The beauty of mathematics

part II: trees, Catalan numbers, bijections

Pope John Paul II
College of Education
Pondicherry, 23 Feb 2012

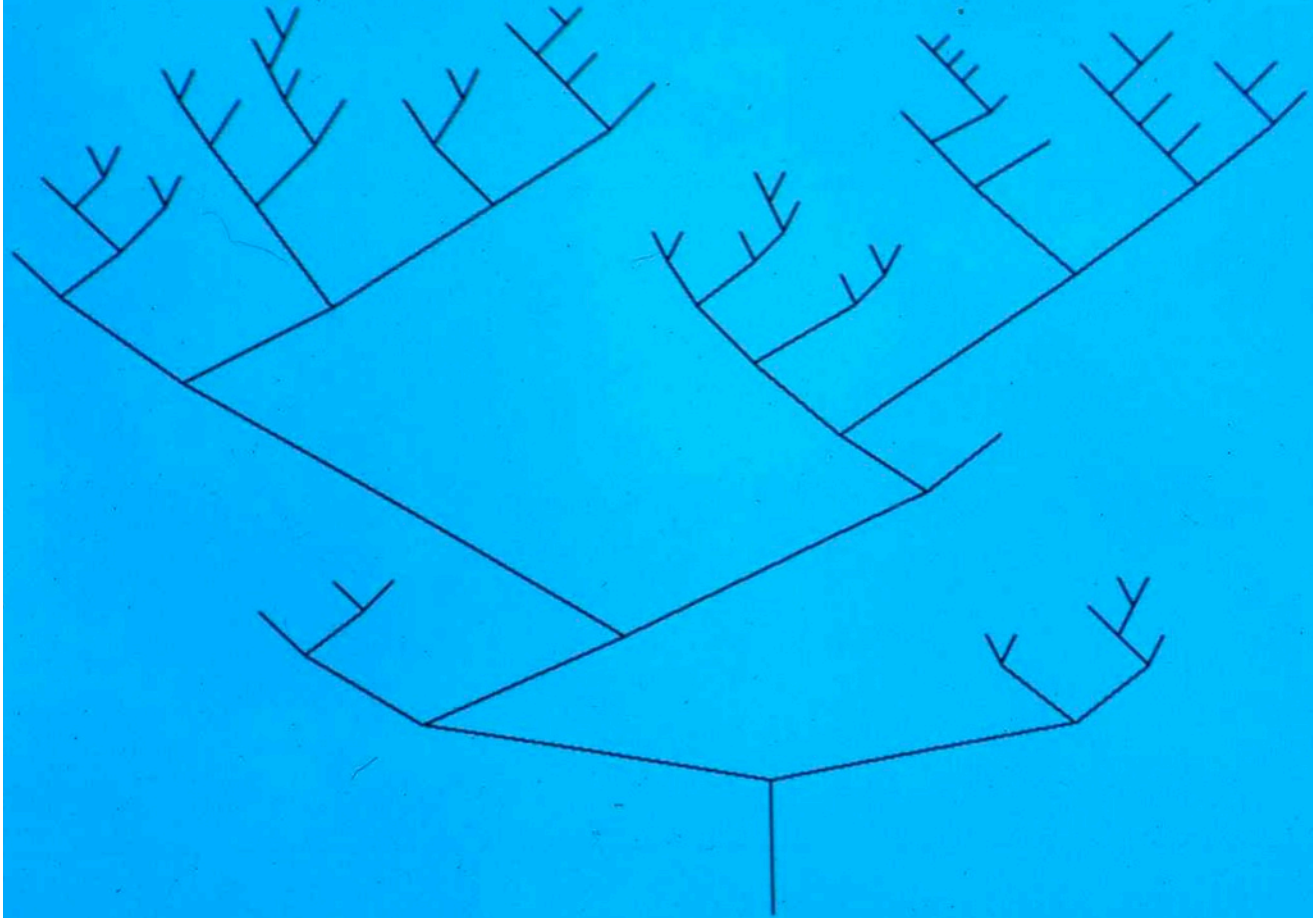
Xavier Viennot
CNRS, Bordeaux, France

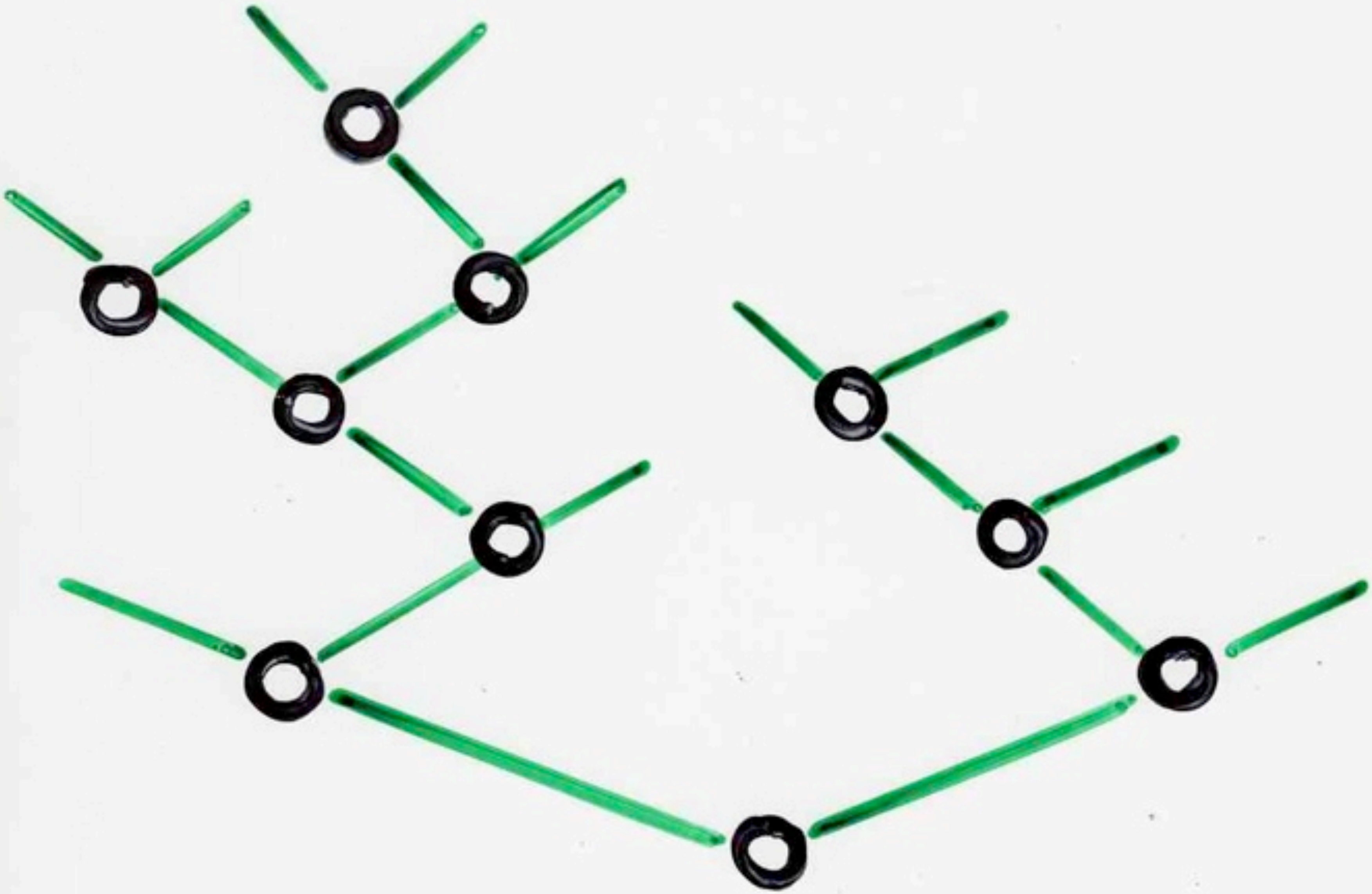
§4 Catalan numbers

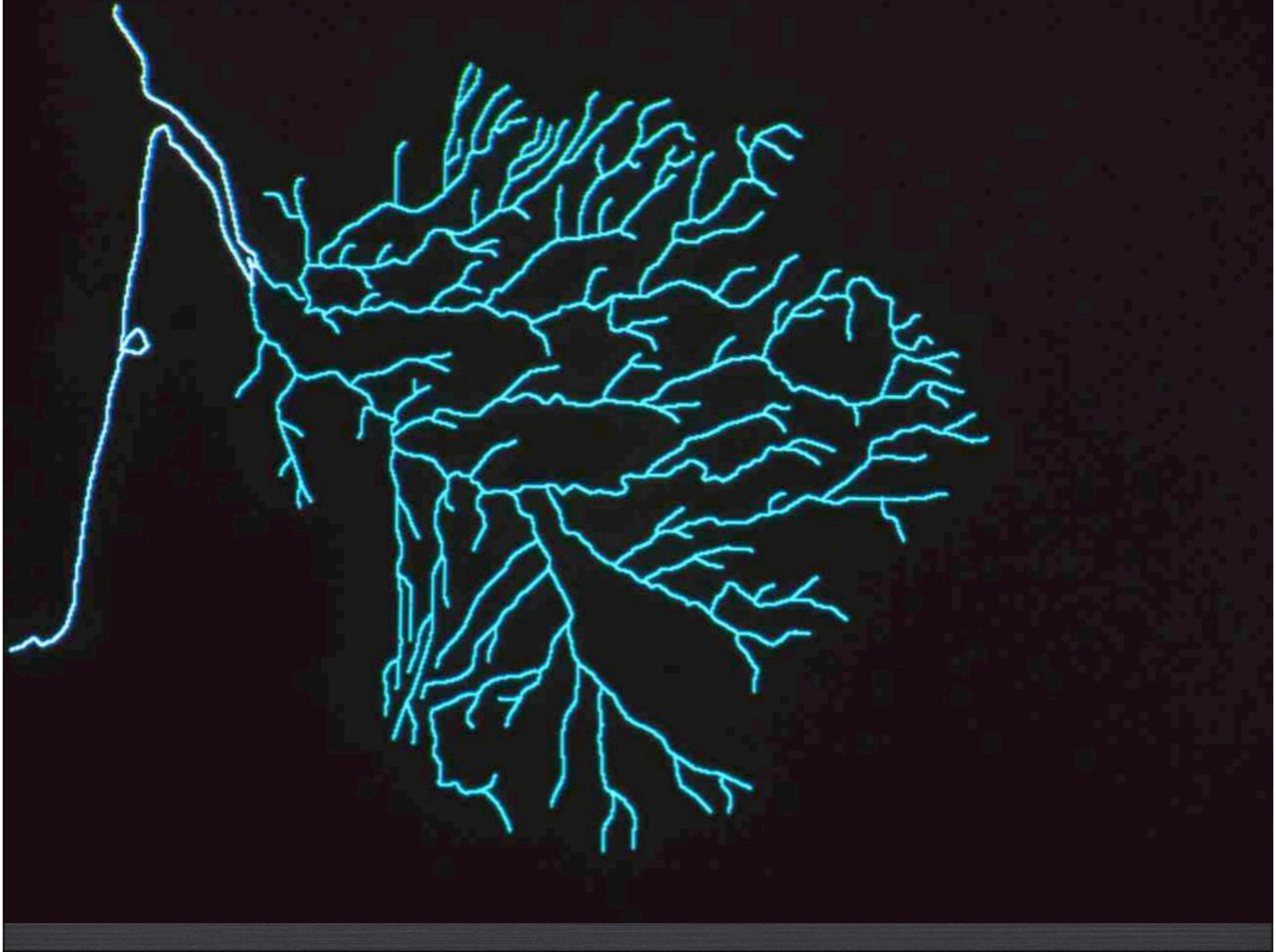
Another beautiful formula ....

From trees in nature

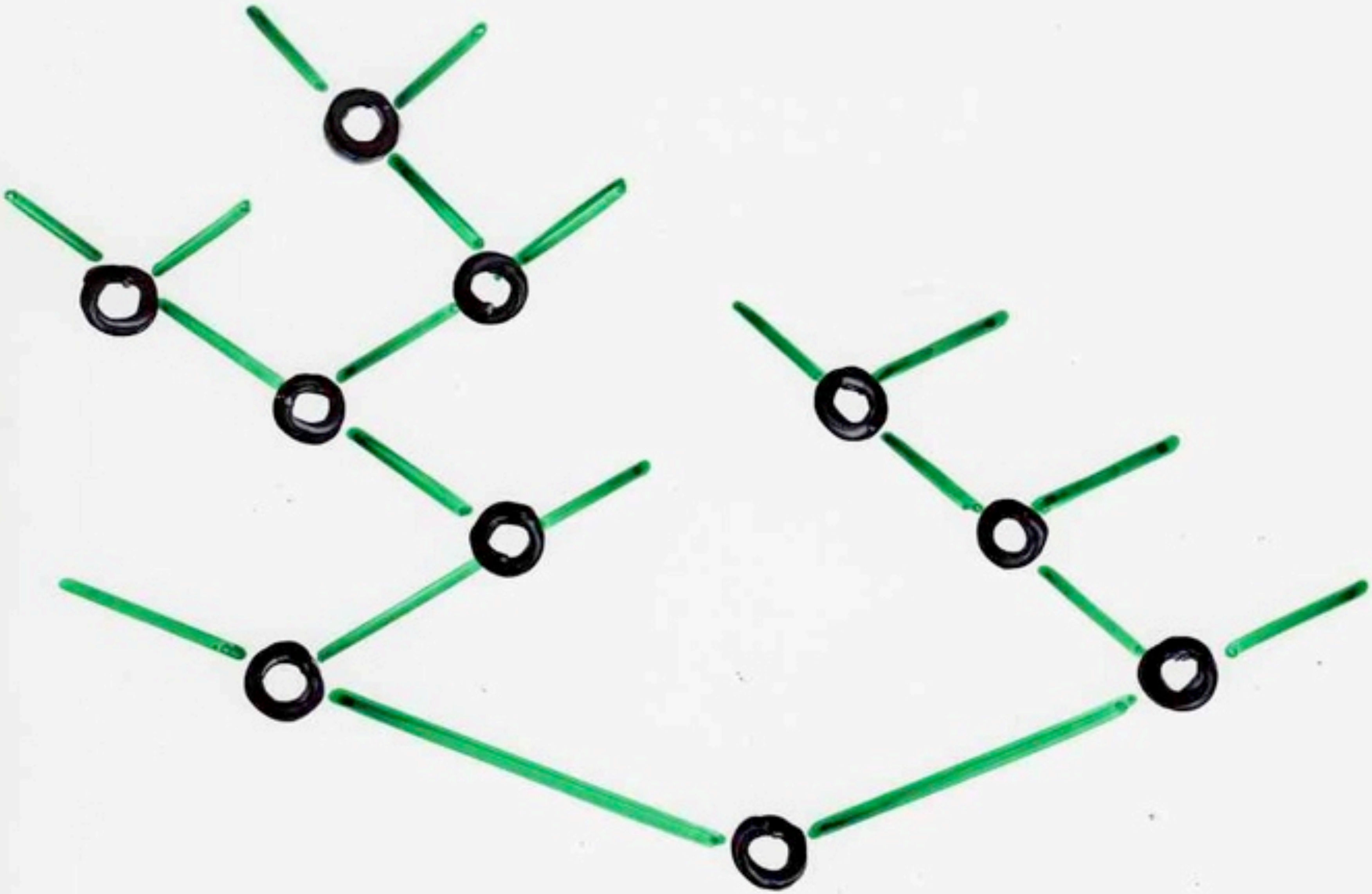
to "mathematical trees"

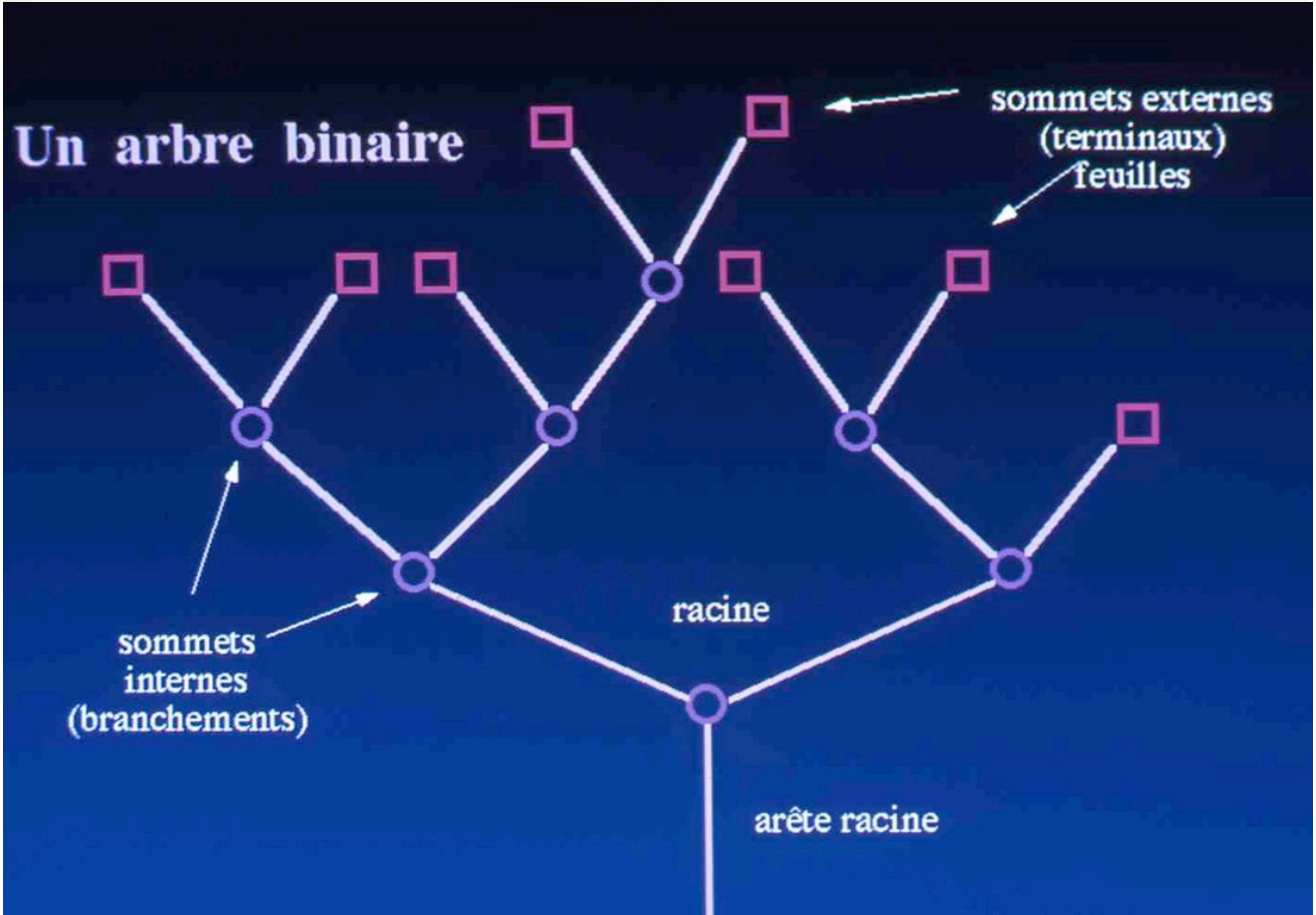






binary tree



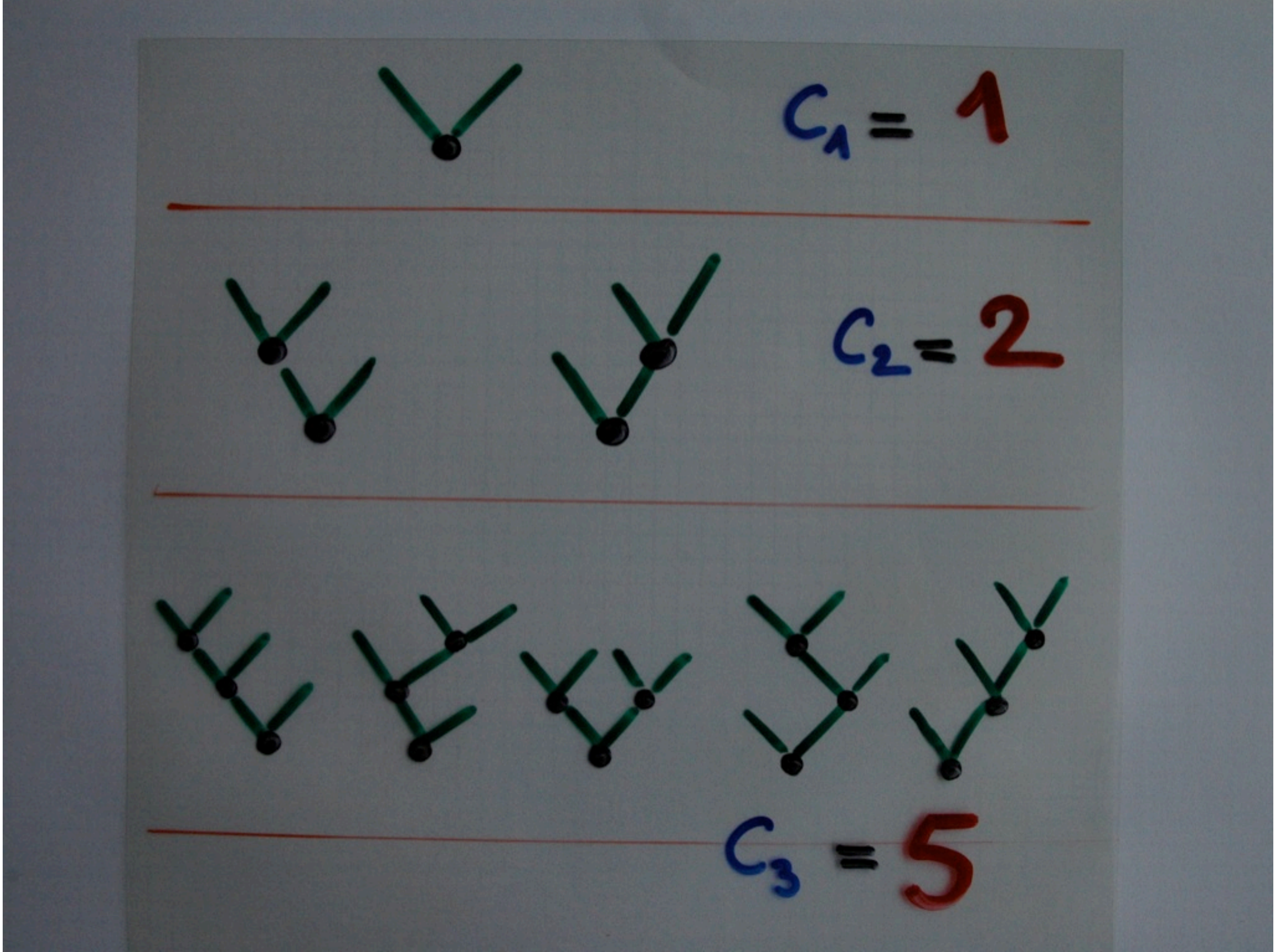

Un arbre binaire
sommets externes
(terminaux)
feuilles
racine
sommets
internes
(branchements)
arête racine

C_n = nombre d'arbres binaires ayant n sommets internes

(et donc $n+1$ feuilles)

nombre de Catalan

number of binary trees
having n internal vertices
(or n+1) leaves (external vertices)


$C_1 = 1$
$C_2 = 2$
$C_3 = 5$


14

addition +

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1


addition +
division :
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1
2
5
14
1
2
3
4
5

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\ n!}$$

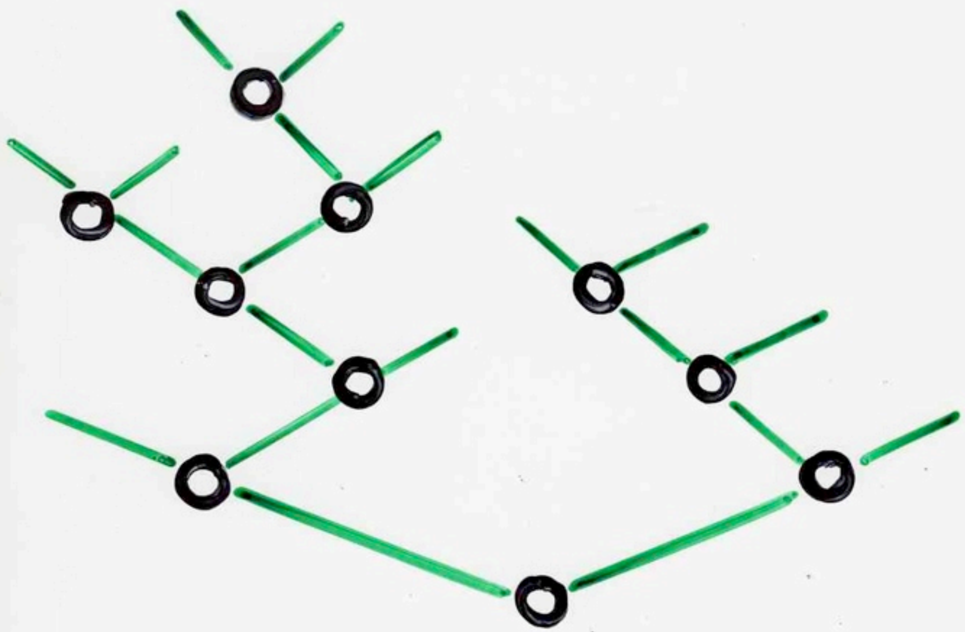
$$n! = 1 \times 2 \times \dots \times n$$

$$C_4 = \frac{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8}{1\times 2\times 3\times 4 \quad 1\times 2\times 3\times 4\times 5}$$

$$= 14$$

$$2(2n+1)\,C_n = (n+2)\,C_{n+1}$$

$$2(2n+1)C_n = (n+2)C_{n+1}$$



1 addition +

1 1 **1**

−

1 2 − 1

1 3 3 1 **2**

1 4 6 − 4 1

1 5 10 10 5 1 **5**

1 6 15 20 − 15 6 1

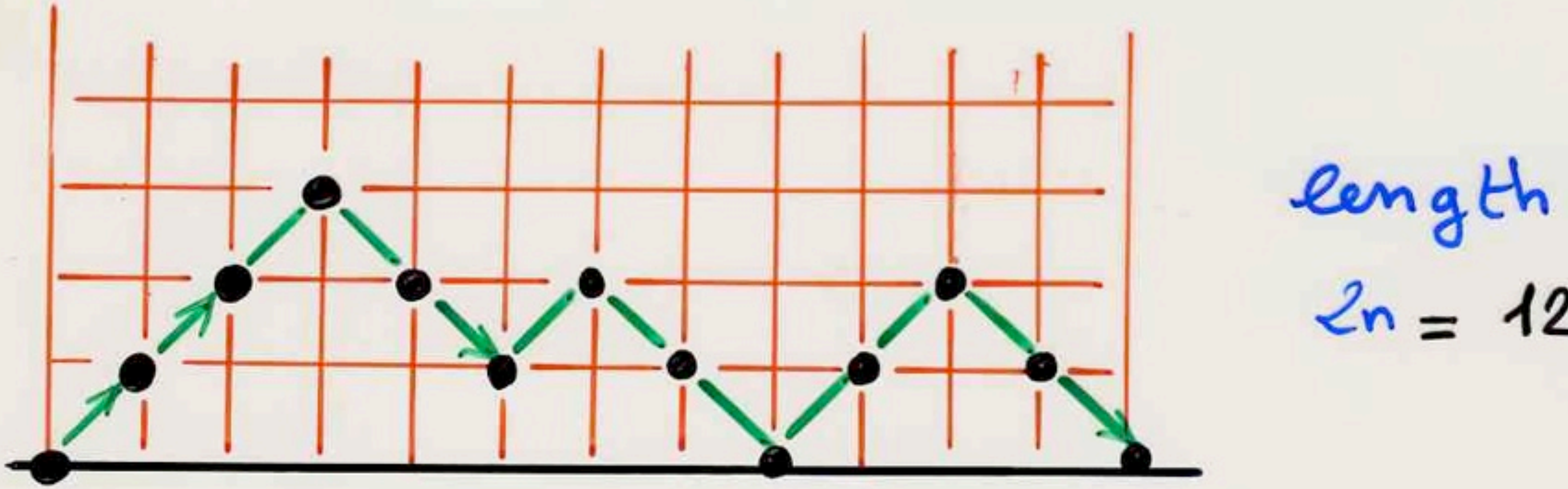
1 7 21 35 35 21 7 1

1 8 28 56 70 − 56 28 8 1

14

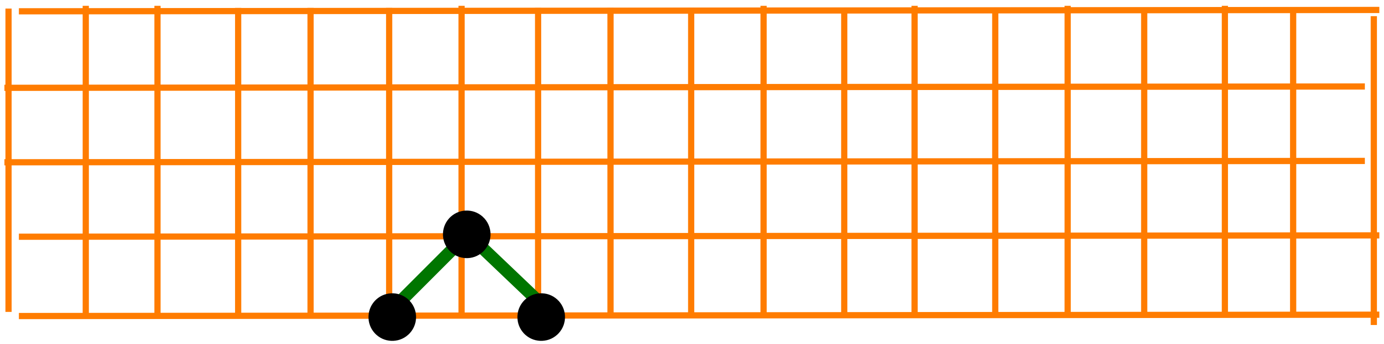
§5 bijections

(one-to-one correspondences)

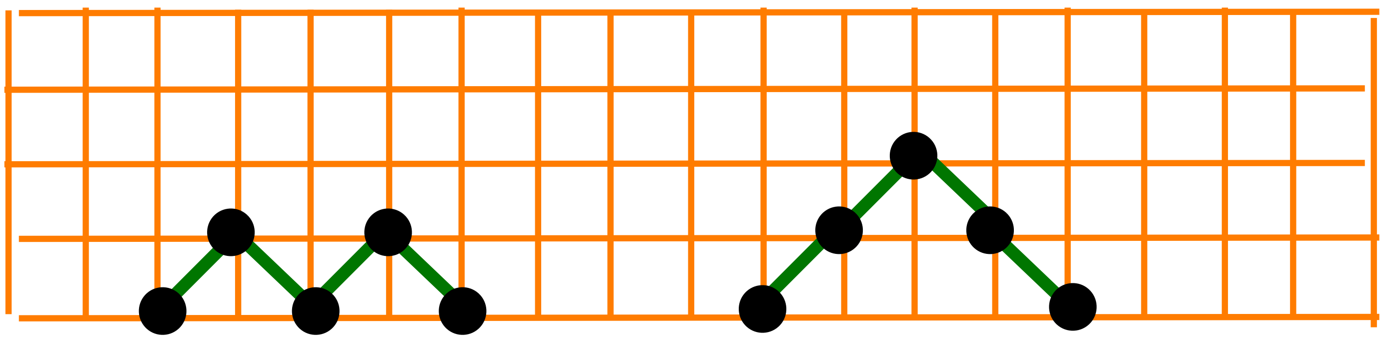


Dyck path

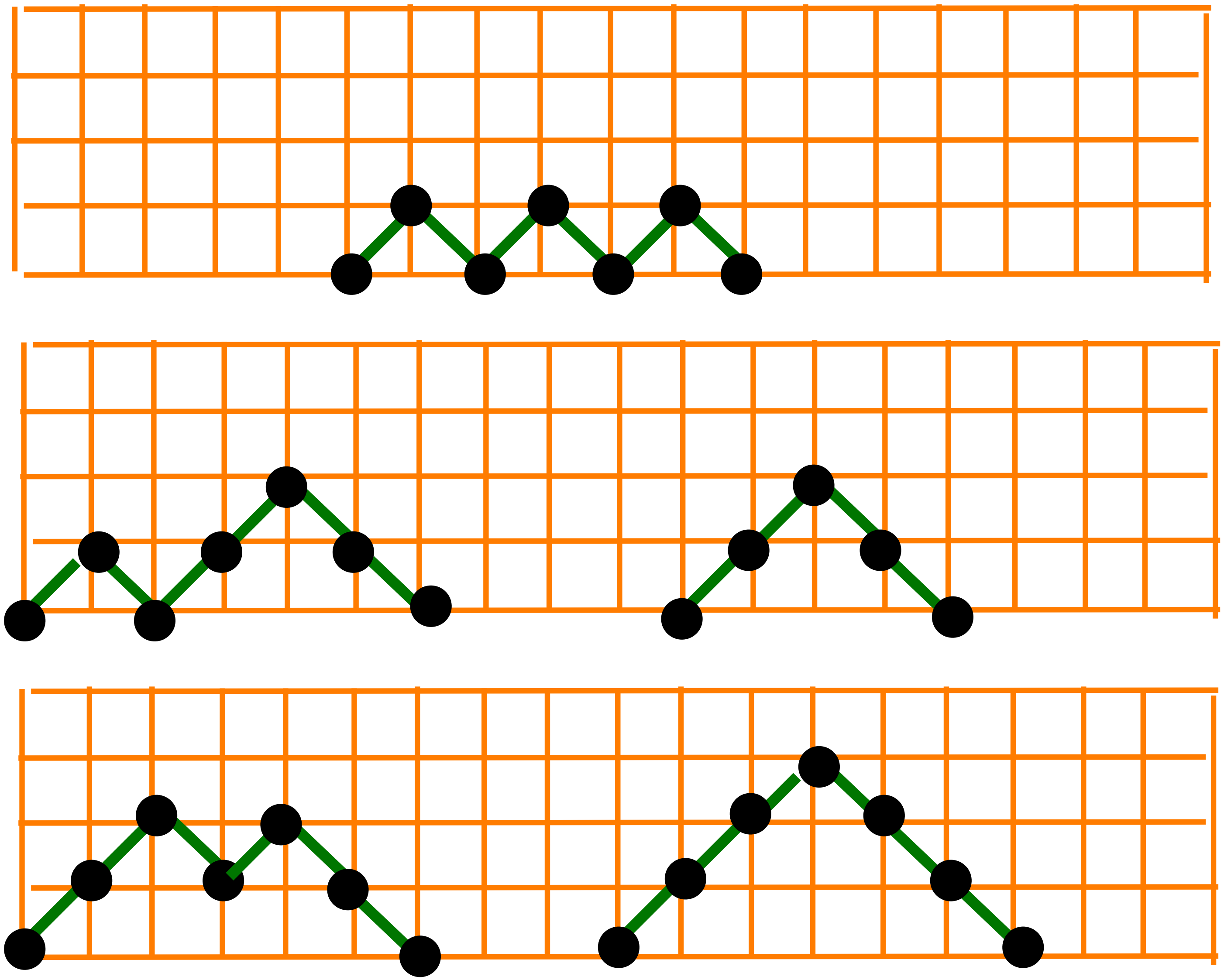
C_n = number of Dyck path of length $2n$



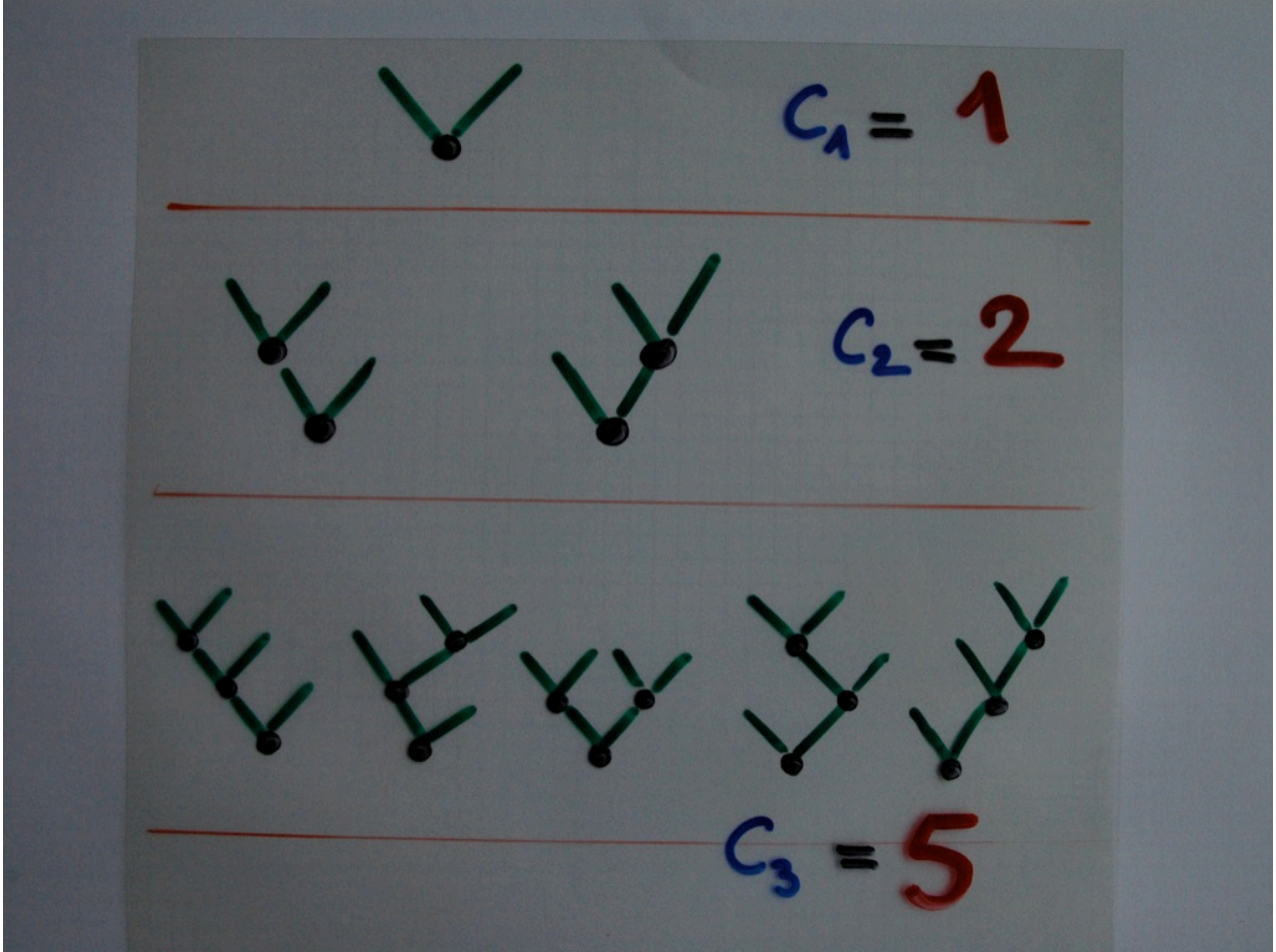
1



2



5


$C_1 = 1$
$C_2 = 2$
$C_3 = 5$

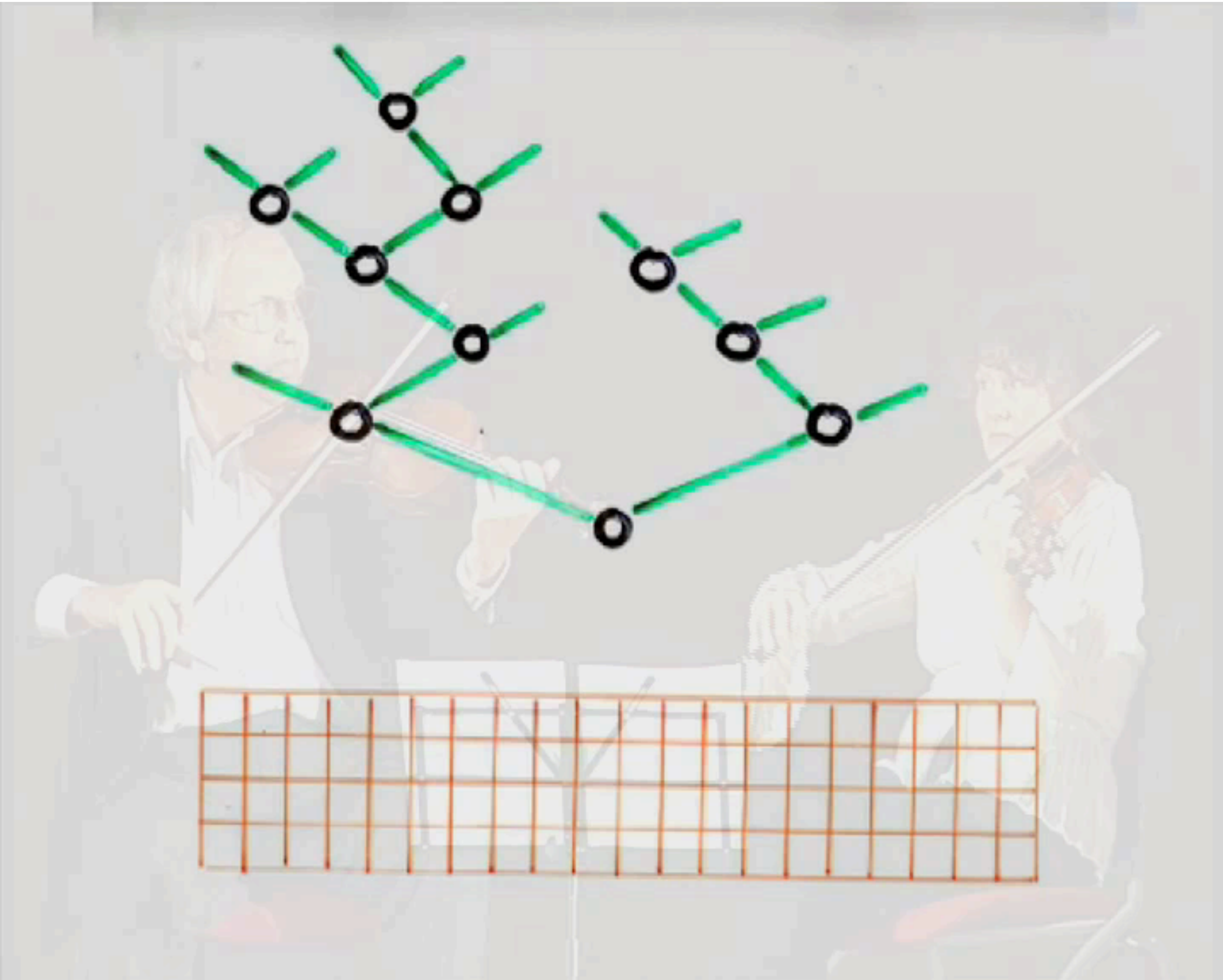
bijection

or one-to-one correspondence

binary trees with n internal vertices

$\updownarrow$

Dyck paths length 2n

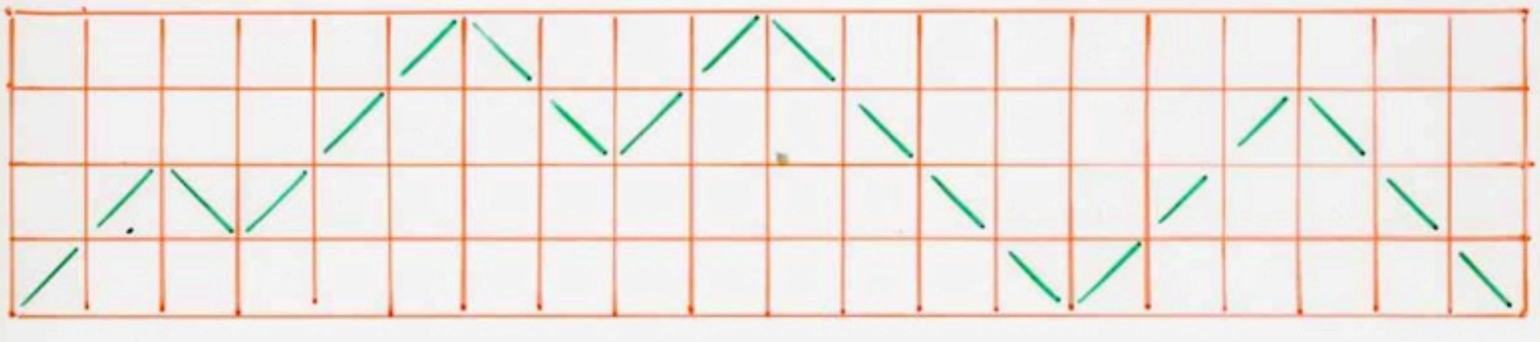


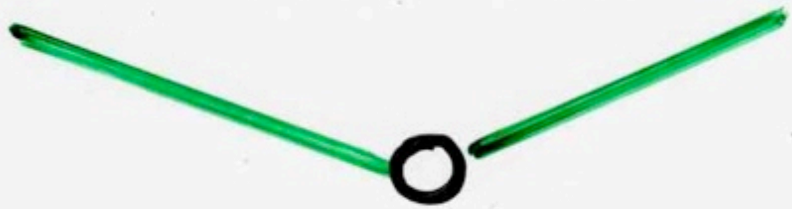
bijection
(one-to-one correspondance)

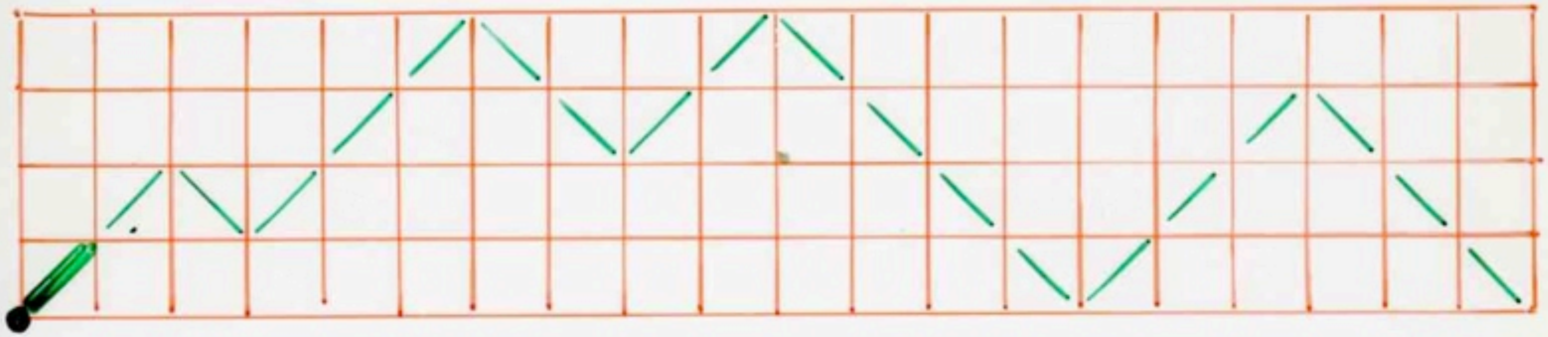
binary with n (internal) vertices

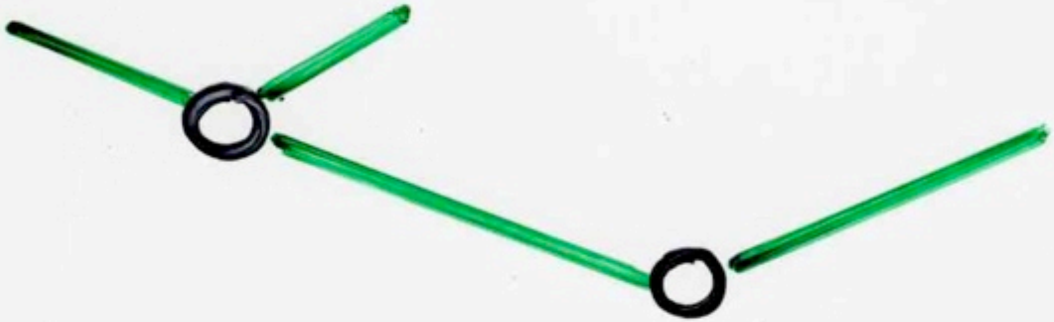
$$\updownarrow$$

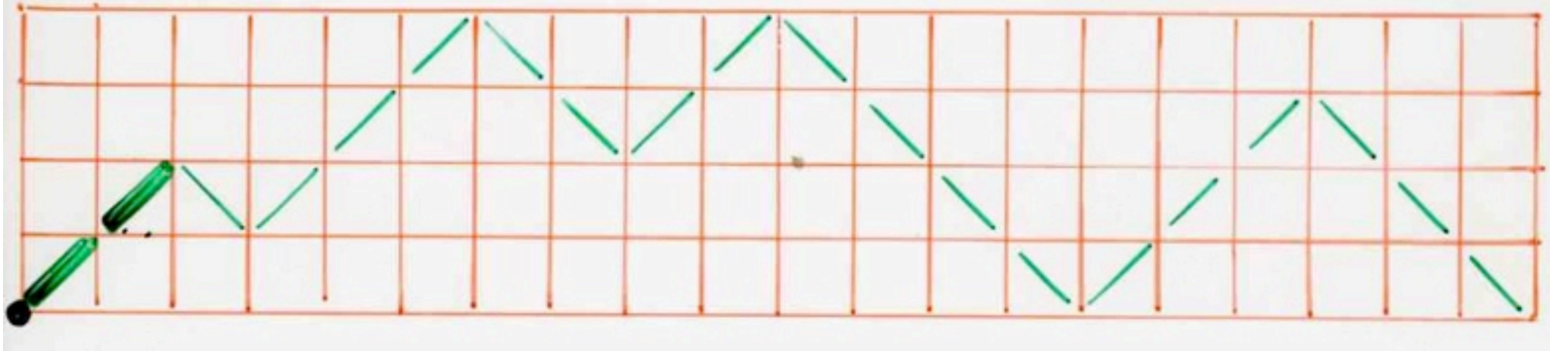
Dyck paths length 2n

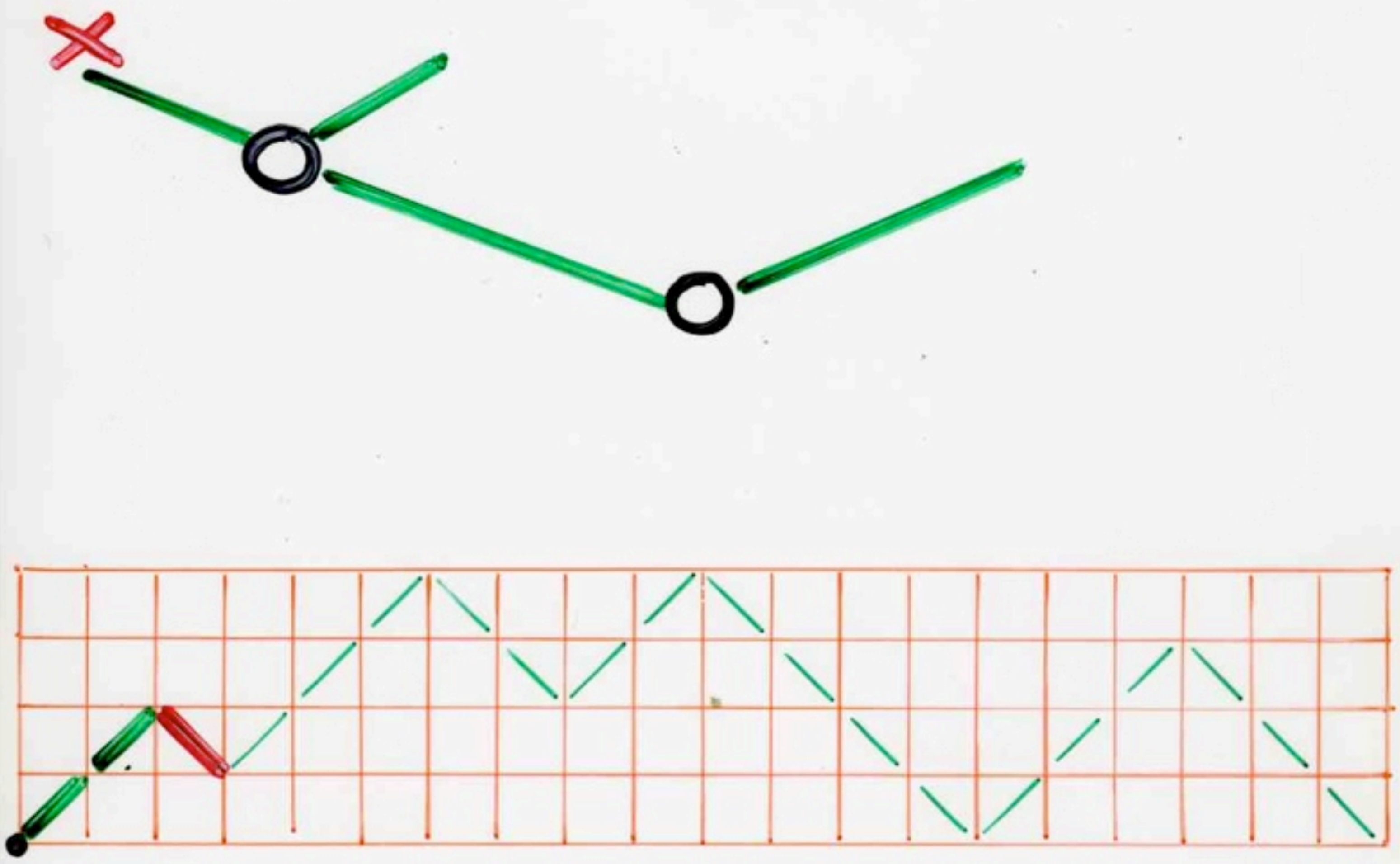


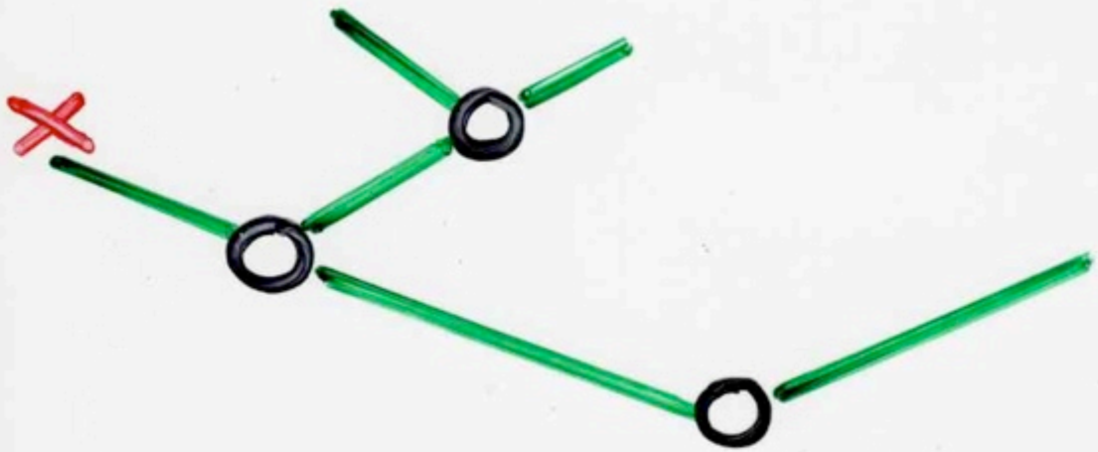


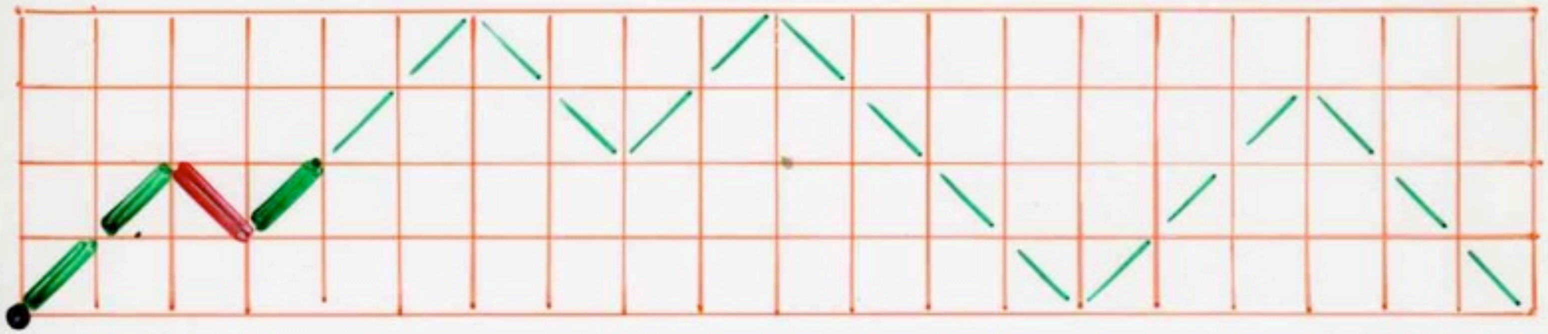


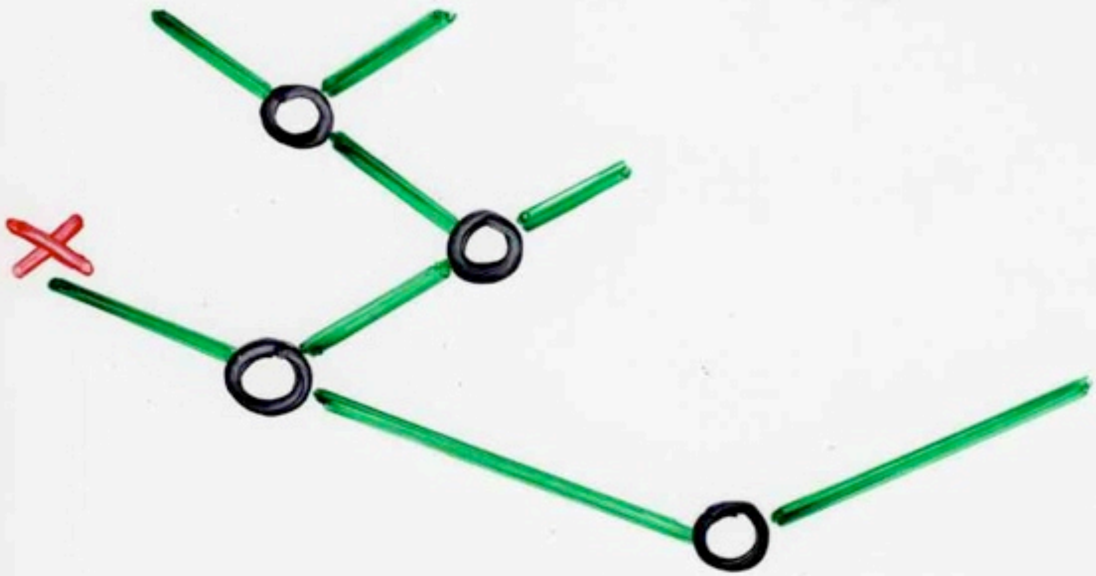


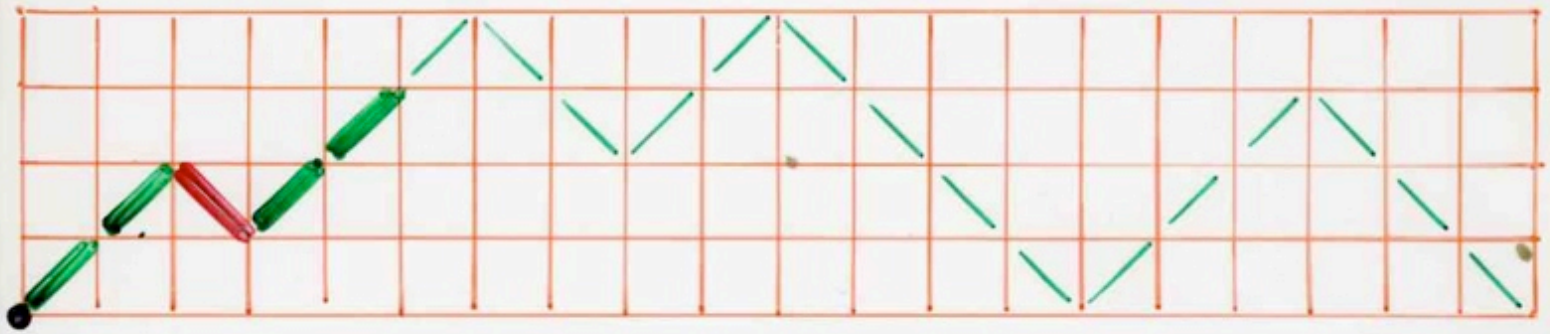


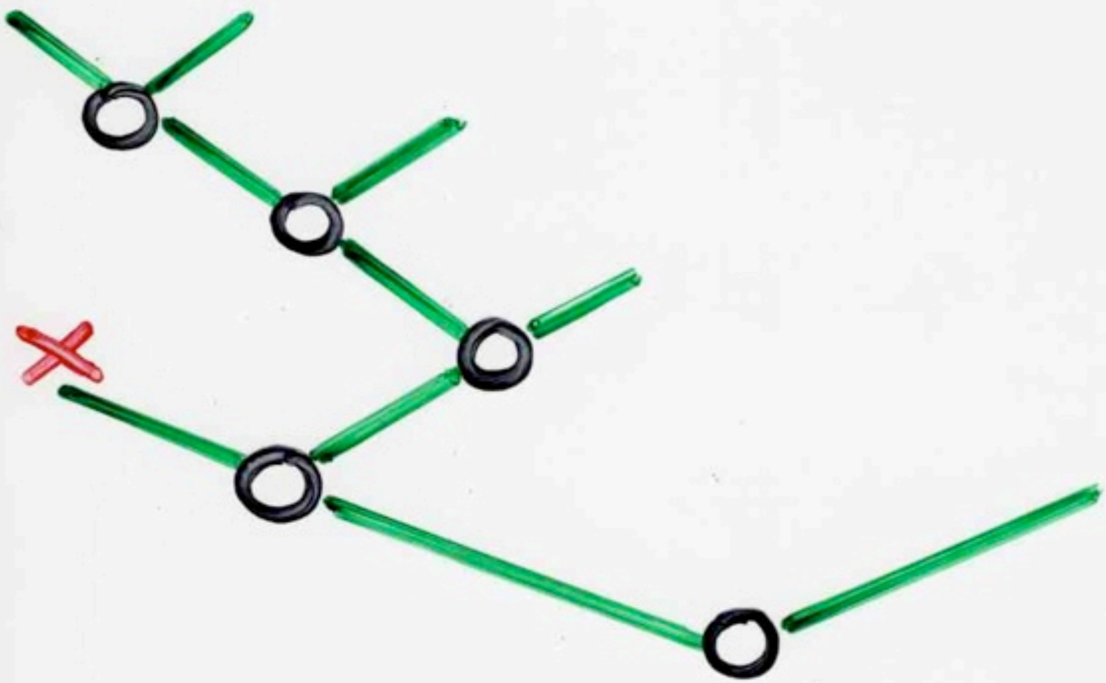


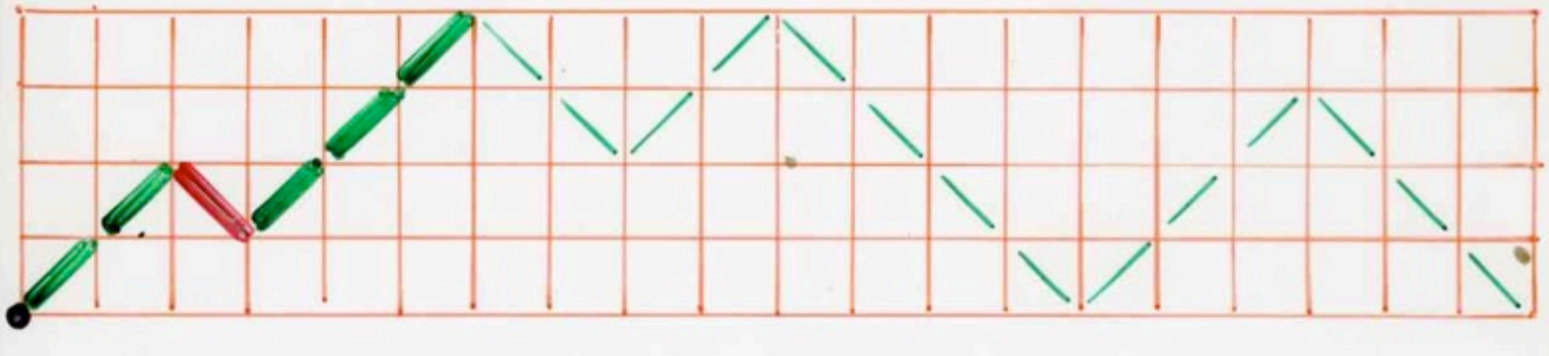


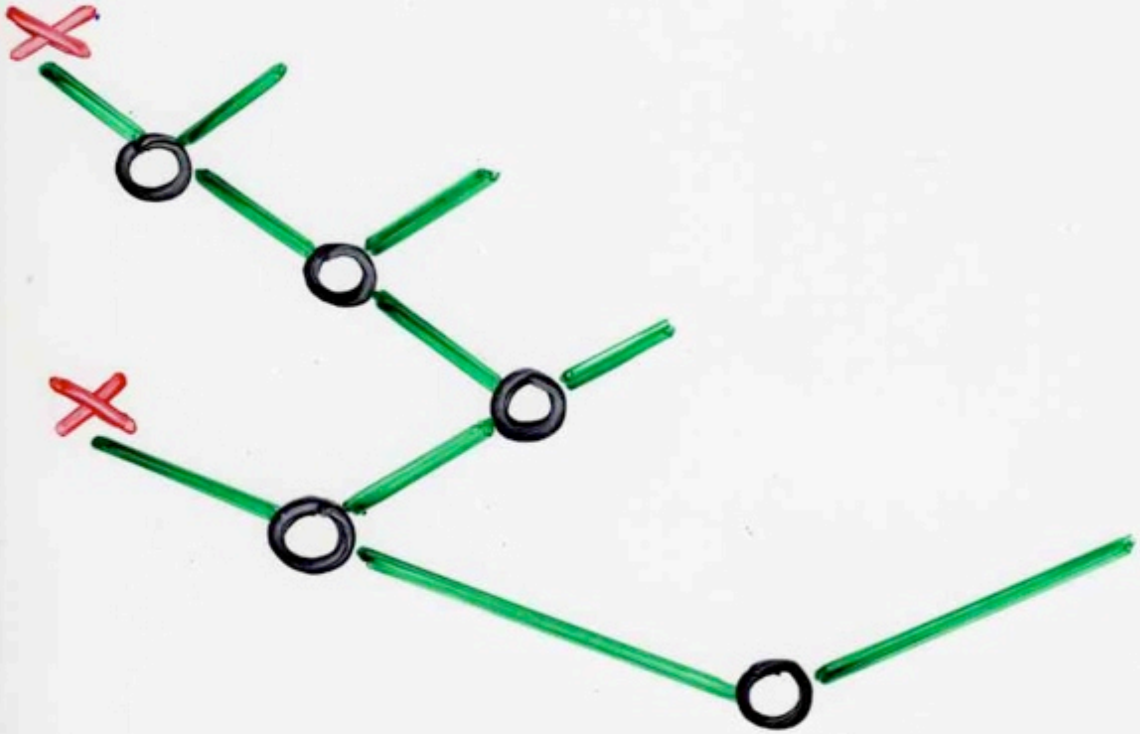


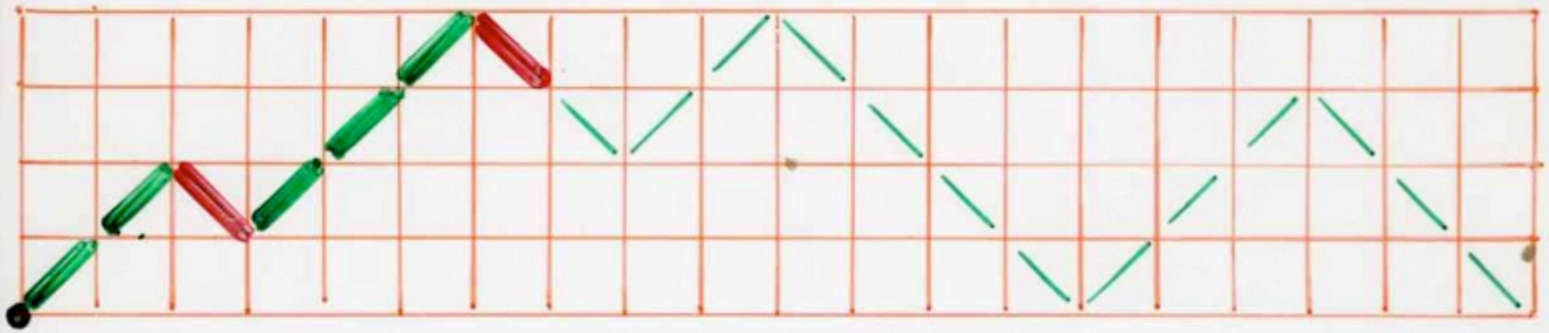


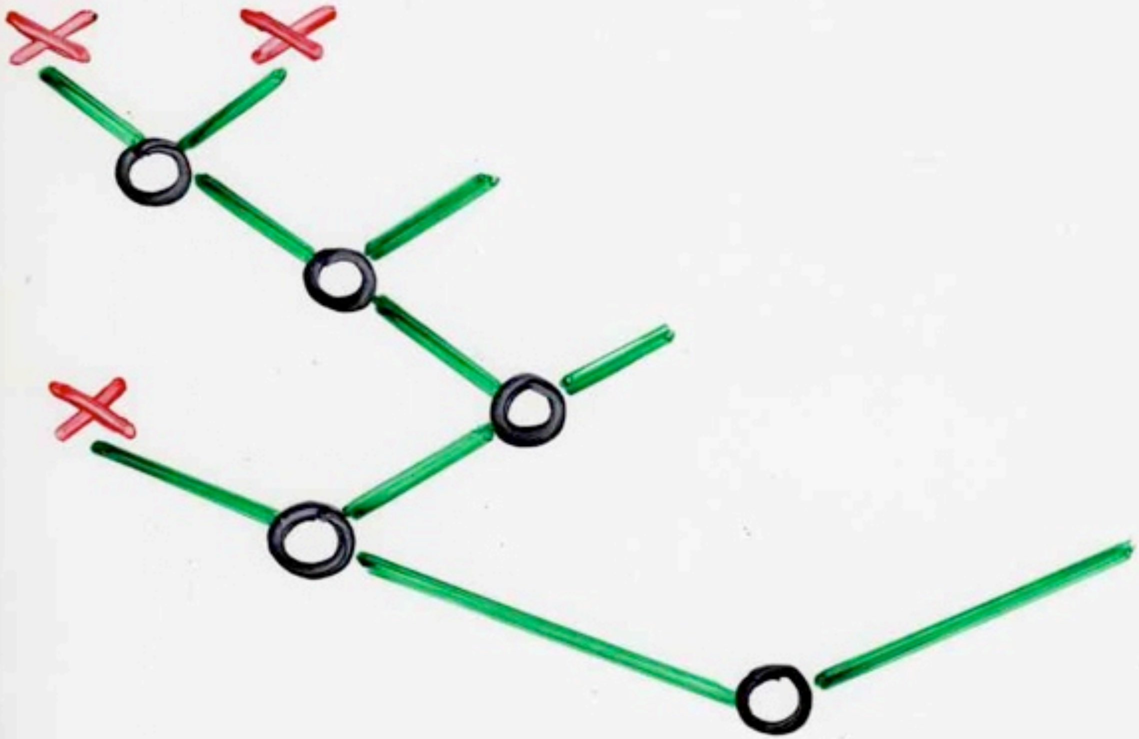


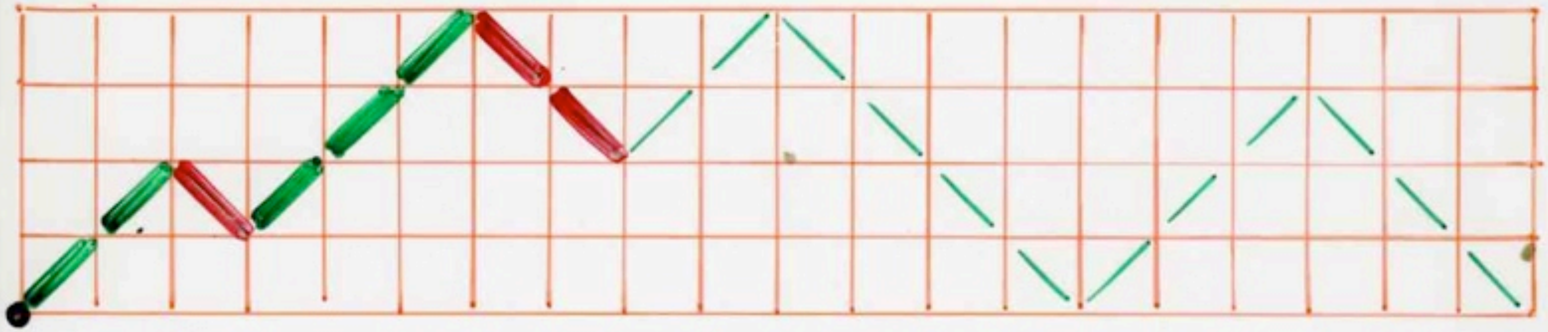


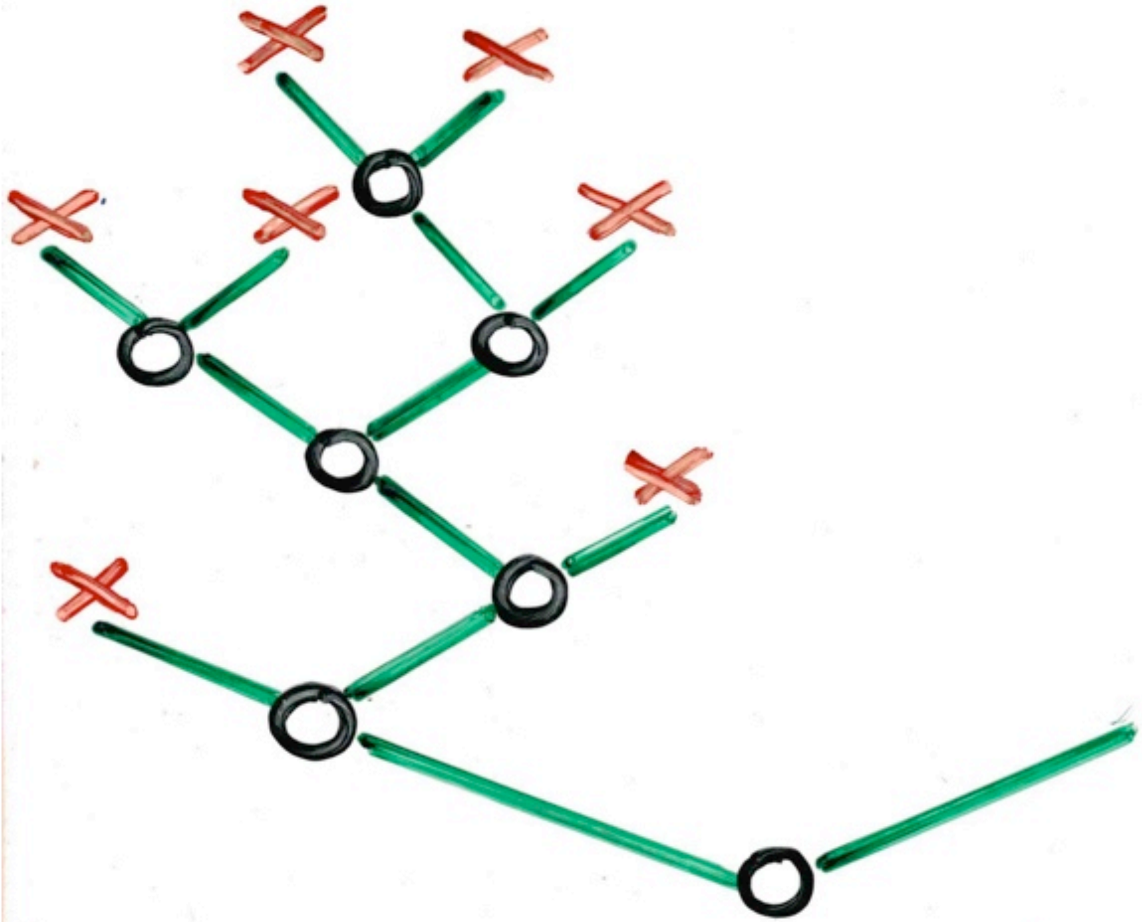


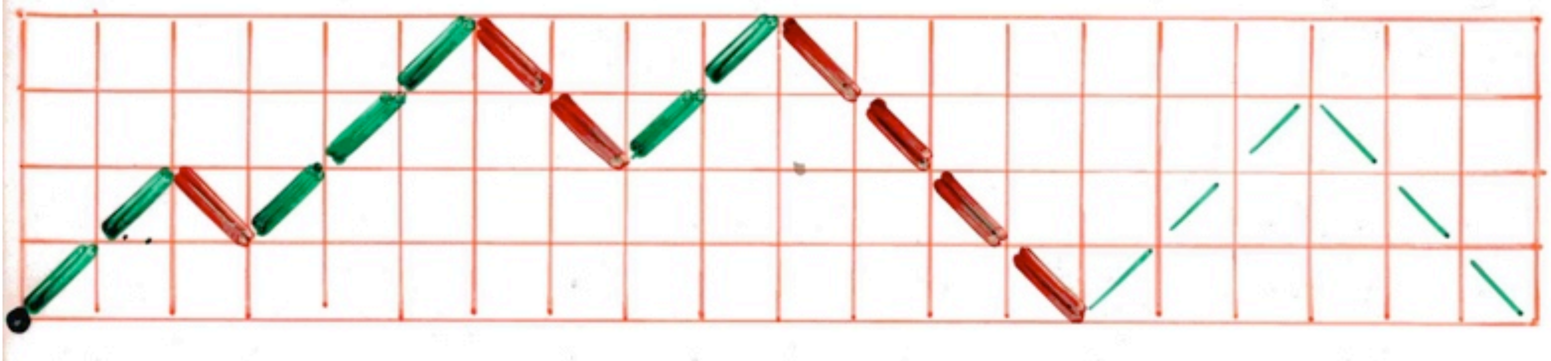


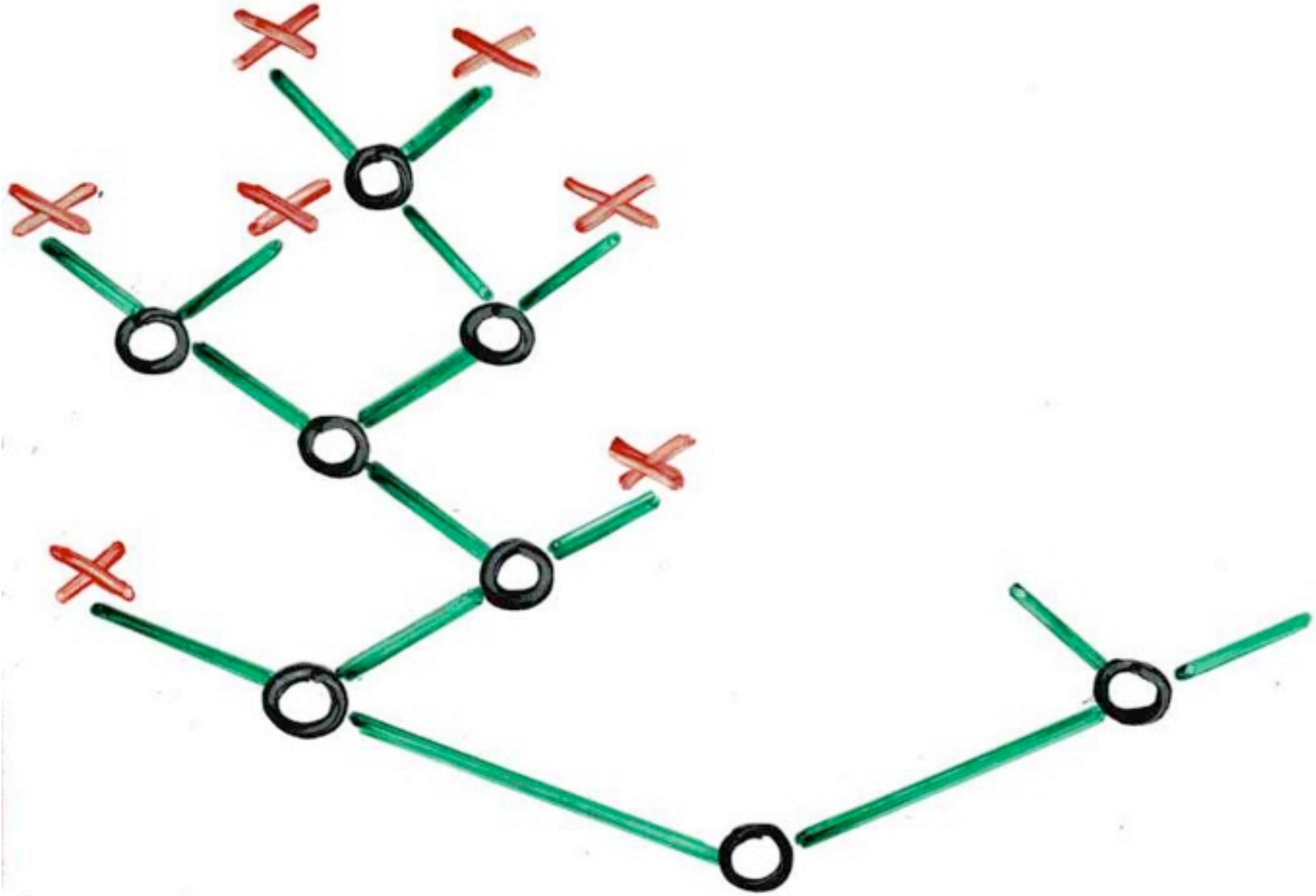


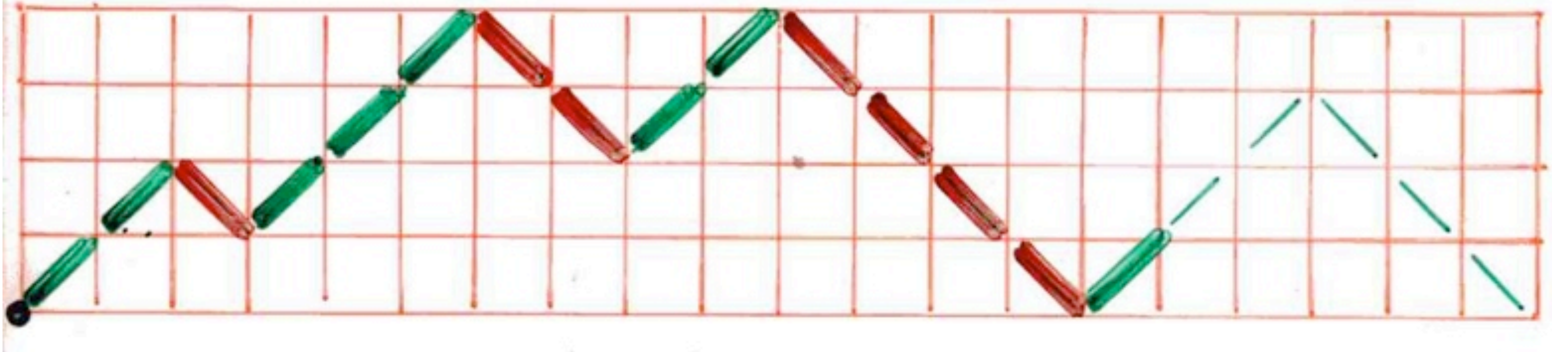


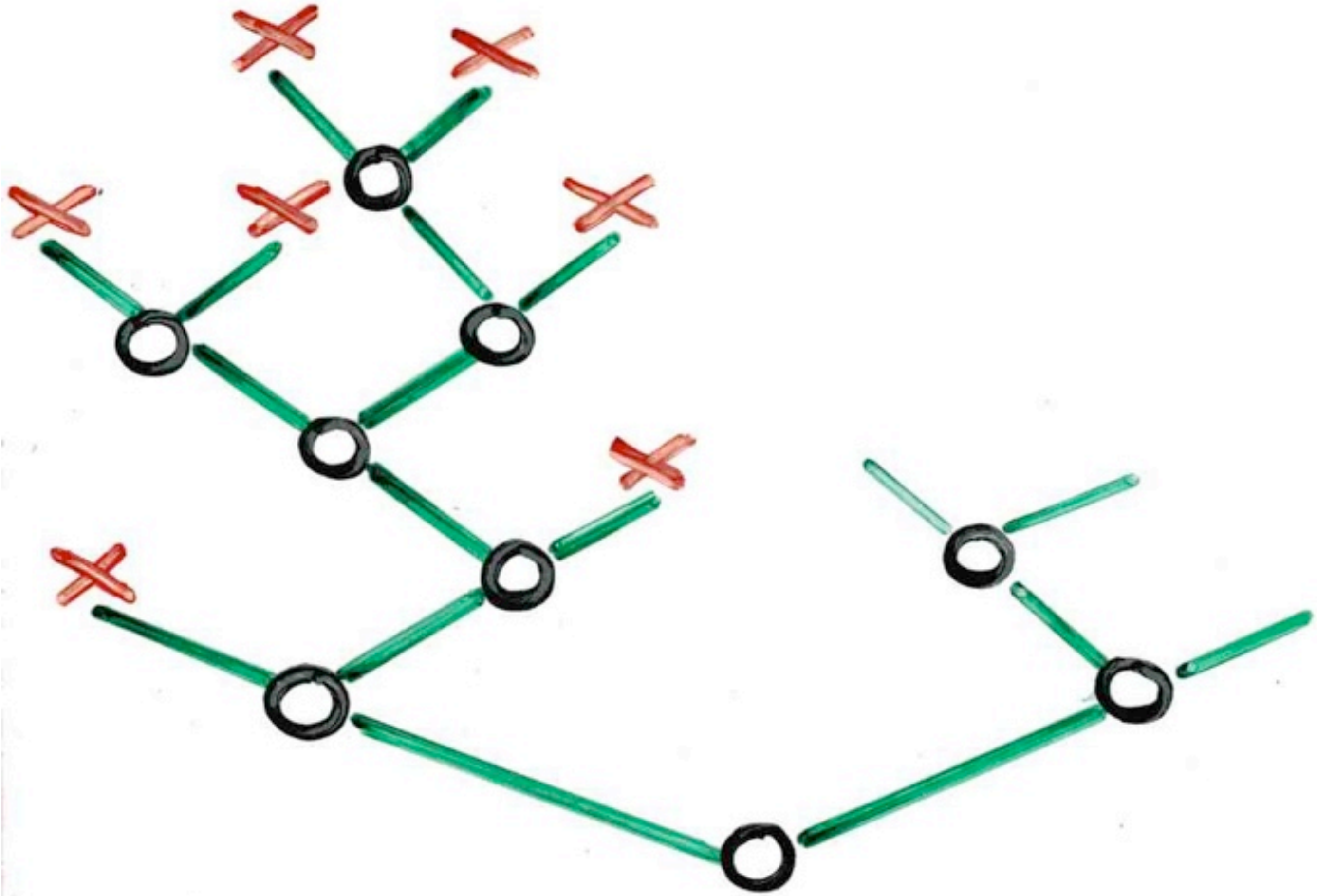


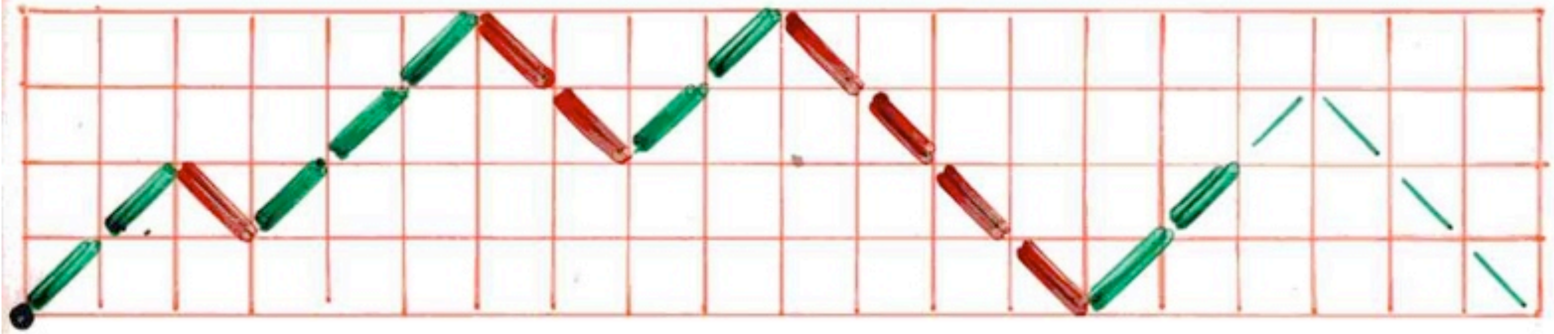


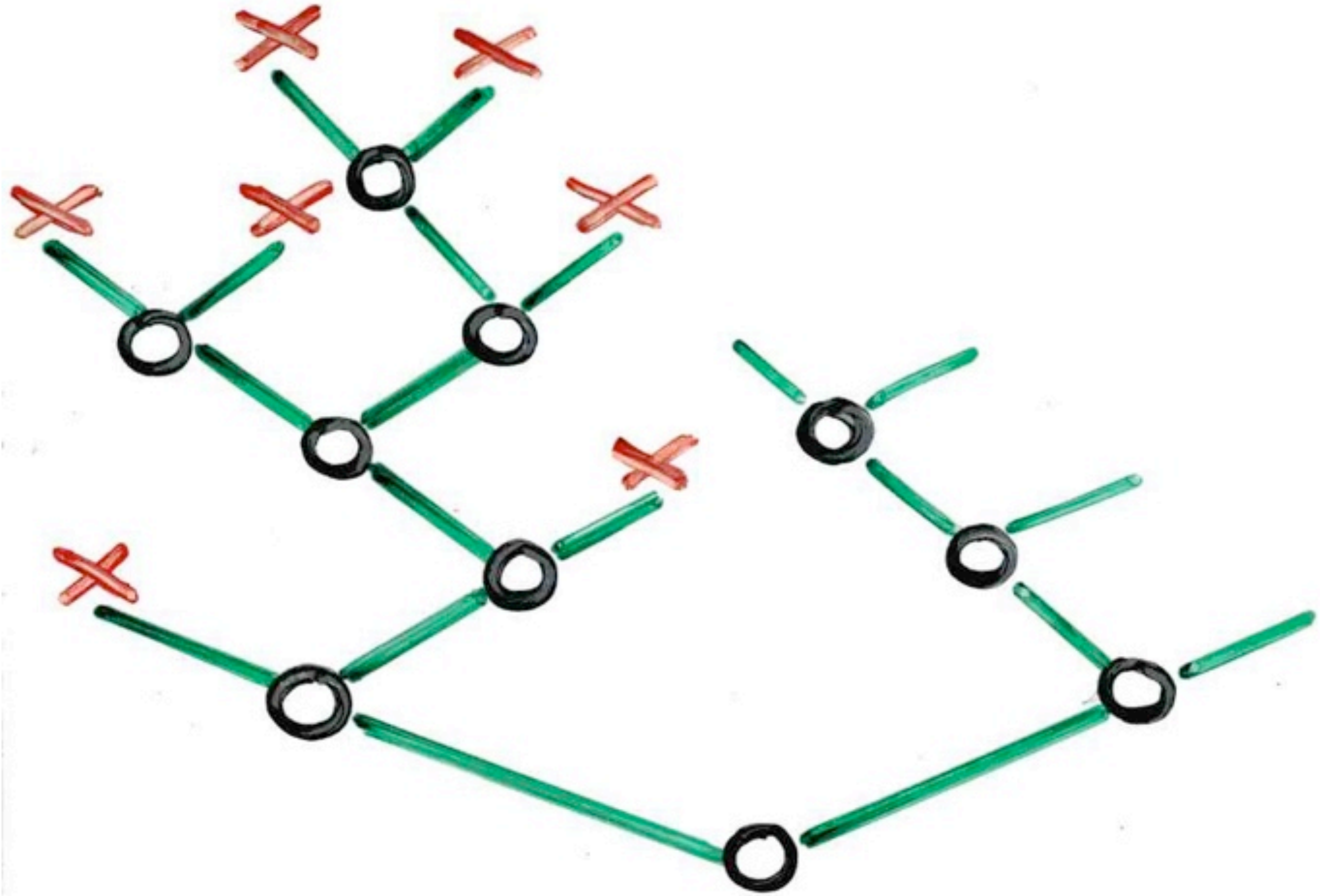


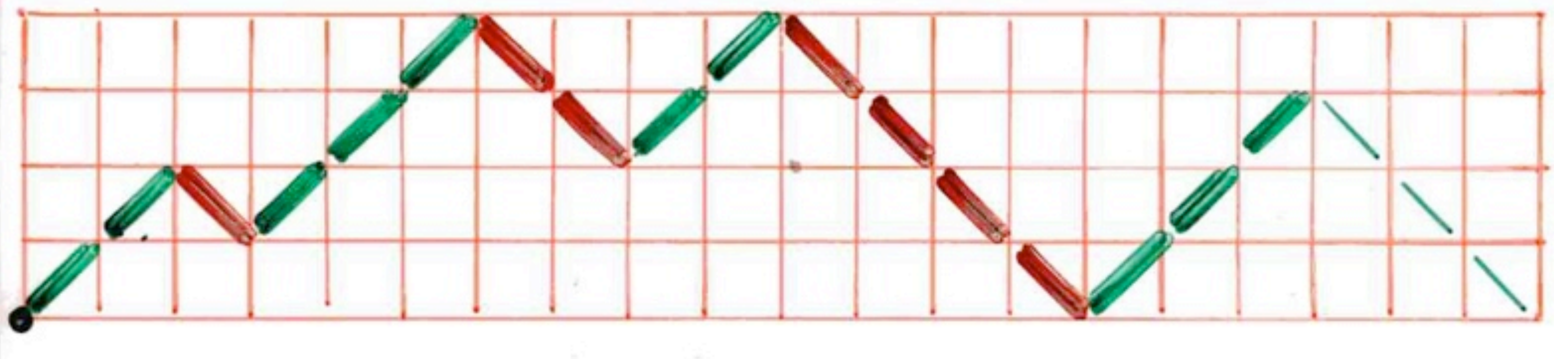


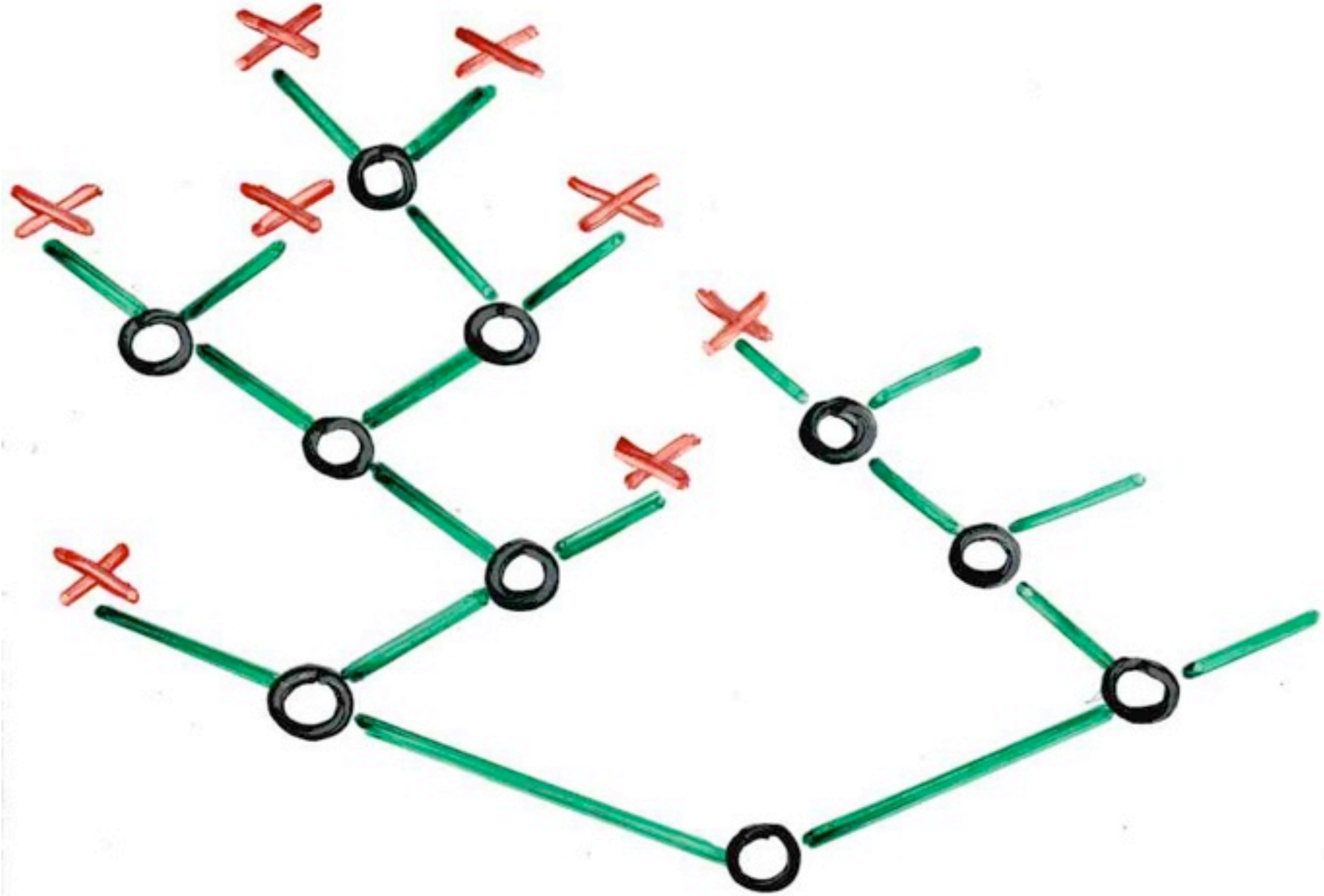


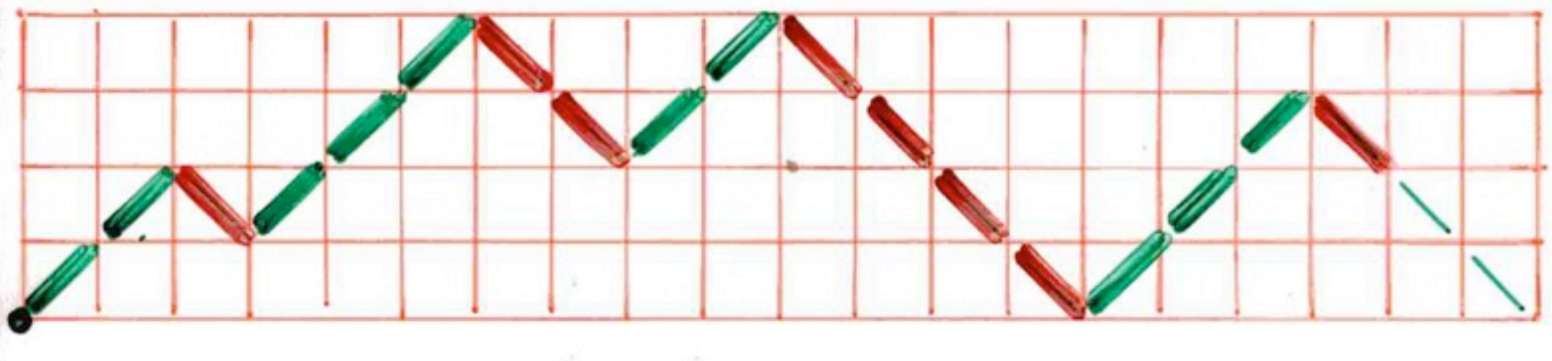


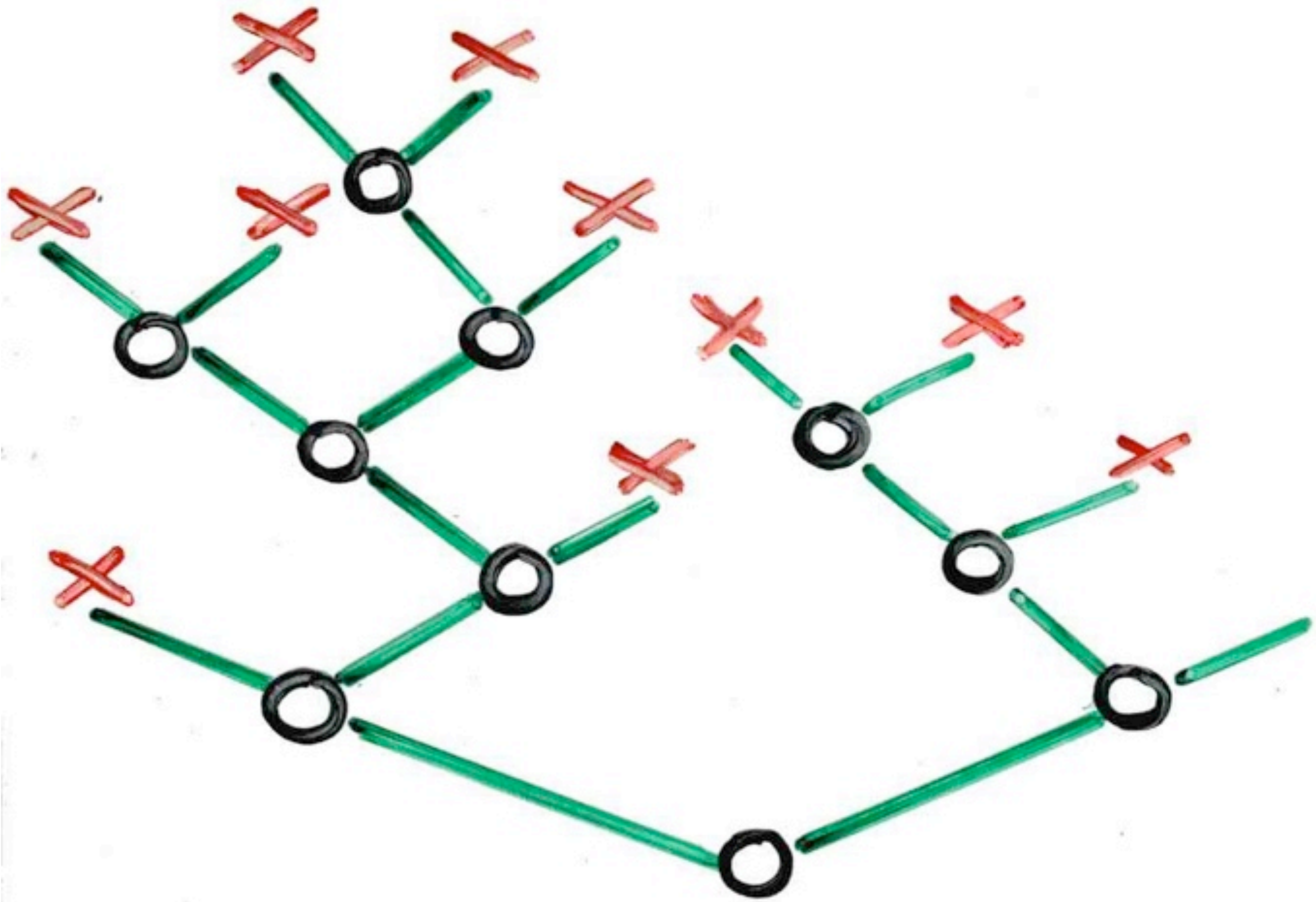
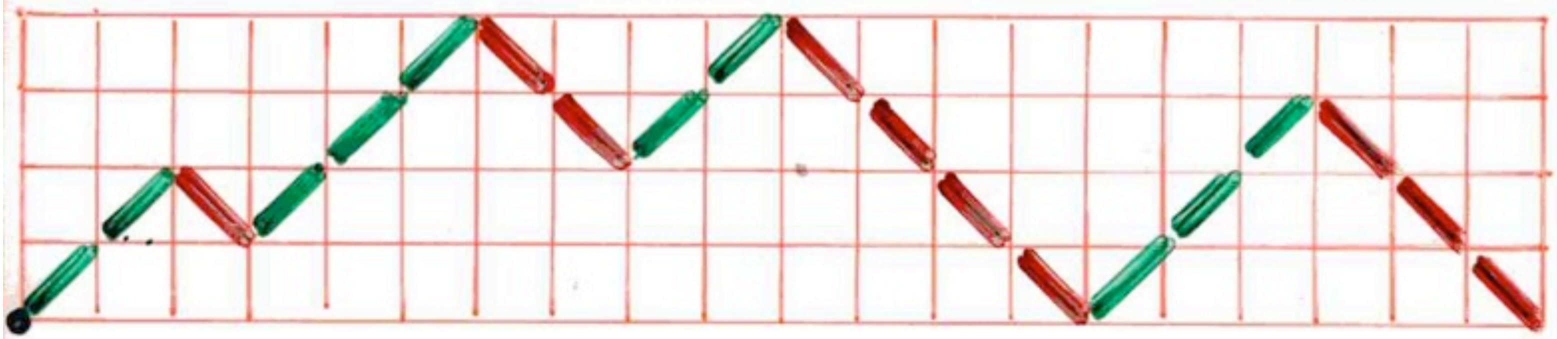


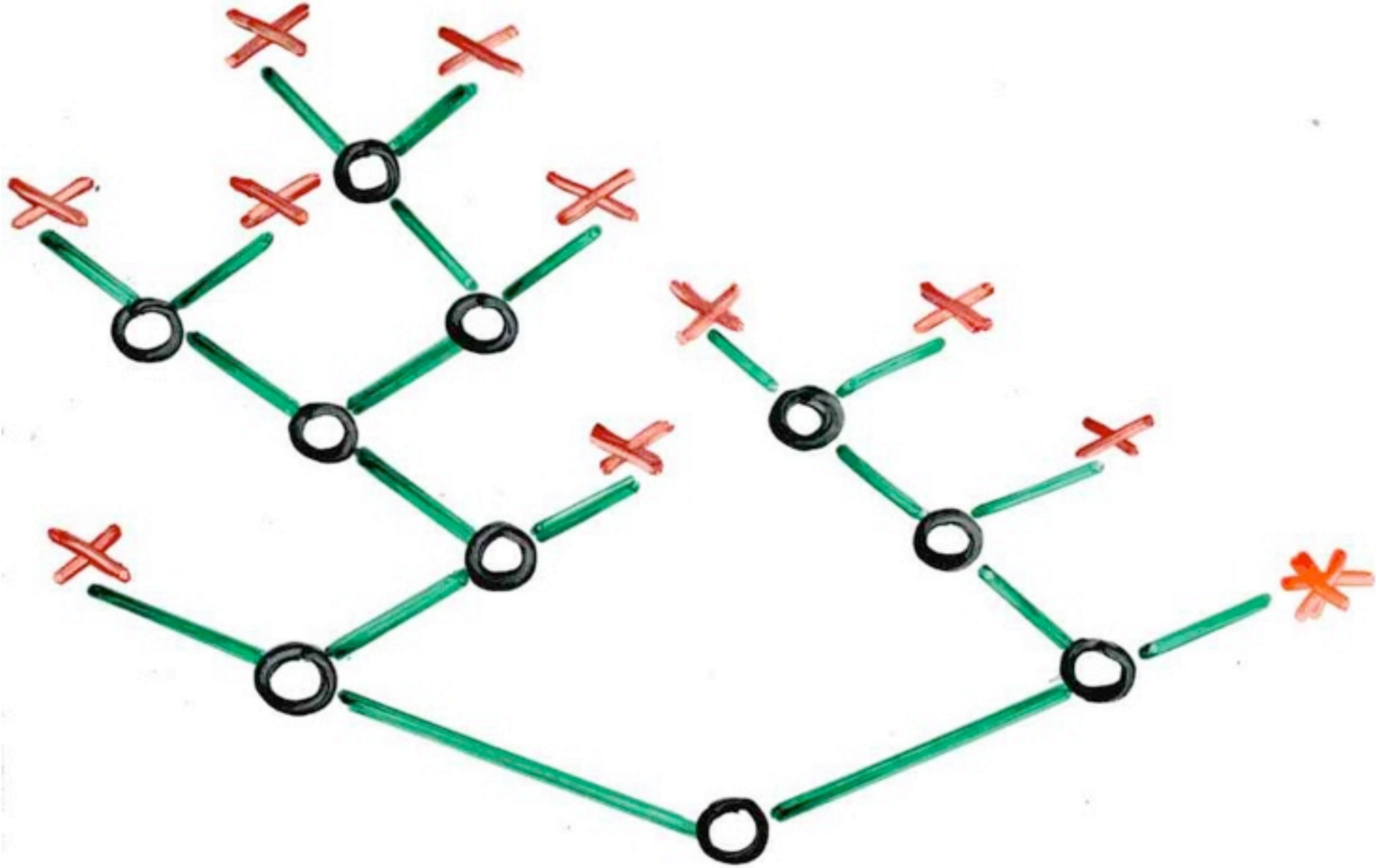


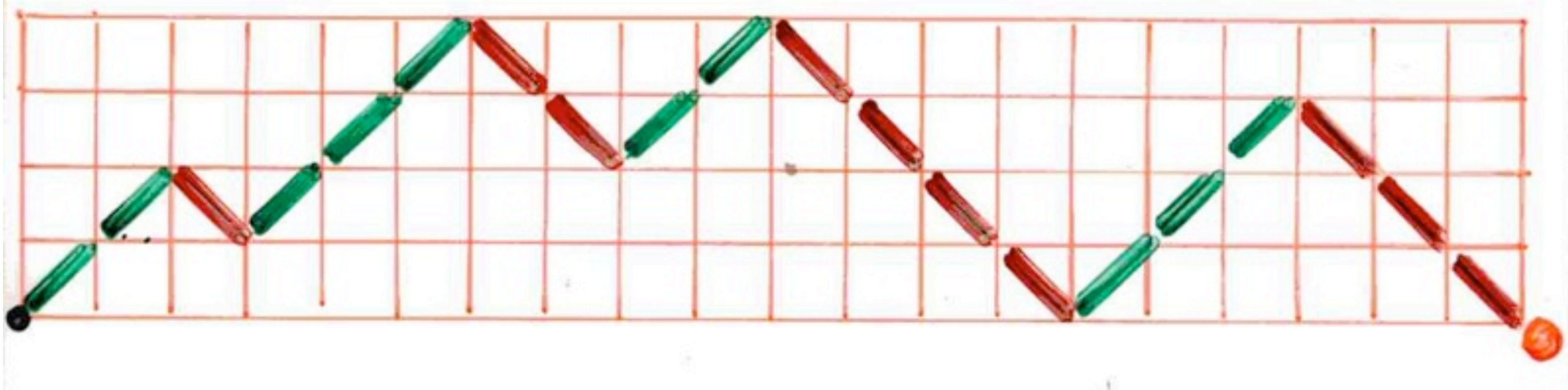




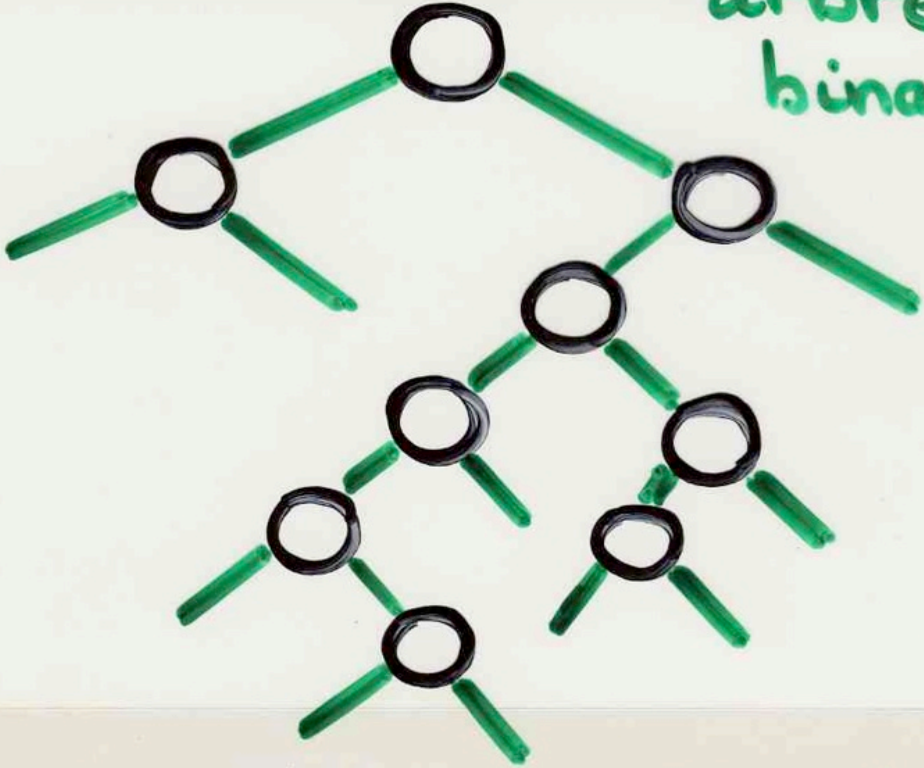


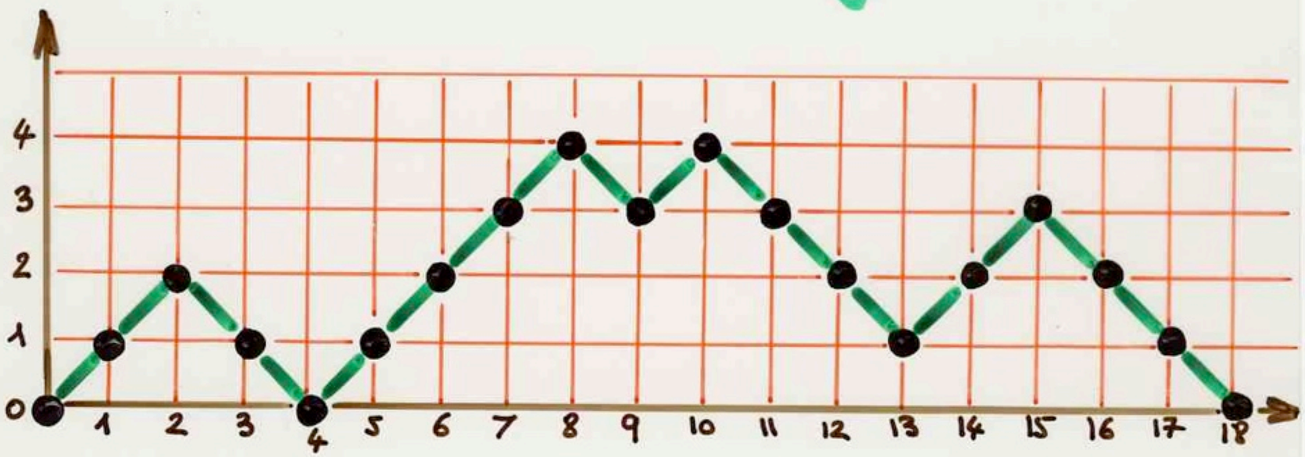




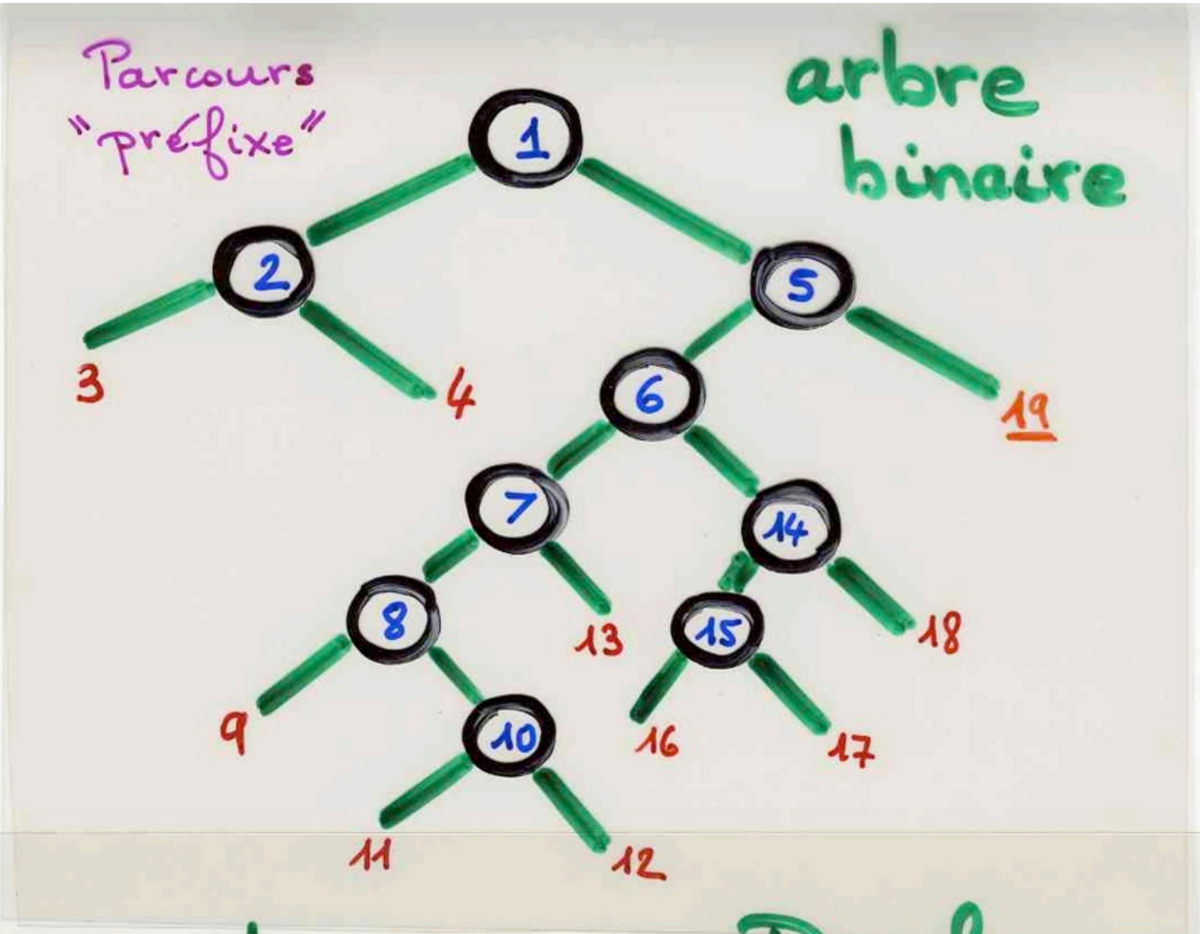
arbre binaire



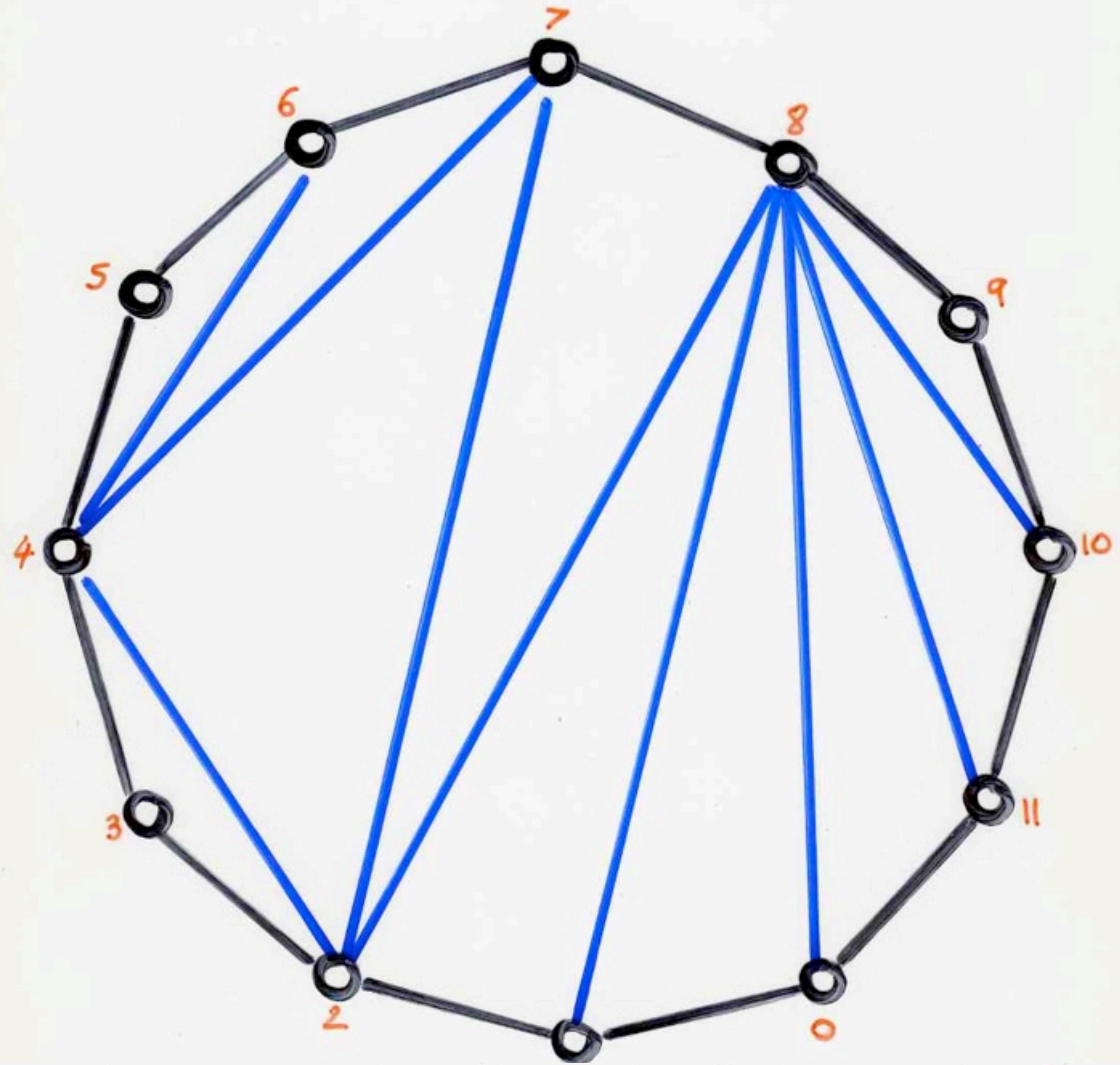
chemin de Dyck



prefix
order



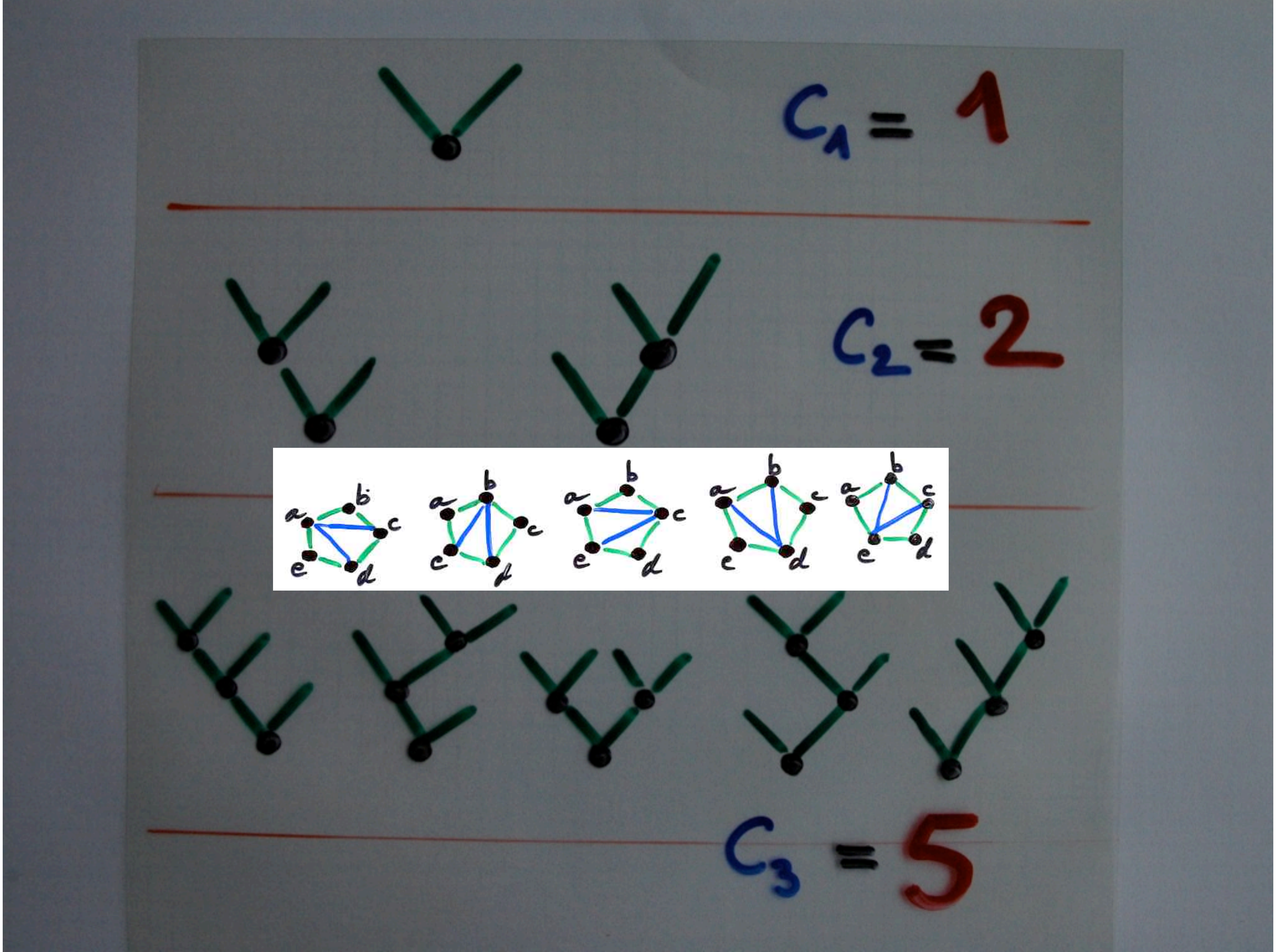
triangulations

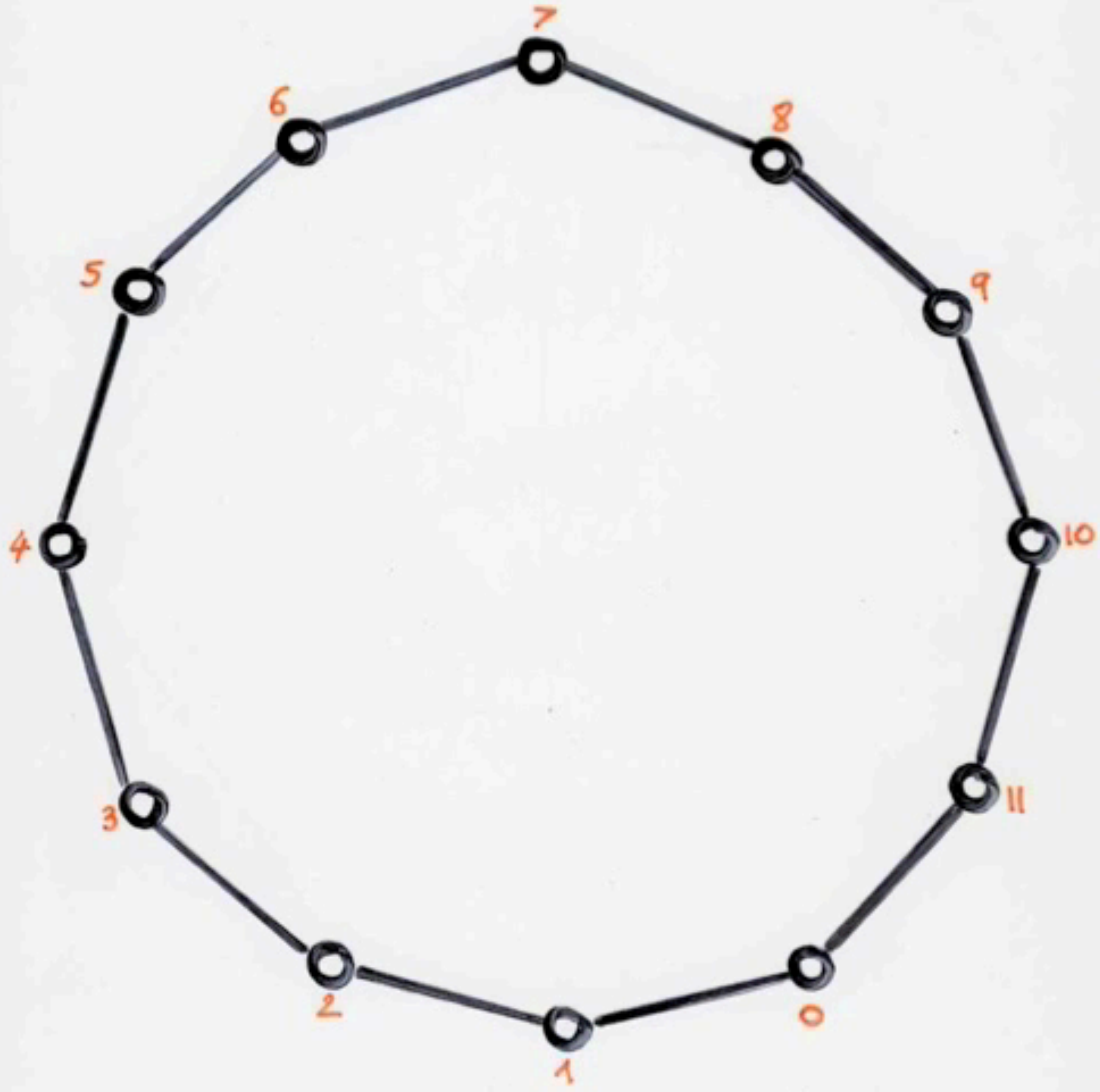


une triangulation d'un polygone

bijection

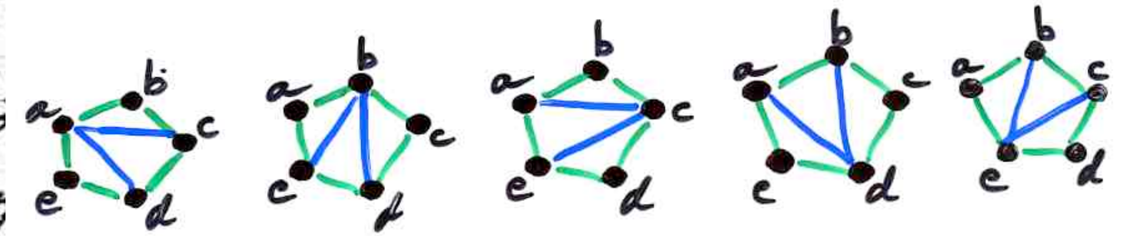
triangulations $\longleftrightarrow$ binary trees


C1 = 1
C2 = 2
C3 = 5
a b c d e
a b c d e
a b c d e
a b c d e
a b c d e


7
6
8
5
9
4
10
3
11
2
0
1


EULER

I. $\frac{ac}{ad}$; II. $\frac{bd}{be}$; III $\frac{ca}{ce}$; IV $\frac{db}{da}$; V $\frac{ec}{eb}$



$n-3$ Diagonalen in $n-2$ Triangula

les nombres "de Catalan"

1, 2, 5, 14, 42, 132, 429, 1430, ...

$x = 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430$

generaliter

$$x = \frac{2.6.10.14.18.22.\ \ldots\ (4n-10)}{2.3.4.5.6.7\ \ldots\ (n-1)} = \frac{(2n)!}{(n+1)!\,n!}$$

$1 = \ldots$; $2 = 1.\frac{6}{3}$; $5 = 2.\frac{10}{4}$; $14 = 5.\frac{14}{5}$; $42 = 14.\frac{18}{6}$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$n! = 1 \times 2 \times 3 \times \ldots \times n$$

Euler introduit les séries génératrices !

$$\frac{1-2a-\sqrt{1-4a}}{2a^2}$$

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc}$$

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$$

$$1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} + \text{etc} = 4.$$

$$a = \frac{1}{4}$$

et même les prémisses de la "combinatoire analytique" !

1751.

4 Sept 1751
Berlin

L. Euler