Tamari lattice and its extensions

(2)

IMSc, Chennai
3 March 2015

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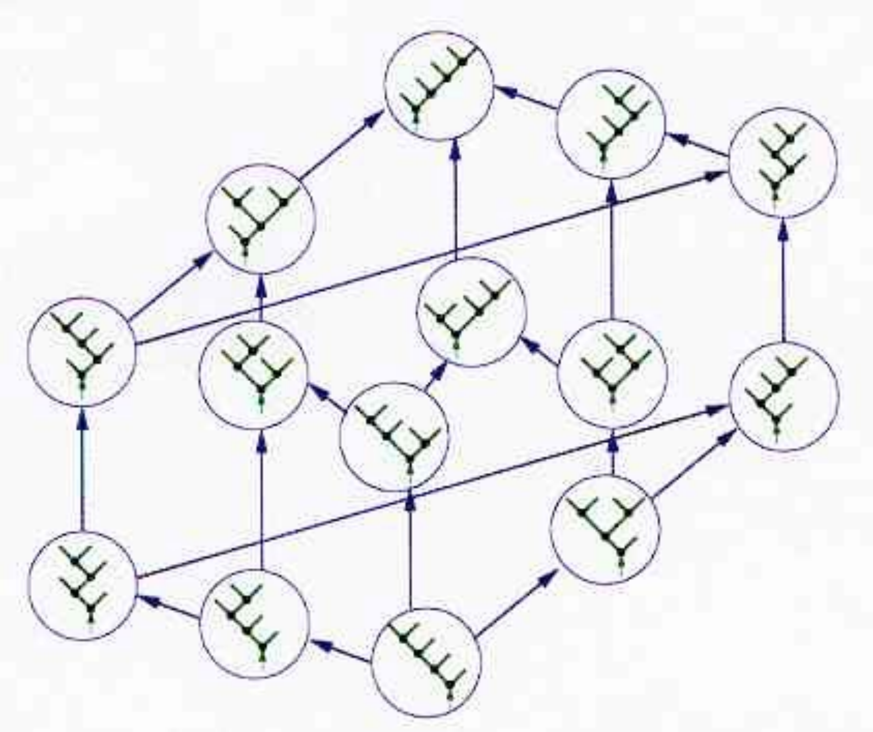
ArXiv: 1406.3787 [math.CO]

to be published in
Transactions A.M.S.

Introduction

Tamari

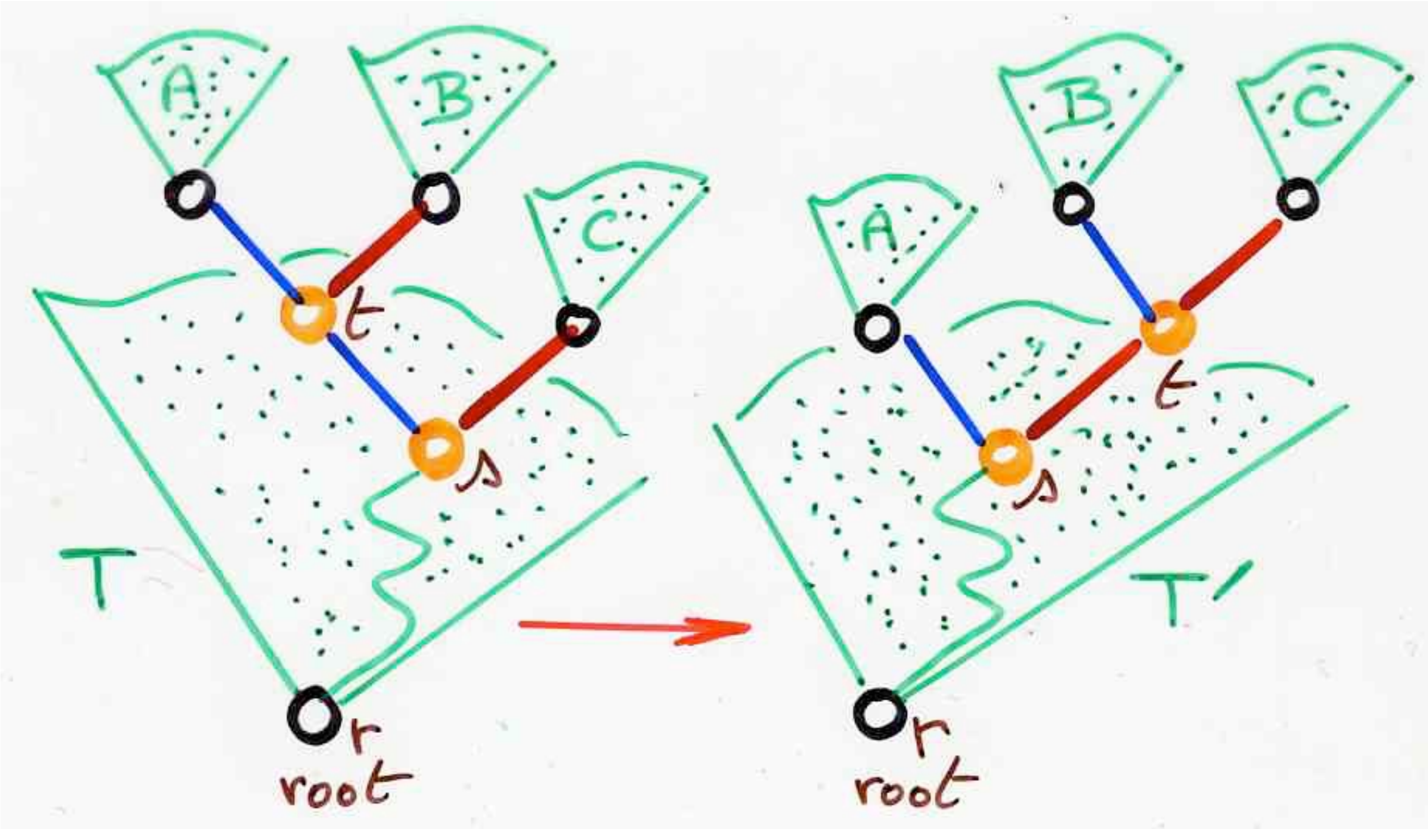
m-Tamari



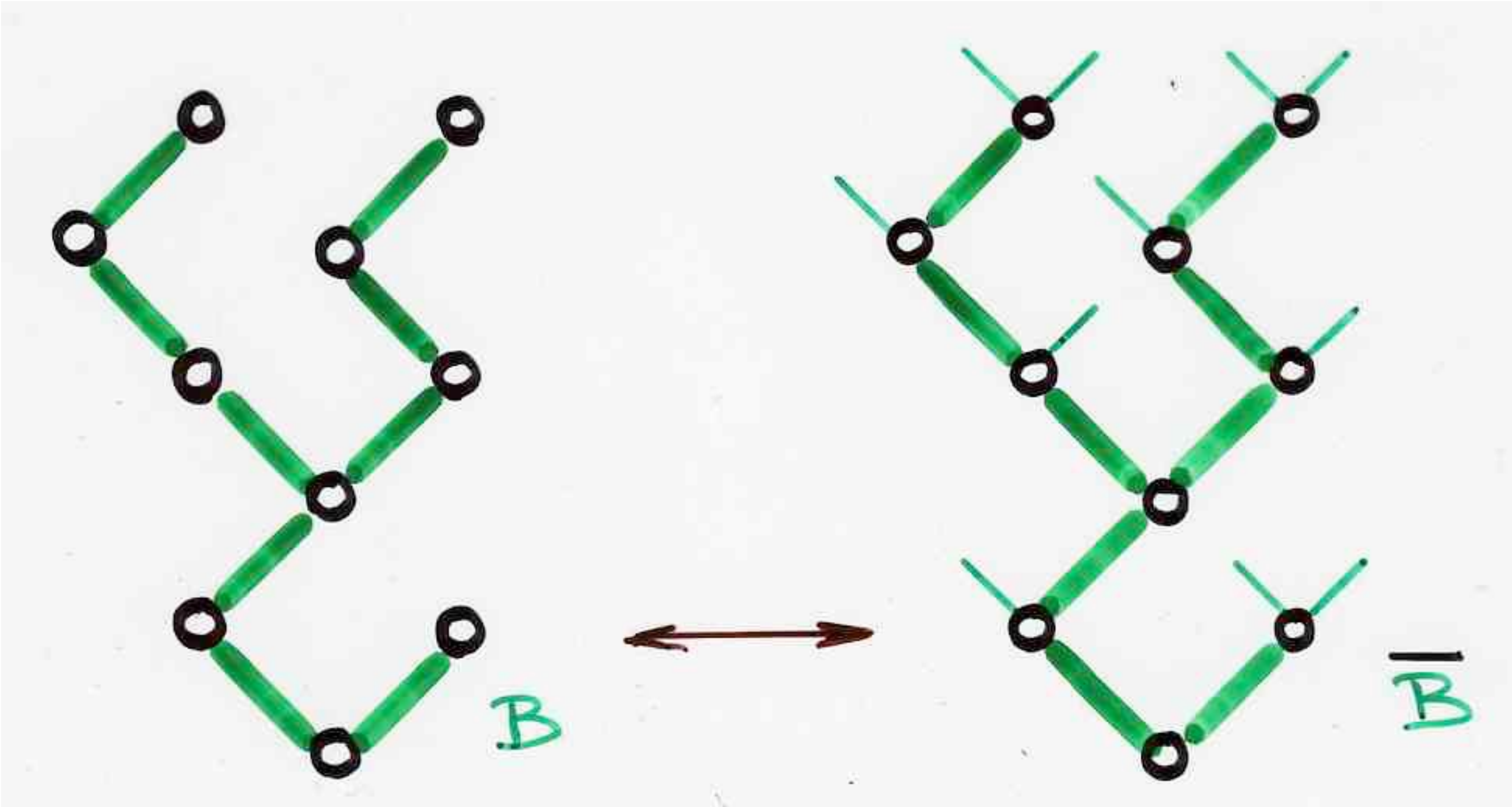
Tamari lattice

$C_4 = 14$
Catalan

Dov Tamari (1951) thèse Sorbone
"Monoides préordonnés et chaînes de Malcev"

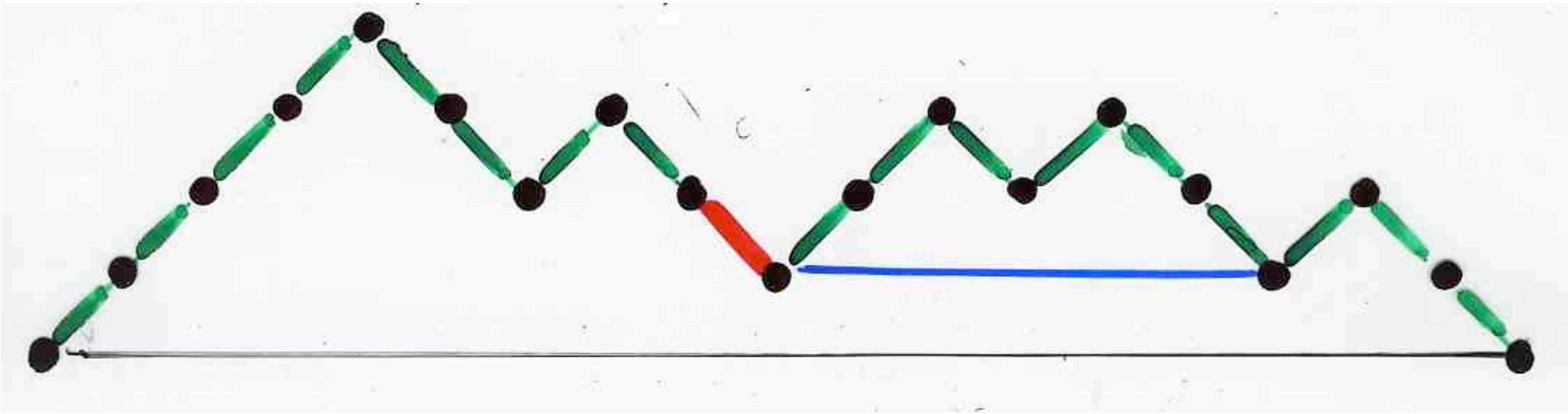


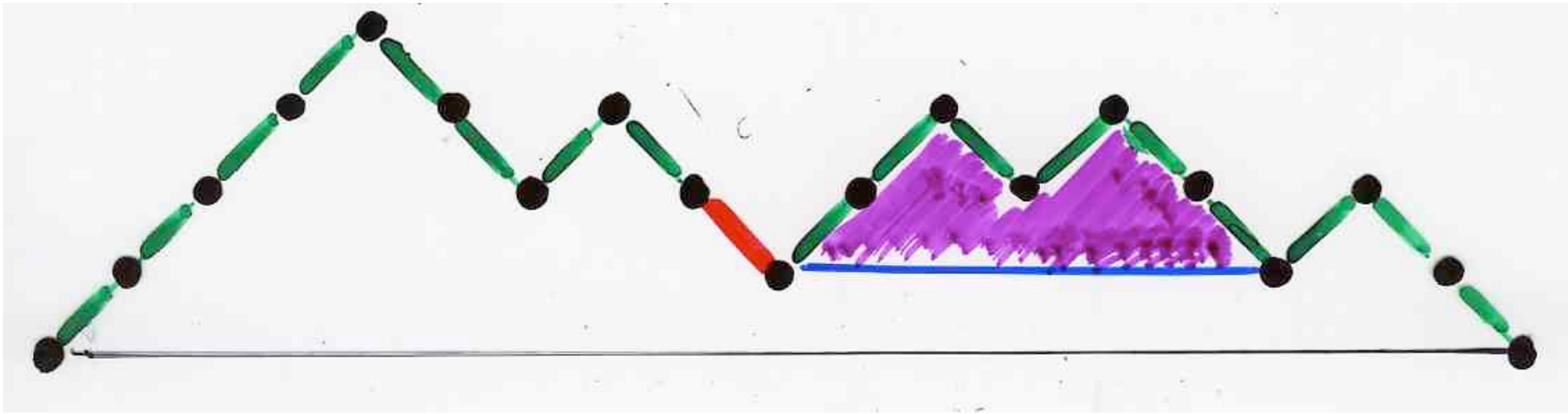
Rotation in a binary tree:
the covering relation in the
Tamari lattice



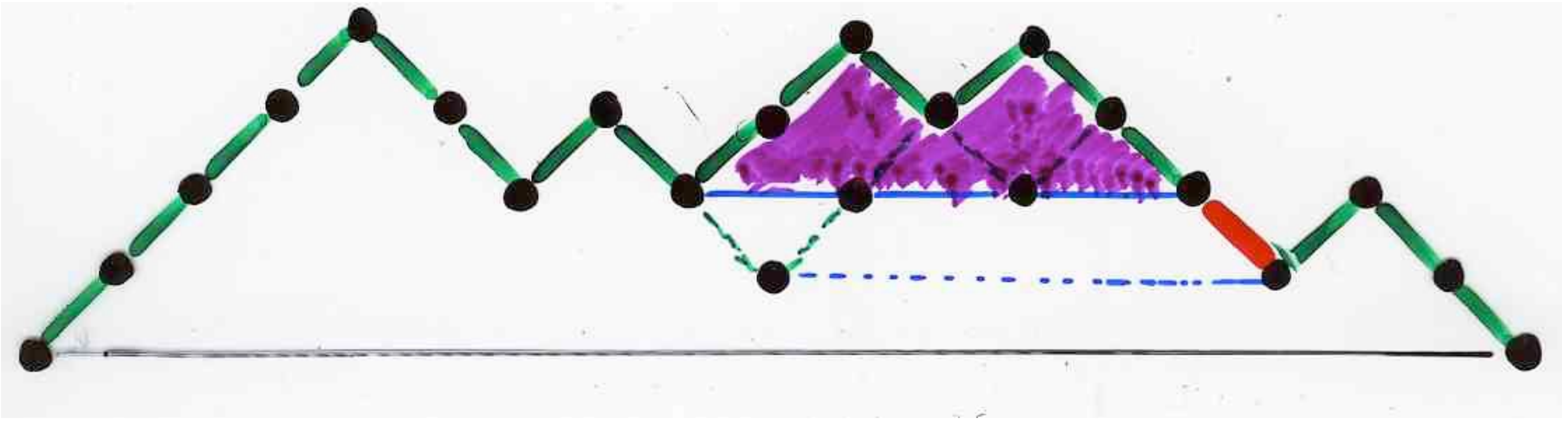
a binary tree B
and its associated complete binary tree $\overline{B}$
(full)

the Tamari lattice
in term
of Dyck paths

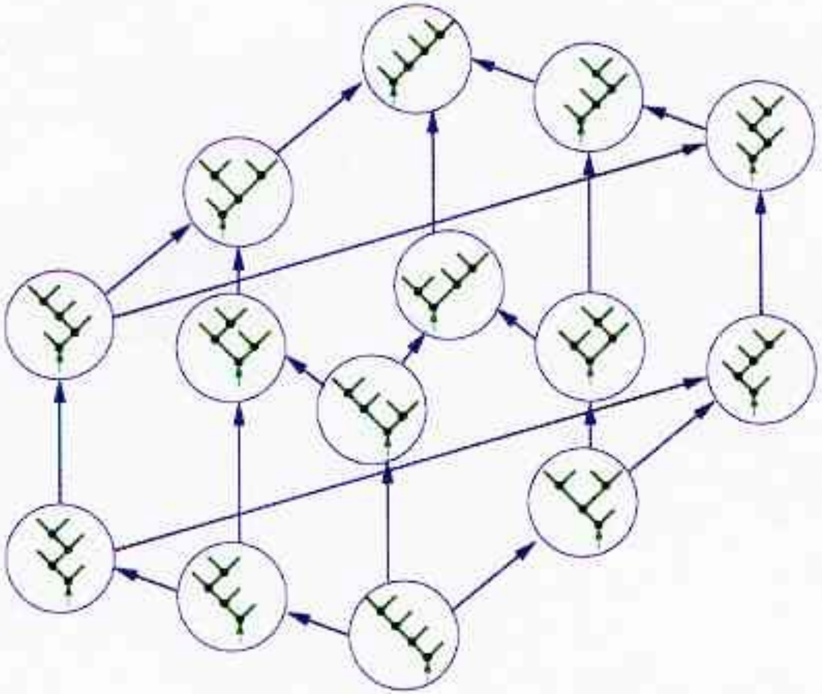


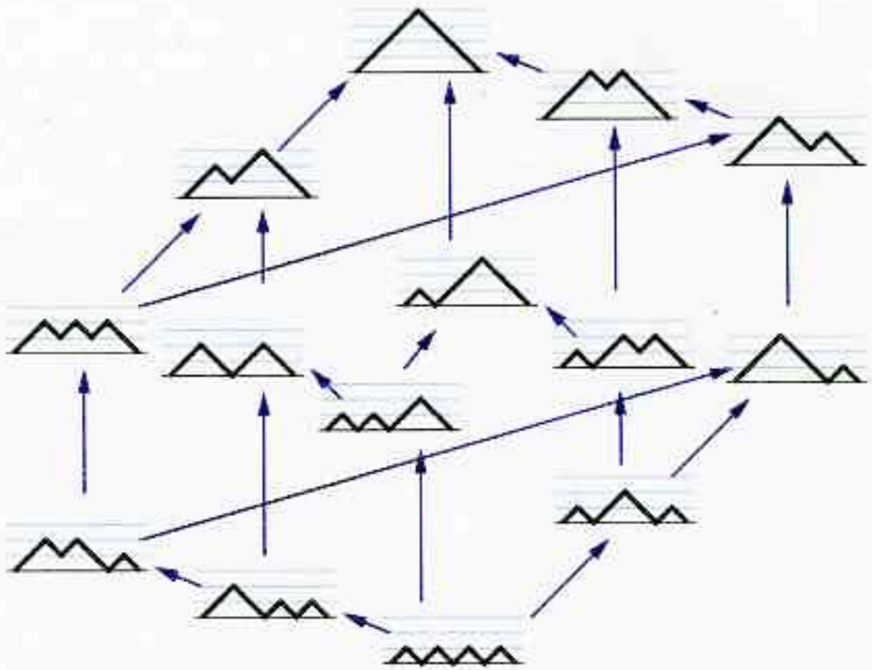


factor Dyck primitif



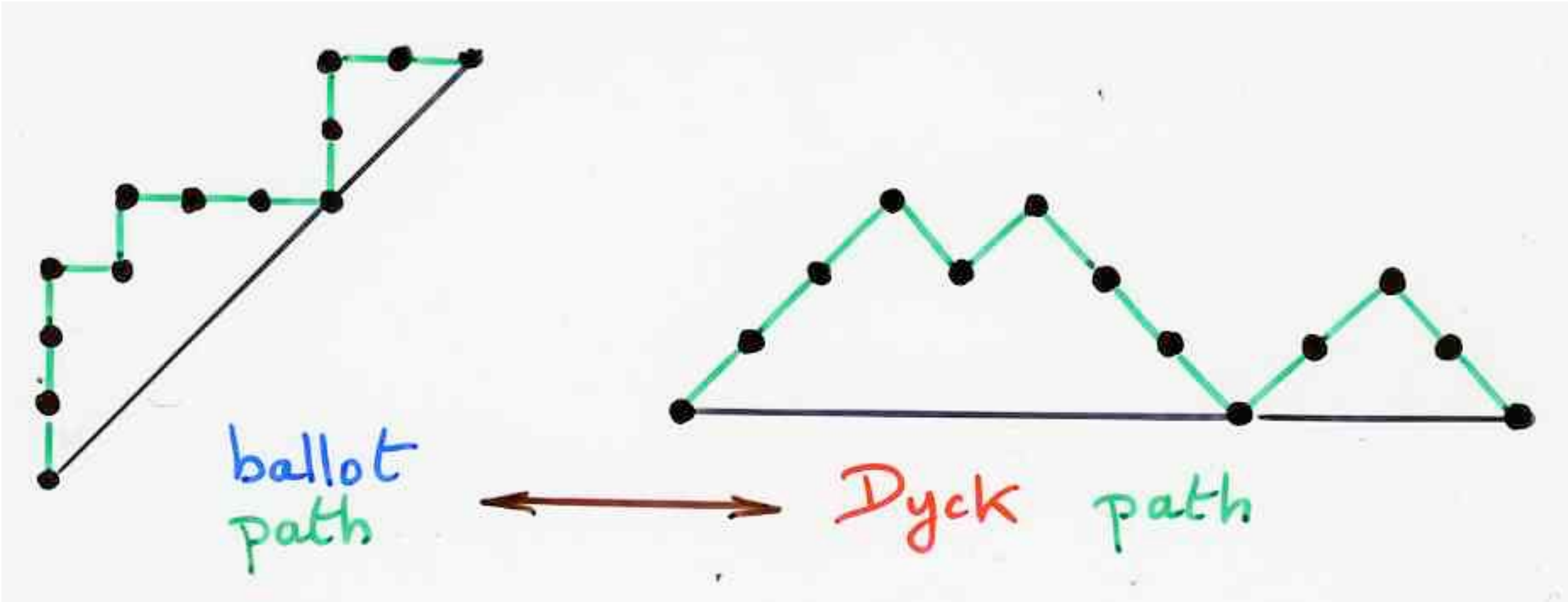
factor Dyck primitif



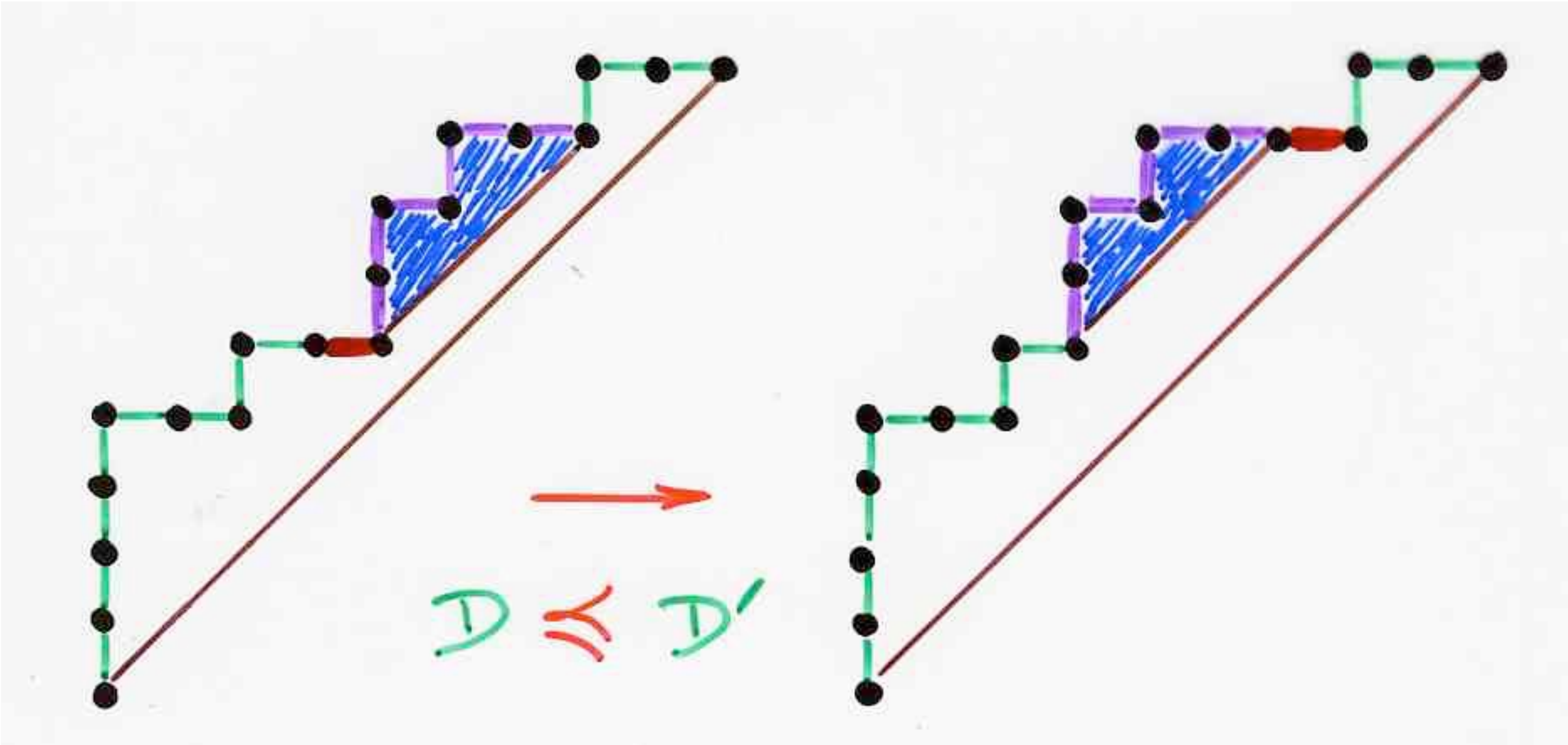


$C_4 = 14$

Catalan



vocabulary: ballot path
Dyck path

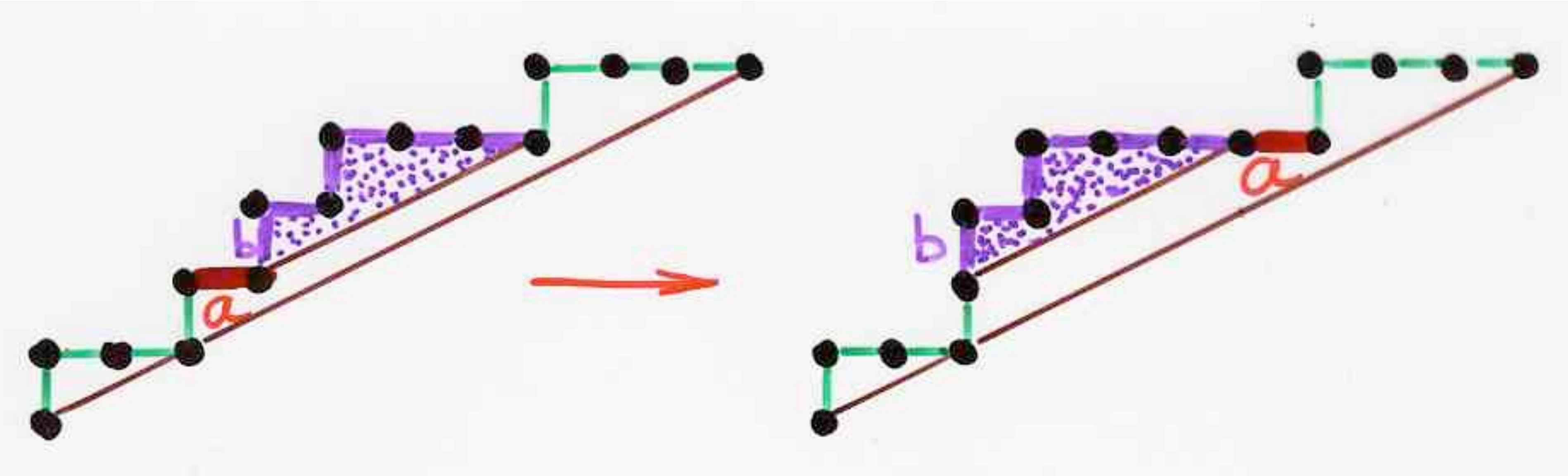


the Tamari covering relation
for ballot (Dyck) path

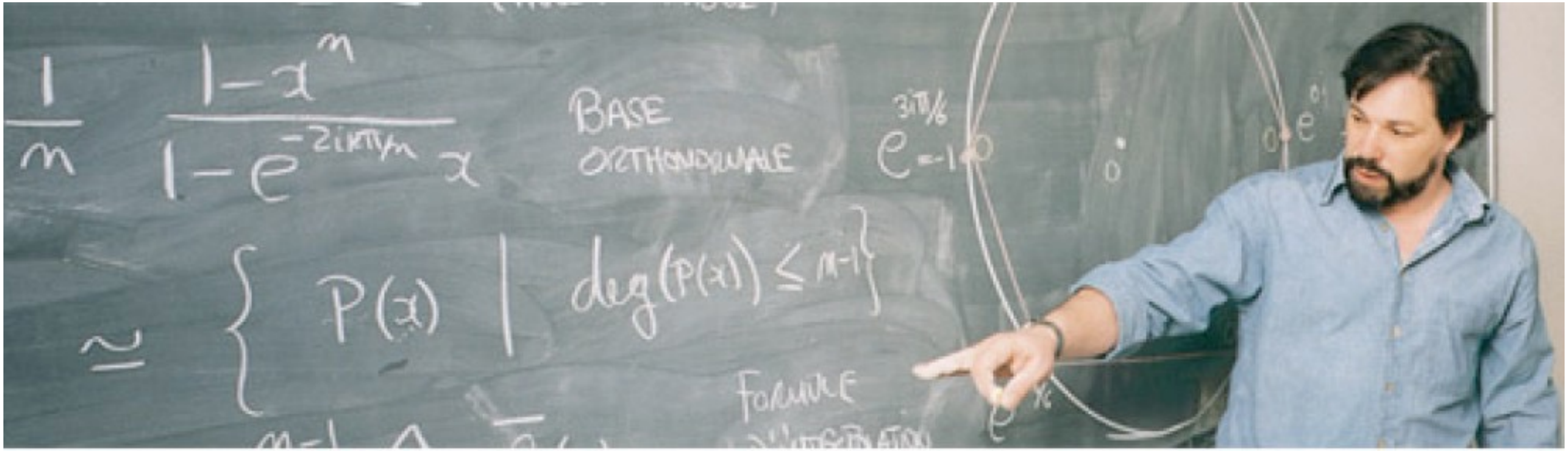
F. Bergeron (2008) introduced the m-Tamari lattice

dimensión $\frac{1}{(m+1)n+1}\binom{(m+1)n+1}{mn}$

m-ballot paths



the covering relation in the
m-Tamari lattice
$(m = 2)$



François Bergeron

diagonal
coinvariant
spaces

Adriano Garsia

$X = (x_{i,j})_{\substack{1\le i\le k\\ 1\le j\le n}}$ matrix of variables

$\sigma \in \mathfrak{S}_n$ symmetric group

$\sigma\cdot(X) = (x_{i,\sigma(j)})_{\substack{1\le i\le k\\ 1\le j\le n}}$ action on $\mathbb{C}[X]$

diagonal coinvariant spaces

$DR_{k,n} = \mathbb{C}[X]/\mathcal{J}$

higher diagonal coinvariant spaces

$DR^{m}_{k,n} = \varepsilon^{m-1} \otimes A^{m-1}/\mathcal{J}A^{m-1}$ alternants

$DR^{m\,\varepsilon}_{k,n}$ subspace of alternants

$k=1$ classical

$k=2$ Garsia, Haiman

→ Macdonald polynomials

$DR^{m}_{2,n}$ ε $DR^{m}_{2,n}$

dimension $\frac{1}{(m+1)n+1}\binom{(m+1)n+1}{mn}$ $(mn+1)^{n-1}$

m-ballot paths

m-parking functions

$DR^{m}_{2,n}$ 20 years of studies

m-shuffle conjecture

Frobenius series (q,t) sum on m-parking

(area dinv

$k=3$ Haiman (conjecture) 1990

$DR^{\varepsilon}_{3,n}$ $DR_{3,n}$

dimension $\dfrac{2}{n(n+1)}\dbinom{4n+1}{n-1}$ $2^n (n+1)^{n-2}$

↑

Chapoton (2006)

number of interval Tamari$_n$

F. Bergeron (2008) introduced the m-Tamari lattice

conjecture $\dfrac{m+1}{n(mn+1)}\dbinom{(m+1)^2 n+m}{n-1}$ $(m+1)^n (mn+1)^{n-2}$

nb of intervals nb of labelled intervals

M. Bousquet-Mélou, E. Fusy, L.-F. Préville-Ratelle (2011)

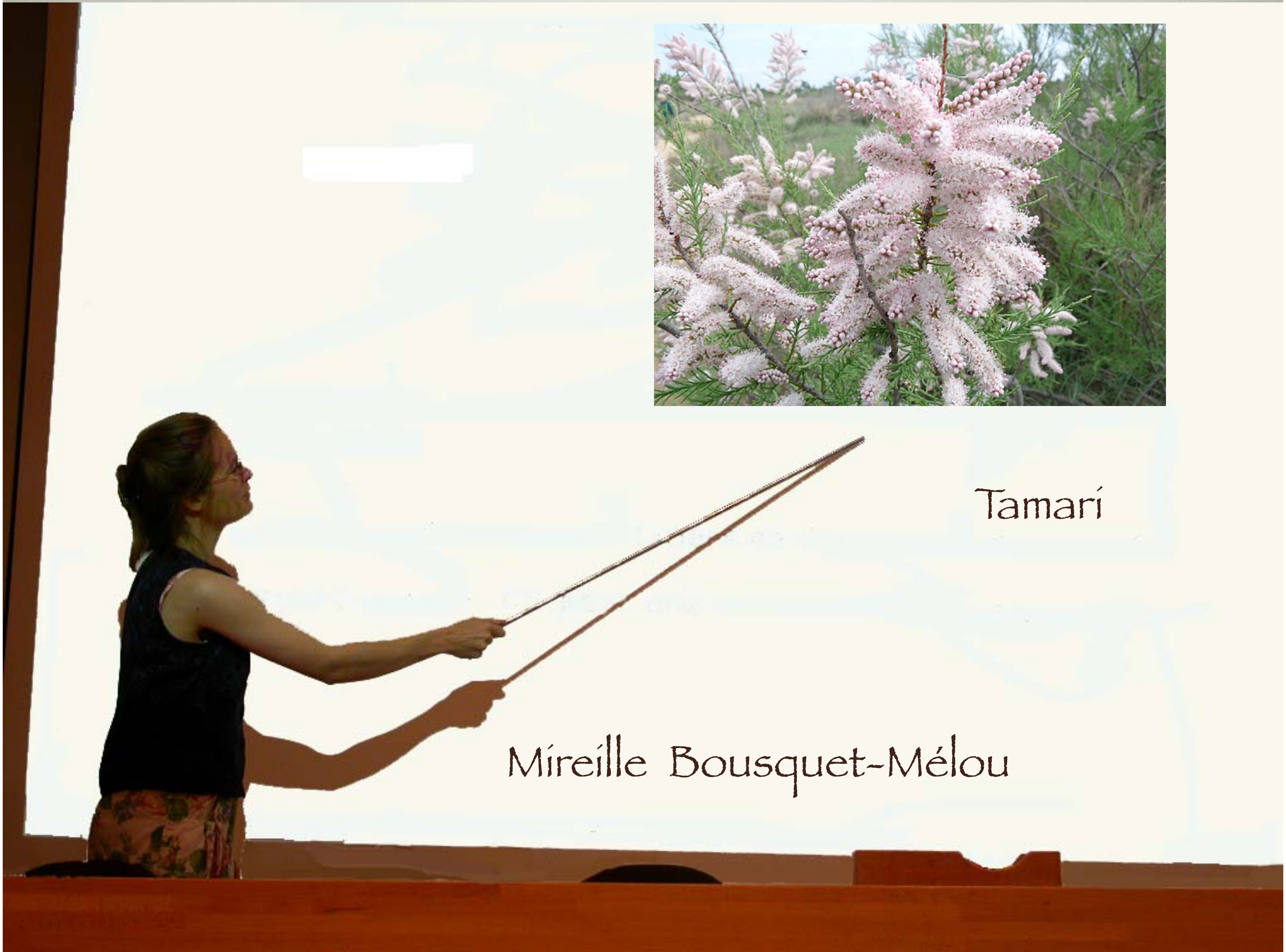
nb of intervals of m-Tamari lattices

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n + m}{n-1}$$

F. Bergeron

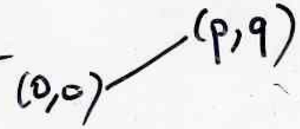
M. Bousquet-Mélou, G. Chapuy, L.-F. Préville-Ratelle (2011)

nb of labelled intervals $(m+1)^n (mn+1)^{n-2}$


Tamari
Mireille Bousquet-Mélou

Hikita, Armstrong

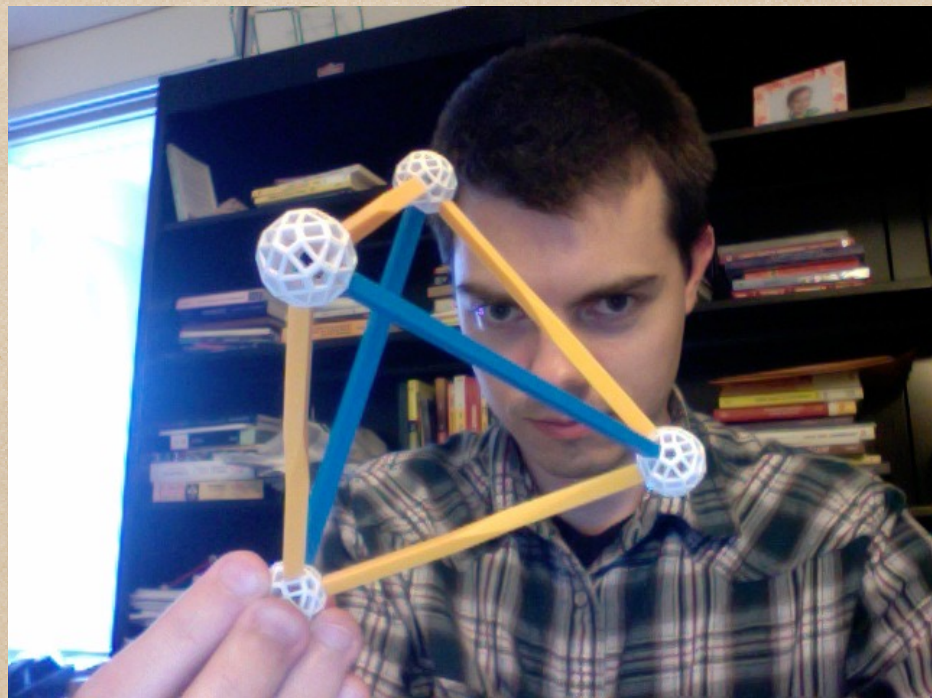
paths and parking functions

above the line 

p, q relatively prime integers

Armstrong, Garsia, Haglund, Haiman, Hicks
Lee, Li, Loehr, Morse, Remmel, Rhoades,
Stout, Xin, Warrington, Zabrocki, ...
+ ......

Rational Catalan Combinatorics



Rational Catalan Combinatorics

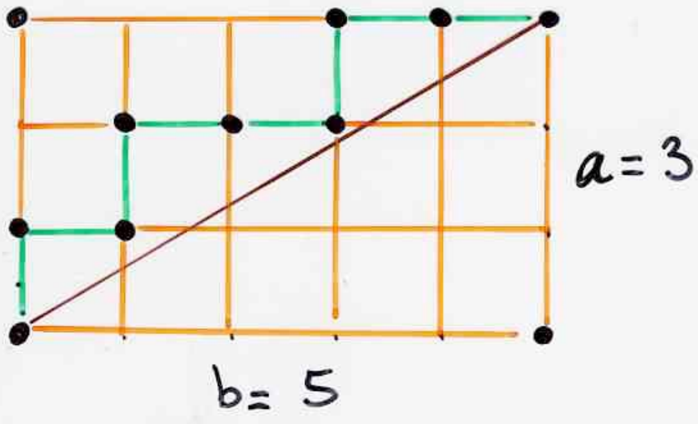
D. Armstrong

$$\mathrm{Cat}(a,b) = \frac{1}{a+b}\binom{a+b}{a,\,b}$$

number of (a,b)- ballot paths $= \mathrm{Cat}(a,b)$

Grossman (1950)
Bizley (1954)

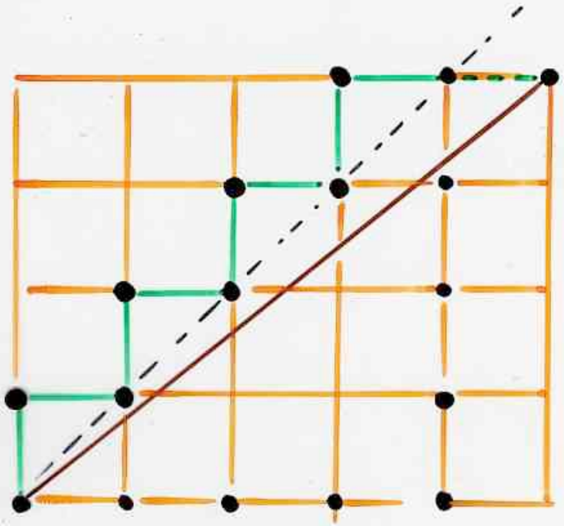
rational ballot (Dyck) paths



$(a,b) = (3,5)$

$(a, b) = (n, n+1) \longrightarrow C_n$ Catalan nb

$(a, b) = (n, mn+1) \longrightarrow \frac{1}{(m+1)n+1} \binom{(m+1)n+1}{n}$ Fuss-Catalan nb

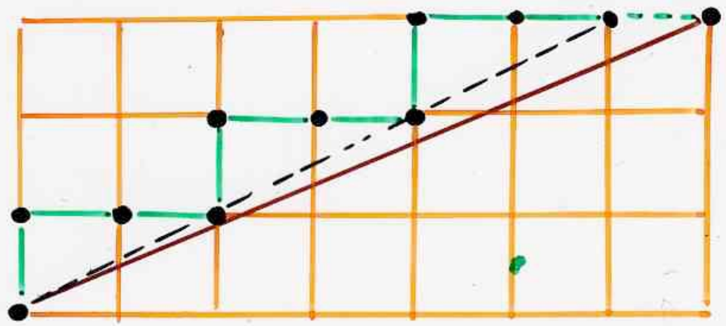


$n = 4$

$n+1 = 5$

$(a, b) = (4, 5)$

Catalan

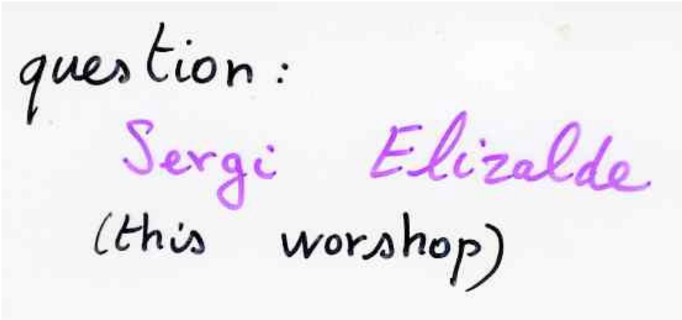


$n = 3$

$m = 2$ $\quad mn+1 = 7$

$(a, b) = (3, 7)$

Fuss-Catalan

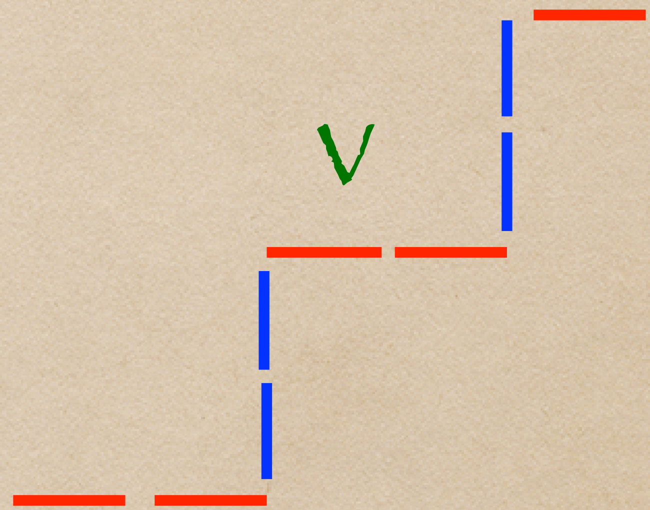


Oberwolfach workshop
on Combinatorics
March 2014

define an (a,b)- Tamari lattice ?

extension: Tamari T_v

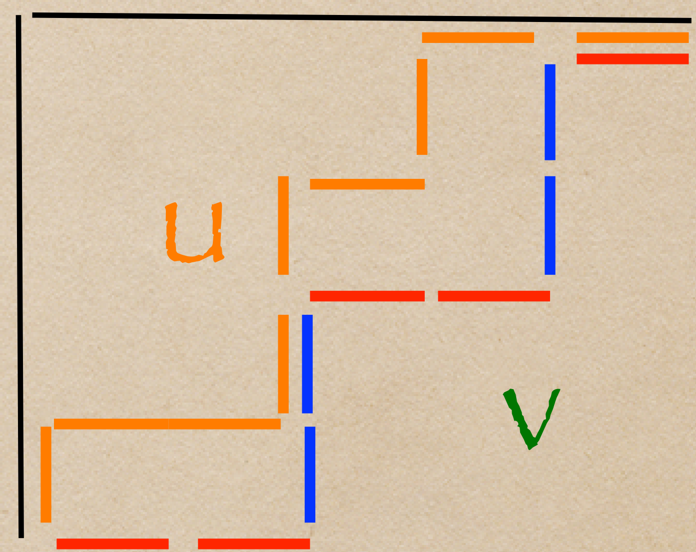
transcental
Catalan
combinatorics ?

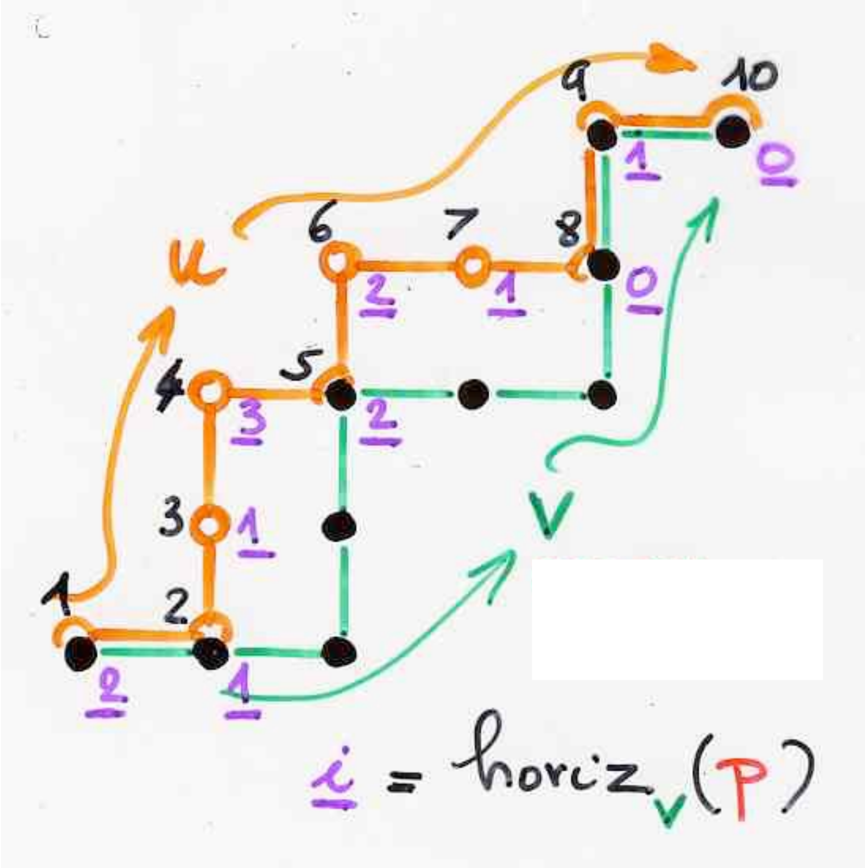


number of pairs (u,v)

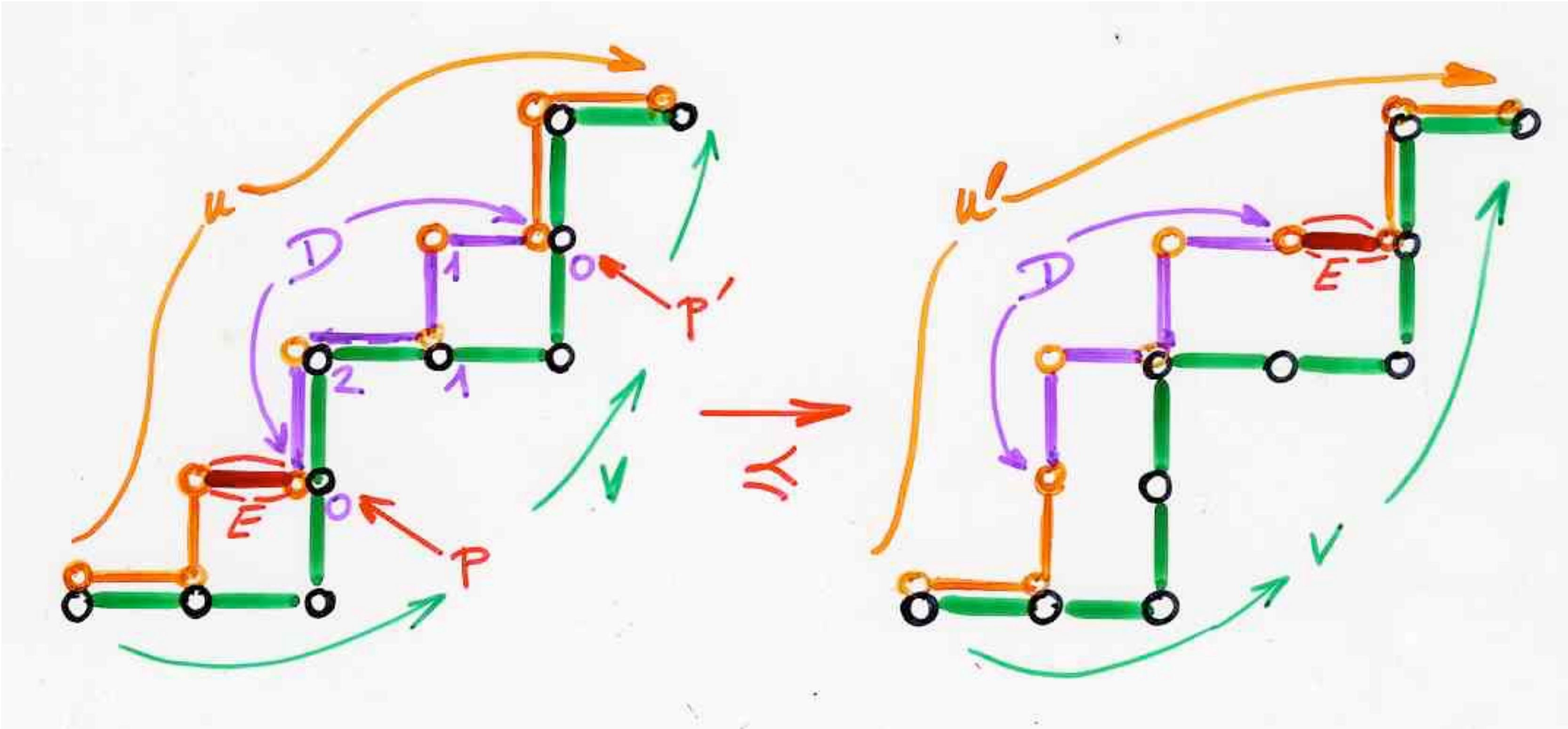
Macmahon determinant

Kreweras determinant



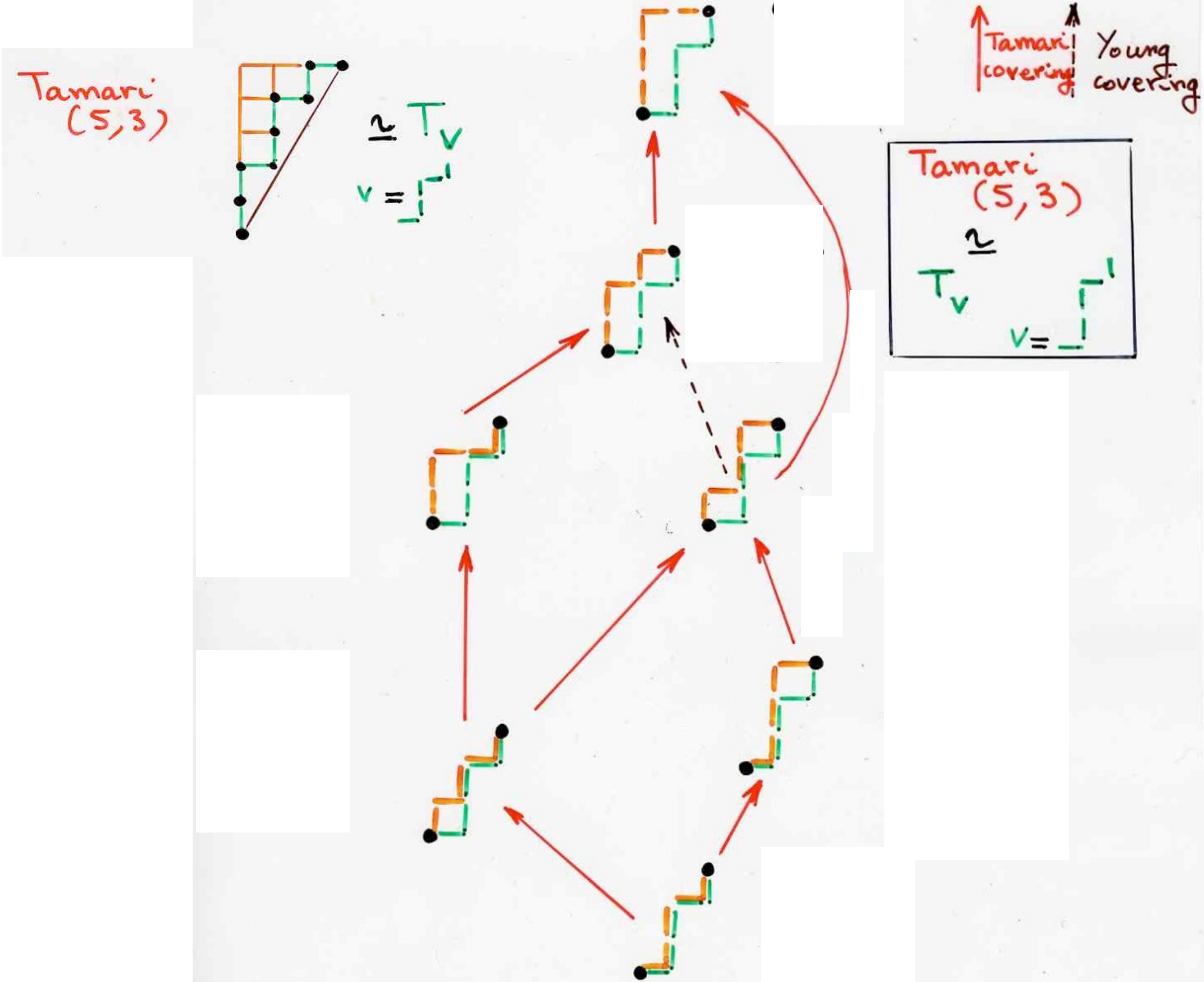


a pair (u, v) of paths
with the "horizontal distance"
$\mathrm{horiz}_v(p)$

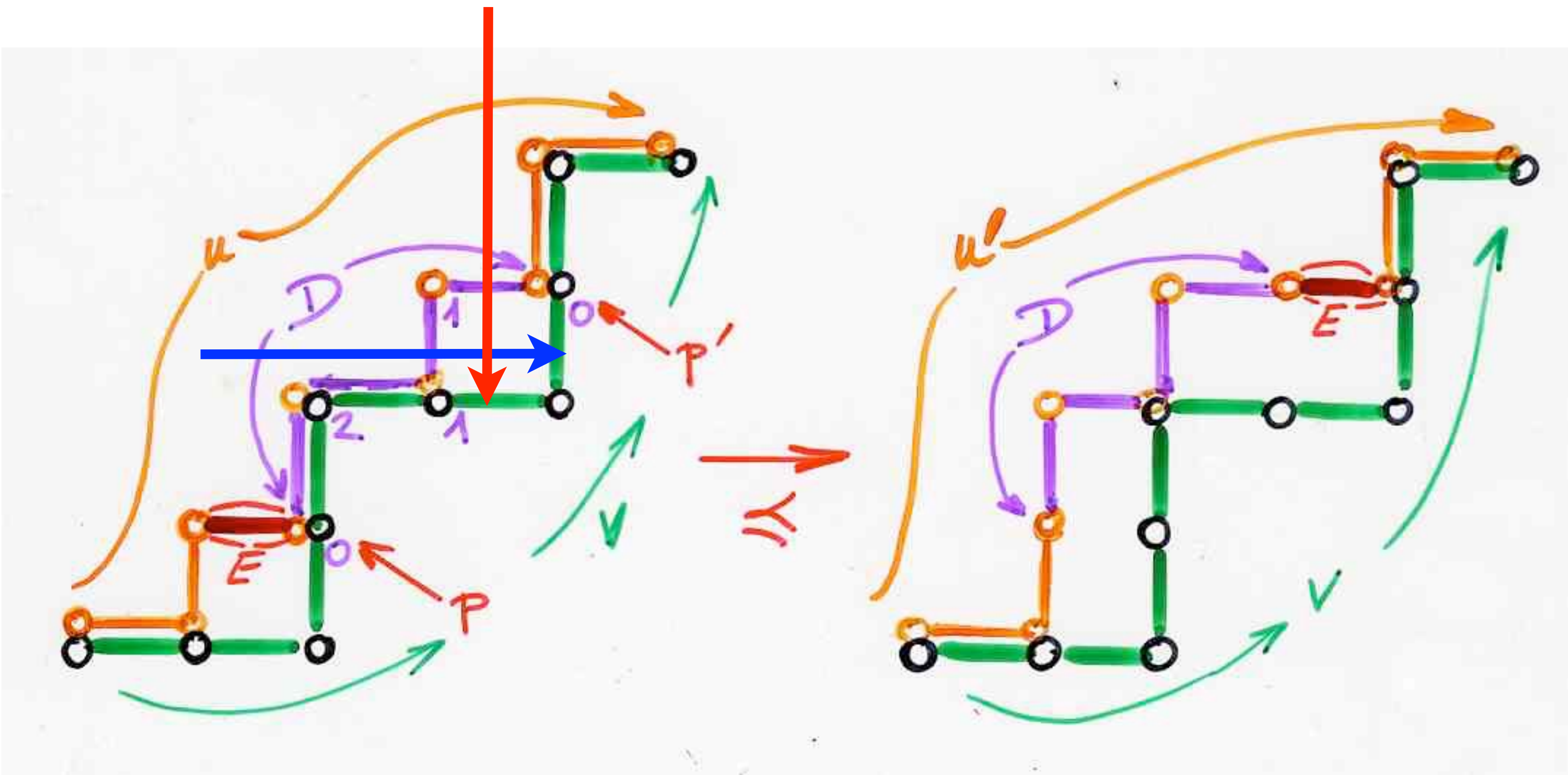


the covering relation in the poset T_v

Thm 1. For any path v, T_v is a lattice



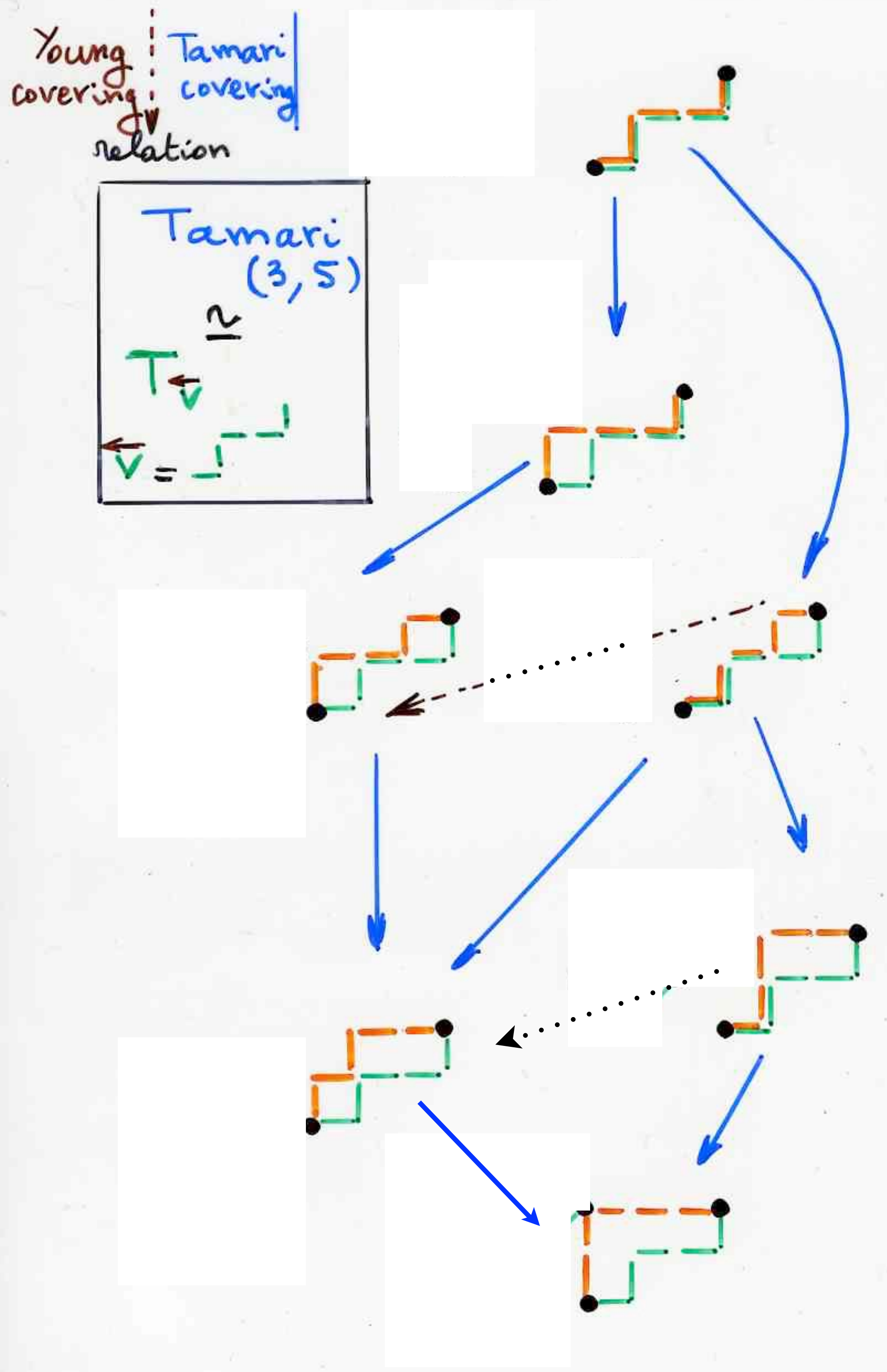
Tamari (5,3)
≃ T_v
v =
Tamari covering
Young covering
Tamari (5,3)
≃
T_v
v =

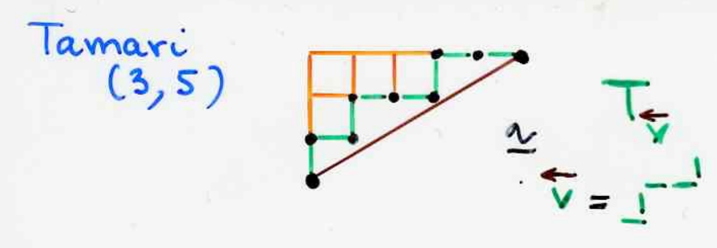


«row covering relation»

«column covering relation»

mirror image, exchange N and E


Young covering relation
Tamari covering
Tamari (3, 5)
≃
T
v =


Tamari (3, 5)
≃
T
v =

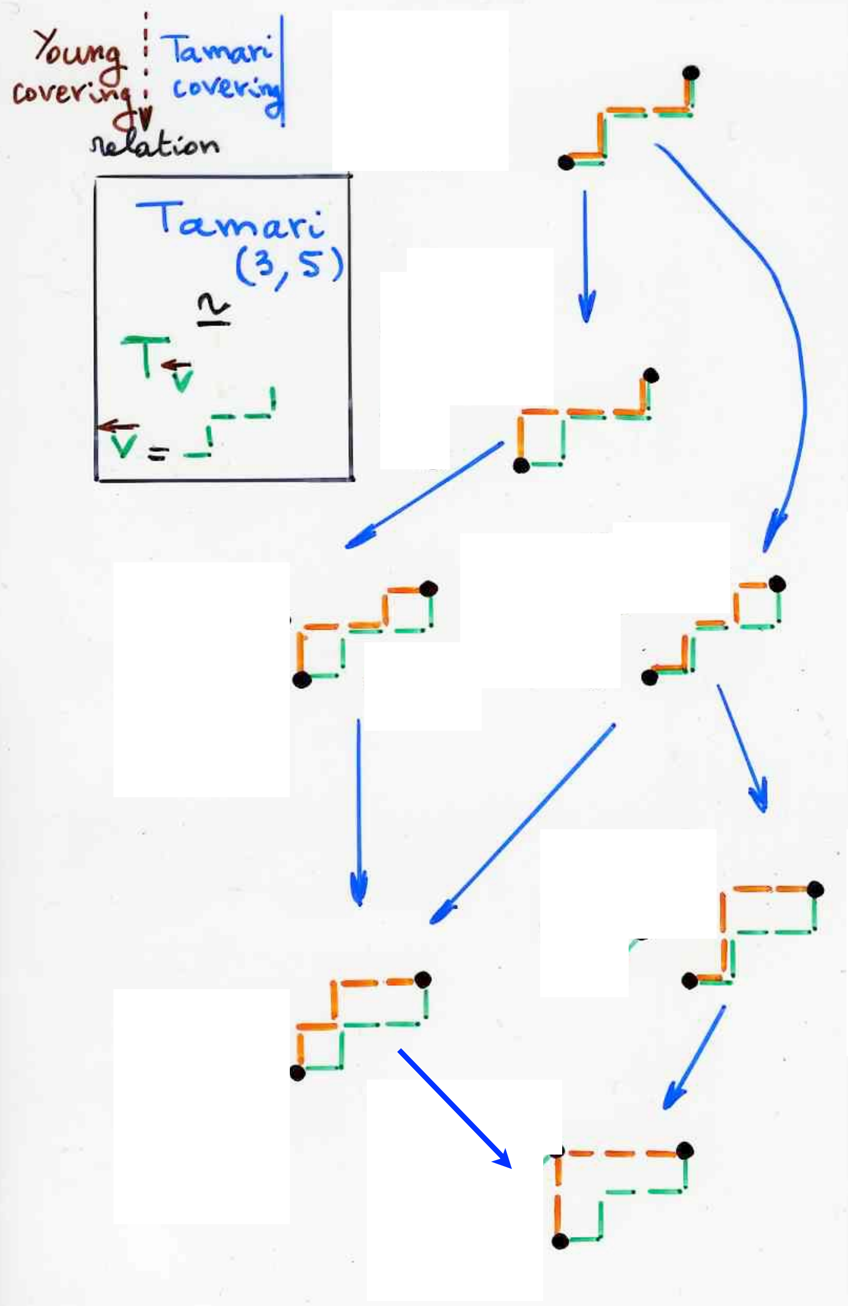
Thm 1. For any path v

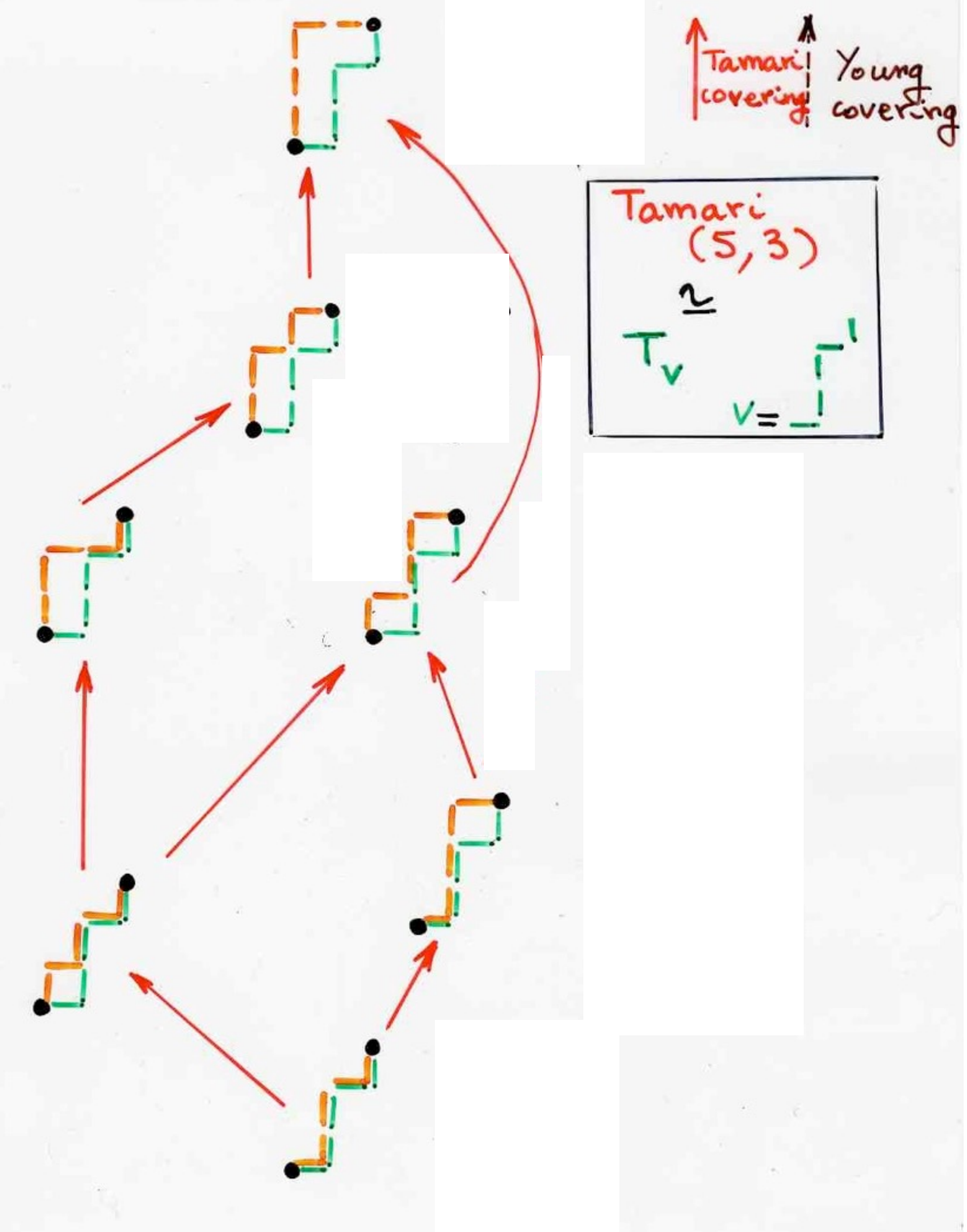
T_v is a *lattice*

Thm 2. The *lattice* T_v

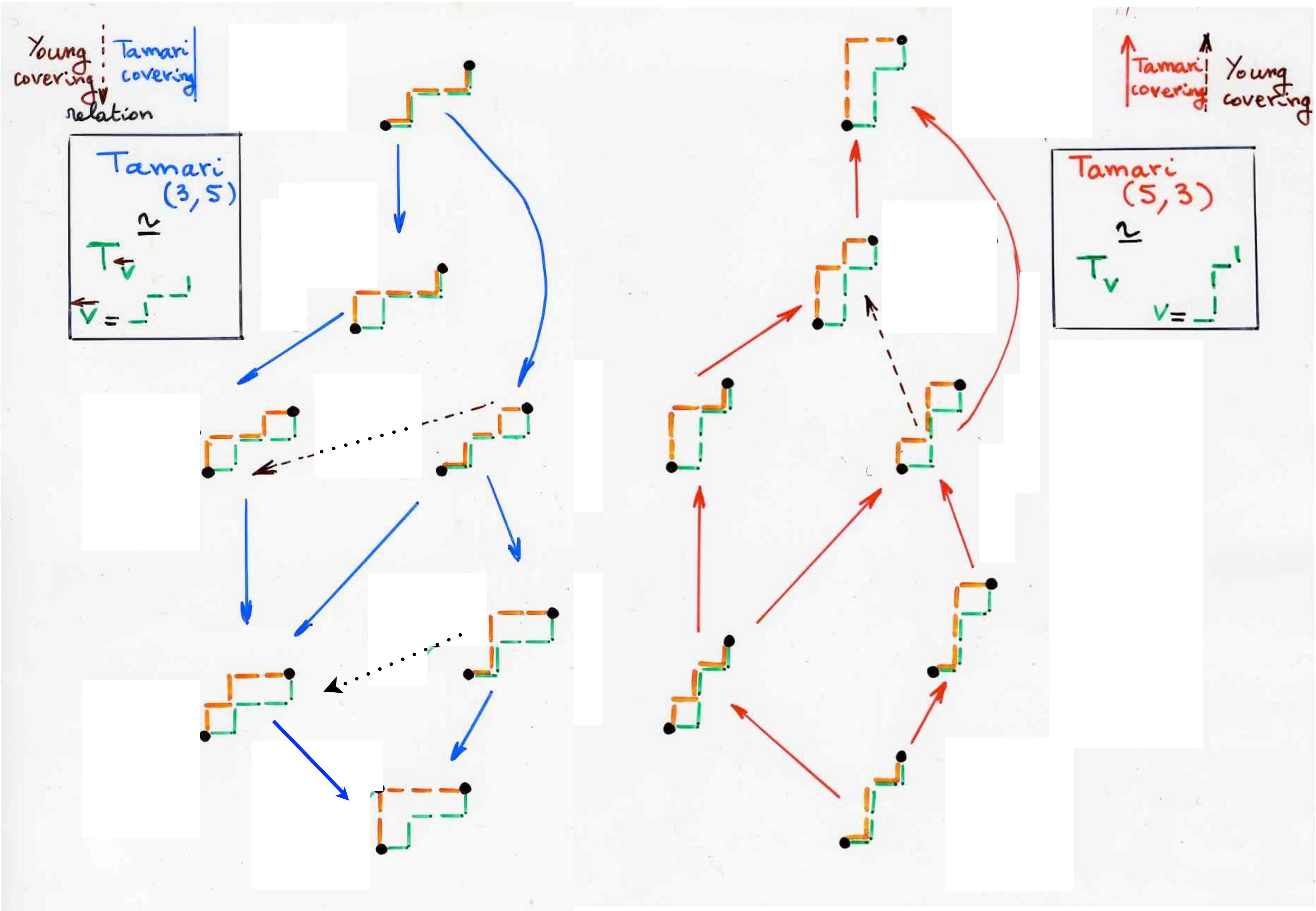
is isomorphic to the dual of $T_{\overleftarrow{v}}$

Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$

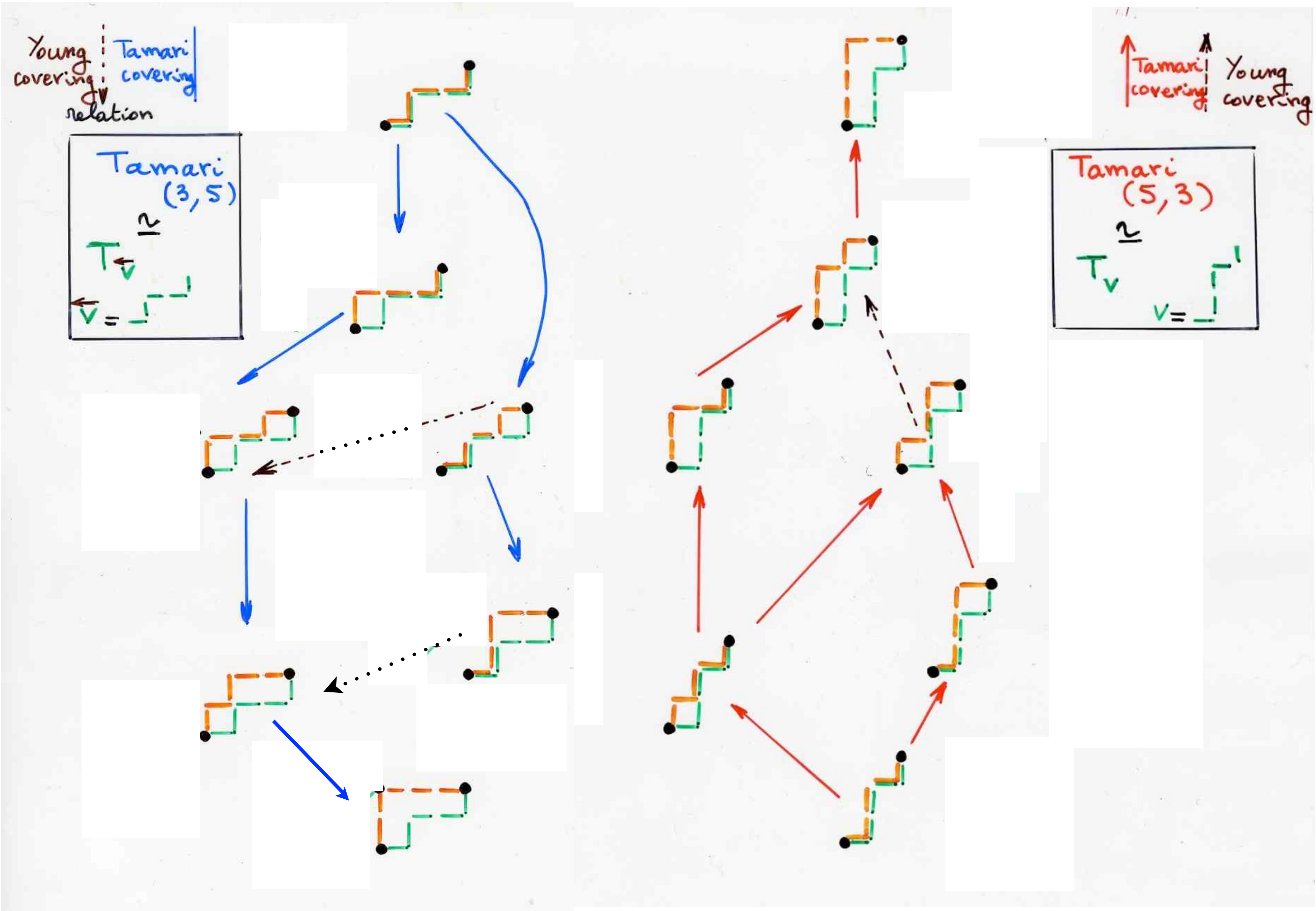




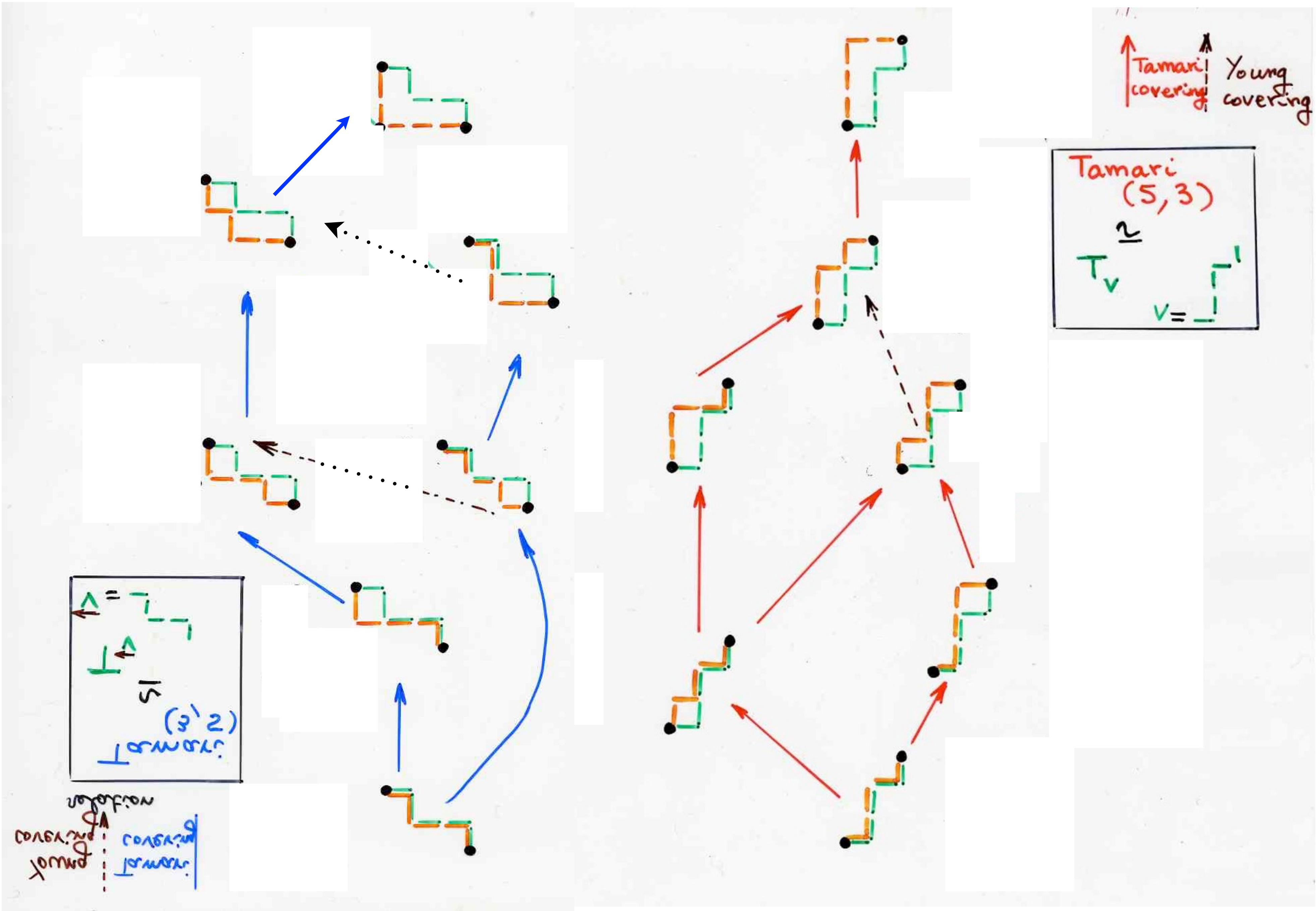
Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$



Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$

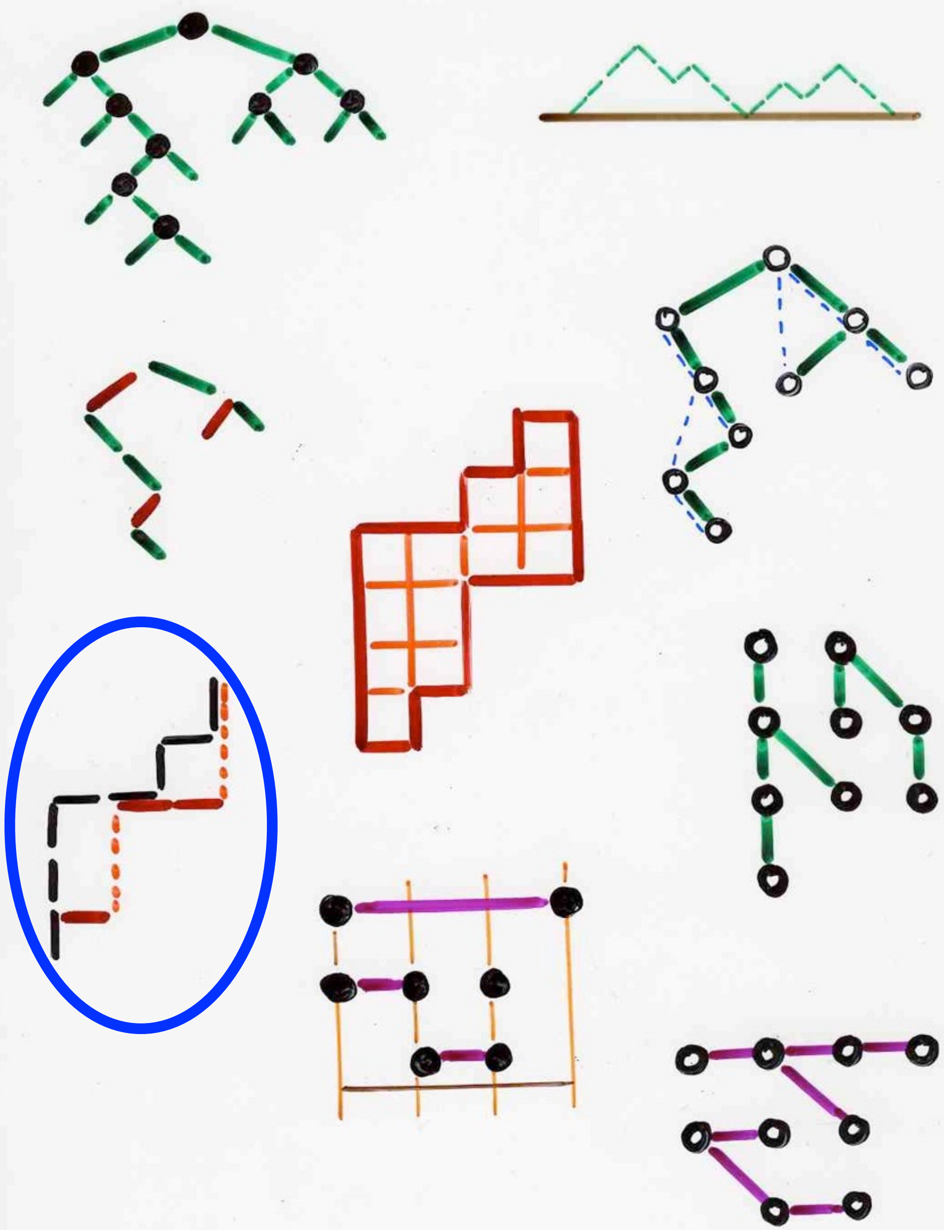


Y_v ; Y_v^{S}

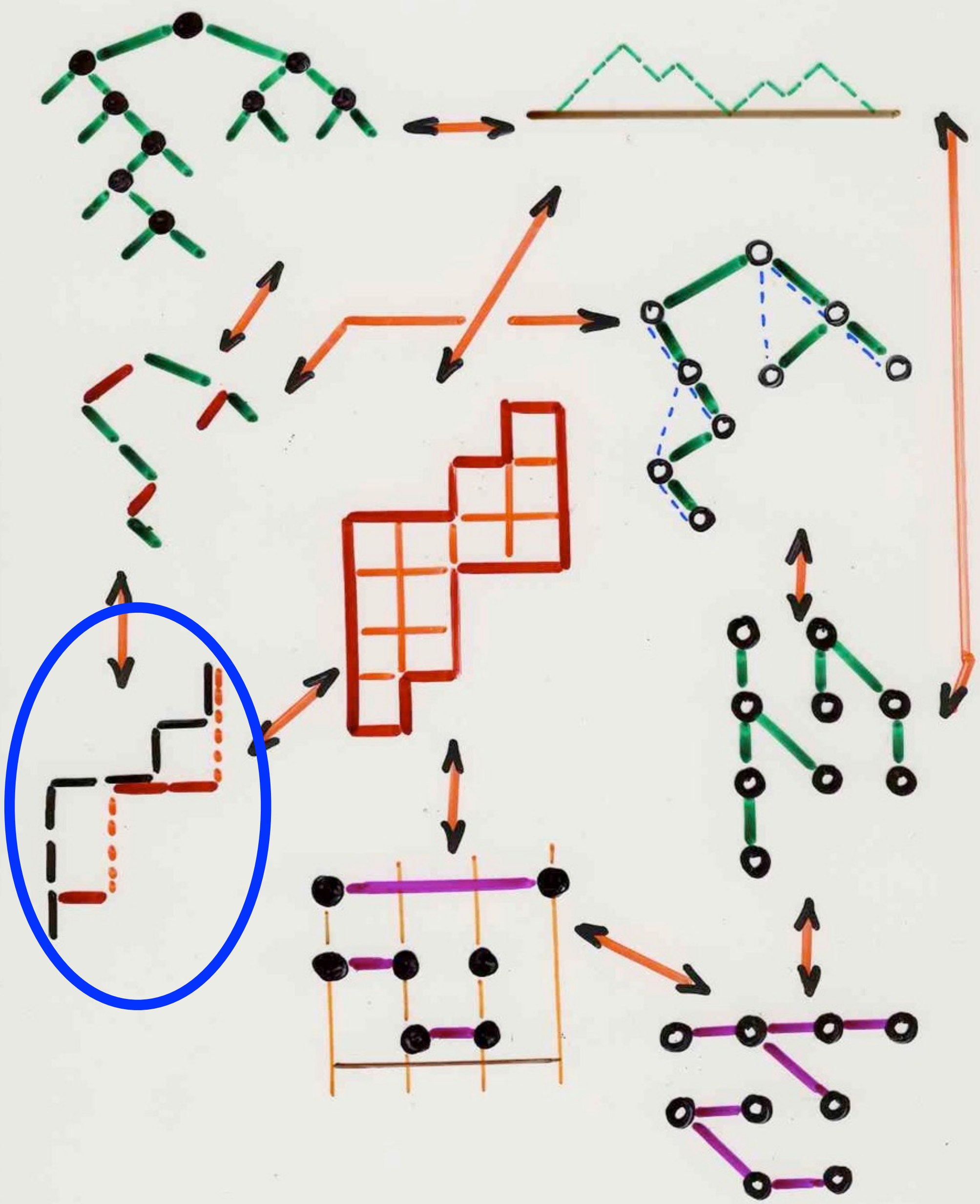


A bijection

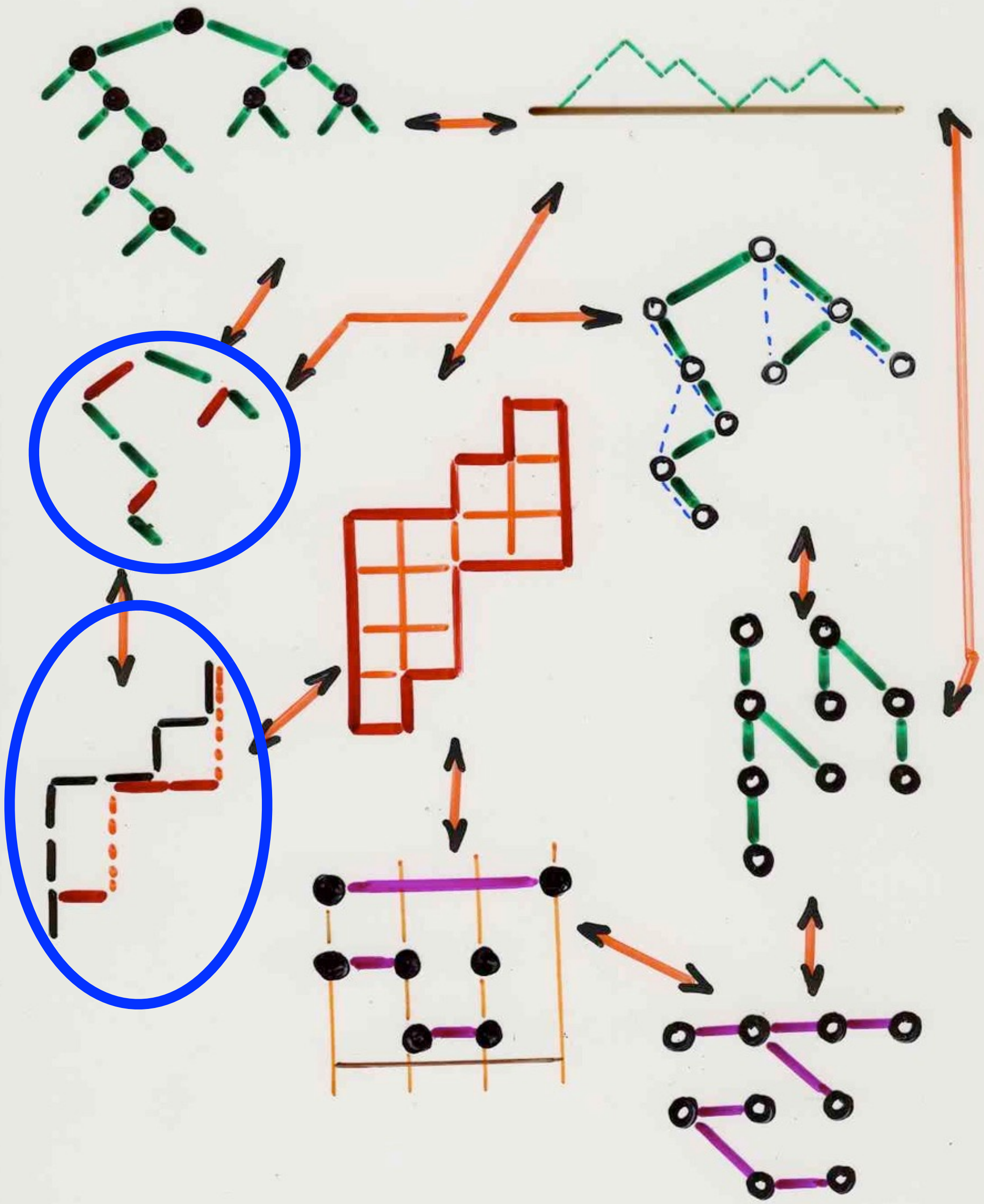
the Catalan garden



the Catalan garden



the Catalan garden



A bijection

binary tree B ⟶ pair of paths (u,v)

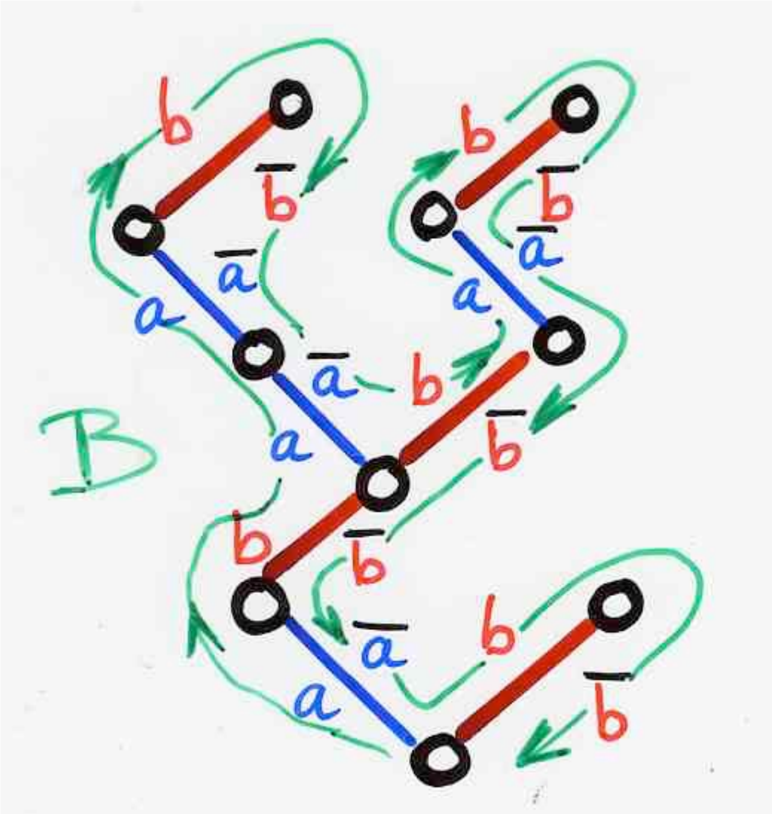
the words $\begin{cases} w(B) \\ u(B) \\ v(B) \end{cases}$

$$w(B) = a\,b\,a\,a\,b\,\bar{b}\,\bar{a}\,\bar{a}\,b\,a\,b\,\bar{b}\,\bar{a}\,\bar{b}\,\bar{b}\,\bar{a}\,b\,\bar{b}$$

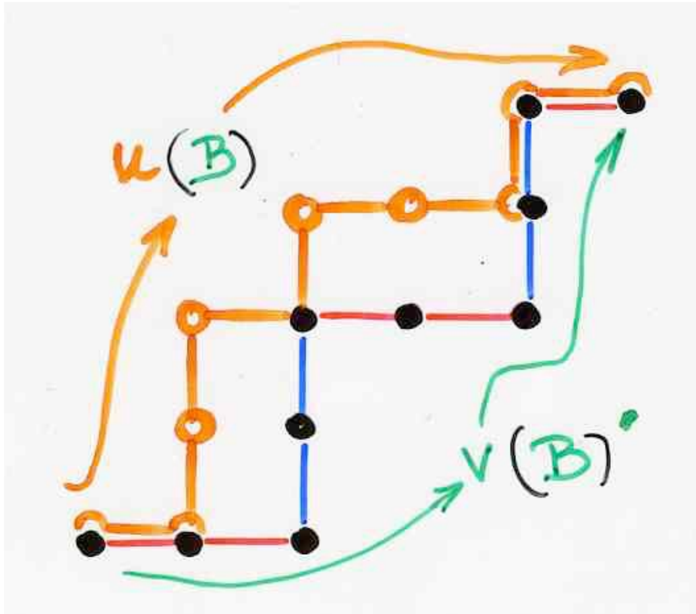
$$u(B) = \bar{b}\,\bar{a}\,\bar{a}\quad \bar{b}\,\bar{a}\,\bar{b}\,\bar{b}\,\bar{a}\quad \bar{b}$$

$$v(B) = b\quad b\quad \bar{a}\,\bar{a}\,b\quad b\quad \bar{a}\quad \bar{a}\,b$$

$$\bar{a} \rightarrow N \qquad \left.\begin{matrix} b \\ \bar{b} \end{matrix}\right\} \rightarrow E$$



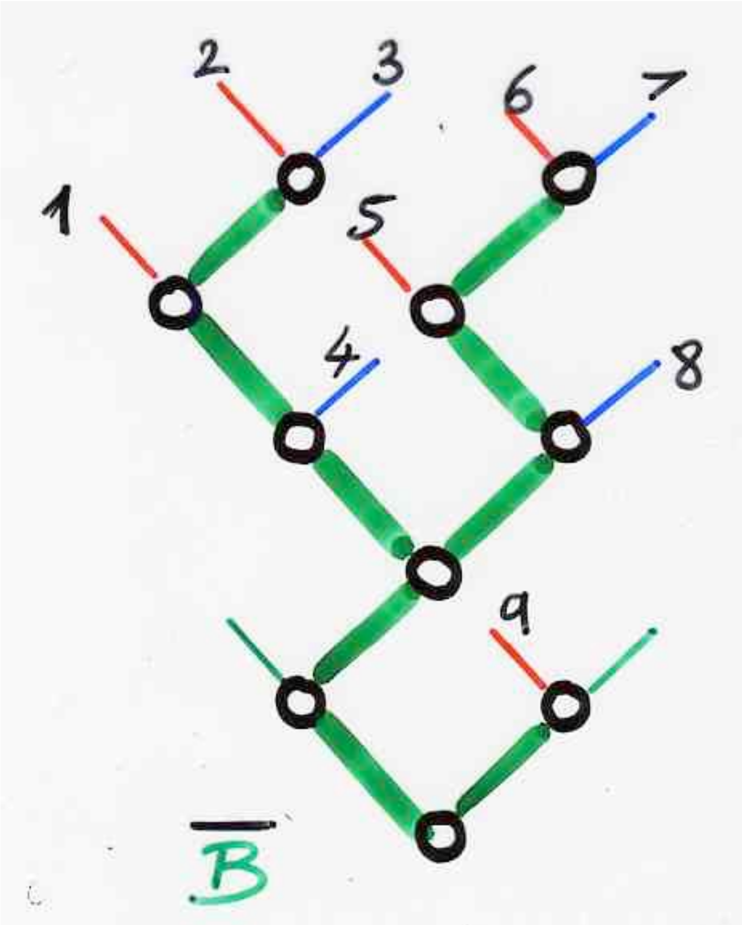
walk around a binary tree B

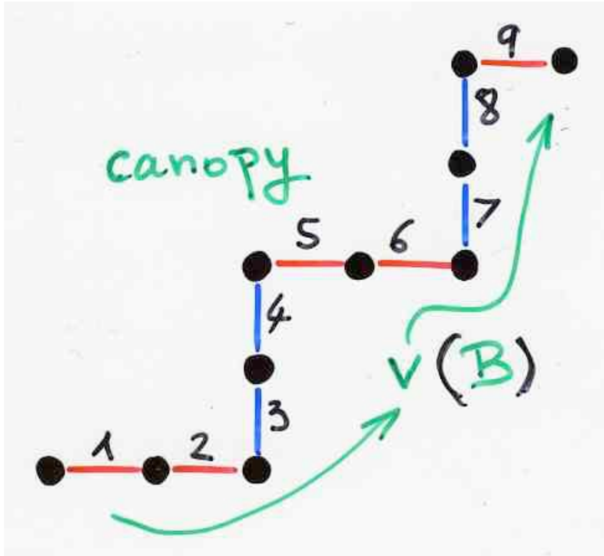


the pair (u, v) of paths associated to a binary tree B

definition:

v (B) Canopy





second definition of the canopy

J.-L.Loday, M. Ronco (1998, 2012)

algebraic structures

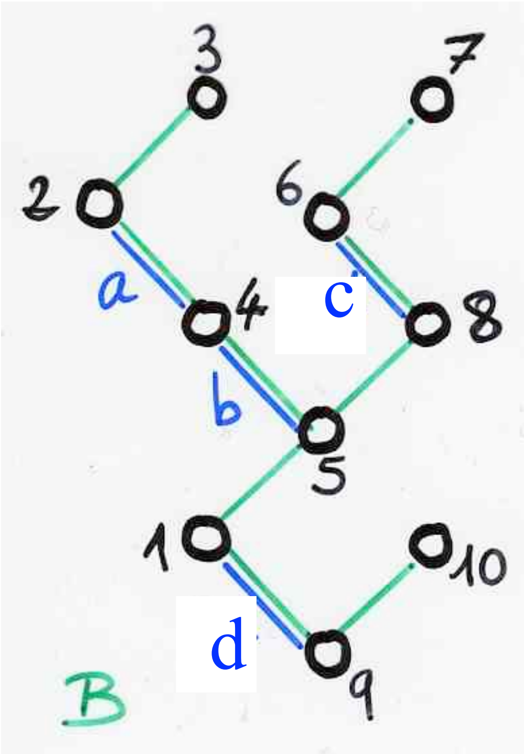
Hopf algebra

dim 2^{n-1} C_n $n!$

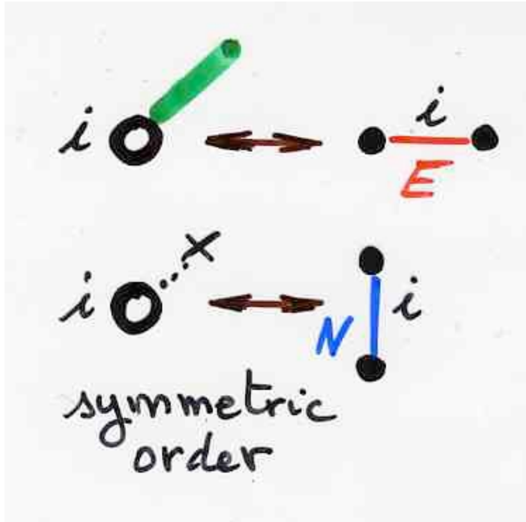
Catalan

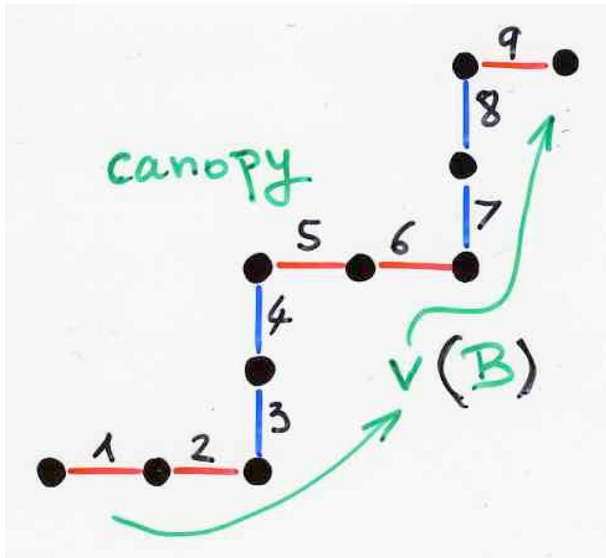
Boolean lattice inclusion ⟷ Tamari order ⟷ weak Bruhat order

J.-L.Loday, M. Ronco (1998, 2012)



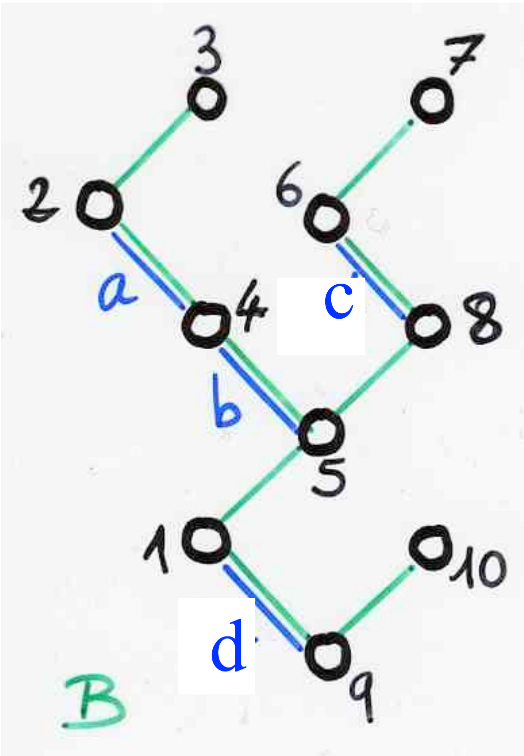
symmetric order

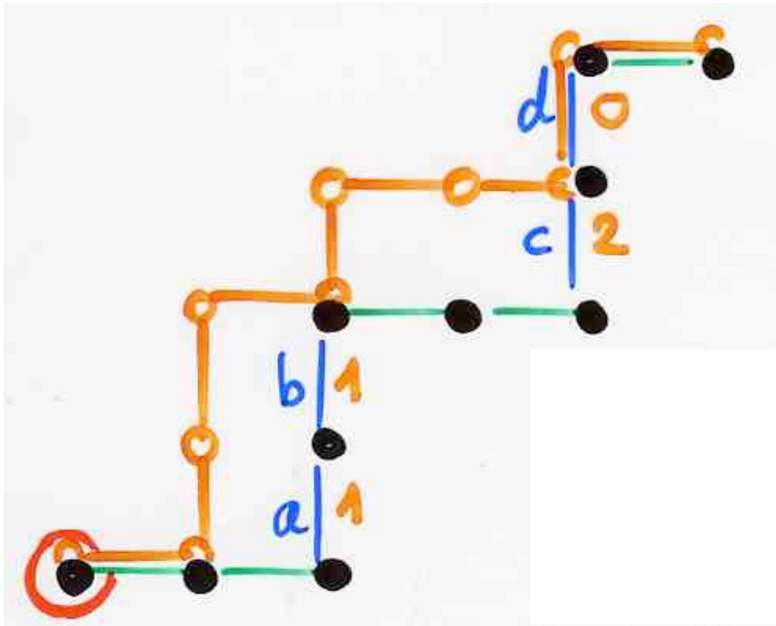




third definition of the canopy

a lemma

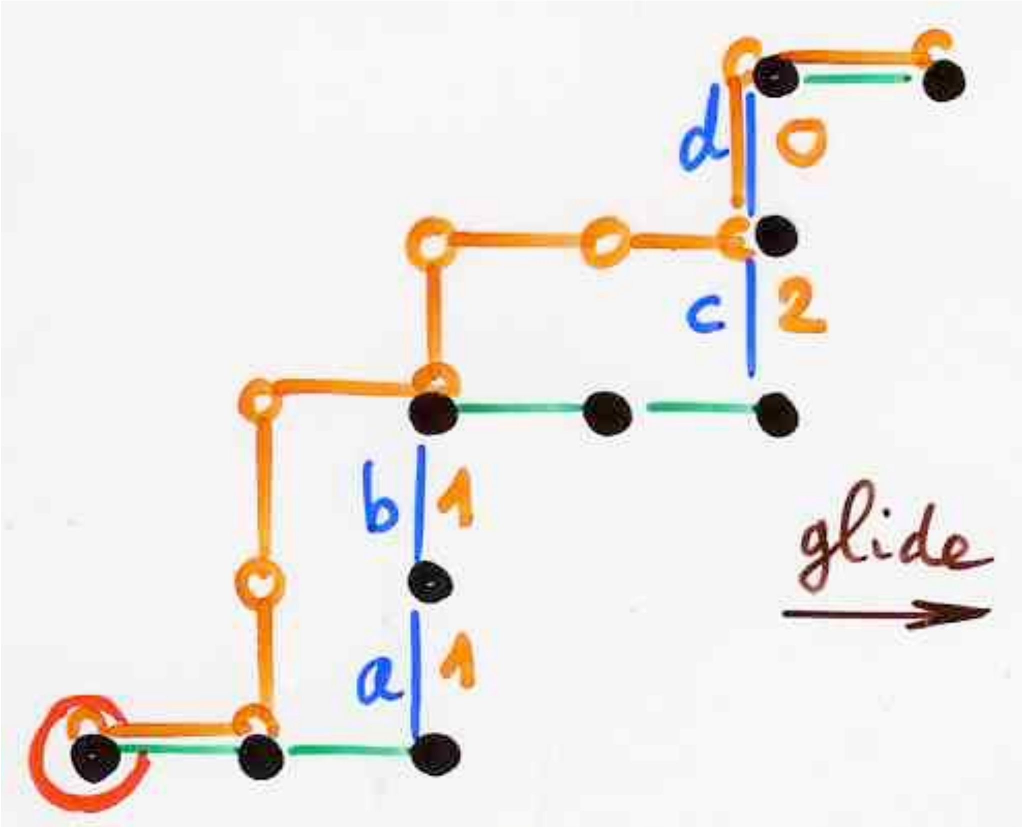




reverse bijection

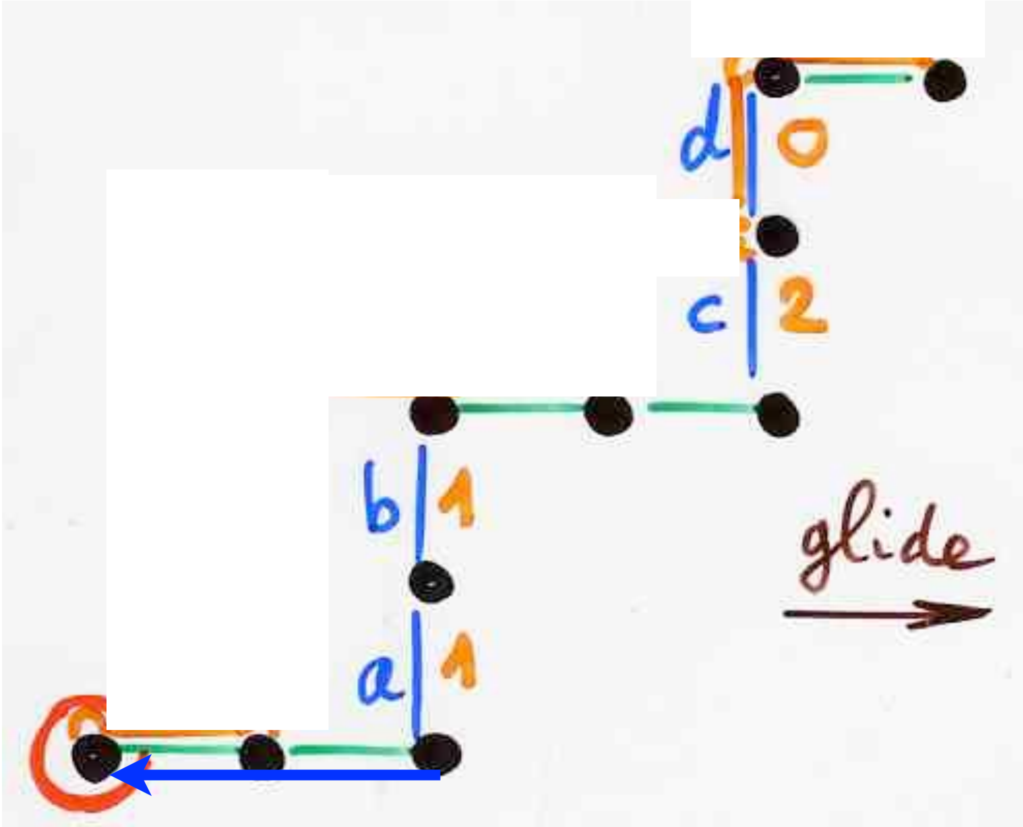
binary tree B $\longleftarrow$ pair of paths (u,v)

the «push-gliding» algorithm



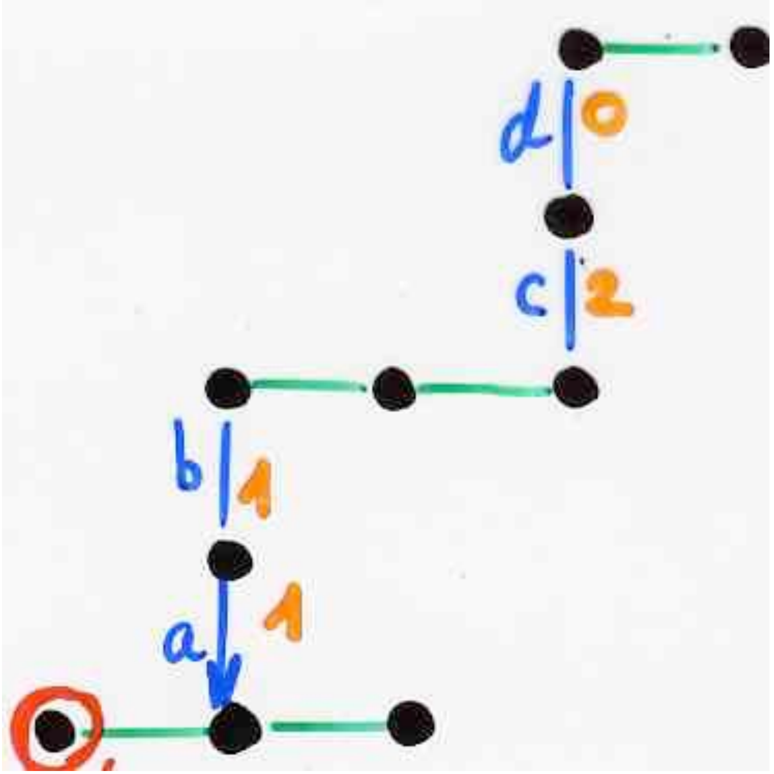
reverse bijection

the "push-gliding" algorithm

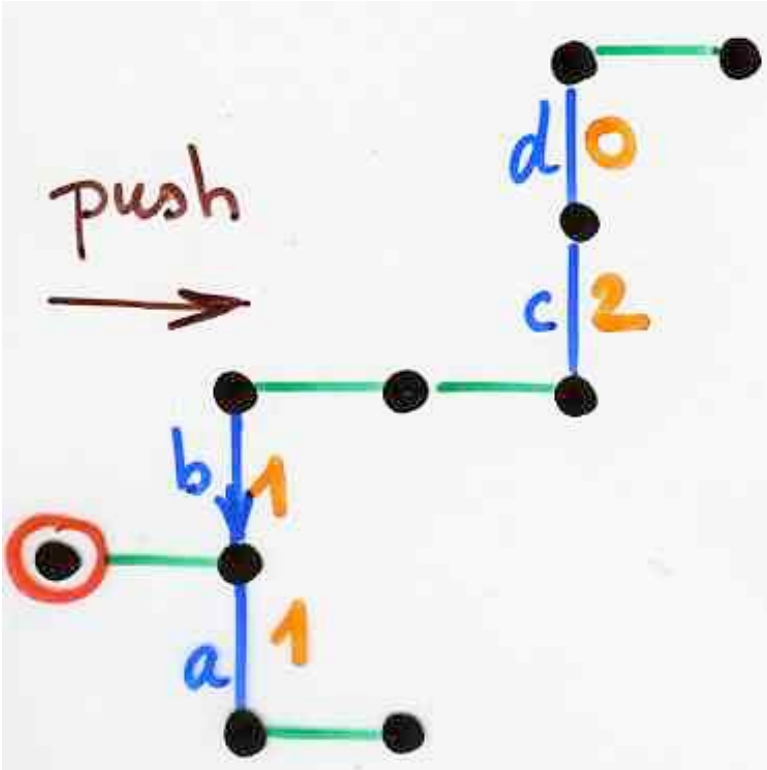


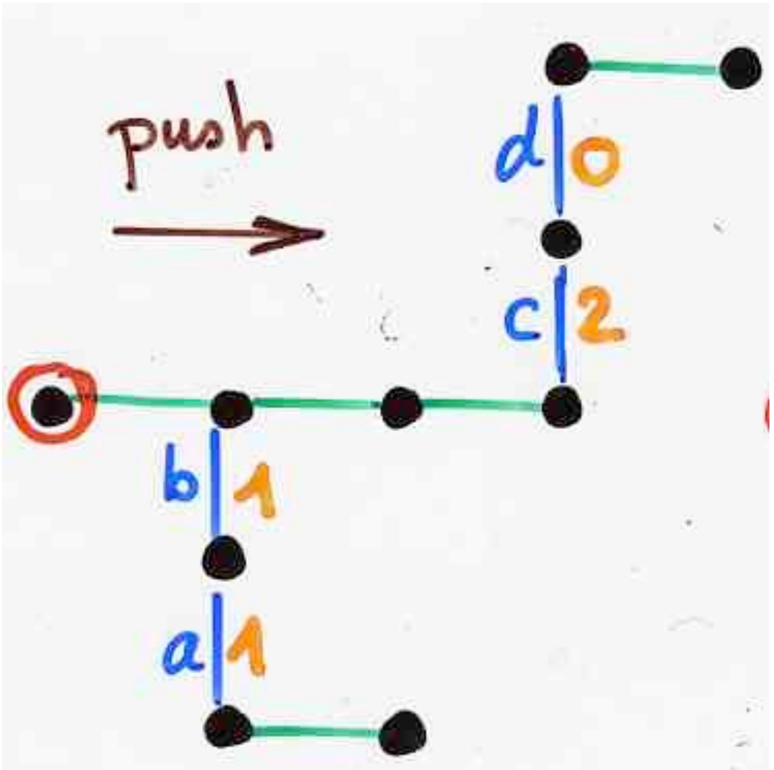
reverse bijection

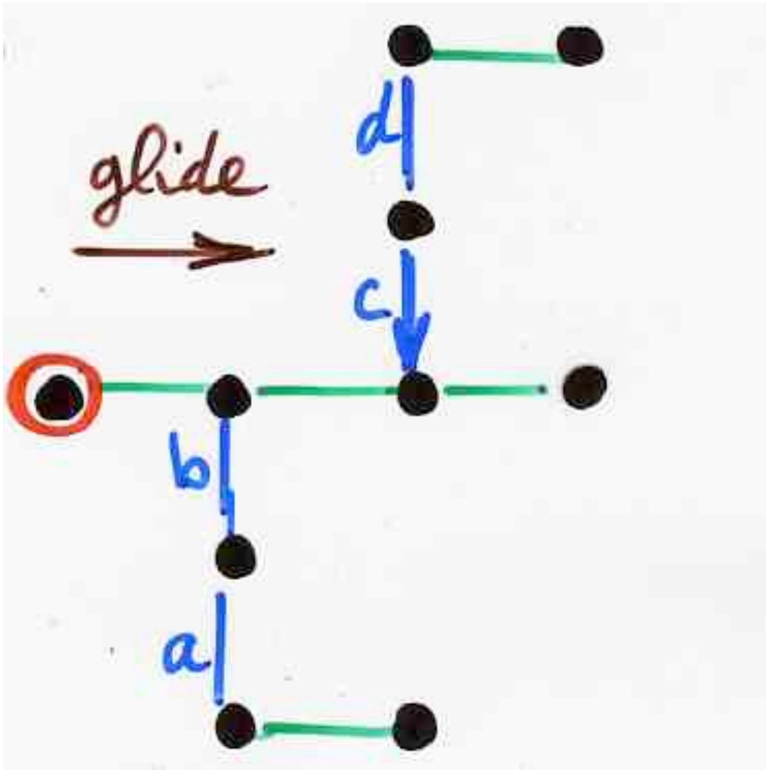
the "push-gliding" algorithm

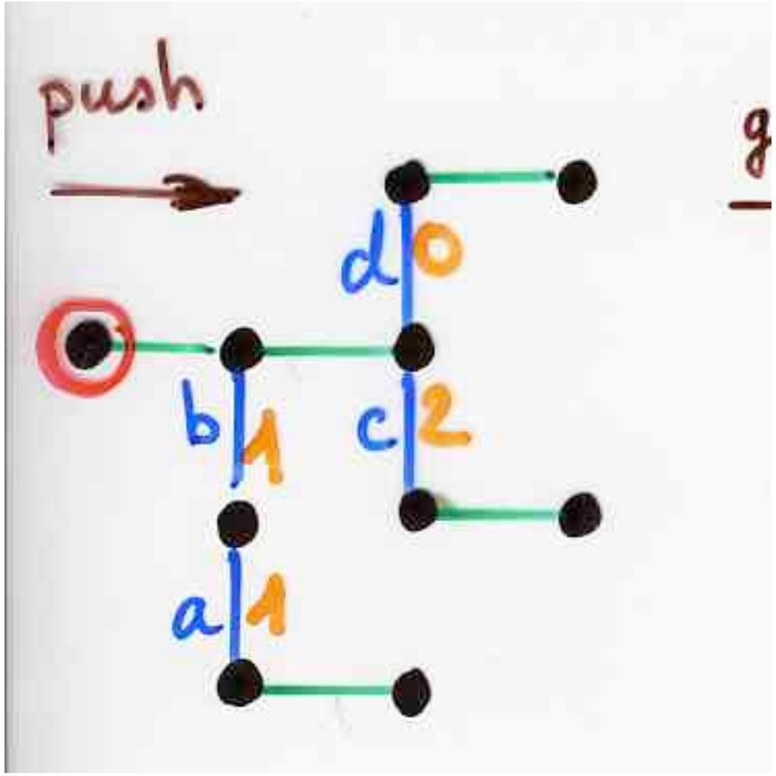


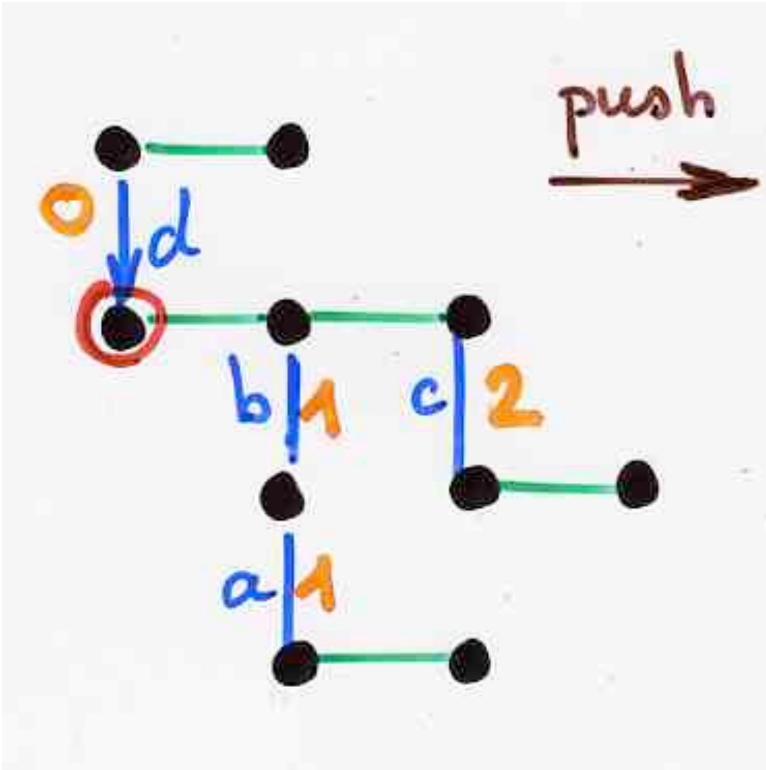
d
0
c
2
b
1
a
1

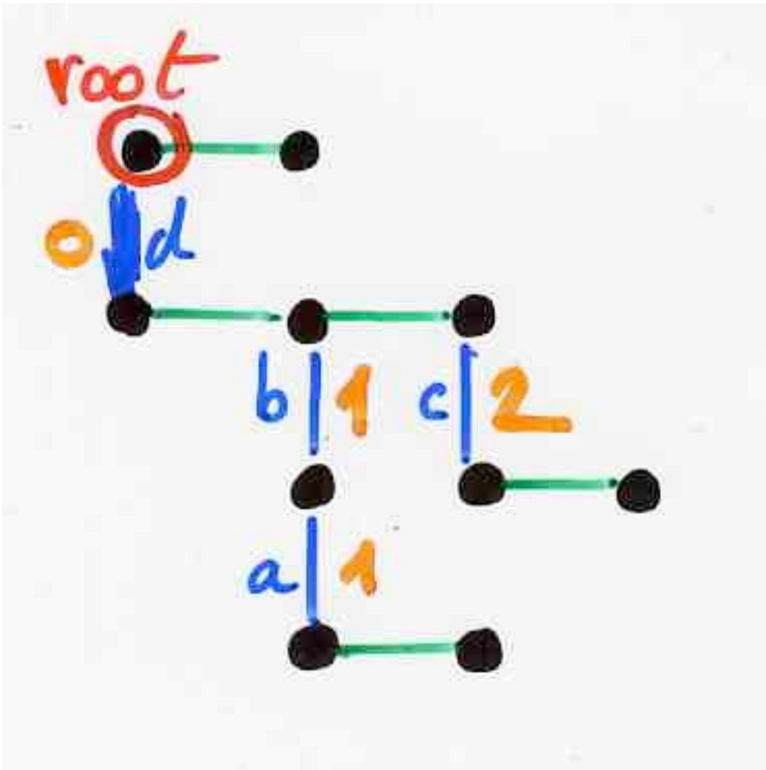

push
d
0
c
2
b
1
a
1

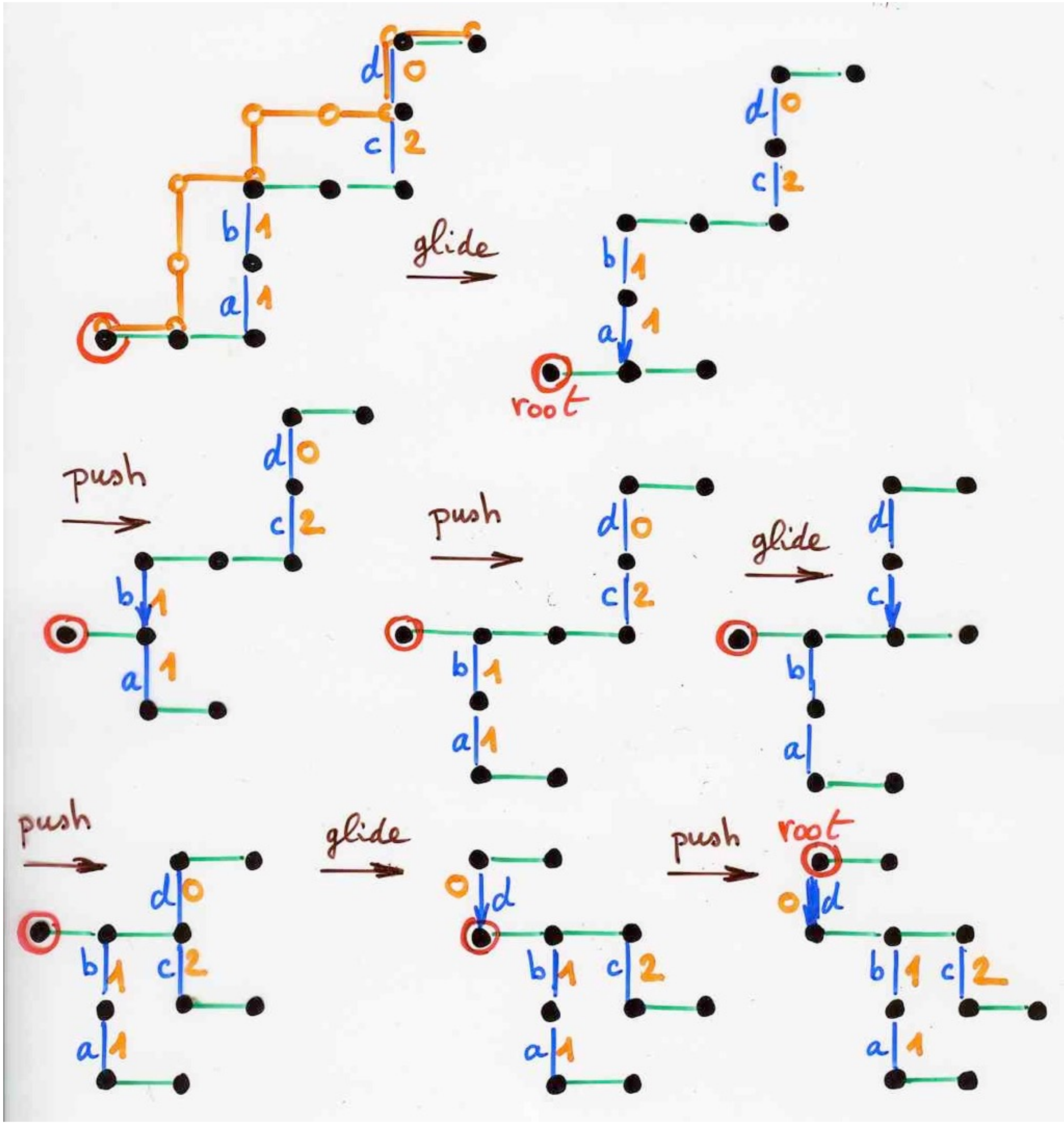

push
d 0
c 2
b 1
a 1


glide
a
b
c
d


push
d
0
b
1
c
2
a
1
g


push
0
d
b
1
c
2
a
1


root
0
d
b
1
c
2
a
1

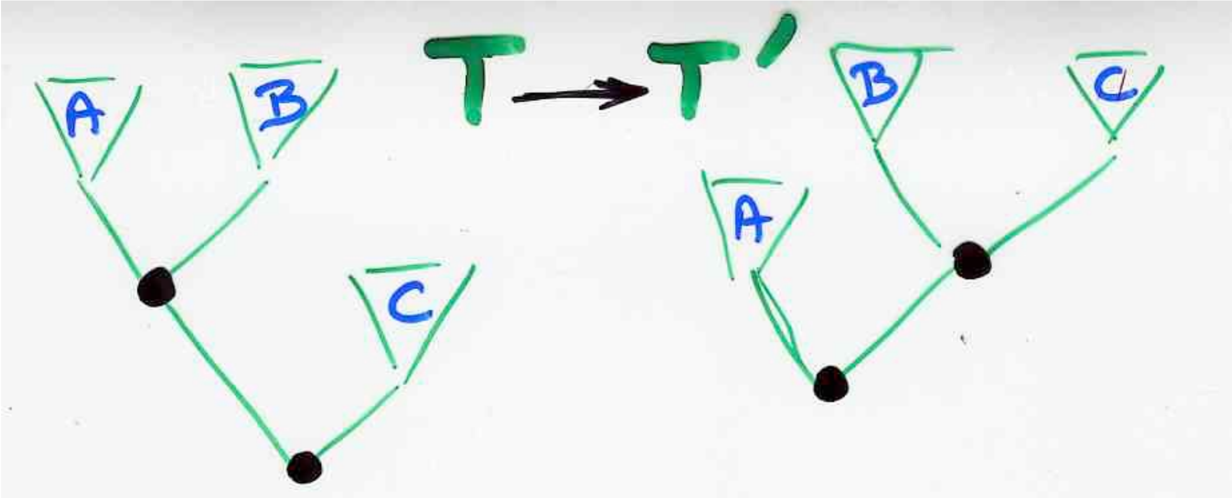


d 0
c 2
b 1
a 1
glide
d 0
c 2
b 1
a 1
root
push
d 0
c 2
b 1
a 1
push
d 0
c 2
b 1
a 1
glide
d
c
b
a
push
d 0
b 1
c 2
a 1
glide
0 d
b 1
c 2
a 1
push
root
0 d
b 1
c 2
a 1

canopy and Tamari lattice

Prop-(i) • The set of binary trees having a given canopy V is an interval $I(V)$ of the Tamari lattice

idea of proof



if $B \neq \bullet$ canopy is invariant

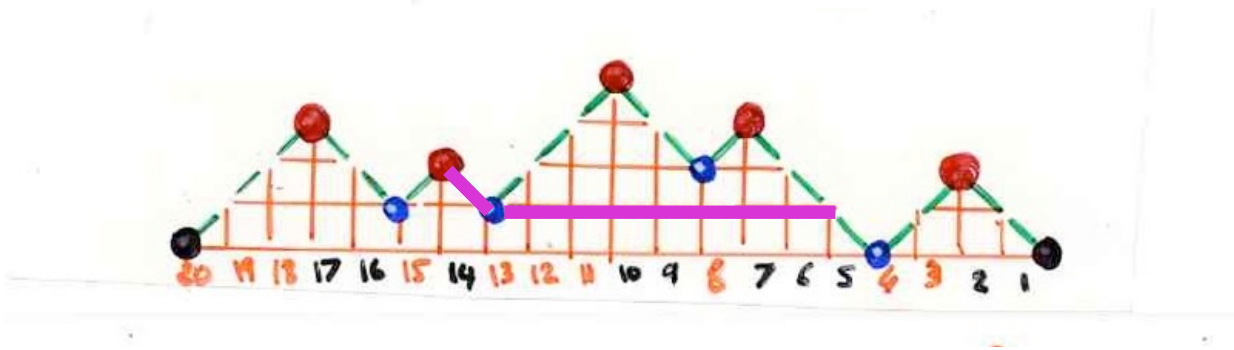
if $B = \bullet$ canopy $c(T')$ not invariant

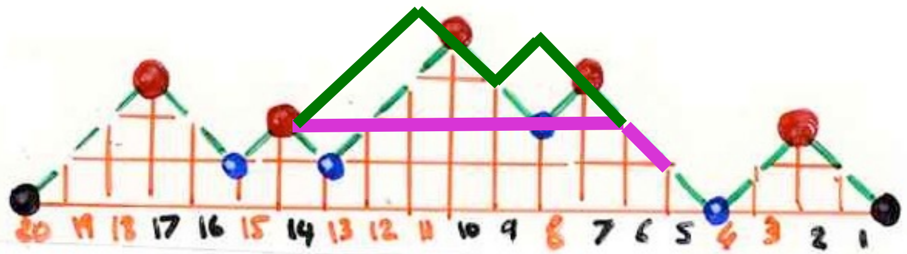
$$c(\overset{\star}{T}) = c(A) \overset{+}{/} c(B)\, c(C)$$

$$c(T') = c(A) \underline{\backslash}\, c(B)\, c(C)$$

$$c(B) = \varnothing$$

forbidden
move

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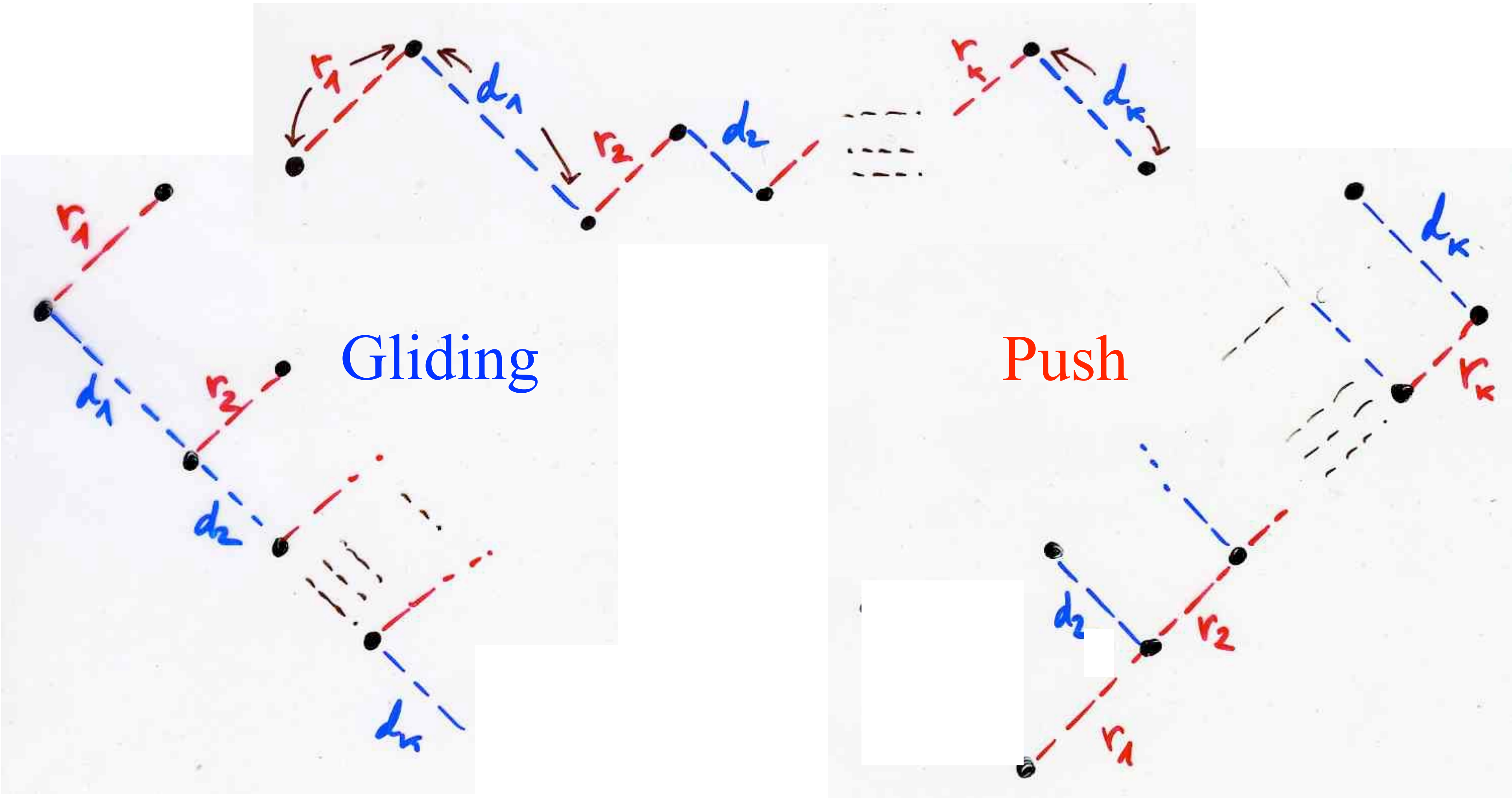
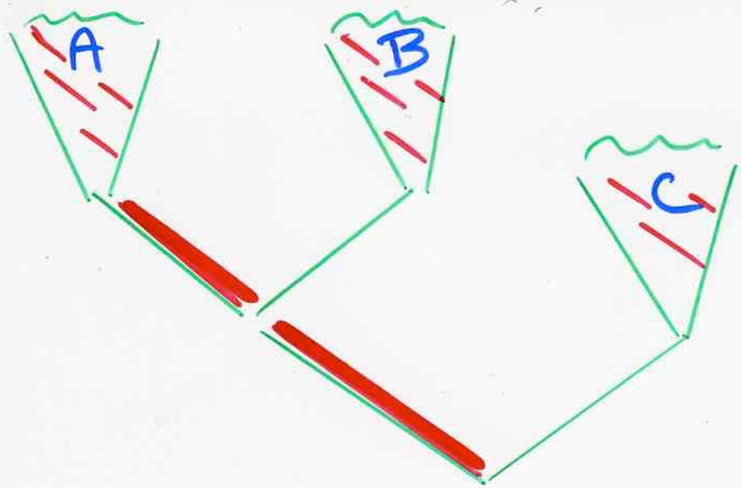
Prop.(i) • The set of binary trees having a given canopy w is an interval $\mathcal{I}(w)$ of the Tamari lattice

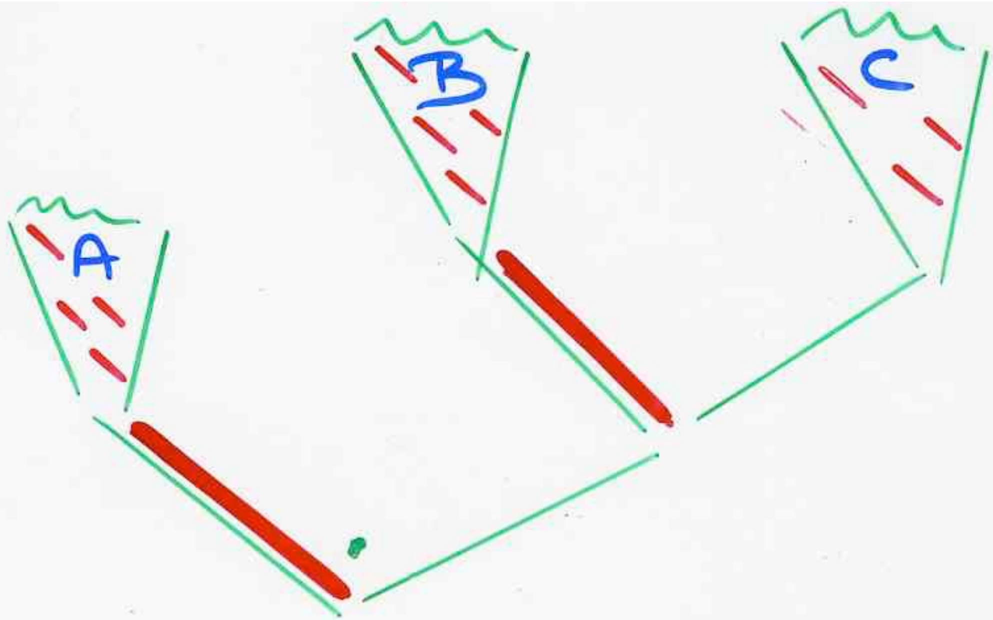


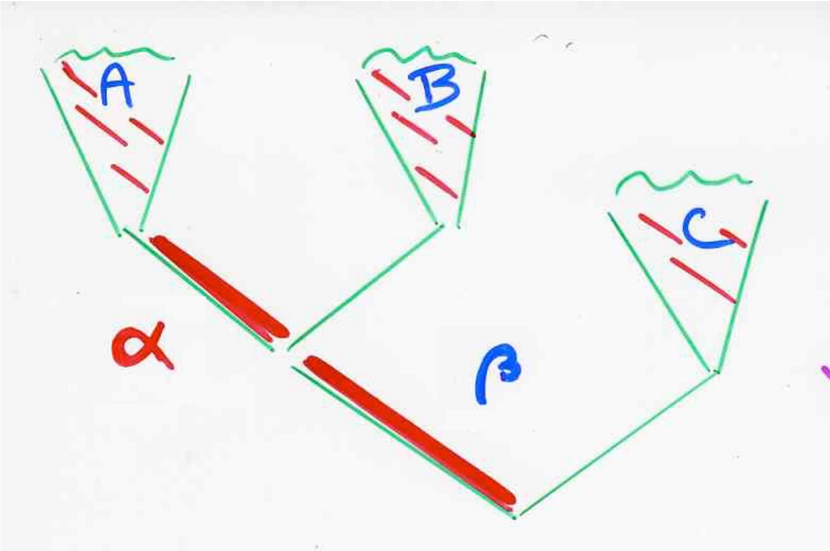
Prop. (i) • The set of binary trees having a given canopy v is an interval I(v) of the Tamari lattice

(ii) This interval I(v) is isomorphic to T_v

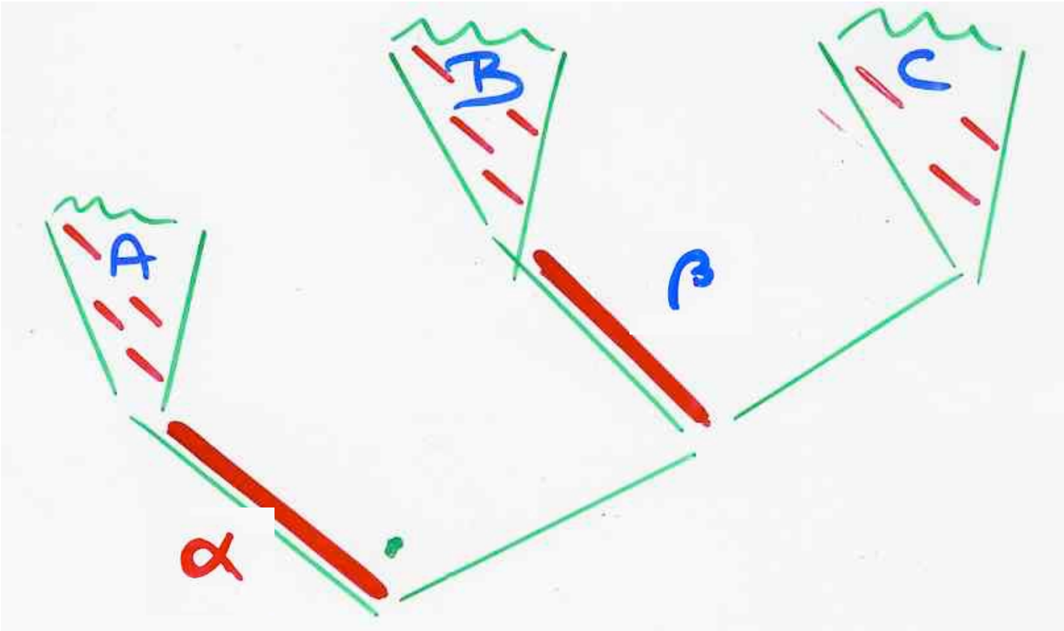


left edges
(in symmetric order)
with their
"right height"



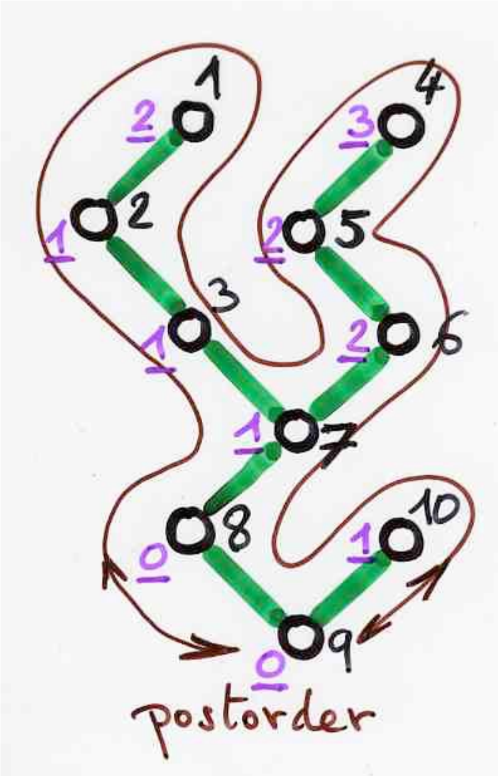


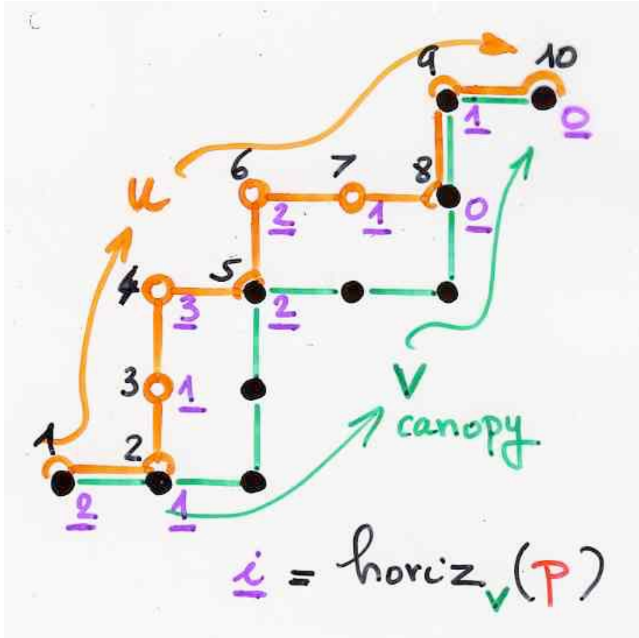
left edges (in symmetric order) with their "right height"



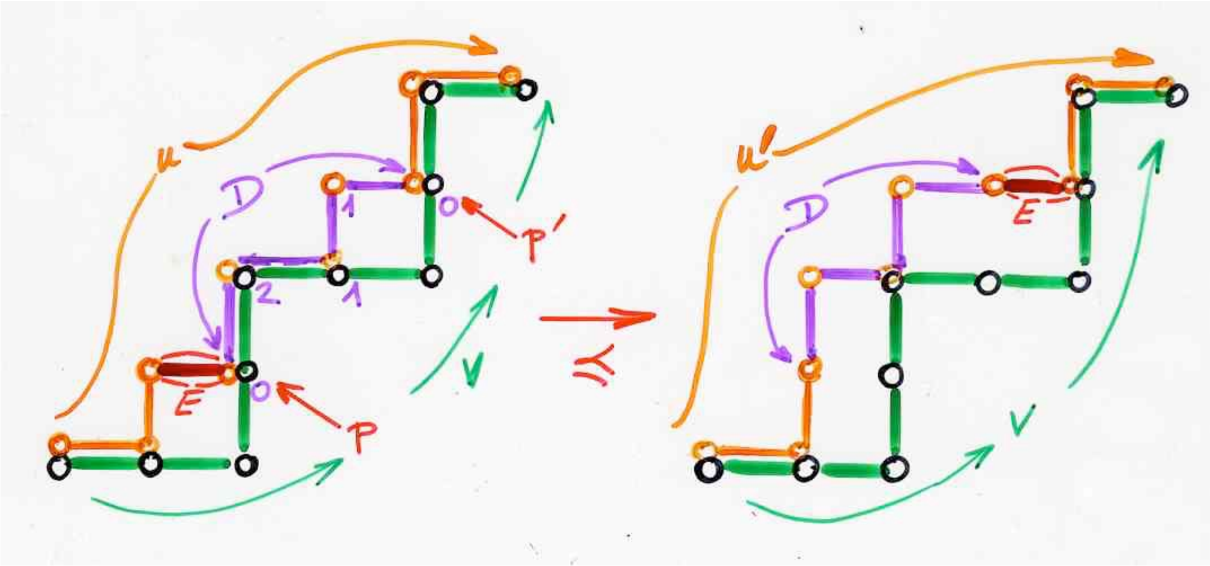
preservation of the symmetric order for left edges

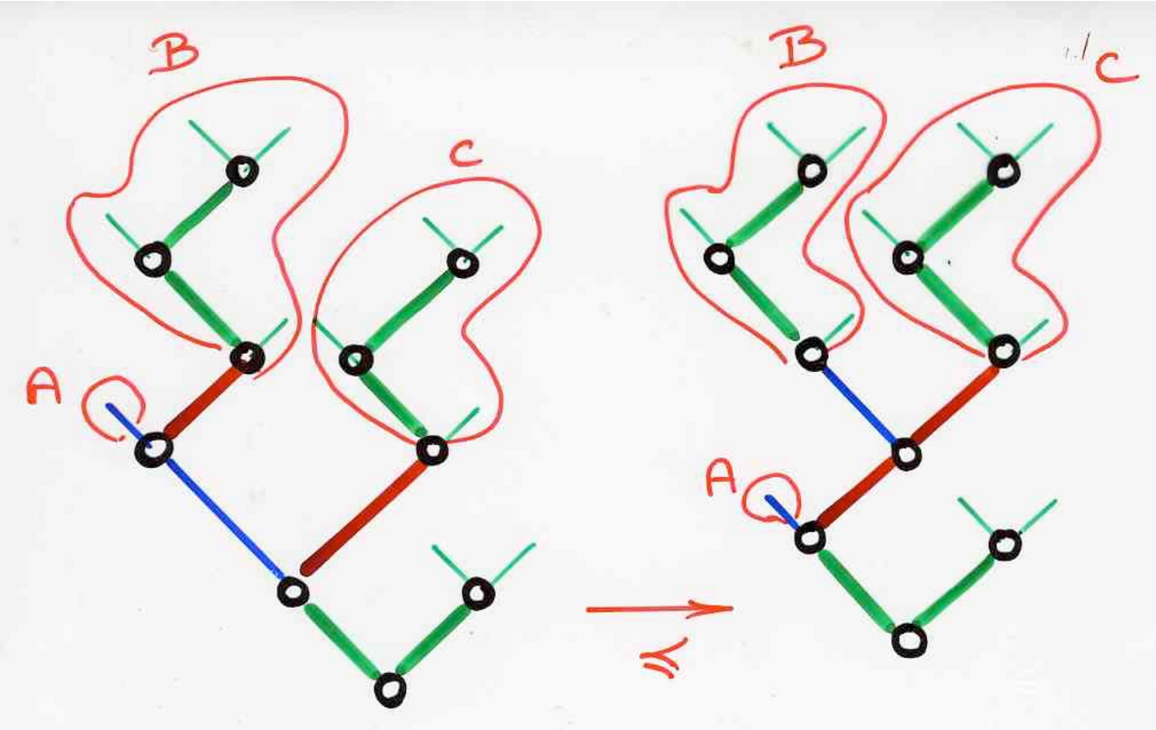
right height: +1 in C and for β





another lemma



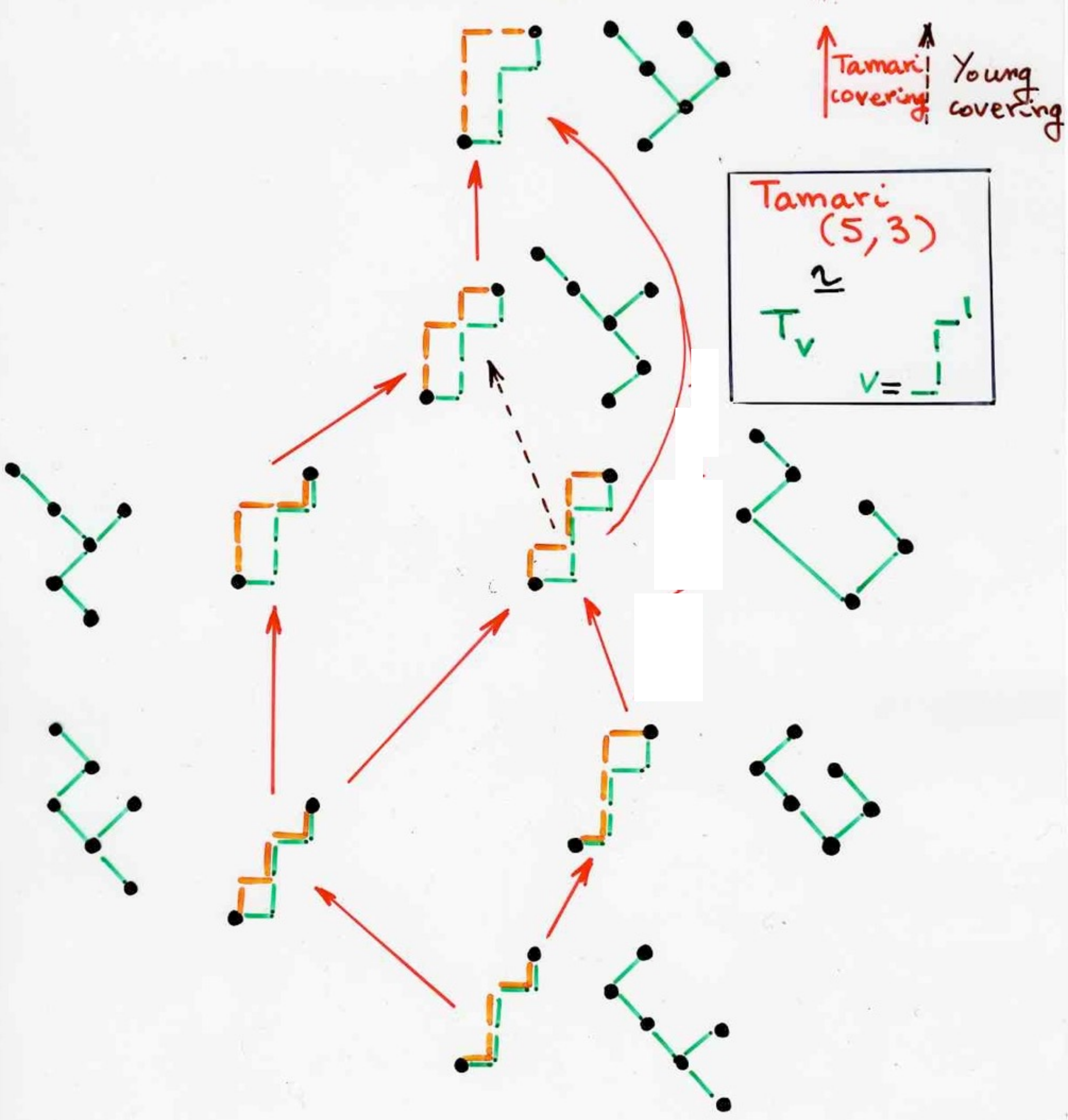


the covering relation in T_V
and the corresponding rotation
in (ordinary) T

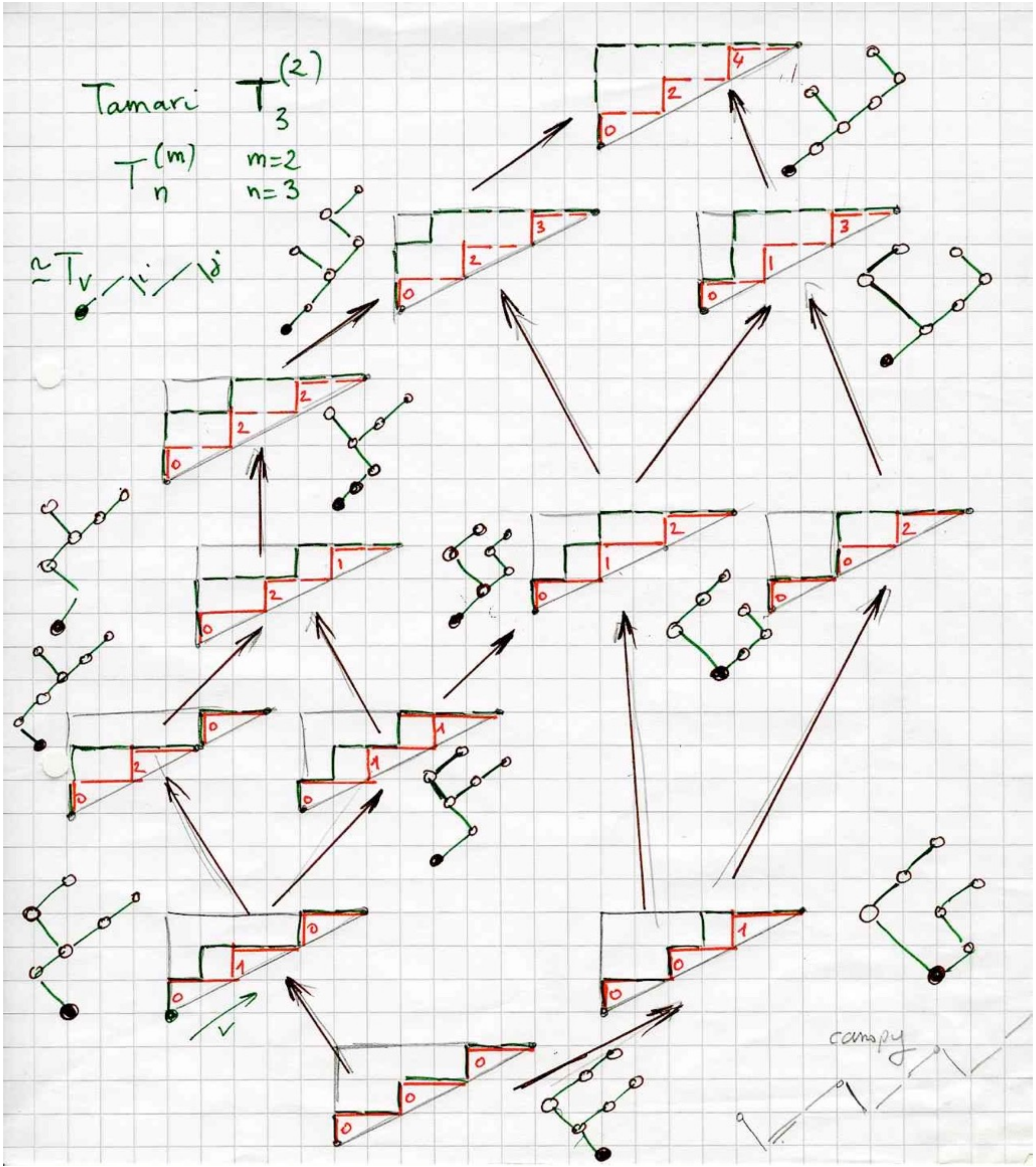
Thm 1. For any path v
T_v is a lattice

is a consequence of (ii)

(ii) This interval $I(v)$ is isomorphic to T_v



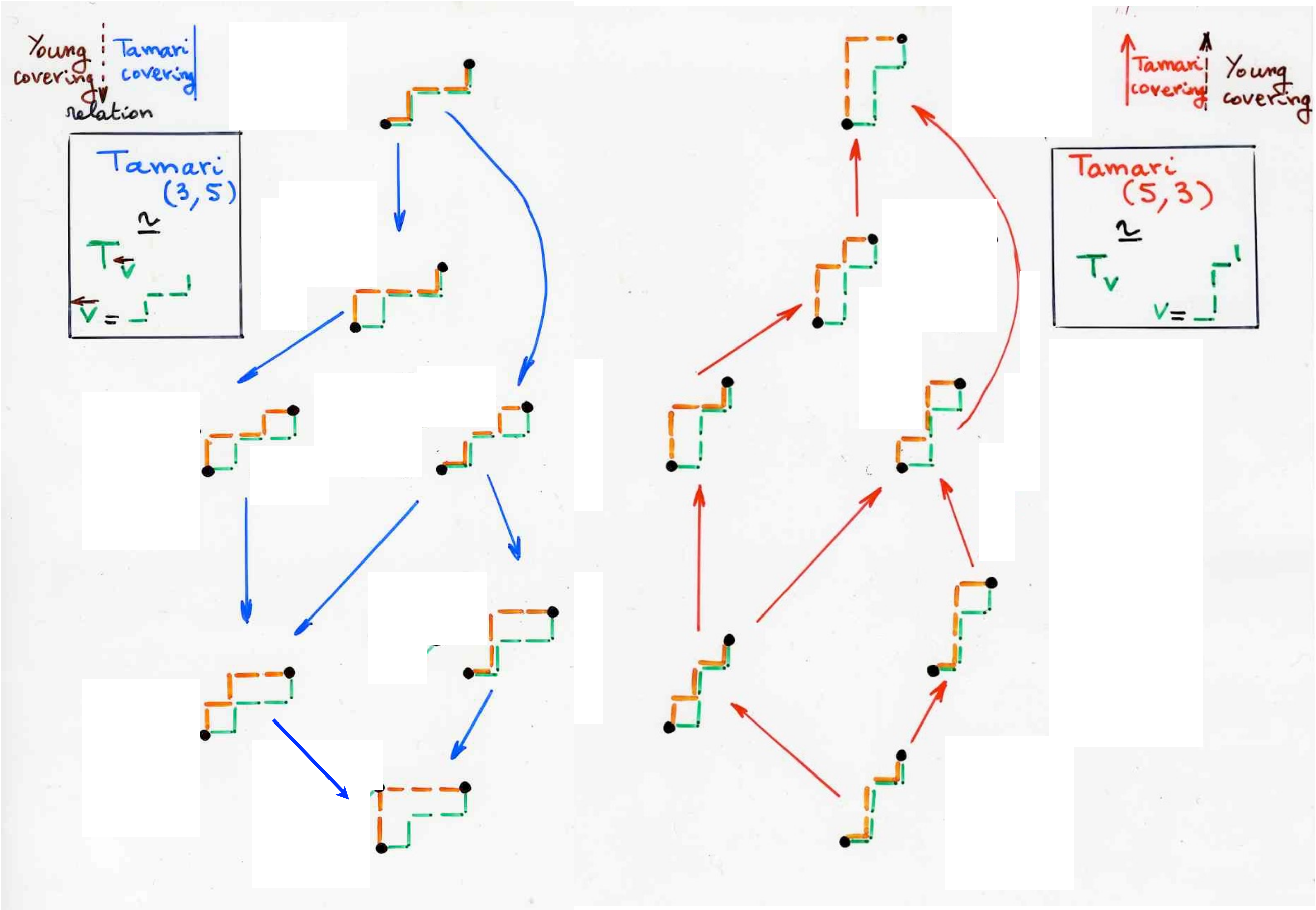
Tamari covering
Young covering
Tamari (5,3)
≃
T
v
v=

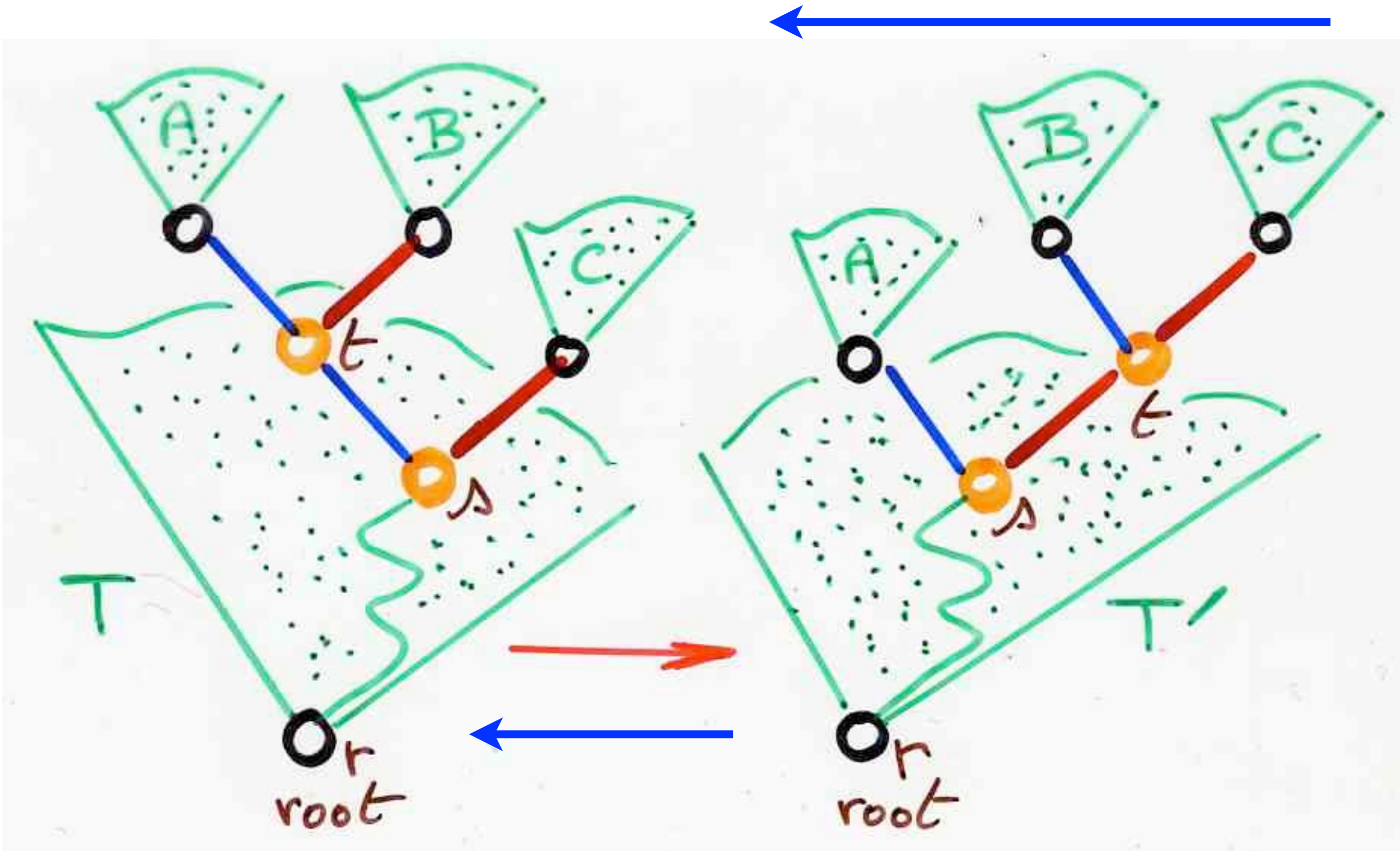


Tamari $T_3^{(2)}$
$T_n^{(m)}$ m=2 n=3
$\simeq T_\nu$
canopy

proof of the duality

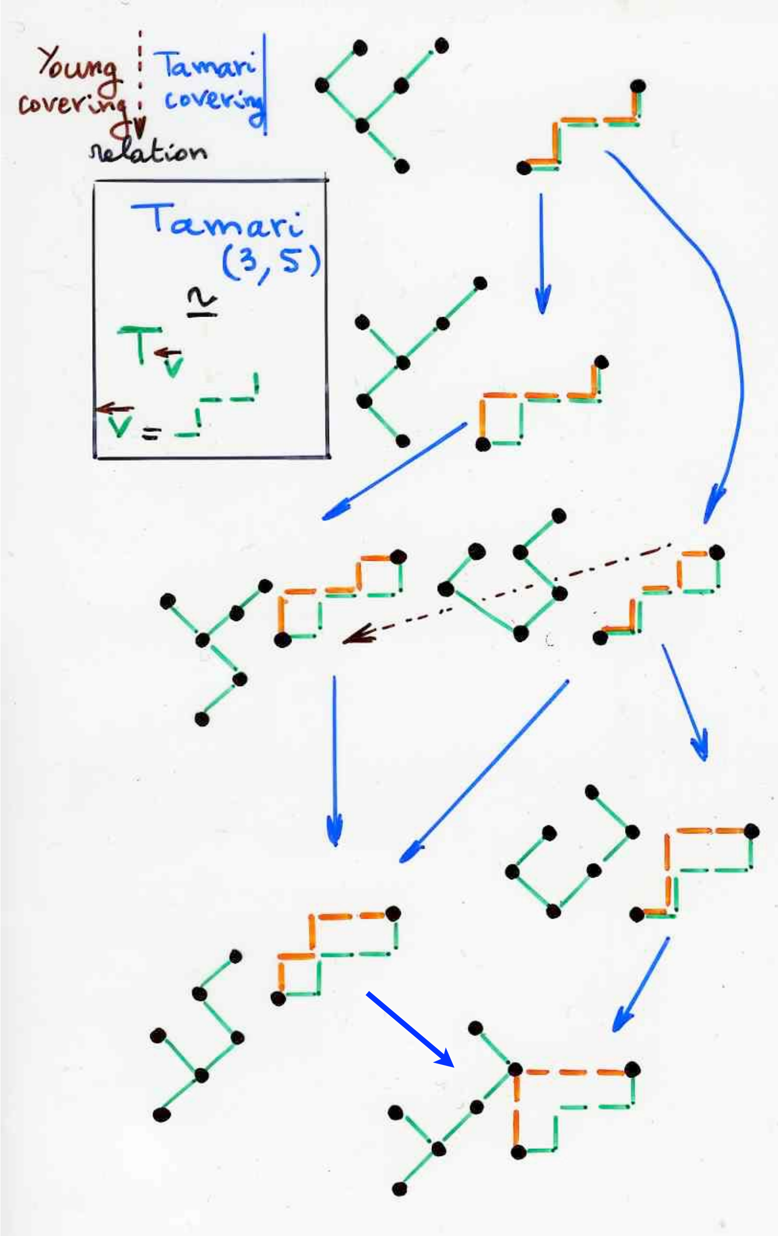
Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$

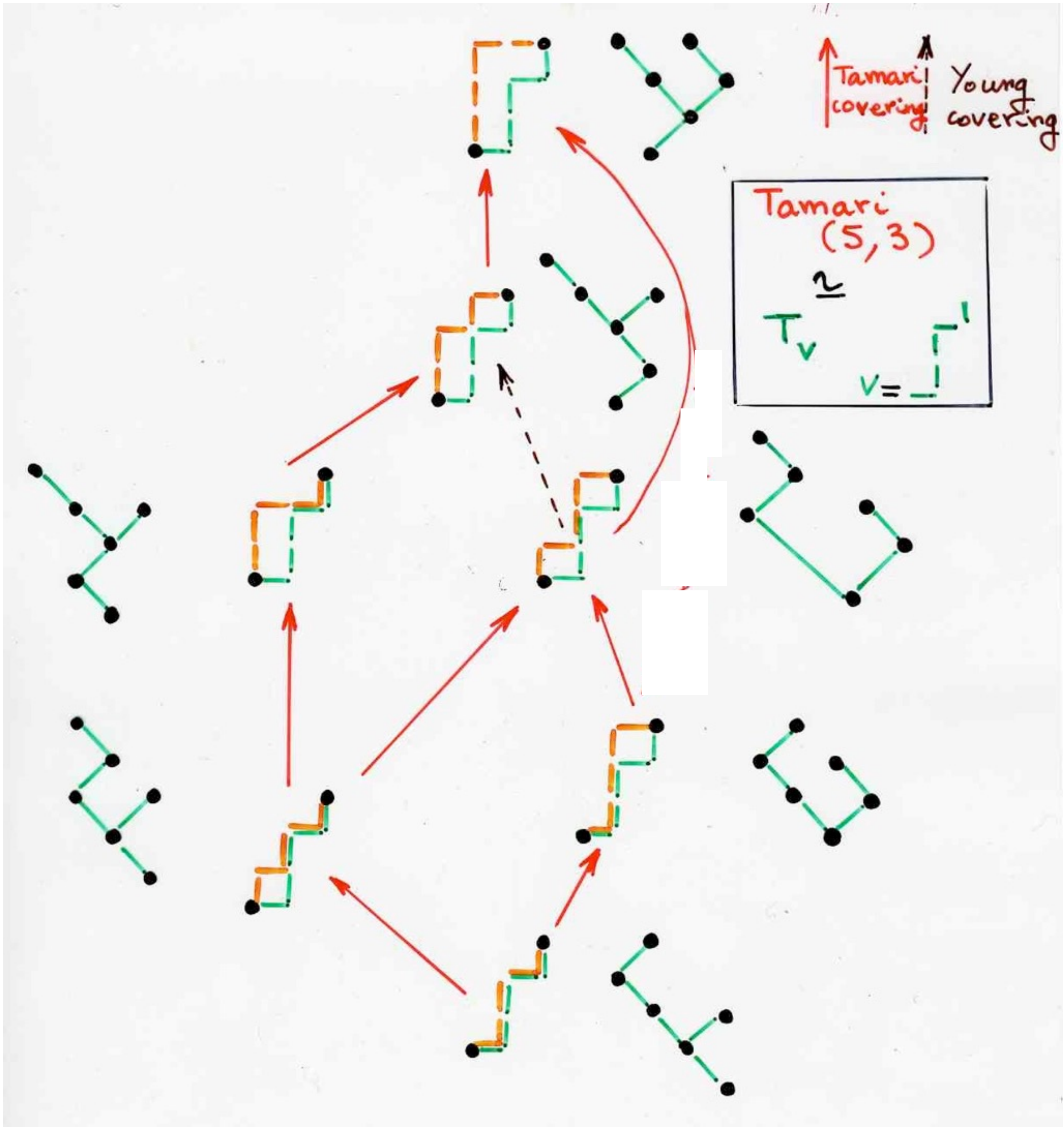


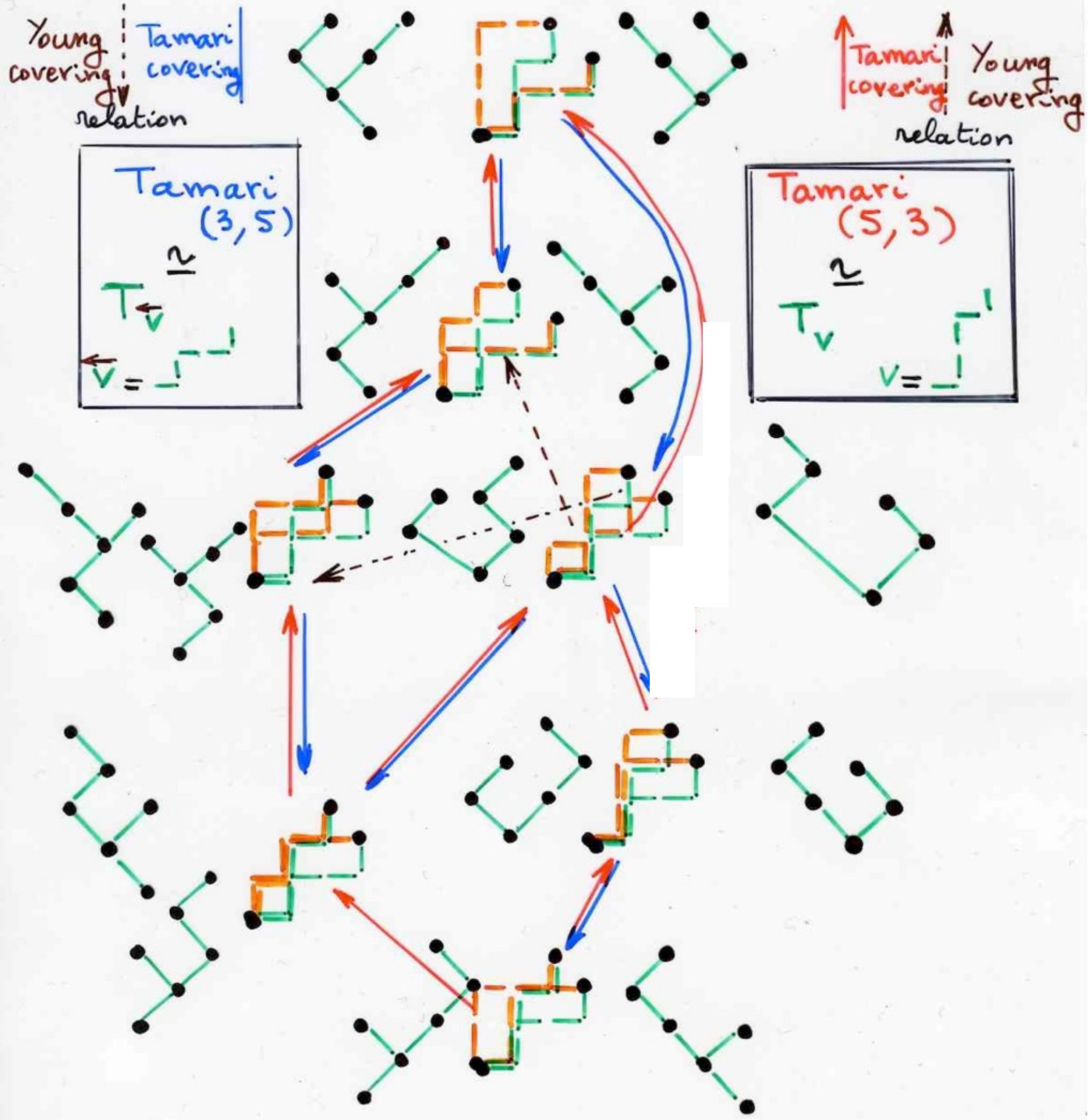


Rotation in a binary tree:
the covering relation in the Tamari lattice

Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$







Young covering relation
Tamari covering
Tamari (3,5)
≃
$T_{\overleftarrow{v}}$
$\overleftarrow{v}$ =
Tamari covering
Young covering relation
Tamari (5,3)
≃
T_v
v =

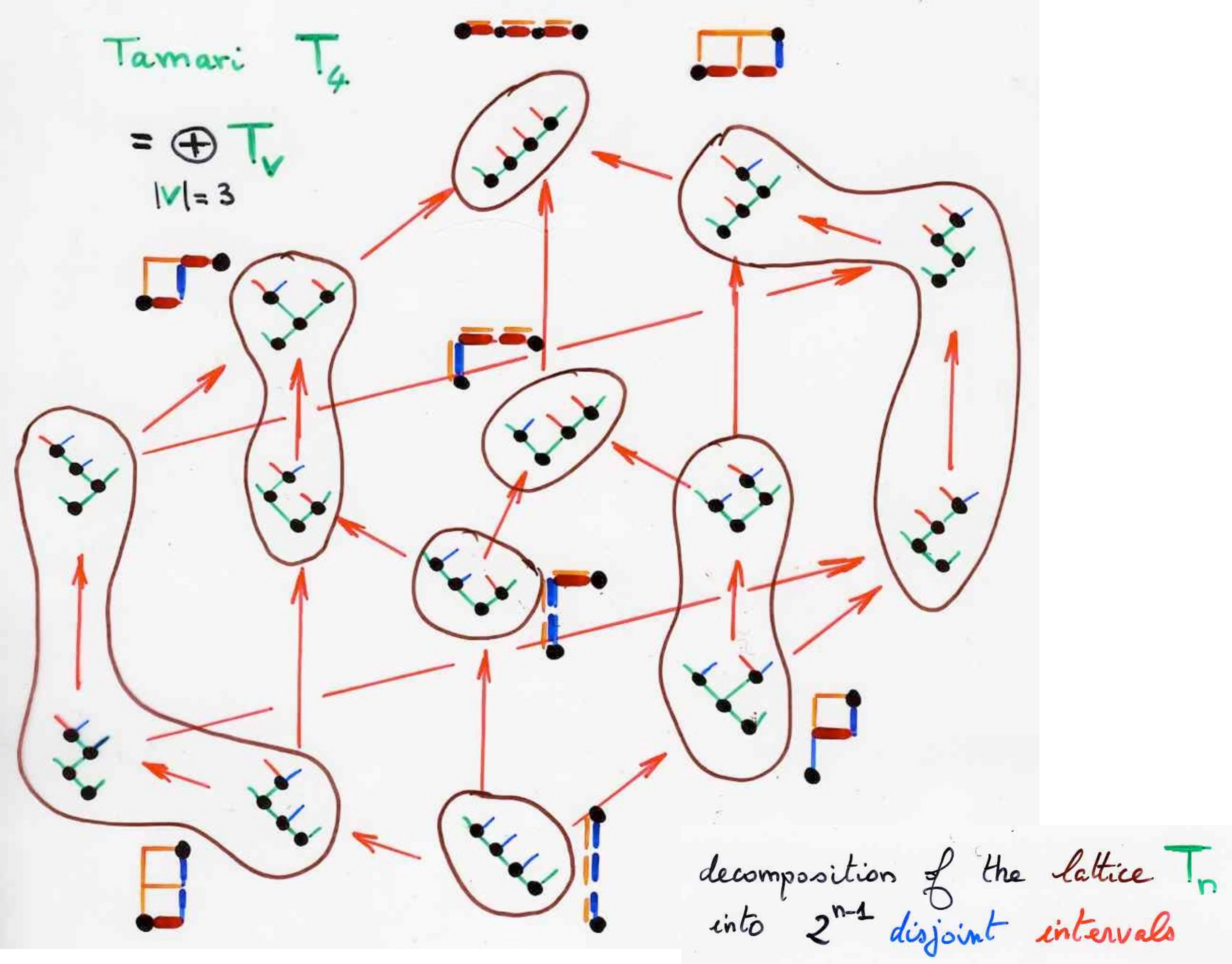
Thm 1. For any path v, T_v is a lattice

Thm 2. The lattice T_v is isomorphic to the dual of $T_{\overleftarrow{v}}$

Thm 3. The usual Tamari lattice T_n can be partitioned into intervals indexed by the 2^{n-1} paths v of length $(n-1)$ with $\{E, N\}$ steps,

$$T_n \cong \biguplus_{|v|=n-1} I_v ,$$

where each $I_v \cong T_v$.



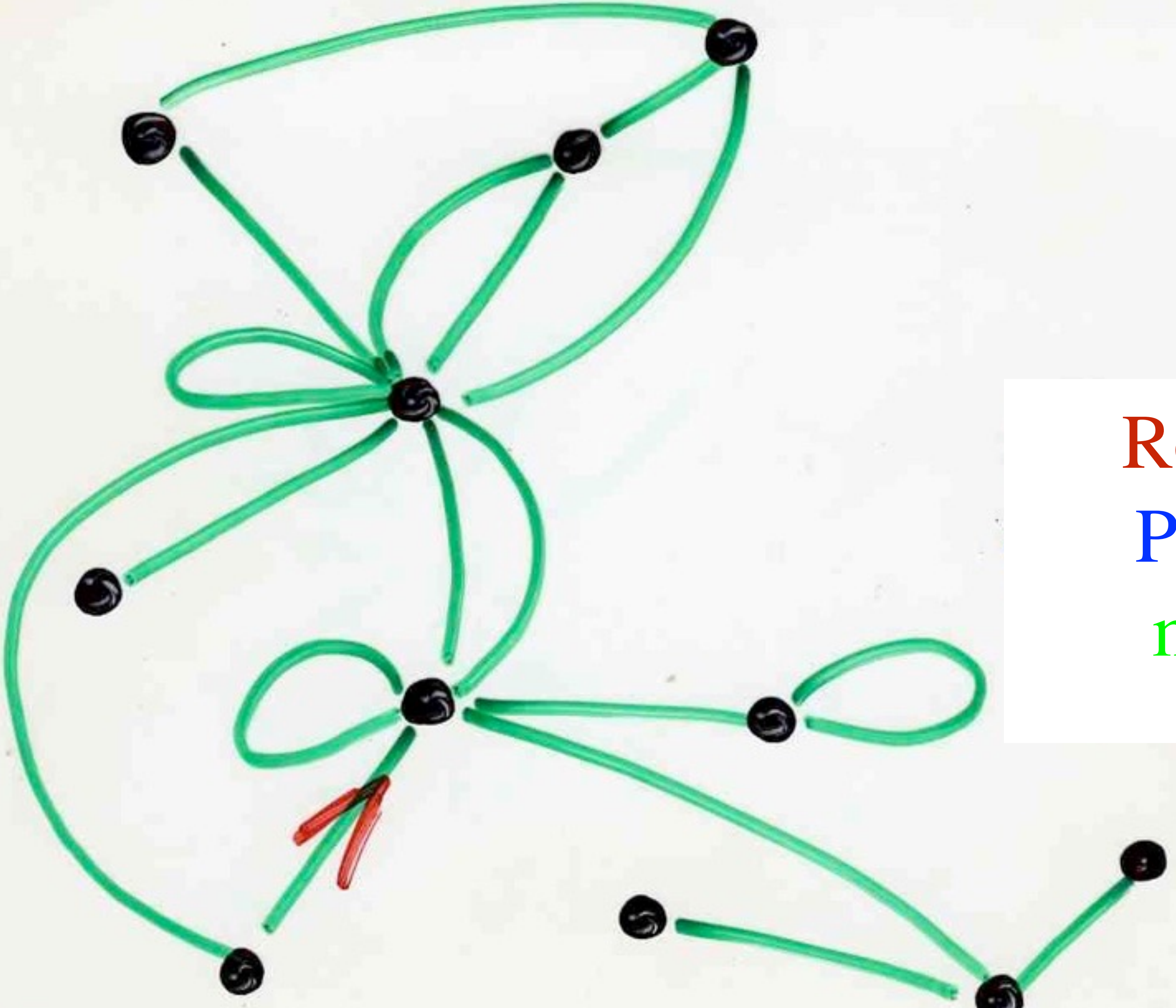
Tamari T_4
$= \bigoplus T_v$
$|v| = 3$
decomposition of the lattice T_n
into 2^{n-1} disjoint intervals

Prop (L.-F. Préville-Ratelle)

The total number of intervals in all T_v $|v|=n$

is the number of non-separable planar maps

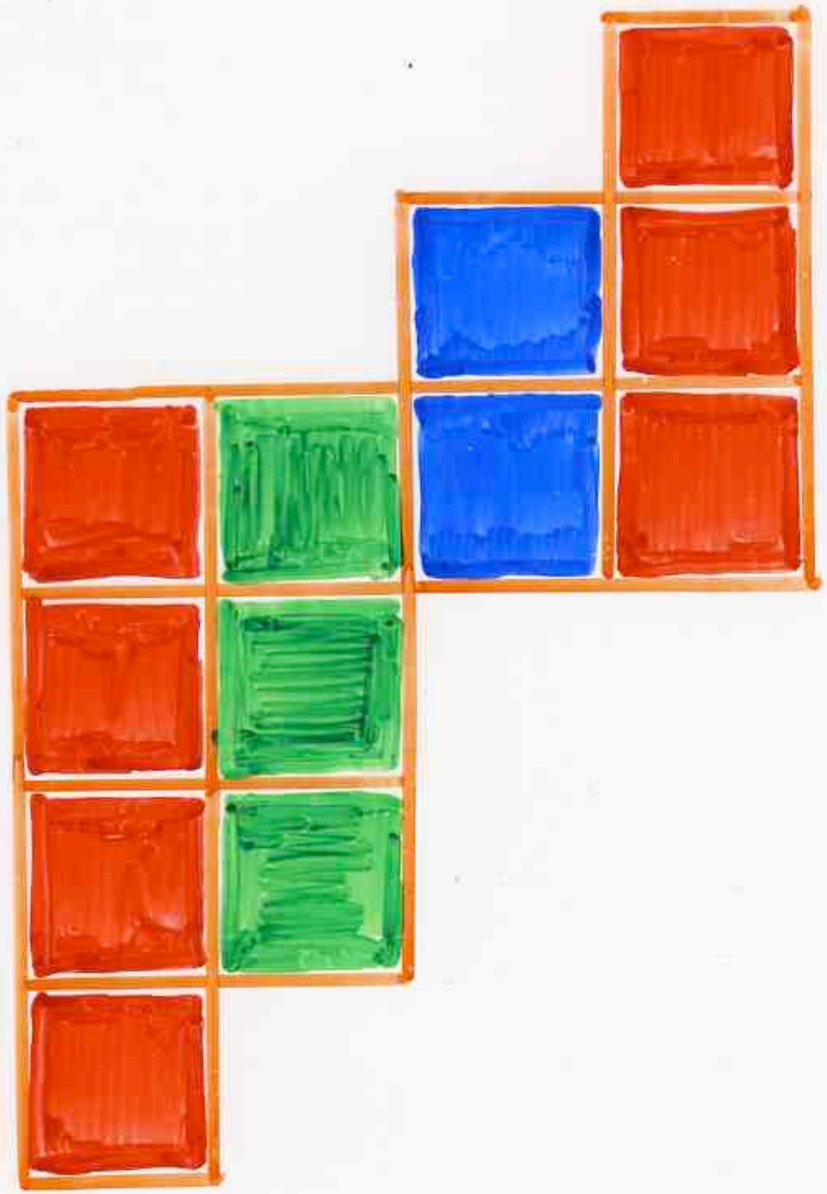
$$\frac{2(3n+3)!}{(n+2)!(n+3)!}$$

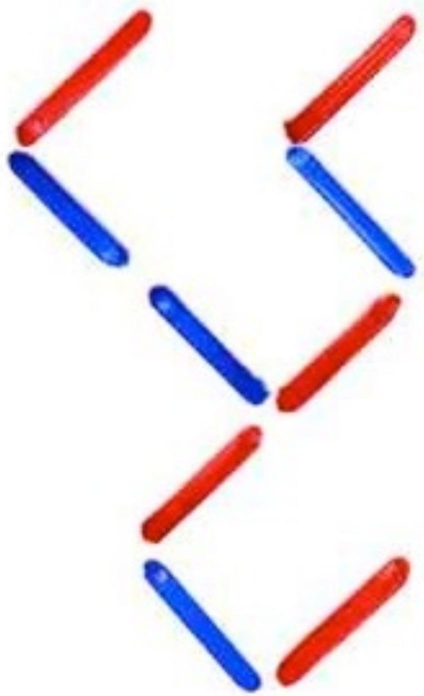


Rooted
Planar
maps

Complements

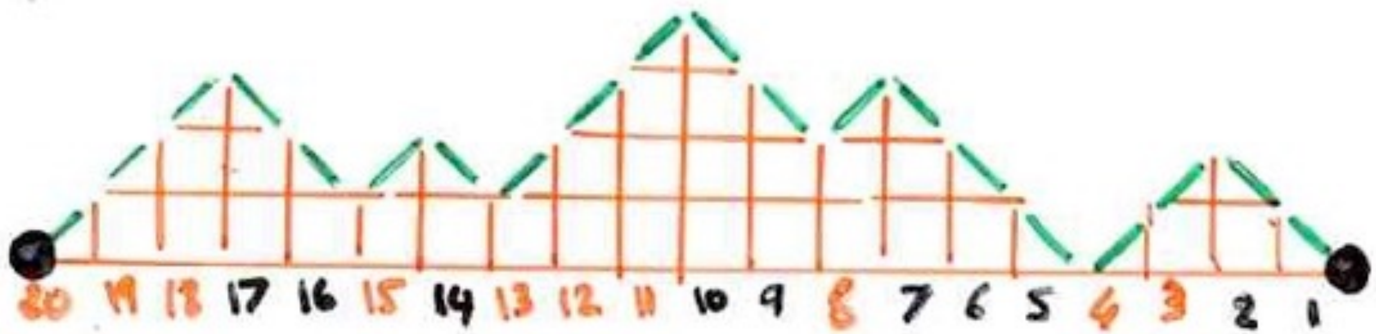
another proof
with staircase polygons



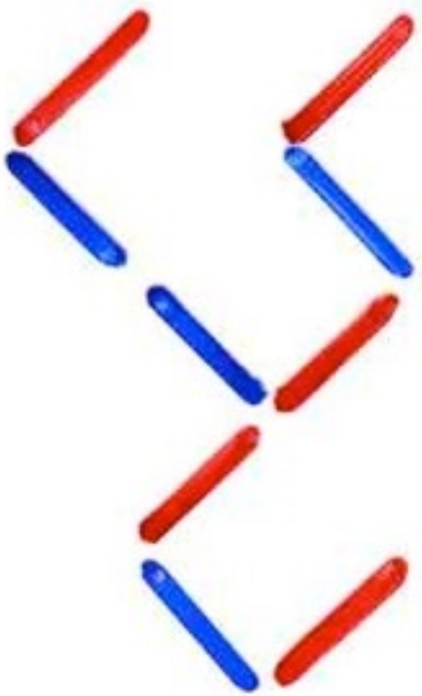


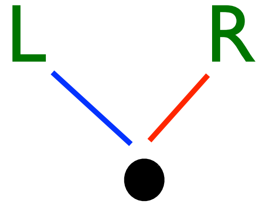
binary
tree

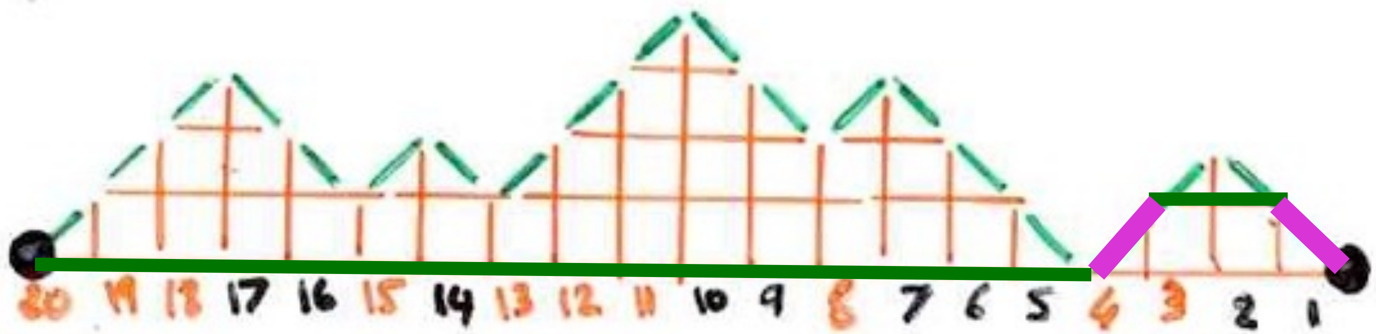
Dyck path

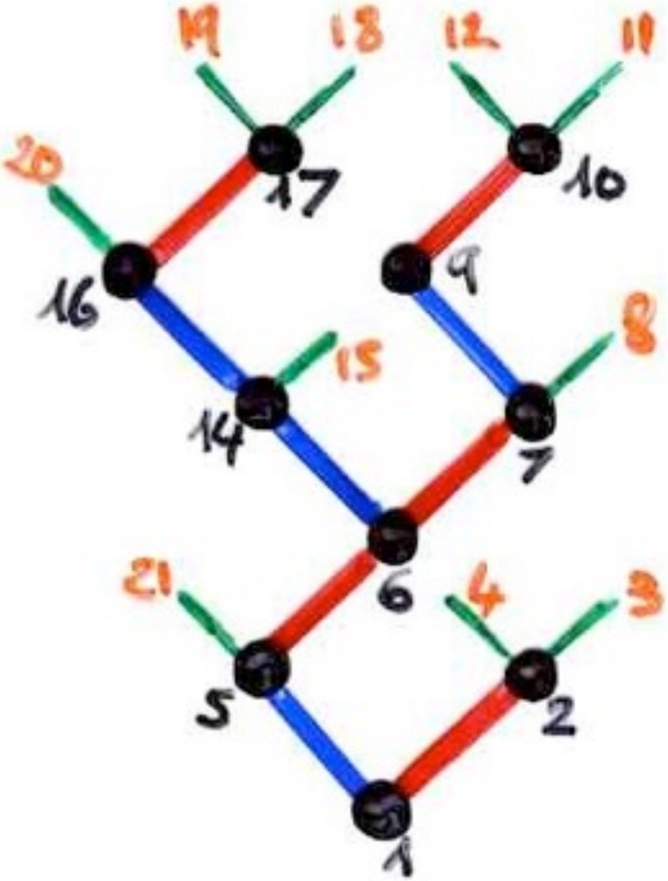


binary
tree





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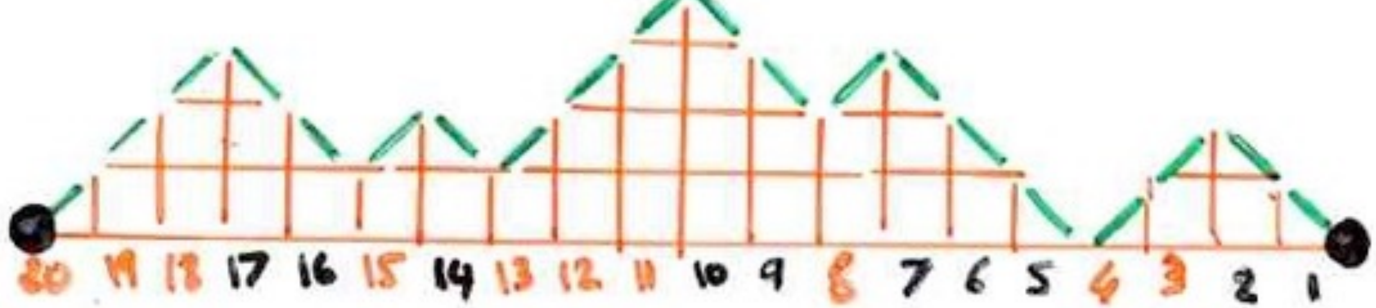
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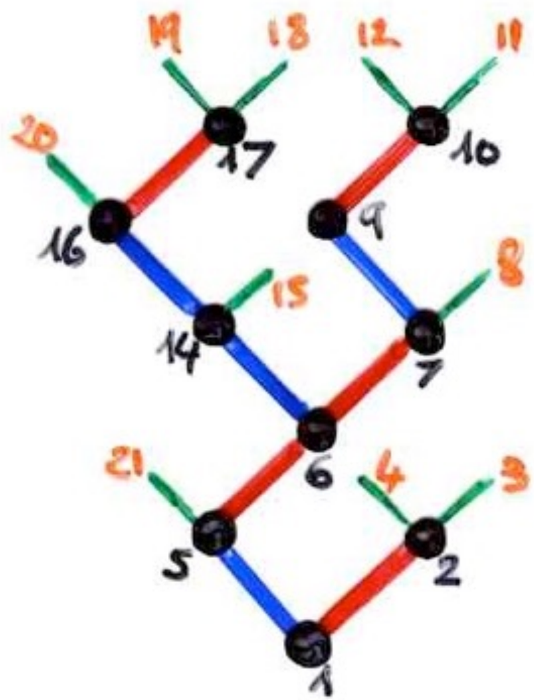
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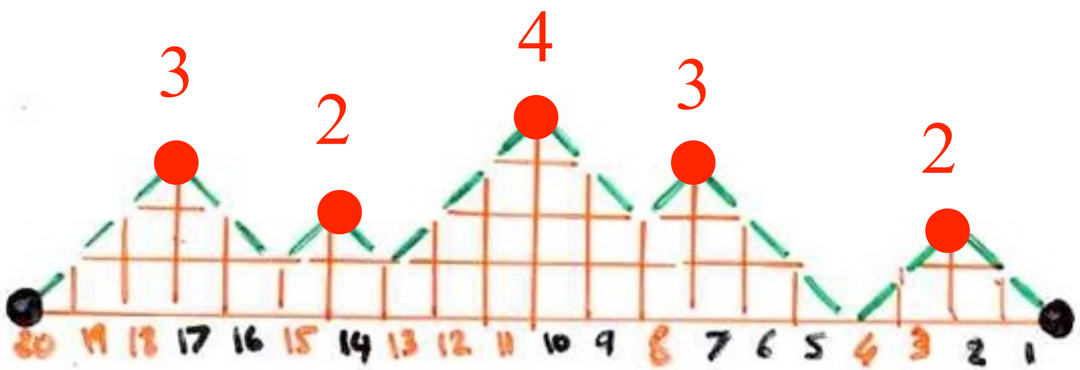
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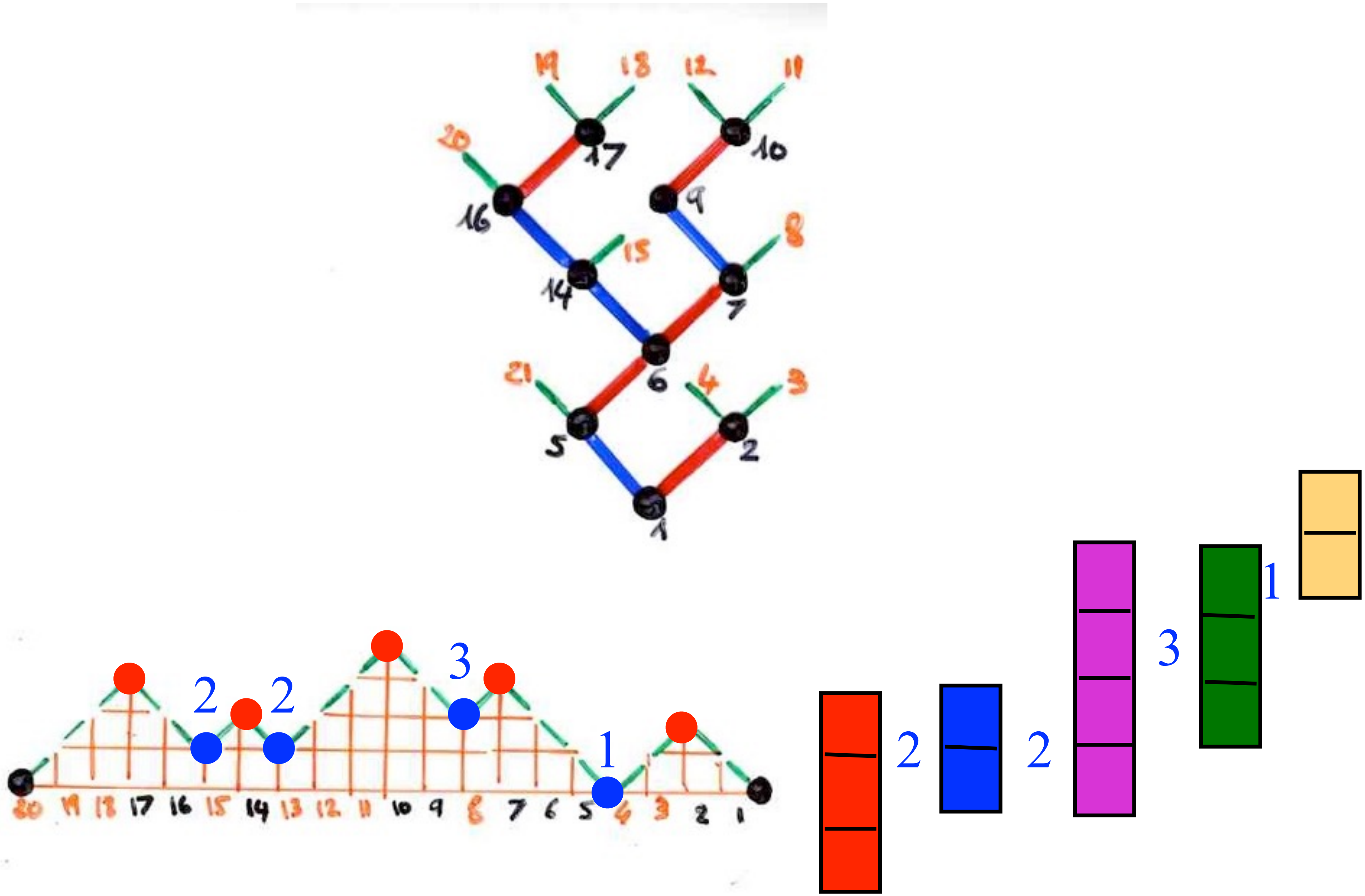
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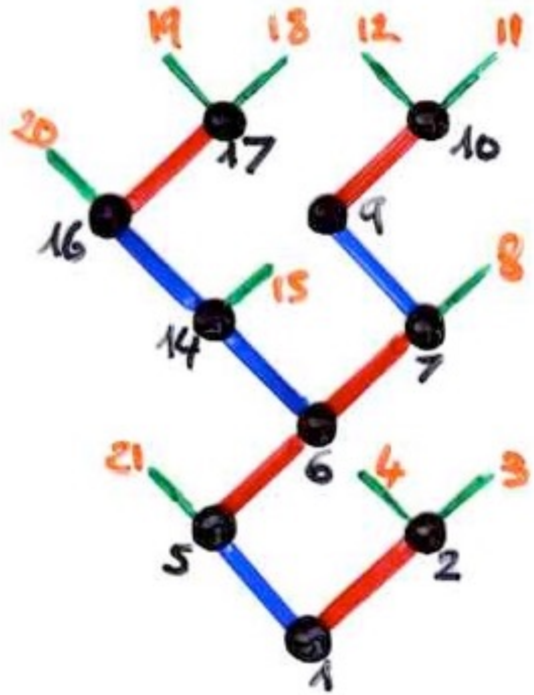
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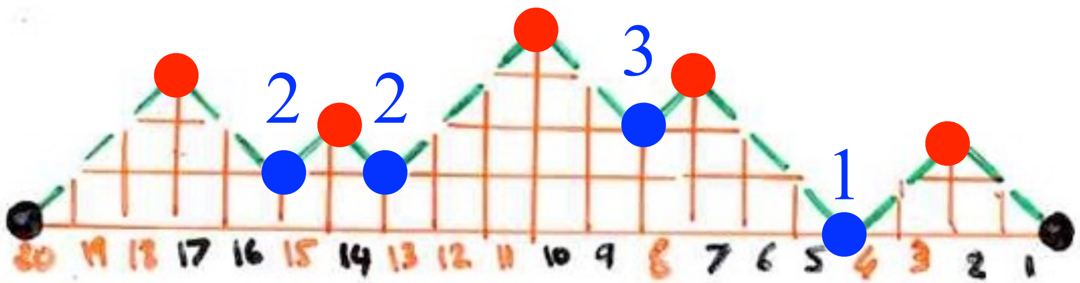
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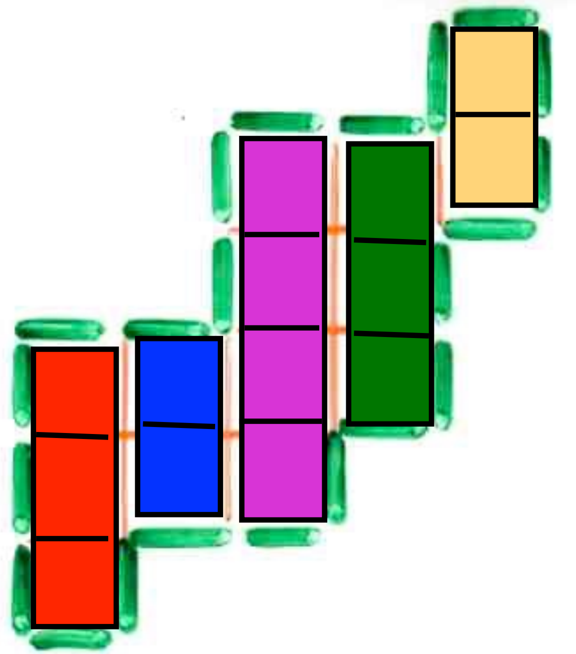

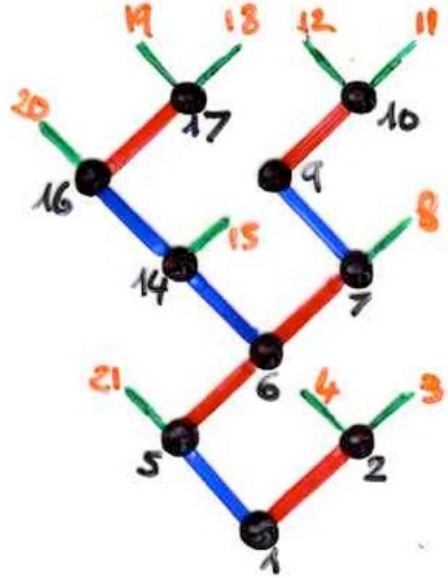

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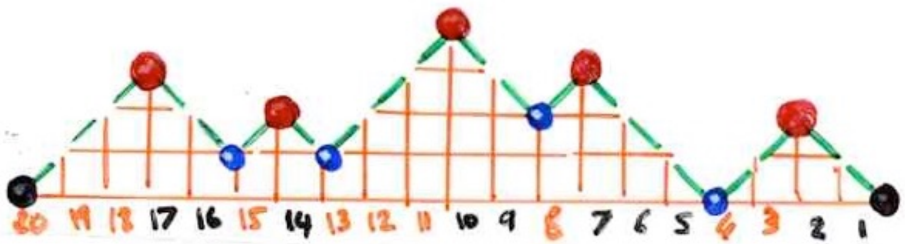

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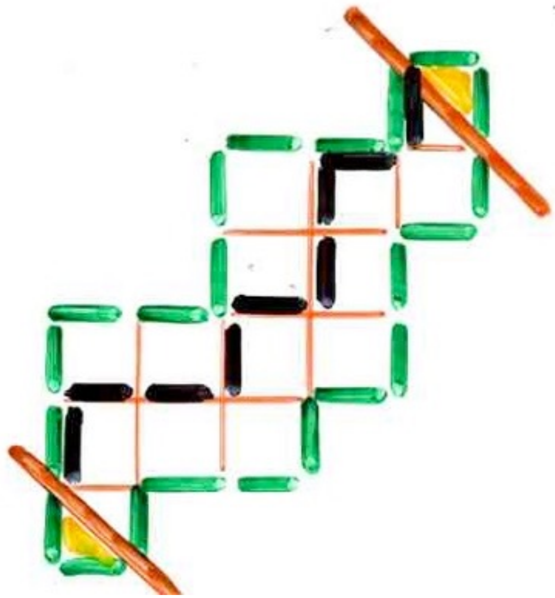
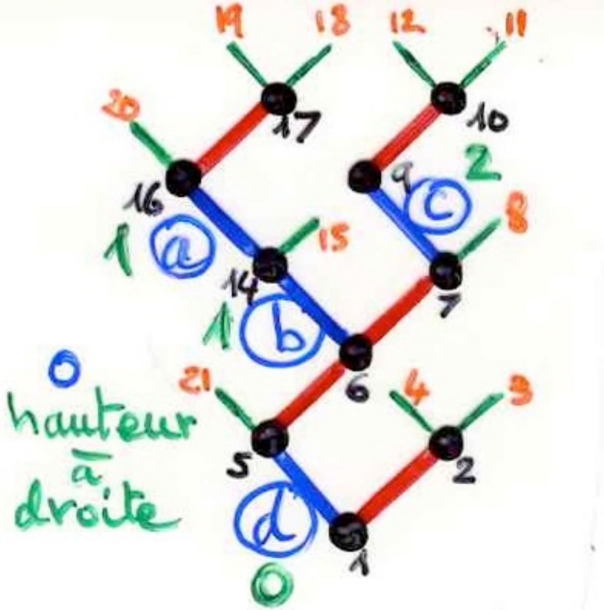


possible covering relation
preserving the canopy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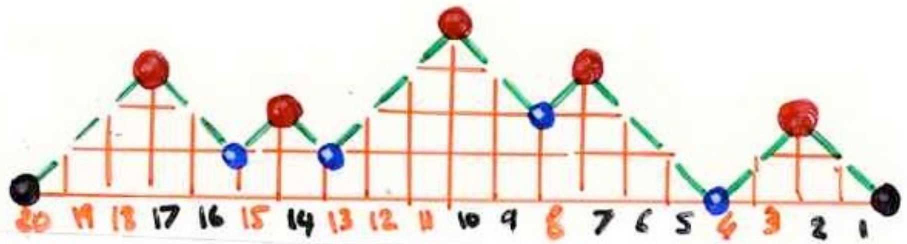
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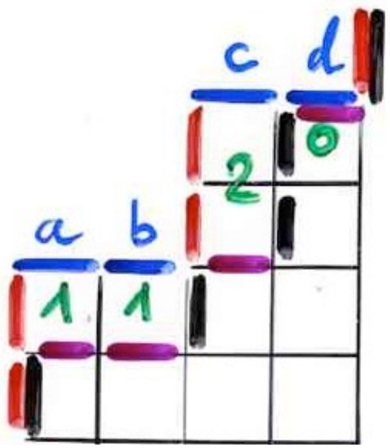
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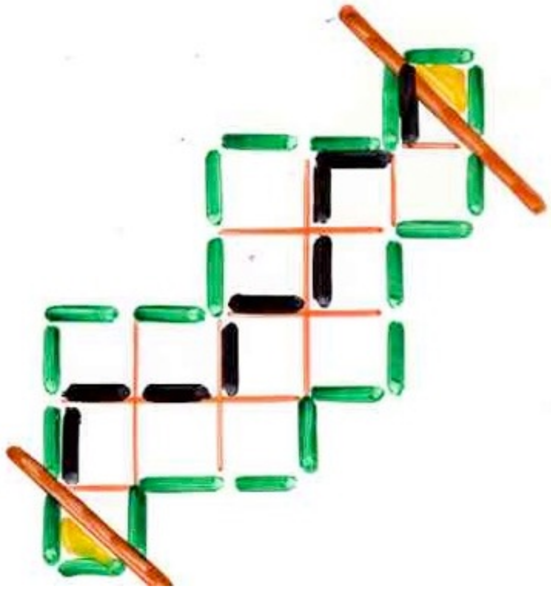

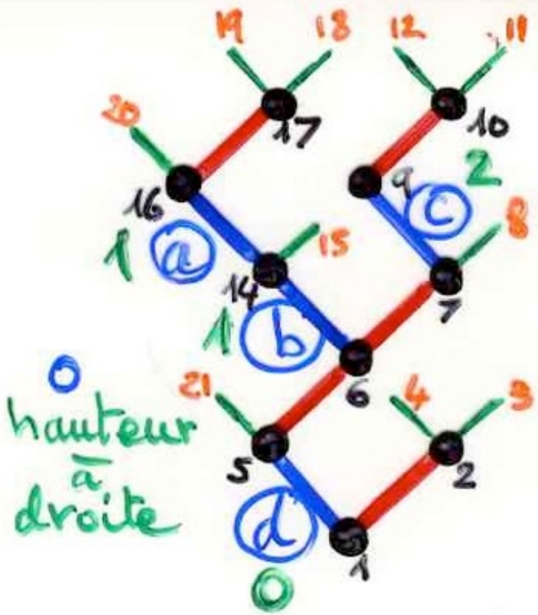

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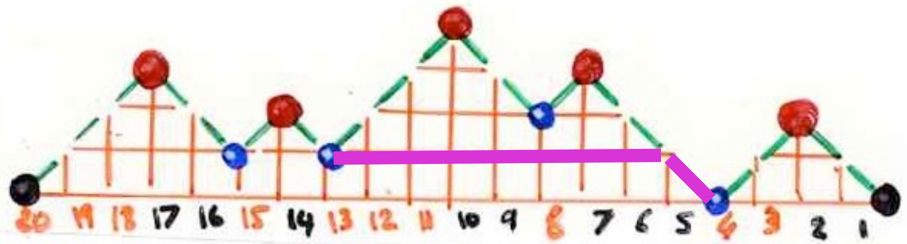

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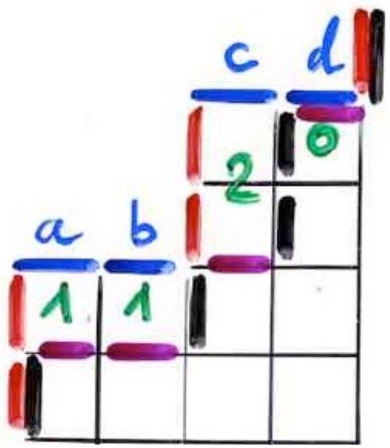

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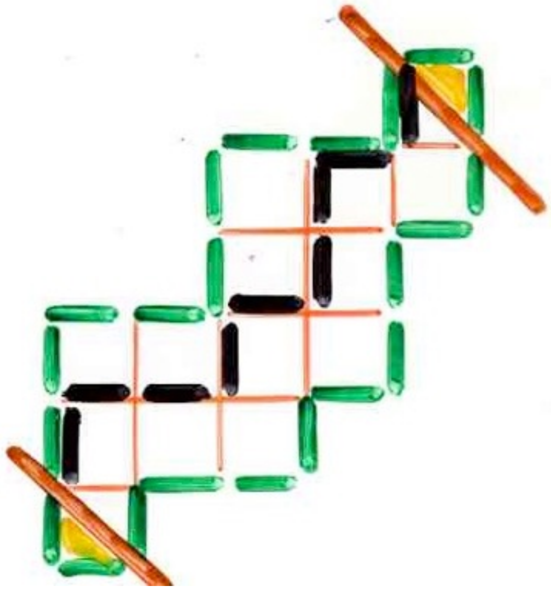

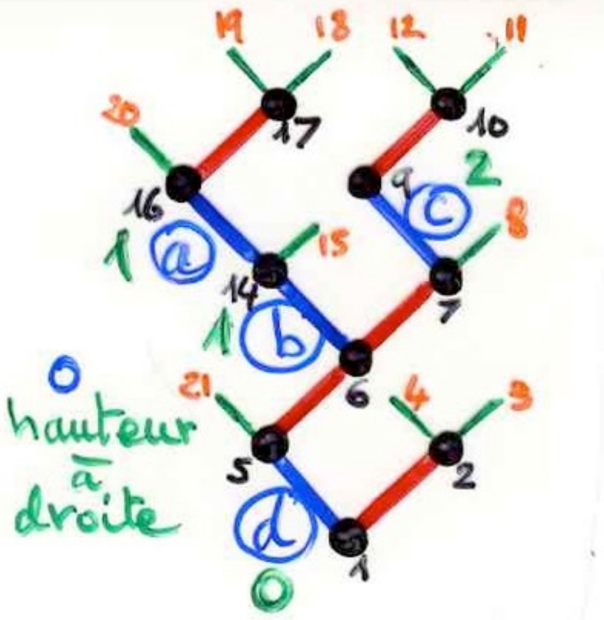
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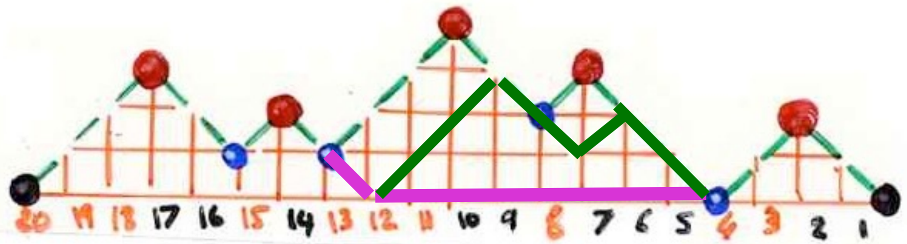
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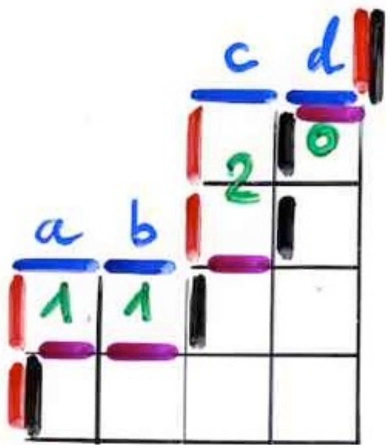
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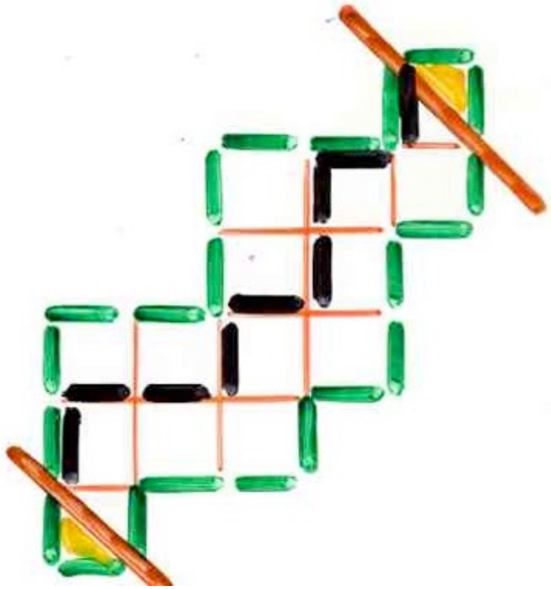

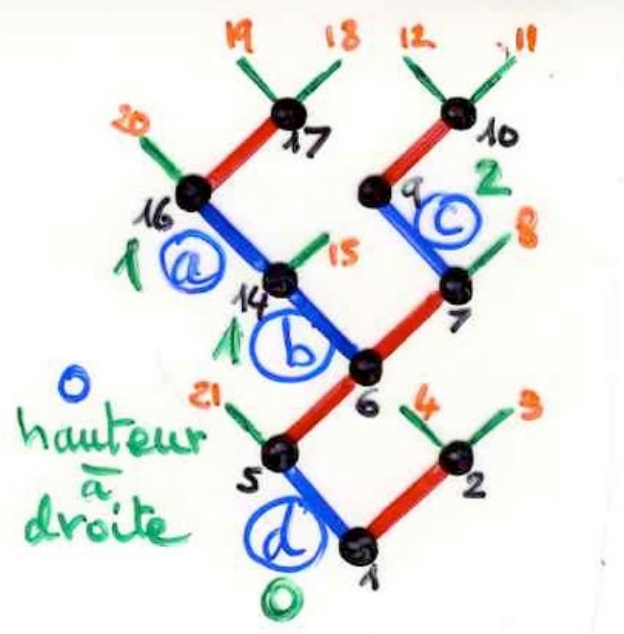

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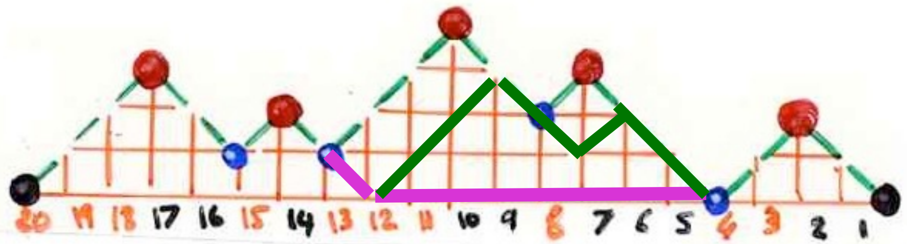

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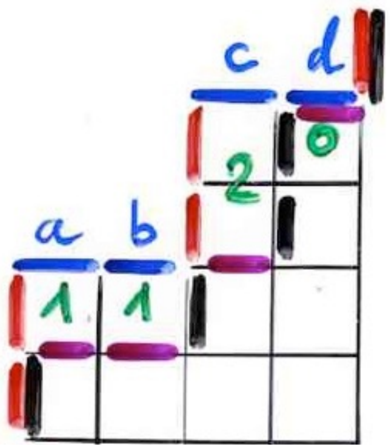

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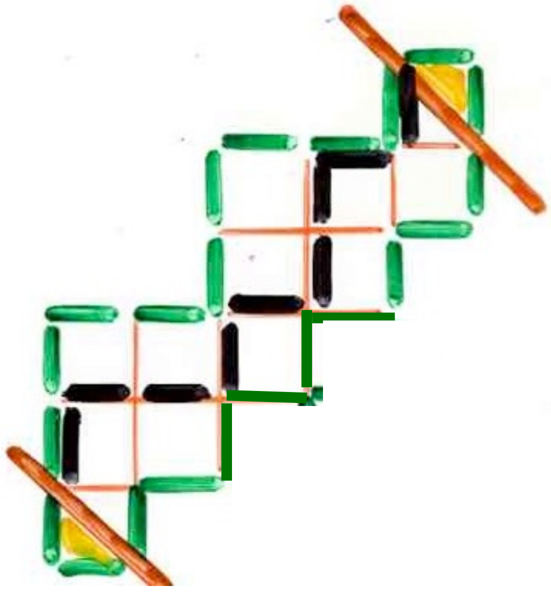

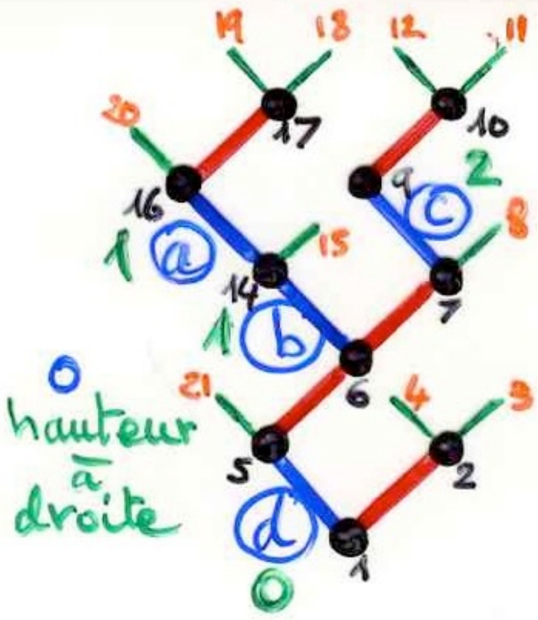
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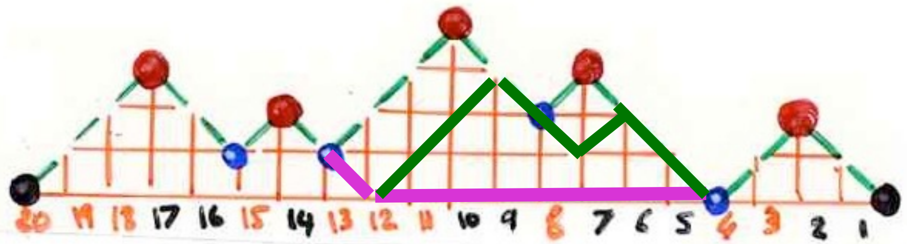
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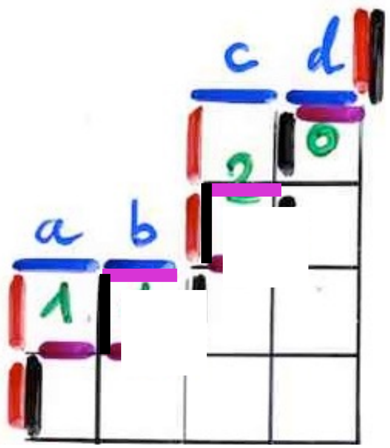
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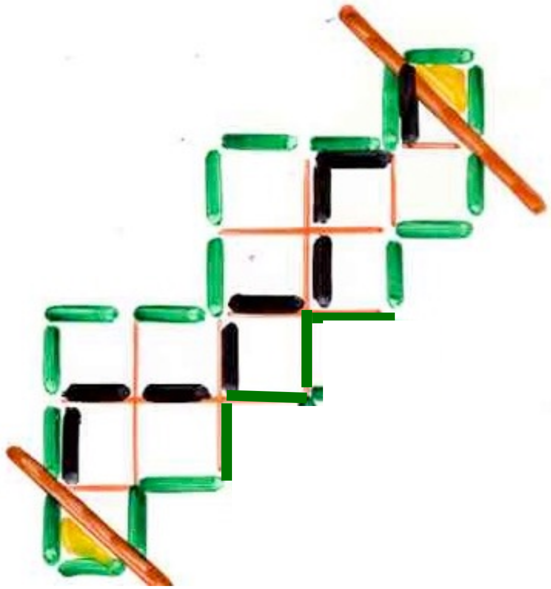

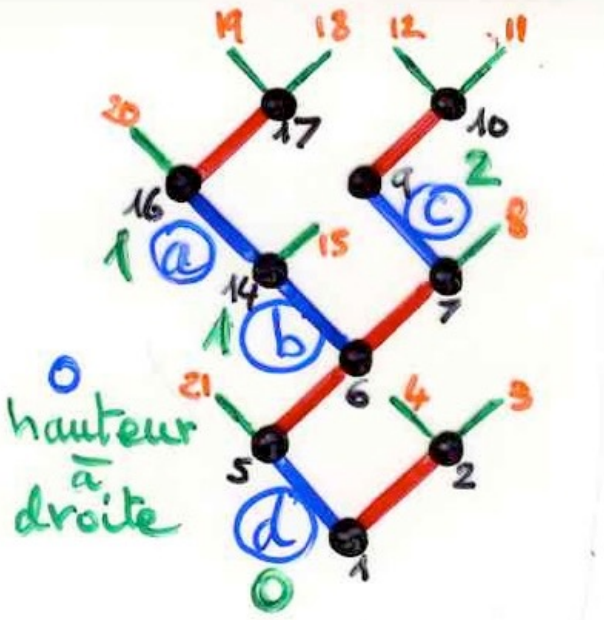

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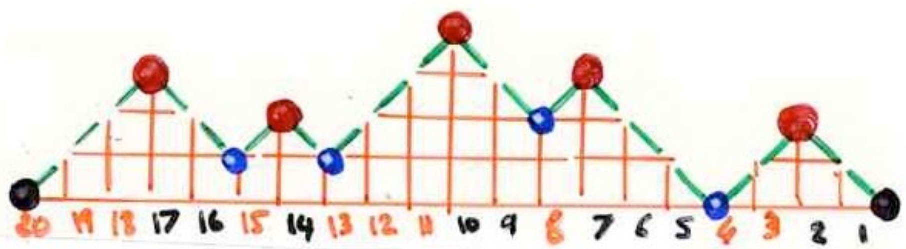

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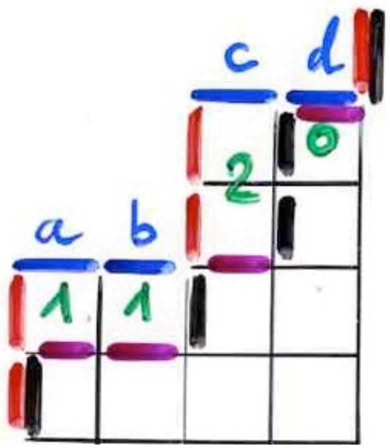

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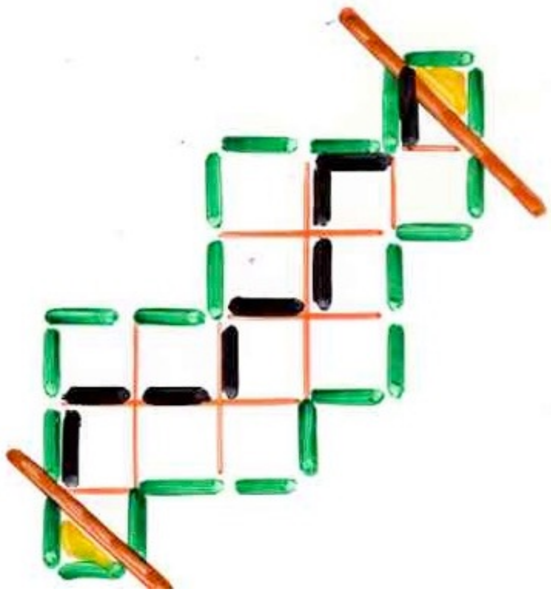
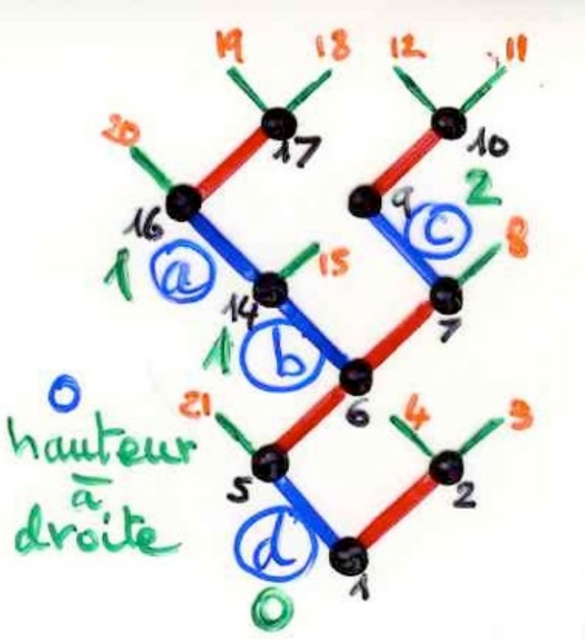
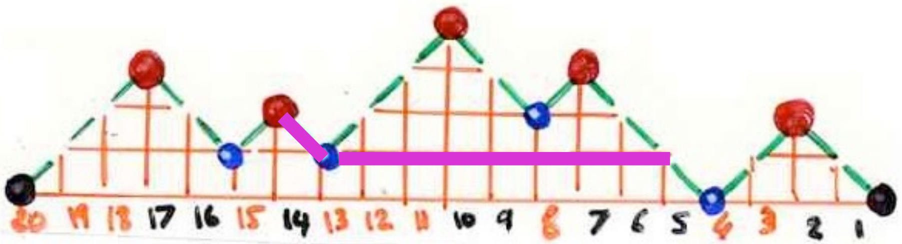
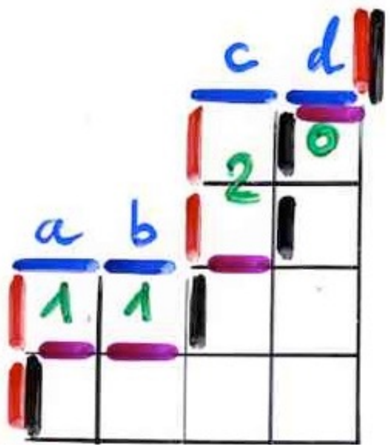
a b
1 1

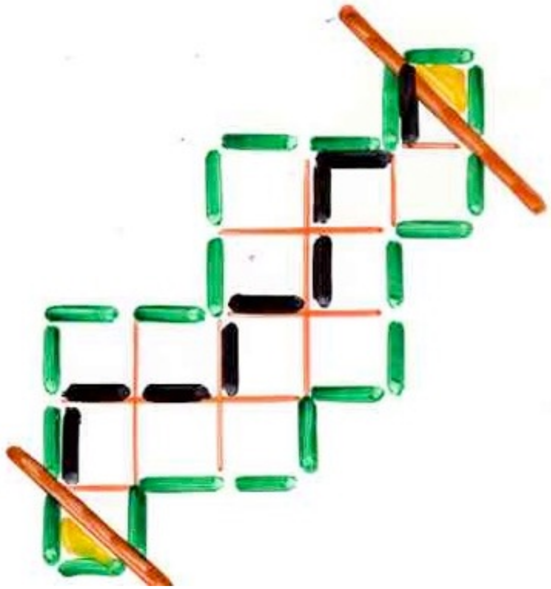


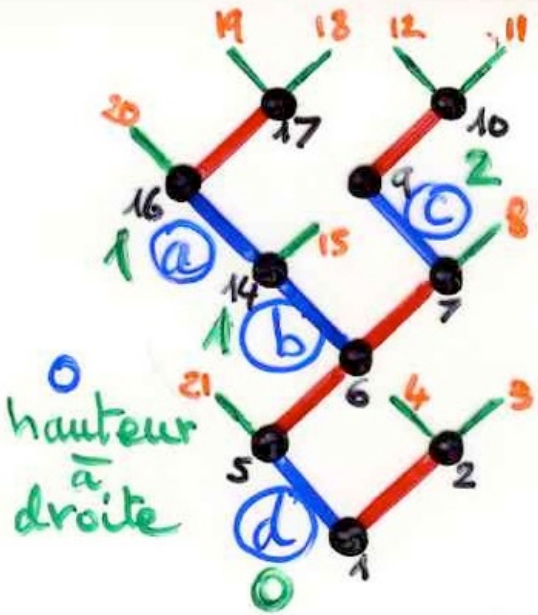
forbidden move

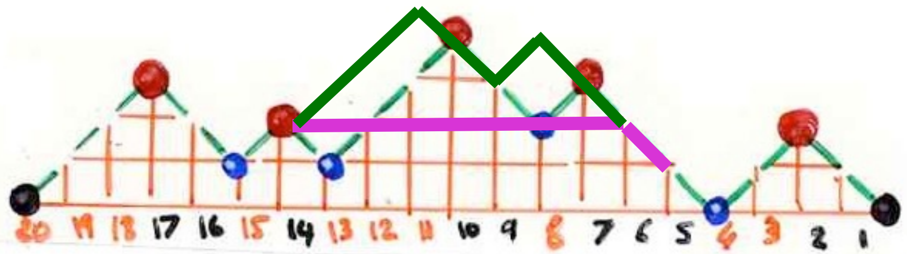


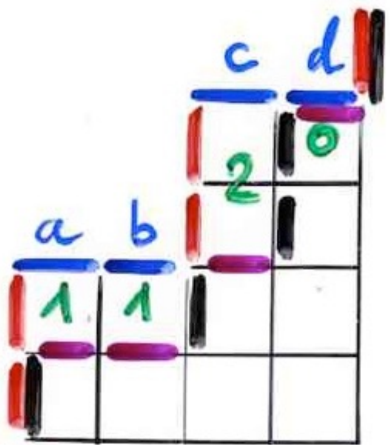


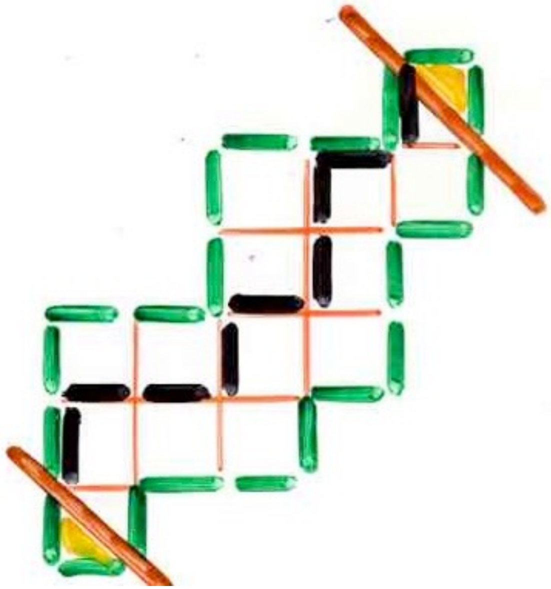




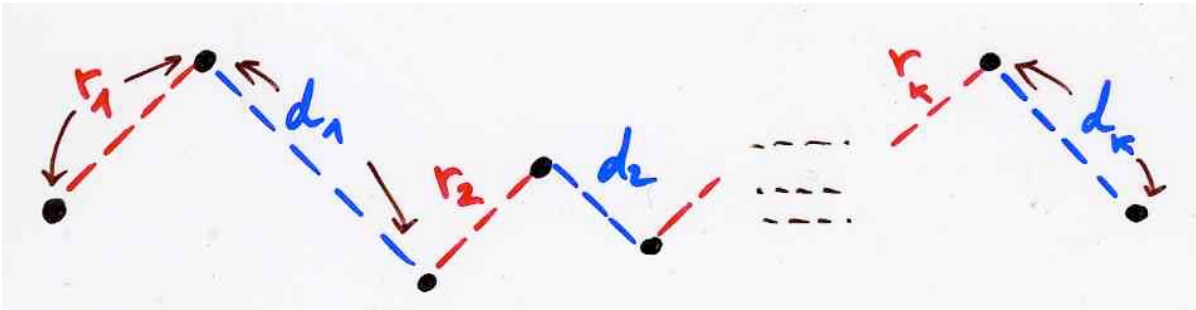

hauteur
à
droite


20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

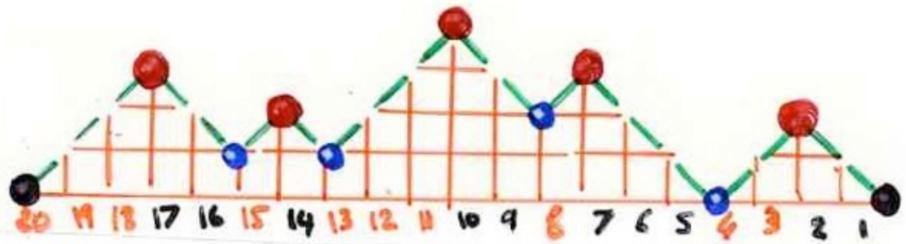

a b c d



the min and max binary trees



r_1
d_1
r_2
d_2
r_k
d_k



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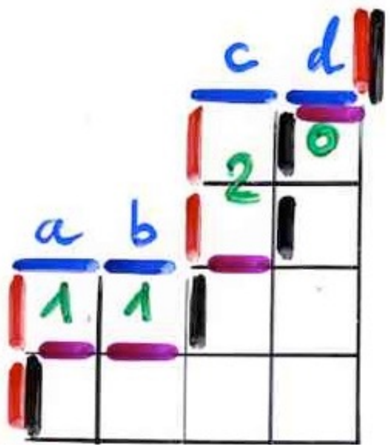
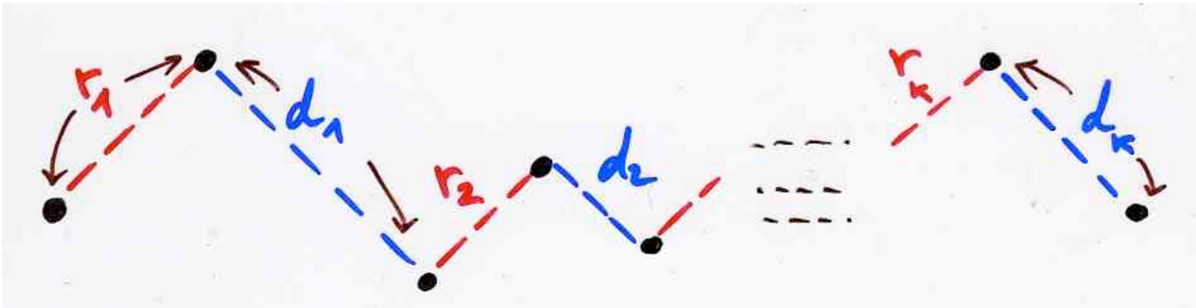
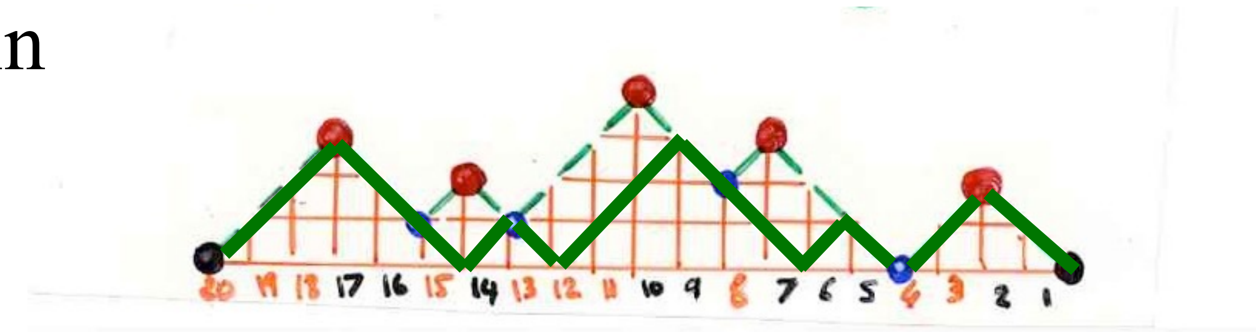


a b c d
1 1 2 0


r1
d1
r2
d2
rk
dk

The min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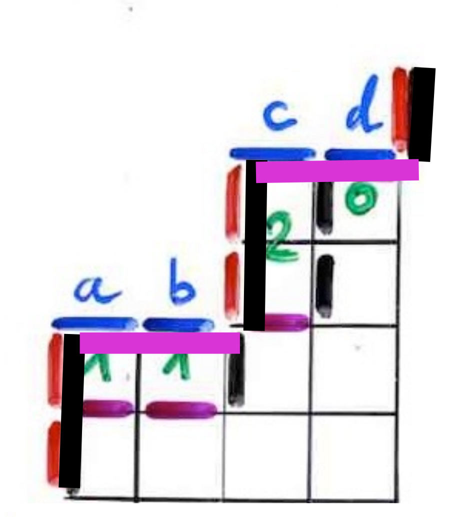
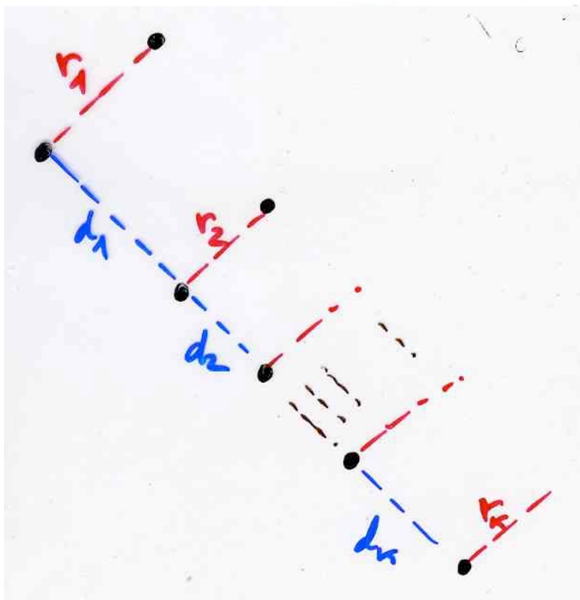
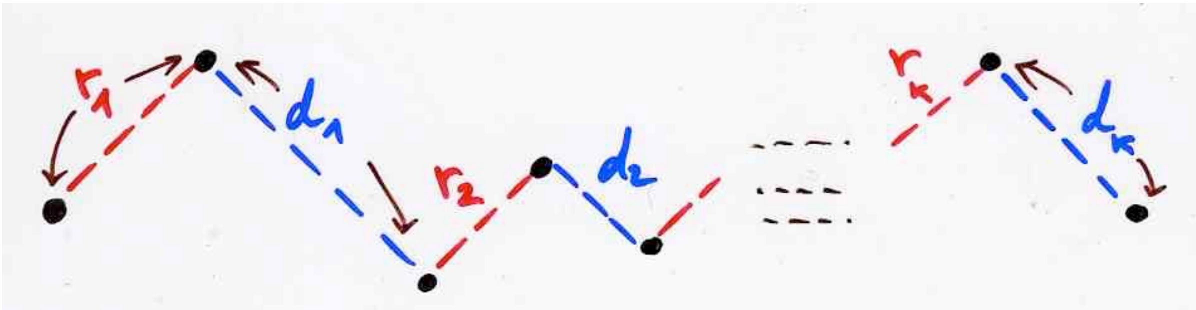
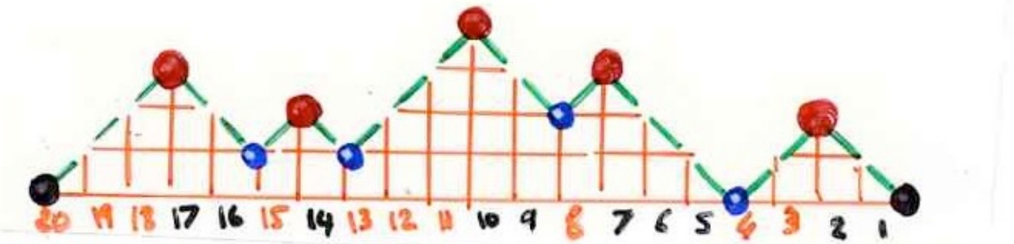

a b
c d
1 1
2
0
1


r1
d1
r2
d2
dk
rk



r_1
d_1
r_2
d_2
r_k
d_k



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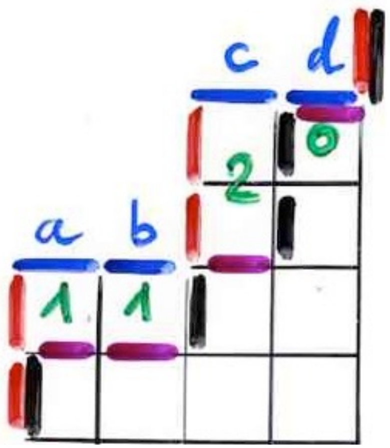
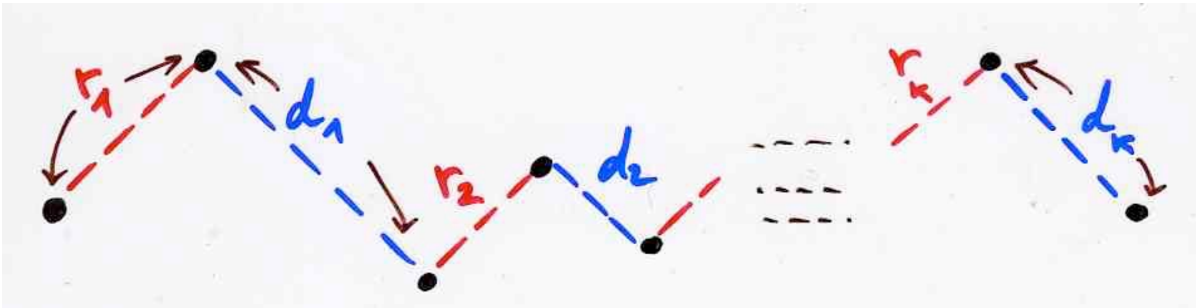
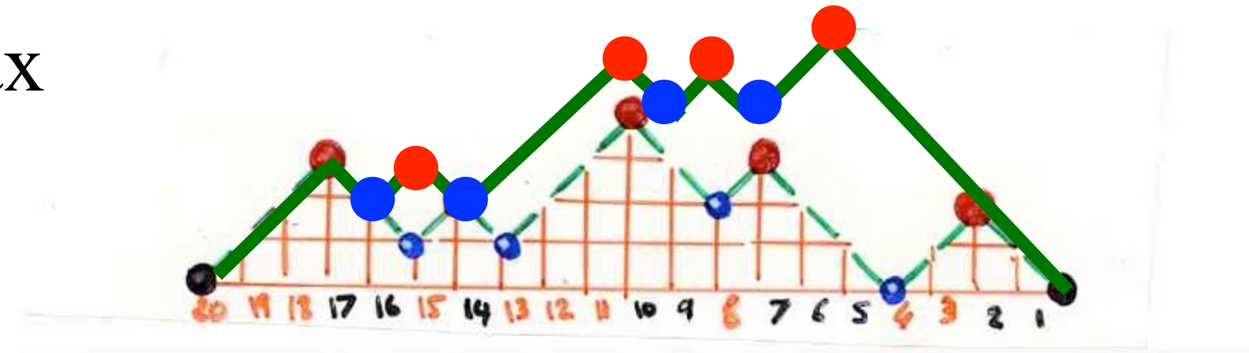


a b c d
1 1 2 0

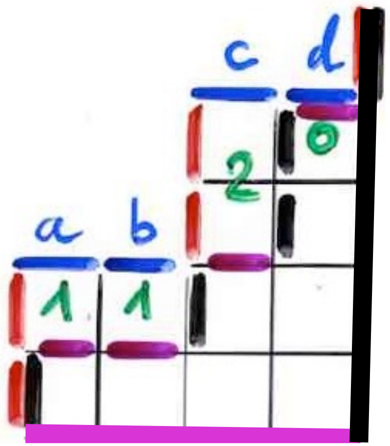


r_1
d_1
r_2
d_2
r_k
d_k

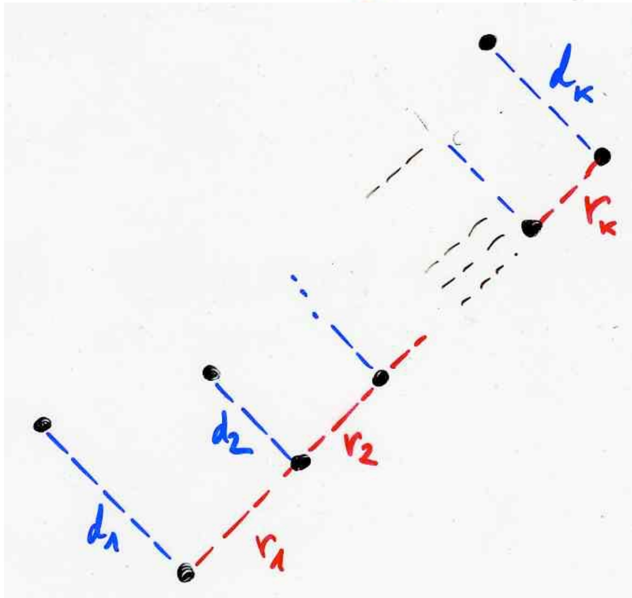
The max



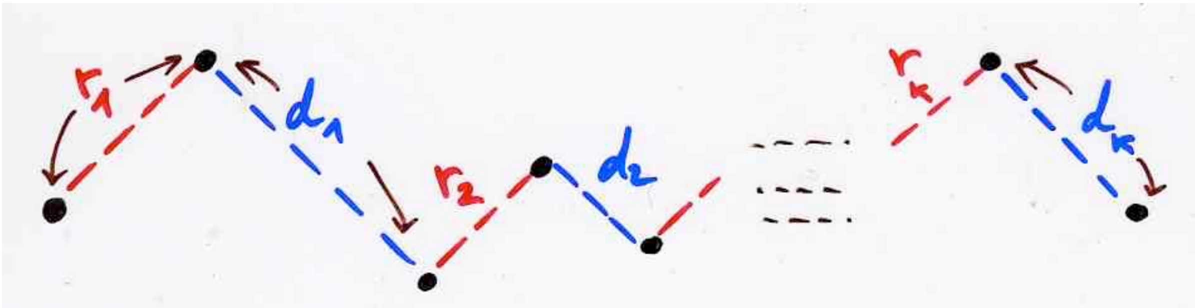
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1



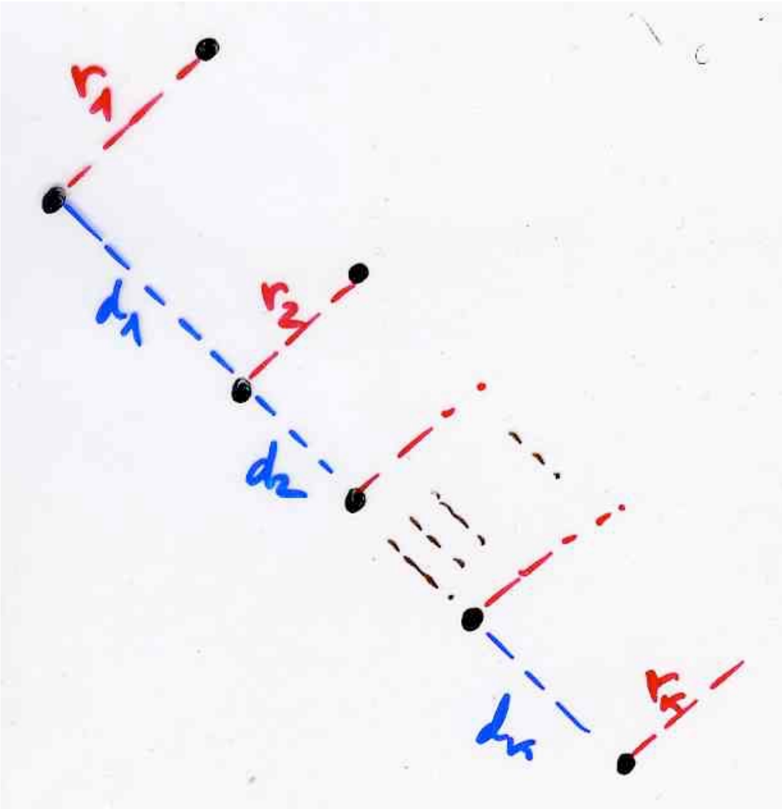
a
b
c
d
1
1
2
0



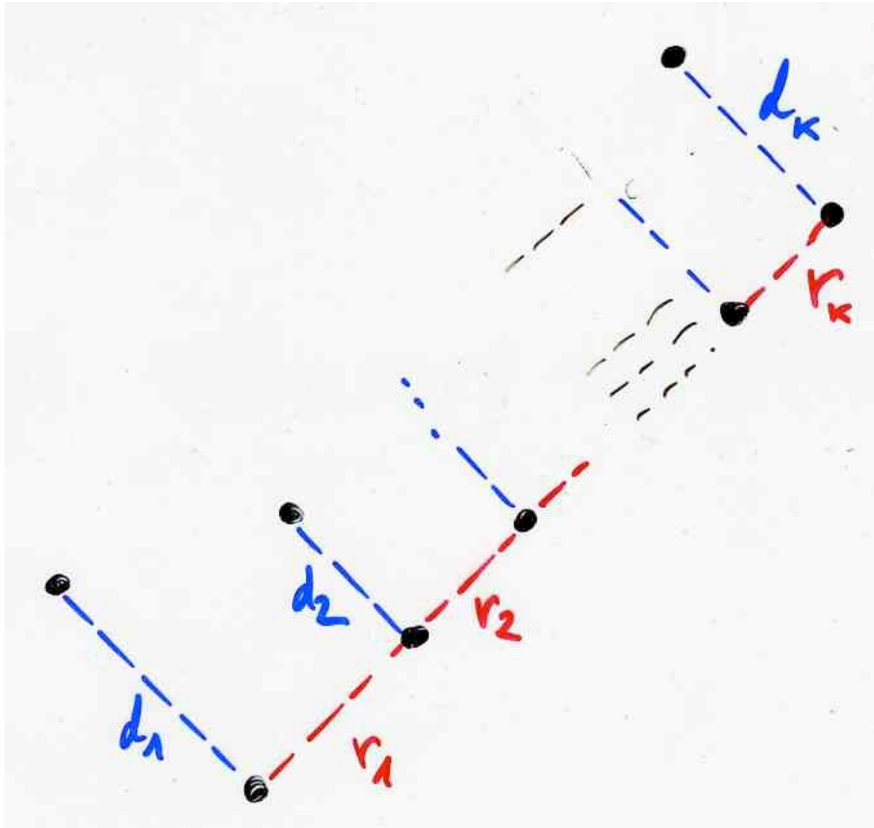
d_k
r_k
d_2
r_2
d_1
r_1



r1
d1
r2
d2
rk
dk

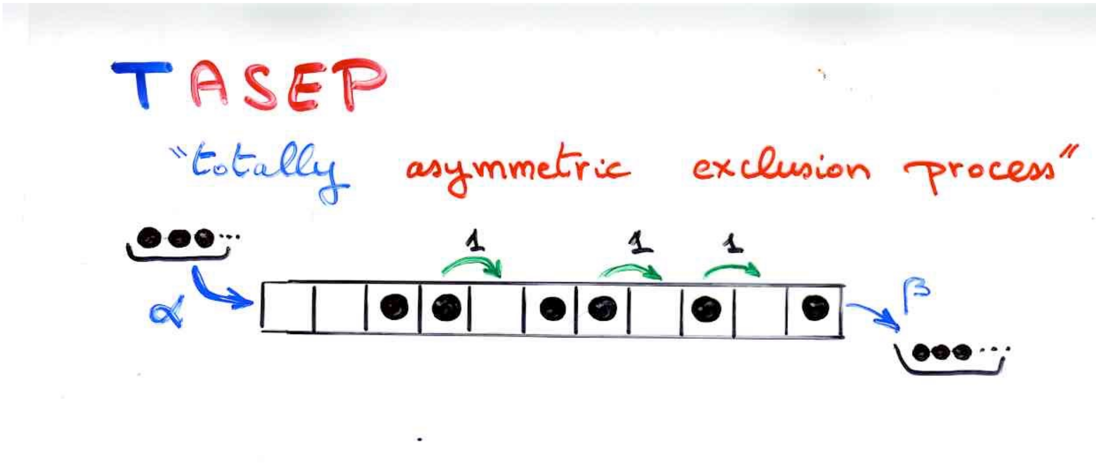


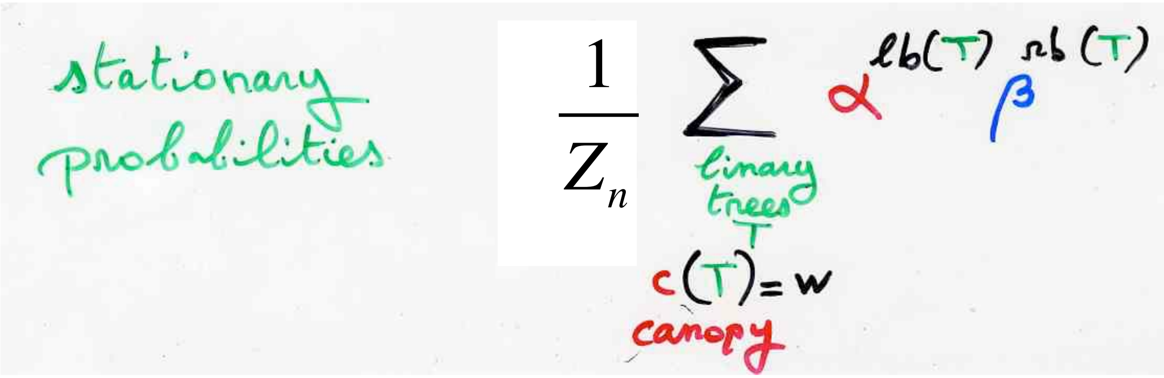
r1
d1
r2
d2
dk
rk



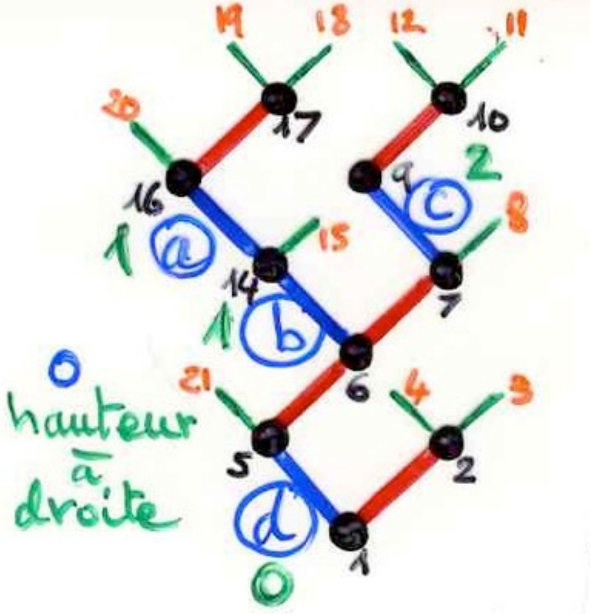
dk
rk
d2
r2
d1
r1

exclusion model in physics:
the TASEP

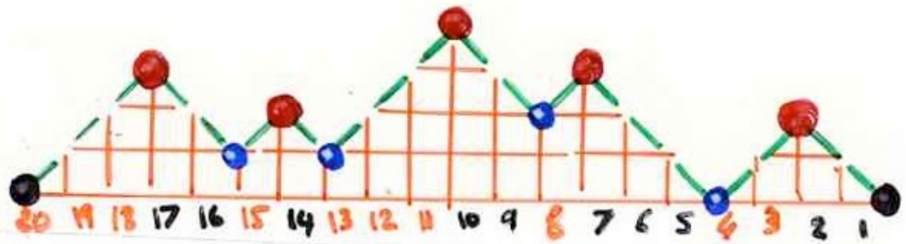




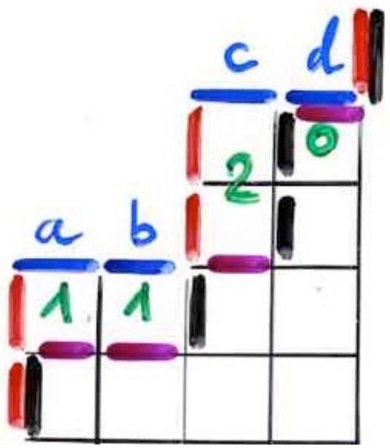
$$Z_n = \sum_{i=1}^{n} \frac{i}{2n-i} \binom{2n-i}{n} \frac{\alpha^{-(i+1)} - \beta^{-(i+1)}}{\alpha^{-1} - \beta^{-1}}$$



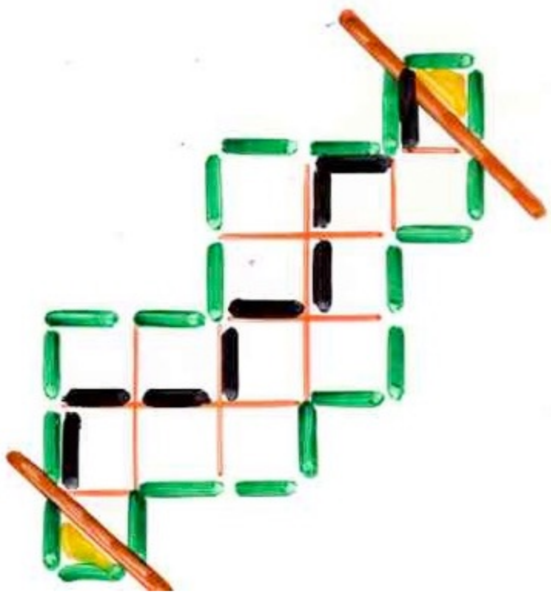
19
18
12
11
20
17
10
16
9
2
a
c
1
15
8
14
7
b
1
0
21
6
4
3
hauteur
à
droite
5
2
d
1
0



20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1



c
d
0
2
a
b
1
1



Kreweras's determinant

Narayana (1955)

$$\det \binom{\lambda_i + 1}{j - i + 1}_{1 \le i, j \le k}$$

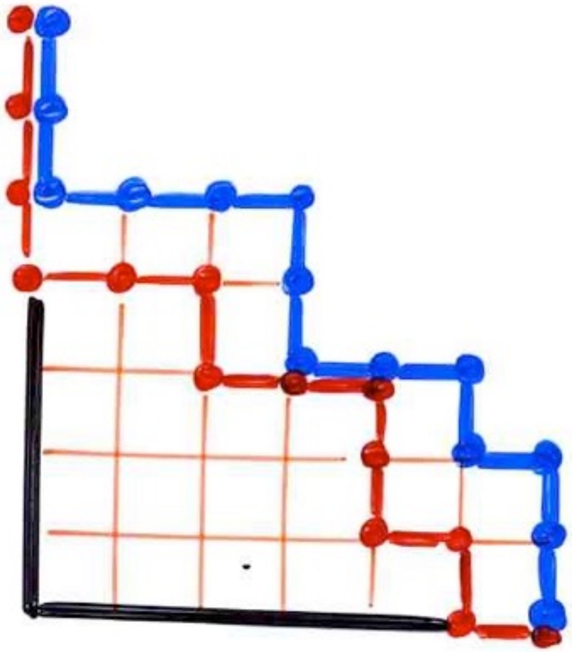
k = nb of 0's in s

λ_i = nb of 1's to the left of the i^{th} zero

déterminant

Kreweras

Narayana



$\lambda = (0, 0, 3, 3, 5, 6, 6)$

Olya Mandelstam

(2013)

(α, β) - analog of Narayana's determinant

TASEP with 2 parameters

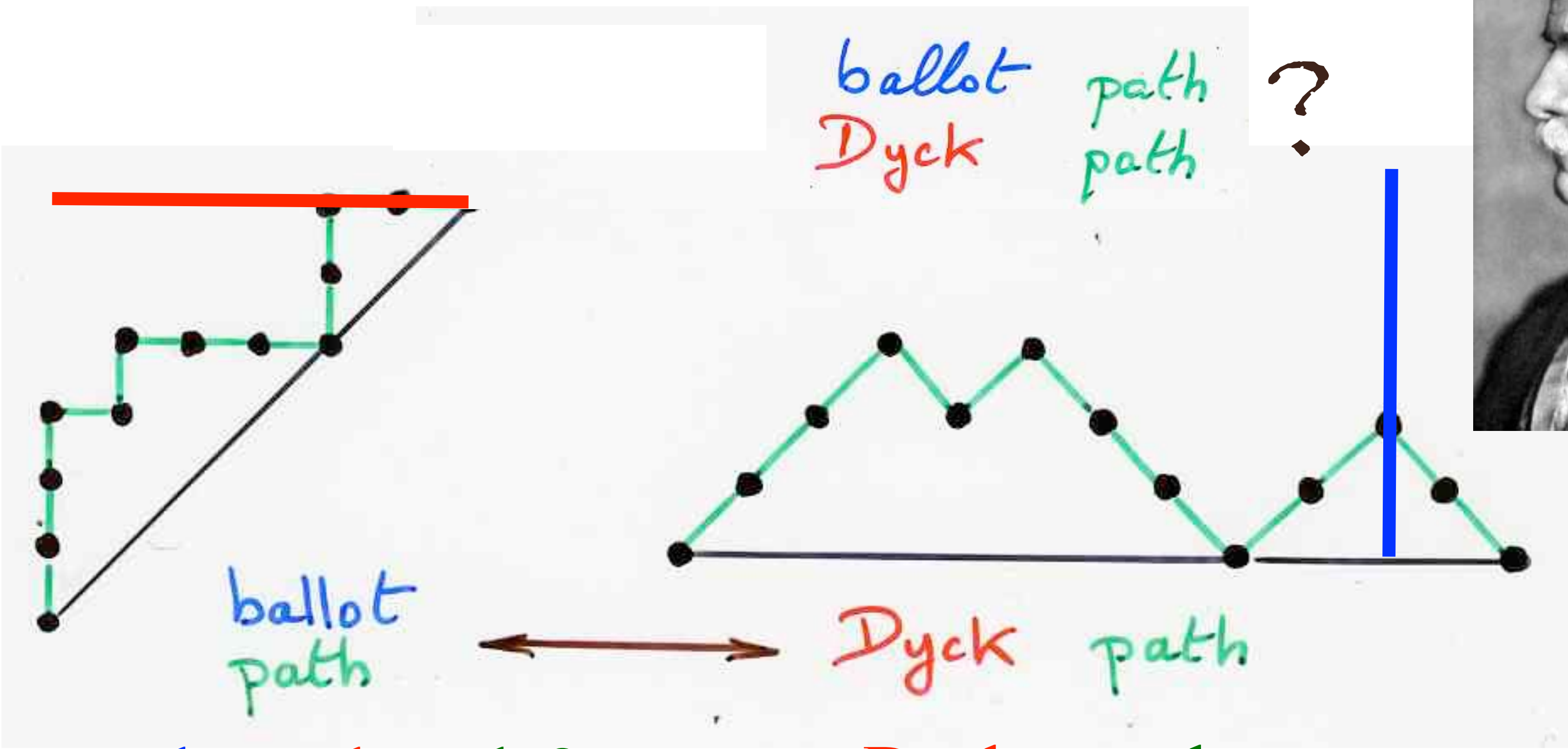
$$\mathcal{P}_{\{\lambda_1, \dots, \lambda_k\}}(\alpha, \beta) = \det A_\lambda^{\alpha, \beta}$$

$$A_\lambda^{\alpha,\beta} = (A_{i,j})$$

$$A_{i,j} = \begin{cases} 0 & \text{for } j < i-1 \\ 1 & \text{for } j = i-1 \\ \beta^{j-i} \alpha^{\lambda_i - \lambda_{j+1}} \left(\binom{\lambda_j + 1}{j-i} + \binom{\lambda_j + 1}{j-i+1} \right) & \\ + \beta^{j-i} \alpha^{\lambda_i - \lambda_j} \cdot \sum_{\ell=0}^{\lambda_j - \lambda_{j+1} - 1} \alpha^{\ell} \left(\binom{\lambda_j - \ell - 1}{j-i-1} + \binom{\lambda_j - \ell - 1}{j-i} \right) & \text{for } j \geq i \end{cases}$$

Complements

Question of vocabulary



Von Dyck

rotated Dyck path ?

Dyck words

Bertrand
D. André
problème du scrutin
(188x)

ballot numbers

G.Kerweras «bridge»
D.Dumont «contraction»

This work was done at Universidad de Talca, Chile
at the invitation of Luc Lapointe,
and in the various following places

Constitution
Isla Negra
Viña del Mar
Valparaiso

Luc Lapointe

Universidad de Talca

La Universidad de Los mejores

www.utalca.cl

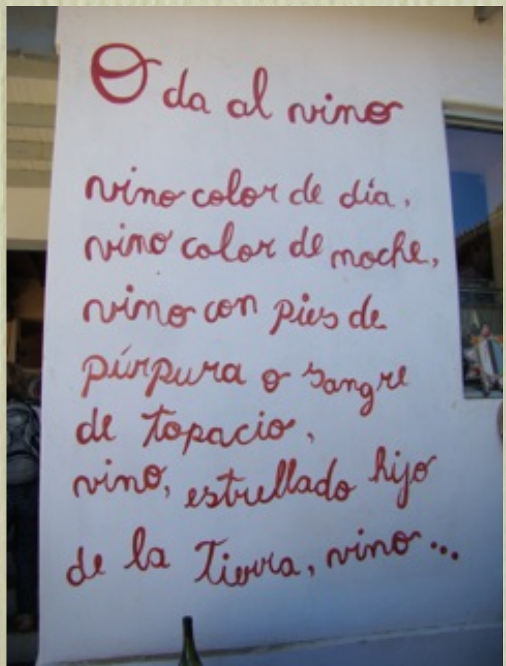
From Talca to Constitution

Maule valley

workshop in Valparaiso

Viña del Mar
(close to Valaparaiso)

Isla Negra
Pablo Neruda



¡ muchas gracias !