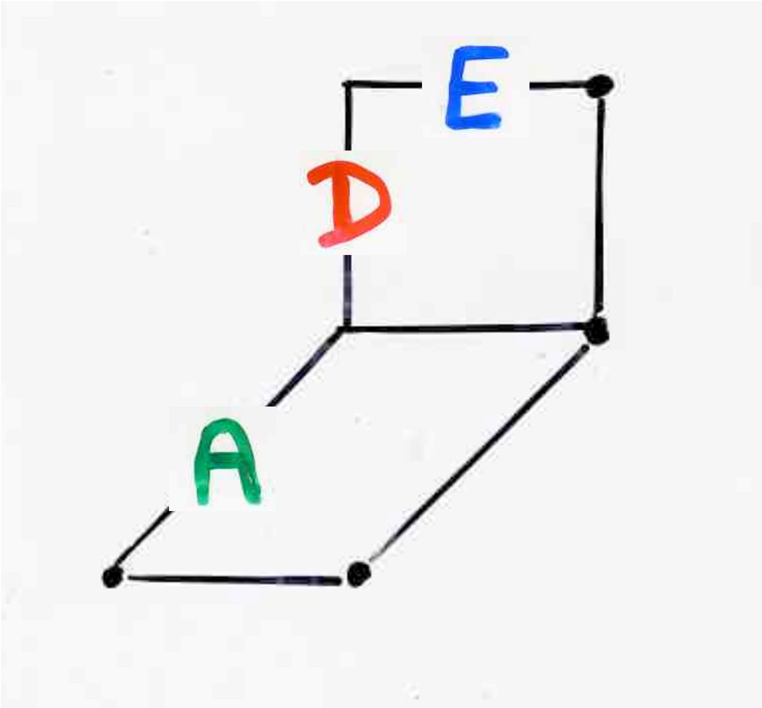
D A E

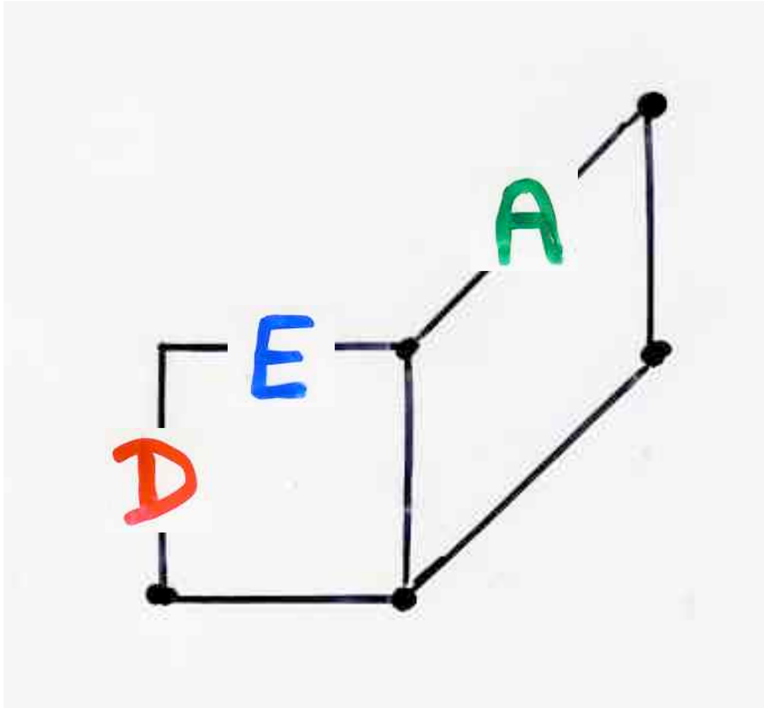


D A E


D
E
A
D A E
D E A

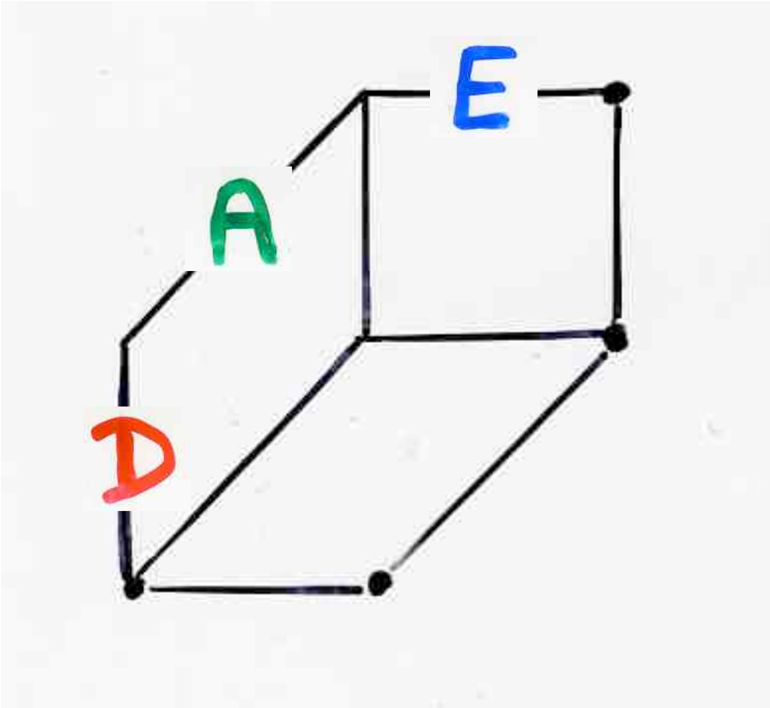

A
D
E
D A E
A D E


E
D
A

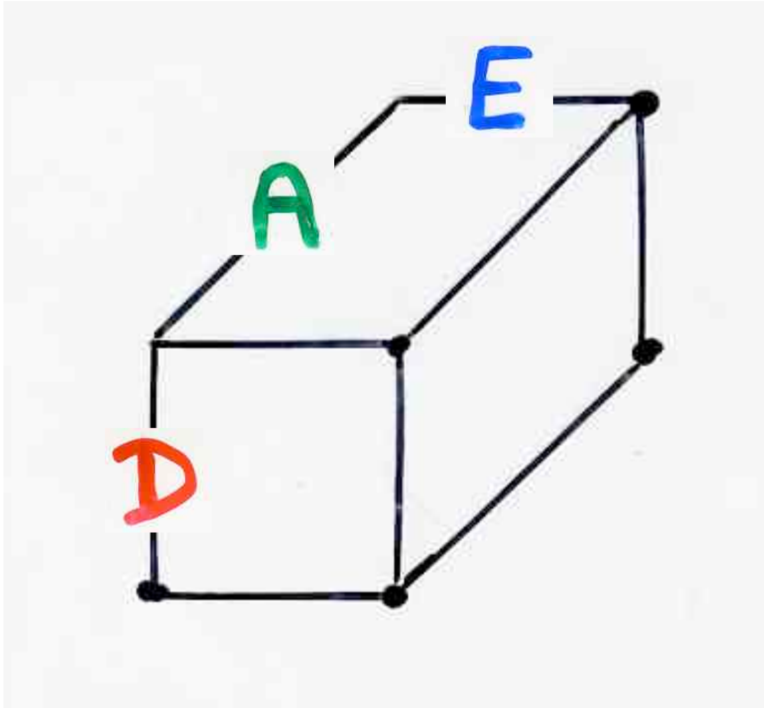

A
E
D


D A E
D E A
E D A


D A E
A D E
A E D

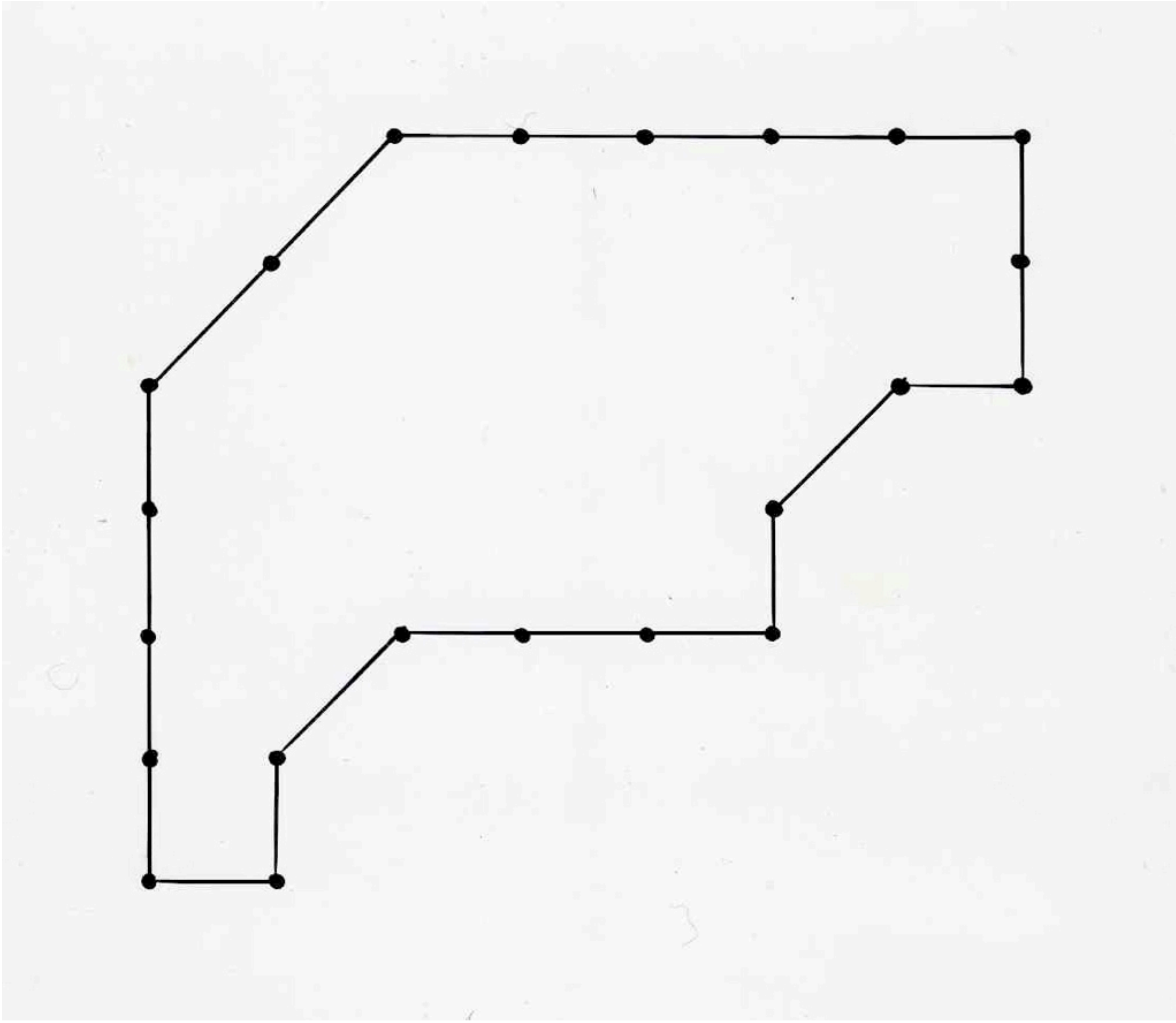


D	A	E
D	E	A
E	D	A
E	A	D

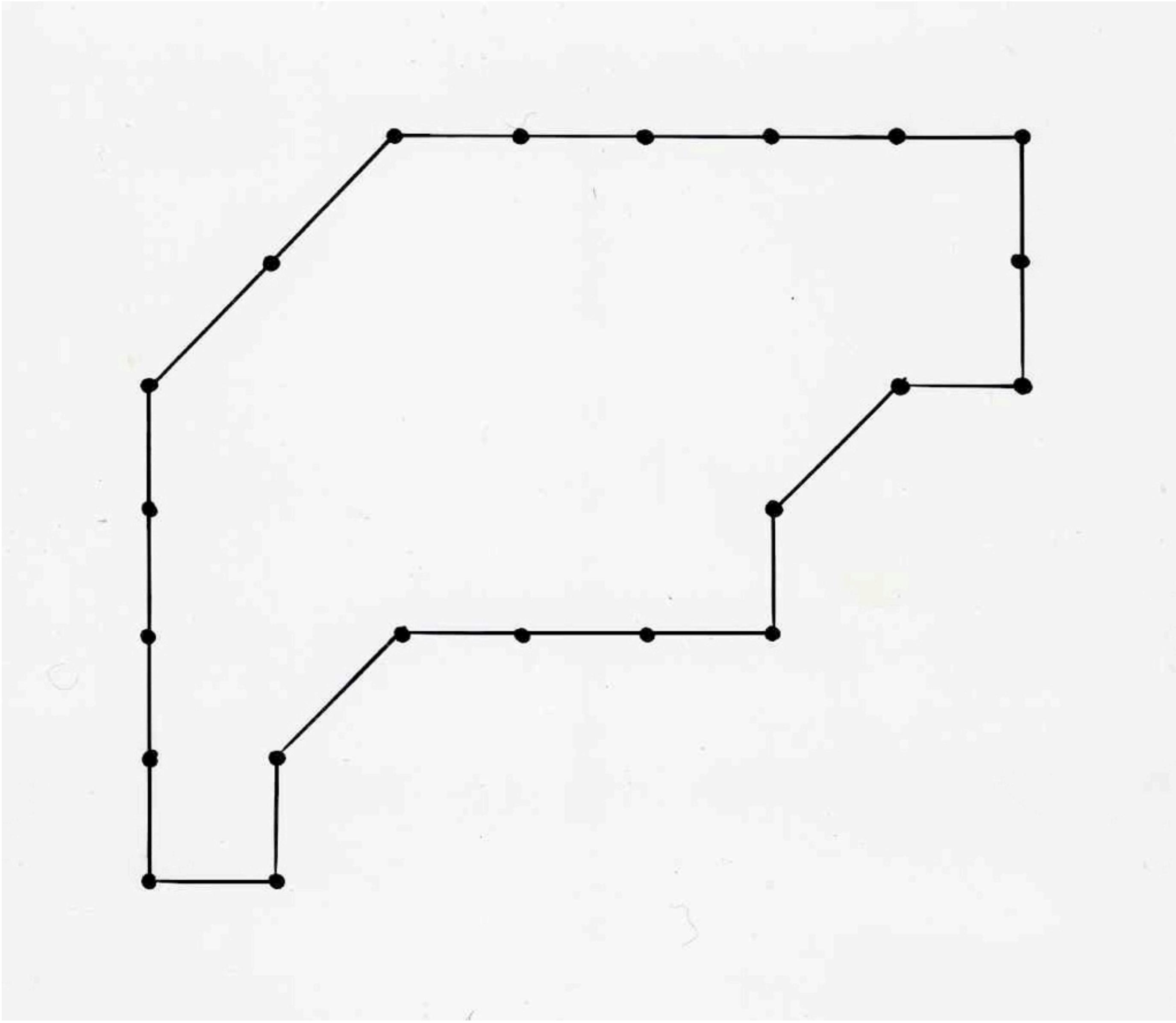


D	A	E
A	D	E
A	E	D
E	A	D

D D E A D E E E A D E



D D E A D E E E A D E



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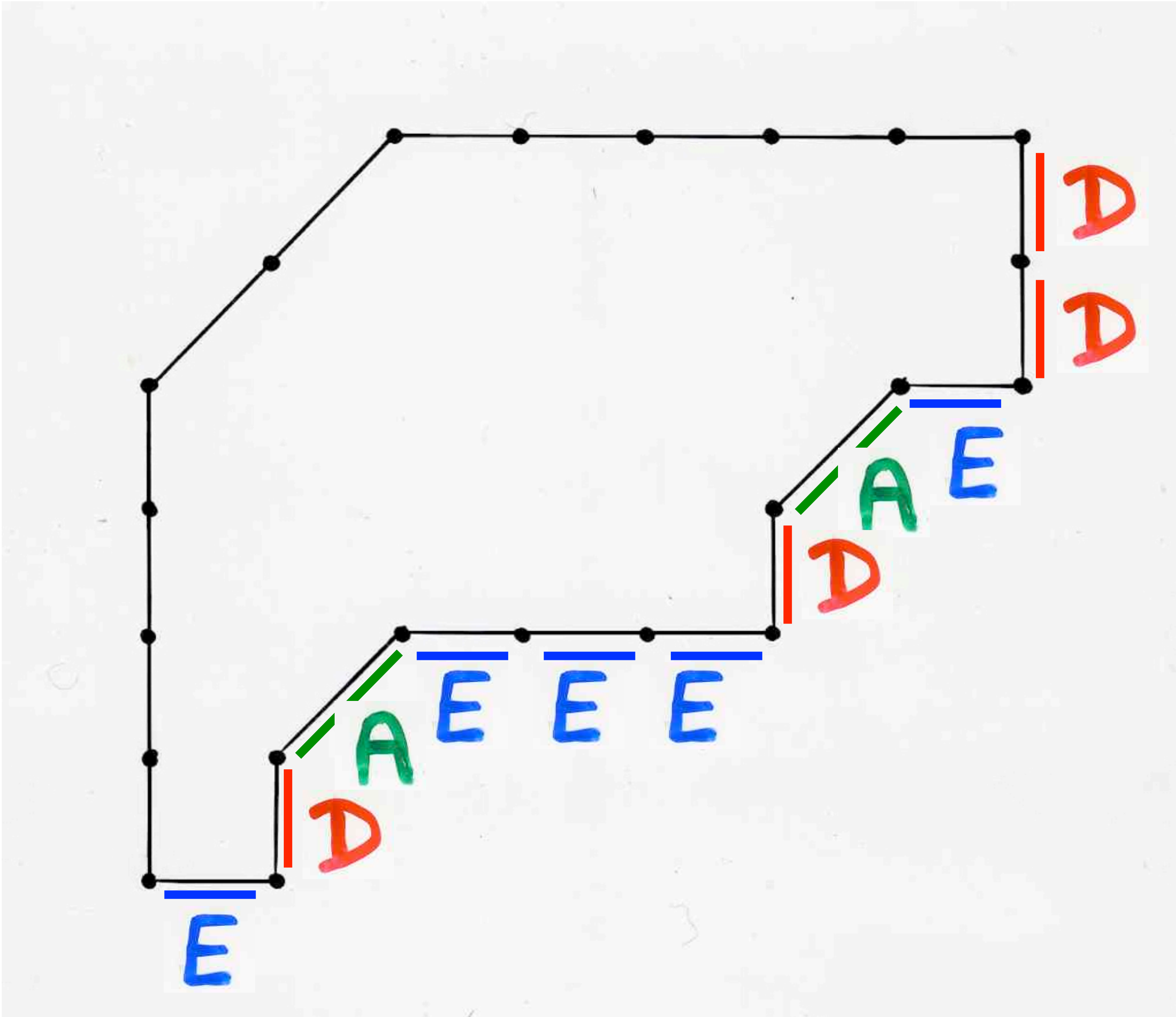

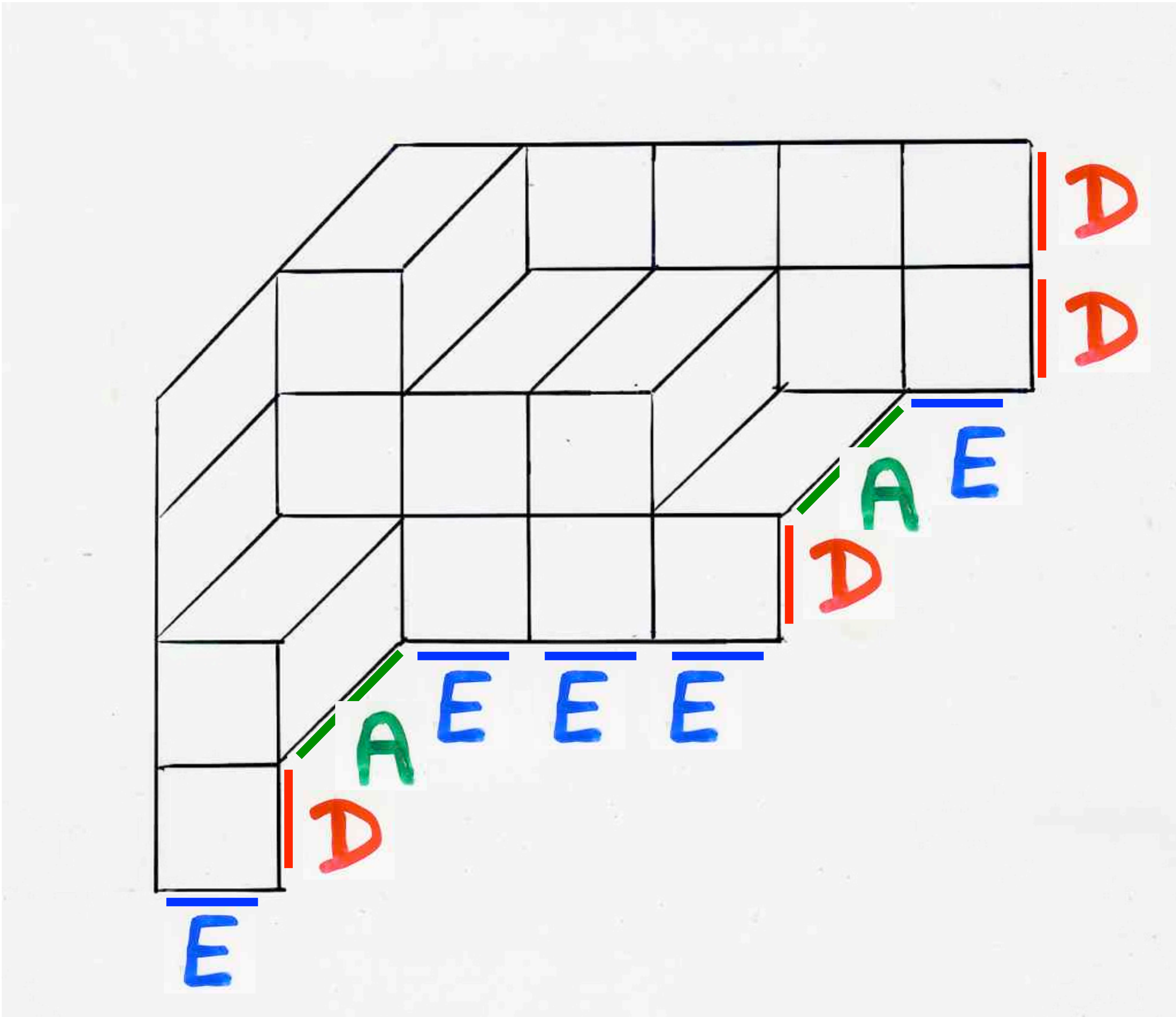
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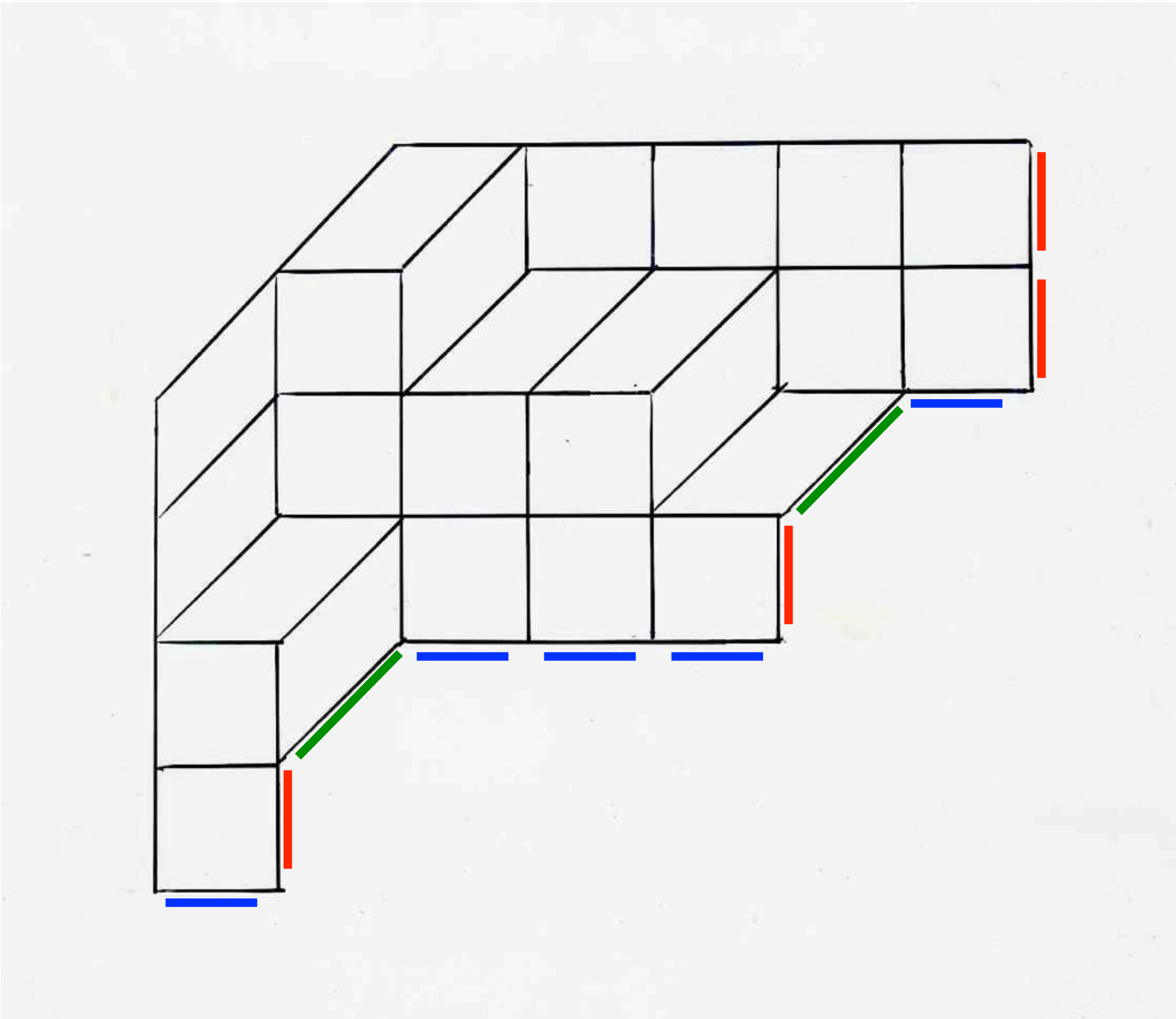

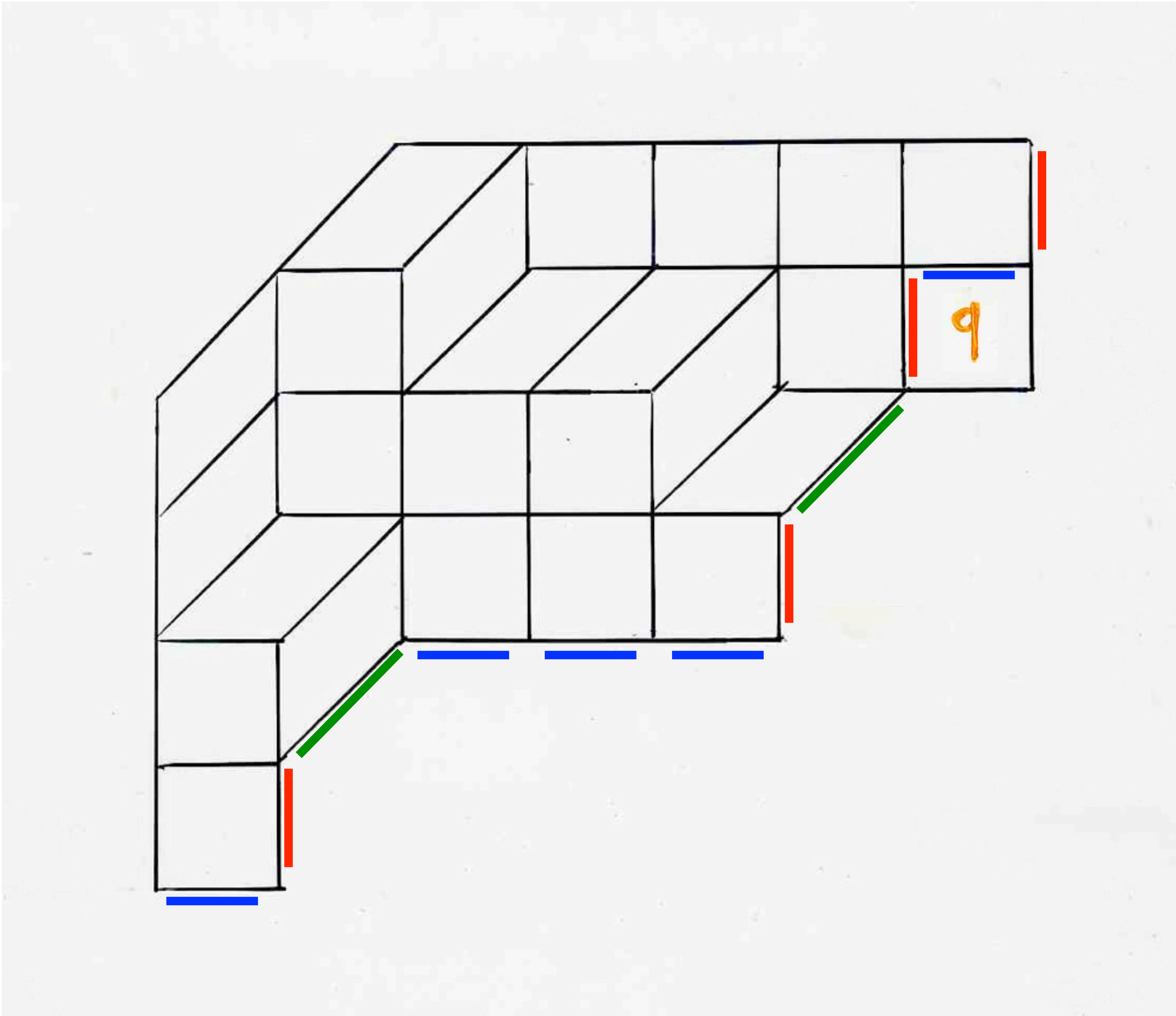
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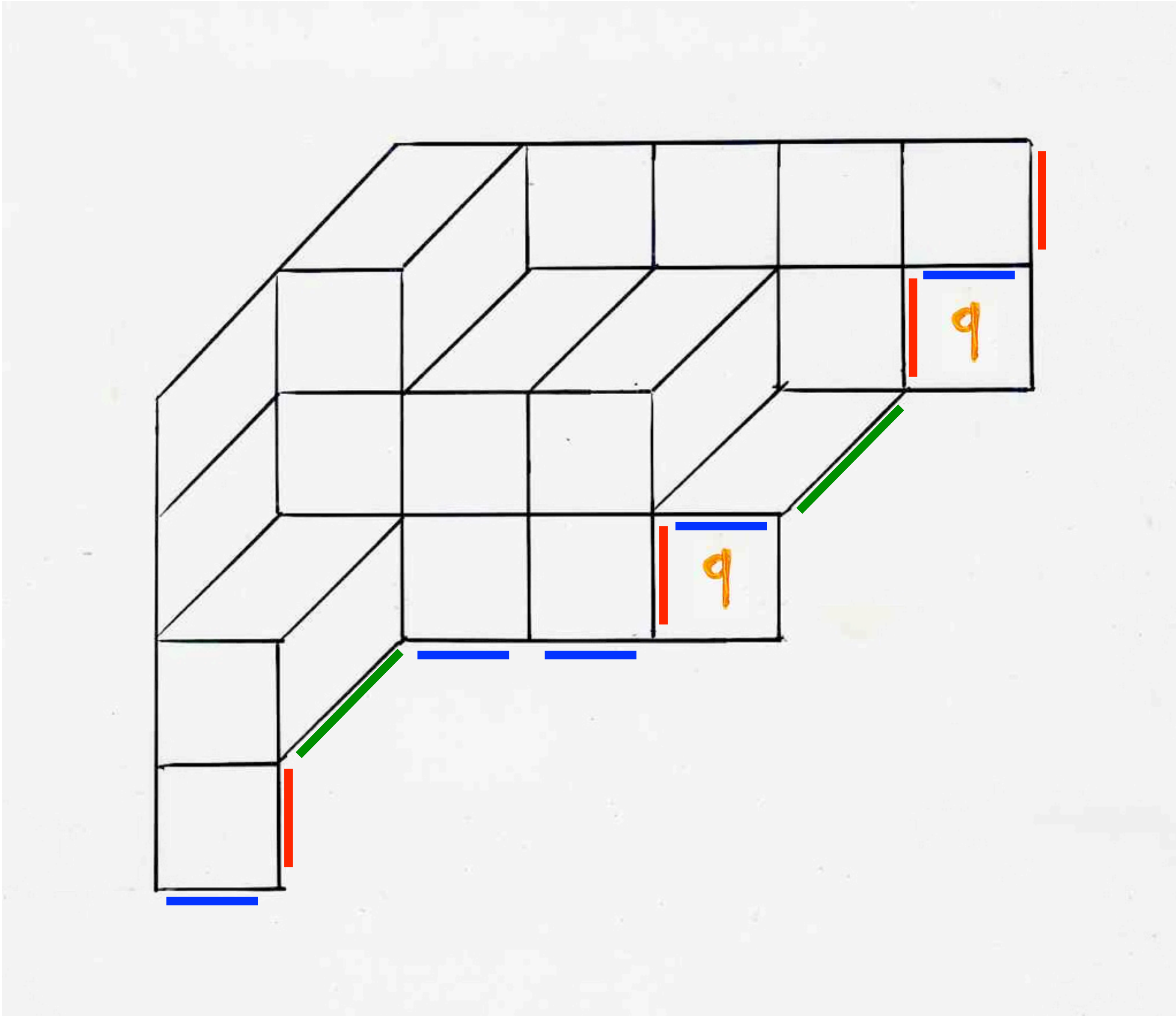
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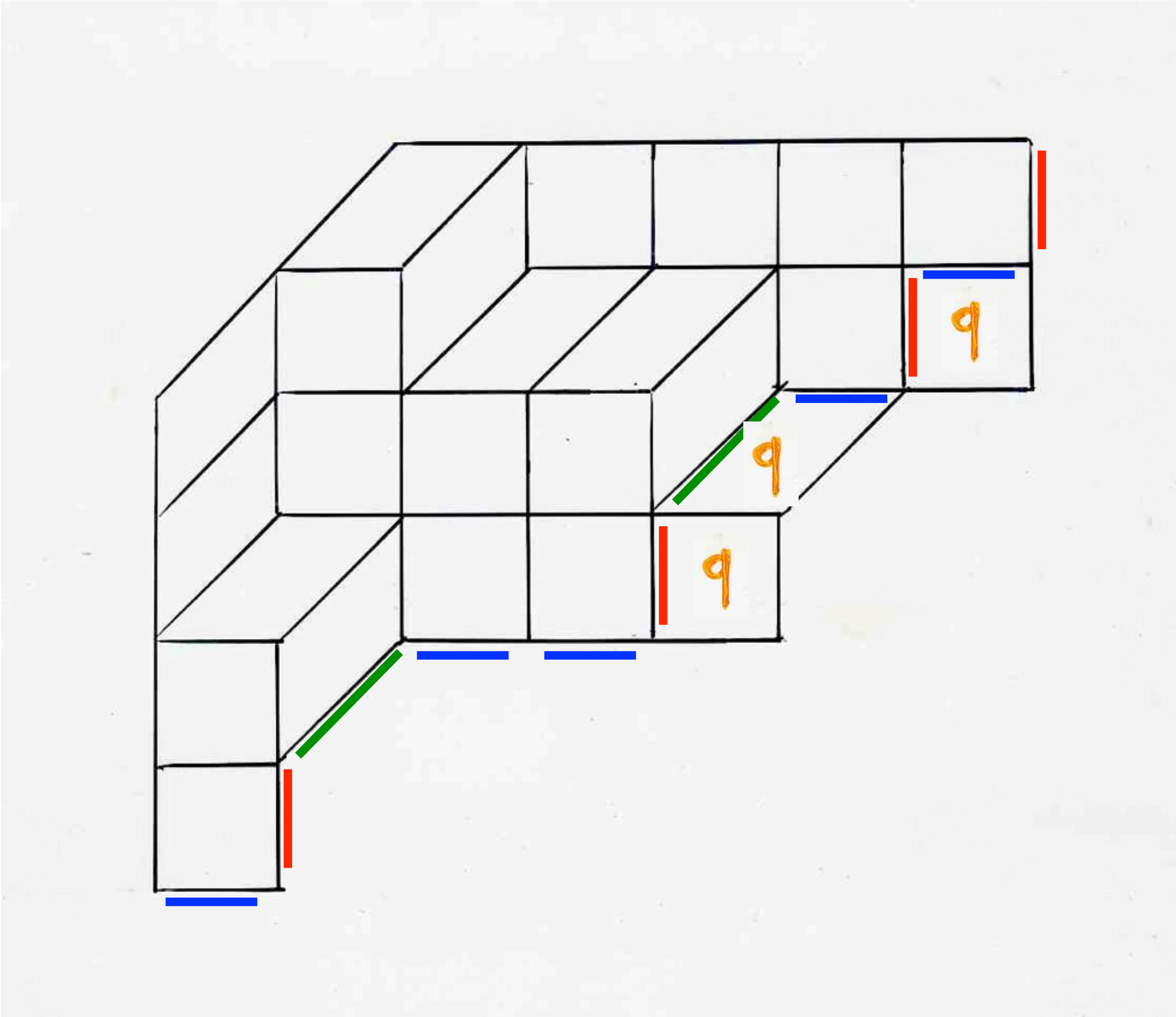
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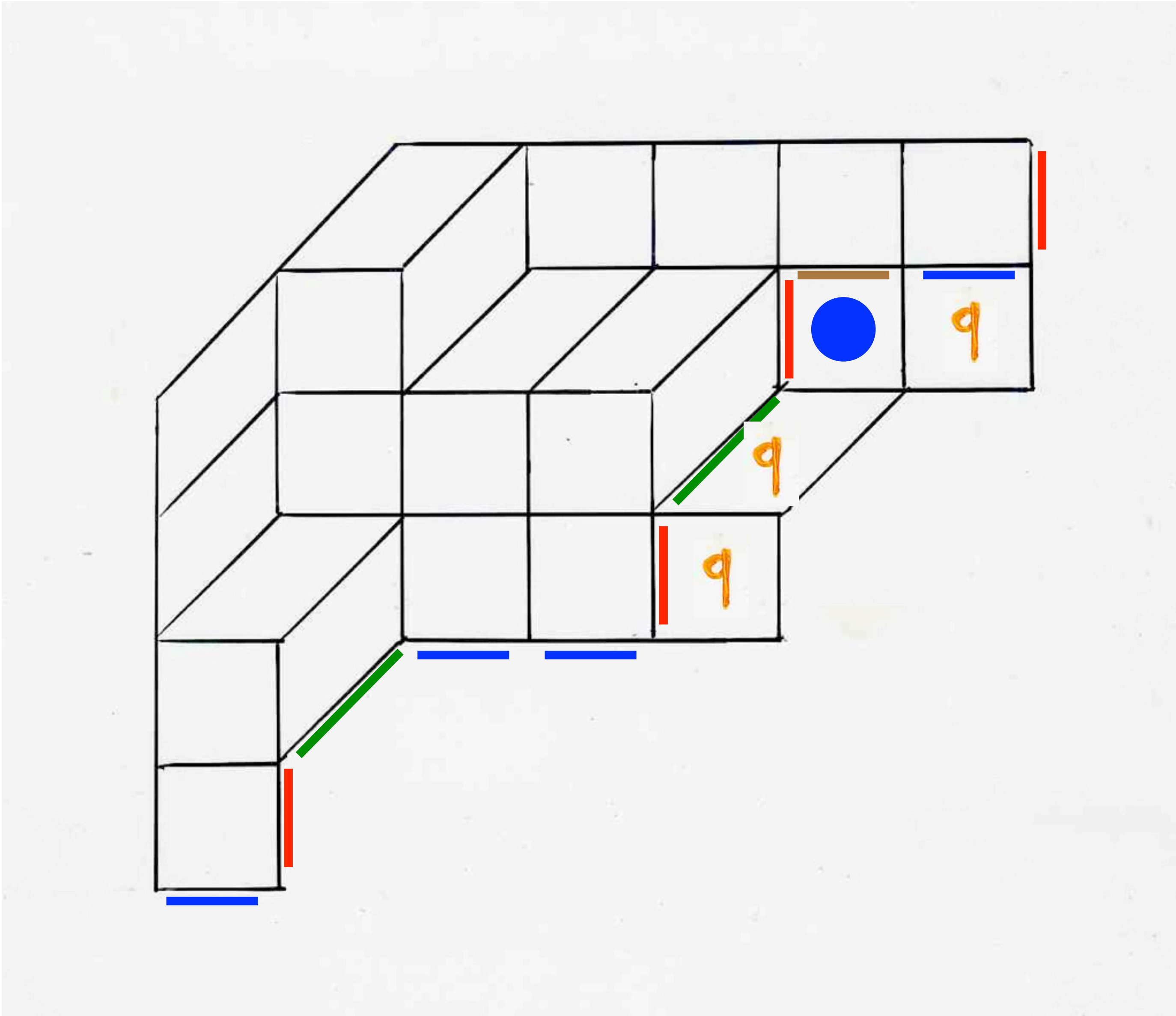

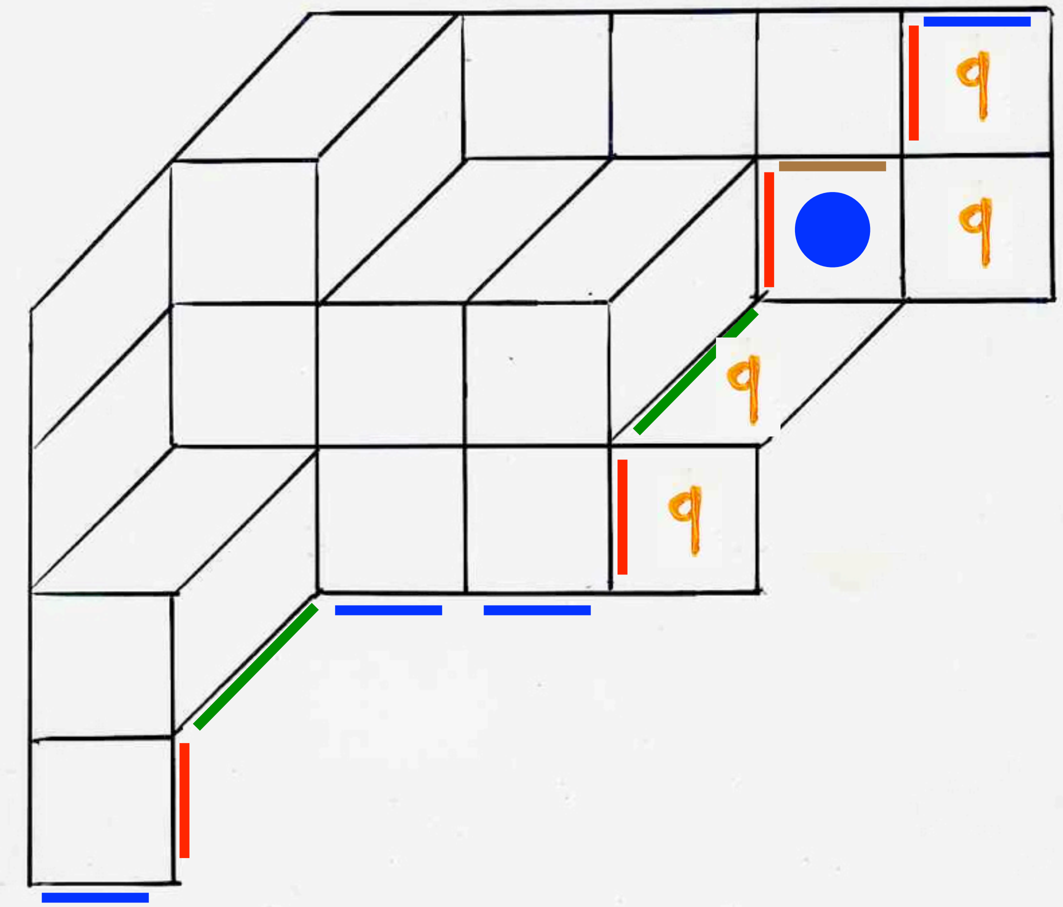
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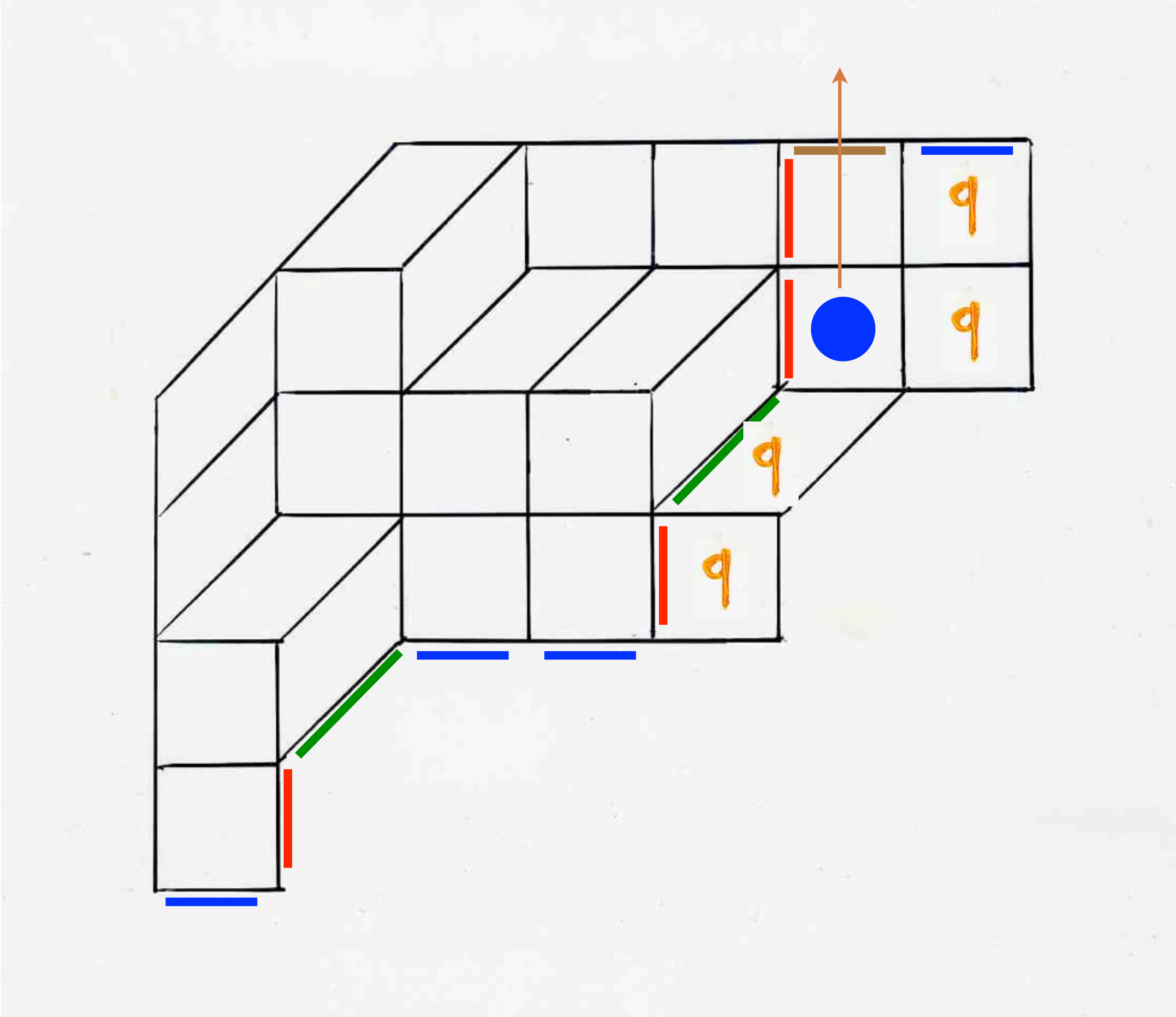
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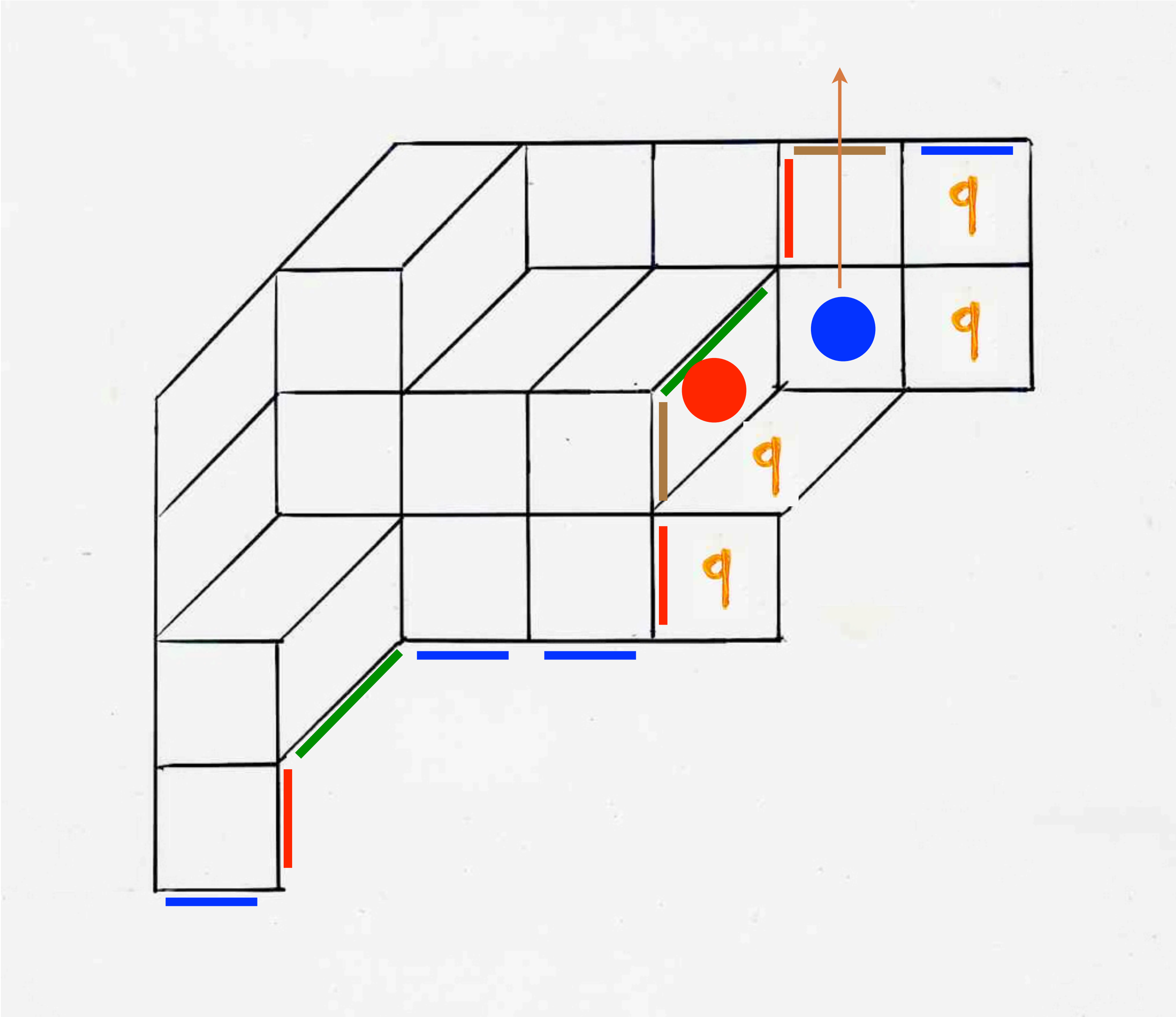
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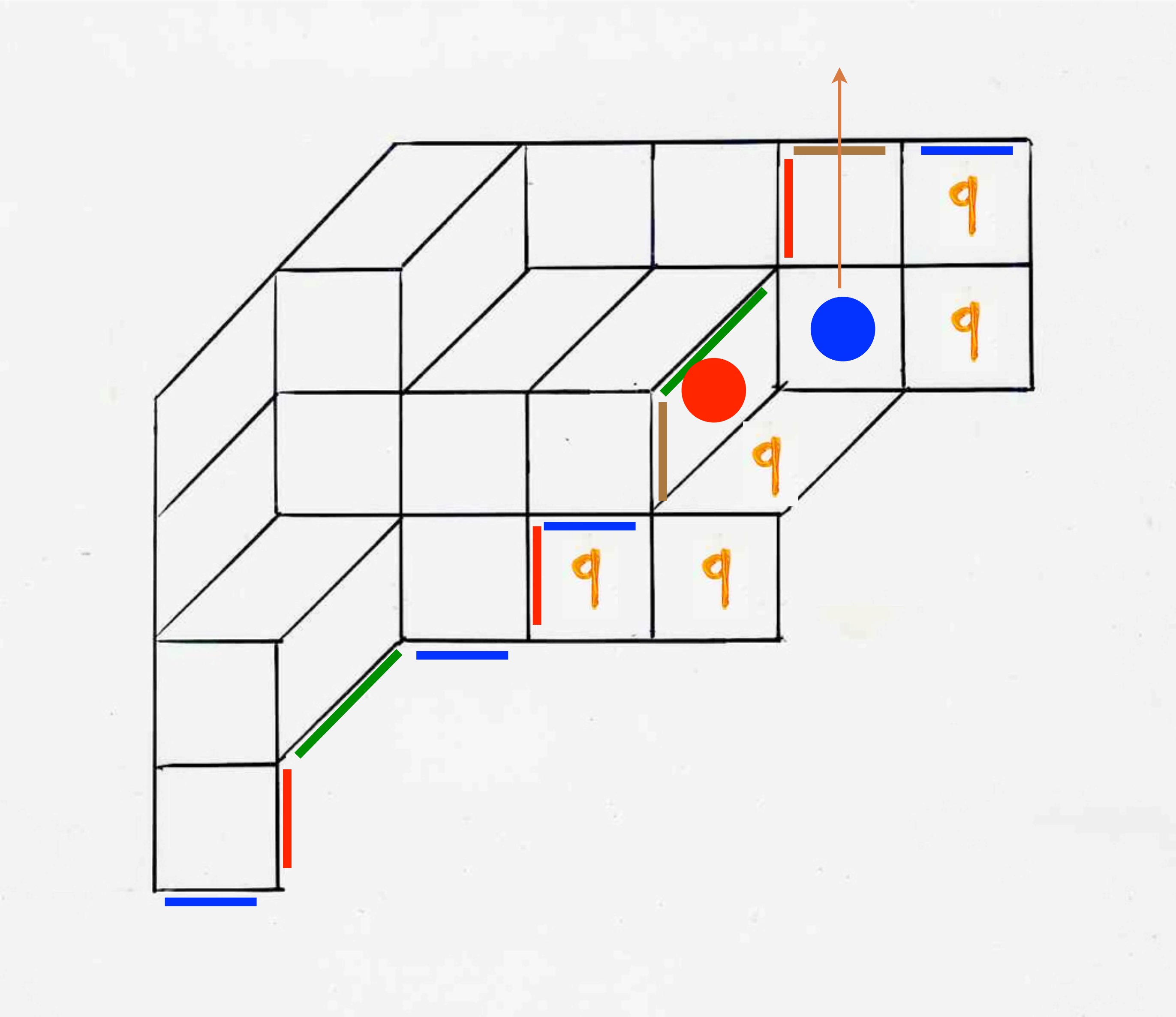
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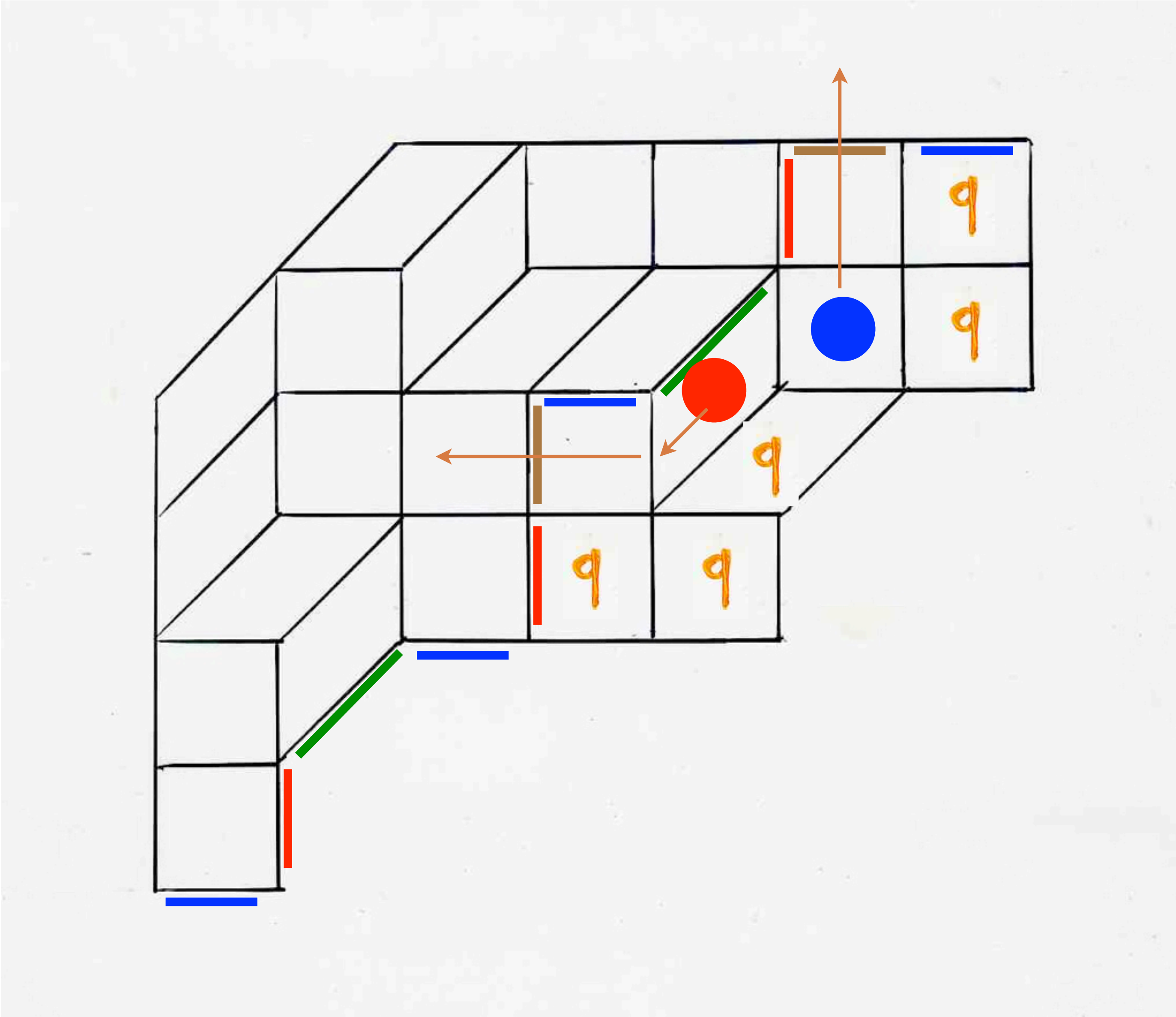
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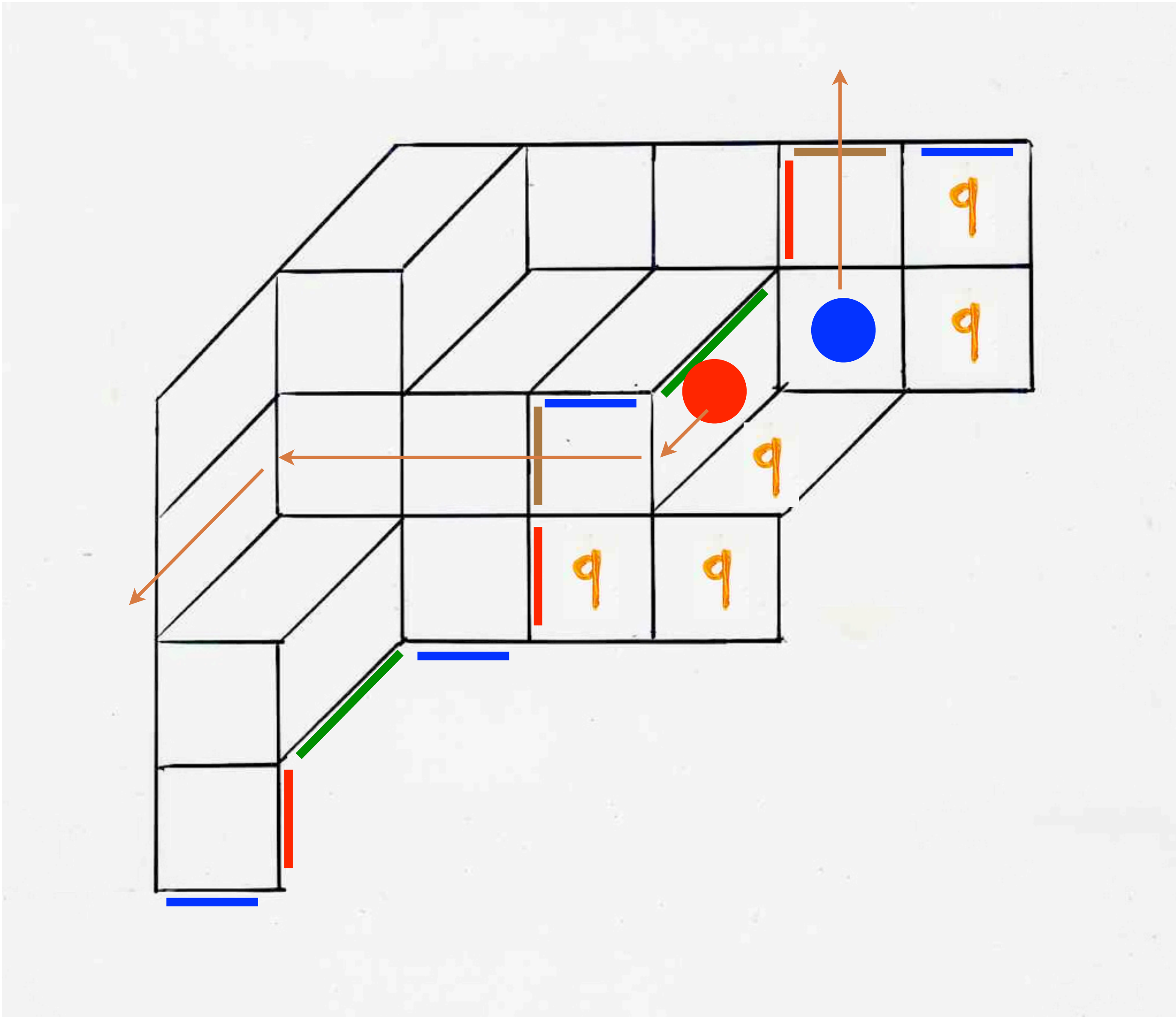

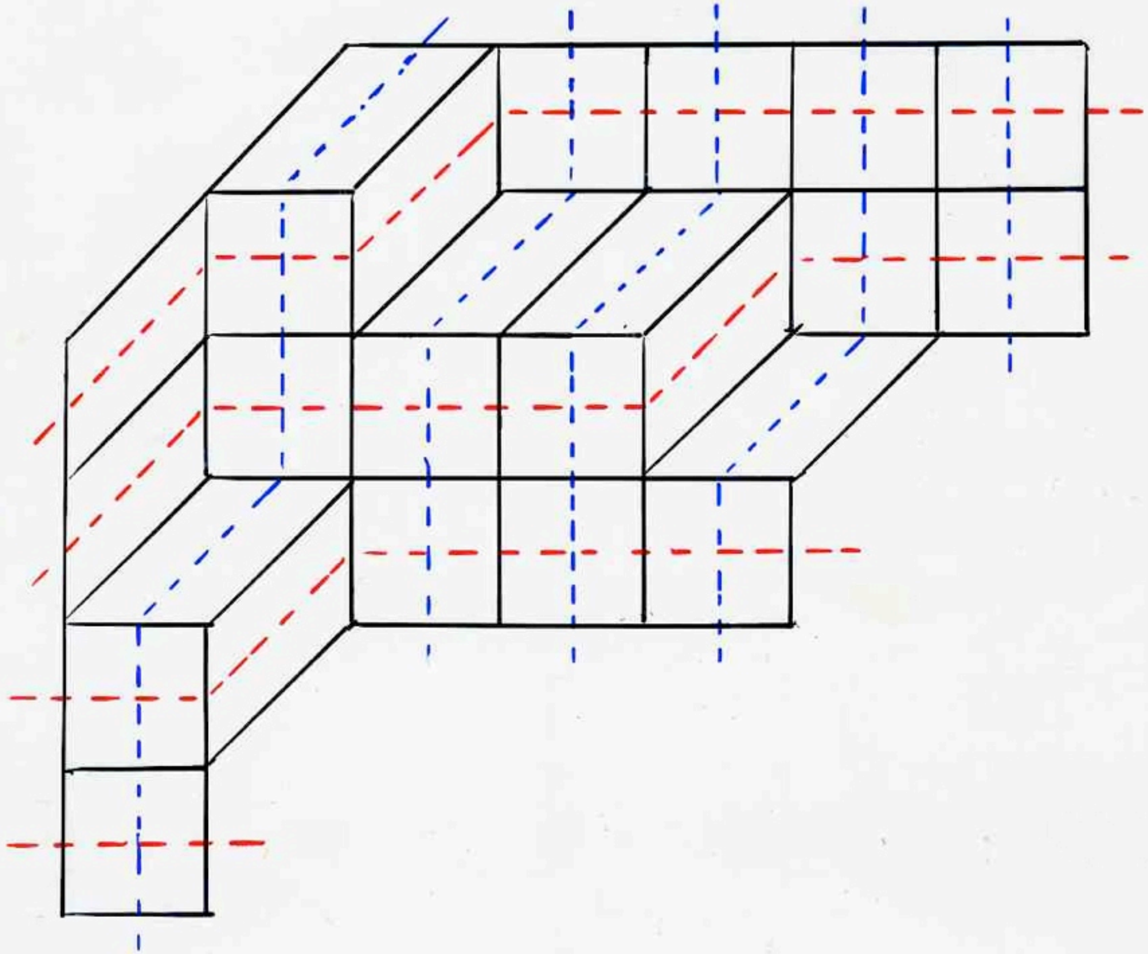

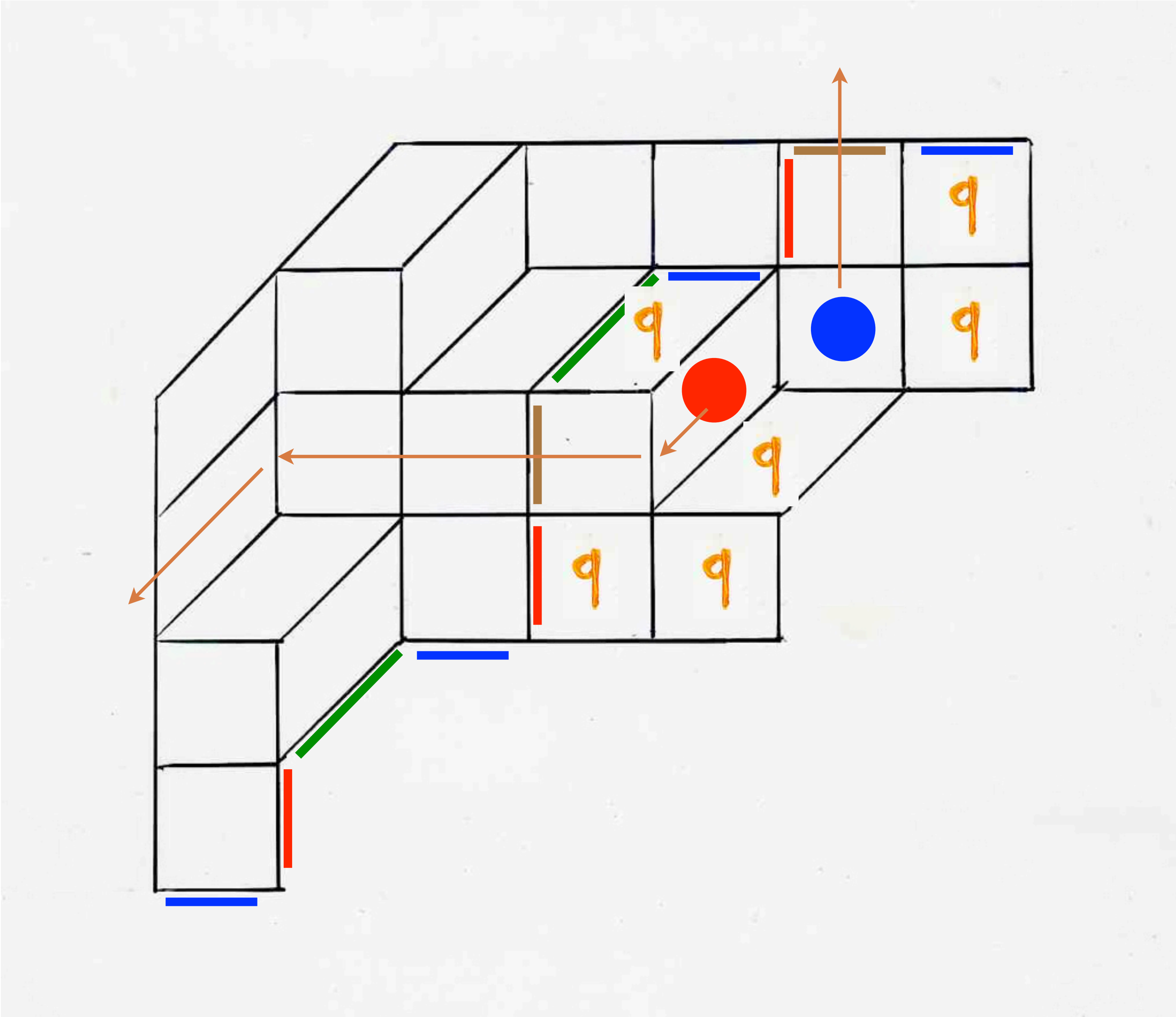
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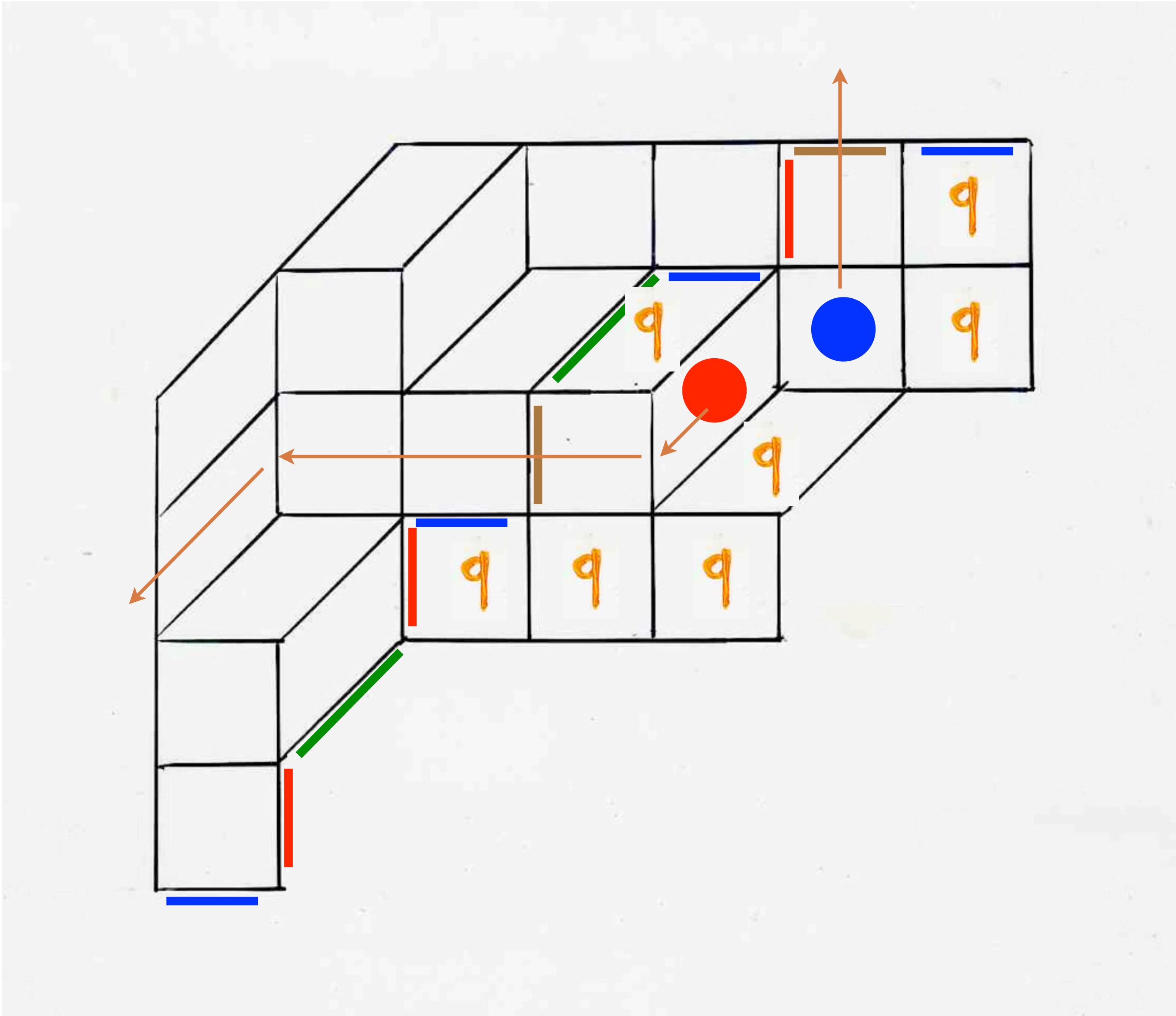
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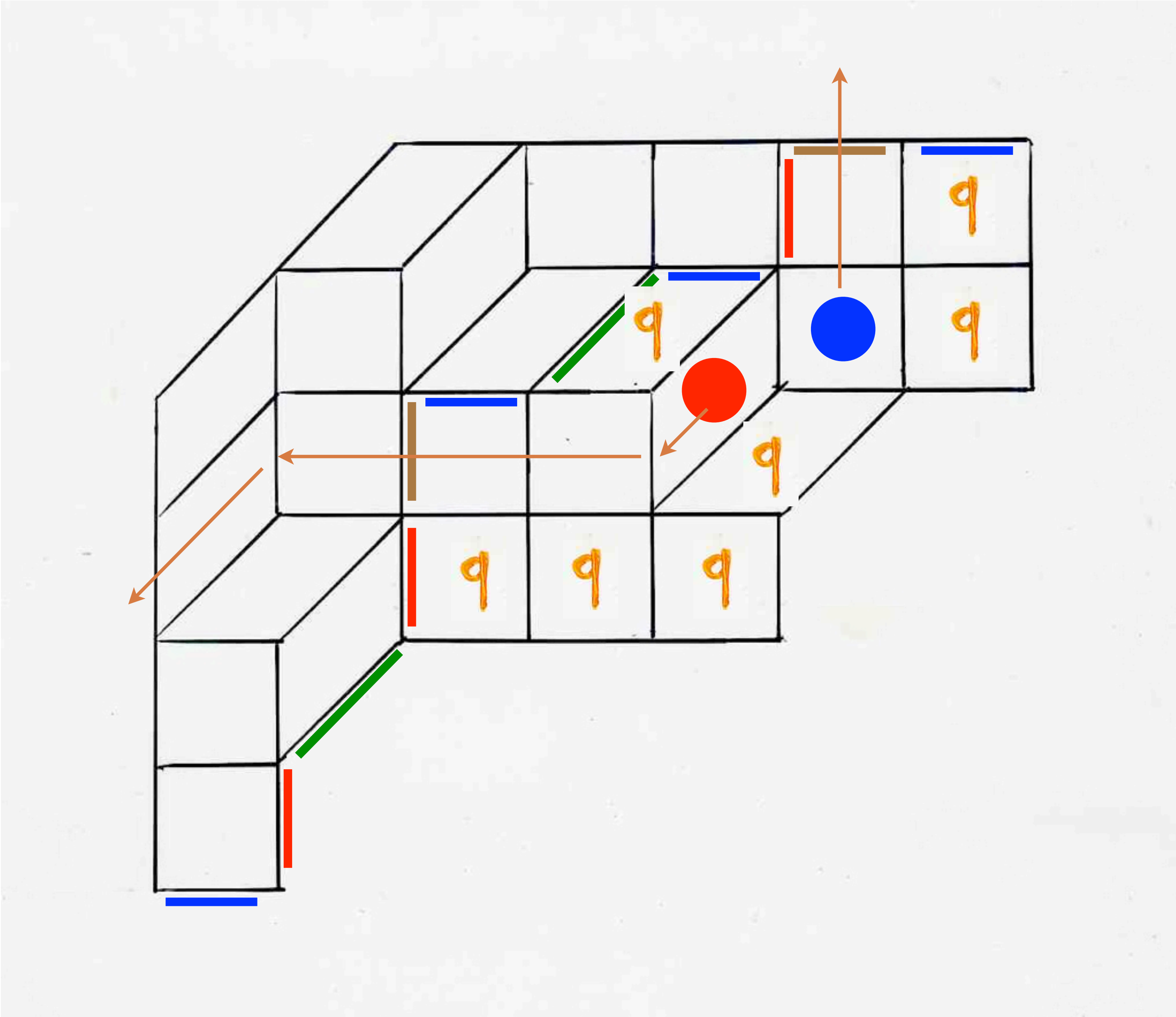

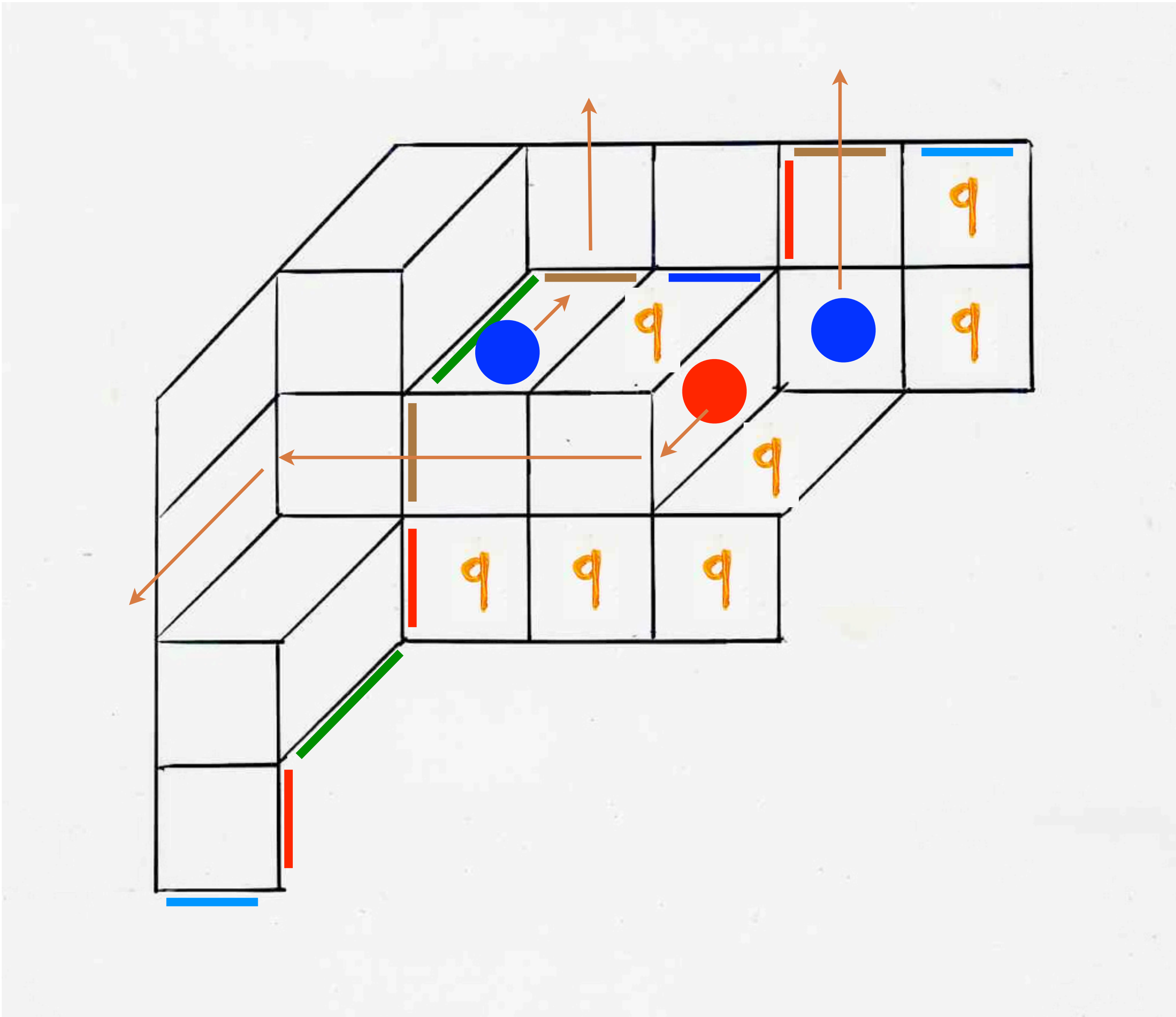
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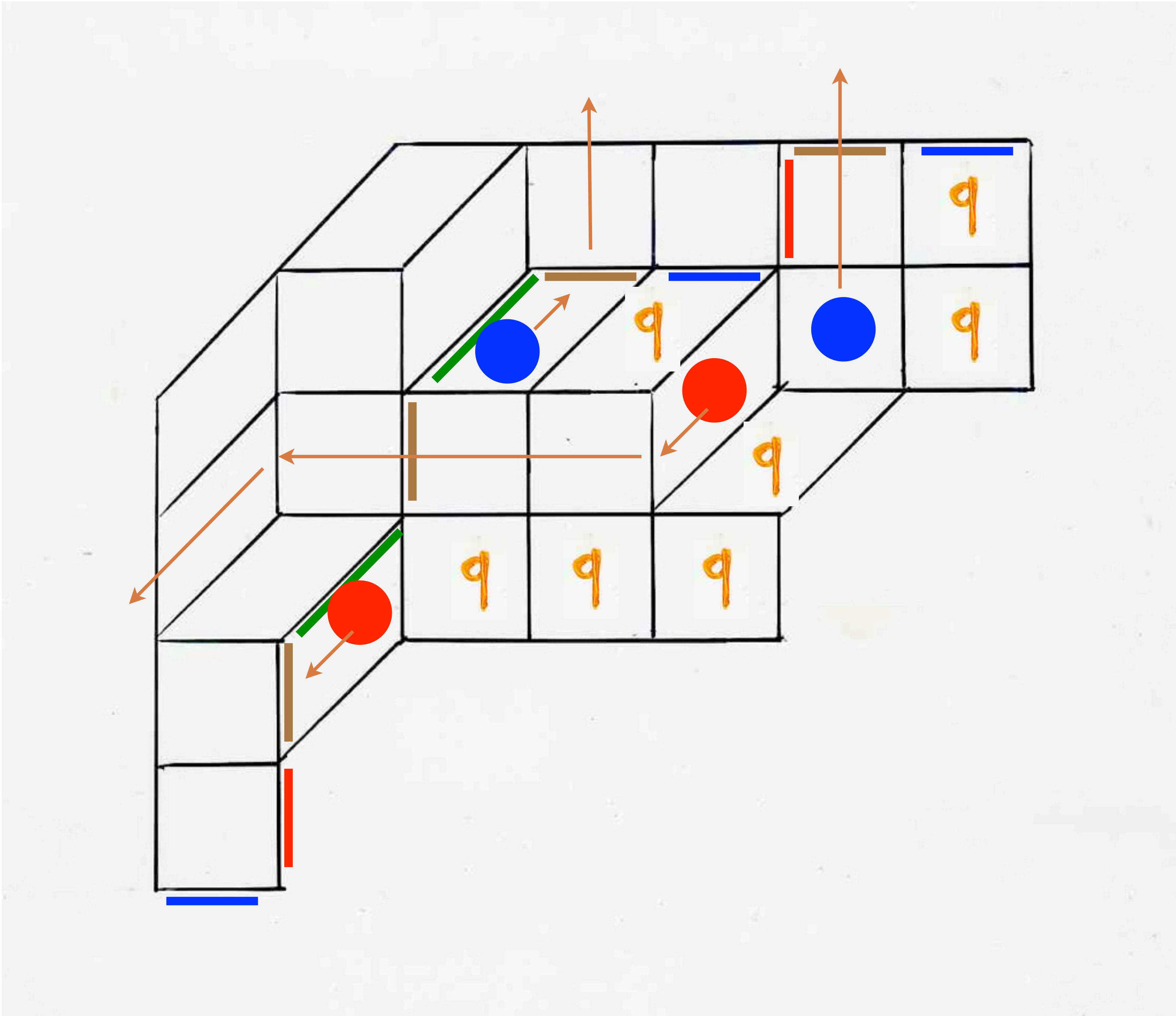
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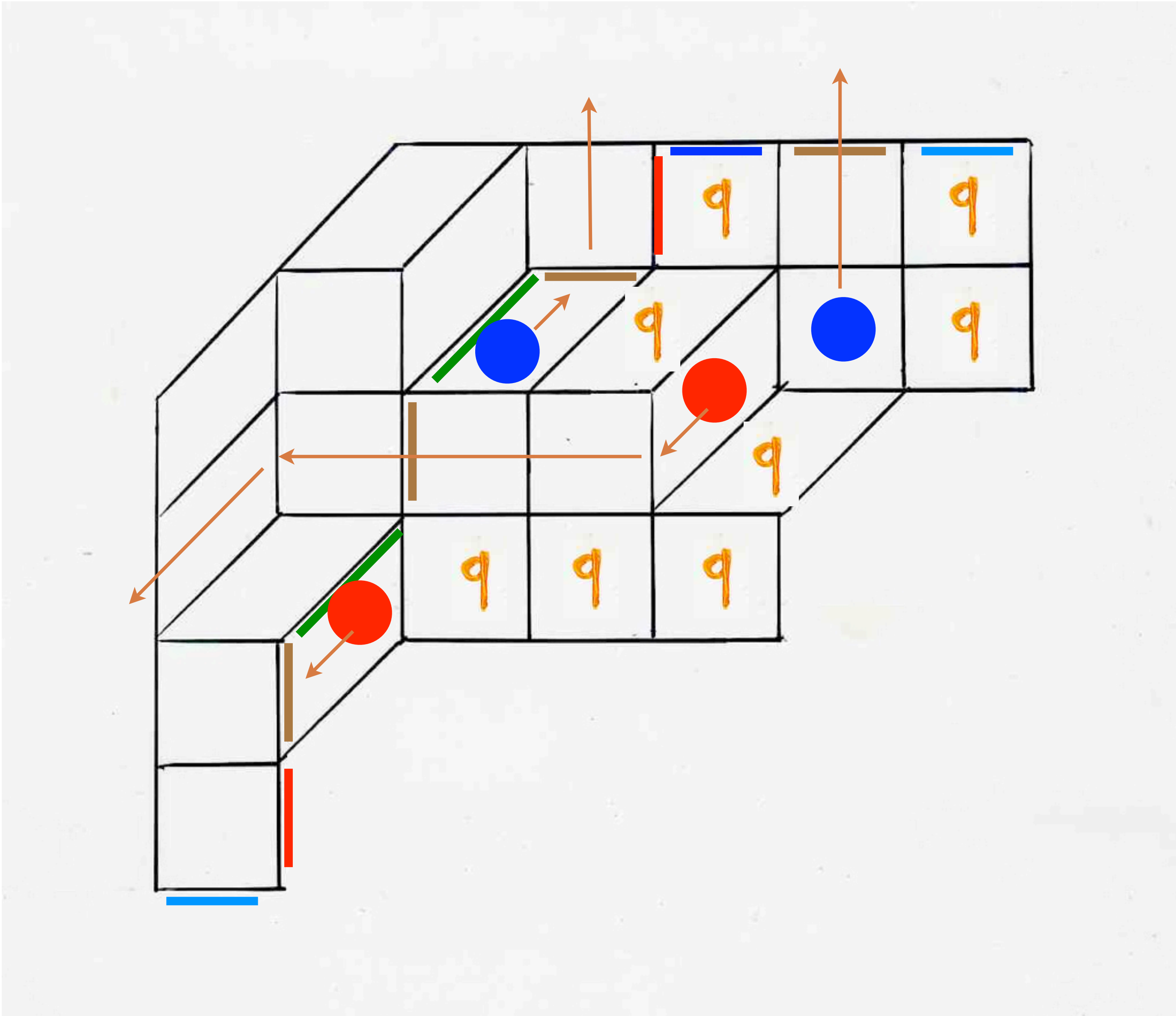
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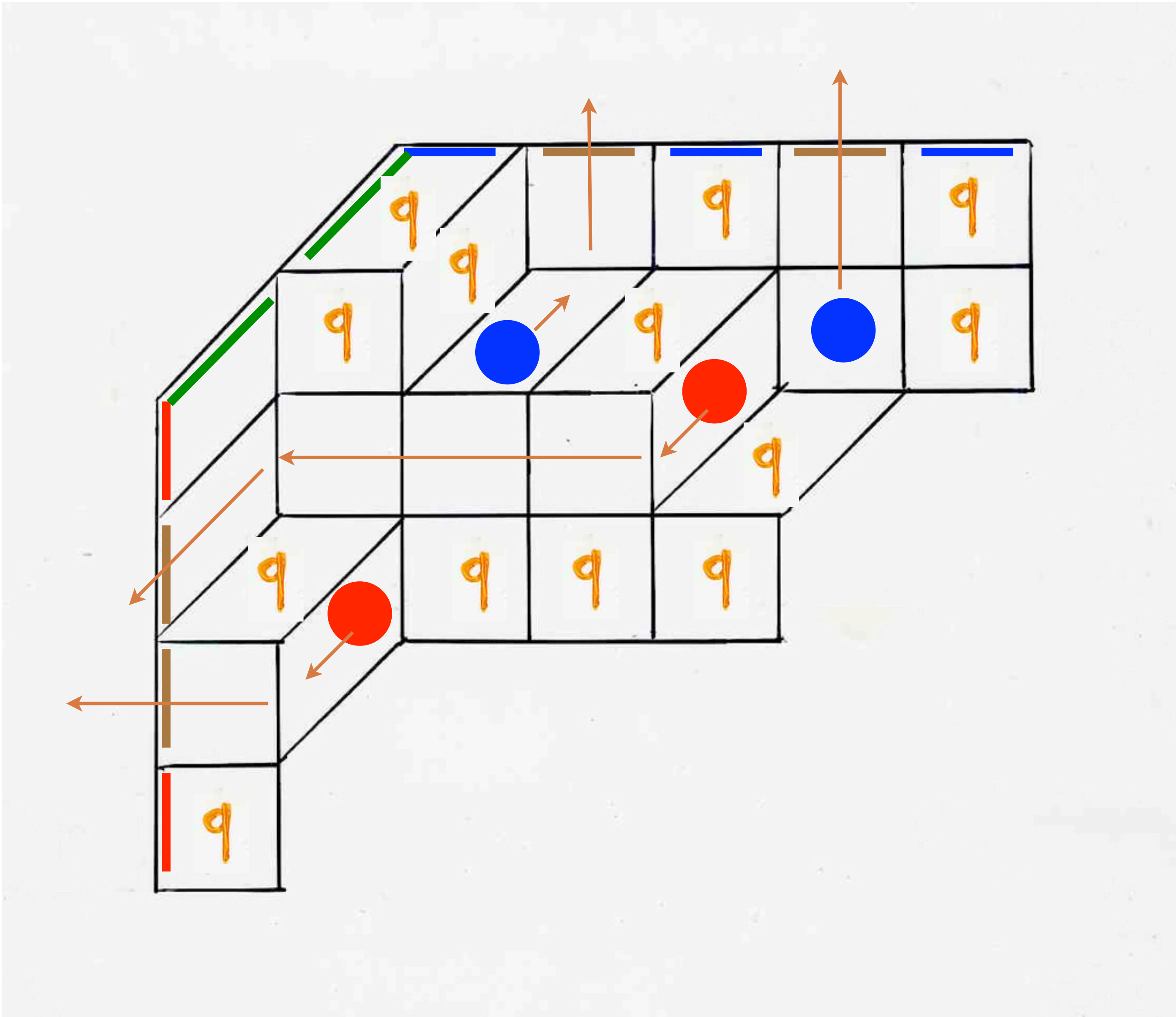

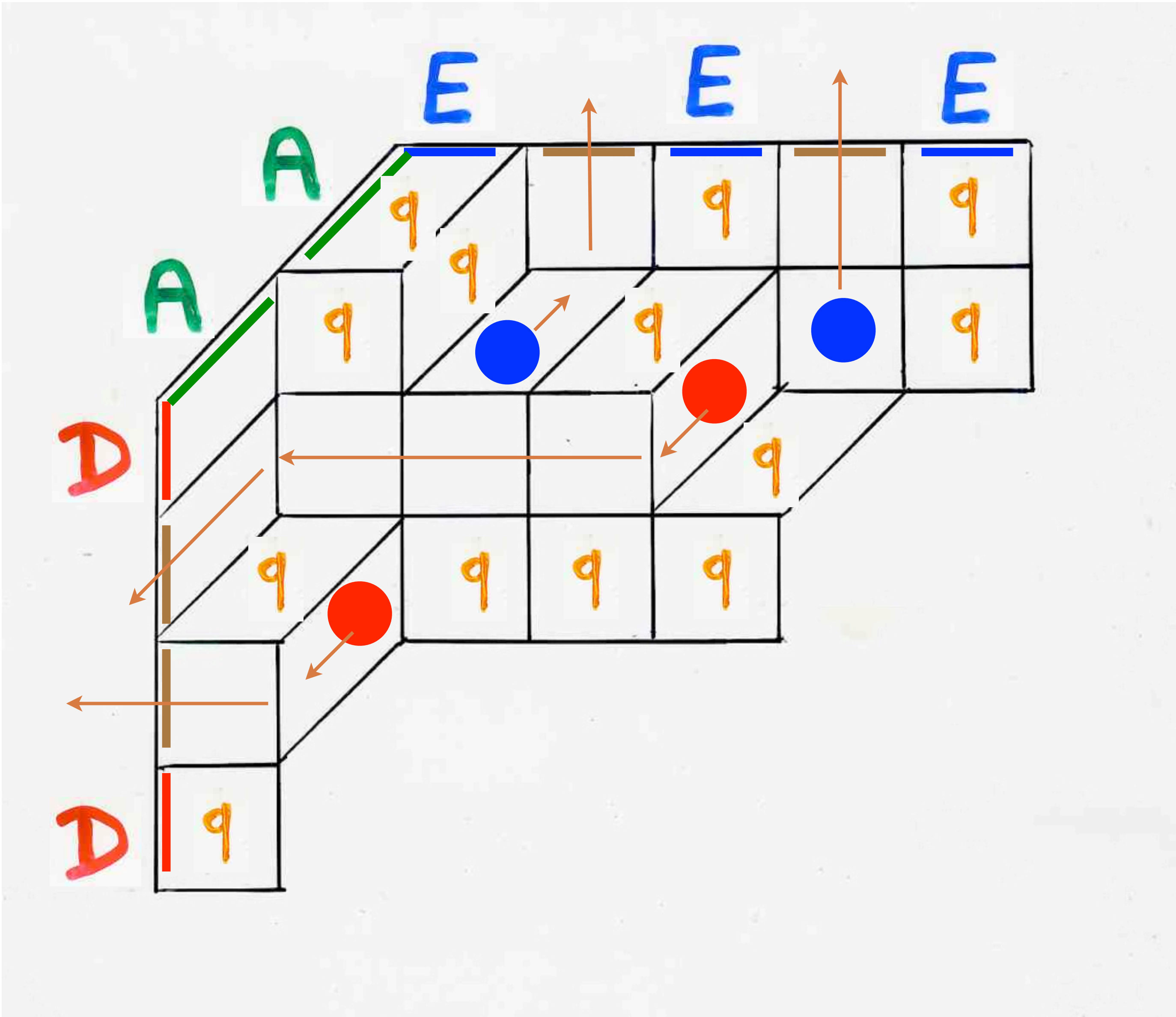

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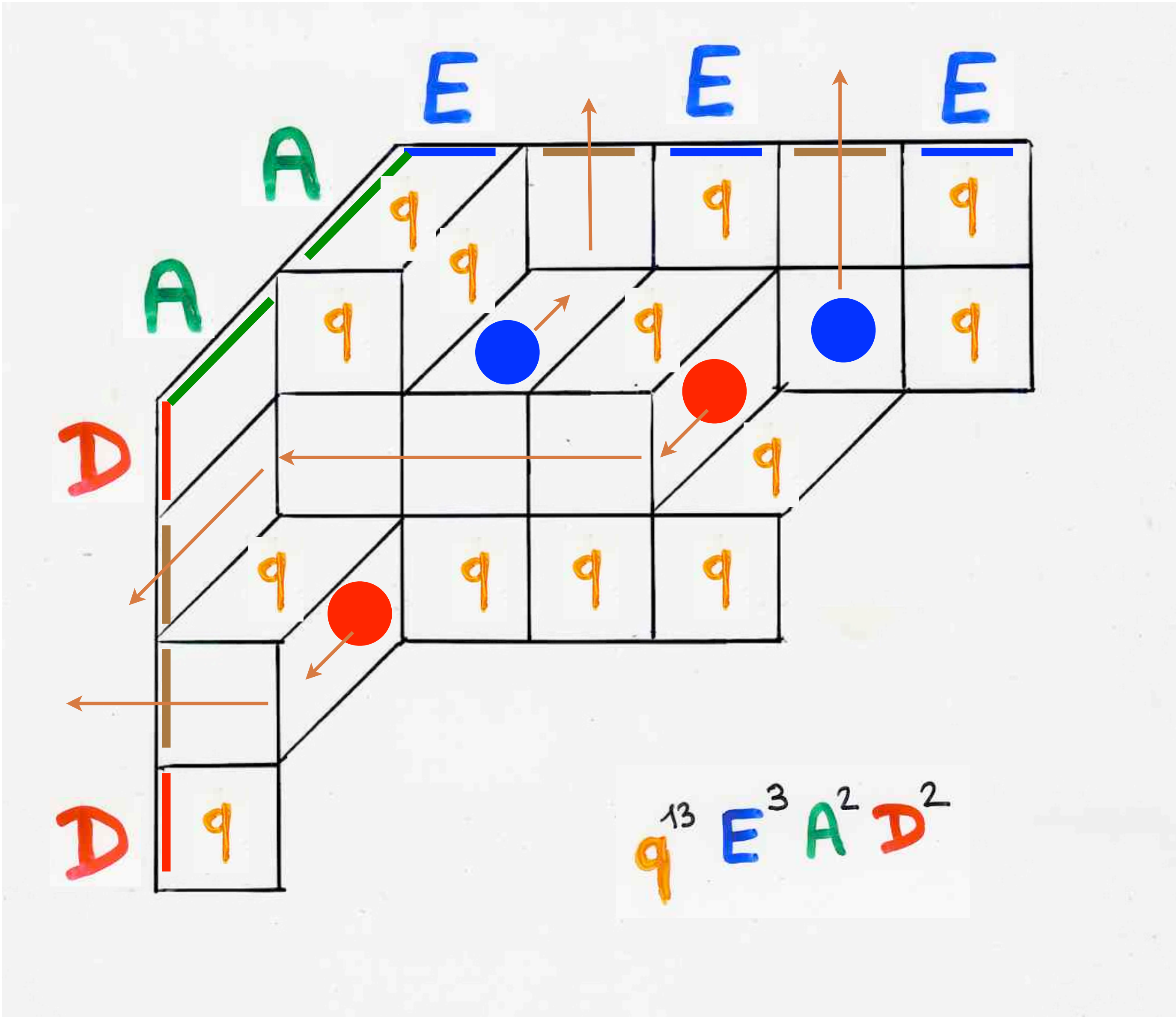

combinatorial interpretation of stationary probabilities

$$\mathrm{Prob}(x) = \frac{1}{Z_{n,r}} \langle W | \prod_{i=1}^{n} D\,\mathbb{1}_{(x_i=\bullet)} + A\,\mathbb{1}_{(x_i=\circledcirc)} + E\,\mathbb{1}_{(x_i=\circ)} | V \rangle$$

$$\langle W | X | V \rangle = \sum_{T \in R(X,\tau)} q^{t} \left(\frac{1}{\alpha}\right)^{i} \left(\frac{1}{\beta}\right)^{j} \langle W | A^{r} | V \rangle$$

i = nb of free north-strips in T
(not containing an ●)

j = nb of free south-strips in T
(= not containing a ●)

t = nb of cells labeled q in T

$$Z_{n,r} = \text{coeff. of } y^r \text{ in } \langle W | (D + yA + E)^n | V \rangle$$

$$Z_{n,r} = Z^*_{n,r} \langle W | A^r | V \rangle$$

$$Z^*_{n,r} = \sum_{X} \sum_{T \in R(X, T_X)} q^t \left(\frac{1}{\alpha}\right)^i \left(\frac{1}{\beta}\right)^j$$

$$\text{Prob}(X) = \frac{1}{Z^*_{n,r}} \sum_{T \in R(X, T_X)} \underbrace{q^t \left(\frac{1}{\alpha}\right)^i \left(\frac{1}{\beta}\right)^j}_{wt(T)}$$

enumeration of rhombic alternative tableaux

$$Z^{*}_{n,r}(\alpha=\beta=q=1) = \binom{n}{r}\frac{(n+1)!}{(r+1)!}$$

Lah numbers

nb of "assemblées" of permutations

$$\{[7,10,5,8]\ [3,1,4]\ [9,2,11,6]\}$$

$$\exp\left(\frac{xt}{1-t}\right)$$

bijective proof

in the next talk

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$$Z^*_{n,r}(\alpha, \beta, q=1) = \binom{n}{r} \prod_{i=r}^{n-1} (\alpha^{-1} + \beta^{-1} + i)$$

arXiv : 1506.01980
[math.CO]

bijective proof
in the next talk
GT Labri 27 Nov 2015