Tamari,
m-Tamari,
(a,b)-Tamari,
and beyond ...

joint work with

Louis-François Préville-Ratelle

U. Talca, Chile

Oberwolfach

6 March 2014

Xavier Viennot

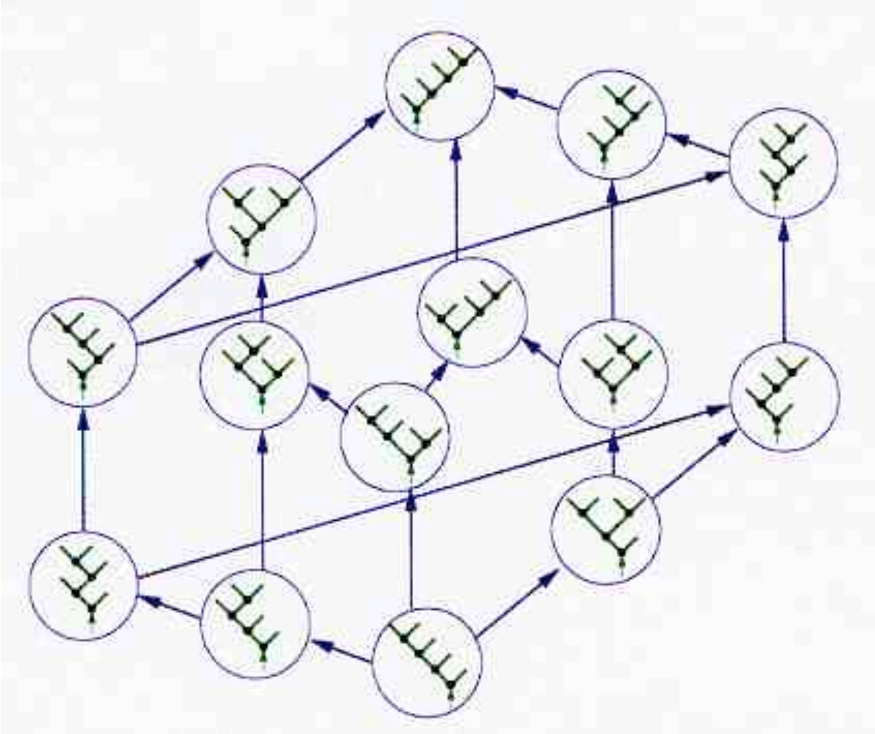
LaBRI, CNRS, Bordeaux

(visiting U. Talca)

Introduction

Tamari

m-Tamari



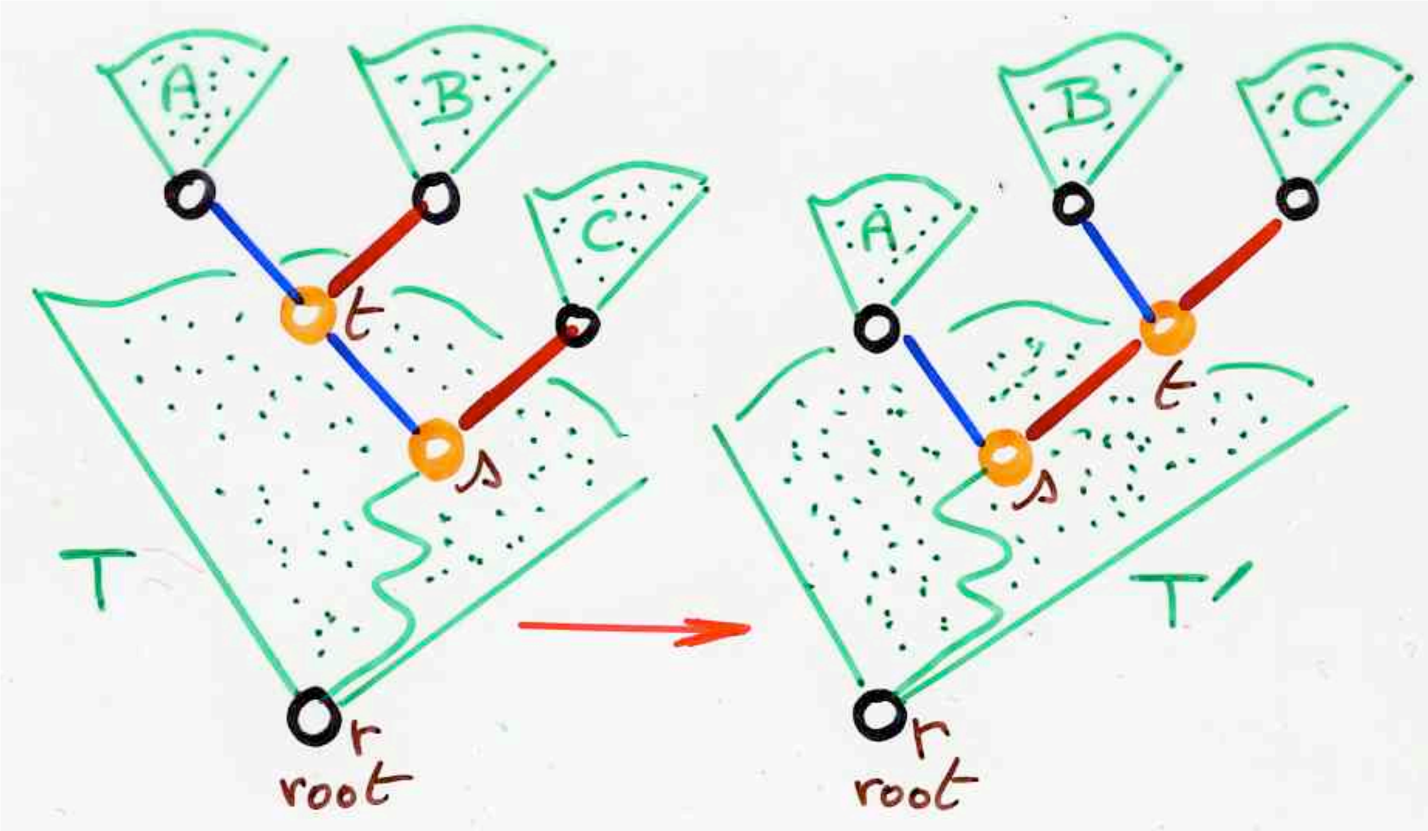
Tamari lattice

$C_4 = 14$

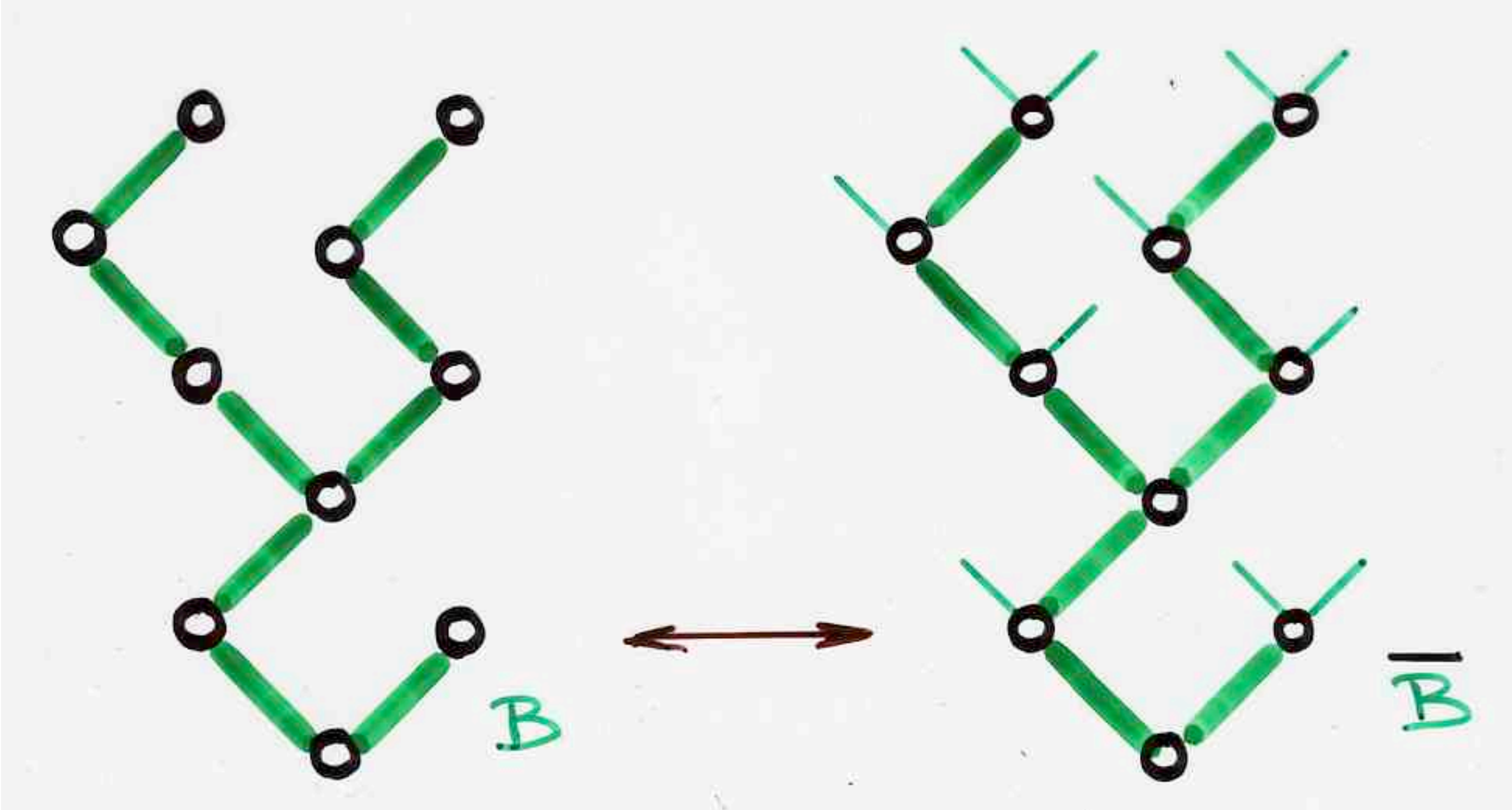
Catalan

Dov Tamari (1951) thèse Sorbone

"Monoides préordonnés et chaînes de Malcev"

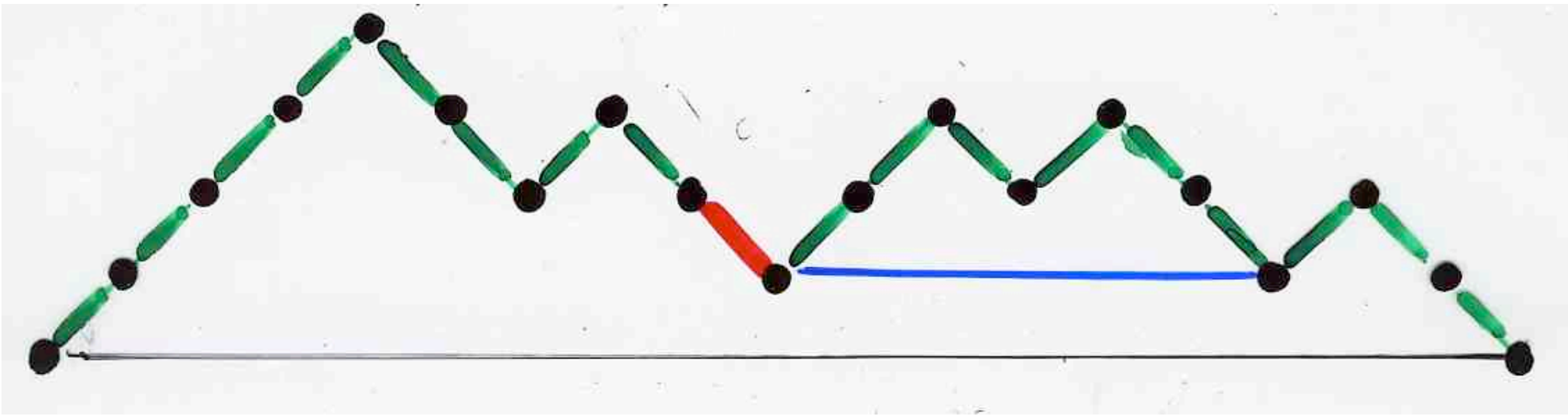


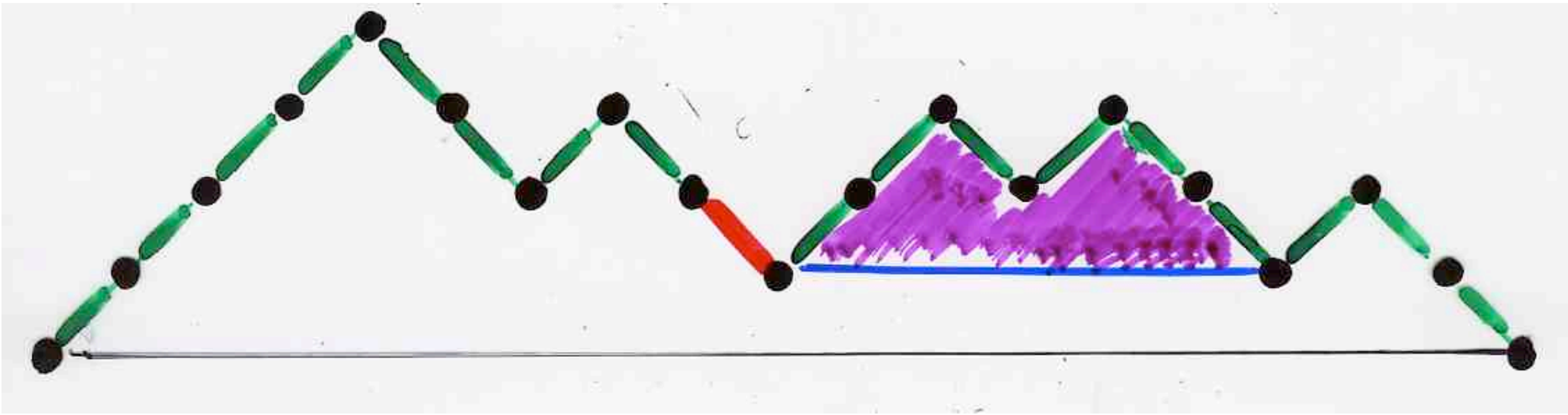
Rotation in a binary tree :
the covering relation in the
Tamari lattice



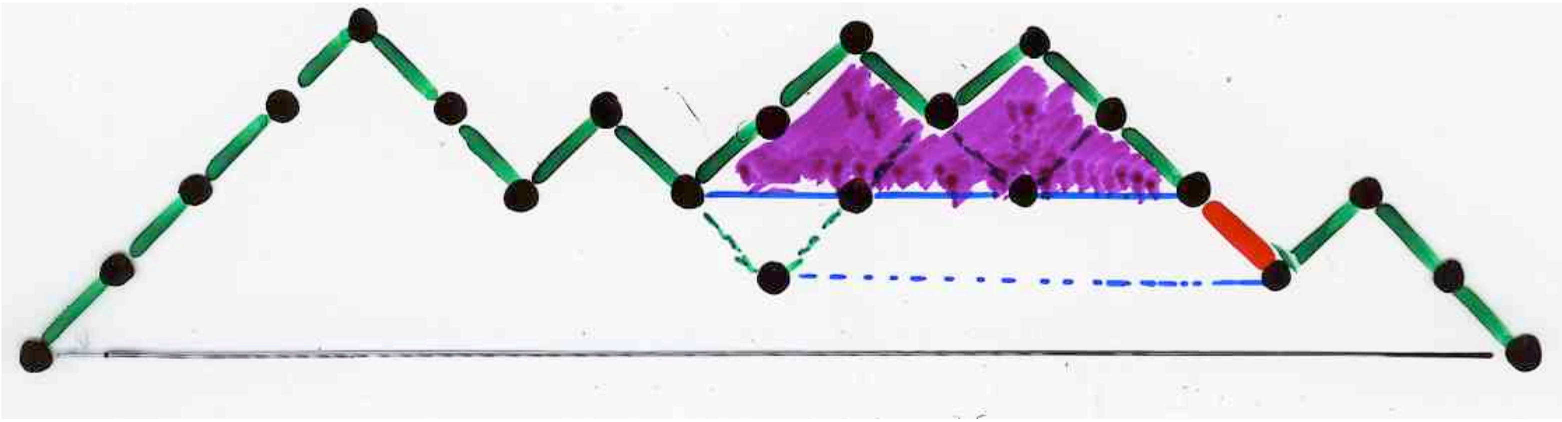
a binary tree B
and its associated complete binary tree $\overline{B}$
(full)

the Tamari lattice
in term
of Dyck paths

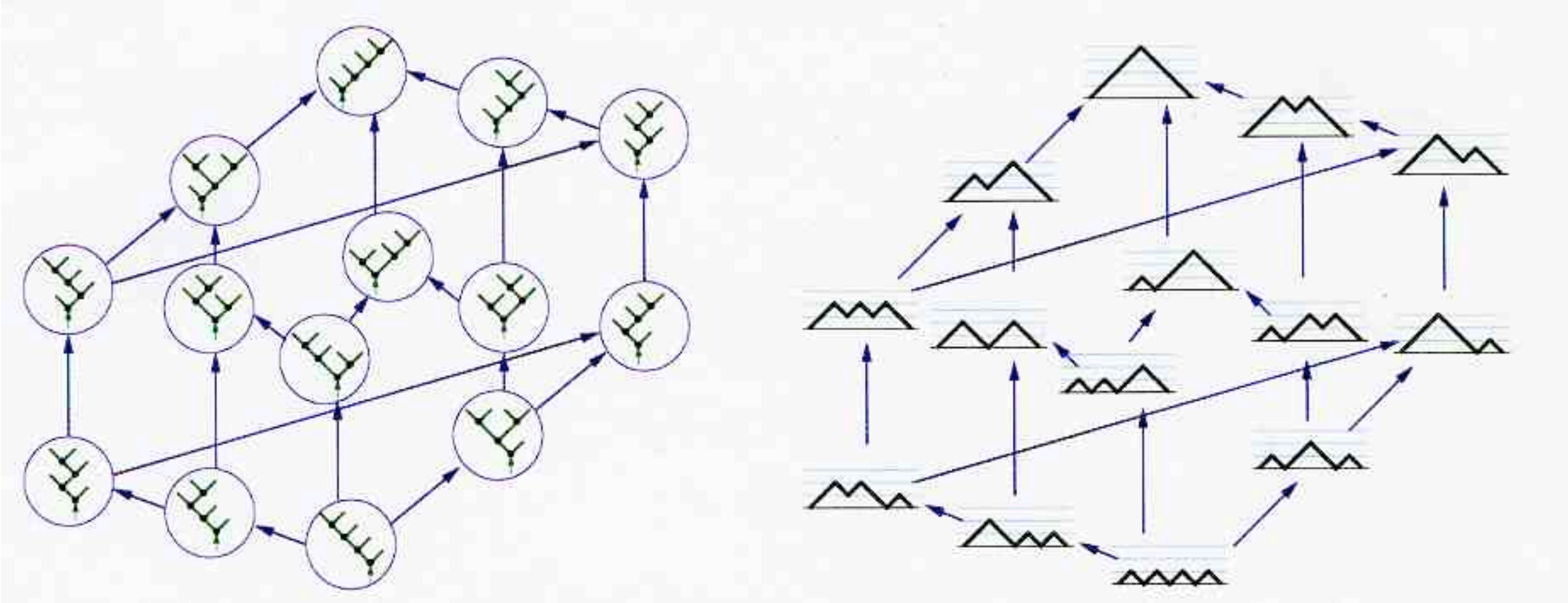




facteur Dyck primitif

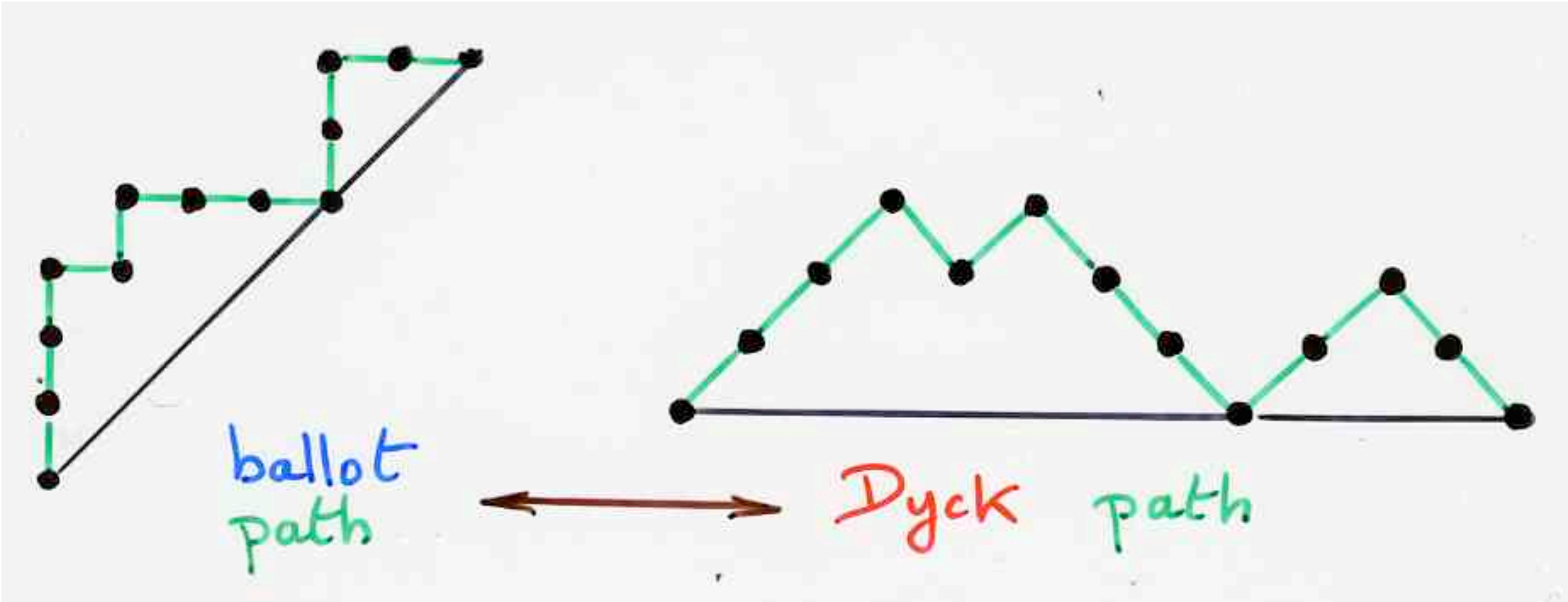


factor Dyck primitif

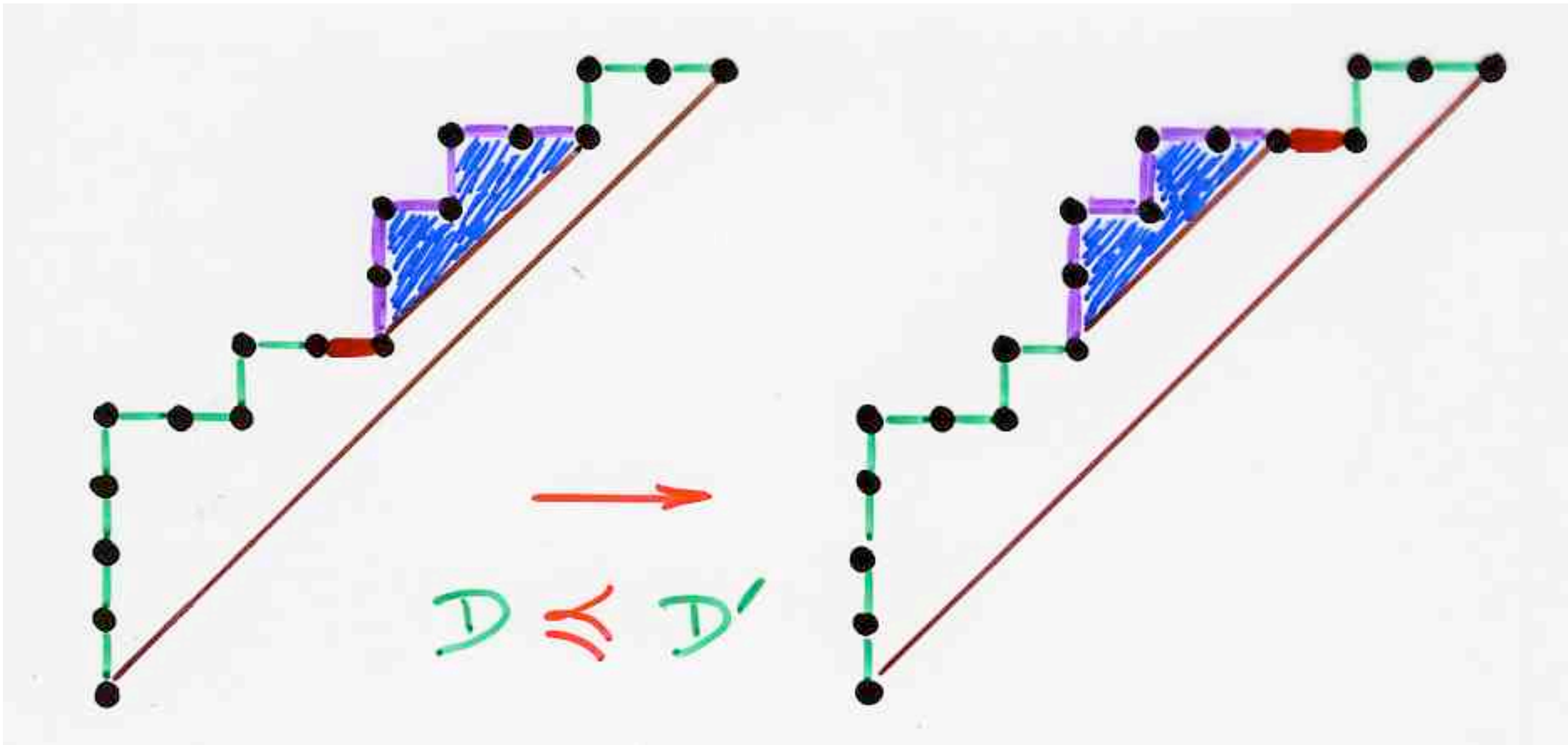


$C_4 = 14$

Catalan



vocabulary: ballot path
Dyck path



the Tamari covering relation
for ballot (Dyck) path

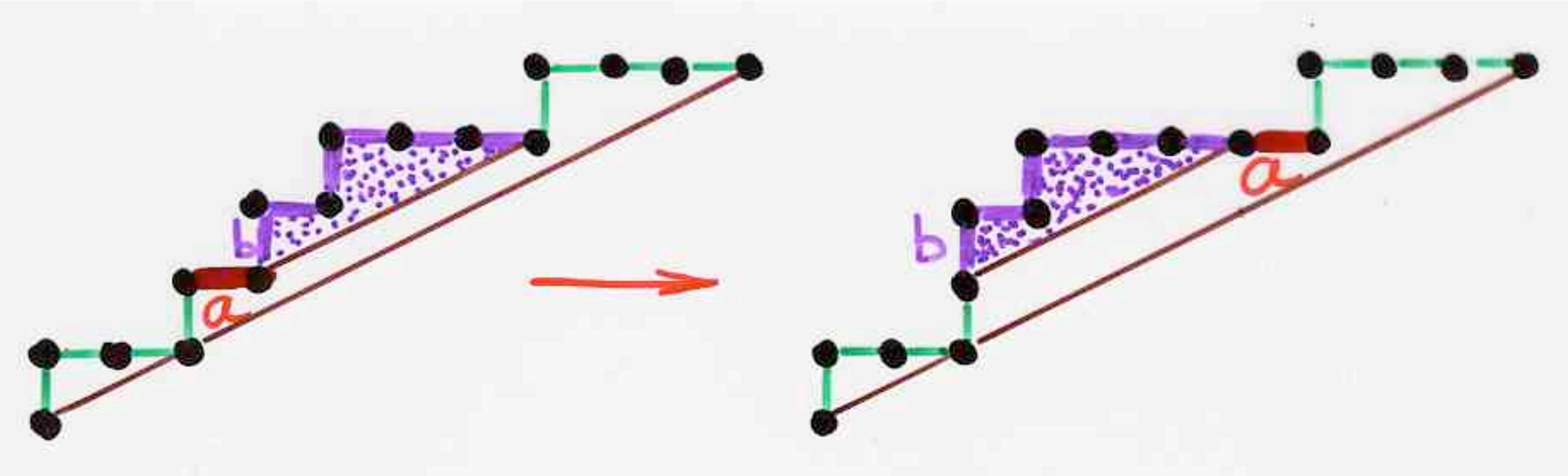
diagonal coinvariant spaces

higher diagonal coinvariant spaces

F. Bergeron (2008) introduced the m-Tamari lattice

dimension $\frac{1}{(m+1)n+1}\binom{(m+1)n+1}{mn}$

m-ballot paths



the covering relation in the
m-Tamari lattice
(m = 2)

diagonal coinvariant spaces

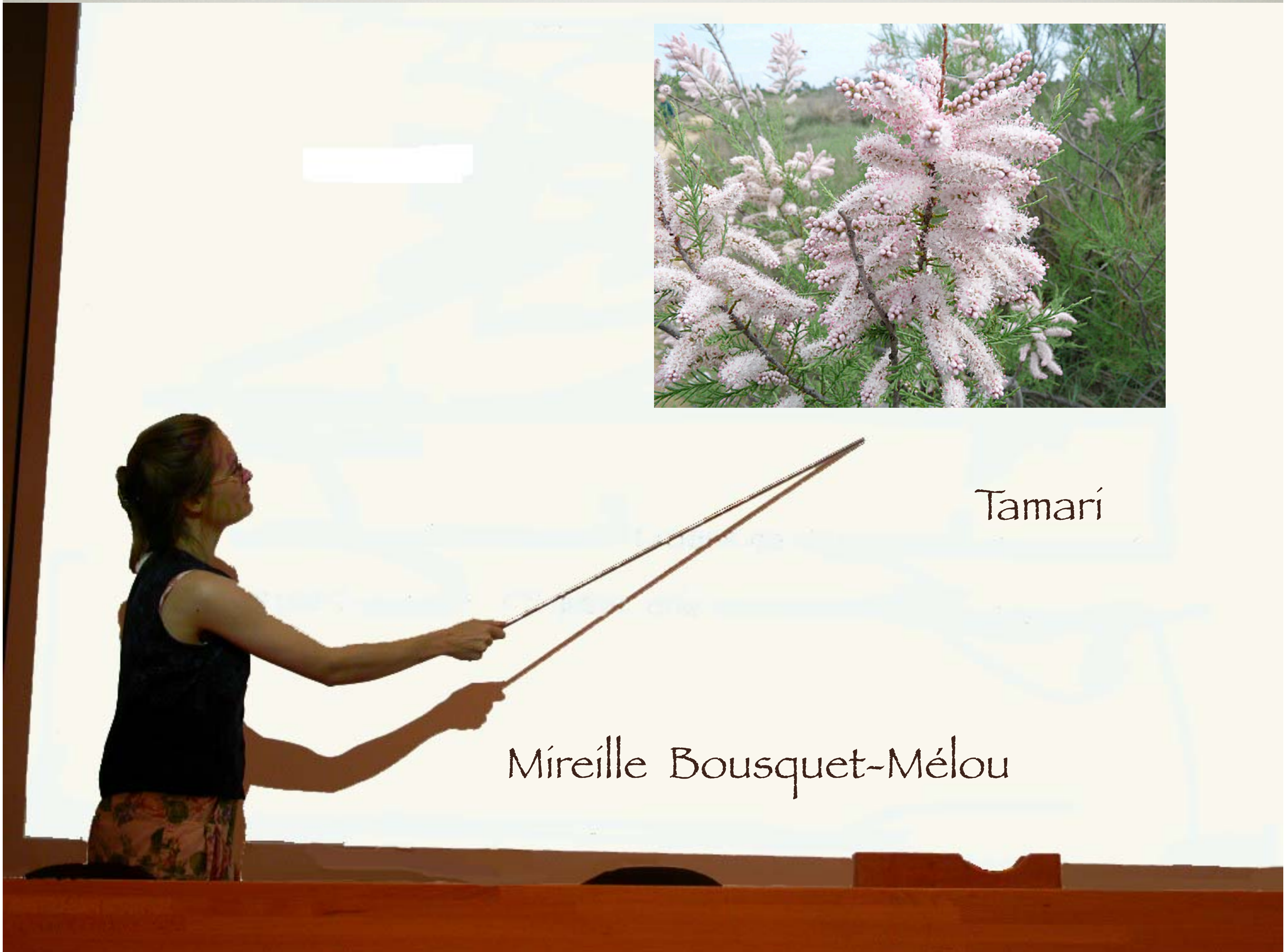
higher diagonal coinvariant spaces

F. Bergeron (2008) introduced the m-Tamari lattice

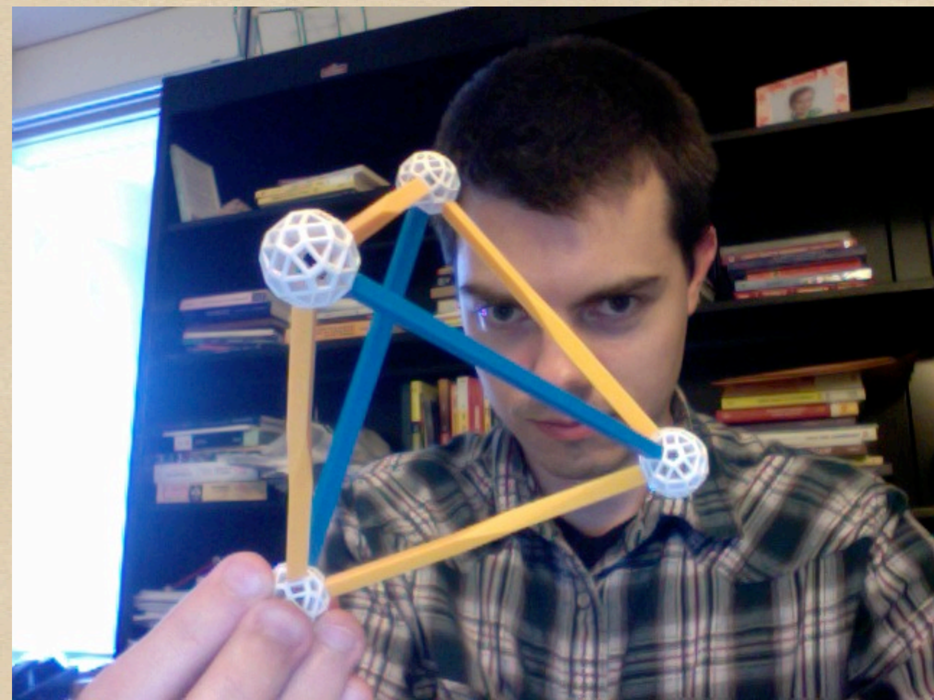
M. Bousquet-Mélou, E. Fusy, L.-F. Préville-Ratelle (2011)
nb of intervals of m-Tamari lattices

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1}$$ F. Bergeron

M. Bousquet-Mélou, G. Chapuy, L.-F. Préville-Ratelle (2011)
nb of labelled intervals $(m+1)^n(mn+1)^{n-2}$


Tamari
Mireille Bousquet-Mélou

Rational Catalan Combinatorics



Rational Catalan Combinatorics

D. Armstrong

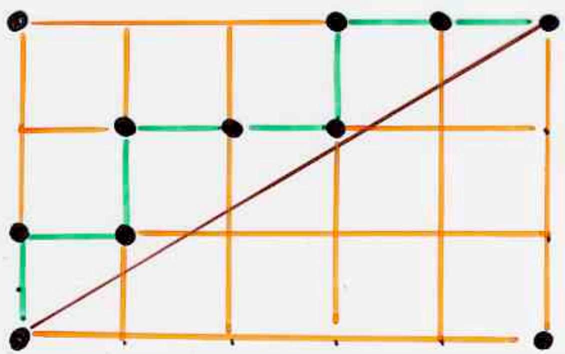
$$Cat(a,b) = \frac{1}{a+b}\binom{a+b}{a,b}$$

number of (a,b)- ballot paths $= Cat(a,b)$

Grossman (1950)

Bizley (1954)

rational ballot (Dyck) paths



$a=3$

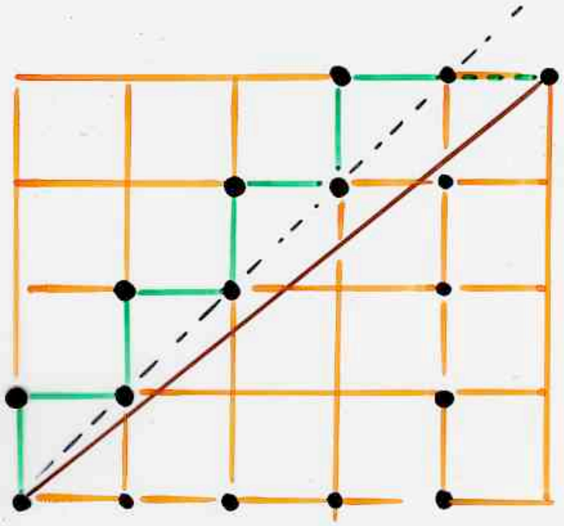
$b=5$

$(a,b) = (3,5)$

$(a,b) = (n, n+1) \longrightarrow C_n$ Catalan nb

$(a,b) = (n, mn+1) \longrightarrow \dfrac{1}{(m+1)n+1}\dbinom{(m+1)n+1}{n}$

Fuss-Catalan nb

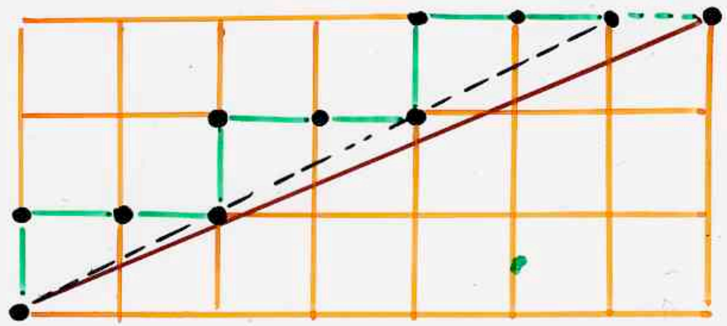


$n=4$

$n+1 = 5$

$(a,b) = (4,5)$

Catalan



$n=3$

$m=2$ $mn+1=7$

$(a,b) = (3,7)$

Fuss-Catalan

question :
Sergi Elizalde
(this worshop)

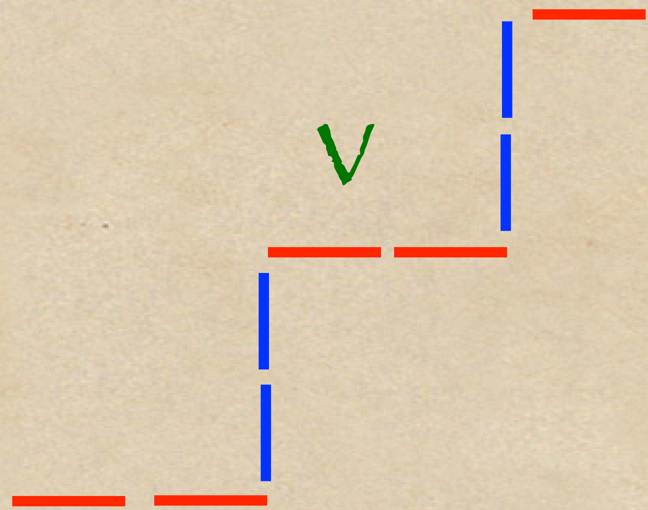
define an (a,b)-Tamari lattice ?

extension: Tamari T_v

transcental

Catalan

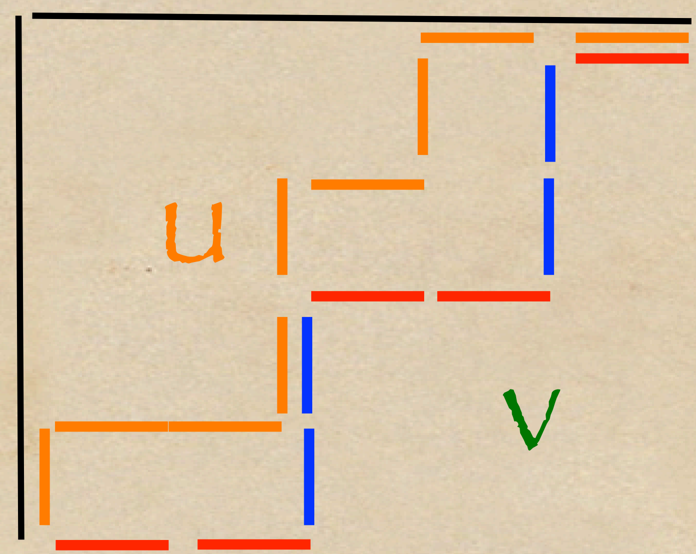
combinatorics ?

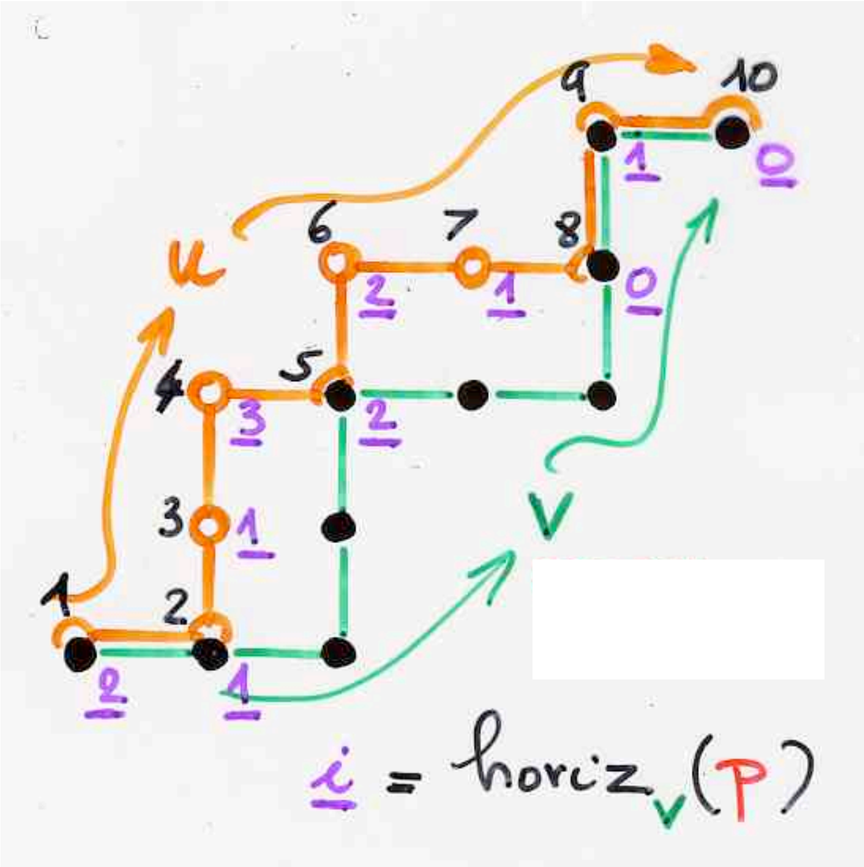


number of pairs (u,v)

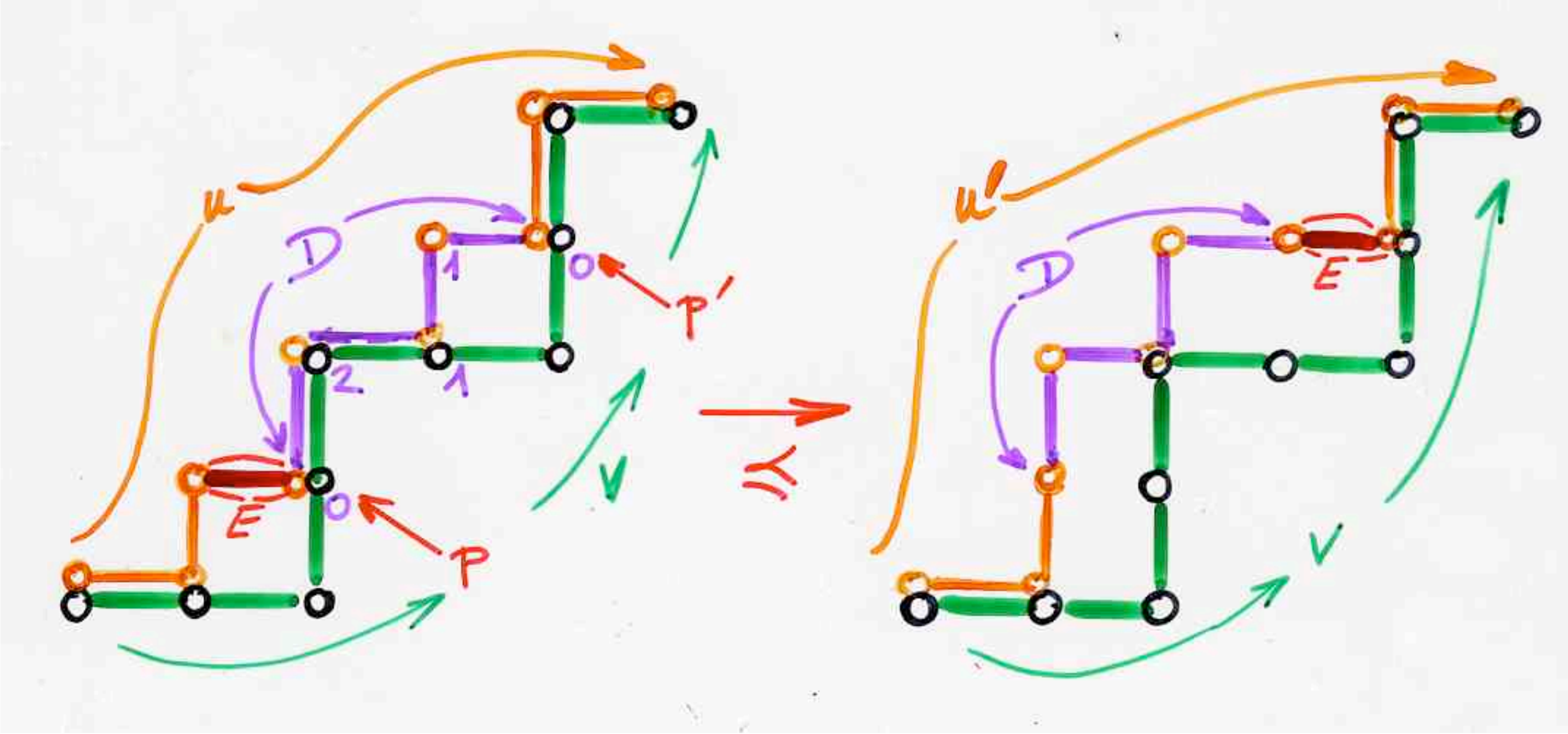
Macmahon determinant

Kreweras determinant





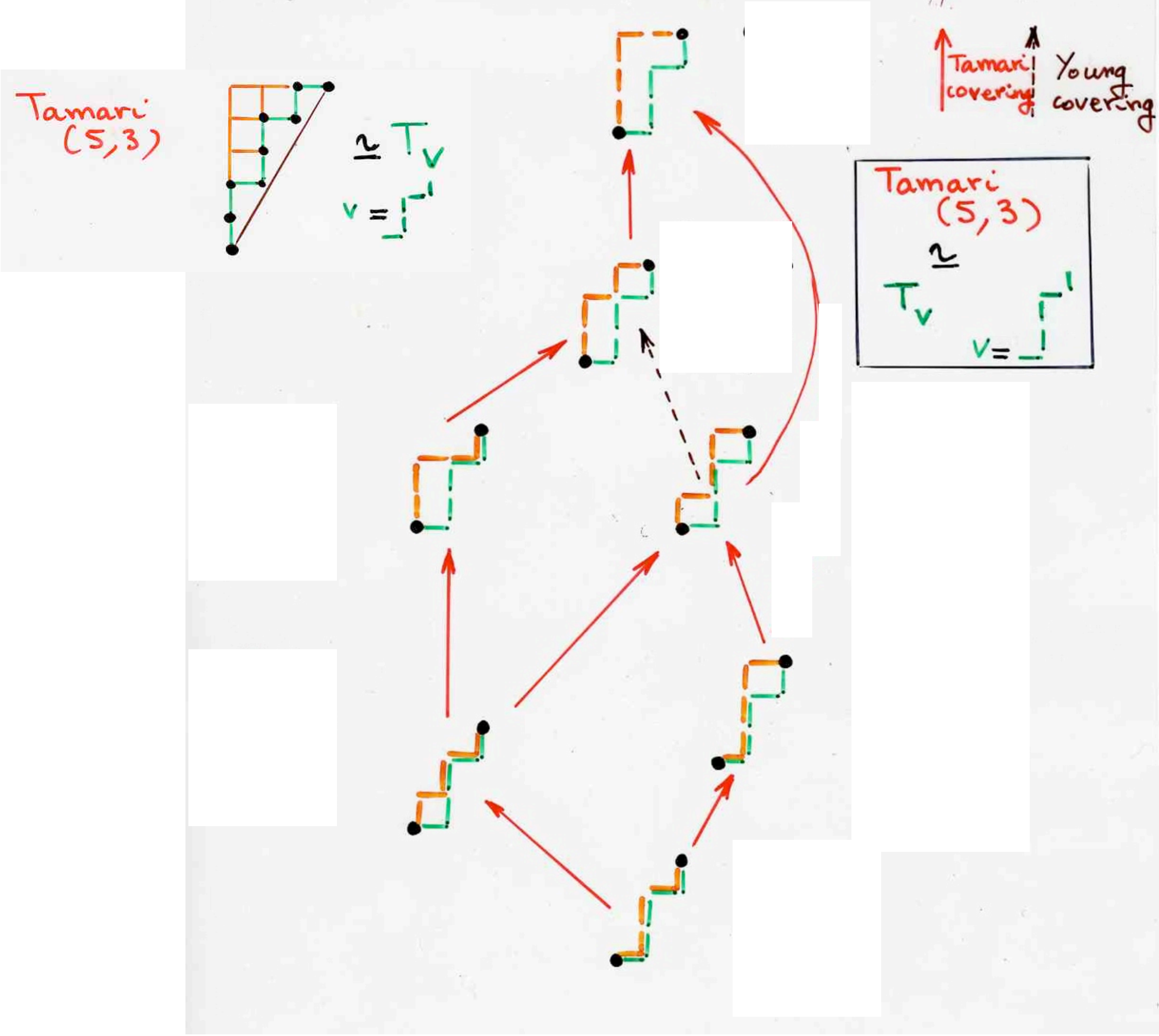
a pair (u, v) of paths
with the "horizontal distance"
$\mathrm{horiz}_v(P)$



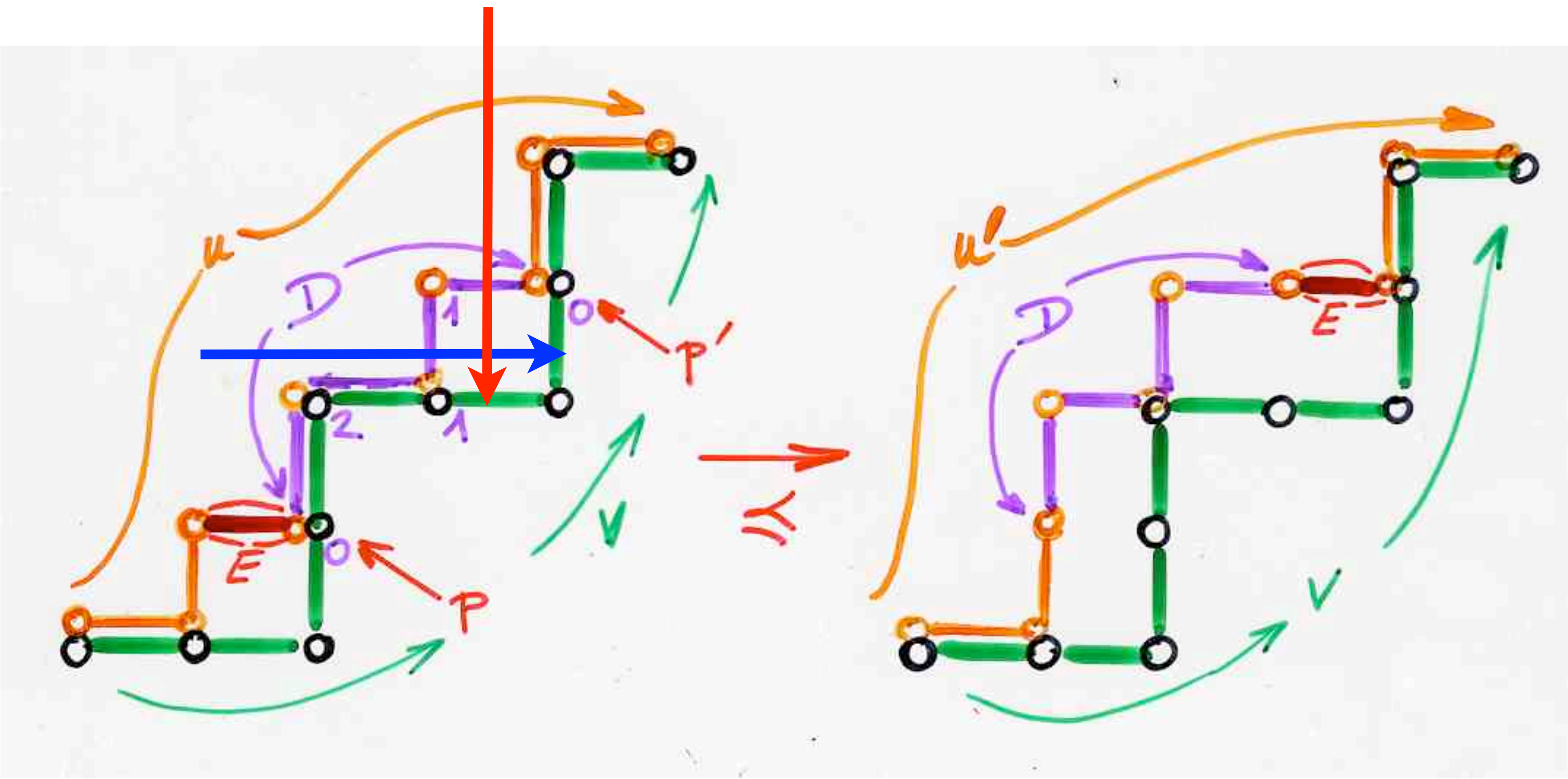
the covering relation
in the poset T_v

Thm 1. For any path v

T_v is a lattice



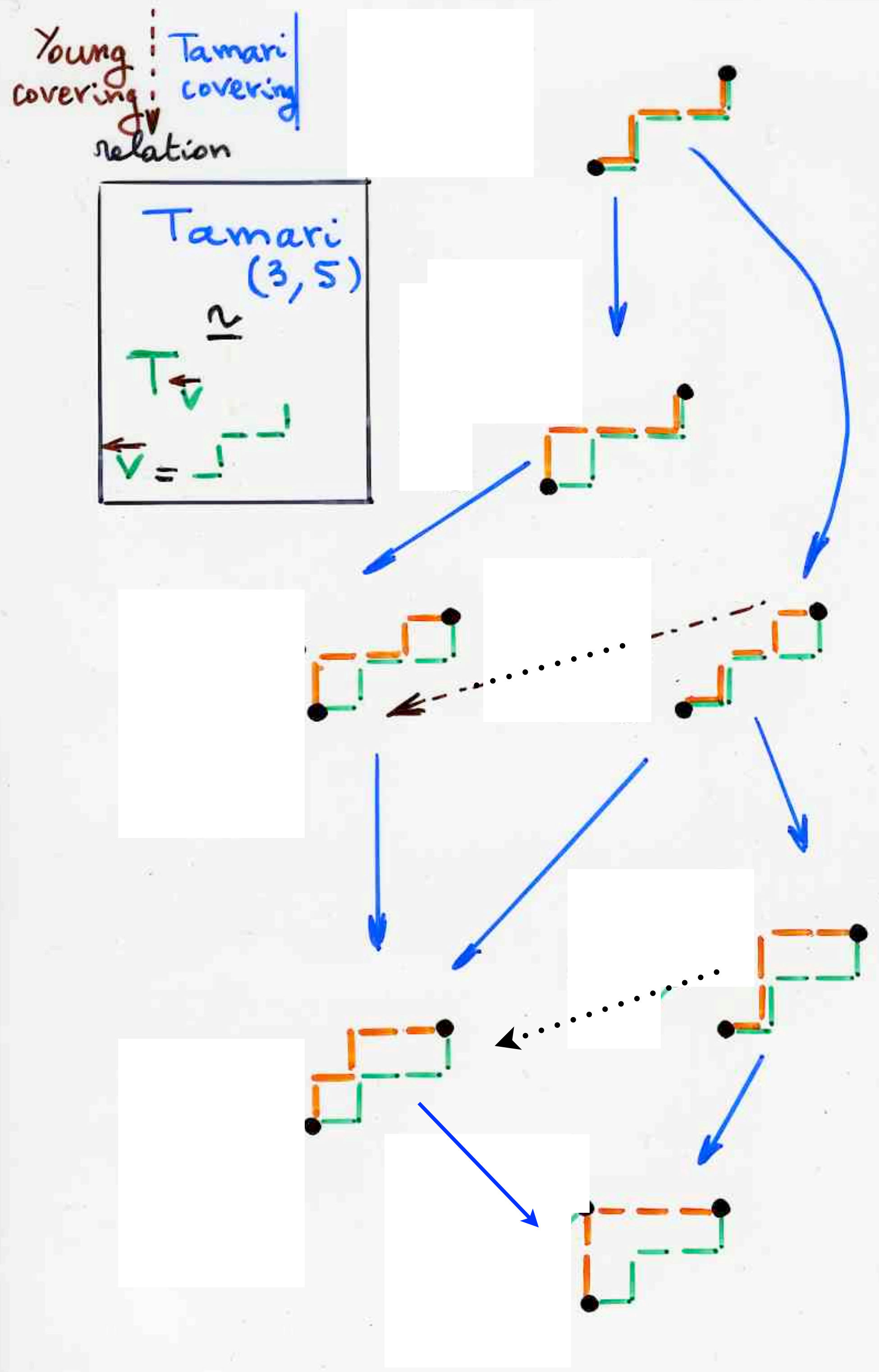
Tamari (5,3)
≃ T
v
v =
Tamari covering
Young covering
Tamari (5,3)
≃
T
v
v=

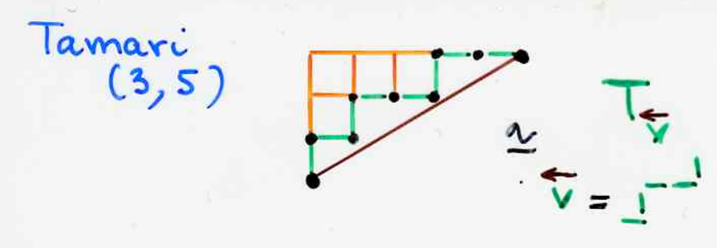


«row covering relation»

«column covering relation»

mirror image, exchange N and E

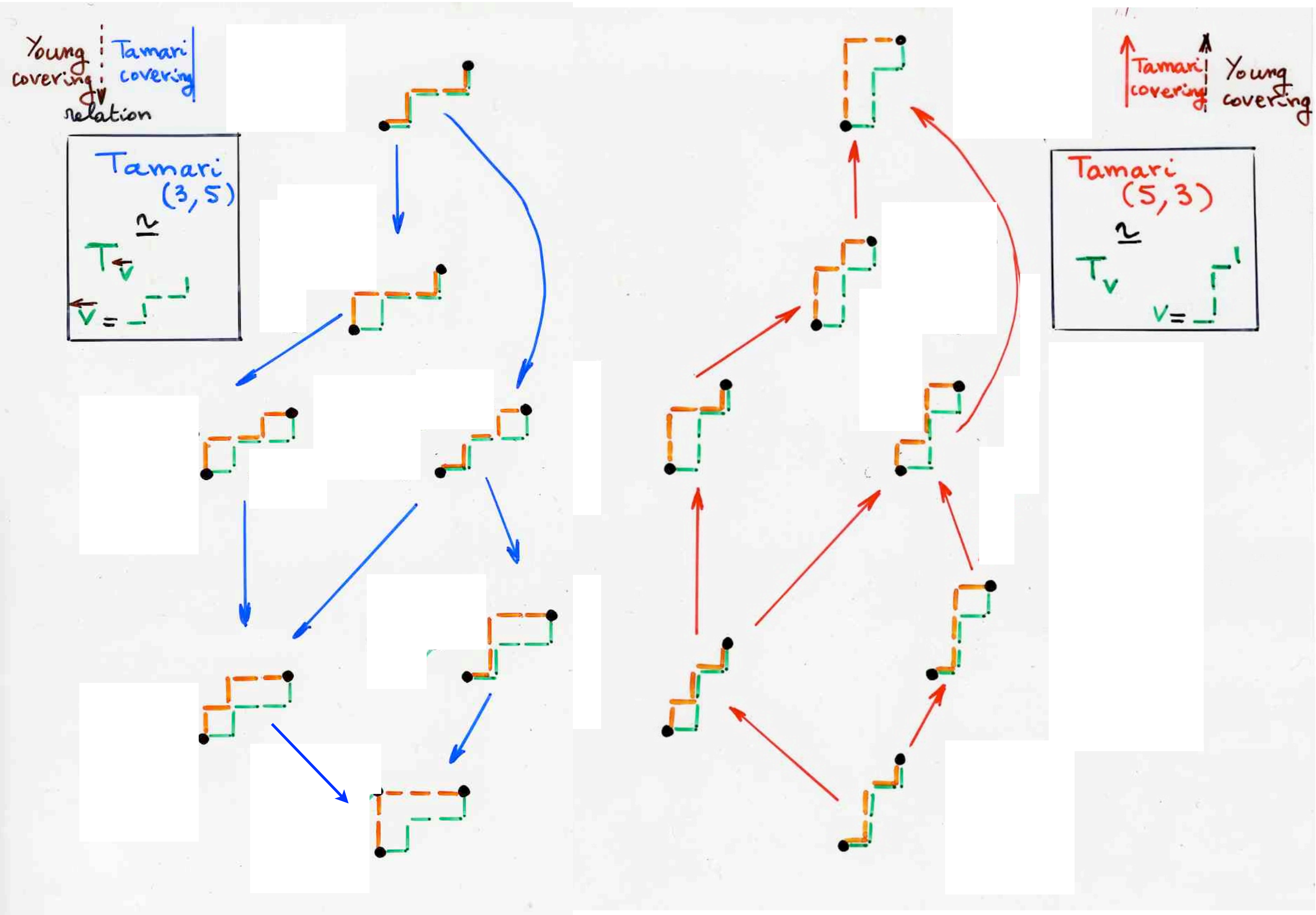

Young covering
Tamari covering
relation
Tamari (3,5)
≃
T←v
←v =


Tamari (3,5)
≃
T←v
←v =

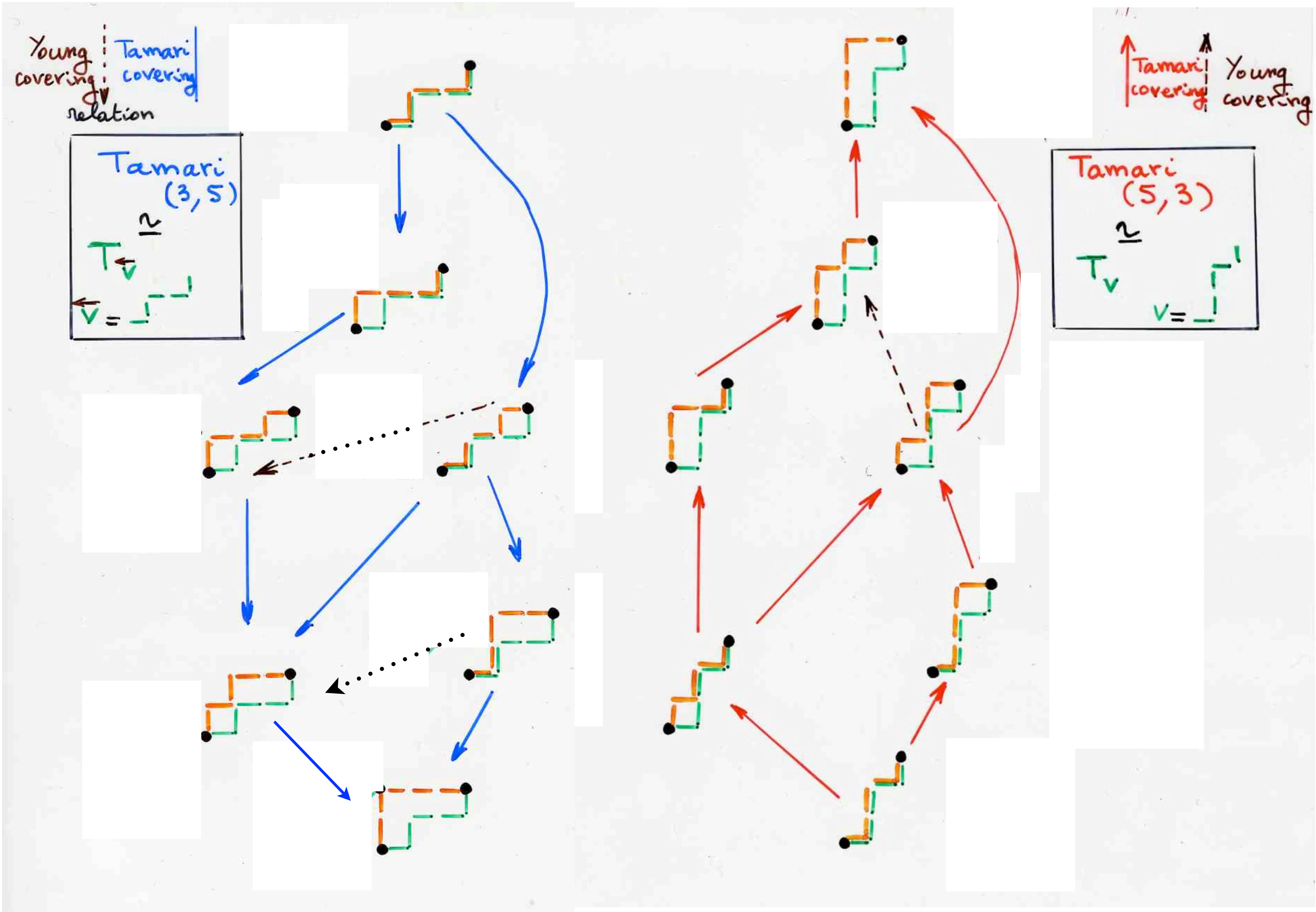
Thm 1. For any path v
T_v is a *lattice*

Thm 2. The *lattice* T_v
is isomorphic to the *dual* of $T_{\overleftarrow{v}}$

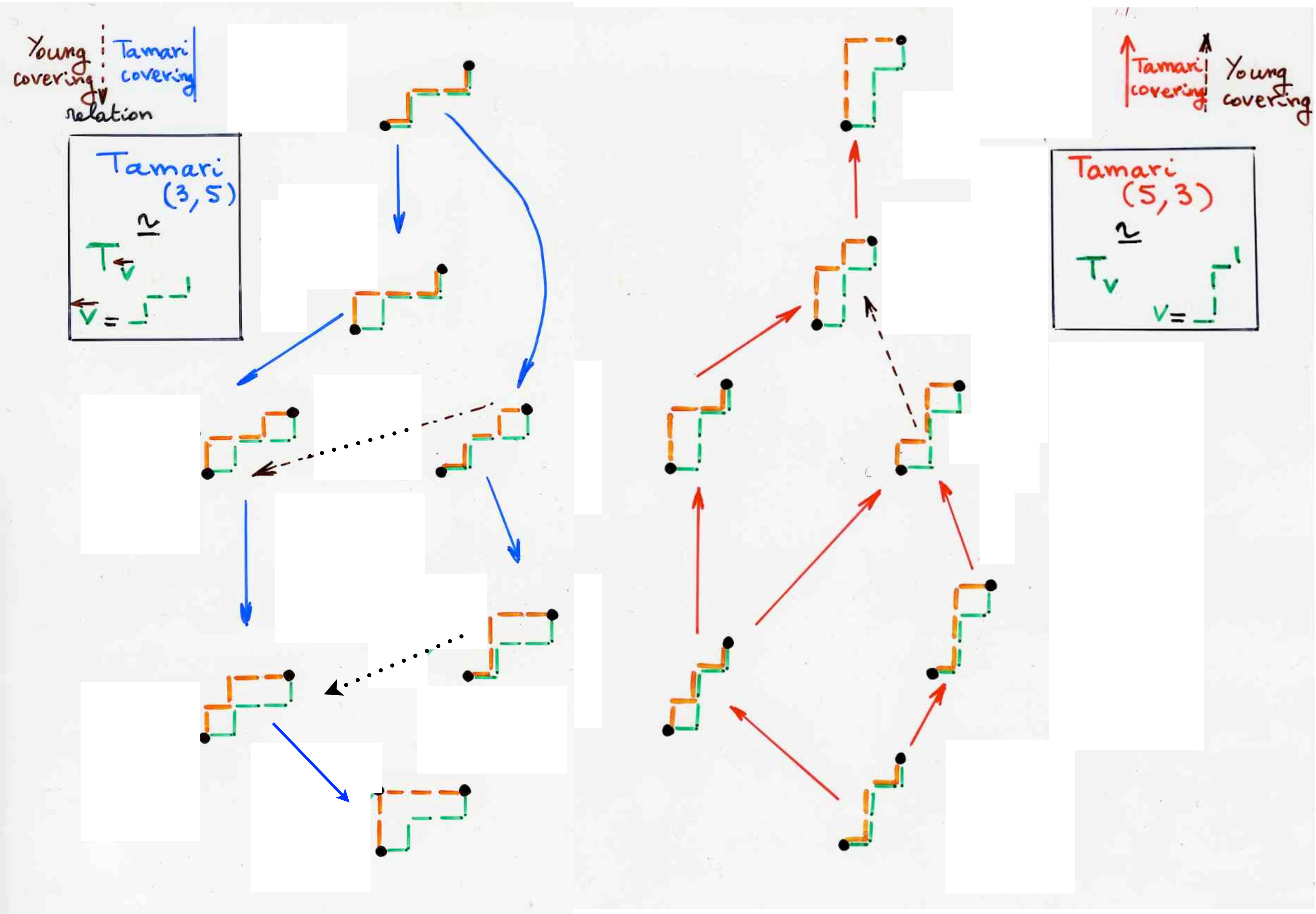
Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$



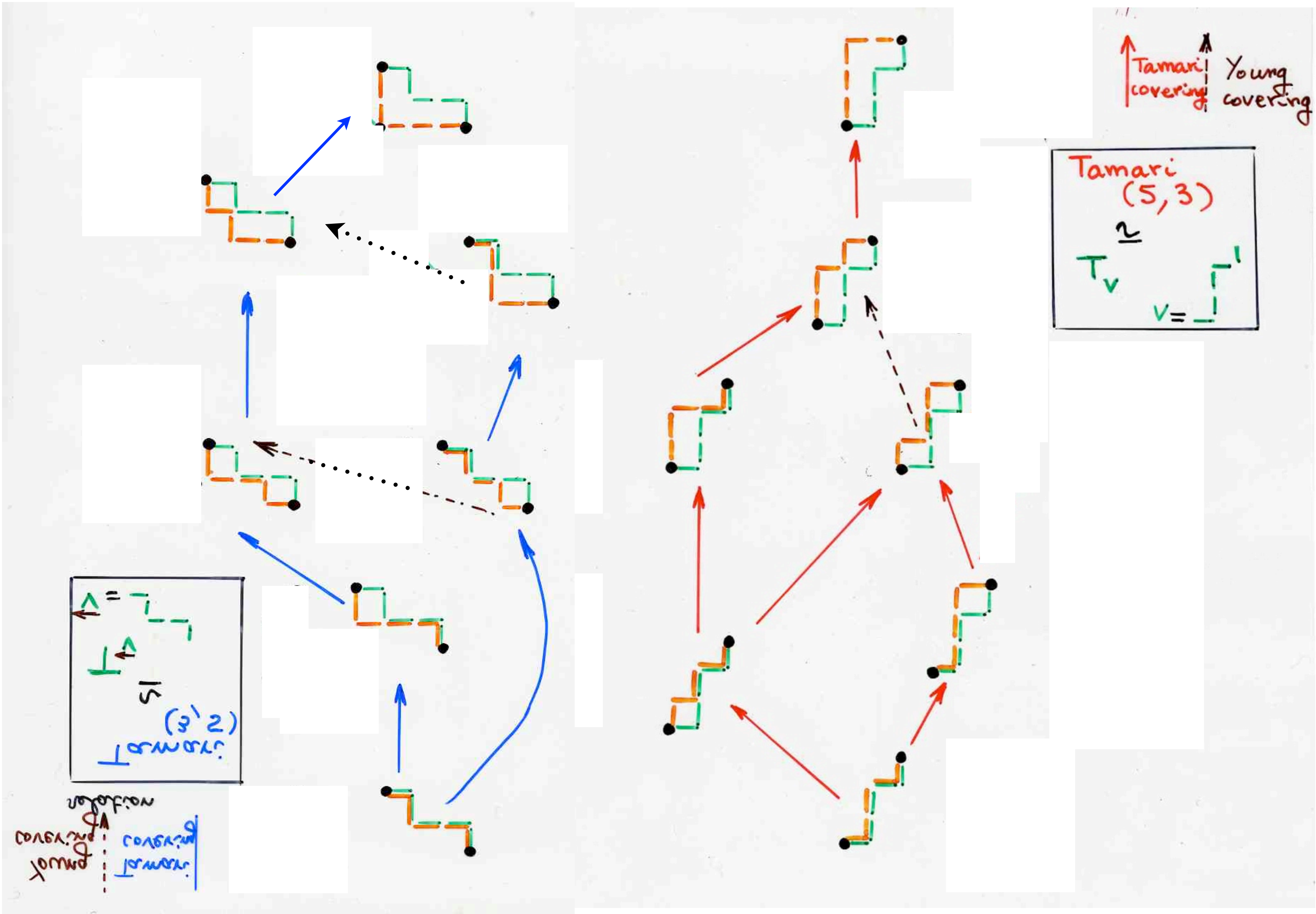
Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$



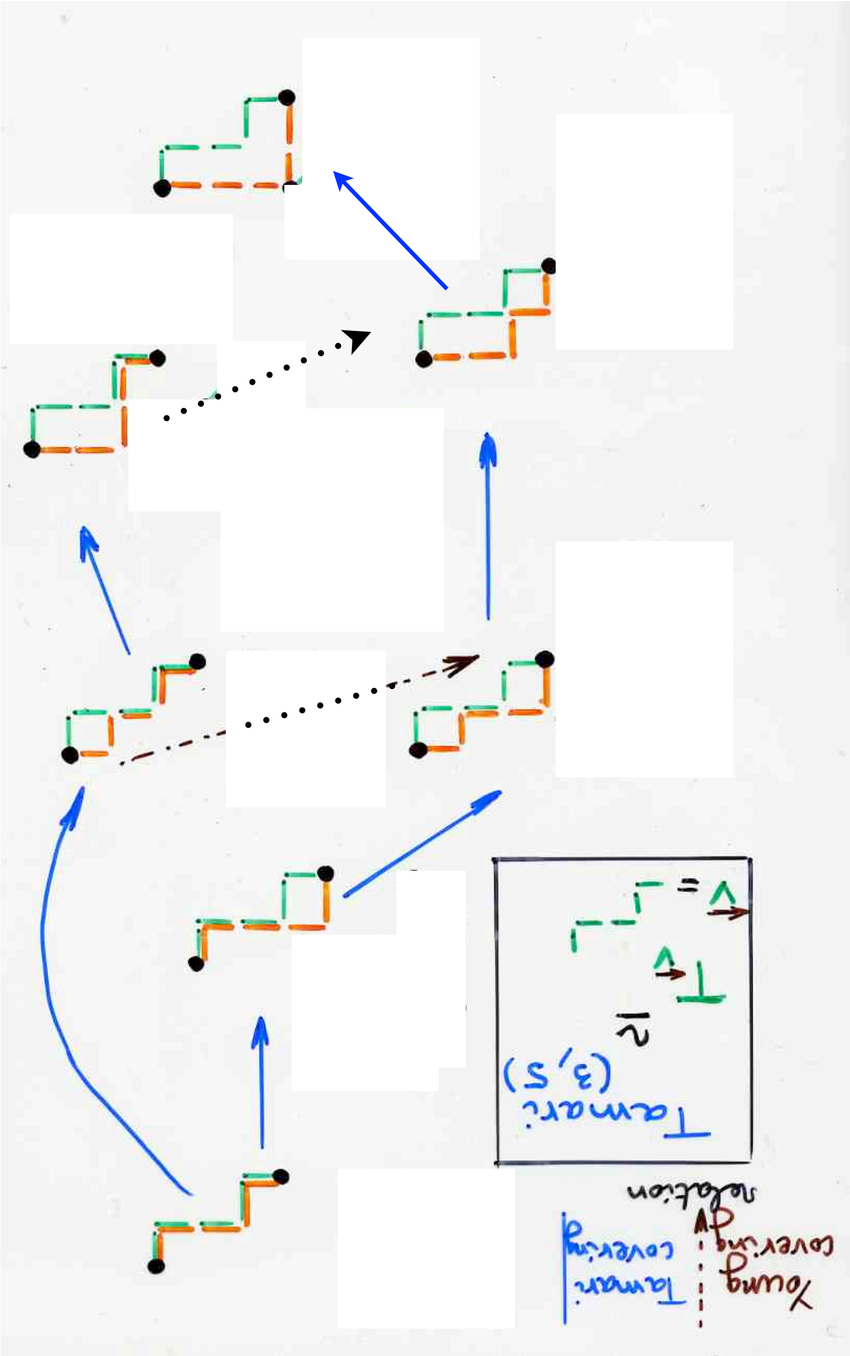
Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$

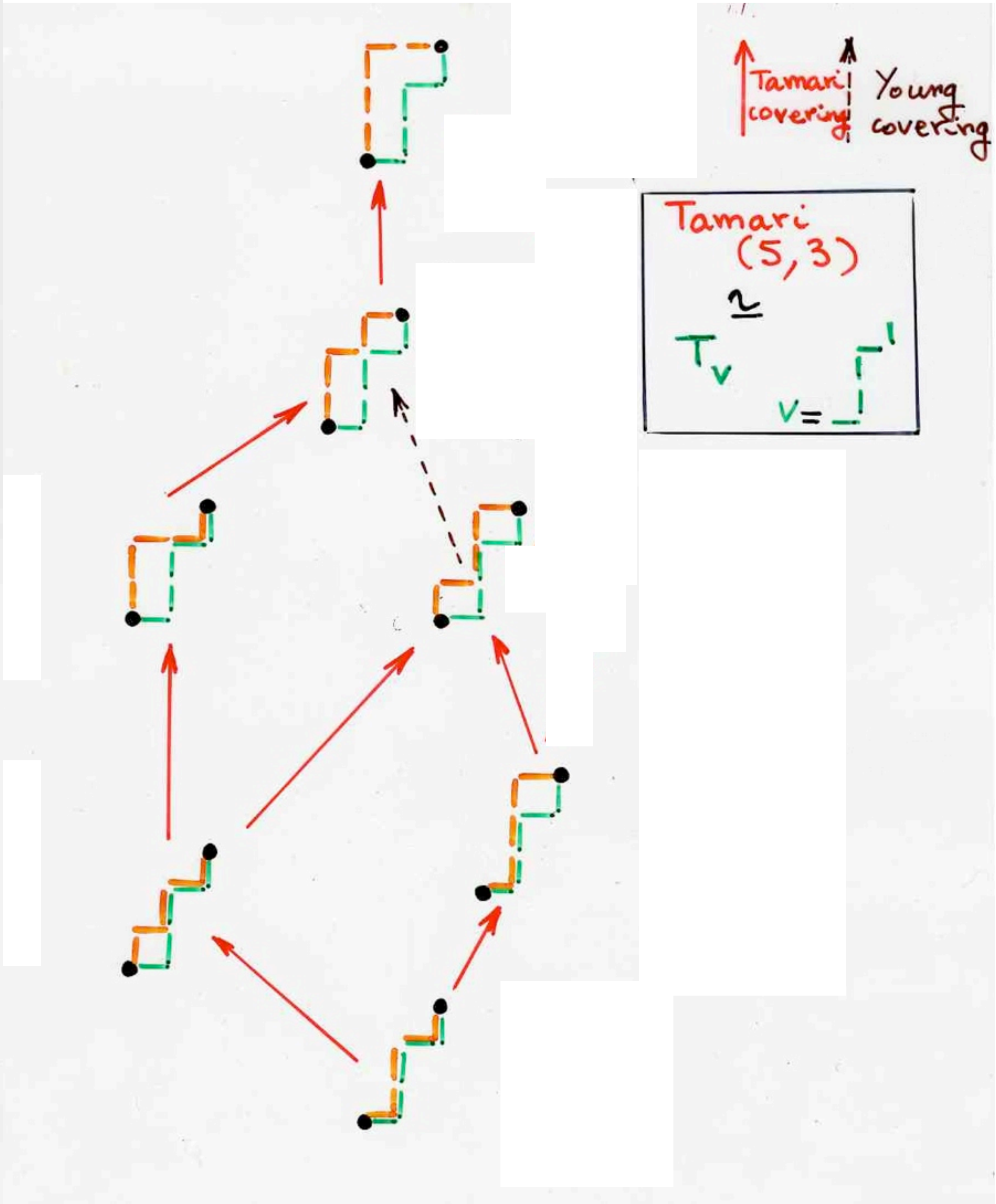


$$Y_v \simeq Y_{\overleftarrow{v}}$$



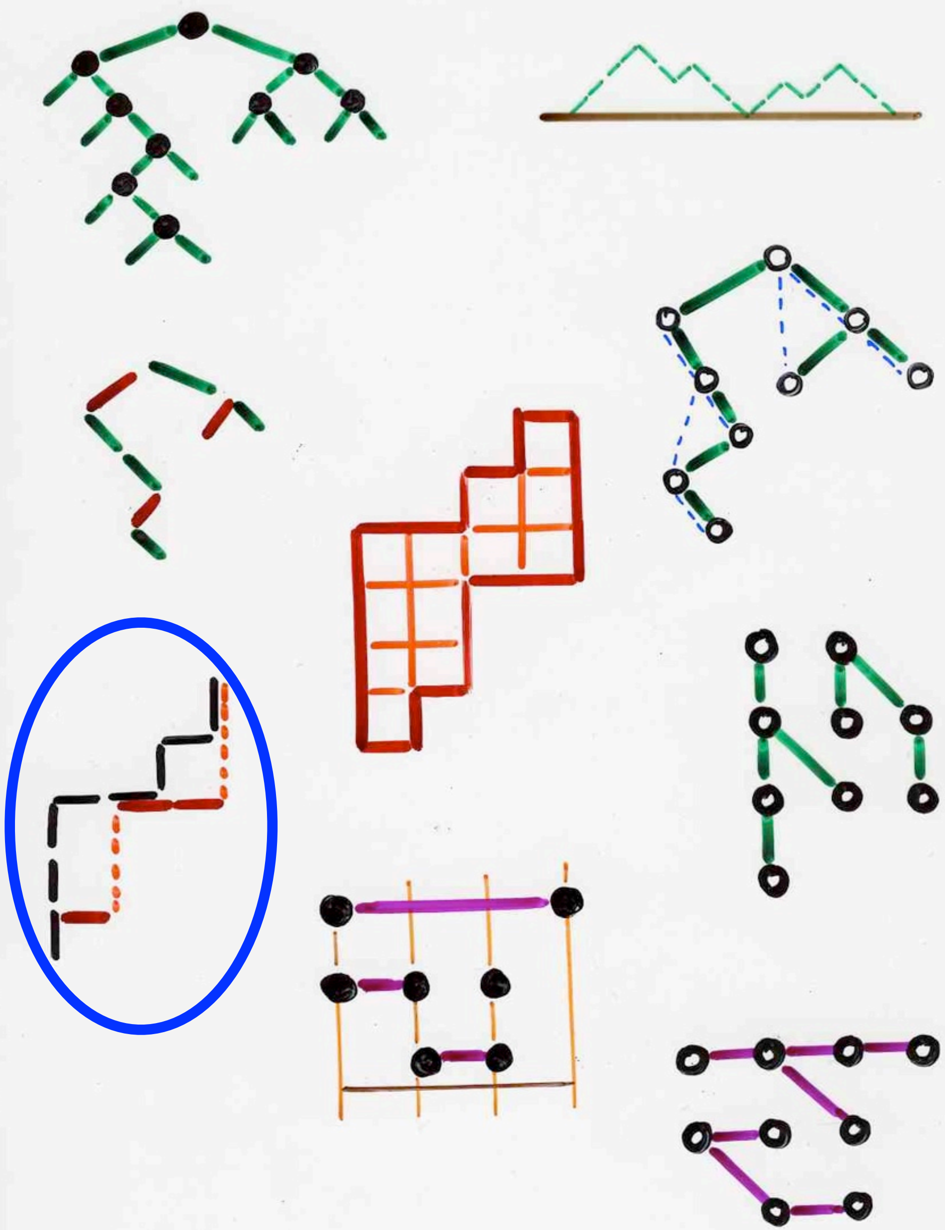
$$Y_v \simeq Y_{\overleftarrow{v}}$$



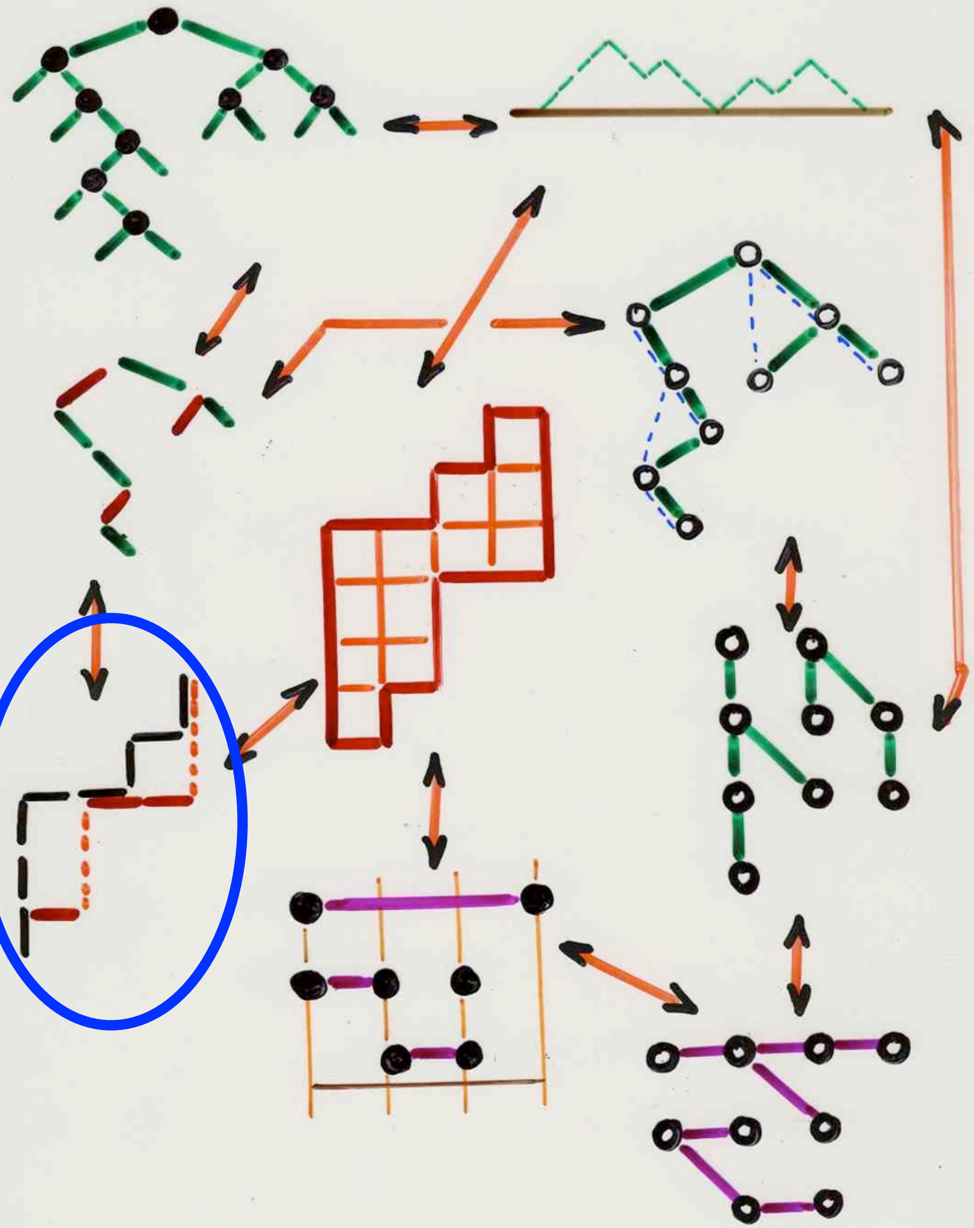


A bijection

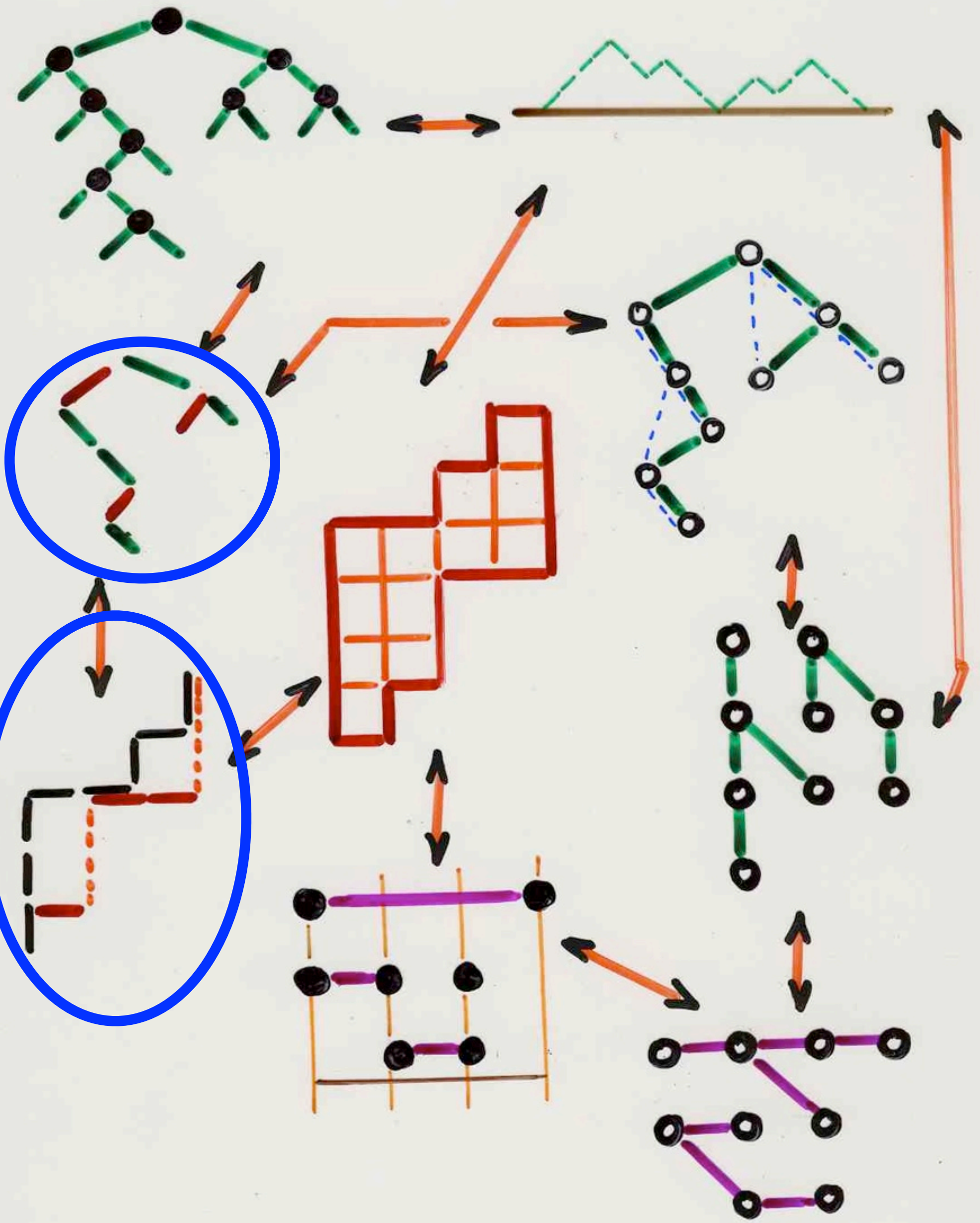
the Catalan garden



the Catalan garden



the Catalan garden



A bijection

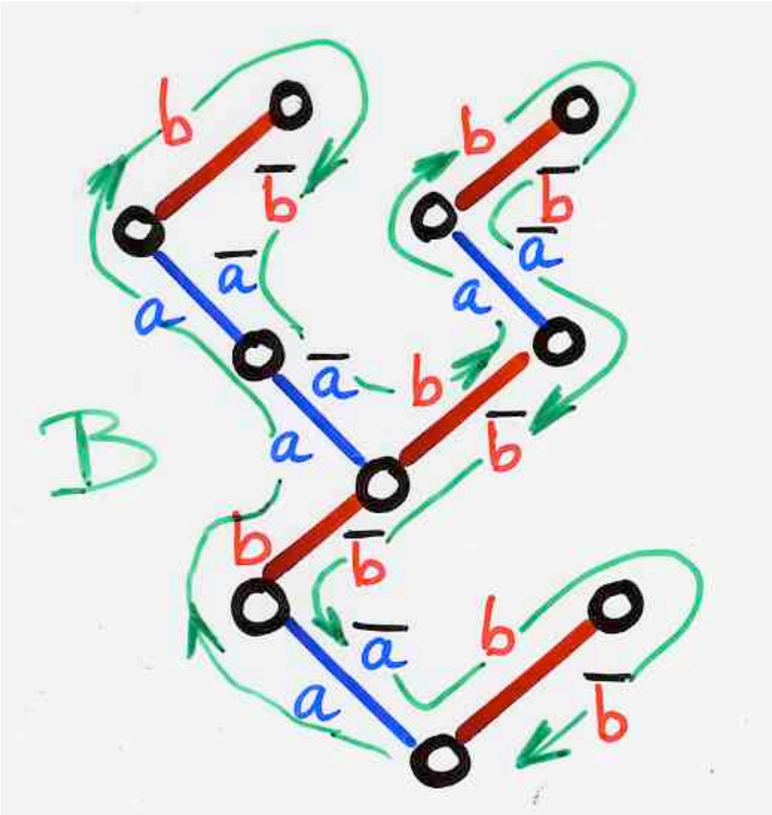
binary tree B $\longrightarrow$ pair of paths (u,v) the words $\left\{ \begin{array}{l} w(B) \\ u(B) \\ v(B) \end{array} \right.$

$$w(B) = a\,b\,a\,a\,b\,\bar{b}\,\bar{a}\,\bar{a}\,b\,a\,b\,\bar{b}\,\bar{a}\,\bar{b}\,\bar{b}\,\bar{a}\,b\,\bar{b}$$

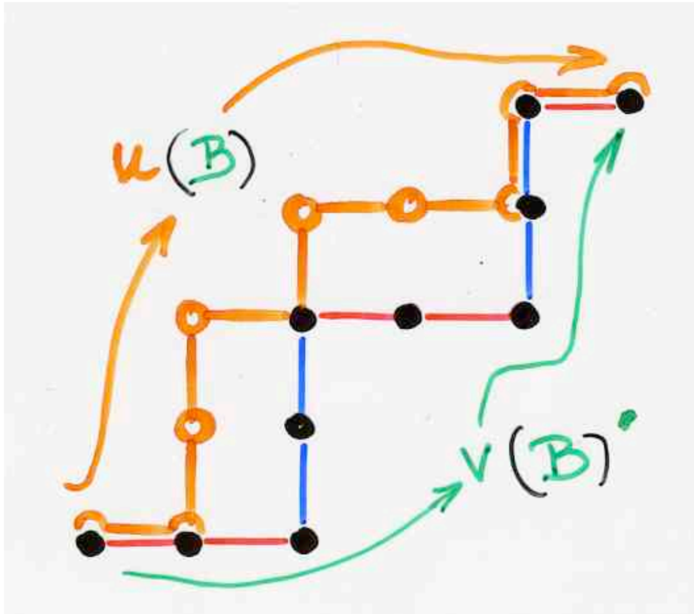
$$u(B) = \bar{b}\,\bar{a}\,\bar{a}\quad \bar{b}\,\bar{a}\,\bar{b}\,\bar{b}\,\bar{a}\quad \bar{b}$$

$$v(B) = b\quad b\quad \bar{a}\,\bar{a}\,b\quad b\quad \bar{a}\quad \bar{a}\,b$$

$$\bar{a} \to N \qquad \left.\begin{array}{c} b \\ \bar{b} \end{array}\right\} \to E$$



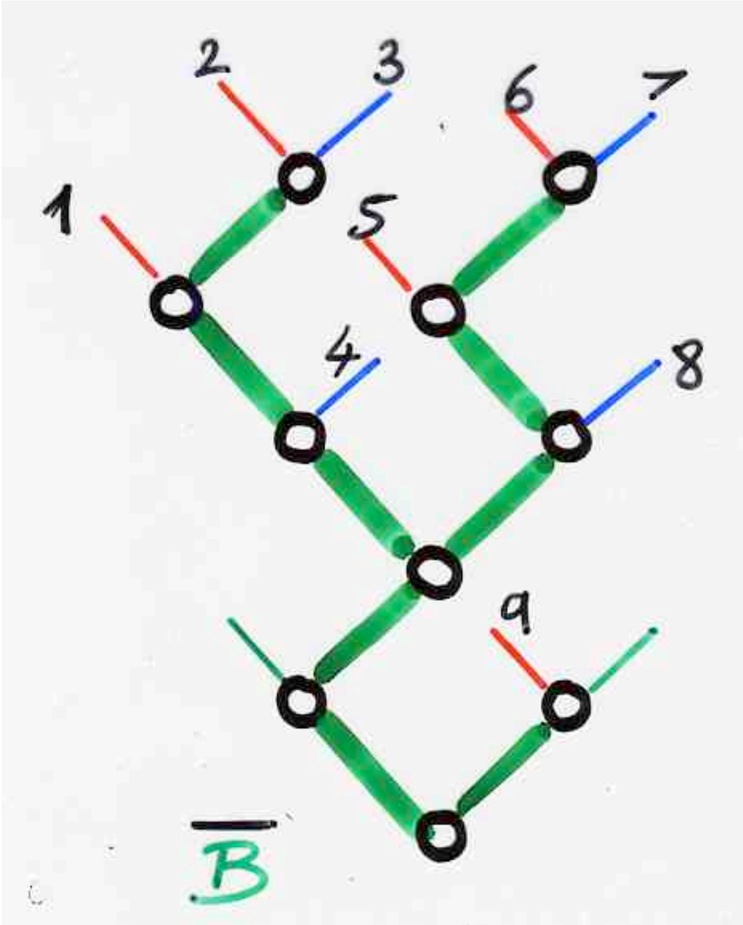
walk around a binary tree B

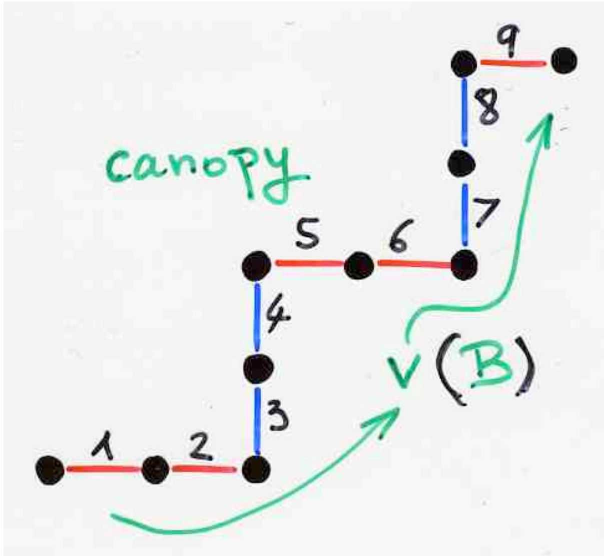


the pair (u, v) of paths associated to a binary tree B

definition:

v (B) Canopy





second definition of the canopy

J.-L.Loday, M. Ronco (1998, 2012)

algebraic structures

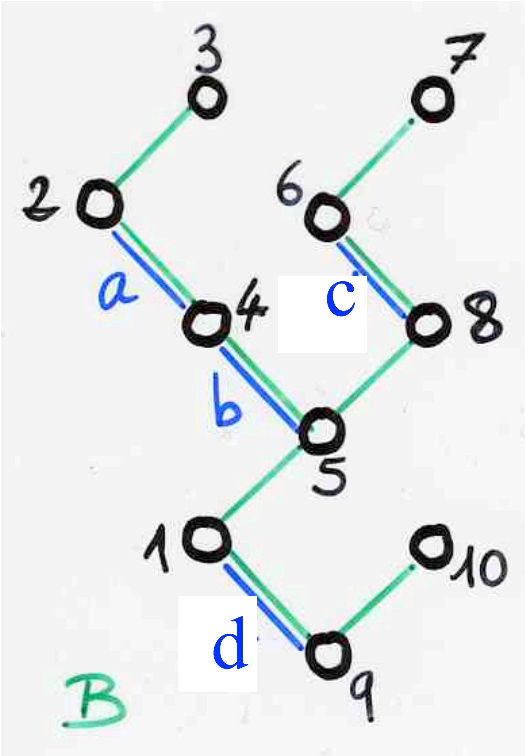
Hopf algebra

dim 2^{n-1} C_n $n!$

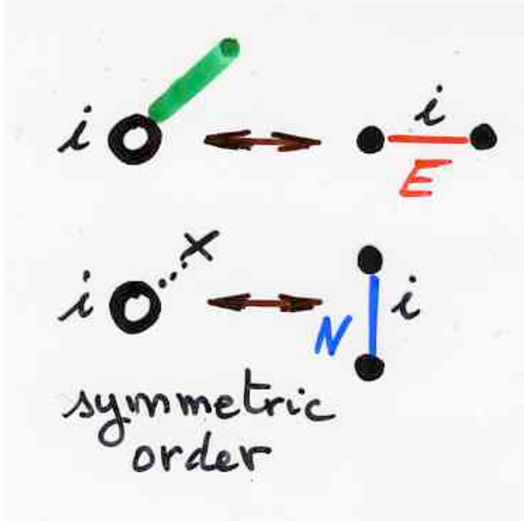
Catalan

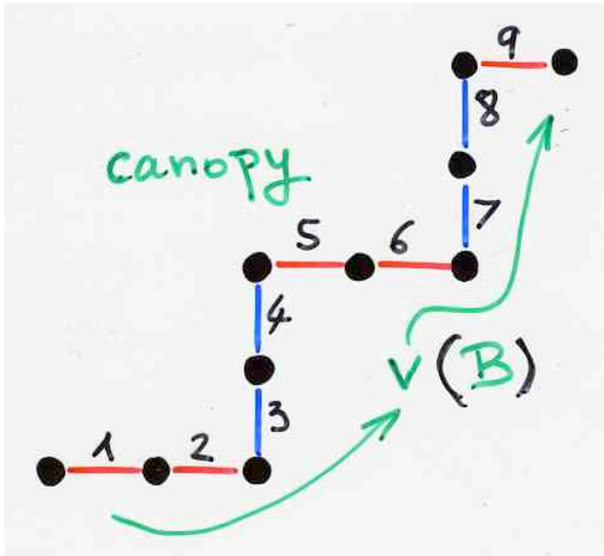
Boolean lattice inclusion $\longleftrightarrow$ Tamari order $\longleftrightarrow$ weak Bruhat order

J.-L.Loday, M. Ronco (1998, 2012)



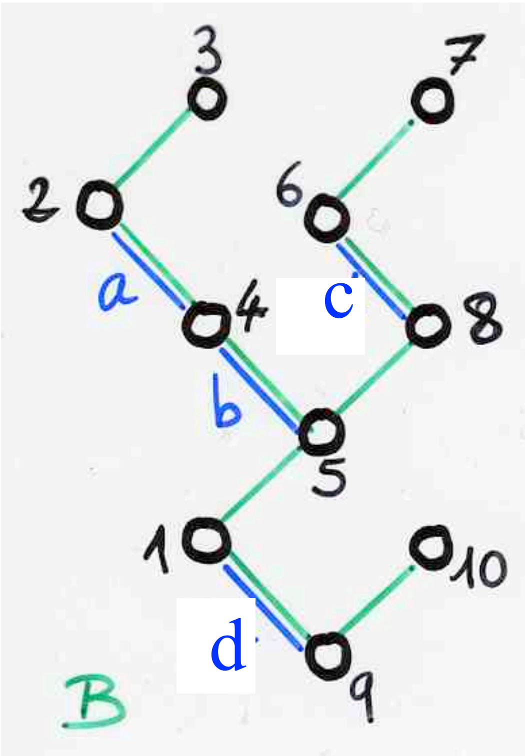
symmetric order

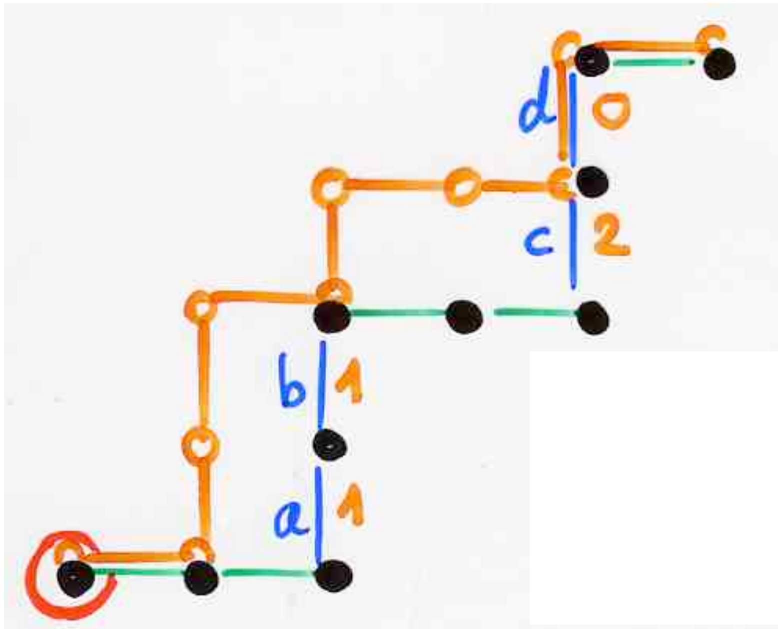




third definition of the canopy

a lemma

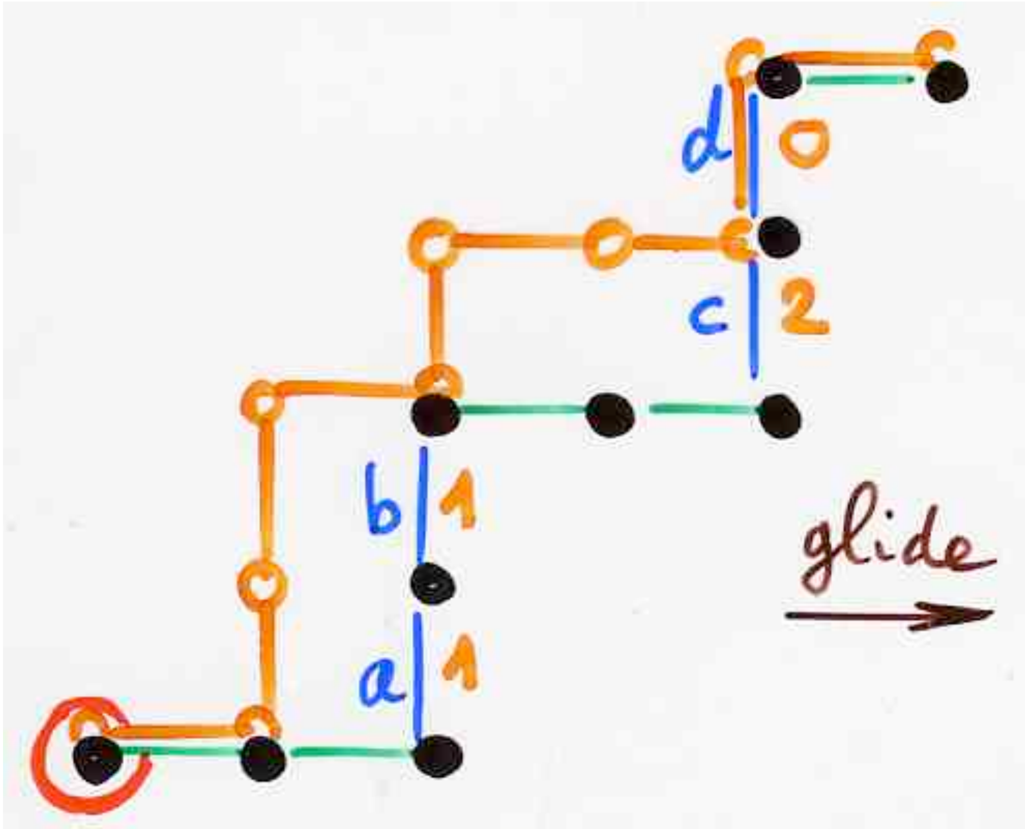




reverse bijection

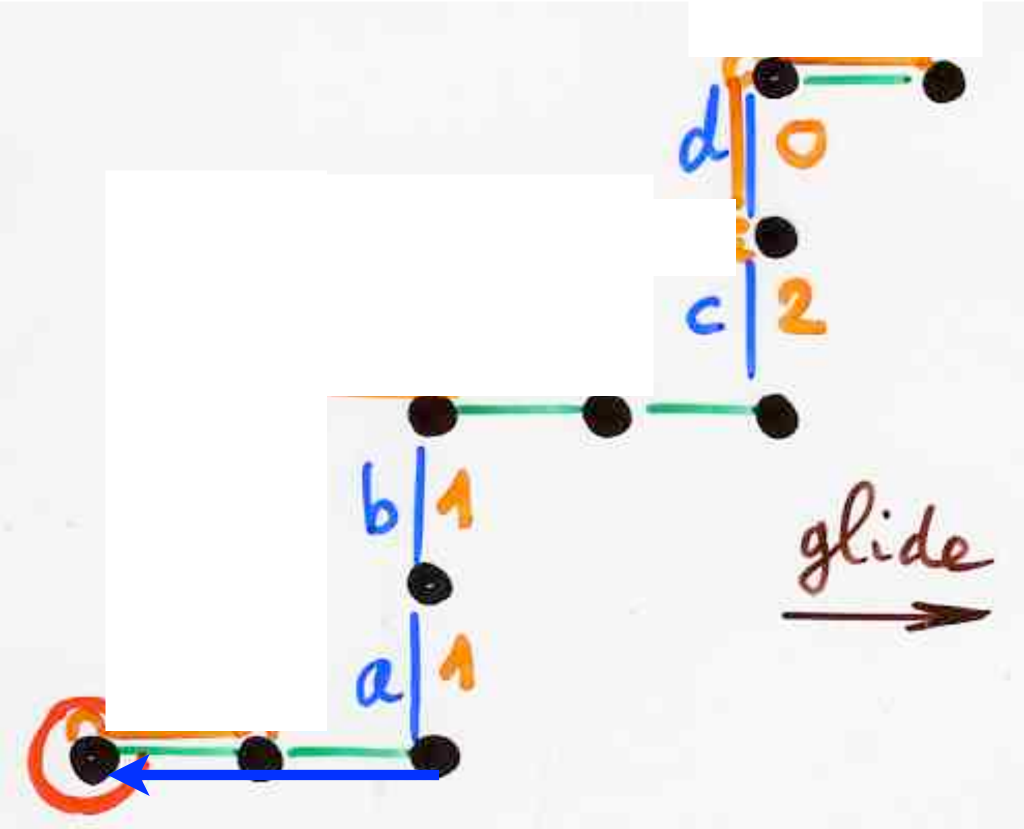
binary tree B $\longleftarrow$ pair of paths (u,v)

the «push-gliding» algorithm



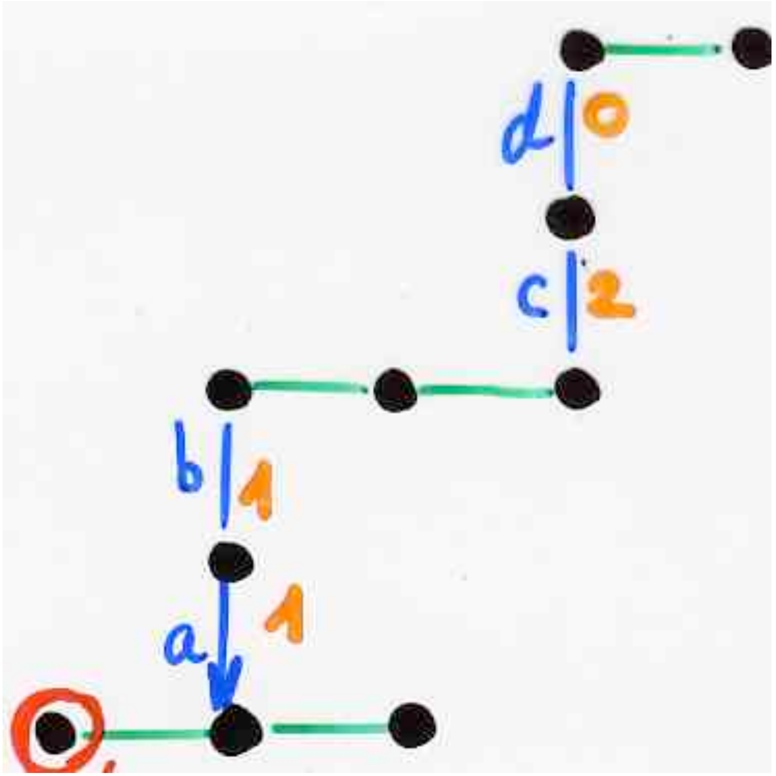
reverse bijection

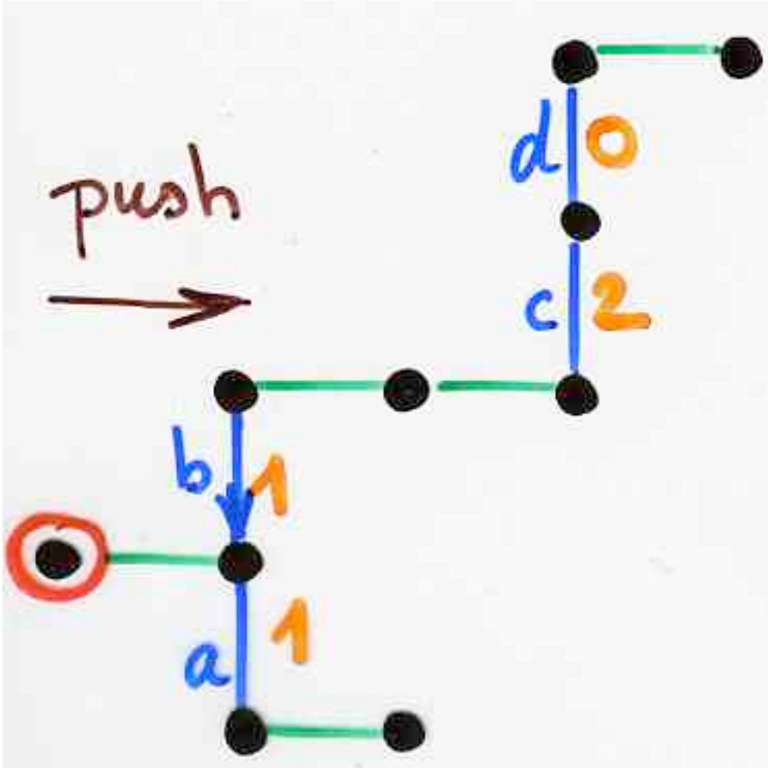
the "push-gliding" algorithm



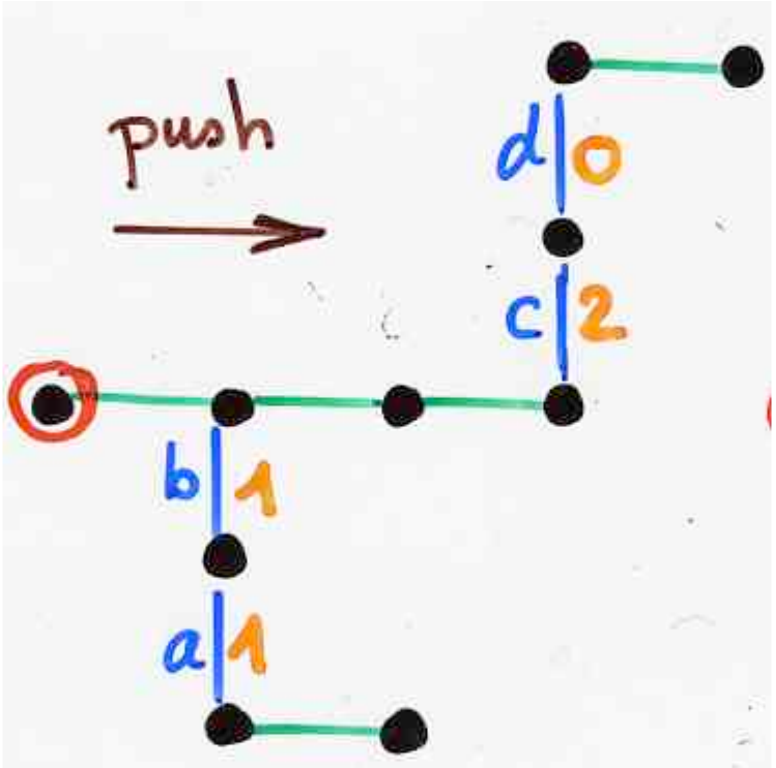
reverse bijection

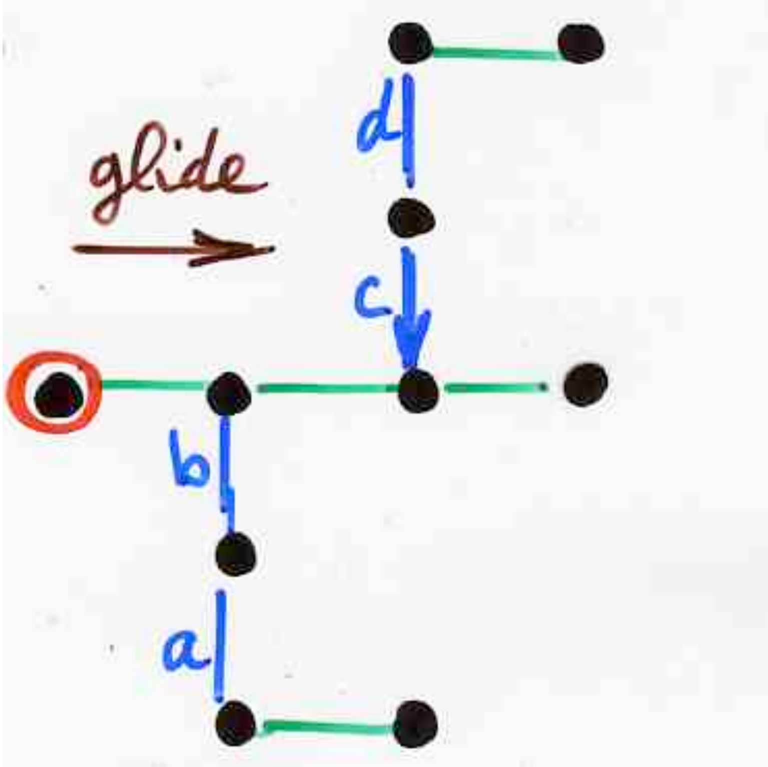
the "push-gliding" algorithm


d
0
c
2
b
1
a
1

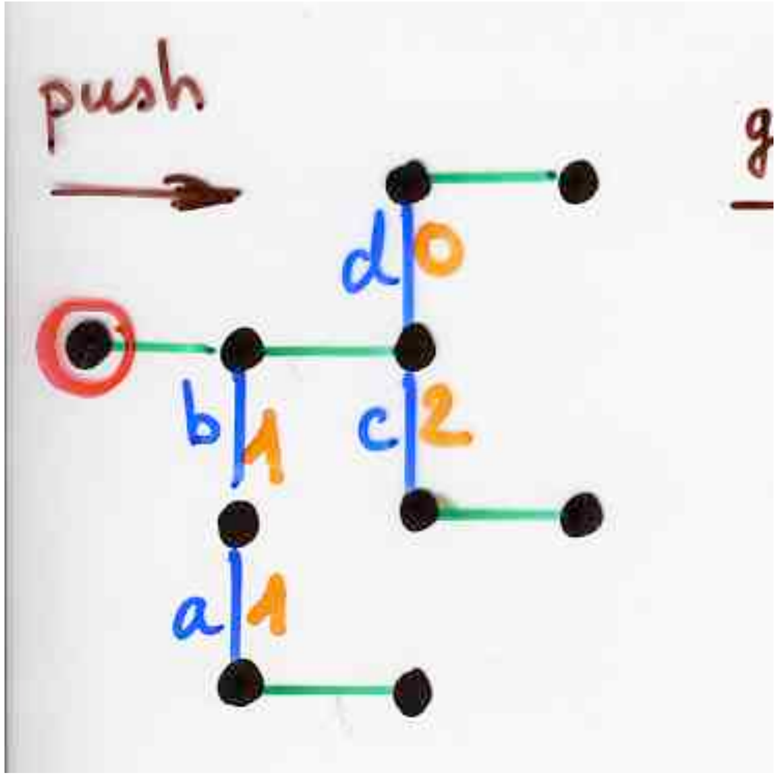


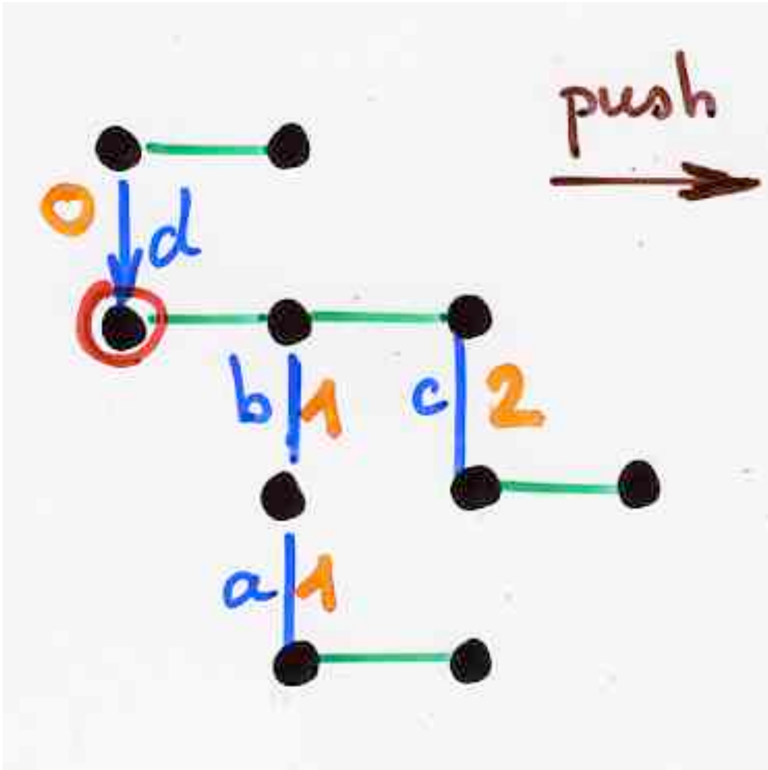
push
d
0
c
2
b
1
a
1

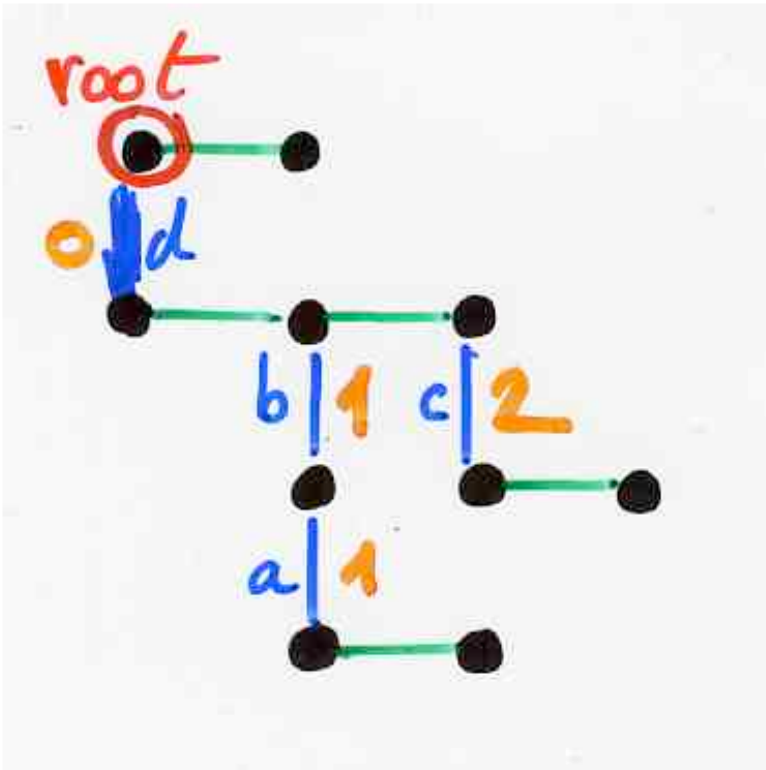

push
d 0
c 2
b 1
a 1

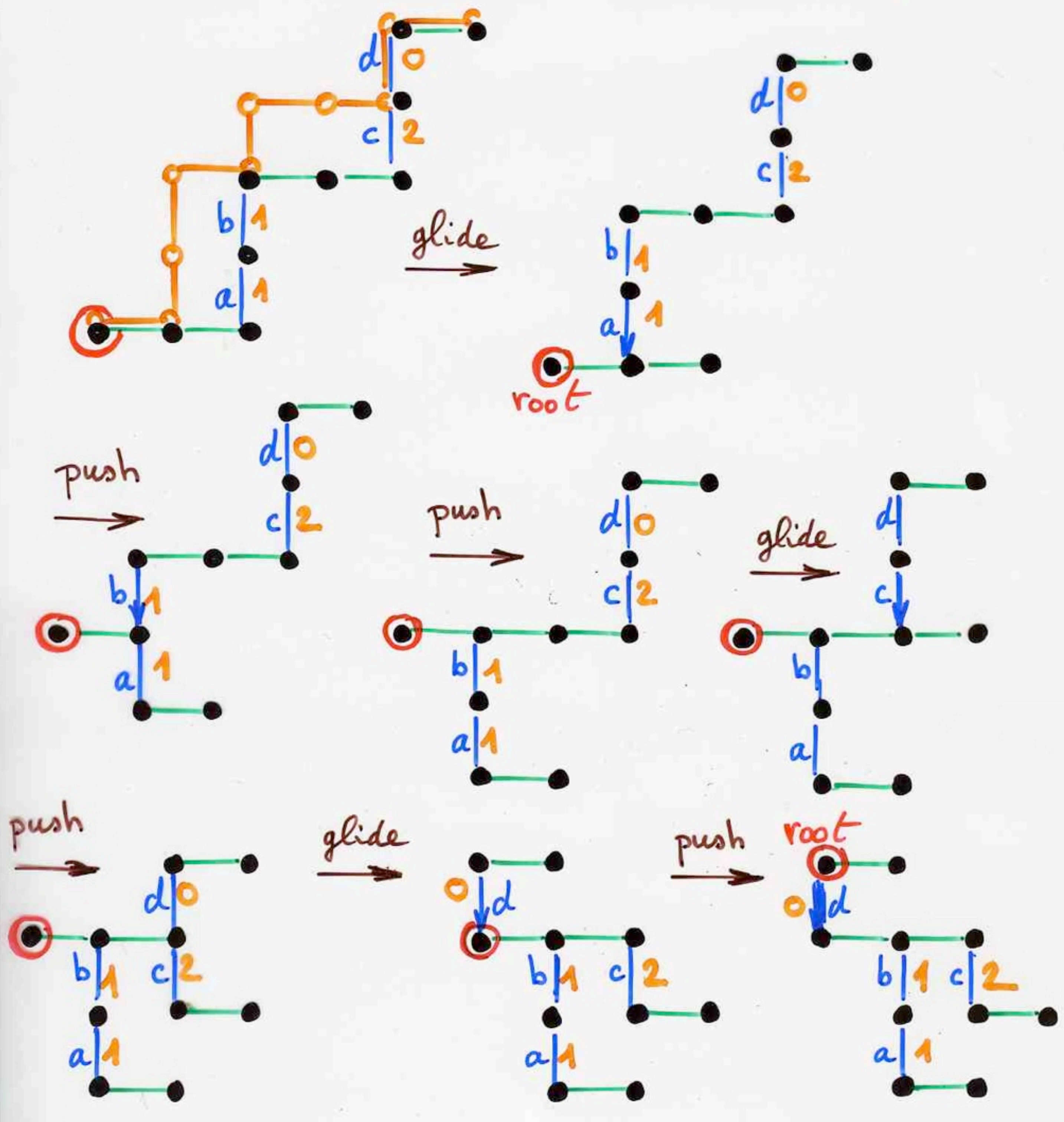


glide
d
c
b
a


push
d 0
b 1
c 2
a 1


push
d
0
b
1
c
2
a
1


root
0
d
b
1
c
2
a
1

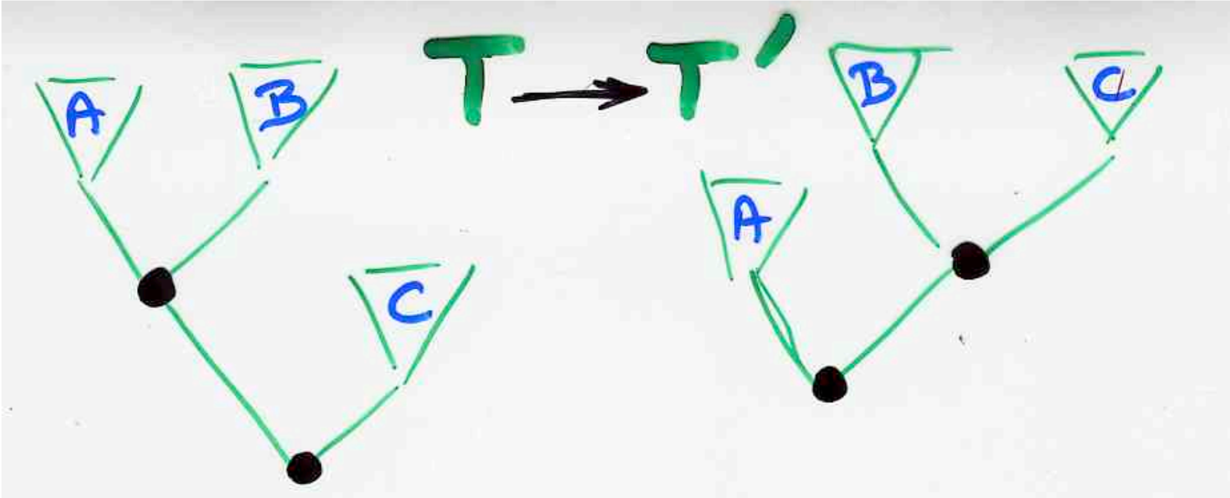


glide
root
push
push
glide
push
glide
push
root

canopy and Tamari lattice

Prop.$^{(i)}$ • The set of binary trees having a given canopy V is an interval $\mathcal{I}$(V) of the Tamari lattice

idea of proof



if $B \neq \bullet$ canopy is invariant

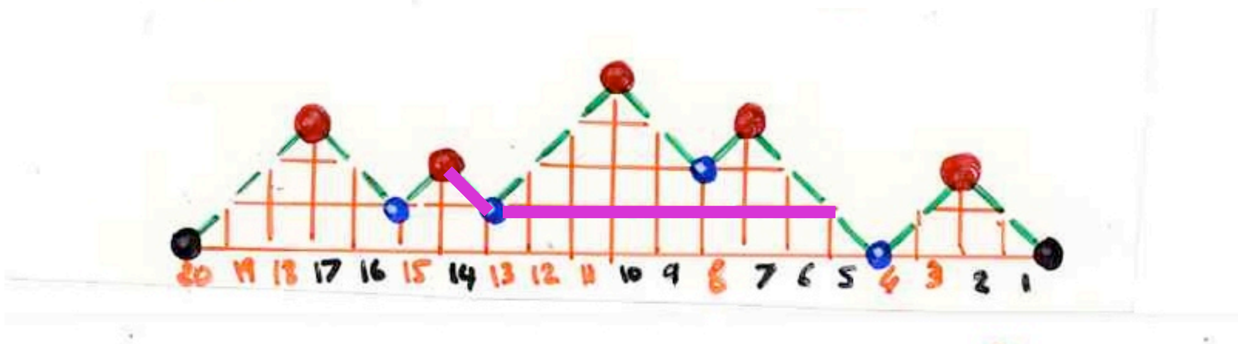
if $B = \bullet$ canopy $c(T')$ not invariant

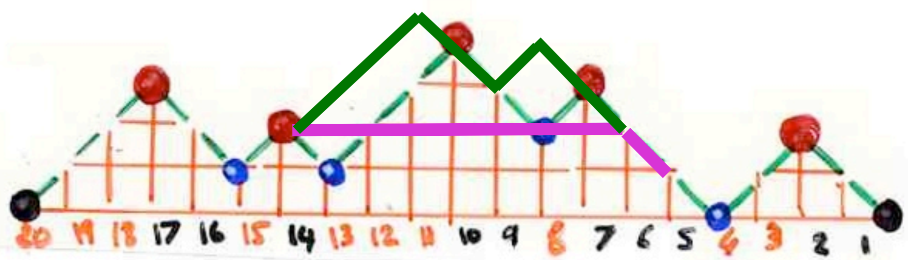
$$c^{*}(T) = c(A) \overset{+}{/} c(B)\, c(C)$$

$$c(T') = c(A) \underset{-}{\backslash} c(B)\, c(C)$$

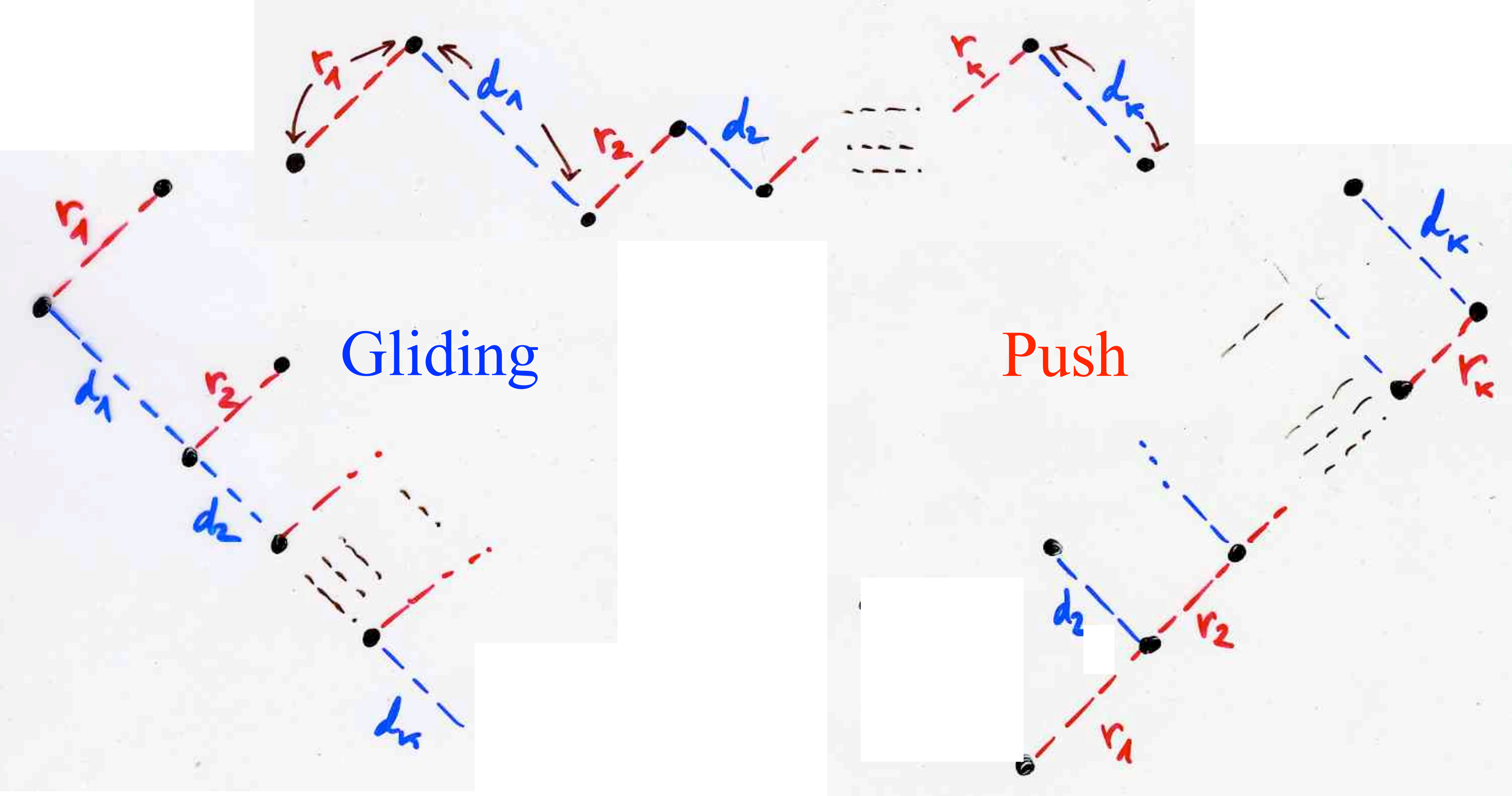
$$c(B) = \varnothing$$

forbidden
move



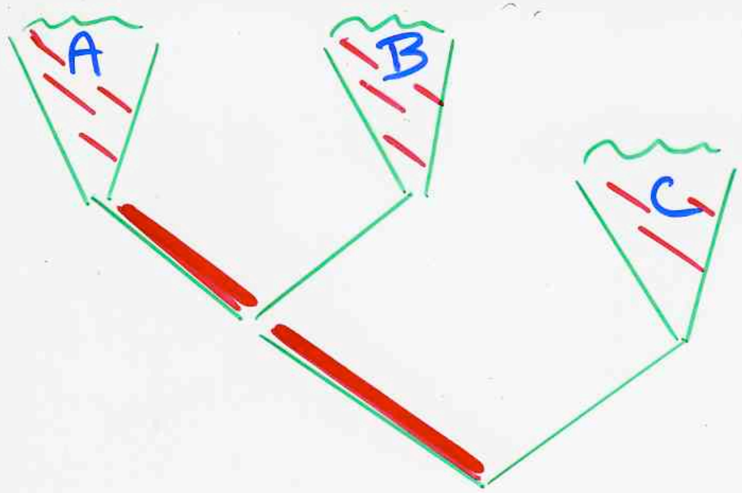

20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

Prop. (i) The set of binary trees having a given canopy w is an interval $I(w)$ of the Tamari lattice

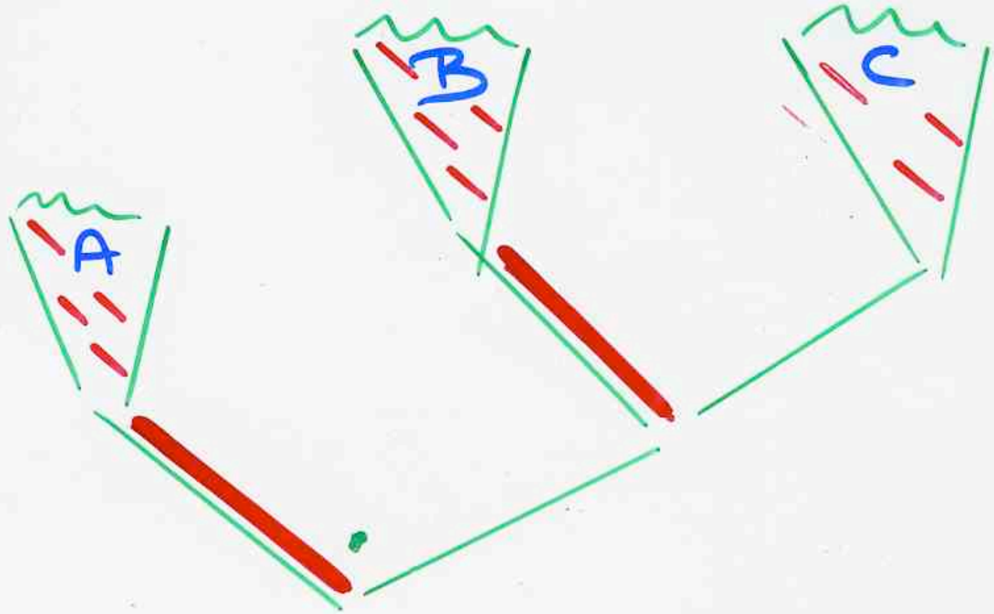


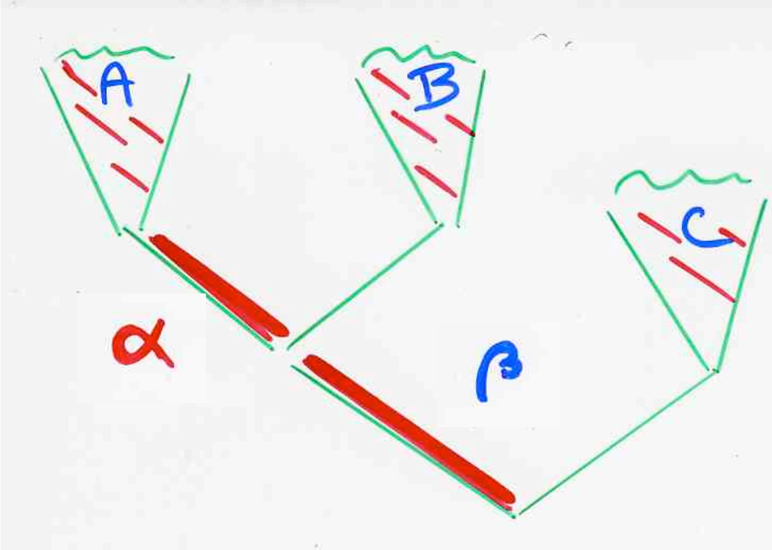
Prop.- (i) The set of binary trees having a given canopy v is an interval $I(v)$ of the Tamari lattice

(ii) This interval $I(v)$ is isomorphic to T_v

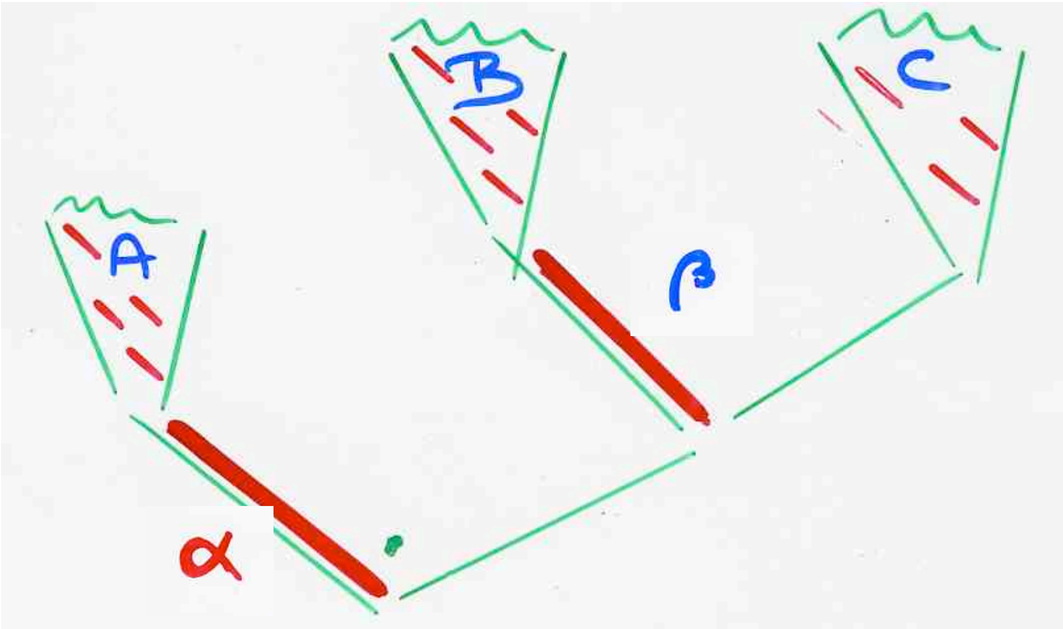


left edges
(in symmetric order)
with their
"right height"



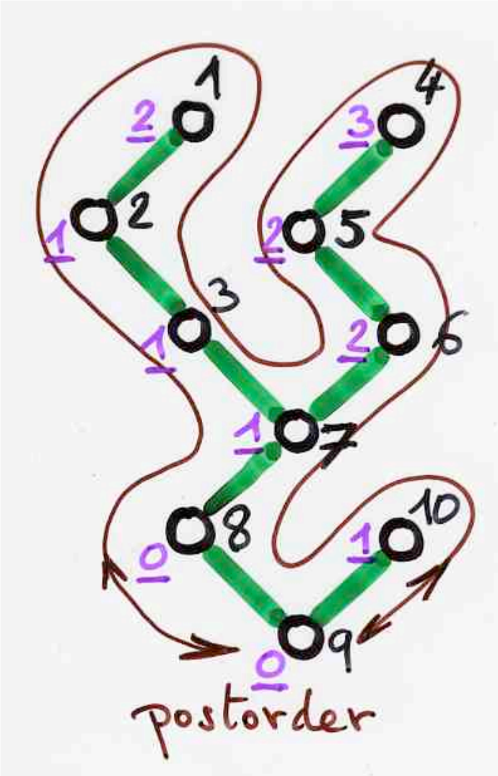


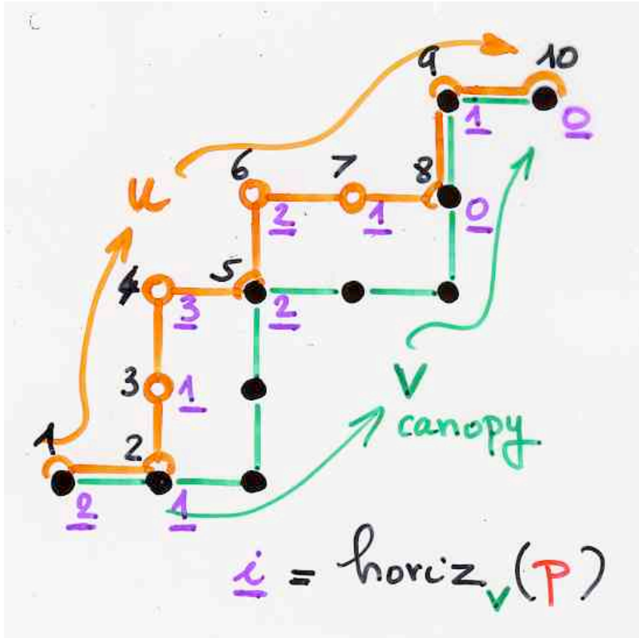
left edges
(in symmetric
order)
with their
"right height"



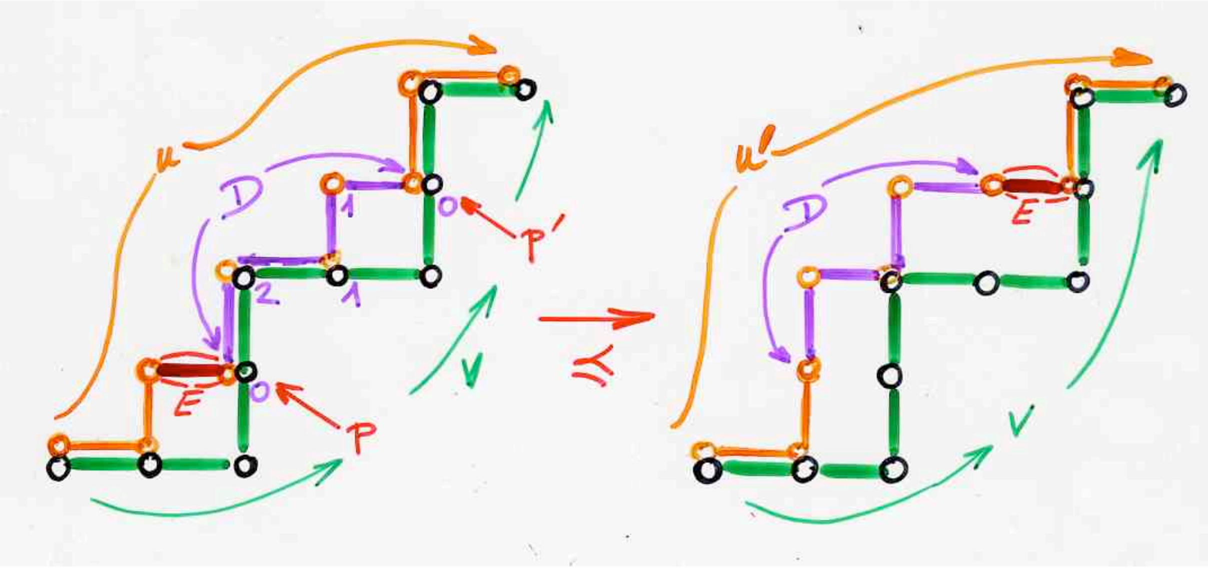
preservation
of the
symmetric order
for left edges

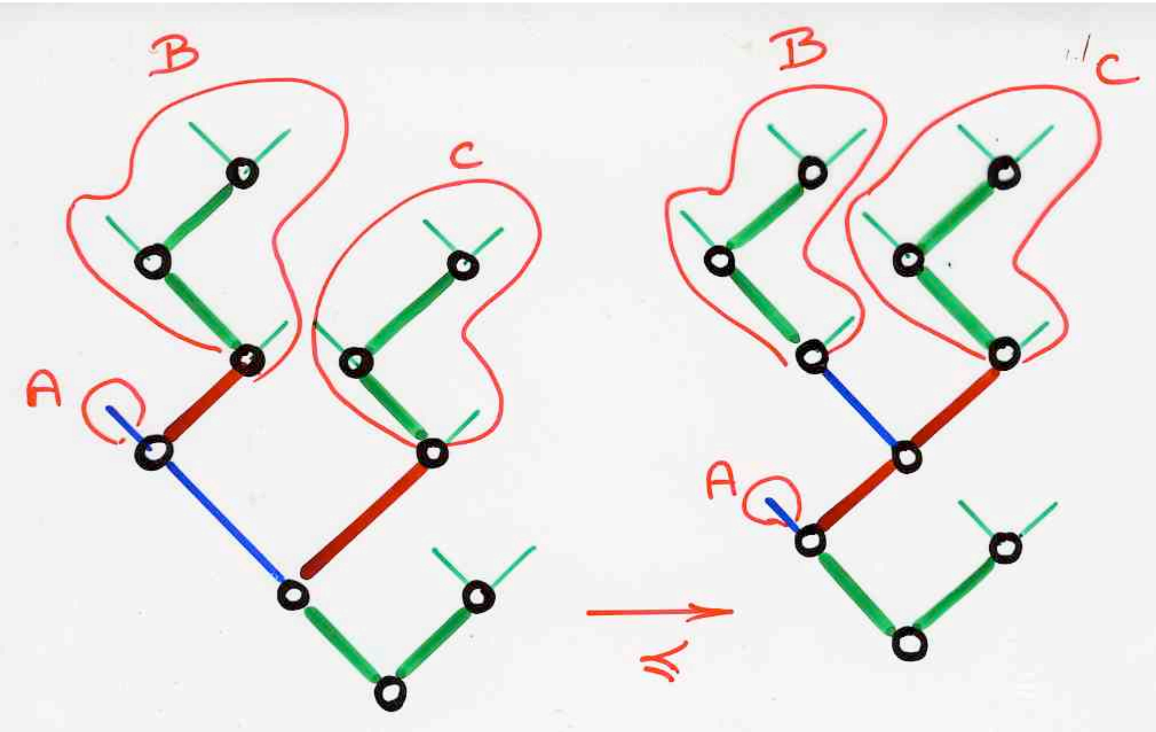
right height:
+1 in C
and for β





another lemma





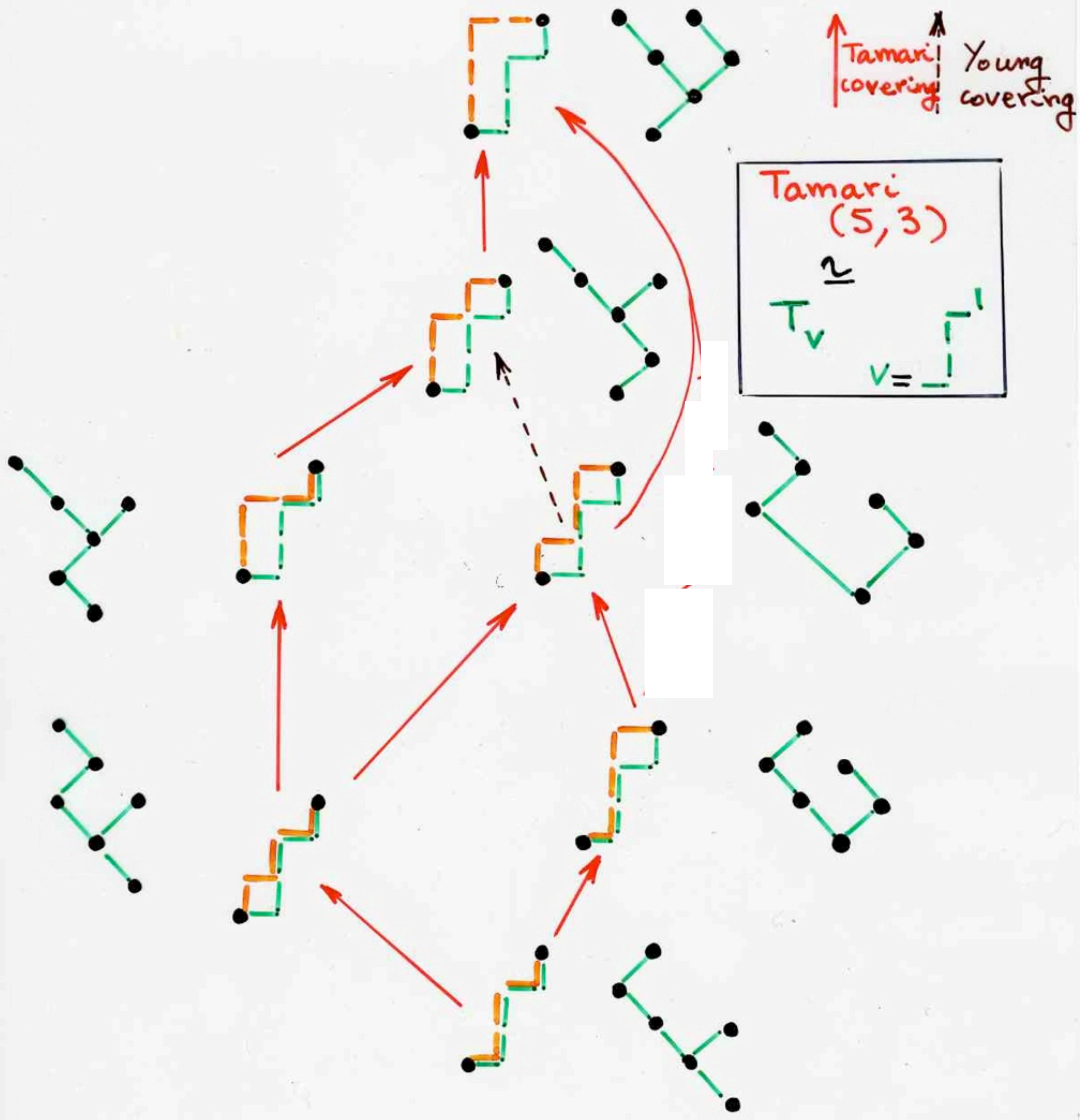
the covering relation in T_V
and the corresponding rotation
in (ordinary) T

Thm 1. For any path v

T_v is a lattice

is a consequence of (ii)

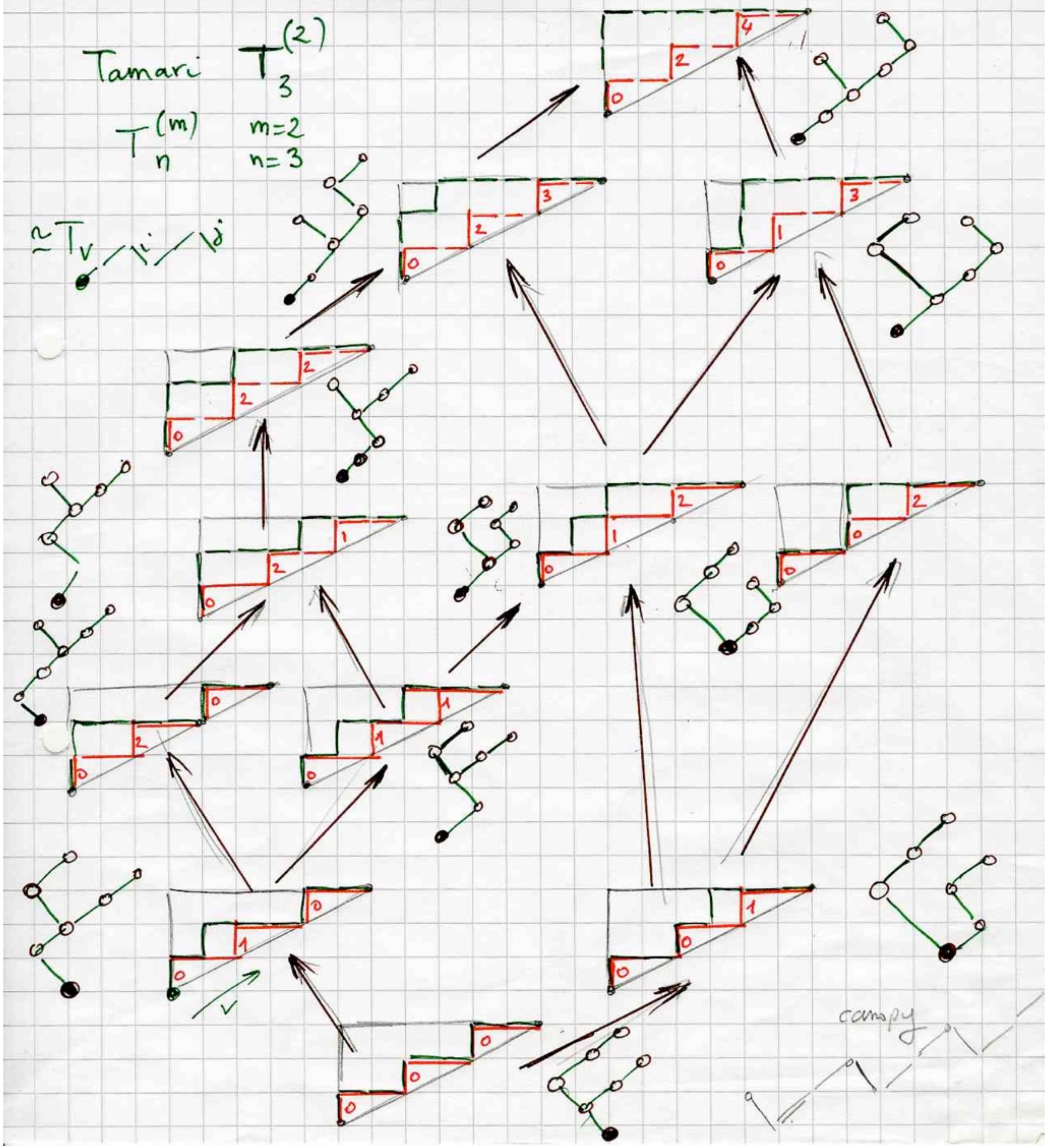
(ii) This interval $I(v)$ is isomorphic to T_v


Tamari covering
Young covering
Tamari (5,3)
≃
T_v
v =

Tamari $T_3^{(2)}$

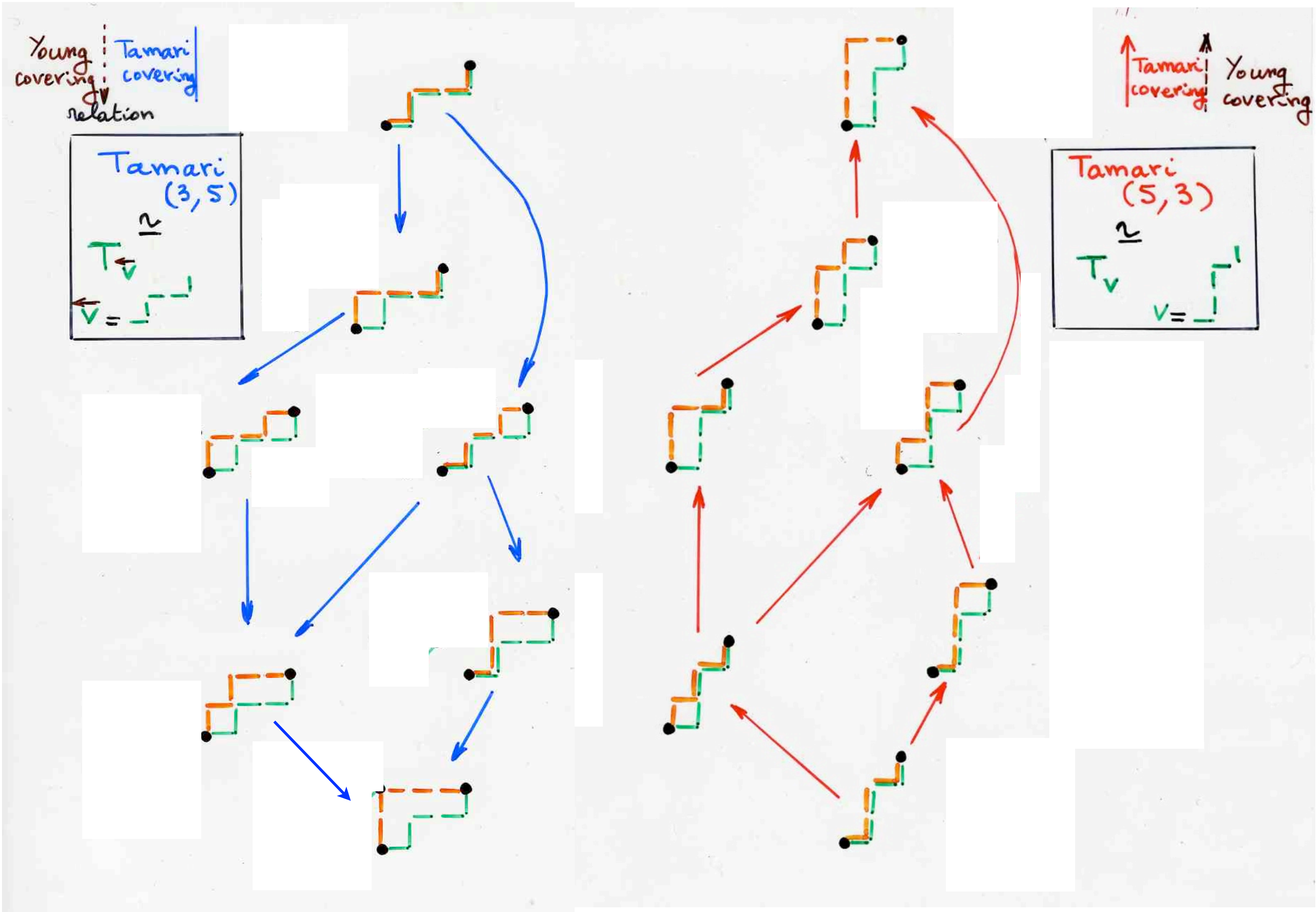
$T_n^{(m)}$ $m=2$ $n=3$

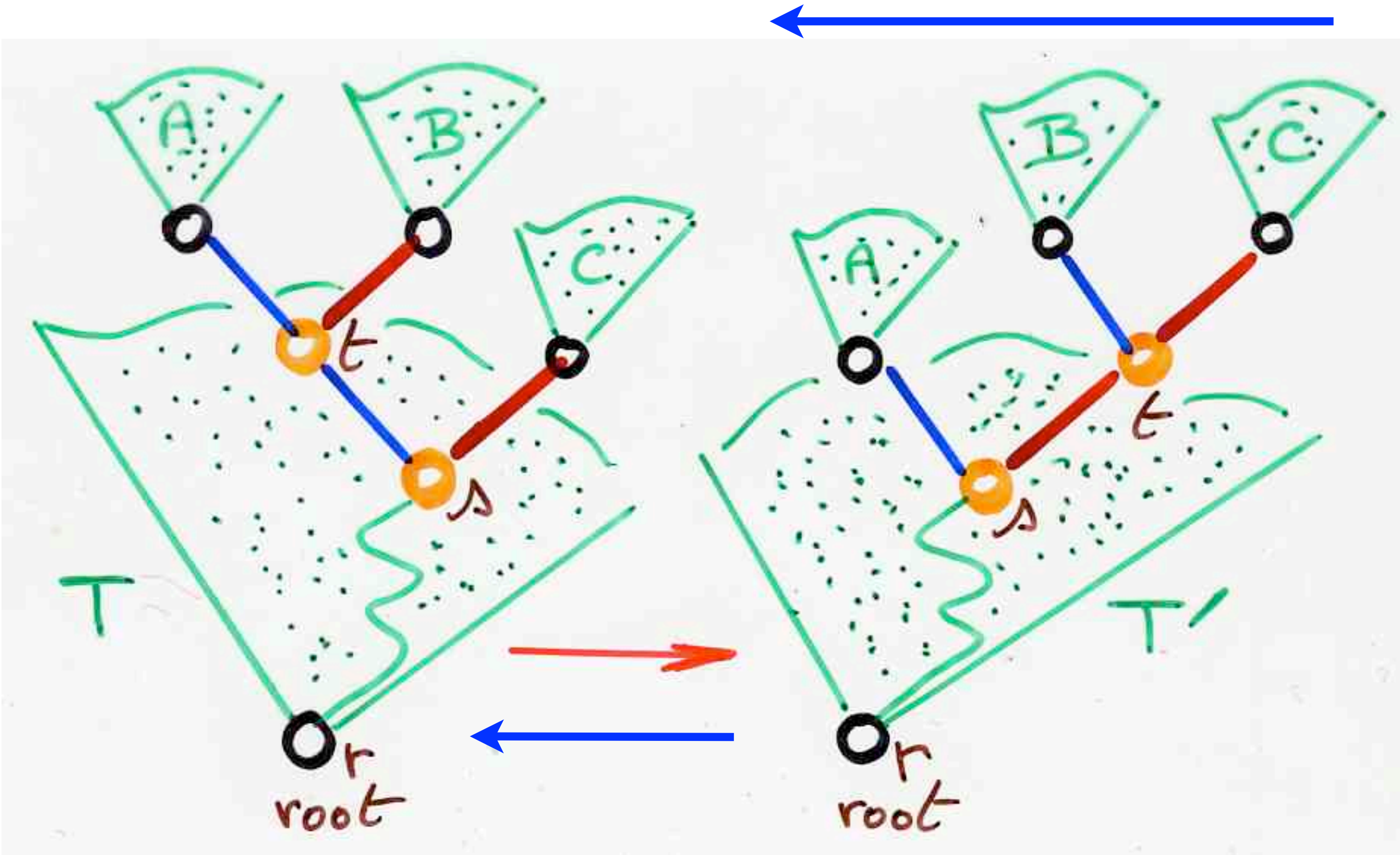
$\simeq T_v$



proof of the duality

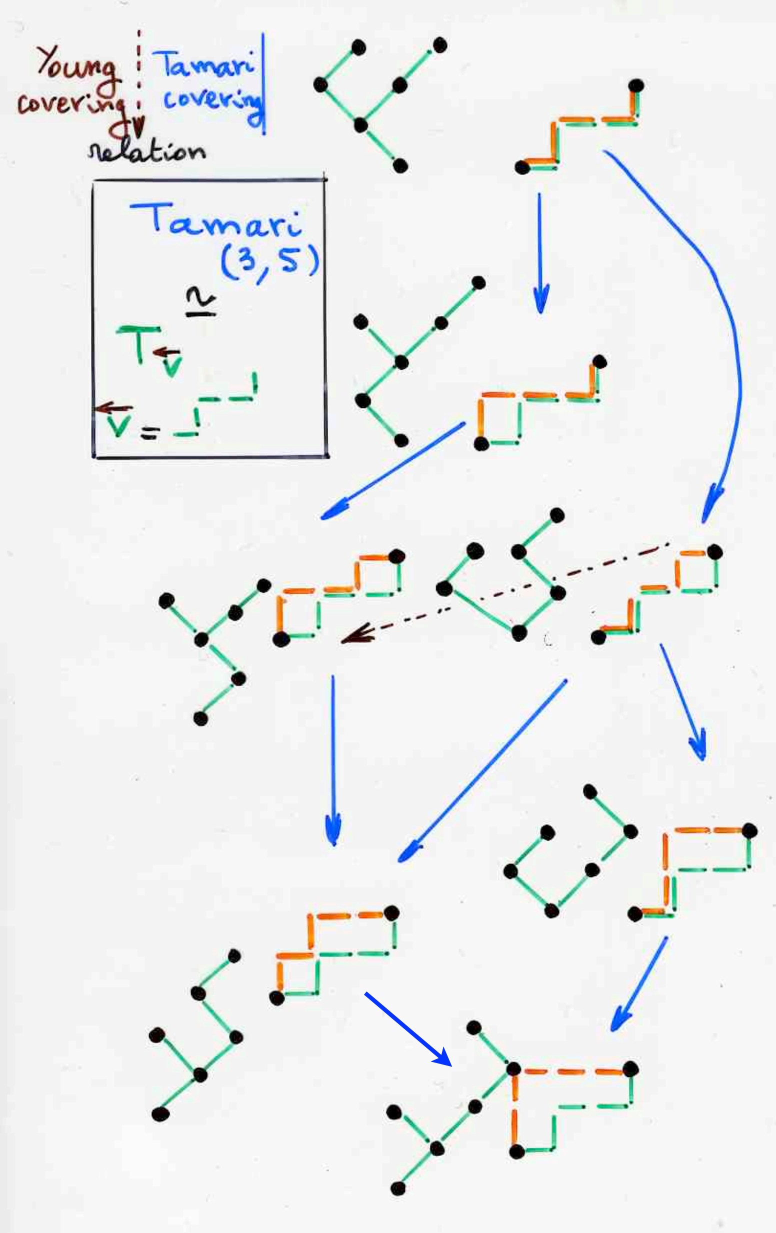
Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$

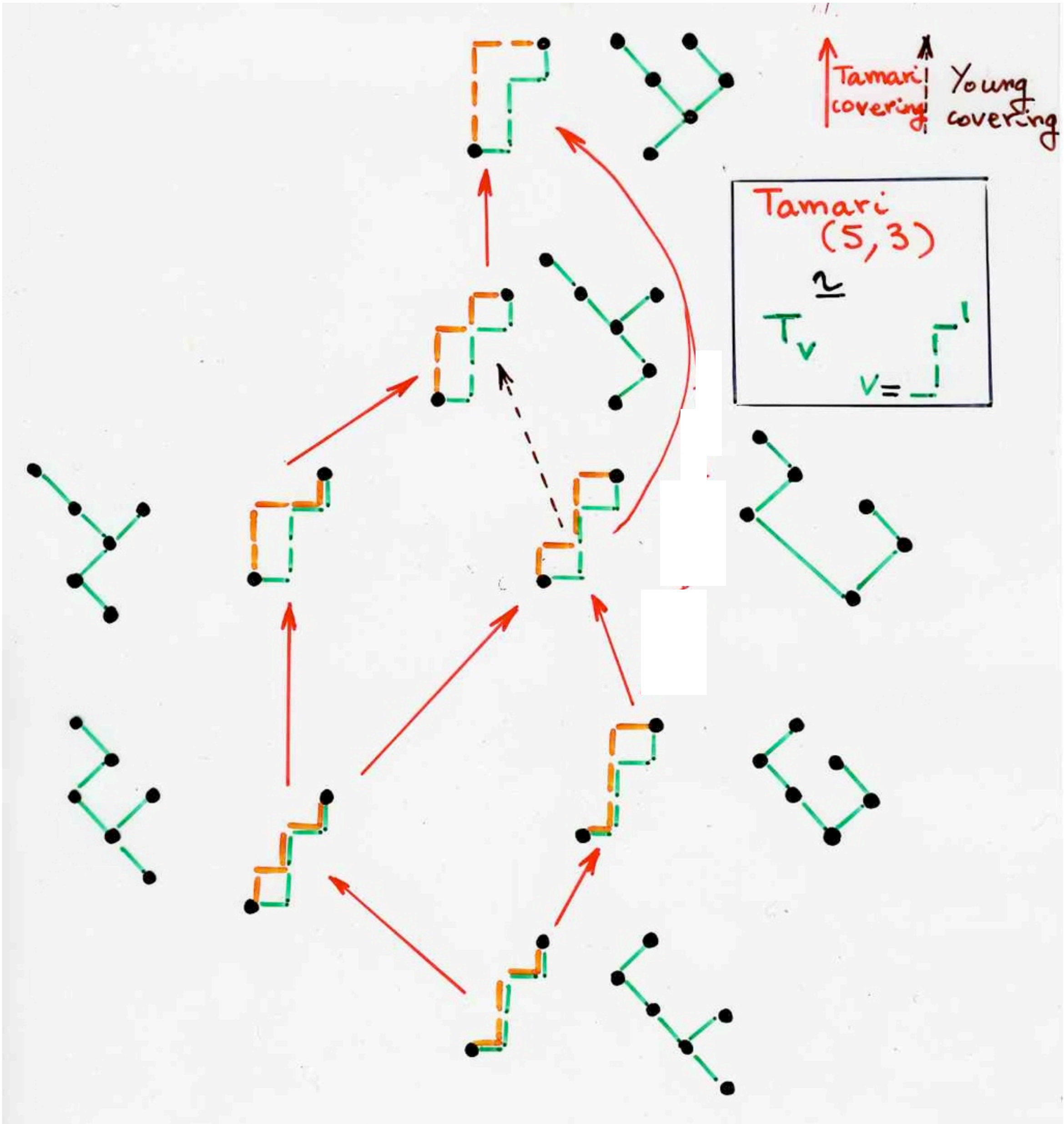


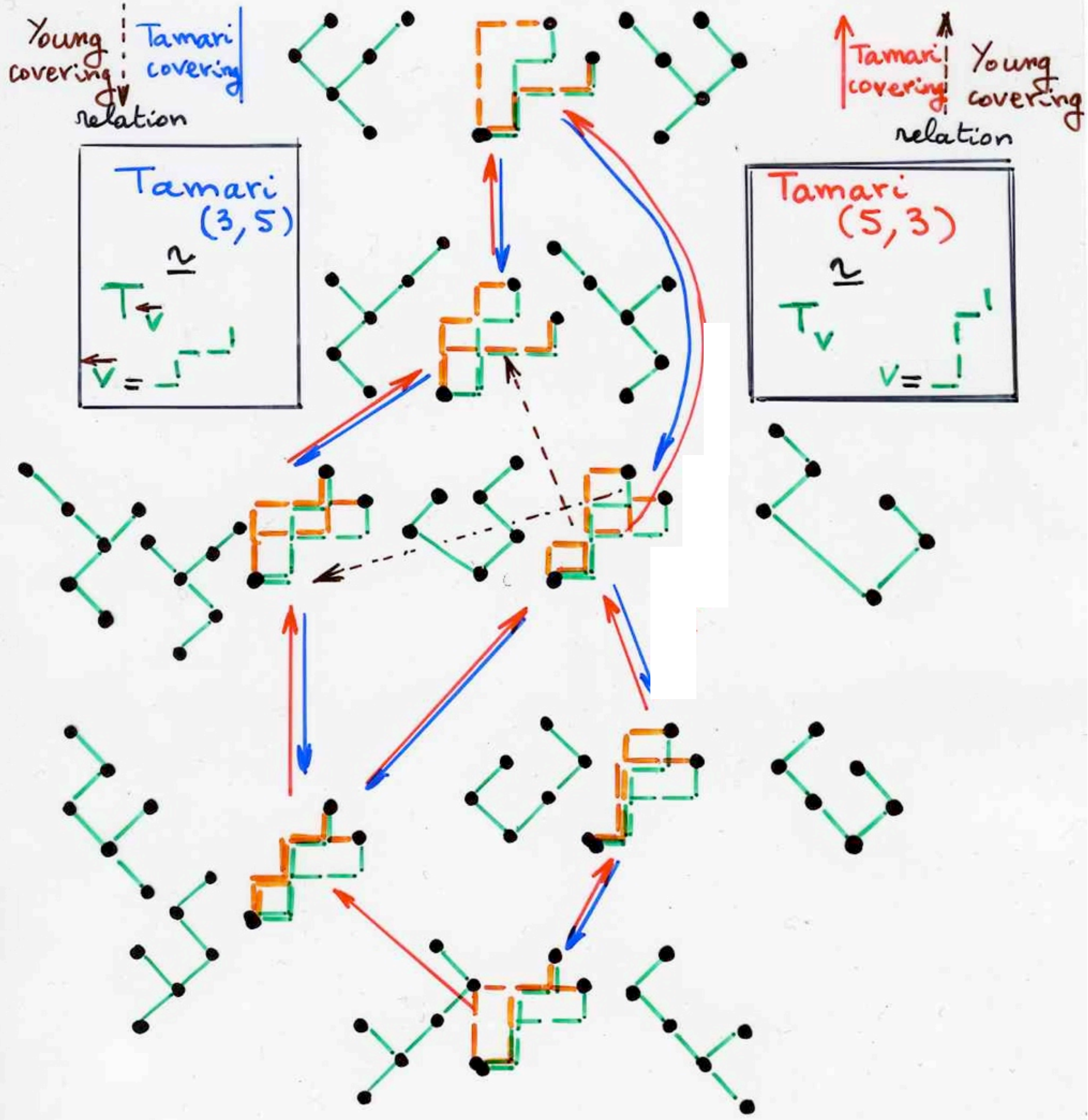


Rotation in a binary tree:
the covering relation in the
Tamari lattice

Duality $T_v \longleftrightarrow T_{\overleftarrow{v}}$







Young covering relation
Tamari covering
Tamari covering
Young covering relation
Tamari (3, 5)
≃
$T_{\overleftarrow{v}}$
$\overleftarrow{v}$ =
Tamari (5, 3)
≃
T_v
v =

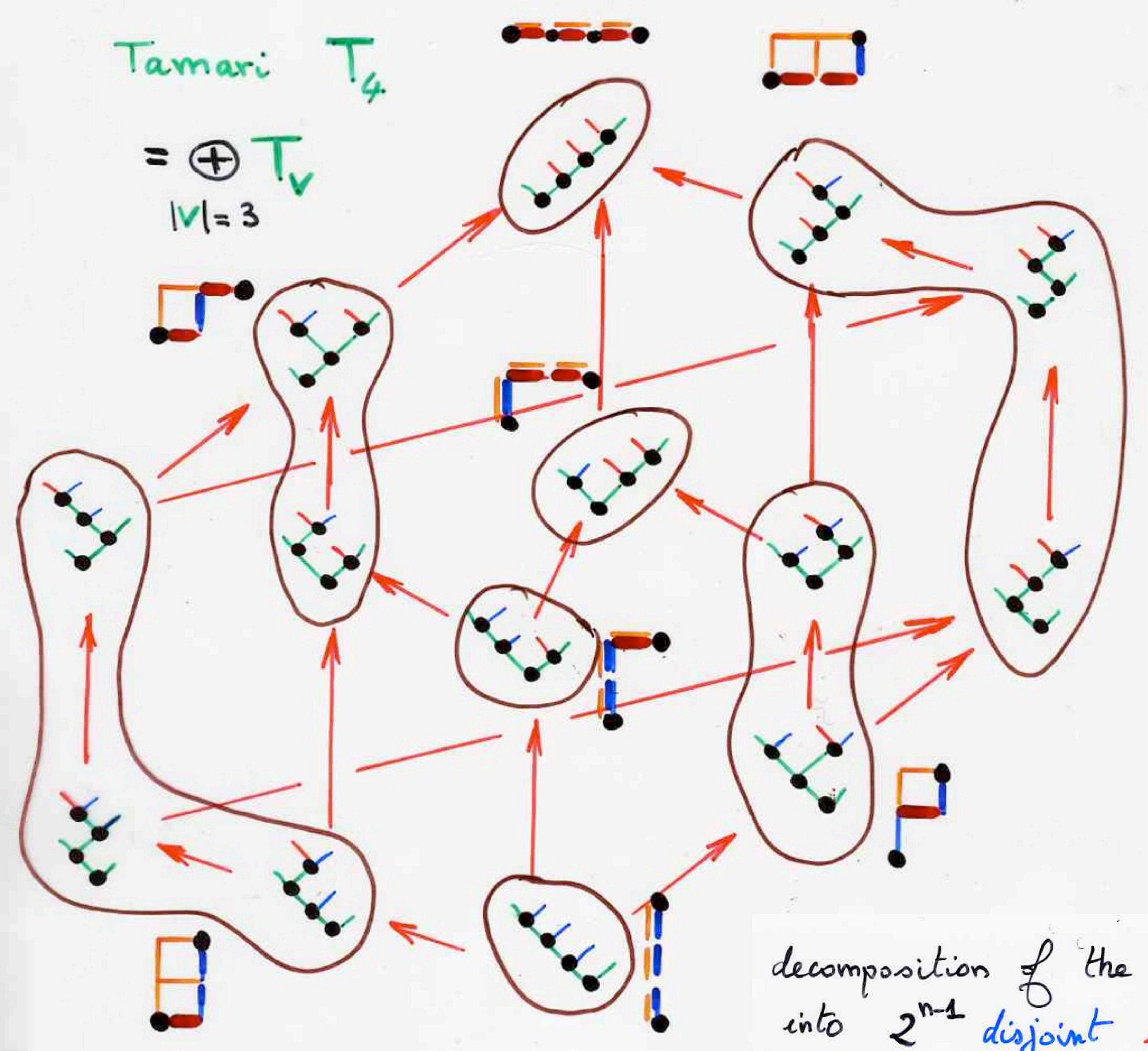
Thm 1. For any path v
T_v is a lattice

Thm 2. The lattice T_v
is isomorphic to the dual of $T_{\overleftarrow{v}}$

Thm 3. The usual Tamari lattice T_n can be partitioned into intervals indexed by the 2^{n-1} paths v of length $(n-1)$ with $\{E, N\}$ steps,

$$T_n \cong \bigsqcup_{|v|=n-1} I_v \,,$$

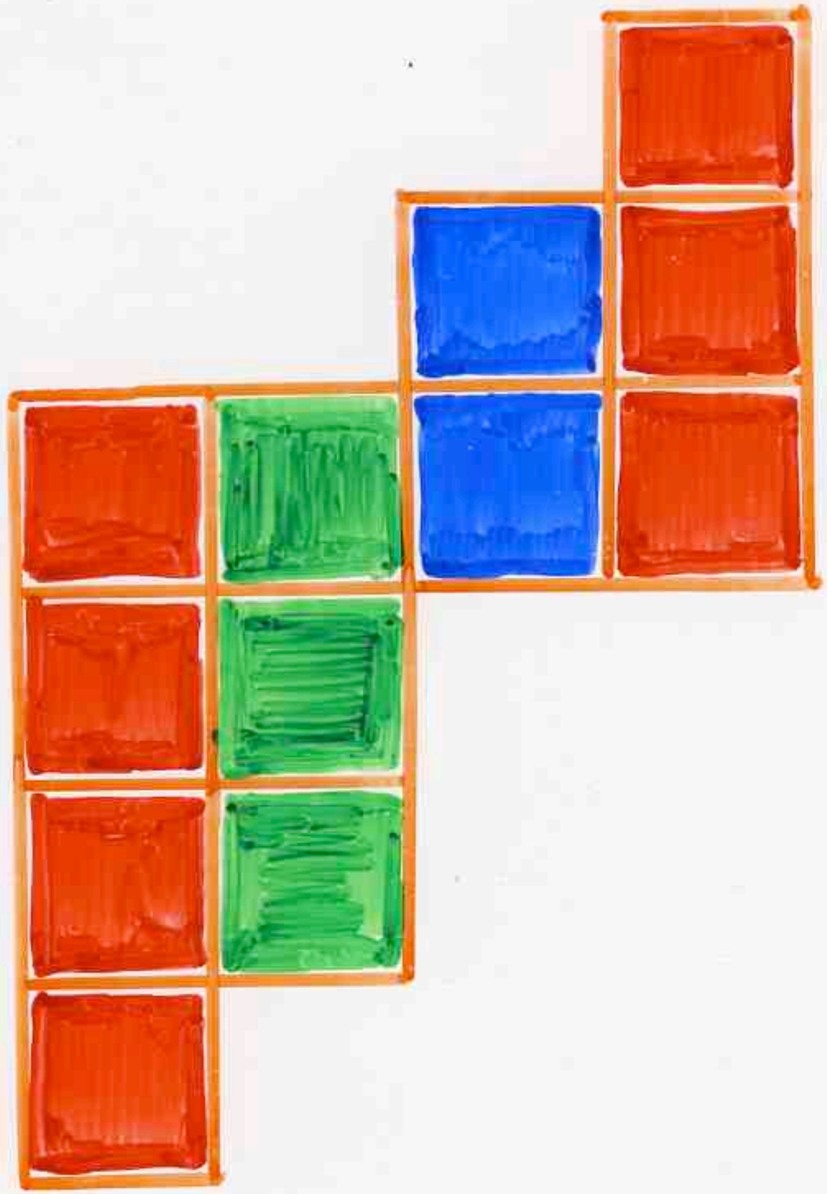
where each $I_v \cong T_v$.

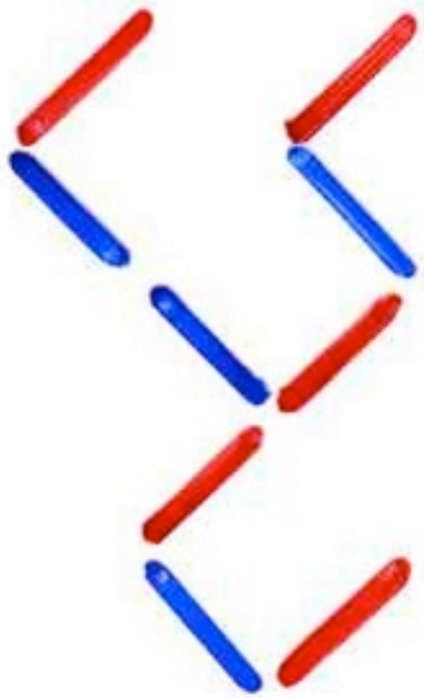


decomposition of the lattice T_n into 2^{n-1} disjoint intervals

Complements

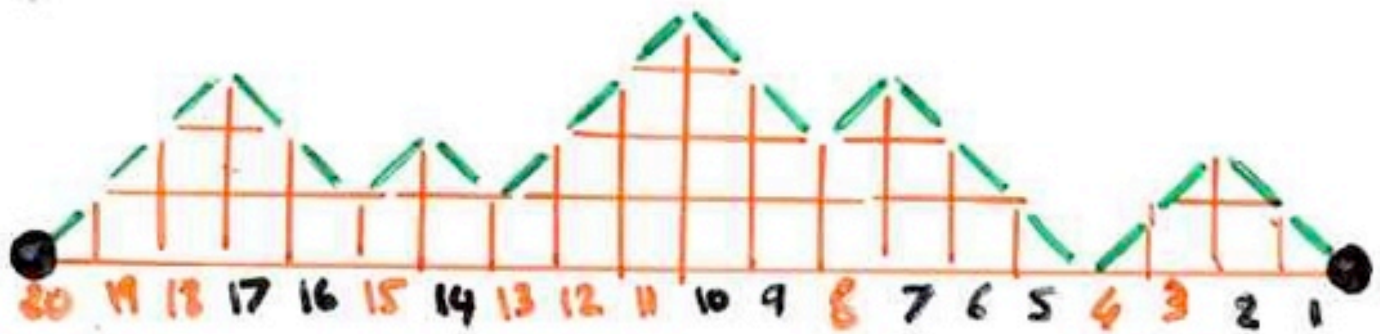
another proof
with staircase polygons

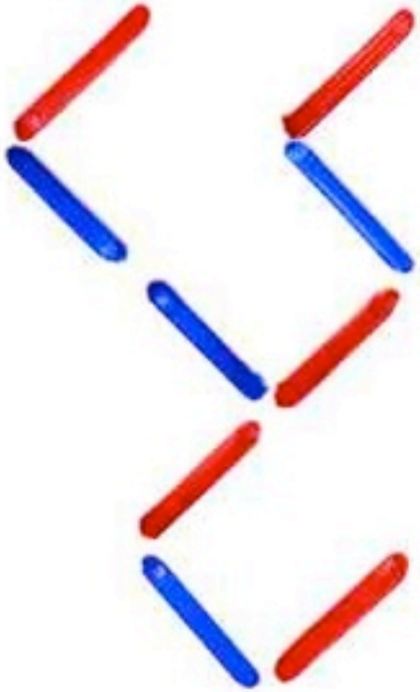




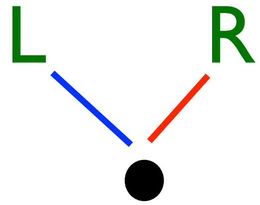
binary
tree

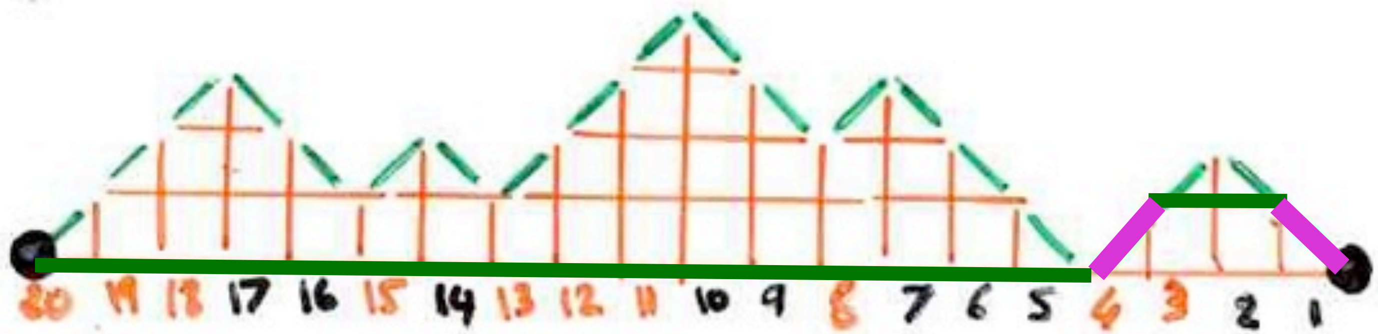
Dyck path

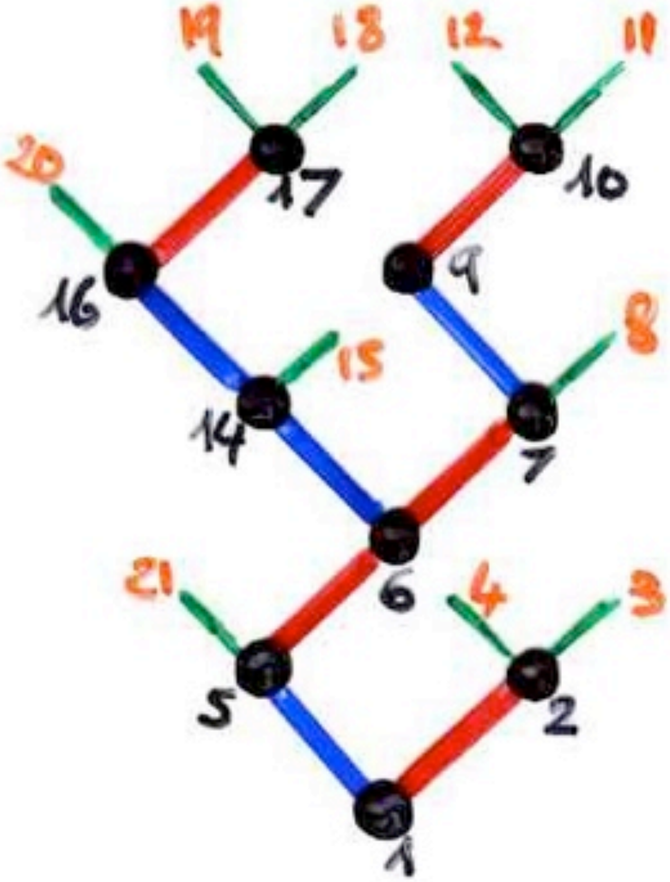




binary
tree



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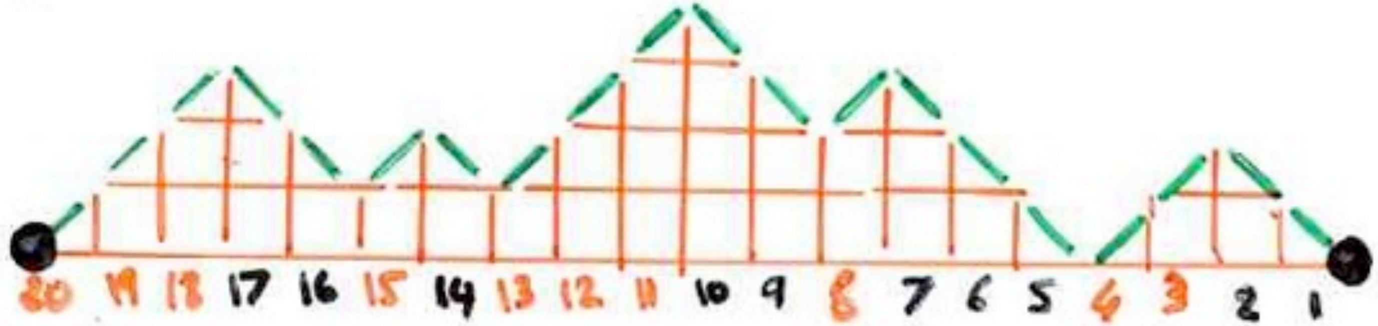
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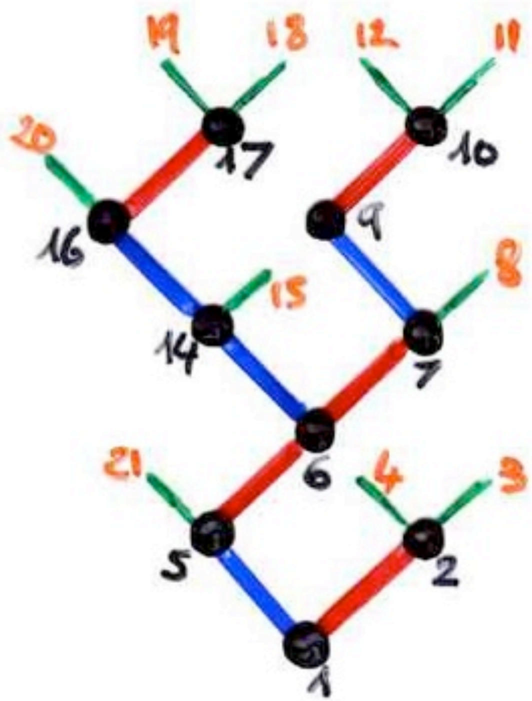

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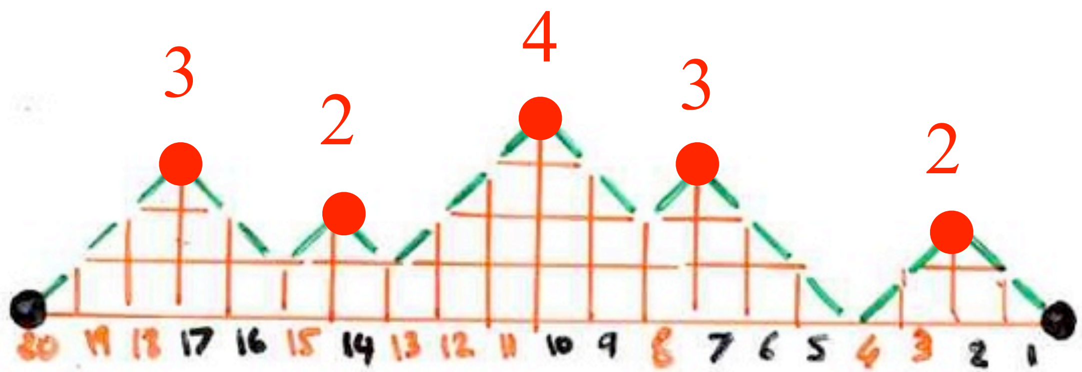

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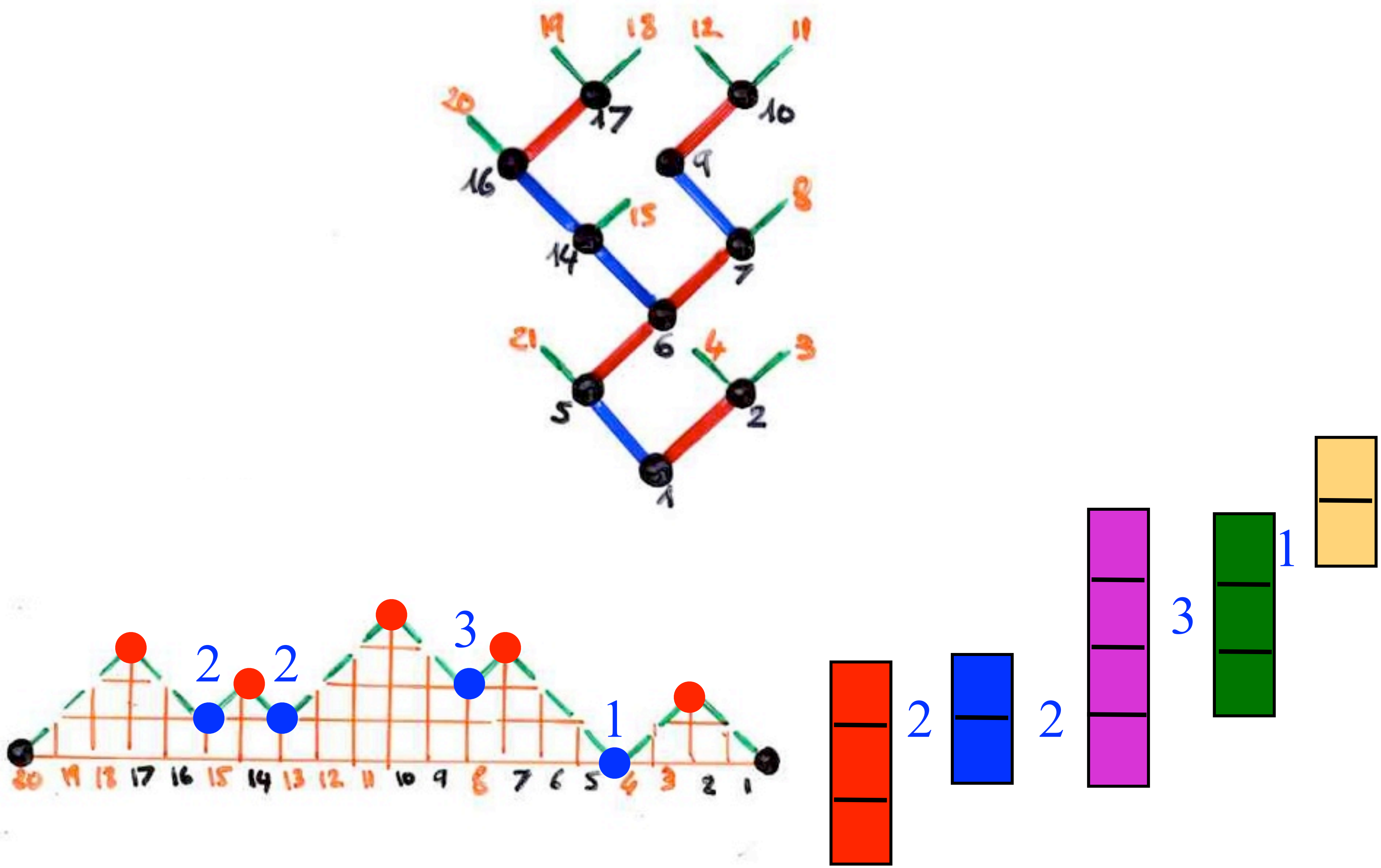
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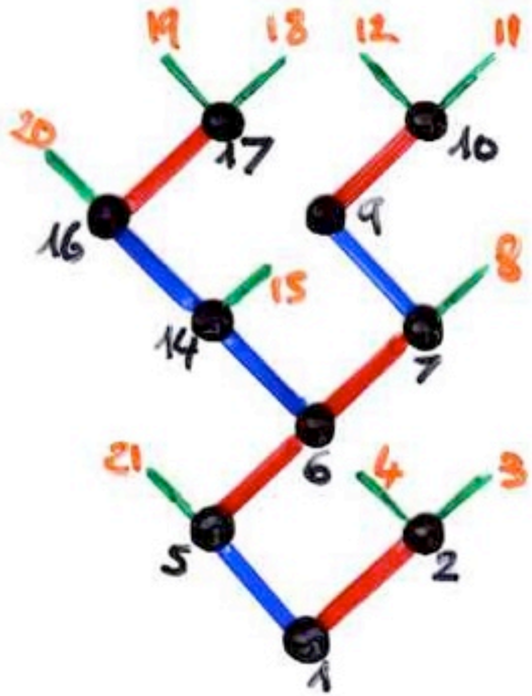
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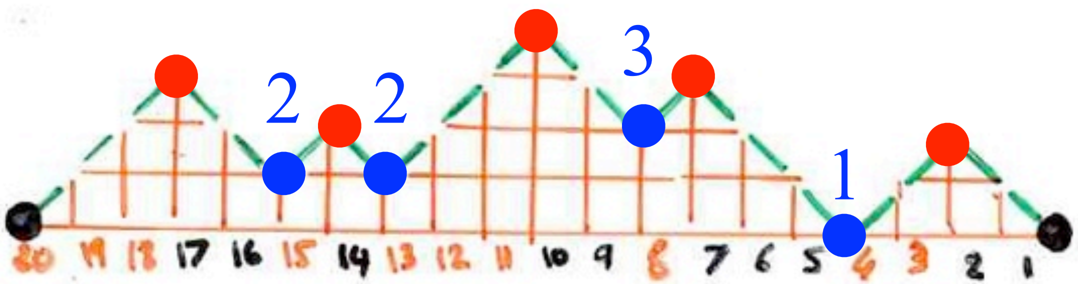
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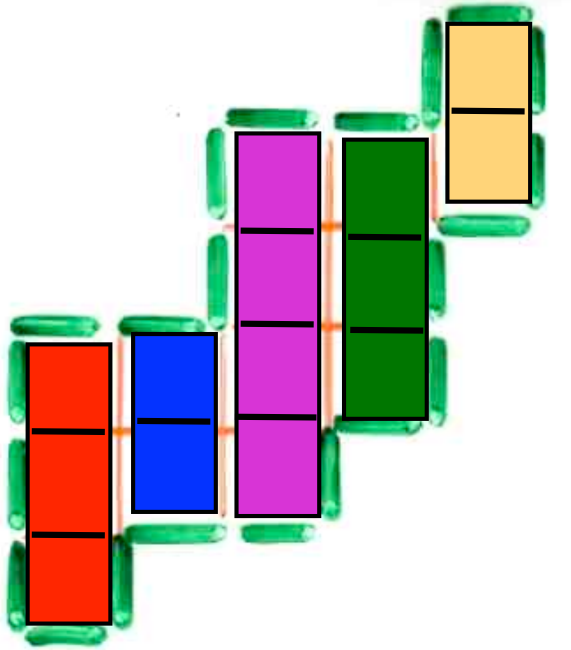

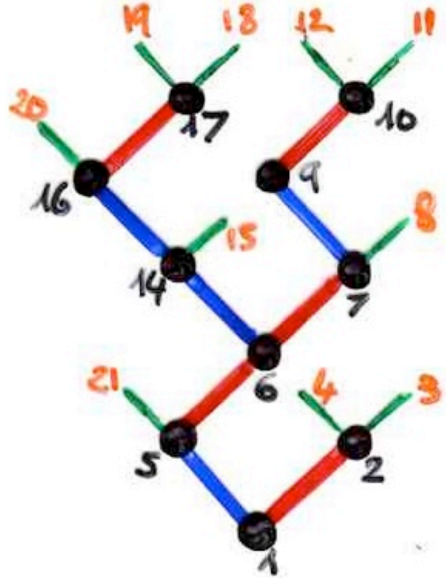
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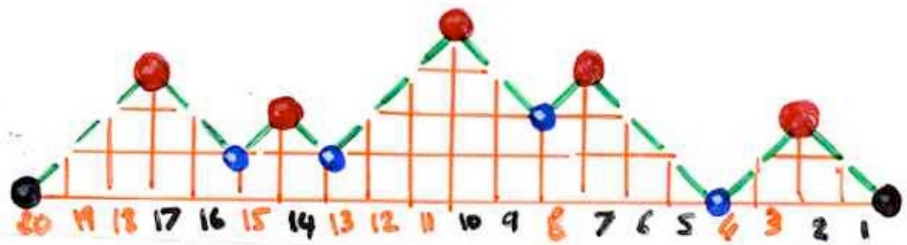
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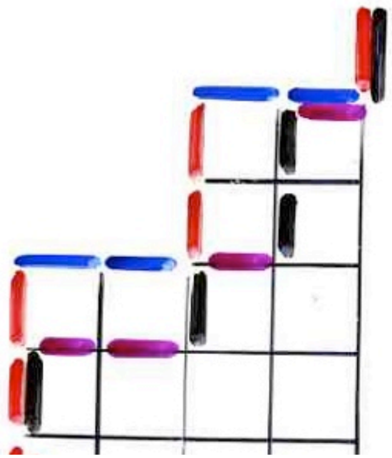
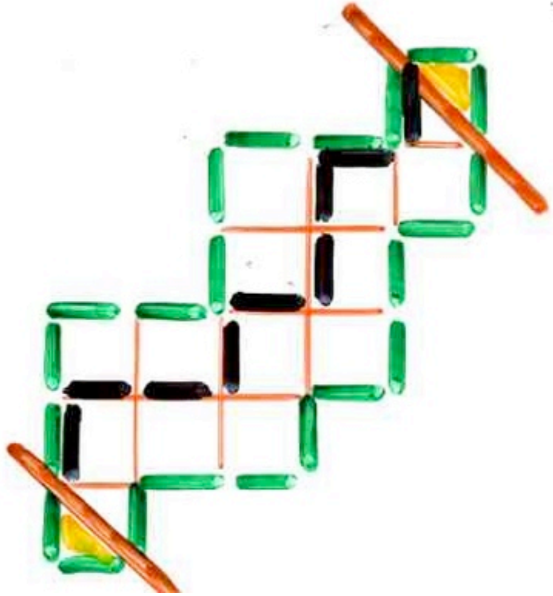
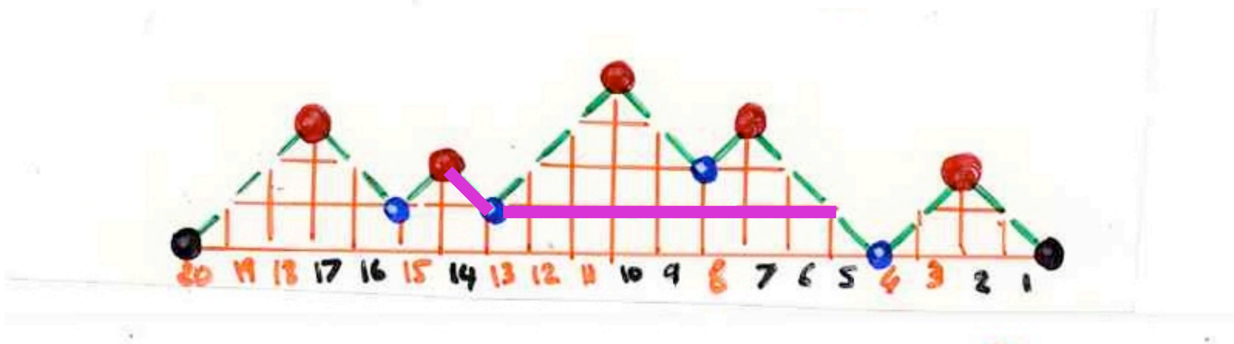
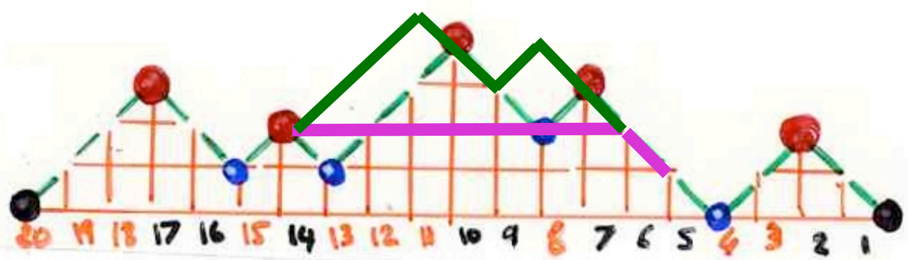




forbidden
move

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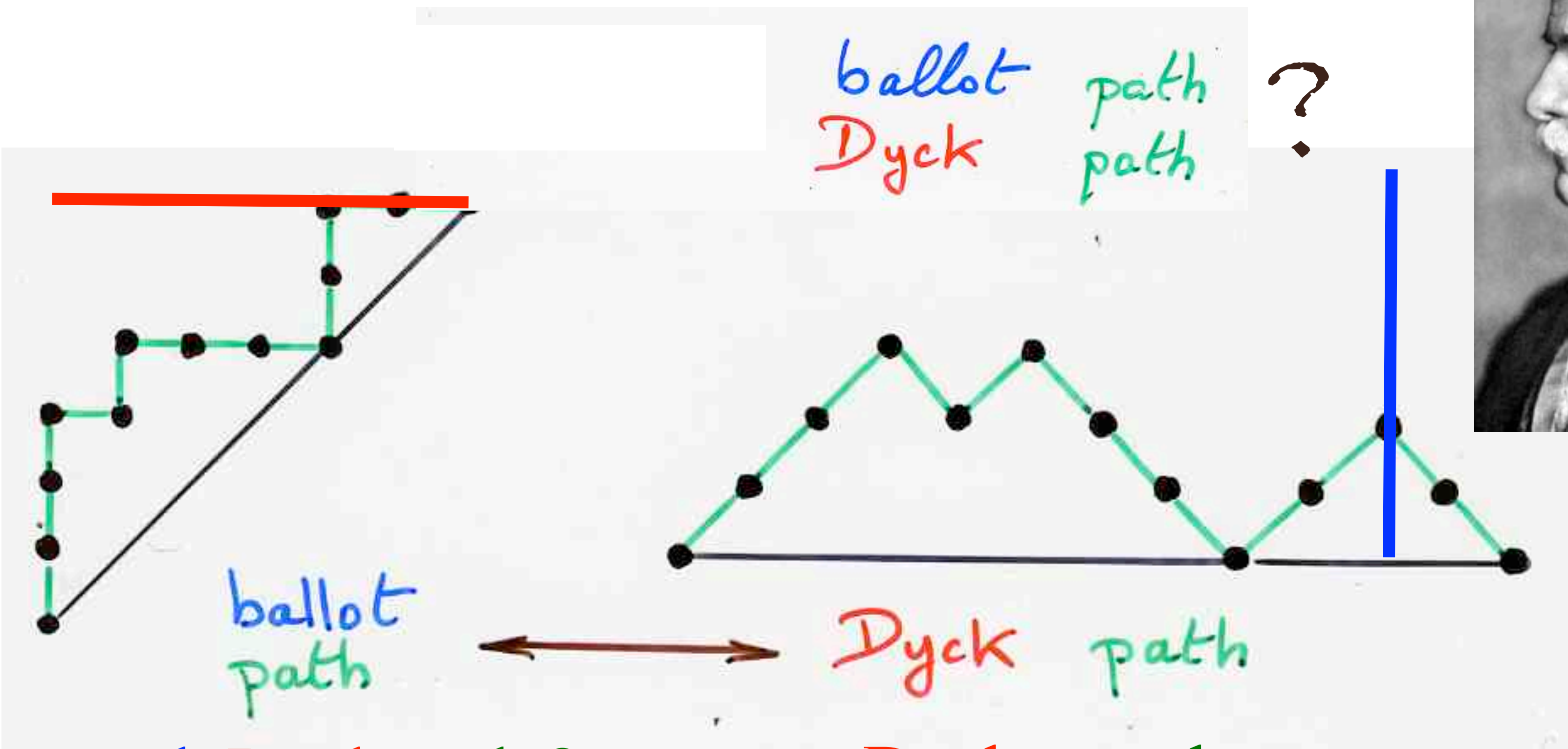
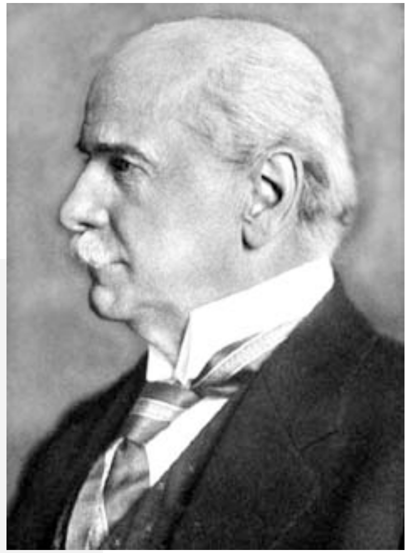
Complements

Question of vocabulary





Von Dyck

rotated Dyck path ?

Dyck words

Bertrand
D. André
problème du scrutin
(188x)

ballot numbers

G.Kerweras «bridge»
D.Dumont «contraction»

$$A_{m,-n} =$$

$$\pm \mathfrak{C}^{l}\{A_{0,0}.\mathfrak{C}^{n}\mathfrak{D}^{m-n}.\mathfrak{C}^{l-n-1} - A_{0,1}.\mathfrak{C}^{n+1}\mathfrak{D}^{m-n-1}.\mathfrak{C}^{l-n-2} + A_{0,2}.\mathfrak{C}^{n+2}\mathfrak{D}^{m-n-2}.\mathfrak{C}^{l-n-3} - \text{etc.}\}.$$

D'où il suit que $A_{m,-n}$ n'est zéro qu'autant que m est $< n$. Ainsi la série récurrente s'étend, sous forme de triangle, dans la quatrième région.

Exemple VI.

246. Étant donné le commencement de la table suivante, où chaque terme est formé de la somme de celui qui le précède dans la même ligne horizontale et de celui qui le suit d'un rang dans la ligne horizontale immédiatement supérieure, avec la condition que chacun des termes de la première ligne horizontale soit égal à l'unité : on demande le terme général de cette table :

1	1	1	1	1	etc.
1	2	3	4	5	etc.
2	5	9	14	20	etc.
5	14	28	48	75	etc.
14	42	90	165	275	etc.
etc.	etc.	etc.	etc.	etc.	etc.

L'équation de relation est $A_{m,n} = A_{m-1,n} + A_{m+1,n-1}$; j'y mets $m - 1$ au lieu de m, et elle devient

ballot numbers in 1800 (!)

DU CALCUL

DES

DÉRIVATIONS;

PAR L. F. A. ARBOGAST,

De l'Institut national de France, Professeur de Mathématiques à Strasbourg.

A STRASBOURG,

DE L'IMPRIMERIE DE LEVRAULT, FRÈRES.

AN VIII (1800).

Donc on a enfin

$$A_{m,n} = \qquad \ldots\ldots(3)$$

$$\pm \beta^{l-m}\{A_{0,n}.\gamma'^{m} + A_{0,n+1}.m\gamma'^{m-1}\beta + A_{0,n-2}.\frac{m.m-1}{1.\ 2}\gamma'^{m-2}\beta^{2}$$
$$+ \text{etc.} + A_{0,n+m-1}.m\gamma'\beta^{m-1} + A_{0,n+m}.\beta^{m}\},$$

le signe supérieur ou inférieur ayant lieu suivant que m est pair ou impair. Ce résultat s'accorde avec celui que l'on peut déduire d'une solution différente du même exemple, donnée par LAPLACE dans les mémoires de Paris, année 1779, n.° XVII, page 267.

Si l'on fait n négatif dans la formule (1) ci-dessus, on trouve, en rejetant les termes et celles de leurs parties où les indices de D sont négatifs et ceux de D' négatifs ou positifs > 0, que cette formule se réduit à la suivante :

$$A_{m,-n} = \qquad \ldots\ldots(4)$$

$$\pm \beta'\{A_{0,0}\underset{c}{\mathrm{D}}^{m}.(\alpha^{n}.\beta^{l-n-1}) - A_{0,1}\underset{c}{\mathrm{D}}^{m}.(\alpha^{n+1}.\beta^{l-n-2}) + A_{0,2}\underset{c}{\mathrm{D}}^{m}.(\alpha^{n+2}.\beta^{l-n-3}) - \text{etc.}\}$$

laquelle, à cause que $\alpha = 0$ et que sa seule dérivée D est β, devient

$$A_{m,-n} = \qquad \ldots\ldots(5)$$

$$\pm \beta'\{A_{0,0}.\beta^{n}\underset{c}{\mathrm{D}}^{m-n}.\beta^{l-n-1} - A_{0,1}.\beta^{n+1}\underset{c}{\mathrm{D}}^{m-n-1}.\beta^{l-n-2} + A_{0,2}.\beta^{n+2}\underset{c}{\mathrm{D}}^{m-n-2}.\beta^{l-n-3} - \text{etc.}\}.$$

D'où il suit que $A_{m,-n}$ n'est zéro qu'autant que m est $< n$. Ainsi la série récurrente s'étend, sous forme de triangle, dans la quatrième région.

EXEMPLE VI.

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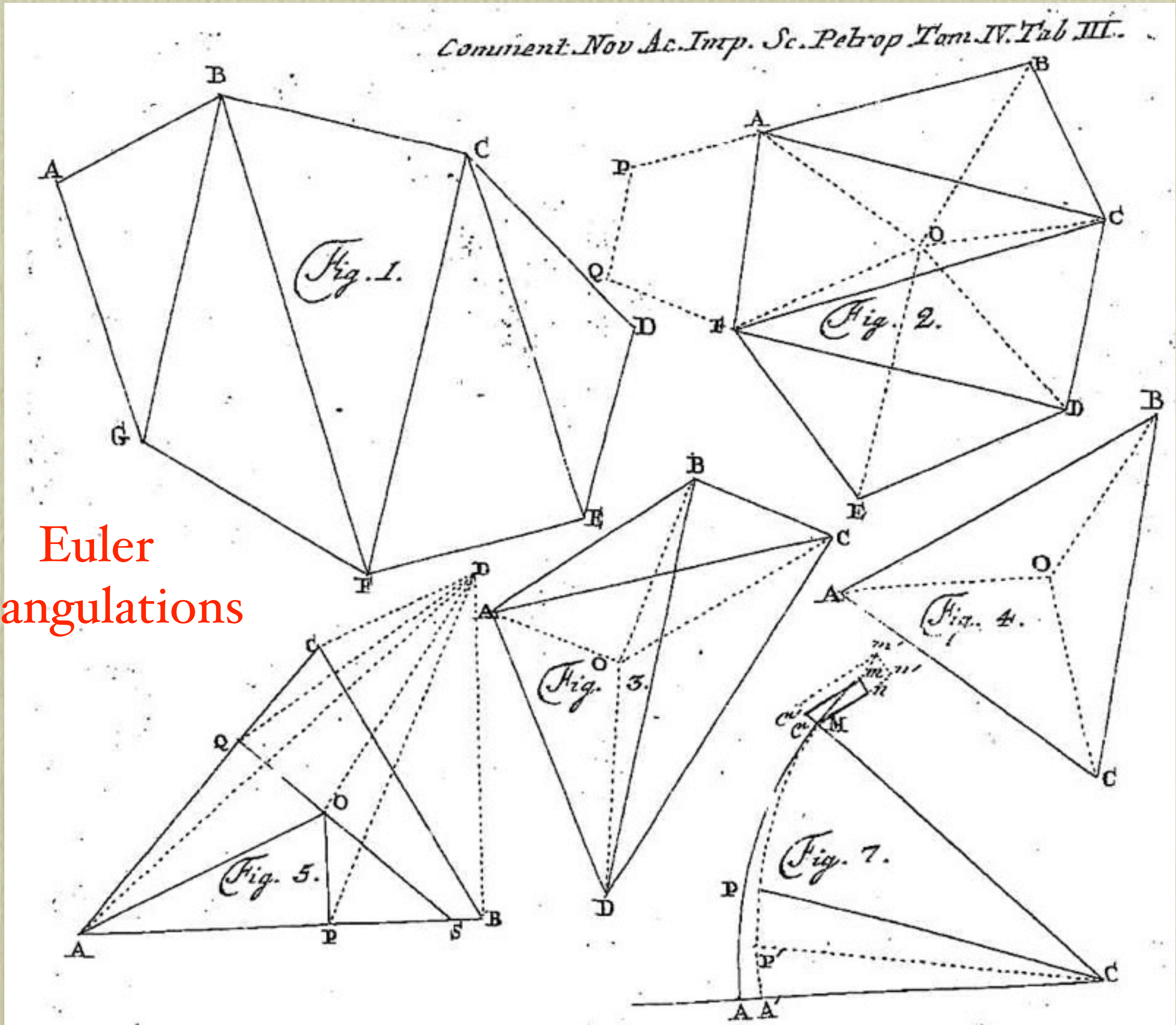
etc.	etc.	etc.	etc.	etc.	etc.

L'équation de relation est $A_{m,n} = A_{m-1,n} + A_{m+1,n-1}$; j'y mets $m - 1$ au lieu de m, et elle devient

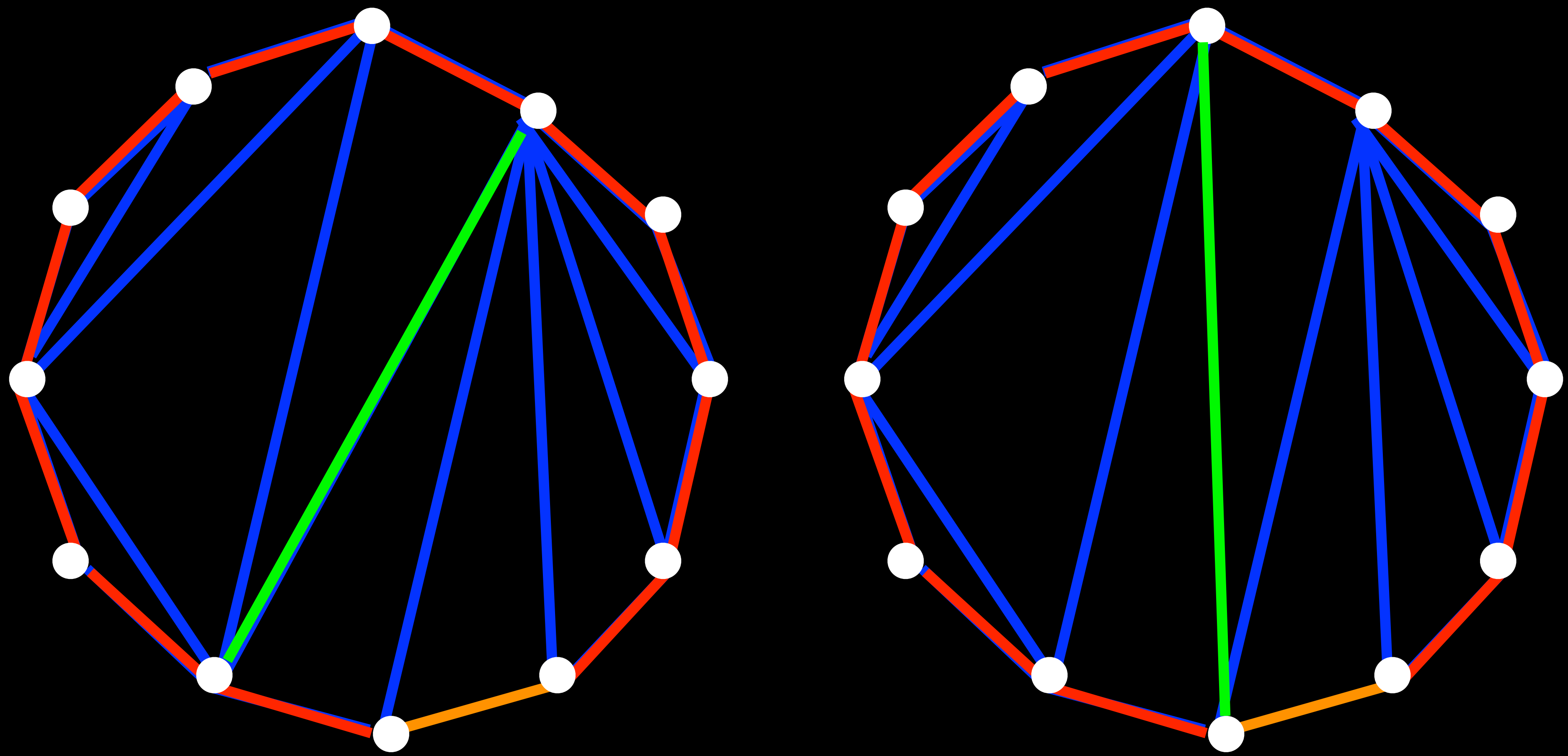
ballot numbers in 1800 (!)


EULER



Euler triangulations

another way to define the associahedron

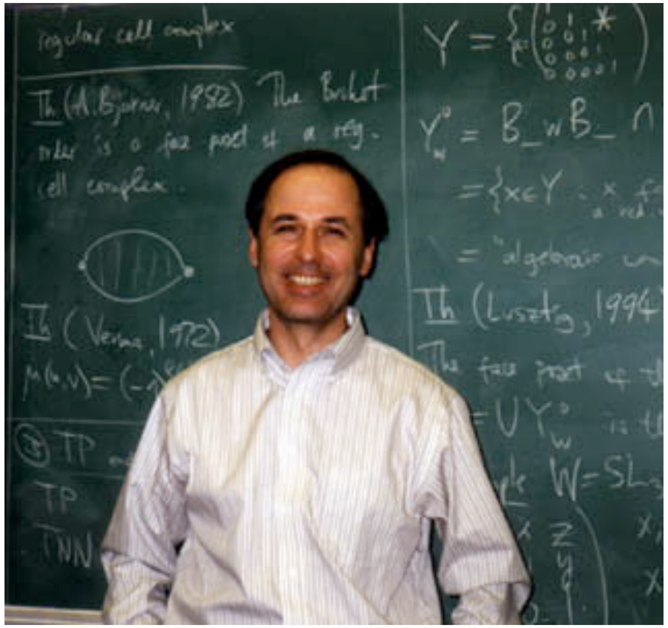


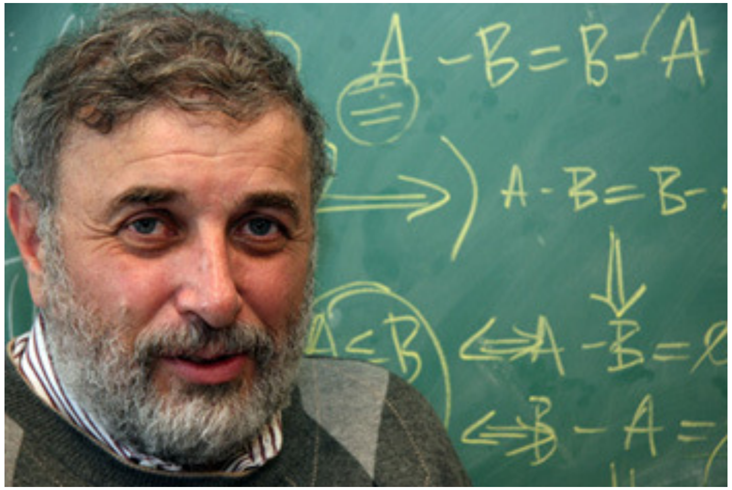
flip in triangulation

pedagogical introduction to .....

root systems

cluster algebras





associahedron

algebraic structures

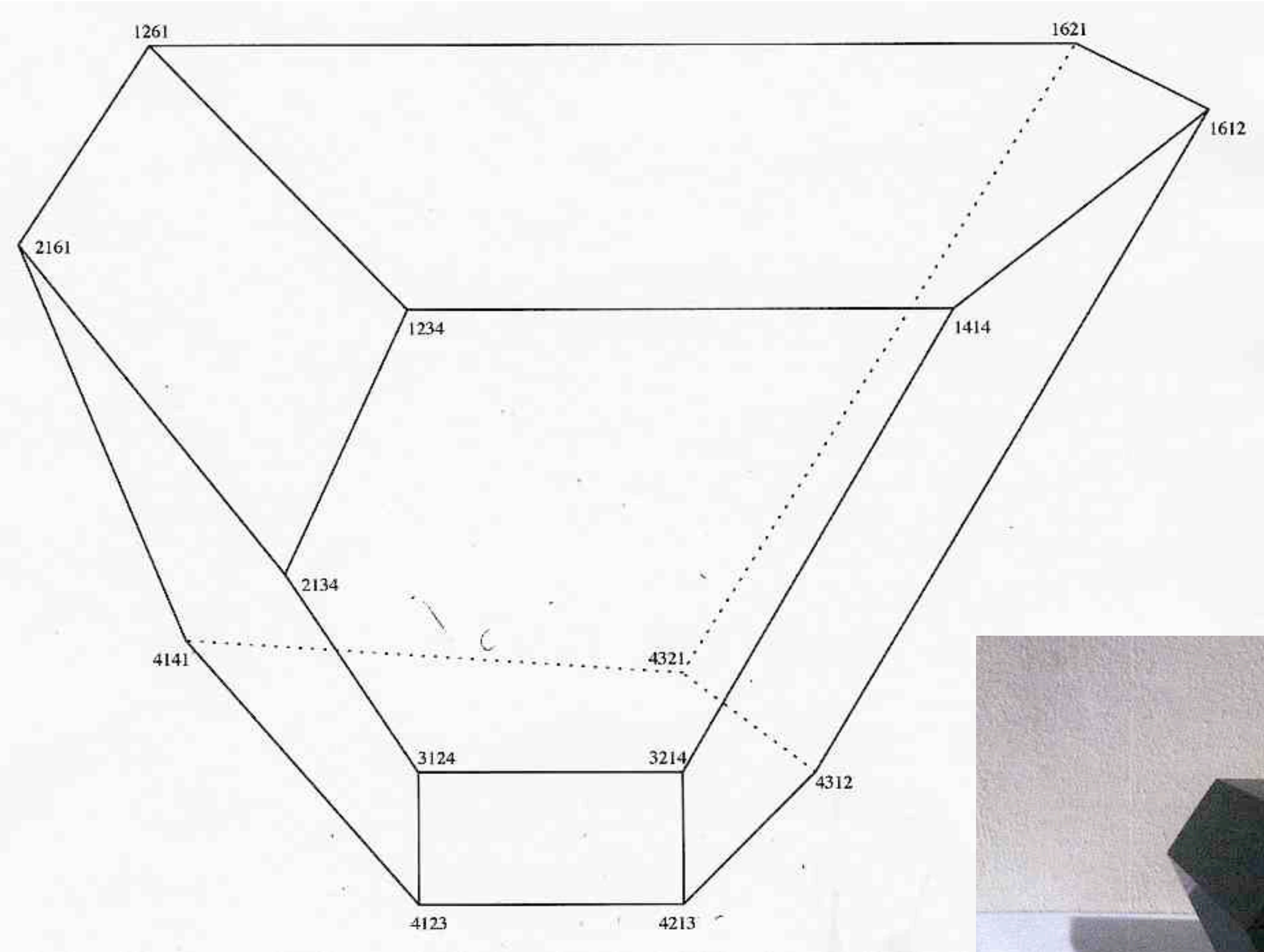
Hopf algebra

	descent algebra	Loday-Ronco algebra	Reutenauer Malvenuto algebra
dim	2^{n-1}	C_n Catalan	$n!$

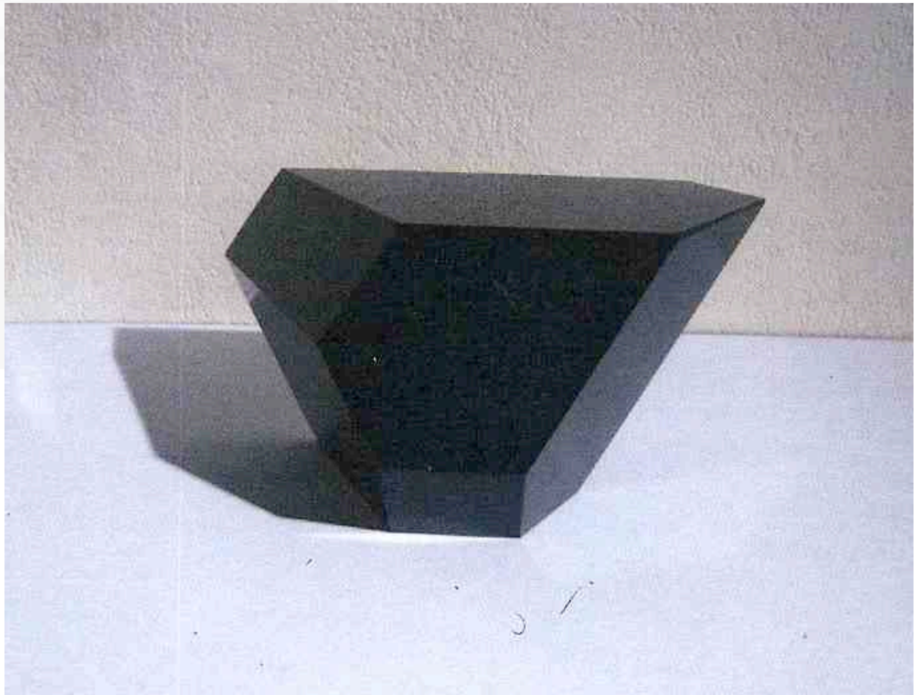
hyperculbe ↔ associahedron ↔ permutahedron

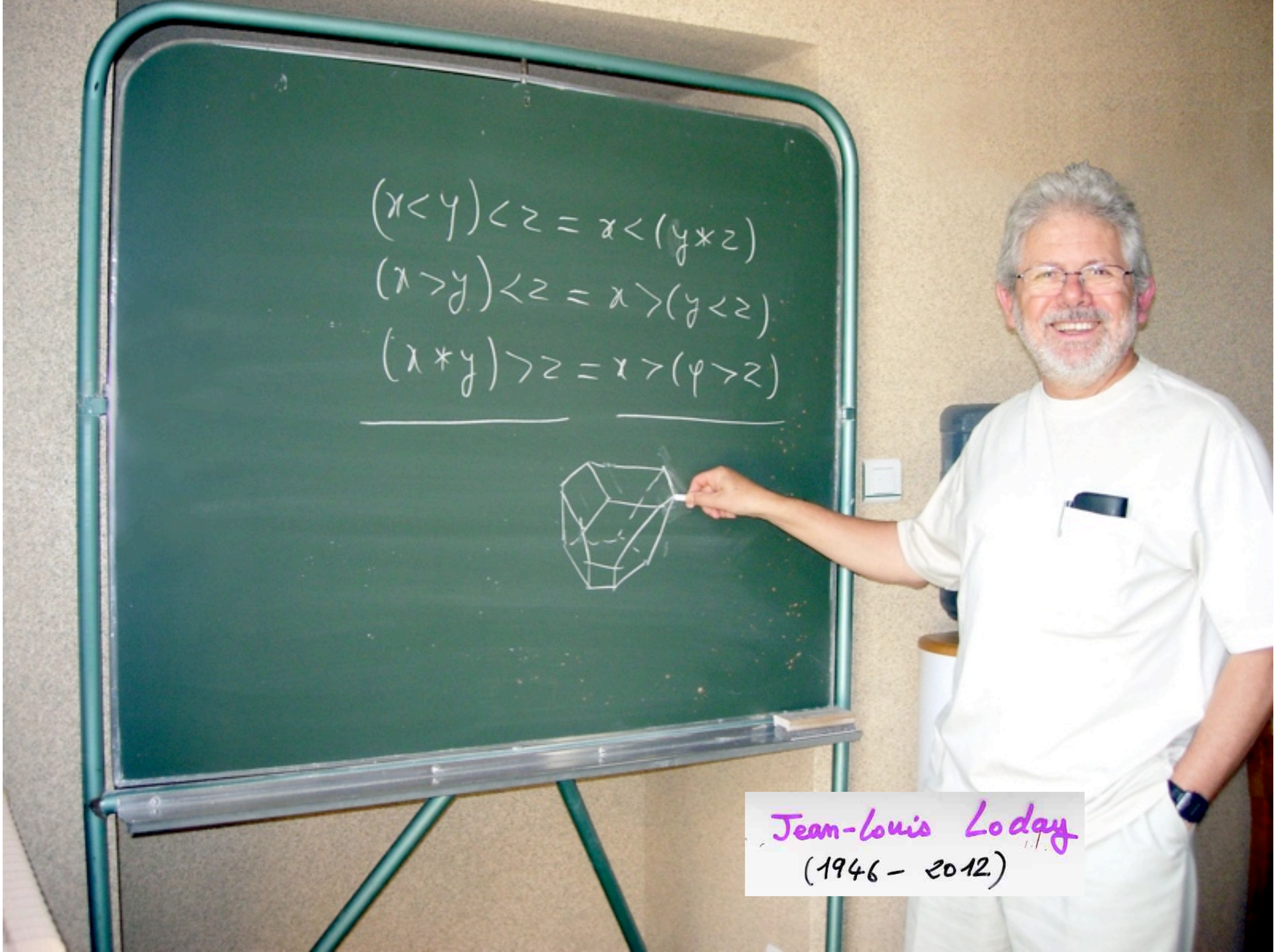
Boolean lattice inclusion ⟷ Tamari order ⟷ weak Bruhat order

J.-L.Loday, M. Ronco (1998, 2012)



associahedron





Jean-Louis Loday
(1946 – 2012)

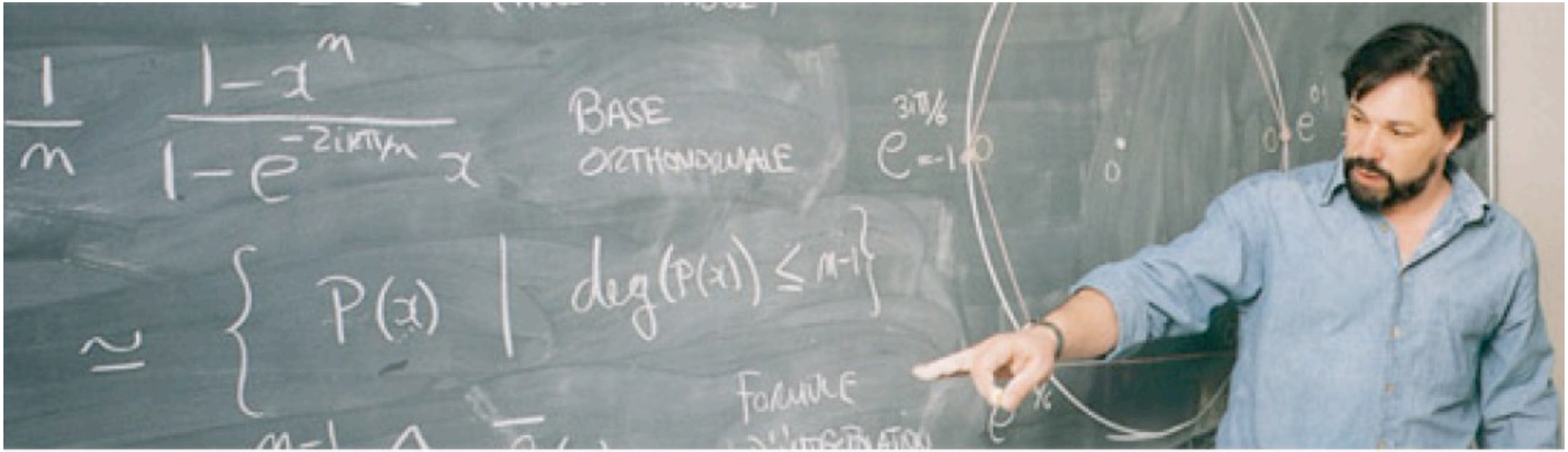
C. Hohlweg, C. Lange (2007)
F. Chapoton, S. Fomin, A. Zelevinsky (2002)

extensions :

C. Ceballos	J.-P. Labbé	C. Stump
V. Pilaud	N. Bergeron	F. Santos
N. Reading	H. Thomas	A. Postnikov
R. Marsh	M. Reineke	C. Athanasiadis
D. Speyer	S. Stella	G. Ziegler

Gil Kalai

and many others ...



François Bergeron

diagonal
coinvariant
spaces

Adriano Garsia

$X = (x_{i,j})_{\substack{1\le i\le k\\ 1\le j\le n}}$ matrix of variables

$\sigma \in \mathfrak{S}_n$ symmetric group

$\sigma(X) = (x_{i,\sigma(j)})_{\substack{1\le i\le k\\ 1\le j\le n}}$ action on $\mathbb{C}[X]$

diagonal coinvariant spaces

$$DR_{k,n} = \mathbb{C}[X]/J$$

higher diagonal coinvariant spaces

$$DR^m_{k,n} = \varepsilon^{m-1} \otimes A^{m-1}/J A^{m-1}$$ alternants

$DR^{m\,\varepsilon}_{k,n}$ subspace of alternants

$k=1$ classical

$k=2$ Garsia, Haiman

$\rightarrow$ Macdonald polynomials

$DR^{m\,\varepsilon}_{2,n}$ $DR^{m}_{2,n}$

dimension $\dfrac{1}{(m+1)n+1}\dbinom{(m+1)n+1}{mn}$ $(mn+1)^{n-1}$

m-ballot paths m-parking functions

$DR^{m}_{2,n}$ 20 years of studies

m-shuffle conjecture

Frobenius series (q,t) sum on m-parking

(area dinv

$k = 3$ Haiman (conjecture) 1990

$DR^{\varepsilon}_{3,n}$ $\qquad\qquad$ $DR_{3,n}$

dimension $\dfrac{2}{n(n+1)}\dbinom{4n+1}{n-1}$ $\qquad$ $2^n (n+1)^{n-2}$

$\uparrow$

Chapoton (2006)

number of interval Tamari$_n$

F. Bergeron (2008) introduced the m-Tamari lattice

conjecture $\dfrac{m+1}{n\,(mn+1)}\dbinom{(m+1)^2 n+m}{n-1}$ $\qquad$ $(m+1)^n (mn+1)^{n-2}$

nb of intervals $\qquad$ nb of labelled intervals

Hikita, Armstrong

paths and parking functions

above the line $(0,0)$ — (p,q) p,q relatively prime integers

Armstrong, Garsia, Haglund, Haiman, Hicks
Lee, Li, Loehr, Morse, Remmel, Rhoades,
Stout, Xin, Warrington, Zabrocki, ...
+ ......

This work was done at Universidad de Talca, Chile
at the invitation of Luc Lapointe,
and in the various following places

Constitution
Isla Negra
Viña del Mar
Valparaiso

Luc Lapointe

Universidad de Talca

La Universidad de Los mejores

www.utalca.cl

From Talca to Constitution

Maule valley

workshop in Valparaiso

Viña del Mar
(close to Valaparaiso)

Isla Negra
Pablo Neruda

¡ muchas gracias !