The beauty of mathematics

part IV: Ramanujan identities and Ramanujan continued fraction with heaps of dimers

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§7 Ramanujan identities

Ramanujan's home
Sarangapani Street
Kumbakonam

Rogers - Ramanujan identities

R_{I}

$$\sum_{n \geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i \equiv 1,4 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

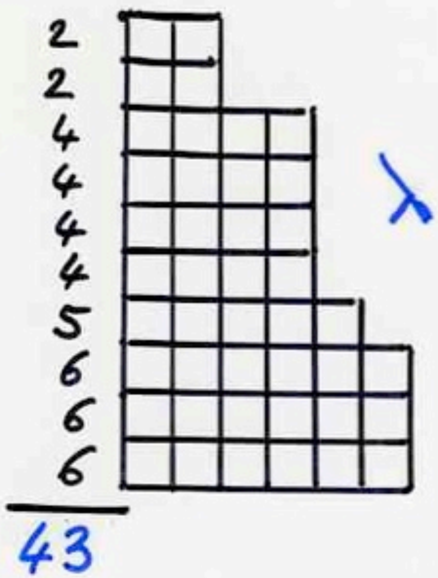
R_{II}

$$\sum_{n \geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i \equiv 2,3 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

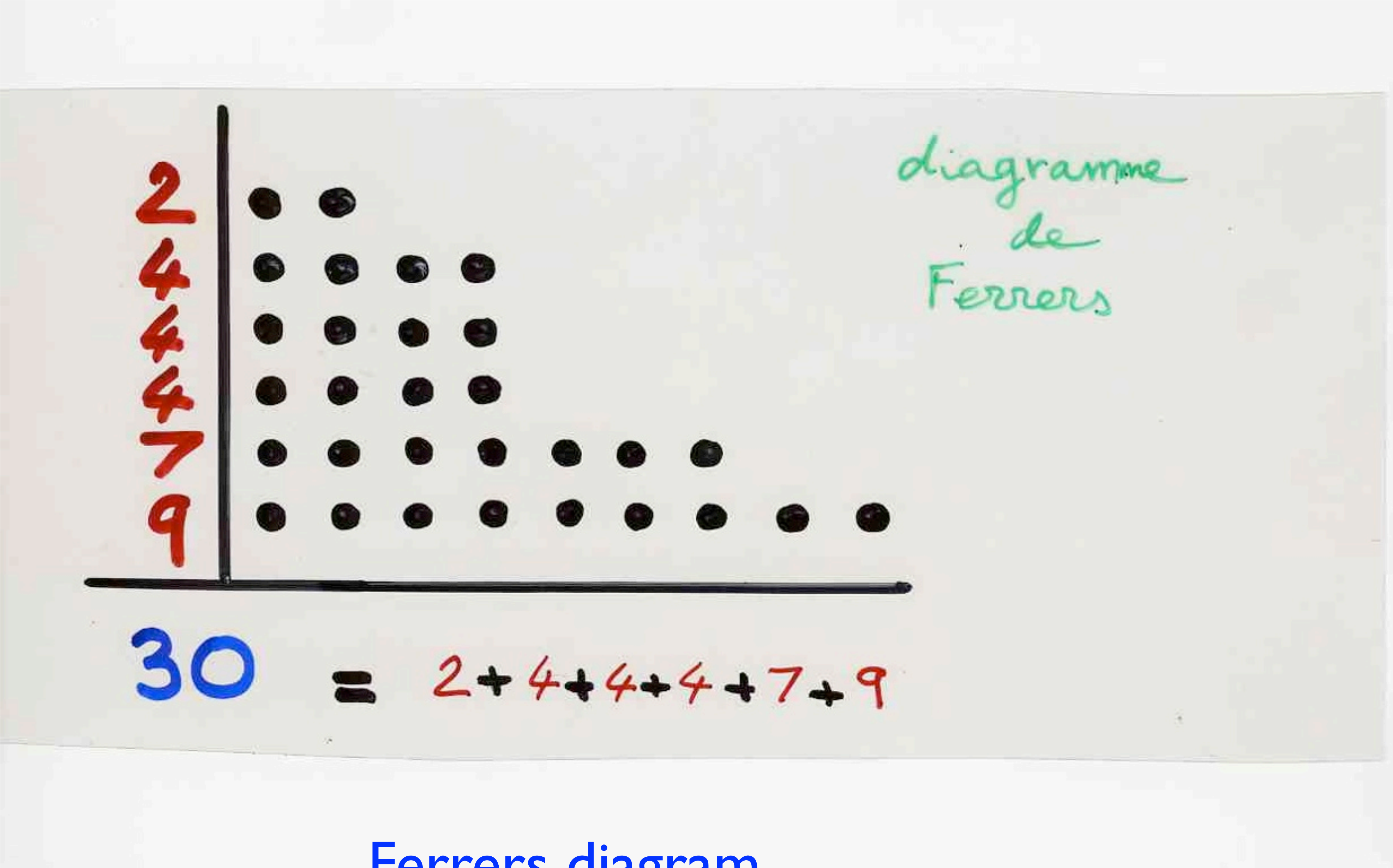
partition of an integer n

$$\lambda = (6, 6, 6, 5, 4, 4, 4, 4, 2, 2)$$

$$n = 43 = 6+6+6+5+4+4+4+4+2+2$$



Ferrers diagram



Ferrers diagram

1	1+1	1+1+1	1+1+1+1	1+1+1+1+1
	2	2+1	2+1+1	2+1+1+1
		3	3+1	2+2+1
			2+2	3+1+1
			4	3+2
				4+1
				5

$$1 \;,\; 2 \;,\; 3 \;,\; 5 \;,\; 7$$

$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$$

Rogers - Ramanujan identities

$$R_I \quad \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

$$1+a_1q+a_2q^2+a_3q^3+....+a_nq^n+...$$

$$=$$

$$1+b_1q+b_2q^2+b_3q^3+....+b_nq^n+...$$

$$a_n = b_n$$

Rogers - Ramanujan identities

$$R_I \quad \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i \equiv 1,4 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

example: $a_9 = b_9$

D-partitions

$$\begin{cases} 9 \\ 8+1 \\ 7+2 \\ 6+3 \\ 5+3+1 \end{cases}$$

partitions

parts $\equiv 1,4$ mod 5

$$\begin{cases} 9 \\ 4+4+1 \\ 6+1+1+1 \\ 4+1+1+1+1+1 \end{cases} \quad \begin{cases} 1+\dots+1 \end{cases}$$

Partition ayant au plus n parts

$$0 \le (\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n)$$

$\Longleftrightarrow$

D-partition ayant exactement n parts

$$(1+\lambda_1, 3+\lambda_2, \dots, (2n-1)+\lambda_n)$$

R_{II} $$\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

D-partitions

parts $\neq 1$

Partitions

parts $\equiv 2, 3$ mod 5

$$\begin{cases} 7+2 \\ 6+3 \\ 9 \end{cases}$$

$$\begin{cases} 2+2+2+3 \\ 3+3+3 \\ 7+2 \end{cases}$$

§8 combinatorial interpretation of an identiy:

from "calculus" to
"visual" combinatorial objects

1	1+1	1+1+1	1+1+1+1	1+1+1+1+1
	2	2+1	2+1+1	2+1+1+1
		3	3+1	2+2+1
			2+2	3+1+1
			4	3+2
				4+1
				5

$$1, \quad 2, \quad 3, \quad 5, \quad 7$$

$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$$

1 2 3 5 7

$$1q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \dots$$

$$1 + 1q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \ldots$$

série génératrice

des partitions (d'entiers)

$$\sum_{n \geq 0} a_n q^n$$

generating function

for (integer) partitions

formal power series

algebra of polynomials

$$(1+t)^2 = 1 + 2t + t^2$$

$$(1+t)^3 = 1 + 3t + 3t^2 + t^3$$

formal power series

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots + t^n + \ldots$$

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n$$
$$+ \dots$$

$$\left(1 + t + t^2 + \dots + t^n + \dots\right)$$
$$\times (1 - t)$$

exercise

$$\frac{1}{1-(t+t^{2})} = ?$$

$$\frac{1}{1-(t+t^2)} = \ ?$$

$$= 1 + t + 2t^2 + 3t^3 + 5t^4$$
$$+ 8t^5 + 13t^6 + 21t^7$$
$$+ 34t^8 + 55t^9 + \ldots$$

$$\sum_{i \geq 0} (t + t^2)^i =$$

$$1 + (t + t^2)$$

$$(t^2 + 2t^3 + t^4)$$

$$(t^3 + 3t^4 + 3t^5 + t^6)$$

$$(t^4 + 4t^5 + 6t^6 + \dots$$

$$+ (t^5 - - - -$$

$$\ddots$$

$$\sum_{i \geq 0} (t + t^2)^i =$$

$$1 + (t + t^2)$$

$$(t^2 + 2t^3 + t^4)$$

$$(t^3 + 3t^4 + 3t^5 + t^6)$$

$$(t^4 + 4t^5 + 6t^6 + \dots$$

$$+ (t^5 \cdots$$

1 2 3 5 8

$$F_{n+1} = F_n + F_{n-1}$$

$$F_0 = F_1 = 1$$

Fibonacci

- $$\frac{1}{1-t-t^2} = \sum_{n \geqslant 0} F_n t^n$$

- $$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

(formule de Binet)

- $$F_n = F_{n-1} + F_{n-2}$$

$$t + t + t + \ldots t + \ldots$$

$$1 + 1 + 1 + \ldots$$

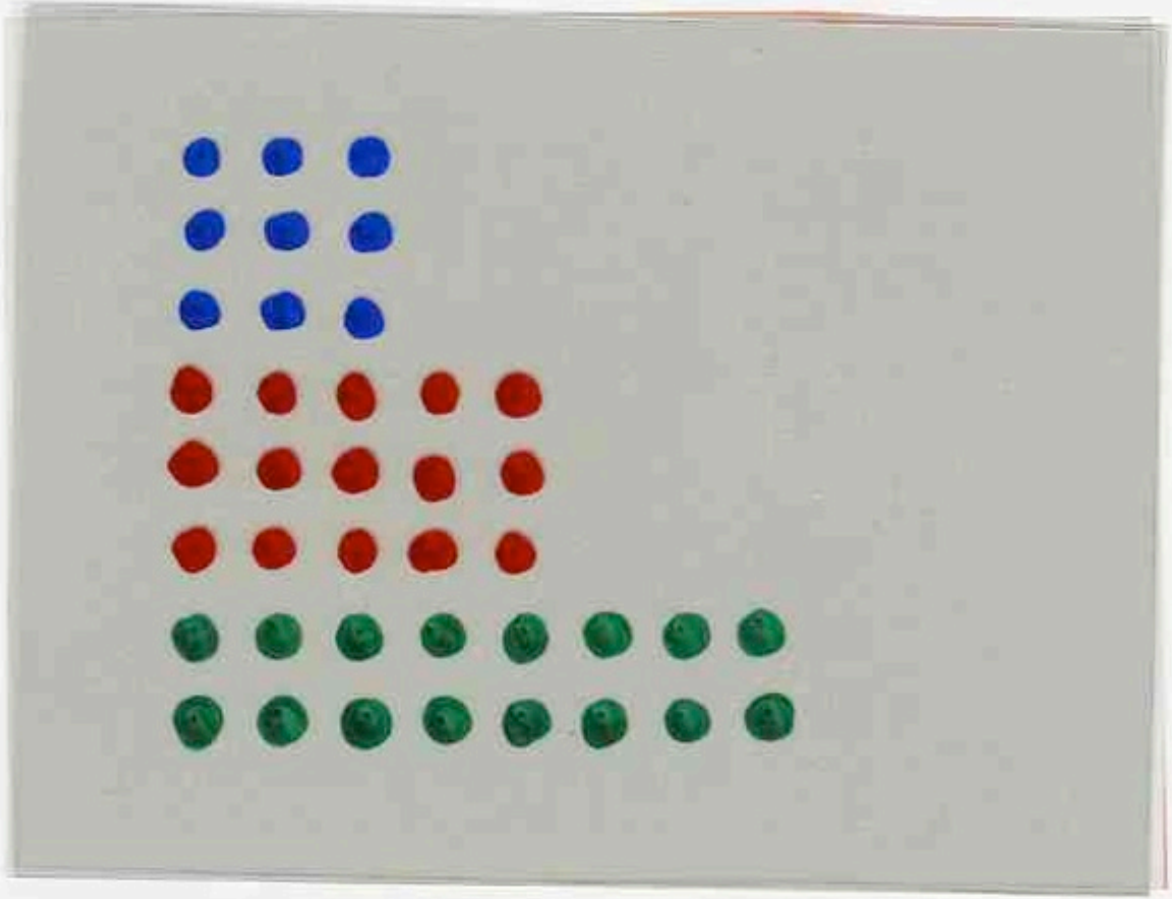
$$\xcancel{t + t + t + \ldots t + \ldots}$$

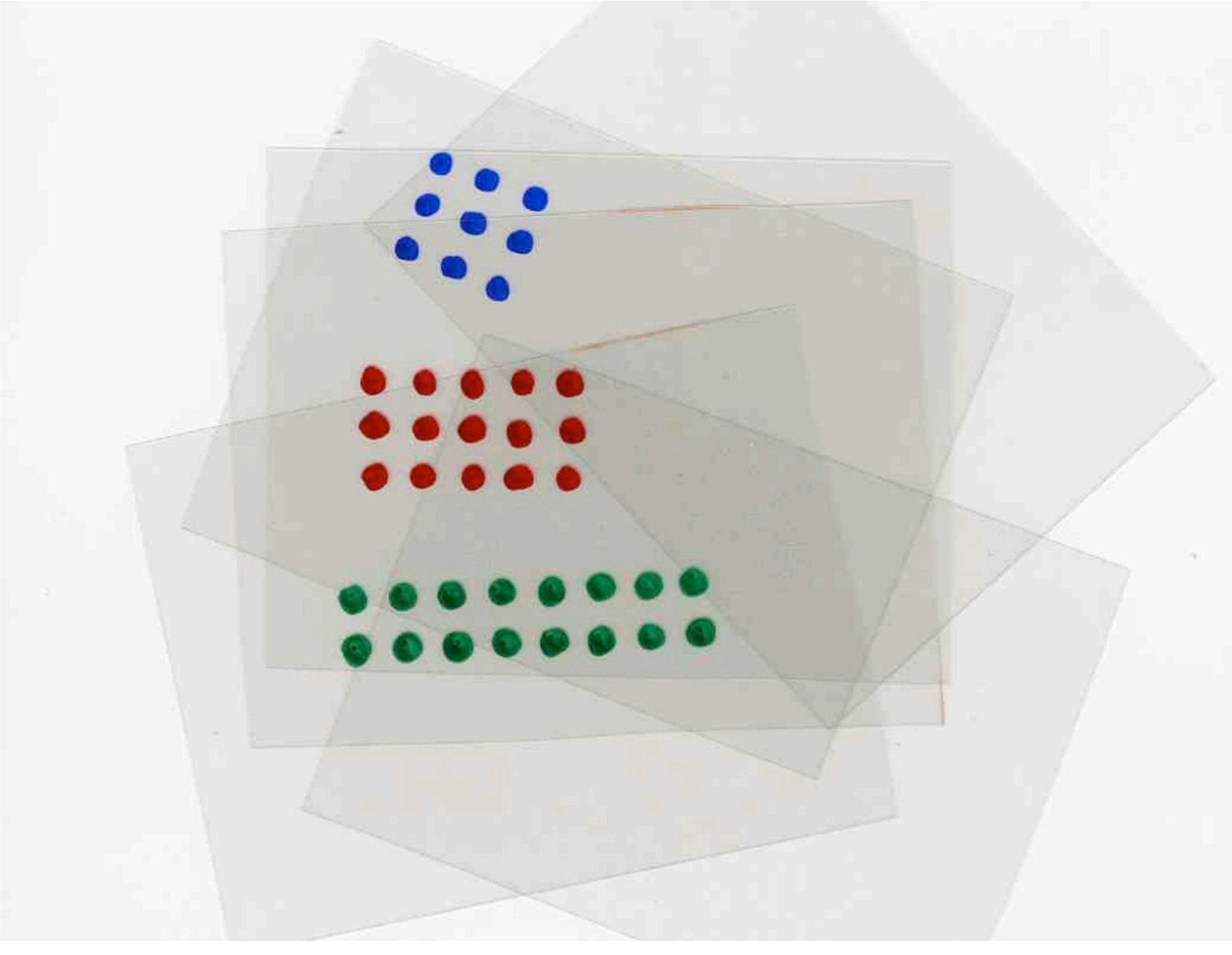
$$\xcancel{1 + 1 + 1 + \ldots}$$

generating function for partitions

$$\underbrace{\bullet\ \bullet\ \bullet\ \bullet\ \bullet}_{i} \qquad\qquad q^{i}$$

$$\frac{1}{1-q^{i}}$$





$$\frac{1}{(1-q)(1-q^2)\cdot\dots\ (1-q^m)}$$

$$\frac{1}{(1-q)(1-q^2)\cdot\cdots(1-q^m)}$$

$$\prod_{i\geq 1}\frac{1}{(1-q^i)}$$

Rogers - Ramanujan identities

$$= \prod_{\substack{i \equiv 1,4 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

partitions

parts $\equiv 1, 4$ mod 5

$$\begin{cases} 9 \\ 4+4+1 \\ 6+1+1+1 \\ 4+1+1+1+1+1 \end{cases} \quad \begin{cases} 1+\dots+1 \end{cases}$$

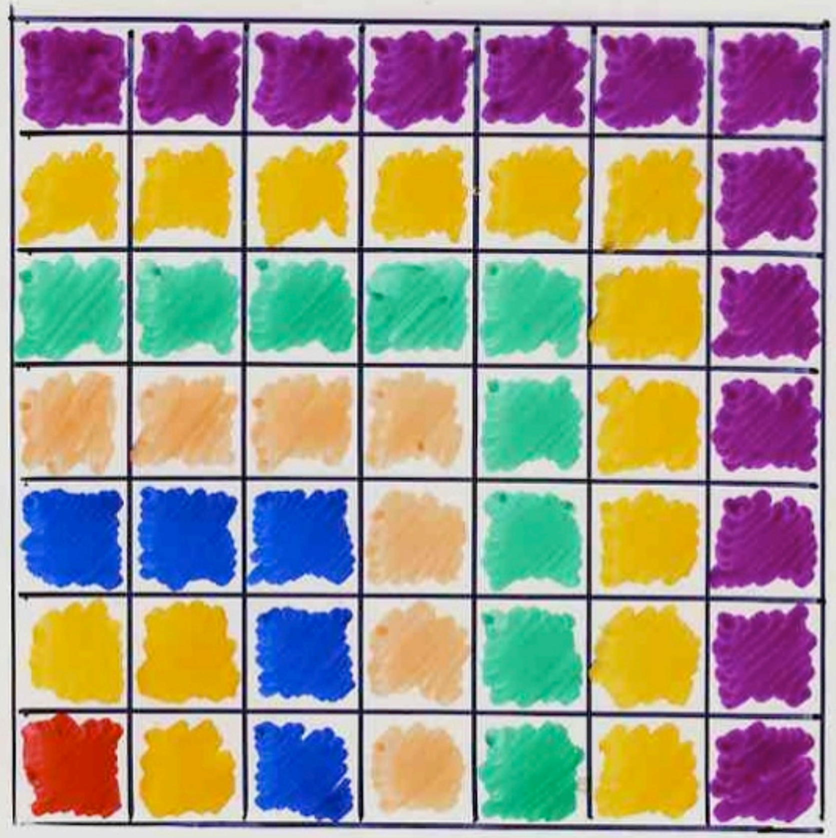
R_{II} $$\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

Partitions

parts $\equiv 2, 3$ mod 5

$$\left\{\begin{array}{l} 2+2+2+3 \\ 3+3+3 \\ 7+2 \end{array}\right.$$

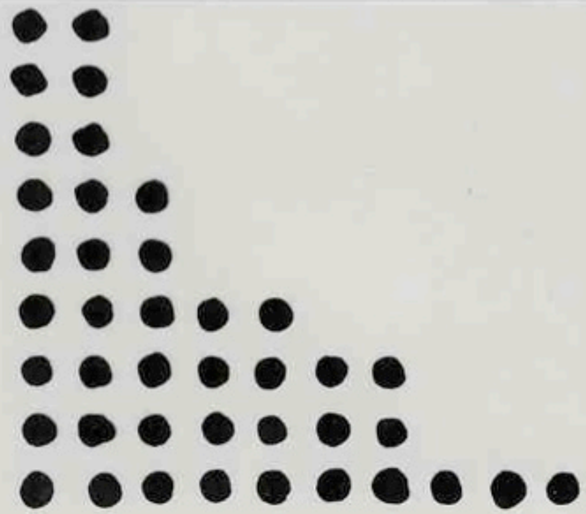
§9 bijective proof of an identity



$$n^2 = 1 + 3 + \ldots + (2n-1)$$

$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1} \frac{1}{(1-q^i)}$$

$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} =$$

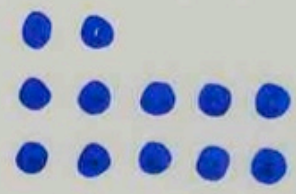


$$\prod_{i\geq 1} \frac{1}{(1-q^i)}$$

$$\sum_{m\geq 1} \frac{\phantom{q^{m^2}}}{[(1-q)(1-q^2)\cdots(1-q^m)]^2} =$$

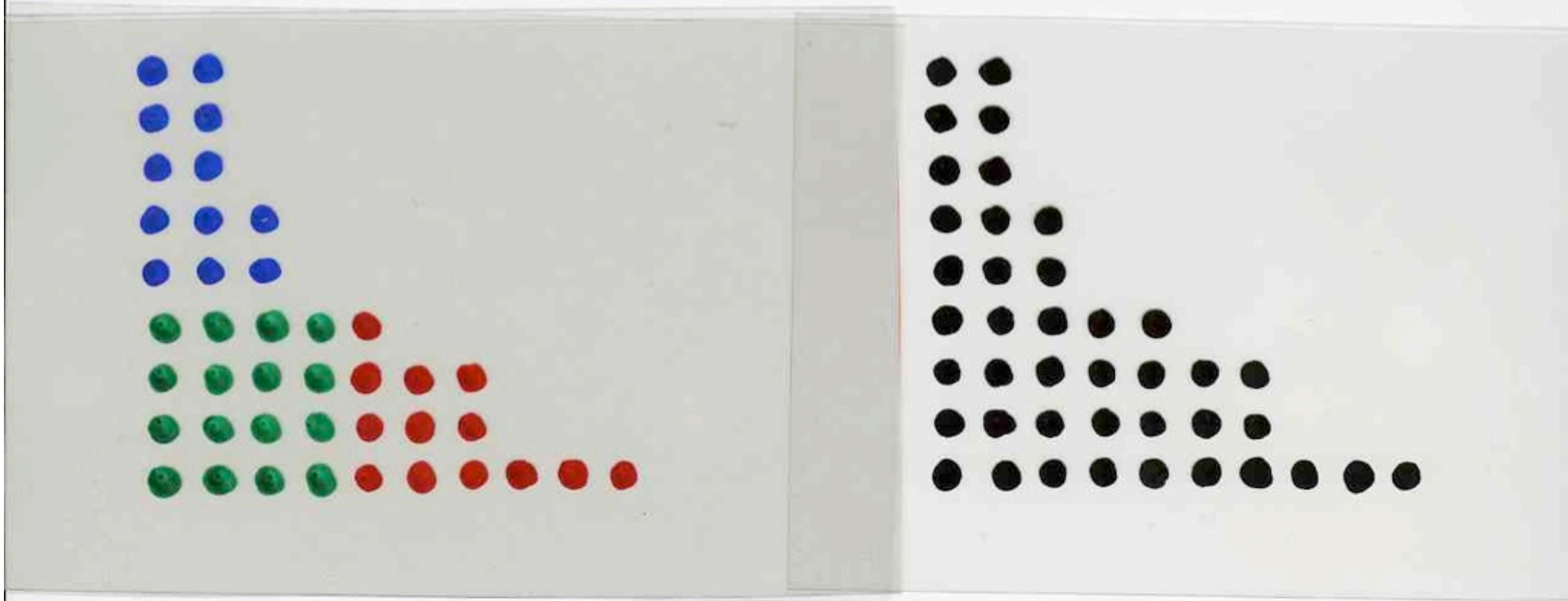
$$q^{m^2}$$

$$\sum_{m\geq 1} \frac{\qquad}{[\qquad]^2} =$$



$$(1-q)(1-q^2)\cdots(1-q^m)$$

$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1} \frac{1}{(1-q^i)}$$



The "bijective" paradigm

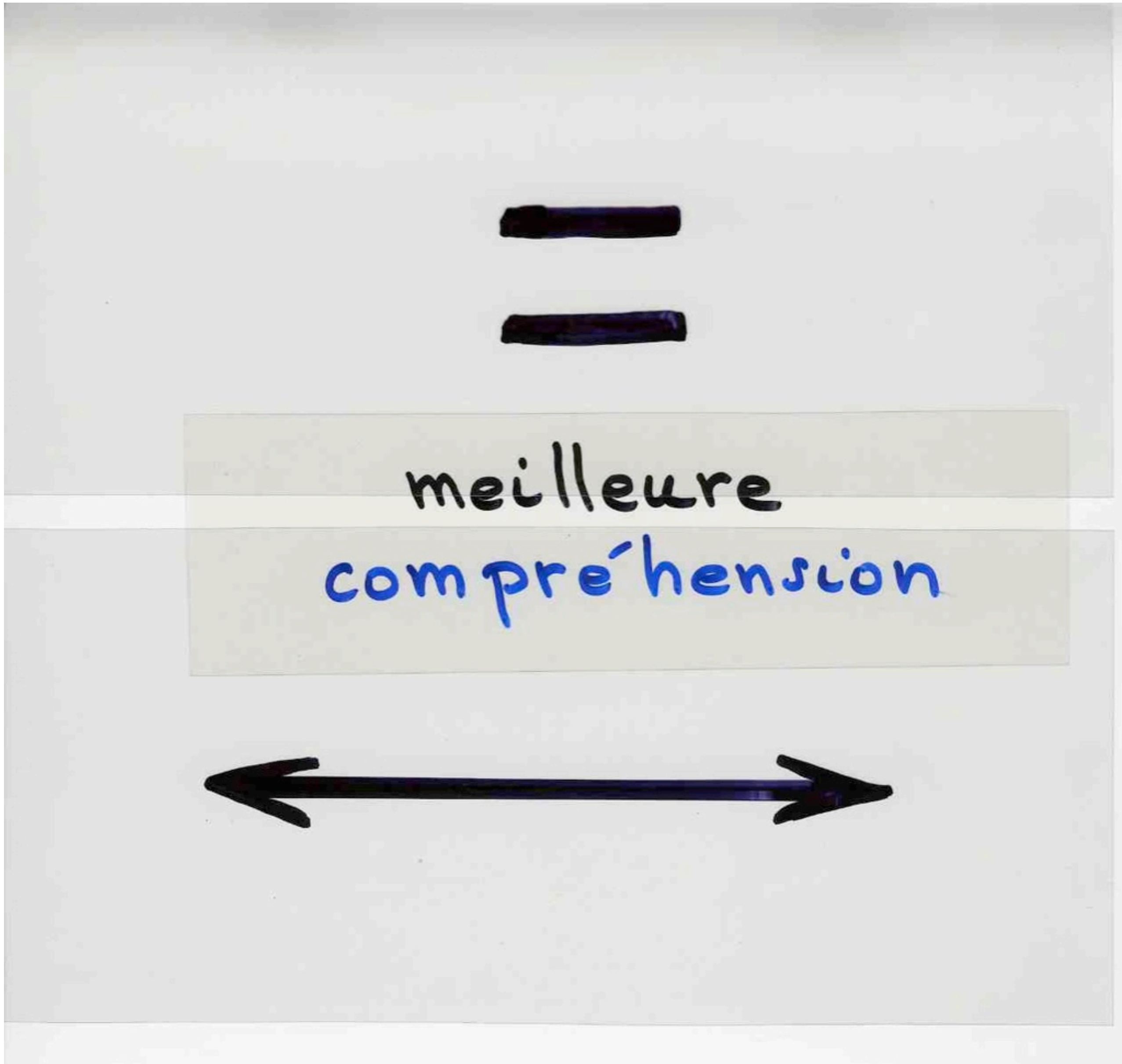
"drawing calculus"

each "pieces" of an identity

$\updownarrow$

combinatorial object

identities $\longleftrightarrow$ correspondences
combinatorial construction
bijections



better understanding

§10 Ramanujan continued fraction

Srinivasa Ramanujan

$$\cfrac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{1+\cdots}}}} = e^{\frac{2\pi}{5}}\left(\left(\frac{5+\sqrt{5}}{2}\right)^{1/2}-\frac{1+\sqrt{5}}{2}\right)$$

....

(1914) G. H. Hardy

" Il suffisait d'un coup d'oeil pour se rendre compte qu'elles n'avaient pu être écrites que par un mathématicien de tout premier rang. Elles sont sûrement vraies, car si elles ne l'étaient pas, personne n'aurait pu avoir assez d'imagination pour les inventer ".

G.H. Hardy

"These theorems defeated me completely; I had never seen anything in the least like them before". Ramanujan's theorems "must be true, because, if they were not true, no one would have the imagination to invent them".

"La fraction continue" de Ramanujan

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots\cdots}}}}}}$$

fractions continues

arithmétiques

ϕ nombre d'or $\frac{1+\sqrt{5}}{2}$

$$t^2 - t - 1 = 0$$

$$\phi - 1 = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}$$

convergent

$$\frac{1}{1+\frac{1}{1+1}}\left.\vphantom{\frac{1}{1+\frac{1}{1+1}}}\right\}2 = \frac{2}{3}$$

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}\left.\vphantom{\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}}\right\}3 = \frac{3}{5}$$

$F_0 \quad F_1 \quad F_2 \quad F_3 \quad F_4 \quad F_5 \;, \ldots$

$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \;, \ldots$ Fibonacci

k ième convergent

$$\left.\frac{1}{1+\cfrac{1}{1+\cdots\cfrac{1}{1+1}}}\right\} k \;=\; \frac{F_k}{F_{k+1}}$$

Rogers - Ramanujan identities

$$R_{I} \quad \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \bmod 5}} \frac{1}{(1-q^i)}$$

$$R_{II} \quad \sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \bmod 5}} \frac{1}{(1-q^i)}$$

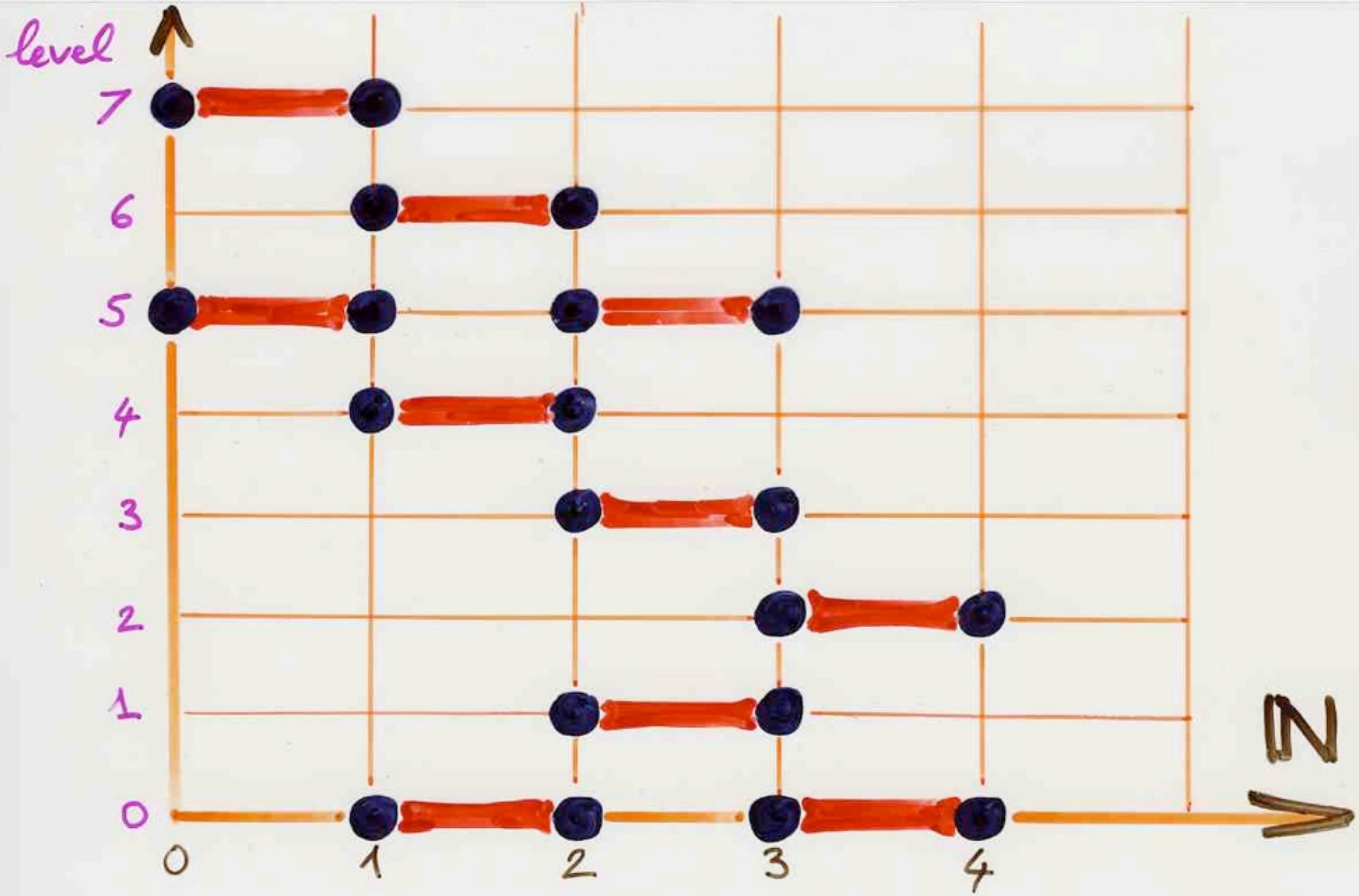
$$\frac{RR_2}{RR_1} = \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{\cdots}}}}}$$

fraction continue

de Ramanujan

"La fraction continue" de Ramanujan

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots\cdots}}}}}} = \frac{\displaystyle\sum_{n\geqslant 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\displaystyle\sum_{n\geqslant 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}$$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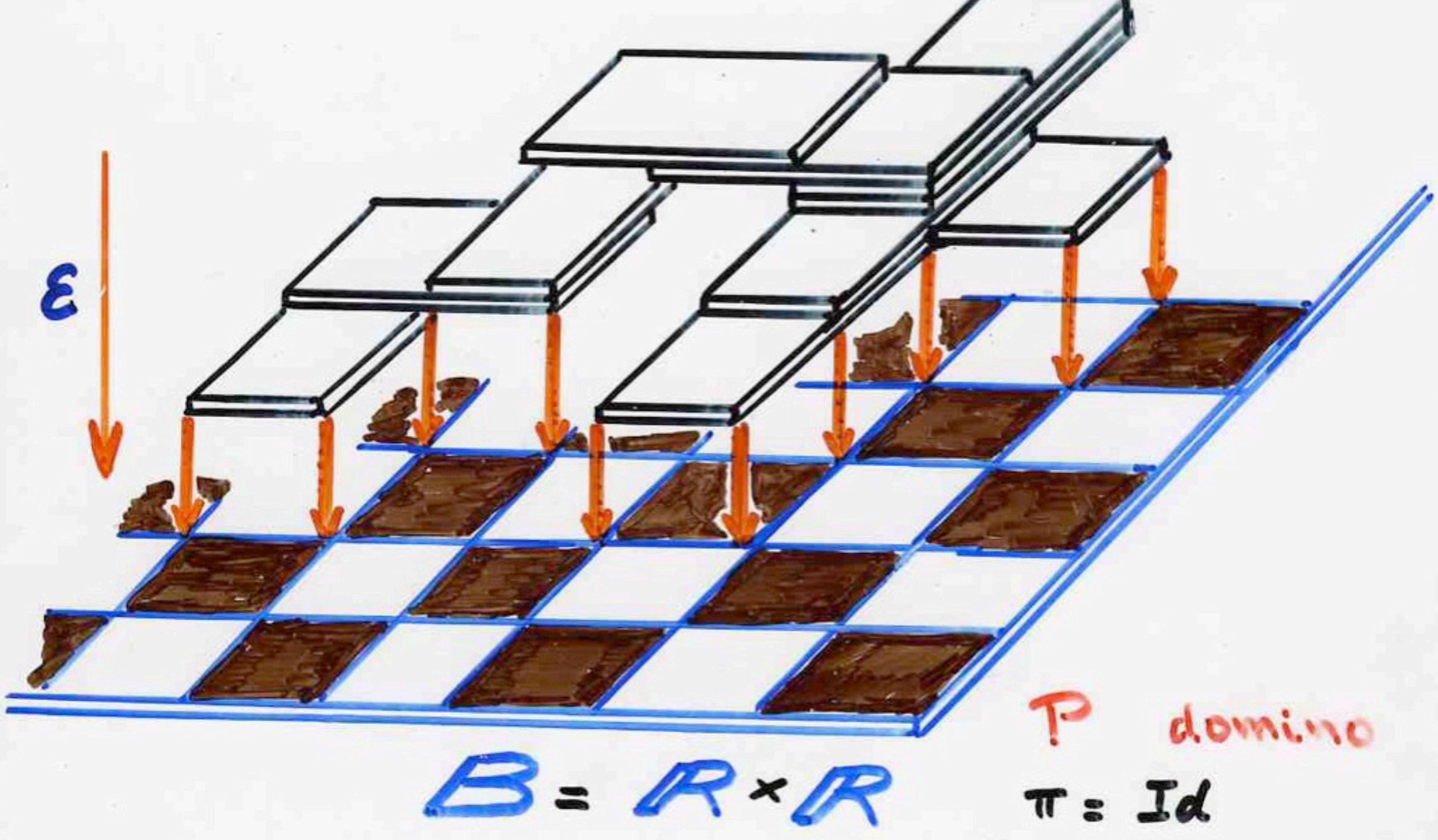
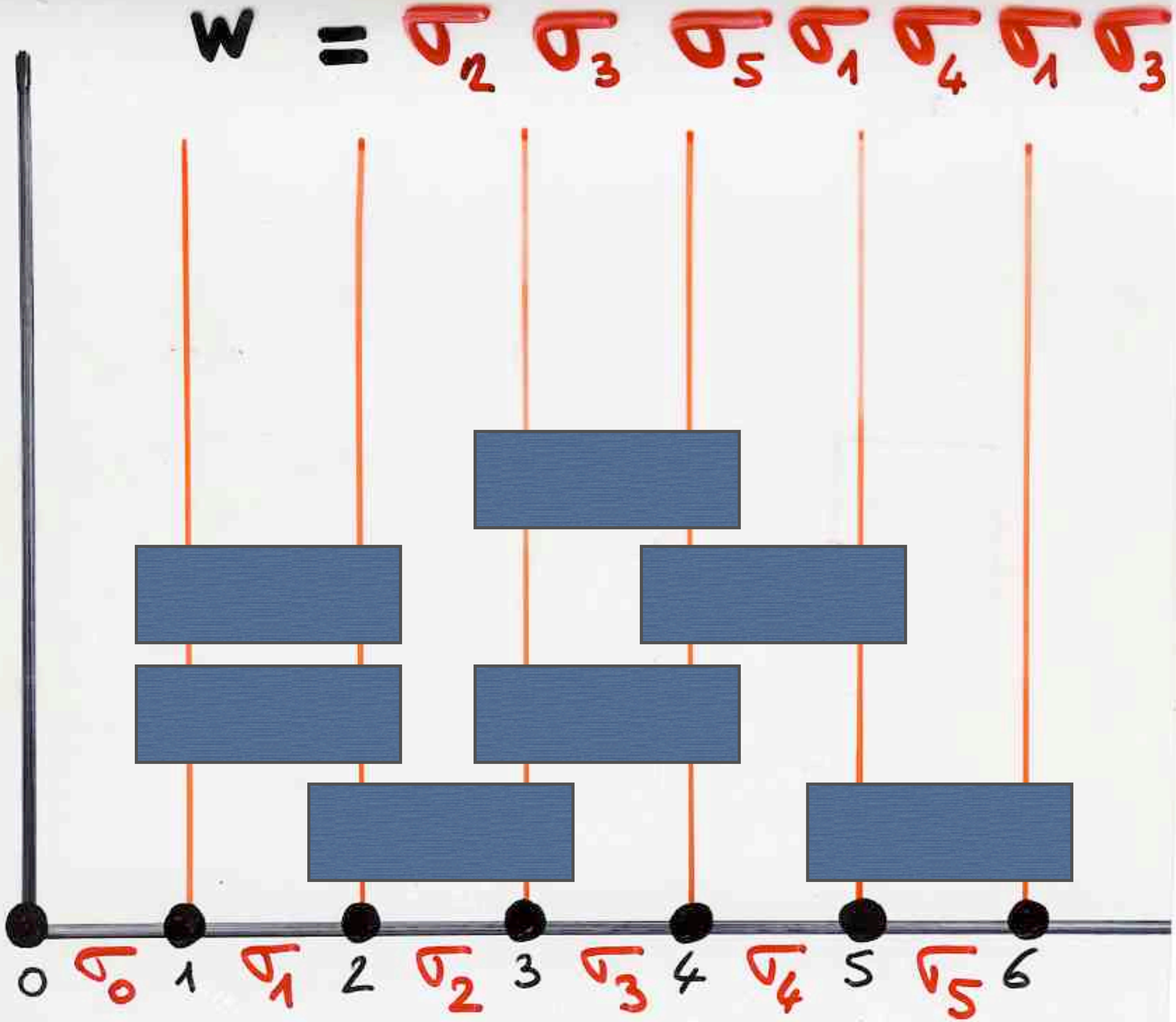
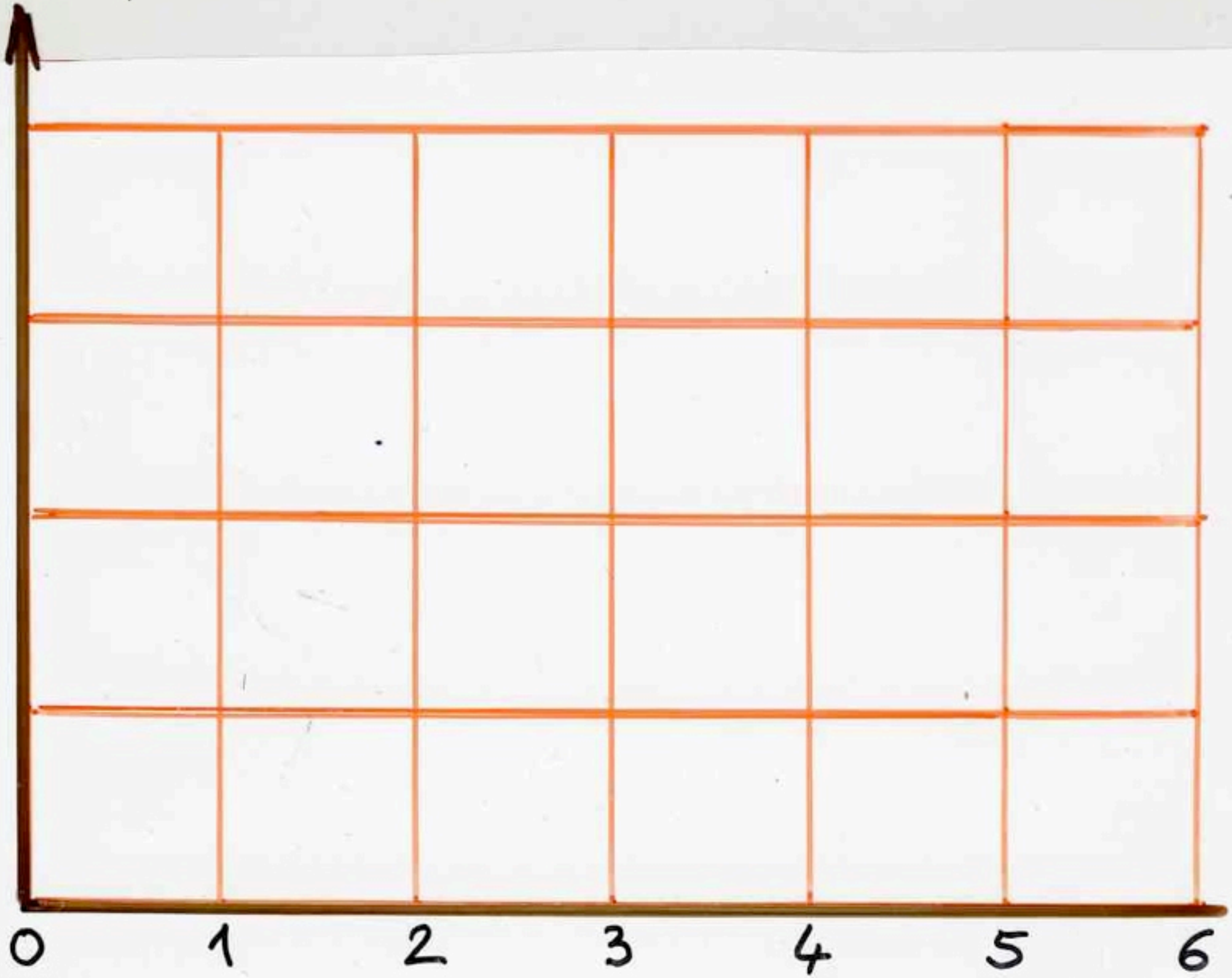
§11 Heaps of dimers



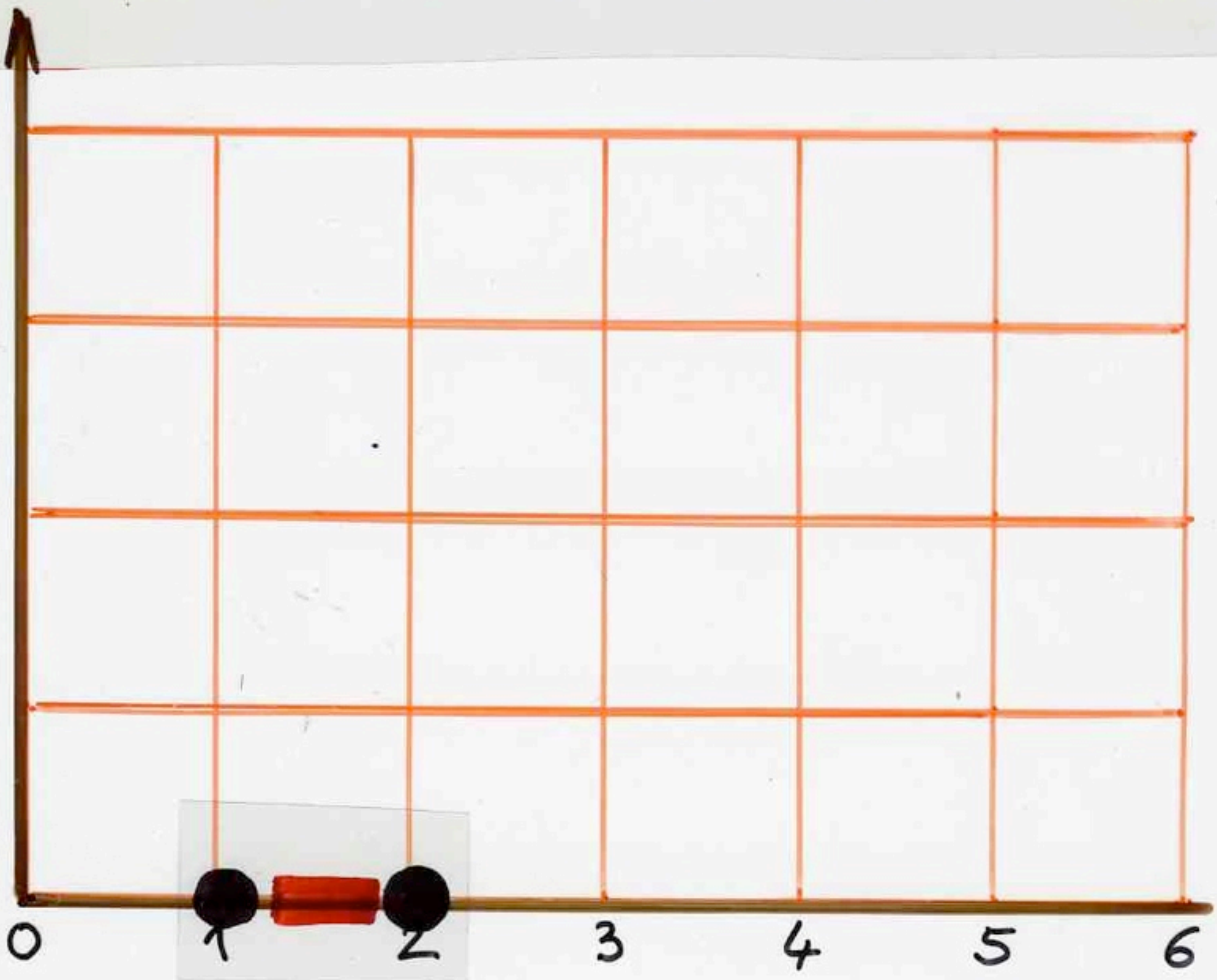
ε
P domino
B = R × R
π = Id


W = σ2 σ3 σ5 σ1 σ4 σ1 σ3
0 σ0 1 σ1 2 σ2 3 σ3 4 σ4 5 σ5 6

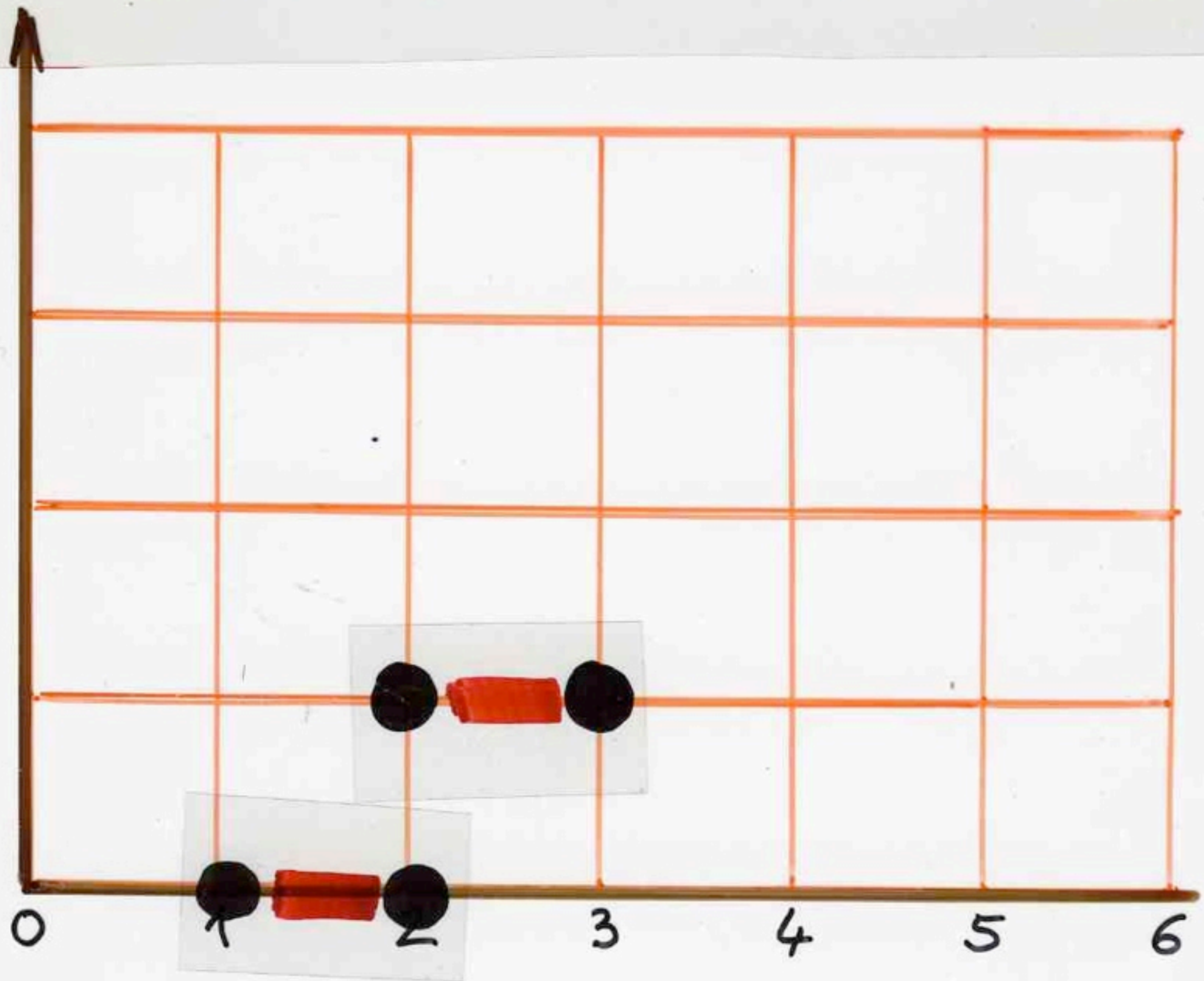
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



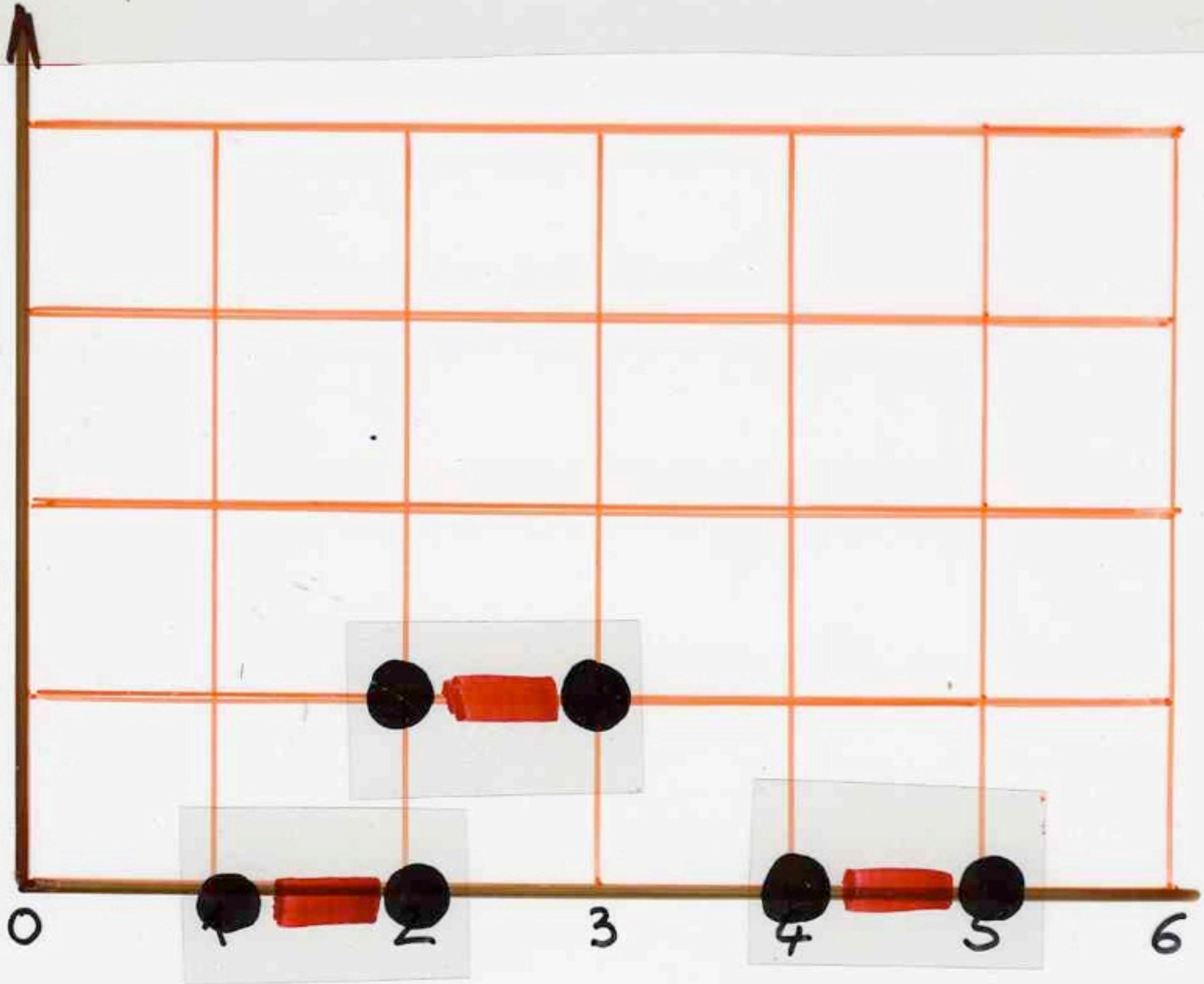
$$W = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



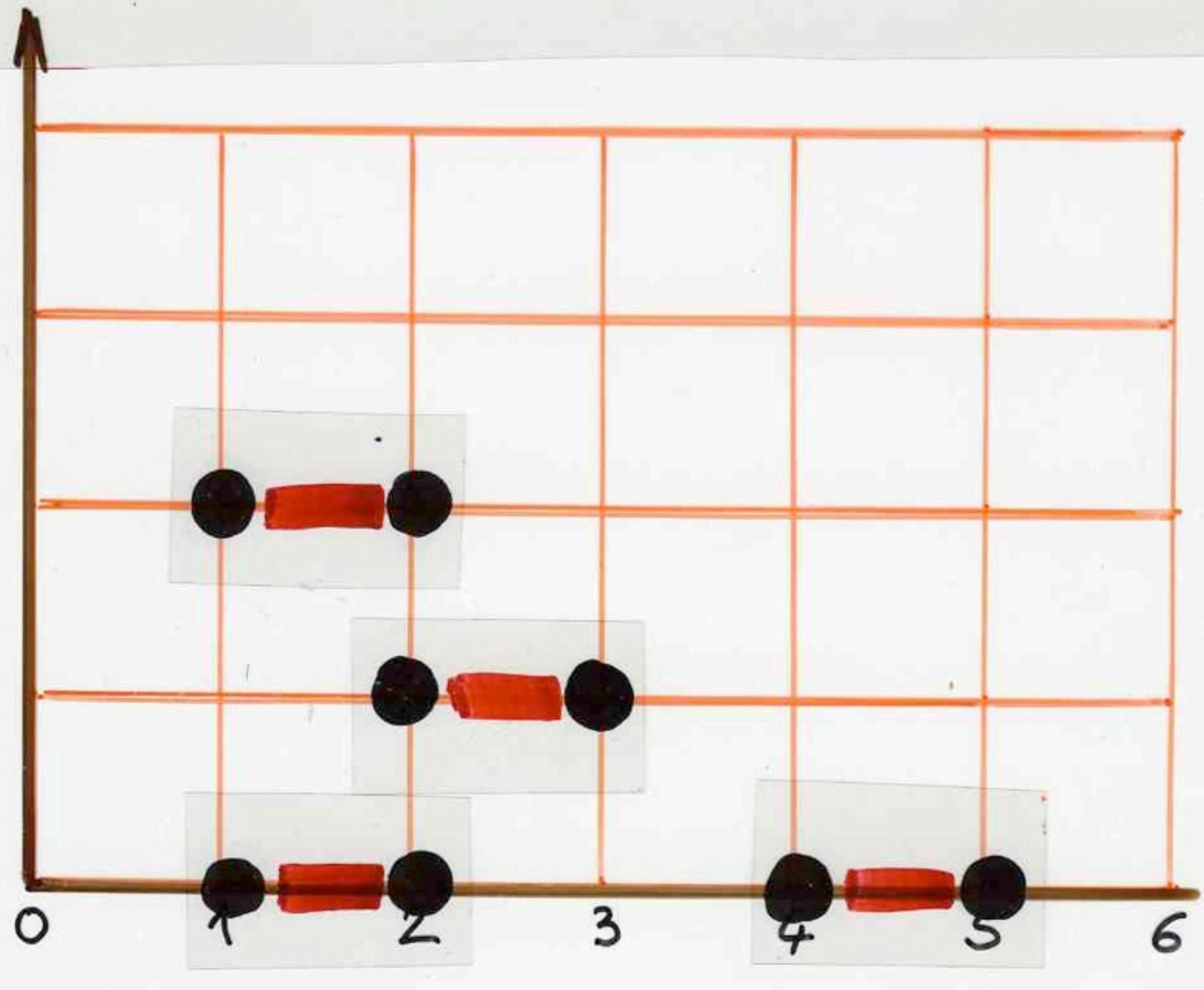
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



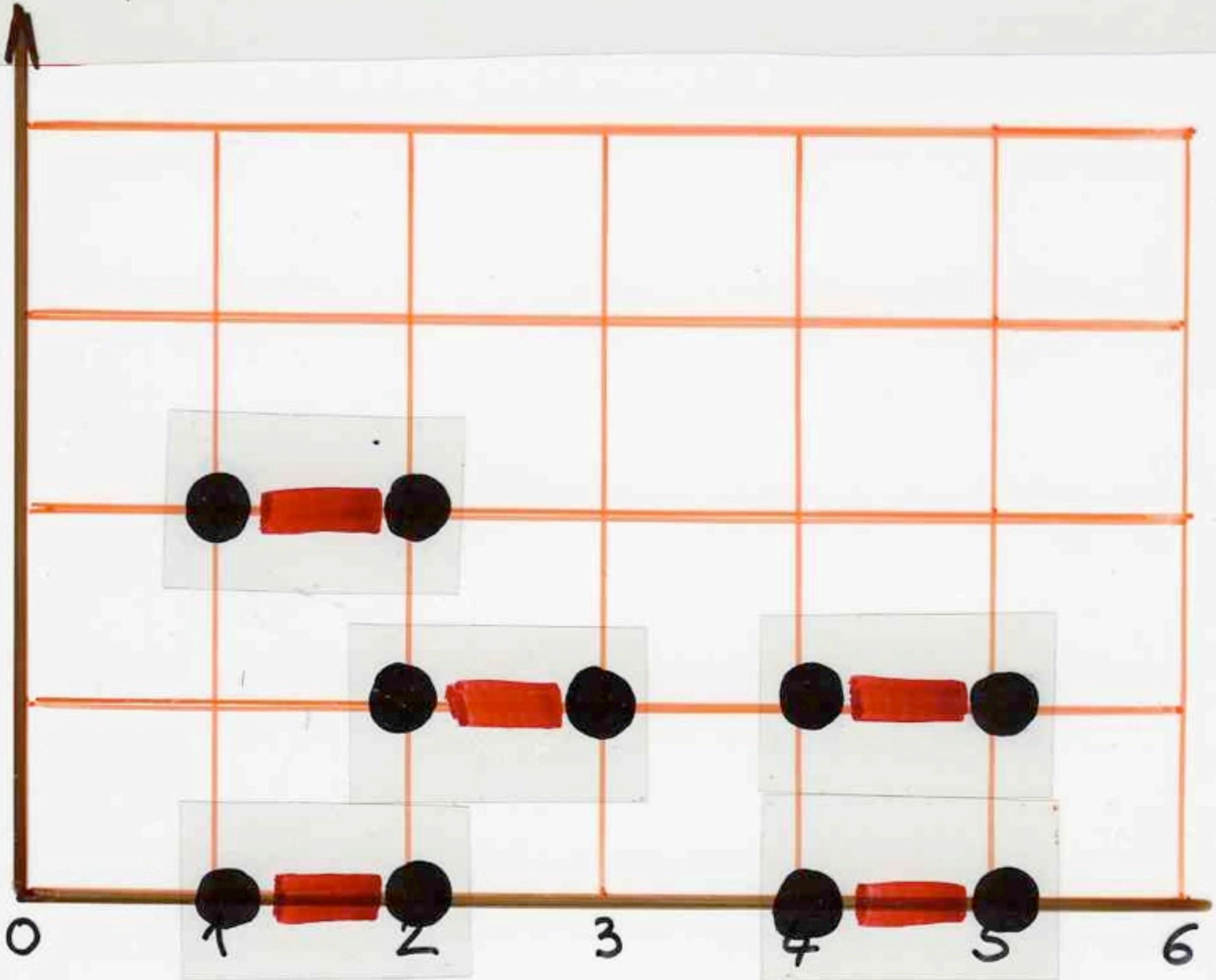
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



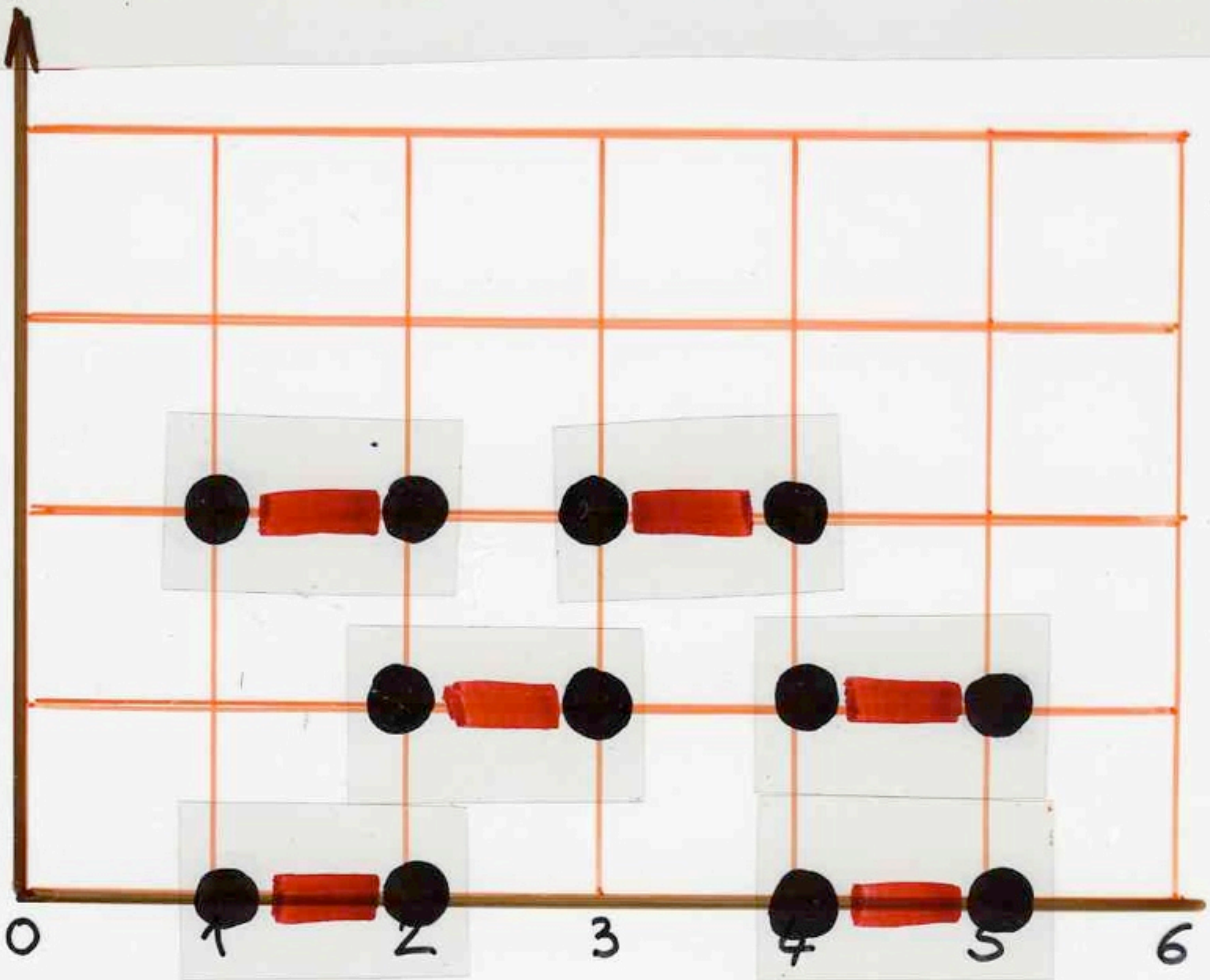
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



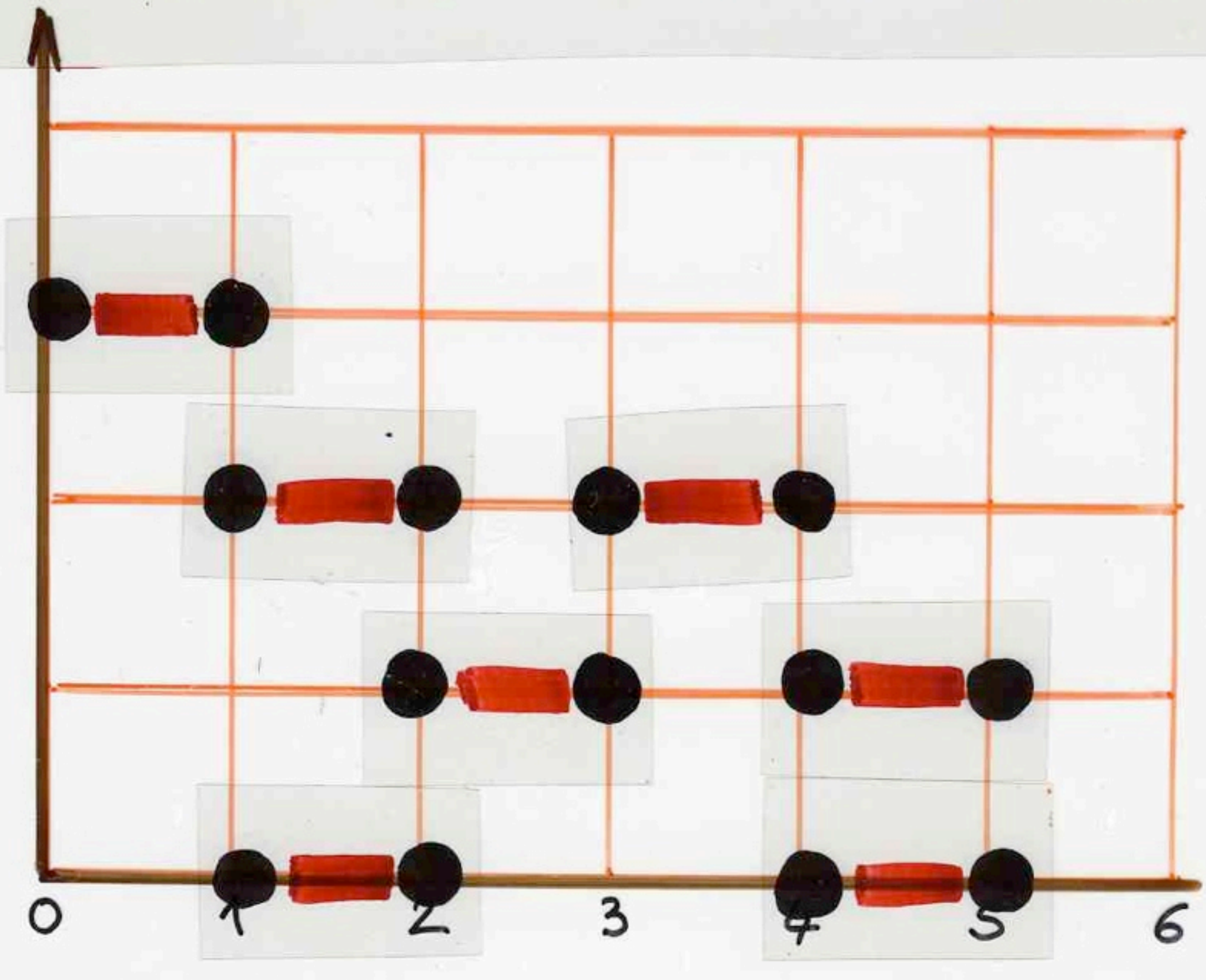
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



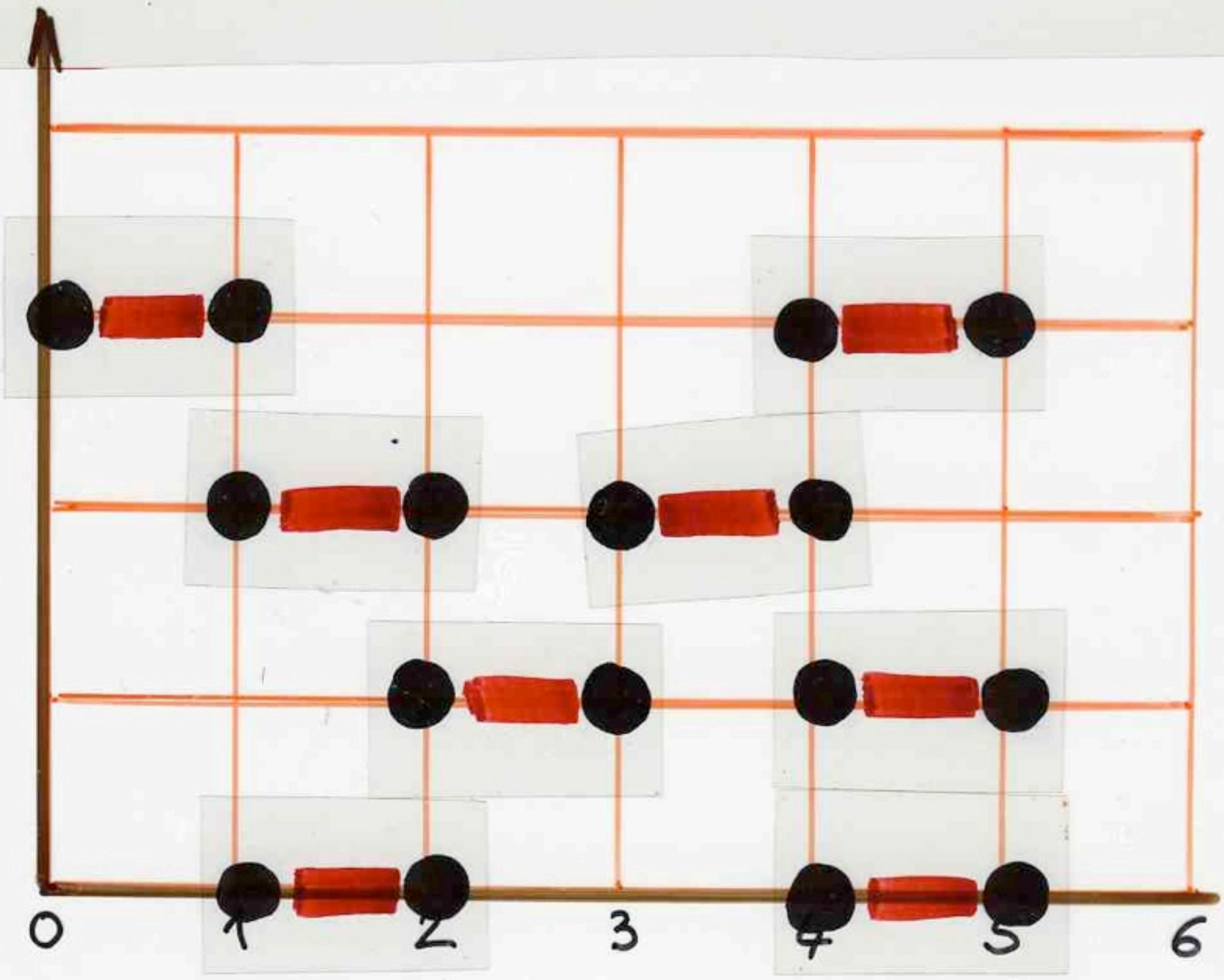
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$

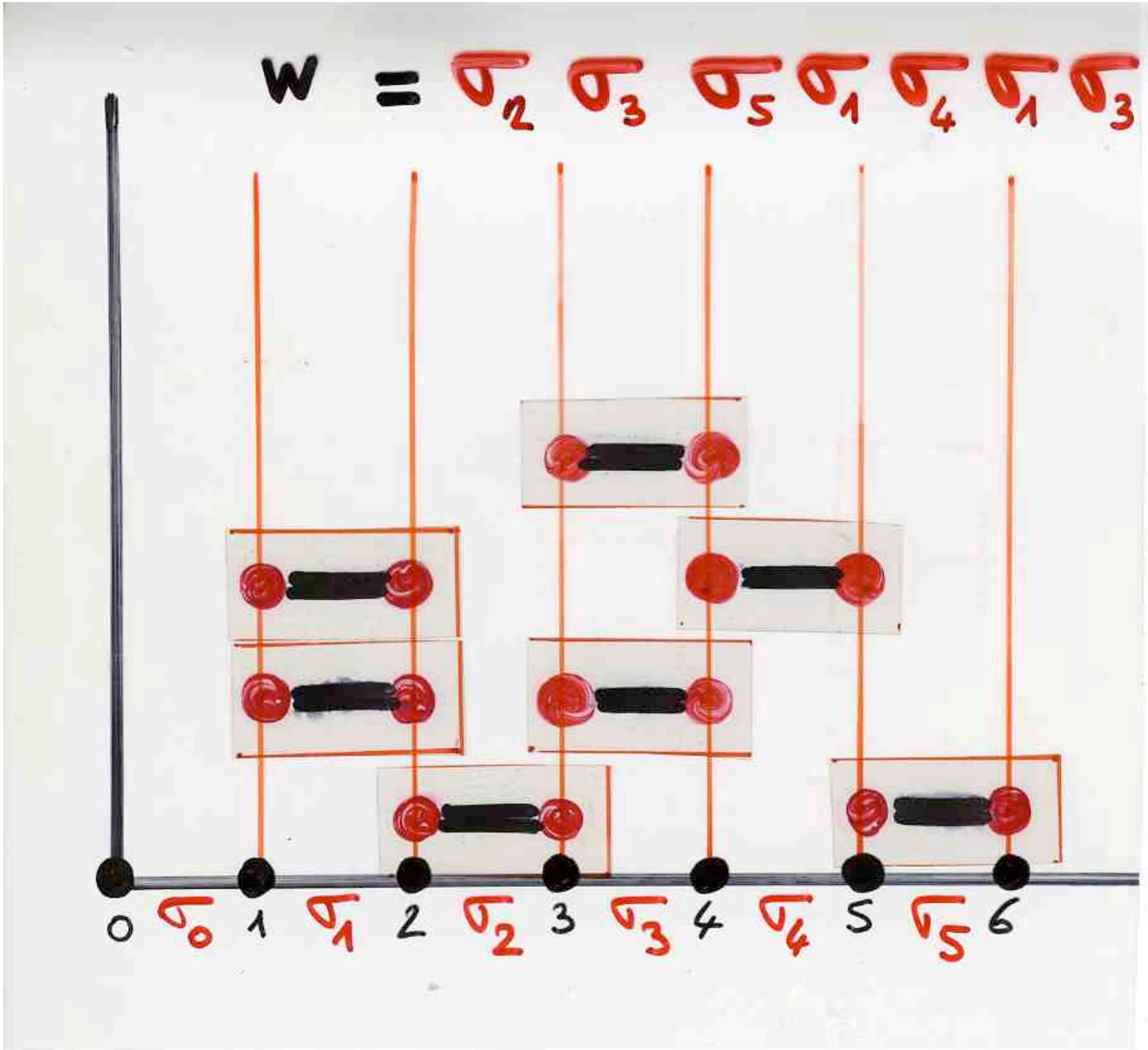


$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$



$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$

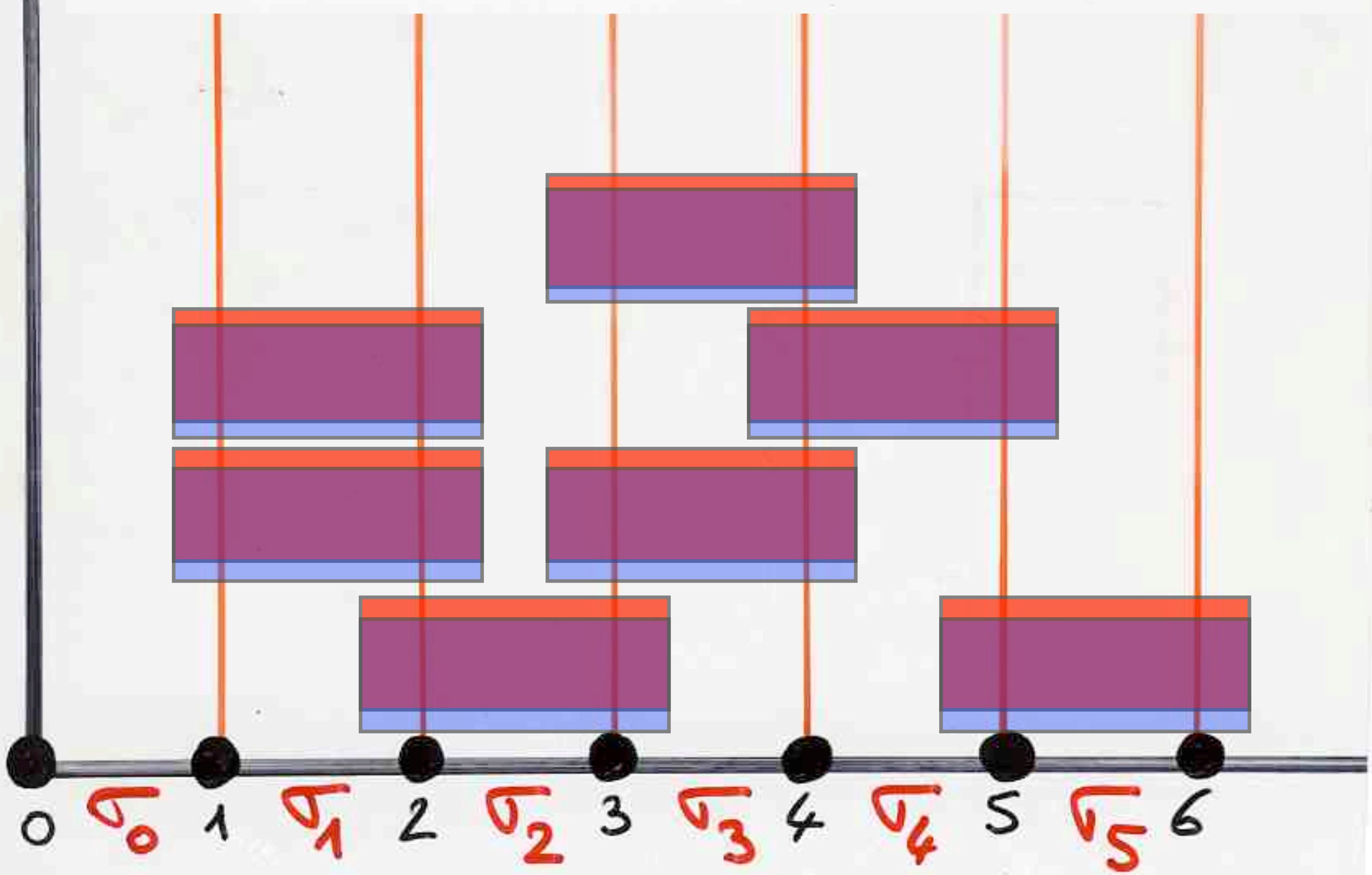




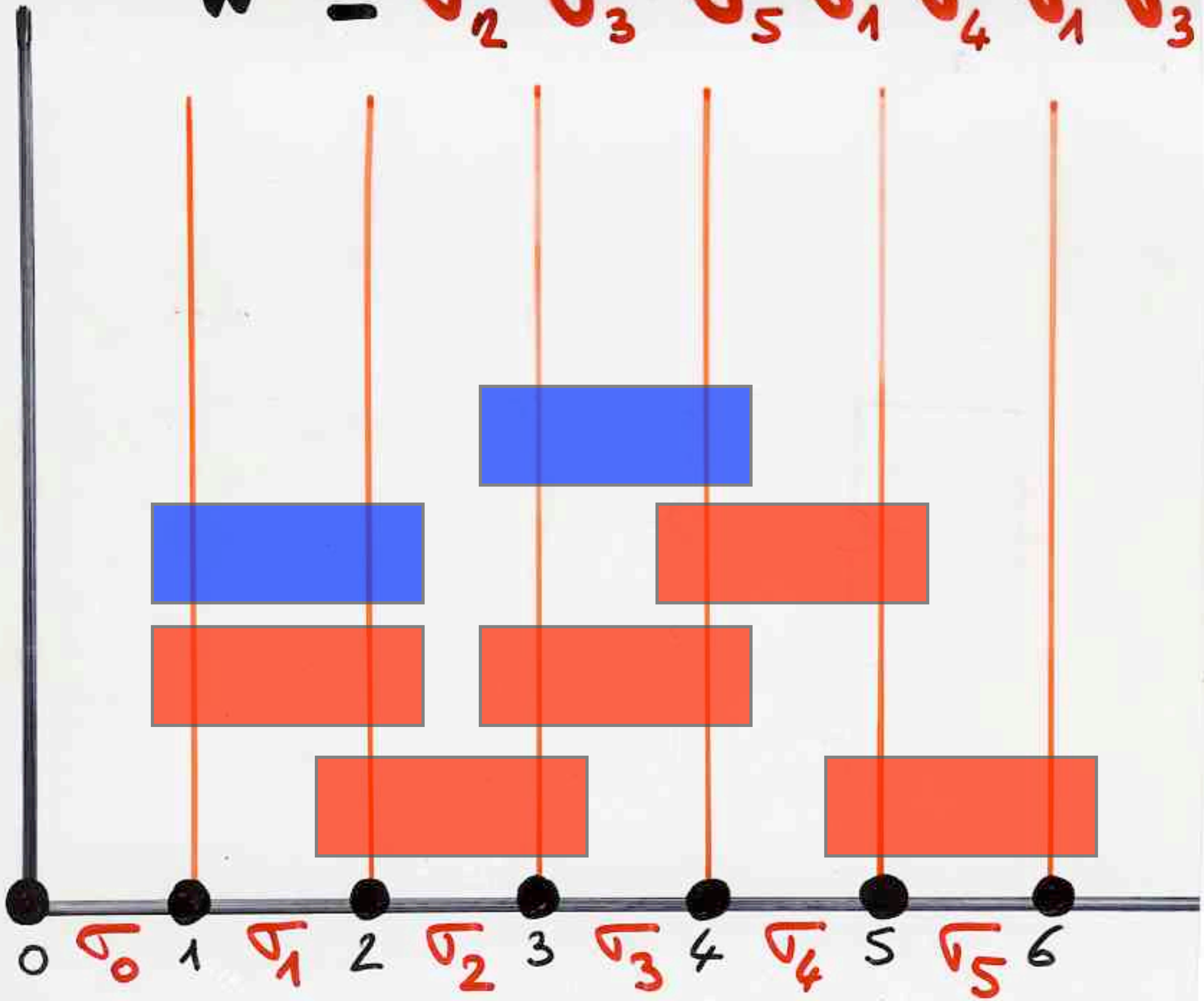
$W = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$
0 σ_0 1 σ_1 2 σ_2 3 σ_3 4 σ_4 5 σ_5 6

$$w = \sigma_2 \, \sigma_3 \, \sigma_5 \, \sigma_1 \, \sigma_4 \, \sigma_1 \, \sigma_3$$

$$w = \sigma_5 \, \sigma_2 \, \sigma_1 \, \sigma_1 \, \sigma_3 \, \sigma_4 \, \sigma_3$$

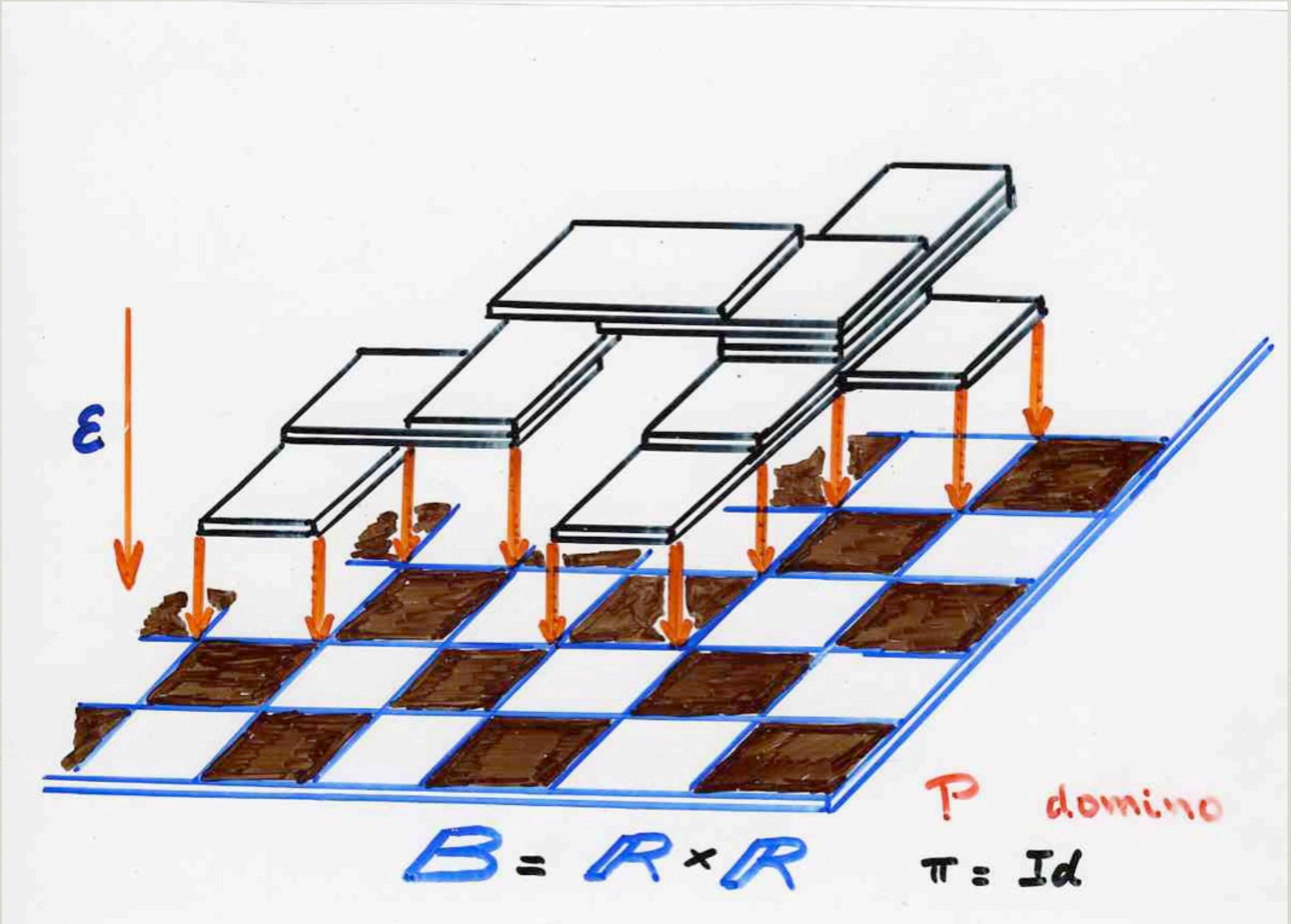


$w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$

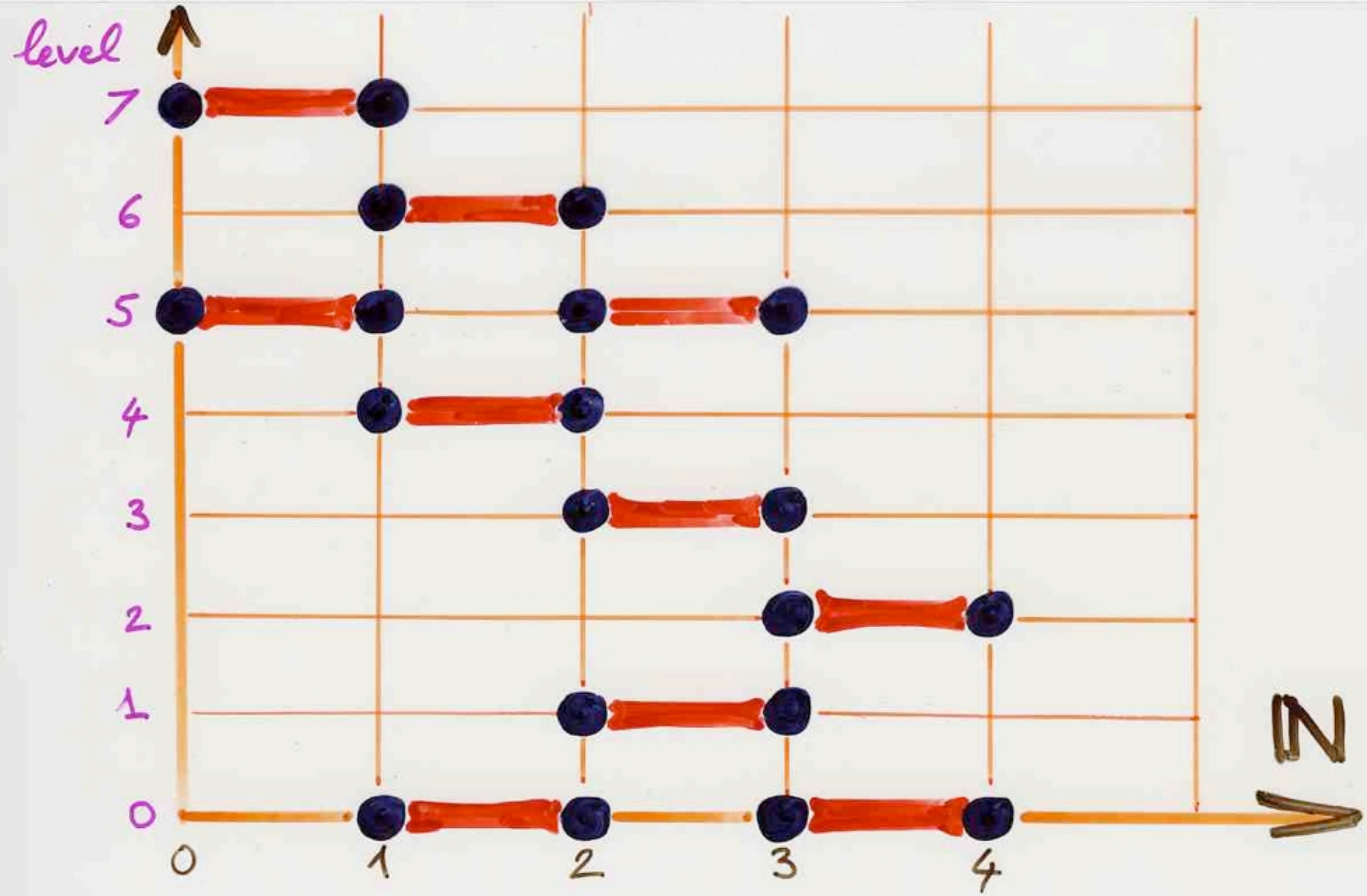


Pyramid

Def- Heap having only one maximal piece



$\mathcal{E}$
P domino
$B = \mathbb{R} \times \mathbb{R}$
$\pi = Id$



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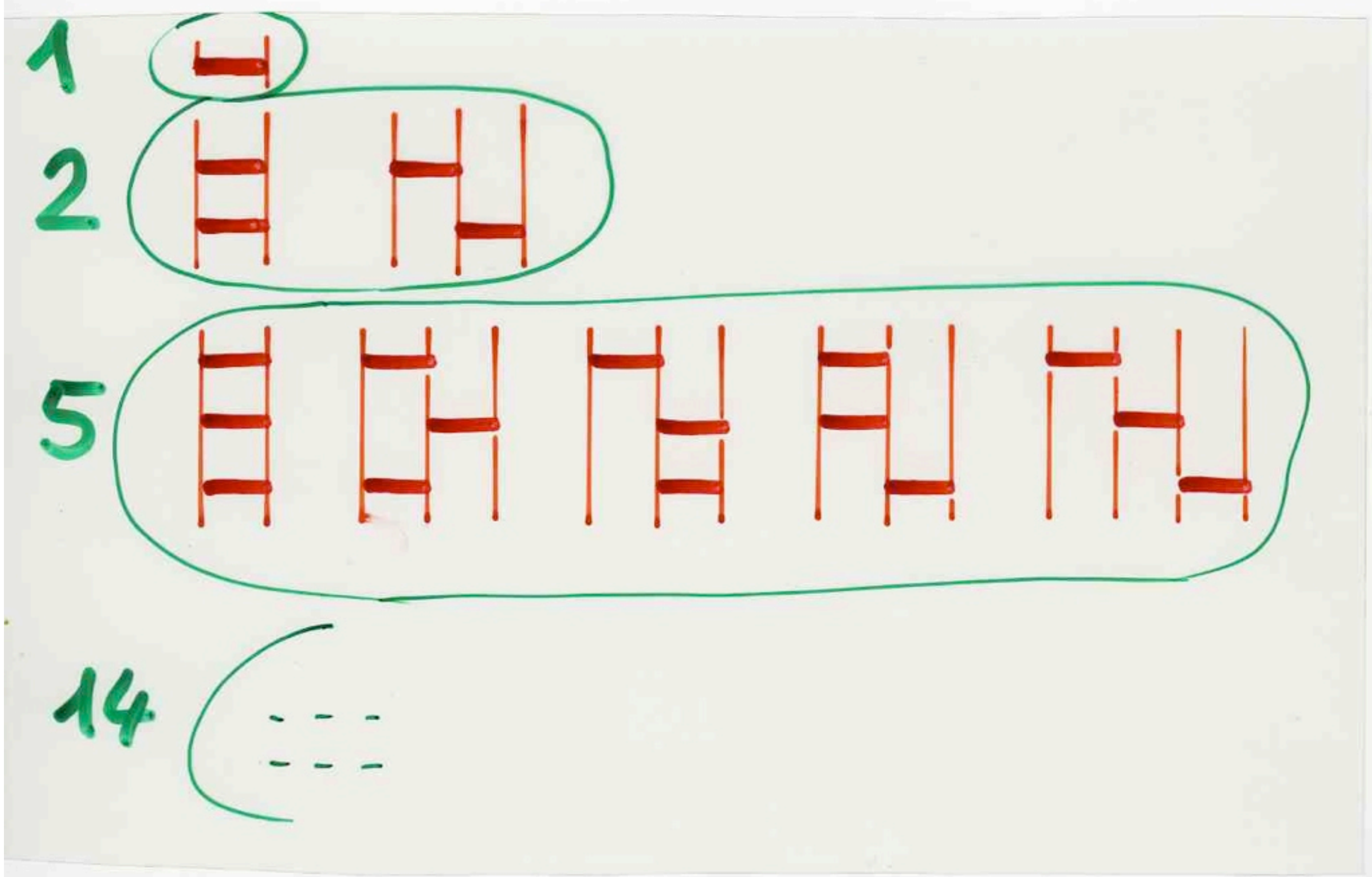
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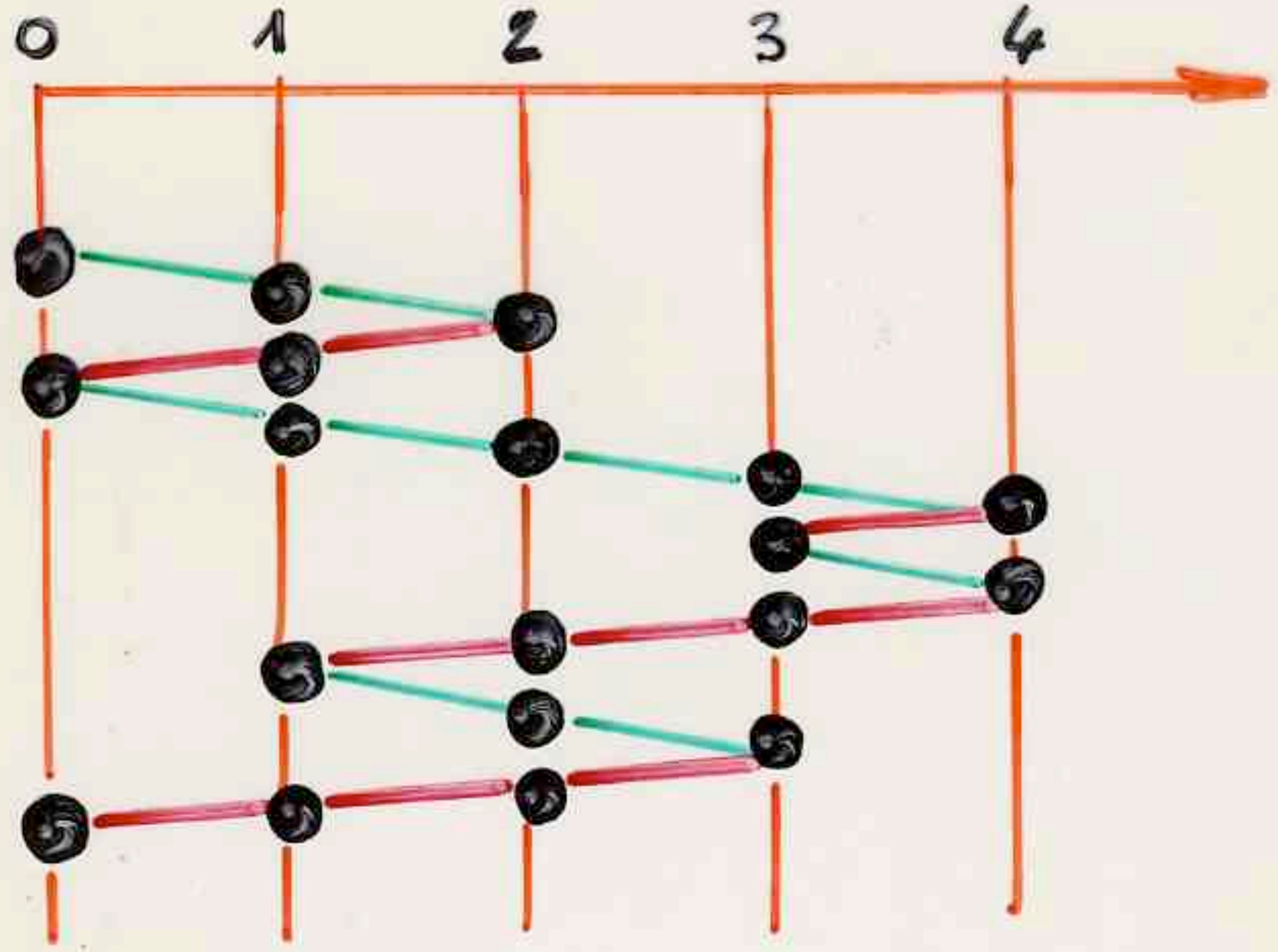
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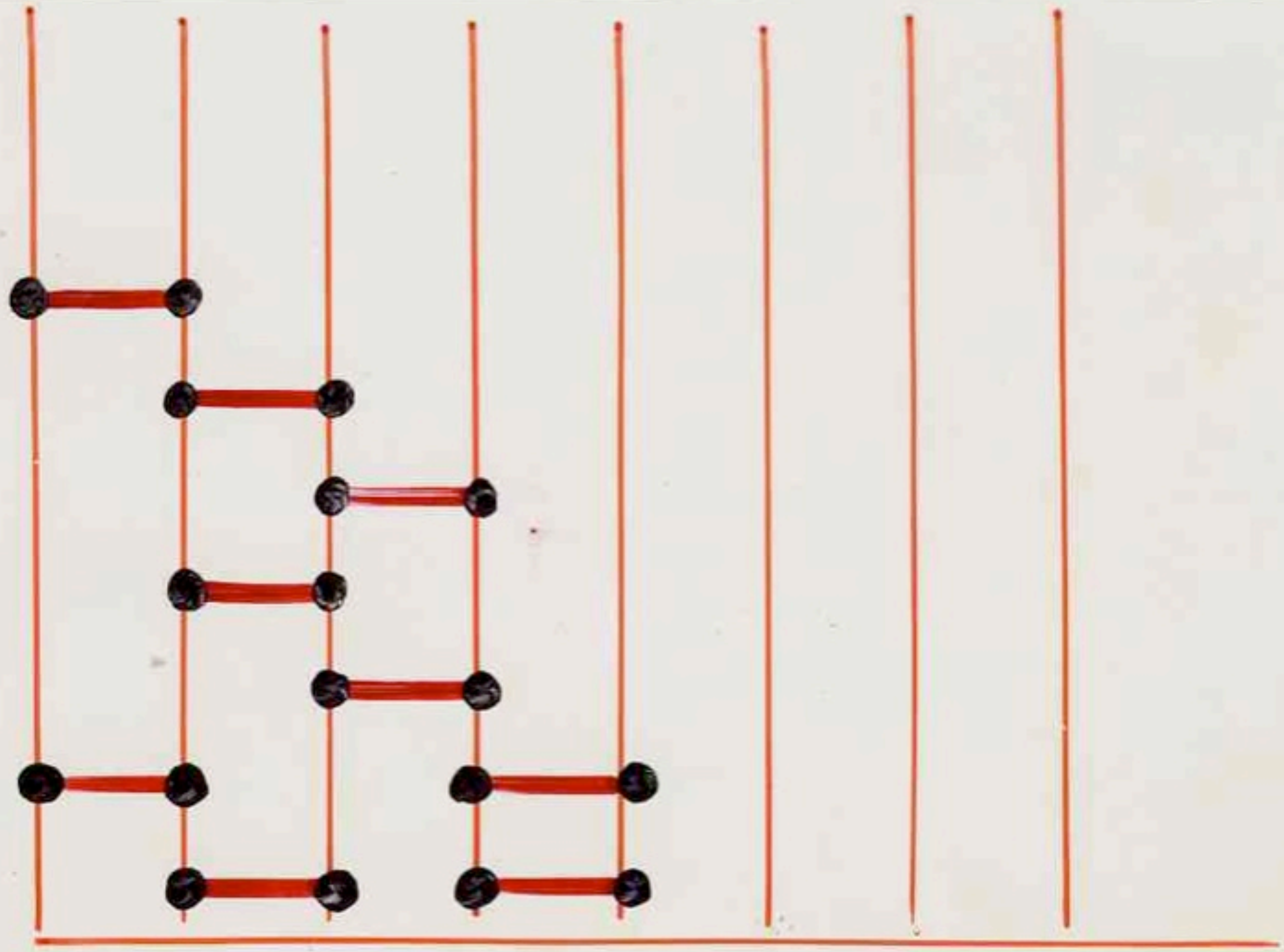
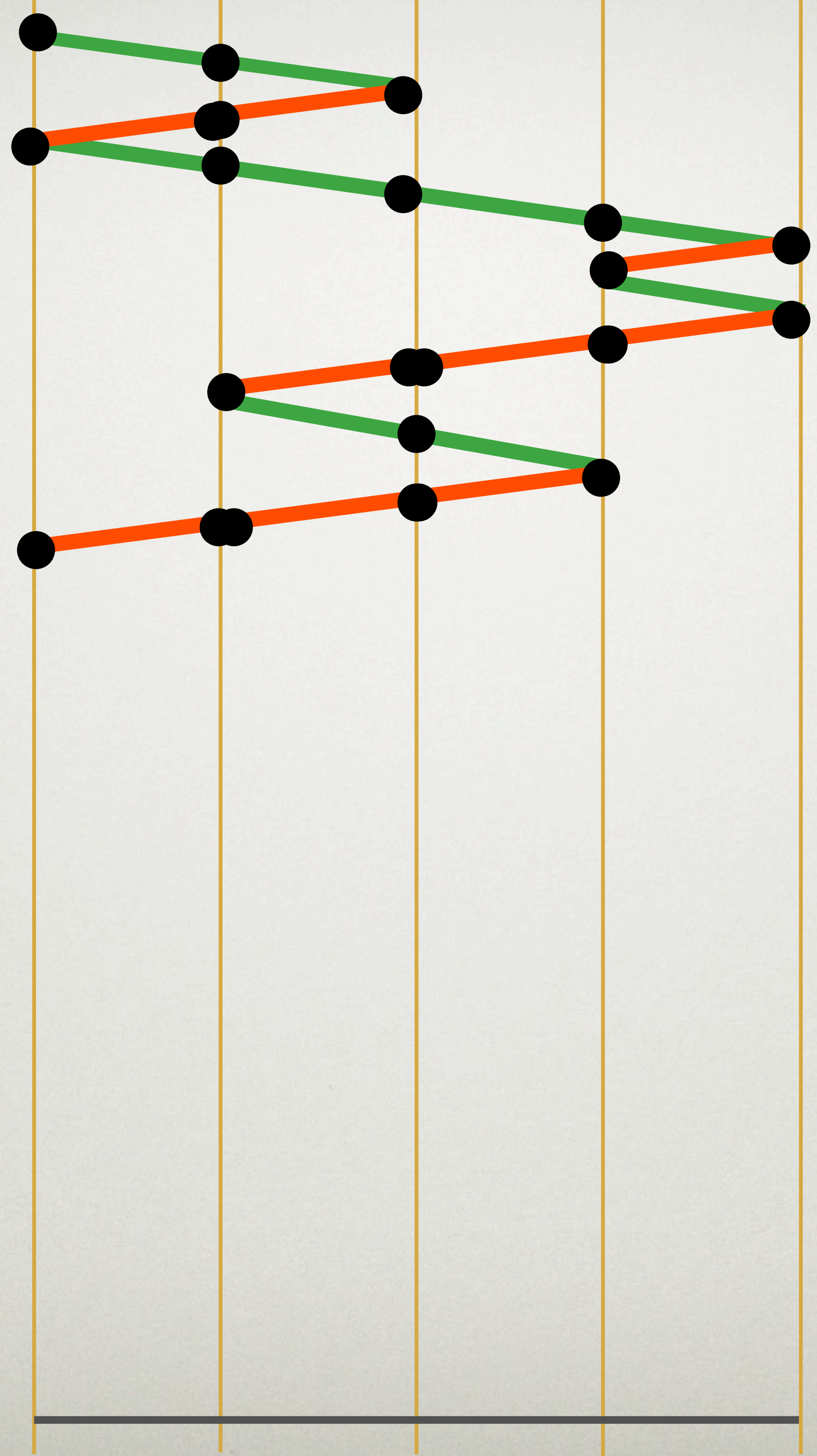
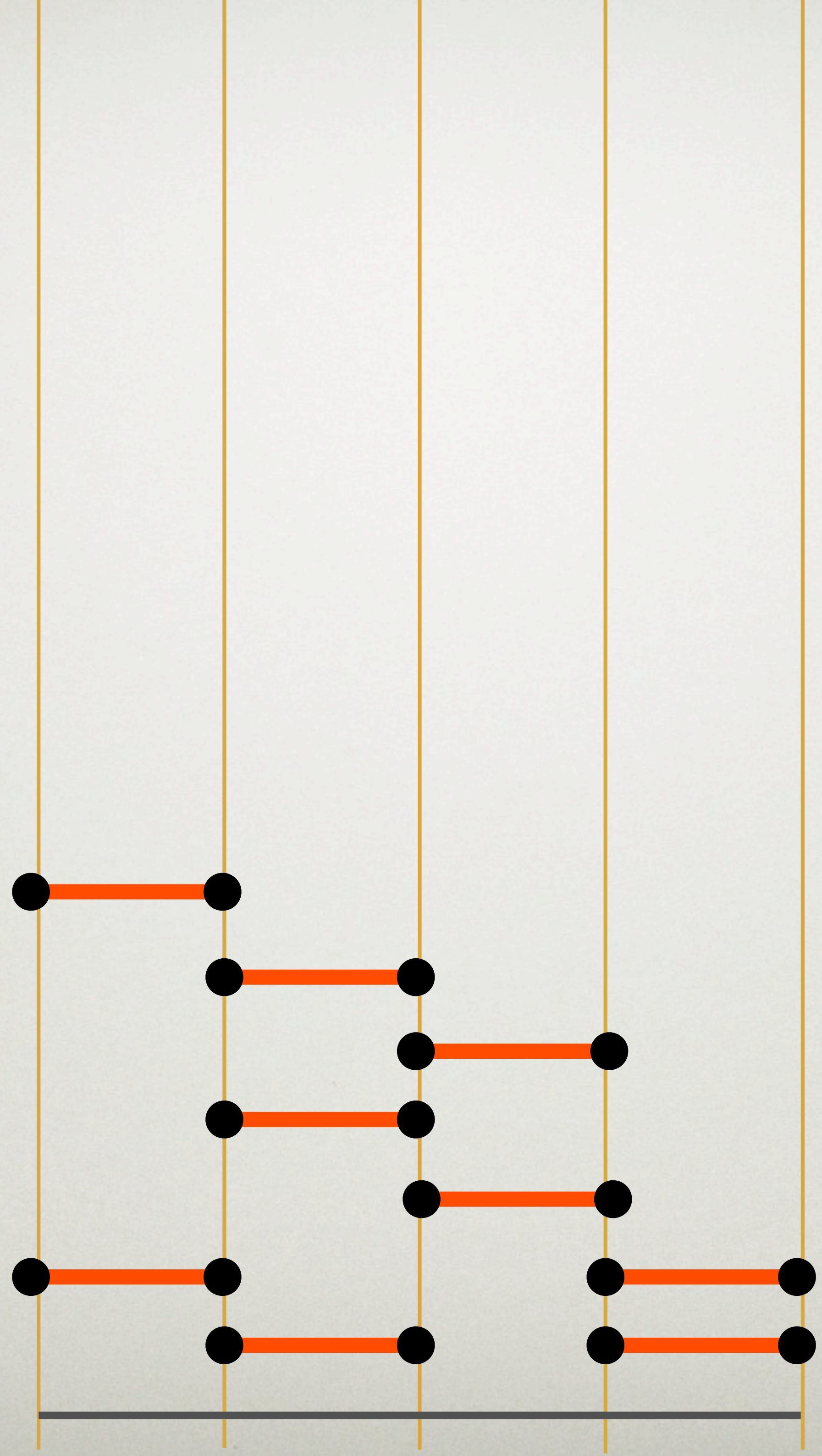


bijection

Dyck paths

semi-pyramid of dimers



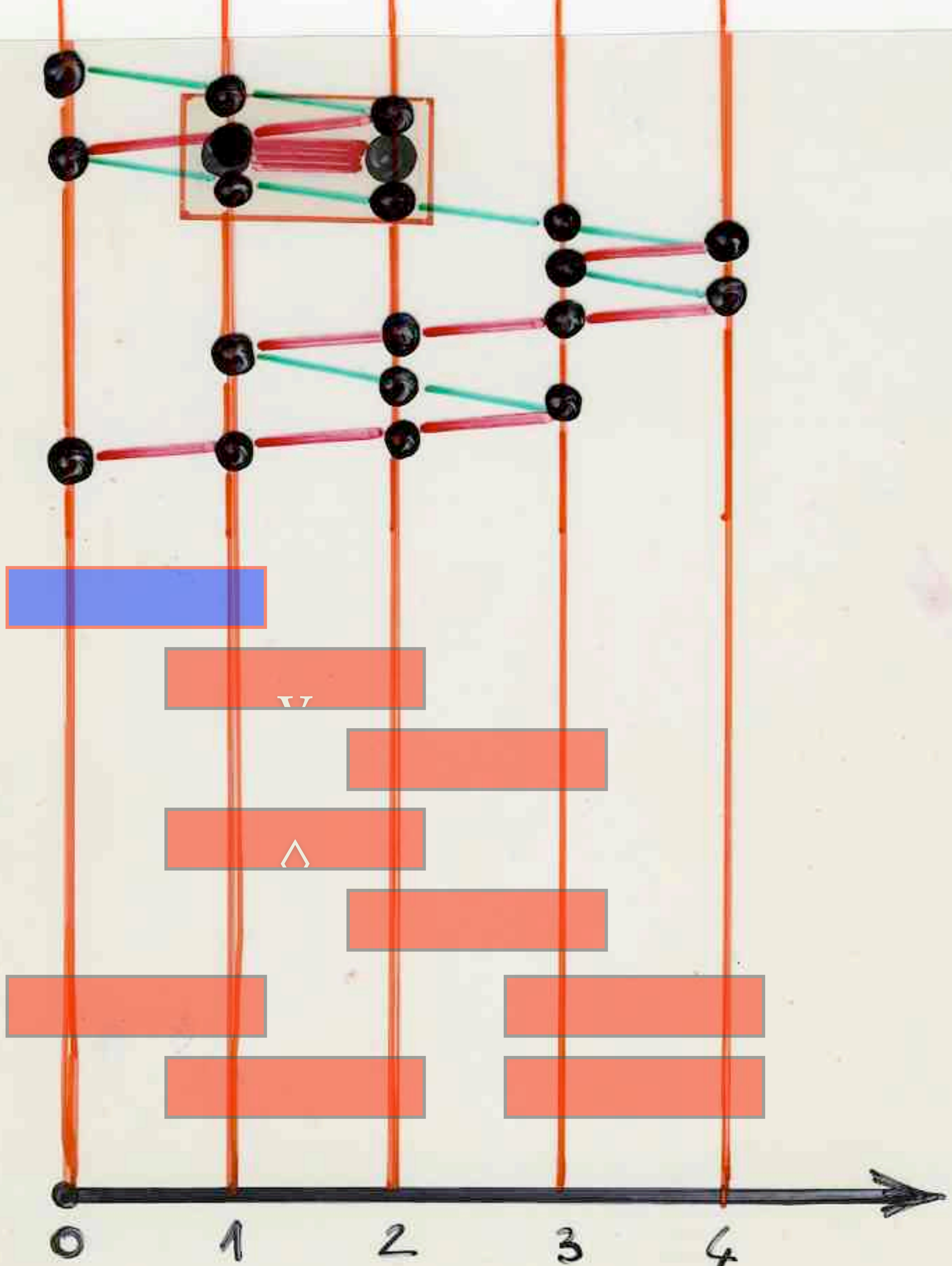


violin:
Gérard
Duchamp

reverse bijection

Dyck paths

semi-pyramid of dimers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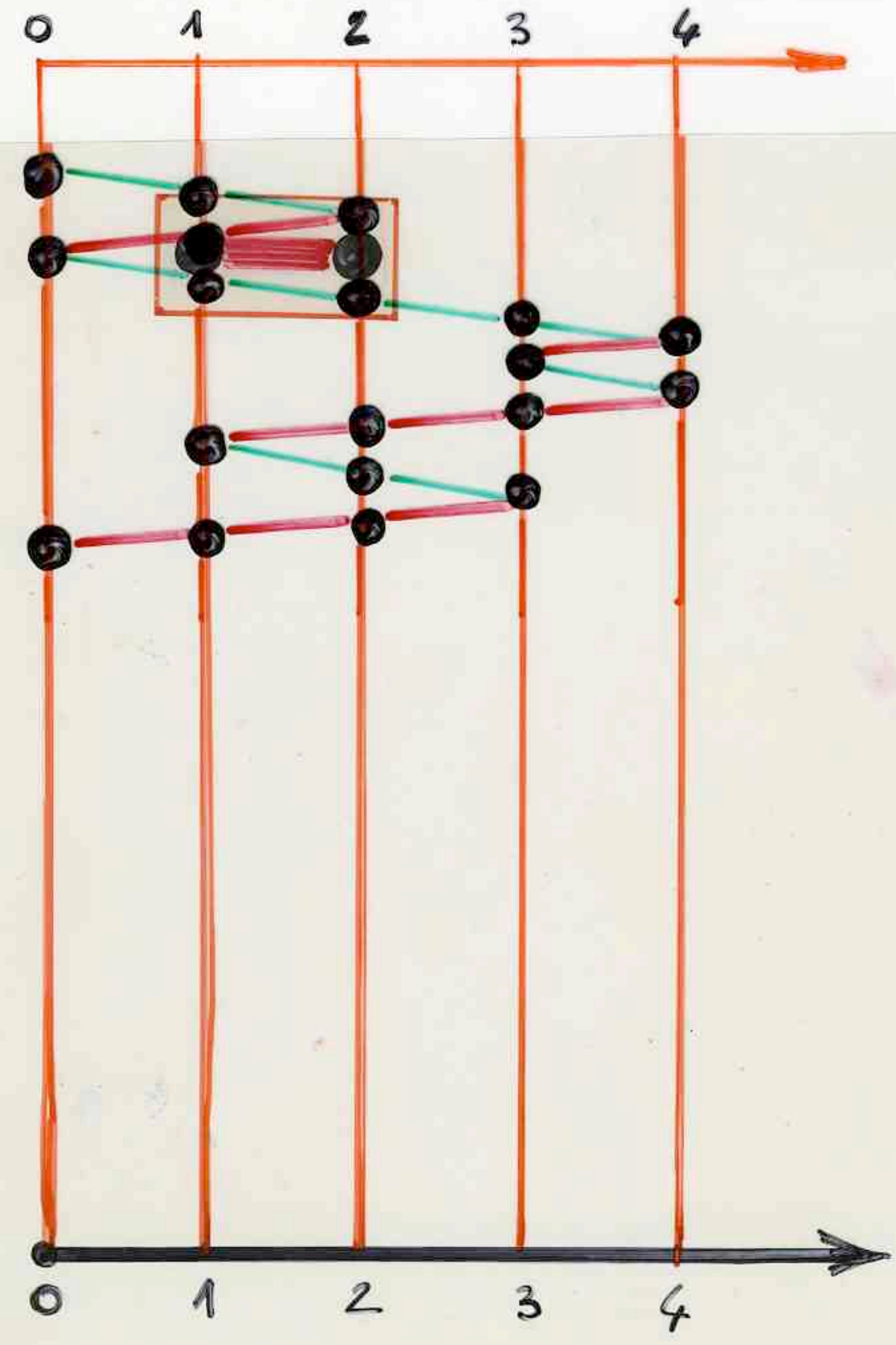
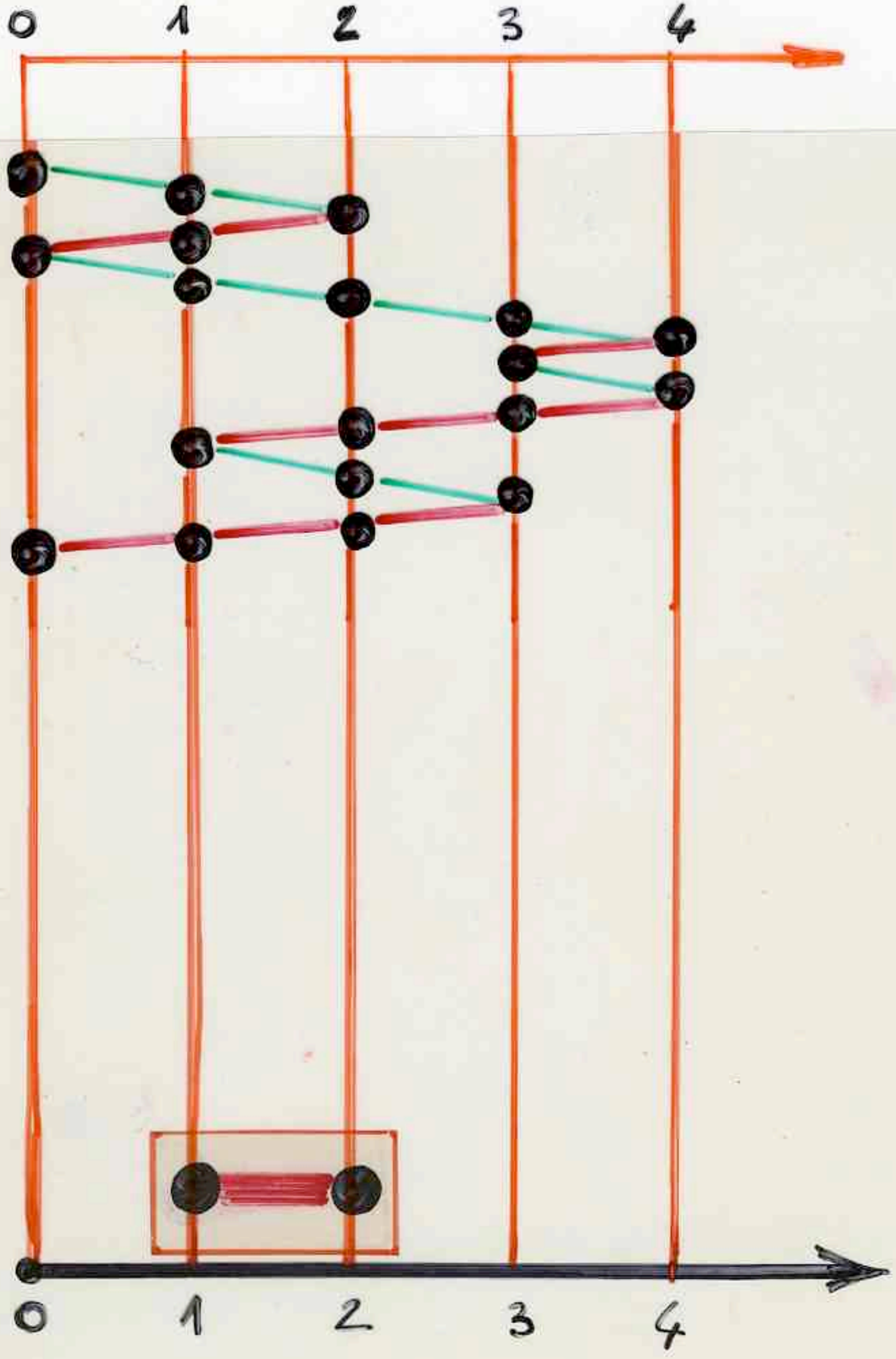


0 1 2 3 4
0 1 2 3 4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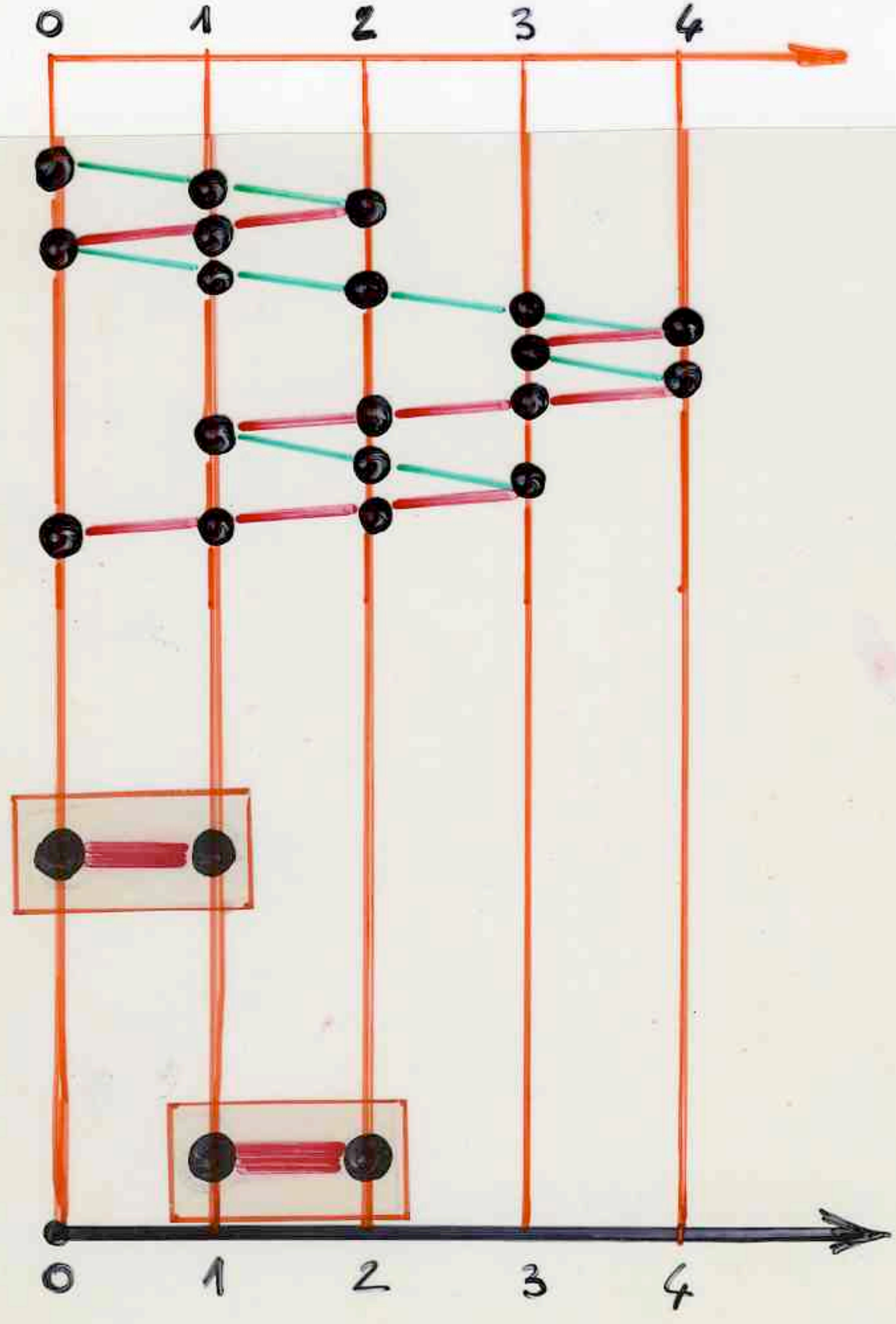

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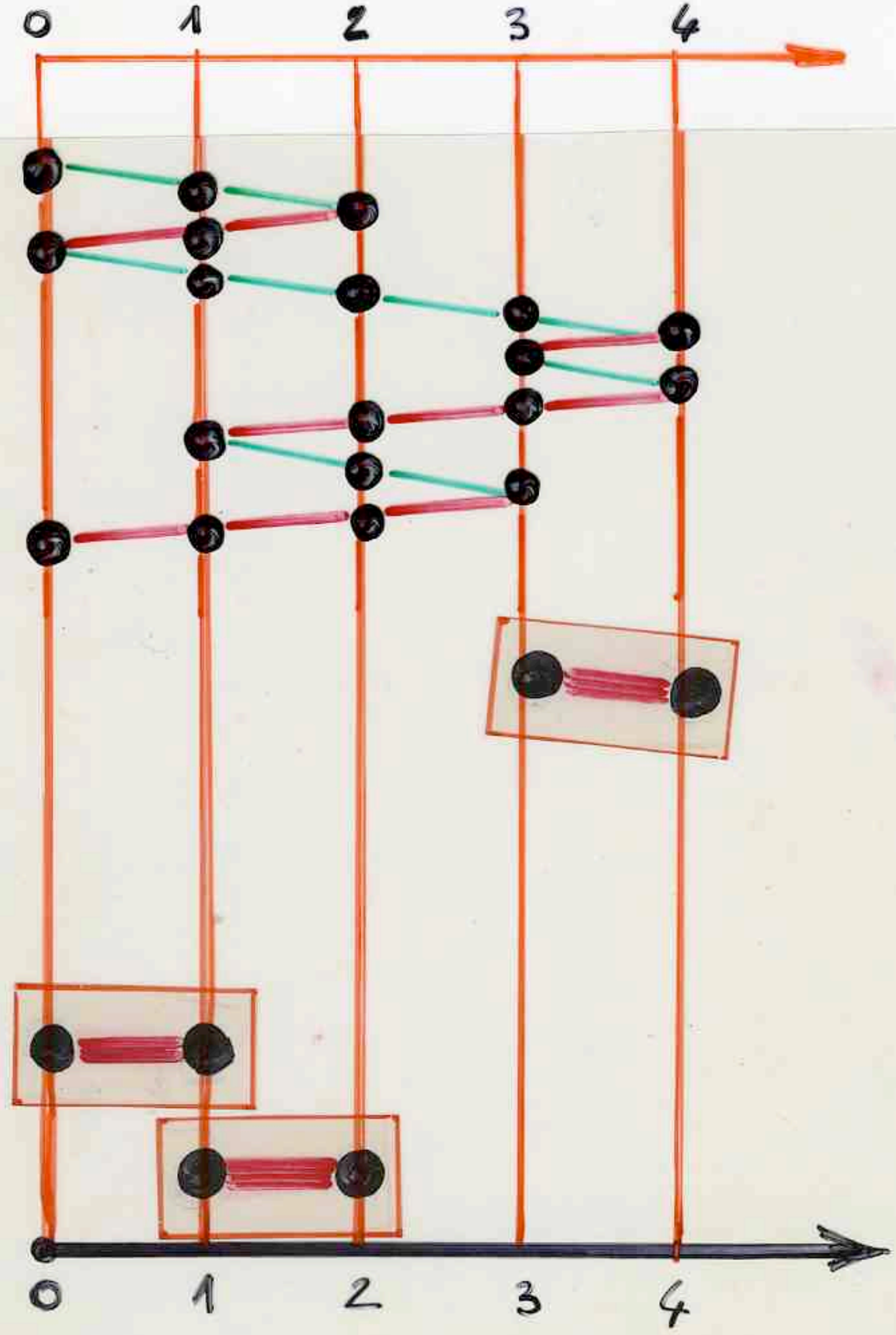
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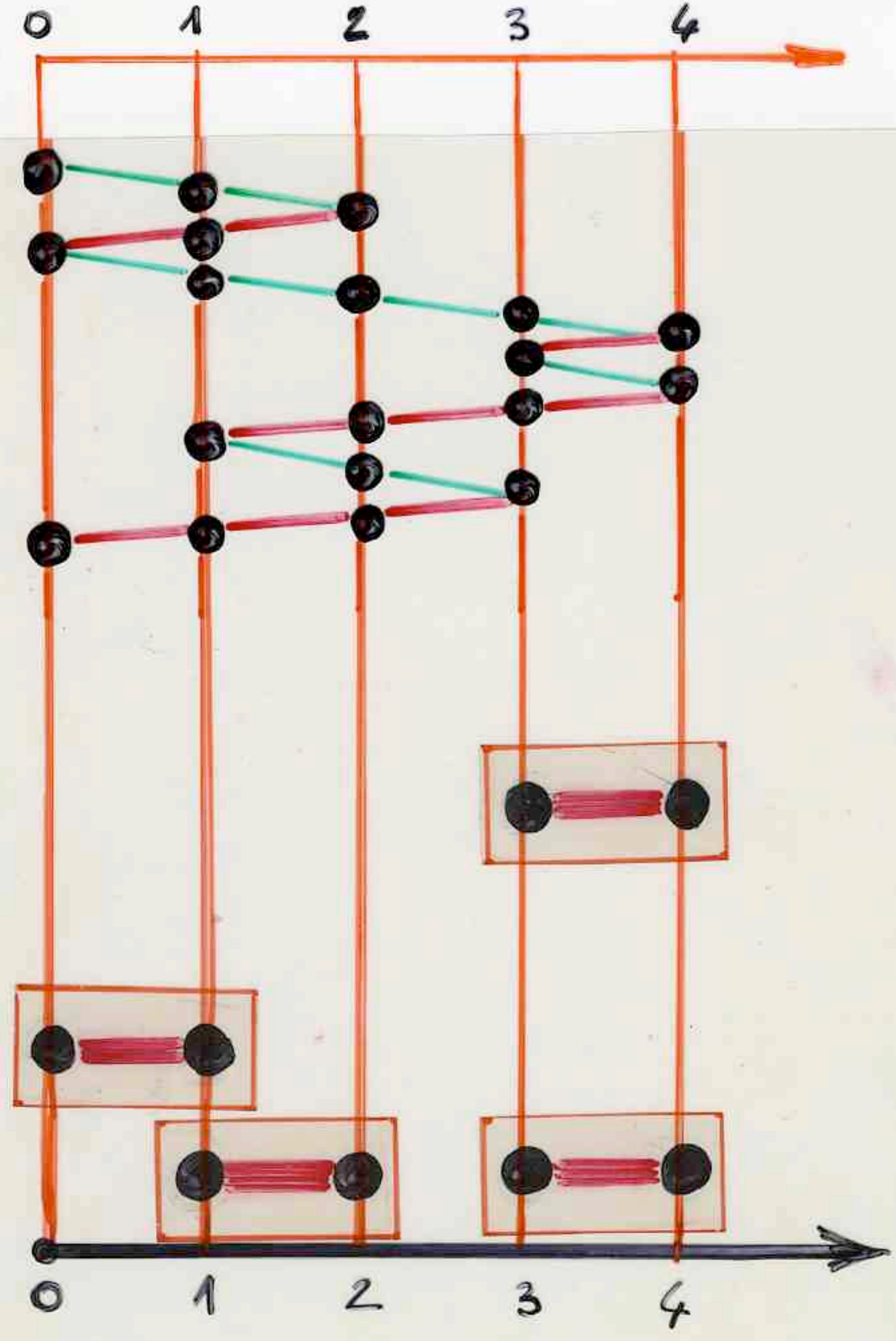

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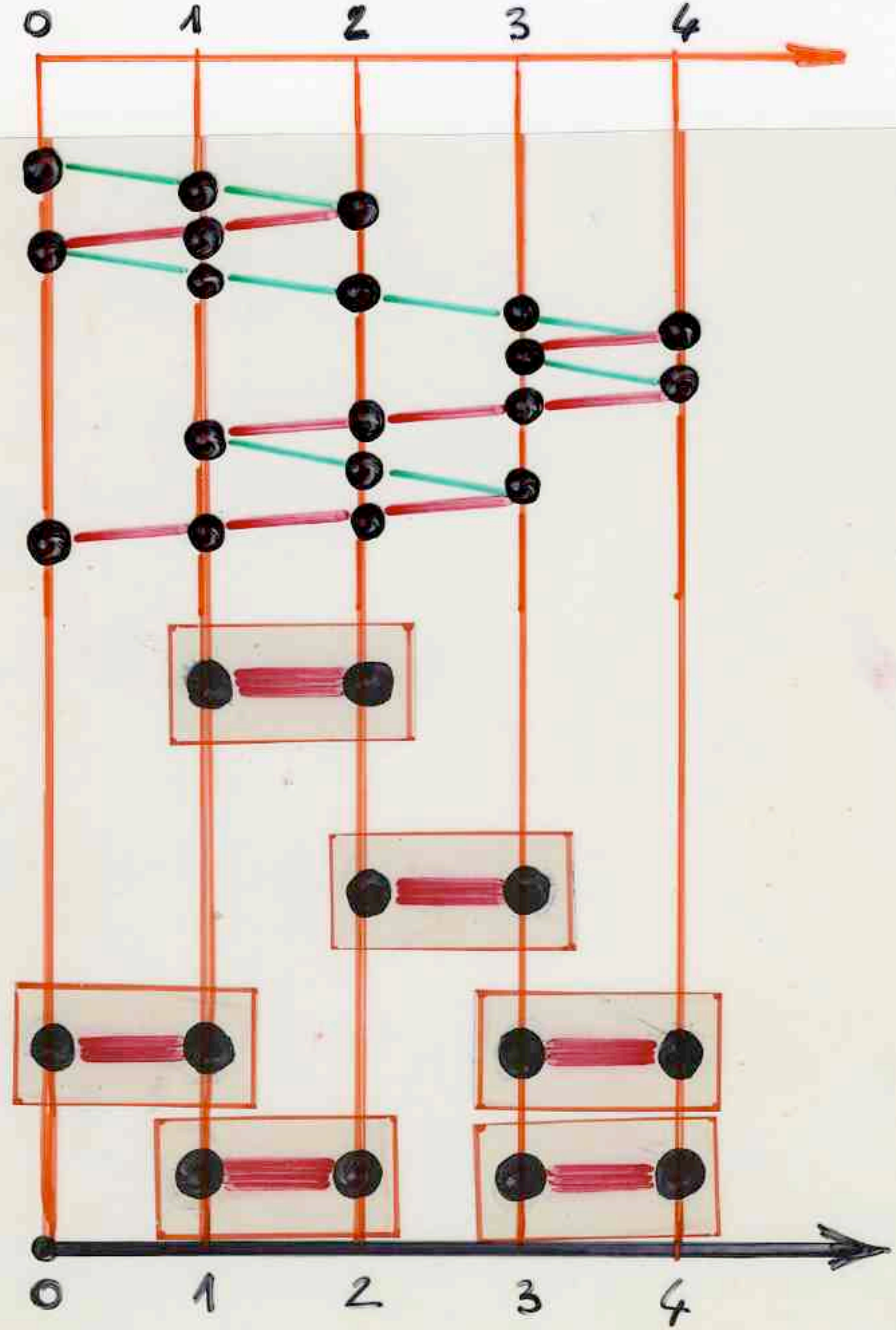
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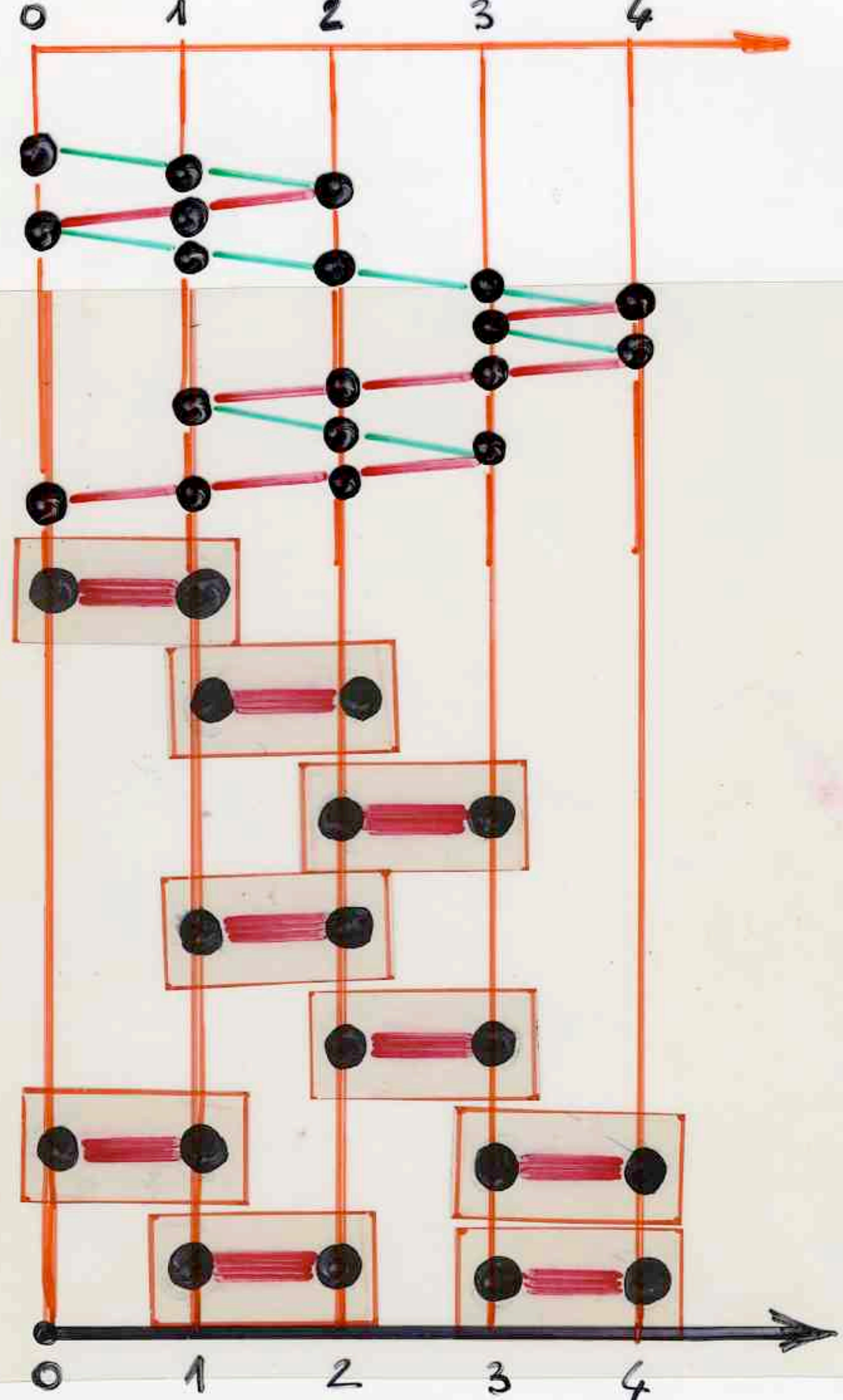
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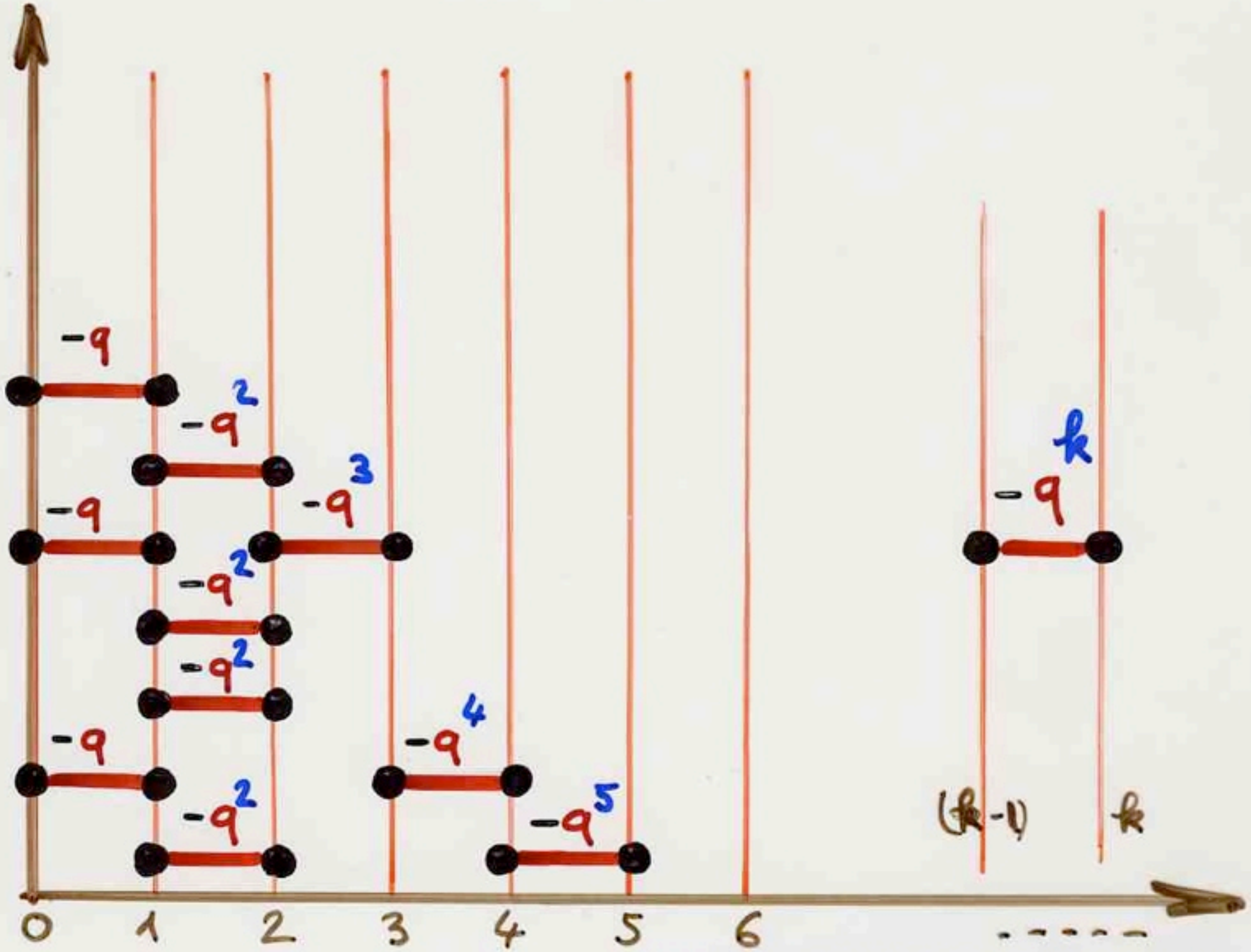
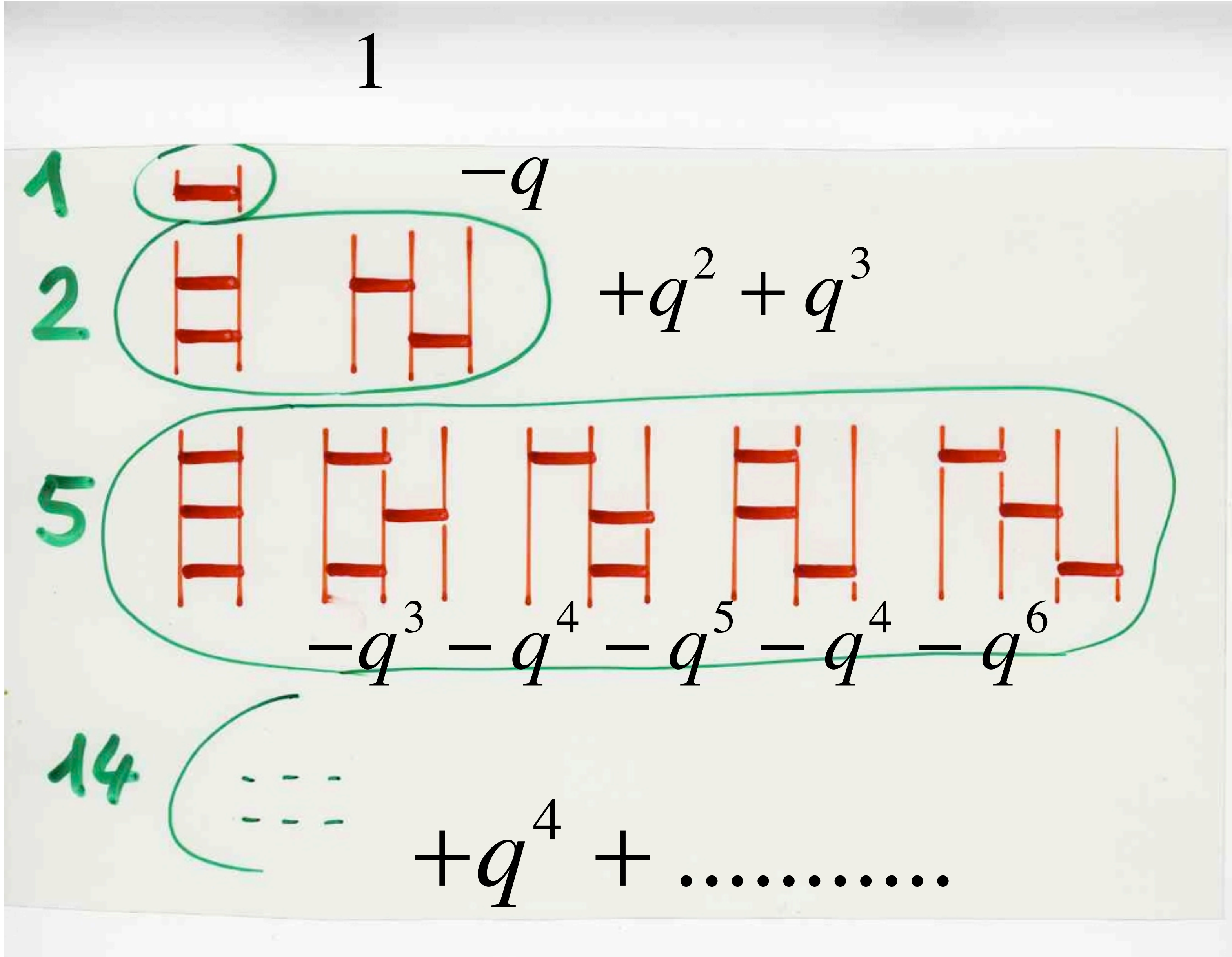
back to
Ramanujan continued fraction



poids total $(-1)^{10}\, q^{1+1+1+2+2+2+2+3+4+5} = q^{23}$



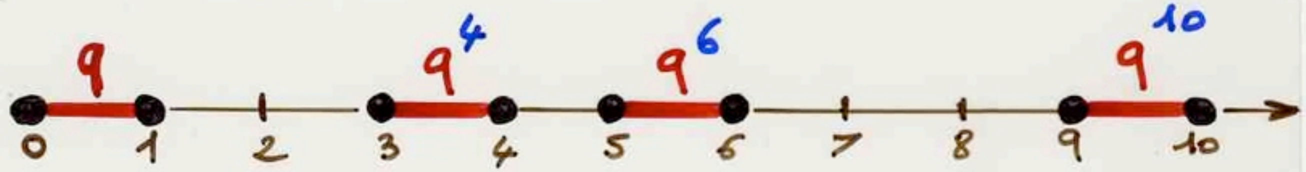
1
1 $-q$
2 $+q^2+q^3$
5 $-q^3-q^4-q^5-q^4-q^6$
14 $+q^4+\ldots\ldots\ldots$

"La fraction continue" de Ramanujan

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\cfrac{\cdots}{1+\cfrac{q^k}{\cdots}}}}}} = \frac{\displaystyle\sum_{n\geqslant 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\displaystyle\sum_{n\geqslant 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}$$

poids total

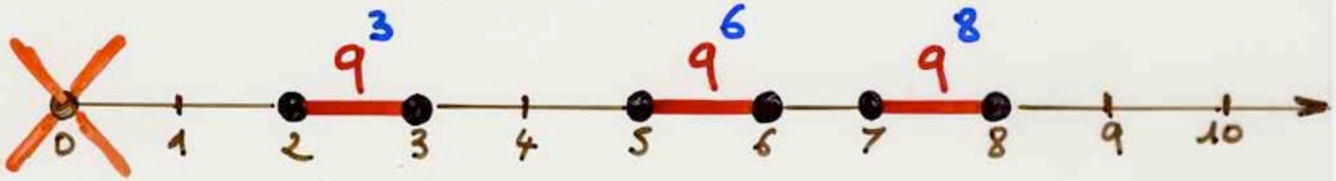
q^{21}



D-partition $\lambda = (10, 6, 4, 1)$

$21 = 10 + 6 + 4 + 1$

q^{17}

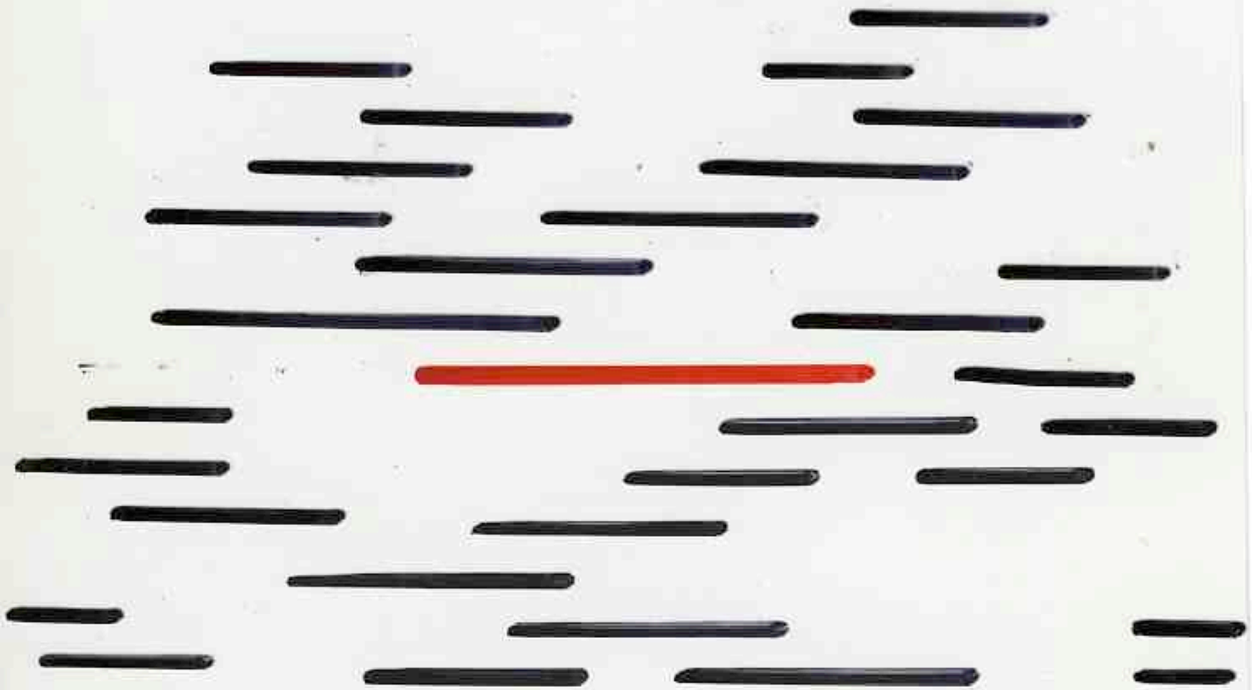


$\lambda = (8, 6, 3)$

$17 = 8 + 6 + 3$

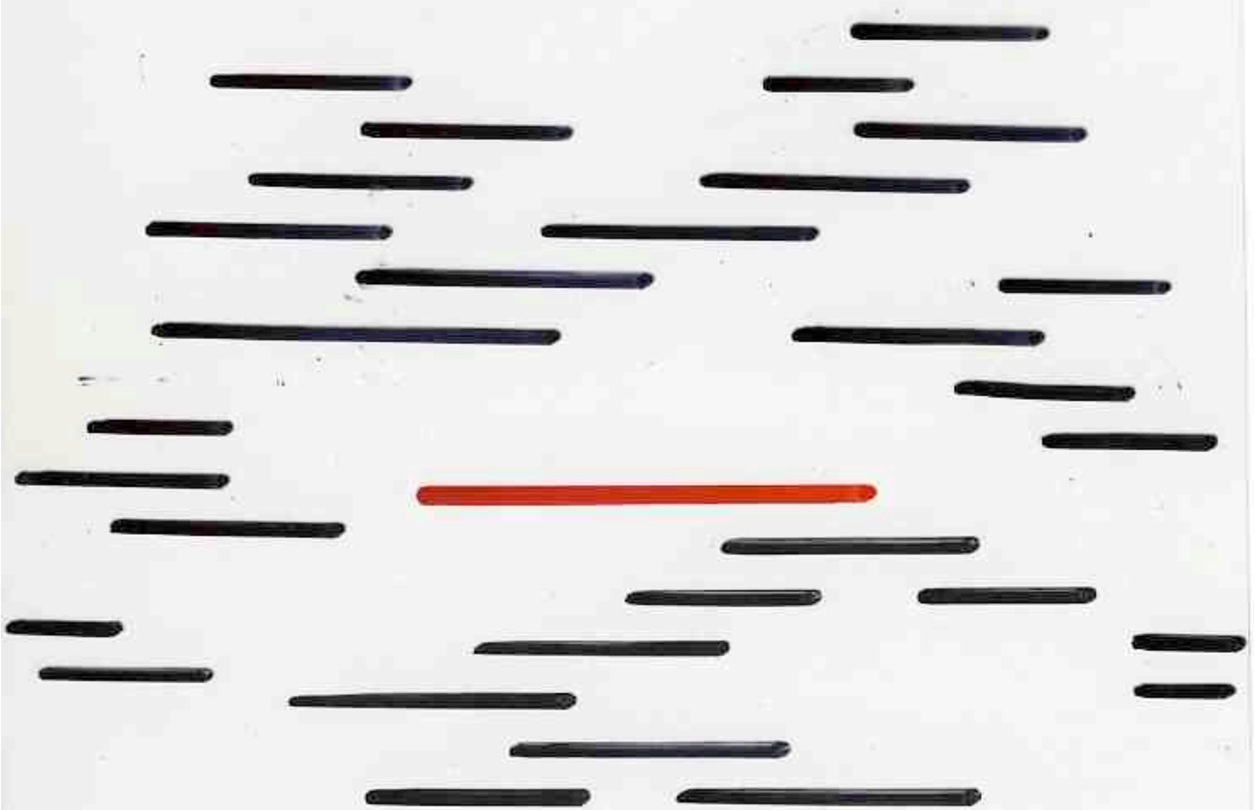
Opérateur

"Poussez" ...

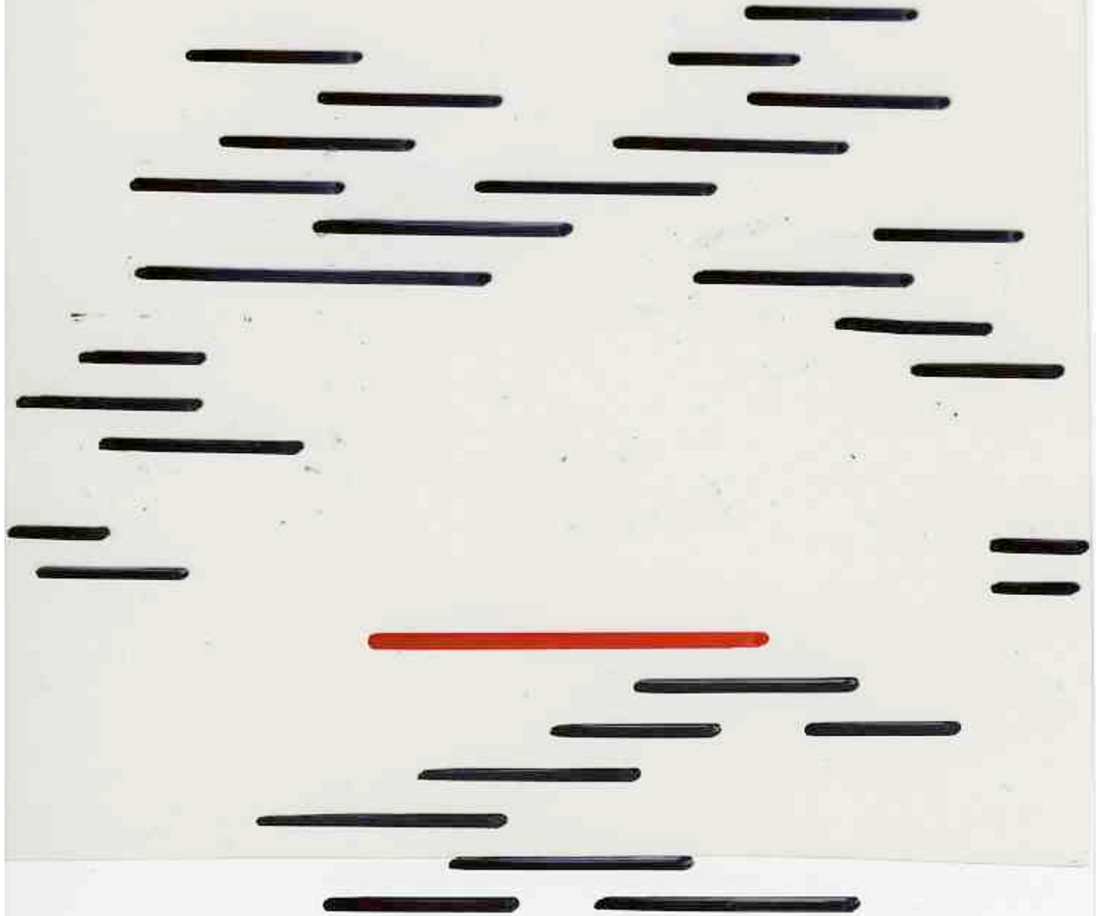


Opérateur

"Poussez" ...

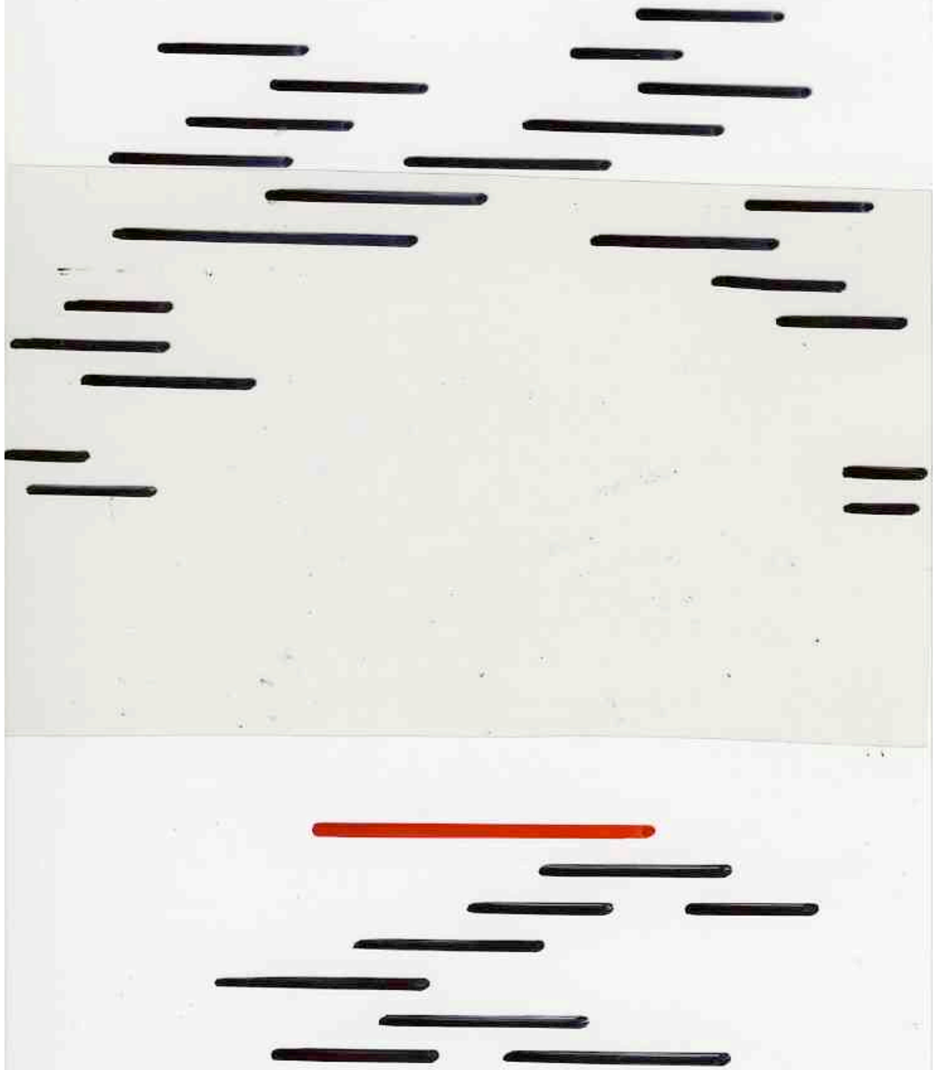


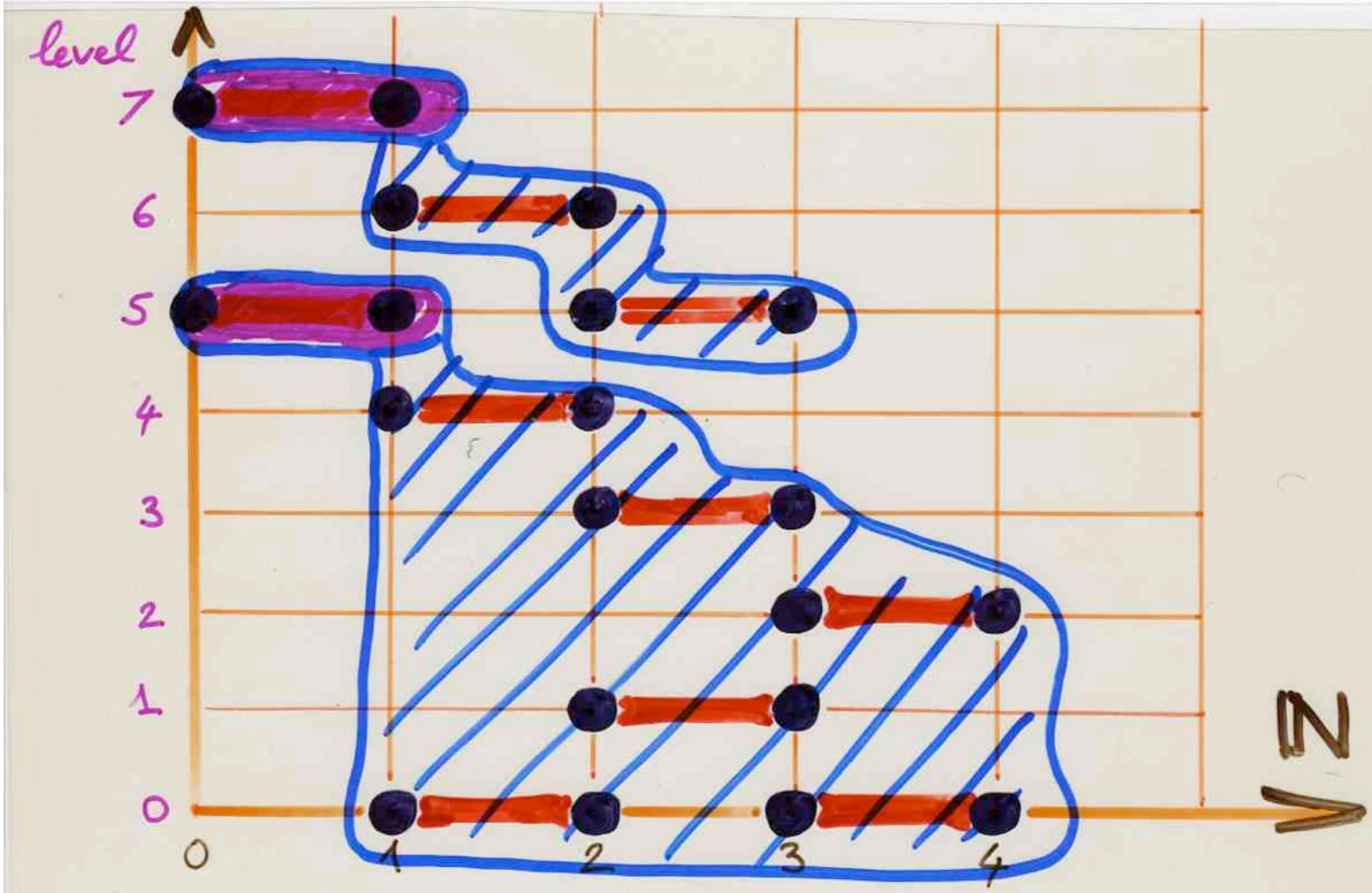
Opérateur "Poussez" ...



Opérateur

"Poussez" ...





level
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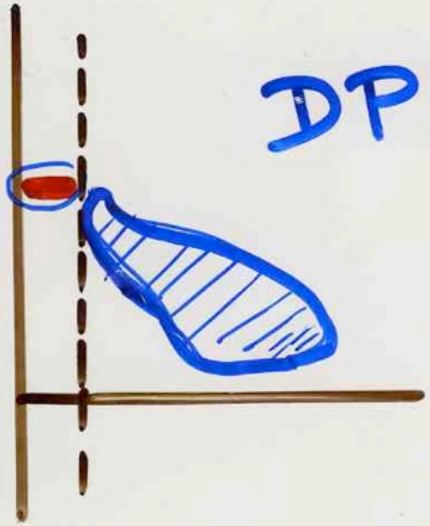
ℕ

$$DP_{yr} = \frac{1}{1 - DPP_{yr}}$$

$$DP_{yr} = \frac{1}{1 - DPP_{yr}}$$



$$DPP_{yr} = t \times \delta(DP_{yr})$$