

Pavages  
formules magiques  
et  
preuves sans mots

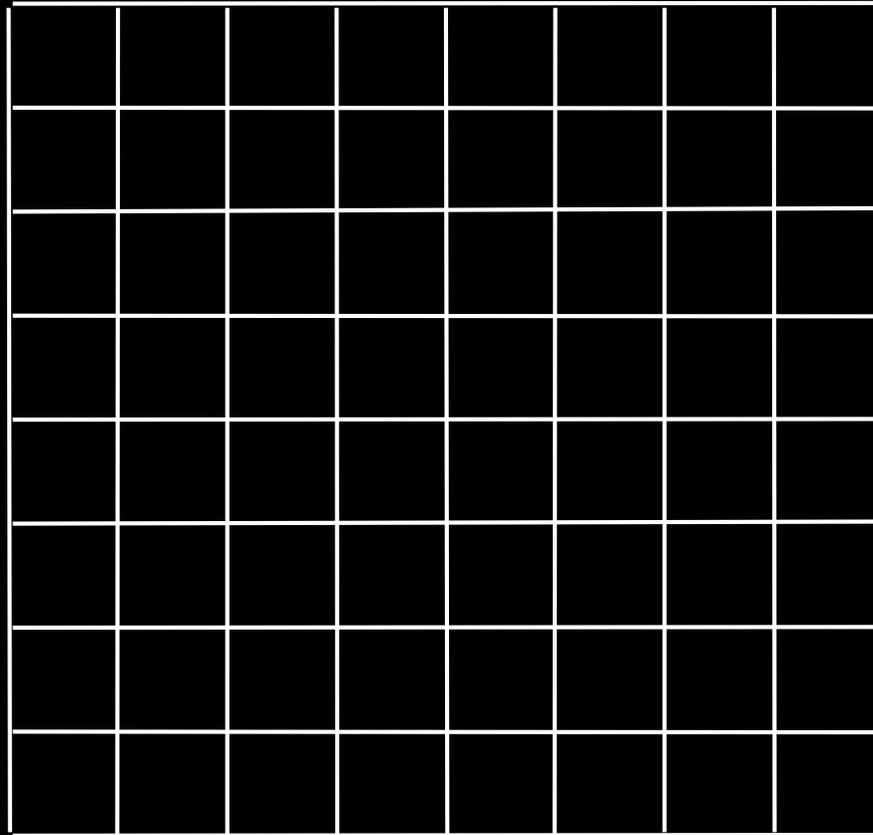
Paris, IHP  
31 mai 2014

Xavier Viennot  
LaBRI, CNRS, Bordeaux  
avec  
Gérard H.E. Duchamp (violon)  
LIPN, Sorbonne Universités

pavages ....



un échiquier

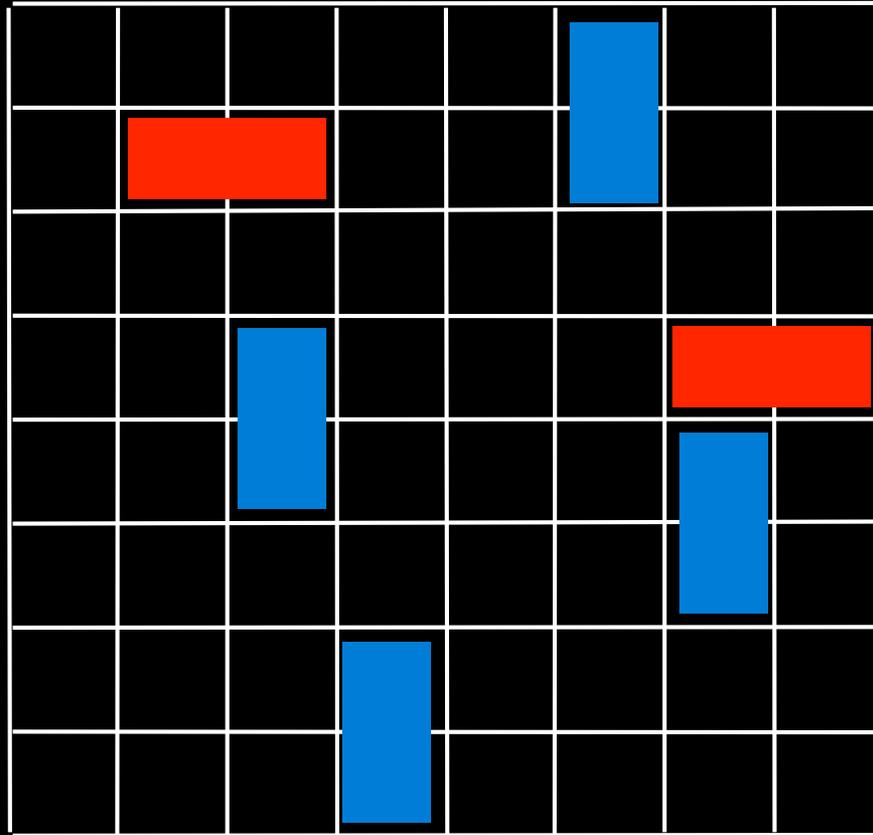


8 lignes  
8 colonnes

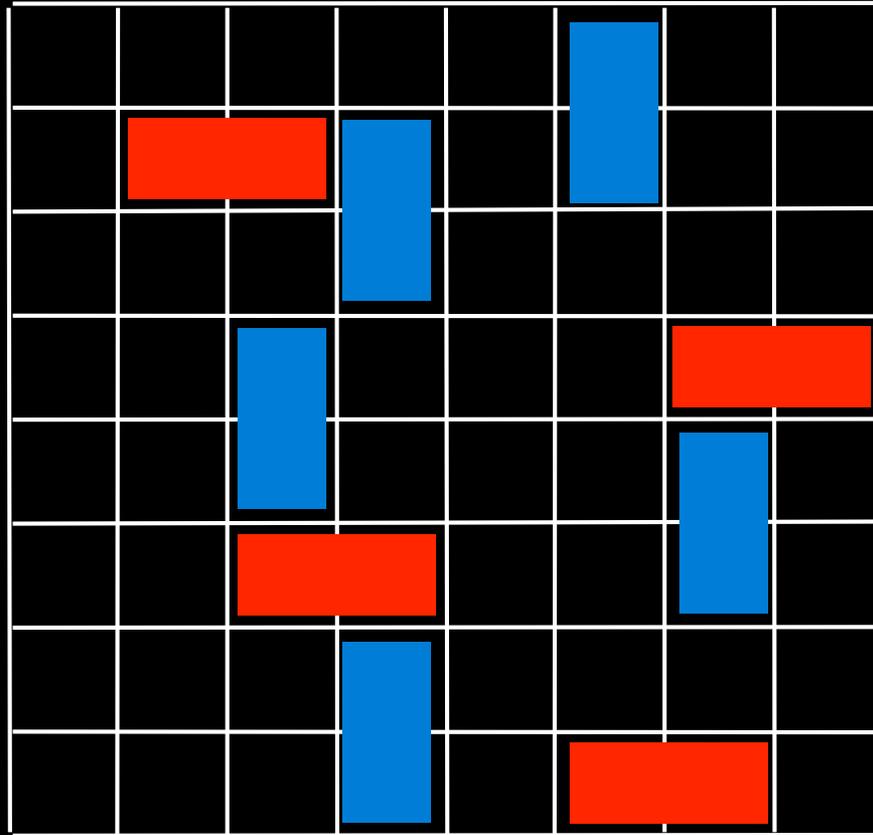




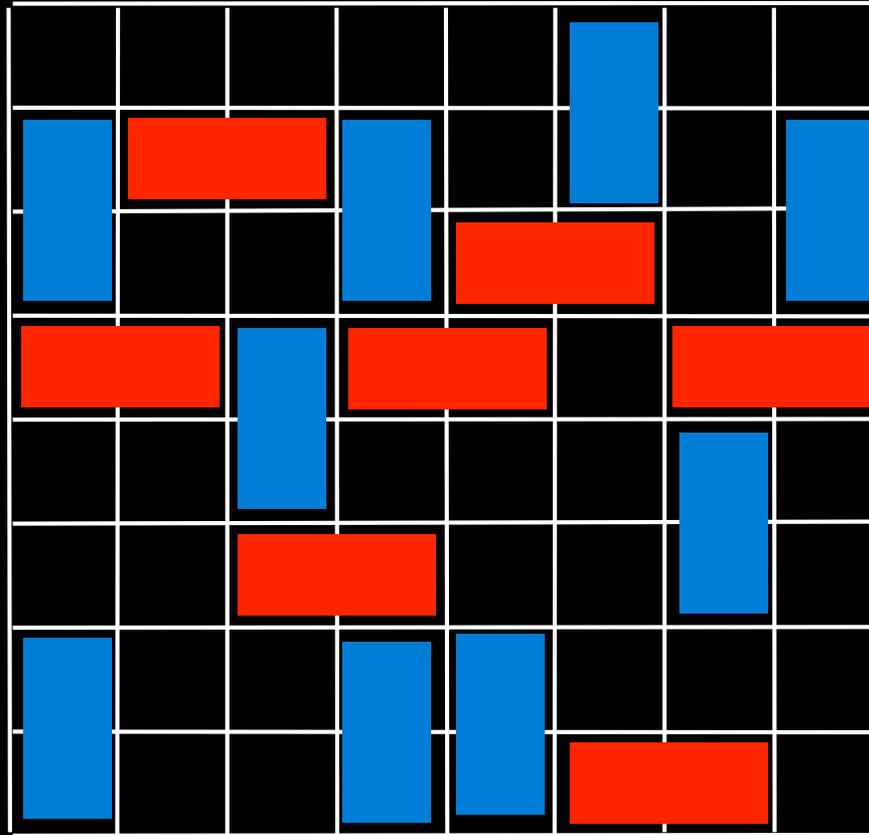
# pavages d'un échiquier avec des dominos



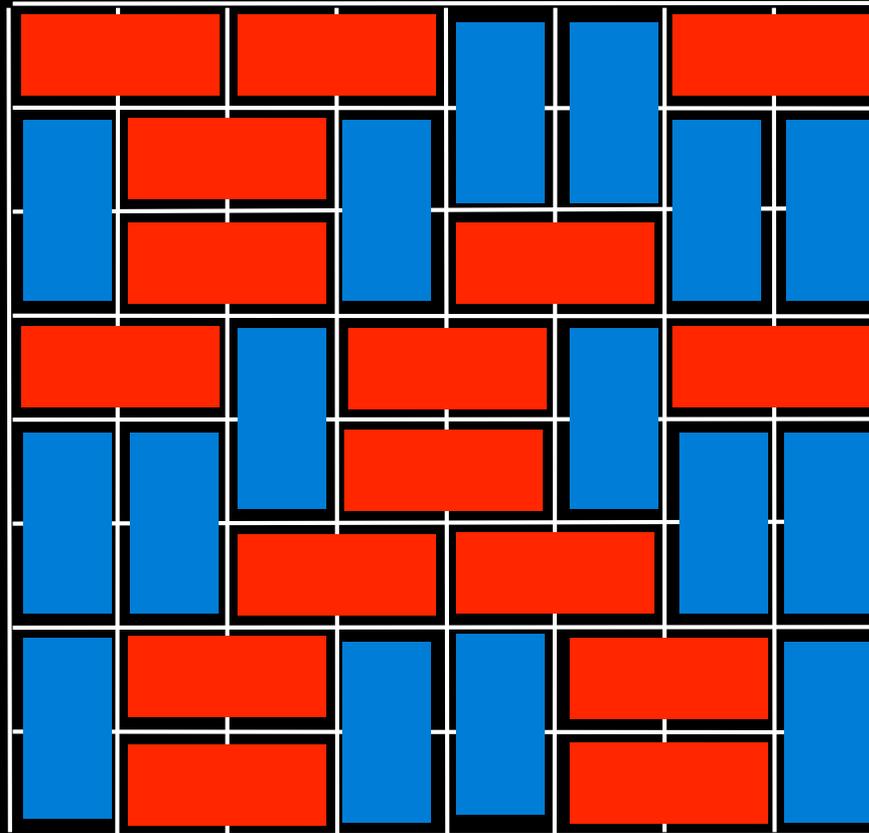
# pavages d'un échiquier avec des dominos



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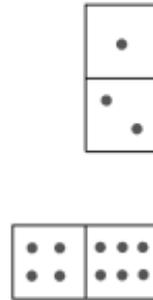
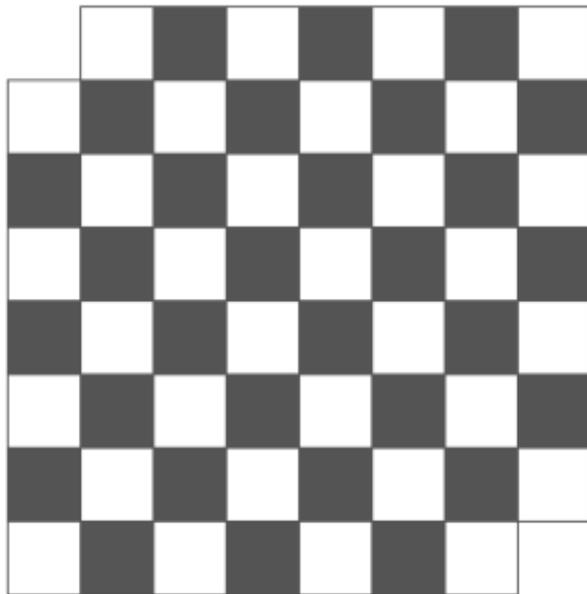


le nombre de pavages d'un échiquier 8 x 8  
= 12 988 816

une «formule» ?

nombre de pavages

pour un rectangle  $m \times n$



existence d'un pavage ?



nombre de pavages

rectangle  
 $2 \times n$



1

2

3

5

8

1, 2, 3, 5, 8, 13, 21, 34, 55, ...

# nombre de Fibonacci

1, 1, 2, 3, 5, 8, 13, 21, 34, 55,...

$$F_{n+1} = F_n + F_{n-1}$$

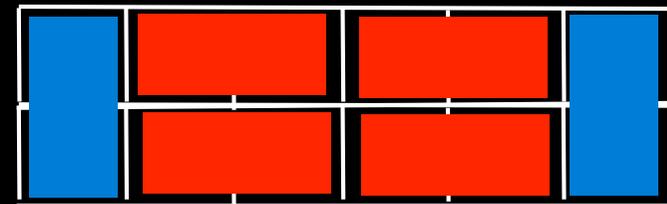
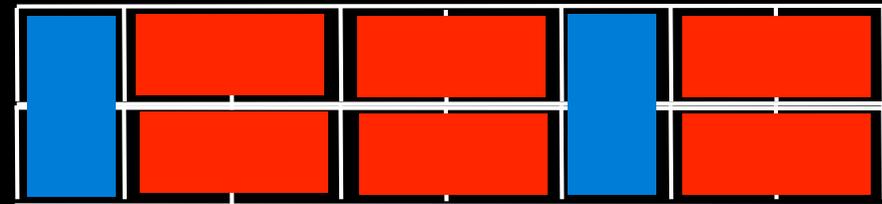
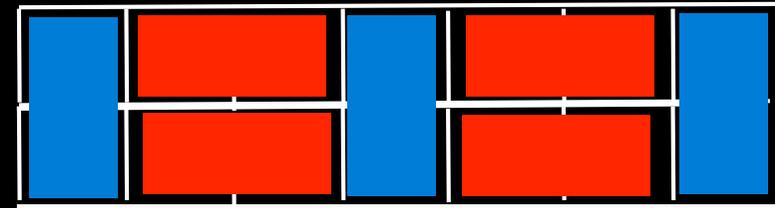
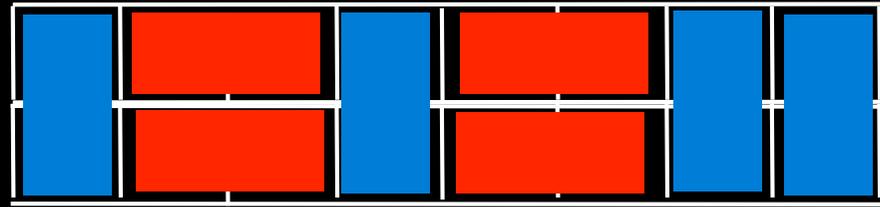
$$F_0 = 1, F_1 = 1$$

nombre total  
de pavages =

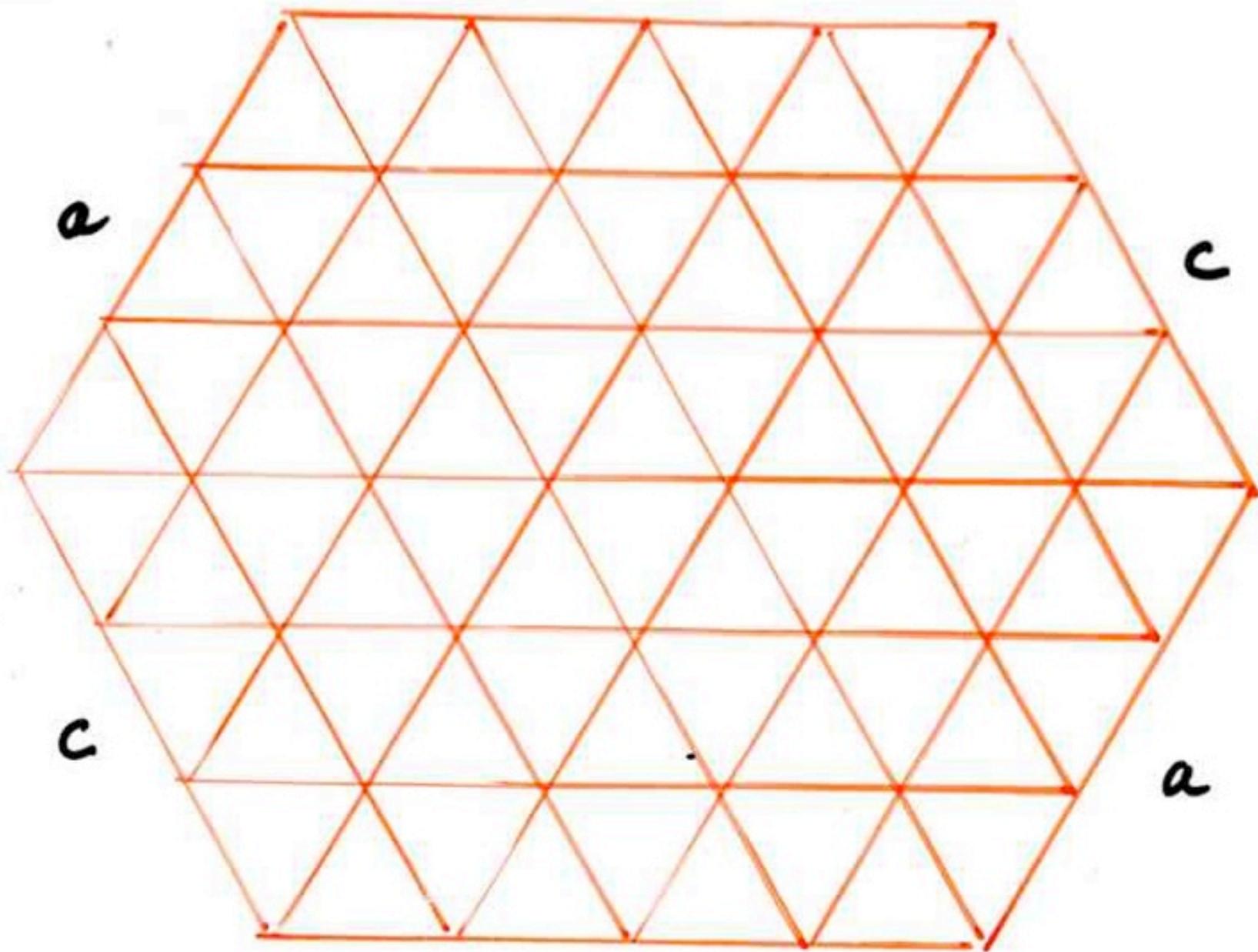
$$F_n$$

nombre de Fibonacci

$$F_{n+1} = F_n + F_{n-1}$$
$$F_1 = 1, F_2 = 2$$



pavages avec des calissons



a

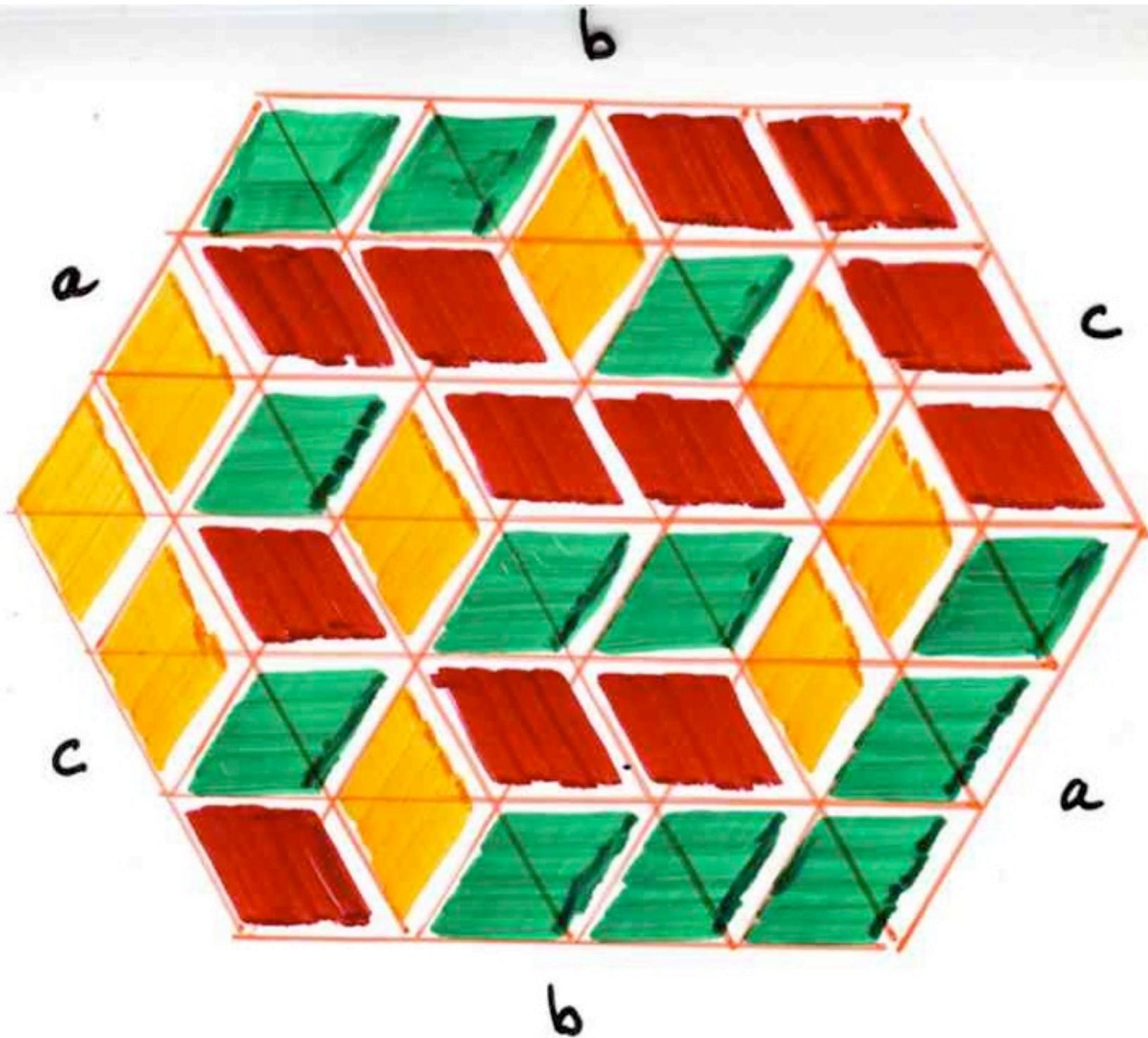
c

c

a

b

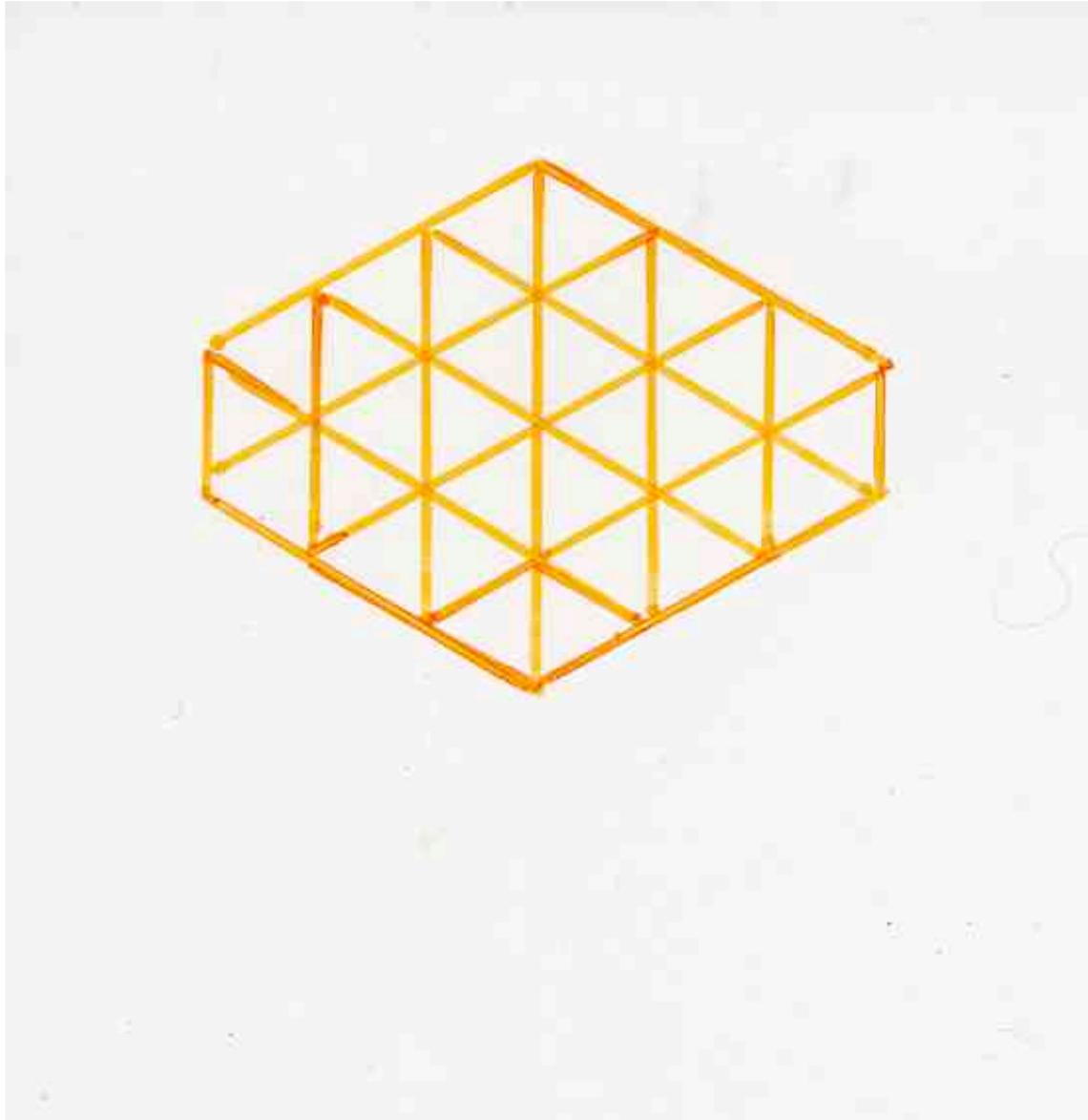
b

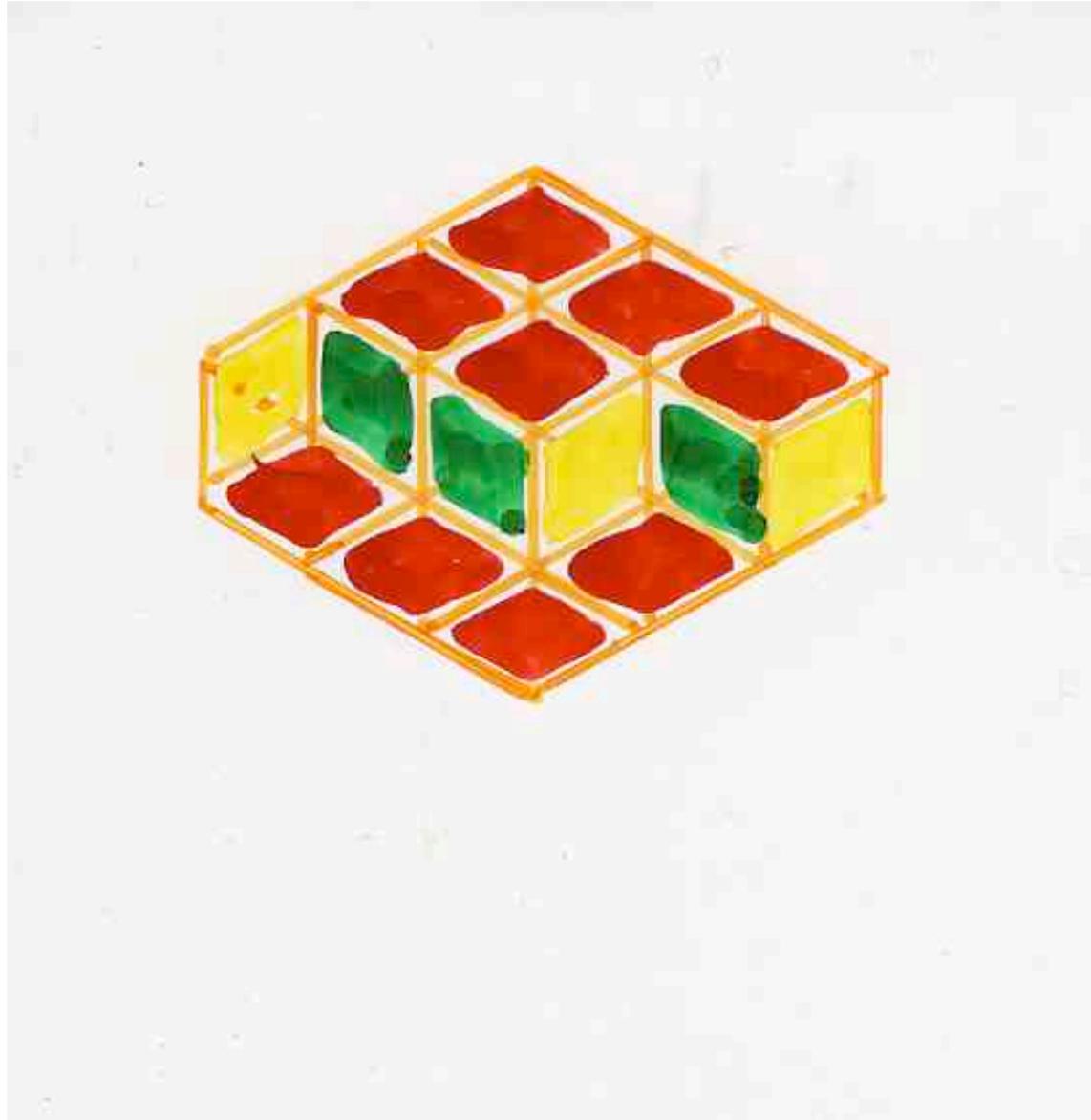


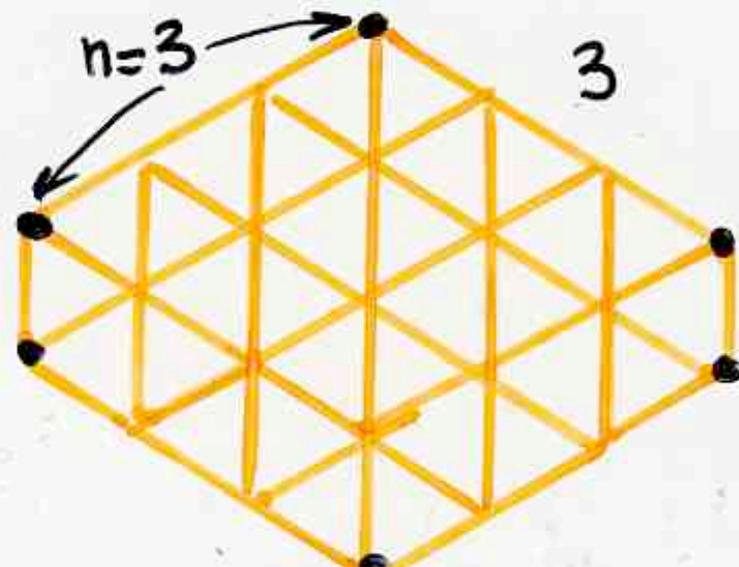
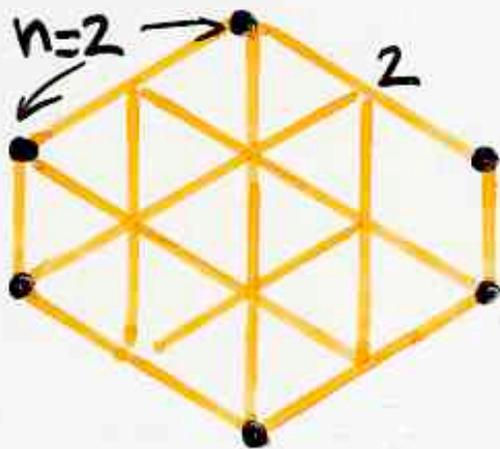
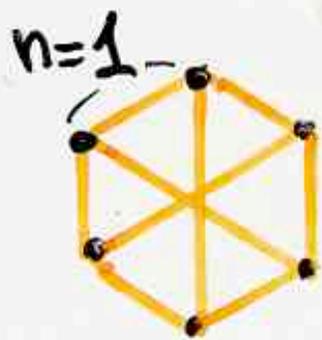
une «formule» ?

nombre de pavages

pour un hexagone  $a \times b \times c$







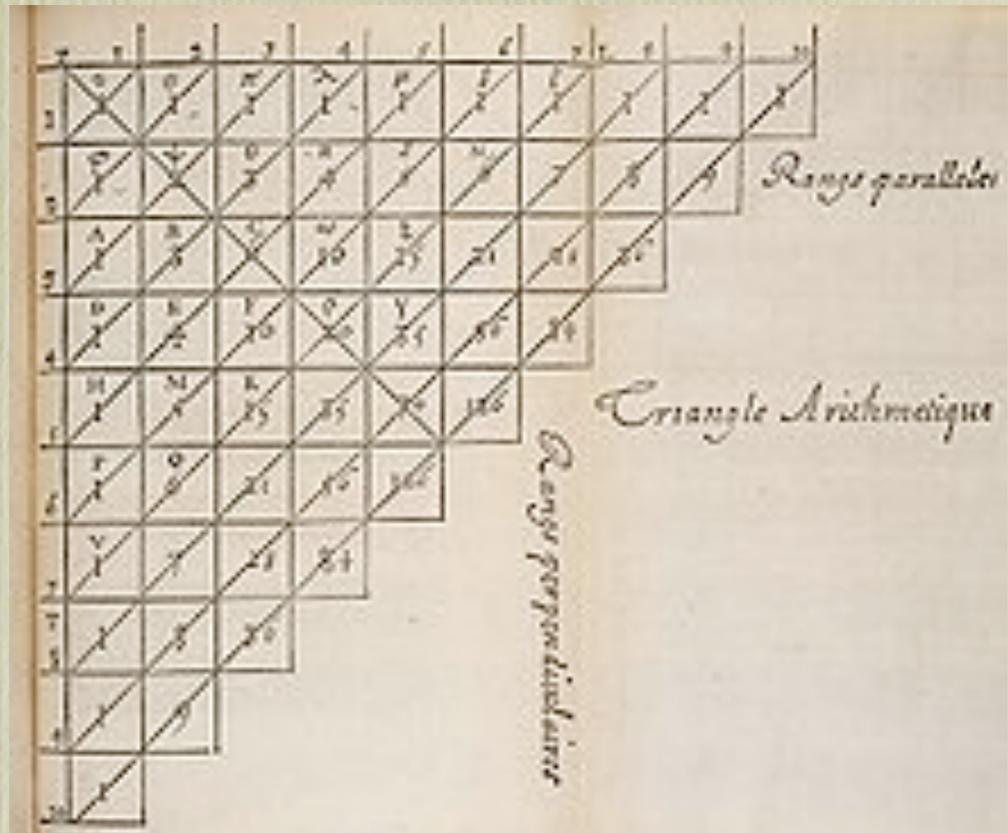
$b_n = 2, 6, 20, \dots$

nombre  
de  
pavages

combinatoire classique



triangle de Pascal  
coefficients binomiaux



# coefficients binomiaux

addition +

1									
1	1								
1	2	1							
1	3	3	1						
1	4	6	4	1					
1	5	10	10	5	1				
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		
1	8	28	56	70	56	28	8	1	

$n=1$  ,    2    3    4    5    6  
 $b_n$  1 ,   2 ,   6 ,   20 ,   70 ,   252 , ...

addition +

1									
1	1								
1	2	1							
1	3	3	1						
1	4	6	4	1					
1	5	10	10	5	1				
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		
1	8	28	56	70	56	28	8	1	

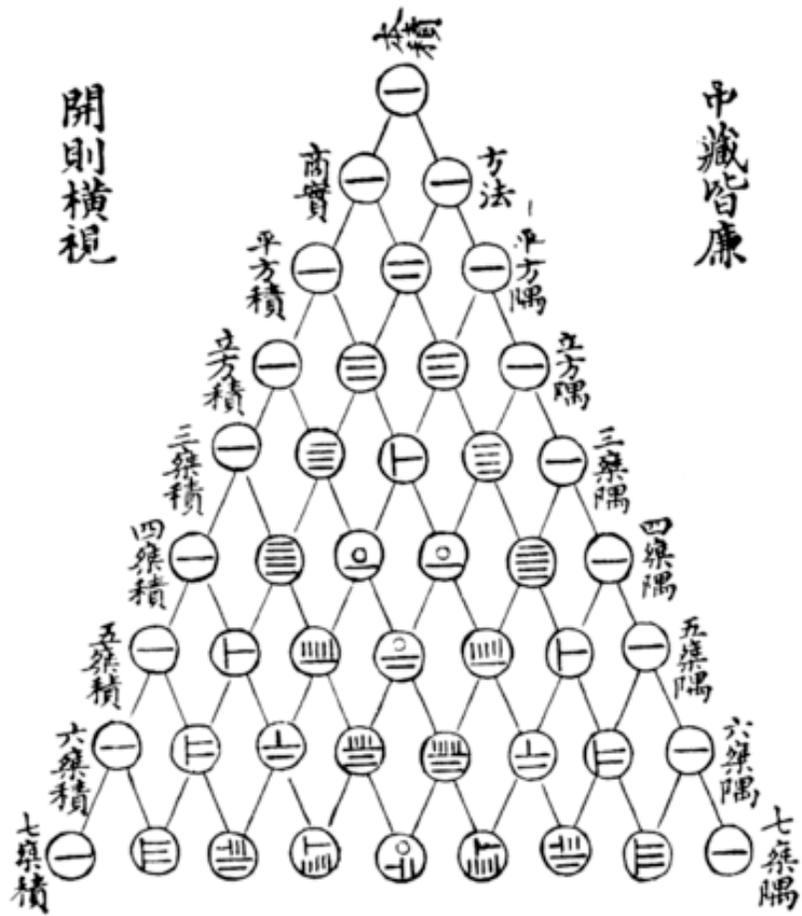
partie à  $k$  éléments  
d'un ensemble ayant  $n$  éléments

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$n! = 1 \times 2 \times \dots \times n$$

$$\binom{2n}{n} = \frac{2n!}{n!(n)!}$$

# 古法七葉方圖

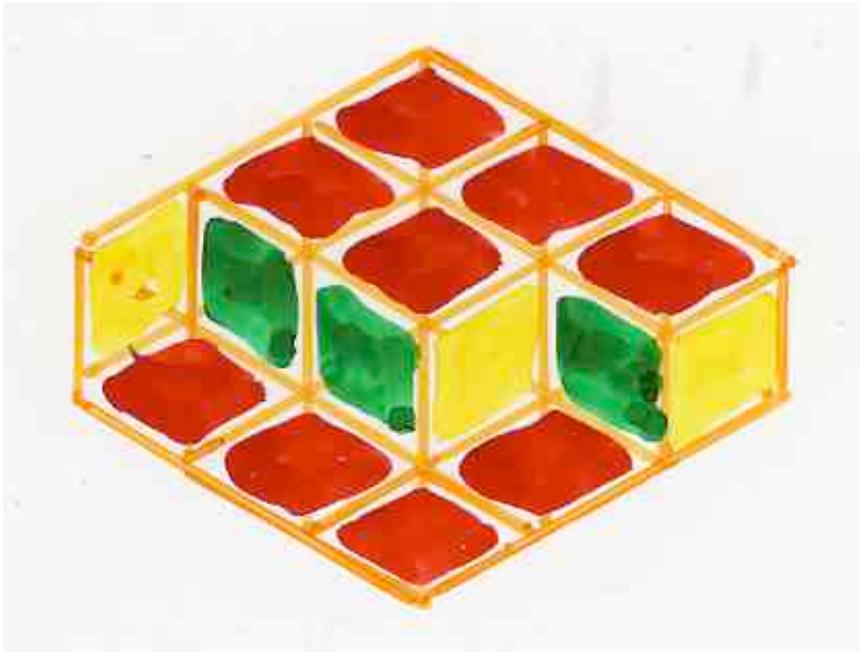


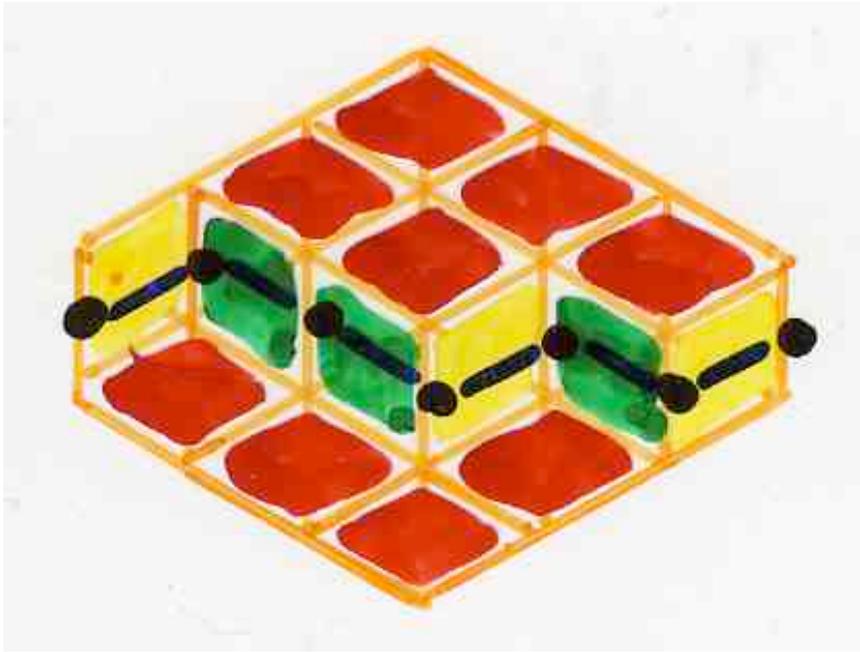
本積	方法	一廉	二廉	三廉	四廉	五廉	六廉	七廉
----	----	----	----	----	----	----	----	----

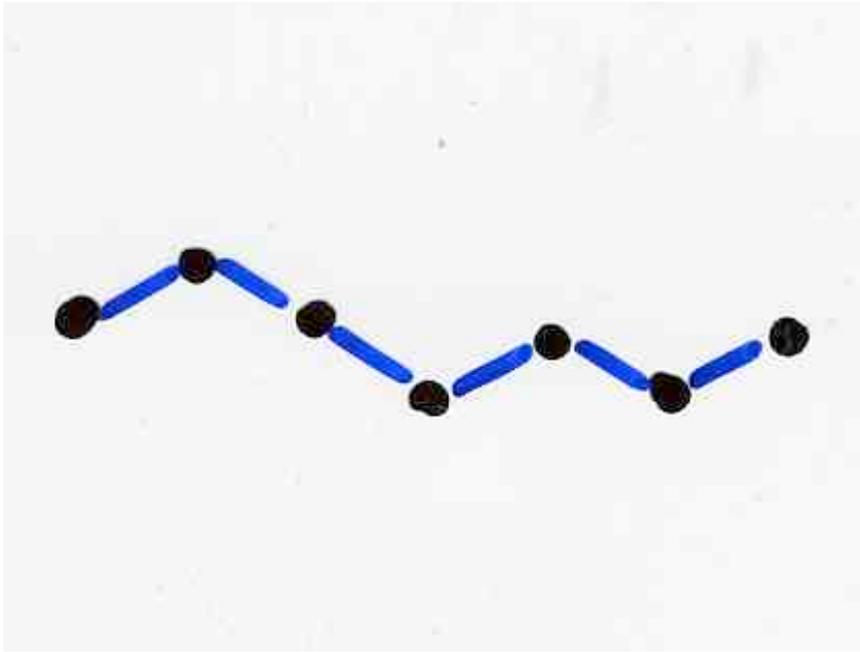
triangle de Yang Hui  
(11th, 12th siècle)

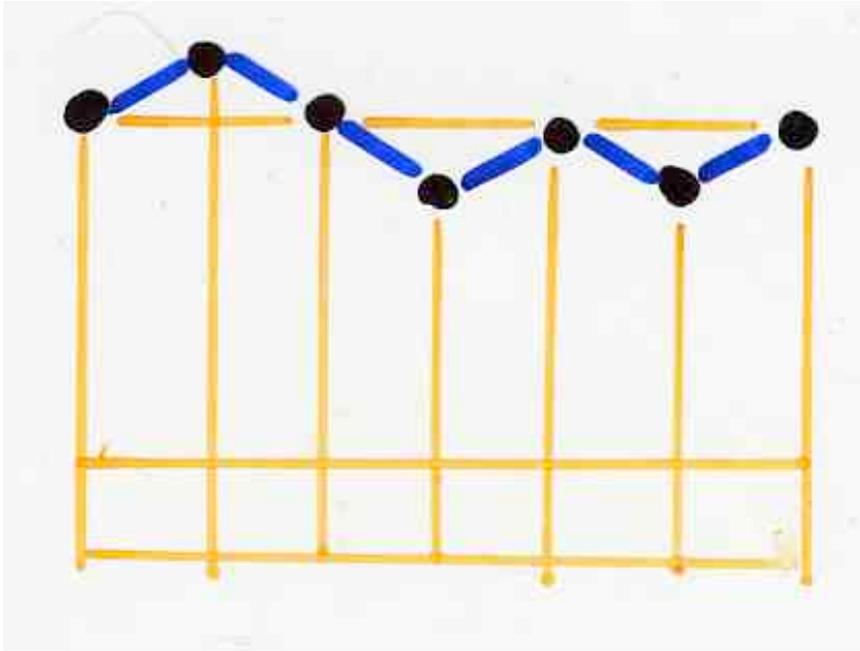
Perse  
Omar Khayyam  
(1048-1131)

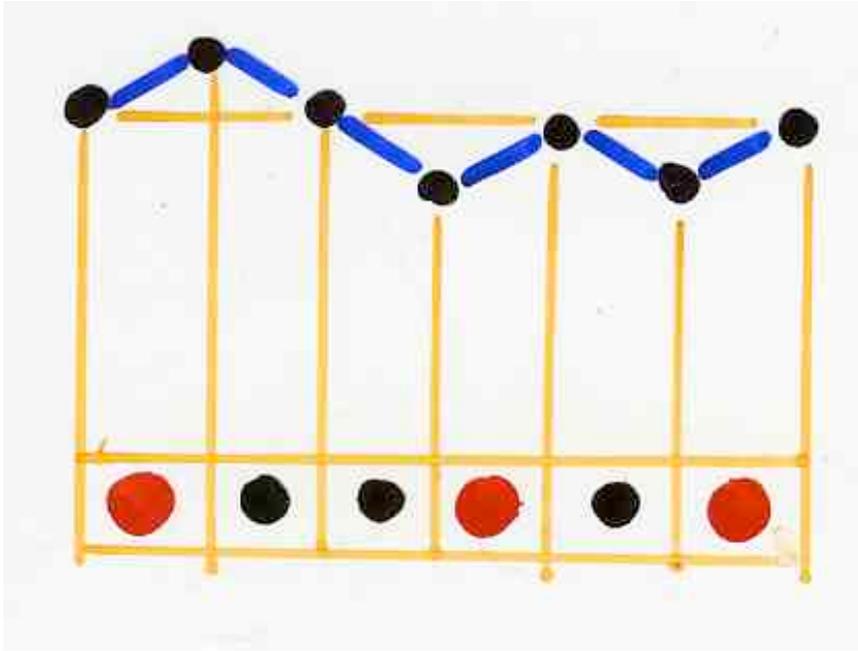
en Inde  
Chandas Shastra by Pingala  
2ème siècle



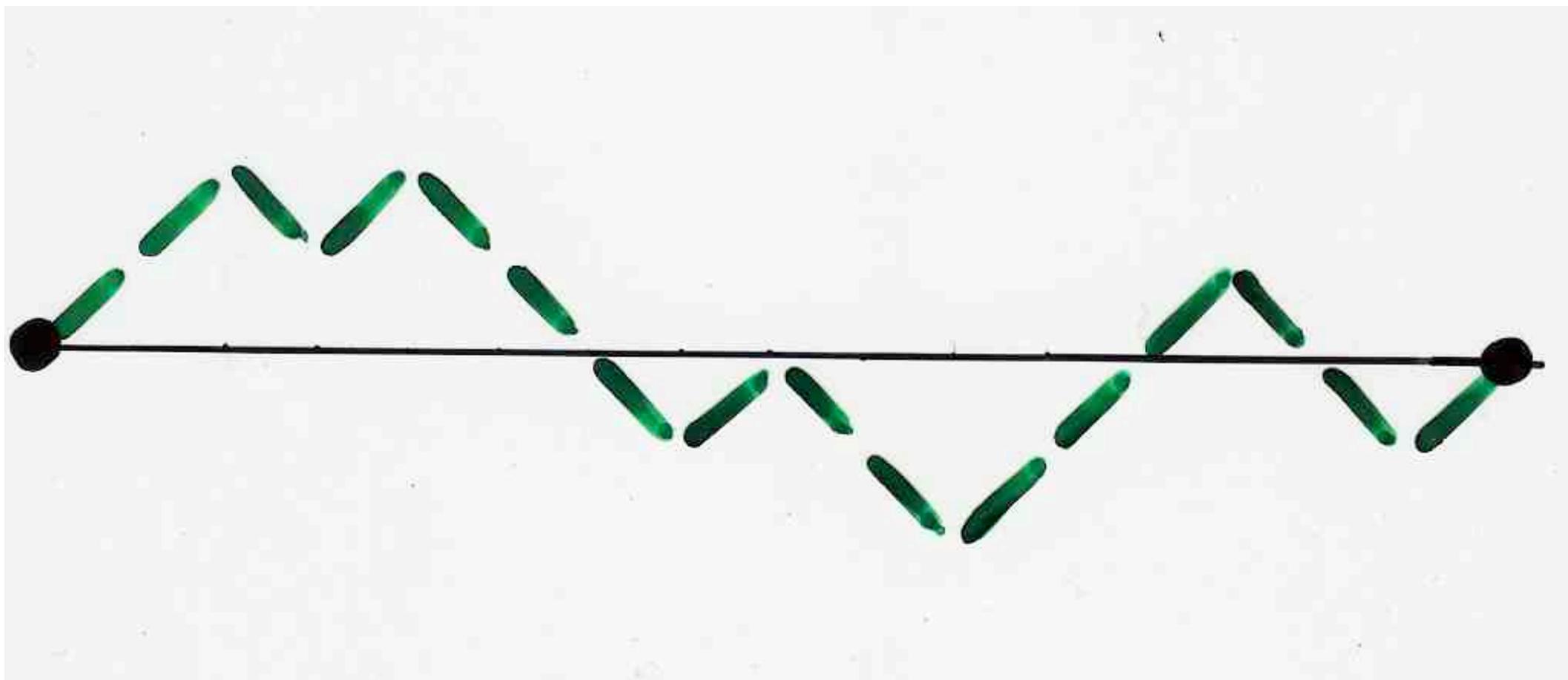


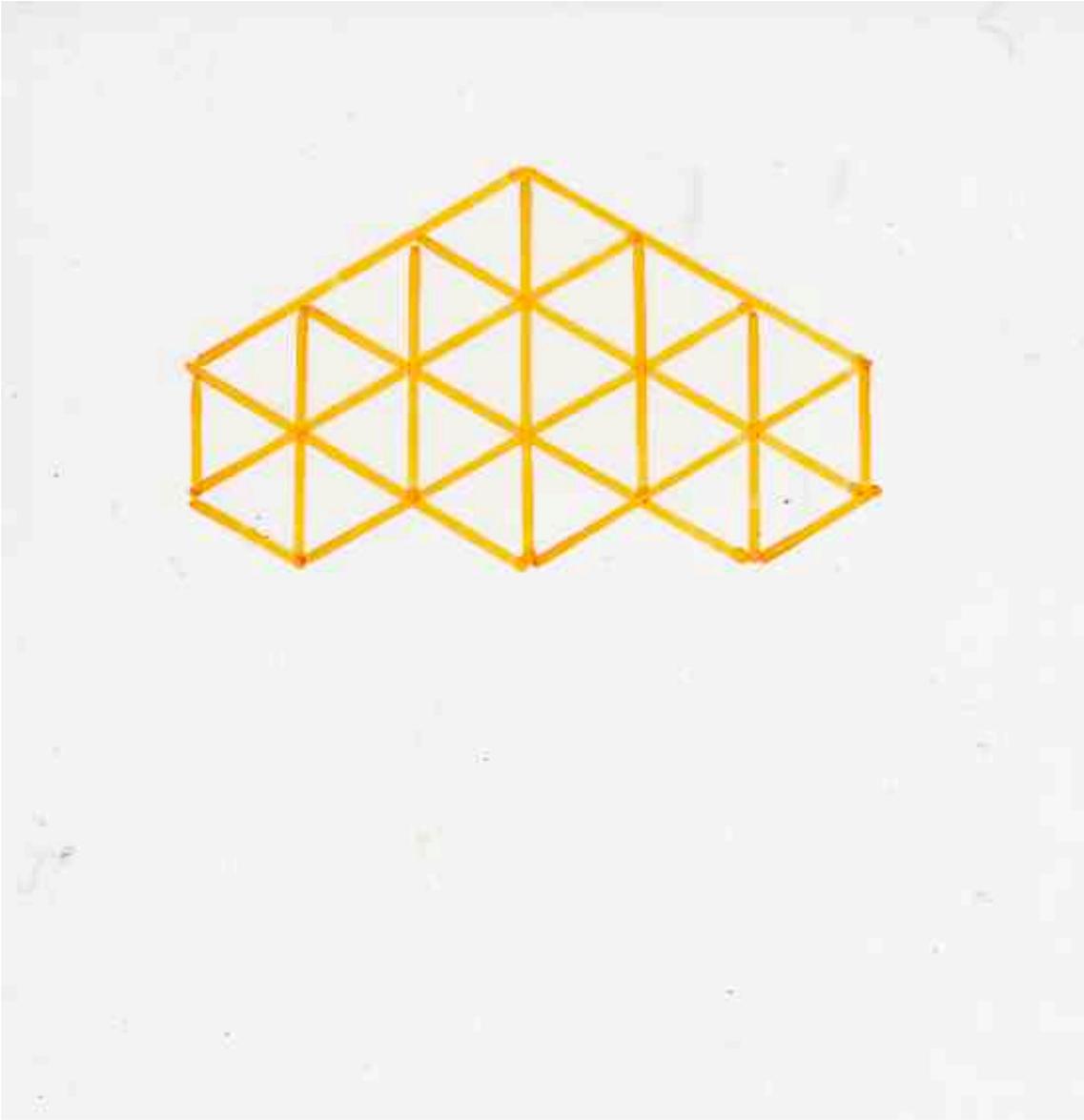


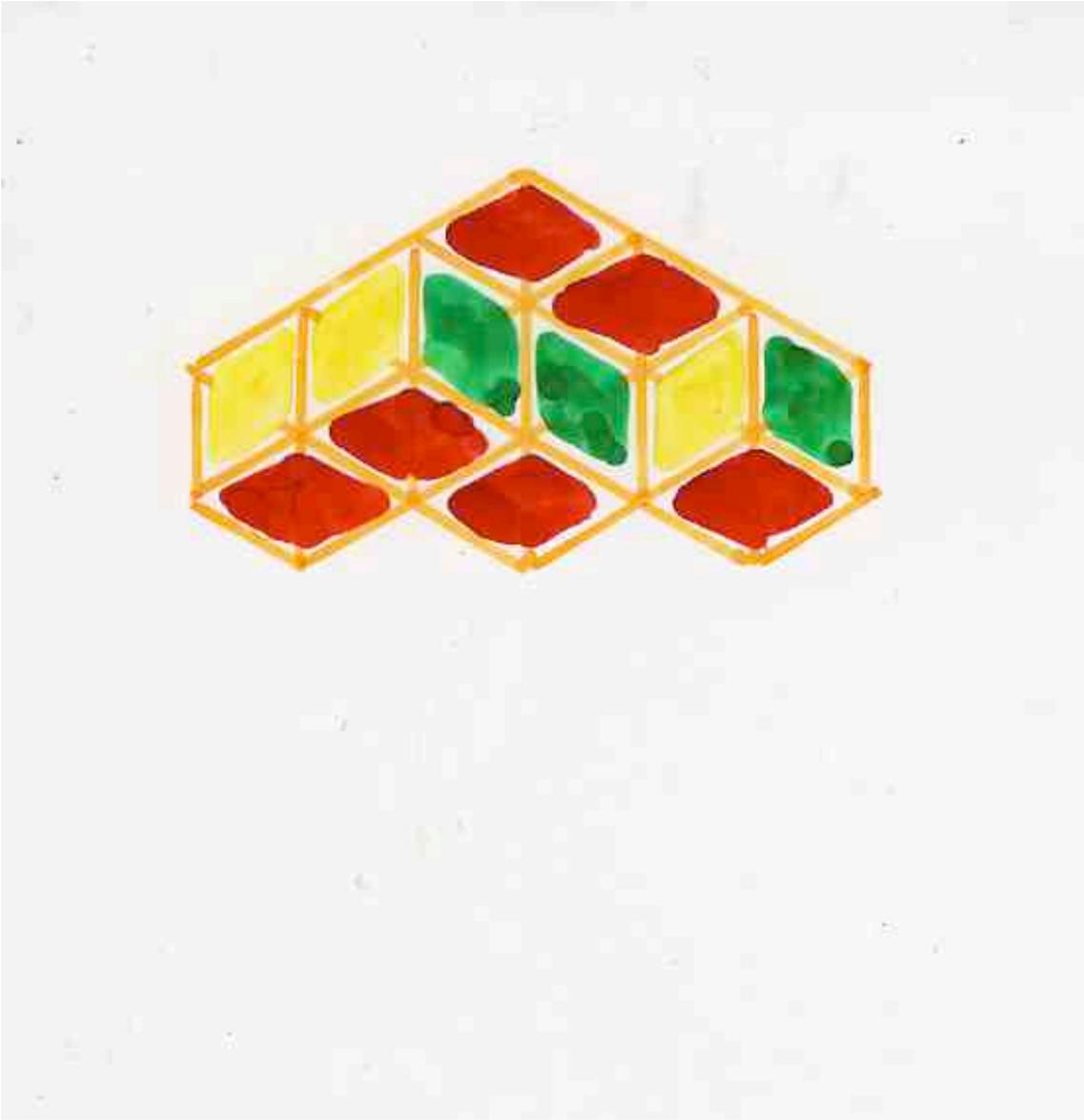


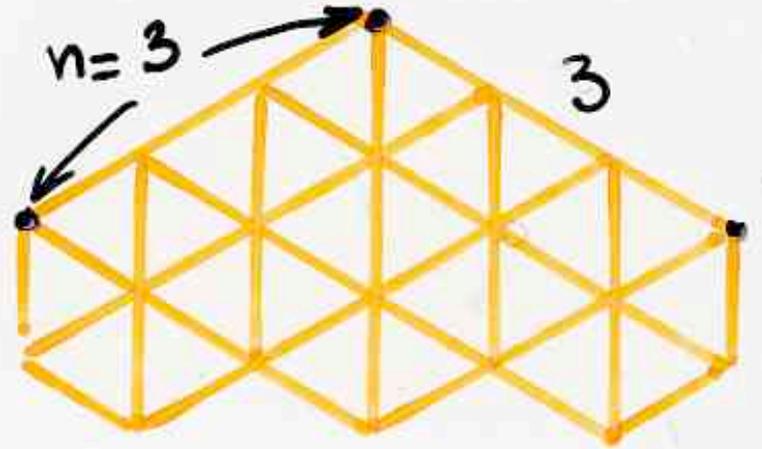
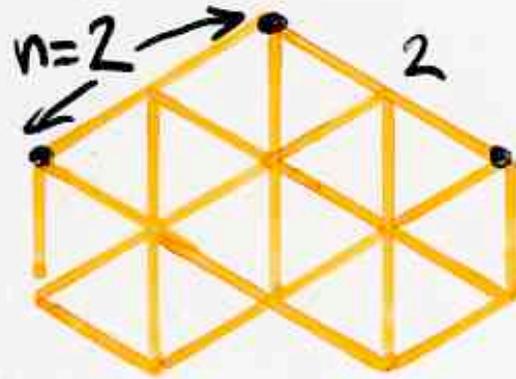
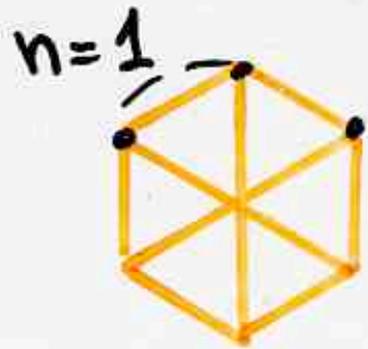


$$\binom{6}{3} = \frac{6!}{3!3!} = 20$$









$$a_n = 2,$$

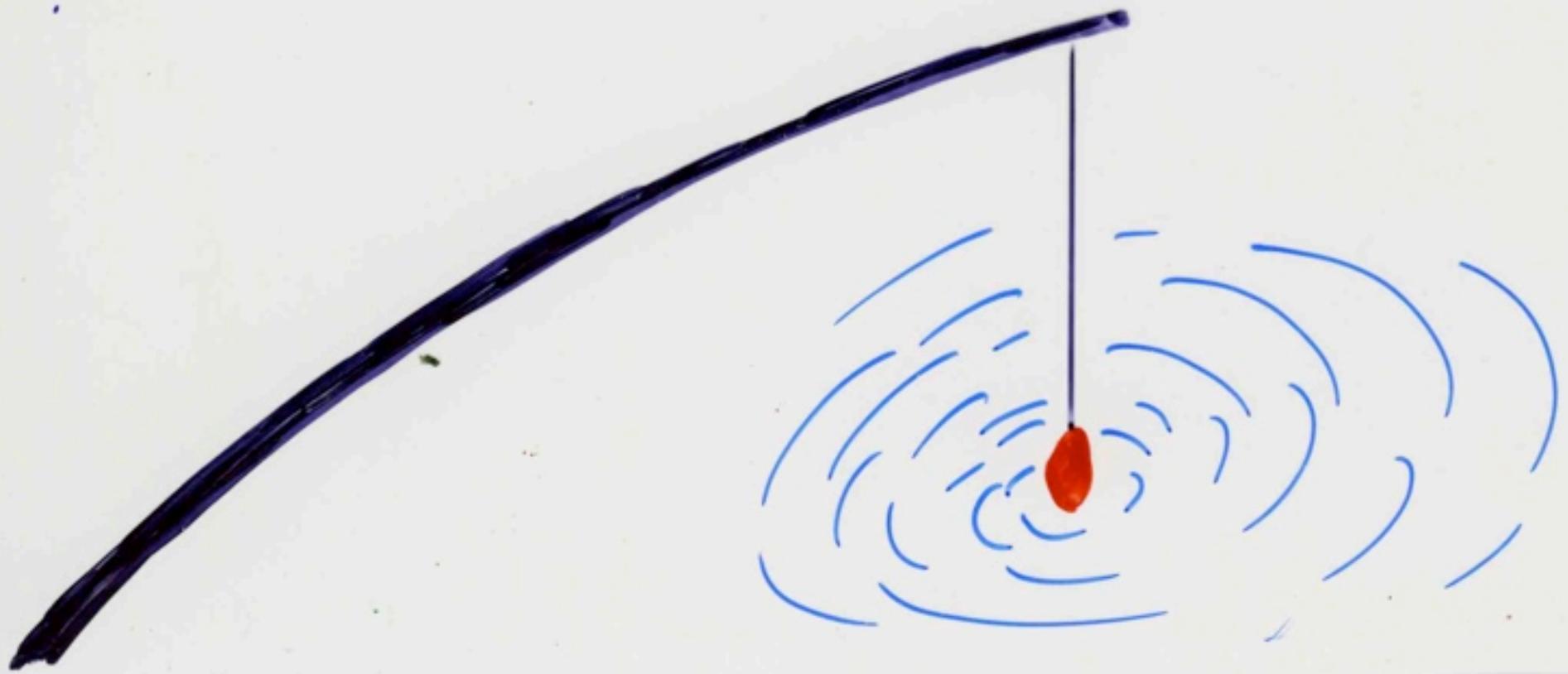
$$5,$$

$$14, \dots$$

nombre  
de  
pavages

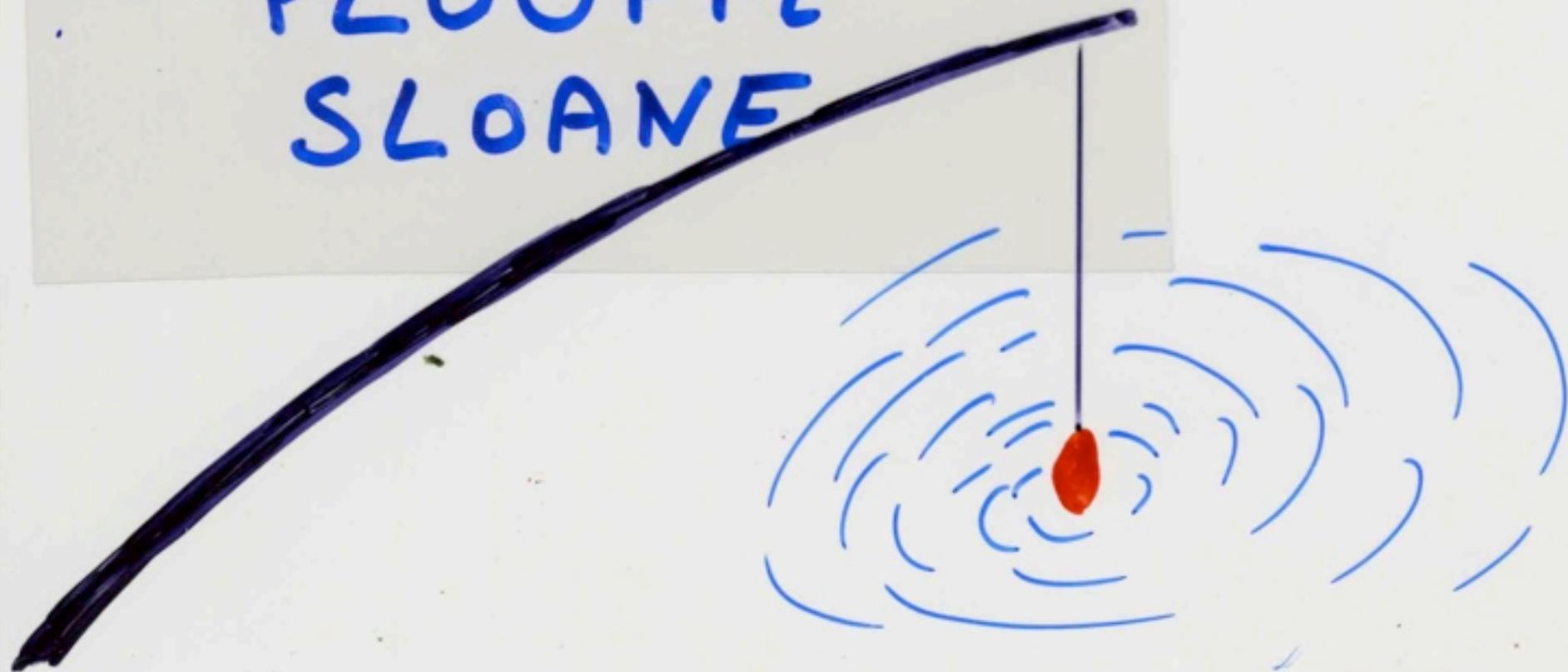
Le côté expérimental  
dans la recherche mathématique

pêche à la ligne.



pêche à la ligne.

PLOUFFE  
SLOANE



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**THE  
ENCYCLOPEDIA  
▼▼▼ OF ▼▼▼  
INTEGER  
SEQUENCES**

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**M1453** 1, 2, 5, 14, 36, 98, 273, 768, 2197, 6360, 18584, 54780, 162672, 486154, 1461197, 4413988, 13393855, 40807290, 124783669, 382842018, 1178140280, 3635626680  
Secondary alcohols with  $n$  carbon atoms. Ref BA76 44. [3,2; A5955]

**M1454** 2, 5, 14, 38, 107  
Domino  $n$ -tuples. Ref JRM 7 324 74. [1,1; A6574]

**M1455** 1, 1, 2, 5, 14, 38, 120, 353, 1148, 3527, 11622, 36627, 121622, 389560, 1301140, 4215748, 13976335, 46235800, 155741571, 512559185, 1732007938, 5732533570  
Folding a strip of  $n$  blank stamps. See Fig M4587. Ref ScAm 209(3) 262 63. JCT 5 151 68. CBUL (2) 3 36 75. [1,3; A1011, N0576]

**M1456** 1, 2, 5, 14, 39, 109  
Paraffins with  $n$  carbon atoms. Ref ZFK 93 437 36. [1,2; A0641, N0575]

**M1457** 1, 1, 2, 5, 14, 40, 128, 369, 1214, 3516, 12776, 40534, 137404, 463232, 1602348, 5216253, 17753898, 58597316, 212150928, 710453534, 2366853608, 8584498376  
Shifts left under l.c.m.-convolution with itself. Ref BeS194. [0,3; A7463]

**M1458** 1, 2, 5, 14, 41, 122, 365, 1094, 3281, 9842, 29525, 88574, 265721, 797162, 2391485, 7174454, 21523361, 64570082, 193710245, 581130734, 1743392201  
 $(3^n + 1)/2$ . Ref BPNR 60. Ribe91 53. HM94. [0,2; A7051]

**M1459** 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020  
Catalan numbers:  $C(n) = C(2n, n)/(n+1)$ . See Fig M1459. Ref AMM 72 973 65. RCI 101. C1 53. PLC 2 109 71. MAG 61 211 88. [0,3; A0108, N0577]

$$\text{G.f.: } \frac{1 - (1 - 4x)^{1/2}}{2x}$$



**Figure M1459.** CATALAN NUMBERS.

The **Catalan numbers**, defined by

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{1}{n+1} C(2n, n)$$

(M1459), are probably the most frequently occurring combinatorial numbers after the binomial coefficients. [GO4] lists over 240 references. See also [MAG 45 199 61], [AMM 72 973 65], [PLC 2 109 71], [C1], [Stan86] and [GKP]. Some of the dozens of interpretations of  $C_n$  are:

(i) The number of ways of dissecting a convex polygon of  $n+2$  sides into  $n$  triangles by drawing  $n-1$  nonintersecting diagonals (Fig. (a)).

(ii) The number of ways of completely parenthesizing a product of  $n+1$  letters (so that there are two factors inside each set of parentheses):

$$\begin{aligned} n = 1 & (ab); & n = 2 & a(bc), (ab)c; \\ n = 3 & (ab)(cd), a((bc)d), ((ab)c)d, a(b(cd)), (a(bc))d. \end{aligned}$$

Domino  $n$ -tuples. Ref JRM 7 324 74. [1,1; A6574]

**M1455** 1, 1, 2, 5, 14, 38, 120, 353, 1148, 3527, 11622, 36627, 121622, 389560, 1301140,  
4215748, 13976335, 46235800, 155741571, 512559185, 1732007938, 5732533570  
Folding a strip of  $n$  blank stamps. See Fig M4587. Ref ScAm 209(3) 262 63. JCT 5 151 68.  
CBUL (2) 3 36 75. [1,3; A1011, N0576]

**M1456** 1, 2, 5, 14, 39, 109  
Paraffins with  $n$  carbon atoms. Ref ZFK 93 437 36. [1,2; A0641, N0575]

**M1457** 1, 1, 2, 5, 14, 40, 128, 369, 1214, 3516, 12776, 40534, 137404, 463232, 1602348,  
5216253, 17753898, 58597316, 212150928, 710453534, 2366853608, 8584498376  
Shifts left under l.c.m.-convolution with itself. Ref BeSI94. [0,3; A7463]

**M1458** 1, 2, 5, 14, 41, 122, 365, 1094, 3281, 9842, 29525, 88574, 265721, 797162,  
2391485, 7174454, 21523361, 64570082, 193710245, 581130734, 1743392201  
 $(3^n + 1)/2$ . Ref BPNR 60. Ribe91 53. HM94. [0,2; A7051]

**M1459** 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440,  
9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020  
Catalan numbers:  $C(n) = C(2n, n)/(n+1)$ . See Fig M1459. Ref AMM 72 973 65. RCI  
101. C1 53. PLC 2 109 71. MAG 61 211 88. [0,3; A0108, N0577]

$$\text{G.f.: } \frac{1 - (1 - 4x)^{1/2}}{2x}$$

Search: seq:1,2,5,14,42

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### [A000108](#)

Catalan numbers:  $C(n) = \text{binomial}(2n,n)/(n+1) = (2n)!/(n!(n+1)!)$ . Also called Segner numbers.

(Formerly M1459 N0577)

+20  
2334

1, **1**, **2**, **5**, **14**, **42**, 132, 429, 1430, 4862, 16796, 58786,  
208012, 742900, 2674440, 9694845, 35357670, 129644790,  
477638700, 1767263190, 6564120420, 24466267020, 91482563640,  
343059613650, 1289904147324, 4861946401452, 18367353072152,  
69533550916004, 263747951750360, 1002242216651368,  
3814986502092304 ([list](#); [graph](#); [refs](#); [listen](#); [history](#); [text](#); [internal format](#))

1 1 2 5 14 42

nombres de Catalan



$n=1$ , 2, 3, 4, 5, 6  
 $b_n$  1, 2, 6, 20, 70, 252,

$a_n$  1, 2, 5, 14, 42,

$n=1$ , 2, 3, 4, 5, 6  
 $b_n$  1, 2, 6, 20, 70, 252,

$a_n$  1, 2, 5, 14, 42,

$$\frac{1}{1}$$

1  
1  
1  
1  
1  
1  
1  
1

$$\frac{2}{2}$$

1  
2  
3  
4  
5  
6  
7  
8

1

$$\frac{6}{3}$$

1  
3  
6  
15  
21  
28

$$\frac{20}{4}$$

1  
4  
10  
20  
35  
56

$$\frac{70}{5}$$

1  
5  
15  
35  
70

addition  
division

+

•

•

14

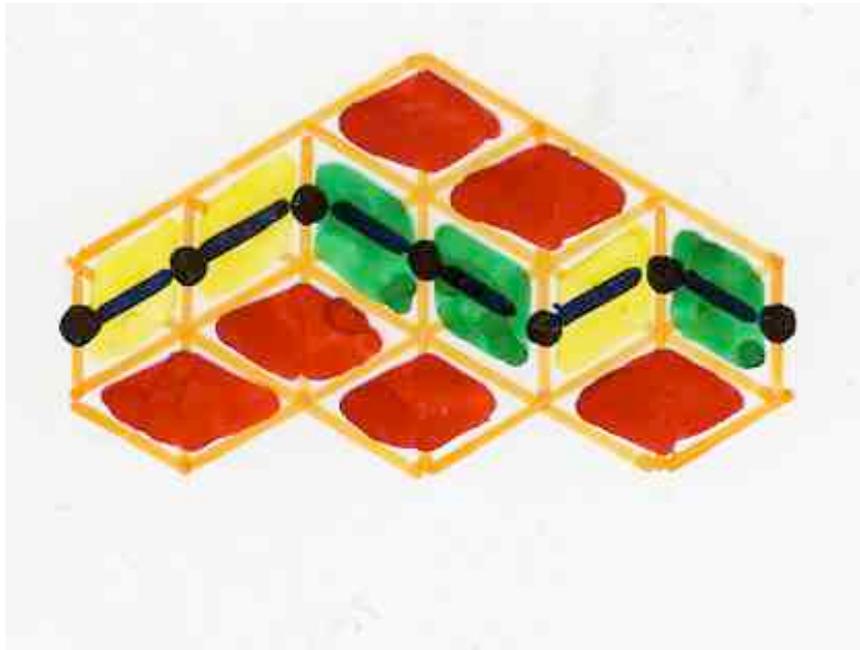
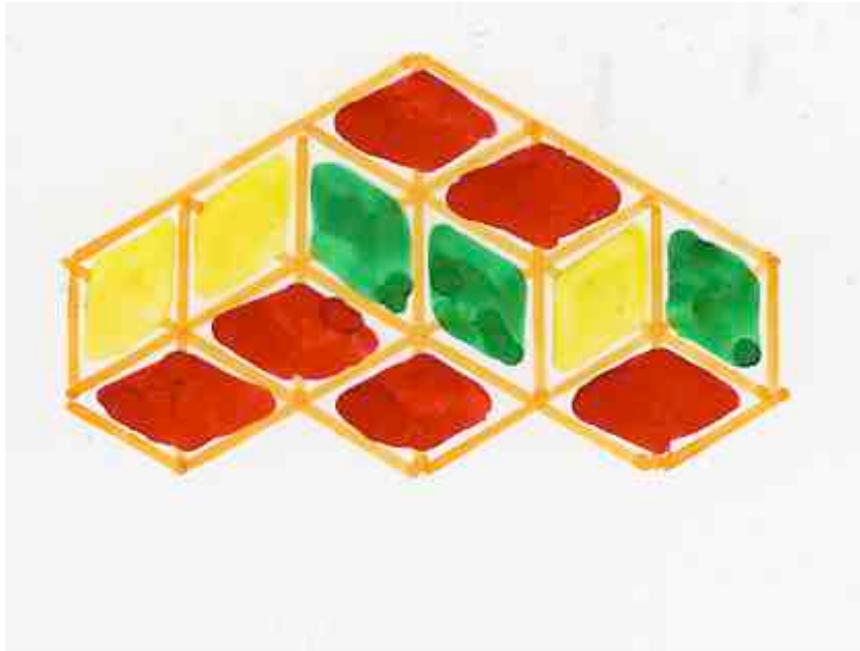
nombre  
de  
Catalan

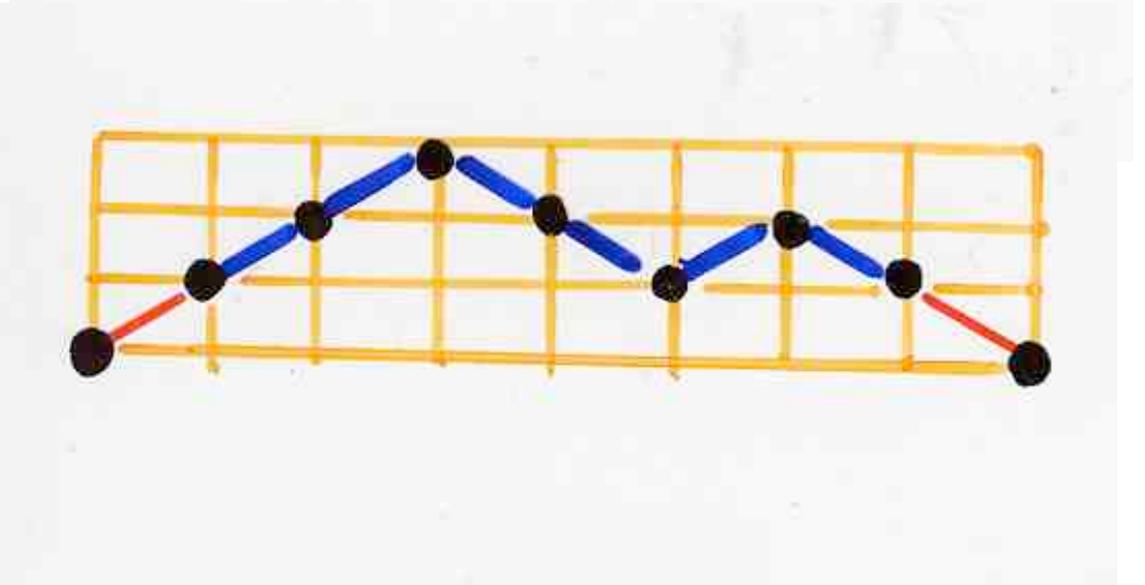
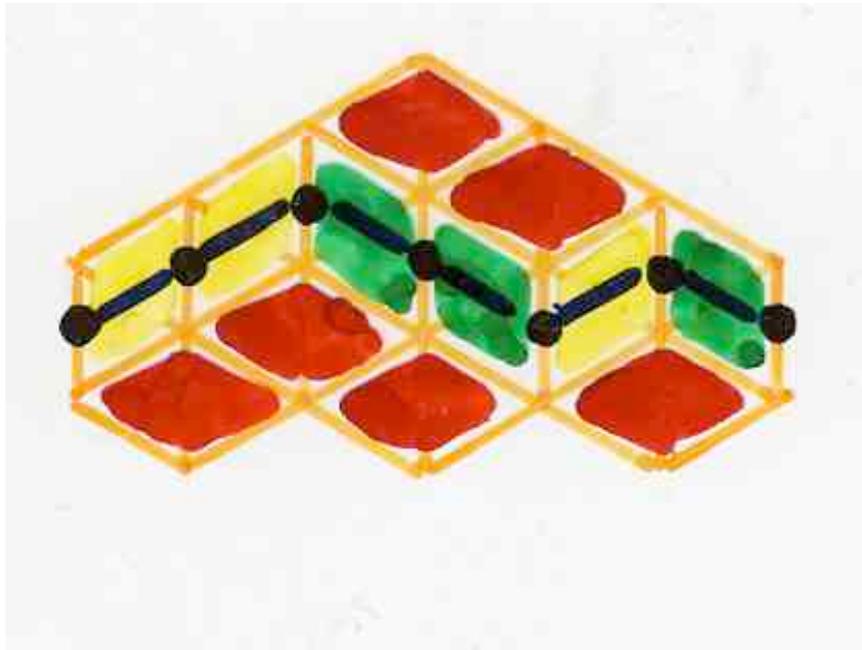
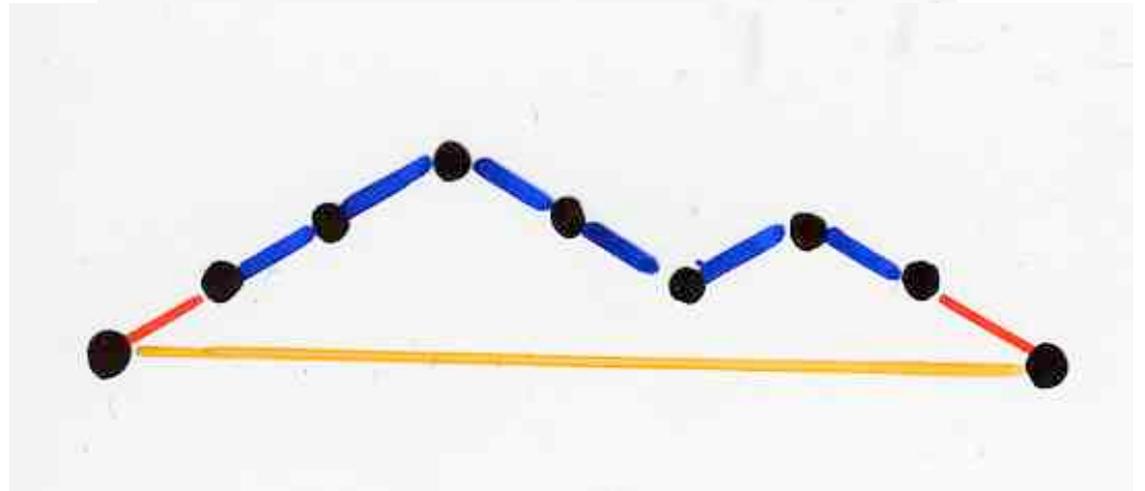
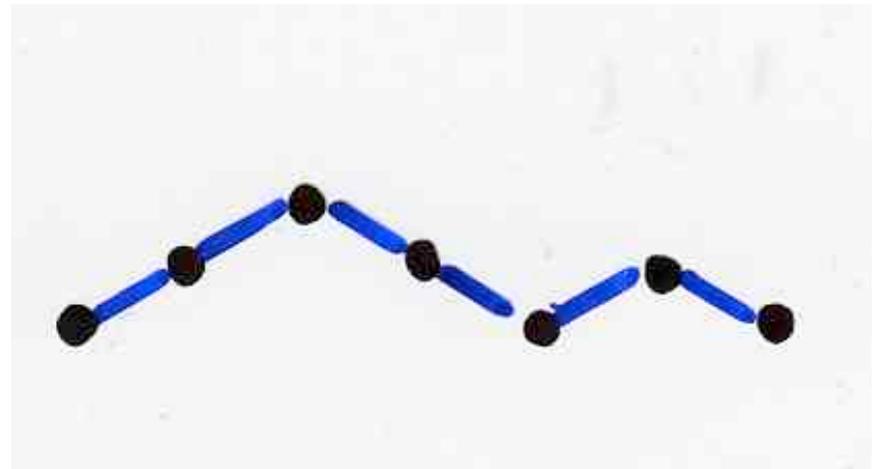
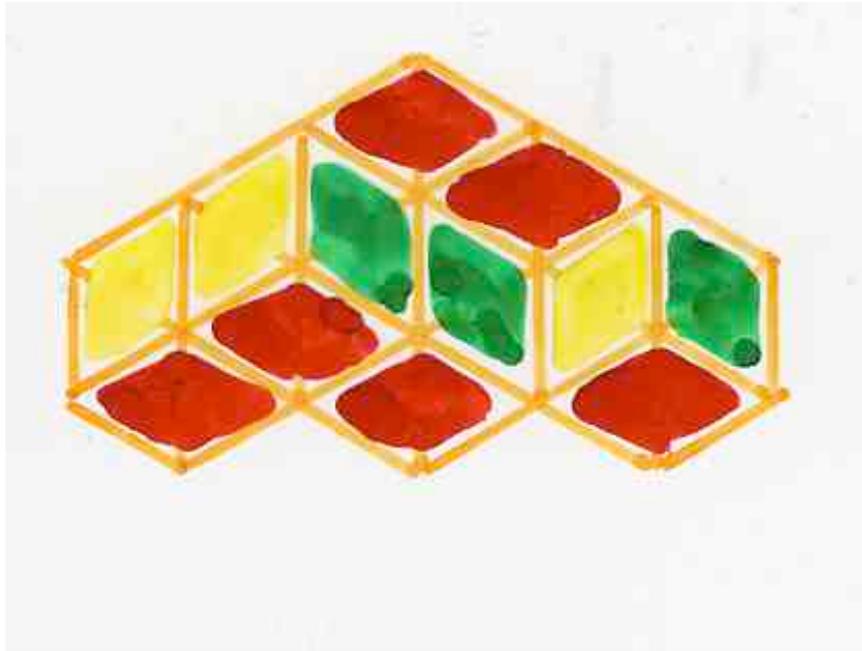
$$C_n = \frac{1}{n+1} \binom{2n}{n}$$
$$= \frac{(2n)!}{(n+1)! n!}$$

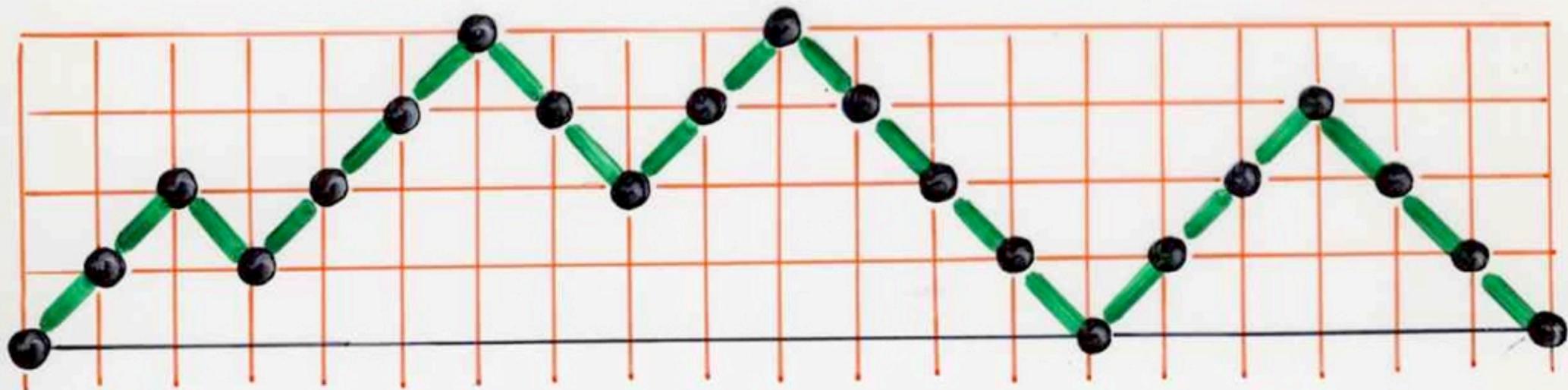
$$n! = 1 \times 2 \times \dots \times n$$

$$C_4 = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}{1 \times 2 \times 3 \times 4 \times 1 \times 2 \times 3 \times 4 \times 5}$$

$$= 14$$







chemin de Dyck

arbres binaires

Des arbres naturels ...

aux arbres mathématiques

1 1 2 5 14 42

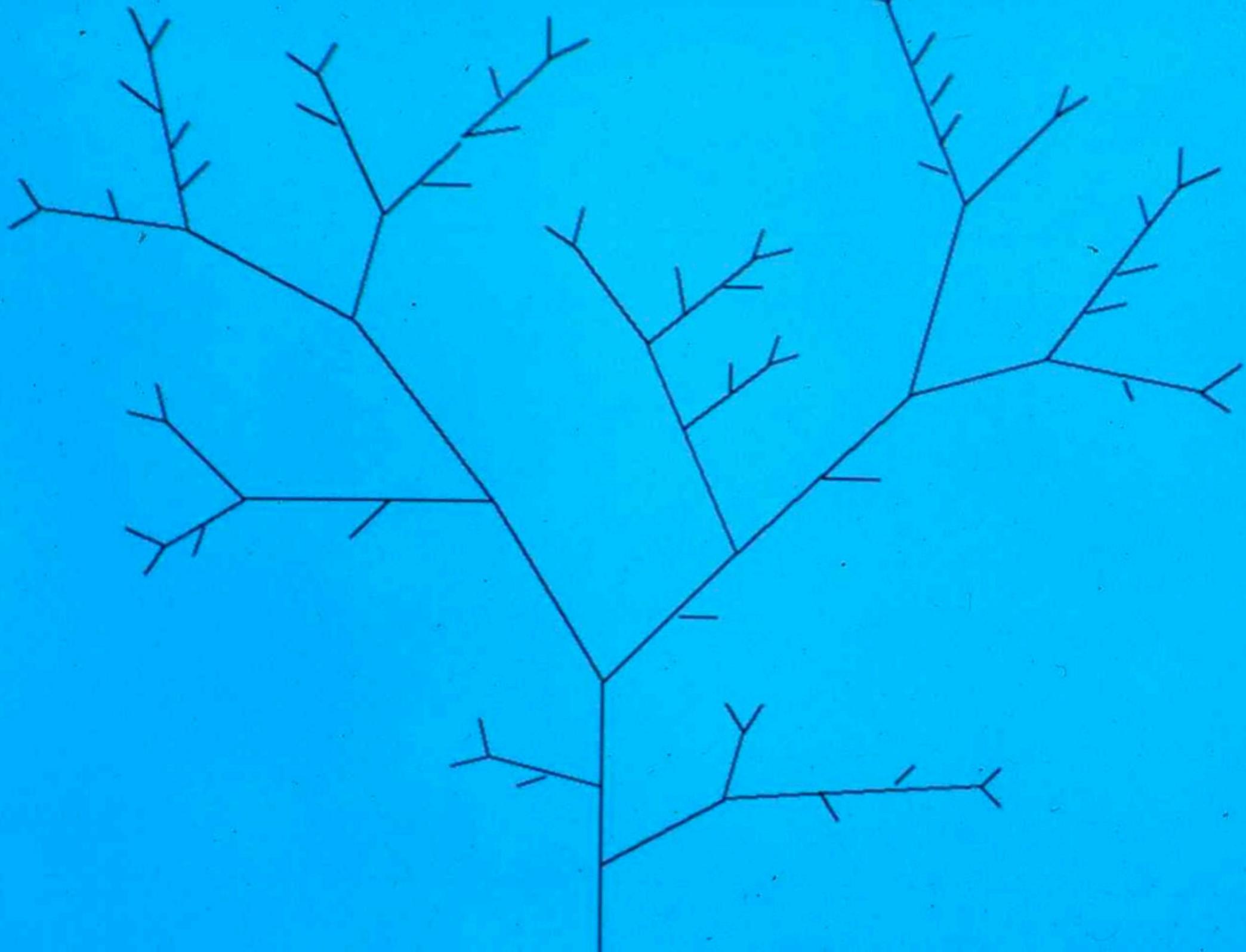
nombres de Catalan

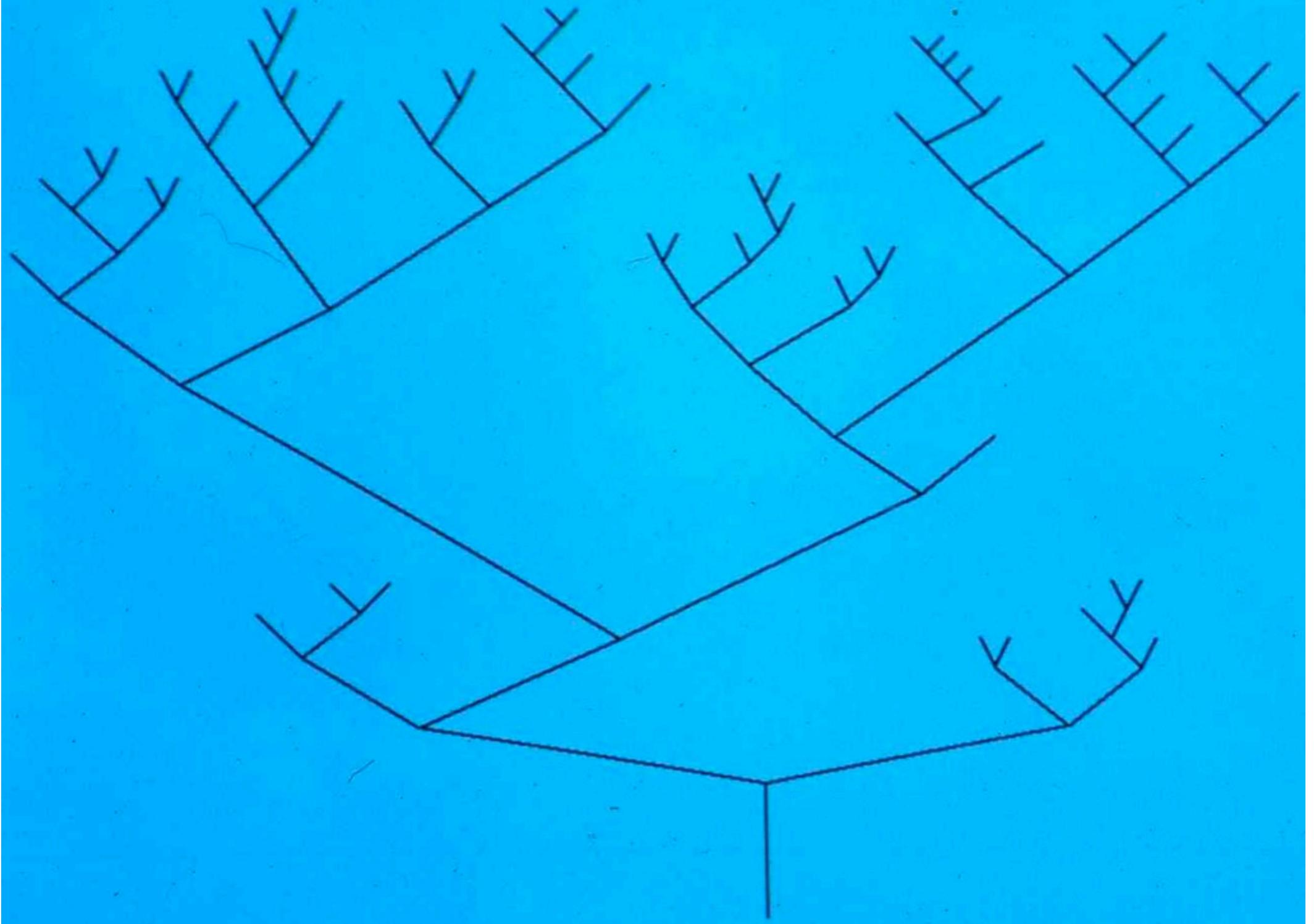


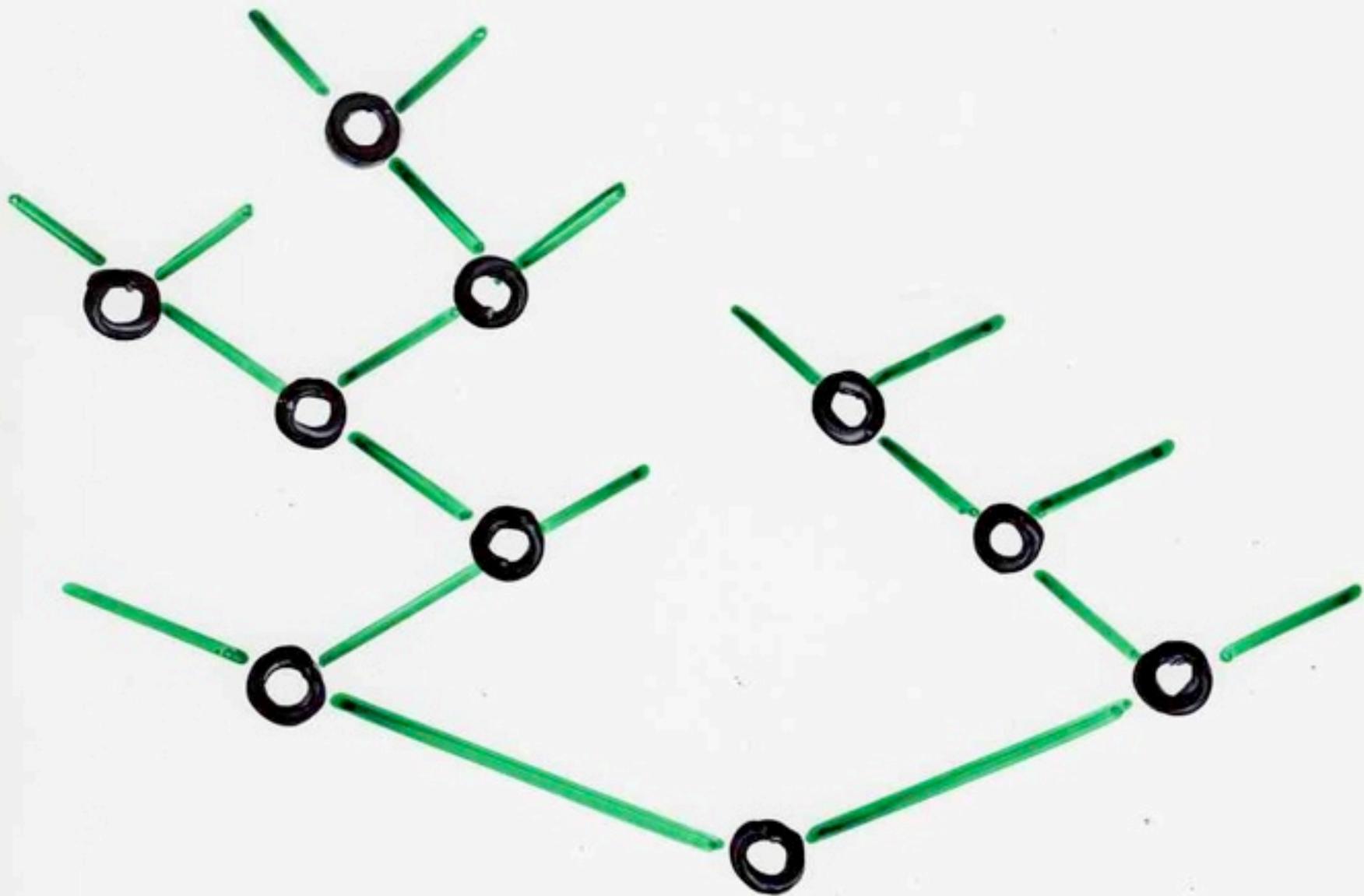


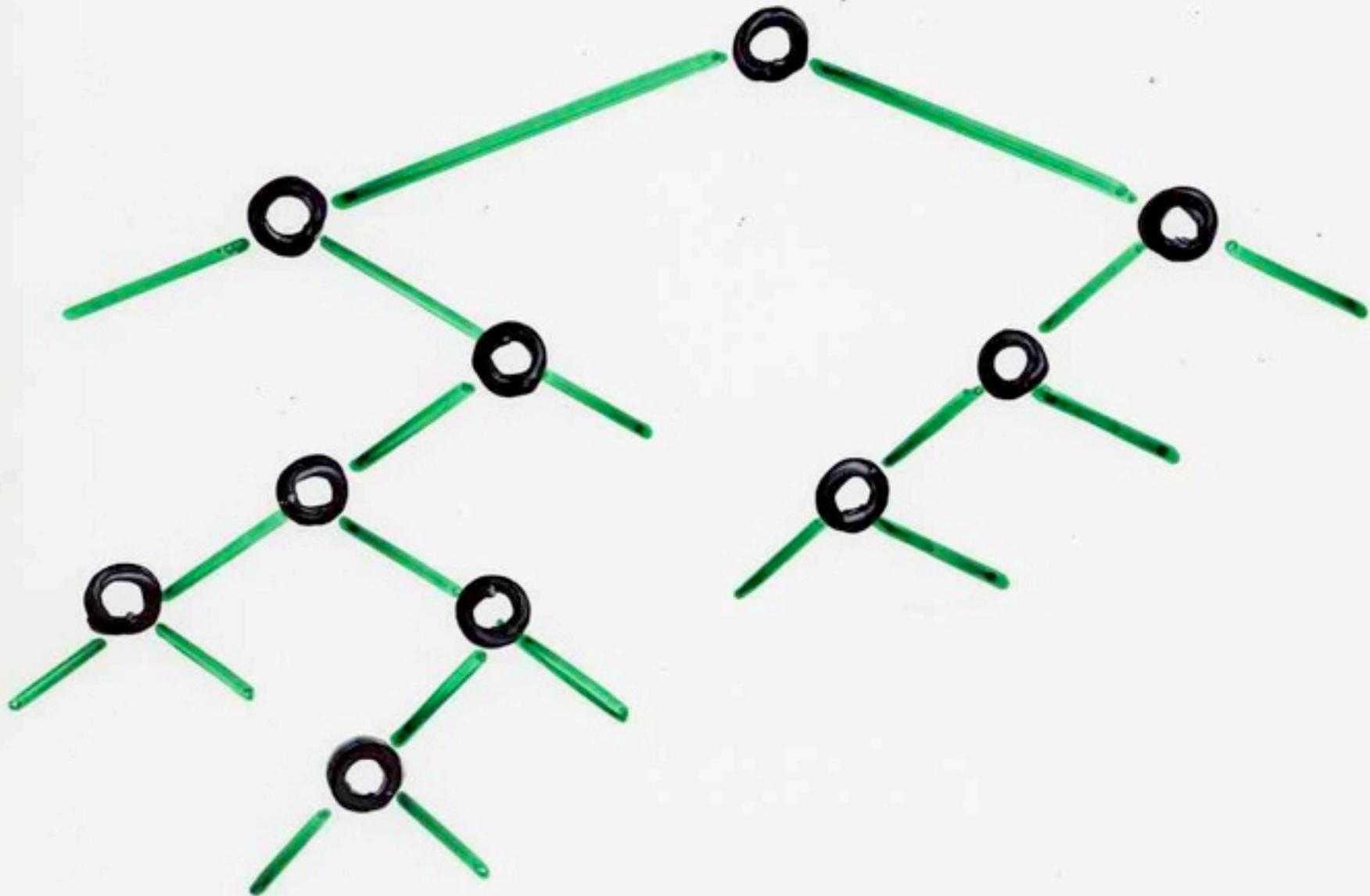






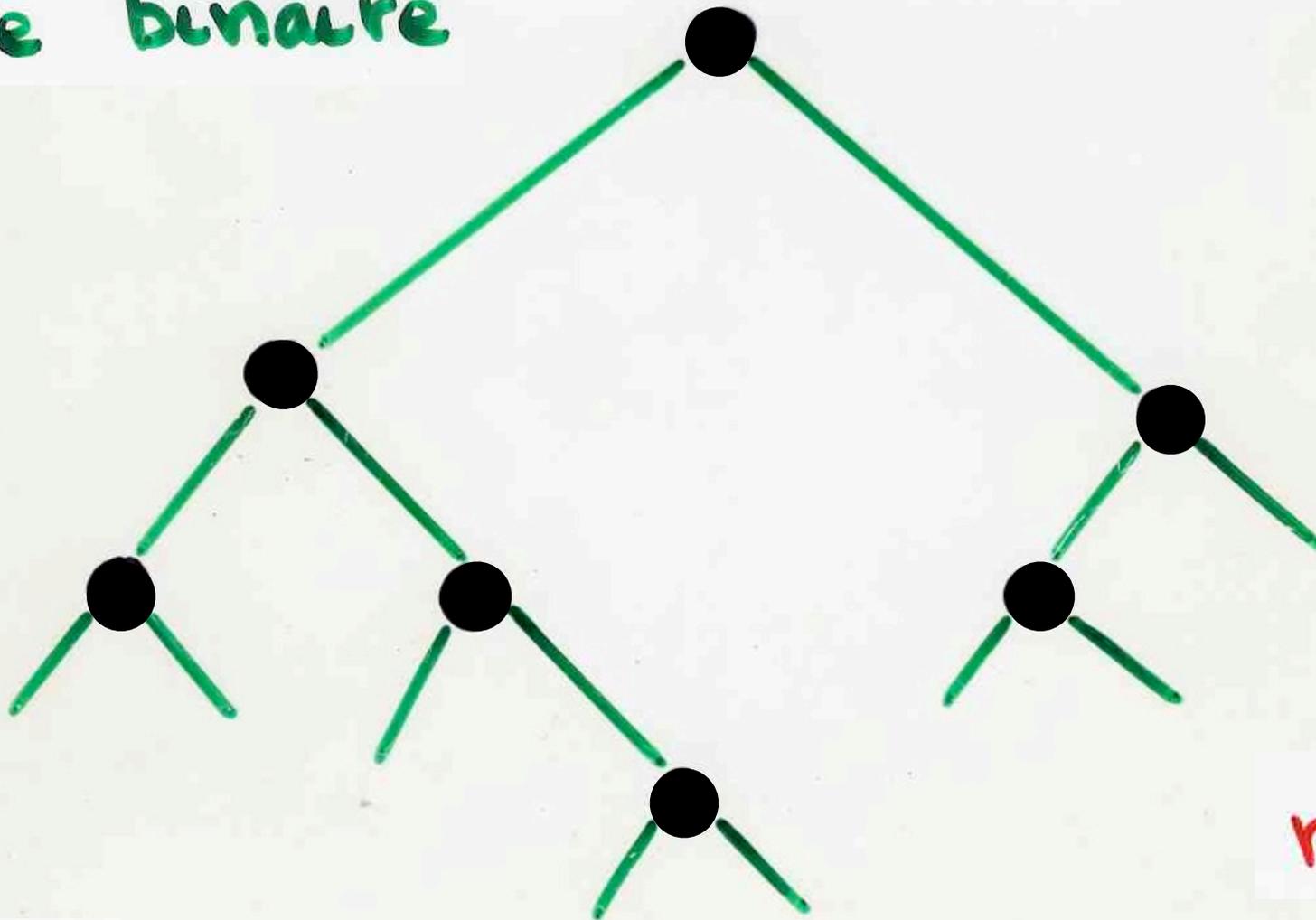






Compter les arbres ...

# arbre binaire



$$n = 7$$

$C_n =$

nombre d'arbres binaires  
ayant  $n$  sommets (internes)

[ou  $n+1$  sommets externes]  
feuilles

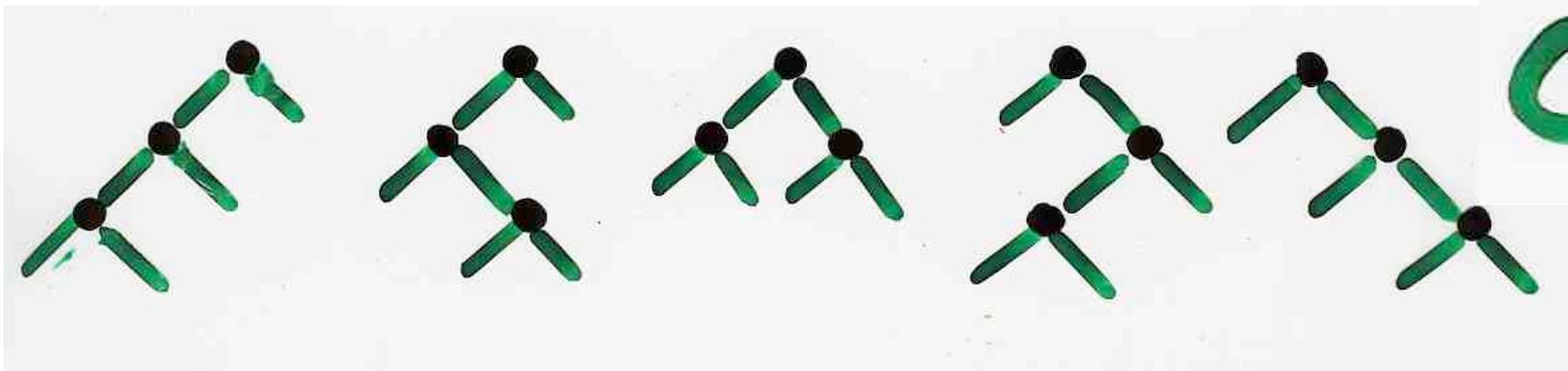




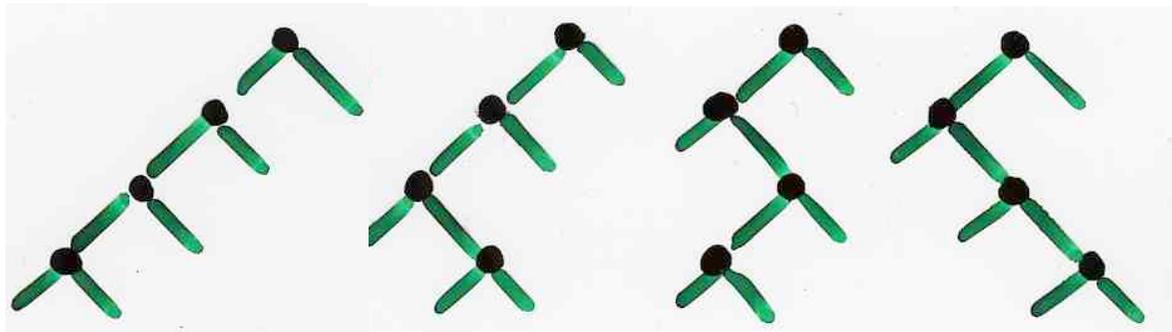
$$C_1 = 1$$



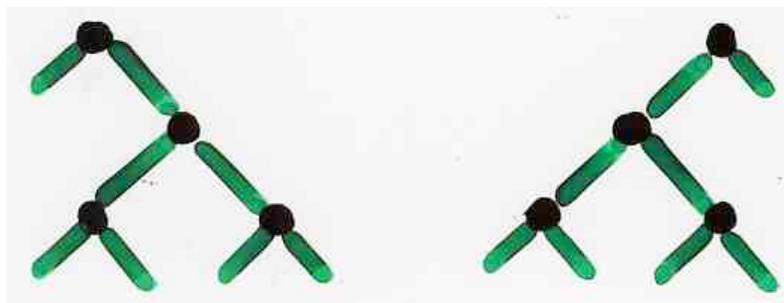
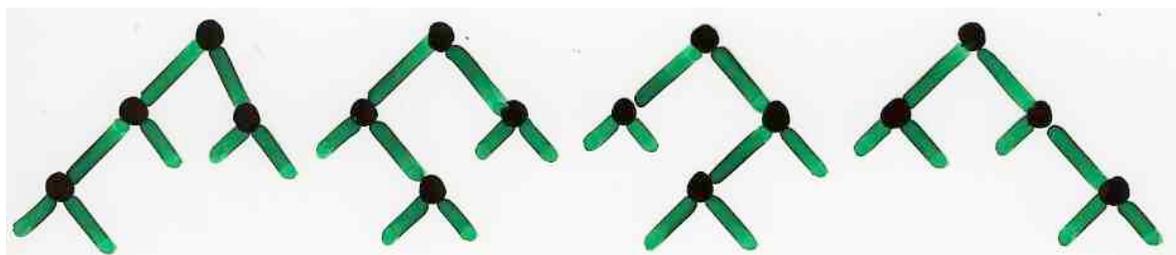
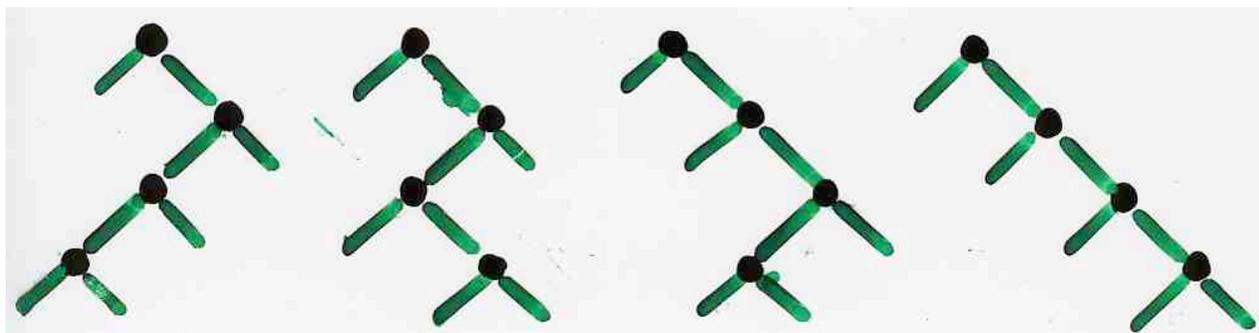
$$C_2 = 2$$

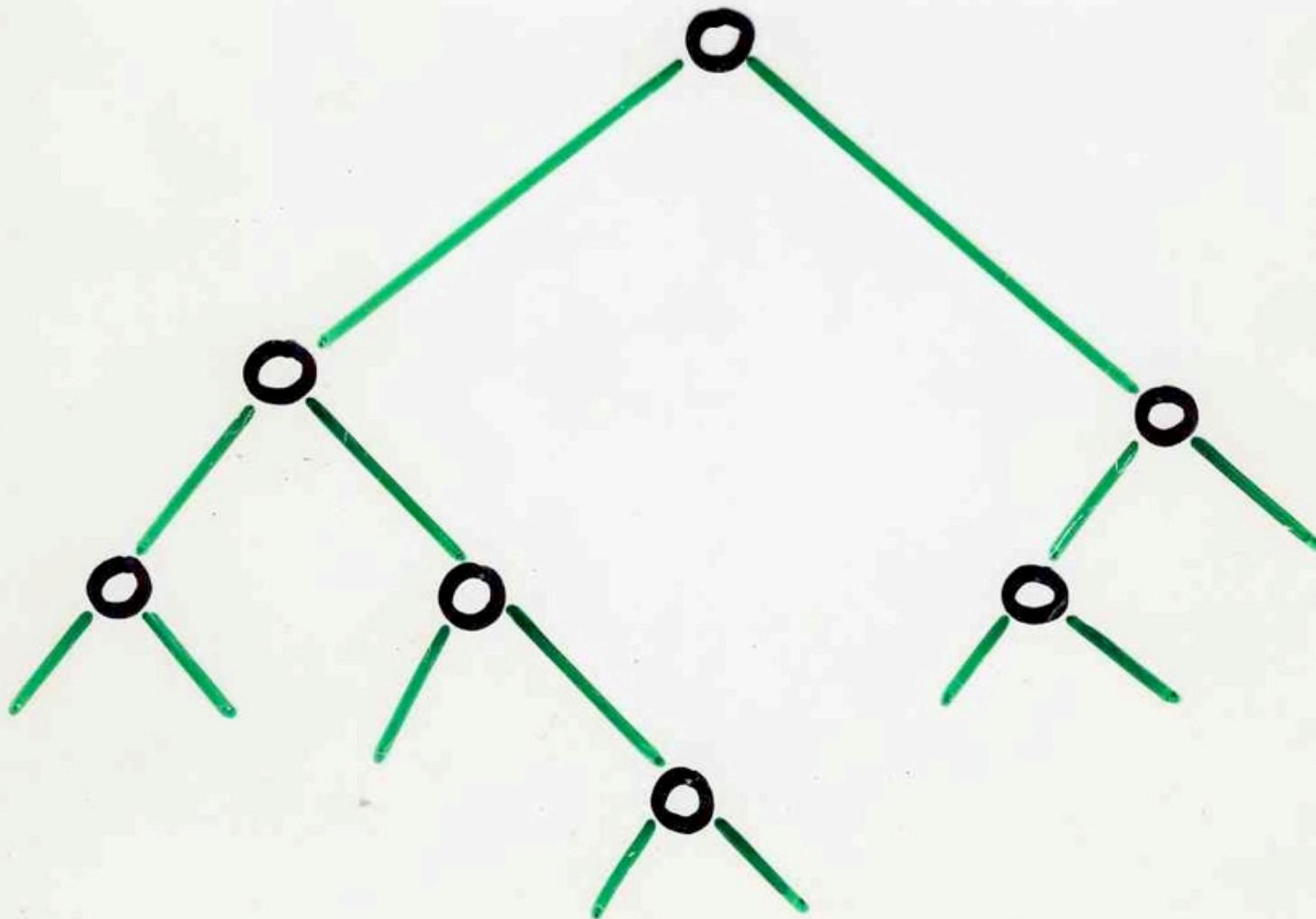


$$C_3 = 5$$



$$C_4 = 14$$





$C_0$   $C_1$   $C_2$   $C_3$   $C_4$   $C_5$   
1, 1, 2, 5, 14, 42, ...

$$C_6 = C_0 C_5 + C_1 C_4 + C_2 C_3 + C_3 C_2 + C_4 C_1 + C_5 C_0$$

132       $1 \times 42 + 1 \times 14 + 2 \times 5 + 5 \times 2 + 14 \times 1 + 42 \times 1$

recurrence

$$C_{n+1} = \sum_{i+j=n} C_i C_j$$

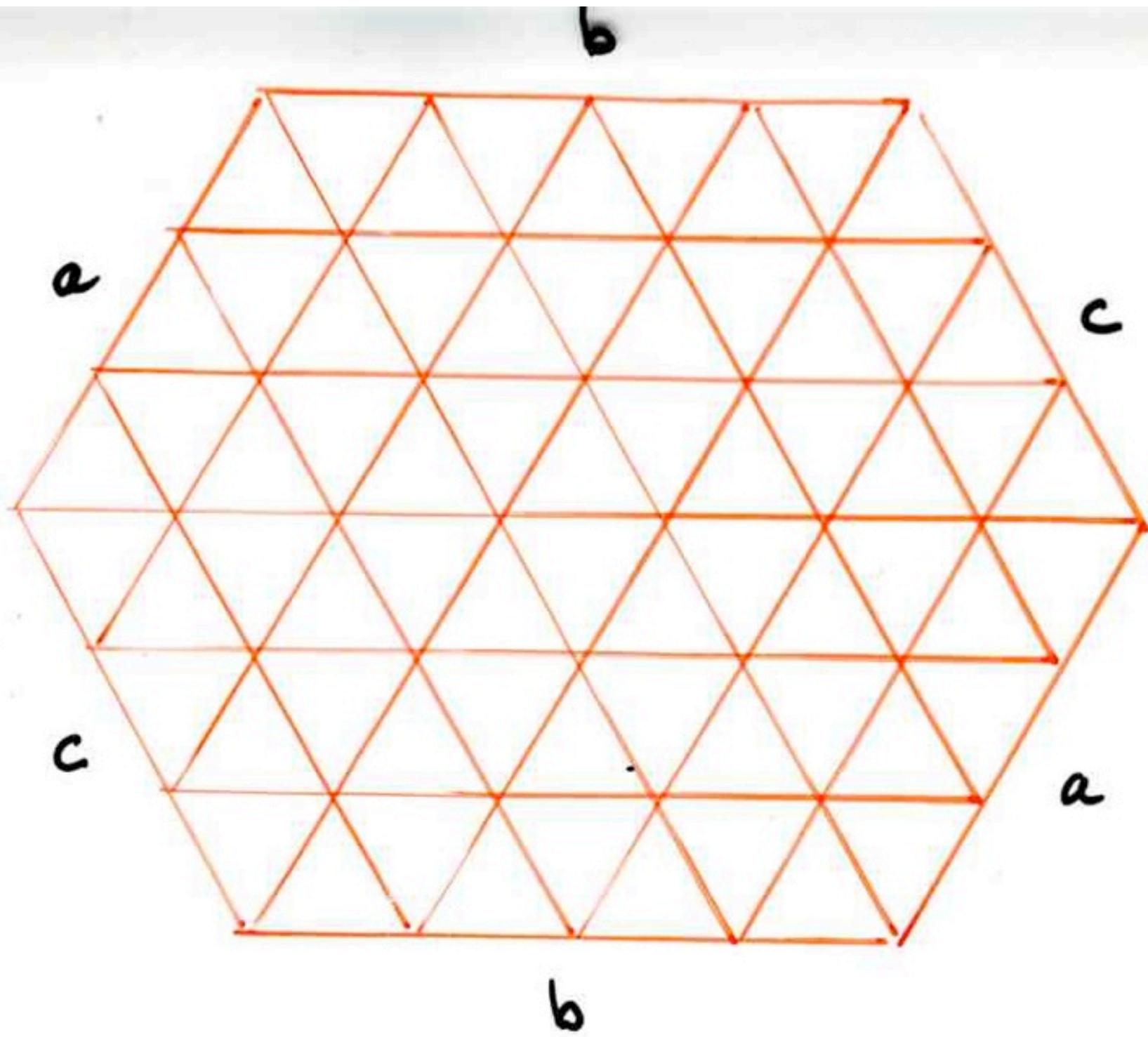
$$C_0 = 1$$

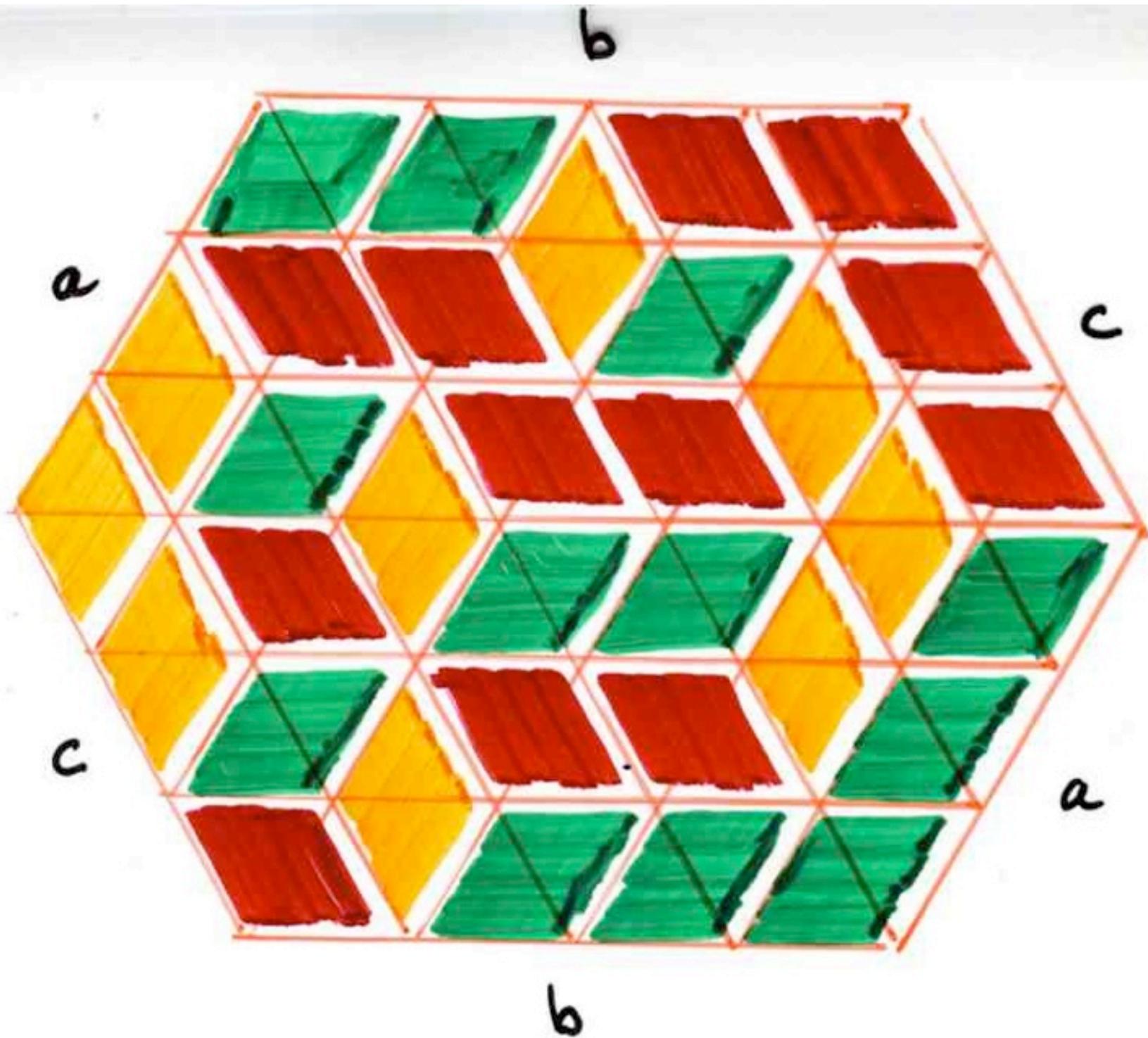
nombres  
de  
Catalan

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$
$$= \frac{(2n)!}{(n+1)! n!}$$

$$n! = 1 \times 2 \times \dots \times n$$

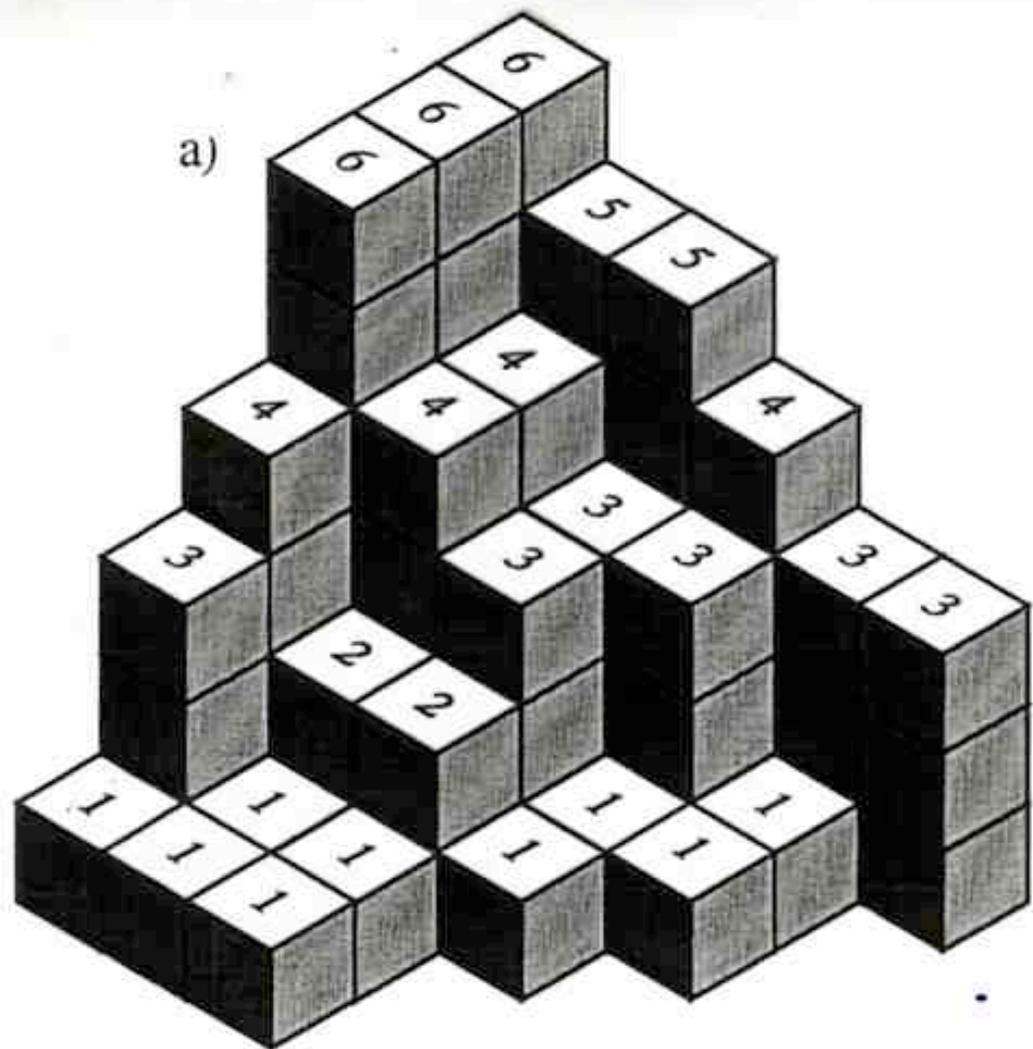
Une belle formule ....





empilements de cubes

(ou partitions planes)



b)

6 5 5 4 3 3

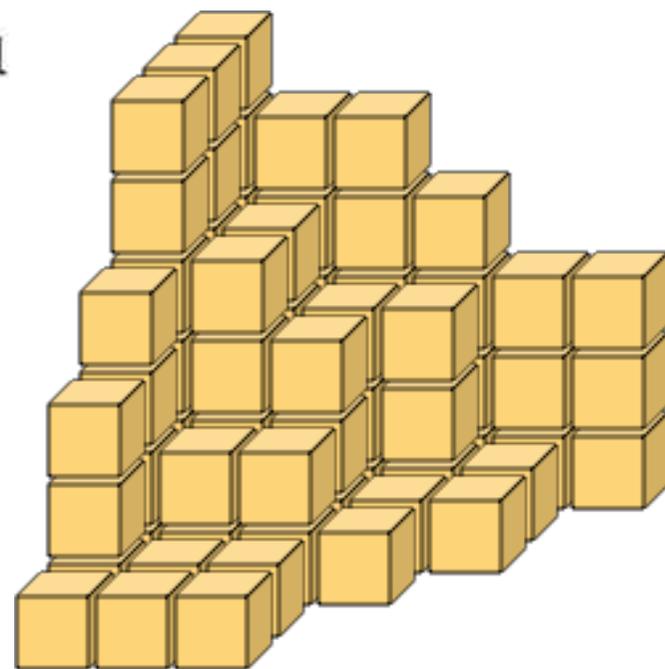
6 4 3 3 1

6 4 3 1 1

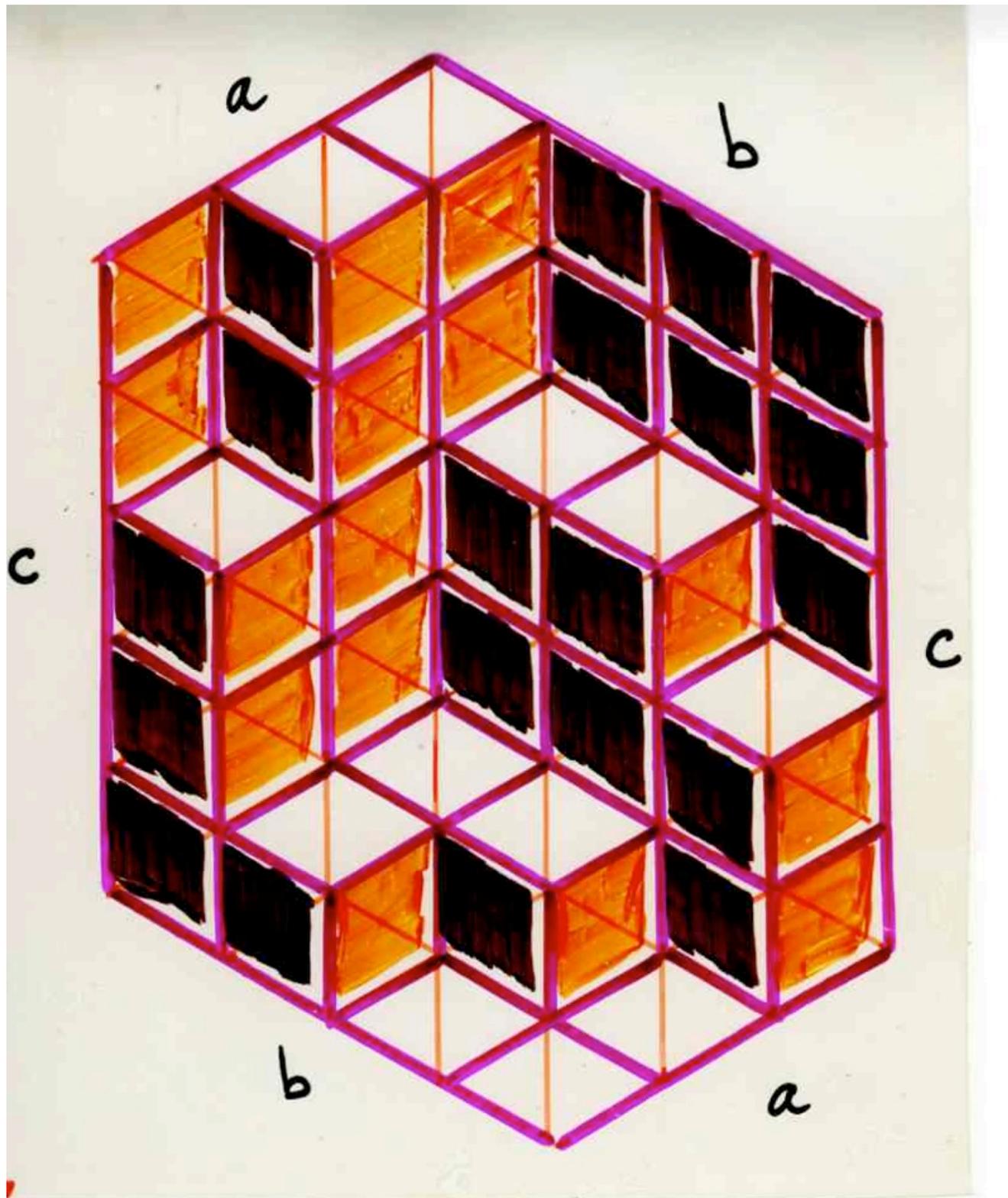
4 2 2 1

3 1 1

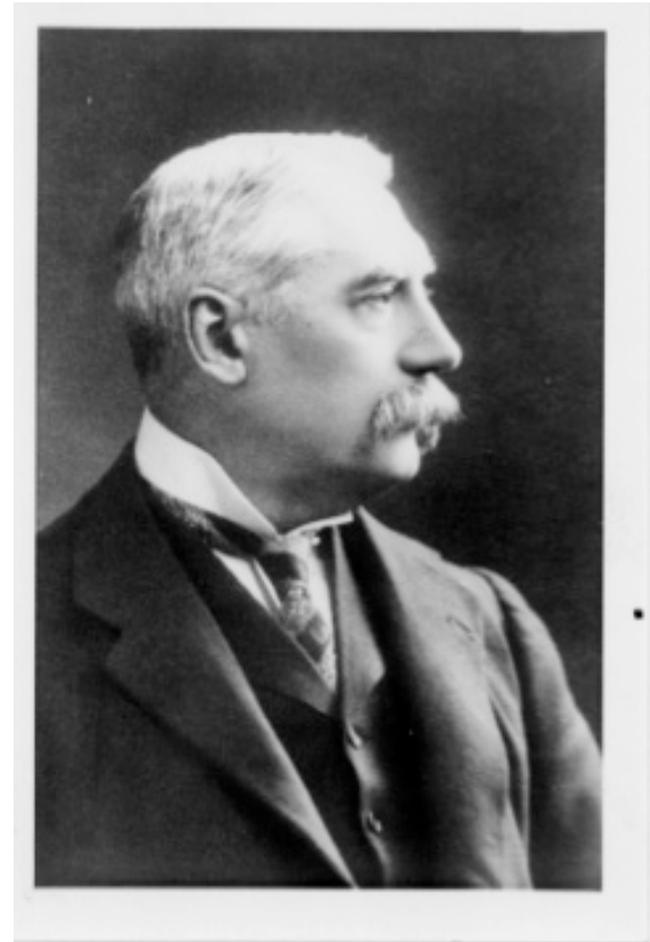
1 1 1

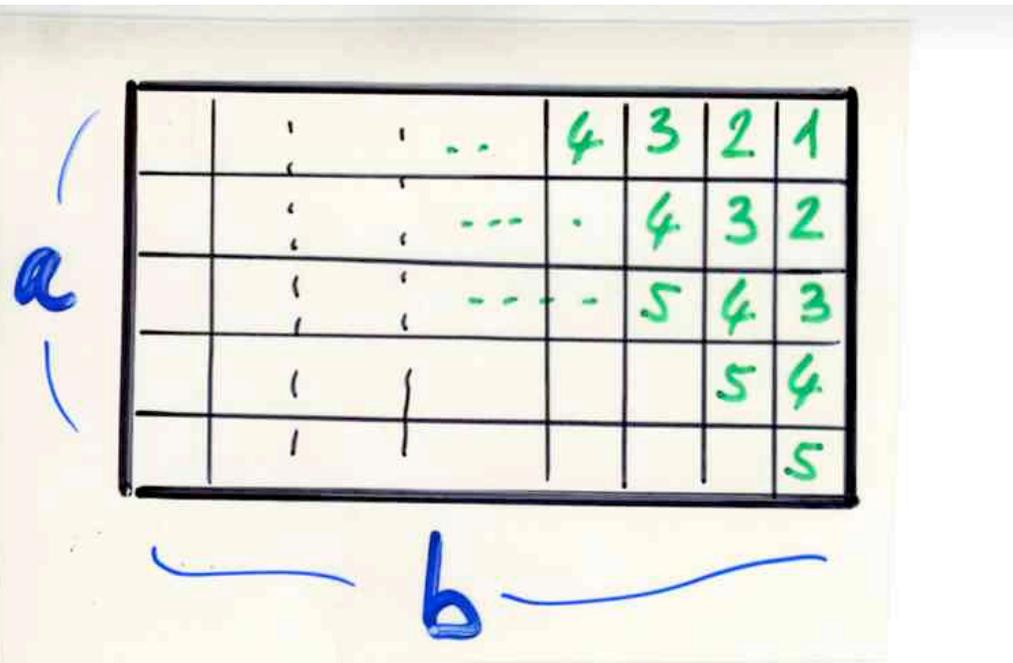
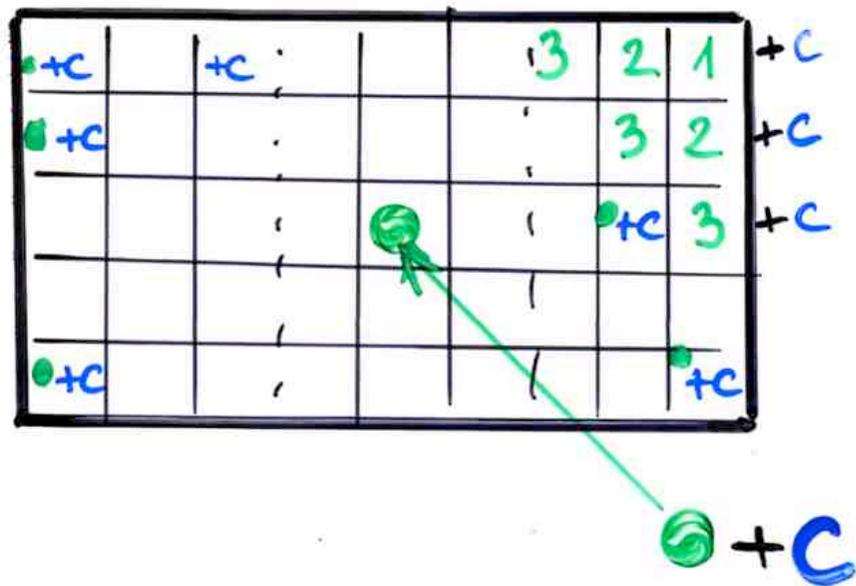


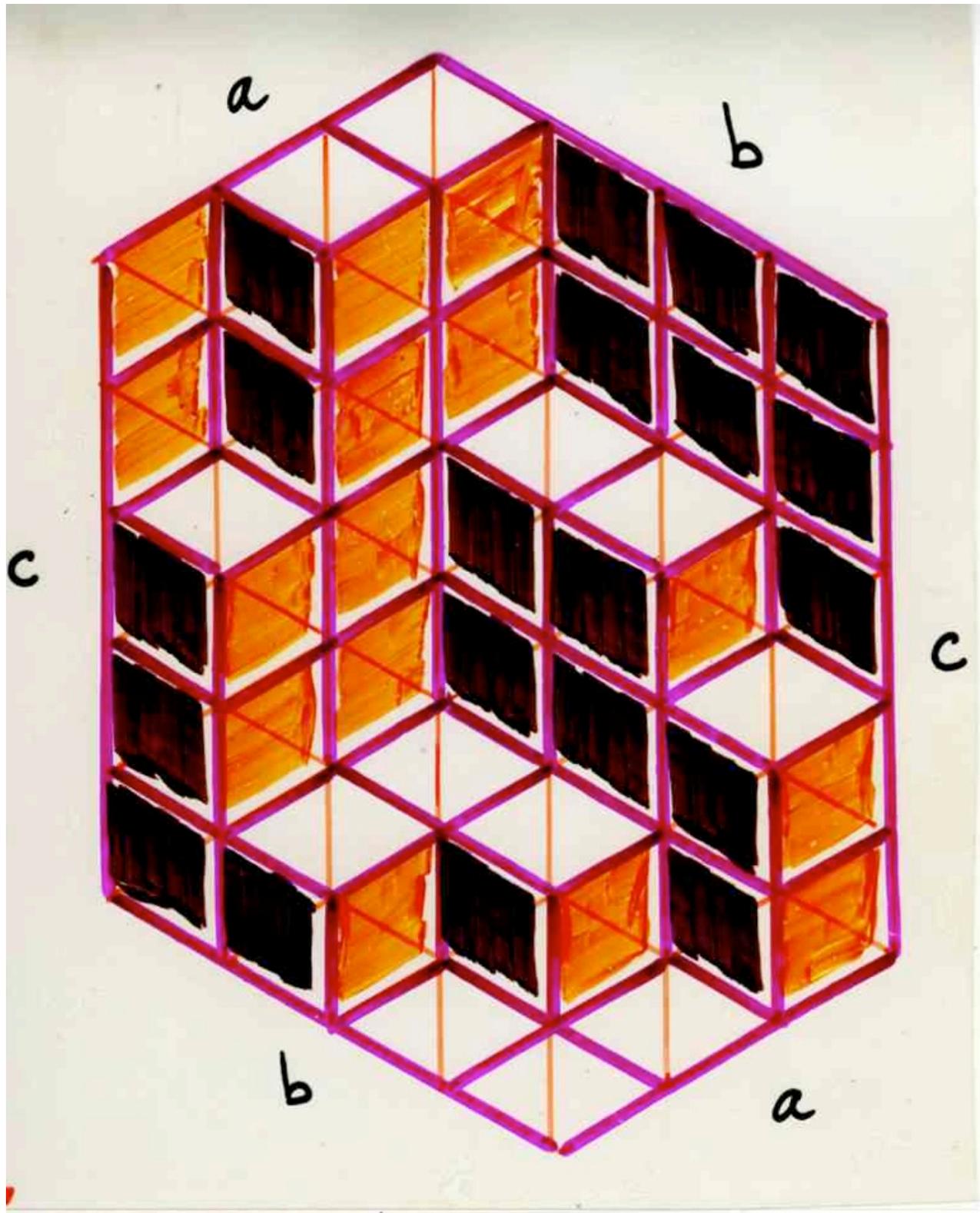
empilements  
de cubes  
dans une  
boîte  
 $a \times b \times c$

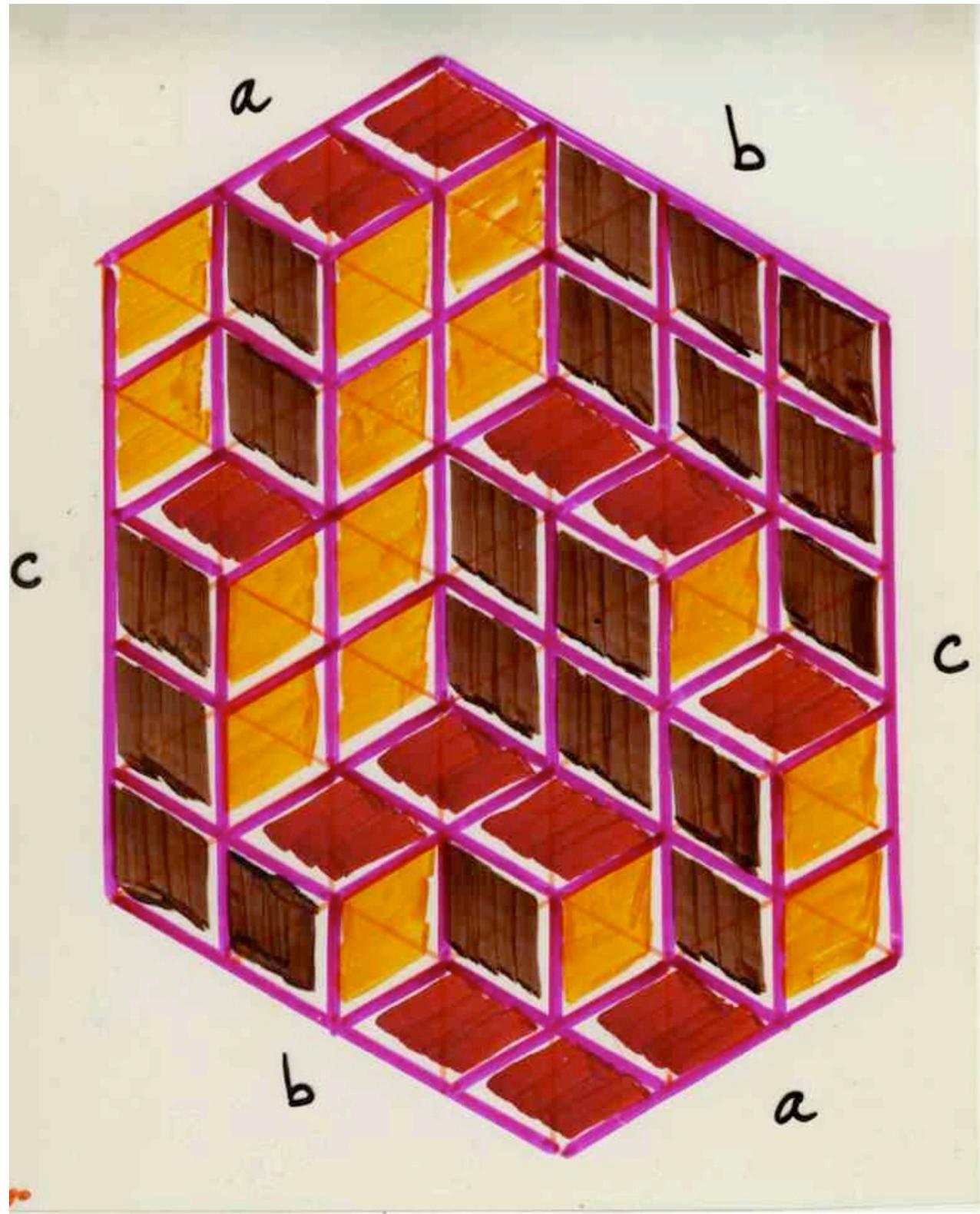


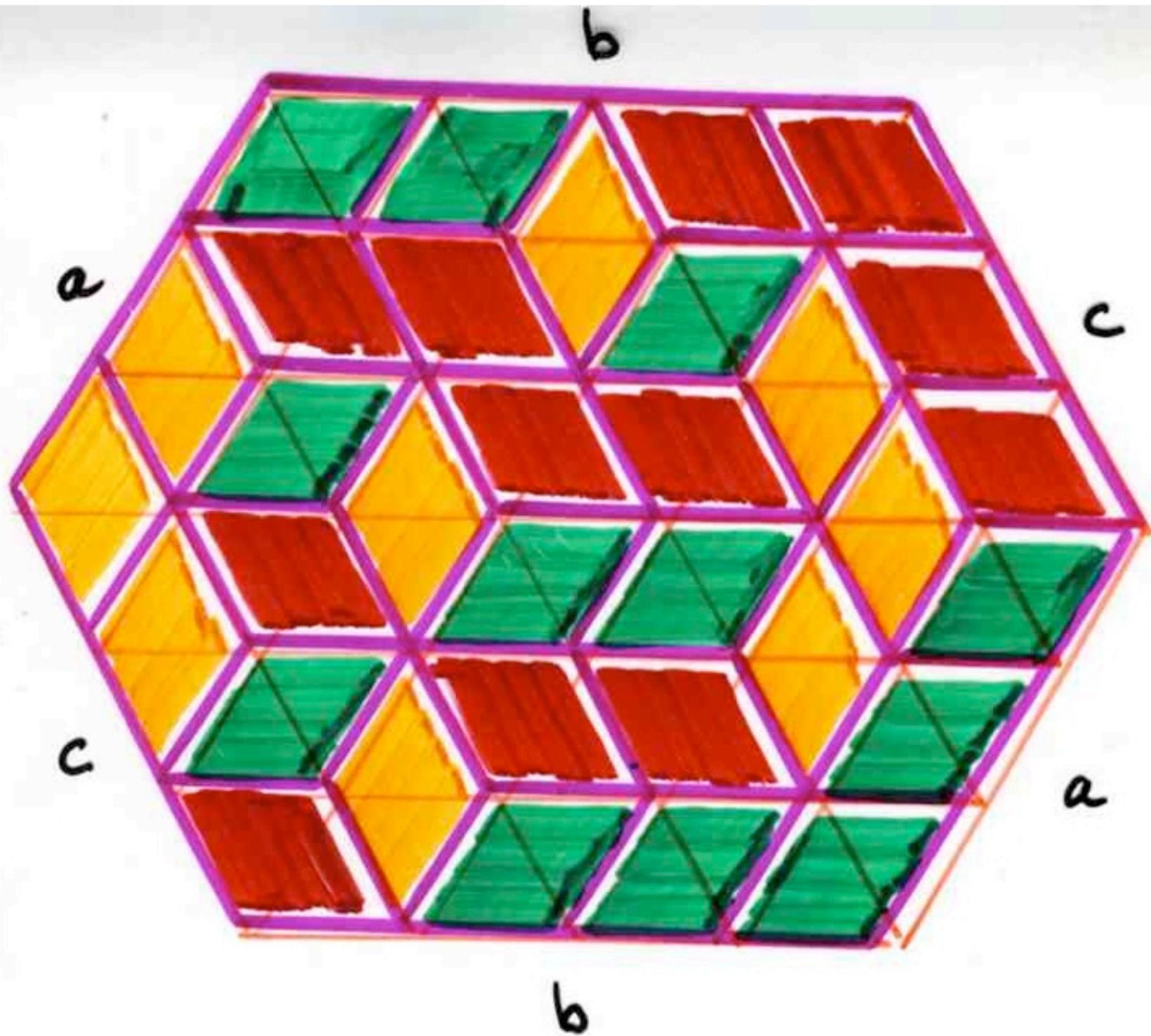
formule de MacMahon

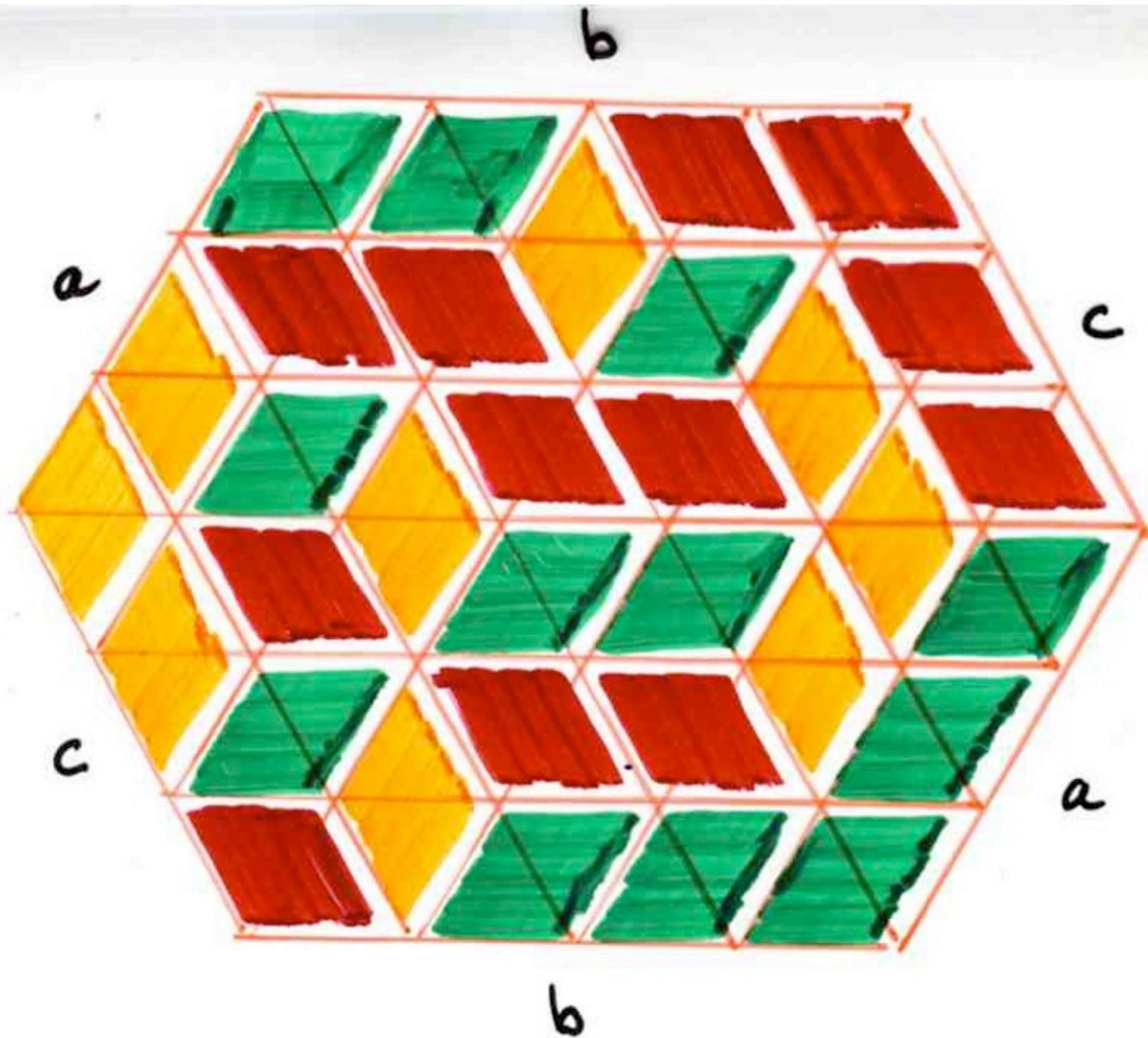


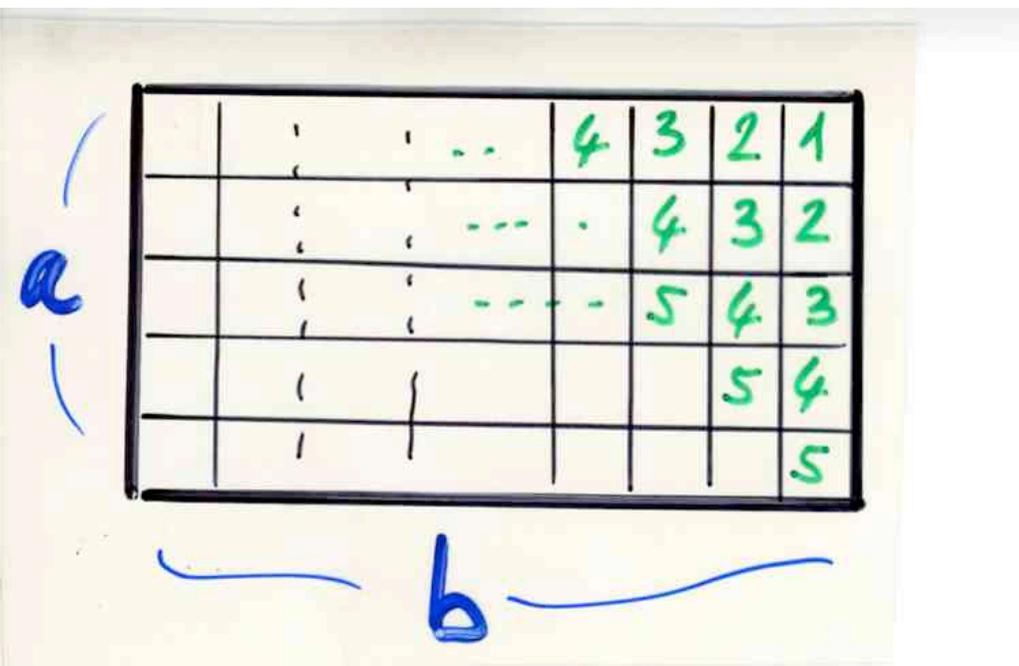
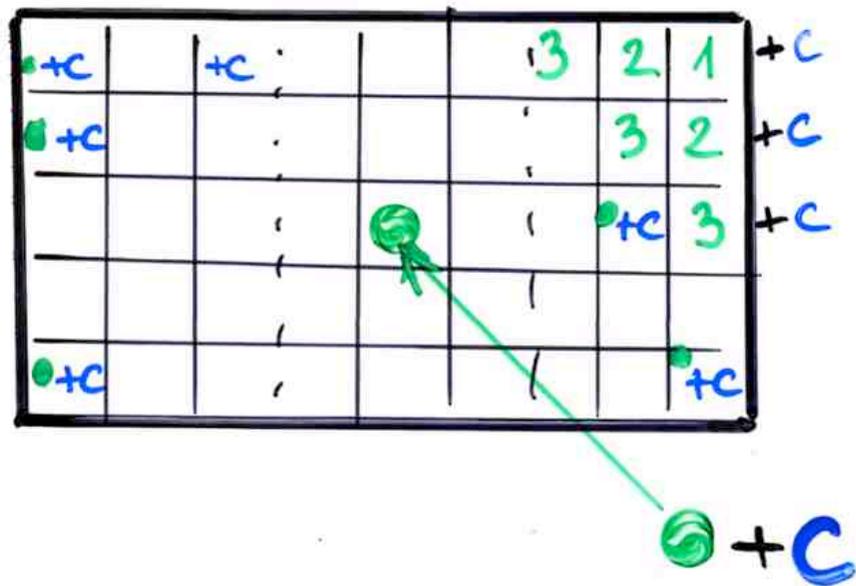












$\prod$ 

$$1 \leq i \leq a$$

$$1 \leq j \leq b$$

$$1 \leq k \leq c$$

$$\frac{i+j+k-1}{i+j+k-2}$$



formule de MacMahon

nombres  
de  
Catalan

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$
$$= \frac{(2n)!}{(n+1)! n!}$$

$$n! = 1 \times 2 \times \dots \times n$$

Une formule magique ....

avec les pavages Aztèques

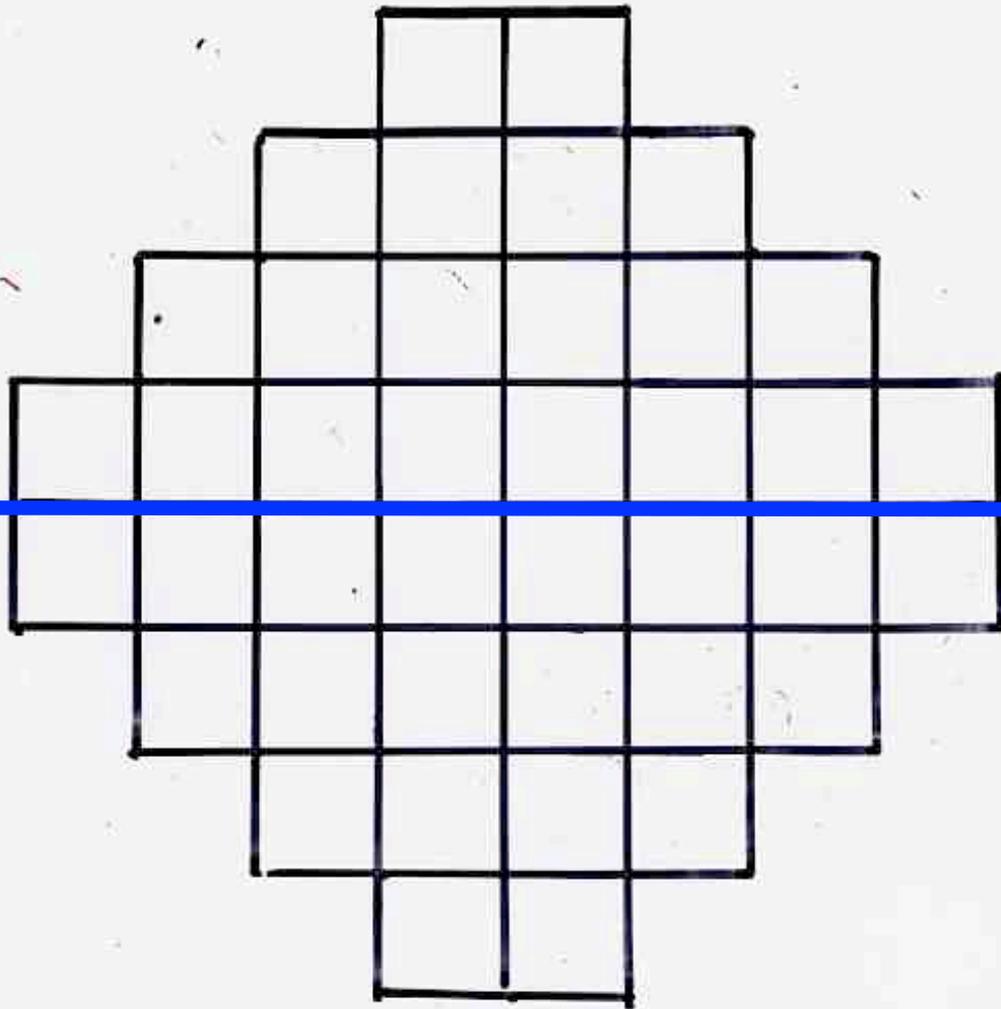
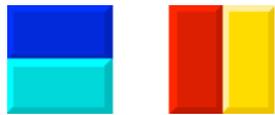
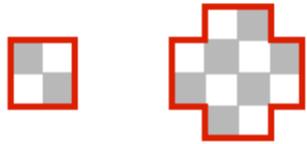


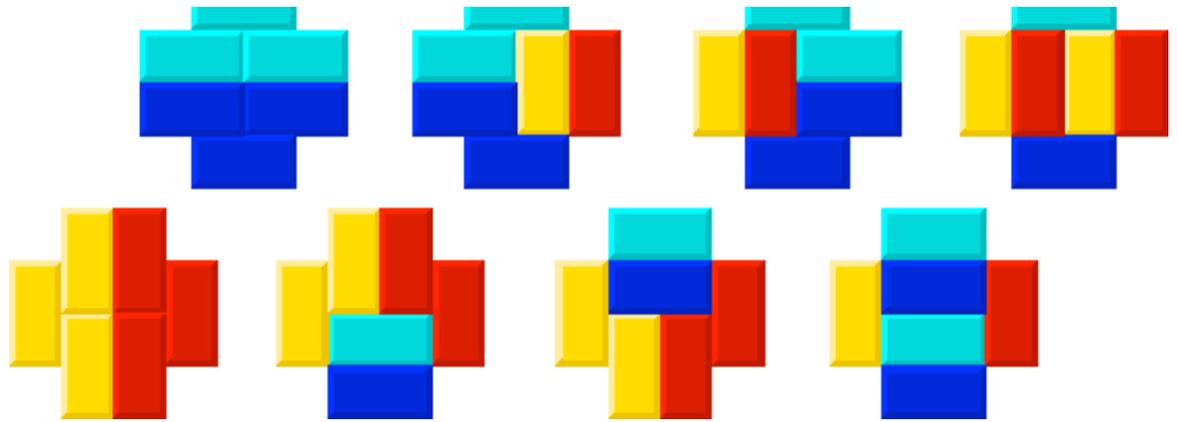
diagramme  
Aztèque



2



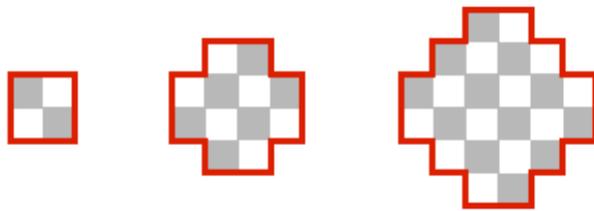
2



8

$$2^m = \underbrace{2 \times 2 \times \dots \times 2}_{m \text{ fois}}$$

nombre  
de  
pavages



2

8

64

1024

$2^1$

$2^3$

$2^6$

$2^{10}$

$2^1$

$2^{(1+2)}$

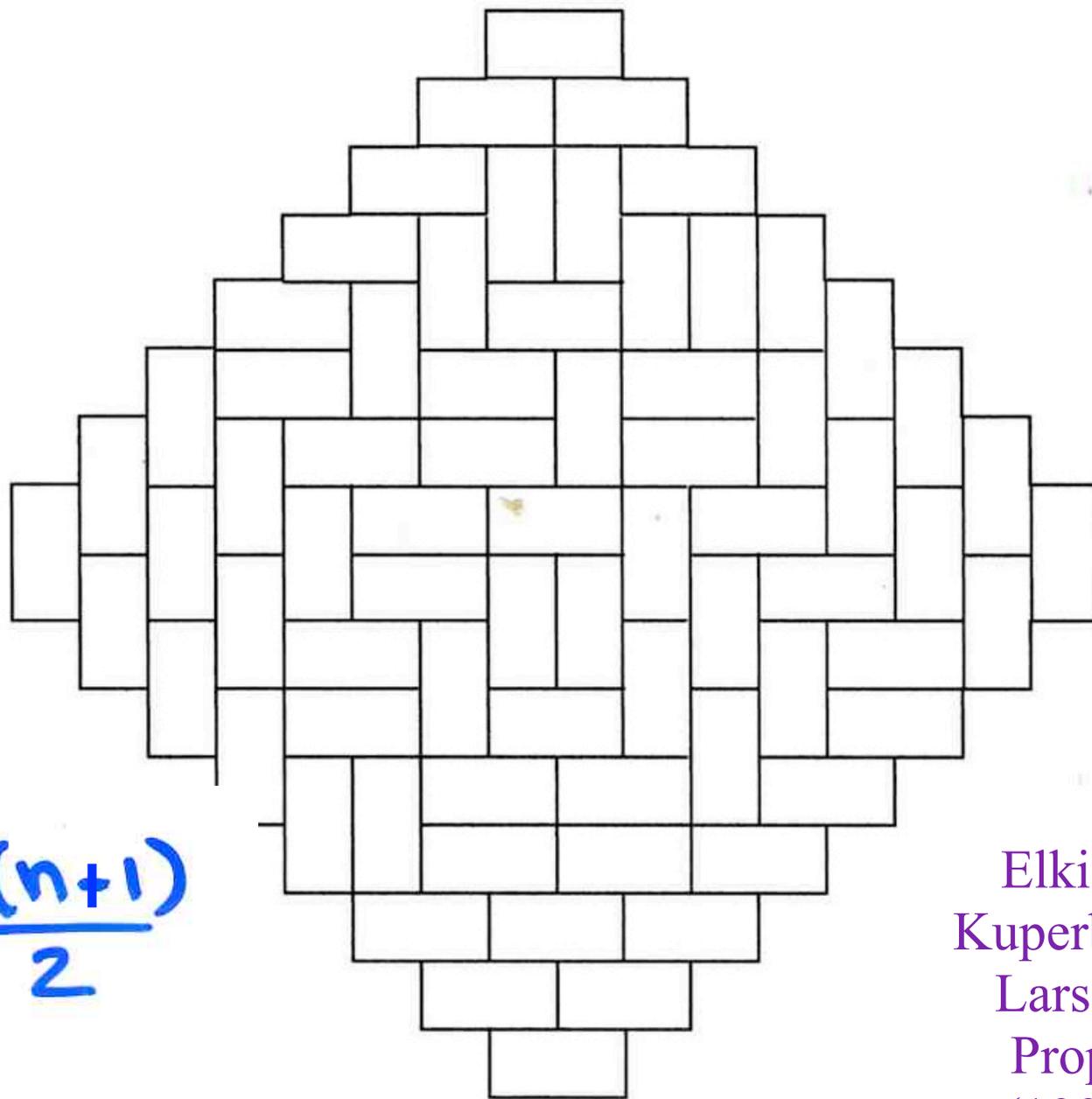
$2^{(1+2+3)}$

$2^{(1+2+3+4)}$

le nombre de  
pavages du  
diagramme  
Azèque  
avec des  
dominos  
est

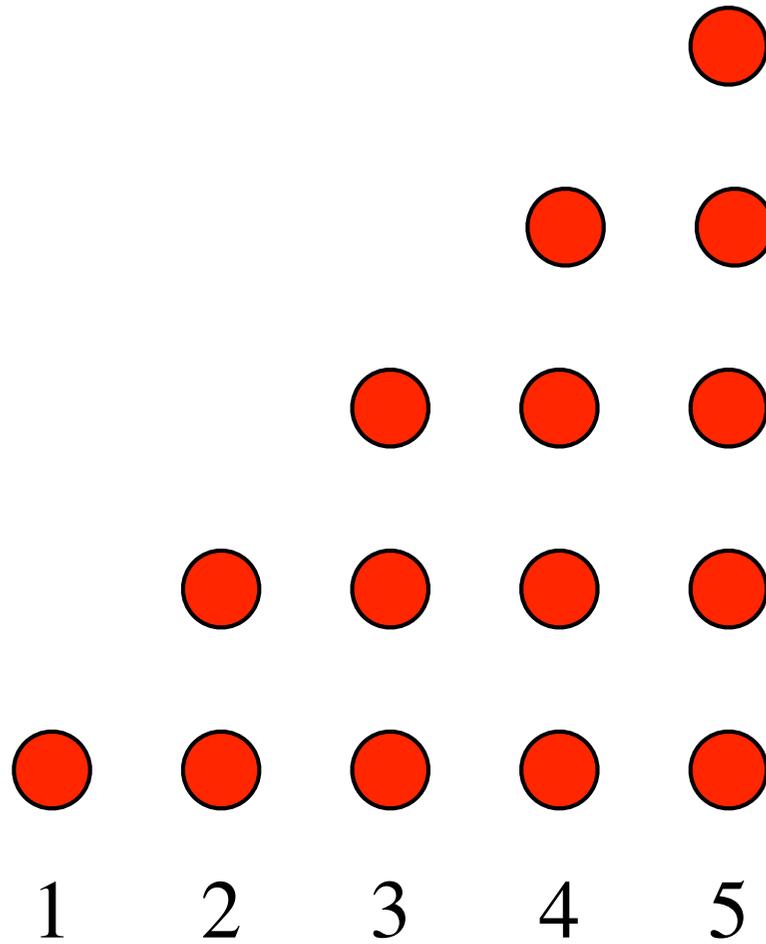
$$2^{(1+2+3+4+\dots+n)}$$

$$2^{\frac{n(n+1)}{2}}$$

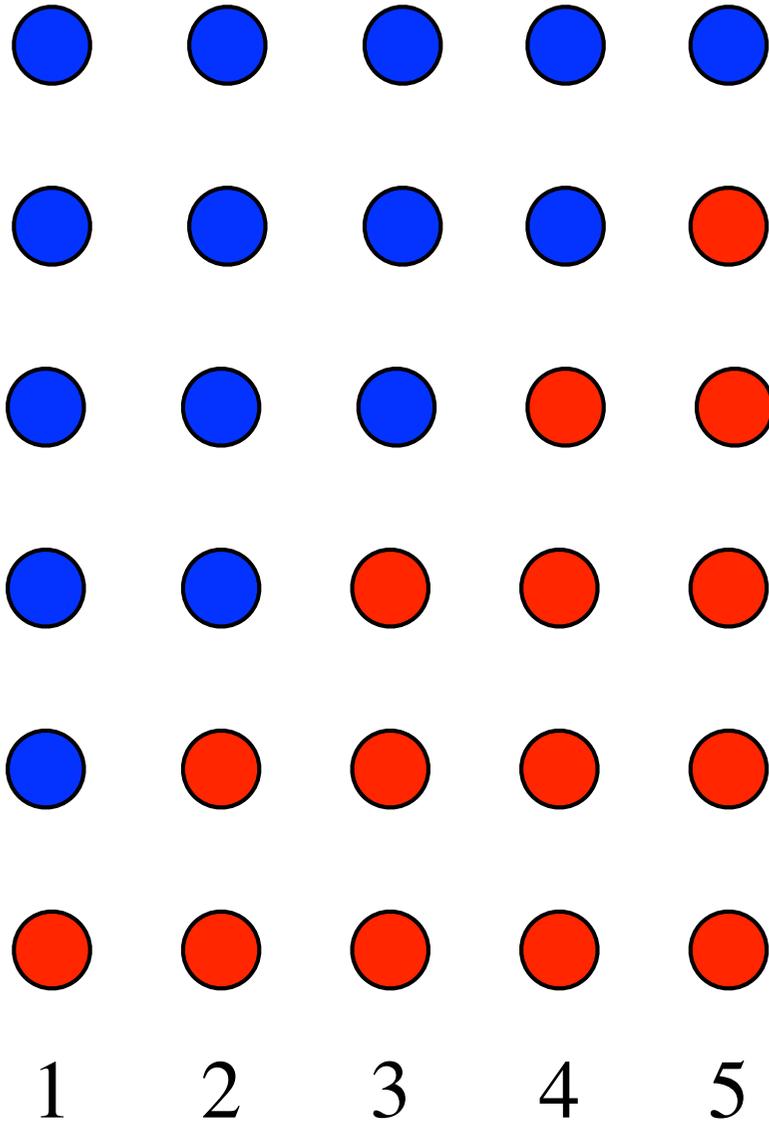


Elkies,  
Kuperberg,  
Larsen,  
Propp  
(1992)

$$1 + 2 + \dots + n = n(n + 1) / 2$$



$$1 + 2 + \dots + n = n(n + 1) / 2$$



«preuve sans mots»

Comment démontrer la miraculeuse

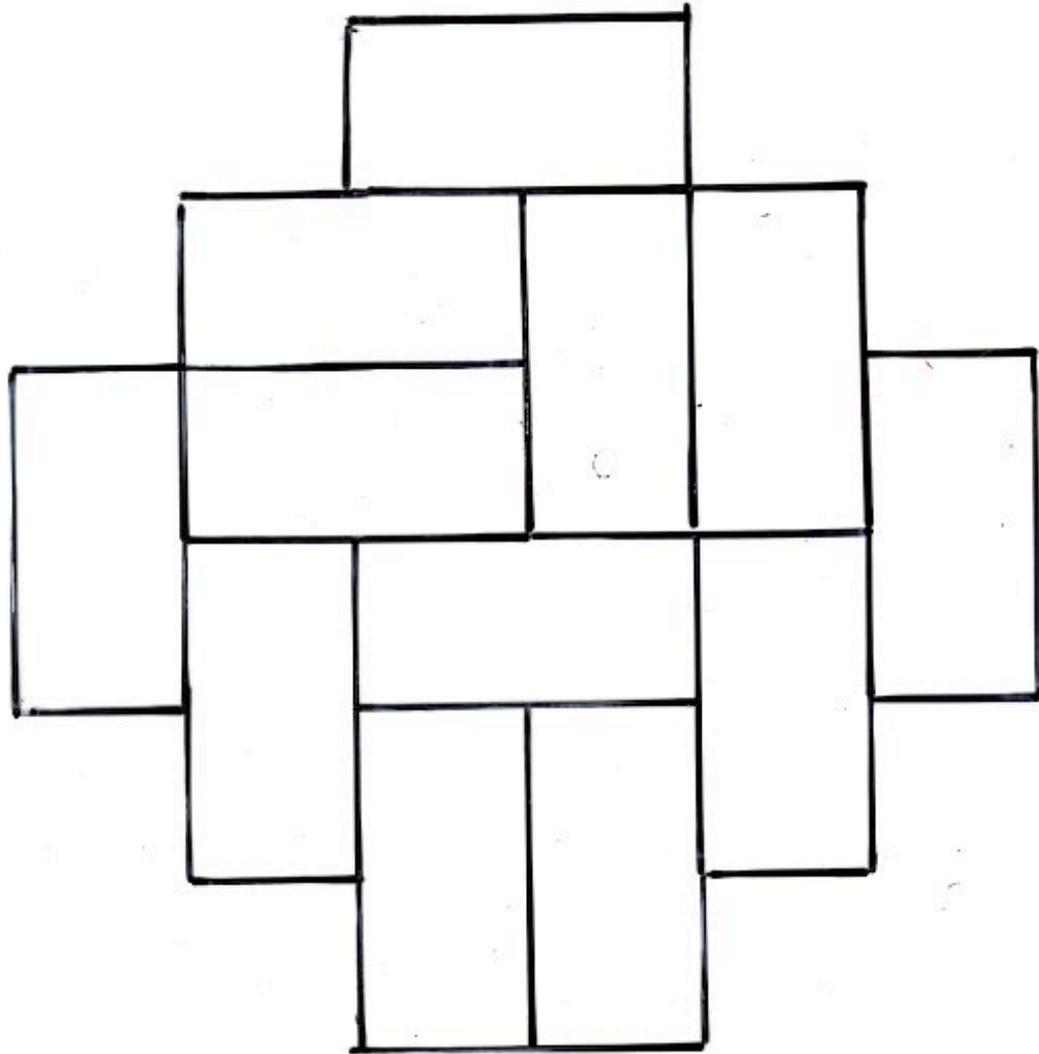
formule du diagramme Astèque ?

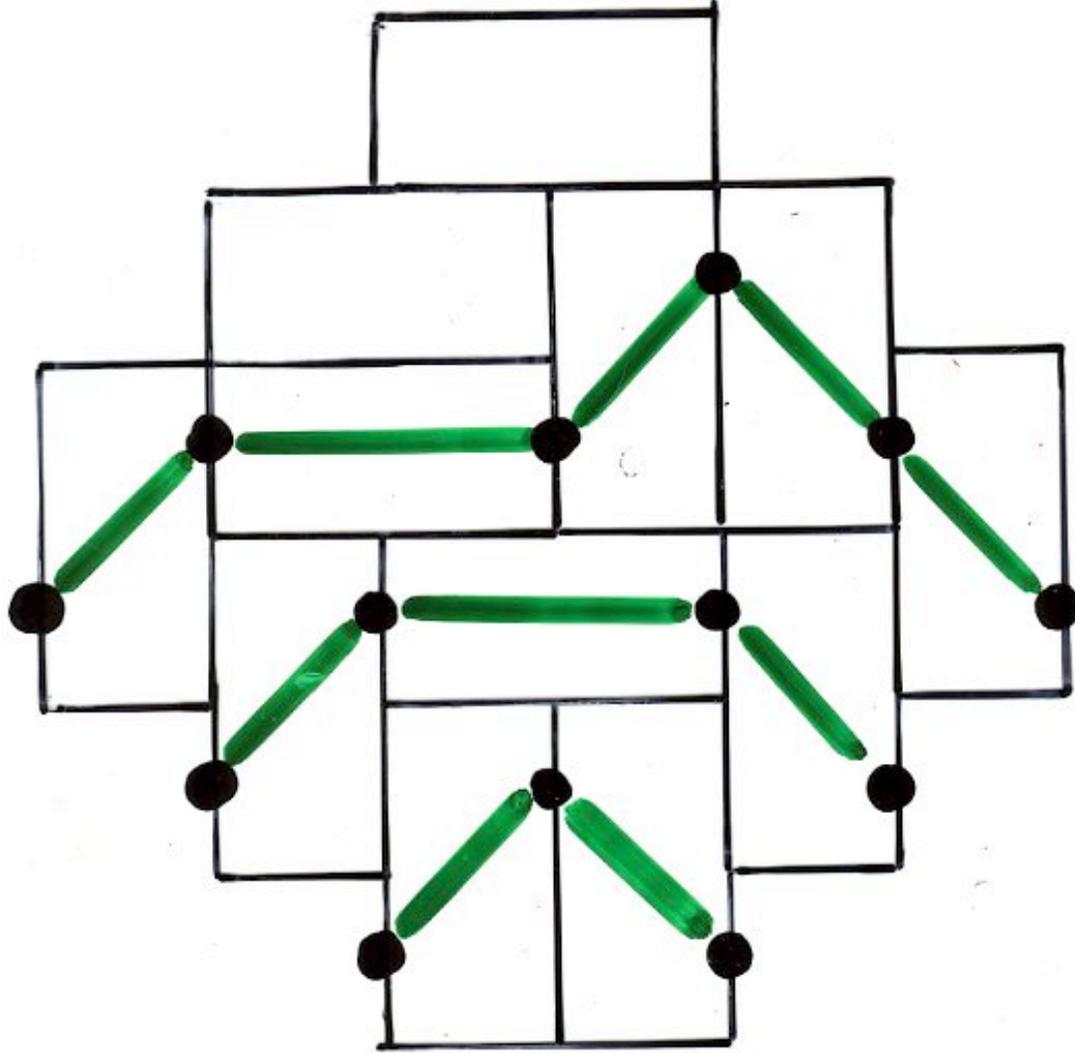
$$2 \frac{n(n+1)}{2}$$

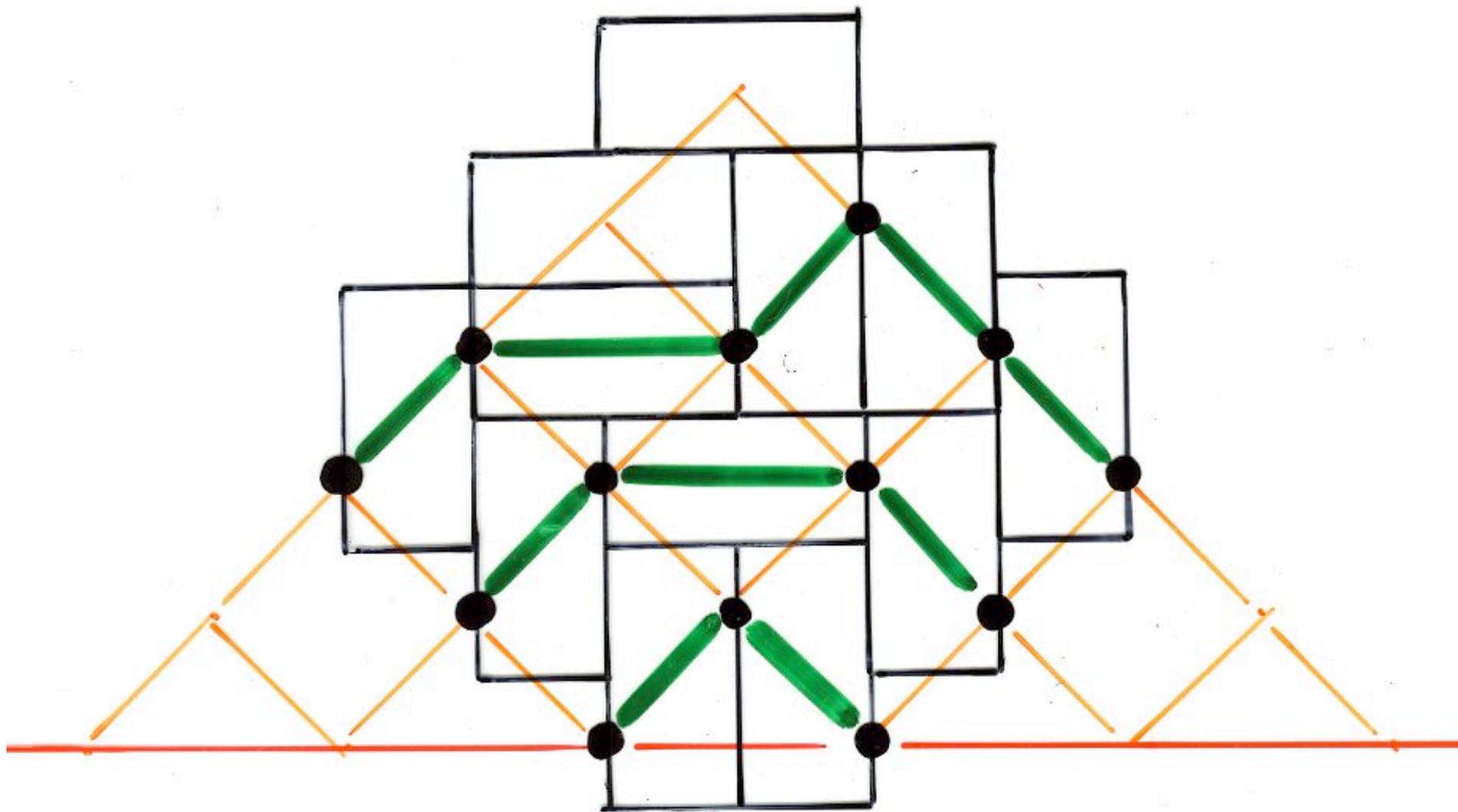
bijection

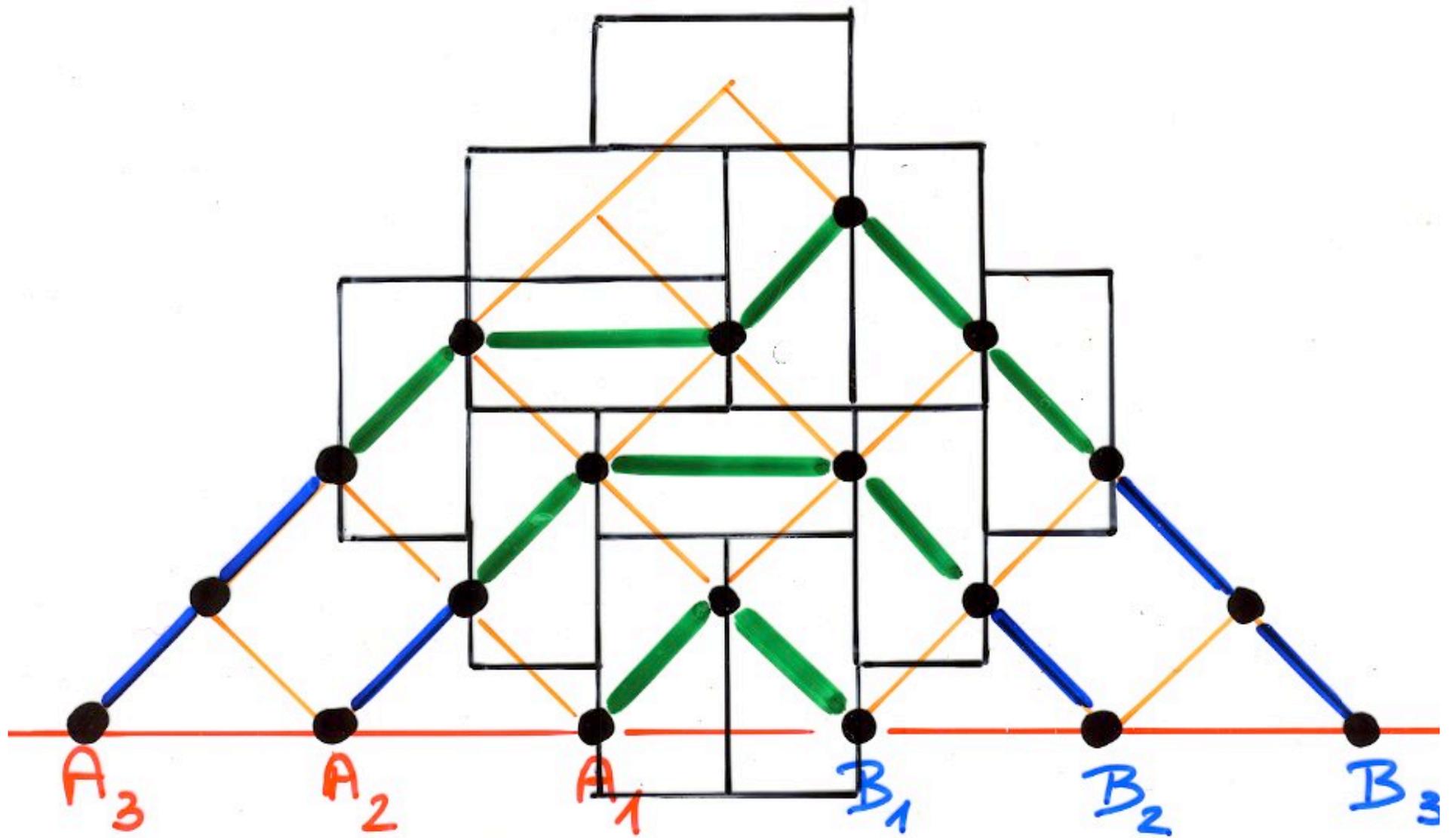
pavages Aztèques

chemins ne se coupant pas

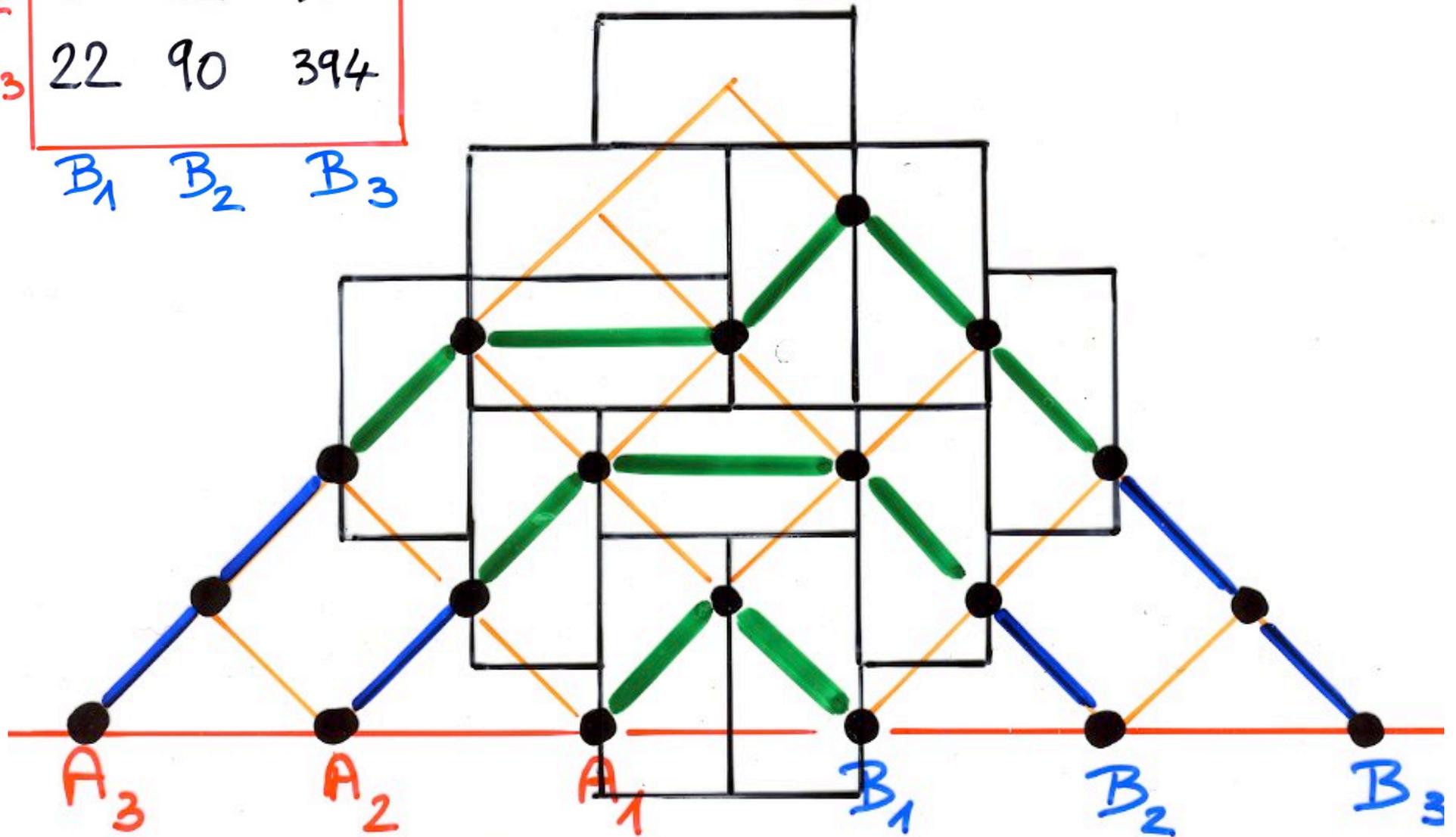








$A_1$	2	6	22
$A_2$	6	22	90
$A_3$	22	90	394
	$B_1$	$B_2$	$B_3$



Le Lemme LGV

chemins  
ne se coupant pas

déterminant

$$\det \begin{pmatrix} 2 & 6 & 22 \\ 6 & 22 & 90 \\ 22 & 90 & 394 \end{pmatrix} =$$

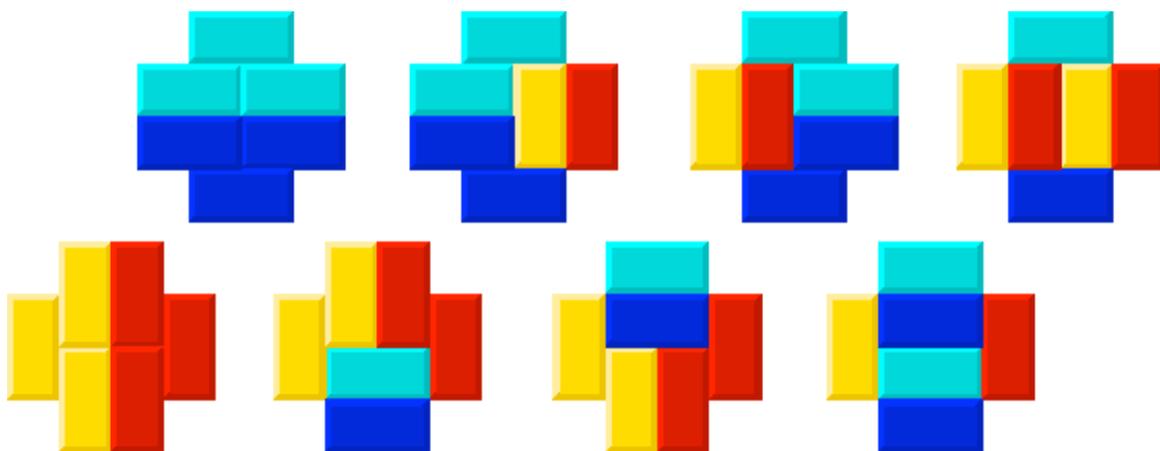
Le Lemme LGV

Linström - Gessel - Viennot

$$\det \begin{pmatrix} 2 & 6 \\ 6 & 22 \end{pmatrix} = (2 \times 22) - (6 \times 6) \\ = 44 - 36$$

déterminant

$$\begin{aligned} \det \begin{pmatrix} 2 & 6 \\ 6 & 22 \end{pmatrix} &= (2 \times 22) - (6 \times 6) \\ &= 44 - 36 \\ &= 8 = 2^3 \end{aligned}$$



$$\det \begin{pmatrix} 2 & 6 & 22 \\ 6 & 22 & 90 \\ 22 & 90 & 394 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & 22 & \cdot \\ \cdot & \cdot & 394 \end{pmatrix} + 17336 \quad \begin{pmatrix} \cdot & \cdot & 22 \\ 6 & \cdot & \cdot \\ \cdot & 90 & \cdot \end{pmatrix} + 11880 \quad \begin{pmatrix} \cdot & 6 & \cdot \\ \cdot & \cdot & 90 \\ 22 & \cdot & \cdot \end{pmatrix} + 11880 \rightarrow 41096$$

$$\begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & \cdot & 90 \\ \cdot & 90 & \cdot \end{pmatrix} - 16200 \quad \begin{pmatrix} \cdot & 6 & \cdot \\ 6 & \cdot & \cdot \\ \cdot & \cdot & 394 \end{pmatrix} - 14184 \quad \begin{pmatrix} \cdot & \cdot & 22 \\ \cdot & 22 & \cdot \\ 22 & \cdot & \cdot \end{pmatrix} - 10648 \rightarrow -41032$$

$$= \frac{64}{2^6} \quad (!!)$$

# Déterminant

d'un tableau de nombres  
(ou *matrice*)

ayant  $n$  lignes et  $n$  colonnes

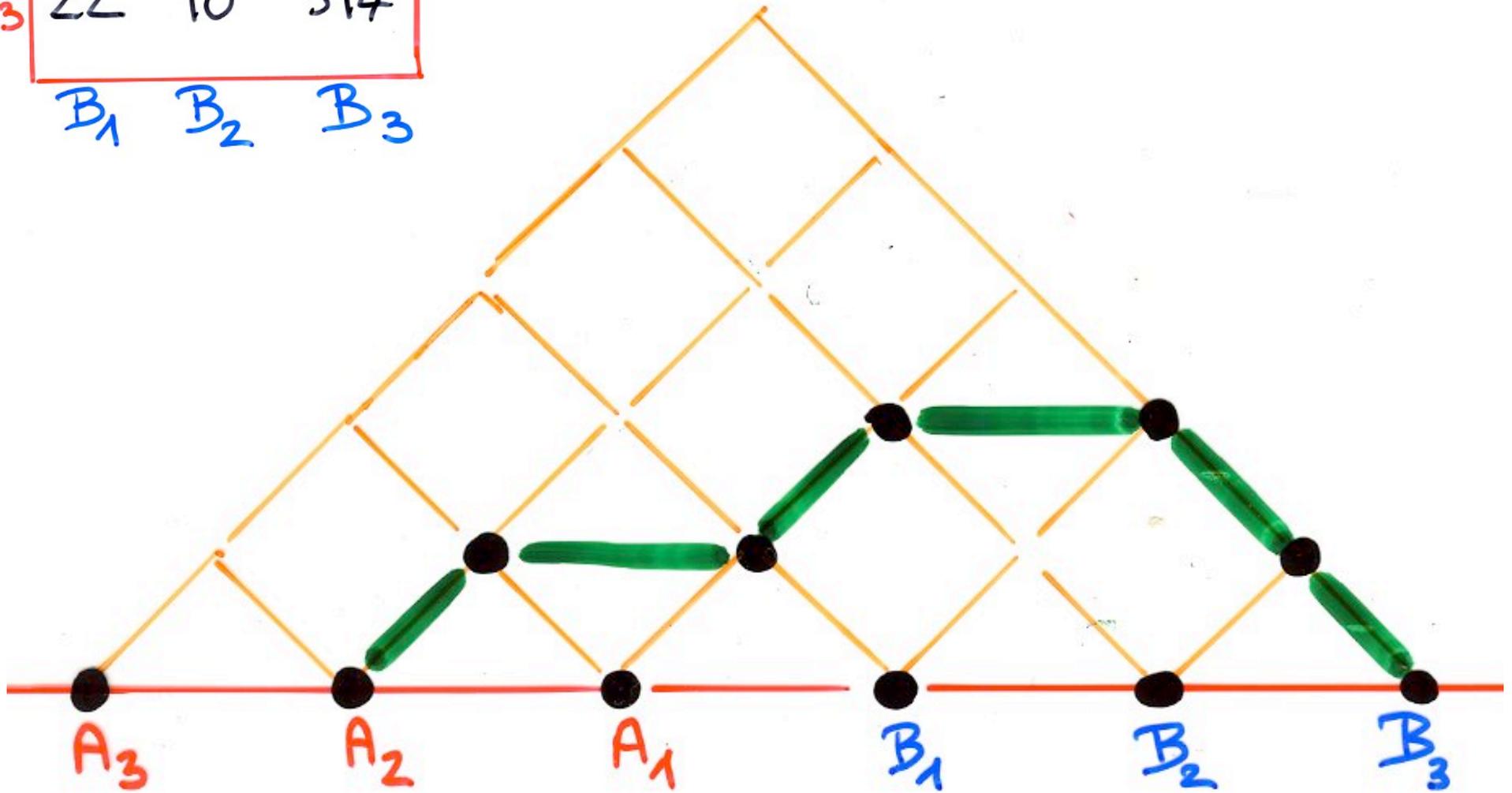
algèbre linéaire

système d'équations linéaires

$$\begin{cases} 2x - 3y + 11z = -2 \\ 4x + y - z = 0 \\ 5x - 7y + 9z = 6 \end{cases}$$

nombre de Schröder

$A_1$	2	6	22
$A_2$	6	22	90
$A_3$	22	90	394
	$B_1$	$B_2$	$B_3$



chemins de Schröder

nombres de Schröder

$$S_n = 1, 2, 6, 22, 90, 394, \dots$$

$$a(n) = 3a(n-1) + 2a(n-2).$$

M2894 1, 1, 3, 11, 41, 153, 571, 2131, 7953, 29681, 110771, 413403, 1542841, 5757961,  
21489003, 80198051, 299303201, 1117014753, 4168755811, 15558008491  
 $a(n) = 4a(n-1) - a(n-2)$ . Ref EUL (1) 1 375 11. MMAG 40 78 67. [0,3; A1835, N1160]

M2895 1, 3, 11, 43, 171, 683, 2731, 10923, 43691, 174763, 699051, 2796203, 11184811,  
44739243, 178956971, 715827883, 2863311531, 11453246123, 45812984491  
 $(2^{2n+1} + 1)/3$ . Ref JGT 17 625 93. [0,2; A7583]

M2896 3, 11, 43, 683, 2731, 43691, 174763, 2796203, 715827883, 2932031007403,  
768614336404564651, 201487636602438195784363  
Primes of form  $(2^p + 1)/3$ . Ref MMAG 27 157 54. [1,1; A0979, N1161]

M2897 1, 3, 11, 44, 186, 814, 3652, 16689, 77359, 362671, 1716033, 8182213, 39267086,  
189492795, 918837374, 4474080844, 21866153748, 107217298977, 527266673134  
Fixed hexagonal polyominoes with  $n$  cells. Ref RE72 97. dhr. [1,2; A1207, N1162]

M2898 1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, 2646723, 13648659,  
71039373, 372693519, 1968801519, 10463578353, 55909013009, 300159426963  
Schroeder's second problem:  $(n+1)a(n+1) = 3(2n-1)a(n) - (n-2)a(n-1)$ . Ref  
EDMN 32 6 40. BAMS 54 359 48. RCI 168. CI 57. VA91 198. [1,3; A1003, N1163]

$$S_n = 1, 2, 6, 22, 90, 394, \dots$$

$$a(n) = 3a(n-1) + 2a(n-2).$$

M2894 1, 1, 3, 11, 41, 153, 571, 2131, 7953, 29681, 110771, 413403, 1542841, 5757961,  
21489003, 80198051, 299303201, 1117014753, 4168755811, 15558008491  
 $a(n) = 4a(n-1) - a(n-2)$ . Ref EUL (1) 1 375 11. MMAG 40 78 67. [0,3; A1835, N1160]

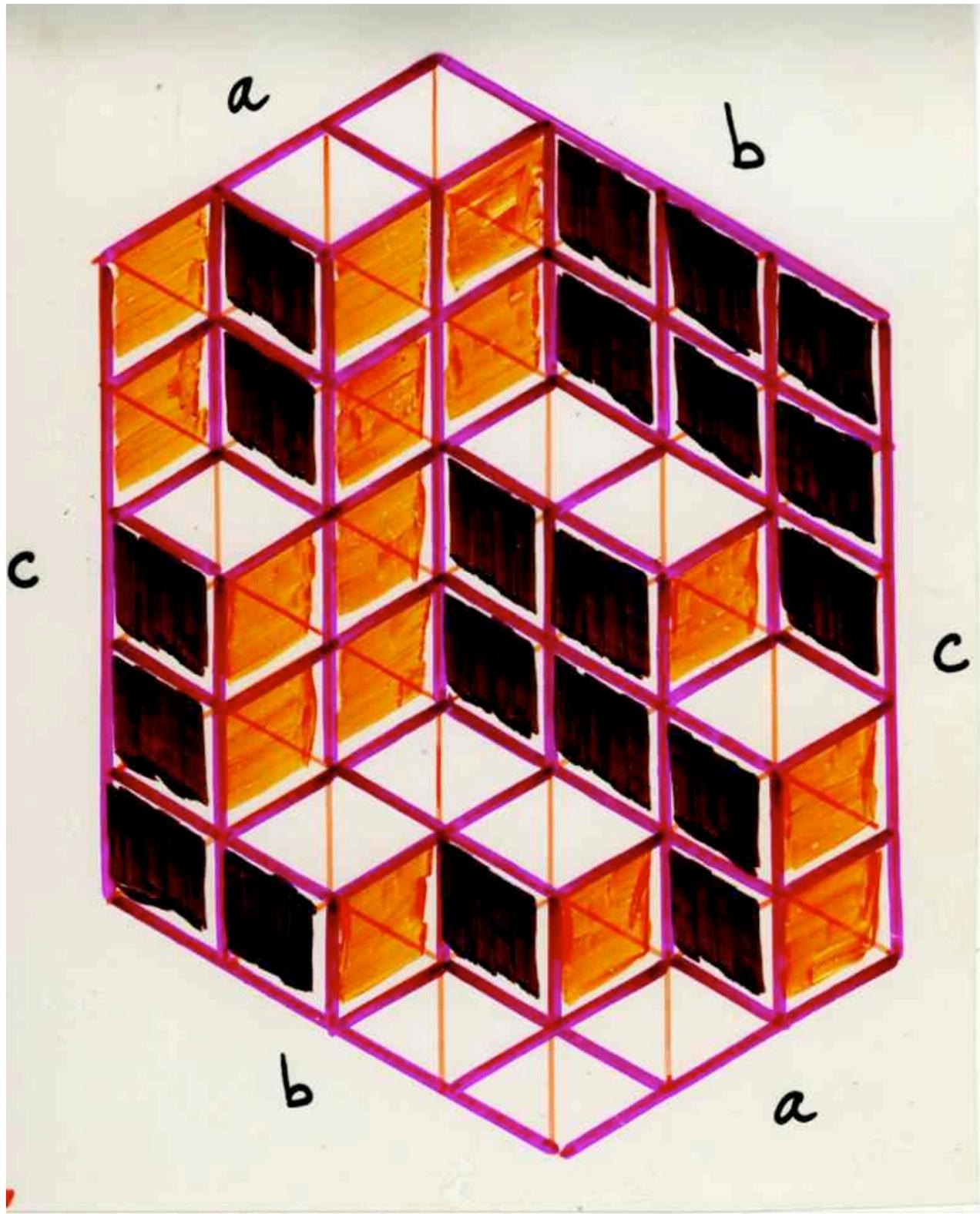
M2895 1, 3, 11, 43, 171, 683, 2731, 10923, 43691, 174763, 699051, 2796203, 11184811,  
44739243, 178956971, 715827883, 2863311531, 11453246123, 45812984491  
 $(2^{2n+1} + 1)/3$ . Ref JGT 17 625 93. [0,2; A7583]

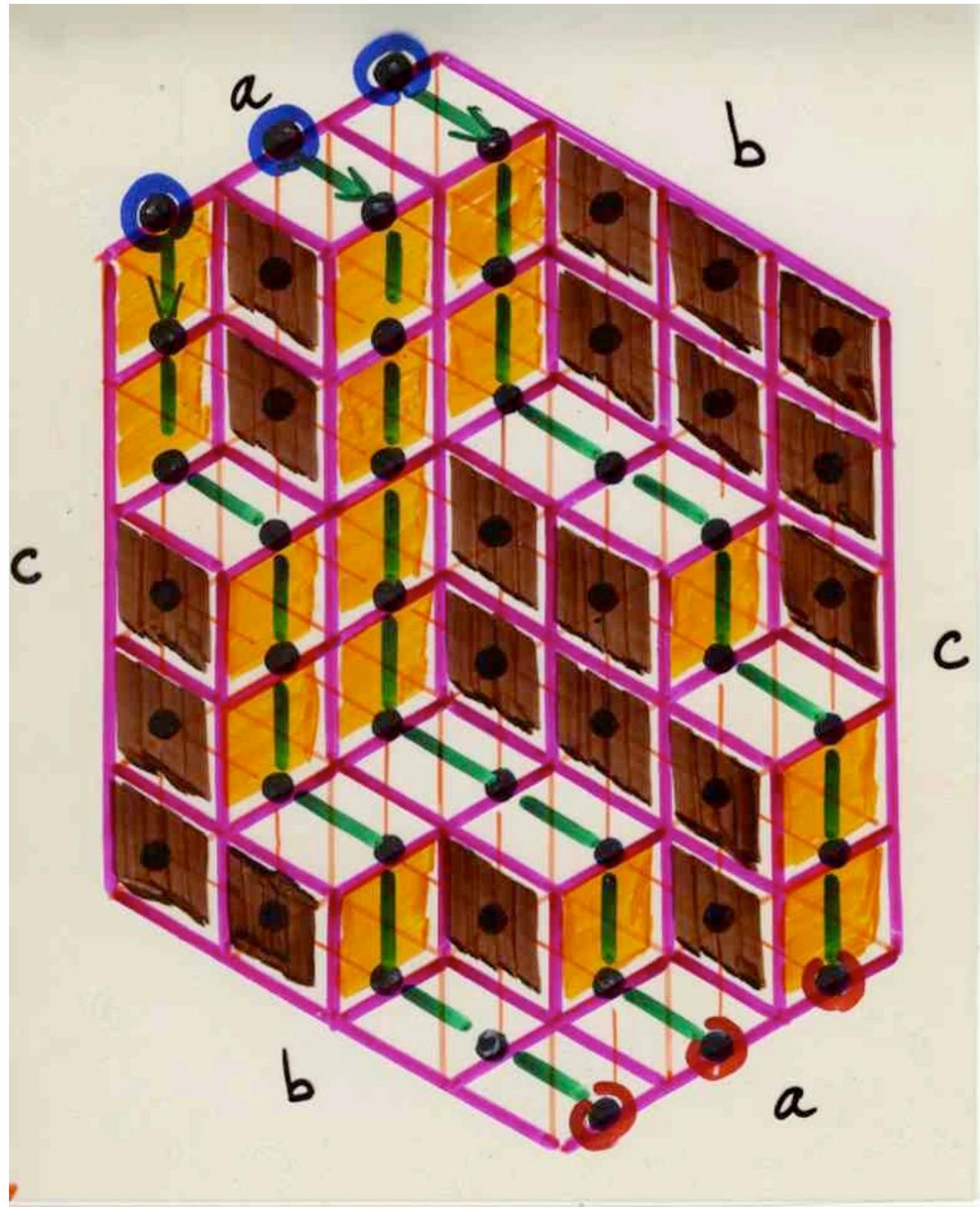
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768614336404564651, 201487636602438195784363  
Primes of form  $(2^p + 1)/3$ . Ref MMAG 27 157 54. [1,1; A0979, N1161]

M2897 1, 3, 11, 44, 186, 814, 3652, 16689, 77359, 362671, 1716033, 8182213, 39267086,  
189492795, 918837374, 4474080844, 21866153748, 107217298977, 527266673134  
Fixed hexagonal polyominoes with  $n$  cells. Ref RE72 97. dhr. [1,2; A1207, N1162]

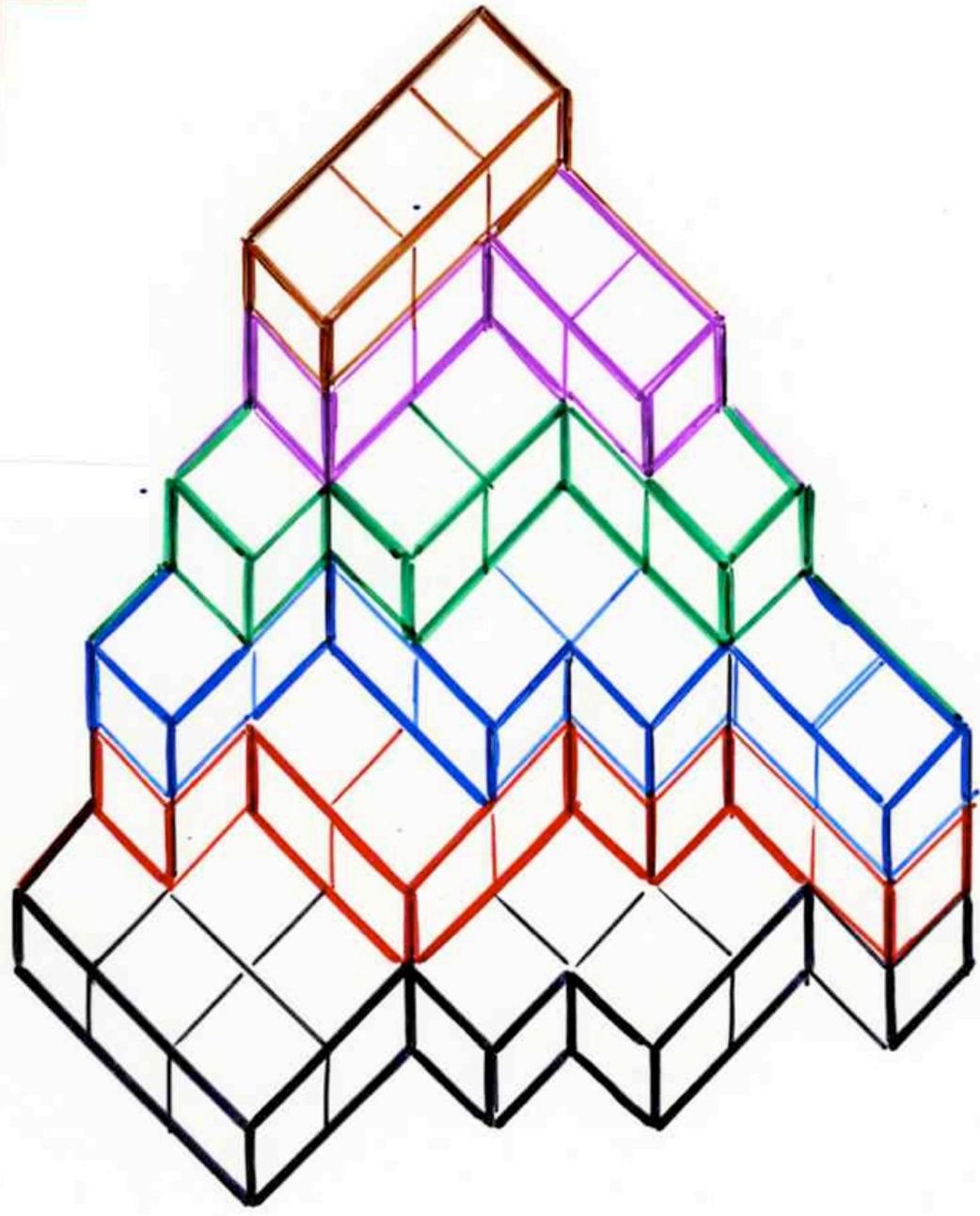
M2898 1, 1, 3, 11, 45, 197, 903, 4279, 20791, 103049, 518859, 2646723, 13648869,  
71039373, 372693519, 1968801519, 10463578253, 55909013009, 300159426963  
Schroeder's second problem:  $(n+1)a(n+1) = 3(2n-1)a(n) - (n-2)a(n-1)$ . Ref  
EDMN 32 6 40. BAMS 54 359 48. RCI 168. C1 57. VA91 198. [1,3; A1003, N1163]

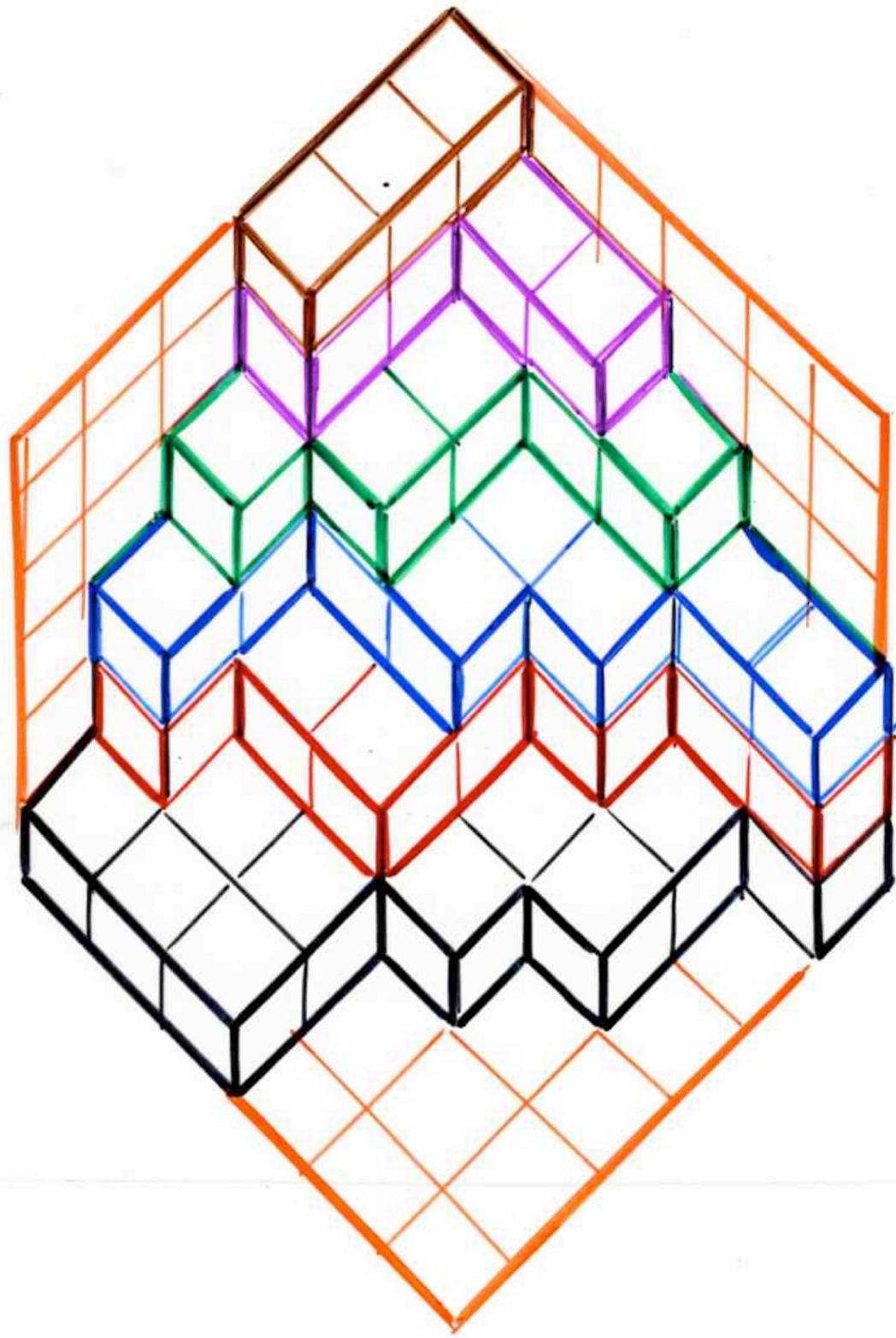
pavages de calissons  
et  
configurations de chemins  
ne se coupant pas

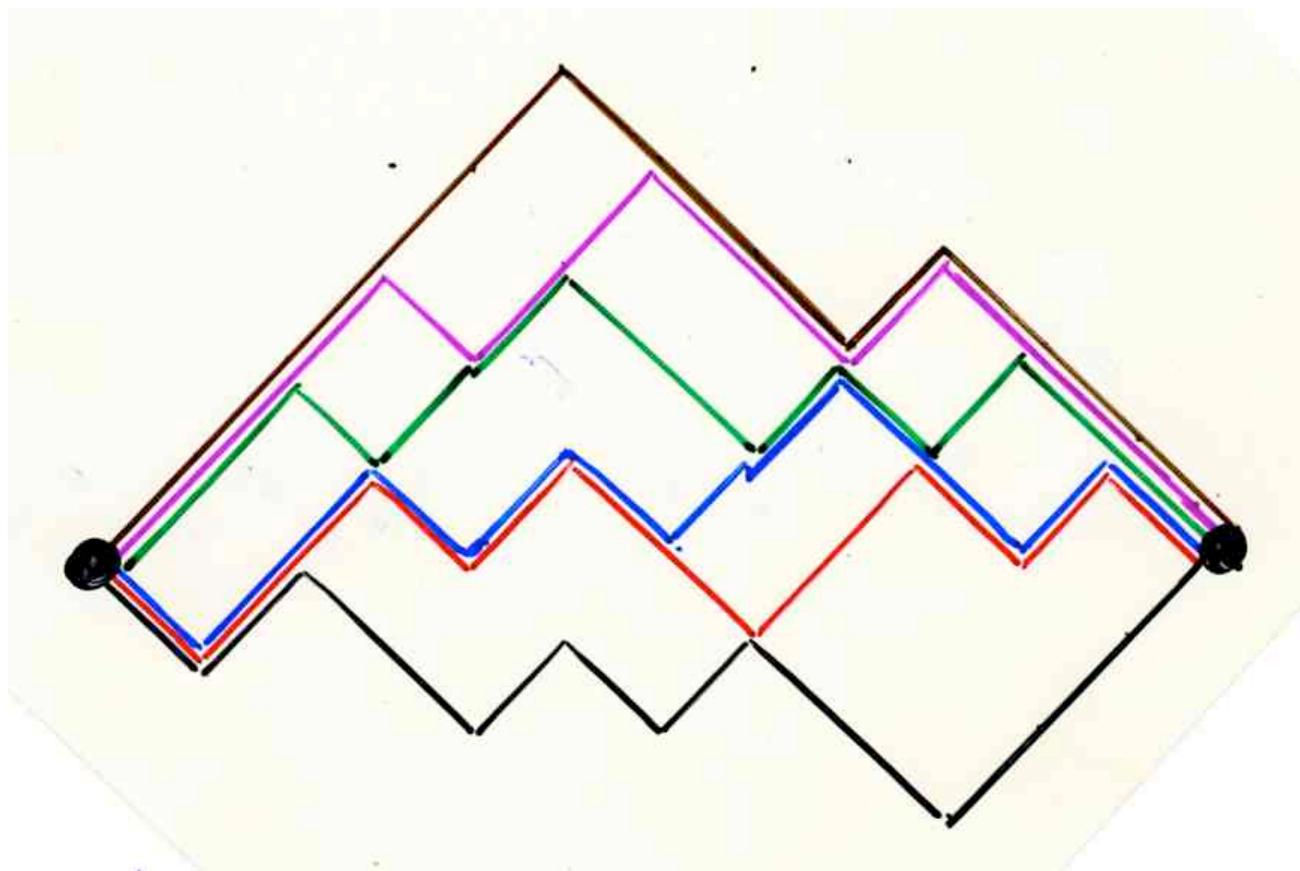


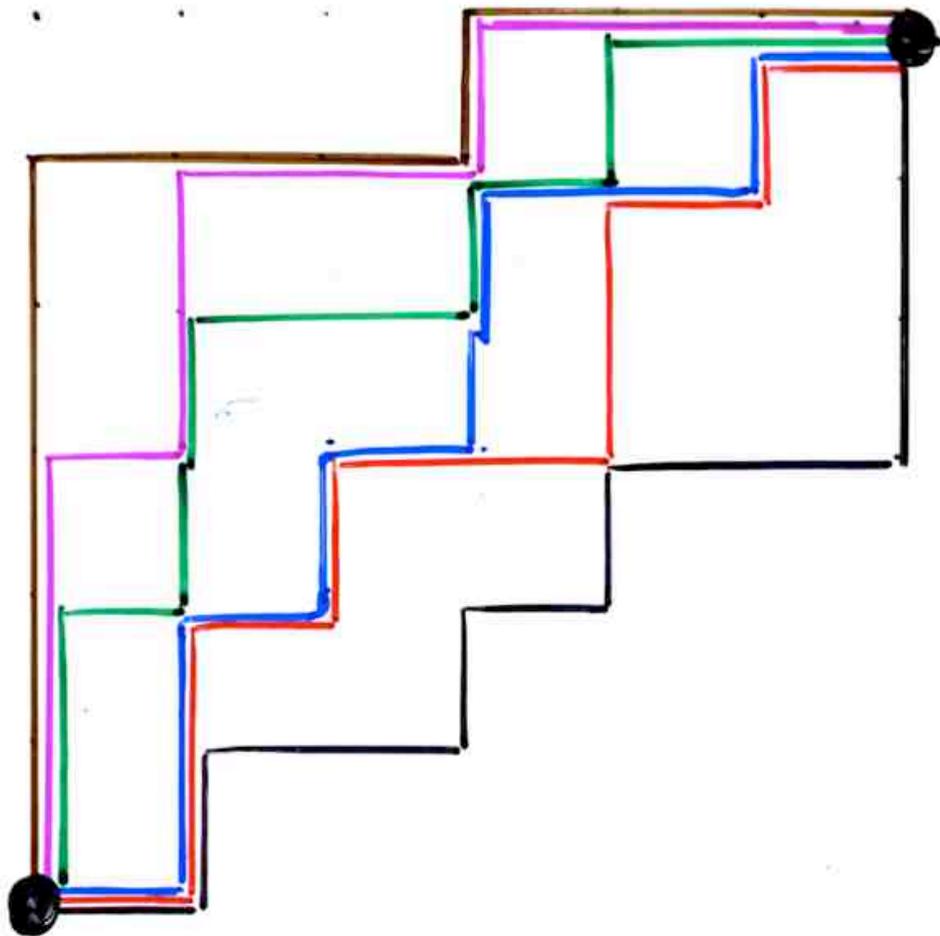


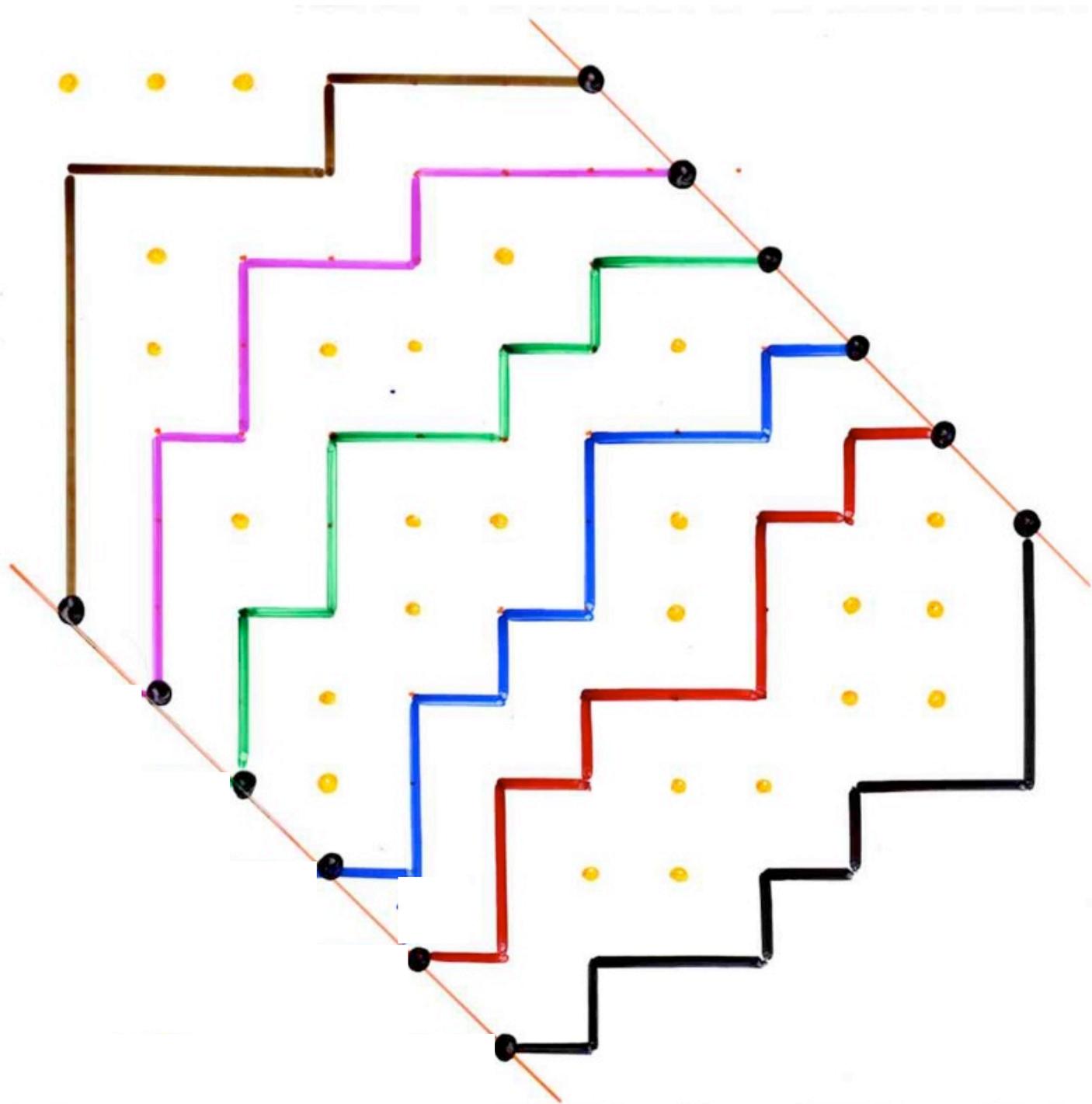
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

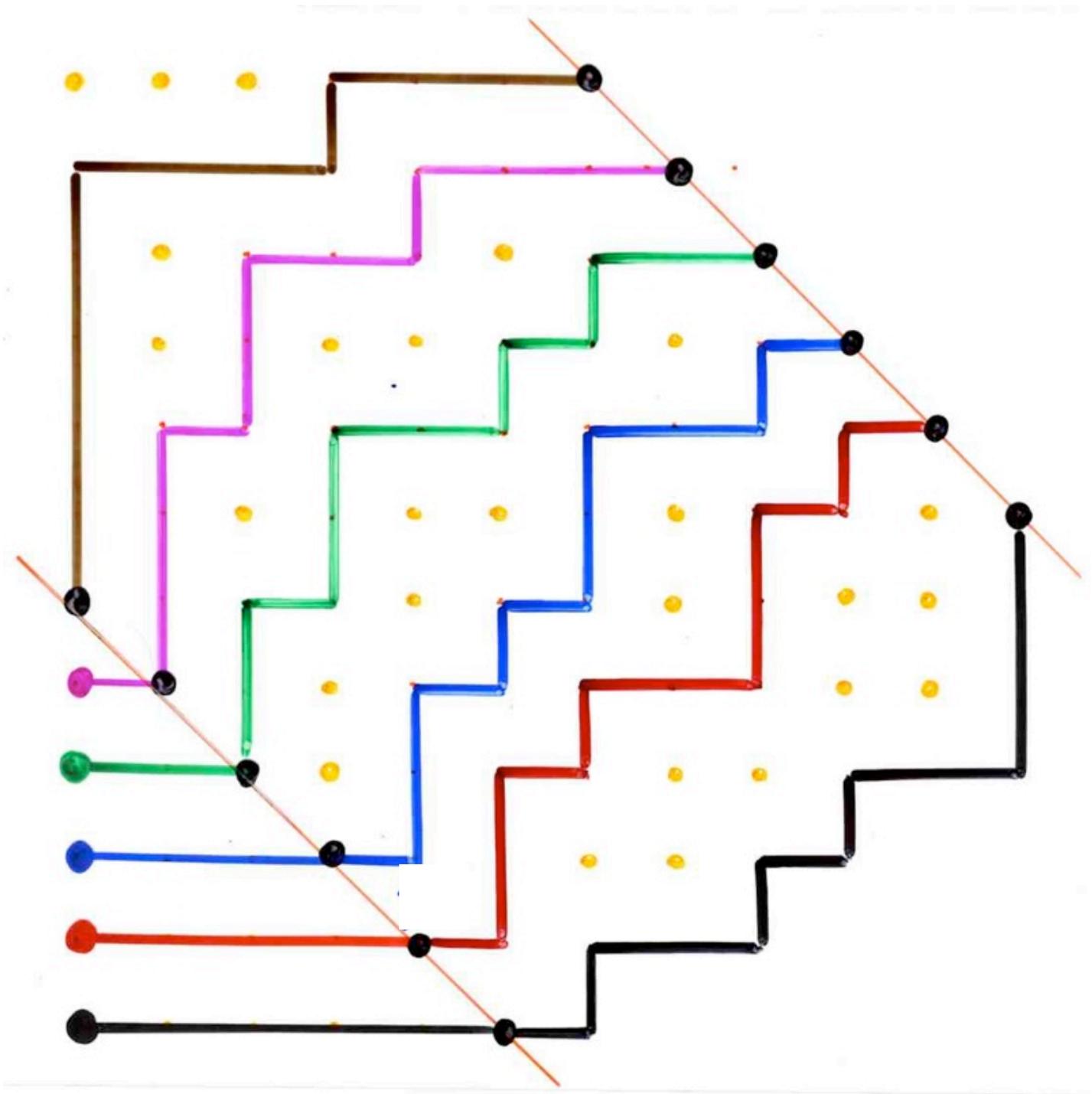


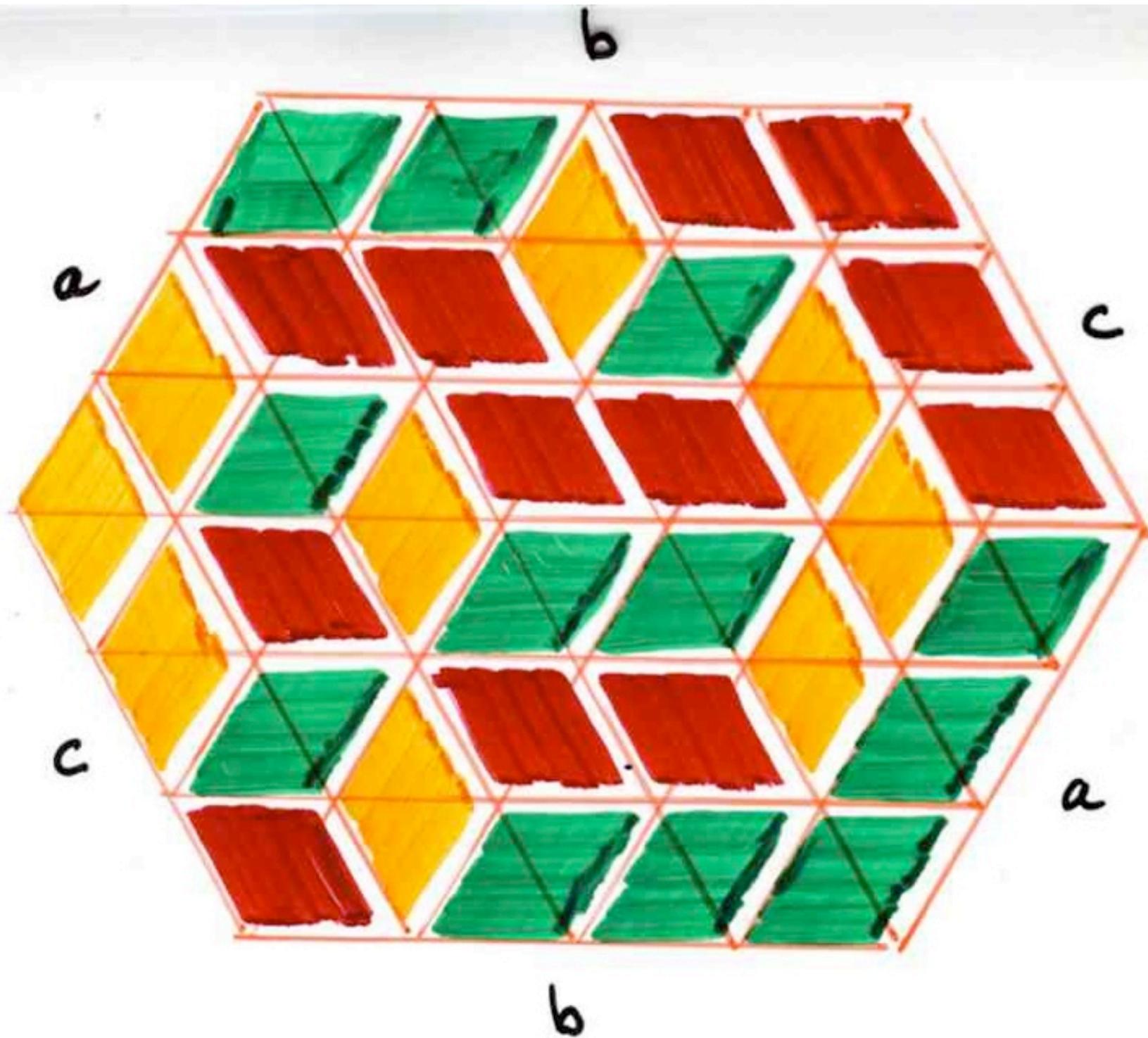


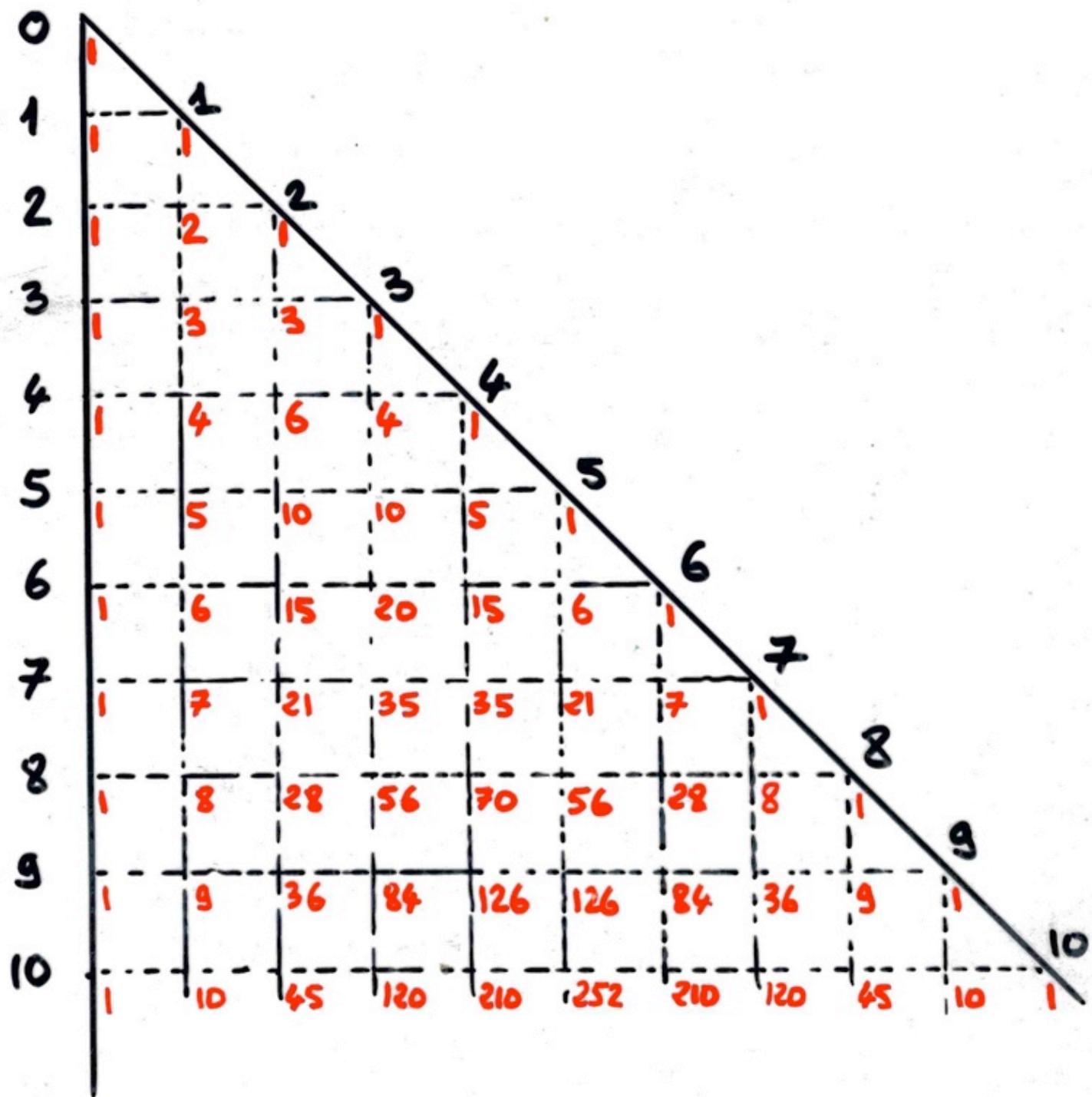


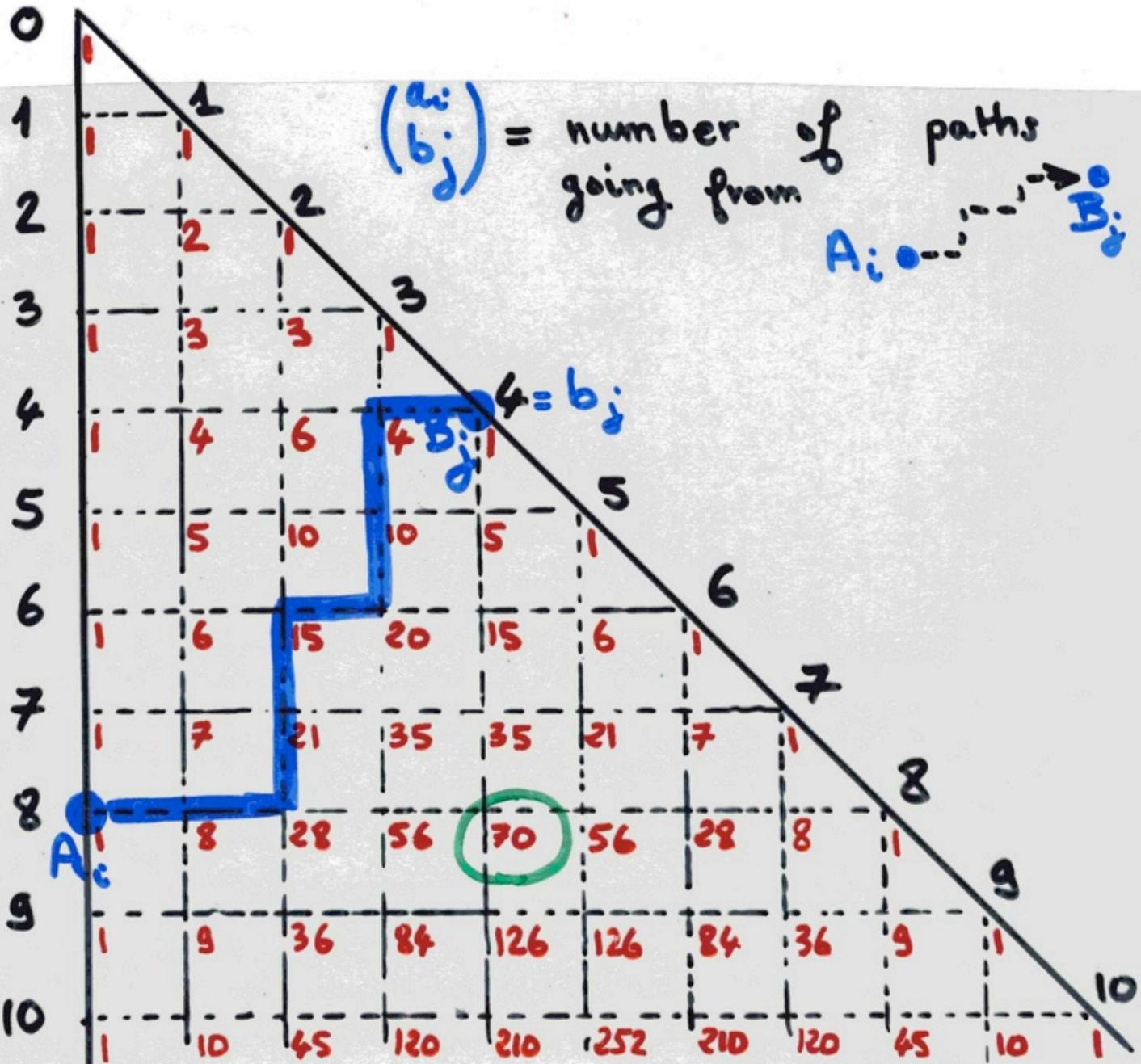






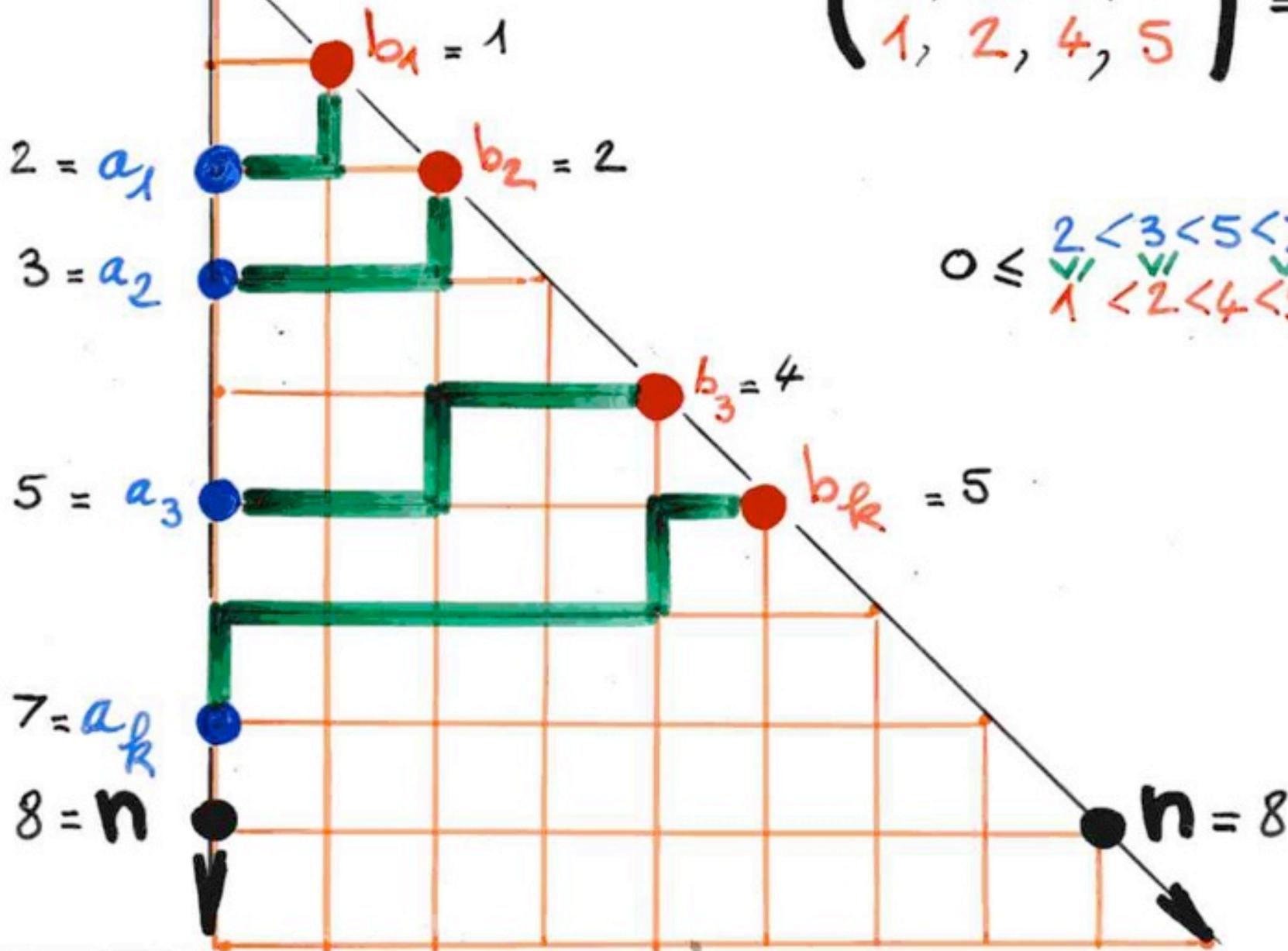




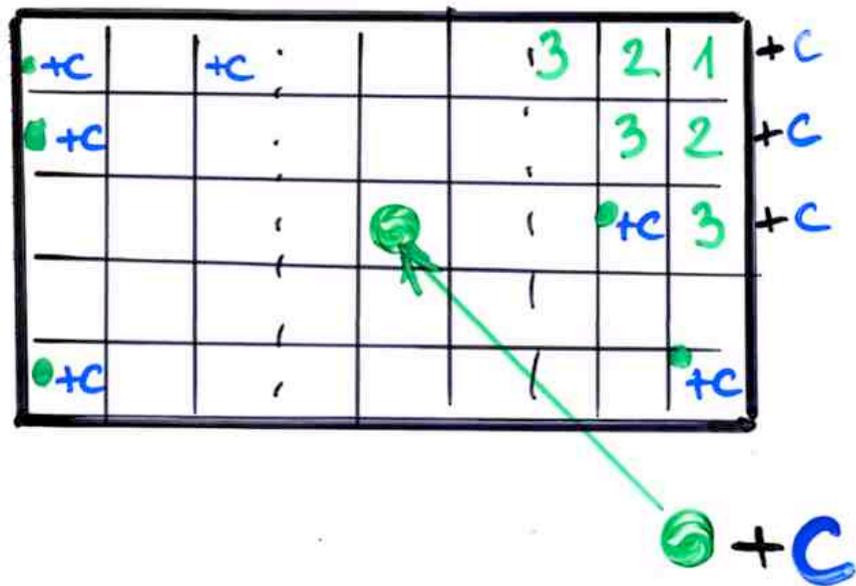


$(0, 0)$

$$\begin{pmatrix} 2, 3, 5, 7 \\ 1, 2, 4, 5 \end{pmatrix} = 210$$



$$0 \leq \begin{matrix} 2 < 3 < 5 < 7 \\ \sqrt{\phantom{x}} & \sqrt{\phantom{x}} & \sqrt{\phantom{x}} \\ 1 < 2 < 4 < 5 \end{matrix} \leq 8 = n$$

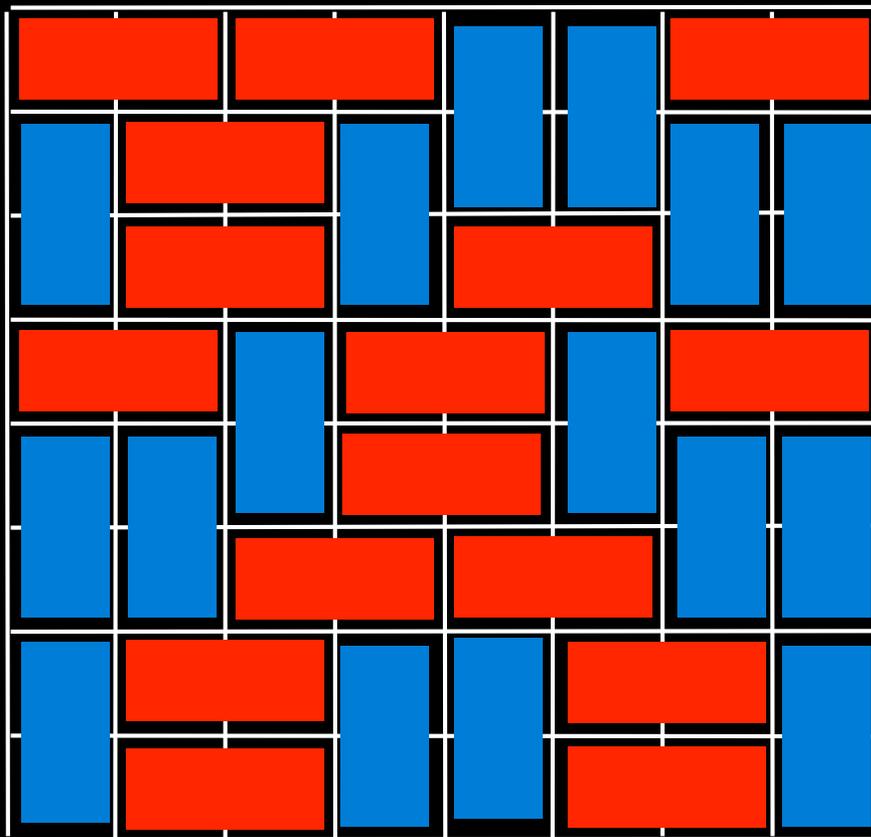


retour

aux pavages de l'échiquier

une autre formule ...

# pavages d'un échiquier avec des dominos

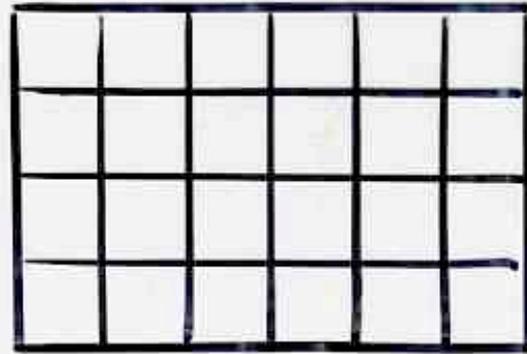


rectangle

$$2m \times 2n$$

$$m = 2$$
$$n = 3$$

$$2m$$
$$= 4$$



$$2n = 6$$

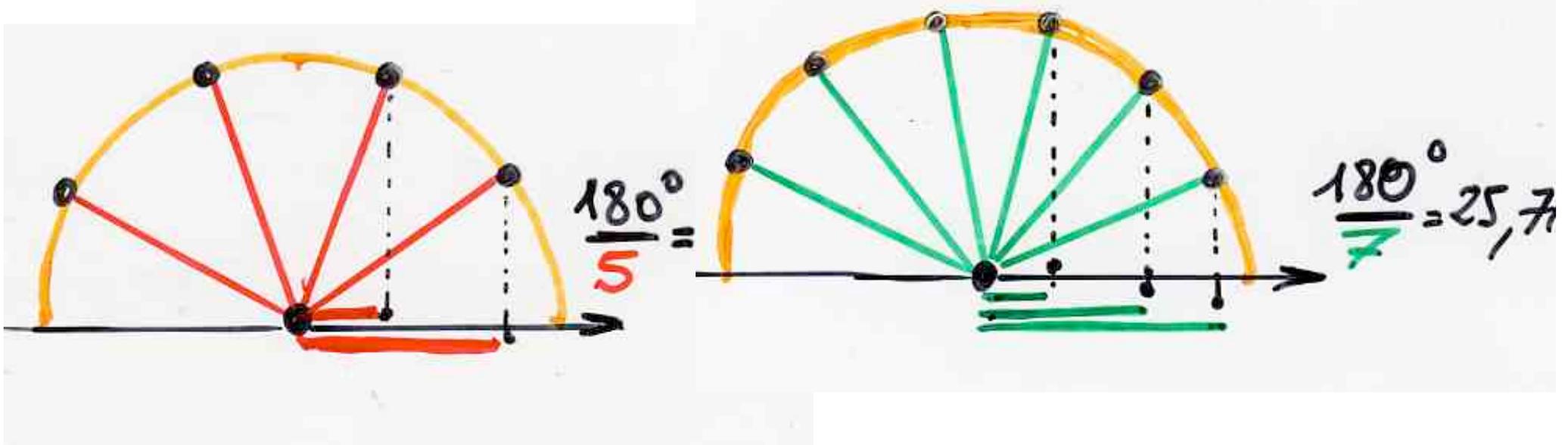
281

pages

$$6 = mn = 2 \times 3$$

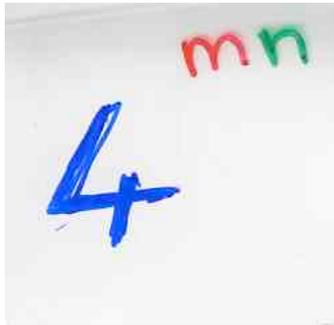
$$4^6$$

$\cos^2\left(\frac{180^\circ}{5}\right)$ $+ \cos^2\left(\frac{180^\circ}{7}\right)$	$\cos^2\left(\frac{180^\circ}{5}\right)$ $+ \cos^2\left(2 \times \frac{180^\circ}{7}\right)$	$\cos^2\left(\frac{180^\circ}{5}\right)$ $+ \cos^2\left(3 \times \frac{180^\circ}{7}\right)$
$\cos^2\left(2 \times \frac{180^\circ}{5}\right)$ $+ \cos^2\left(\frac{180^\circ}{7}\right)$	$\cos^2\left(2 \times \frac{180^\circ}{5}\right)$ $+ \cos^2\left(2 \times \frac{180^\circ}{7}\right)$	$\cos^2\left(2 \times \frac{180^\circ}{5}\right)$ $+ \cos^2\left(3 \times \frac{180^\circ}{7}\right)$



$$4^6 = \underbrace{4 \times 4 \times 4 \times 4 \times 4 \times 4}_{6 \text{ fois}}$$
$$= 4096$$

le nombre de pavages avec des dominos d'un rectangle  $m \times n$  est égal au produit:



$$\prod_{i=1}^{m/2} \prod_{j=1}^{n/2} \left( 4 \cos^2 \frac{i\pi}{m+1} + 4 \cos^2 \frac{j\pi}{n+1} \right)$$

Kasteleyn (1961)

c'est un entier !!

pour l'échiquier  $m=8, n=8$ : 12 988 816

- dénombrement de  
couplages parfaits

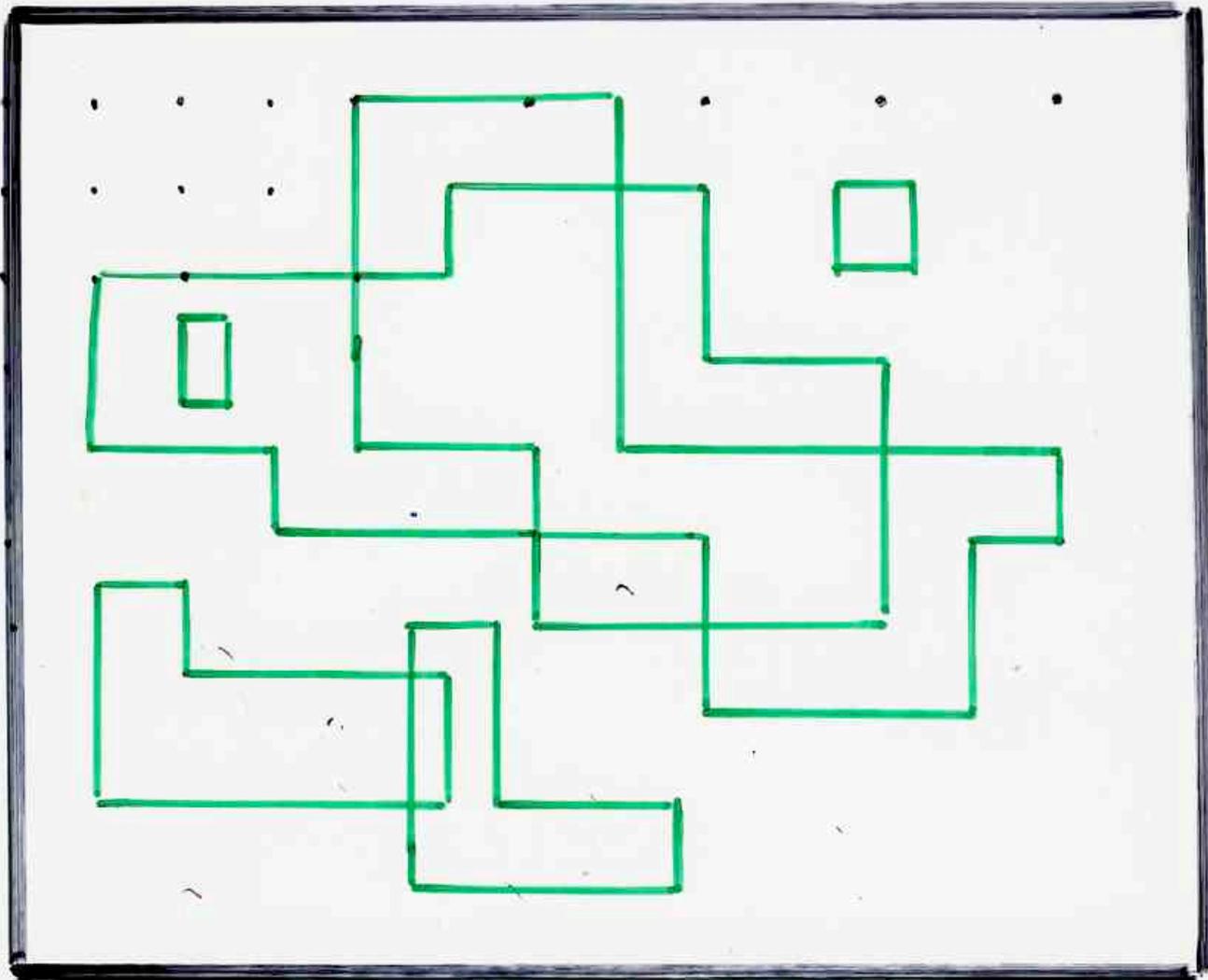
- graphe planaire

méthode du Pfaffien

- modèle d'Ising (1925)

Kasteleyn, Fisher, Temperley  
(1961, ....)

Onsager (1944)



"closed" graph

Ising model

$w$

$||$

$B^m$

$A^n$

$uv$

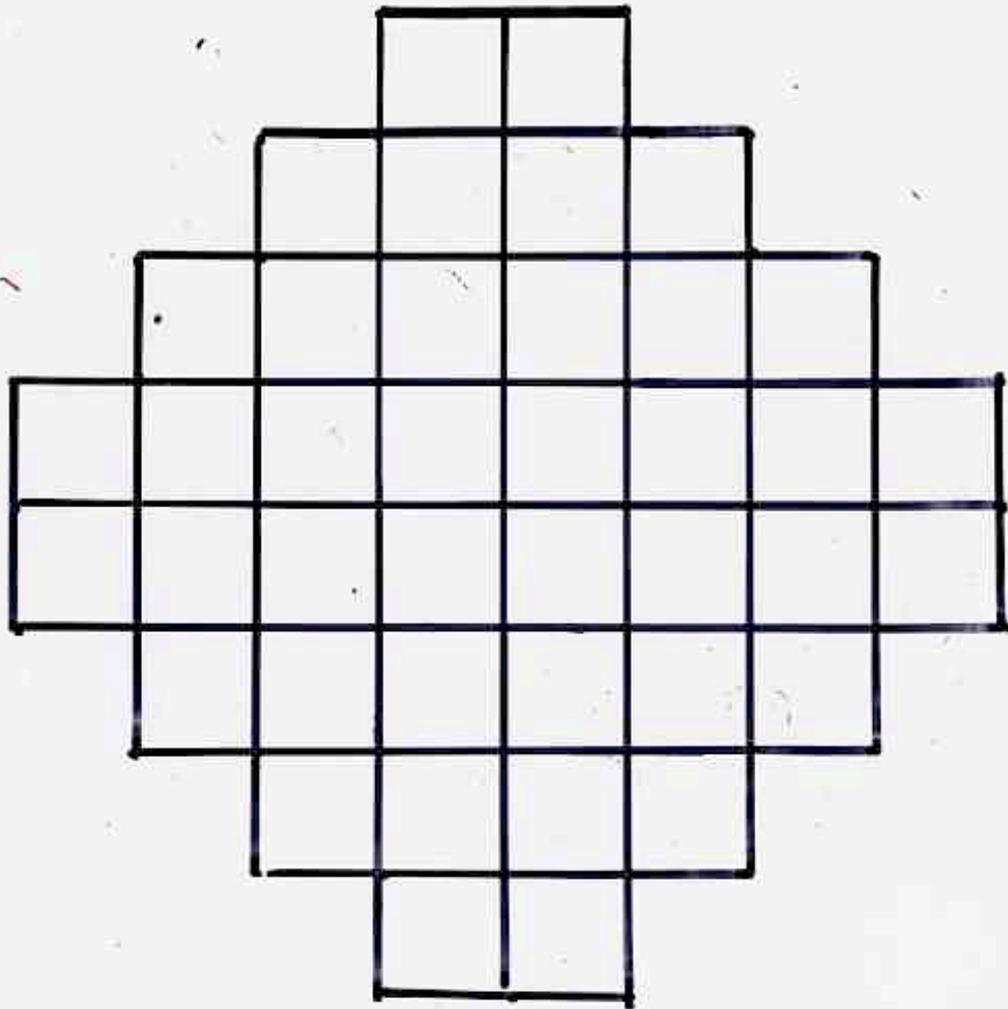
$||$

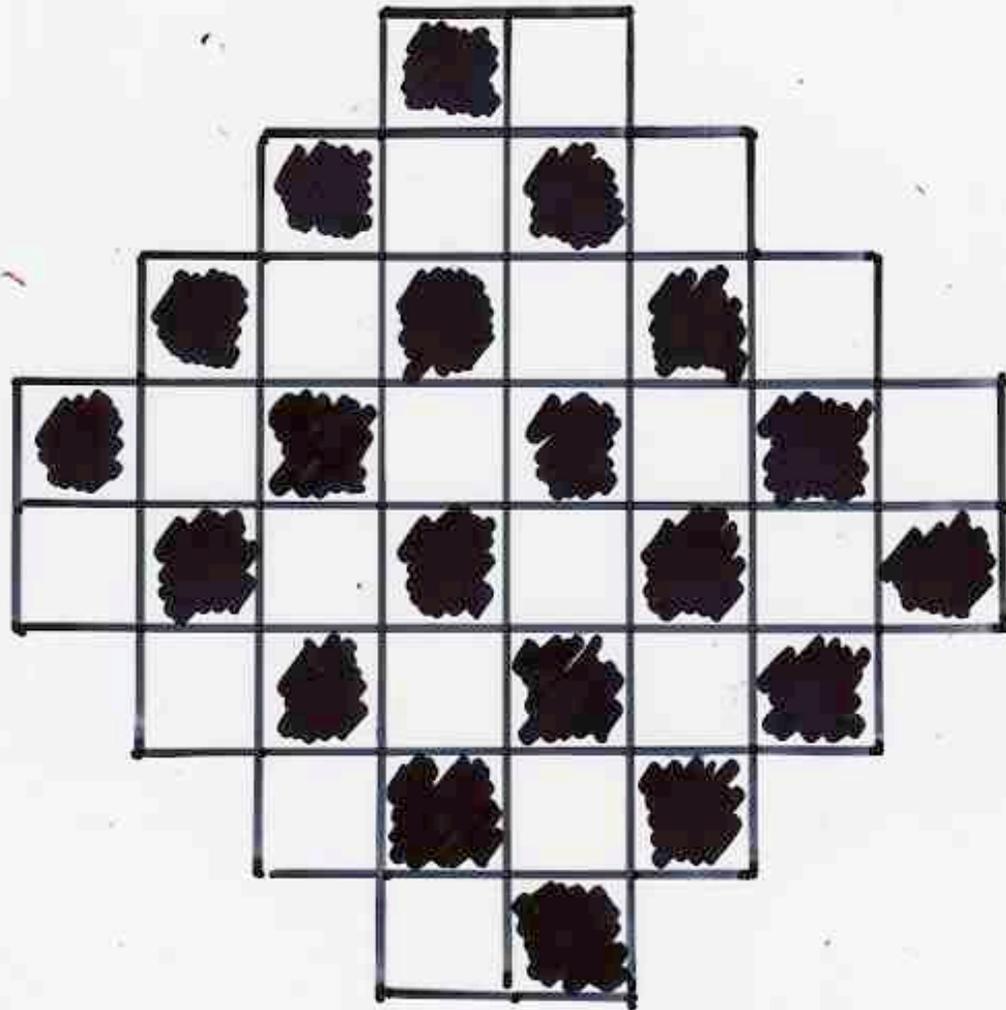
$A^n$

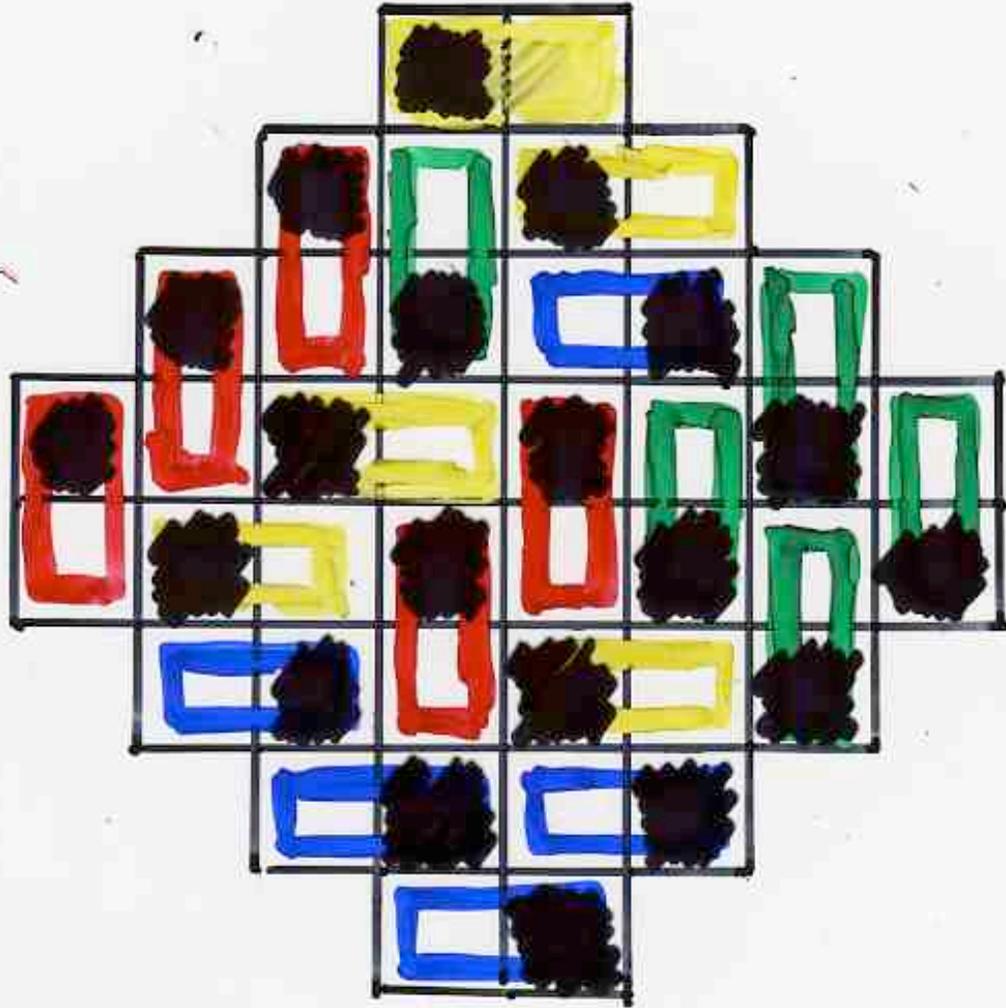
$B^m$

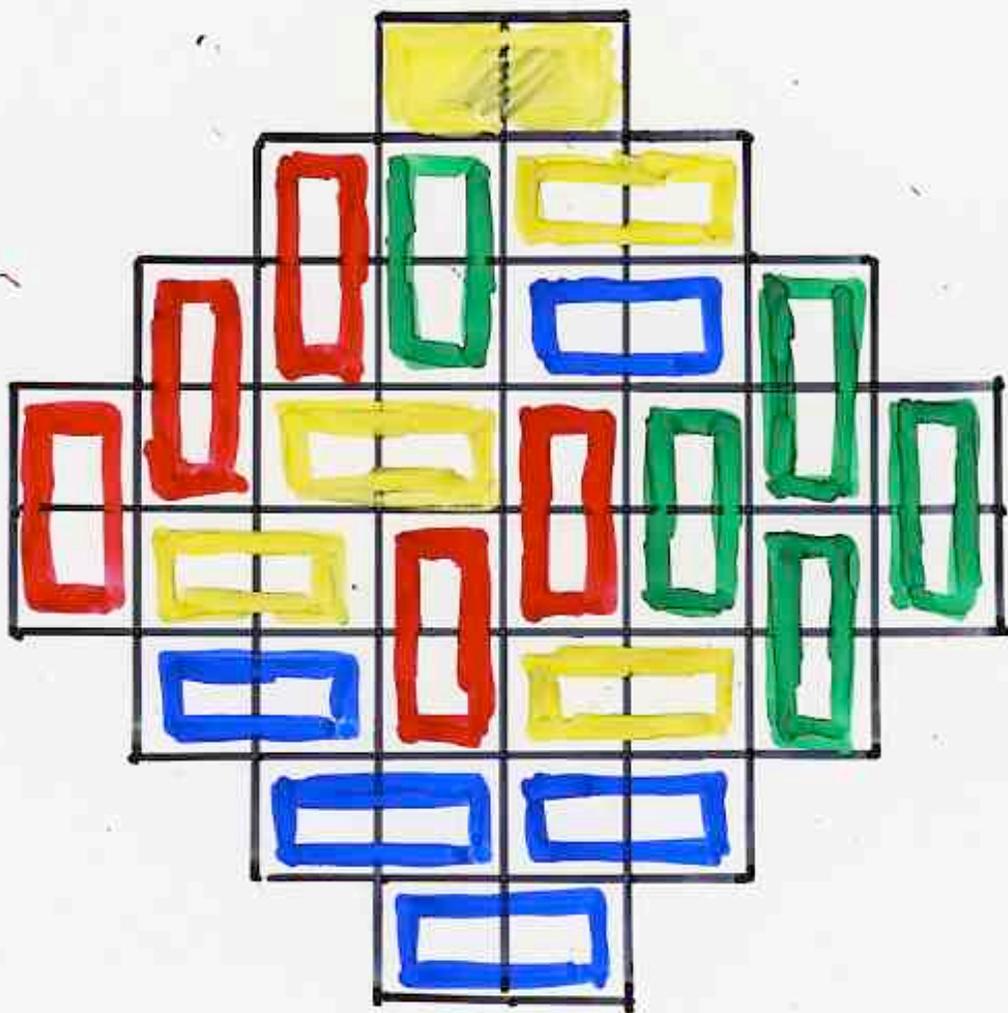
génération aléatoire  
d'objets combinatoires

le théorème du cercle arctique

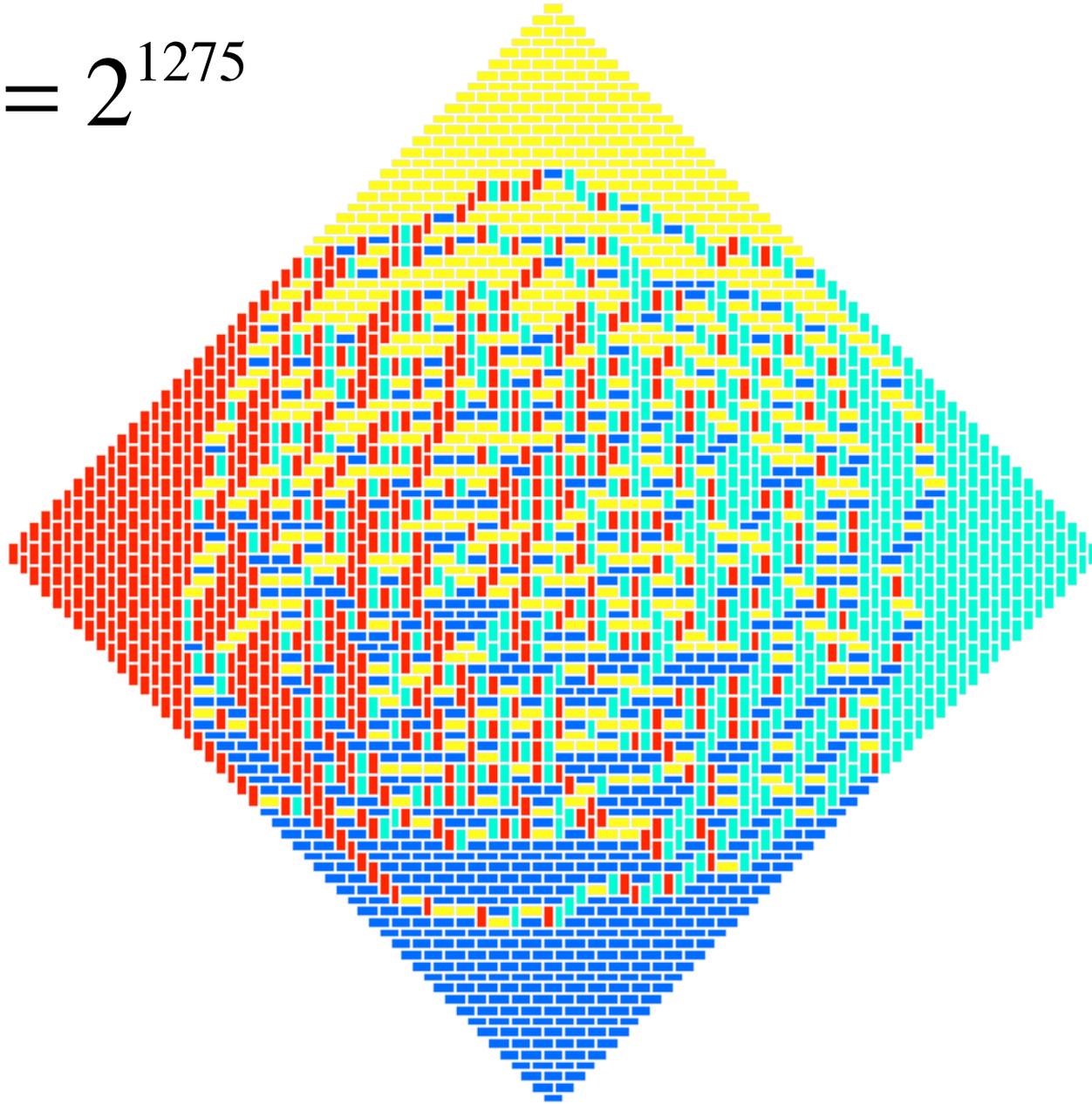




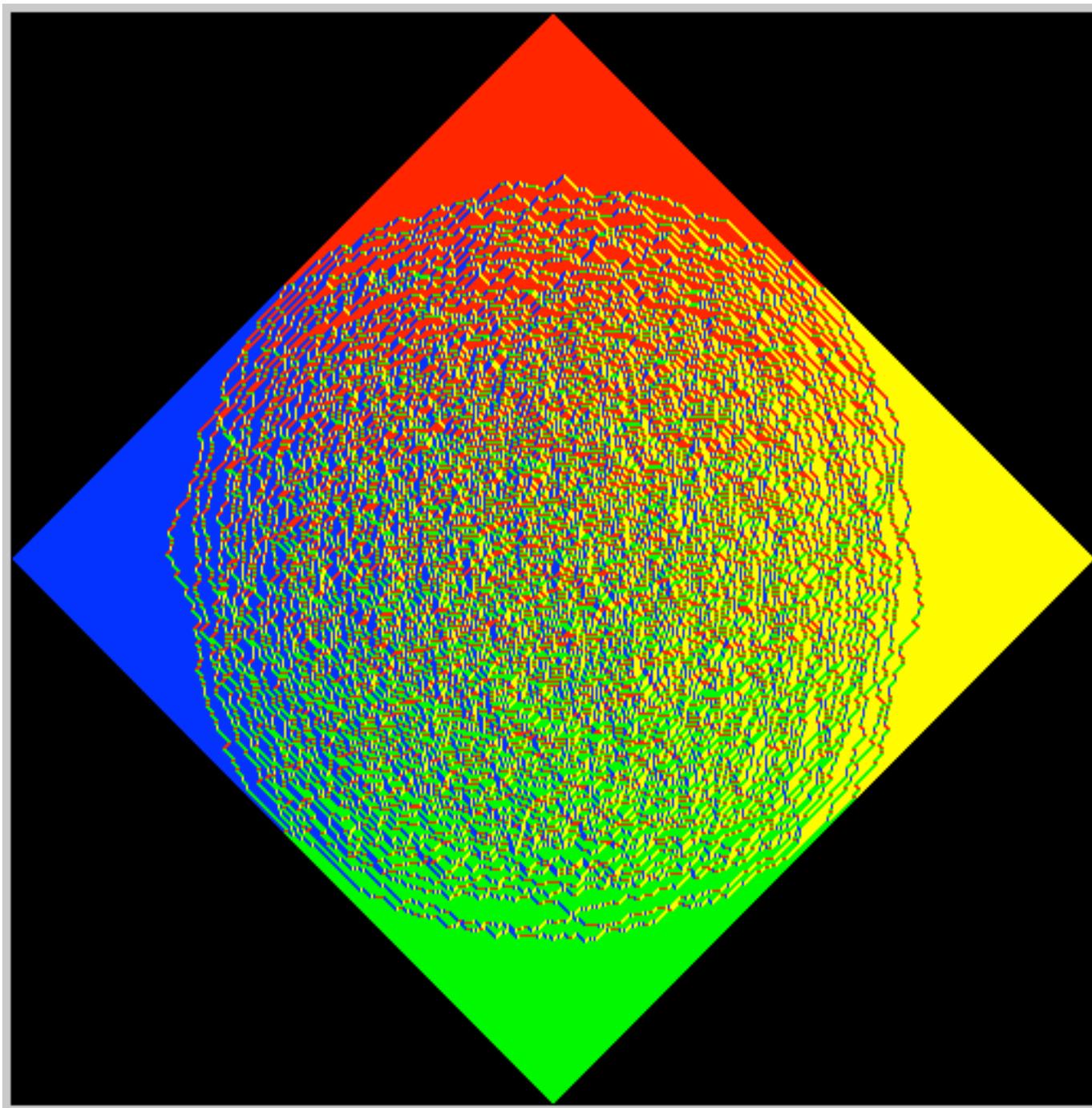




$$2^{(50 \times 51)/2} = 2^{1275}$$



le  
théorème  
du  
«arctique  
cercle»



algorithme de  
génération aléatoire  
d'un pavage aztèque

avec une preuve bijective  
de la formule

$$2^{\frac{n(n+1)}{2}}$$

$$w = w_1$$

$$w_1 = 0$$

$$w_1 = 1$$



$2^1$





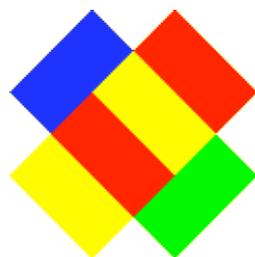
$$w = w_1 w_2$$

$$w_i = 0$$

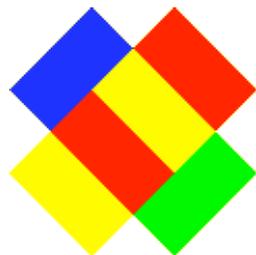
$$w_i = 1$$



$$2^{(1+2)}$$



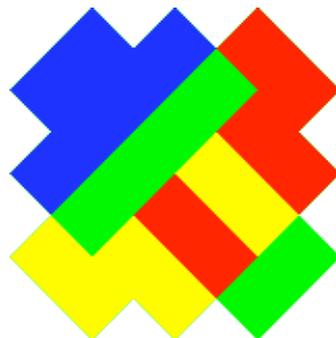
$$2^3 = 8$$



$$W = W_1 W_2 W_3$$

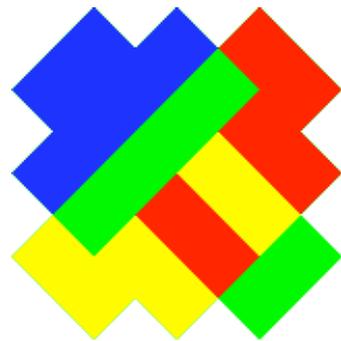
$$w_i = 0$$

$$w_i = 1$$



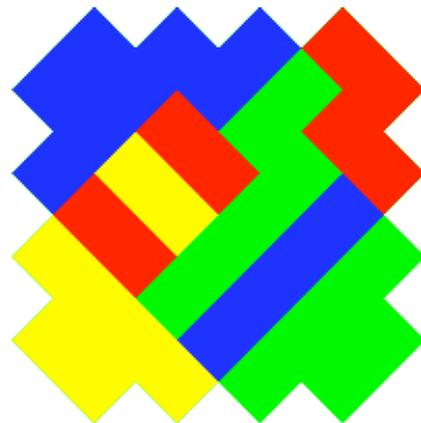
$$2^{(1+2+3)}$$

$$2^6 = 64$$



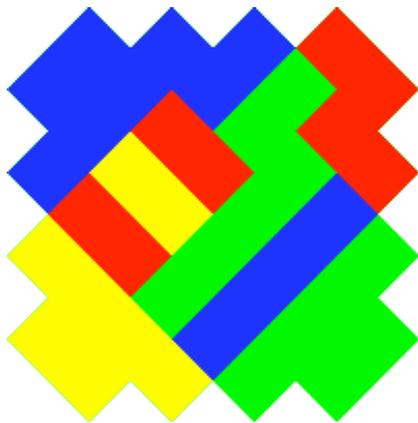
$$W = w_1 w_2 w_3 w_4$$

$$w_i = 0$$
$$w_i = 1$$



$$2^{(1+2+3+4)}$$

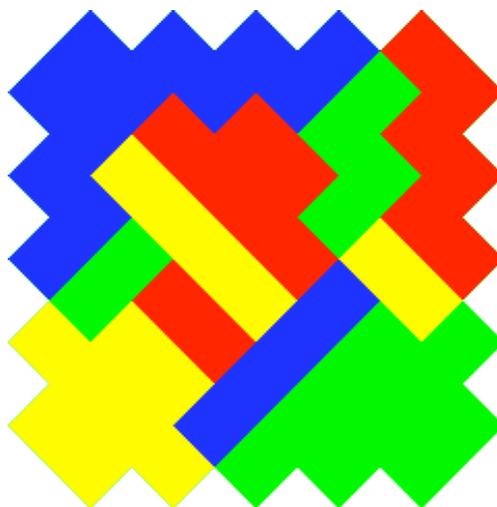
$$2^{10} = 1024$$



$$W = w_1 w_2 w_3 w_4 w_5$$

$$w_i = 0$$

$$w_i = 1$$



$$2^{(1+2+3+4+5)}$$

$$2^{15} = 32.768$$



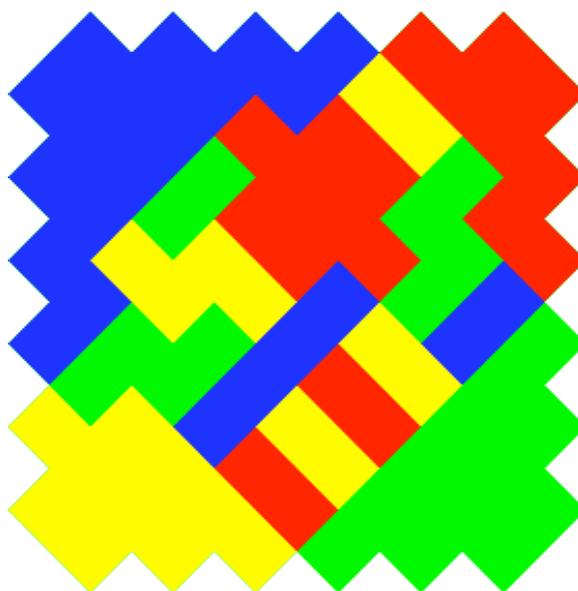
$$W = w_1 w_2 w_3 w_4 w_5 w_6$$



$$w_i = 0$$

$$w_i = 1$$

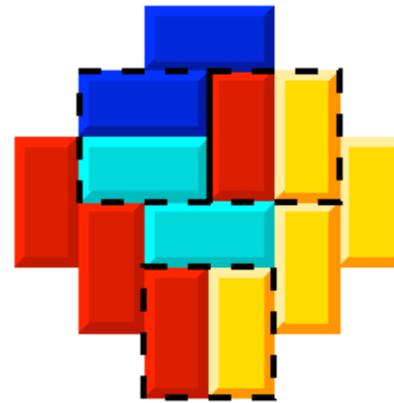
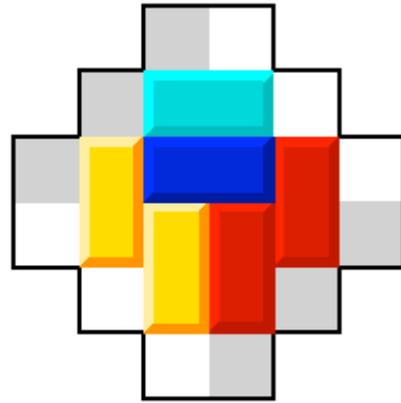
$$2^{(1+2+3+4+5+6)}$$



$$2^{21} = 2.097.152$$

$$2^{(1+2)}$$

$$2^3 = 8$$



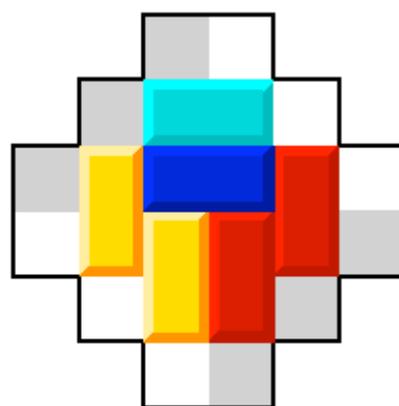
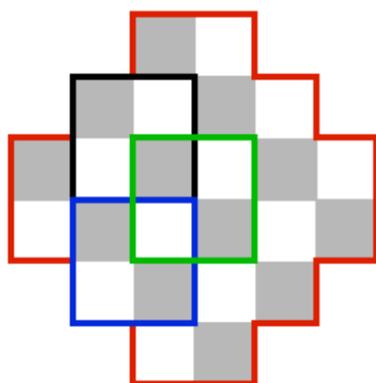
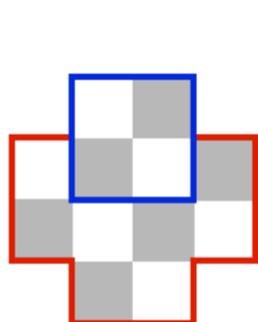
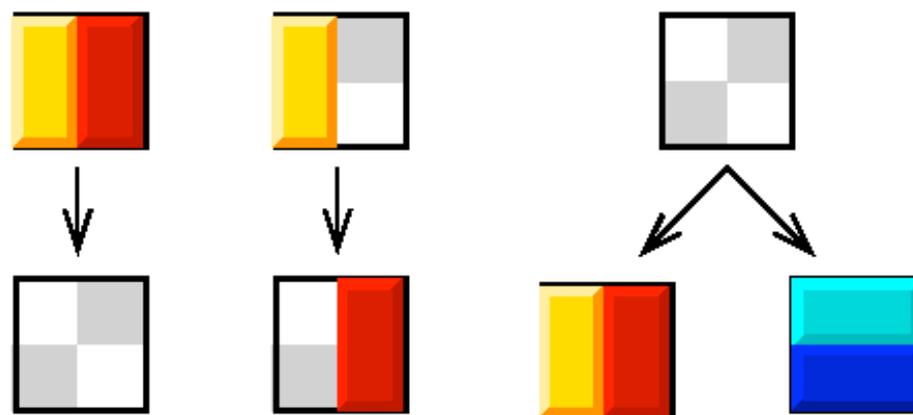
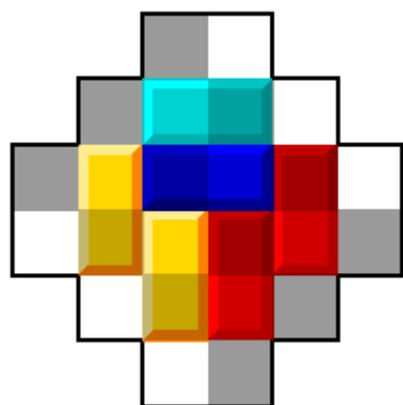
$$2^{(1+2+3)}$$

$$2^6 = 64$$

$$w = w_1 w_2 w_3$$

$$w_i = 0$$

$$w_i = 1$$



combinatoire bijective

«preuves sans mots»

«Le paradigme bijectif»



**Elise Janvresse** et **Thierry de la Rue** ,

« Pavages aléatoires par touillage de dominos » —

*Images des Mathématiques*, CNRS, 2009.

**<http://images.math.cnrs.fr/Pavages-aleatoires-par-touillage.html>**

**Elise Janvresse** et **Thierry de la Rue**,

« Pavages aléatoires par touillage de dominos » —

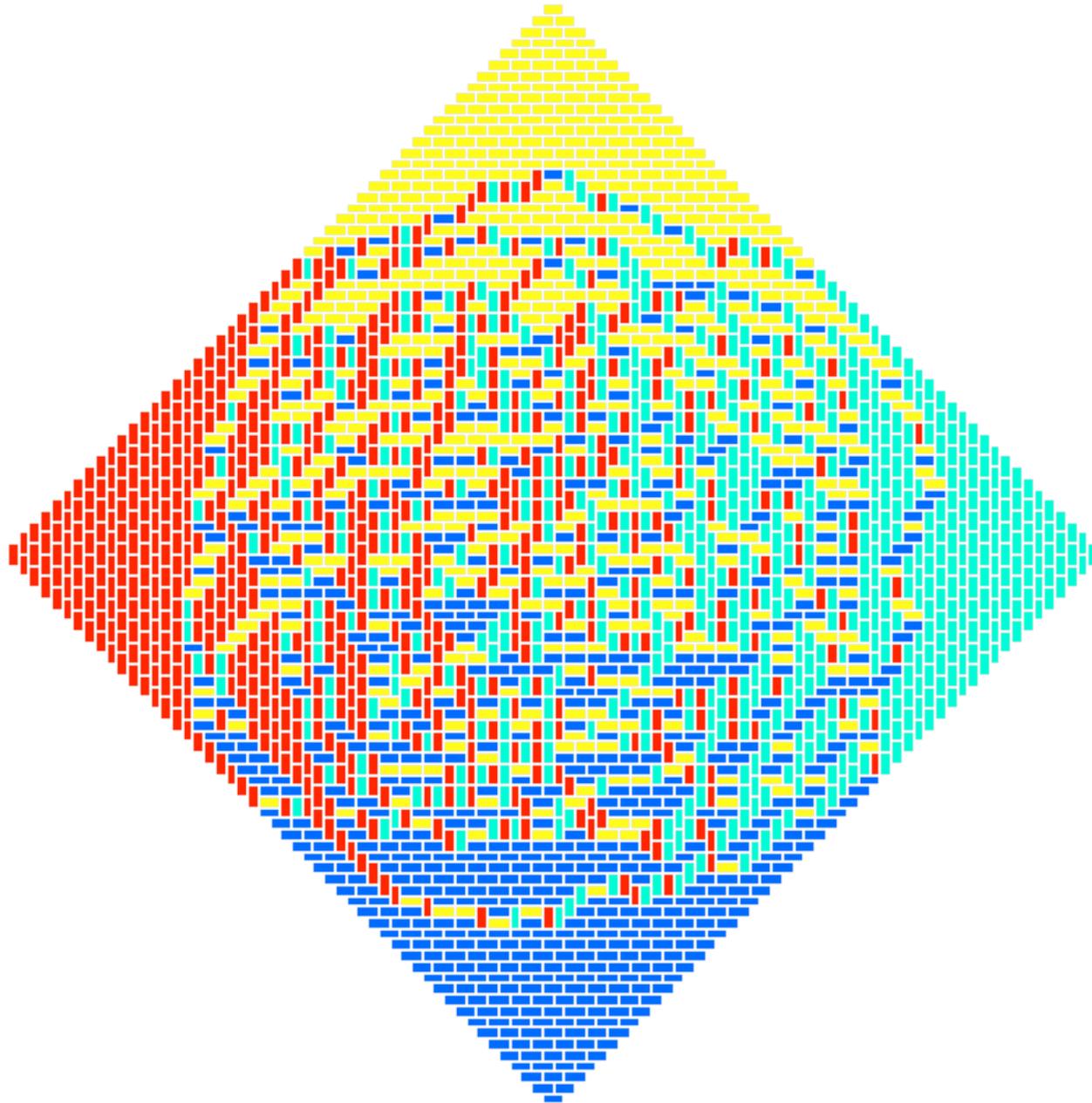
*Images des Mathématiques*, CNRS, 2009.

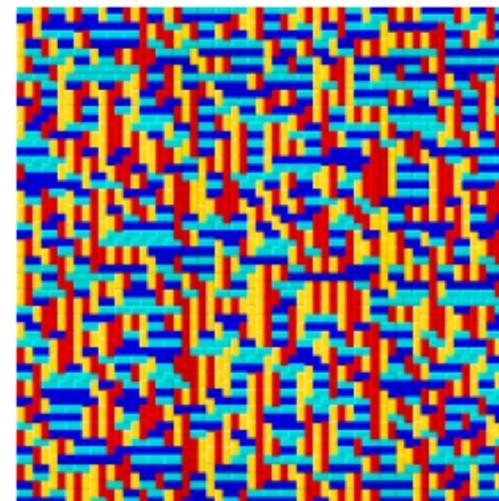
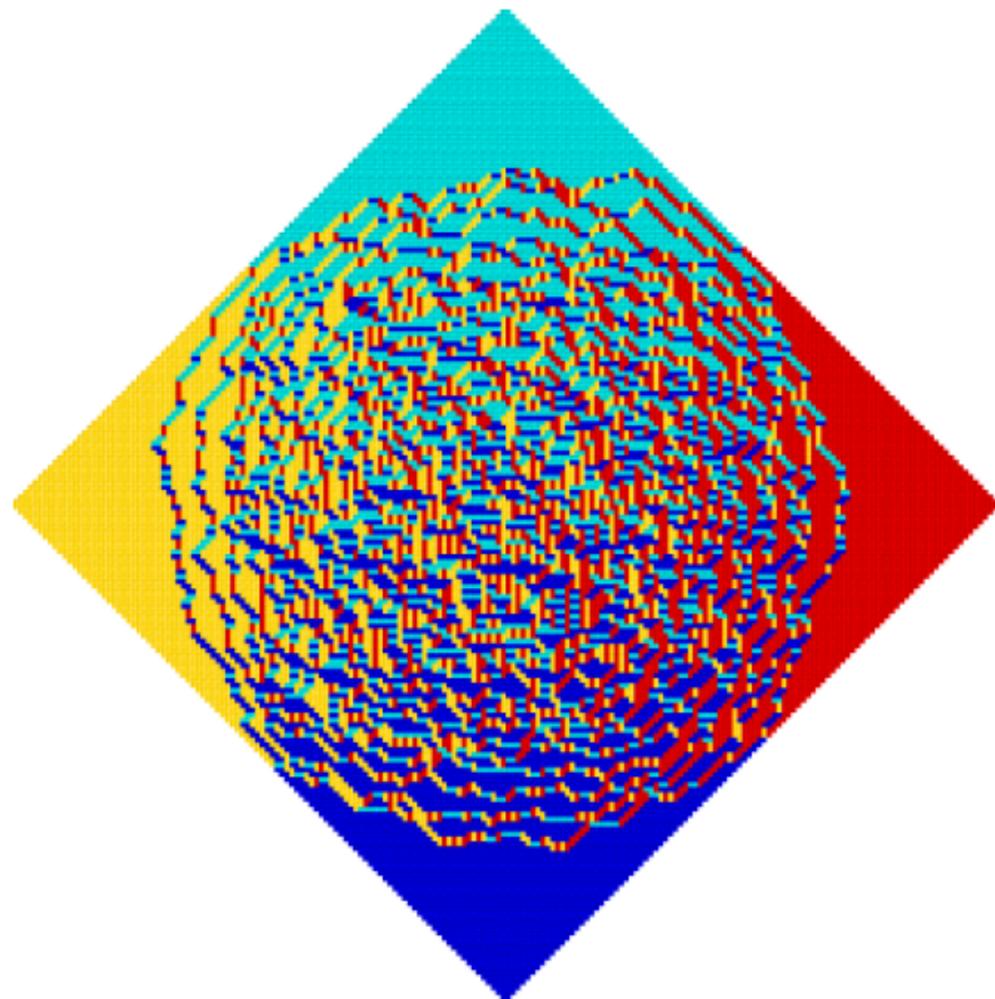
**<http://images.math.cnrs.fr/Pavages-aleatoires-par-touillage.html>**

random

Aztec

tilings





### **Effets de bords**

À gauche : un pavage typique du diamant aztèque d'ordre 100, sur lequel on observe le phénomène du cercle arctique. À droite : un pavage typique du carré de côté 60.

