Tamari lattice and its extensions

(1)

IMSc, Chennai
24 February 2015

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LaBRI, CNRS, Bordeaux

Dov Tamari (1951) thèse Sorbone
"Monoïdes préordonnés et chaînes de Malcev"

((a,(b, c)),(d, e))
(a (b c))(d e)

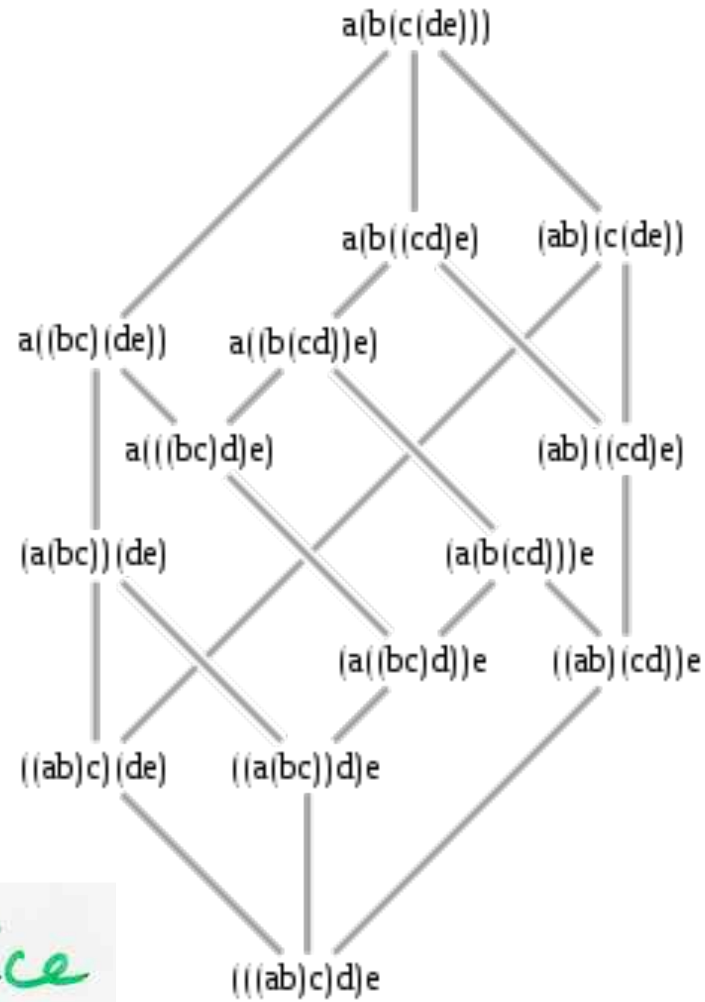
well parenthesis expression

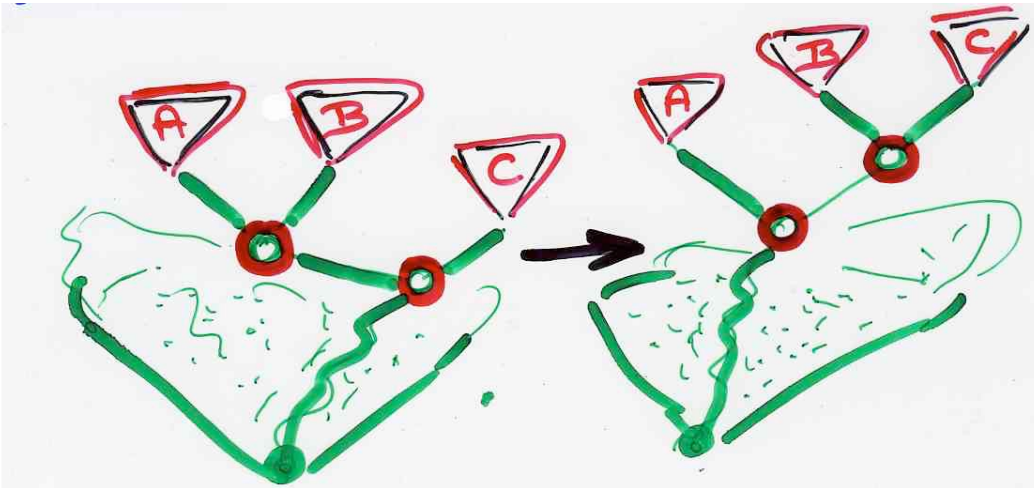
associativity

...((u v) w)....
....(u (v w))---

order relation

Tamari lattice



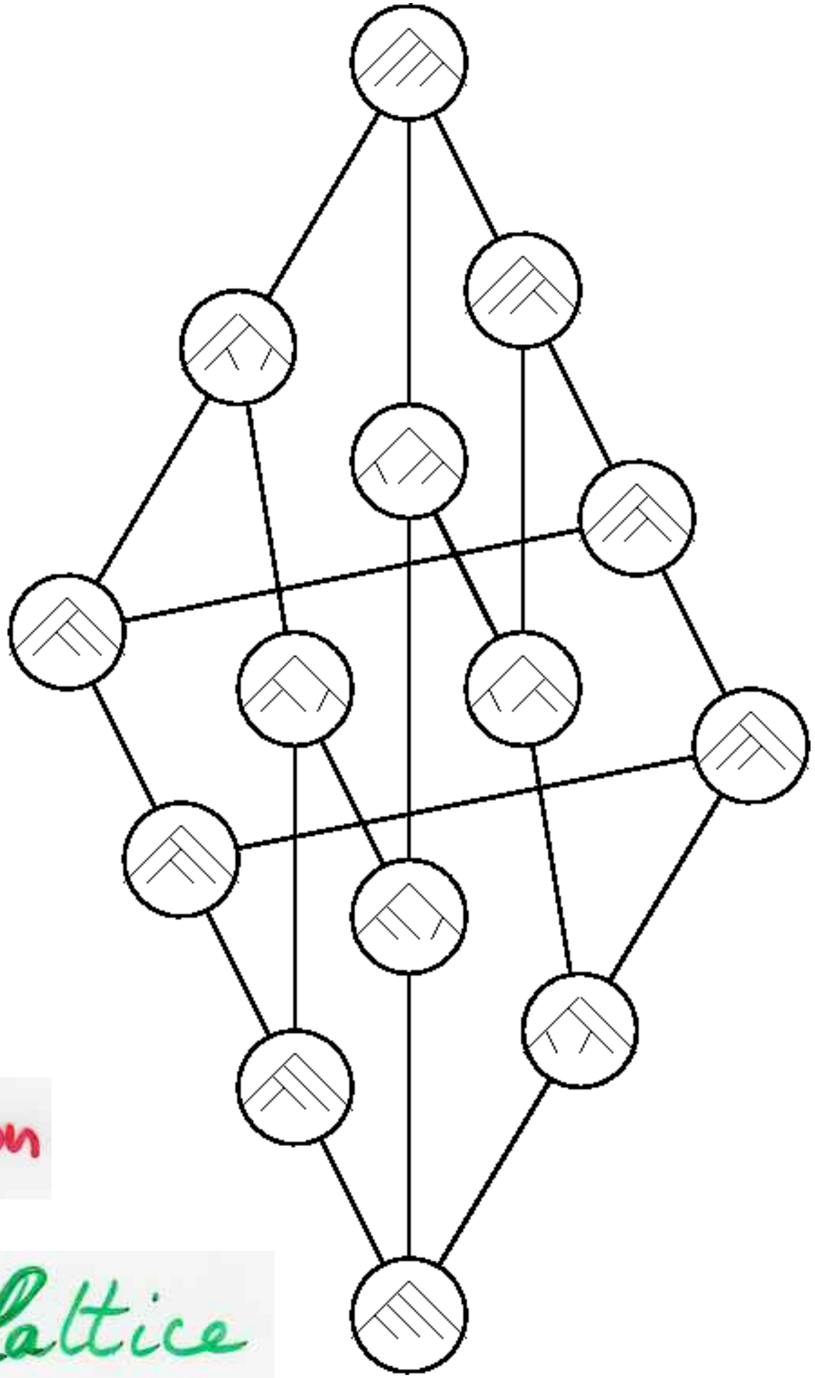


... ((u v) w) ....

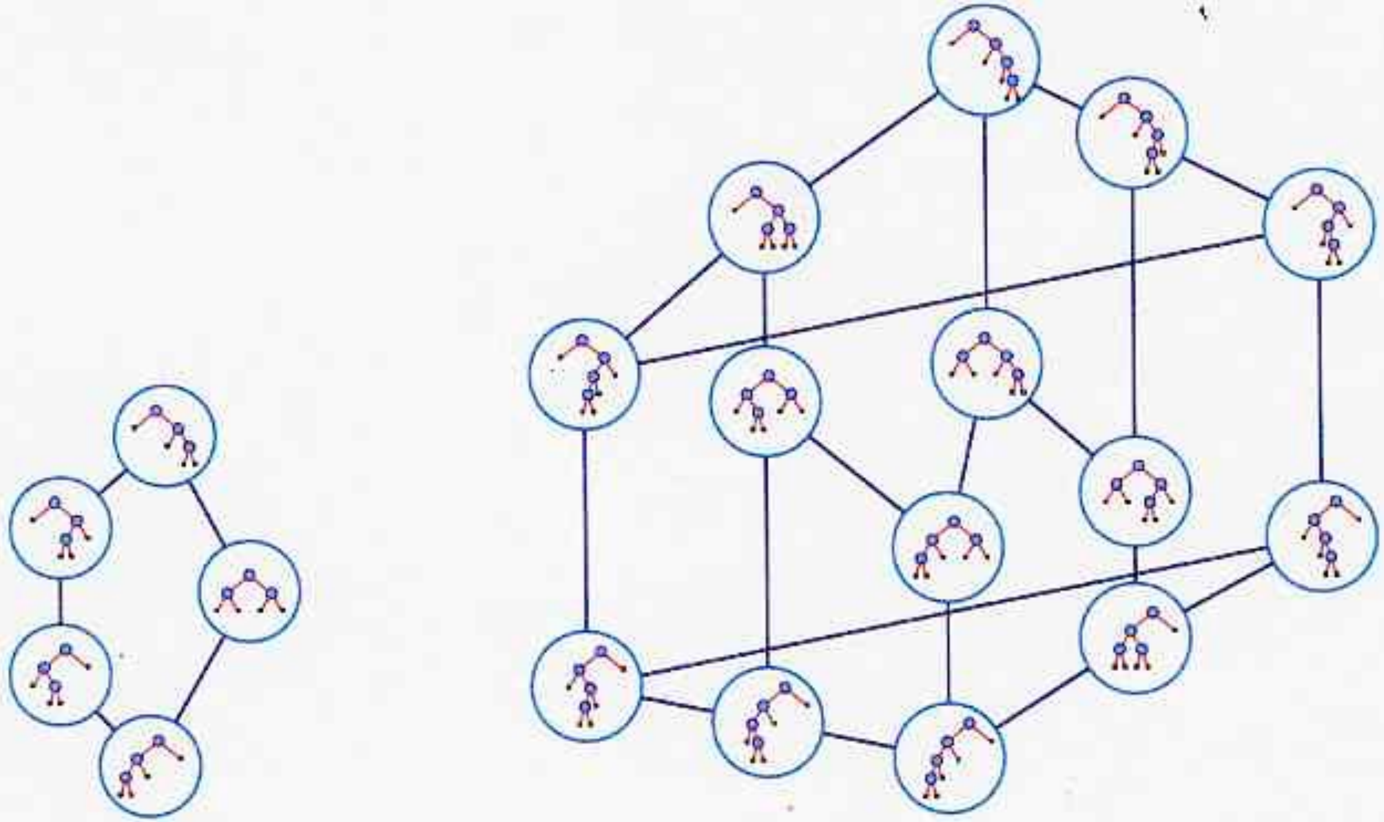
.... (u (v w)) ...

associativity

order relation



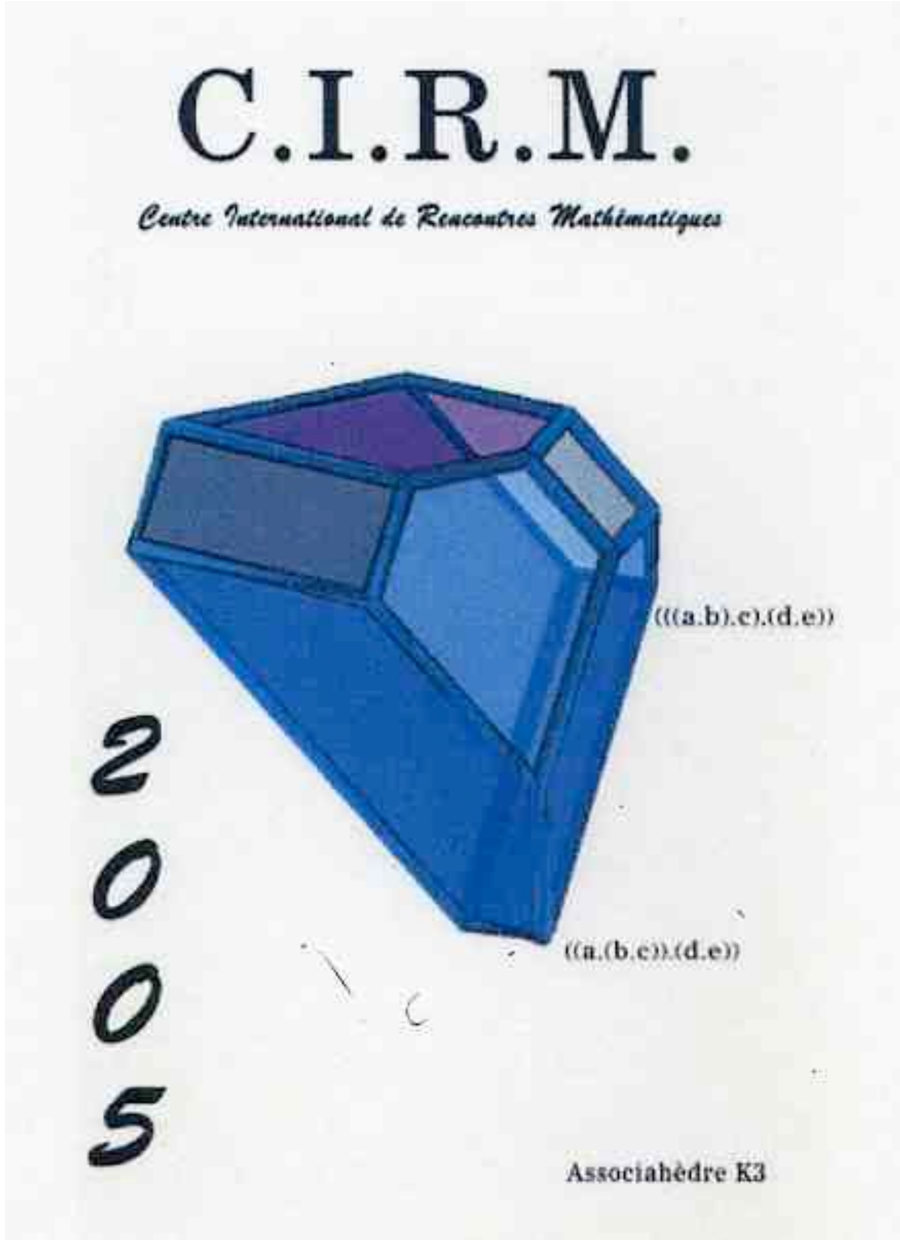
Tamari lattice



(a) $\mathbb{T}_3$ (b) $\mathbb{T}_4$

Fig. 3: The Tamari lattices $\mathbb{T}_3$ and $\mathbb{T}_4$.

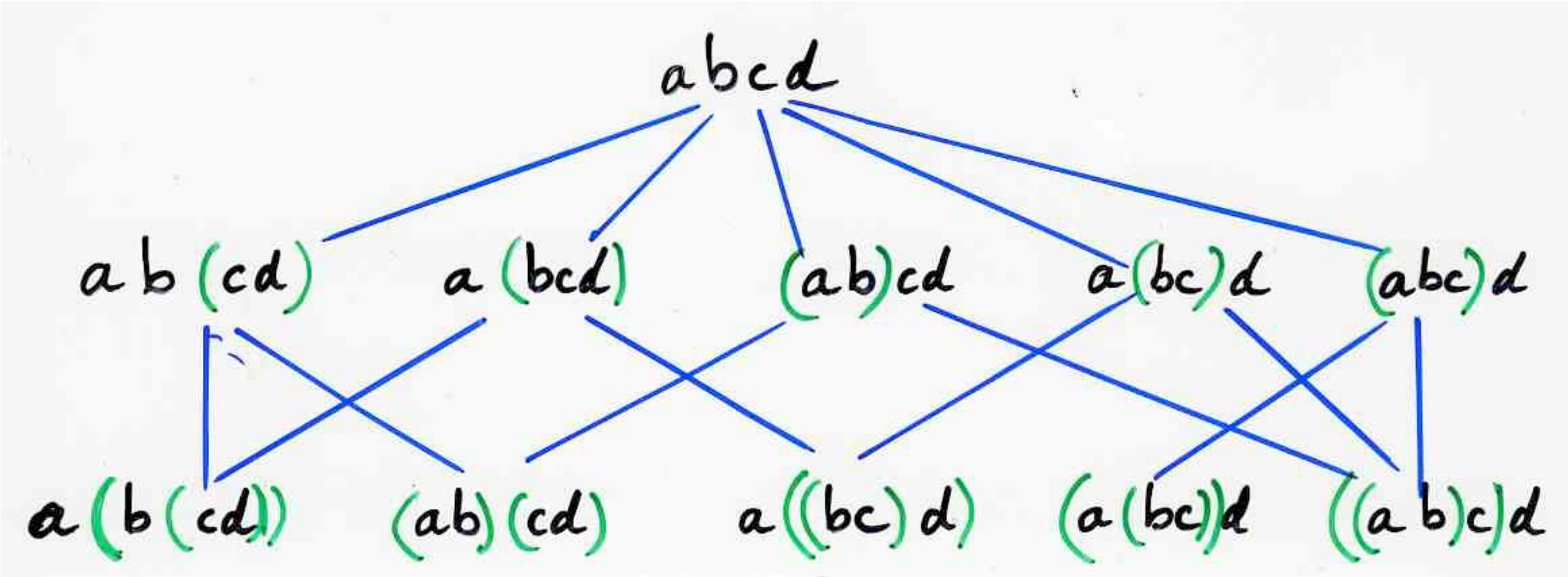
associahedron

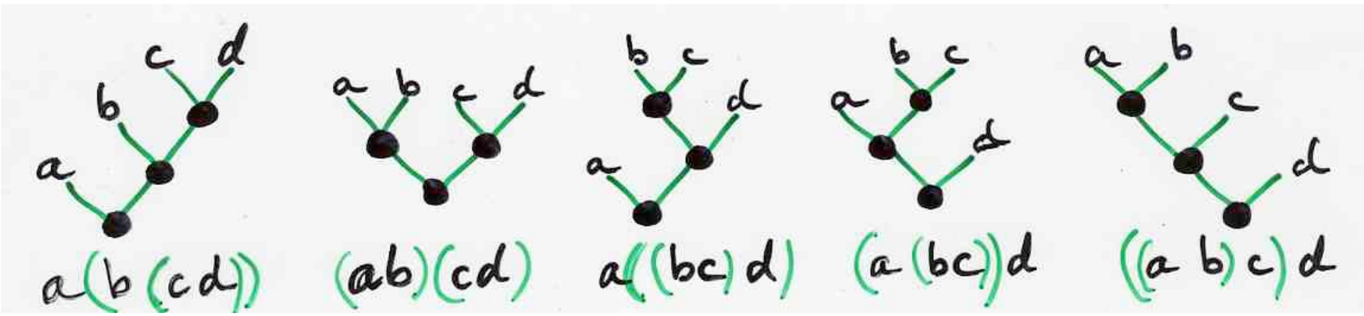


cells structure

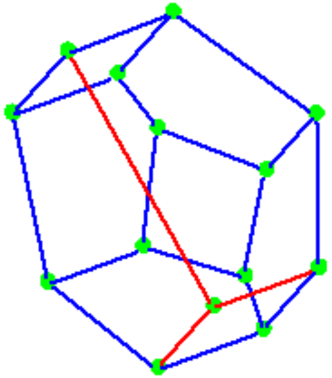
simplicial complex

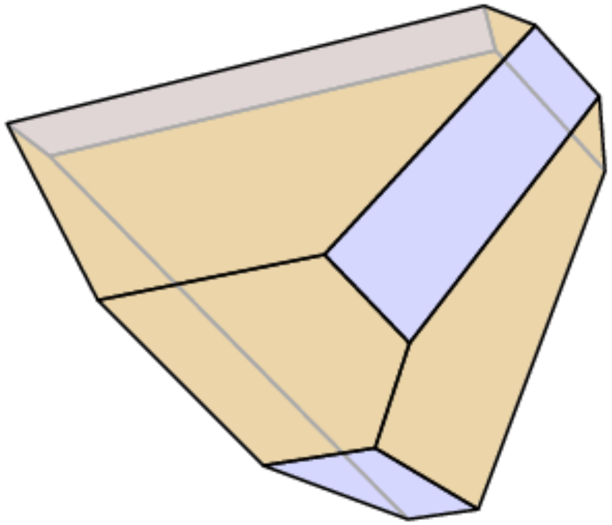
polytope

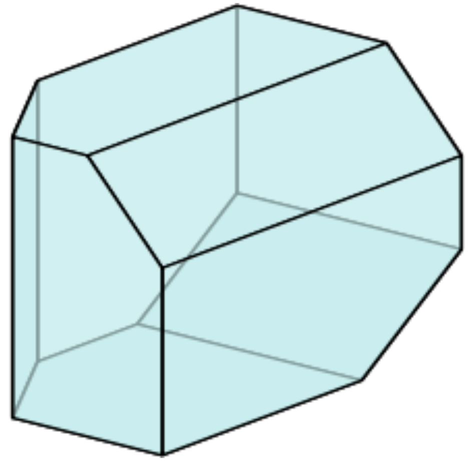




Is it possible to realize the cells structure of the associahedron as the cells of a convex polytope ?







S. Huang, D. Tamari JCTA (1972)
"a simple proof of the lattice property"

D. Huguet, D. Tamari (1978)
face-graph of a polytope ? no proof

M. Haiman (1984) manuscript

C. Lee (1989) first published proof

I. Gelfand, M. Kapranov, A. Zelevinsky (1994)

"secondary polytope" ← fiber polytope

Constructing the Associahedron

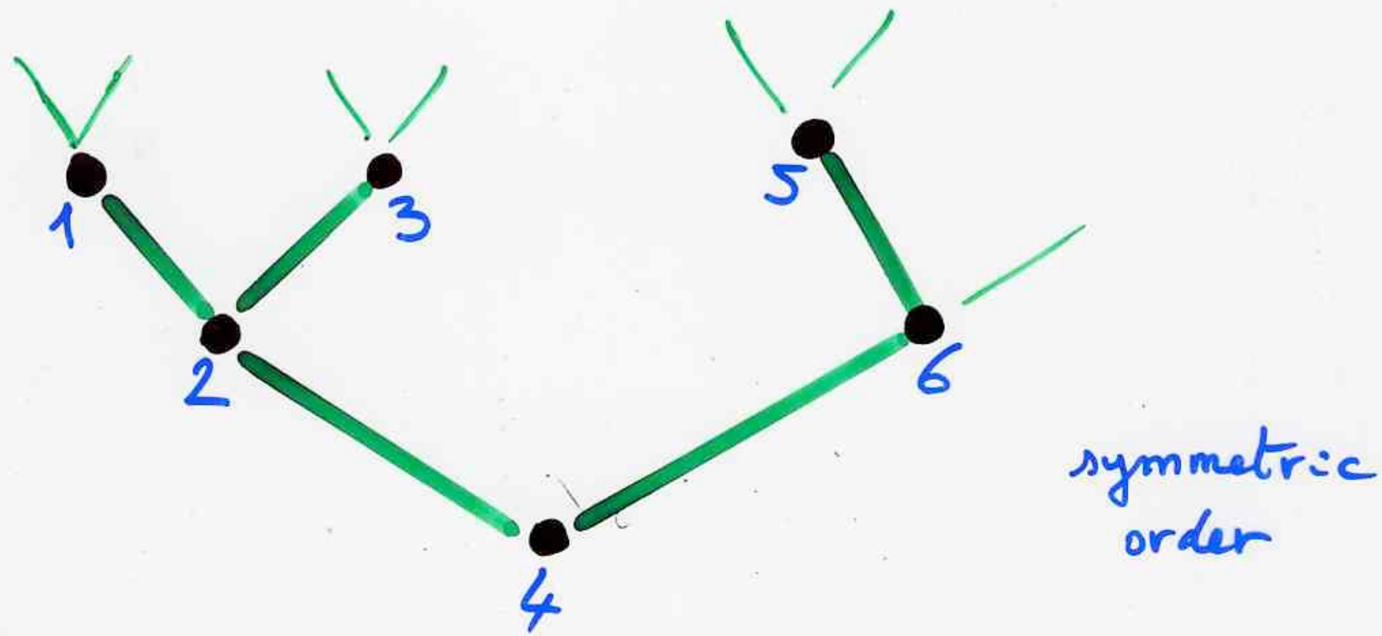
Mark Haiman

The associahedron is a mythical polytope whose face structure represents the lattice of partial parenthesizations of a sequence of variables, in a way to be made precise below. The purpose of these notes is to give an explicit construction of such a polytope.

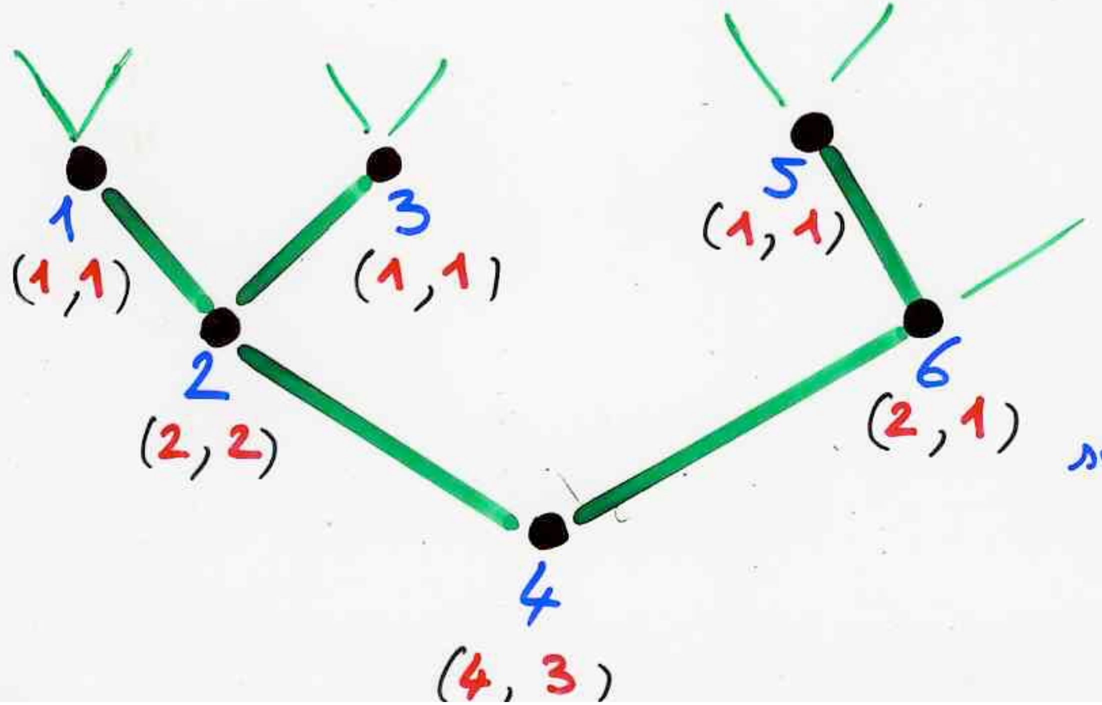
Let $x_1, \ldots, x_n$ be variables. A bracket is a consecutive subsequence of the x_i. Denote the bracket $\{x_i, \ldots, x_j\}$ by $[i, j]$. A grouping G is a set of

J.-L. Loday (2004) arXiv: déc 2002

"Realization of the Stasheff polytope"



J.-L. Loday (2004) arXiv: déc 2002
"Realization of the Stasheff polytope"



symmetric order

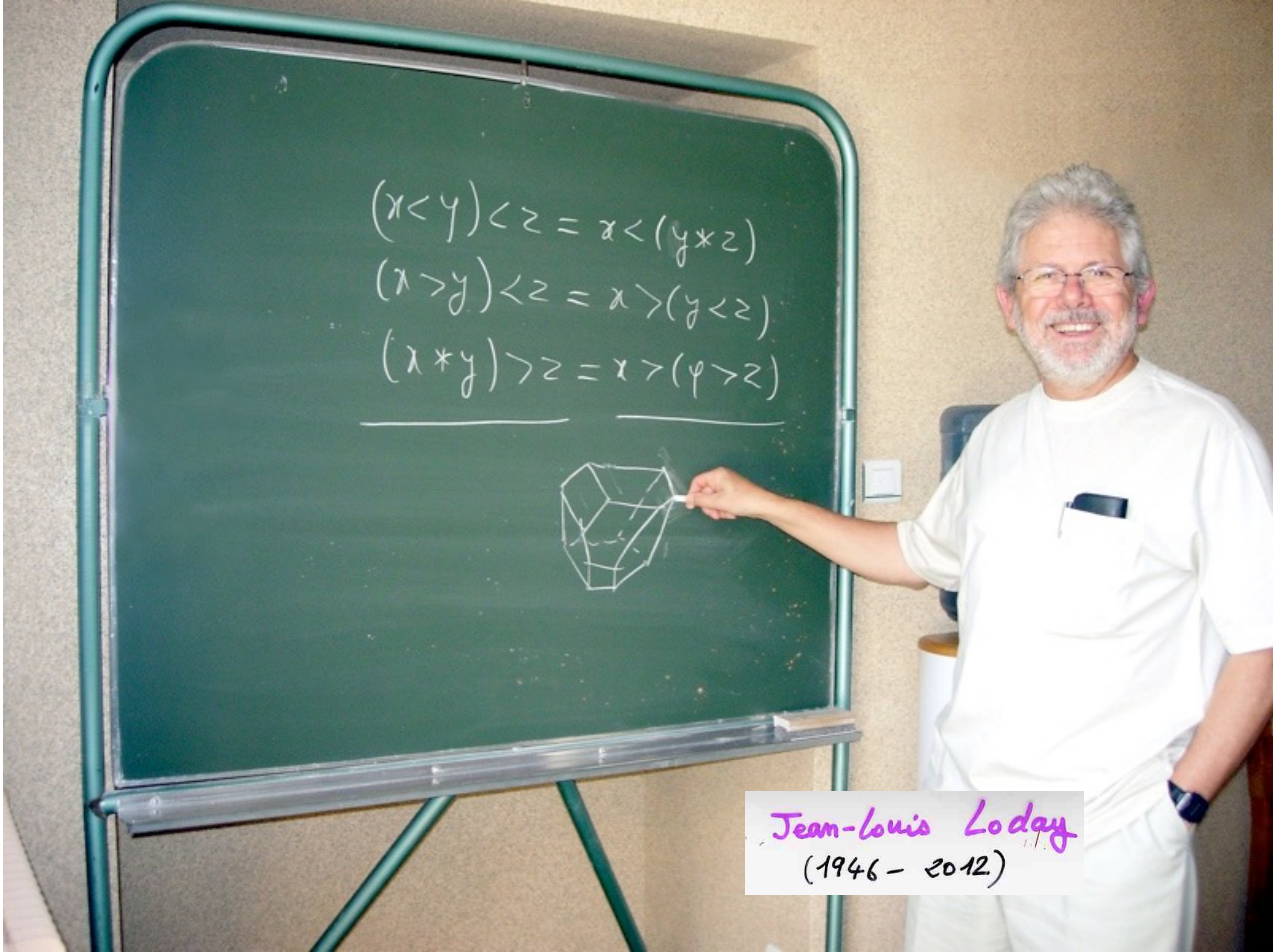
$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ (1, & 4, & 1, & 12, & 1, & 2) \end{array} \xrightarrow[\text{sum}]{} 21$$

$$\frac{n(n+1)}{2}$$

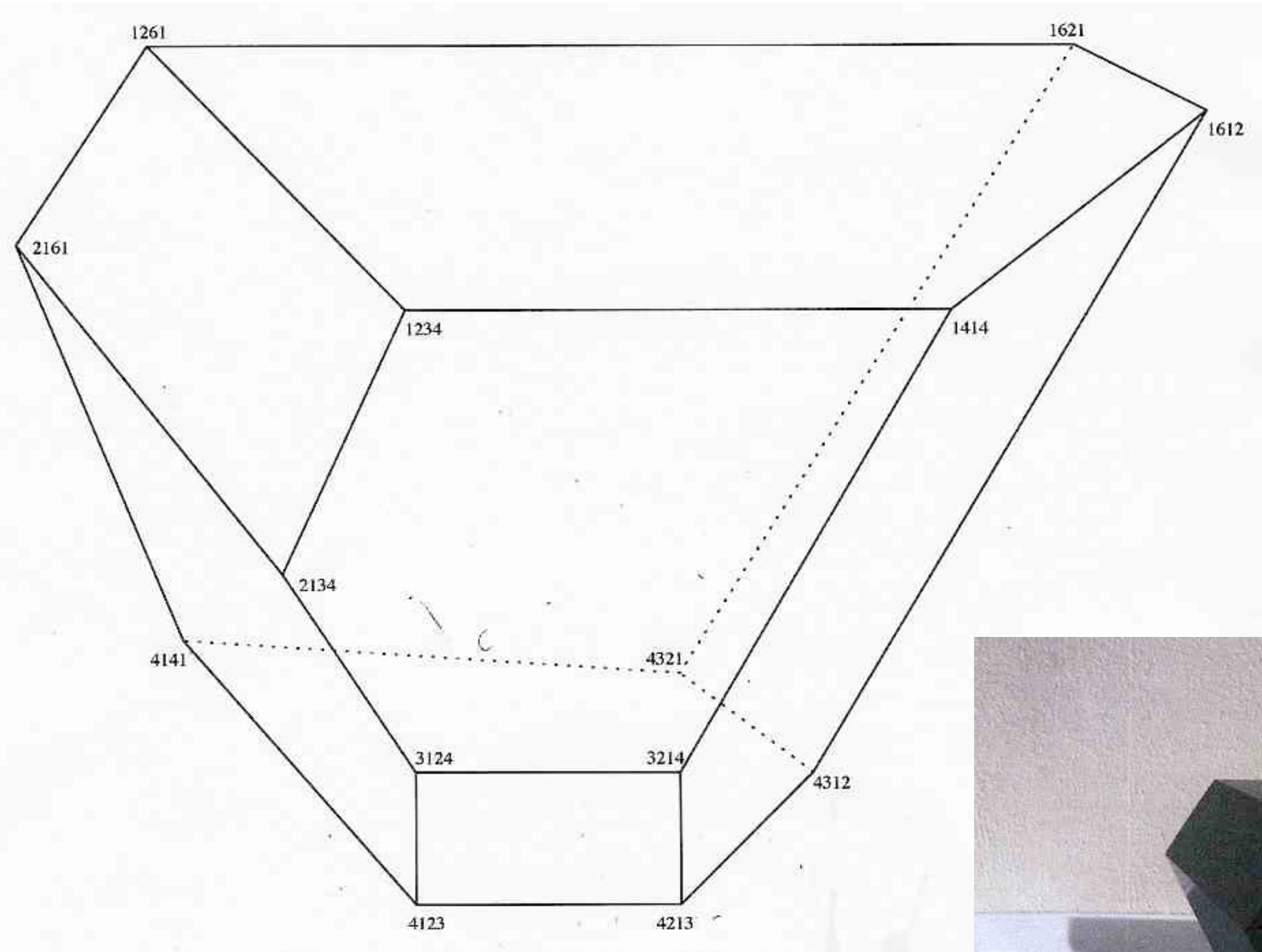
convex hull of the points

hyperplan

$$x_1 + \dots + x_n = \frac{n(n+1)}{2}$$



Jean-Louis Loday
(1946 – 2012)


1261
1621
1612
2161
1234
1414
2134
4141
4321
3124
3214
4312
4123
4213

C. Hohlweg, C. Lange (2007)
F. Chapoton, S. Fomin, A. Zelevinsky (2002)

extensions :

C. Ceballos	J.-P. Labbé	C. Stump
V. Pilaud	N. Bergeron	F. Santos
N. Reading	H. Thomas	A. Postnikov
R. Marsh	M. Reineke	C. Athanasiadis
D. Speyer	S. Stella	G. Ziegler

Gil Kalai

and many others ...

combinatorial structures

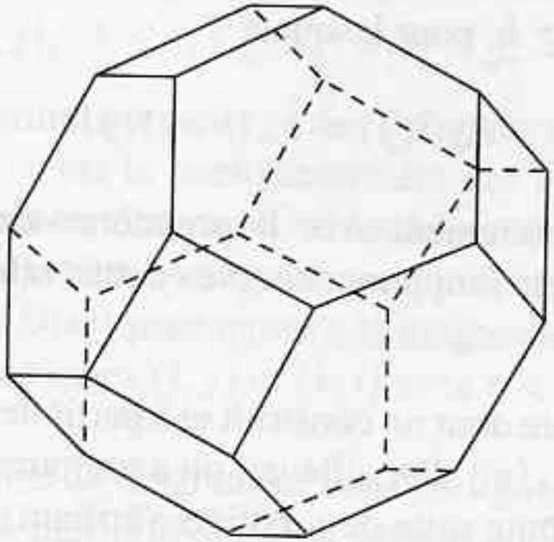
hyperacube — associahedron — permutahedron

lexicographic order $\leftrightarrow$ Tamari order $\leftrightarrow$ weak Bruhat order

(boolean lattice
inclusion)

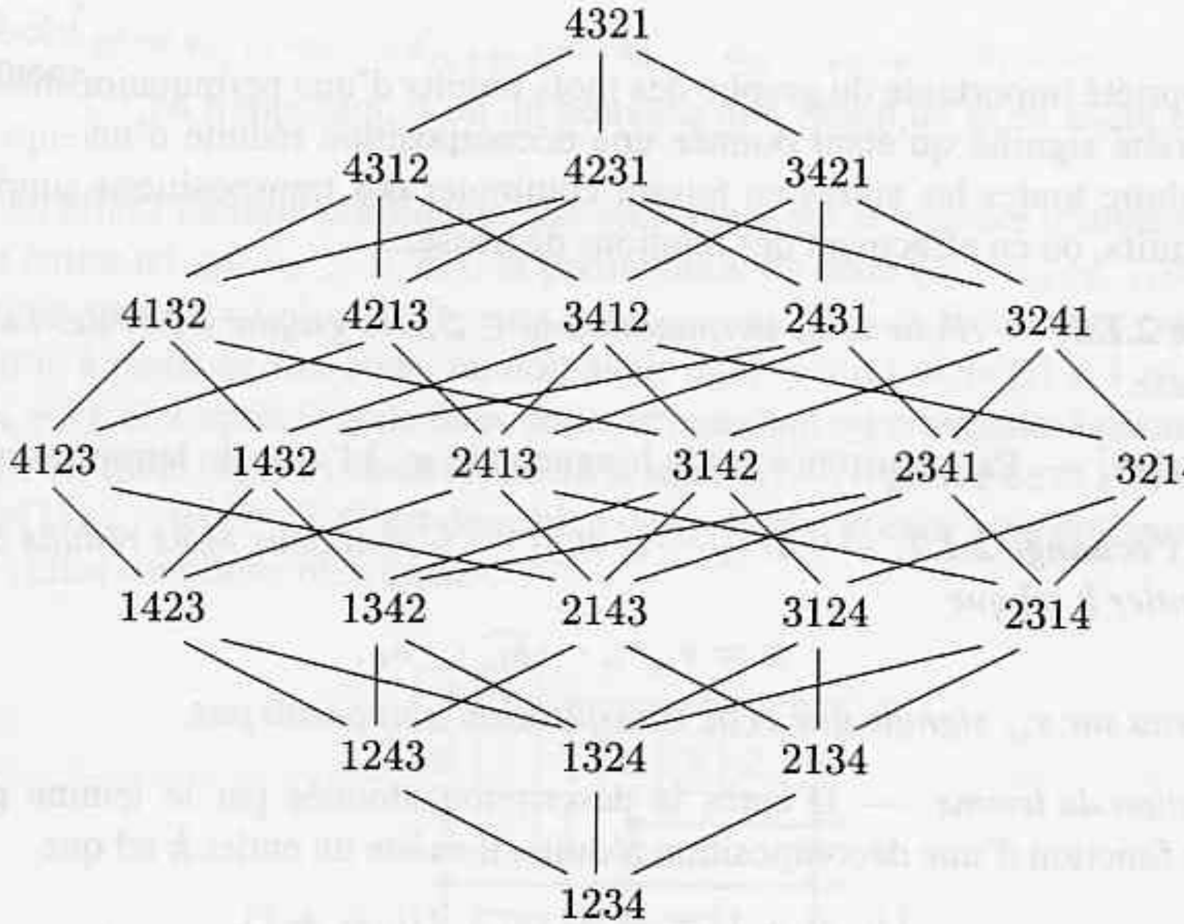
dim $\quad 2^n \qquad C_n \qquad n!$

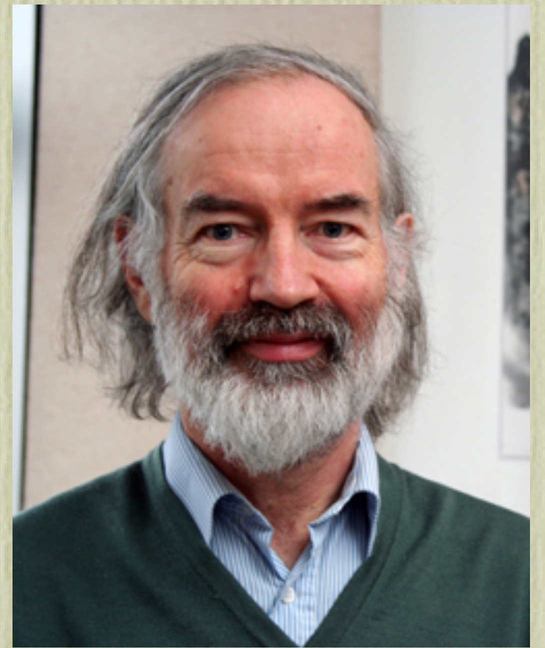
Catalan



2. Le permutoèdre Π_3.

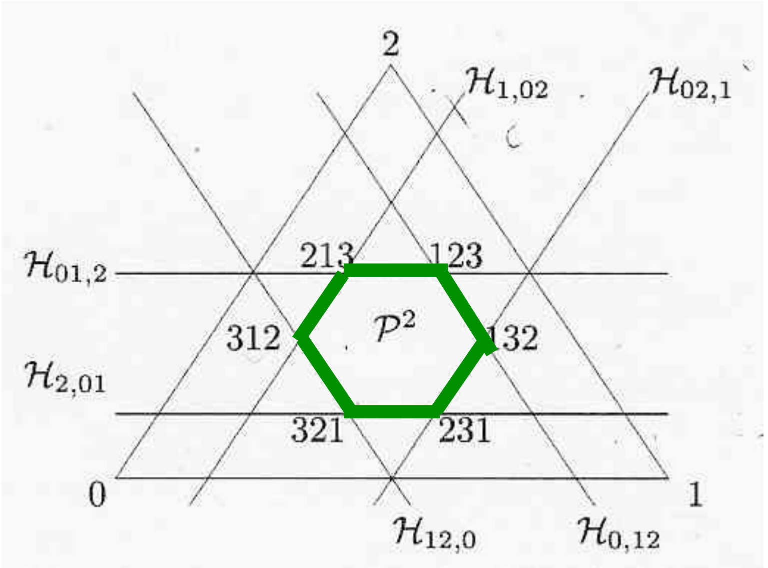
permutohedron



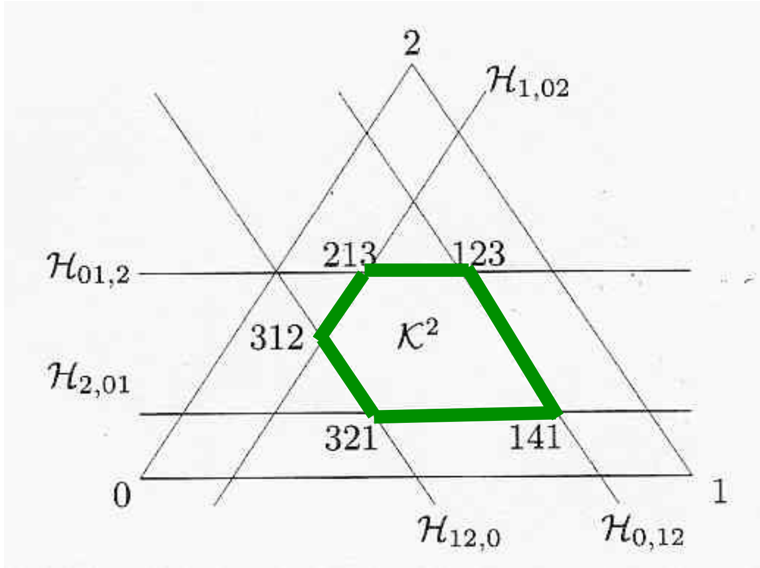


permutohedron

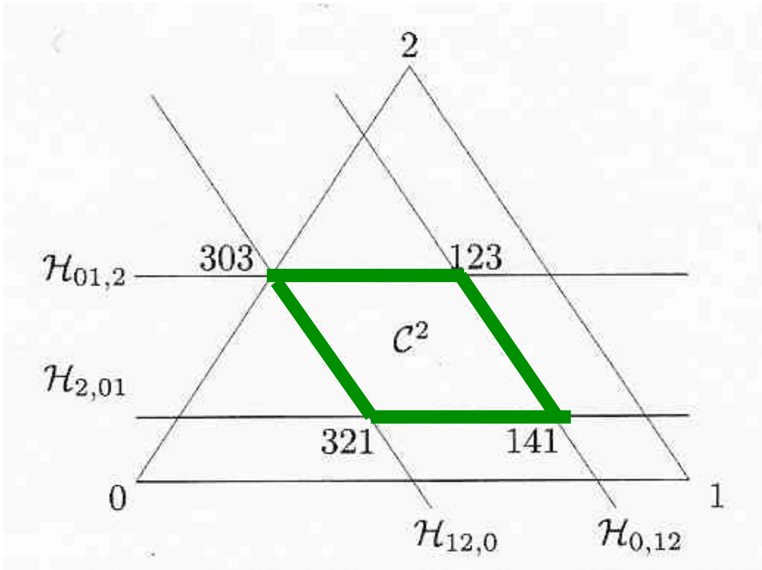
Alain Lascoux
(1944–2013)



2
$\mathcal{H}_{1,02}$
$\mathcal{H}_{02,1}$
$\mathcal{H}_{01,2}$
213
123
312
$\mathcal{P}^2$
132
$\mathcal{H}_{2,01}$
321
231
0
1
$\mathcal{H}_{12,0}$
$\mathcal{H}_{0,12}$



2
$\mathcal{H}_{1,02}$
$\mathcal{H}_{01,2}$
213
123
312
$\mathcal{K}^2$
$\mathcal{H}_{2,01}$
321
141
0
1
$\mathcal{H}_{12,0}$
$\mathcal{H}_{0,12}$



2
$\mathcal{H}_{01,2}$
303
123
$\mathcal{C}^2$
$\mathcal{H}_{2,01}$
321
141
0
1
$\mathcal{H}_{12,0}$
$\mathcal{H}_{0,12}$

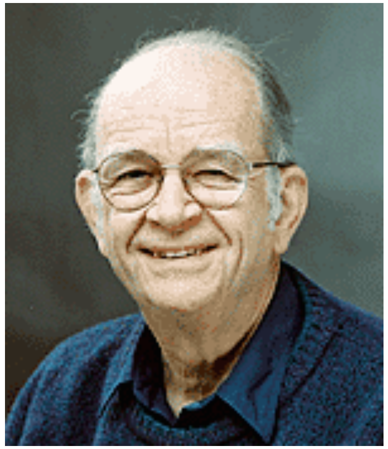
algebraic structures
Hopf algebra

	descent algebra	Loday-Ronco algebra	Reutenauer Malvenuto algebra
dim	2^n	C_n Catalan	$n!$

hypercube — lexicographic order (boolean lattice inclusion) ⟷ associahedron — Tamari order ⟷ permutahedron — weak Bruhat order

Stasheff polytope (1963)

thesis (1961)



J. Milnor

Homotopy theory

loops

$[0,1] \xrightarrow{f} X$ continuous

topological space

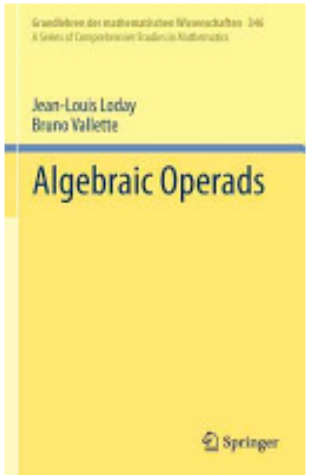
$[0,\tfrac{1}{2}] \xrightarrow{f} X$

$[\tfrac{1}{2},1] \xrightarrow{g} X$

Operads

book J.-L. Loday
B. Vallette
(xxiv, 634 p)

Algebraic Operads
(Springer, 2012)



Physics

A. Dimakis, F. Müller-Hoissen

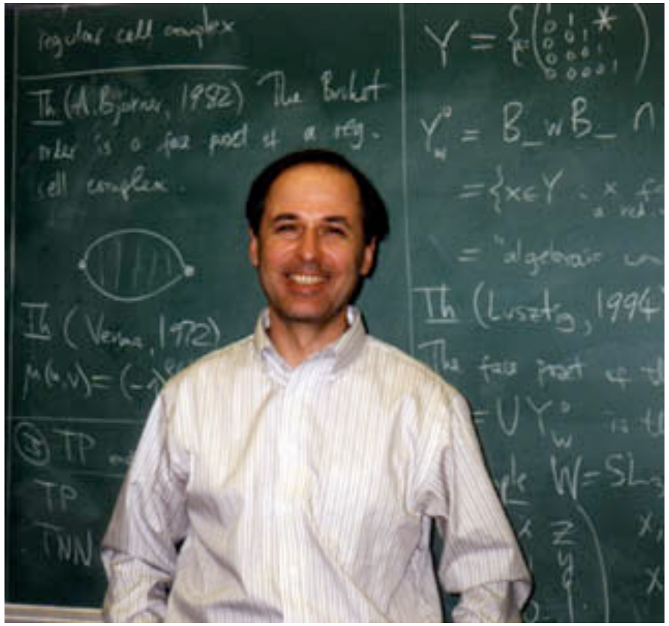
soliton KP-equation
waves in shallow water

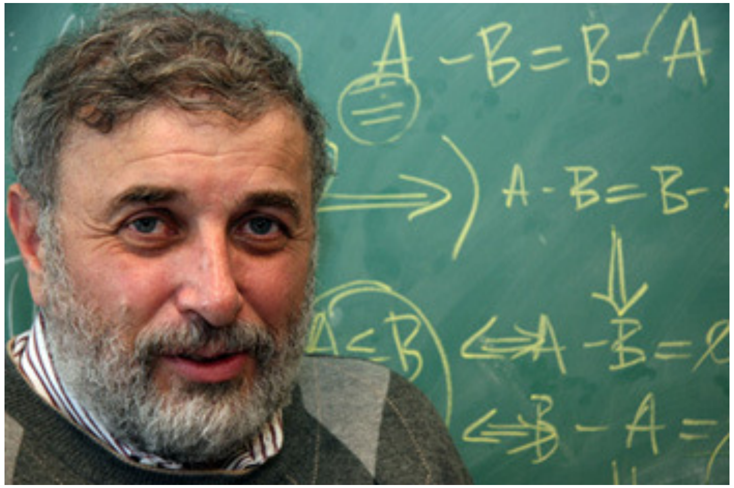
$\longleftrightarrow$ maximal chains in the Tamari lattice

"Associahedra, Tamari lattice, and related structure", Progress in Math vol 299
Birkhauser (2012)

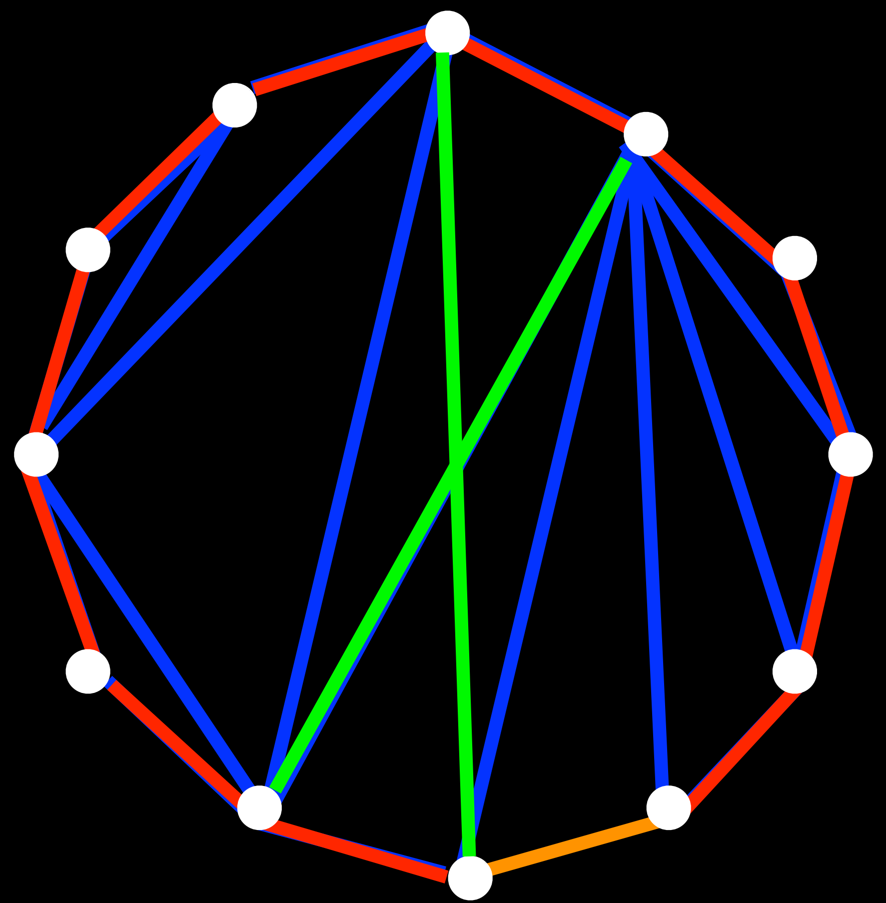
P. Dehornoy
symmetric Thompson monoid
F Thompson group flip distance, diameter

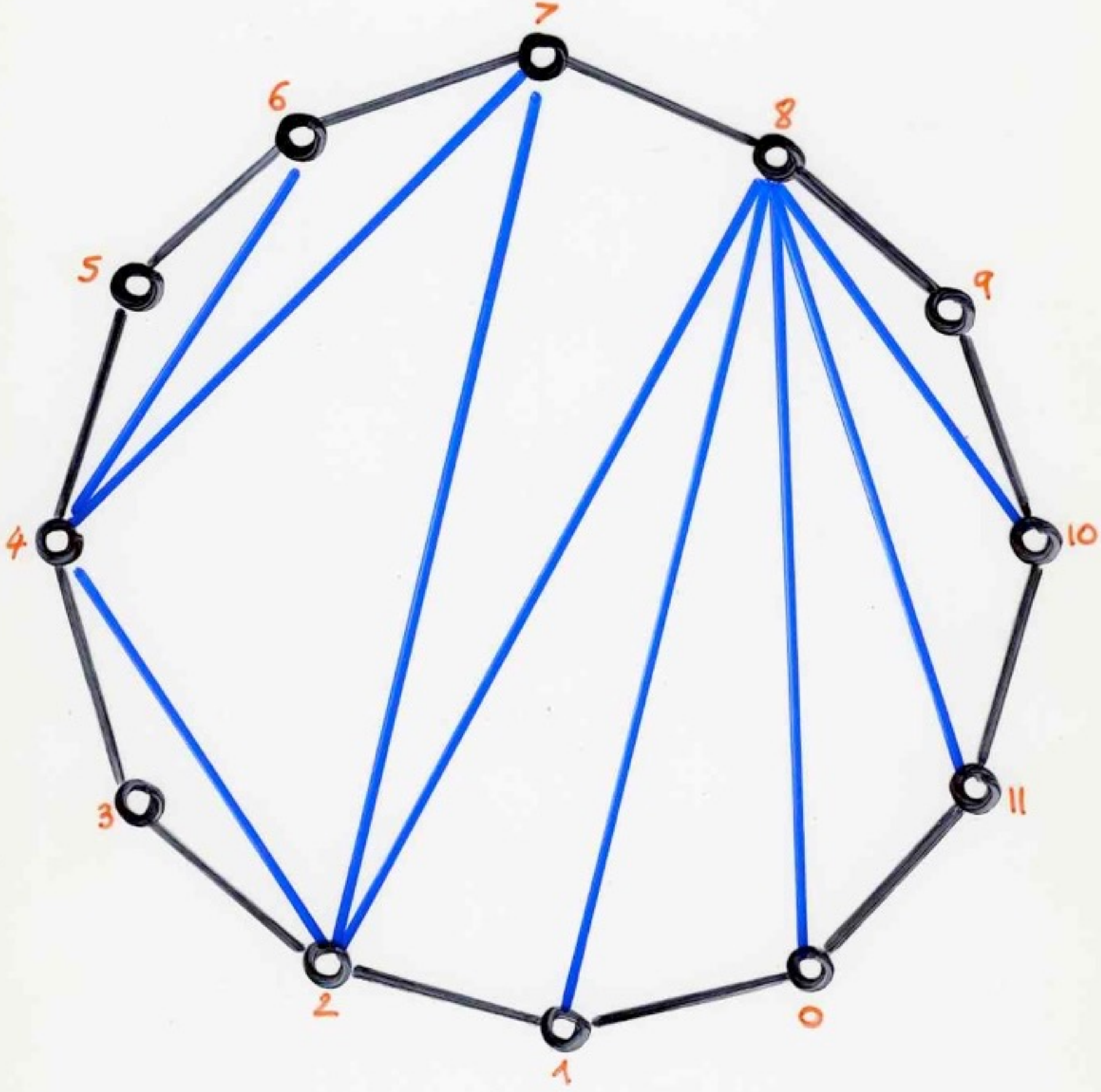
root systems cluster algebras



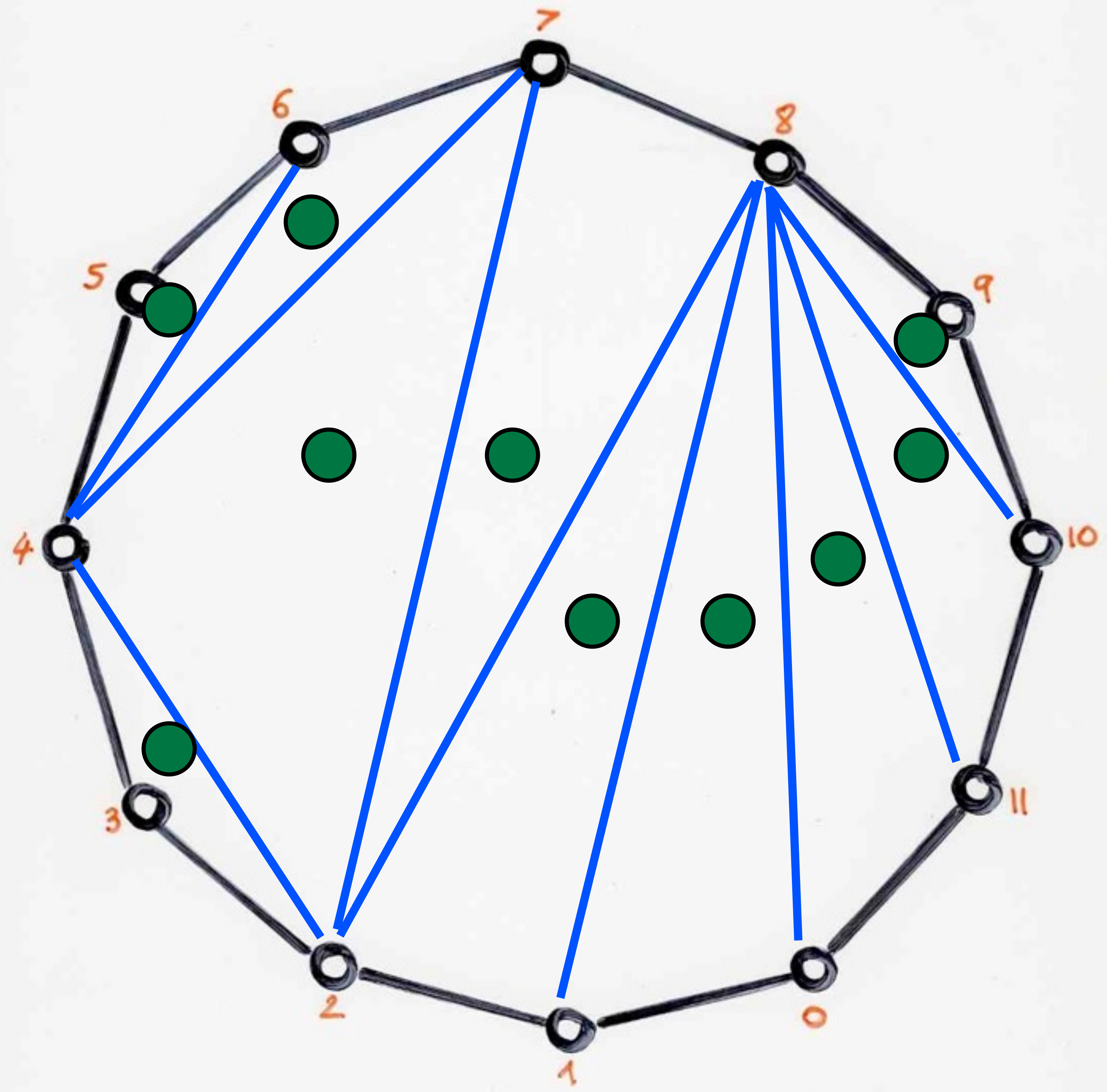


associahedron





7
6
8
5
9
4
10
3
11
2
0
1



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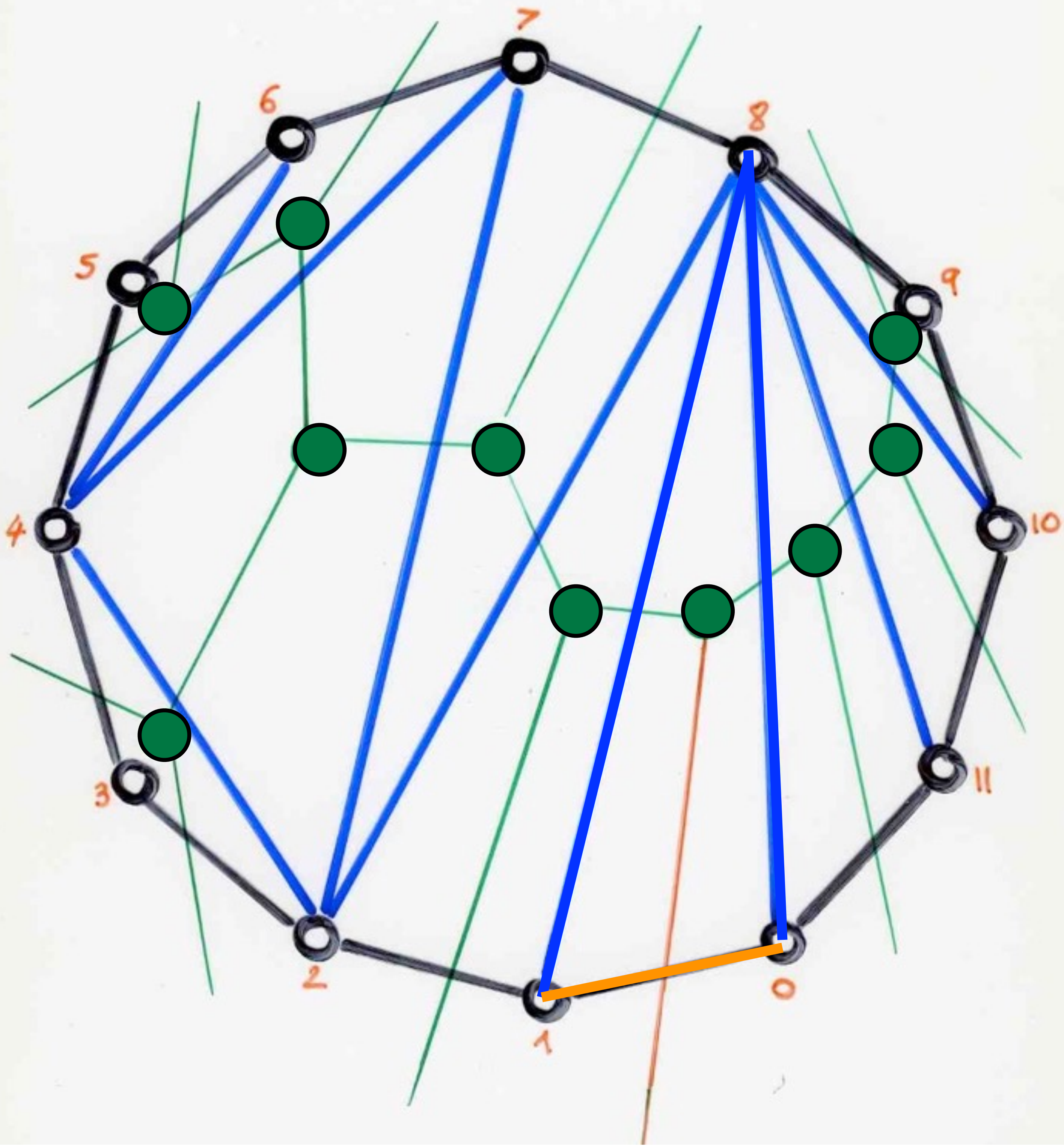
2
0
1



7
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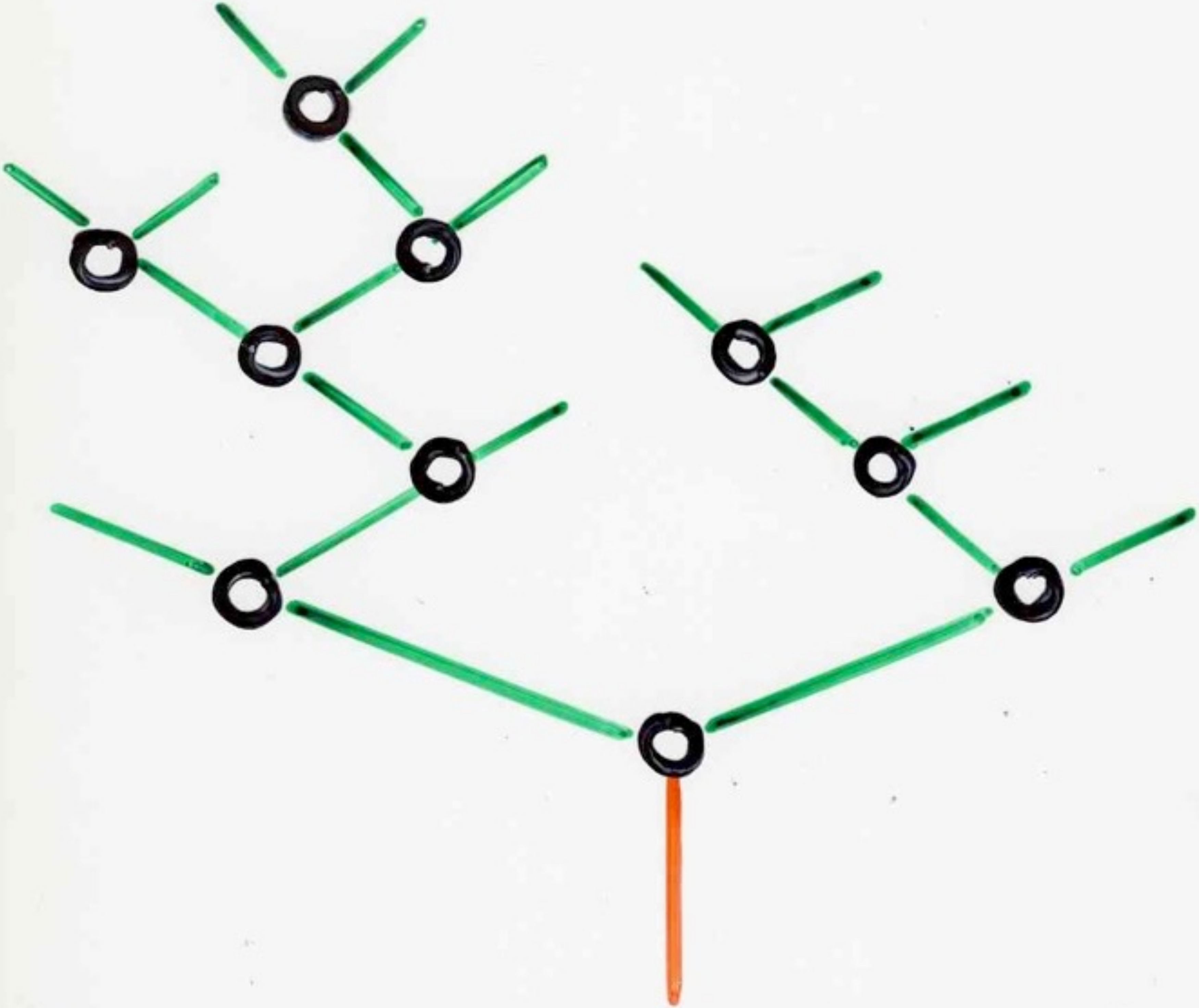
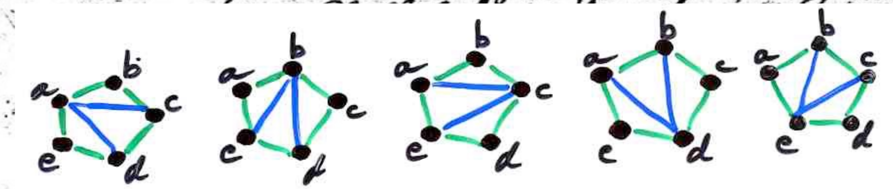
11
2
0
1



theilet, und solches kann auf 5 verschiedene Arten geschehen vermittelst der Diagonalen I. $\frac{ac}{ad}$; II. $\frac{bd}{be}$; III. $\frac{ca}{ce}$; IV. $\frac{da}{db}$; V. $\frac{ec}{eb}$



Setze ich nun die Anzahl dieser verschiedenen Arten $= x$ so habe ich per inductionem gefunden

wenn $n = 3.\ 4.\ 5.\ 6.\ 7.\ 8.\ 9.\ 10$

so ist $x = 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430$

Hieraus habe ich nun den Schluss gemacht, dass generaliter sey

$$x = \frac{2.\,6.\,10.\,14.\,18.\,22.\ \ldots\ (4n-10)}{2.\,3.\,4.\,5.\,6.\,7.\ \ldots\ (n-1)} = \frac{(2n)!}{(n+1)!\,n!}$$

$1 = \frac{2}{2}$; $2 = 1.\frac{6}{3}$; $5 = 2.\frac{10}{4}$; $14 = 5.\frac{14}{5}$, $42 =$

dass also aus einem jeden Gliede das folgende leicht gefunden wird. Die Induction aber, so ich gebraucht, war ziemlich mühsam, doch zweifle ich nicht, dass diese Sach nicht solte weit leichter entwickelt werden können. Ueber die Progression der Zahlen $1, 2, 5, 14, 42, 132$, etc. habe ich auch diese Eigenschaft angemerkt, dass

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc.} = \frac{1 - 2a - \sqrt{(1-4a)}}{2aa}$$

also wann $a = \frac{1}{4}$, so ist $1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} + \text{etc.} = 4.$

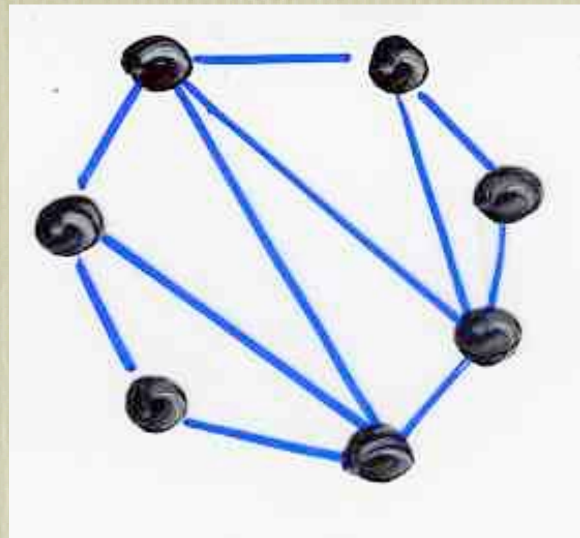
[illegible] Ew. Hochwohlgeb. [illegible]

Ew. Hochwohlgebohren

Berlin d. 4ten Sept 1751.

gehorsamster Diener

L. Euler



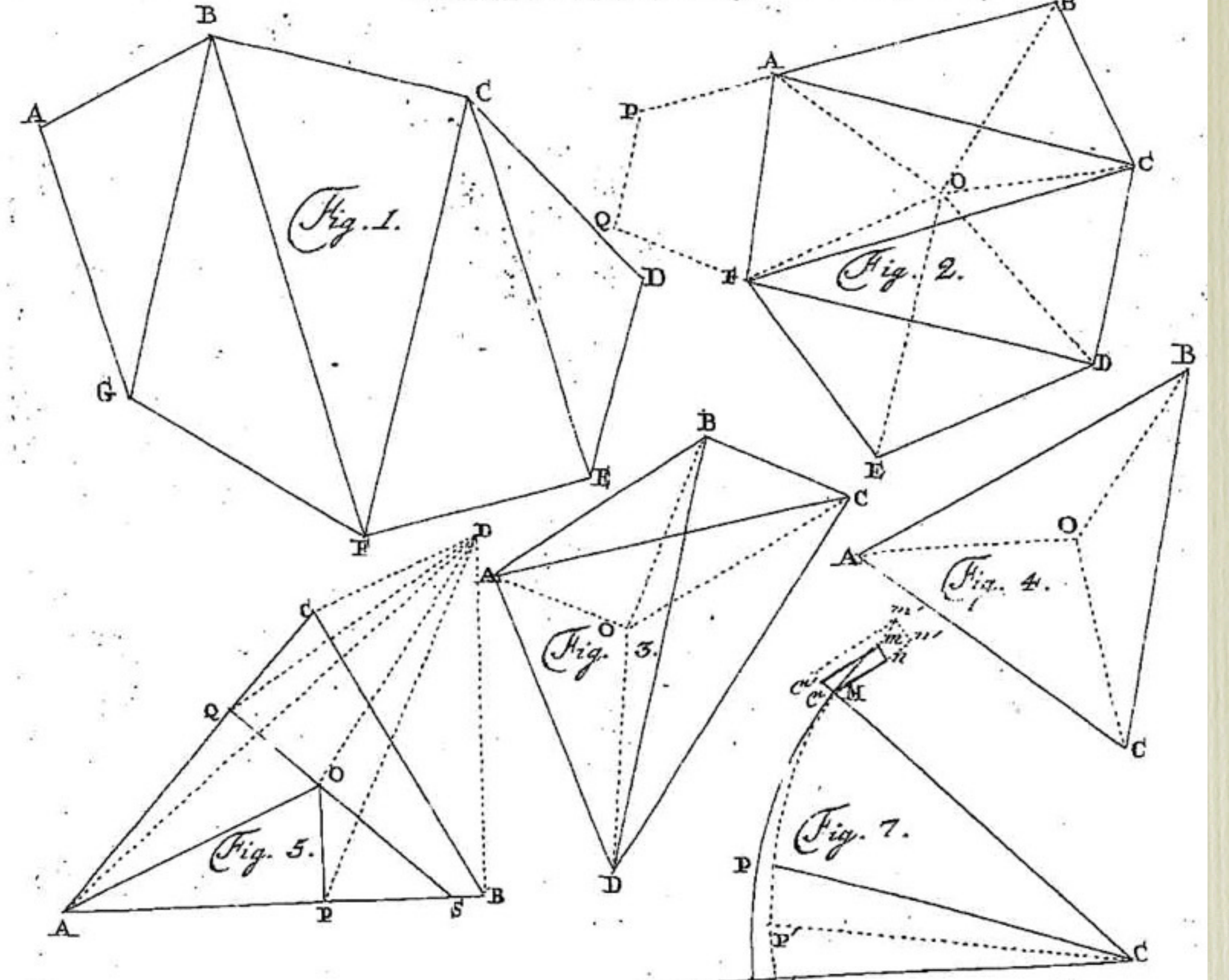
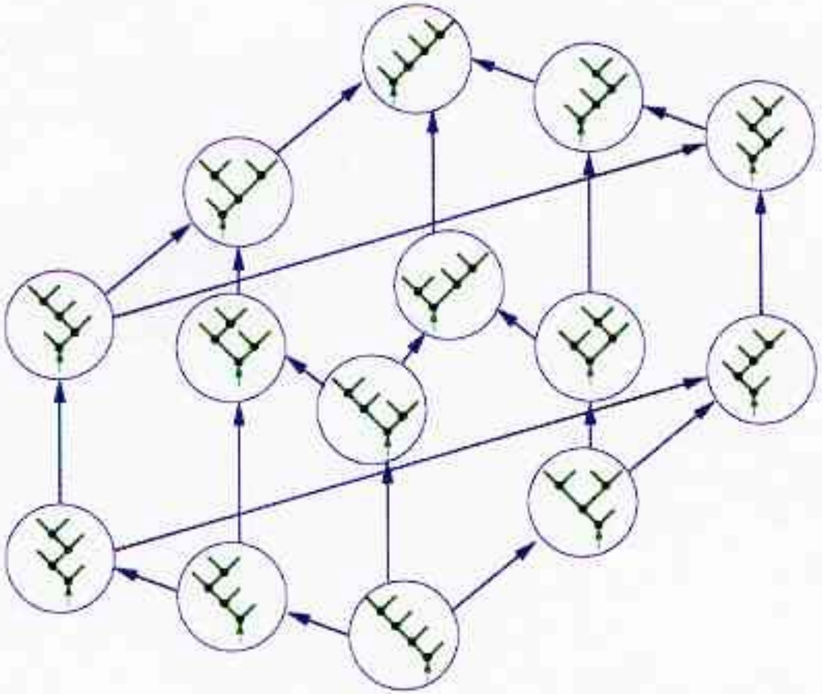
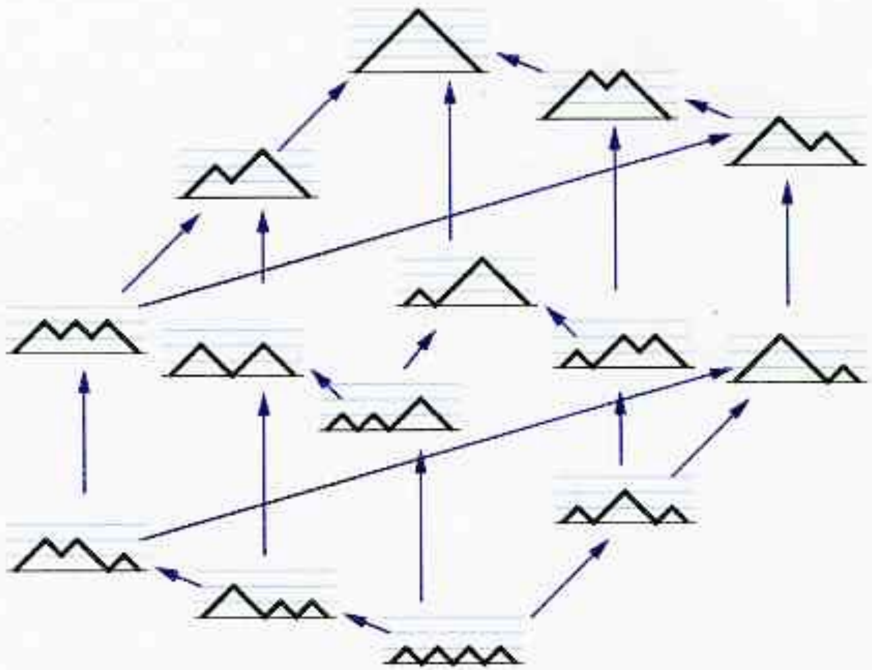


Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 7.


4000 BASEL
e-k+f=2
Leonhard Euler
1707-1783
AUSGABETAG 6. MÄRZ 2007
LEONHARD EULER
1707–1783
130
300
300. Geburtstag
300ème anniversaire
300° anniversario
300th anniversary

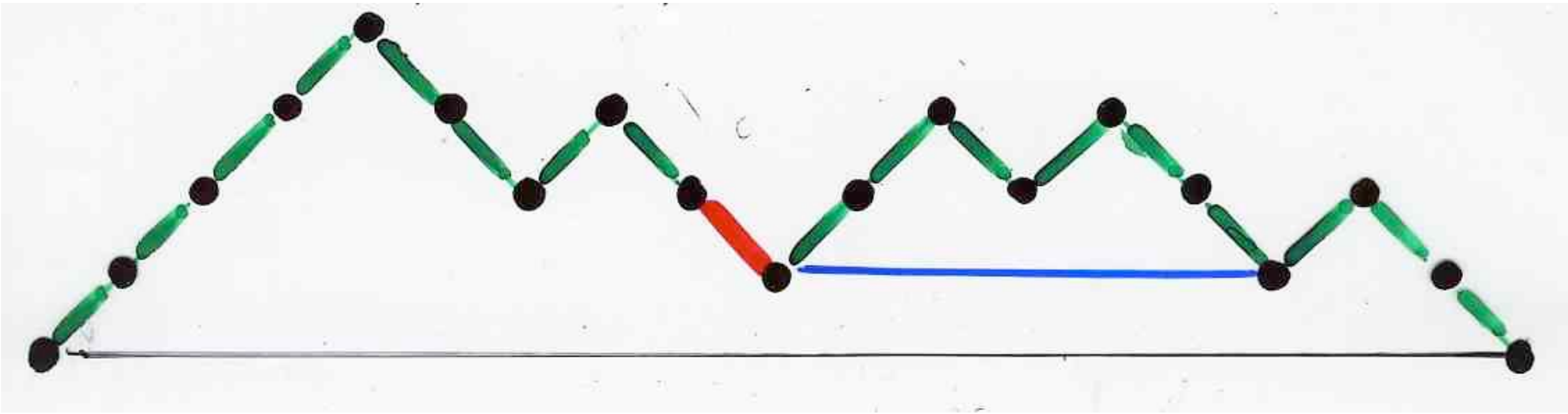
Dyck lattice

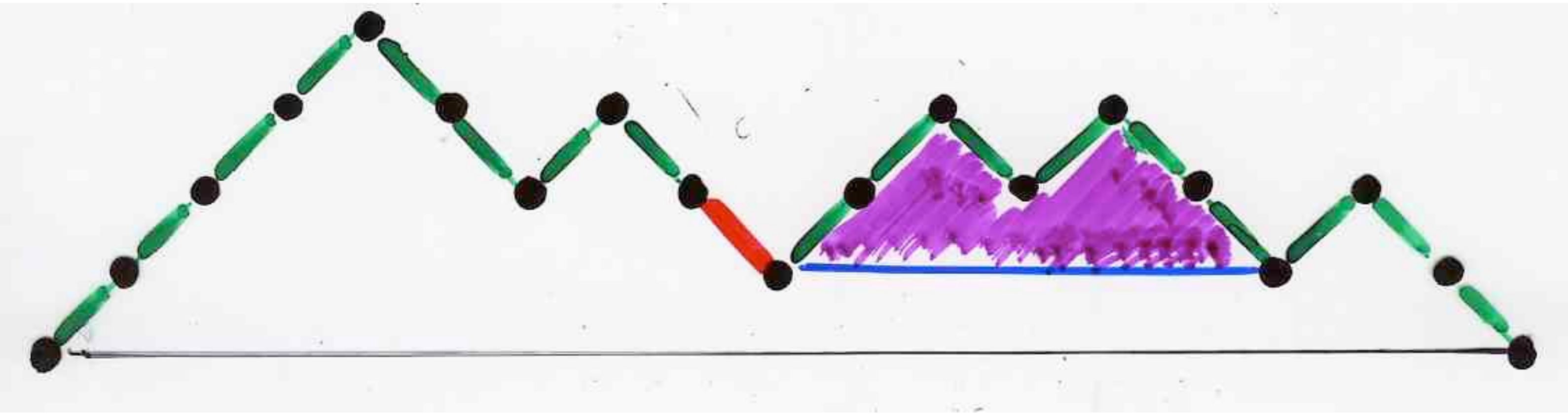




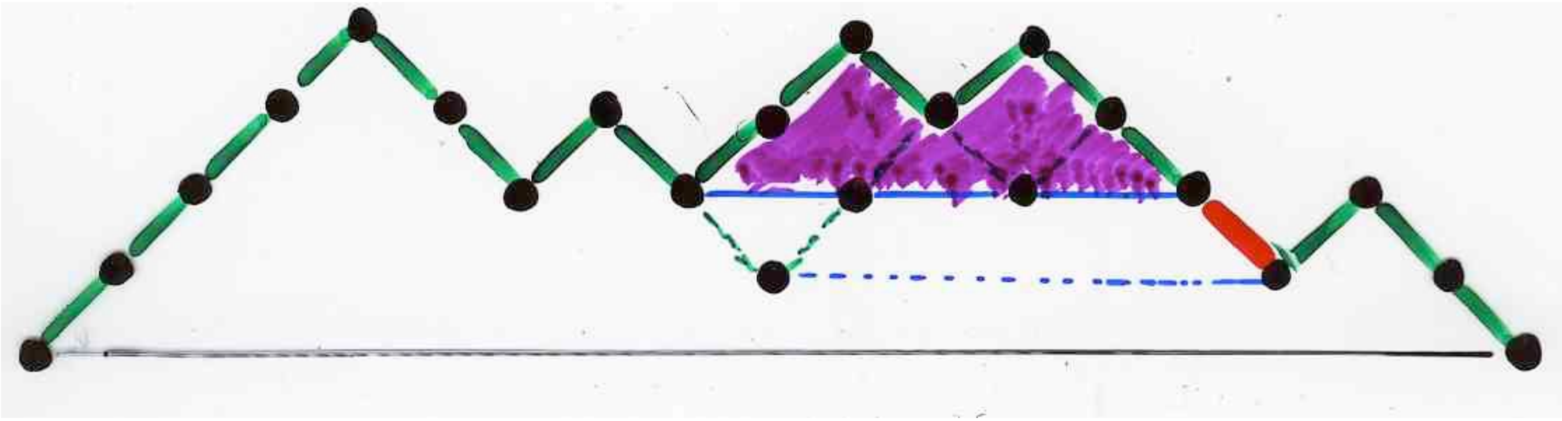
$C_4 = 14$

Catalan





facteur Dyck primitif



factor Dyck primitif

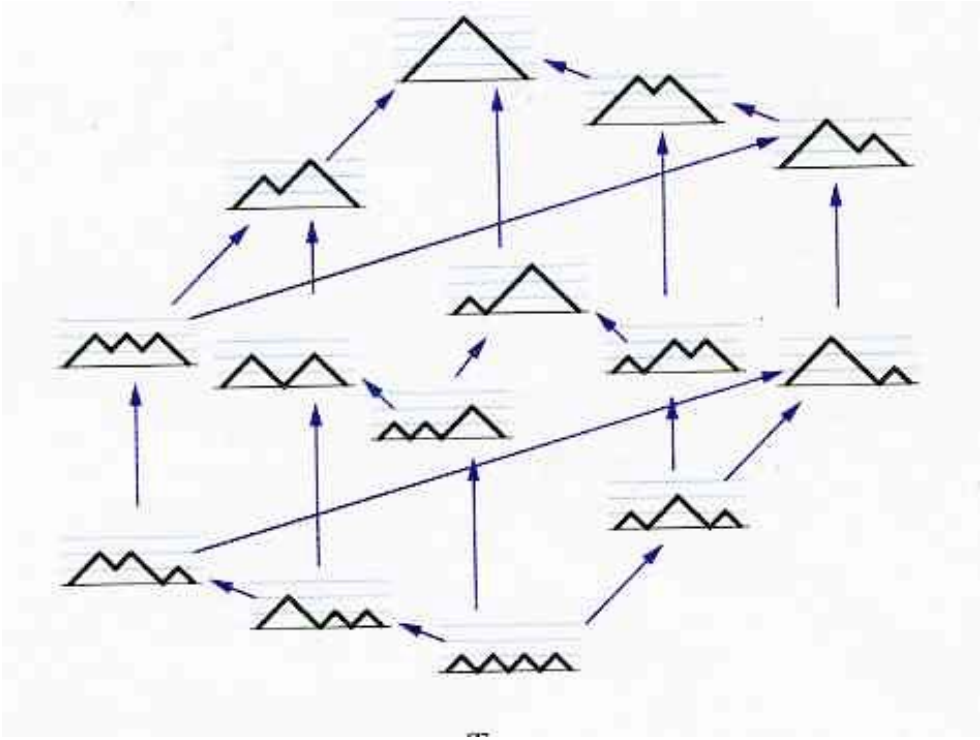
If $T \leqslant T'$ in $(\text{Tamari})_n$ lattice

then $T \leqslant T'$ in $(\text{Dyck})_n$ lattice

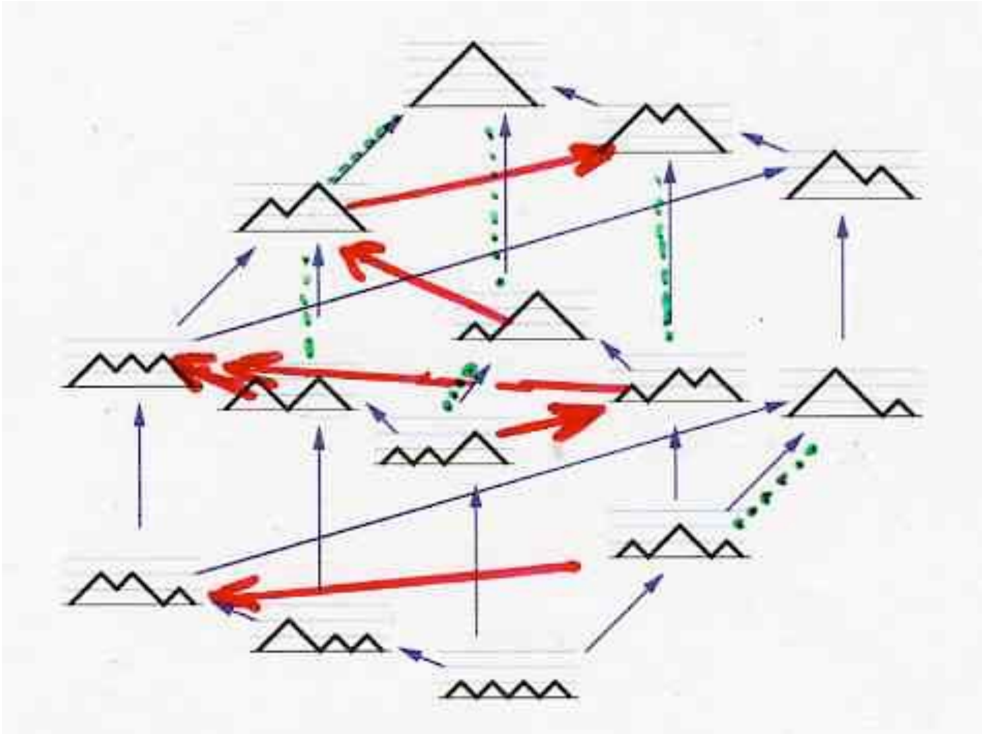
[i.e. T below T']

converse not true

$(\text{Dyck})_n$ extension of $(\text{Tamari})_n$



$\left(\text{Tamari lattice}\right)_4$



(Dyck lattice)$_4$

Combinatorial problems in lattices

Combinatorial
enumerative
problems
in lattices

poset

Partially
Ordered
Set

lattice

inf

sup

nb of intervals

nb of maximal chains

nb of intervals

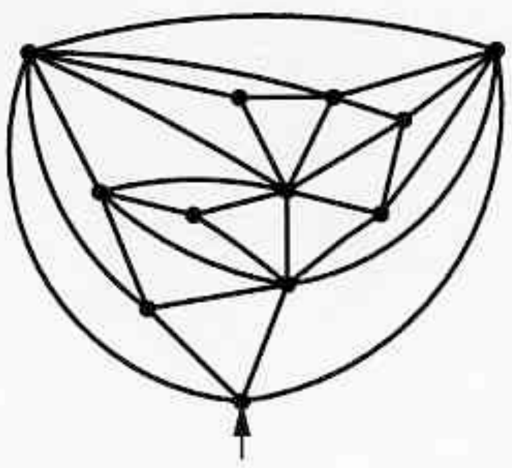
F. Chapoton (2006) $\frac{2(4n+1)!}{(n+1)!\,(3n+2)!}$

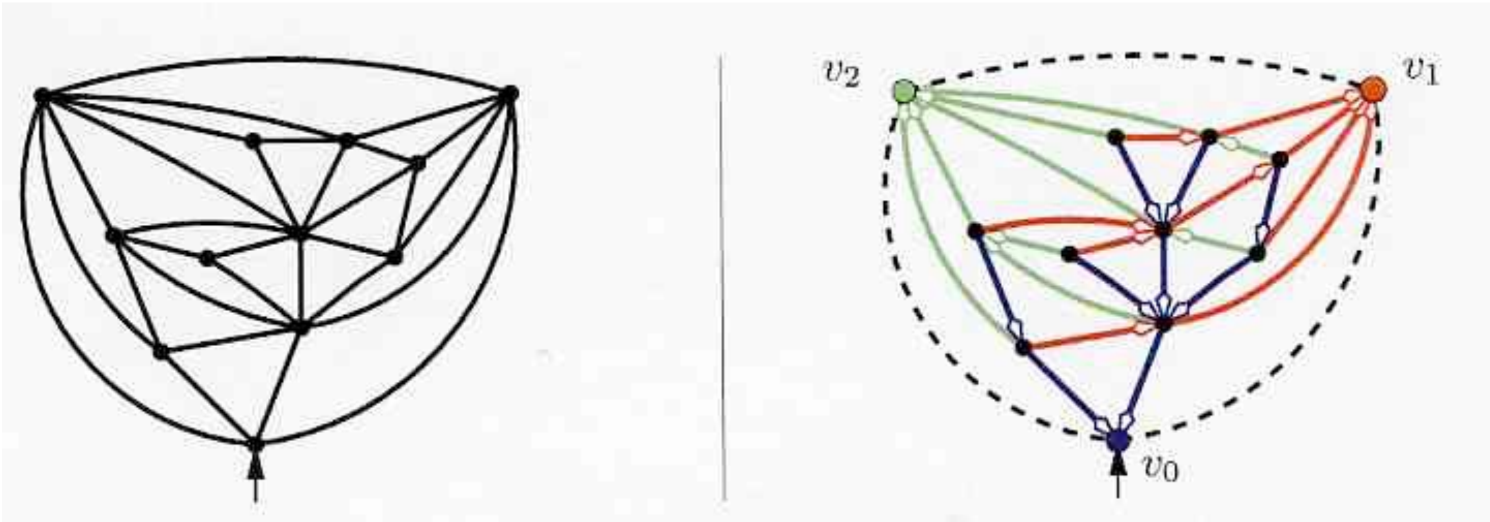
1, 3, 13, 68, 399, ...

triangulation

Bijective proof FPSAC 2007

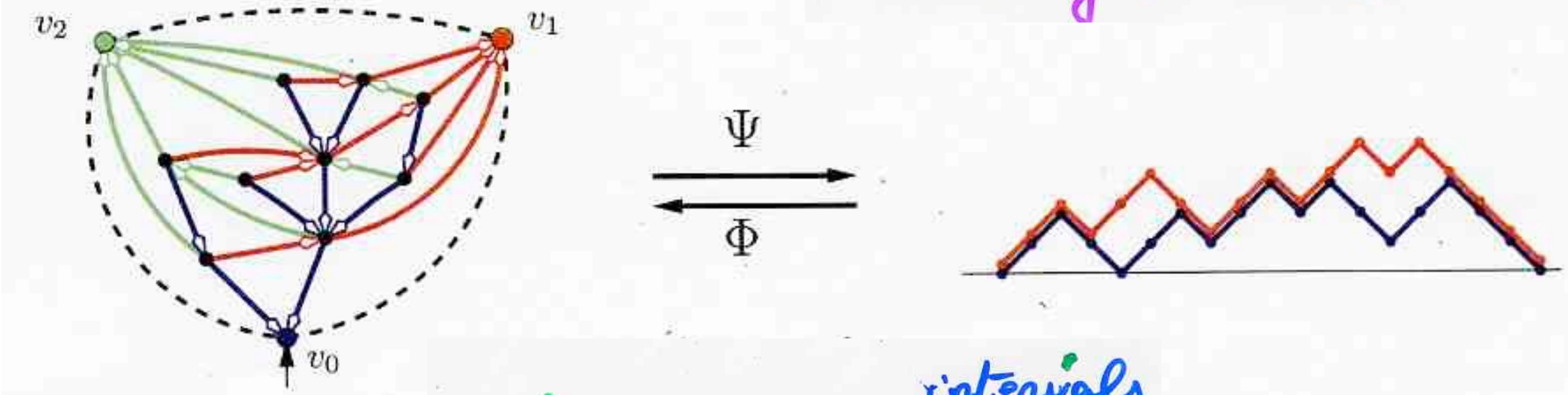
Bernardi, N. Bonichon





triangulation

realizers
or Schnyder woods



realizers $\longleftrightarrow$ intervals $(\text{Dyck})_n$

minimal realizers $\longleftrightarrow$ intervals $(\text{Tamari})_n$

$\updownarrow$

triangulation

lattice as a polytope

nb of faces

nb of simplex in a triangulation

for the associahedron

$(n+1)^{n-1}$ parking functions

bijection J.-L. Loday

S. Giraudo FPSAC 2010

set of balanced binary tree

- closed by interval
- $[T_1, T_2]$ interval $\simeq$ $\mathbb{H}_k$ hypercube

product of two binary trees

- in the Loday-Ronco algebra "is" an interval
- in the # algebra "is" an interval

Aval, X.V. SLC
Aval, Novelli, Thibon FPSAC 2011

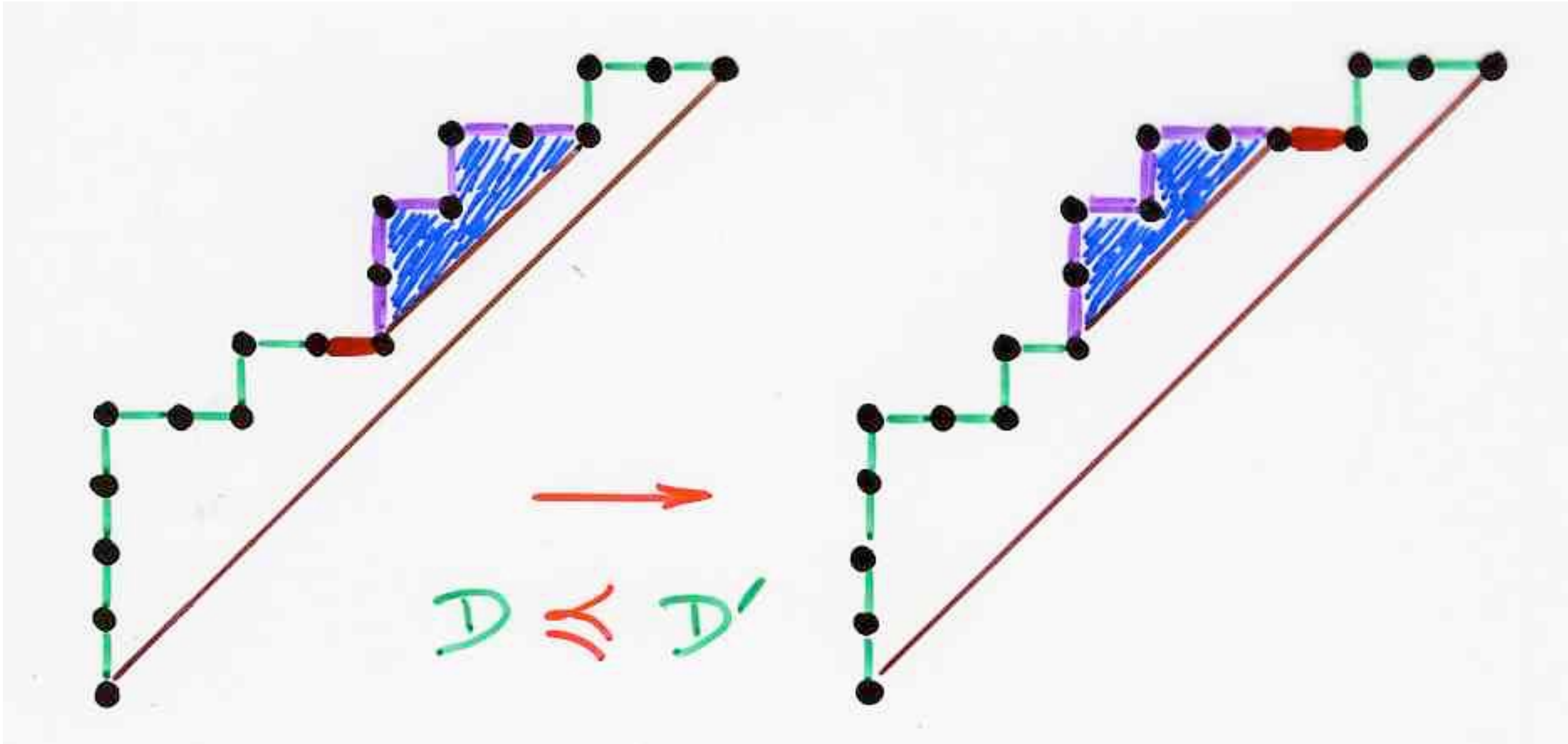
V. Pons

Tamari interval-posets $\Longleftrightarrow$ Tamari intervals
(FPSAC, 2013) thesis (Oct. 2013)

G. Chatel

bijections on Tamari intervals
$\longleftrightarrow$ closed flow of an ordered forest
$\rightarrow$ Pre-Lie operad F. Chapoton

m-Tamari



the Tamari covering relation
for ballot (Dyck) path

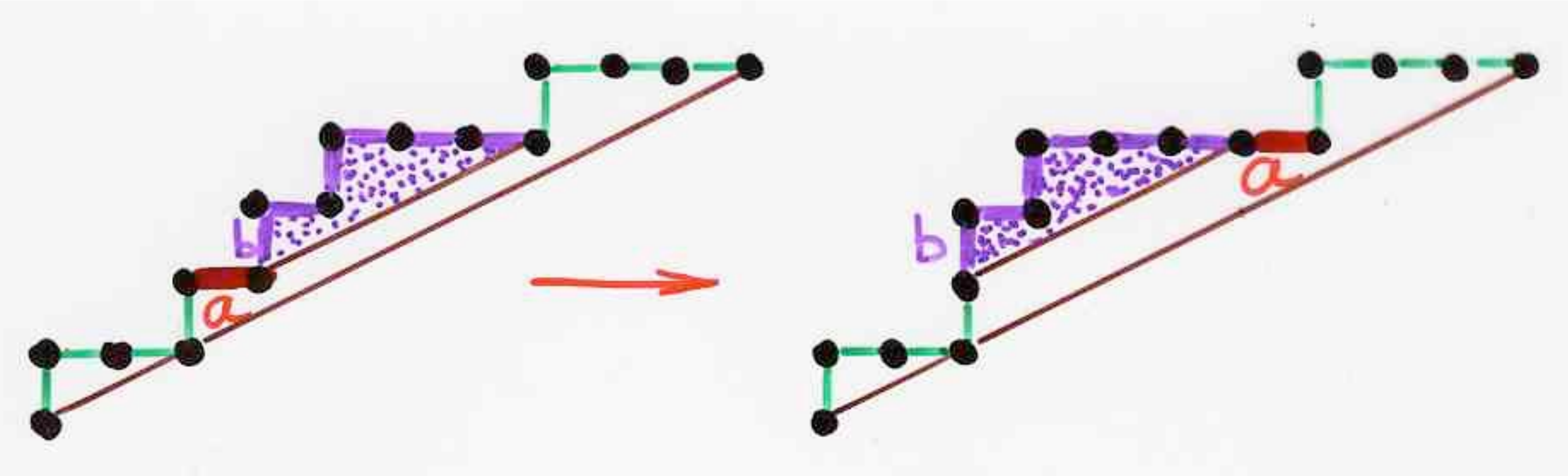
diagonal coinvariant spaces

higher diagonal coinvariant spaces

F. Bergeron (2008) introduced the m-Tamari lattice

dimension $\frac{1}{(m+1)n+1}\binom{(m+1)n+1}{mn}$

m-ballot paths



the covering relation in the
m-Tamari lattice
($m = 2$)

diagonal coinvariant spaces

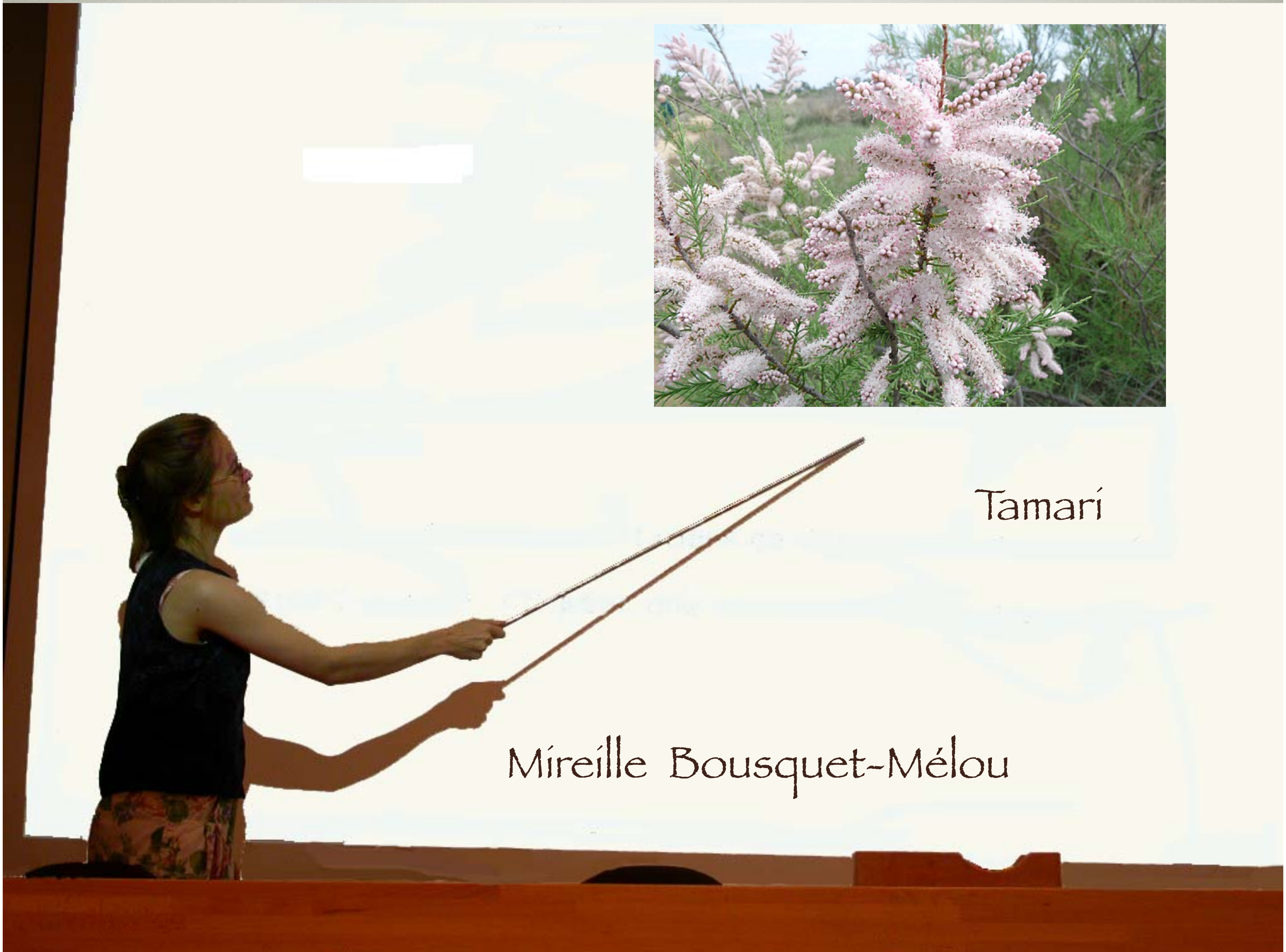
higher diagonal coinvariant spaces

F. Bergeron (2008) introduced the m-Tamari lattice

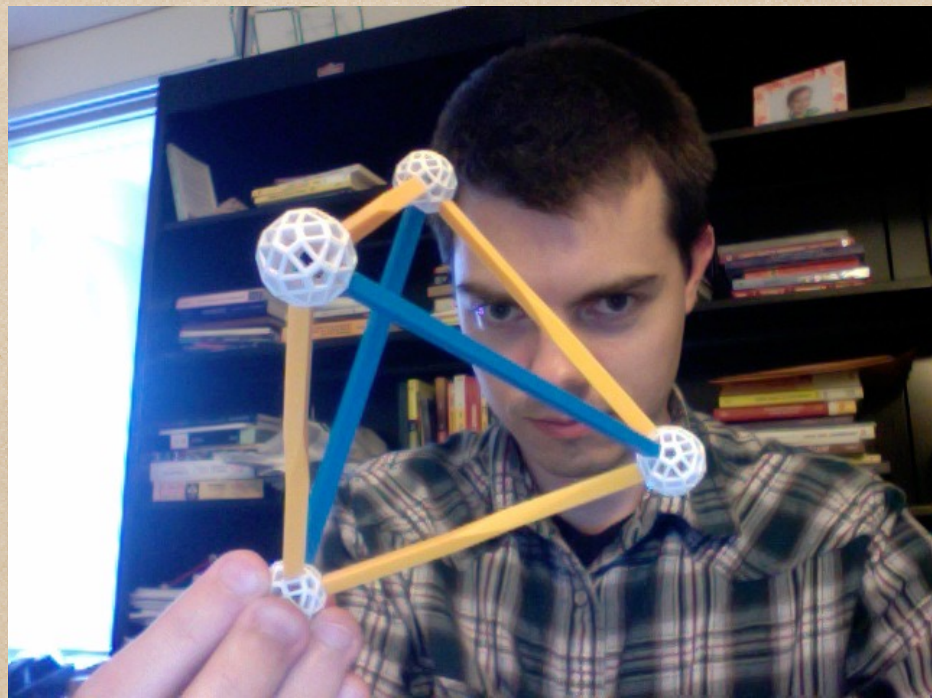
M. Bousquet-Mélou, E. Fusy, L.-F. Préville-Ratelle (2011)
nb of intervals of m-Tamari lattices

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1}$$ F. Bergeron

M. Bousquet-Mélou, G. Chapuy, L.-F. Préville-Ratelle (2011)
nb of labelled intervals $(m+1)^n(mn+1)^{n-2}$


Tamari
Mireille Bousquet-Mélou

Rational Catalan Combinatorics



Rational Catalan Combinatorics

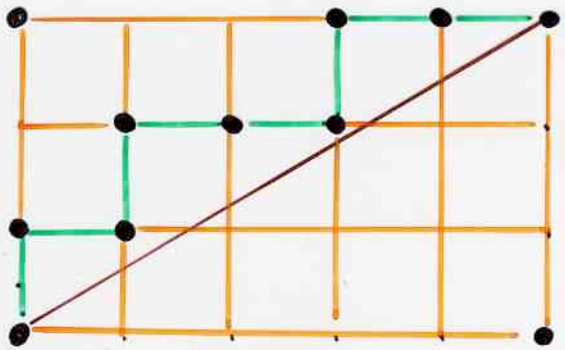
D. Armstrong

$$\mathrm{Cat}(a,b) = \frac{1}{a+b}\binom{a+b}{a,b}$$

number of (a,b)- ballot paths $= \mathrm{Cat}(a,b)$

Grossman (1950)
Bizley (1954)

rational ballot (Dyck) paths



$a=3$

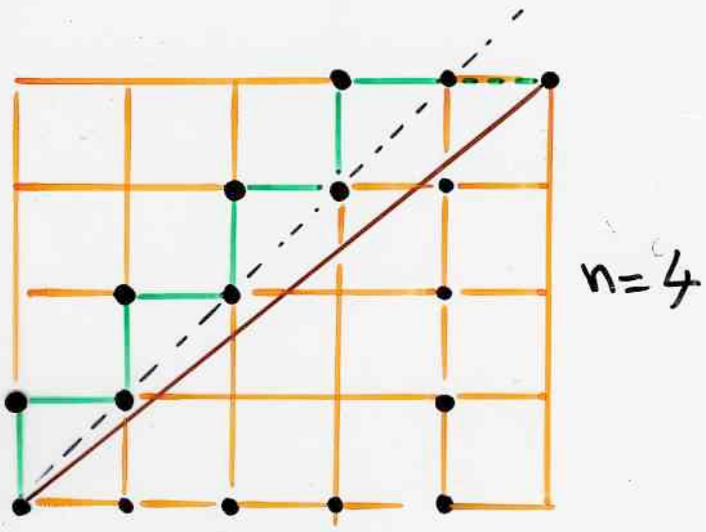
$b=5$

$(a,b) = (3,5)$

$(a, b) = (n, n+1) \longrightarrow C_n$ Catalan nb

$(a, b) = (n, mn+1) \longrightarrow \dfrac{1}{(m+1)n+1} \dbinom{(m+1)n+1}{n}$

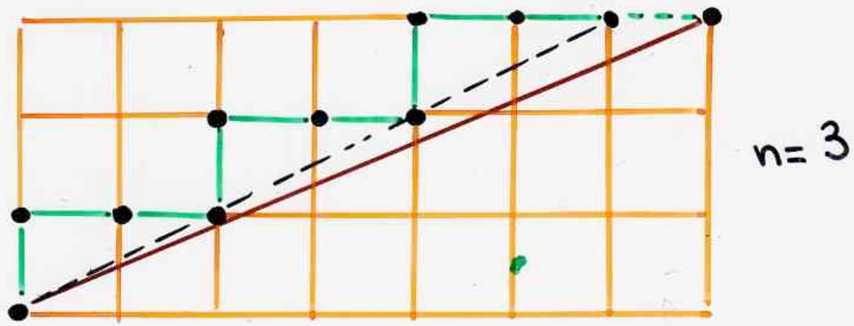
Fuss-Catalan nb



n+1 = 5

(a, b) = (4, 5)

Catalan



m = 2 mn+1 = 7

(a, b) = (3, 7)

Fuss-Catalan

question :
Sergi Elizalde
(this worshop)

Oberwolfach workshop
on Combinatorics
March 2014

define an (a,b)-Tamari lattice ?

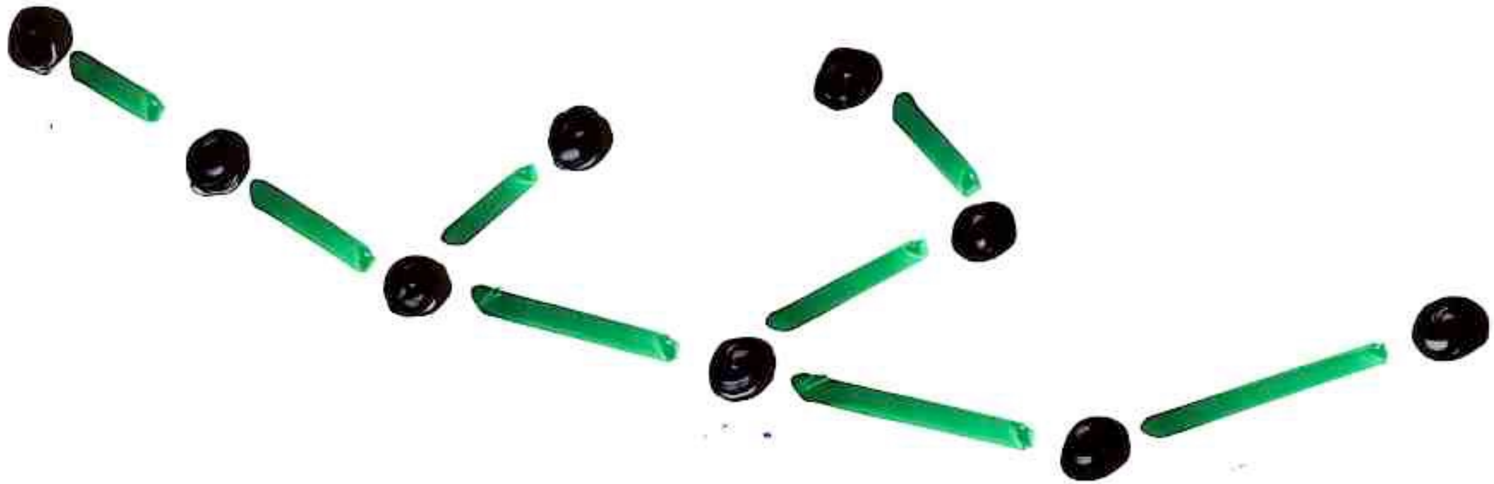
Tamari,
m-Tamari,
(a,b)-Tamari,
and beyond ...

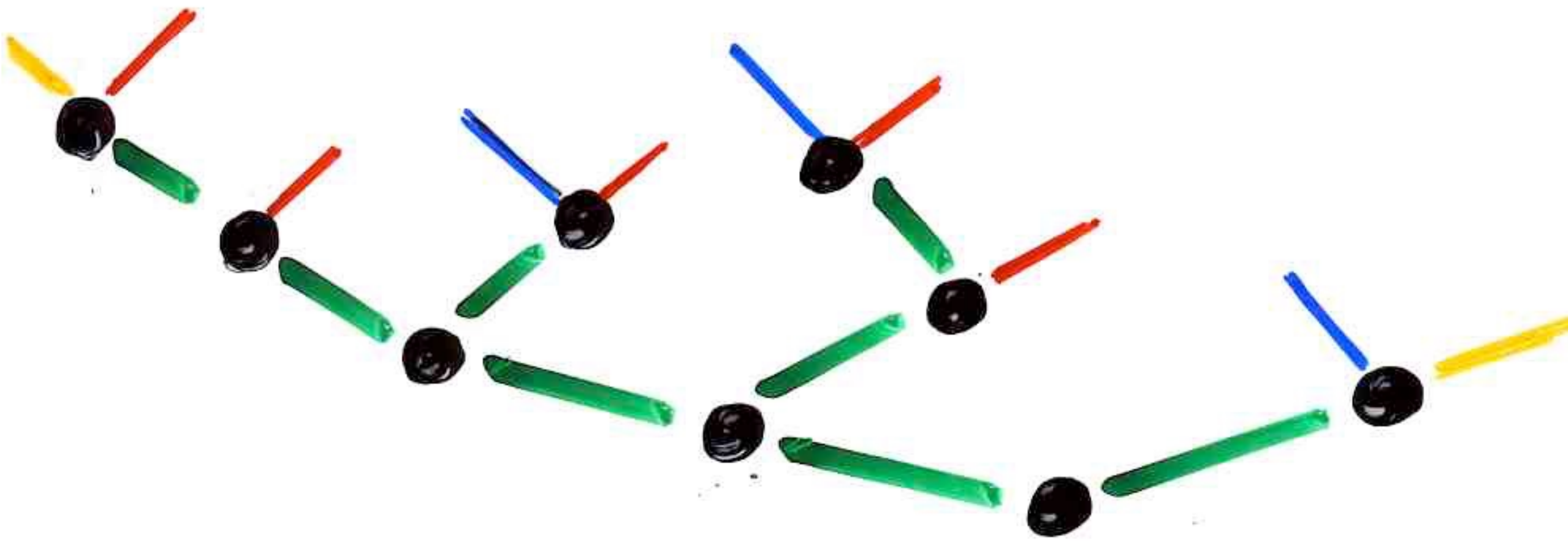
joint work with
Louis-François Préville-Ratelle
U. Talca, Chile

Oberwolfach
6 March 2014

to be published in
Transactions A.M.S.

canopy of a binary tree





canopy of a binary tree

$C(B) = - - + - + - - +$

montée $\sigma(i) < \sigma(i+1)$

$1 \leqslant i < n$

descente $\sigma(i) > \sigma(i+1)$

$\sigma \in \mathfrak{S}_n$

forme$(\sigma) = w_1 w_2 \cdots w_{n-1}$, $\quad w_i = \begin{cases} + & \text{montée} \\ - & \text{descente} \end{cases}$ en i

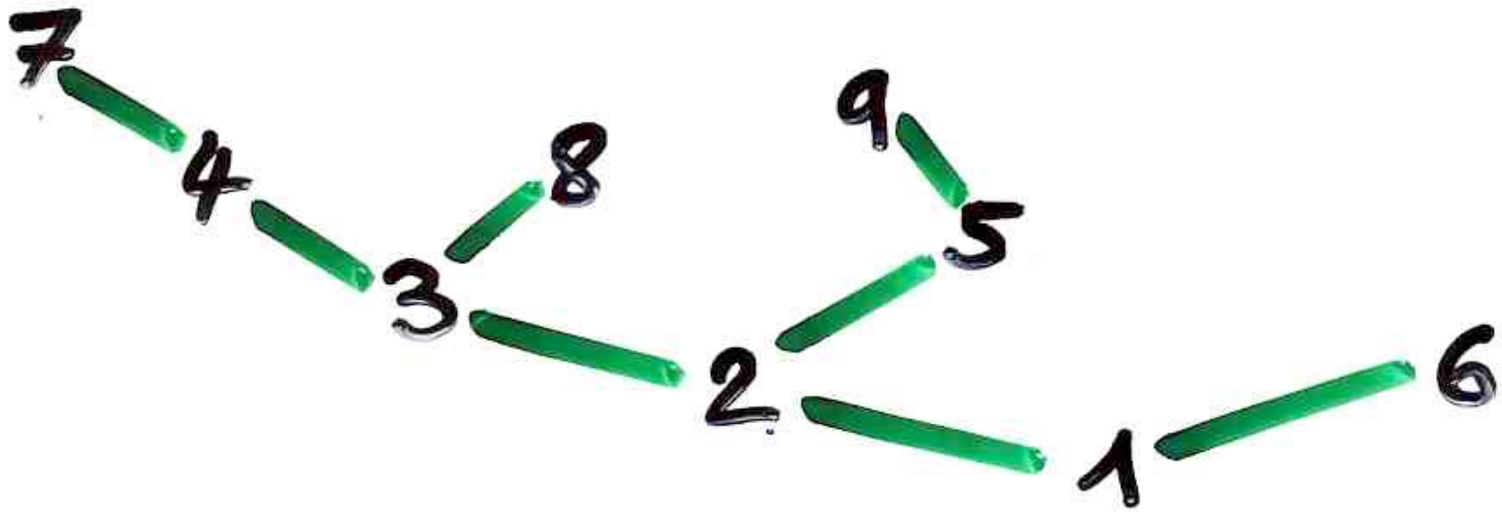
UD(σ) mot $w \in \{+, -\}^*$

$$\sigma = 5\ 8\ 2\ 9\ 6\ 1\ 4\ 3$$

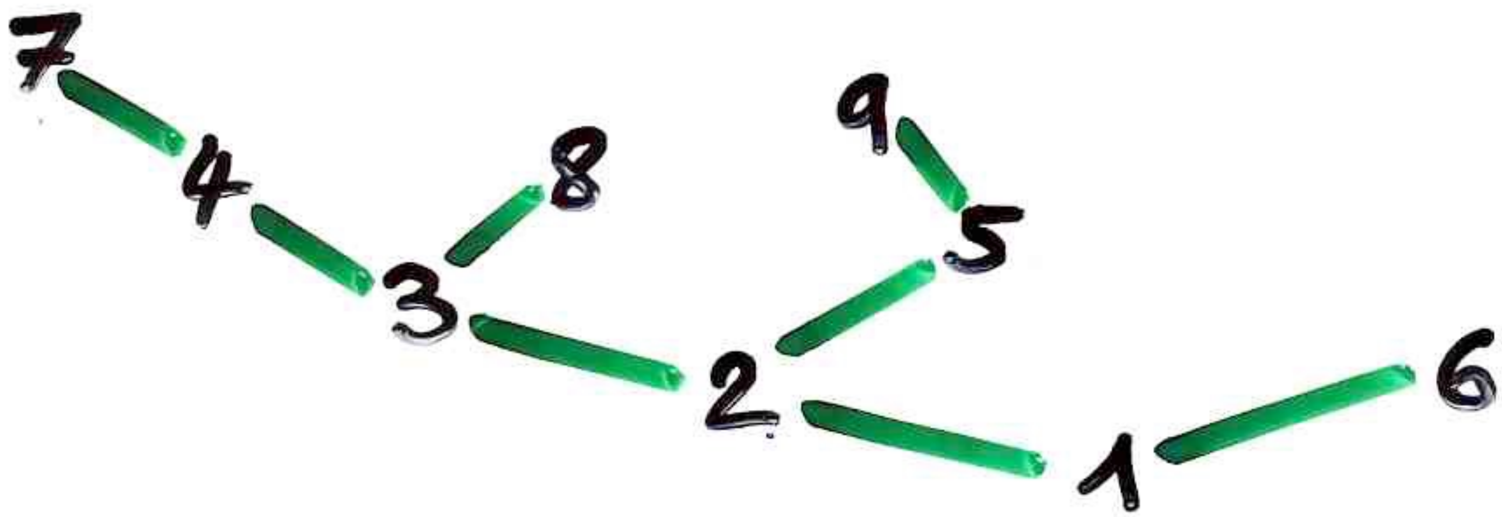
$$\text{forme}(\sigma) = + - + - - + -$$

Bijection increasing binary tree $\longleftrightarrow$ permutation

T σ



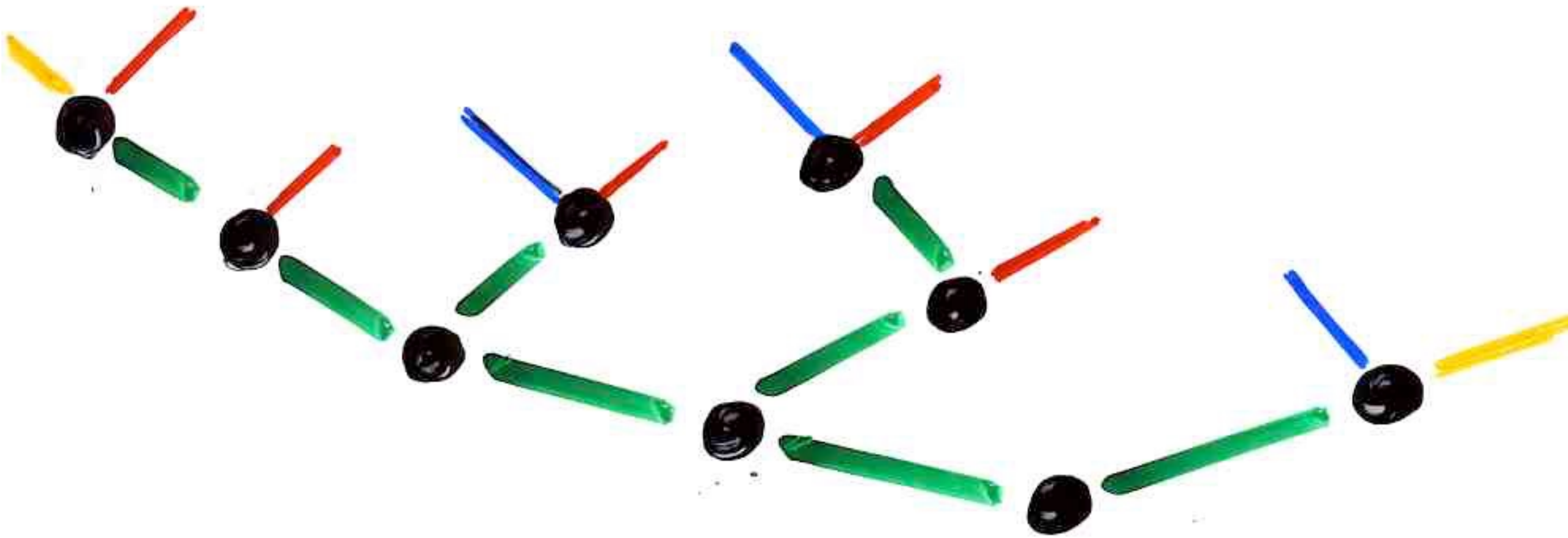
$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$



$$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6\ \dots$$

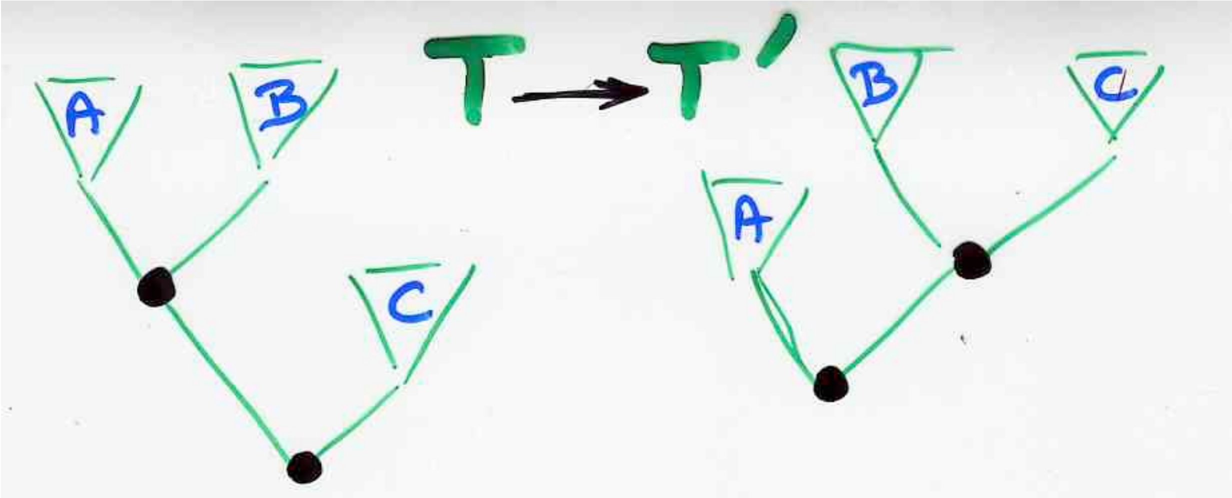
up-down sequence $\quad -\quad -\quad +\quad -\quad +\quad -\quad -\quad +$

$$\begin{tikzcd} \underset{\text{permutation}}{\sigma} \arrow[r] \arrow[dd] & T(\sigma) \arrow[d] & \text{increasing binary tree} \\ & B \arrow[d] & \text{binary tree} \\ UD(\sigma) \arrow[r, phantom, "="] & c(B) & \\ \text{up-down sequence} & \text{canopy} & \end{tikzcd}$$



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6 \ldots$

up-down sequence $\quad - \quad - \quad + \quad - \quad + \quad - \quad - \quad +$



if $B \neq \bullet$ canopy is invariant

if $B = \bullet$ canopy $c(T')$ not invariant

$$c^{*}(T) = c(A) \overset{+}{/}\, c(B)\, c(C)$$

$$c(T') = c(A) \underset{-}{\backslash}\, c(B)\, c(C)$$

the theorem

relating canopy and Tamari lattice

Prop. (i) • The set of binary trees having a given canopy w is an interval $J(w)$ of the Tamari lattice

(ii) • this interval can be extended to an initial interval of the Young lattice

i.e. $\exists$ (integer) partition μ such that $J(w)$ is in bijection with $I(\mu)$, the set of partition $\emptyset \leqslant \lambda \leqslant \mu$
(inclusion of Ferrers Young diagrams)

with $\underset{\text{Tamari lattice}}{T \leqslant T'} \Rightarrow \underset{\text{Young lattice}}{f(T) \geqslant f(T')}$

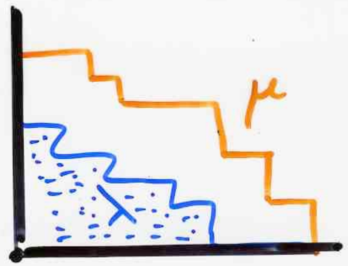
$$J(w) \xrightarrow[\text{bijection}]{f} I(\mu)$$

Young lattice

partition μ

$I(\mu)$

$\lambda \leqslant \mu$



initial segment in the Young lattice (or lower ideal)