The beauty of mathematics

Part 1 : Tilings, classical combinatorics, trigonometry

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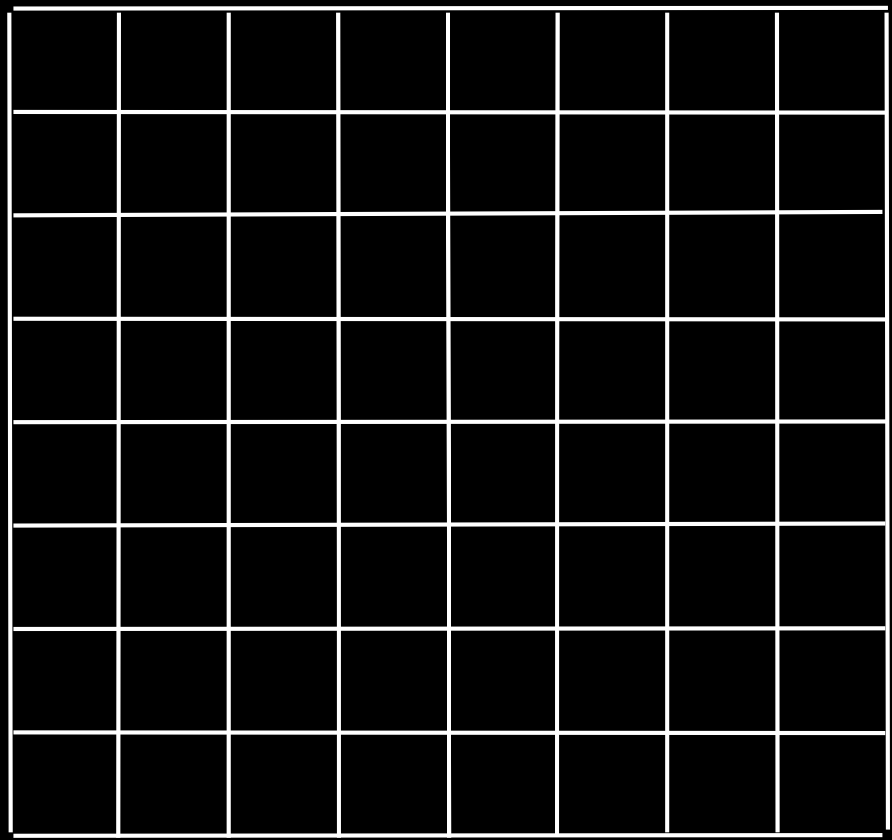
§1 Tilings

some "beautiful" formulae

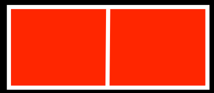
tiling the floor

(from Greg Kuperberg)

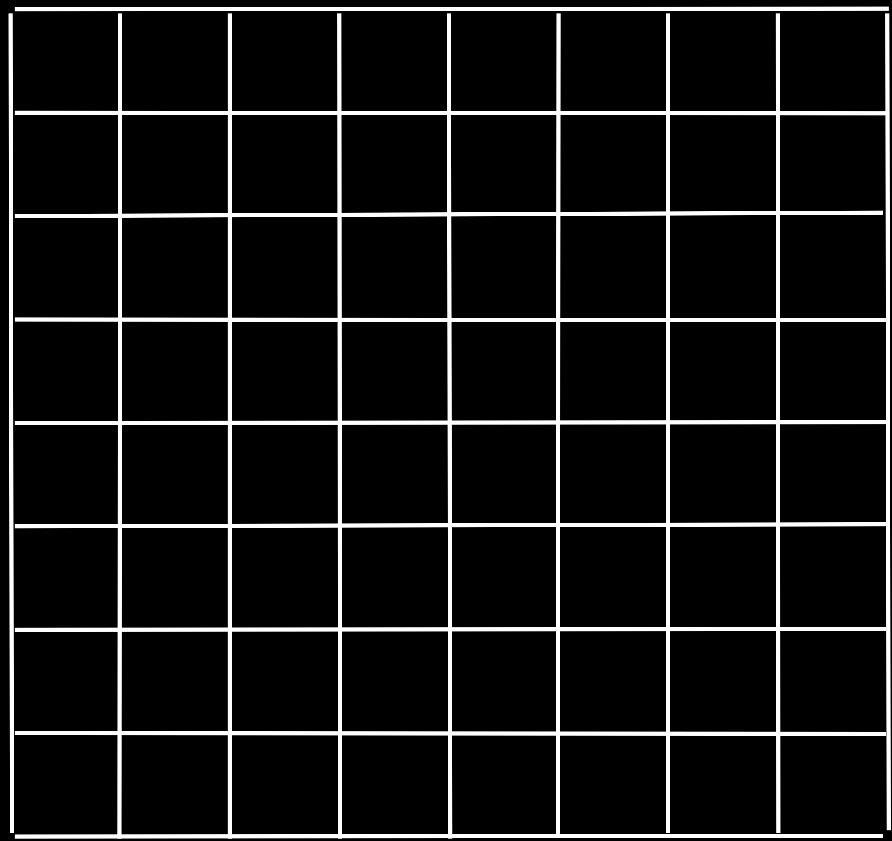
chessboard



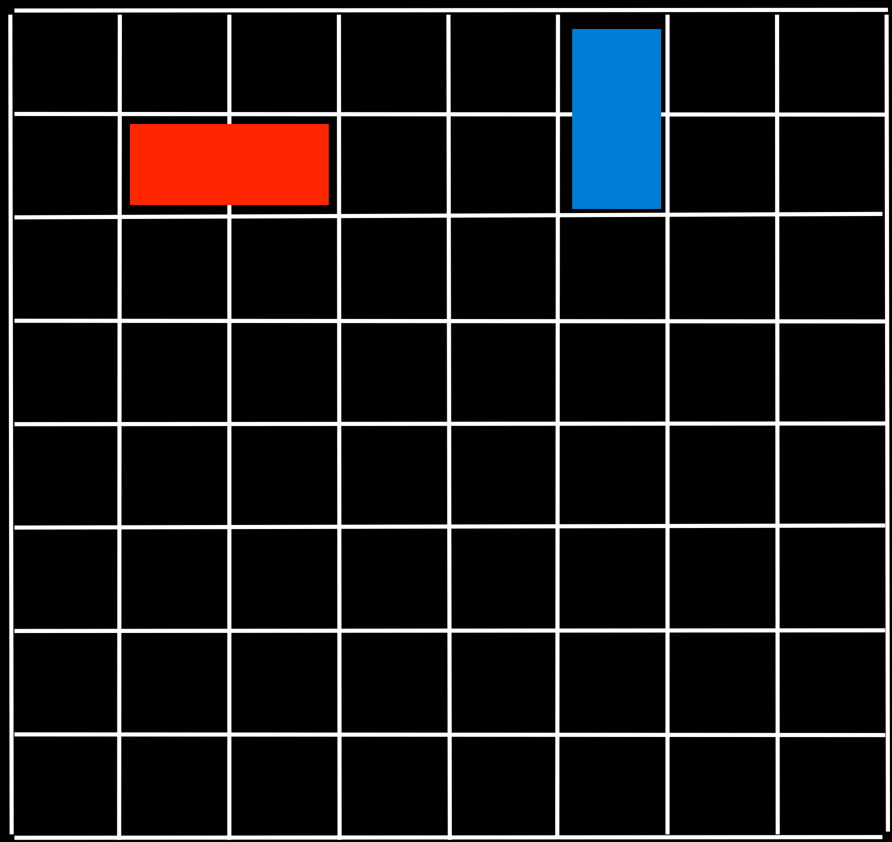
8 rows
8 columns



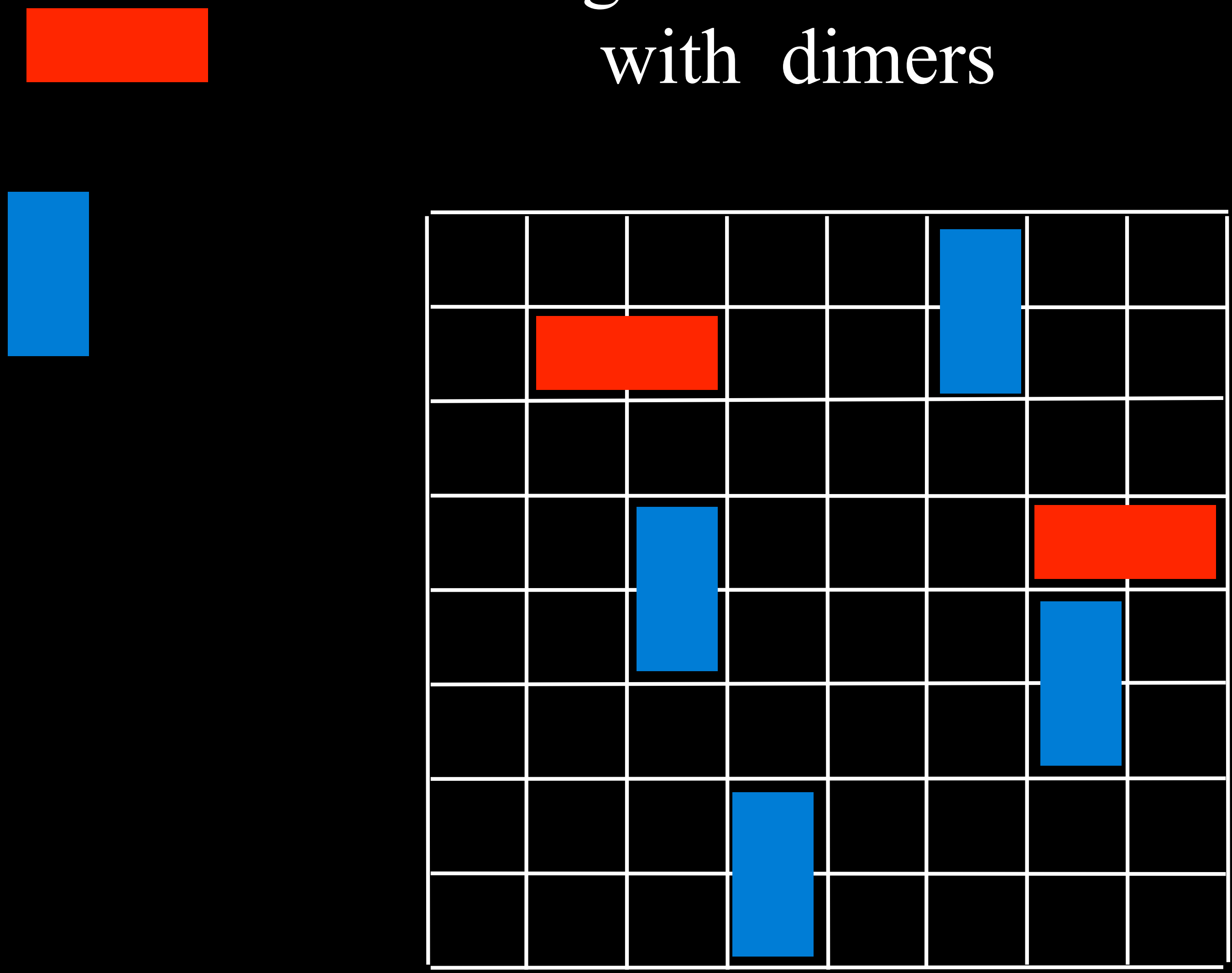
dimers



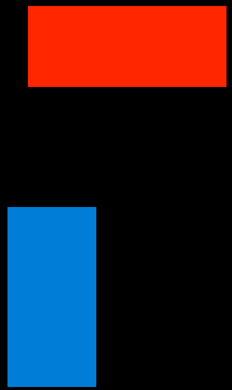
Tilings of a chessboard with dimers

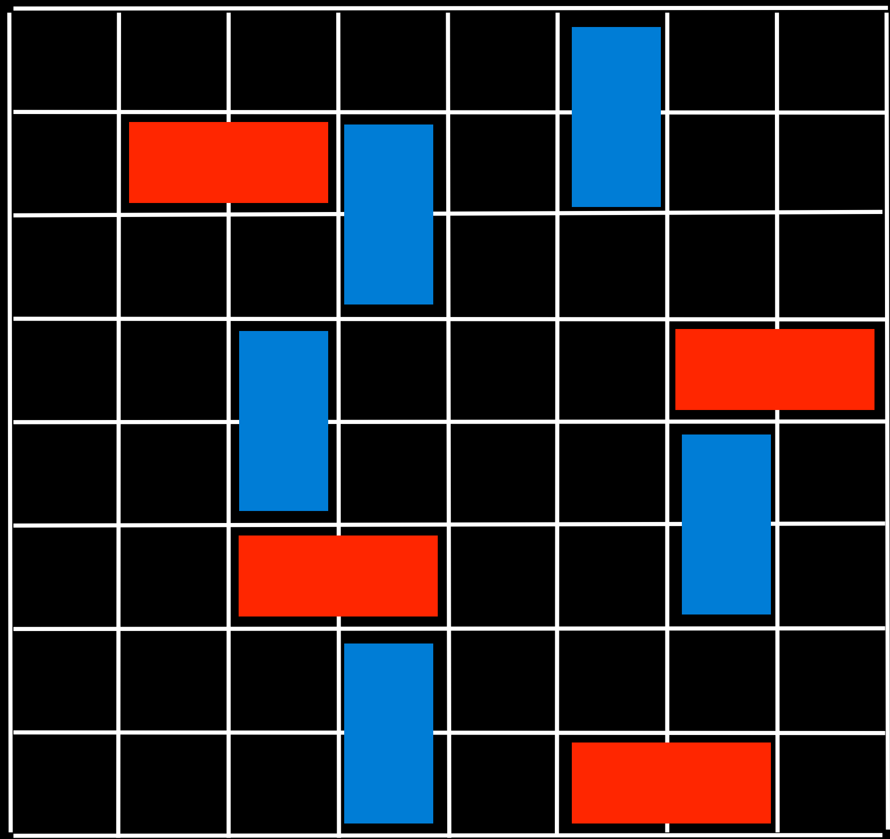


Tilings of a chessboard with dimers

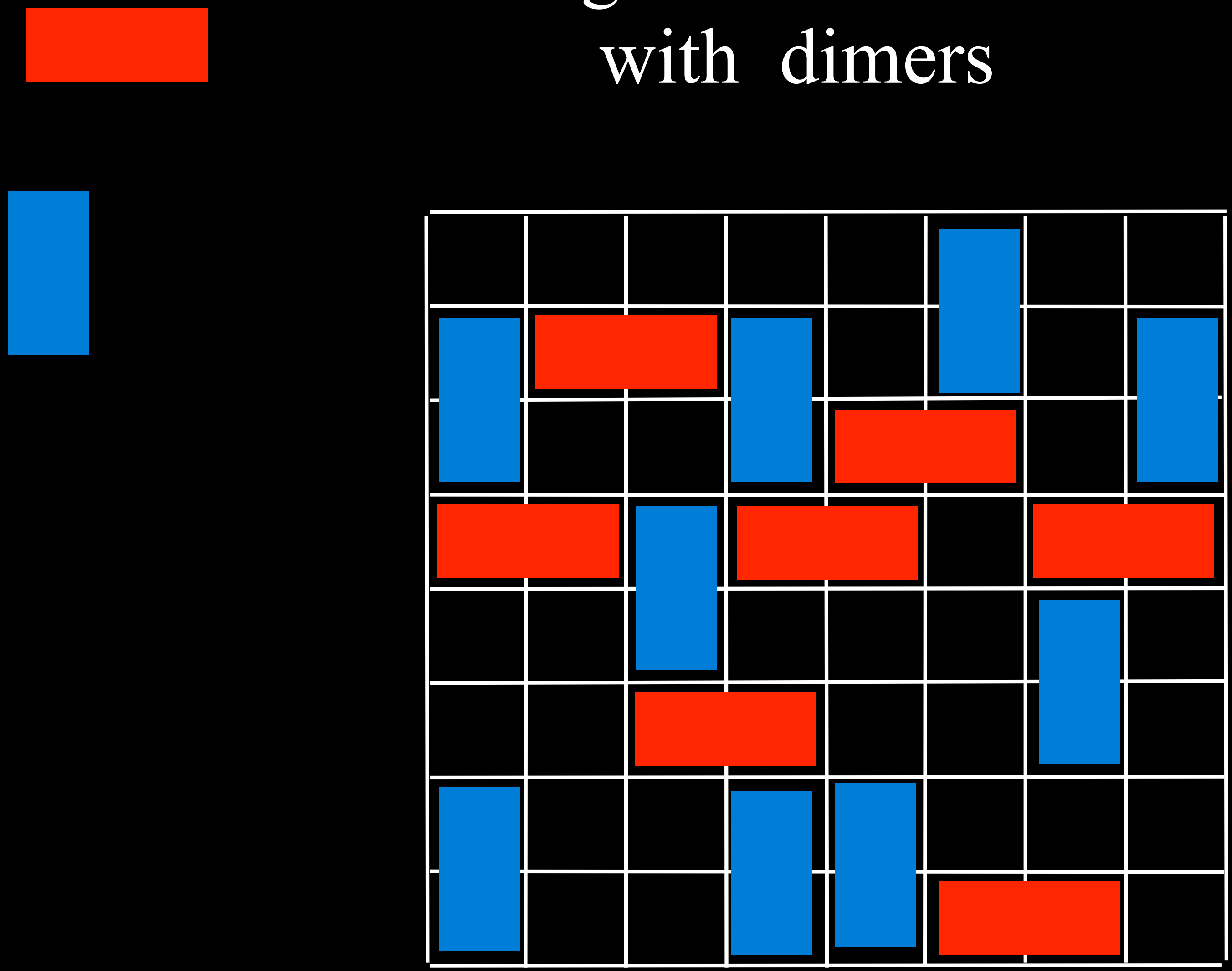


Tilings of a chessboard with dimers

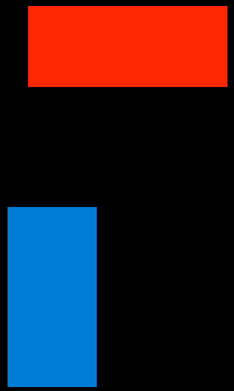


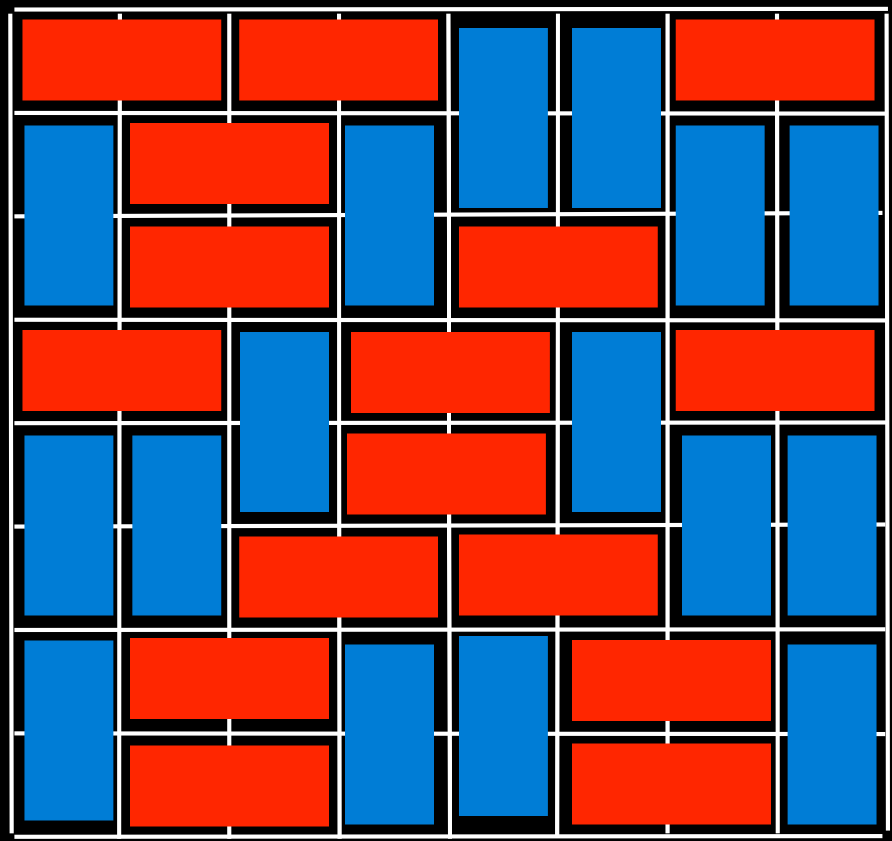


Tilings of a chessboard with dimers



Tilings of a chessboard with dimers

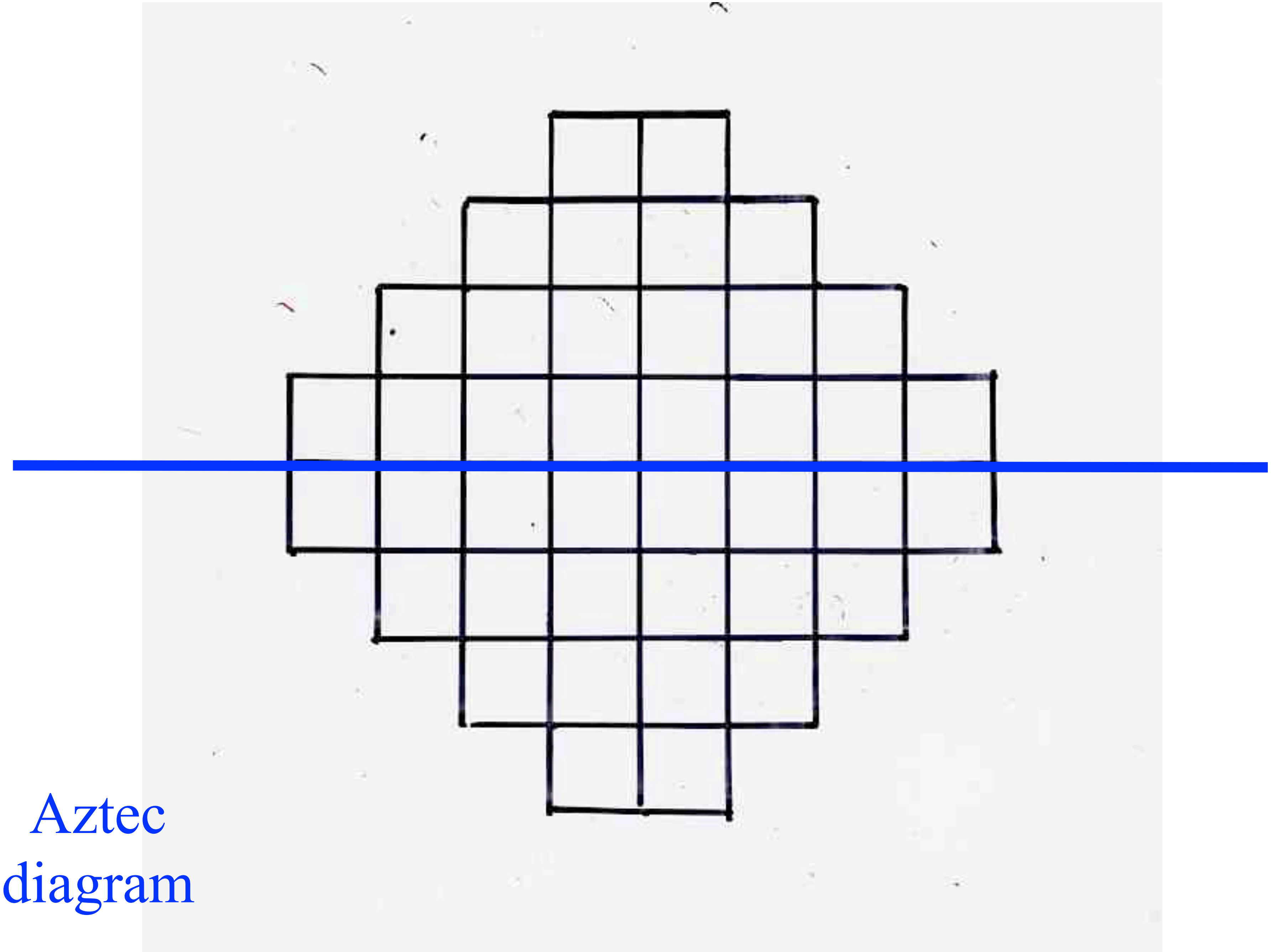




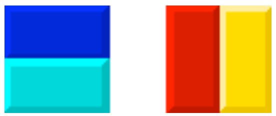
the number of tilings for the 8 x 8 chessboard
= 12 988 816

A beautiful formula ....

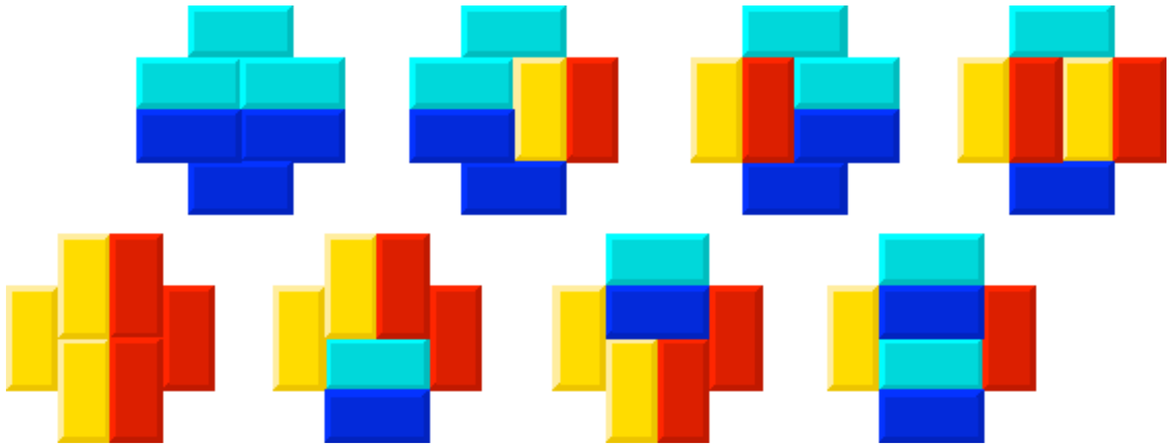
with Aztec tilings



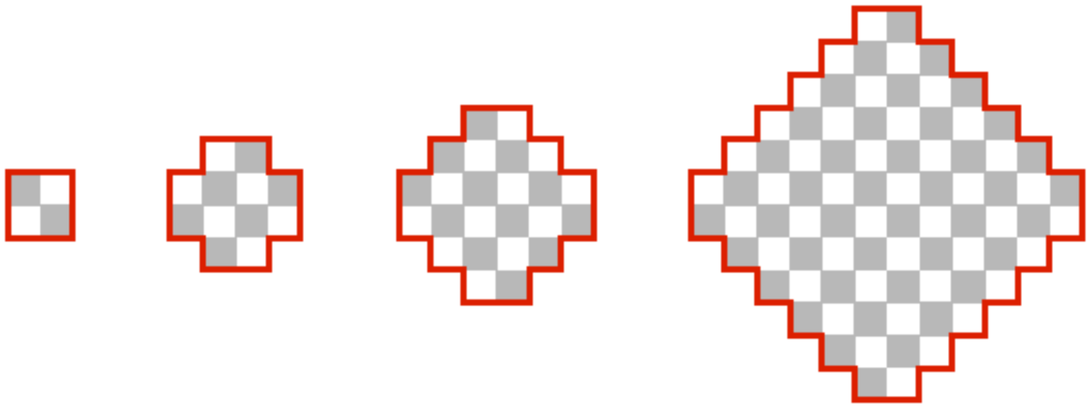
Aztec
diagram



2



8



number of tilings

2 8 64 1024

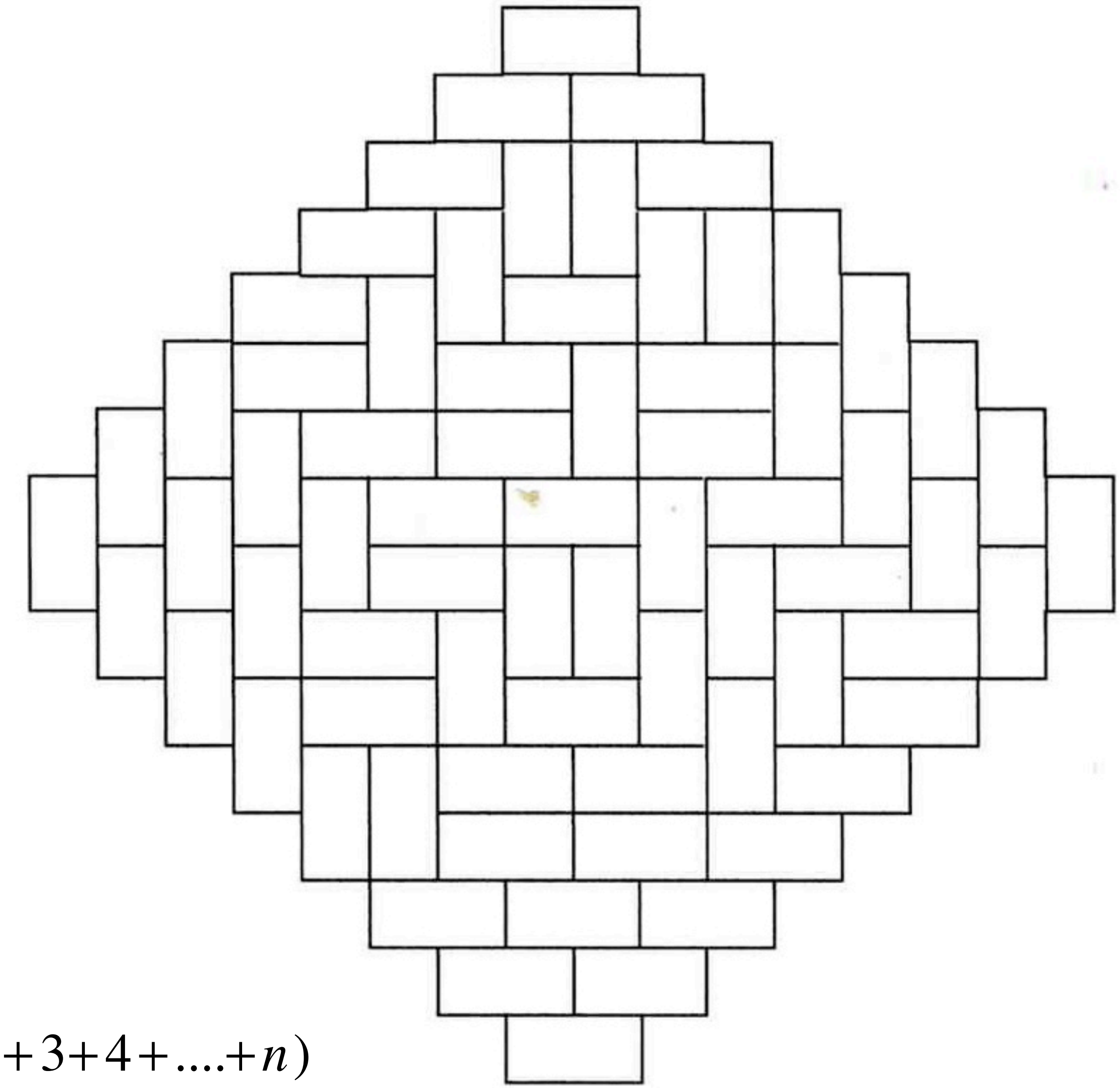
2^1 2^3 2^6 2^{10}

2^1 $2^{(1+2)}$ $2^{(1+2+3)}$ $2^{(1+2+3+4)}$

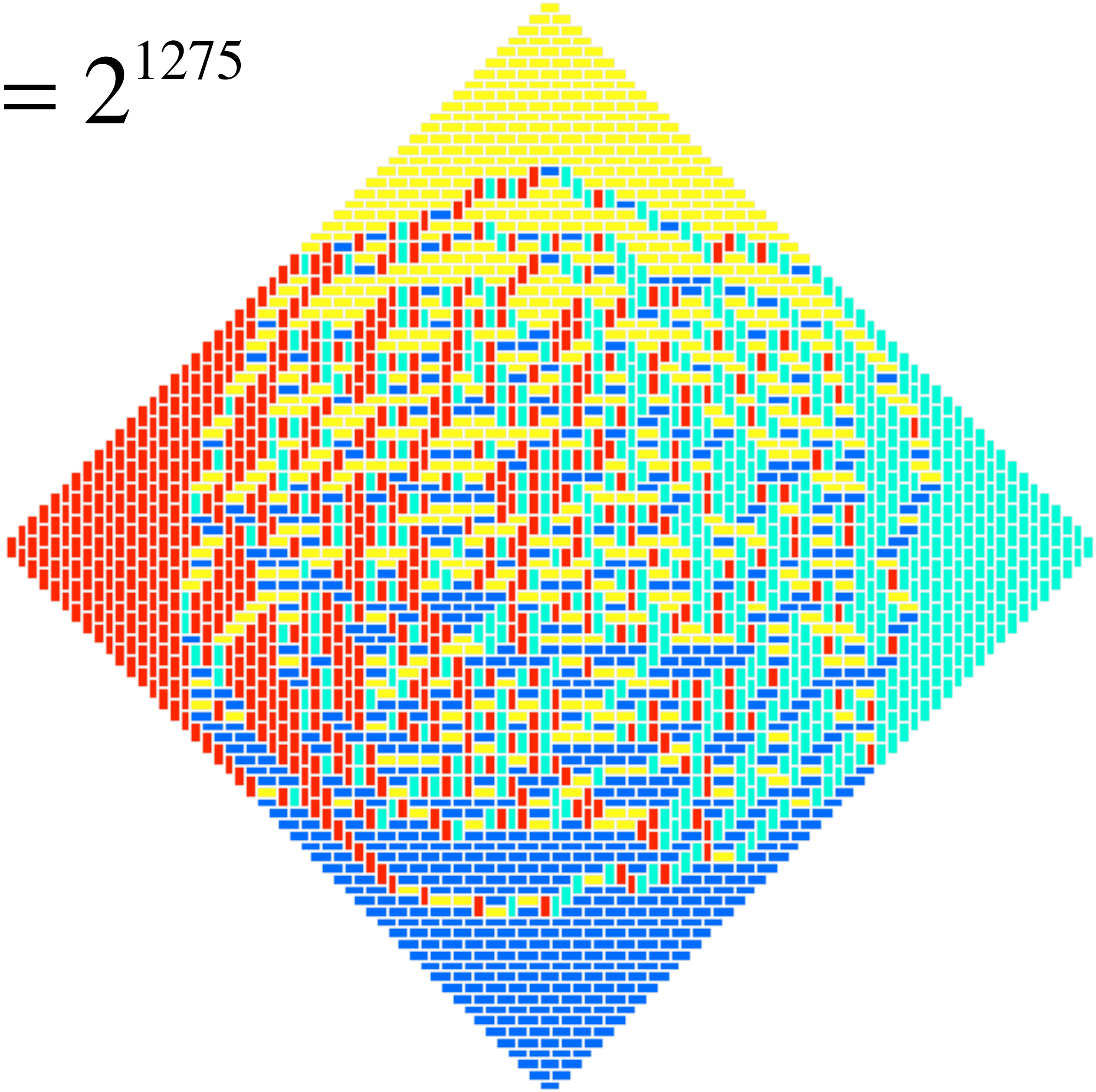
the number
of
tilings of the
Aztec diagram
with dimers
is

$$2^{\frac{n(n+1)}{2}}$$

$$2^{(1+2+3+4+....+n)}$$



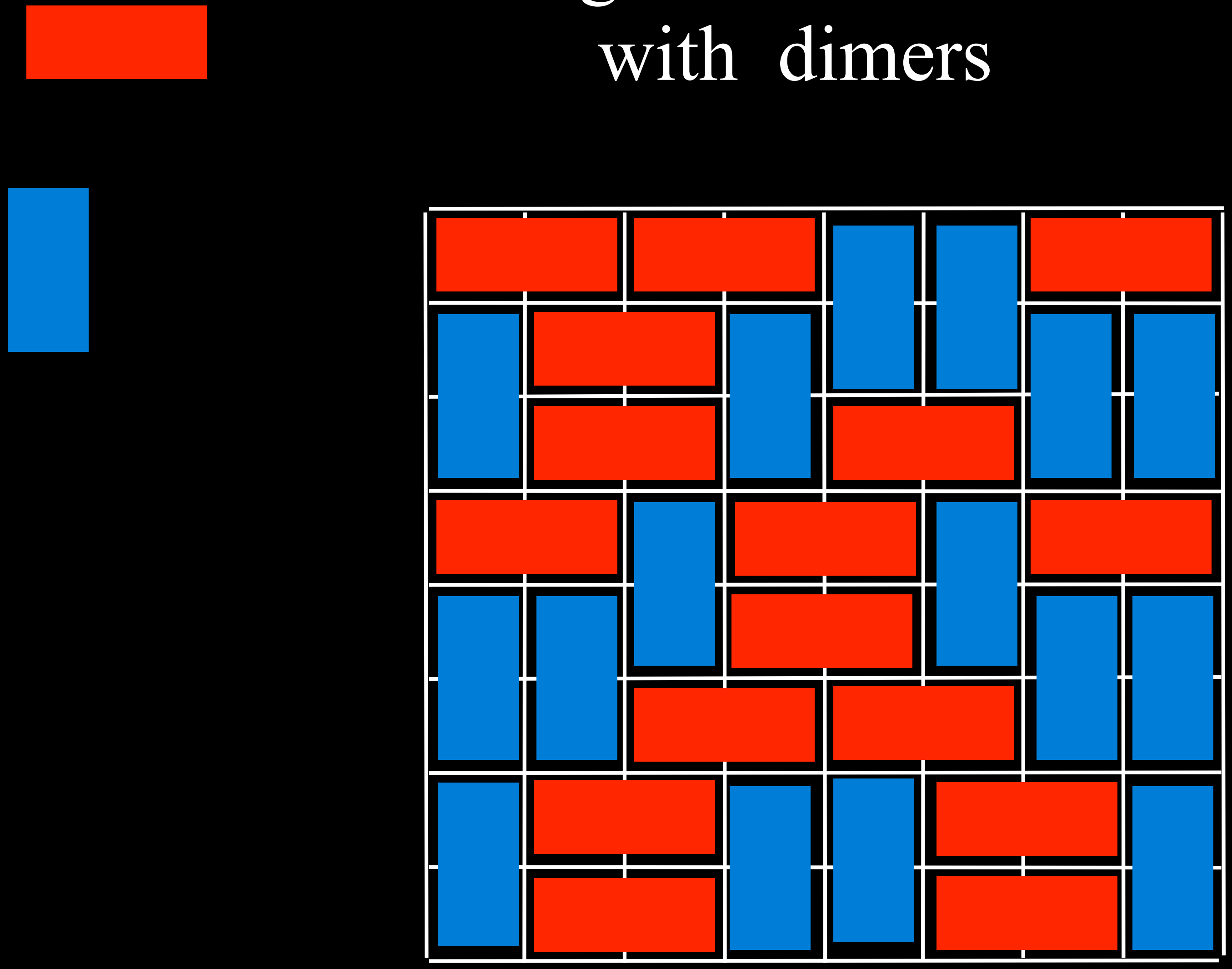
$$2^{(50\times 51)/2} = 2^{1275}$$



going back

to tilings of the chessboard

Tilings of a chessboard with dimers



the number of tilings with dimers a

$m \times n$ rectangle is

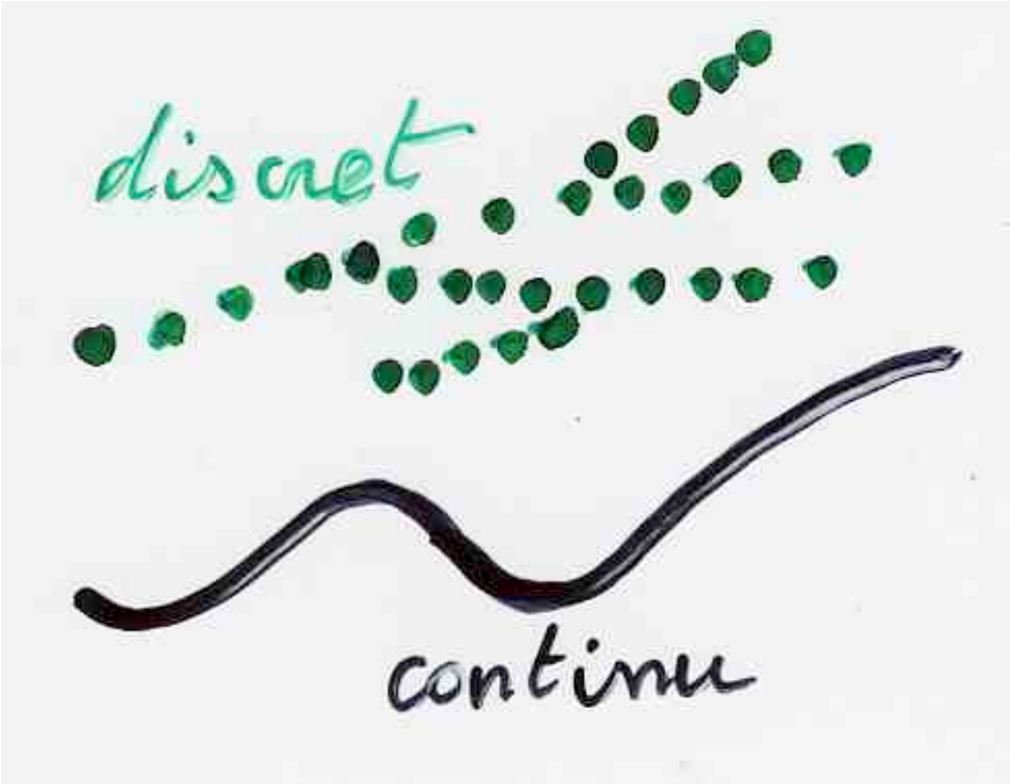
$$\prod_{i=1}^{m/2} \prod_{j=1}^{n/2} \left(4 \cos^2 \frac{i\pi}{m+1} + 4 \cos^2 \frac{j\pi}{n+1} \right)$$

Kasteleyn (1961)

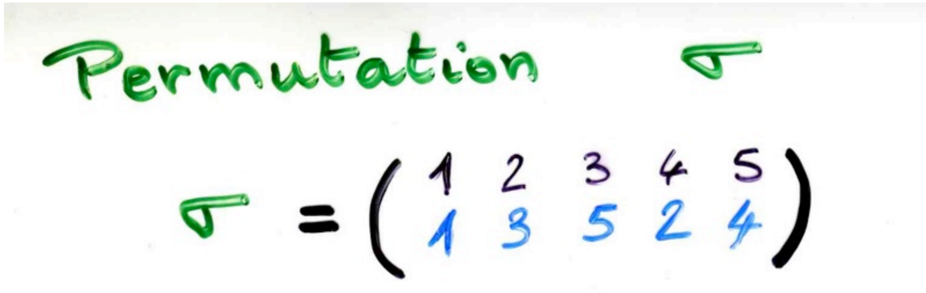
it is an integer !

for a chessboard m=8, n=8: 12 988 816

§2 classical combinatorics


discret
continu

permutations



$$n! = 1 \times 2 \times \ldots \times n$$

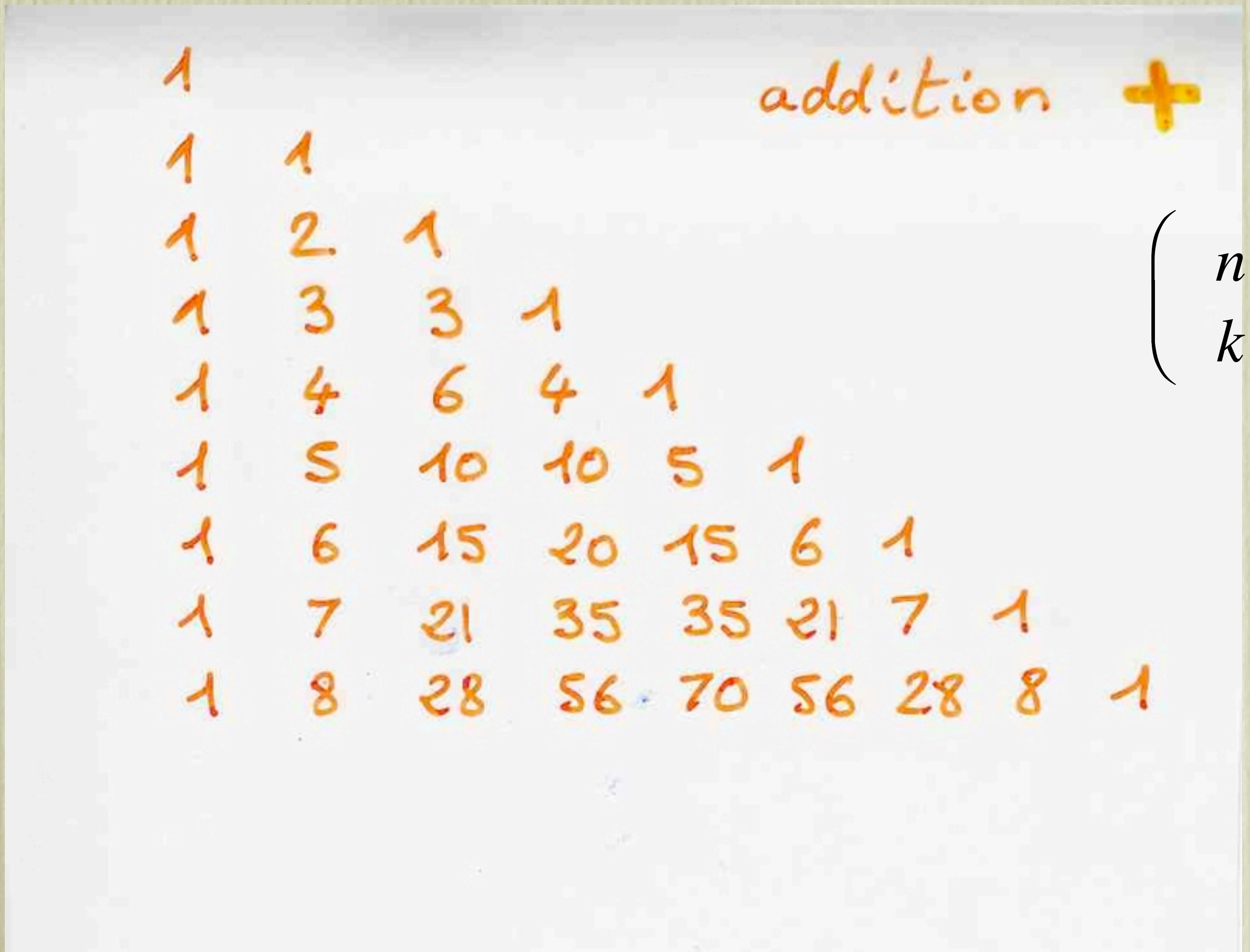
1, 2, 6, 24, 120, 720, 5040, 40320,...

k elements subset
of a set having n elements

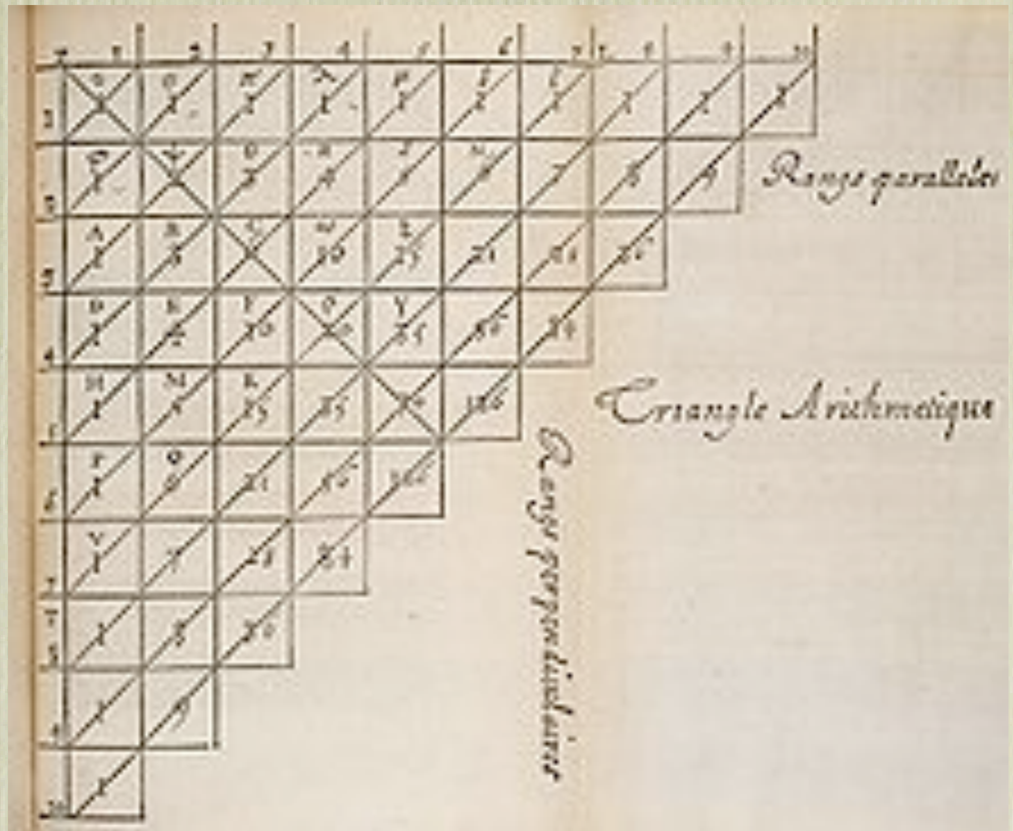
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

binomial coefficients

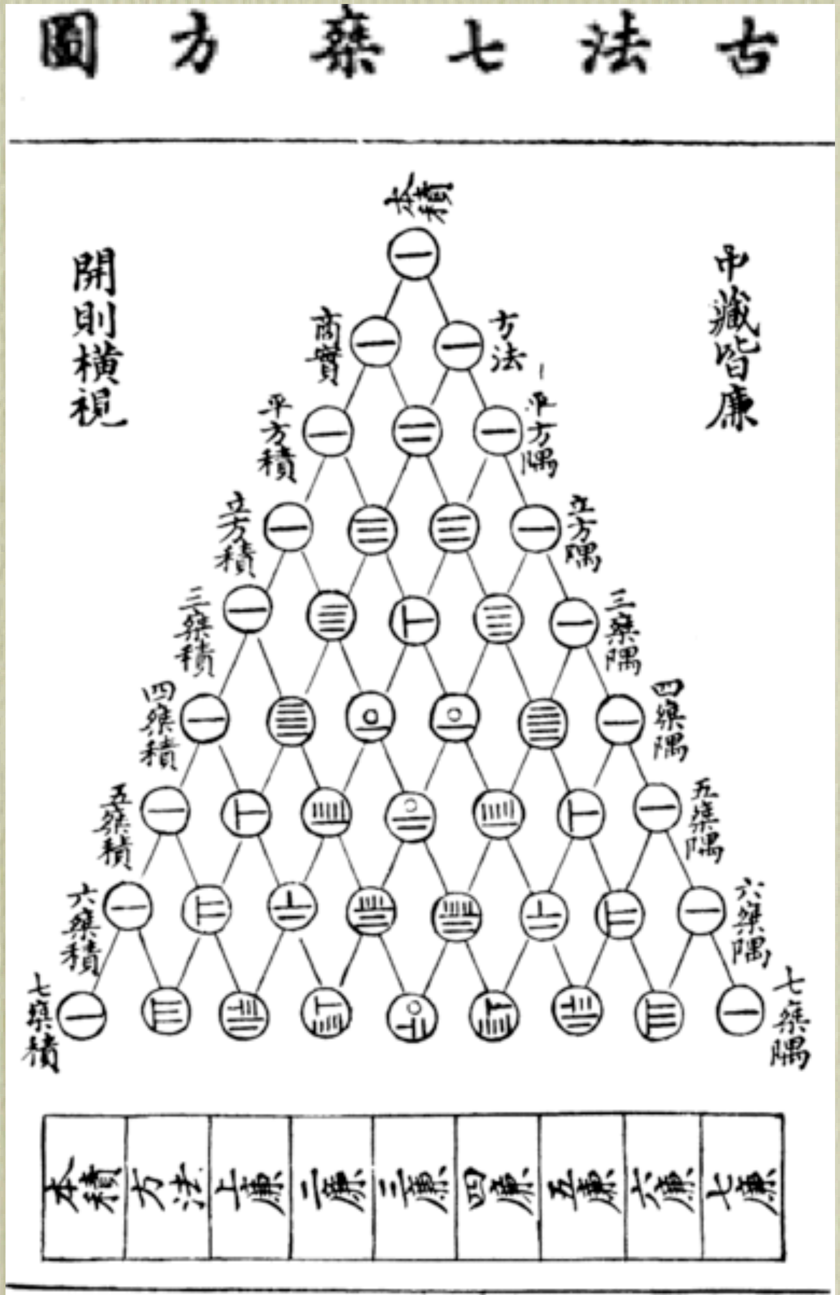
$$(1+t)^n = 1 + \binom{n}{1} t + \binom{n}{2} t^2 + \binom{n}{3} t^3 + \ldots + \binom{n}{k} t^k + \ldots + \binom{n}{n} t^n$$



$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$



Pascal triangle
binomial coefficients

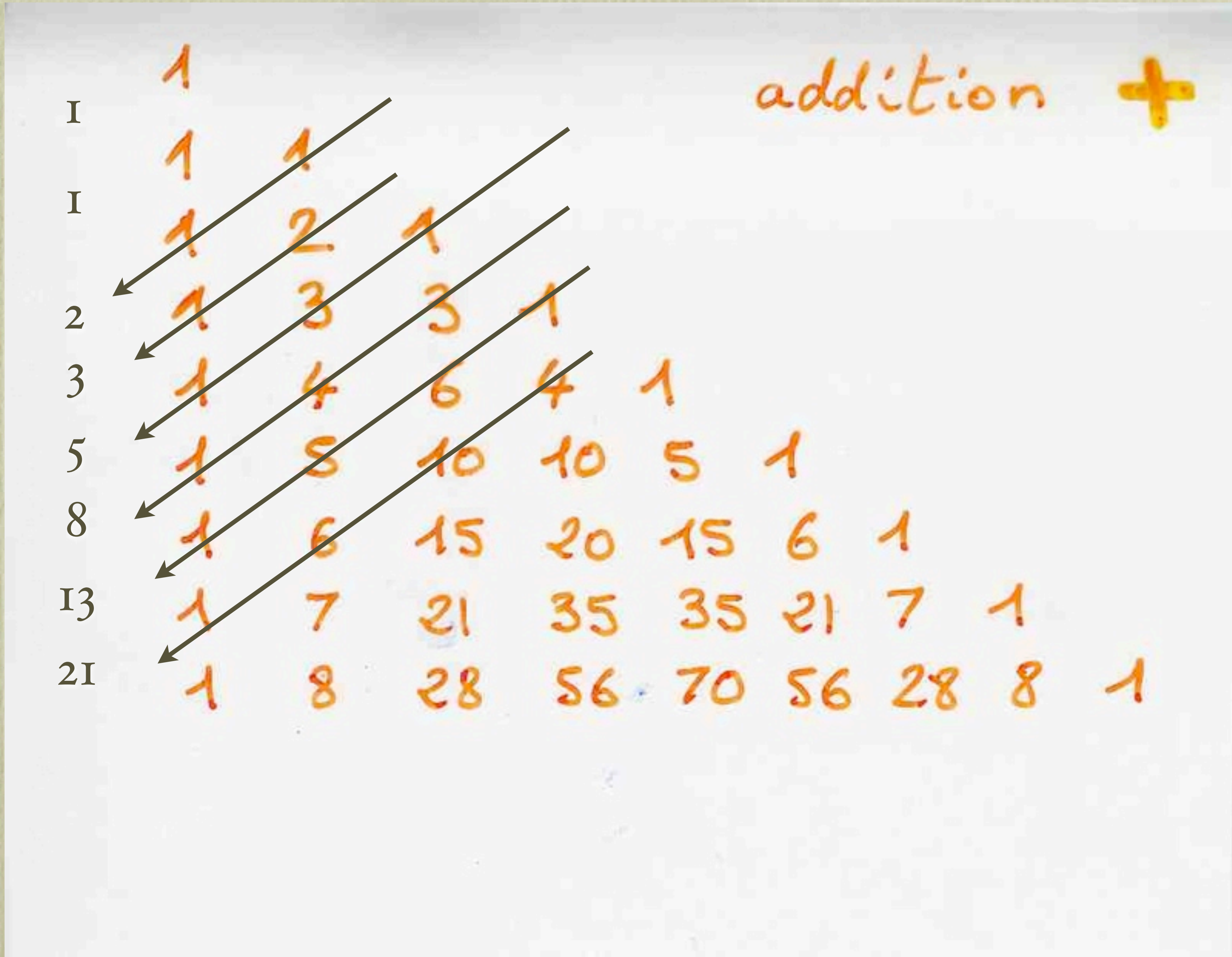


Yang Hui triangle
(11th, 12th century)

in Persia
Omar Khayyam
(1048-1131)

in India
Chandas Shastra by Pingala
2nd century BC

relation with Fibonacci numbers
(10th century)

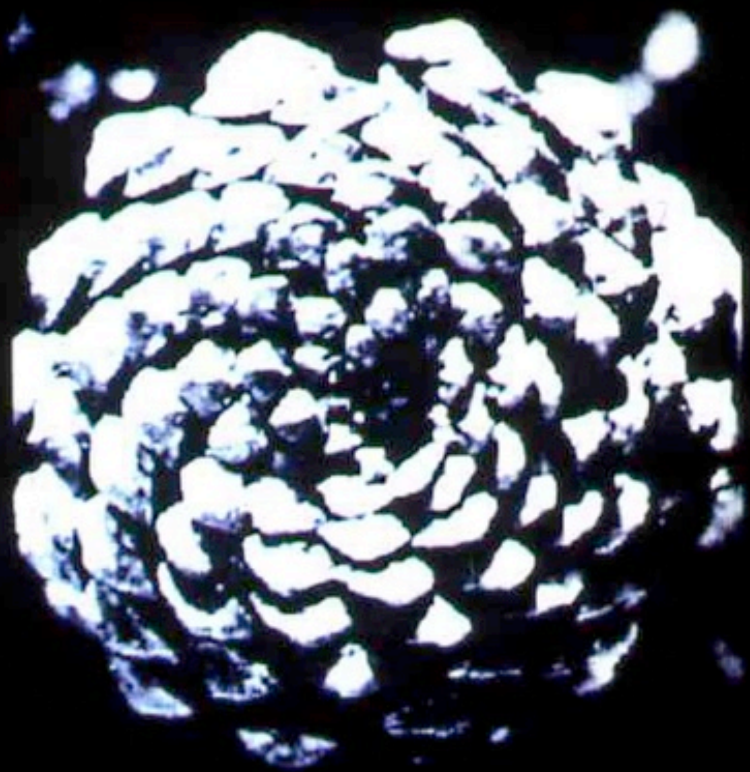
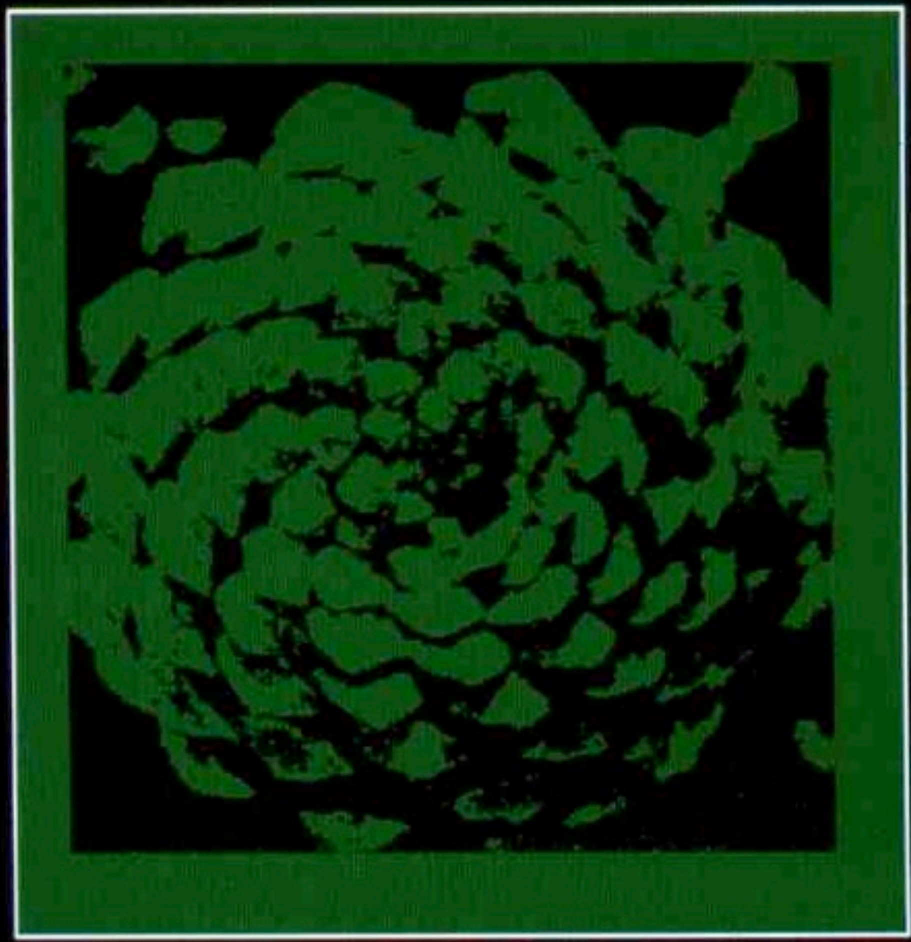

addition +
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1
1
2
3
5
8
13
21

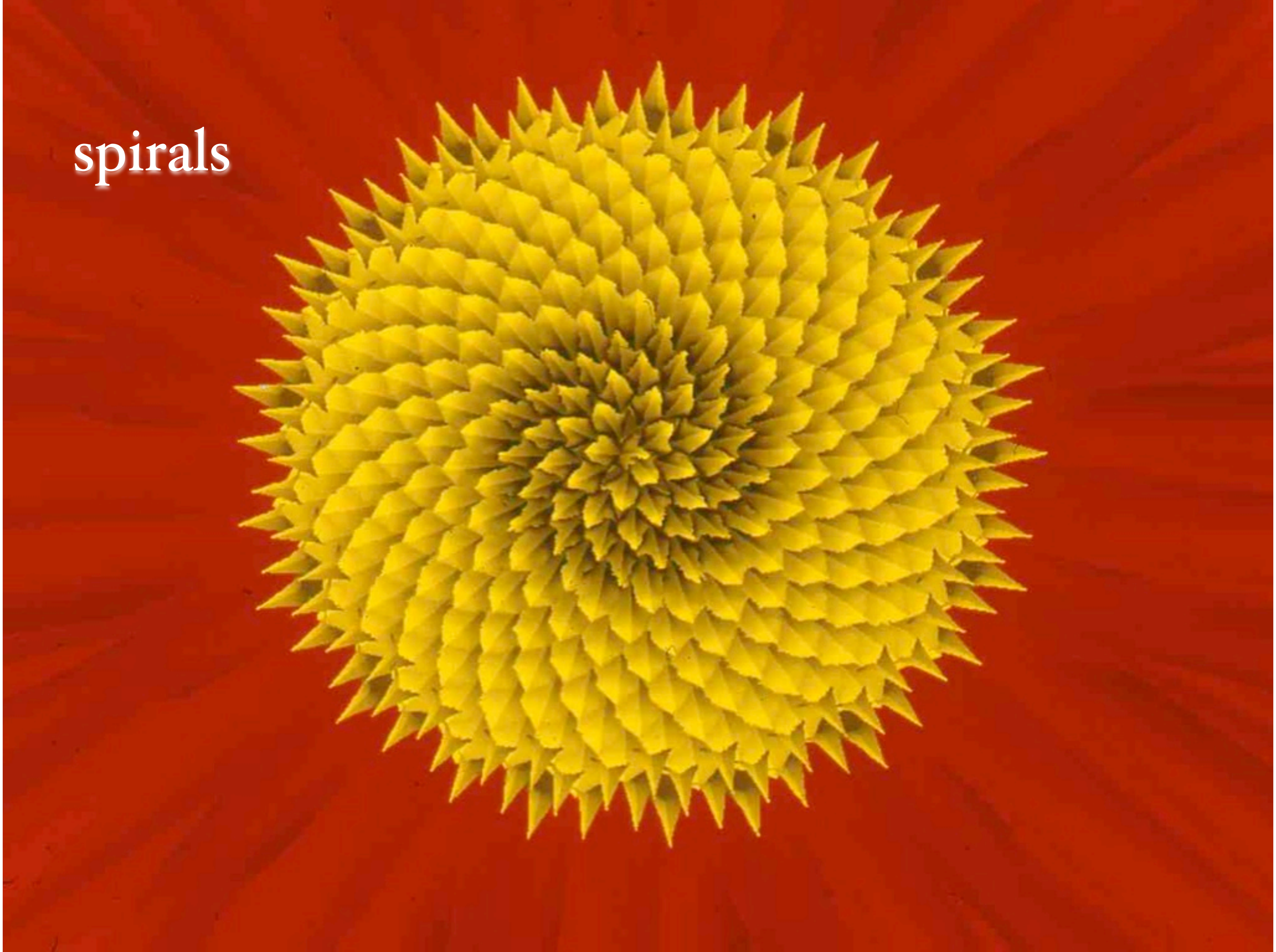
Fibonacci numbers

1, 1,2, 3, 5, 8, 13, 21, 34, 55,...

$$F_{n+1} = F_n + F_{n-1}$$

$$F_0 = 1, F_1 = 1$$

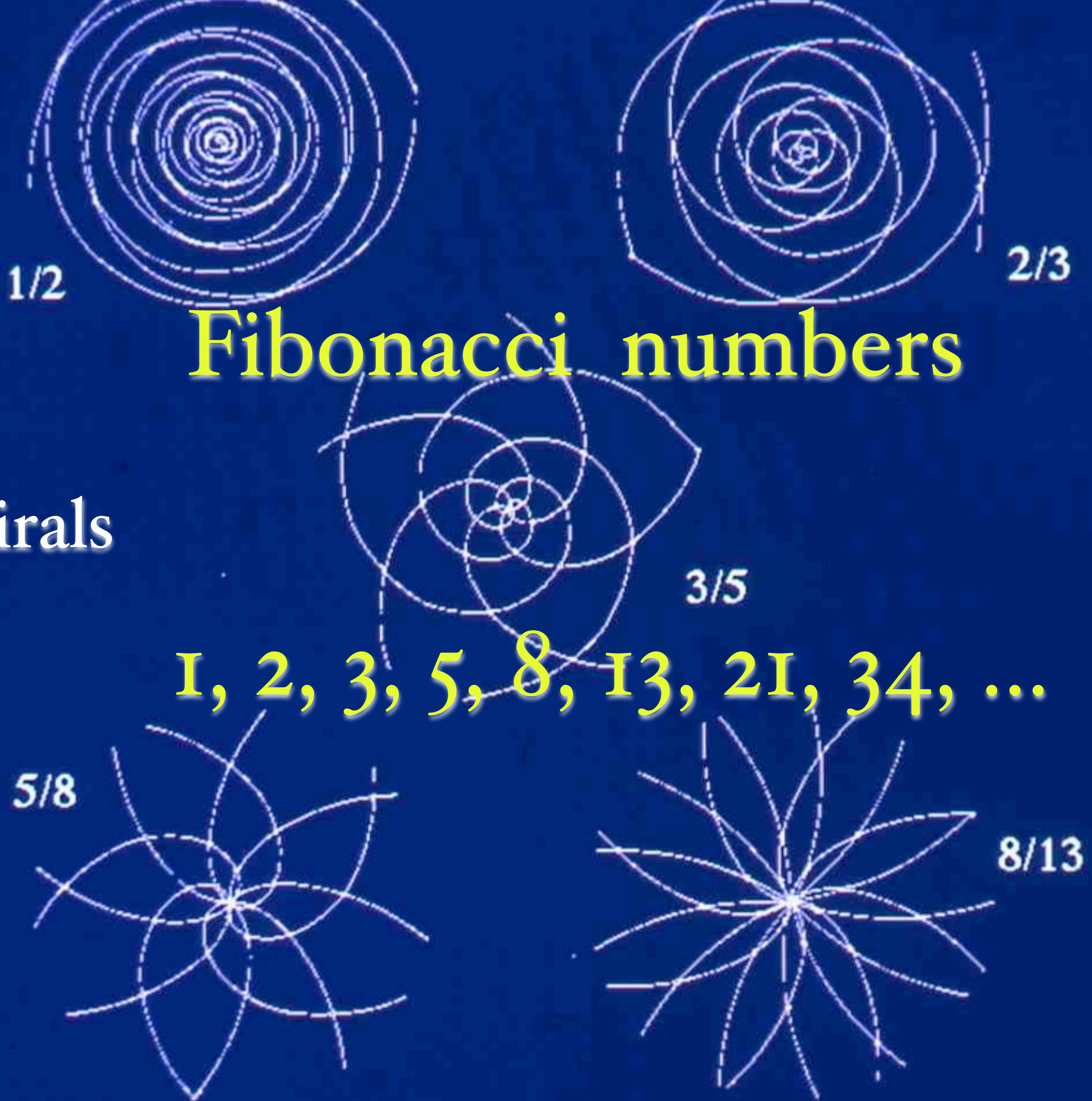




spirals

Fibonacci numbers

spirals

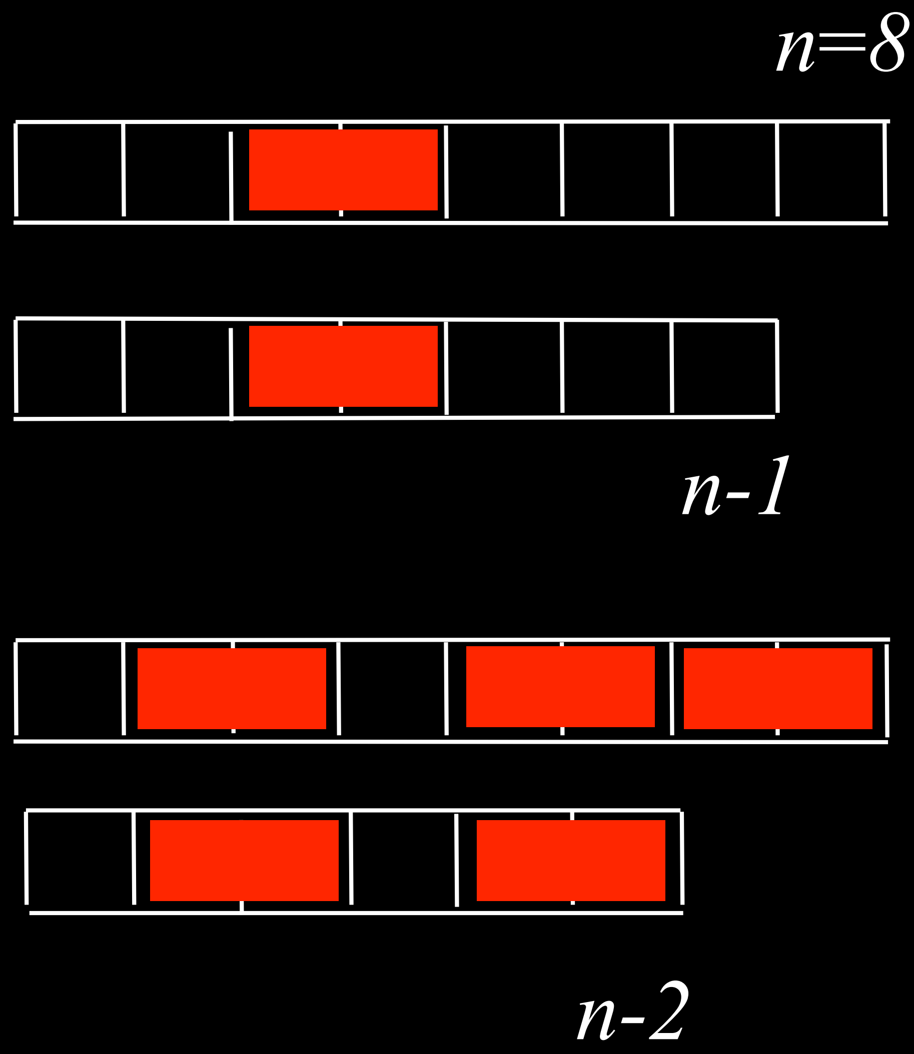
1, 2, 3, 5, 8, 13, 21, 34, ...




dimers
matching
tiling

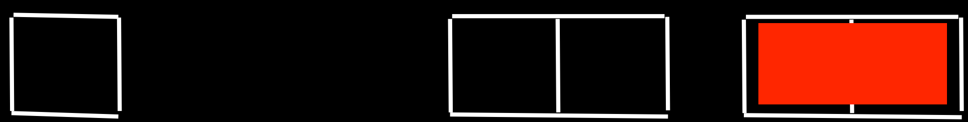
dimers

total number of matchings = F_n



Fibonacci number

$$F_{n+1} = F_n + F_{n-1}$$
$$F_1 = 1, F_2 = 2$$

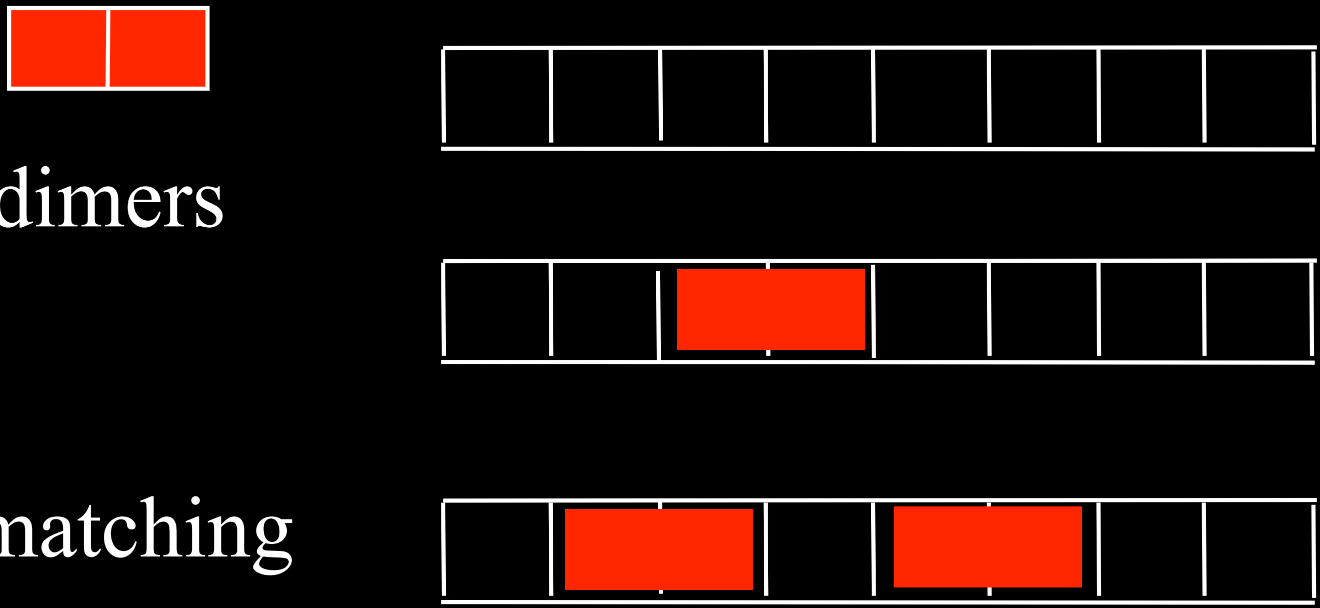


§3 Combinatorial interpretations of some formulae

trigonometry

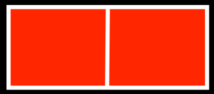
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\sin 3\theta = \sin\theta\left(4\cos^2\theta - 1\right)$$
$$\sin 4\theta = \sin\theta\left(8\cos^3\theta - 4\cos\theta\right)$$
$$\cdots\cdots\cdots\cdot$$

$$\cos 2\theta = \left(2\cos^2\theta - 1\right)$$
$$\cos 3\theta = \left(4\cos^3\theta - 3\cos\theta\right)$$
$$\cos 4\theta = \left(8\cos^4\theta - 8\cos^2\theta + 1\right)$$

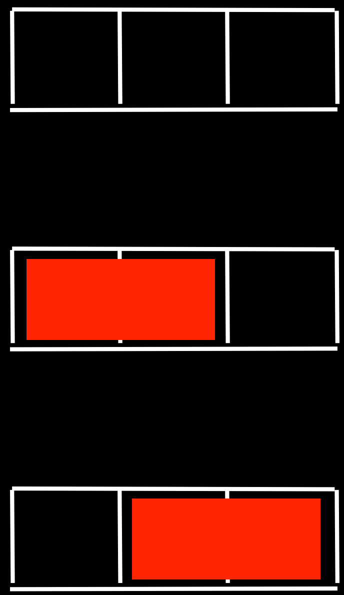


dimers

matching

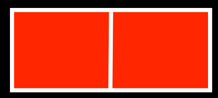


dimers

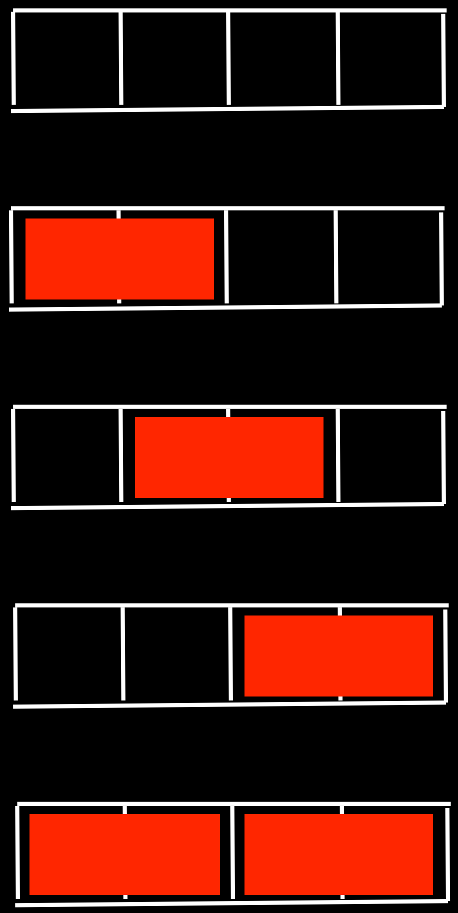


x^3	$(2\cos\theta)^3$
$-x$	$(2\cos\theta)$
$-x$	

$$\sin 4\theta = \sin\theta\left(8\cos^3\theta - 4\cos\theta\right)$$

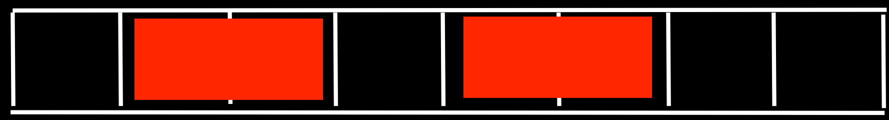


dimers



x^4	$(2\cos\theta)^4$
$-x^2$	$(2\cos\theta)^2$
$-x^2$	
$-x^2$	
1	1

$$\sin 5\theta = \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right)$$

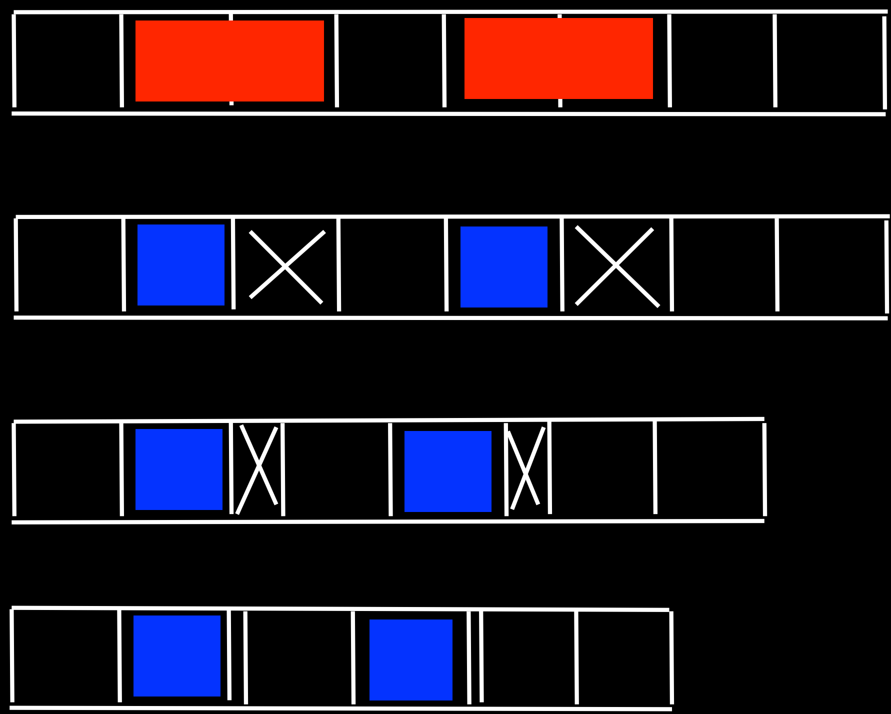


matching

k dimers

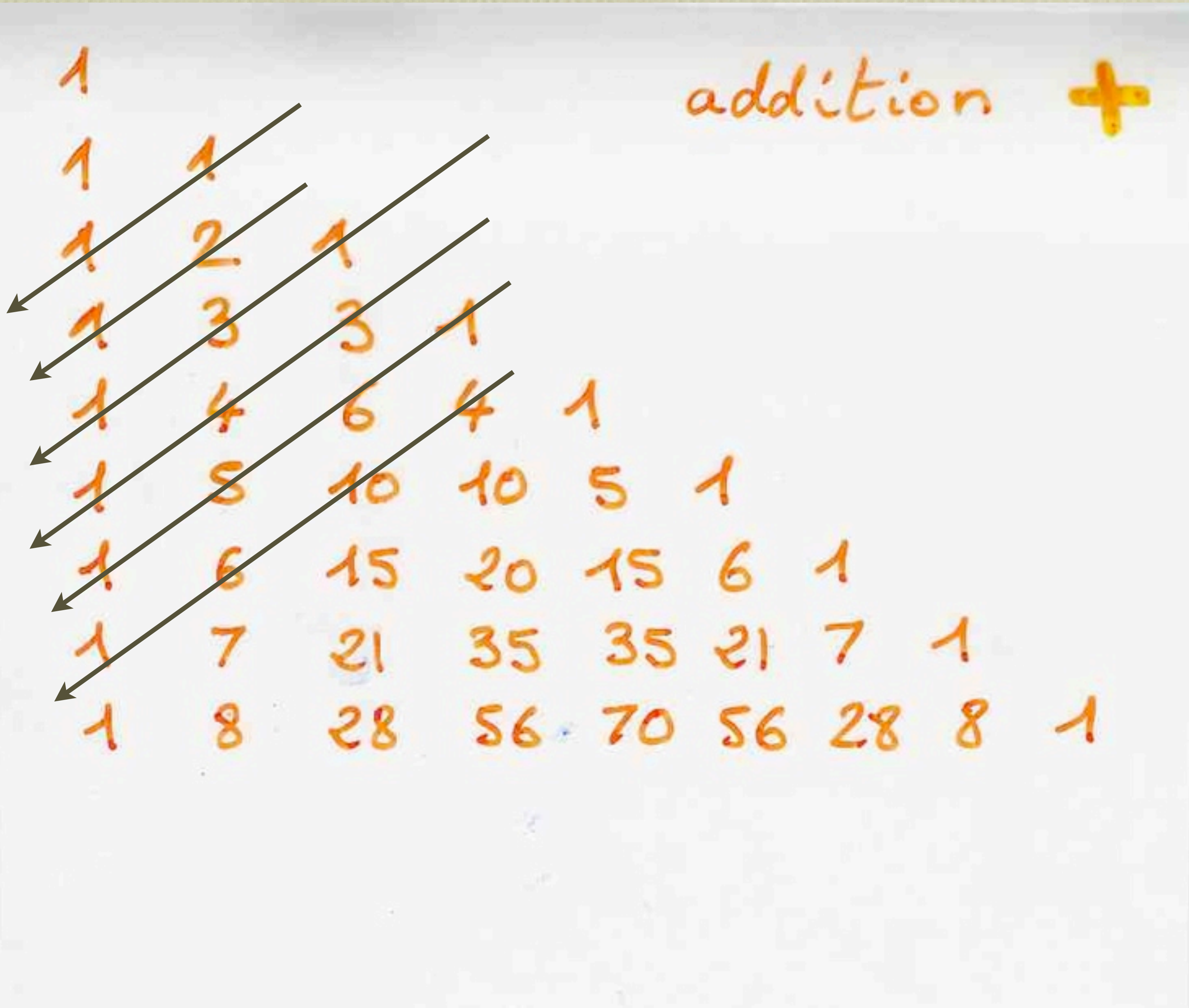
$n = 8$

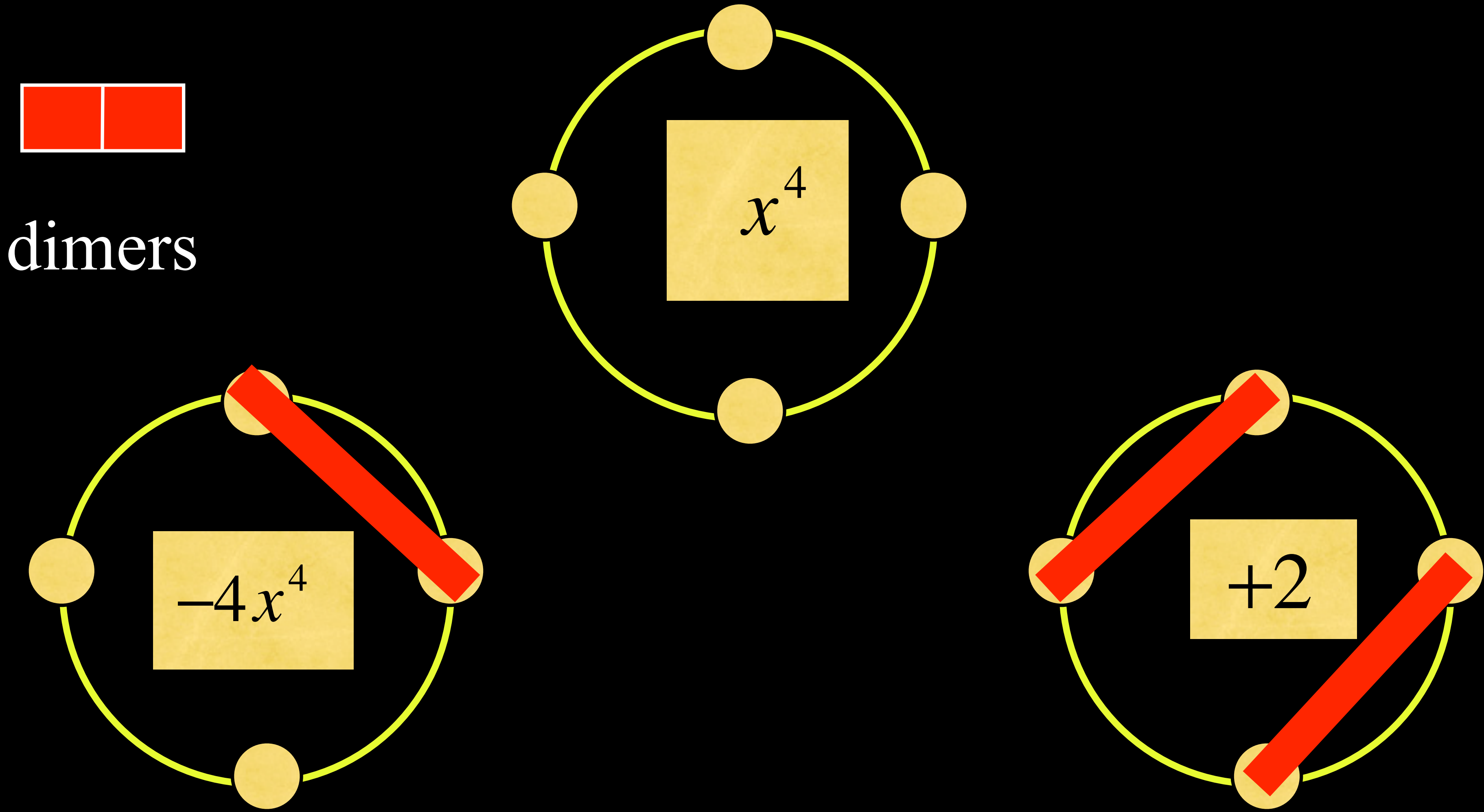
$k = 2$



$$\binom{n-k}{k} x^{n-2k}$$

1
1
2
3
5
8
13
21


addition +
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1



$$x \rightarrow 2\cos\theta$$

$$\left(16\cos^4\theta - 16\cos^2\theta + 2\right)$$

$$\cos 4\theta = \left(8\cos^4\theta - 8\cos^2\theta + 1\right)$$