The combinatorics of some exclusion model in physics

Orsay
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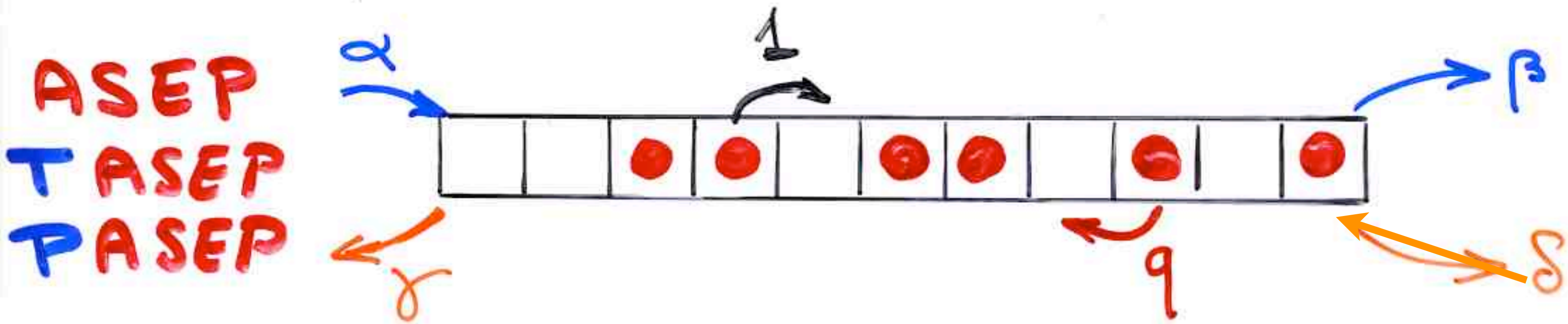
The PASEP

Partialy asymmetric exclusion process


ASEP
TASEP
PASEP
α
1
β
γ
q
δ

non-equilibrium statistical mechanics

... relaxation → stationary state

states $\tau = (\tau_1, \tau_2, \dots, \tau_n)$

$$\tau_i = \begin{cases} 1 & \text{site } i \text{ occupied} \\ 0 & \text{site } i \text{ empty} \end{cases}$$

unique stationary state

$$\frac{d}{dt} P_n(\tau_1, \dots, \tau_n) = 0$$

Derrida, Evans, Hakim, Pasquier (1993)

boundary induced phase transitions

molecular diffusion

linear array of enzymes

biopolymers

traffic flow

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formation of shocks

$$\mathbb{P}_n(\tau_1, \dots, \tau_n) = \mathcal{f}_n(\tau_1, \dots, \tau_n) / Z_n$$

$$Z_n = \sum_{\tau} \mathcal{f}_n(\tau_1, \dots, \tau_n)$$

partition function

The Matrix Ansatz

Derrida, Evans, Hakim, Pasquier (1993)

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

$$\begin{cases} DE = qED + D + E \\ (\beta D - \delta E)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \gamma D) = \langle W| \end{cases}$$

Then

$f_n(\tau_1, \ldots, \tau_n)$

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

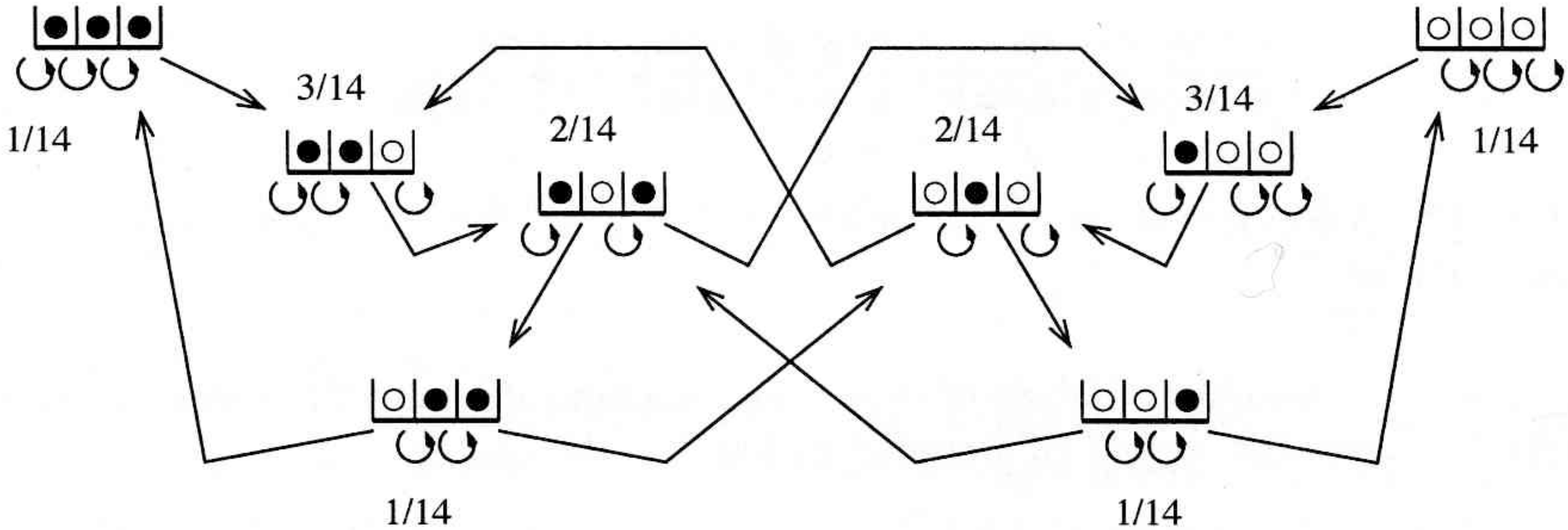
TASEP

q=0

$$\begin{cases} DE = \boxed{} + D + E \\ (\beta D - \boxed{})\,|V\rangle = |V\rangle \\ \langle W|\,(\alpha E - \boxed{}) = \langle W| \end{cases}$$

Then

$$f_n(\tau_1, \dots, \tau_n) = \langle W | \prod_{i=1}^{n} \left(\tau_i D + (1-\tau_i) E\right) | V \rangle$$


3/14
1/14
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2/14
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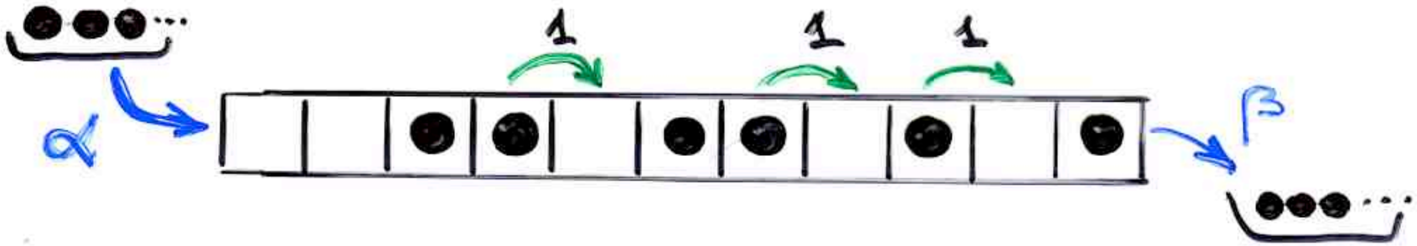
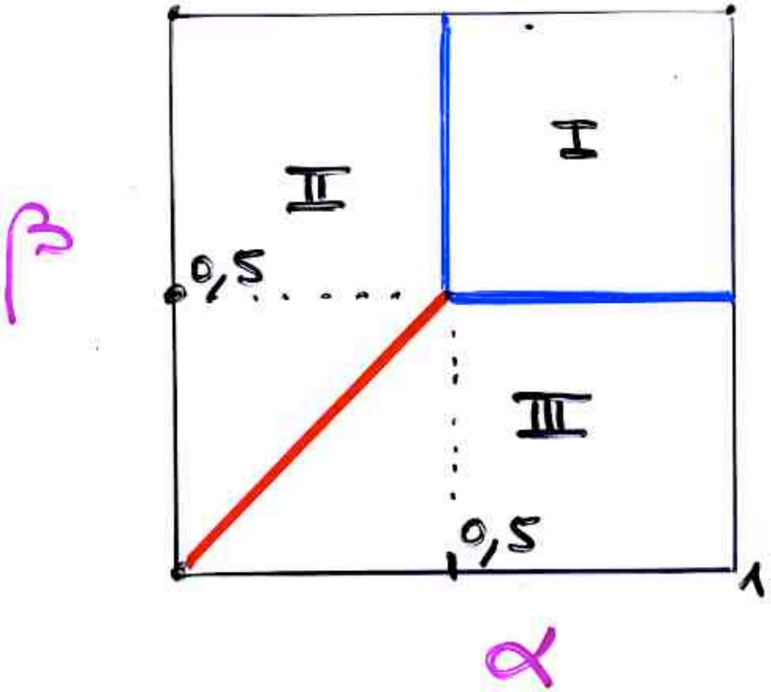
1/14

TASEP

"totally asymmetric exclusion process"





$n \to \infty$

$\rho = \langle \tau_i \rangle$ = taux moyen d'occupation

i loin des bords

(I) $\rho = 1/2$

(II) $\rho = \alpha$

(III) $\rho = 1-\beta$

transitions de phase

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

$\alpha, \beta, q \qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

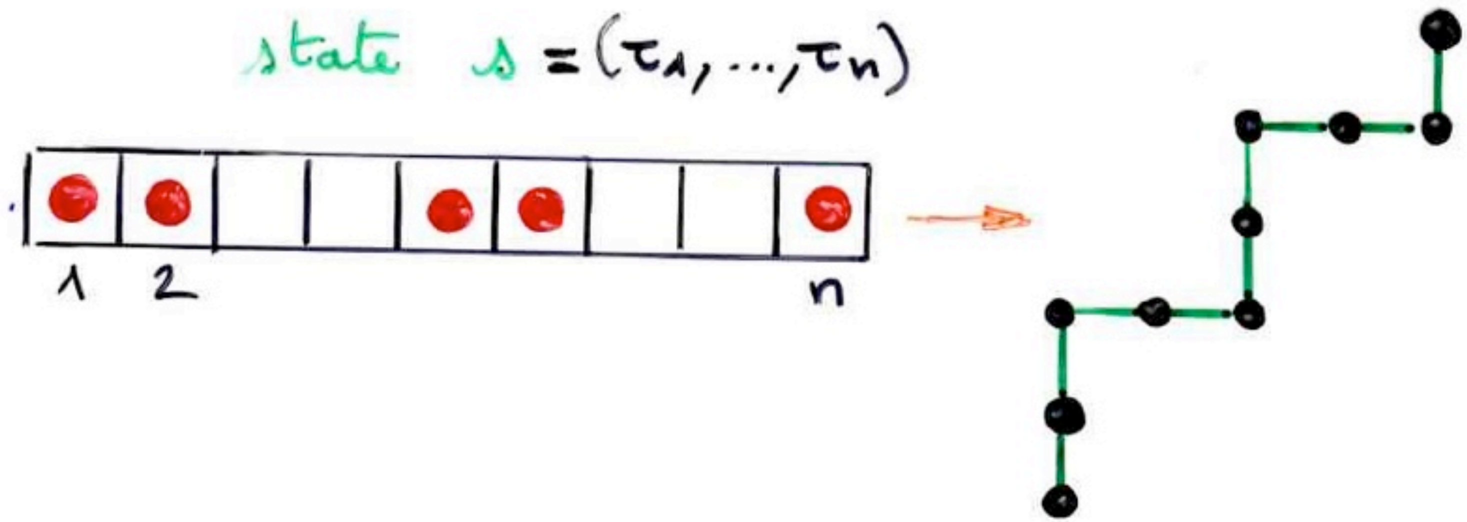
$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \qquad q$

Askey-Wilson polynomials

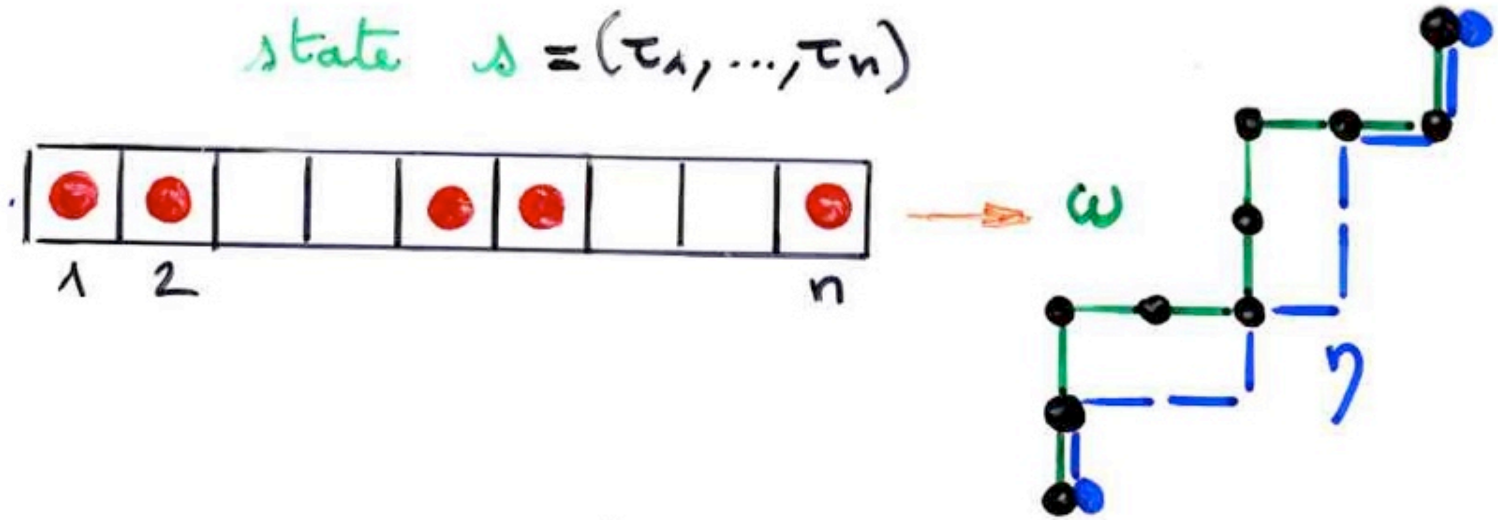
Combinatorics of the TASEP



$\mathcal{P}_n(s) =$

Shapiro, Zeilberger, 1982

Combinatorics of the TASEP



$$P_n(s) = \frac{1}{C_{n+1}} \left(\begin{array}{c} \text{number of paths } \eta \\ \text{below the path } \omega \\ \text{associated to } s \end{array} \right)$$

Shapiro, Zeilberger, 1982

TASEP

Shapiro, Zeilberger (1982)
Brak, Essam (2003), Duchi, Schaeffer, (2004),
Angel (2005), XGV (2007)

(P) ASEP

Brak, Corteel, Essam, Parviainen, Rechnitzer (2006)
Corteel, Williams (2006,..., 2010)
Corteel, Stanton, Stanley, Williams (2011)
Josuat-Vergès (2008,..., 2010)

Derrida, ...
Mallick, ... Golinelli, Mallick (2006)

....

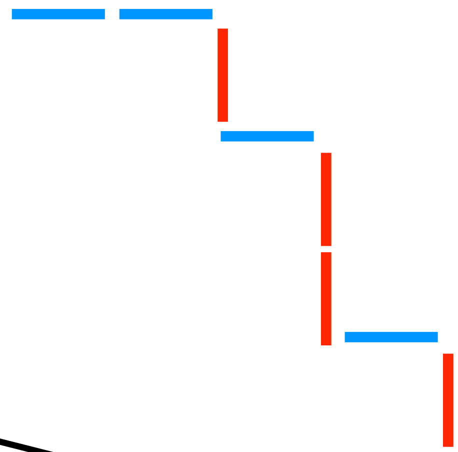
The PASEP algebra

$$DE = qED + E + D$$

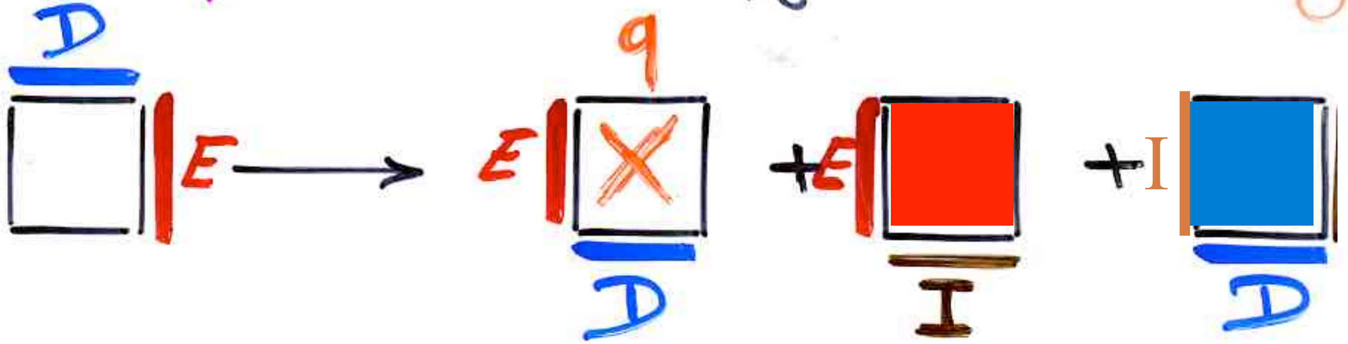
D D E D E E D E

D DE (D E) E D E

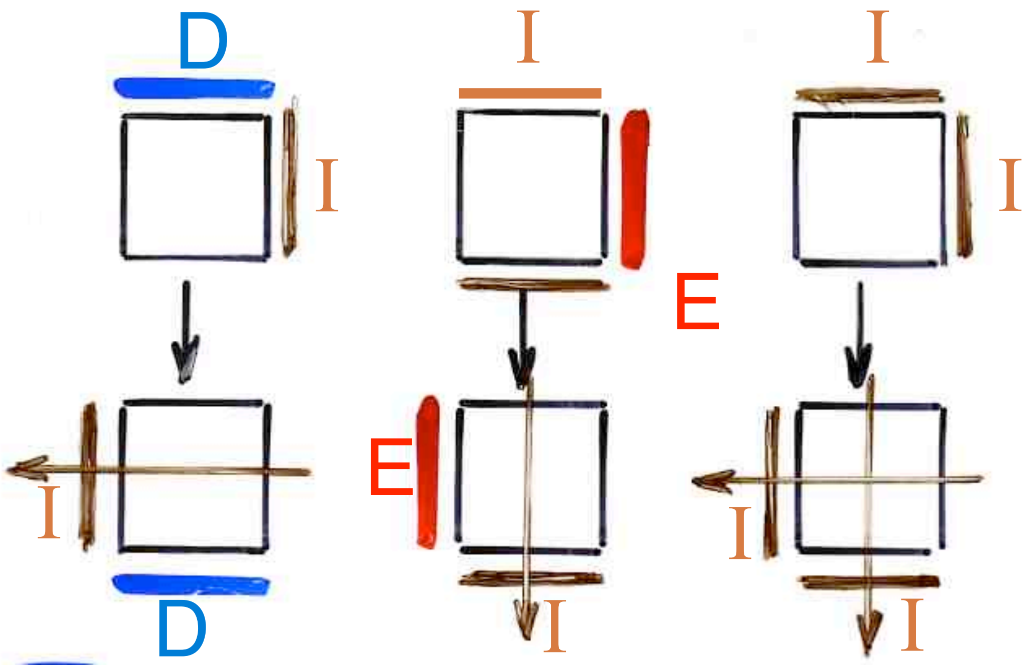
DDE(E)EDE + DDE(ED)EDE + DDE(D)EDE

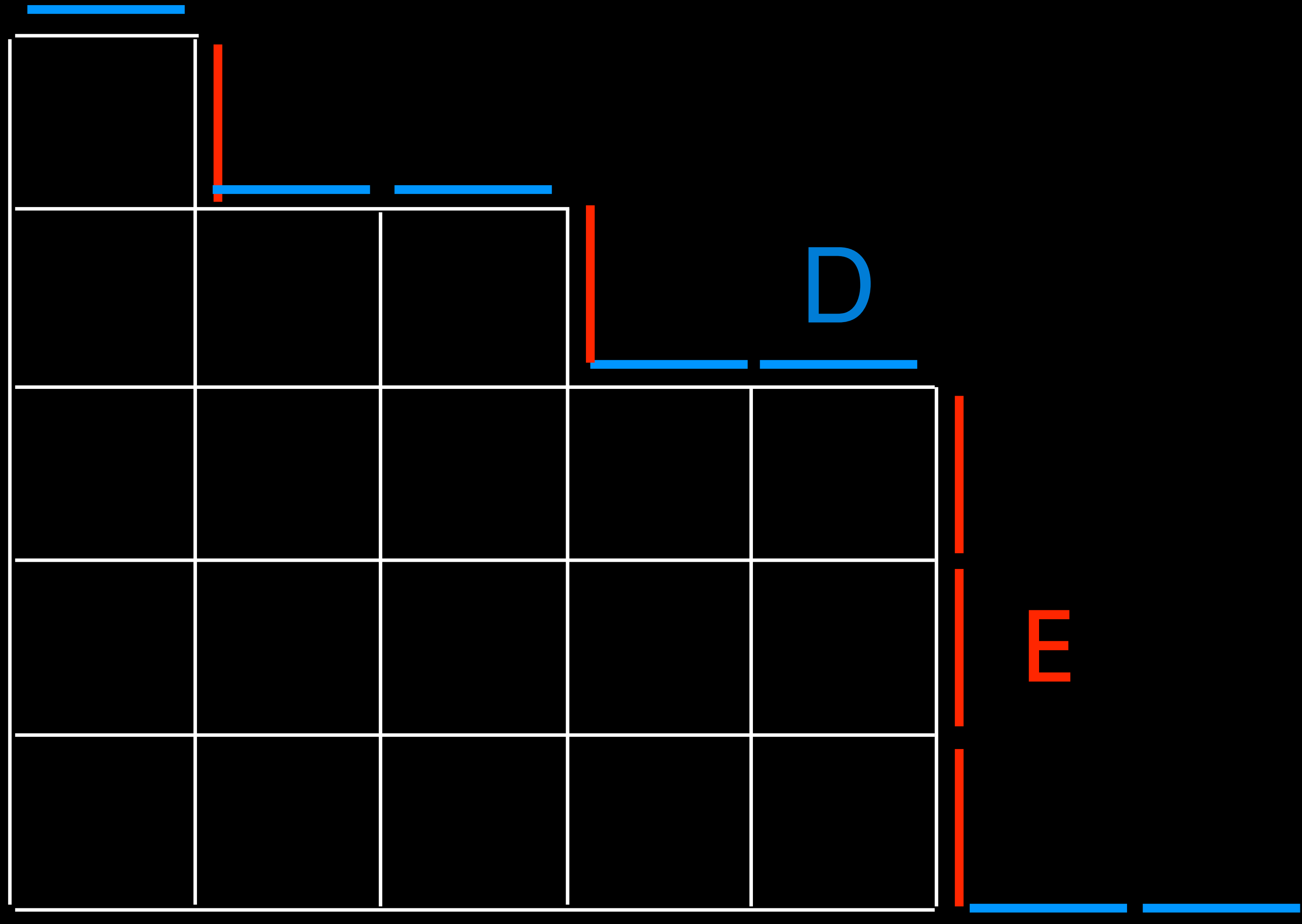


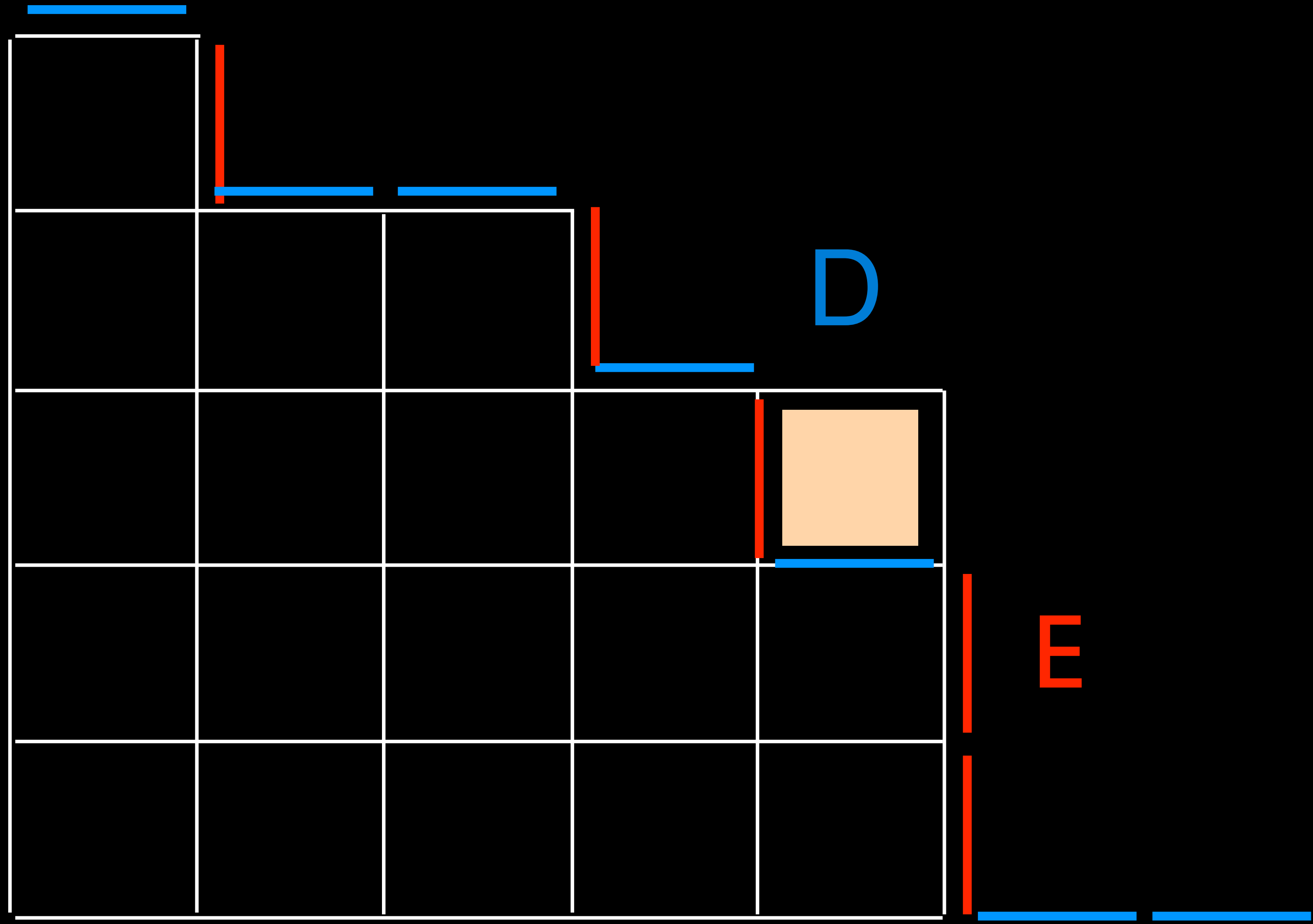
Proof: "planarization" of the rewriting rules



I identity




D
E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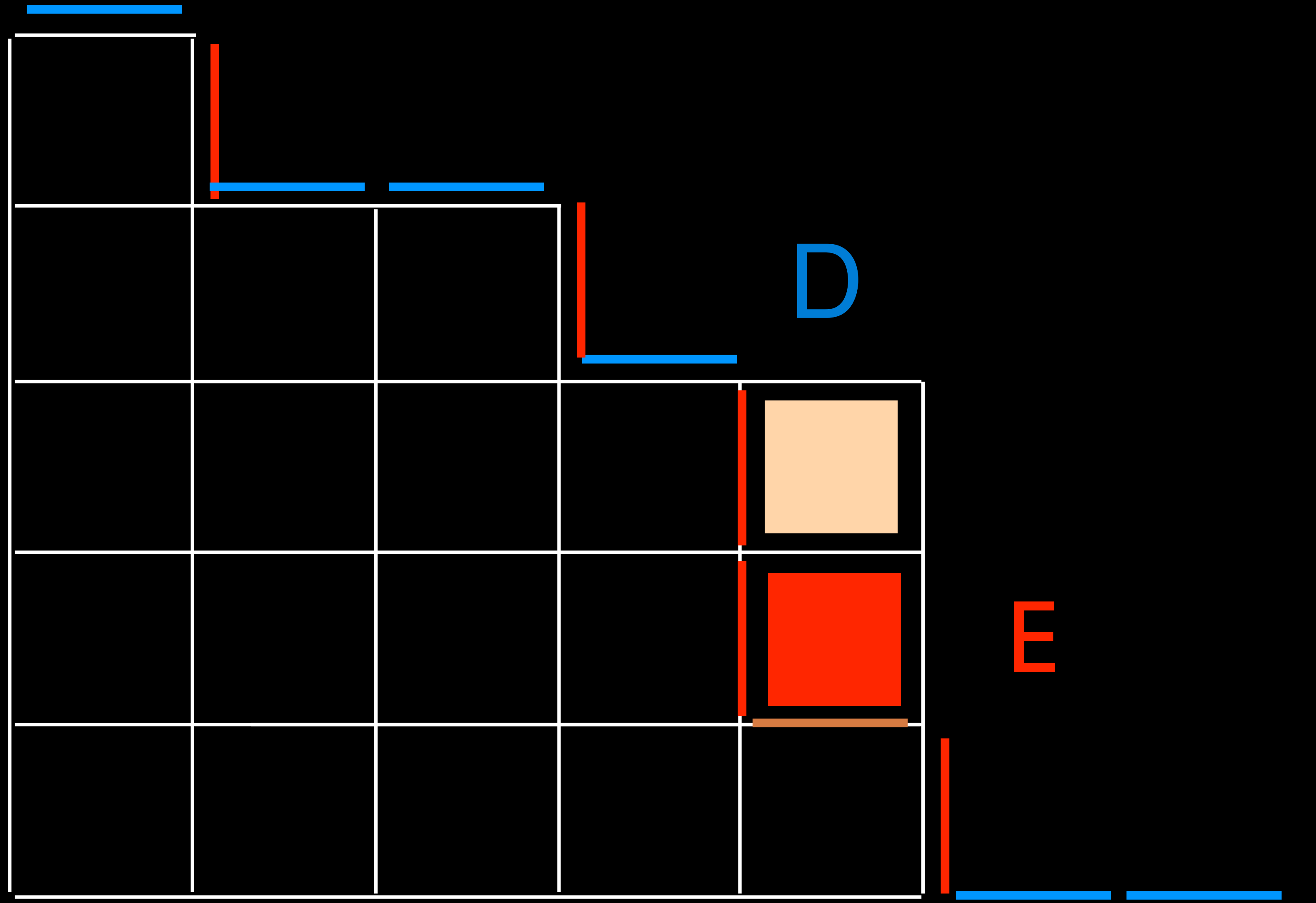
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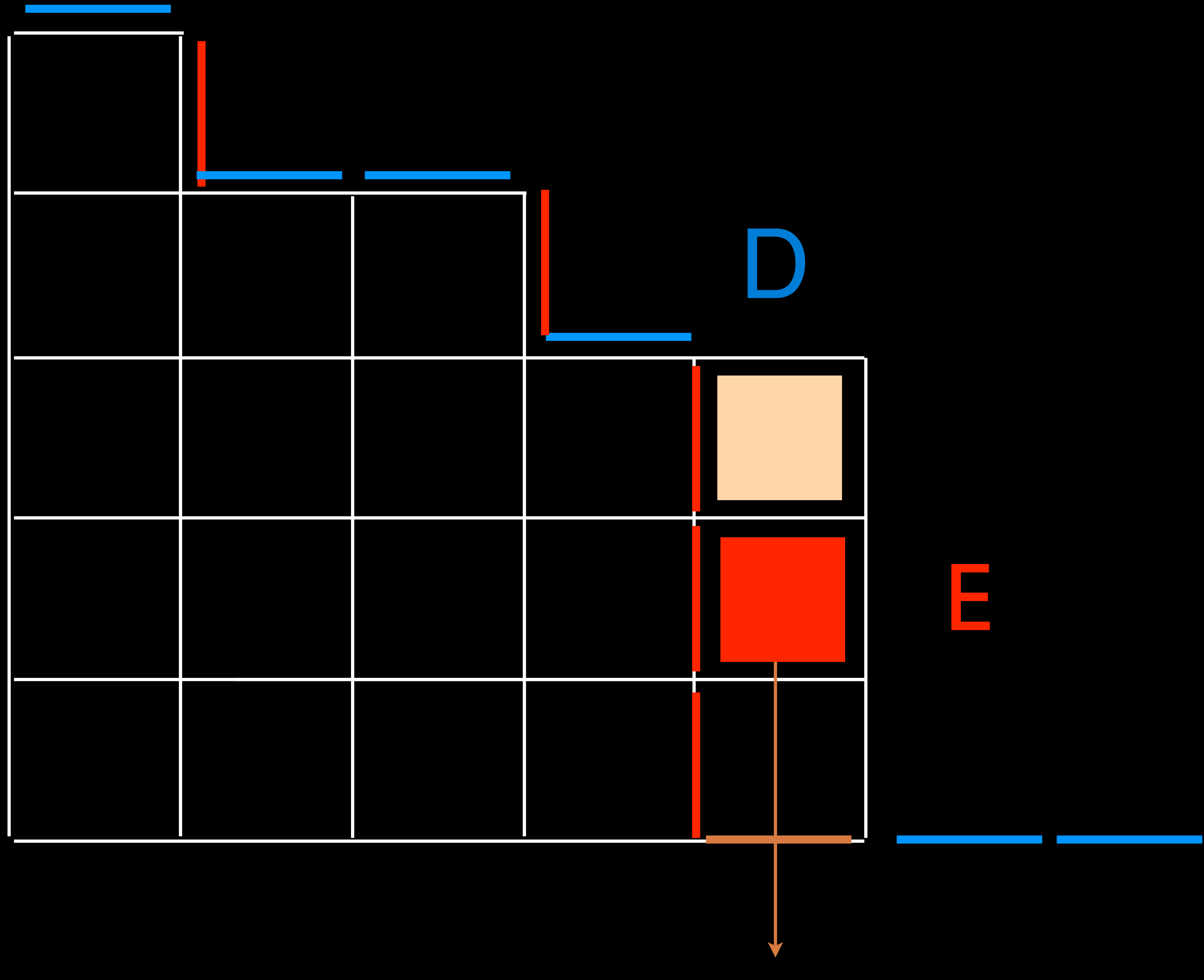
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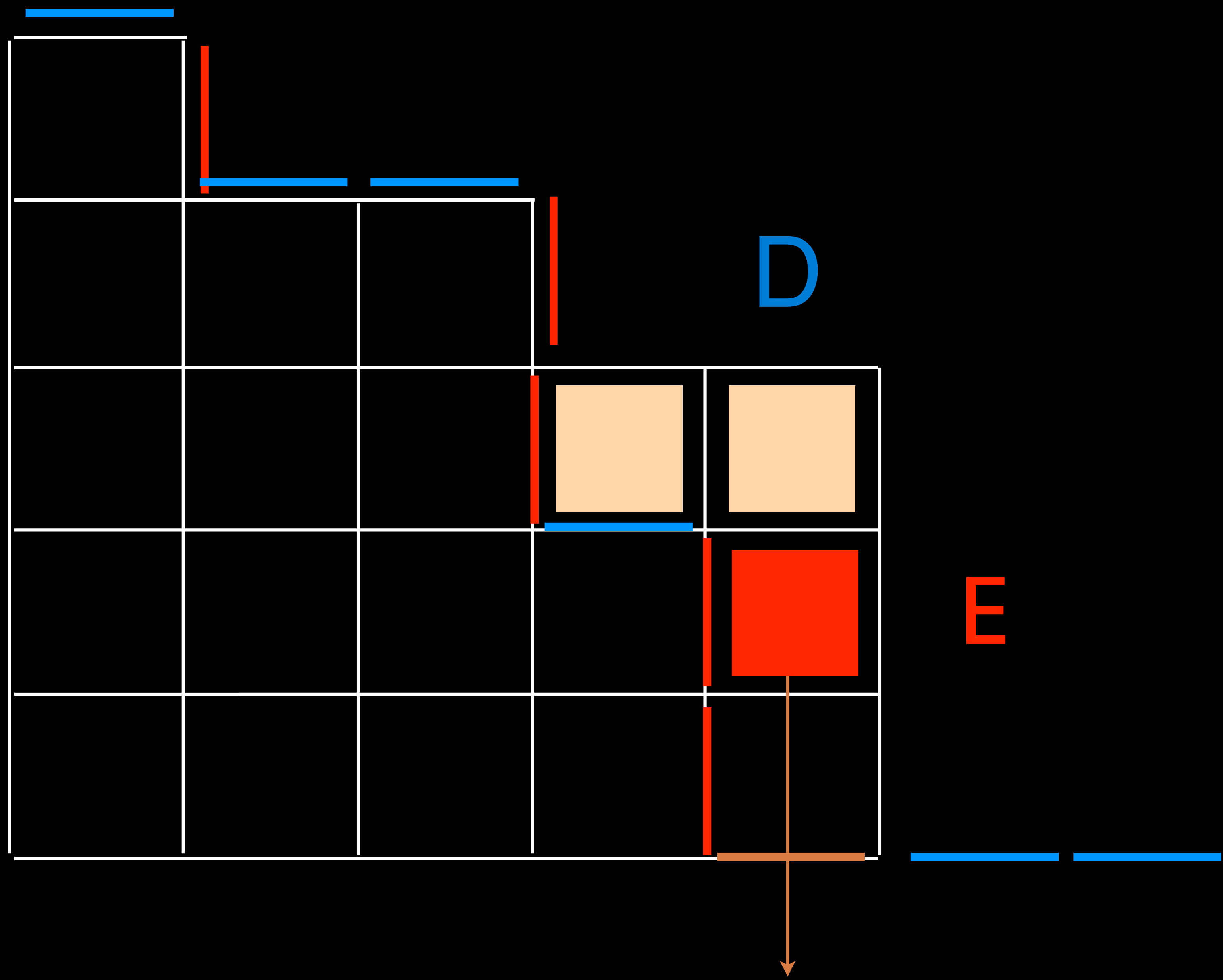
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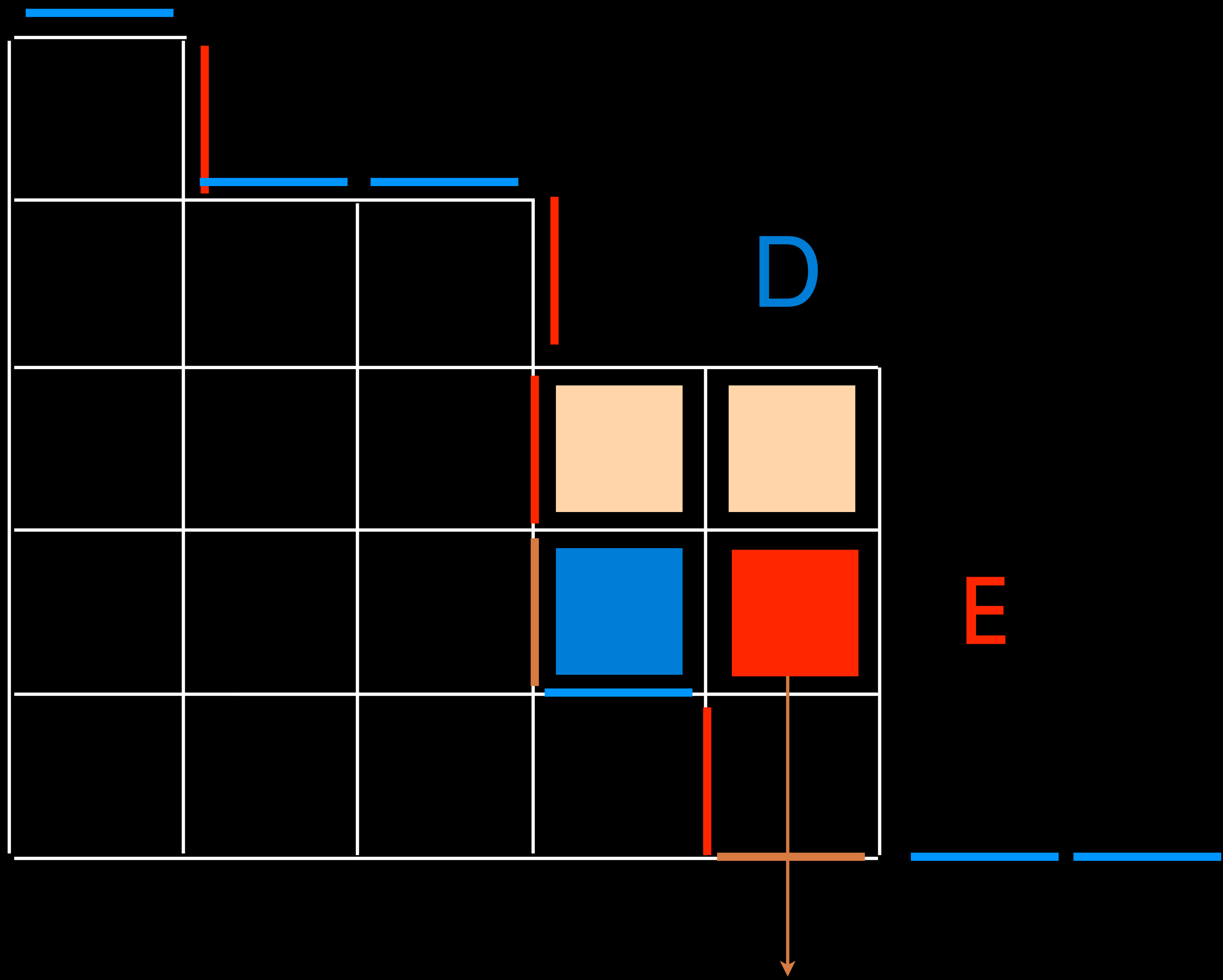

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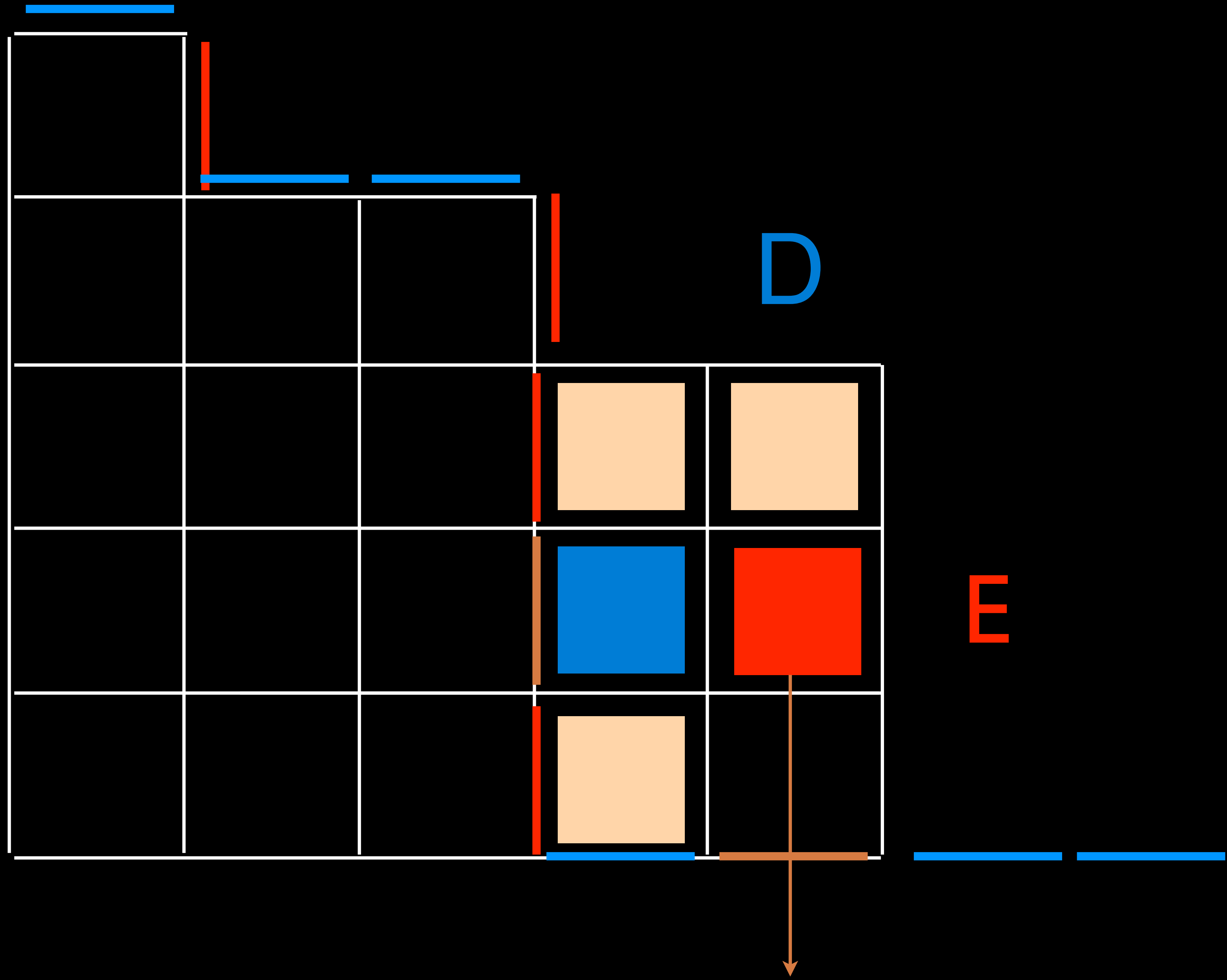
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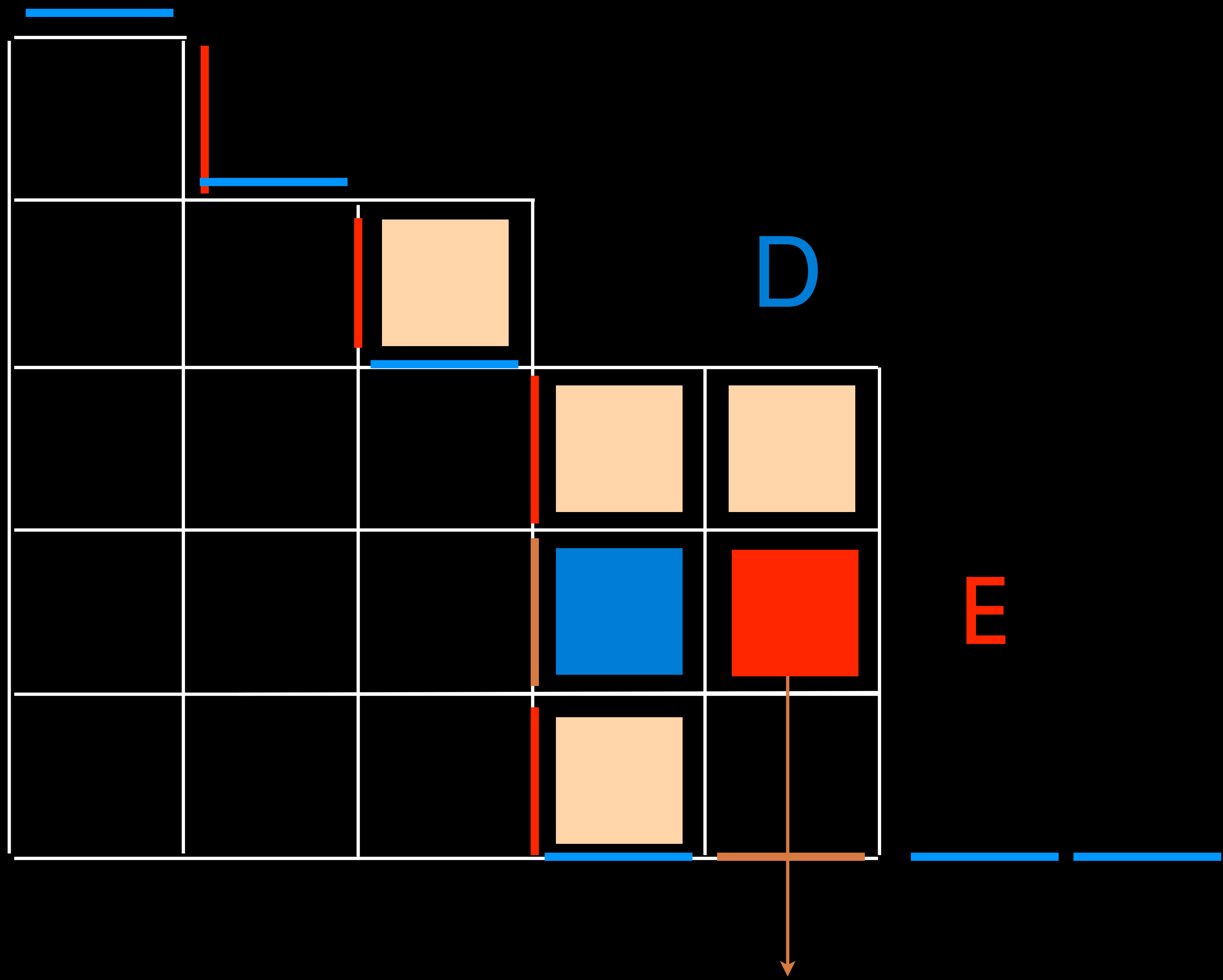
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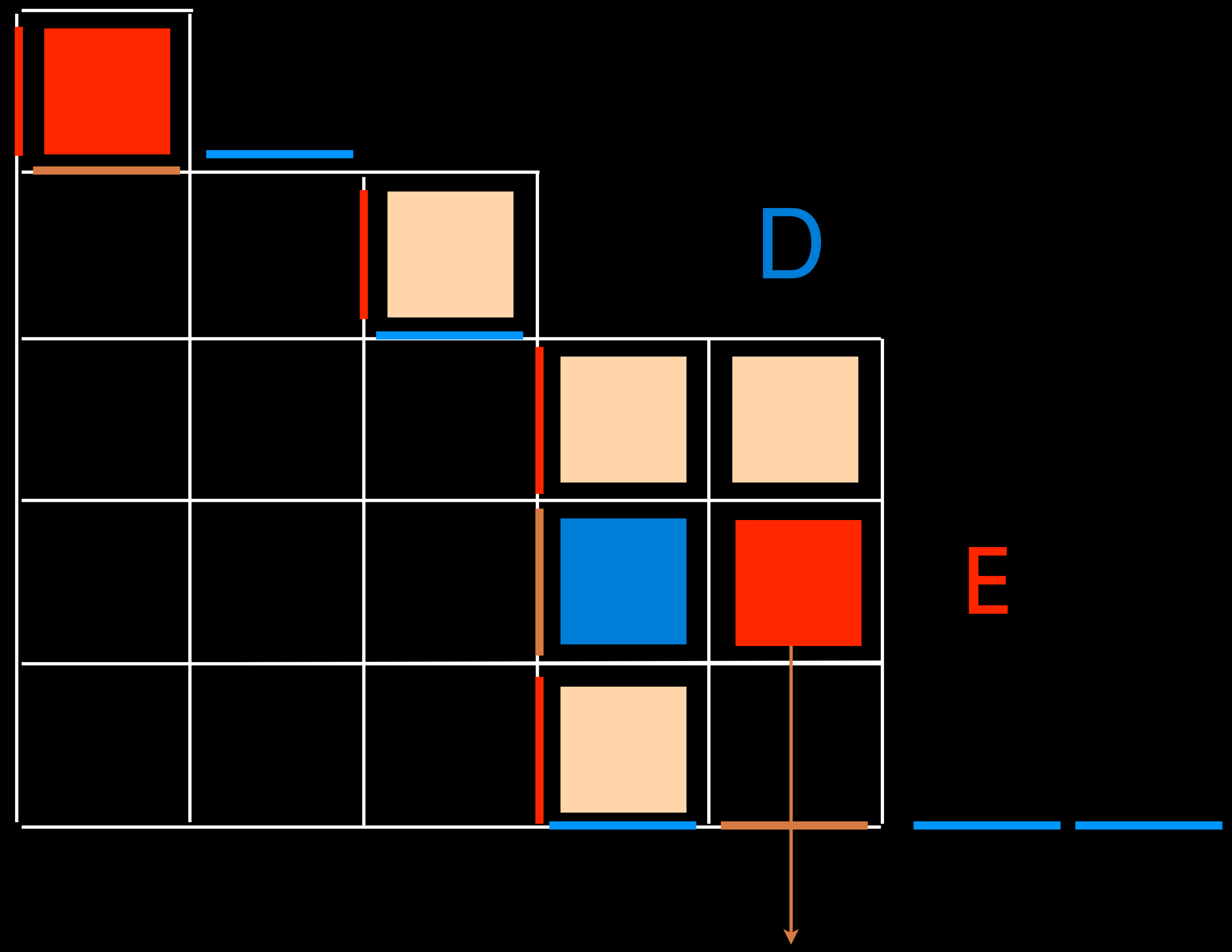
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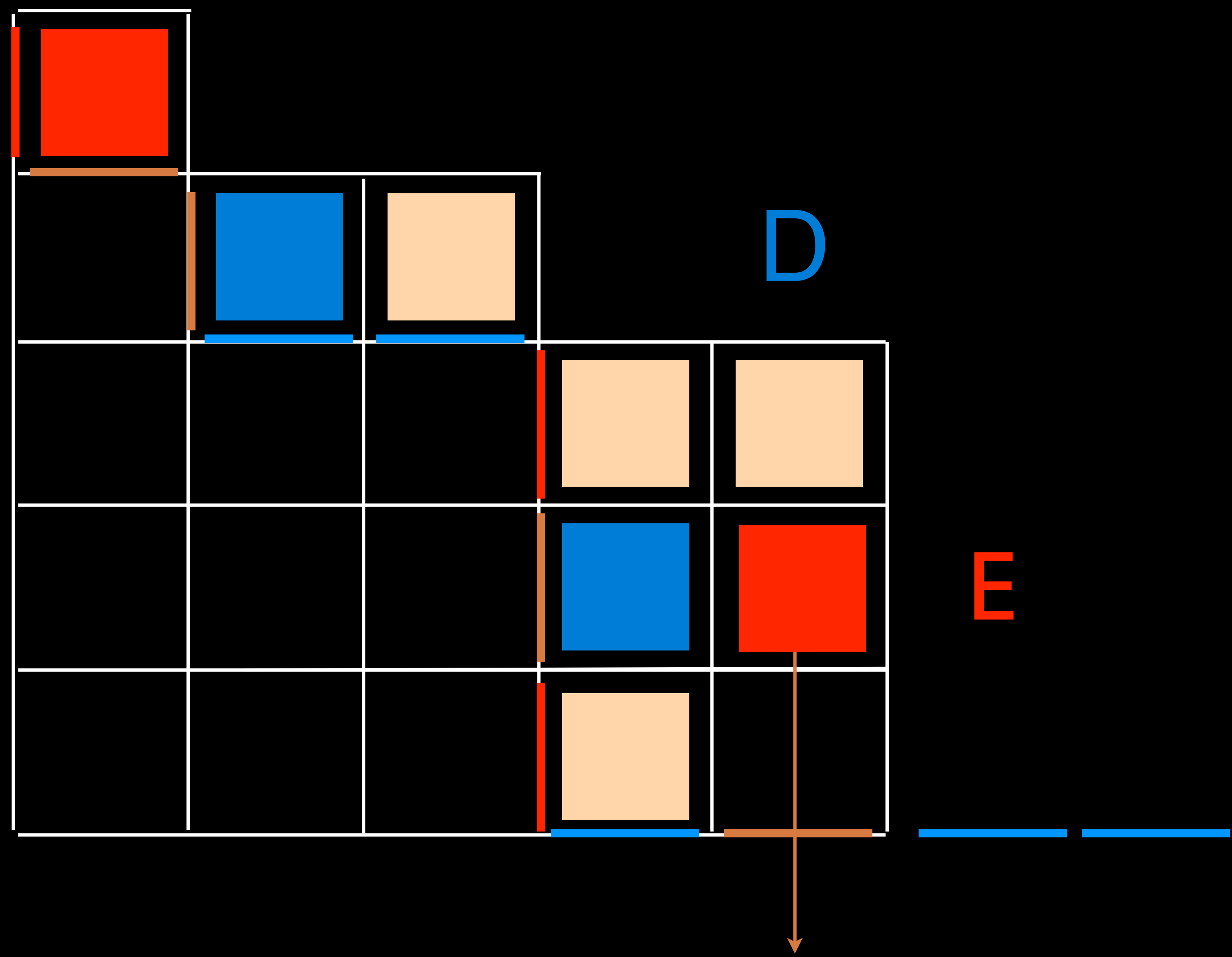
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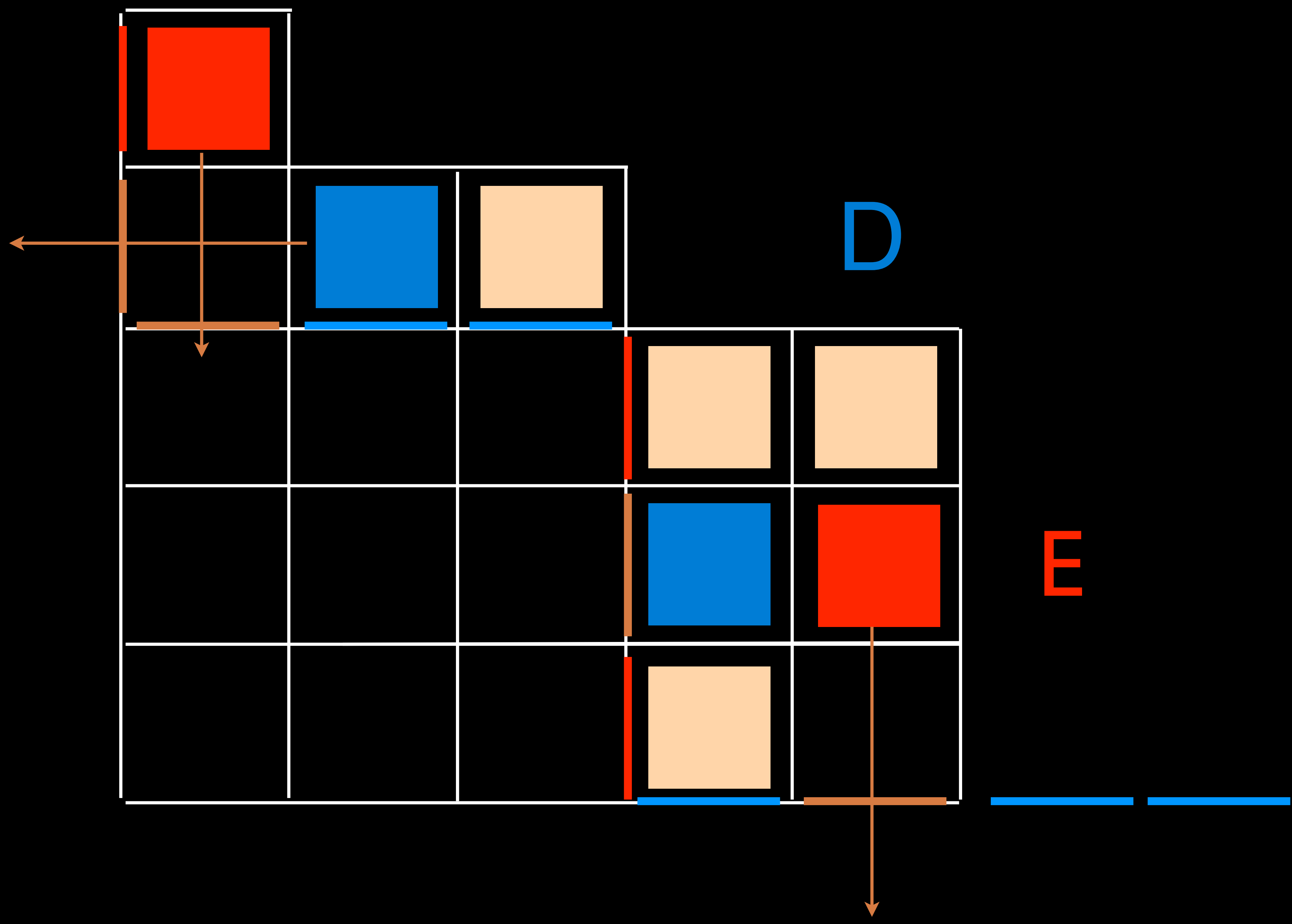
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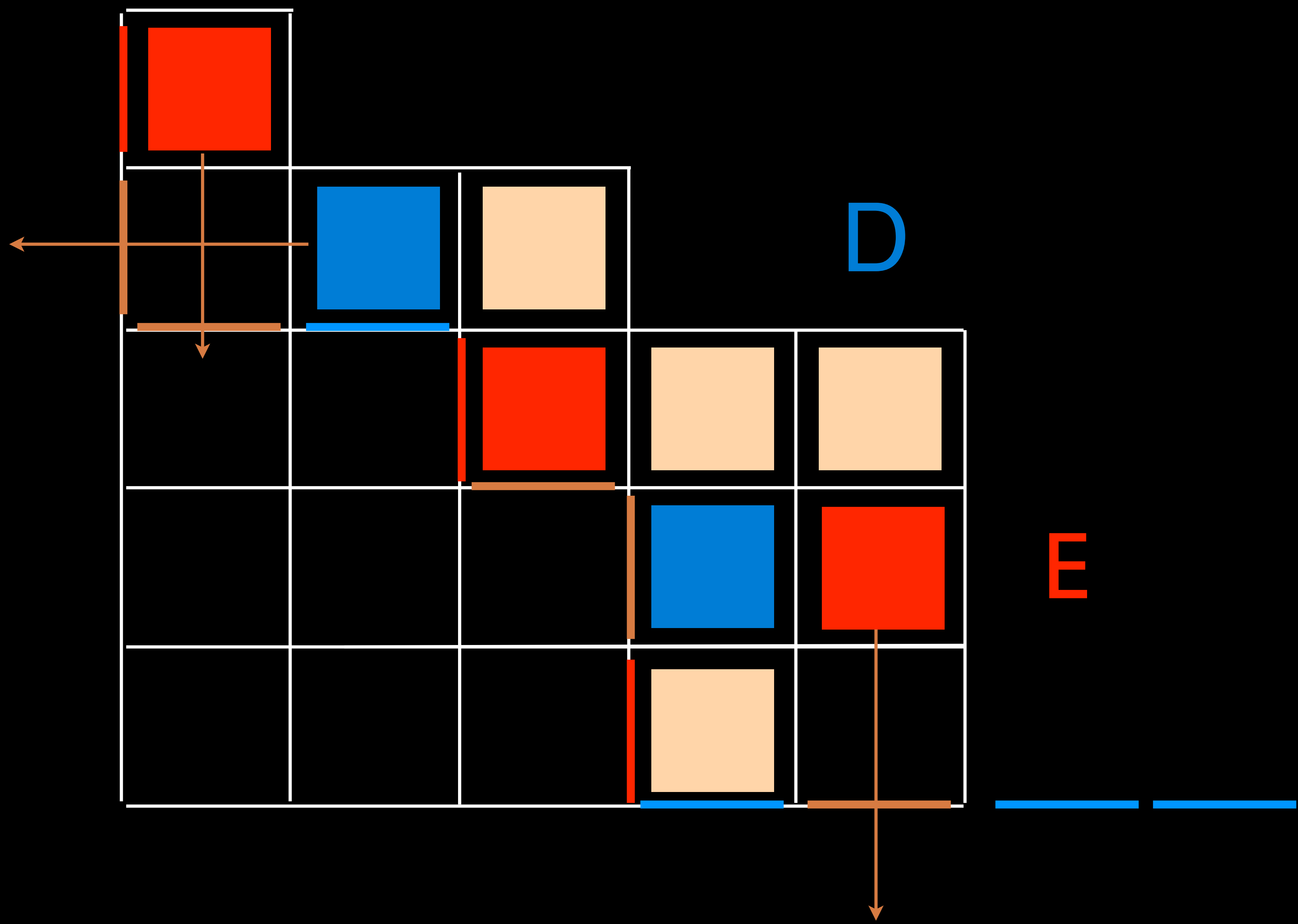
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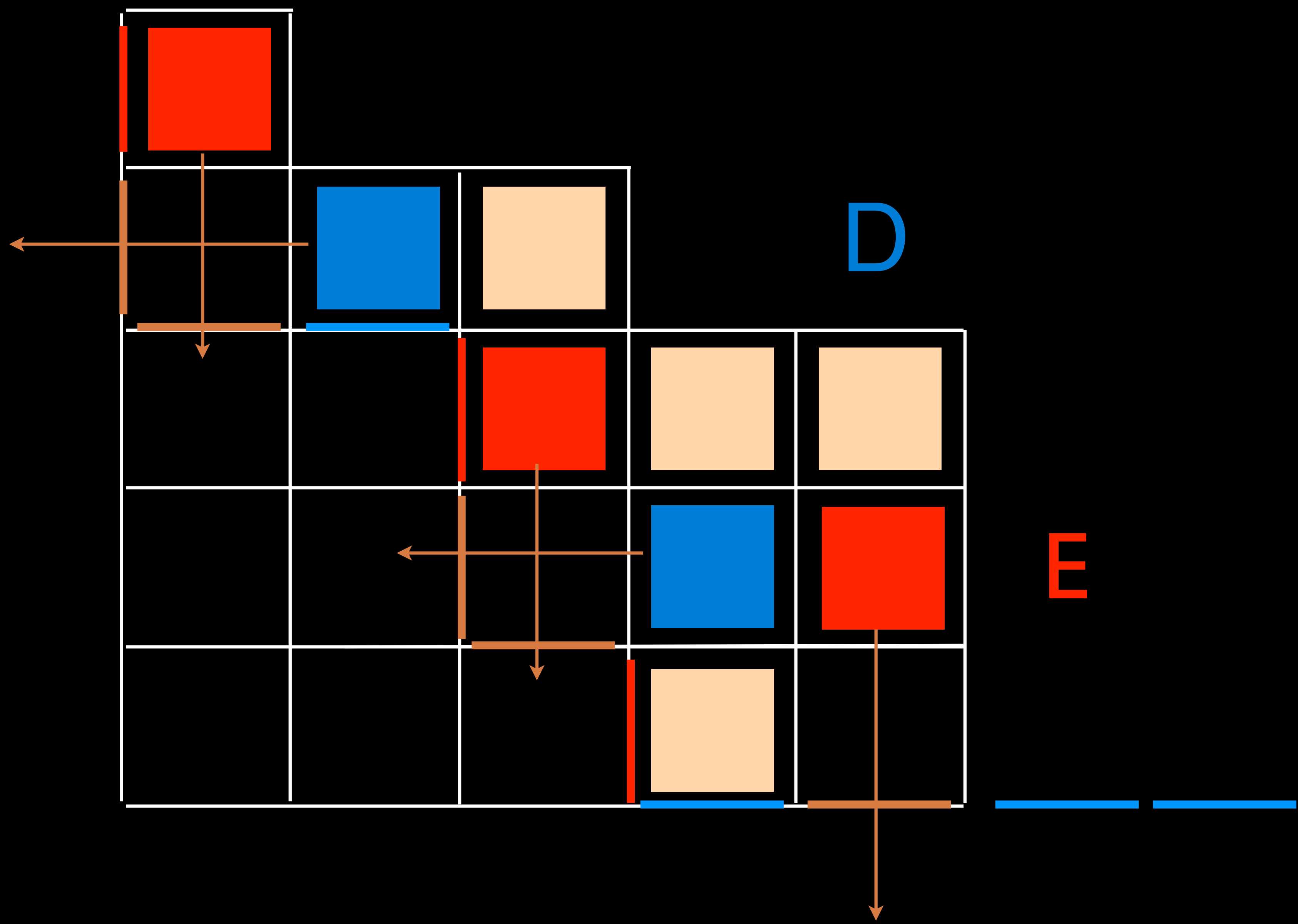
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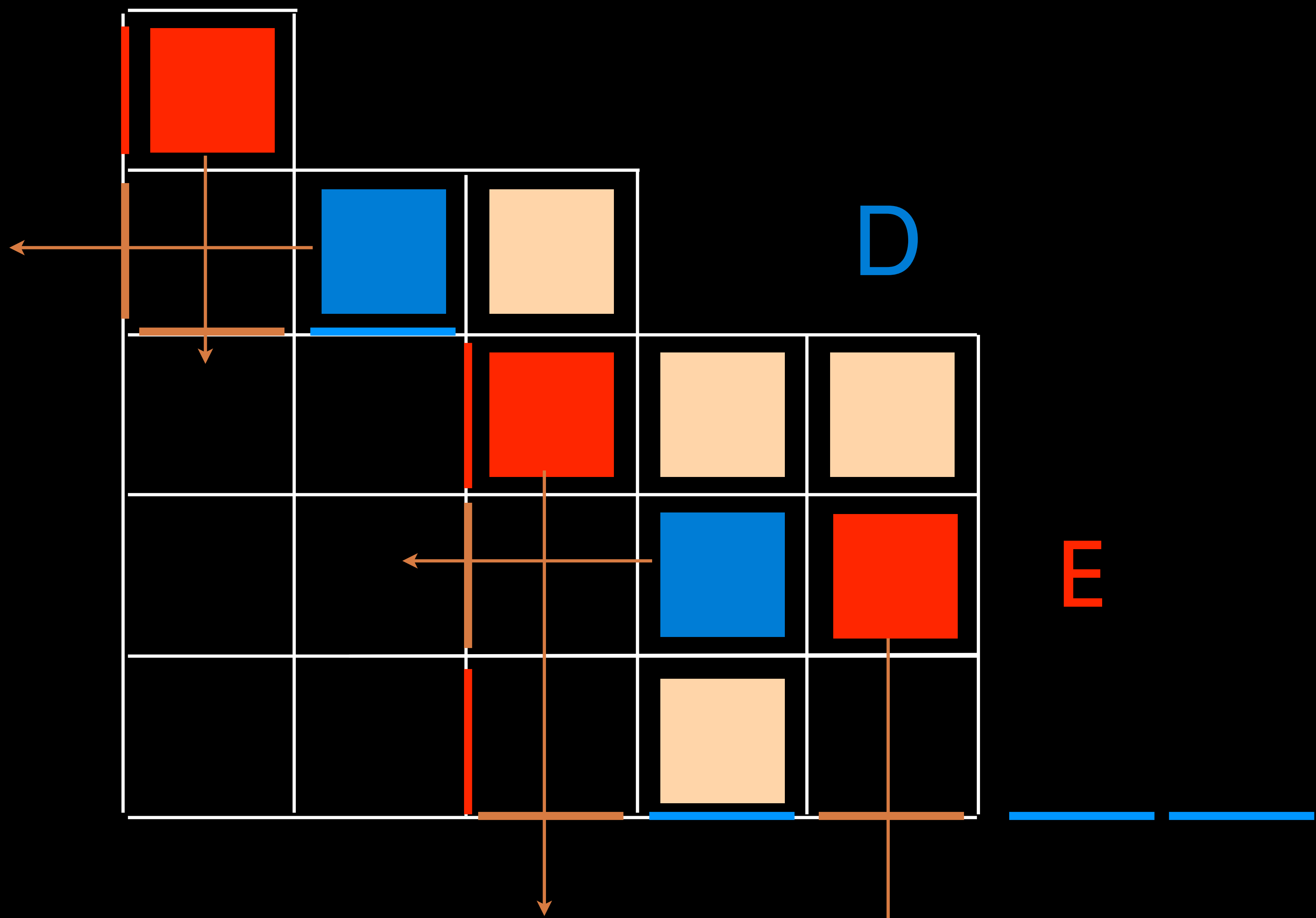
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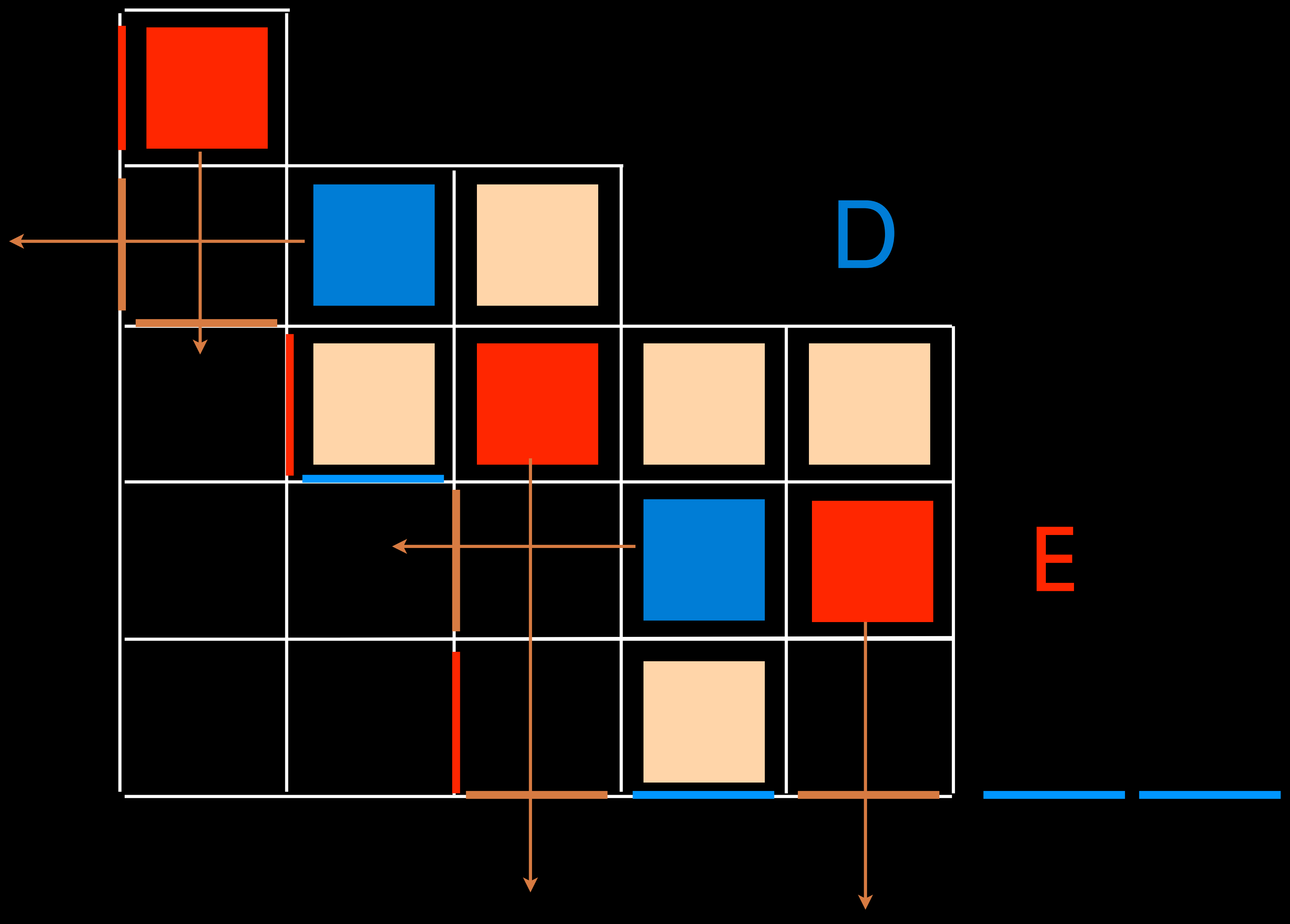
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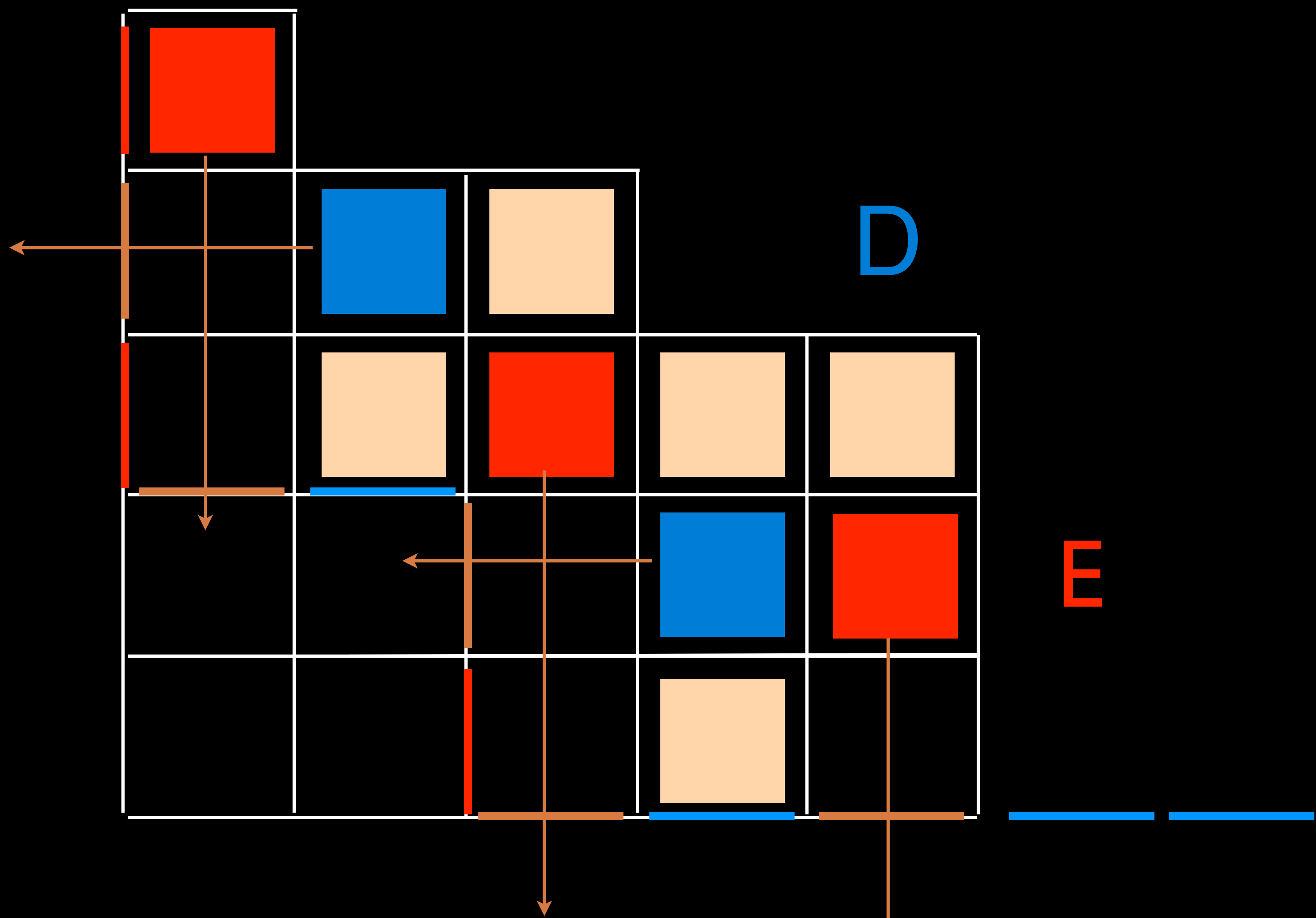
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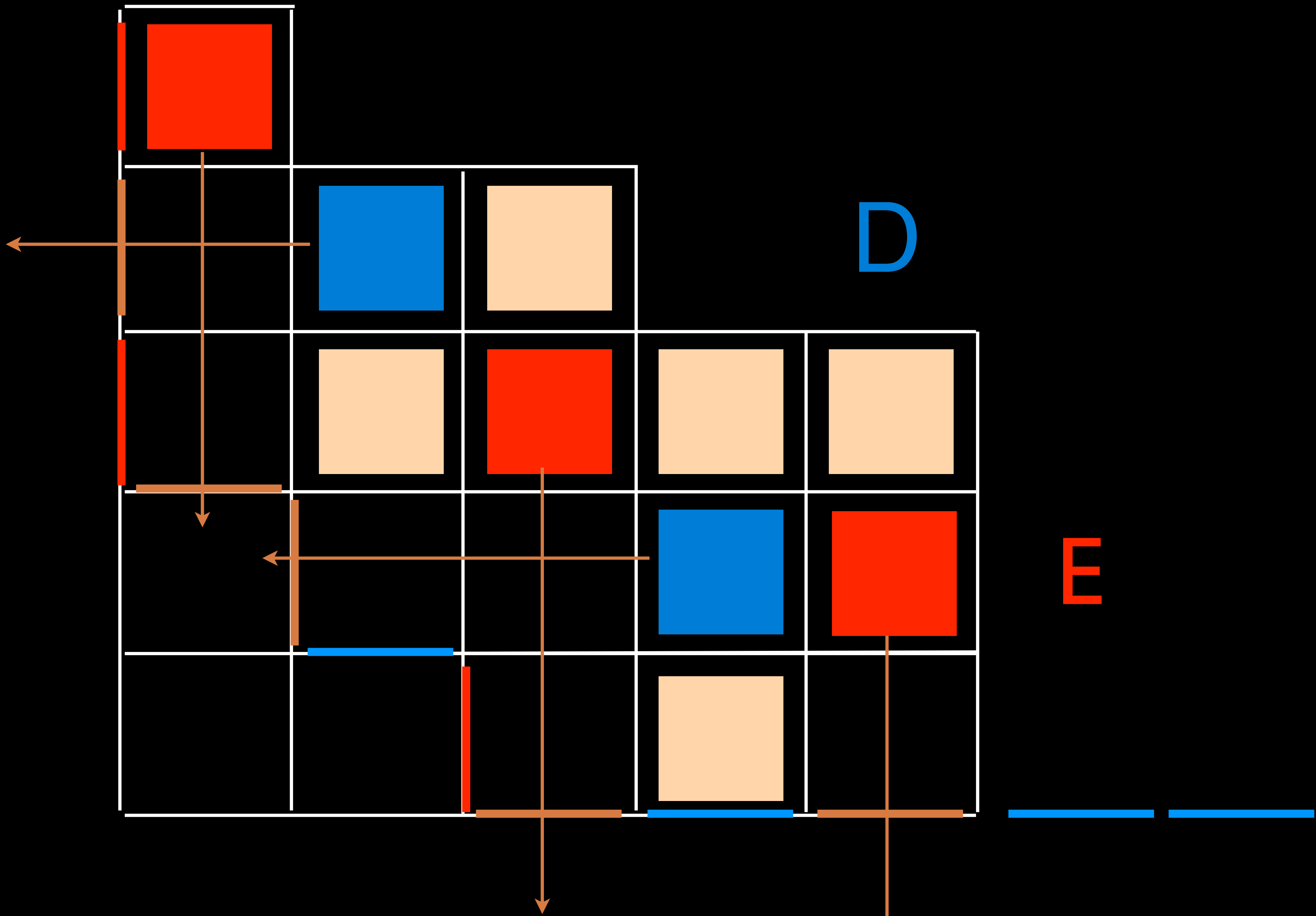
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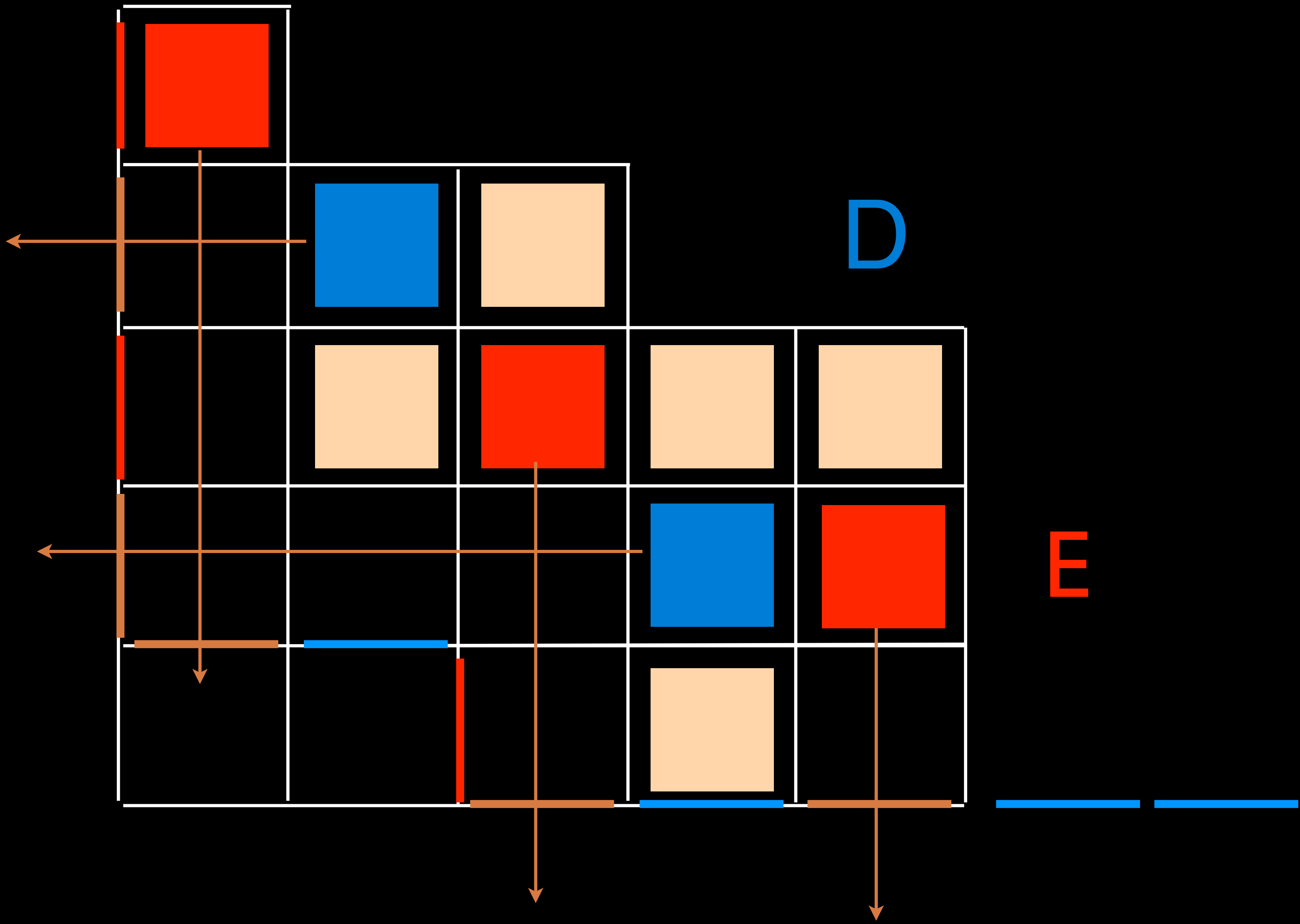
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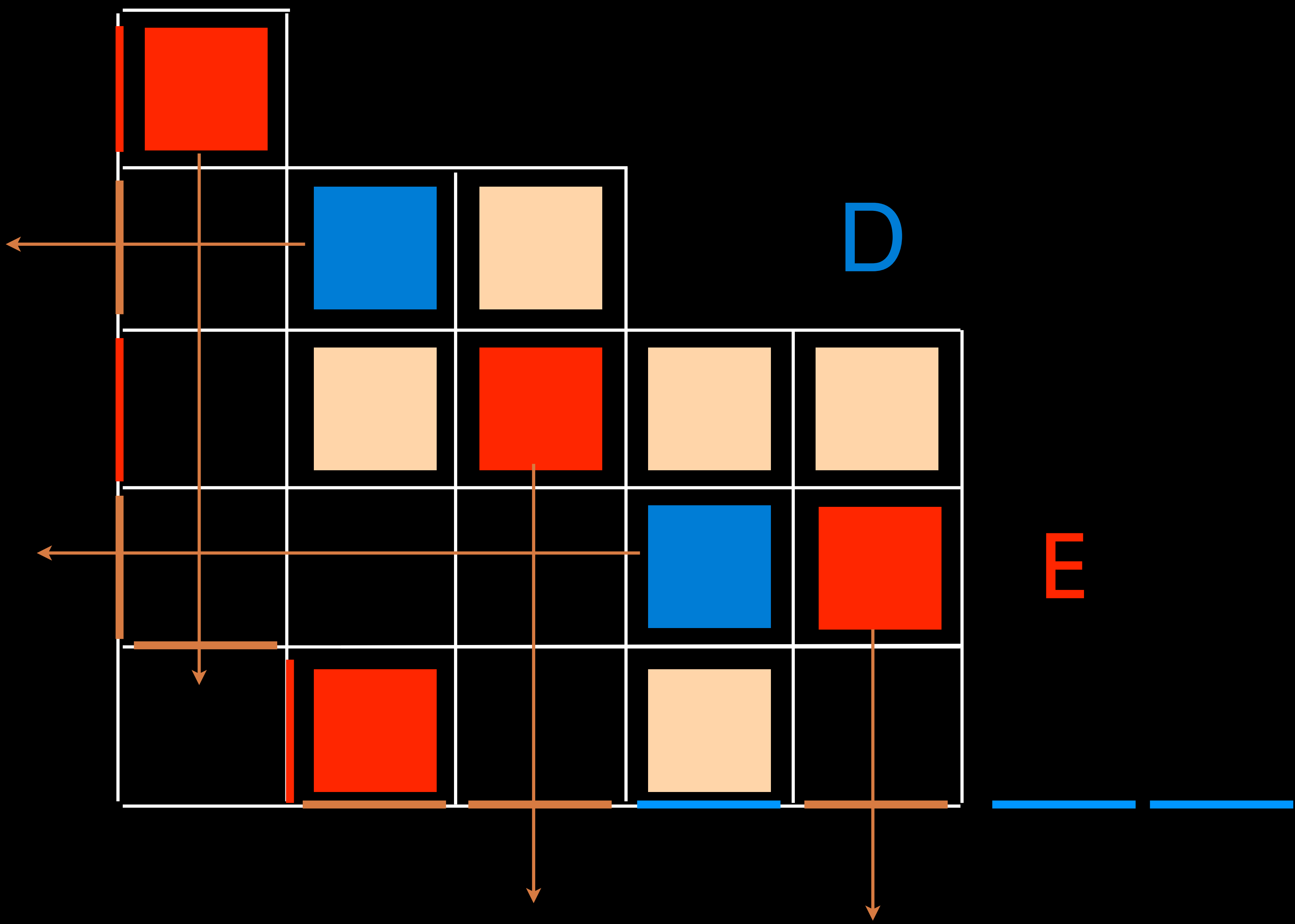
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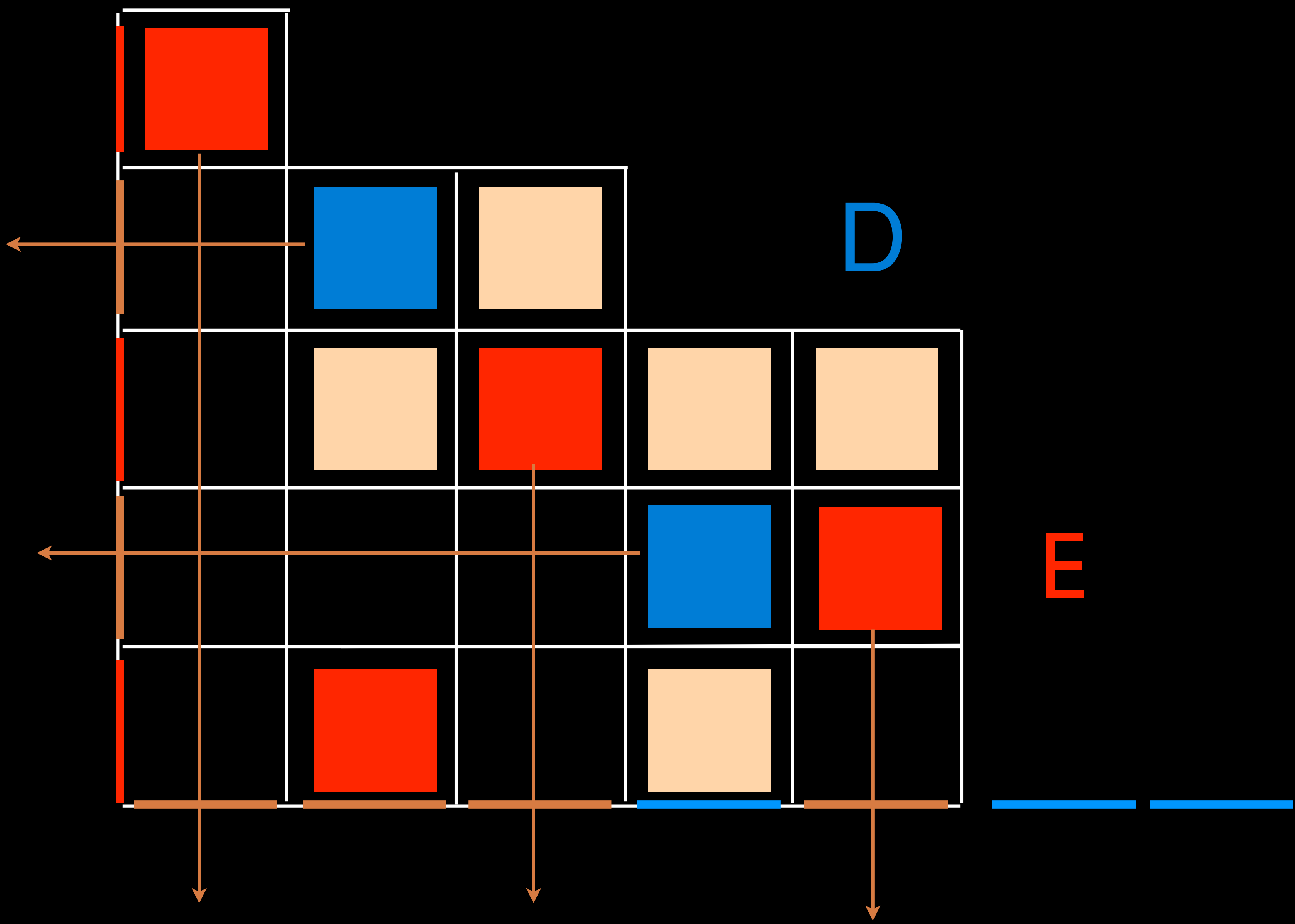
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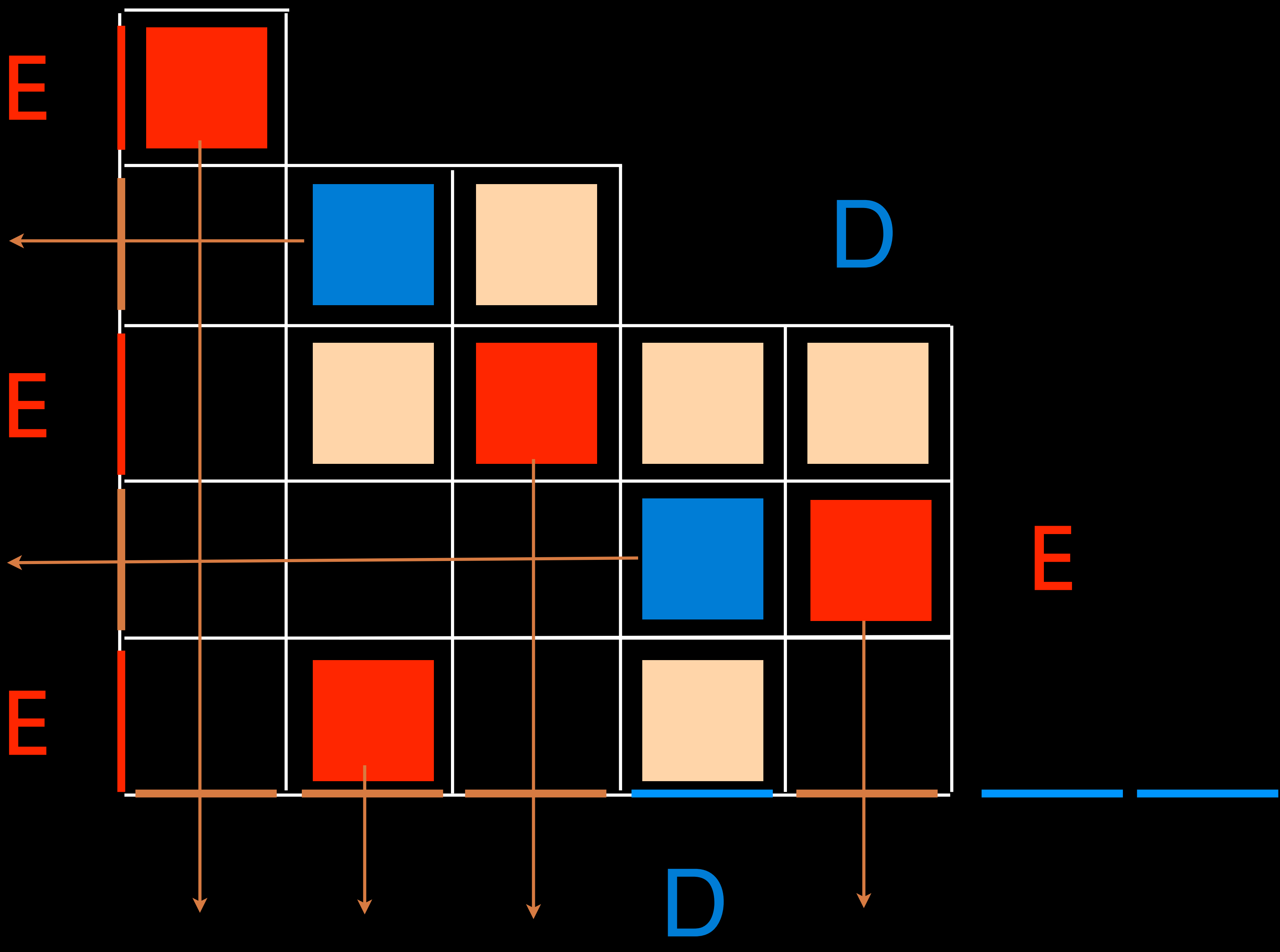
E
D
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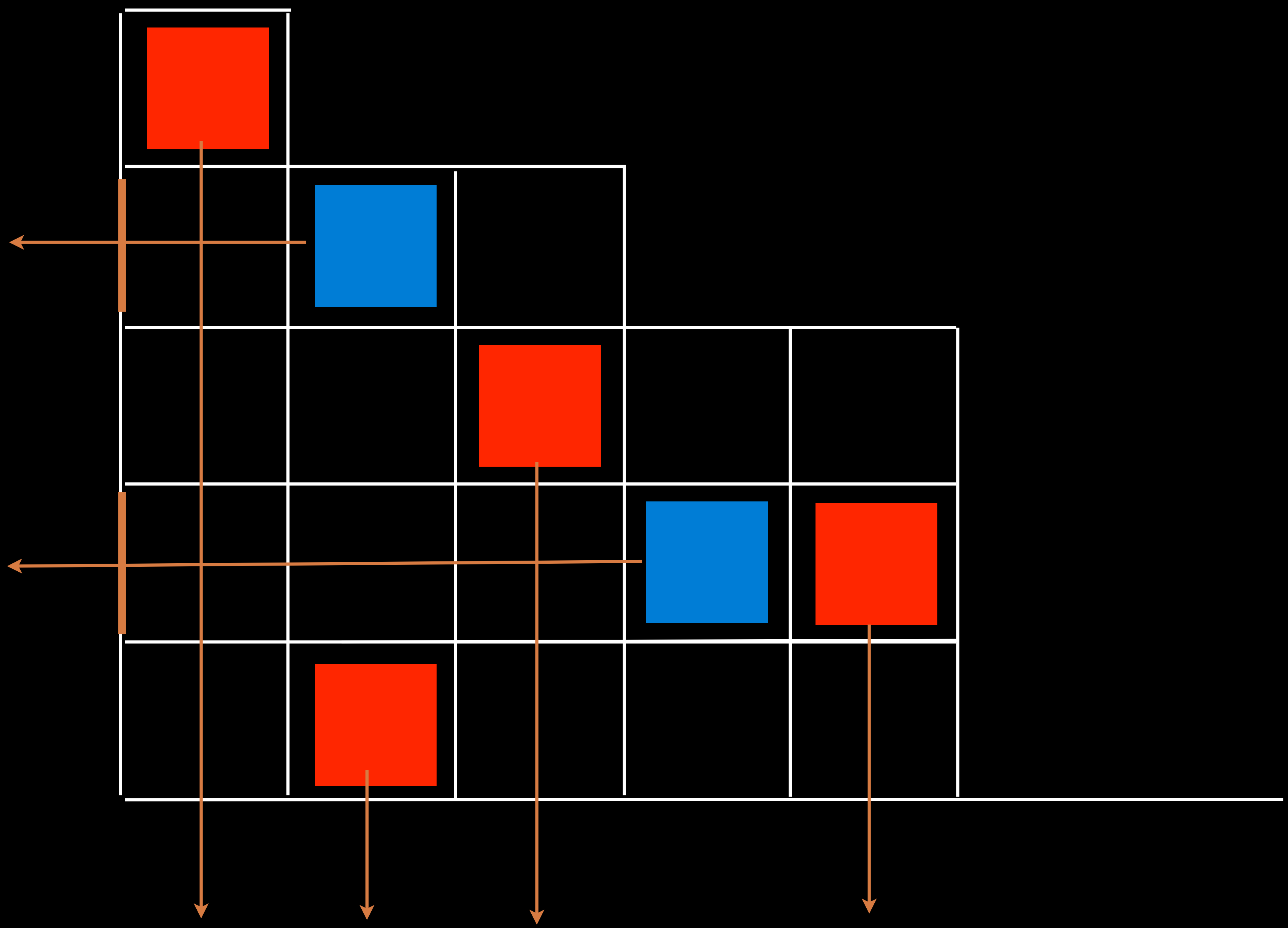
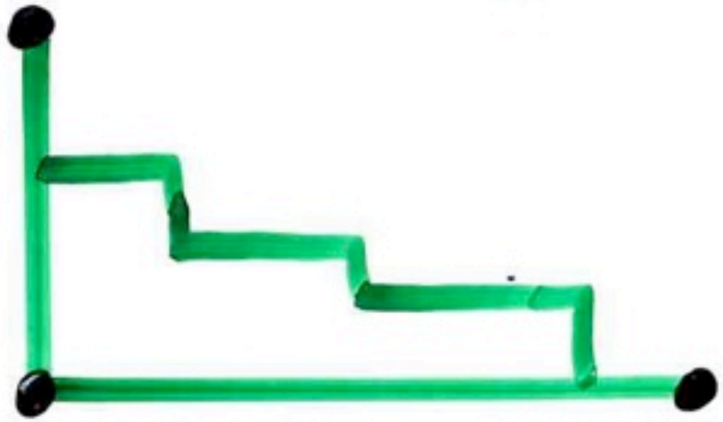
E
D



alternative tableau

- Ferrers diagram F

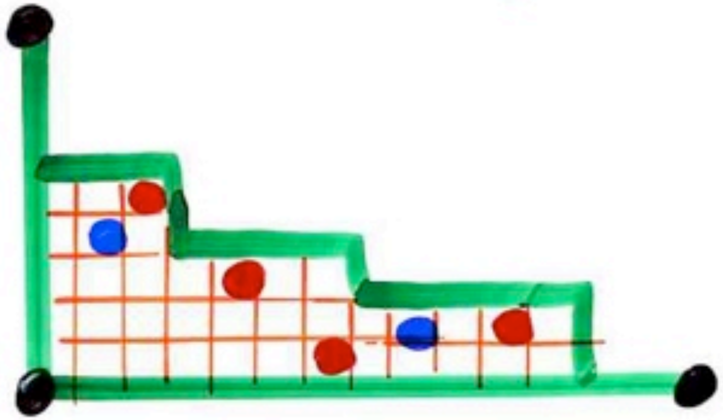


(possibly empty rows or column)

(nb of rows) + (nb of columns) $= n$

alternative tableau

- Ferrers diagram **F**



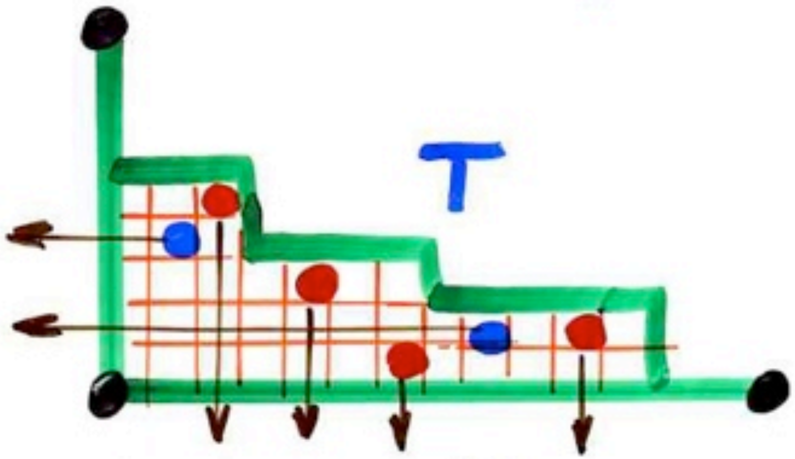
(possibly empty rows or column)

(nb of rows) + (nb of columns) $= n$

- some cells are coloured **red** or **blue**

alternative tableau T

- Ferrers diagram F (possibly empty rows or column)



(nb of rows) + (nb of columns) = n

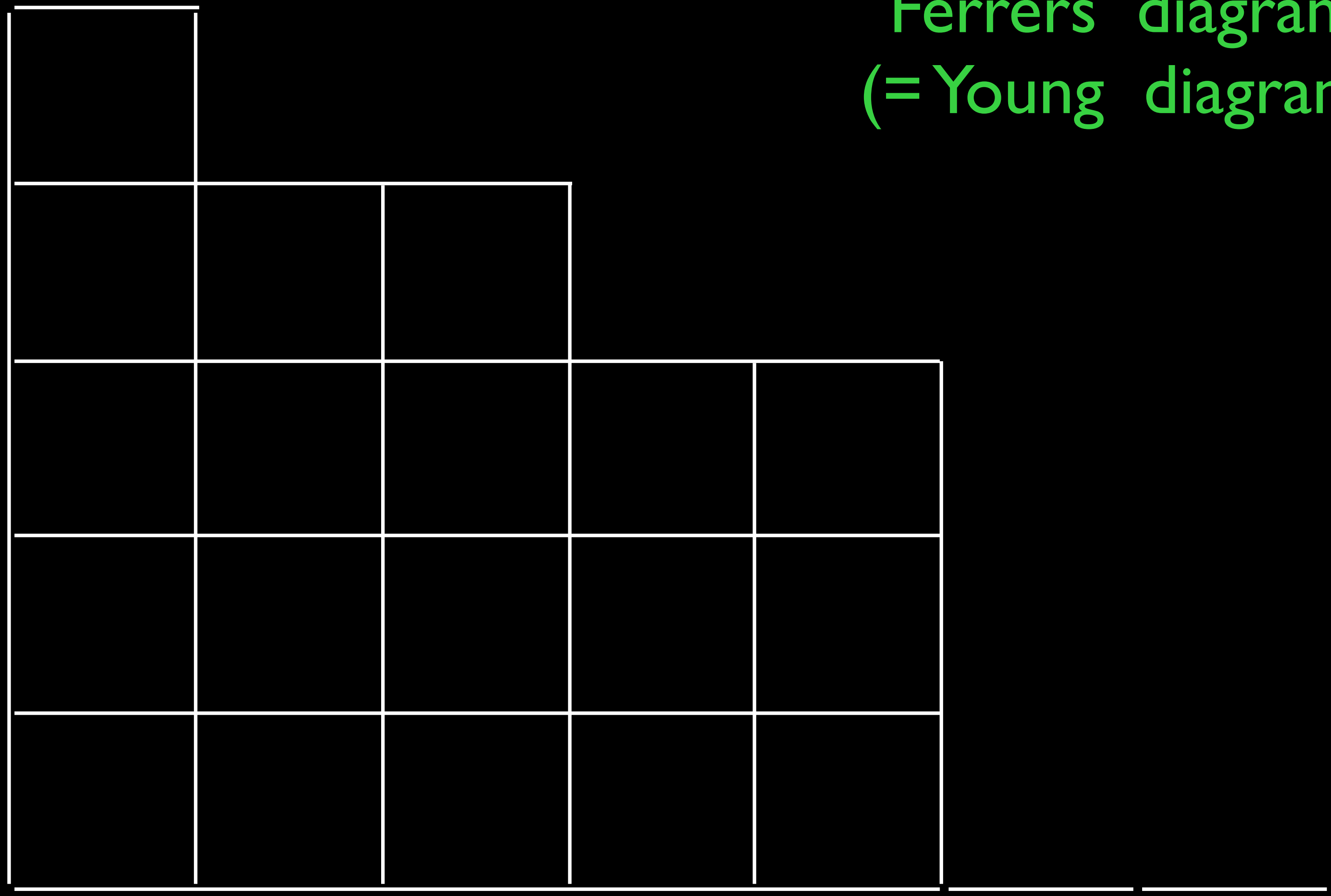
- some cells are coloured **red** or **blue**

- { no coloured cell at the left of [blue cell]
  { no coloured cell below [red cell]

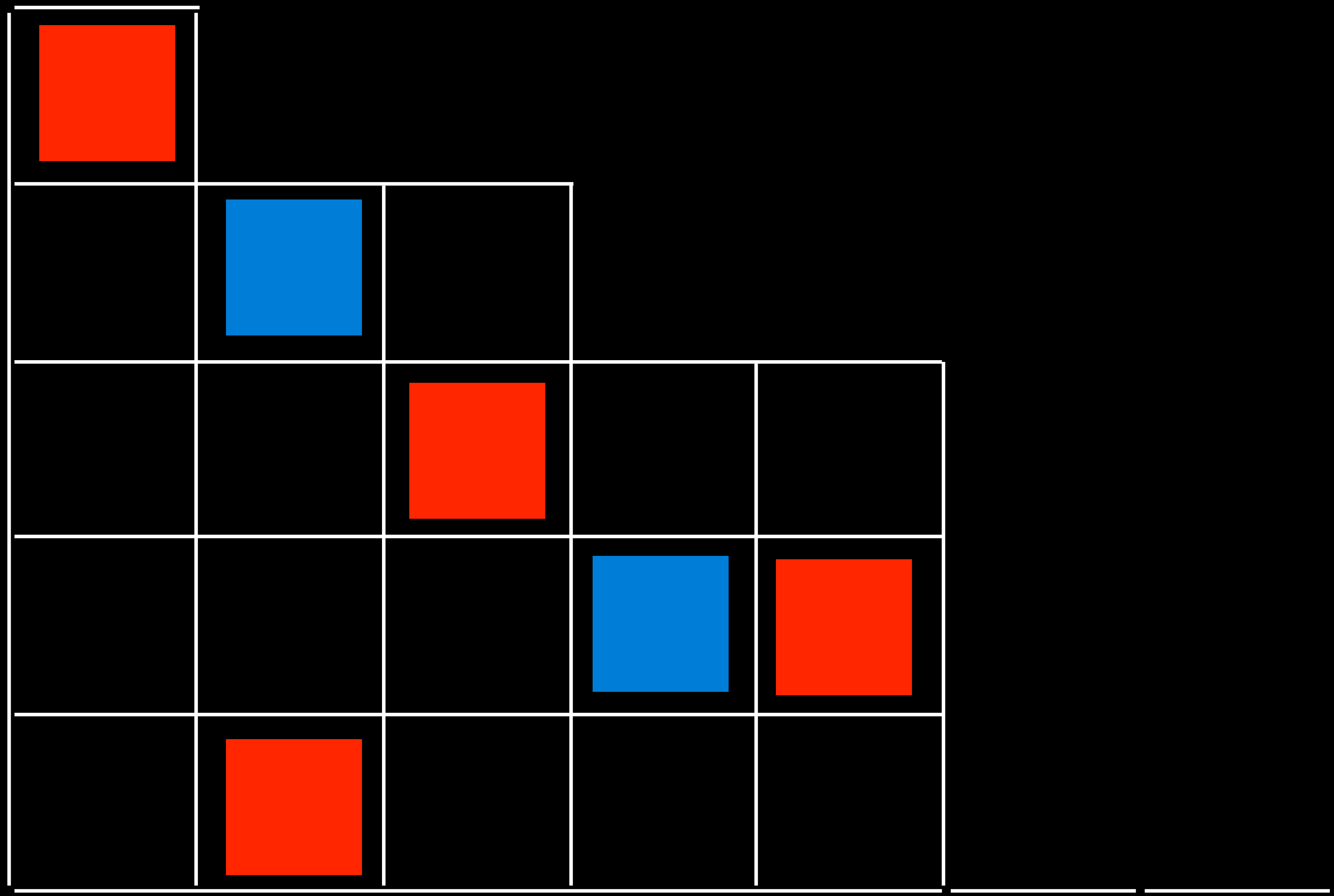
n size of T

alternative tableau

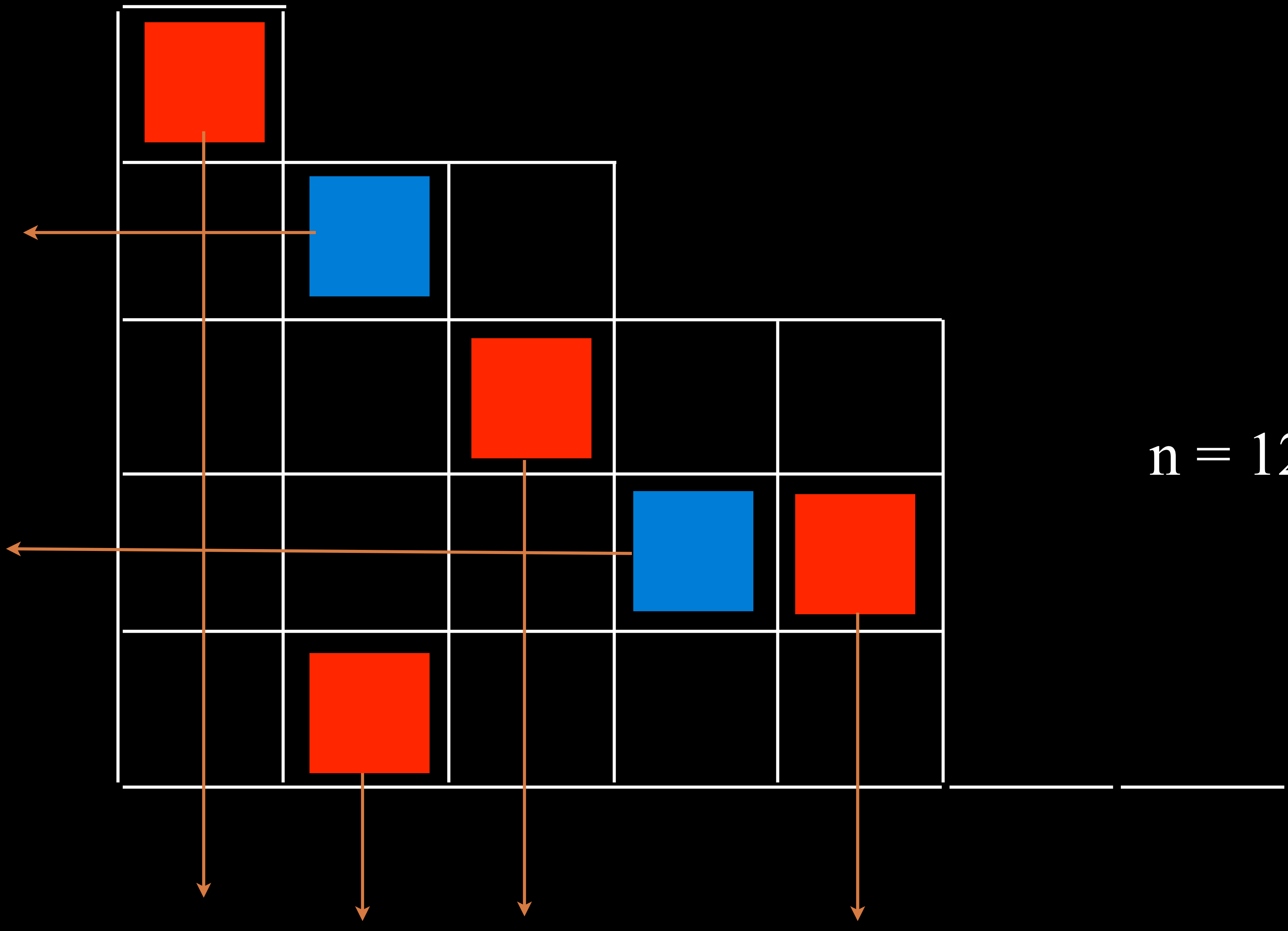
Ferrers diagram
(= Young diagram)



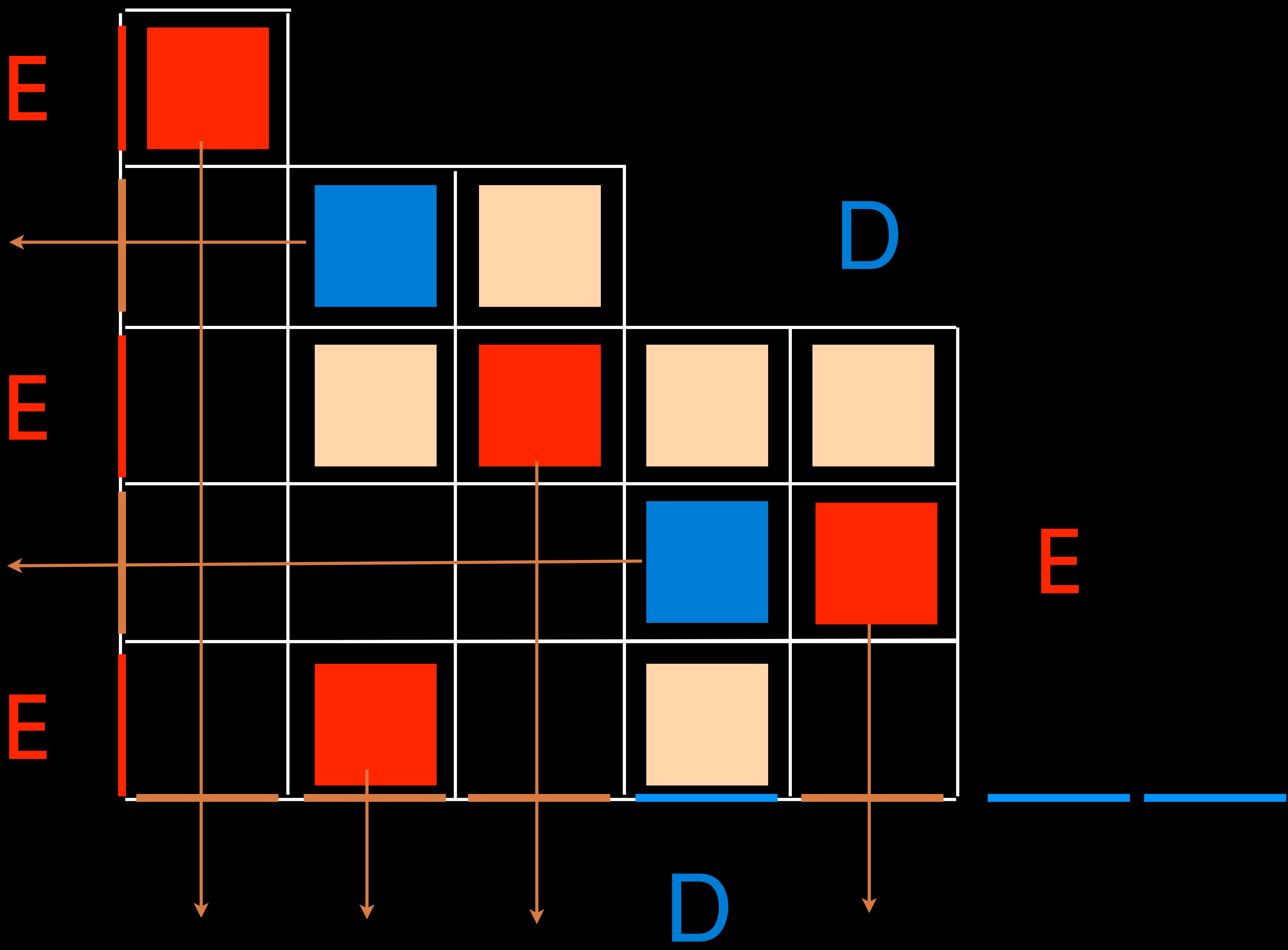
alternative tableau



alternative tableau



$n = 12$


E
D
E
E
E
D

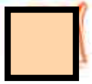
Def - profile of an alternative tableau

word $w \in \{E, D\}^*$

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

T alternative tableau with profile w

$k(T)$ = nb of 

$i(T)$ = nb of rows without blue cell

$j(T)$ = nb of columns without red cell

"normal ordering"

Heisenberg operators U, D

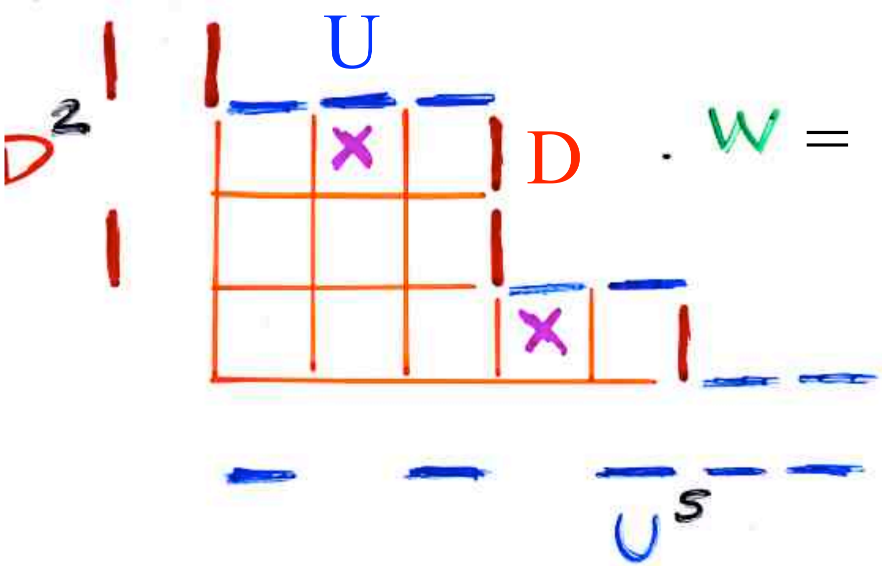
$$UD = DU + 1$$

$$UD = DU + I$$

Lemma

Every word w with letters U and D can be written in a unique way

$$w = \sum_{i,j \geq 0} c_{ij}(w) D^i U^j$$



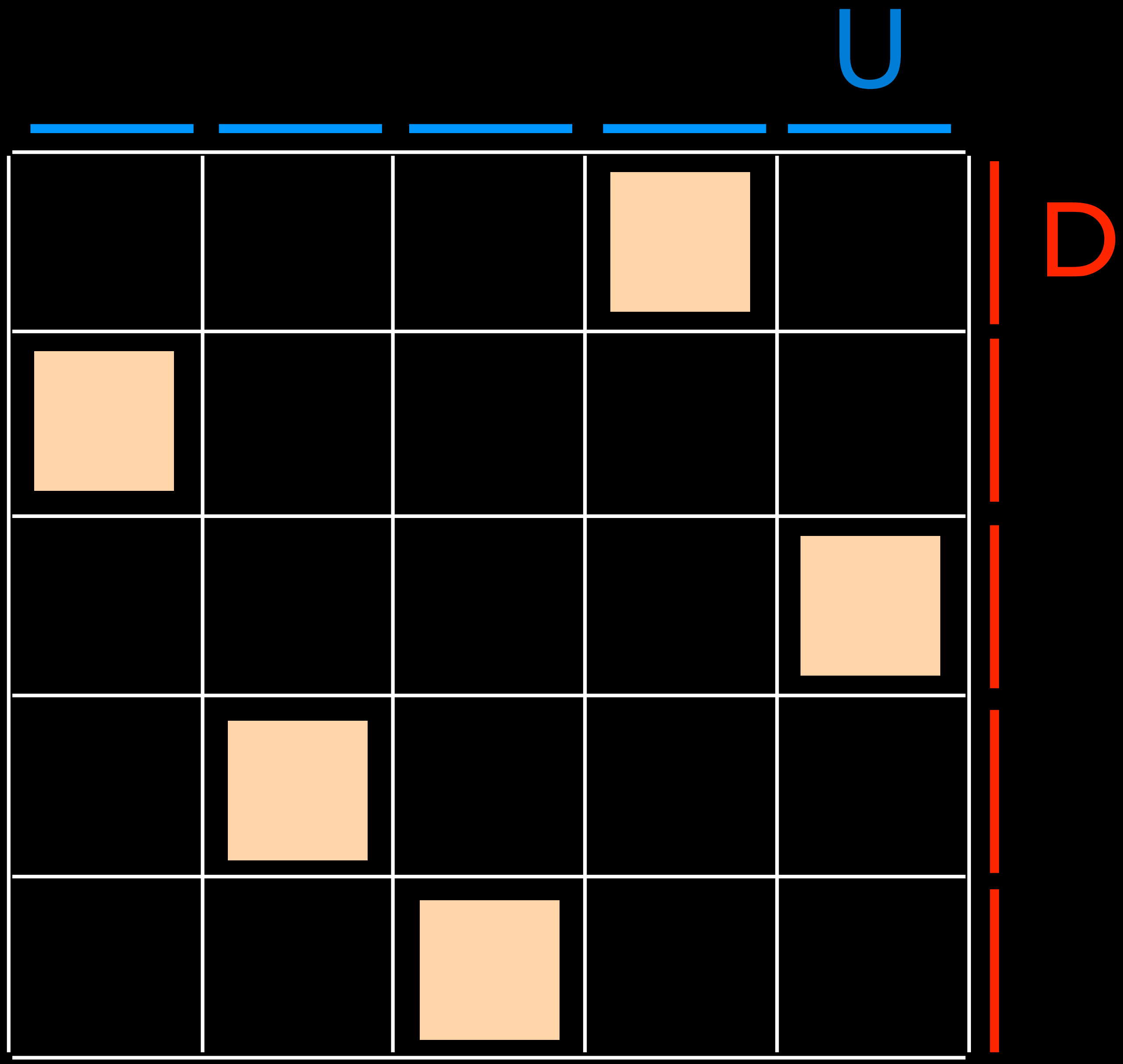
. w = D U U U D D U U D U U

Towers
placements on a
Ferrers diagram

$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} \, D^i U^i$$

normal ordering

$$c_{n,0} = n!$$



permutations n!

stationary probabilities for the PASEP

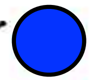
$$\begin{cases} DE = qED + D + E \\ DV = \bar{\beta} V & \bar{\beta} = 1/\beta \\ WE = \bar{\alpha} W & \bar{\alpha} = 1/\alpha \end{cases}$$

$$W E^i D^j V = \bar{\alpha}^i \bar{\beta}^j \underbrace{WV}_{1}$$

Cor. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ (PASEP) is

$$\text{proba}_\tau(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{T} q^{L(T)} \alpha^{-f(T)} \beta^{-u(T)}$$

(T: alternative tableaux, profile τ)

$f(T)$ nb of rows without  cell
$u(T)$ nb of columns without  cell
$L(T)$ nb of cells 

permutation tableau

S. Corteel, L. Williams (2007) (2008) (2009) permutation tableau

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)

Corteel, Williams (2006) PASEP

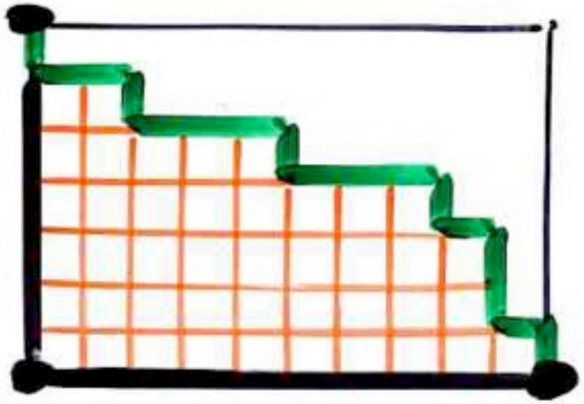
Partially Asymmetric Exclusion Process

M. Josuat-Vergès (2007)

permutation tableaux

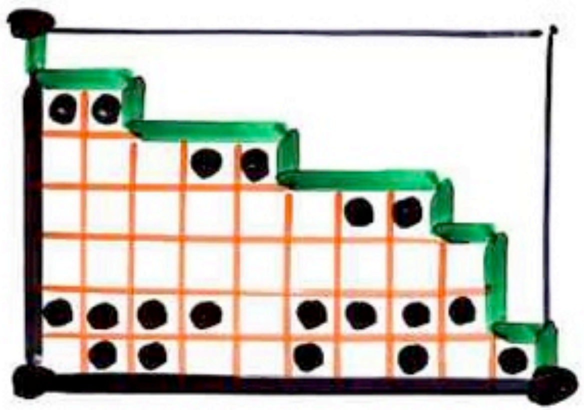
Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

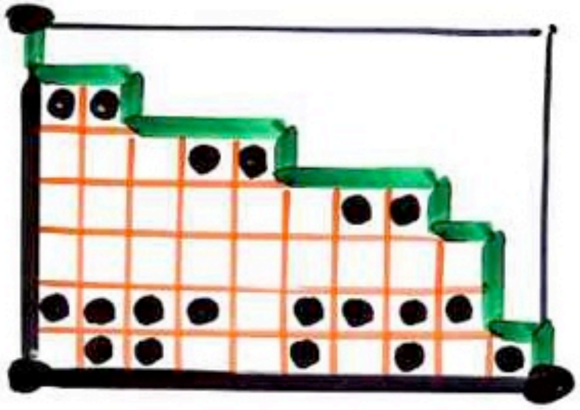
(i)

(ii)

□ = 0   ▣ = 1

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



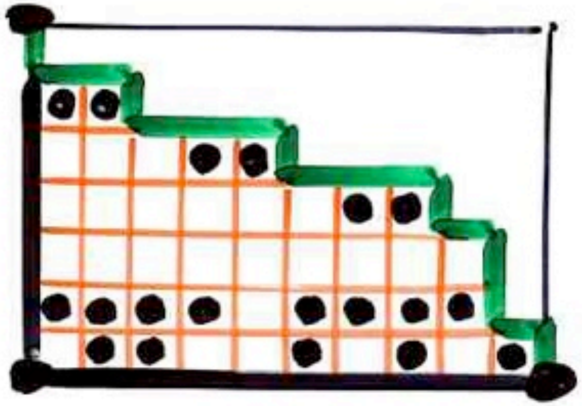
filling of the cells with 0 and 1

(i) in each column : at least one 1

(ii)

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



□ = 0   ▣ = 1

filling of the cells with 0 and 1

(i) in each column : at least one 1

(ii)

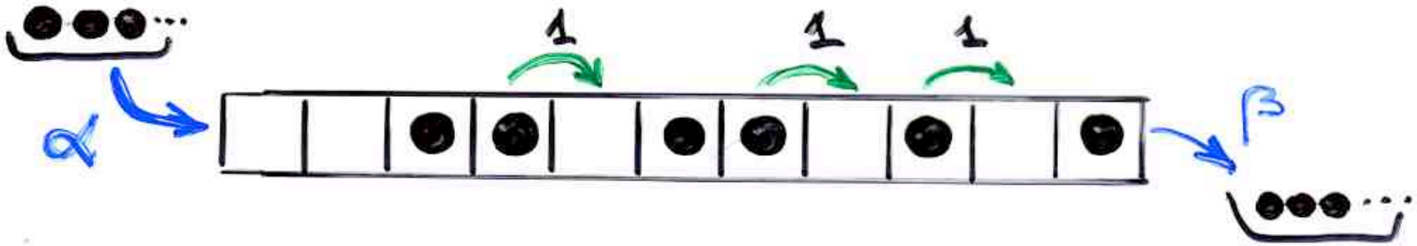
forbidden

TASEP

Totally asymmetric exclusion process

TASEP

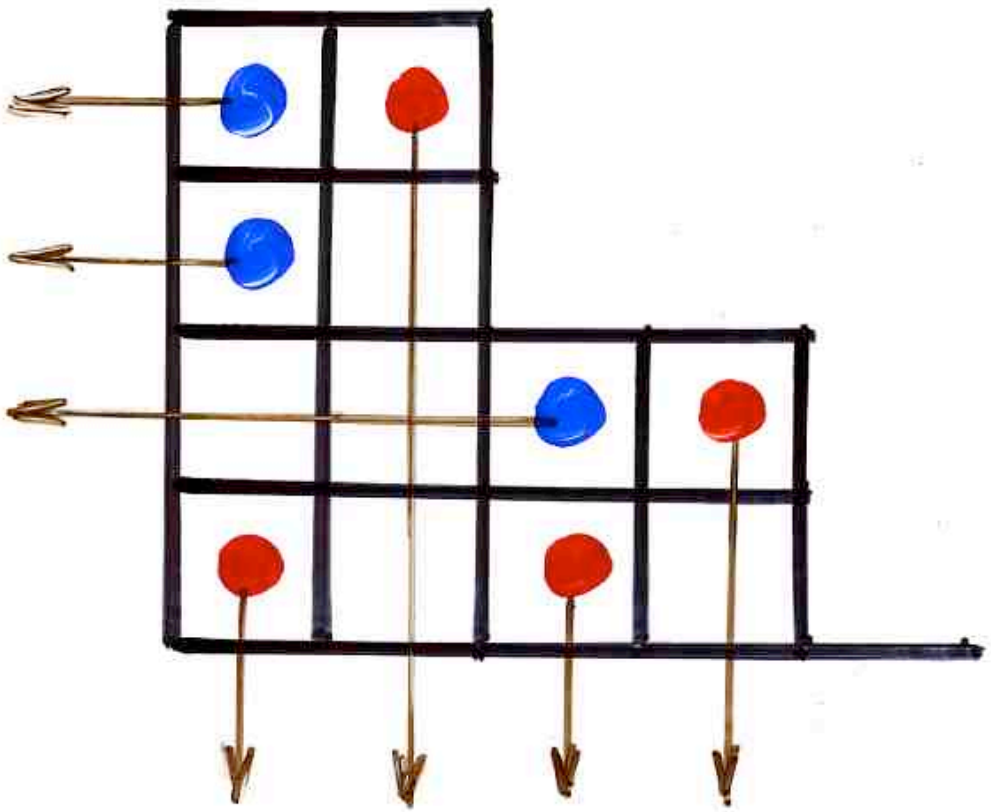
"totally asymmetric exclusion process"



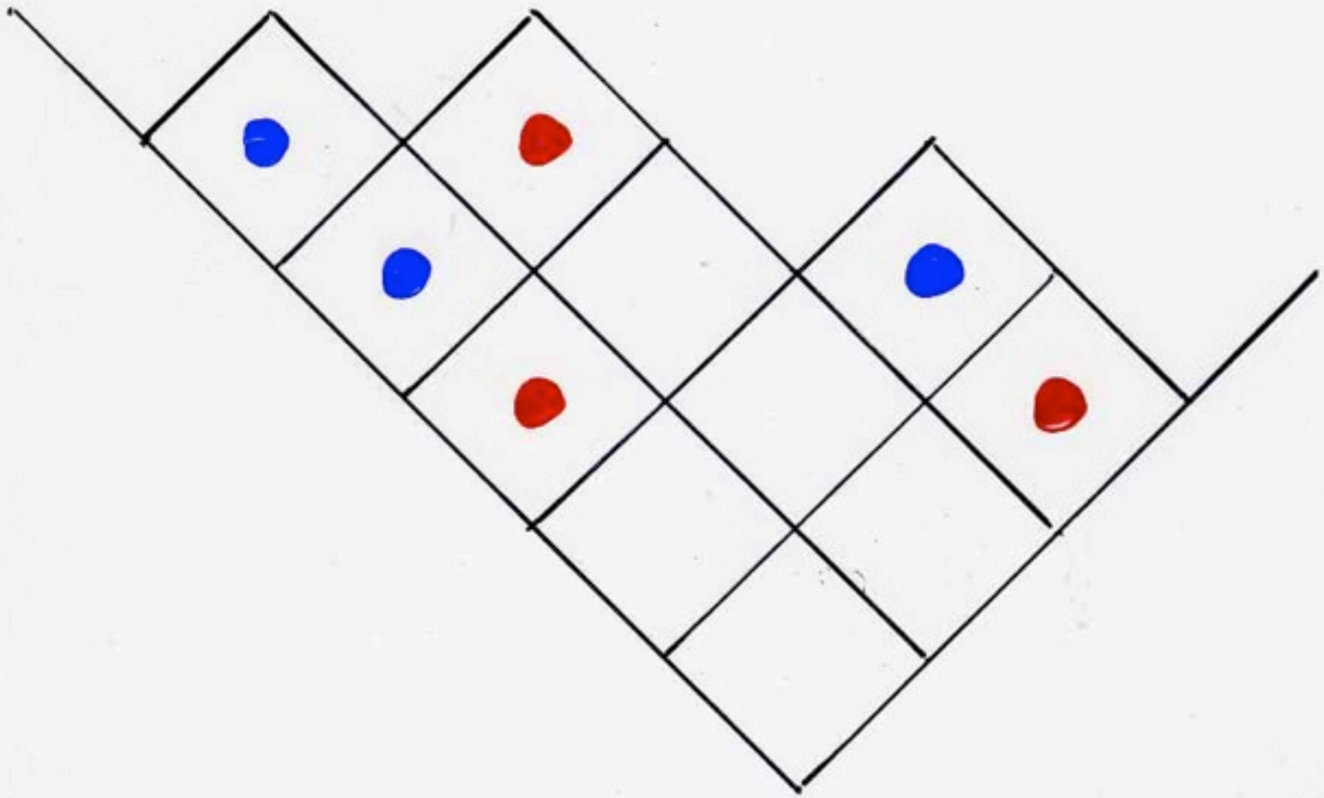
Def Catalan alternative tableau T

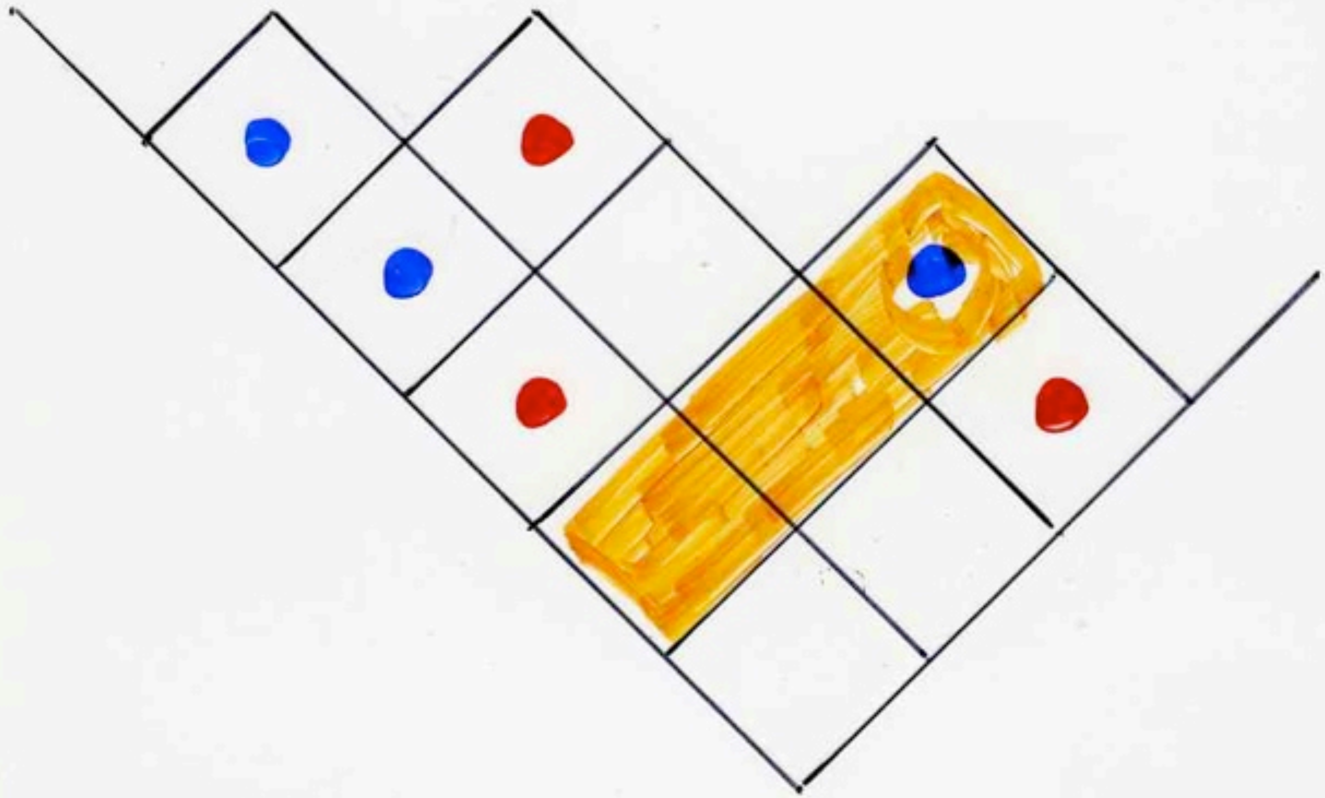
alt. tab. without cells ☒

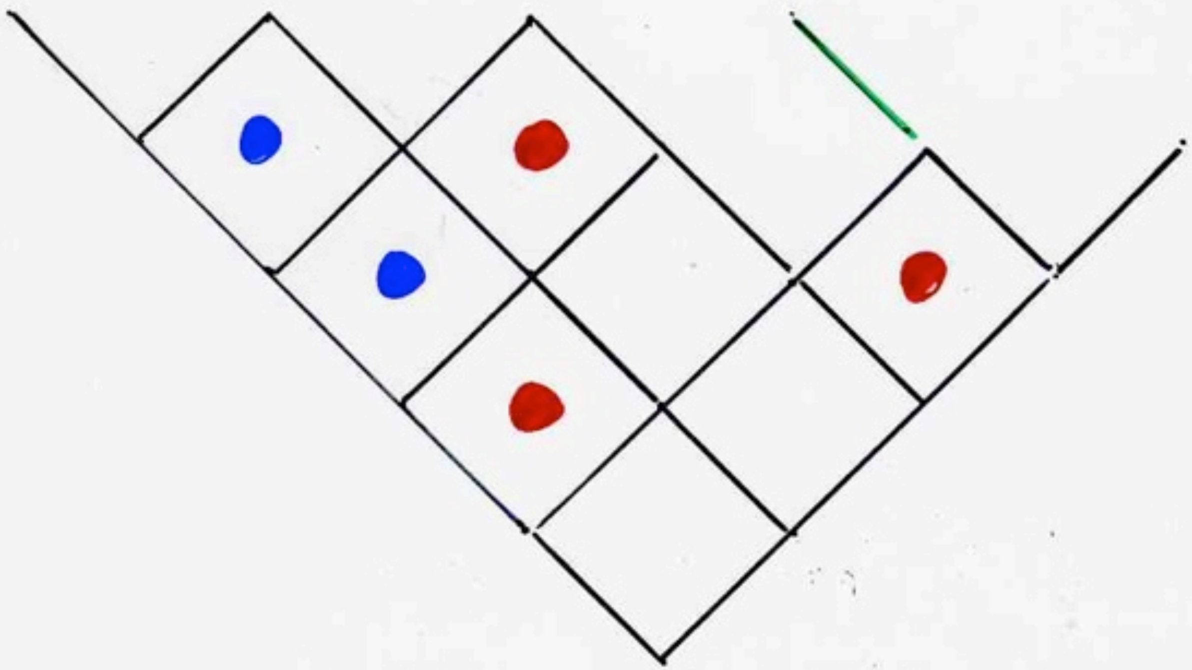
i.e: every empty cell is below a red cell or on the left of a blue cell

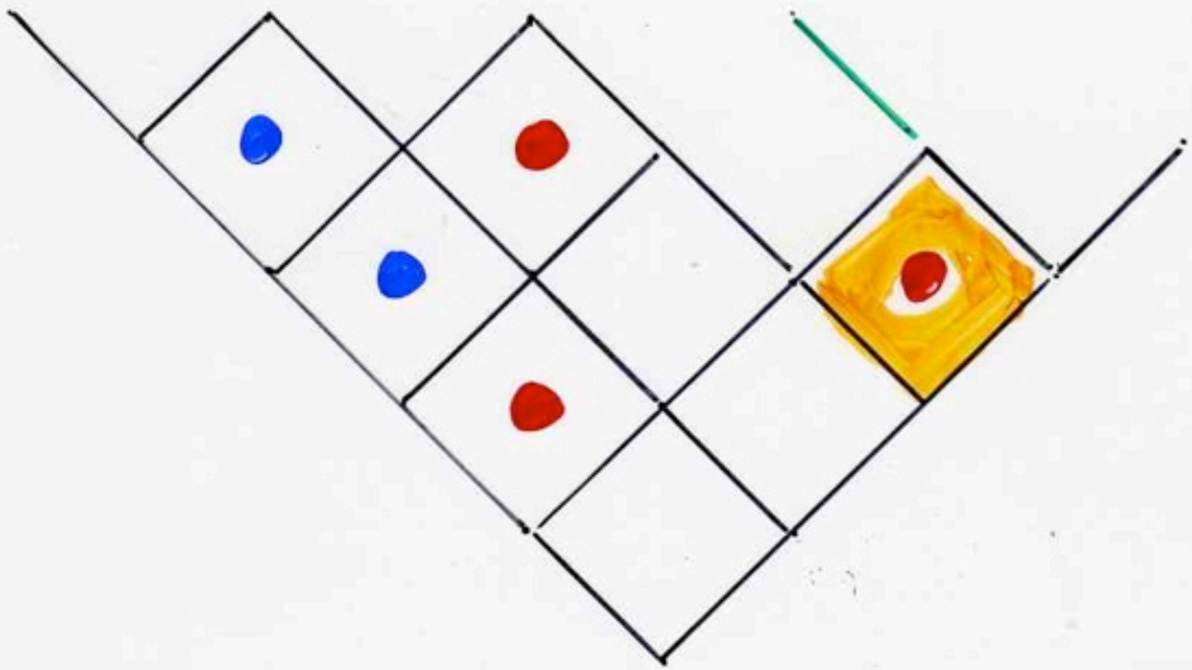


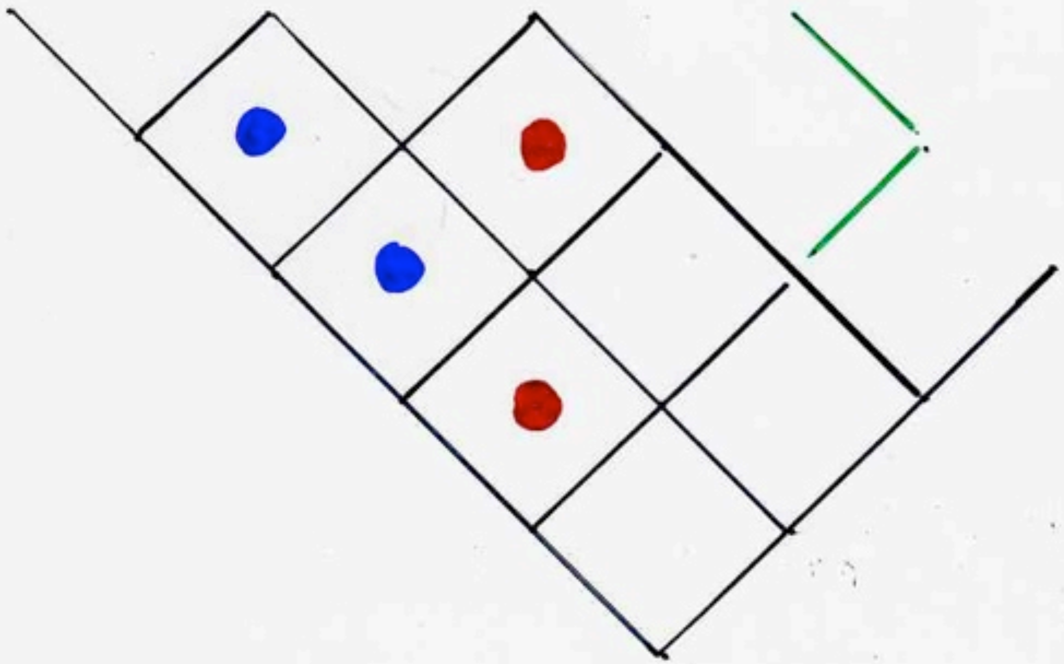
Bijection
alternative Catalan tableaux
binary trees

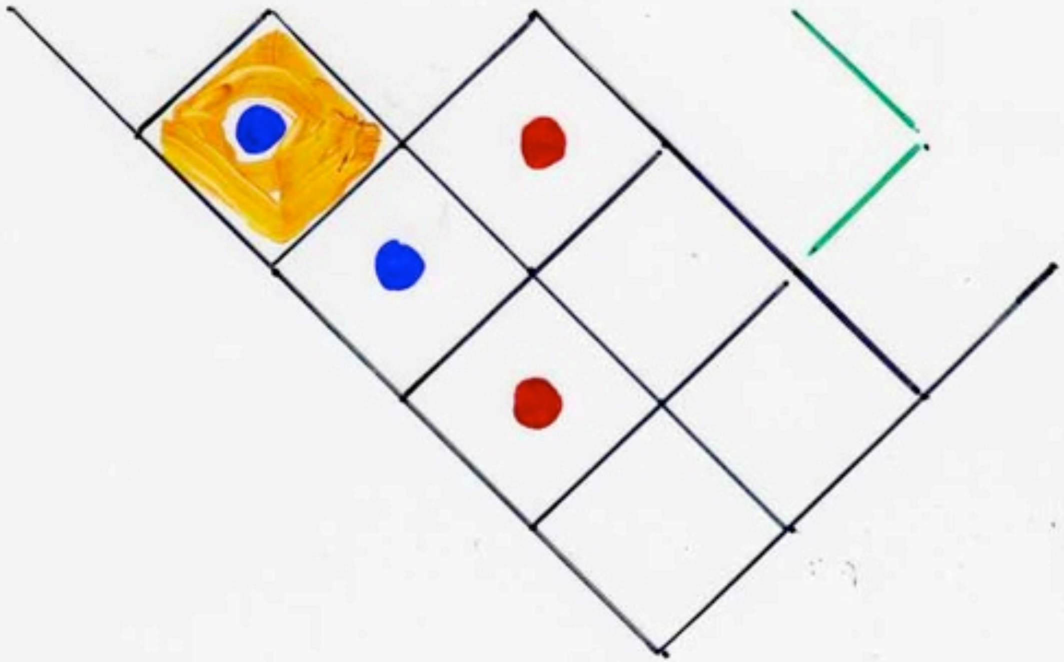


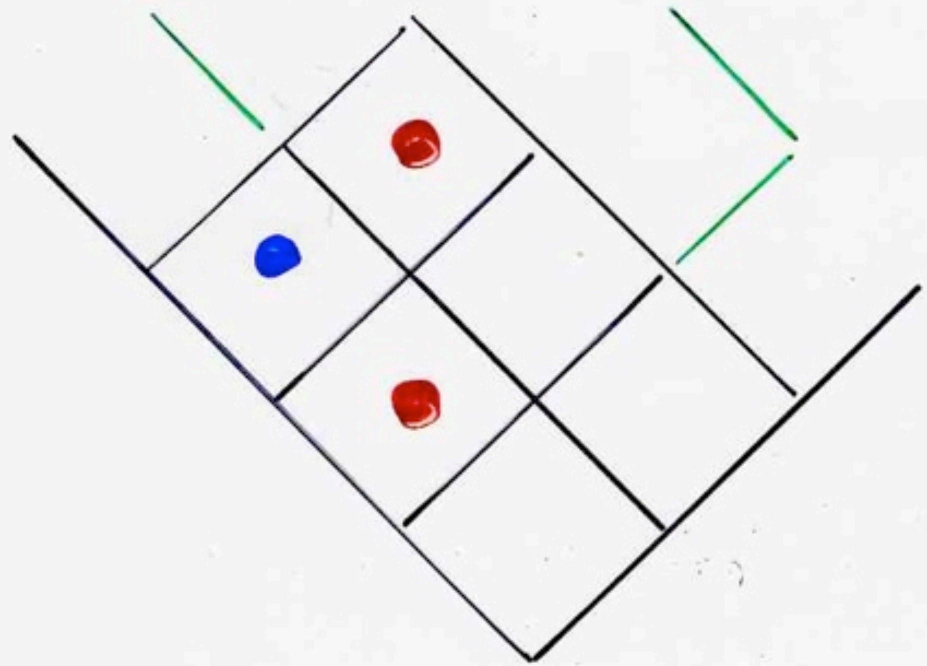


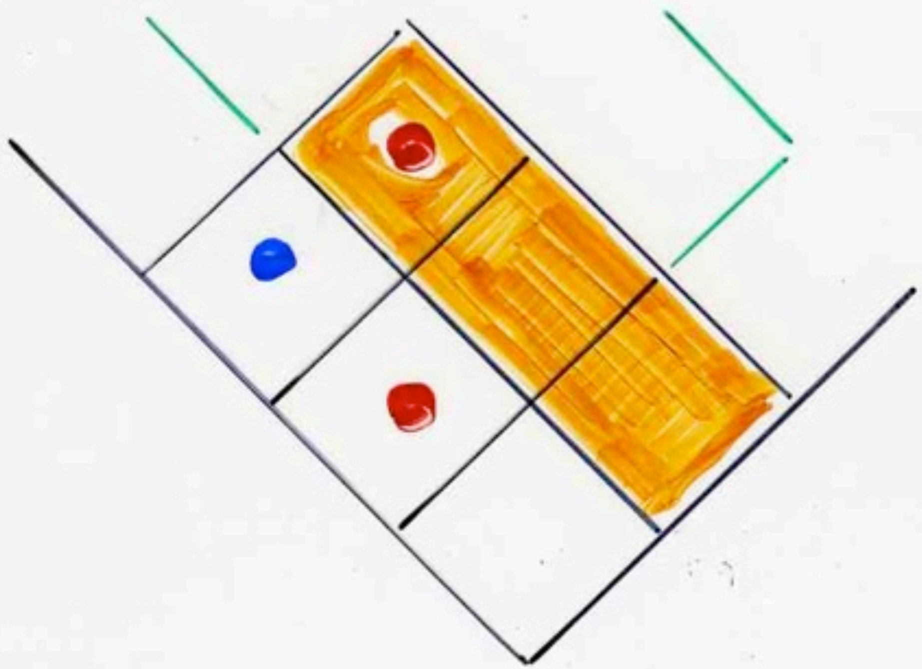


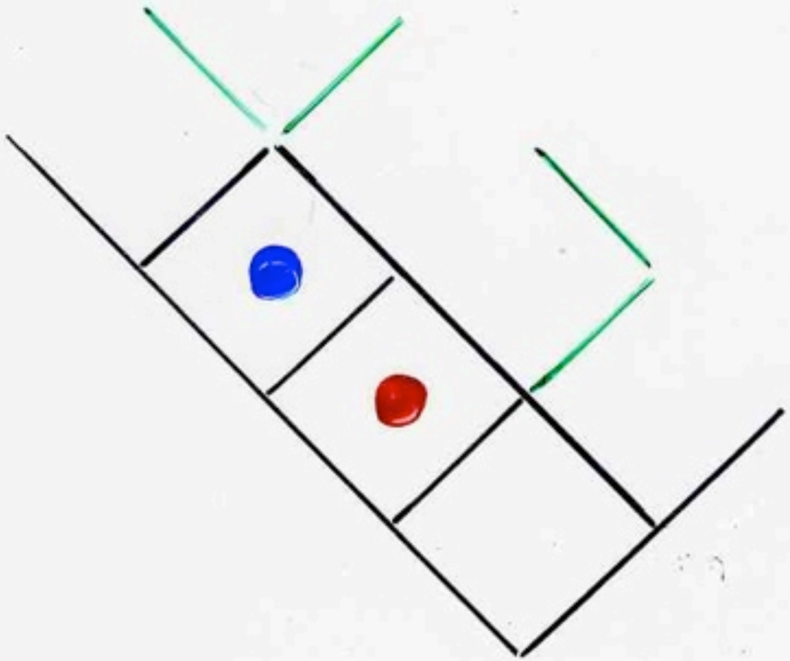


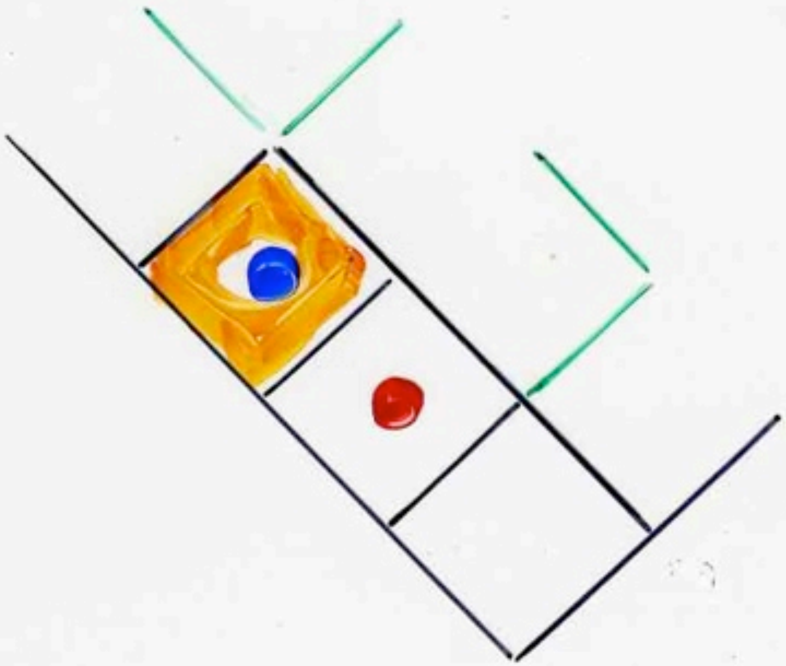


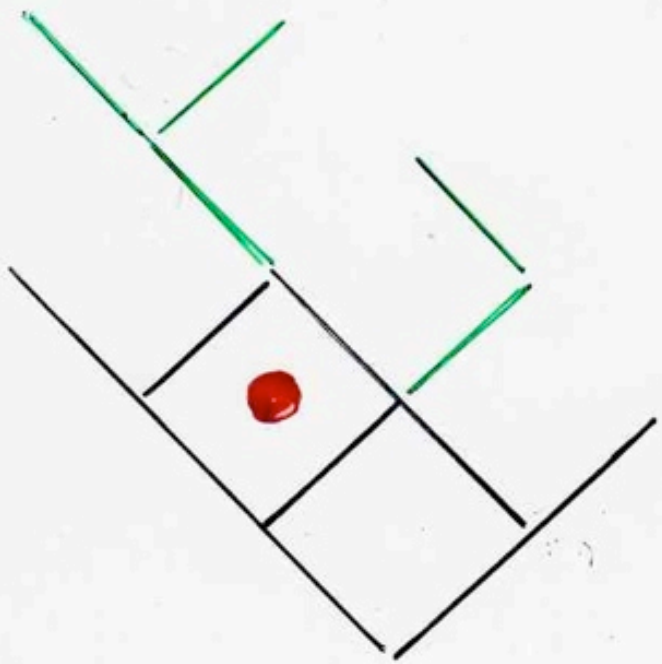


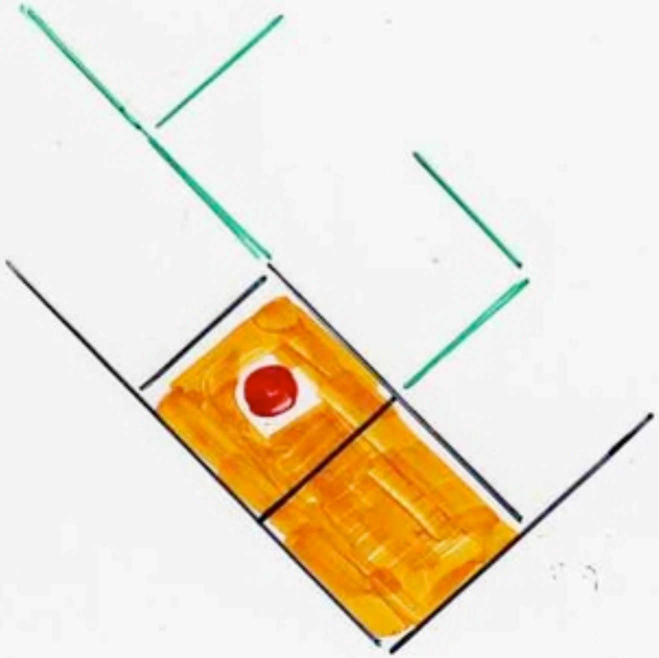


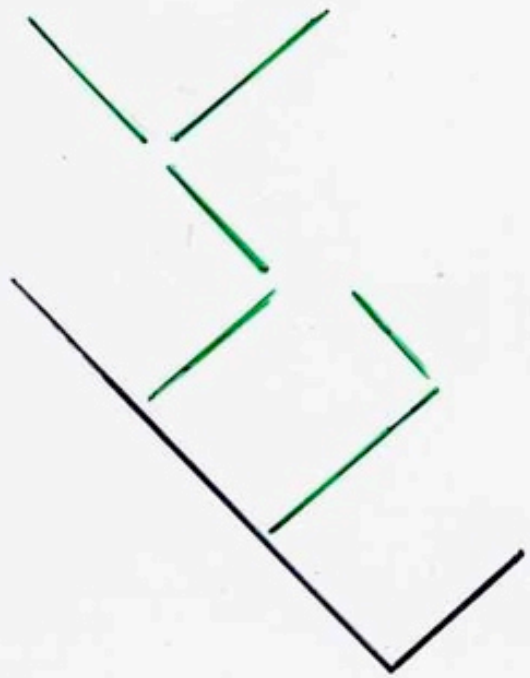






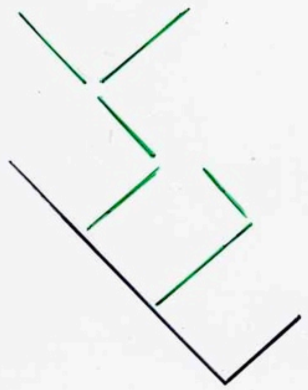


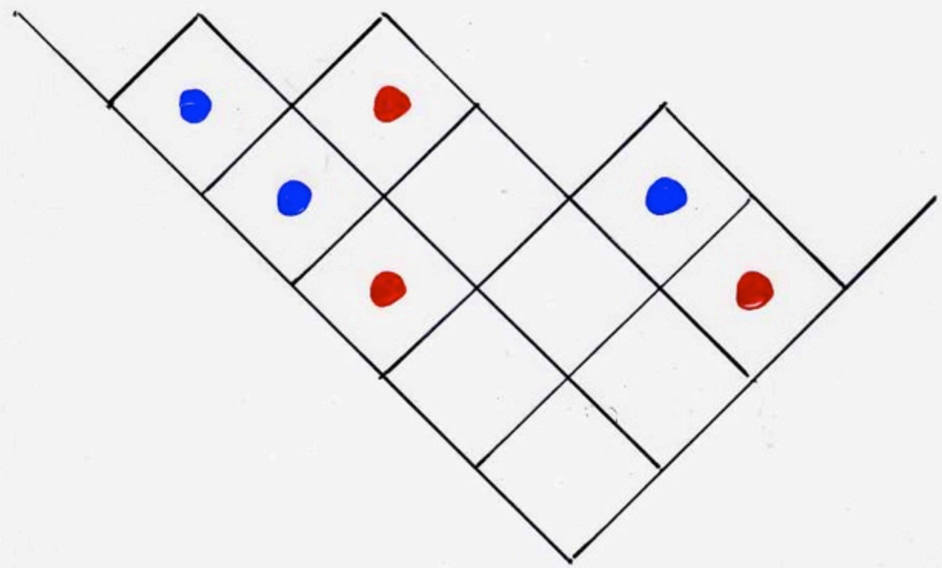




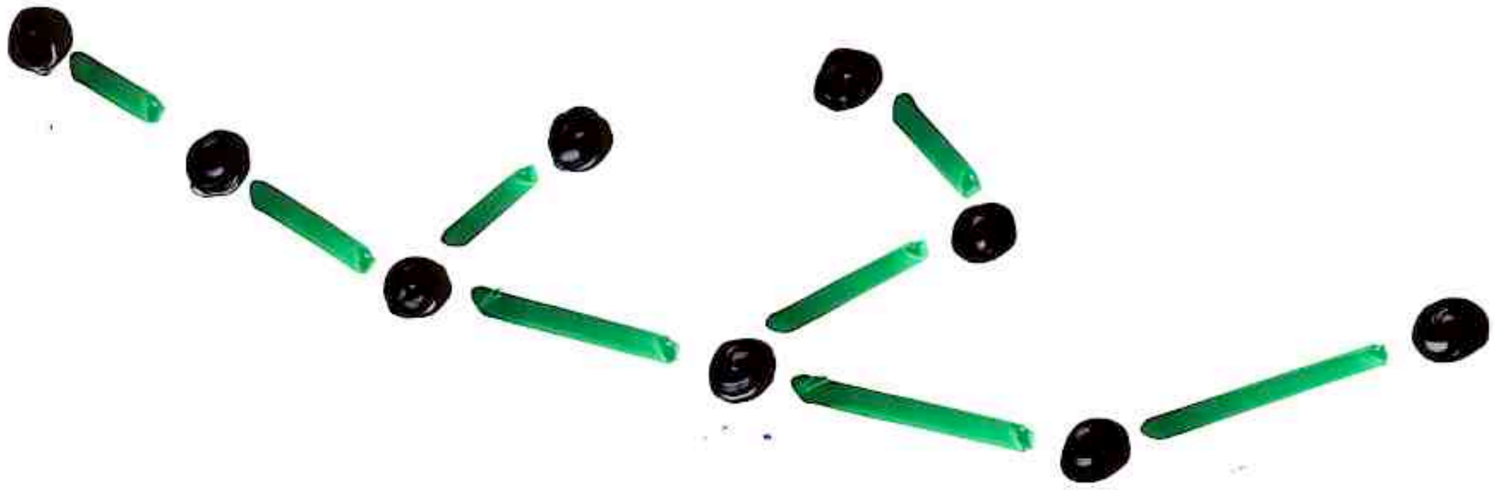
Bijection

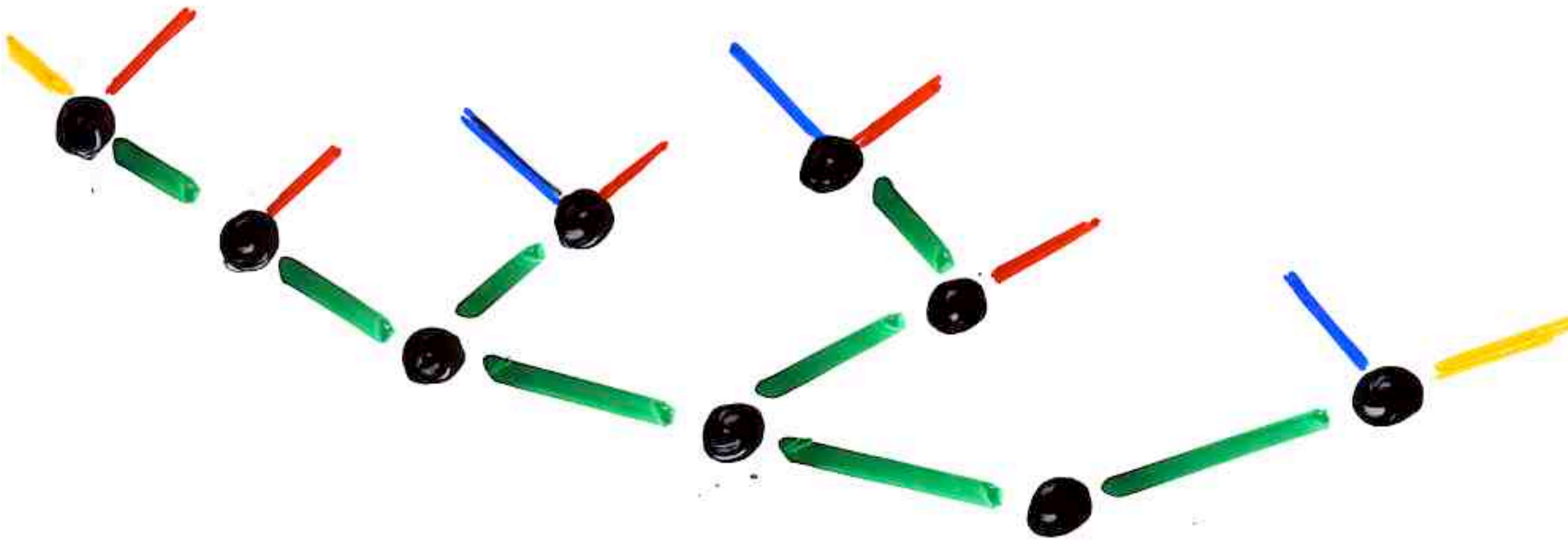
tableaux alternatifs de Catalan taille n ⟷ arbres binaires n arêtes





profil (bord) du diagramme de Ferrers ⟵---⟶ canopée





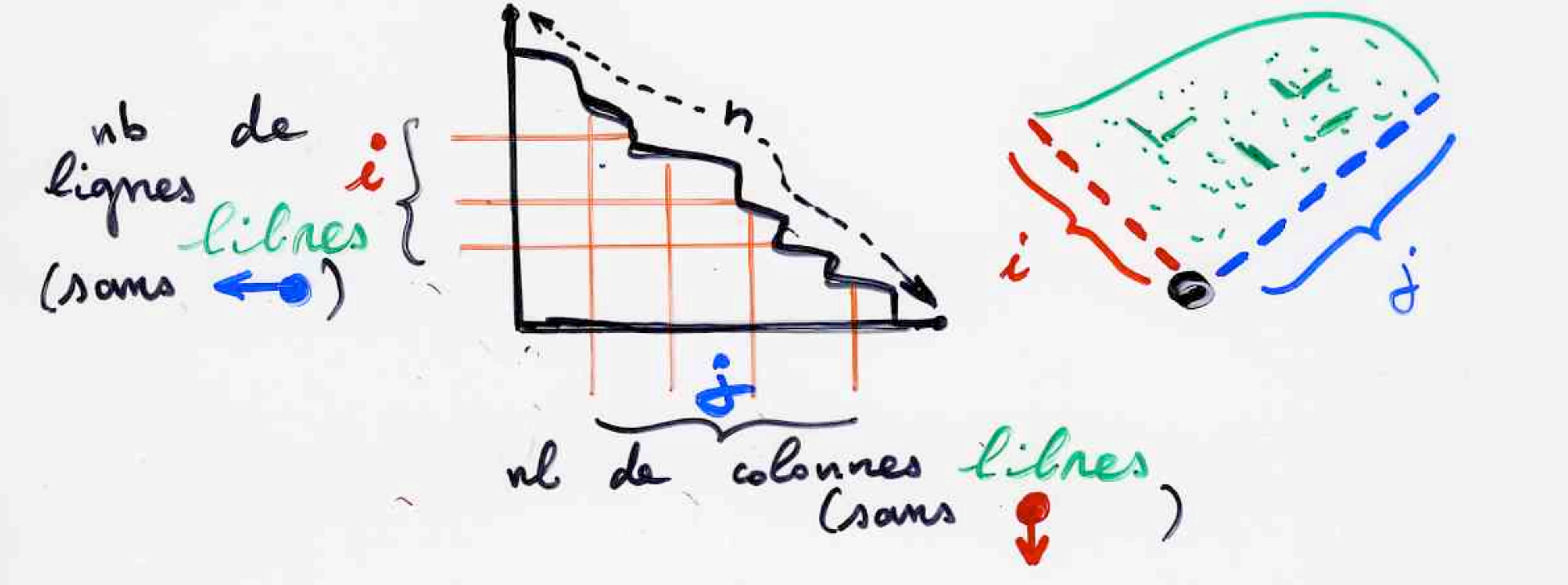
canopy of a binary tree

$c(B) = - - + - + - - +$

Bijection

tableaux alternatifs de Catalan taille n ⟷ arbres binaires n arêtes

profil (bord) du diagramme de Ferrers ⟷ canopée



$$s = (\tau_1, \dots, \tau_n)$$

$$P_n(s;\alpha,\beta) = \frac{1}{Z_n} \sum_{B} \bar{\alpha}^{\,lb(B)} \; \bar{\beta}^{\,rb(B)}$$

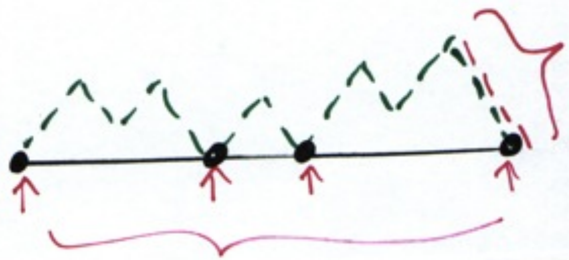
binary trees
canopy s

Derrida, Evans, Hakim, Pasquier (1993)

$$Z_n = \sum_{i=1}^{n} \frac{i}{2n-i} \binom{2n-i}{n} \frac{\bar{\alpha}^{(i+1)} - \bar{\beta}^{(i+1)}}{\bar{\alpha} - \bar{\beta}}$$

partition function

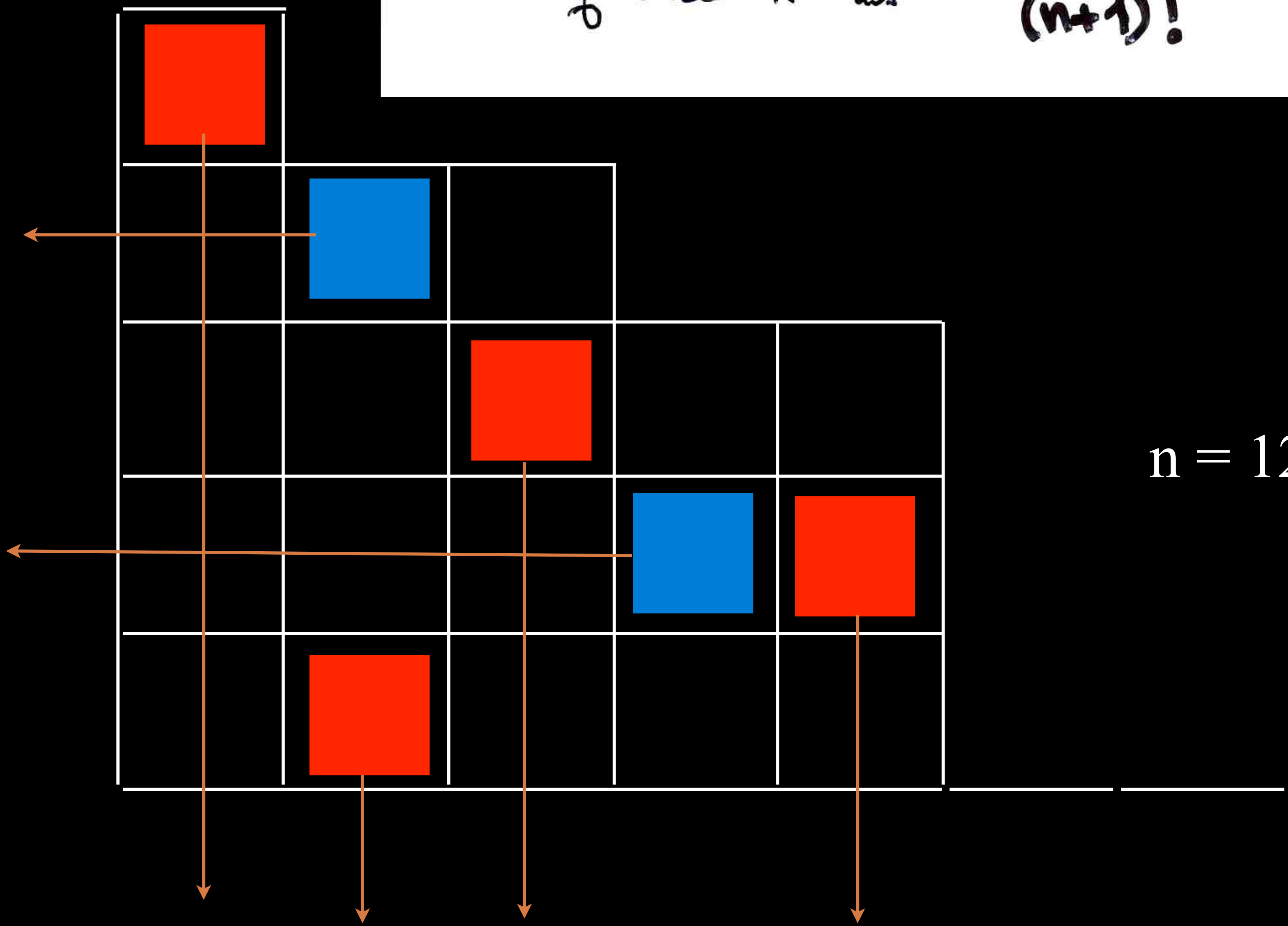
"ballot" numbers



combinatorial physics
or
integrable combinatorics

number of
alternative tableaux

Prop. The number of alternative tableaux of size n is (n+1)!



n = 12

ex: $n=2$

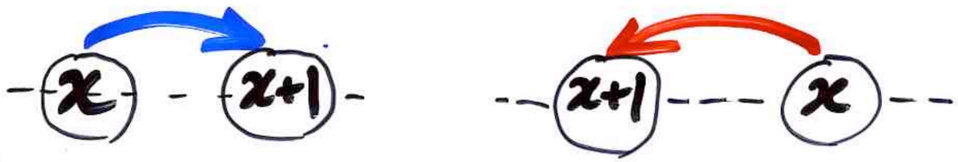
The “exchange-fusion” algorithm

Déf — Permutation $\sigma = \sigma(1) \cdots \sigma(n)$

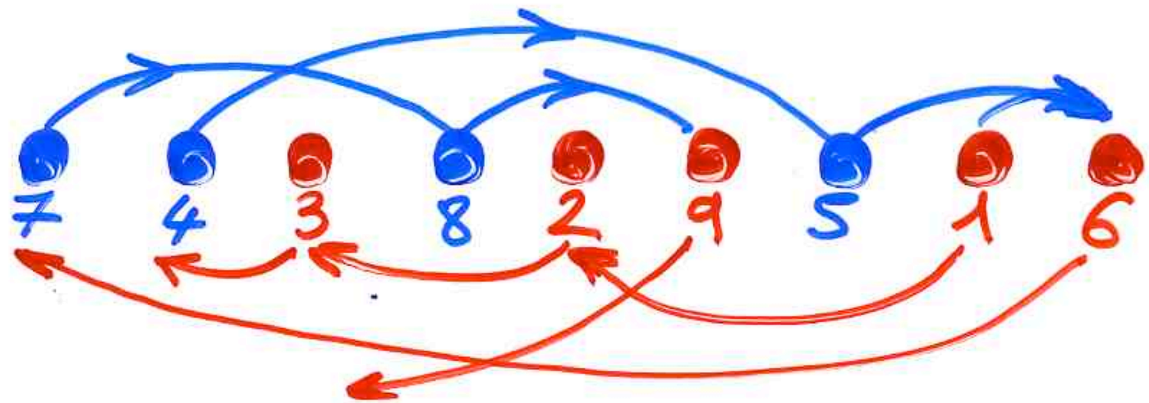
$x = \sigma(i)$, $1 \leq x < n$

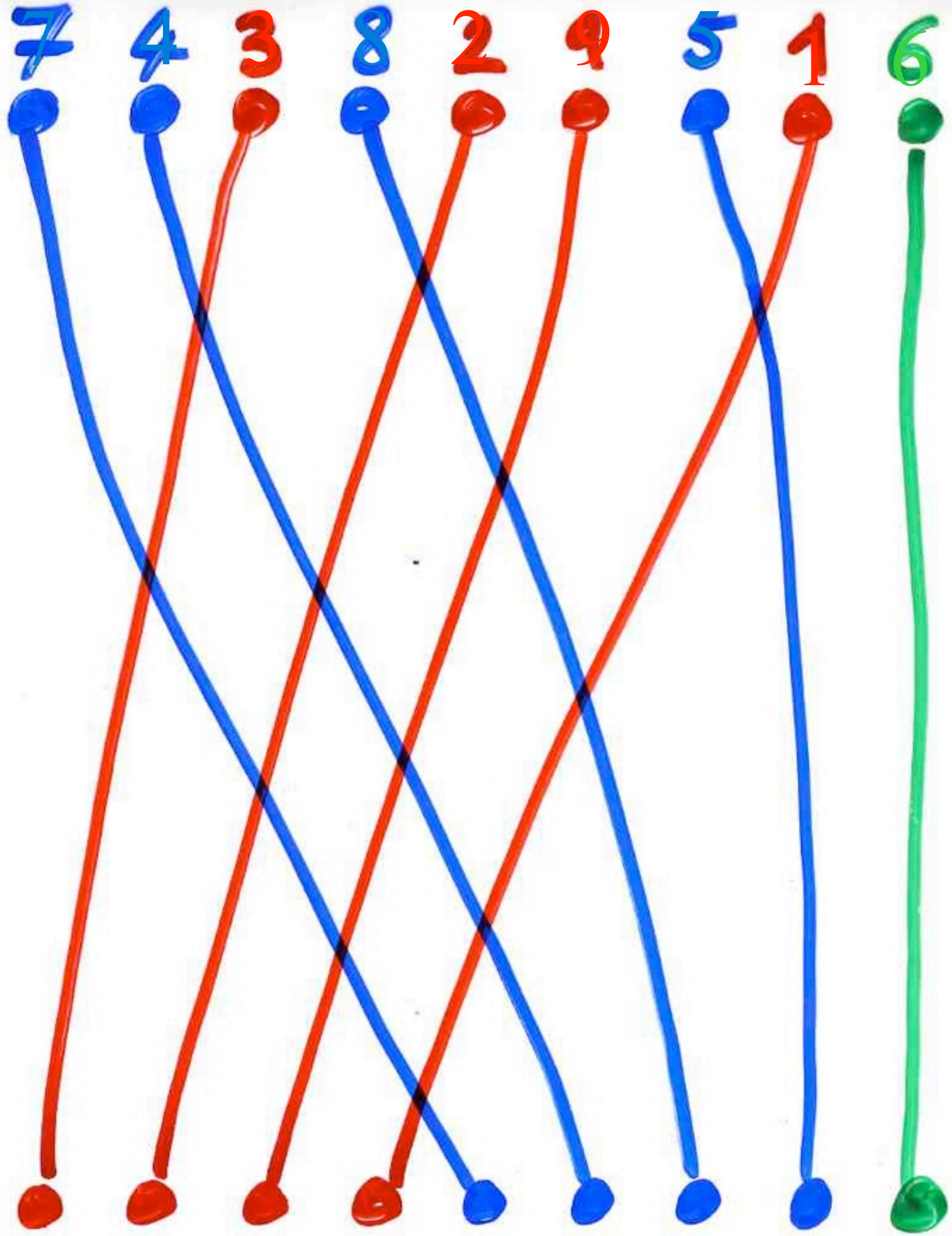
(valeur) x $\begin{cases} \text{avance} \\ \text{recul} \end{cases}$ $x+1 = \sigma(j)$, $\begin{cases} i < j \\ j < i \end{cases}$

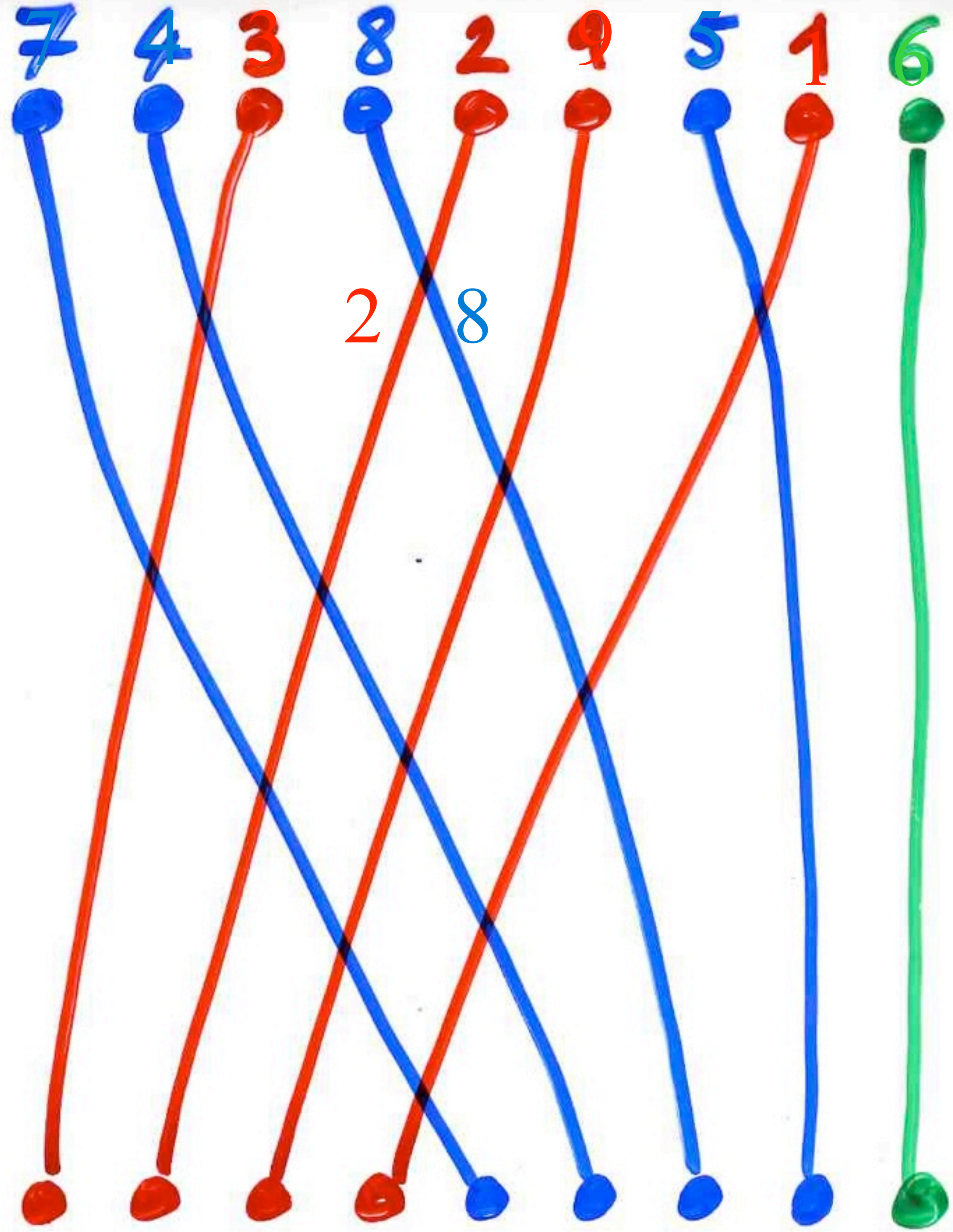
- convention $x = n$ est un recul



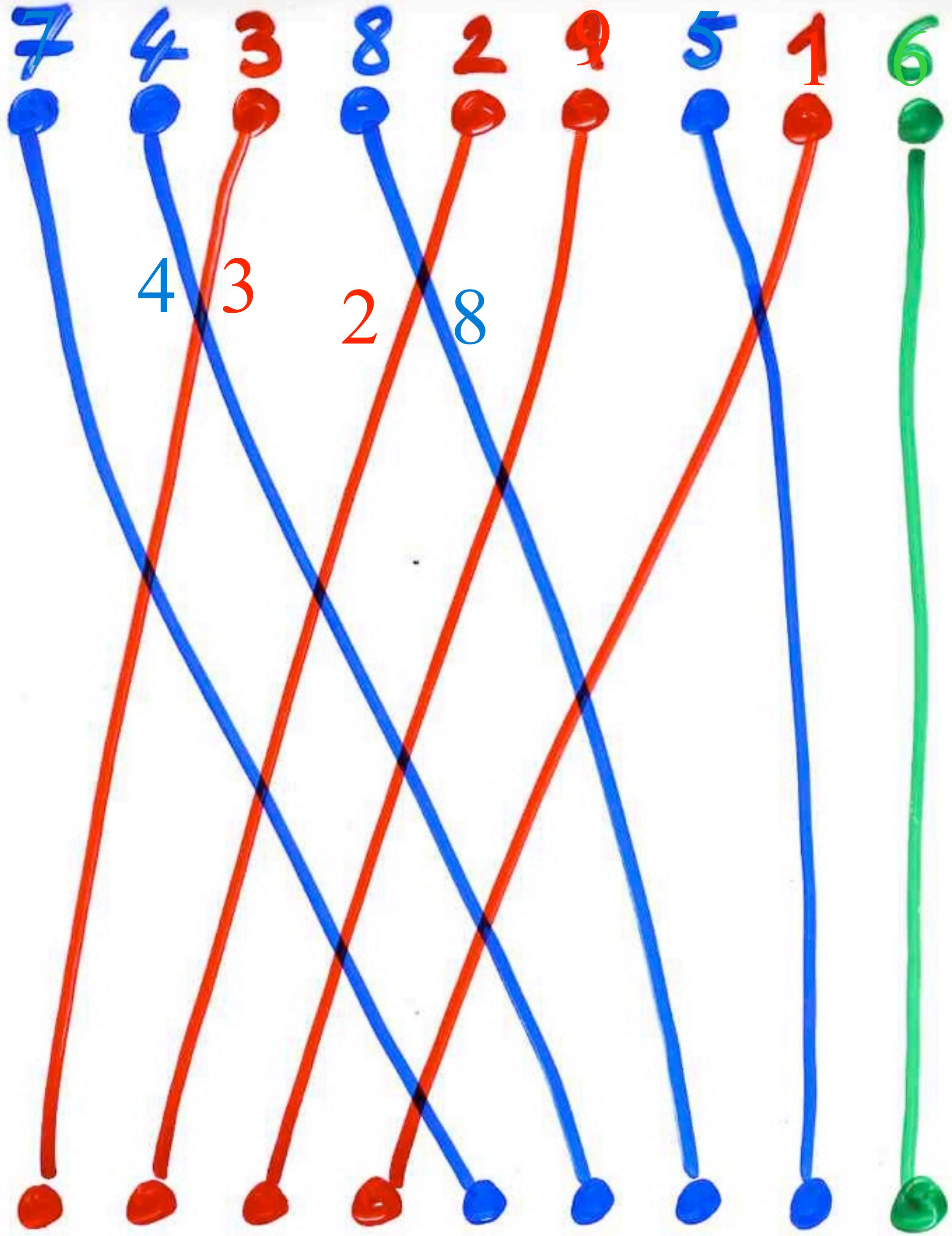
$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$



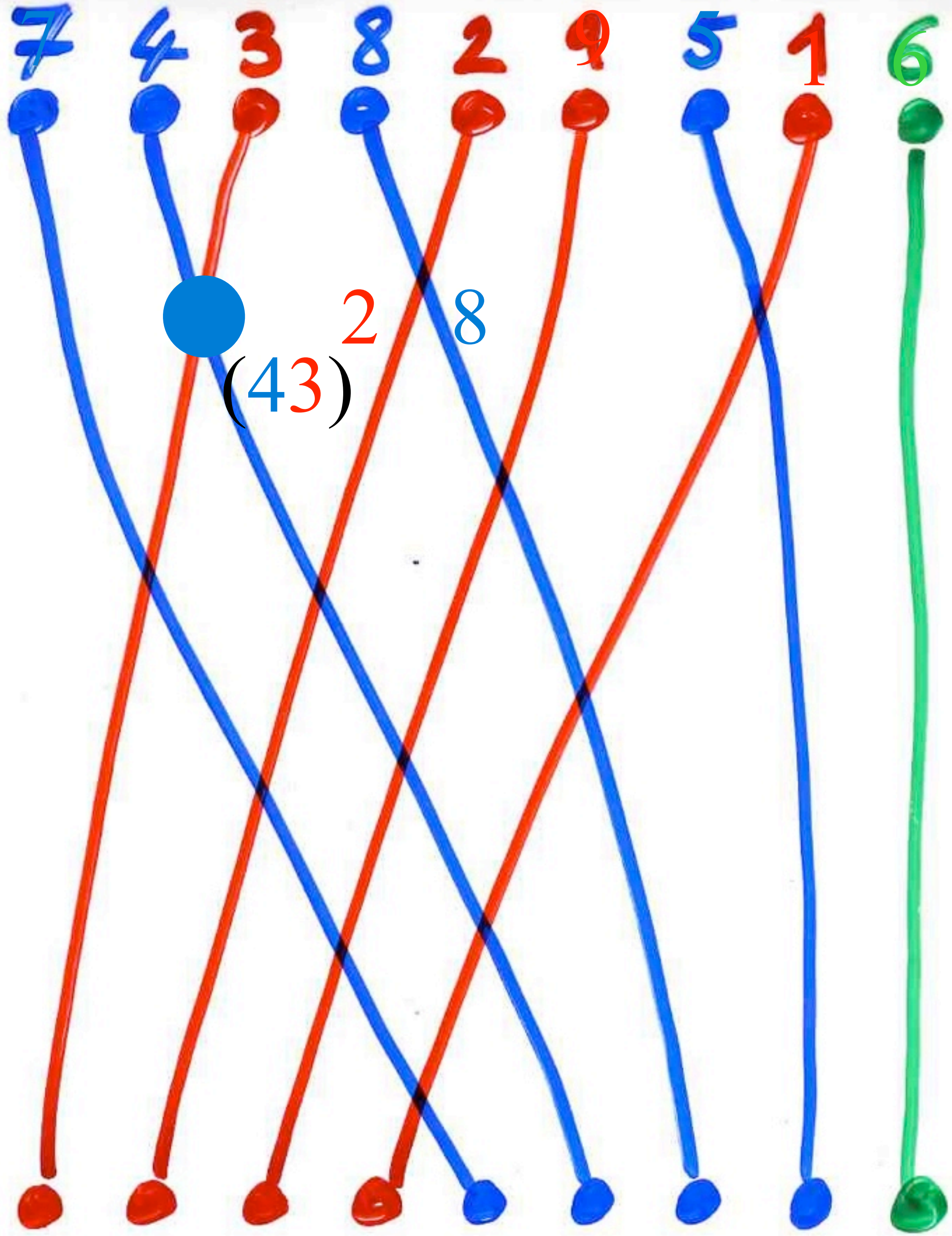

7 4 3 8 2 9 5 1 6

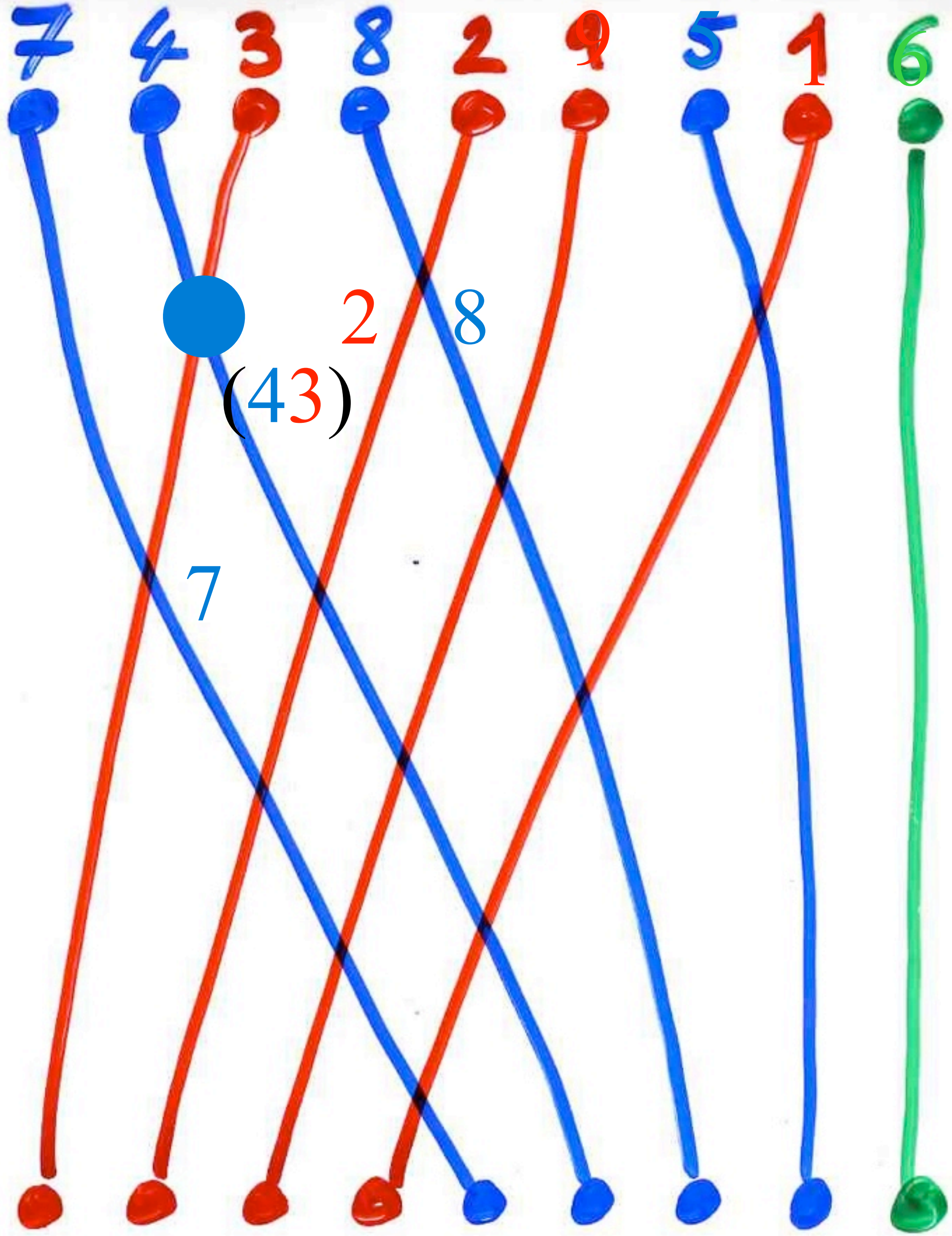


7 4 3 8 2 9 5 1 6
2 8

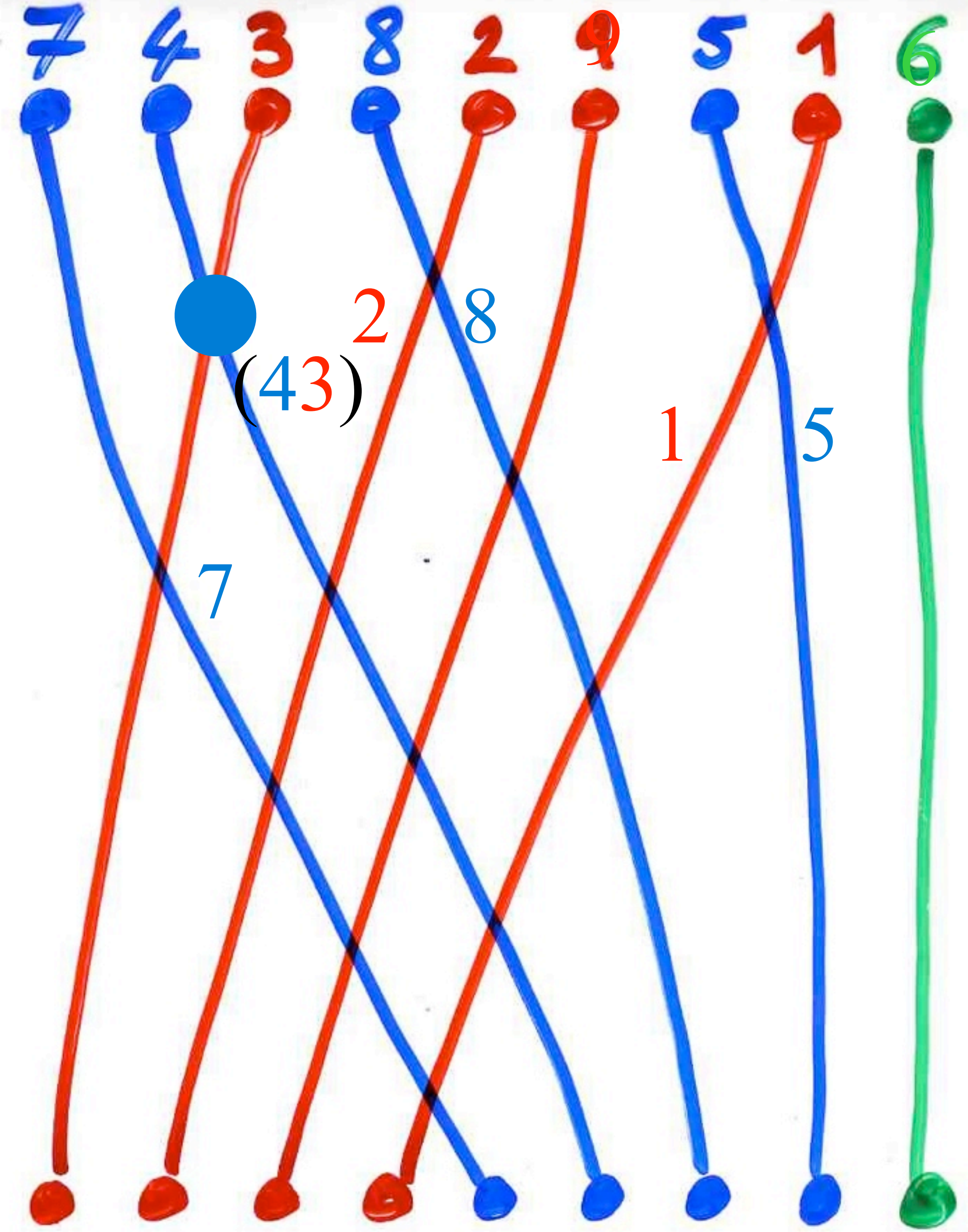


7 4 3 8 2 9 5 1 6
4 3 2 8

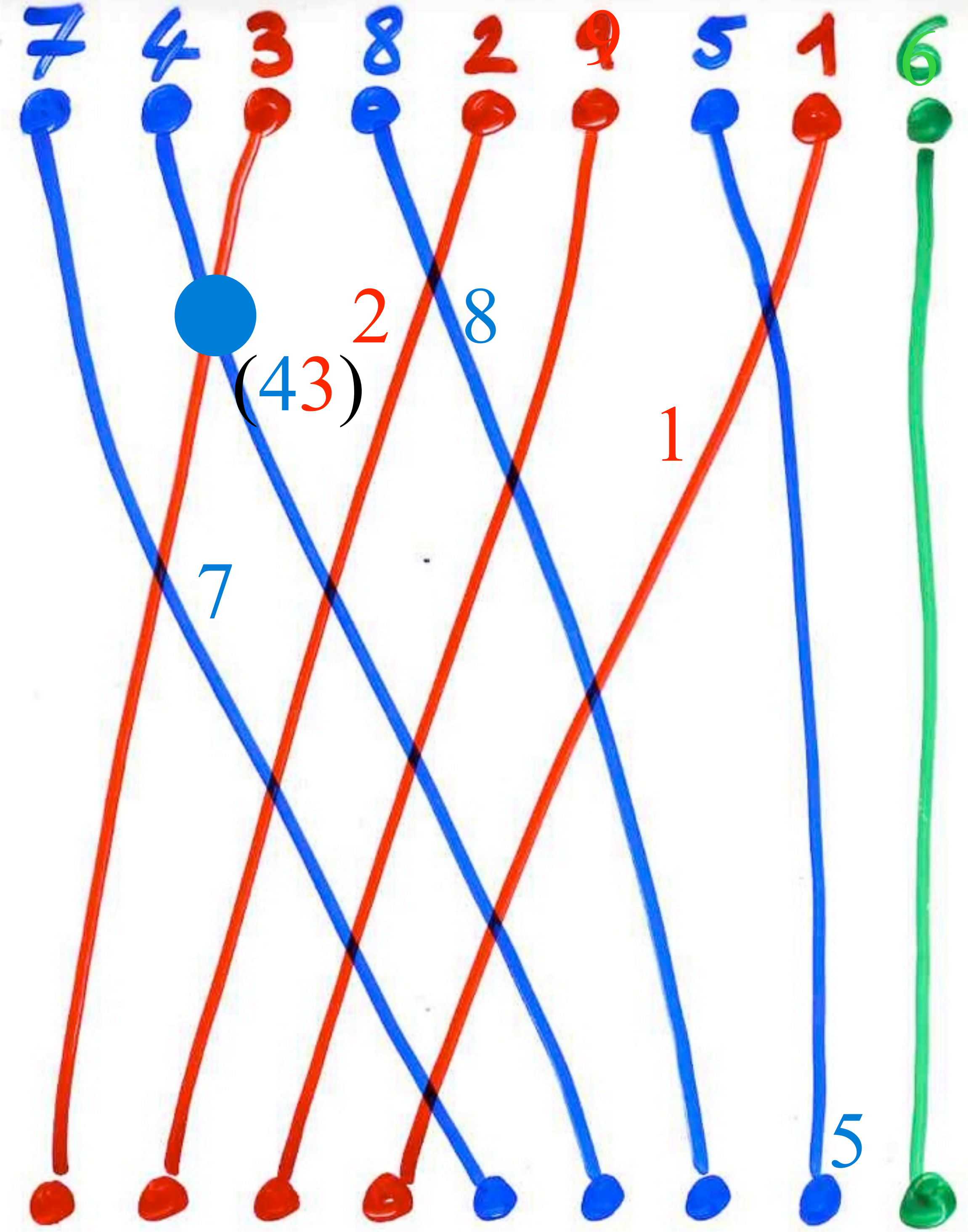

7 4 3 8 2 9 5 1 6
(43)
2
8



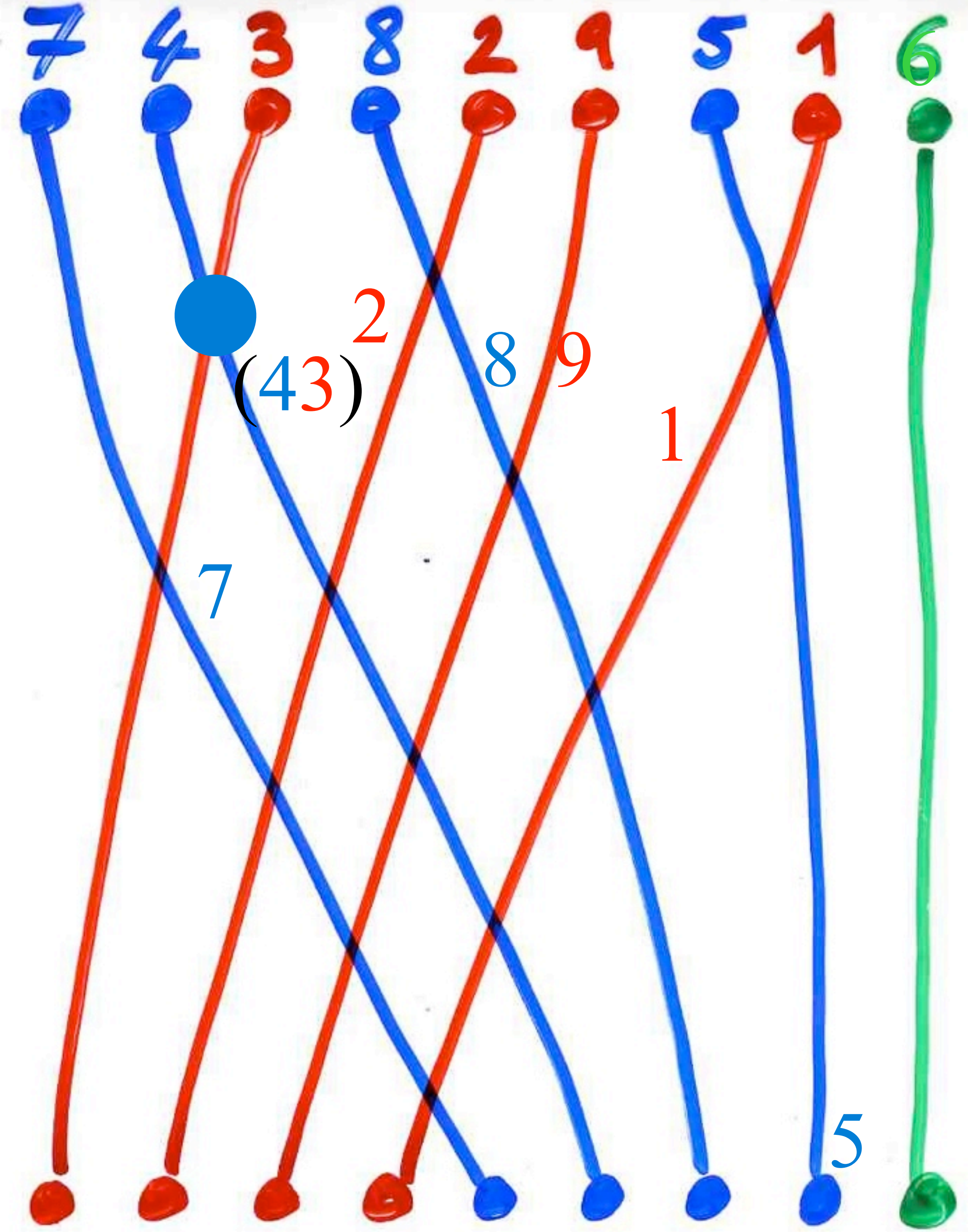
7 4 3 8 2 9 5 1 6
2
8
(43)
7

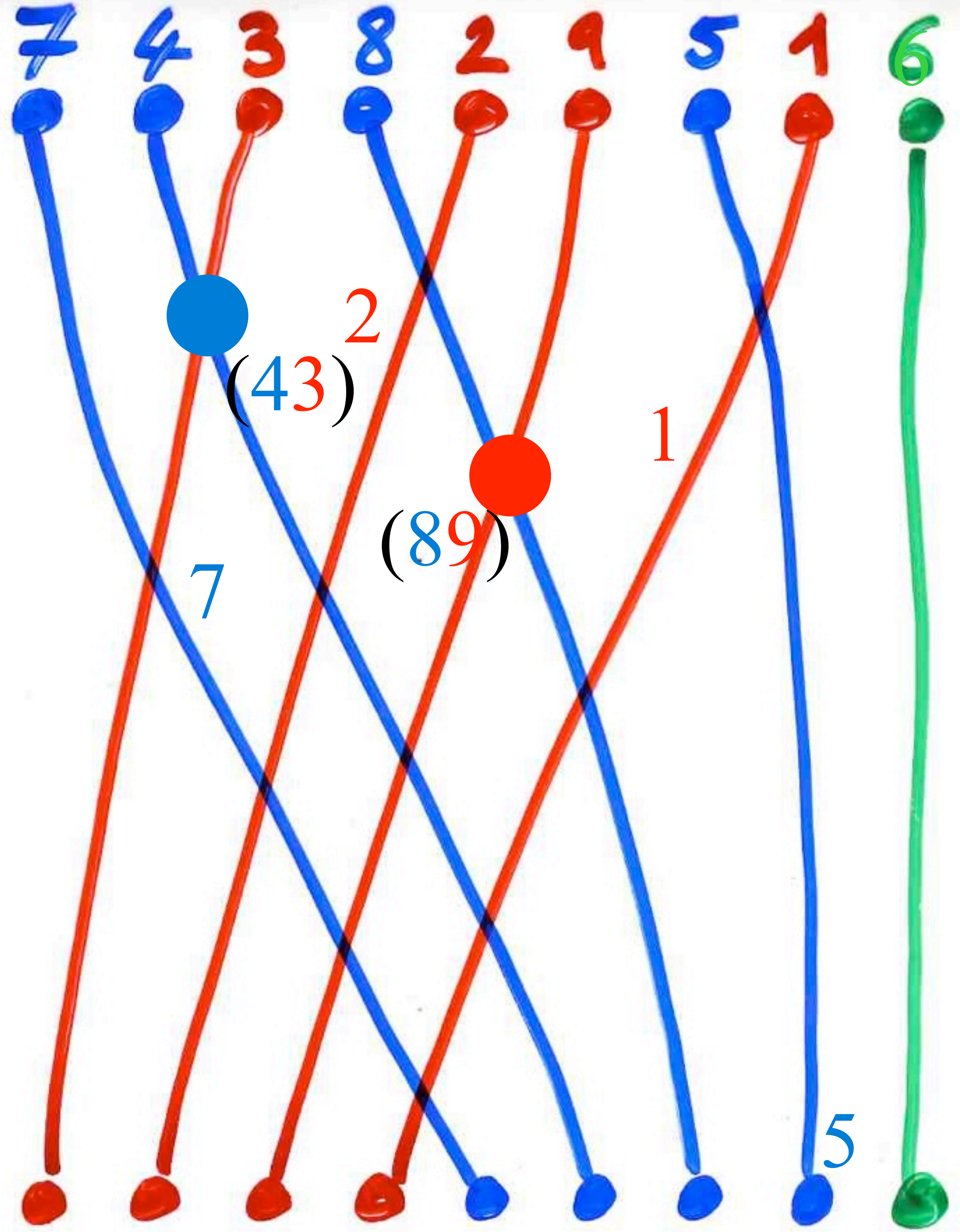


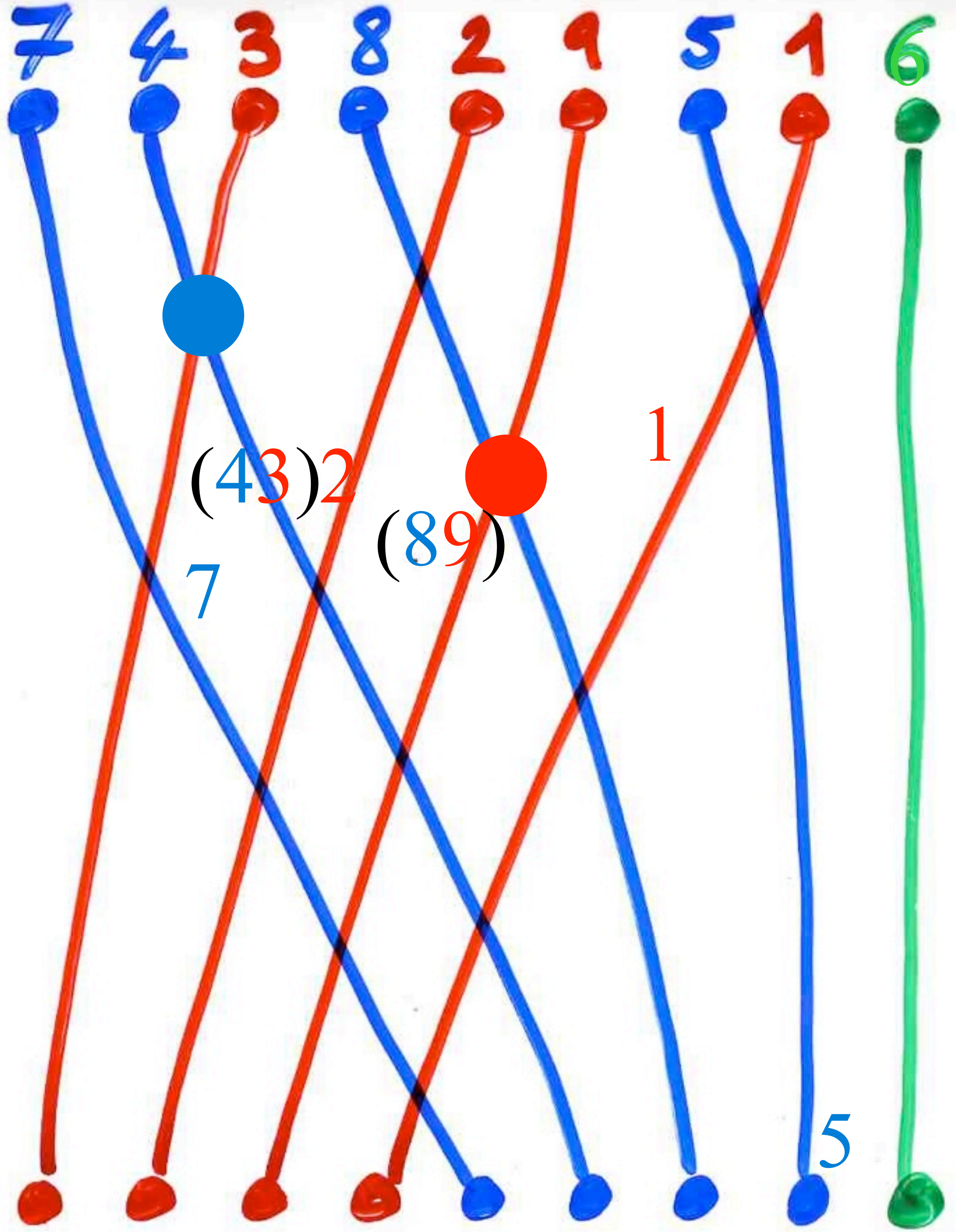
7 4 3 8 2 9 5 1 6
2
8
(43)
1
5
7

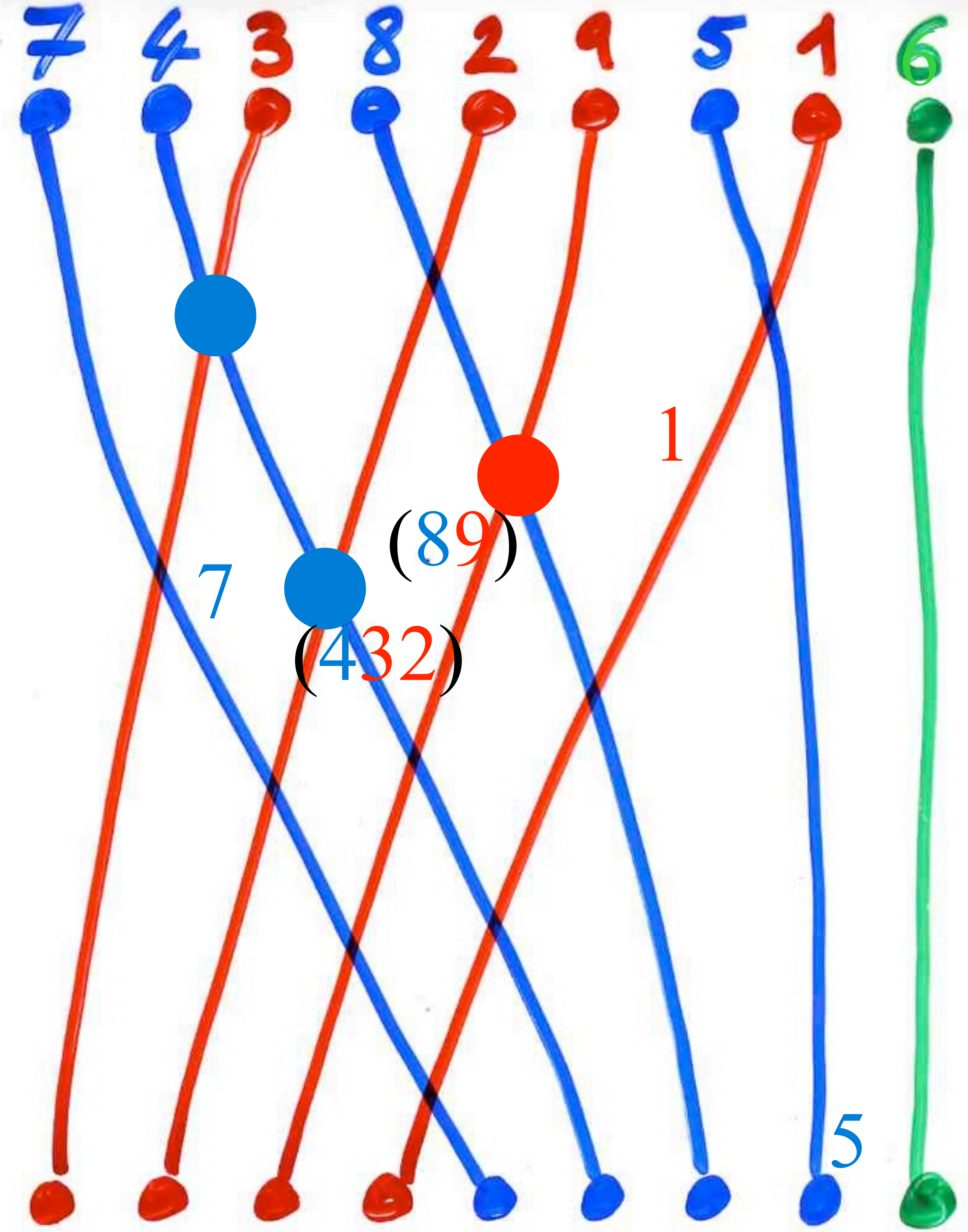


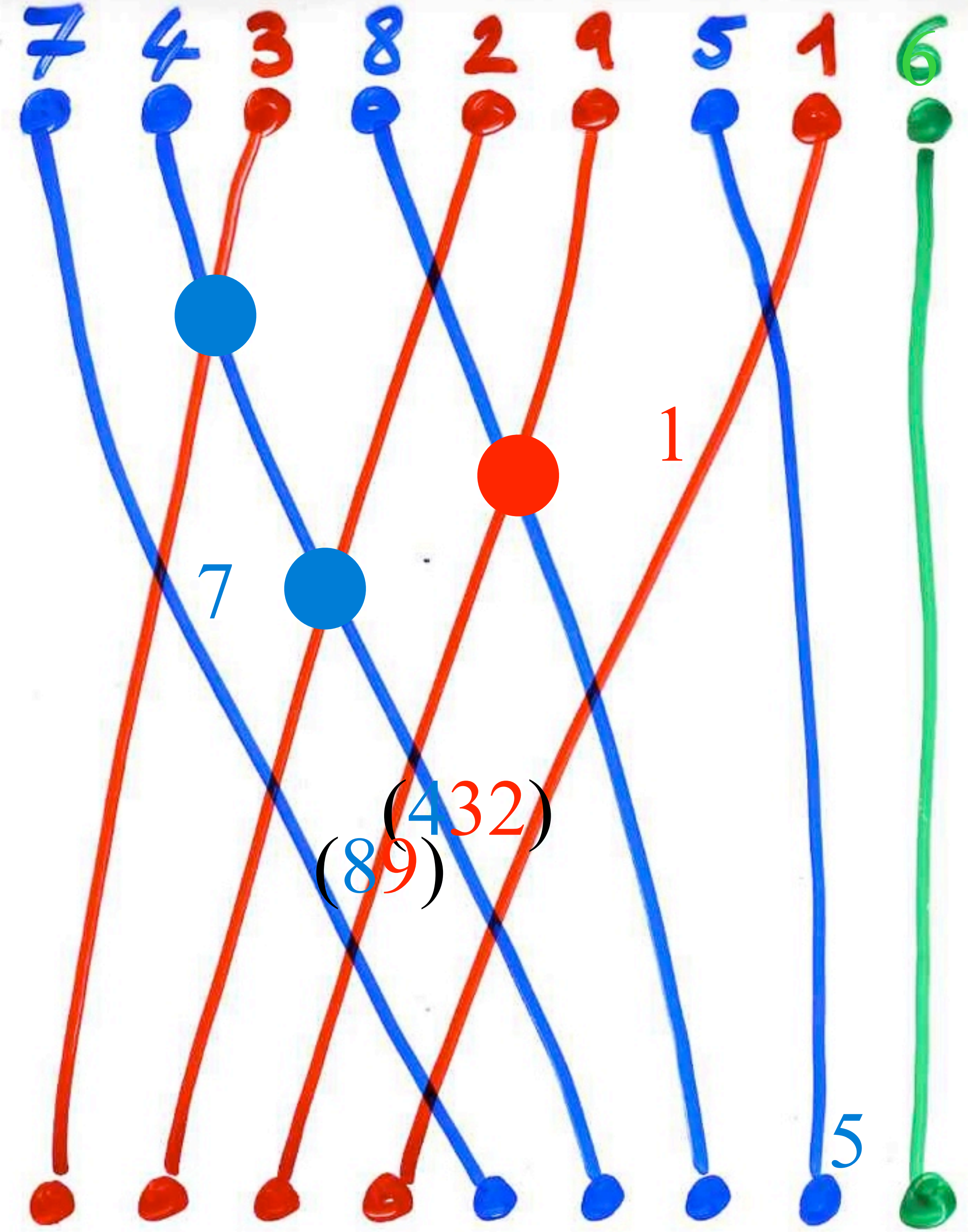
7 4 3 8 2 9 5 1 6
2
8
(43)
1
7
5


7 4 3 8 2 9 5 1 6
(43)
2
8
9
1
7
5

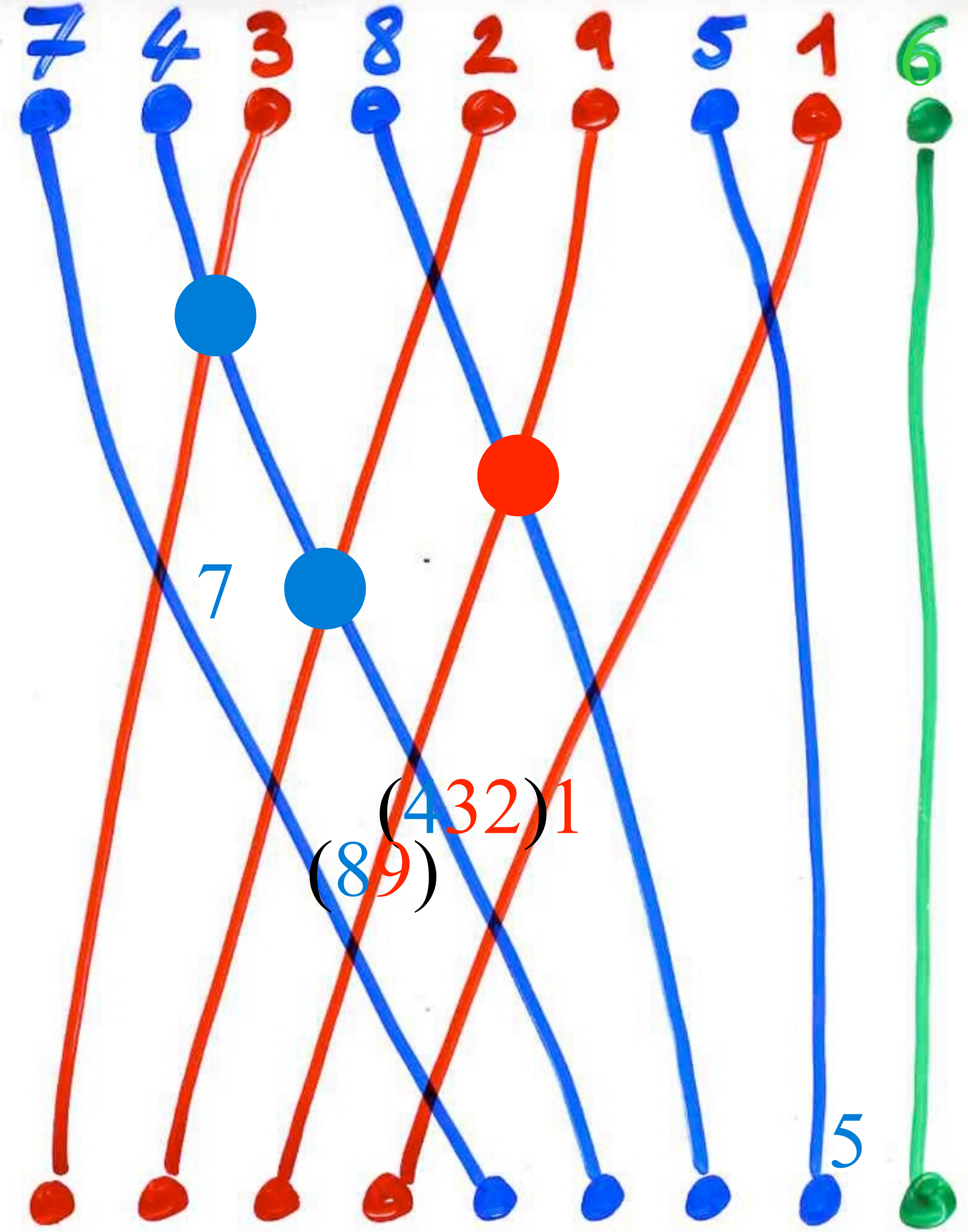

7 4 3 8 2 9 5 1 6
2
(43)
1
(89)
7
5

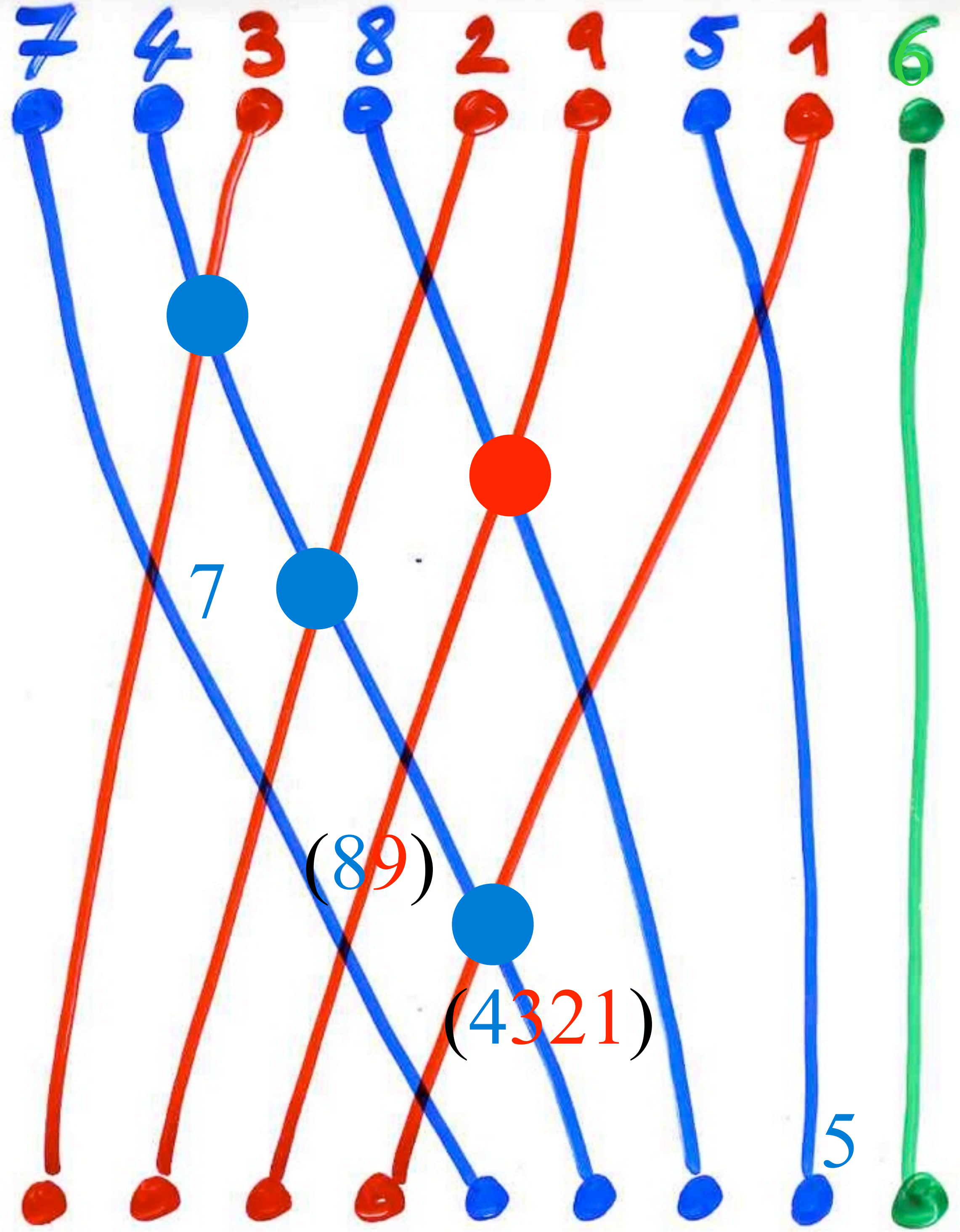

7 4 3 8 2 9 5 1 6
(43)2
(89)
1
7
5

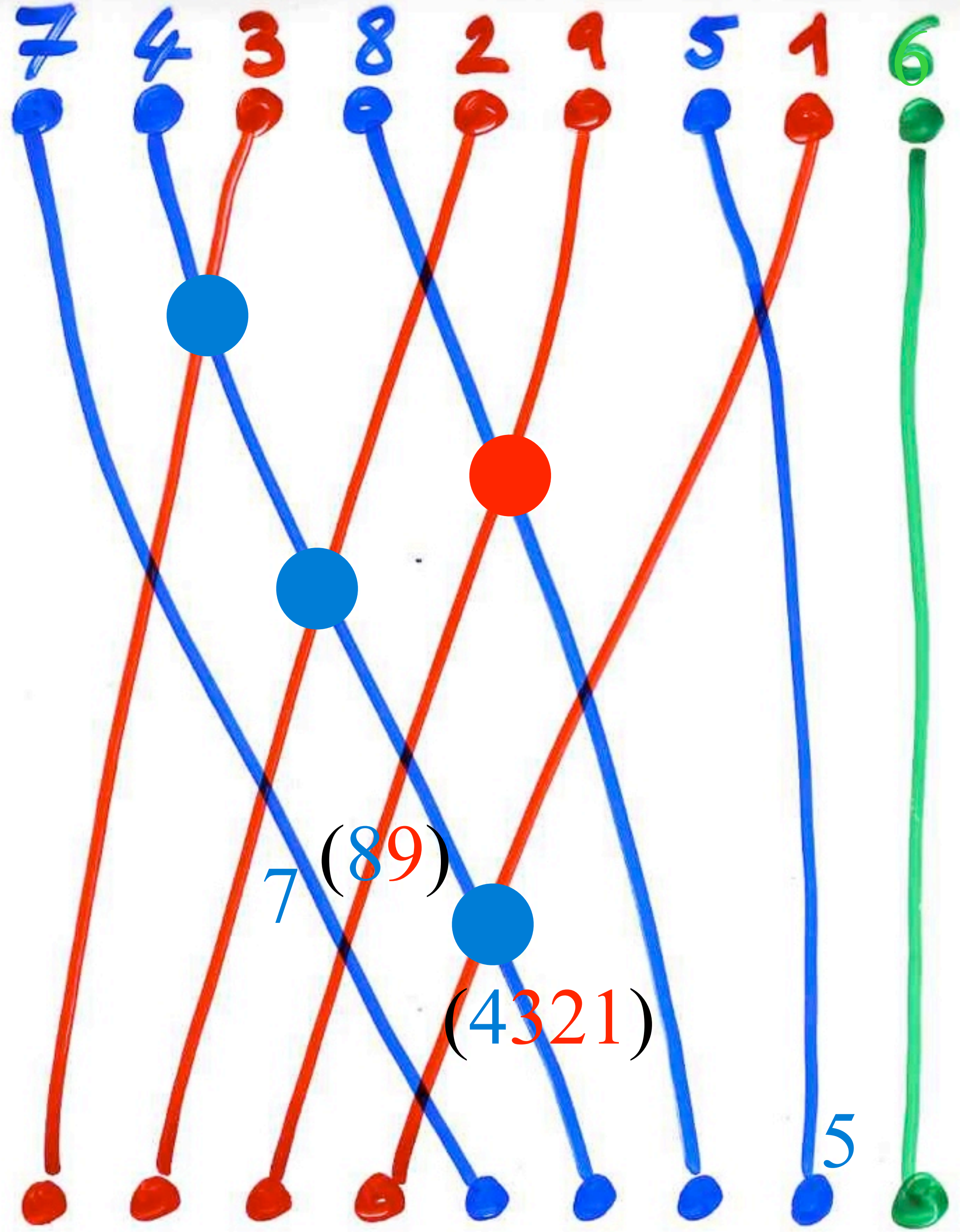

7 4 3 8 2 9 5 1 6
1
(89)
7
(432)
5

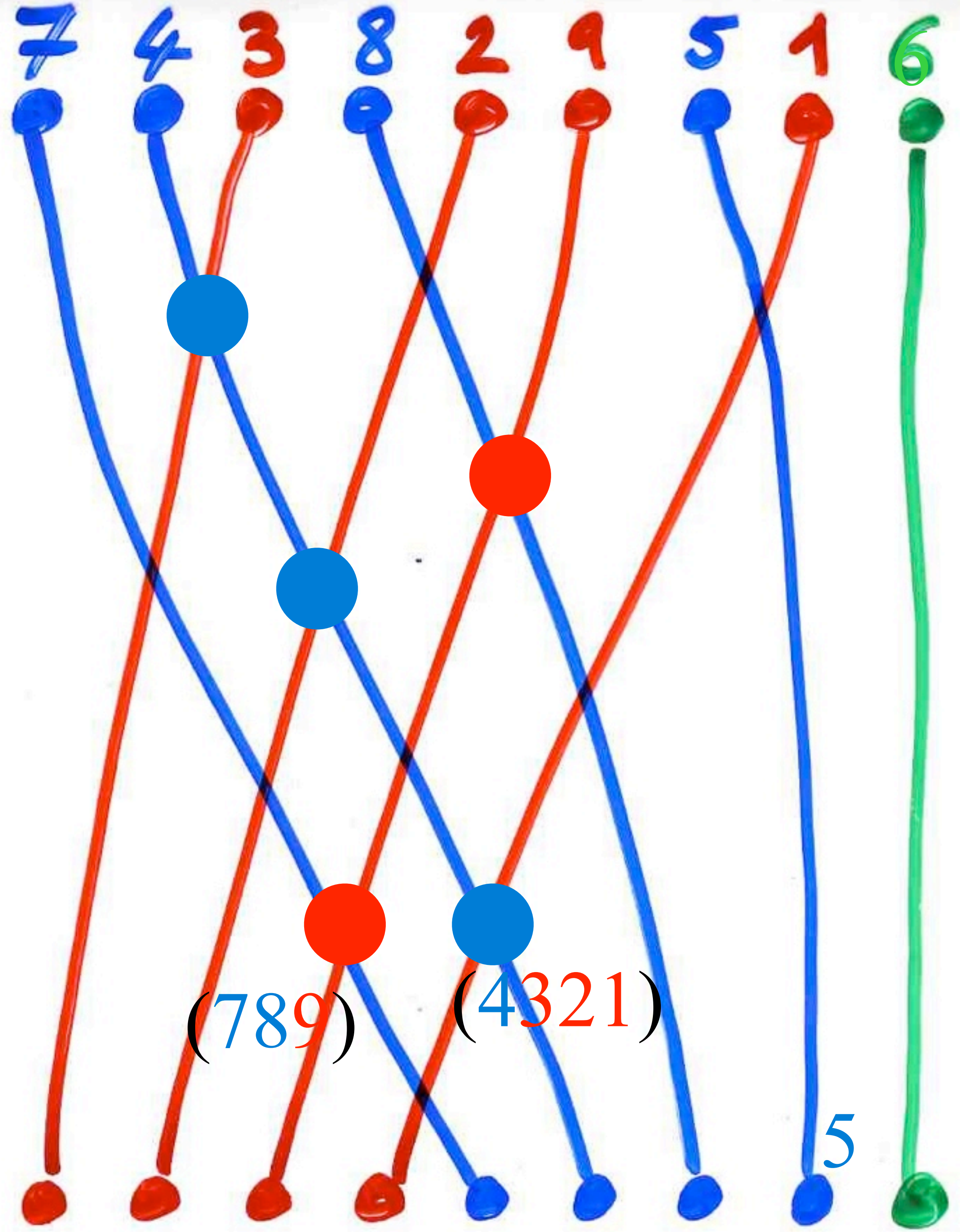


7 4 3 8 2 9 5 1 6
1
7
(432)
(89)
5

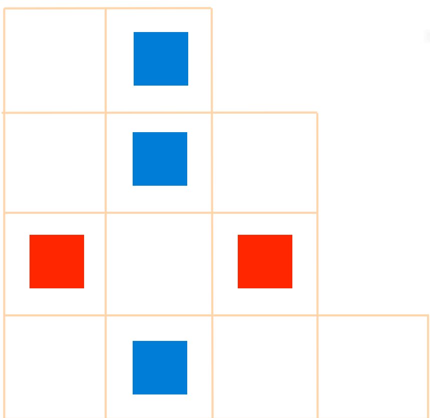

7 4 3 8 2 9 5 1 6
7
(432)1
(89)
5

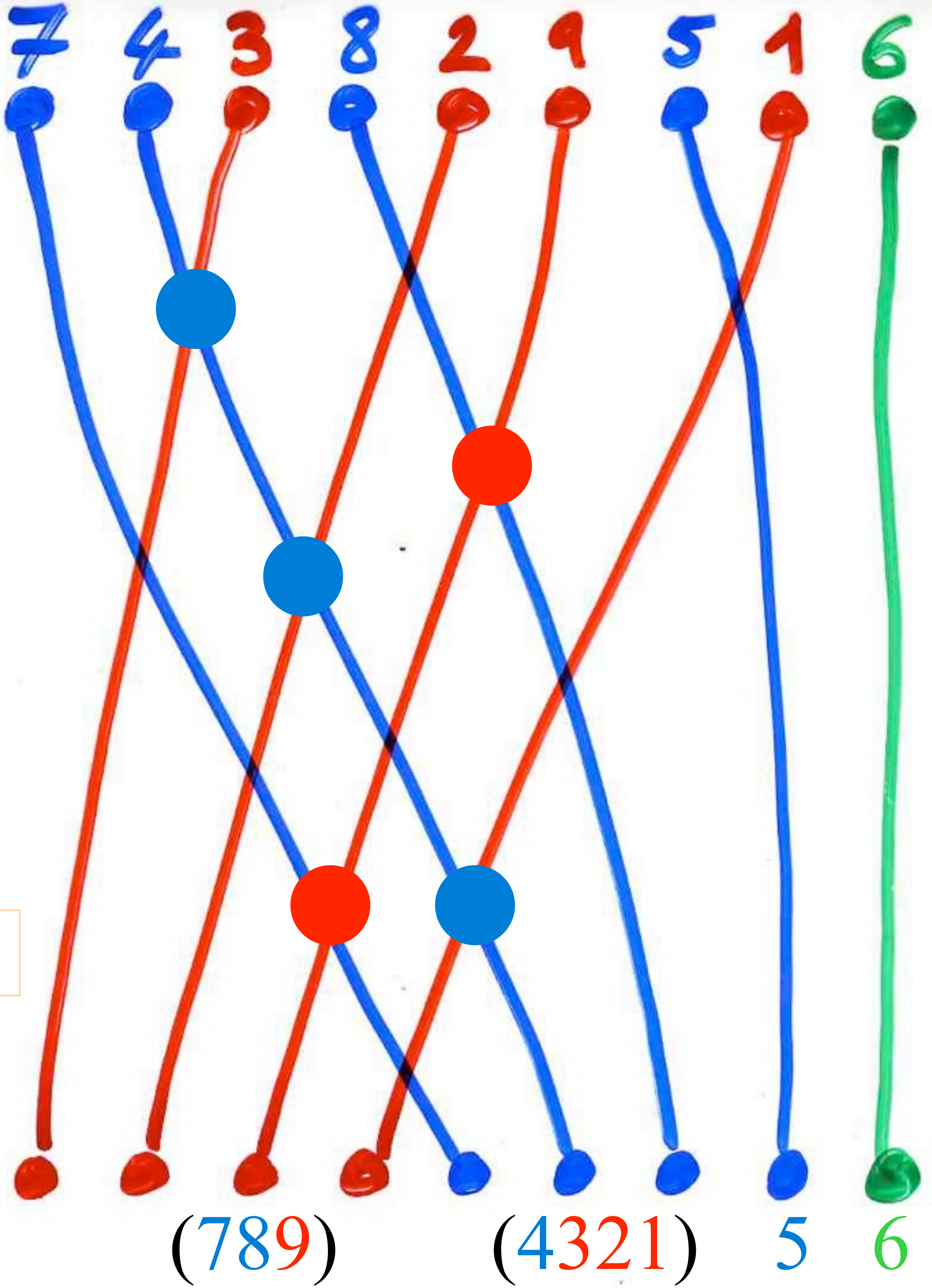

7 4 3 8 2 9 5 1 6
7
(89)
(4321)
5


7 4 3 8 2 9 5 1 6
(89)
7
(4321)
5

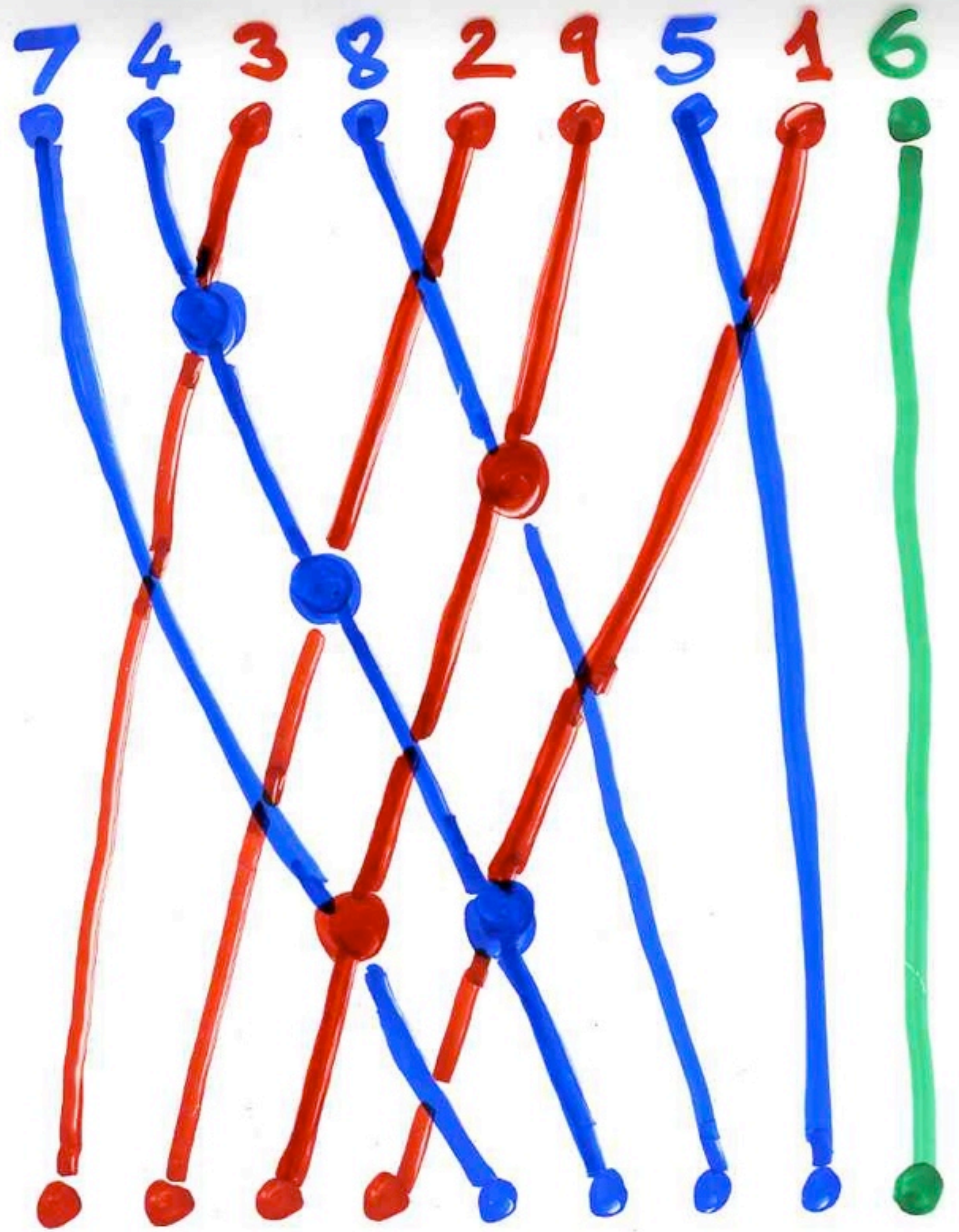

7 4 3 8 2 9 5 1 6
(789)
(4321)
5

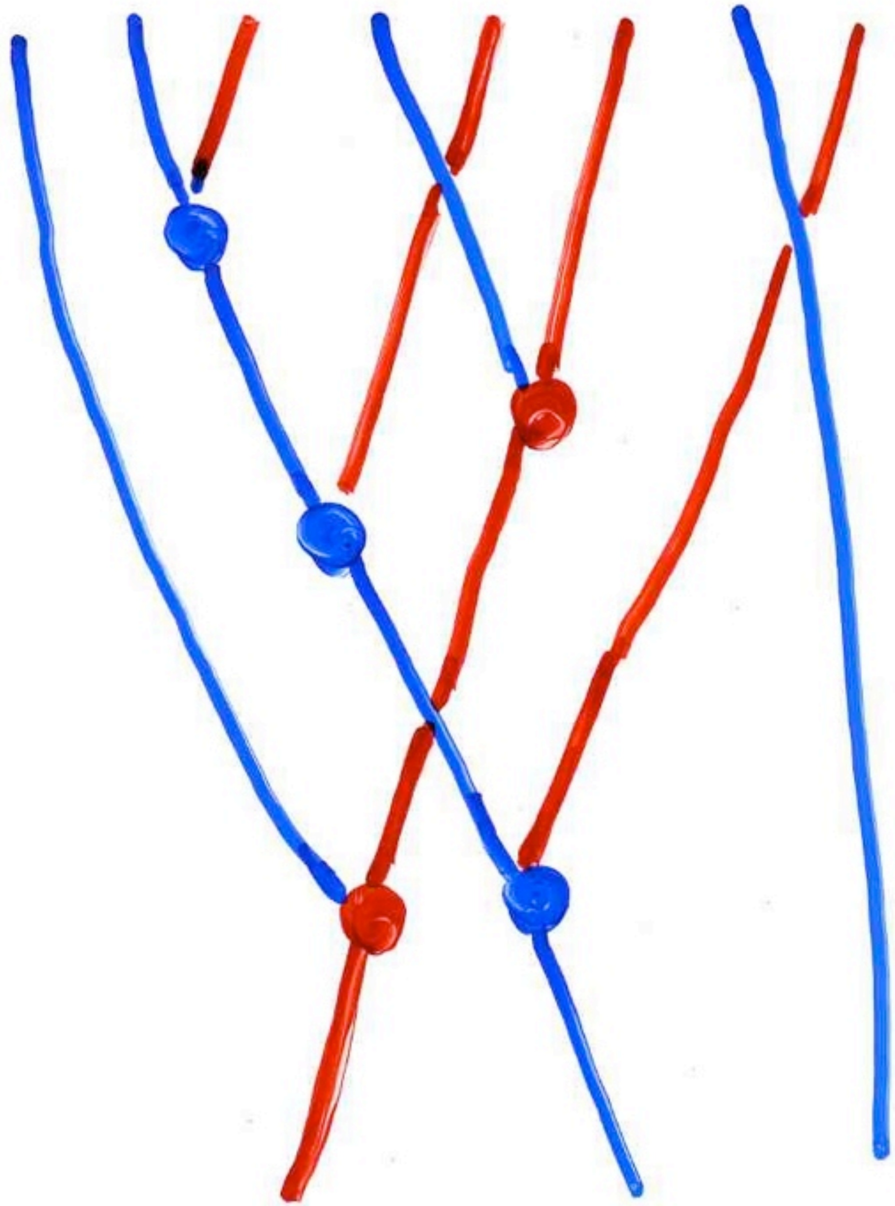
“exchange-fusion” algorithm

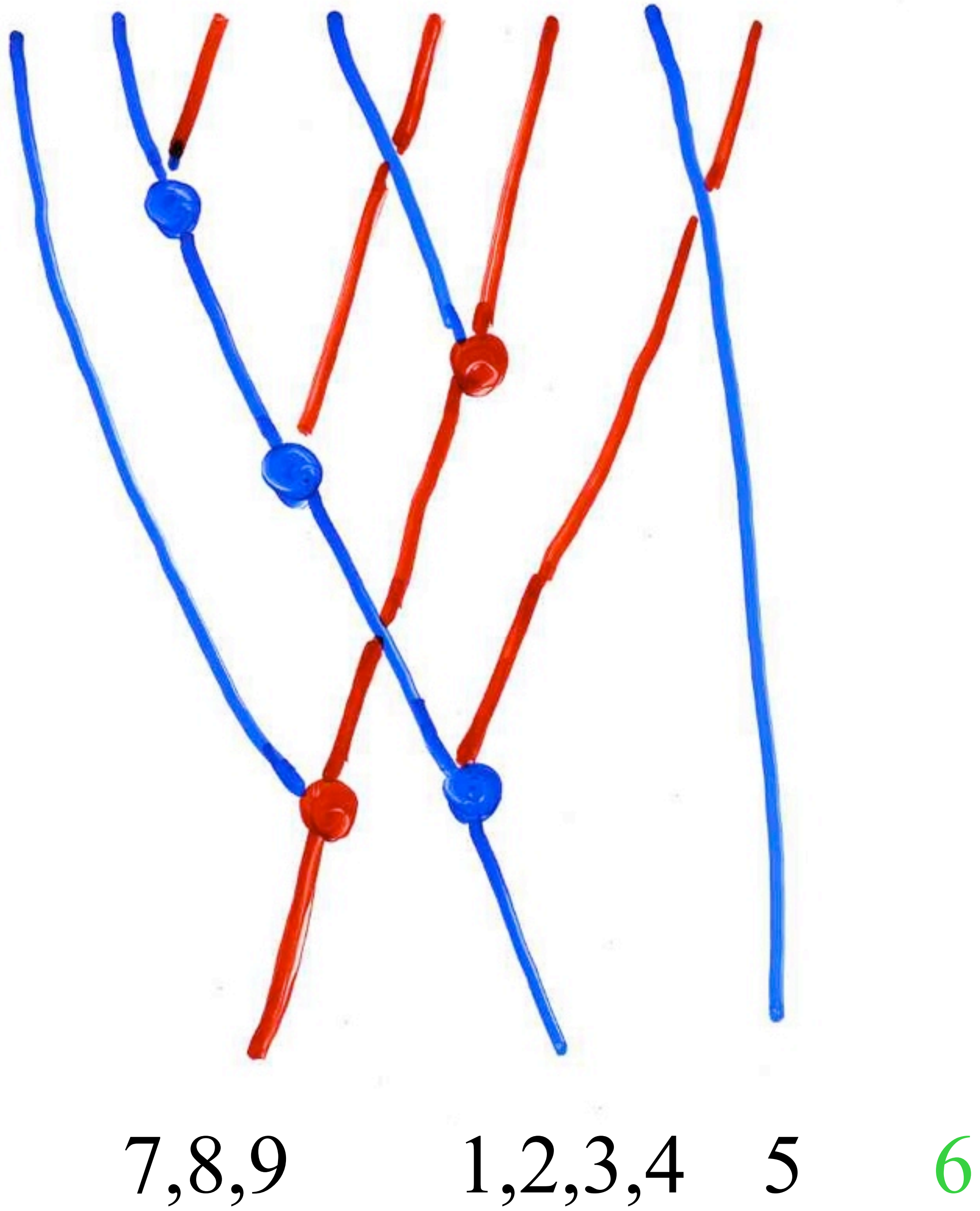


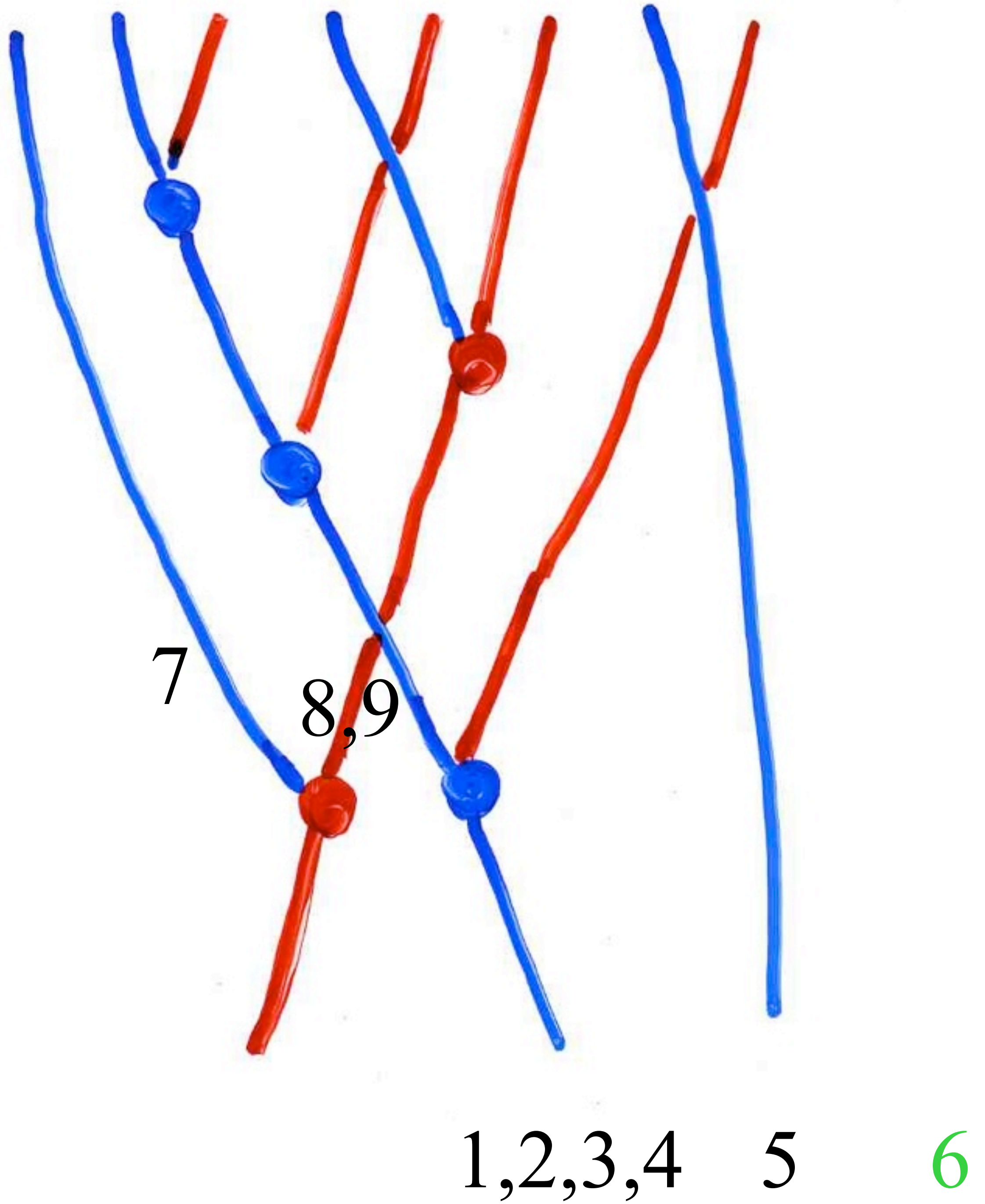


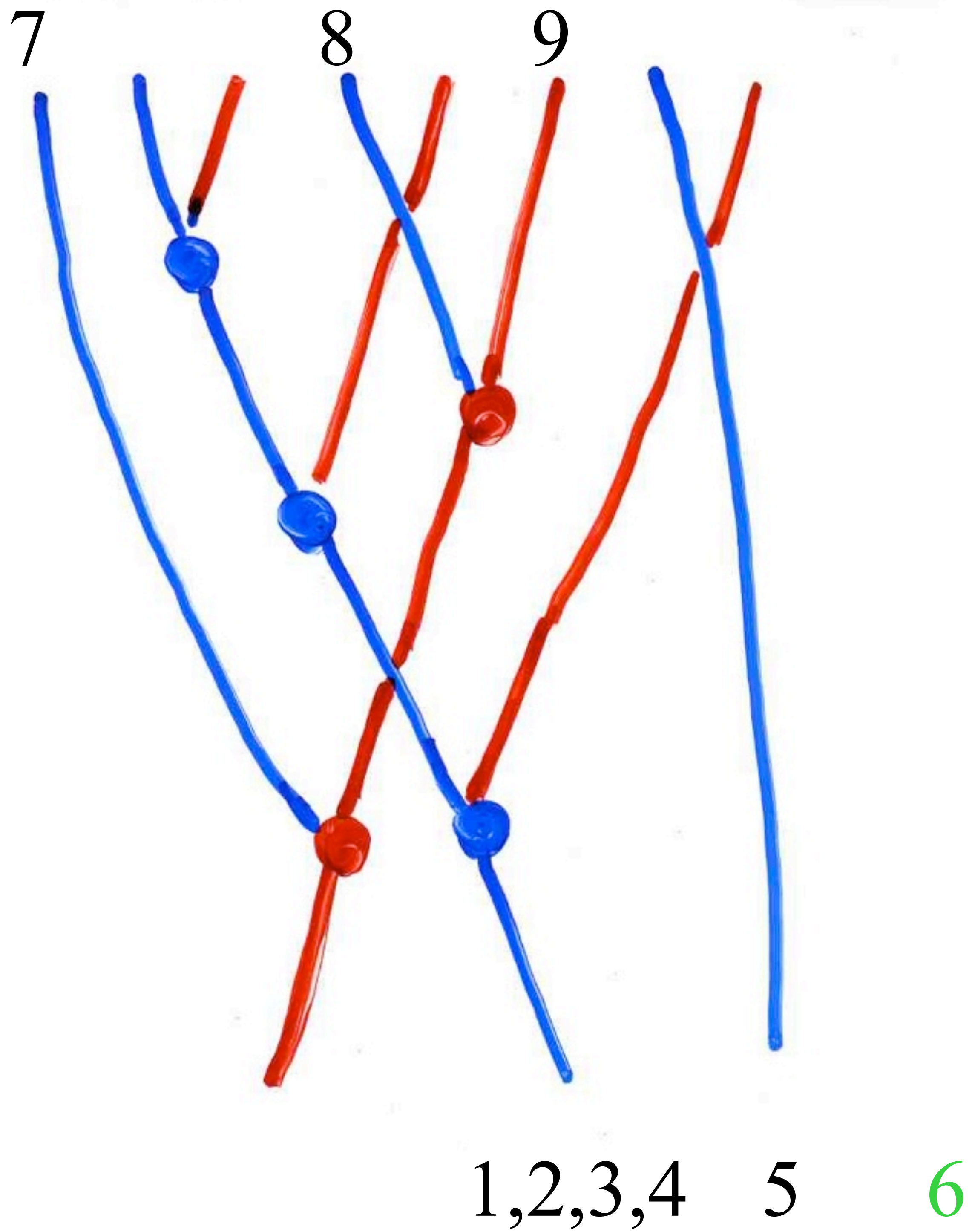
The inverse “exchange-fusion” algorithm

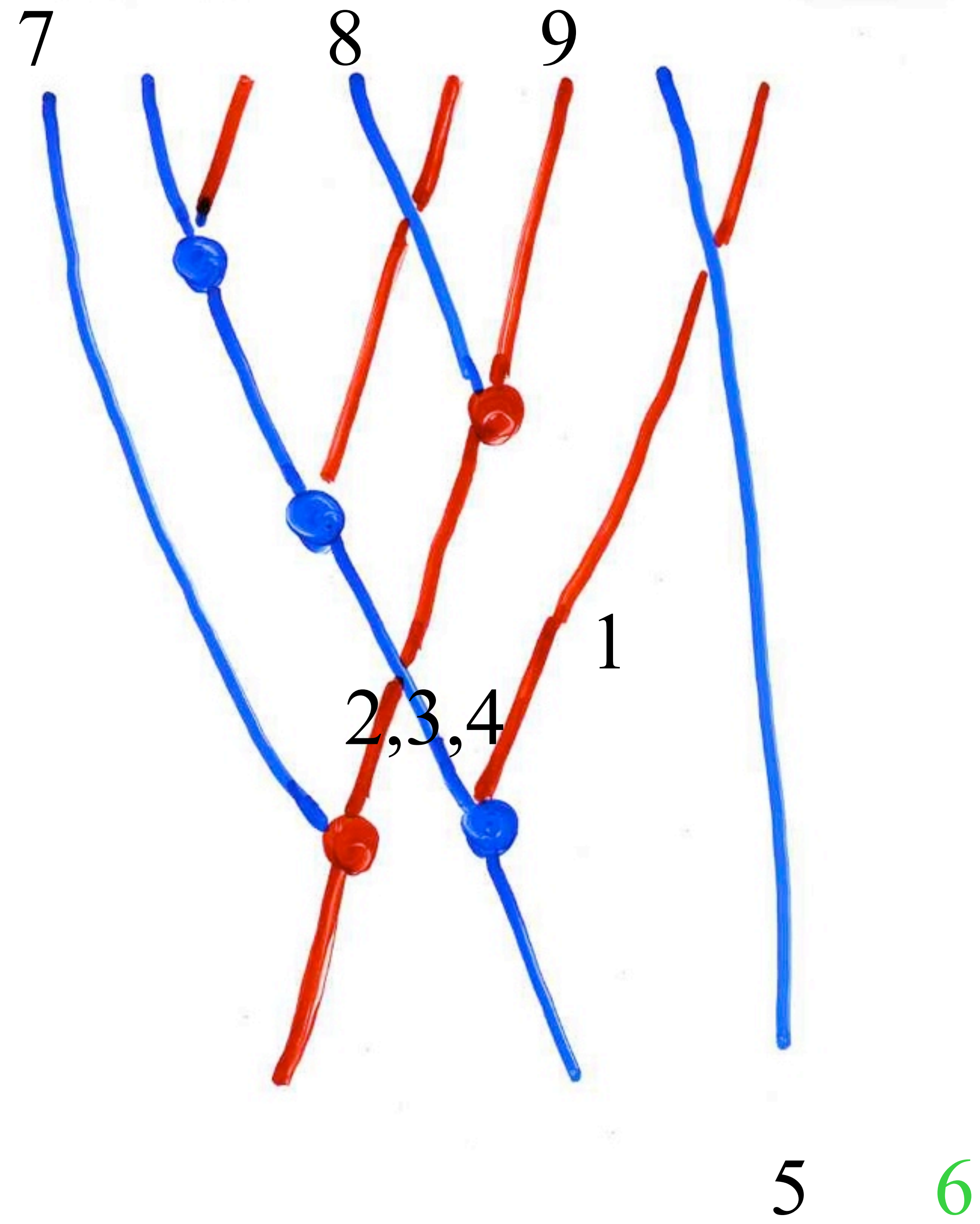

7 4 3 8 2 9 5 1 6

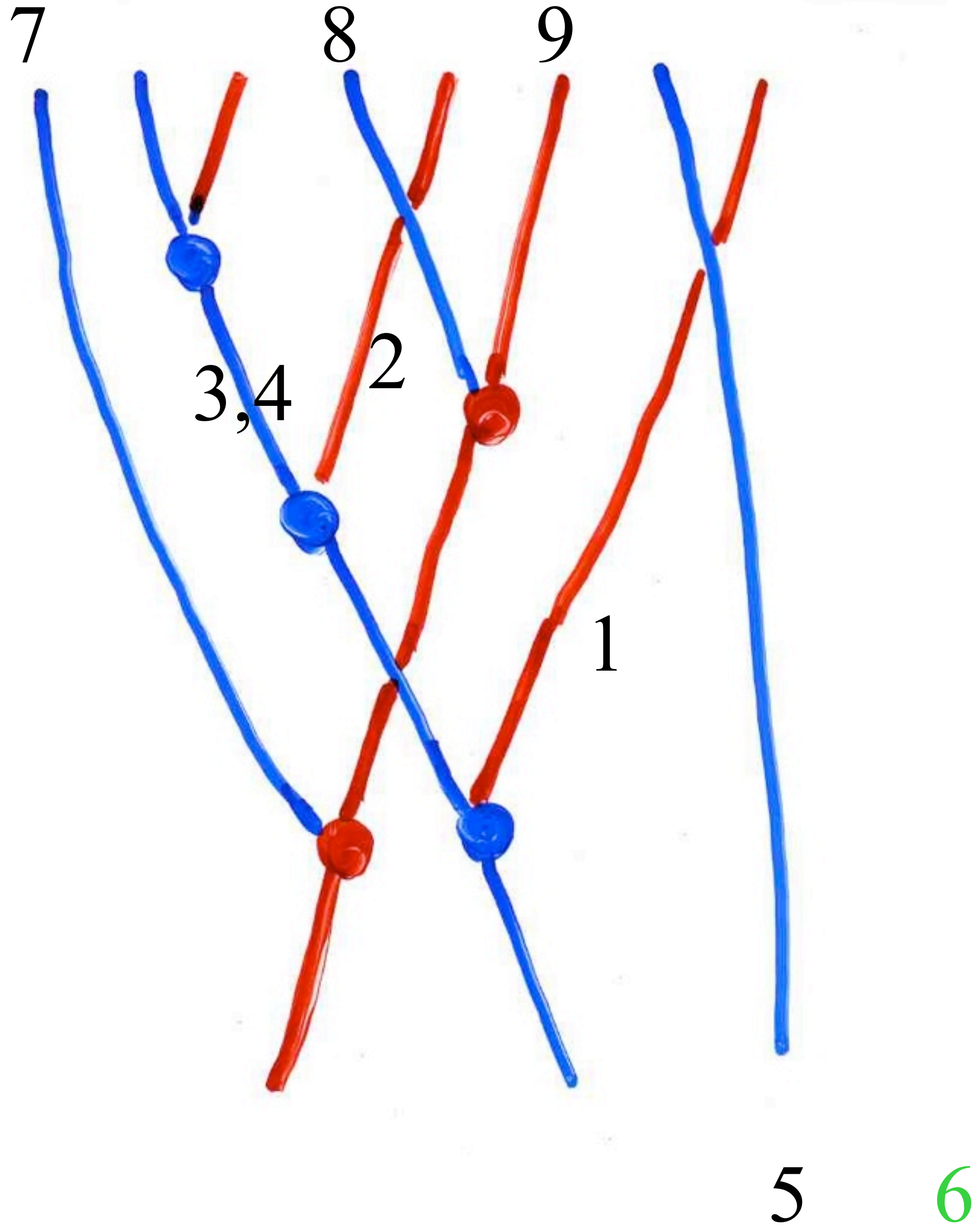


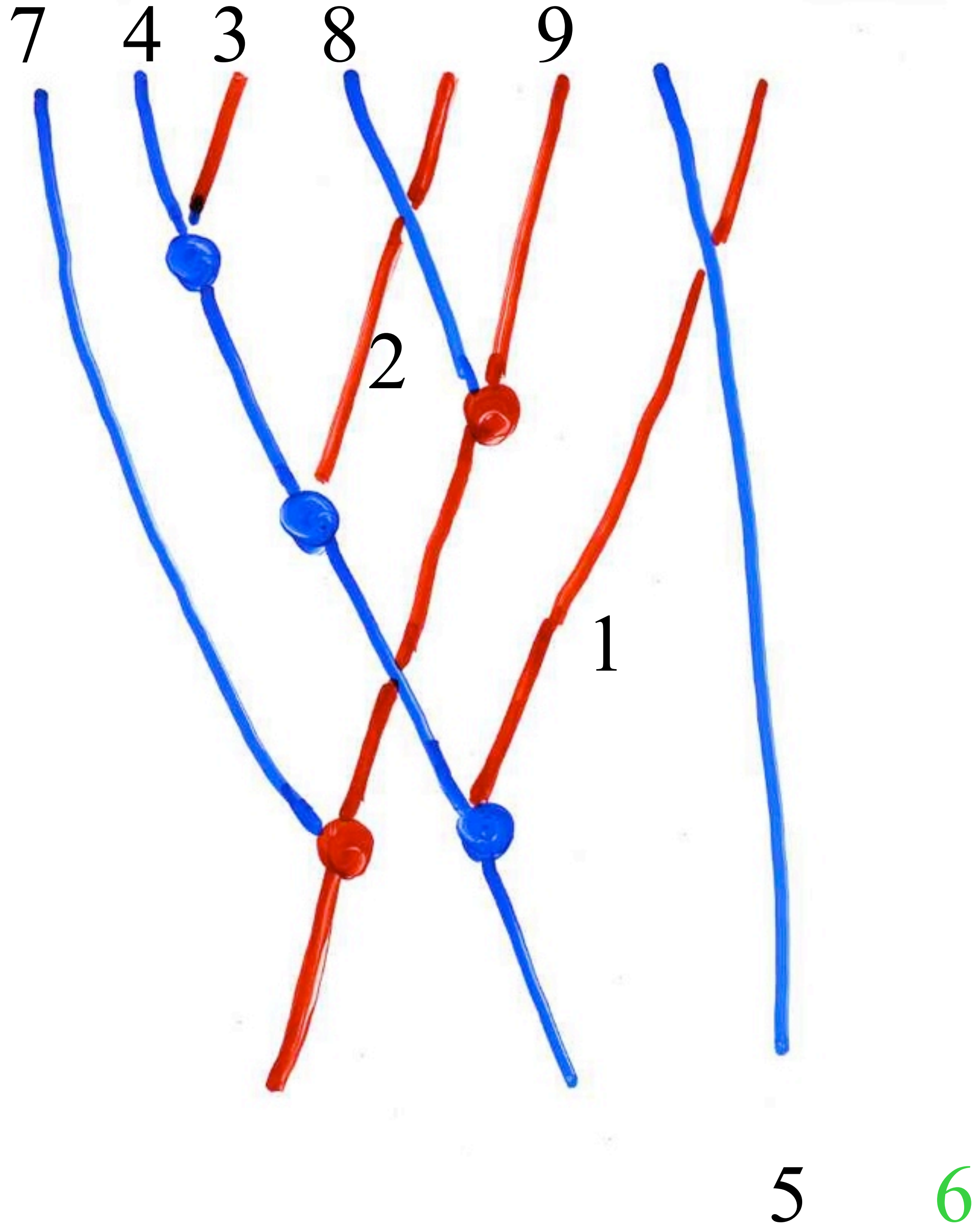

7,8,9
1,2,3,4
5
6

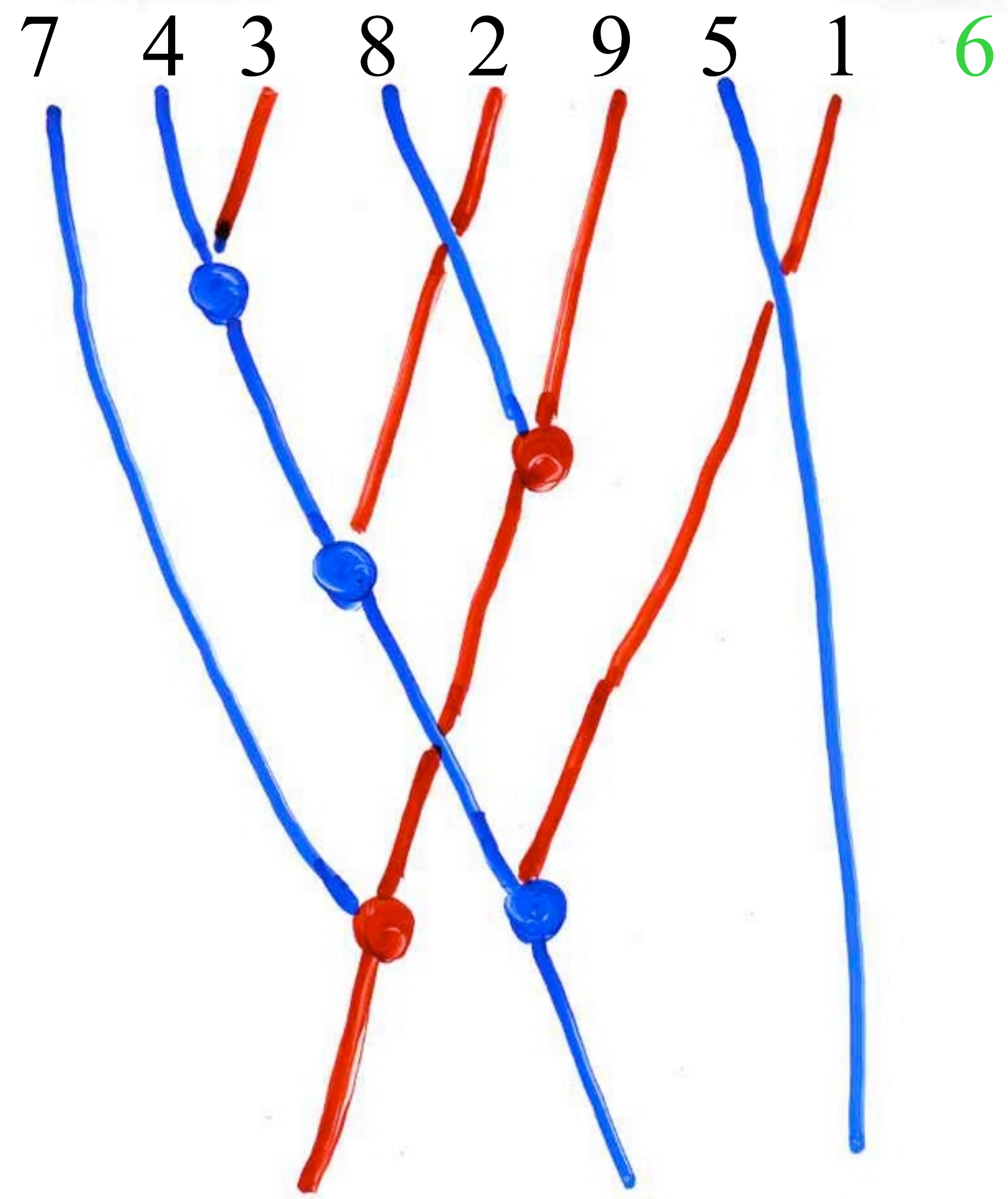

7
8,9
1,2,3,4
5
6


7
8
9
1,2,3,4
5
6


7
8
9
1
2,3,4
5
6


7
8
9
2
3,4
1
5
6


7
4
3
8
9
2
1
5
6


7 4 3 8 2 9 5 1 6

The "cellular Ansatz"

representation
of the
operators
E and D

DE = ED + E + D

V vector space generated by B basis

B alternating words two letters {○, ●}

(no occurrences of ○○ or ●●)

4 operators A, S, J, K

4 operators A, S, J, K, $u \in B$

$\langle u | A = \sum_{\text{letter } \circ \text{ of } u} v$, v obtained by: ○ → ○●○

$\langle u | S = \sum_{\circ \text{ of } u} v$ v obtained by: ○ → ●
(and ●● → ● ●●● → ●)

$\langle u | J = \sum_{\circ \text{ of } u} v$ v, ○ → ●○ (and ●● → ●)

$\langle u | K = \sum_{\circ \text{ of } u} v$ v, ○ → ○● (and ●● → ●)

○●○●|S = ●○● + ○●

Lemma.

$$AS = SA + J + K$$

$$AK = KA + A$$

$$JS = SJ + S$$

$$JK = KJ$$

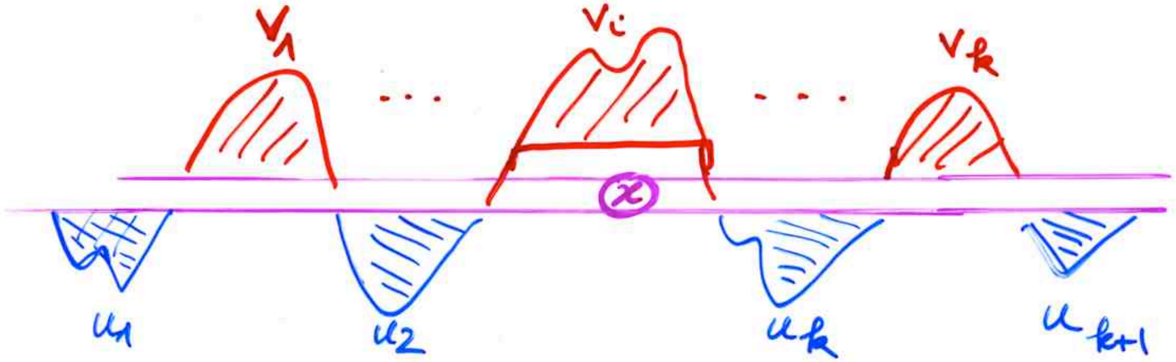
$$D = A + J$$

$$E = S + K$$

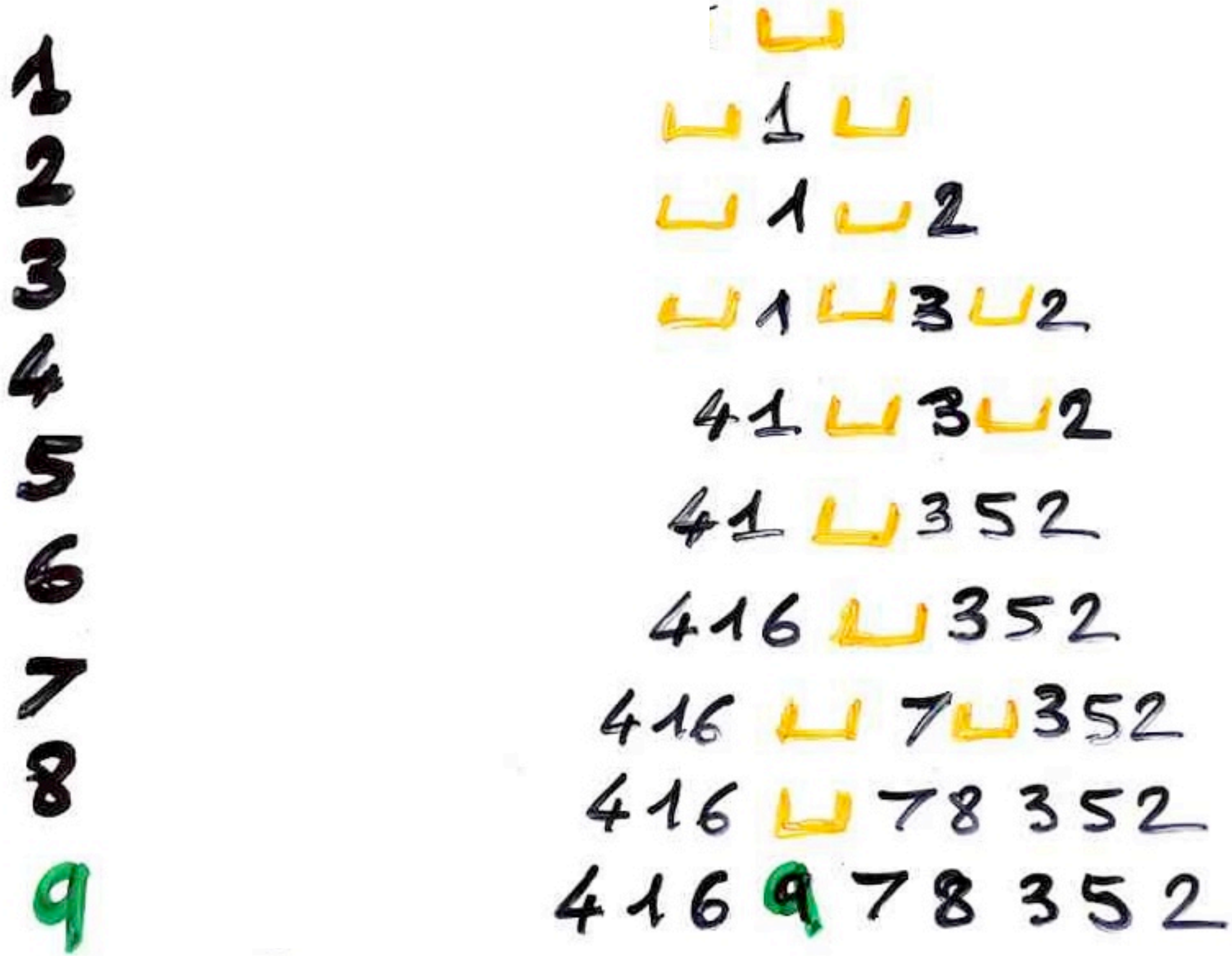
$$D = A + J$$

$$E = S + K$$

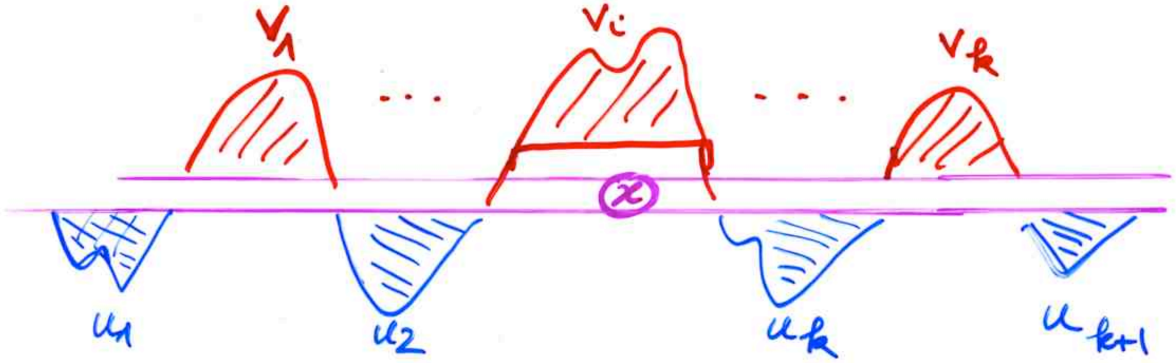
$$\begin{aligned} DE &= (A + J)(S + K) \\ &= AS + AK + JS + JK \\ &= \underbrace{\underbrace{(SA + KA + SJ + KJ)}_{(S+K)(A+J)}}_{ED} + \underbrace{J + K + A + S}_{E + D} \end{aligned}$$

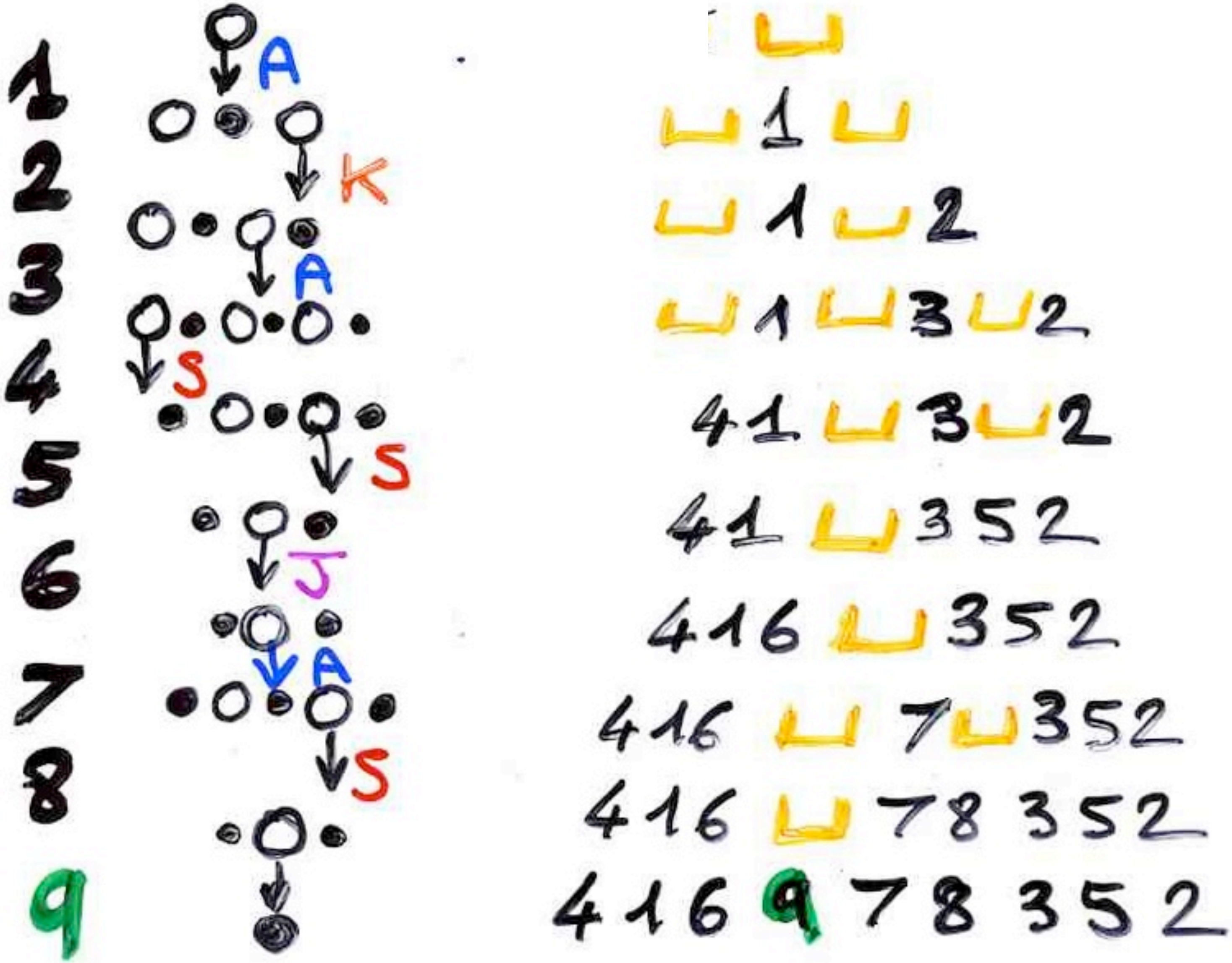


v1
vi
vk
x
u1
u2
uk
uk+1



⊔
1 ⊔ 1 ⊔
2 ⊔ 1 ⊔ 2
3 ⊔ 1 ⊔ 3 ⊔ 2
4 4 1 ⊔ 3 ⊔ 2
5 4 1 ⊔ 3 5 2
6 4 1 6 ⊔ 3 5 2
7 4 1 6 ⊔ 7 ⊔ 3 5 2
8 4 1 6 ⊔ 7 8 3 5 2
9 4 1 6 9 7 8 3 5 2


v1
vi
vk
x
u1
u2
uk
uk+1


1 2 3 4 5 6 7 8 9
A
K
A
S
S
J
A
S
1
1 2
1 3 2
4 1 3 2
4 1 3 5 2
4 1 6 3 5 2
4 1 6 7 3 5 2
4 1 6 7 8 3 5 2
4 1 6 9 7 8 3 5 2

The "cellular Ansatz"

"planar construction" of a bijection

permutations $\longleftrightarrow$ alternative tableaux

416978352

●

●1 ●

●1●2 ●

●1●3●2

41● 3 ●2

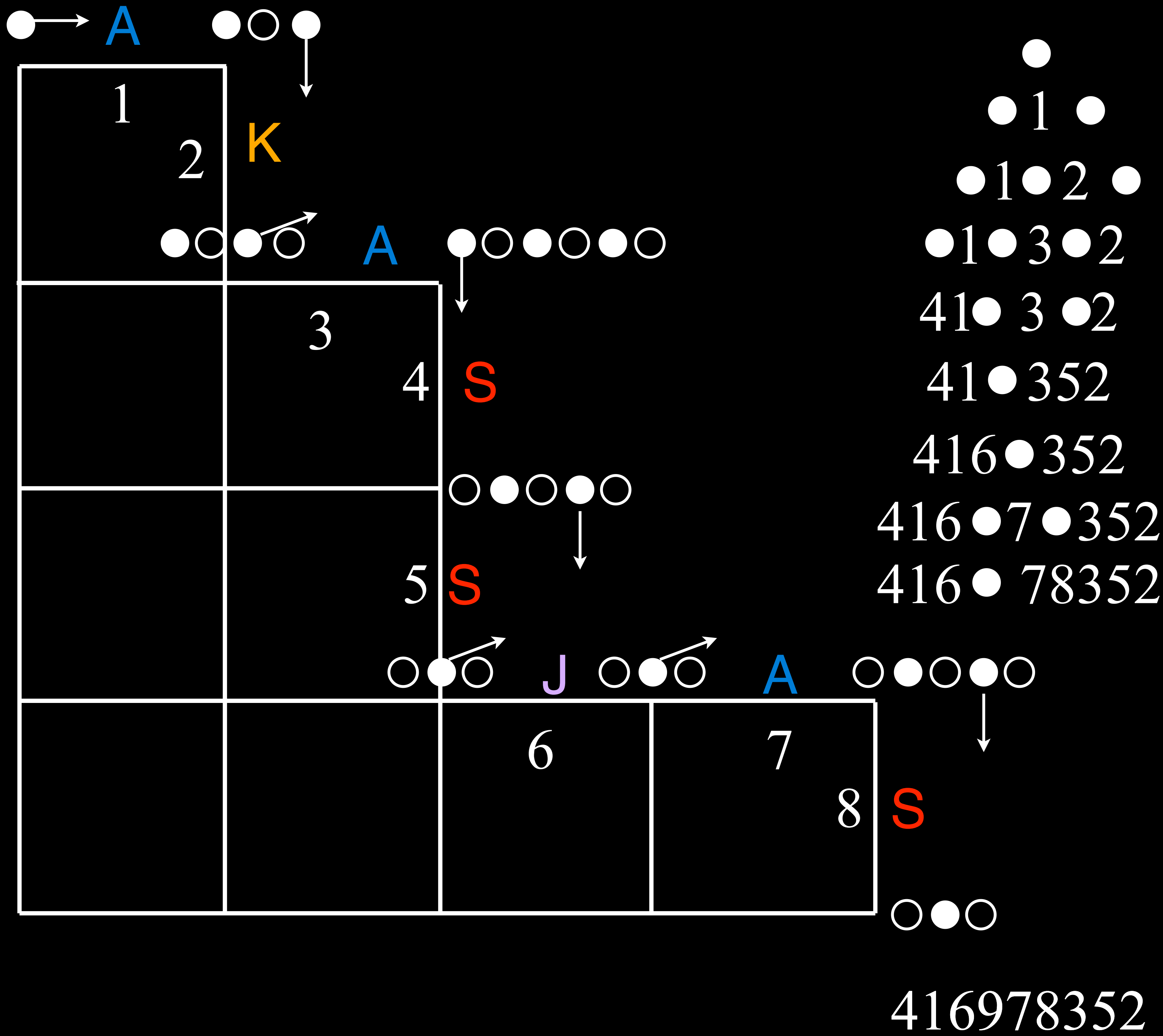
41●352

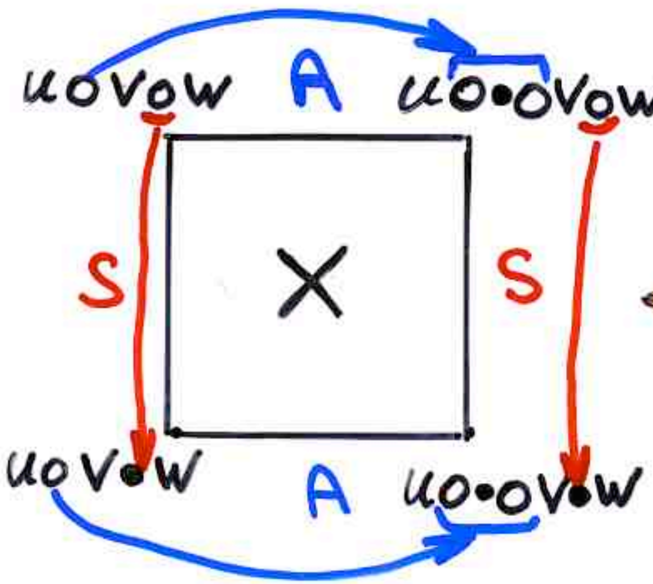
416●352

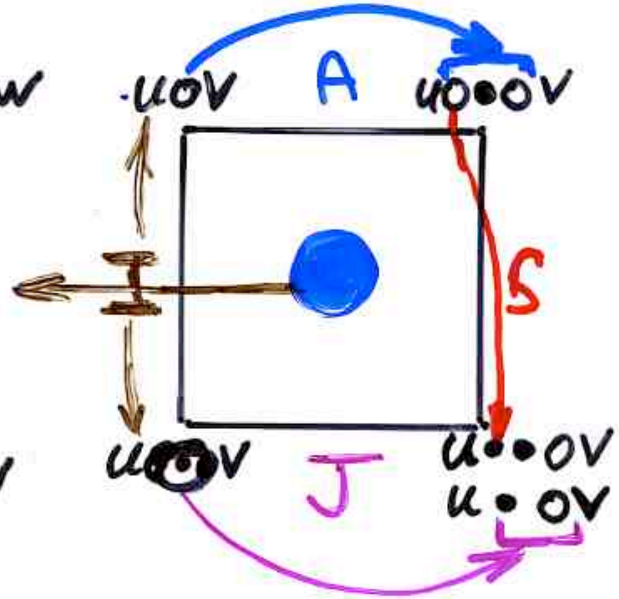
416●7●352

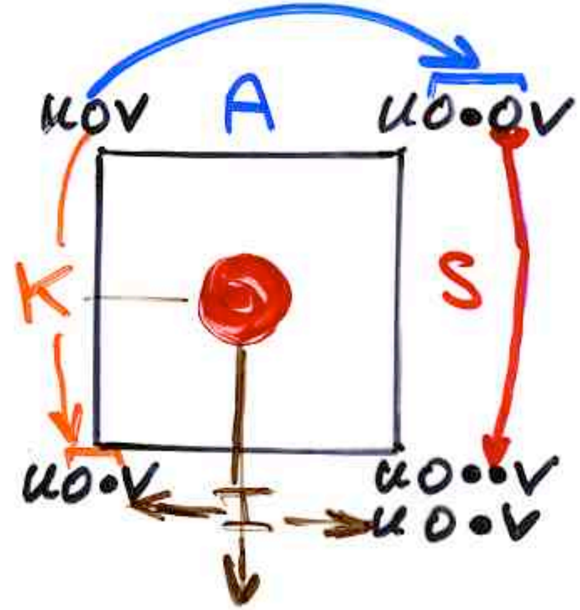
416● 78352

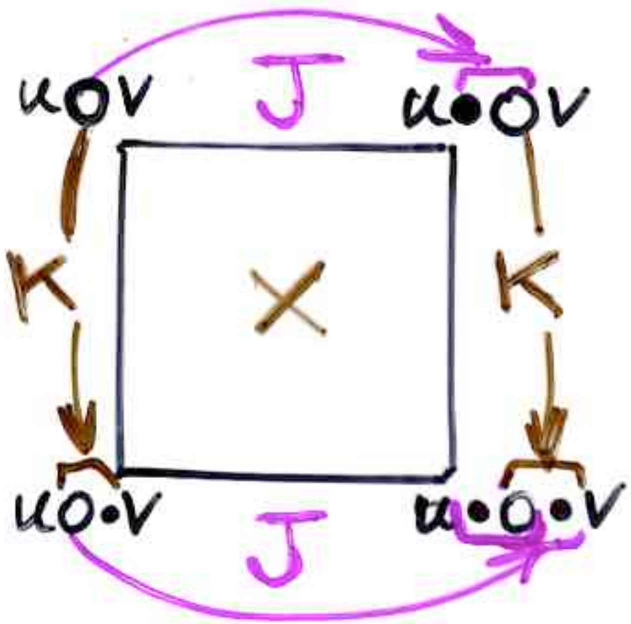
416978352

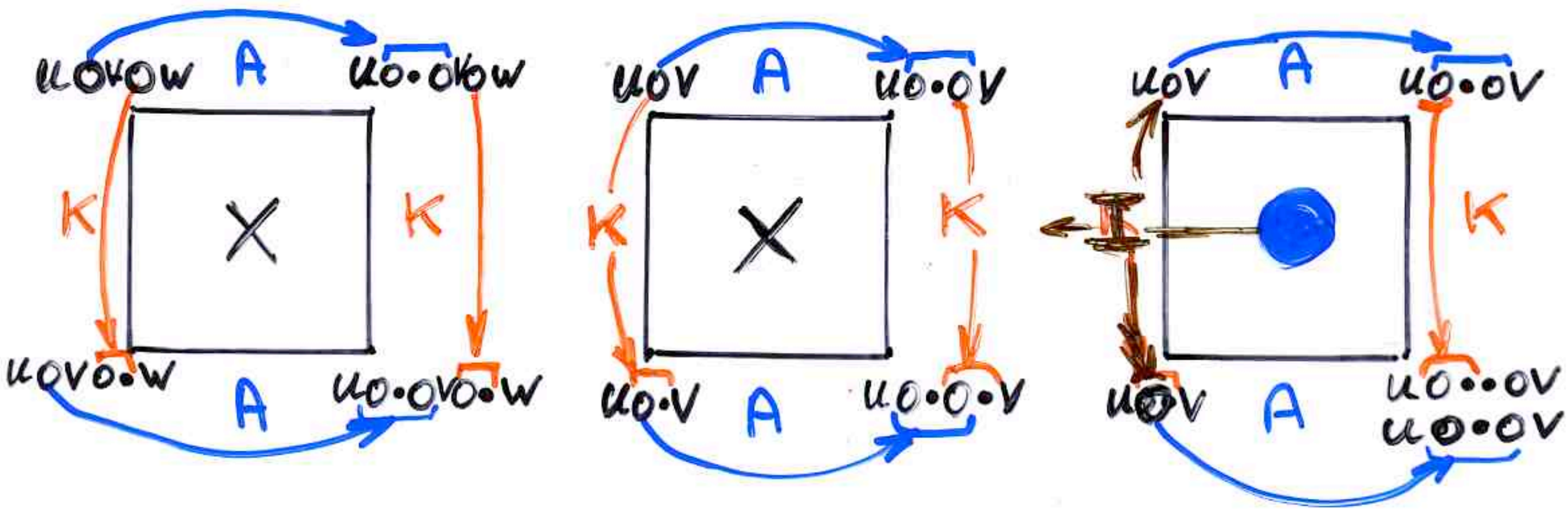

A
1
2
K
A
3
4
S
5
S
J
A
6
7
8
S
1
1 2
1 3 2
41 3 2
41 352
416 352
416 7 352
416 78352
416978352

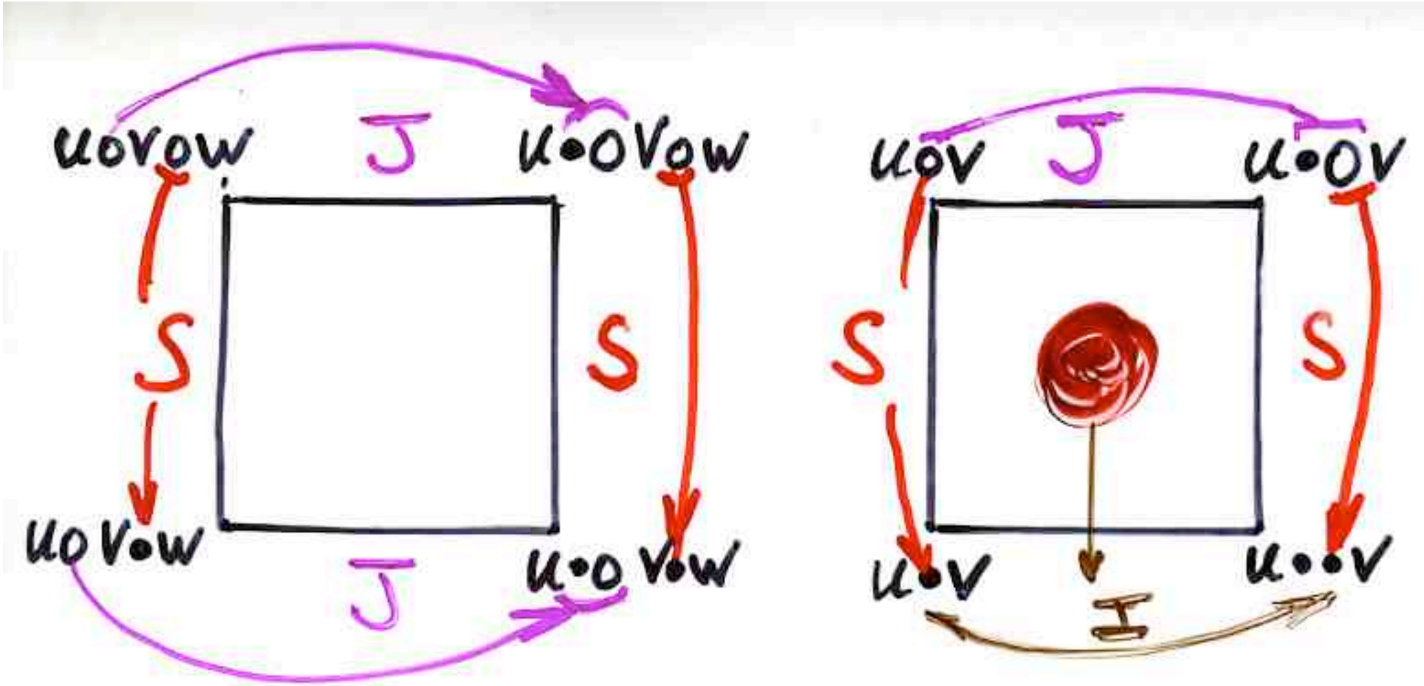

uovow
A
uo•ovow
S
S
uov•w
A
uo•ov•w

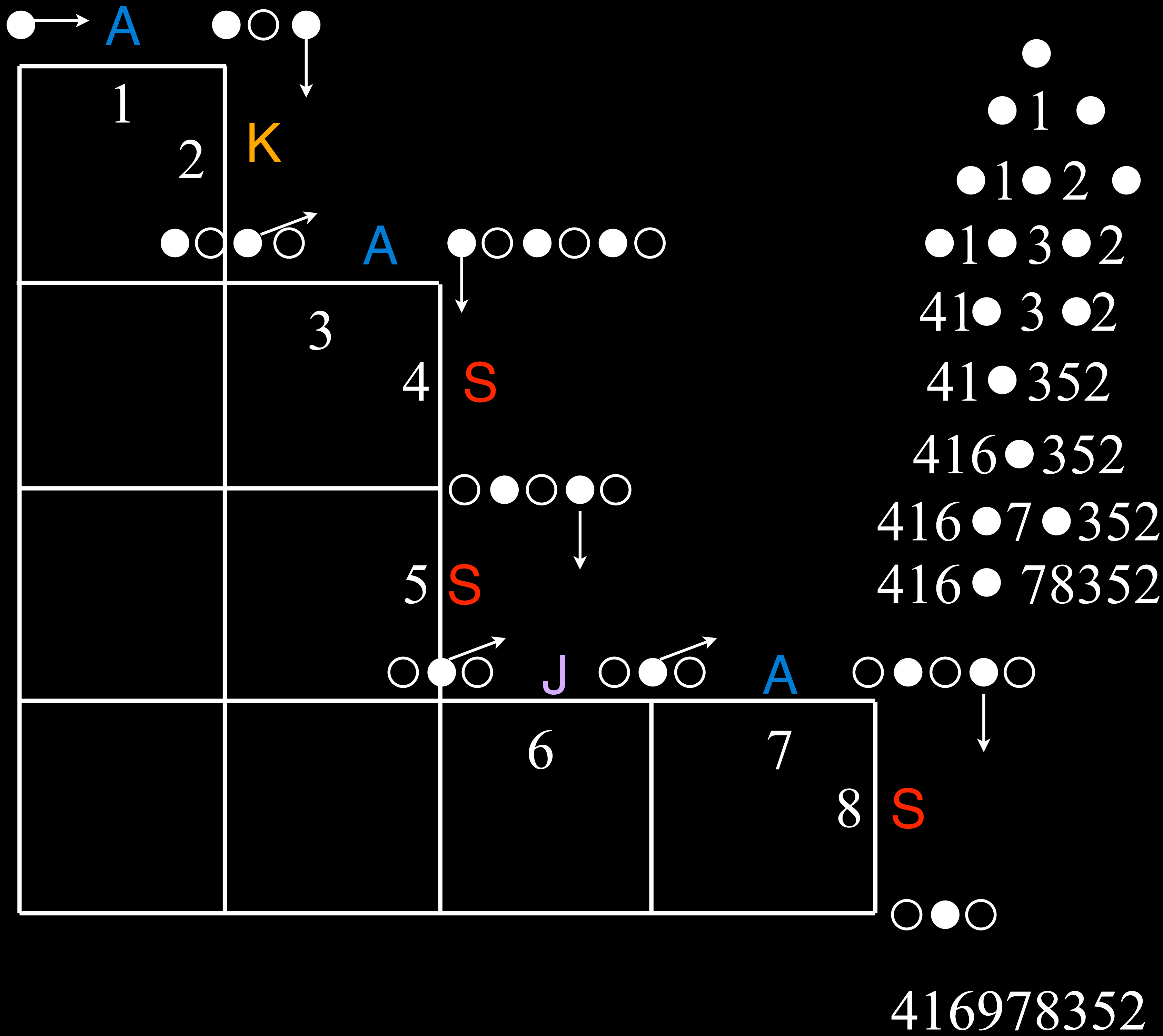

uov
A
uo•ov
S
uov
J
u••ov
u•ov


uov
A
uo•ov
K
S
uo•v
uo••v
uo•v


uov
J
u•ov
K
K
uo•v
J
u•o•v


A
K
A
K
A
K
A
I
A
K
A


J
S
S
J
J
S
S
I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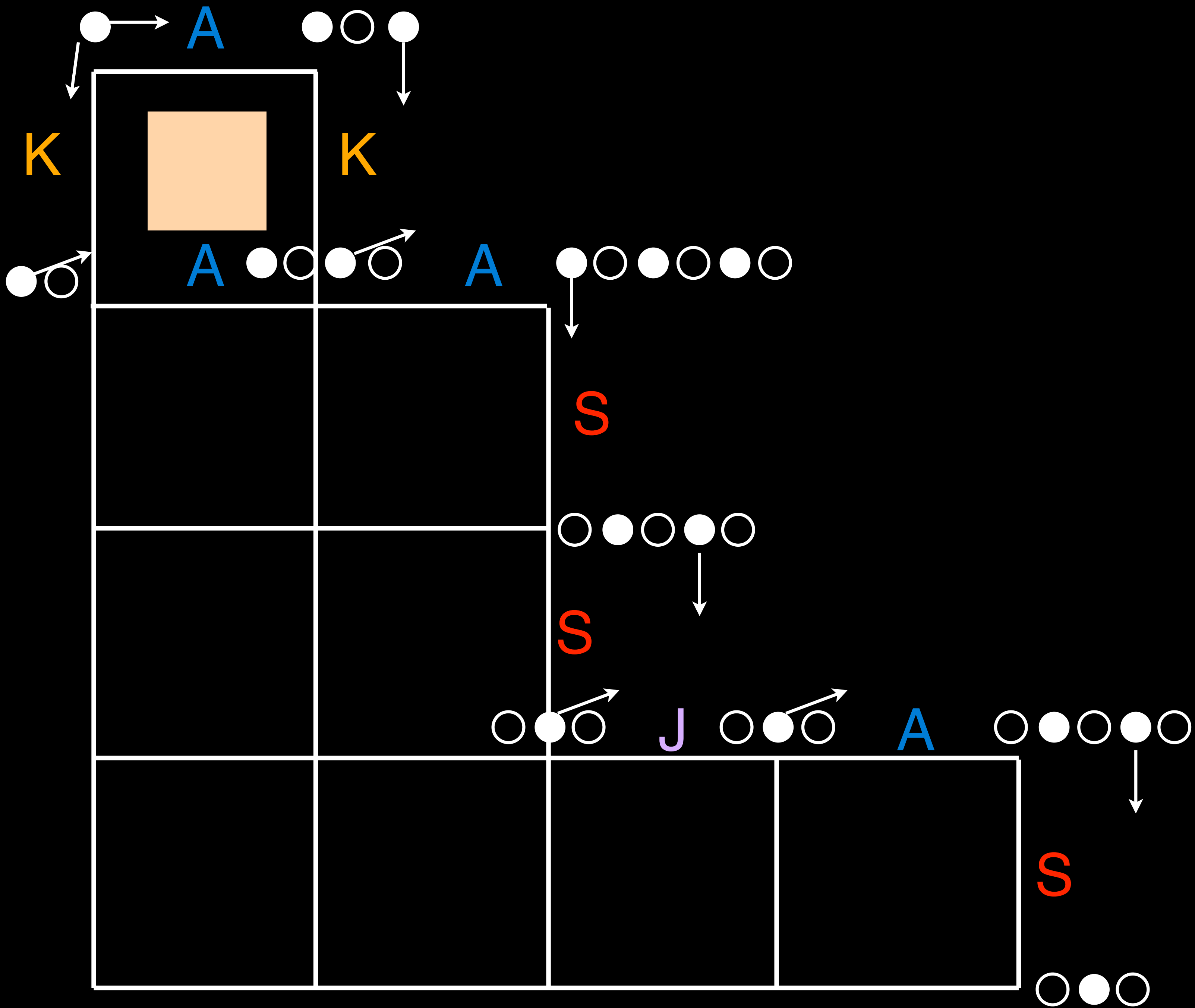
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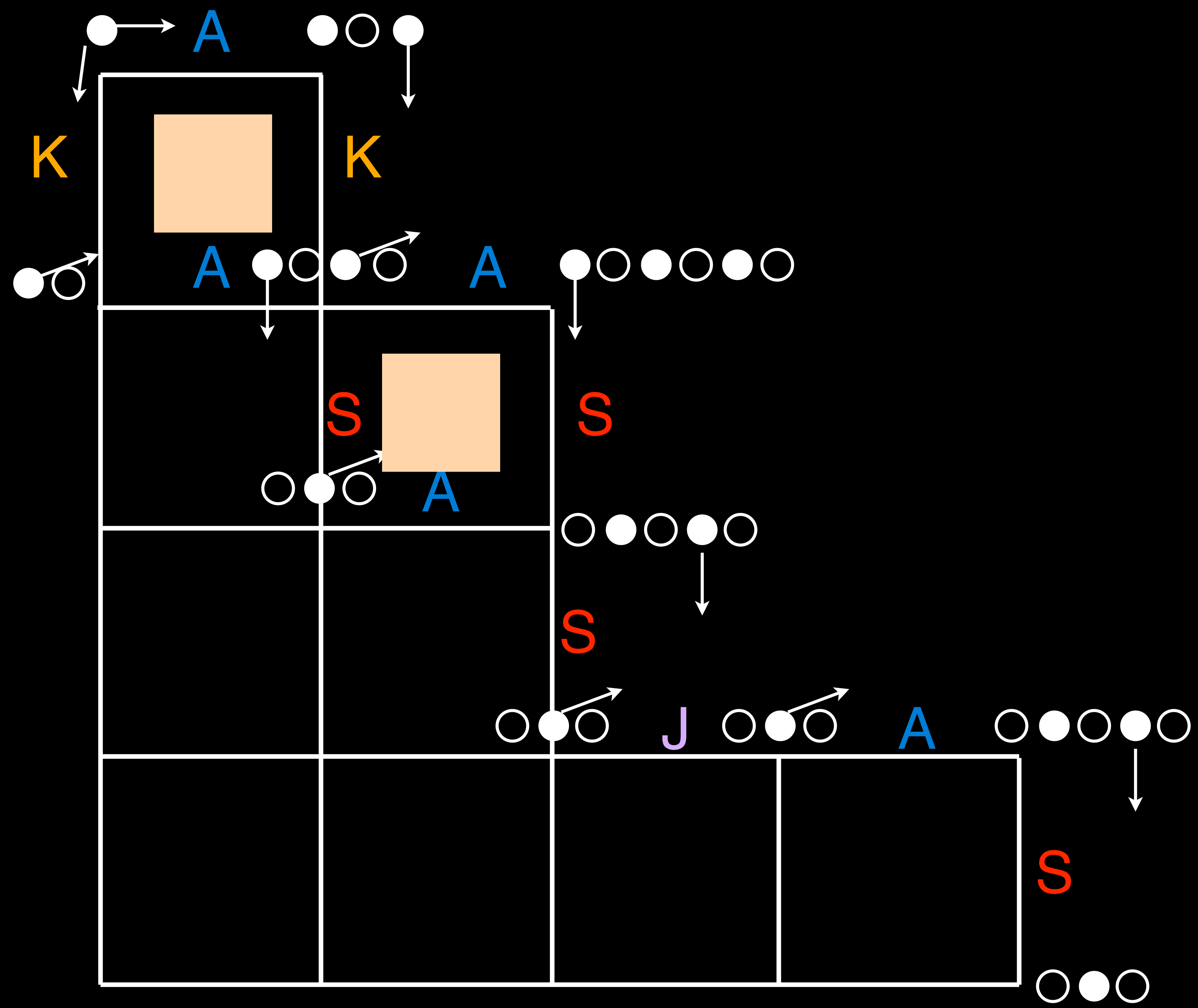
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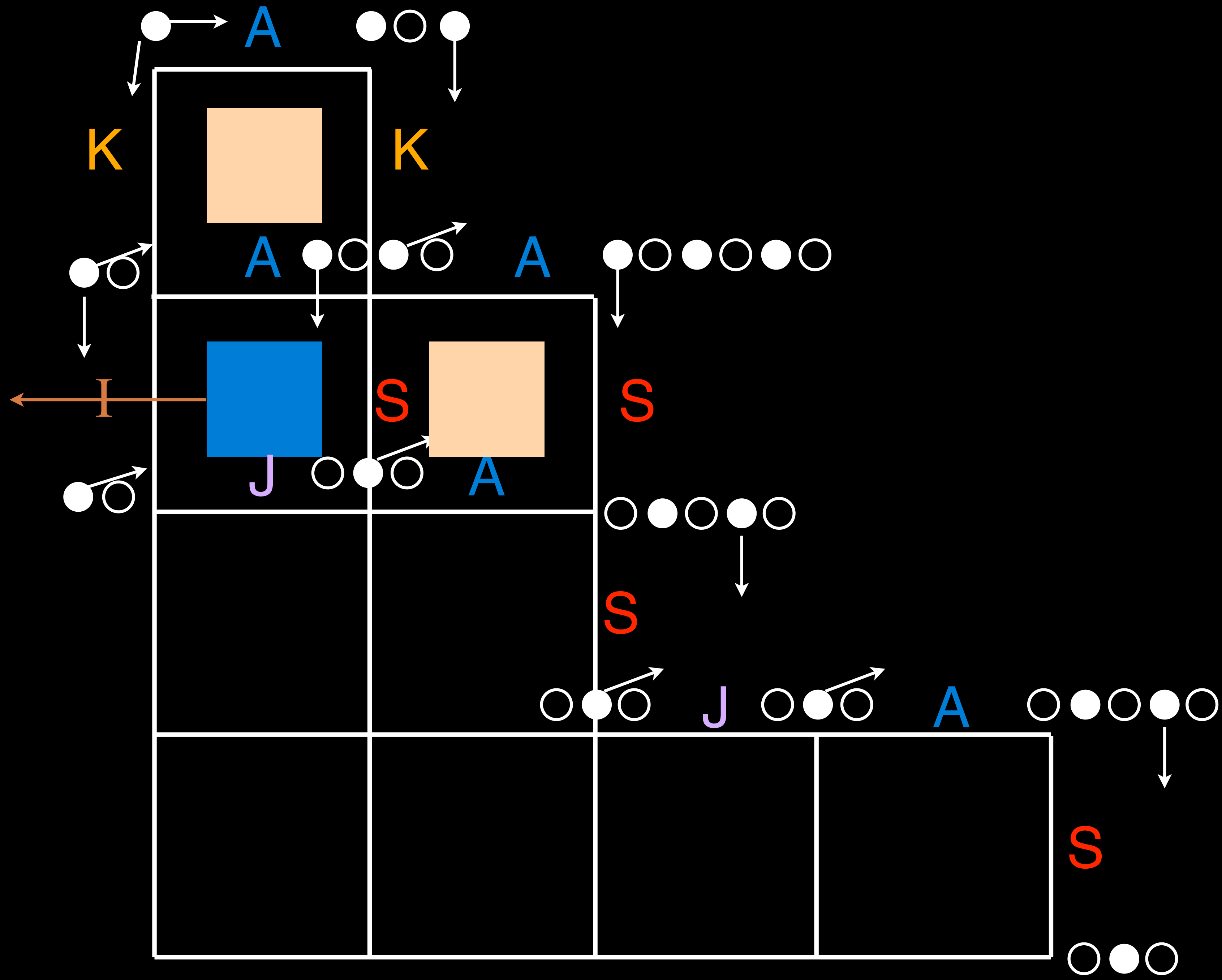
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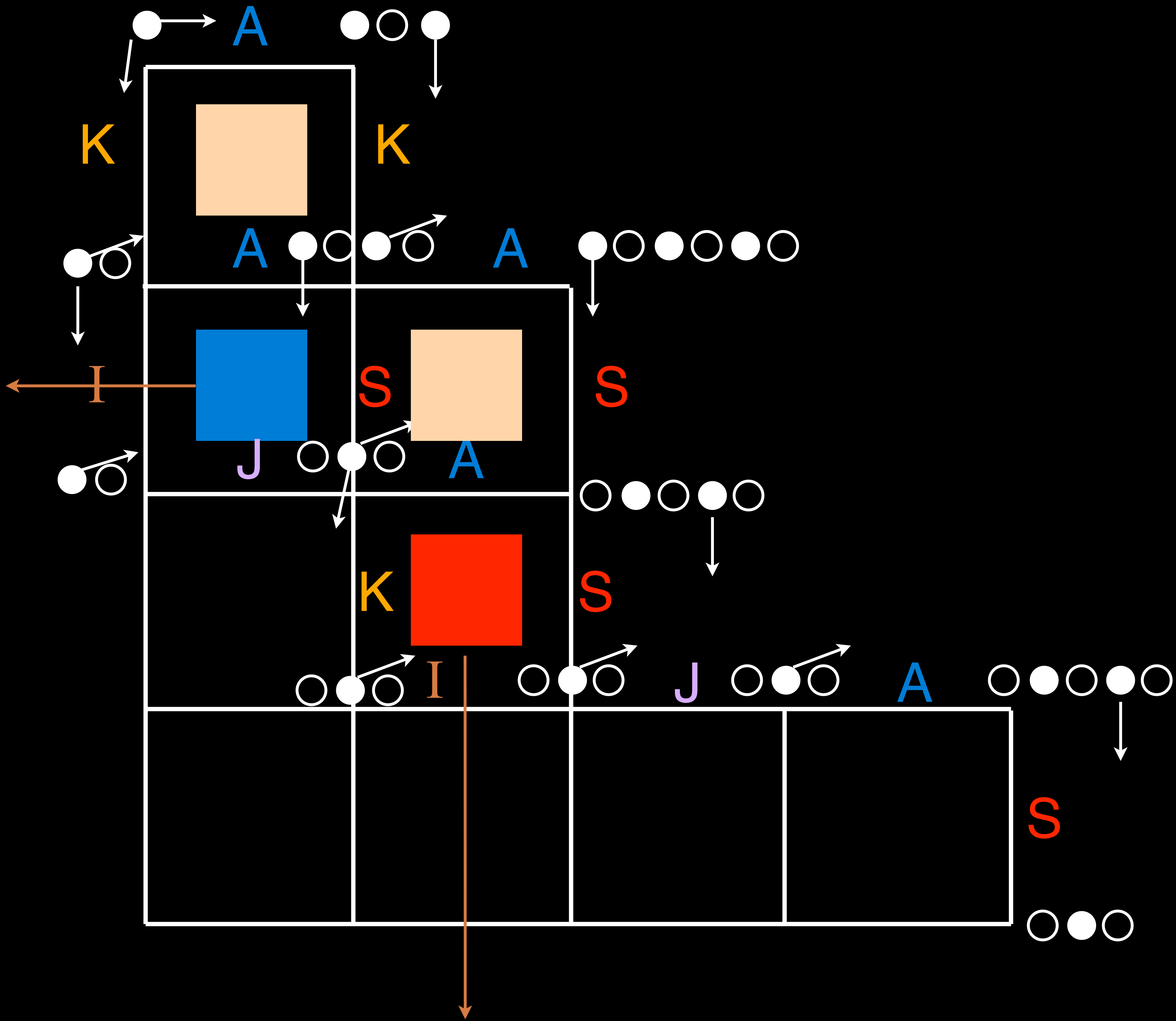

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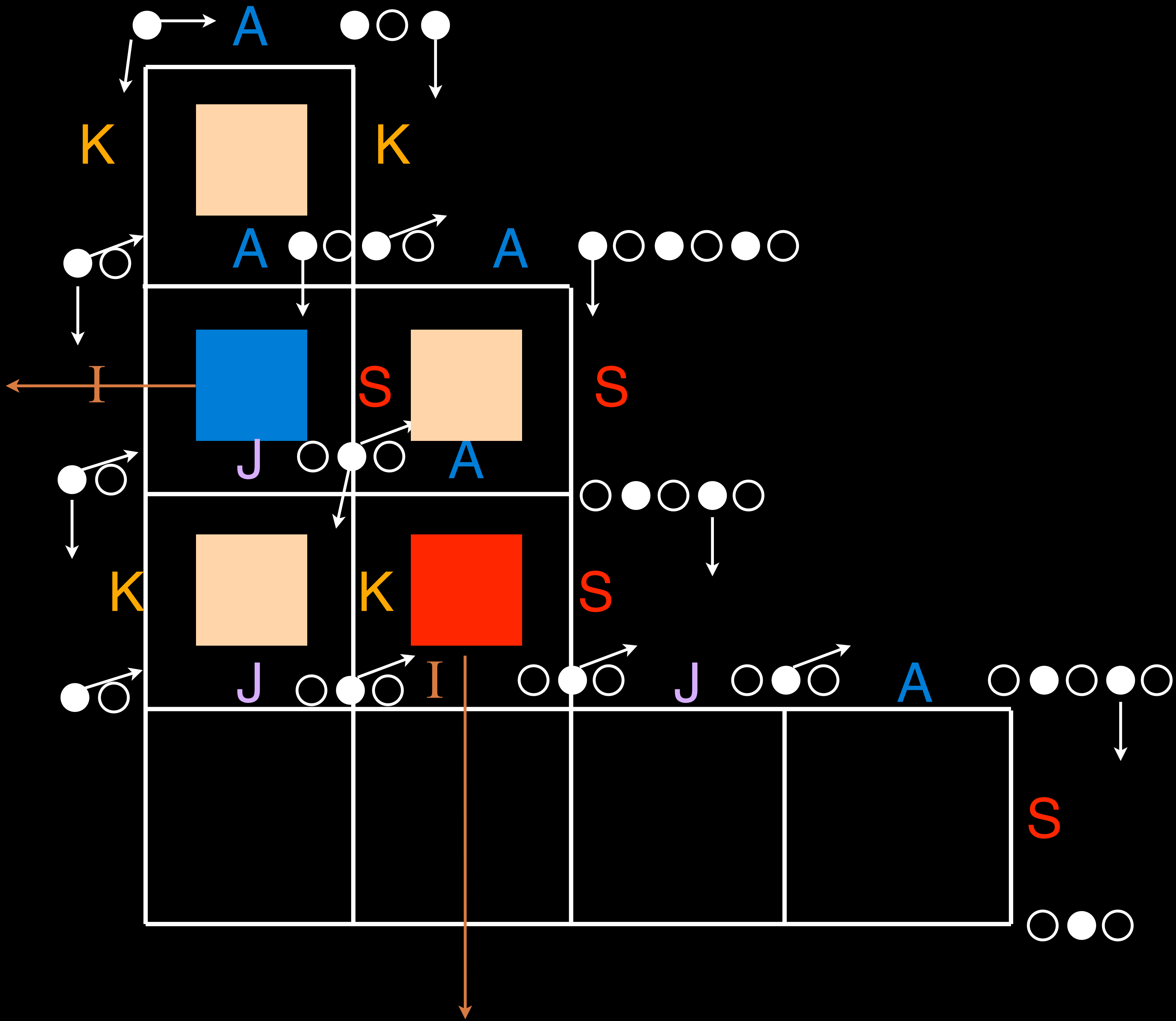

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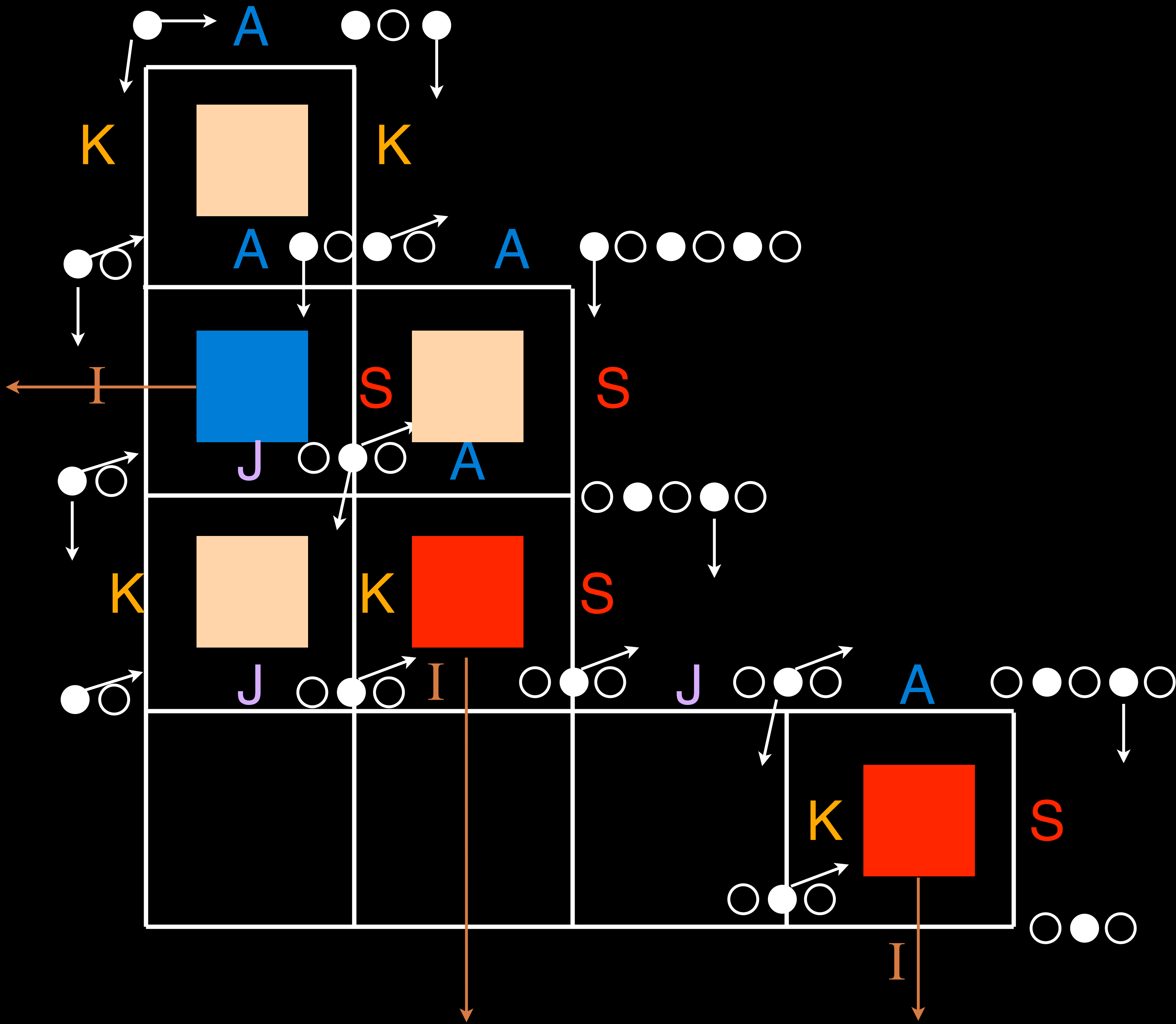
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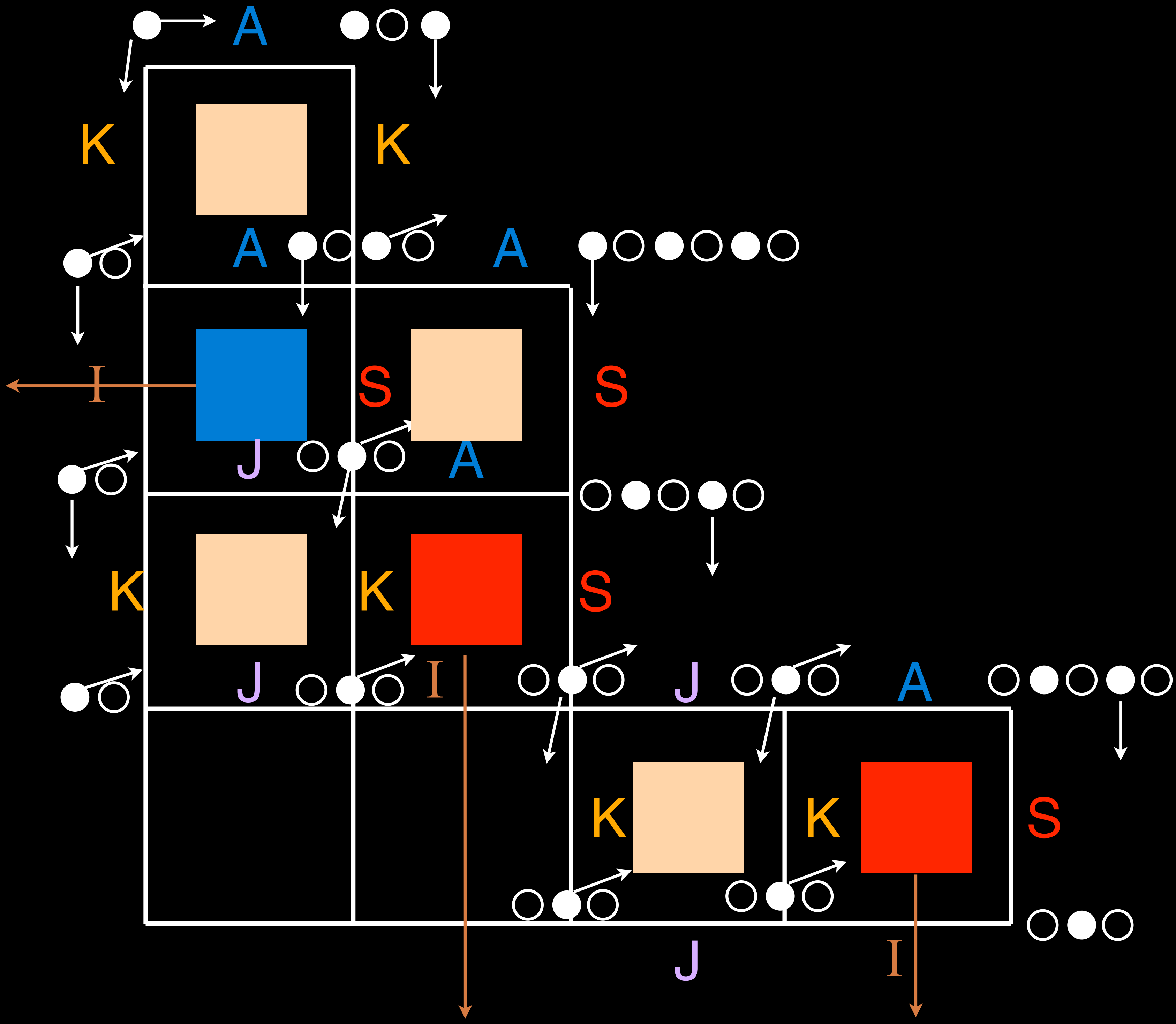
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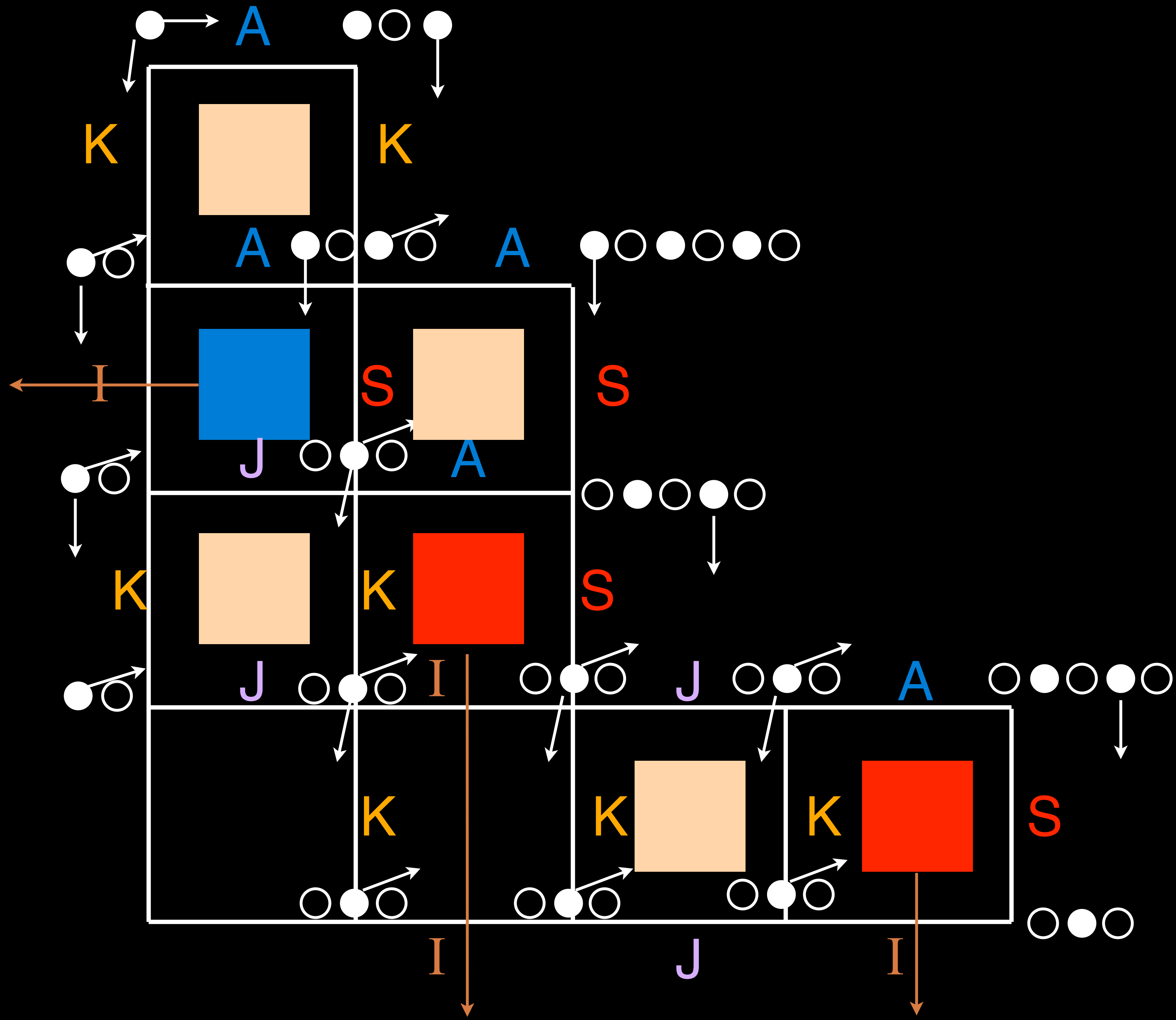
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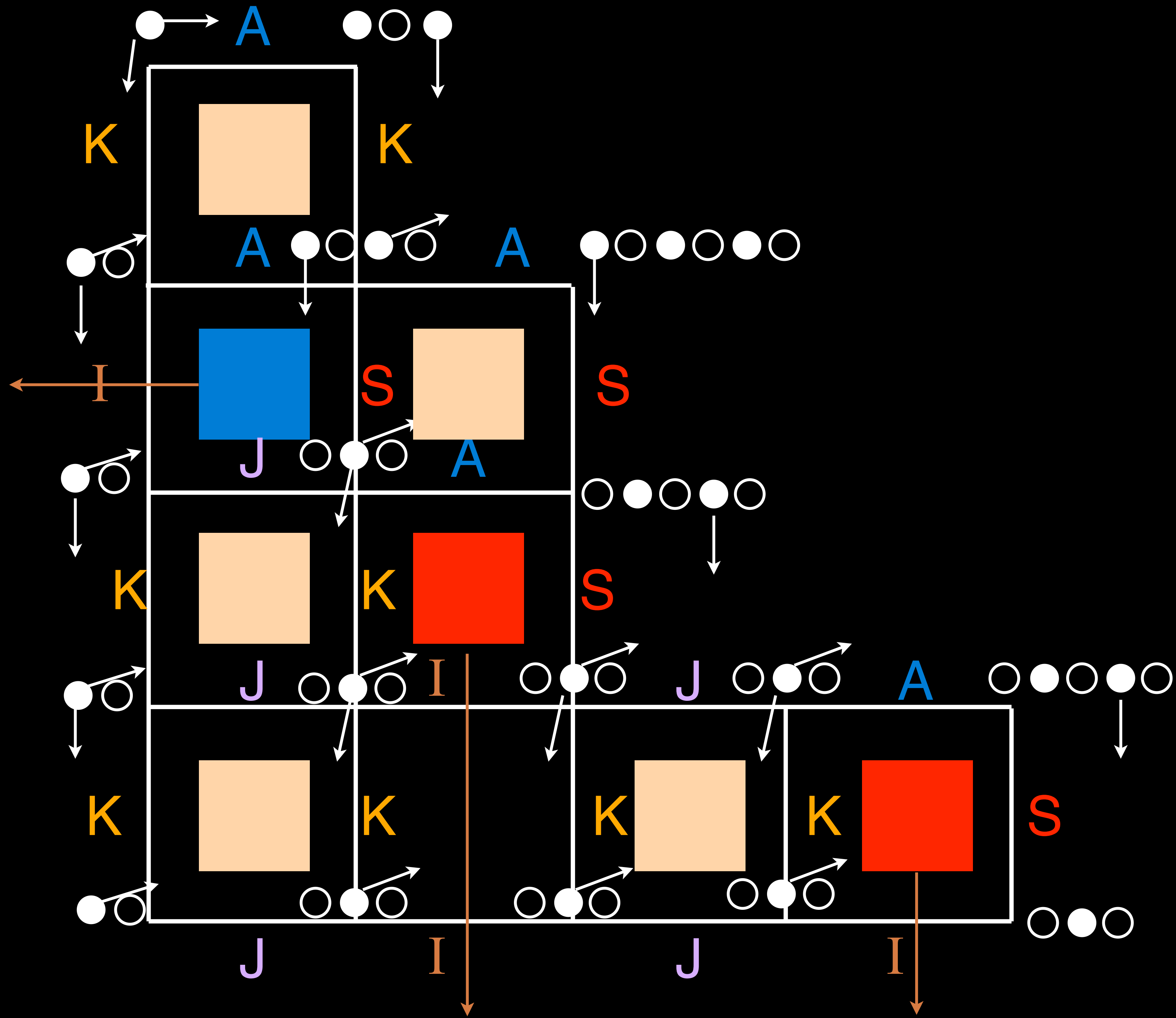
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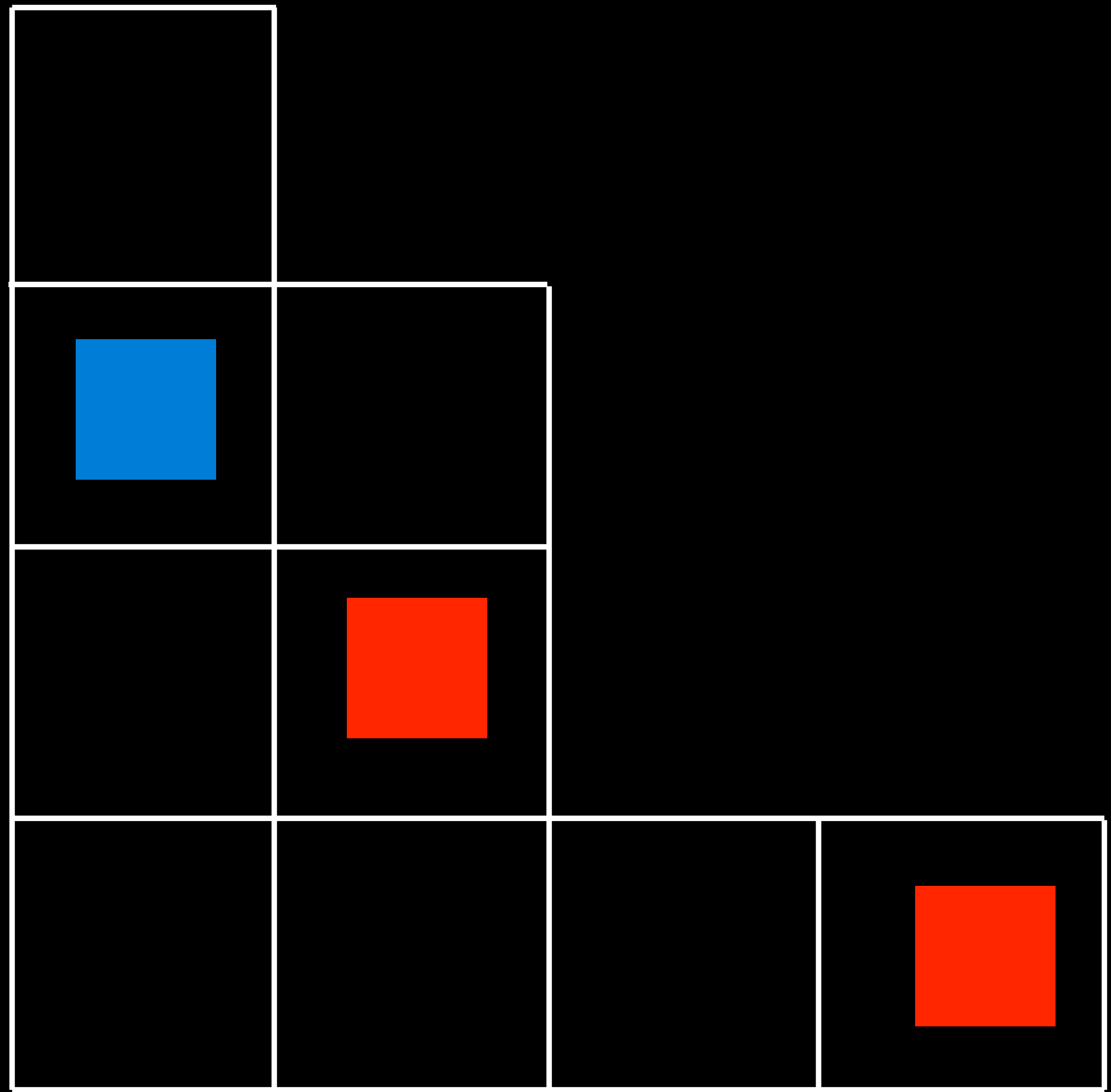


416978352

Laguerre histories

combinatorial theory of orthogonal polynomials

The FV bijection

Françon-XV 1978

Bijection Permutations $n+1$

$\updownarrow$

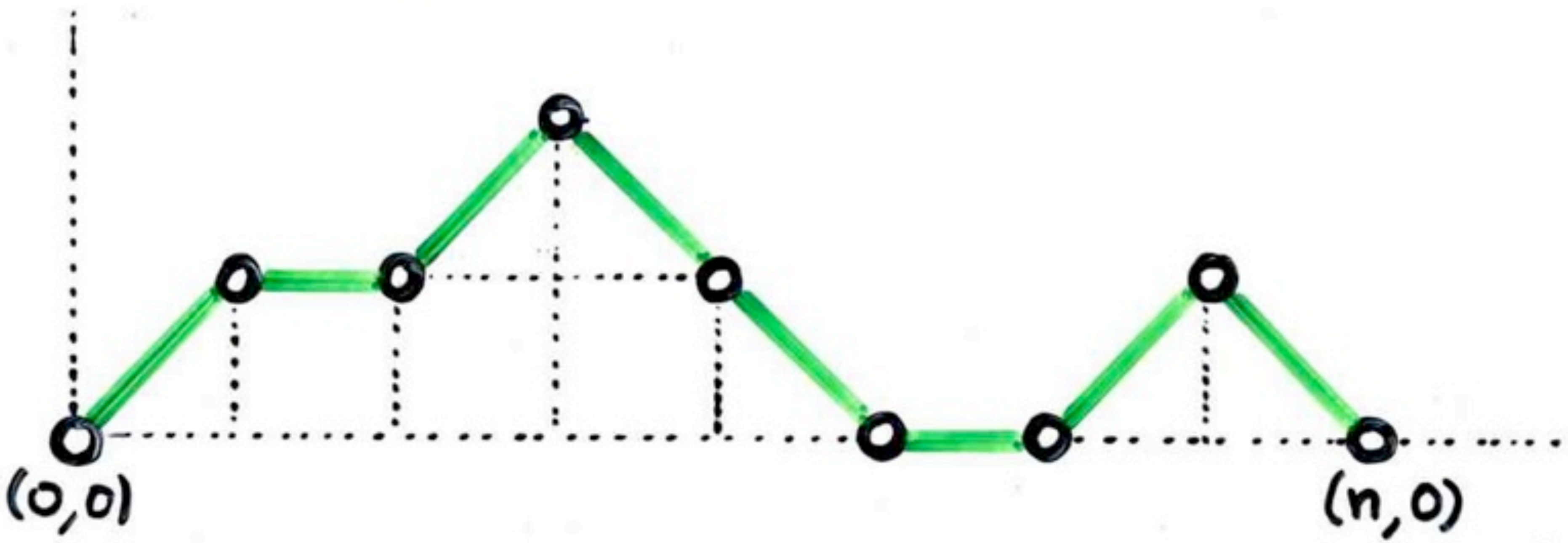
Histoires de Laguerre $(\gamma_c ; f)$ n

Bijection **Permutations** $n+1$

Histoires de Laguerre $(\gamma_c\ ;\ f)$ n

Chemin de Motzkin

$n-1$

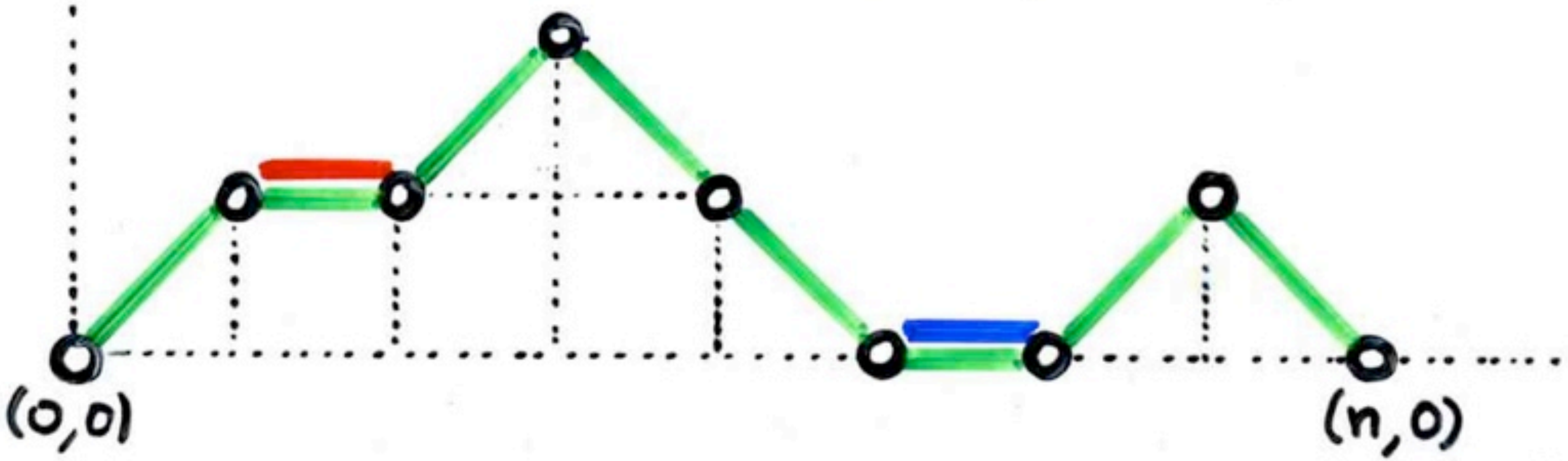


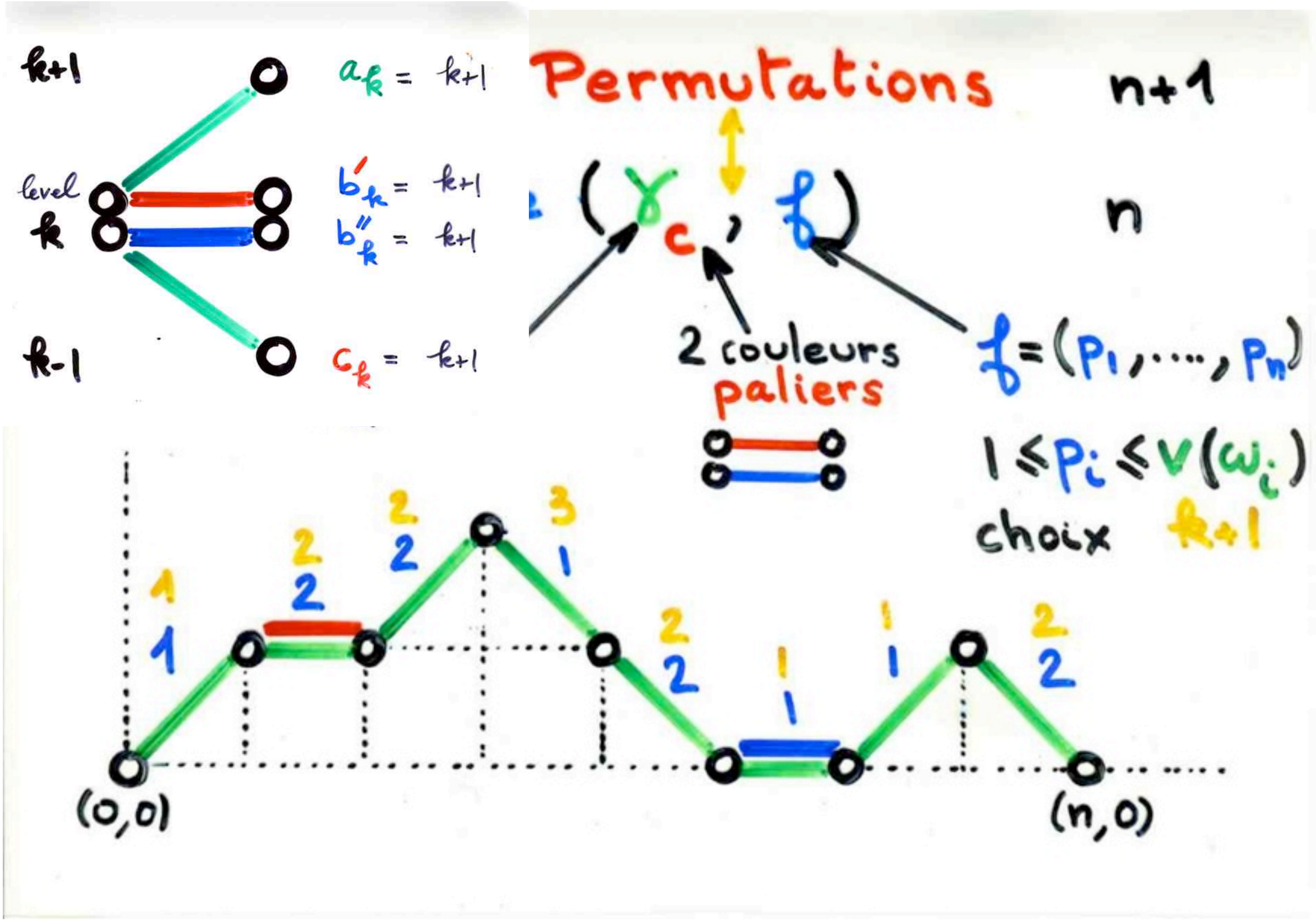
Bijection **Permutations** $n+1$

Histoires de Laguerre $(\gamma_c \, ; \, f)$ n

Chemin de Motzkin $n-1$ → γ

2 couleurs paliers → c

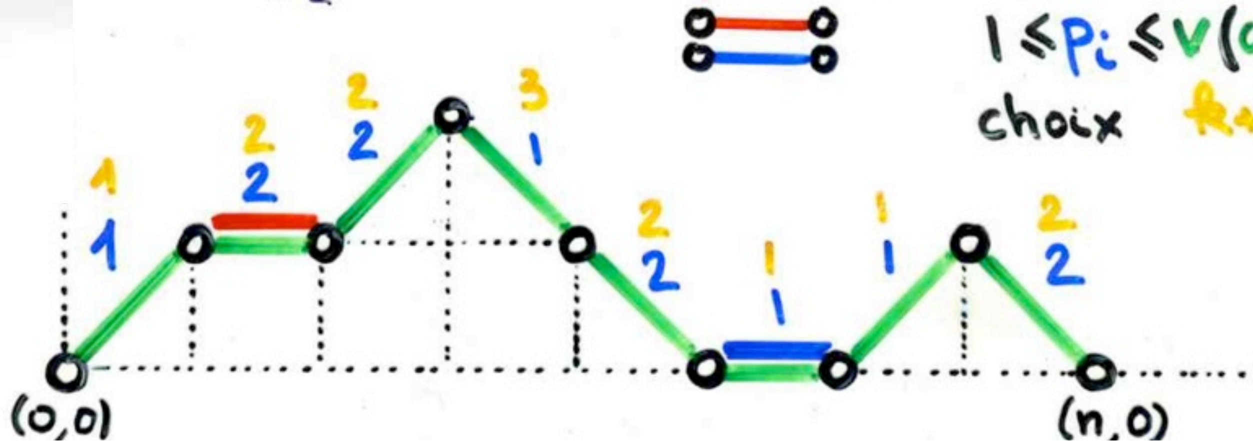



Permutations
n+1
n
k+1
level k
k-1
$a_k = k+1$
$b'_k = k+1$
$b''_k = k+1$
$c_k = k+1$
(γ_c , f)
2 couleurs
paliers
$f = (p_1, \ldots, p_n)$
$1 \leq p_i \leq v(\omega_i)$
choix k+1
(0,0)
(n,0)

Bijection

Histoires de Laguerre $(\omega ; (p_1, \dots, p_n)) \longleftrightarrow$ permutations $(n+1)!$

$h = (\omega_c ; (p_1, \dots, p_n))$



$1 \leq p_i \leq v(\omega_i)$

choix k+1

x	ω_c	pos	v
1	↗	1	1
2	— (red)	2	2
3	↗	2	2
4	↘	1	3
5	↘	2	2
6	— (blue)	1	1
7	↗	1	1
$n =$ 8	↘	2	2
9	•		

␣

␣ 1 ␣

␣ 1 ␣ 2

␣ 1 ␣ 3 ␣ 2

4 1 ␣ 3 ␣ 2

4 1 ␣ 3 5 2

4 1 6 ␣ 3 5 2

4 1 6 ␣ 7 ␣ 3 5 2

4 1 6 ␣ 7 8 3 5 2

4 1 6 9 7 8 3 5 2 $= \sigma \in \mathfrak{S}_{n+1}$

(formal) orthogonal polynomials

Orthogonal polynomials

Def. $\{P_n(x)\}_{n\geq 0}$ $\quad P_n(x) \in \mathbb{K}[x]$

orthogonal iff $\exists\ f : \mathbb{K}[x] \to \mathbb{K}$

linear functional

$$\begin{cases} \text{(i)} & \deg(P_n(x)) = n \quad (\forall n \geq 0) \\ \text{(ii)} & f(P_k P_l) = 0 \quad \text{for } k \neq l \geq 0 \\ \text{(iii)} & f(P_k^2) \neq 0 \quad \text{for } k \geq 0 \end{cases}$$

$$\ell(x^n) = \mu_n \quad (n \geq 0)$$

moments

$$\ell(PQ) = \int_a^b P(x)\,Q(x)\,d\mu$$

mesure

combinatorial interpretation of the moments

Thm. (Favard)

- $\{P_n(x)\}_{n\geqslant 0}$ sequence of monic polynomials, $\deg(P_n)=n$
- $\{b_k\}_{k\geqslant 0}$, $\{\lambda_k\}_{k\geqslant 1}$ coeff. in $\mathbb{K}$

orthogonality $\Longleftrightarrow$

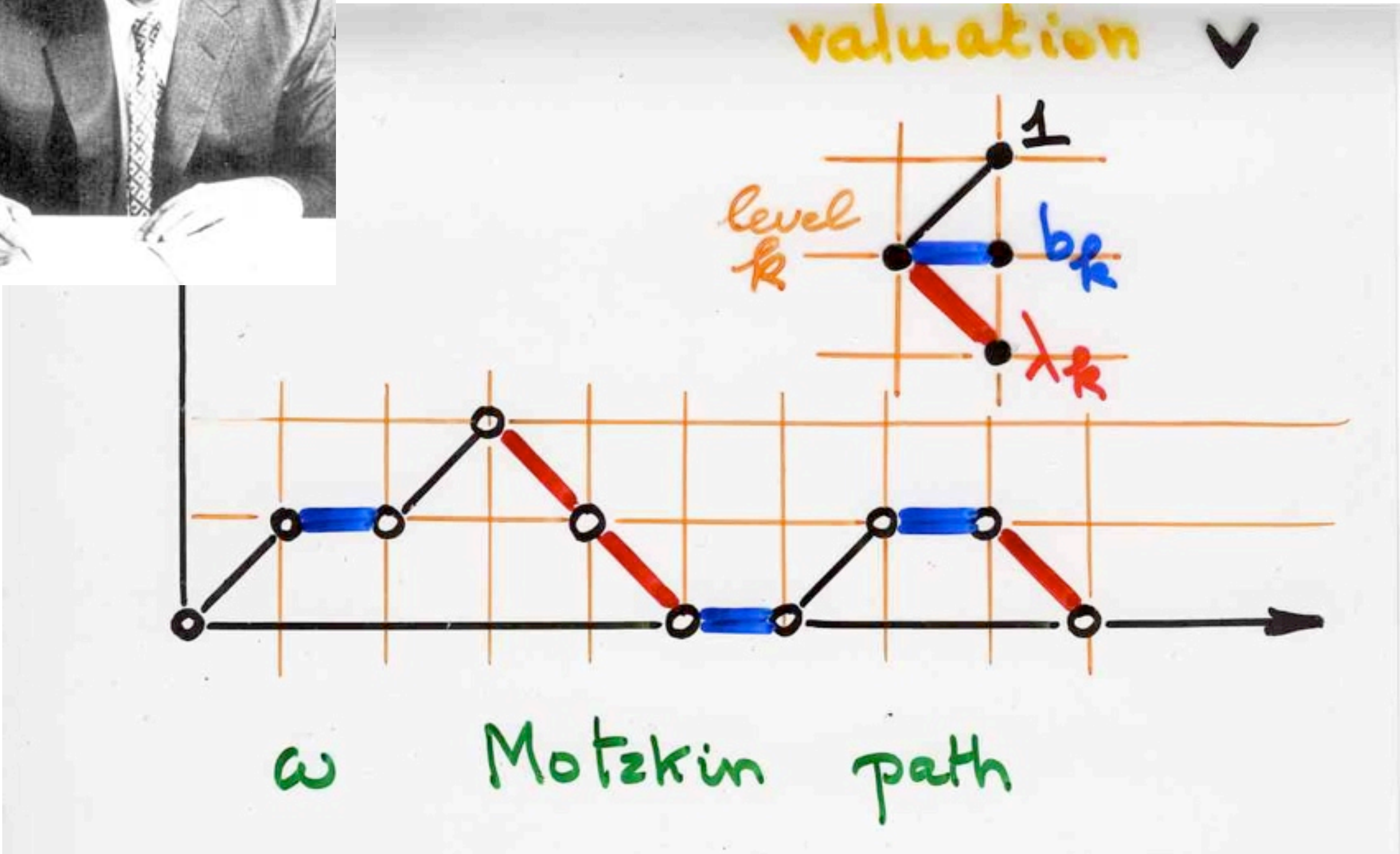
$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$(\forall k\geqslant 1)$

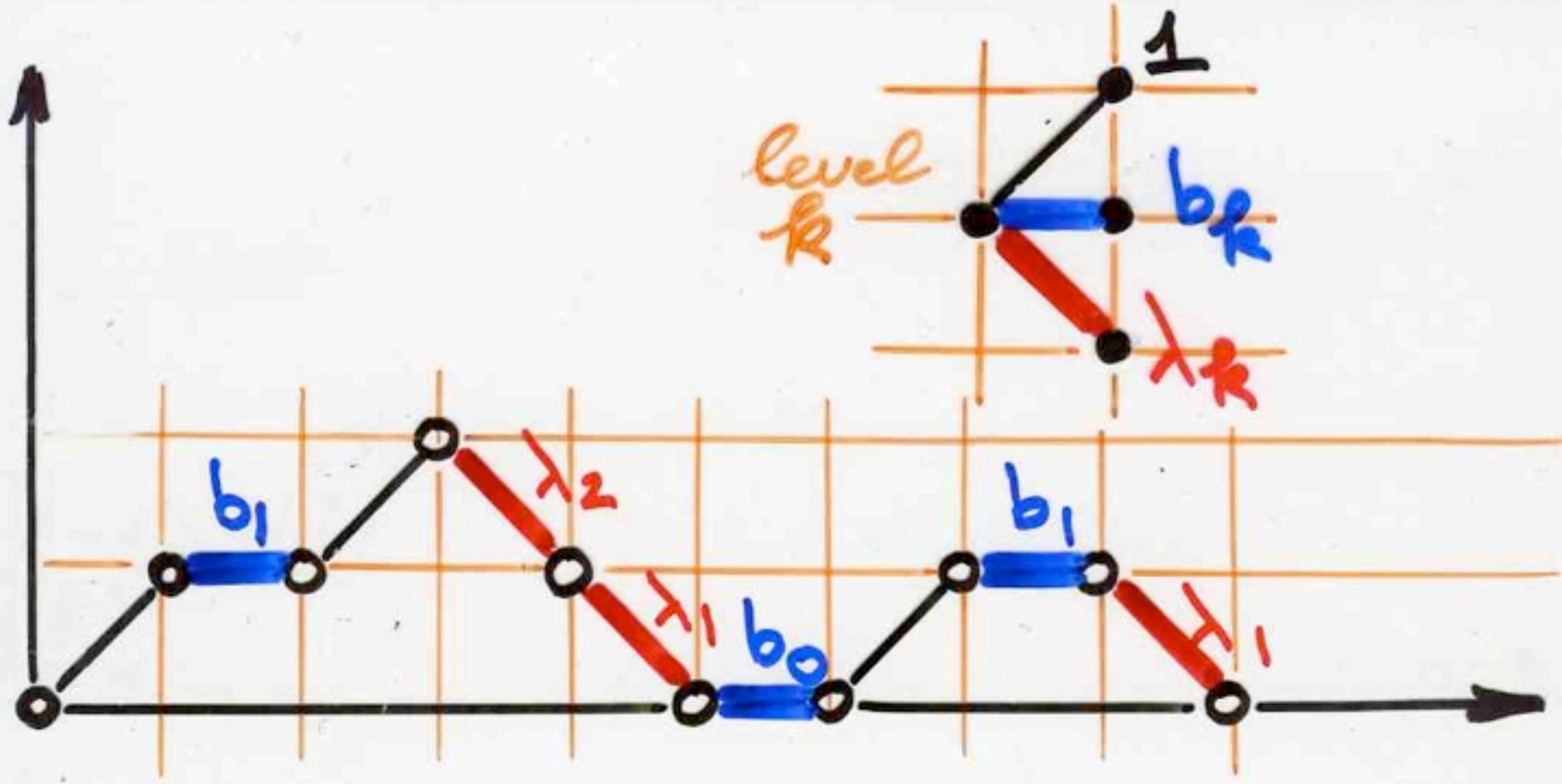
3 terms linear recurrence relation

$\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$

$b_k, \lambda_k \in \mathbb{K}$ ring



valuation v



ω Motzkin path

$$v(\omega) = b_0 \, b_1^2 \, \lambda_1^2 \, \lambda_2$$

$$f(x^n) = \mu_n \quad (n \geq 0)$$

moments

$$\mu_n = \sum_{\omega} v(\omega)$$

ω Motzkin path

$|\omega| = n$

continued fractions 1980

orthogonal polynomials

P. Flajolet

Lecture Note X.V. 1983

Laguerre histories
and
Laguerre polynomials

The FV bijection

Françon-XV 1978

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

$$P_0 = 1 \qquad P_1 = x - b_0$$

$$\mu_n = (n+1)!$$

$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

Laguerre polynomial

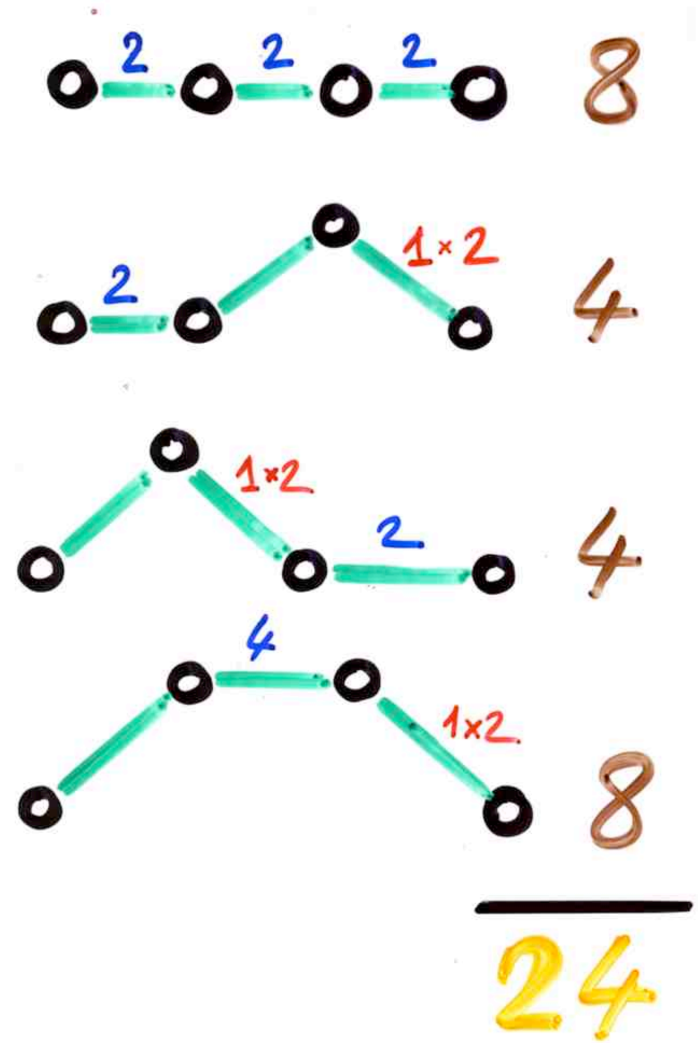
$$J(t) = \cfrac{1}{1 - 2t - \cfrac{1\cdot 2t^2}{1 - 4t - \cfrac{2\cdot 3t^2}{\cdots}}}$$

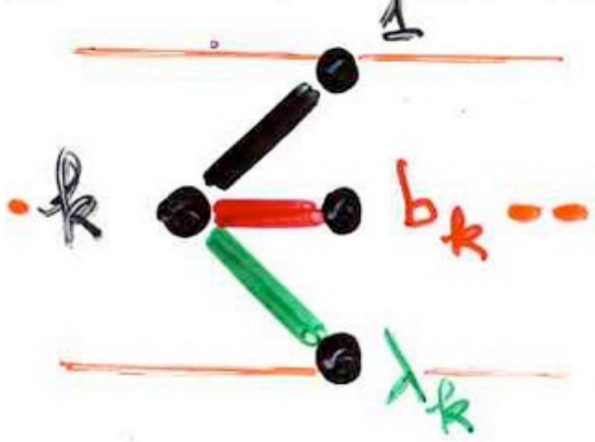
Laguerre $L_n^{(1)}(x)$

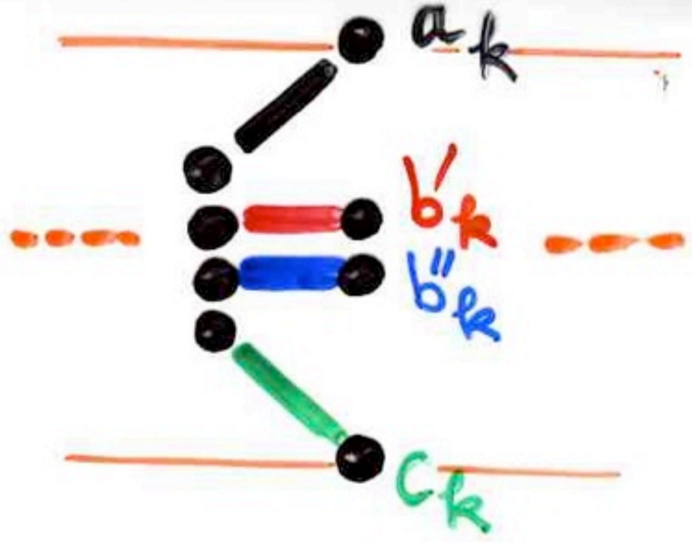
moment $\mu_n = (n+1)!$

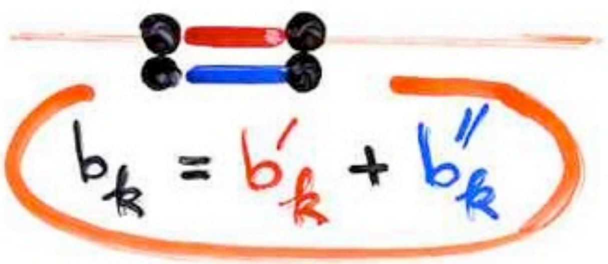
$b_k = 2k+2$

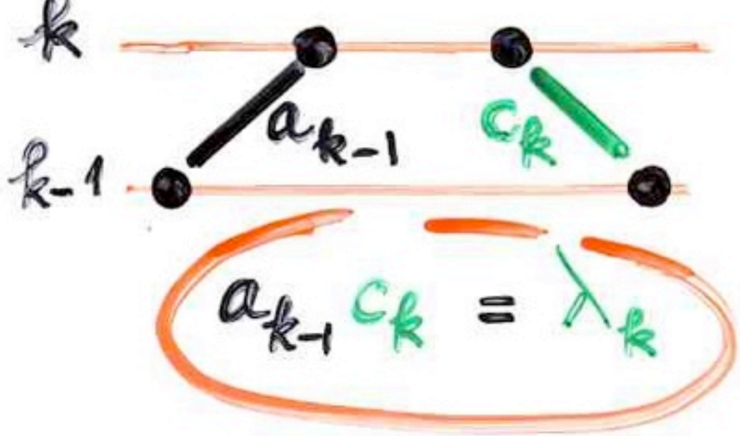
$\lambda_k = k(k+1)$




1
k
b_k
λ_k


a_k
b'_k
b''_k
c_k

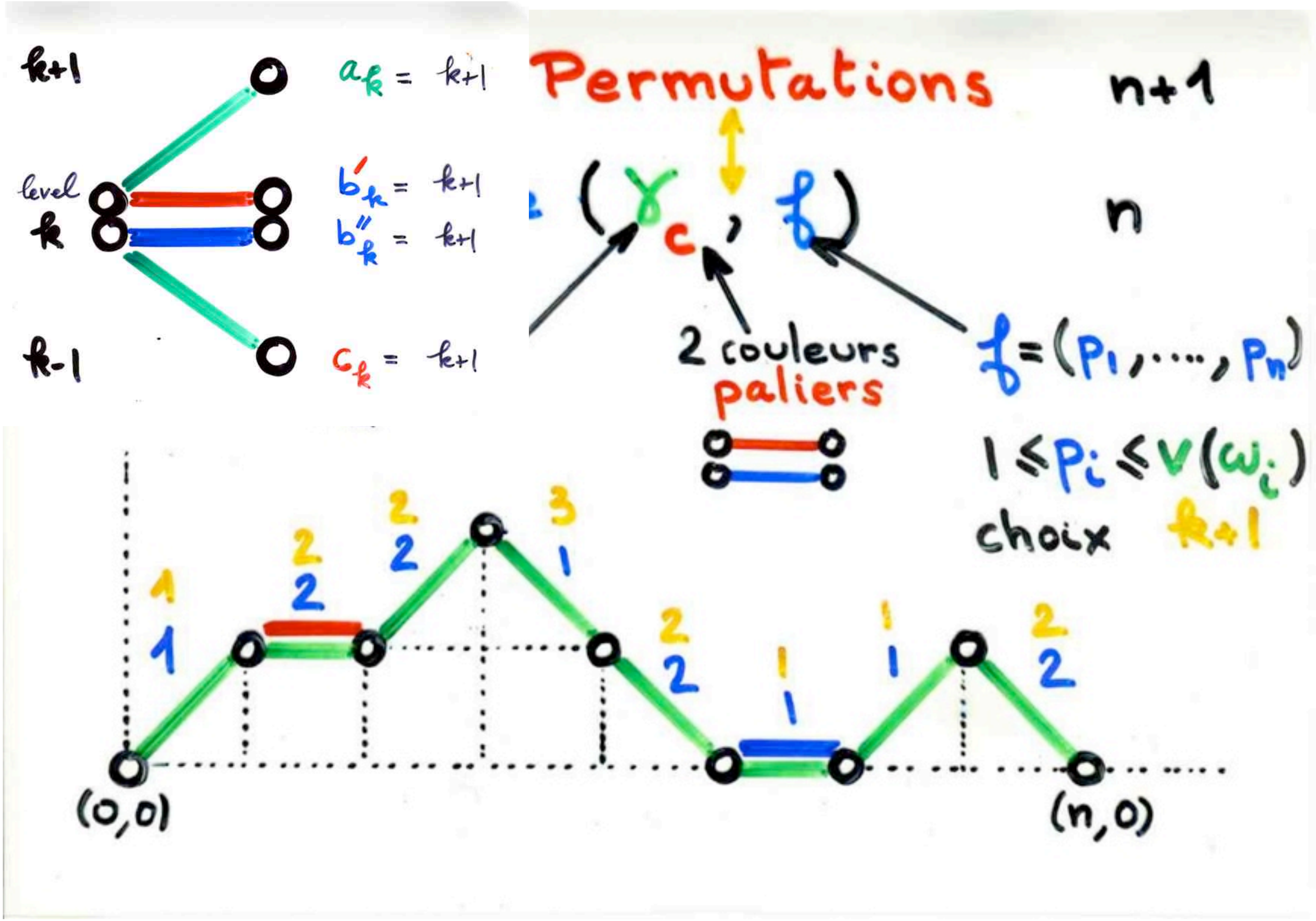

b_k = b'_k + b''_k


k
k-1
a_{k-1}
c_k
a_{k-1} c_k = λ_k

Laguerre $L_n^{(1)}(x)$

$$\mu_n = (n+1)!$$

$$a_k = k+1$$
$$b'_k = k+1$$
$$b''_k = k+1$$
$$c_k = k+1$$


k+1
level k
k-1
a_k = k+1
b'_k = k+1
b''_k = k+1
c_k = k+1
Permutations
n+1
(γ_c , f)
n
2 couleurs
paliers
f = (p_1, ..., p_n)
1 ≤ p_i ≤ v(ω_i)
choix k+1
(0,0)
(n,0)

3 parameters

Cor. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ (PASEP) is

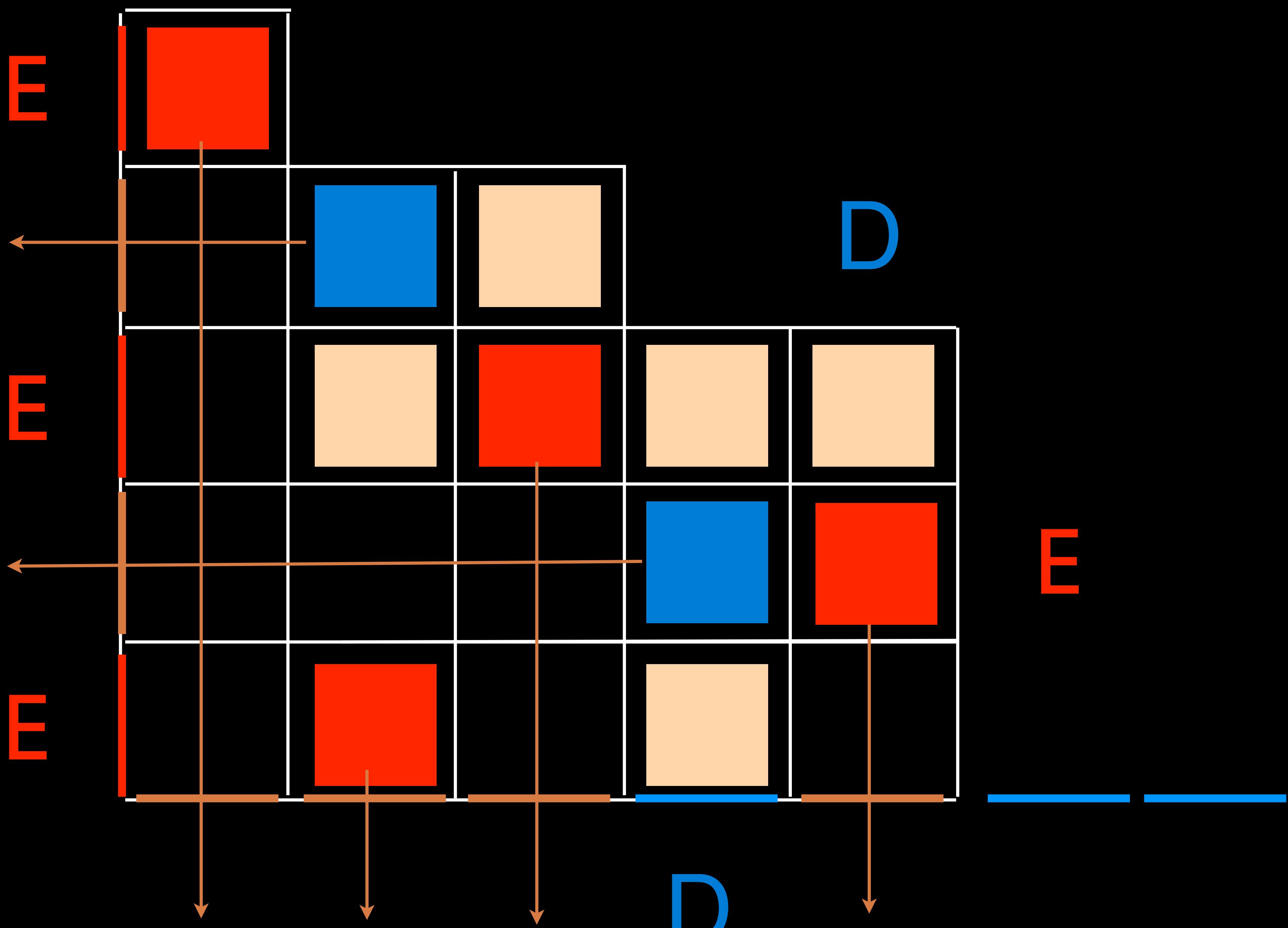
$$\mathrm{proba}_\tau(q; \alpha, \beta) = \frac{1}{Z_n} \sum_{T} q^{L(T)} \alpha^{-f(T)} \beta^{-u(T)}$$

T alternative tableaux profile τ

$f(T)$ nb of rows without cell
$u(T)$ nb of columns without cell
$L(T)$ nb of cells

permutation tableau

S. Corteel, L. Williams (2007) (2008) (2009)


E
D
E
E
E
D

total order

$\{1, 2, \dots, n\}$

$\sigma = 7\ 2\ 3\ 9\ 6\ 8\ 5\ 1\ 4$

word

left-to-right

right-to-left

minimum elements

$\sigma = ⓻\ ⓶\ 3\ 9\ 6\ 8\ 5\ [⓵]\ [4]$

left-to-right
right-to-left

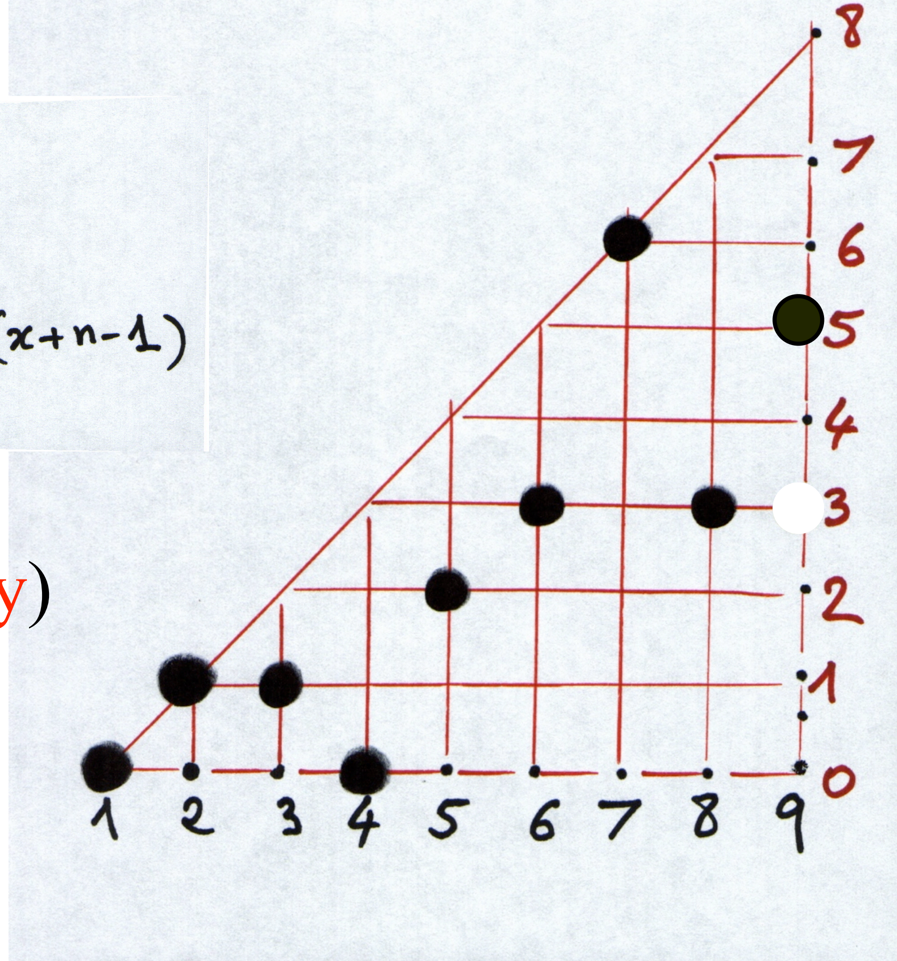
minimum elements

$\sigma = ⑦\ ②\ 3\ 9\ 6\ 8\ 5\ ①\ 4$

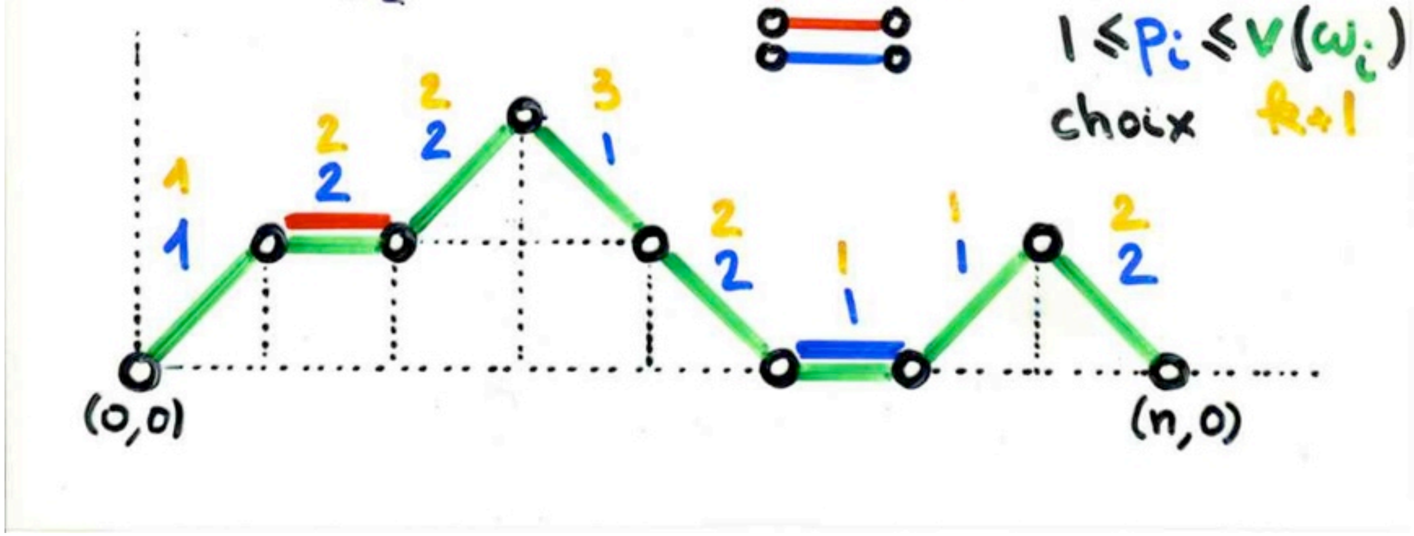
Stirling numbers $s_{n,k}$

$$\sum_{k\geq 1} s_{n,k}\, x^k = x(x+1)\ldots(x+n-1)$$

$$xy(x+y)(x+1+y)\ldots(x+n-2+y)$$



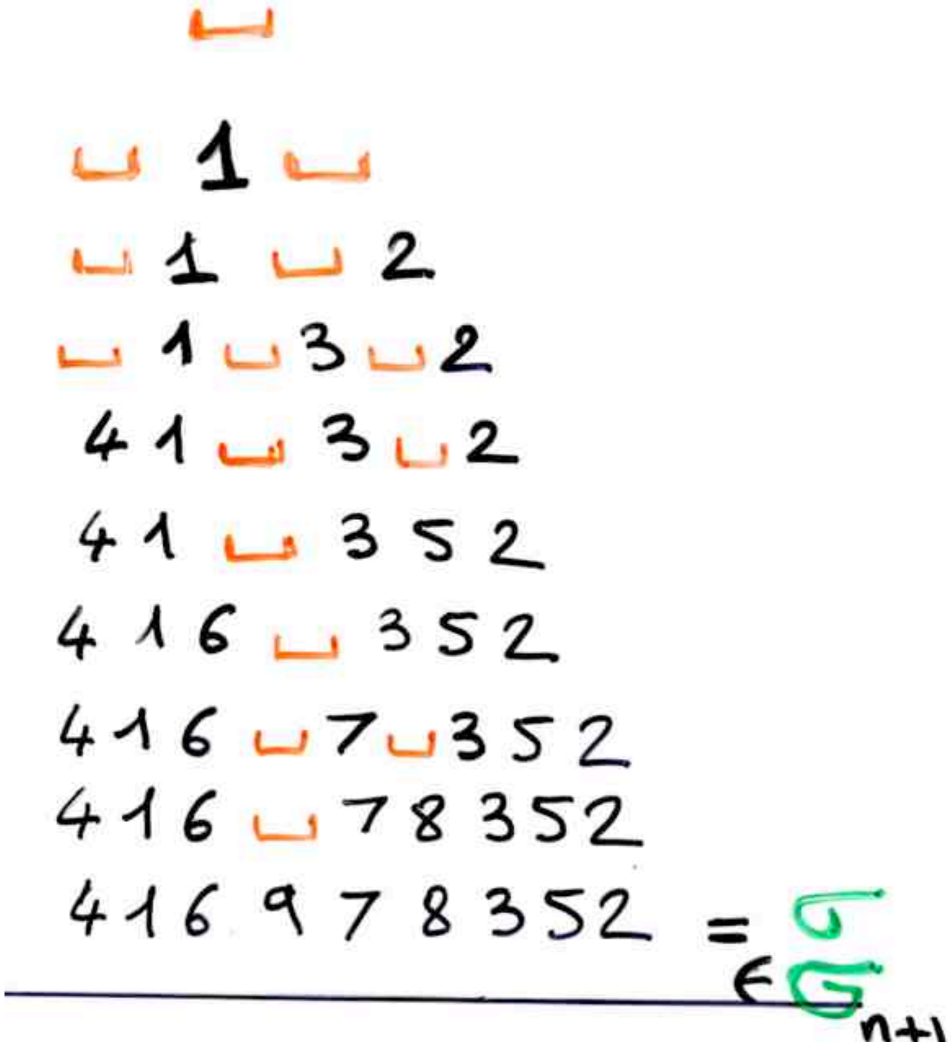
"q-analogue" of Laguerre histories



choices function

1	2	3	4	5	6	7	8
1	2	2	1	2	1	1	2
0	1	1	0	1	0	0	1

q-Laguerre : q^4



q-Laguerre

$L_n^{I\,\beta}(x;q)$

$\beta = \alpha + 1$

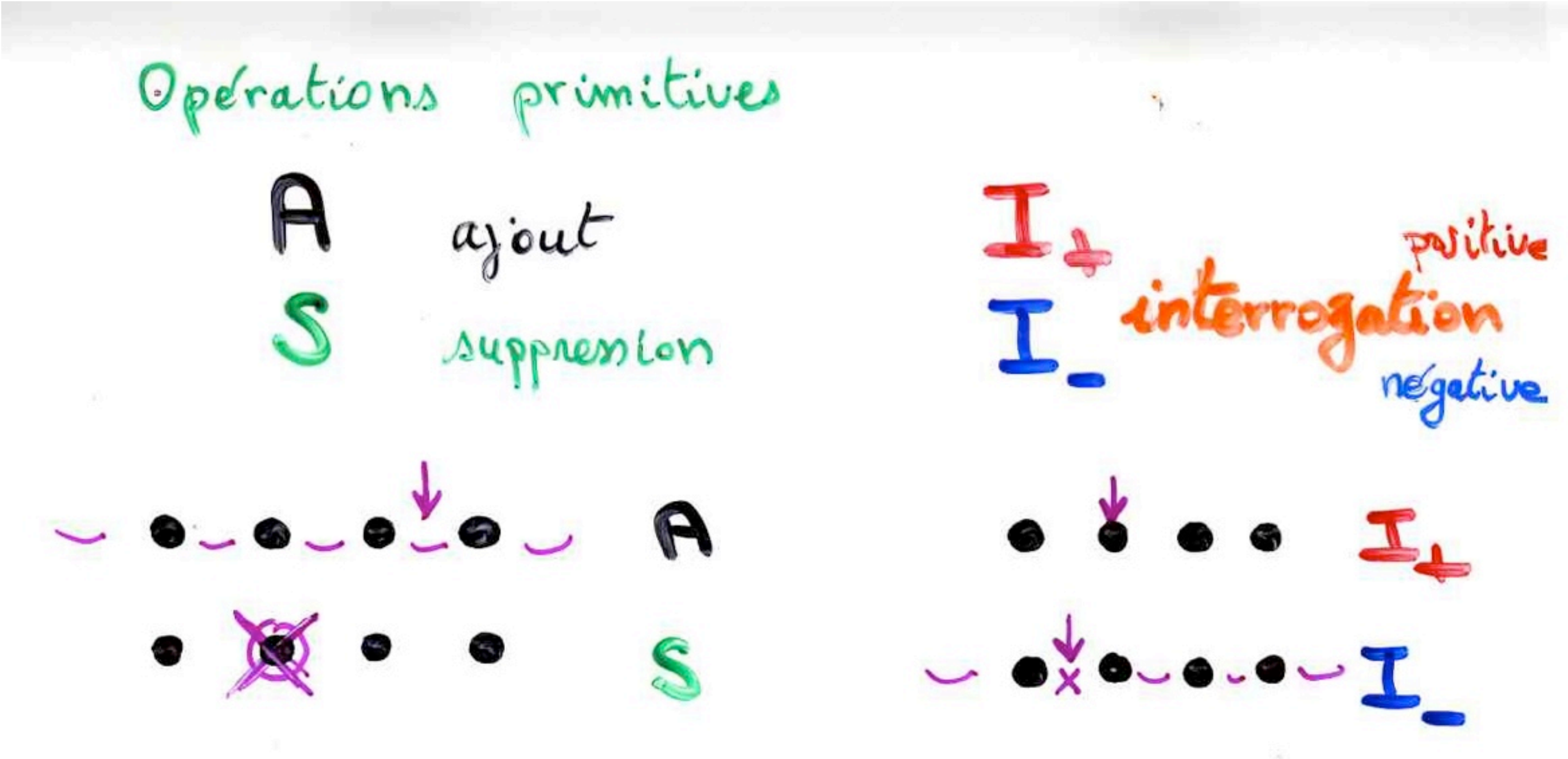
$$\begin{cases} b_{k,q}^{(\beta)} = [k]_q + [k+1;\beta]_q \\ \lambda_{k,q}^{(\beta)} = [k]_q \cdot [k;\beta]_q \end{cases}$$

$$[k;\beta]_q = \beta + q + q^2 + \dots + q^{k-1}$$

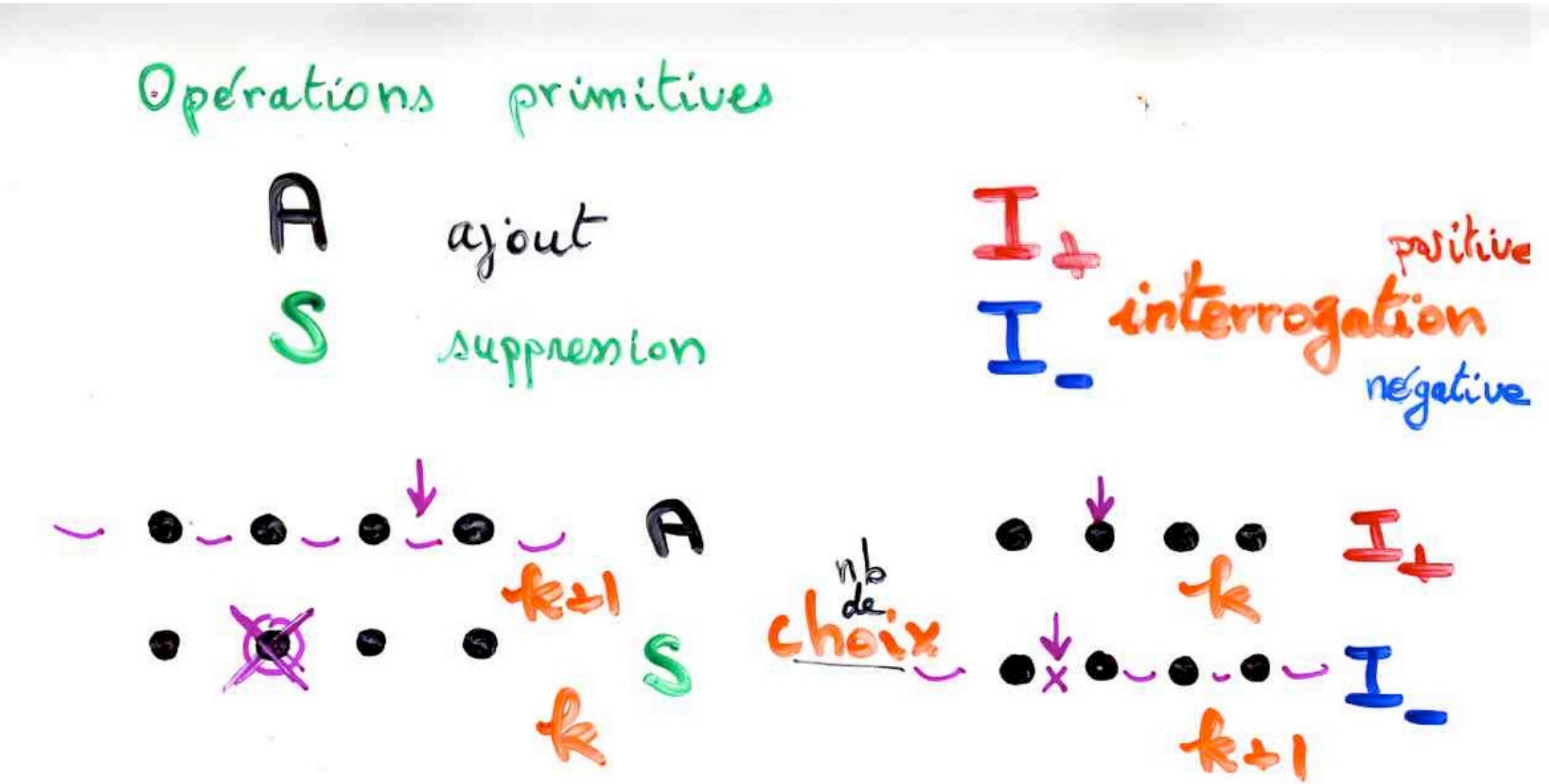
$$\mu_n = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\binom{2n}{n-k} - \binom{2n}{n-k-2} \right) \left(\sum_{i=0}^{k} q^{i(k+1-i)} \right)$$

Corteel, Josuat-Vergès y Prellberg, Rubey (2008)

quadratic algebra
operators
data structures
and
orthogonal polynomials

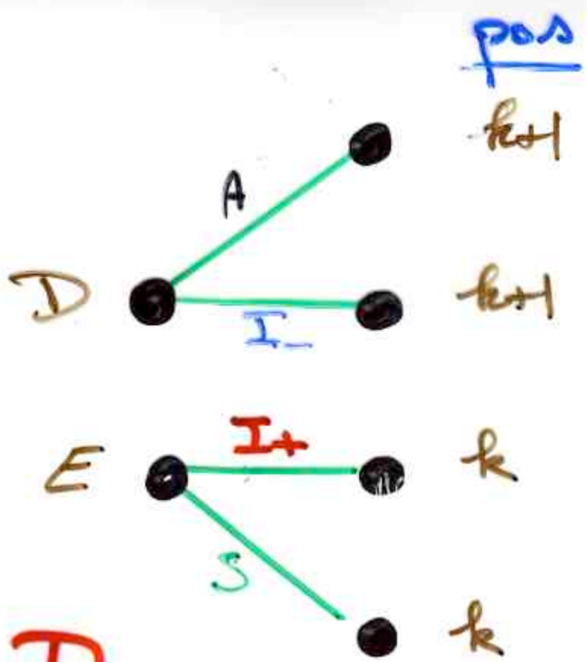


Primitive operations
for “dictionnaries” data structure:
add or delete any elements, asking questions (with positive or negative answer)



number of choices for each
primitive operations

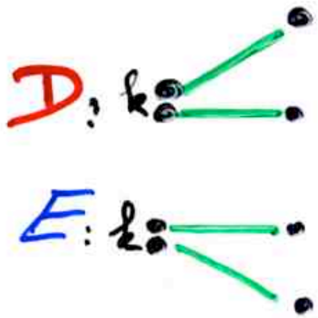
$$\begin{cases} D = A + I_- \\ E = S + I_+ \end{cases}$$



this corresponds to the n!
“restricted Laguerre histories”

$$DE = ED + E + D$$

aussi :



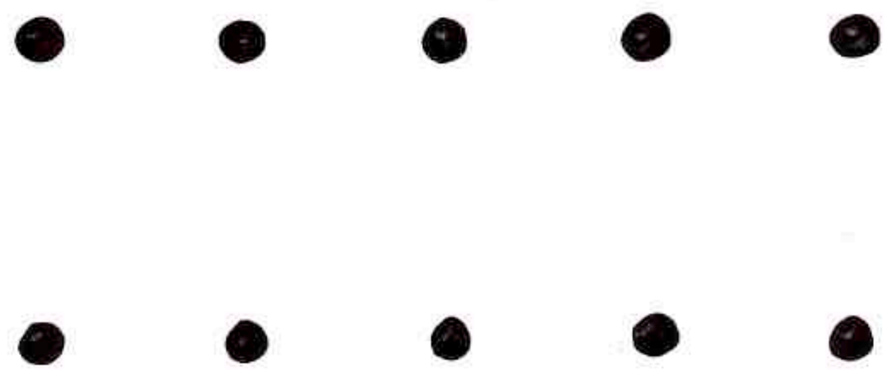
(k+1) possibilités partout

(histoires de Laguerre “larges”)

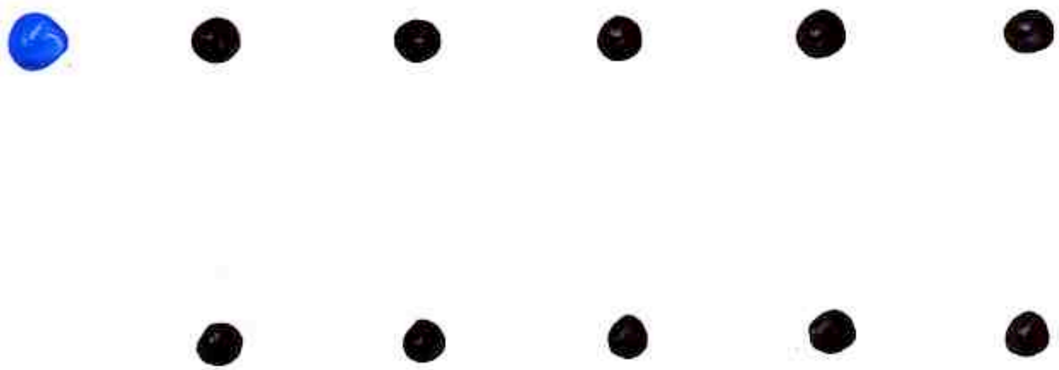
this valuation corresponds to the (n+1)!
“enlarged Laguerre histories”

priority queue

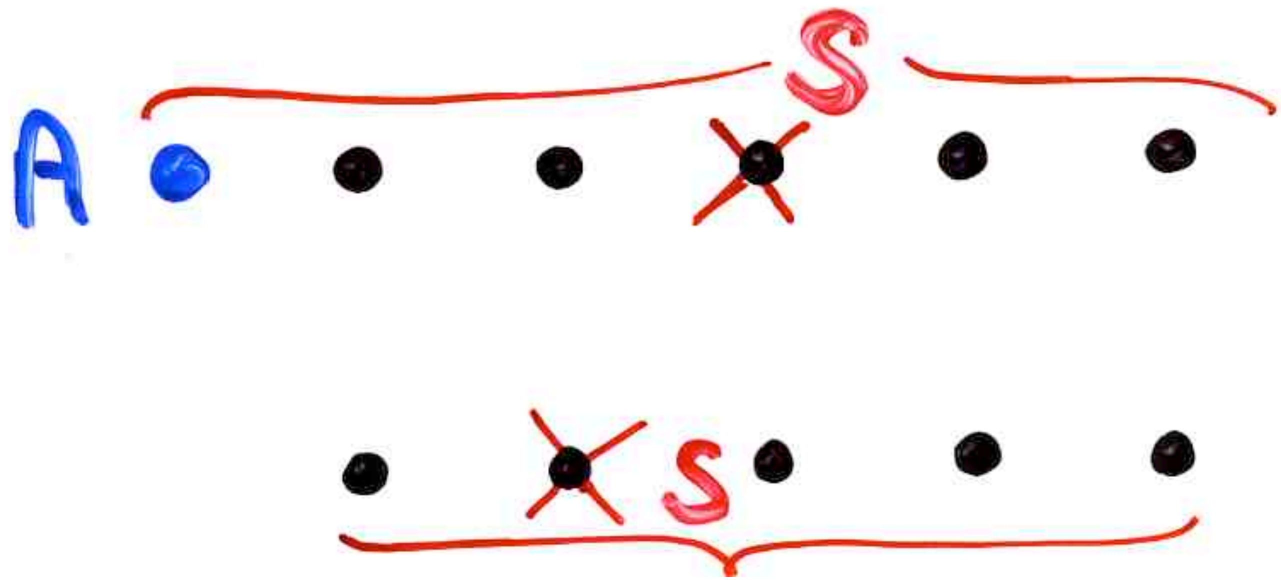
Polya urn



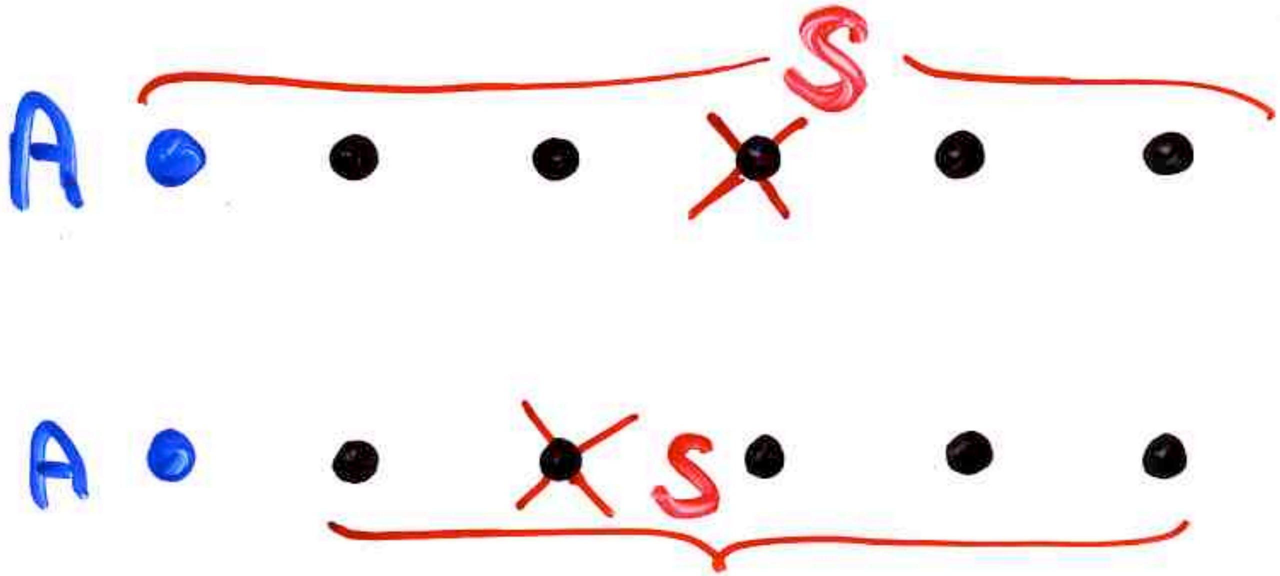
A




A
S


S
A
S

$AS - SA = I$



A produit par x

S $\frac{d}{dx}(\ \ )$

polynôme d'Hermite $H_n(x)$

$\lambda_k = k$; $b_k = 0$

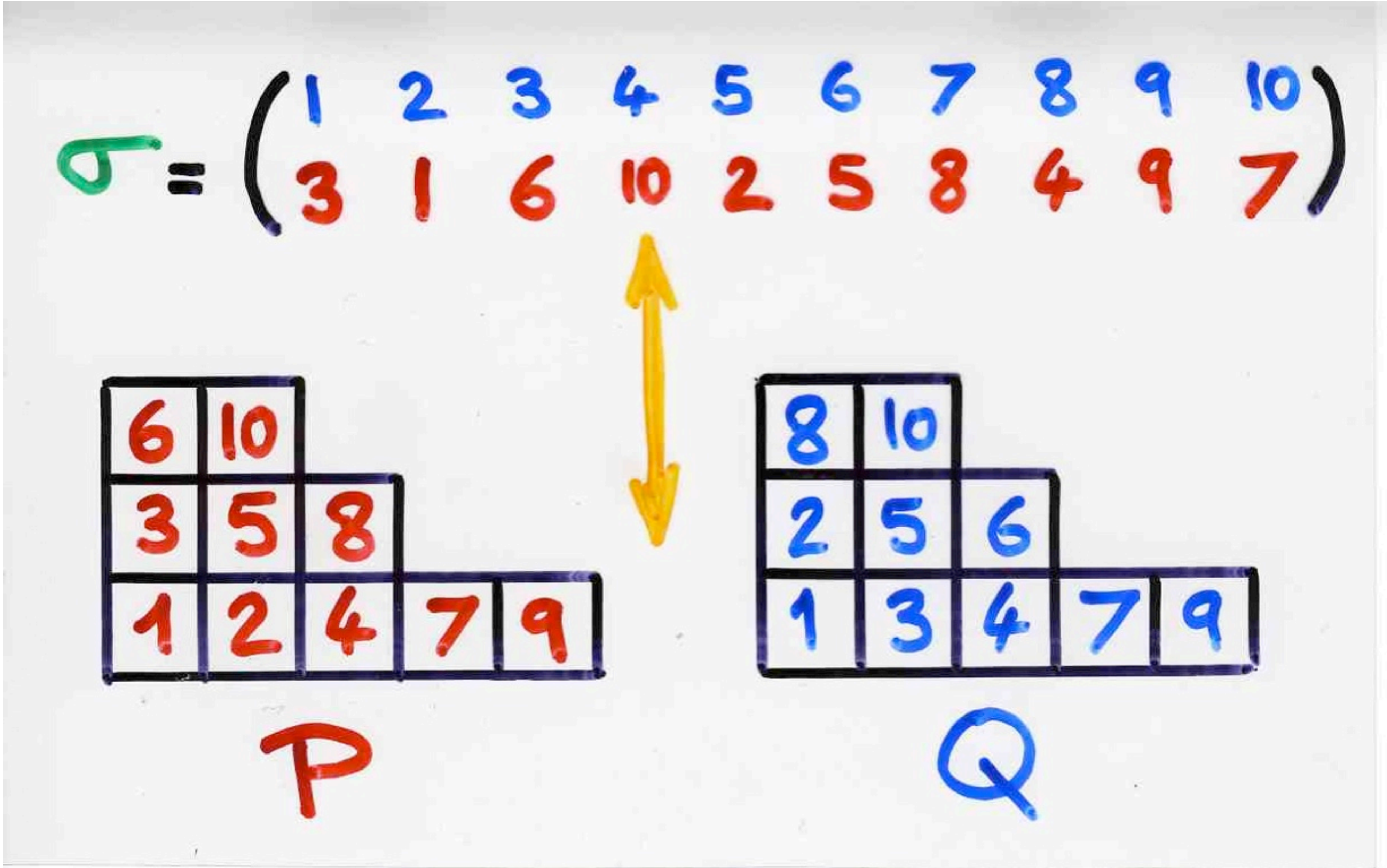
$(k \geq 1)$ $(k \geq 0)$

$a_k = 1$ $\begin{cases} b'_k = 0 \\ b''_k = 0 \end{cases}$ $c_k = k$

Histoires d'Hermite

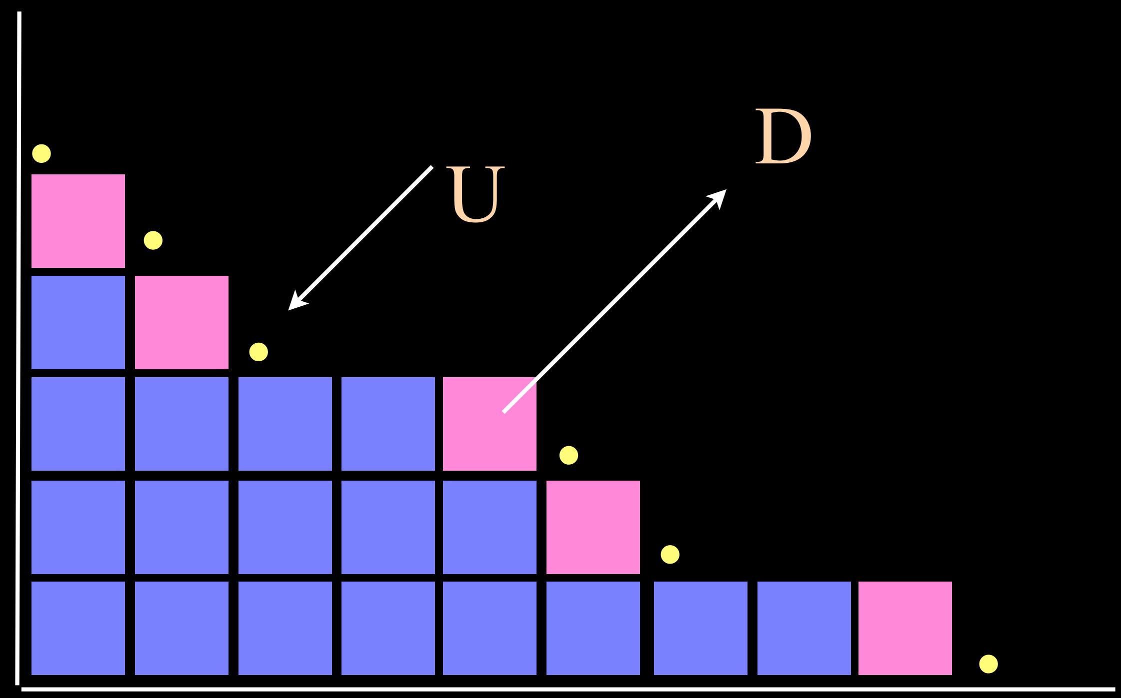
The cellular Ansatz

From quadratic algebra Q
to combinatorial objects (Q-tableaux)
and bijections



The Robinson-Schensted correspondence between permutations and pair of (standard) Young tableaux with the same shape

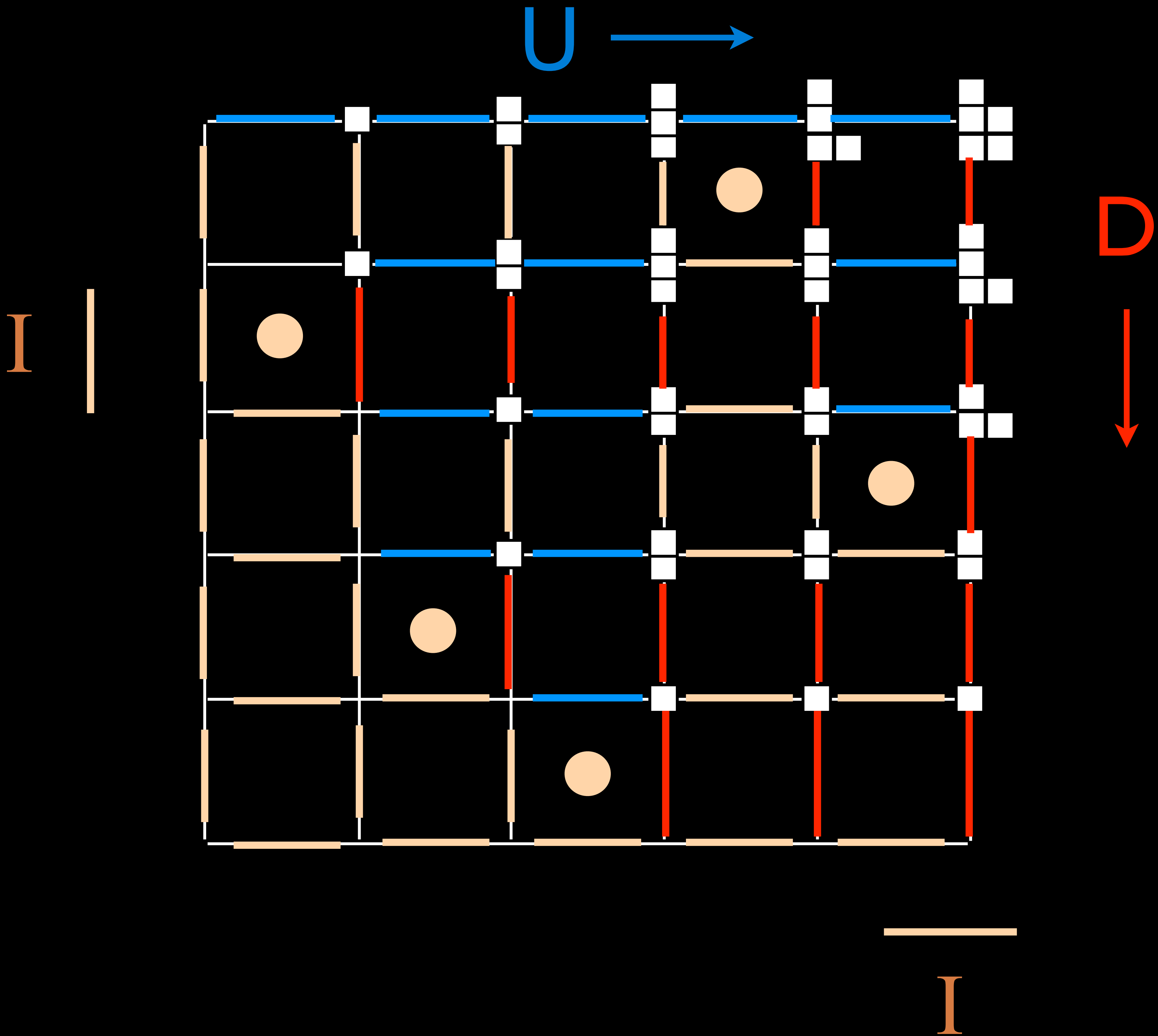
Operators U and D



adding
or deleting
a cell in
a Ferrers
diagram

Young lattice

$$UD = DU + I$$



U
D
I
I

"The cellular Ansatz"

combinatorial objects on a 2d lattice

representation by operators

Physics

data structures "histories"

bijections

orthogonal polynomials

"normal ordering"

UD = DU + Id

Weyl-Heisenberg

DE = qED + E + D

PASEP

towers placements

permutations ⟷ RSK ⟷ pairs of Tableaux Young

tableaux alternatifs ⟷ permutations

Laguerre histories

Q-tableaux

ex: ASM, (alternating sign matrices)

FPL(fully packed loops)

tilings, 8-vertex

quadratic algebra Q

?

commutations

rewriting rules

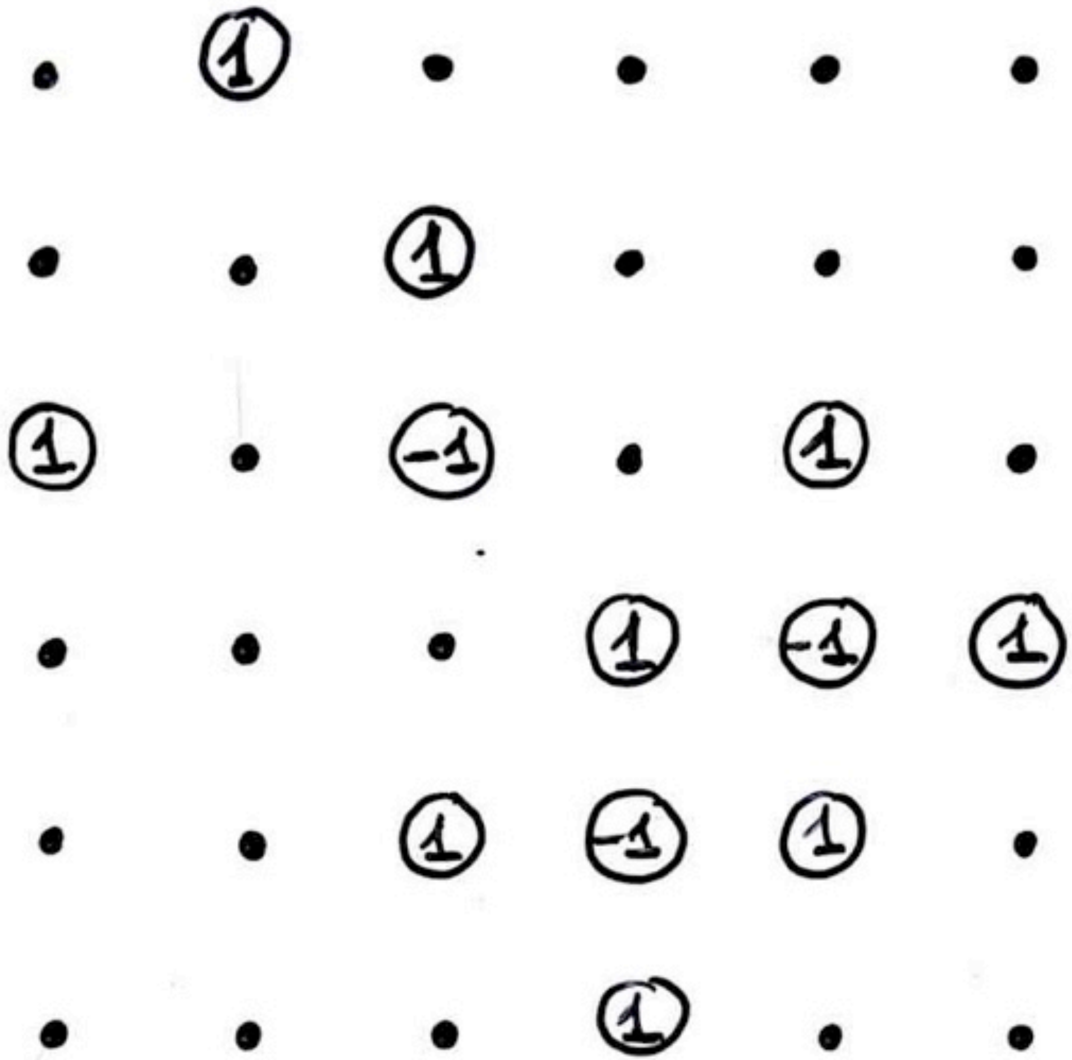
planarisation

planar automata

Koszul algebras

duality

ASM



Alternating sign matrices

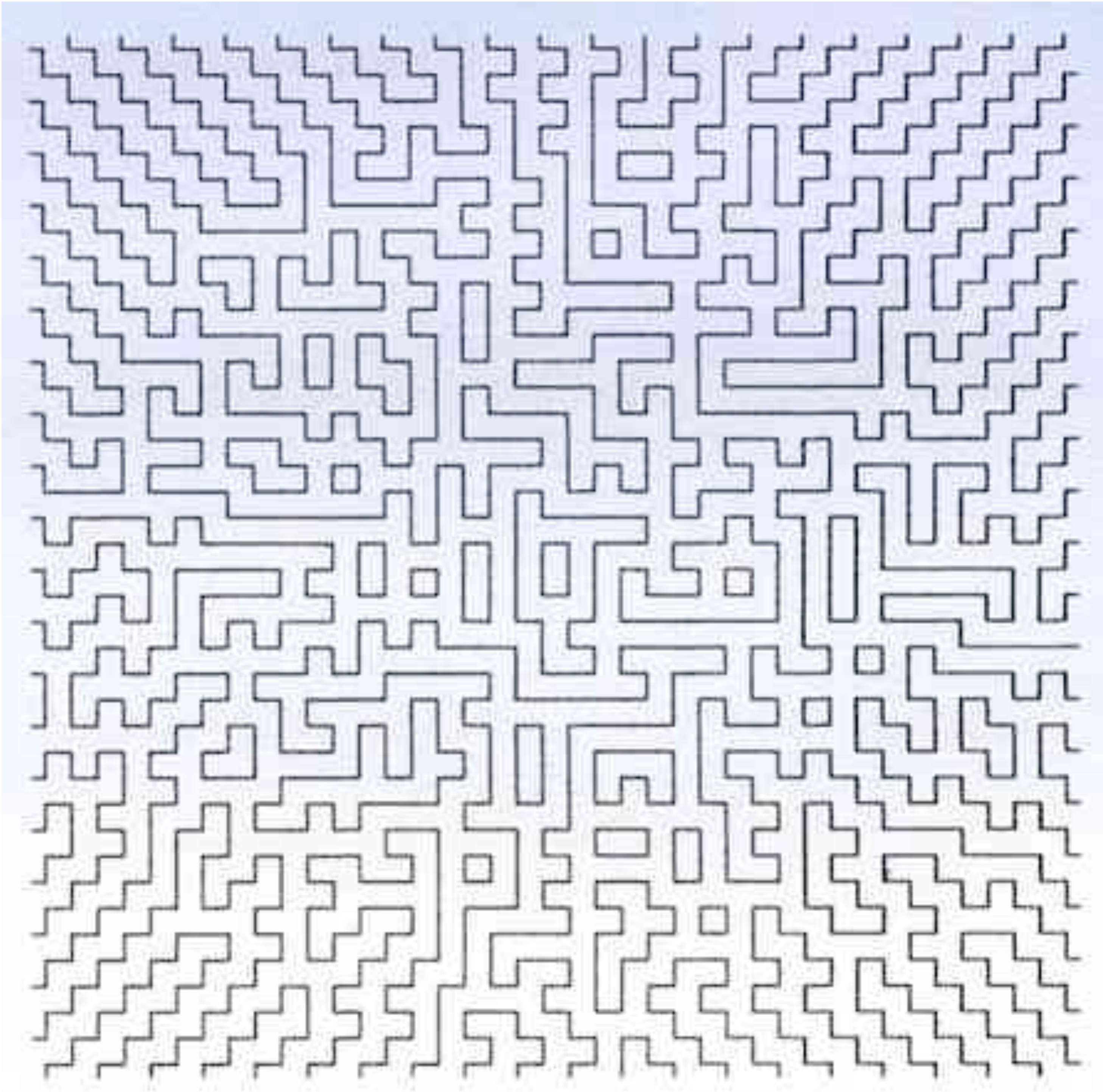
$A, A', B, B',$

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

random
FPL



Razumov - Stroganov (ex)- conjecture 2000-2001

proof by :

L. Cantini and A.Sportiello (March 2010)

arXiv: 1003.3376 [math.CO]

completely combinatorial proof

Around the Razumov-Stroganov conjecture

Philippe Di Francesco, Paul Zinn-Justin (2005 - 2009)

De Gier, Pyatov (2007)

Knizhnik - Zamolodchikov
equation

qKZ

TSSCPP

ASM

The 8-vertex algebra
(or XYZ - algebra)
(or Z - algebra)

The quadratic algebra $\mathbb{Z}$

4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} B\,A = q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, A\,B \\ B_\bullet A = q_{\bullet\circ}\, A\,B_\bullet + t_{\bullet\circ}\, A_\bullet B \\ B\,A_\bullet = q_{\circ\bullet}\, A_\bullet B + t_{\circ\bullet}\, A\,B_\bullet \end{cases}$$

$$\begin{cases} t_{\bullet\circ} = t_{\circ\bullet} = 0 \\ q_{\bullet\bullet} = 0 \end{cases} \qquad \text{(ASM)}$$

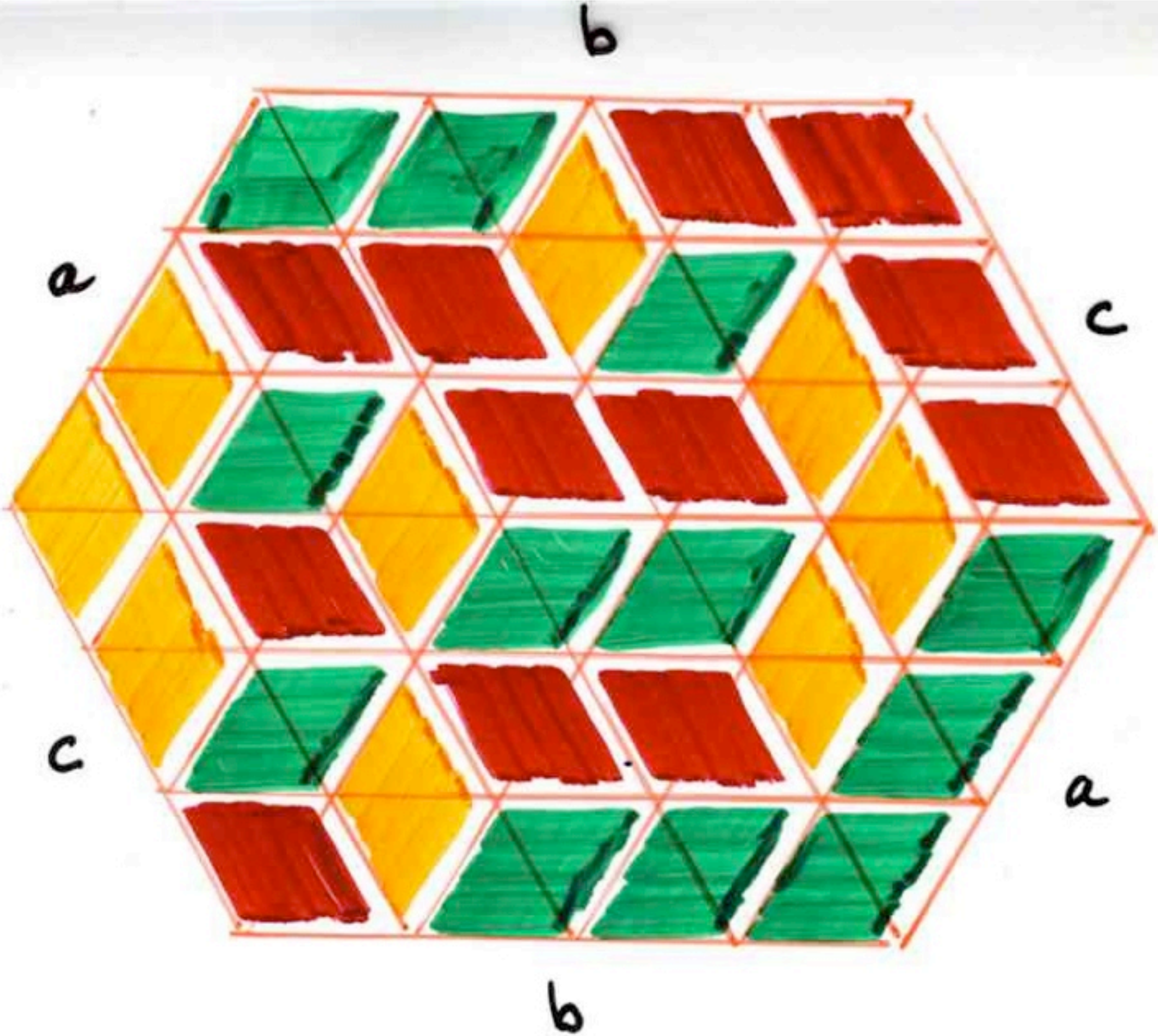
Rhombus tilings

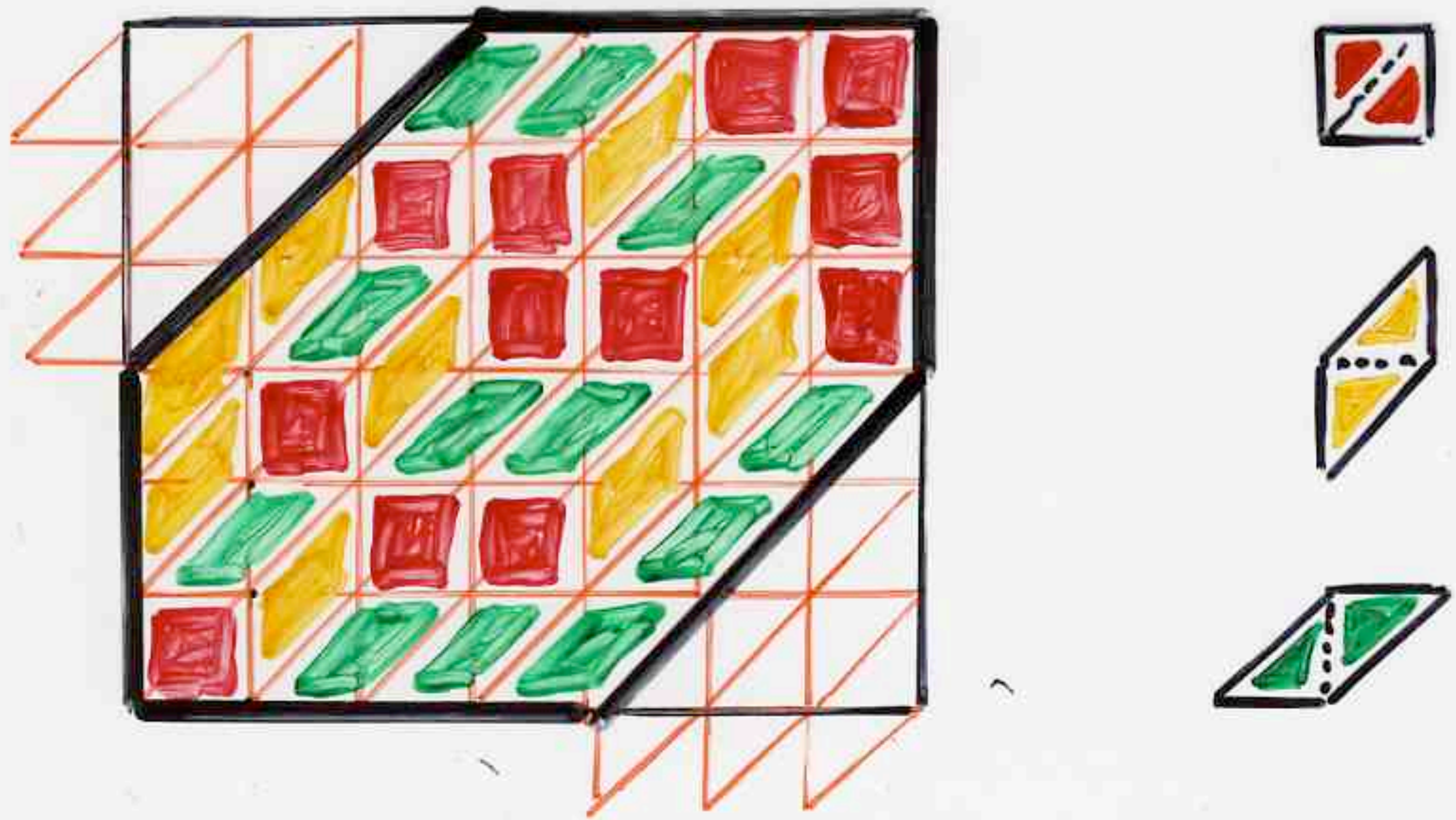
The quadratic algebra Z

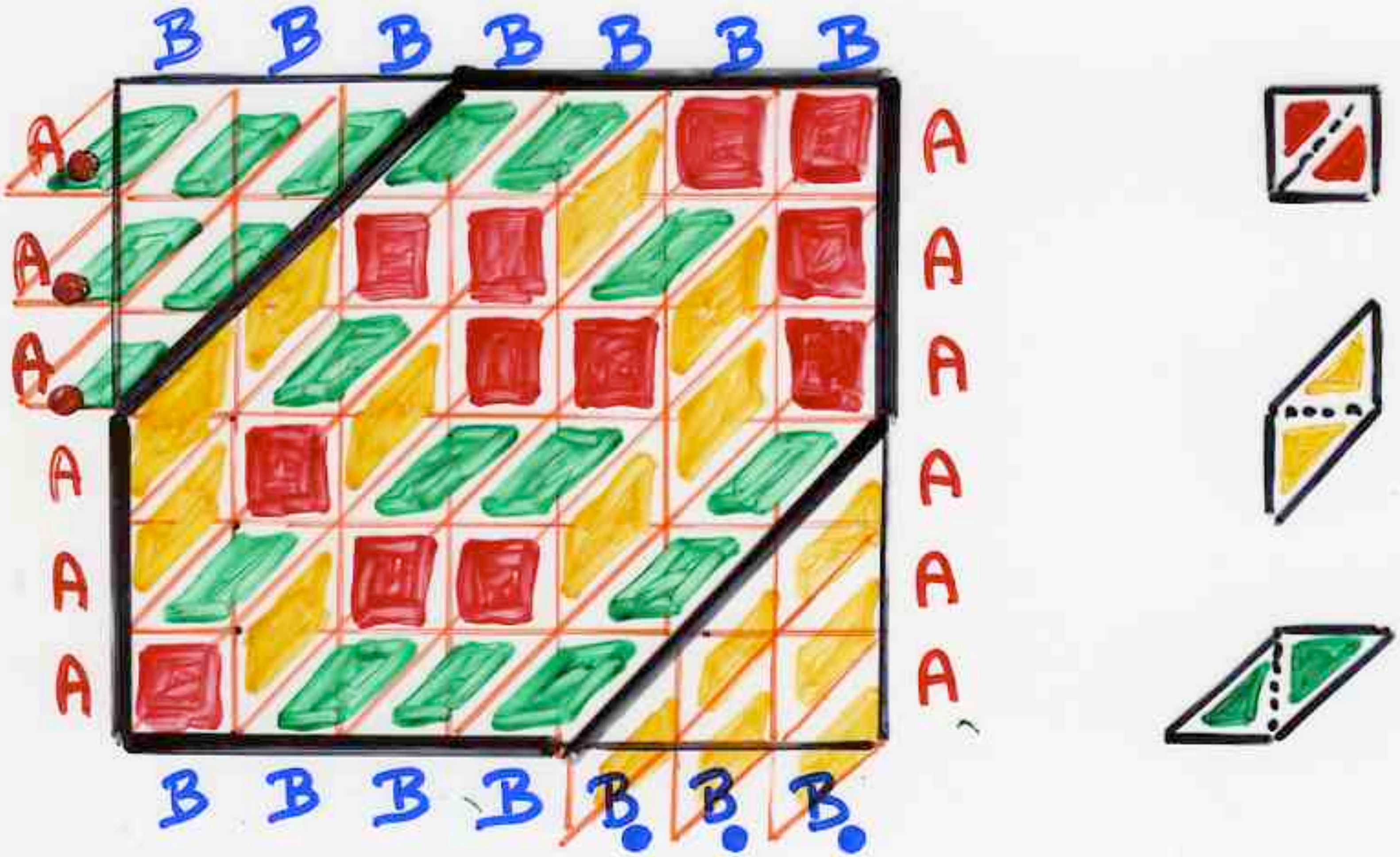
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} B A = q_{\circ\circ} A B + t_{\circ\circ} A_\bullet B_\bullet \\ B_\bullet A_\bullet = 0 \, A_\bullet B_\bullet + t_{\bullet\bullet} A B \\ B_\bullet A = q_{\bullet\circ} A B_\bullet + 0 \, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} A_\bullet B + 0 \, A B_\bullet \end{cases}$$


b
a
c
c
a
b




B B B B B B B
A A A A A A
A A A A A A
B B B B B B B

Aztec tilings

$$t_{\bullet\circ} = t_{\circ\bullet} = 0 \qquad \text{(ASM)}$$

$$t_{\bullet\bullet} = 2 \qquad \text{(nb of } -1 \text{ in ASM)}$$

The quadratic algebra $\mathbb{Z}$

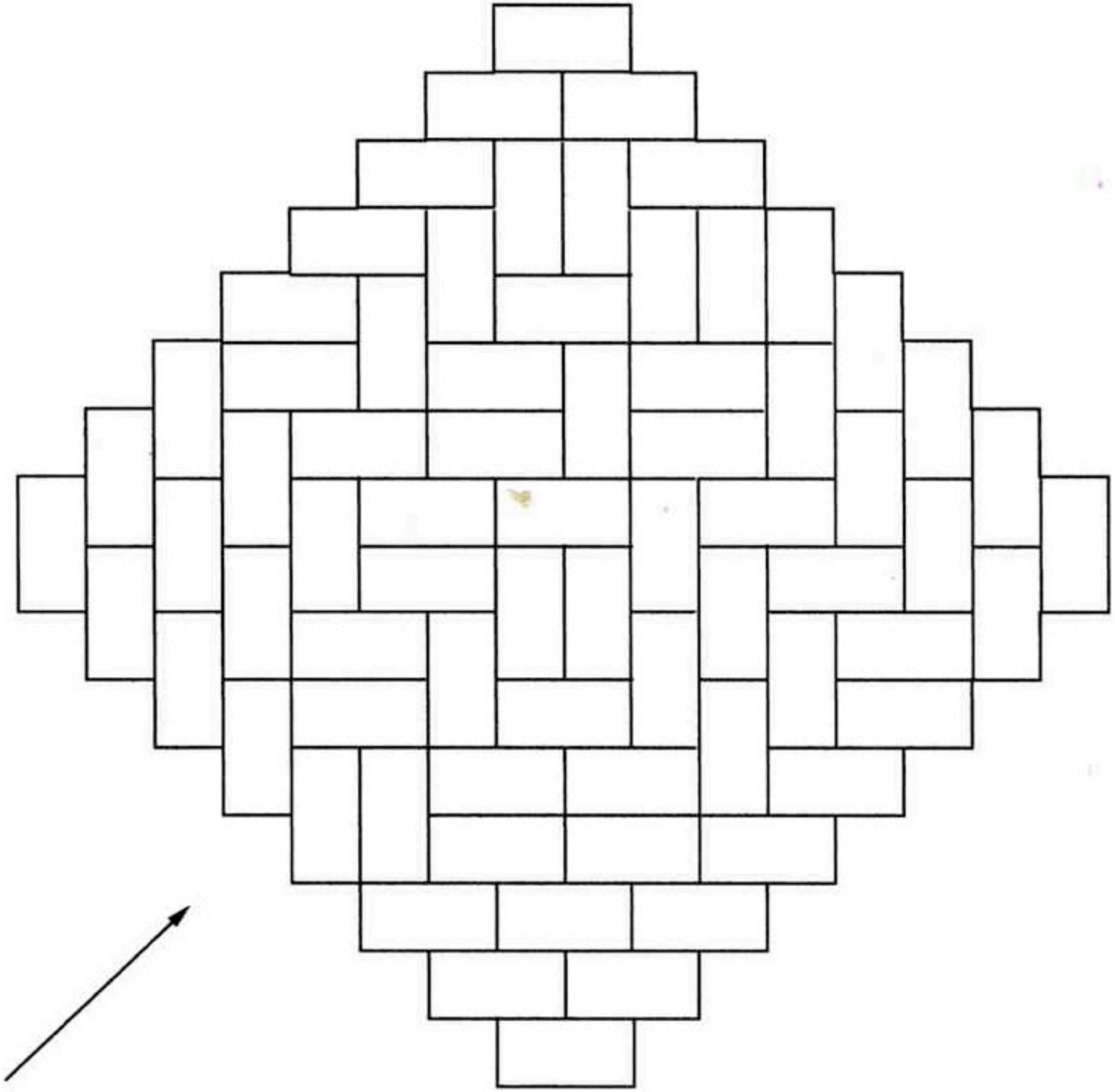
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} B\,A = q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + 2\, A\,B \\ B_\bullet A = q_{\bullet\circ}\, A\,B_\bullet + 0\, A_\bullet B \\ B\,A_\bullet = q_{\circ\bullet}\, A_\bullet B + 0\, A\,B_\bullet \end{cases}$$

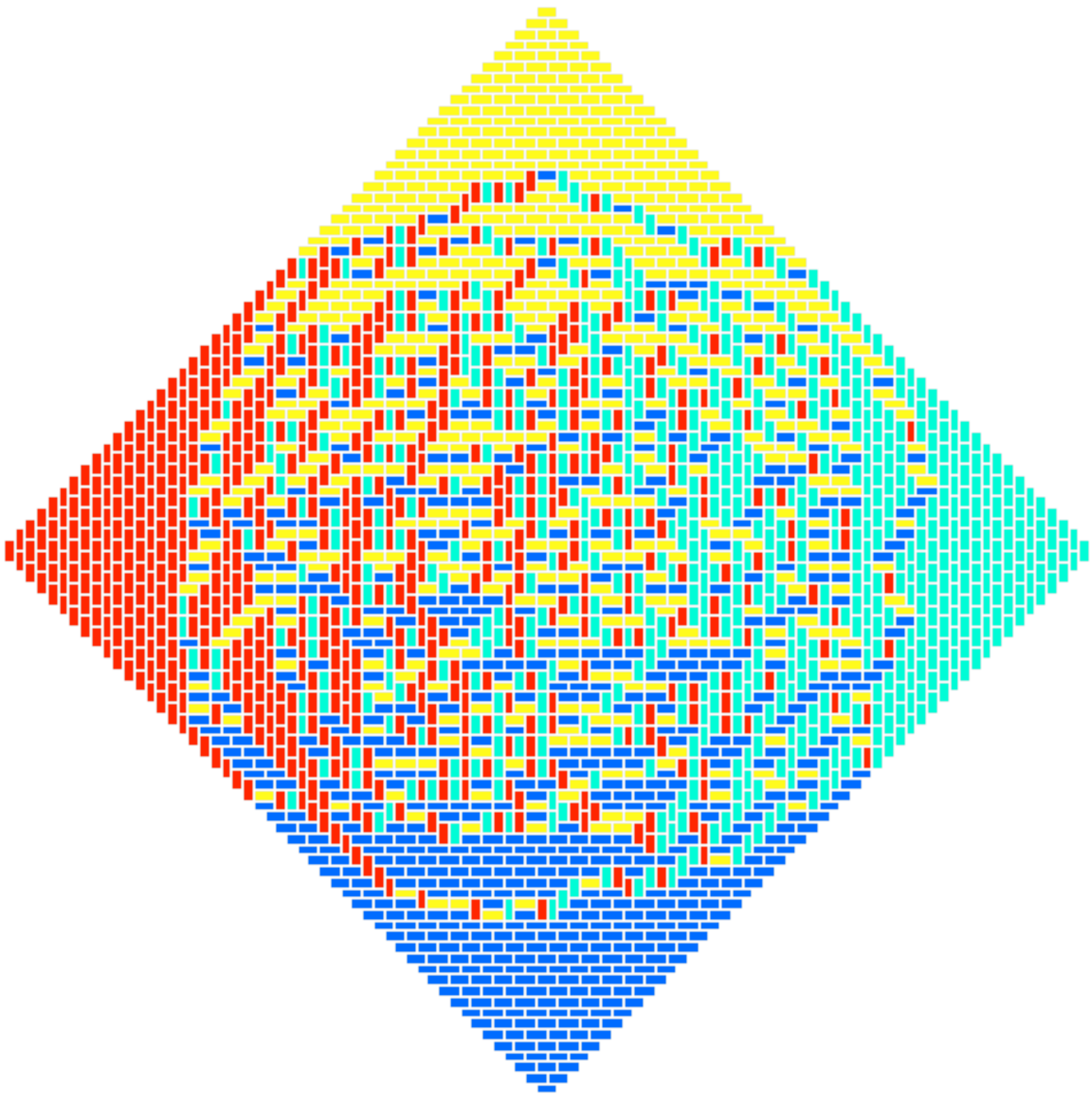
$2^{n(n-1)/2}$

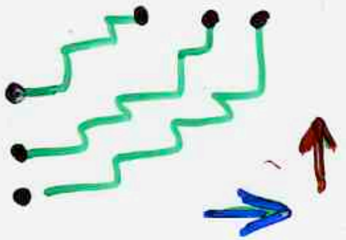
$A_n(2)$



Elkies,
Kuperberg,
Larsen,
Propp
(1992)

random
Aztec
tilings





$$\begin{cases} t_{\circ\circ} = 0 \\ q_{\bullet\bullet} = t_{\bullet\bullet} = 0 \end{cases}$$

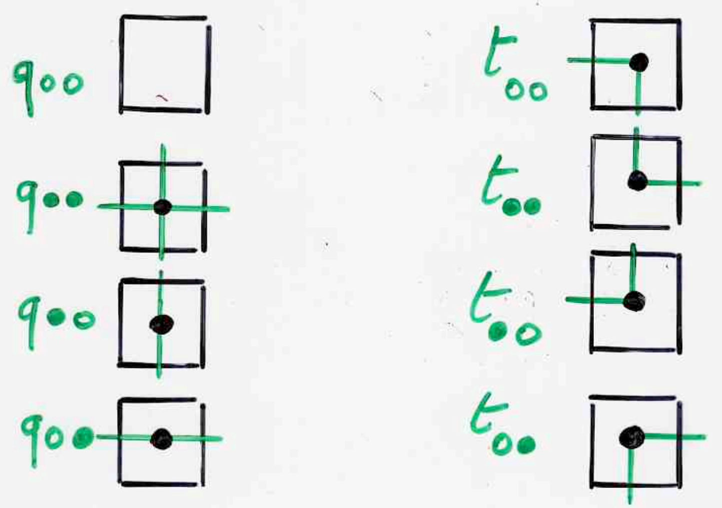
$A \leftrightarrow A_{\bullet}$
exchanging

$$\begin{cases} t_{\circ\bullet} = 0 \\ q_{\bullet\circ} = t_{\bullet\circ} = 0 \end{cases}$$

The quadratic algebra Z

4 generators $B_{\bullet}\ A_{\bullet}\ B\ A$
8 parameters $q_{\cdots},\ t_{\cdots}$

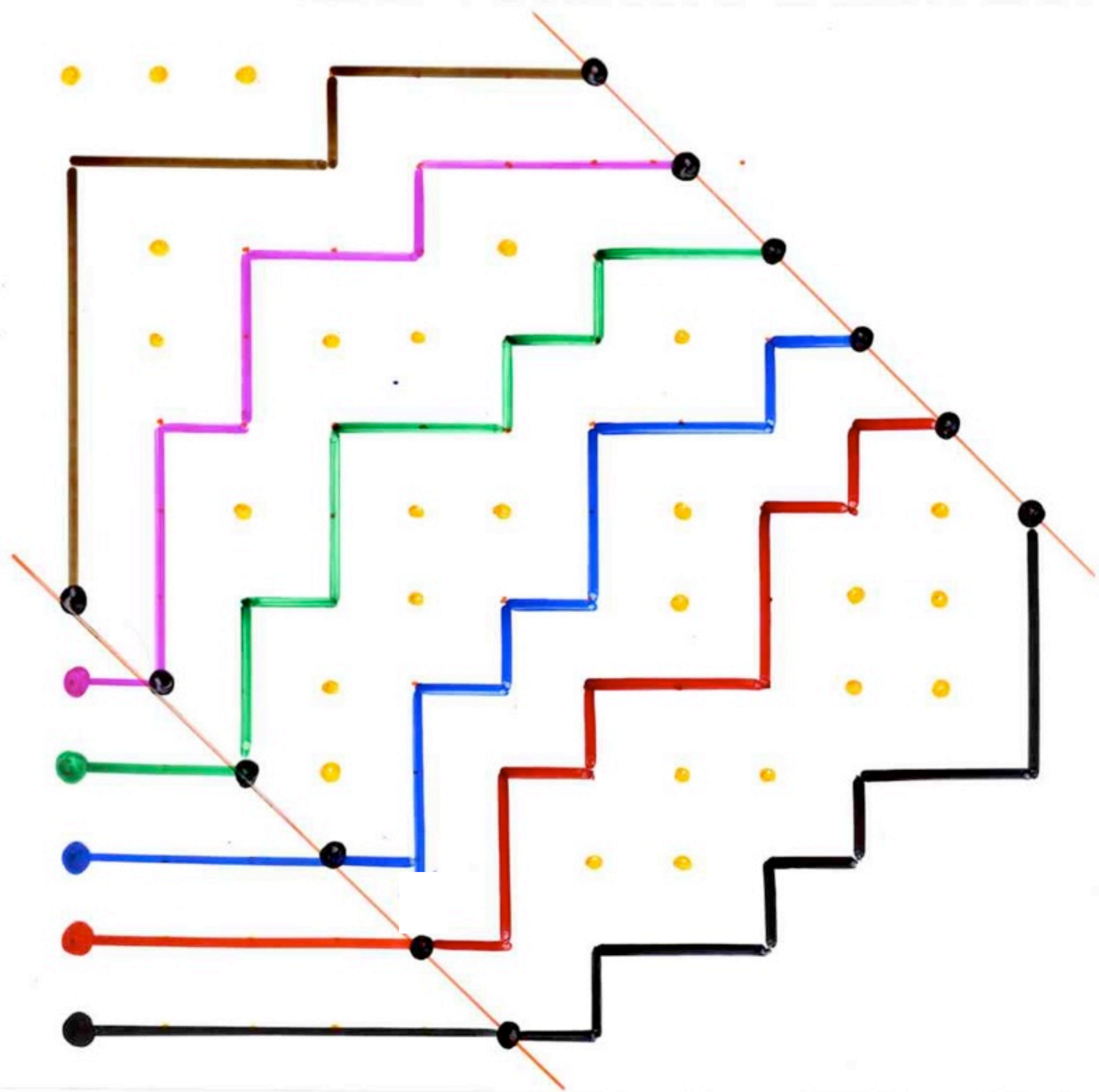
$$\begin{cases} BA = q_{\circ\circ}\, AB + \bigcirc\, A_{\bullet}B_{\bullet} \\ B_{\bullet}A_{\bullet} = \bigcirc\, A_{\bullet}B_{\bullet} + \bigcirc\, AB \\ B_{\bullet}A = q_{\bullet\circ}\, AB_{\bullet} + t_{\bullet\circ}\, A_{\bullet}B \\ BA_{\bullet} = q_{\circ\bullet}\, A_{\bullet}B + t_{\circ\bullet}\, AB_{\bullet} \end{cases}$$

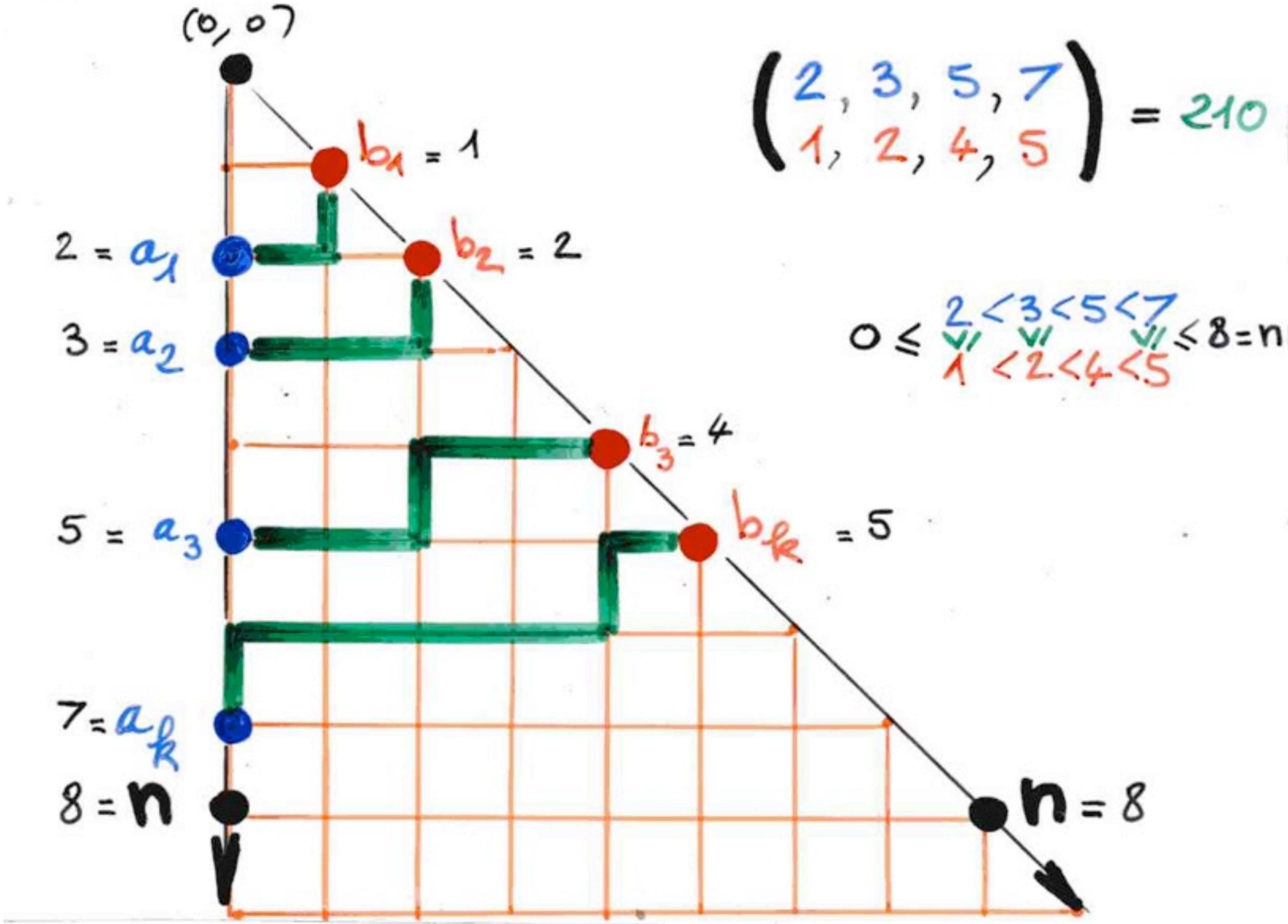


The quadratic algebra Z

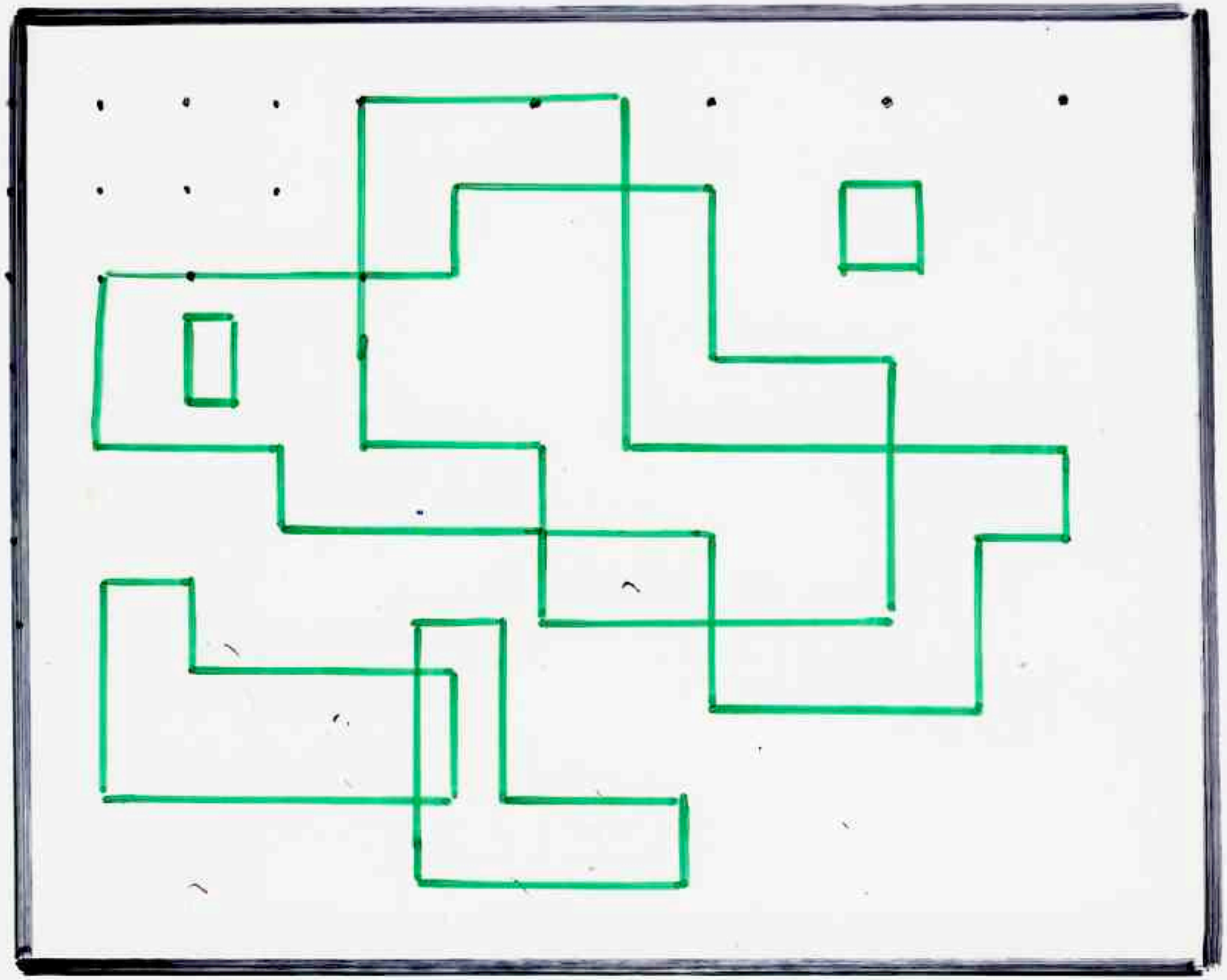
4 generators $B_{\bullet}\ A_{\bullet}\ B\ A$
8 parameters $q_{\cdots},\ t_{\cdots}$

$$\begin{cases} BA = q_{\circ\circ}\, AB + t_{\circ\circ}\, A_{\bullet}B_{\bullet} \\ B_{\bullet}A_{\bullet} = q_{\bullet\bullet}\, A_{\bullet}B_{\bullet} + t_{\bullet\bullet}\, AB \\ B_{\bullet}A = \bigcirc\, AB_{\bullet} + \bigcirc\, A_{\bullet}B \\ BA_{\bullet} = q_{\circ\bullet}\, A_{\bullet}B + \bigcirc\, AB_{\bullet} \end{cases}$$





example: binomial determinant



"closed" graph

Ising model

$$w = B^m A^n$$

$$uv = A^n B^m$$

general PASEP

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

$\alpha, \beta, q \qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\,\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \qquad q$

Askey-Wilson polynomials

Askey-Wilson integral

L'intégrale de Askey-Wilson

$$(a)_\infty = \prod_i (1 - aq^i)$$

$$w(\cos\theta, a, b, c, d \mid q) = \frac{(e^{2i\theta})_\infty (e^{-2i\theta})_\infty}{(ae^{i\theta})_\infty (ae^{-i\theta})_\infty (be^{i\theta})_\infty (be^{-i\theta})_\infty (ce^{i\theta})_\infty (ce^{-i\theta})_\infty (de^{i\theta})_\infty (de^{-i\theta})_\infty}$$

$$\frac{(q)_\infty}{2\pi} \int_0^\pi w(\cos\theta, a, b, c, d \mid q)\, d\theta = \frac{(abcd)_\infty}{(ab)_\infty (ac)_\infty (ad)_\infty (bc)_\infty (bd)_\infty (cd)_\infty}$$

Askey, Wilson (1985)
Ismail, Stanton, Viennot (1986)
Rahman (1984), Ismail, Stanton (1989)
Gasper, Rahman (1989)

Integral of the product of

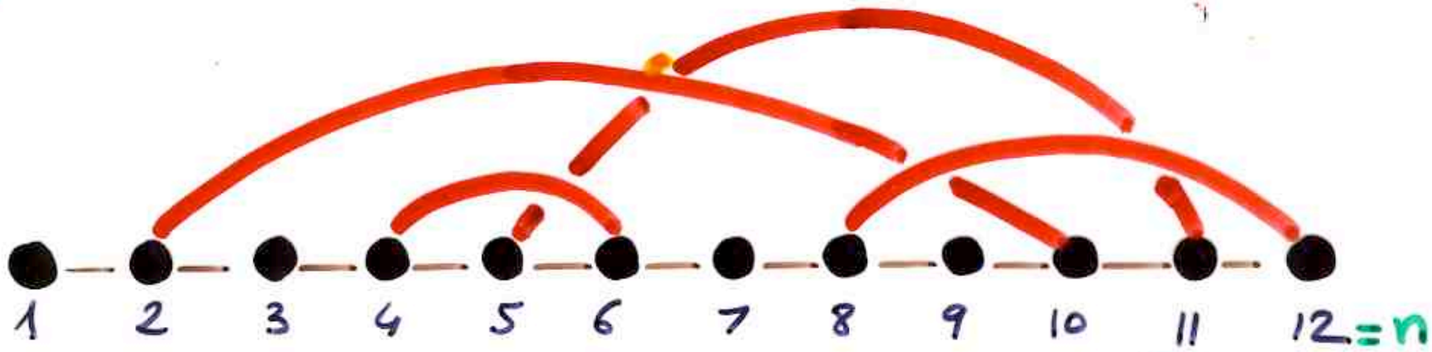
4 q-Hermite polynomials

$$\frac{(q)_\infty}{2\pi}\int_0^\pi H_n(\cos\theta|q)\,H_m(\cos\theta|q)\,(e^{2i\theta})_\infty(e^{-2i\theta})_\infty = (q)_n\,\delta_{nm}$$

q-moments

perfect matchings

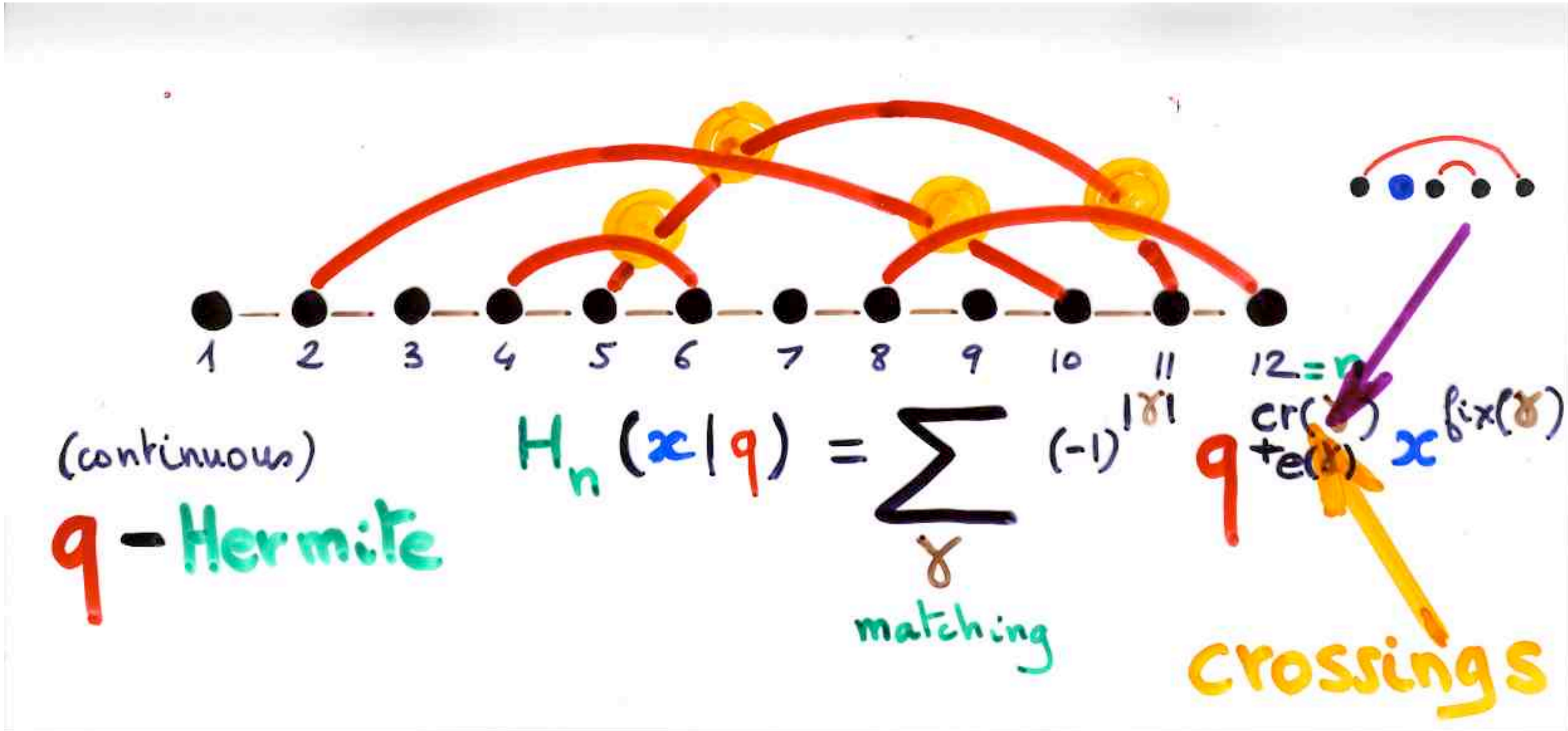
number of crossings



(continuous)

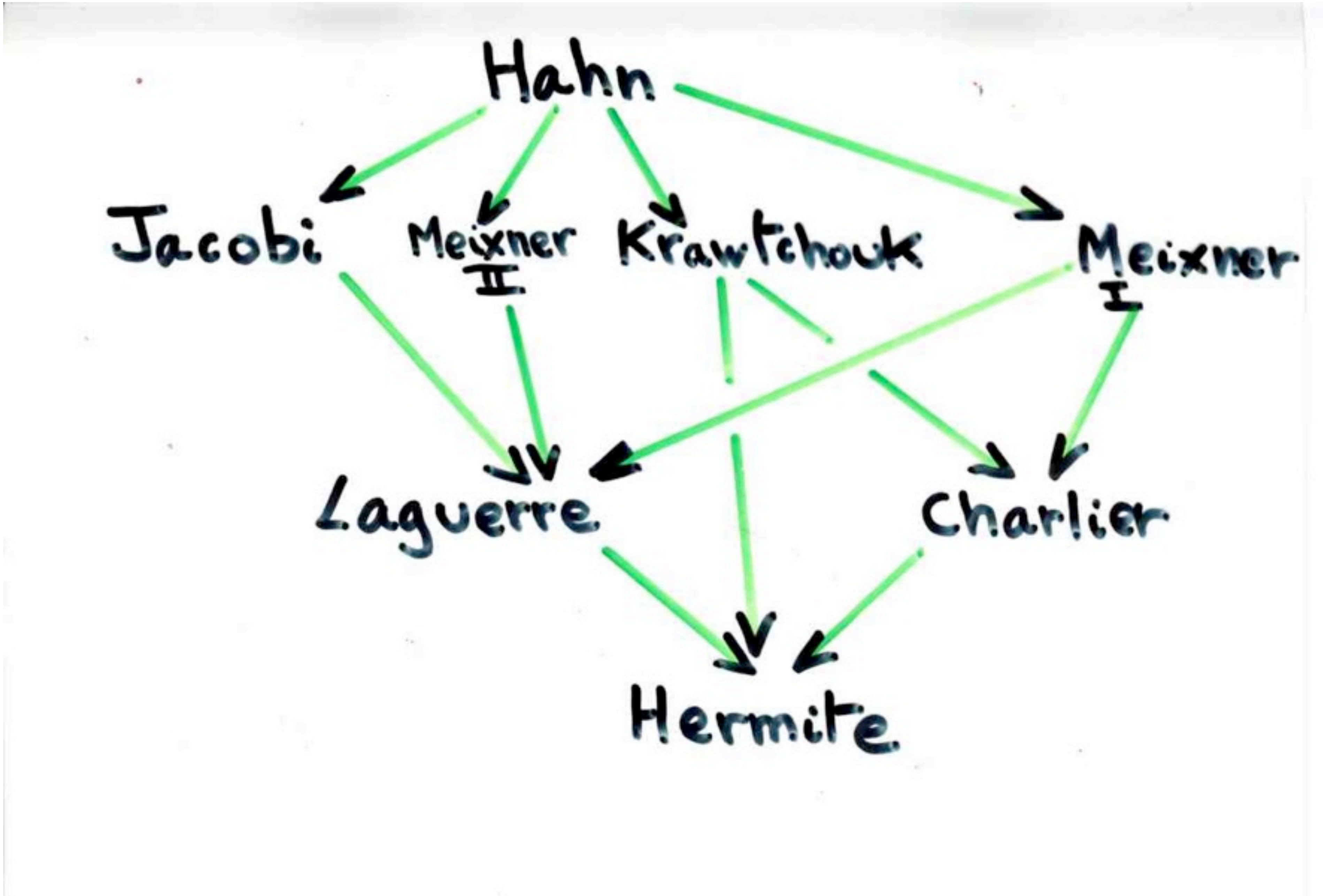
q-Hermite

$$H_n(x|q) = \sum_{\substack{\gamma \\ \text{matching}}} (-1)^{|\gamma|} \, q^{cr(\gamma)+e(\gamma)} \, x^{fix(\gamma)}$$

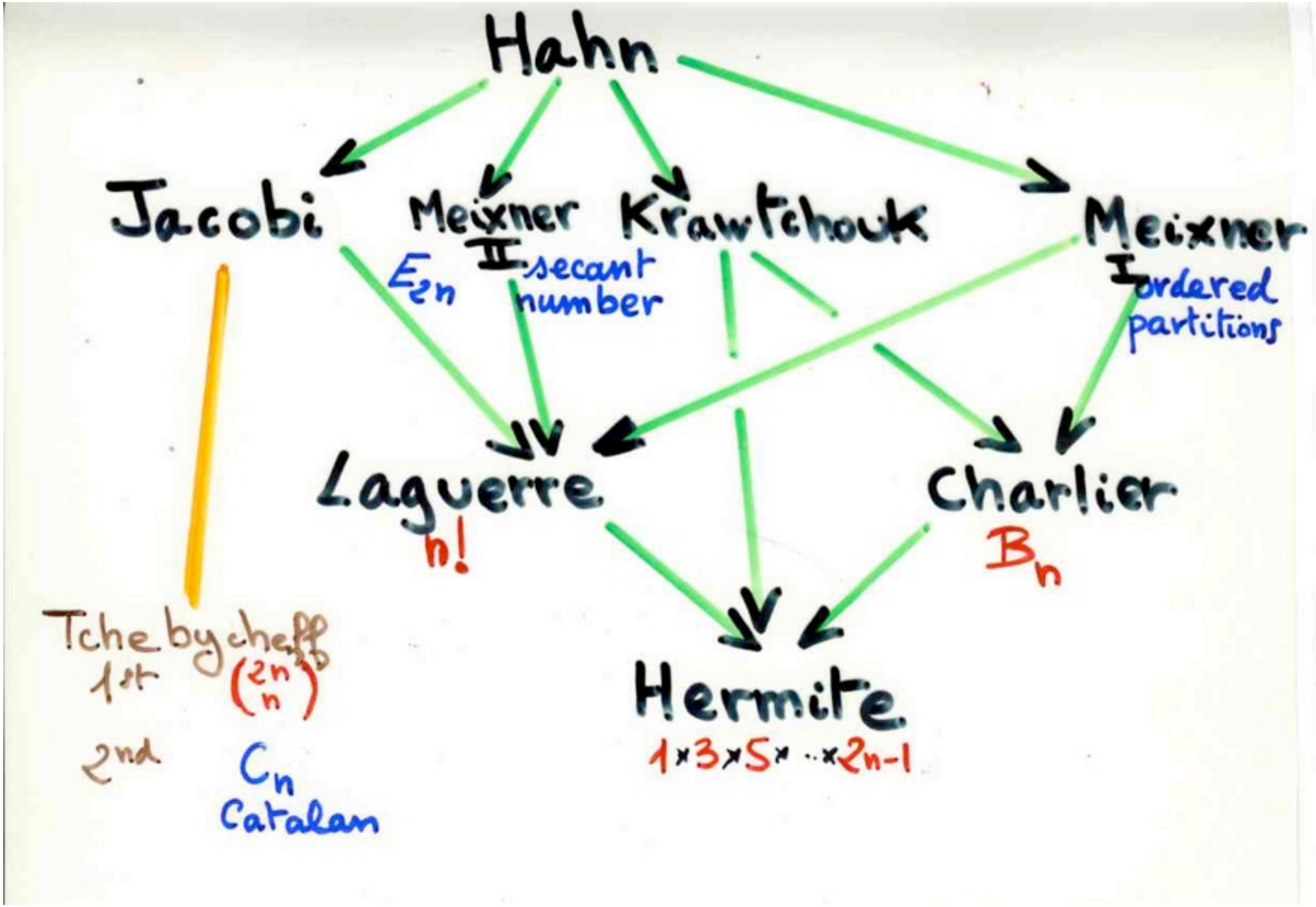

1 2 3 4 5 6 7 8 9 10 11 12 = n
(continuous)
q-Hermite
H_n(x|q) = Σ_γ (-1)^|γ| q^(cr(γ)+e(γ)) x^fix(γ)
matching
crossings

Askey tableau

Askey-Wilson



Askey-Wilson



orthogonal polynomials

(binomial type)

Scheffer type

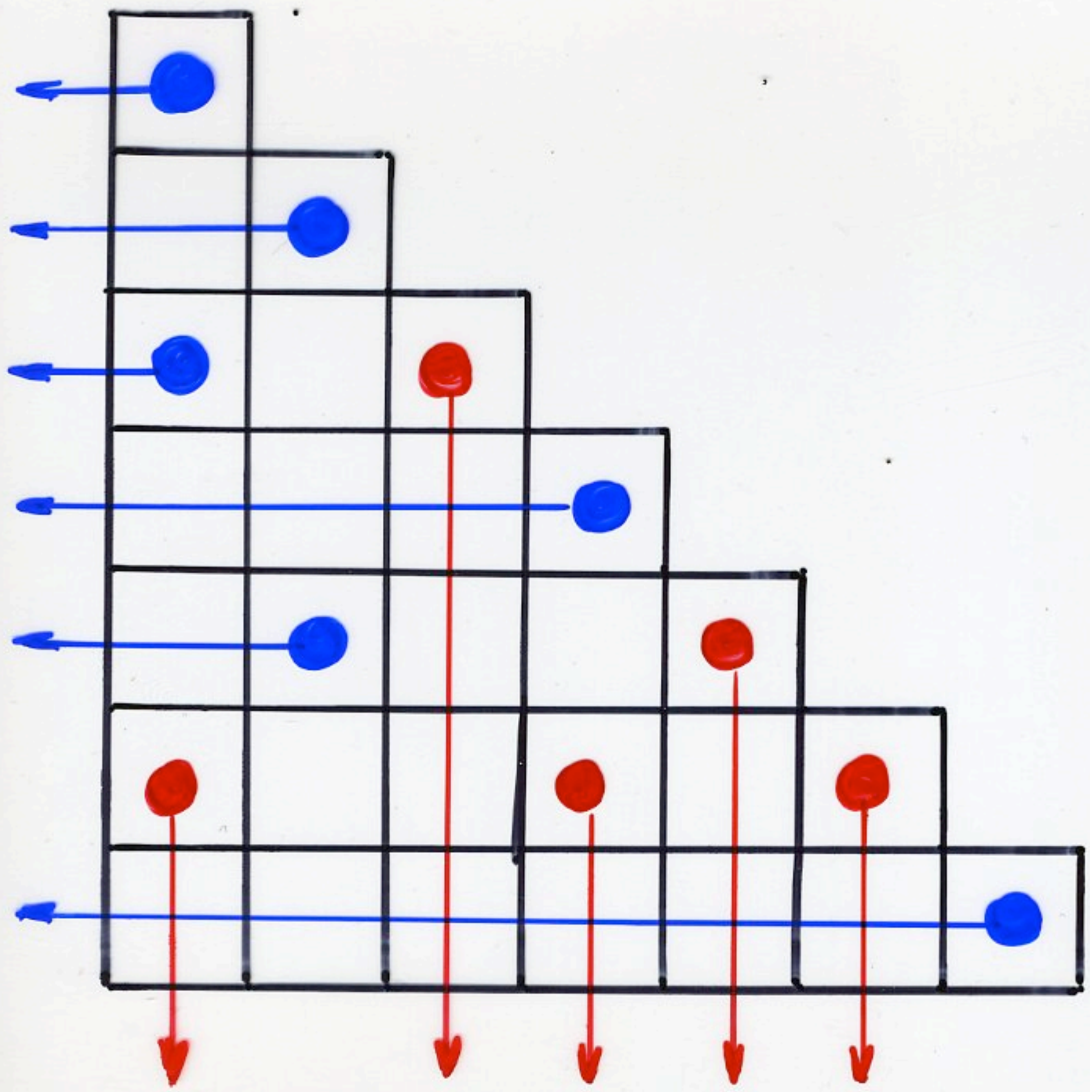
$$\sum P_n(x) \frac{t^n}{n!} = g(t) e^{x f(t)}$$

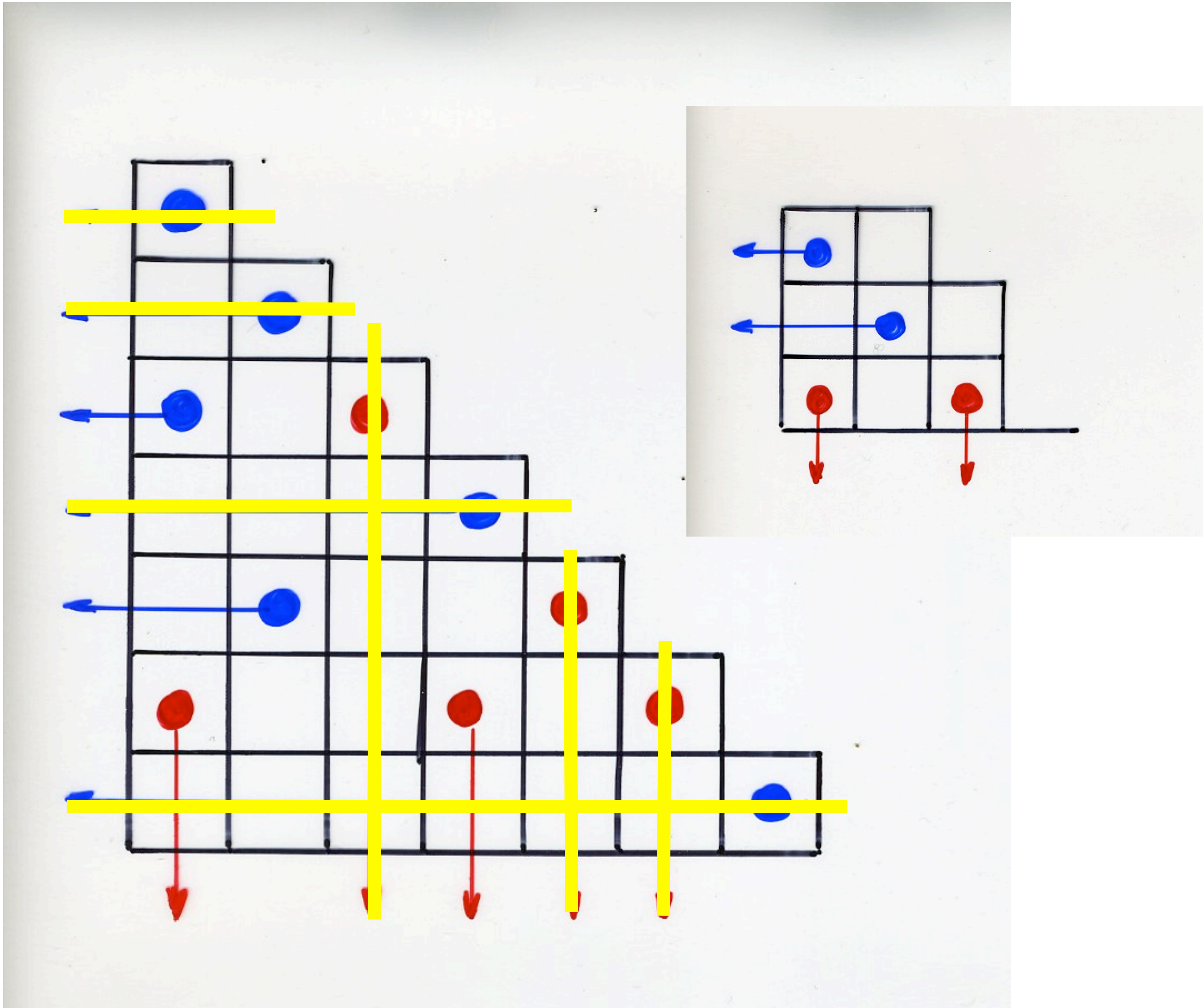
- Hermite H_n
- Laguerre $L_n^{(\alpha)}$
- Charlier $C_n^{(a)}$
- Meixner I $M_n^{I\,(\alpha,)}$
- Meixner II $M_n^{II\,(\delta,\eta)}$

staircase tableaux

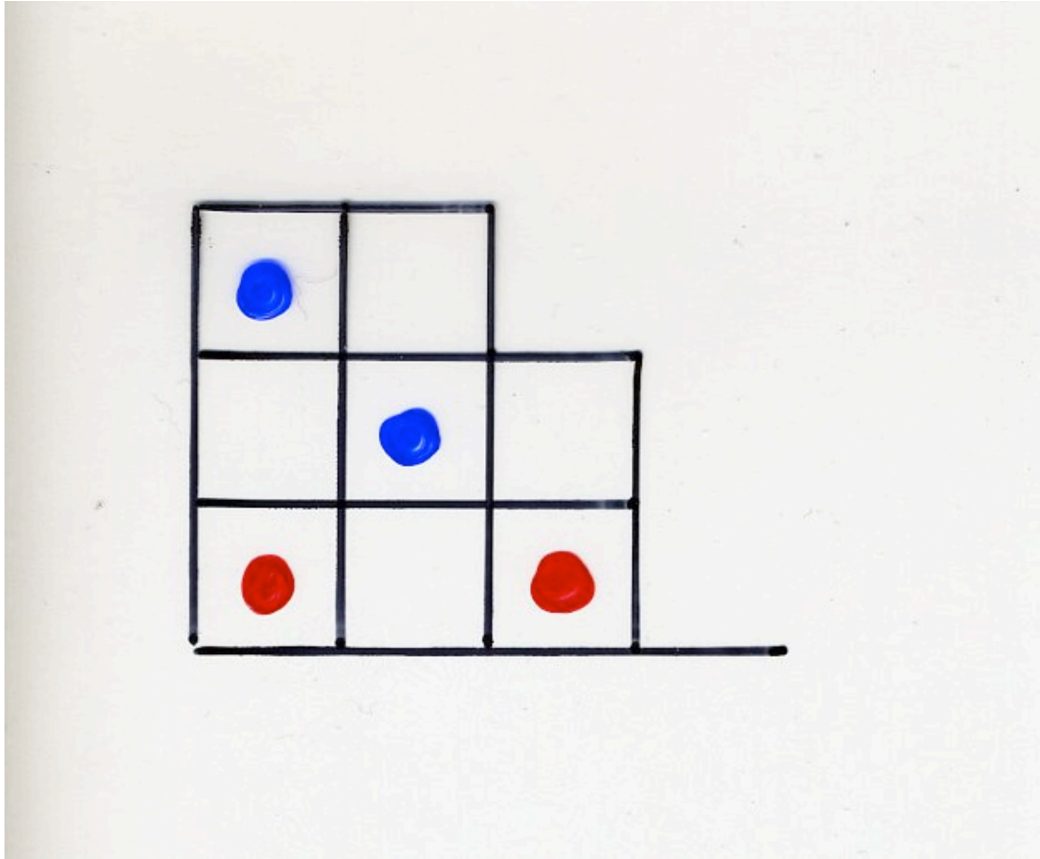
Corteel, Williams, 2009

Corteel, Stanley, Stanton, Williams, 2010

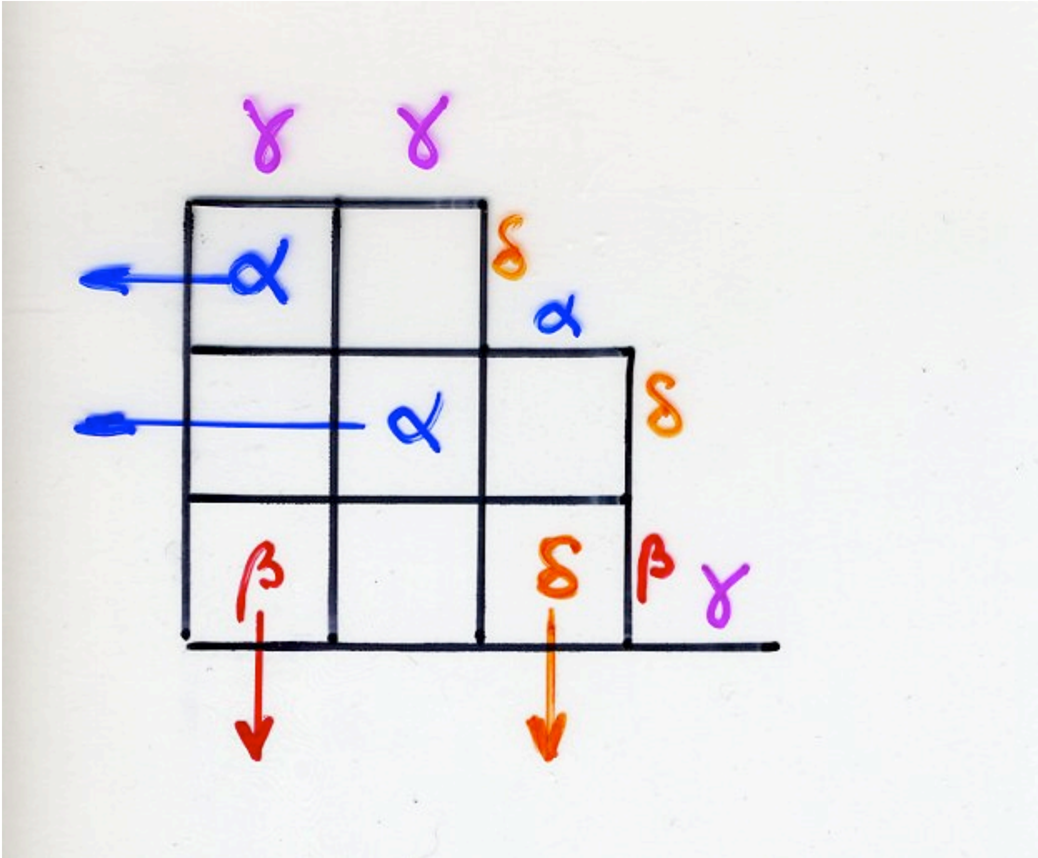




nb of 2-colored alternative tableaux $= 2^n \cdot n!$

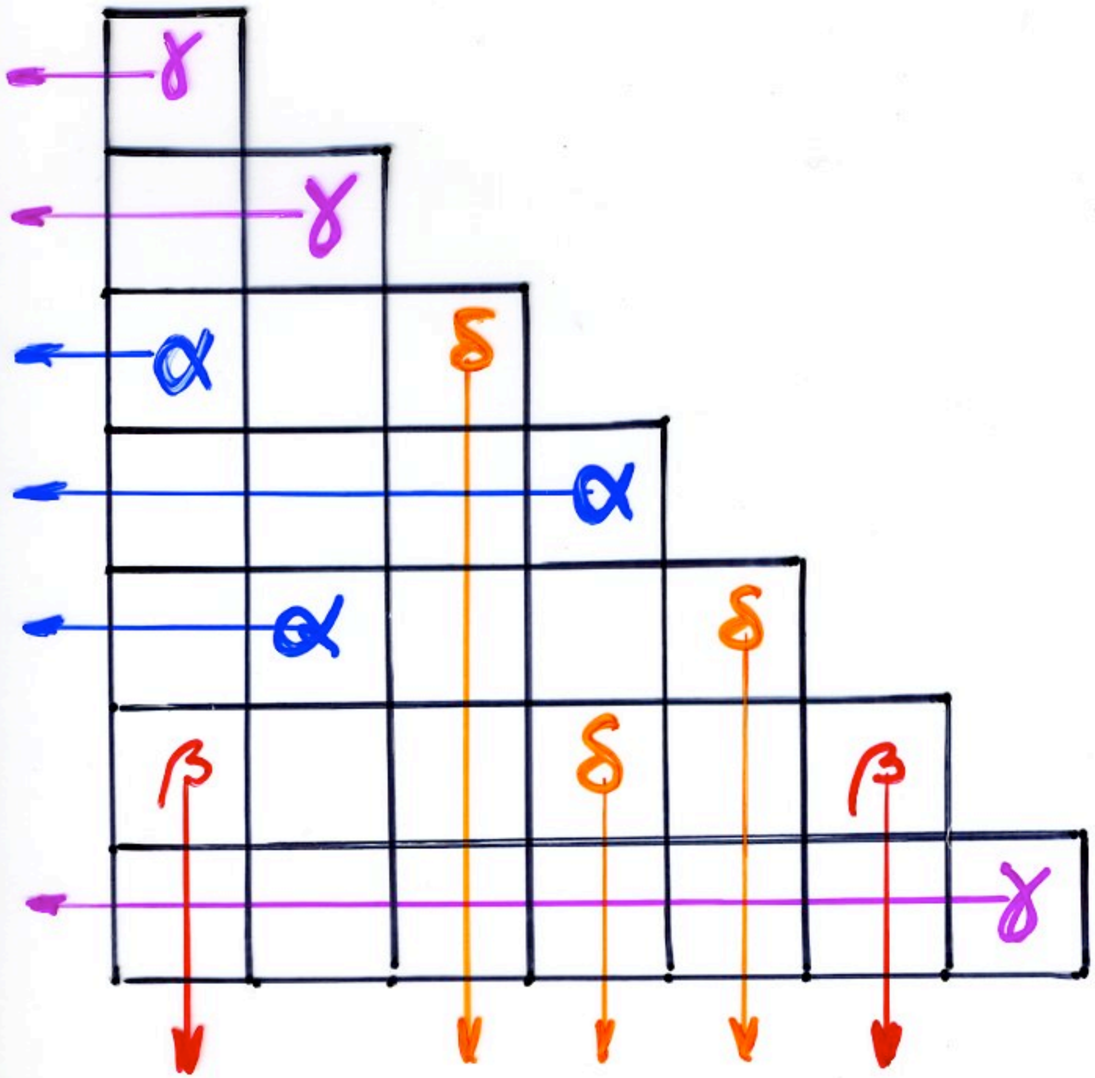


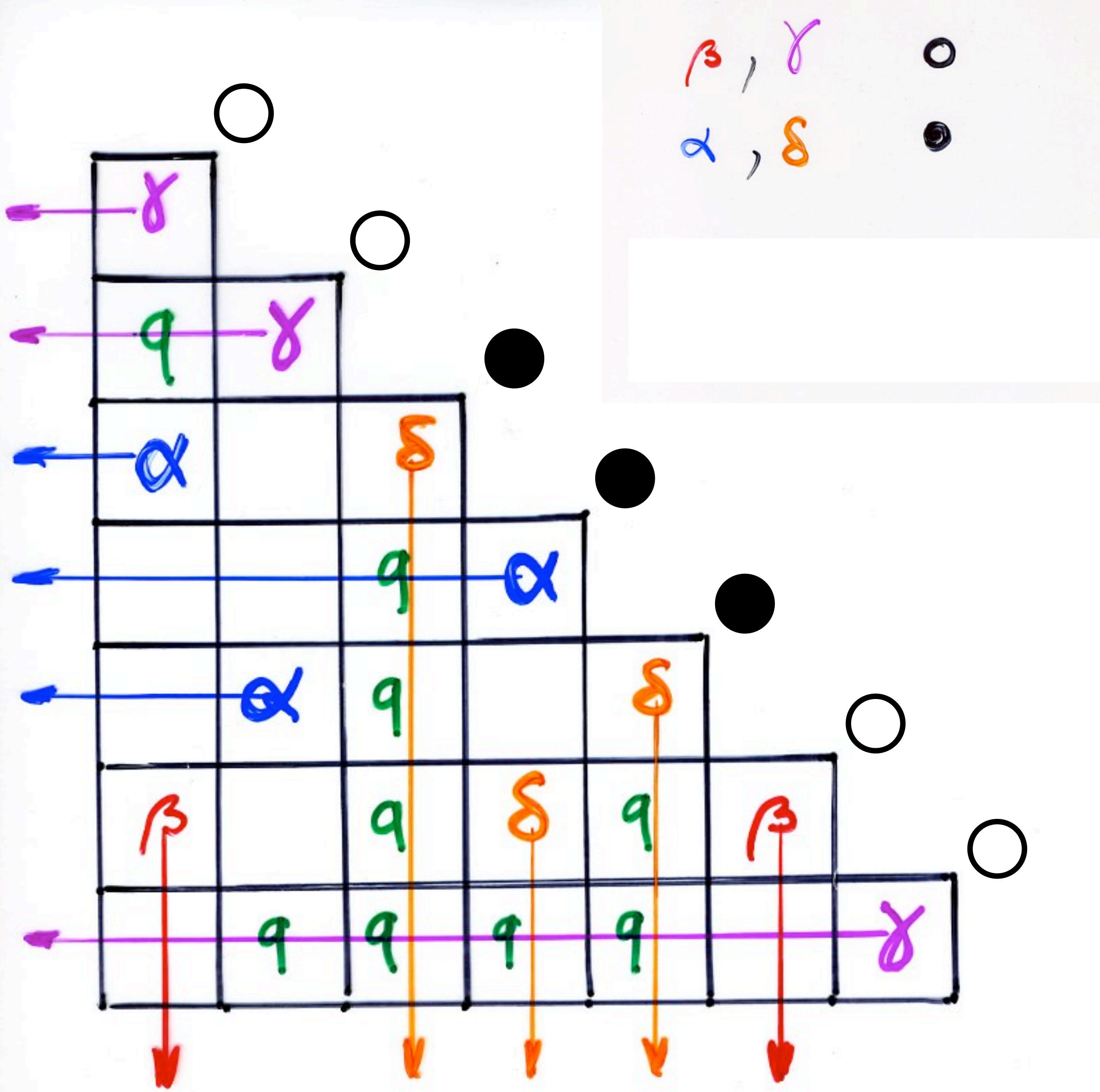
nb of 2-colored alternative tableaux $= 2^n \cdot n!$

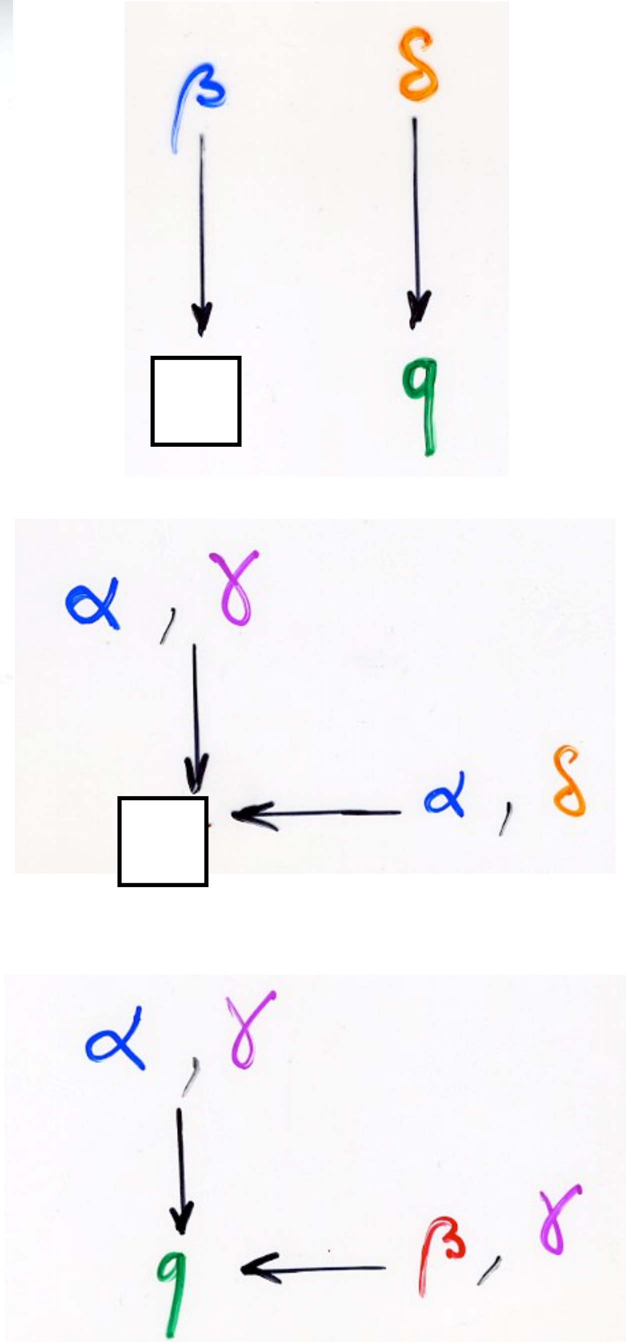


nb of staircase tableaux $= 4^n \cdot n!$

staircase tableaux




β , γ
α , δ
γ
q γ
α δ
q α
α q δ
β q δ q β
q q q q γ


β δ
q
α , γ
α , δ
α , γ
q
β , γ

steady state probability PASEP

$$\frac{1}{Z_n} Z_\tau(\alpha, \beta, \gamma, \delta; q)$$

$$Z_n = \sum_\tau Z_\tau$$

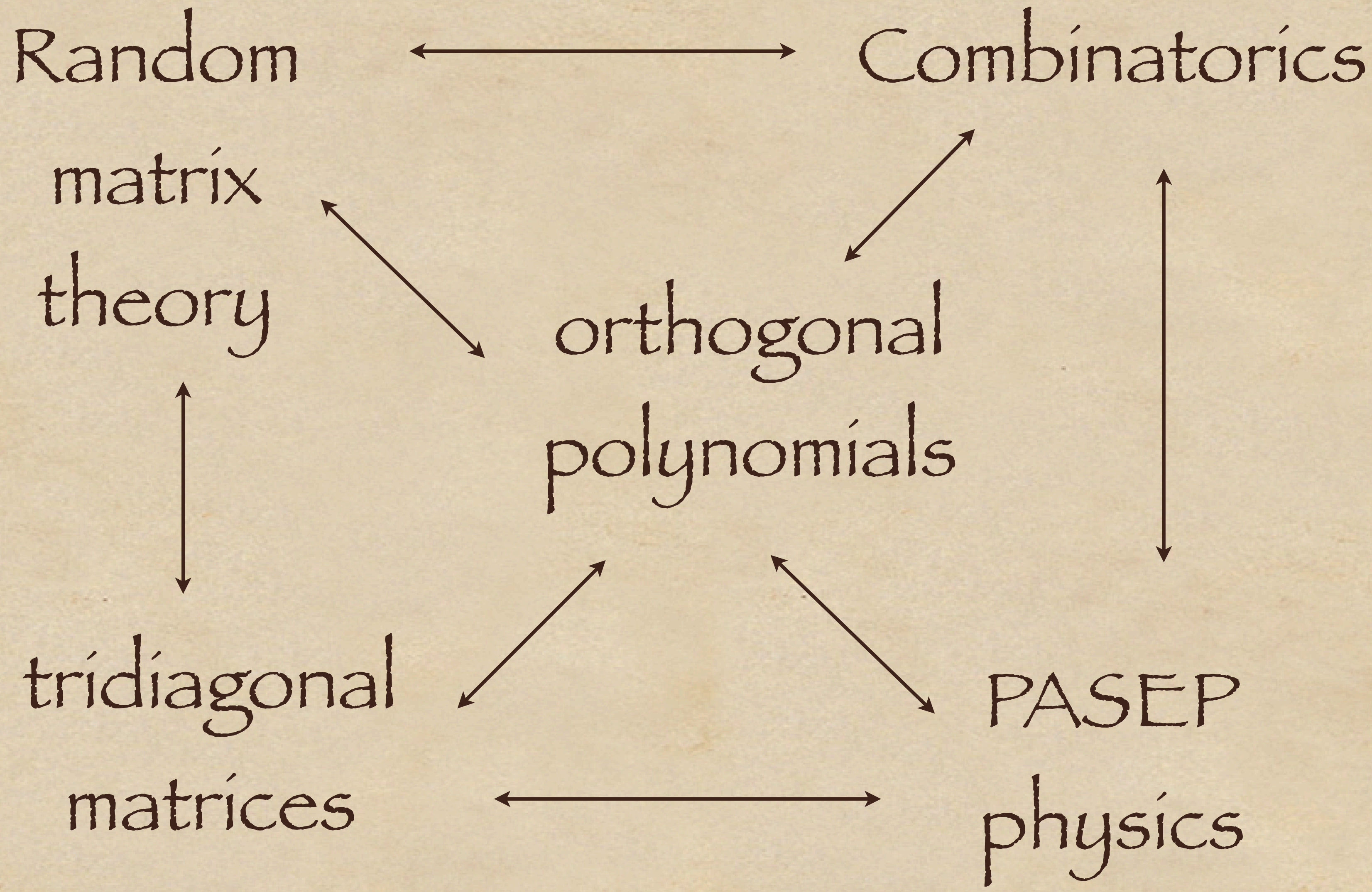
$\tau = (\tau_1, \dots, \tau_n)$

state

relation with moments of Askey-Wilson polynomials

Corteel, Williams, 2009

Corteel, Stanley, Stanton, Williams, 2010


Random matrix theory
Combinatorics
orthogonal polynomials
tridiagonal matrices
PASEP physics

"The cellular Ansatz"

combinatorial objects on a 2d lattice

representation by operators

Physics

data structures "histories"

bijections

"normal ordering"

orthogonal polynomials

$UD = DU + Id$

Weyl-Heisenberg

towers placements

permutations

RSK

$\longleftrightarrow$ pairs of Tableaux Young

$DE = qED + E + D$

PASEP

tableaux alternatifs $\longleftrightarrow$ permutations

Laguerre histories

Q-tableaux

ex: ASM, (alternating sign matrices)

FPL(fully packed loops)

tilings, 8-vertex

quadratic algebra Q

?

commutations

rewriting rules

planarisation

planar automata

Koszul algebras

duality

Thank you !

pour plus de détails
voir les diaporamas du cours donné à Talca:

Cours XGV, Universidad de Talca
(December 2010 - January 2011)
Combinatorics and interactions (with physics) (24h)
«The Cellular Ansatz»

accessible sur les sites:

http://www.labri.fr/perso/viennot/

Recherche, cv, publications, exposés, diaporamas, livres, petite école, photos: voir ma page personnelle ici

Vulgarisation scientifique voir la page de l'association Cont'Science

http://web.me.com/xgviennot/Xavier_Viennot/

http://web.me.com/xgviennot/Xavier_Viennot2/

Ch 0 Introduction

Cours XGV
Universidad de Talca

(December 2010 - January 2011)
24 h

Combinatorics and interactions
(with physics)
«The Cellular Ansatz»

Ch 1 Ordinary generating function, the Catalan garden

Ch1a (1/12/2010, 54 p.)
Ch 1b (7/12/2010, 81 p.)
Ch 1c (7/12/2010, 30 p.) **algebraic complements in relation with physics**

Ch 2 Exponential generating functions, permutations

Ch 2a (22/12/2010, 40 p.)
Ch 2b (4/01/2010, 63 p.)
Ch 2c (4/01/2010, 33 p.) **Permutations: Laguerre histories**

Ch 3 Permutations and Young tableaux, the Robinson-Schensted correspondence (RSK)

Ch 3a (6/01/2011, 117 p.)
Ch 3b (6, 11/01/2011, 121 p.) **RSK and operators**

Ch 4 Alternative tableaux and the PASEP (partially asymmetric exclusion process)

Ch 4a (13/01/2011, 98p.)
Ch 4b (13, 18/01/2011, 102 p.) **alternative tableaux and the PASEP**
Ch 4c (18/01/2011, 81 p.) **complements**

Ch 5 Combinatorial theory of orthogonal polynomials

(20/01/2011, 110 p.)

Ch 6 "jeu de taquin" for binary trees, Catalan tableaux and the TASEP

Ch 6a (24/01/2011, 98 p.)
Ch 6b (24/01/2011, 111 p.) **alternative tableaux and increasing/alternative binary trees**
Ch 6c (24/01/2011, 21 p.) **Catalan tableaux and the Loday-Ronco algebra**

Ch 7 The cellular Ansatz

Ch 7a (25/01/2011, 117 p.)
Ch 7b (25/01/2011, 49 p.) **complements**

PASEP Matrix Ansatz PASEP algebra AT

"normal ordering" stationary probabilities

permutation tableaux TASEP Bijection CAT - BT

The “exchange-fusion” algorithm

representation E , D The "cellular Ansatz"

FV bijection formal OP Laguerre polynomials

3 parameters data structures

cellular Ansatz Q ASM R-S 8-vertex algebra

general PASEP A-W integral Askey tableau

staircase tableaux