The birth of a new domain: combinatorial physics

IMSc, Chennai
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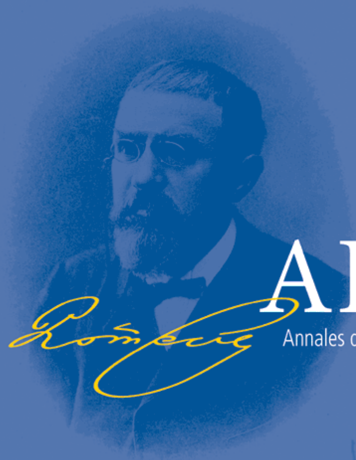


AIHPD

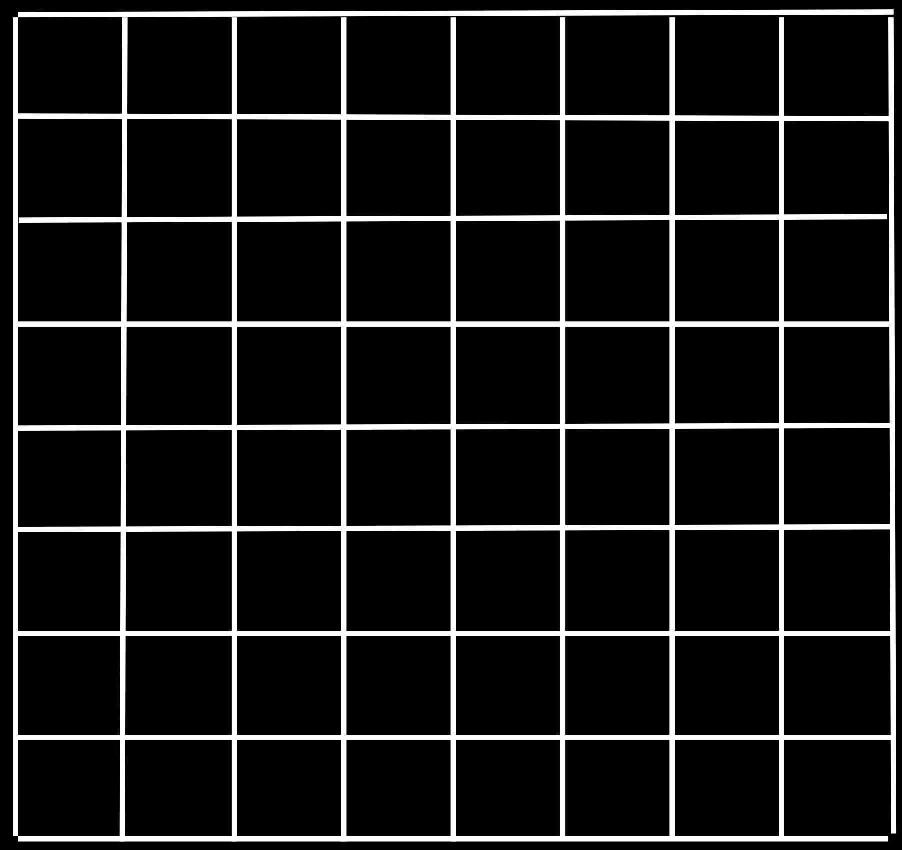
Annales de l'Institut Henri Poincaré – D

Combinatorics, Physics and their Interactions

European Mathematical Society

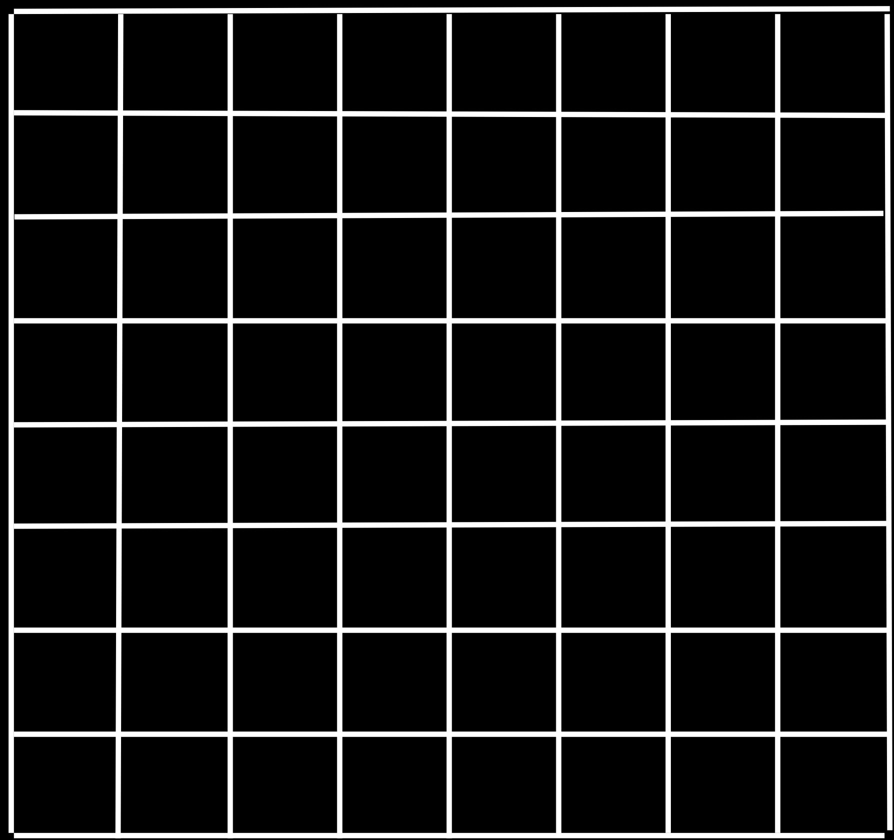
counting problems

chessboard

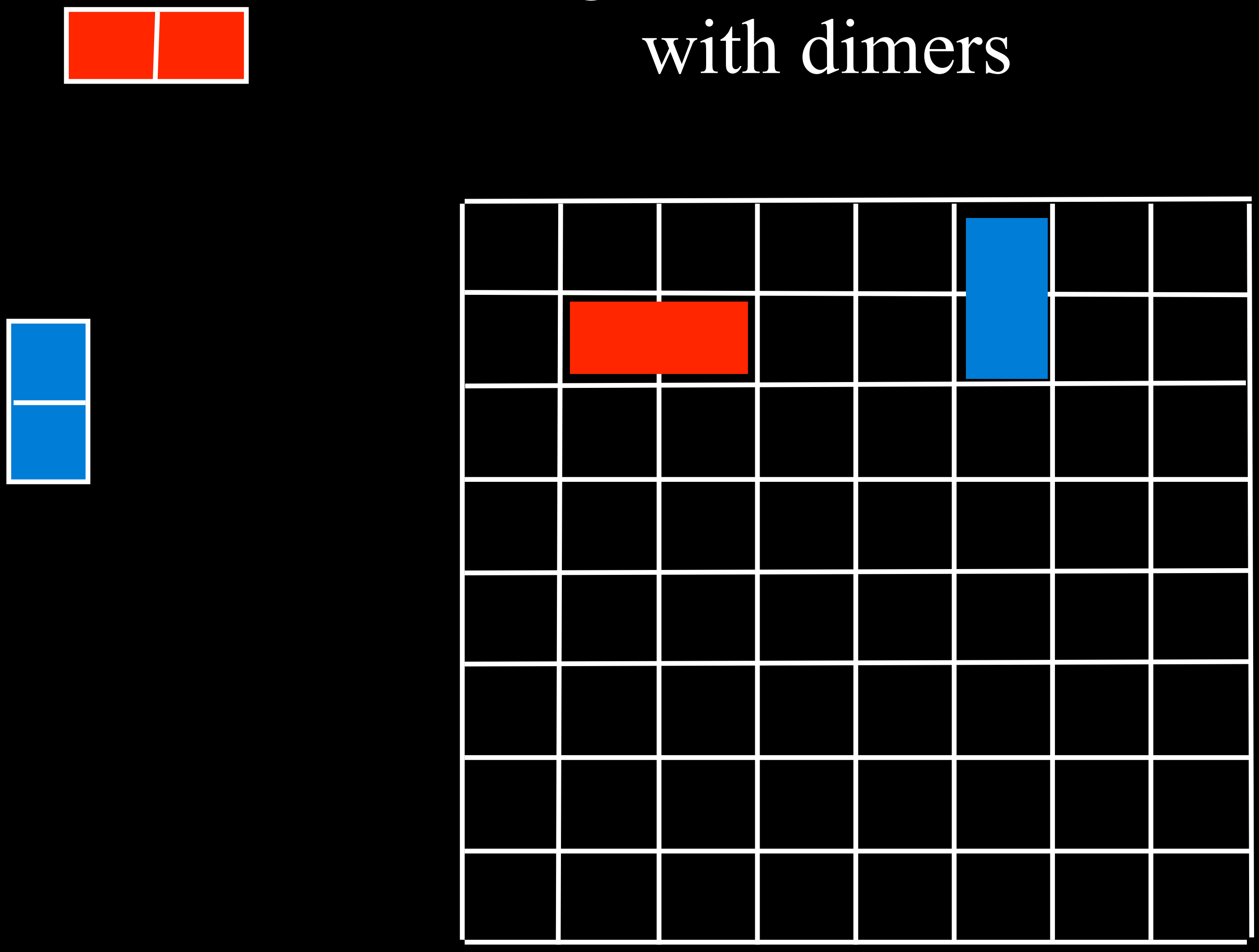


8 rows
8 columns

dimers



tiling of the chesboard
with dimers



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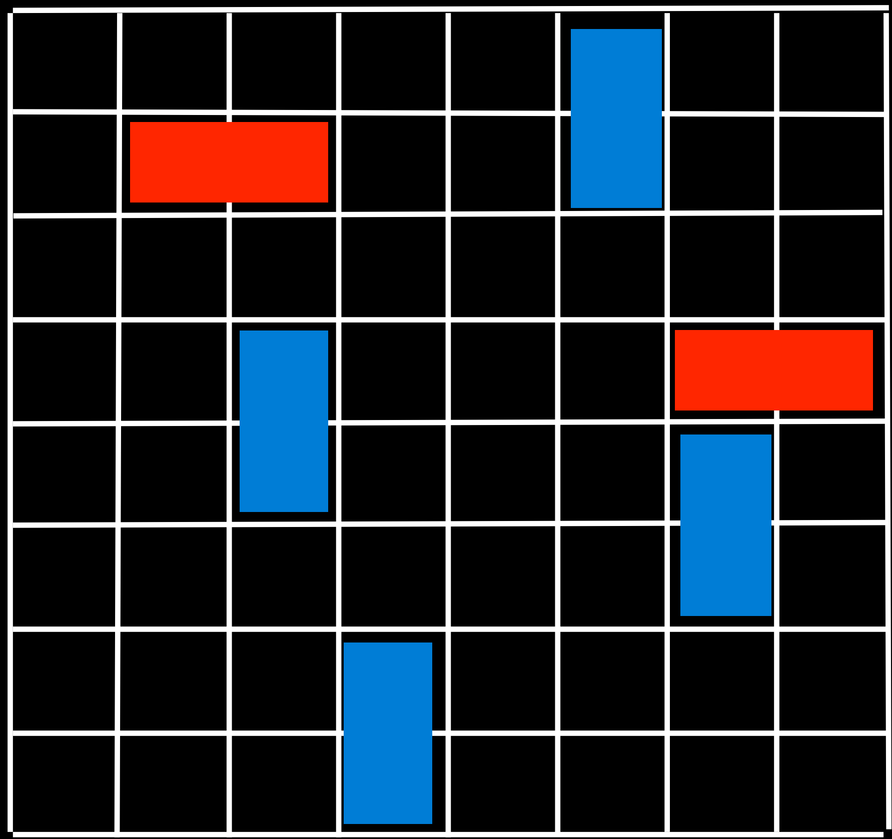

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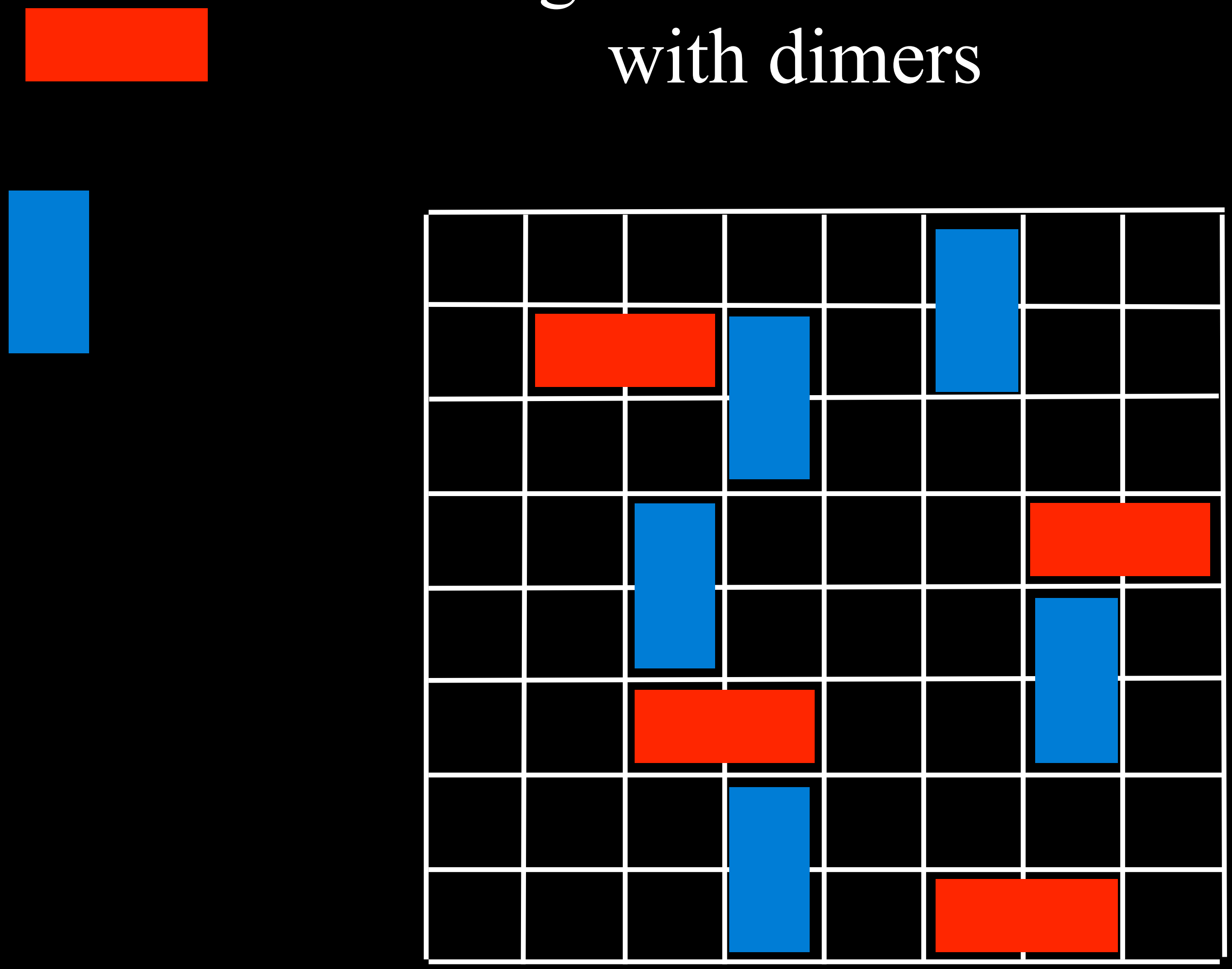

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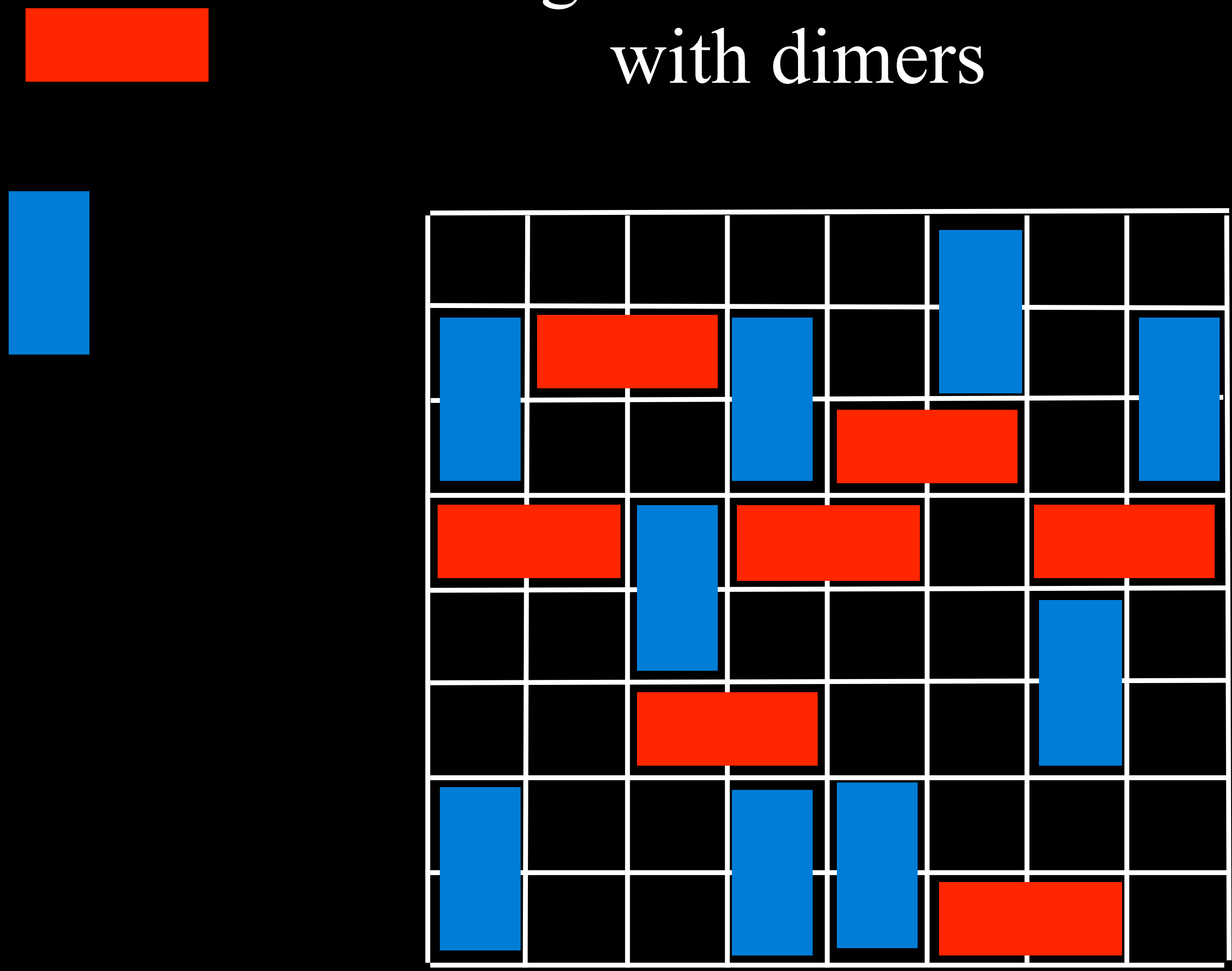

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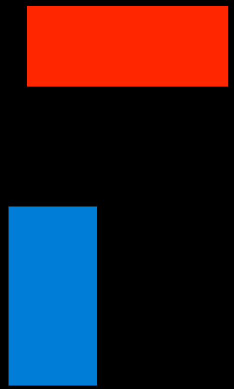
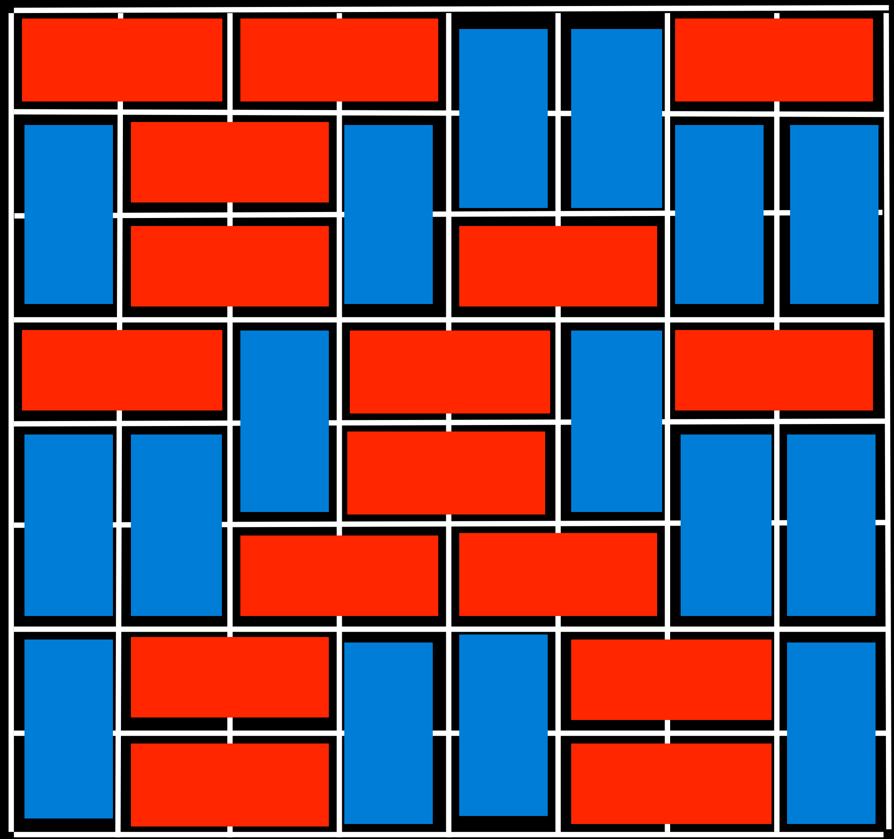




the number of tilings of a 8 x 8 chessboard is 12 988 816 formula

for the number of tilings

of a $m \times n$ rectangle ?

enumerative combinatorics

the number of tilings with dimers of a $m \times n$ rectangle is equal to the product

$$4^{mn} \prod_{i=1}^{m/2} \prod_{j=1}^{n/2} \left(4 \cos^2 \frac{i\pi}{m+1} + 4 \cos^2 \frac{j\pi}{n+1} \right)$$

Kasteleyn (1961)

it is an integer !!

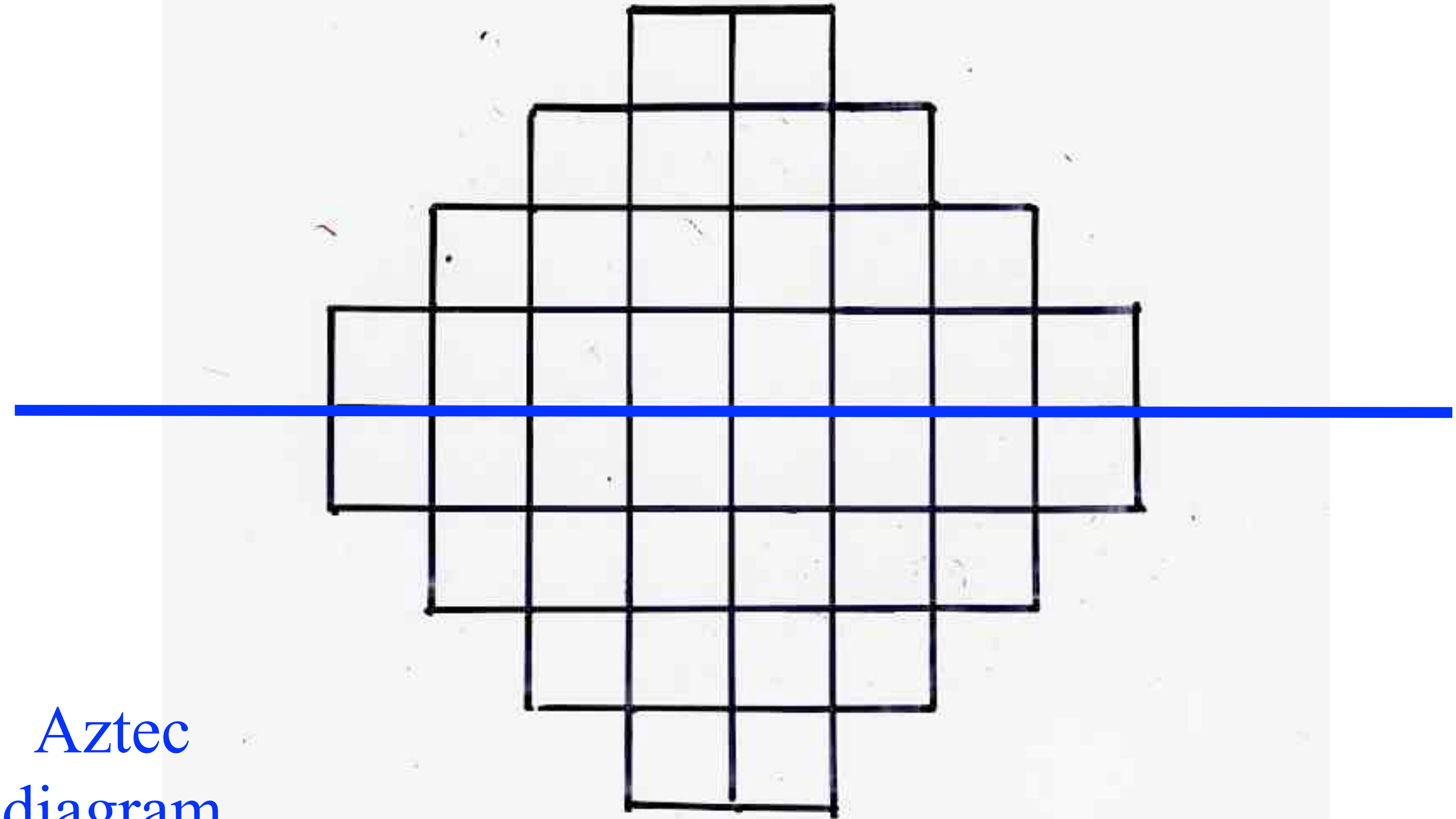
for the chessboard m=8, n=8: 12 988 816

enumerative combinatorics

$$a_n = ?$$

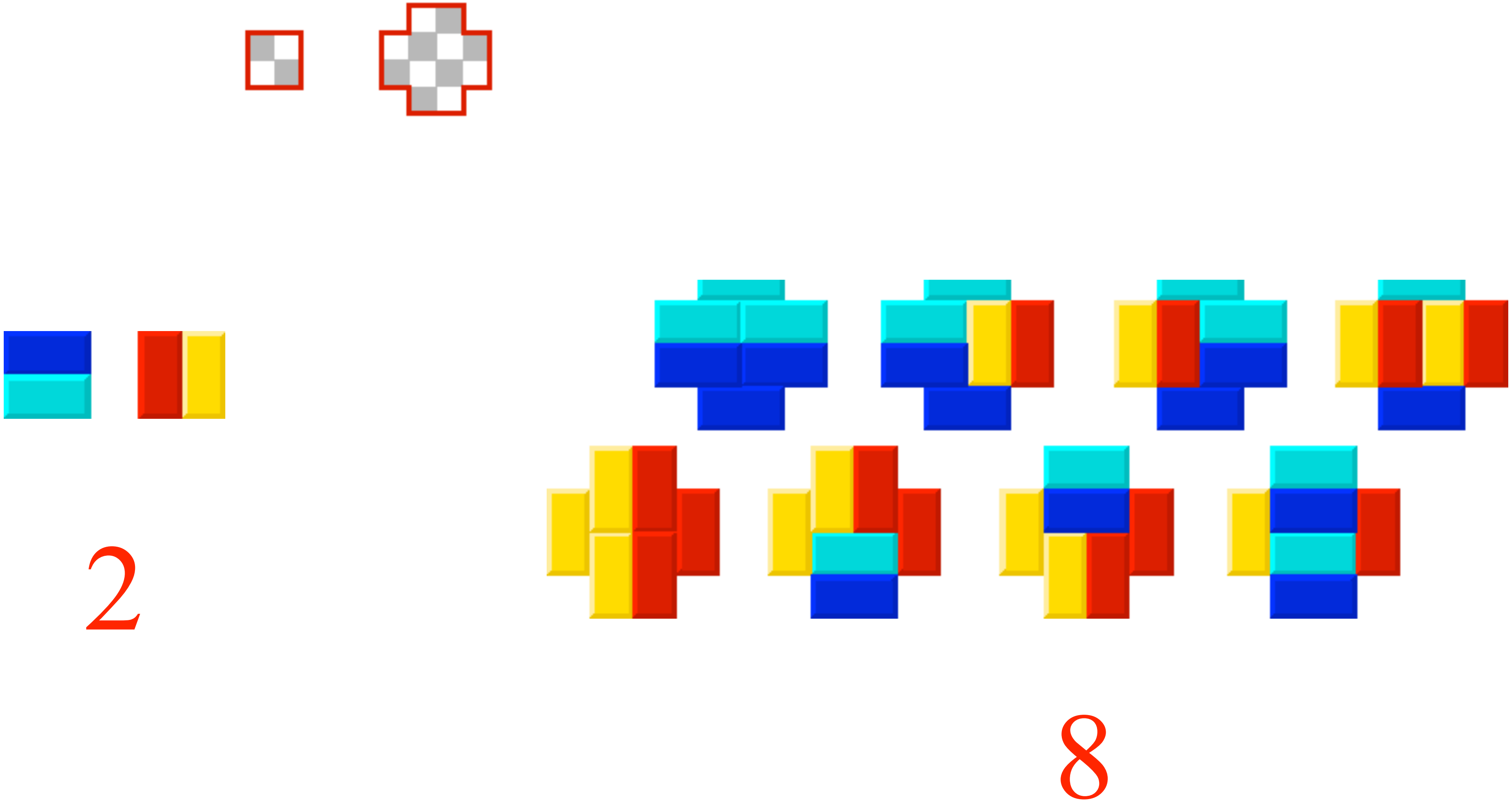
A nother formula

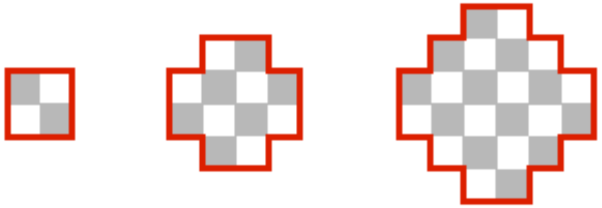
for Aztec tilings



Aztec diagram

2


2
8

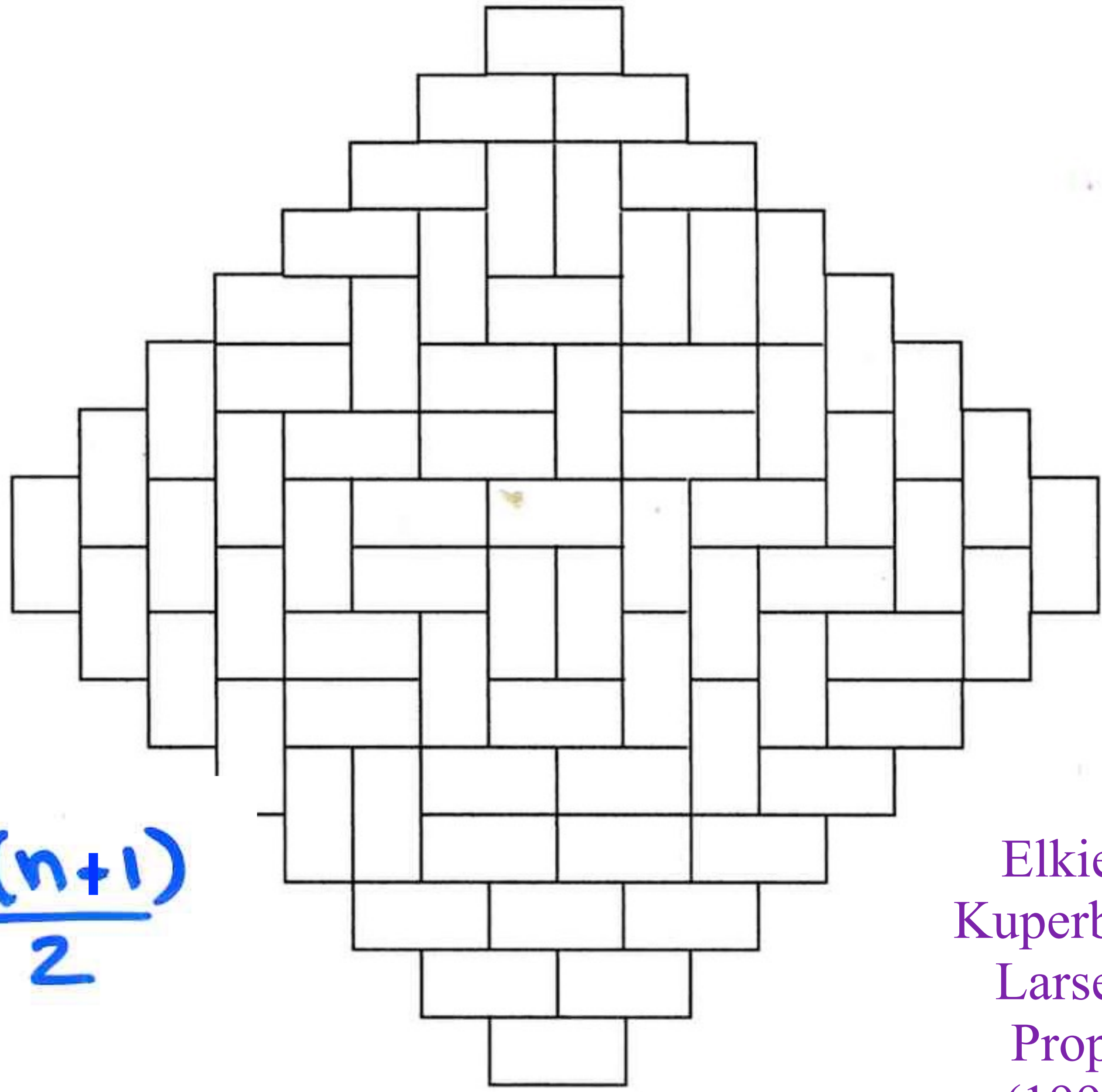


number of tilings	2	8	64	1024
	2^1	2^3	2^6	2^{10}
	2^1	$2^{(1+2)}$	$2^{(1+2+3)}$	$2^{(1+2+3+4)}$

the number of
tilings of
the Aztec
diagram
with dimers
is

$$2^{(1+2+3+4+....+n)}$$

$$2^{\frac{n(n+1)}{2}}$$



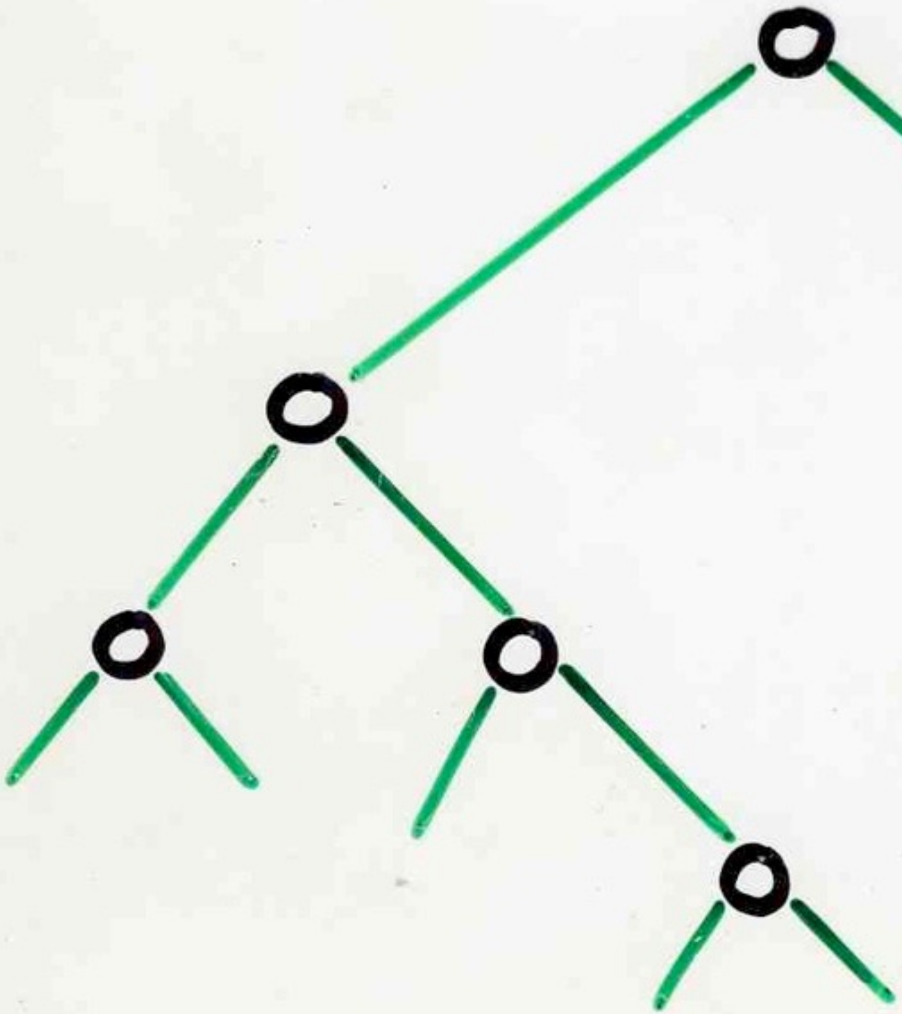
Elkies,
Kuperberg,
Larsen,
Propp
(1992)

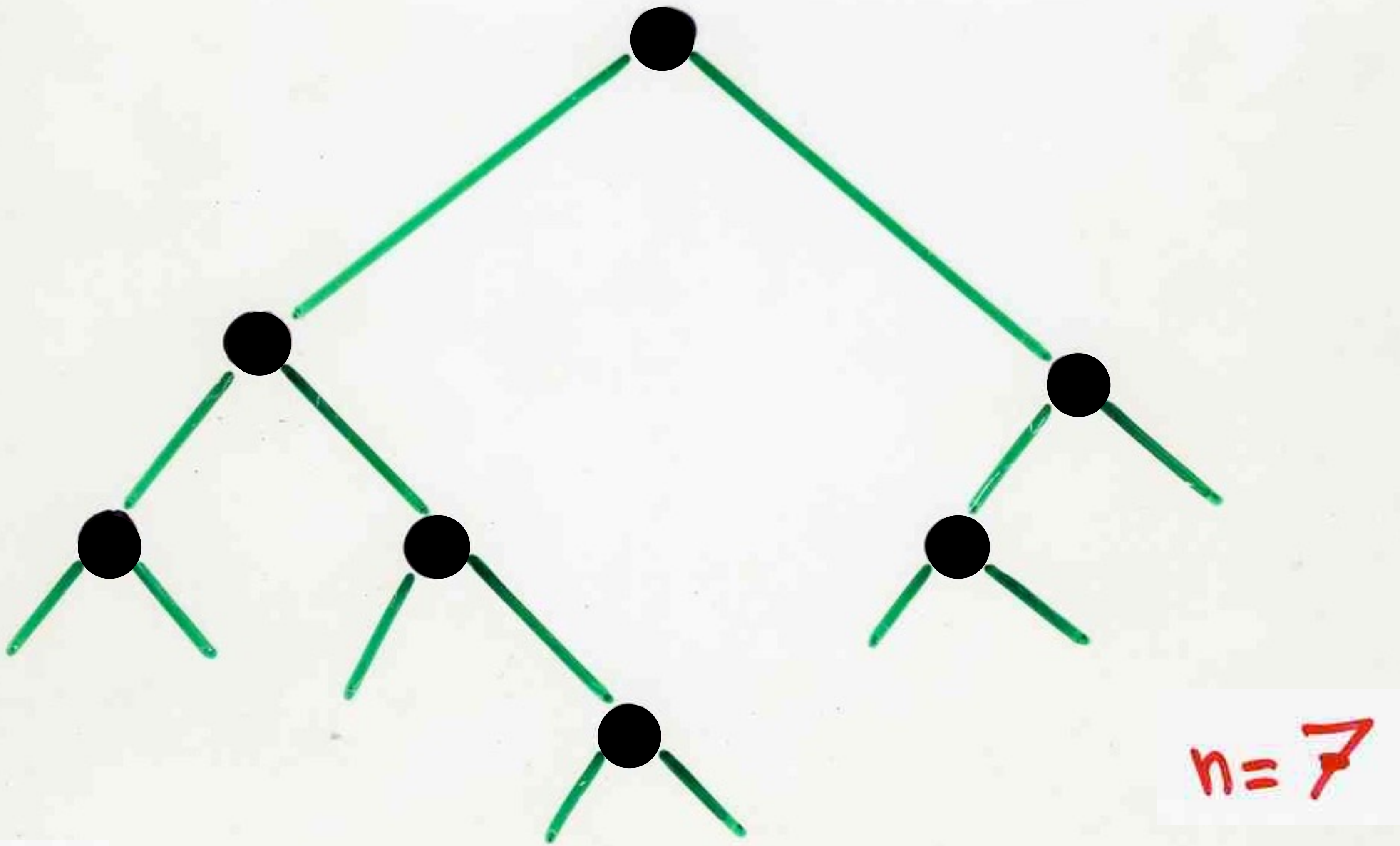
enumerative combinatorics

generating function

example: binary tree

Binary tree

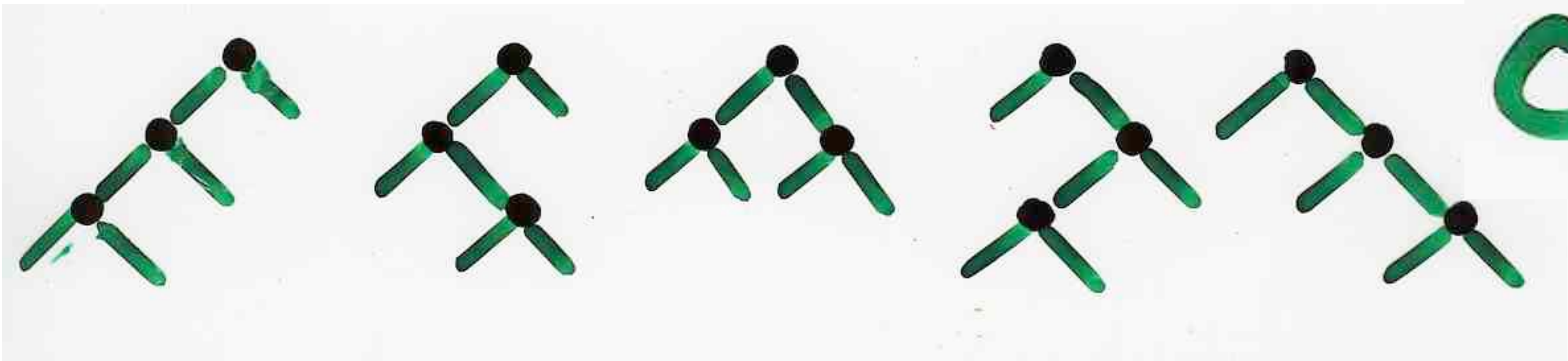




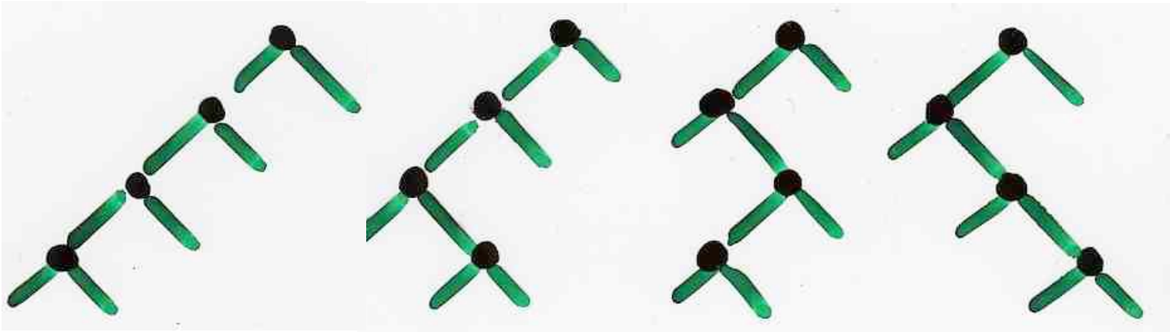
C_n = number of binary trees
having n internal vertices
(or n+1) leaves (external vertices)

$C_1 = 1$

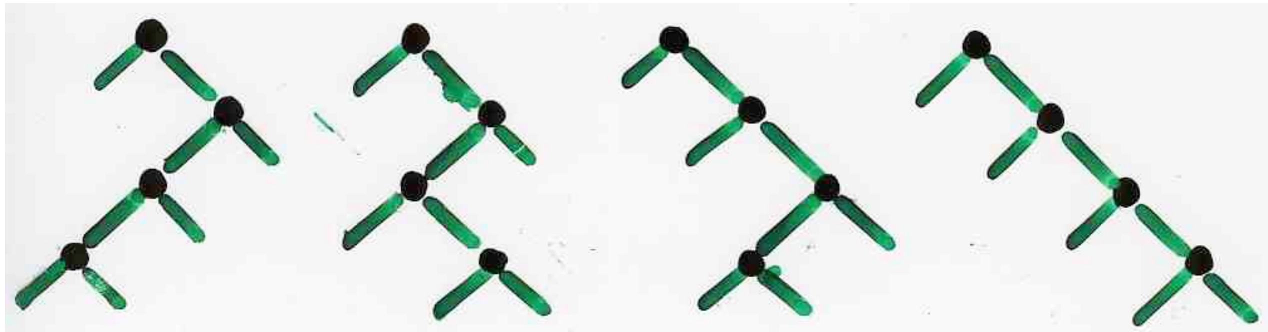
$C_2 = 2$

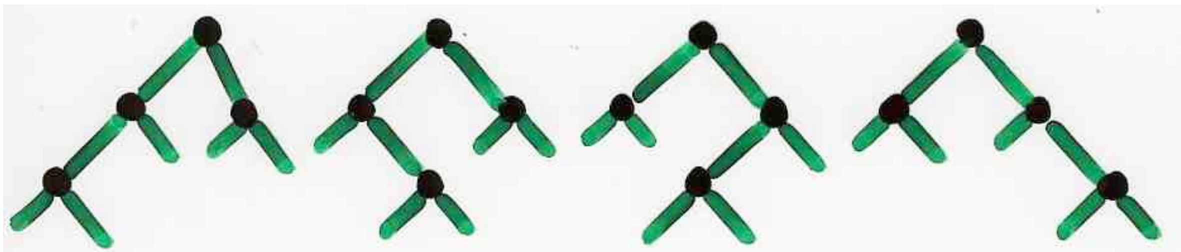


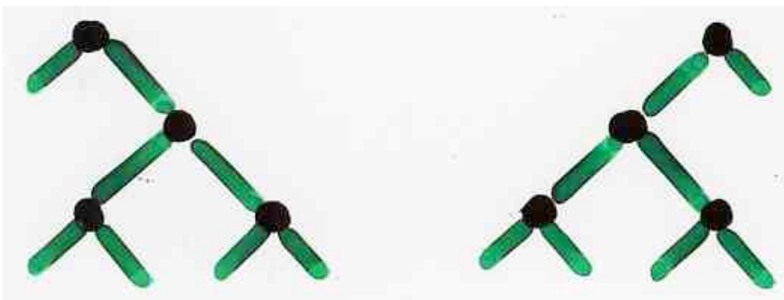
$C_3 = 5$

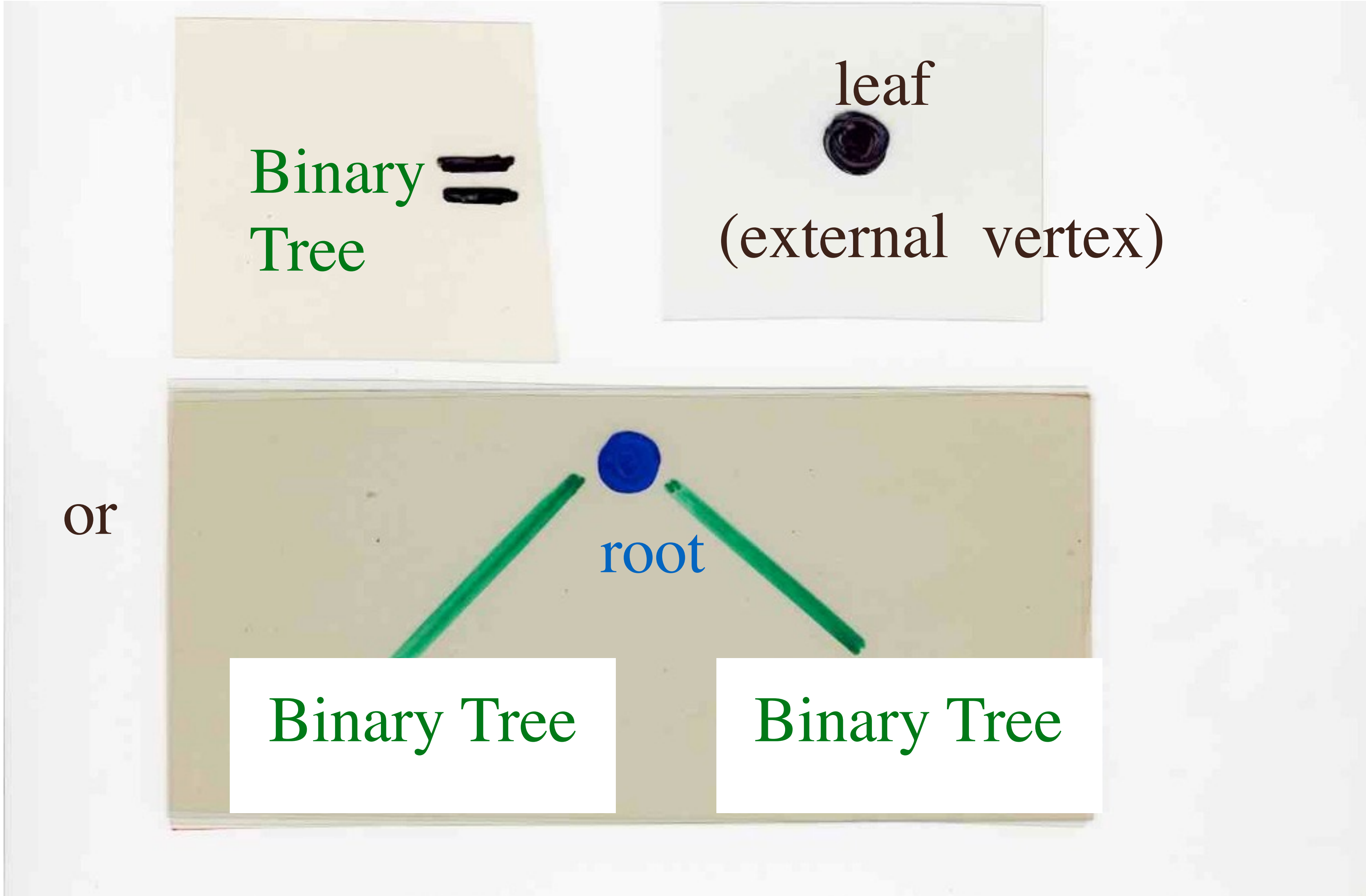


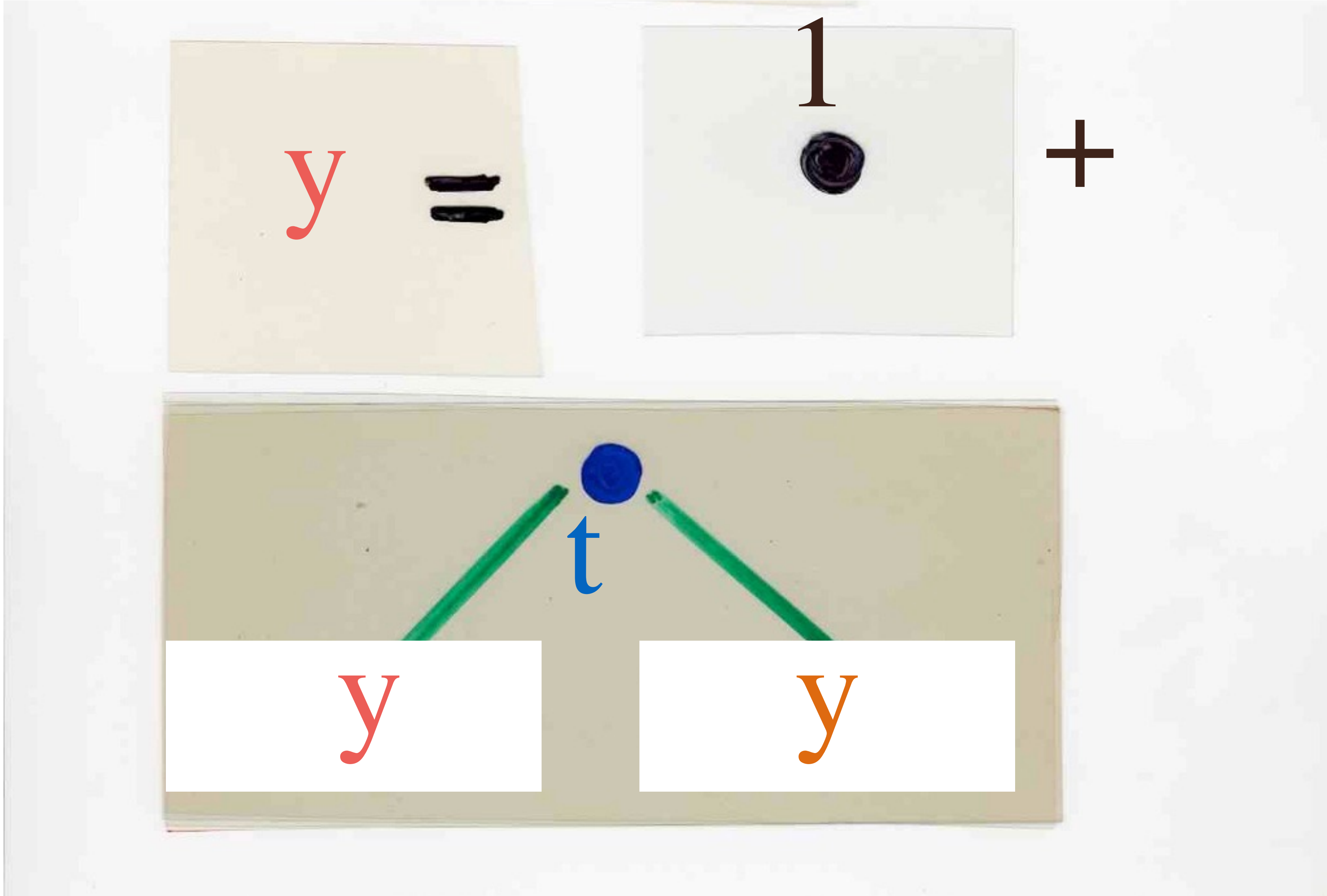
$C_4 = 14$



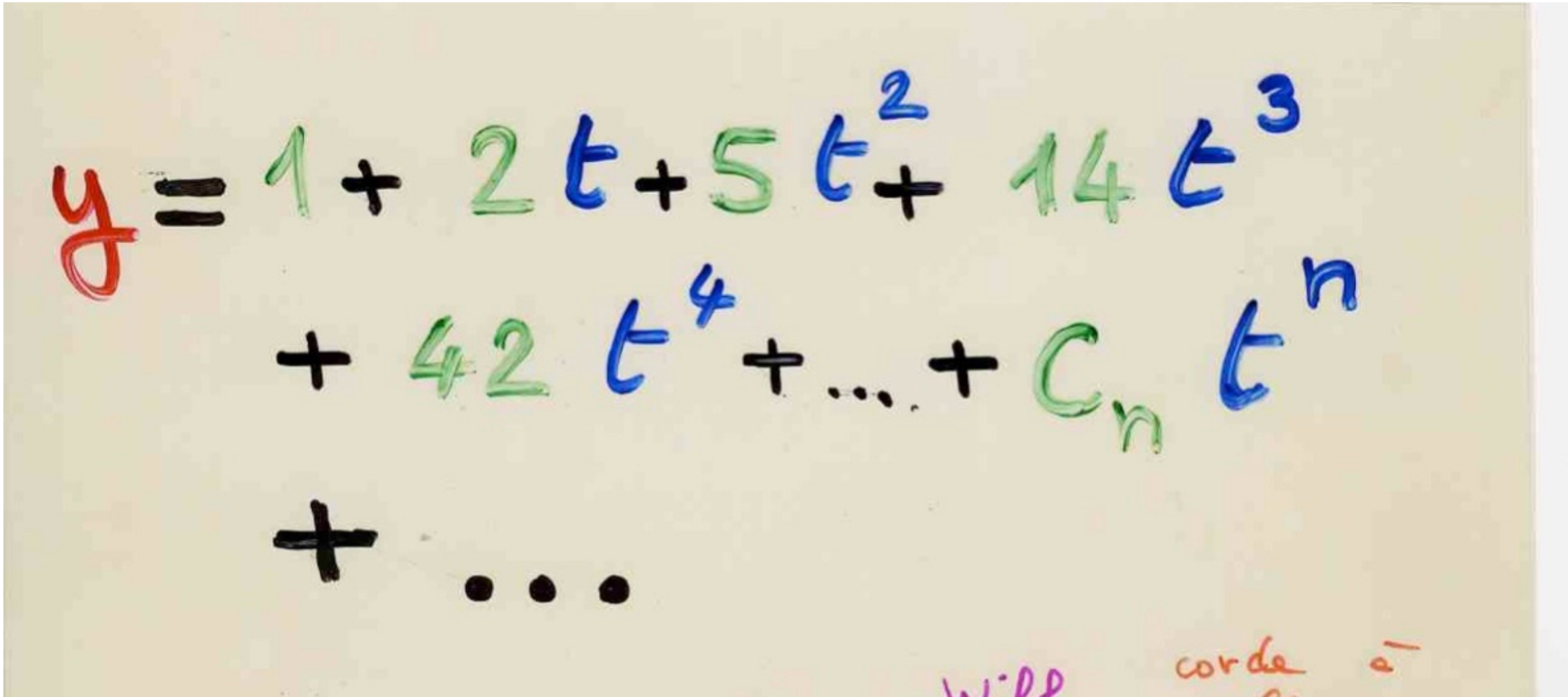





Binary Tree =
leaf
(external vertex)
or
root
Binary Tree
Binary Tree


y =
1
+
t
y
y

$$y = 1 + t\,y^2$$



$$y = 1 + t\,y^2$$

$$y = \frac{1 - (1 - 4t)^{1/2}}{2t}$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\ n!}$$

$$n! = 1 \times 2 \times \dots \times n$$

statistical mechanics

Ising model

phase transition

critical phenomena

Physics

exactly solved model

Baxter (1982)

Ising model

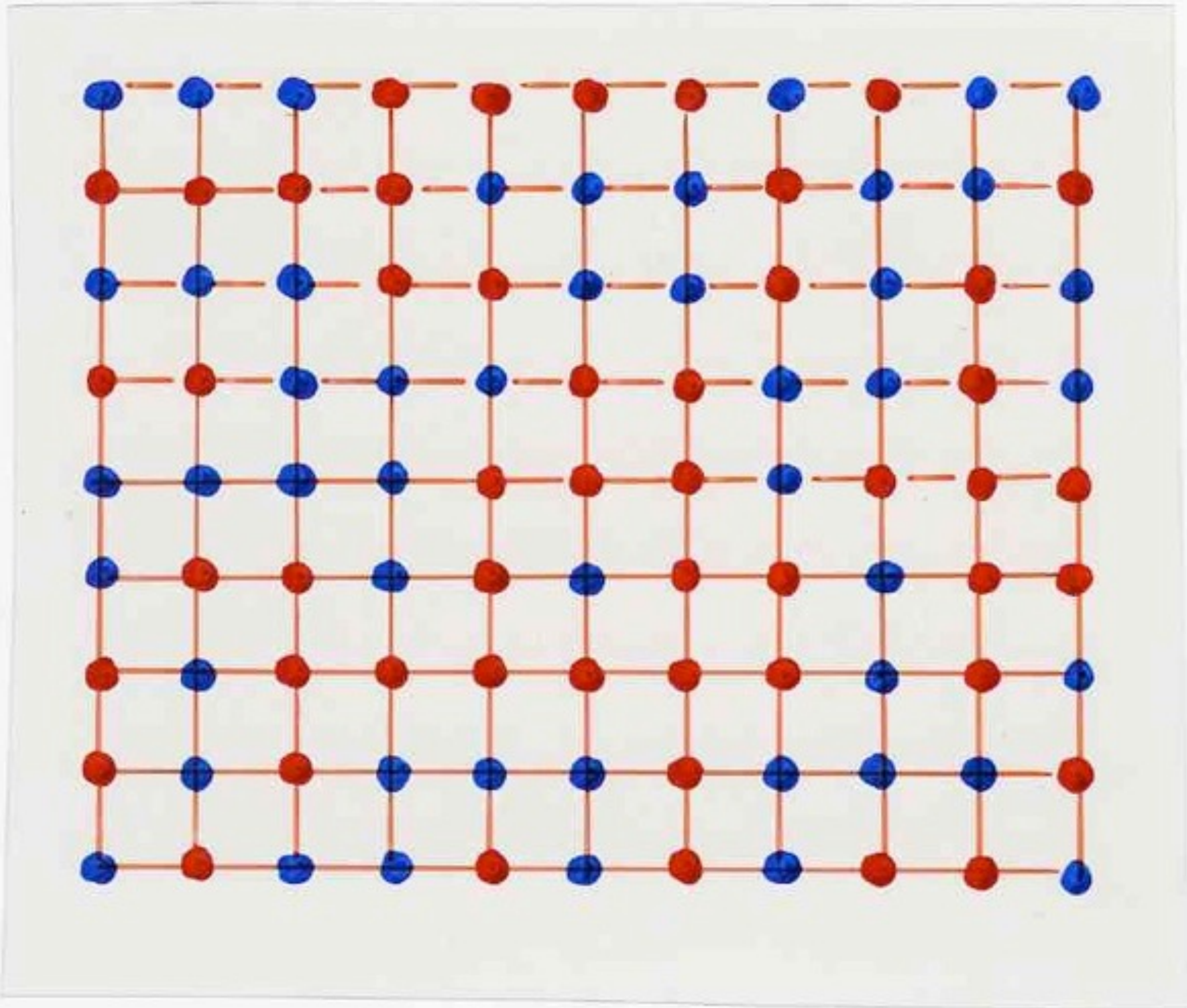
Onsager (1944)

Potts, ice model

Temperley-Lieb (1971)

Baxter (1982)

exactly solved models

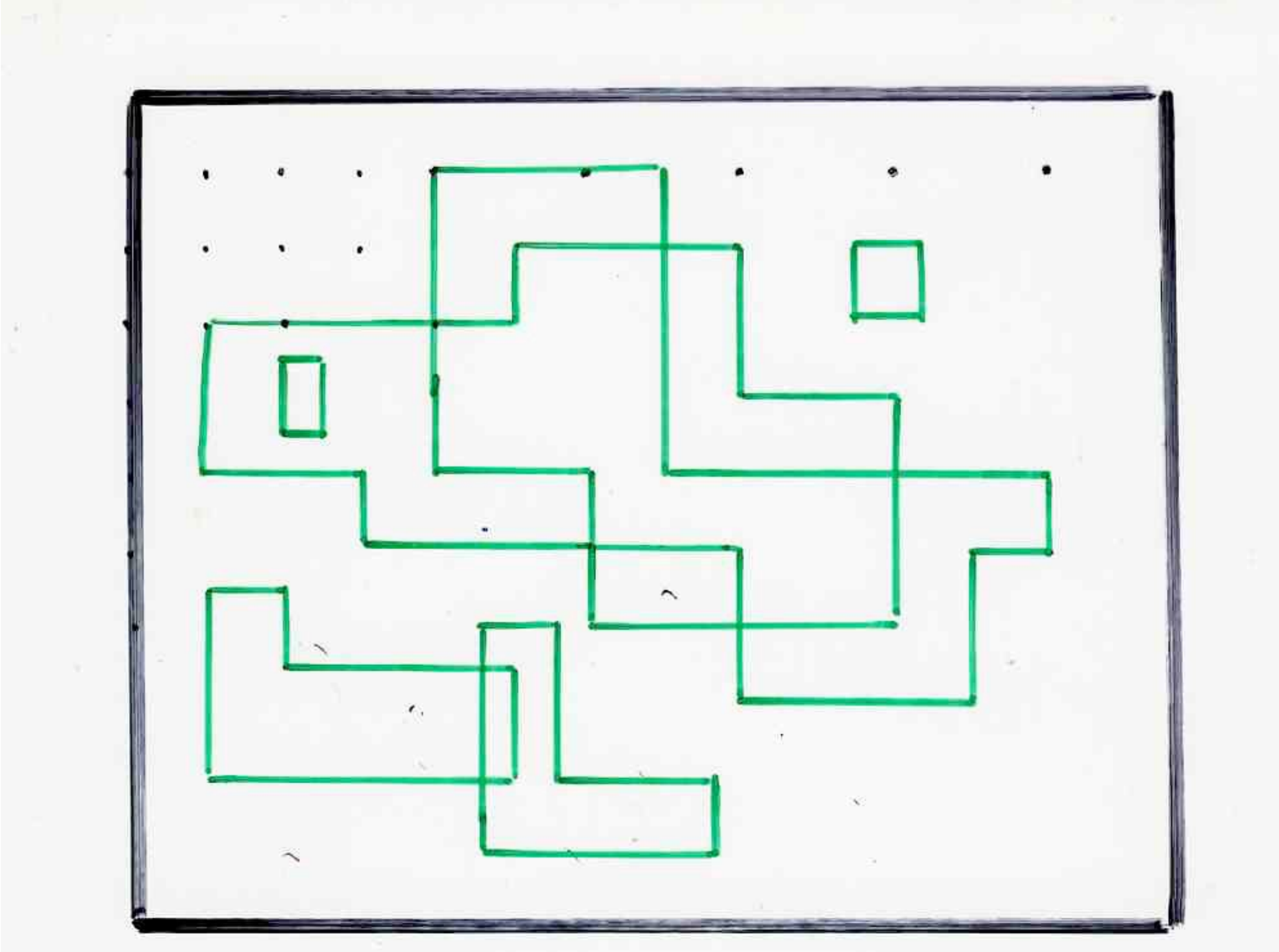


Partition function

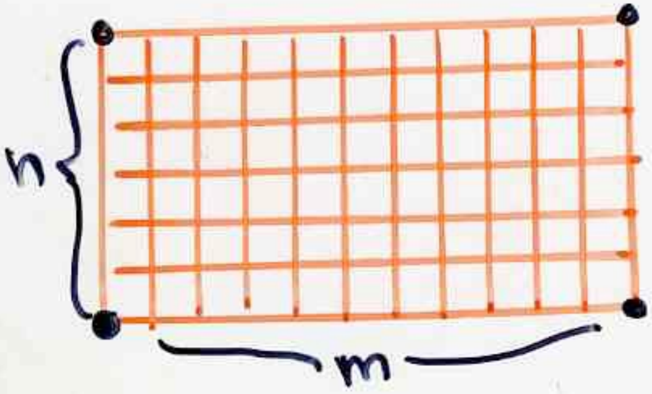
$$Z_L = \sum_{\sigma} \exp\left(-E_{\sigma}/kT\right)$$

k Boltzmann constant

T temperature



combinatorial resolution:
dimers tilings and Pfaffian methodology



thermodynamic limit

$N = nm$

"$N \to \infty$"

$$Z = \lim_{\text{"}N\to\infty\text{"}} Z^{1/N}$$

.

generating function in combinatorics
and
thermodynamical function in
statistical mechanics

Statistical physics

$$F(T) \cong \frac{1}{(T - T_c)^{\alpha}}$$

thermodynamic function

temperature

critical temperature

critical exponent

- $F(t) = \sum_{n \geqslant 0} a_n t^n$

  ↑ a_n: number of ....

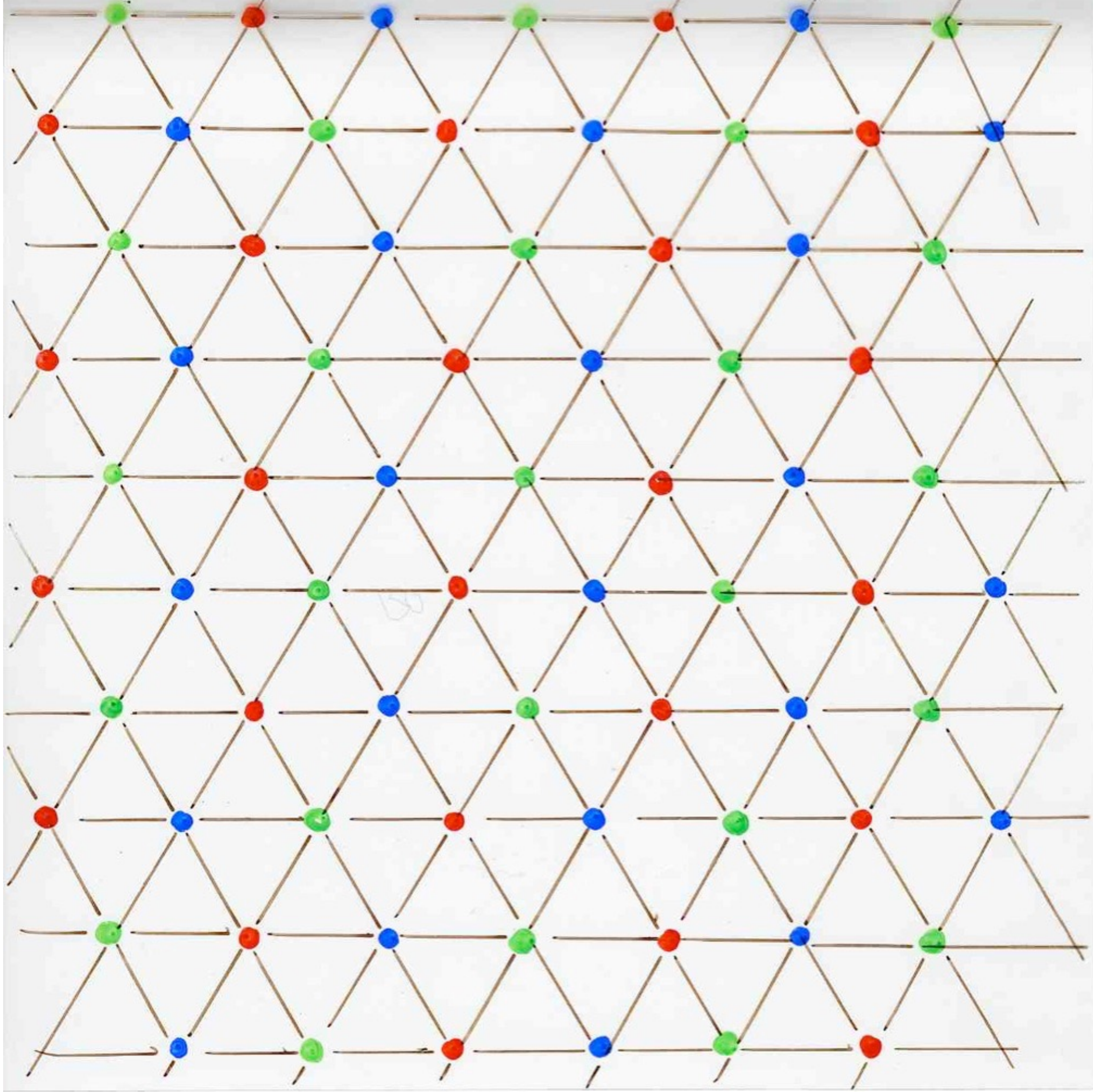
- $a_n \simeq \mu^n n^{-\theta}$

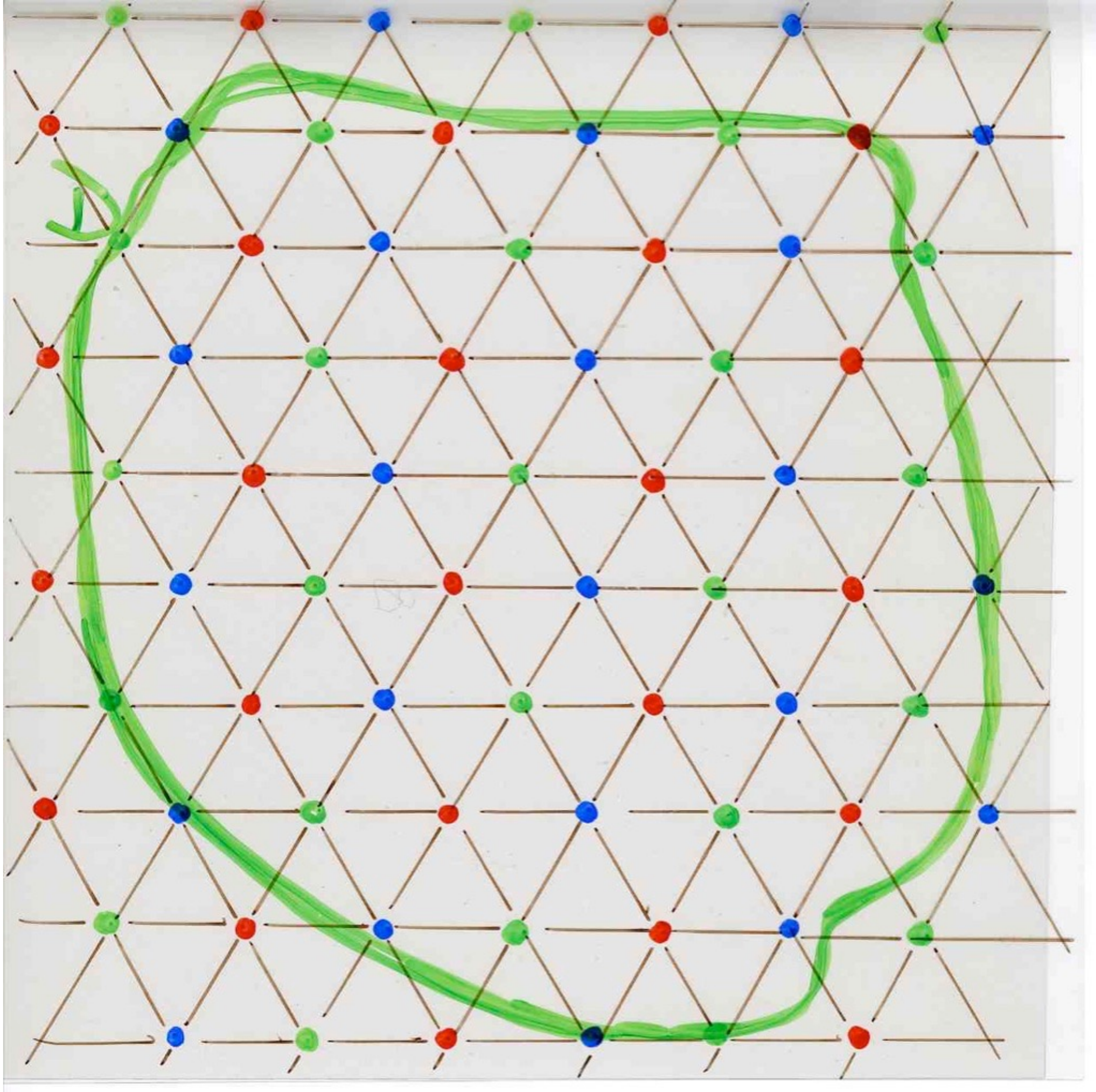
  μ: connective constant

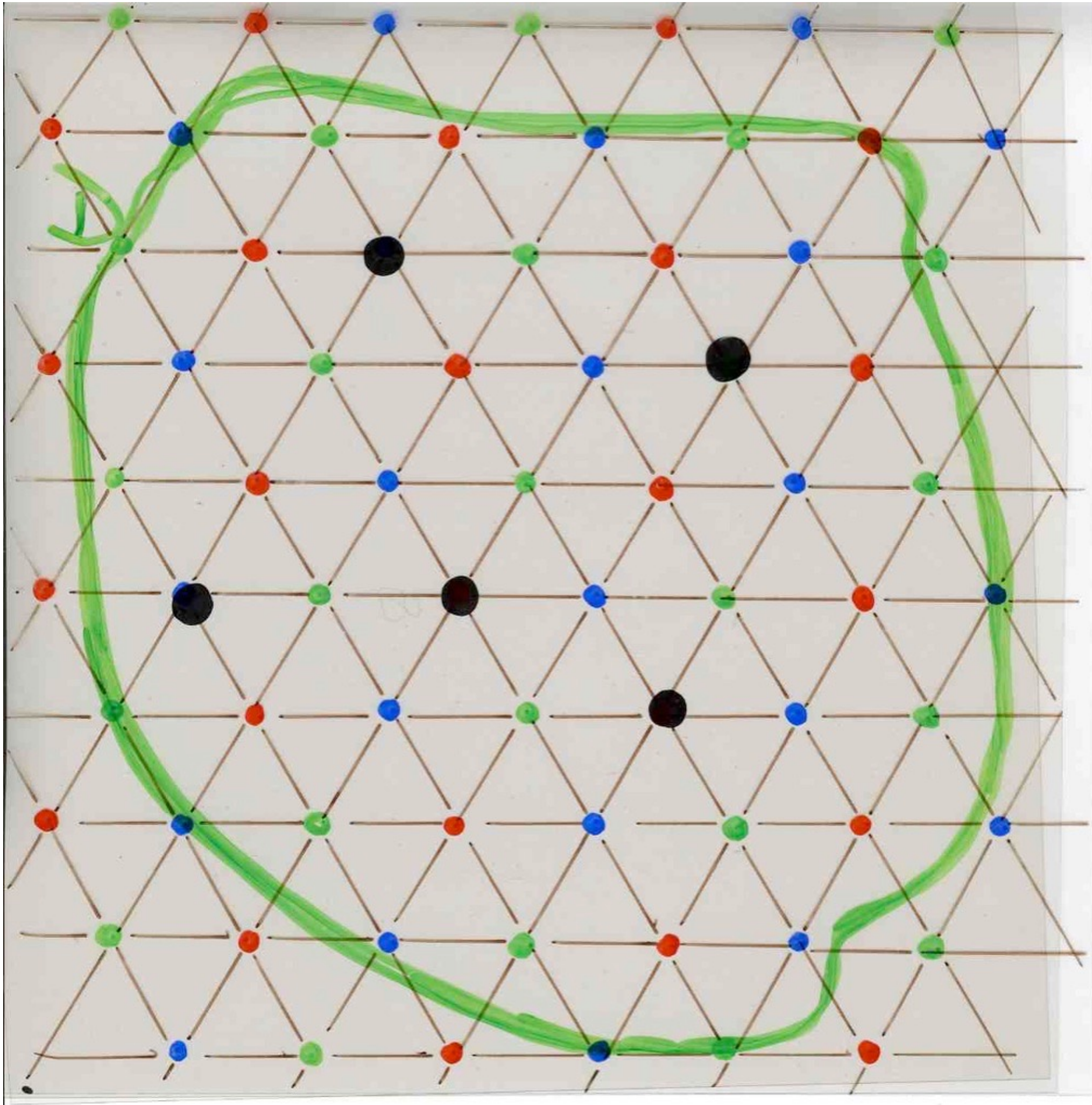
  θ: critical exponent

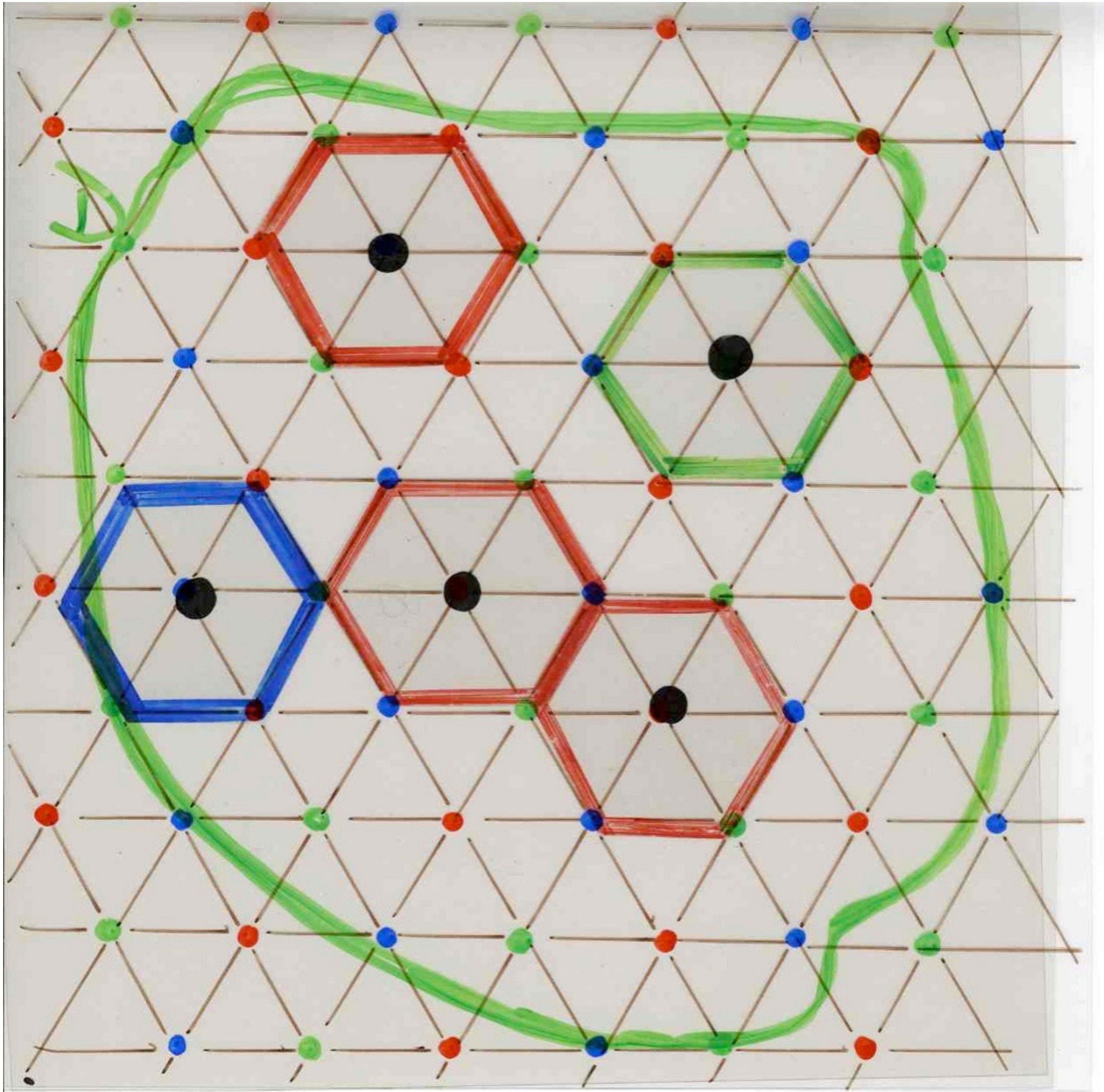
  $$\mu = \frac{1}{t_c}$$

the density of a hard gas model
is the
generating function for the number of
certain « heaps of hexagons »









partition function

$$Z_D(t) = \sum_{n \geq 0} a_{n,D}\, t^n$$

$$Z(t) = \lim_{``D \to \infty"} \left(Z_D(t) \right)^{1/|D|}$$

thermodynamic limit

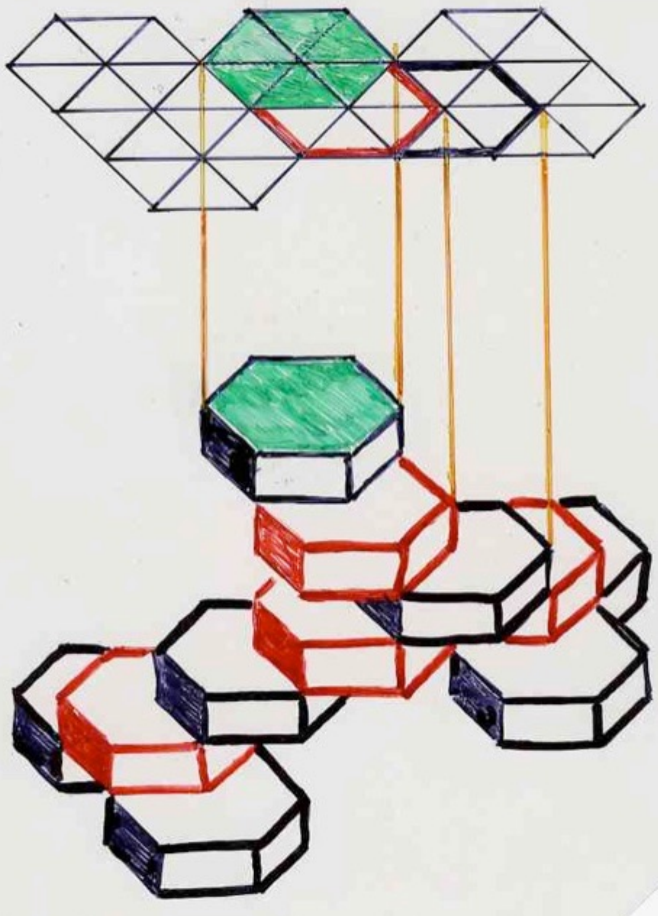
$$\rho(t) = t \frac{d}{dt} \log Z(t)$$

density of a

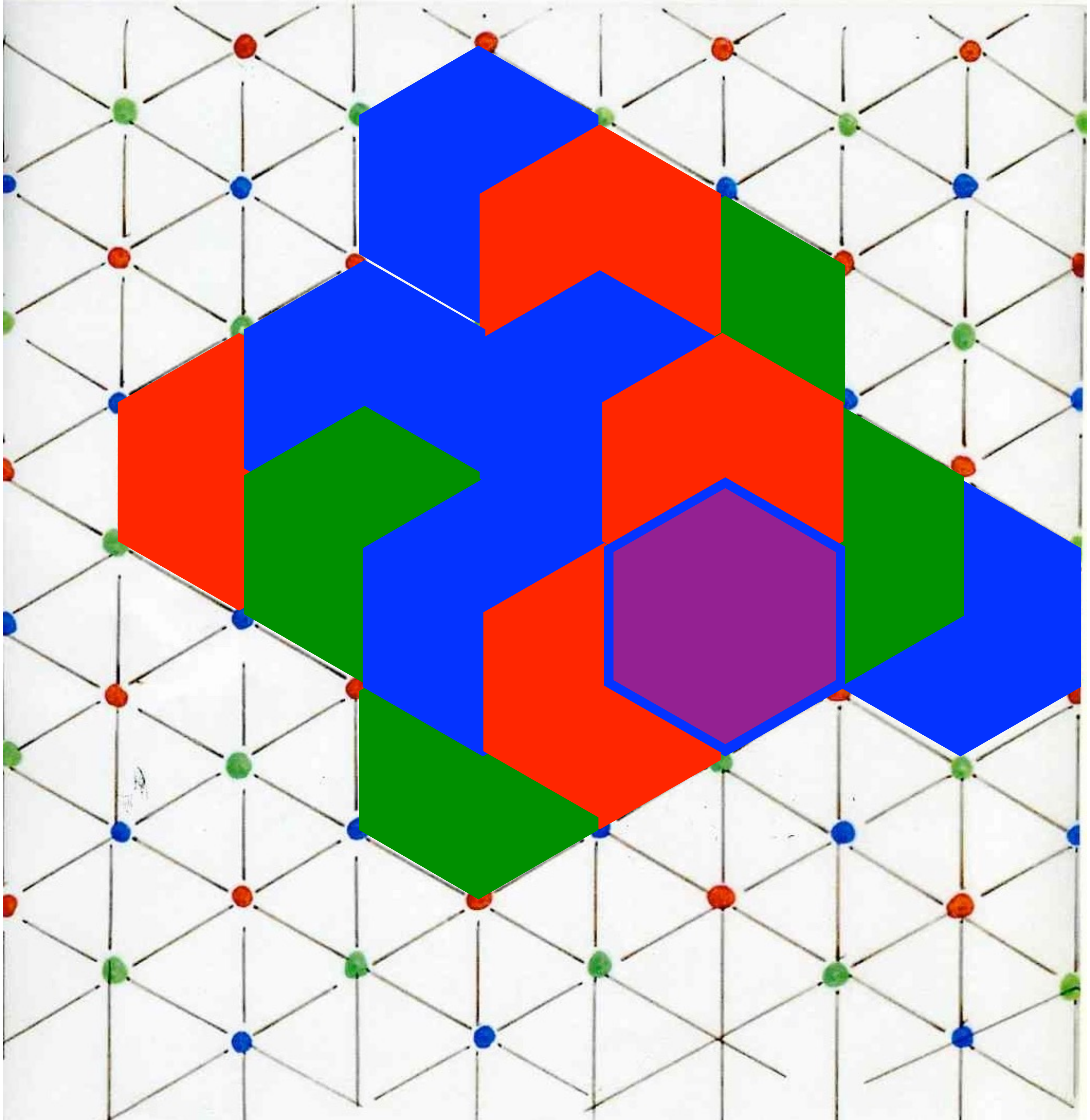
"hard-core" lattice gas model

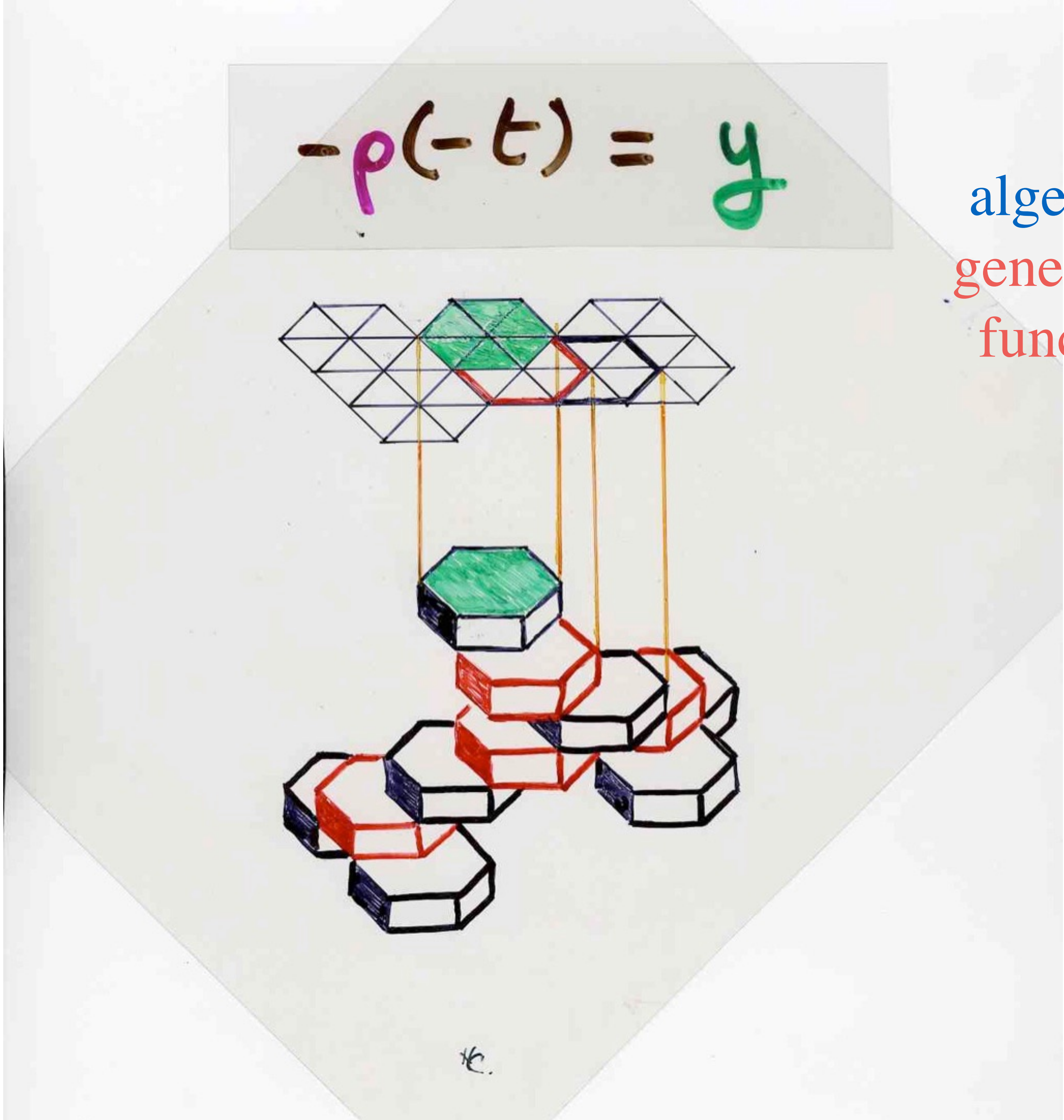
t is the "activity" of the gas

$$-p(-t) = y$$



KC.

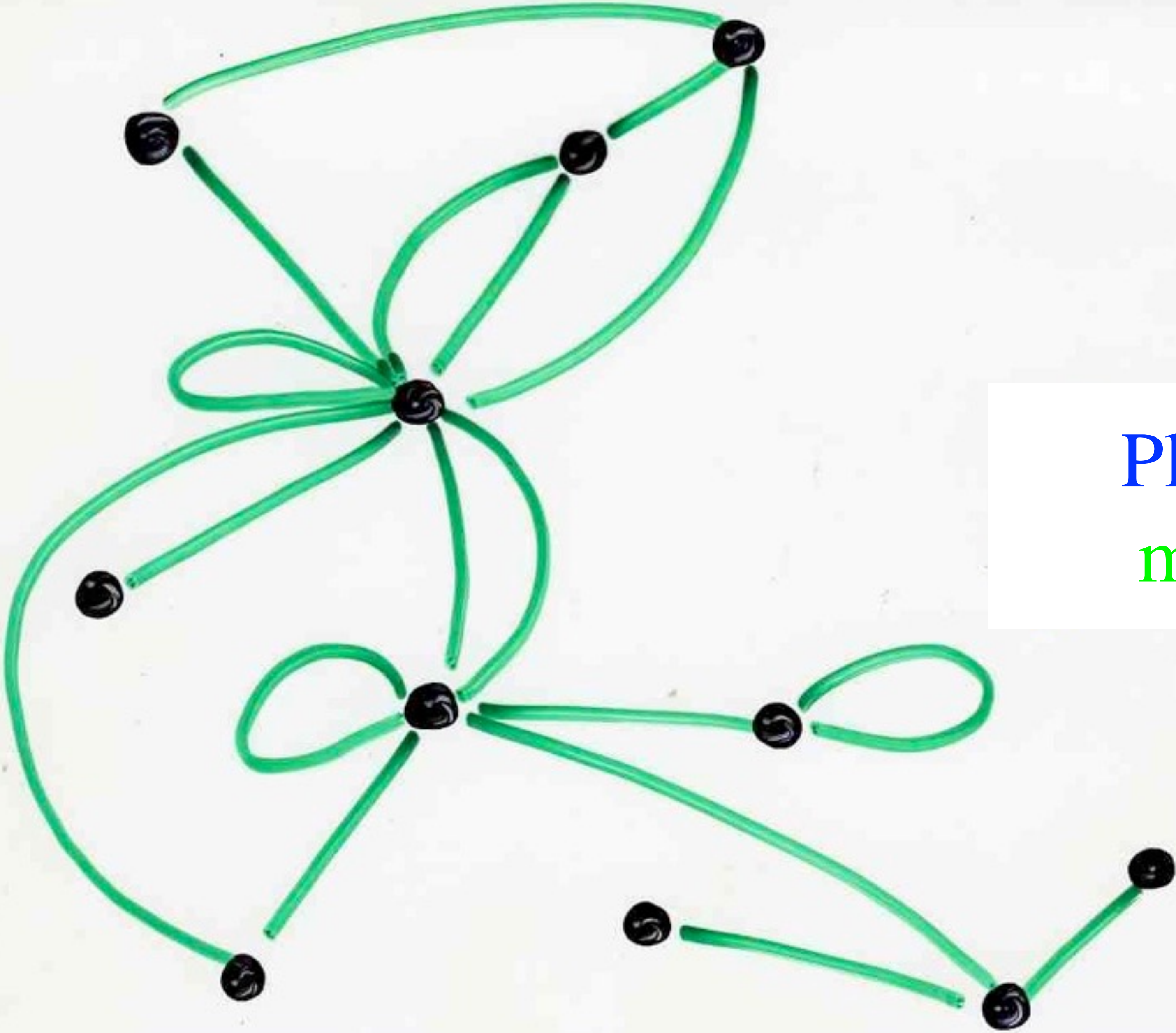




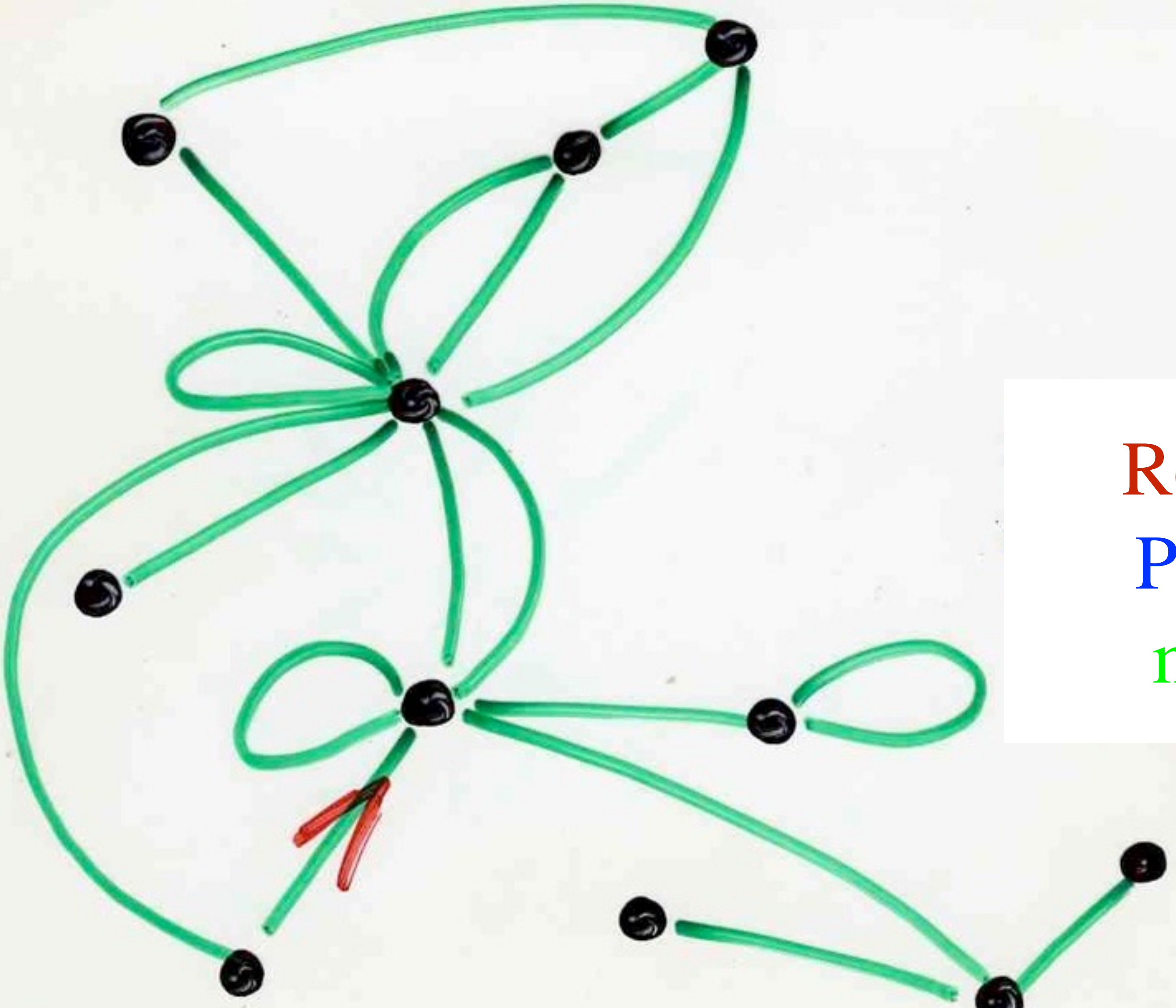
algebraic
generating
function

bijective combinatorics

example: planar maps



Planar
maps



Rooted Planar maps

$$y = A - tA^3$$

$$A = 1 + 3tA^2$$

Tutte (1968)

$$y = A - tA^3$$

Tutte (1968)

$$A = 1 + 3tA^2$$

Tutte (1968)

$$\frac{2 . 3^m}{(n+2)} C_m$$

Catalan

m arêtes

$$y = A - tA^3 \qquad \text{Tutte (1968)}$$
$$A = 1 + 3tA^2$$

Tutte (1968)

$$\frac{2 \cdot 3^m}{(n+2)} C_m$$

Catalan

m arêtes

Cori, Vauquelin (1970, ---)

Arquès (1980, ---)

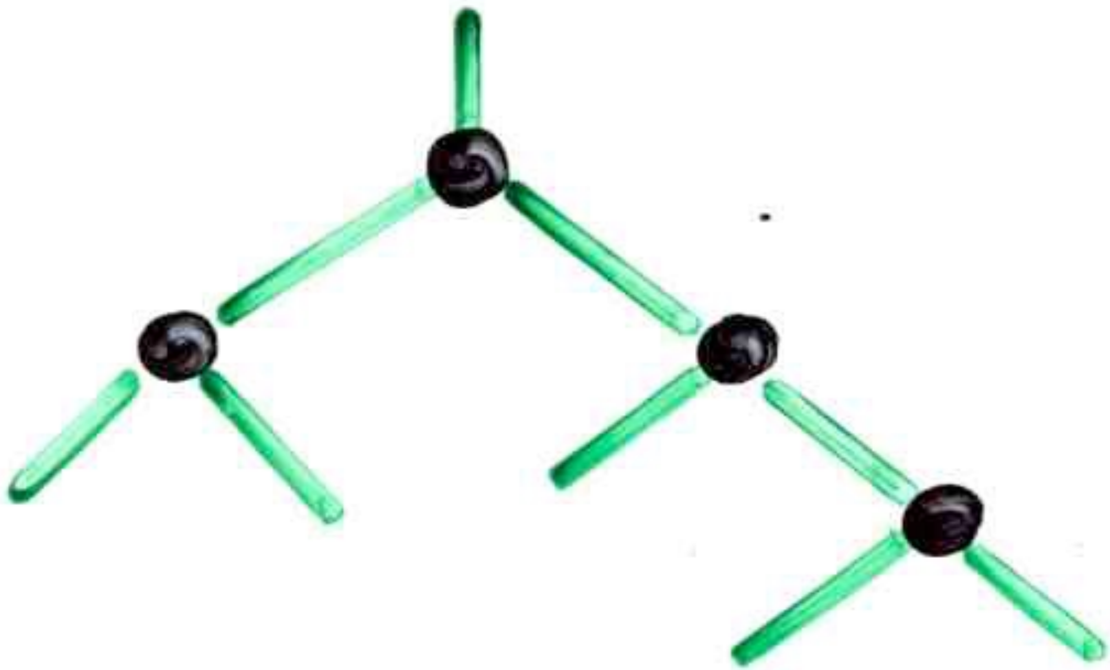
Schaeffer (1997, ---)

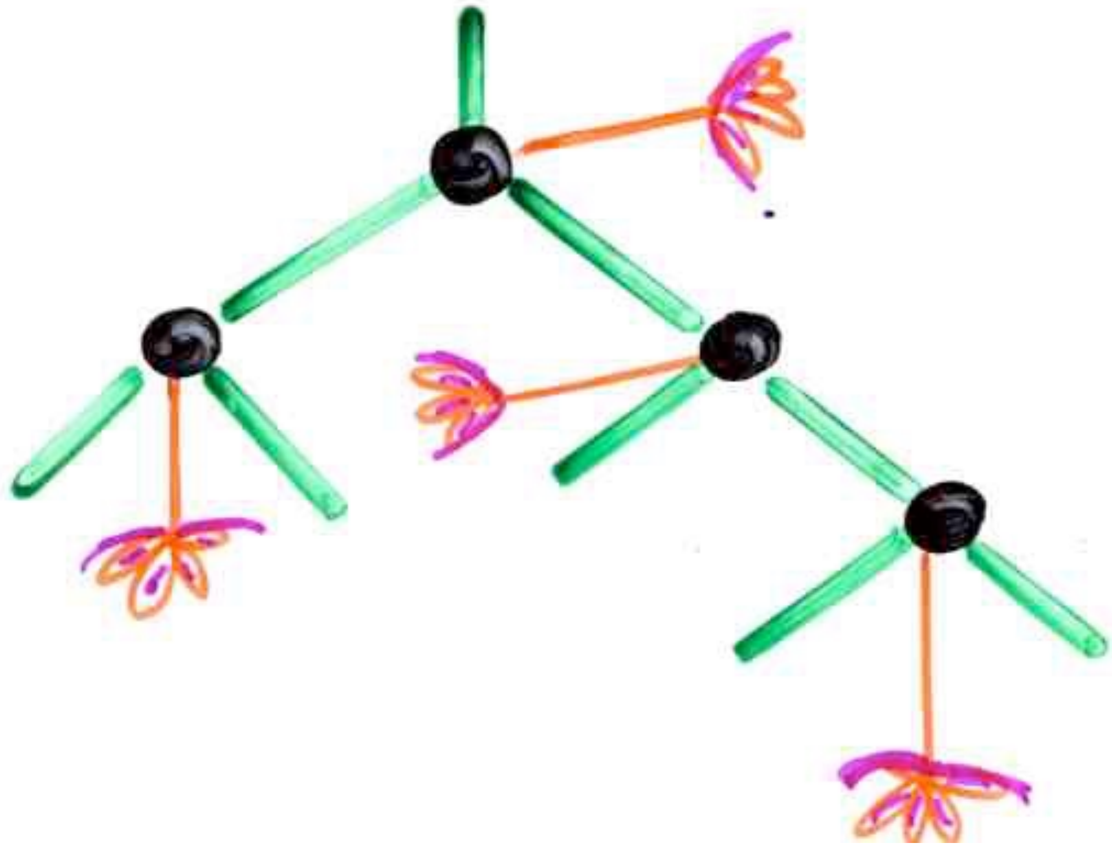
Bouttier, Di Francesco, Guitter (2002, --)

bijection

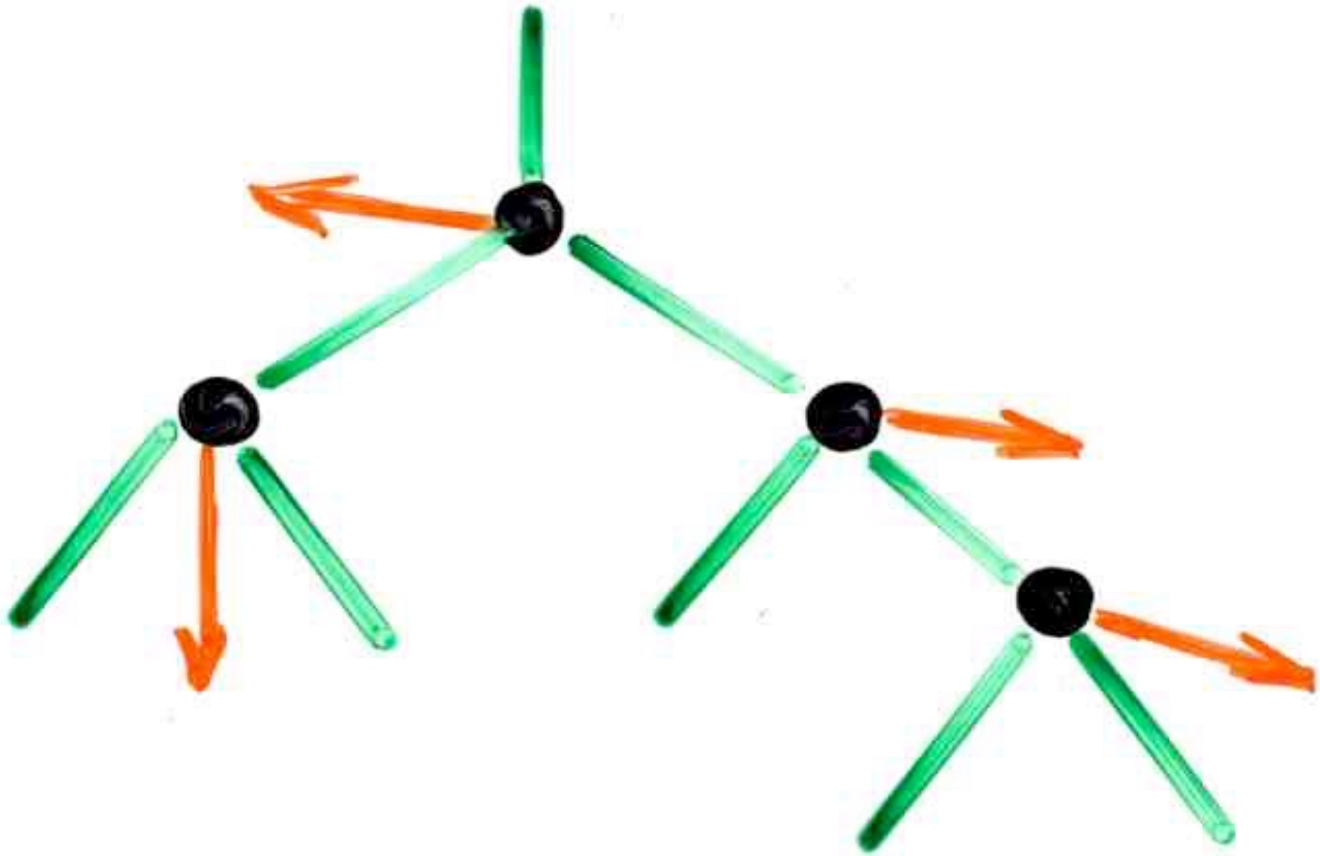
- planar maps (n edges)
- balanced blossoming trees (n nodes)

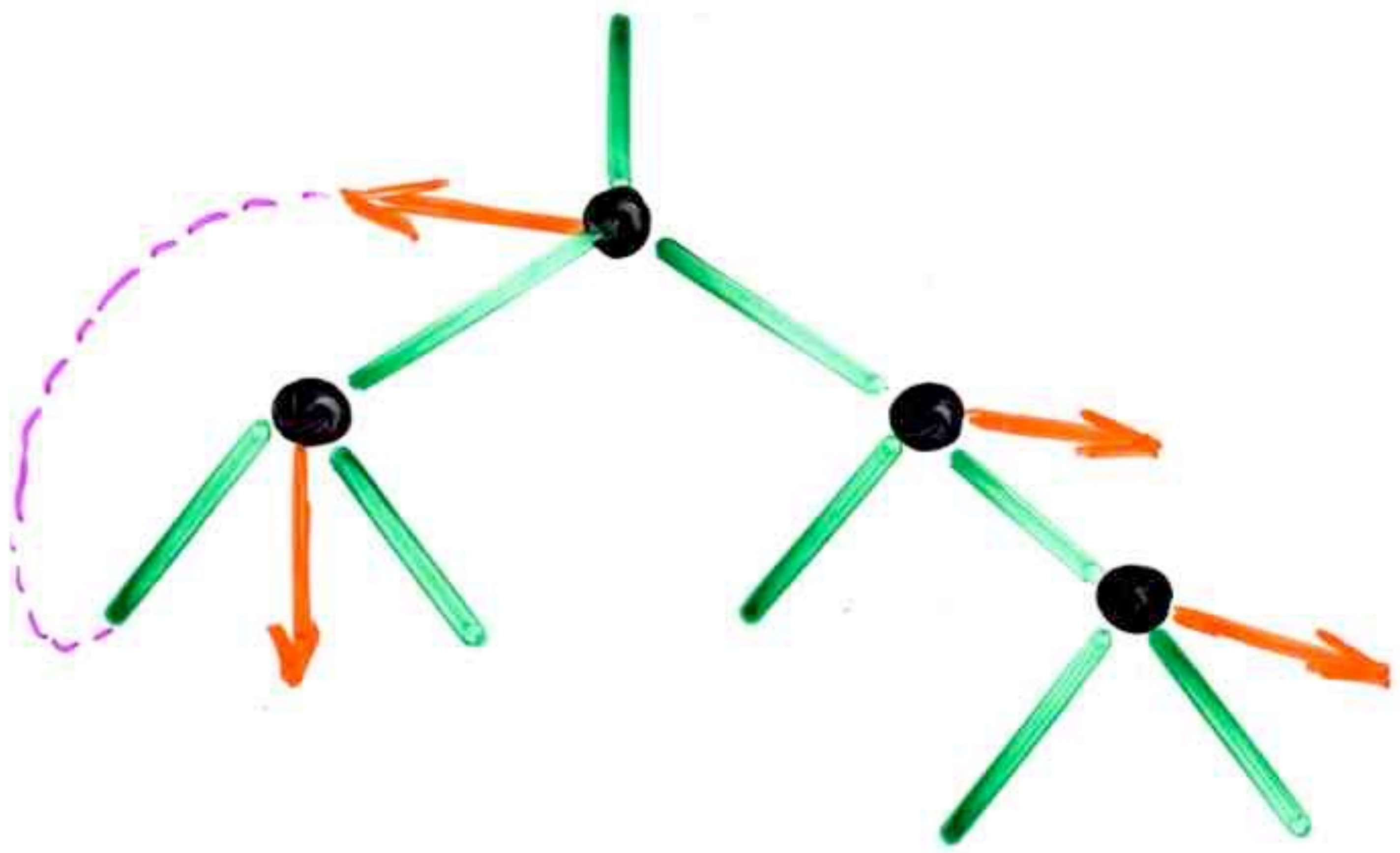
Schaeffer (1997)

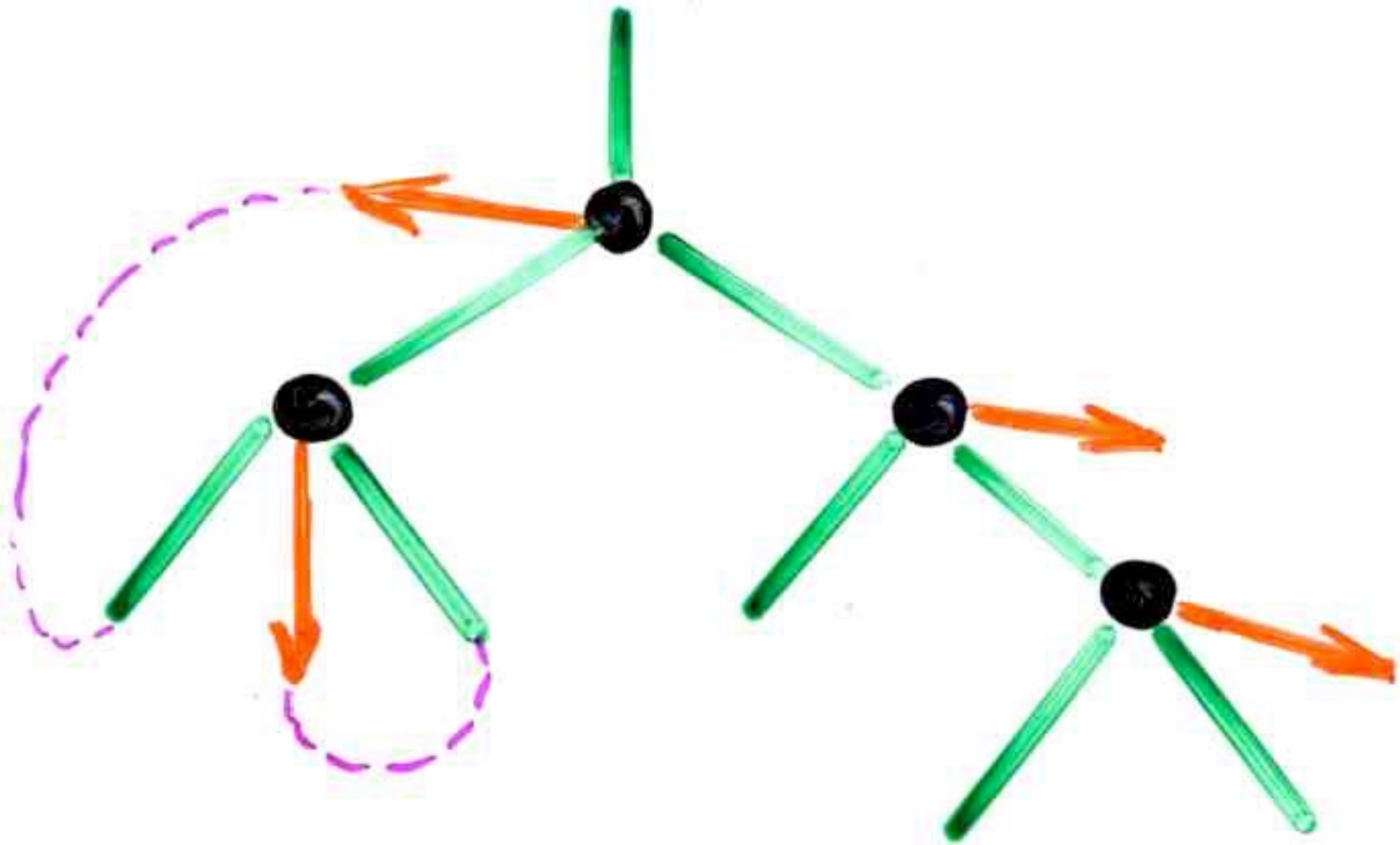


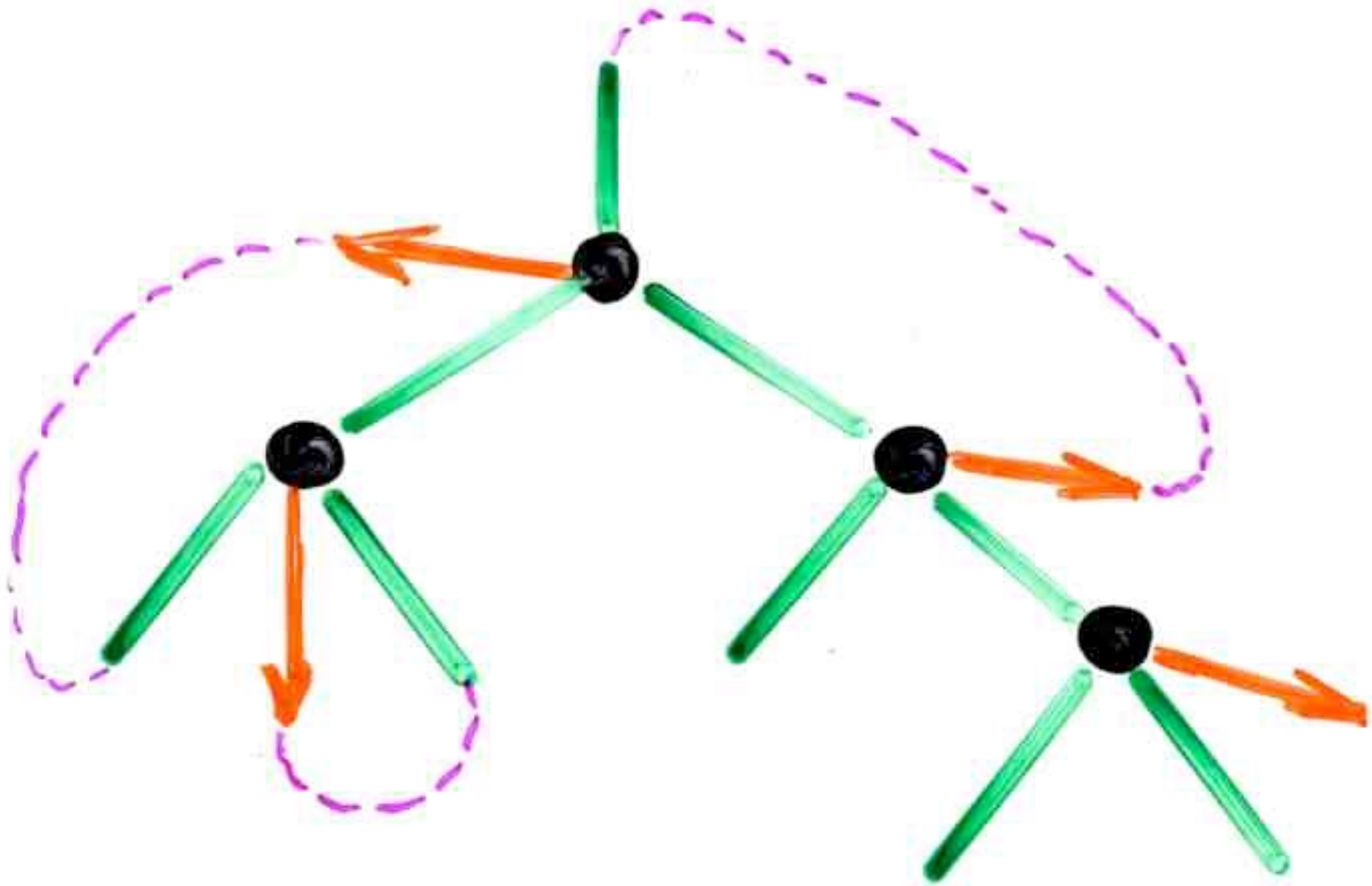


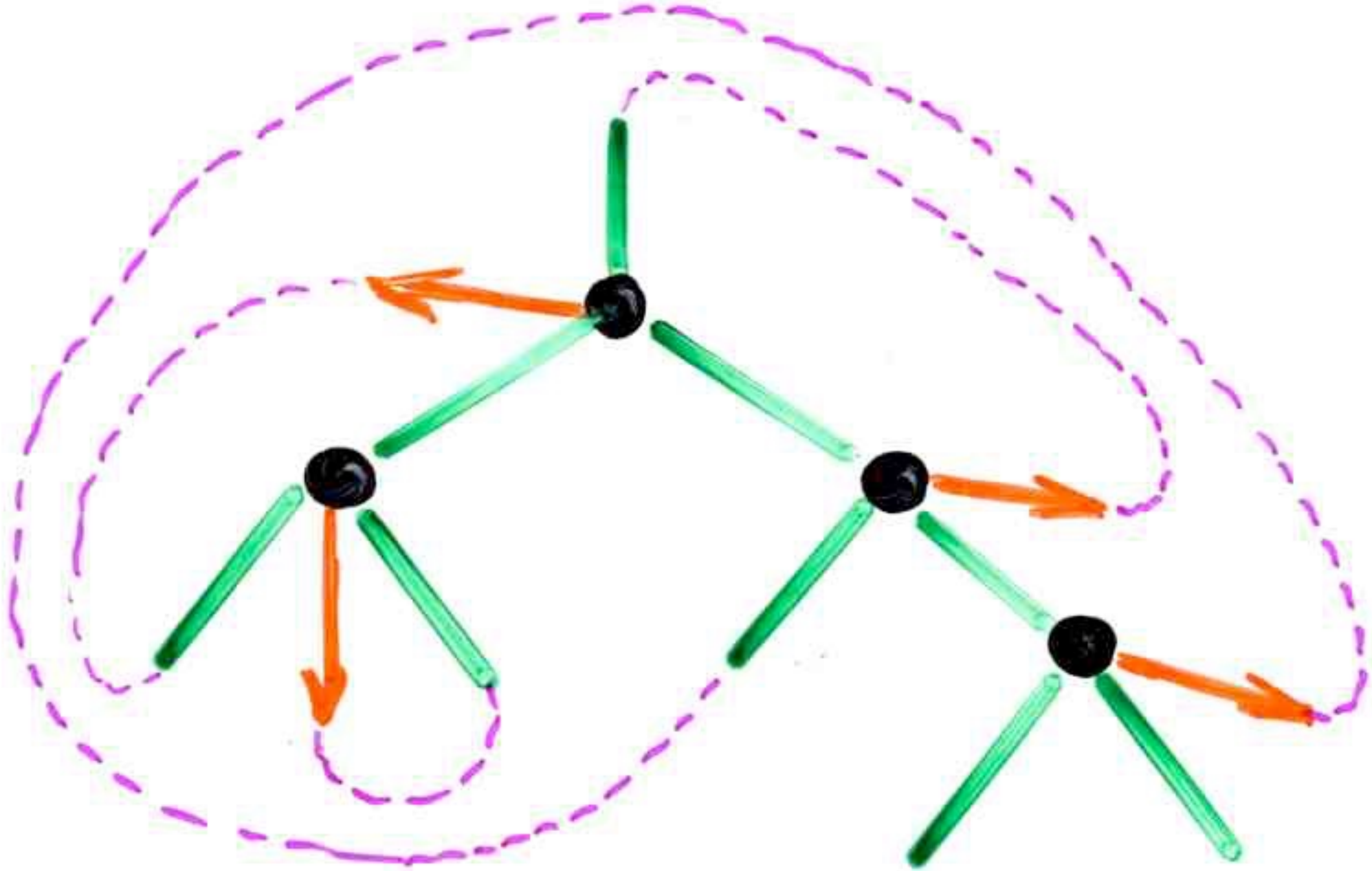
$$A = 1 + 3t\,A^2$$

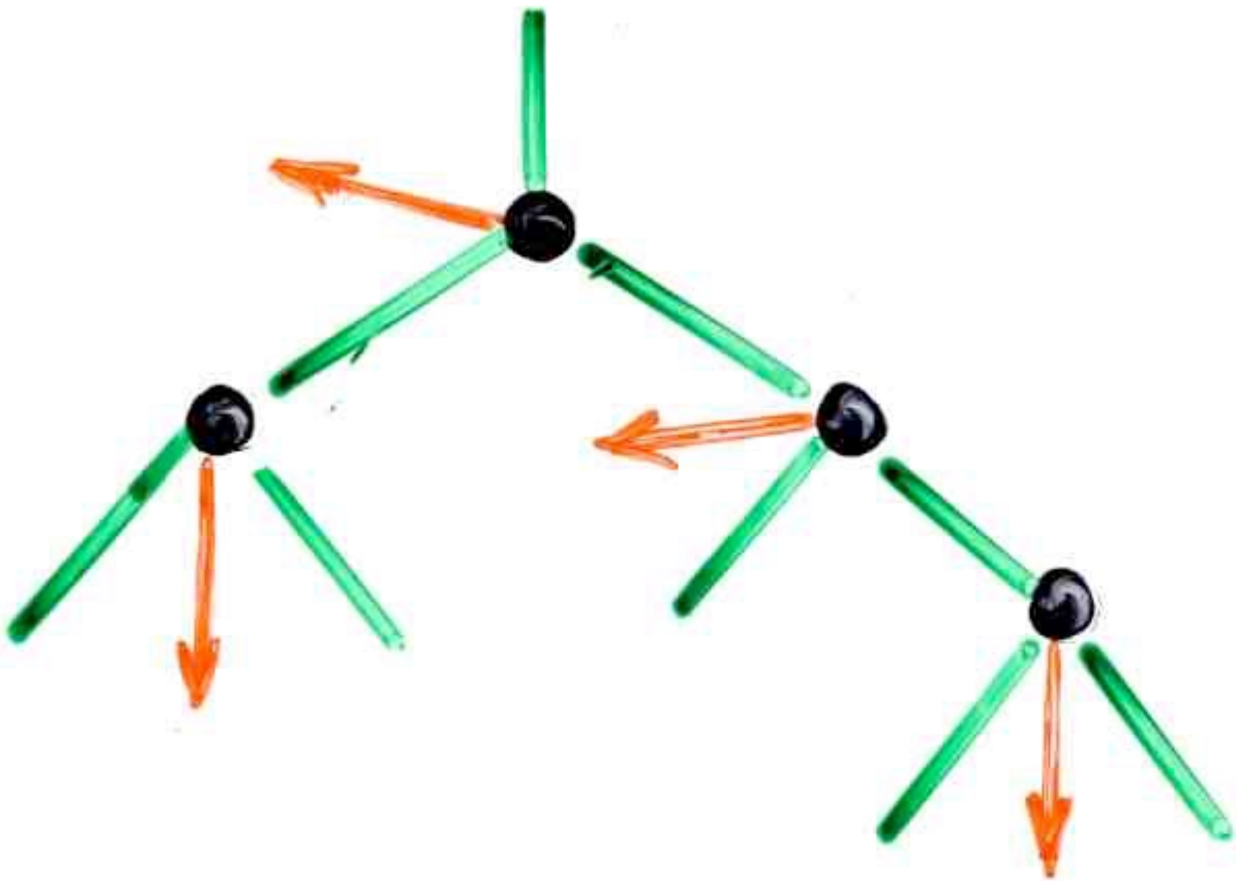


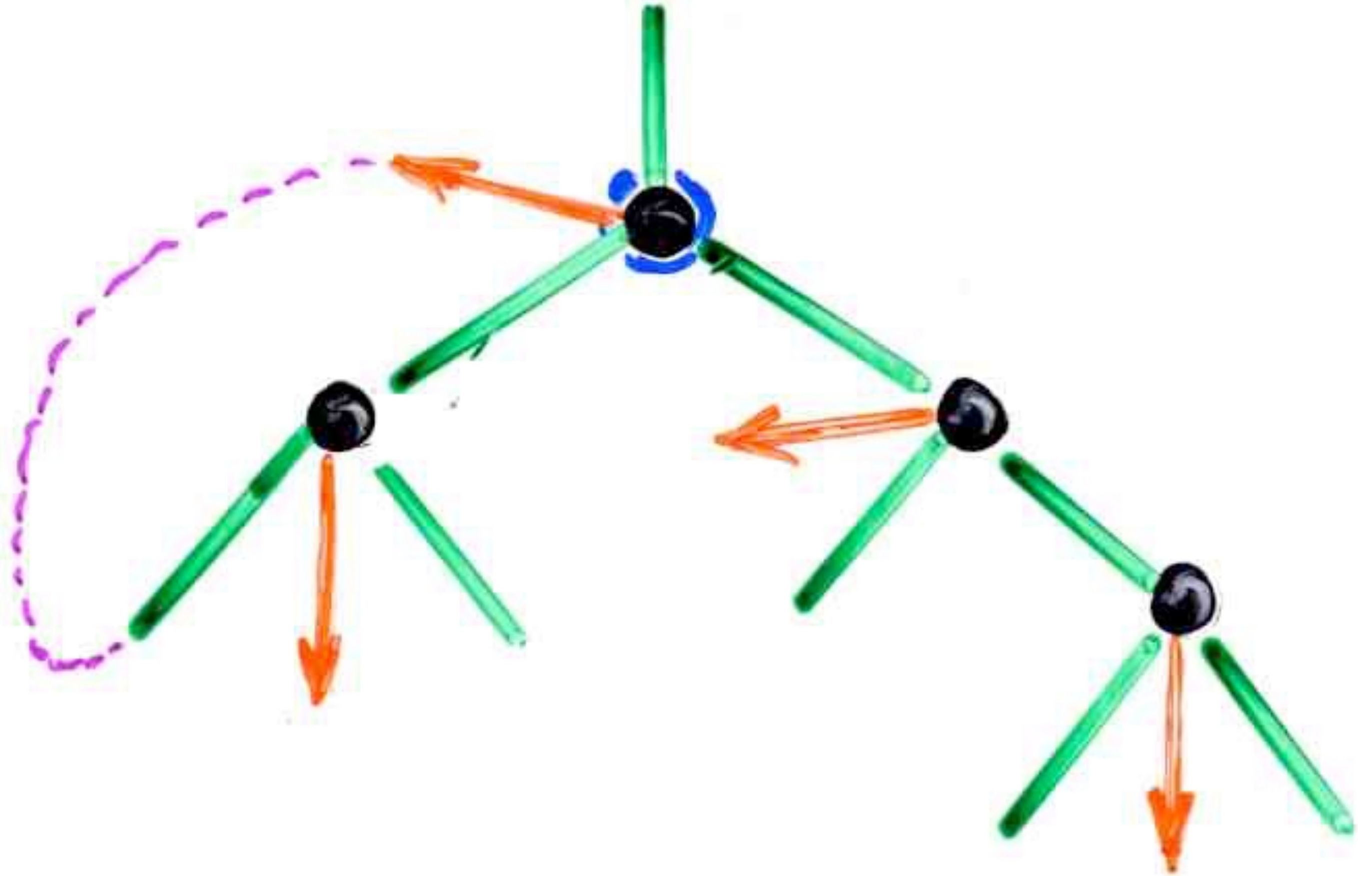


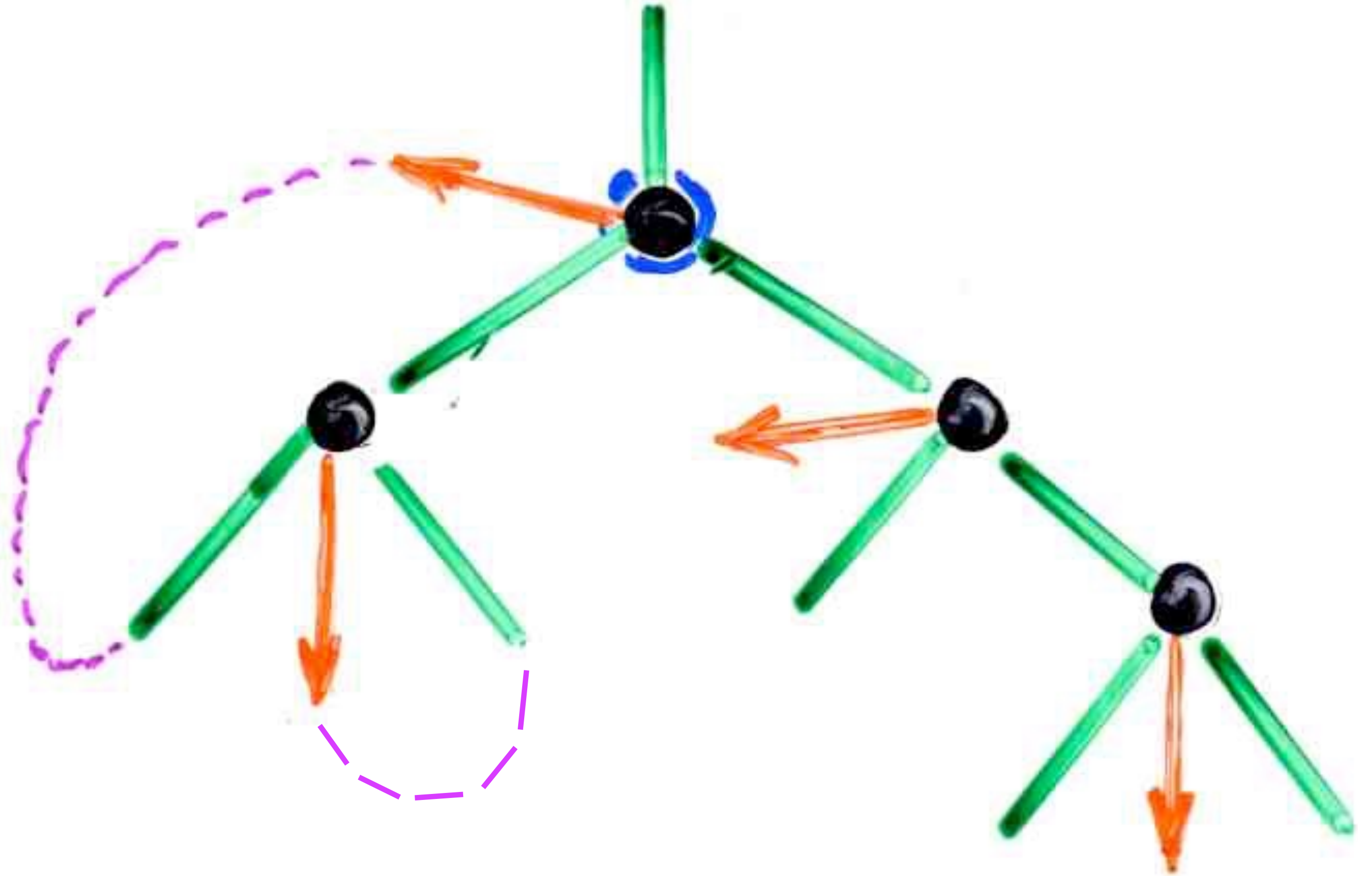


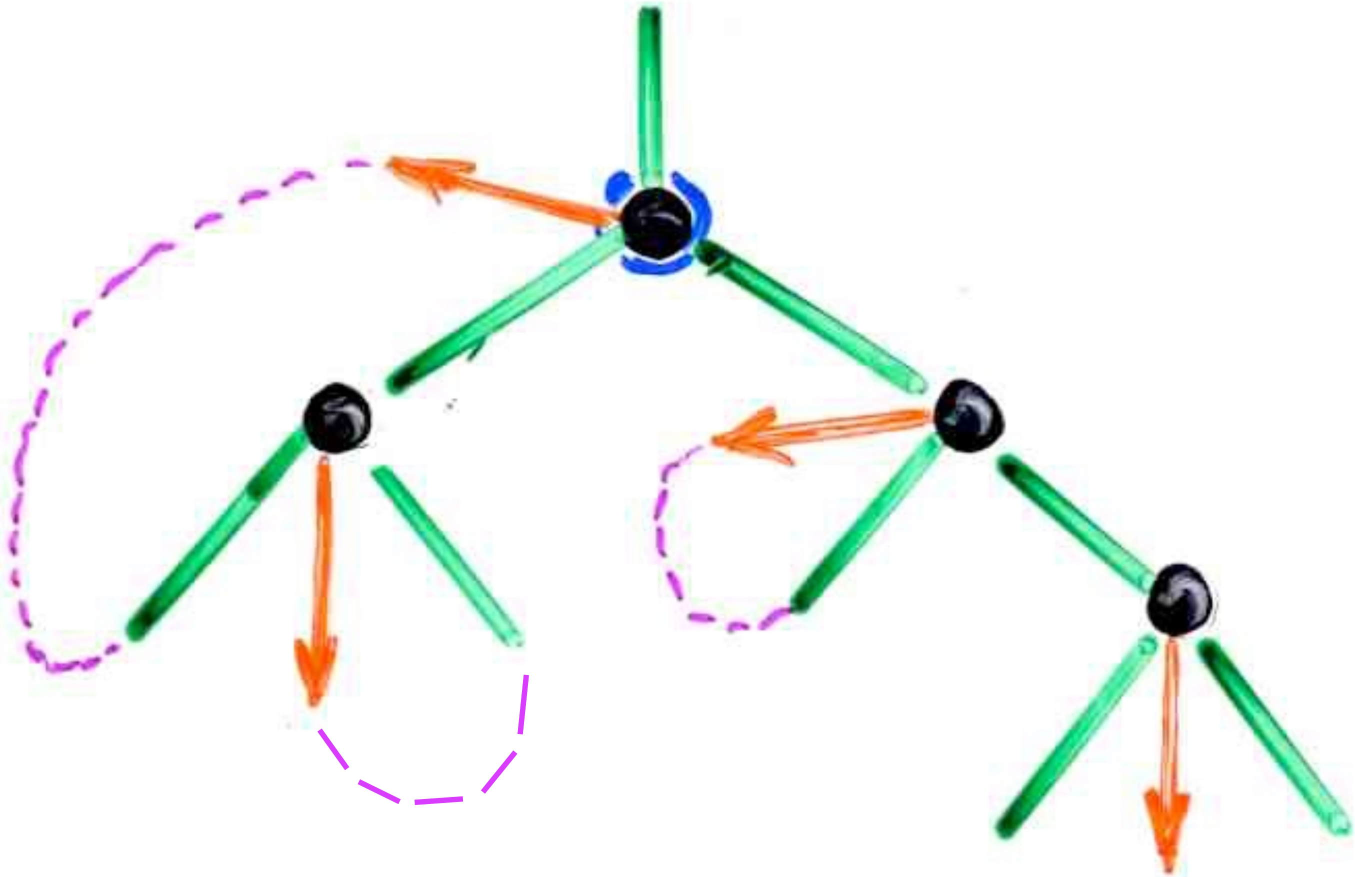


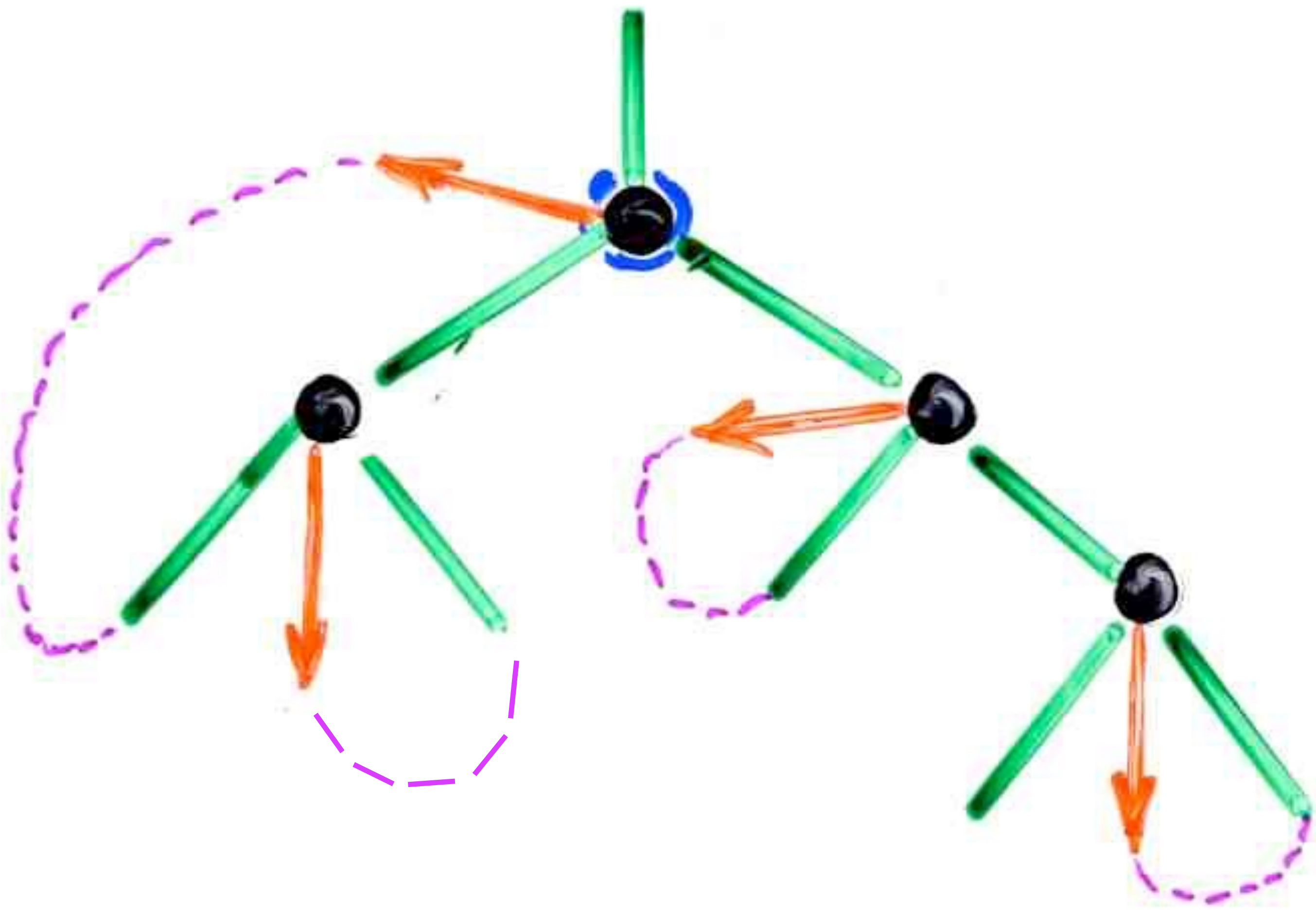


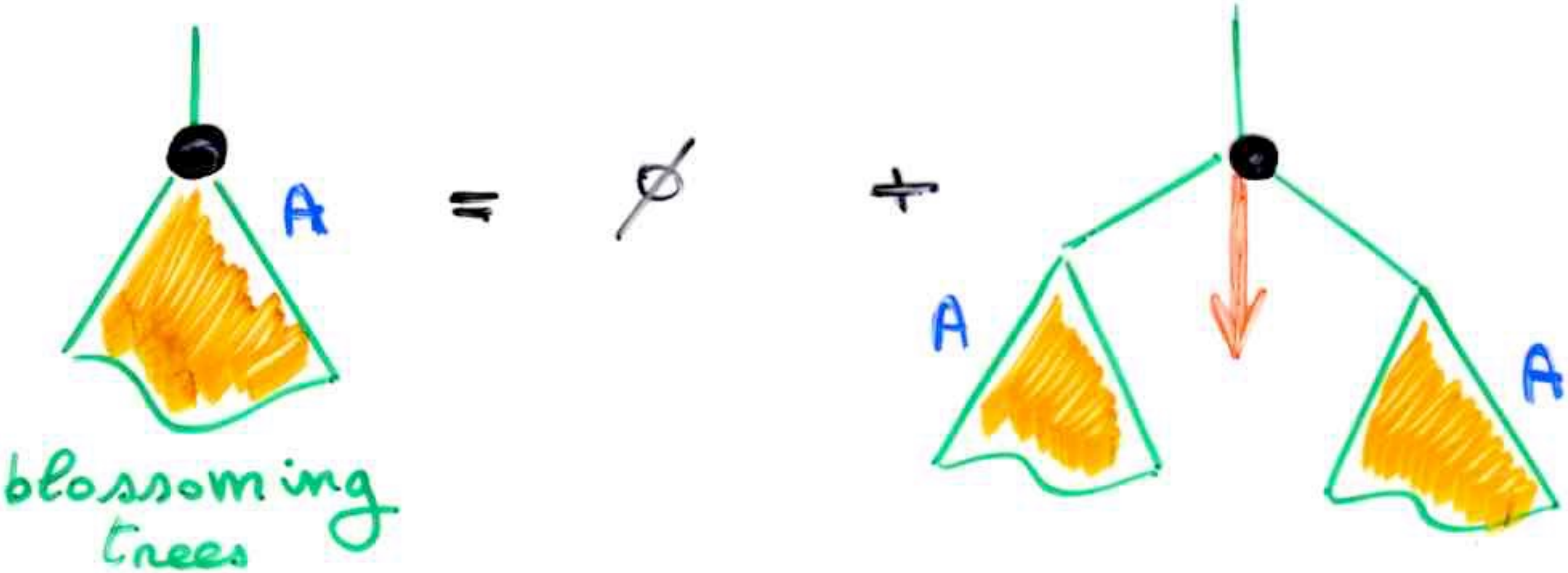






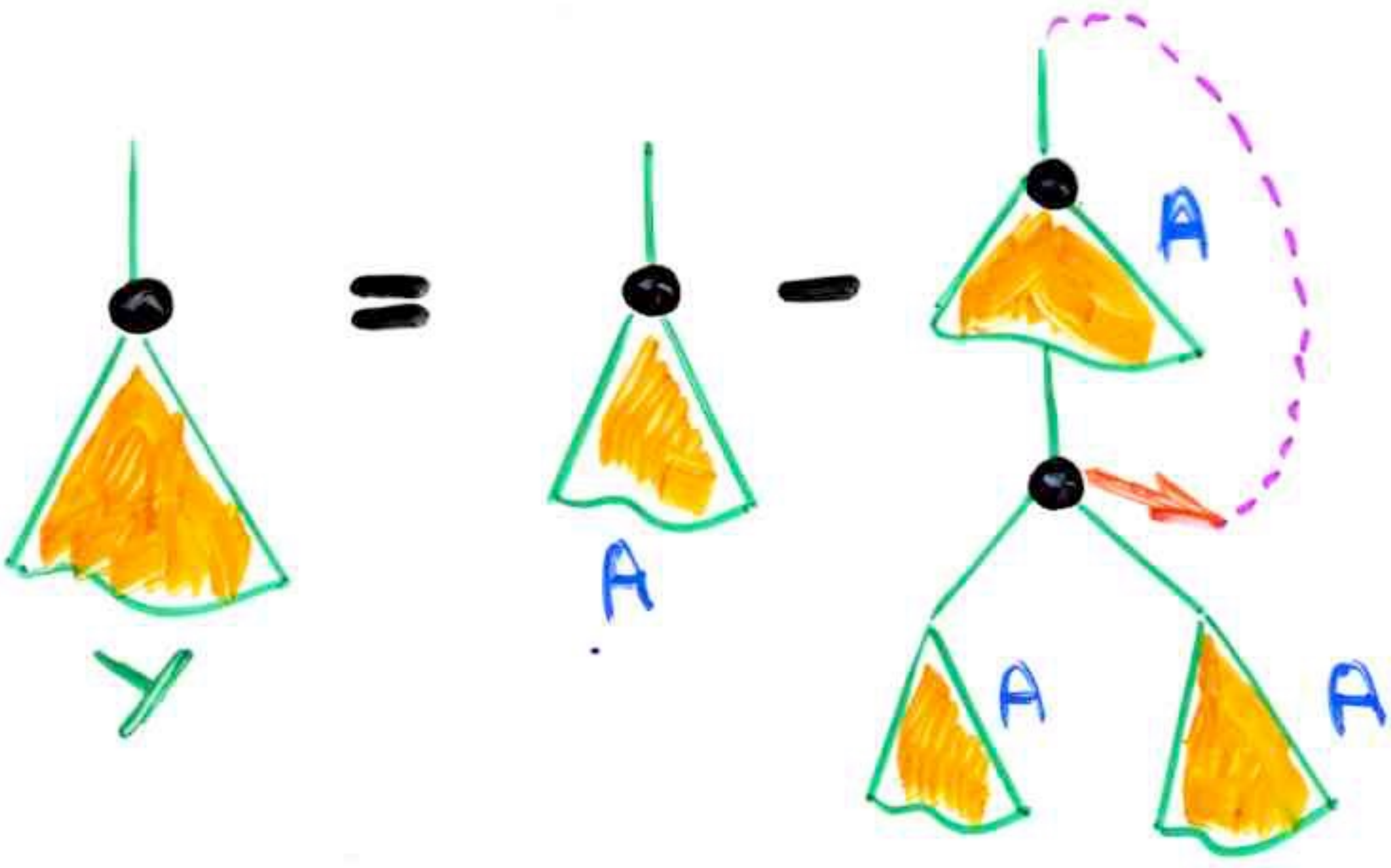






$$A = 1 + 3t\,A^2$$

Boutier, Di Francesco, Guitter (2002)



$$Y = A - tA^3$$

$$y = A - tA^3$$
$$A = 1 + 3tA^2$$

Tutte (1968)

Tutte (1968)

$$\frac{2 . 3^m}{(n+2)} C_m$$

Catalan

m arêtes

Cori, Vauquelin (1970, ...)

Arquès (1980, ...)

Schaeffer (1997, ...)

Bouttier, Di Francesco, Guitter (2002, ...)

quantum gravity

random maps

geodesic in random maps

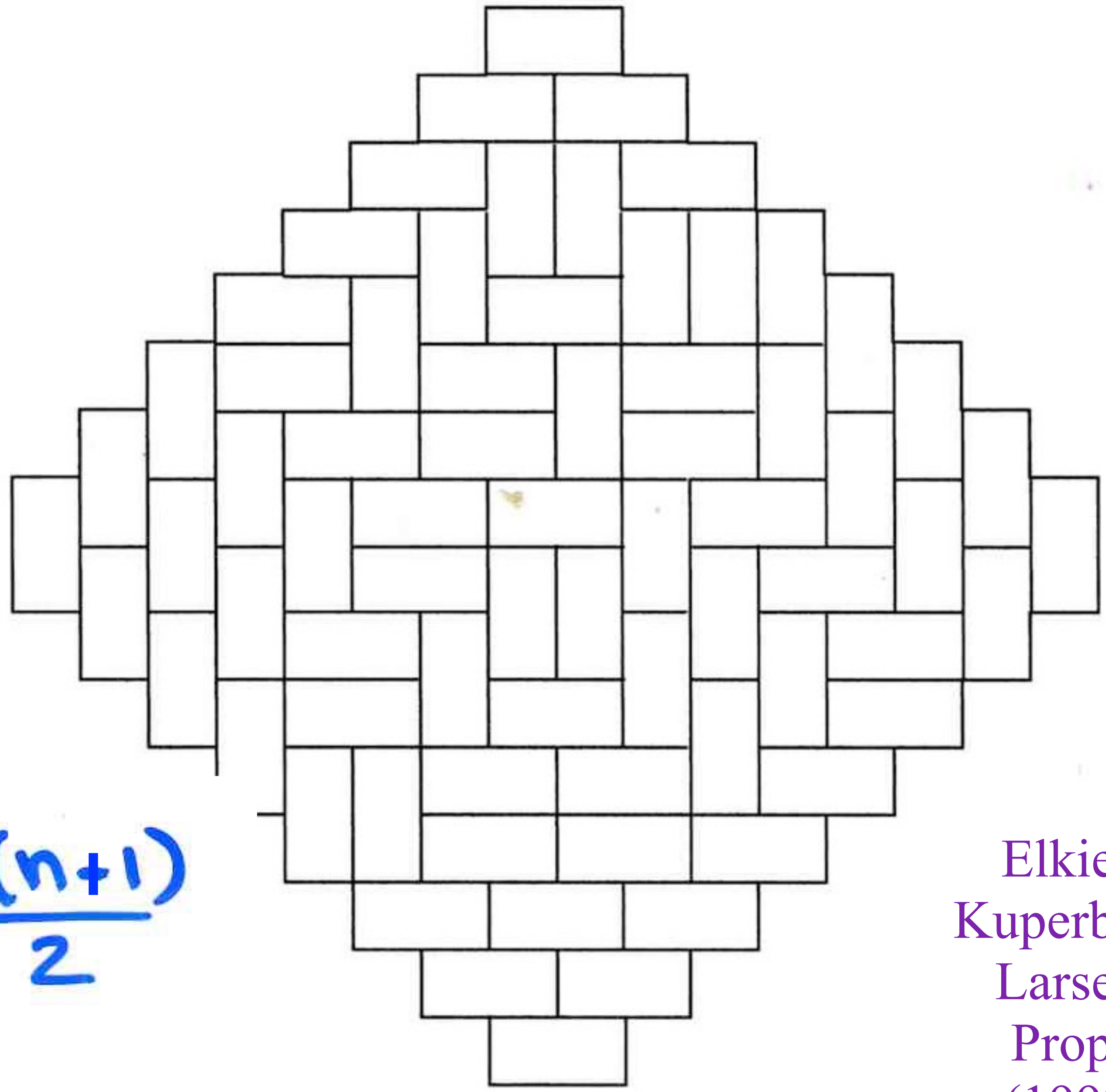
…………

understand formulae ....

bijective combinatorics

the number of
tilings of
the Aztec
diagram
with dimers
is

$$2^{(1+2+3+4+....+n)}$$



$$2^{\frac{n(n+1)}{2}}$$

Elkies,
Kuperberg,
Larsen,
Propp
(1992)

$$2^{(1+2+3+4+....+n)}$$

tilings of
the Aztec
diagram order n

+

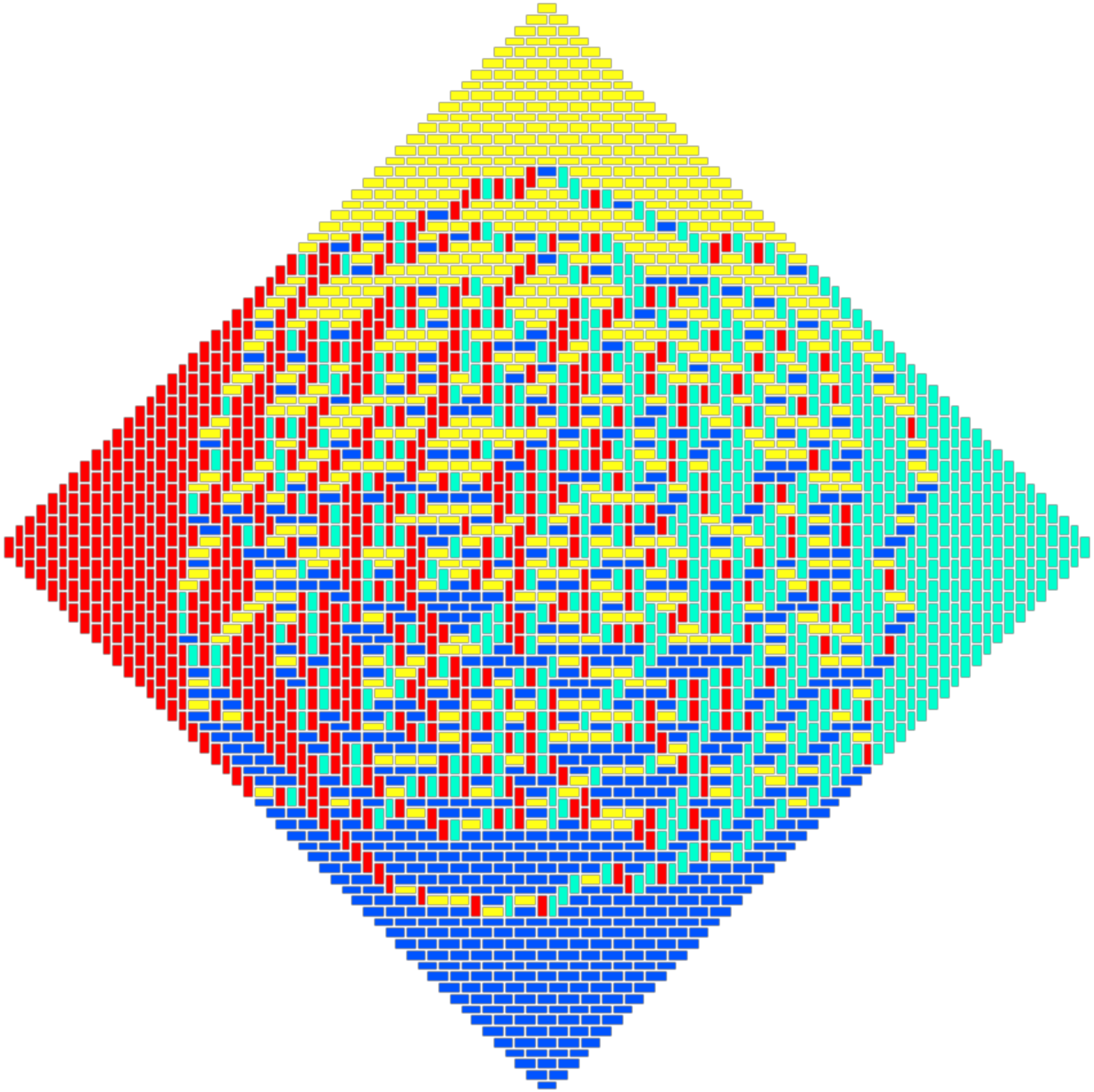
word in 2 letters
length (n+1)

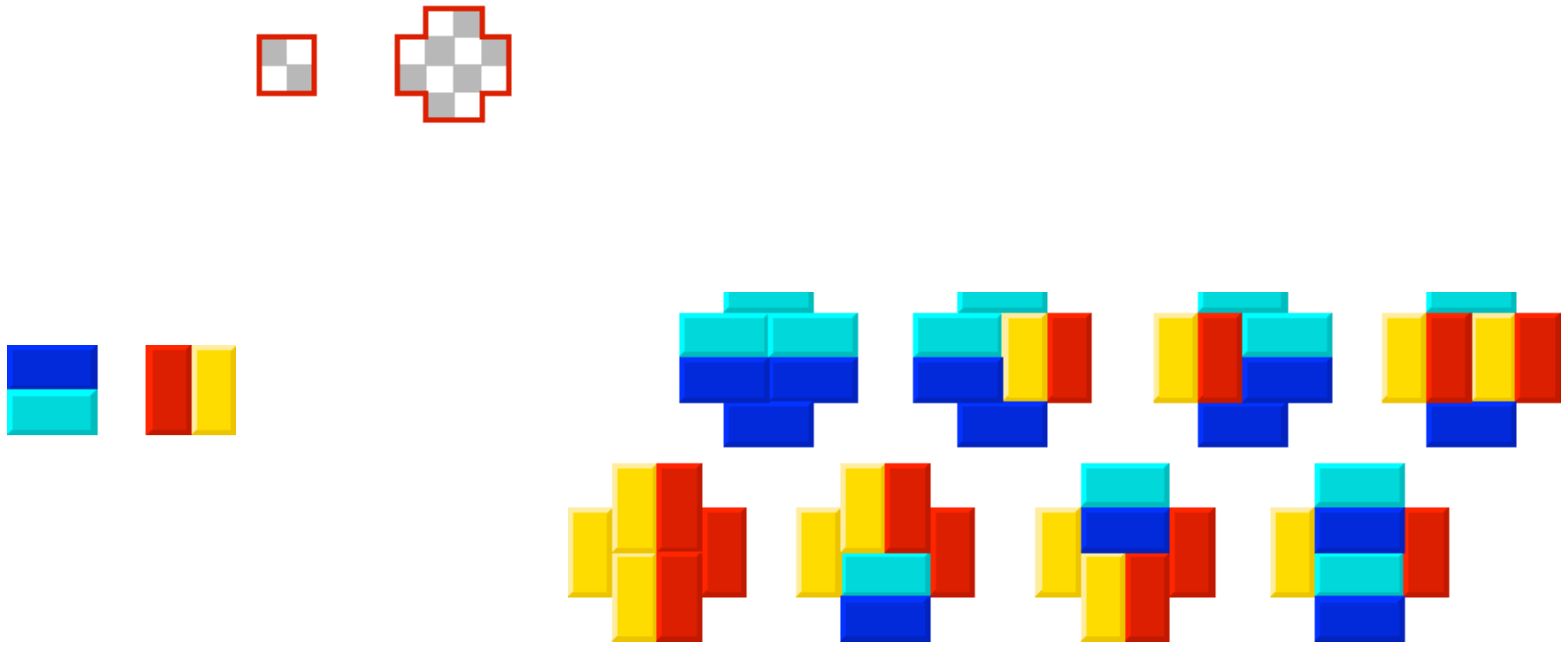
tilings of
the Aztec
diagram order (n+1)

« dominos shuffling »

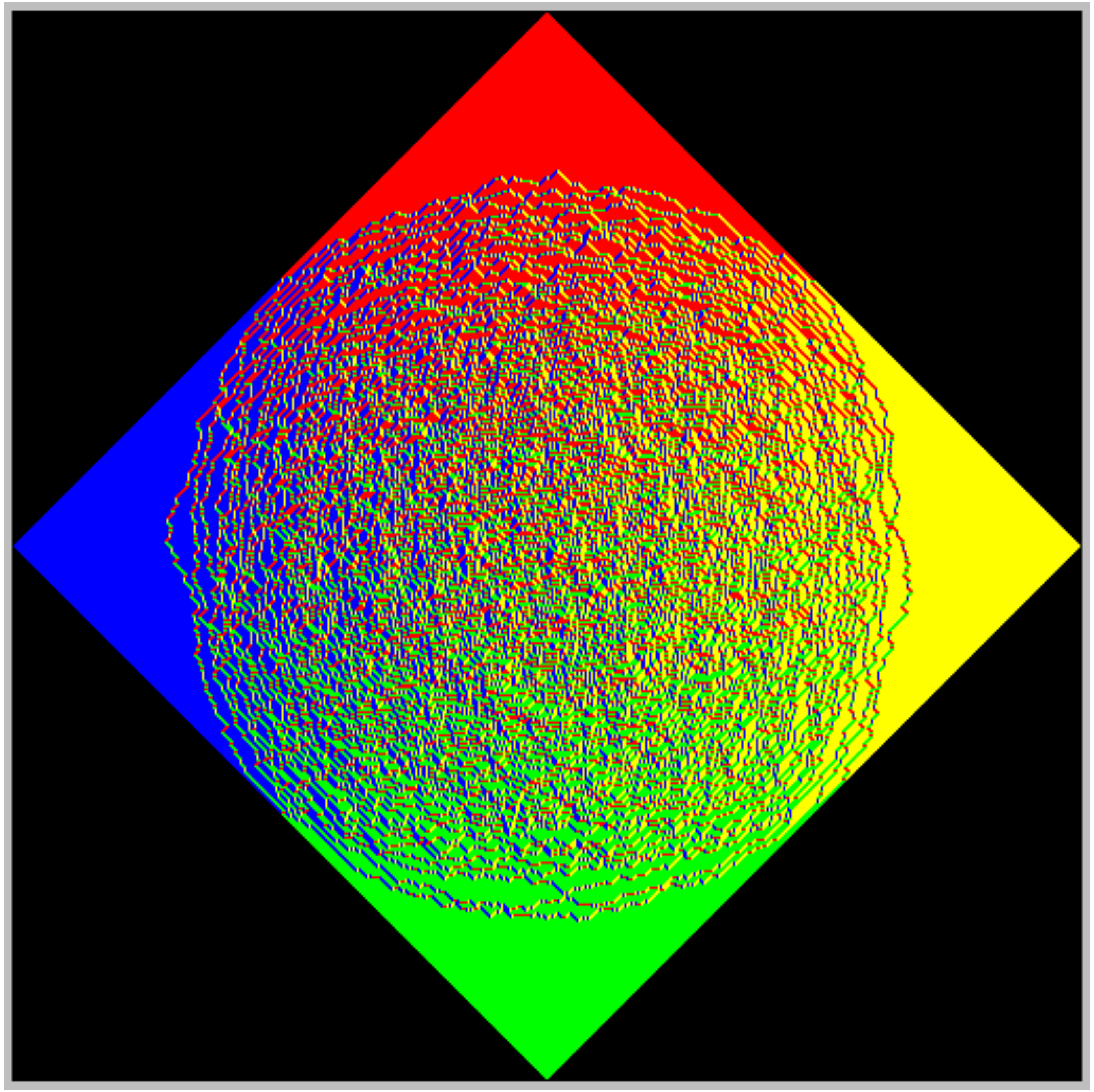
Elkies,
Kuperberg,
Larsen,
Propp
(1992)

random
Aztec
tilings





the
«artic
circle»
theorem



conversely: solving a combinatorial problems with methods from physics

ASM

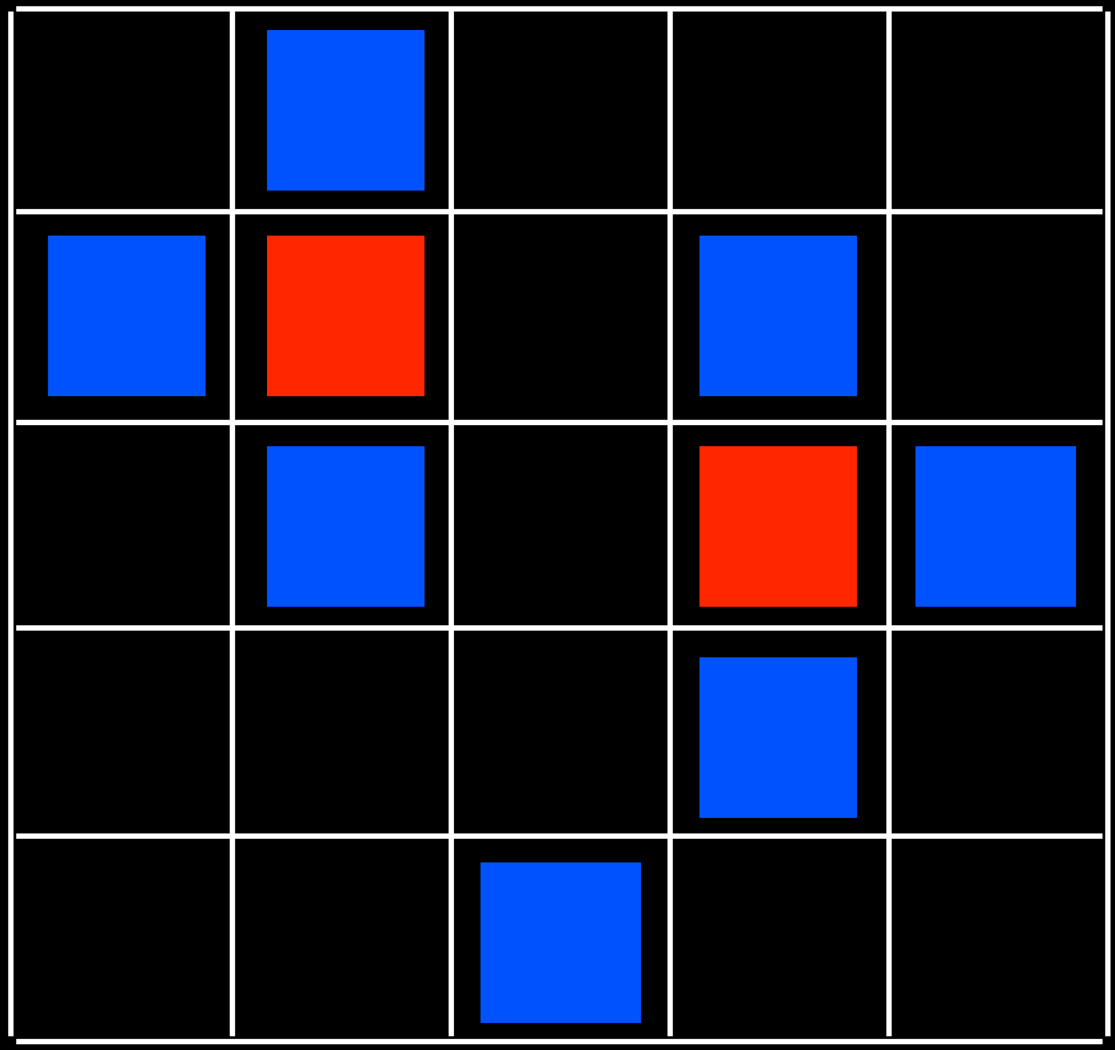
Alternating sign matrices

Alternating sign matrices

- entries: 0, 1, -1
- sum in rows and columns = 1
- non 0 entries alternate in sign
  in each row and column

ex:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



Permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} + 6$$

permutations

$$1,\ 2,\ 7,\ 42,\ 429,\ \ldots.$$

" What else have you got in your pocket ?" he went on, turning to A

" Only a thimble,"

" Hand it over her

Then they all crow

while the Dodo solem

Lewis Carroll

"Alice aux pays des merveilles"

C. I. Dodgson (1866)

Condensation of determinants

$$det(M) = \frac{M_{NO} M_{SE} - M_{NE} M_{SO}}{M_C}$$

$$1,\ 2,\ 7,\ 42,\ 429,\ \ldots$$

$$\frac{1!\,4!\cdots(3n-2)!}{n!\,(n+1)!\cdots(n+n-1)!}$$

alternating sign matrices conjecture

Mills, Robbins, Rumsey (1982)

Robbins

The Mathematical Intelligencer (1991)

“These conjectures are of such compelling simplicity that it is hard to understand how any mathematician can bear the pain of living without understanding why they are true”

D. Zeilberger (1992- 1995)
(+ 90 checkers)

Proof of the A. S. M. conj.

D. Zeilberger

livre A=B

PROOF OF THE ALTERNATING SIGN MATRIX CONJECTURE [1]

Doron ZEILBERGER[2]

Checked by[3]: David Bressoud and

Gert Almkvist, Noga Alon, George Andrews, Anonymous, Dror Bar-Natan, Francois Bergeron, Nantel Bergeron, Gaurav Bhatnagar, Anders Björner, Jonathan Borwein, Mireille Bousquest-Mélou, Francesco Brenti, E. Rodney Canfield, William Chen, Chu Wenchang, Shaun Cooper, Kequan Ding, Charles Dunkl, Richard Ehrenborg, Leon Ehrenpreis, Shalosh B. Ekhad, Kimmo Eriksson, Dominique Foata, Omar Foda, Aviezri Fraenkel, Jane Friedman, Frank Garvan, George Gasper, Ron Graham, Andrew Granville, Eric Grinberg, Laurent Habsieger, Jim Haglund, Han Guo-Niu, Roger Howe, Warren Johnson, Gil Kalai, Viggo Kann, Marvin Knopp, Don Knuth, Christian Krattenthaler, Gilbert Labelle, Jacques Labelle, Jane Legrange, Pierre Leroux, Ethan Lewis, Daniel Loeb, John Majewicz, Steve Milne, John Noonan, Kathy O'Hara, Soichi Okada, Craig Orr, Sheldon Parnes, Peter Paule, Bob Proctor, Arun Ram, Marge Readdy, Amitai Regev, Jeff Remmel, Christoph Reutenauer, Bruce Reznick, Dave Robbins, Gian-Carlo Rota, Cecil Rousseau, Bruce Sagan, Bruno Salvy, Isabella Sheftel, Rodica Simion, R. Jamie Simpson, Richard Stanley, Dennis Stanton, Volker Strehl, Walt Stromquist, Bob Sulanke, X.Y. Sun, Sheila Sundaram, Raphaële Supper, Nobuki Takayama, Xavier G. Viennot, Michelle Wachs, Michael Werman, Herb Wilf, Celia Zeilberger, Hadas Zeilberger, Tamar Zeilberger, Li Zhang, Paul Zimmermann .

Dedicated to my Friend, Mentor, and Guru, Dominique Foata.

Two stones build two houses. Three build six houses. Four build four and twenty houses. Five build hundred and twenty houses. Six build Seven hundreds and twenty houses. Seven build five thousands and forty houses. From now on, [exit and] ponder what the mouth cannot speak and the ear cannot hear.

(Sepher Yetsira IV,12)

Abstract: The number of $n \times n$ matrices whose entries are either -1, 0, or 1, whose row- and column- sums are all 1, and such that in every row and every column the non-zero entries alternate in sign, is proved to be $[1!4!\dots(3n-2)!]/[n!(n+1)!\dots(2n-1)!]$, as conjectured by Mills, Robbins, and Rumsey.

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[1] To appear in Electronic J. of Combinatorics (Foata's 60th Birthday issue). Version of July 31, 1995; original version written December 1992. The Maple package ROBBINS, accompanying this paper, can be downloaded from the www address in footnote 2 below.

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E-mail:`zeilberg@math.temple.edu`. WWW:`http://www.math.temple.edu/~zeilberg`. Anon. ftp: `ftp.math.temple.edu`, directory `/pub/zeilberg`. Supported in part by the NSF.

[3] See the Exodion for affiliations, attribution, and short bios.

Subsublemma 1.1.3:

$$\sum_{\pi \in \mathcal{S}_k} \mathrm{sgn}(\pi) \cdot \pi \left[\frac{x_1 x_2^2 \dots x_k^k}{(1 - x_k)(1 - x_k x_{k-1}) \cdots (1 - x_k x_{k-1} \dots x_1)} \right] = \frac{x_1 \cdots x_k \prod_{1 \le i < j \le k} (x_j - x_i)}{\prod_{i=1}^k (1 - x_i) \prod_{1 \le i < j \le k} (1 - x_i x_j)} \quad . \tag{Issai}$$

[Type 'S113(k);' in ROBBINS, for specific k.]

Proof : See [PS], problem VII.47. Alternatively, (Issai) is easily seen to be equivalent to Schur's identity that sums all the Schur functions ([Ma], ex I.5.4, p. 45). This takes care of subsublemma 1.1.3. □

Inserting $(Issai)$ into $(Stanley)$, expanding $\prod_{1 \le i < j \le k} (x_j - x_i)$ by Vandermonde's expansion,

$$\sum_{\pi \in \mathcal{S}_k} \mathrm{sgn}(\pi) \cdot \pi(x_1^0 x_2^1 \cdots x_k^{k-1}) \quad ,$$

using the antisymmetry of Δ_k once again, and employing crucial fact $\aleph_4$, we get the following string of equalities:

$$b_k(n) = \frac{1}{k!} CT_{x_1,\dots,x_k} \left\{ \frac{\Delta_k(x_1, \dots, x_k)}{\prod_{i=1}^k (\bar{x}_i)^{k+n} x_i^{n+k-1}} \left(\frac{x_1 \cdots x_k \prod_{1 \le i < j \le k} (x_j - x_i)}{\prod_{i=1}^k (1 - x_i) \prod_{1 \le i < j \le k} (1 - x_i x_j)} \right) \right\}$$

$$= \frac{1}{k!} CT_{x_1,\dots,x_k} \left\{ \frac{\Delta_k(x_1, \dots, x_k)}{\prod_{i=1}^k (\bar{x}_i)^{k+n+1} x_i^{n+k-2} \prod_{1 \le i < j \le k} (1 - x_i x_j)} \left(\sum_{\pi \in \mathcal{S}_k} \mathrm{sgn}(\pi) \cdot \pi(x_1^0 x_2^1 \dots x_k^{k-1}) \right) \right\}$$

$$= \frac{1}{k!} \sum_{\pi \in \mathcal{S}_k} CT_{x_1,\dots,x_k} \left\{ \pi \left[\frac{\Delta_k(x_1, \dots, x_k)}{\prod_{i=1}^k (\bar{x}_i)^{k+n+1} x_i^{n+k-2} \prod_{1 \le i < j \le k} (1 - x_i x_j)} \left(\prod_{i=1}^k x_i^{i-1} \right) \right] \right\}$$

$$= \frac{1}{k!} \sum_{\pi \in \mathcal{S}_k} CT_{x_1,\dots,x_k} \left\{ \pi \left[\frac{\Delta_k(x_1, \dots, x_k)}{\prod_{i=1}^k (\bar{x}_i)^{k+n+1} x_i^{n+k-i-1} \prod_{1 \le i < j \le k} (1 - x_i x_j)} \right] \right\}$$

$$= \frac{1}{k!} \sum_{\pi \in \mathcal{S}_k} CT_{x_1,\dots,x_k} \left\{ \frac{\Delta_k(x_1, \dots, x_k)}{\prod_{i=1}^k (\bar{x}_i)^{k+n+1} x_i^{n+k-i-1} \prod_{1 \le i < j \le k} (1 - x_i x_j)} \right\}$$

$$= \frac{1}{k!} \left(\sum_{\pi \in \mathcal{S}_k} 1 \right) CT_{x_1,\dots,x_k} \left\{ \frac{\Delta_k(x_1, \dots, x_k)}{\prod_{i=1}^k (\bar{x}_i)^{k+n+1} x_i^{n+k-i-1} \prod_{1 \le i < j \le k} (1 - x_i x_j)} \right\}$$

$$= CT_{x_1,\dots,x_k} \left\{ \frac{\Delta_k(x_1, \dots, x_k)}{\prod_{i=1}^k (\bar{x}_i)^{k+n+1} x_i^{n+k-i-1} \prod_{1 \le i < j \le k} (1 - x_i x_j)} \right\} , \tag{George'''}$$

where in the last equality we have used Levi Ben Gerson's celebrated result that the number of elements in $\mathcal{S}_k$ (the symmetric group on k elements,) equals $k!$. The extreme right of $(George''')$ is exactly the right side of $(MagogTotal)$. This completes the proof of sublemma 1.1. □

"EXTREME UGLYNESS CAN BE BEAUTIFUL"

Doron Zeilberger

Bordeaux, May, 1991

3rd FPSAC

1, 2, 7, 42, 429, 7436,

G. Andrews (1979)

- descending plane partitions

M. Mills, D.P. Robbins, H. Rumsey (1983)

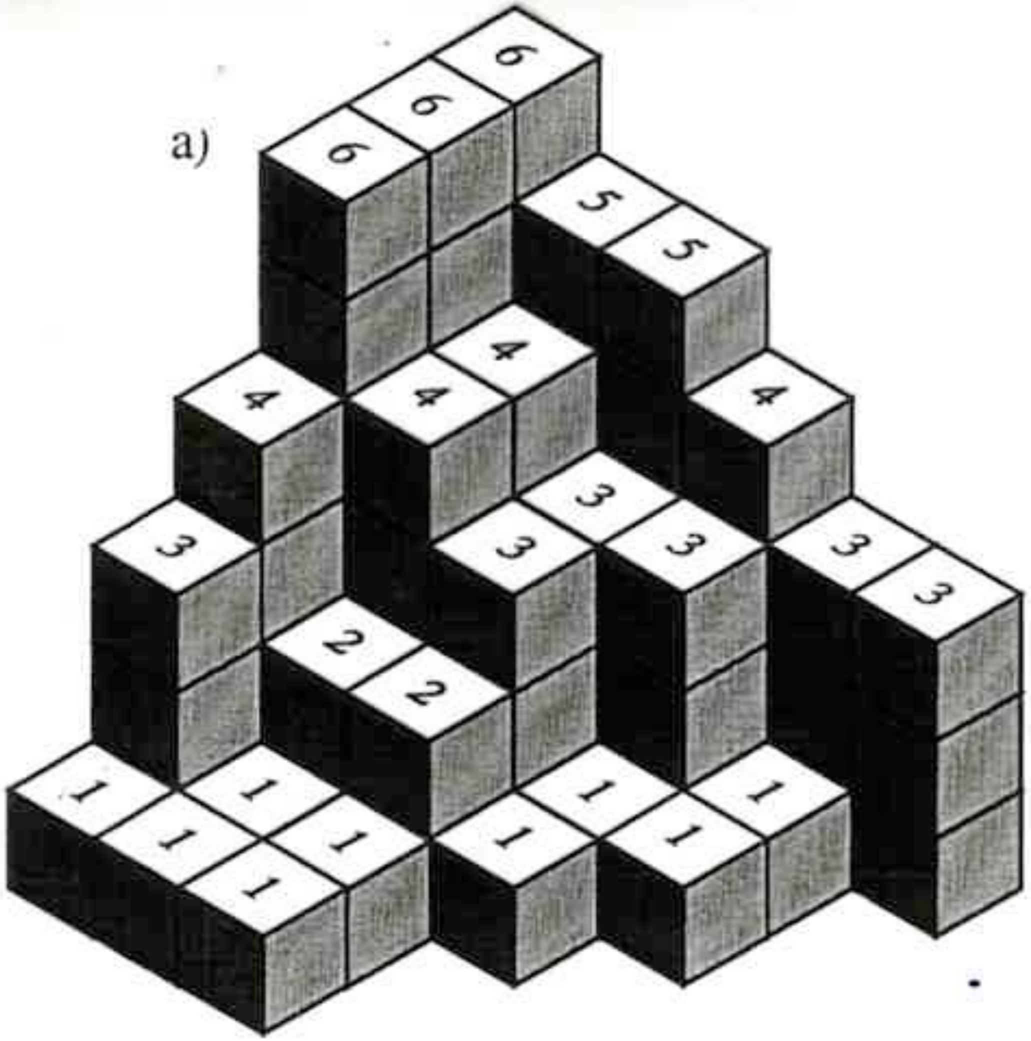
- alternating sign matrices

- totally symmetric self-complementary plane partition

(T.S.S.C.P.P.)

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = \frac{1!\,4!\,\cdots\,(3n-2)!}{n!\,(n+1)!\,\cdots\,(2n-1)!}$$

a)



b)

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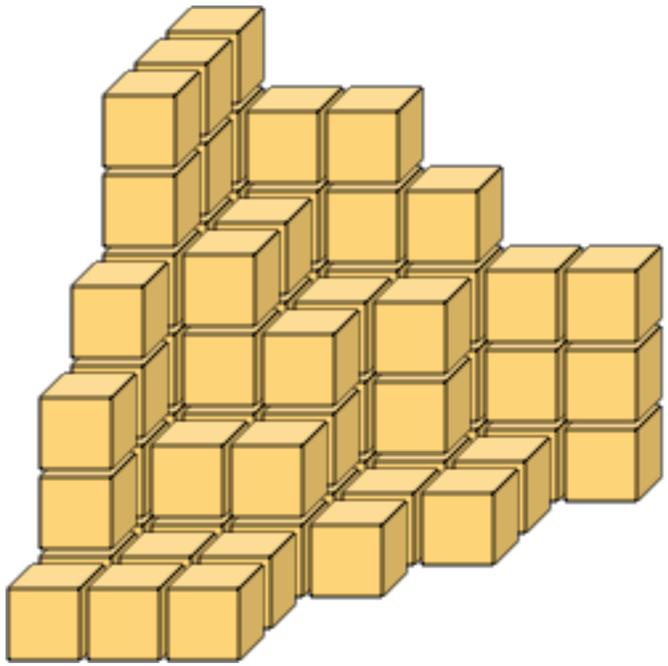
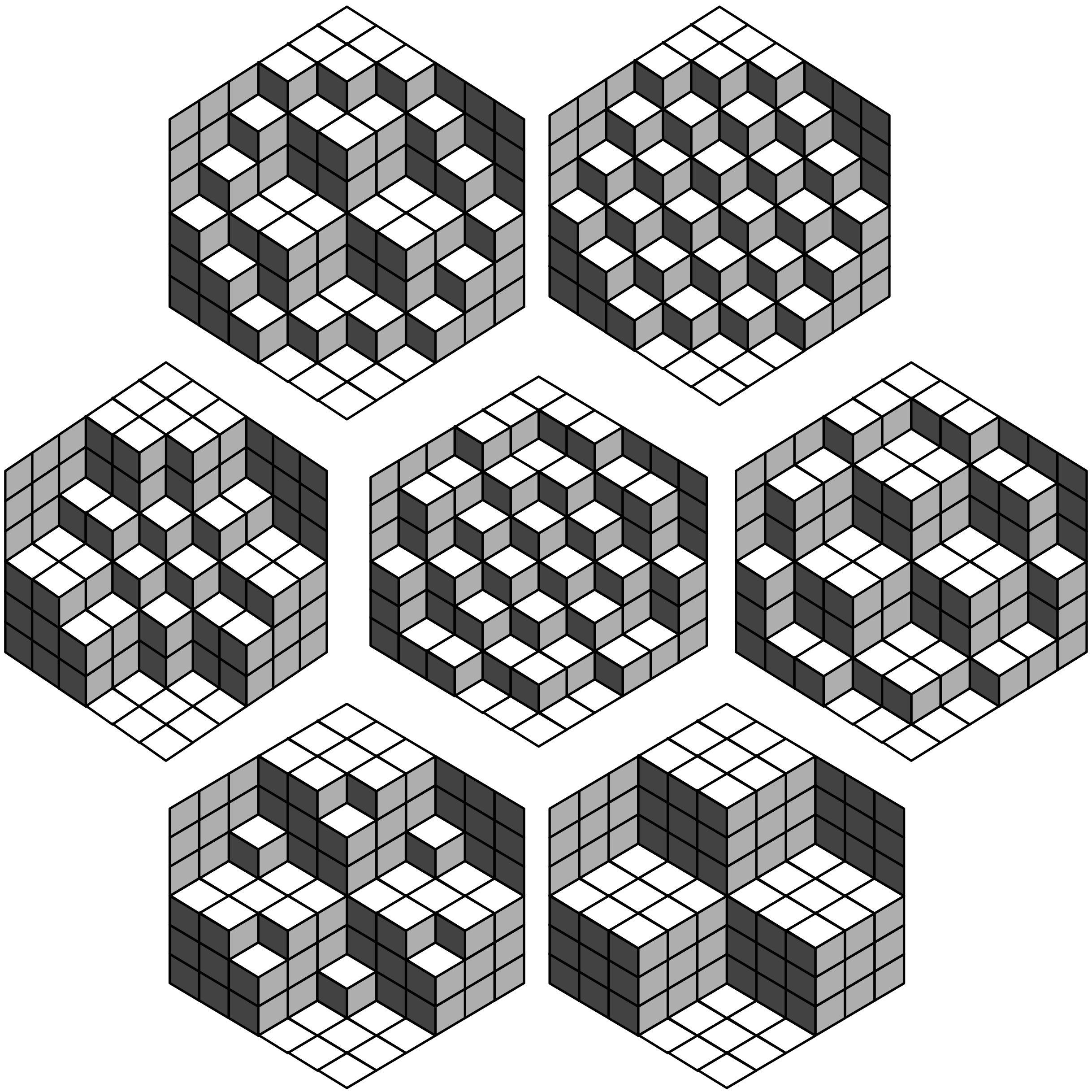
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			





Proofs and Confirmations
The story of the alternating sign matrix conjecture

David M. Bressoud
Macalester College
Saint Paul, MN

July 28, 1997

Kuperberg (1995)

6- vertex model

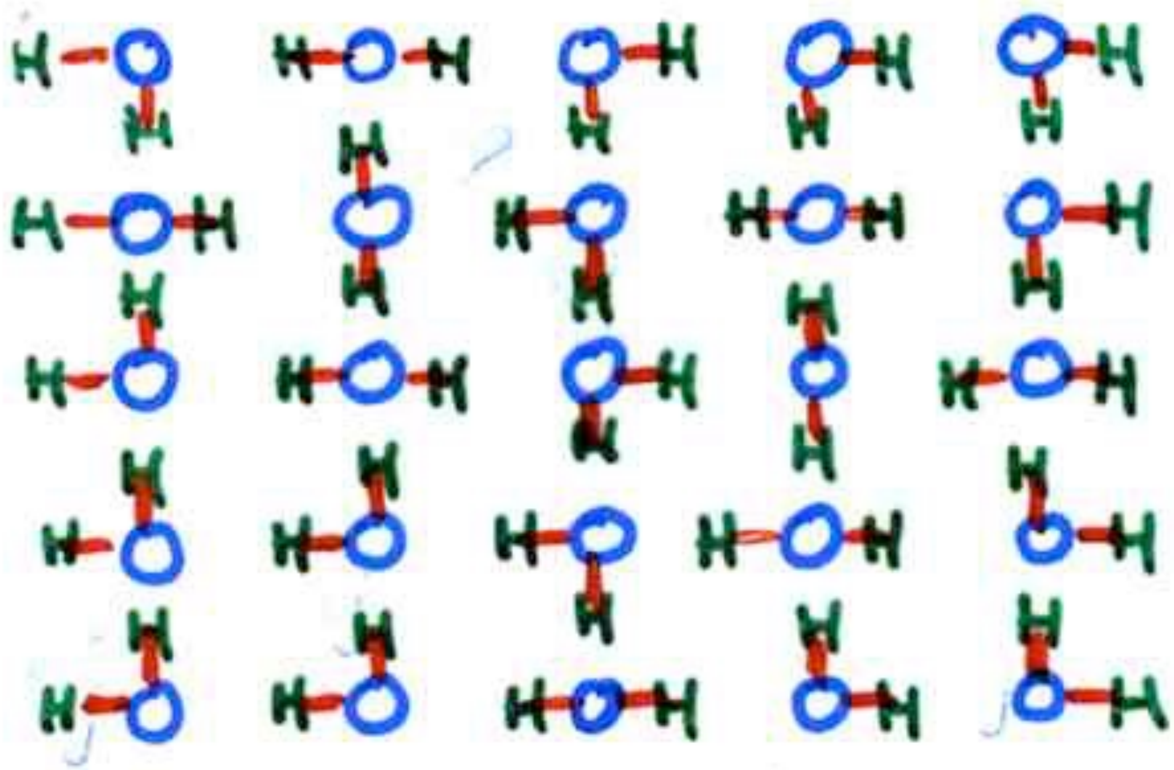
(ice model)

with domain wall boundary condition

ice model

or

six-vertex model



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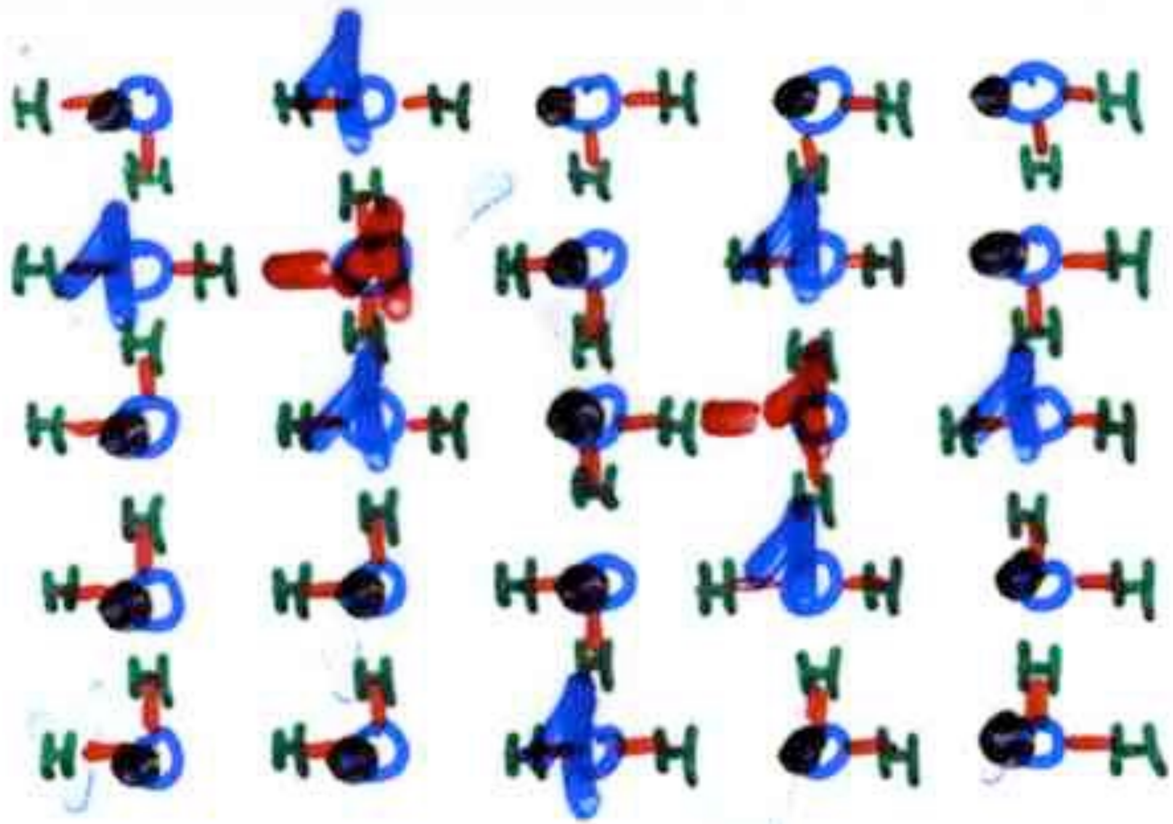
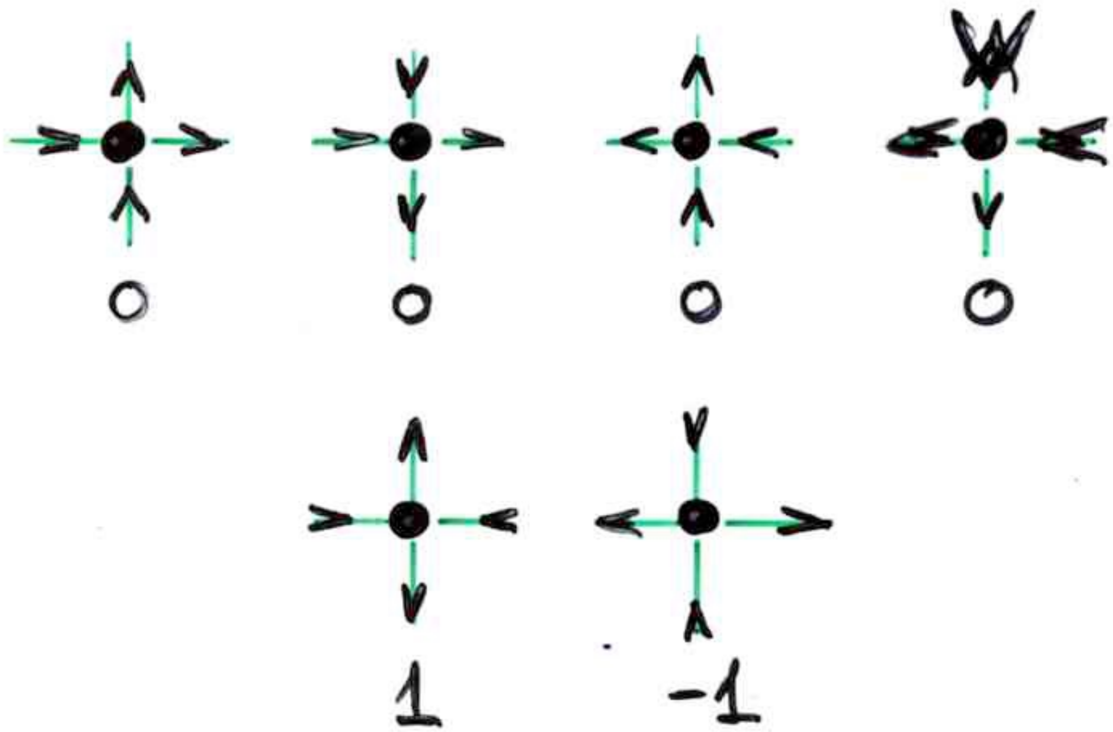
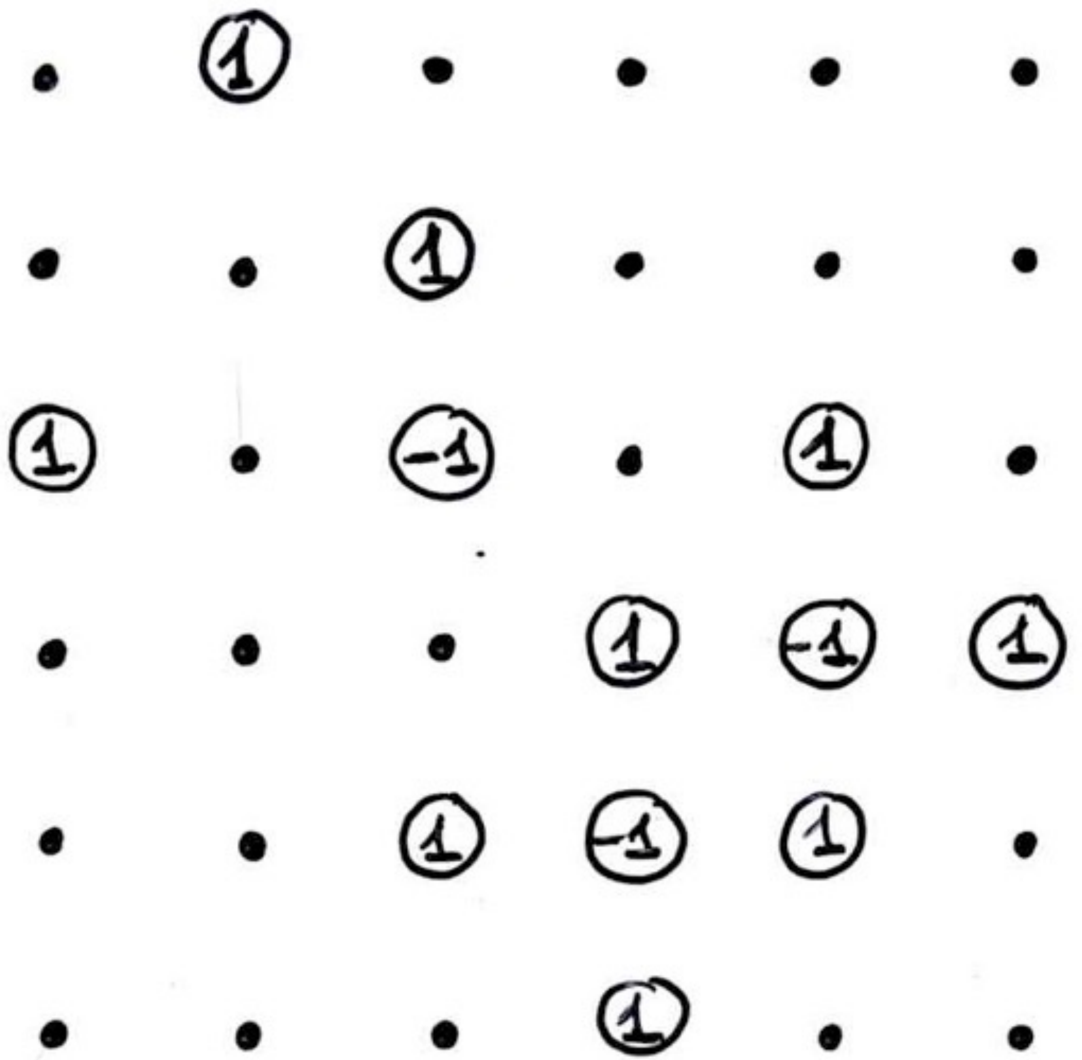
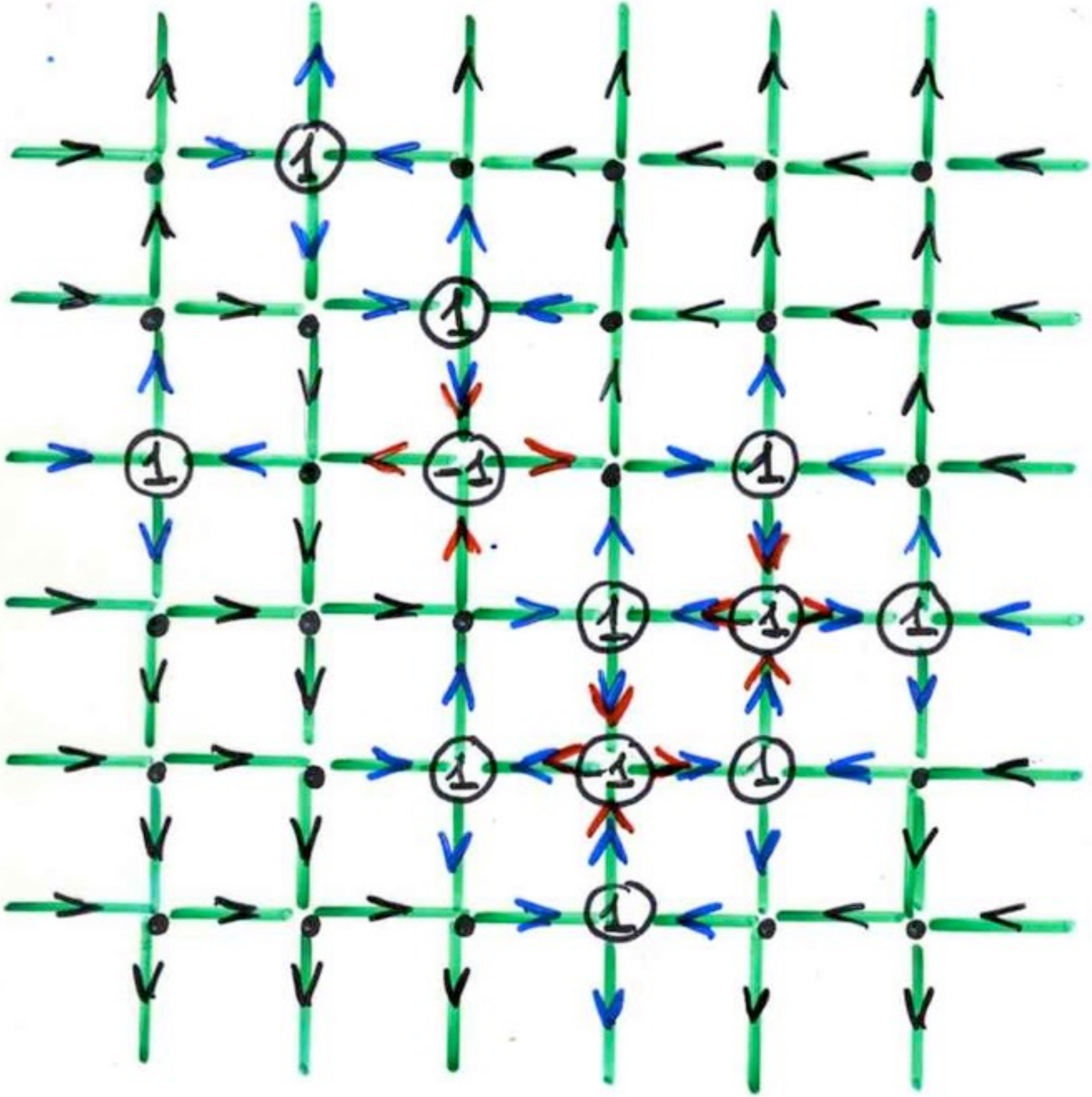
·	1	·	-1	1
·	·	·	1	·
·	·	1	·	·





6-vertex model

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fonction de partition

Gaudin

Korepin, Bogoliubov, Izergin

"Quantum Inverse Scattering Method and Correlation Functions" (1993)

$$Z_n(\vec{x}; \vec{y}; a) = \frac{\prod_{i=1}^{n} x_i/y_i \prod_{1 \le i, j \le n} (x_i/y_j)(ax_i/y_j)}{\prod_{1 \le i < j \le n} (x_i/x_j)(y_j/y_i)} \det(M)$$

$$M = \frac{1}{(x_i/y_j)(ax_i/y_j)}$$

équation Yang-Baxter

Razumov -Stroganov conjecture

(2001, ...)

quantum mechanics:
spin chain model

Spin chains and combinatorics

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(J. R. R. Tolkien)

The XXZ quantum spin chain model with periodic boundary conditions is one of the most popular integrable models which has been investigating by the Bethe Ansatz method during the last 35 years [3]. It is described by the Hamiltonian

$$H_{XXZ} = -\sum_{j=1}^{N} \left\{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right\}, \qquad \vec{\sigma}_{N+1} = \vec{\sigma}_1. \tag{1}$$

The nonzero wave function components are

$$N = 3: \quad \psi_{001} = 1;$$
$$N = 5: \quad \psi_{00011} = 1, \quad \psi_{00101} = 2;$$
$$N = 7: \quad \psi_{0000111} = 1, \quad \psi_{0001101} = \psi_{0001011} = 3, \quad \psi_{0010011} = 4 \quad \psi_{0010101} = 7.$$

All components not included in the list can be obtained by shifting. Notice that the components of the ground state are positive in accordance with the Perron–Frobenius theorem.

Let us continue the list. For $N = 9$ the components of the eigenvector with the energy $-27/2$ and $S_z = -1/2$ are

$$\psi_{000001111} = 1, \quad \psi_{000010111} = 4, \quad \psi_{000011011} = 6, \quad \psi_{000100111} = 7,$$
$$\psi_{000101011} = 17, \quad \psi_{000101101} = 14, \quad \psi_{000110011} = 12, \quad \psi_{001001011} = 21,$$
$$\psi_{001010011} = 25, \quad \psi_{001010101} = 42.$$

Let us continue the list. For $N = 9$ the components of the eigenvector with the energy $-27/2$ and $S_z = -1/2$ are

$$\psi_{000001111} = 1, \quad \psi_{000010111} = 4, \quad \psi_{000011011} = 6, \quad \psi_{000100111} = 7,$$
$$\psi_{000101011} = 17, \quad \psi_{000101101} = 14, \quad \psi_{000110011} = 12, \quad \psi_{001001011} = 21,$$
$$\psi_{001010011} = 25, \quad \psi_{001010101} = 42.$$

We omit nonzero components which can be obtained by the reflection of the order of sites since this transformation is a symmetry of our state, as it is for the ground state. For example, we have

$$\psi_{000011101} = \psi_{000010111} = 4.$$

1 , 2 , 7 , 42 , 429 , ...

?

M1791 0, 1, 2, 7, 32, 181, 1214, 9403, 82508, 808393, 8743994, 103459471, 1328953592, 18414450877, 273749755382, 4345634192131, 73362643649444, 1312349454922513
$a(n)=n.a(n-1)+(n-2)a(n-2)$. Ref R1 188. [0,3; A0153, N0706]

$$\text{E.g.f.: } (1-x)^{-3}\, e^{-x}.$$

M1792 1, 1, 2, 7, 32, 181, 1232, 9787, 88832, 907081, 10291712, 128445967, 1748805632, 25794366781, 409725396992, 6973071372547, 126585529106432
Expansion of $1/(1-\sinh x)$. Ref ARS 10 138 80. [0,3; A6154]

M1793 0, 1, 1, 2, 7, 32, 184, 1268, 10186, 93356, 960646, 10959452, 137221954, 1870087808, 27548231008, 436081302248, 7380628161076, 132975267434552
Stochastic matrices of integers. Ref DUMJ 35 659 68. [0,4; A0987, N0707]

M1794 1, 2, 7, 33, 192
Permutations of length n with n in second orbit. Ref C1 258. [2,2; A6595]

M1795 1, 2, 7, 34, 209, 1546, 13327, 130922, 1441729, 17572114, 234662231, 3405357682, 53334454417, 896324308634, 16083557845279, 306827170866106
$a(n)=2n.a(n-1)-(n-1)^2a(n-2)$. Ref SE33 78. [0,2; A2720, N0708]

M1796 1, 2, 7, 34, 257, 2606, 32300, 440564, 6384634
Polyhedra with n nodes. Ref GR67 424. UPG B15. Dil92. [4,2; A0944, N0709]

M1797 2, 7, 35, 219, 1594, 12935, 113945, 1070324, 10586856, 109259633, 1168384157, 12877168147, 145656436074, 1685157199175, 19886174611045
Two-rowed truncated monotone triangles. Ref JCT A42 277 86. Zei93. [1,1; A6947]

M1798 1, 1, 2, 7, 35, 228, 1834, 17382, 195866, 2487832, 35499576, 562356672, 9794156448, 186025364016, 3826961710272, 84775065603888, 2011929826983504
Coefficients of iterated exponentials. Ref SMA 11 353 45. [0,3; A0154, N0710]

M1799 1, 2, 7, 35, 228, 1834, 17582, 195866, 2487832, 35499576, 562356672, 9794156448, 186025364016, 3826961710272, 84775065603888, 2011929826983504
Expansion of $\ln(1+\ln(1+x))$. [0,2; A3713]

M1800 1, 0, 1, 2, 7, 36, 300, 3218, 42335, 644808
Circular diagrams with n chords. Ref BarN94. [0,4; A7474]

M1801 1, 2, 7, 36, 317, 5624, 251610, 33642660, 14685630688
$n \times n$ binary matrices. Ref CPM 89 217 64. SLC 19 79 88. [0,2; A2724, N0711]

M1802 2, 7, 37, 216, 1780, 32652
Semigroups of order n with 2 idempotents. Ref MAL 2 2 67. SGF 14 71 77. [2,1; A2787, N0712]

M1803 1, 2, 7, 37, 266, 2431, 27007, 353522, 5329837, 90960751, 1733584106, 36496226977, 841146804577, 21065166341402, 569600638022431
$a(n)=(2n-1)a(n-1)+a(n-2)$. Ref RCI 77. [0,2; A1515, N0713]

M1804 1, 1, 2, 7, 38, 291, 2932, 36961, 561948, 10026505, 205608536, 4767440679, 123373203208, 3525630110107, 110284283006640, 3748357699560961
Forests of labeled trees with n nodes. Ref JCT 5 96 68. SIAD 3 574 90. [0,3; A1858, N0714]

M1805 1, 1, 2, 7, 40, 357, 4824, 96428, 2800472, 116473461
n-element partial orders contained in linear order. Ref nbh. [0,3; A6455]

M1806 1, 2, 7, 41, 346, 3797, 51157, 816356, 15050581, 314726117, 7359554632, 190283748371, 5389914888541, 165983936096162, 5521346346543307
Planted binary phylogenetic trees with n labels. Ref LNM 884 196 81. [1,2; A6677]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \ldots n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 64539836938720749739356
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

M1810 0, 1, 2, 7, 44, 361, 3654, 44207, 622552, 10005041, 180713290, 3624270839, 79914671748, 1921576392793, 50040900884366, 1403066801155039
Modified Bessel function $K_n(1)$. Ref AS1 429. [0,3; A0155, N0716]

M1811 0, 1, 2, 7, 44, 447, 6749, 142176, 3987677, 143698548, 6470422337, 356016927083, 23503587609815, 1833635850492653, 166884365982441238
$a(n)=n(n-1)a(n-1)/2+a(n-2)$. [0,3; A1046, N0717]

M1812 1, 2, 7, 44, 529, 12278, 565723, 51409856, 9371059621, 3387887032202, 2463333456292207, 3557380311703796564, 10339081666350180289849
Sum of Gaussian binomial coefficients $[n,k]$ for $q=4$. Ref TU69 76. GJ83 99. ARS A17 328 84. [0,2; A6118]

M1813 2, 7, 52, 2133, 2590407, 3374951541062, 5695183504479116640376509, 16217557574922386301420514191523784895639577710480
Free binary trees of height n. Ref JCIS 17 180 92. [1,1; A5588]

M1814 1, 1, 2, 7, 56, 2212, 2595782, 3374959180831, 5695183504489239067484387, 16217557574922386301420531277071365103168734284282
Planted 3-trees of height n. Ref RSE 59(2) 159 39. CMB 11 87 68. JCIS 17 180 92. [0,3; A2658, N0718]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \,.\,.\, n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

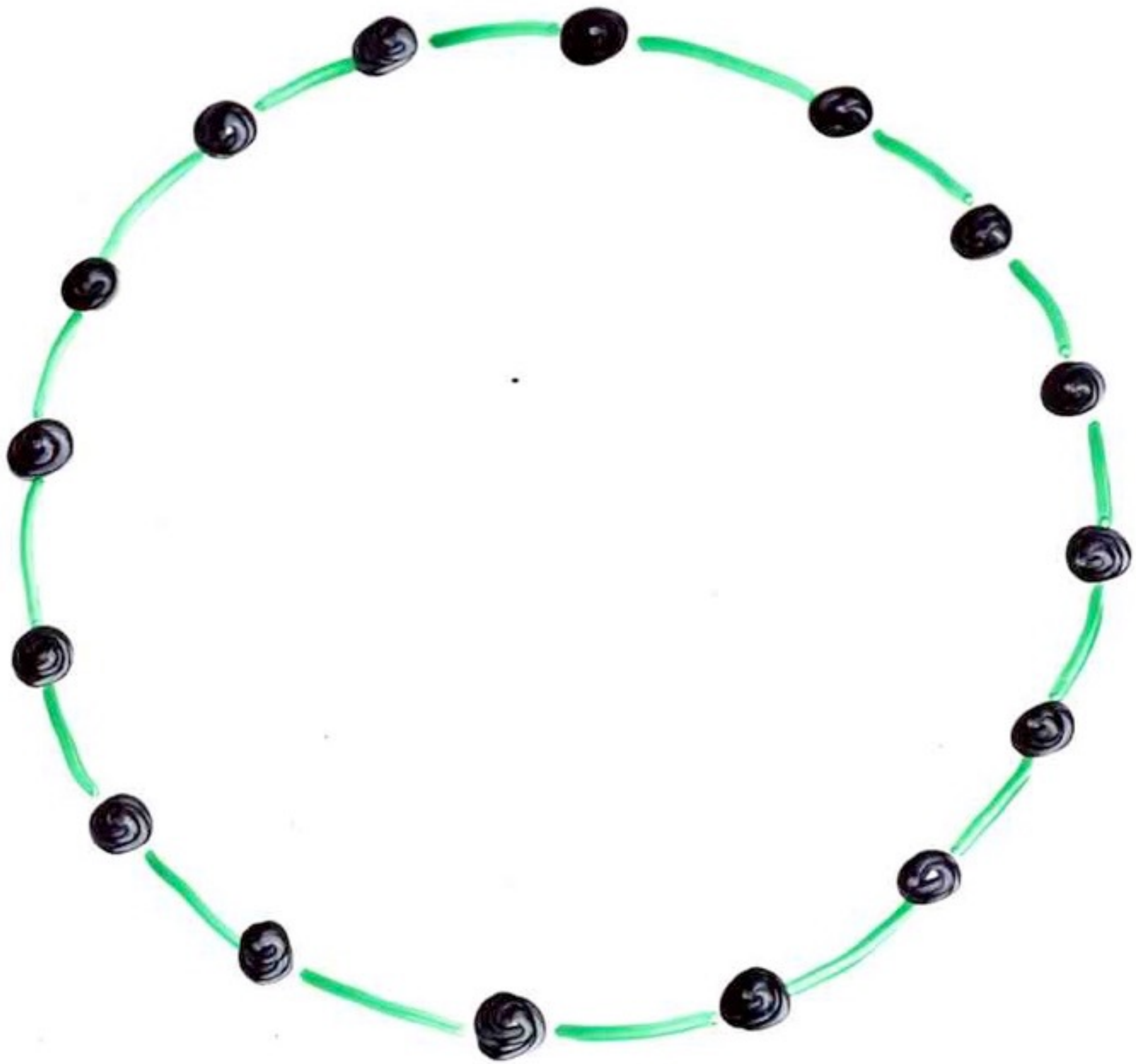
M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 64539836938720749739356
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

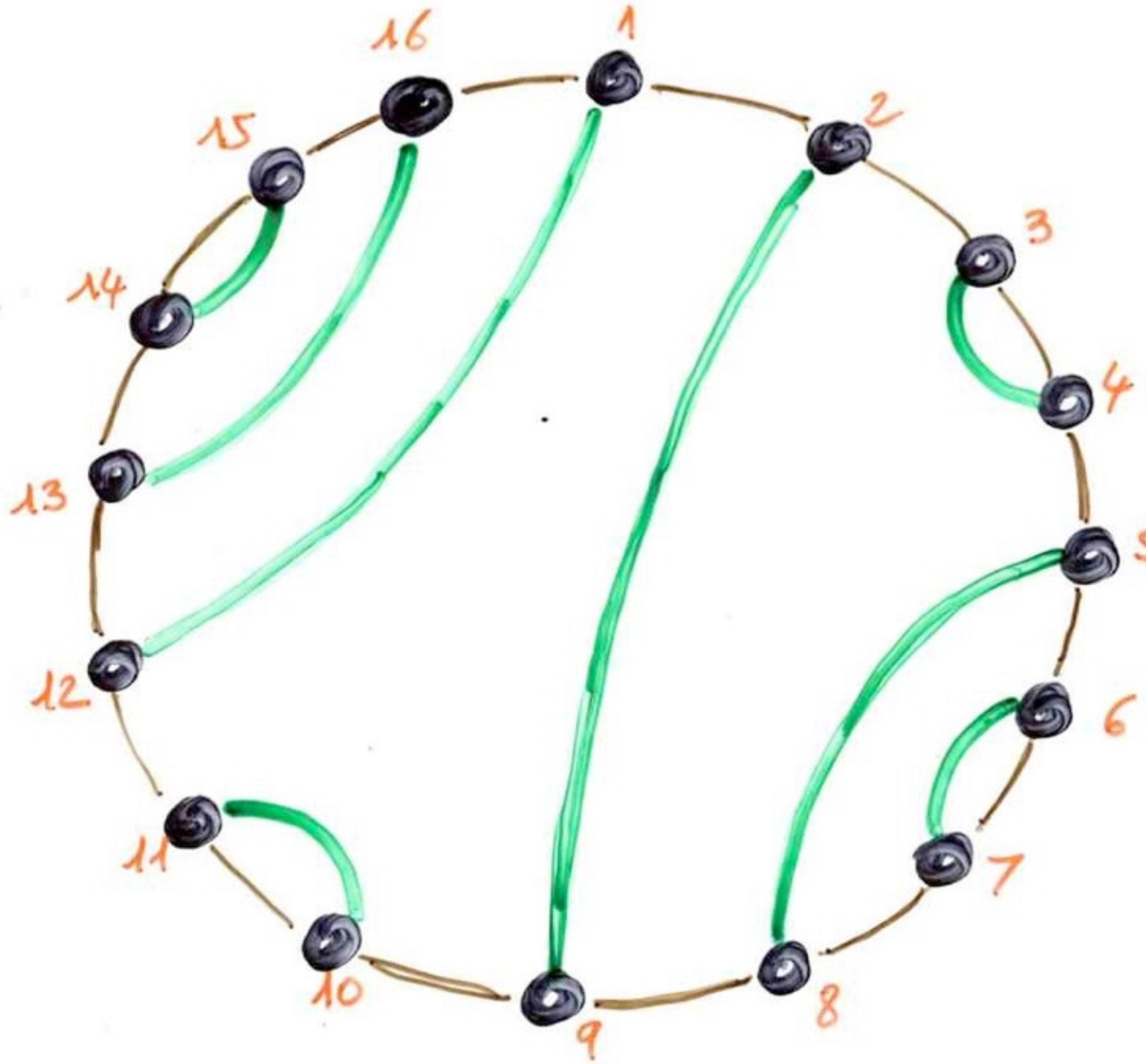
M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \, . \, . \, n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 64539836938720749739356
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

Markov chain
on
chord diagrams

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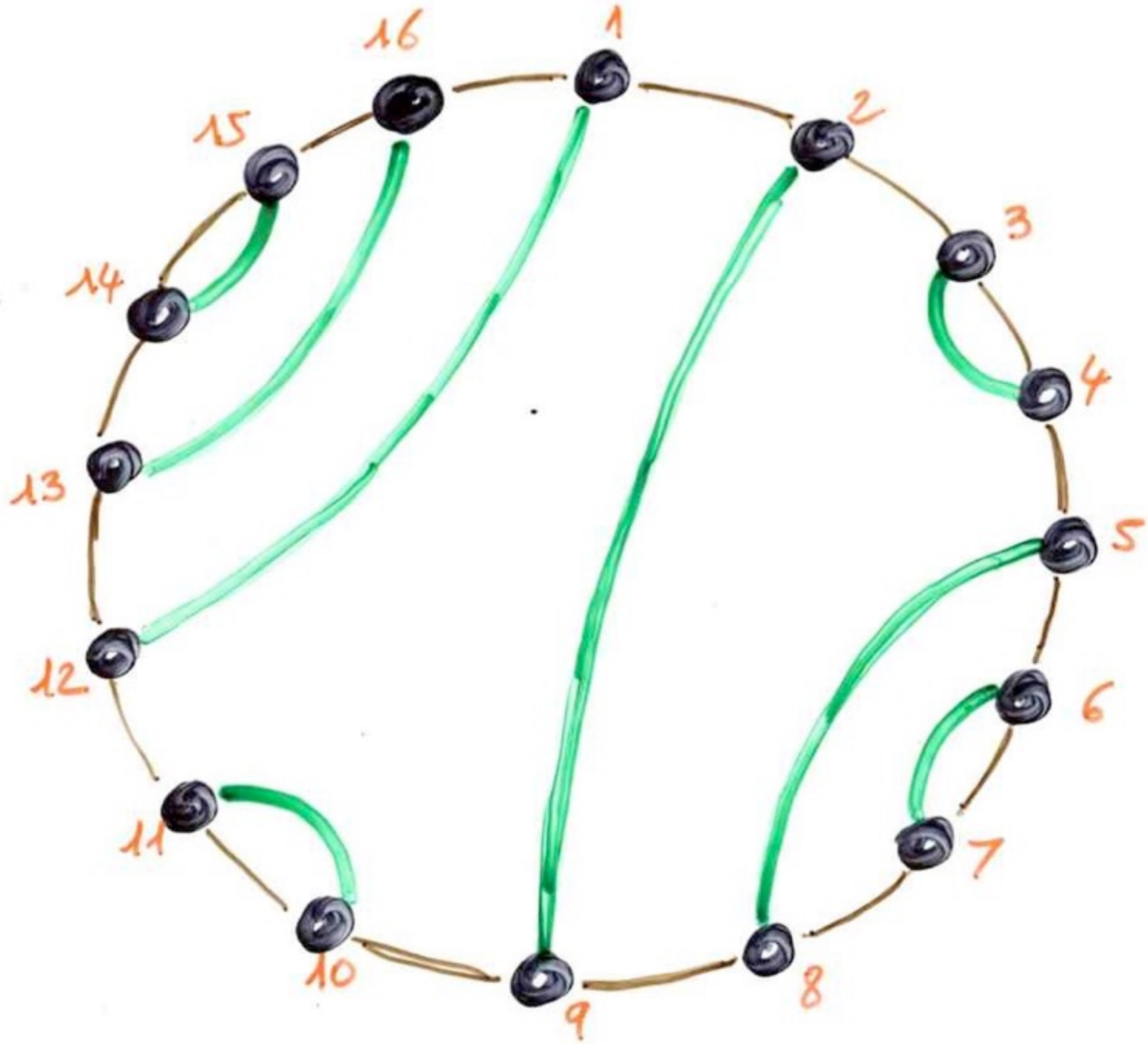
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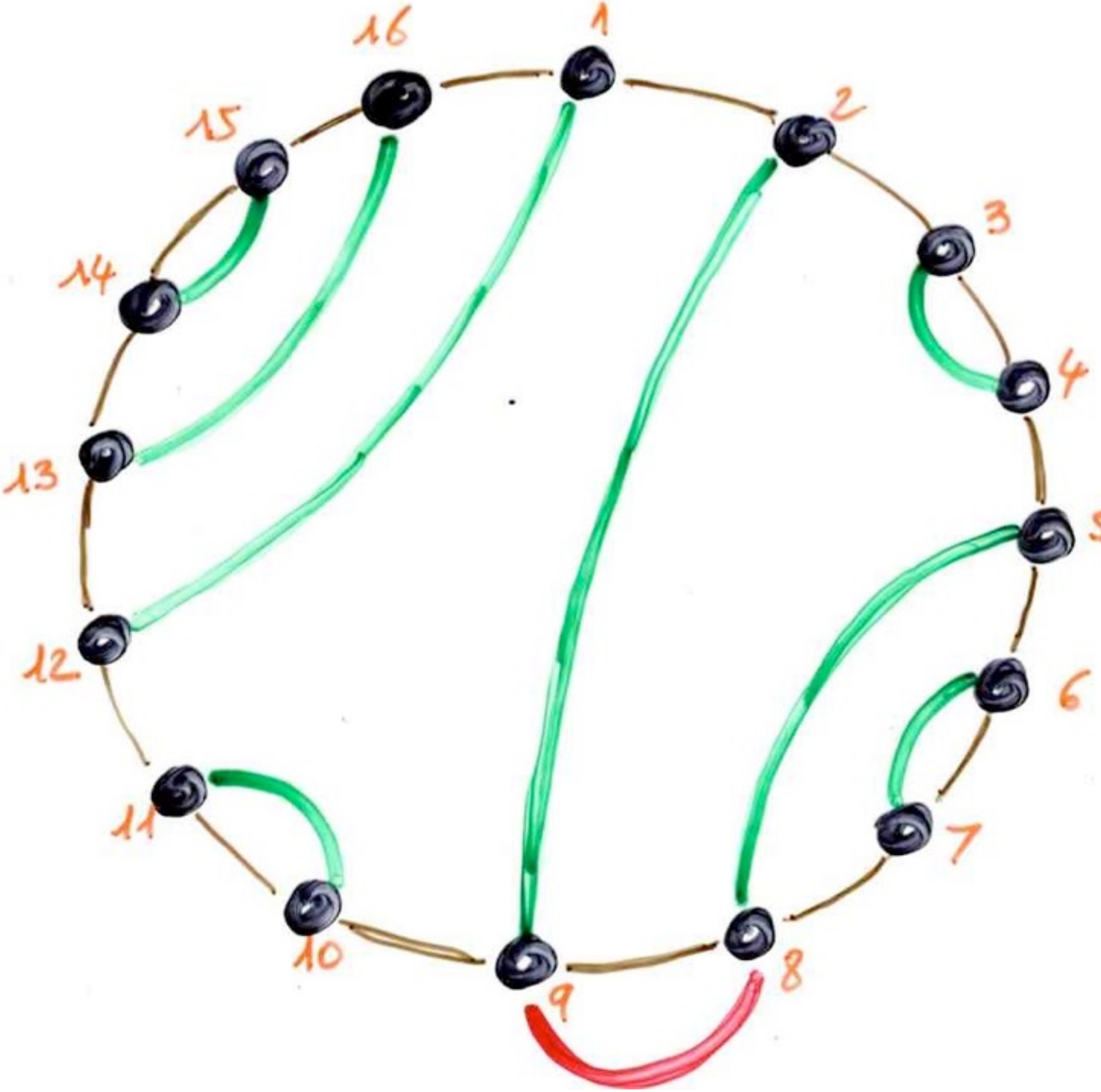
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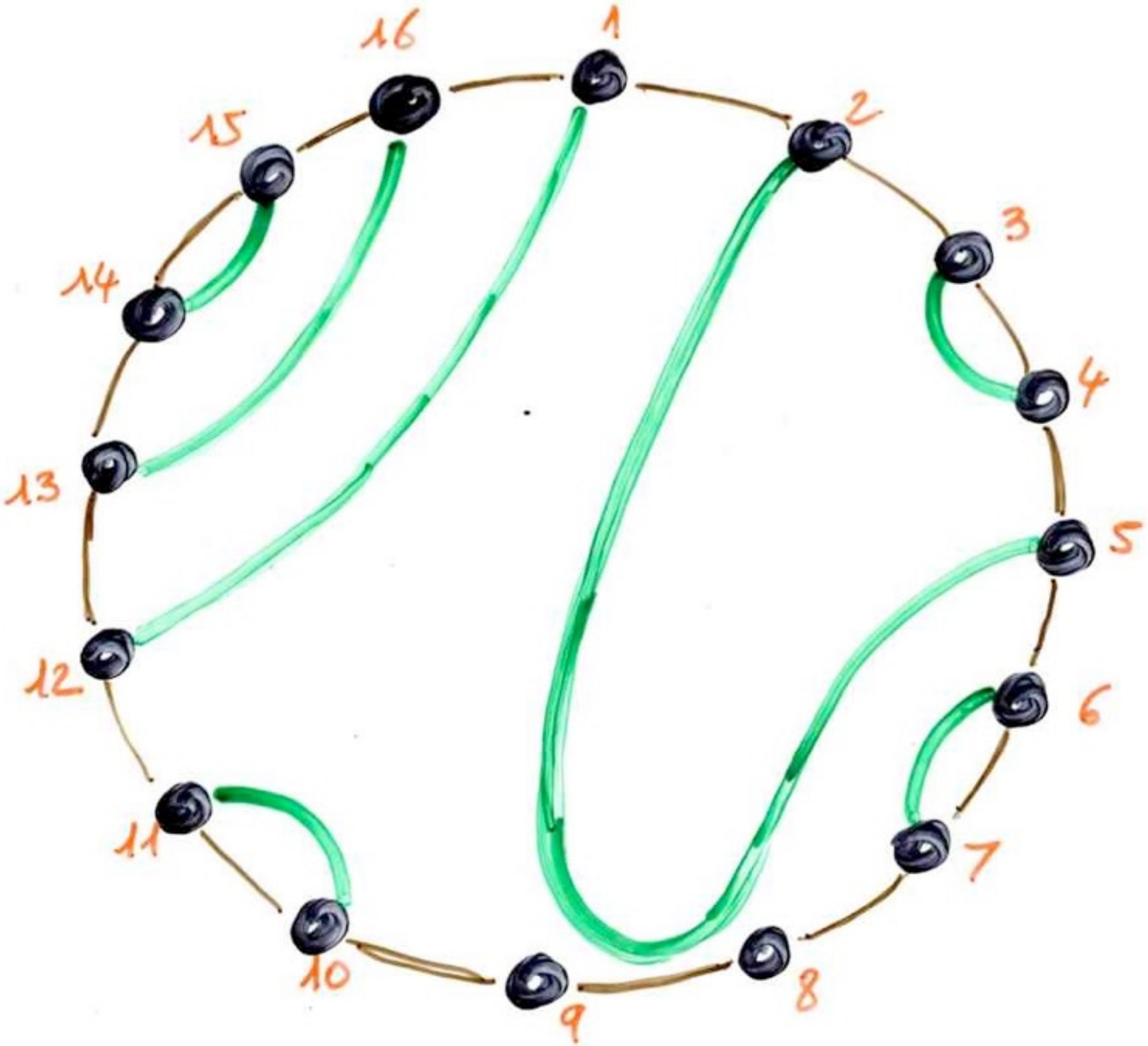
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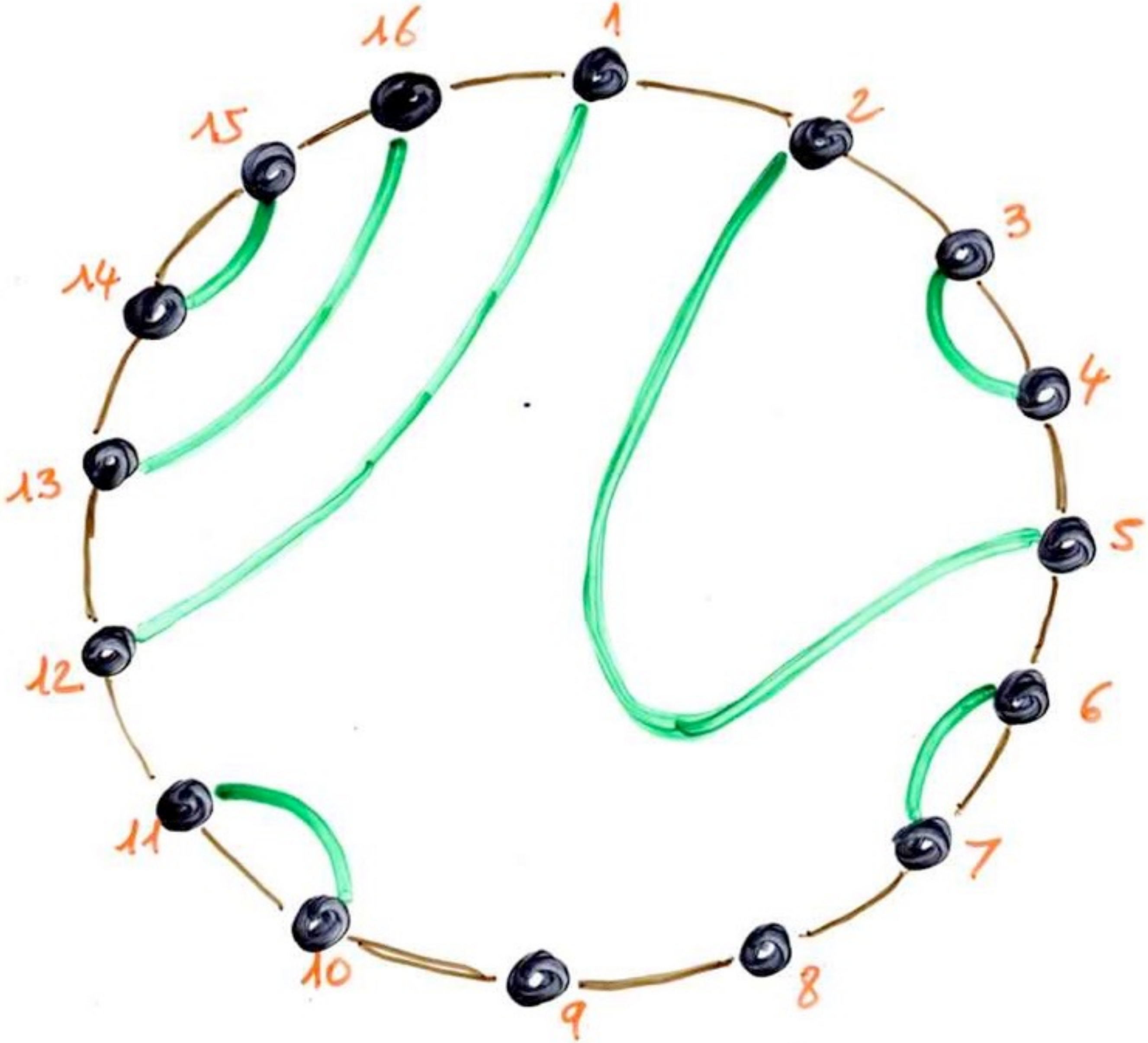
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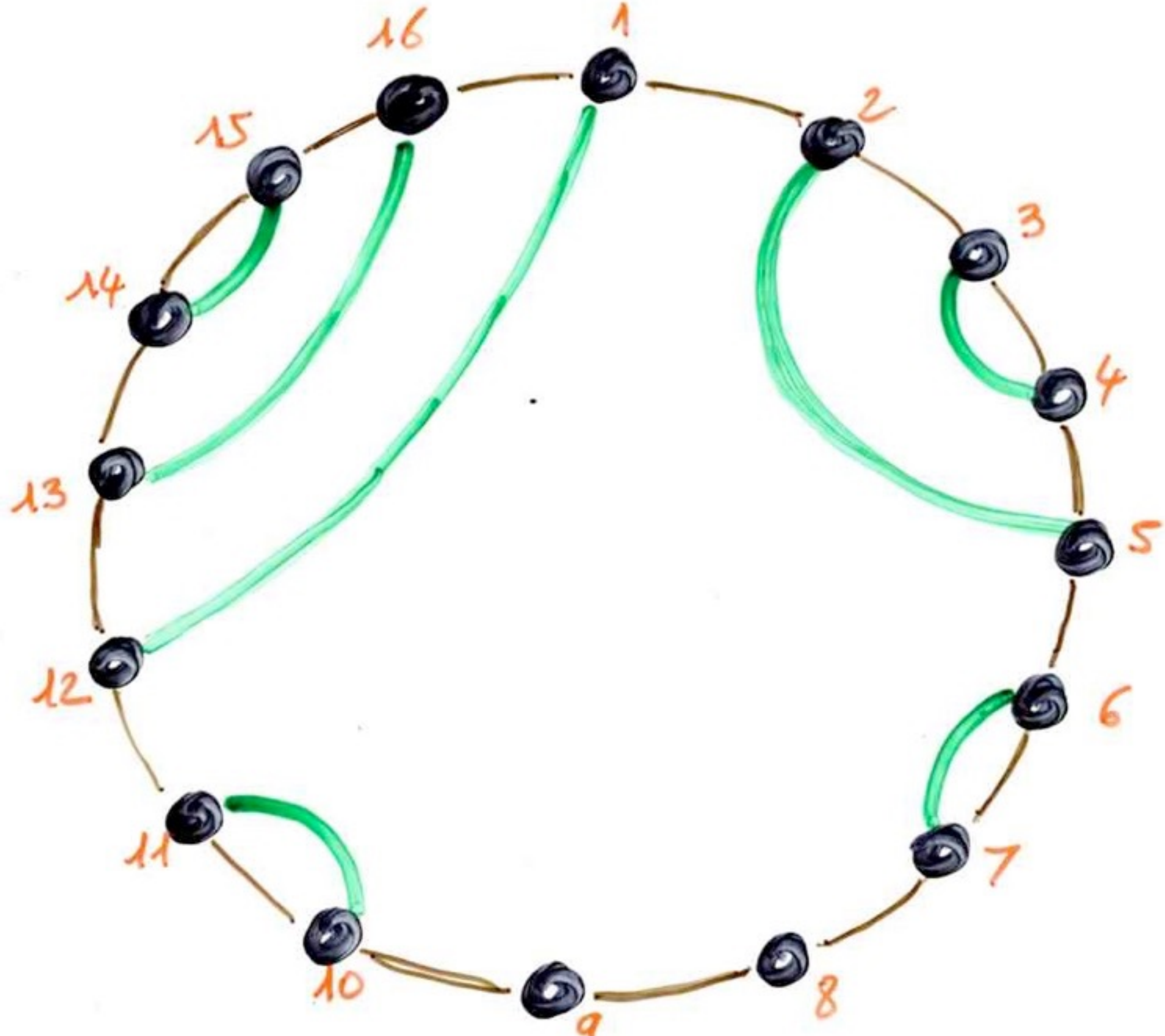
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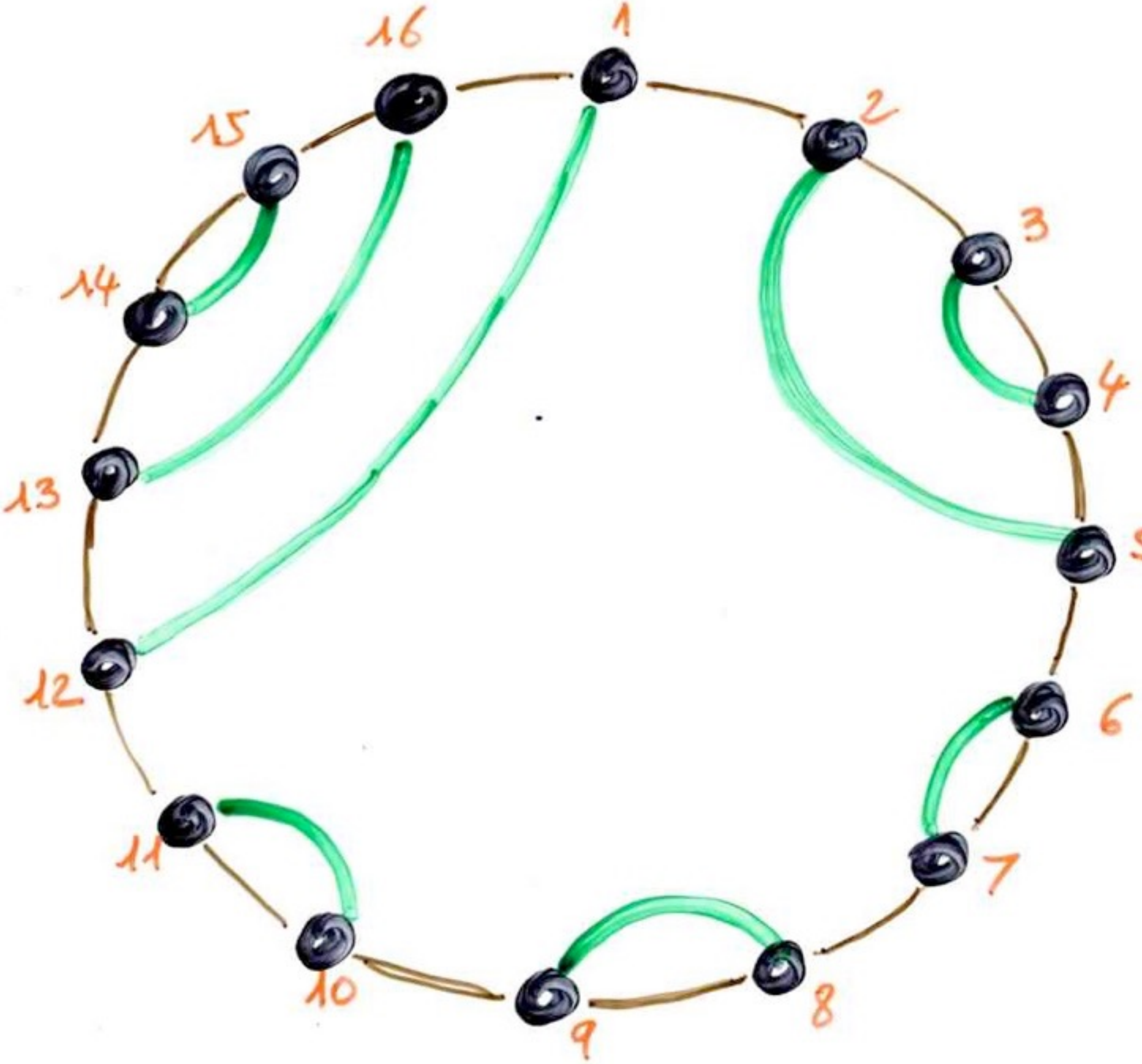
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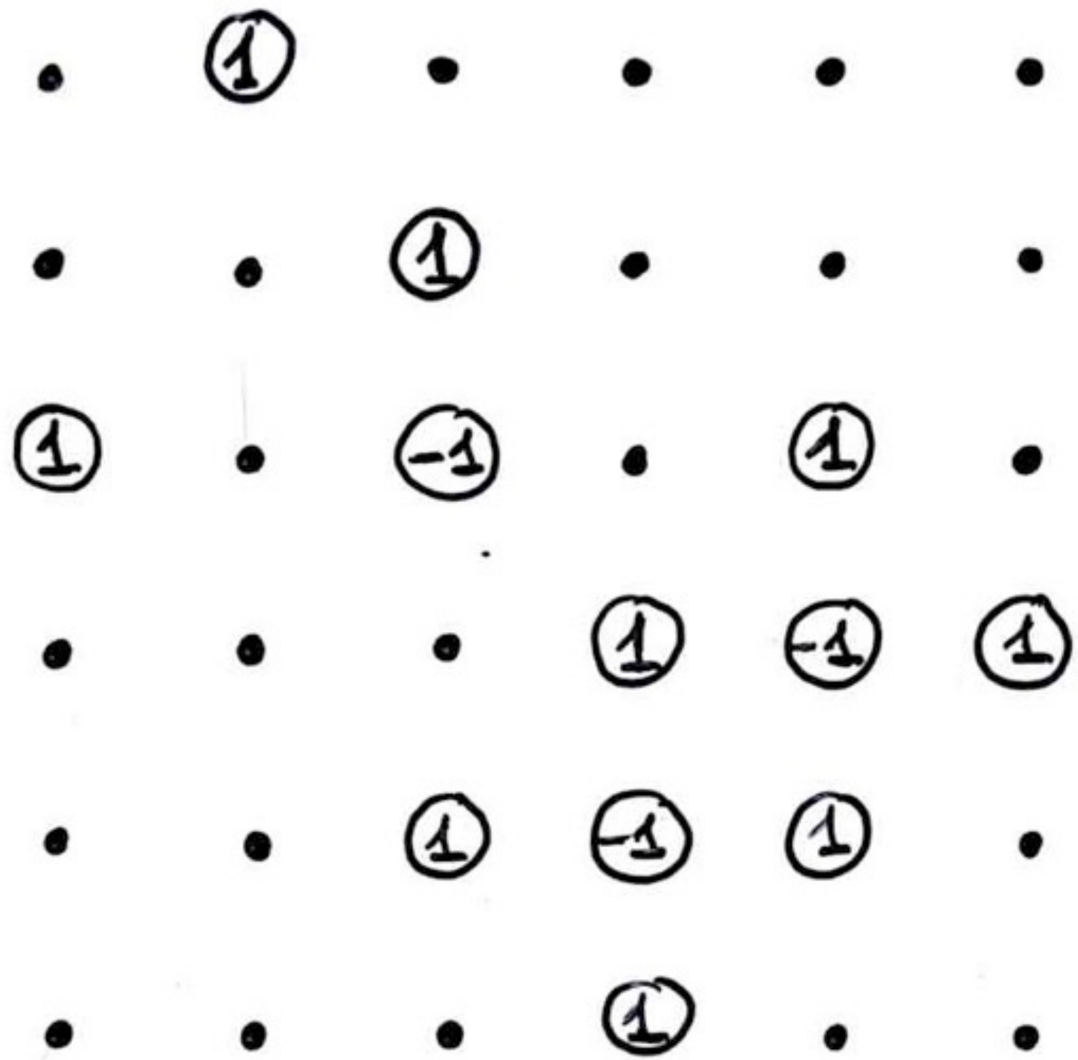
stationary probabilities

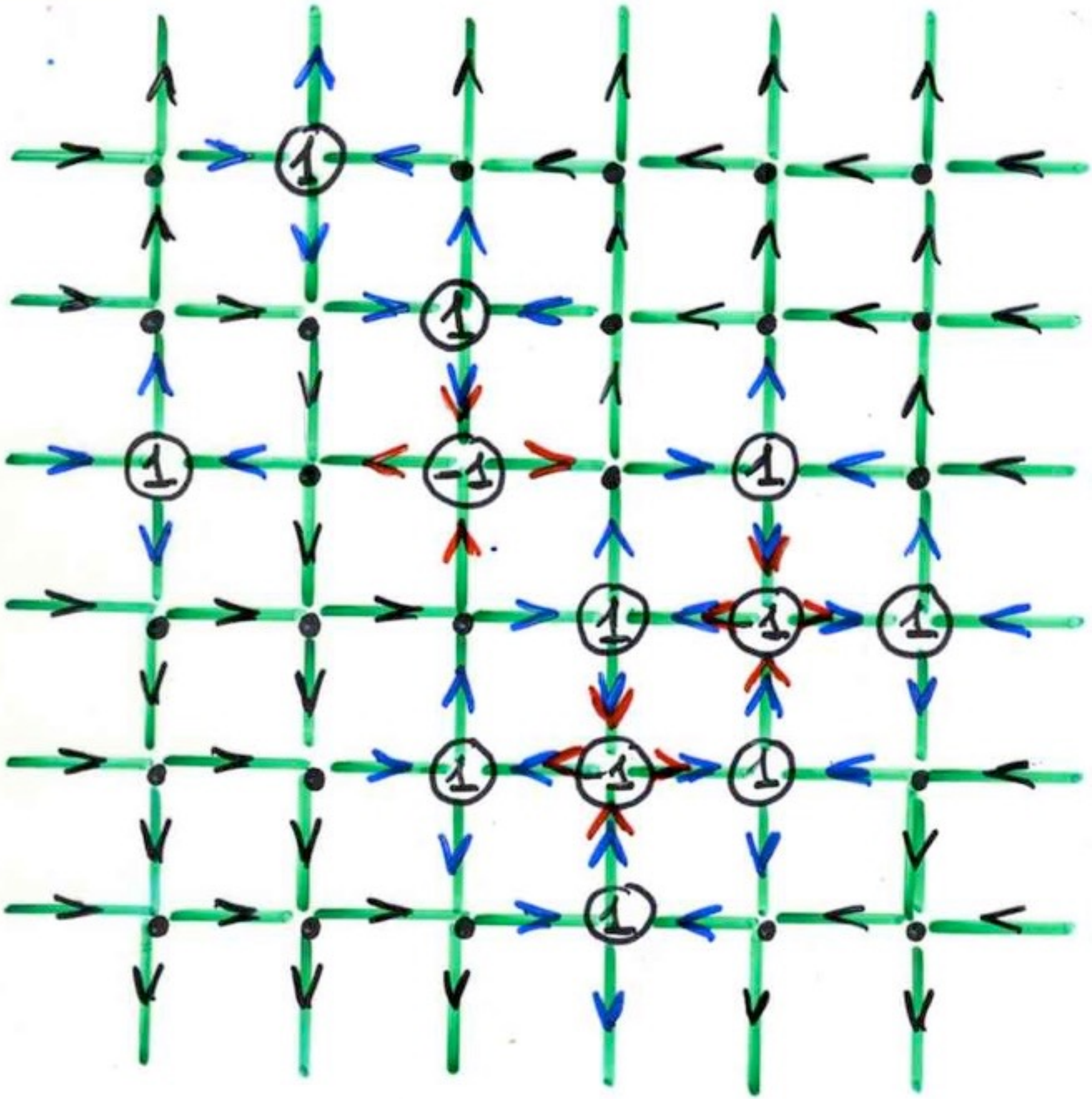
FPL

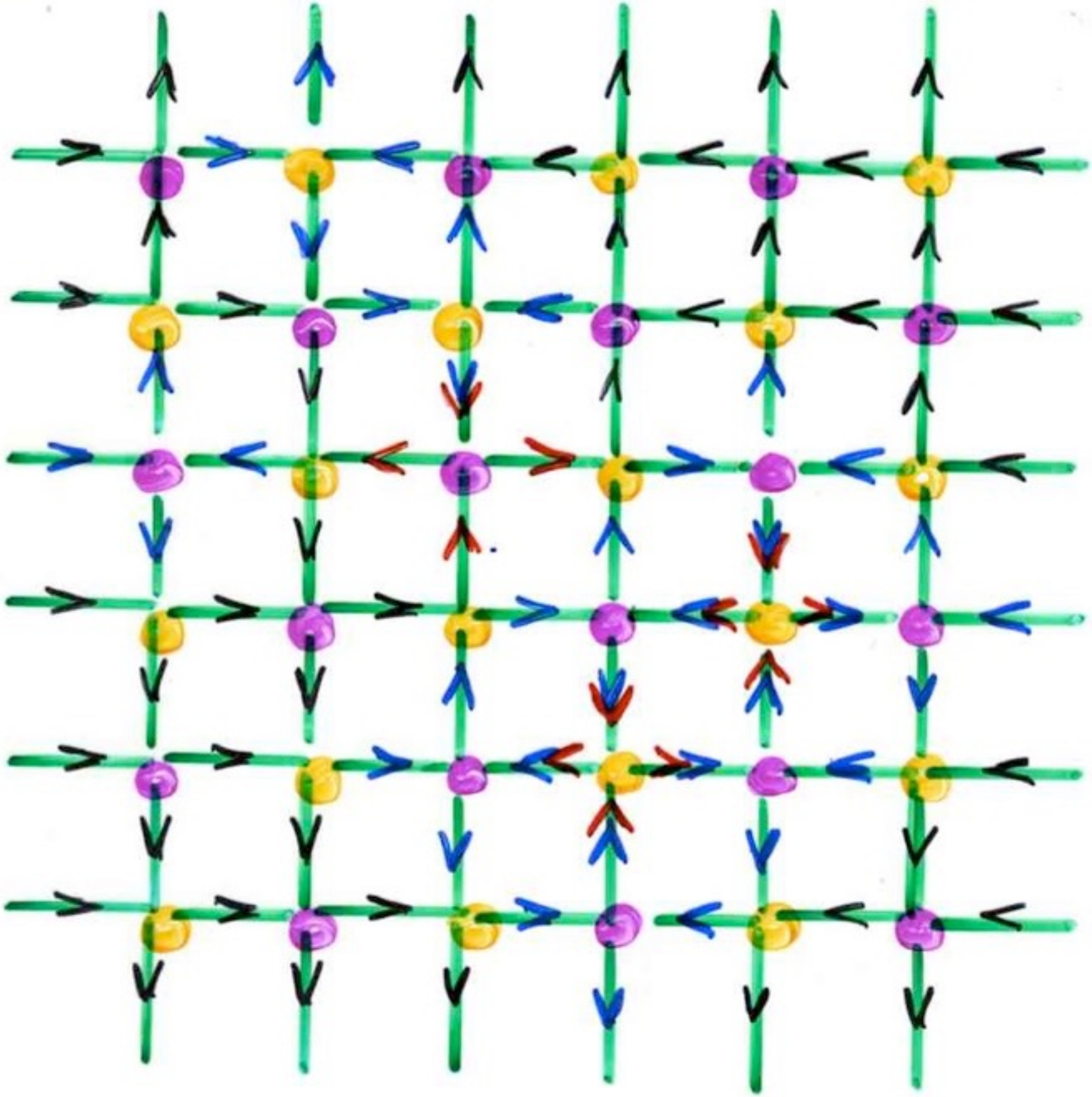
“Fully packed loop configurations”

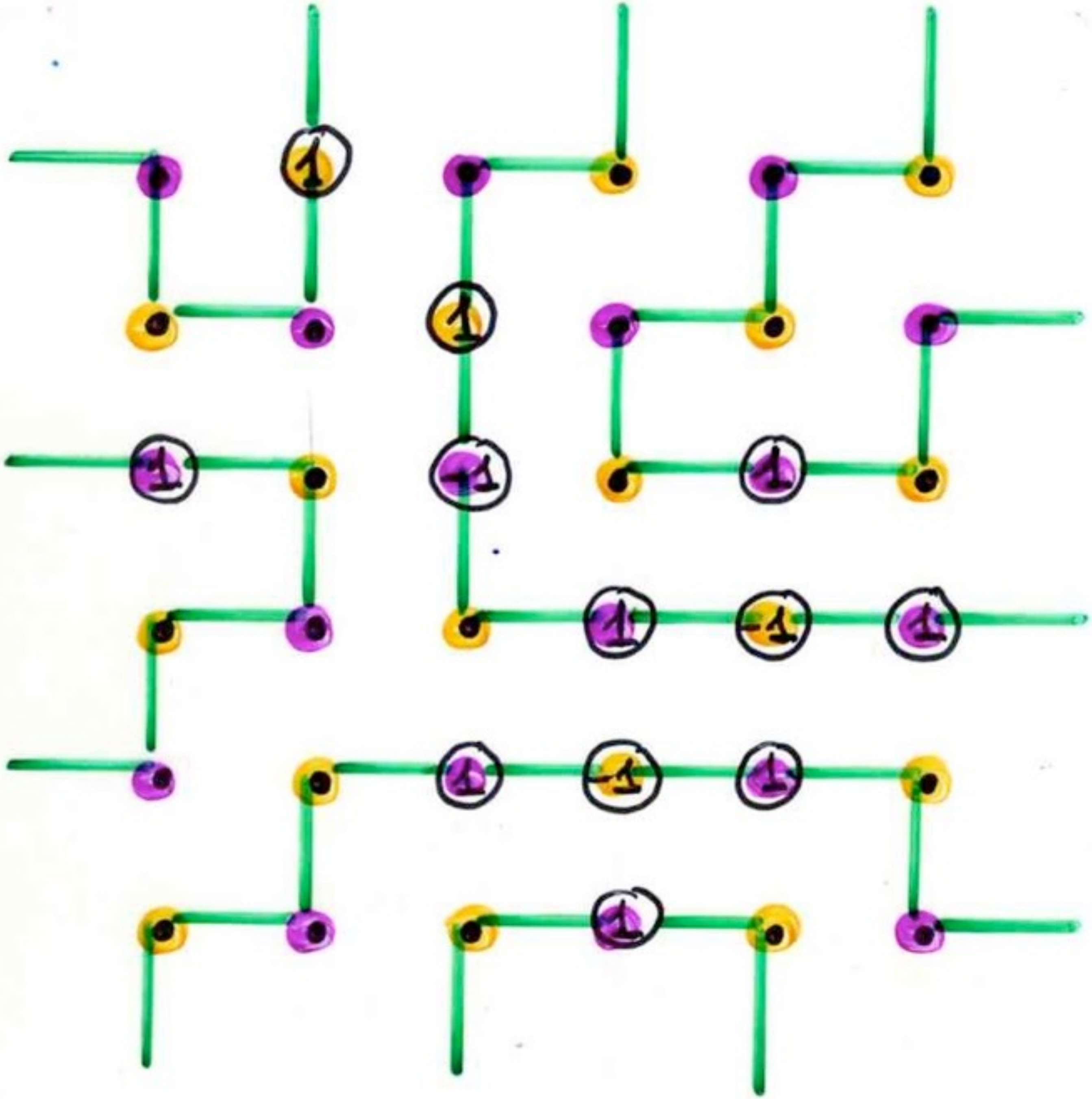
The
bijection
AMS
FPL



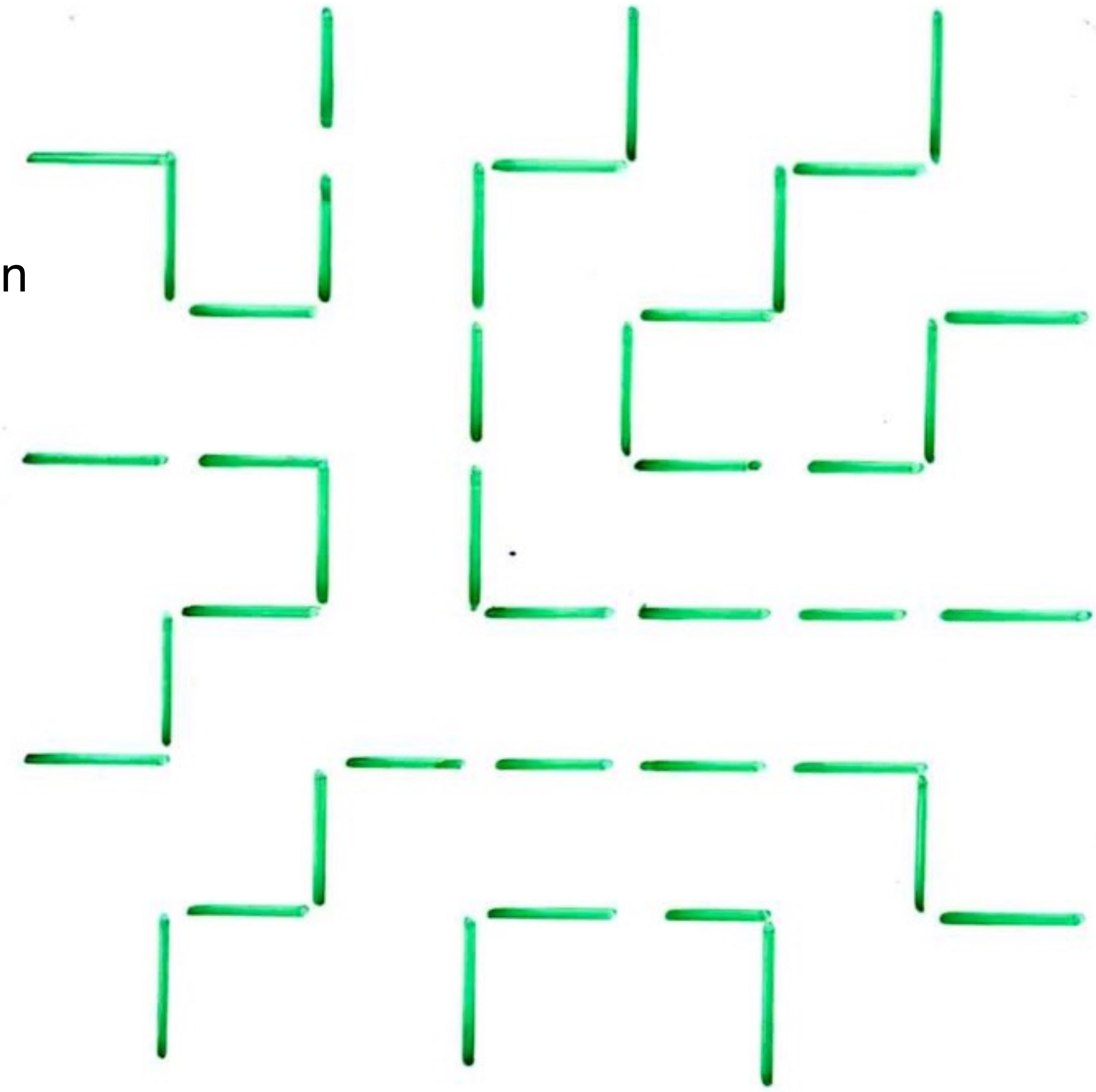
The 6-vertex model

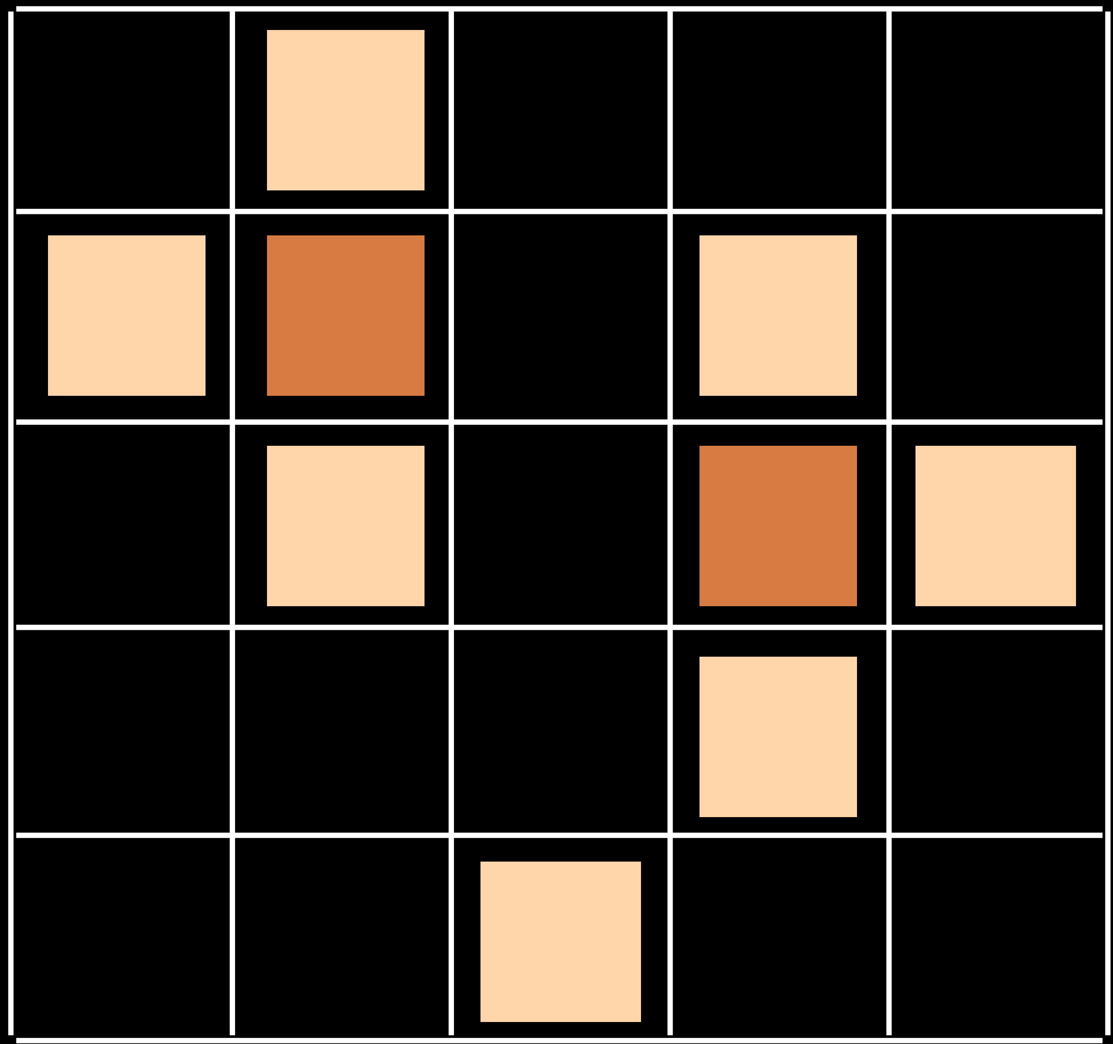


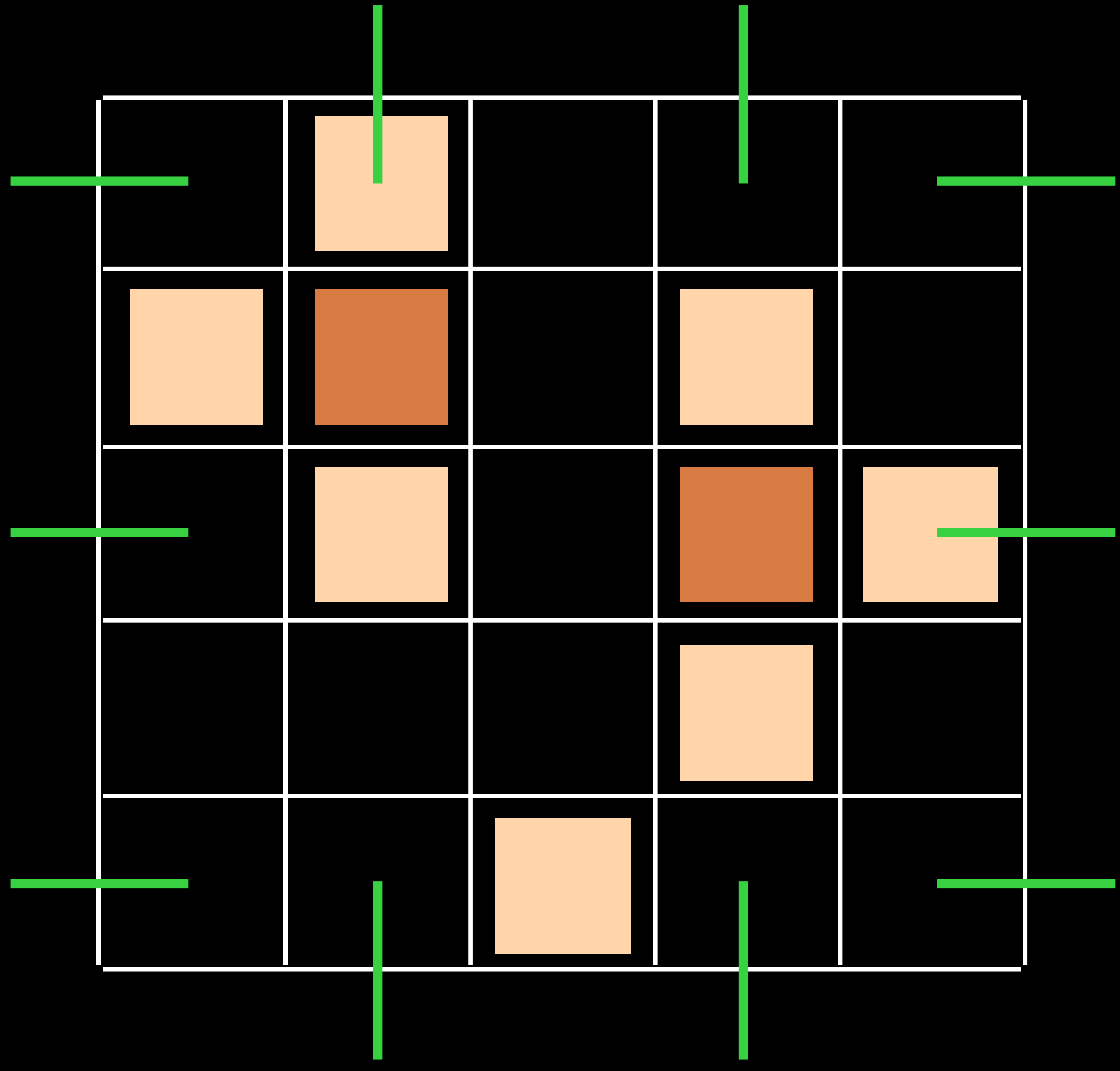


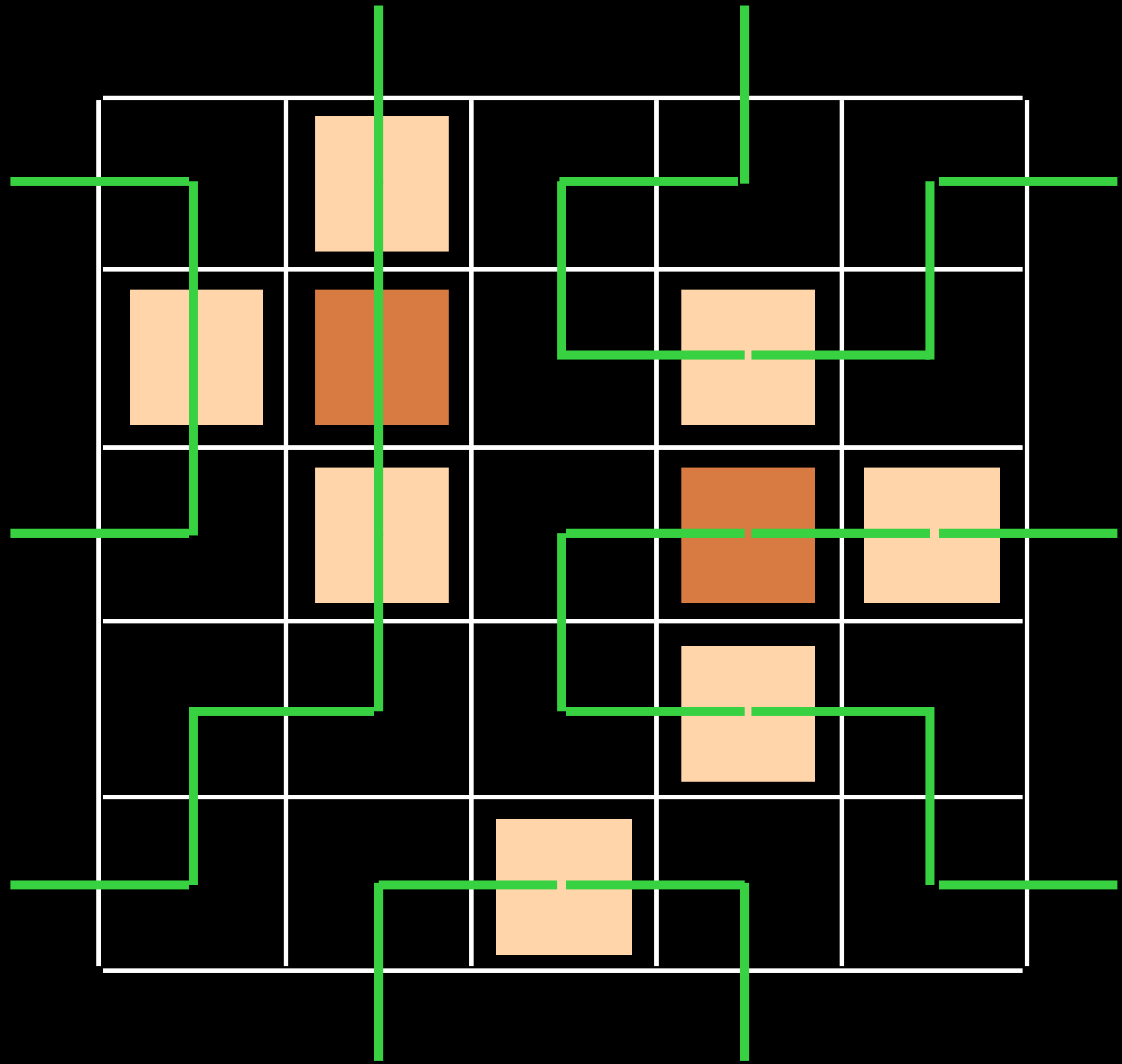


FPL
“Fully
Packed
Loop”
configuration

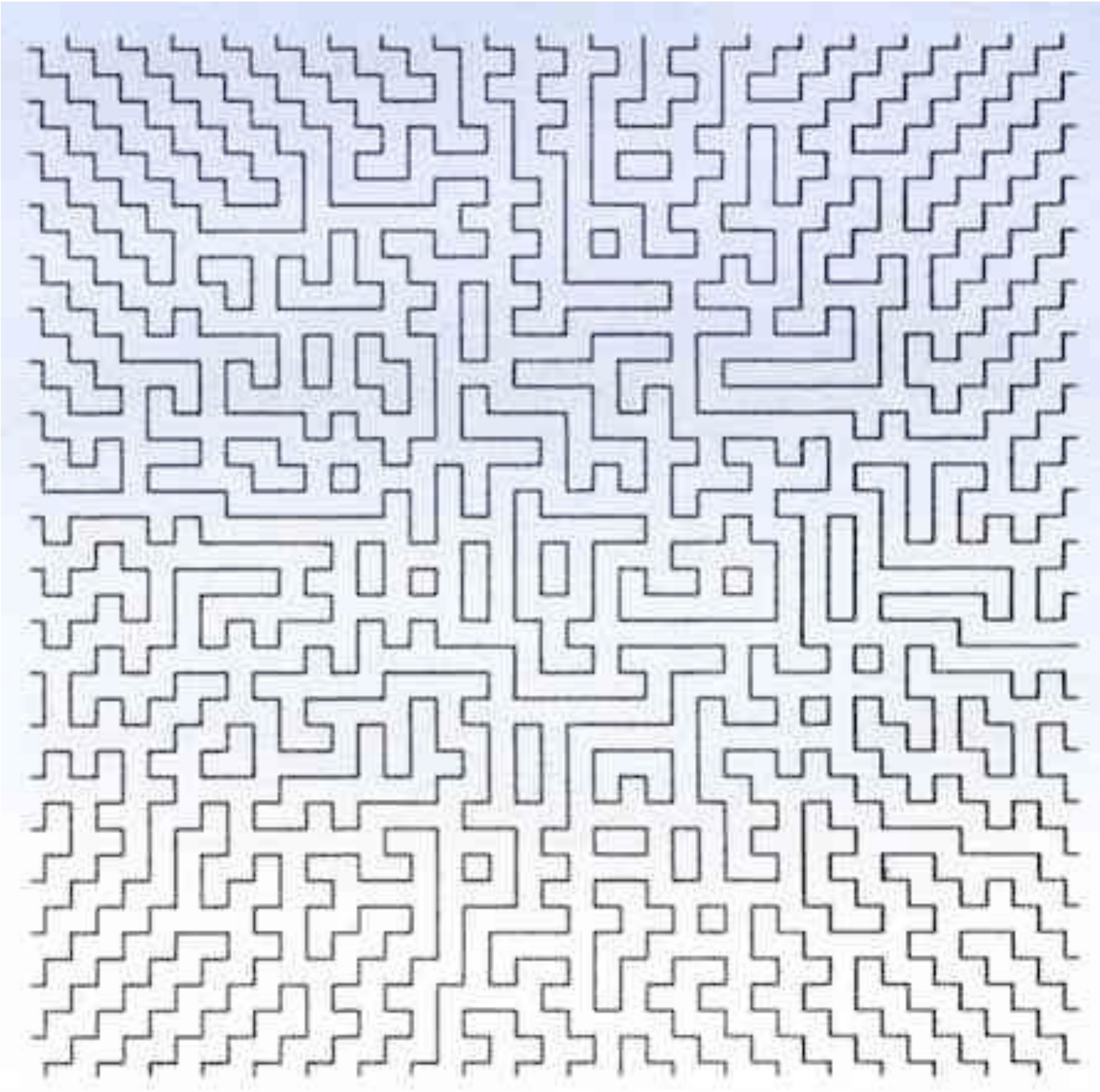




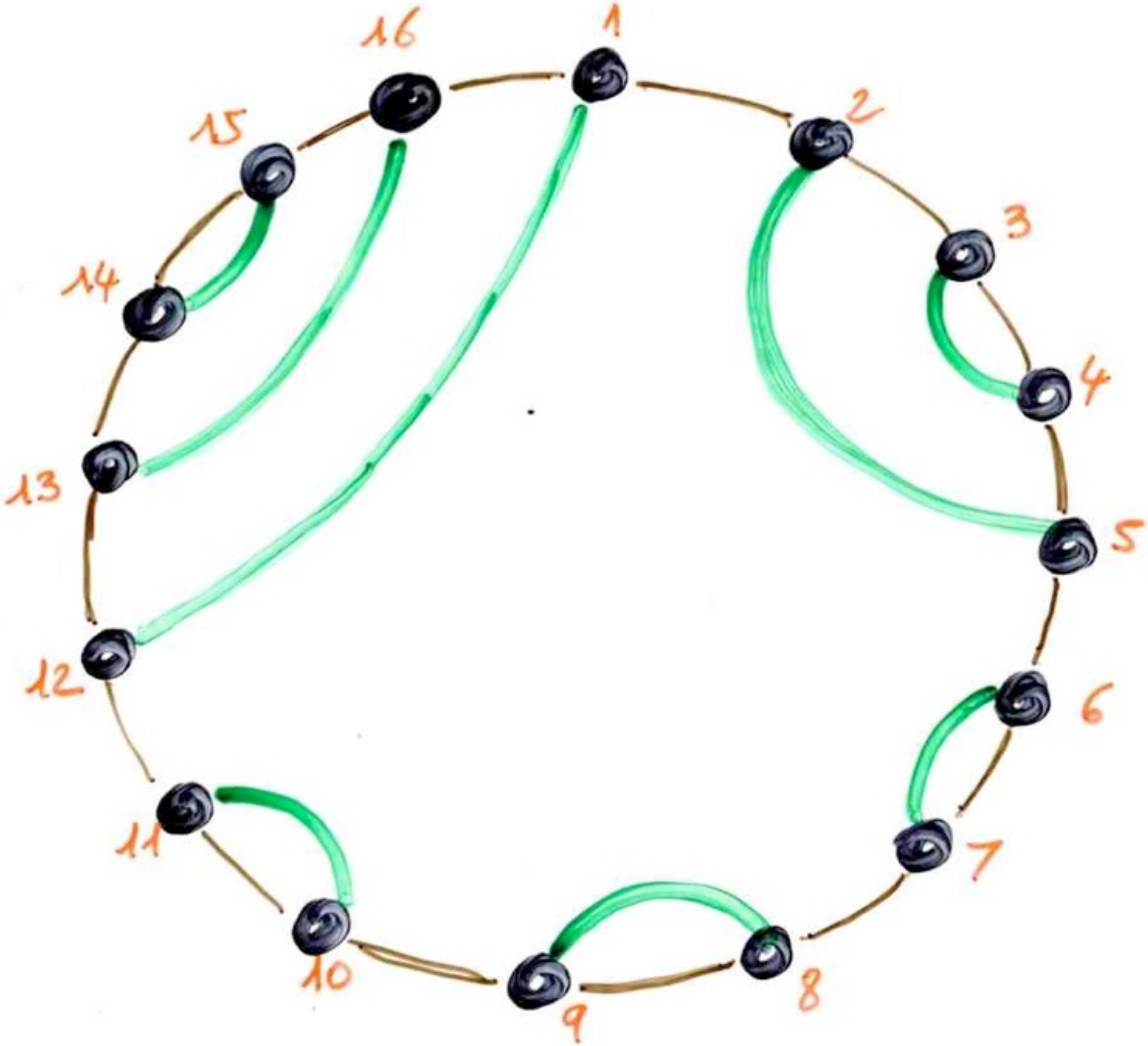




random
FPL



Razumov-Stroganov conjecture



stationary probabilities

L. Cantini, A.Sportielo (2011)

algebraic combinatorics

Around
the Razumov-Stroganov conjecture
and alternating sign matrices

Around the Razumov-Stroganov conjecture

Philippe Di Francesco, Paul Zinn-Justin (2005 - 2009)

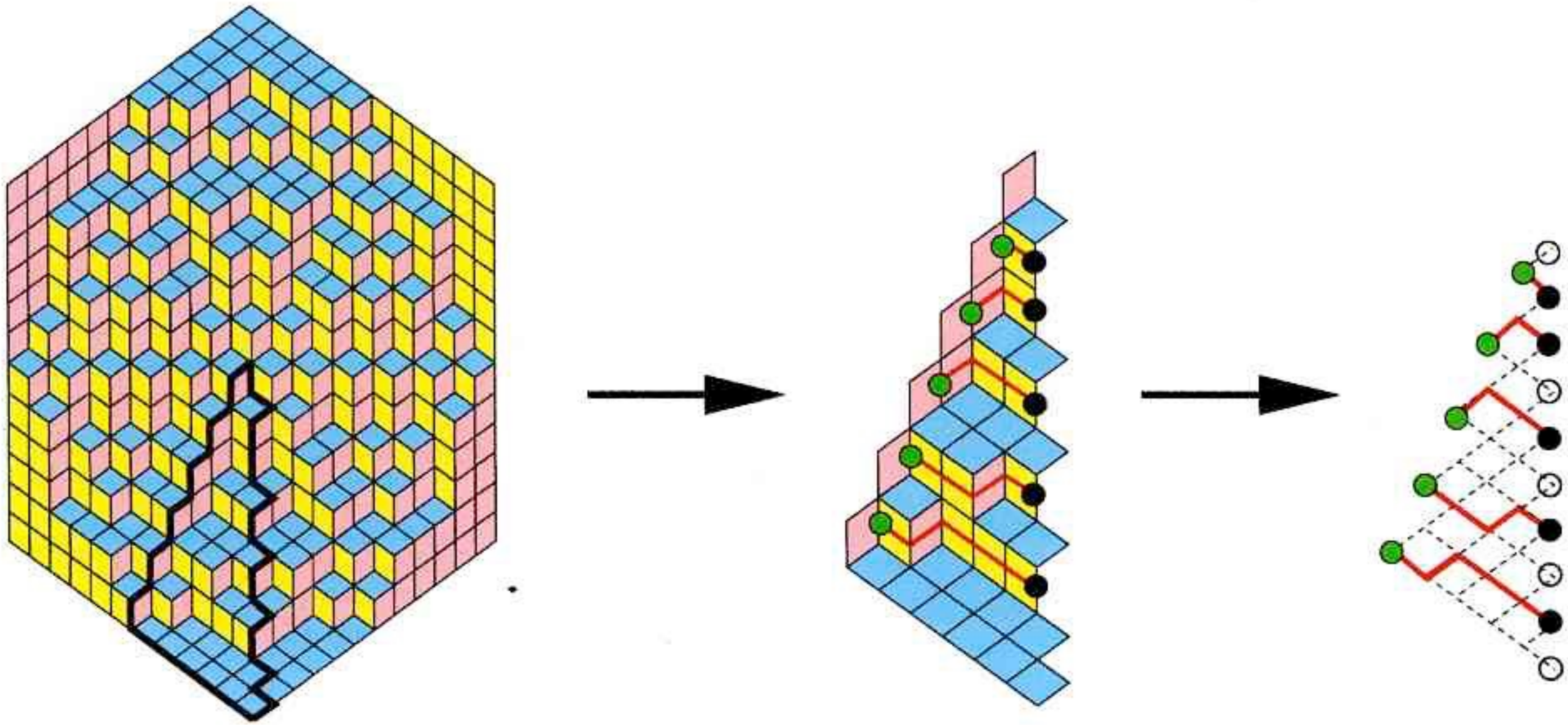
De Gier, Pyatov (2007)

Knizhnik - Zamolodchikov
equation

qKZ

TSSCPP

ASM



Di Francesco (2006)

ASM

1-, 2-, 3- enumeration $A_n(x)$

Colomo, Pronco, (2004)

Hankel determinants

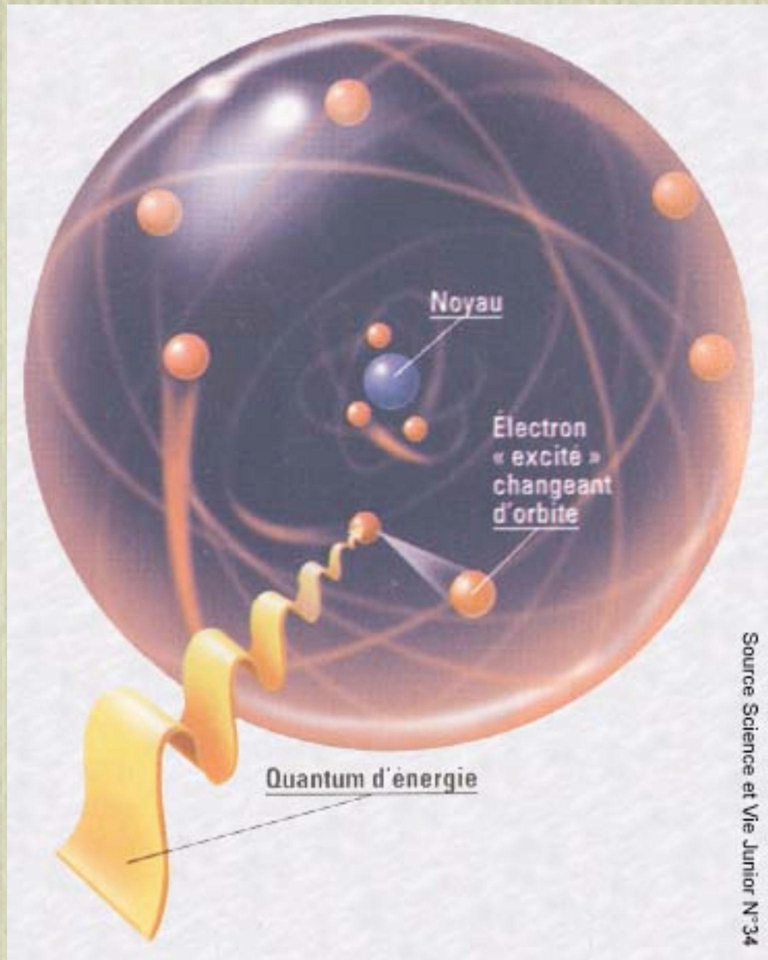
(continuous) Hahn, Meixner-Pollaczek,
(continuous) dual Hahn orthogonal polynomials

Ismail, Lin, Roan (2004)
XXZ spin chains and Askey-Wilson operator

Schubert and Grothendick polynomials
Lascoux, Schützenberger

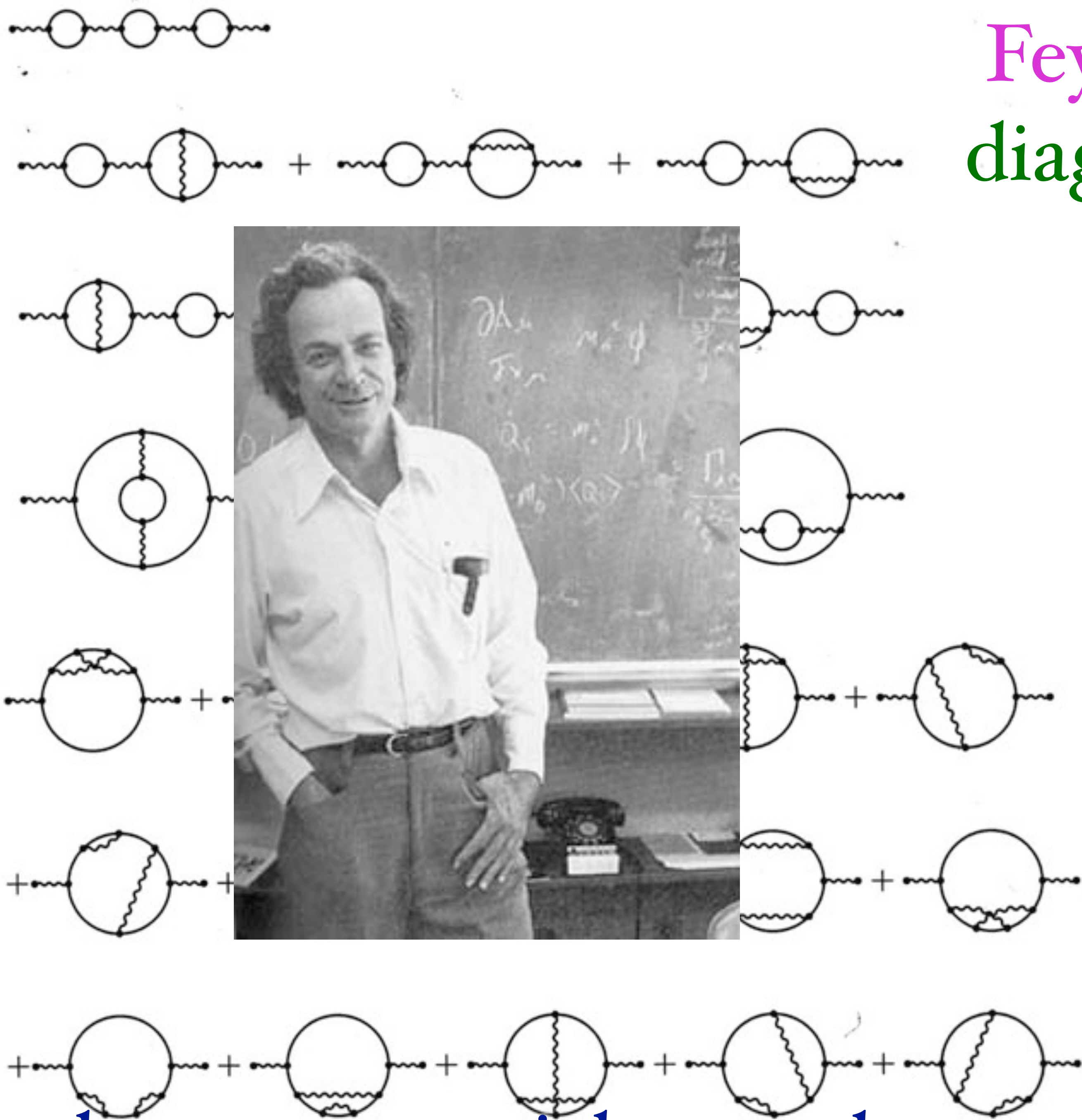
algebraic combinatorics

the quantum world



Feyman

diagrams



interactions between particles, photons

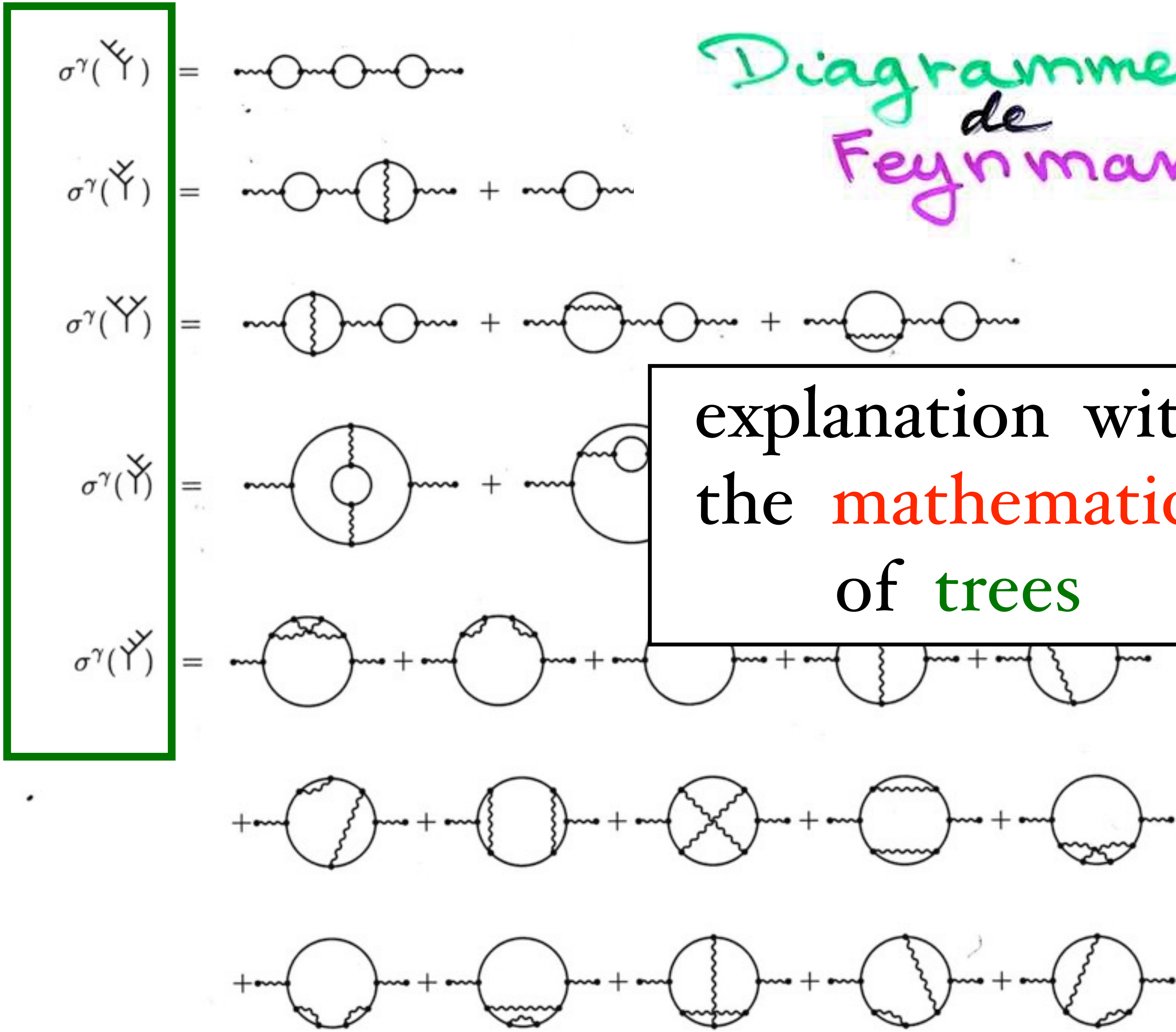
infinite sums of infinite quantities ?!?

deleting the double infinite ...

quantum renormalization

recipe for cooking

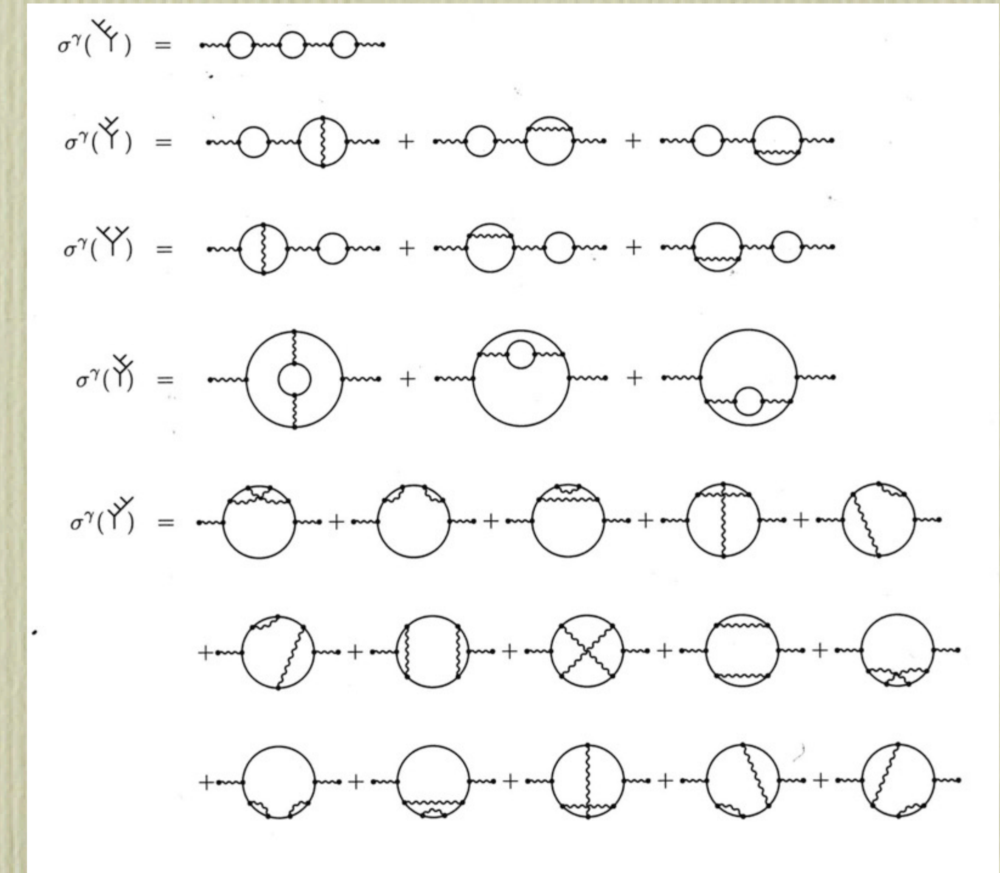
Diagrammes de Feynman



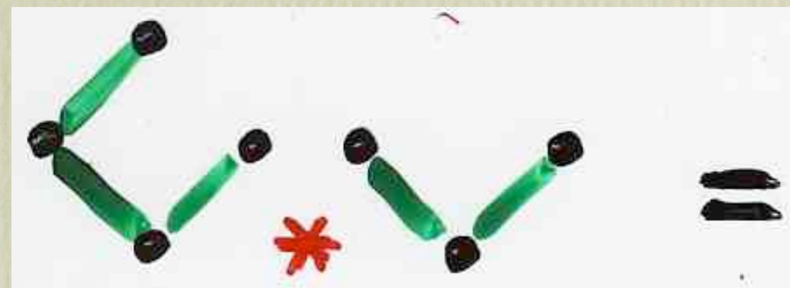
explanation with the mathematics of trees

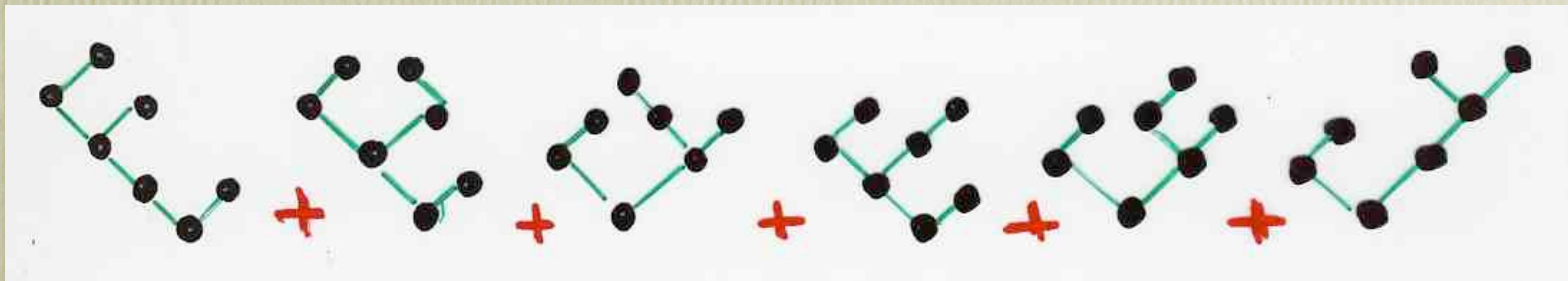
Alain Connes

today,
apparition of «figures»,
but on another level



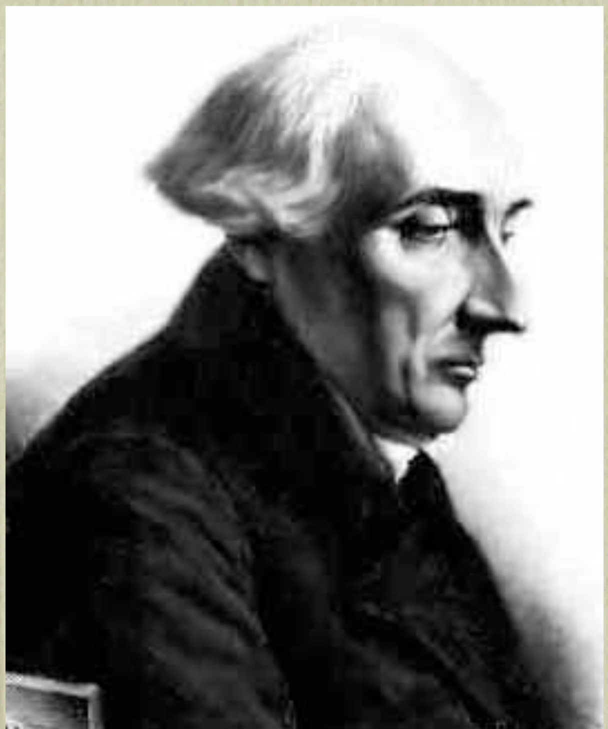
product of two binary trees





Euclide mathematics, many figures until Newton
after, elimination of figures

Lagrange, treatise on mechanics: not a single figure
equations, identities, pure abstraction



Joseph-Louis Lagrange
1736 - 1813

AVERTISSEMENT

DE LA DEUXIÈME ÉDITION.

On a déjà plusieurs Traités de Mécanique, mais le plan de celui-ci est entièrement neuf. Je me suis proposé de réduire la théorie de cette Science, et l'art de résoudre les problèmes qui s'y rapportent, à des formules générales, dont le simple développement donne toutes les équations nécessaires pour la solution de chaque problème.

Cet Ouvrage aura d'ailleurs une autre utilité : il réunira et présentera sous un même point de vue les différents principes trouvés jusqu'ici pour faciliter la solution des questions de Mécanique, en montrera la liaison et la dépendance mutuelle, et mettra à portée de juger de leur justesse et de leur étendue.

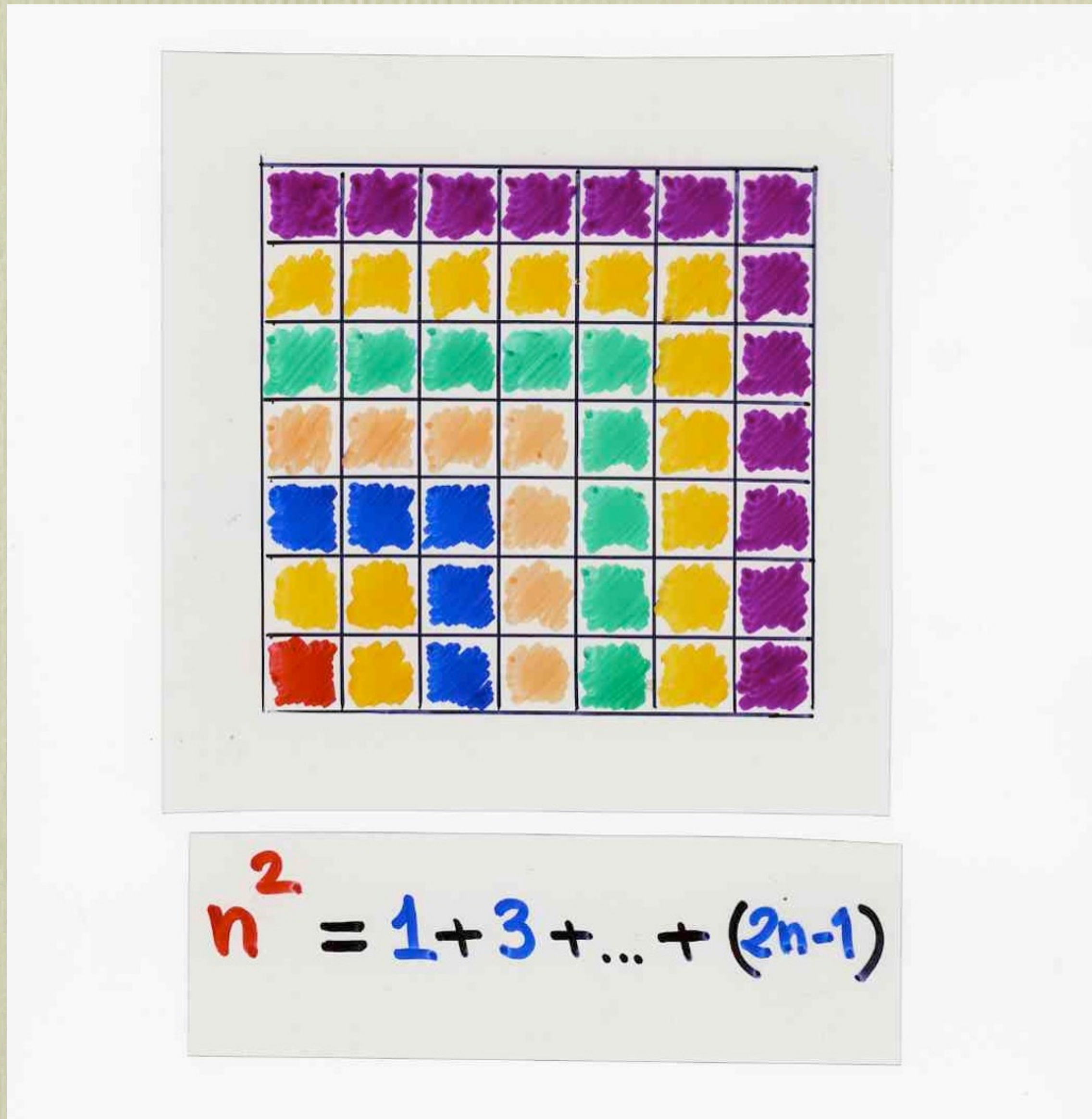
Je le divise en deux Parties : la Statique ou la Théorie de l'Équilibre, et la Dynamique ou la Théorie du Mouvement; et, dans chacune de ces Parties, je traite séparément des corps solides et des fluides.

On ne trouvera point de Figures dans cet Ouvrage. Les méthodes que j'y expose ne demandent ni constructions, ni raisonnements géométriques ou mécaniques, mais seulement des opérations algébriques, assujetties à une marche régulière et uniforme. Ceux qui aiment l'Analyse verront avec plaisir la Mécanique en devenir une nouvelle branche, et me sauront gré d'en avoir étendu ainsi le domaine.

proofs with «figures»

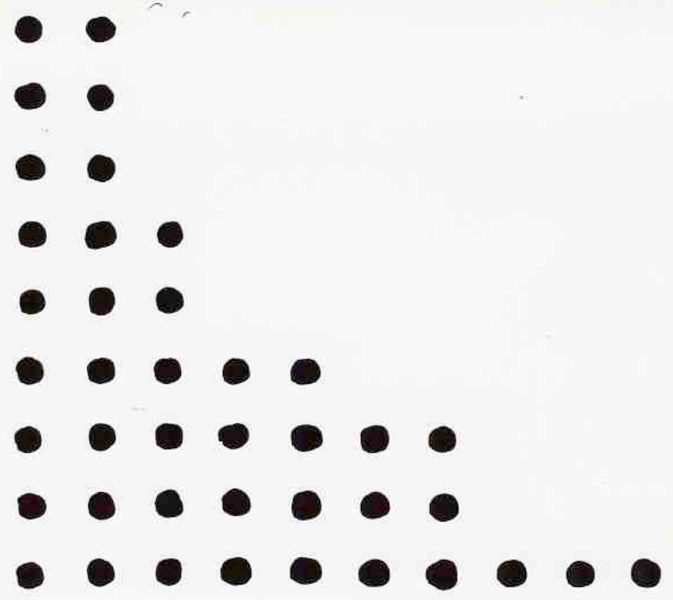
Combinatorial
proofs

«combinatorial proof» of some identities
with bijections, correspondences
combinatorial interpretations



$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1} \frac{1}{(1-q^i)}$$

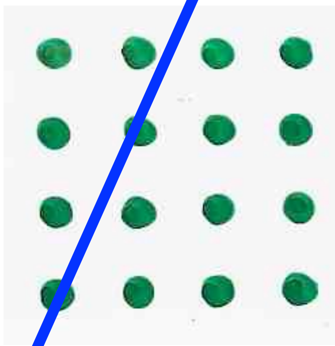
$$\sum_{m \geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i \geq 1} \frac{1}{(1-q^i)}$$



$$= \prod_{i \geq 1} \frac{1}{(1-q^i)}$$

$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1} \frac{1}{(1-q^i)}$$

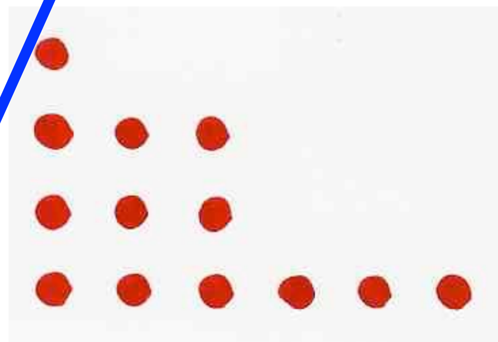
$$q^{m^2}$$



square

m X m

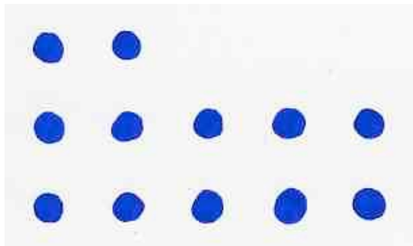
$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$



at most

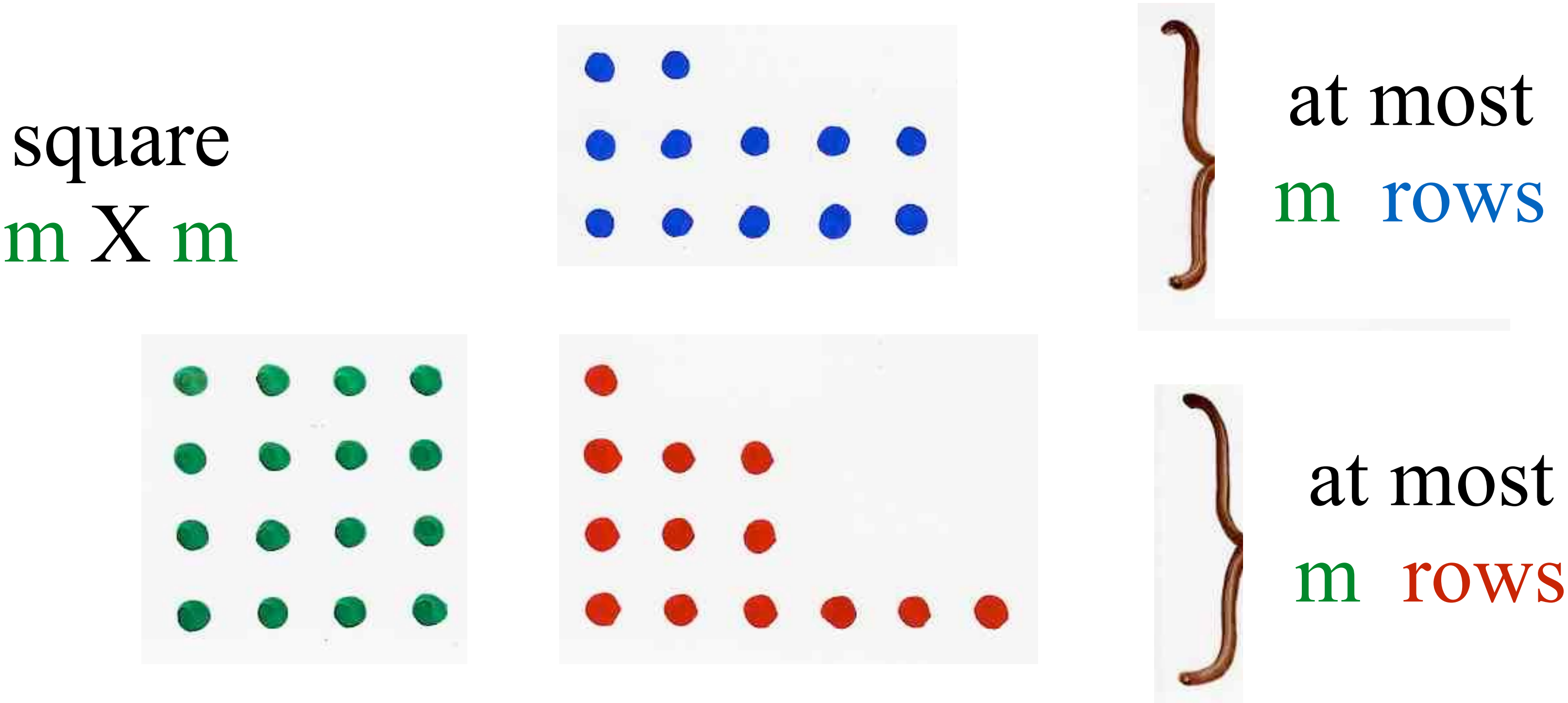
m rows

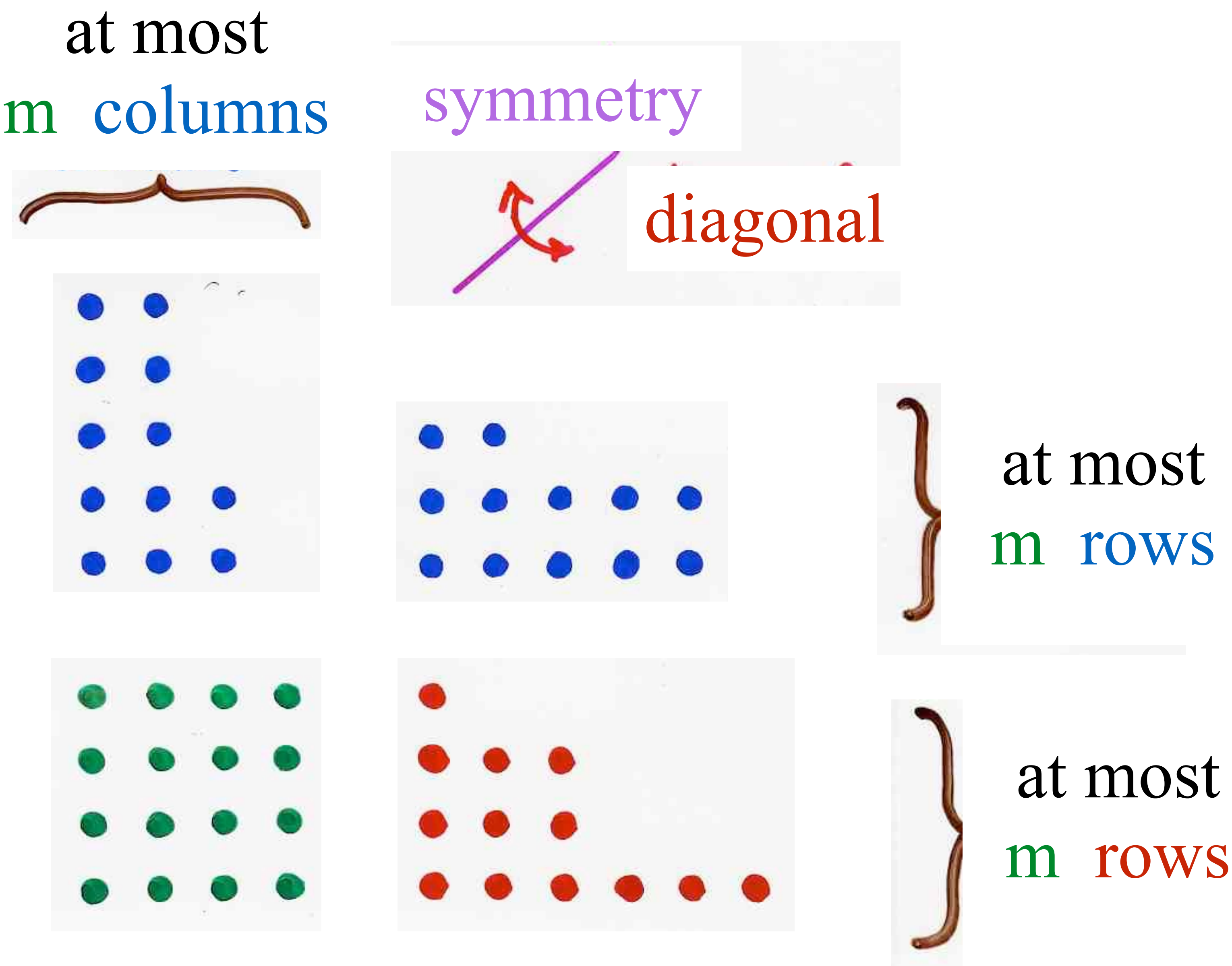
$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$



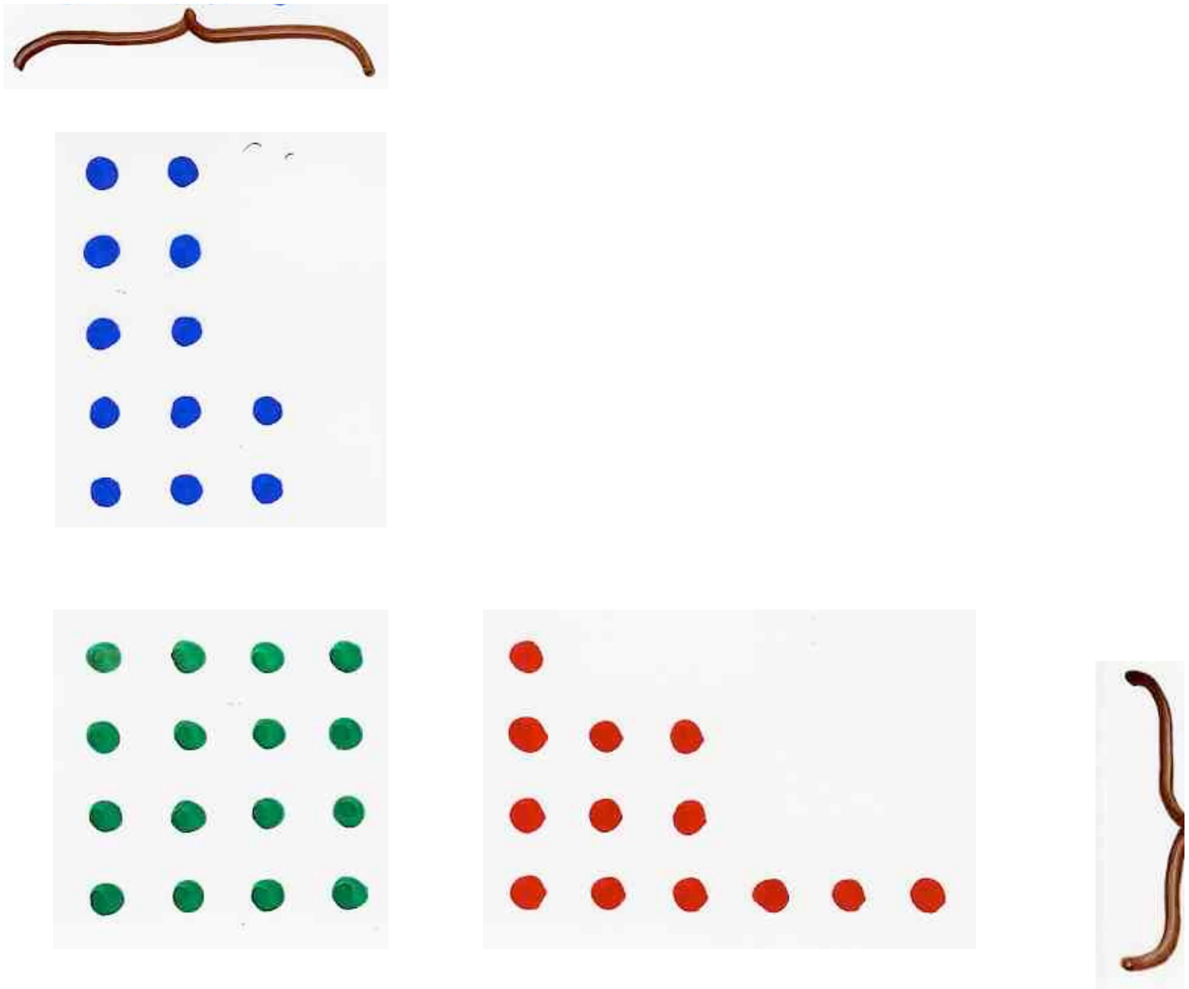
at most

m rows

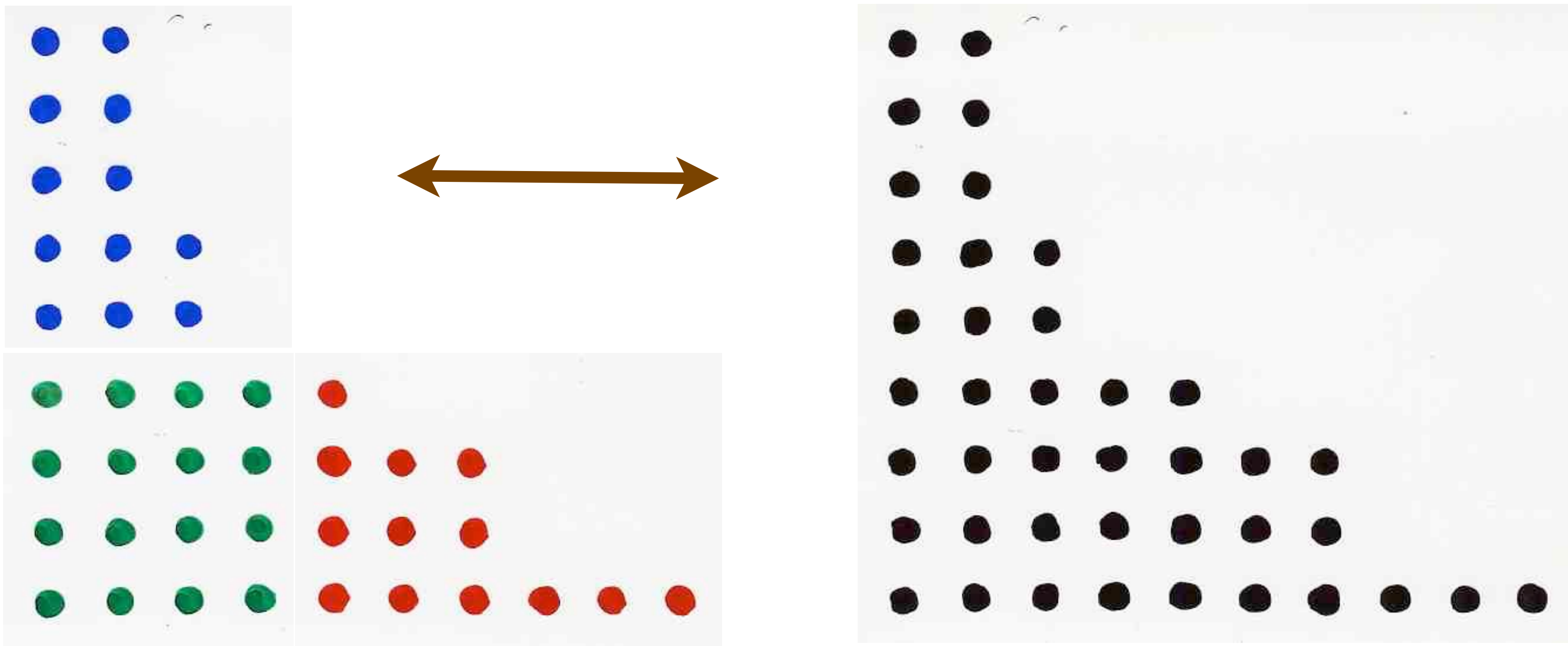

square
m X m
at most
m rows
at most
m rows


at most
m columns
symmetry
diagonal
at most
m rows
at most
m rows

at most
m columns

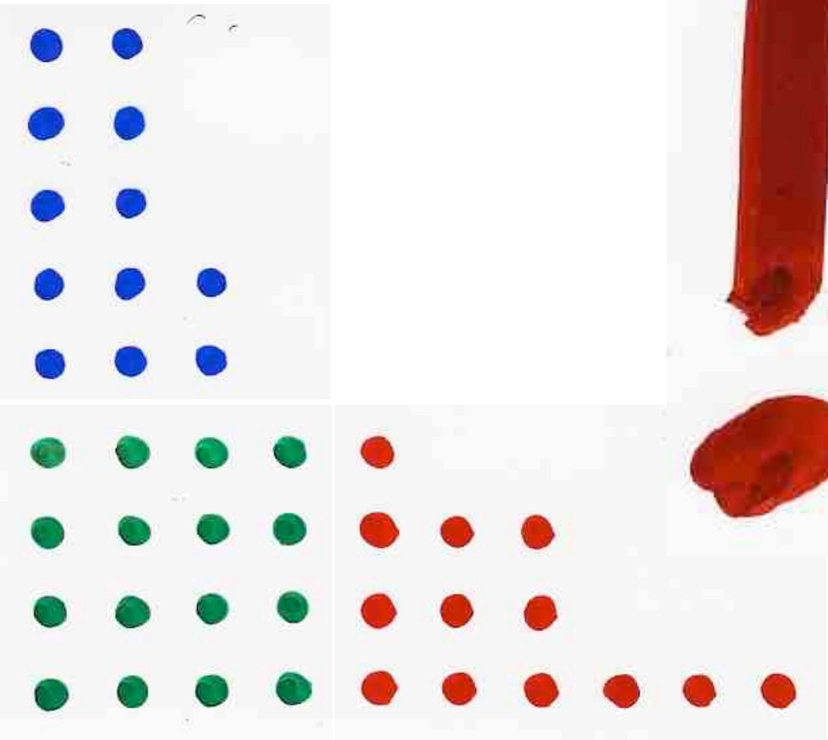


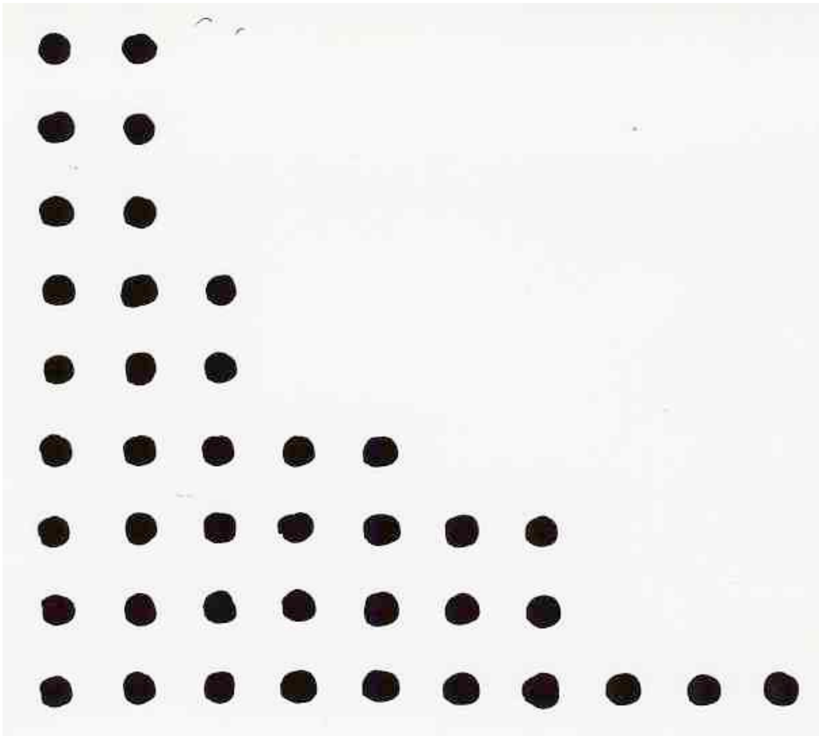
at most
m rows



$$\sum_{m\geq 1} \frac{q^{m^2}}{[(1-q)(1-q^2)\cdots(1-q^m)]^2} = \prod_{i\geq 1}\frac{1}{(1-q^i)}$$

!

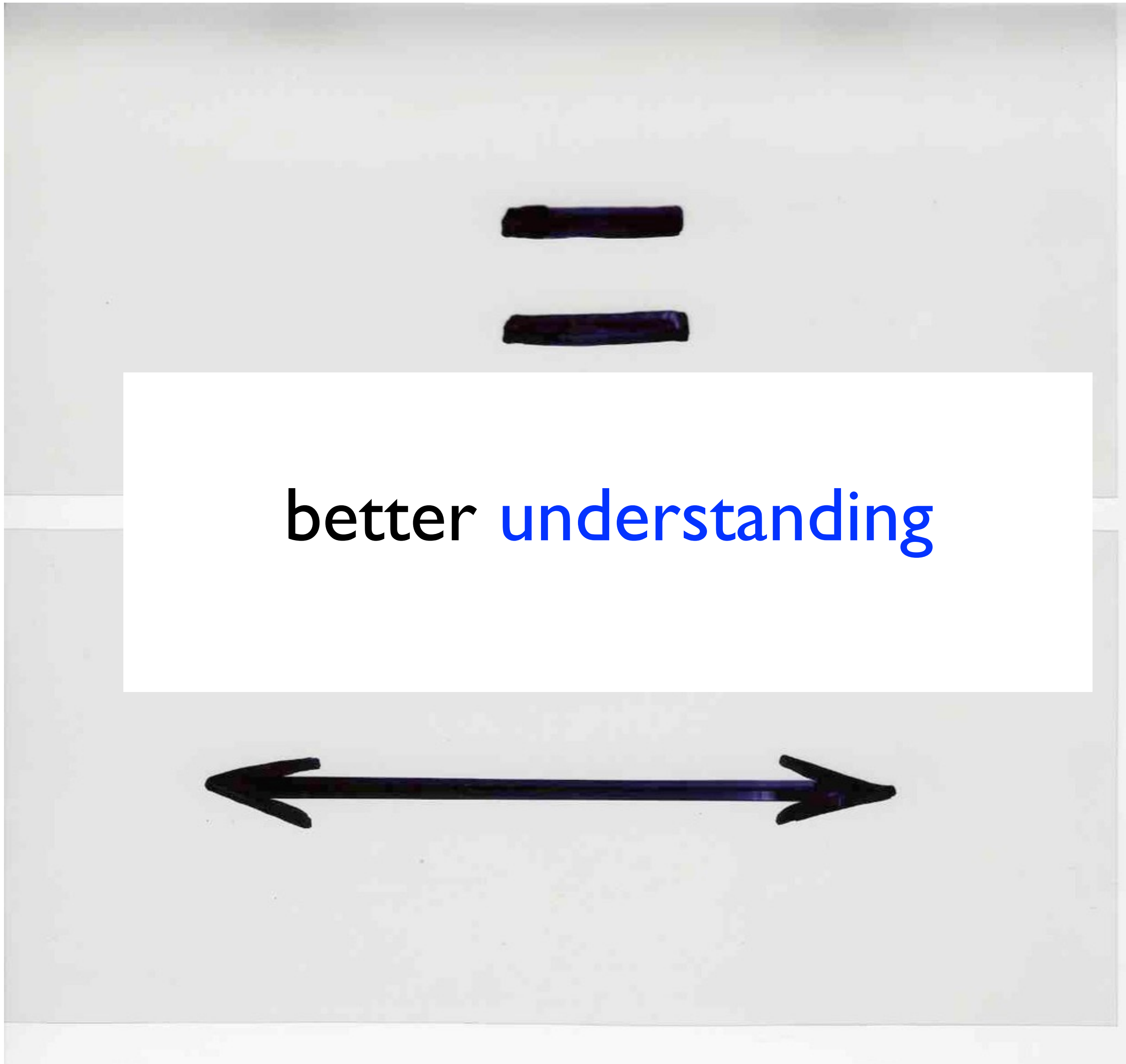




drawing calculus

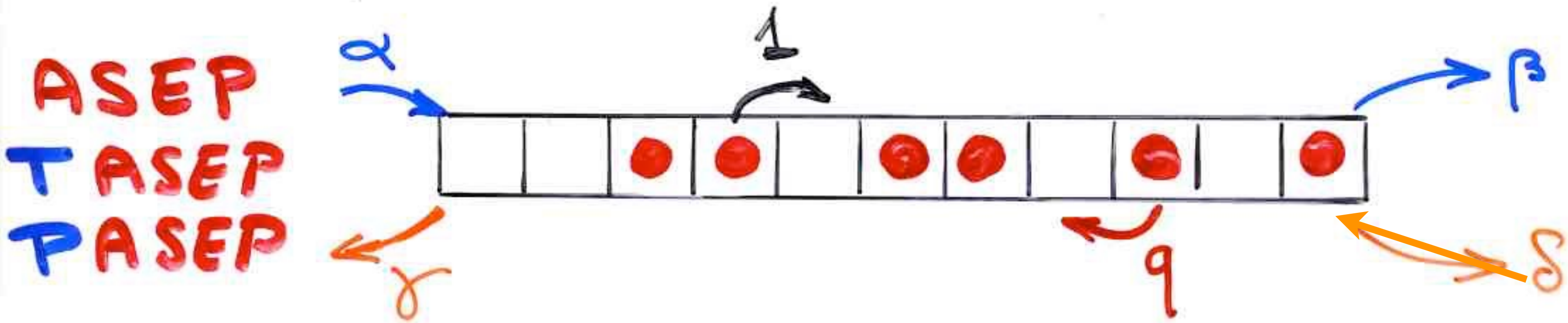
...

computing drawings


better understanding

The PASEP
(ASEP)

(Partially) ASymmetric Exclusion Process


ASEP
TASEP
PASEP
α
1
β
γ
q
δ

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\, \hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\, \hat{a}^+$$

$$\hat{a}\,\hat{a}^+ - q\,\hat{a}^+\hat{a} = 1$$

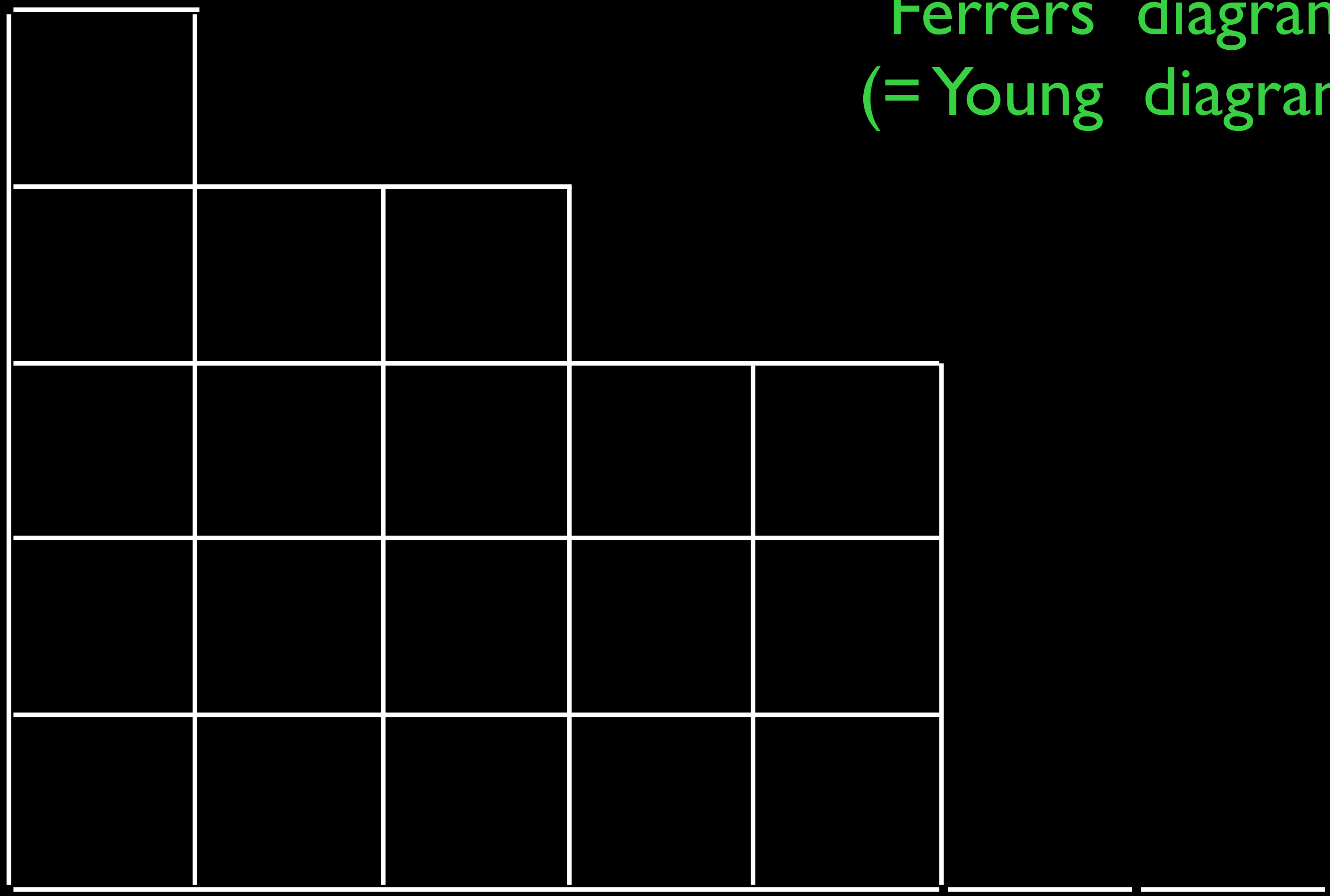
→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

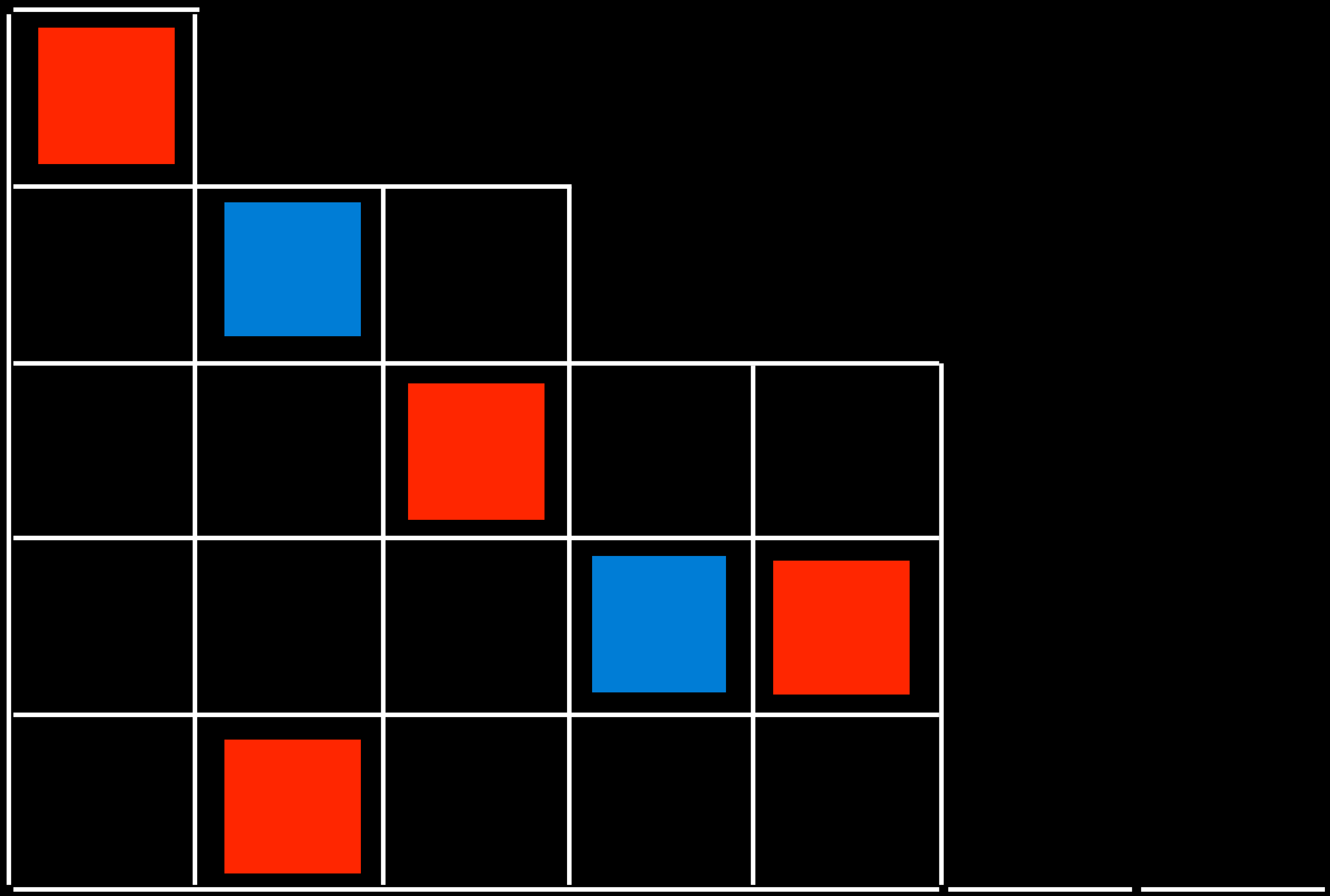
Askey-Wilson polynomials

alternative tableau

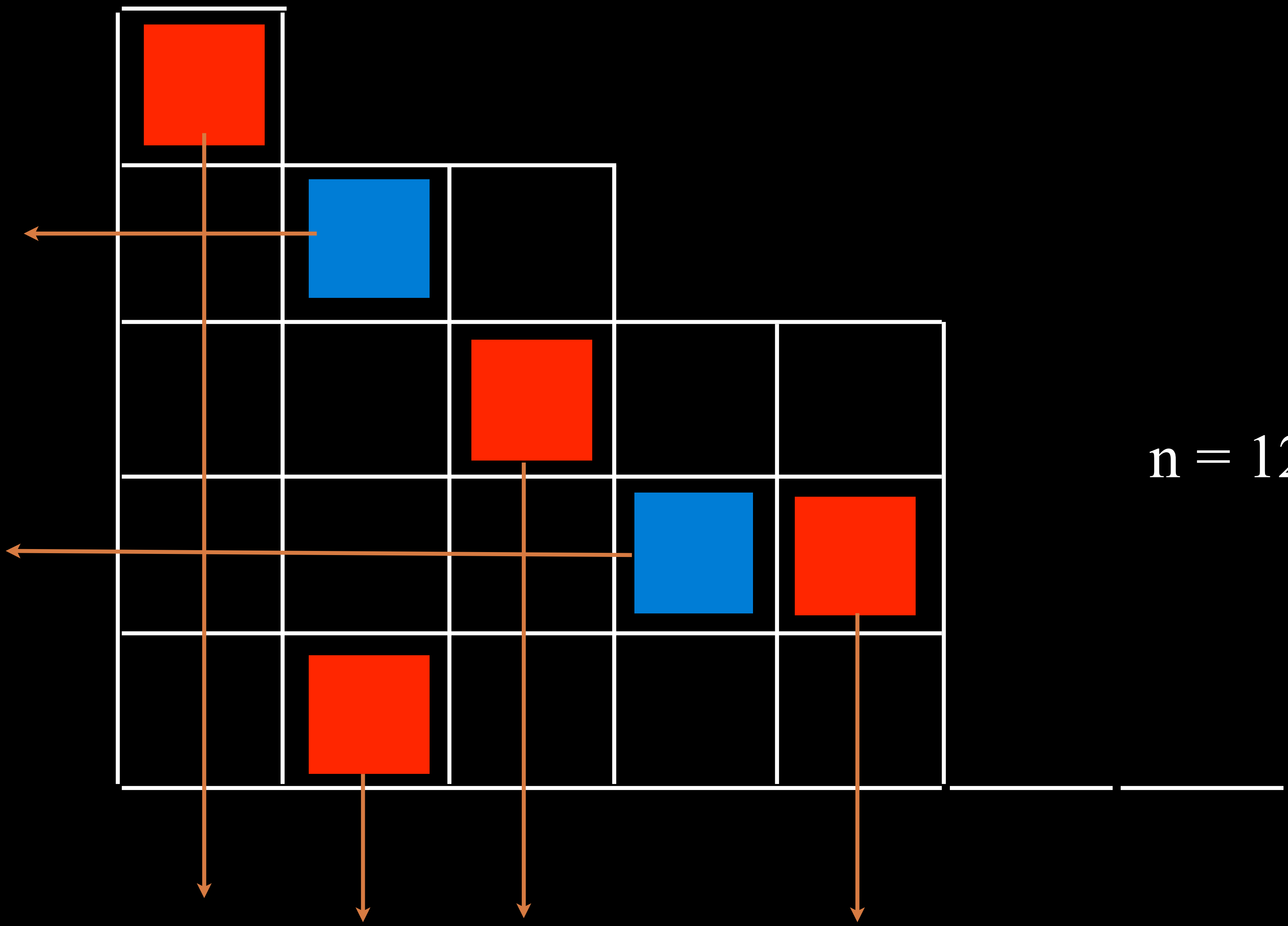
Ferrers diagram
(= Young diagram)



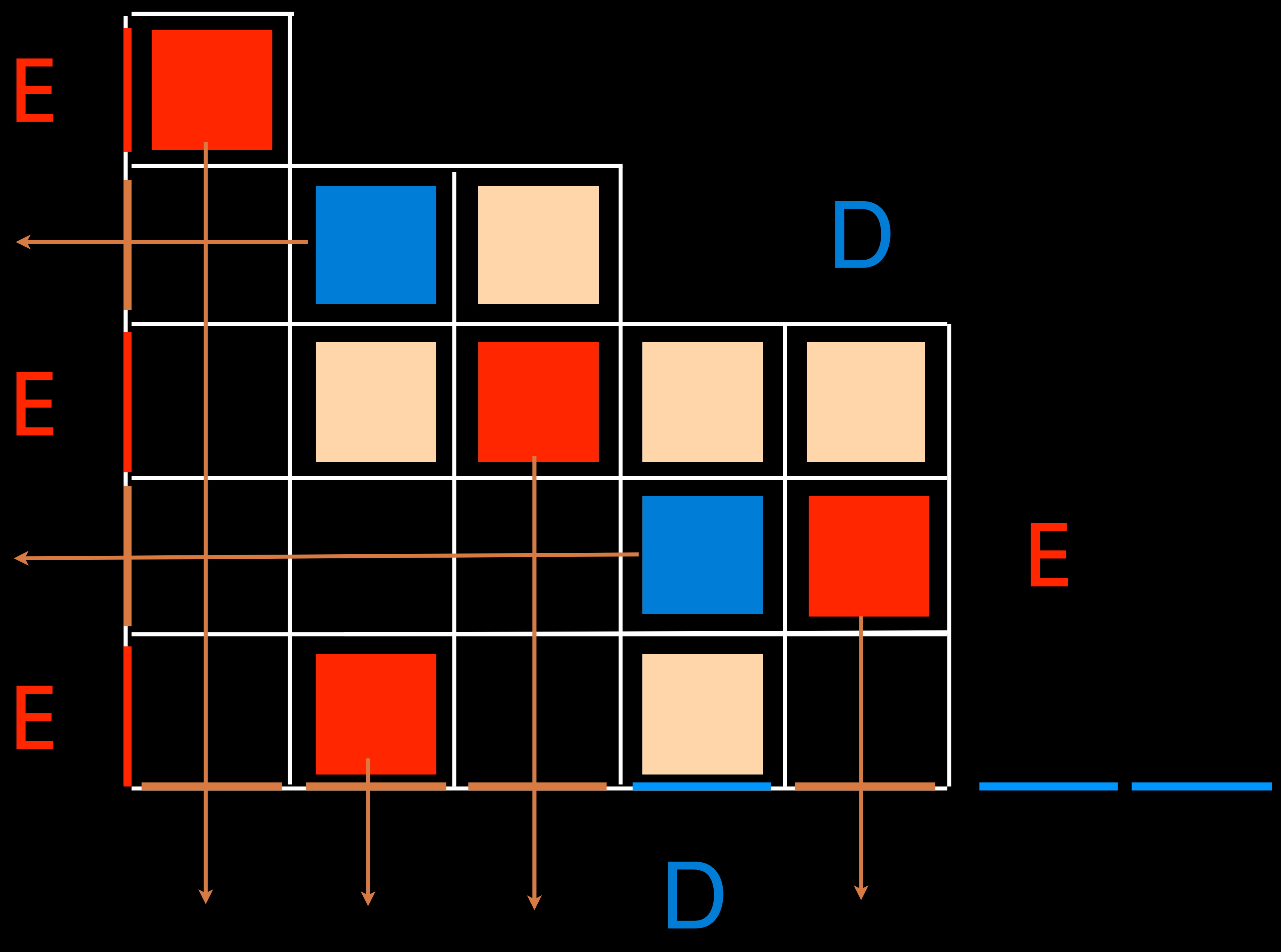
alternative tableau



alternative tableau



$n = 12$


E
D
E
E
E
D

Cor. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ (PASEP) is

$$\text{prob}_\tau(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{T} q^{L(T)} \alpha^{-f(T)} \beta^{-u(T)}$$

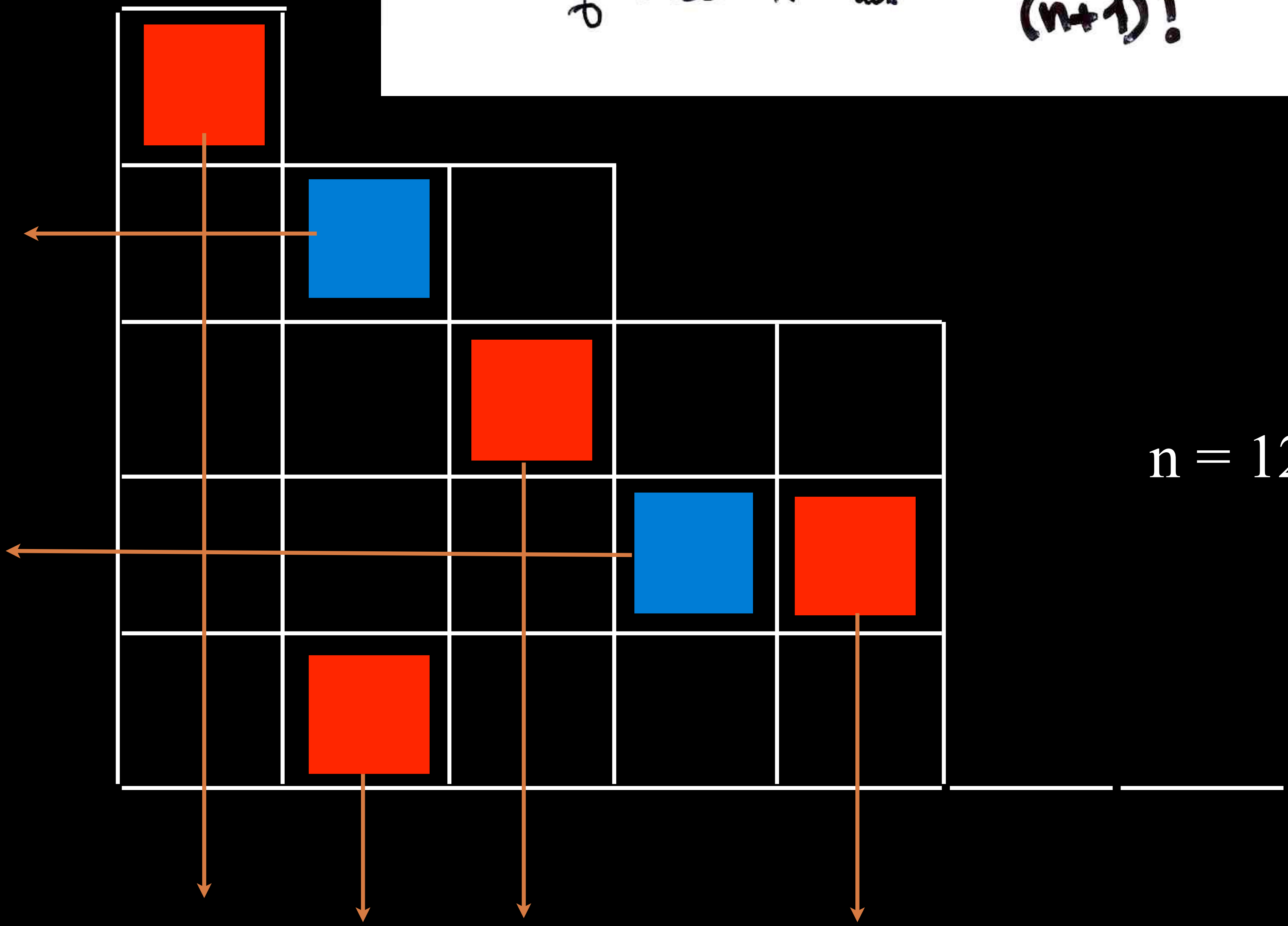
T: alternative tableaux, profile τ

$f(T)$ nb of rows without cell
$u(T)$ nb of columns without cell
$L(T)$ nb of cells

permutation tableau

S. Corteel, L. Williams
(2007) (2008) (2009)

Prop. The number of alternative tableaux of size n is $(n+1)!$



$n = 12$

the generating function of alternating tableaux
with variables (q,α,β)
are the moments of some q-Laguerre polynomials

Orthogonal polynomials

Def. $\{P_n(x)\}_{n\geq 0}$ $\quad P_n(x) \in \mathbb{K}[x]$

orthogonal iff $\exists\ f : \mathbb{K}[x] \longrightarrow \mathbb{K}$

linear functional

$$\begin{cases} \text{(i)} & \deg(P_n(x)) = n \quad (\forall n \geq 0) \\ \text{(ii)} & f(P_k P_l) = 0 \quad \text{for } k \neq l \geq 0 \\ \text{(iii)} & f(P_k^2) \neq 0 \quad \text{for } k \geq 0 \end{cases}$$

$$\mathcal{L}(x^n) = \mu_n \qquad (n \geqslant 0)$$

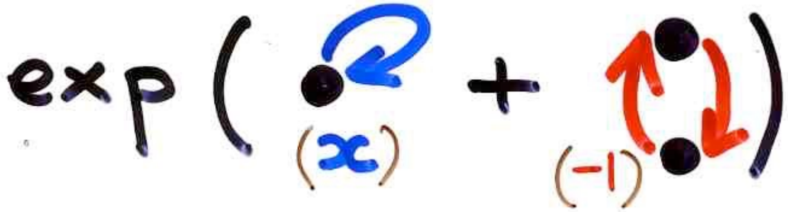
moments

$$\mathcal{L}(PQ) = \int_a^b P(x)\,Q(x)\,d\mu$$

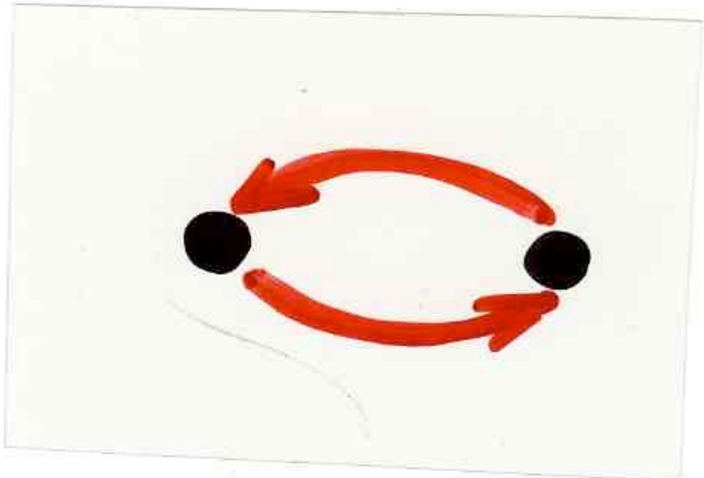
mesure

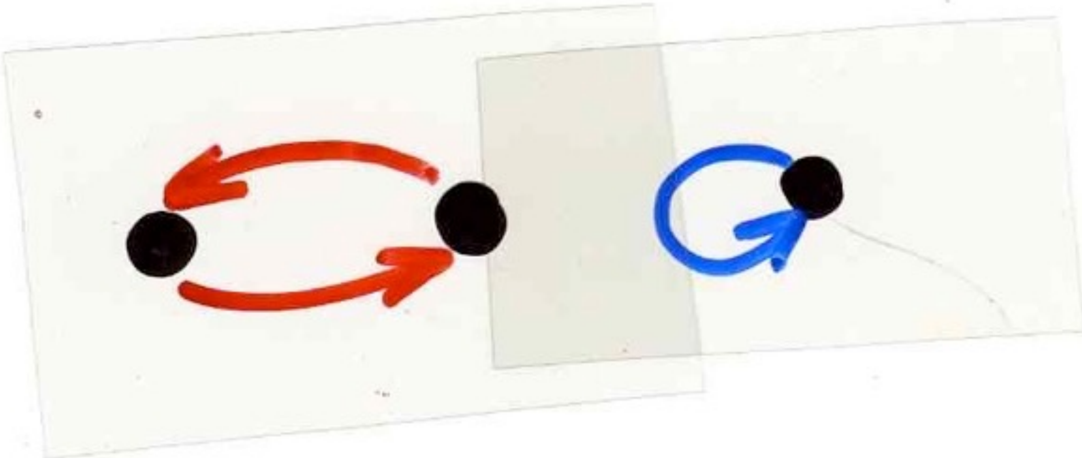
Combinatorial theory of orthogonal polynomials

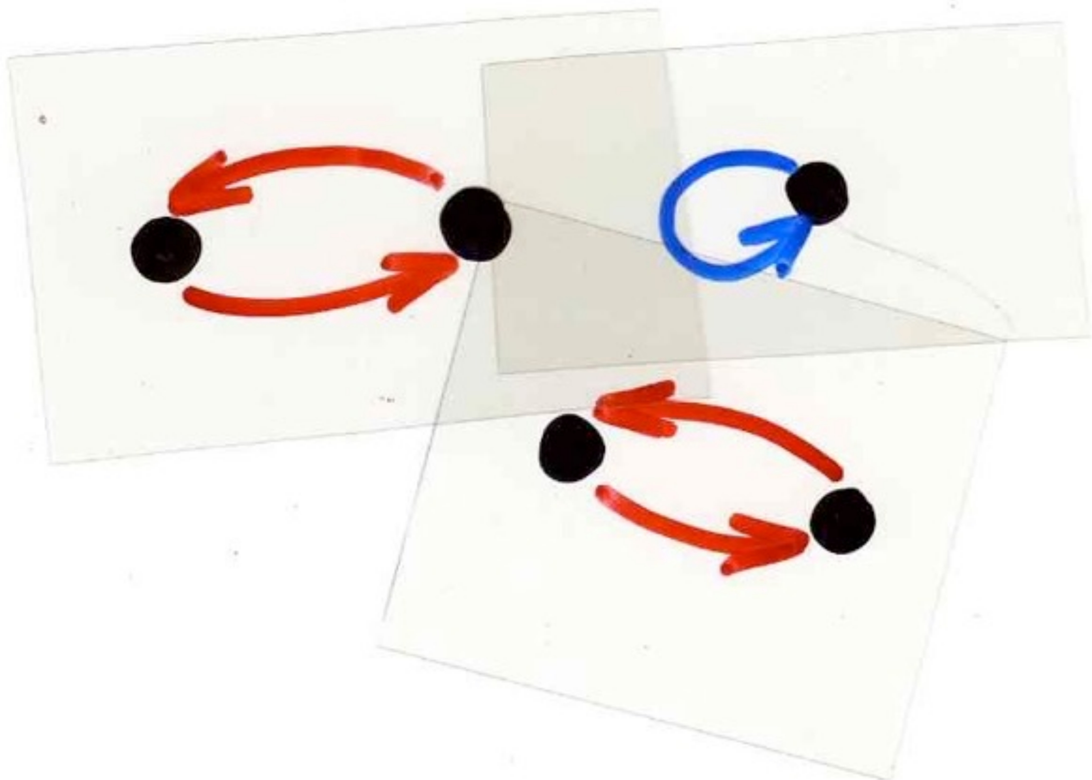
combinatorial interpretations of the polynomials
combinatorial interpretations of the moments

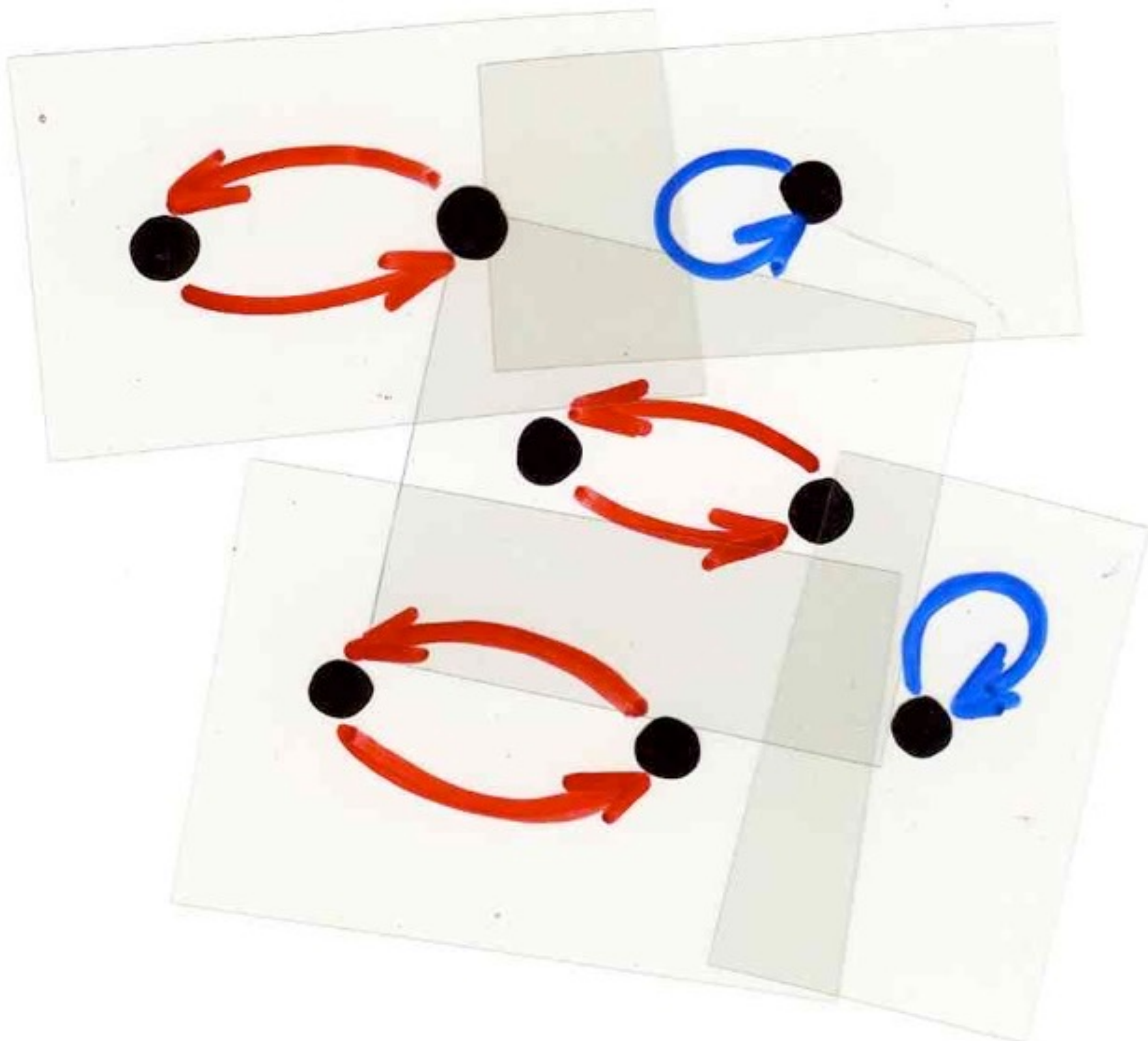


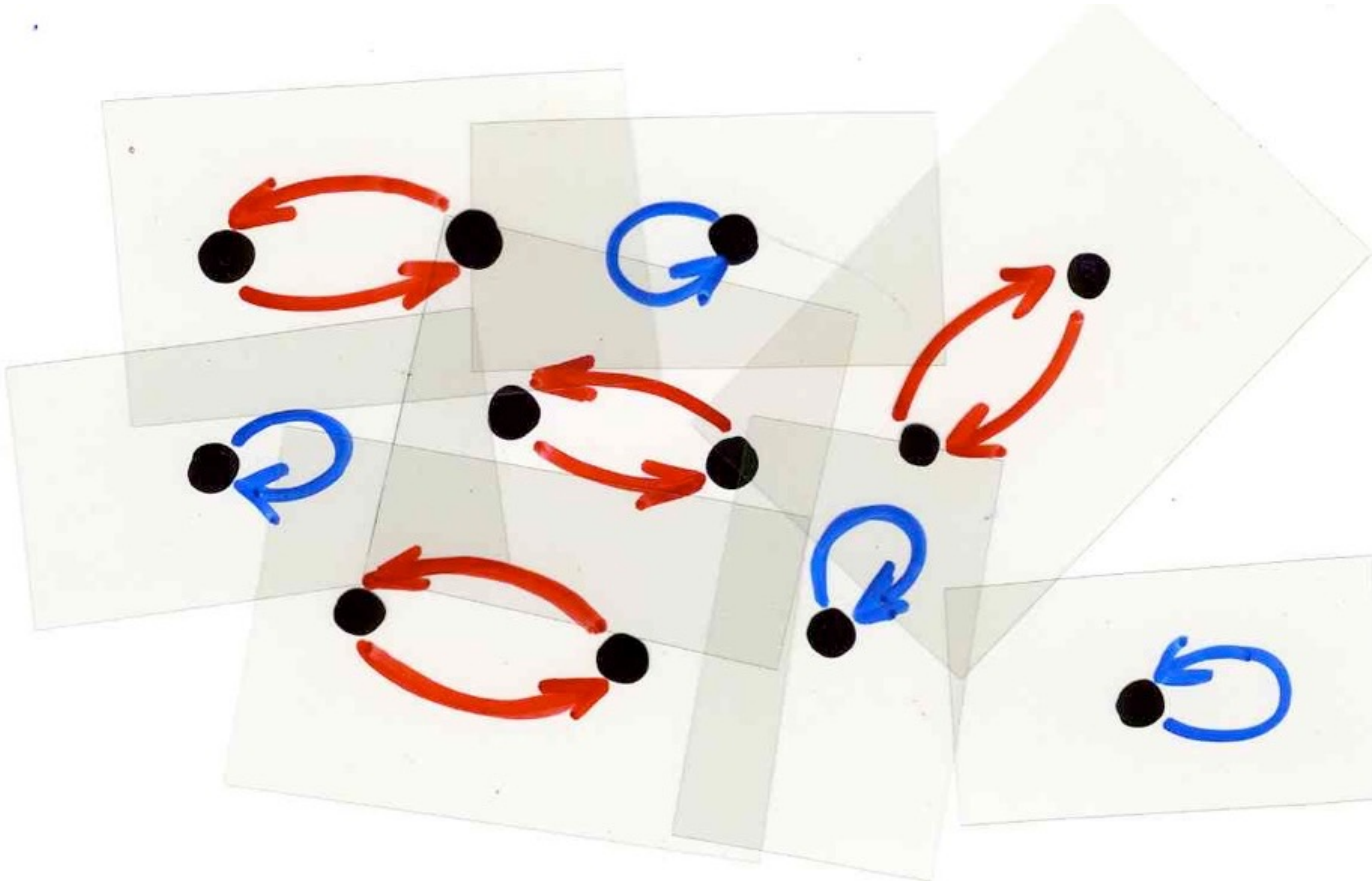
$$\sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = \exp\left(xt - \frac{t^2}{2}\right)$$

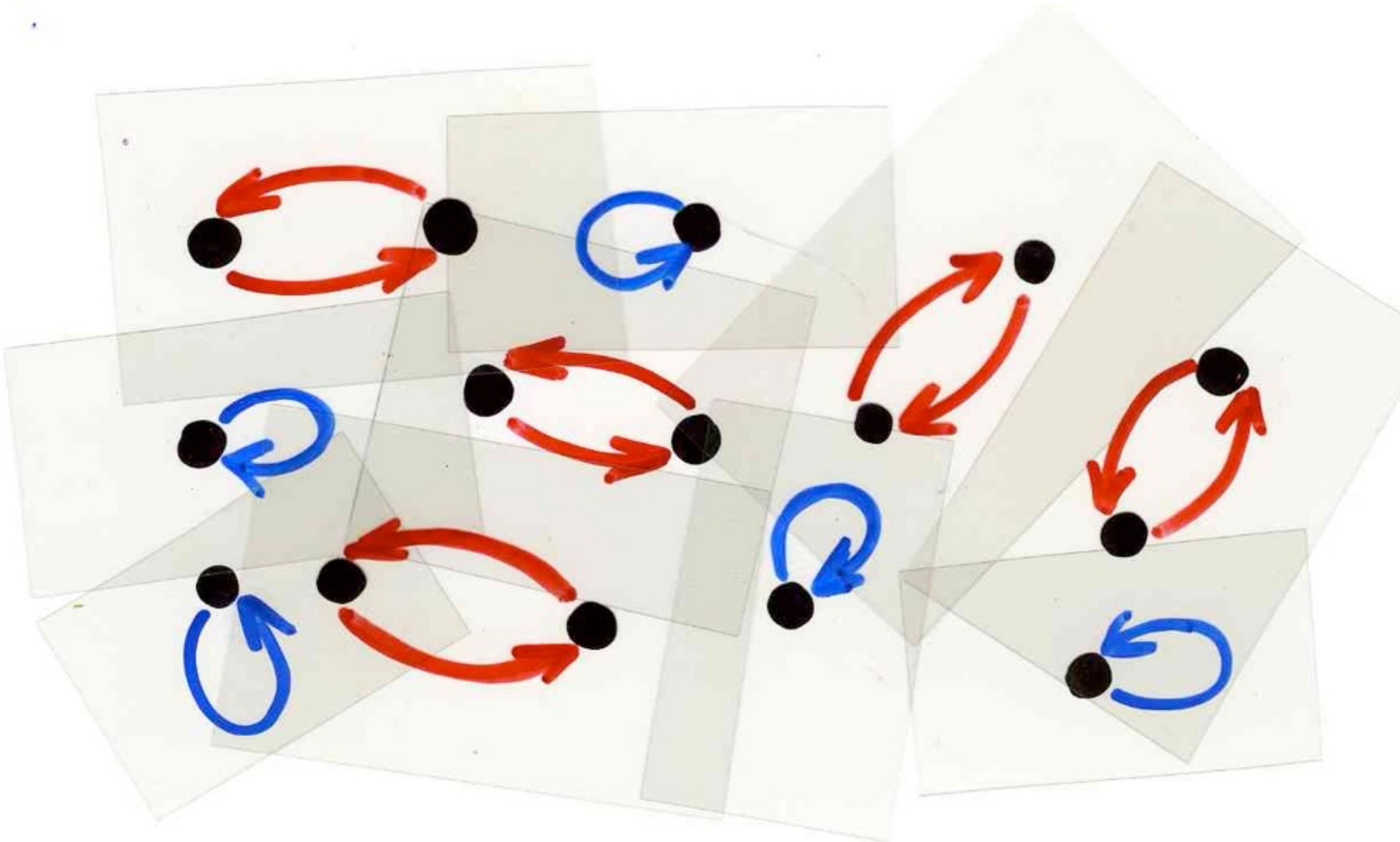




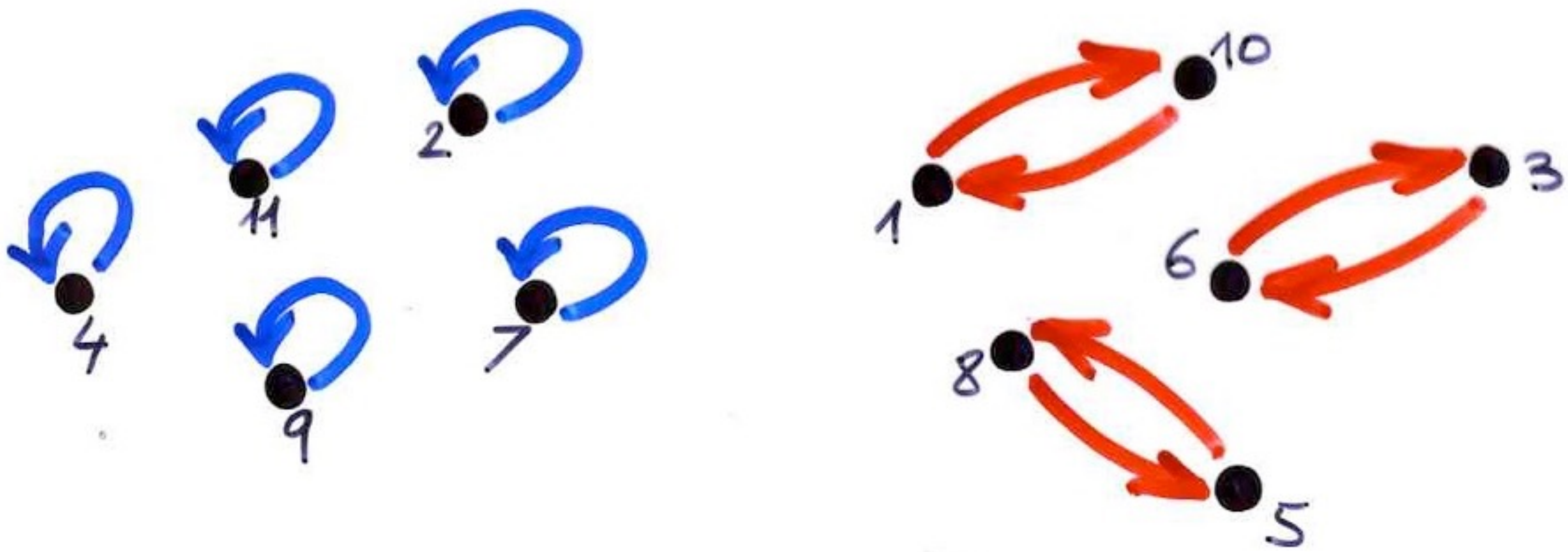




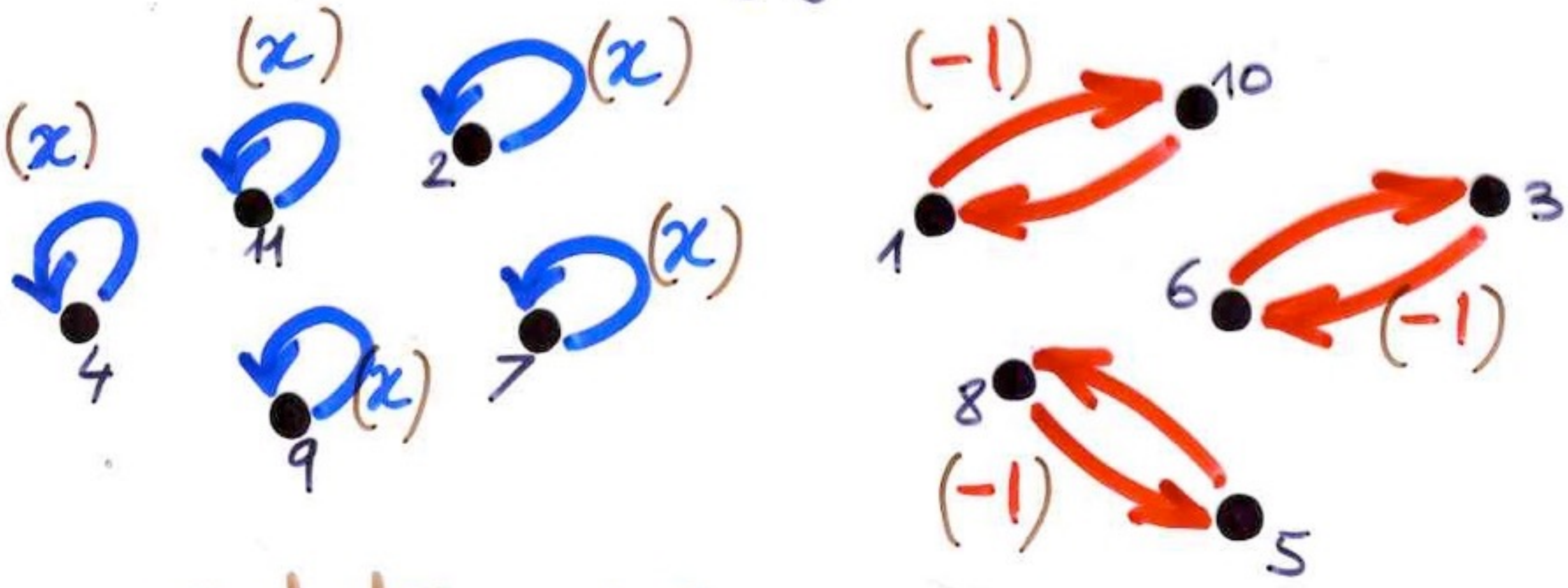




Hermite configurations



Hermite configurations

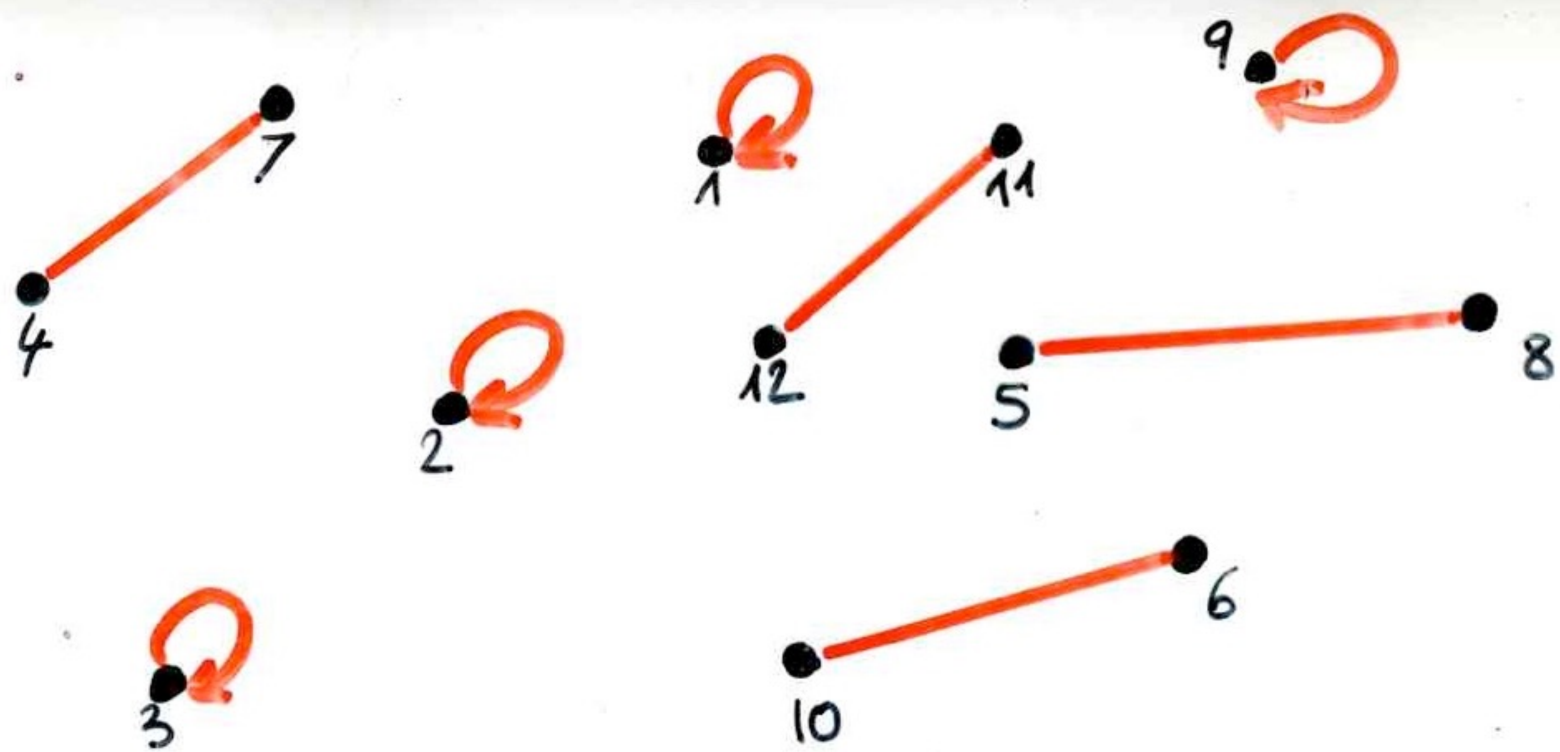


weight (x)

(-1)

$$\sum_{n\geq 0} H_n(x) H_n(y) \frac{t^n}{n!} = (1-4t^2)^{-1/2} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\sum_{n\geq 0} H_n(x) \, H_n(y) \frac{t^n}{n!} =$$



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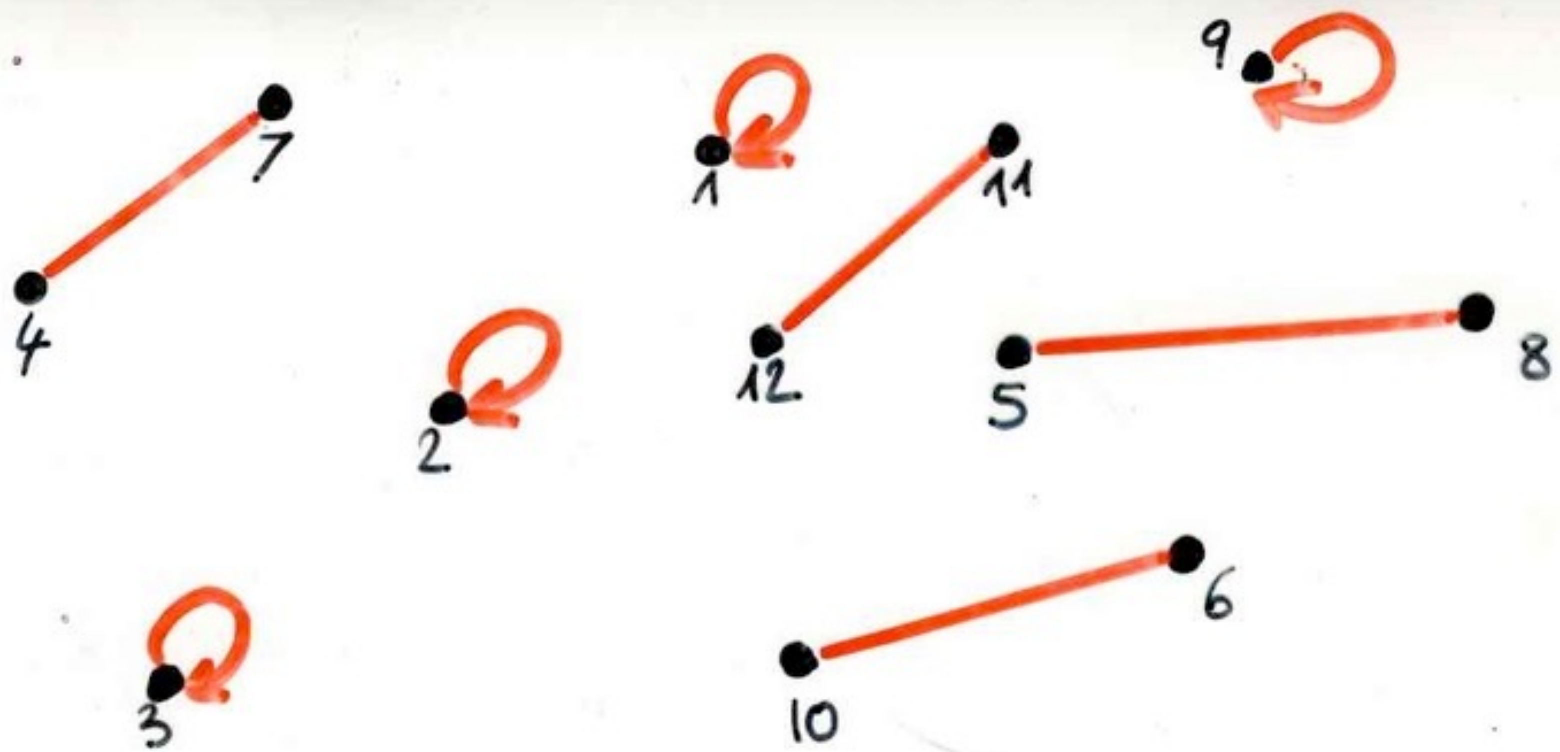
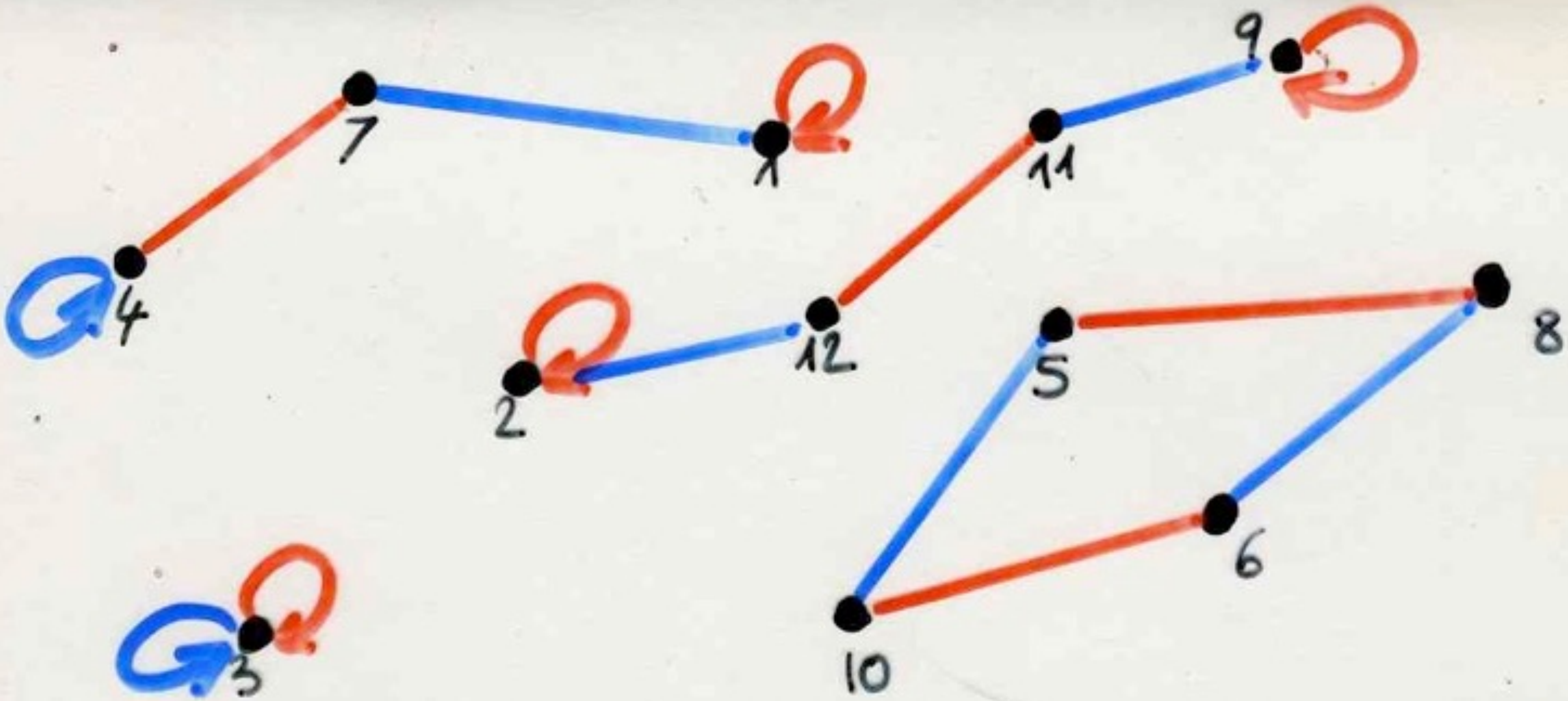


$$\sum_{n\geq 0} H_n(x) \frac{t^n}{n!} =$$



$$\sum_{n\geq 0} H_n(x)\, H_n(y) \frac{t^n}{n!} =$$

$$(1-4t^2)^{-1/2}\exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\exp\left[\frac{1}{2}\log\frac{1}{(1-4t^2)}+\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\exp\left[\frac{1}{2}\log\frac{1}{(1-4t^2)}+\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\frac{-4\,y^2\,t^2}{1-4t^2}$$

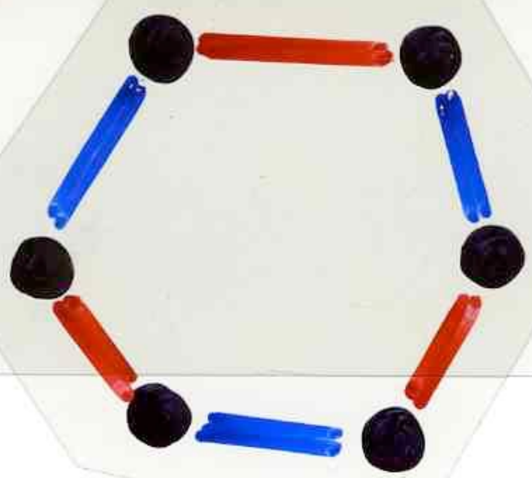
$$\frac{1}{2}\log\frac{1}{(1-4t^2)}\qquad\frac{-4\,x^2\,t^2}{1-4t^2}$$

$$\frac{4xyt}{1-4t^2}$$

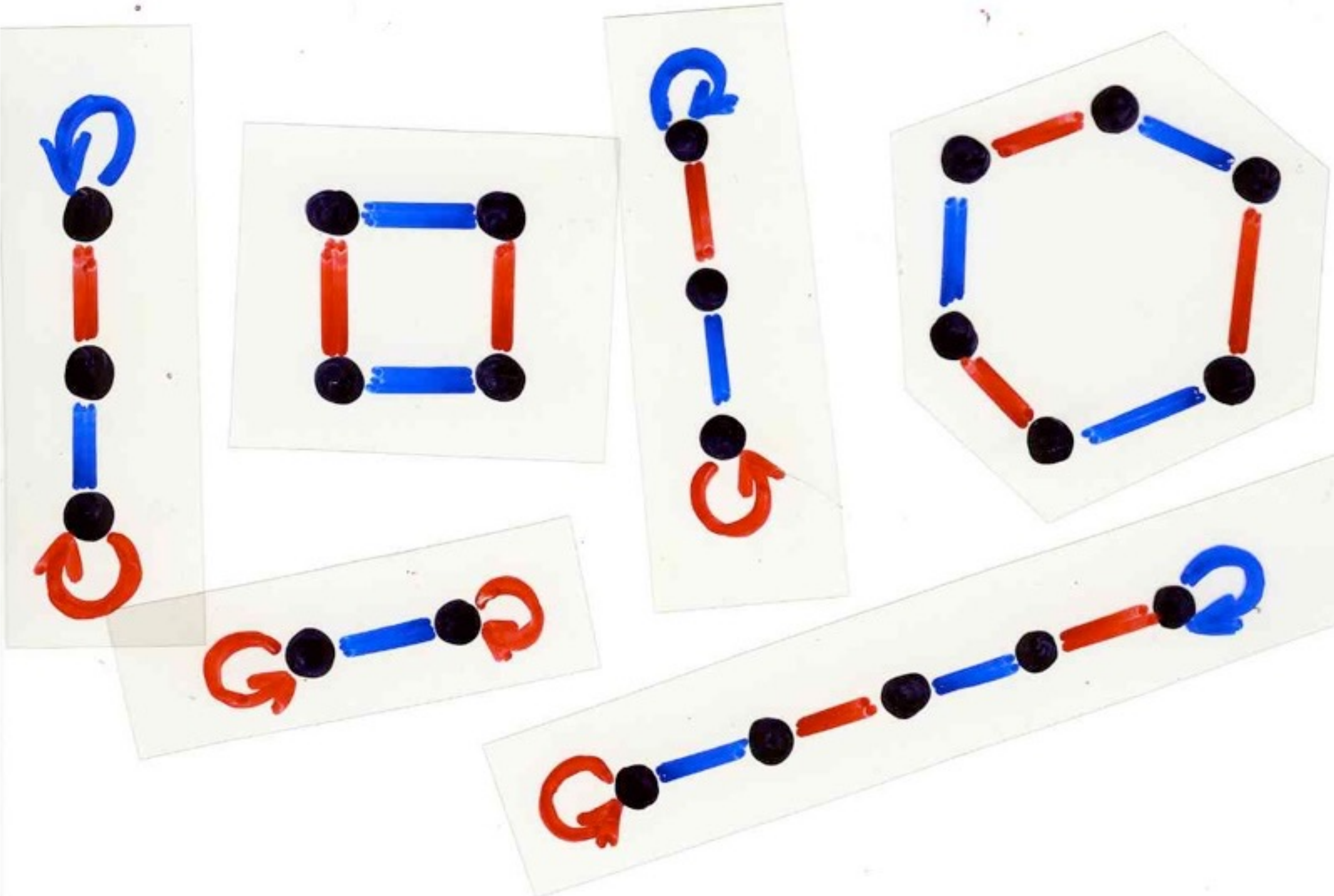
$$\frac{-4x^2\, t^2}{1-4t^2}$$

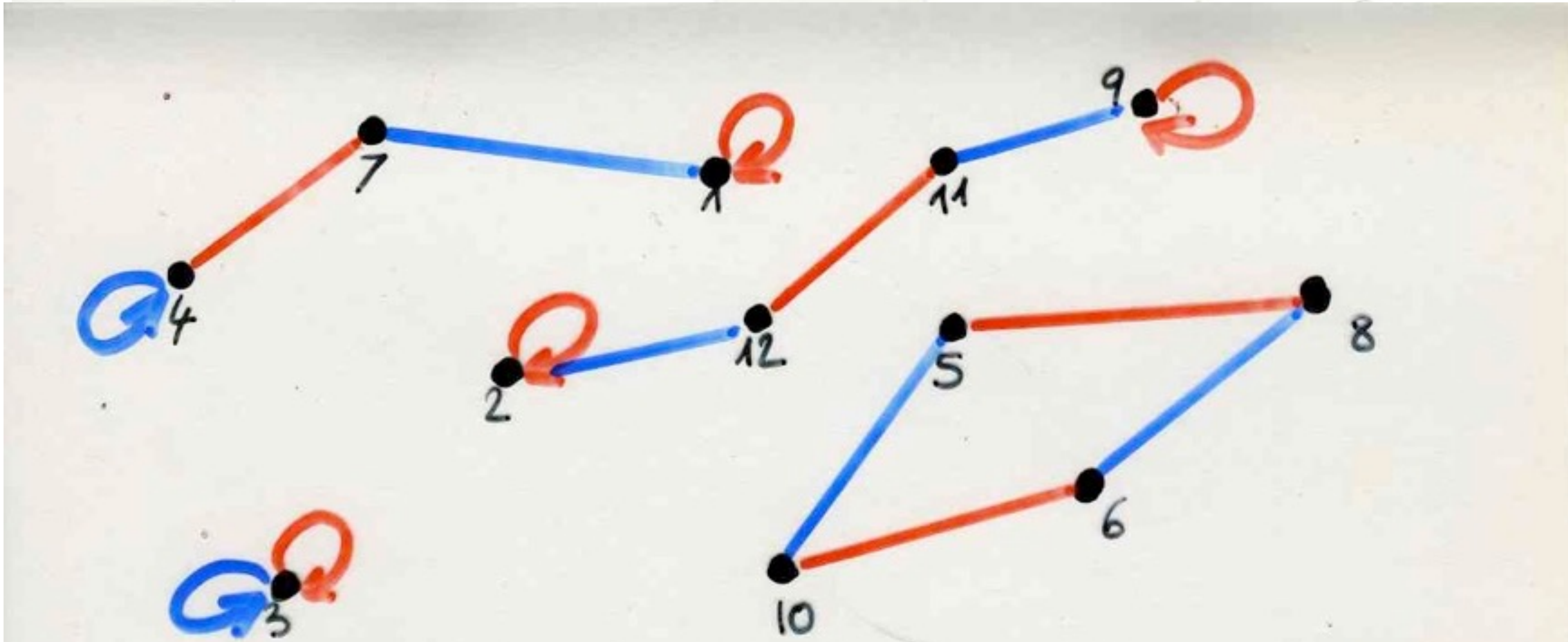
$$\frac{-4\, y^2 t^2}{1-4t^2}$$

$$\frac{4xyt}{1-4t^2}$$



$$\frac{1}{2}\log\frac{1}{(1-4t^2)}$$




7
1
9
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4
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12
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8
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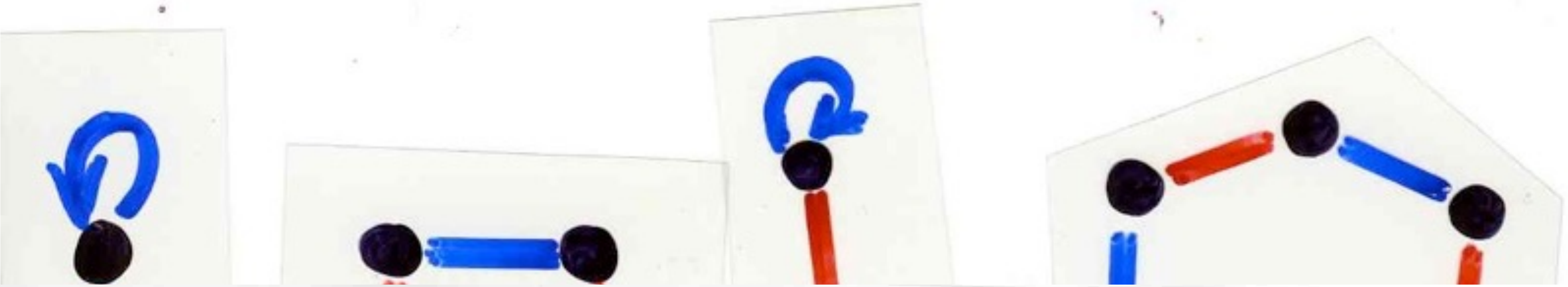
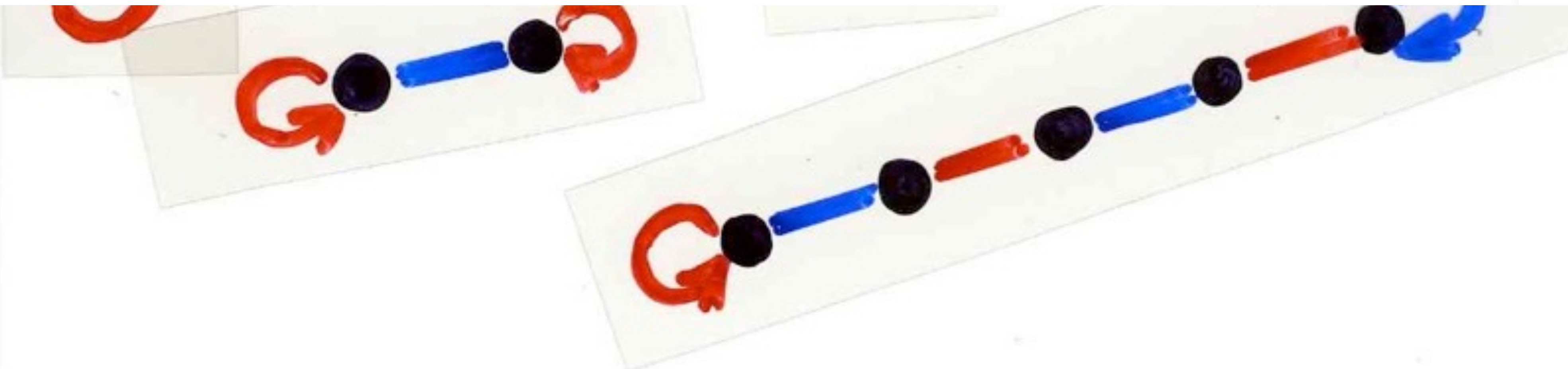
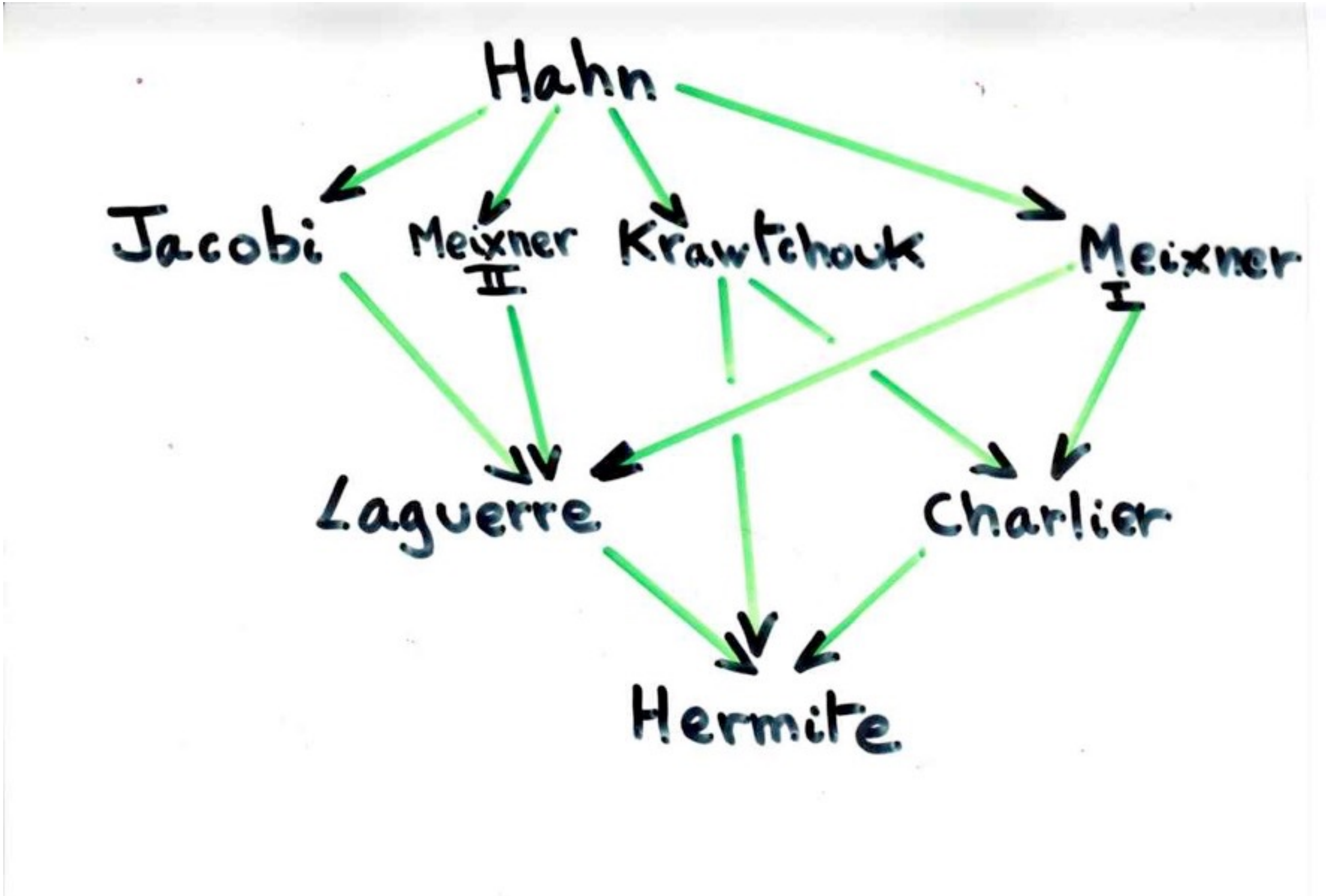


$$\sum_{n\geq 0} H_n(x)\,H(y)\,\frac{t^n}{n!} = (1-4t^2)^{-1/2}\exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

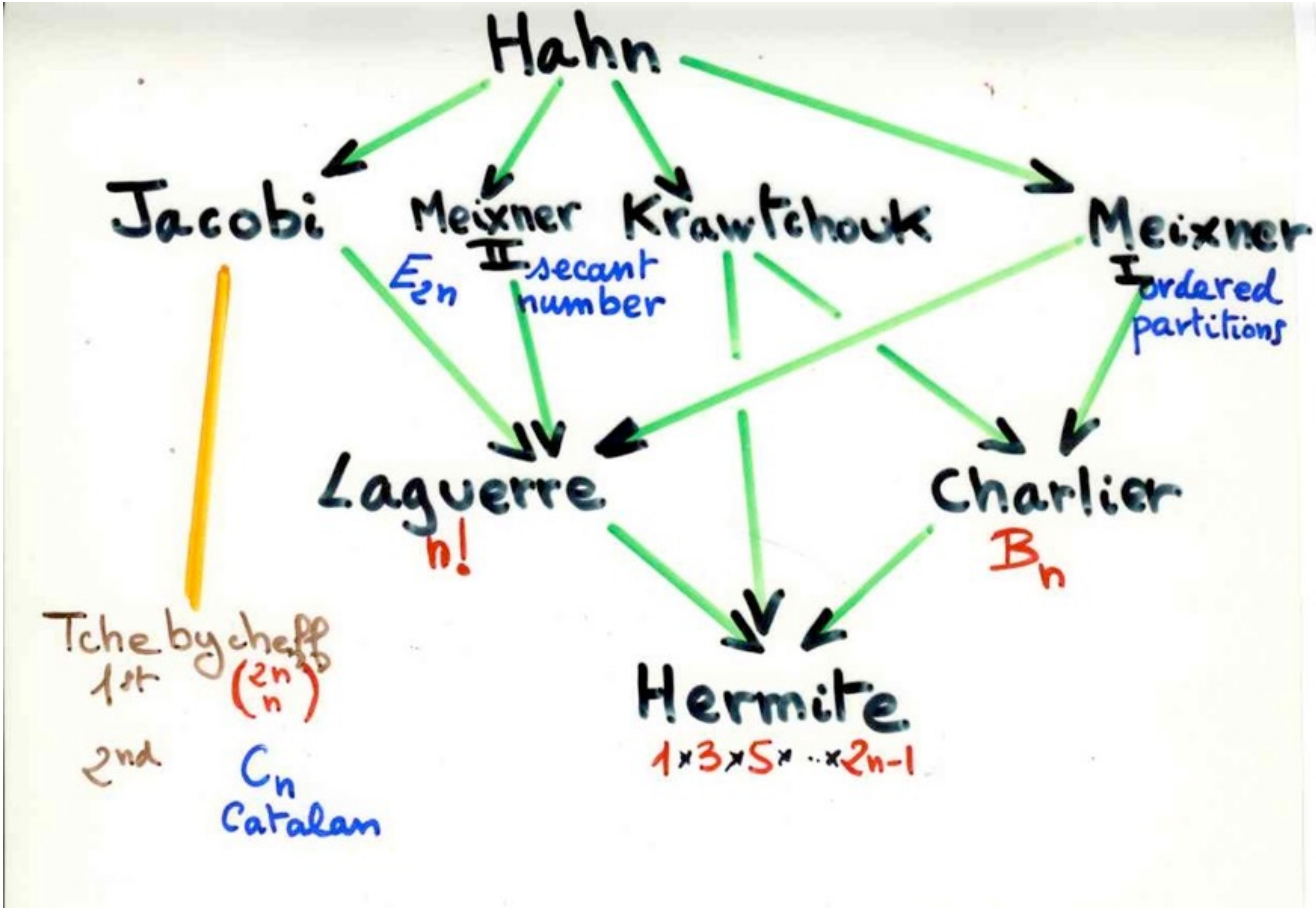


Askey tableau

Askey-Wilson



Askey-Wilson



- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^+$$

$$\hat{a}\,\hat{a}^+ - q\,\hat{a}^+\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta,$ q

Askey-Wilson polynomials

steady state
probability
PASEP

$$\frac{1}{Z_n} Z_\tau(\alpha, \beta, \gamma, \delta ; q)$$

$$Z_n = \sum_\tau Z_\tau$$

$\tau = (\tau_1, \dots, \tau_n)$
state

combinatorial interpretation of the
moments of the Askey-Wilson polynomials
with some « staircase tableaux »

Corteel, Williams, 2009
Corteel, Stanley, Stanton, Williams, 2010

main website www.xavierviennot.org

secondary website: Courses cours.xavierviennot.org

- course IIT Madras « Combinatorics and Physics »

THANK

YOU !

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