

The beauty of mathematics

part II: trees,
Catalan numbers, bijections

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College of Education
Pondicherry, 23 Feb 2012

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§4 Catalan numbers

Another beautiful formula

From trees in nature

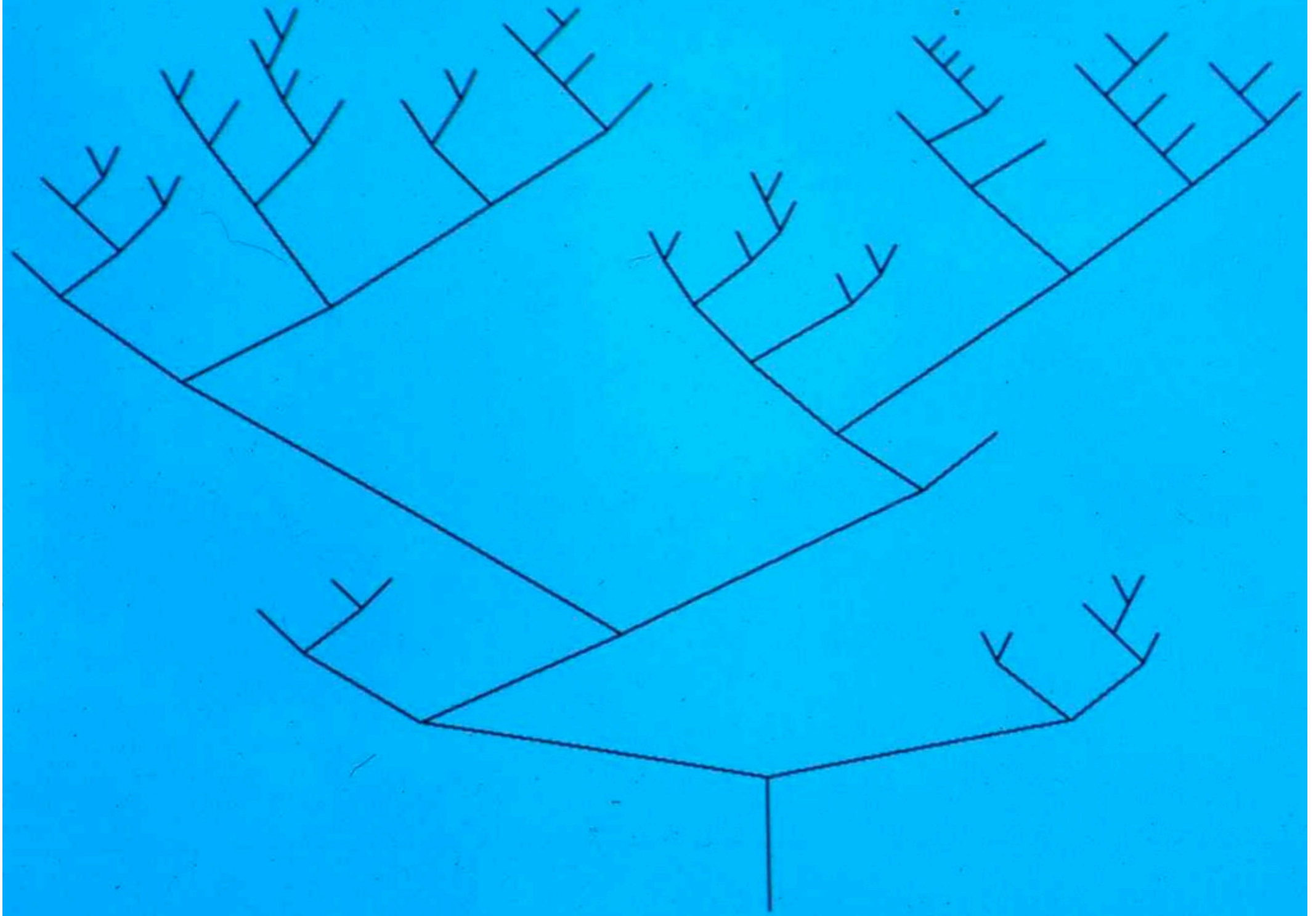
to "mathematical trees"

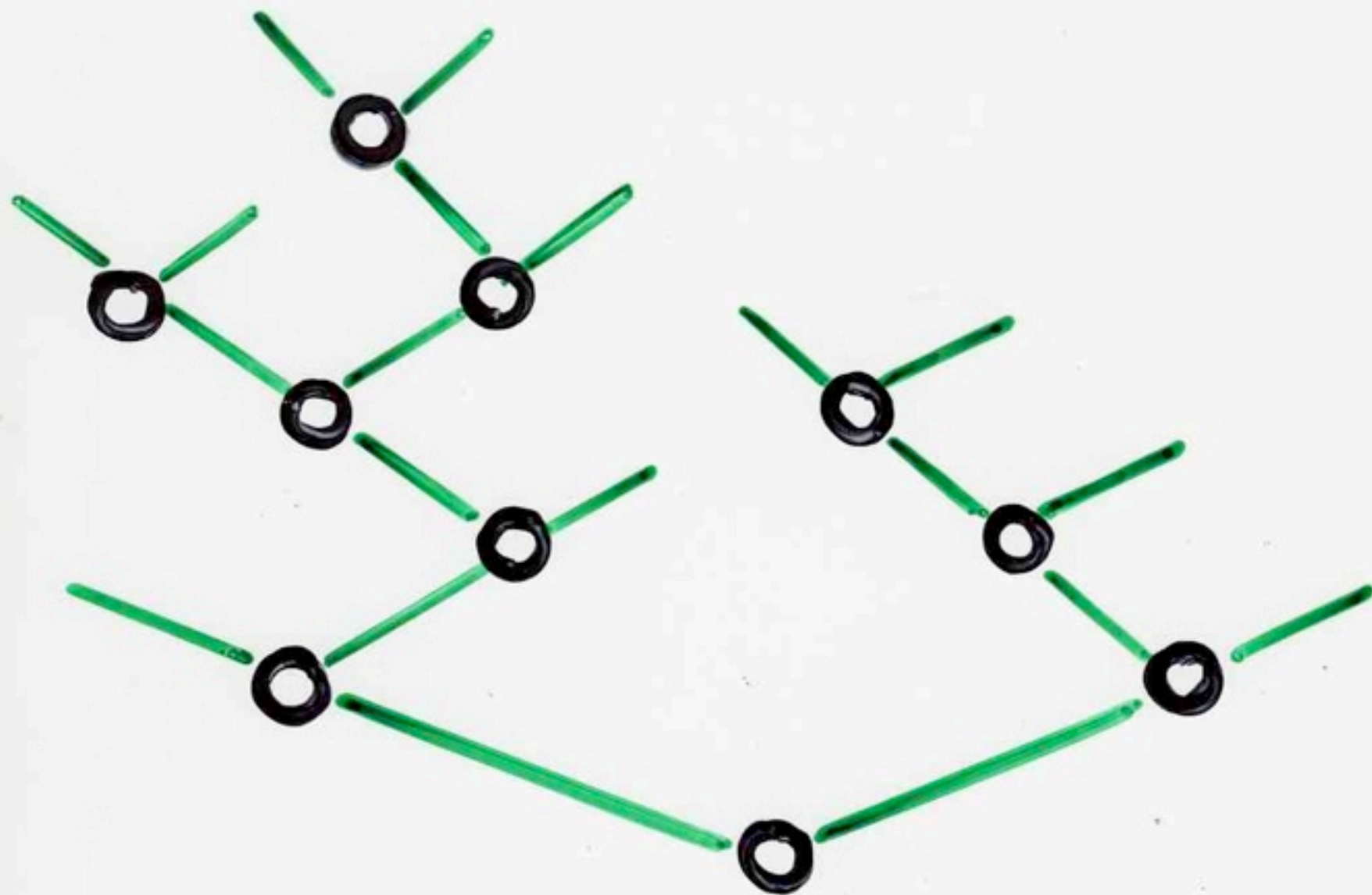


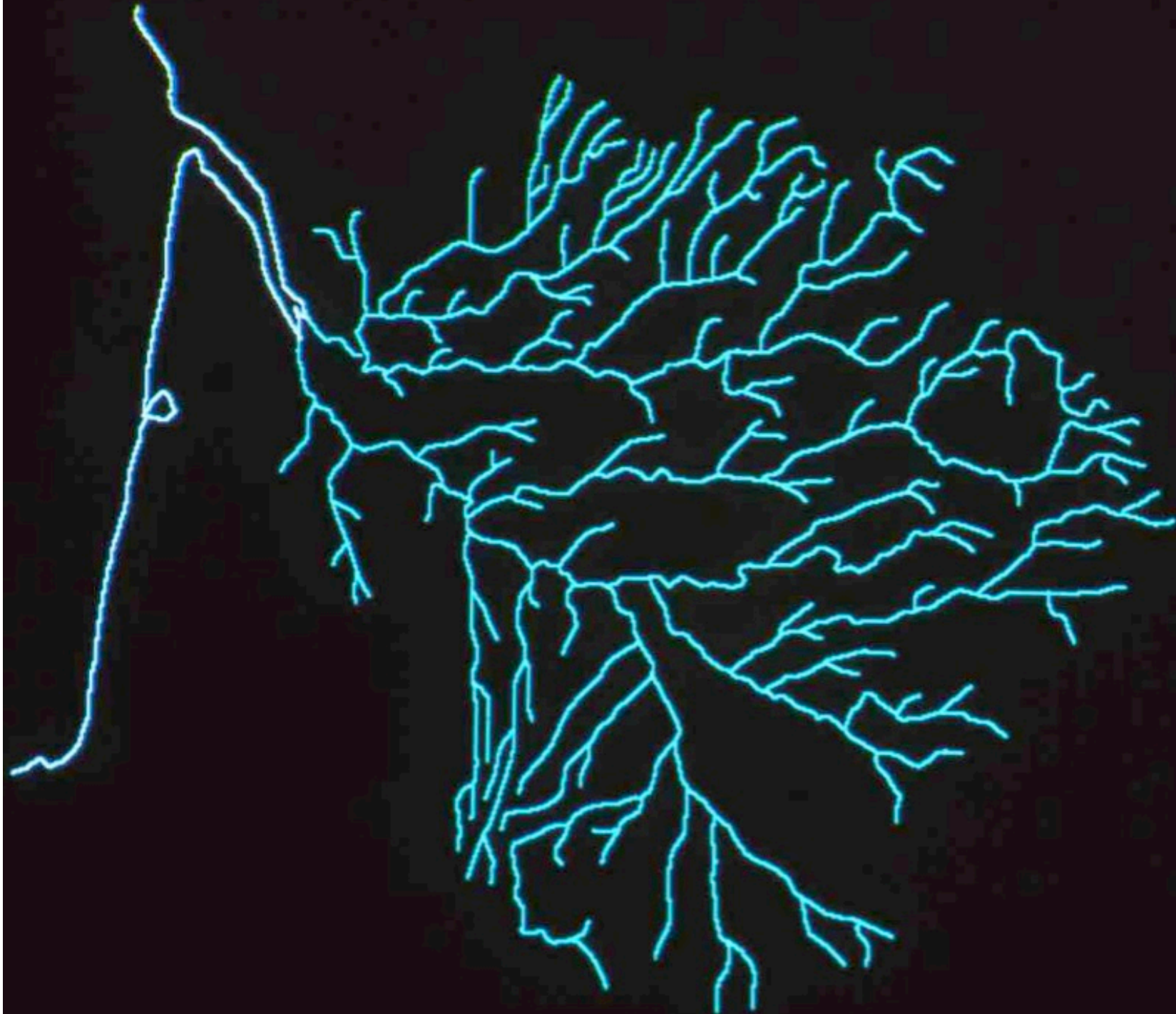




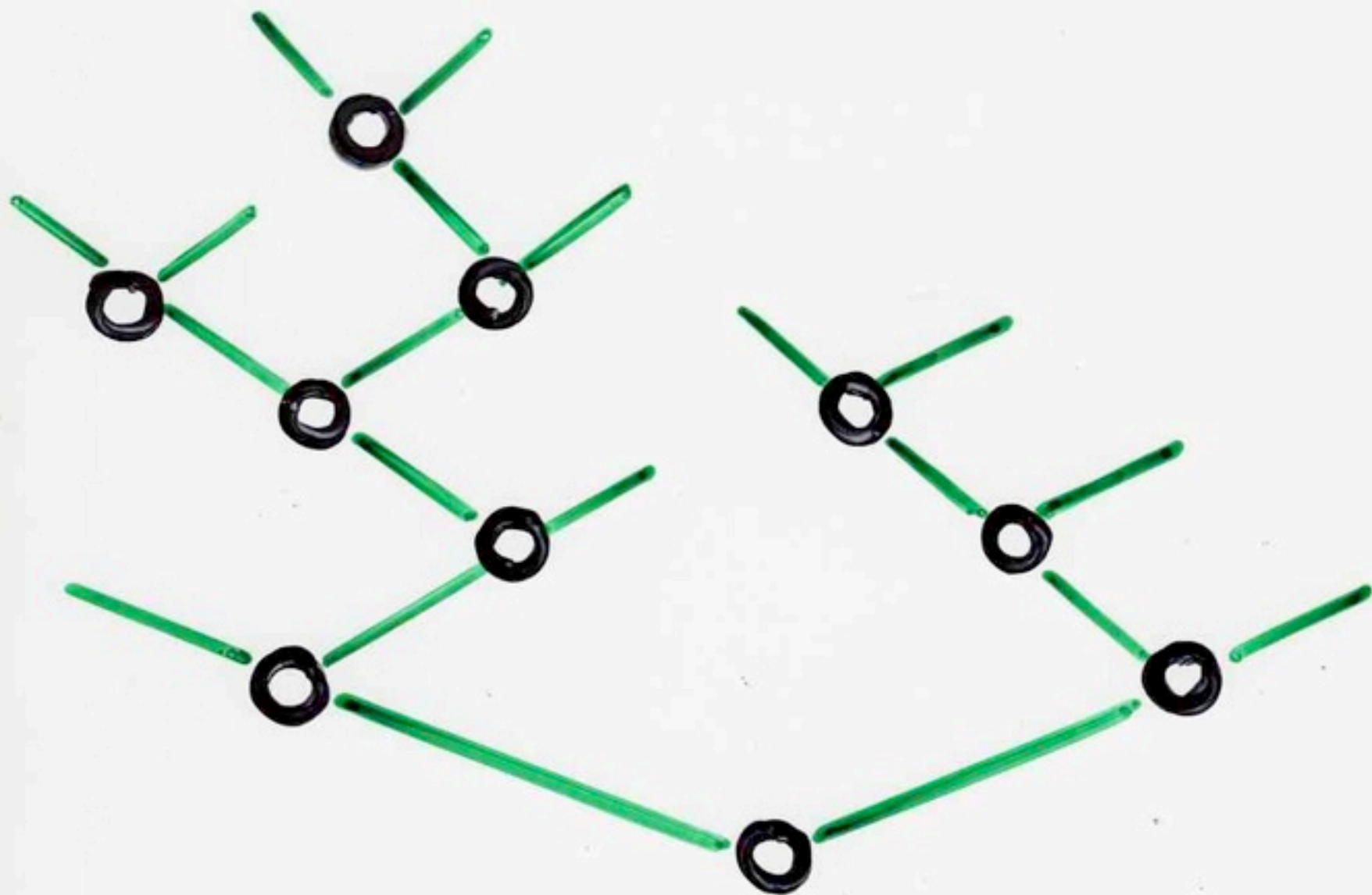




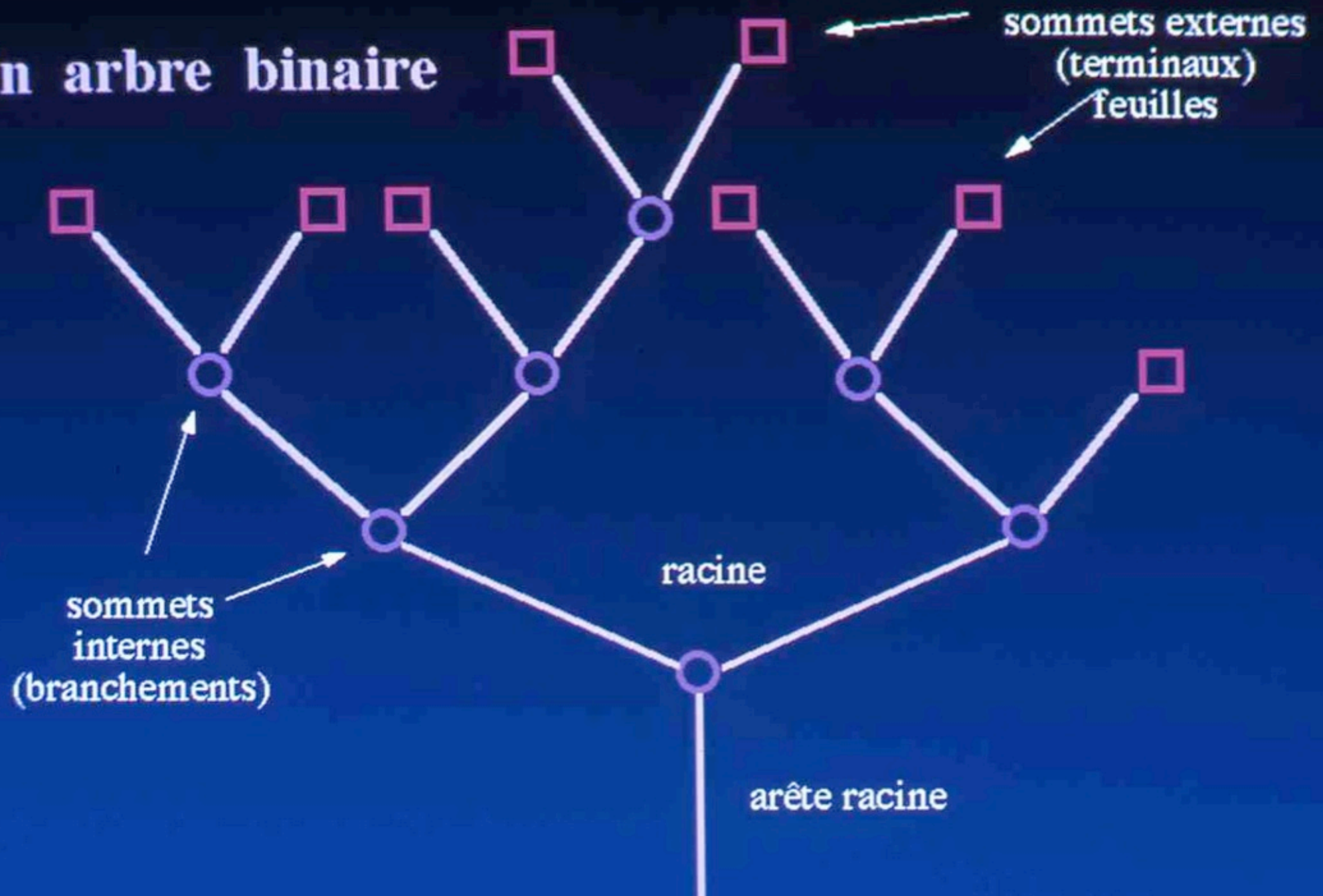




binary tree



Un arbre binaire

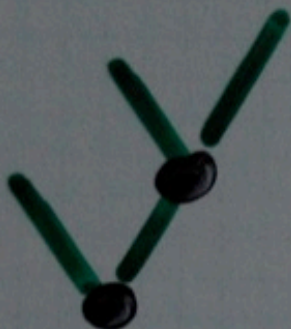
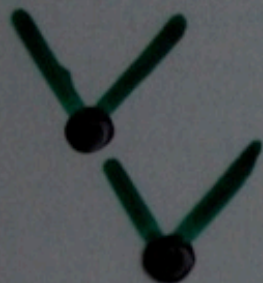


C_n = nombre
d'arbres binaires
ayant n
sommets internes
(et donc $n+1$ feuilles)
nombre de Catalan

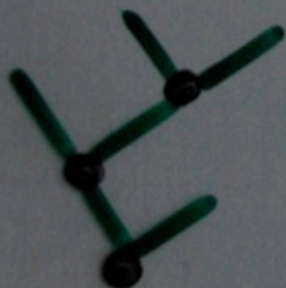
number of binary trees
having n internal vertices
(or $n+1$) leaves (external vertices)



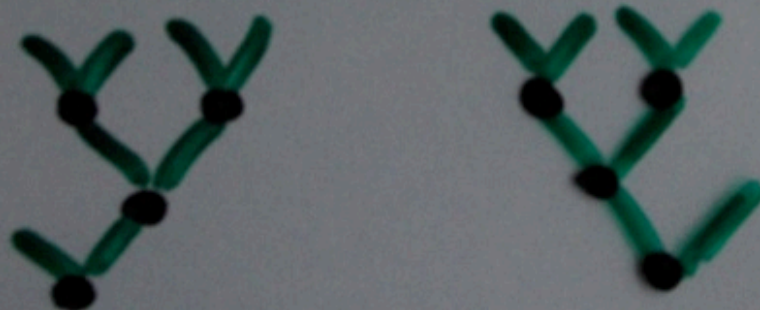
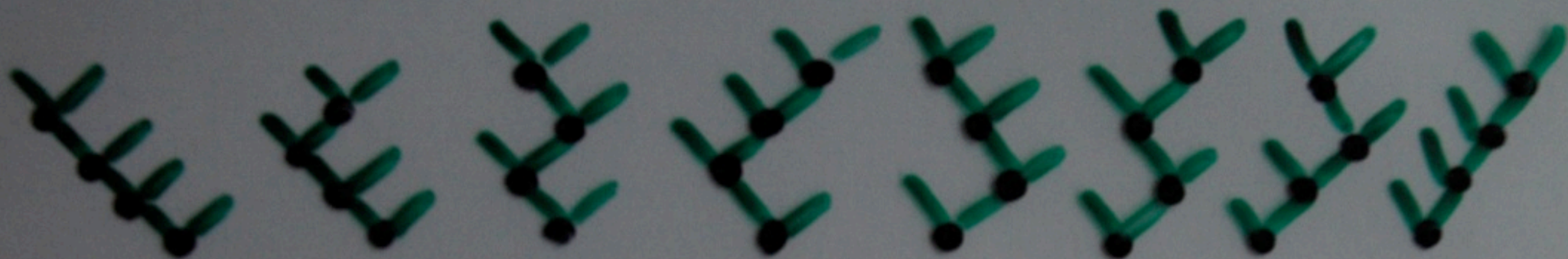
$$C_1 = 1$$



$$C_2 = 2$$



$$C_3 = 5$$



14

1

addition +

1

1

1

2

1

1

3

3

1

1

4

6

4

1

1

5

10

10

5

1

1

6

15

20

15

6

1

1

7

21

35

35

21

7

1

1

8

28

56

70

56

28

8

1

$$\frac{1}{1}$$

$$\frac{2}{2}$$

$$\frac{6}{3}$$

$$\frac{20}{4}$$

$$\frac{70}{5}$$

addition
division

+

:

1

2

5

14

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)! n!}$$

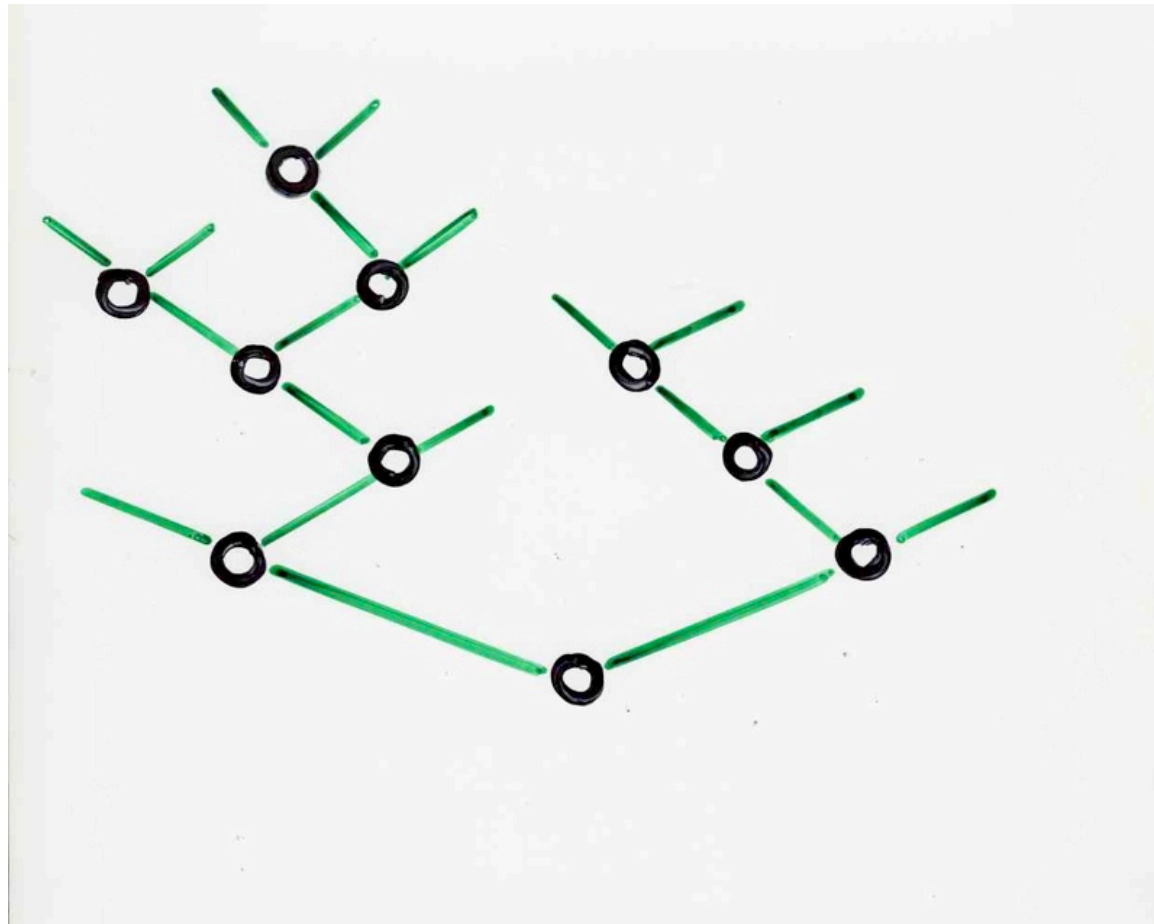
$$n! = 1 \times 2 \times \dots \times n$$

$$C_4 = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}{1 \times 2 \times 3 \times 4 \times 1 \times 2 \times 3 \times 4 \times 5}$$

$$= 14$$

$$2(2n+1)C_n = (n+2)C_{n+1}$$

$$2(2n+1)C_n = (n+2)C_{n+1}$$



①

addition +

1

1

1

1

②

-

①

1

3

3

1

2

1

4

⑥

-

④

1

1

5

10

10

5

1

1

6

15

②0

-

①5

6

1

1

7

21

35

35

21

7

1

1

8

28

56

⑦0

-

⑤6

28

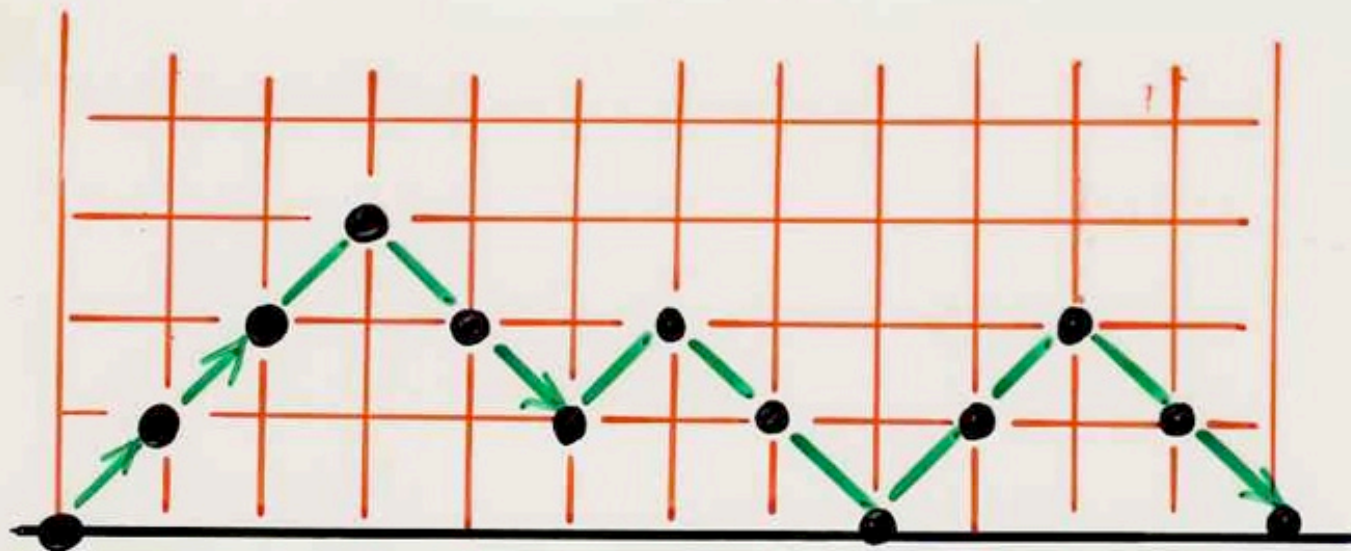
8

1

14

§5 bijections

(one-to-one correspondences)

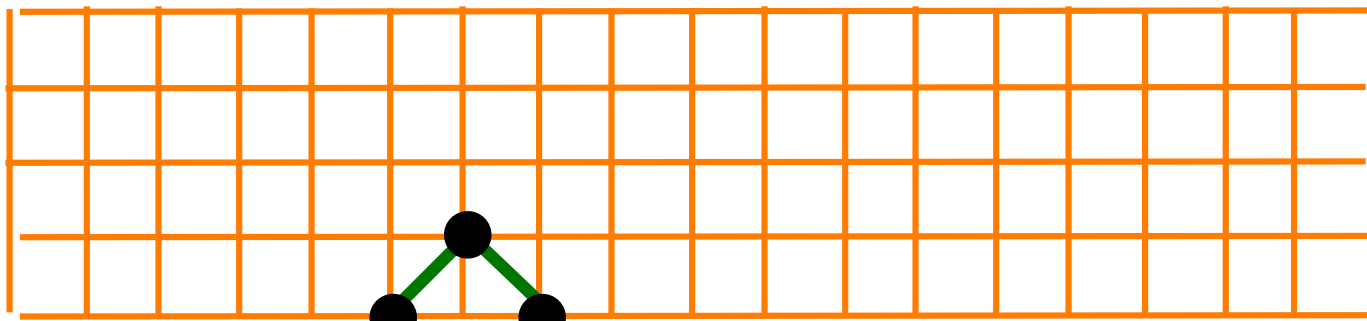


length

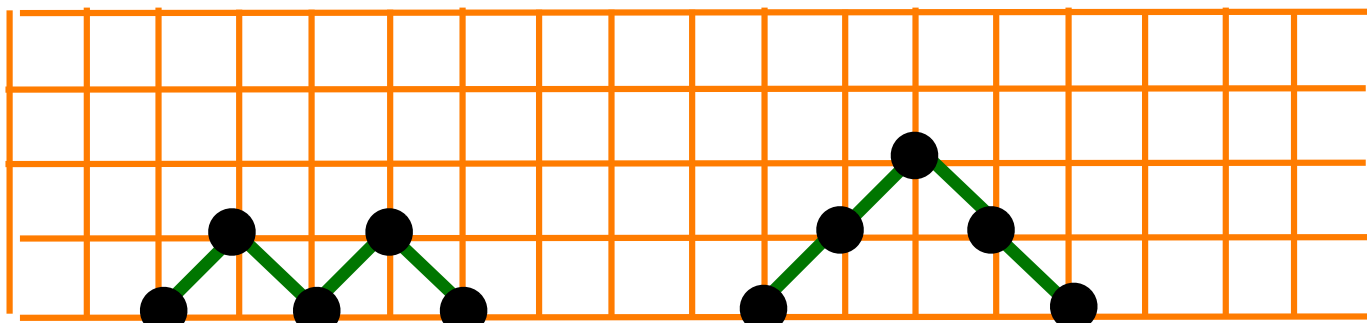
$$2n = 12$$

Dyck path

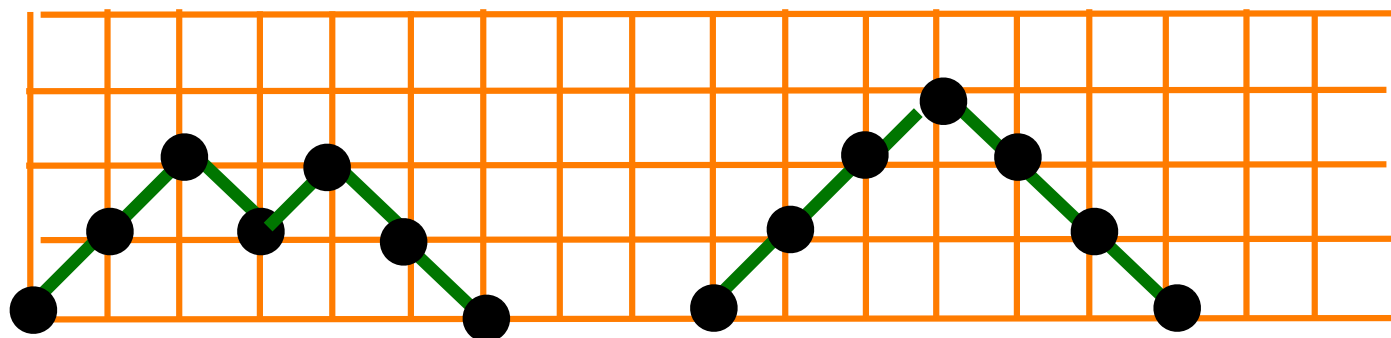
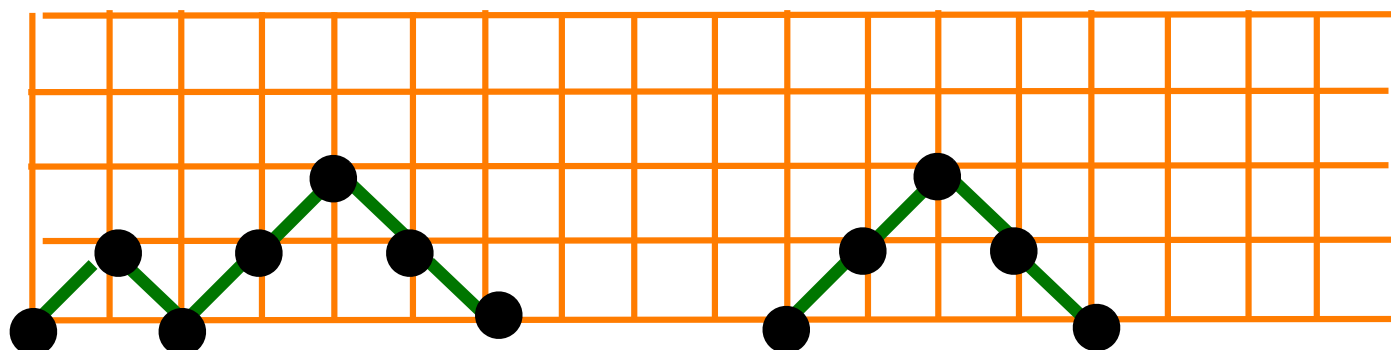
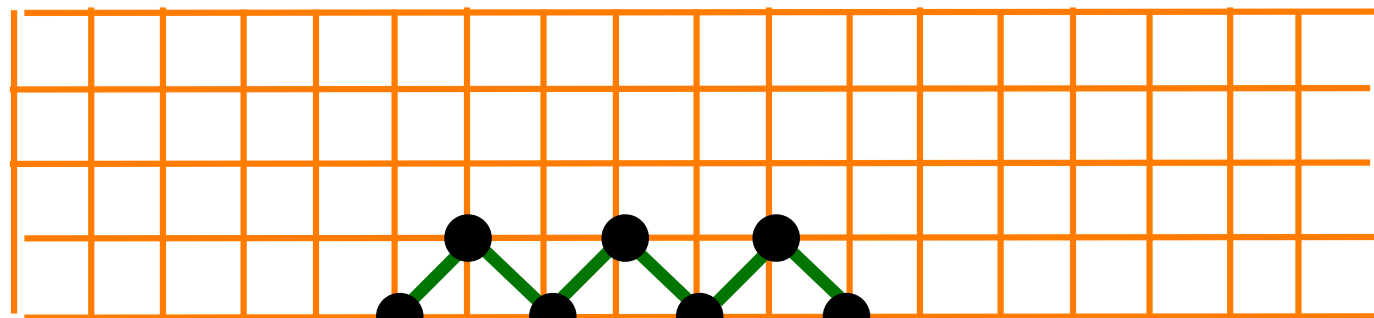
C_n = number of Dyck path
of length $2n$



1



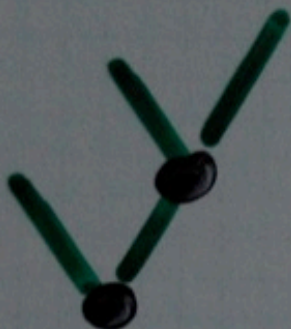
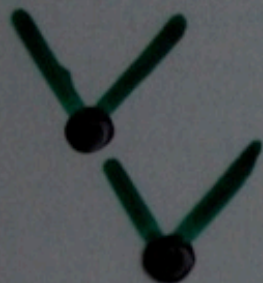
2



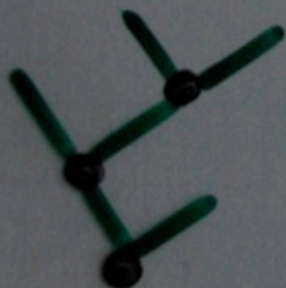
5



$$C_1 = 1$$



$$C_2 = 2$$



$$C_3 = 5$$

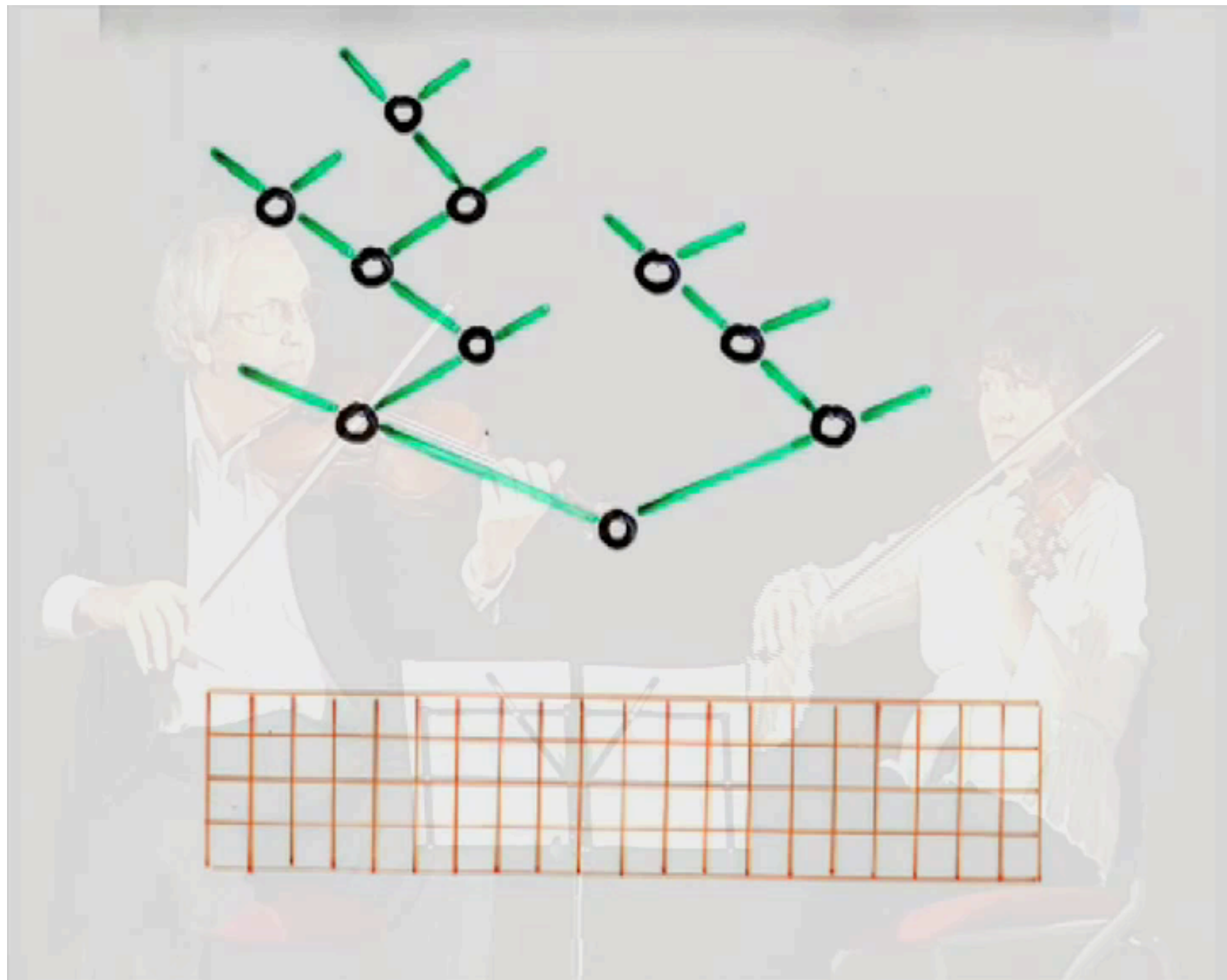
bijection

or one-to-one correspondence

binary trees with n internal vertices



Dyck paths length $2n$



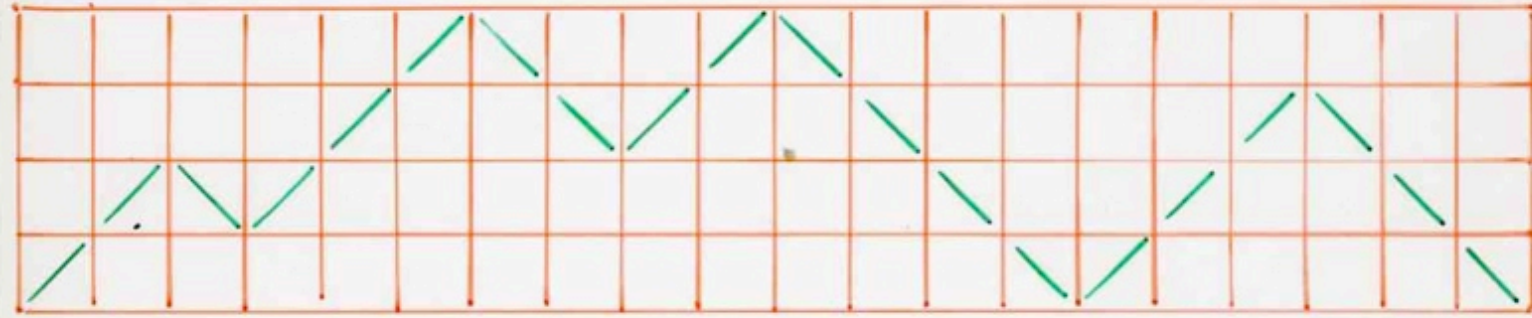
bijection

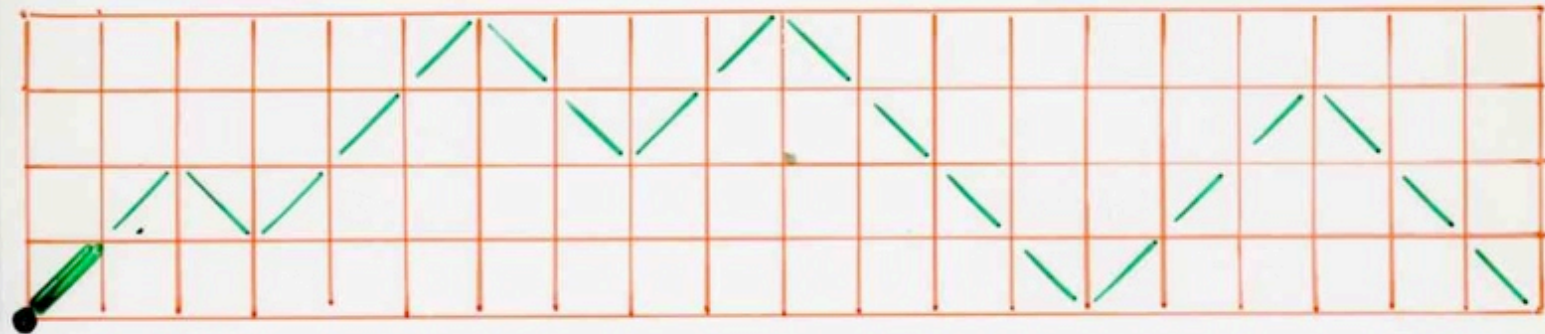
(one-to-one correspondance)

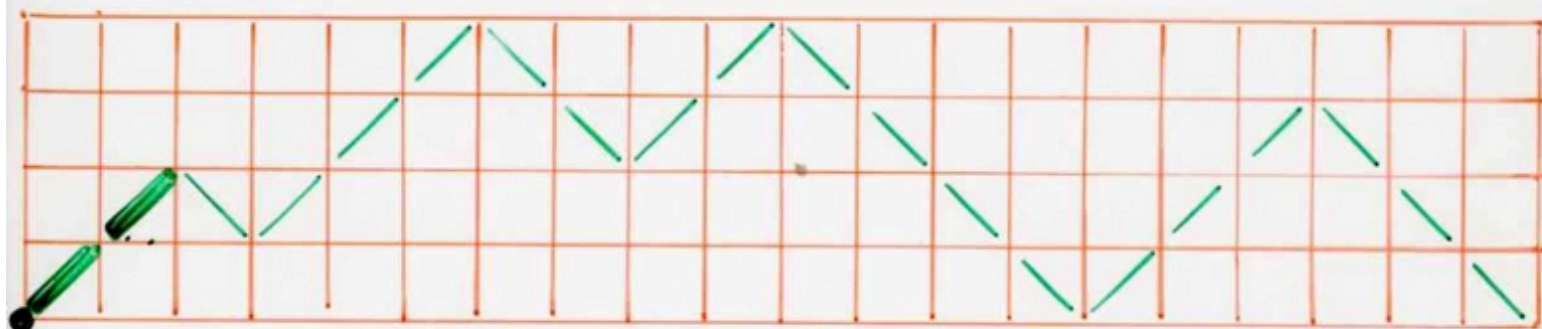
binary with n (internal) vertices

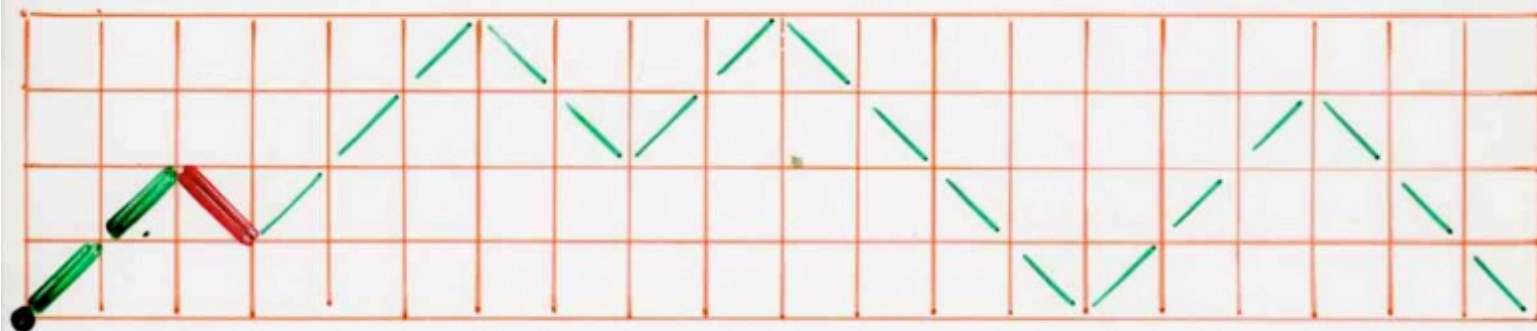
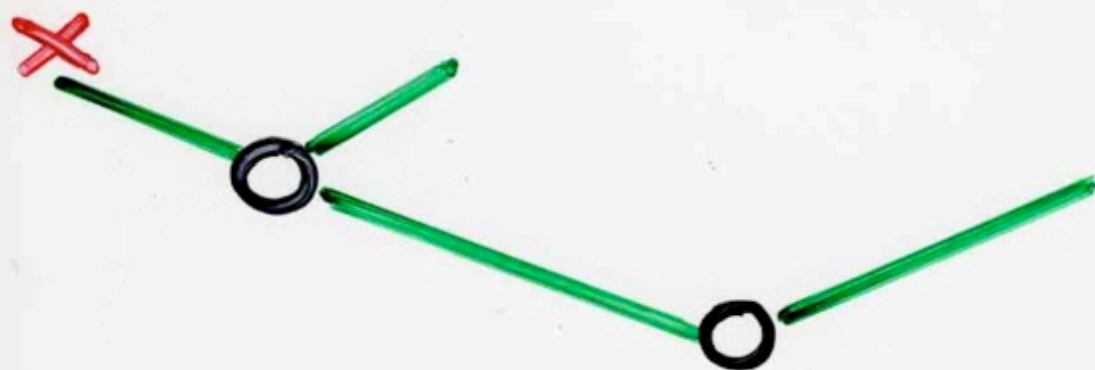


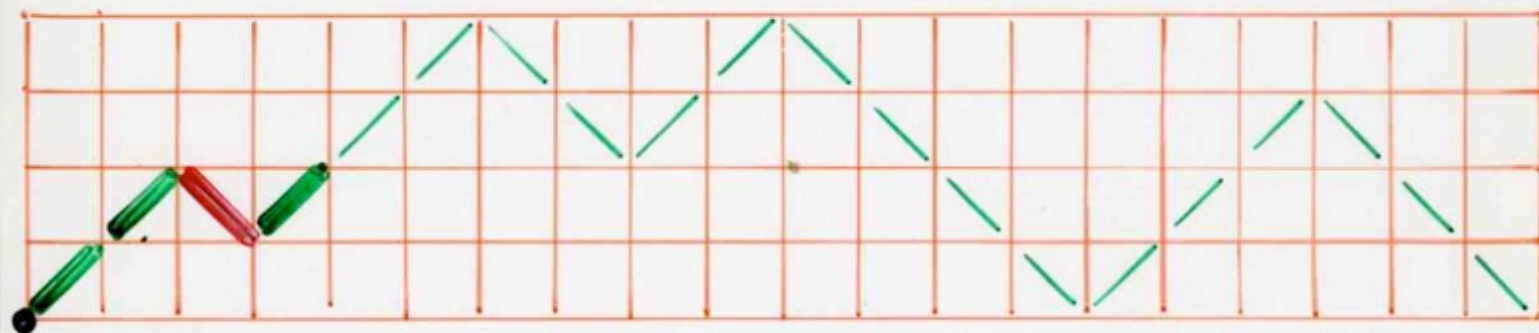
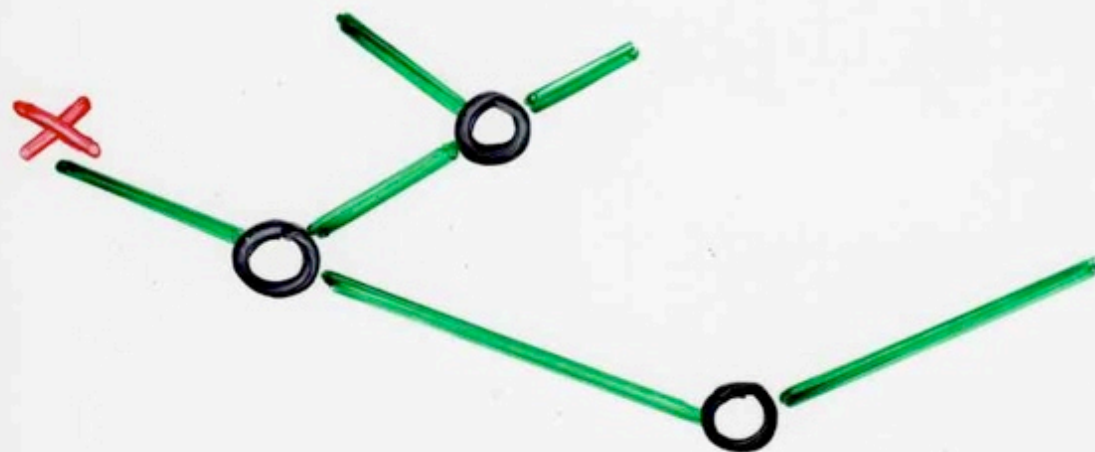
Dyck paths length $2n$

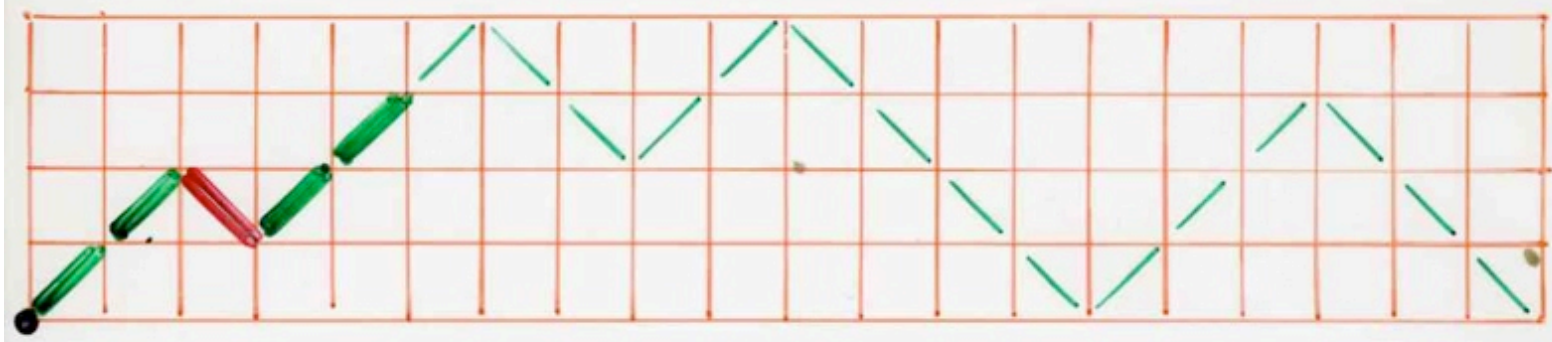
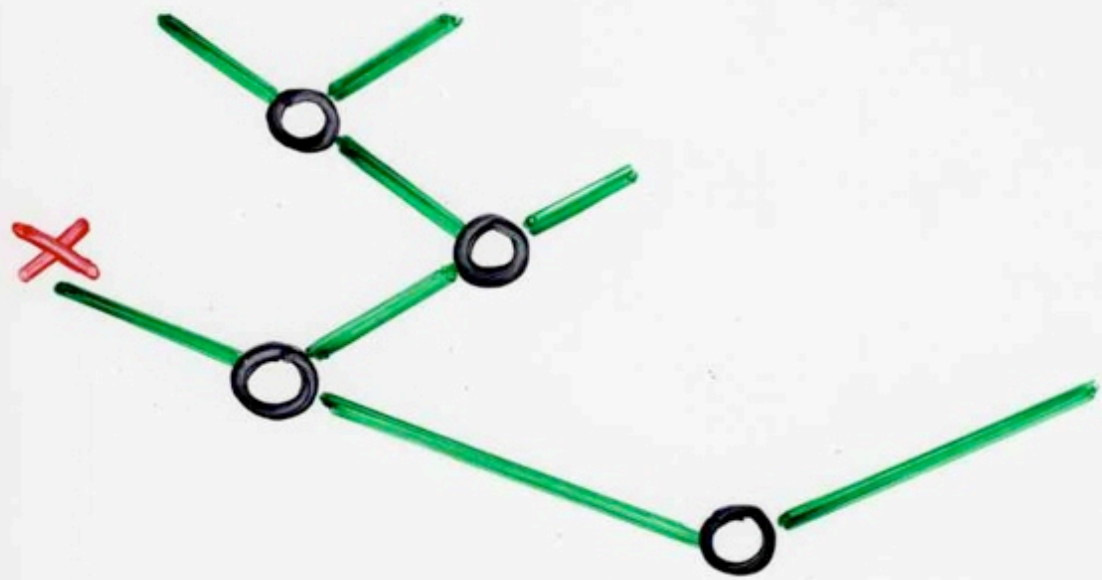


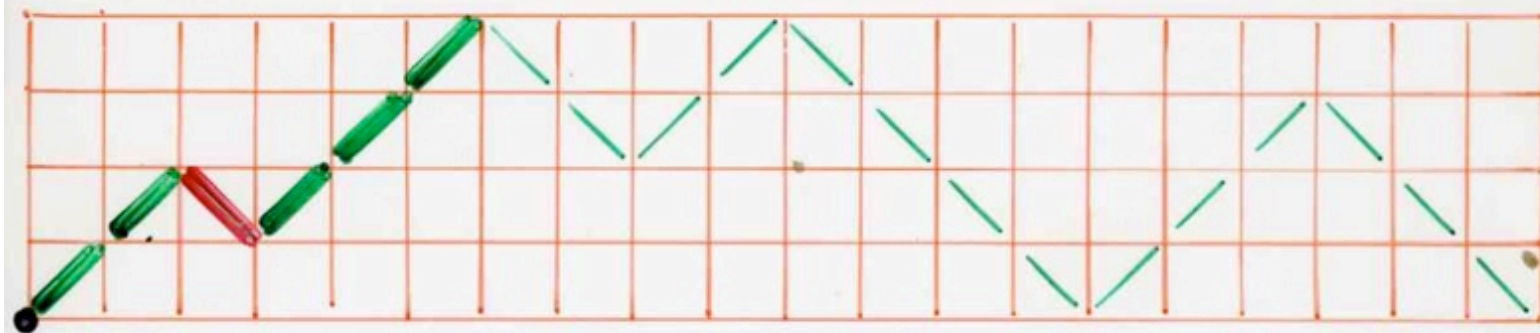
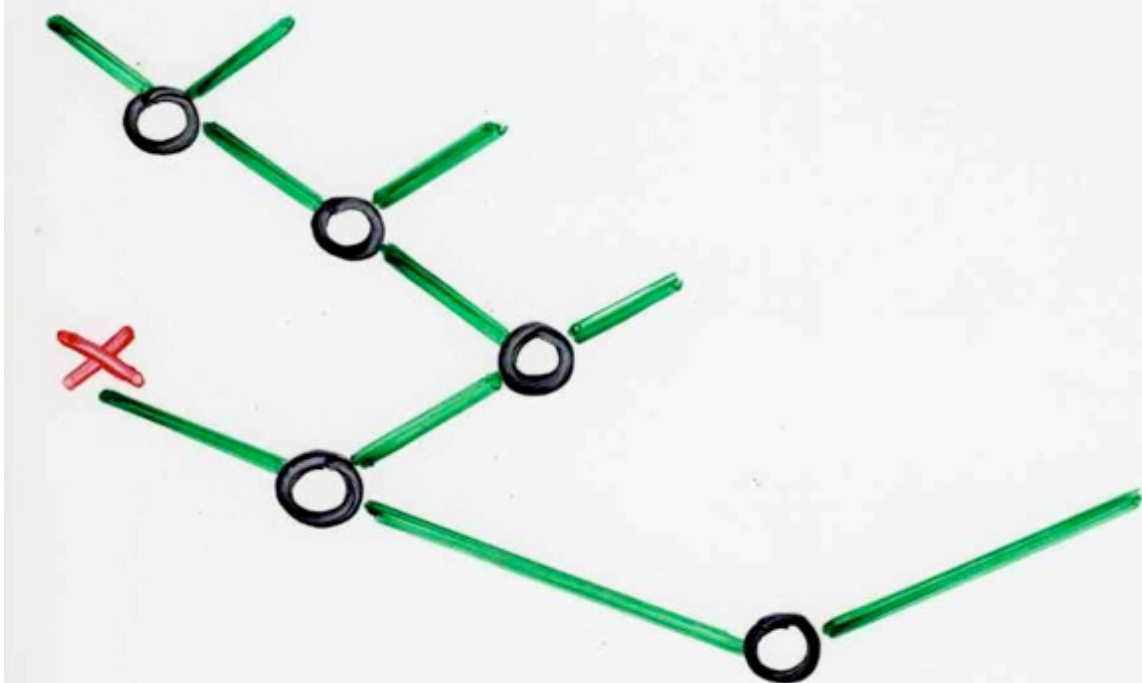


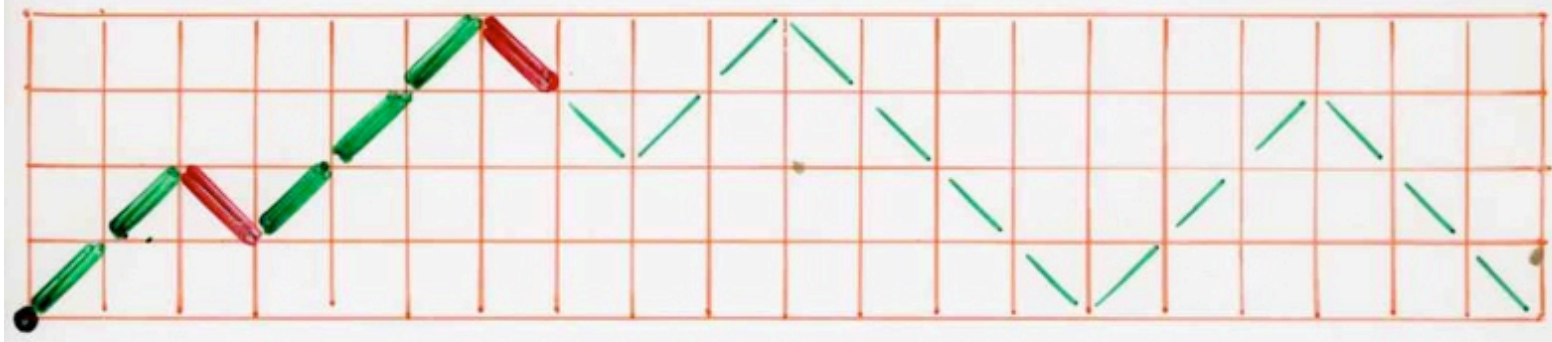
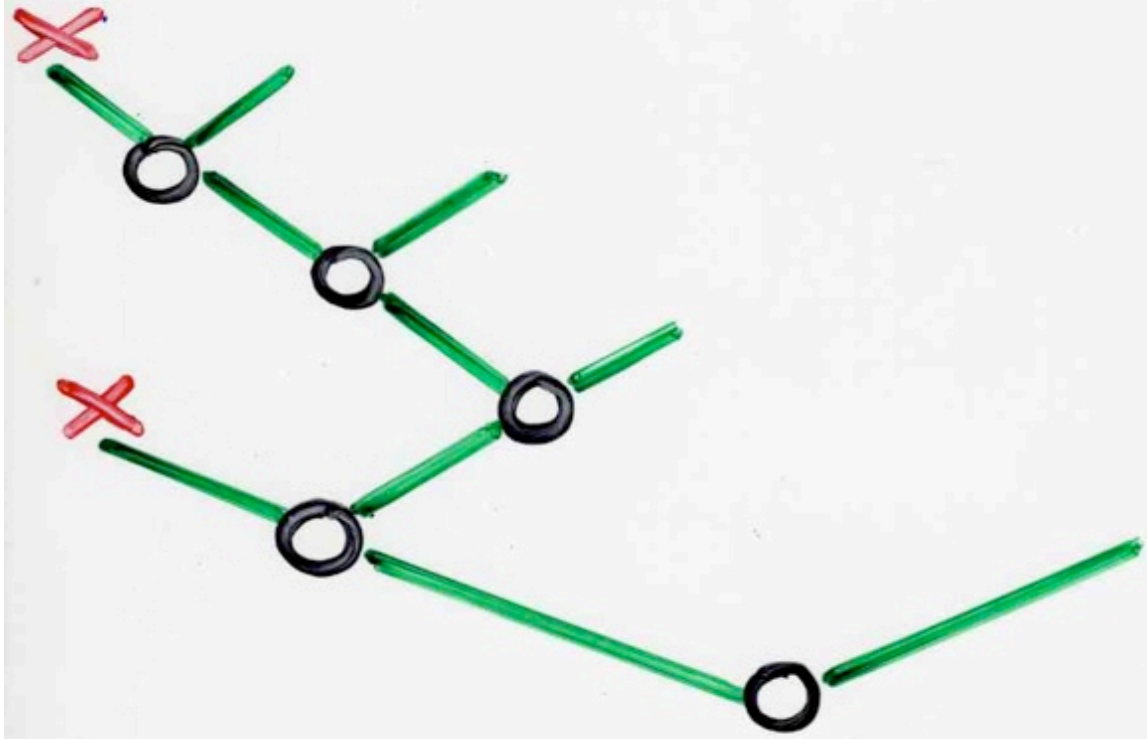


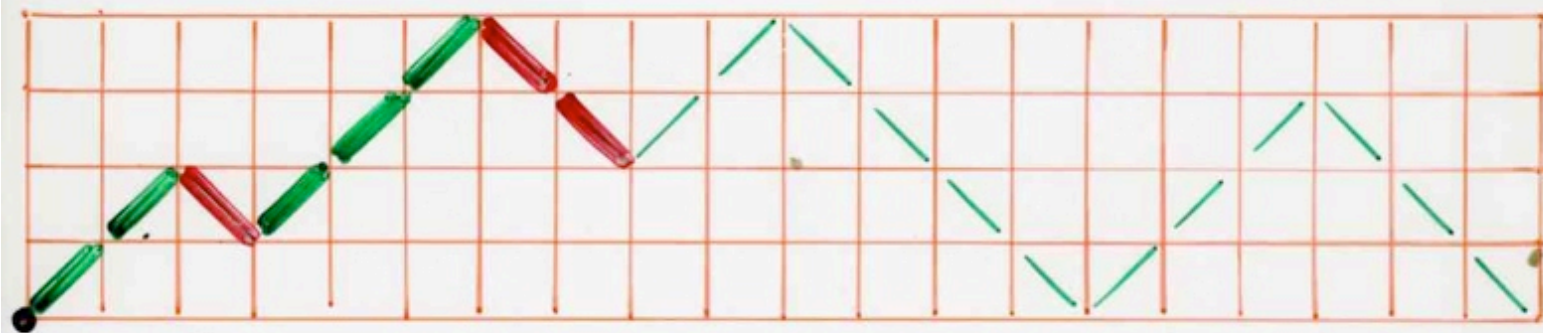
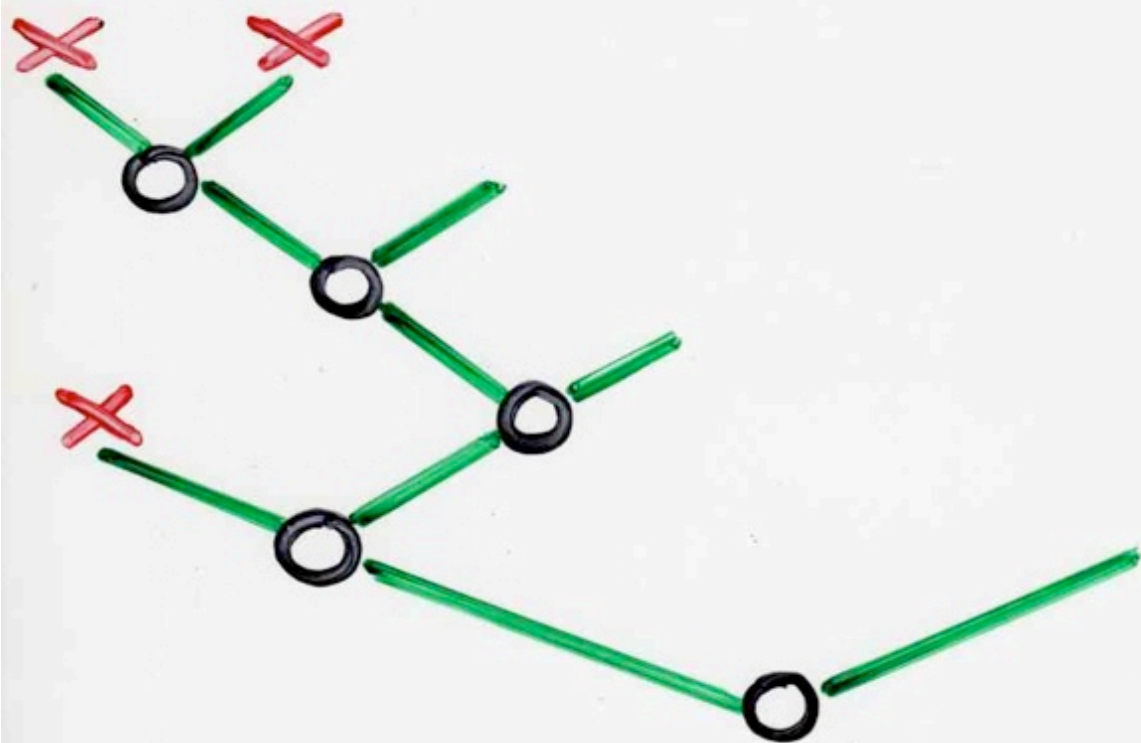


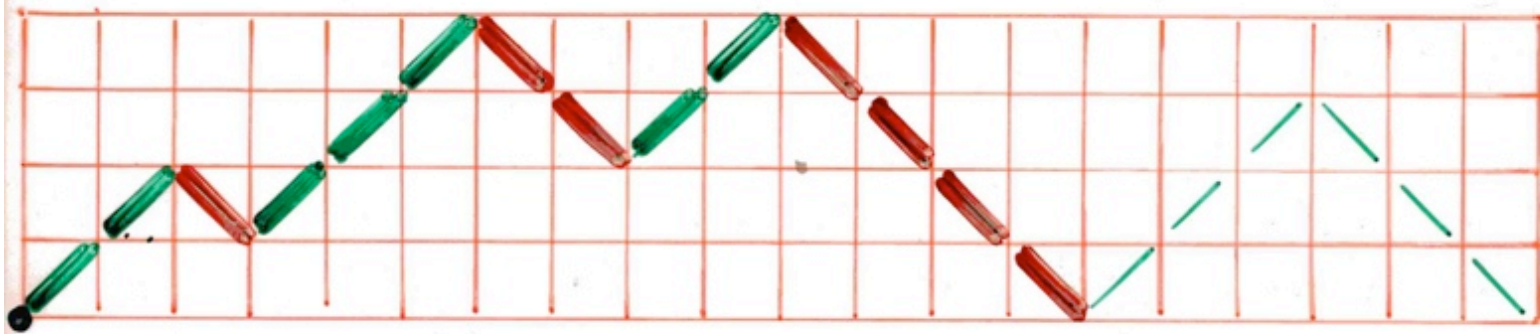
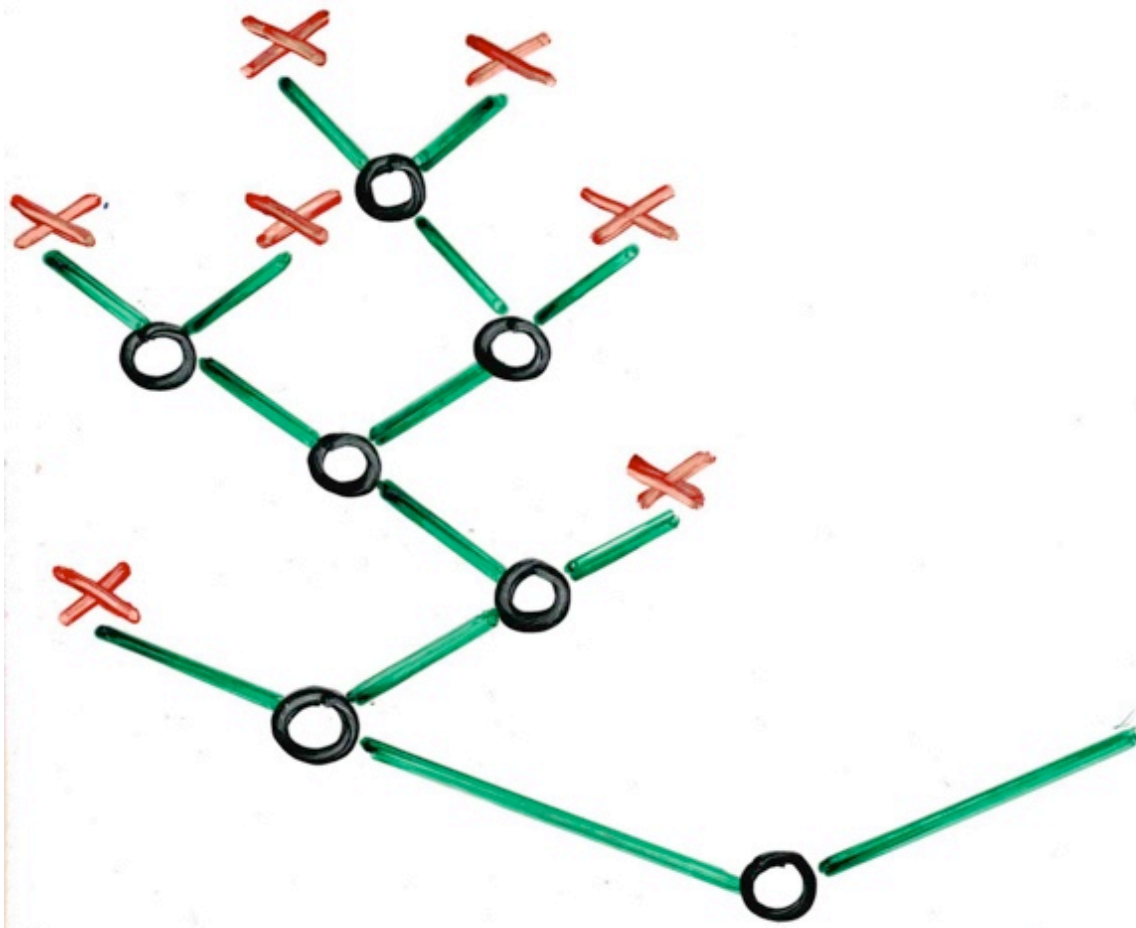


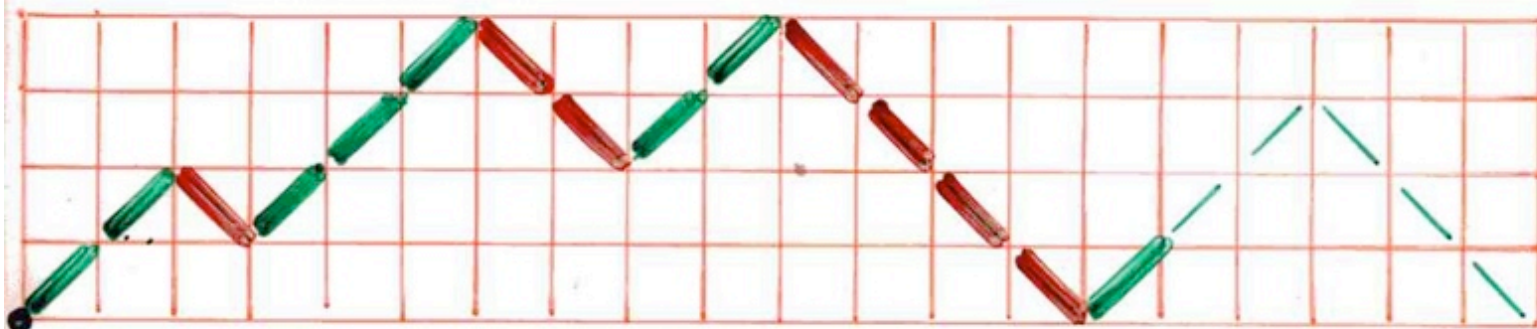
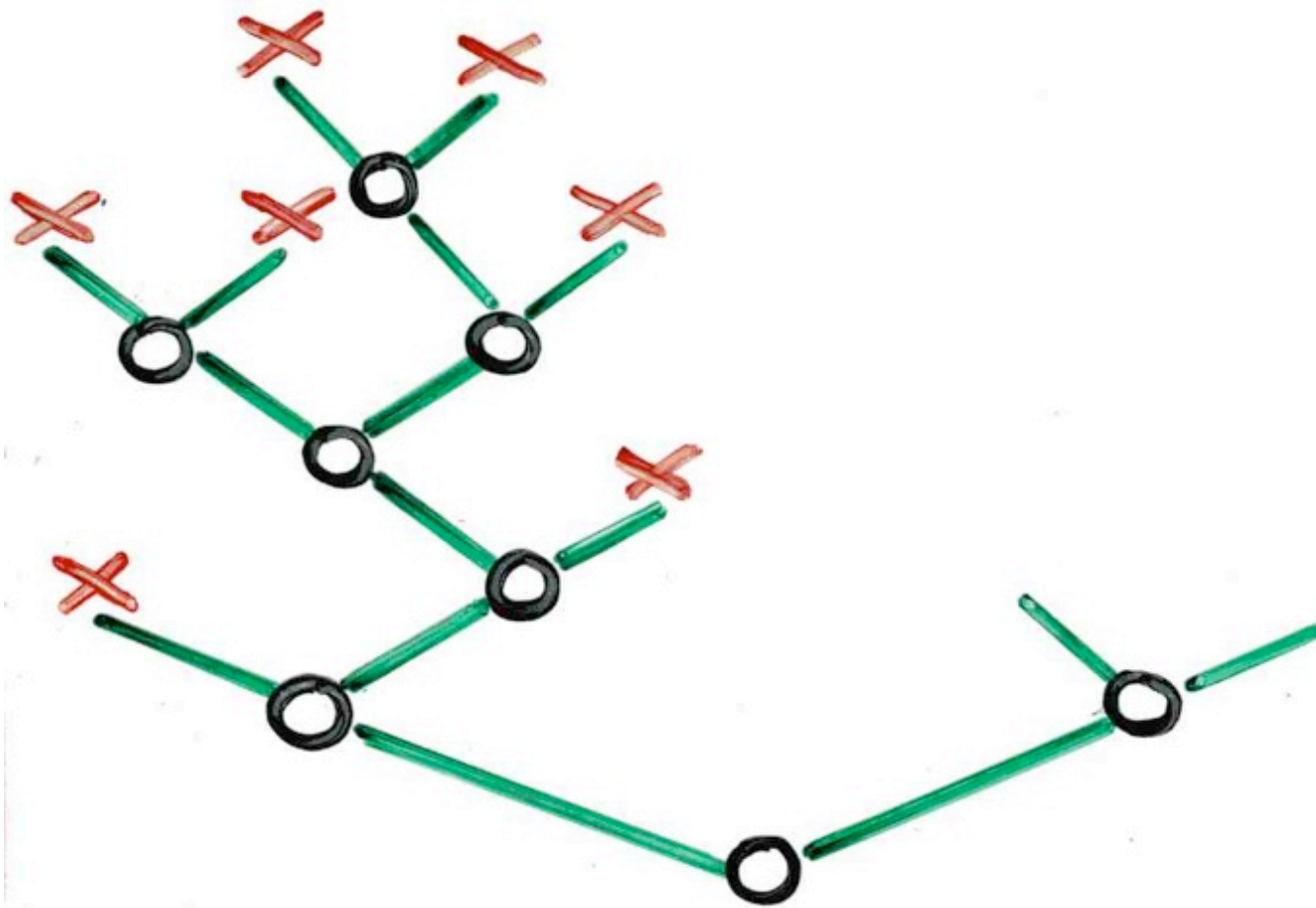


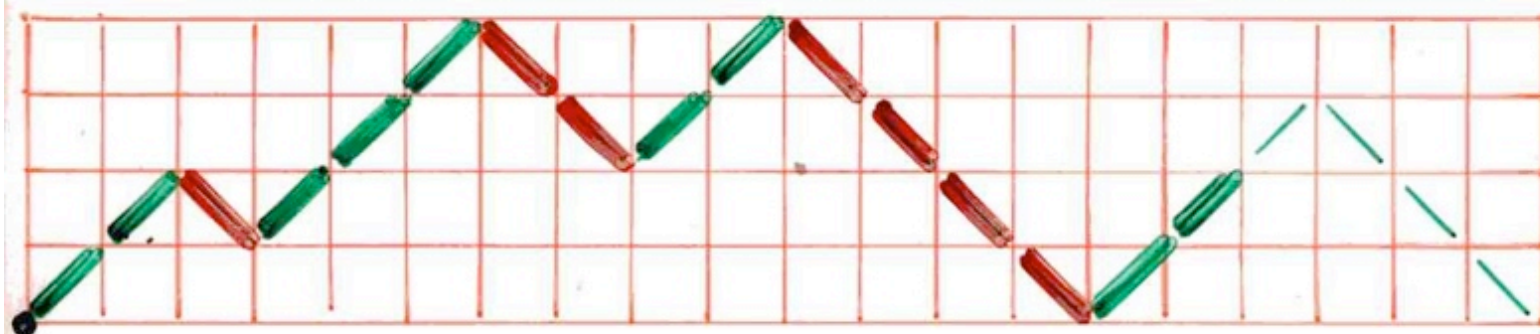
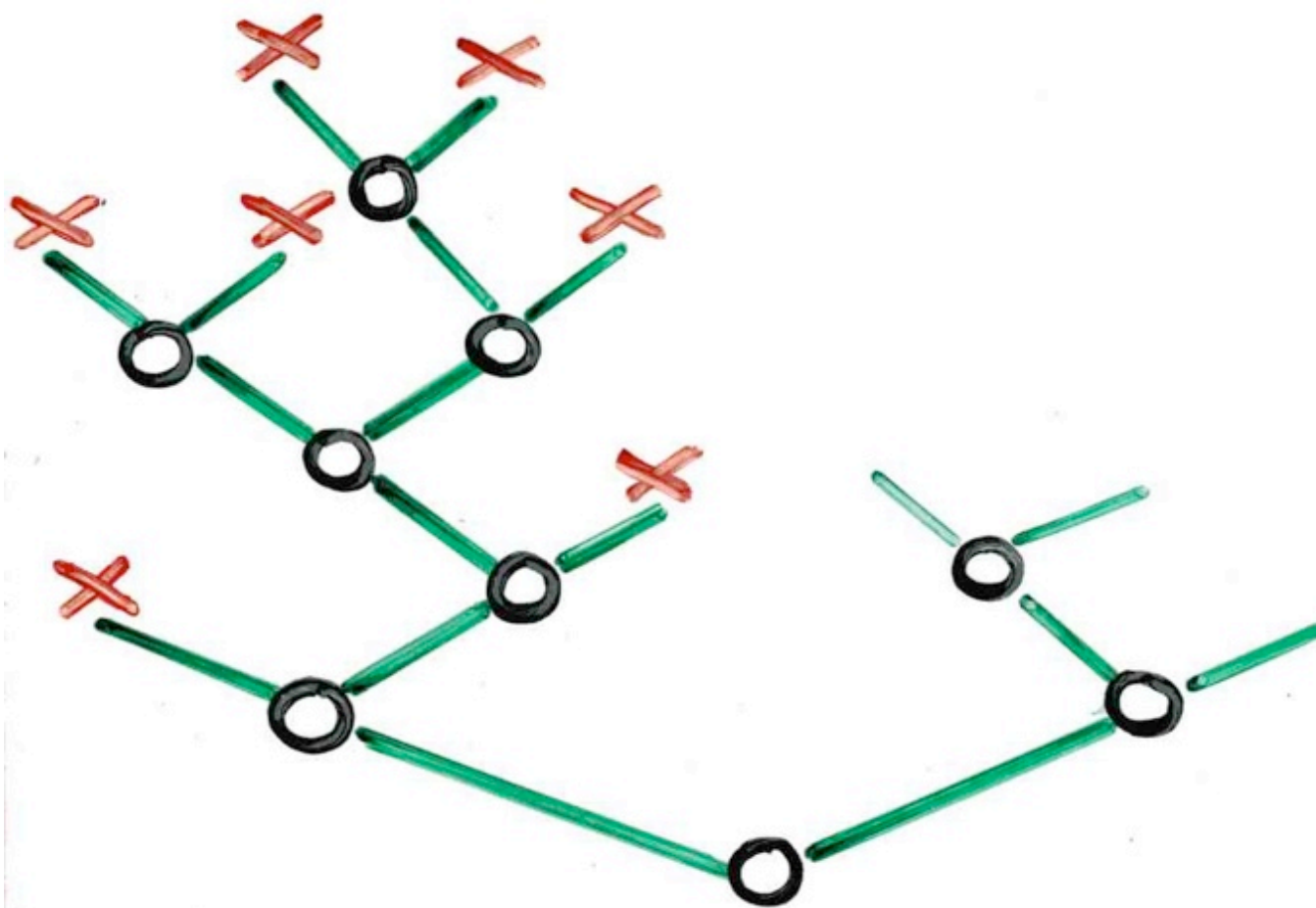


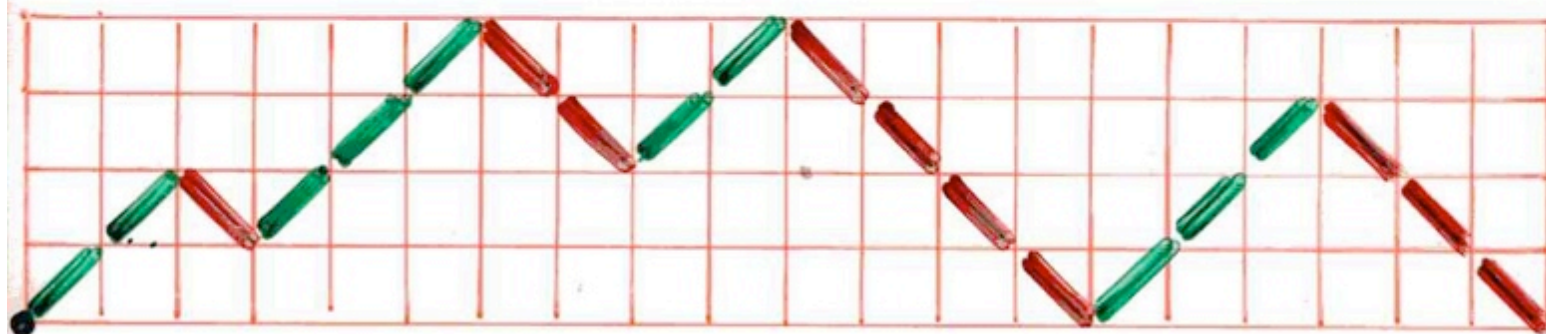
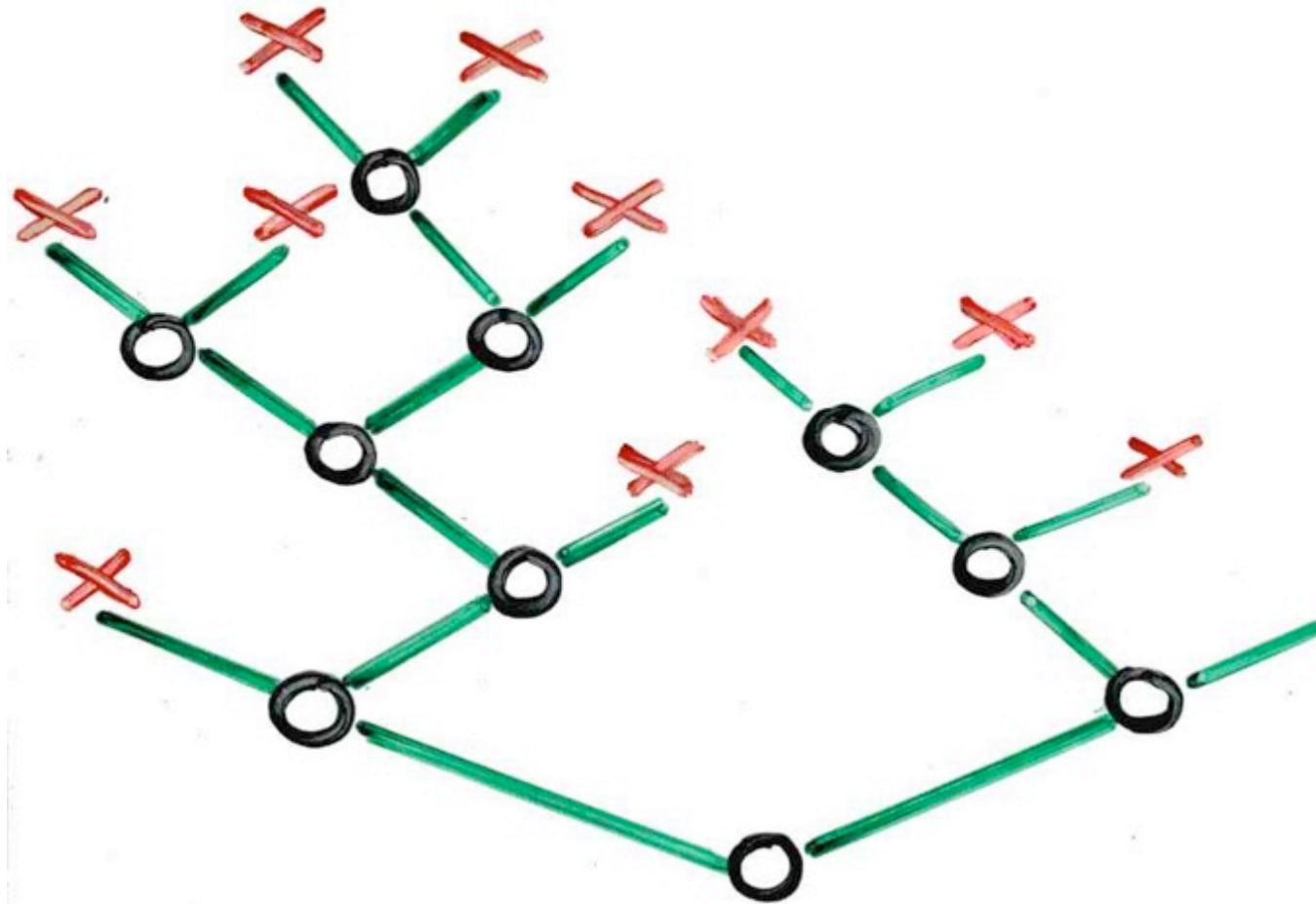


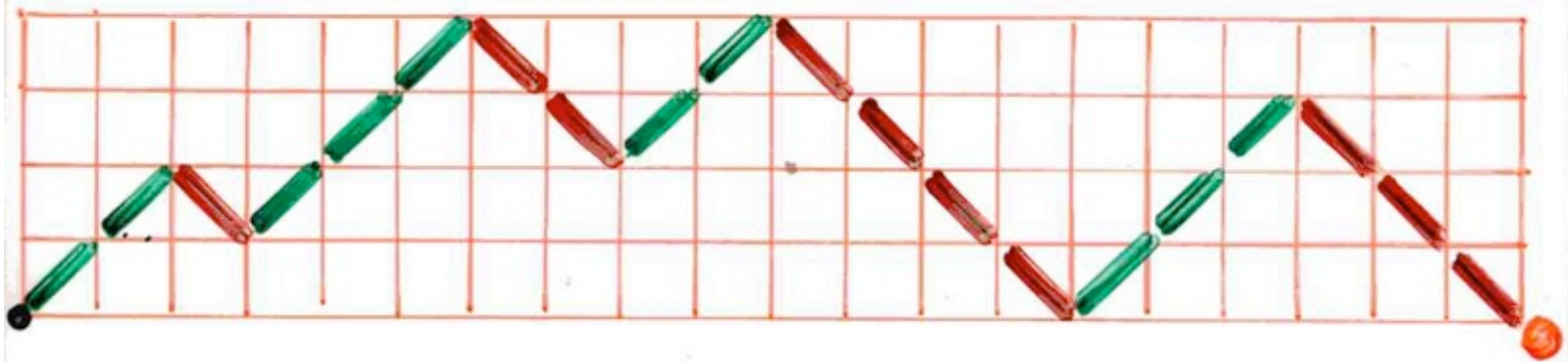
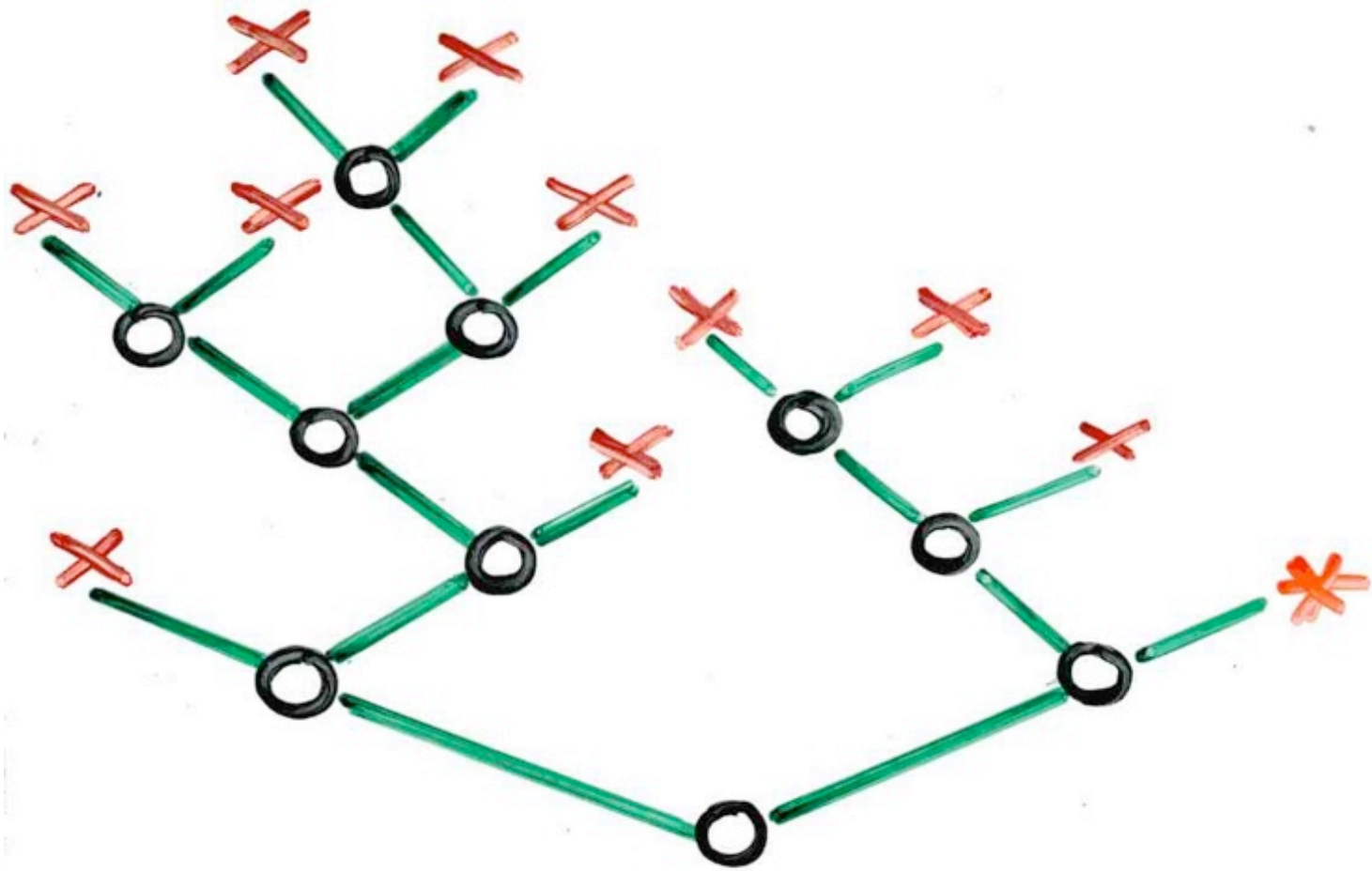












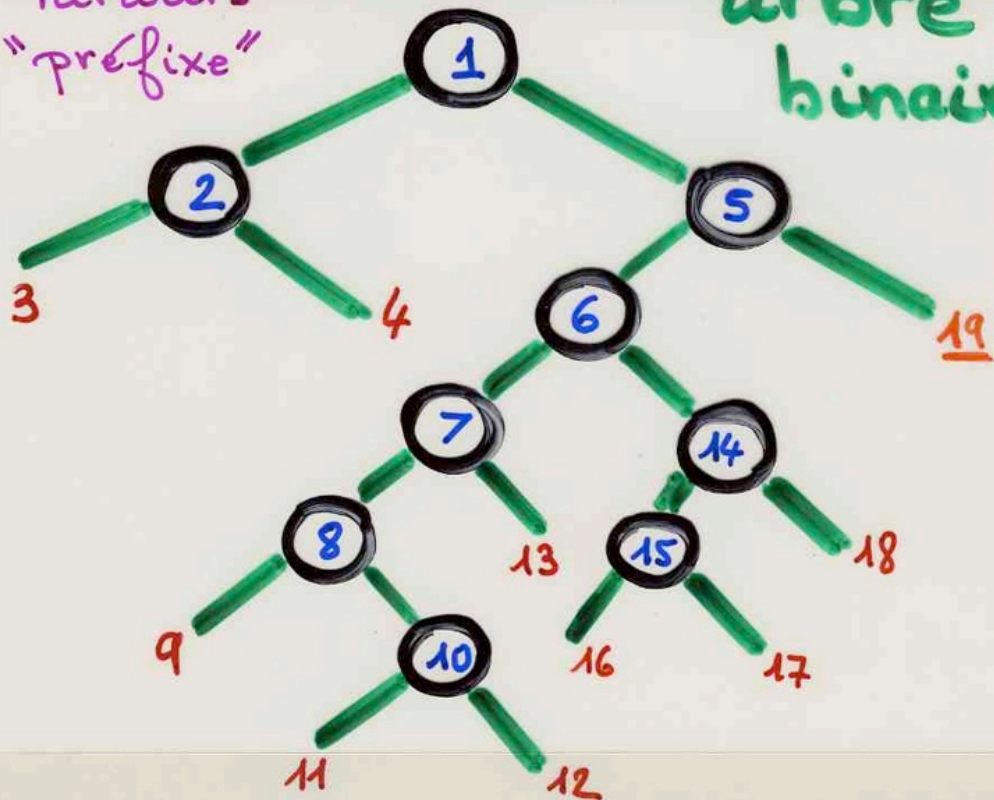
A diagram showing a network of black circular nodes connected by green lines, representing a molecular structure or a network graph. The nodes are arranged in a hierarchical, branching pattern. The top node connects to two nodes below it. The left node of this pair connects to two more nodes, while the right node connects to one. The bottom-most node connects to two nodes, which then connect to two more nodes at the top of the lower section. The green lines represent bonds or connections between the nodes. In the top right corner, the text "Arbeits- bindung" is partially visible in green.

Period	Value
0	0
1	1
2	2
3	1
4	0
5	1
6	2
7	3
8	4
9	3
10	4
11	3
12	2
13	1
14	2
15	3
16	2
17	1
18	0

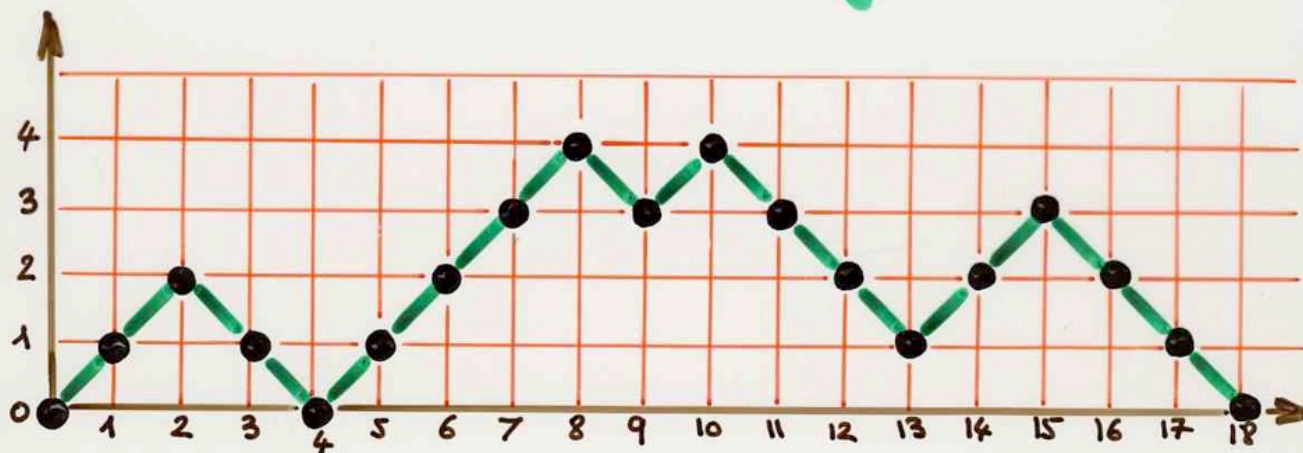
prefix
order

Parcours
"préfixe"

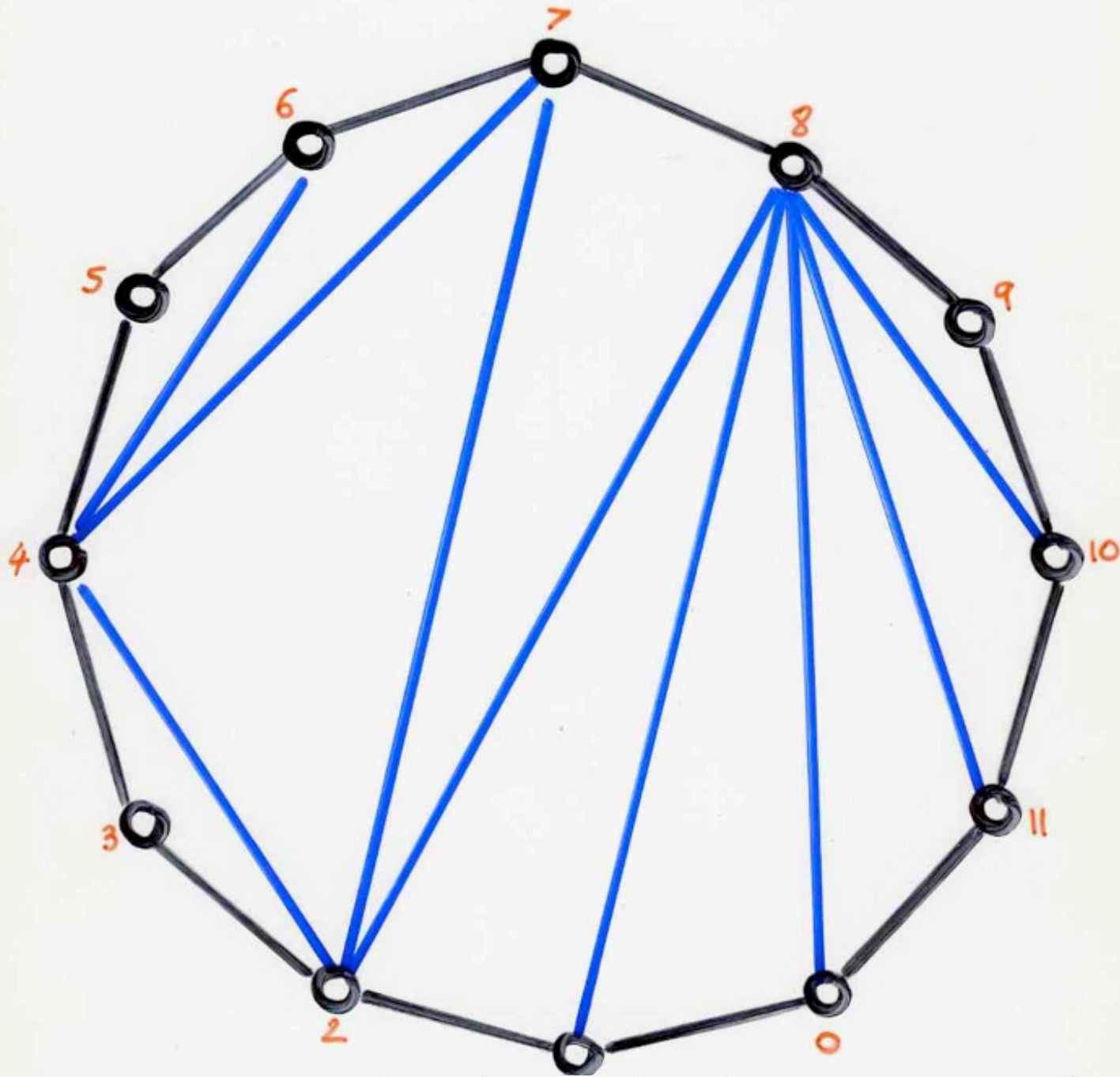
arbre
binaire



chemin de Dyck



triangulations



une triangulation d'un polygone

bijection

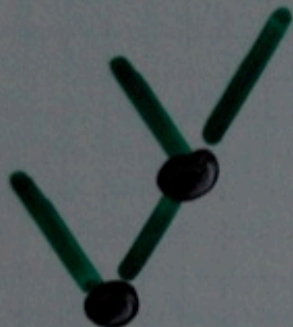
triangulations



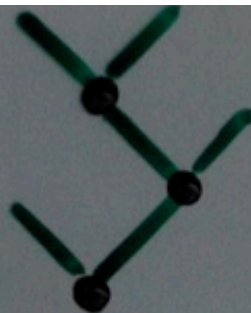
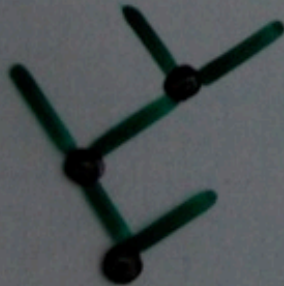
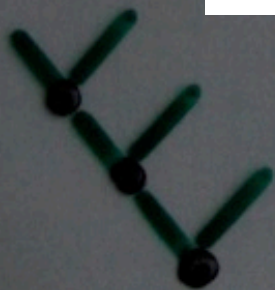
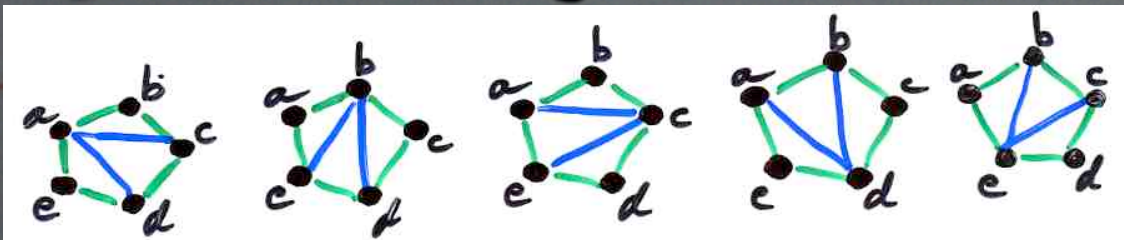
binary trees



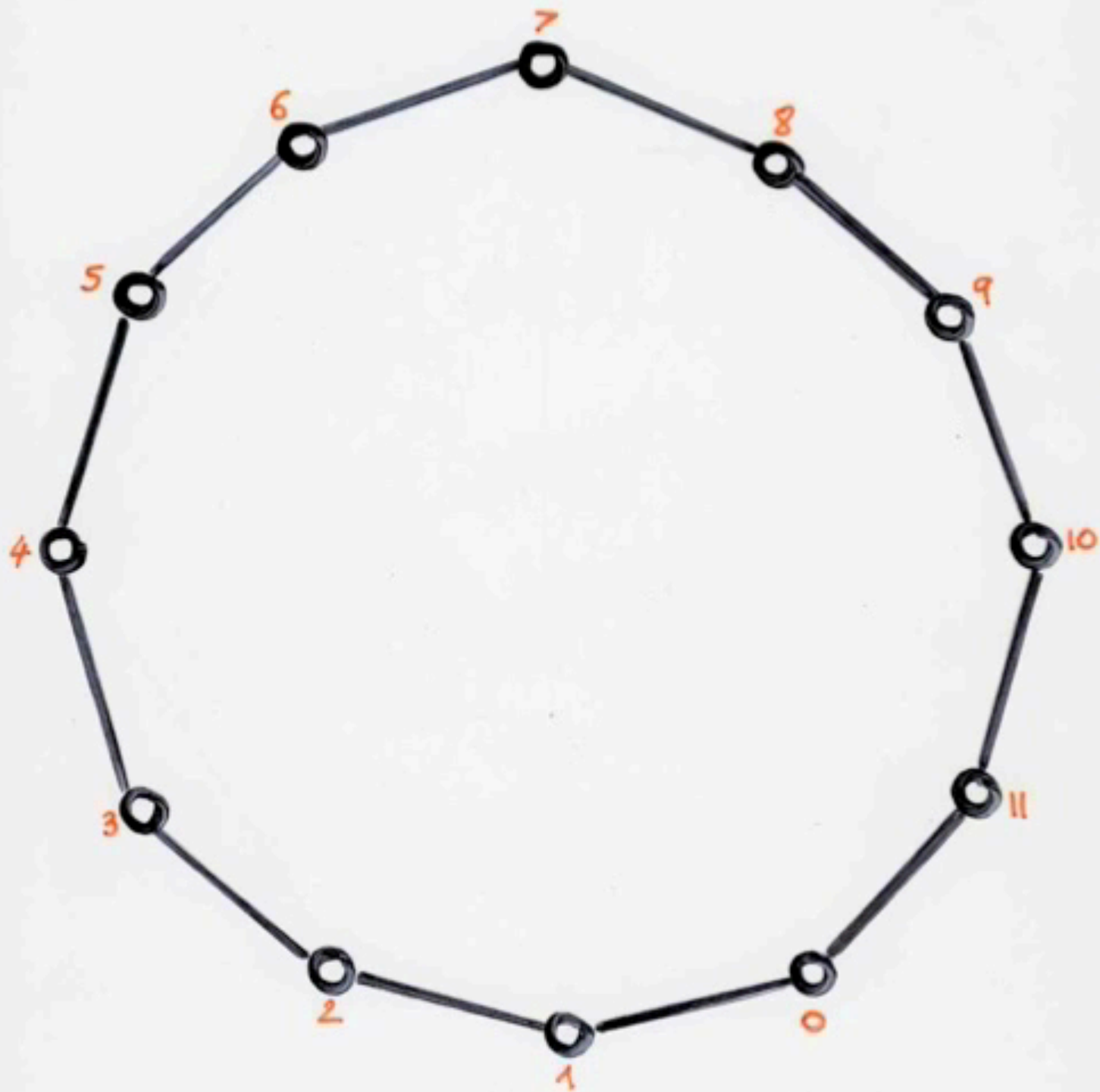
$$C_1 = 1$$



$$C_2 = 2$$

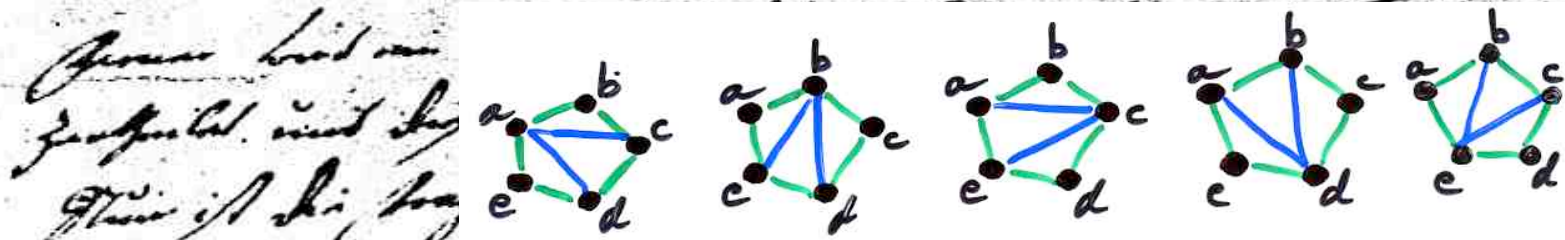


$$C_3 = 5$$





Geht, und steht hier auf 8 nicht liegenden Seiten gegeben. Auf 5 Diagonalen: I. $\frac{a}{d}$; II. $\frac{b}{e}$; III. $\frac{c}{f}$; IV. $\frac{d}{a}$; V. $\frac{e}{b}$



Hier ist die Folge $n-3$ Diagonalen in $n-2$ Triangula gegeben, an
bei beliebig liegenden Seiten, selbst gegeben.

Schließlich sind **les nombres "de Catalan"**

Wenn $n = 1, 2, 5, 14, 42, 132, 429, 1430, \dots$

ist $x = 1, 2, 5, 14, 42, 132, 429, 1430, \dots$

Hieraus sieht man den Zusammenhang. In generaliter
sich

$$x = \frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \cdot \dots \cdot (4n-10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot \dots \cdot (n-1)} = \frac{(2n)!}{(n+1)!n!}$$

$5 = 2 \cdot \frac{10}{4}$; $14 = 5 \cdot \frac{14}{3}$; $42 = 14 \cdot \frac{18}{6}$; $132 = 42 \cdot \frac{22}{7}$

$C_n = \frac{1}{n+1} \binom{2n}{n}$ heißt die $n!$ = $1 \times 2 \times 3 \times \dots \times n$

Euler introduit les séries génératrices!

$$\frac{1 - 2a - \sqrt{1 - 4a^2}}{2a^2}$$

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc}$$

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc} = \frac{1 - 2a - \sqrt{1 - 4a^2}}{2a^2}$$

$$\text{all. pour } a = \frac{1}{4} \quad 1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} + \text{etc} = 4$$

$$a = \frac{1}{4}$$

et même les prémisses

de la "combinatoire analytique"!

Alon. Joseph Lebeuvre

le 4^e sept
1751.

4 Sept 1751
Berlin

grosjean
Euler