Couplages dans les graphes

Journées jeunes chercheurs lycéens

Université de Pau

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Xavier Viennot

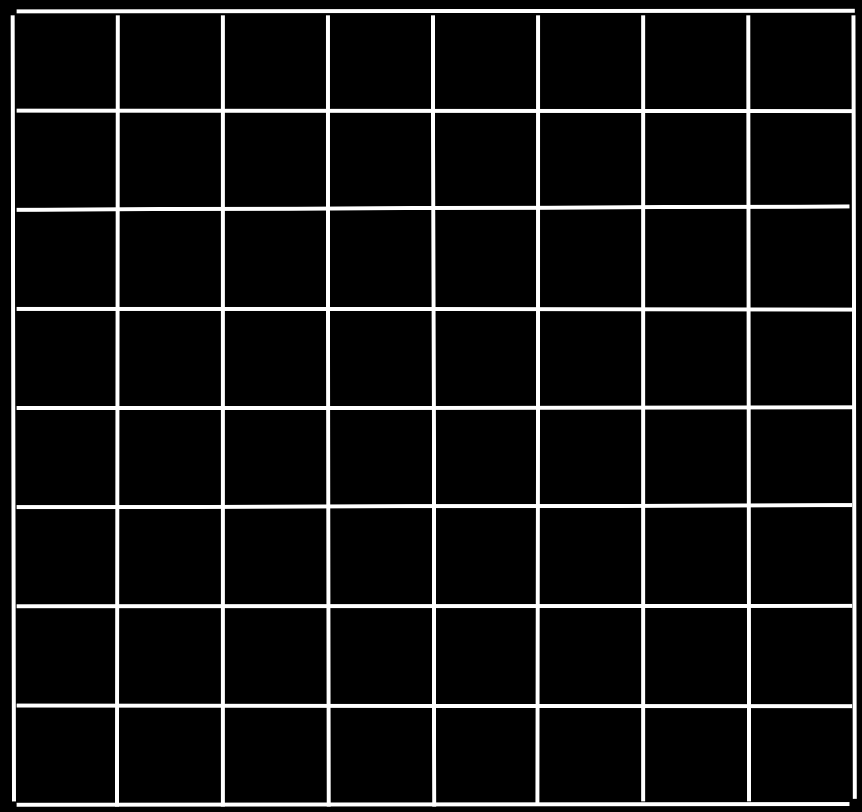
LaBRI, CNRS, Bordeaux

www.xavierviennot.org

pavages ....

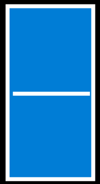
pavage apériodique de R.Ammann
(salle de bain de G.Kuperberg)

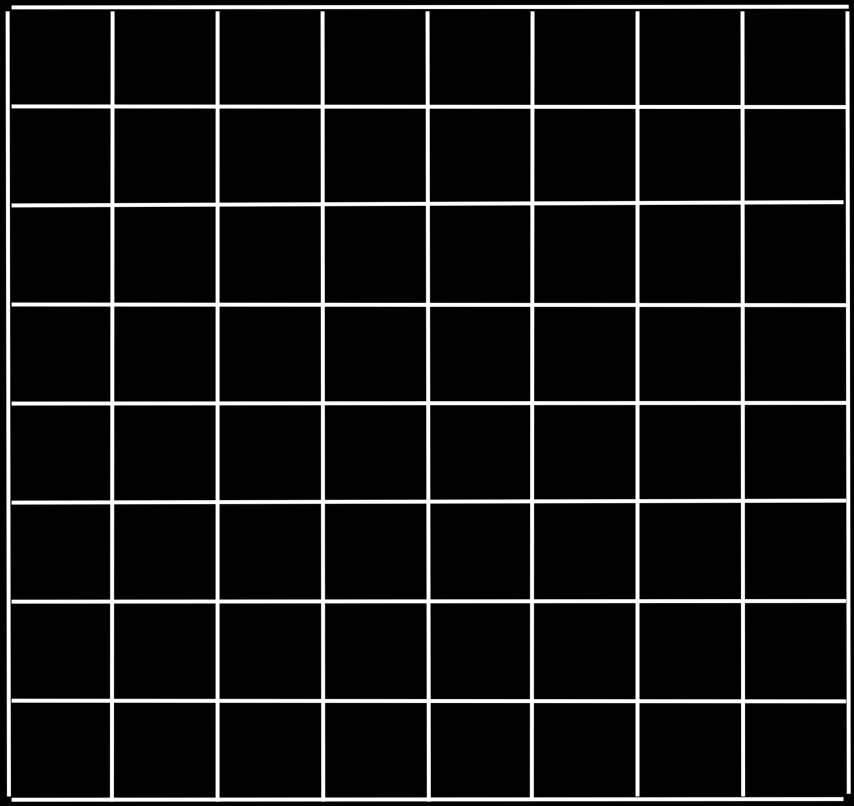
un échiquier



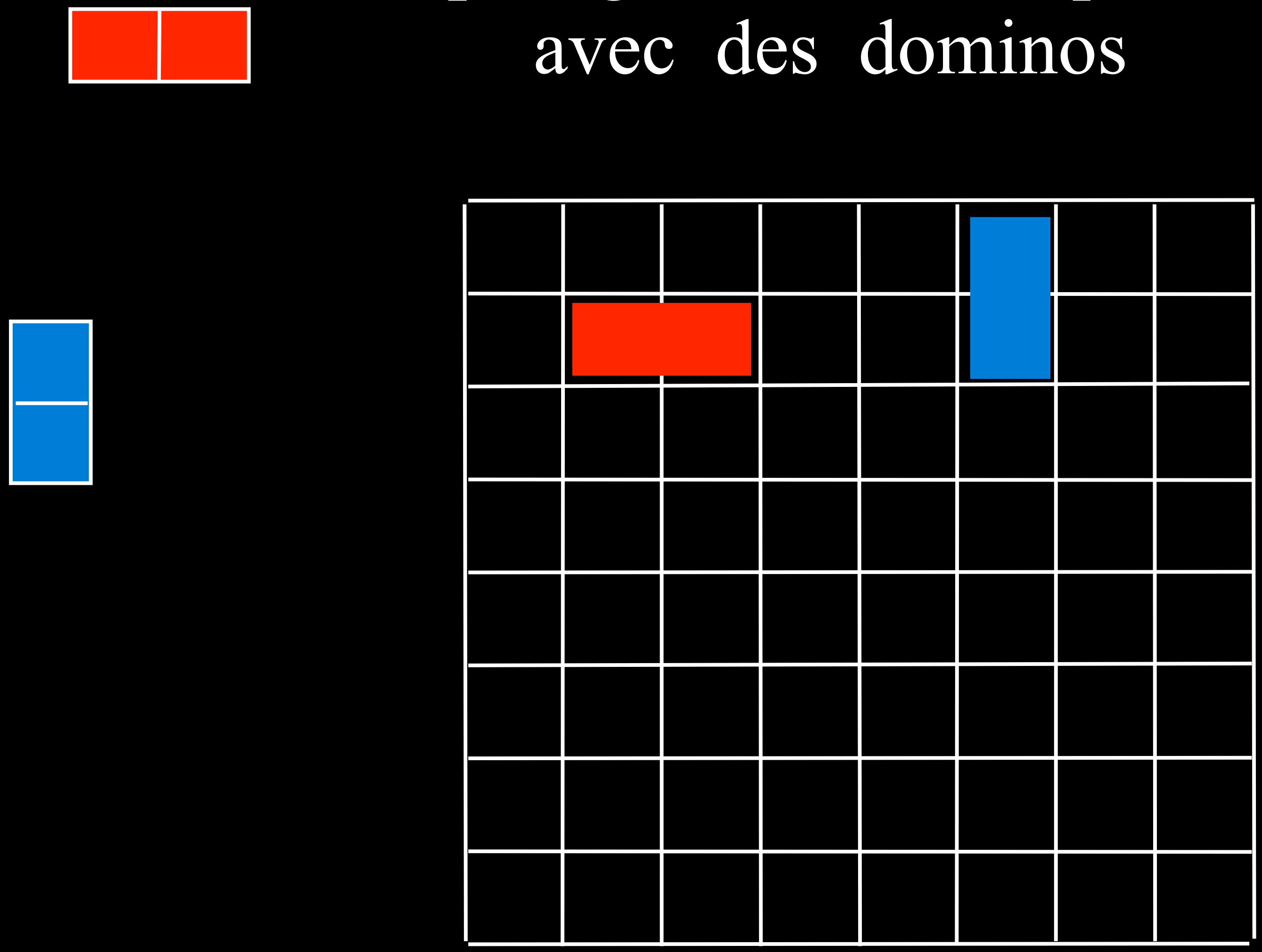
8 lignes
8 colonnes

dominos





pavages d'un échiquier avec des dominos



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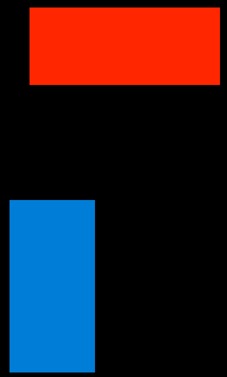
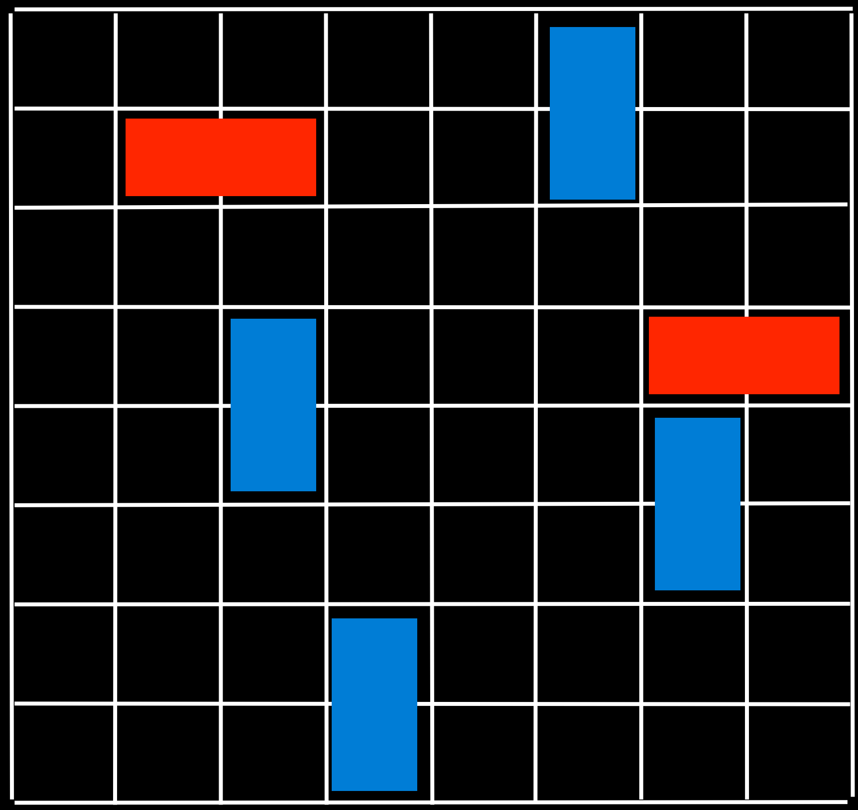
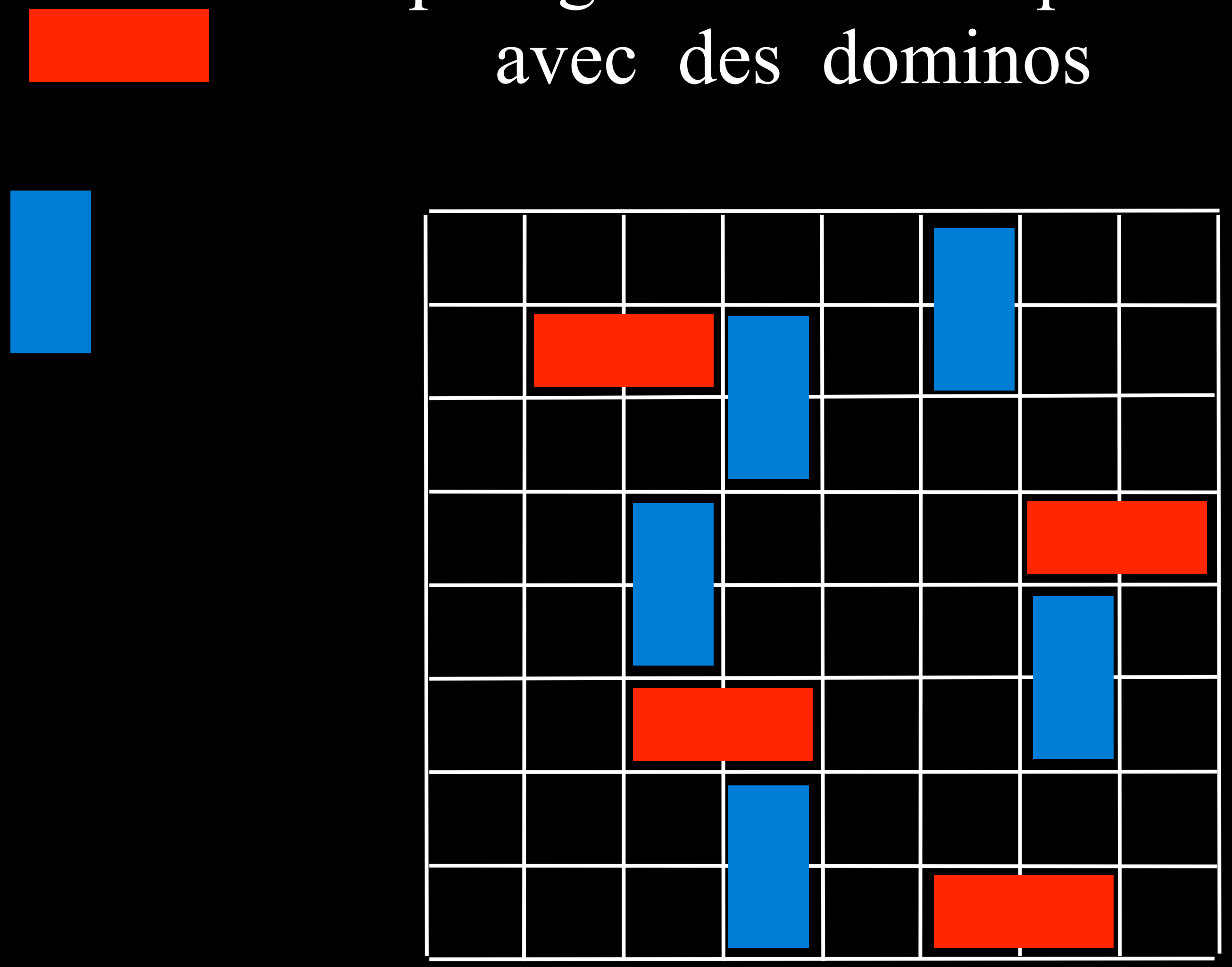




pavages d'un échiquier
avec des dominos



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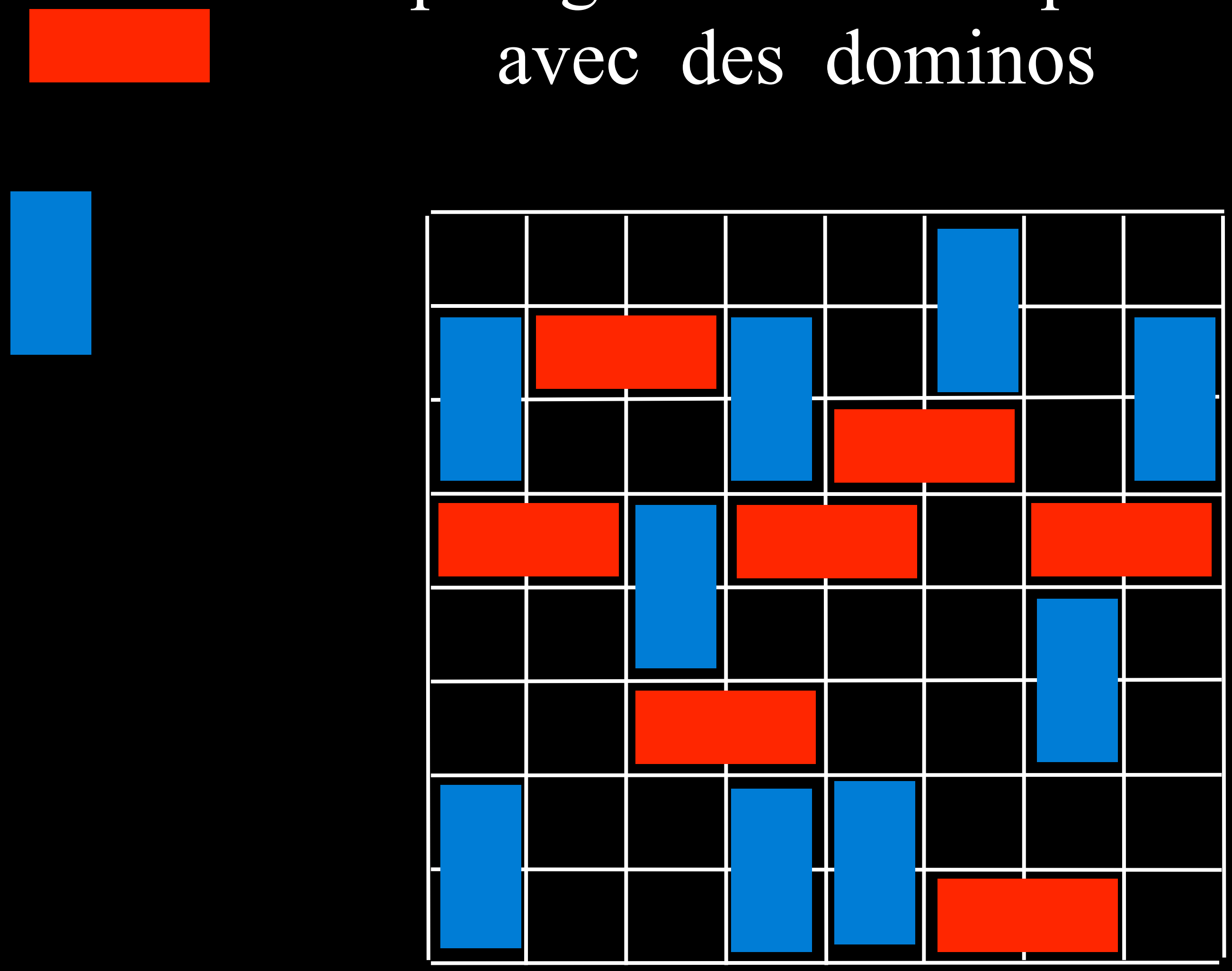
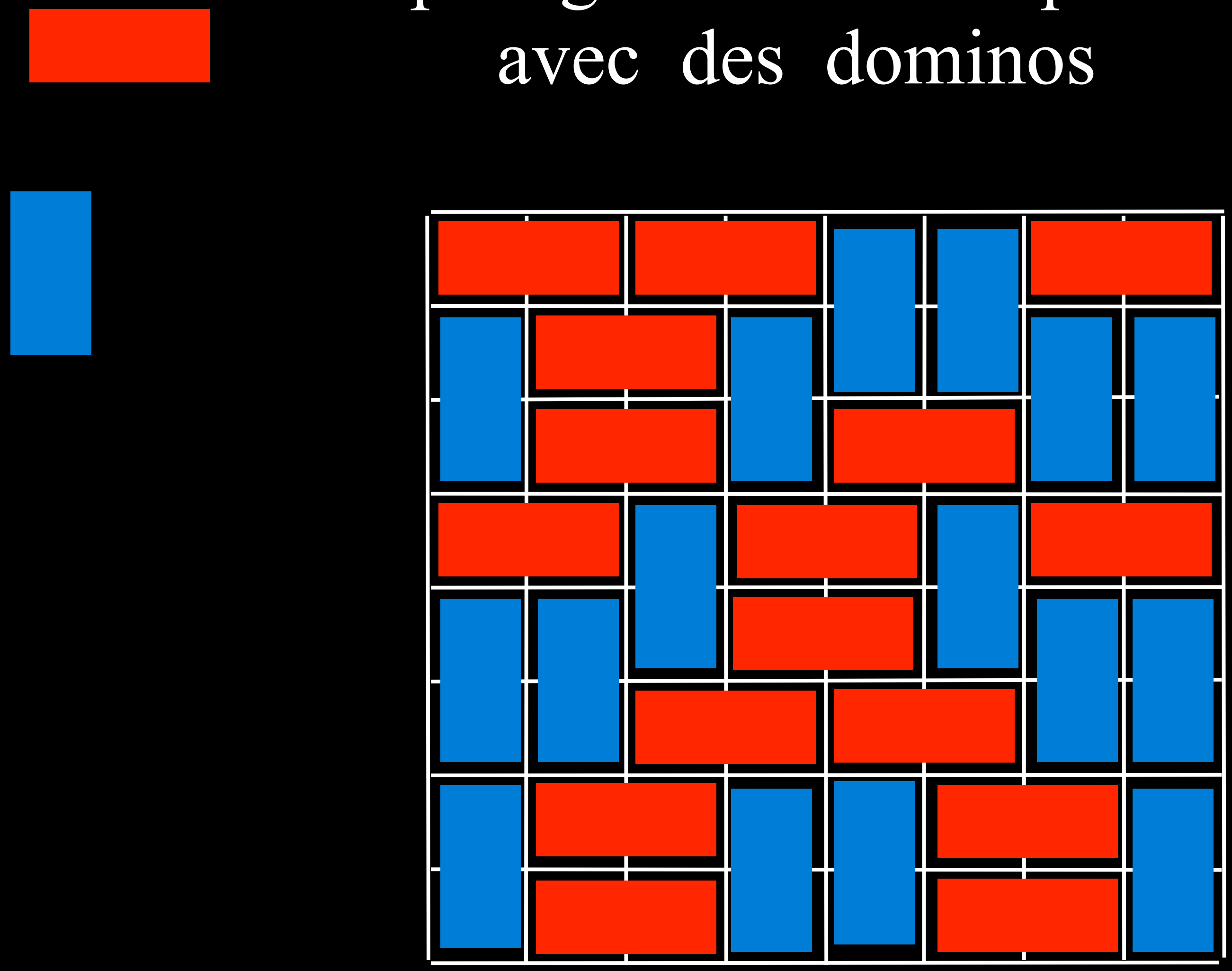


pavages d'un échiquier
avec des dominos



le nombre de pavages d'un échiquier 8 x 8
= 12 988 816

une «formule» ?

nombre de pavages

pour un rectangle $m \times n$

Une belle formule ....

avec les pavages Aztèques

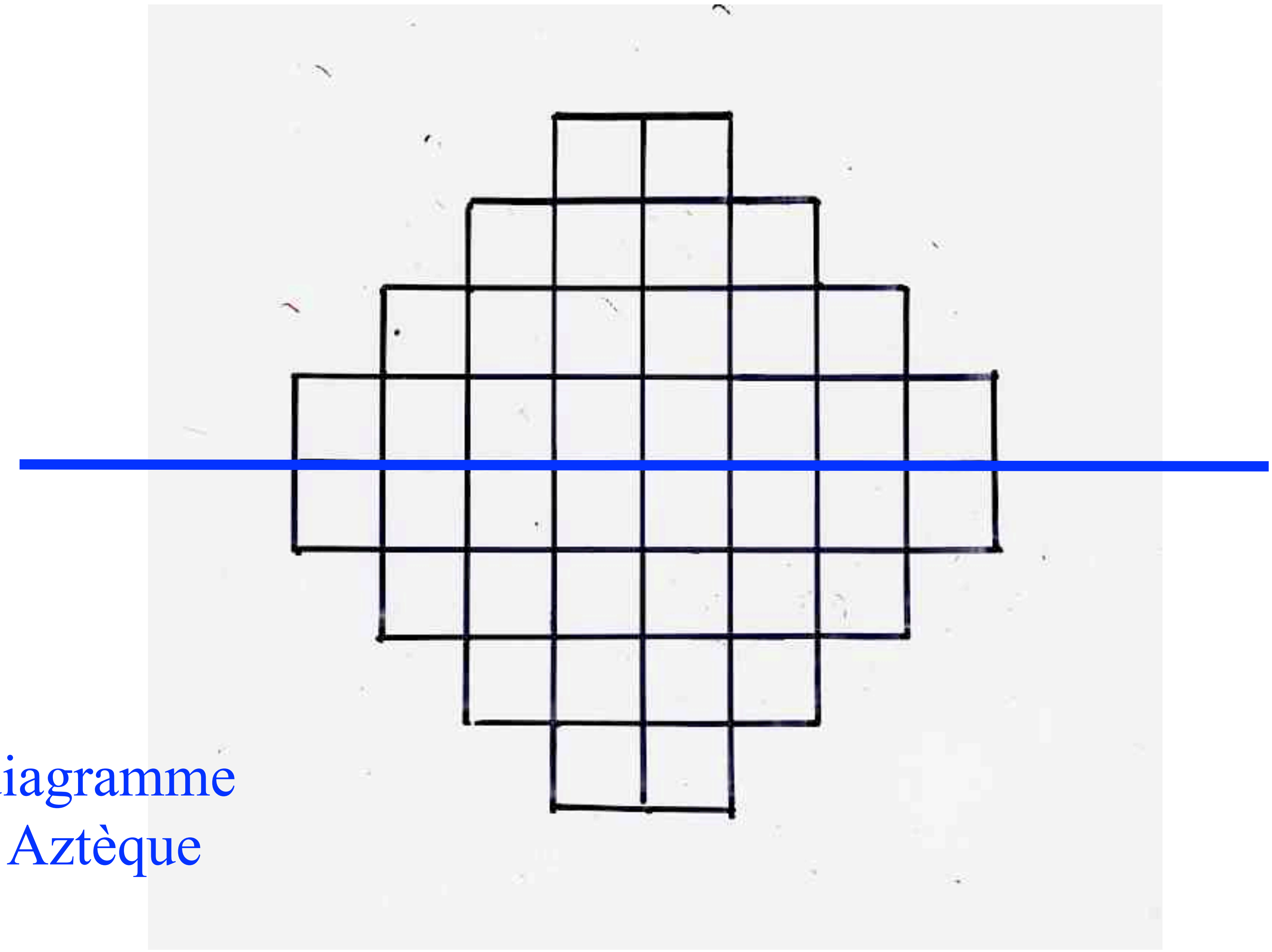
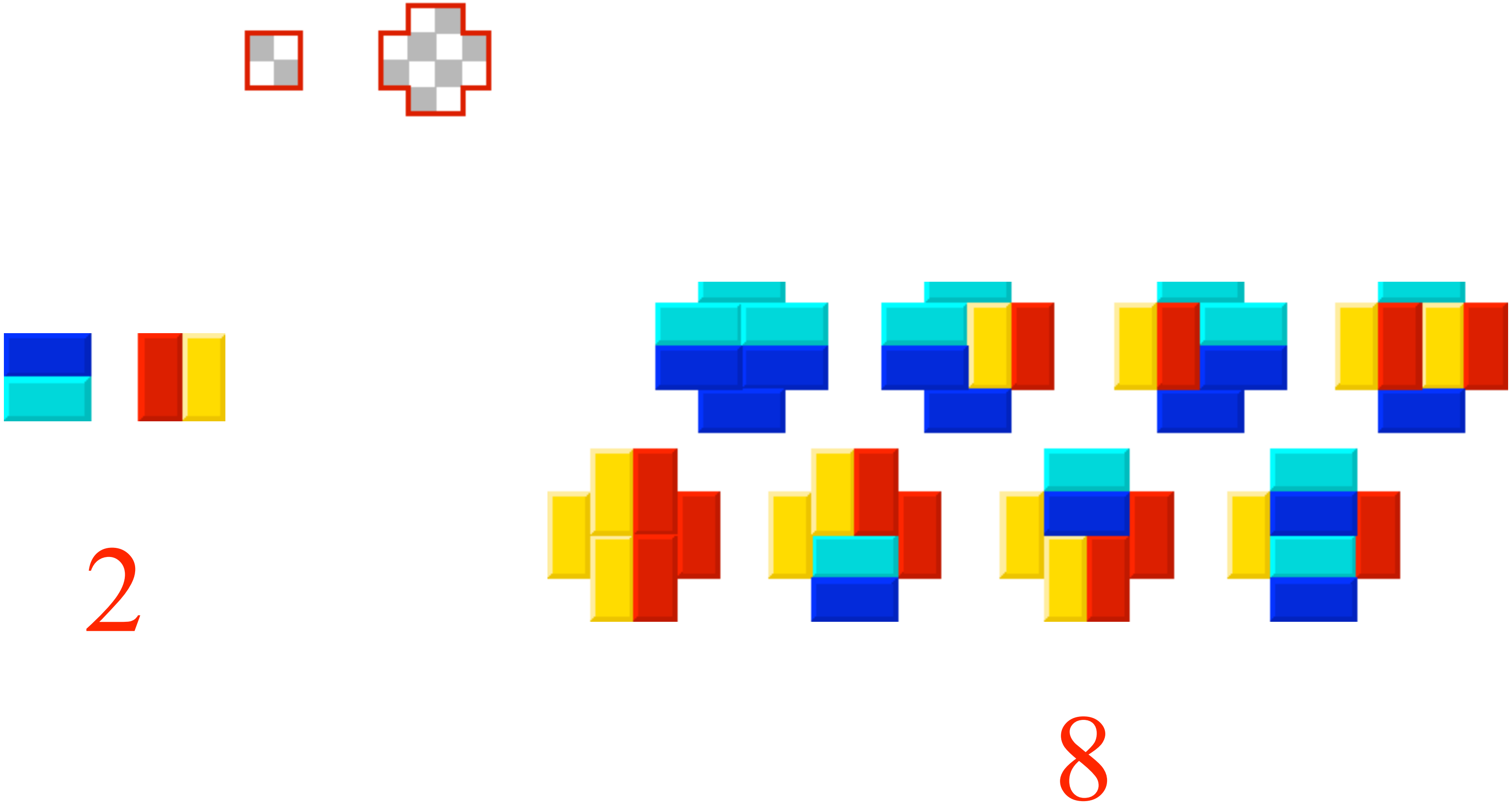
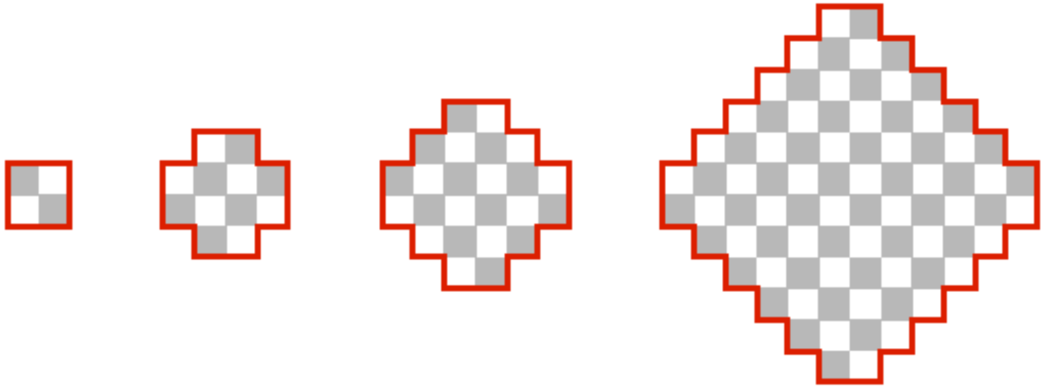


diagramme
Aztèque

2


2
8

Le côté expérimental dans la recherche mathématique

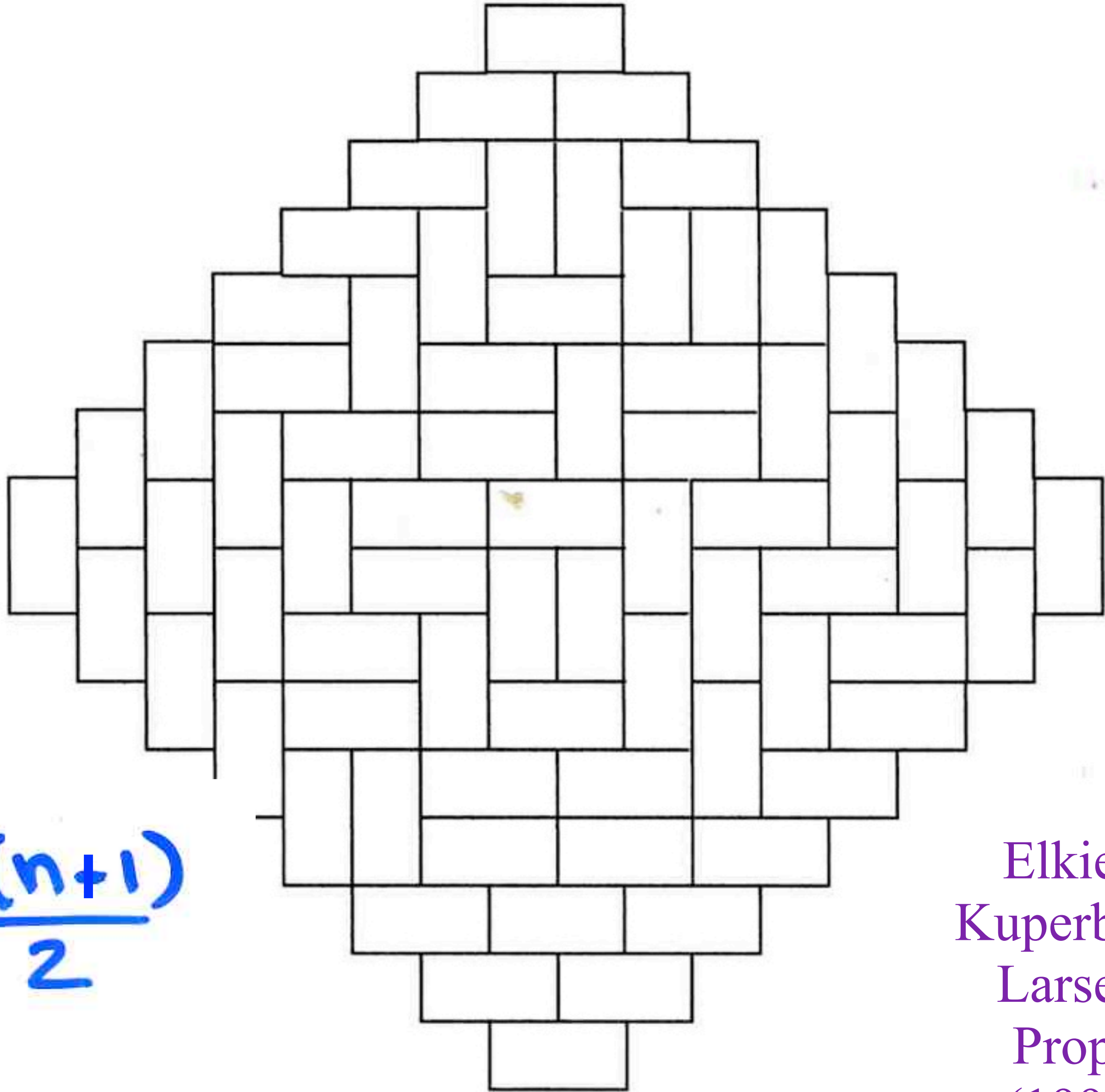


nombre de pavages

2	8	64	1024
2^1	2^3	2^6	2^{10}
2^1	$2^{(1+2)}$	$2^{(1+2+3)}$	$2^{(1+2+3+4)}$

le nombre de
pavages du
diagramme
Aztèque
avec des
dominos
est

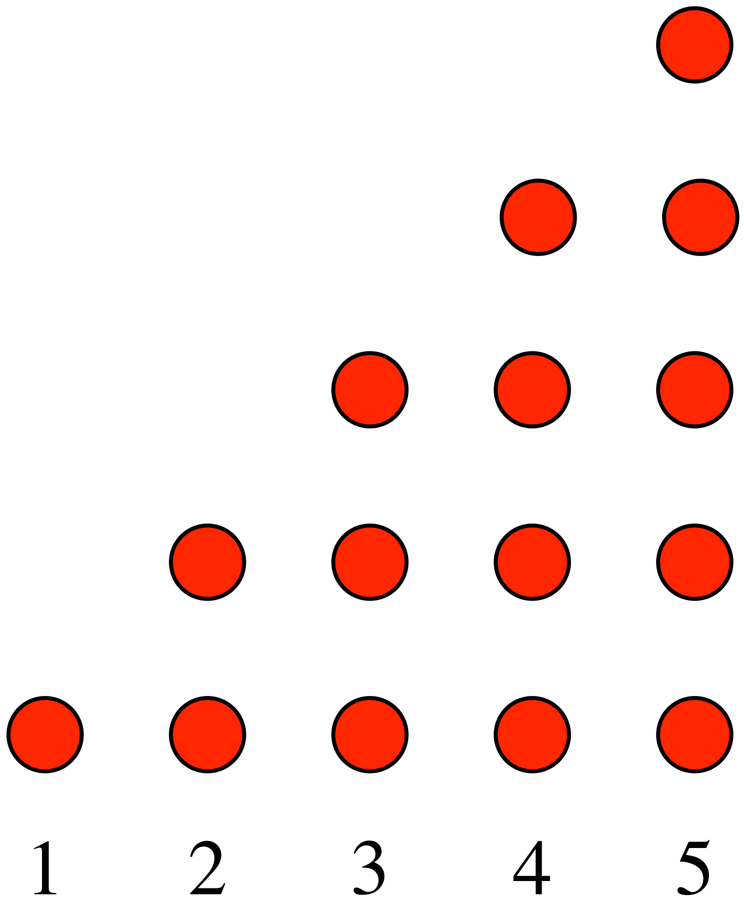
$$2^{(1+2+3+4+....+n)}$$



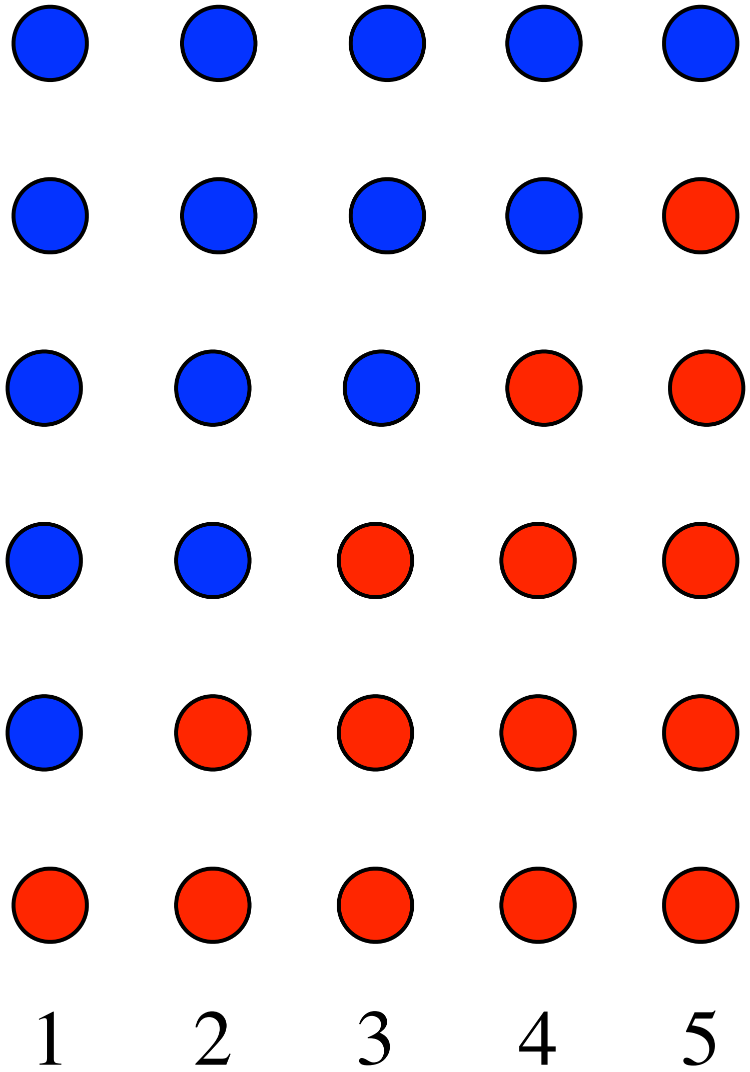
$$2^{\frac{n(n+1)}{2}}$$

Elkies,
Kuperberg,
Larsen,
Propp
(1992)

$$1 + 2 + \ldots + n = n(n+1)/2$$



$$1+2+\ldots+n=n(n+1)/2$$



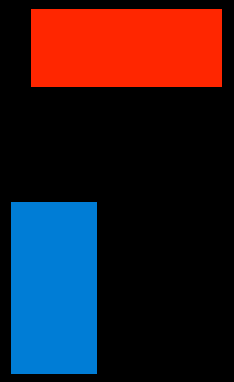
«preuve sans mots»

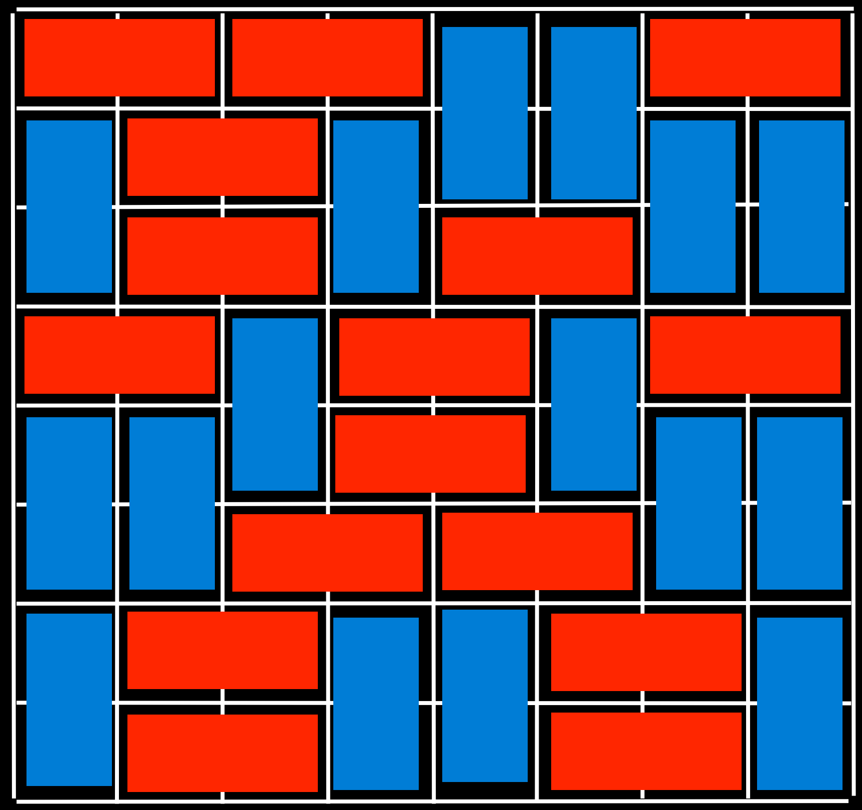
retour

aux pavages de l'échiquier

une autre formule ...

pavages d'un échiquier avec des dominos





le nombre de pavages avec des dominos d'un rectangle $m \times n$ est égal au produit:

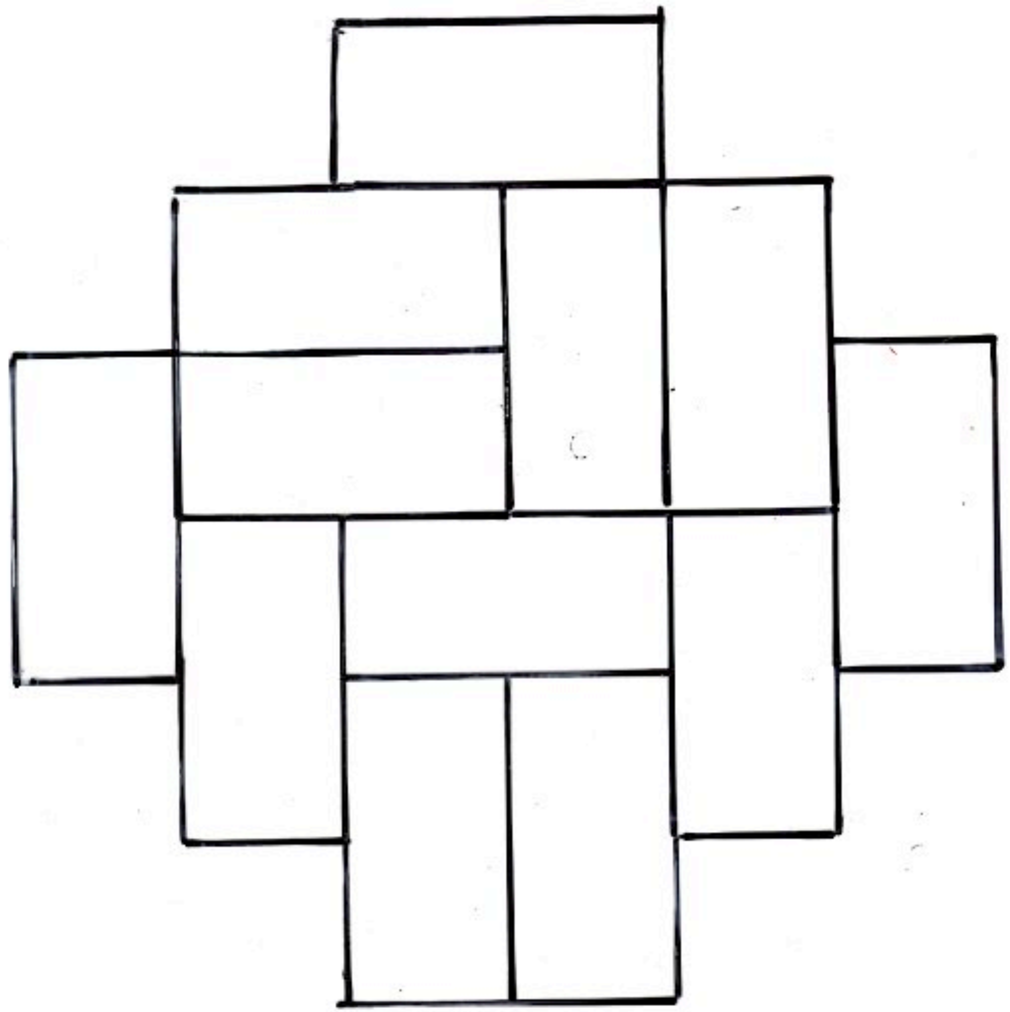
$$\prod_{i=1}^{m/2}\prod_{j=1}^{n/2}\left(4\cos^2\frac{i\pi}{m+1}+4\cos^2\frac{j\pi}{n+1}\right)$$

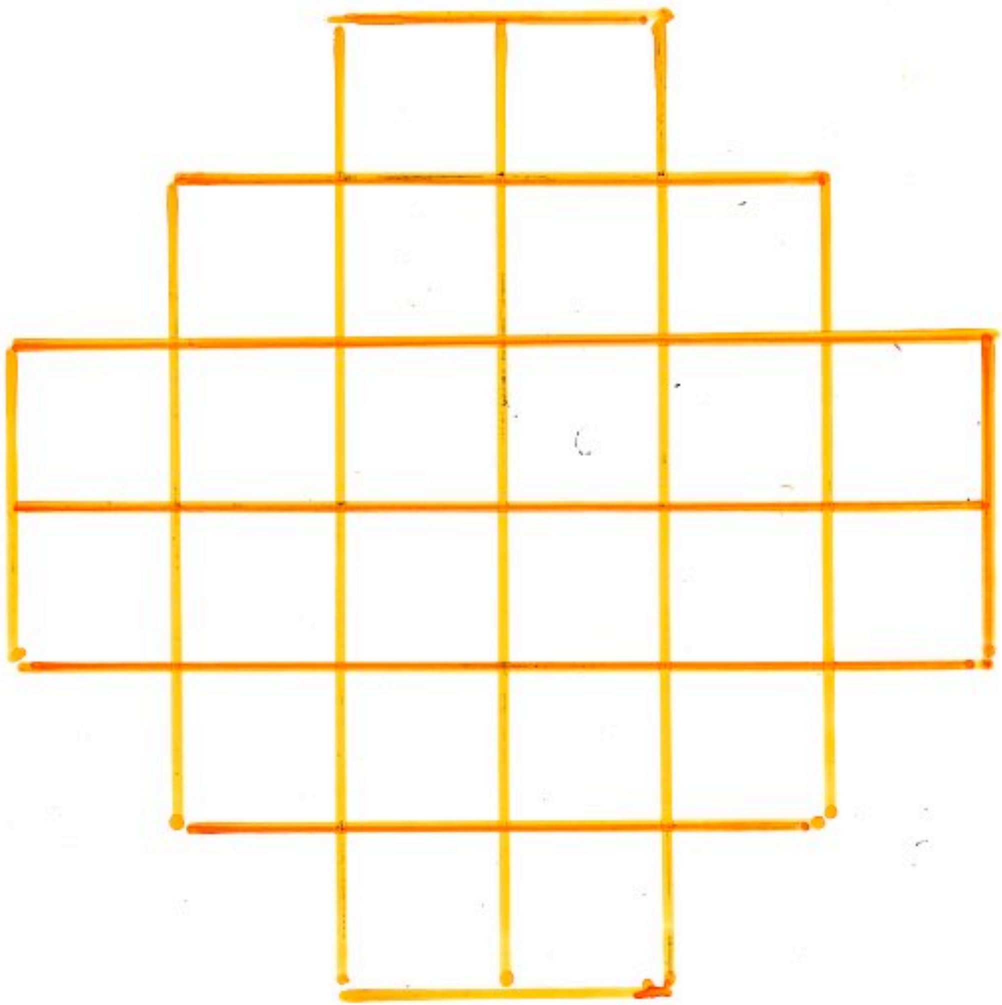
Kasteleyn (1961)

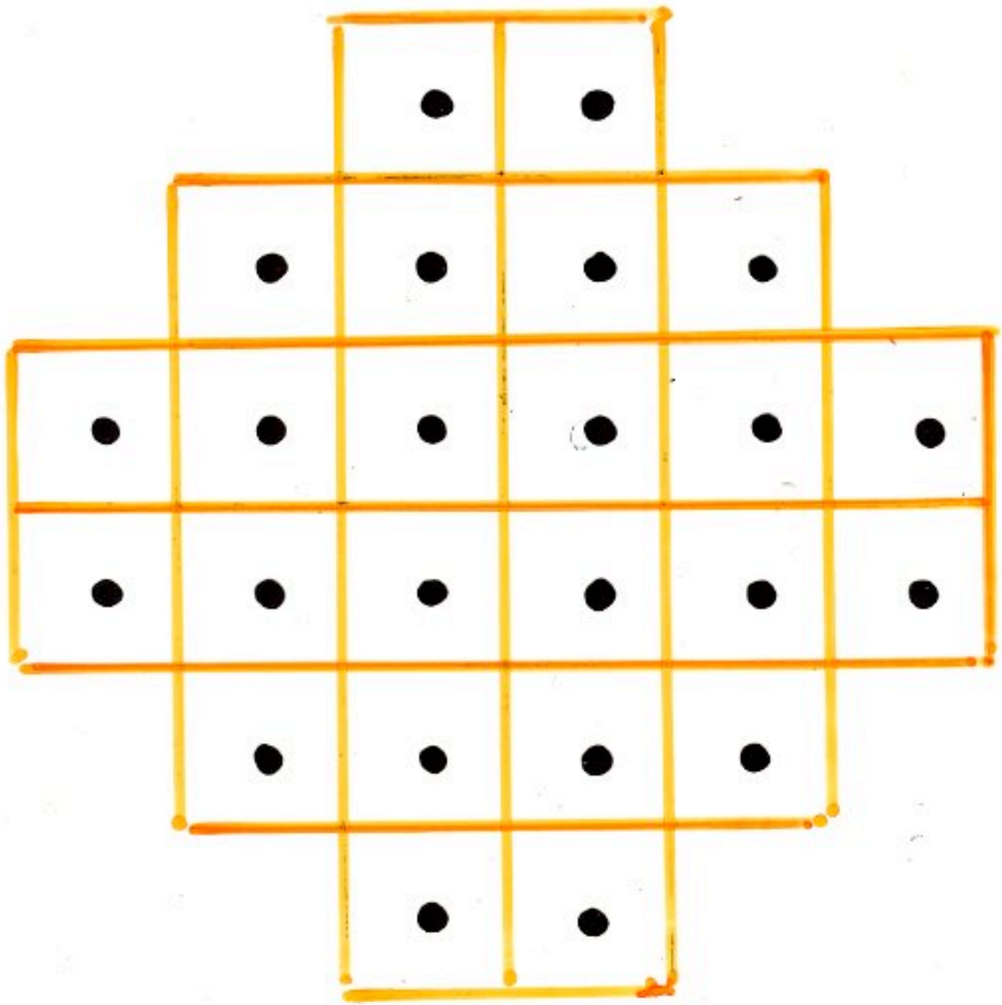
c'est un entier !!

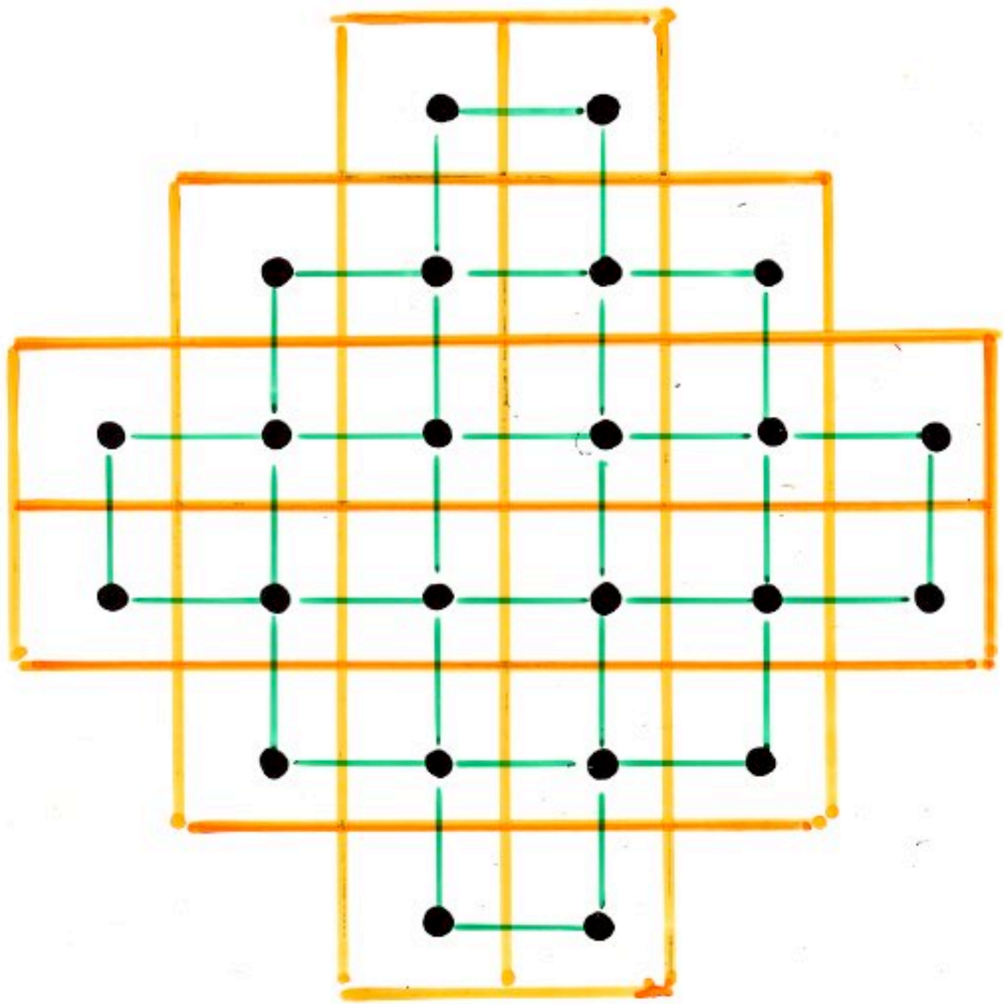
pour l'échiquier $m=8$, $n=8$: 12 988 816

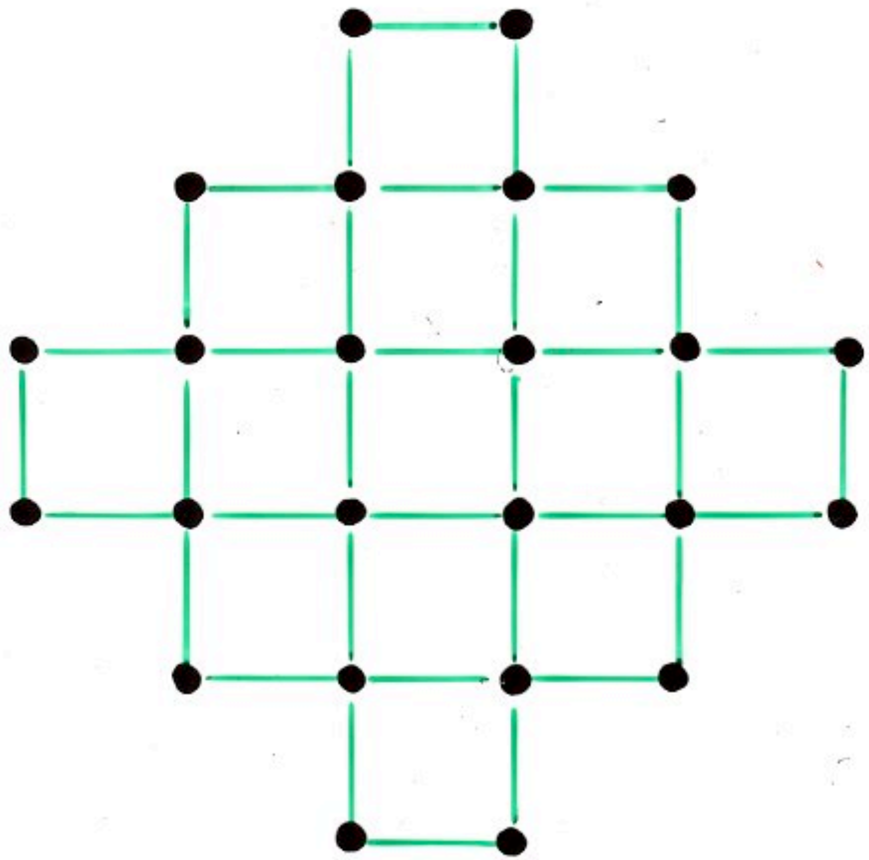
pavages et couplages parfaits

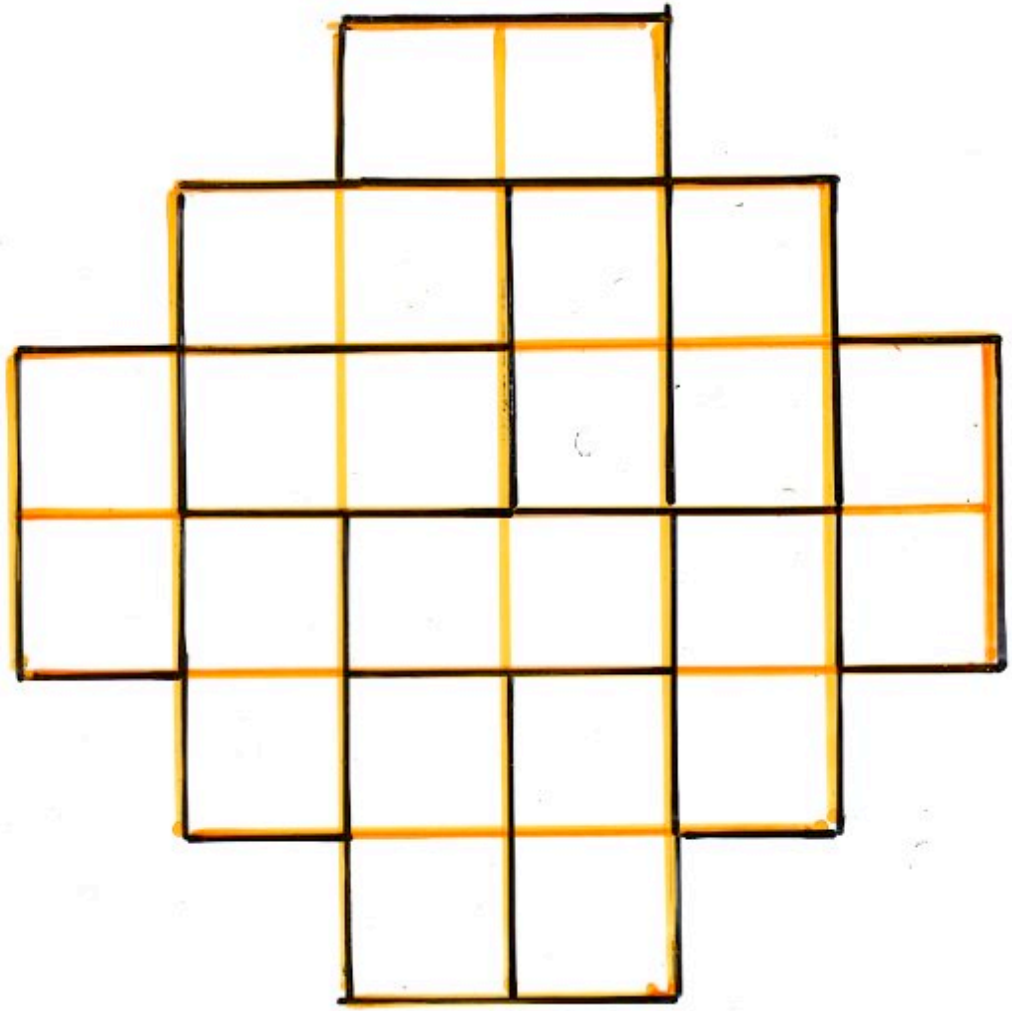


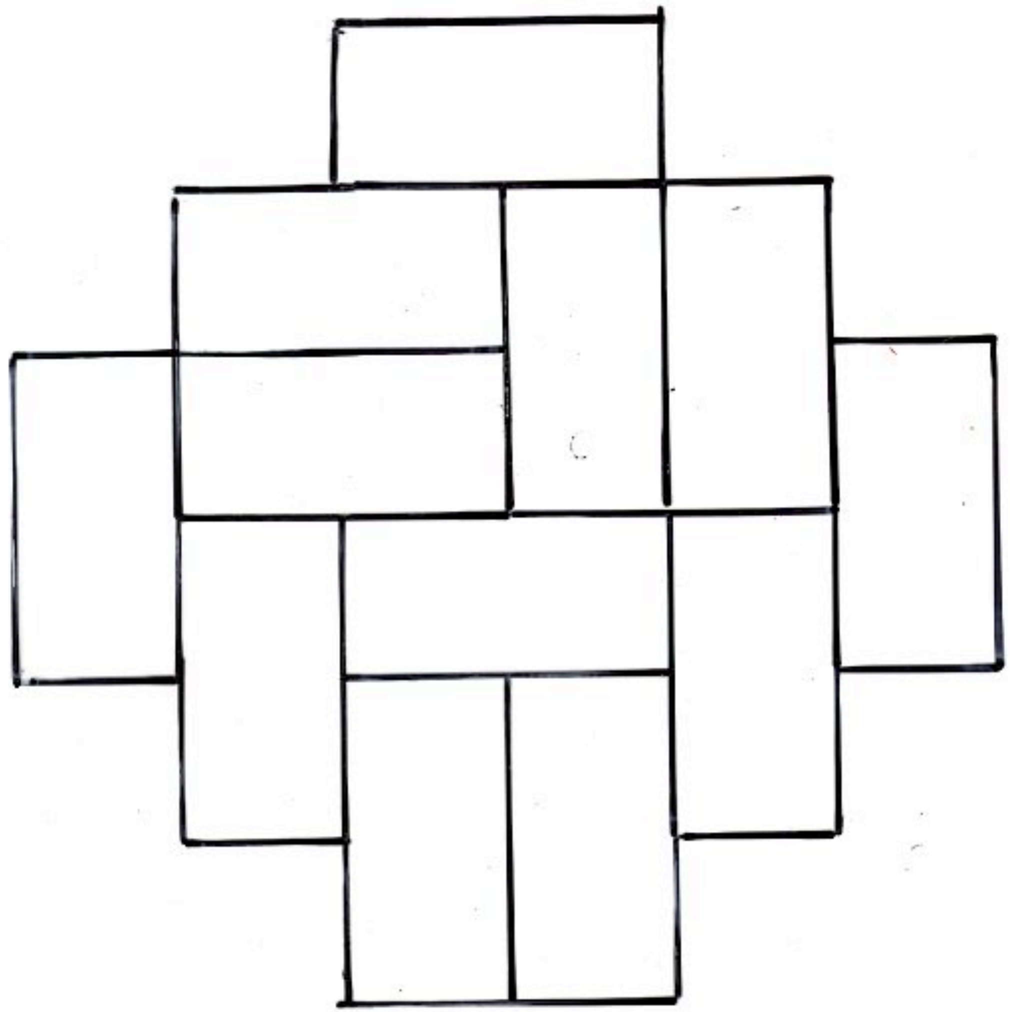


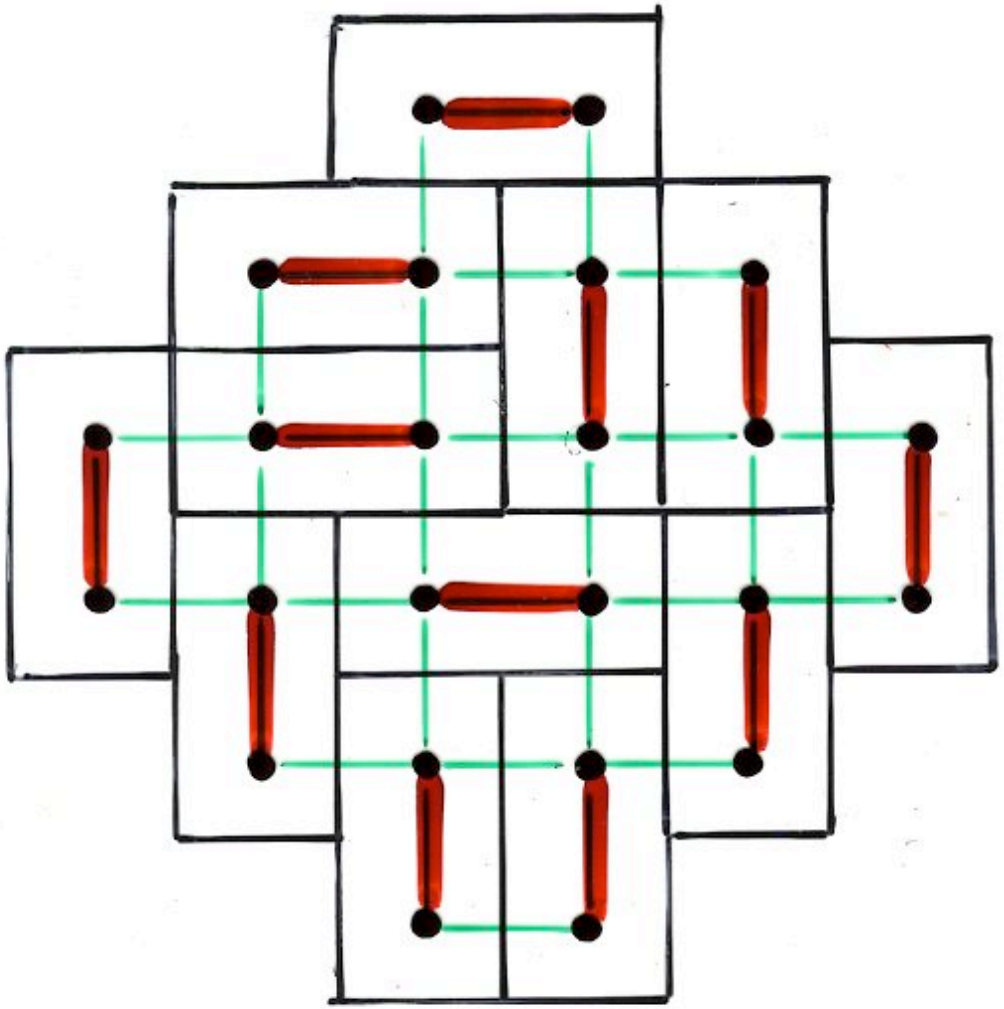


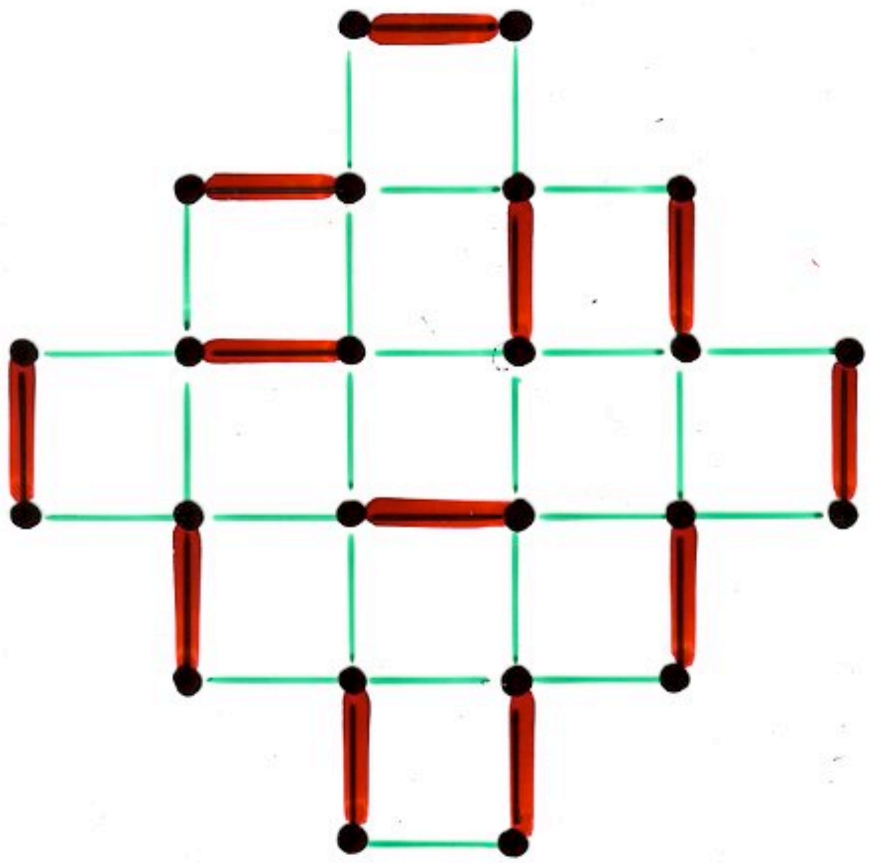










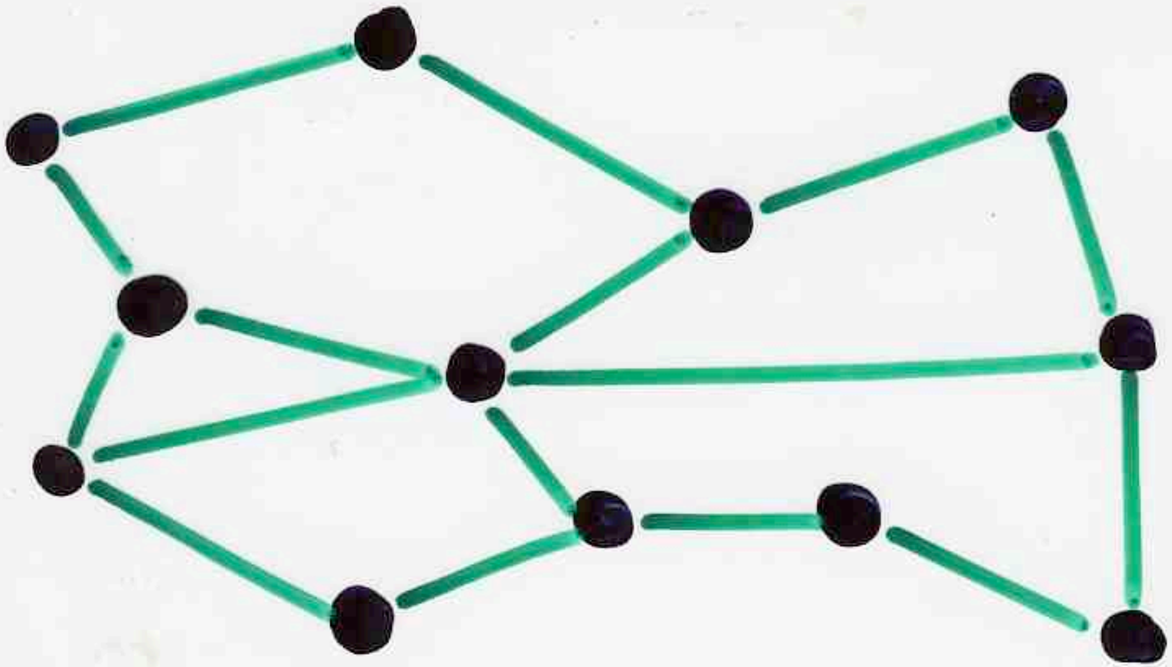


couplages d'un graphe $G = (S,A)$

Déf.- Couplage dans un graphe $G = (S, A)$

sommets   arêtes

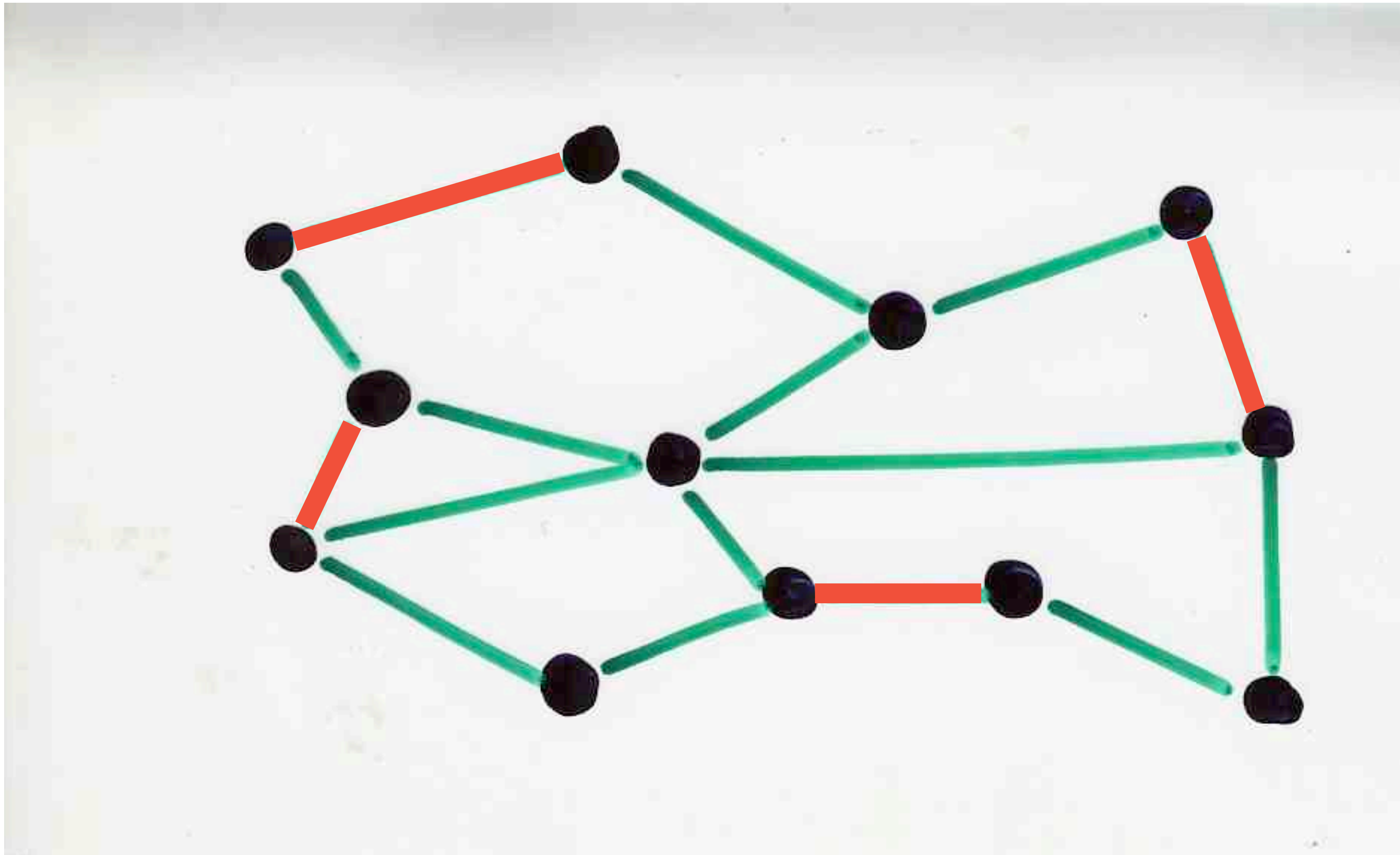
ensemble d'arêtes deux à deux disjointes



Def- Couplage dans un graphe $G = (S, A)$

sommets arêtes

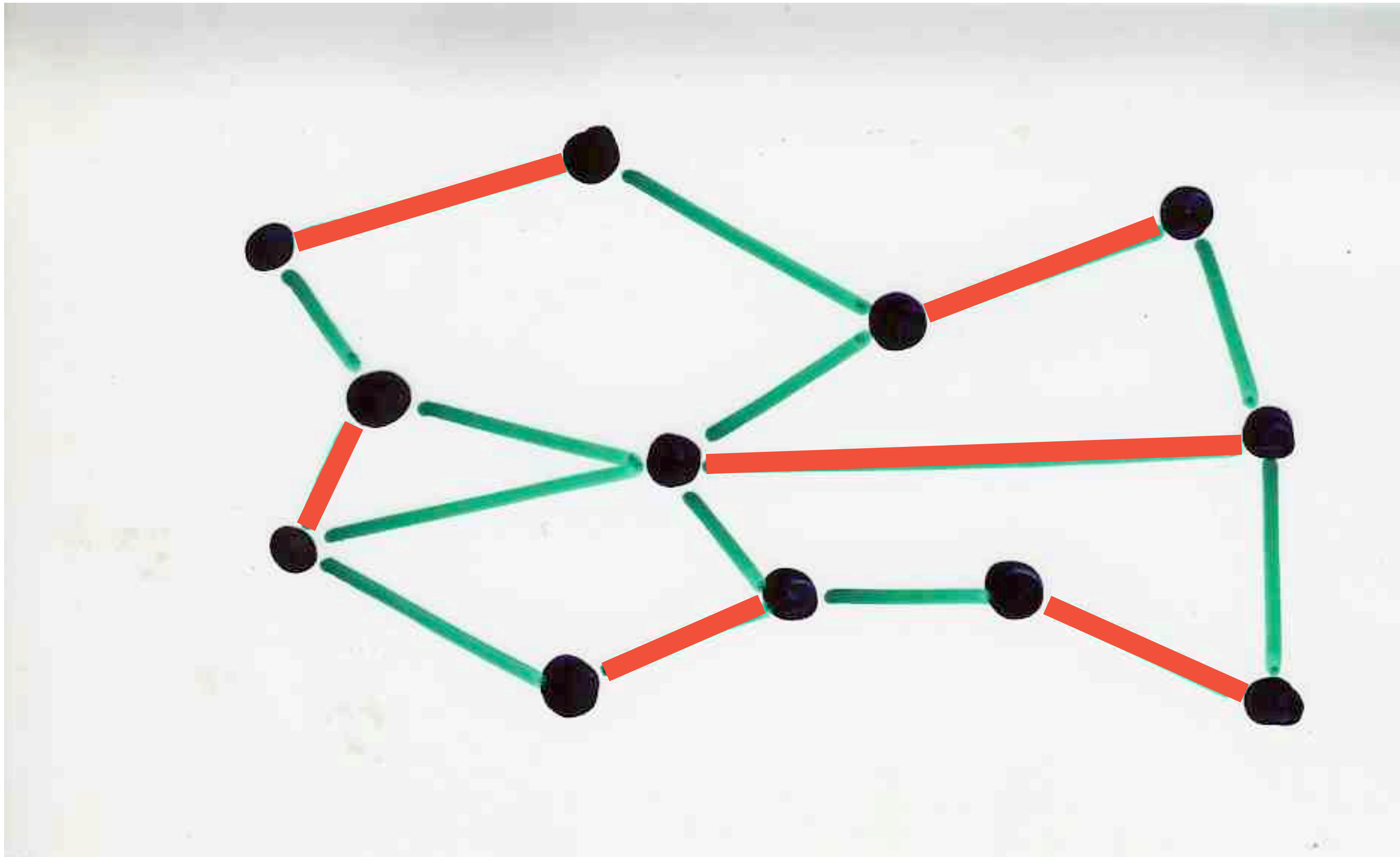
ensemble d'arêtes deux à deux disjointes



Def- Couplage parfait

tous les sommets S de G
sont couverts par une arête
du couplage

= pas de sommets isolés



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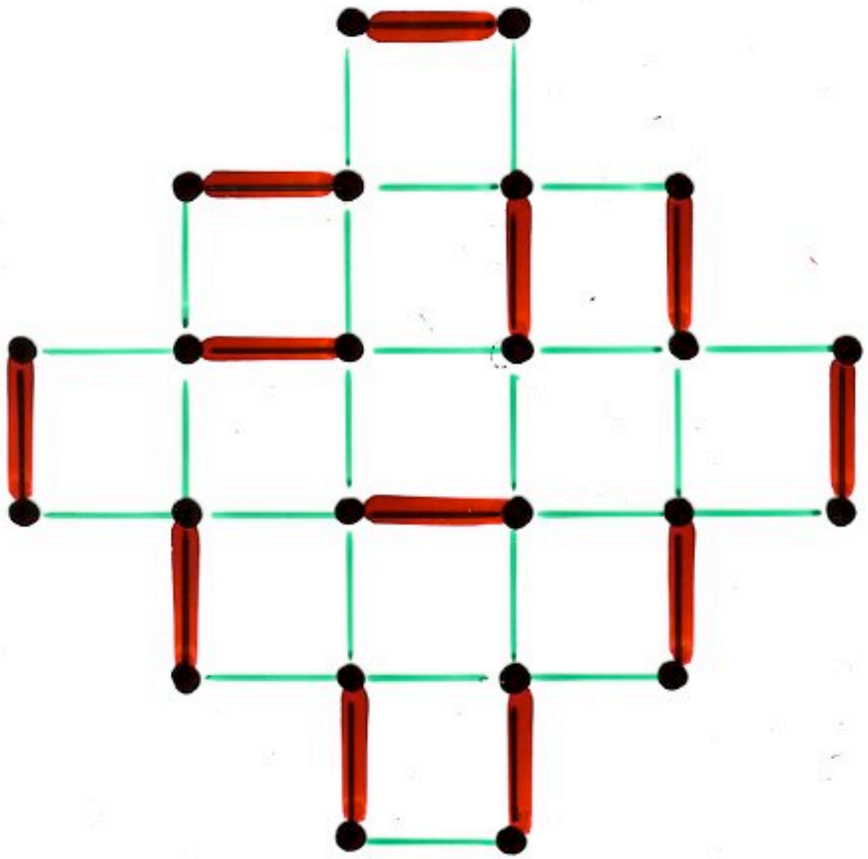


existence de couplages parfaits

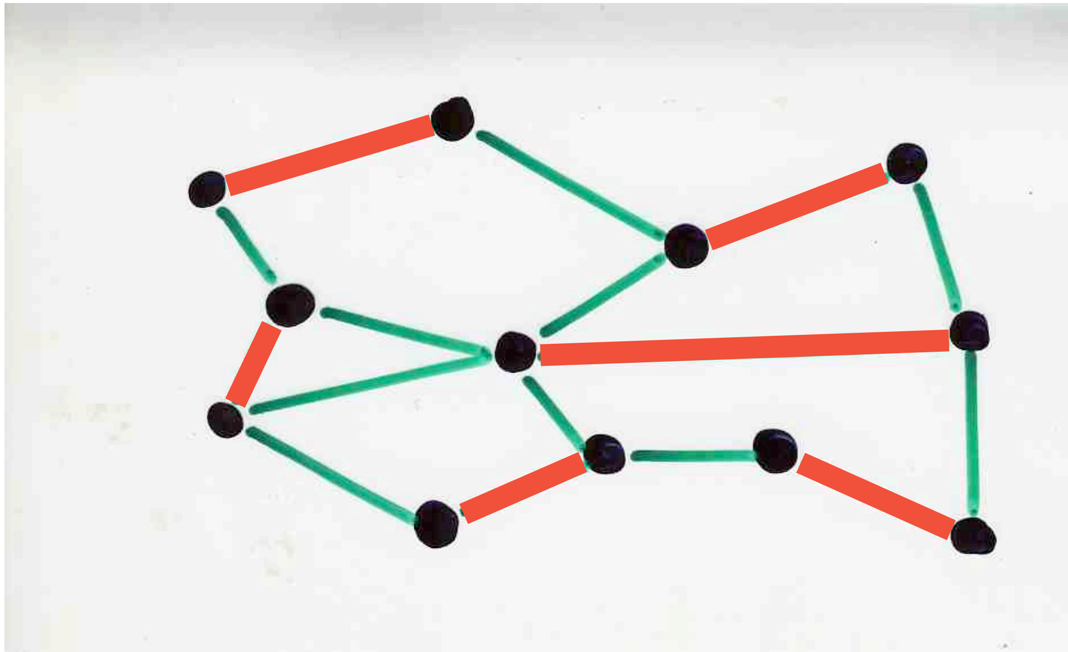
$n = |S|$

nombre de sommets du graphe $G = (S, A)$

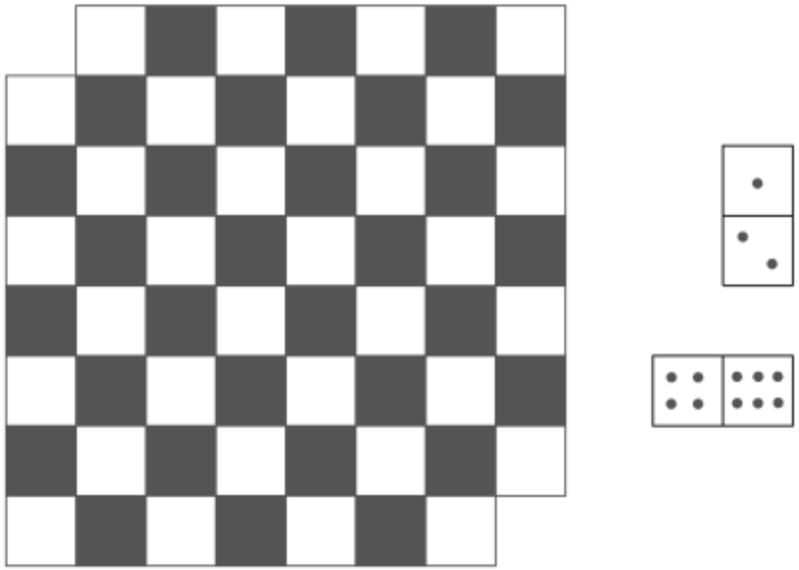
doit être pair

condition nécessaire

pas suffisante

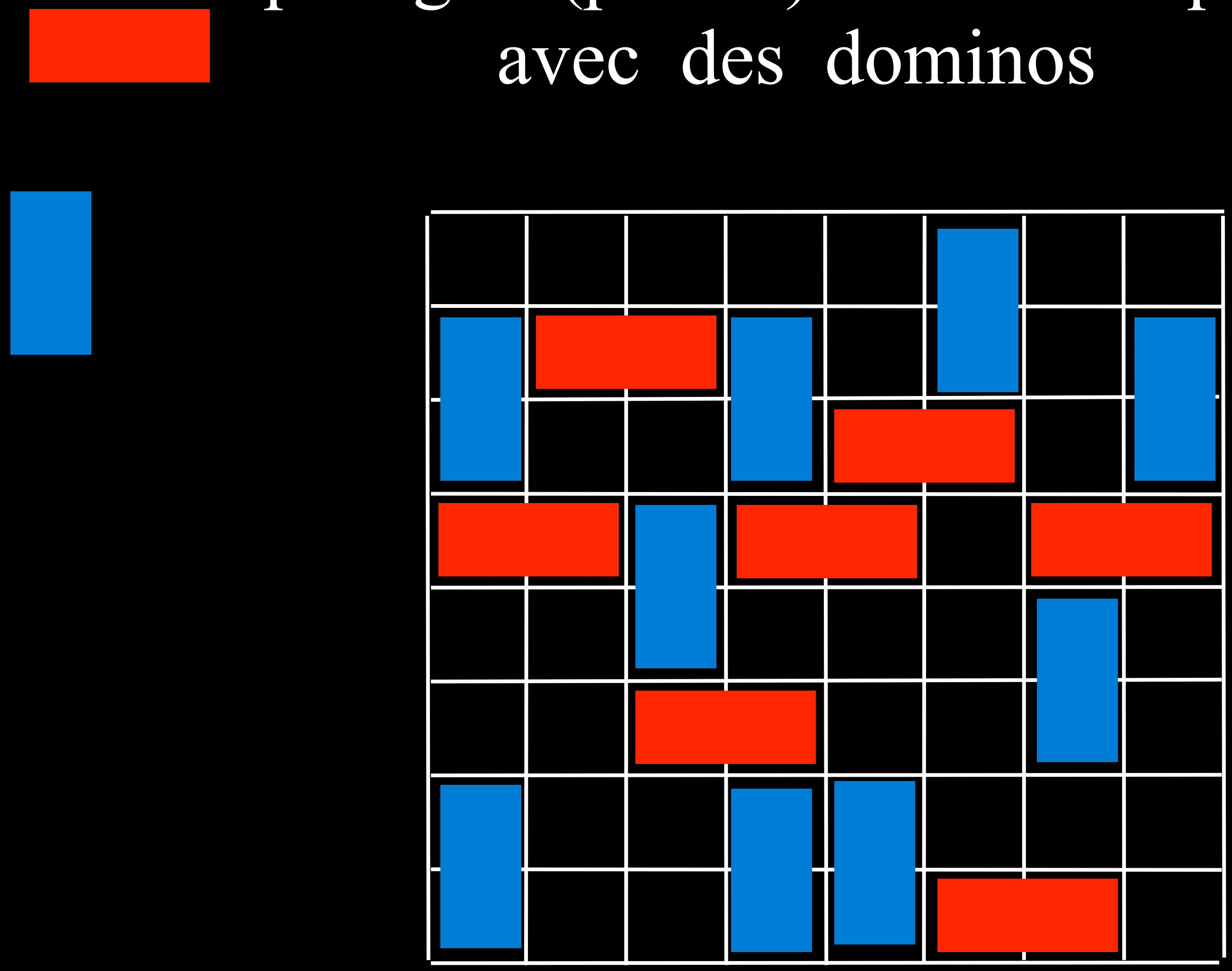


existence de couplages parfaits

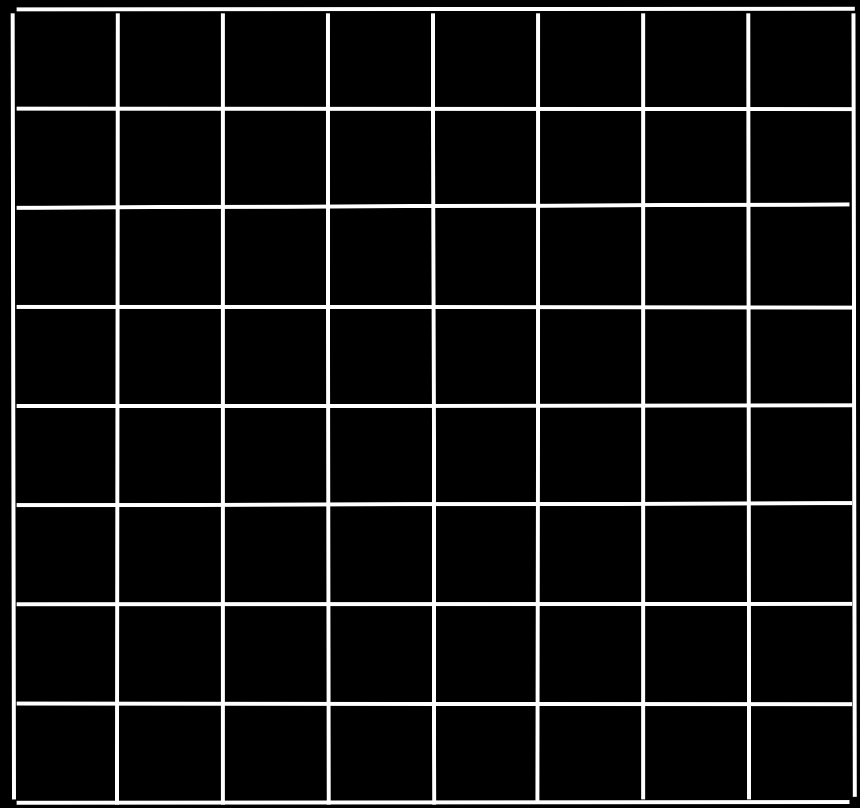


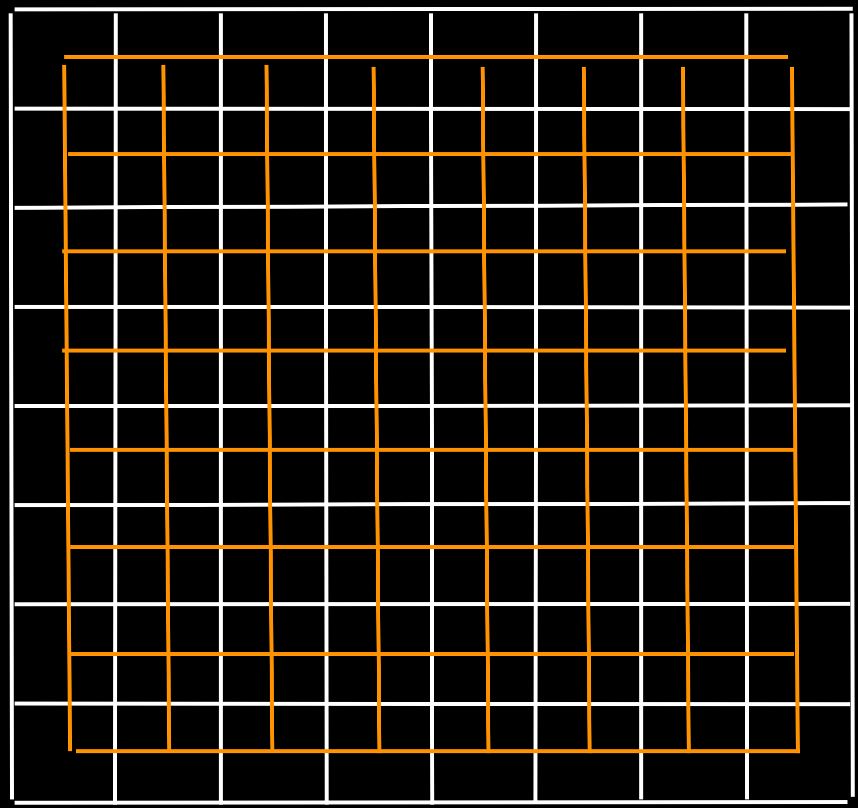
existence d'un pavage ?

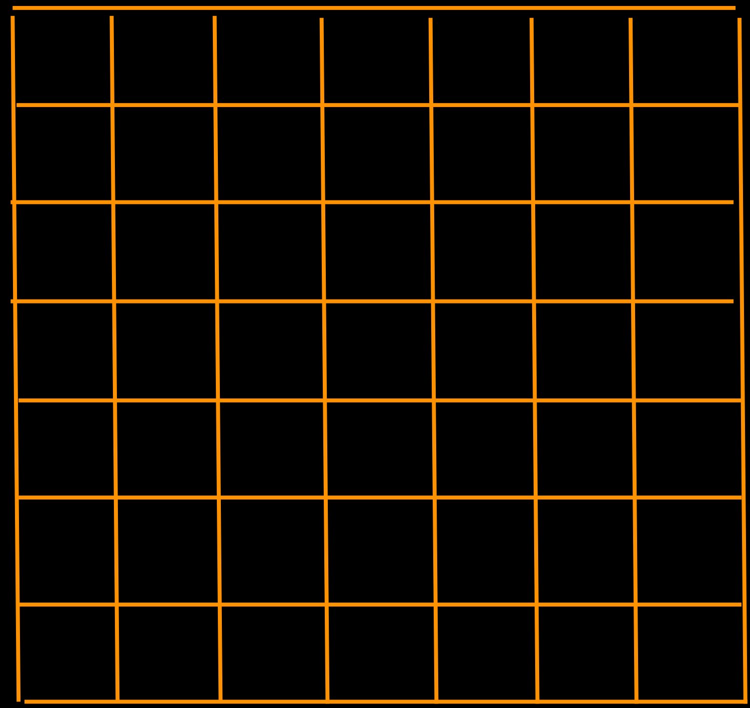
pavages (partiel) d'un échiquier
avec des dominos



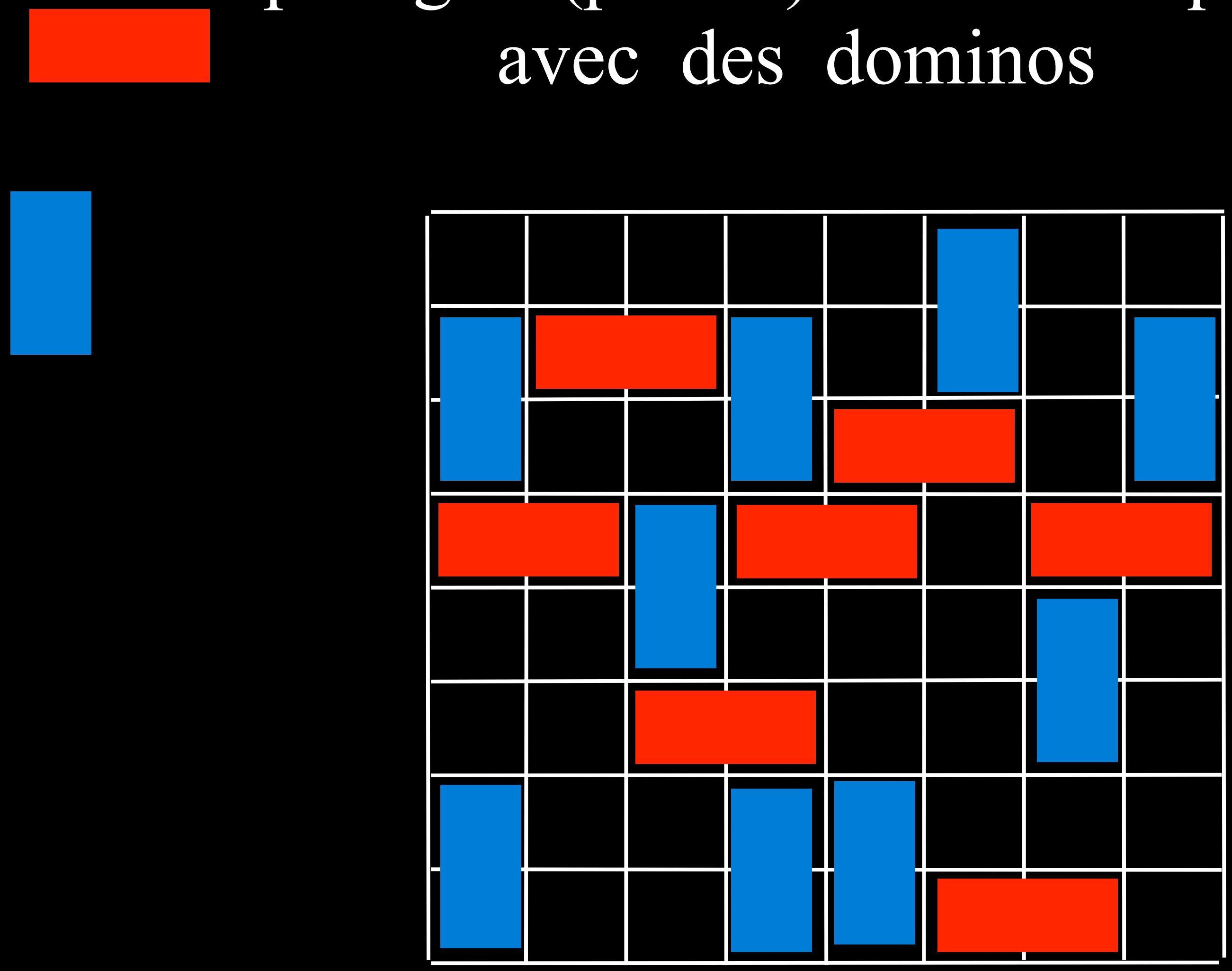
couplage du graphe associé





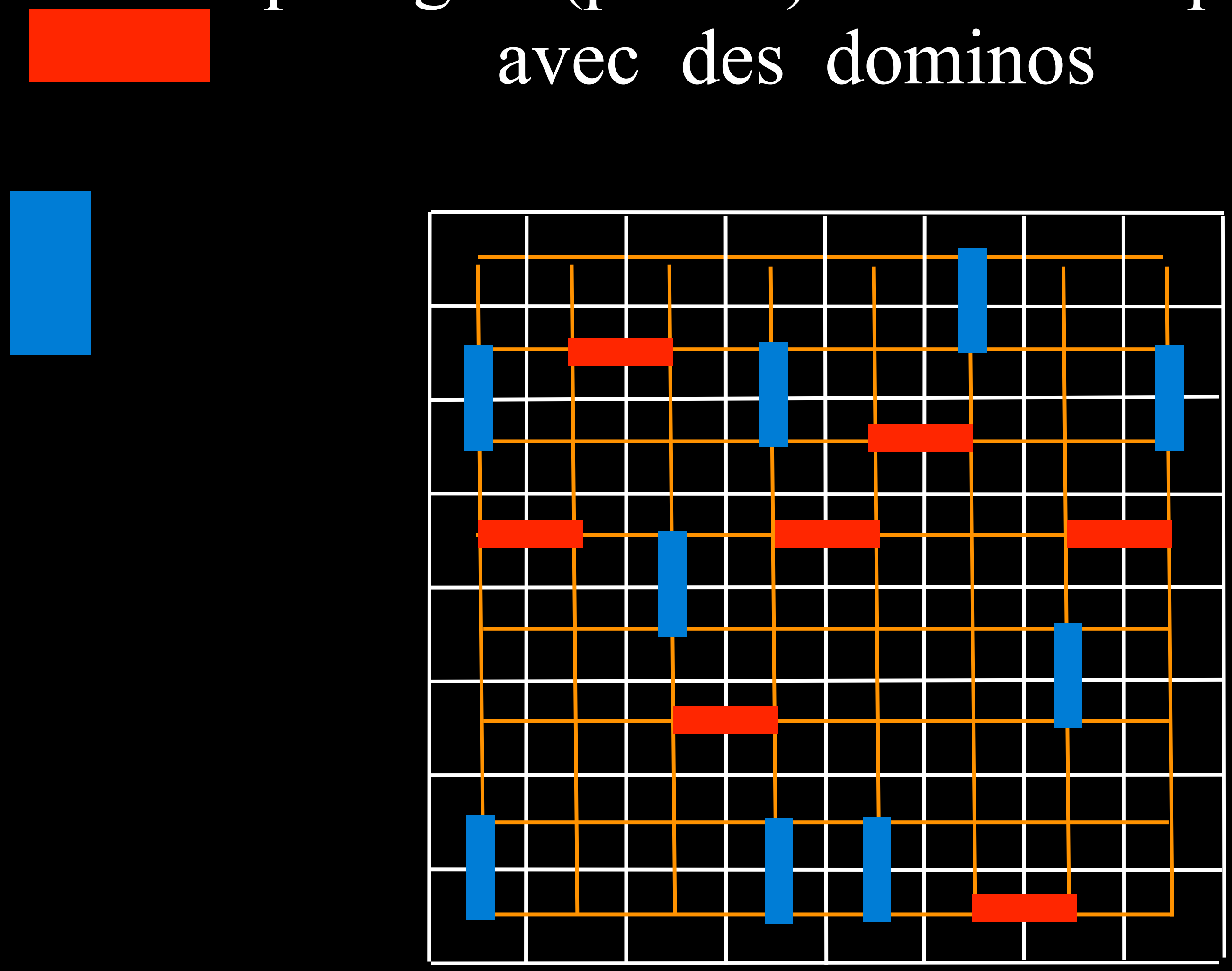


pavages (partiel) d'un échiquier
avec des dominos



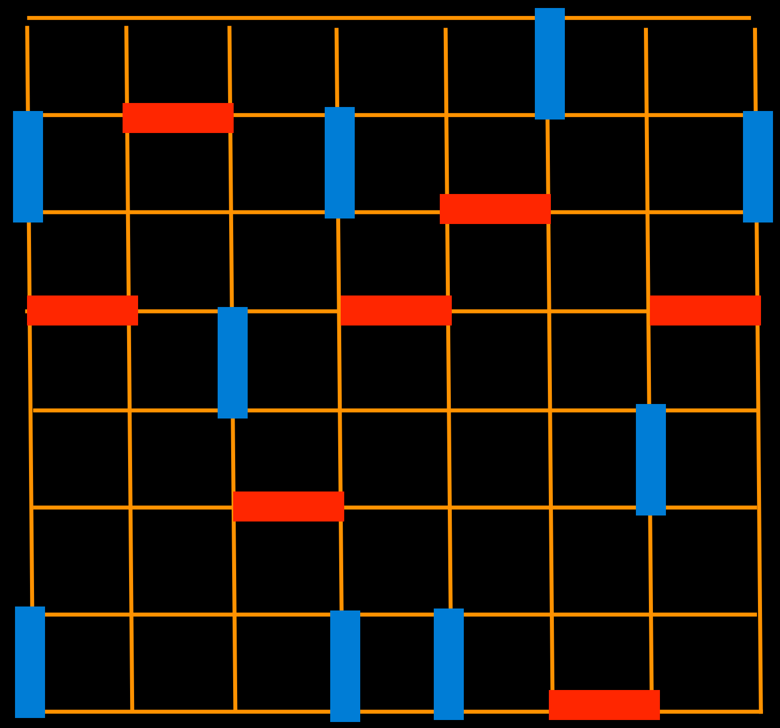
couplage du graphe associé

pavages (partiel) d'un échiquier
avec des dominos



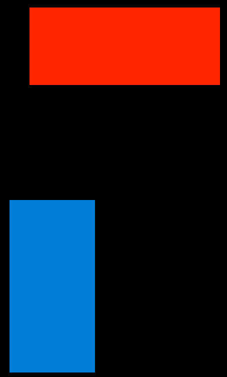
couplage du graphe associé

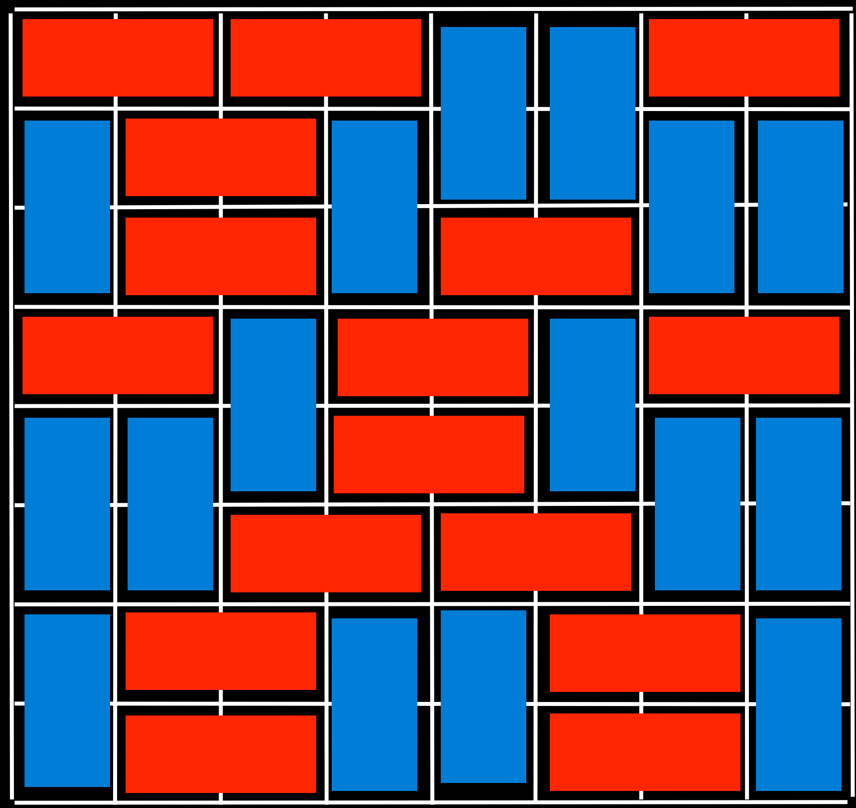
pavages (partiel) d'un échiquier
avec des dominos

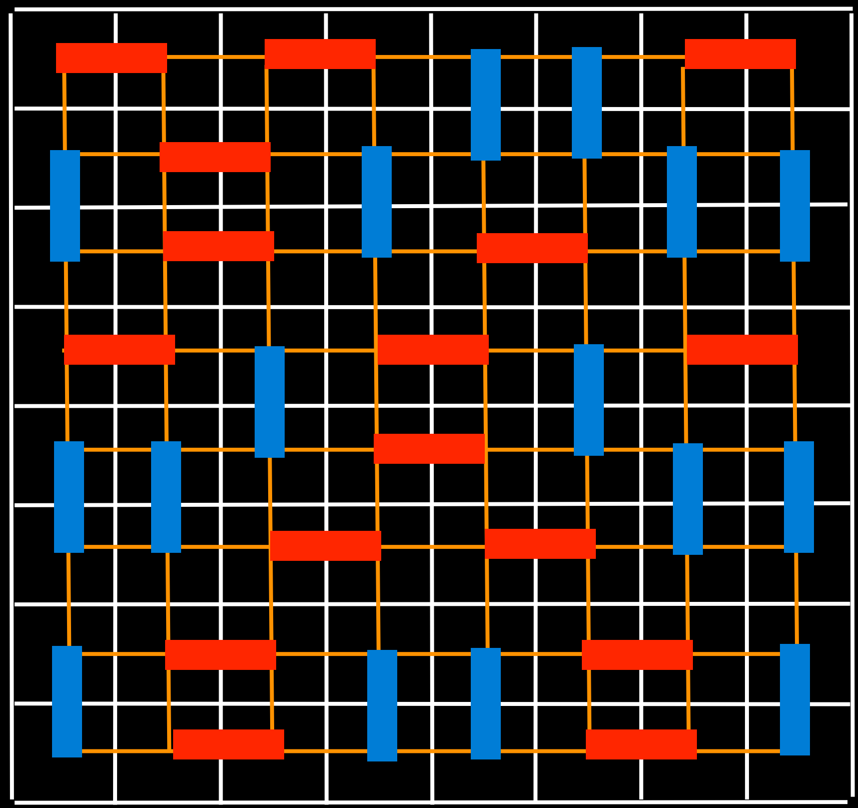


couplage du graphe associé

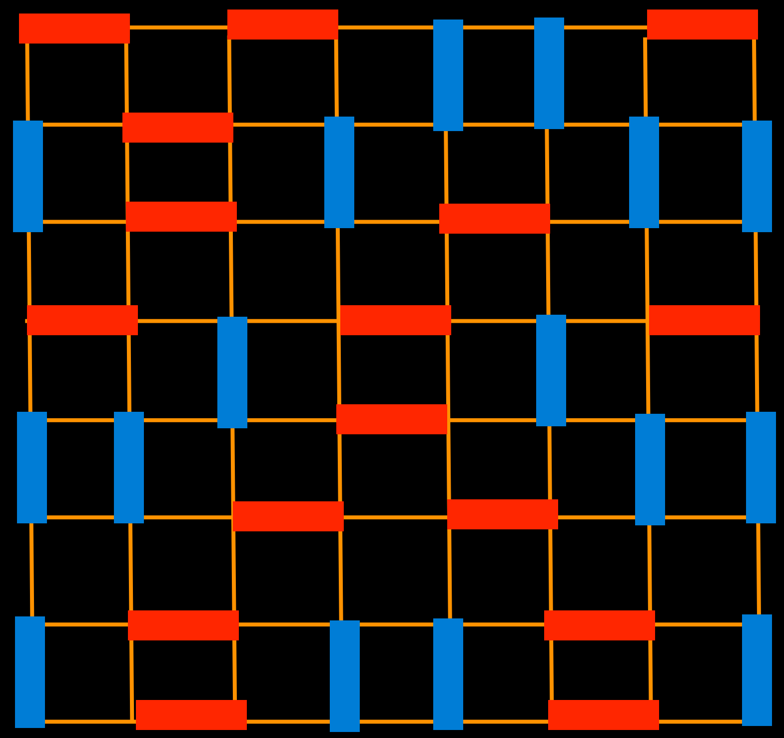
pavages d’un échiquier avec des dominos







couplage parfait du graphe associé



couplage parfait du graphe associé

$$\prod_{i=1}^{m/2} \prod_{j=1}^{n/2} \left(4 \cos^2 \frac{i\pi}{m+1} + 4 \cos^2 \frac{j\pi}{n+1} \right)$$

Kasteleyn (1961)

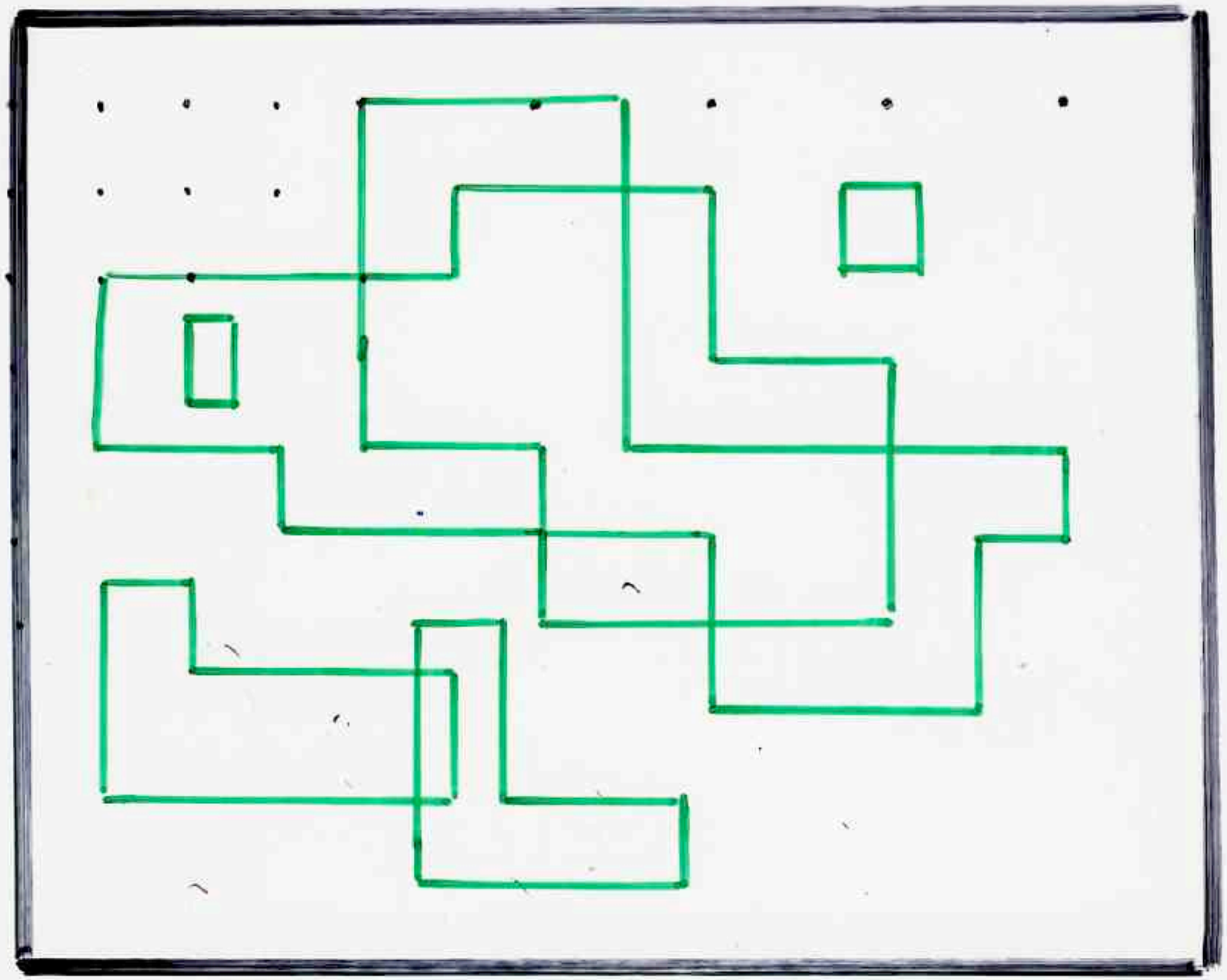
- dénombrement de
  couplages parfaits
- graphe planaire

  méthode du Pfaffien

- modèle d'Ising (1925)

  Kasteleyn, Fisher, Temperley
  (1961, ...)

  Onsager (1944)



"closed" graph

Ising model

Enumérer tous les couplages

d'un graphe $G = (S,A)$

Polynôme de couplage

polynôme

$$(1+x)^2 = 1+2x+x^2$$

$$(1+x)^3 = 1+3x+3x^2+x^3$$

degré d'un polynôme

terme constant

$$(x-1)(x-2) = x^2-3x+2$$

$$(x-1)(x-2)(x-3) = x^3-6x^2+11x-6$$

$P(x) = 0$ racines

équation du polynôme $P(x)$

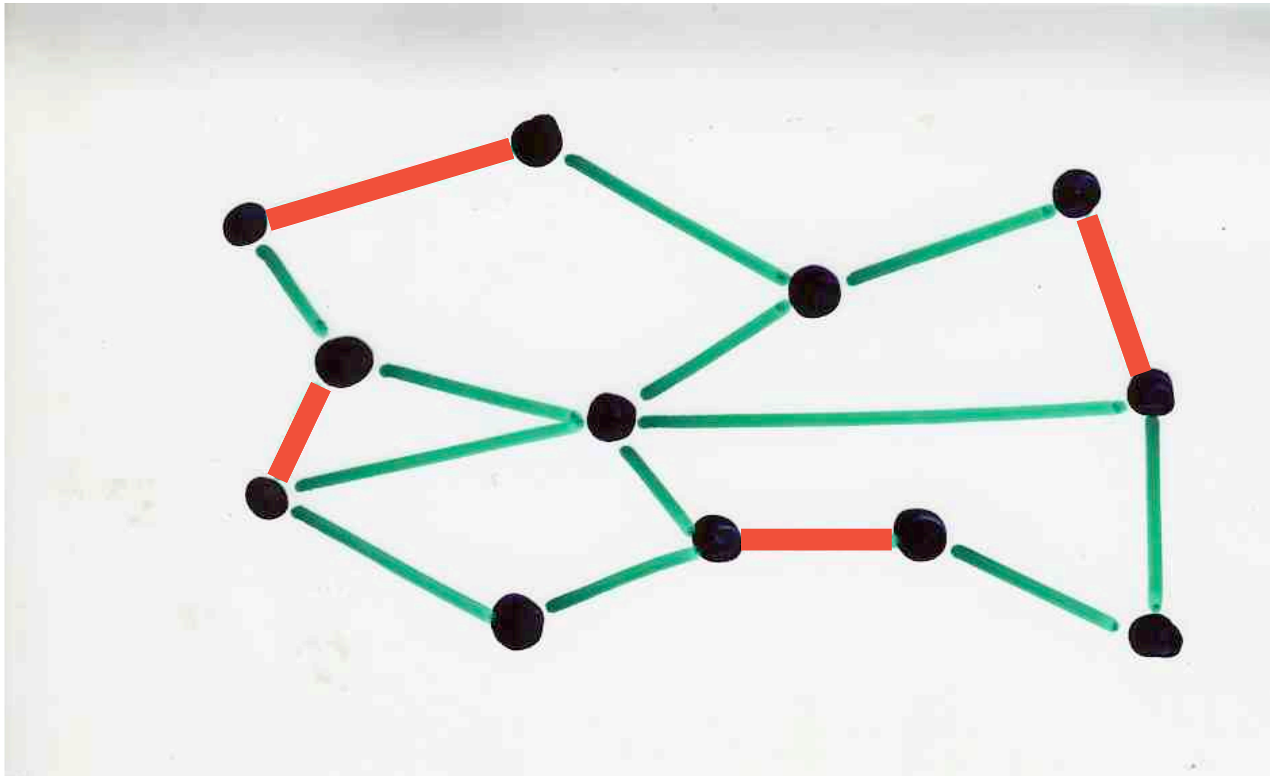
x^2-2 racines $\sqrt{2}$, $-\sqrt{2}$

x^2+1 i, $-i$

Def. Polynôme de couplage du graphe $G = (S, A)$ $|S| = n$

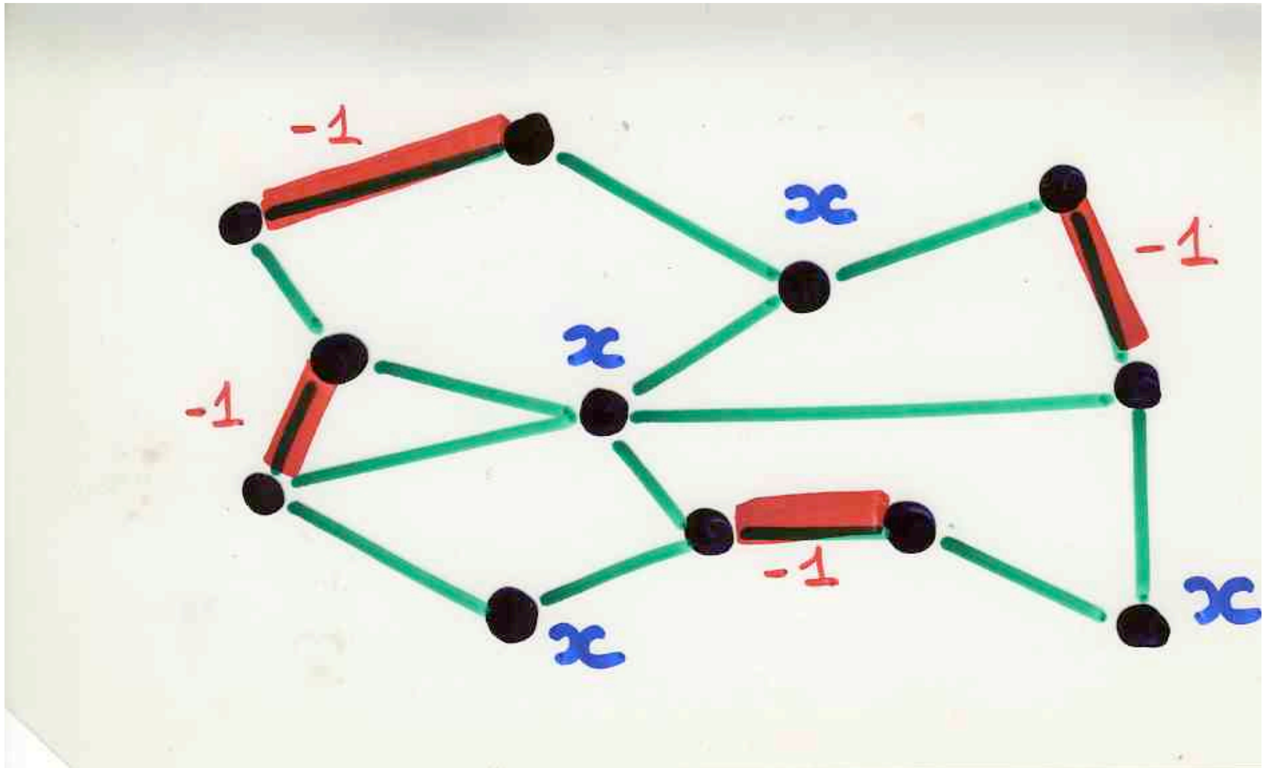
$$P_G(x) = \sum_{0 \leq k \leq \frac{n}{2}} (-1)^k a_{G,k} \, x^{n-2k}$$

nombre de couplages du graphe G ayant k arêtes



$$P_G(x) = \sum_{\substack{C \\ \text{couplages}}} (-1)^{|C|}\, x^{i(C)}$$

$i(C)$ = nombre de sommets isolés du couplage C



$$P_G(x) = x^n + (\quad)\,x^{n-2} + \dots + c$$

nombre d'arêtes du graphe G $\quad |A|$

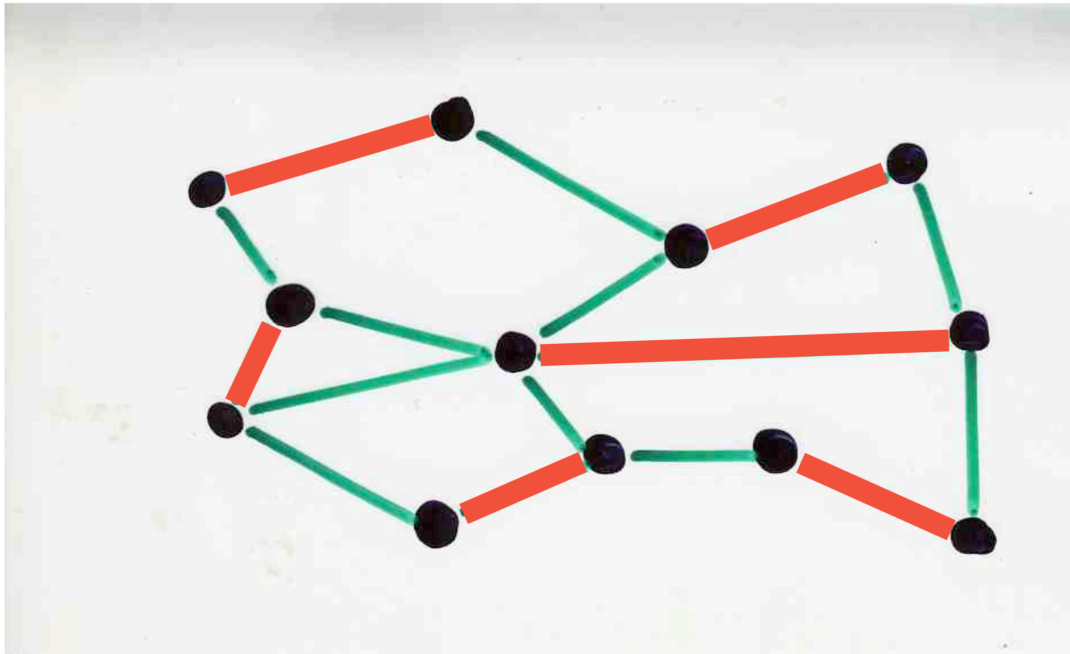
$$P_G(x) = x^n + (\ )\,x^{n-2} + \dots + c$$

nombre d'arêtes du graphe G

$|A|$

$n = 2p$ $\quad n$ pair

$c = (-1)^p$ [nombre de couplages parfaits du graphe $G = (S, A)$]



combinatoire classique

coefficients binomiaux

$$(1+t)^n = 1 + \binom{n}{1} t + \binom{n}{2} t^2 + \binom{n}{3} t^3 + \ldots + \binom{n}{k} t^k + \ldots + \binom{n}{n} t^n$$

addition +

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

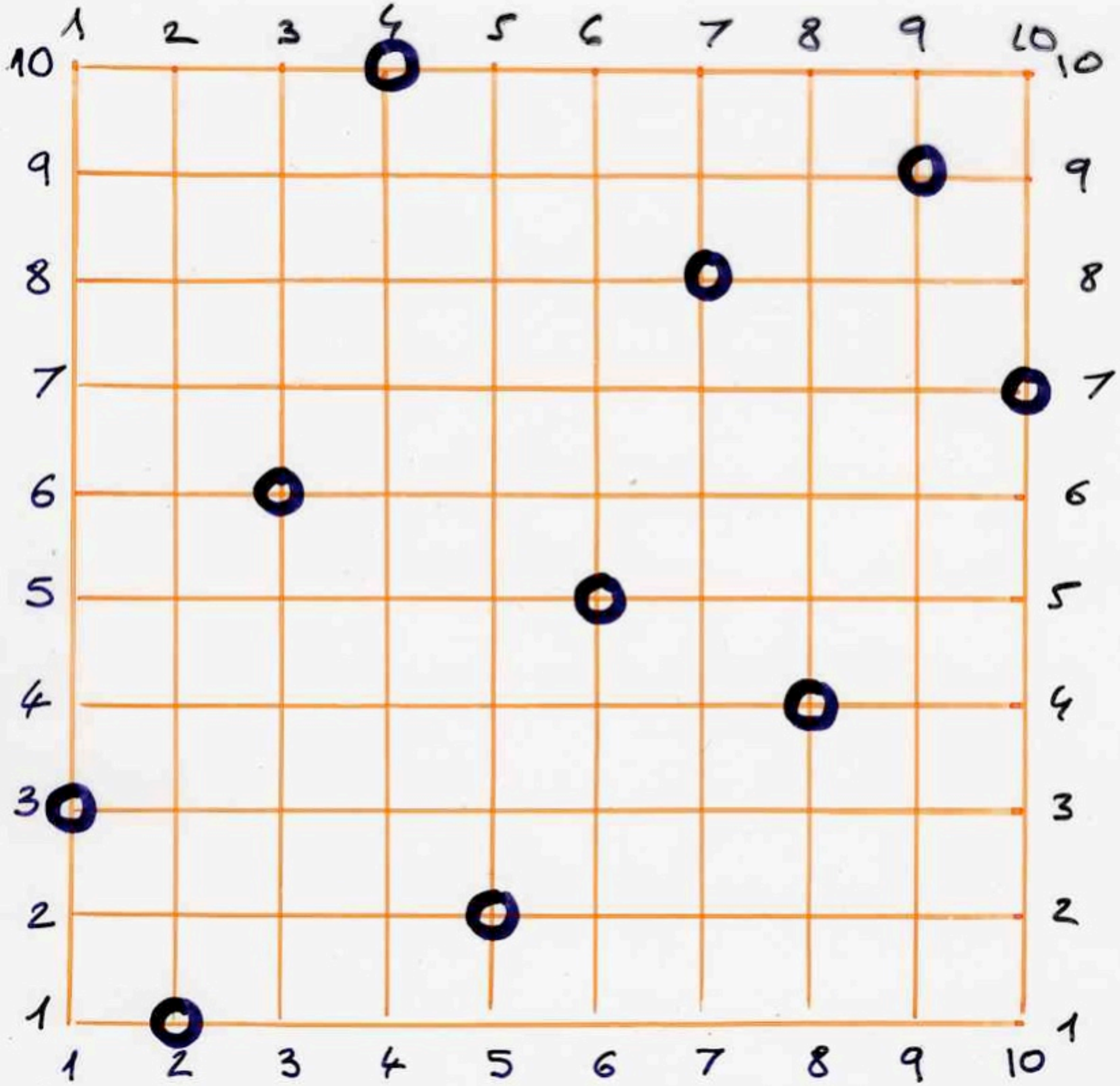
permutations

Permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$n! = 1 \times 2 \times \ldots \times n$$

1, 2, 6, 24, 120, 720, 5040, 40320,...



$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$

permutations

Permutation σ

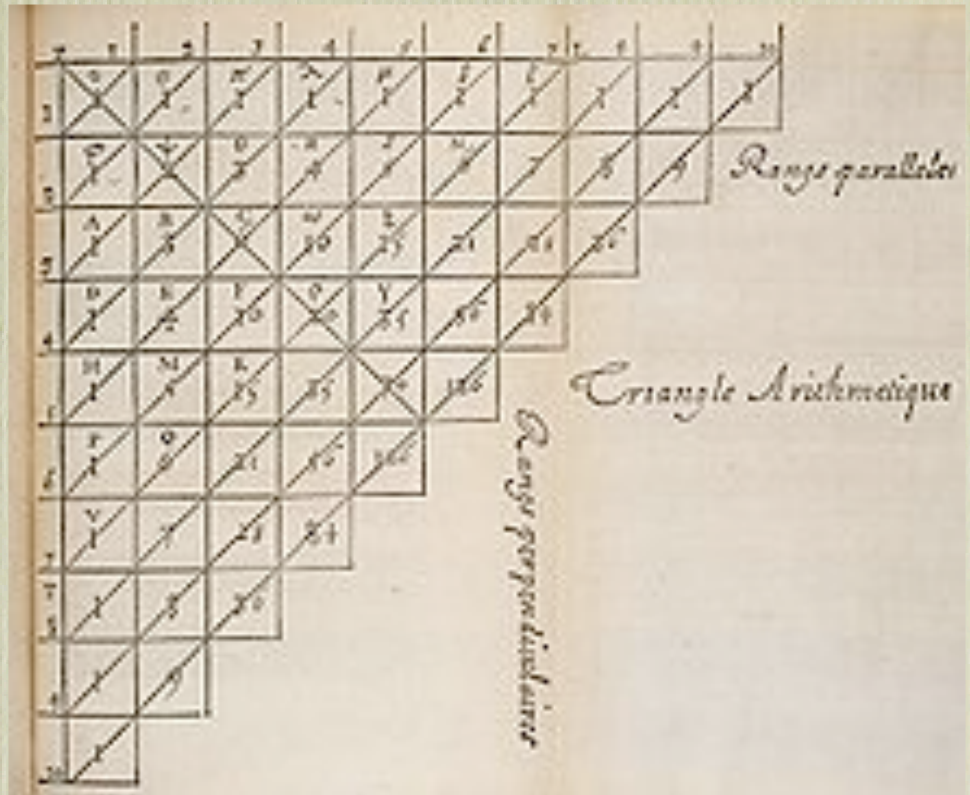
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$n! = 1 \times 2 \times \ldots \times n$$

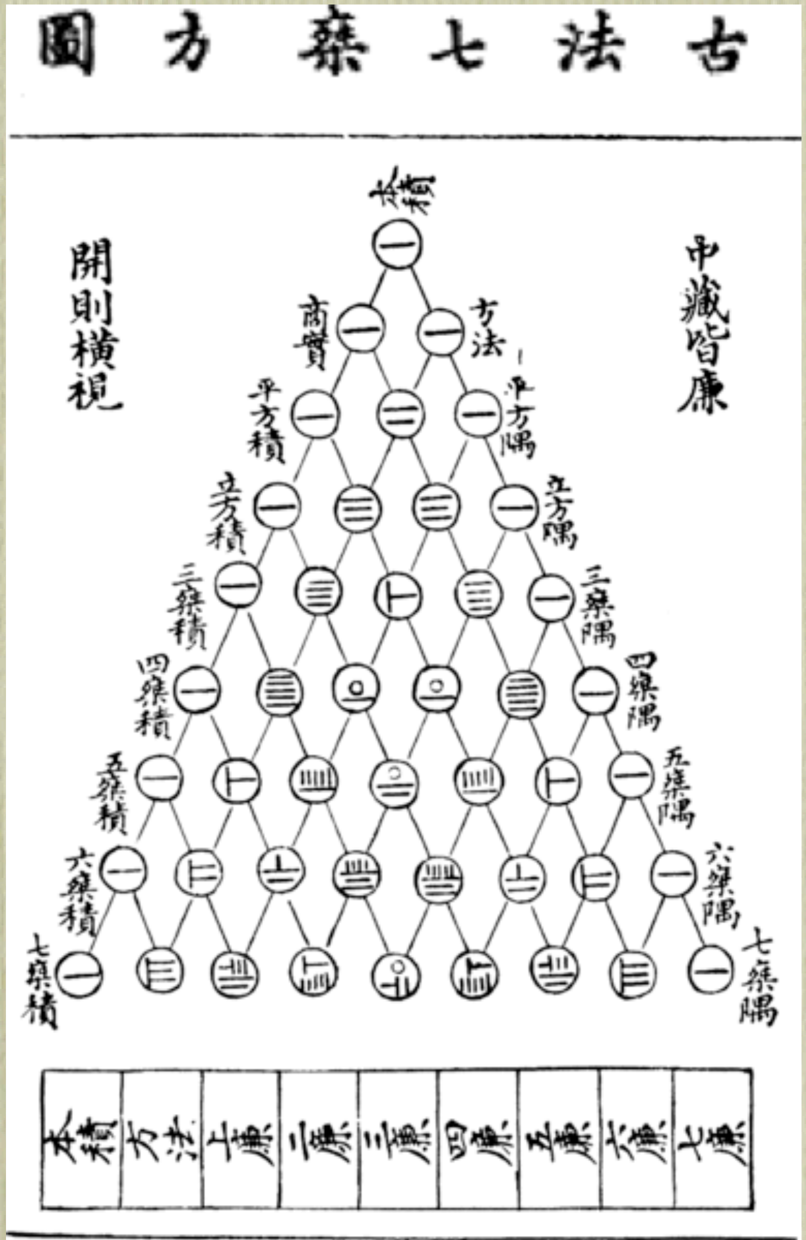
1, 2, 6, 24, 120, 720, 5040, 40320,...

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

partie à k éléments
d'un ensemble ayant n éléments



triangle de Pascal
coefficients binomiaux



triangle de Yang Hui
(11th, 12th siècle)

Perse
Omar Khayyam
(1048-1131)

en Inde
Chandas Shastra by Pingala
2ème siècle AC

polynôme de couplage

exemples

graphe "segment" $\qquad G_n \quad [1, n]$

$$F_n(x) = \sum_{k=1}^{n} (-1)^k a_{n,k} \, x^{n-2k}$$

$a_{n,k}$ = nb de couplages du graphe segment de longueur n ayant k arêtes

x^3

$-x$

$-x$

$$F_3(x) = x^3 - 2x$$

graphe "segment"

G_n $[1,n]$

x^4

$-x^2$

$-x^2$

$-x^2$

1

$$P_{G_4}(x) = x^4 - 3x^2 + 1$$

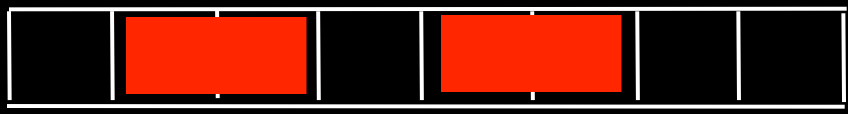
$$F_4(x)$$

$$a_{n,k} = \binom{n-k}{k}$$

nombre de couplages du graphe segment $[1, n]$ ayant k arêtes

$$\binom{n-k}{k} x^{n-2k}$$

$$F_n(x) = \sum_{k=1}^{n} (-1)^k a_{n,k}\, x^{n-2k}$$

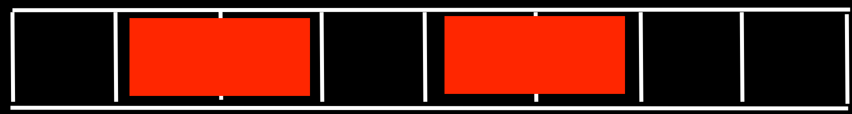


couplage

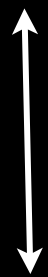
k dominos

$n = 8$

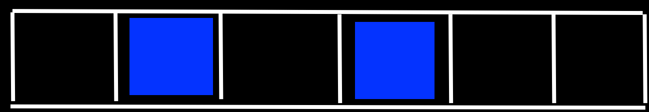
$k = 2$

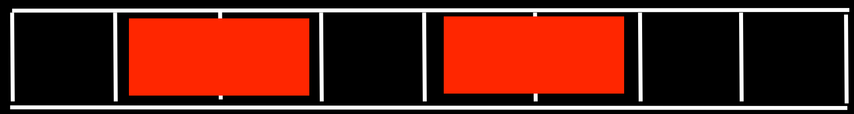


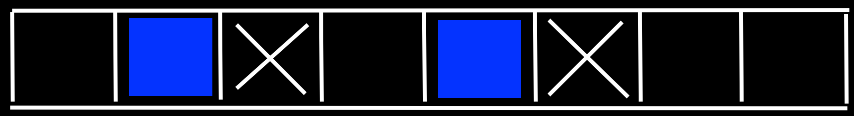
$$\binom{n-k}{k} x^{n-2k}$$

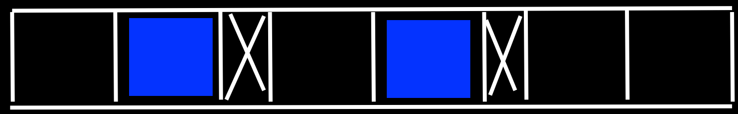


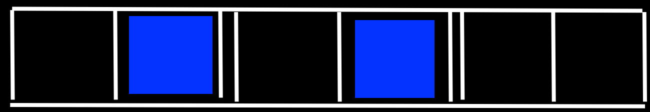
bijection

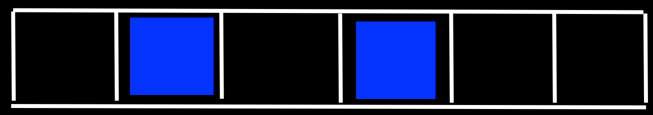








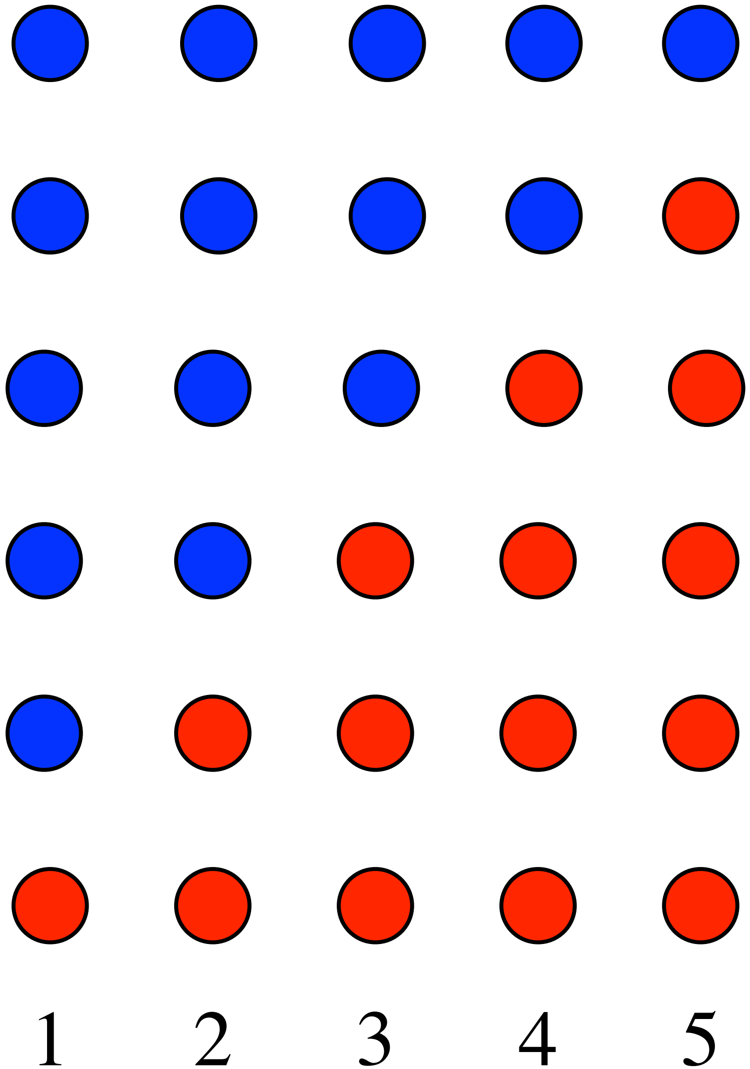


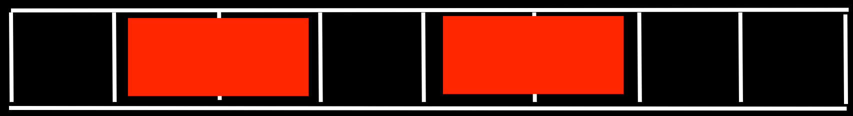


combinatoire bijective

«preuves sans mots»

$$1+2+\ldots+n=n(n+1)/2$$





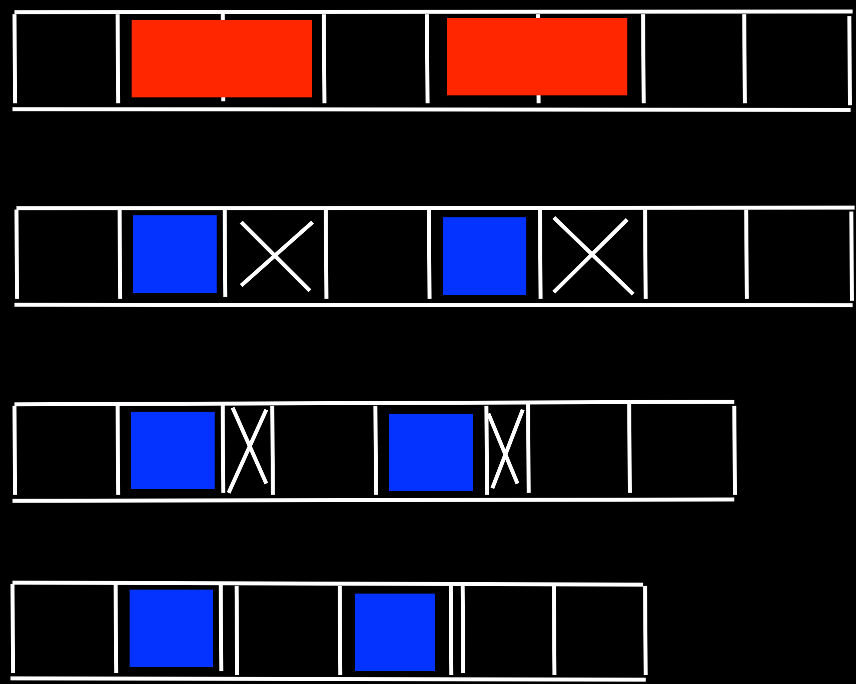
couplage

k dominos

$n = 8$

$k = 2$

$$\binom{n-k}{k} x^{n-2k}$$



addition +

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

1 addition +

	1								
1	1	1							
1	1	2	1						
2	1	3	3	1					
3	1	4	6	4	1				
5	1	5	10	10	5	1			
8	1	6	15	20	15	6	1		
13	1	7	21	35	35	21	7	1	
21	1	8	28	56	70	56	28	8	1

relation avec les nombres de Fibonacci
(10ème siècle)

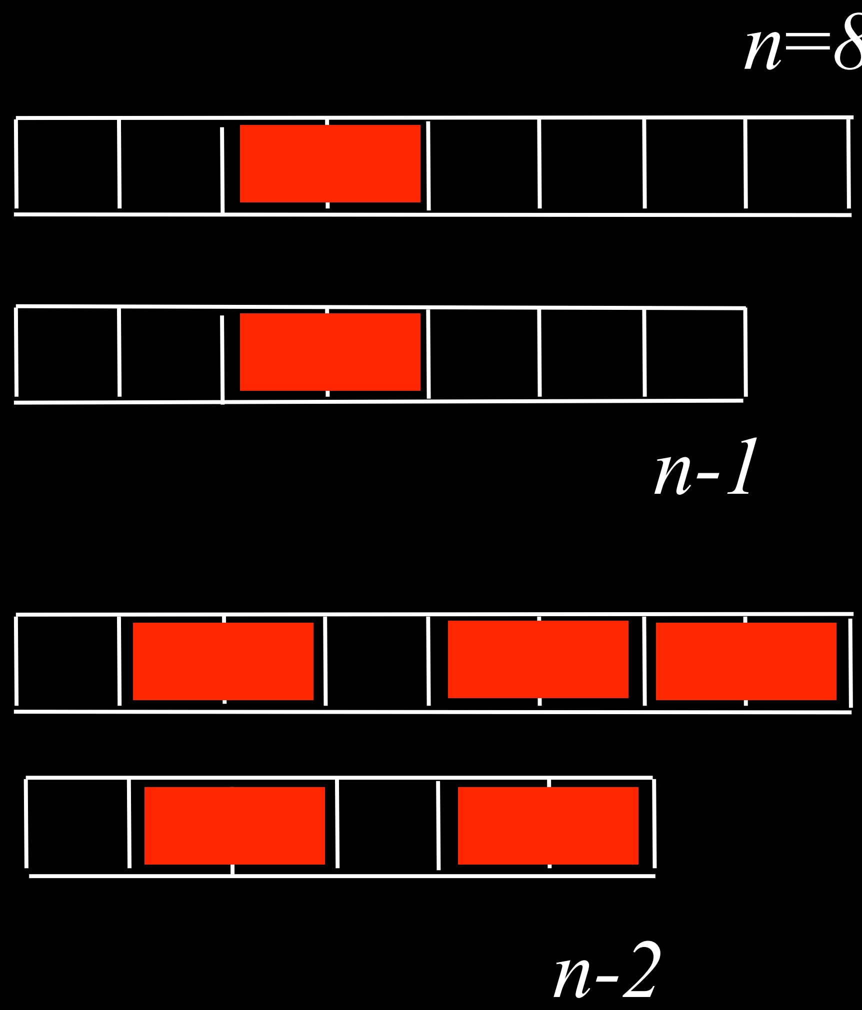
nombres de Fibonacci

1, 1, 2, 3, 5, 8, 13, 21, 34, 55,...

$$F_{n+1} = F_n + F_{n-1}$$

$$F_0 = 1, F_1 = 1$$

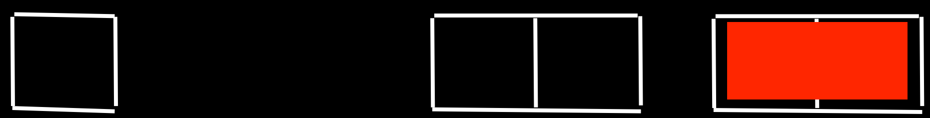
nombre total de couplages = F_n

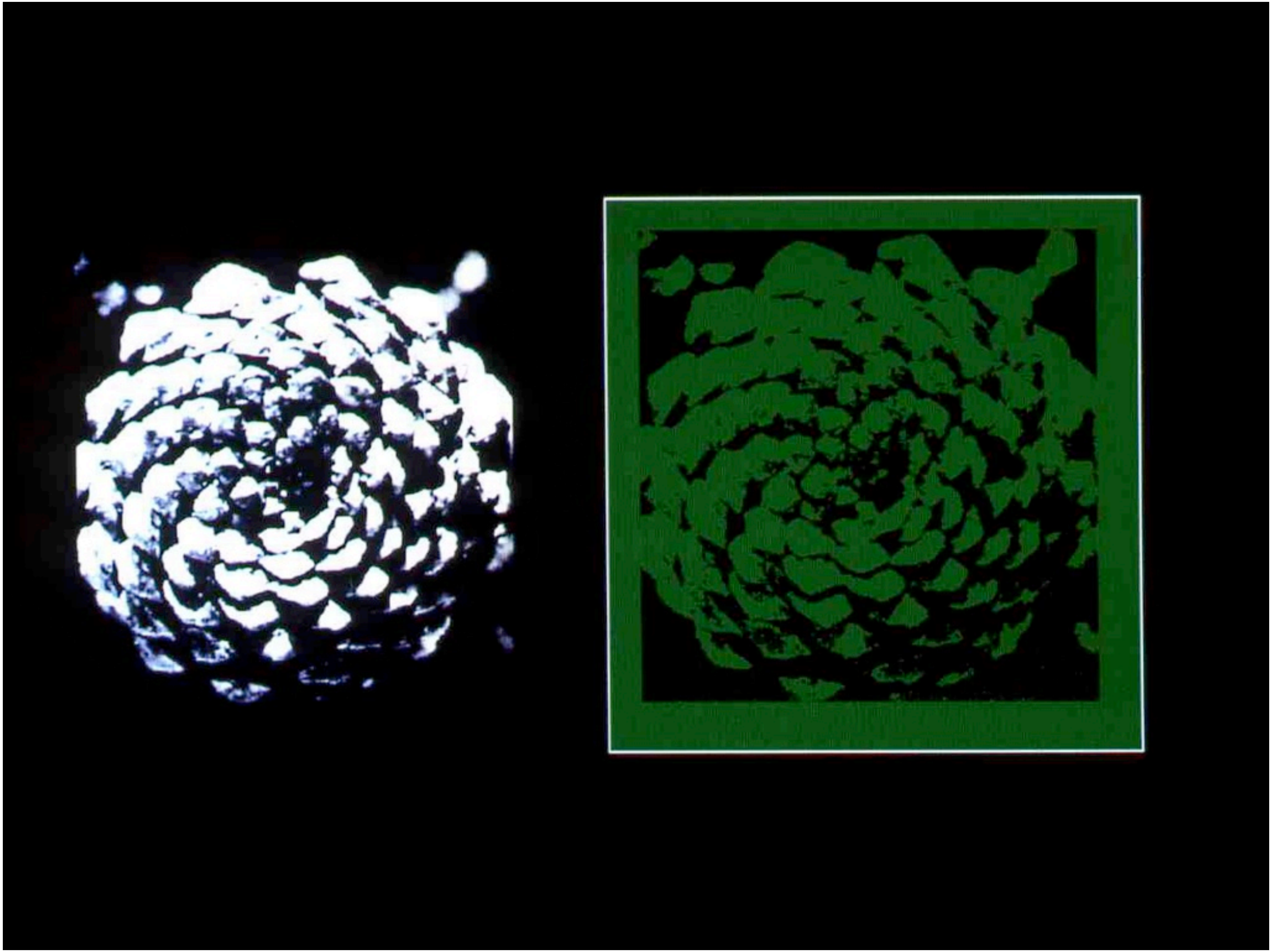


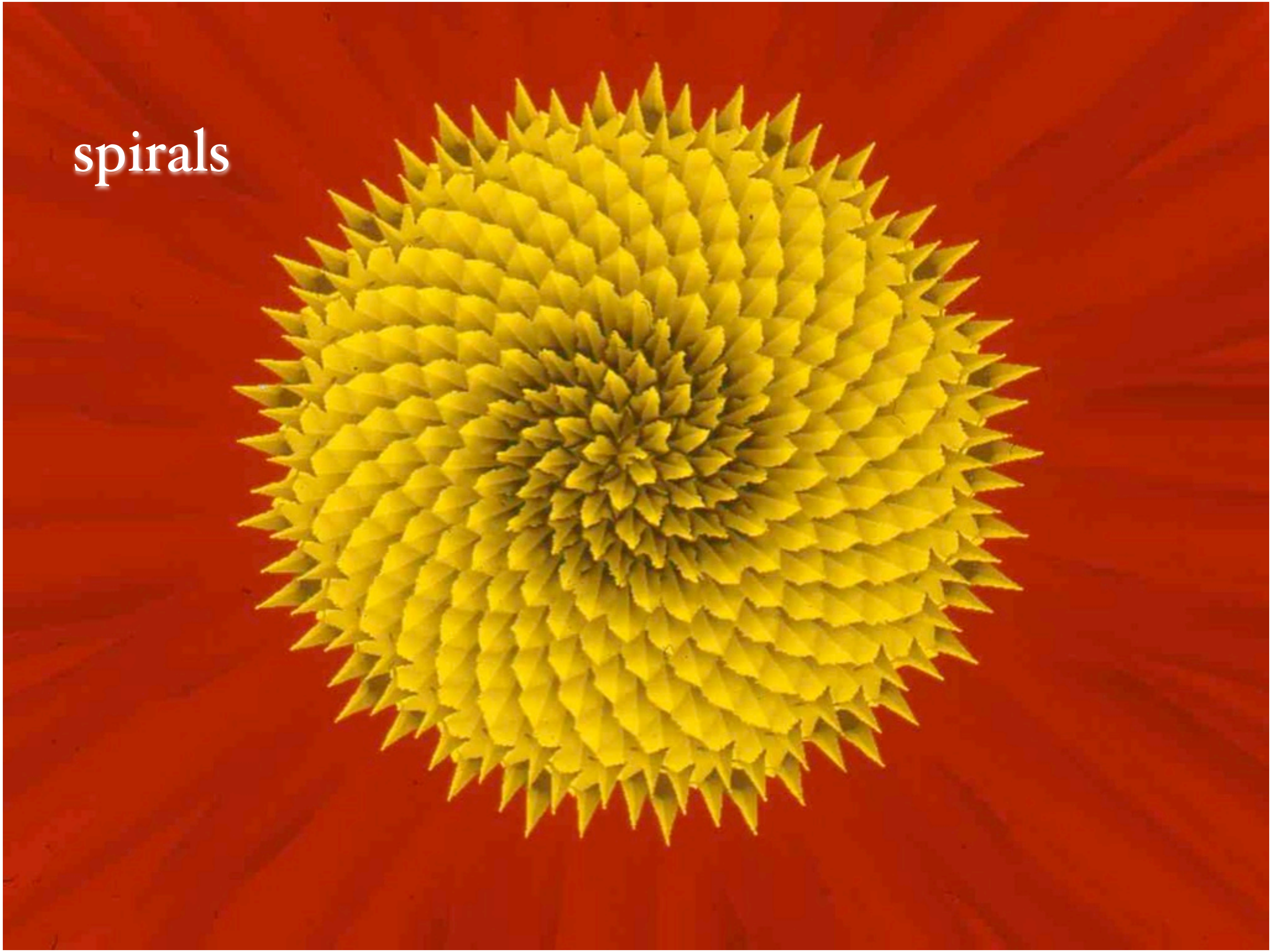
nombre de Fibonacci

$$F_{n+1} = F_n + F_{n-1}$$

$$F_1 = 1, F_2 = 2$$



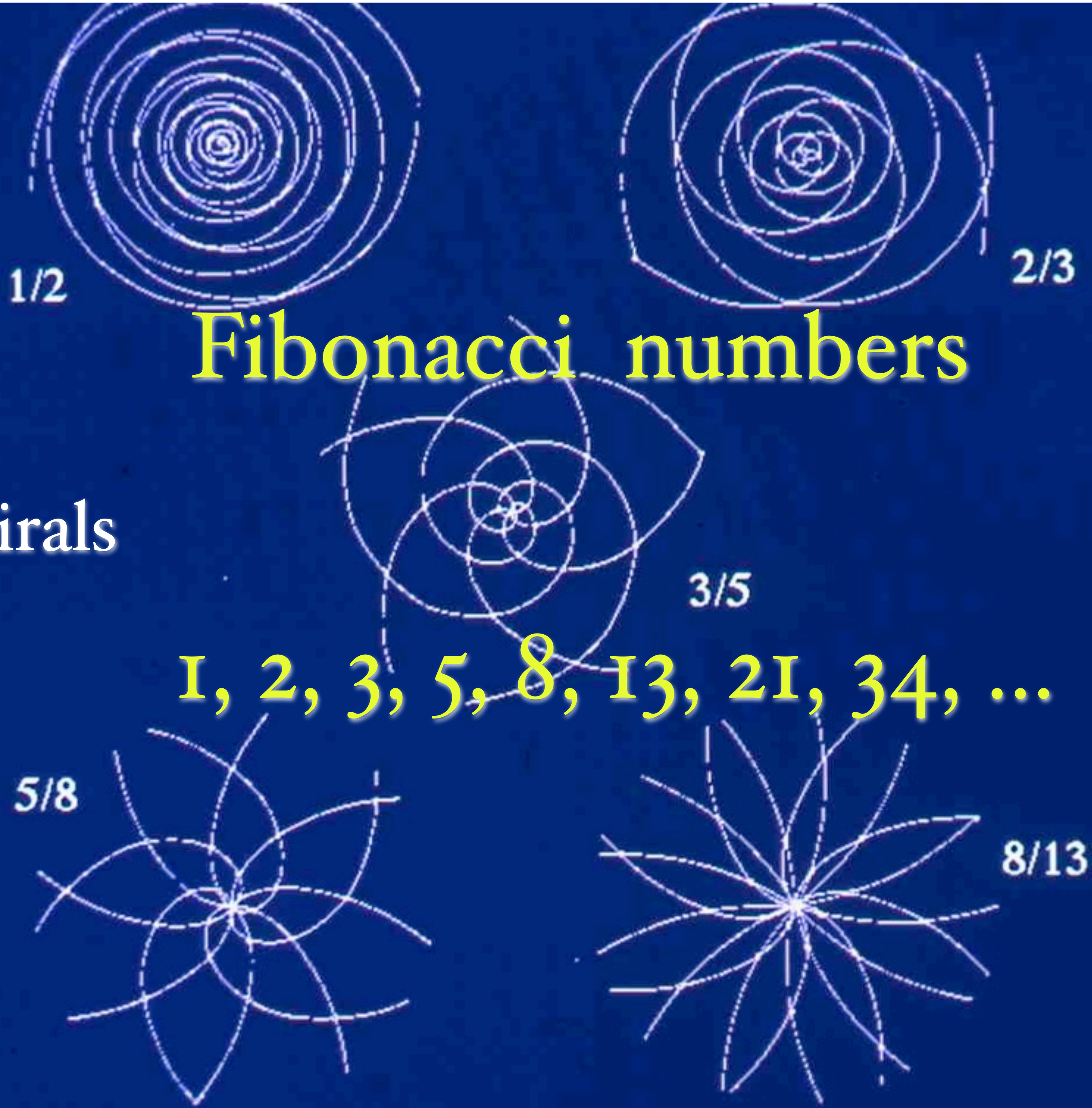



spirals

Fibonacci numbers

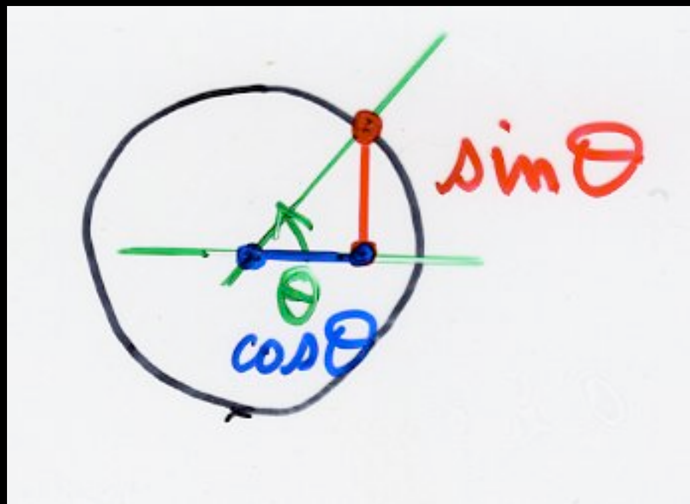
spirals

1, 2, 3, 5, 8, 13, 21, 34, ...



un peu de trigonométrie ....

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\sin 3\theta = \sin\theta\left(4\cos^2\theta - 1\right)$$
$$\sin 4\theta = \sin\theta\left(8\cos^3\theta - 4\cos\theta\right)$$
$$\cdots\cdots\cdots\cdots$$



$$\sin((n+1)\theta) = \sin\theta \; U_n(\cos\theta)$$

$$U_n(x) = F_n(2x)$$

$$F_n(x) = \sum_{k=1}^{n} (-1)^k a_{n,k} \, x^{n-2k}$$

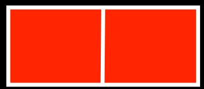
x^3

$-x$

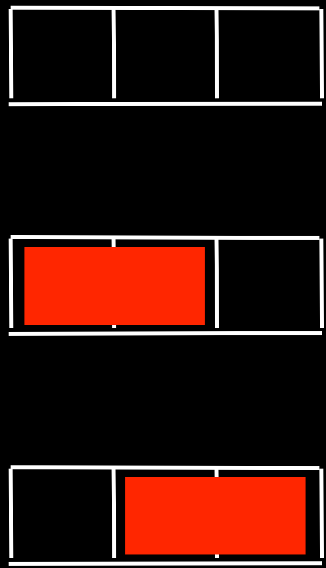
$-x$

$$F_3(x) = x^3 - 2x$$

$$\sin(4\theta) = \sin\theta\left[8\cos^3\theta - 4\cos\theta\right]$$



dimers

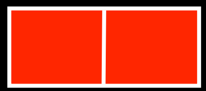


$$x^3 \qquad (2\cos\theta)^3$$

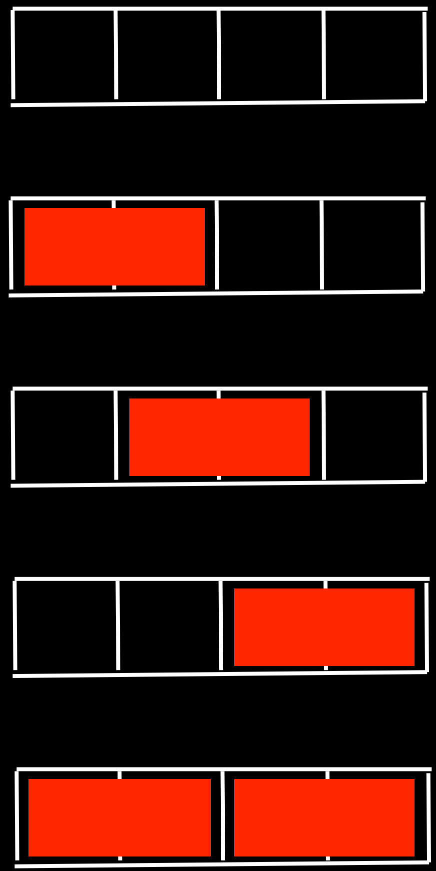
$$-x \qquad (2\cos\theta)$$

$$-x$$

$$\sin 4\theta = \sin\theta\left(8\cos^3\theta - 4\cos\theta\right)$$



dimers



x^4	$(2\cos\theta)^4$
$-x^2$	$(2\cos\theta)^2$
$-x^2$	
$-x^2$	
1	1

$$\sin 5\theta = \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right)$$

$$\cos 2\theta = \left(2\cos^2\theta - 1\right)$$

$$\cos 3\theta = \left(4\cos^3\theta - 3\cos\theta\right)$$

$$\cos 4\theta = \left(8\cos^4\theta - 8\cos^2\theta + 1\right)$$

$$\cos(n\theta) = T_n(\cos\theta)$$

polynôme de Tchebychef

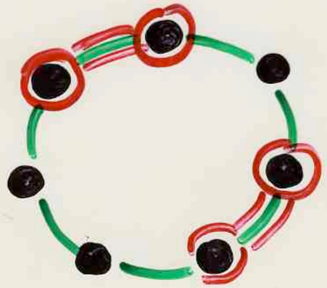
(1ère espèce)

$$\sin((n+1)\theta) = \sin\theta \; U_n(\cos\theta)$$

polynôme de Tchebychef

(2ème espèce)

$$\cos(n\theta) = T_n(\cos\theta)$$



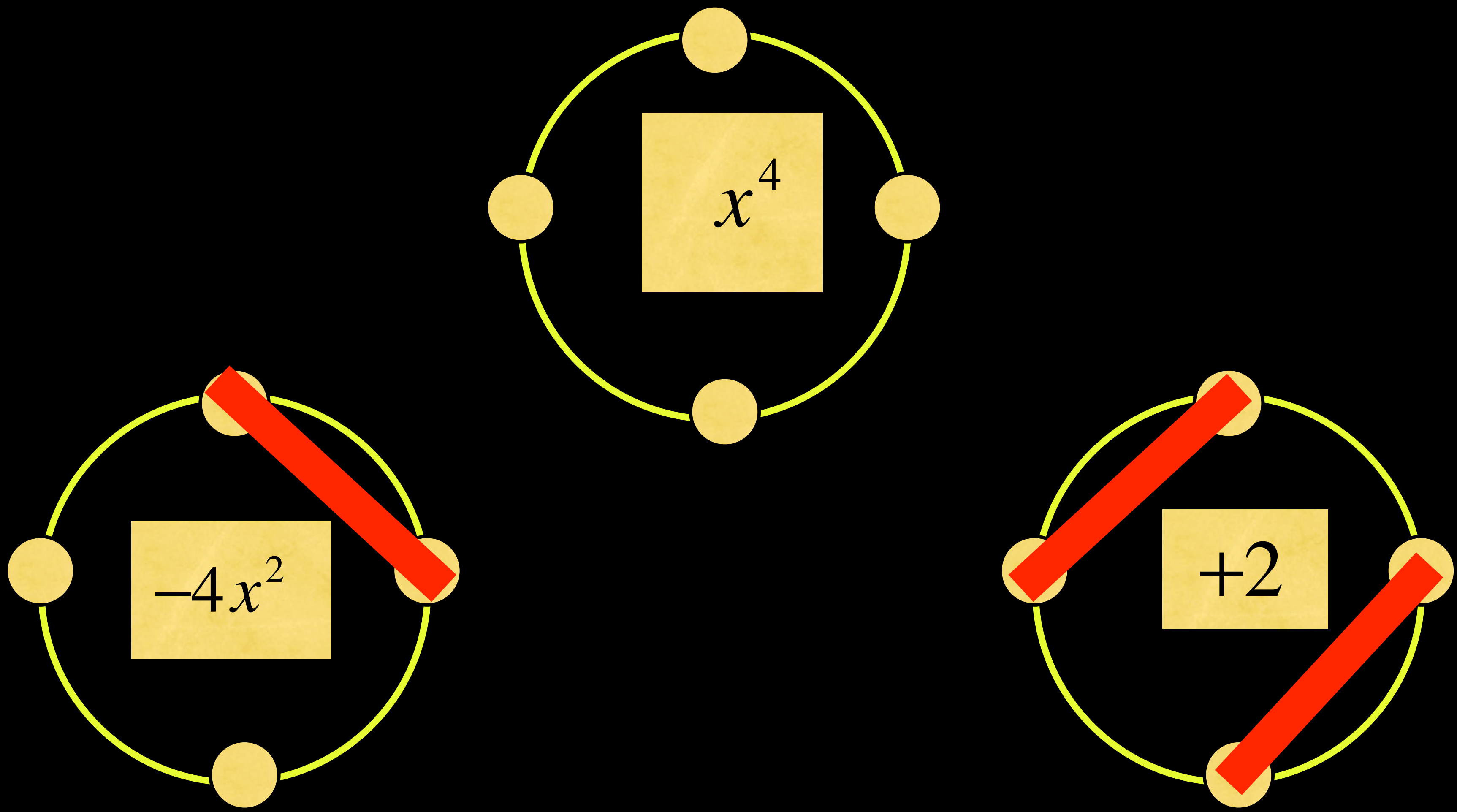
polynôme de Tchebychef

(1ère espèce)

$$T_n(x) = \frac{1}{2}\, G_n(2x)$$

$G_n(x)$

polynôme de couplage du graphe "cycle" de longueur n



$$x \rightarrow 2\cos\theta$$

$$\left(16\cos^4\theta - 16\cos^2\theta + 2\right)$$

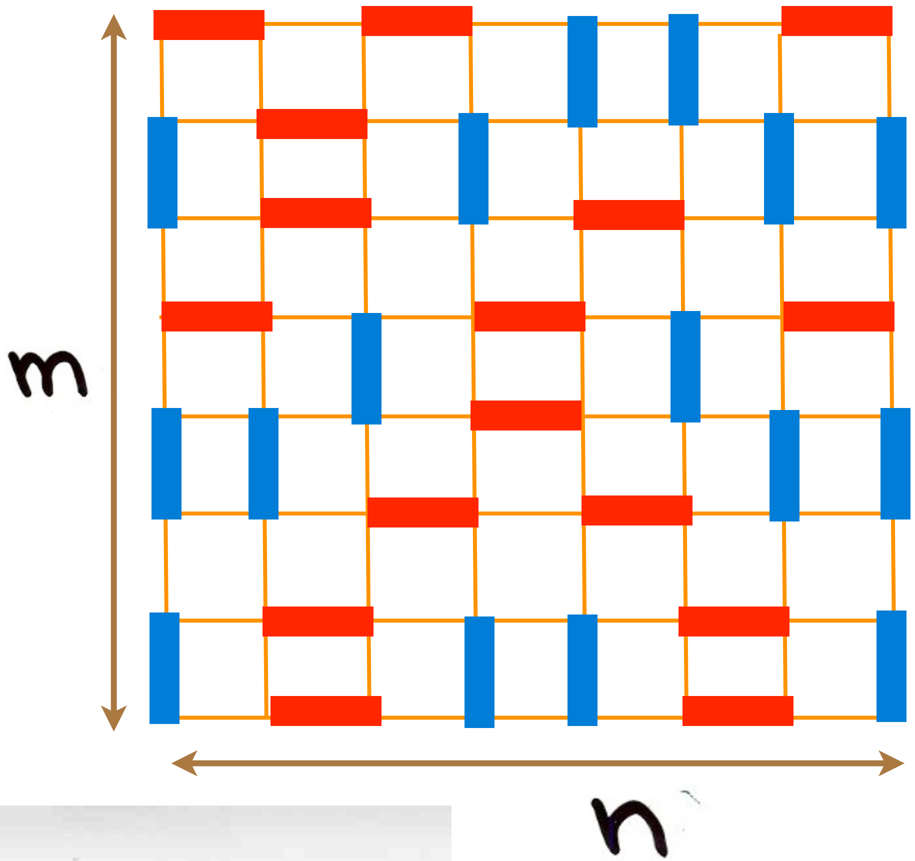
$$\cos 4\theta = \left(8\cos^4\theta - 8\cos^2\theta + 1\right)$$

$$\sin((n+1)\theta) = \sin\theta \; U_n(\cos\theta)$$

racines
des
polynômes
de couplage

$F_n(x)$

$F_m(x)$



$$\prod_{i=1}^{m/2} \prod_{j=1}^{n/2} \left(4 \cos^2 \frac{i\pi}{m+1} + 4 \cos^2 \frac{j\pi}{n+1} \right)$$

Kasteleyn (1961)

«résultant» de deux polynômes

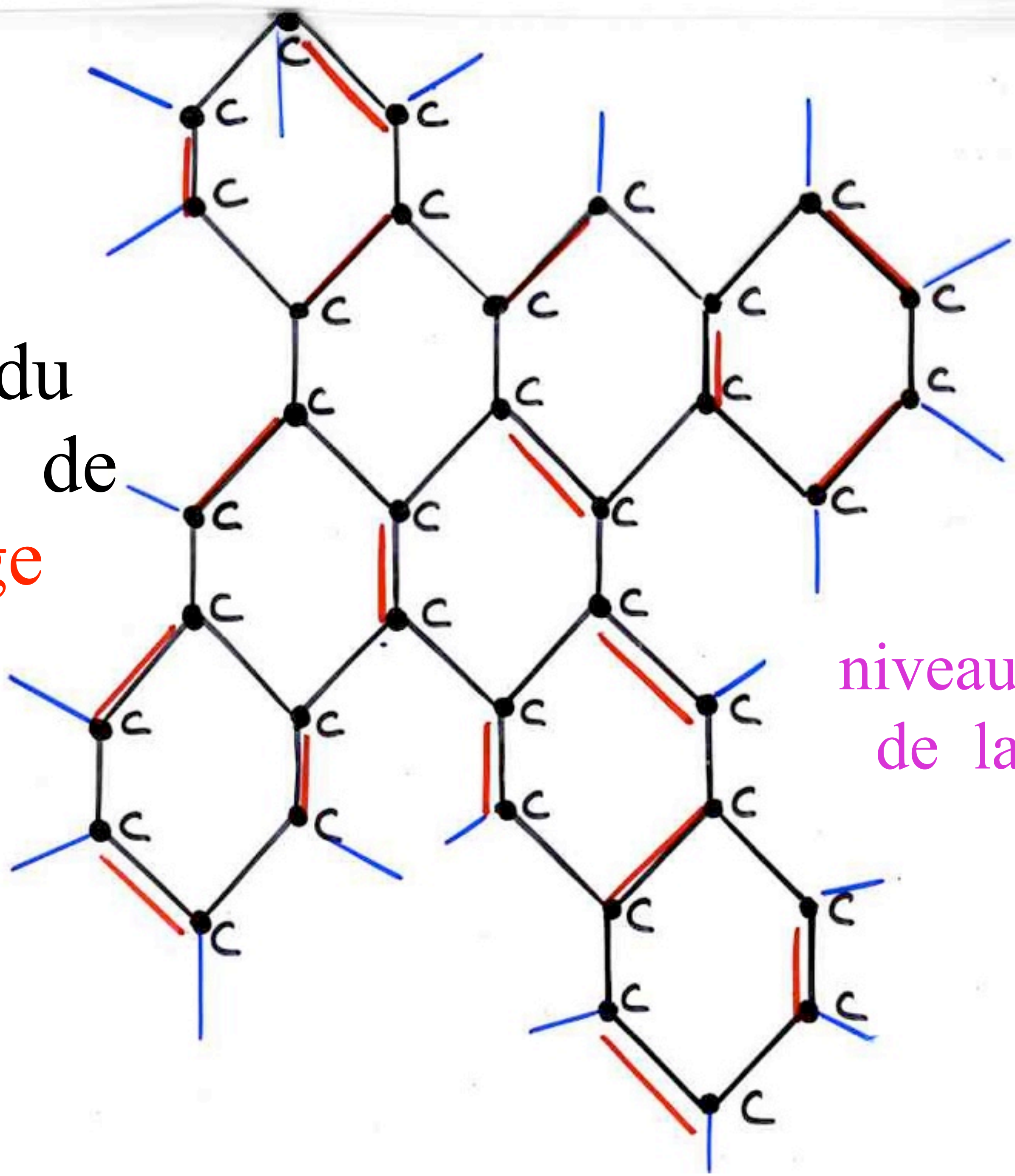
Pour tout graphe G ayant n sommets,

le polynôme de couplage associé a

toutes ses n racines réelles

Chimie

racines du
polynôme de
couplage

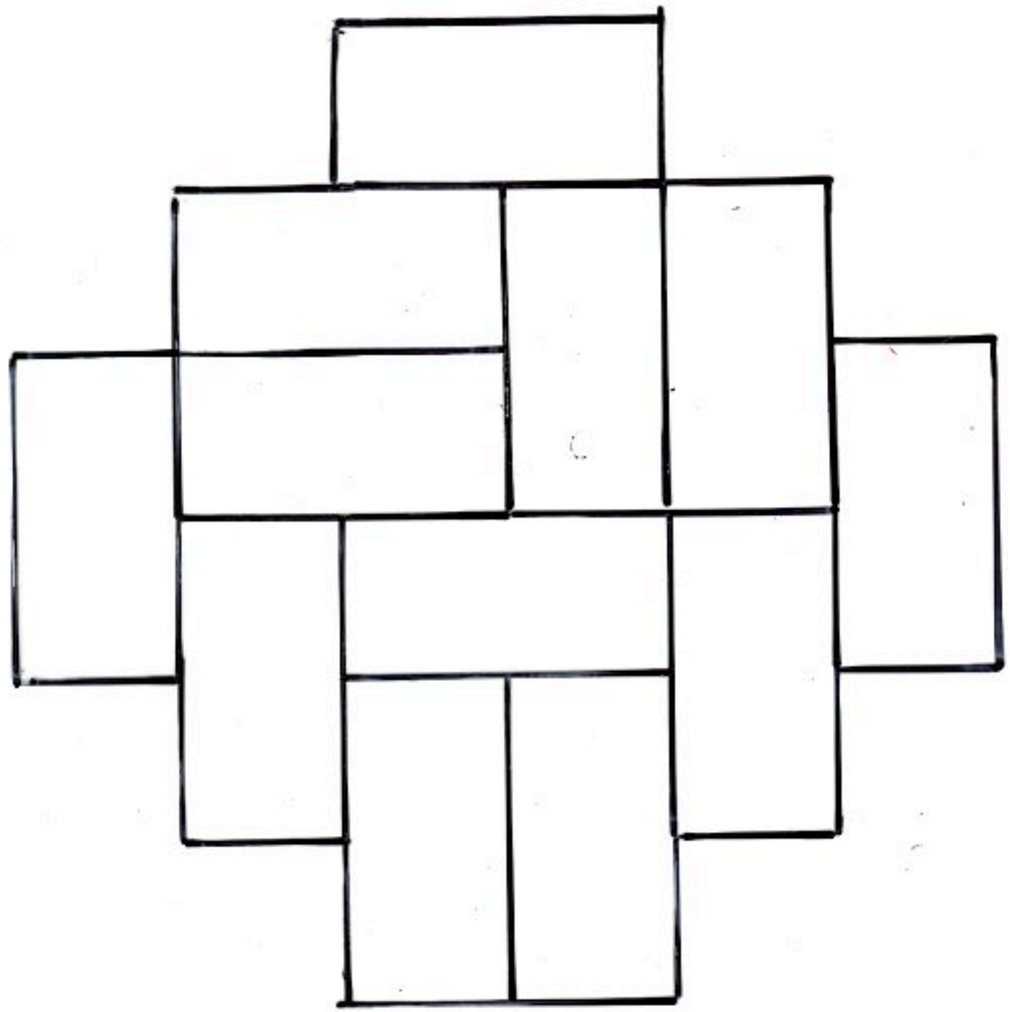


niveaux d'énergie
de la molécule

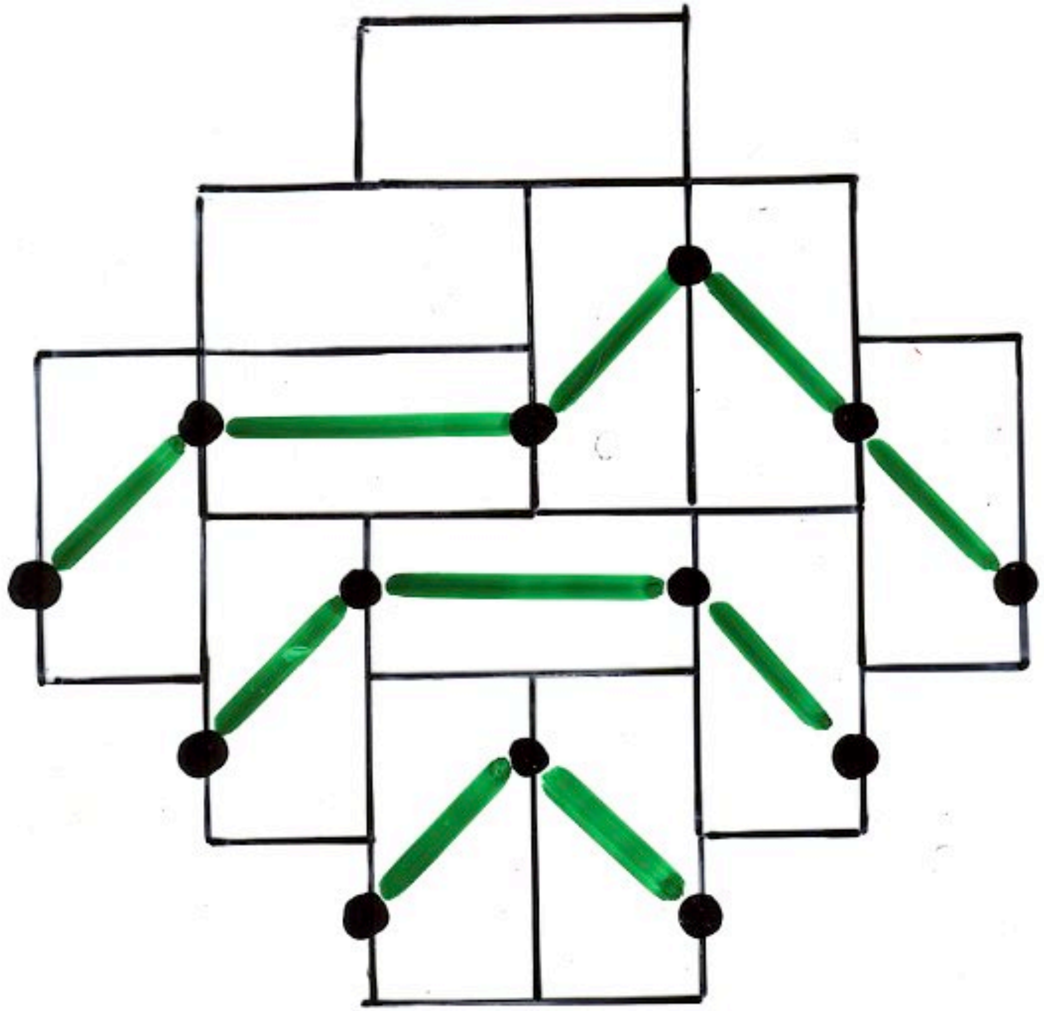
Comment démontrer la miraculeuse

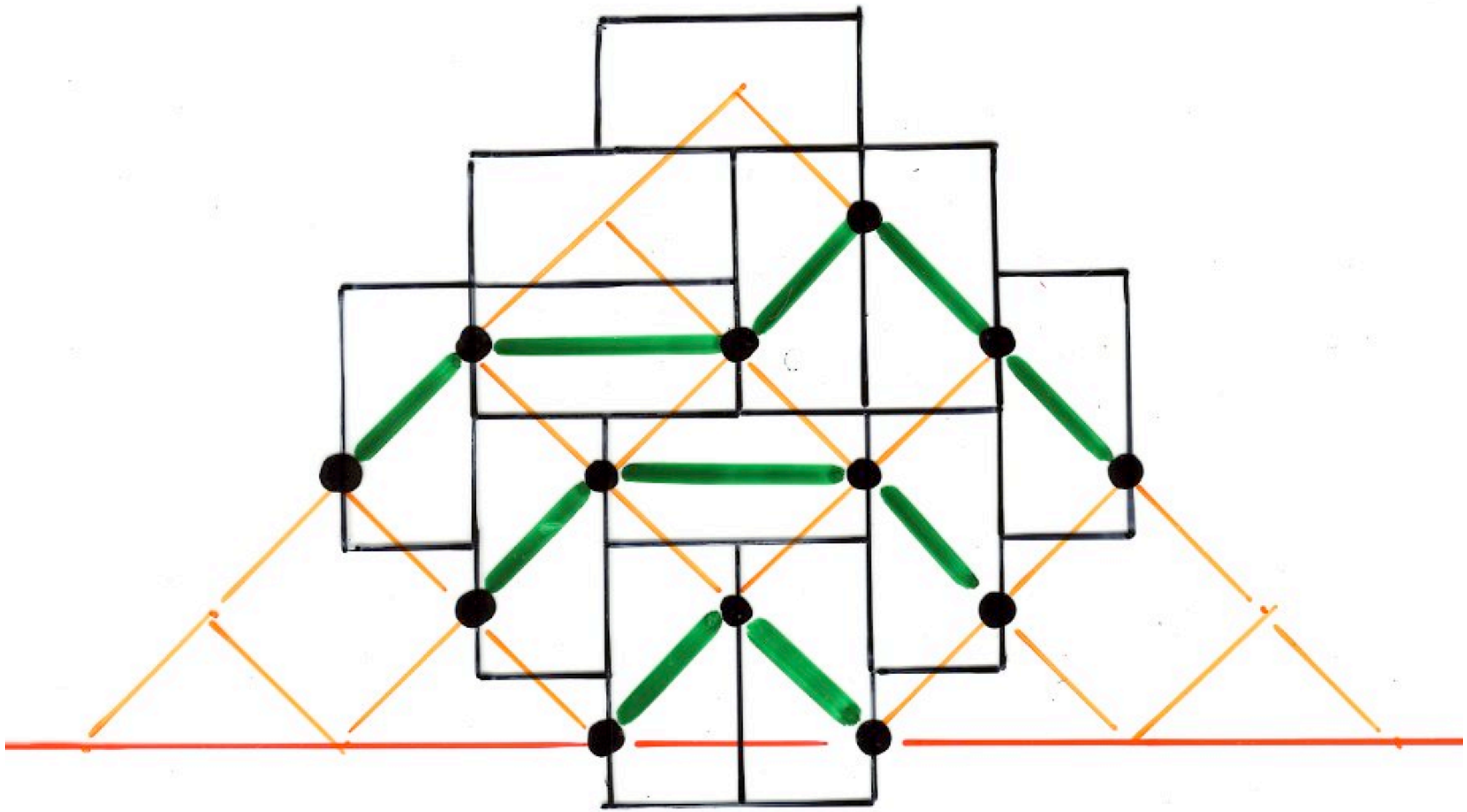
formule du diagramme Astèque ?

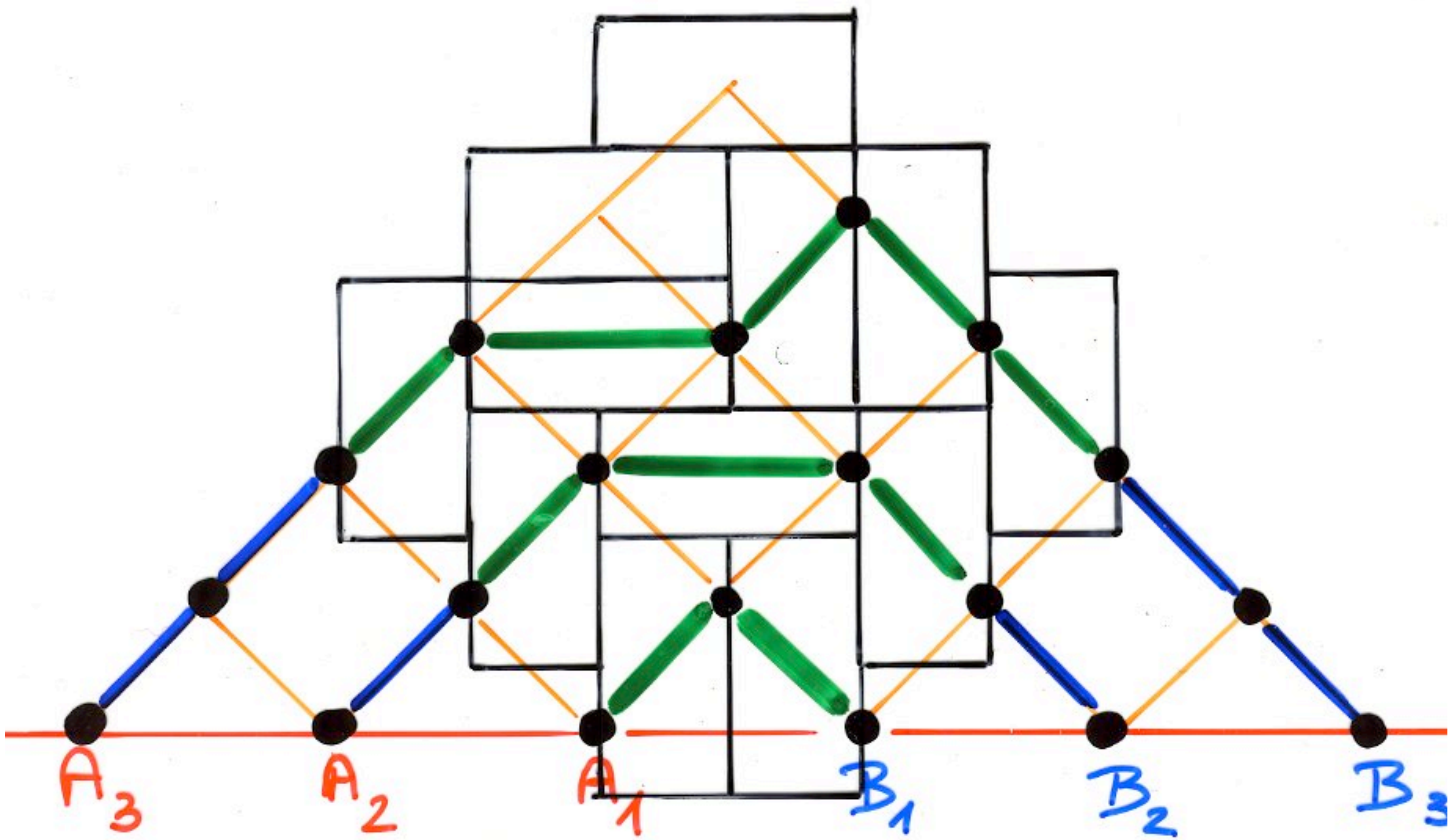
$$2^{\frac{n(n+1)}{2}}$$



bijection avec des chemins





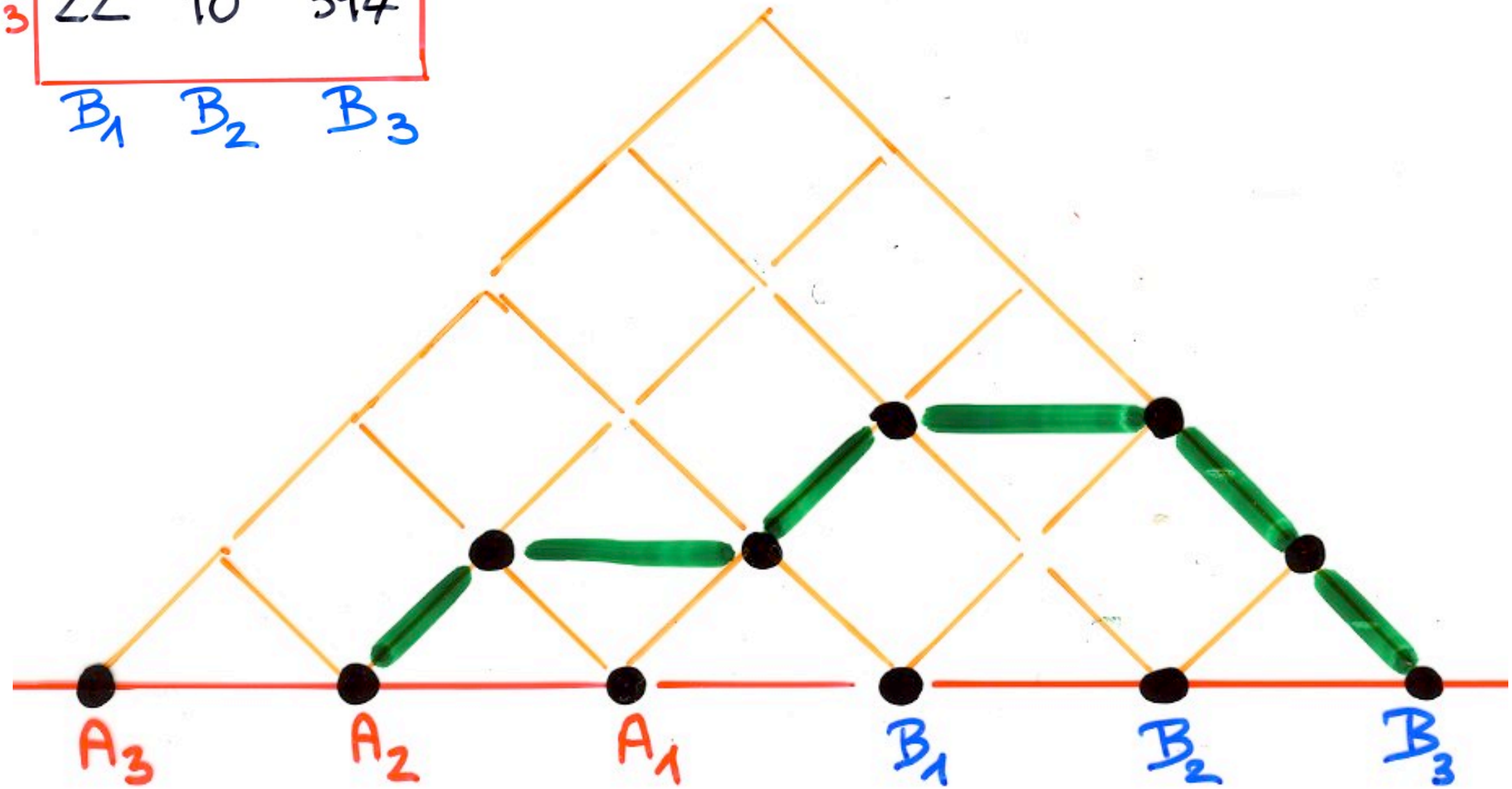

A3
A2
A1
B1
B2
B3

chemins de Schröder

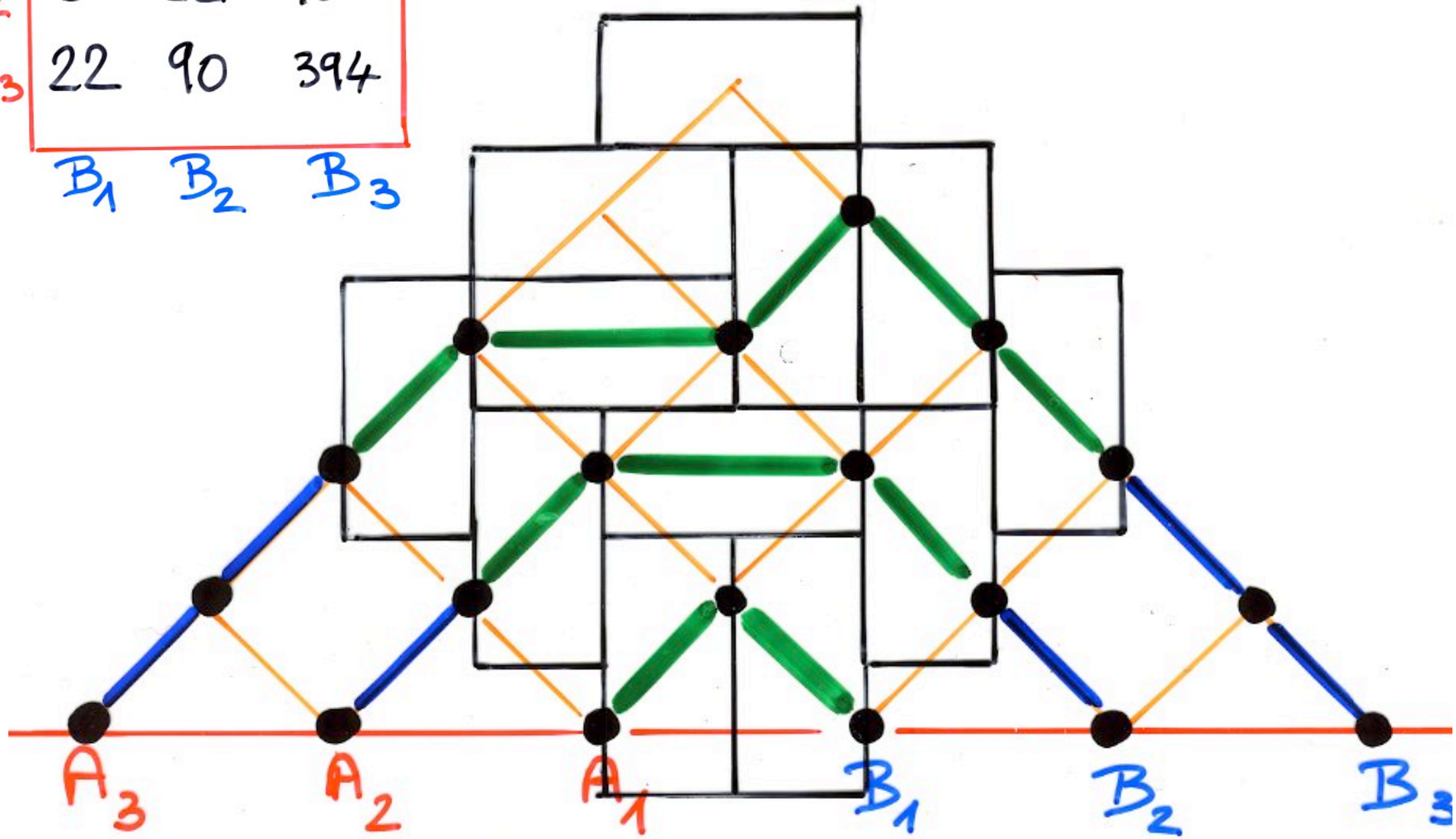
nombres de Schröder

$S_n = 1, 2, 6, 22, 90, 394, \ldots$

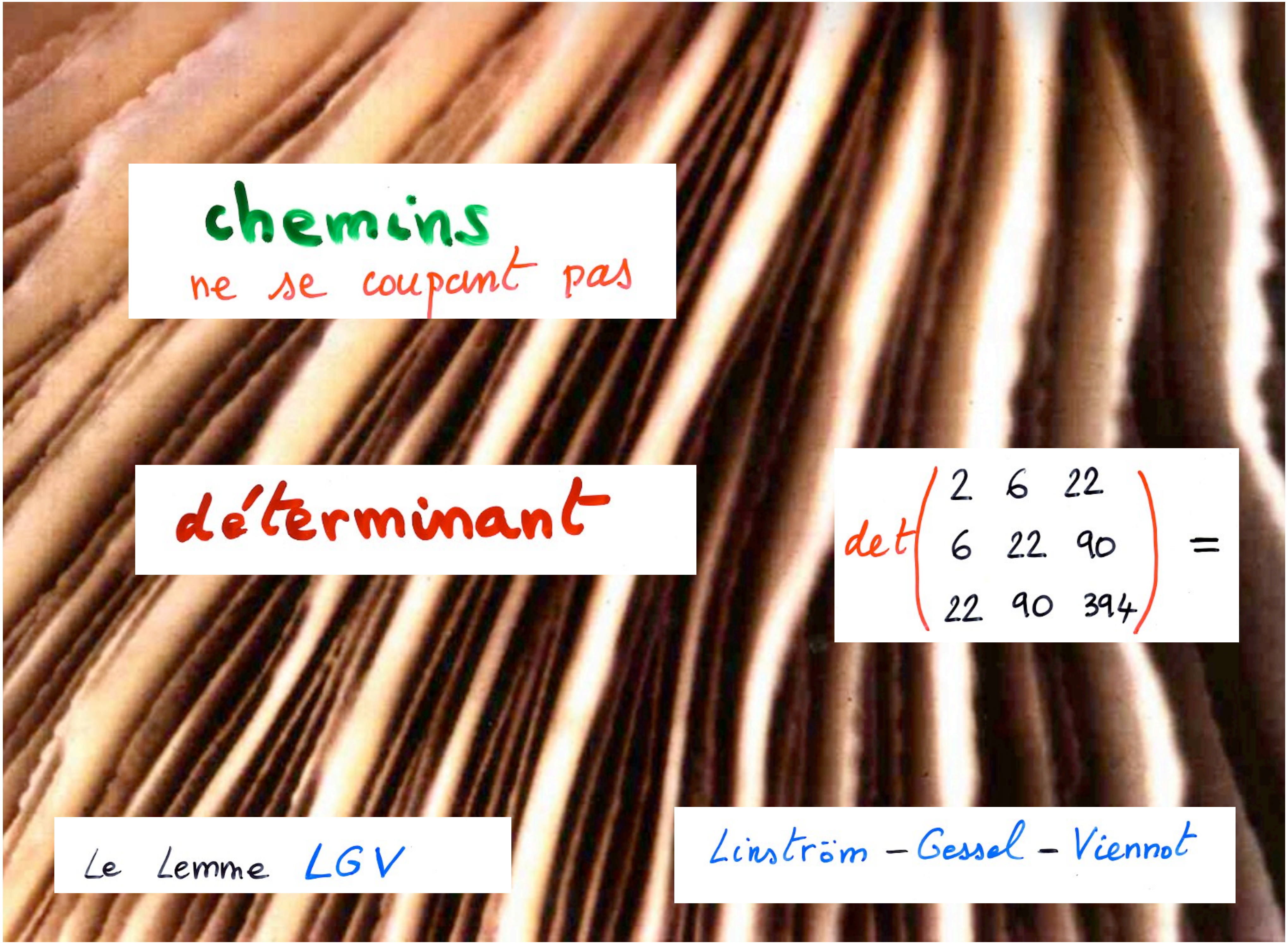
	B_1	B_2	B_3
A_1	2	6	22
A_2	6	22	90
A_3	22	90	394



	B_1	B_2	B_3
A_1	2	6	22
A_2	6	22	90
A_3	22	90	394



Le Lemme LGV



chemins
ne se coupant pas

déterminant

$$\det\begin{pmatrix} 2 & 6 & 22 \\ 6 & 22 & 90 \\ 22 & 90 & 394 \end{pmatrix} =$$

Le Lemme LGV

Linström - Gessel - Viennot

Déterminant

d'un tableau de nombres
(ou matrice)
ayant n lignes et n colonnes

algèbre linéaire

système d'équations linéaires

$$\begin{cases} 2x - 3y + 11z = -2 \\ 4x + y - z = 0 \\ 5x - 7y + 9z = 6 \end{cases}$$

$$\det\begin{pmatrix} 2 & 6 \\ 6 & 22 \end{pmatrix} = (2 \times 22) - (6 \times 6)$$
$$= 44 - 36$$

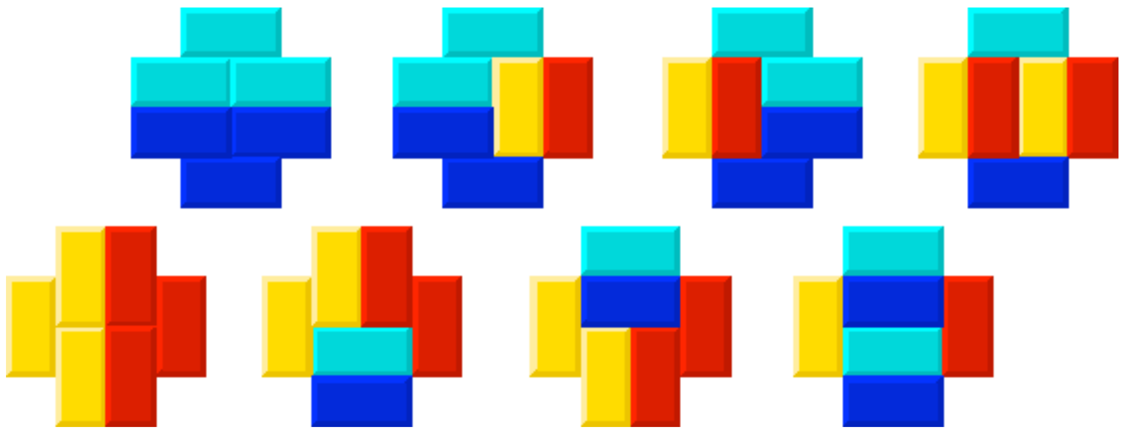
déterminant

$$\det\begin{pmatrix}2 & 6\\ 6 & 22\end{pmatrix} = (2\times 22) - (6\times 6)$$

$$= 44 - 36$$

$$= 8 = 2^3$$

(!)



déterminant

$$\begin{matrix} a & b & c \\ x & y & z \\ u & v & w \end{matrix}$$

$$\begin{matrix} a & b & c \\ x & y & z \\ u & v & w \end{matrix} \qquad \begin{matrix} a & b & c \\ x & y & z \\ u & v & w \end{matrix} \qquad \begin{matrix} a & b & c \\ x & y & z \\ u & v & w \end{matrix}$$

$+ayw \qquad +cxv \qquad +bzu$

$$\begin{matrix} a & b & c \\ x & y & z \\ u & v & w \end{matrix} \qquad \begin{matrix} a & b & c \\ x & y & z \\ u & v & w \end{matrix} \qquad \begin{matrix} a & b & c \\ x & y & z \\ u & v & w \end{matrix}$$

$-azv \qquad -bxw \qquad -cyu$

déterminant

$$\begin{matrix} a & b & c \\ x & y & z \\ u & v & w \end{matrix}$$

$$\begin{matrix} a & b & c \\ x & y & z \\ u & v & w \end{matrix}$$

$+ a y w$

$$\begin{matrix} a & b & c \\ x & y & z \\ u & v & w \end{matrix}$$

$+ c x v$

$$\begin{matrix} a & b & c \\ x & y & z \\ u & v & w \end{matrix}$$

$+ b z u$

$$\begin{matrix} a & b & c \\ x & y & z \\ u & v & w \end{matrix}$$

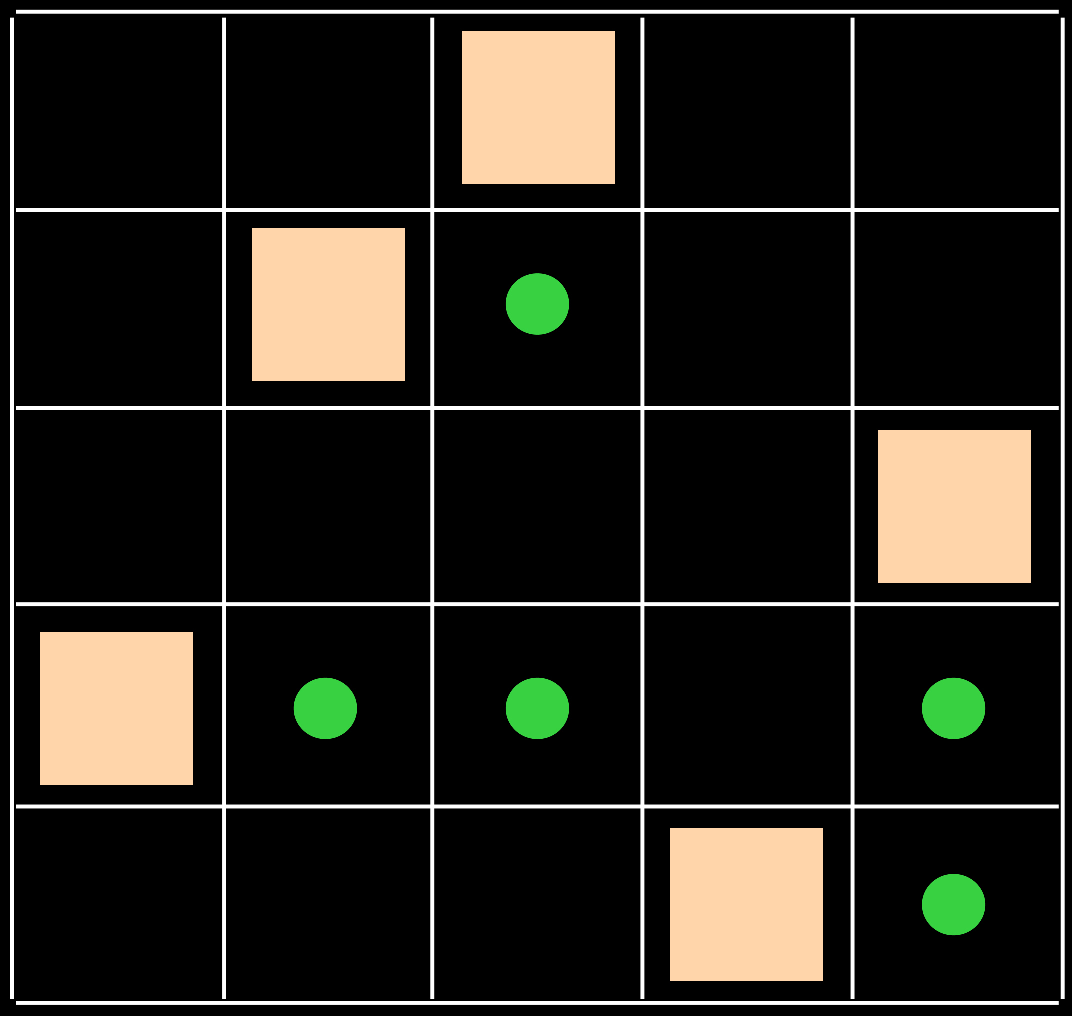
$- a z v$

$$\begin{matrix} a & b & c \\ x & y & z \\ u & v & w \end{matrix}$$

$- b x w$

$$\begin{matrix} a & b & c \\ x & y & z \\ u & v & w \end{matrix}$$

$- c y u$



$$\det\begin{pmatrix} 2 & 6 & 22 \\ 6 & 22 & 90 \\ 22 & 90 & 394 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & 22 & \cdot \\ \cdot & \cdot & 394 \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & 22 \\ 6 & \cdot & \cdot \\ \cdot & 90 & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & 6 & \cdot \\ \cdot & \cdot & 90 \\ 22 & \cdot & \cdot \end{pmatrix}$$

$$+\,17336 \qquad +\,11880 \qquad +\,11880 \;\rightarrow\; 41096$$

$$\begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & \cdot & 90 \\ \cdot & 90 & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & 6 & \cdot \\ 6 & \cdot & \cdot \\ \cdot & \cdot & 394 \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & 22 \\ \cdot & 22 & \cdot \\ 22 & \cdot & \cdot \end{pmatrix}$$

$$-\,16200 \qquad -\,14184 \qquad -\,10648 \;\rightarrow\; -41032$$

$$\overline{64}$$

$$= 2^6 \quad (!!)$$

$$\prod_{i=1}^{m/2} \prod_{j=1}^{n/2} \left(4 \cos^2 \frac{i\pi}{m+1} + 4 \cos^2 \frac{j\pi}{n+1} \right)$$

Kasteleyn (1961)

- dénombrement de
  couplages parfaits
- graphe planaire

  méthode du Pfaffien

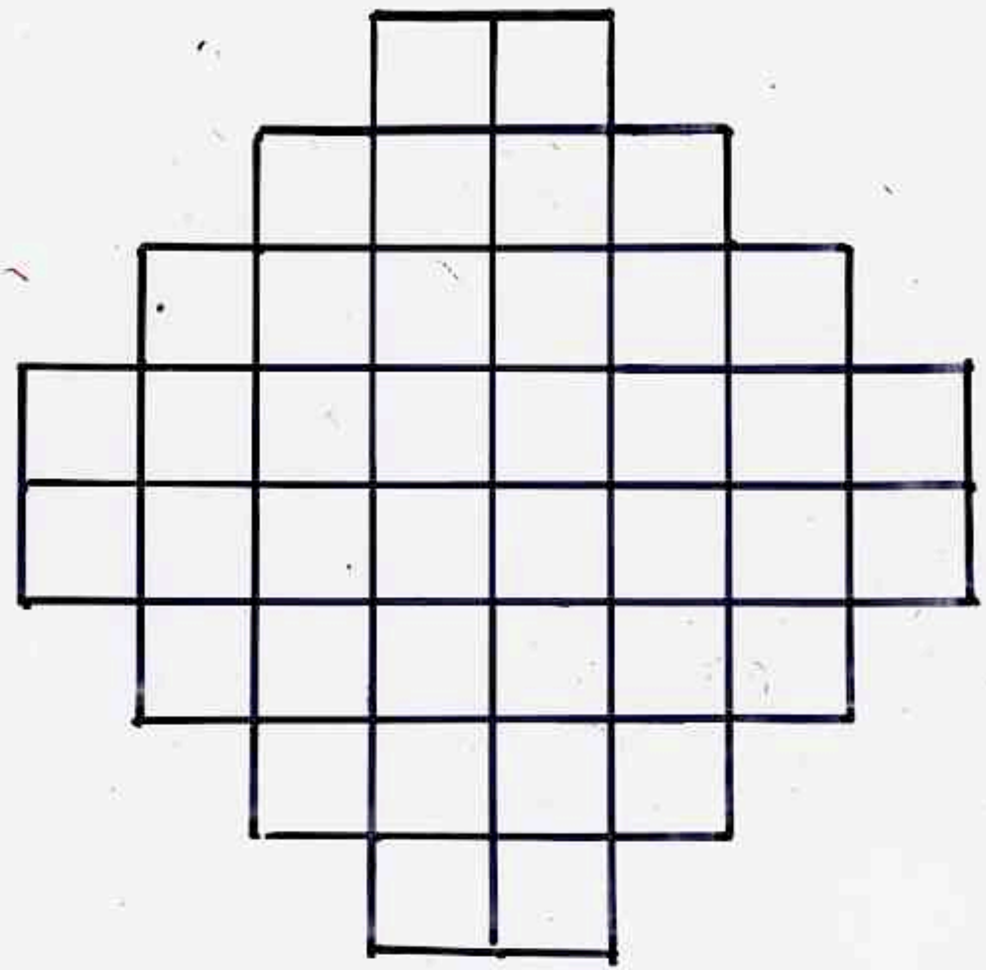
- modèle d'Ising (1925)

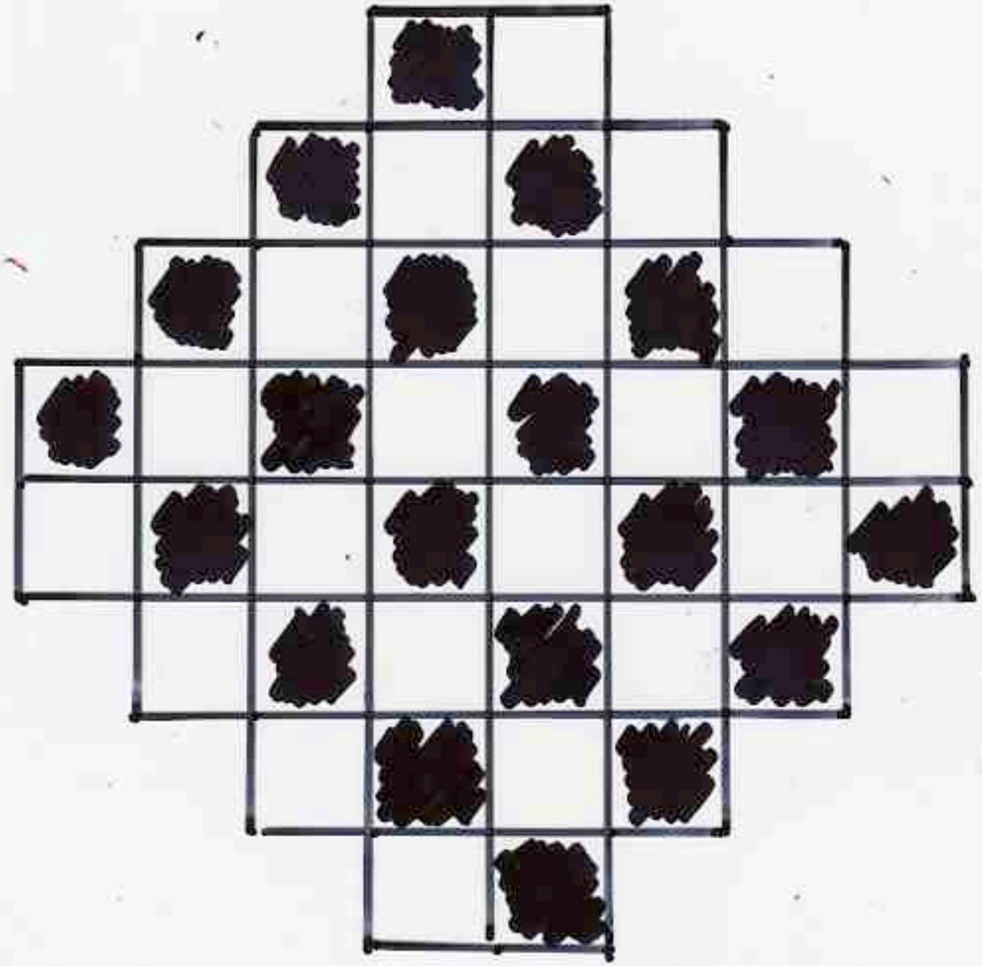
  Kasteleyn, Fisher, Temperley
  (1961, ...)

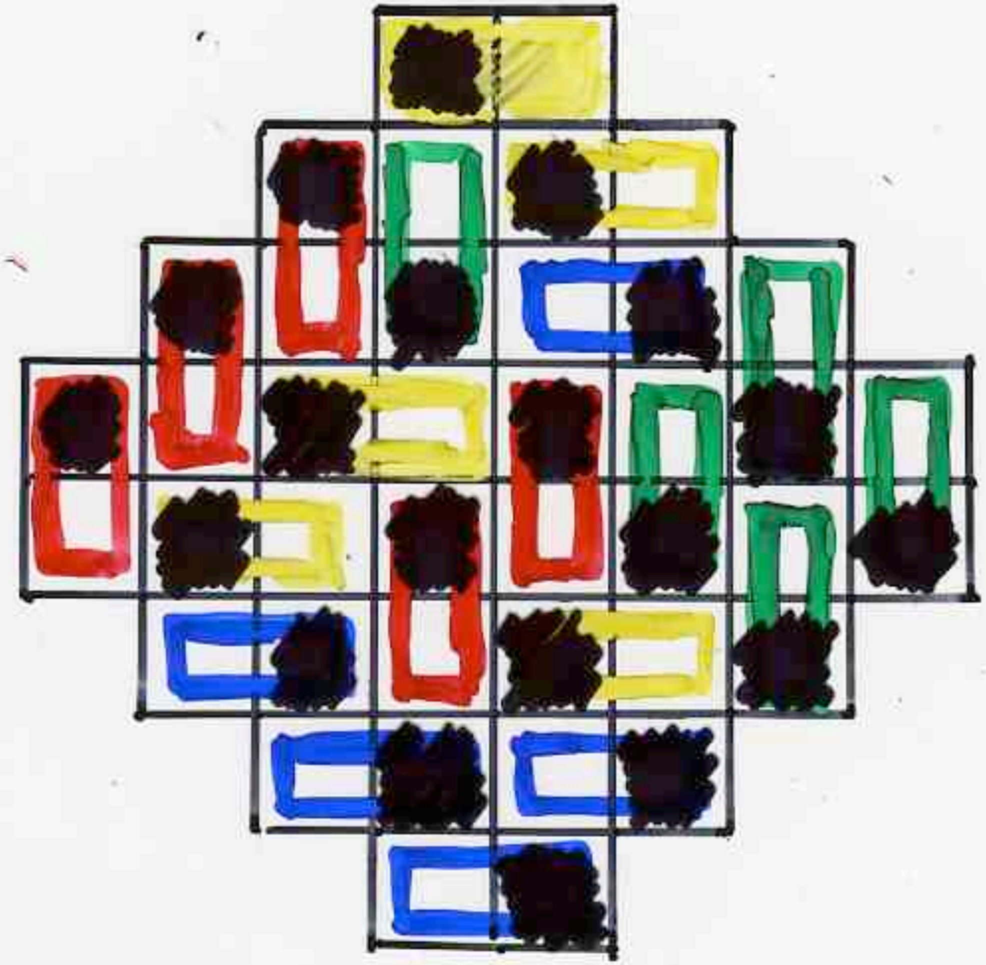
  Onsager (1944)

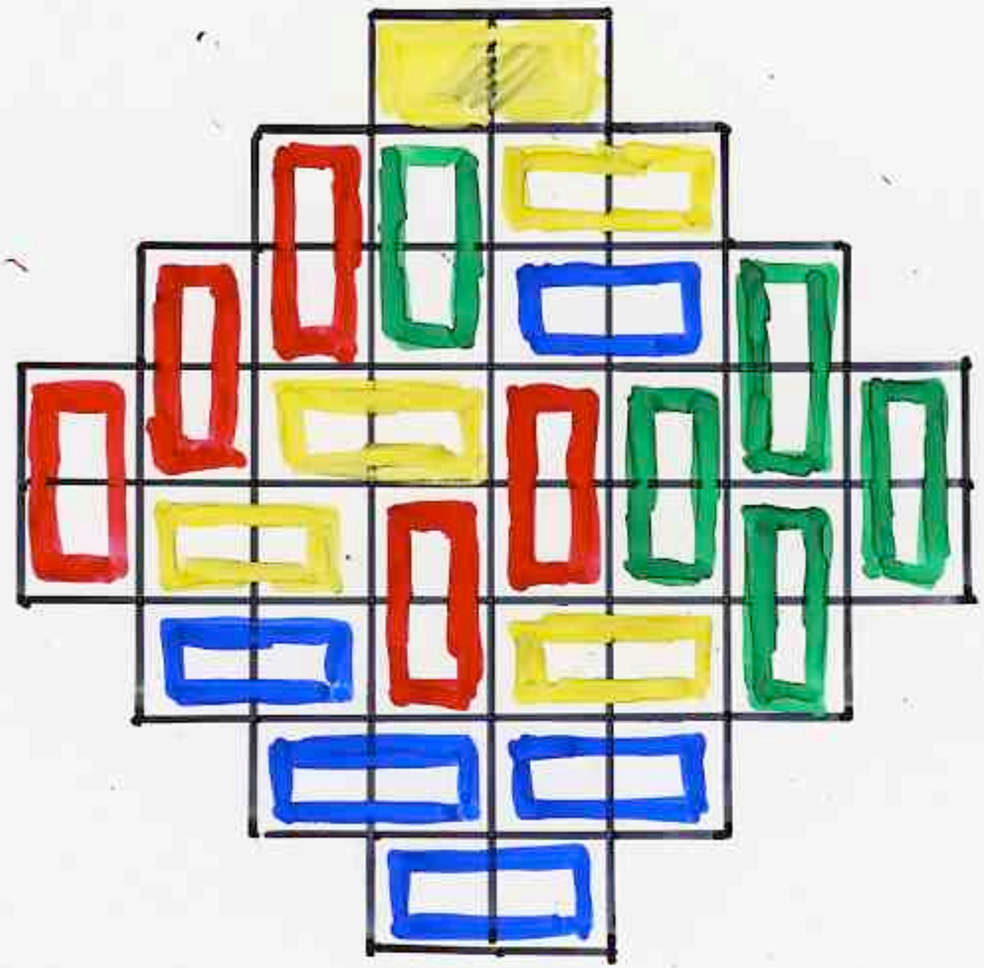
génération aléatoire
d'objets combinatoires

le théorème du cercle arctique

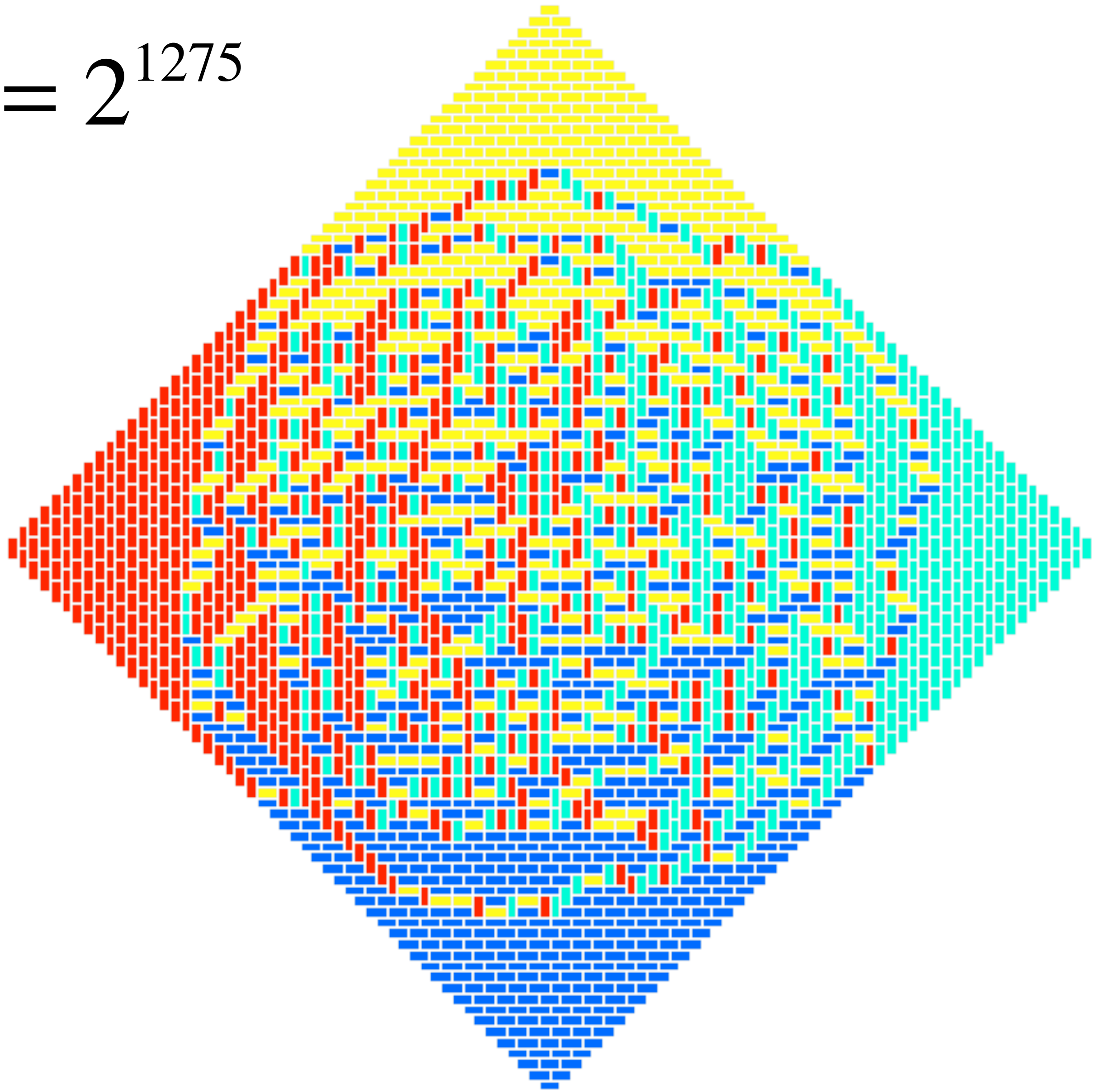




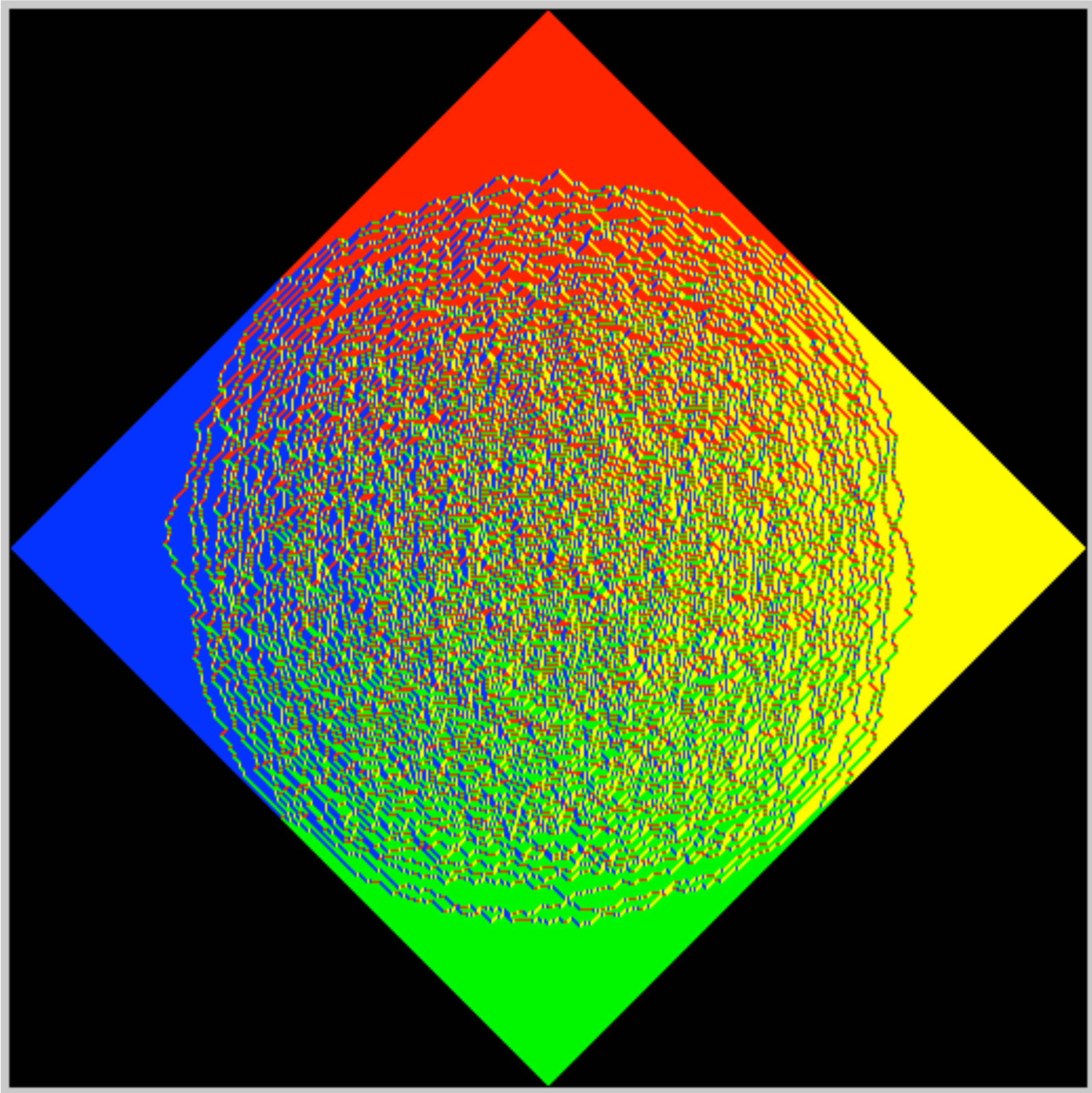




$$2^{(50\times 51)/2} = 2^{1275}$$



le
théorème
du
«arctique
cercle»



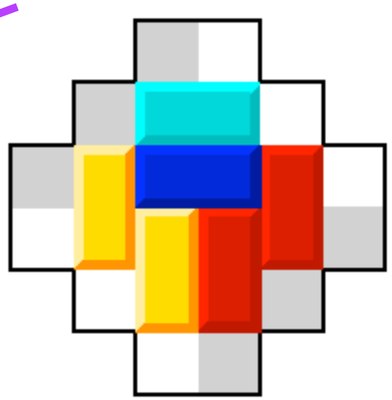
Joskush
Propp
Shor
(1998)

algorithme de
génération aléatoire
d'un pavage aztèque

avec une preuve bijective
de la formule $2^{\frac{n(n+1)}{2}}$

$$2^{(1+2)}$$

$$2^3 = 8$$



$$w = w_1 w_2 w_3$$

$$w_i = 0$$

$$w_i = 1$$

bijection

$$2^{(1+2+3)}$$

$$2^6 = 64$$

$$w = w_1$$

$w_1 = 0$

$w_1 = 1$

2^1

$$w = w_1 w_2$$

$$w_i = 0$$

$$w_i = 1$$

$$2^{(1+2)}$$

$$2^3 = 8$$

$$w = w_1 w_2 w_3$$

$$w_i = 0$$

$$w_i = 1$$

$$2^{(1+2+3)}$$

$$2^6 = 64$$

$$w = w_1 w_2 w_3 w_4$$

$$w_i = 0$$

$$w_i = 1$$

$$2^{(1+2+3+4)}$$

$$2^{10} = 1024$$

$$w = w_1 w_2 w_3 w_4 w_5$$

$$w_i = 0$$

$$w_i = 1$$

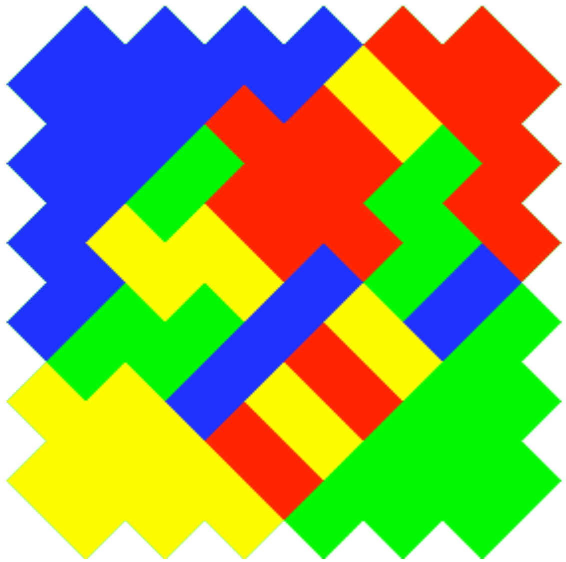
$$2^{(1+2+3+4+5)}$$

$$2^{15} = 32.768$$

$$w = w_1 w_2 w_3 w_4 w_5 w_6$$

$$w_i = 0$$

$$w_i = 1$$



$$2^{(1+2+3+4+5+6)}$$

$$2^{21} = 2.097.152$$

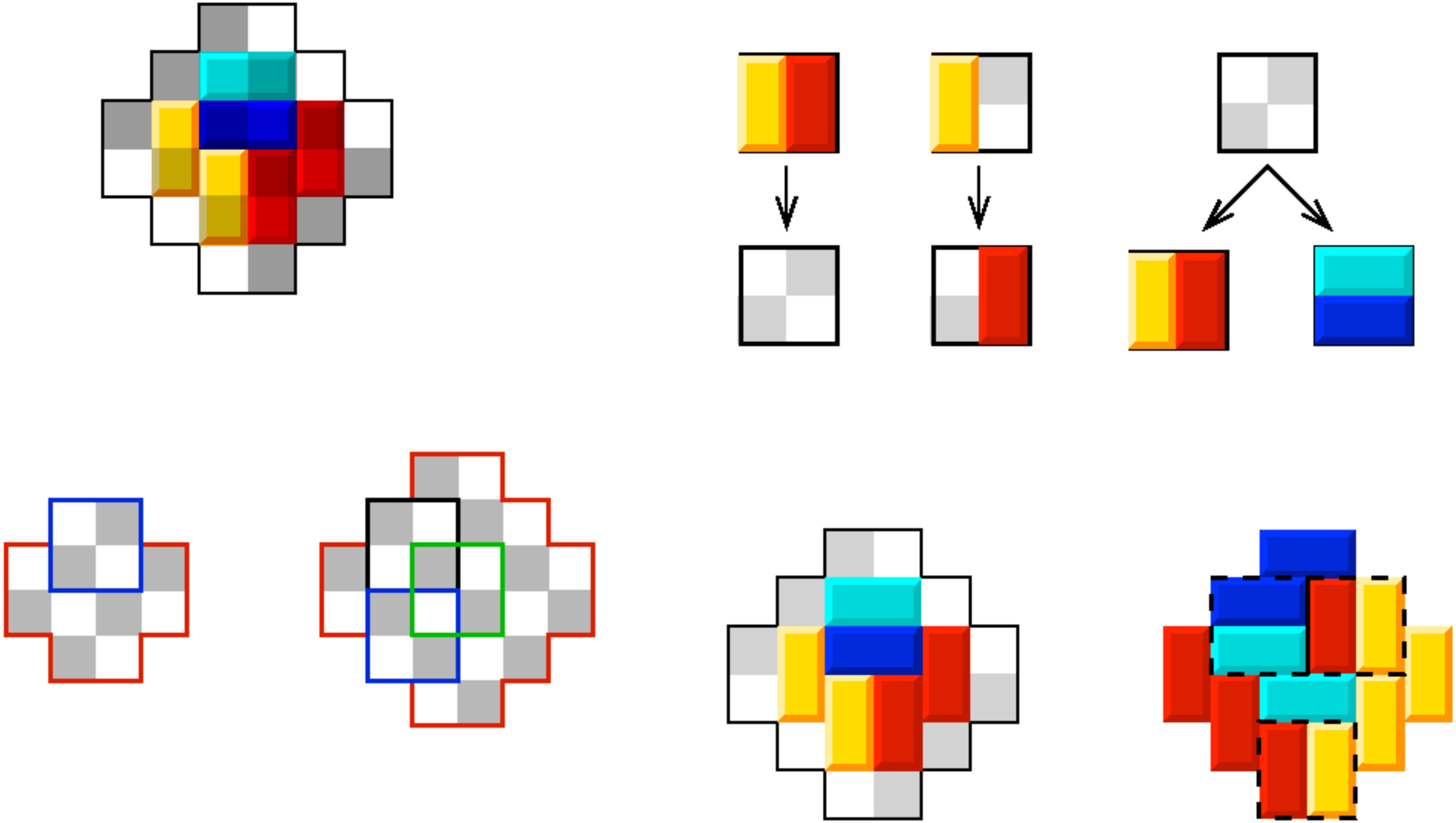


figure: Elise Janvresse et Thiery de la Rue combinatoire bijective

«preuves sans mots»

«Le paradigme bijectif»

$$y\left(1+14t+97t^2+415t^3+1180t^4+2321t^5+3247t^6+3300t^7+2475t^8+1375t^9+550t^{10}+143t^{11}+18t^{12}\right)+$$

série génératrice algébrique de la densité d'un gaz

$$y^2\left(1+17t+83t^2+601t^3+1647t^4+4606t^5+7809t^6+710t^7+124t^8-608t^9-440t^{10}-92t^{11}-36t^{12}\right)+$$

modèle des hexagones durs

$$y^3\left(3+50t+381t^2+1715t^3+5040t^4+10130t^5+14062t^6+13002t^7+6930t^8+715t^9-1595t^{10}-488t^{11}-198t^{12}\right)+$$

preuve bijective ?

$$y^4\left(1+17t+131t^2+595t^3+1765t^4+3574t^5+4939t^6+4356t^7+1815t^8-605t^9-1210t^{10}-616t^{11}-126t^{12}\right)$$

avec des empilements d'hexagones ?

$$=\left(t+11t^2+55t^3+165t^4+330t^5+462t^6+462t^7+330t^8+165t^9+55t^{10}+11t^{11}+t^{12}\right)$$

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Elise Janvresse et Thierry de la Rue ,

« Pavages aléatoires par touillage de dominos » —

Images des Mathématiques, CNRS, 2009.

http://images.math.cnrs.fr/Pavages-aleatoires-par-touillage.html

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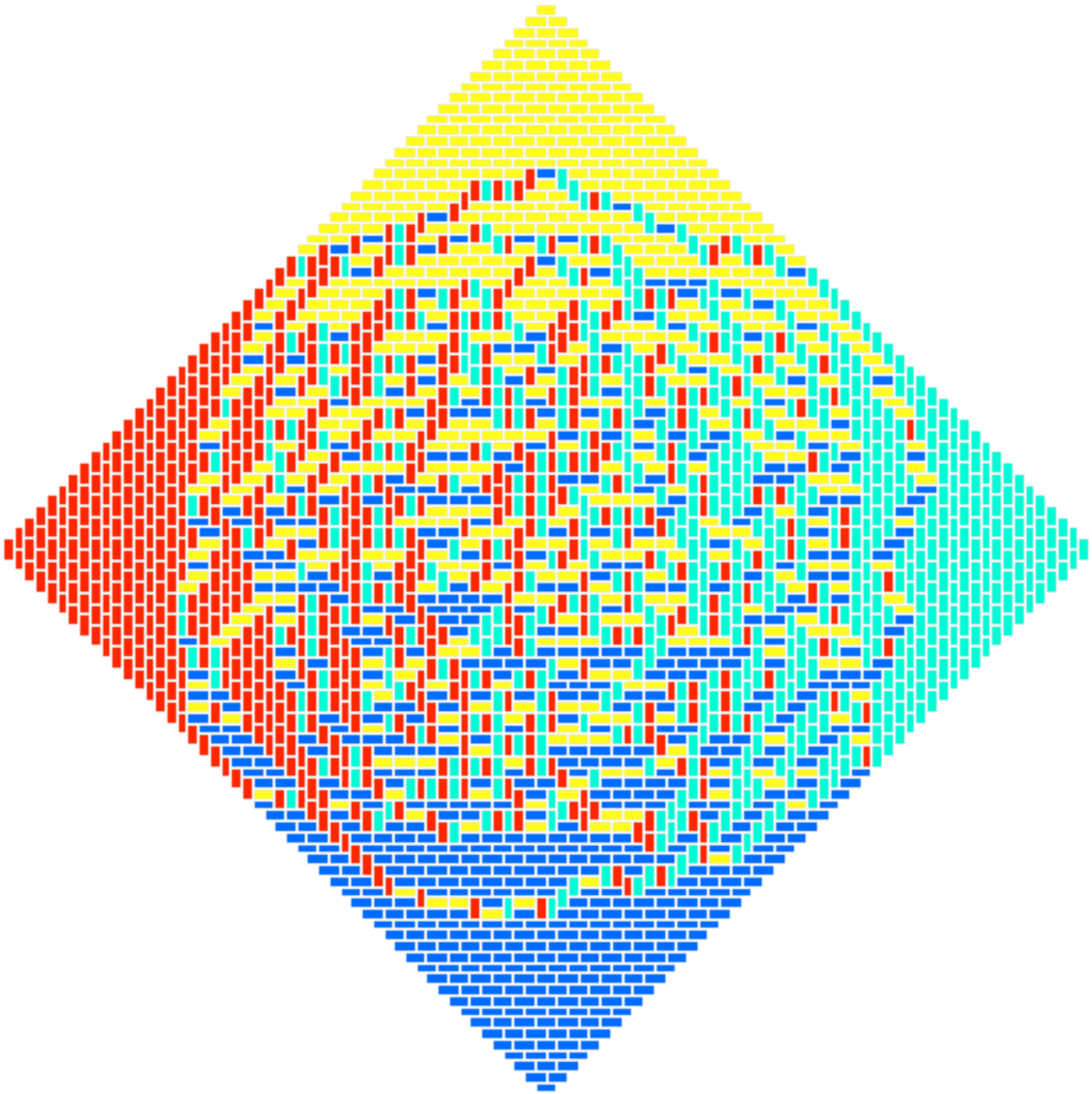
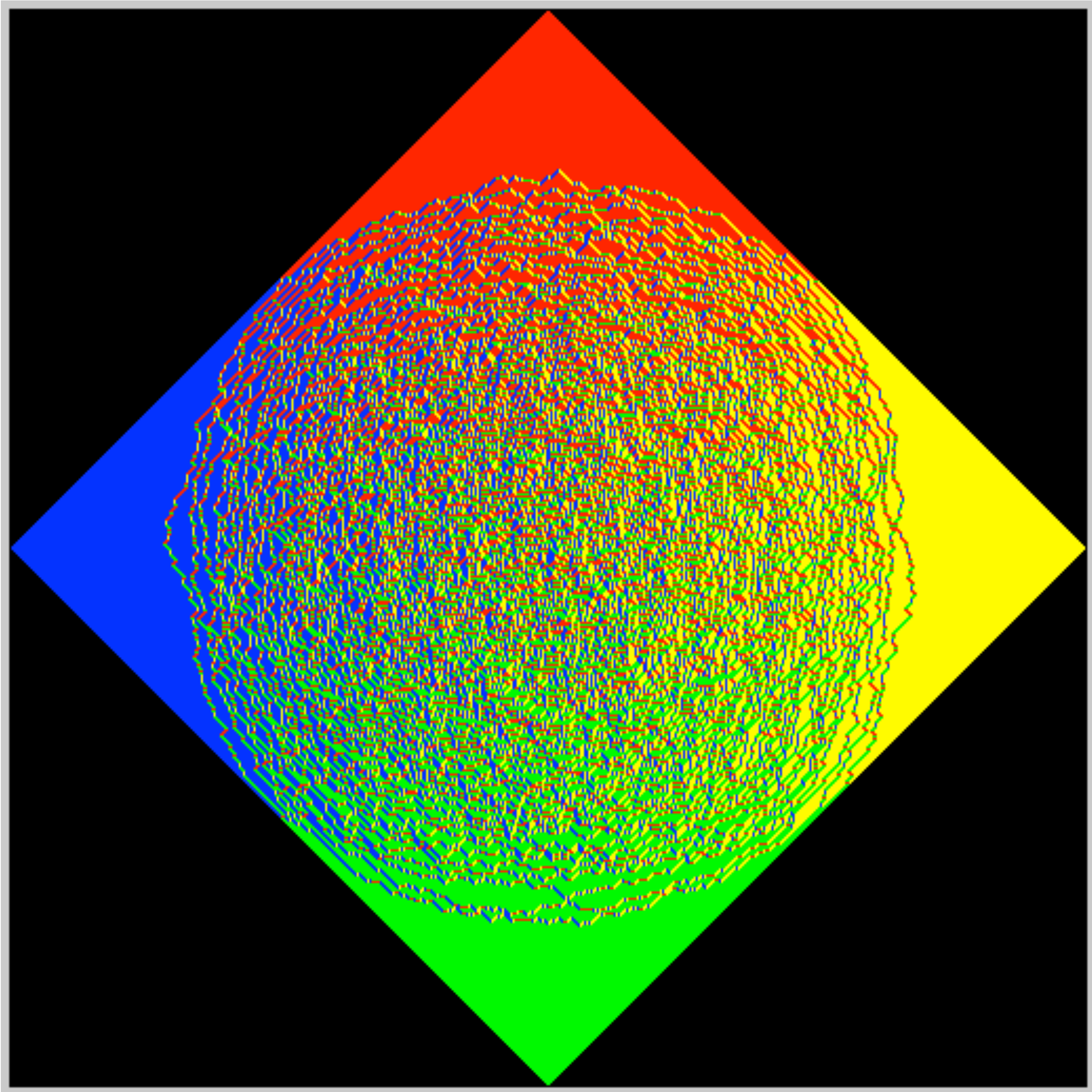
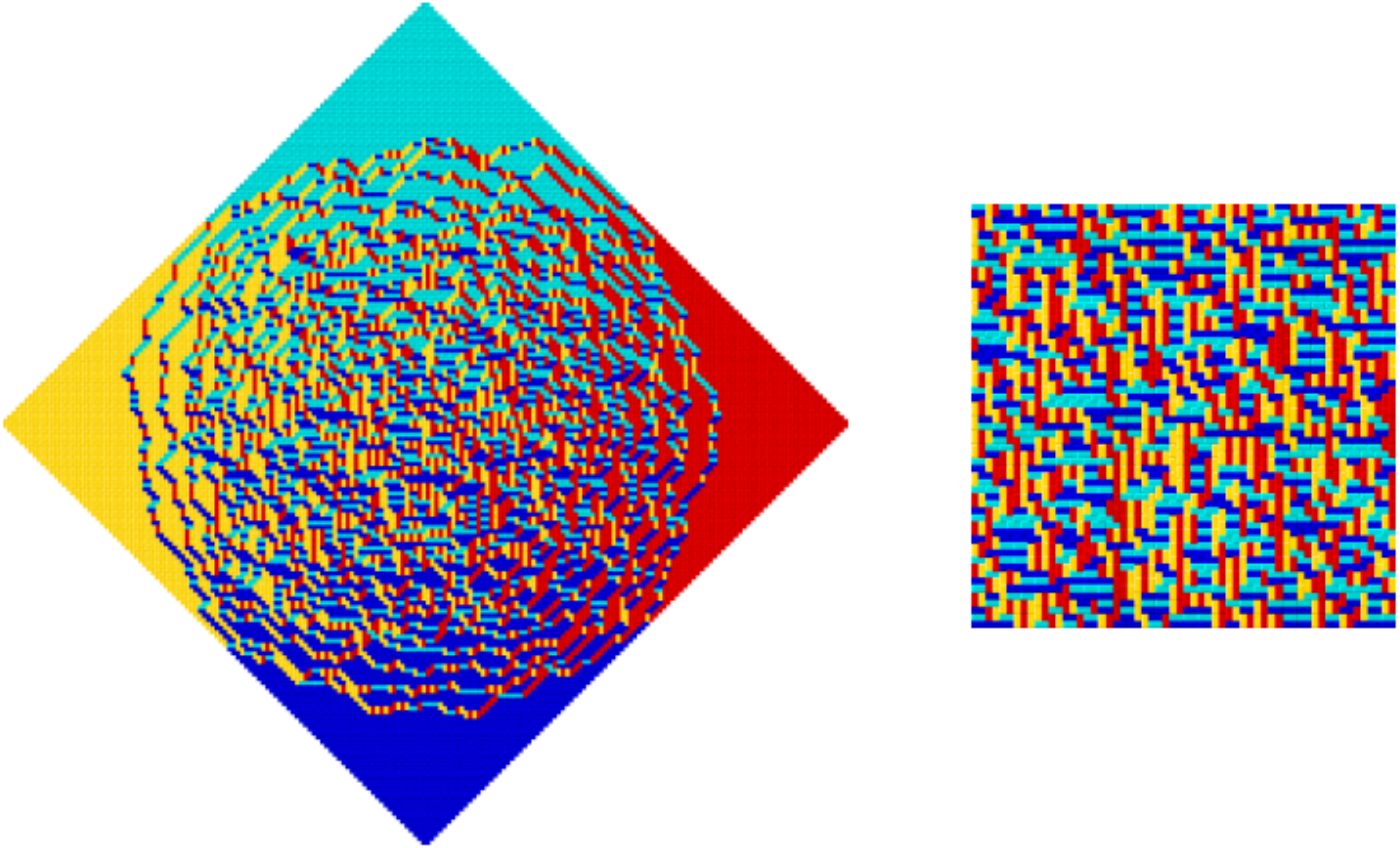
pavage
Aztèque
aléatoire



le
théorème
du
«arctique
cercle»



Joskush
Propp
Shor
(1998)



Effets de bords
À gauche : un pavage typique du diamant aztèque d'ordre 100, sur lequel on observe le phénomène du cercle arctique. À droite : un pavage typique du carré de côté 60.

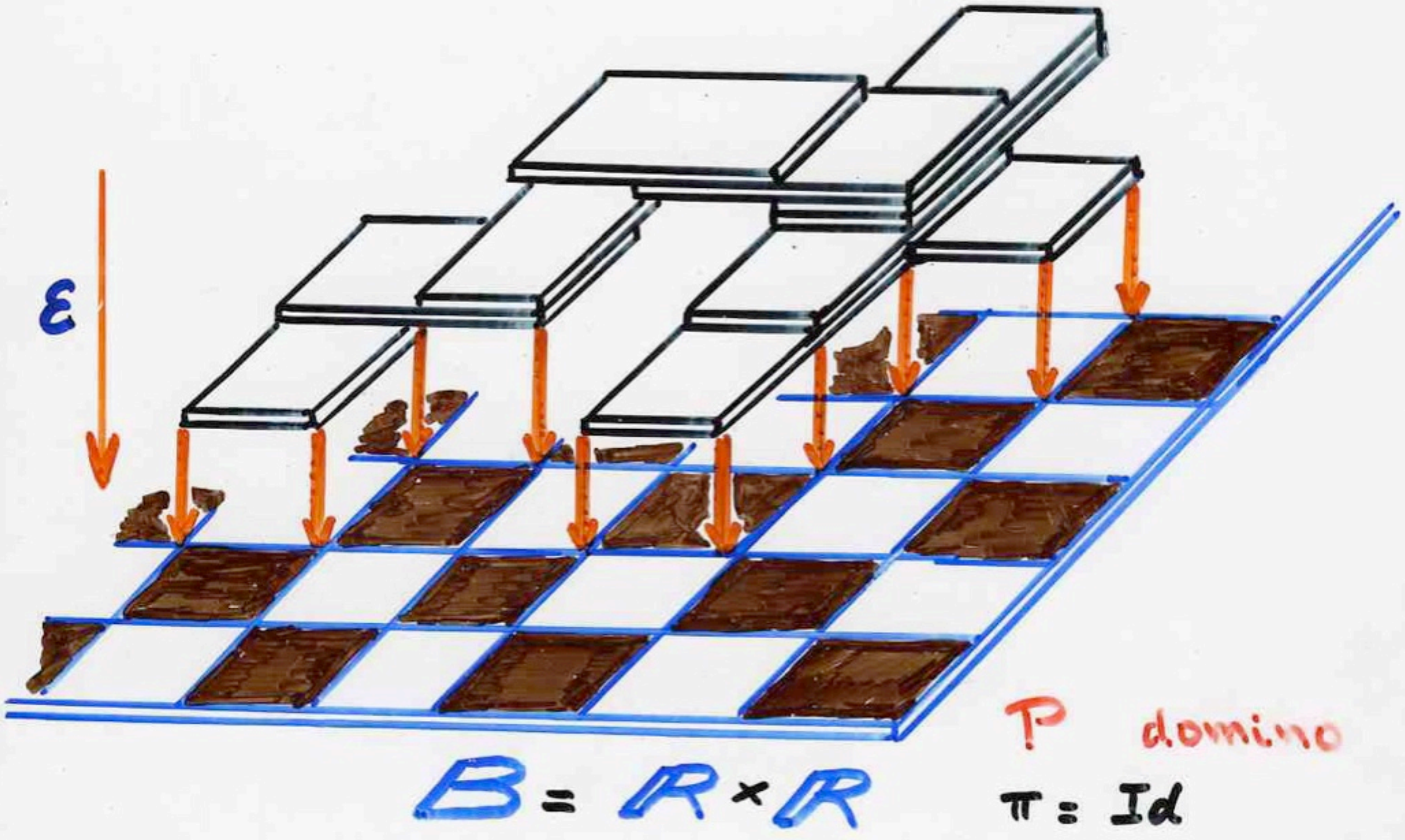
Compléments

Pour tout graphe G ayant n sommets,

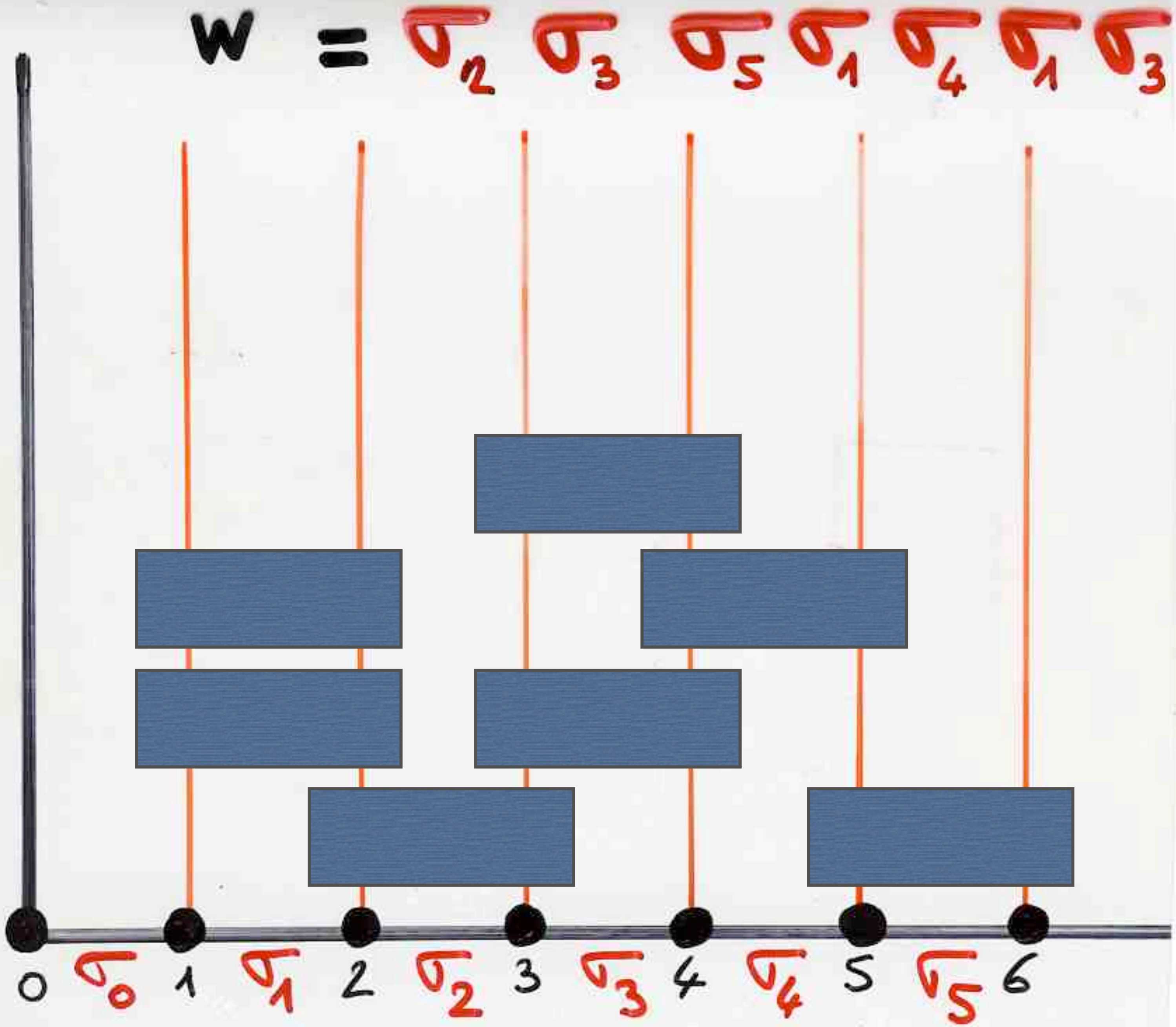
le polynôme de couplage associé a

toutes ses n racines réelles

Empilements de dominos

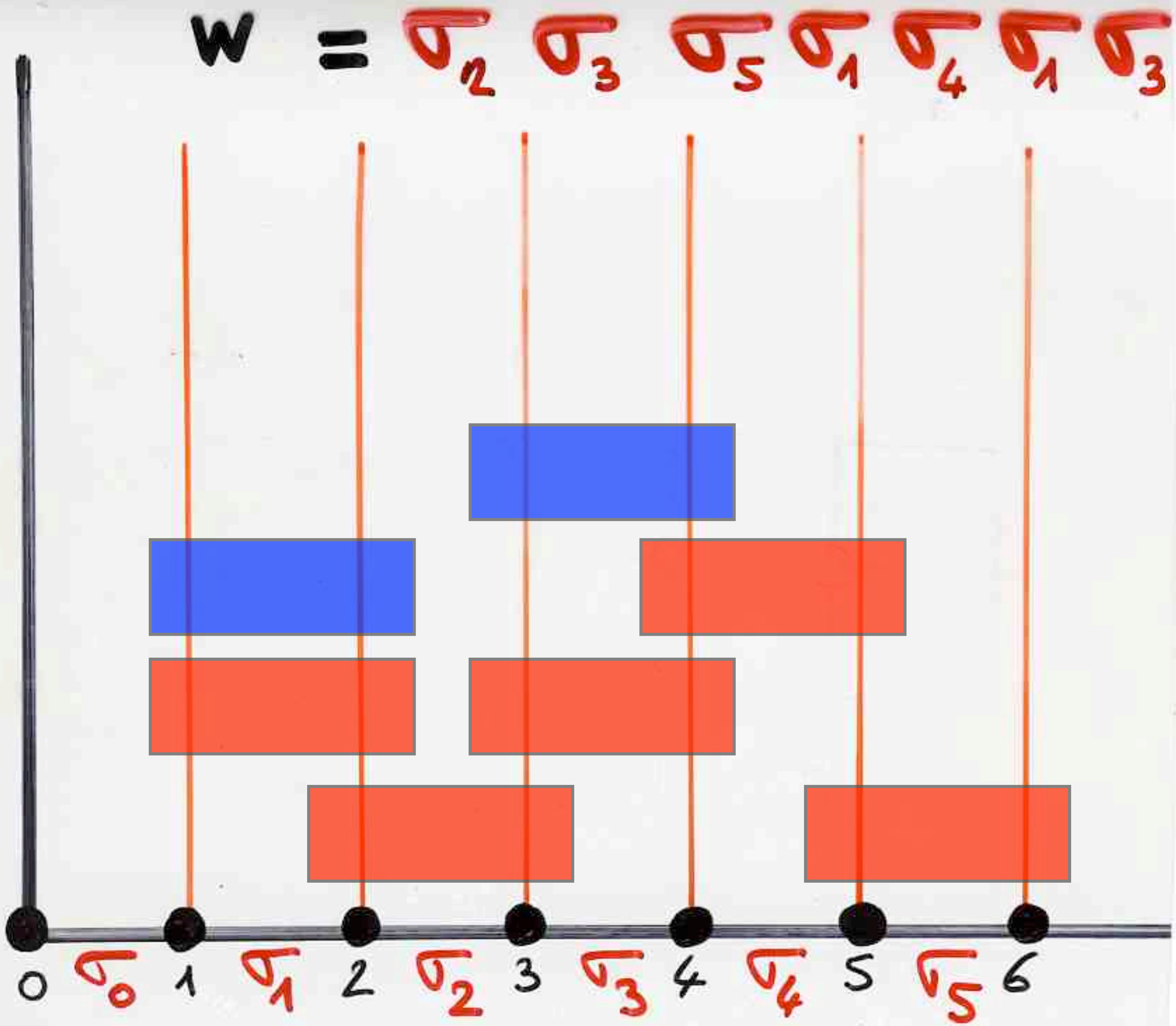


ε
P domino
B = ℝ × ℝ
π = Id



chant:
Bombay
S.Jayashri

chant:
Bombay
S.Jayashri

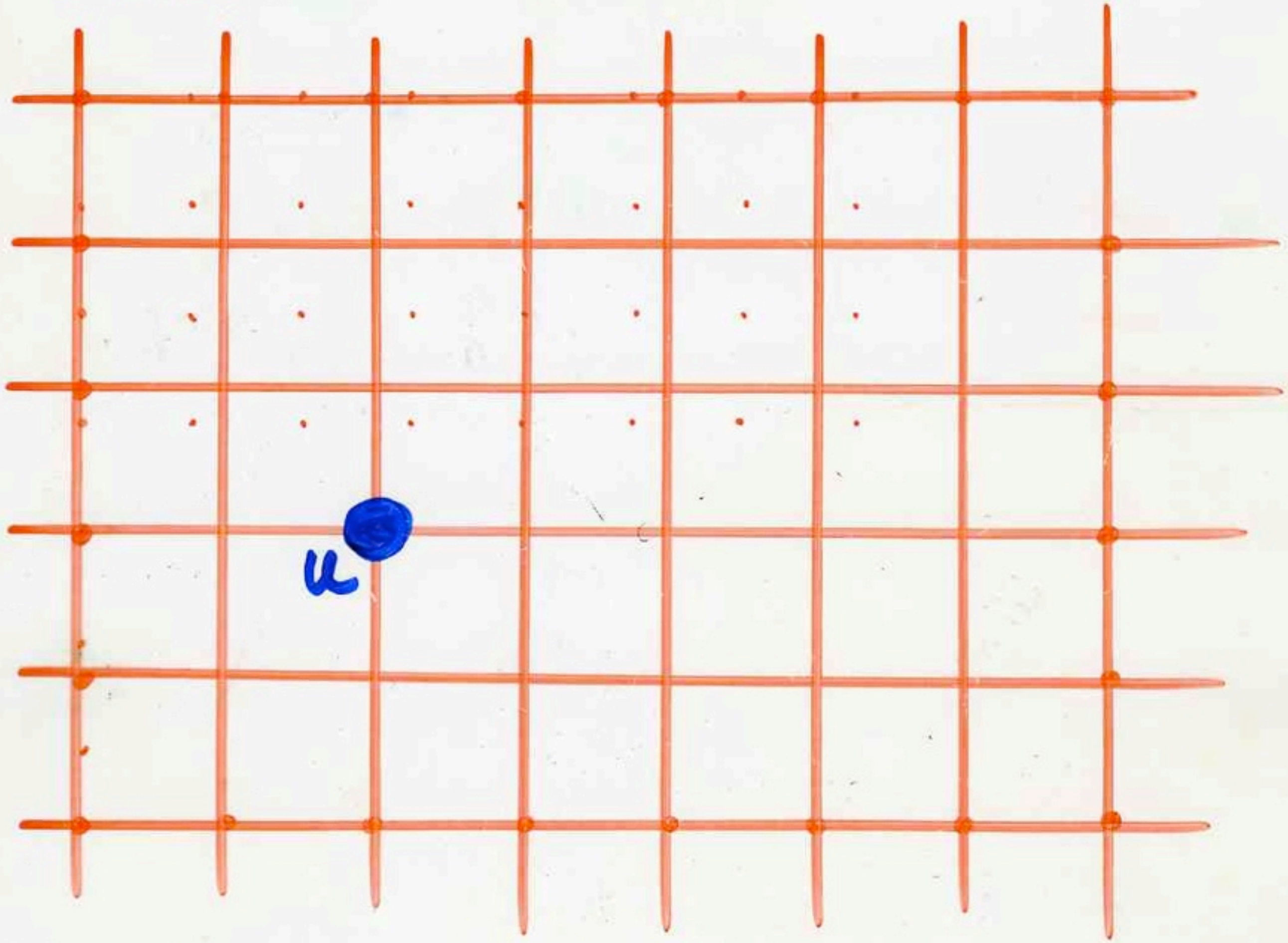


Pyramid

Def- Heap having only
one maximal piece

Tree-like paths on a graph G

Godsil (1981)



Tree-like paths on a graph G

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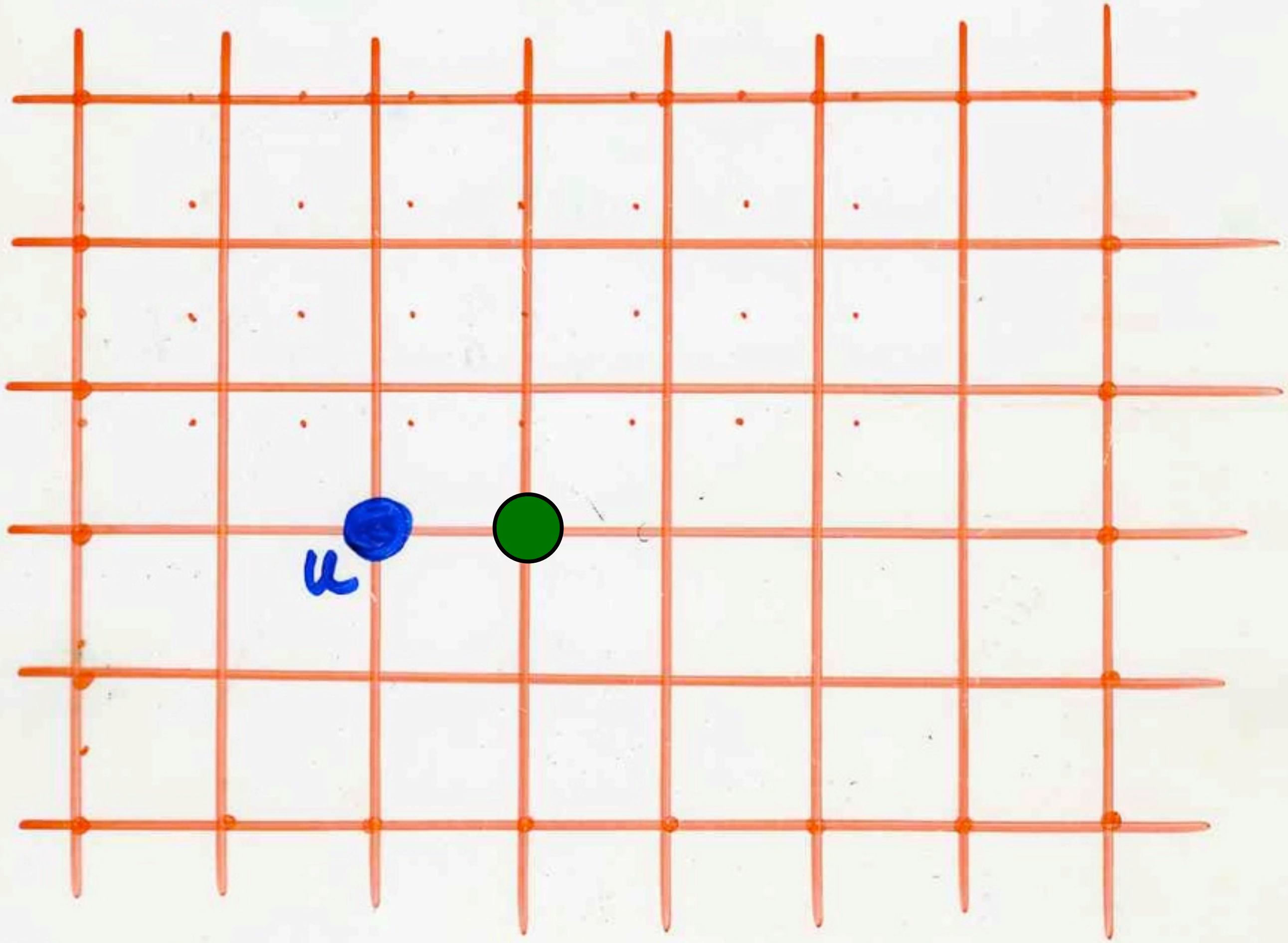

Tree-like paths on a graph G

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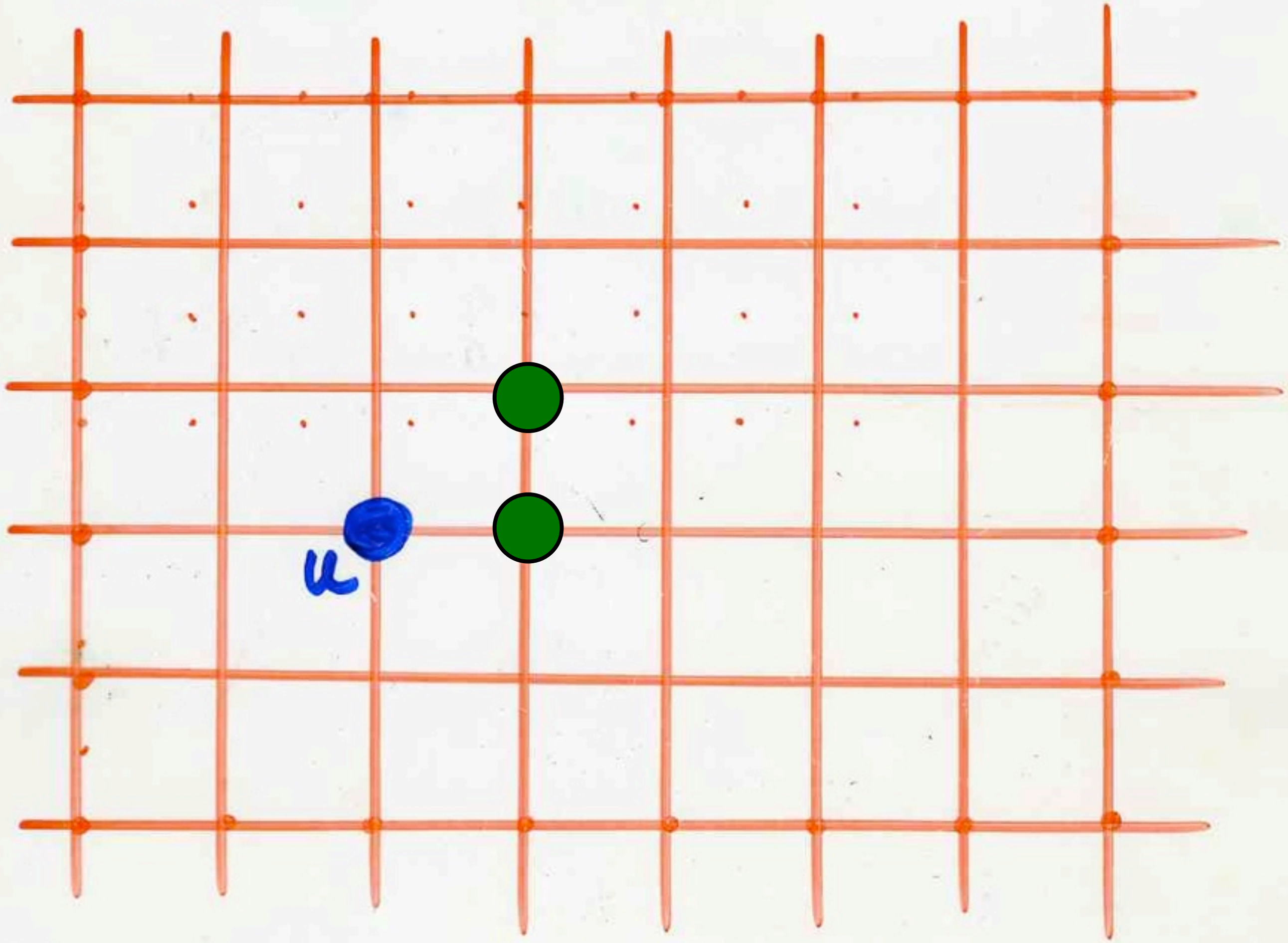

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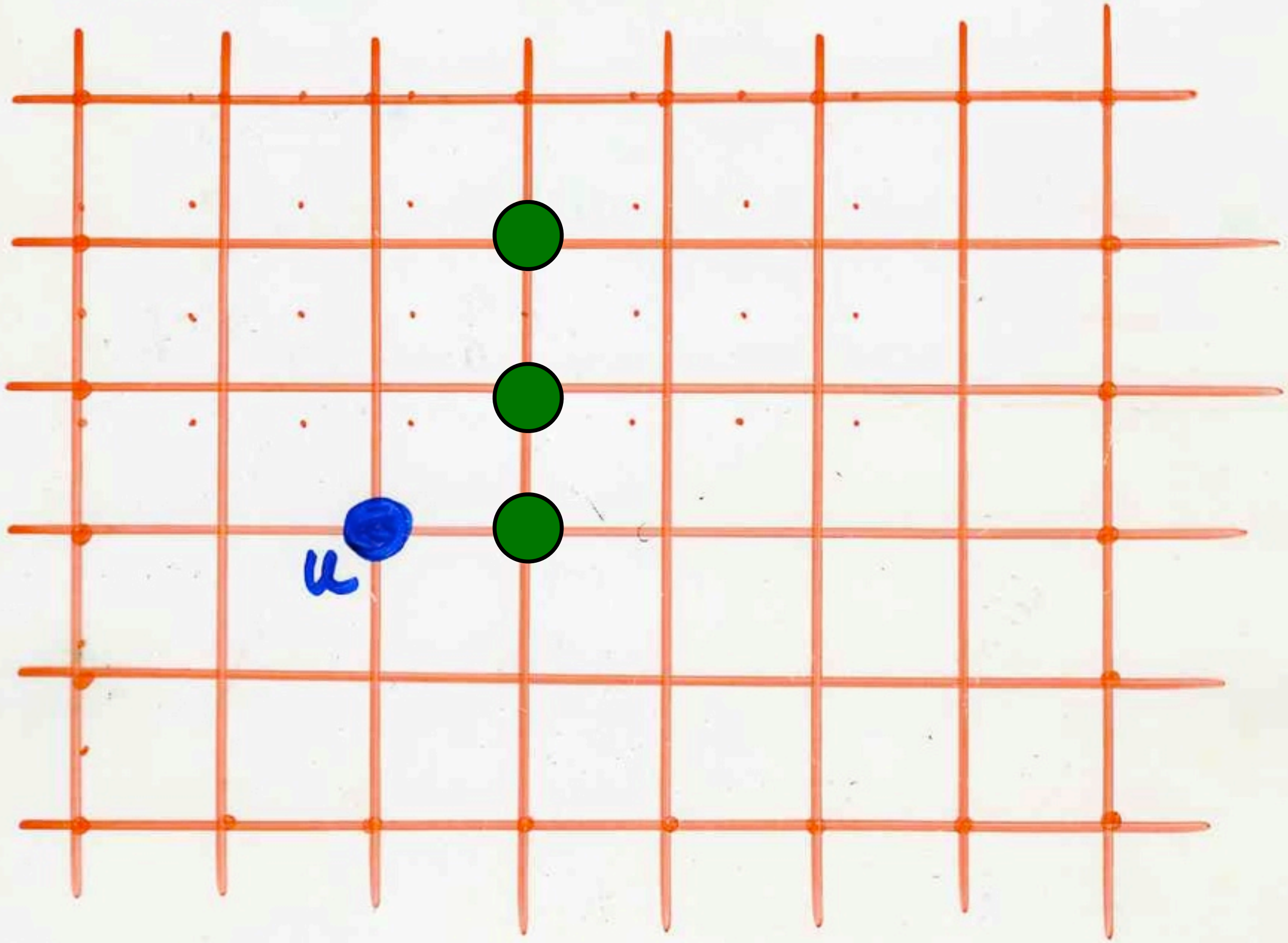

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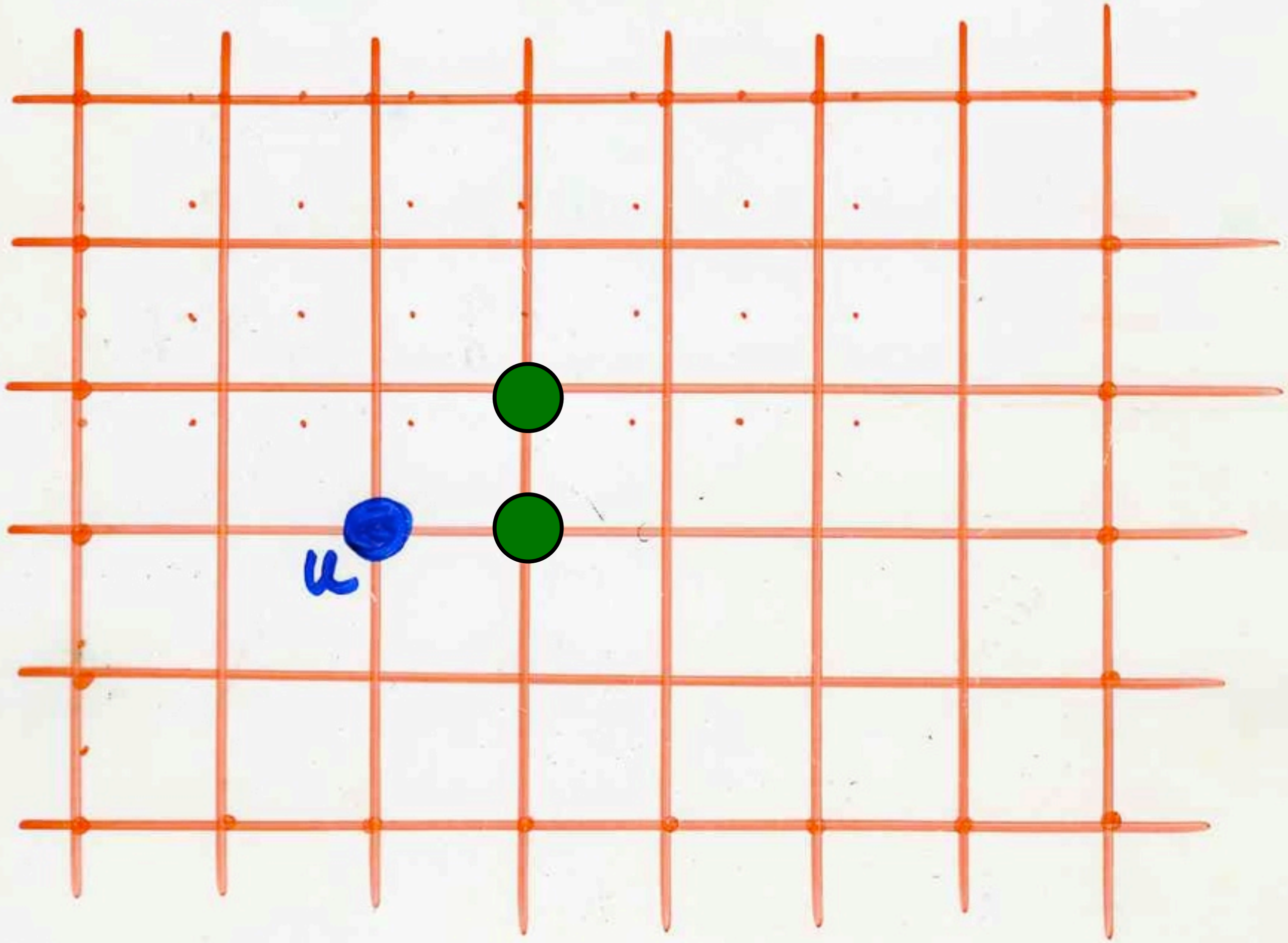

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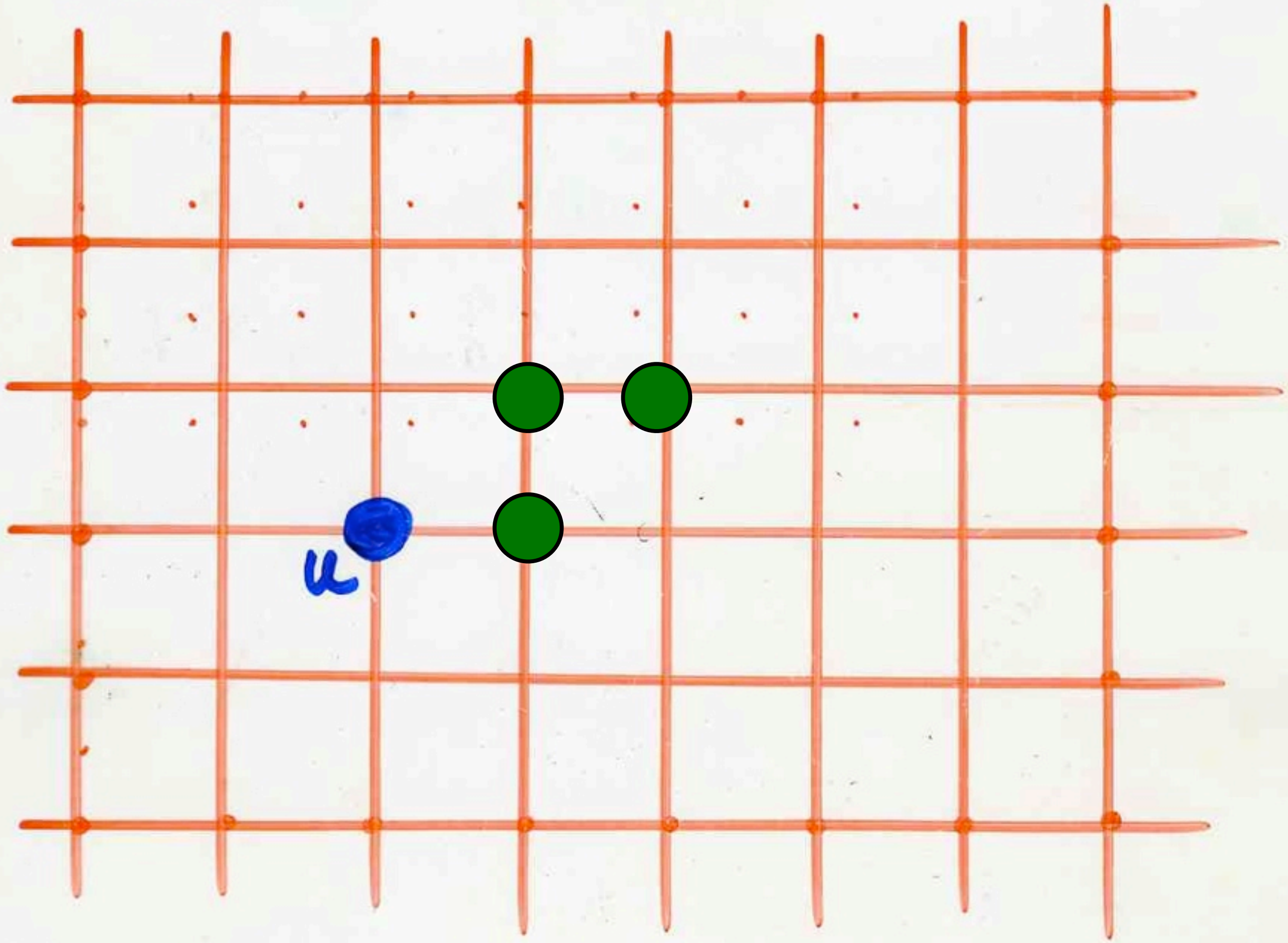

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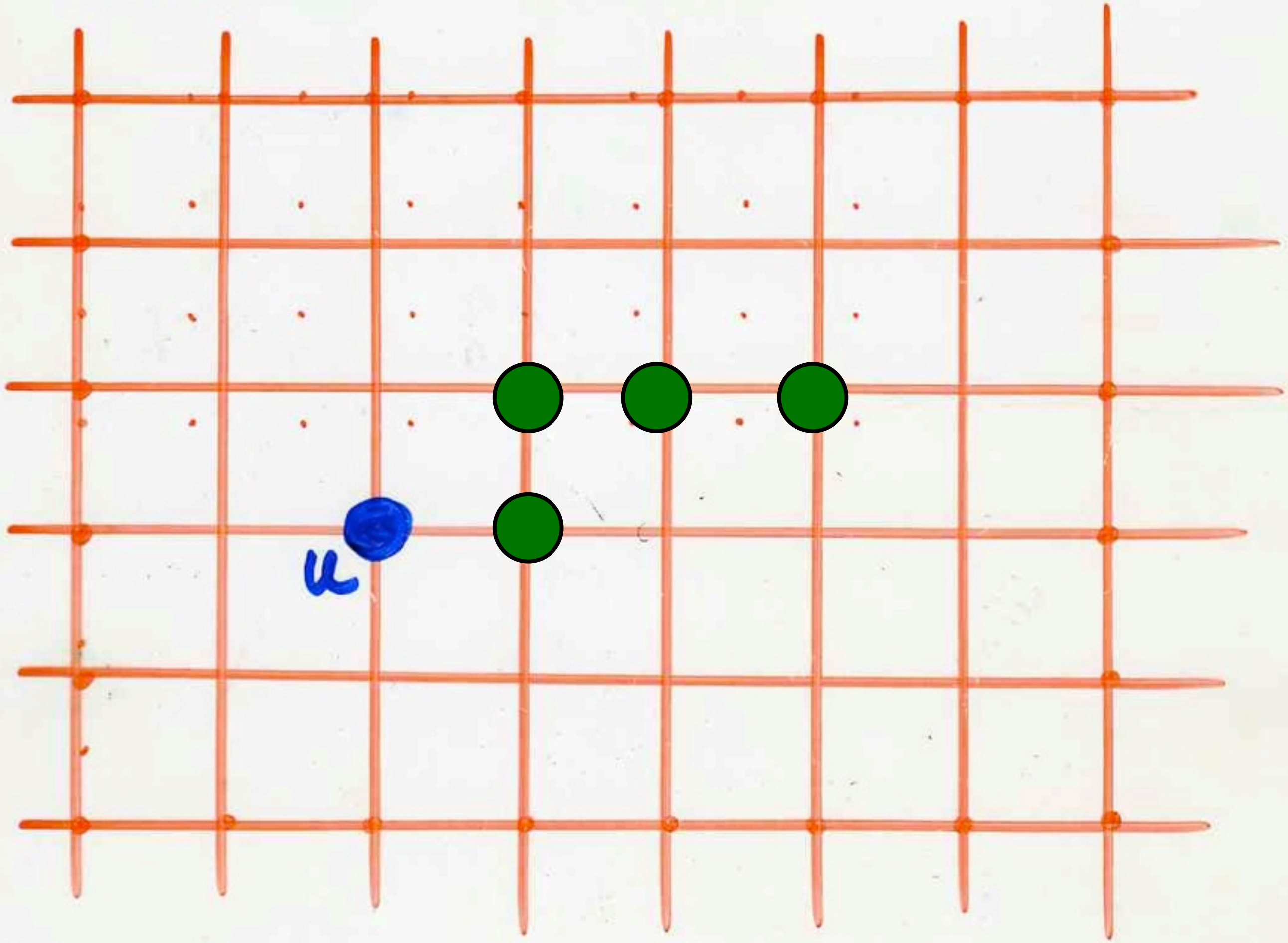

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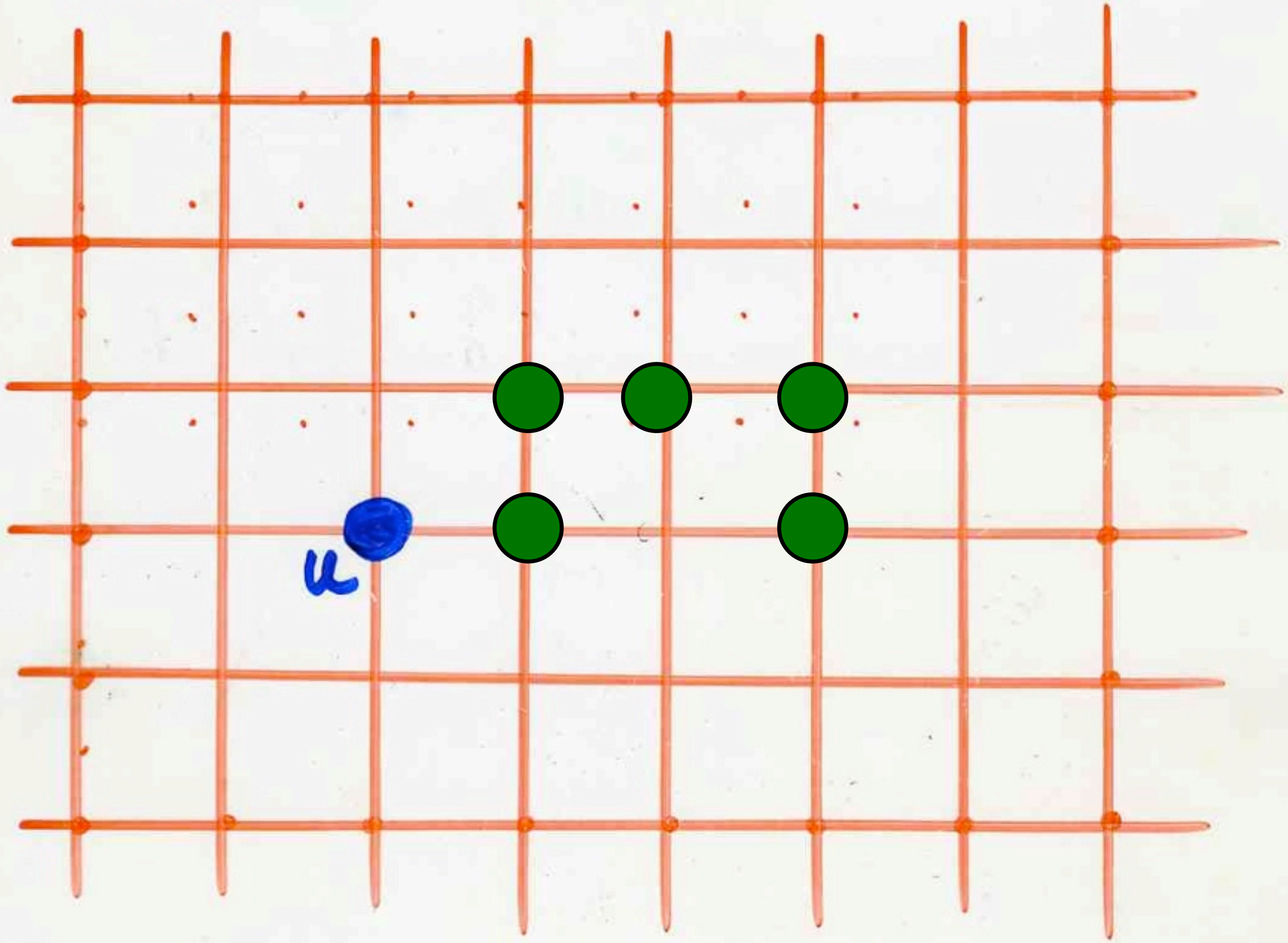

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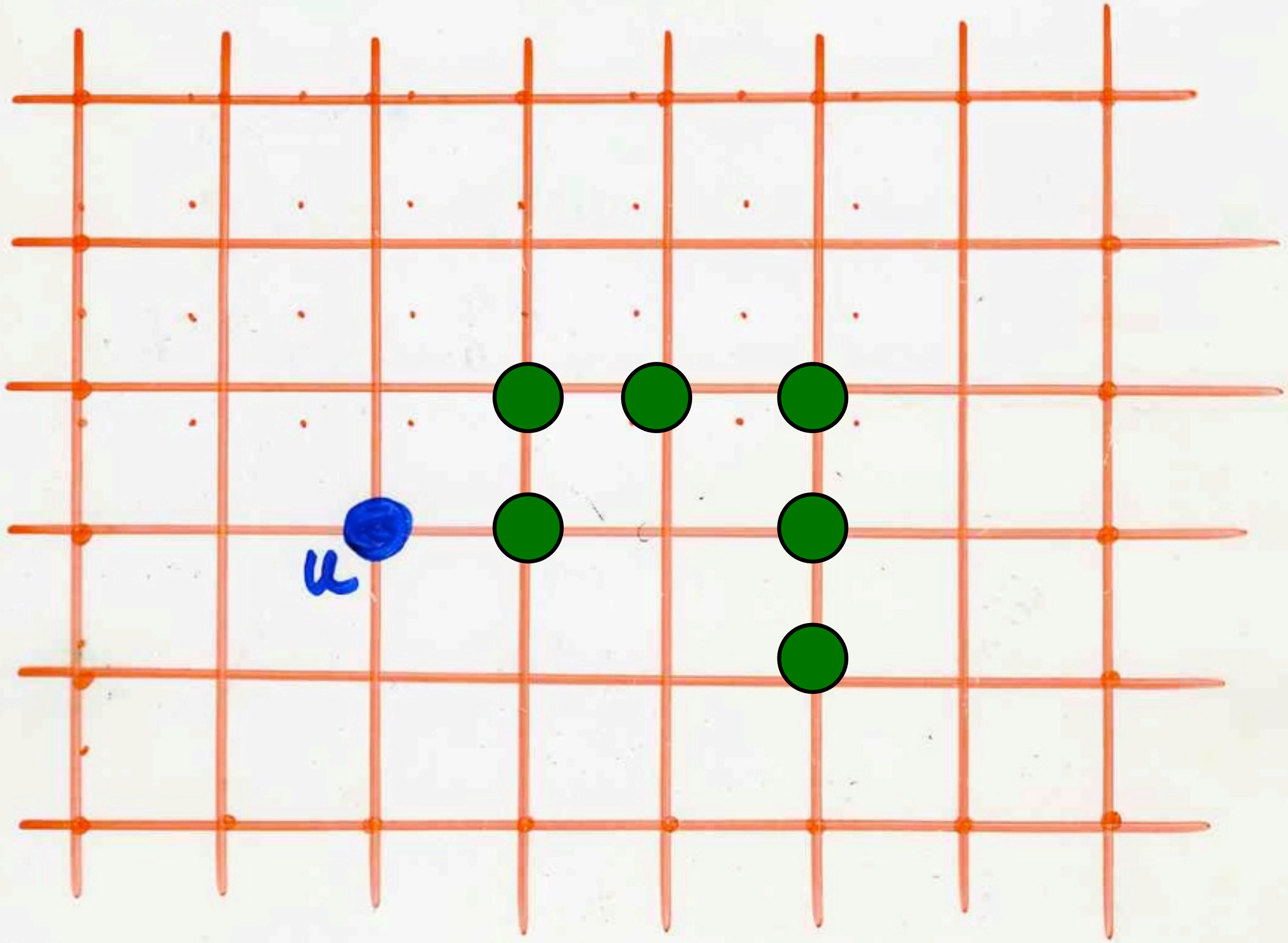

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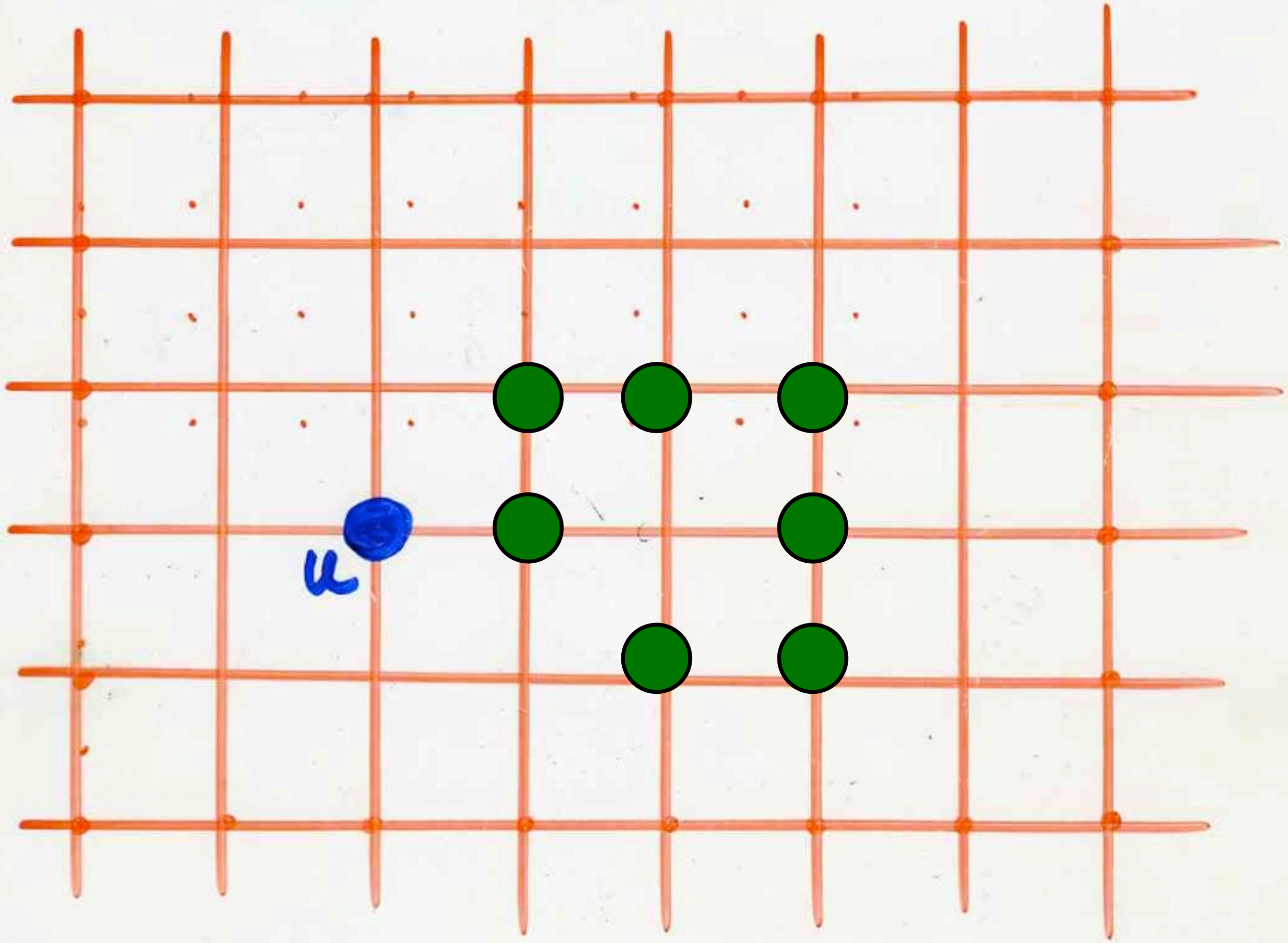

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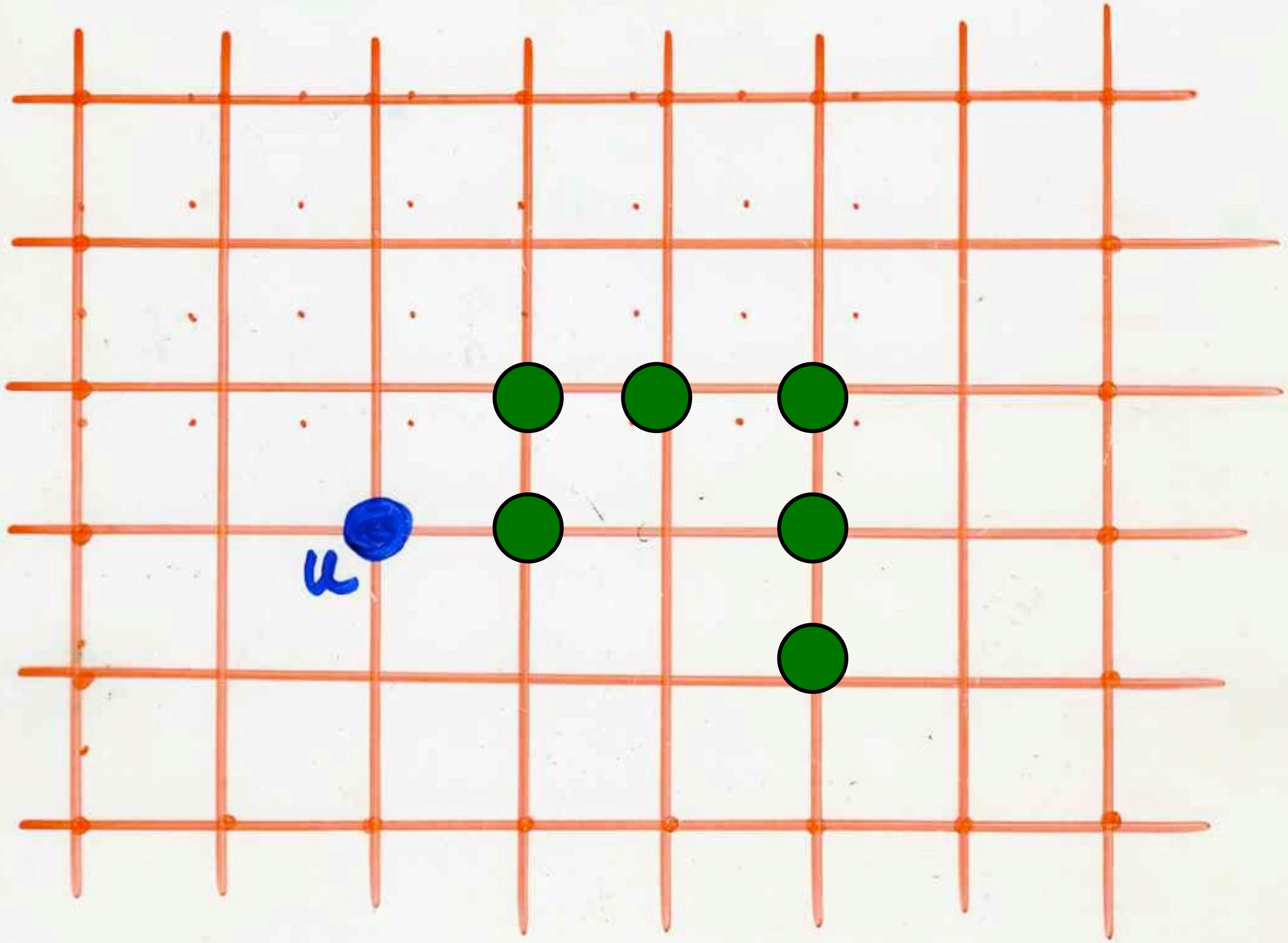

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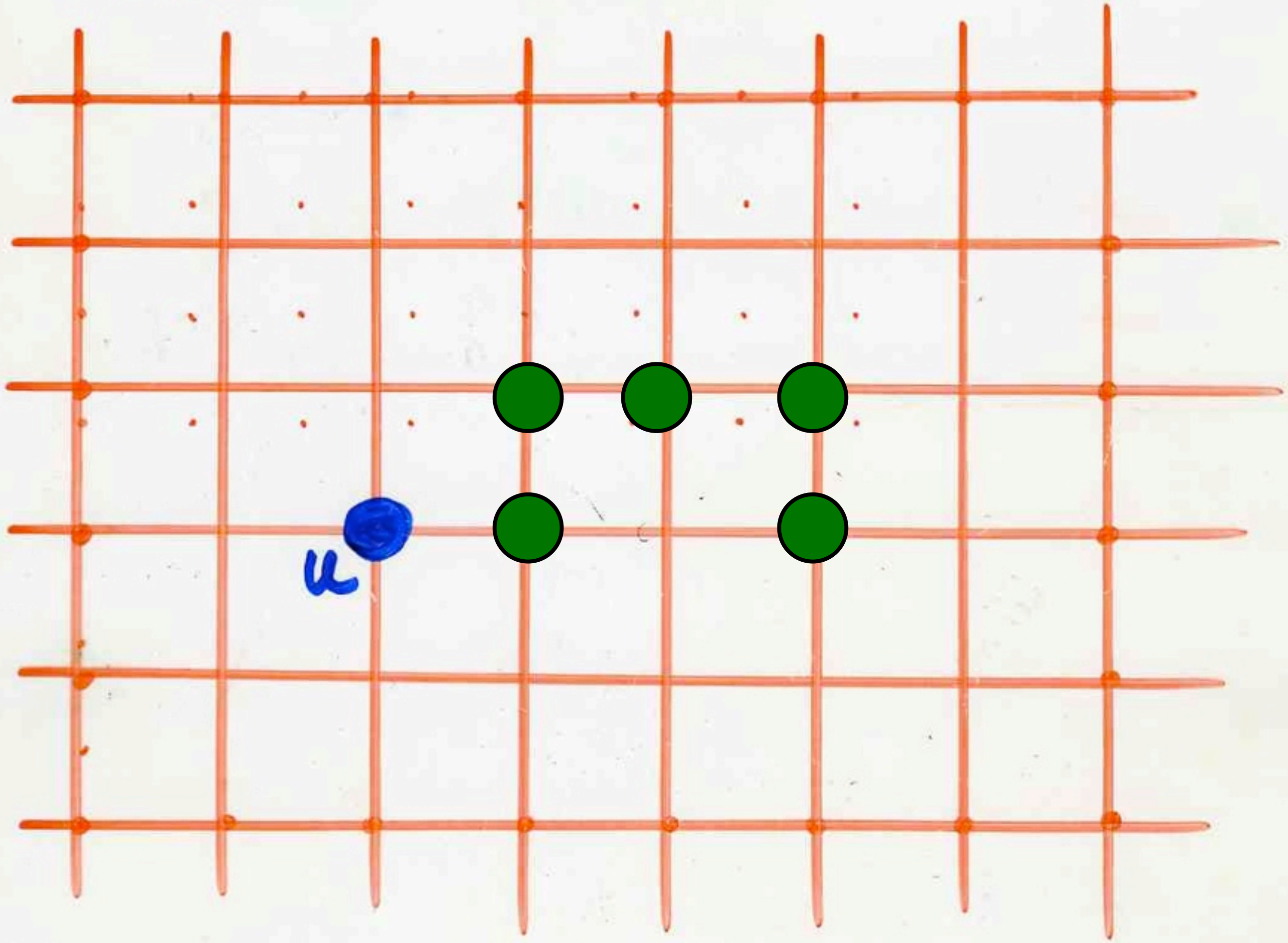
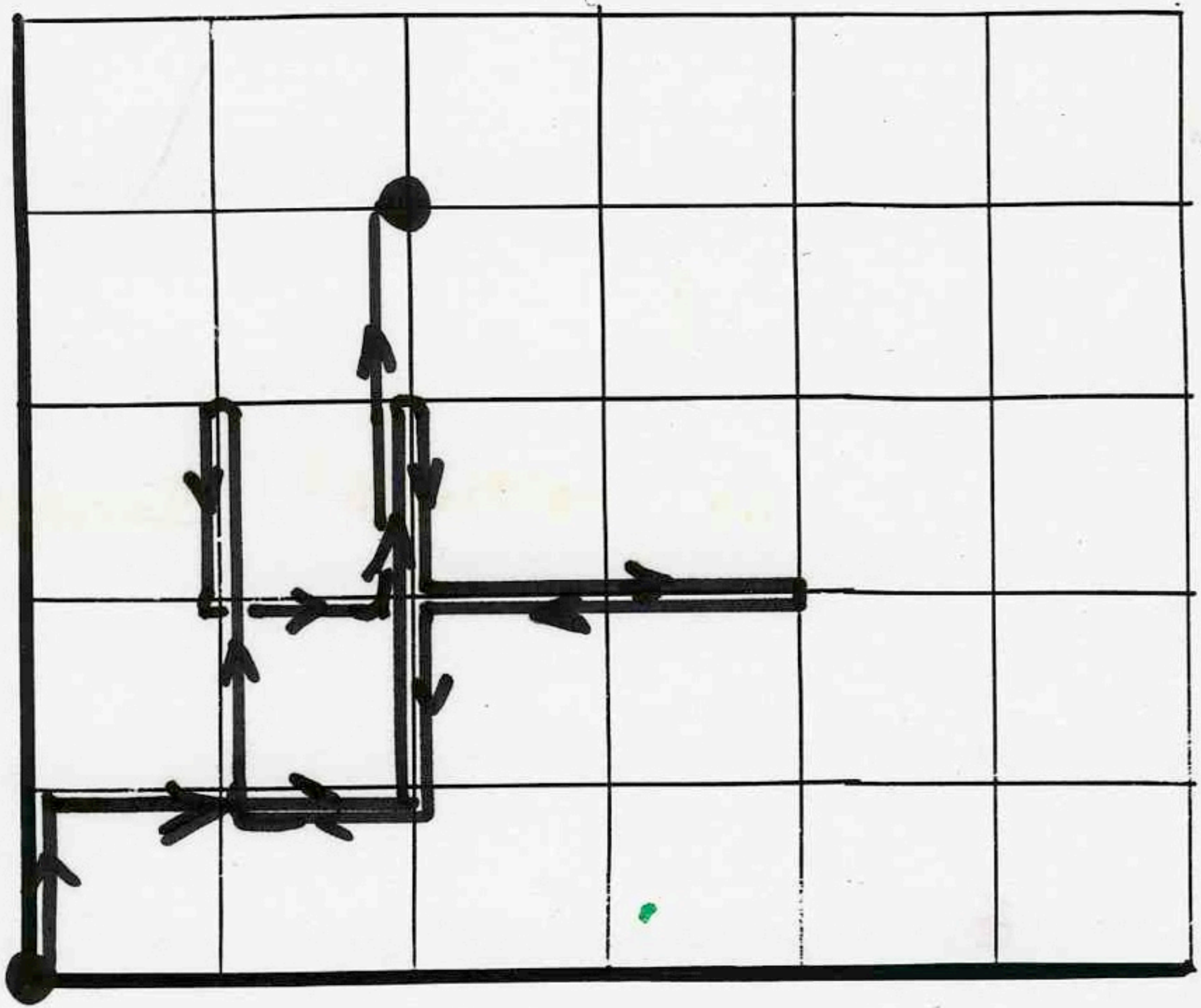
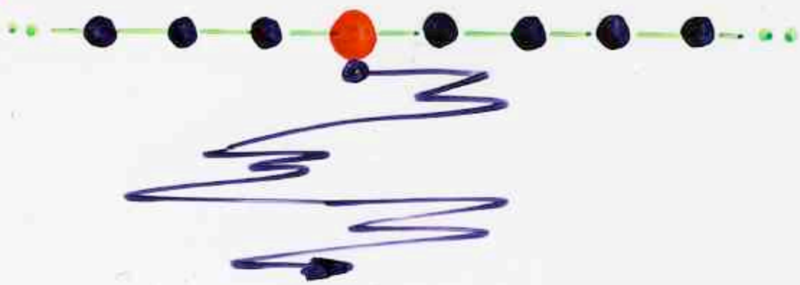




Particular cases.

- **Dyck** path

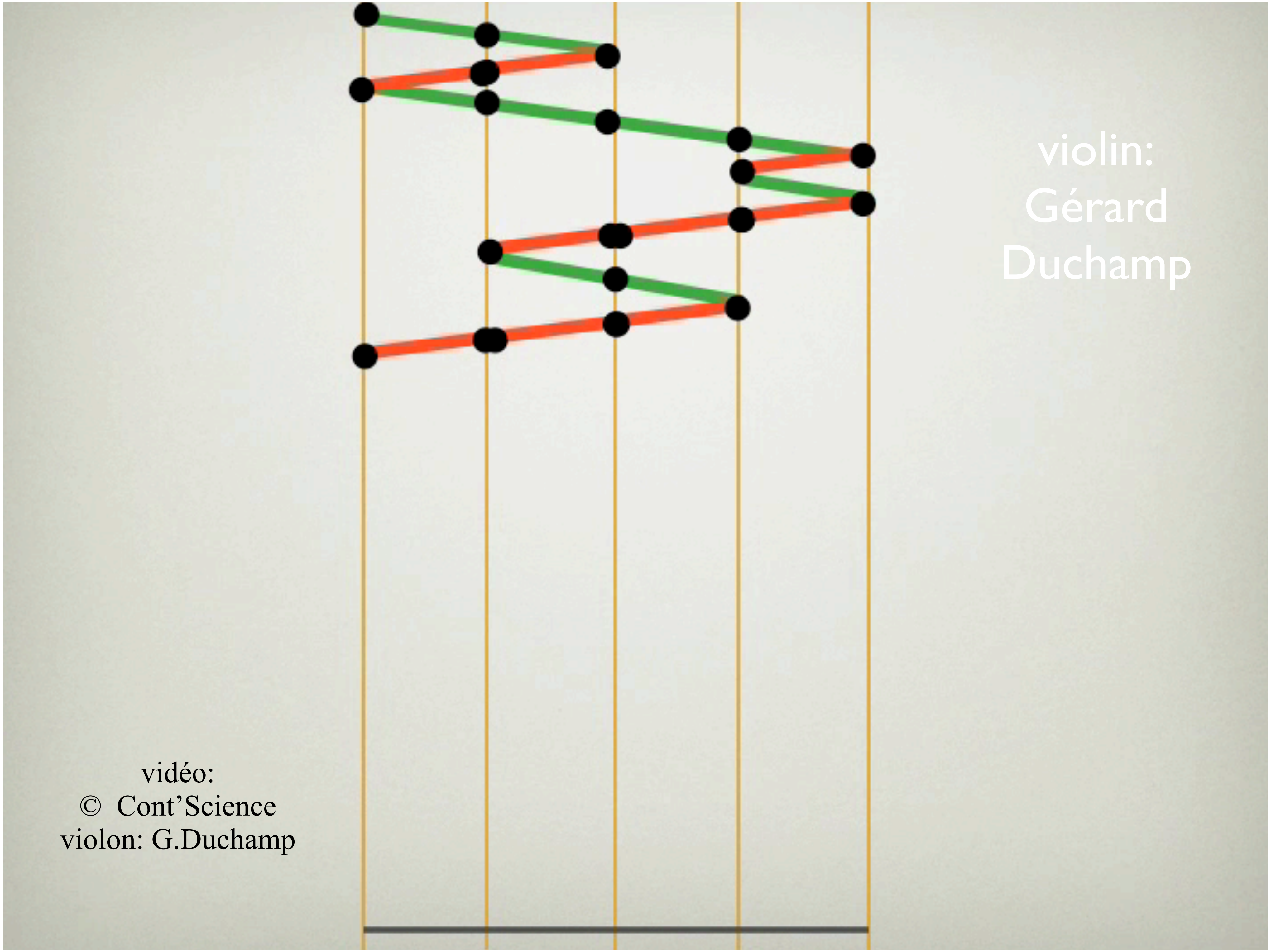
- bilateral **Dyck** paths



bijection

chemins de Dyck

pyramide de dominos



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Merci !