Canopy of binary trees, intervals in associahedra and exclusion model in physics

ICMAT Workshop
Algebraic and Geometric Combinatorics
Madrid, 28 november 2013

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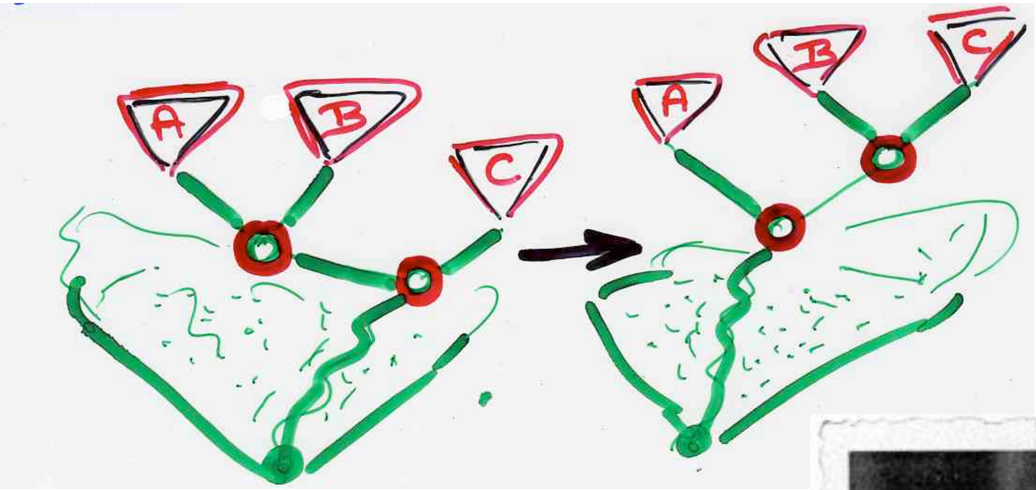
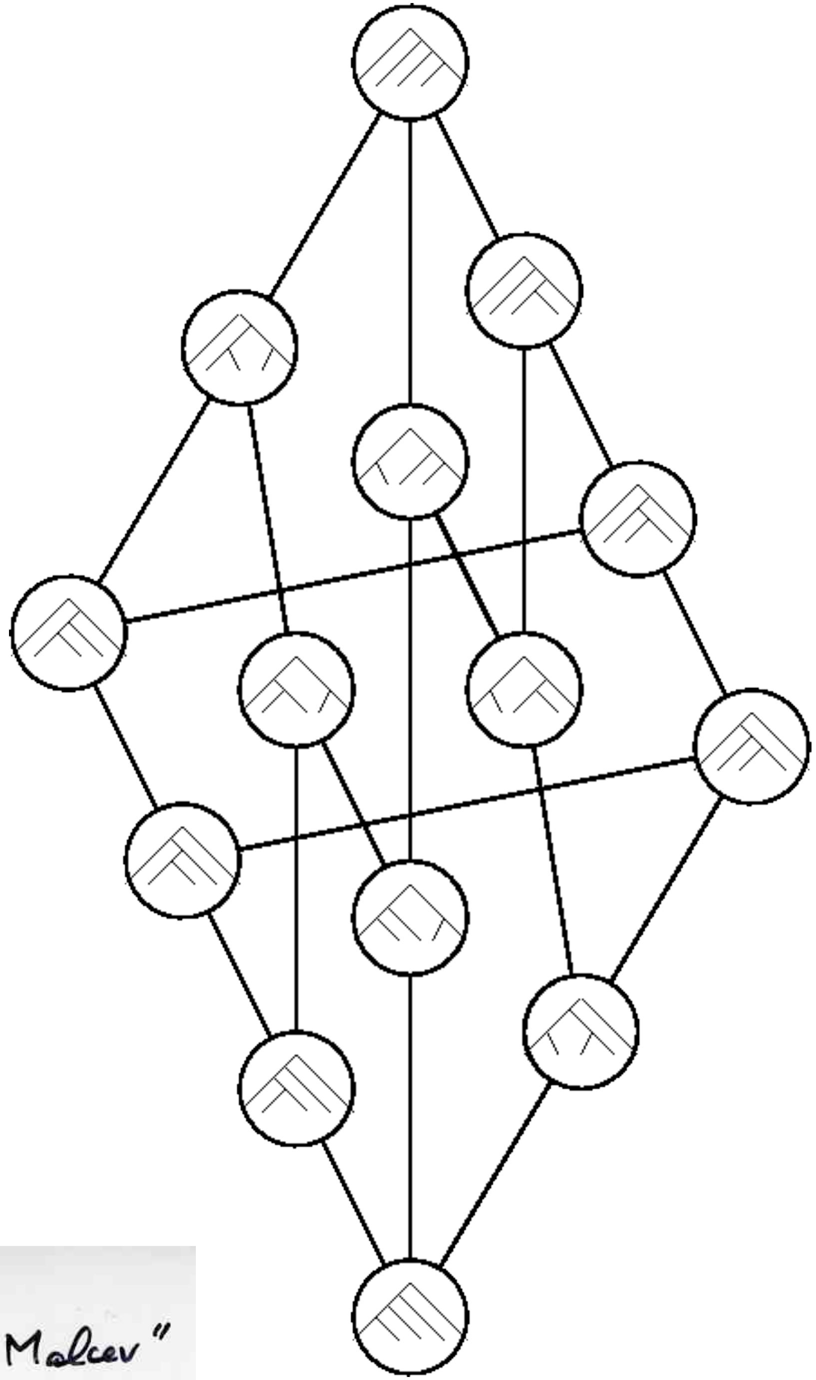
LaBRI, CNRS, Bordeaux



order relation

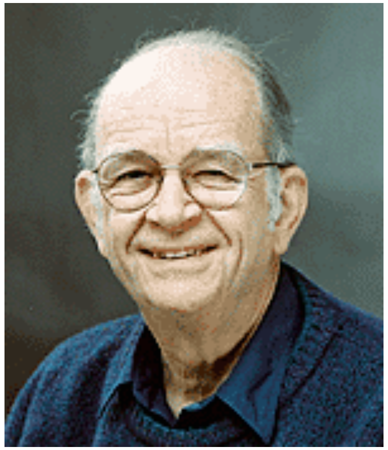
Tamari lattice



Dov Tamari (1951) thèse Sorbone
"Monoïdes préordonnés et chaînes de Malcev"

Stasheff polytope (1963)

thesis (1961)

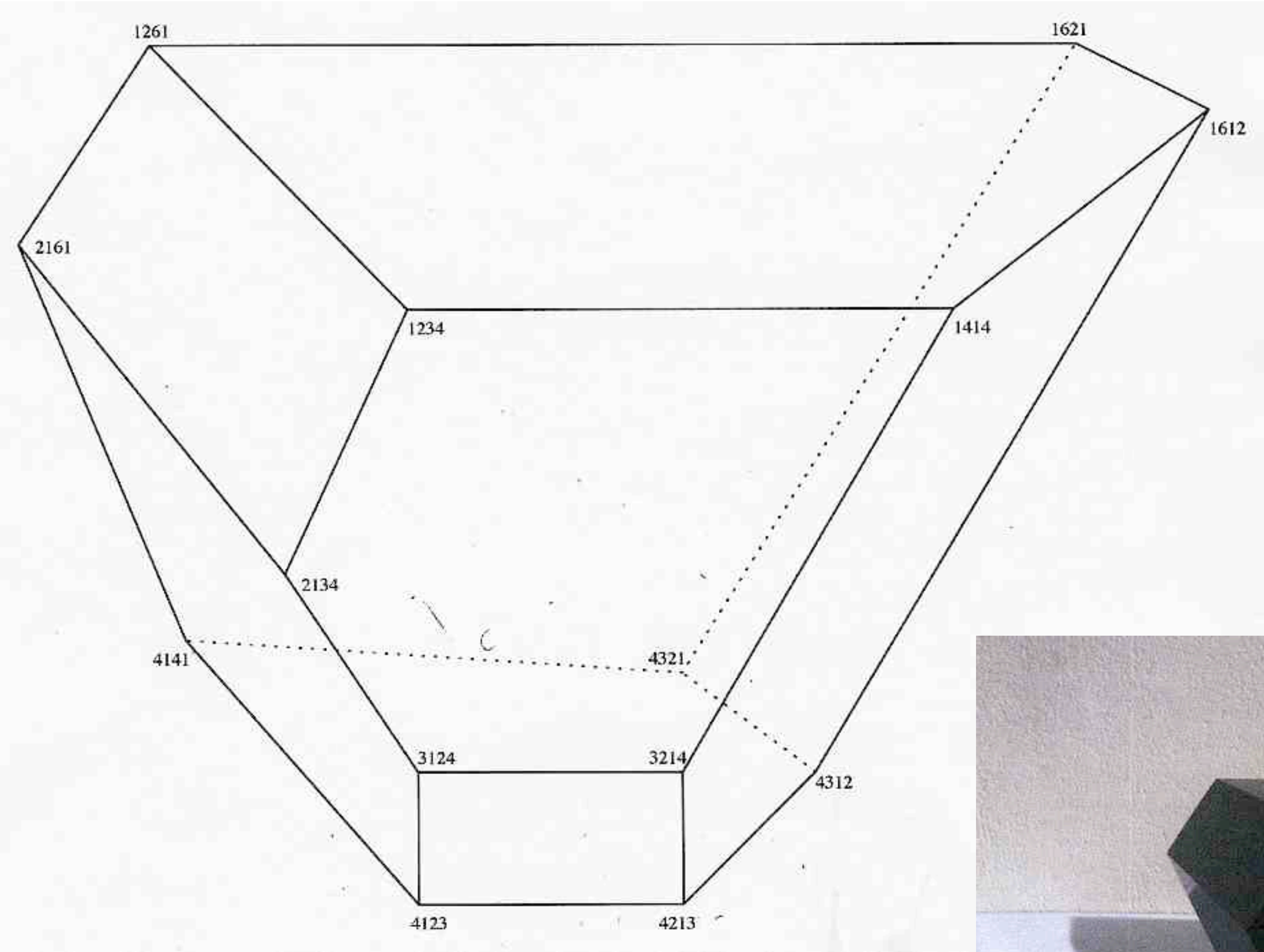


Homotopy theory

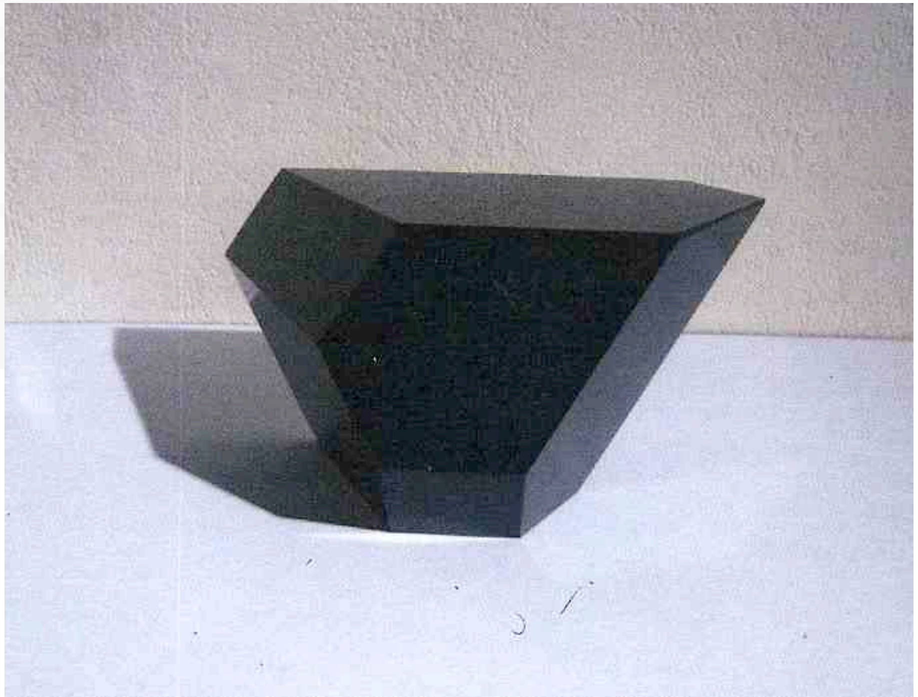
loops

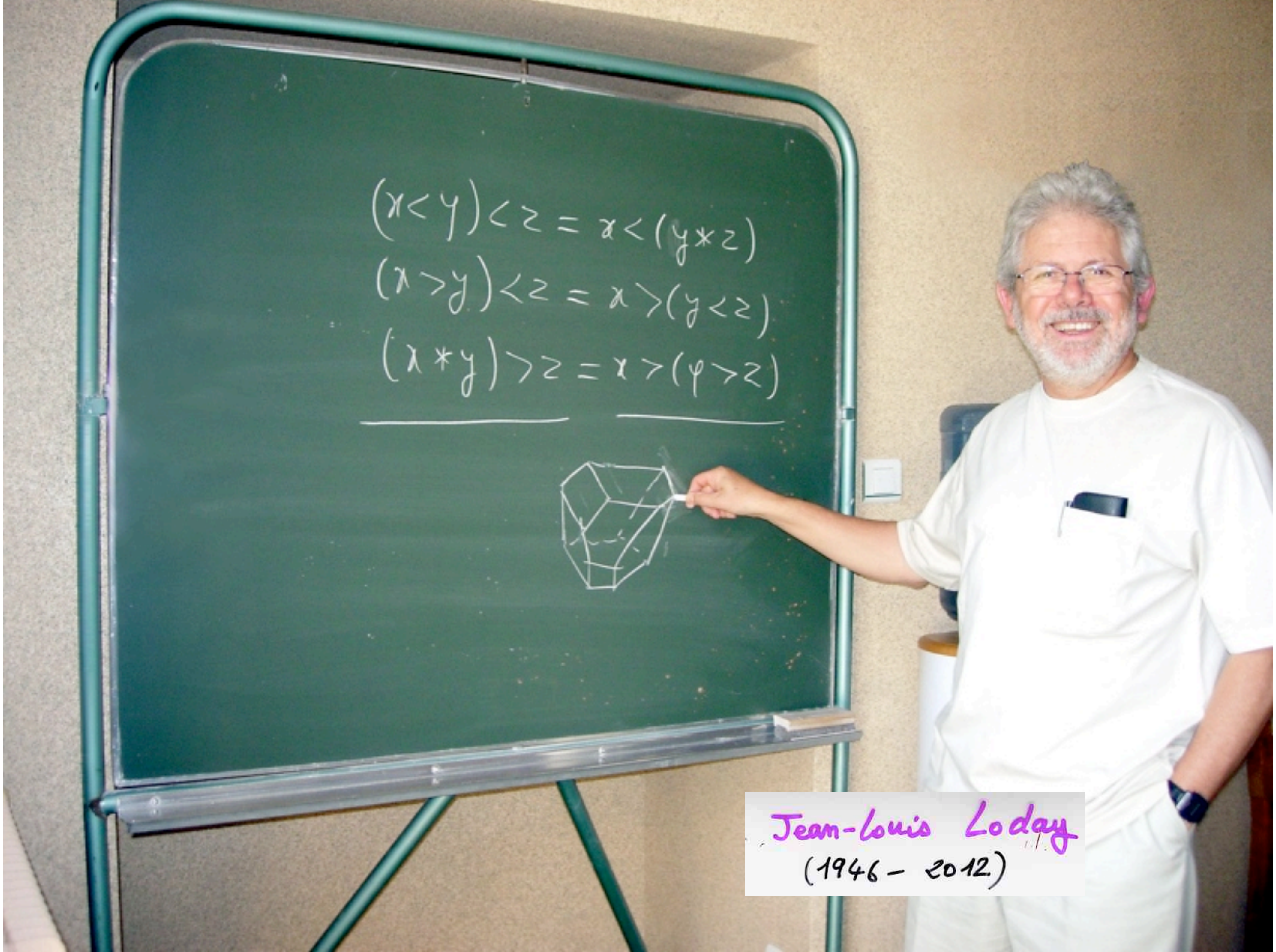
$[0,1] \xrightarrow{f} X$ continuous

topological space

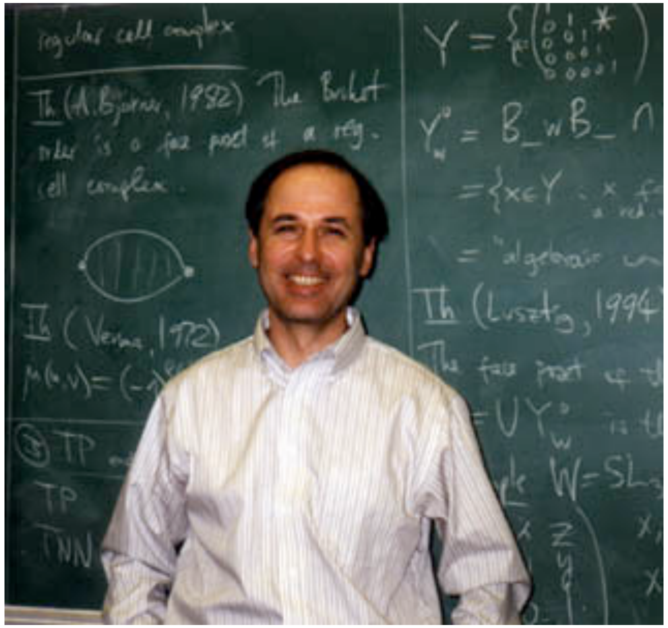


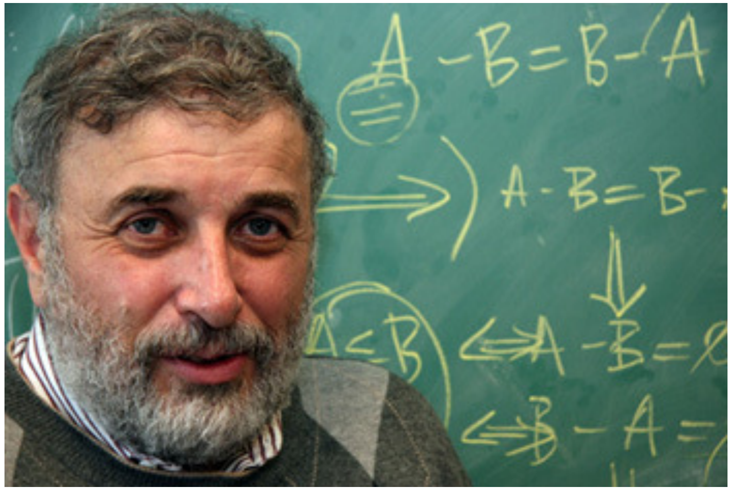
associahedron



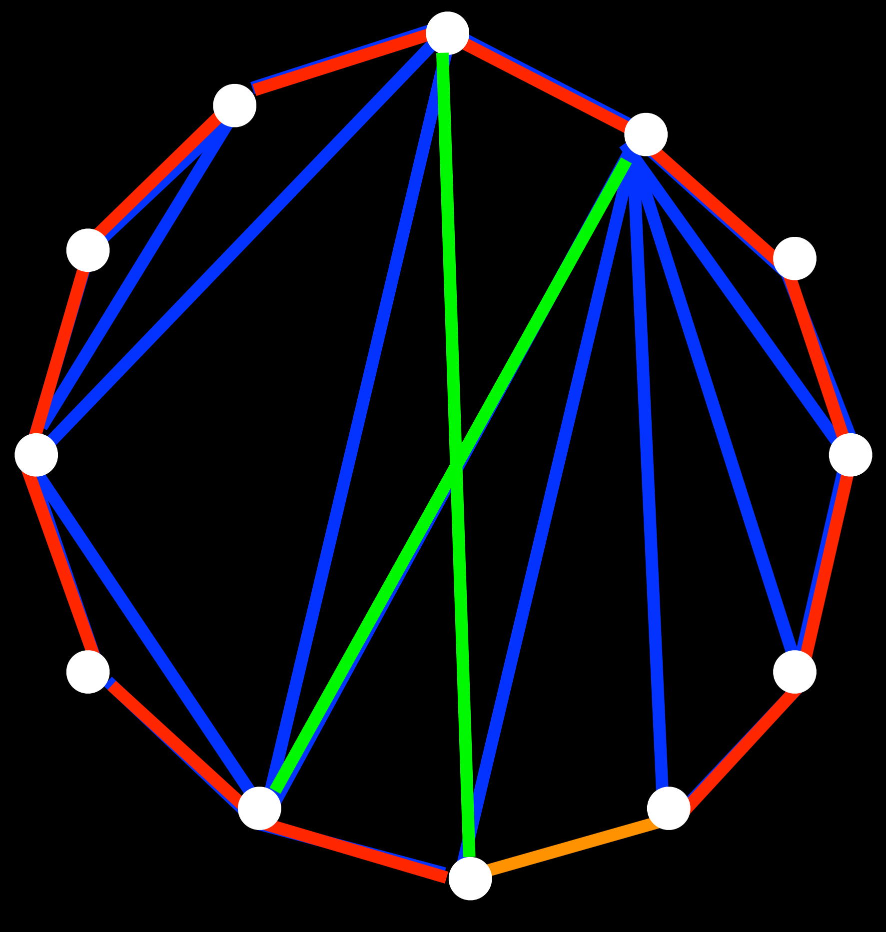

$(x<y)<z = x<(y*z)$
$(x>y)<z = x>(y<z)$
$(x*y)>z = x>(\varphi>z)$
Jean-Louis Loday
(1946 – 2012)

root systems cluster algebras





associahedron



C. Hohlweg, C. Lange (2007)
F. Chapoton, S. Fomin, A. Zelevinsky (2002)

extensions :

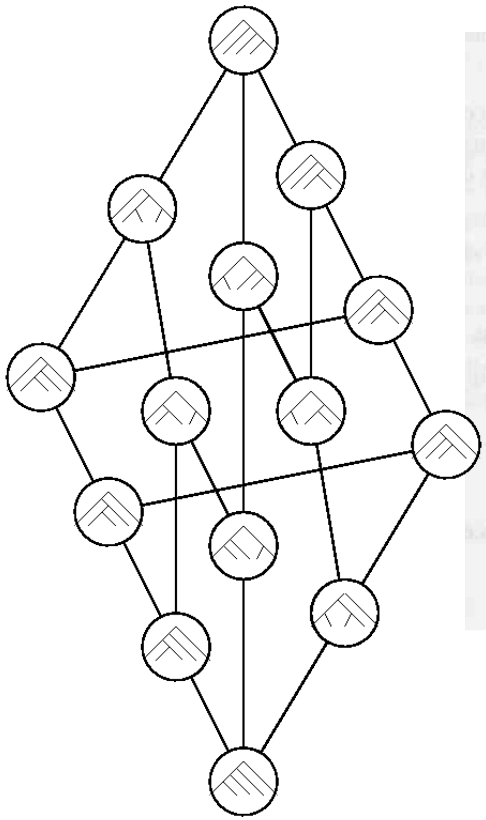
C. Ceballos	J.-P. Labbé	C. Stump
V. Pilaud	N. Bergeron	F. Santos
N. Reading	H. Thomas	A. Postnikov
R. Marsh	M. Reineke	C. Athanasiadis
D. Speyer	S. Stella	G. Ziegler

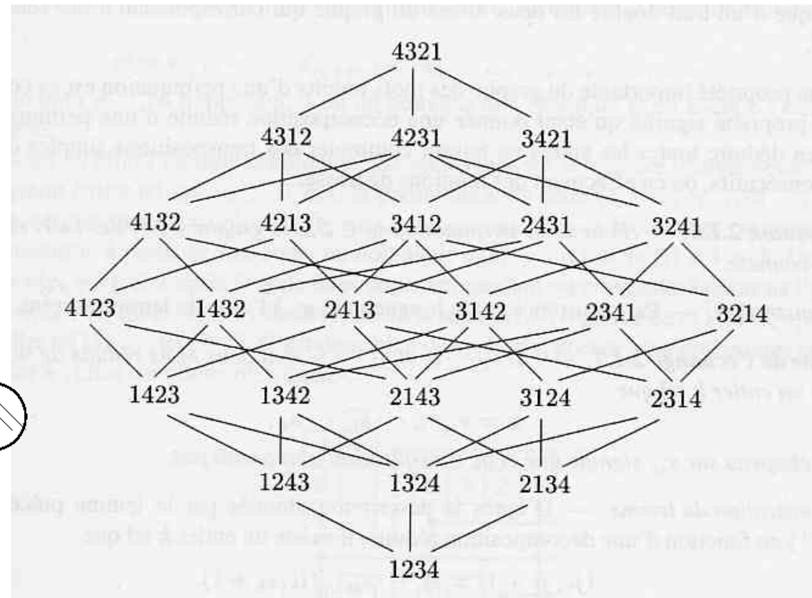
Gil Kalai

and many others ...

combinatorial structures

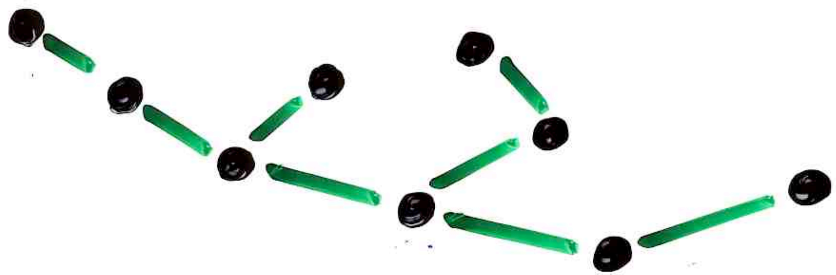
hypercube	associahedron	permutahedron
Boolean lattice inclusion $\longleftrightarrow$	Tamari order $\longleftrightarrow$	weak Bruhat order
dim 2^{n-1}	C_n	$n!$
	Catalan	

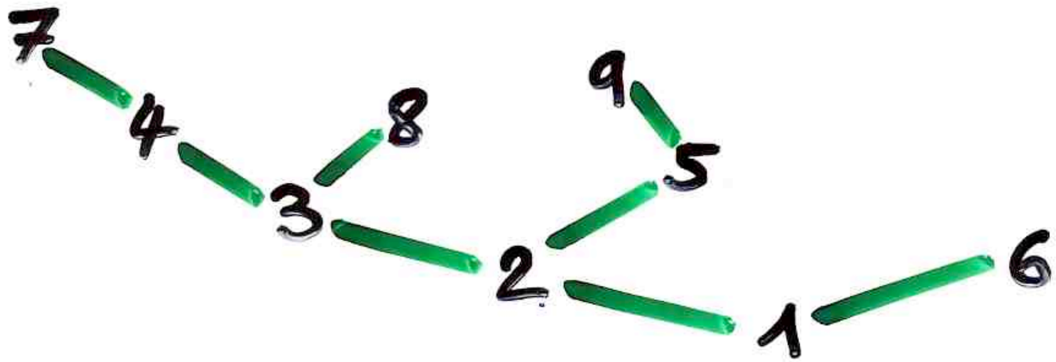




Bijection increasing binary tree ⟷ permutation

T σ





$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$

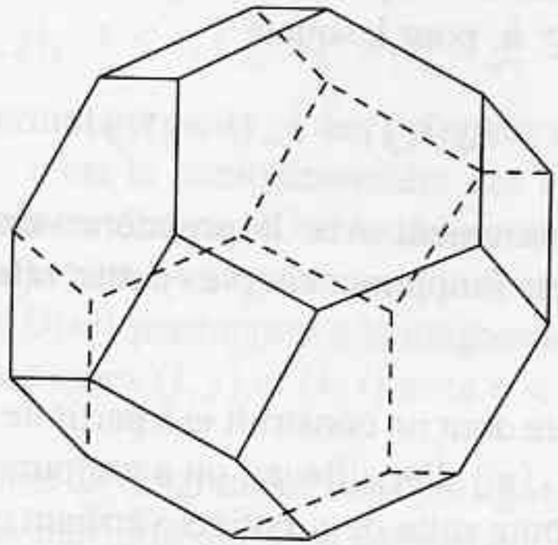
A. Björner, M. Wachs (1991)

$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6 \ldots$

up-down sequence $-\ -\ +\ -\ +\ -\ -\ +$

Alain Lascoux
(1944–2013)

permutohedron



2. Le permutoèdre Π_3.

Gil Kalai

LYCEE d'Etat " LOUIS LE GRAND "
- P a r i s -

Année Scolaire
1963-1964

LYCEE d'Etat " LOUIS LE GRAND "

- P a r i s -

Année Scolaire
1963-1964

Binohedron

00110100110001100000

associahedron

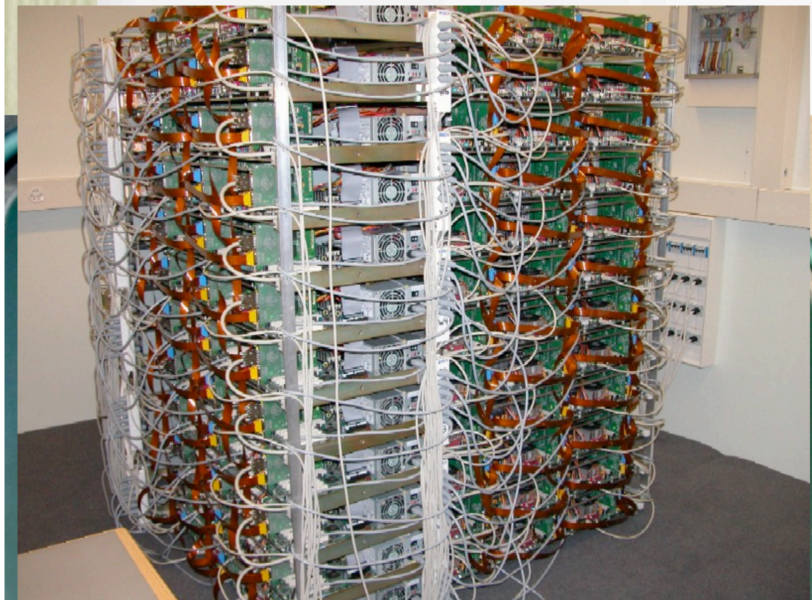
Tamari order

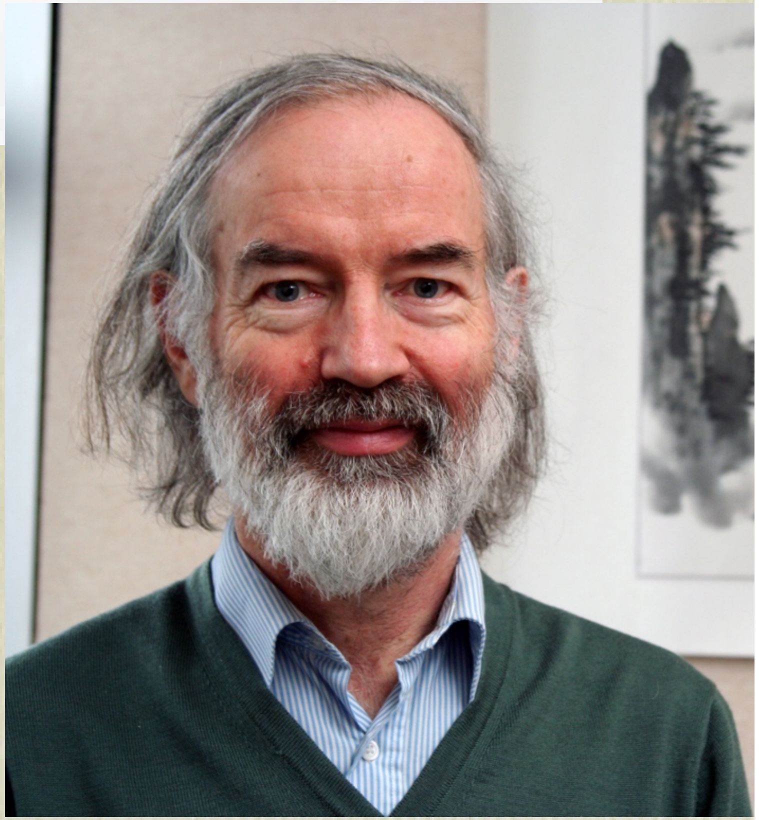
permutohedron

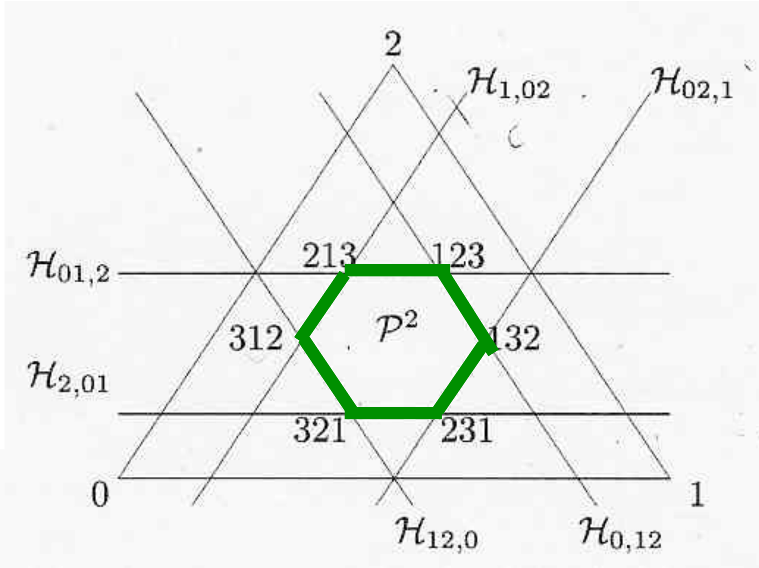
weak Bruhat order

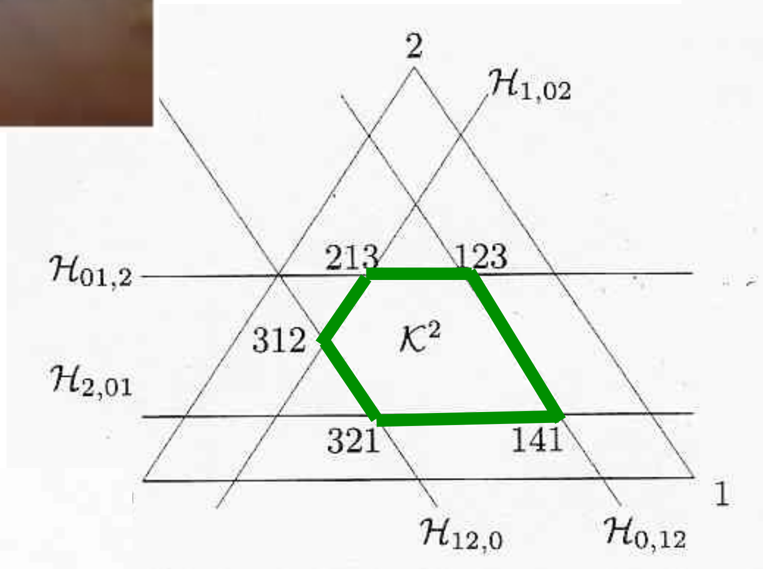
dim 2^{n-1} C_n $n!$

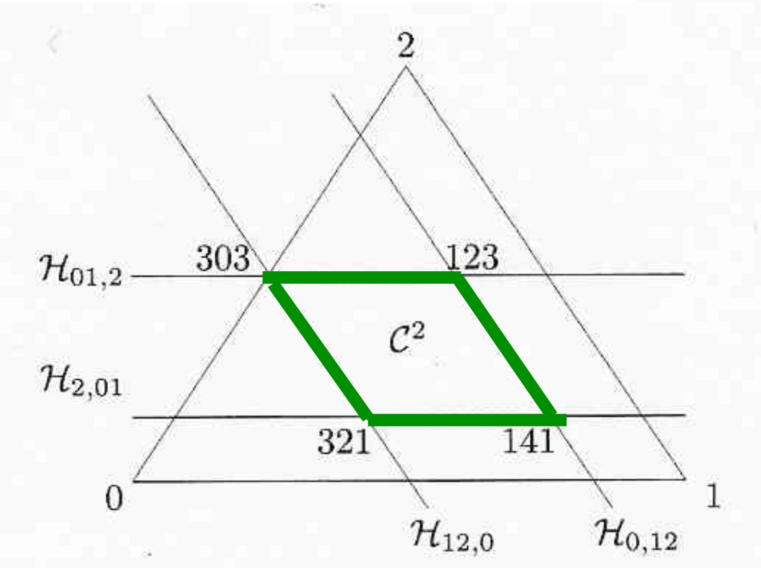
Catalan






2
$\mathcal{H}_{1,02}$
$\mathcal{H}_{02,1}$
$\mathcal{H}_{01,2}$
213
123
312
$\mathcal{P}^2$
132
$\mathcal{H}_{2,01}$
321
231
0
1
$\mathcal{H}_{12,0}$
$\mathcal{H}_{0,12}$


2
$\mathcal{H}_{1,02}$
$\mathcal{H}_{01,2}$
213
123
312
$\mathcal{K}^2$
$\mathcal{H}_{2,01}$
321
141
1
$\mathcal{H}_{12,0}$
$\mathcal{H}_{0,12}$


2
$\mathcal{H}_{01,2}$
303
123
$\mathcal{C}^2$
$\mathcal{H}_{2,01}$
321
141
0
1
$\mathcal{H}_{12,0}$
$\mathcal{H}_{0,12}$

algebraic structures

Hopf algebra

	descent algebra	Loday-Ronco algebra	Reutenauer Malvenuto algebra
dim	2^{n-1}	c_n Catalan	$n!$

hypercube Boolean lattice inclusion ↔ associahedron Tamari order ↔ permutahedron weak Bruhat order

J.-L.Loday, M. Ronco (1998, 2012)

nb of intervals

F. Chapoton (2006) $\dfrac{2(4n+1)!}{(n+1)!\,(3n+2)!}$

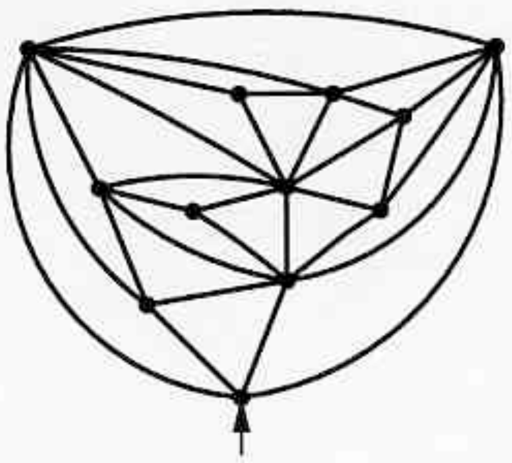
1, 3, 13, 68, 399, ...

Tamari lattice

triangulation

Bijective proof FPSAC 2007

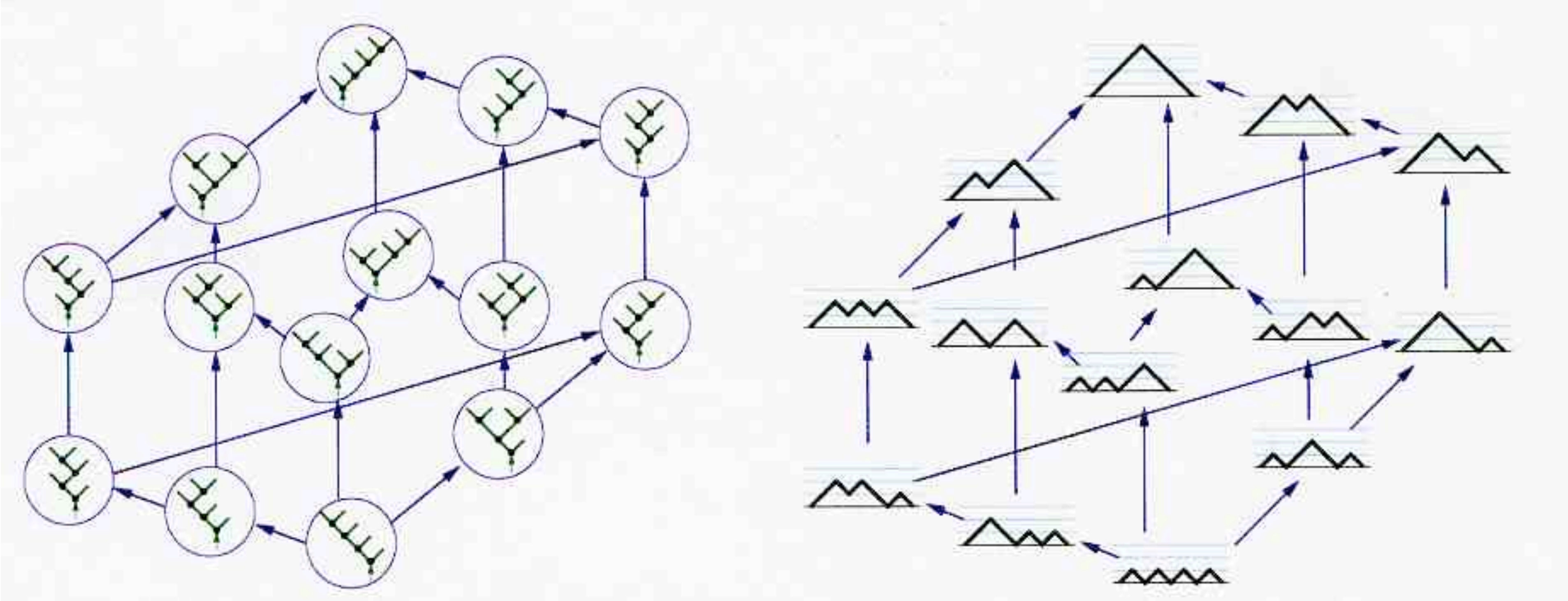
Bernardi, N. Bonichon



the Tamari lattice

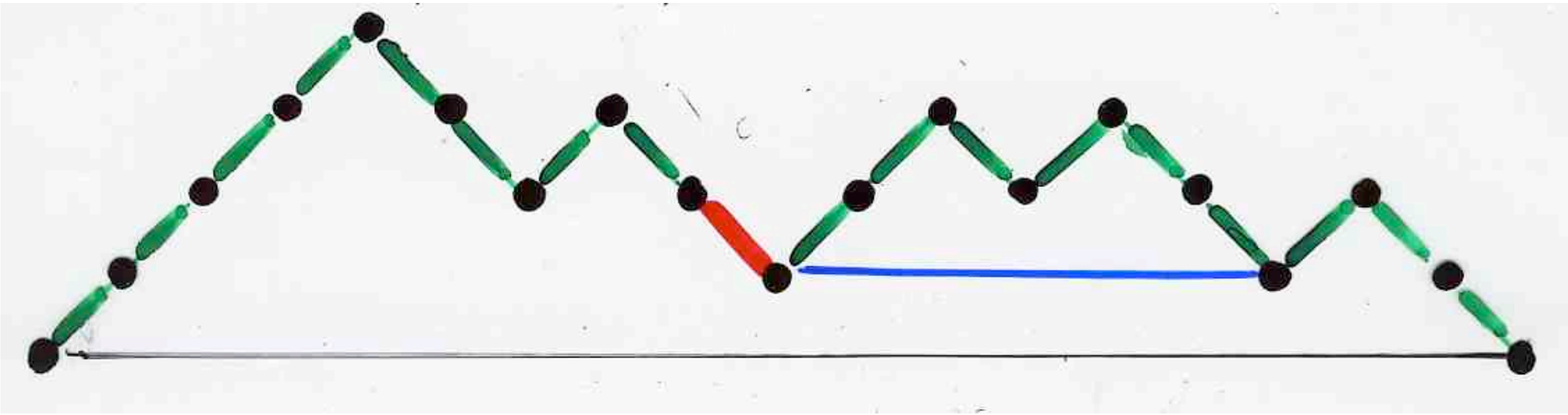
in term

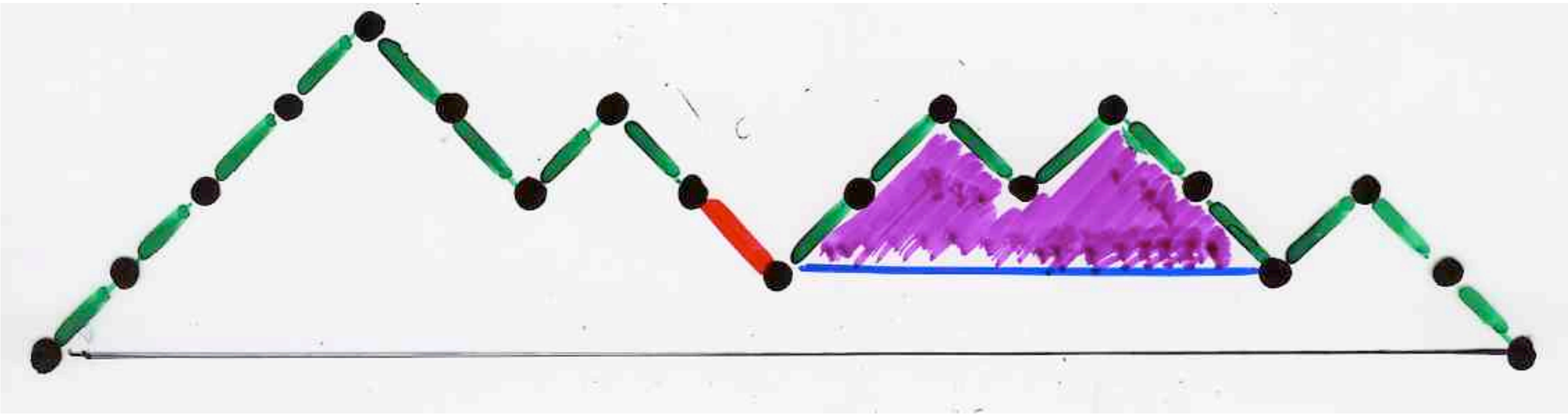
of Dyck paths



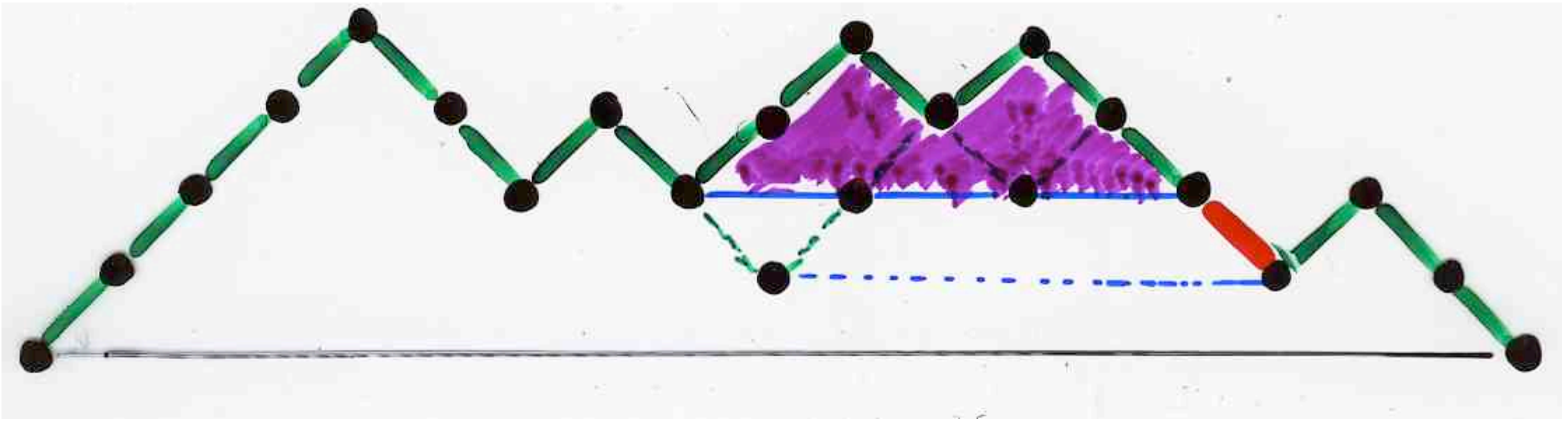
$C_4 = 14$

Catalan





facteur Dyck primitif



facteur Dyck primitif

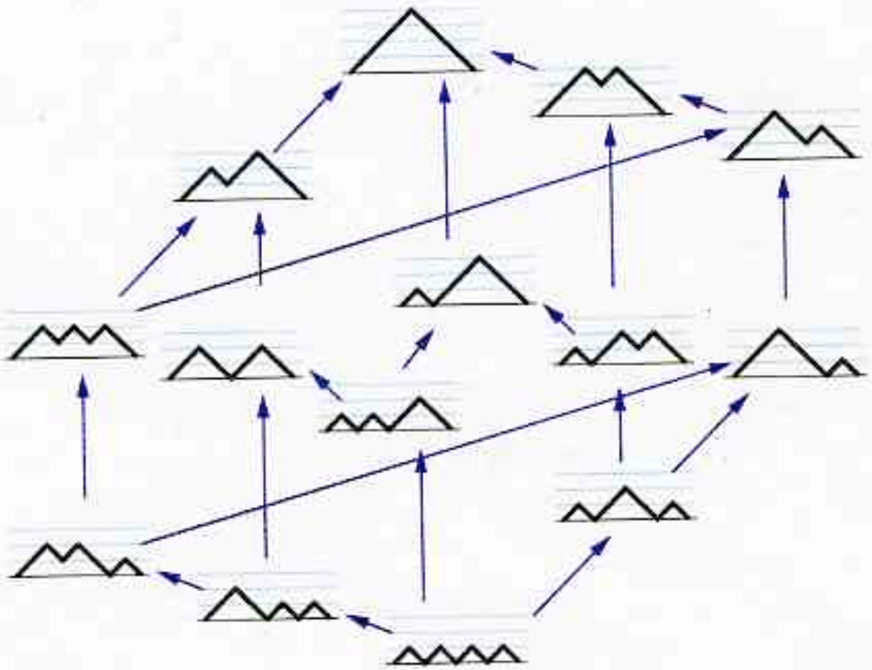
If $T \leqslant T'$ in $(\text{Tamari})_n$ lattice

then $T \leqslant T'$ in $(\text{Dyck})_n$ lattice

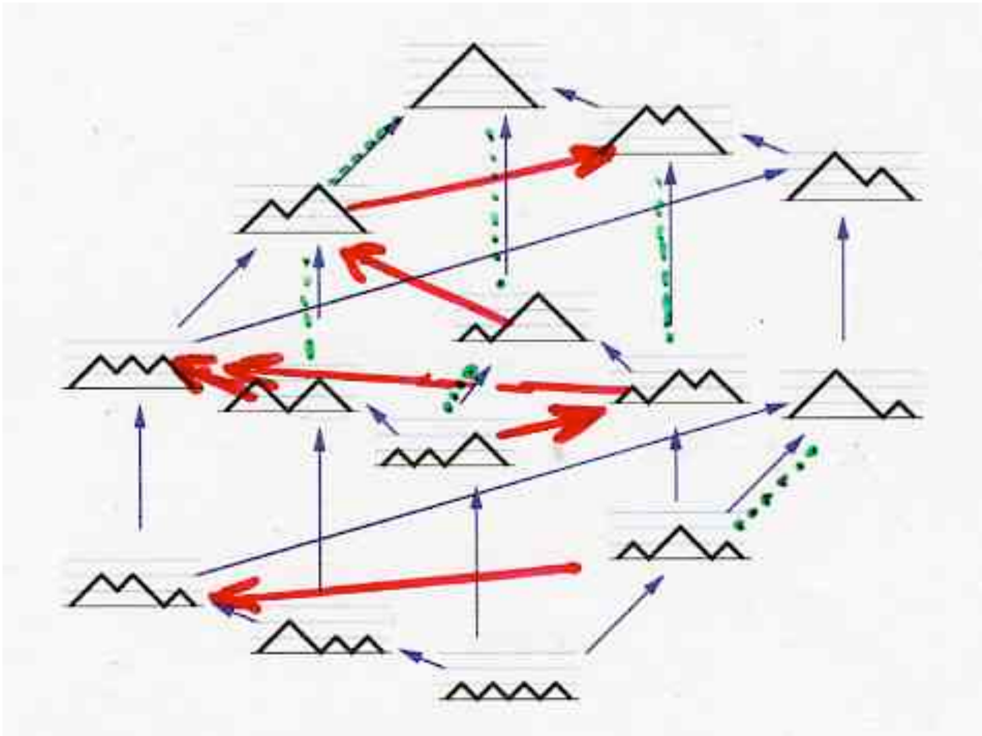
[i.e. T below T']

converse not true

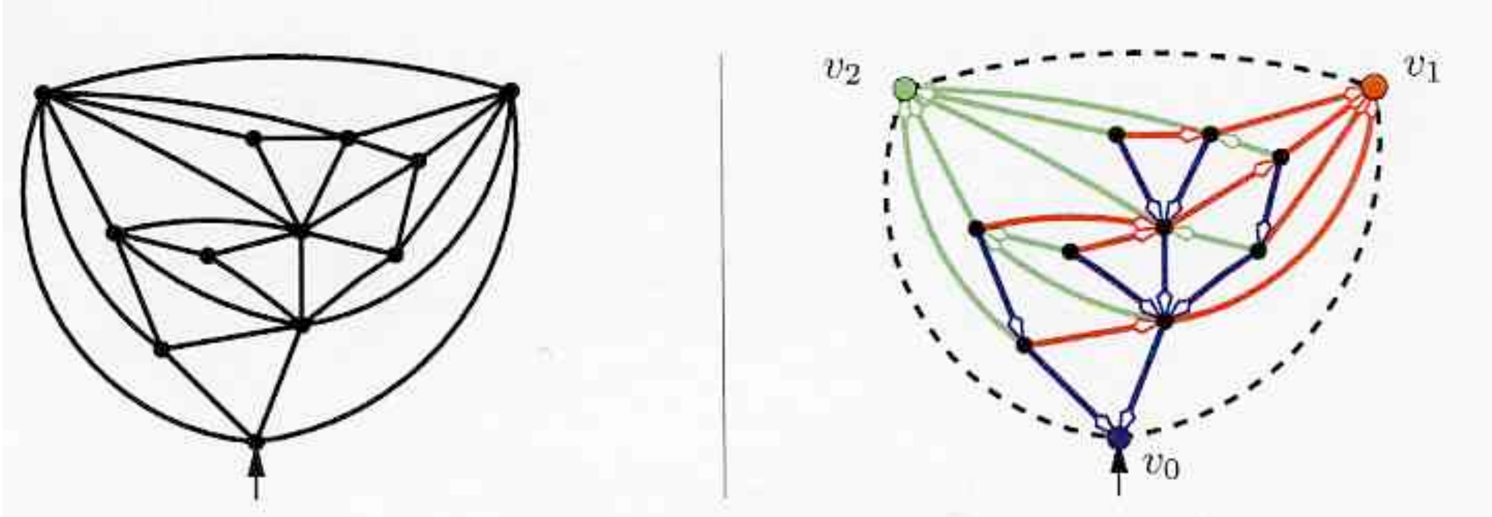
$(\text{Dyck})_n$ extension of $(\text{Tamari})_n$



(Tamari lattice)$_4$



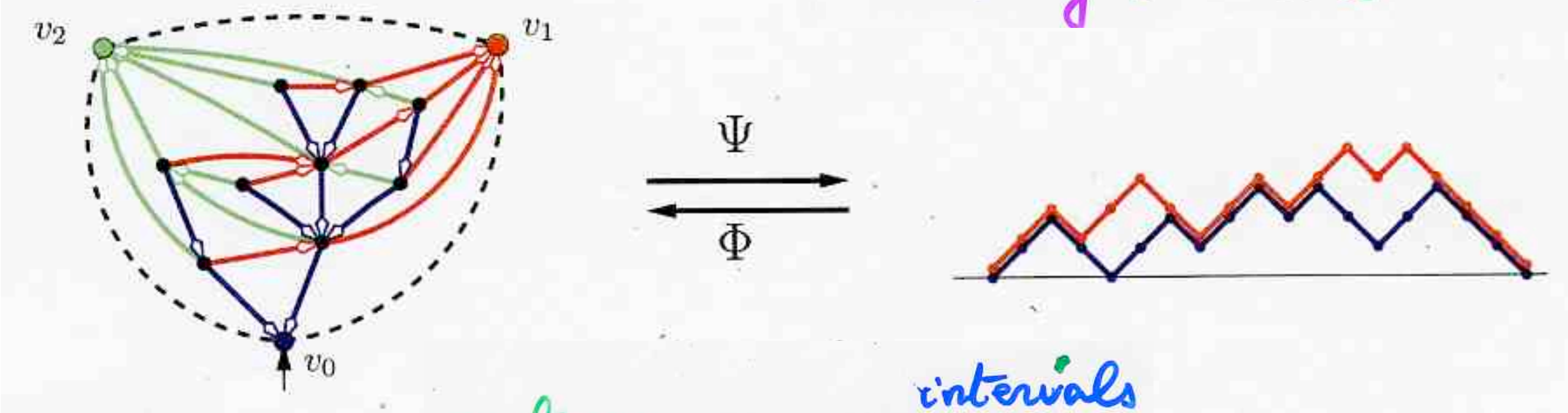
(Dyck lattice)$_4$



triangulation

realizers

or Schnyder woods



realizers $\longleftrightarrow$ intervals $(\text{Dyck})_n$

minimal realizers $\longleftrightarrow$ intervals $(\text{Tamari})_n$

$\updownarrow$

triangulation

Bijective proof FPSAC 2007
Bernardi, N. Bonichon

M. Bousquet-Mélou, E. Fusy, L.-F. Préville-Ratelle (2011)

nb of intervals of m-Tamari lattices

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n + m}{n-1}$$ F. Bergeron

M. Bousquet-Mélou, G. Chapuy, L.-F. Préville-Ratelle (2011)

nb of labelled intervals $(m+1)^n (mn+1)^{n-2}$

V. Pons

Tamari interval-posets $\Longleftrightarrow$ Tamari intervals

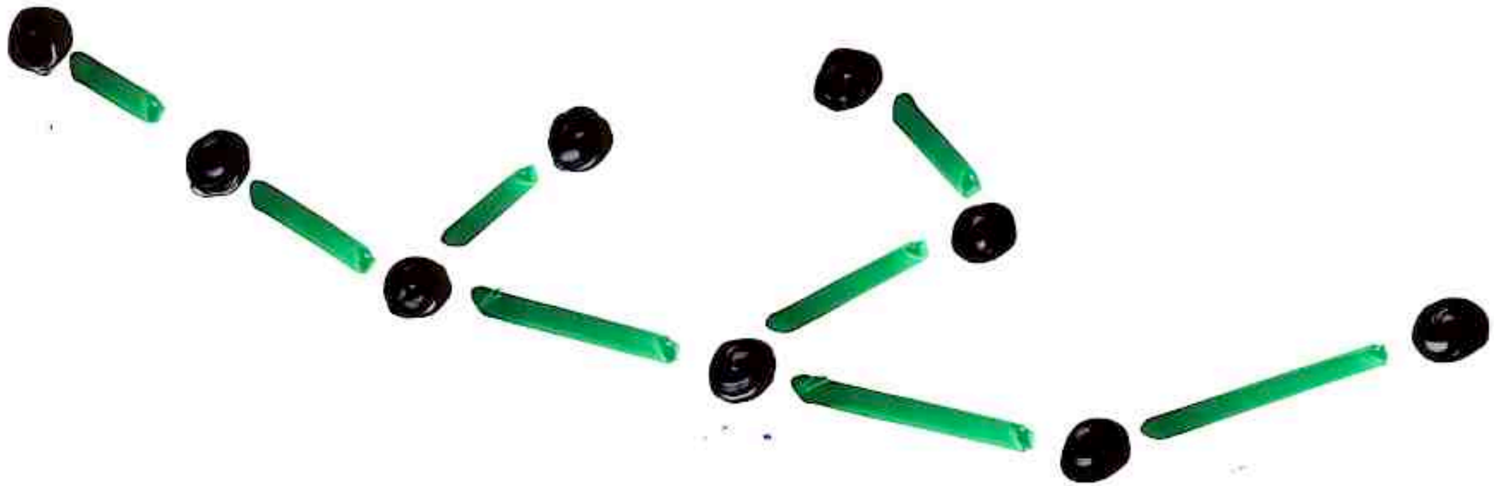
(FPSAC, 2013) thesis (Oct. 2013)

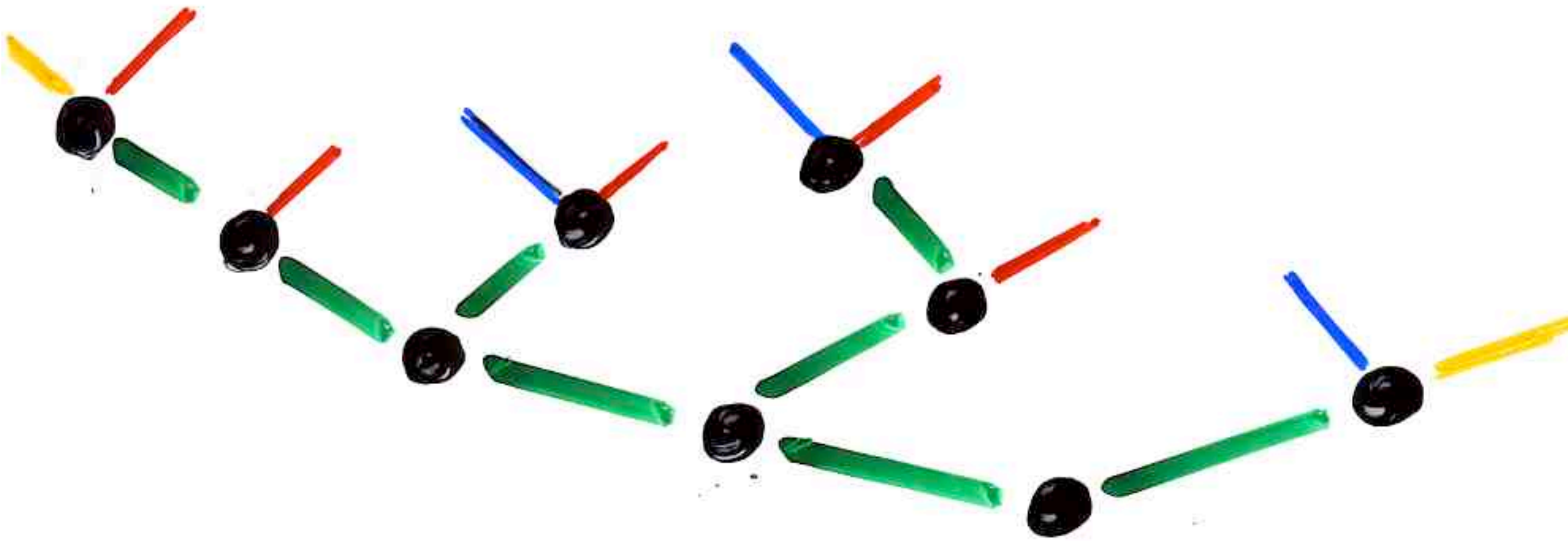
G. Chatel

bijections on Tamari intervals

$\longleftrightarrow$ closed flow of an ordered forest

$\rightarrow$ Pre-Lie operad F. Chapoton canopy of a binary tree





canopy of a binary tree

$c(B) = - - + - + - - +$

J.-L.Loday, M. Ronco (1998, 2012)

montée $\sigma(i) < \sigma(i+1)$

$1 \leq i < n$

descente $\sigma(i) > \sigma(i+1)$

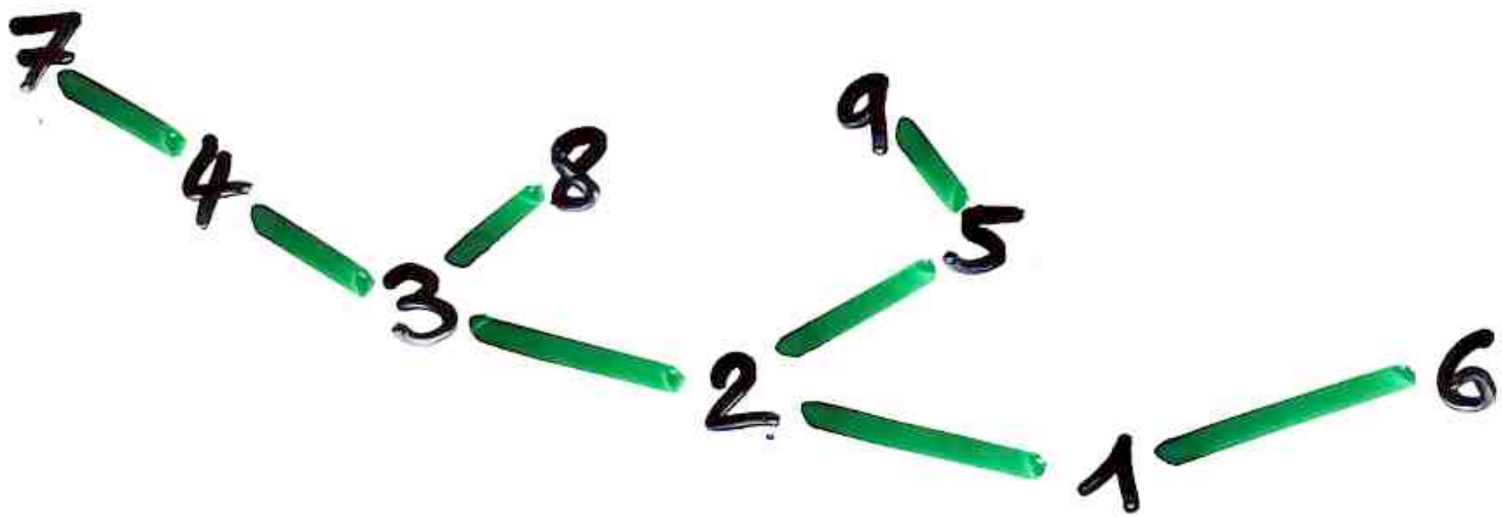
$\sigma \in \mathfrak{S}_n$

forme$(\sigma) = w_1 w_2 \cdots w_{n-1}$, $w_i = \begin{cases} + & \text{montée} \\ - & \text{descente} \end{cases}$ en i

UD(σ) mot $w \in \{+, -\}^*$

$\sigma = 5\ 8\ 2\ 9\ 6\ 1\ 4\ 3$

forme$(\sigma) = + - + - - + -$



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6\ \dots$

up-down sequence $\quad - \quad - \quad + \quad - \quad + \quad - \quad - \quad +$

$$
\begin{tikzcd}
\sigma \arrow[r] \arrow[dd] & T(\sigma) \arrow[d] \\
 & B \arrow[d] \\
UD(\sigma) \arrow[r, phantom, "="] & c(B)
\end{tikzcd}
$$

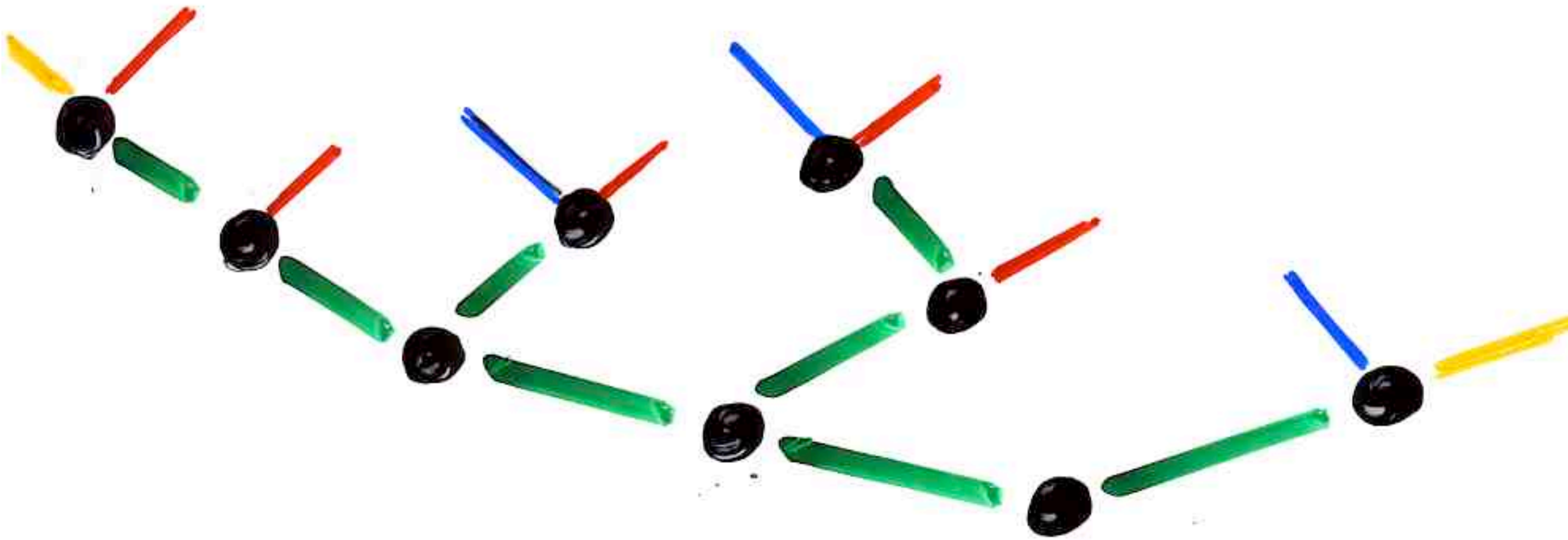
σ permutation

$T(\sigma)$ increasing binary tree

B binary tree

$UD(\sigma)$ up-down sequence

$c(B)$ canopy

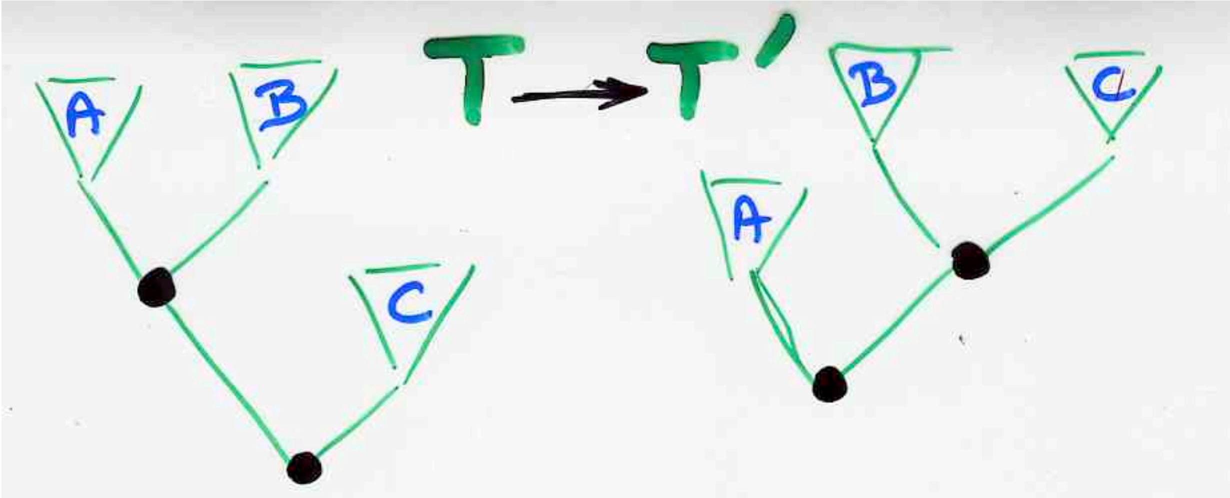


$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6\ \ldots$

up-down sequence $\quad - \ - \ + \ - \ + \ - \ - \ +$

a proposition

relating canopy and Tamari lattice



if $B \neq \bullet$ canopy is invariant

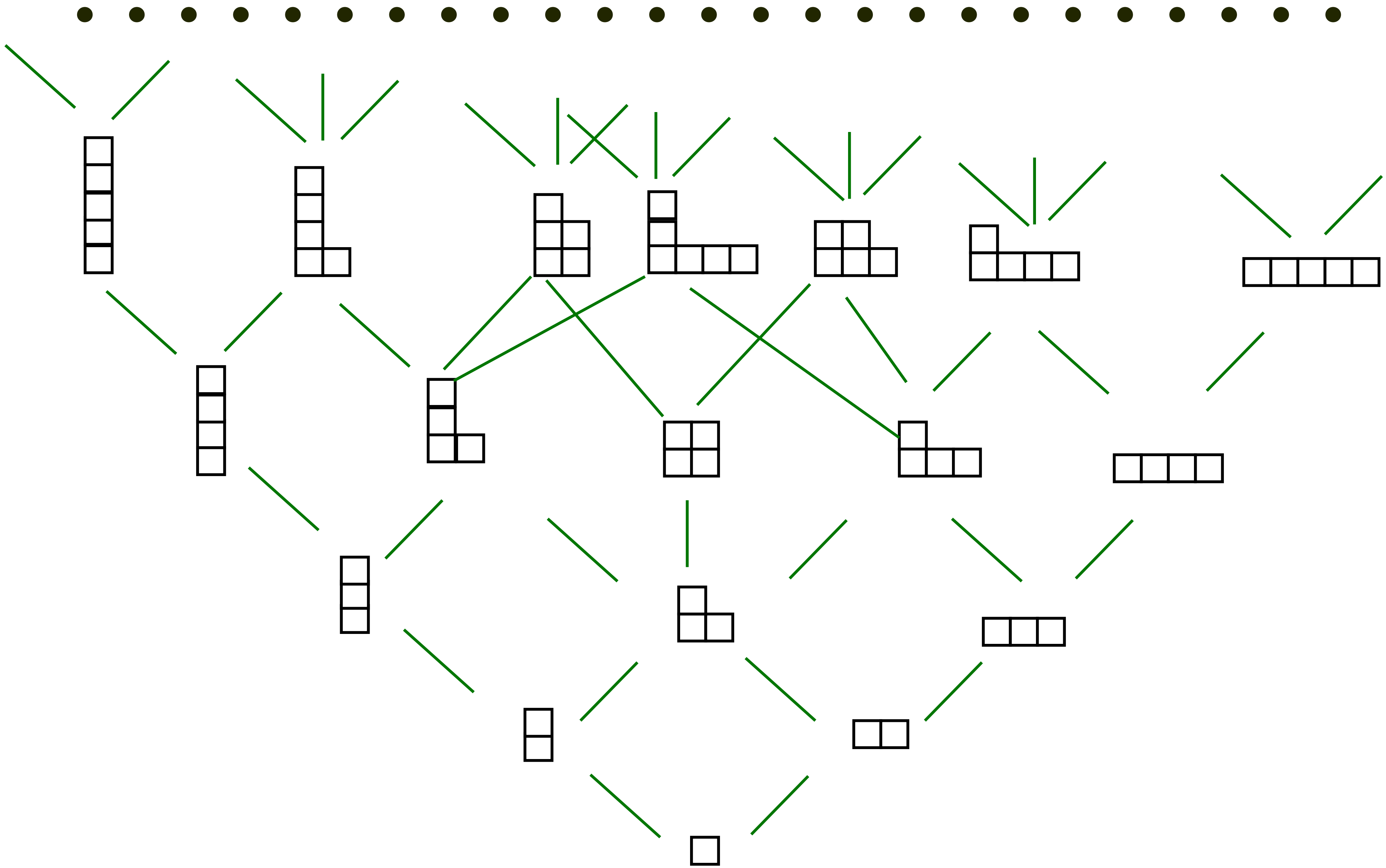
if $B = \bullet$ canopy $c(T')$

not invariant

$$c^{*}(T) = c(A) \overset{+}{/} c(B)\, c(C)$$

$$c(T') = c(A) \underset{-}{\backslash} c(B)\, c(C)$$

Young lattice



Prop. (i) • The set of binary trees having a given canopy w is an interval $J(w)$ of the Tamari lattice

(ii) • this interval can be extended to an initial interval of the Young lattice

i.e. $\exists$ (integer) partition μ such that $J(w)$ is in bijection with $I(\mu)$, the set of partition $\emptyset \leqslant \lambda \leqslant \mu$

(inclusion of Ferrers diagrams Young)

with $\underset{\text{Tamari lattice}}{T \leqslant T'} \Rightarrow \underset{\text{Young lattice}}{f(T) \geqslant f(T')}$

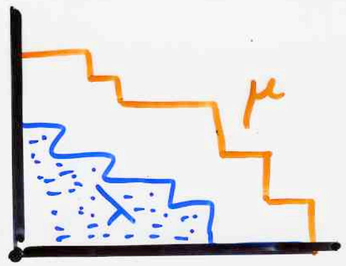
$$J(w) \xrightarrow[\text{bijection}]{f} I(\mu)$$

Young lattice

partition μ

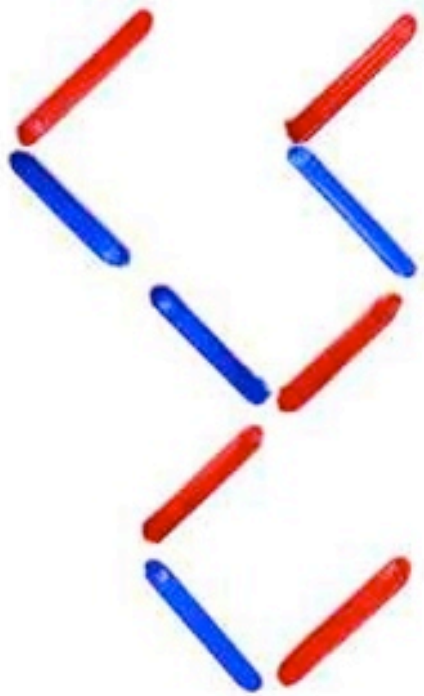
$I(\mu)$

$\lambda \leqslant \mu$



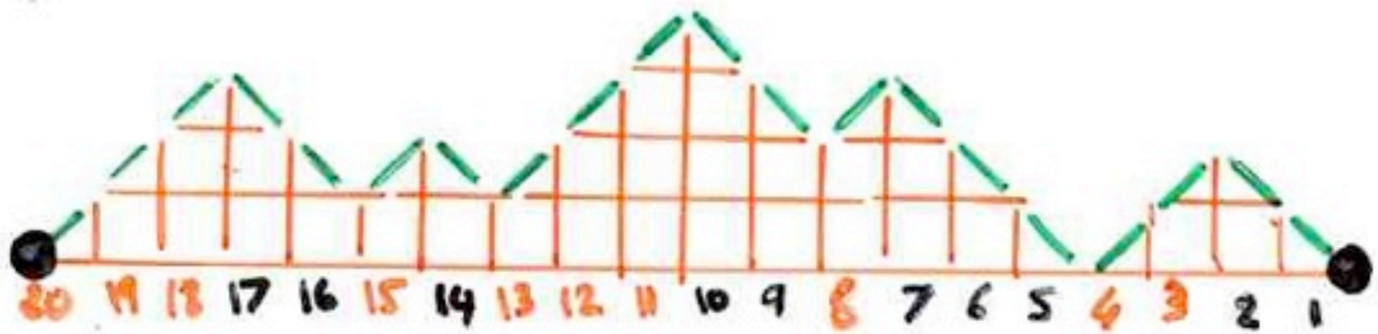
initial segment in the Young lattice (or lower ideal)

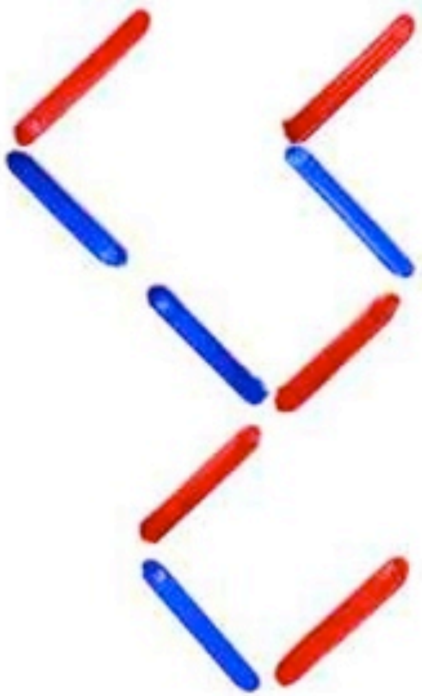
a Catalan bijection



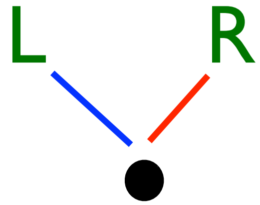
binary
tree

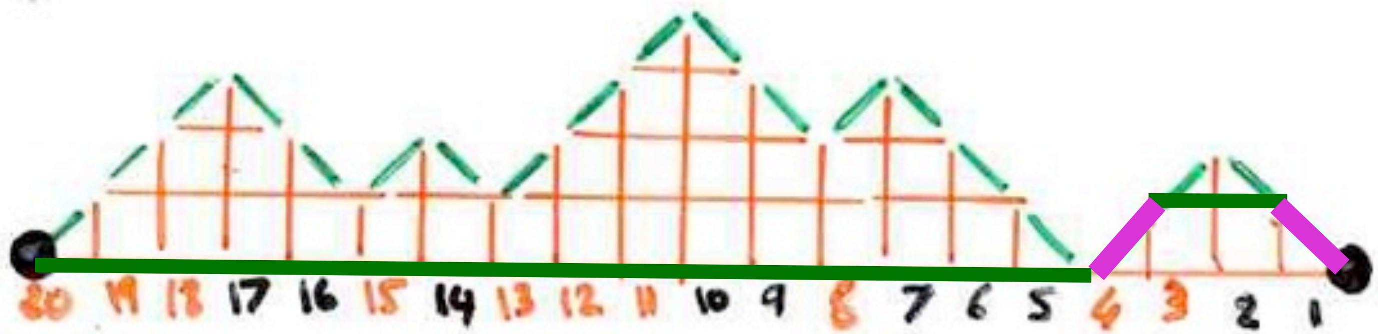
Dyck path



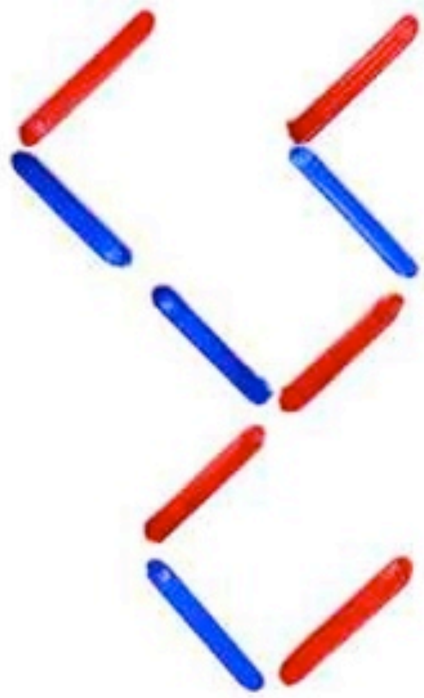


binary
tree

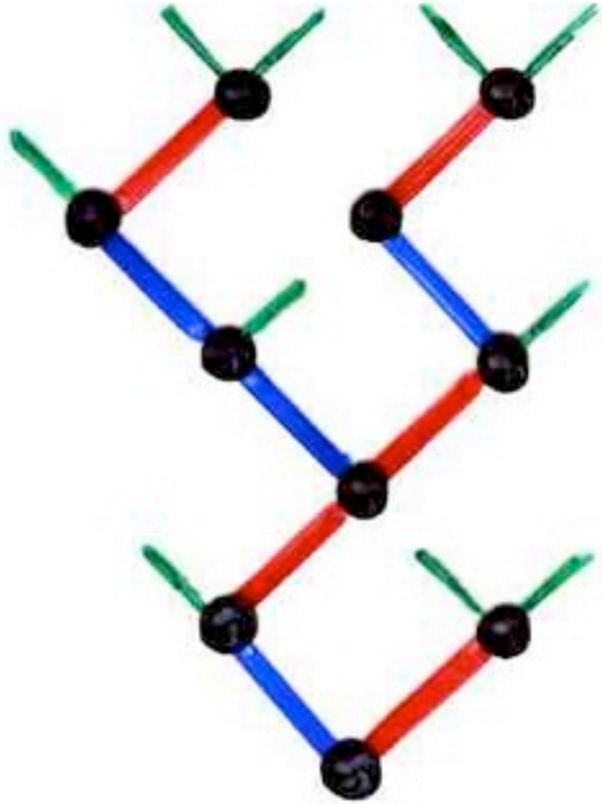


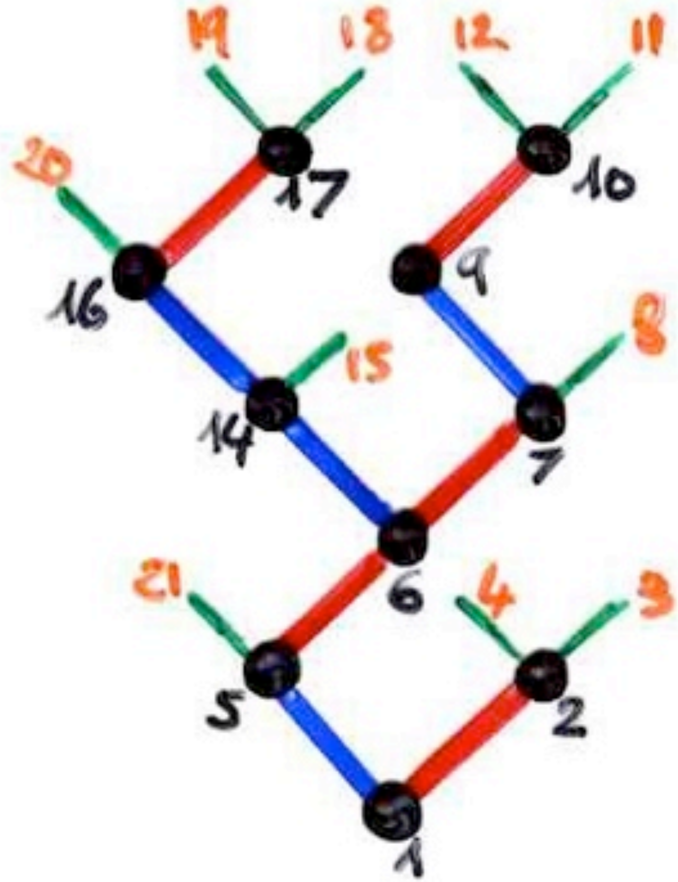


binary
tree



complete
binary
tree





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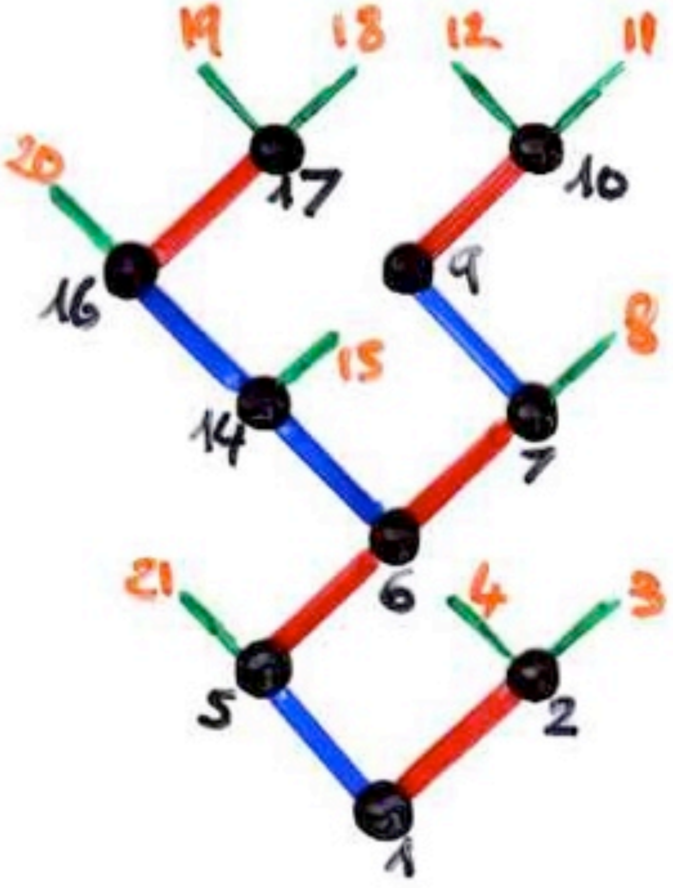
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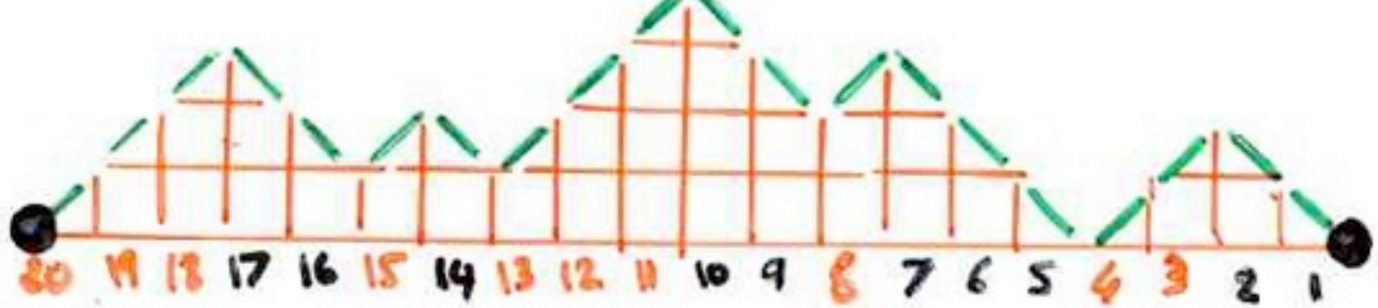
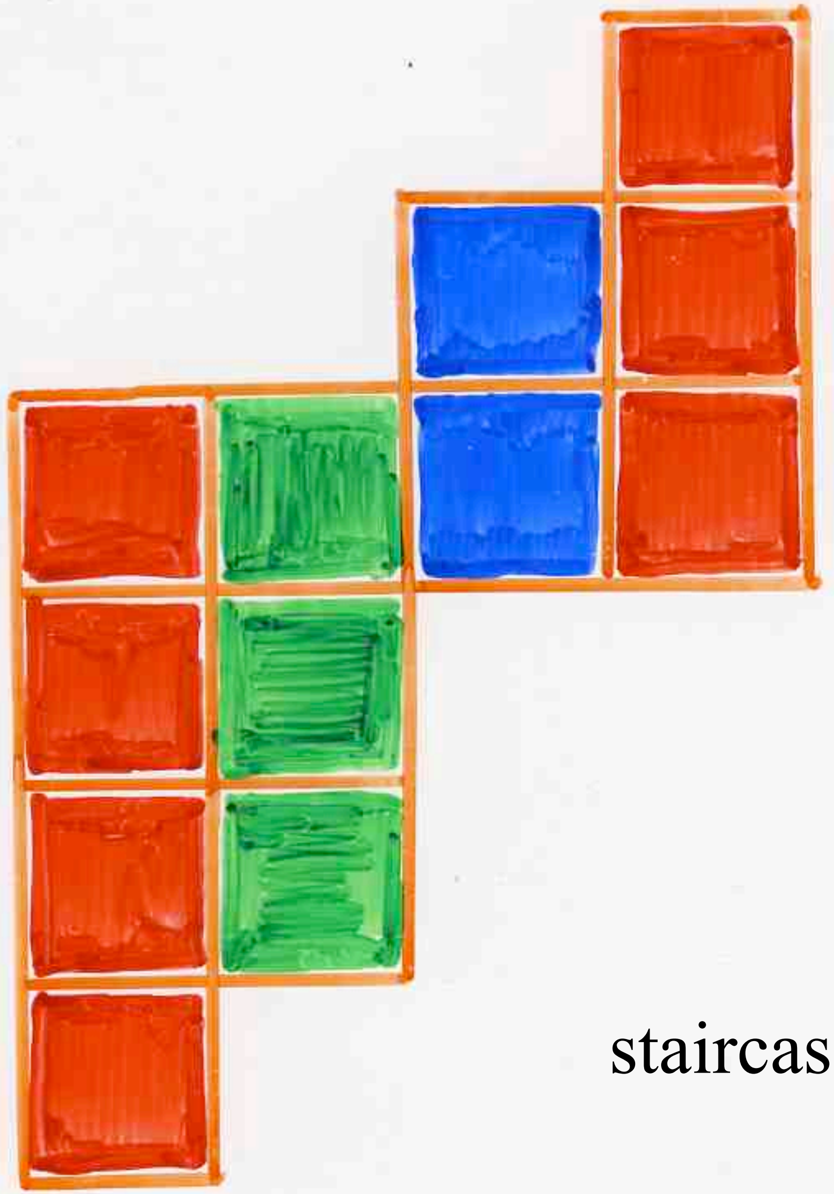
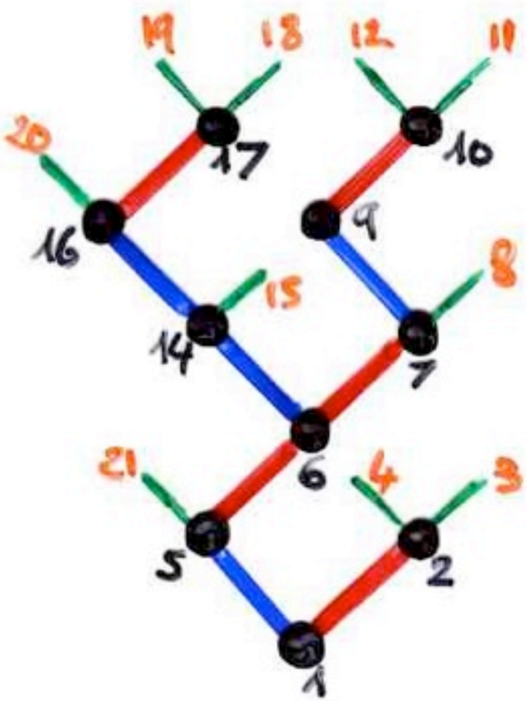

20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1



staircase polygon
or
parallelogram polyomino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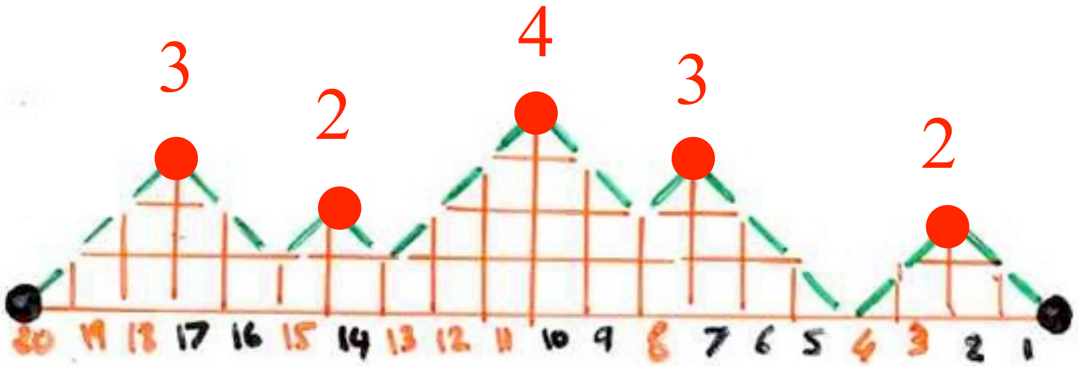
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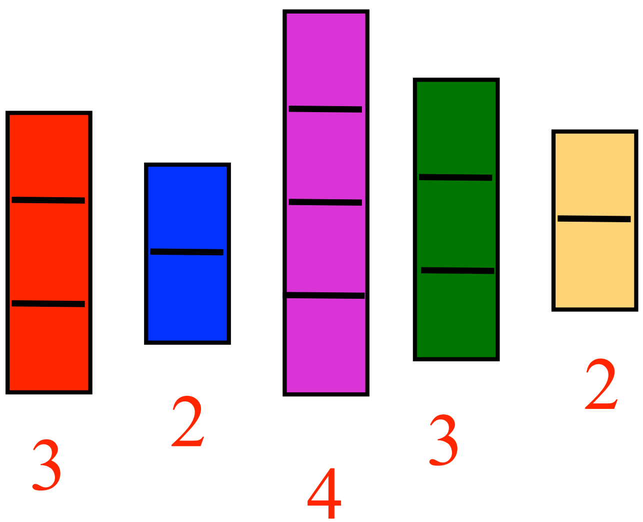
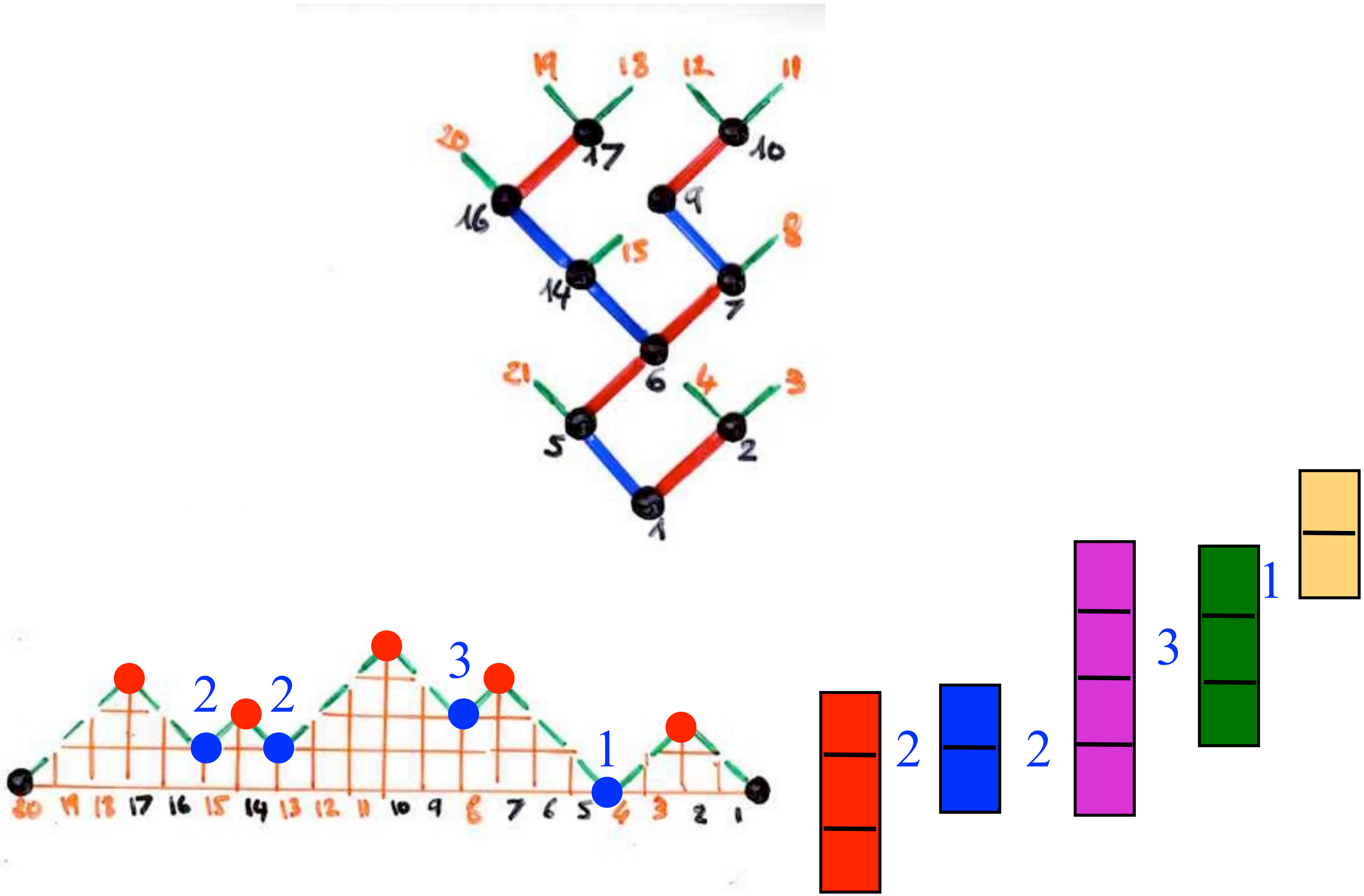
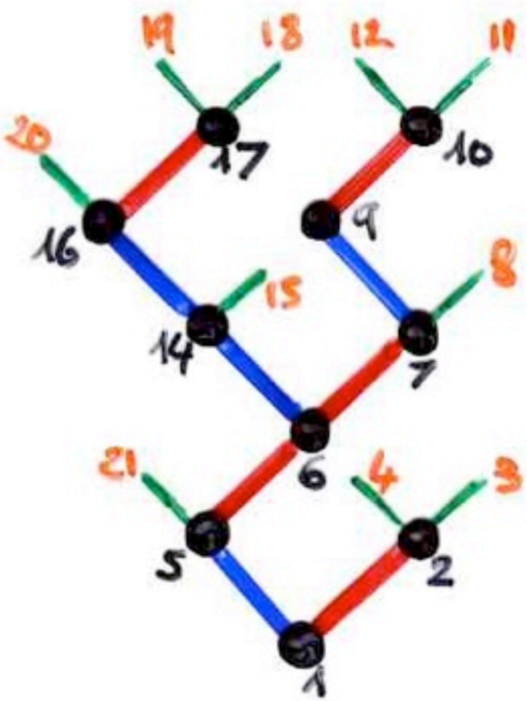
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1


3
2
4
3
2



2 2 3 1
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1
2 2 3 1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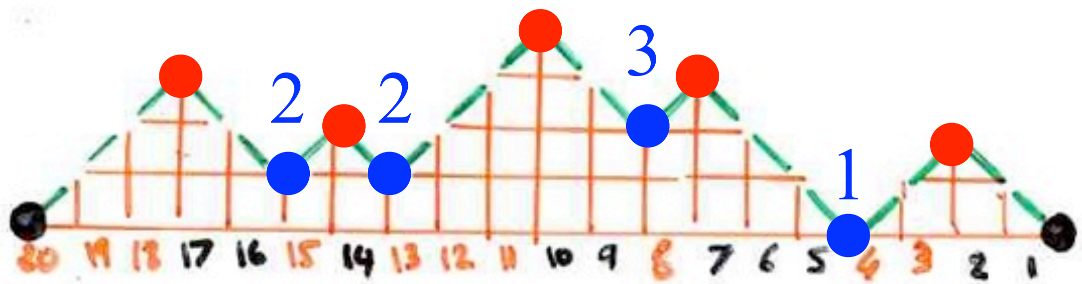
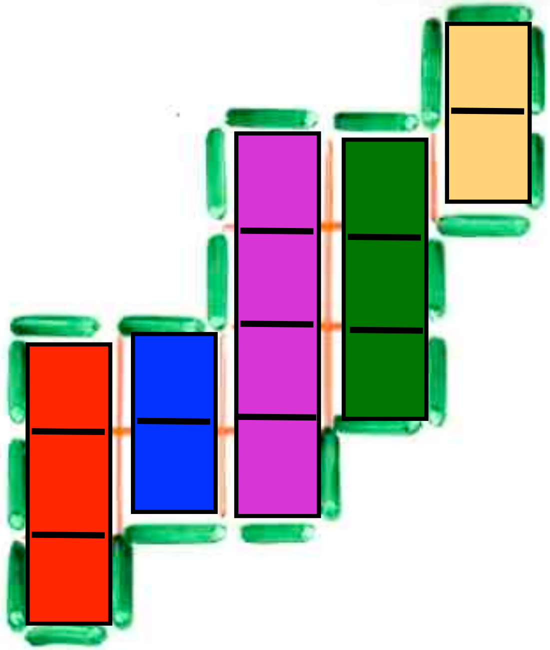
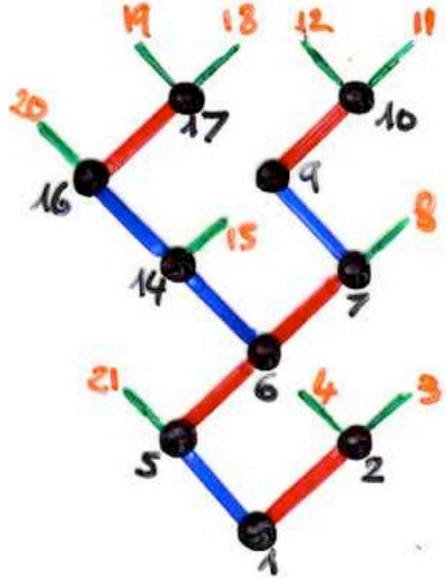

2 2 3 1
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

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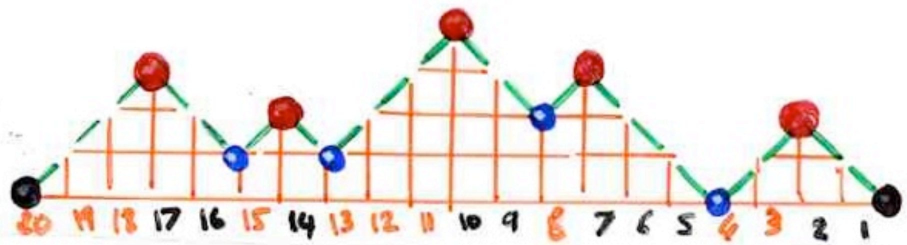
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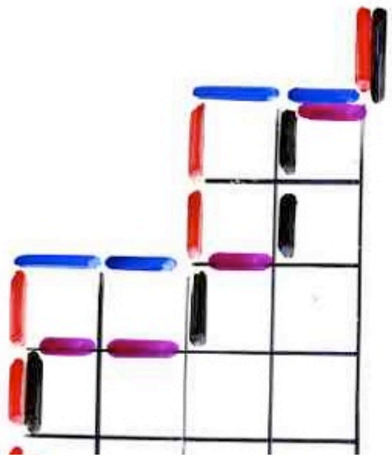
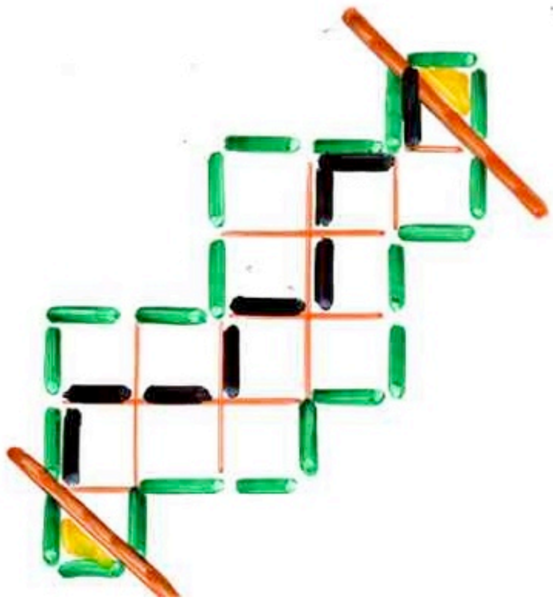
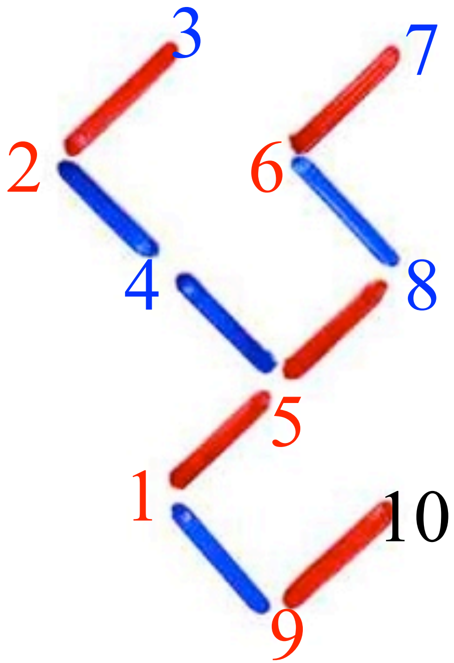




equivalent definition
of this Catalan bijection



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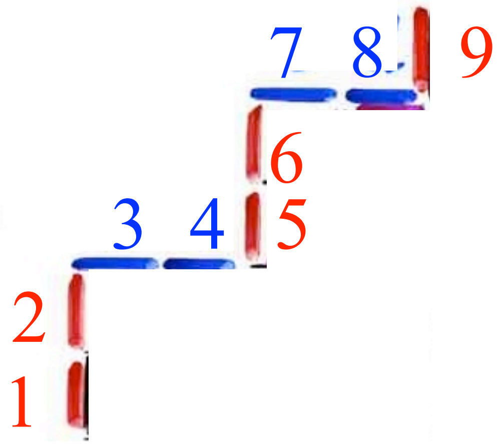

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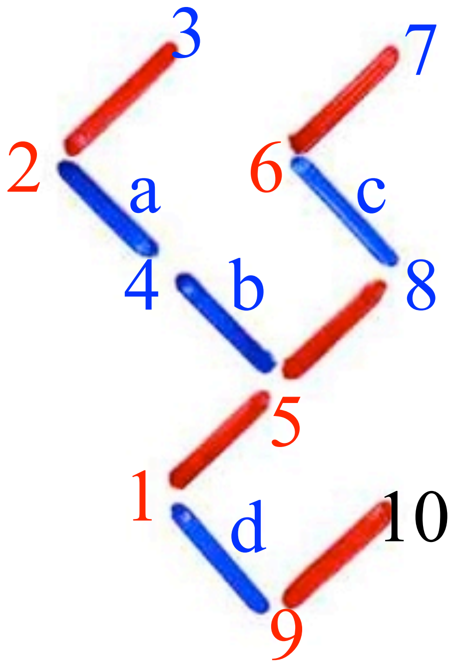
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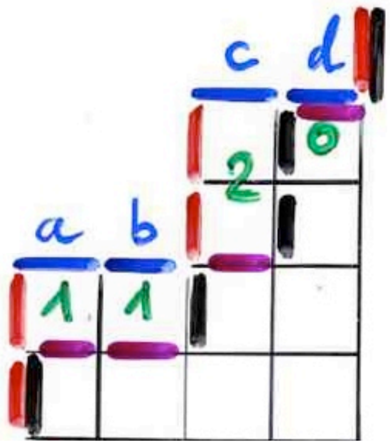
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a
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1
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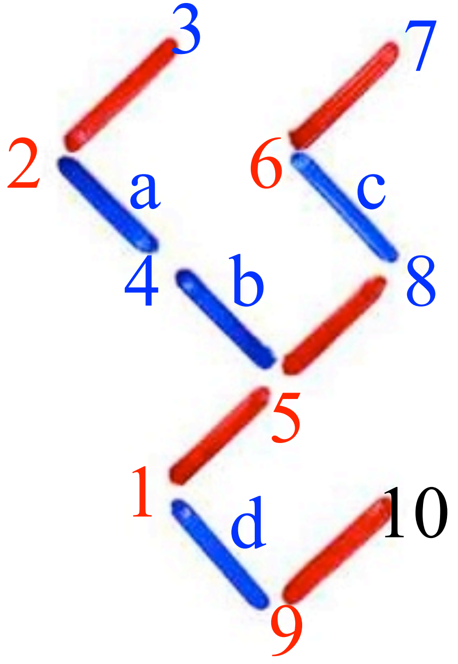
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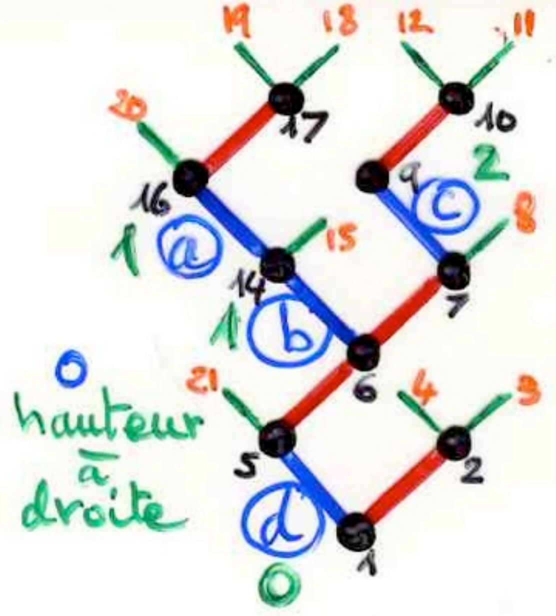
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hauteur à droite 21 4 3
5 2
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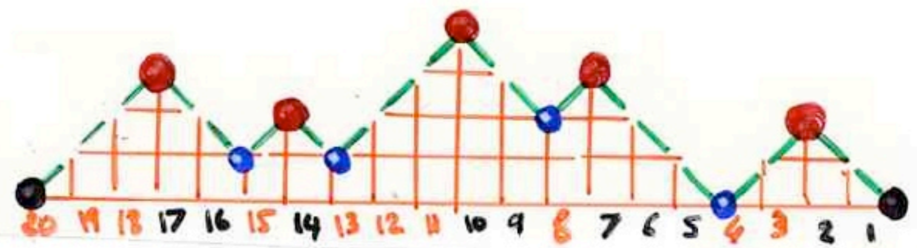
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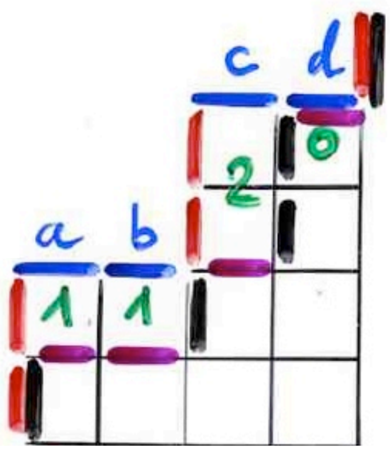
c d
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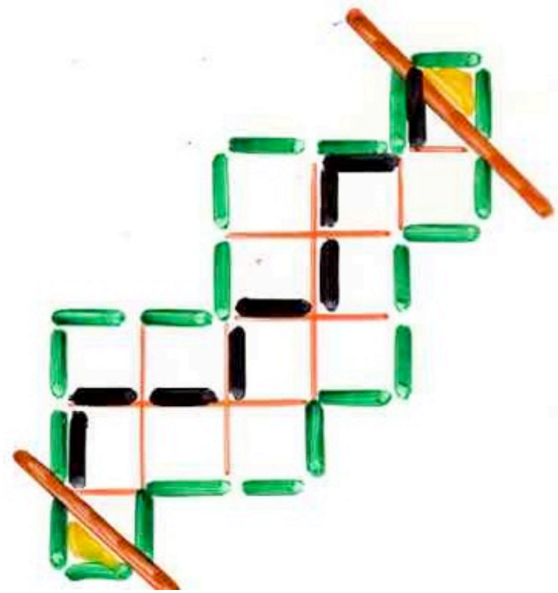
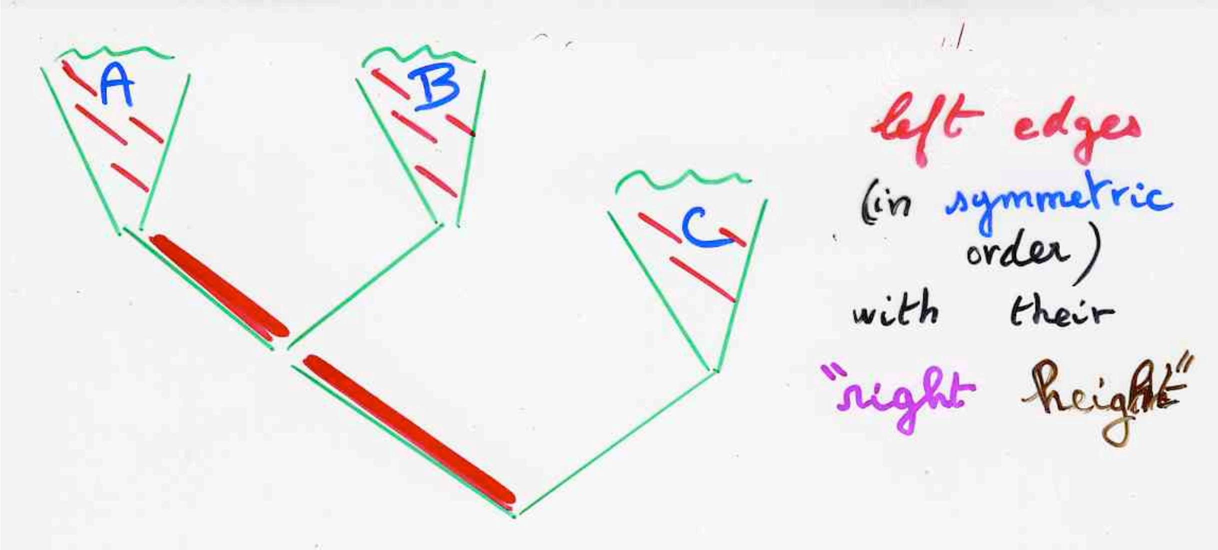
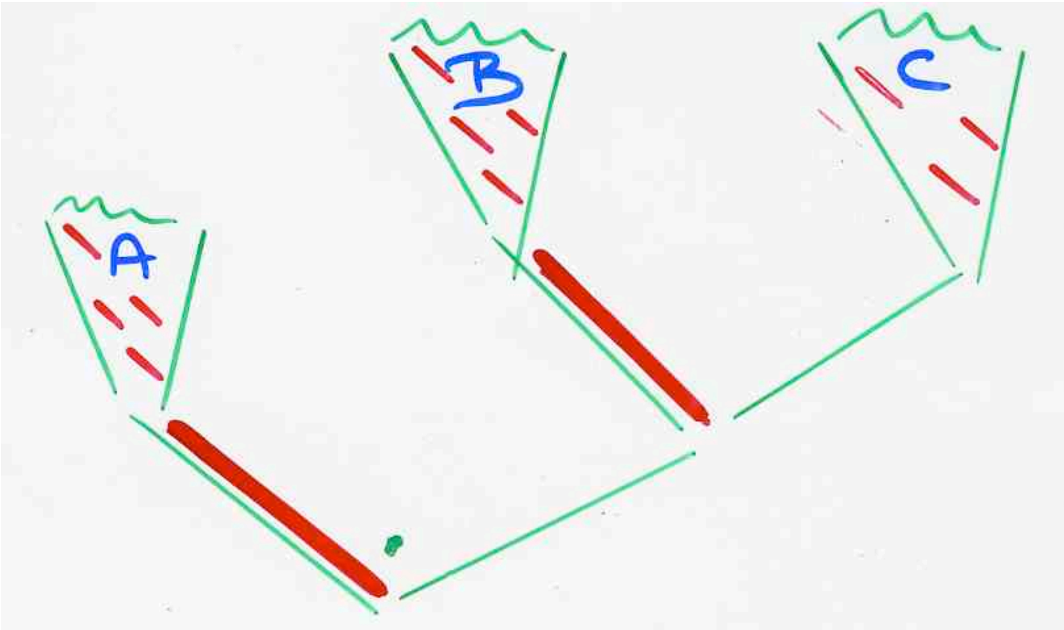
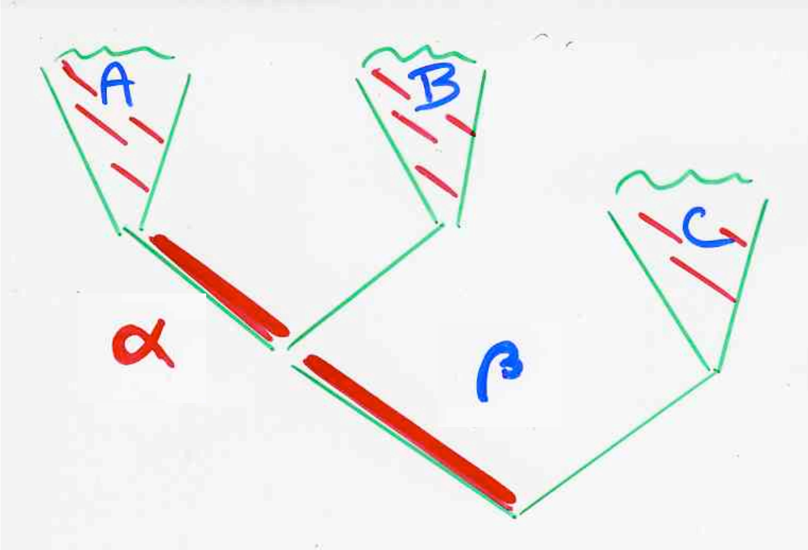
2
a b
1 1



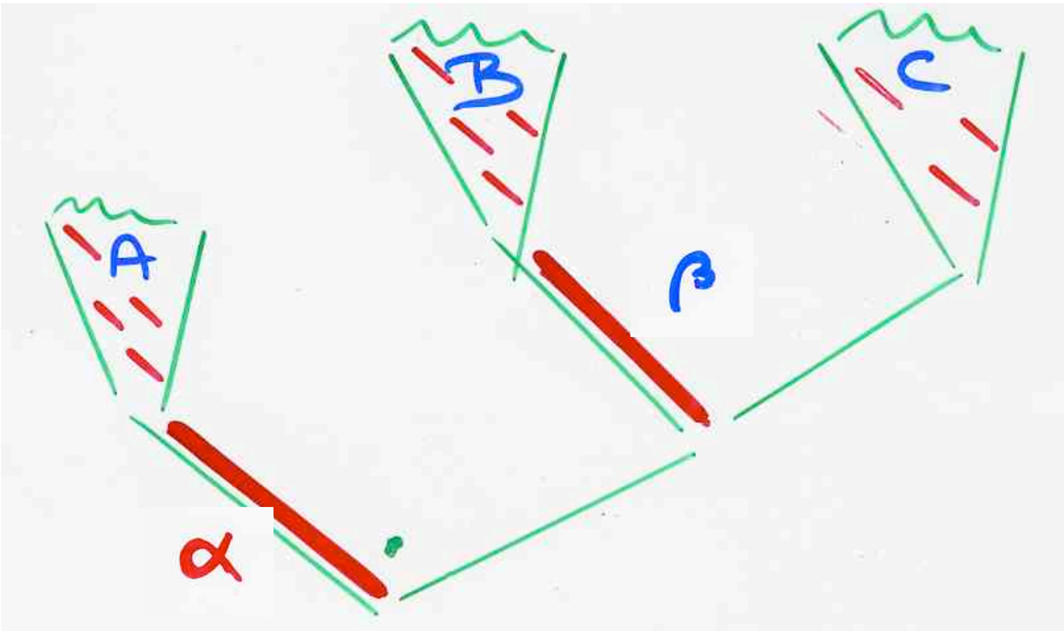
proof of the proposition


A
B
C
left edges
(in symmetric order)
with their
"right height"


A
B
C

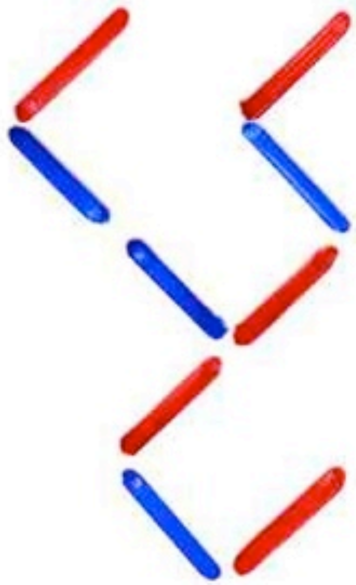


left edges (in symmetric order) with their "right height"



preservation of the symmetric order for left edges

right height:
+1 in C
and for β



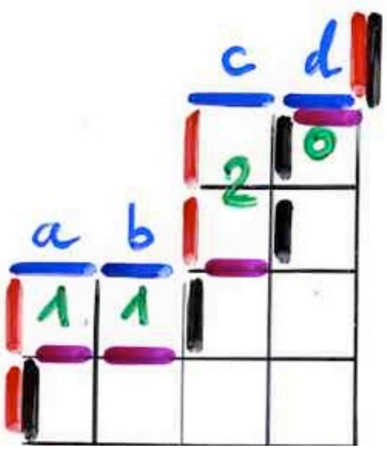
i.e. $\exists$ (integer) partition μ such that $J(w)$ is in bijection with $I(\mu)$, the set of partition $\emptyset \leqslant \lambda \leqslant \mu$

(inclusion of Ferrers Young diagrams)

with $T \leqslant T' \Rightarrow f(T) \geqslant f(T')$

Tamari lattice — Young lattice

$$J(w) \xrightarrow[\text{bijection}]{f} I(\mu)$$

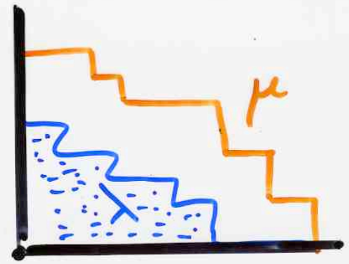


Young lattice

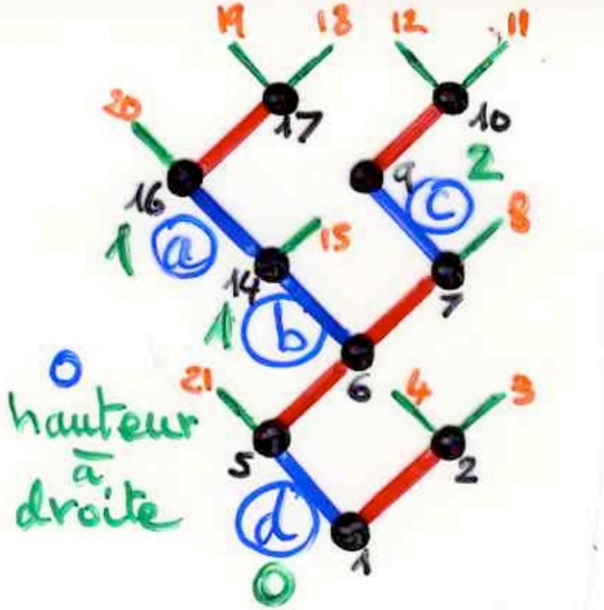
partition μ

$I(\mu)$

$\lambda \leqslant \mu$



possible covering relation
preserving the canopy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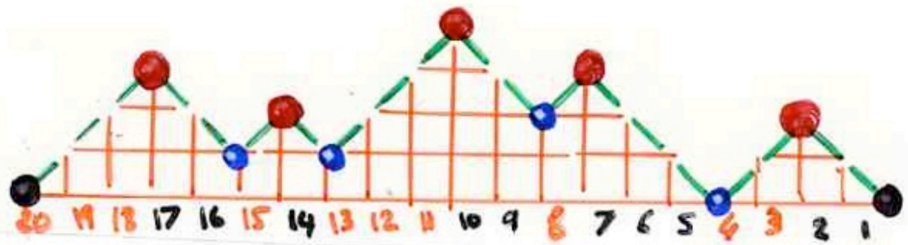
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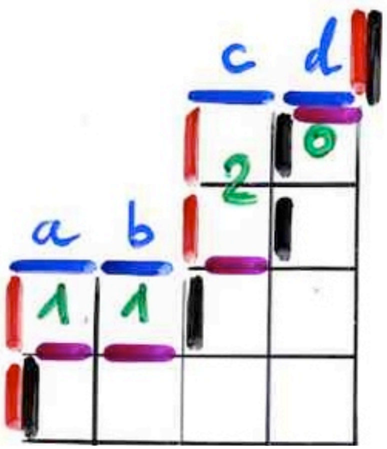
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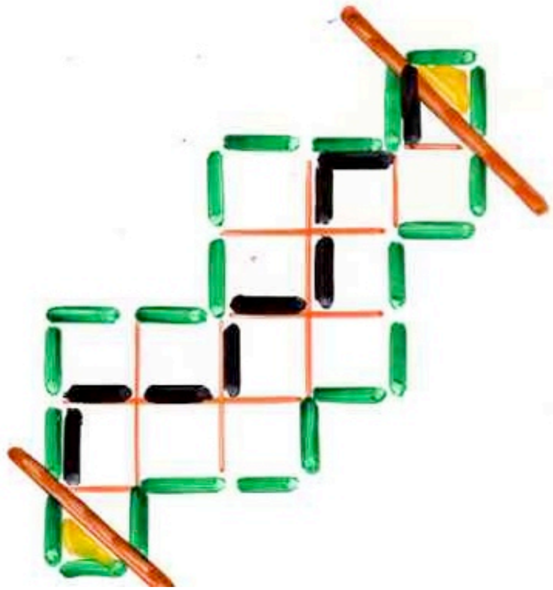

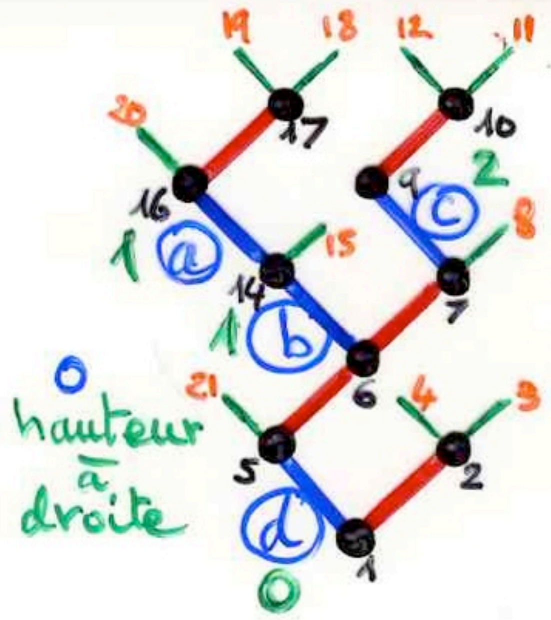
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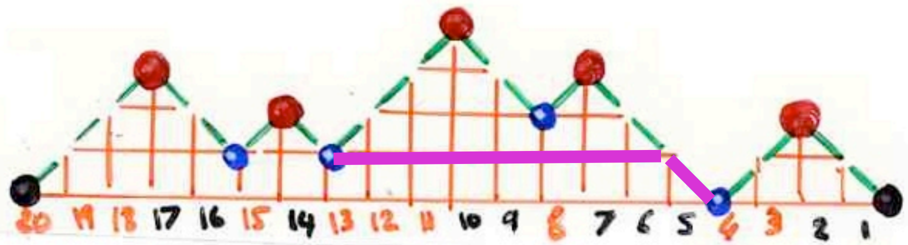
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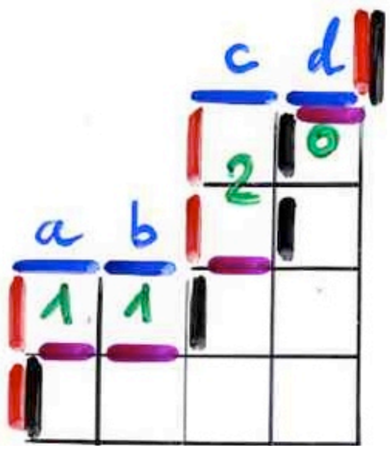
c
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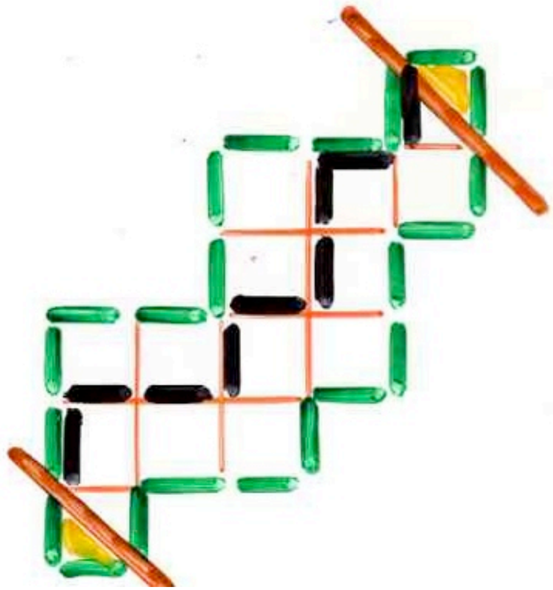

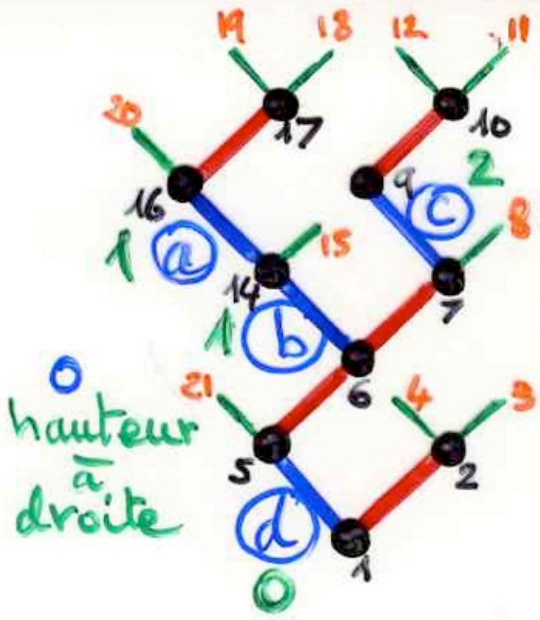
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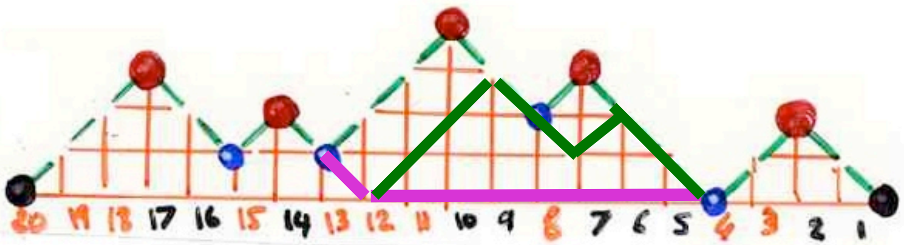
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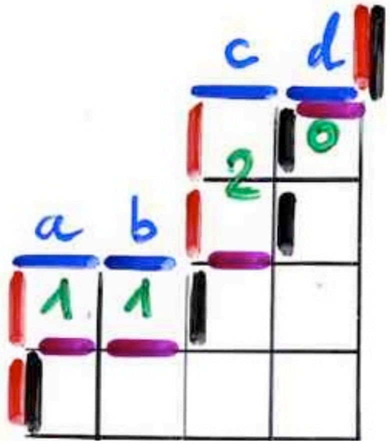
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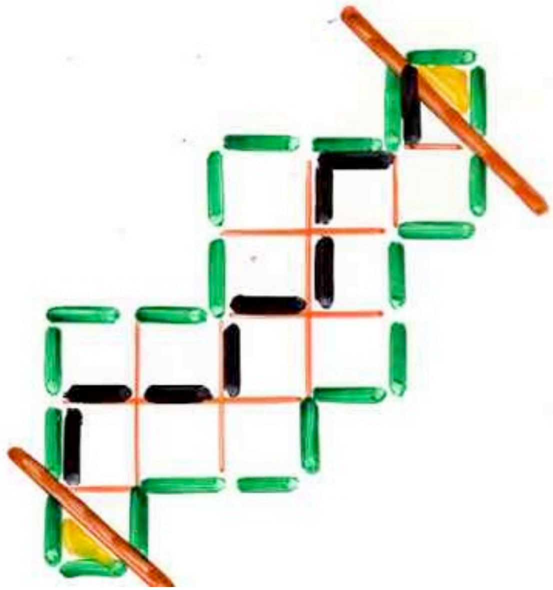

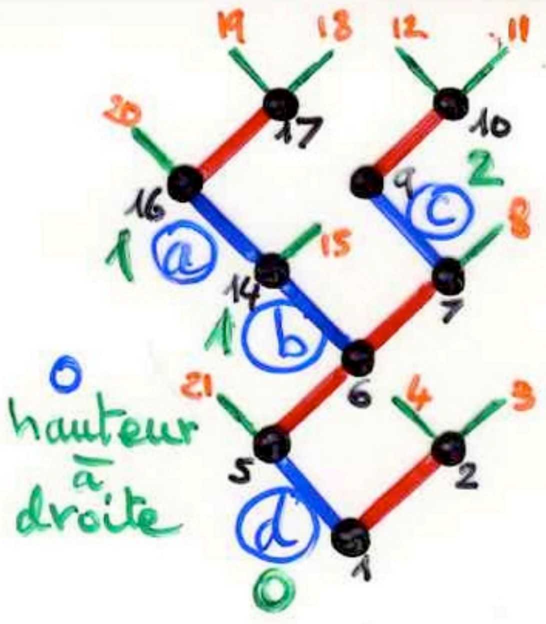
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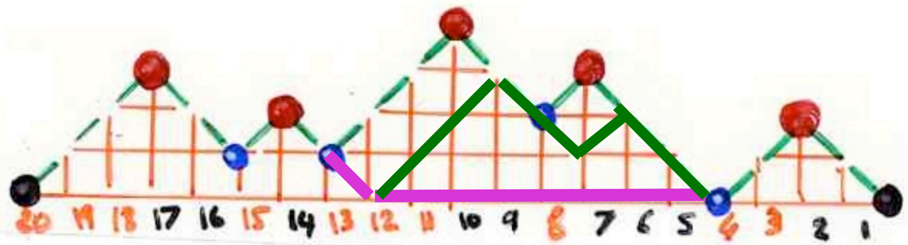
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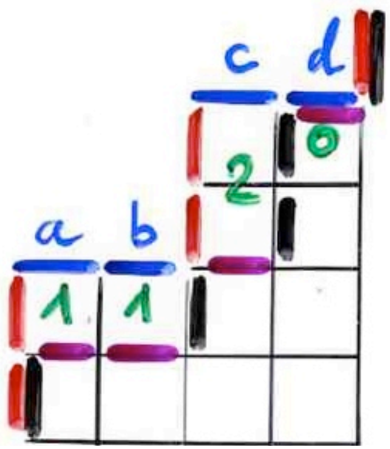
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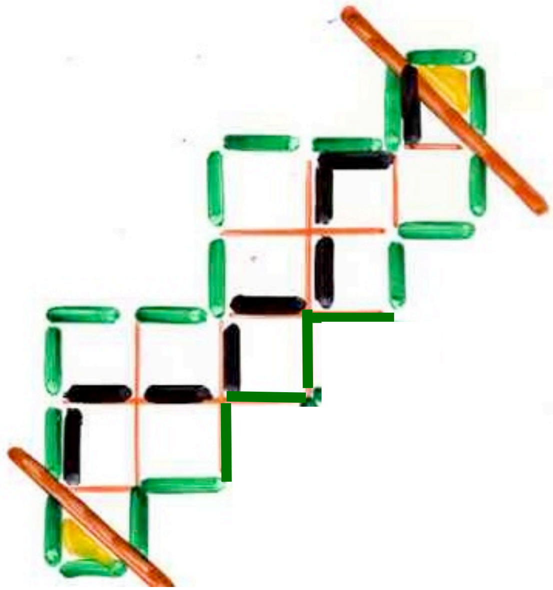

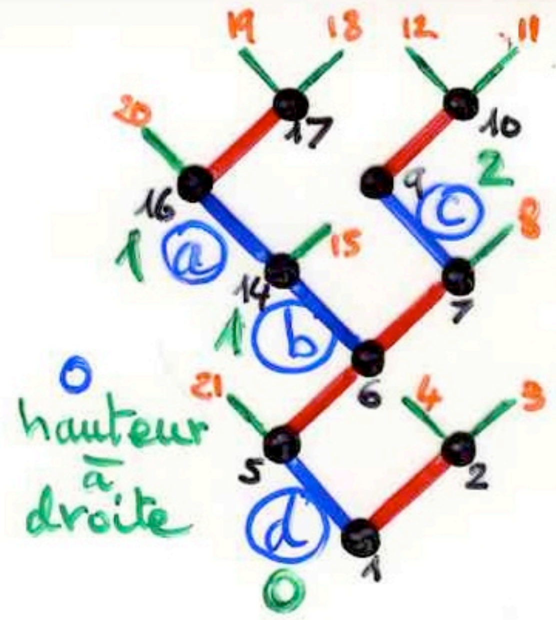
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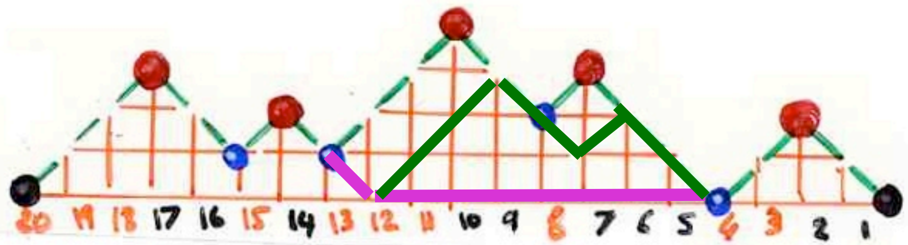
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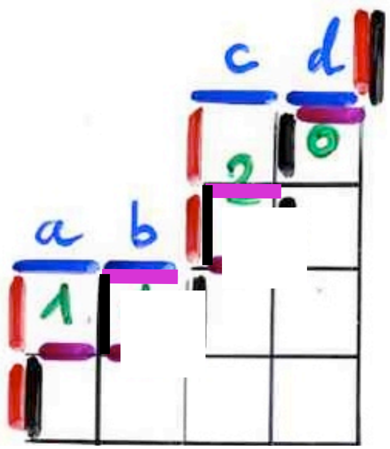
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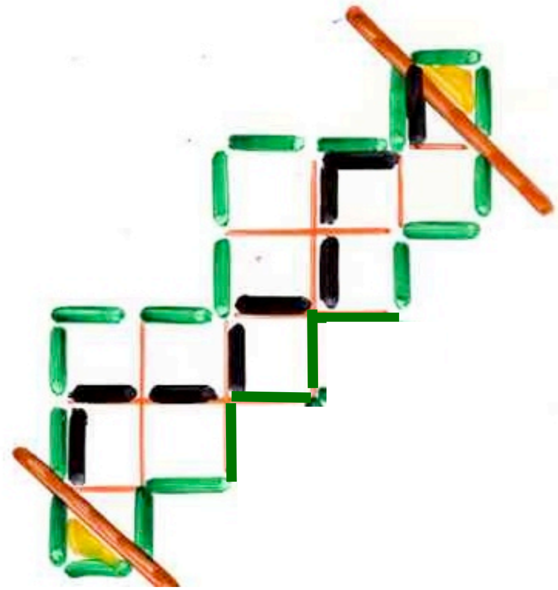

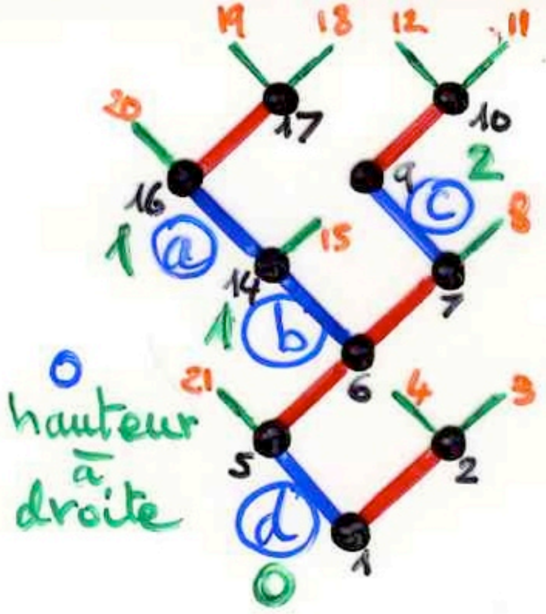
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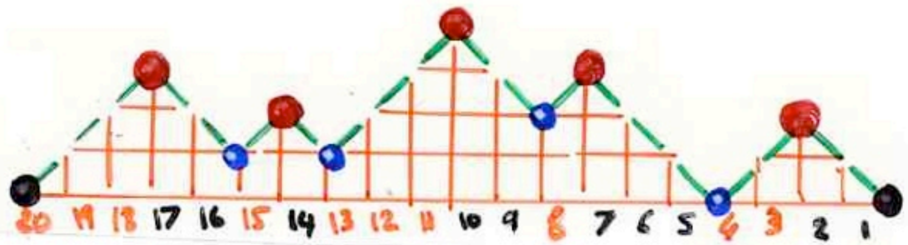
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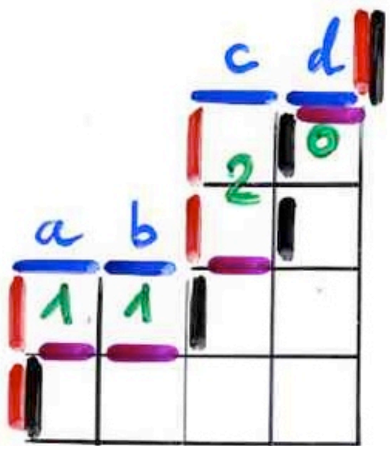
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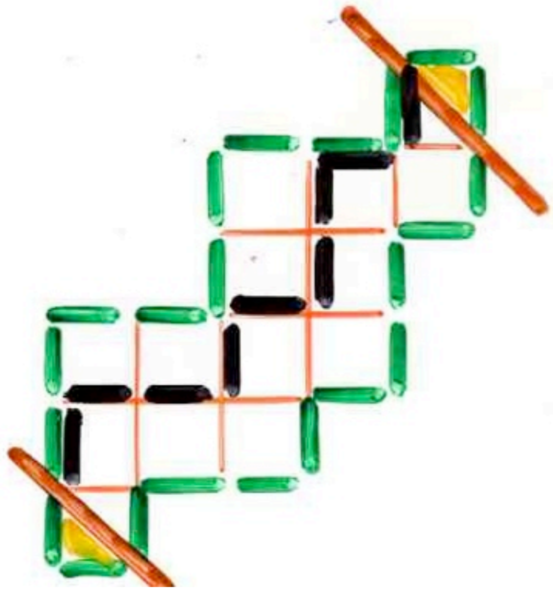
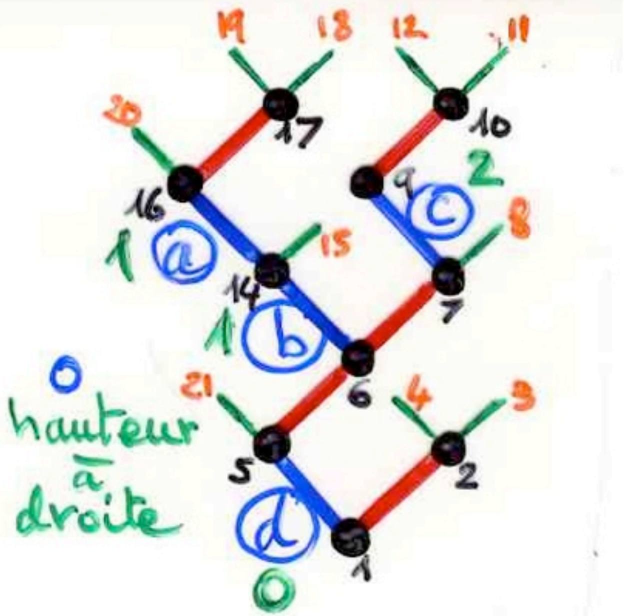
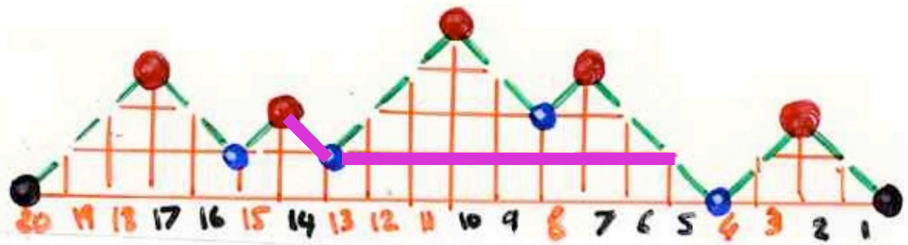
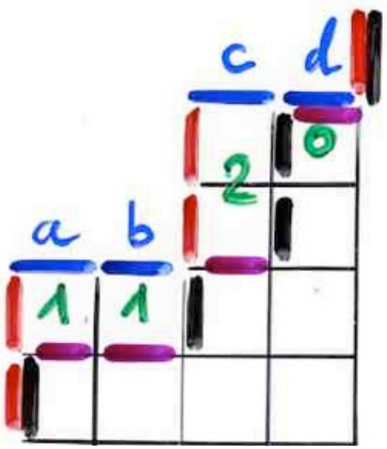
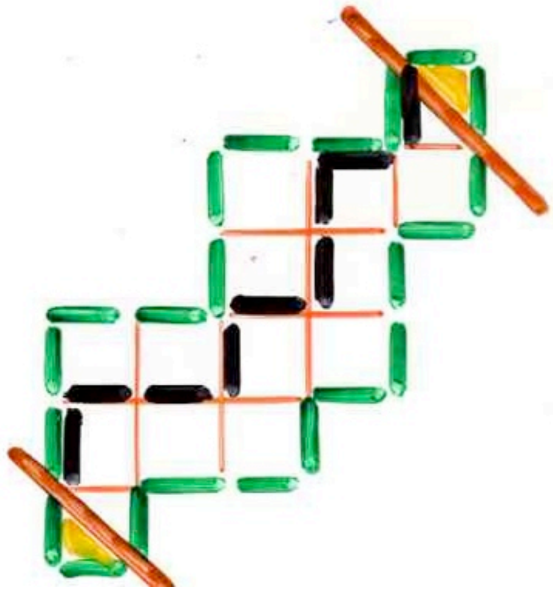


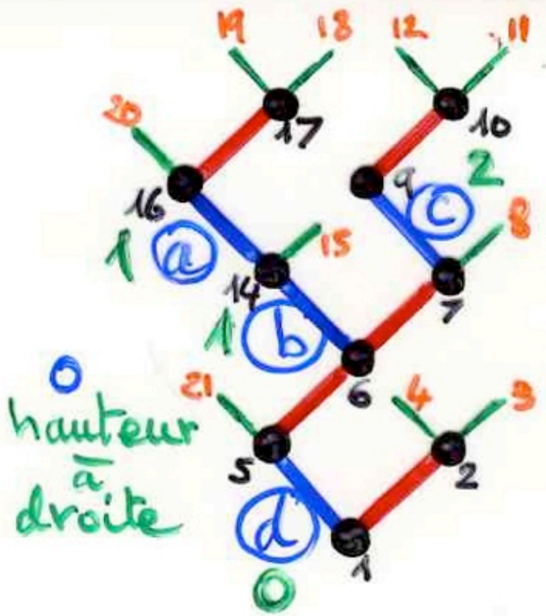
forbidden move







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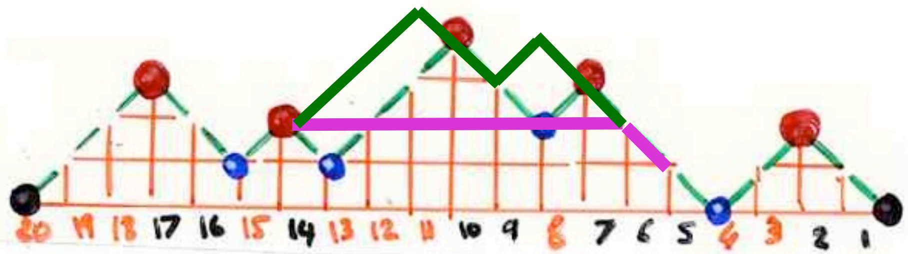
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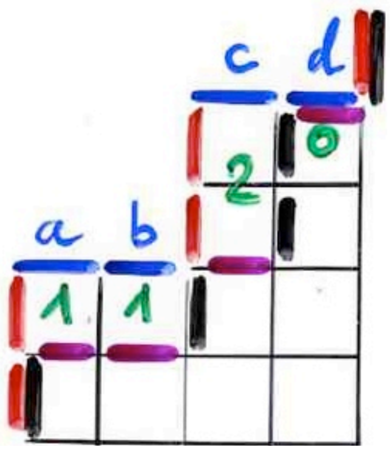
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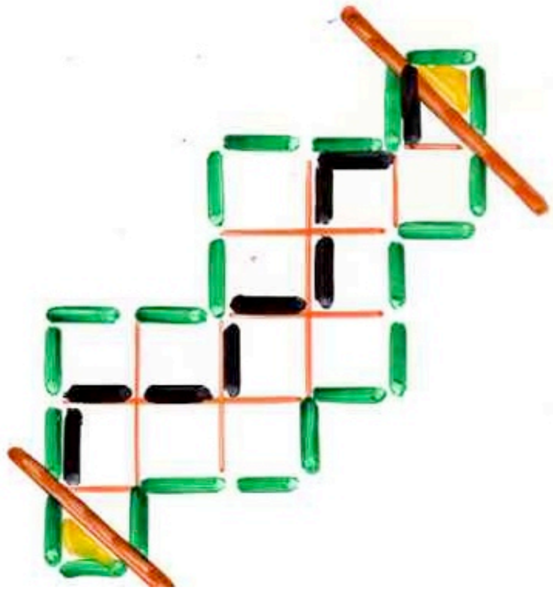
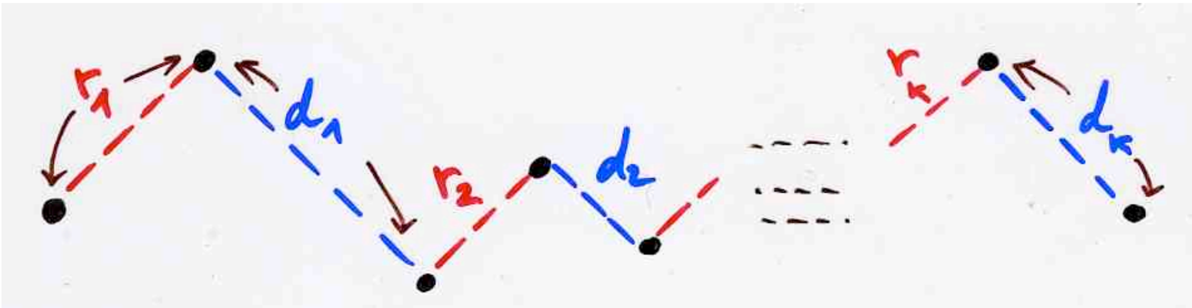
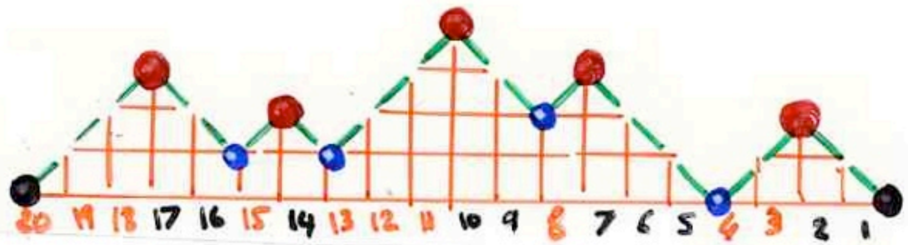


the min and max binary trees



r1
d1
r2
d2
rk
dk



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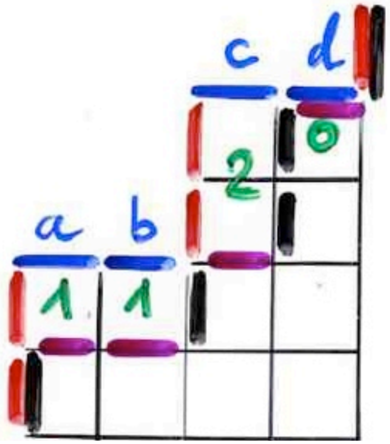

c d
a b
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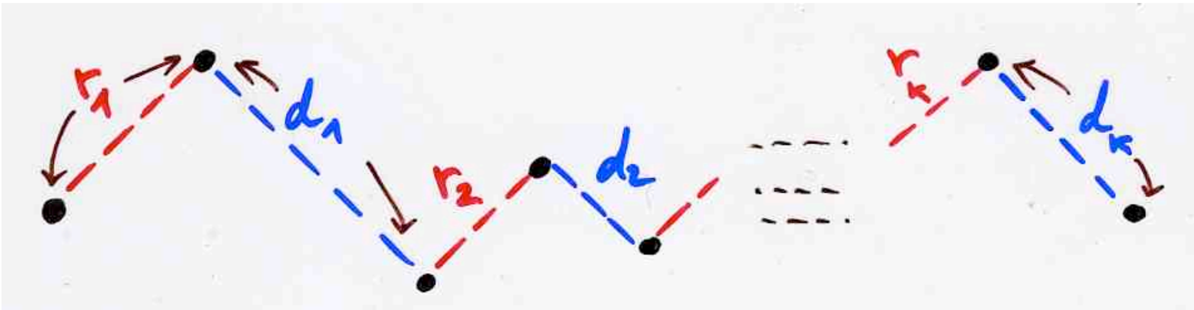
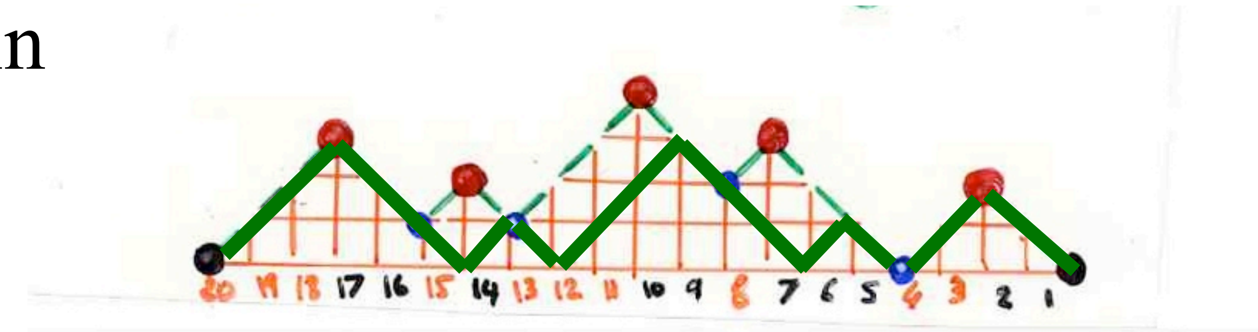

r_1
d_1
r_2
d_2
r_k
d_k

The min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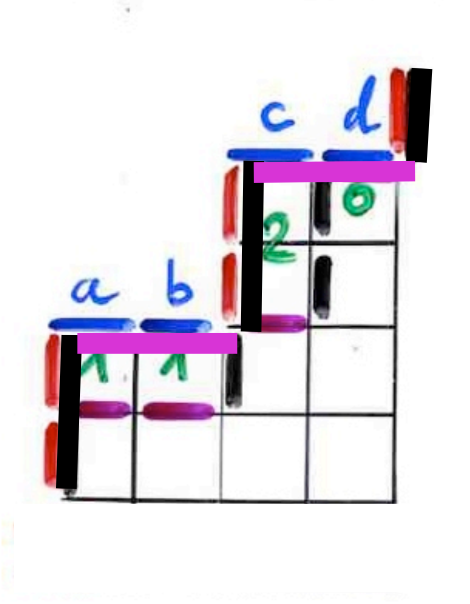
a
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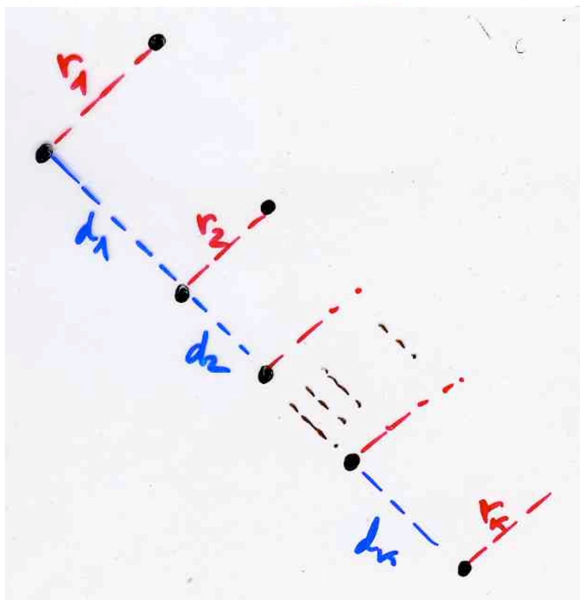
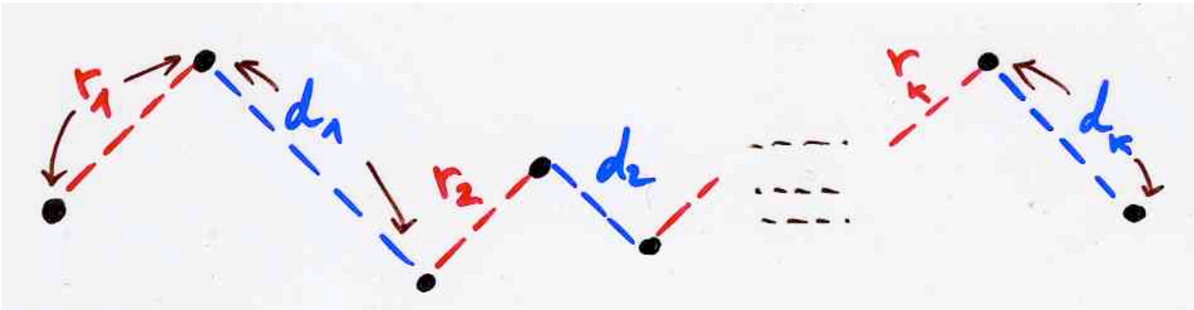
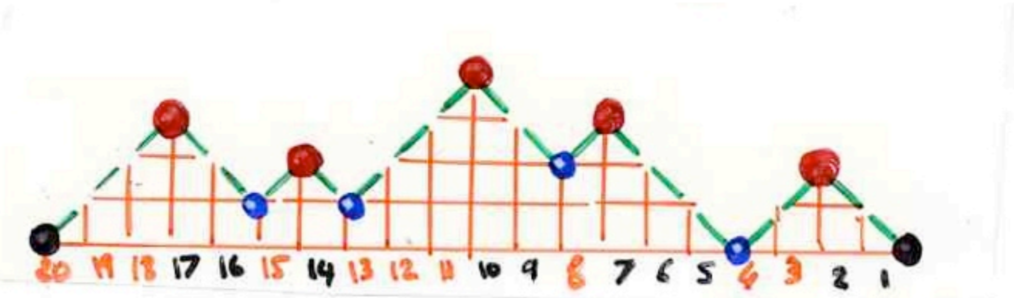

r_1
d_1
r_2
d_2
d_k
r_k



r_1
d_1
r_2
d_2
r_k
d_k



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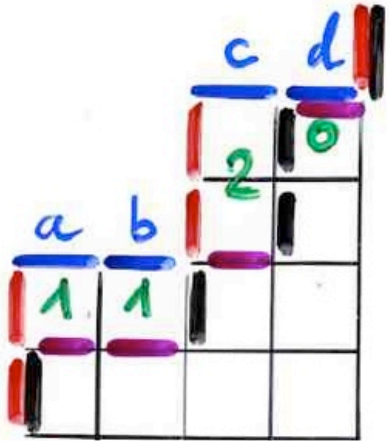

a
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c
d
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2
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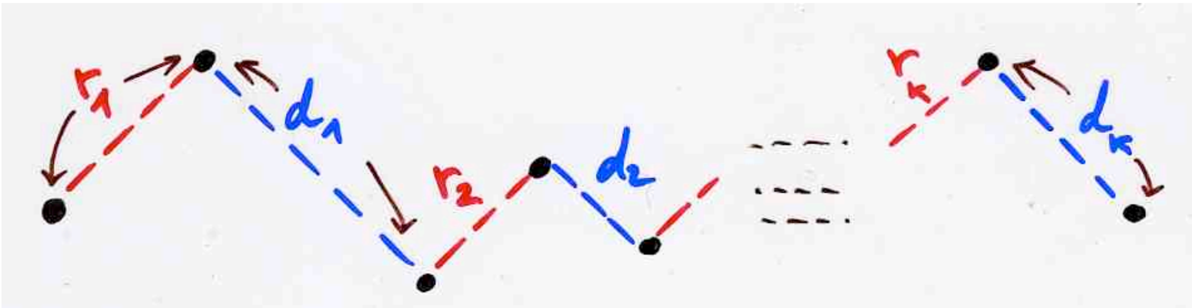
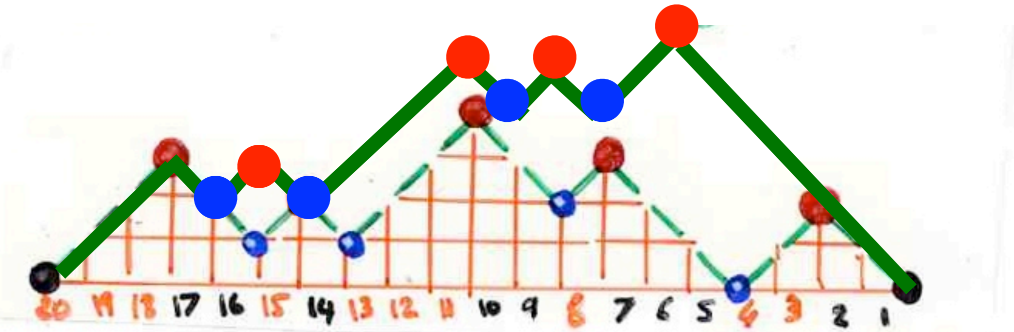


r1
d1
r2
d2
rk
dk

The max



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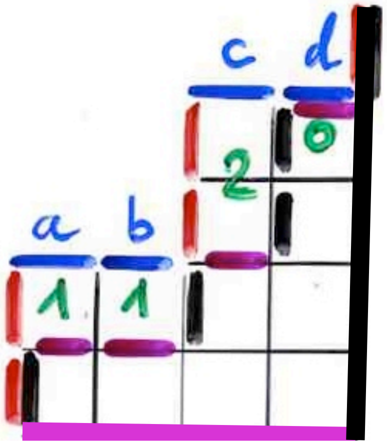

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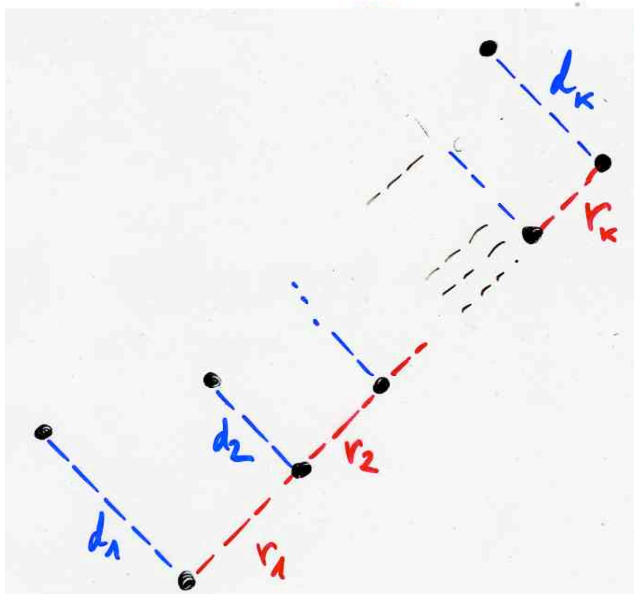

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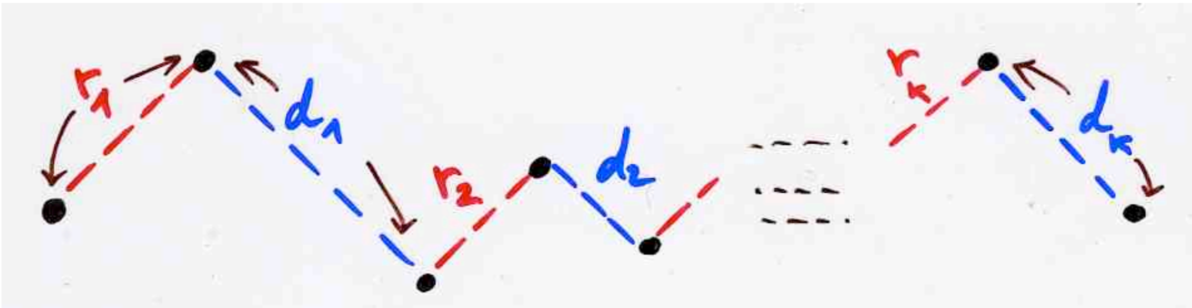
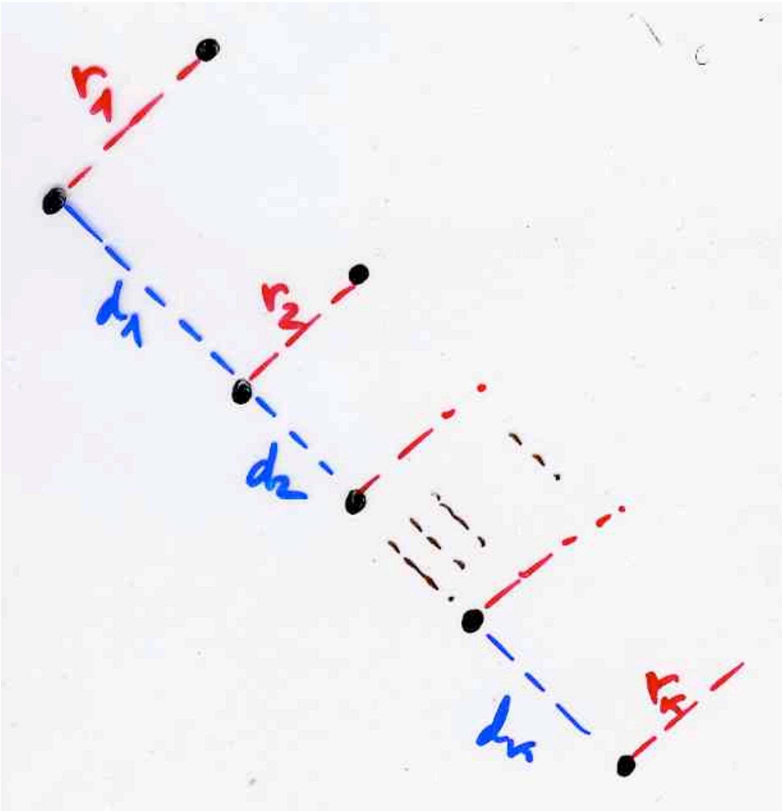
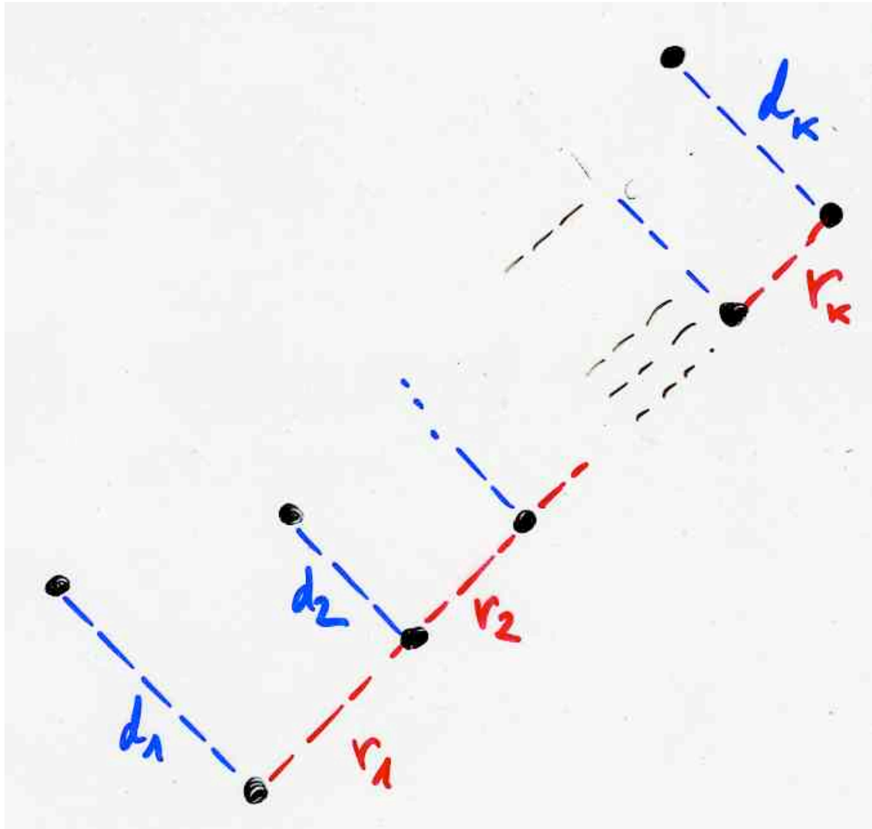
r1
d2
r2
rk
dk



r_1
d_1
r_2
d_2
r_k
d_k



r_1
d_1
r_2
d_2
d_k
r_k



d_k
r_k
d_2
r_2
d_1
r_1

Prop. (i) • The set of binary trees having a given canopy w is an interval $J(w)$ of the Tamari lattice

(ii) • this interval can be extended to an initial interval of the Young lattice

i.e. $\exists$ (integer) partition μ such that $J(w)$ is in bijection with $I(\mu)$, the set of partition $\emptyset \leqslant \lambda \leqslant \mu$ (inclusion of Ferrers diagrams)
Young

with $T \leqslant T' \Rightarrow f(T) \geqslant f(T')$
Tamari lattice — Young lattice

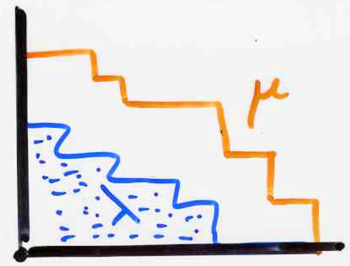
$$J(w) \xrightarrow[\text{bijection}]{f} I(\mu)$$

Young lattice

partition μ

$I(\mu)$

$\lambda \leqslant \mu$



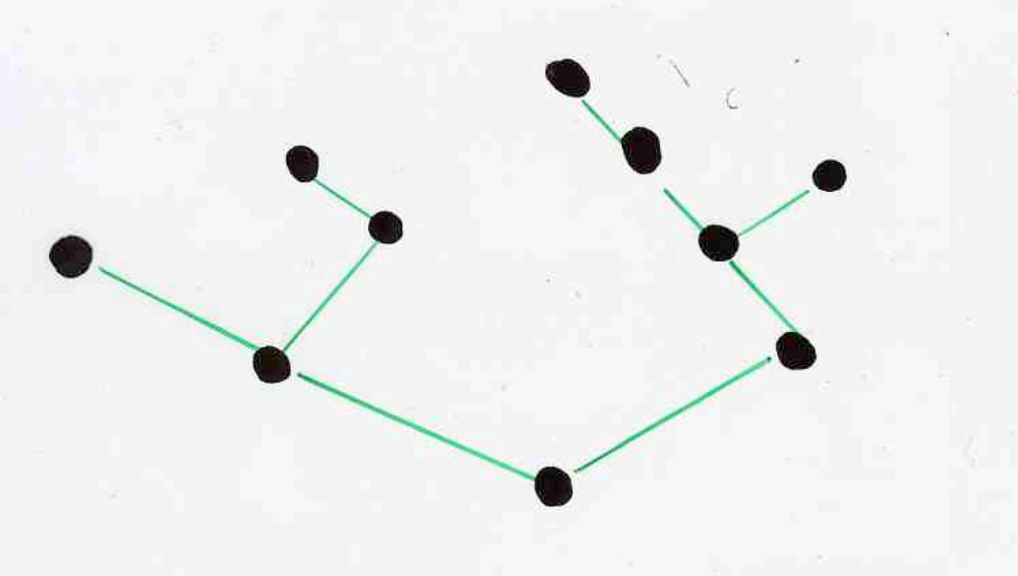
initial segment in the Young lattice (or lower ideal) canopy,

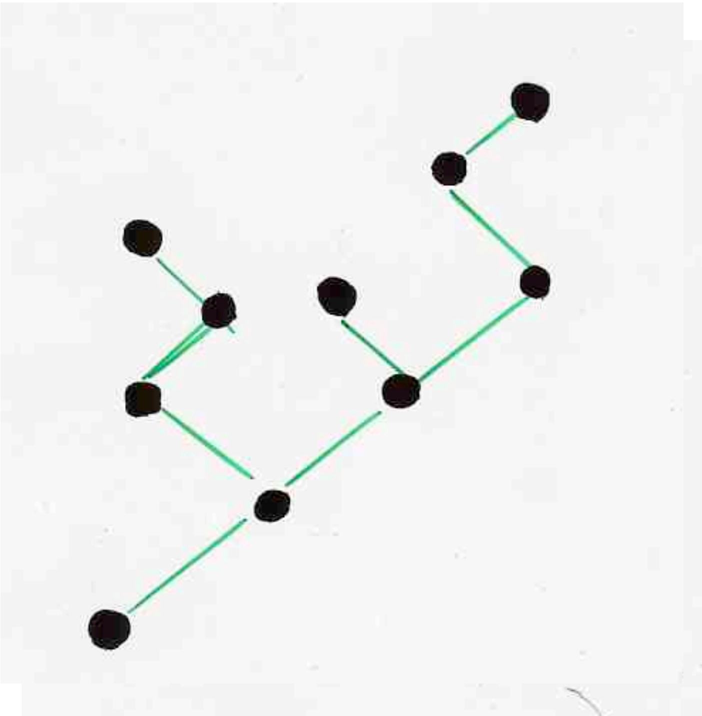
interval in the Tamari lattice

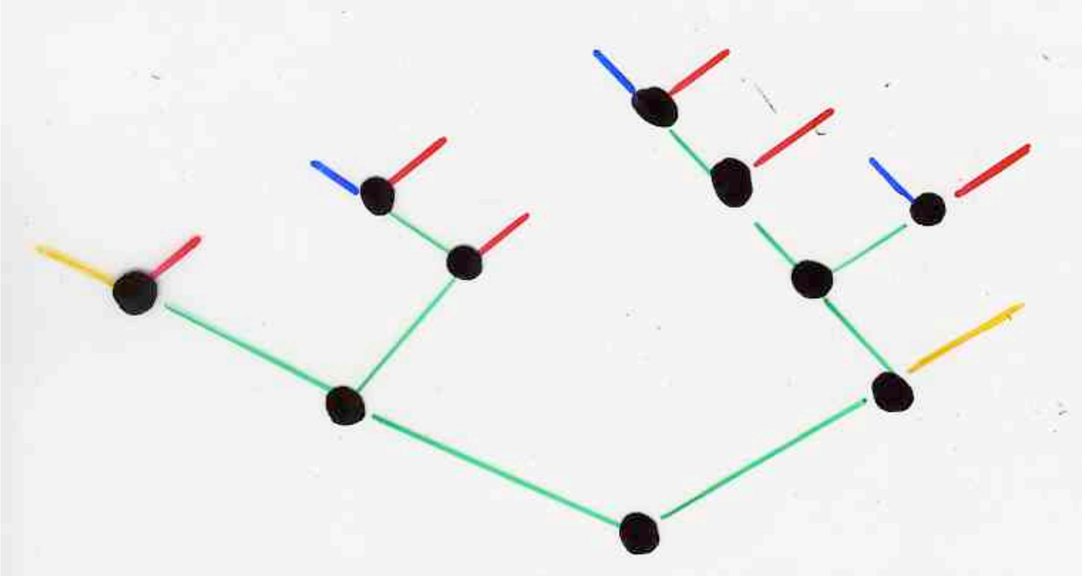
up-down seqence

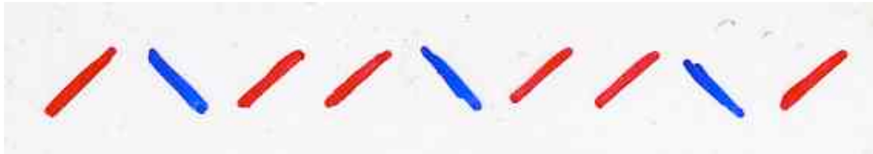
and

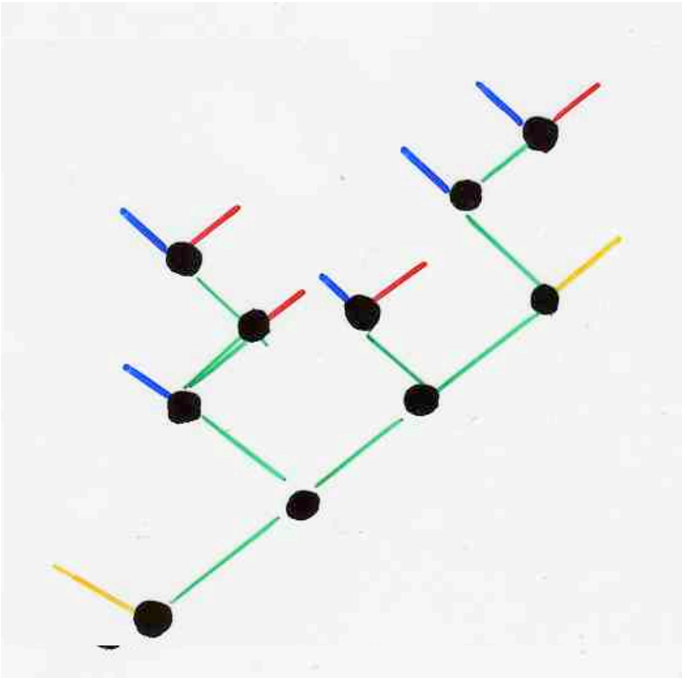
intervals in the permutations lattice





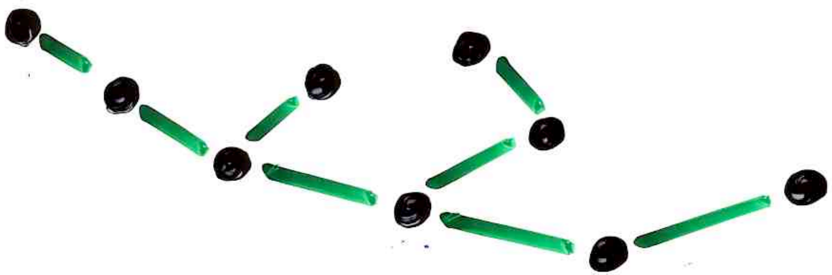


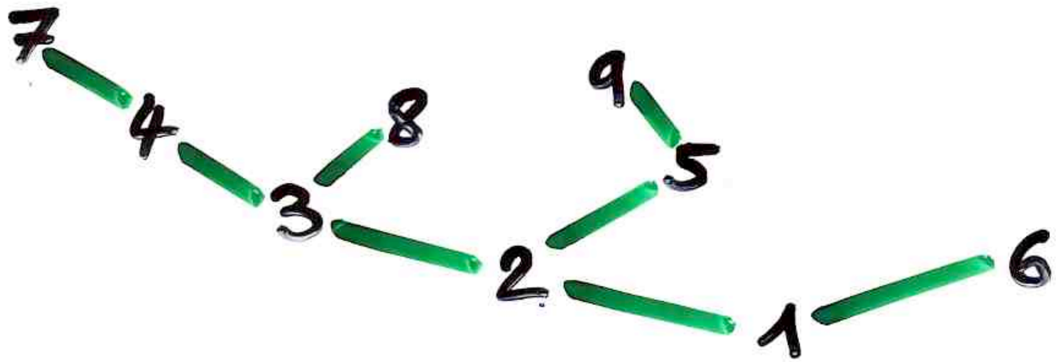




Bijection increasing binary tree $\longleftrightarrow$ permutation

T σ



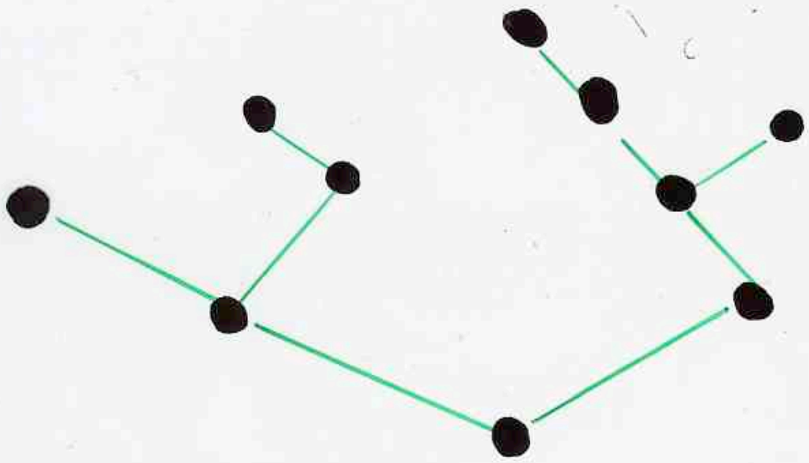


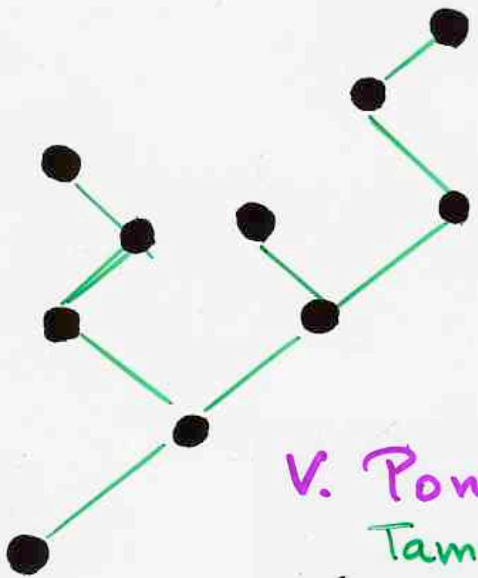
$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$

A. Björner, M. Wachs (1991)

$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6 \ldots$

up-down sequence $-\ -\ +\ -\ +\ -\ -\ +$



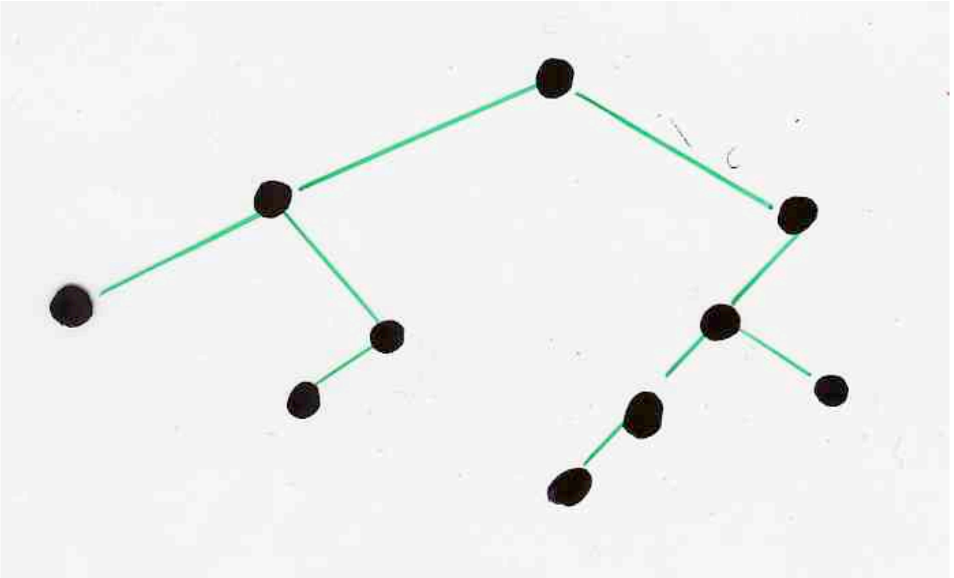


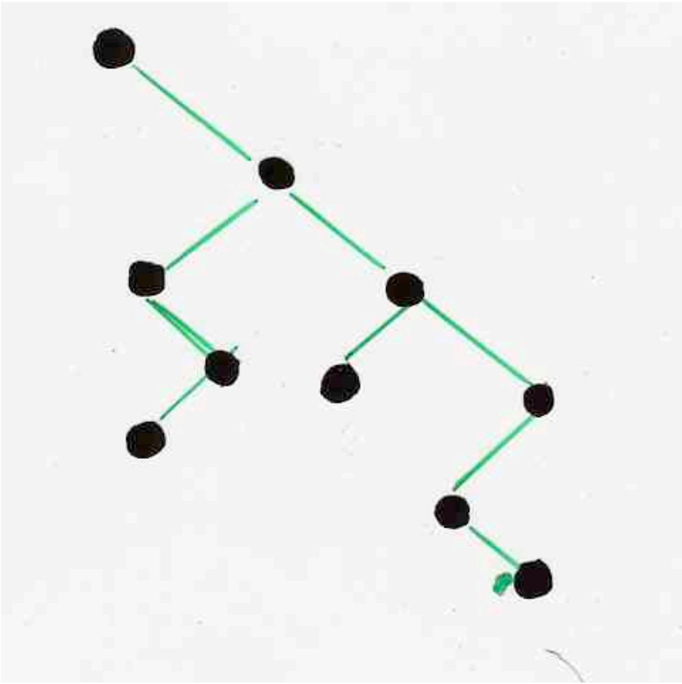
V. Pons

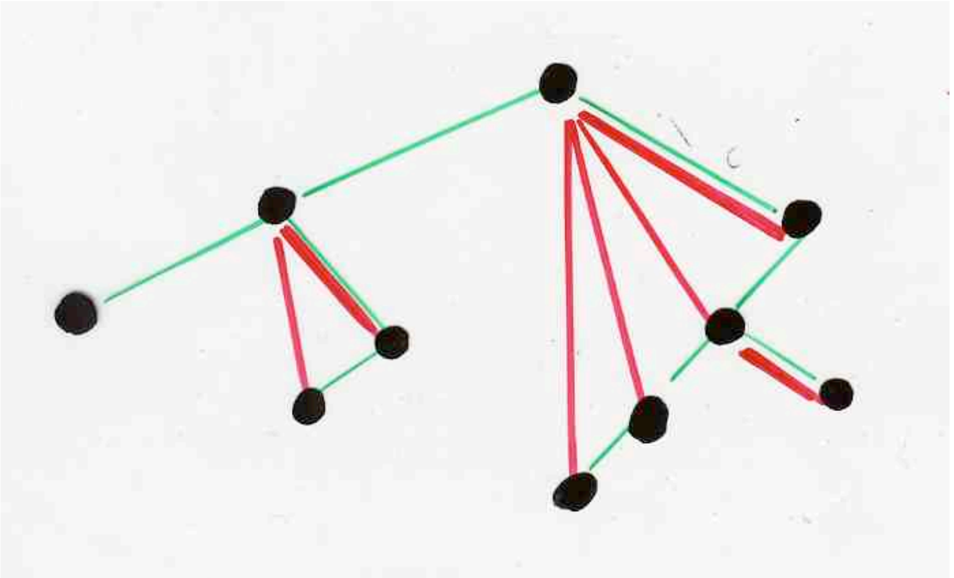
Tamari interval-posets $\Longleftrightarrow$ Tamari intervals

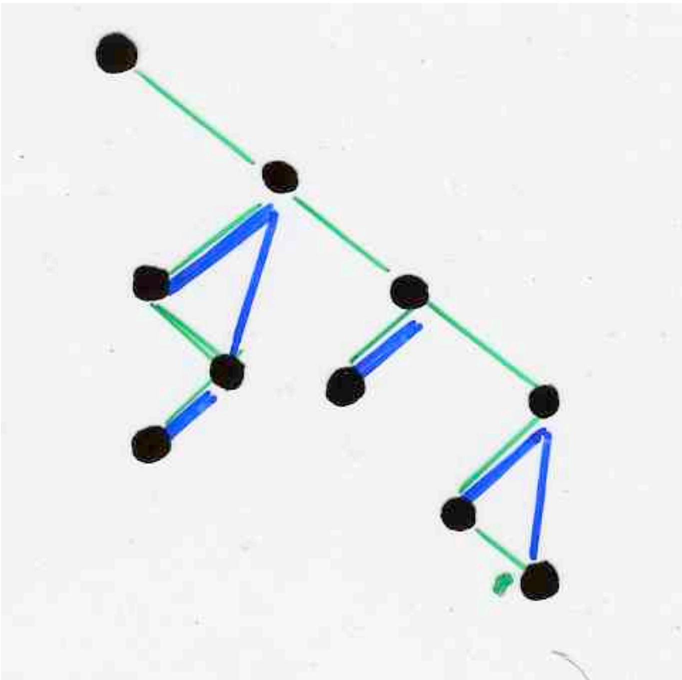
(FPSAC, 2013) thesis (Oct. 2013)

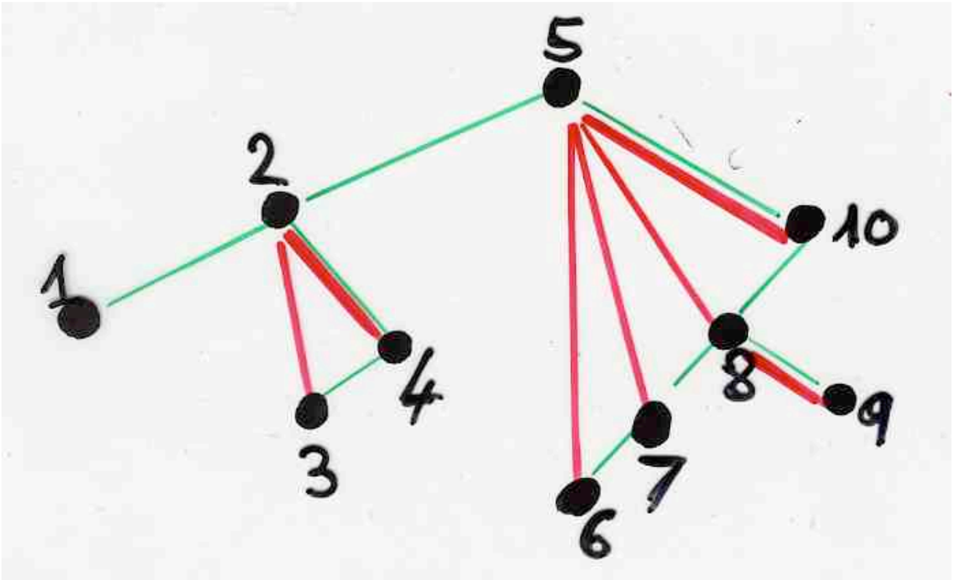
G. Chatel







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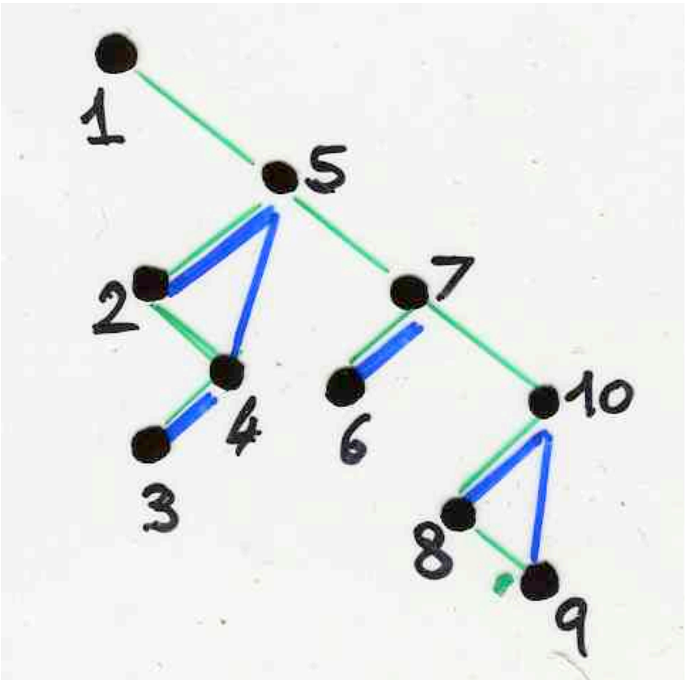
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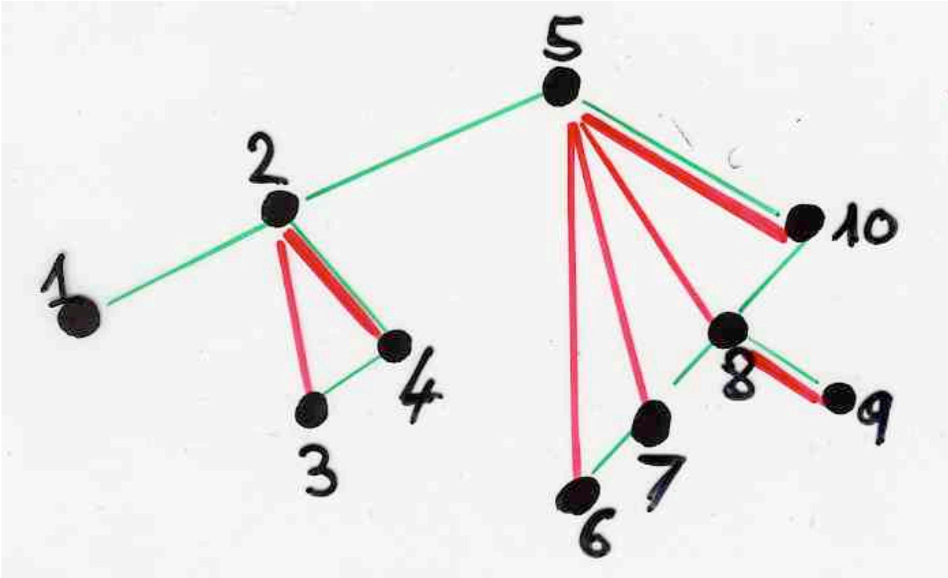
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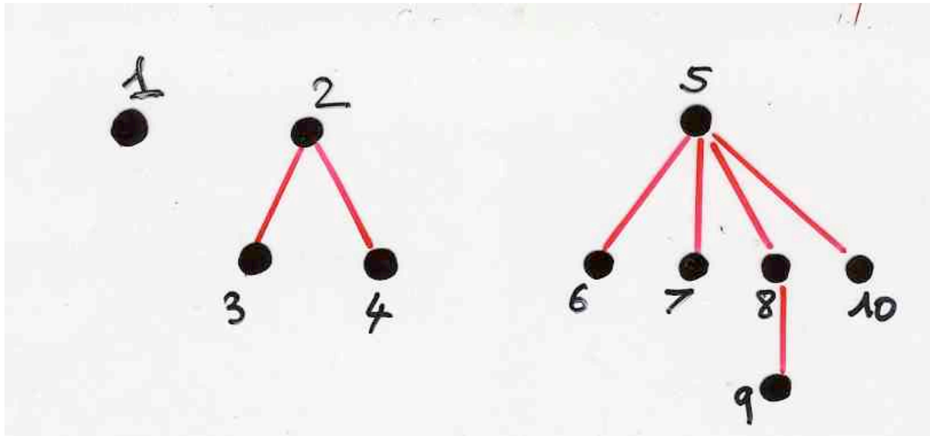
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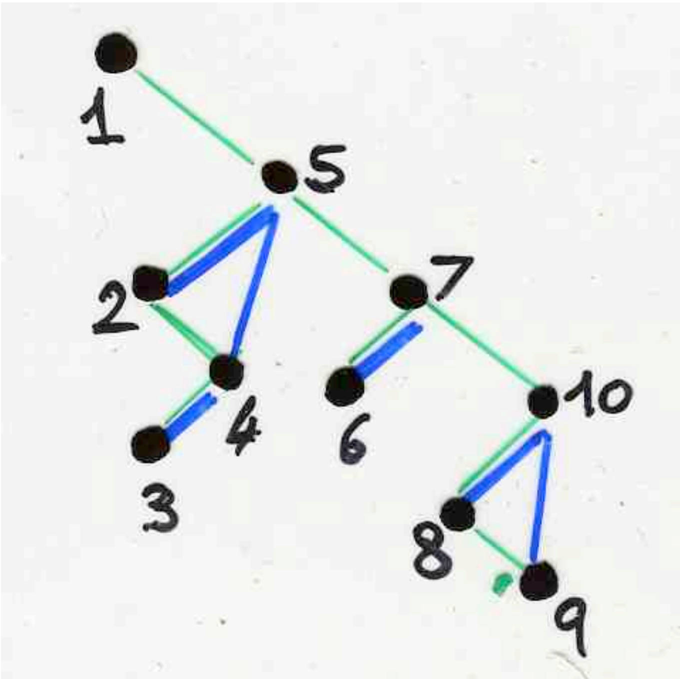
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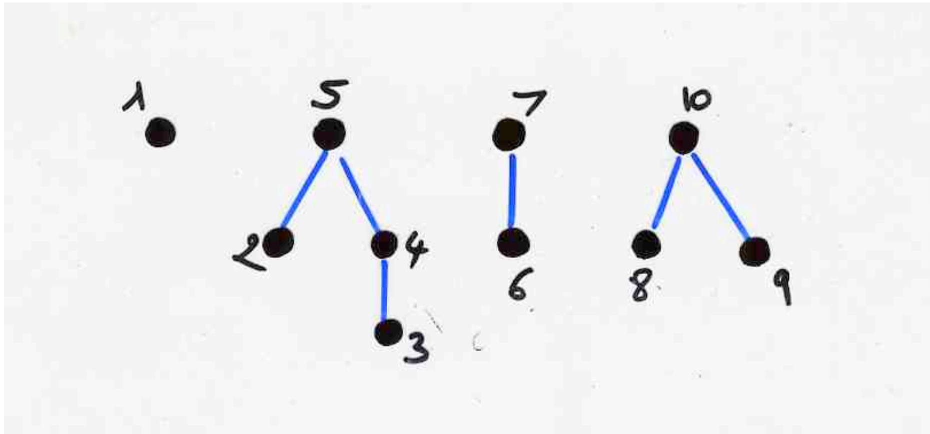
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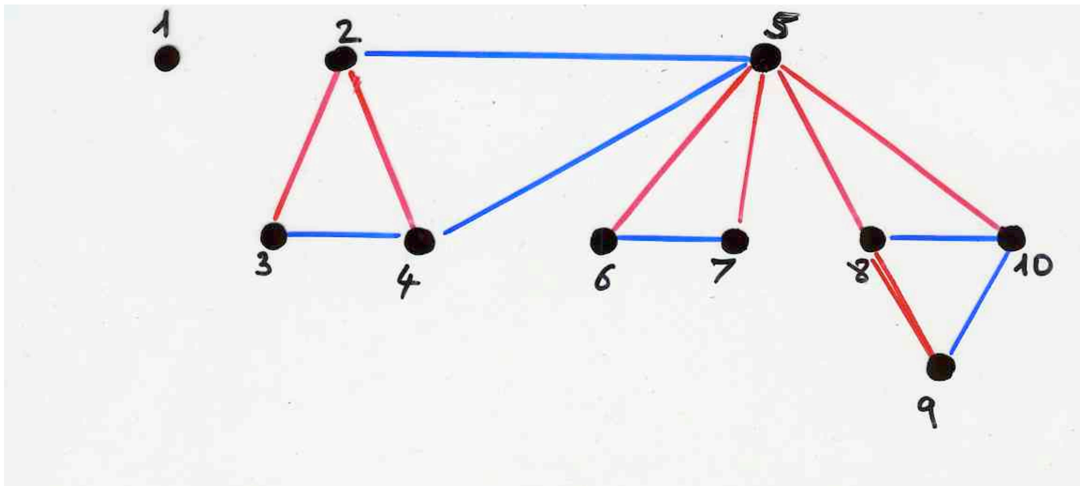
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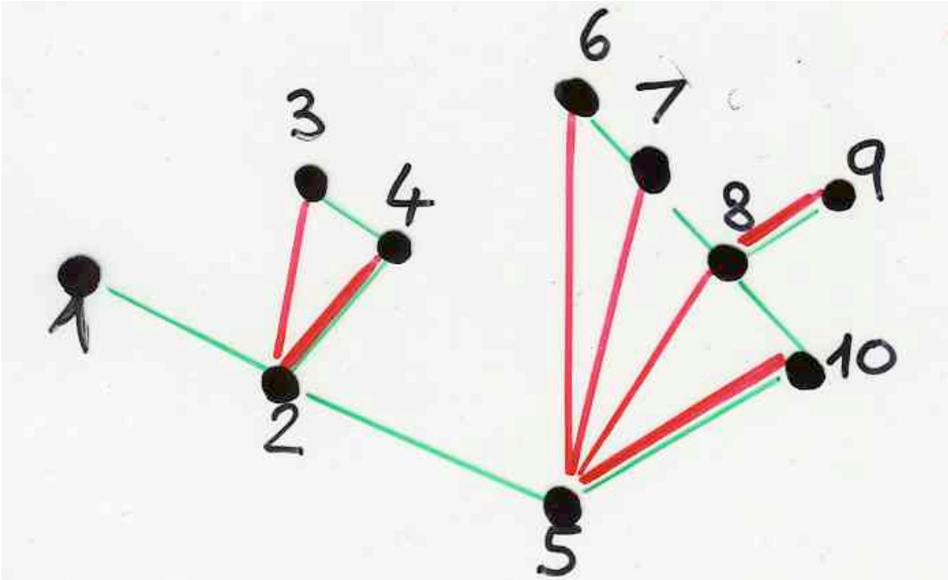
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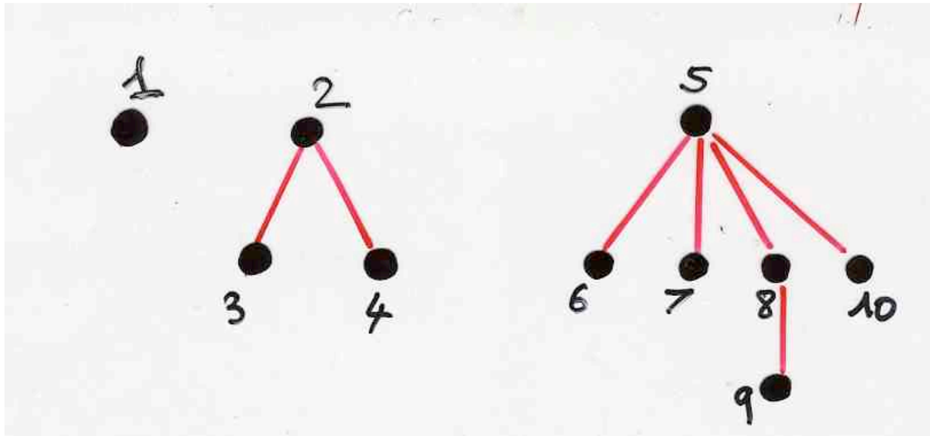
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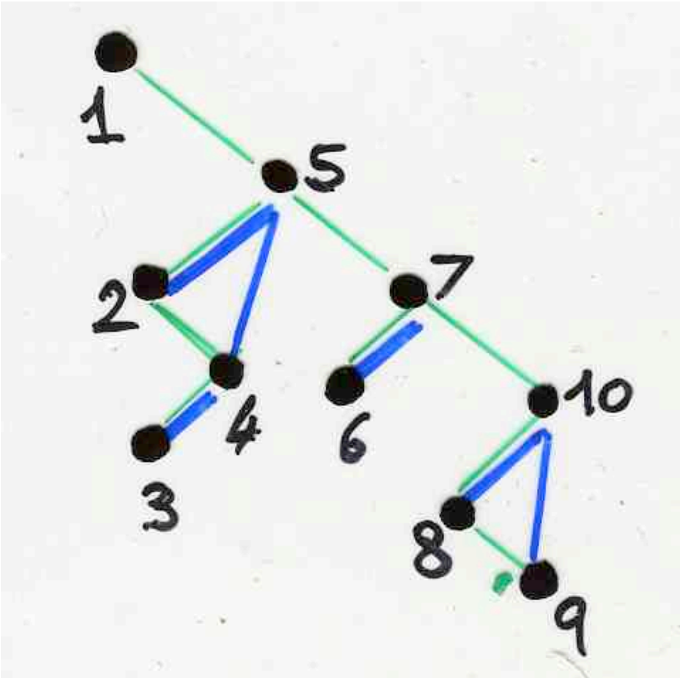
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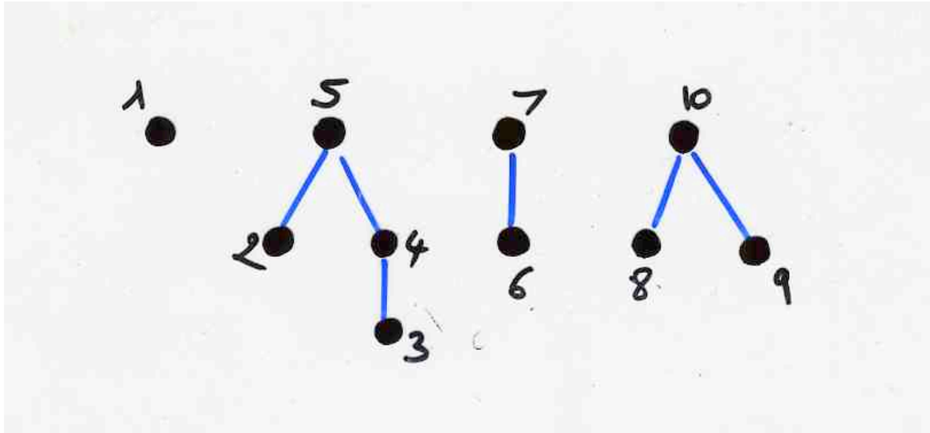
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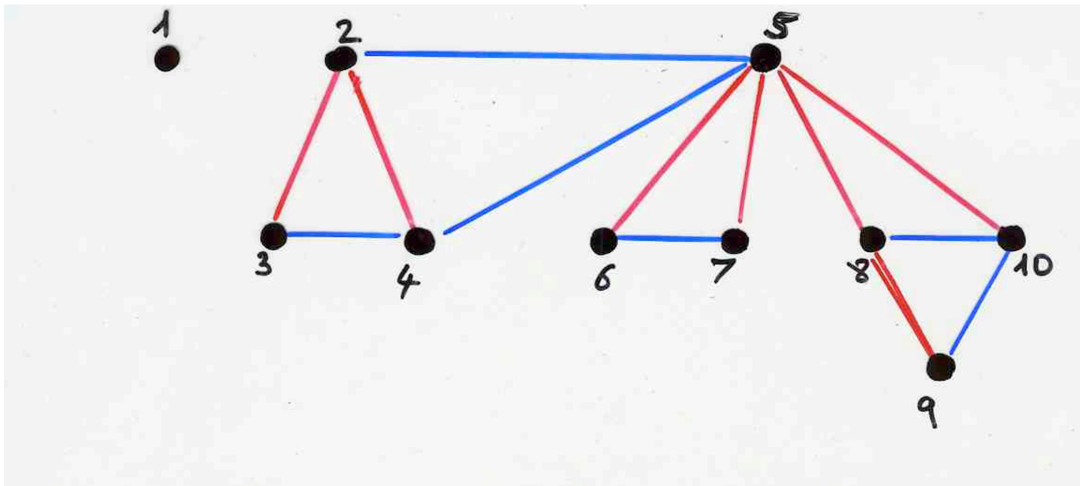
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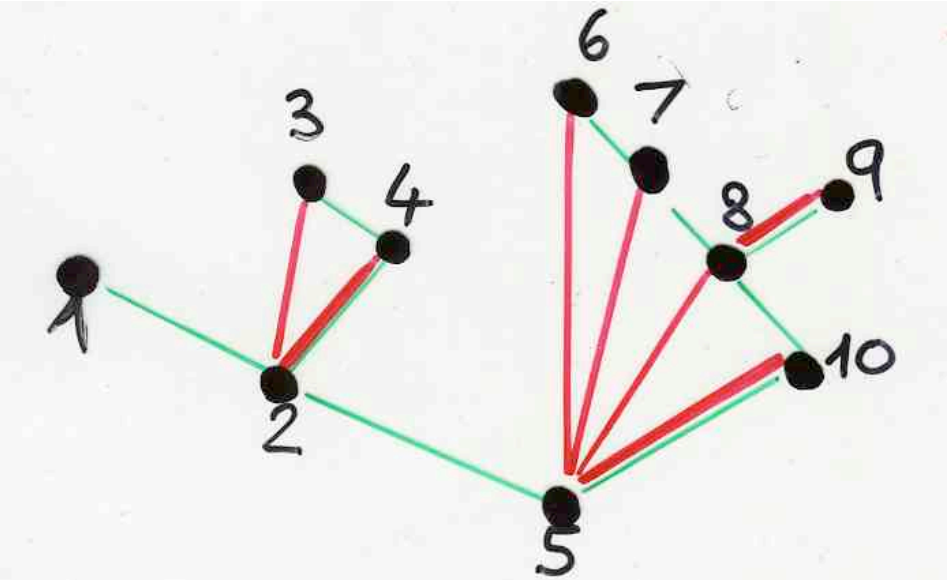
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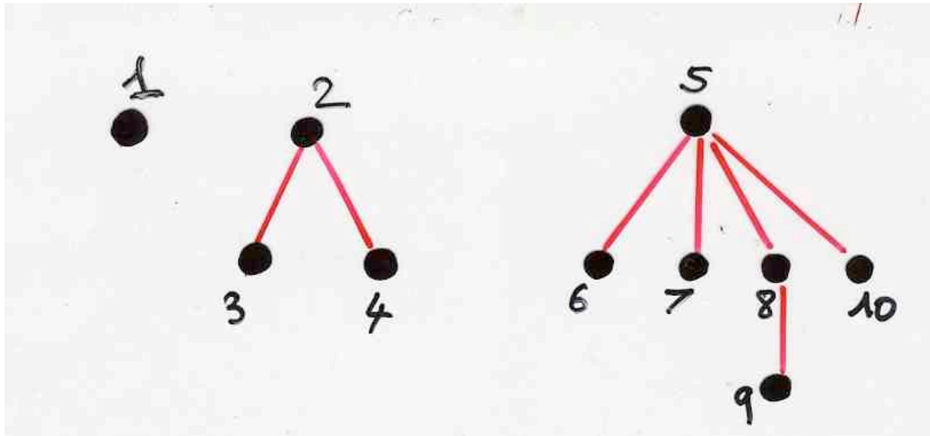
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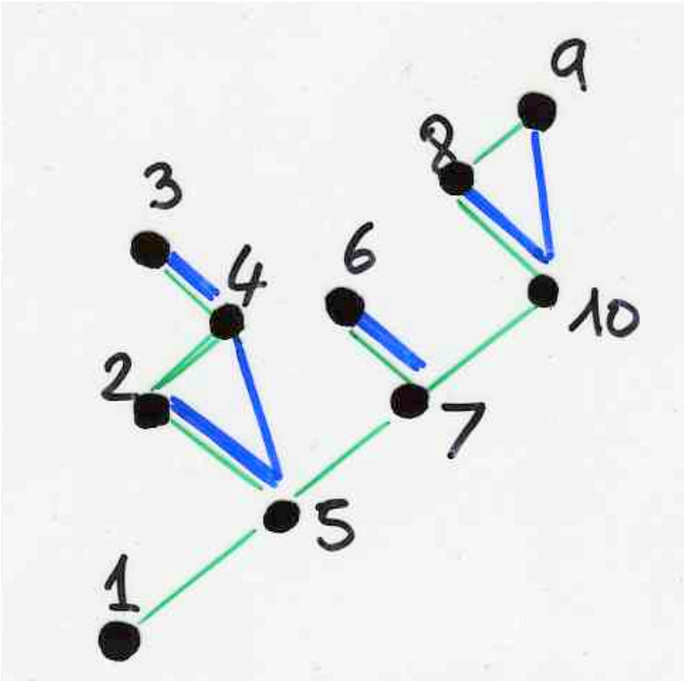
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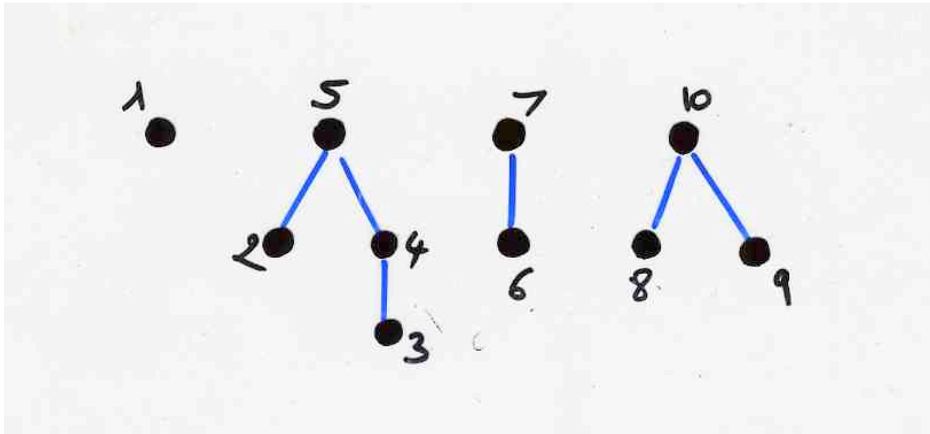
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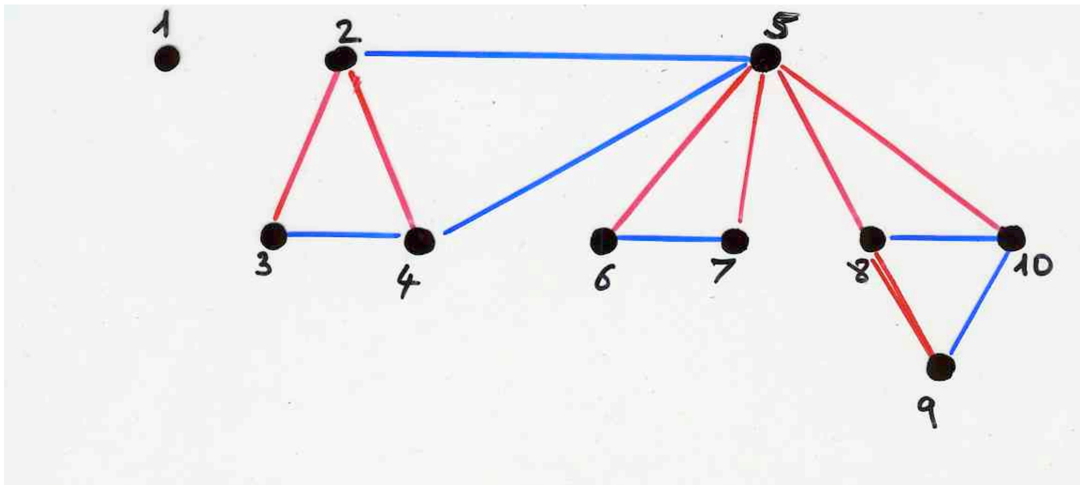
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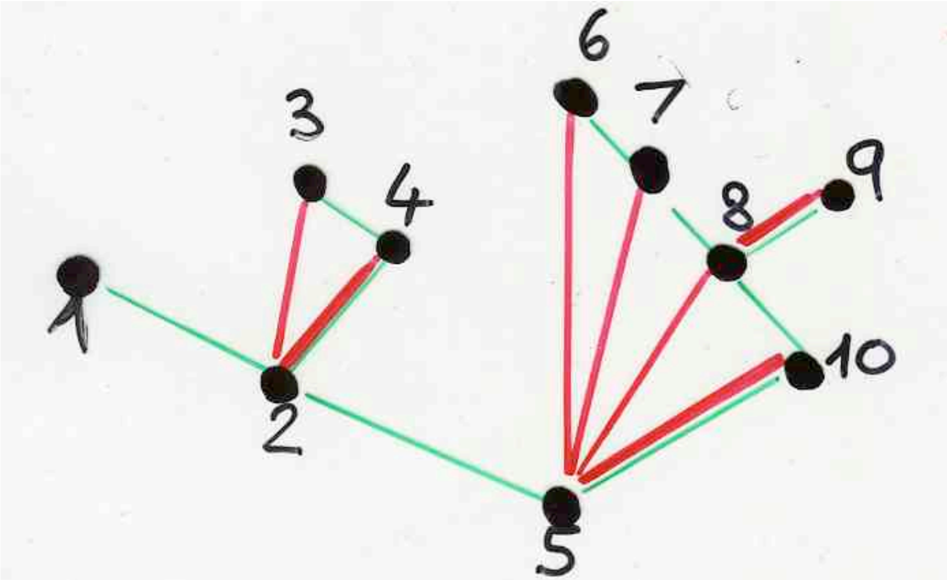
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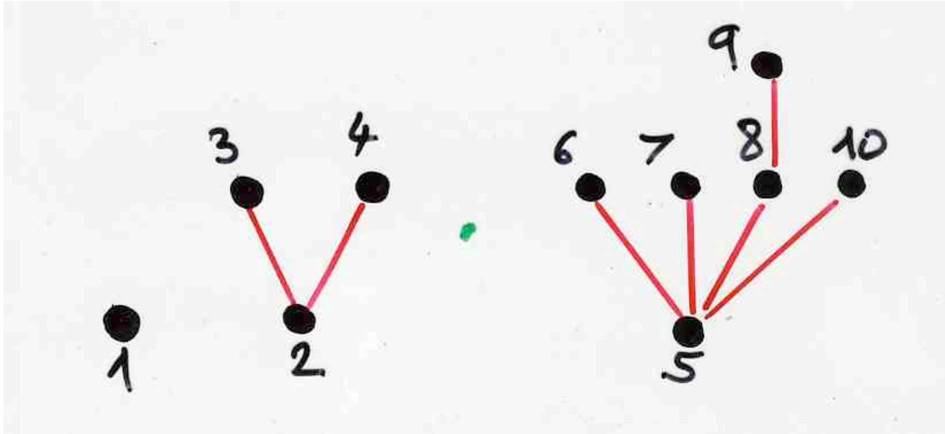
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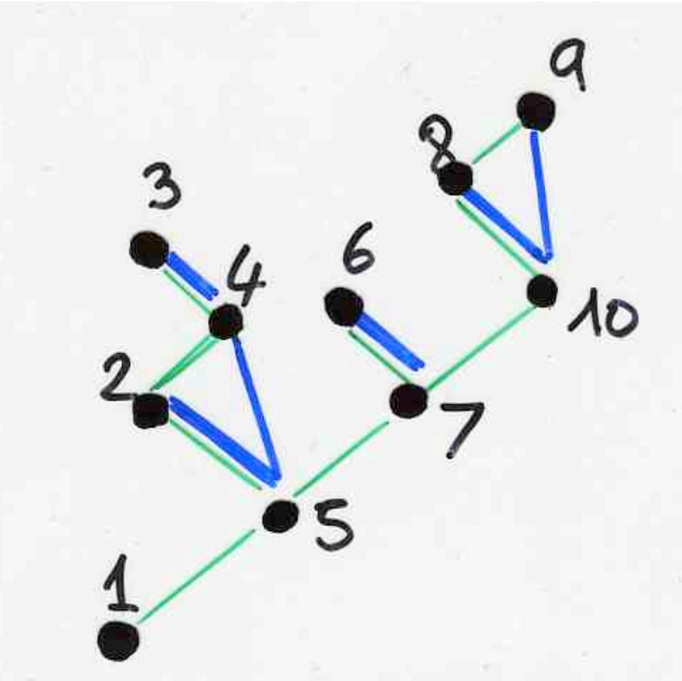
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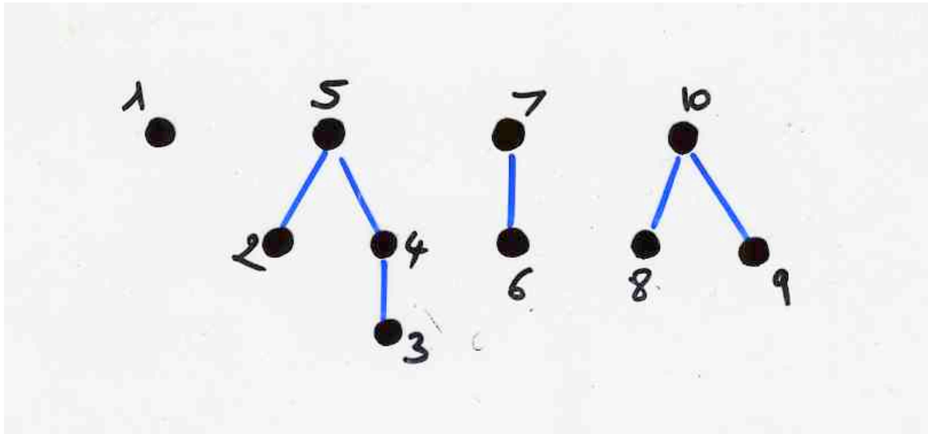
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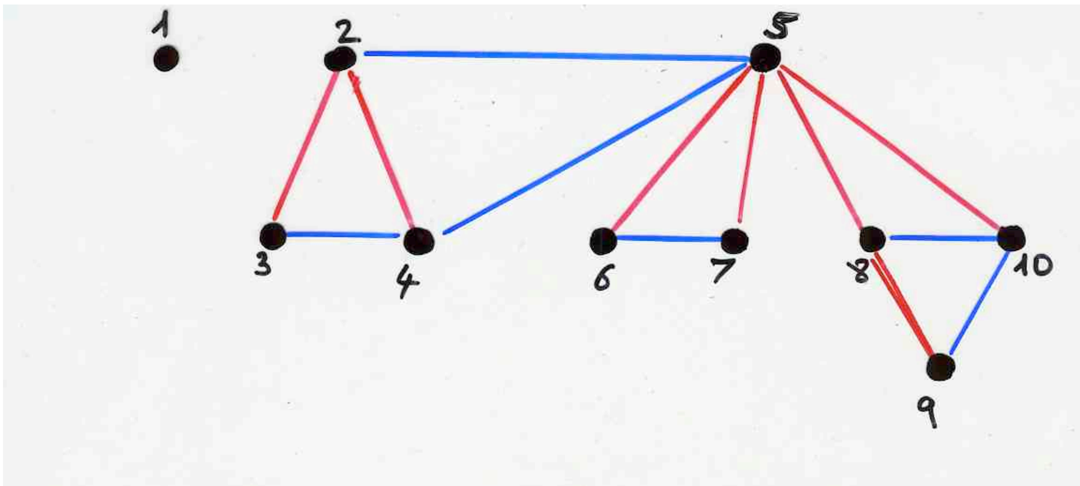
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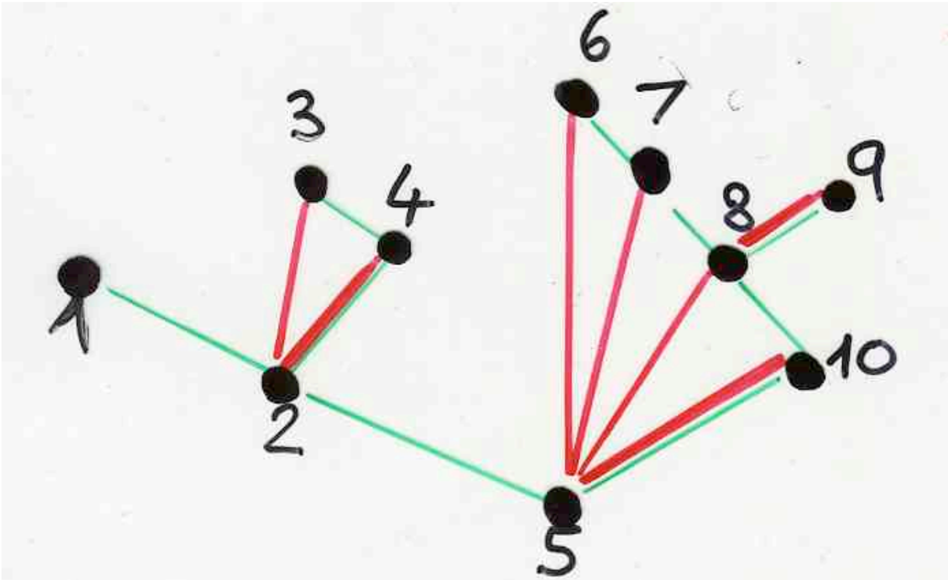
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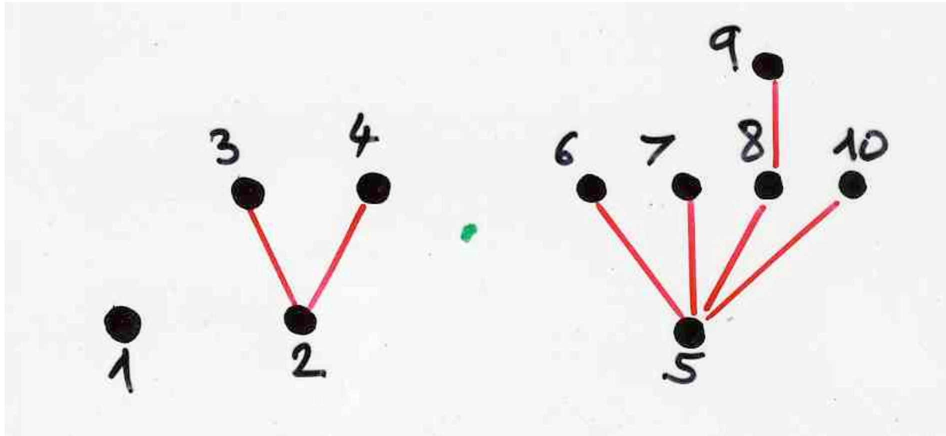
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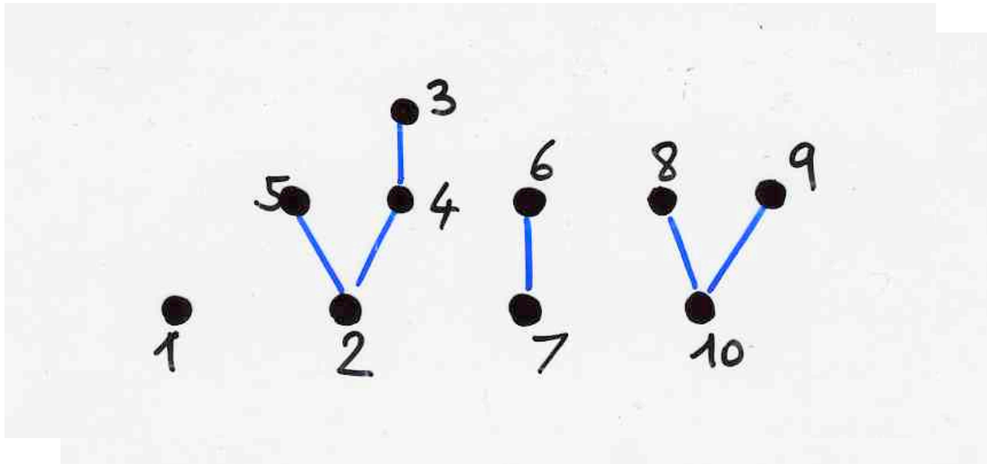
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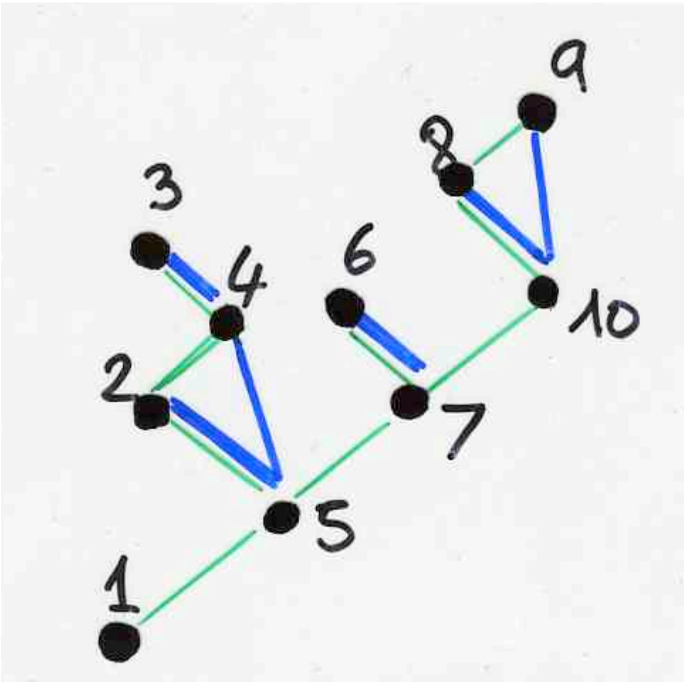
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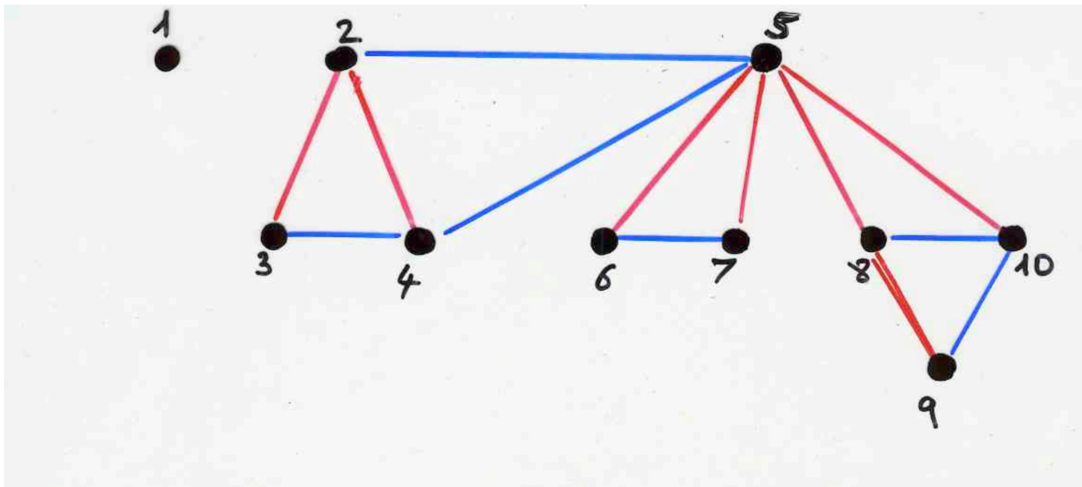
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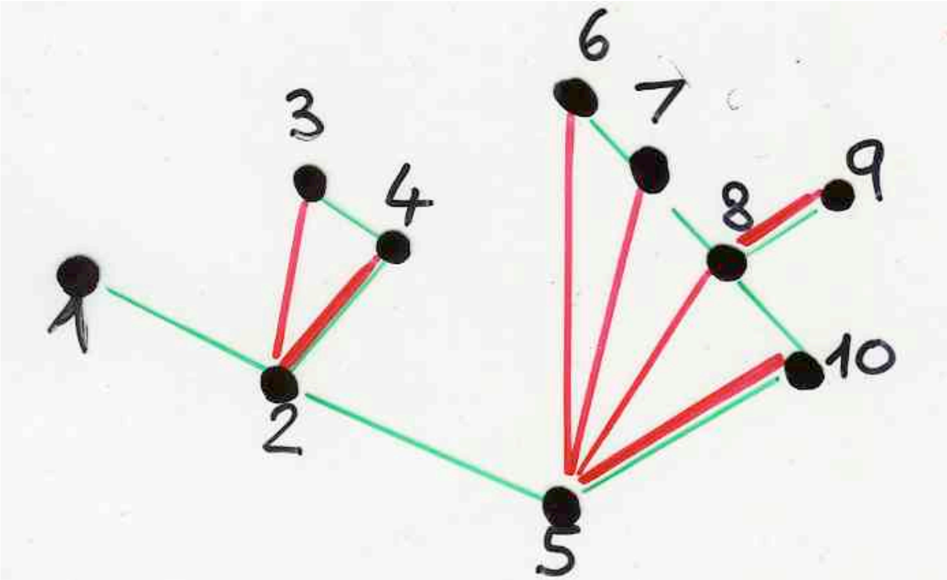

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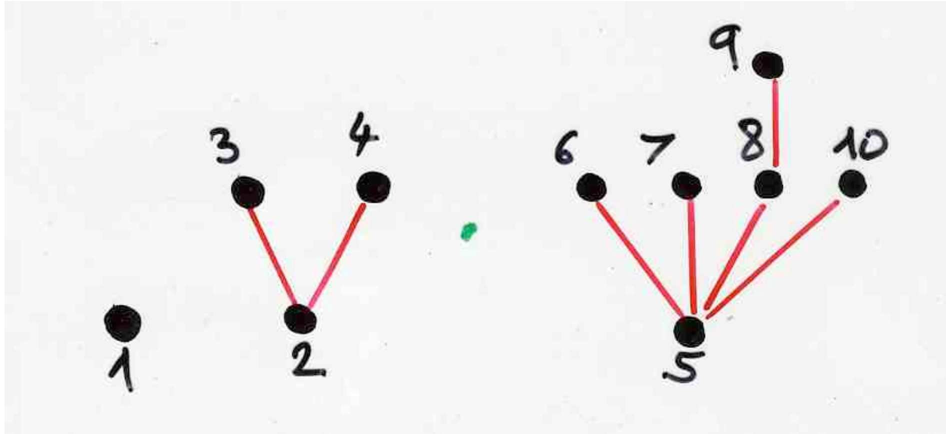

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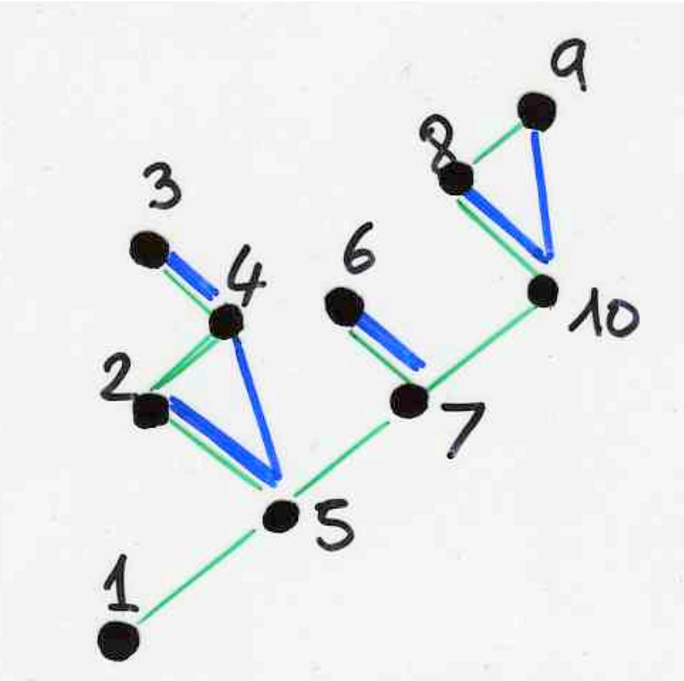

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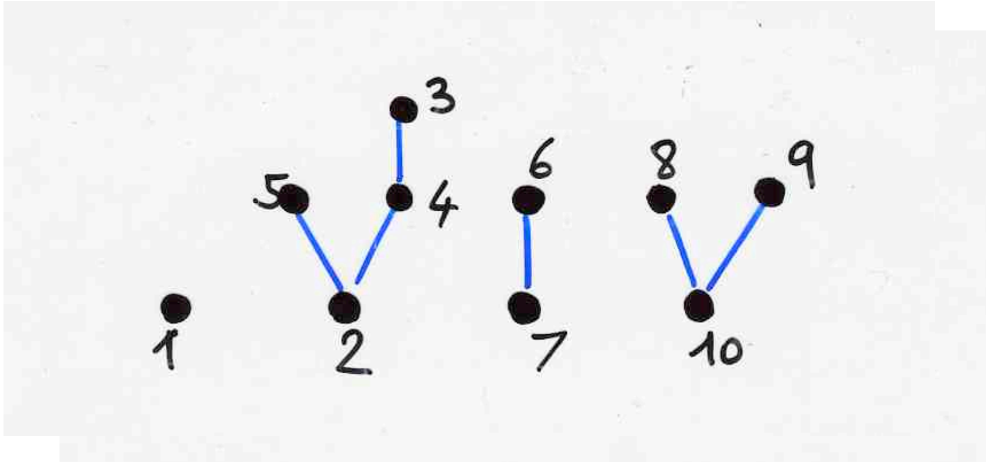

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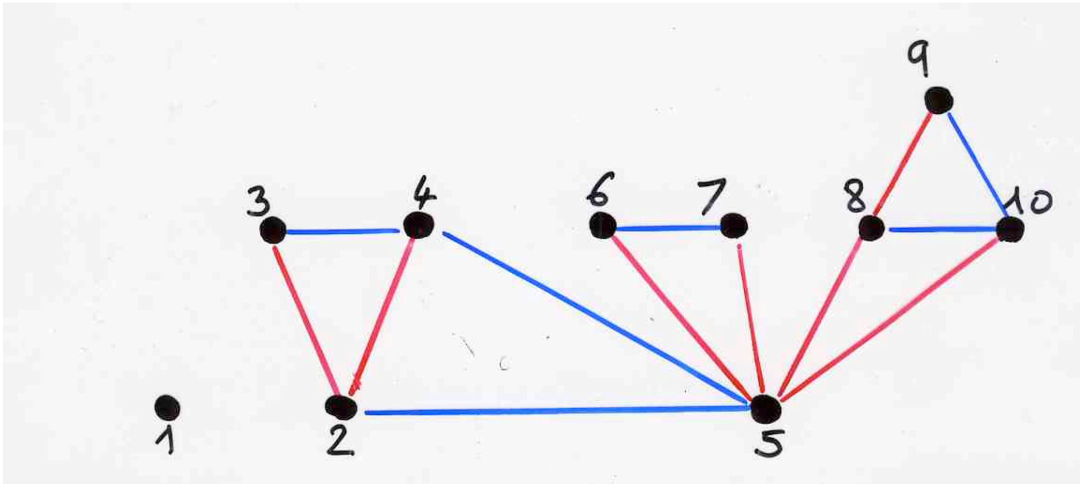

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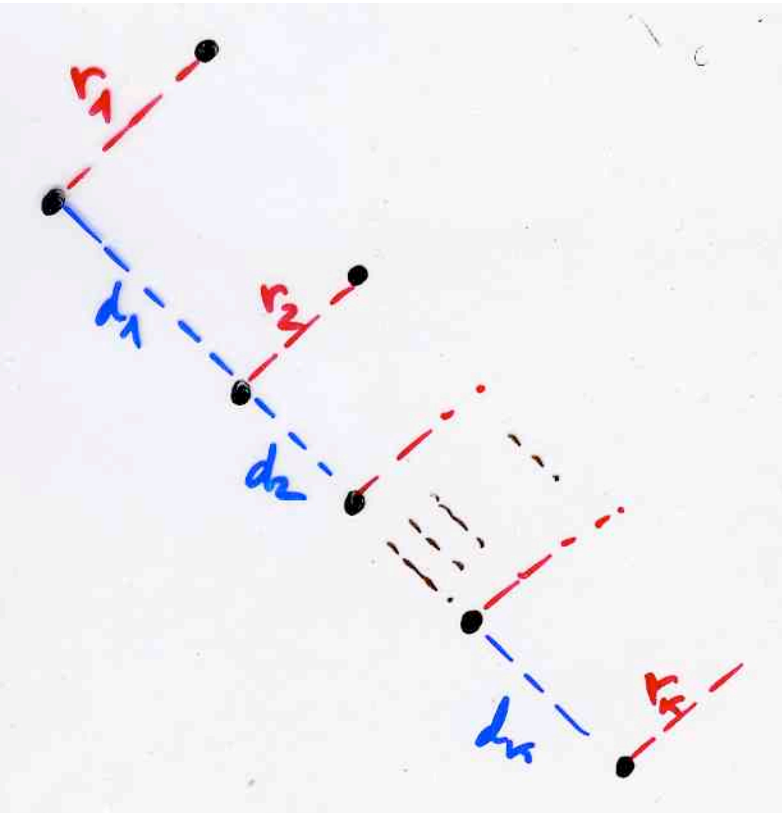

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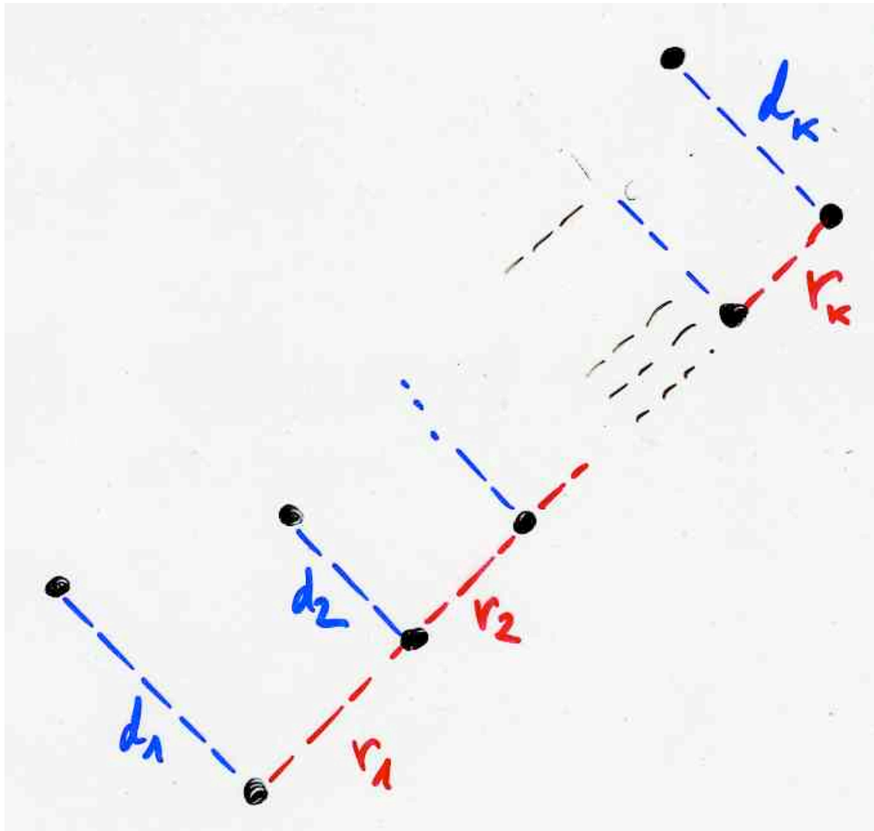

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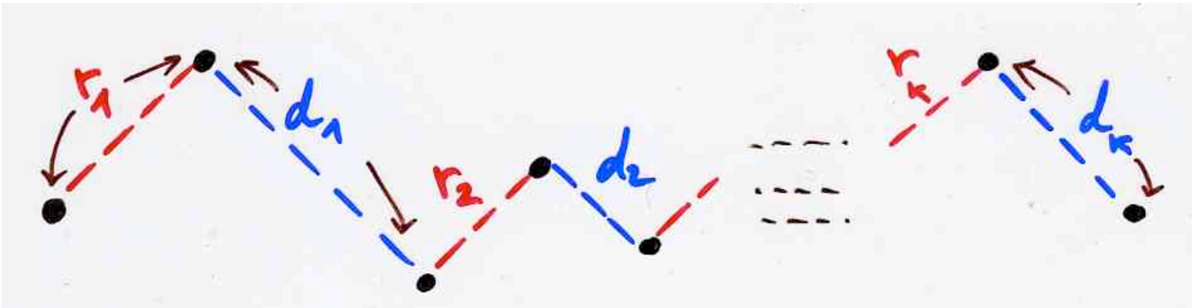
d1
r1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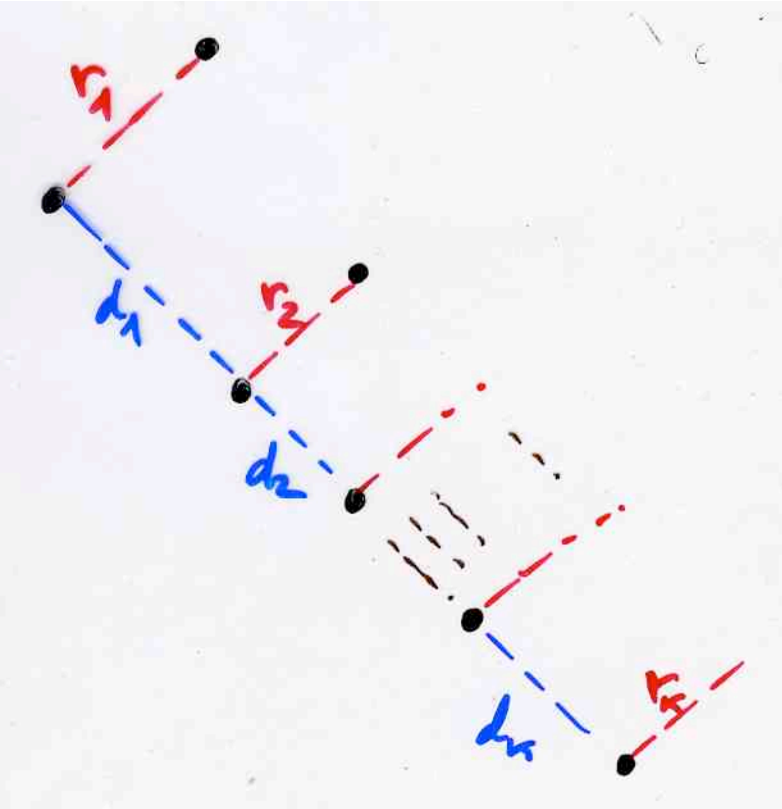
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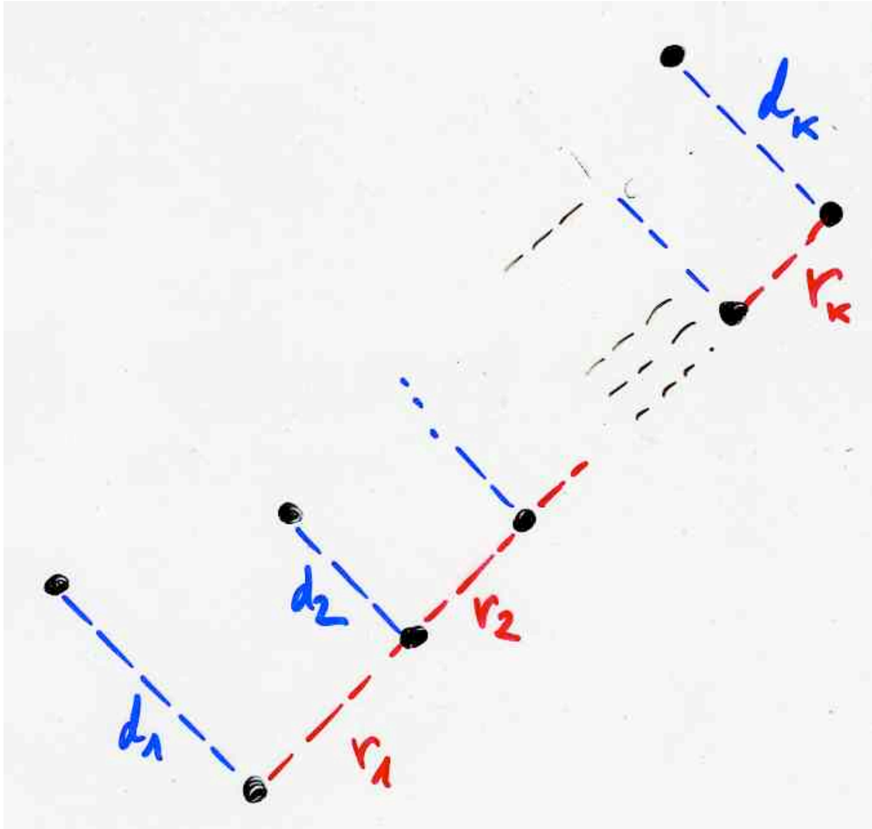
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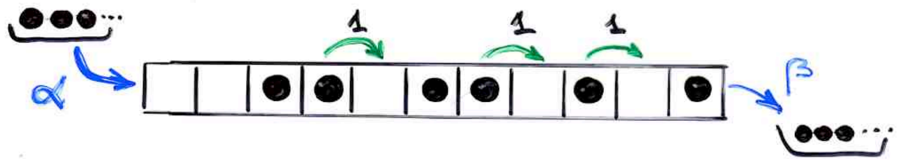
exclusion model in physics:
the TASEP

TASEP

"totally asymmetric exclusion process"

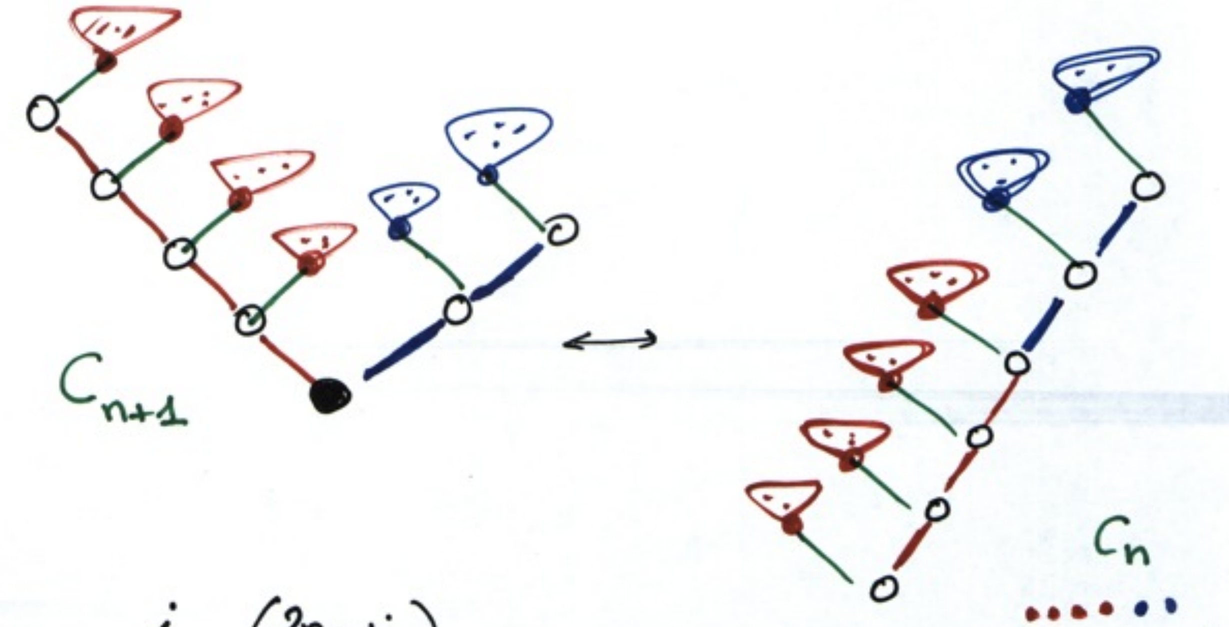


stationary probabilities

$$\frac{1}{Z_n} \sum_{\substack{\text{binary trees } T \\ c(T) = w}} \alpha^{lb(T)} \beta^{rb(T)}$$

canopy

$$Z_n = \sum_{i=1}^{n} \frac{i}{2n-i} \binom{2n-i}{n} \frac{\alpha^{-(i+1)} - \beta^{-(i+1)}}{\alpha^{-1} - \beta^{-1}}$$



$$\frac{i}{2n-i}\binom{2n-i}{n}$$

$$\left[\bar{\alpha}^{(i)} + \bar{\alpha}^{(i-1)}\bar{\beta} + \dots + \alpha\,\bar{\beta}^{(i-1)} + \bar{\beta}^{(i)}\right]$$

Olya Mandelstam

(2013)

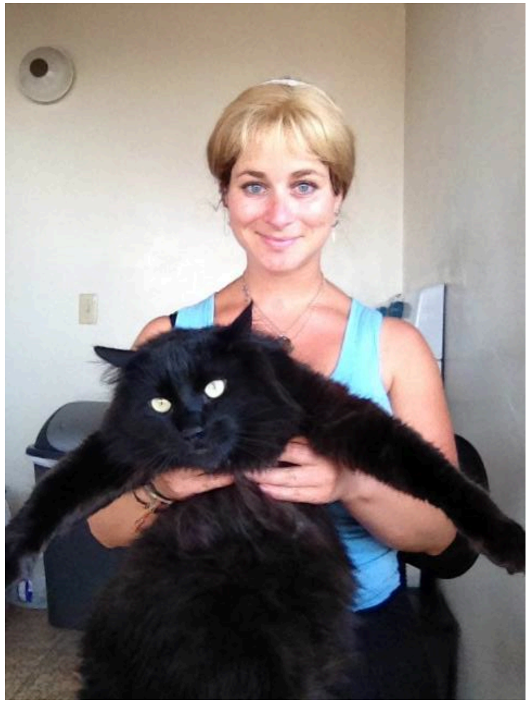
(α, β) - analog of Narayana's determinant

TASEP with 2 parameters

$$\mathcal{P}_{\{\lambda_1, \dots \lambda_k\}}(\alpha, \beta) = \det A_\lambda^{\alpha, \beta}$$

$$A_\lambda^{\alpha,\beta} = (A_{i,j})$$

$$A_{i,j} = \begin{cases} 0 & \text{for } j < i-1 \\ 1 & \text{for } j = i-1 \\ \beta^{j-i} \alpha^{\lambda_i - \lambda_{j+1}} \left(\binom{\lambda_{j+1}}{j-i} + \binom{\lambda_j + 1}{j-i+1} \right) & \\ + \beta^{j-i} \alpha^{\lambda_i - \lambda_j} \cdot \sum_{\ell=0}^{\lambda_j - \lambda_{j+1} - 1} \alpha^{\ell} \left(\binom{\lambda_j - \ell - 1}{j-i-1} + \binom{\lambda_j - \ell - 1}{j-i} \right) & \text{for } j \geq i \end{cases}$$

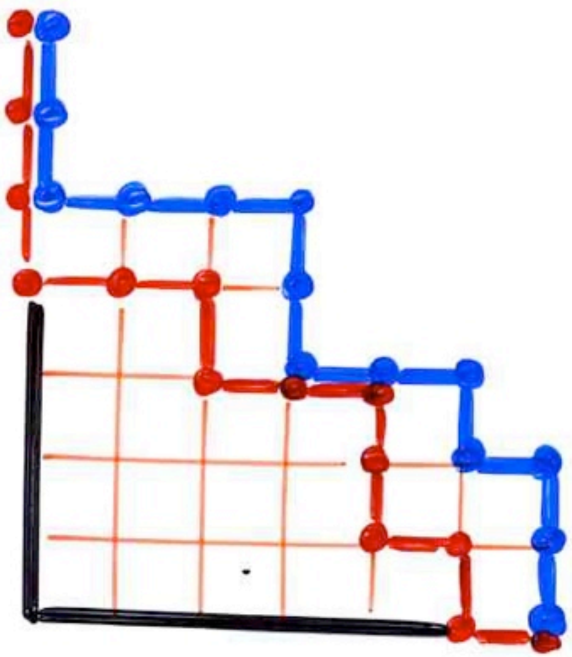


Kreweras's determinant
Narayana (1955)

$$\det \begin{pmatrix} \lambda_i + 1 \\ j-i+1 \end{pmatrix}_{1 \le i,j \le k}$$

$k =$ nb of 0's in s

$\lambda_i =$ nb of 1's to the left of the i^{th} zero



$\lambda = (0, 0, 3, 3, 5, 6, 6)$

déterminant
Kreweras
Narayana

Tamari polynomials

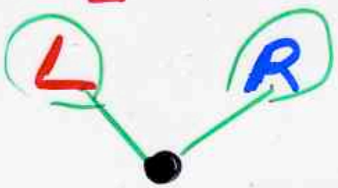
nb of intervals

F. Chapoton (2006) $\dfrac{2(4n+1)!}{(n+1)!\,(3n+2)!}$

1, 3, 13, 68, 399, ...

$B_{\emptyset} = 1$

$$B_T(x) = x\, B_L(x)\, \frac{x B_R(x) - B_R(1)}{x-1}$$

$T =$ 

Tamari polynomials

¡ muchas gracias !

Complements

enumeration
of the size of these intervals

bijective proof of Narayana determinant
with LGV lemma and duality of paths

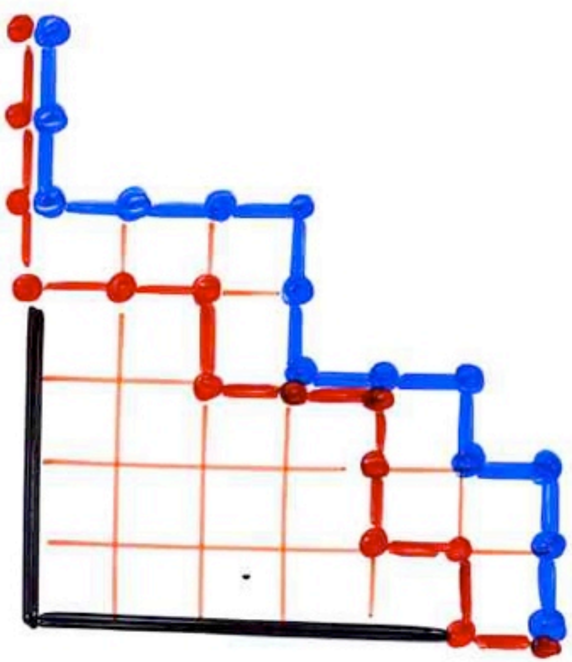
Kreweras's determinant
Narayana (1955)

$$\det \begin{pmatrix} \lambda_i + 1 \\ j - i + 1 \end{pmatrix}_{1 \le i, j \le k}$$

k = nb of 0's in s

λ_i = nb of 1's to the left of the i^{th} zero

détermiant
Kreweras
Narayana



$\lambda = (0, 0, 3, 3, 5, 6, 6)$

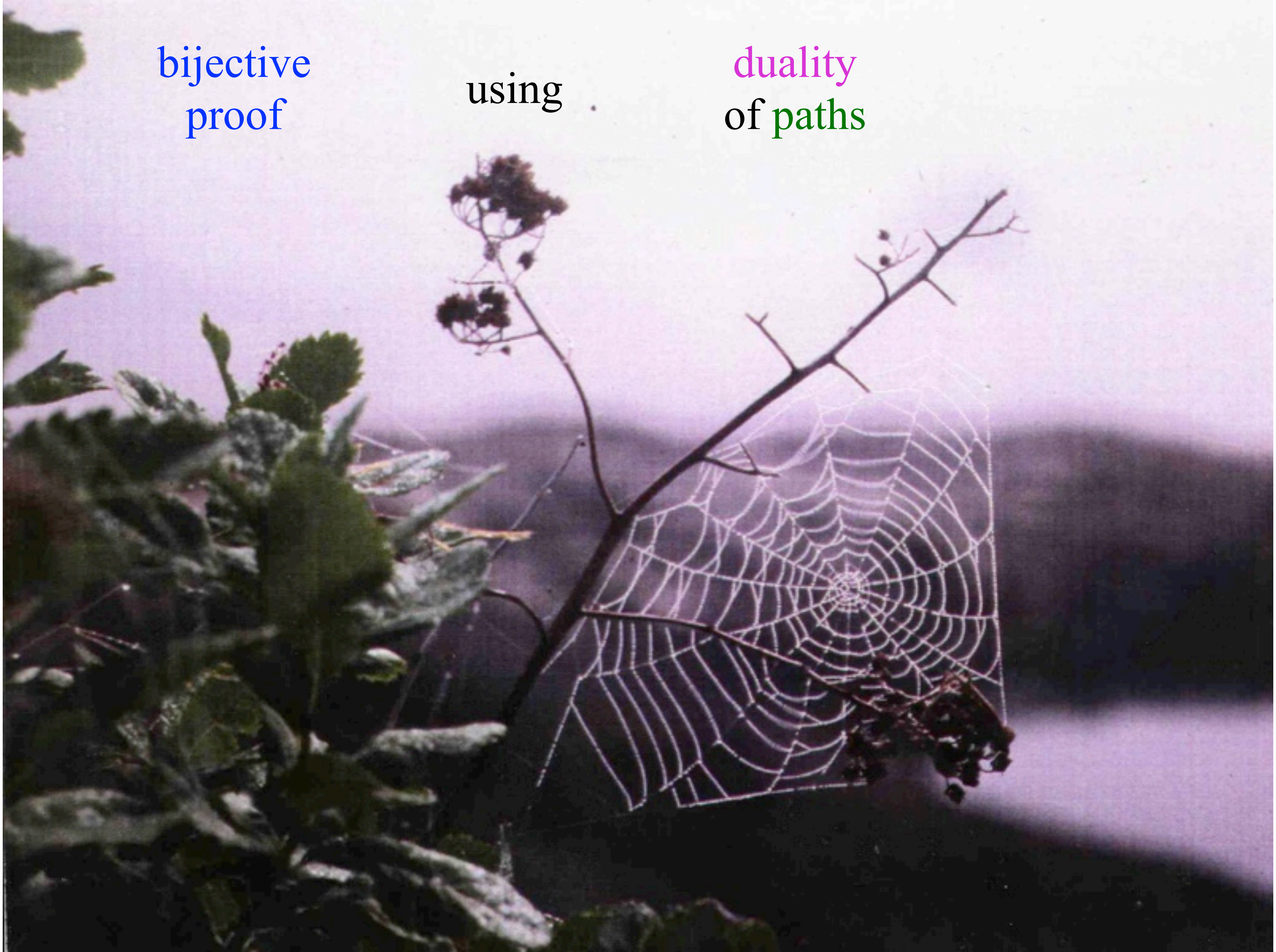
ex: $s = (10100)$ $\qquad \lambda = (1,2,2)$

$\omega =$ 

$$\det\left(\vdots \right) = \begin{vmatrix} \binom{2}{1} & \binom{2}{2} & \binom{2}{3} \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} \\ \binom{3}{-1} & \binom{3}{0} & \binom{3}{1} \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 3 \\ 0 & 1 & 3 \end{vmatrix} = 9$$

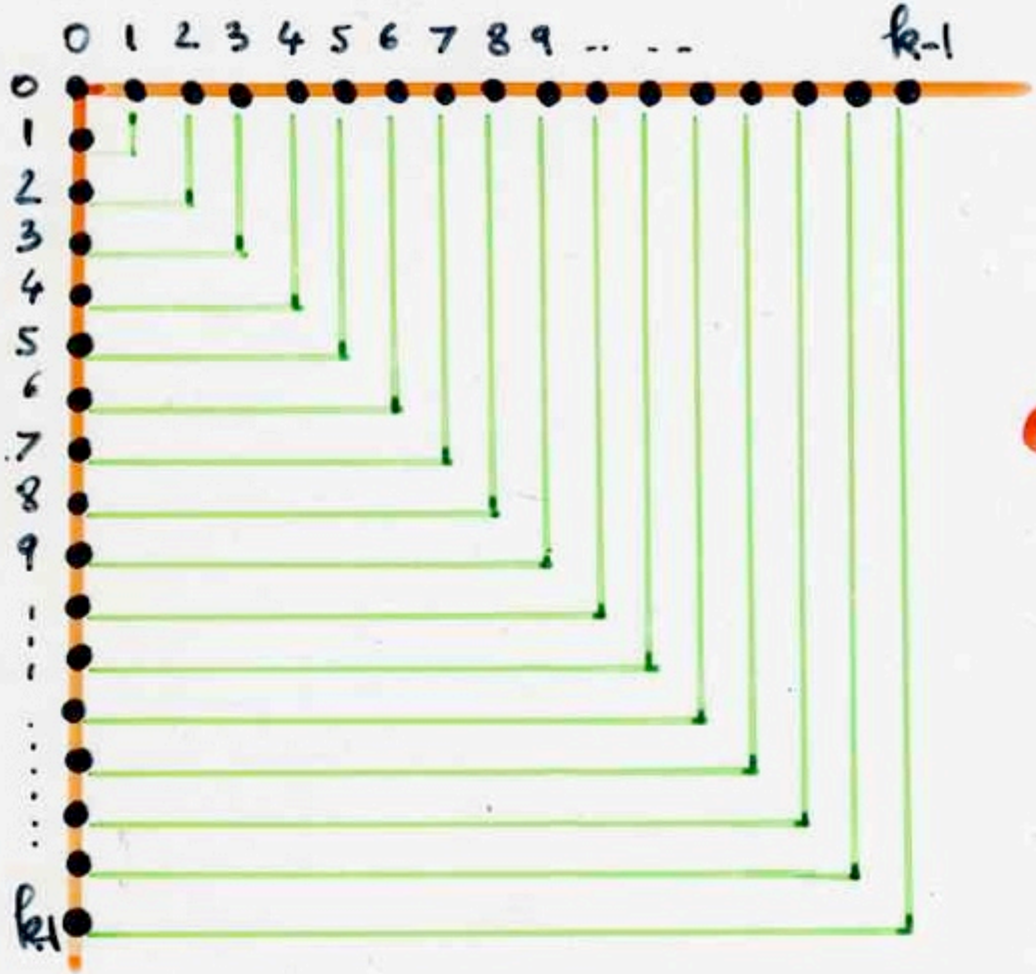
$$P(10100) = \frac{9}{132}$$

bijective proof using duality of paths



an example of duality of paths:

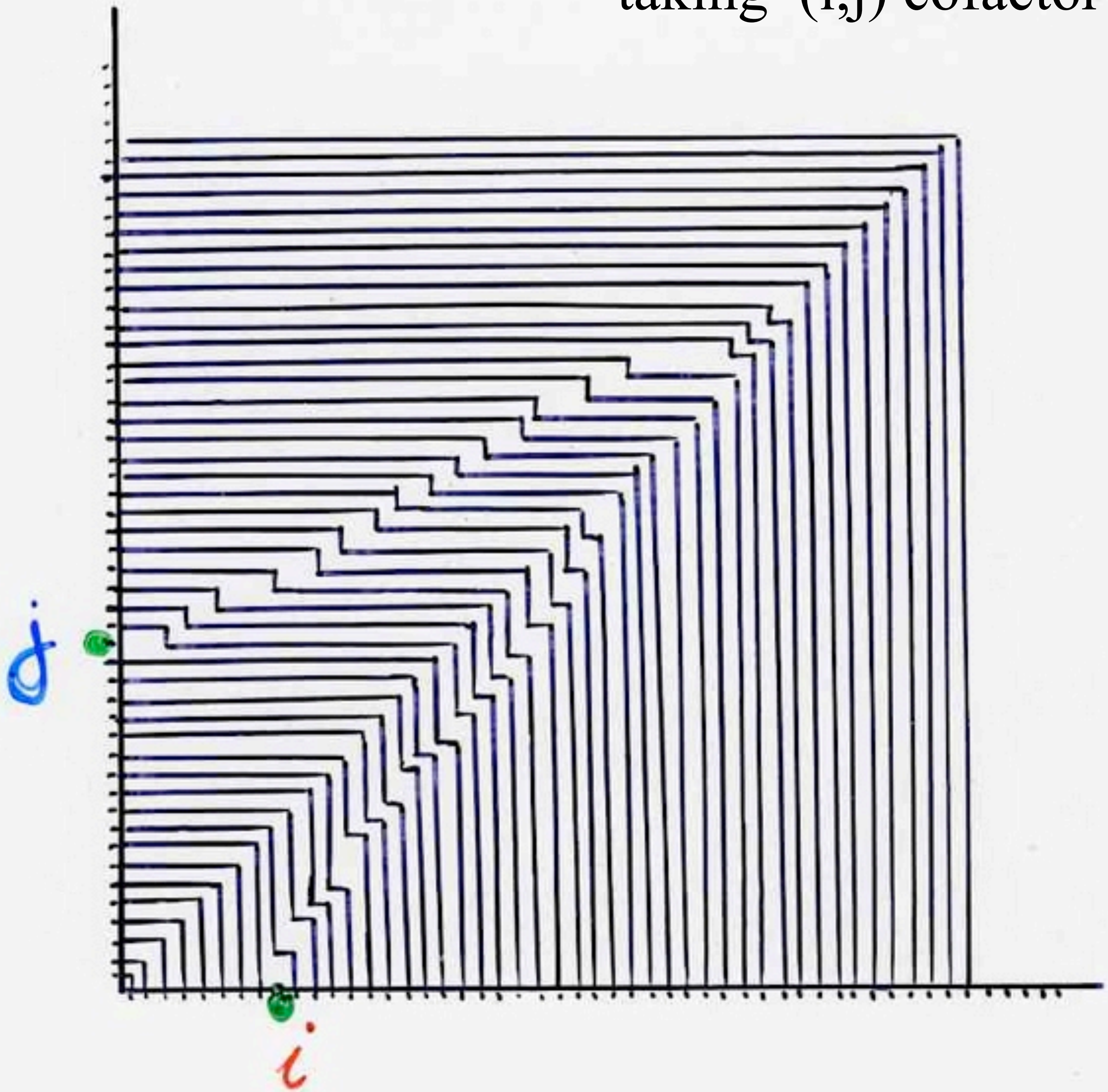
with paths for binomial coefficients



$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots & \\ 1 & 3 & 6 & 10 & \cdots & & \\ 1 & 4 & 10 & \cdots & & & \\ 1 & 5 & \cdots & & \binom{i+j}{i} & & \\ 1 & \cdots & & & & & \end{bmatrix}_{k \times k} = 1$$

by LGV Lemma

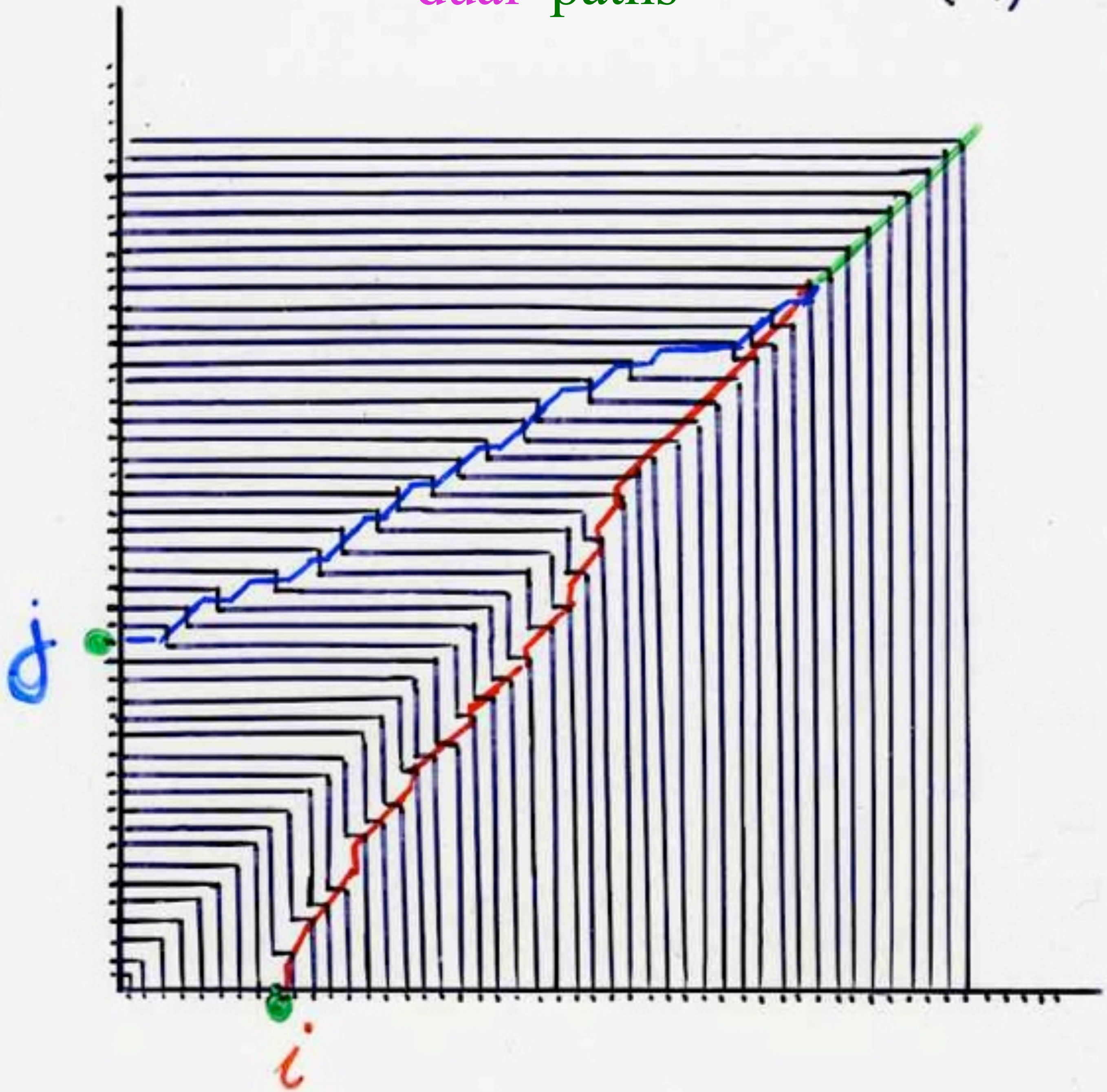
taking (i,j) cofactor ....



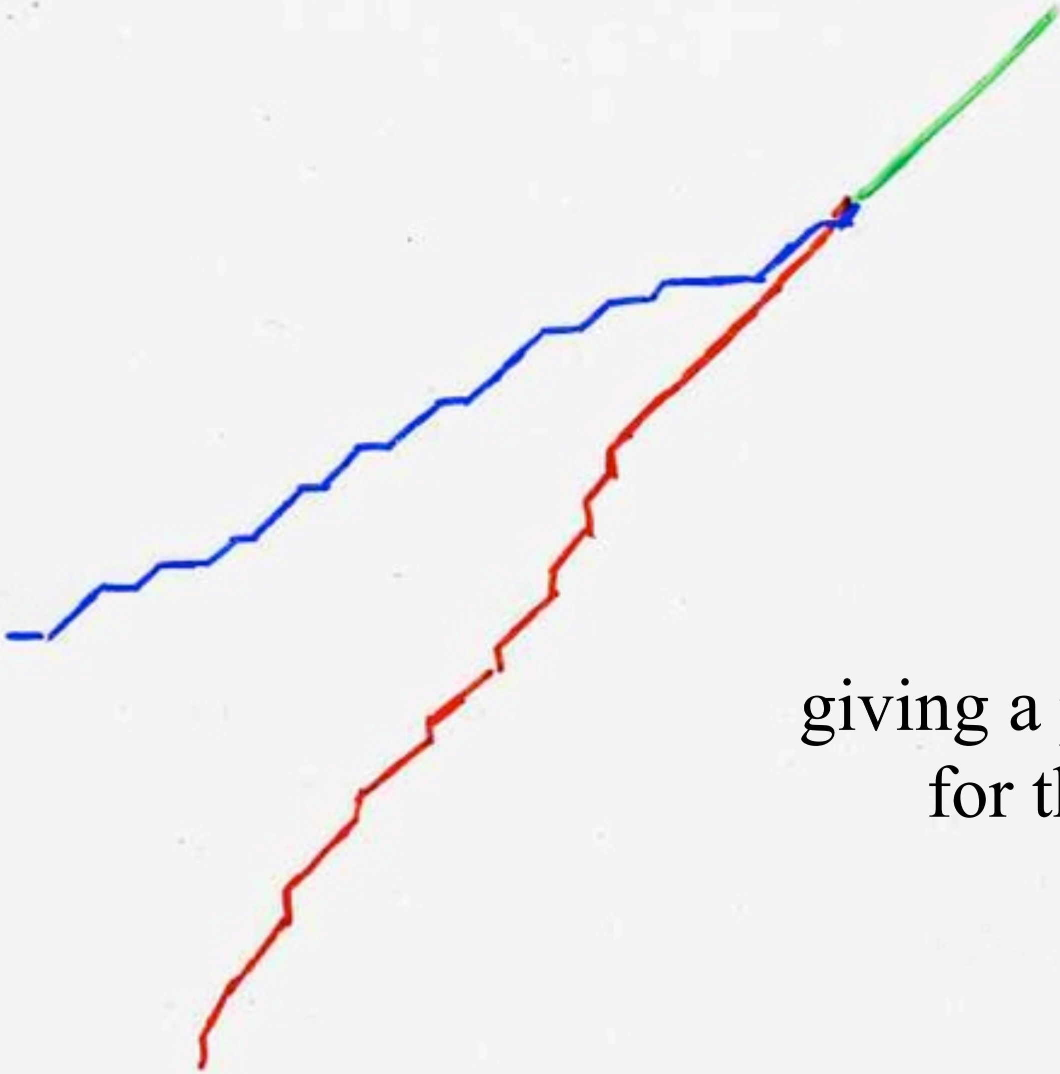
LGV lemma:
determinant
and
non-crossing
configurations
of paths

dual paths

$$(-1)^{i+j} \sum_k \binom{k}{i}\binom{k}{j}$$

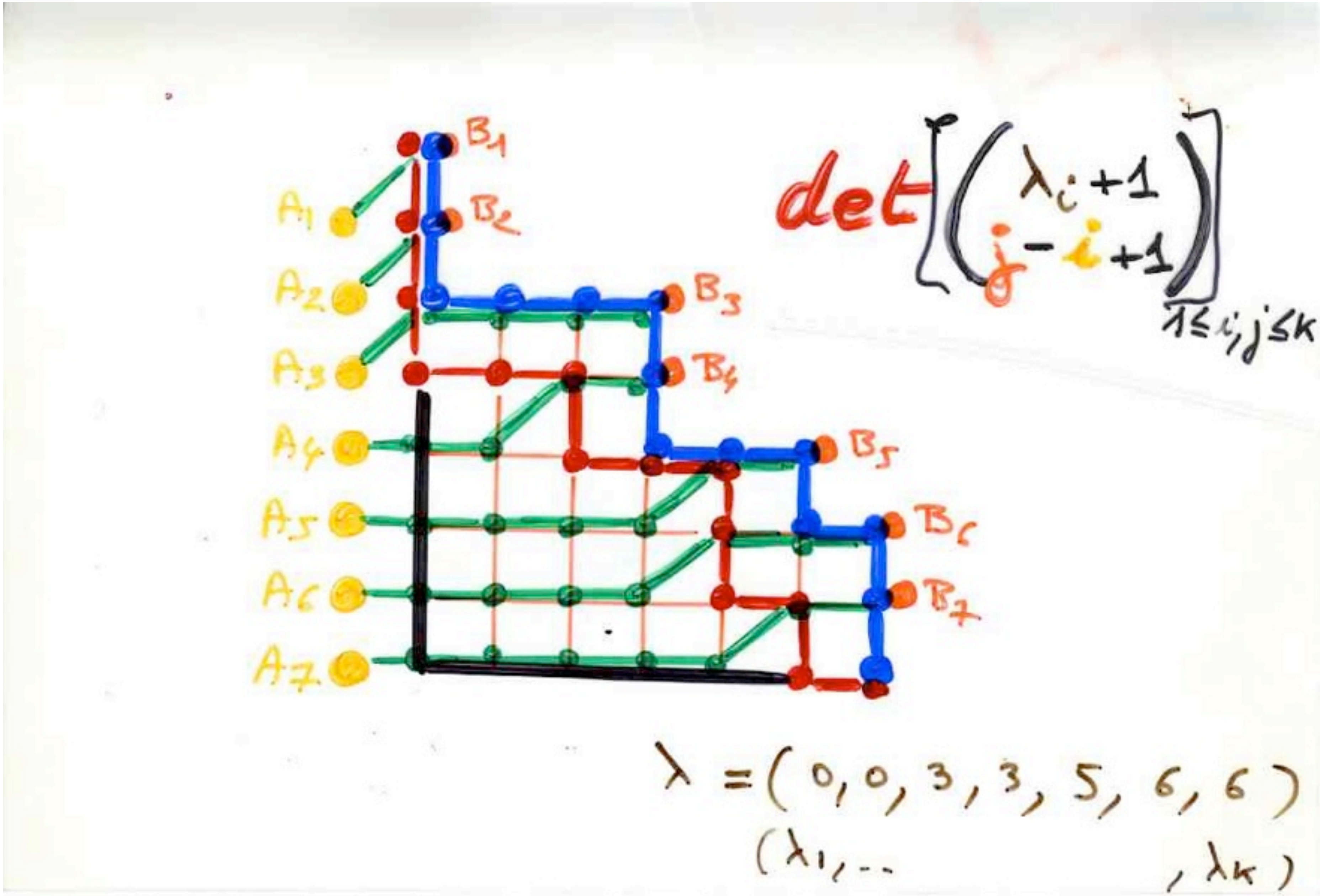


$$(-1)^{i+j}\sum_{k}\binom{k}{i}\binom{k}{j}$$



giving a proof of the formula
for the (i,j) co-factor

bijection with dual configuration of paths
giving a bijective proof of Narayana determinant



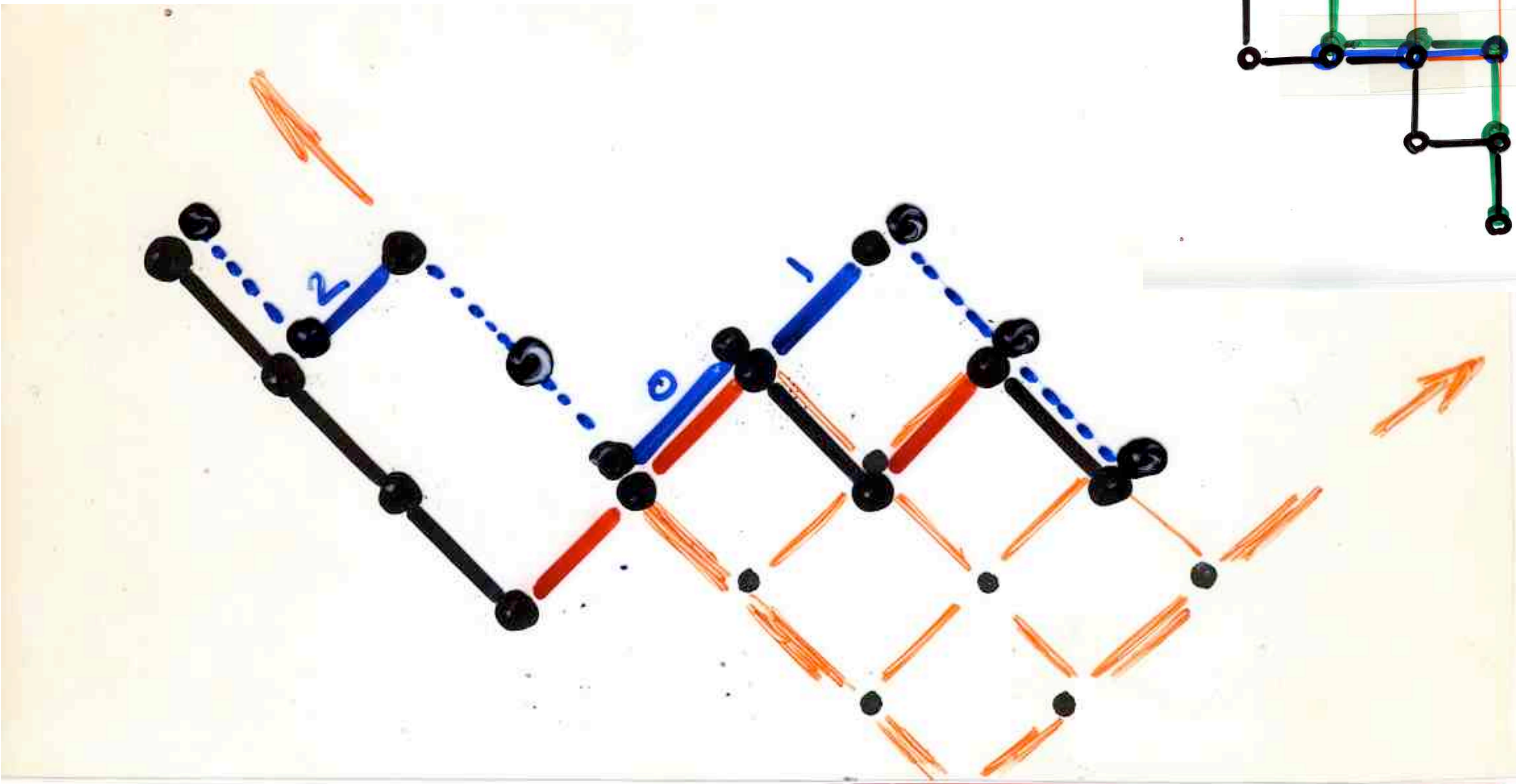
another description of the bijection between pair of paths and binary trees

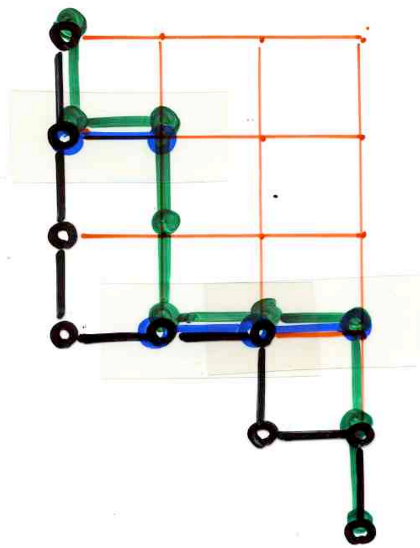
the algorithm
«sliding and pushing»

«jeu de taquin» for binary trees
is an extension of this bijection

l' algorithme

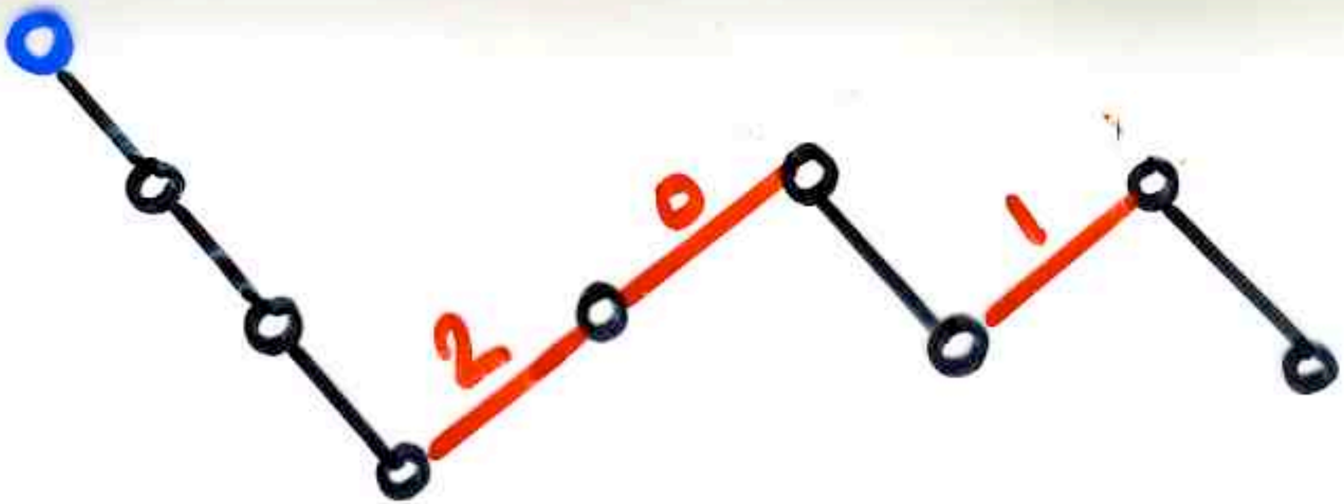
"glisser - pousser"

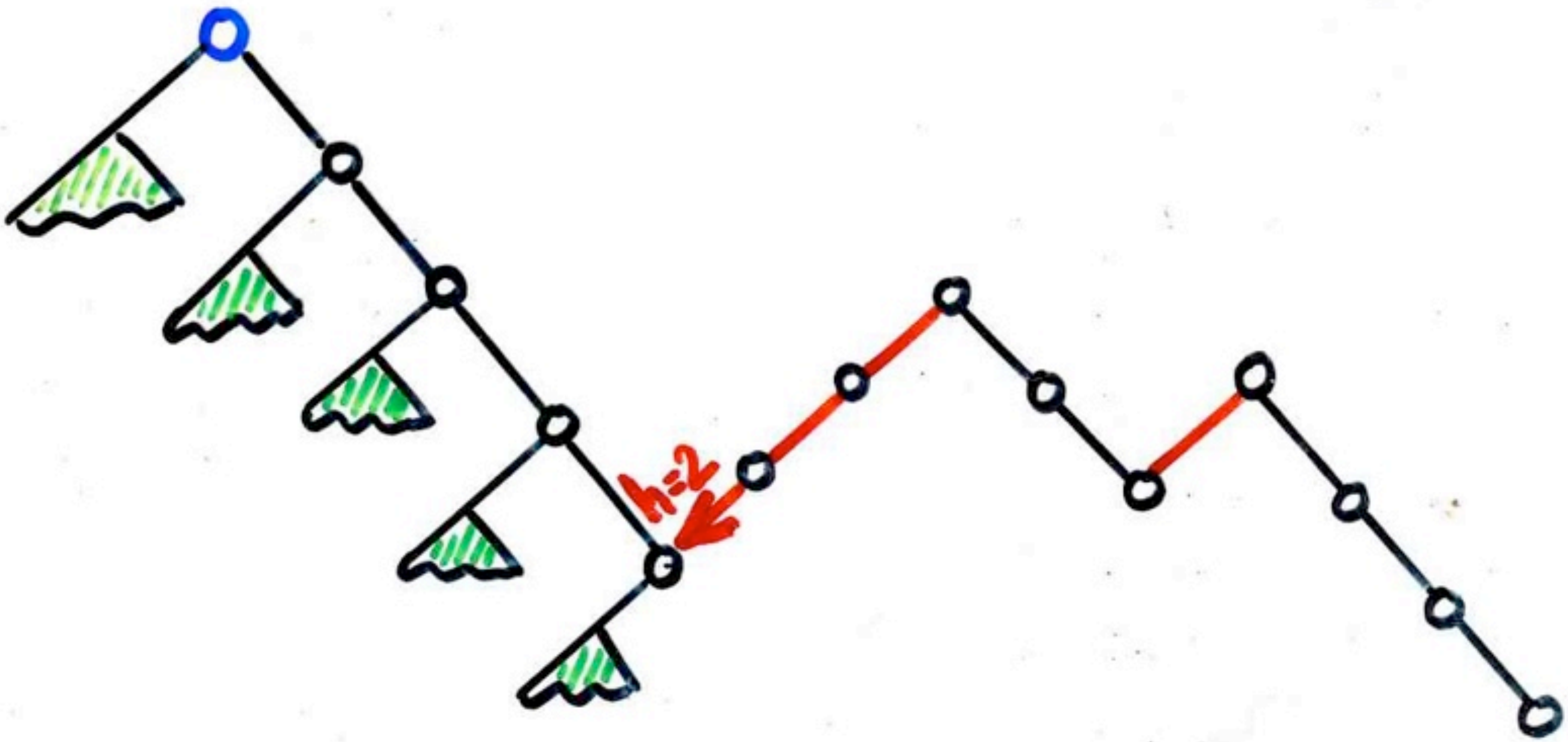




l'algorithme

"glisser - pousser"

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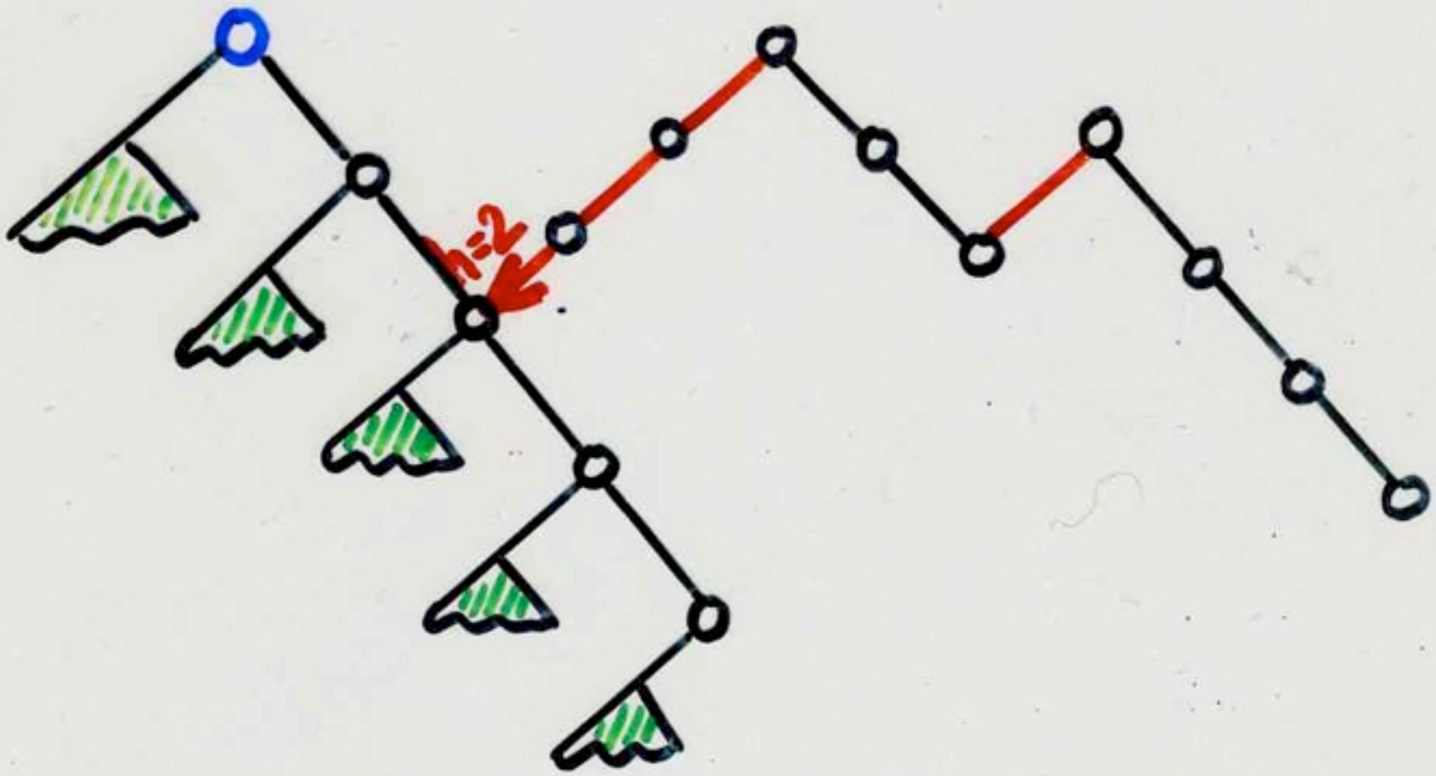
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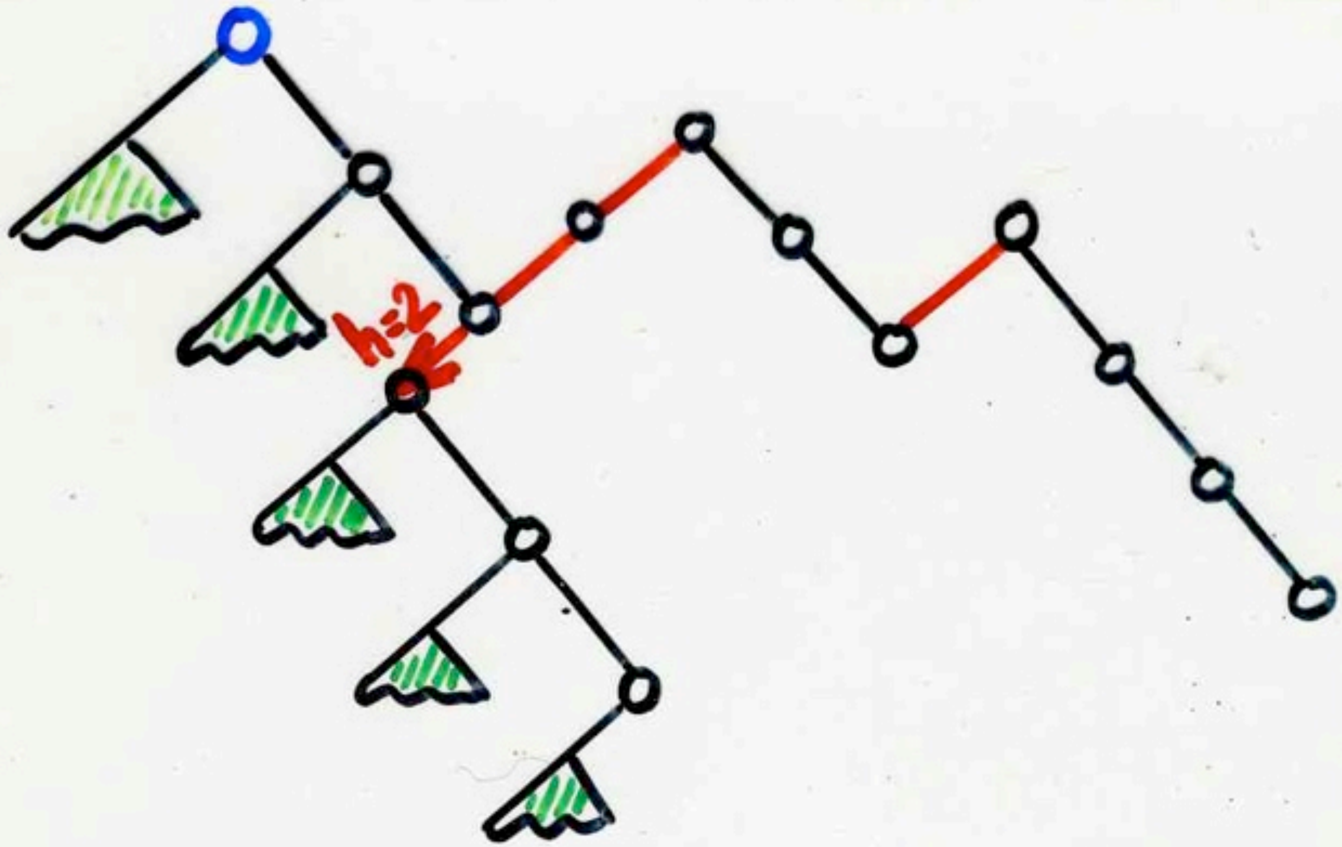
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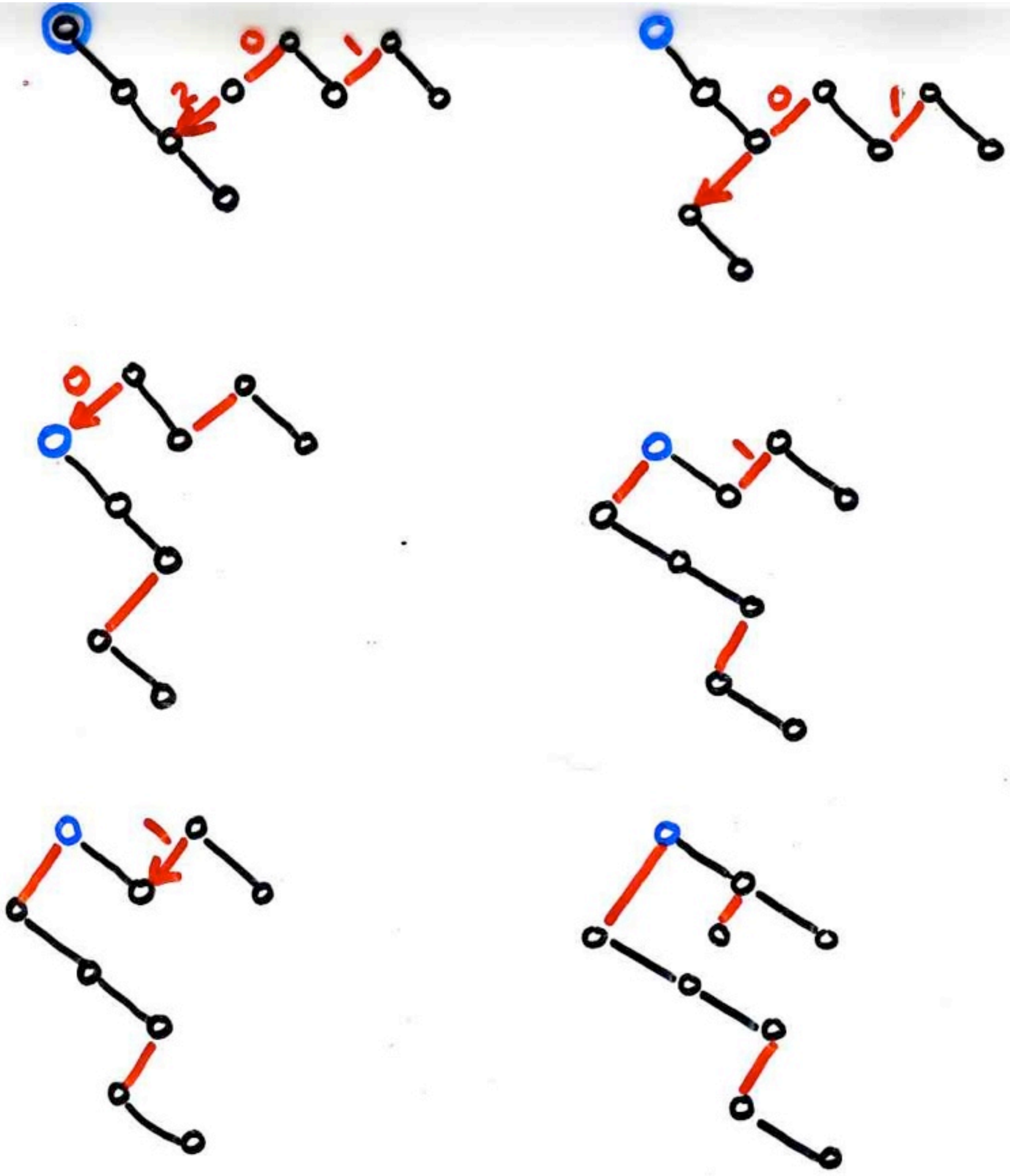

0
1
2