Les nouvelles mathématiques combinatoires: renaissance ou révolution ?

Quimper
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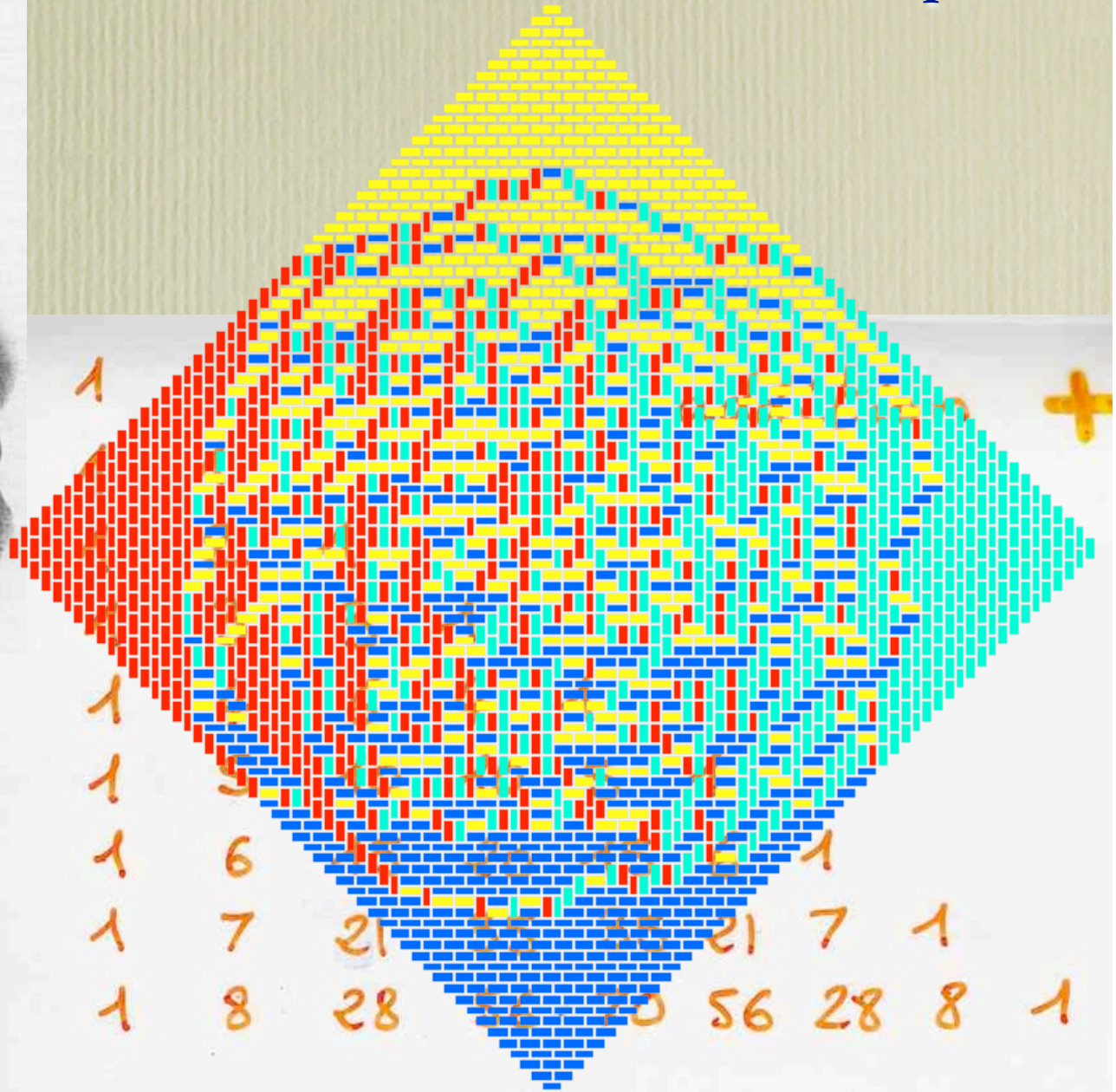
introduction

addition +

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

Triangle de Pascal
Coefficients binomiaux

théorème du cercle arctique



Triangle de Pascal
Coefficients binomiaux

récréations mathématiques

problèmes amusants et délectables

carrés latins, carrés magiques, ...
problèmes des ménages, ...
problème des ponts de Koenigsberg,

problème des quatre couleurs

permutations, combinaisons,
arrangements ....

$$n! = 1 \times 2 \times \ldots \times n$$

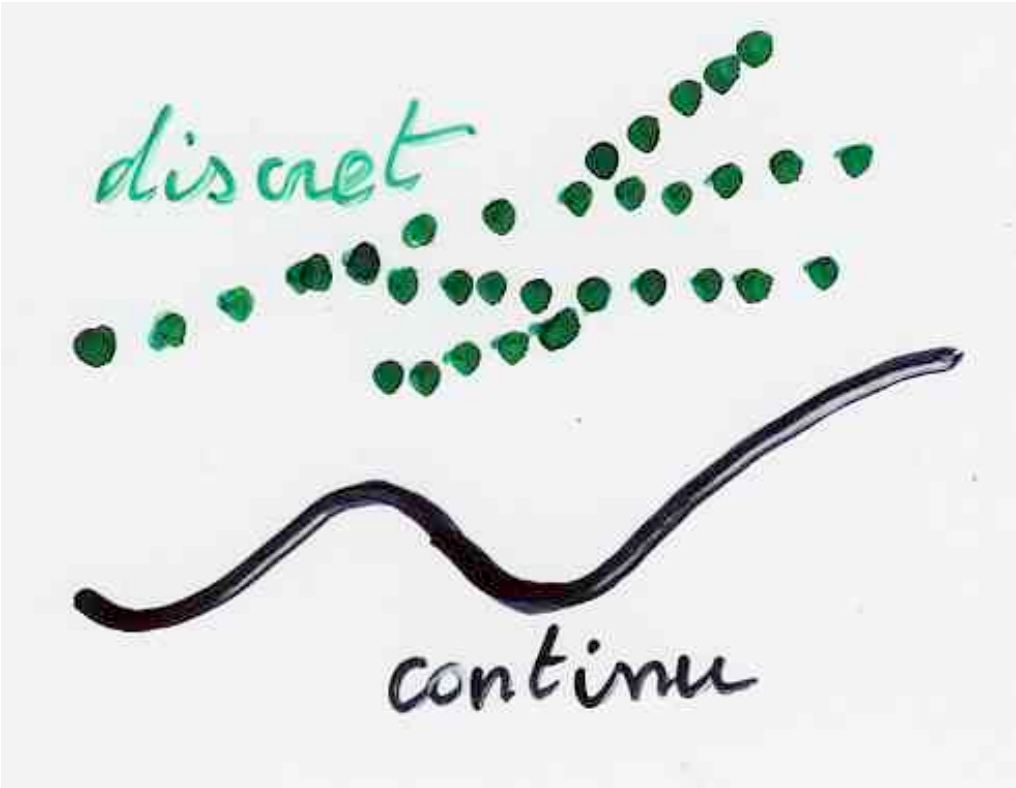
Permutation σ

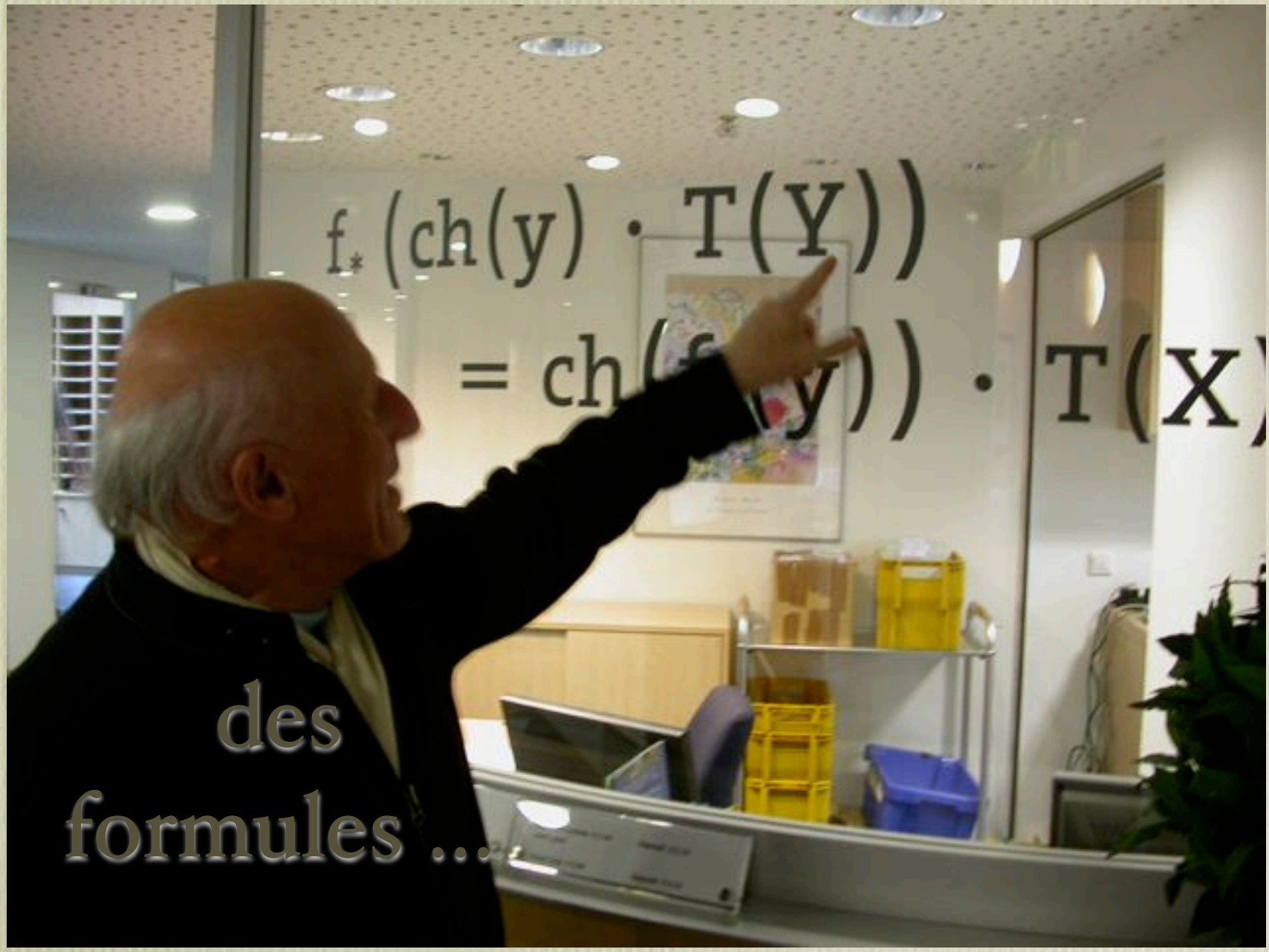
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

1, 2, 6, 24, 120, 720, 5040, 40320,...

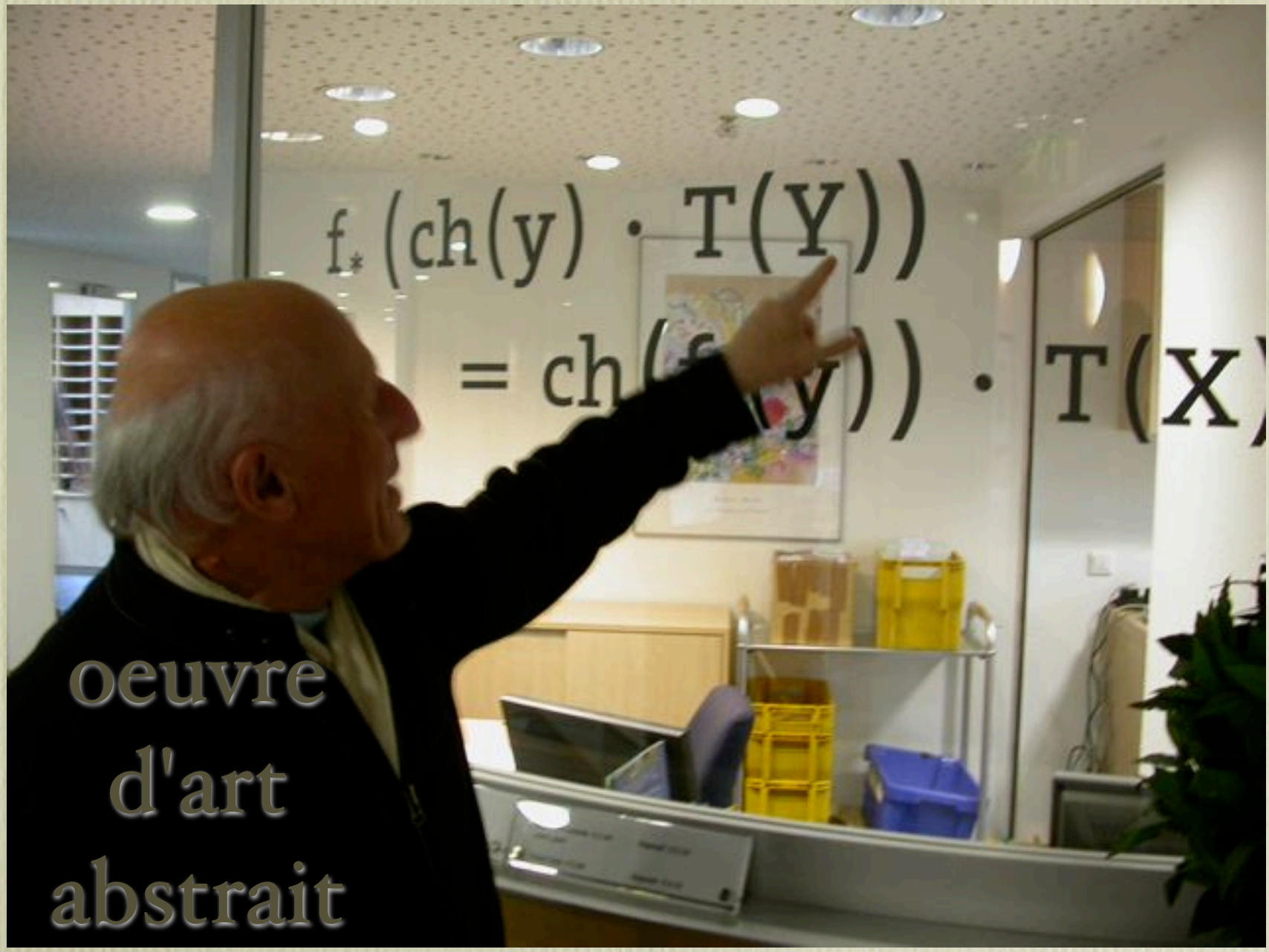
nombres de Fibonacci

1, 2, 3, 5, 8, 13, 21, 34, 55,...

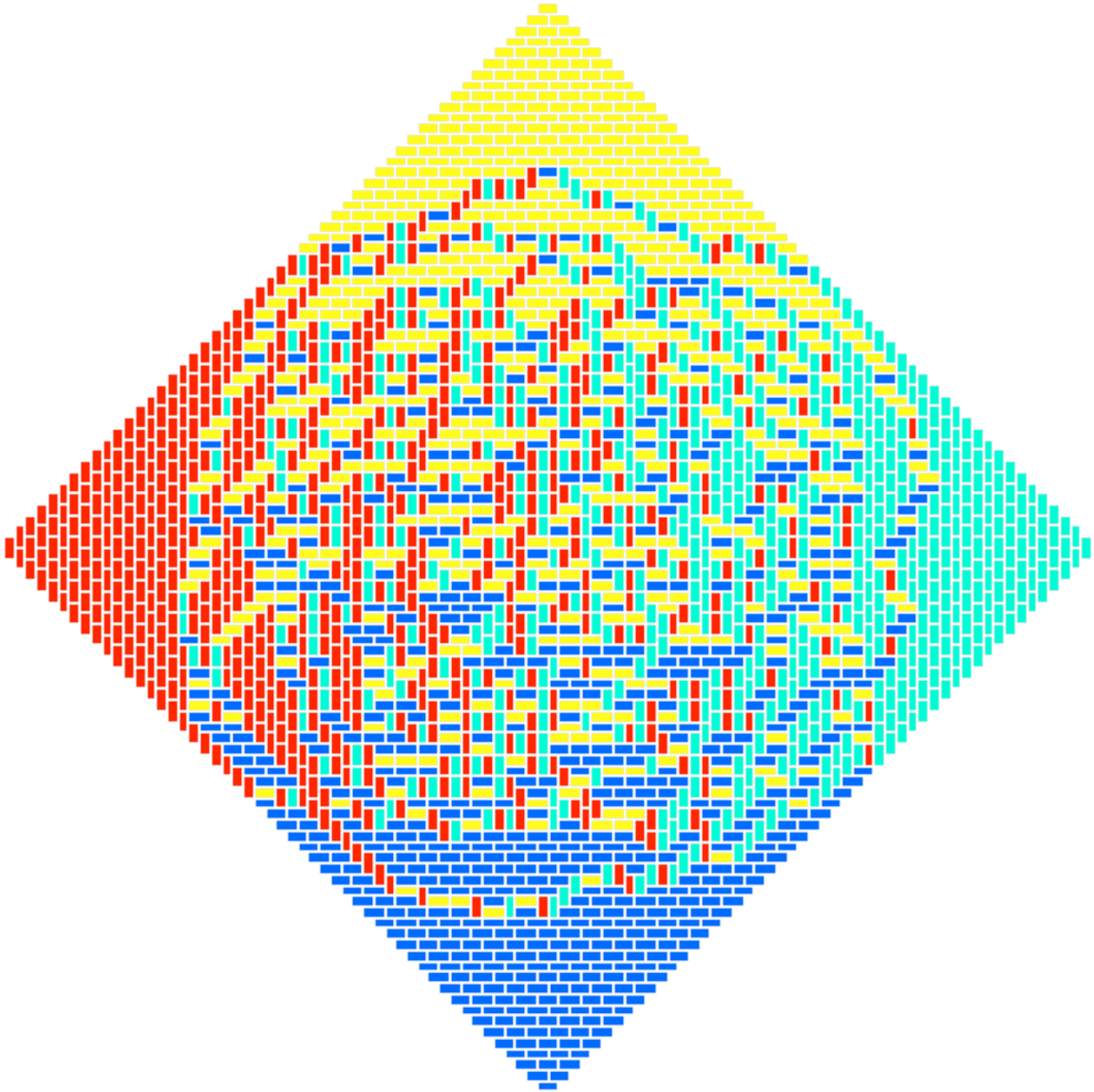

discret
continu



des formules ...


$f_*(ch(y) \cdot T(Y))$
$= ch(f_*(y)) \cdot T(X)$
oeuvre
d'art
abstrait

Combinatoire énumérative



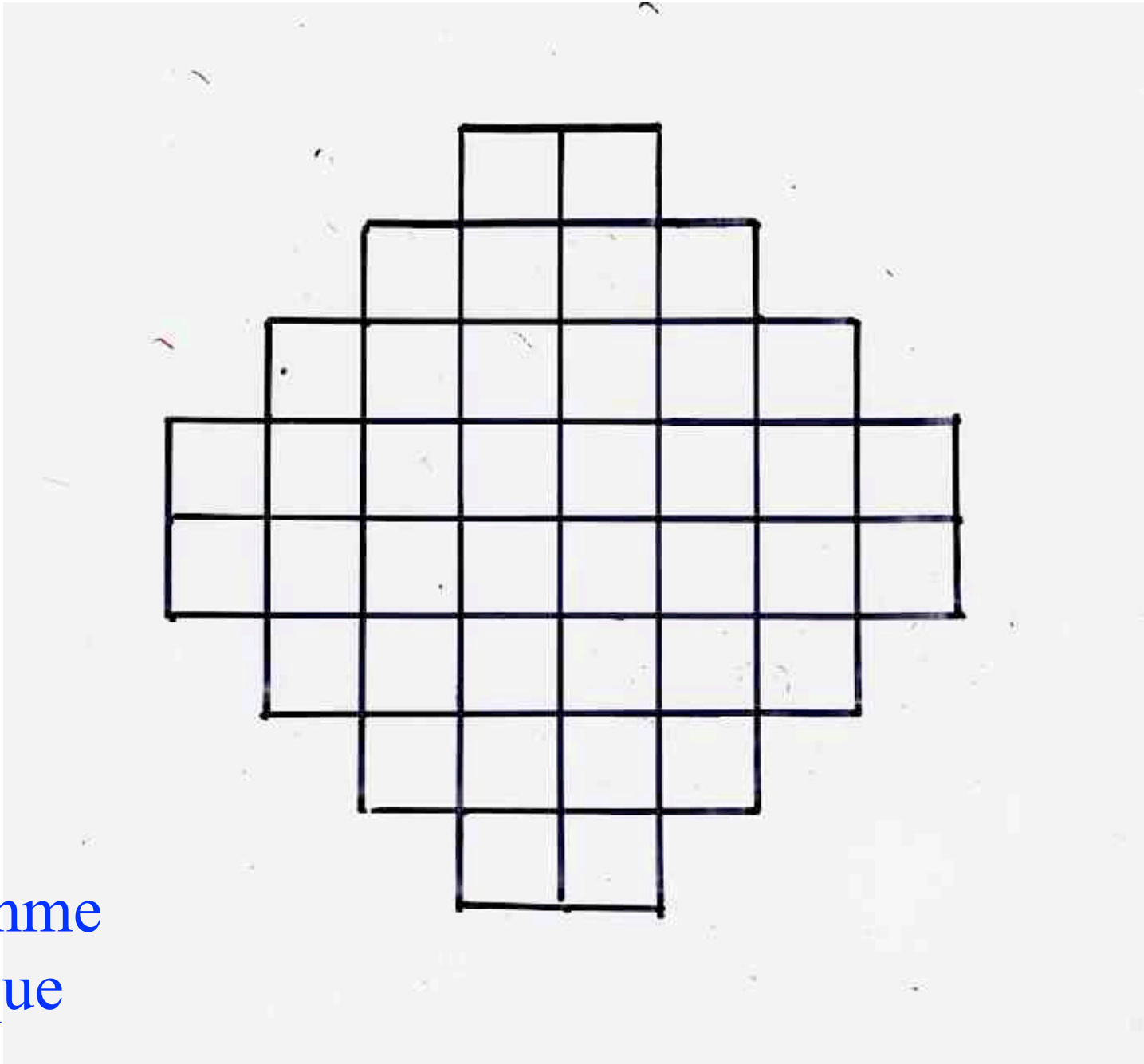
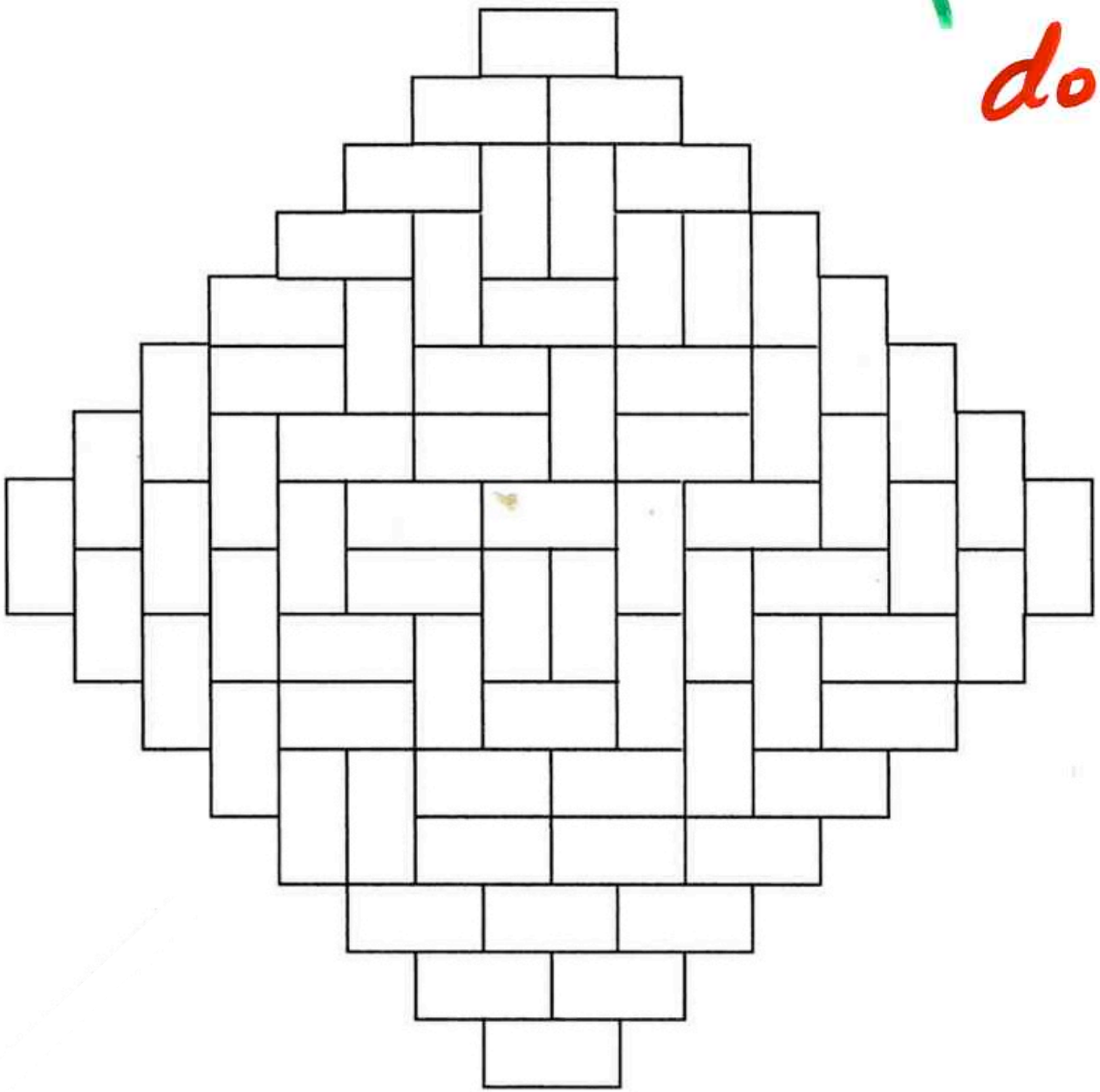
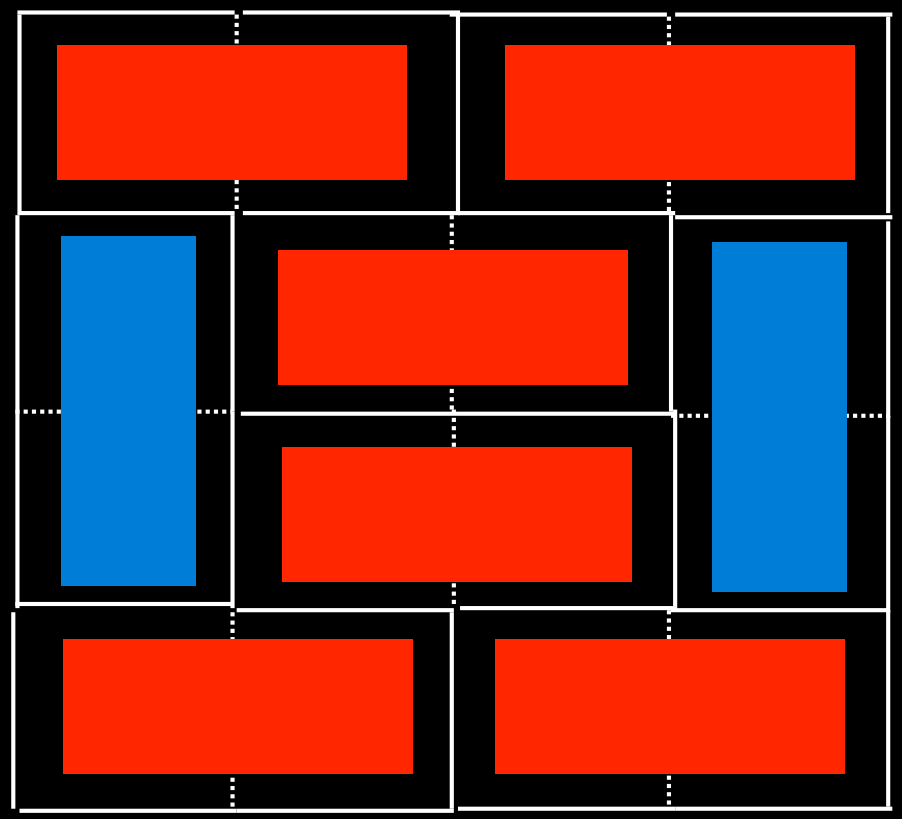


diagramme
Aztèque

Pavage

dominos





pavage d'un rectangle par des dominos

$$\prod_{i=1}^{m/2} \prod_{j=1}^{n/2} \left(4 \cos^2 \frac{i\pi}{m+1} + 4 \cos^2 \frac{j\pi}{n+1} \right)$$

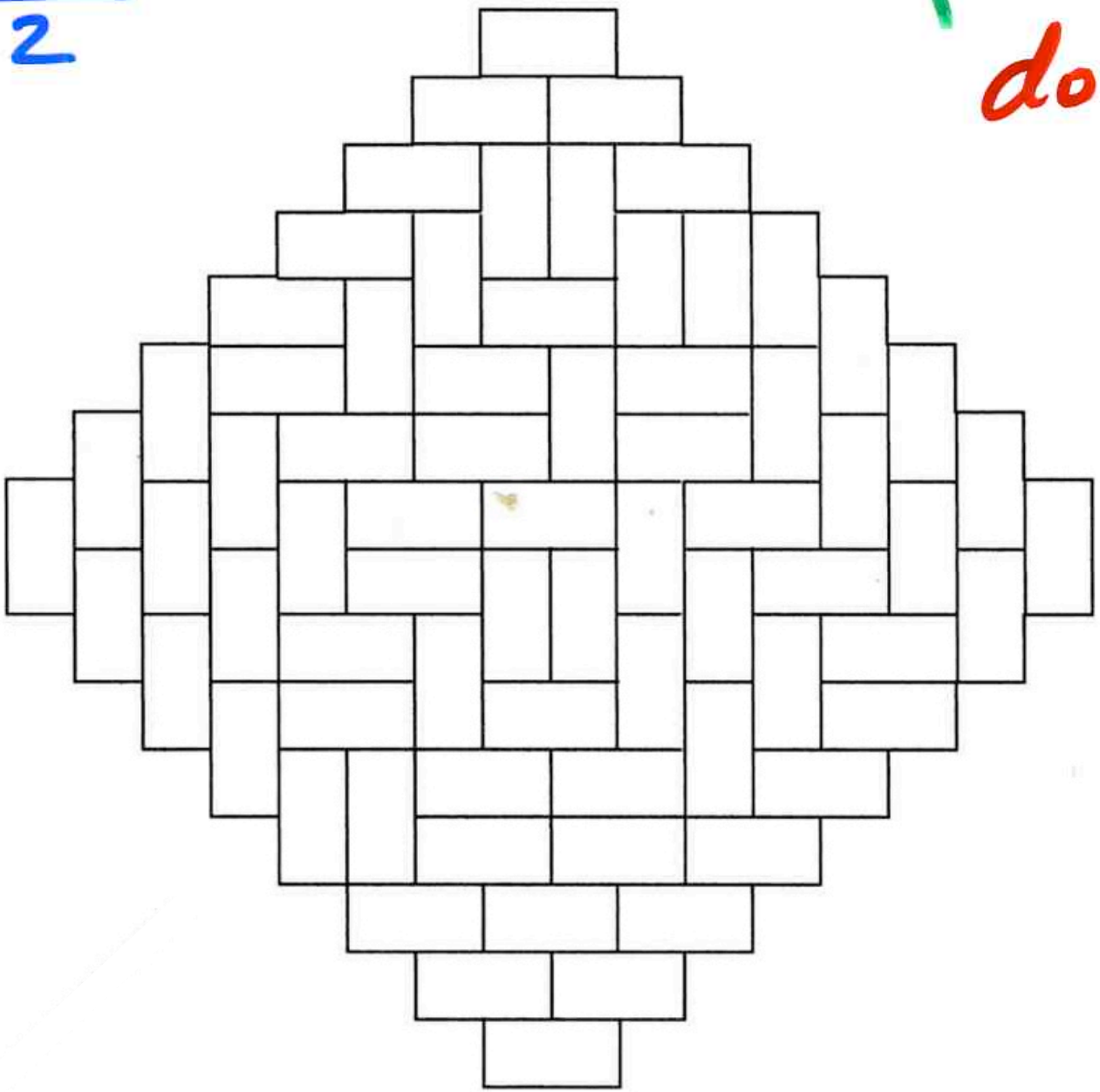
Kasteleyn (1961)

C'est un nombre entier !

$$2^{\frac{n(n+1)}{2}}$$

pavage

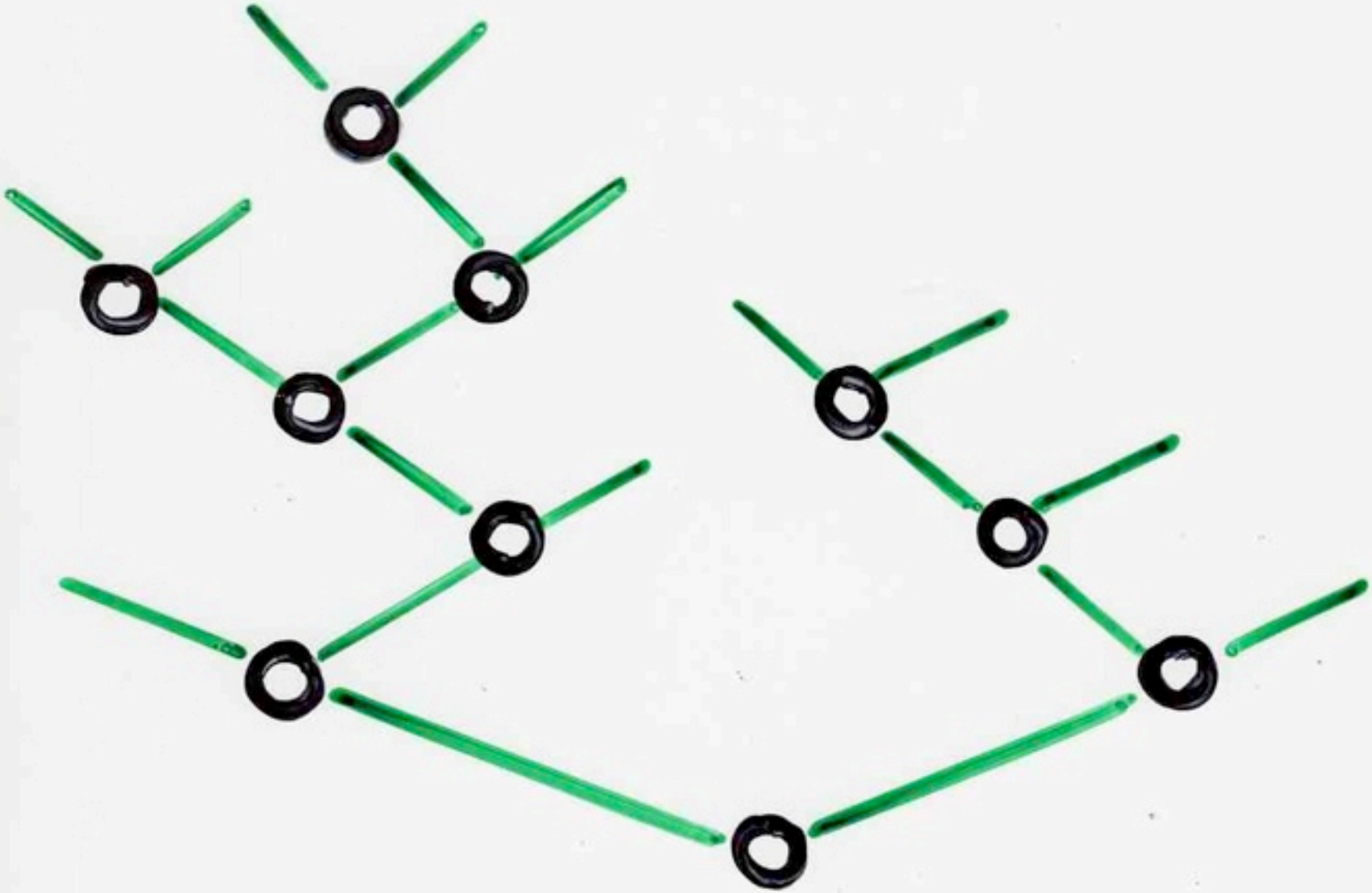
dominos

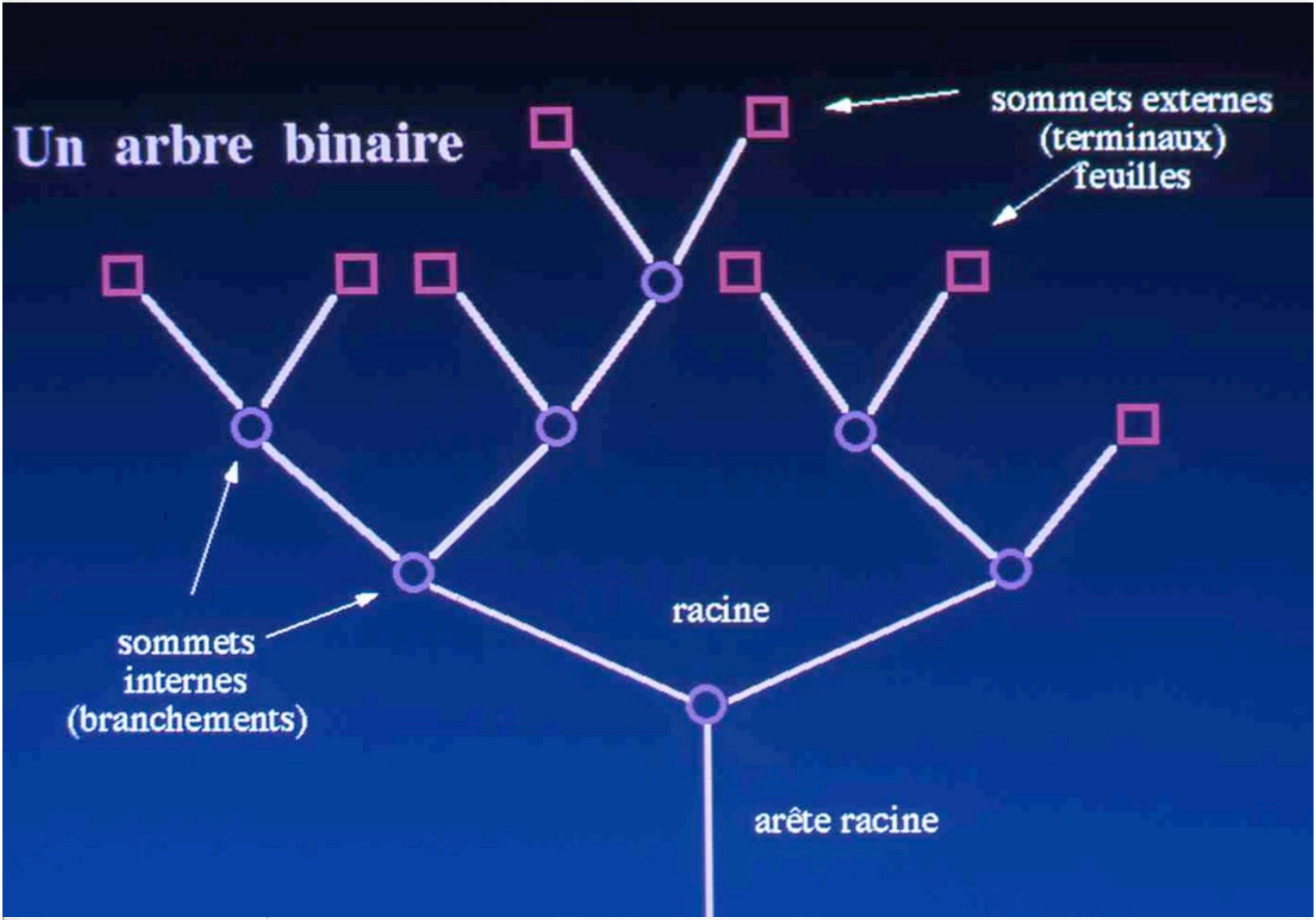


comme la naissance d'une étoile ...

renaissance de la combinatoire

arbre binaire



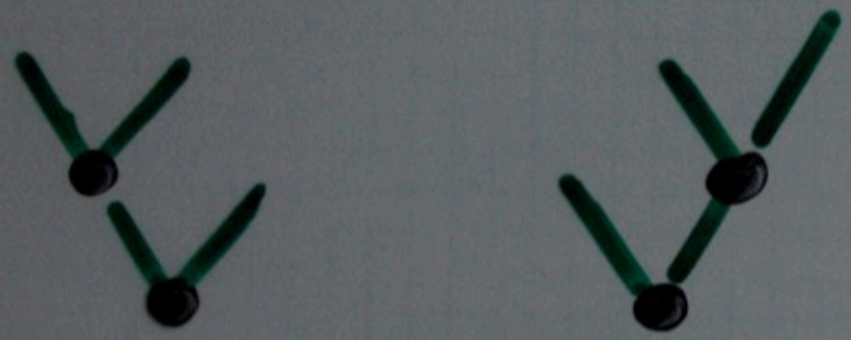

Un arbre binaire
sommets externes
(terminaux)
feuilles
sommets
internes
(branchements)
racine
arête racine

C_n = nombre d'arbres binaires ayant n sommets internes

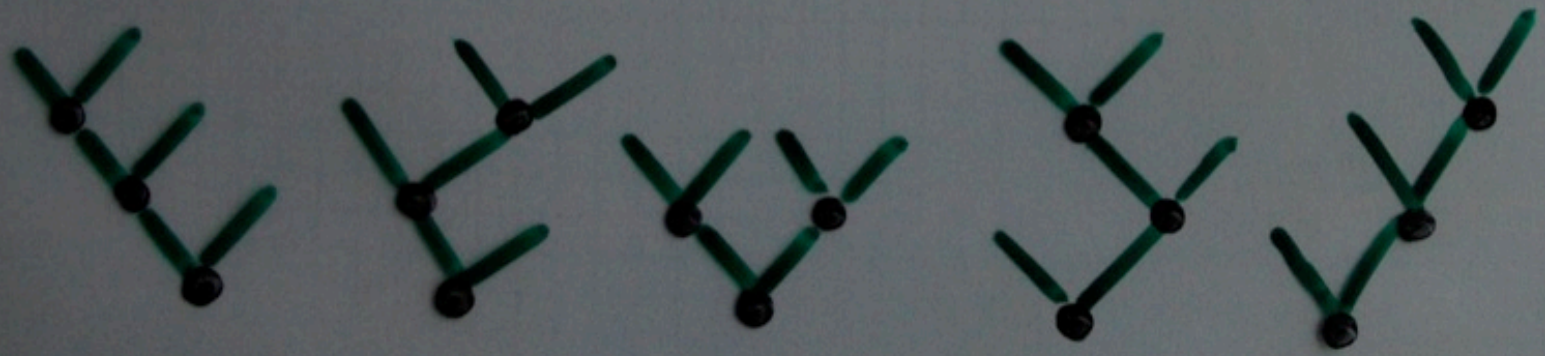
(et donc $n+1$ feuilles)

<u>nombre de Catalan</u>

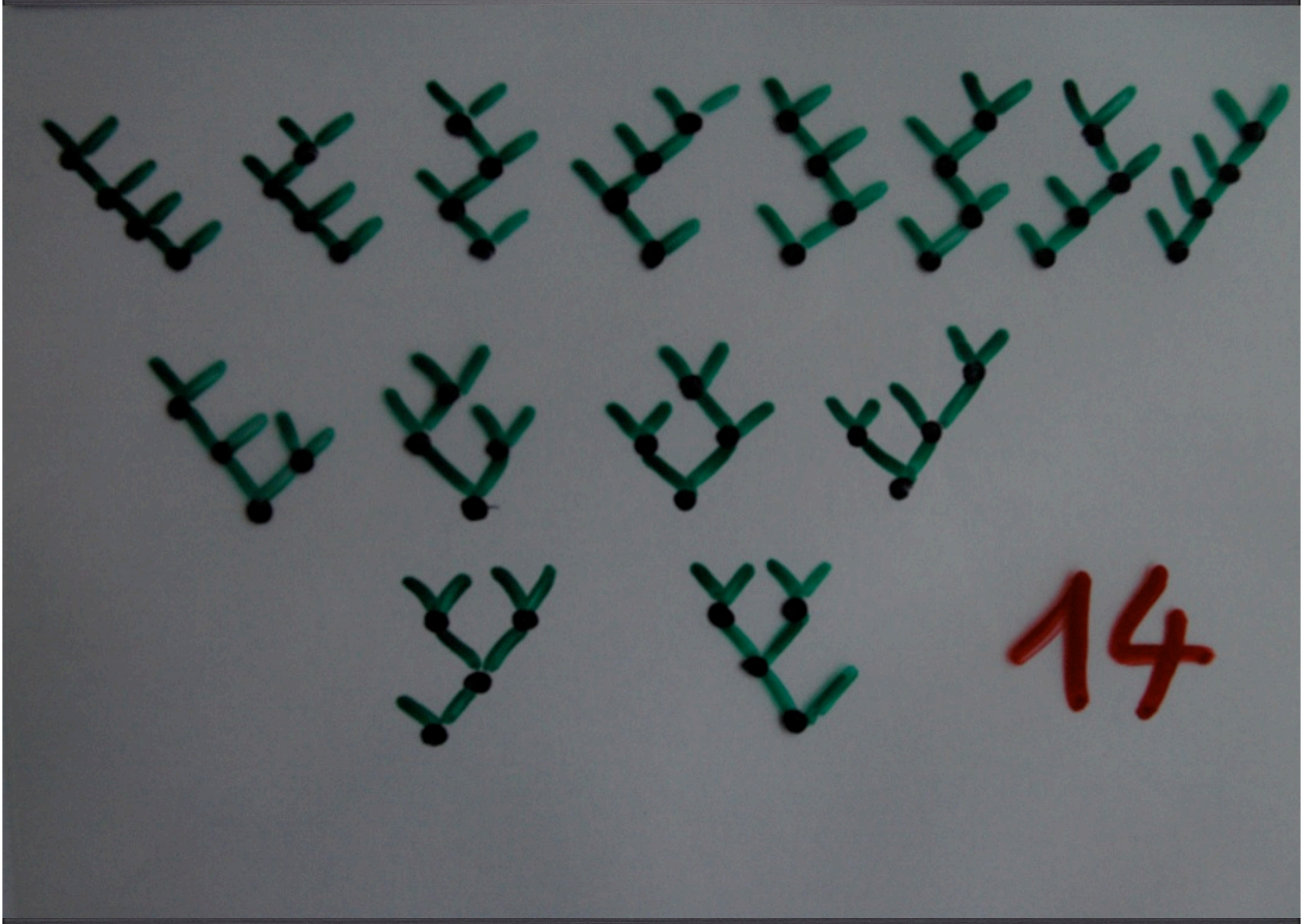
$C_1 = 1$



$C_2 = 2$



$C_3 = 5$


14

recurrence

$$C_{n+1} = \sum_{i+j=n} C_i \, C_j$$

$$C_0 = 1$$

$$\begin{array}{cccccc} C_0 & C_1 & C_2 & C_3 & C_4 & C_5 \\ 1, & 1, & 2, & 5, & 14, & 42, \ldots \end{array}$$

$$C_6 = C_0 C_5 + C_1 C_4 + C_2 C_3 + C_3 C_2 + C_4 C_1 + C_5 C_0$$

$$132 \qquad 1 \times 42 + 1 \times 14 + 2 \times 5 + 5 \times 2 + 14 \times 1 + 42 \times 1$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\ n!}$$

$$n! = 1 \times 2 \times \dots \times n$$

$$C_4 = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}{1 \times 2 \times 3 \times 4 \quad 1 \times 2 \times 3 \times 4 \times 5}$$

$$= 14$$

addition +

division :

1									
1	1								
1	2	1							
1	3	3	1						
1	4	6	4	1					
1	5	10	10	5	1				
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		
1	8	28	56	70	56	28	8	1	

1 2 5 14

vérifier "bêtement" une identité

démontrer une identité

Série génératrice

algèbre des polynômes

$$(1+t)^2 = 1+2t+t^2$$

$$(1+t)^3 = 1+3t+3t^2+t^3$$

$$1 + 1t + 2t^2 + 5t^3 + 14t^4 + 42t^5$$

polynôme

$$1 + 1t + 2t^2 + 5t^3 + 14t^4 + 42t^5$$

$$+ \cdots$$

$$y = 1 + 1t + 2t^2 + 5t^3 + 14t^4 + \dots + C_n t^n + \dots$$

série génératrice des nombres de Catalan C_n

$$t + t + t + \dots t + \dots$$

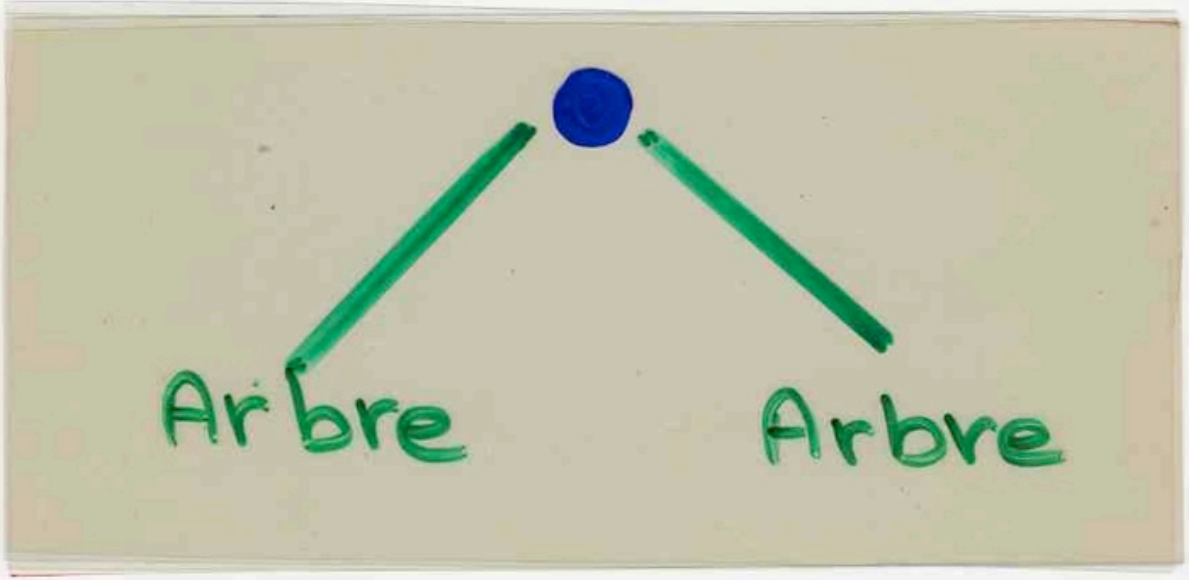
$$1 + 1 + 1 + \dots$$

$$\cancel{t + t + t + \dots t + \dots}$$

$$\cancel{1 + 1 + 1 + \dots}$$

Arbre

=



$$y = 1 + \begin{array}{c} t \\ y \quad y \end{array}$$

$$y = 1 + t(y)^2$$

$$y = 1 + t y^2$$

equation algébrique

$$y = \frac{1-(1-4t)^{1/2}}{2t}$$

$$(1+u)^m =$$

$$1 + \frac{m}{1!}u + \frac{m(m-1)}{2!}u^2 + \frac{m(m-1)(m-2)}{3!}u^3 +$$

$$+ \dots$$

$$m = \frac{1}{2}$$

$$u = -4t$$

$$y = 1 + 1t + 2t^2 + 5t^3 + 14t^4 + \dots + C_n t^n + \dots$$

série génératrice des nombres de Catalan C_n

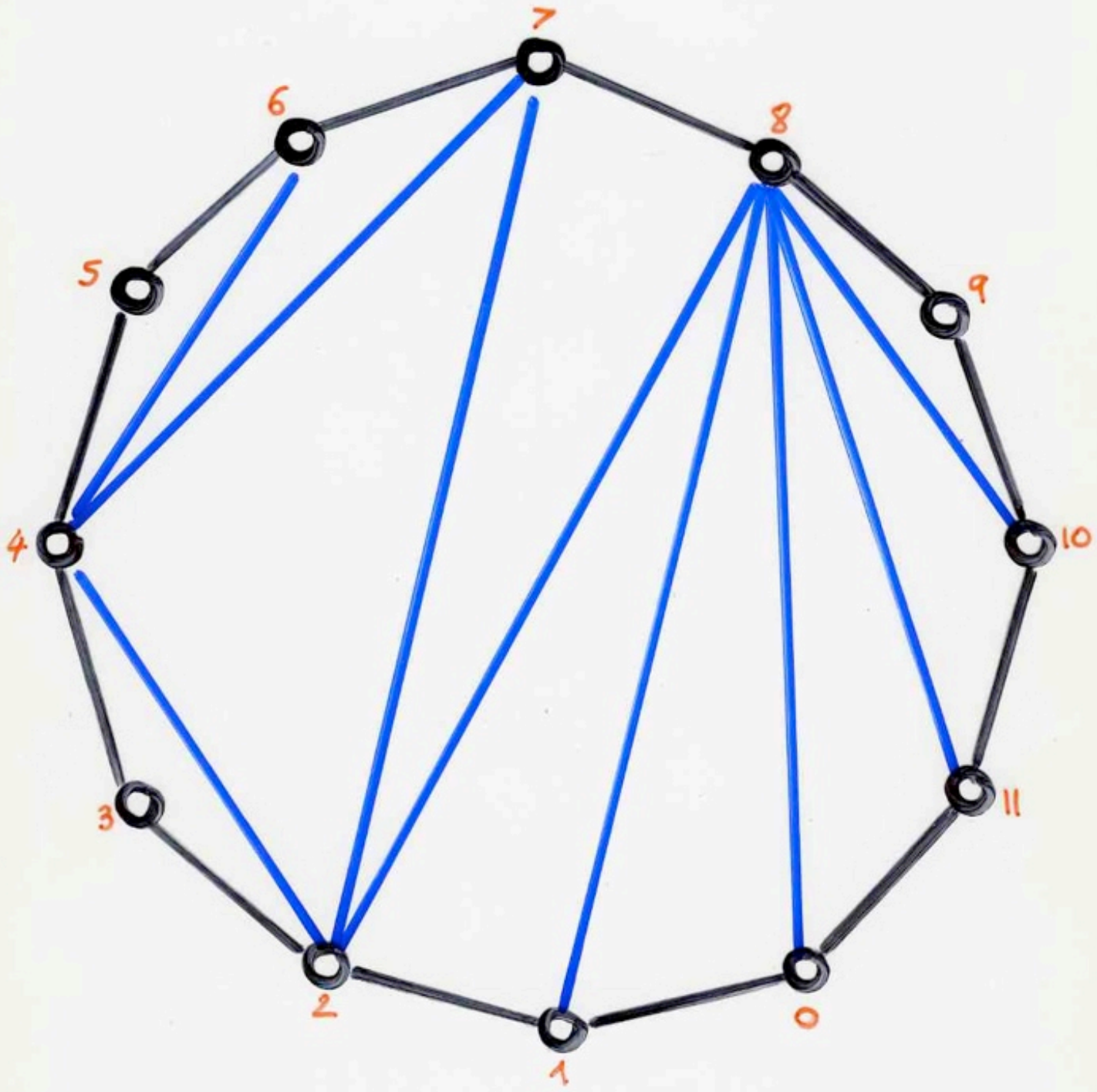
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\ n!}$$

$$n! = 1 \times 2 \times \dots \times n$$

triangulations

Leonhard
Euler
1707 - 1783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Theil, und solches kann auf 5 mahl verschiedene Arten geschehen nemlich
durch die Diagonales I. ac, ad; II. bd, be; III. ce, ca; IV. de, da; V. ea, eb

Wenn hier ein Theil durch 3 Diagonales in 4 Triangula getheilt, und solches kann auf 14 verschiedene Arten geschehen.
Nun ist die ganze Generalitet. Da ein Polygonum von n Seiten durch n-3 Diagonales in n-2 Triangula getheilt wird, auf wie vielerley verschiedene Arten solches geschehen kann.
Setze ich nun die Anzahl dieser verschiedenen Arten = x
so habe ich per Inductionem gefunden

wenn $n = 3 . 4 . 5 . 6 . 7 . 8 . 9 . 10$
so ist $x = 1 , 2 , 5 , 14 . 42 , 132 , 429 , 1430$

Hieraus habe ich nun den Schluß gemacht, die generalitet sey

$$x = \frac{2 . 6 . 10 . 14 . 18 . 22 . \ldots (4n-10)}{2 . 3 . 4 . 5 . 6 . 7 \ldots (n-1)}$$ oder es ist

$1 = \frac{2}{2}$; $2 = 1 . \frac{6}{3}$; $5 = 2 . \frac{10}{4}$; $14 = 5 . \frac{14}{5}$; $42 = 14 . \frac{18}{6}$; $132 = 42 . \frac{22}{7}$;

daß alle auf einerley Art die Regel durch gefunden wird. Die Induction aber ist gar leicht, weil sich nur eine Art Theilung ist nicht. Daß diese Art nicht alle und letzte Art Theilung sey. Also die Progression der Zahlen 1, 2, 5, 14, 42, 132, etc. habe ich auch diese Regel zu generiren. Daß

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc.} = \frac{1 - 2a - \sqrt{(1-4a)}}{2a}$$

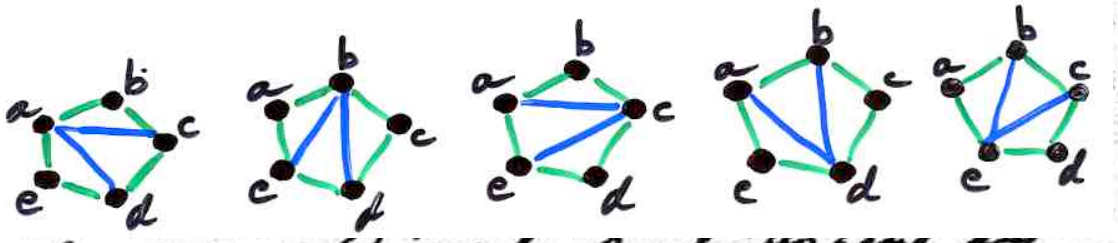
also wenn $a = \frac{1}{4}$ so ist $1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} + \text{etc.} = 4$.

Alles dies wenige bestehet ich für die Hochwohlgebohrne und stundige Abgunstigkeit gutherzigst anzunehmen und habe die Ehre mich der fortdauernden Hochachtung zu empfehlen

Ew. Hochwohlgebohrnen

Berlin d. 4. Sept. 1751.

gehorsamster Diener
L. Euler



$$1, 2, 5, 14, 42, 132, 429, 1430, \ldots$$

$$x = 1, 2, 6, 14, 42, 132, 429, \ldots$$

$$x = \frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \cdots (4n-10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots (n-1)} = \frac{(2n)!}{(n+1)!\,n!}$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$n! = 1 \times 2 \times 3 \times \ldots \times n$$

$$\frac{1-2a-\sqrt{1-4a^2}}{2a^2}$$

$$1+2a+5a^2+14a^3+42a^4+132a^5+\text{etc}$$

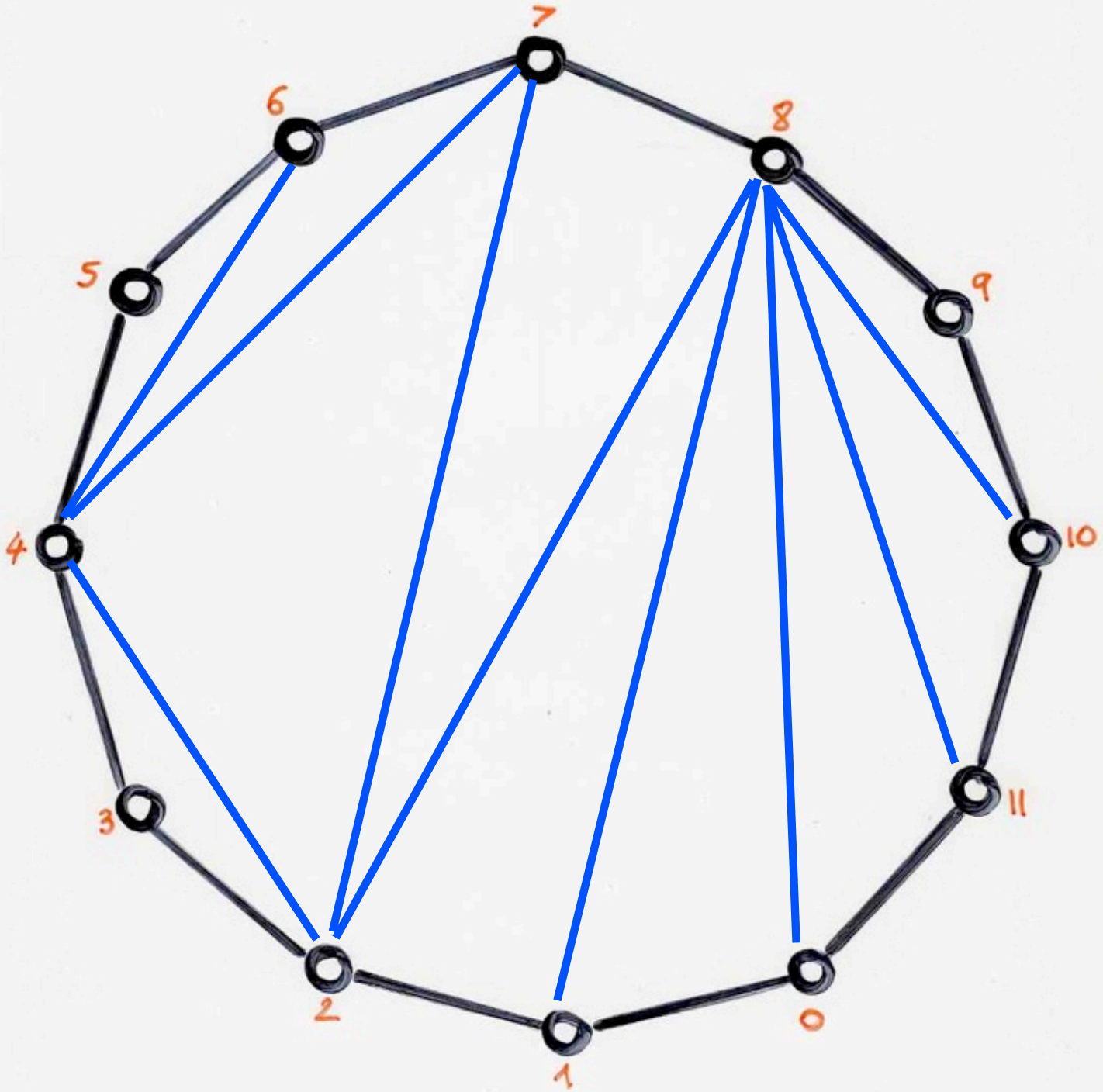
$$1+2a+5a^2+14a^3+42a^4+132a^5+\text{etc} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$$

$$1+\frac{2}{4}+\frac{5}{4^2}+\frac{14}{4^3}+\frac{42}{4^4}+\text{etc} = 4.$$

4 Sept 1751
Berlin

bijection

Des triangulations
aux arbres binaires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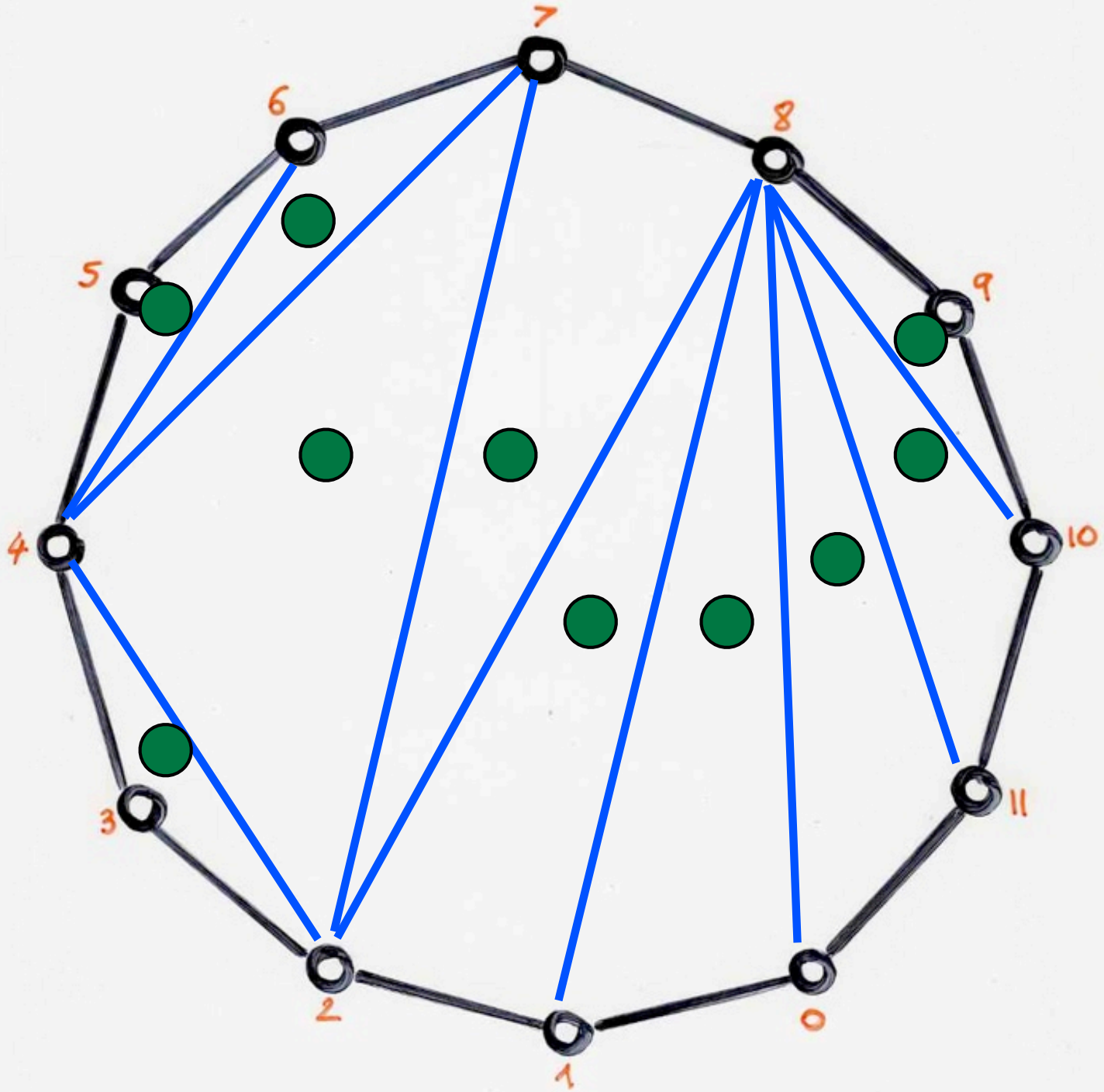
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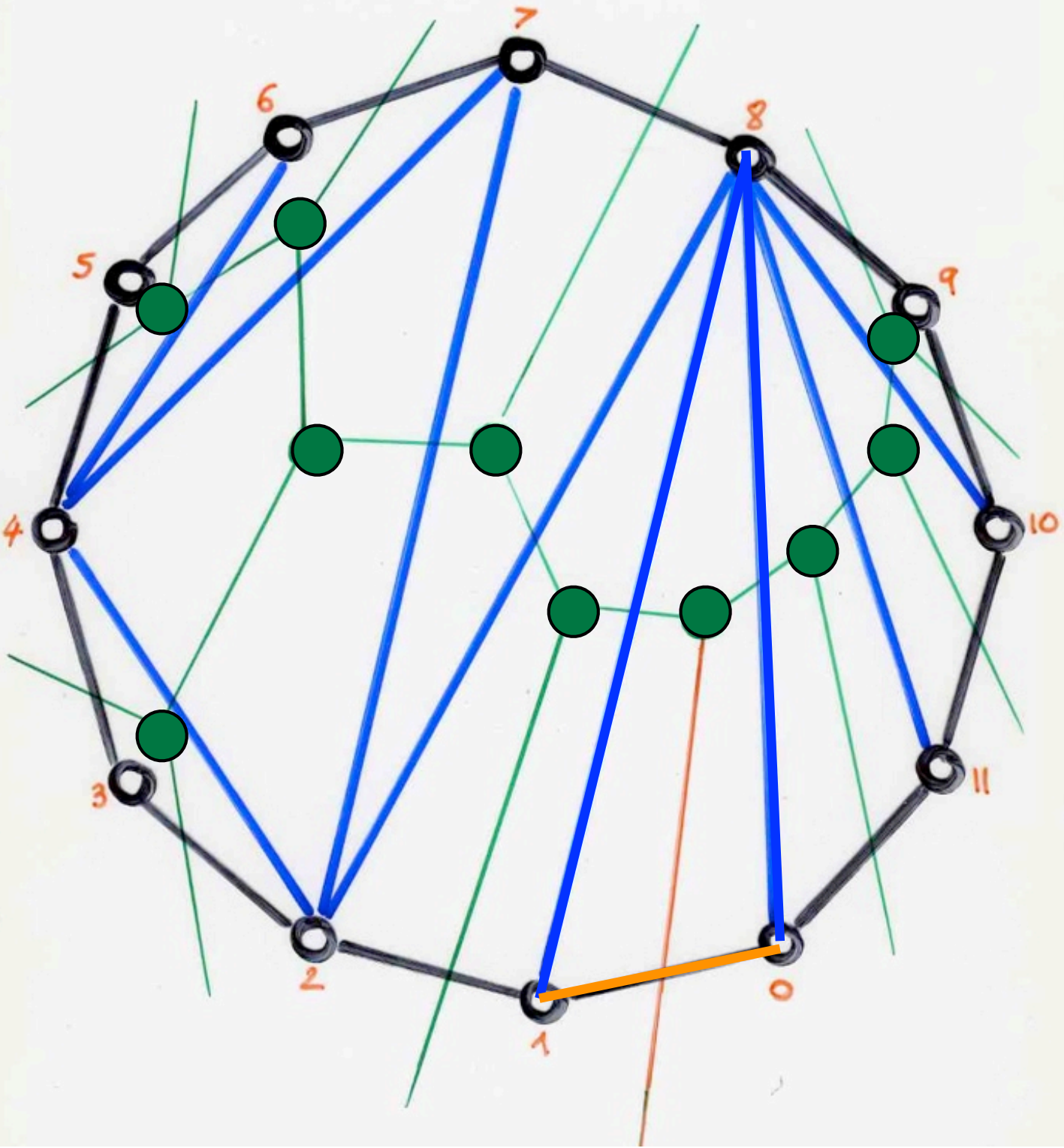
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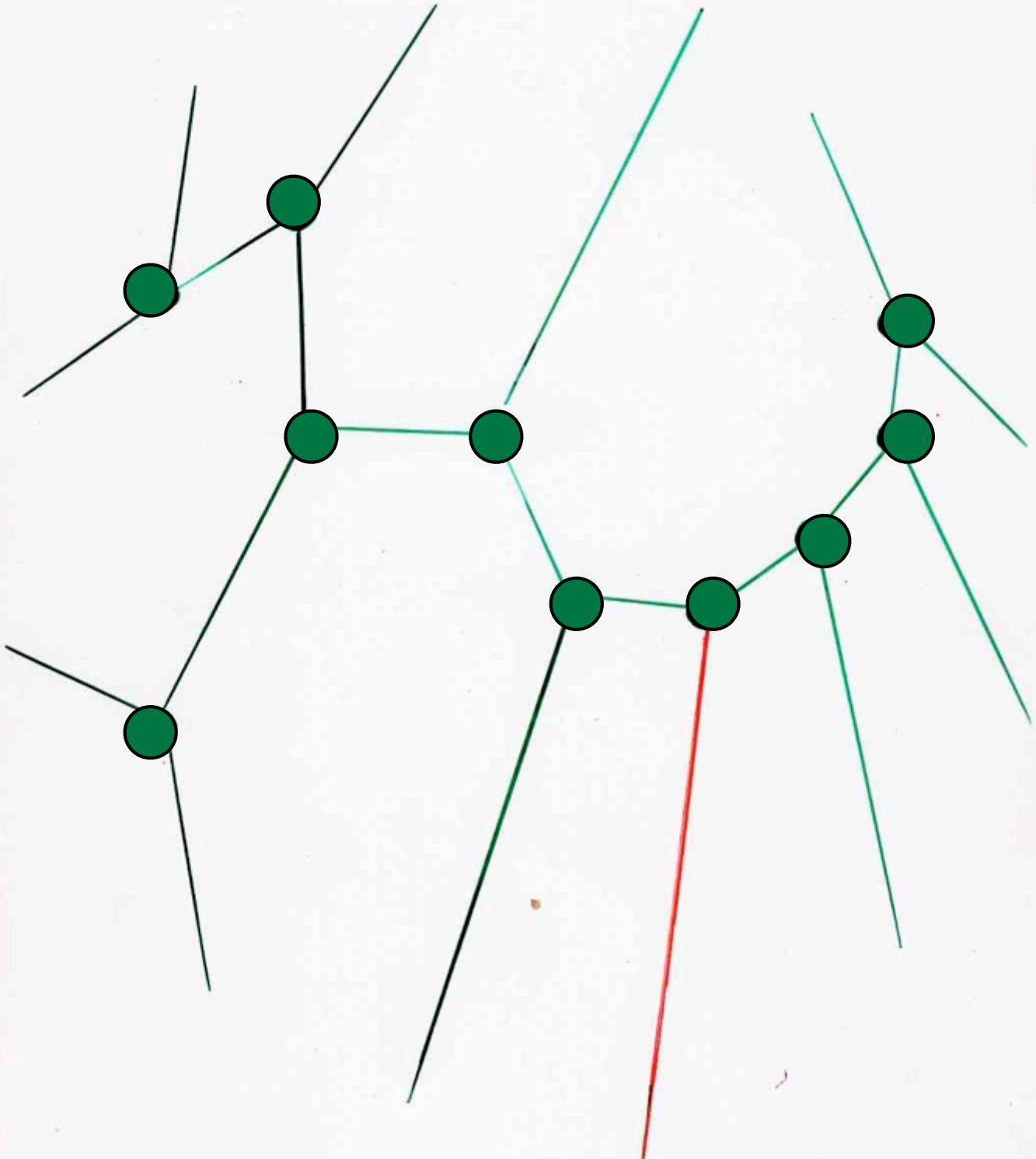
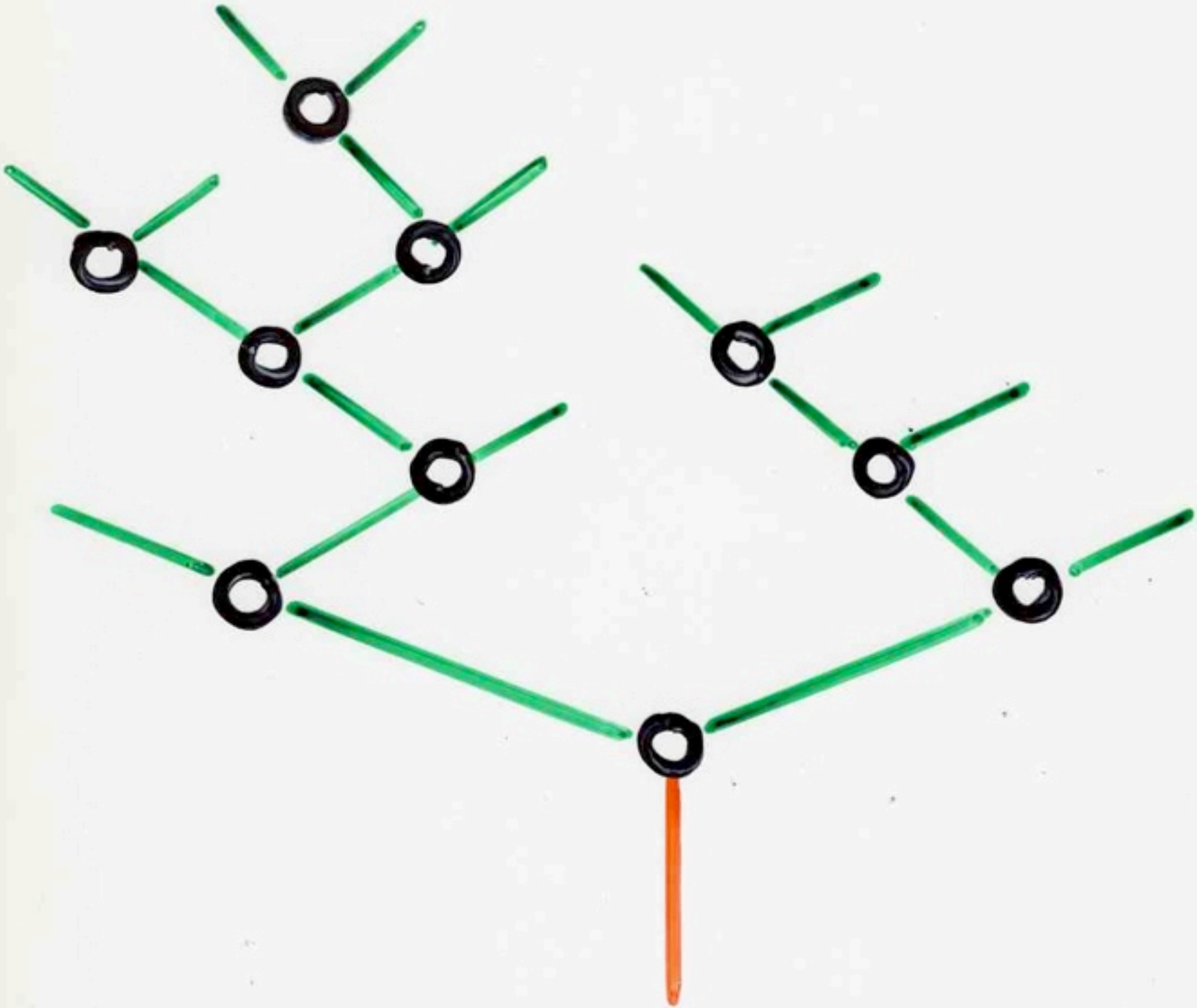
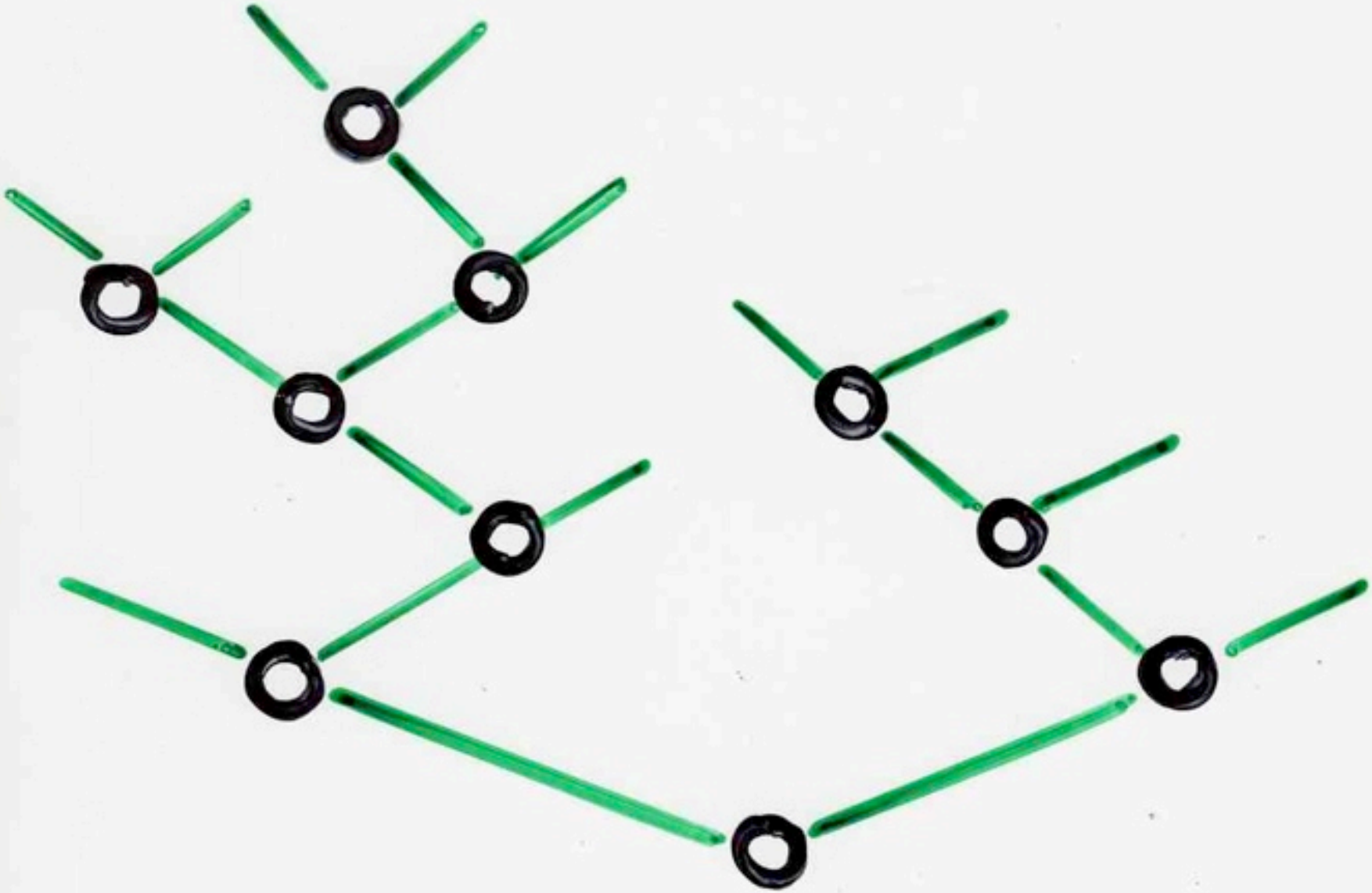






combinatoire bijective

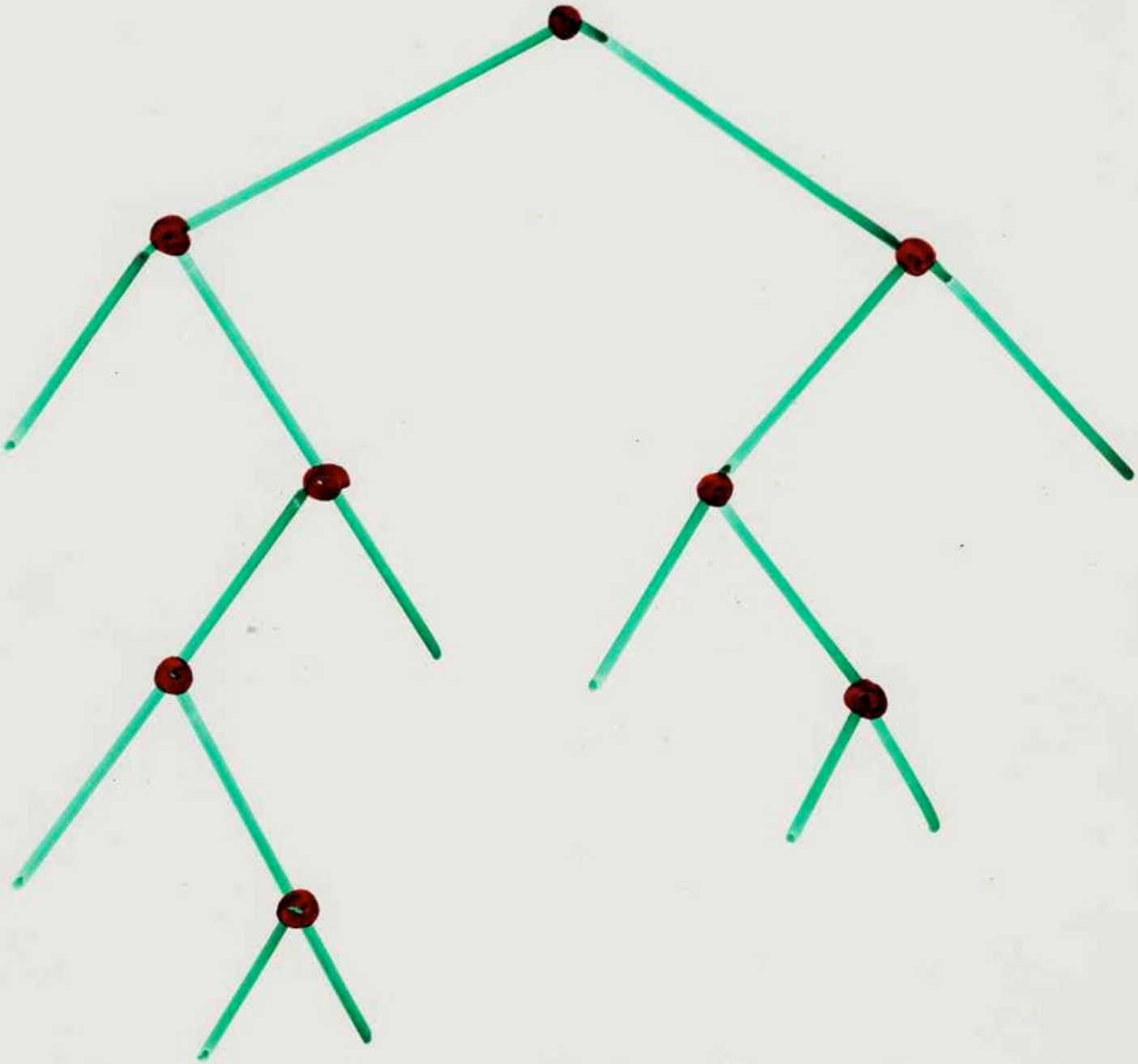
vérifier "bêtement"

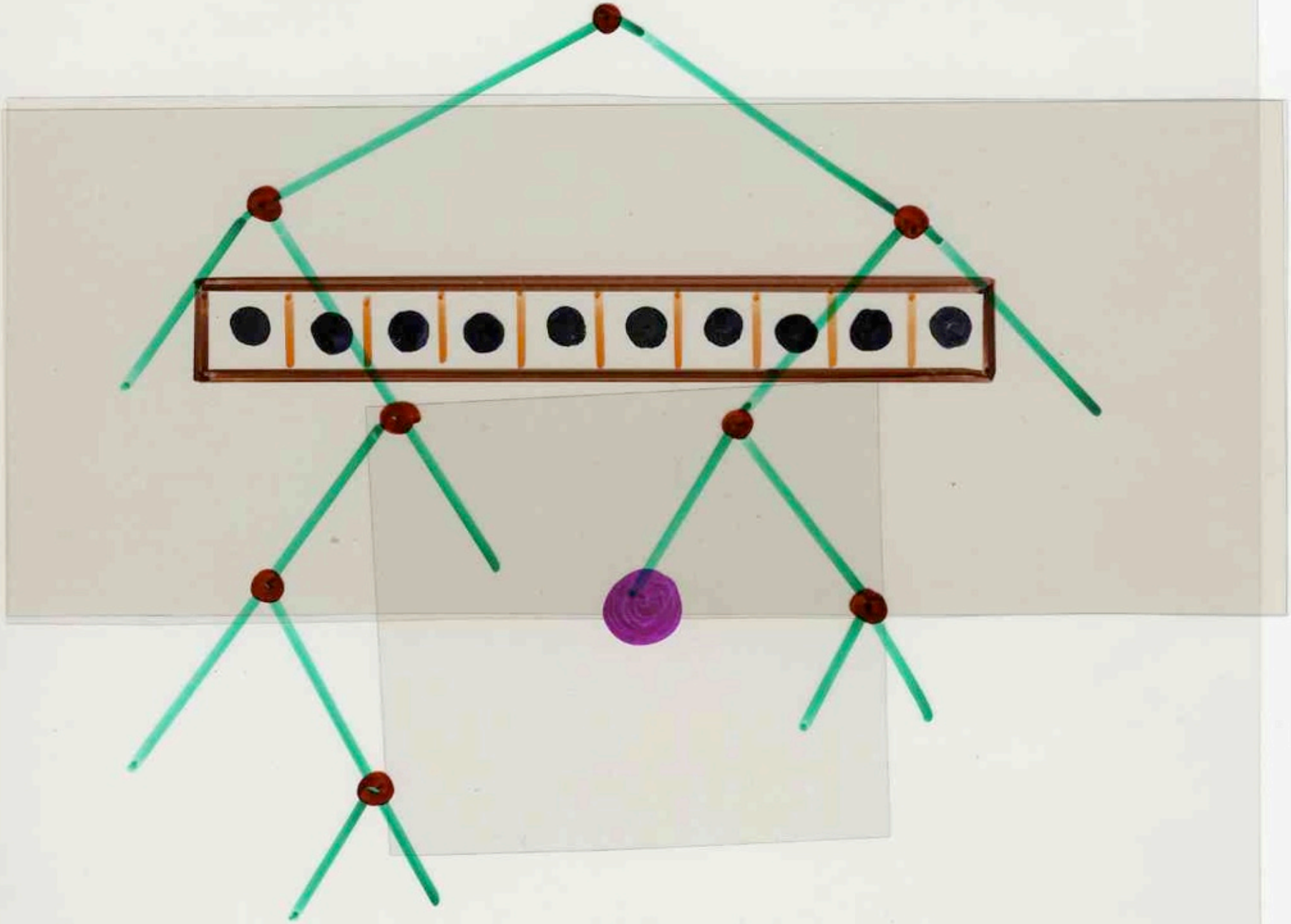
démontrer une identité

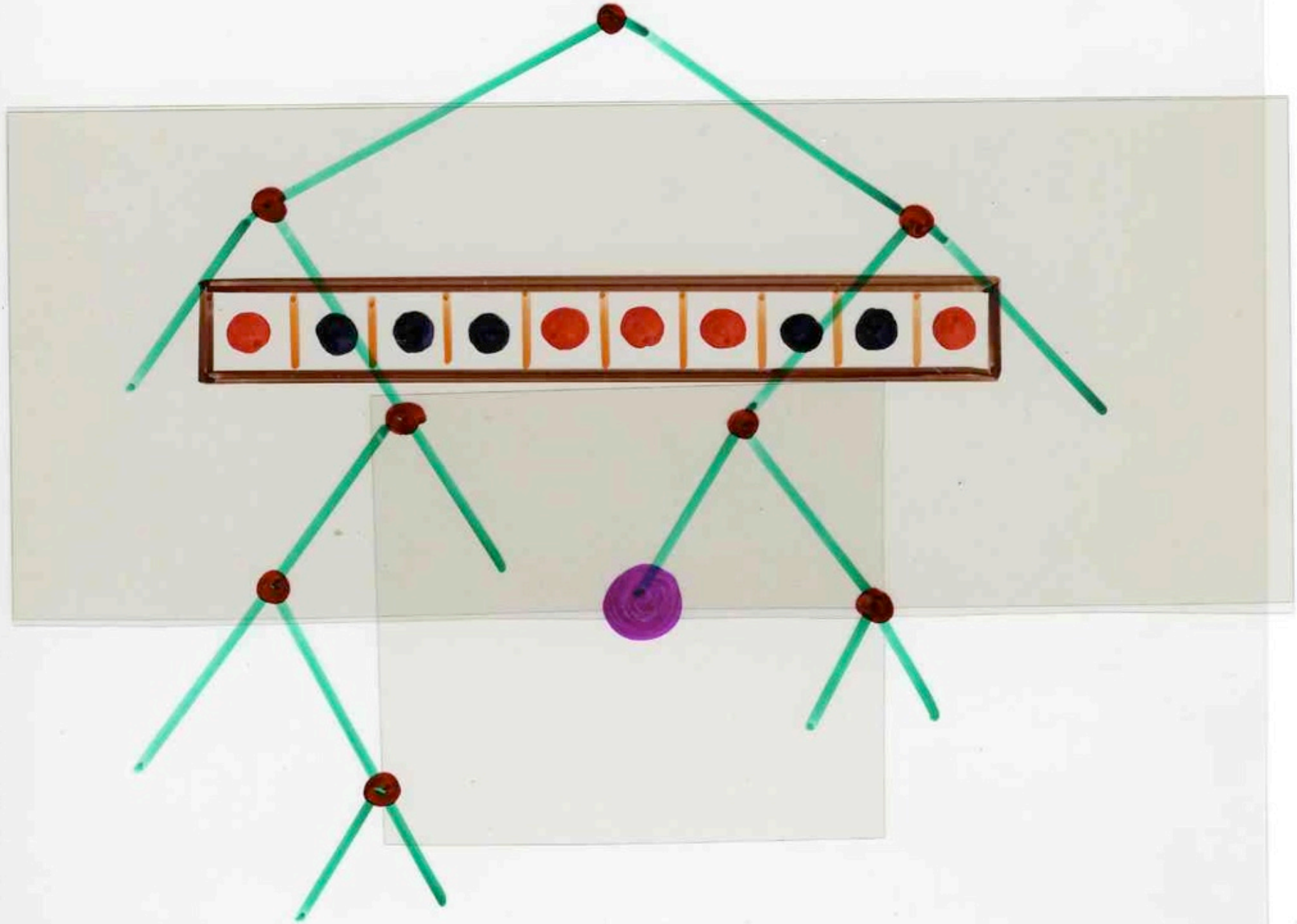
"comprendre" l'identité

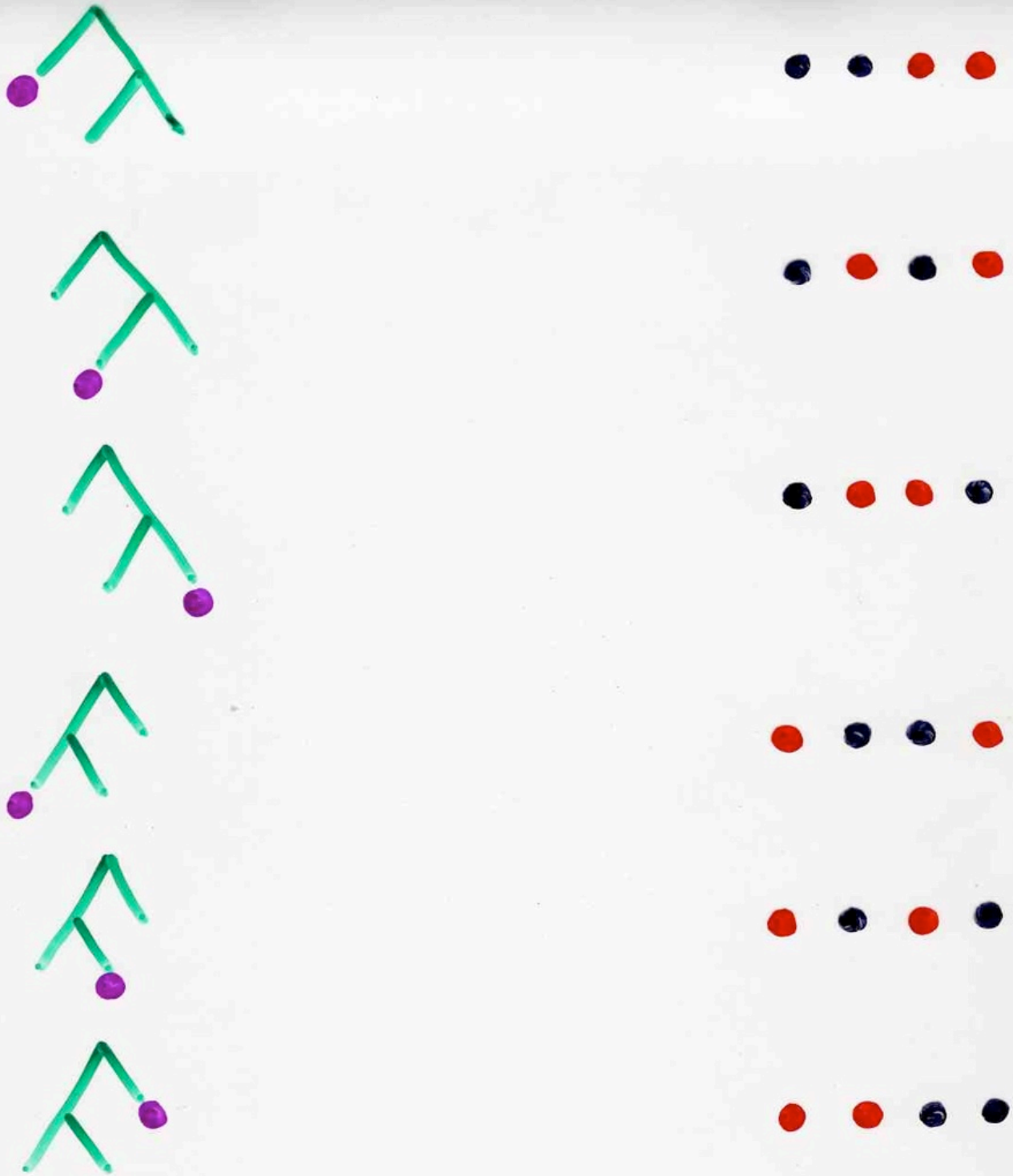
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

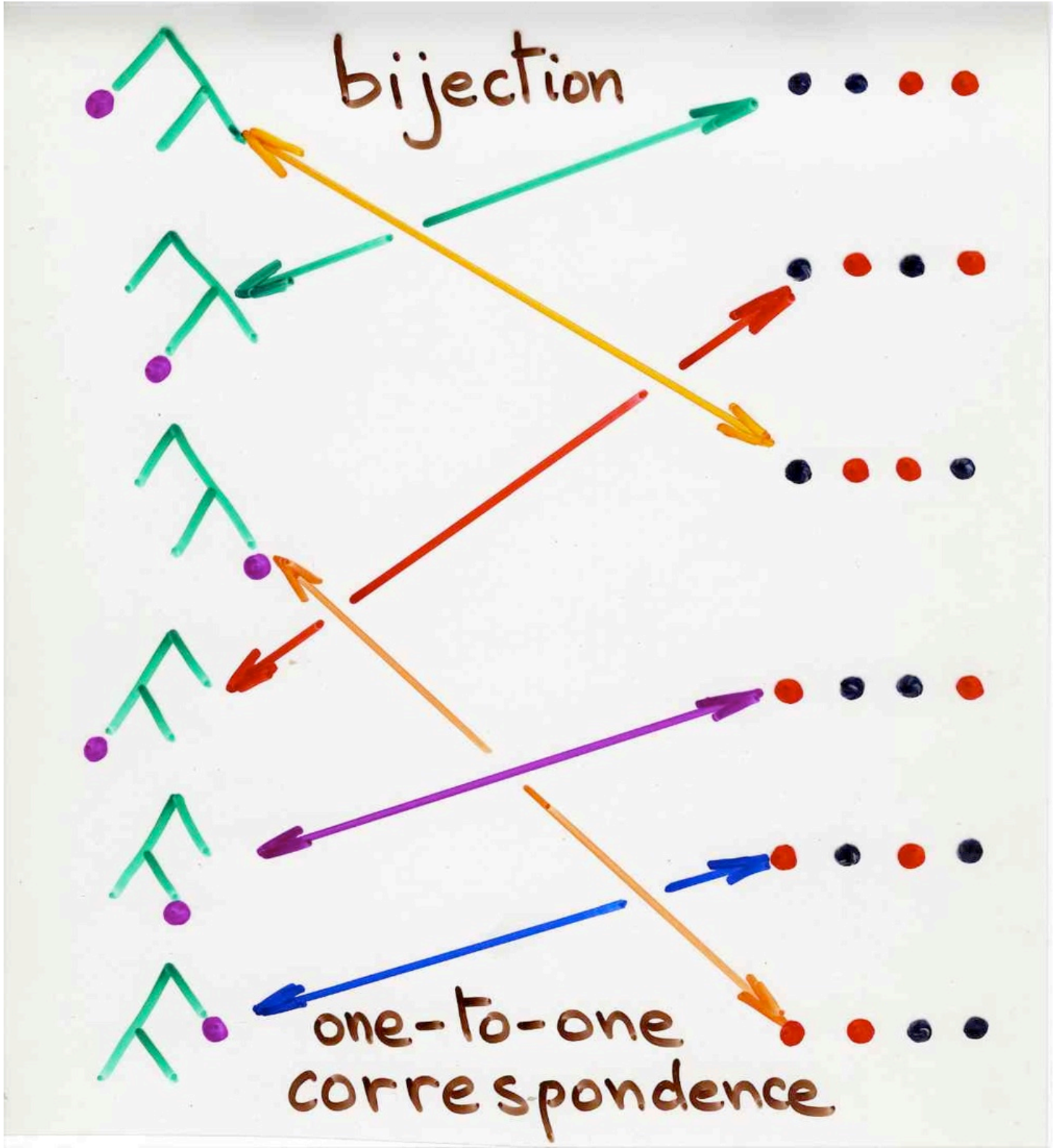
$$(n+1)\,C_n = \binom{2n}{n}$$



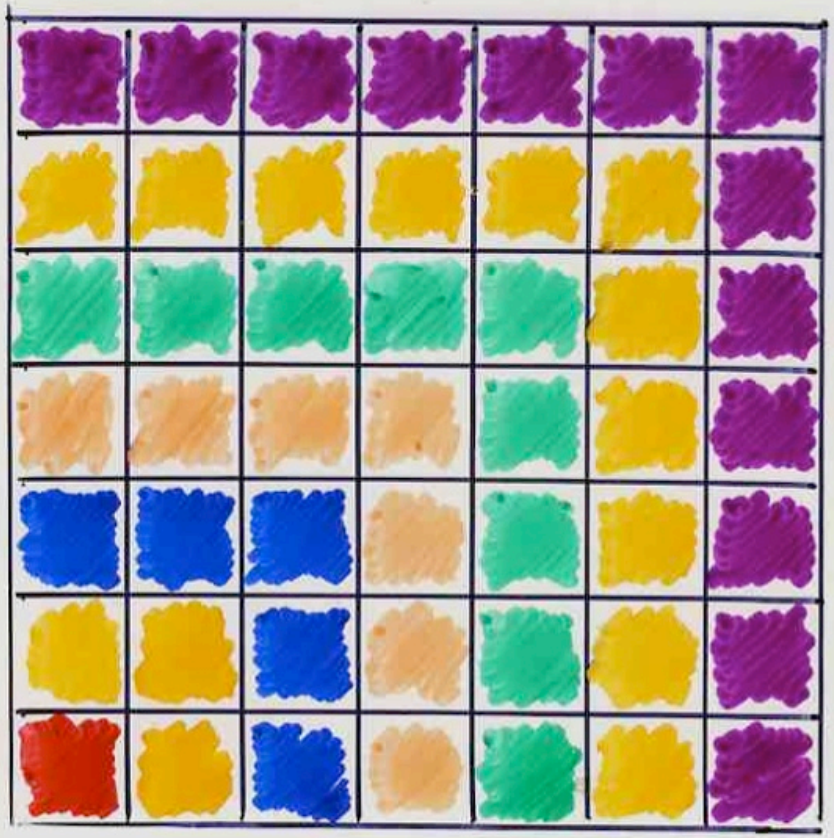







bijection
one-to-one
correspondence

preuve bijective d'une identité



$$n^2 = 1 + 3 + \ldots + (2n-1)$$

un exemple:

identité de Mehler

pour les polynômes d'Hermite

$$\sum_{n\geq 0} H_n(x)\,H(y)\,\frac{t^n}{n!} = (1-4t^2)^{-1/2}\exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

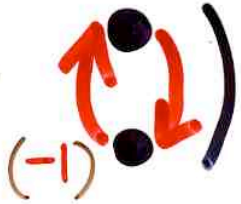
"dessiner des calculs"

à chaque "morceau" d'une identité

$\updownarrow$

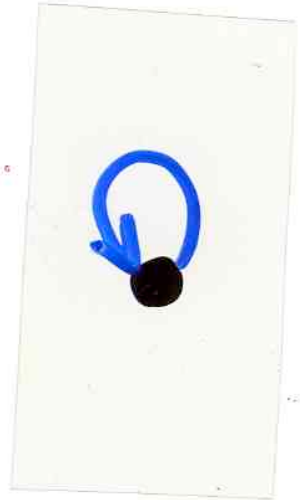
objet combinatoire

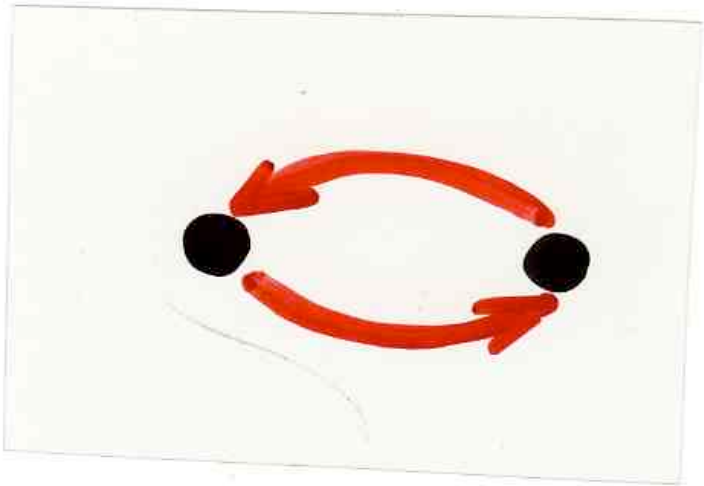
identité $\longleftrightarrow$ correspondances
construction combinatoire
bijection

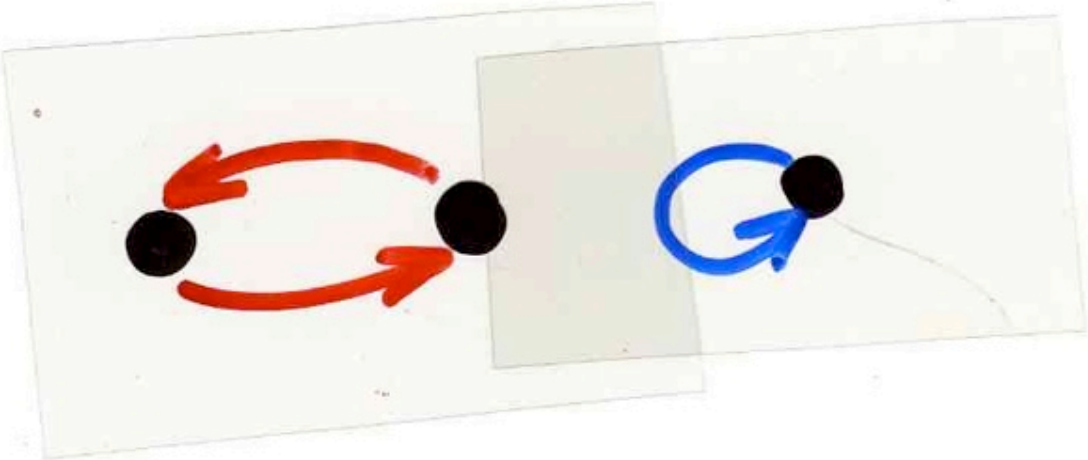
$\exp\Big($  $+$  $\Big)$

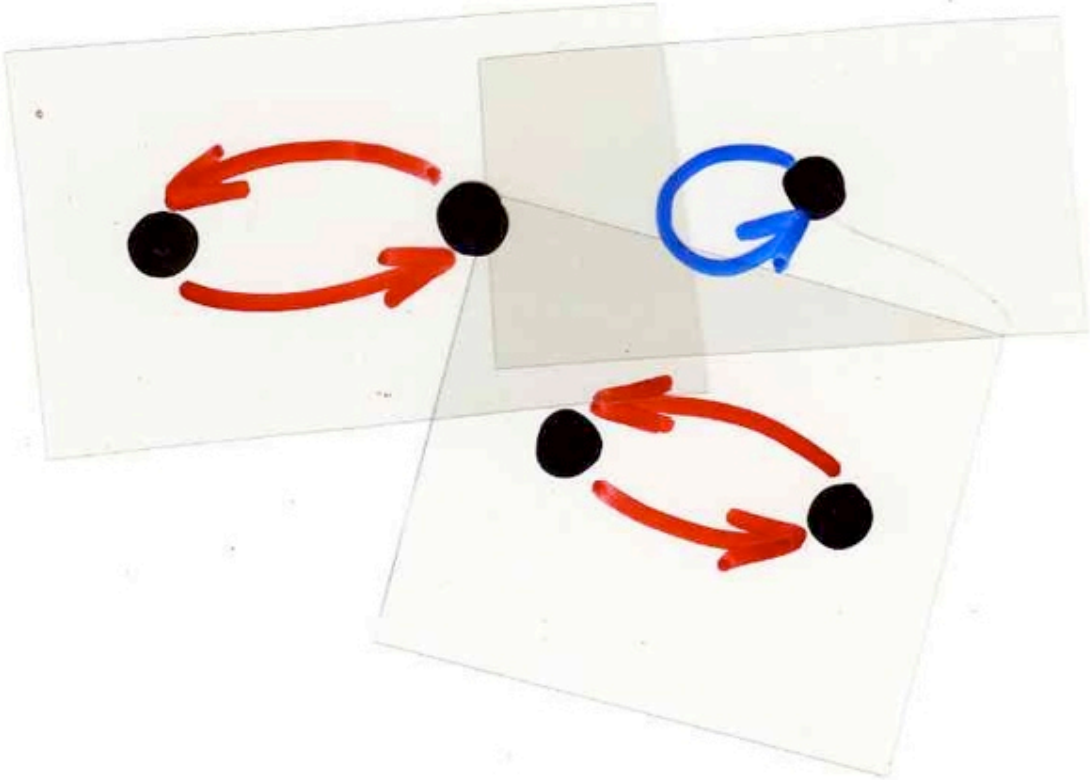
(x) (-1)

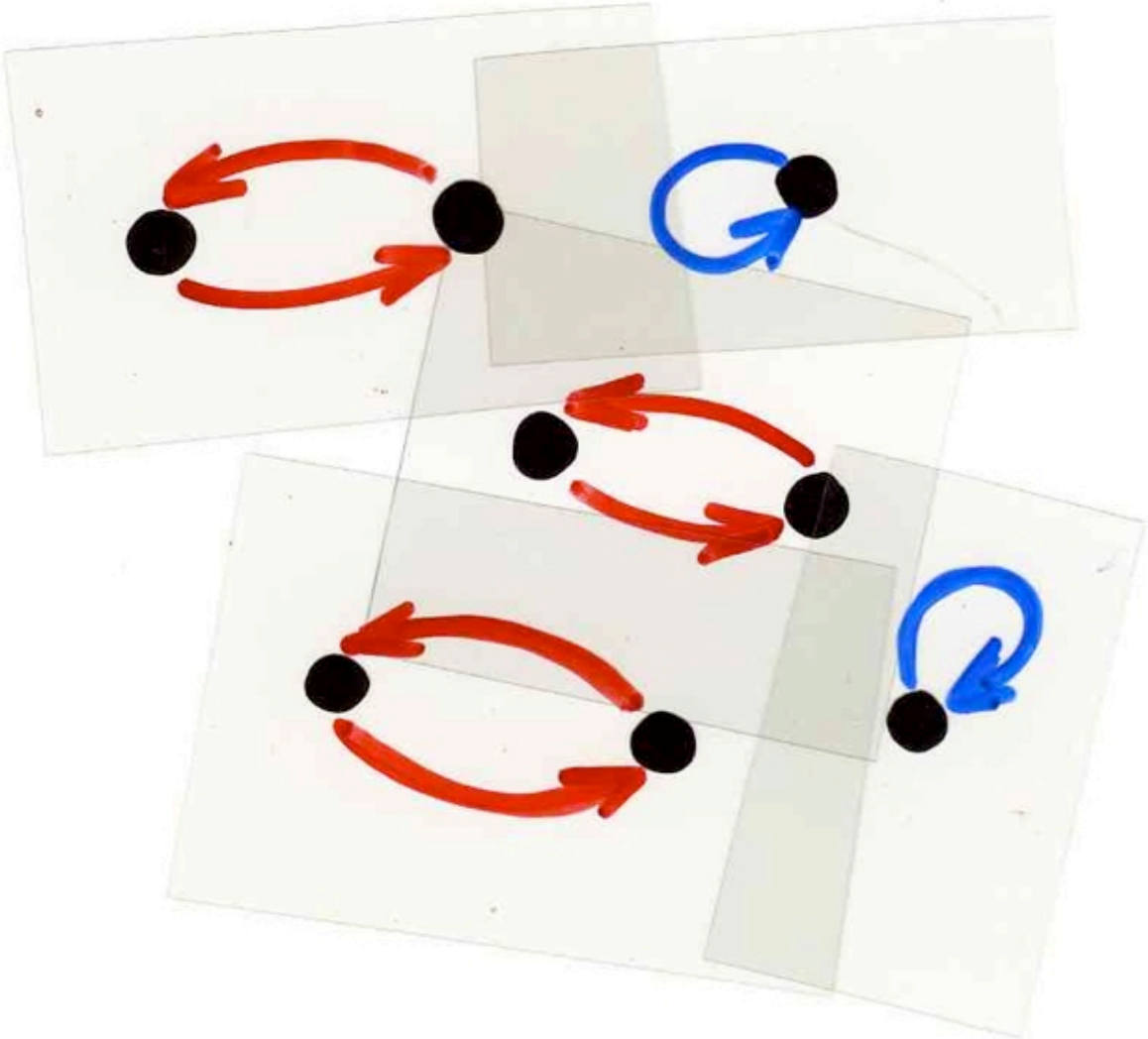
$$\sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = \exp\left(xt - \frac{t^2}{2}\right)$$

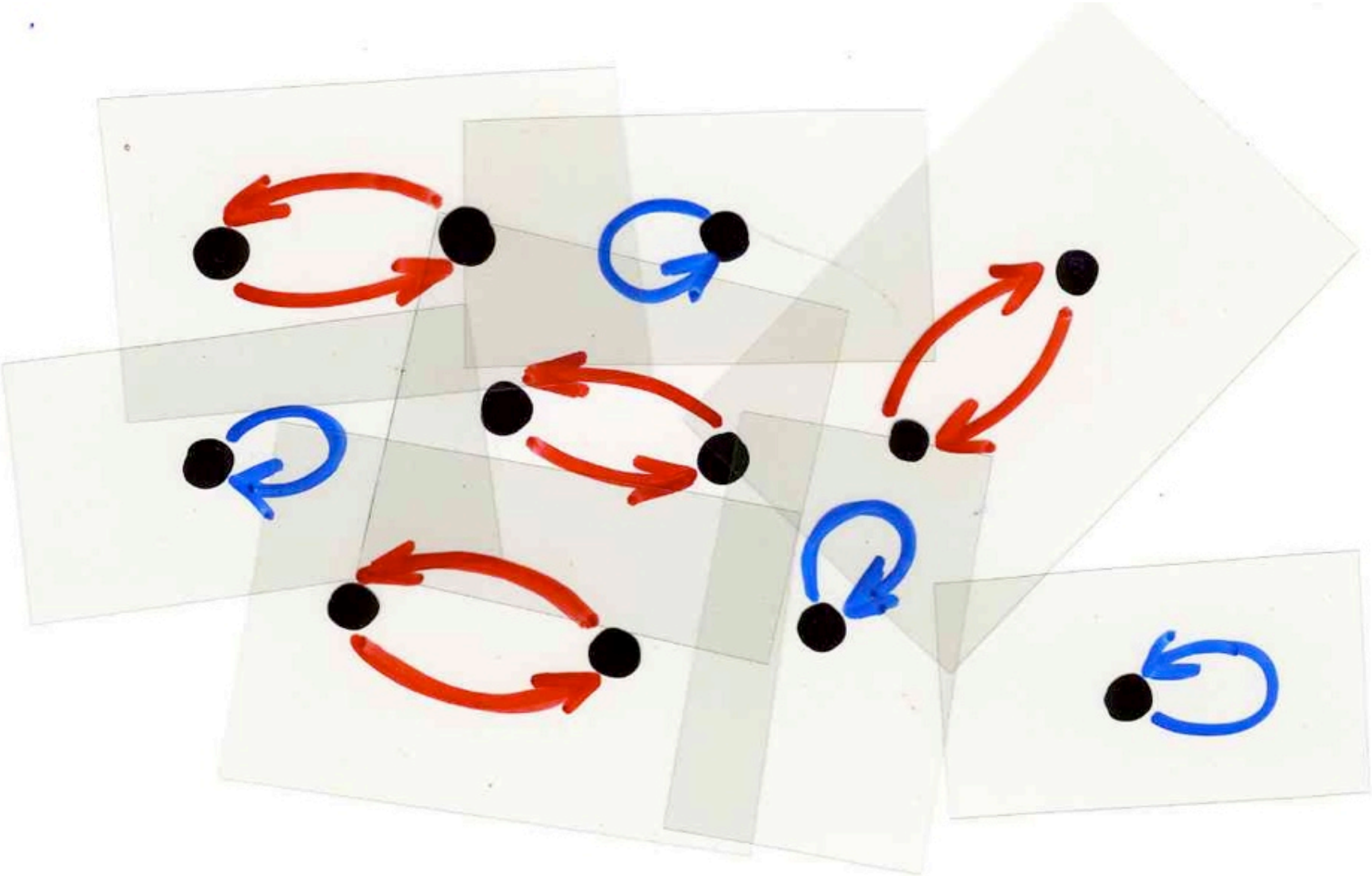


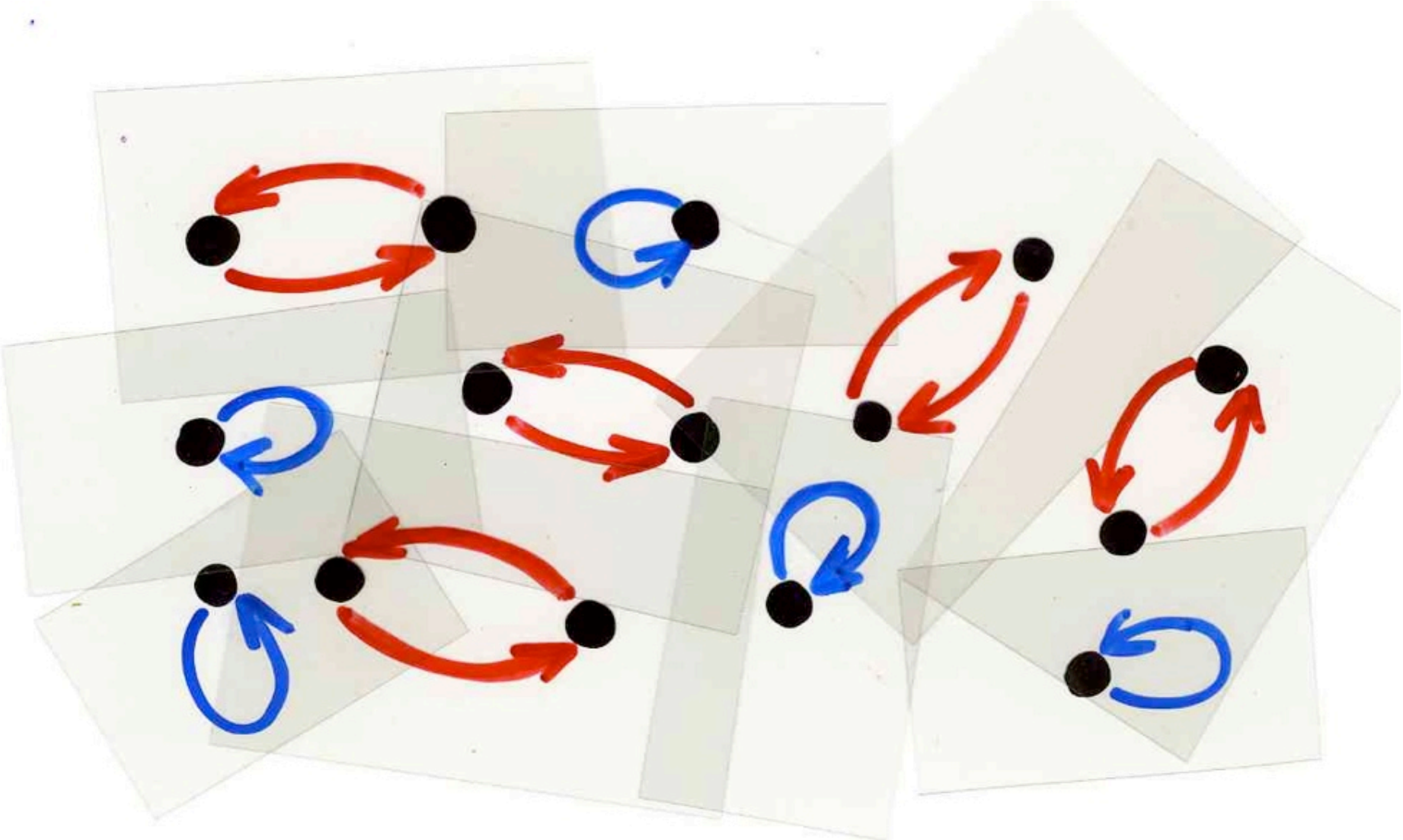




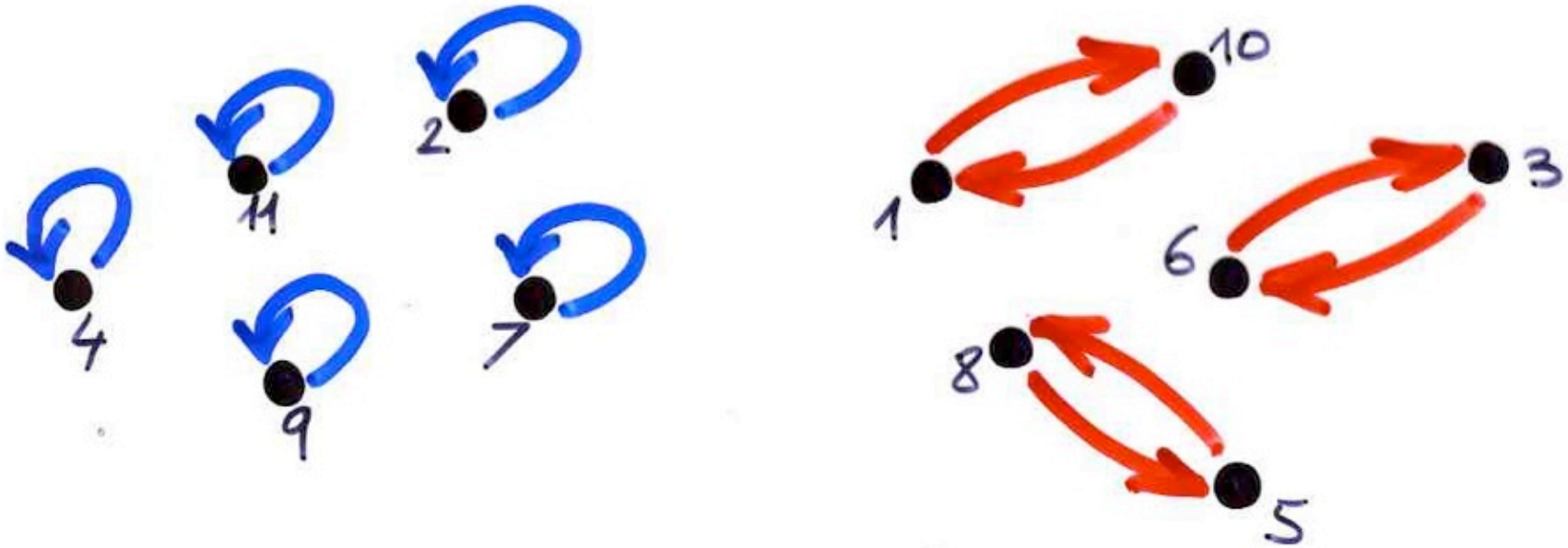




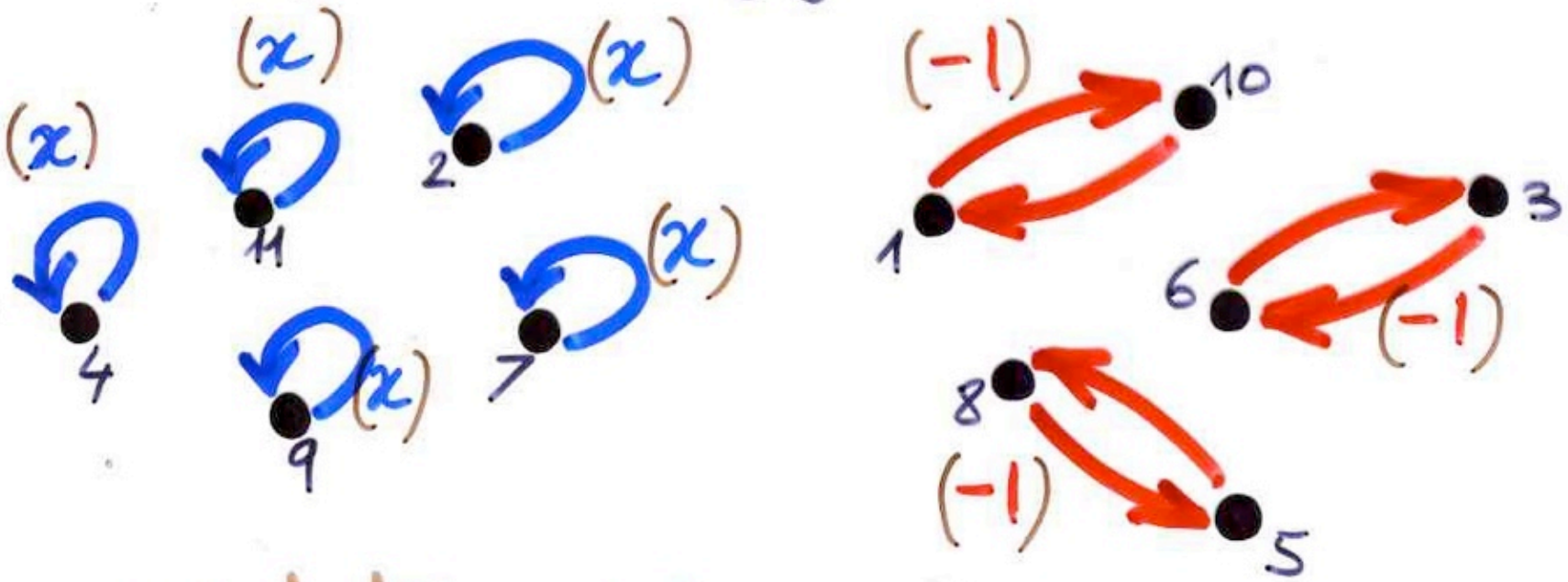




Hermite configurations



Hermite configurations

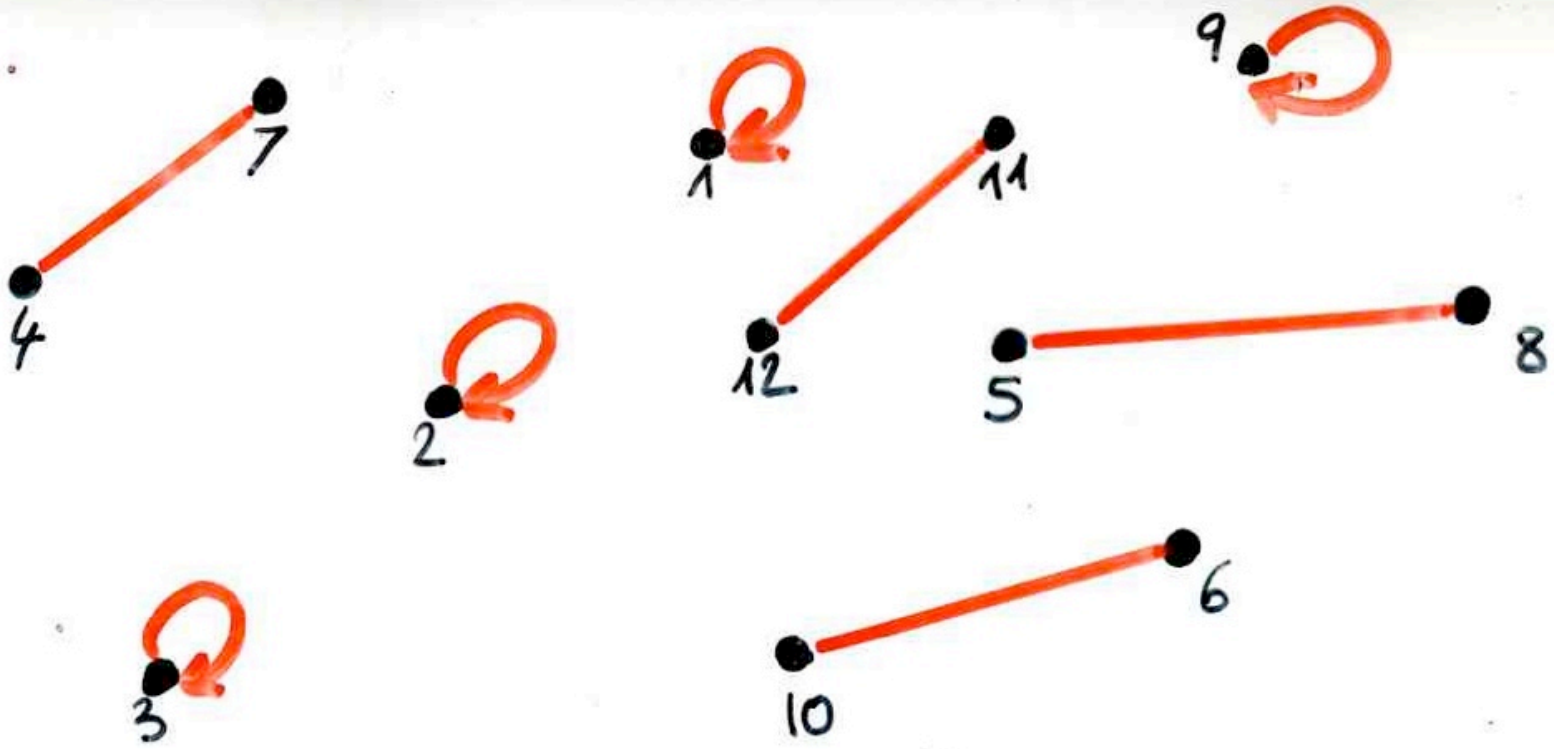


weight (x)

(-1)

$$\sum_{n\geq 0} H_n(x) H_n(y) \frac{t^n}{n!} = (1-4t^2)^{-1/2} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\sum_{n\geq 0} H_n(x) \quad H_n(y) \quad \frac{t^n}{n!} =$$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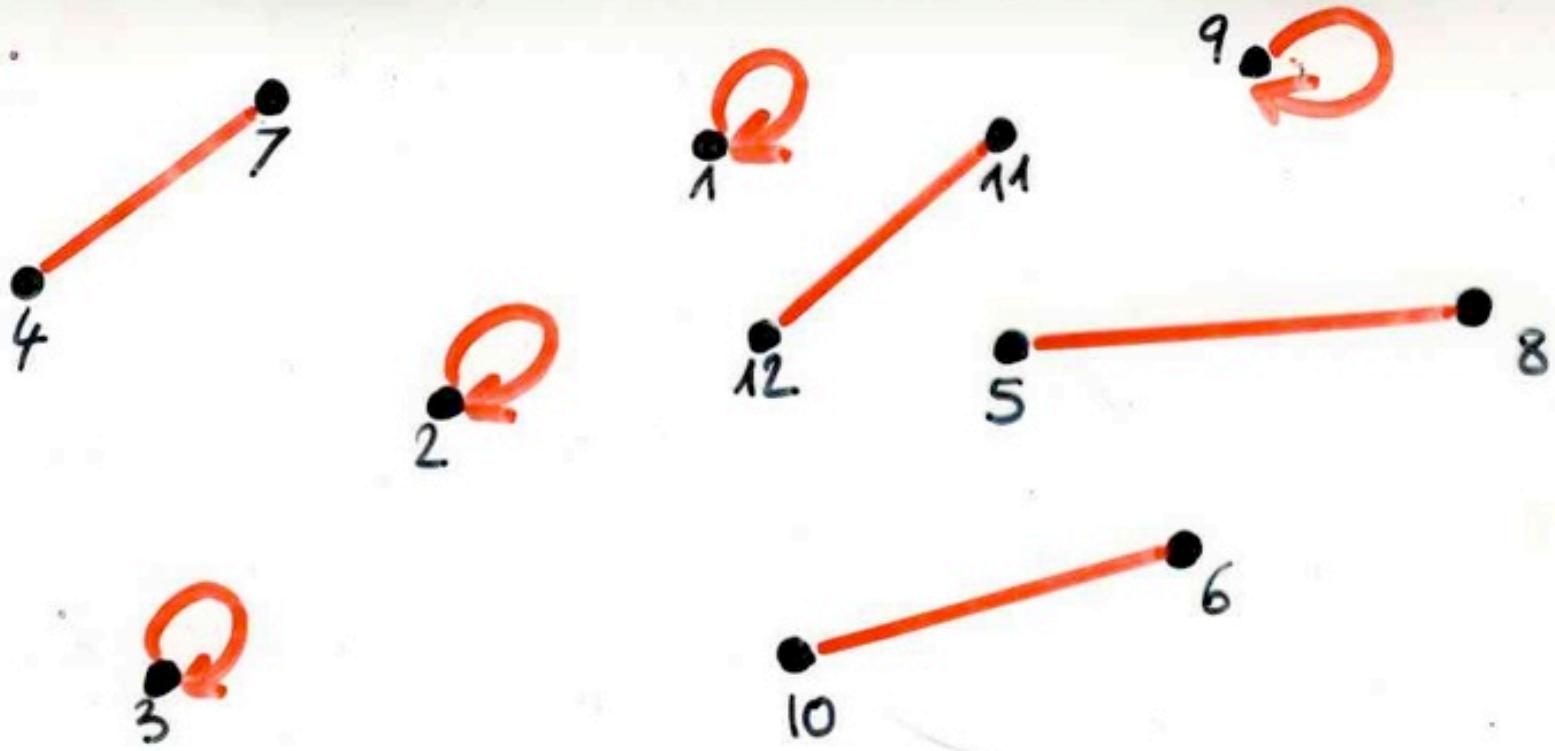
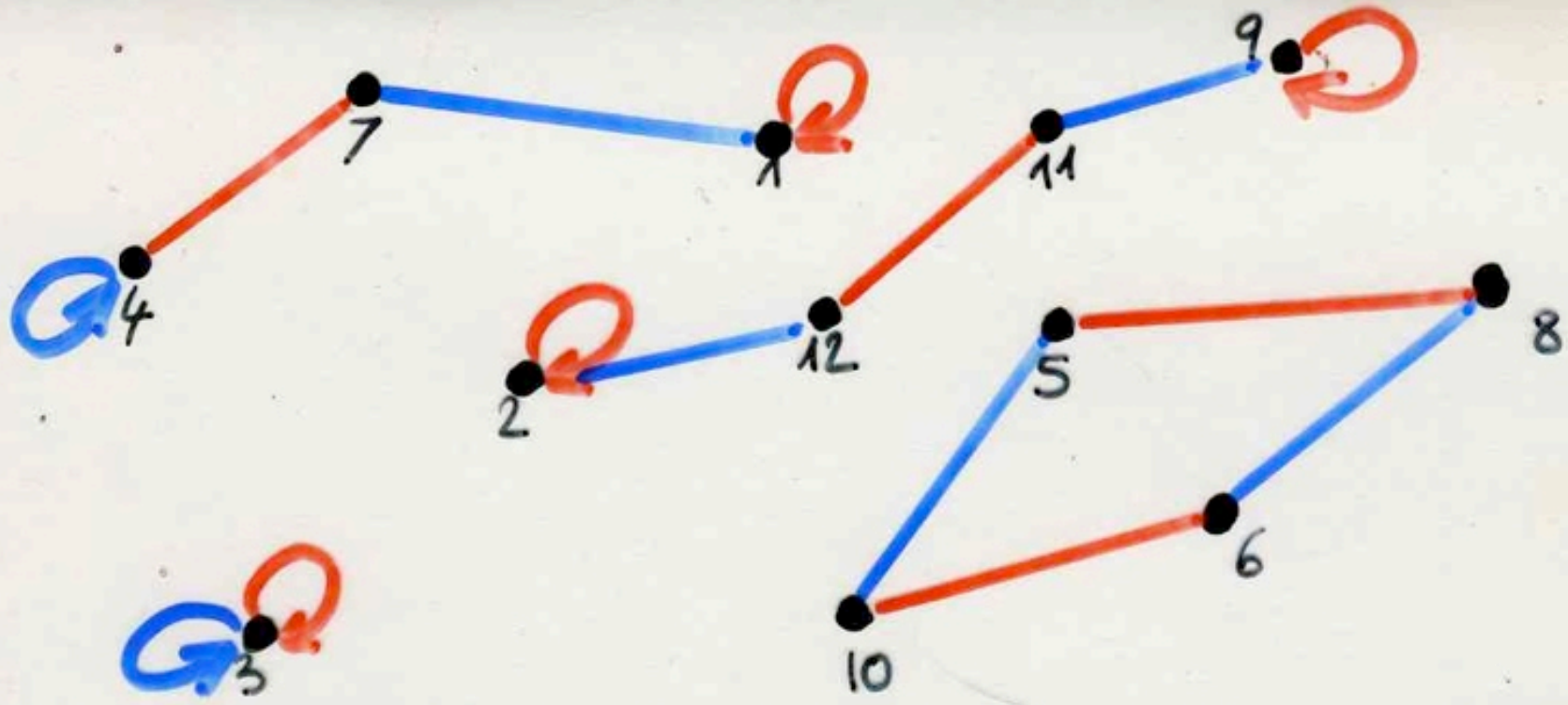


$$\sum_{n\geq 0} H_n(x) \frac{t^n}{n!} =$$



$$\sum_{n\geq 0} H_n(x)H_n(y)\frac{t^n}{n!} =$$

$$(1-4t^2)^{-1/2} \exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\exp\left[\frac{1}{2}\log\frac{1}{(1-4t^2)} + \frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

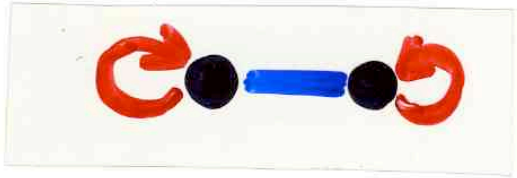
$$\exp\left[\frac{1}{2}\log\frac{1}{(1-4t^2)} + \frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\frac{-4\,y^2\,t^2}{1-4t^2}$$

$$\frac{1}{2}\log\frac{1}{(1-4t^2)}$$

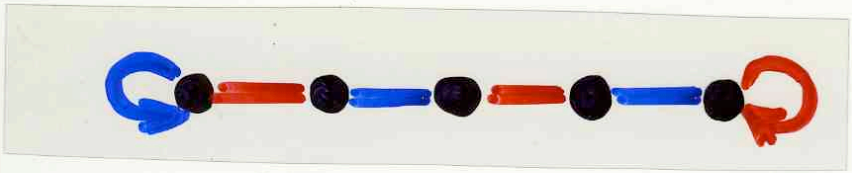
$$\frac{-4\,x^2\,t^2}{1-4t^2}$$

$$\frac{4xyt}{1-4t^2}$$

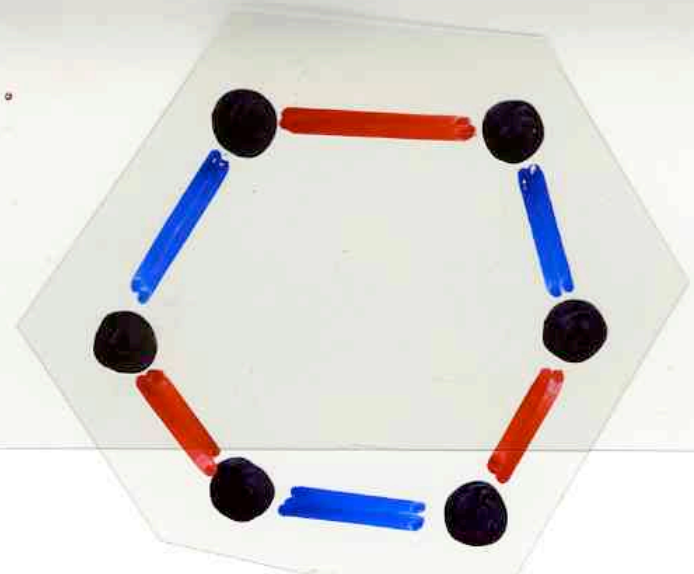


$$\frac{-4x^2 \quad t^2}{1-4t^2}$$

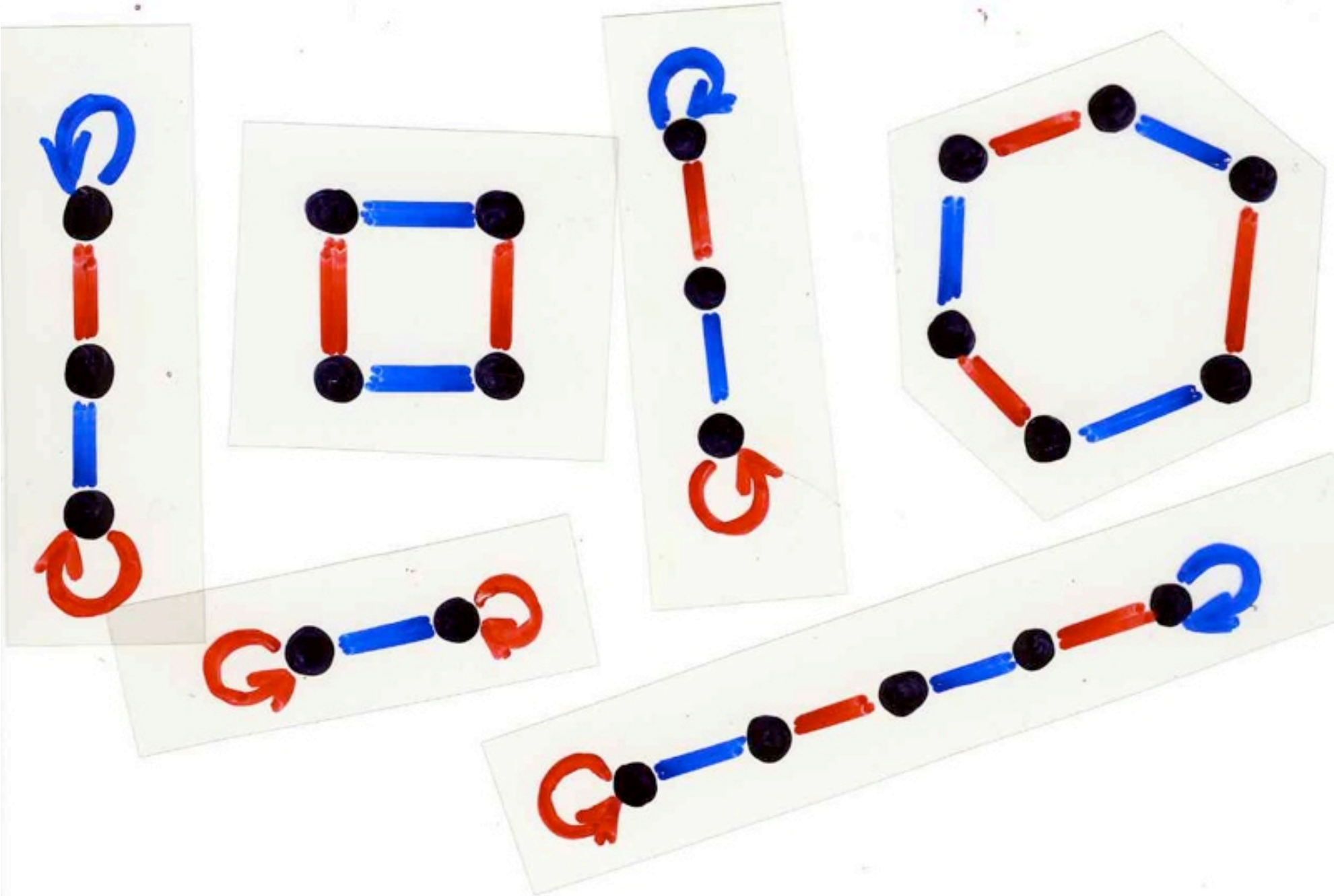
$$\frac{-4 \quad y^2 t^2}{1-4t^2}$$

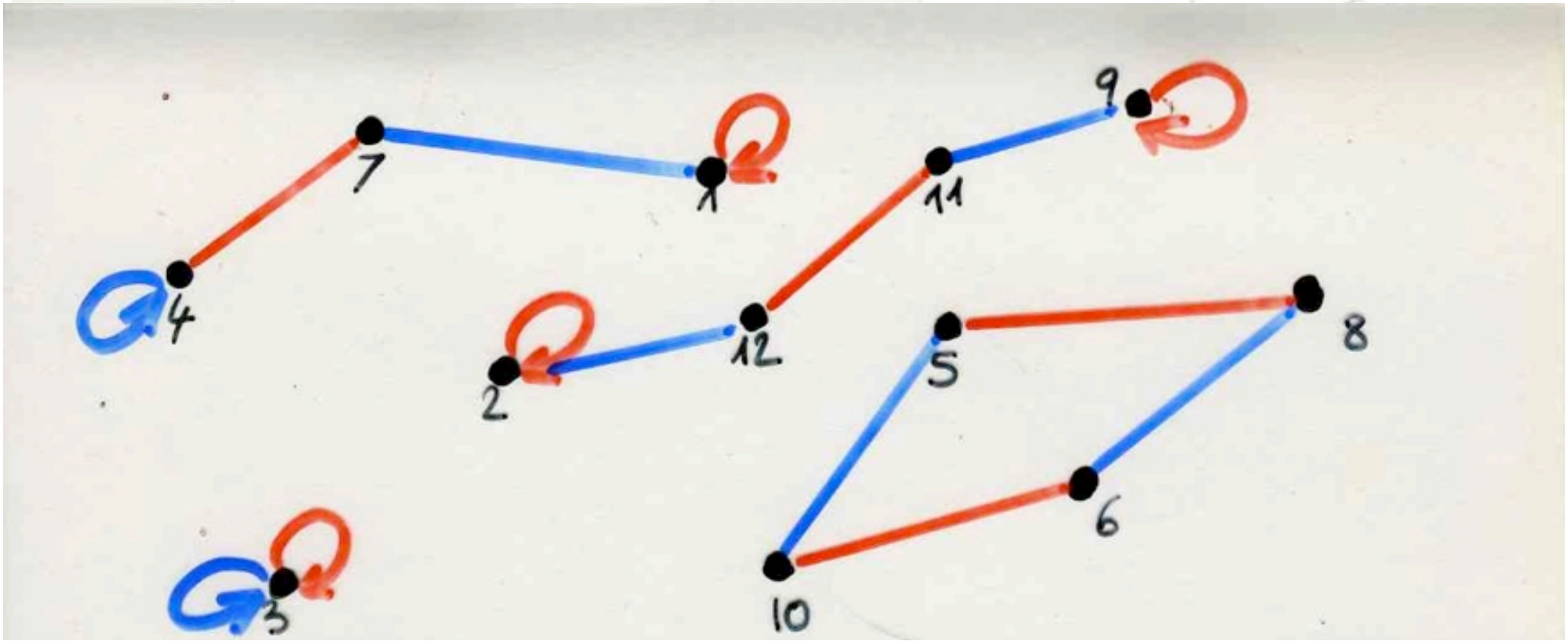


$$\frac{4xyt}{1-4t^2}$$



$$\frac{1}{2}\log\frac{1}{(1-4t^2)}$$

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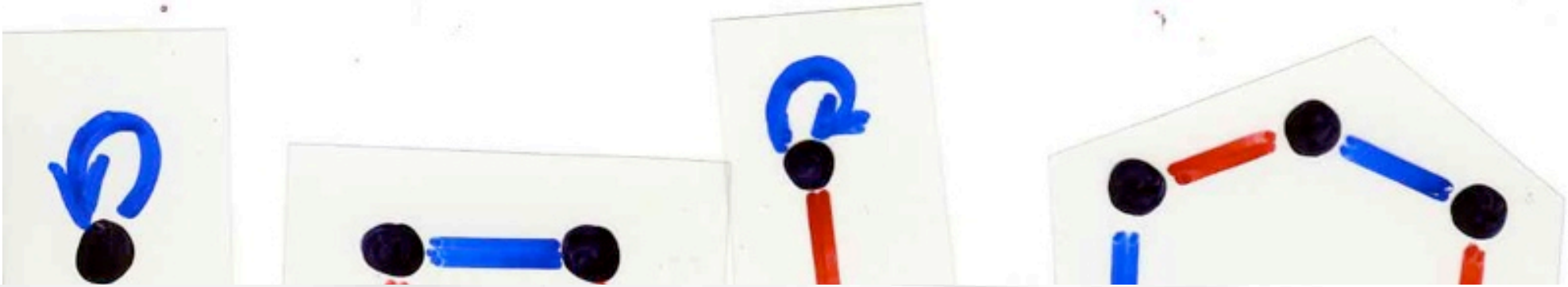
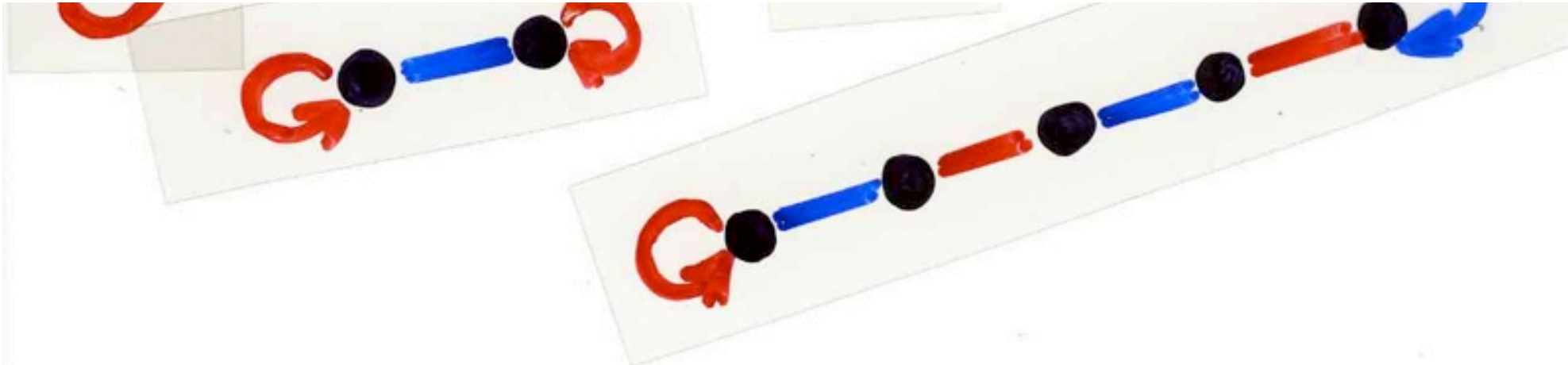


$$\sum_{n\geq 0} H_n(x)\,H(y)\,\frac{t^n}{n!} = (1-4t^2)^{-1/2}\exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$



"comprendre" une identité

interprétation combinatoire

une suite de nombres célèbres ....

$$1t + 2\frac{t^3}{3!} + 16\frac{t^5}{5!} + 272\frac{t^7}{7!} + \dots$$

$$y = \operatorname{tg} t$$

tangente

$$y = \operatorname{tg} t$$

tangente

$$y = \sum_{n \geq 0} a_{2n+1} \frac{t^{2n+1}}{(2n+1)!}$$

$$y' = 1 + y^2 \; ; \quad y(0) = 0$$

Permutations alternantes

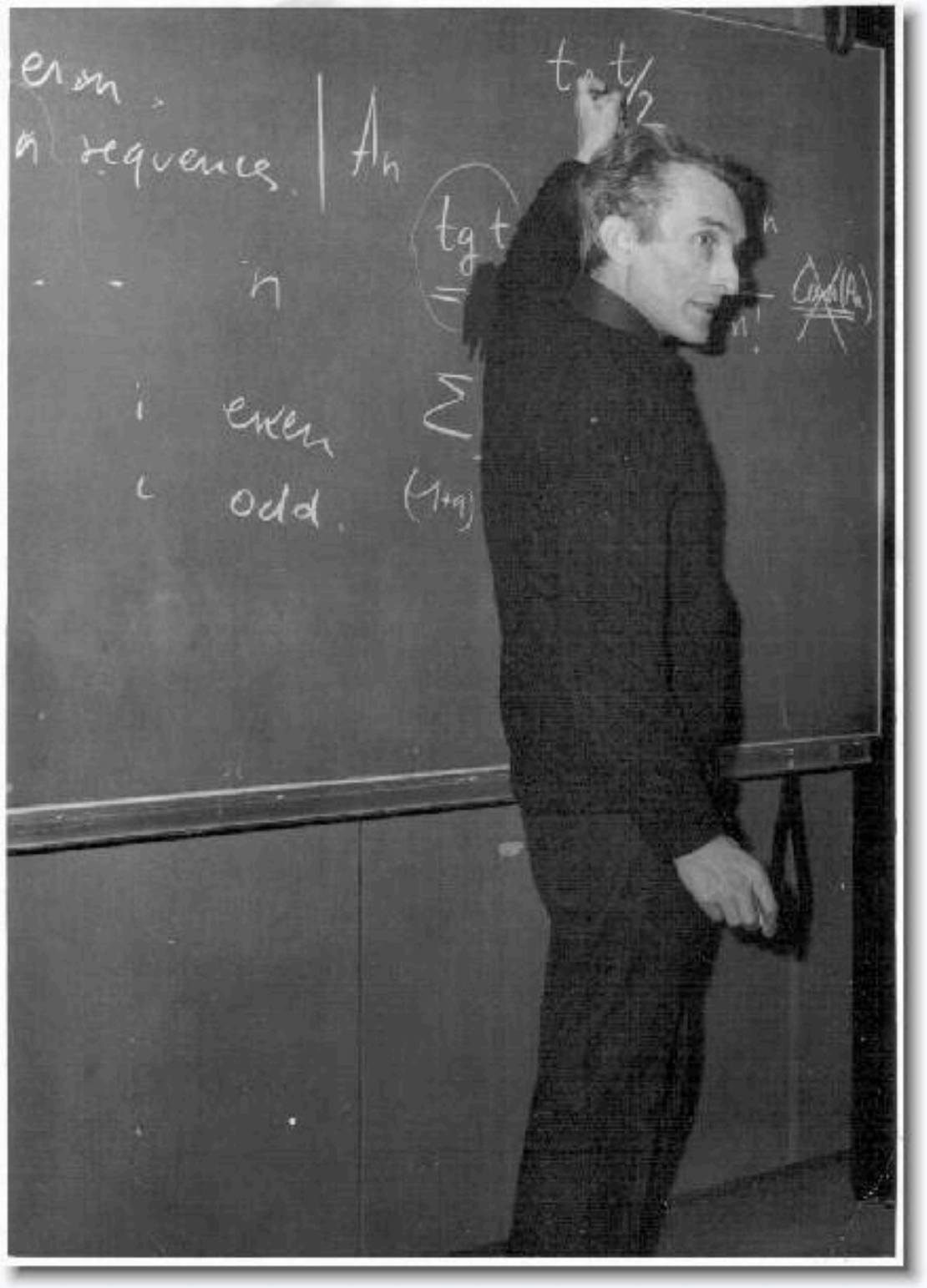
D. André (1880)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 2 & 9 & 7 & 8 & 4 & 5 & 1 & 3 \end{pmatrix}$$

$$T_{2n+1} = \sum_{i+j=n-1} \binom{2n+1}{2i-1} T_{2i-1} \, T_{2j-1}$$

D. Foata
M.P. Schützenberger

"Théorie géométrique des polynômes Eulériens" (1970)

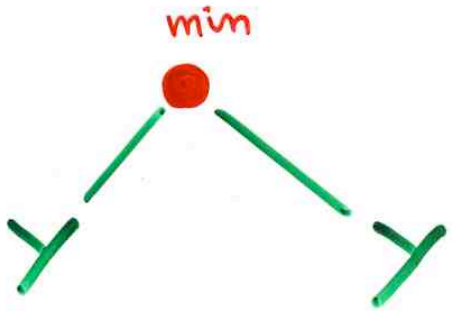


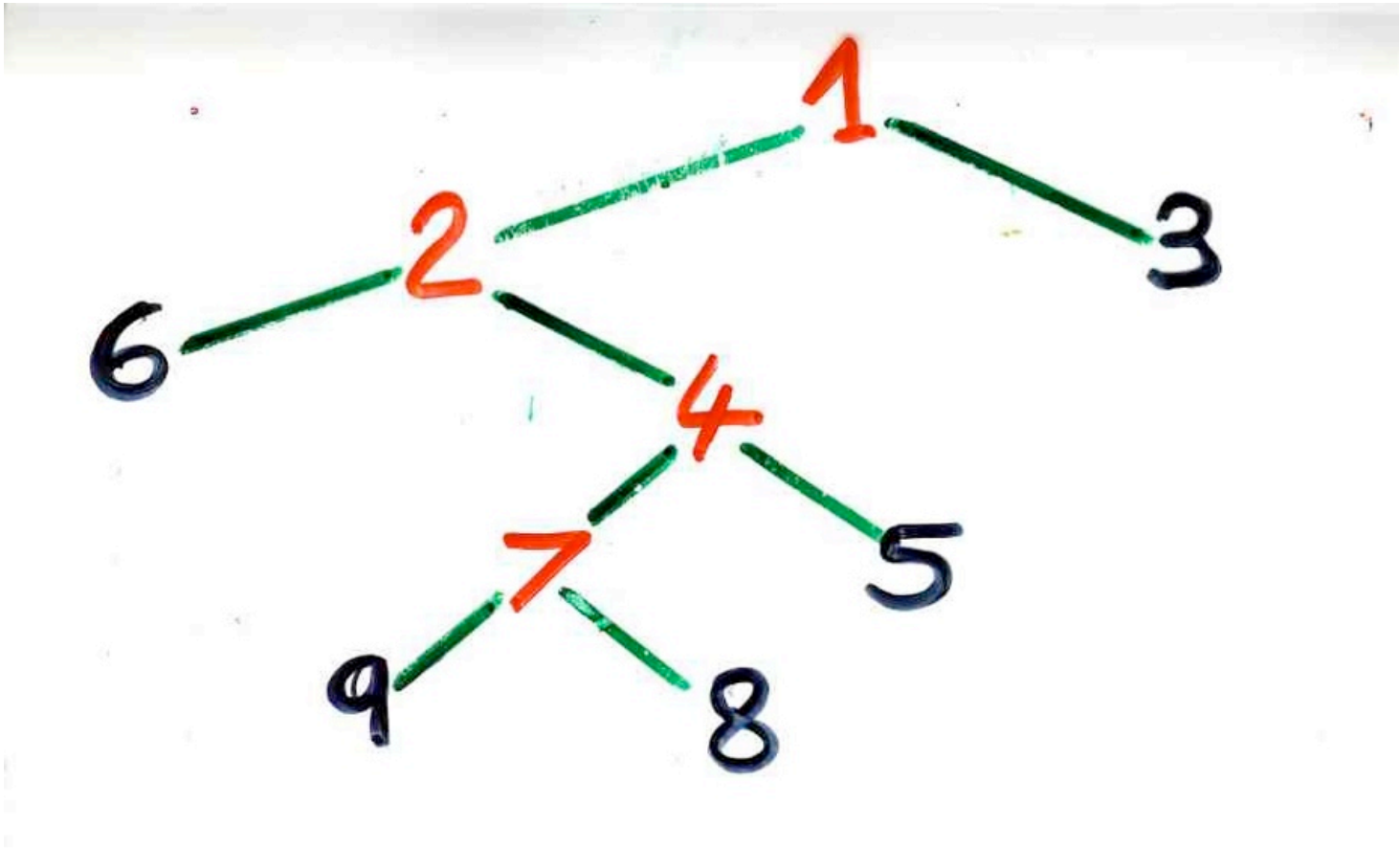
Marcel Paul
Schützenberger

1920 - 1996

$$y = t + \int_0^t y^2 \, dt$$

$$Y = T + \int_0^T Y^2(T) \, dT$$

$Y =$  ou 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6 2 9 7 8 4 5 1 3

théorie combinatoire

équations différentielles

calcul intégral

P. Leroux, X.G.V.

théorie du contrôle

Automatique non-linéaire

$$y' = y^2 + u(t)$$

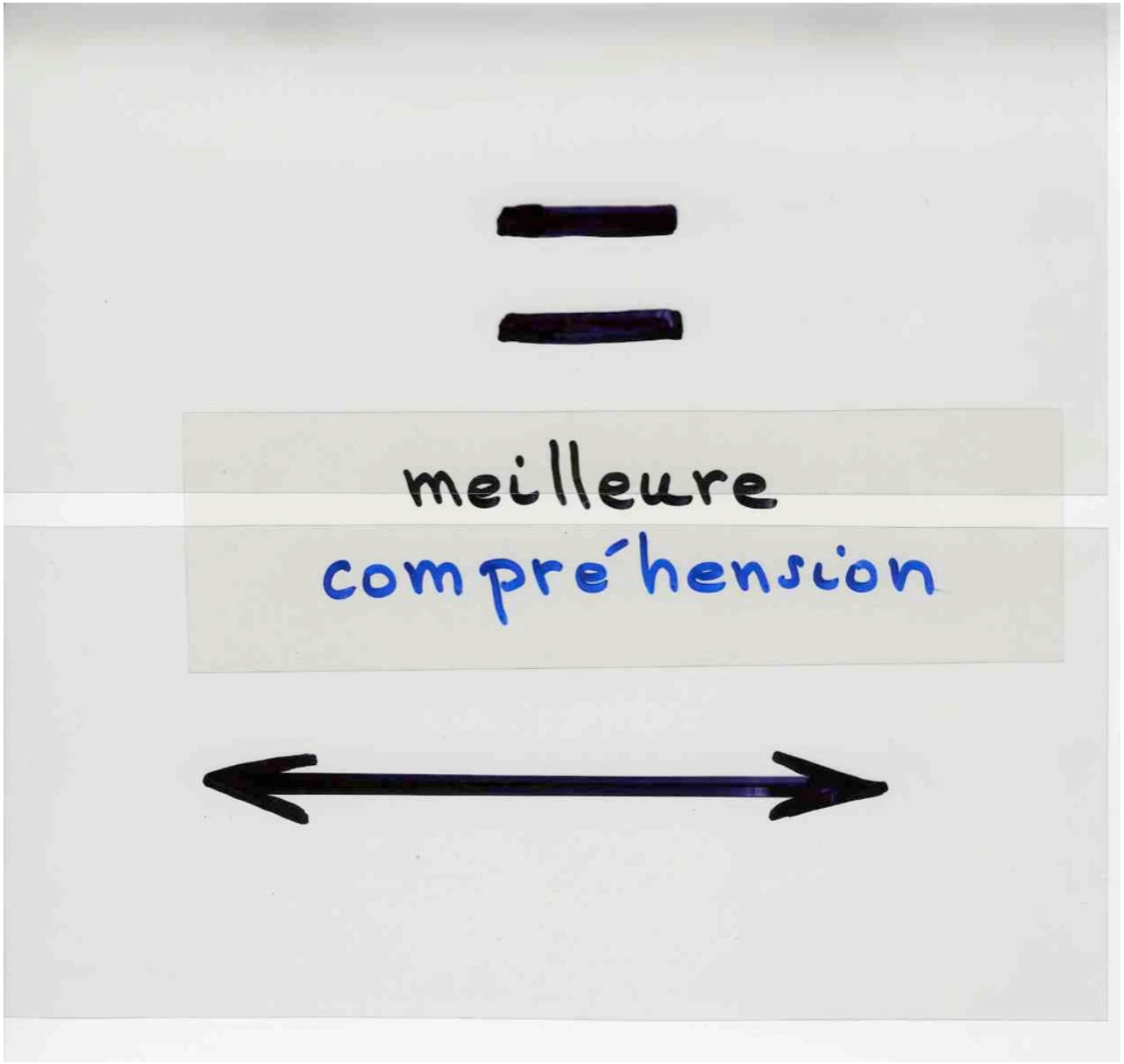
Equations différentielles

en régime forcé

dessiner des calculs

...

calculer des dessins

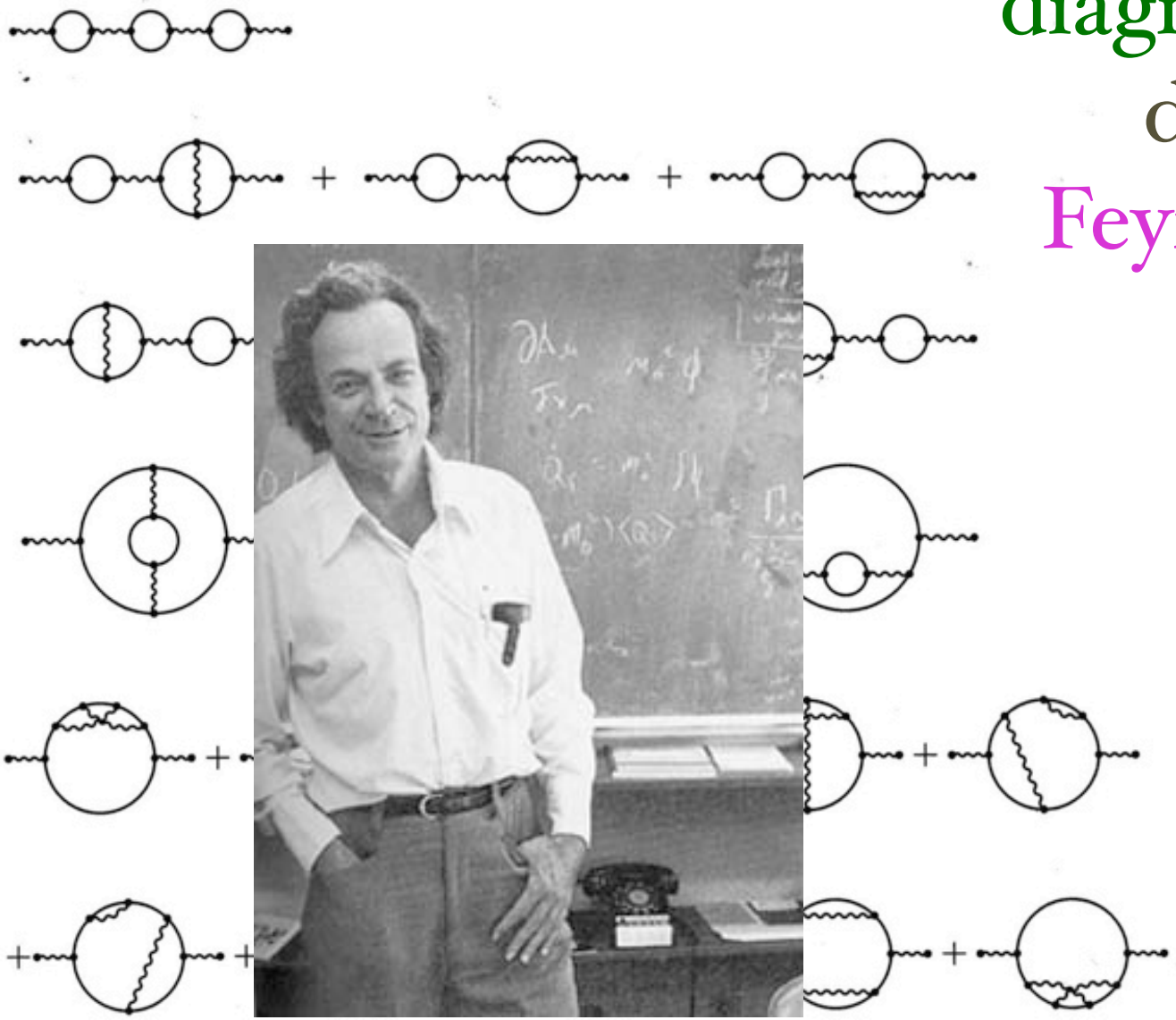

meilleure
compréhension

"calculer des dessins"

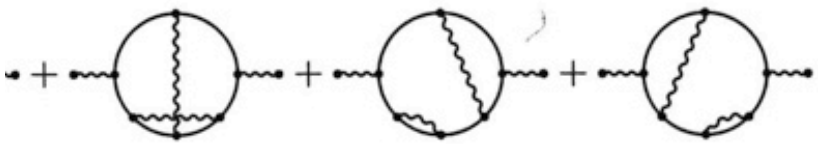
"calculs" sur des
figures, objets combinatoires, ...

construction de structures algébriques

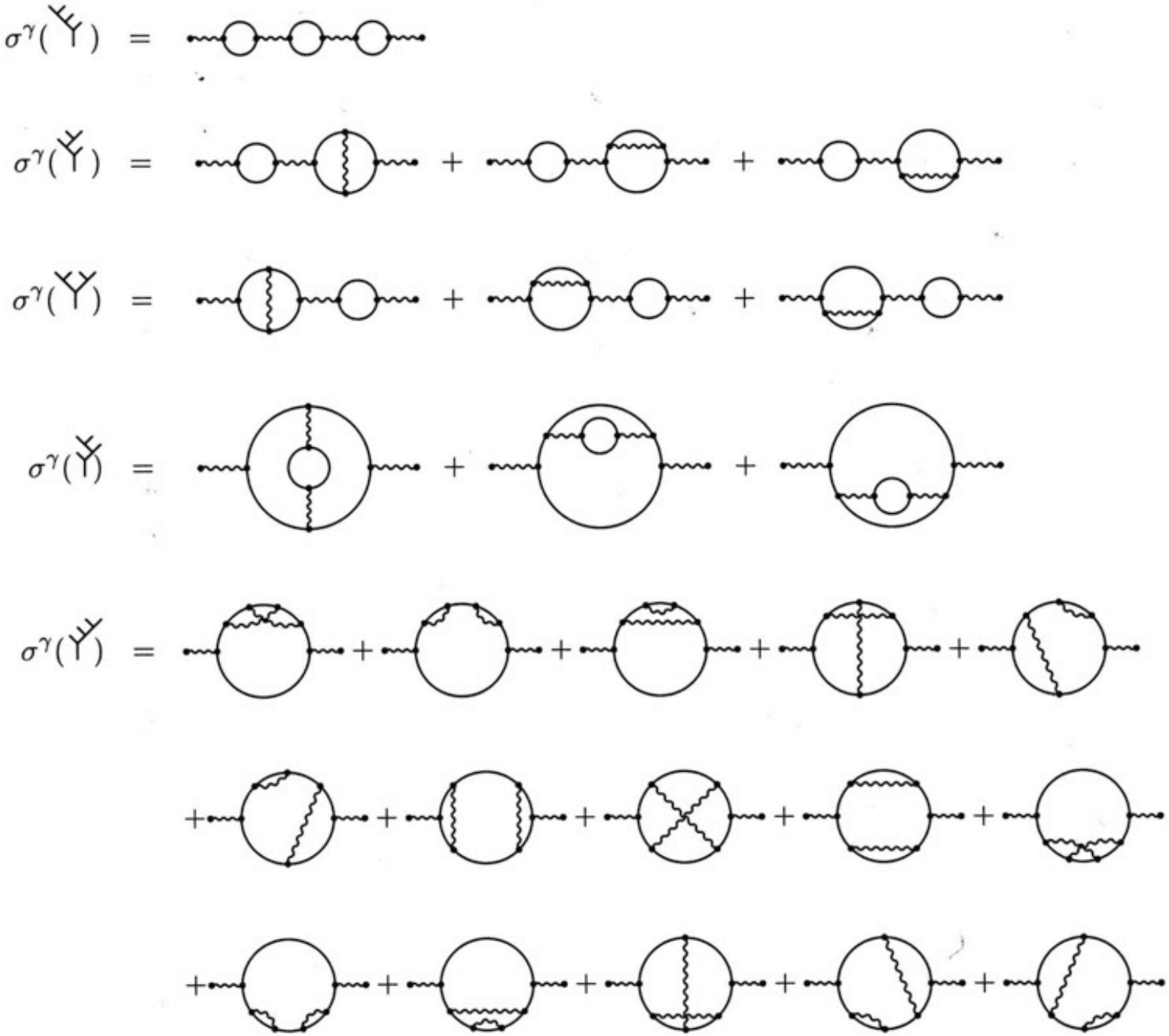
diagramas de Feyman



Renormalisation quantique



les arbres
algèbres de Hopf


σ^γ( ) =
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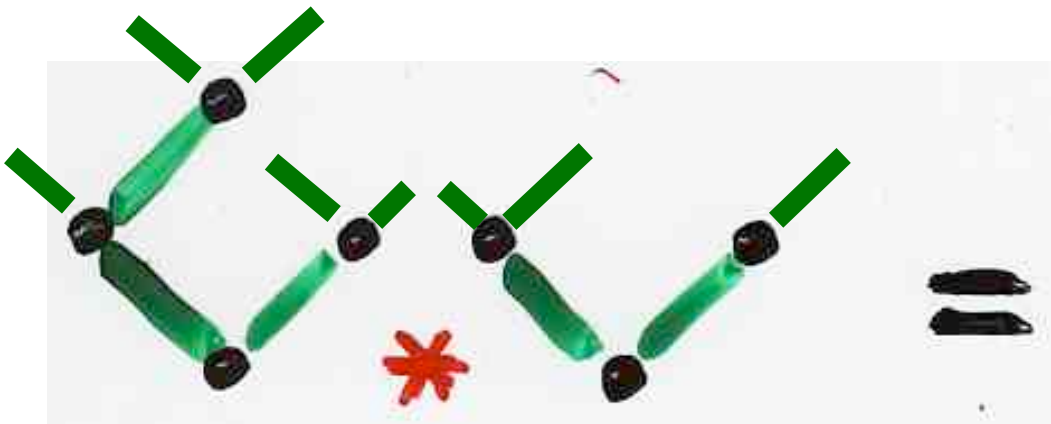
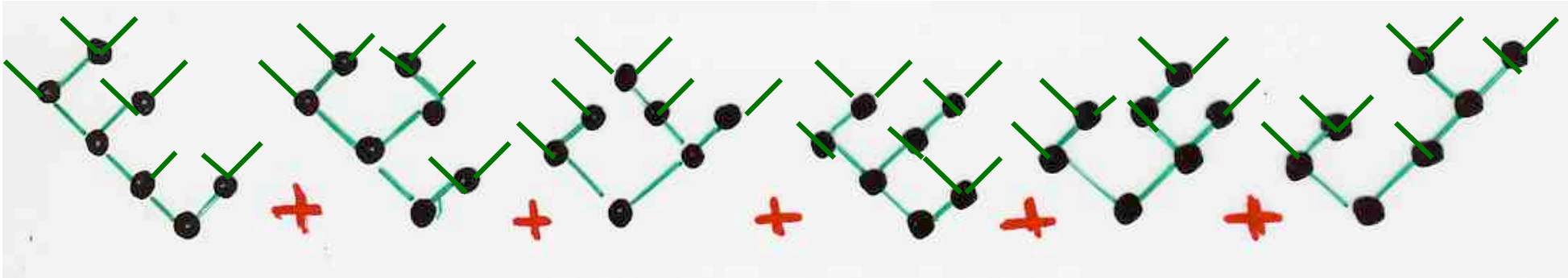
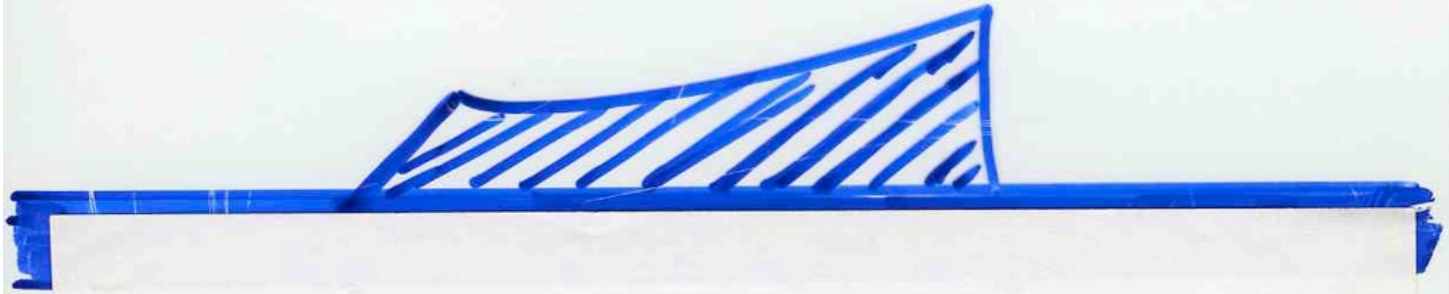
σ^γ( ) =
σ^γ( ) =





produit d'arbres (Loday-Ronco)

“Le paradigme bijectif”



fonctions spéciales

polynômes orthogonaux

fonctions elliptiques

représentation des groupes

fonction symétrique

équations différentielles

approximants de Padé

fractions continues

partitions d'entiers

systèmes dynamiques

- - - - - - - - - - - - - -

Renaissance ou révolution ?

combinatoire algébrique

LGV

Déterminants binomiaux

déterminant d'une matrice
k lignes et k colonnes

3 1 4 2

permutation

2	1	●	0
●	3	0	0
5	10	5	●
7	●	35	21

déterminant d'une matrice
k lignes et k colonnes

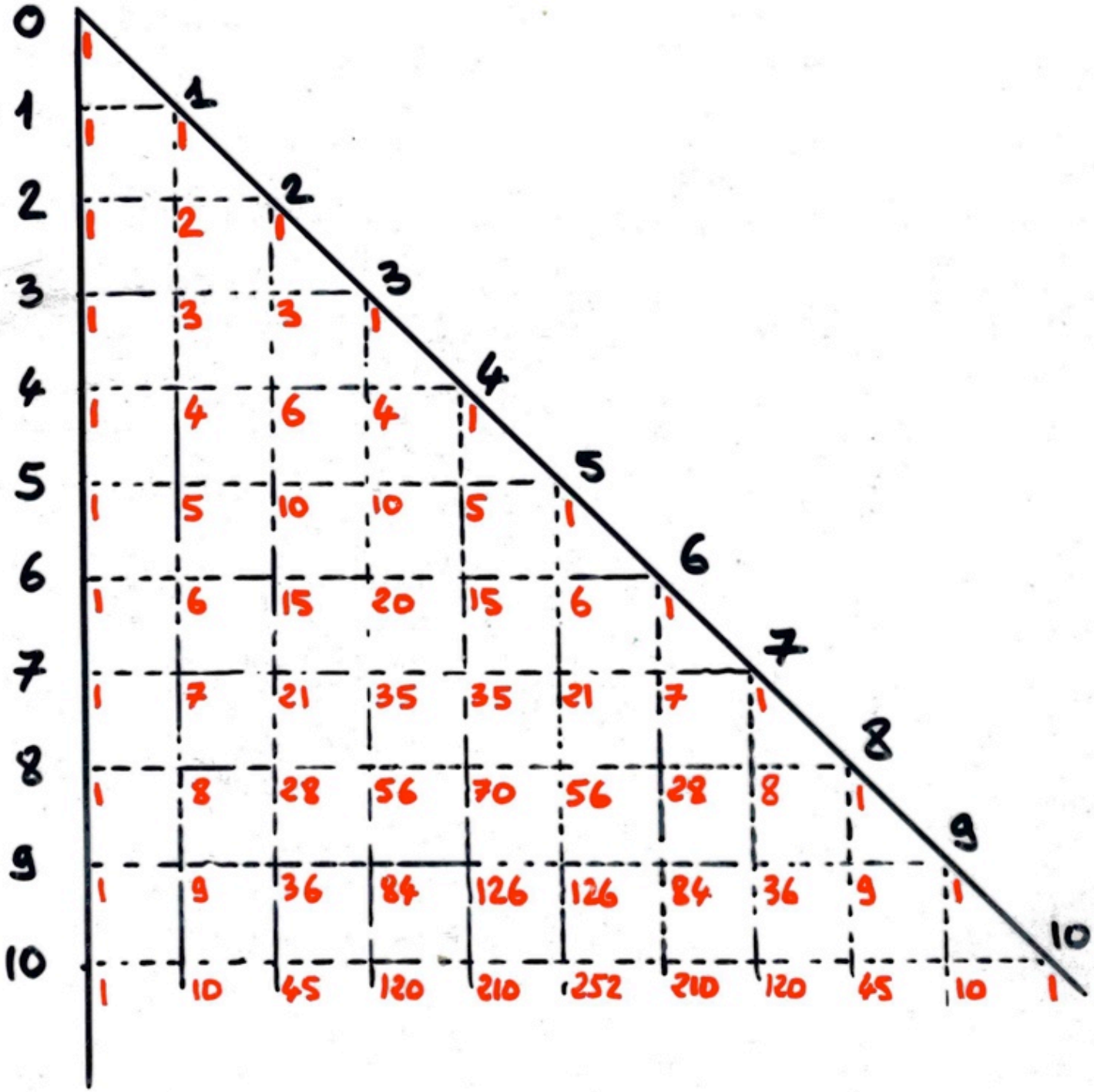
3 1 4 2

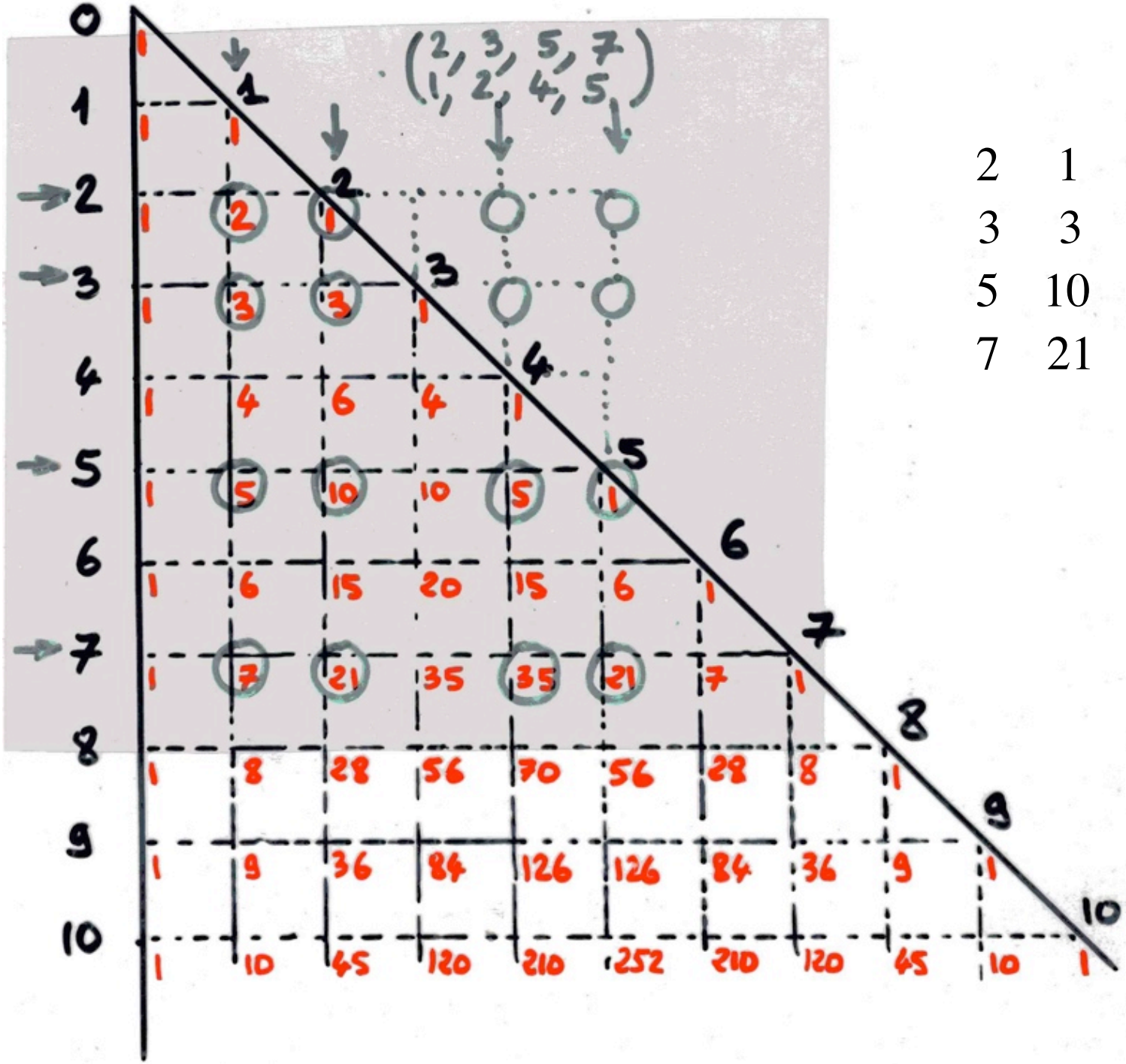
permutation

2	1	0	0
3	3	0	0
5	10	5	1
7	21	35	21

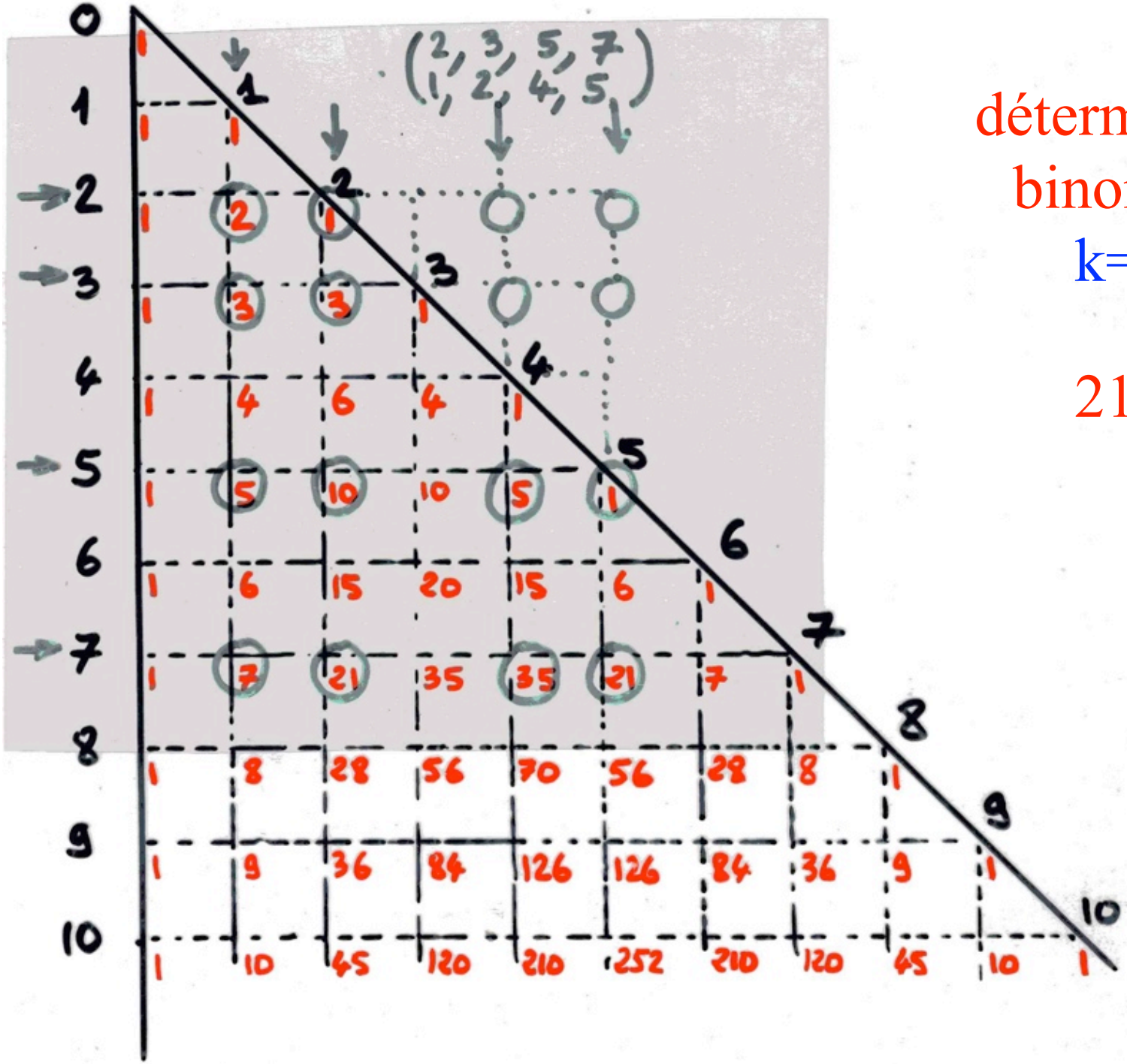
produit: (+ou -) 3 x 21 x 0 x 1

somme des produits sur toutes les permutations


0 1 2 3 4 5 6 7 8 9 10
1 1 2 3 4 5 6 7 8 9 10
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1



2	1	0	0
3	3	0	0
5	10	5	1
7	21	35	21



déterminant binomial

k=4

210

$$0 \leq a_1 < \cdots < a_k$$

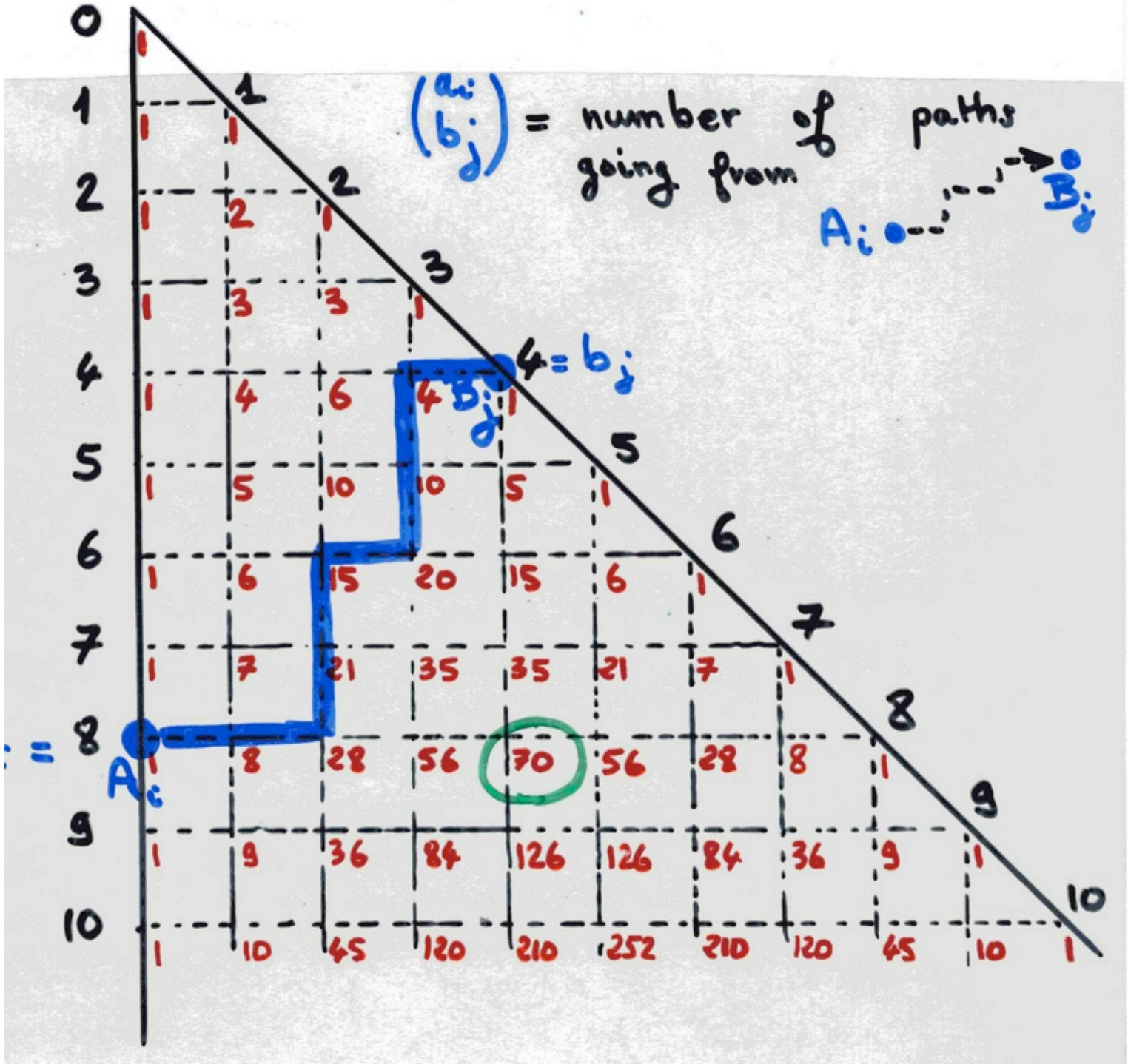
$$0 \leq b_1 < \cdots < b_k$$

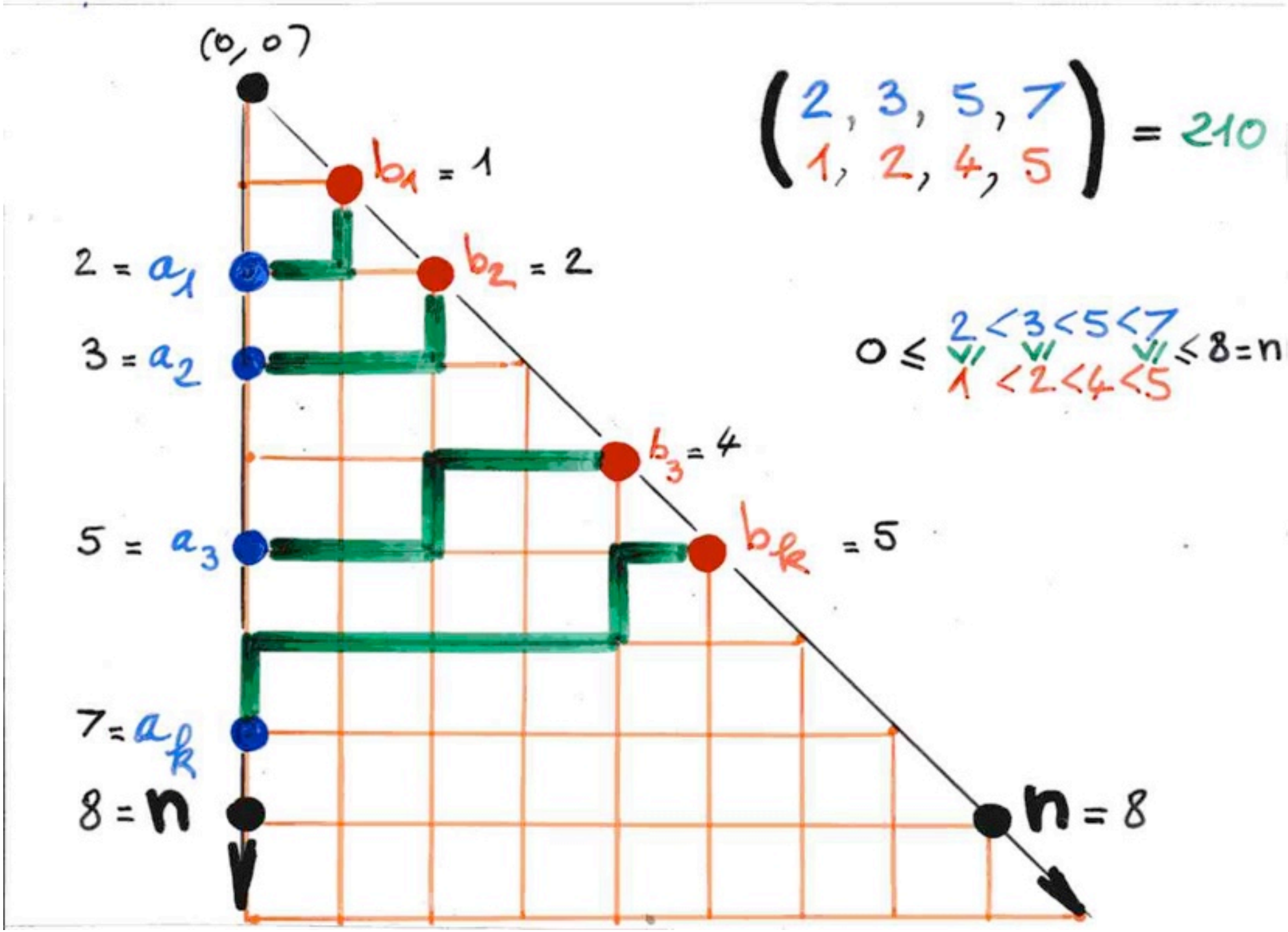
déterminant binomial

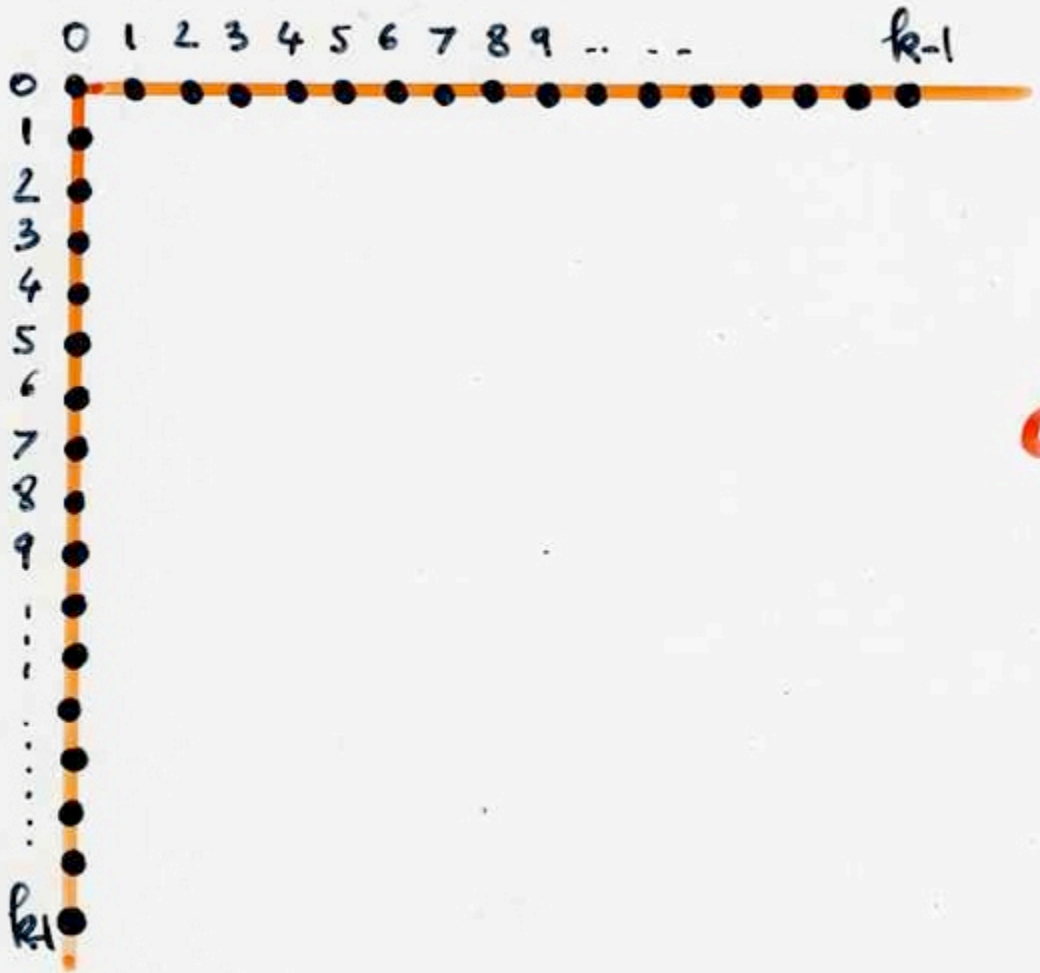
$$\begin{pmatrix} a_1, \ldots, a_k \\ b_1, \ldots, b_k \end{pmatrix}$$

$$= \text{dét}\left(\binom{a_i}{b_j}\right)_{1 \leq i, j \leq k}$$

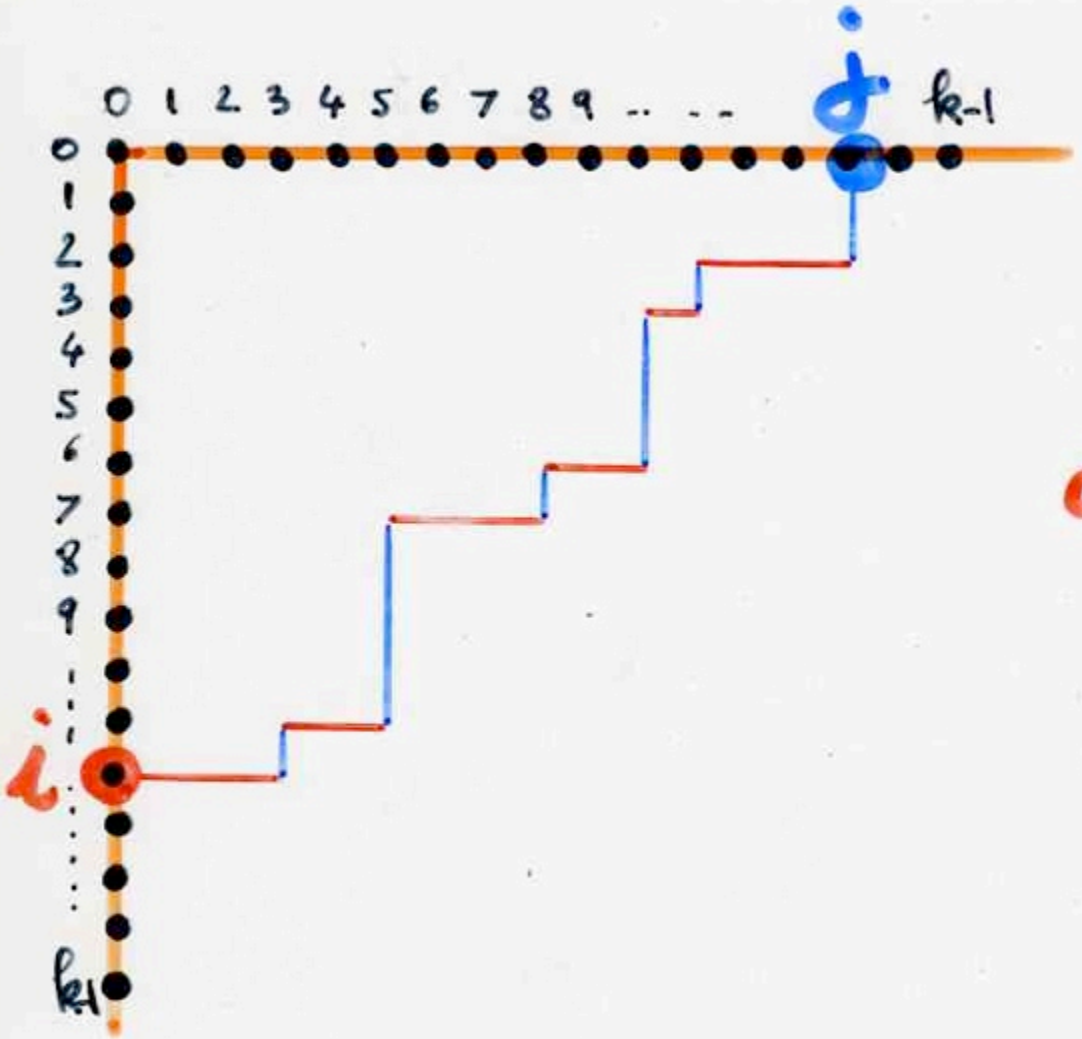
ils sont tous positifs ou nuls !


$\binom{a_i}{b_j}$ = number of paths going from
A_i
B_j
$= b_j$
B_j
A_i
0 1 2 3 4 5 6 7 8 9 10
1 1 1 1 1 1 1 1 1 1 1
1 2 3 4 5 6 7 8 9 10
1 3 6 10 15 21 28 36 45
1 4 10 20 35 56 84 120
1 5 15 35 70 126 210
1 6 21 56 126 252
1 7 28 84 210
1 8 36 120
1 9 45
1 10

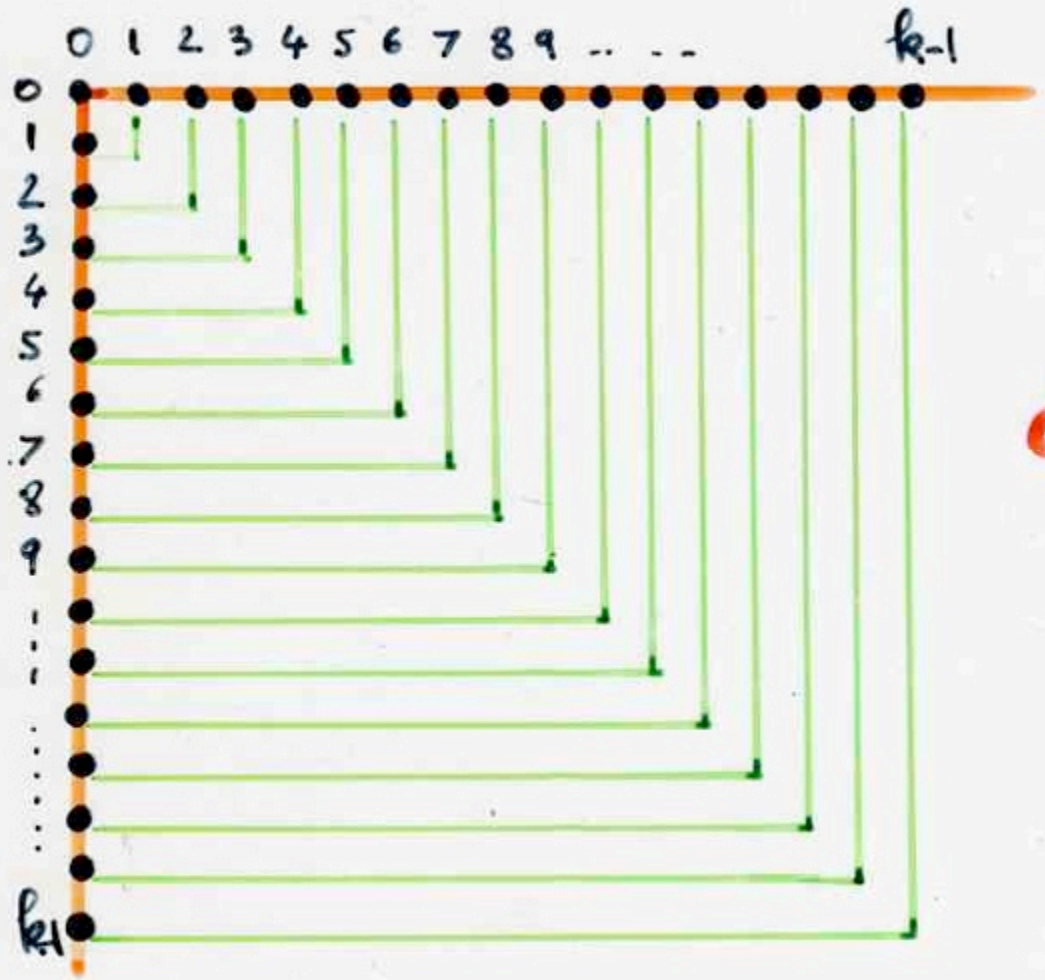

(0, 0)
$b_1 = 1$
$b_2 = 2$
$b_3 = 4$
$b_k = 5$
$2 = a_1$
$3 = a_2$
$5 = a_3$
$7 = a_k$
$8 = n$
$n = 8$
$\begin{pmatrix} 2, 3, 5, 7 \\ 1, 2, 4, 5 \end{pmatrix} = 210$
$0 \leq \begin{matrix} 2 < 3 < 5 < 7 \\ \vee\!\!| \quad \vee\!\!| \quad \vee\!\!| \\ 1 < 2 < 4 < 5 \end{matrix} \leq 8 = n$



$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots & \\ 1 & 3 & 6 & 10 & \cdots & & \\ 1 & 4 & 10 & \cdots & & & \\ 1 & 5 & \cdots & & & & \\ 1 & \cdots & & & & & \\ & & & & & & \\ & & & & & & \end{bmatrix}_{k \times k} = \qquad \binom{i+j}{i}$$



$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots & \\ 1 & 3 & 6 & 10 & \cdots & & \\ 1 & 4 & 10 & \cdots & & & \\ 1 & 5 & \cdots & & & & \\ 1 & \cdots & & & & & \\ & & & & & & \\ & & & & & & \end{bmatrix}_{k \times k} = \qquad \binom{i+j}{i}$$



$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots & \\ 1 & 3 & 6 & 10 & \cdots & & \\ 1 & 4 & 10 & \cdots & & & \\ 1 & 5 & \cdots & & & & \\ 1 & \cdots & & & & & \\ & & & & & & \end{bmatrix}_{k \times k} = 1$$

$$\binom{i+j}{i}$$

le retour de l'oeil en mathématiques

combinatoire quantique

Spin chains and combinatorics

A. V. Razumov, Yu. G. Stroganov

Institute for High Energy Physics
142284 Protvino, Moscow region, Russia

The XXZ quantum spin chain model with periodic boundary conditions is one of the most popular integrable models which has been investigating by the Bethe Ansatz method during the last 35 years [3]. It is described by the Hamiltonian

$$H_{XXZ} = -\sum_{j=1}^{N} \left\{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right\}, \qquad \vec{\sigma}_{N+1} = \vec{\sigma}_1. \tag{1}$$

The nonzero wave function components are

$$
\begin{aligned}
&N=3: \quad \psi_{001}=1;\\
&N=5: \quad \psi_{00011}=1, \quad \psi_{00101}=2;\\
&N=7: \quad \psi_{0000111}=1, \quad \psi_{0001101}=\psi_{0001011}=3, \quad \psi_{0010011}=4 \quad \psi_{0010101}=7.
\end{aligned}
$$

All components not included in the list can be obtained by shifting. Notice that the components of the ground state are positive in accordance with the Perron–Frobenius theorem.

Let us continue the list. For $N = 9$ the components of the eigenvector with the energy $-27/2$ and $S_z = -1/2$ are

$$
\begin{array}{llll}
\psi_{000001111}=1, & \psi_{000010111}=4, & \psi_{000011011}=6, & \psi_{000100111}=7,\\
\psi_{000101011}=17, & \psi_{000101101}=14, & \psi_{000110011}=12, & \psi_{001001011}=21,\\
 & \psi_{001010011}=25, & \psi_{001010101}=42. &
\end{array}
$$

Let us continue the list. For $N = 9$ the components of the eigenvector with the energy $-27/2$ and $S_z = -1/2$ are

$$
\begin{array}{llll}
\psi_{000001111}=1, & \psi_{000010111}=4, & \psi_{000011011}=6, & \psi_{000100111}=7,\\
\psi_{000101011}=17, & \psi_{000101101}=14, & \psi_{000110011}=12, & \psi_{001001011}=21,\\
 & \psi_{001010011}=25, & \psi_{001010101}=42. &
\end{array}
$$

We omit nonzero components which can be obtained by the reflection of the order of sites since this transformation is a symmetry of our state, as it is for the ground state. For example, we have

$$\psi_{000011101}=\psi_{000010111}=4.$$

1, 2, 7, 42, 429, ...

?

M1791 0, 1, 2, 7, 32, 181, 1214, 9403, 82508, 808393, 8743994, 103459471, 1328953592, 18414450877, 273749755382, 4345634192131, 73362643649444, 1312349454922513
$a(n)=n.a(n-1)+(n-2)a(n-2)$. Ref R1 188. [0,3; A0153, N0706]

E.g.f.: $(1-x)^{-3}\,e^{-x}$.

M1792 1, 1, 2, 7, 32, 181, 1232, 9787, 88832, 907081, 10291712, 128445967, 1748805632, 25794366781, 409725396992, 6973071372547, 126585529106432
Expansion of $1/(1-\sinh x)$. Ref ARS 10 138 80. [0,3; A6154]

M1793 0, 1, 1, 2, 7, 32, 184, 1268, 10186, 93356, 960646, 10959452, 137221954, 1870087808, 27548231008, 436081302248, 7380628161076, 132975267434552
Stochastic matrices of integers. Ref DUMJ 35 659 68. [0,4; A0987, N0707]

M1794 1, 2, 7, 33, 192
Permutations of length n with n in second orbit. Ref C1 258. [2,2; A6595]

M1795 1, 2, 7, 34, 209, 1546, 13327, 130922, 1441729, 17572114, 234662231, 3405357682, 53334454417, 896324308634, 16083557845279, 306827170866106
$a(n)=2n.a(n-1)-(n-1)^2a(n-2)$. Ref SE33 78. [0,2; A2720, N0708]

M1796 1, 2, 7, 34, 257, 2606, 32300, 440564, 6384634
Polyhedra with n nodes. Ref GR67 424. UPG B15. Dil92. [4,2; A0944, N0709]

M1797 2, 7, 35, 219, 1594, 12935, 113945, 1070324, 10586856, 109259633, 1168384157, 12877168147, 145656436074, 1685157199175, 19886174611045
Two-rowed truncated monotone triangles. Ref JCT A42 277 86. Zei93. [1,1; A6947]

M1798 1, 1, 2, 7, 35, 228, 1834, 17382, 195866, 2487832, 35499576, 562356672, 9794156448, 186025364016, 3826961710272, 84775065603888, 2011929826983504
Coefficients of iterated exponentials. Ref SMA 11 353 45. [0,3; A0154, N0710]

M1799 1, 2, 7, 35, 228, 1834, 17582, 195866, 2487832, 35499576, 562356672, 9794156448, 186025364016, 3826961710272, 84775065603888, 2011929826983504
Expansion of $\ln(1+\ln(1+x))$. [0,2; A3713]

M1800 1, 0, 1, 2, 7, 36, 300, 3218, 42335, 644808
Circular diagrams with n chords. Ref BarN94. [0,4; A7474]

M1801 1, 2, 7, 36, 317, 5624, 251610, 33642660, 14685630688
$n \times n$ binary matrices. Ref CPM 89 217 64. SLC 19 79 88. [0,2; A2724, N0711]

M1802 2, 7, 37, 216, 1780, 32652
Semigroups of order n with 2 idempotents. Ref MAL 2 2 67. SGF 14 71 77. [2,1; A2787, N0712]

M1803 1, 2, 7, 37, 266, 2431, 27007, 353522, 5329837, 90960751, 1733584106, 36496226977, 841146804577, 21065166341402, 569600638022431
$a(n)=(2n-1)a(n-1)+a(n-2)$. Ref RCI 77. [0,2; A1515, N0713]

M1804 1, 1, 2, 7, 38, 291, 2932, 36961, 561948, 10026505, 205608536, 4767440679, 123373203208, 3525630110107, 110284283006640, 3748357699560961
Forests of labeled trees with n nodes. Ref JCT 5 96 68. SIAD 3 574 90. [0,3; A1858, N0714]

M1805 1, 1, 2, 7, 40, 357, 4824, 96428, 2800472, 116473461
n-element partial orders contained in linear order. Ref nbh. [0,3; A6455]

M1806 1, 2, 7, 41, 346, 3797, 51157, 816356, 15050581, 314726117, 7359554632, 190283748371, 5389914888541, 165983936096162, 5521346346543307
Planted binary phylogenetic trees with n labels. Ref LNM 884 196 81. [1,2; A6677]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \,.\,.\, n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 64539836938720749739356
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

M1810 0, 1, 2, 7, 44, 361, 3654, 44207, 622552, 10005041, 180713290, 3624270839, 79914671748, 1921576392793, 50040900884366, 1403066801155039
Modified Bessel function $K_n(1)$. Ref AS1 429. [0,3; A0155, N0716]

M1811 0, 1, 2, 7, 44, 447, 6749, 142176, 3987677, 143698548, 6470422337, 356016927083, 23503587609815, 1833635850492653, 166884365982441238
$a(n)=n(n-1)a(n-1)/2+a(n-2)$. [0,3; A1046, N0717]

M1812 1, 2, 7, 44, 529, 12278, 565723, 51409856, 9371059621, 3387887032202, 2463333456292207, 3557380311703796564, 1033908166635018028984 9
Sum of Gaussian binomial coefficients $[n,k]$ for $q=4$. Ref TU69 76. GJ83 99. ARS A17 328 84. [0,2; A6118]

M1813 2, 7, 52, 2133, 2590407, 3374951541062, 569518350447911664037 6509, 162175575749223863014205141915237848956395775710480
Free binary trees of height n. Ref JCIS 17 180 92. [1,1; A5588]

M1814 1, 1, 2, 7, 56, 2212, 2595782, 3374959180831, 569518350448923906748 4387, 162175575749223863014205312770713651031687342842 82
Planted 3-trees of height n. Ref RSE 59(2) 159 39. CMB 11 87 68. JCIS 17 180 92. [0,3; A2658, N0718]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \,.\,.\, n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 64539836938720749739356
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727

Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500

Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \,..\, n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 64539836938720749739356

Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

Physique combinatoire

Matrices à signes alternants

"Matrices à signes alternés"

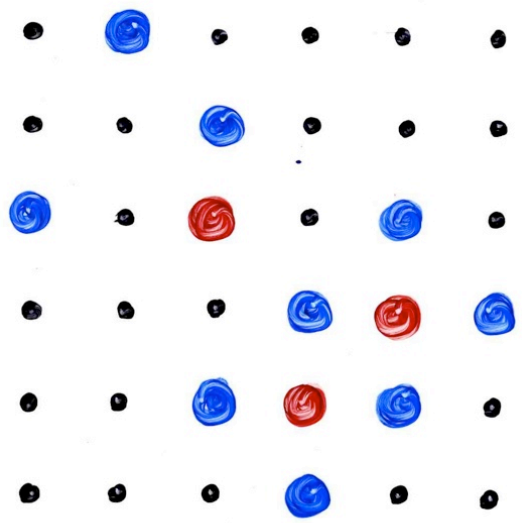
(alternating sign matrices)

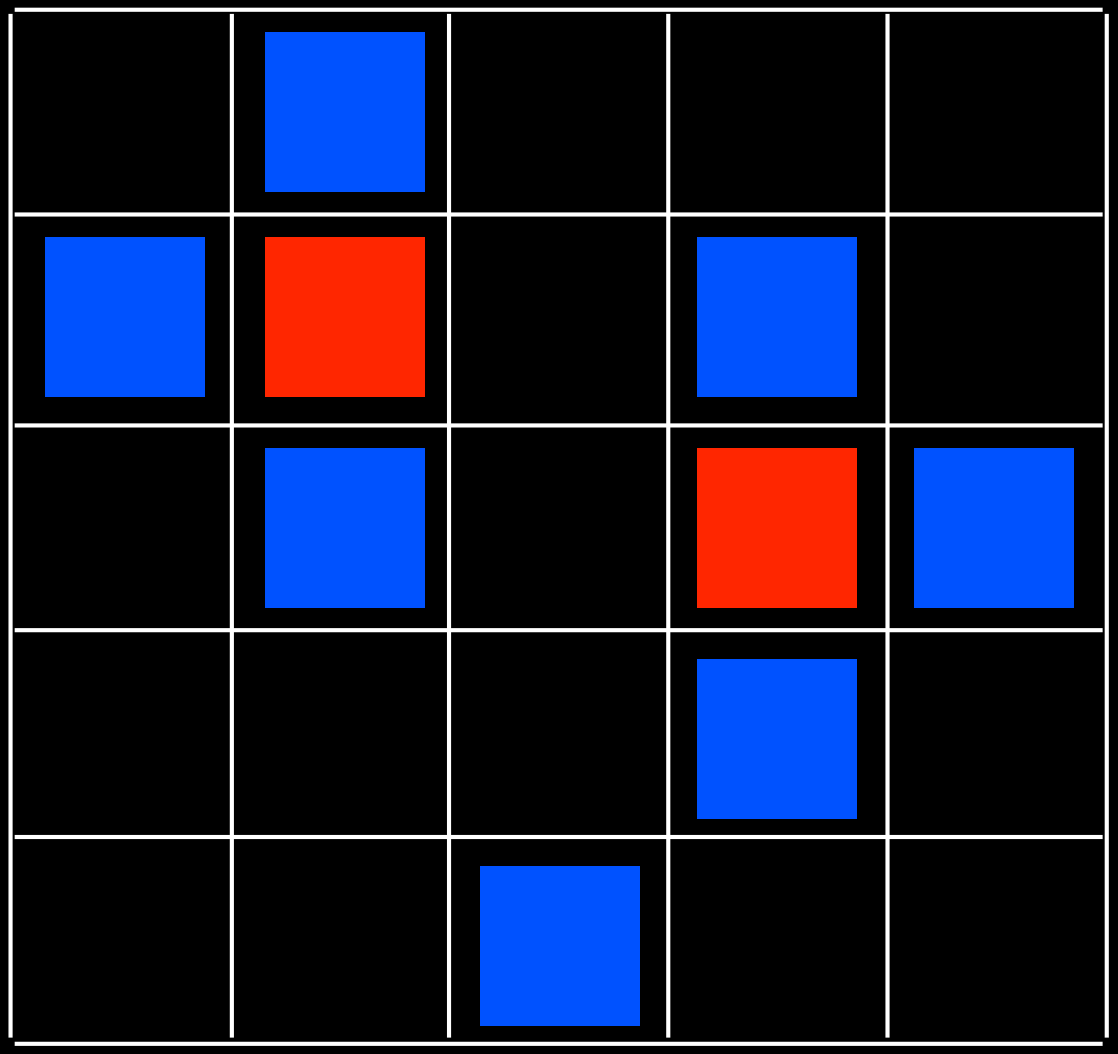
- entrées : 0, 1, −1
- somme des entrées, ligne, colonne = 1
- entrées ≠ 0 alternent en signe ligne, colonne

alternating sign matrices (ASM)

ex :

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$





Permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} + 6$$

permutations

1, 2, 7, 42, 429, ...

Razumov - Stroganov (ex)- conjecture 2000-2001

proof by :
L. Cantini and A.Sportiello (March 2010)
arXiv: 1003.3376 [math.CO]
completely combinatorial proof

énumération des matrices à signes alternants

$$1, 2, 7, 42, 429, \ldots$$

$$\frac{1!\ 4!\ 7!\ \cdots\ (3n-2)!}{n!\ (n+1)!\ (n+2)!\ \cdots\ (n+n-1)!}$$

conjecture des matrices à signes alternants

Mills, Robbins, Rumsey (1982)

$$1,\ 2,\ 7,\ 42,\ 429,\ \ldots$$

$$\frac{1!\ 4!}{n!\ (n+1)} \quad ? \quad \frac{(3n-2)!}{(n+n-1)!}$$

conjecture des matrices à signes alternants

Mills, Robbins, Rumsey (1982)

formule "miraculeuse"

des perles

des diamants ....

un problème simple

une formule simple

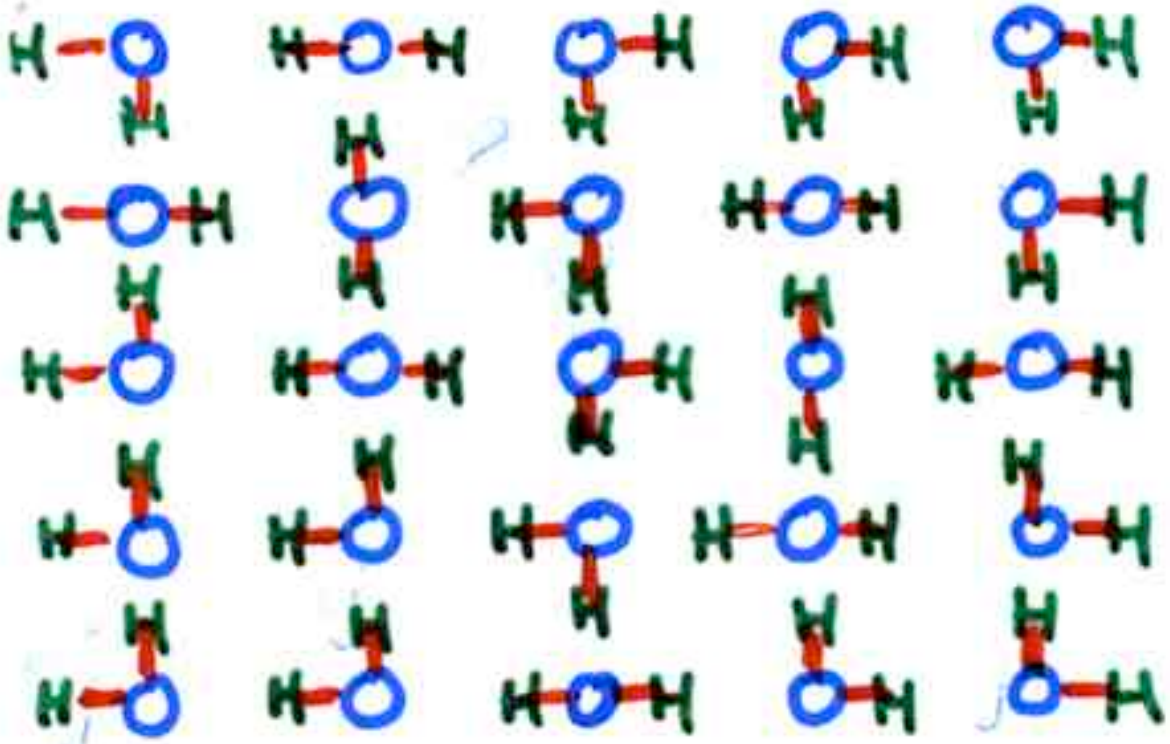
avec une démonstation apocalyptique ...

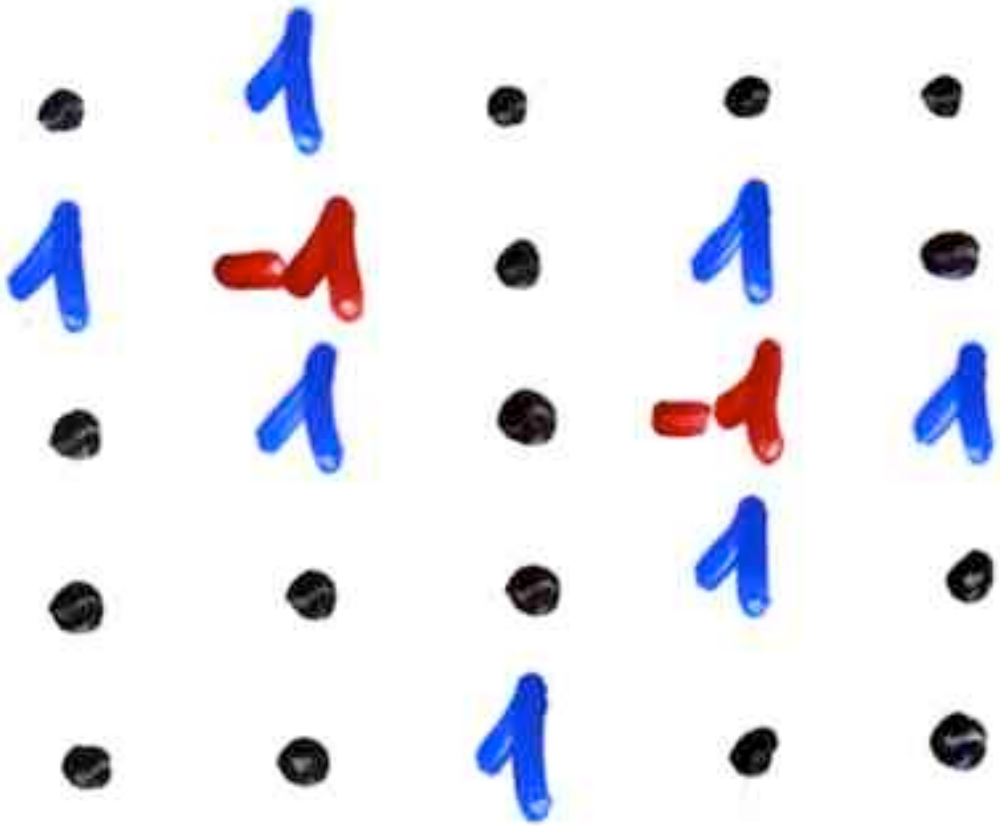
Robbins

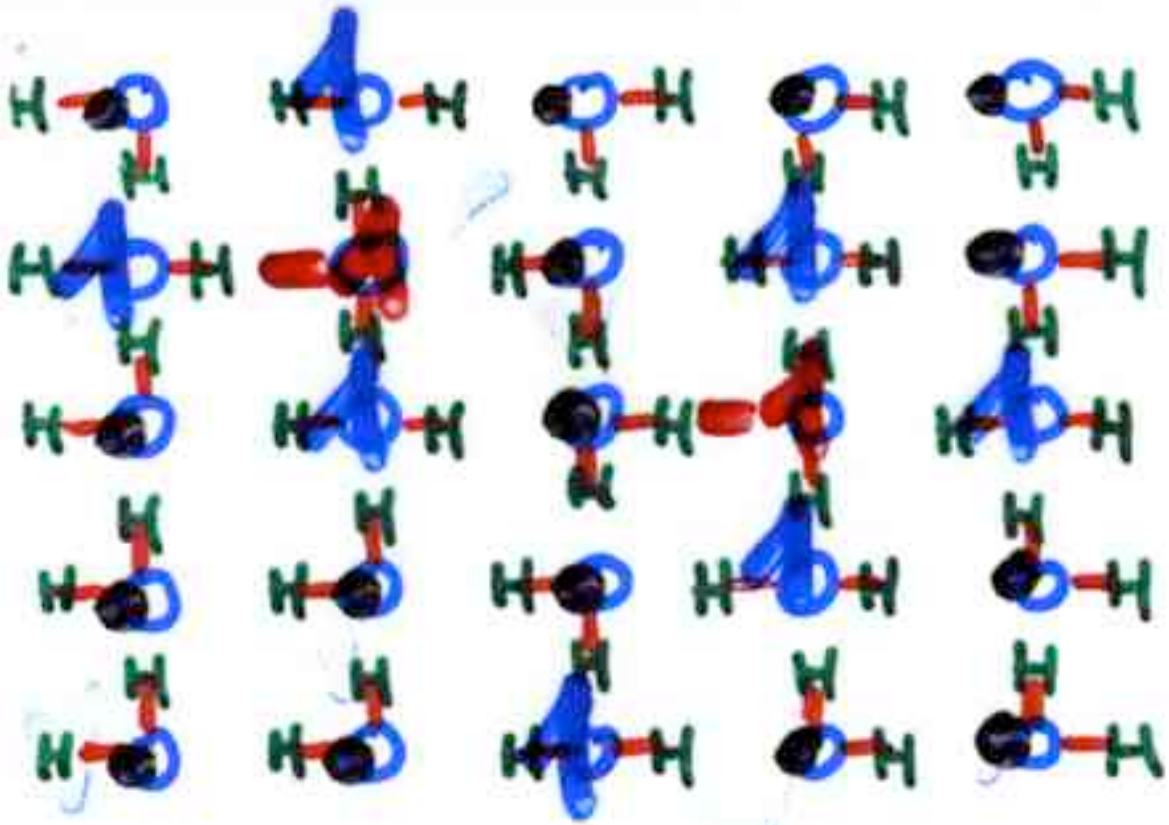
The Mathematical Intelligencer (1991)

“These conjectures are of such compelling simplicity that it is hard to understand how any mathematician can bear the pain of living without understanding why they are true”

Modèle de la glace



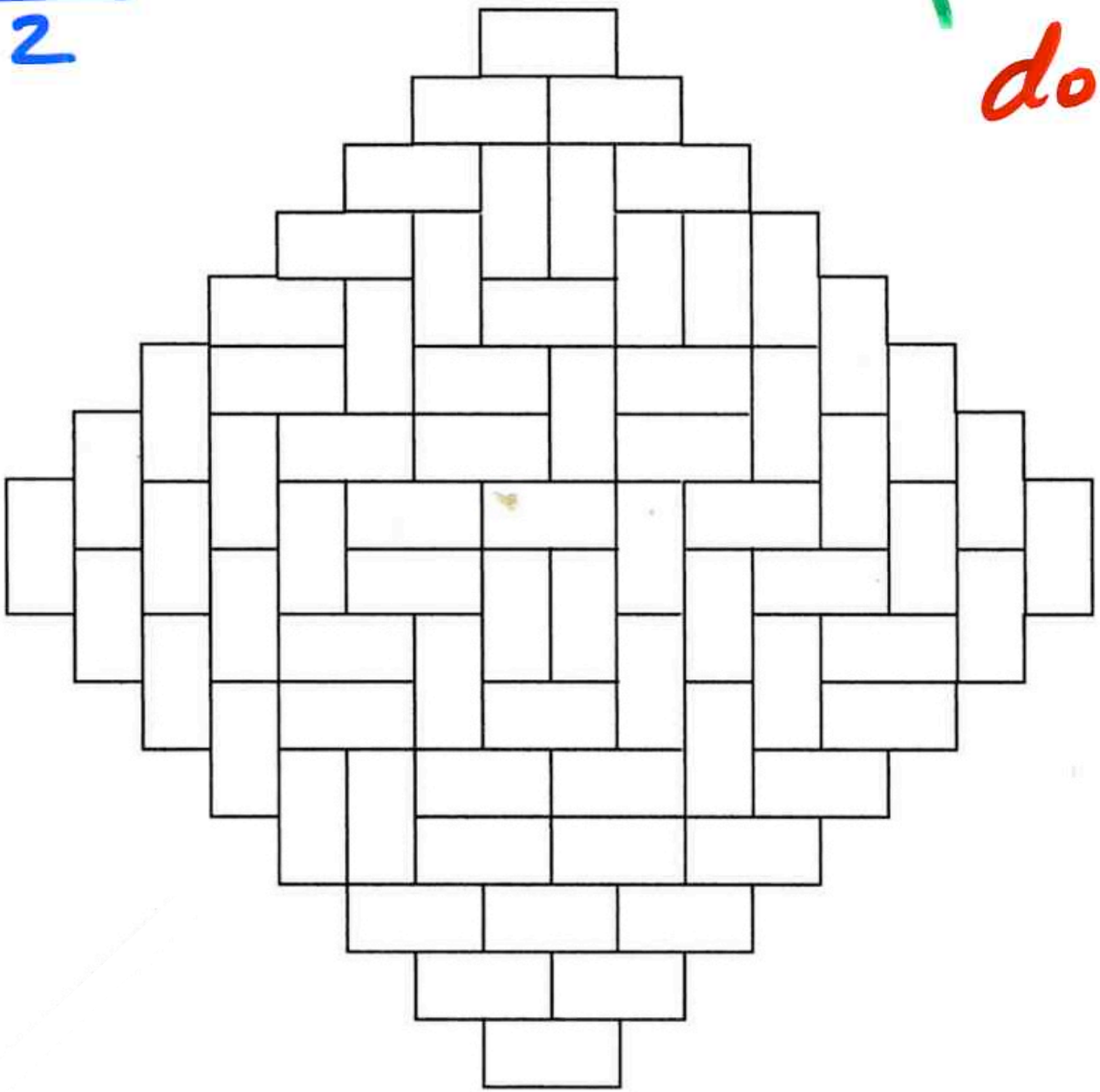

1
1 -1 1
1 -1 1
1
1
1

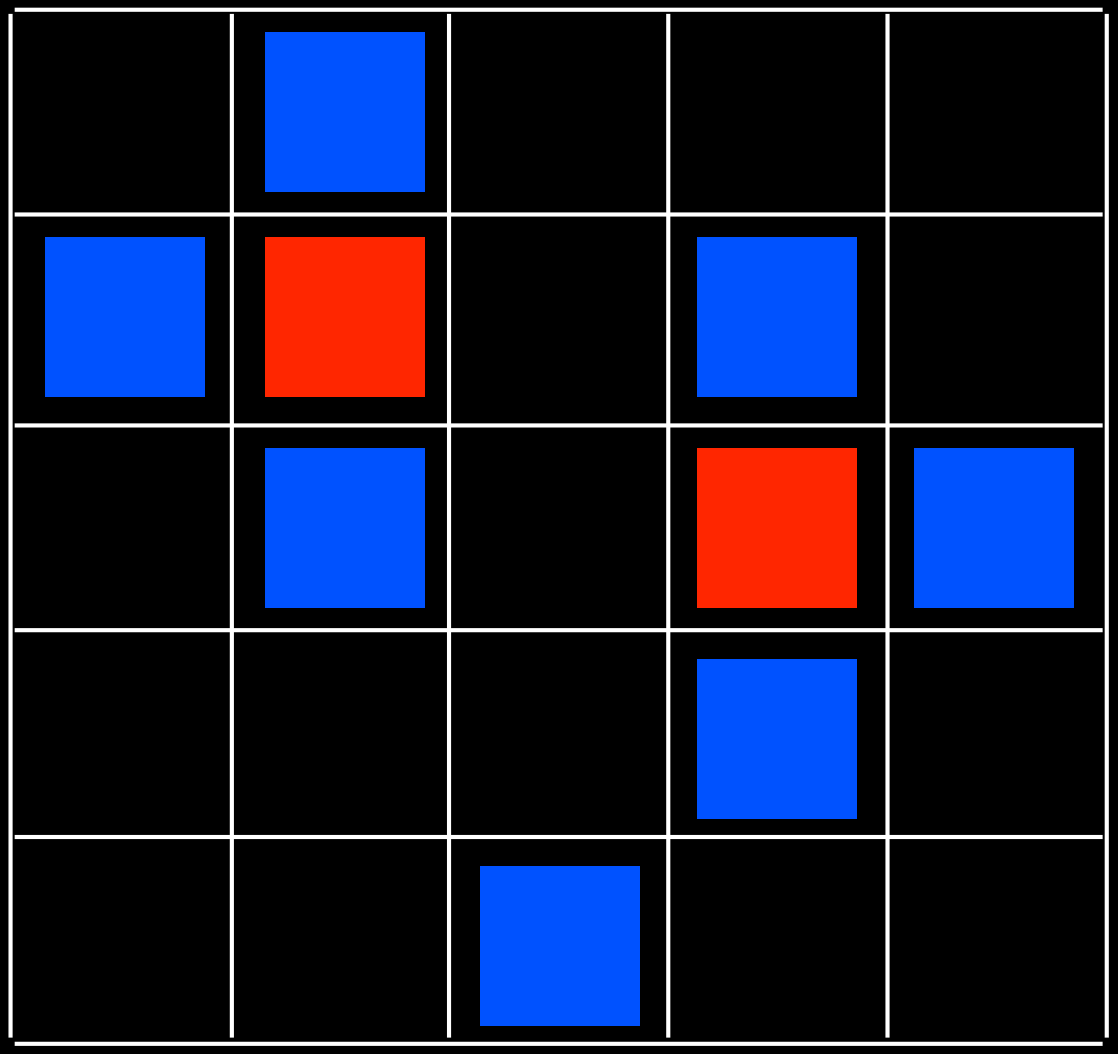


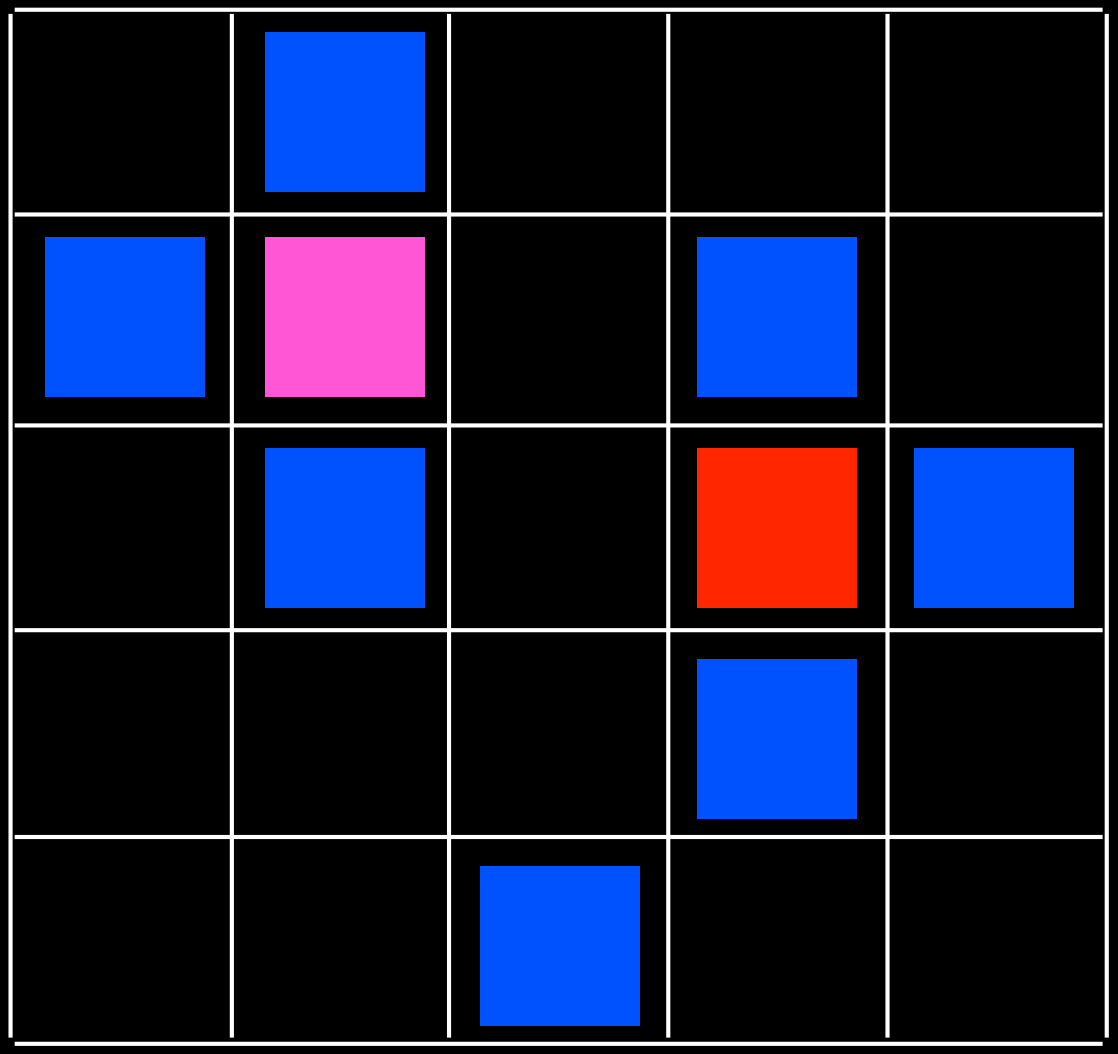
$$2^{\frac{n(n+1)}{2}}$$

pavage

dominos







La combinatoire "plurielle"

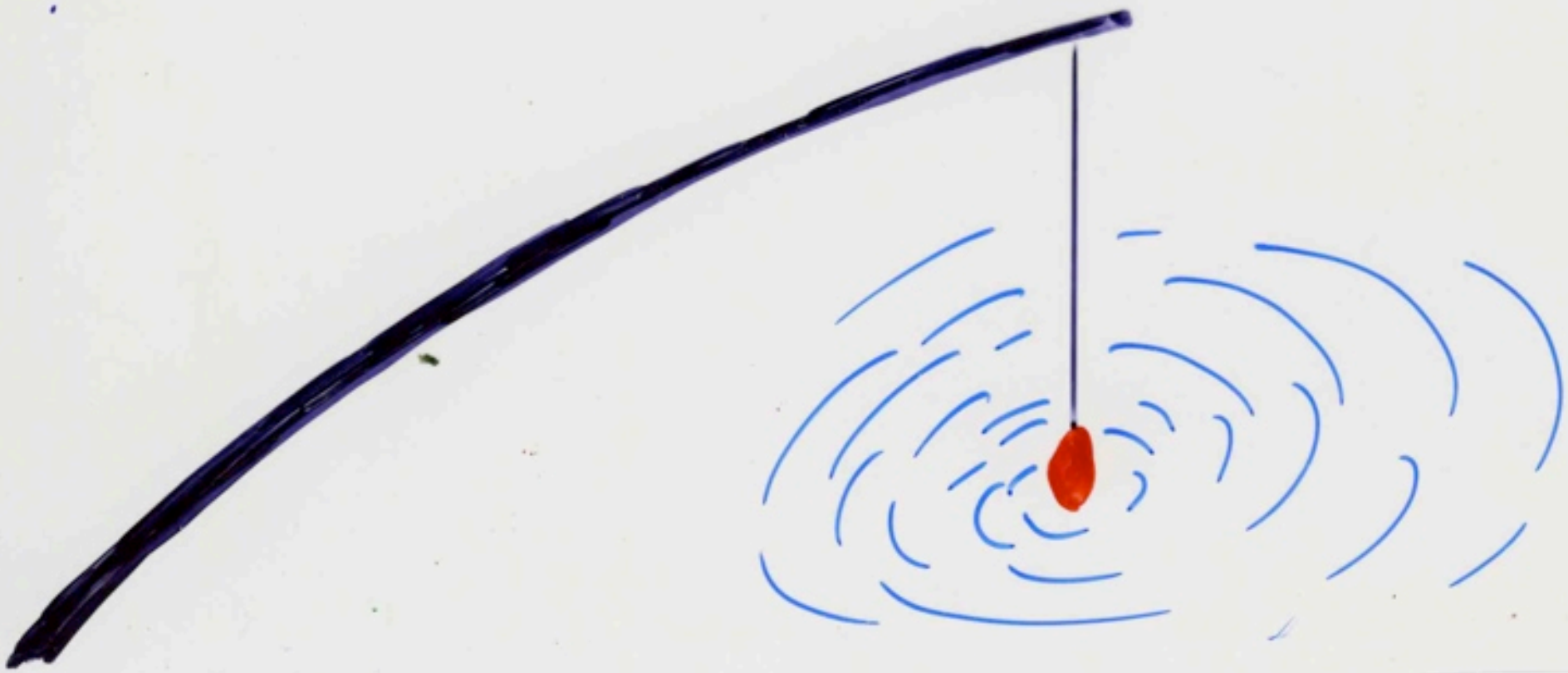
combinatoire classique
énumérative,
bijective,
algébrique,

existentielle,
analytique,
quantique,
expérimentale,
nostalgique,

magique ....

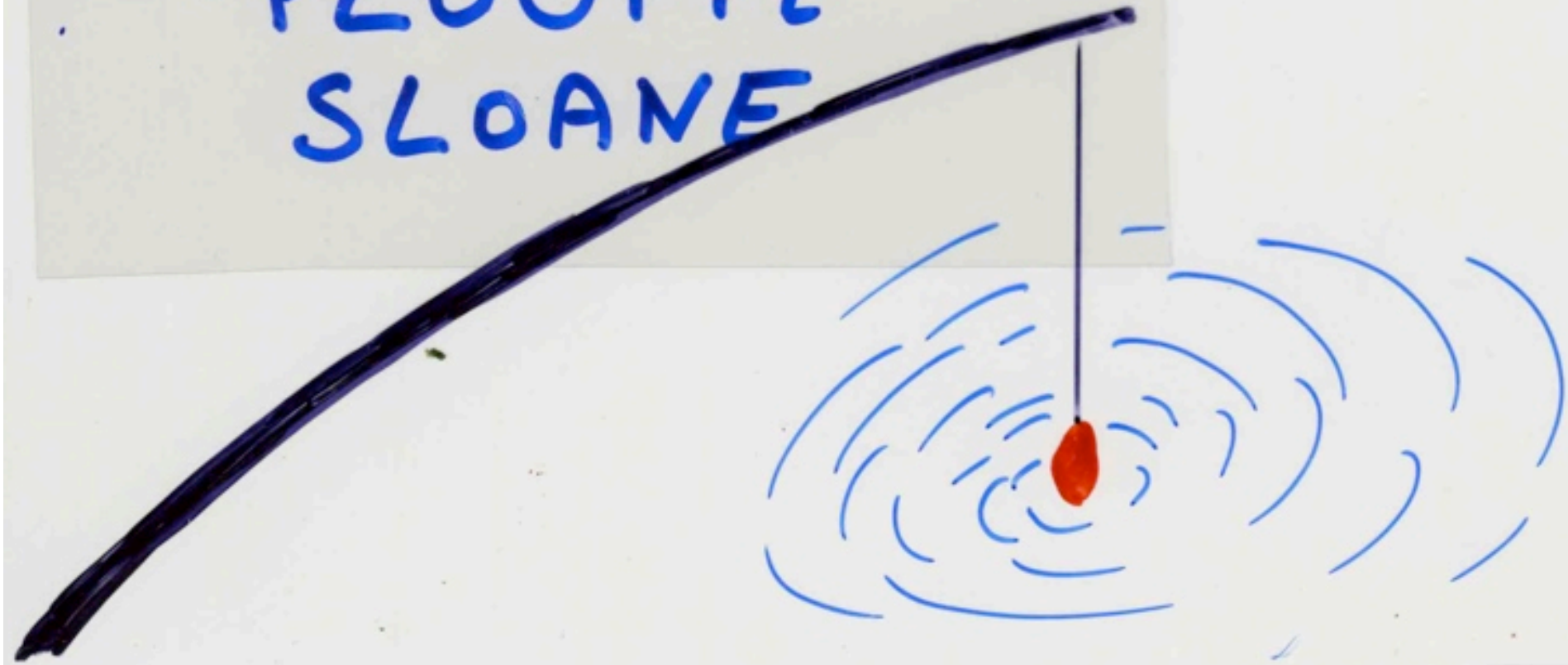
Combinatoire expérimentale

pêche à la ligne



pêche à la ligne

PLOUFFE
SLOANE



THE ENCYCLOPEDIA ▼▼▼ OF ▼▼▼ INTEGER SEQUENCES

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ACADEMIC PRESS
San Diego New York Boston London Sydney Tokyo Toronto

M2819 1, 3, 9, 35, 178
Van der Waerden numbers. Ref Loth83 49. [1,2; A5346]

M2820 1, 3, 9, 35, 201, 1827
Coefficients of Bell's formula. Ref NMT 10 65 62. [2,2; A2575, N1134]

M2821 1, 3, 9, 37, 153, 951, 5473, 42729, 353937, 3455083, 30071001, 426685293, 4707929449, 59350096287, 882391484913, 15177204356401, 205119866263713
Sums of logarithmic numbers. Ref TMS 31 79 63. jos. [0,2; A2751, N1135]

M2822 1, 1, 1, 3, 9, 37, 177, 959, 6097, 41641, 325249, 2693691, 24807321, 241586893, 2558036145, 28607094455, 342232522657, 4315903789009, 57569080467073
Expansion of $e^{\tan x}$. Ref JO61 150. [0,4; A6229]

M2823 1, 3, 9, 42, 206, 1352, 10168
Regular semigroups of order n. Ref PL65. MAL 2 2 67. SGF 14 71 77. [1,2; A1427, N1136]

M2824 0, 1, 1, 3, 9, 45, 225, 1575, 11025, 99225, 893025, 9823275, 108056025, 1404728325, 18261468225, 273922023375, 4108830350625, 69850115960625
Expansion of $1 / (1-x)(1-x^2)^{1/2}$. Ref R1 87. [1,4; A0246, N1137]

M2825 1, 1, 1, 3, 9, 48, 504, 14188, 1351563
Threshold functions of n variables. Ref PGEC 19 821 70. MU71 38. [0,4; A1530, N1138]

M2826 3, 9, 54, 450, 4725, 59535, 873180, 14594580
Expansion of an integral. Ref C1 167. [2,1; A1194, N1139]

M2827 1, 3, 9, 89, 1705, 67774
Superpositions of cycles. Ref AMA 131 143 73. [3,2; A3225]

M2828 1, 3, 9, 93, 315, 3855, 13797, 182361, 9256395, 34636833, 1857283155, 26817356775, 102280151421, 1497207322929, 84973577874915, 4885260612740877
Fermat quotients: $(2^{p-1}-1)/p$. Ref Well86 70. [0,2; A7663]

M2829 3, 10, 4, 5, 10, 2, 5, 3, 2, 3, 6, 6, 6, 3, 5, 6, 10, 5, 5, 10, 6, 6, 6, 2, 5, 8, 2, 6, 8, 4, 6, 6, 4, 5, 10, 2, 4, 7, 11, 5, 7, 9, 10, 7, 1, 6, 7, 11, 7, 10, 0, 6, 8, 9, 6, 4, 11, 7, 13, 2, 6, 4, 4
Iterations until $3n$ reaches 153 under x goes to sum of cubes of digits map. Ref Robe92 13. [1,1; A3620]

M2830 1, 3, 10, 13, 62, 75, 187, 512, 848, 6296, 13361, 73011, 597449, 1865358,

système de calcul formel
MAPLE, ...

deviner une identité

démontrer une identité par un ordinateur

“Combinatoire nostalgique”

Theil, und solches kann auf 5 mahl verschiedene Arten geschehen nemlich
durch die Diagonales I. ac, ad; II. bd, be; III. ce, ca; IV. de, da; V. ea, eb

Wenn hier ein Theil durch 3 Diagonales in 4 Triangula getheilt, und solches kann auf 14 verschiedene Arten geschehen.
Nun ist die ganze Generalitet. Da ein Polygonum von n Seiten durch n-3 Diagonales in n-2 Triangula getheilt wird, auf wie vielerley verschiedene Arten solches geschehen kann.
Setze ich nun die Anzahl dieser verschiedenen Arten = x
so habe ich per Inductionem gefunden

wenn $n = 3 . 4 . 5 . 6 . 7 . 8 . 9 . 10$
so ist $x = 1 , 2 , 5 , 14 . 42 , 132 , 429 , 1430$

Hieraus habe ich nun den Schluß gemacht, die generalitet sey

$$x = \frac{2 . 6 . 10 . 14 . 18 . 22 . \ldots (4n-10)}{2 . 3 . 4 . 5 . 6 . 7 \ldots (n-1)}$$ oder es ist

$1 = \frac{2}{2}$; $2 = 1 . \frac{6}{3}$; $5 = 2 . \frac{10}{4}$; $14 = 5 . \frac{14}{5}$; $42 = 14 . \frac{18}{6}$; $132 = 42 . \frac{22}{7}$;

daß alle auf einerley Art die Regel durch gefunden wird. Die Induction aber ist gar leicht, weil sich nur eine Art Theilung ist nicht. Daß diese Art nicht alle und letzte Art Theilung sey. Also die Progression der Zahlen 1, 2, 5, 14, 42, 132, etc. habe ich auch diese Regel zu generiren. Daß

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc.} = \frac{1 - 2a - \sqrt{(1-4a)}}{2a}$$

also wenn $a = \frac{1}{4}$ so ist $1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} + \text{etc.} = 4$.

Alles dies wenige bestehet ich für die Hochwohlgebohrne und stundige Abgunstigkeit gutherzigst anzunehmen und habe die Ehre mich der fortdauernden Hochachtung zu empfehlen

Ew. Hochwohlgebohrnen

Berlin d. 4. Sept. 1751.

gehorsamster Diener
L. Euler

Monsieur Hipparque,

Vous avez dit 103049 ?

M2819 1, 3, 9, 35, 178
Van der Waerden numbers. Ref Loth83 49. [1,2; A5346]

M2820 1, 3, 9, 35, 201, 1827
Coefficients of Bell's formula. Ref NMT 10 65 62. [2,2; A2575, N1134]

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Iterations until $3n$ reaches 153 under x goes to sum of cubes of digi[illegible] [1,1; A3620]

M2830 1, 3, 10, 13, 62, 75, 437, 512, 949, 6206, 13361, 7301[illegible] 6193523, 26639450, 59472423, 383473988, 1593368375, [illegible]
Convergents to cube root of 3. Ref AMP 46 105 1866. L1 67[illegible]

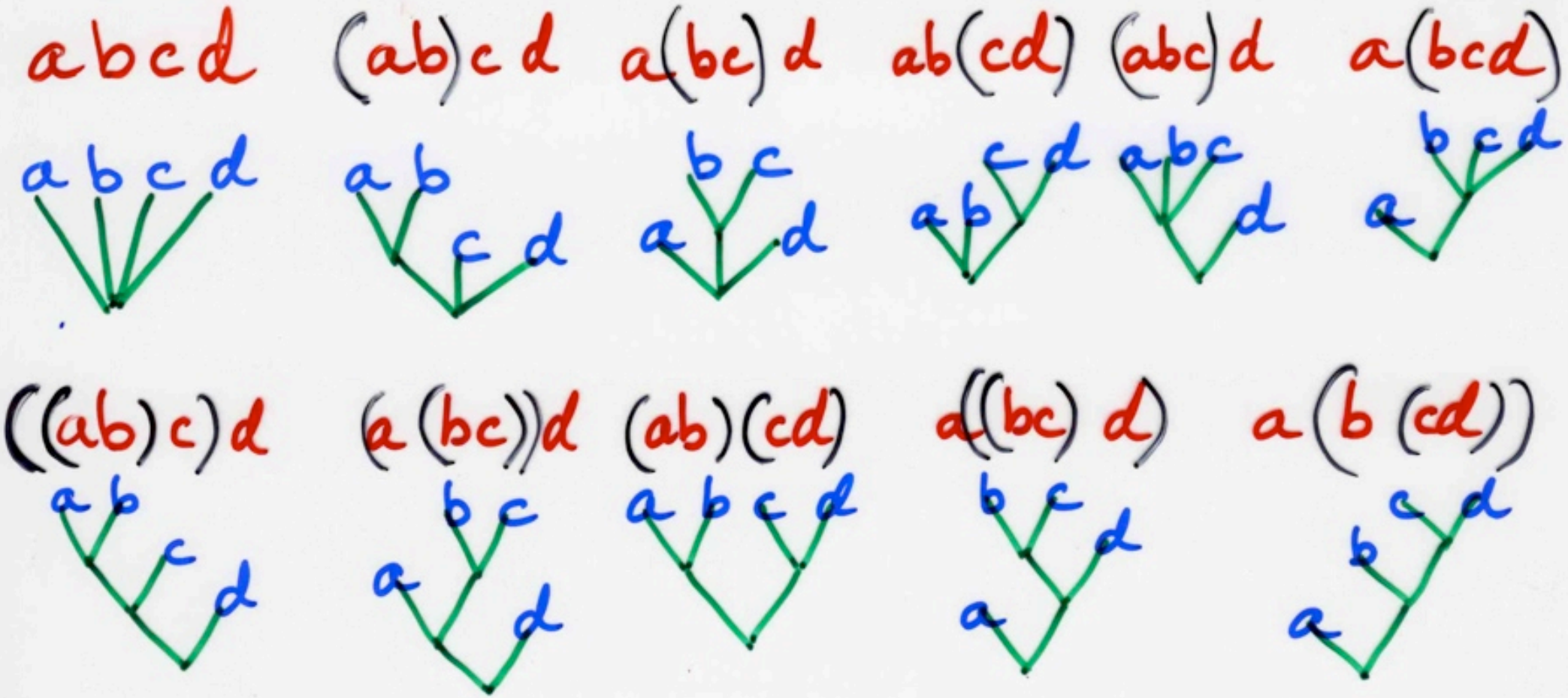
21489003, 80198051, 299303201, 1117014753, 4168755811, 15558008491
$a(n)=4a(n-1)-a(n-2)$. Ref EUL (1) 1 375 11. MMAG 40 78 67. [0,3; A1835, N1160]

M2895 1, 3, 11, 43, 171, 683, 2731, 10923, 43691, 174763, 699051, 2796203, 11184811, 44739243, 178956971, 715827883, 2863311531, 11453246123, 45812984491
$(2^{2n+1}+1)/3$. Ref JGT 17 625 93. [0,2; A7583]

M2896 3, 11, 43, 683, 2731, 43691, 174763, 2796203, 715827883, 2932031007403, 768614336404564651, 201487636602438195784363
Primes of form $(2^p+1)/3$. Ref MMAG 27 157 54. [1,1; A0979, N1161]

M2897 1, 3, 11, 44, 186, 814, 3652, 16689, 77359, 362671, 1716033, 8182213, 39267086, 189492795, 918837374, 4474080844, 21866153748, 107217298977, 527266673134
Fixed hexagonal polyominoes with n cells. Ref RE72 97. dhr. [1,2; A1207, N1162]

M2898 1, 1, 3, 11, 45, 197, 903, 4279, 207[illegible], 103049, [illegible]18859, 2646723, 13648869, 71039373, 372693519, 1968801519, 104635[illegible]8353, [illegible]5909013009, 300159426963
Schroeder's second problem: $(n+1)a(n+[illegible]-1)a(n)-(n-2)a(n-1)$. Ref EDMN 32 6 40. BAMS 54 359 48. RCI 168. C[illegible] 198. [1,3; A1003, N1163]



$S_4 = 11$ arbres de Schröder

génération aléatoire d'objets combinatoires

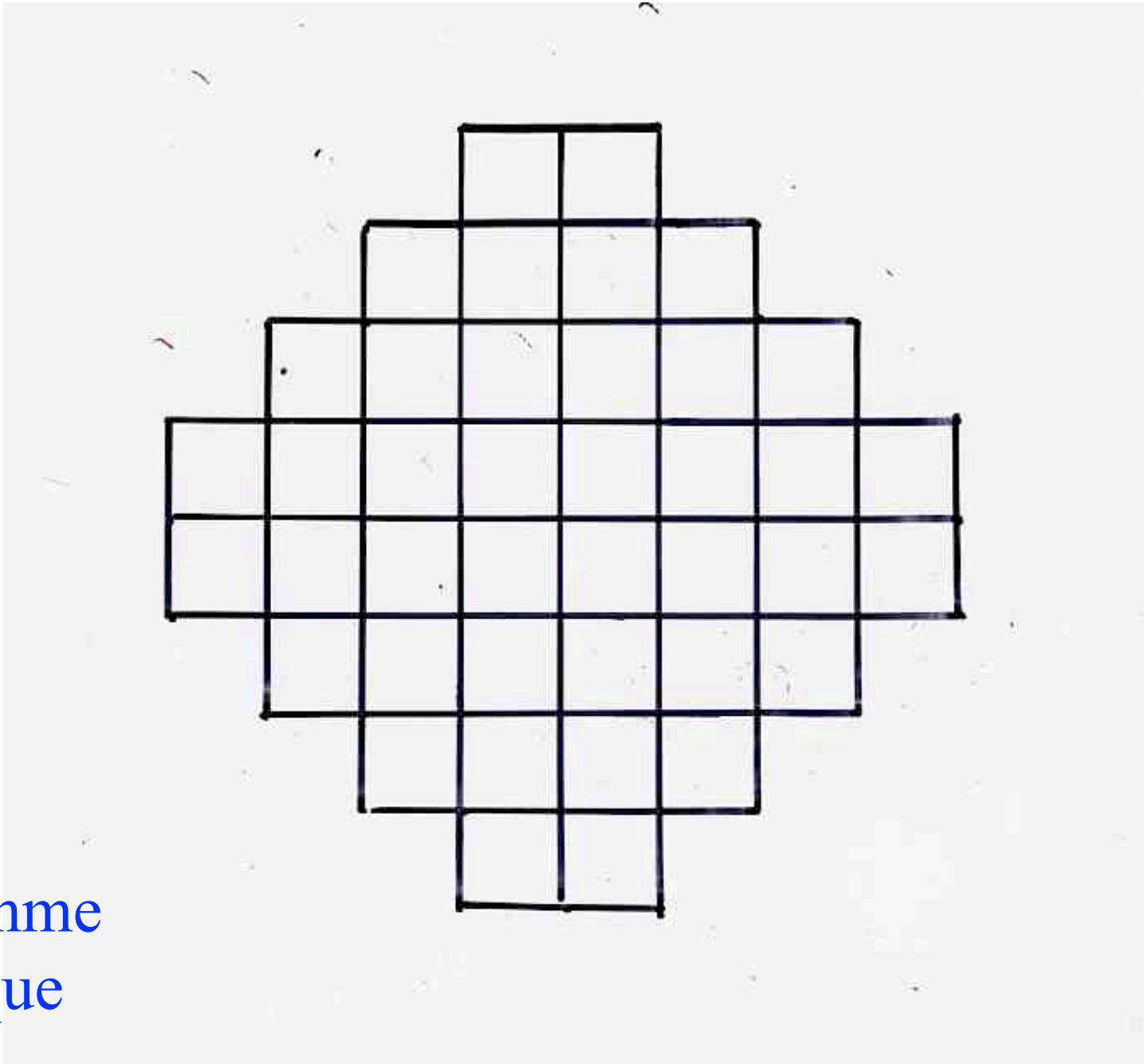
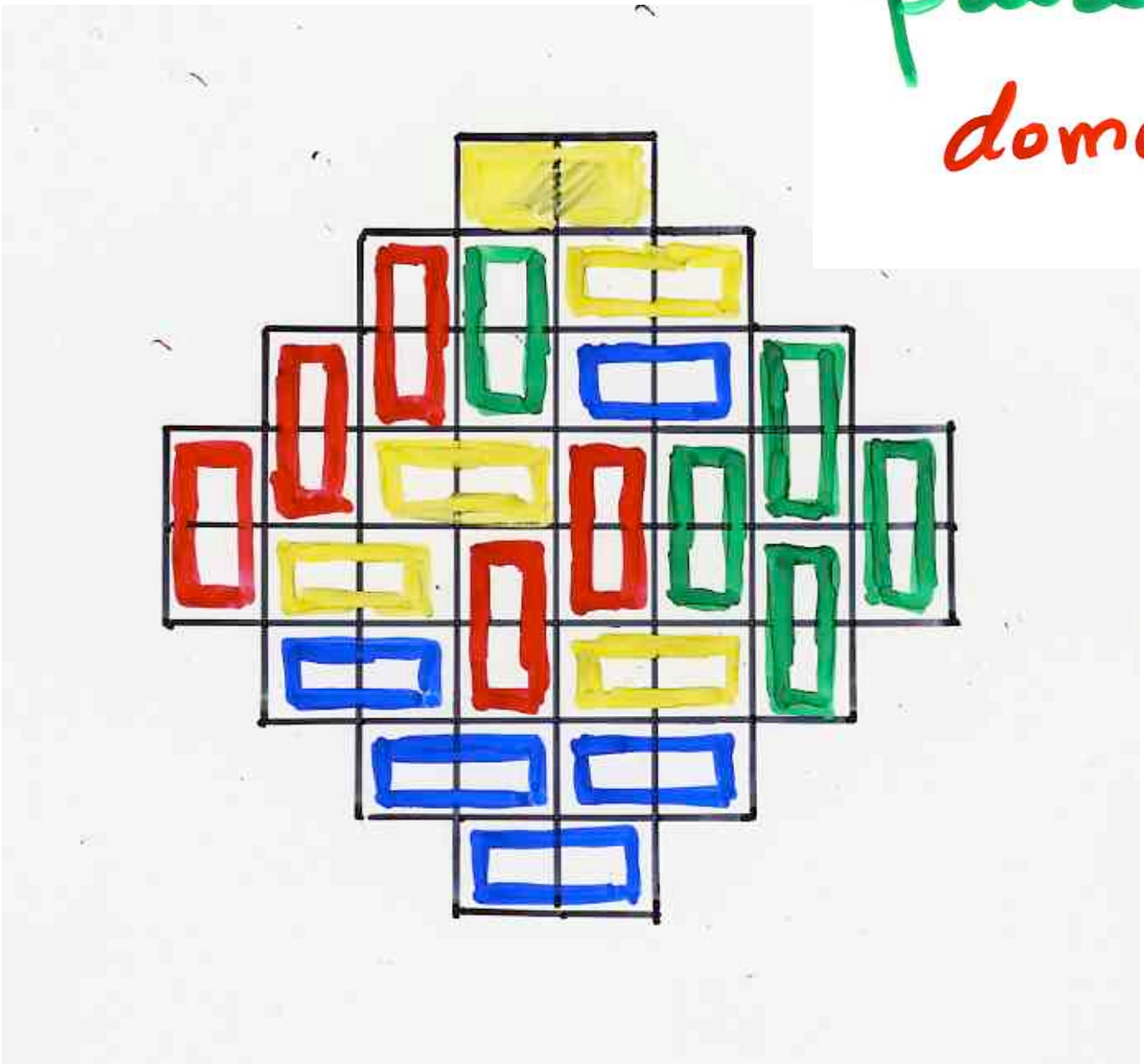
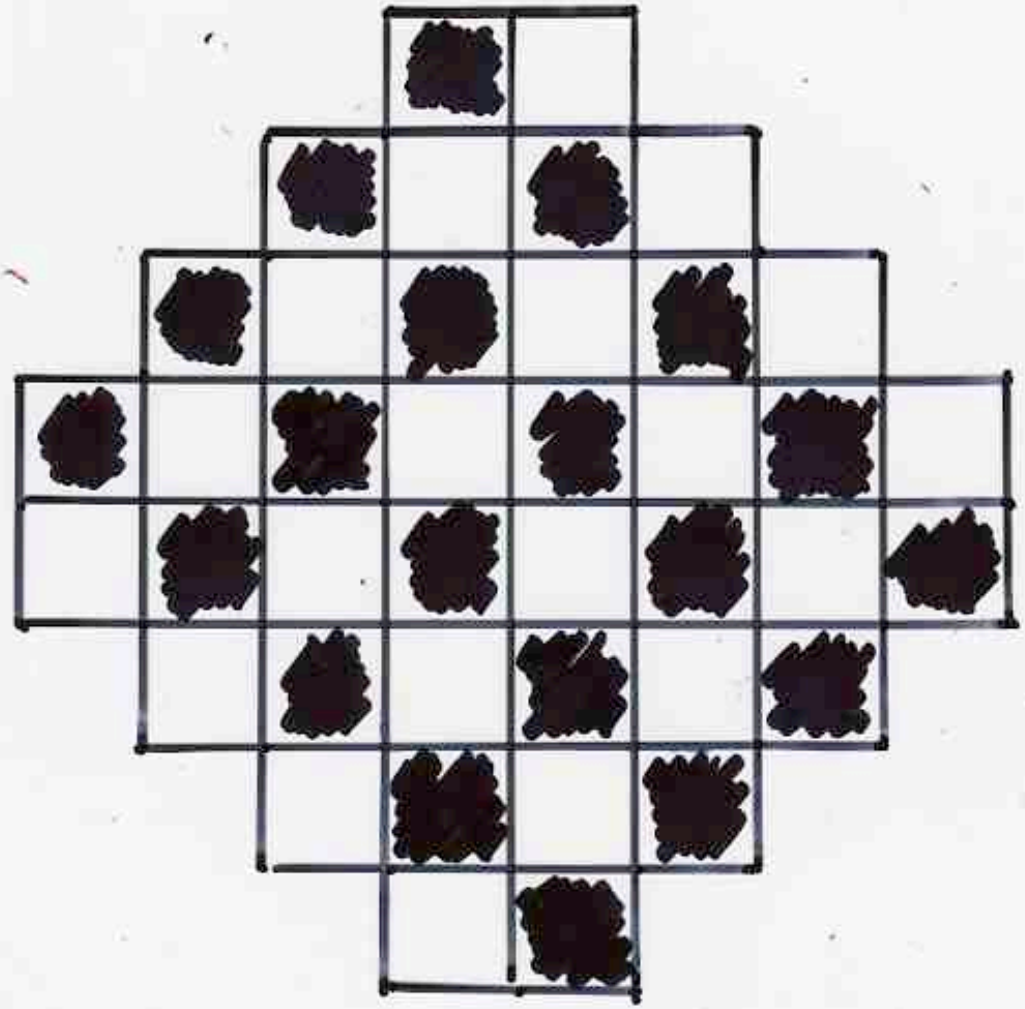
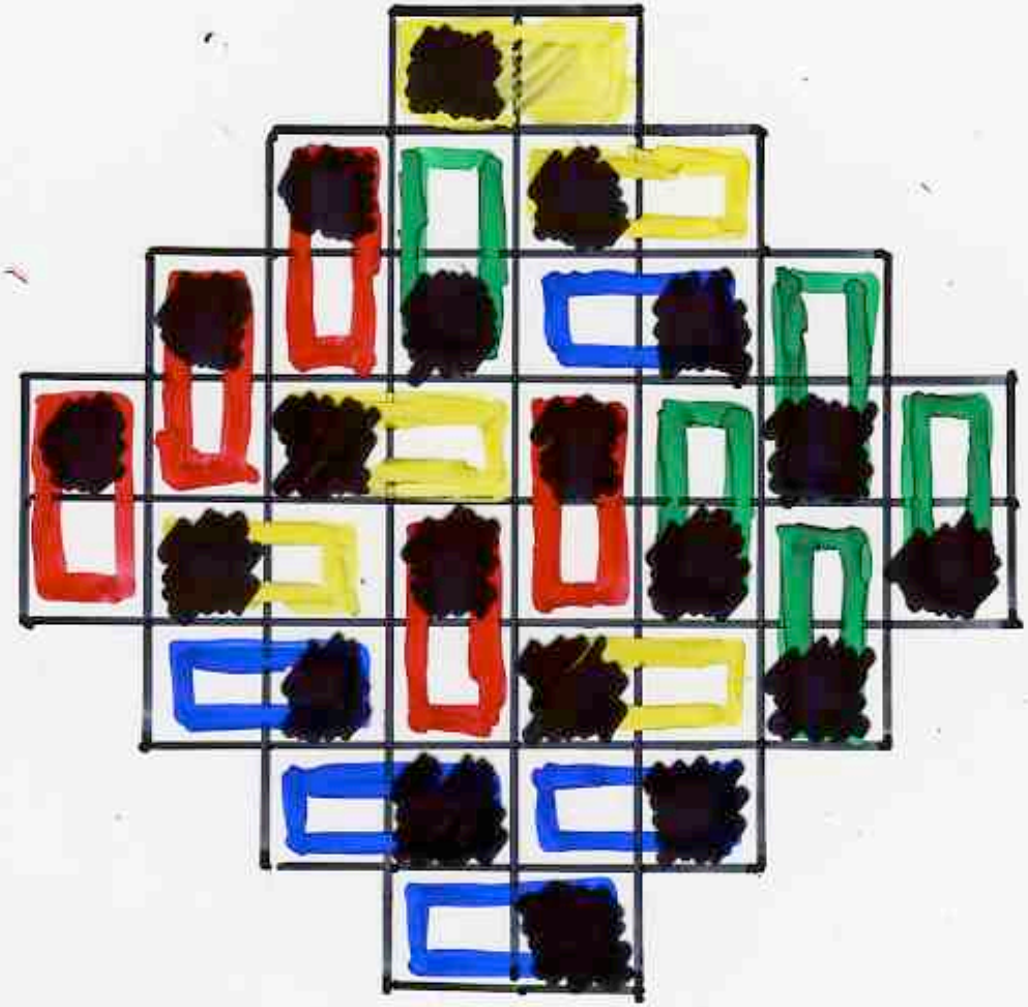


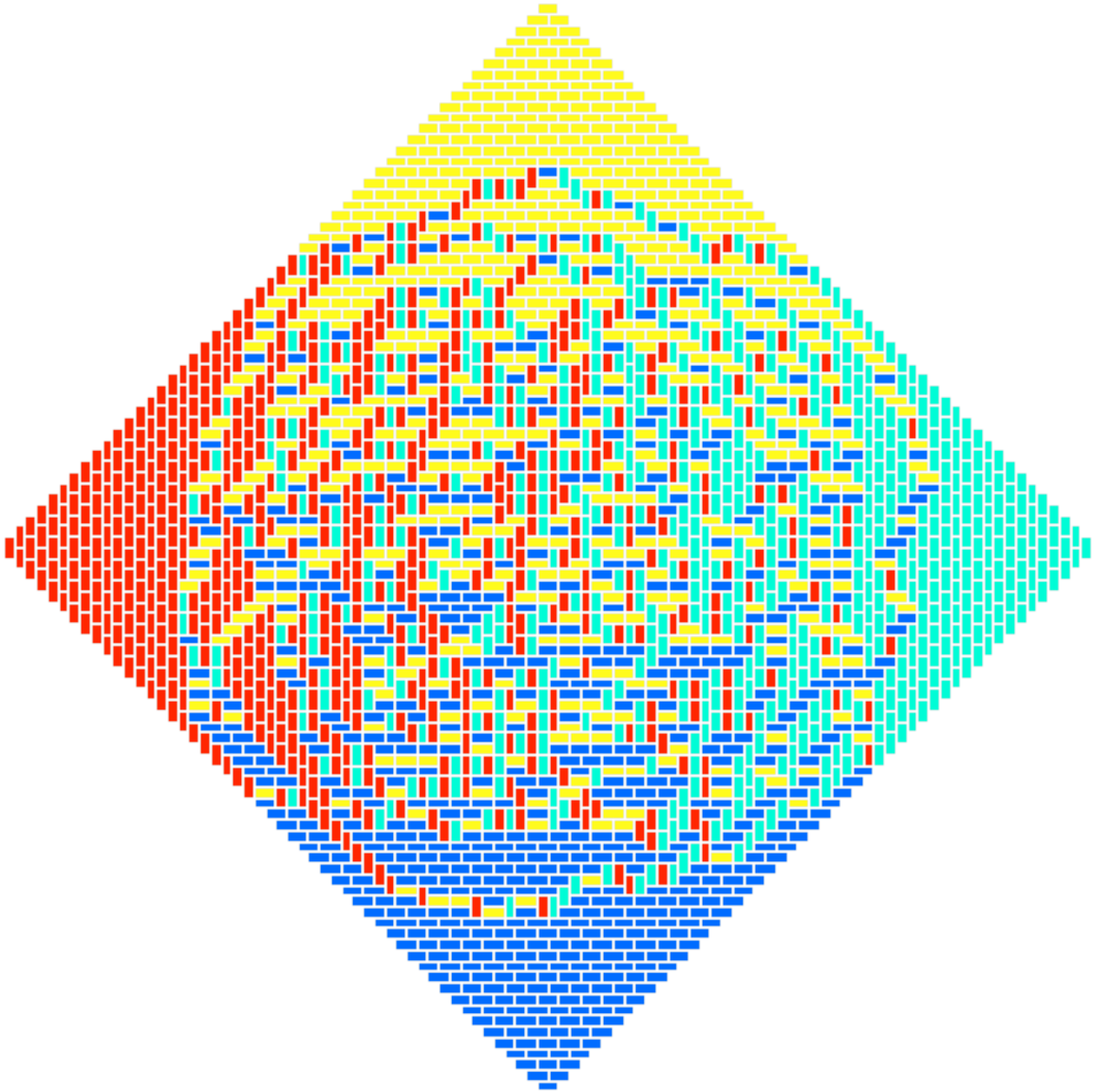
diagramme
Aztèque

pavage dominos

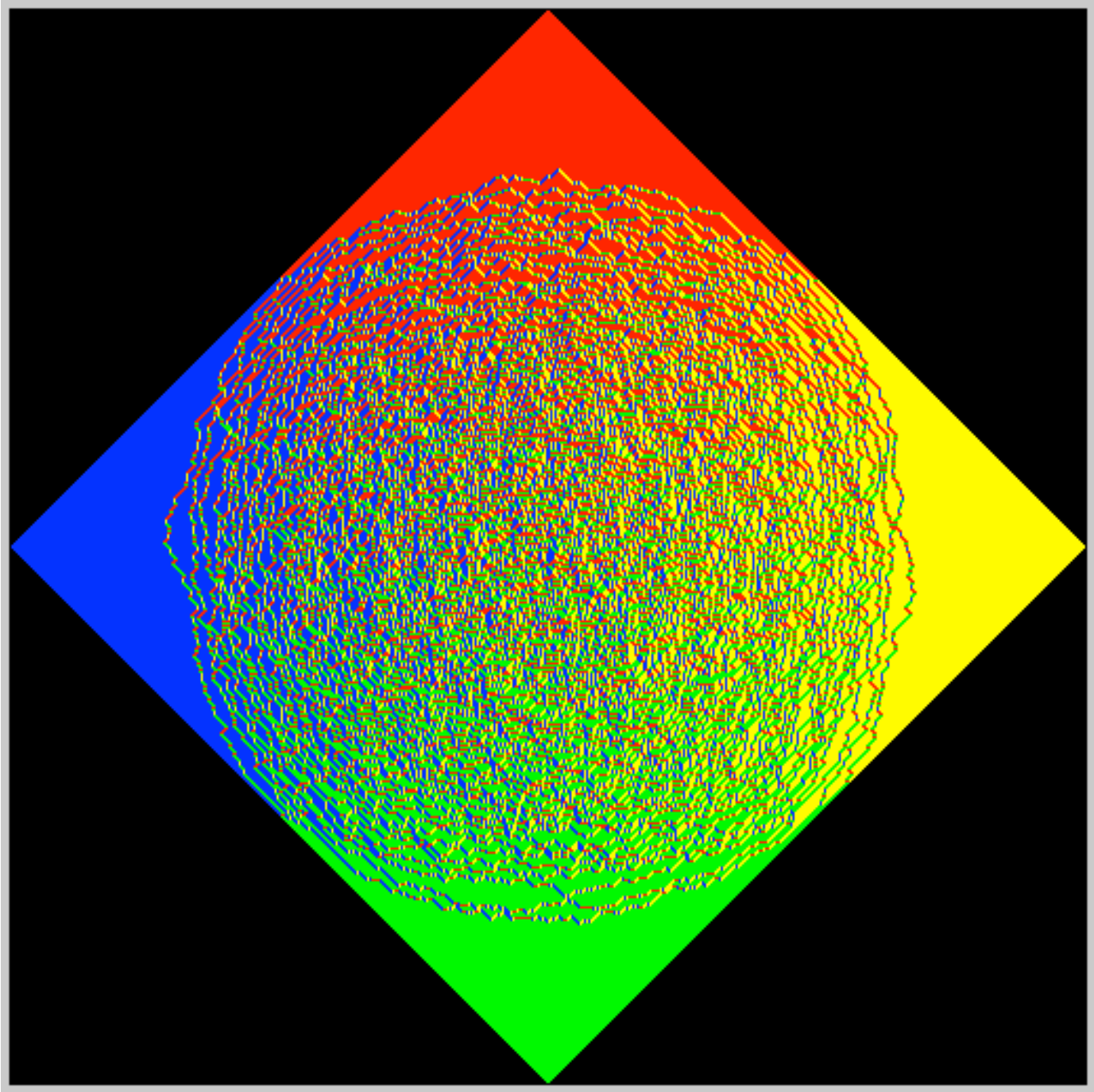








le théorème du cercle antique



Il y a des arbres dans les étoiles,
des arbres dans les grains de lumière.

Les théories mathématiques s'interpellent,
s'entrecroisent, renaissent, se fondent entre elles.

Les grands Maîtres se parlent à travers les siècles
dans le jardin merveilleux des Mathématiques.