Combinatorial operators
and
quadratic algebras

part III: bijection alternative tableaux -- permutations
from a combinatorial representation of
the PASEP algebra DE=qED+E+D

IMSc, Chennai
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combinatorial
representation
of the
operators
E and D

PASEP algebra

$$DE = qED + E + D$$

V vector space generated by B basis

B alternating words two letters {○, ●}

(no occurence of ○○ or ●●)

4 operators A, S, J, K

4 operators A, S, J, K, $u \in B$

$\langle u | A = \sum_{\text{letter } \circ \text{ of } u} v$, v obtained by: $\circ \to \circ\bullet\circ$

$\langle u | S = \sum_{\circ \text{ of } u} v$ v obtained by: $\circ \to \bullet$ (and $\bullet\bullet \to \bullet$ $\bullet\bullet\bullet \to \bullet$)

$\langle u | J = \sum_{\circ \text{ of } u} v$ v, $\circ \to \bullet\circ$ (and $\bullet\bullet \to \bullet$)

$\langle u | K = \sum_{\circ \text{ of } u} v$ v, $\circ \to \circ\bullet$ (and $\bullet\bullet \to \bullet$)

$$\circ\bullet\circ\bullet | S = \bullet\circ\bullet + \circ\bullet$$

Lemma.

$$AS = SA + J + K$$

$$AK = KA + A$$

$$JS = SJ + S$$

$$JK = KJ$$

$$D = A + J$$

$$E = S + K$$

$$D = A + J$$

$$E = S + K$$

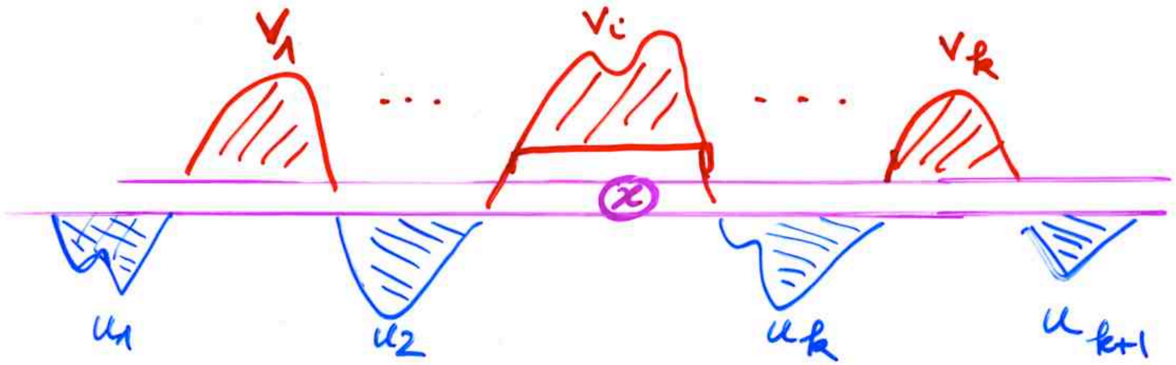
$$DE = (A+J)(S+K)$$

$$= AS + AK + JS + JK$$

$$= \underbrace{(SA + KA + SJ + KJ)}_{\underbrace{(S+K)(A+J)}_{ED}} + \underbrace{J + K + A + S}_{E + D}$$

Bijection
Laguerre histories
permutations

Françon-X.V., 1978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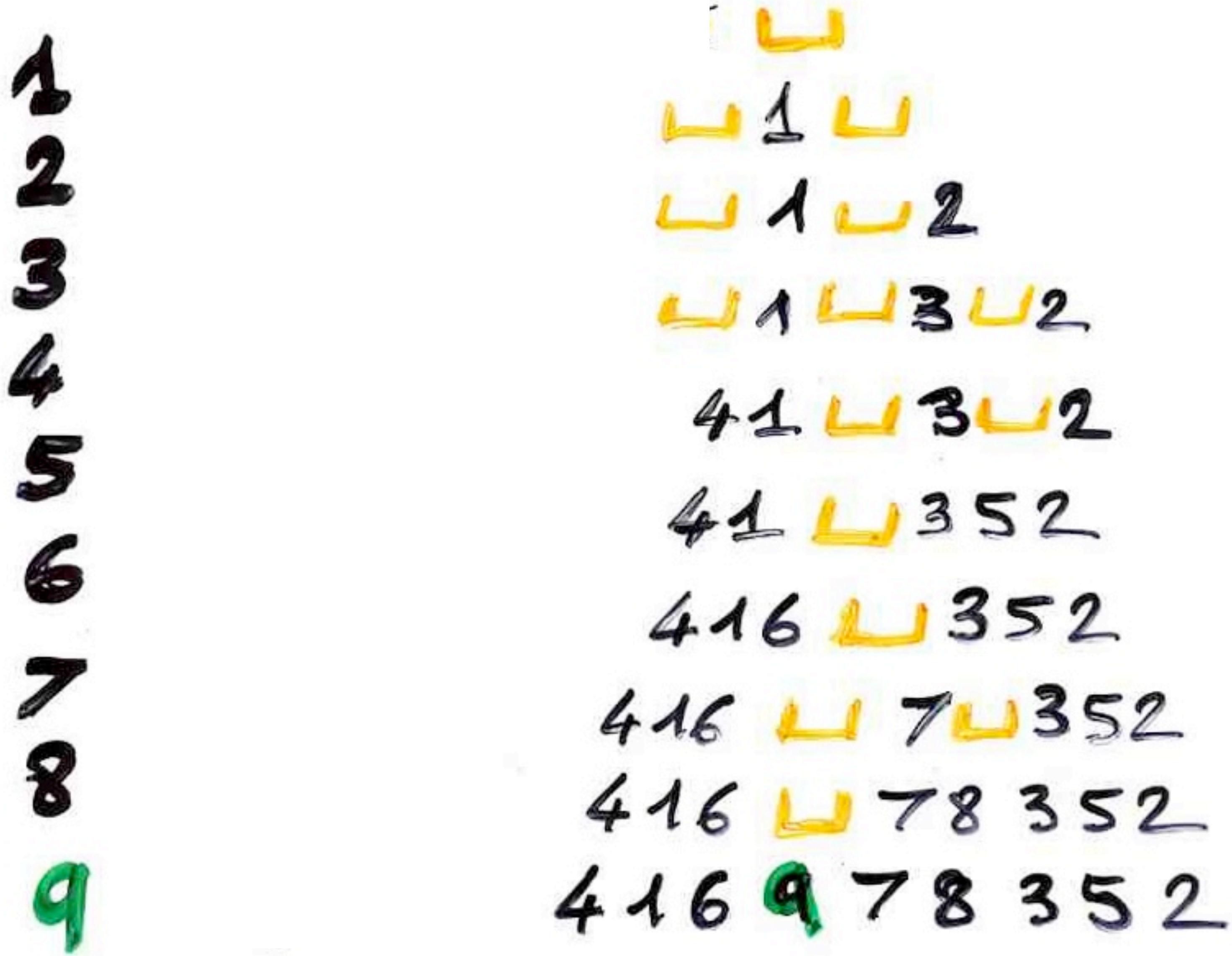
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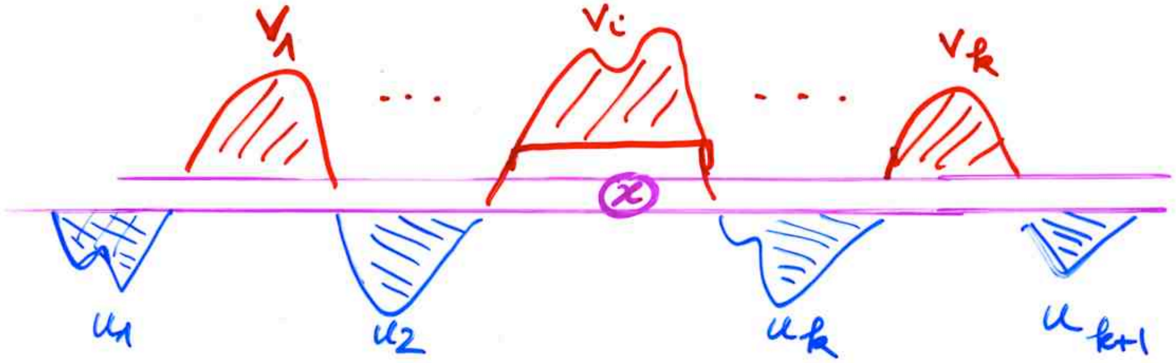
4 1 6 ␣ 7 8 3 5 2
4 1 6 9 7 8 3 5 2



v1
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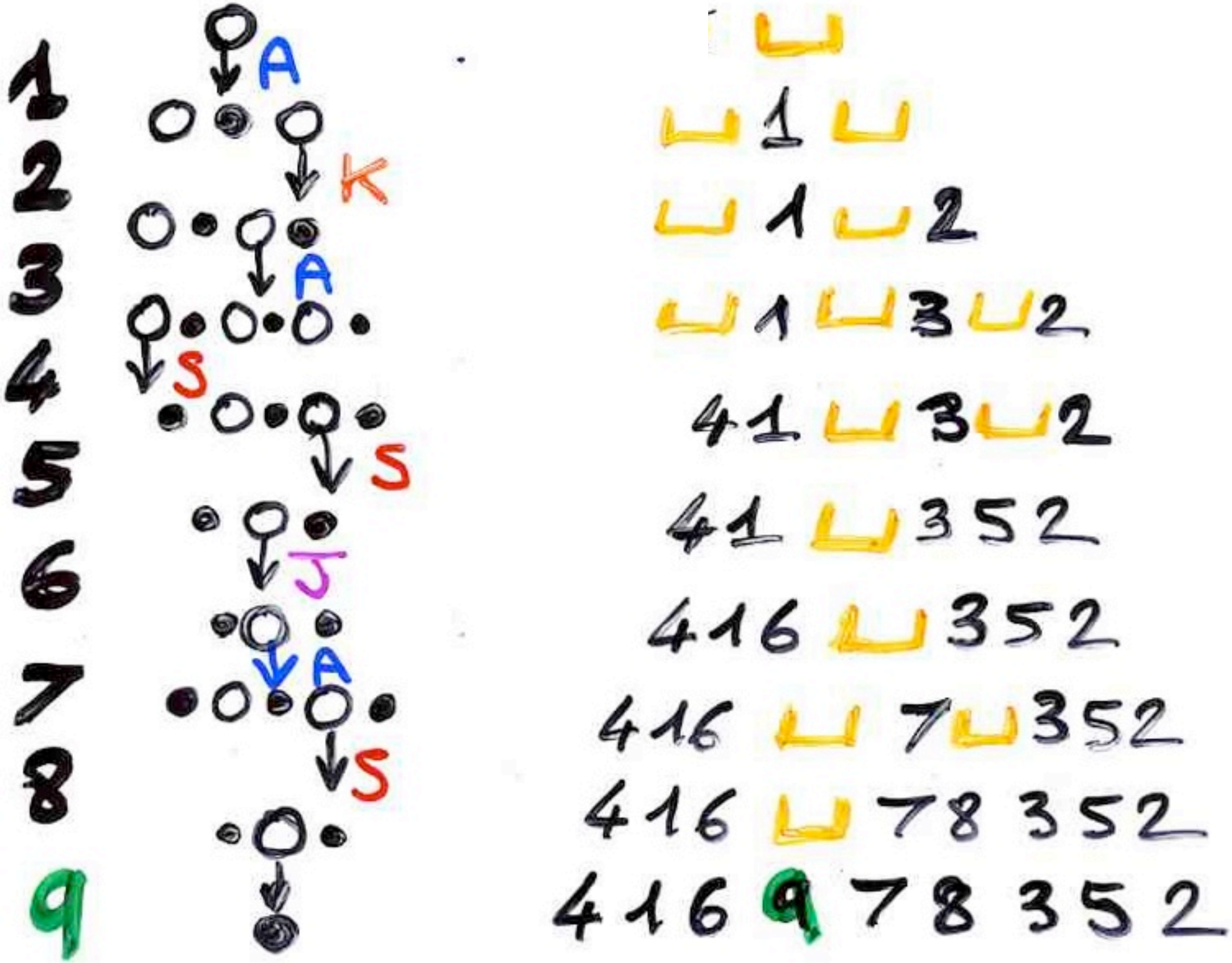
uk+1



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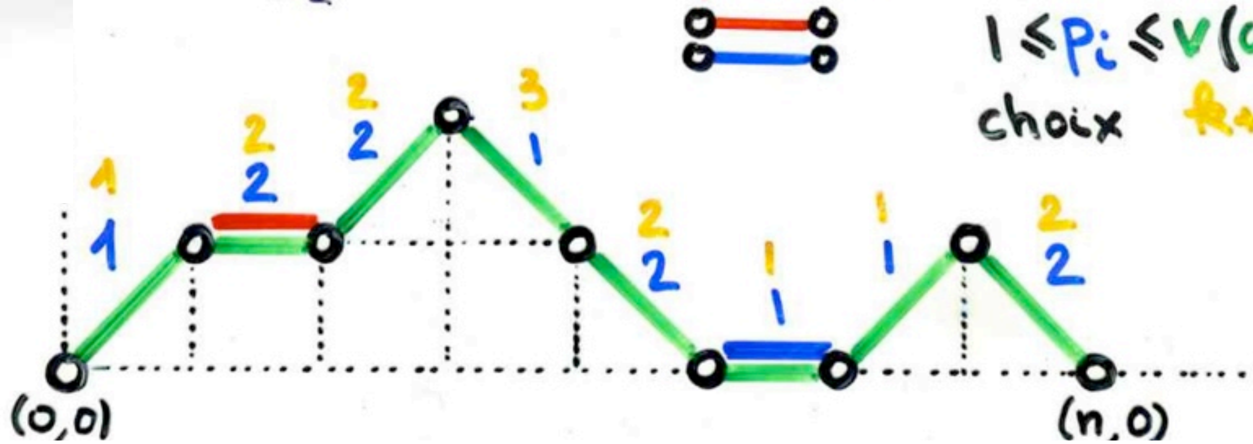
$h = (\omega_c ; (p_1, \dots, p_n))$



$1 \leq p_i \leq v(\omega_i)$

choix k+1

x	ω_c	pos	v
1	(see drawing)	1	1
2	(see drawing)	2	2
3	(see drawing)	2	2
4	(see drawing)	1	3
5	(see drawing)	2	2
6	(see drawing)	1	1
7	(see drawing)	1	1
$n=$ 8	(see drawing)	2	2
9	•		

␣

␣ 1 ␣

␣ 1 ␣ 2

␣ 1 ␣ 3 ␣ 2

4 1 ␣ 3 ␣ 2

4 1 ␣ 3 5 2

4 1 6 ␣ 3 5 2

4 1 6 ␣ 7 ␣ 3 5 2

4 1 6 ␣ 7 8 3 5 2

4 1 6 9 7 8 3 5 2 $= \sigma \in \mathfrak{S}_{n+1}$

bijection
permutations
alternative
tableaux

416978352

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●1●

●1●2●

●1●3●2

41●3●2

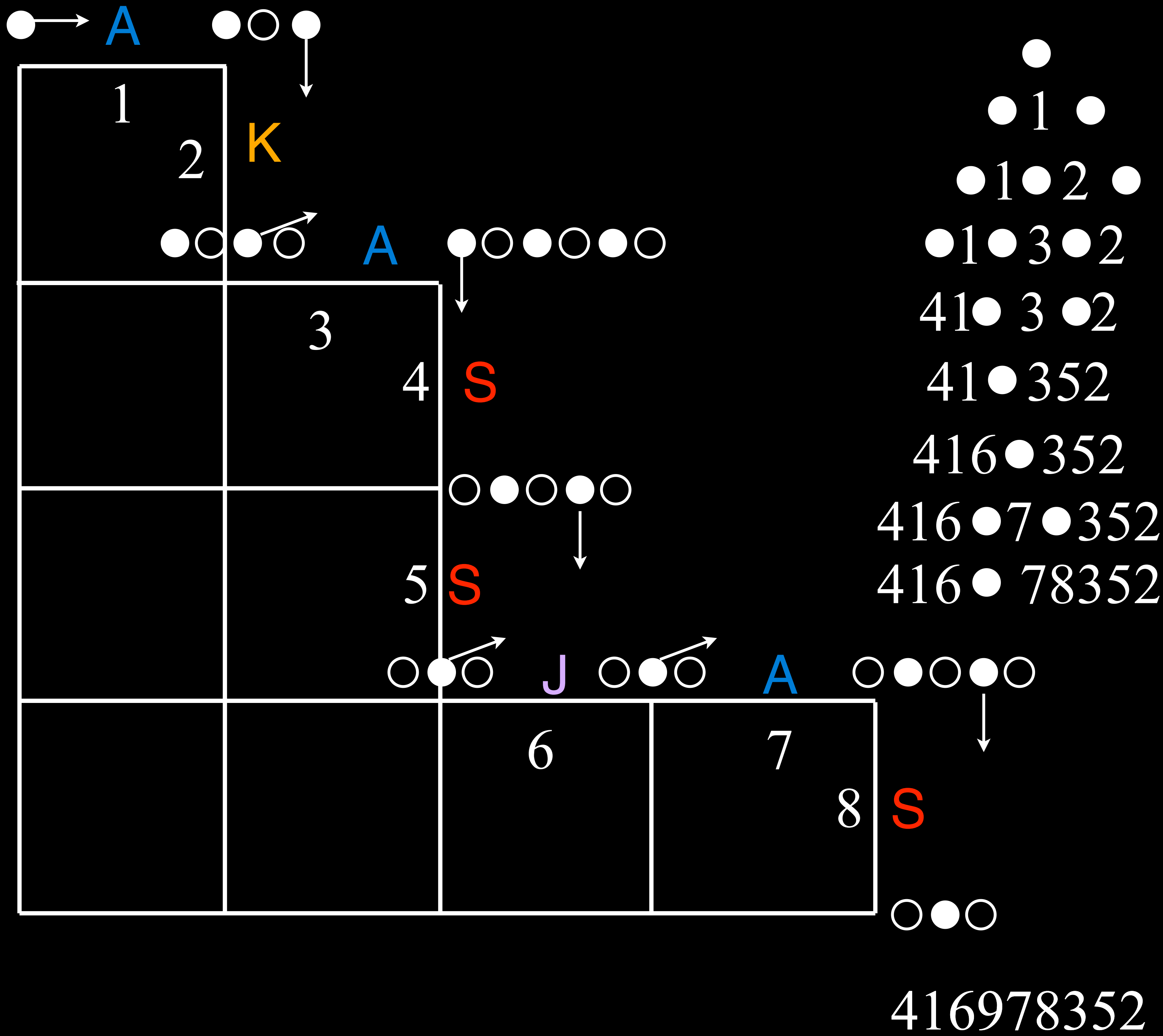
41●352

416●352

416●7●352

416●78352

416978352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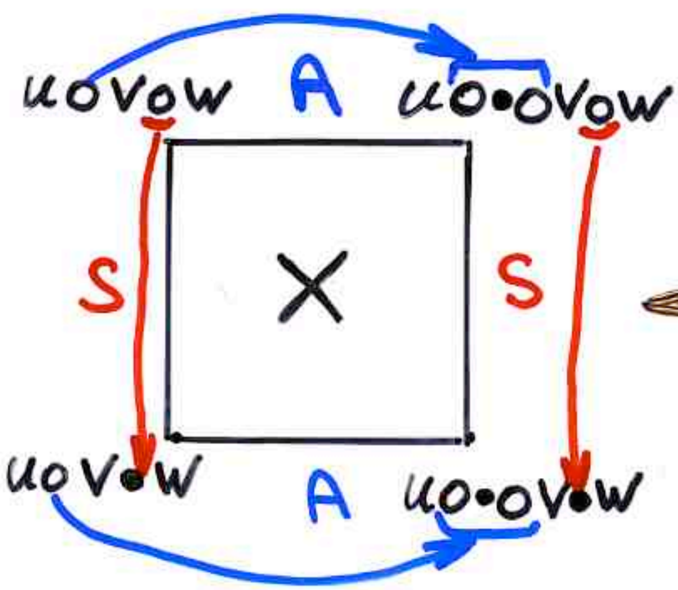
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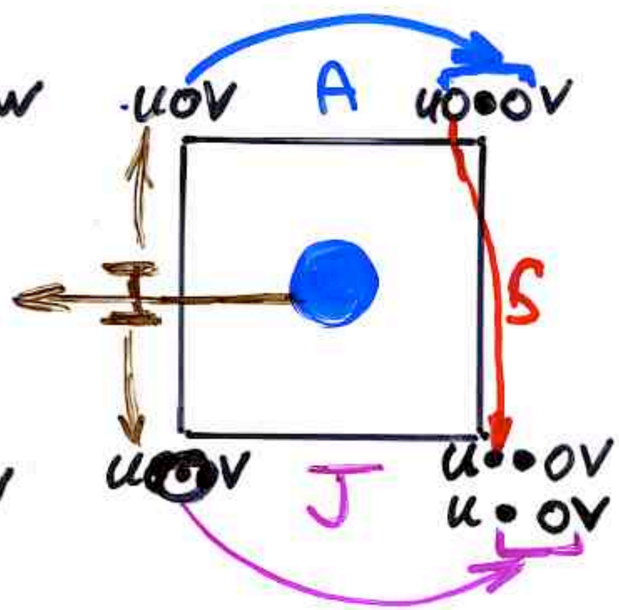
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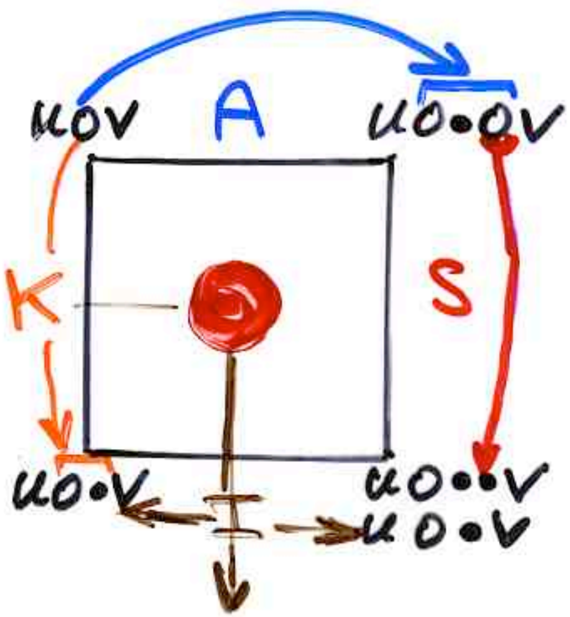
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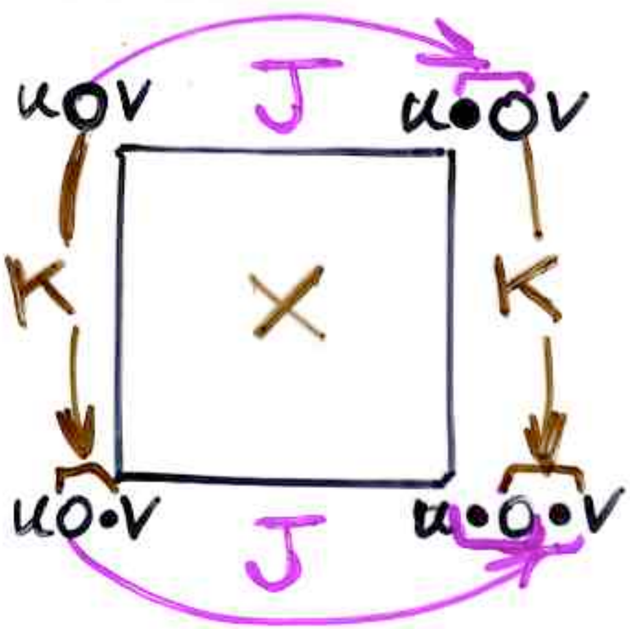
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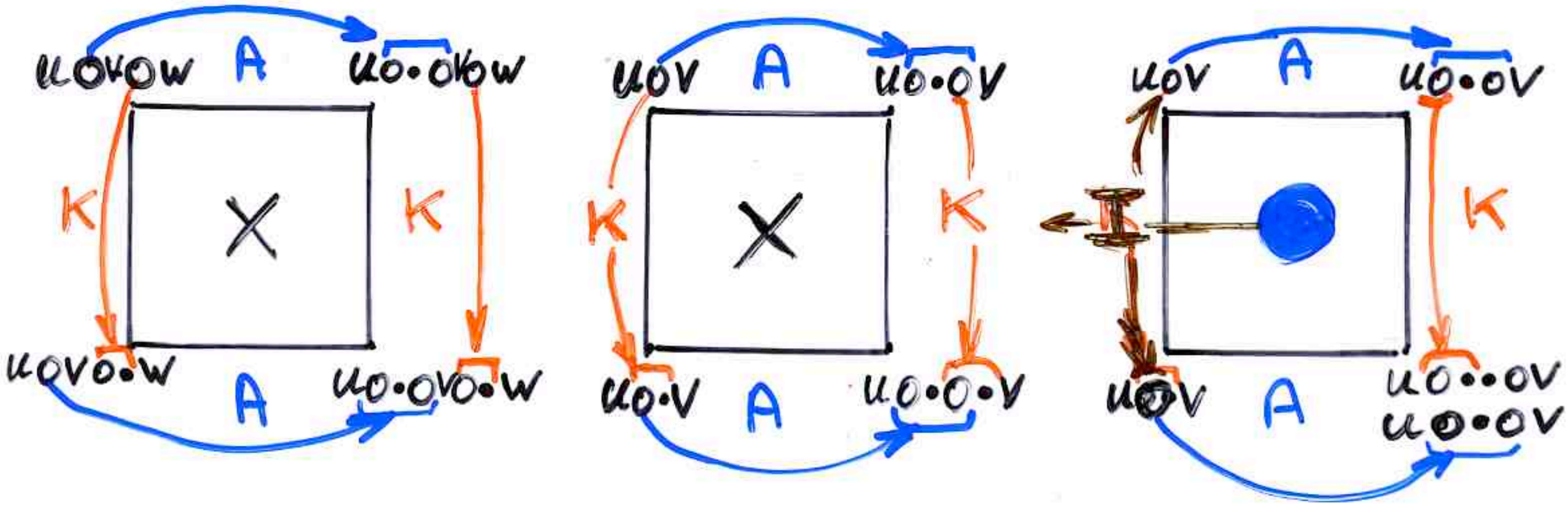
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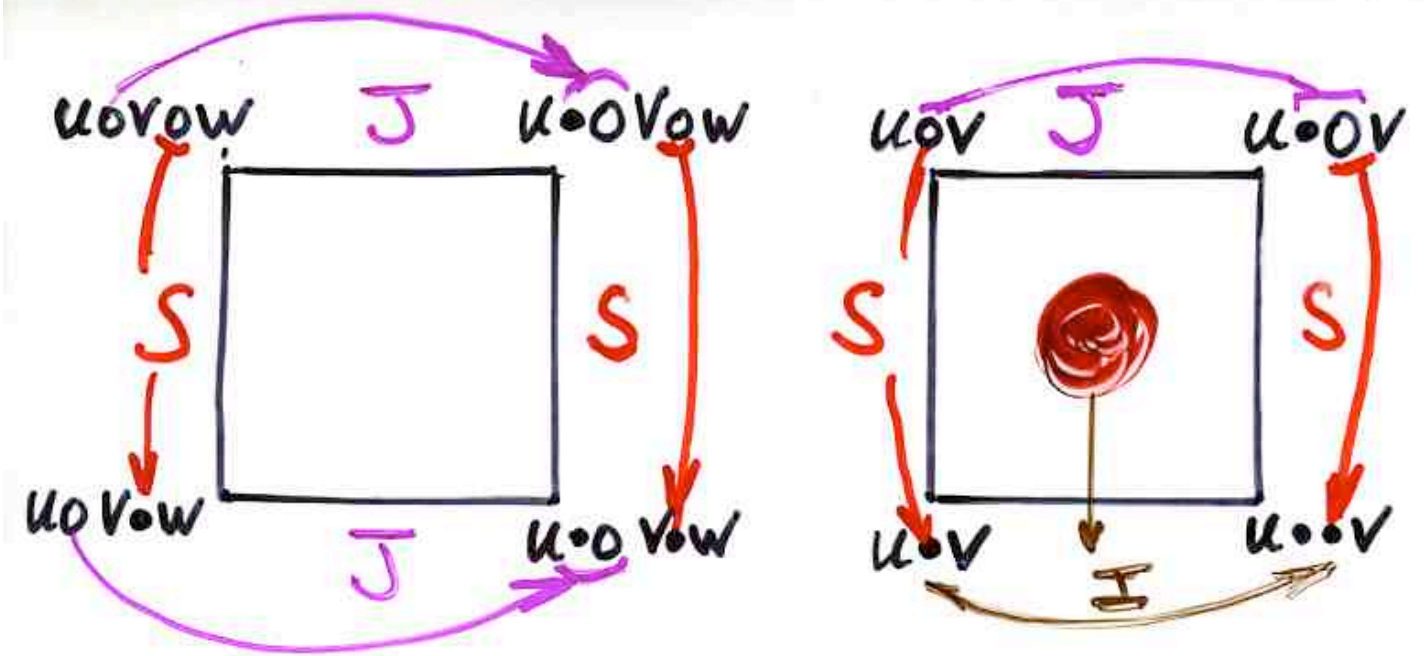
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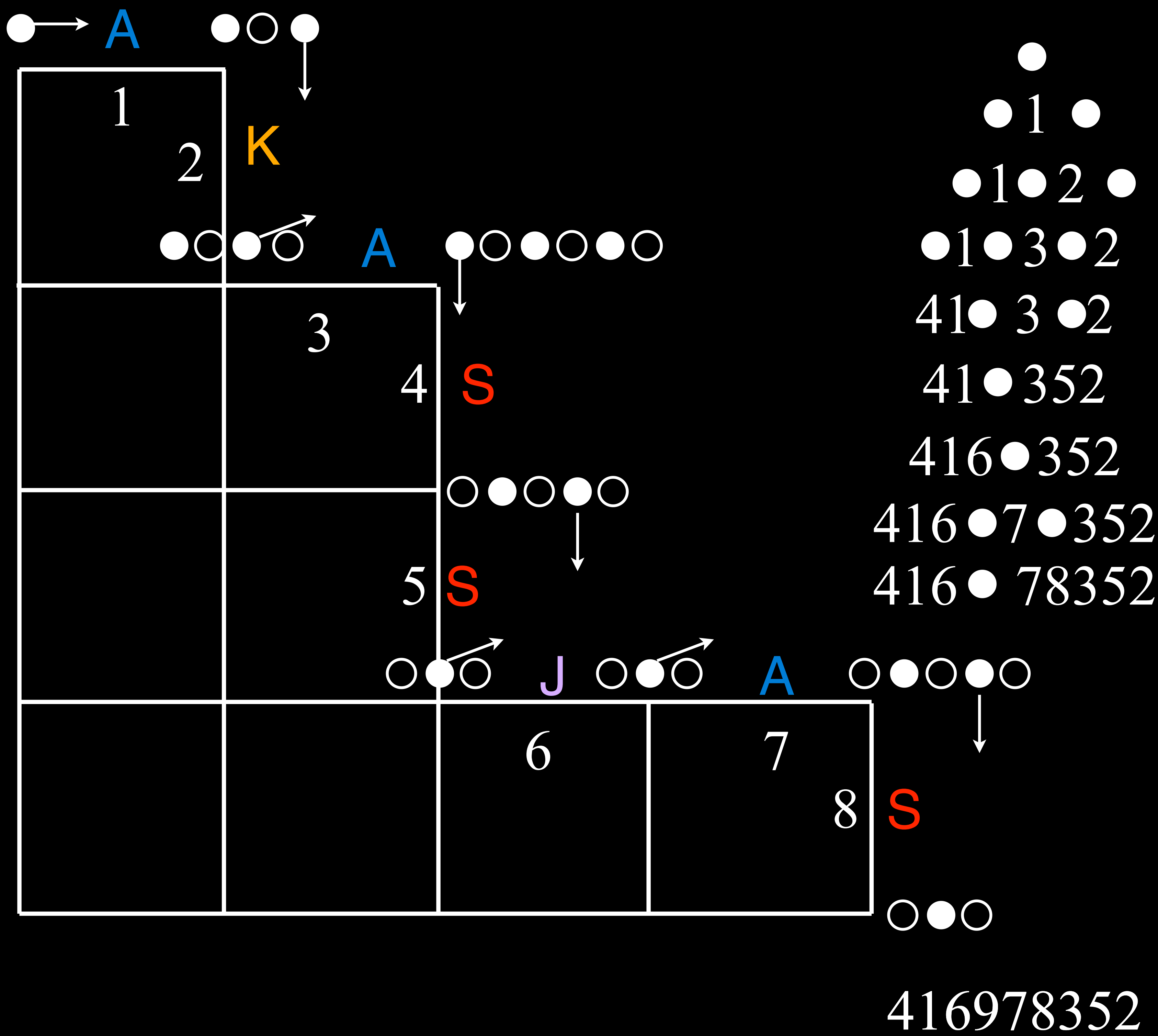

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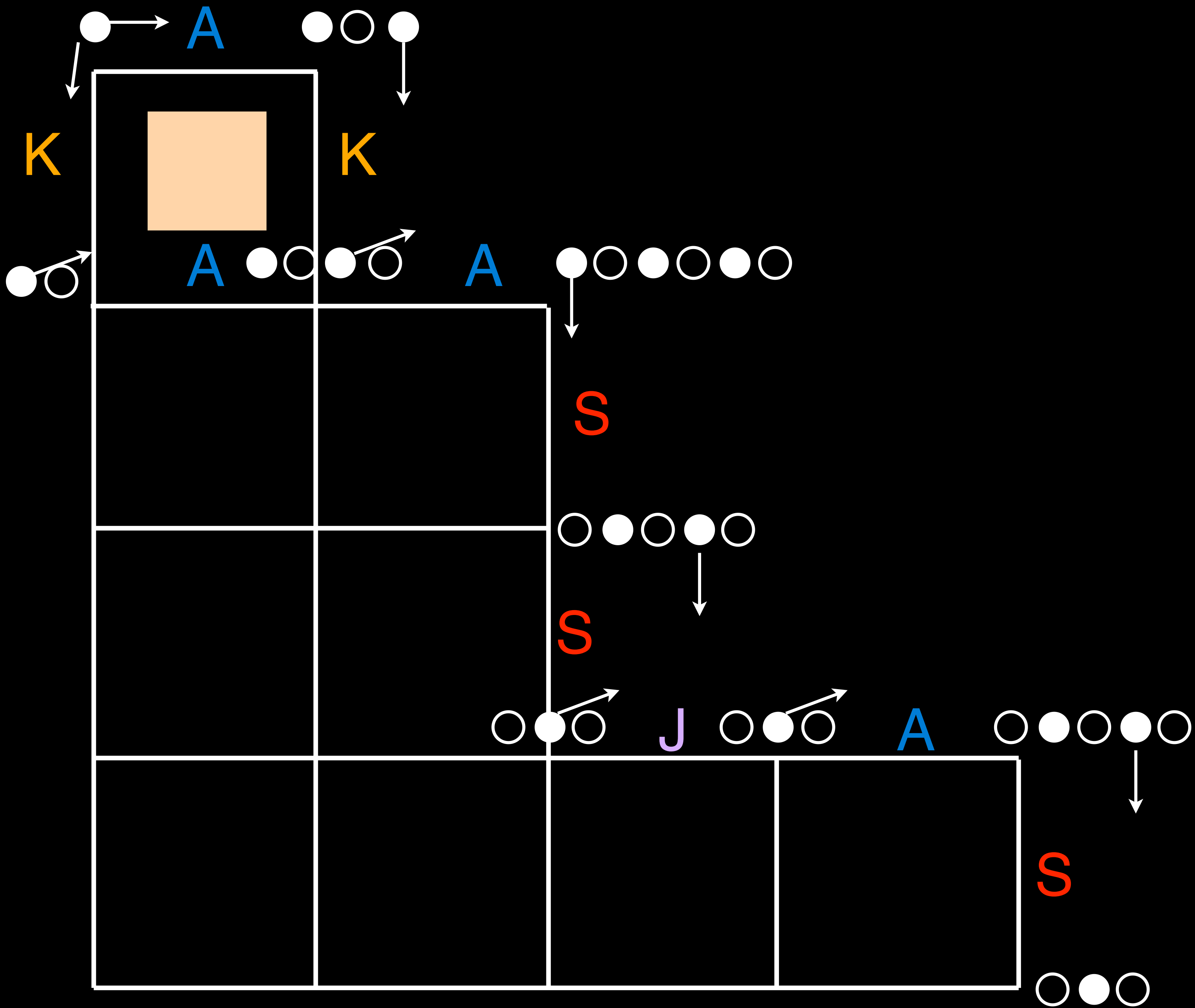
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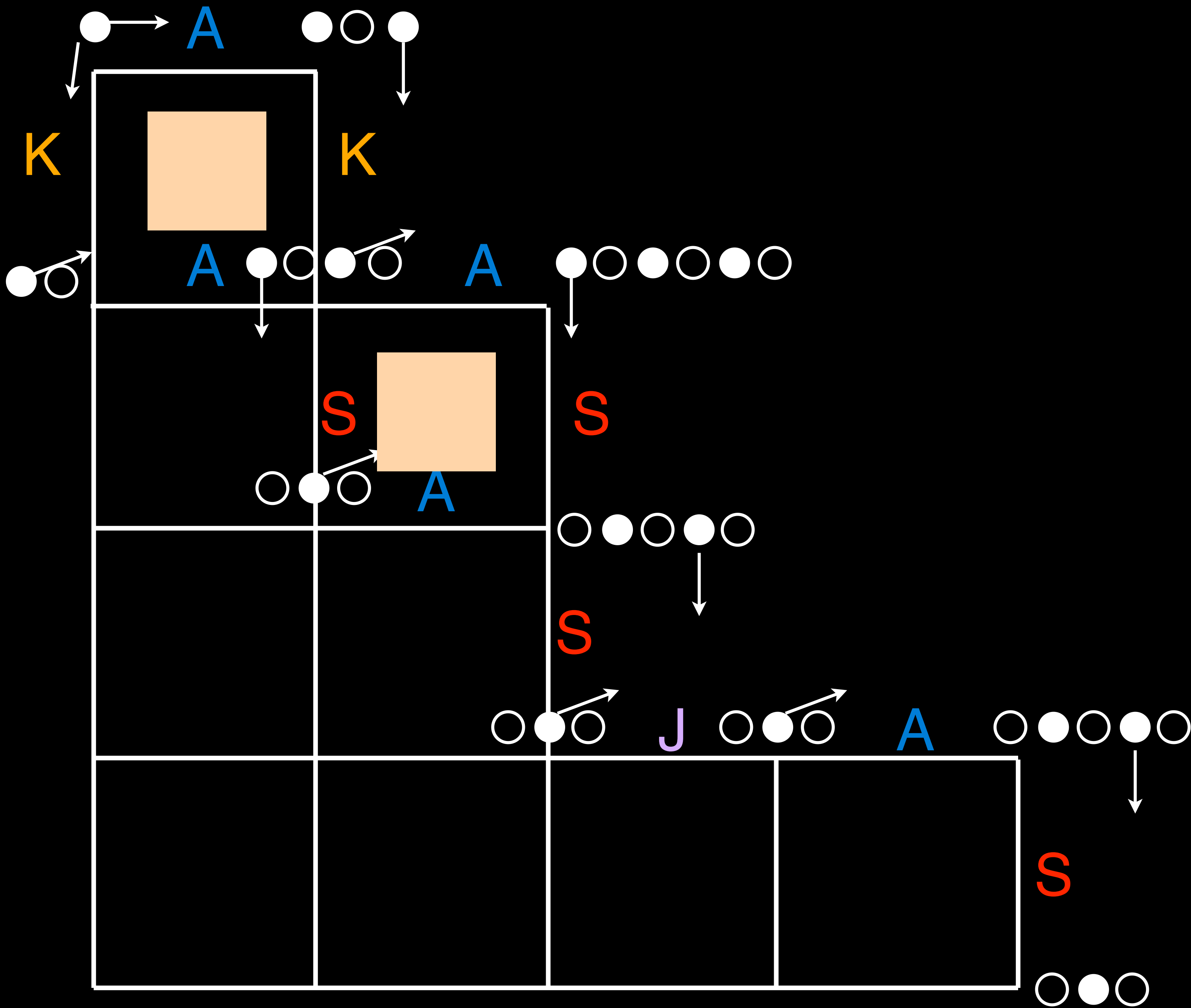
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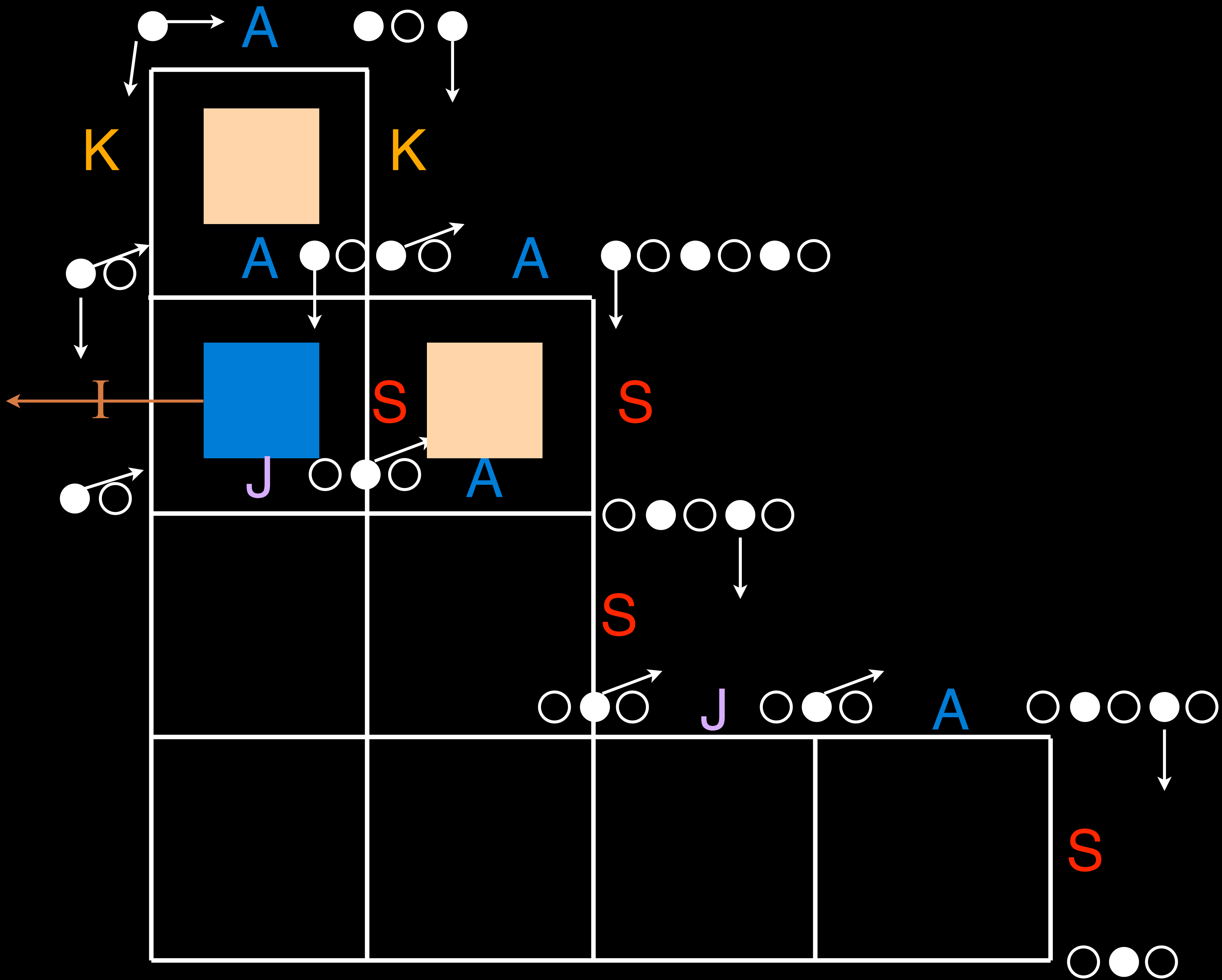
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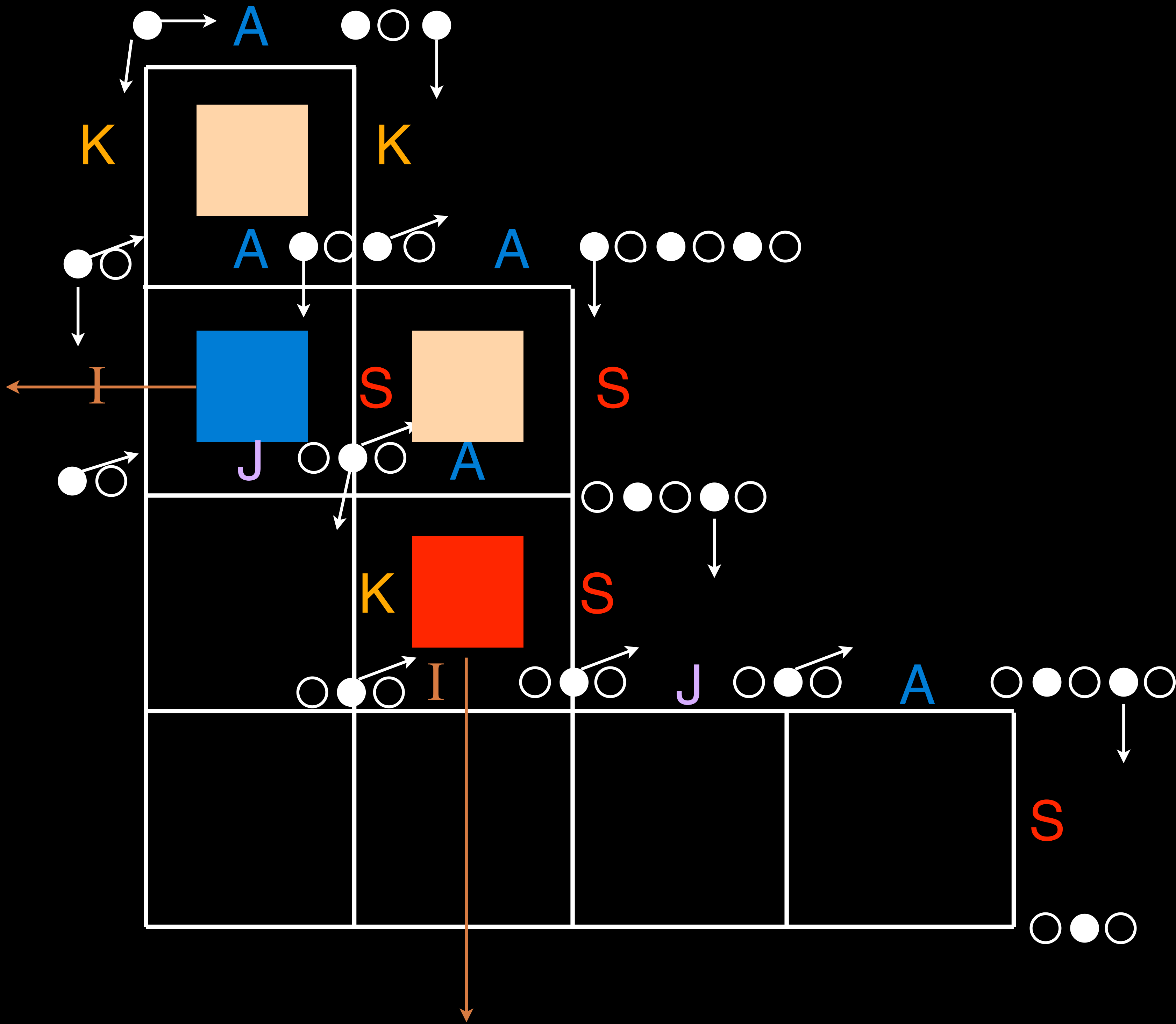
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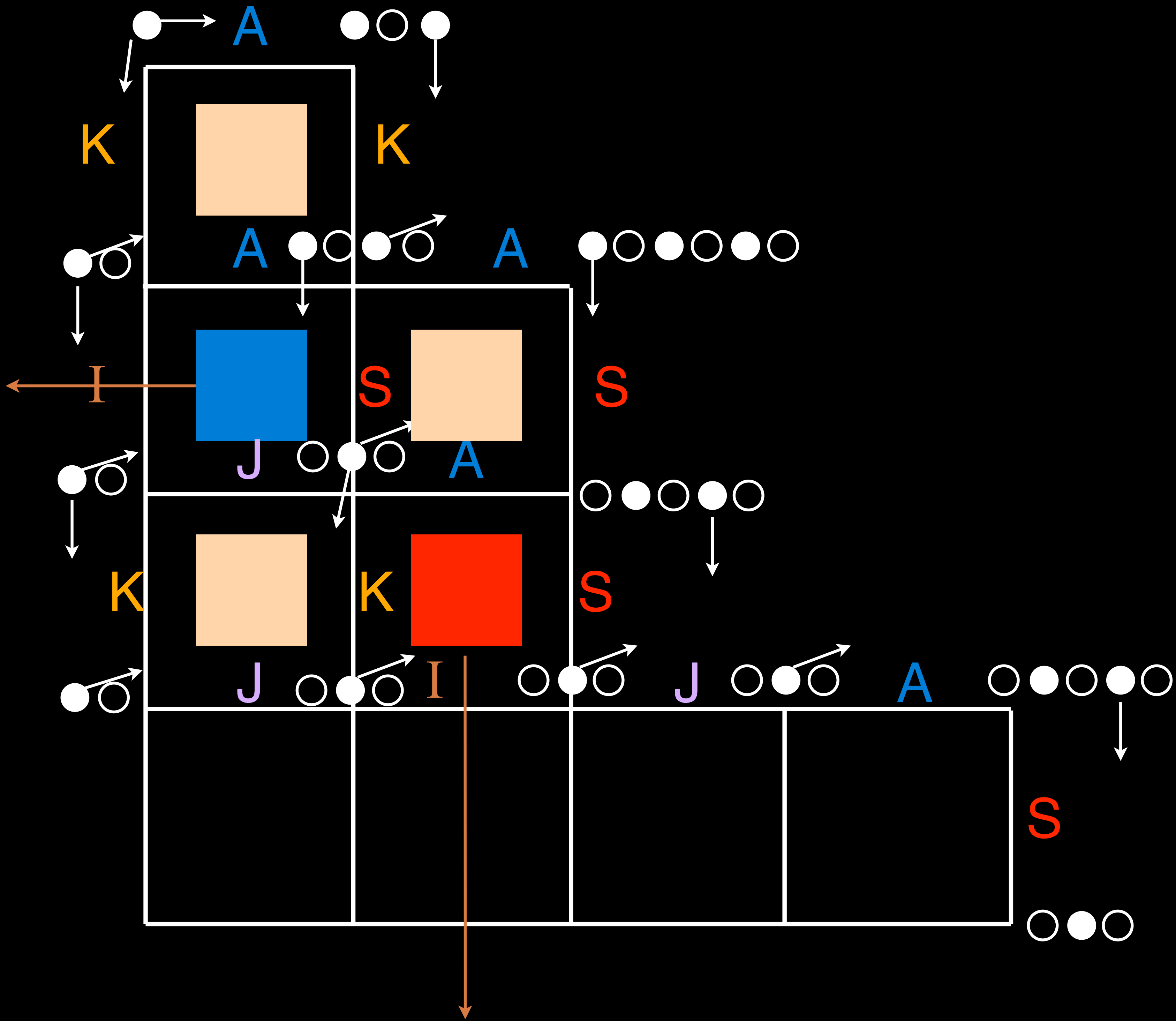

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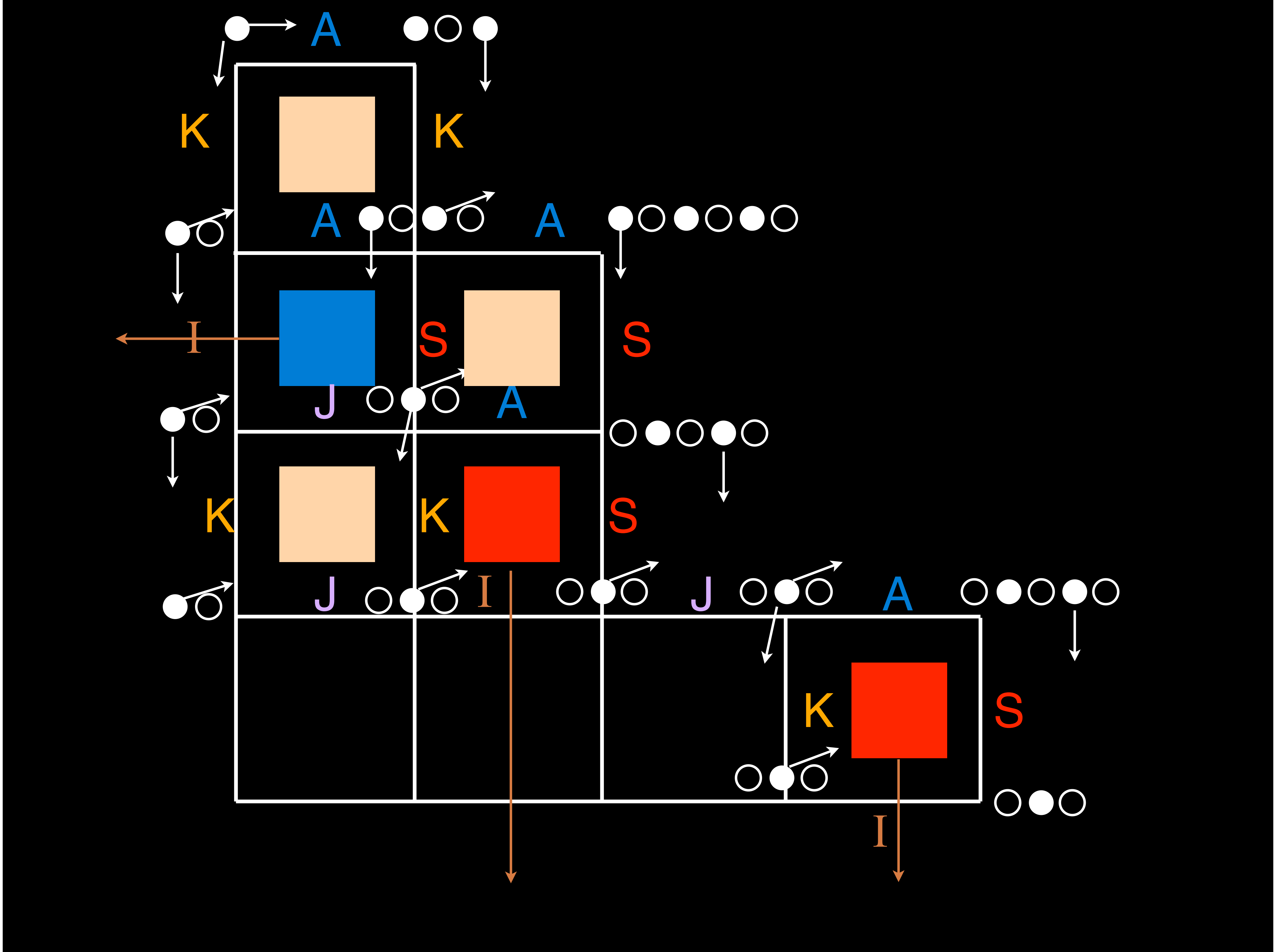
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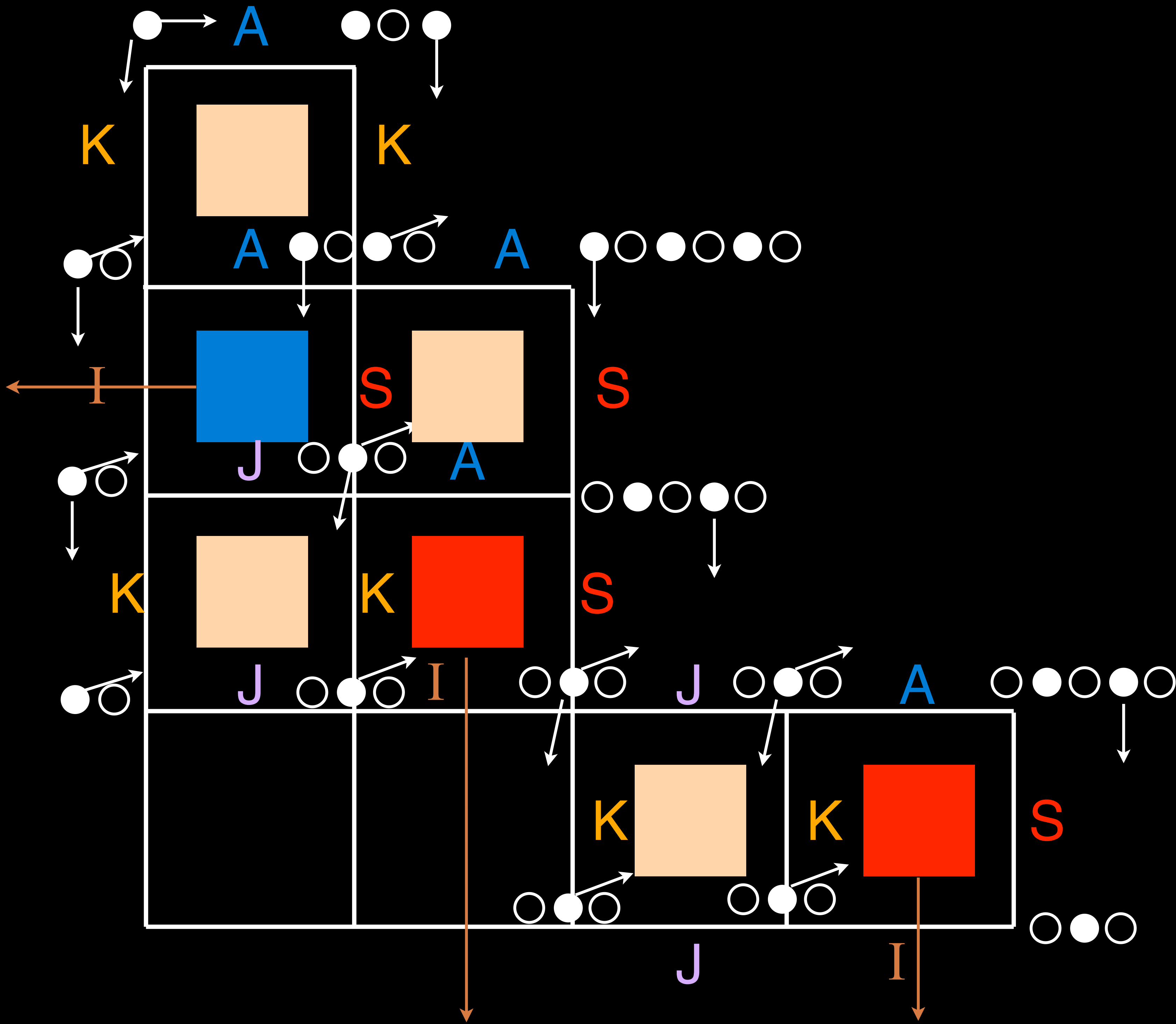
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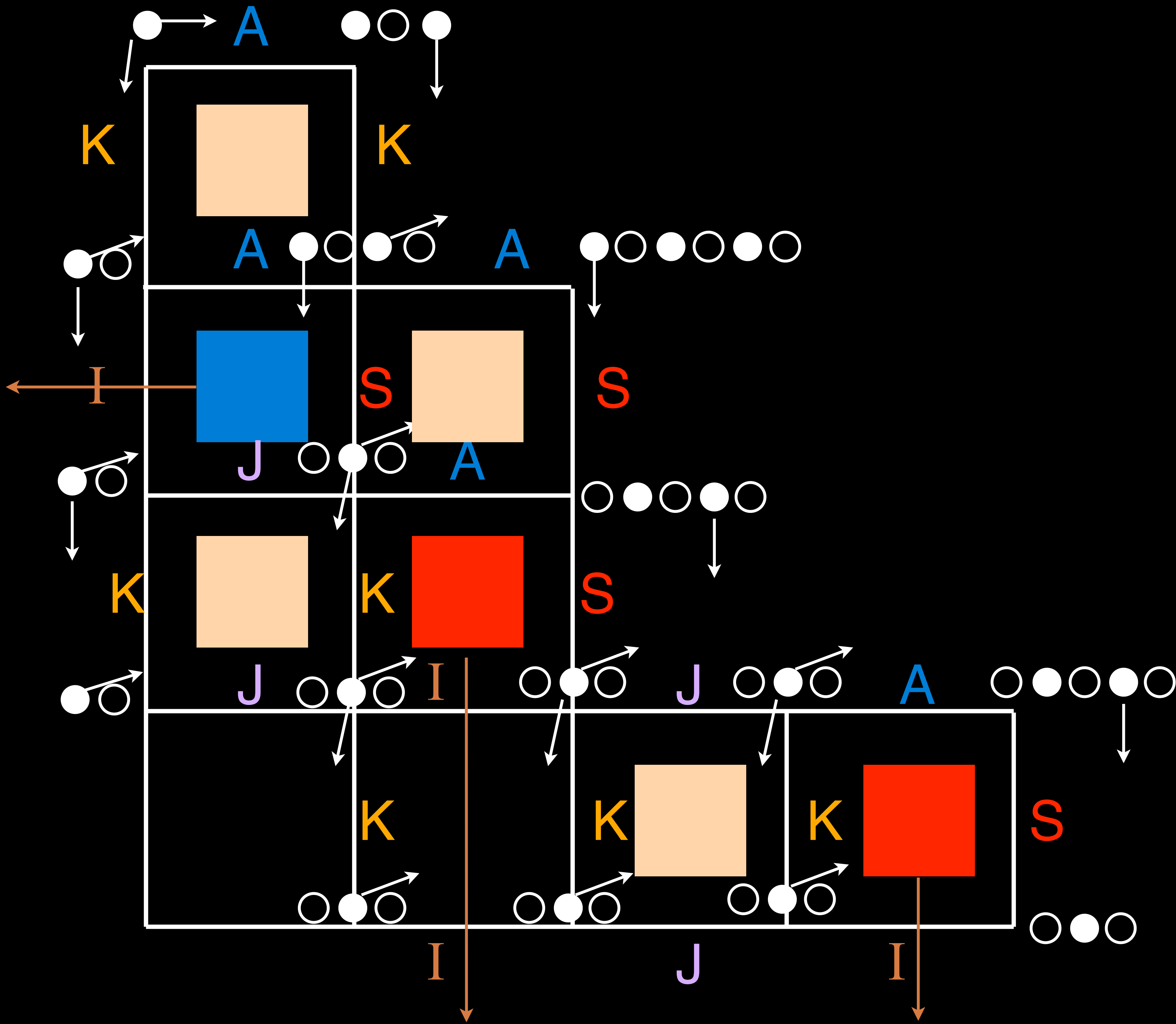

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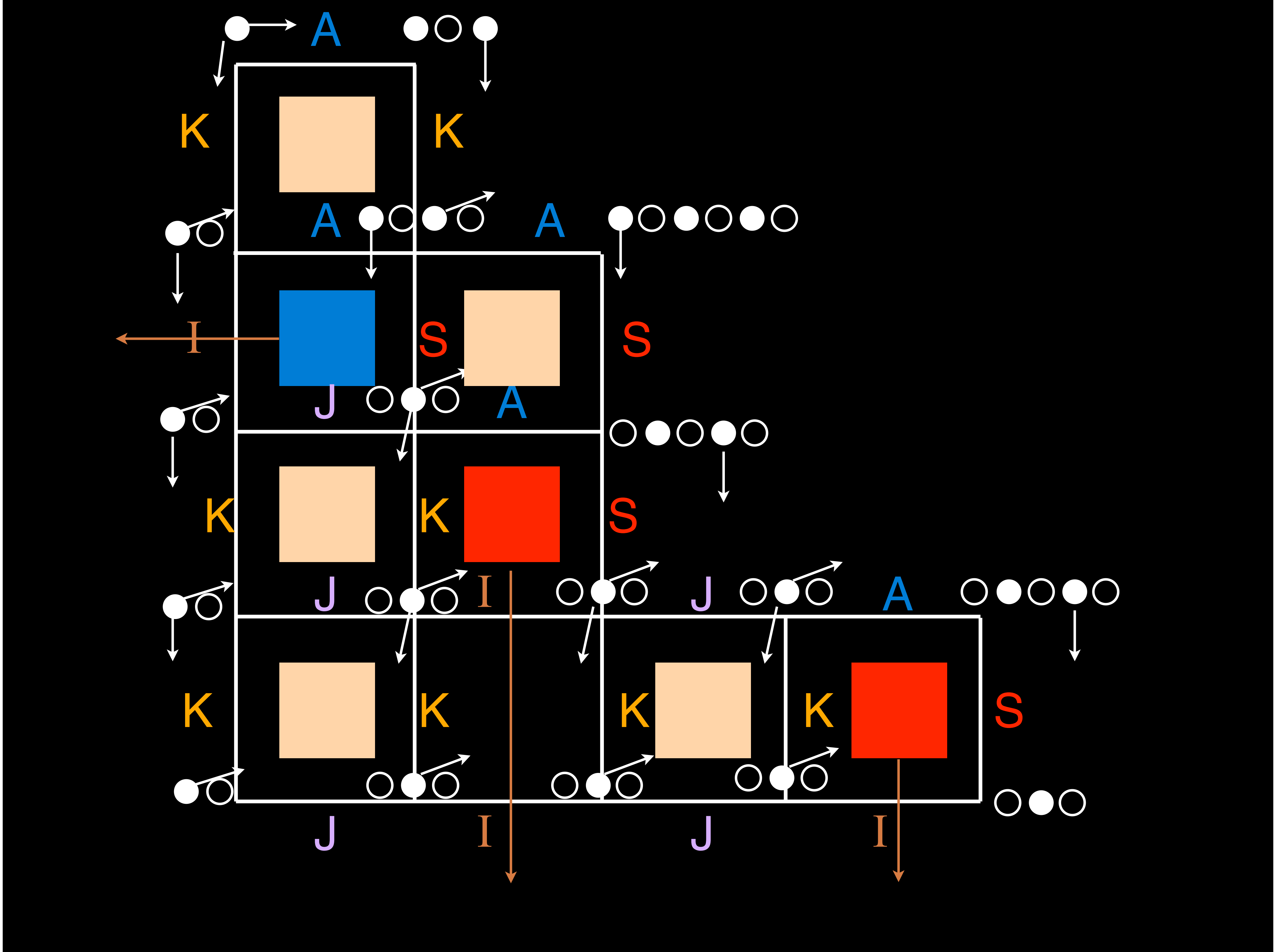
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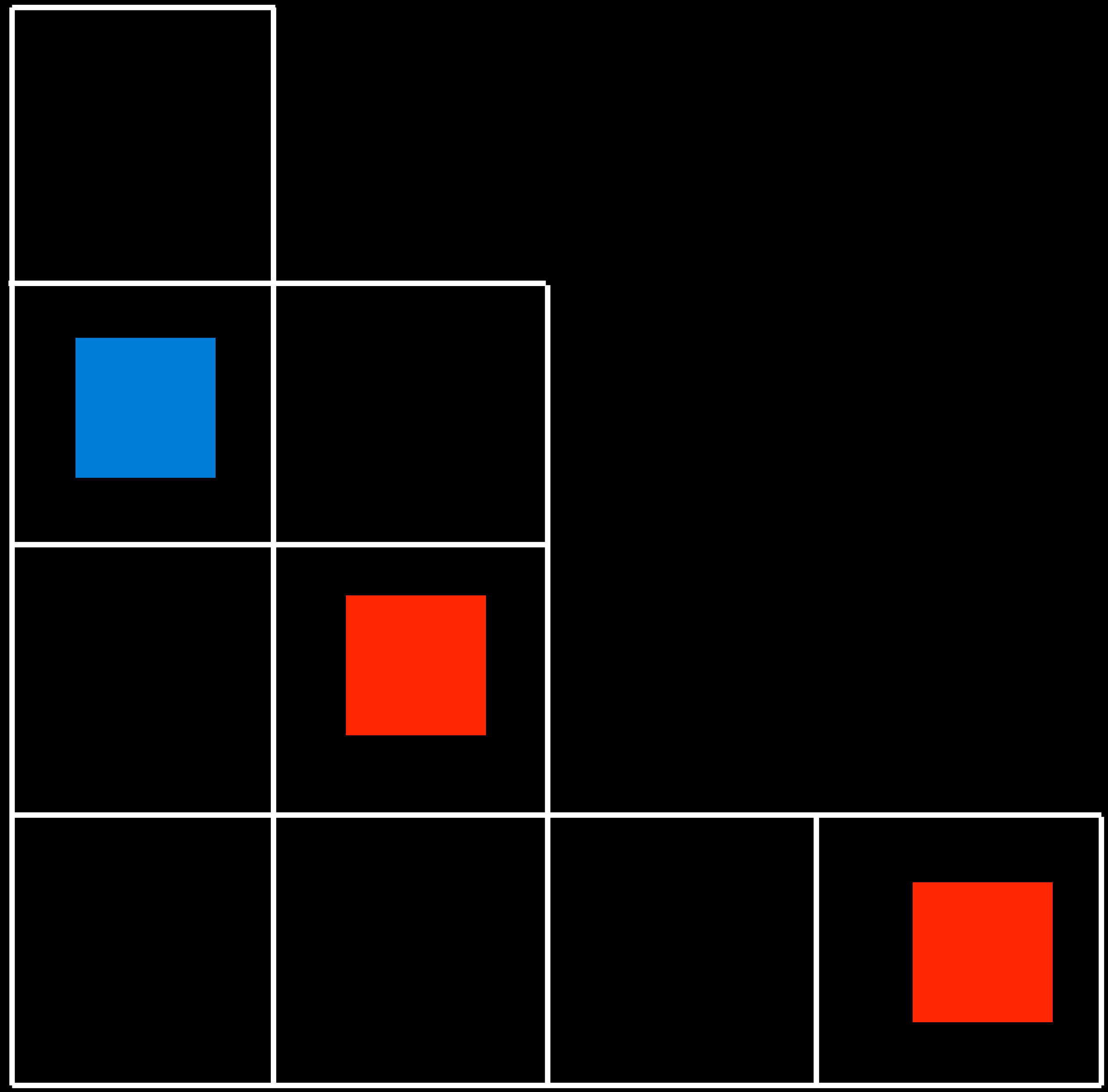

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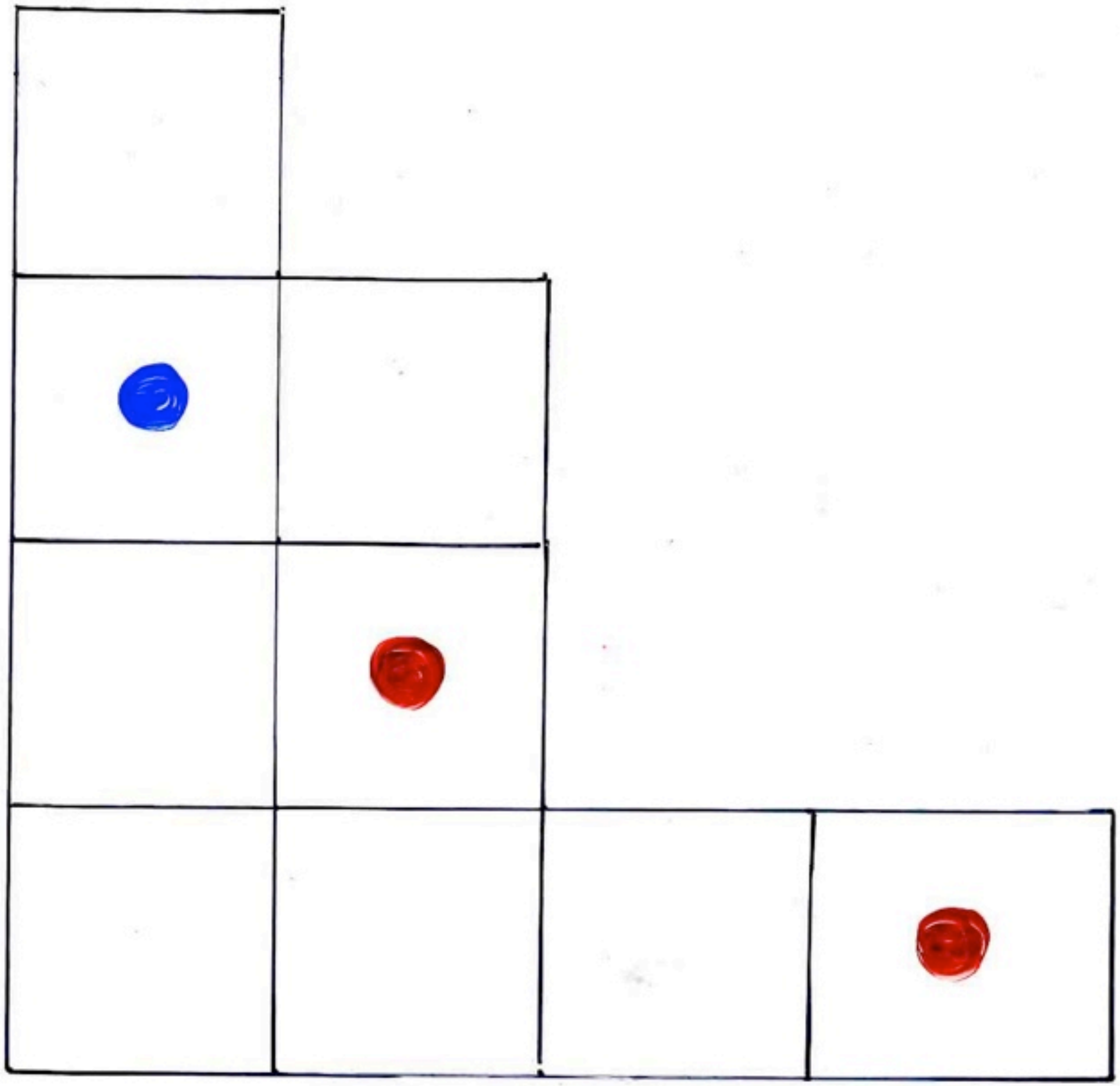
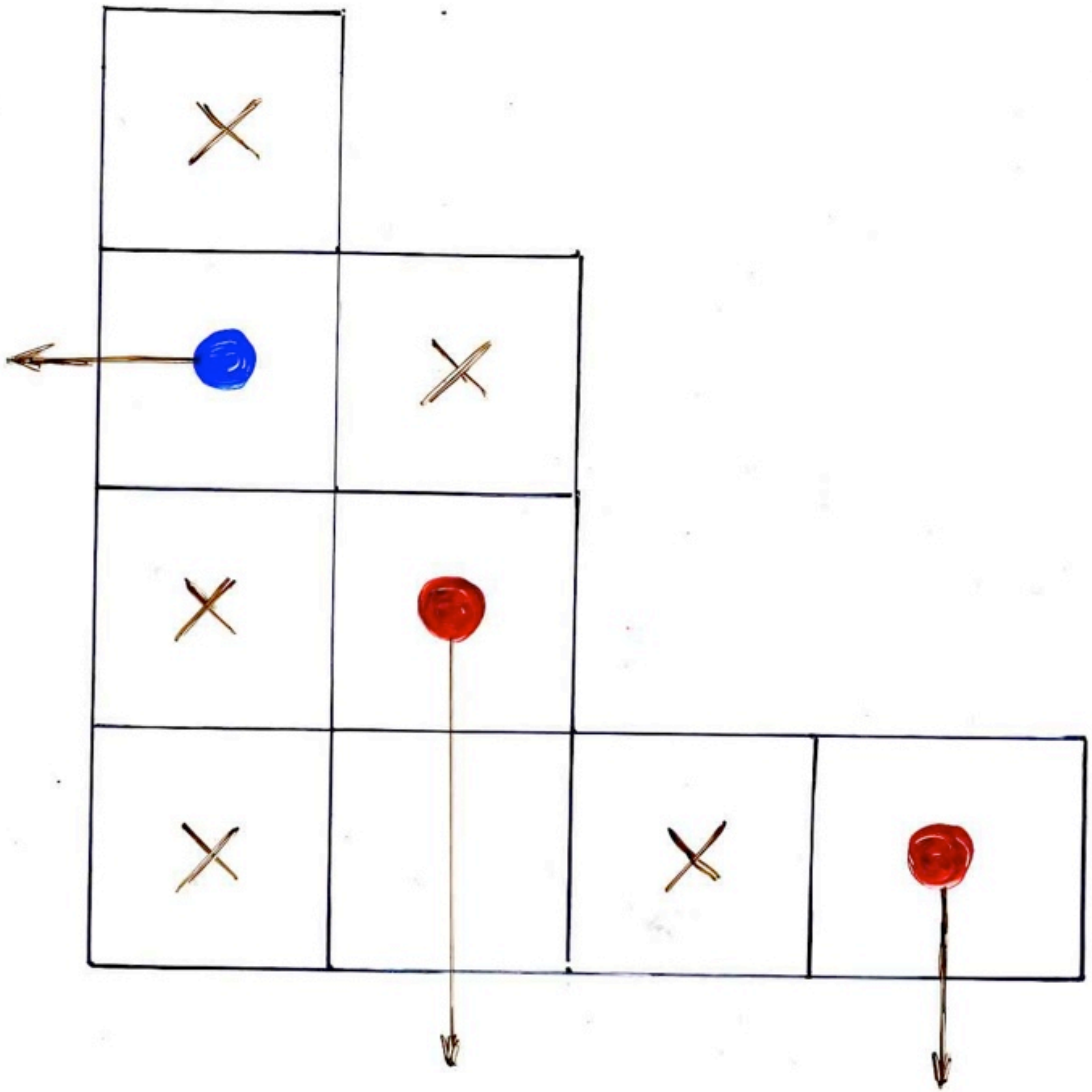
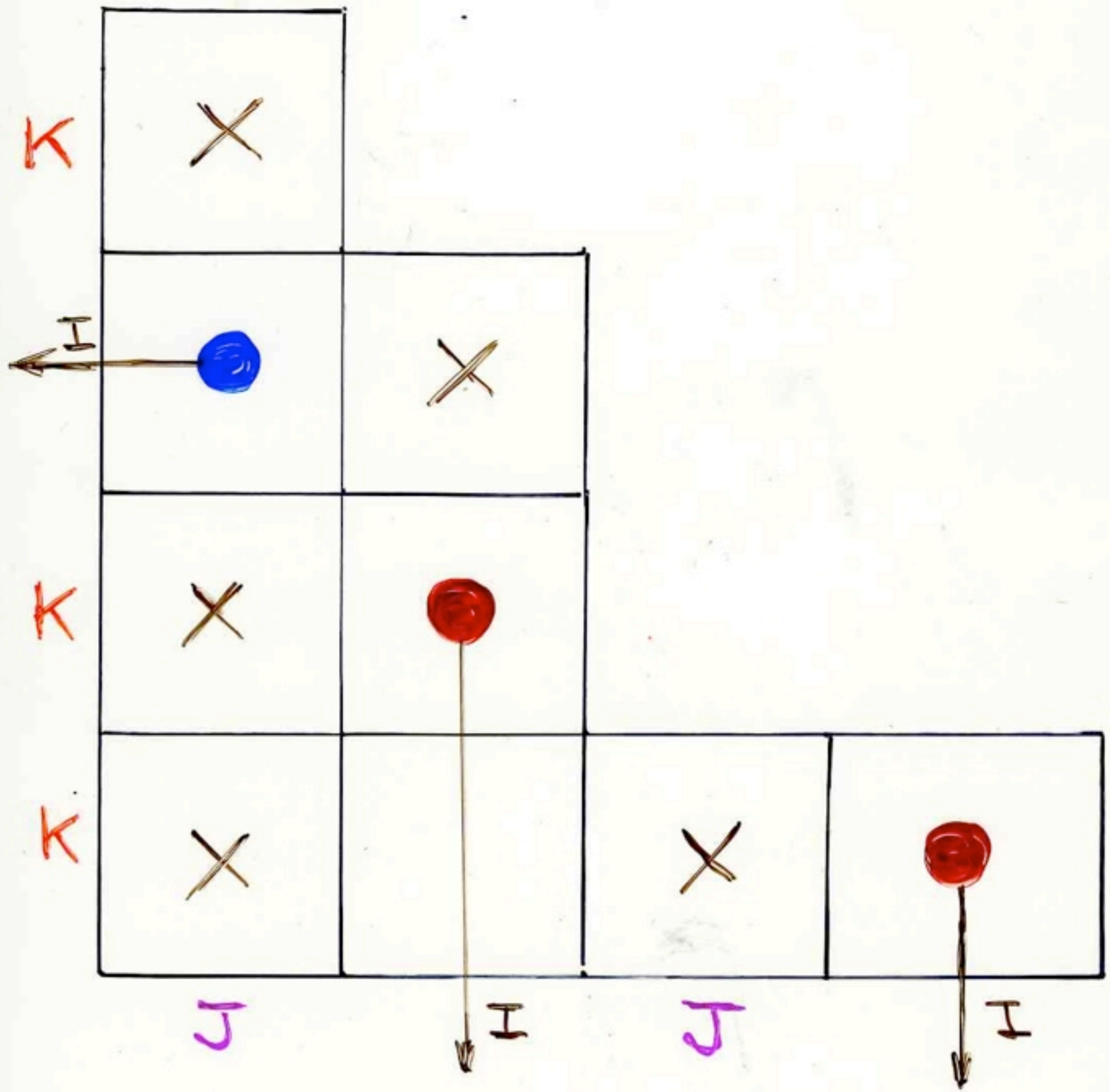
inverse bijection



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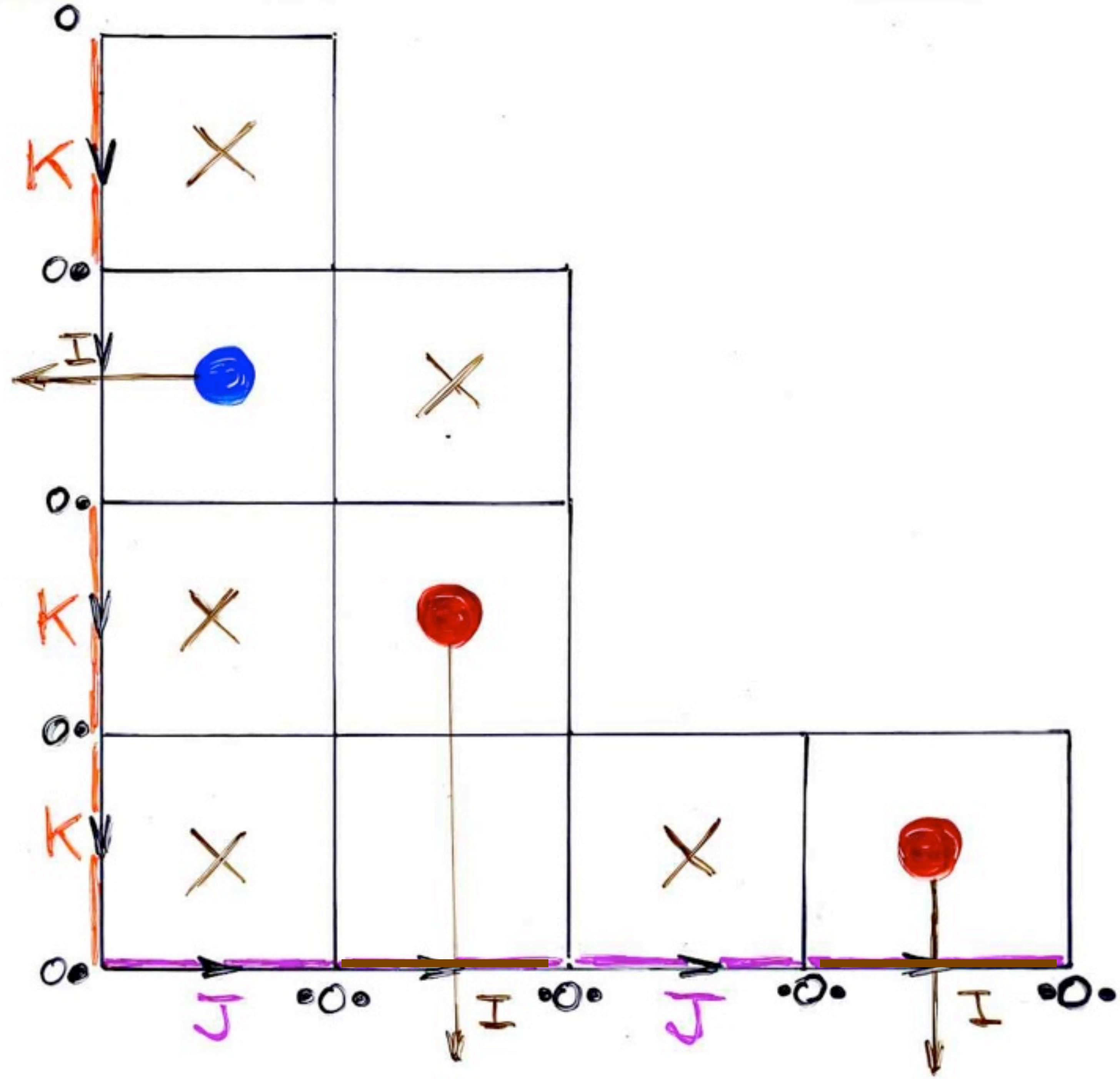

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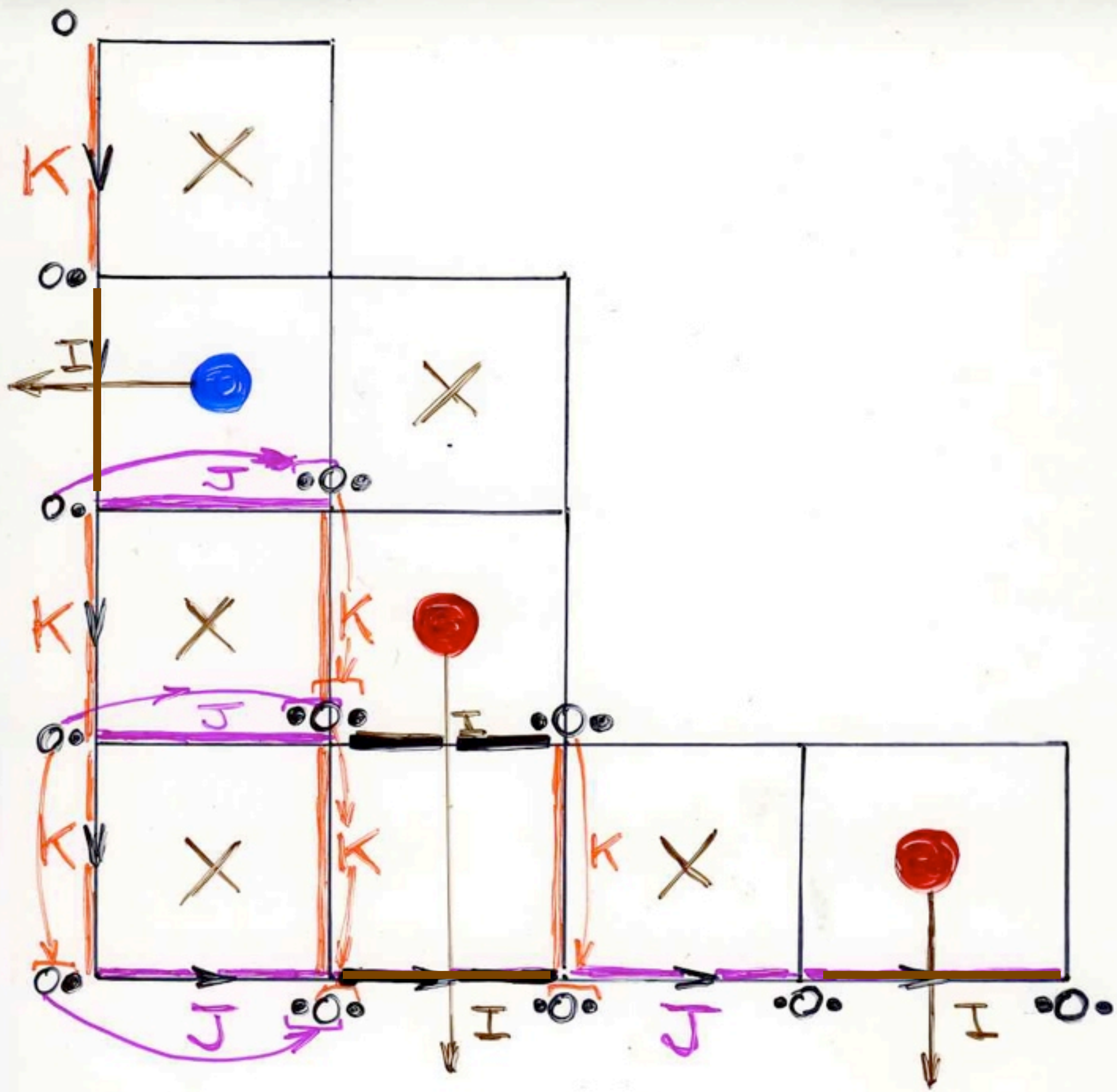
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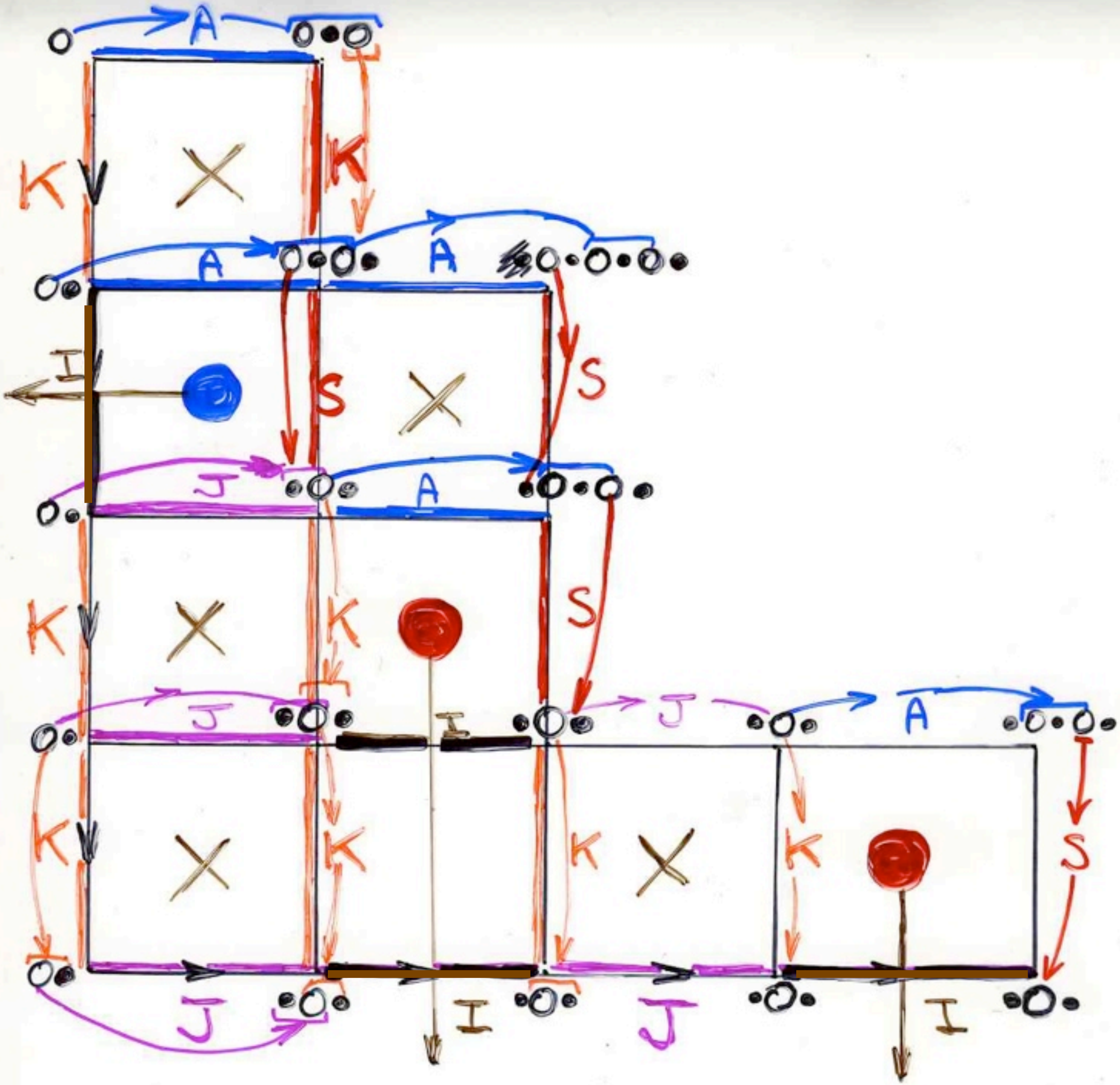
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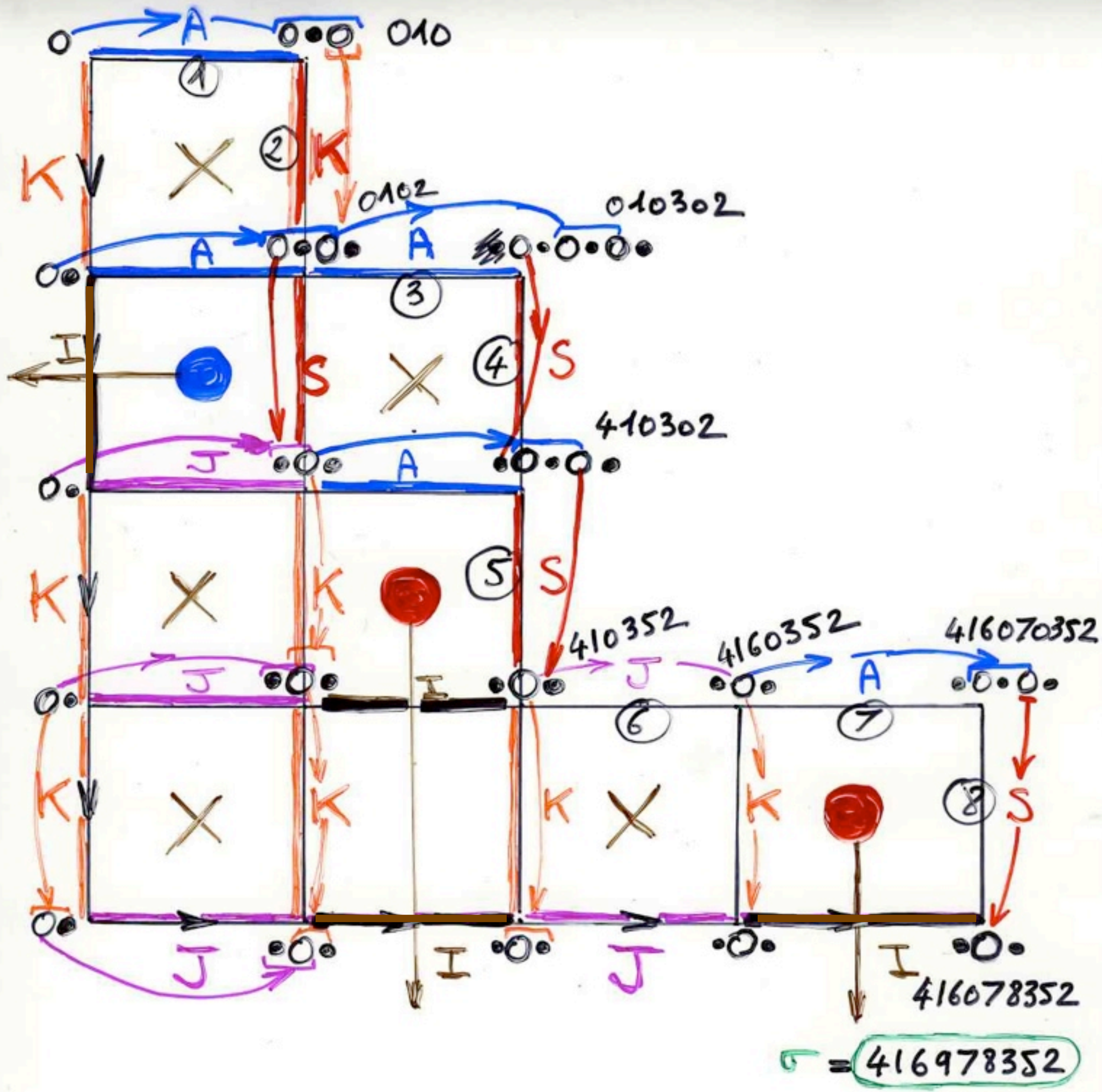

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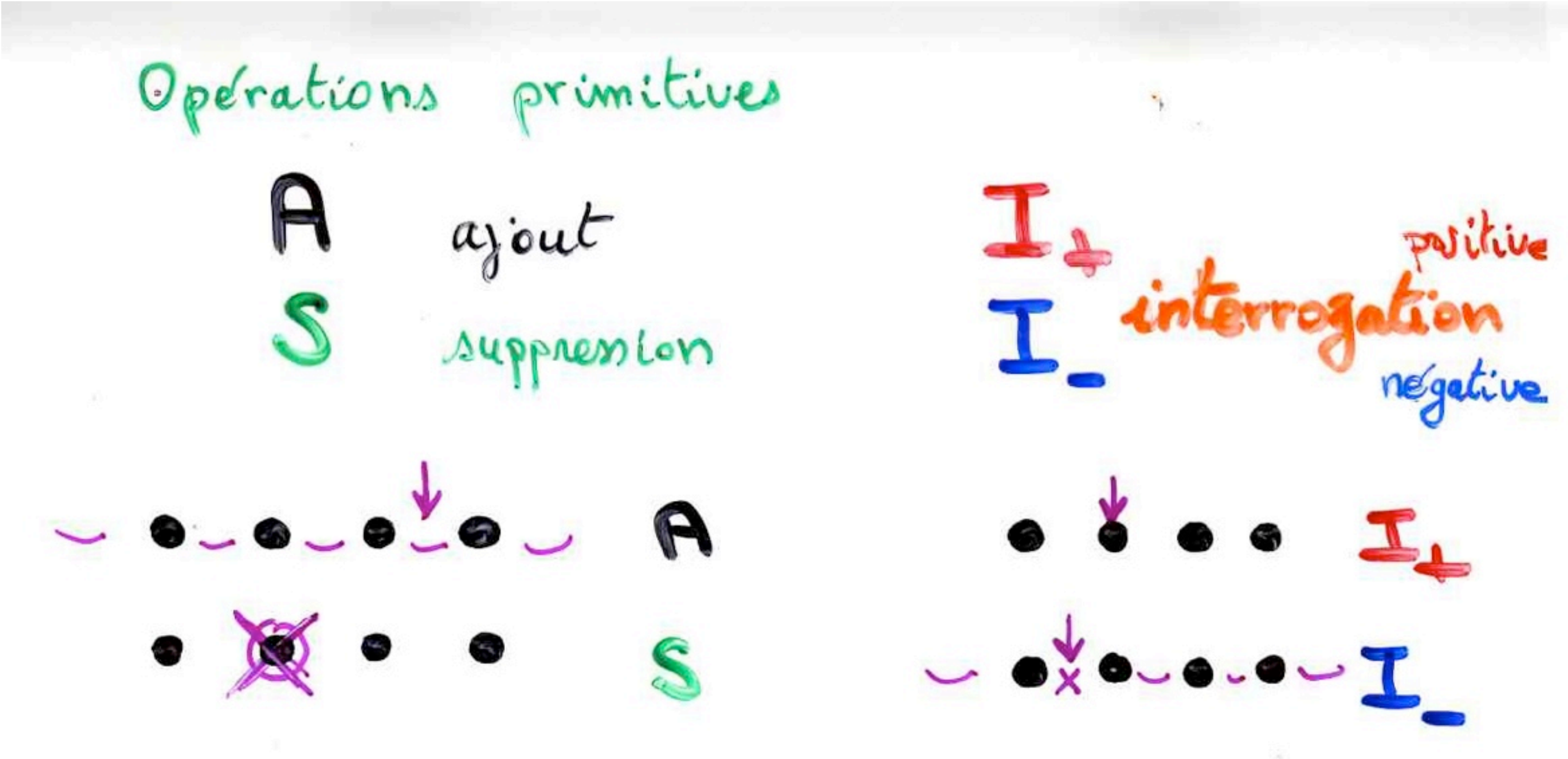
σ = 416978352

quadratic algebra

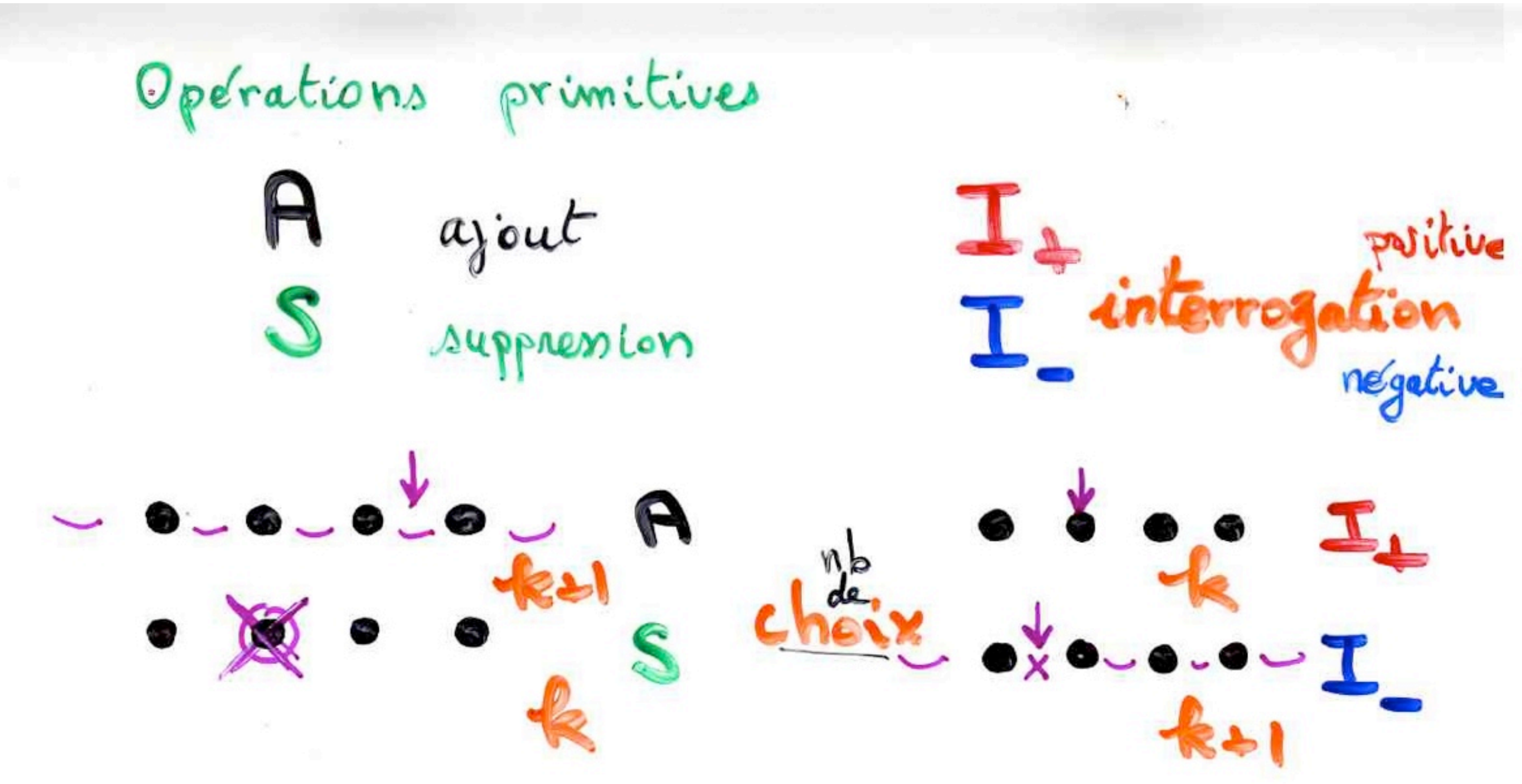
operators

data structures

and orthogonal polynomials



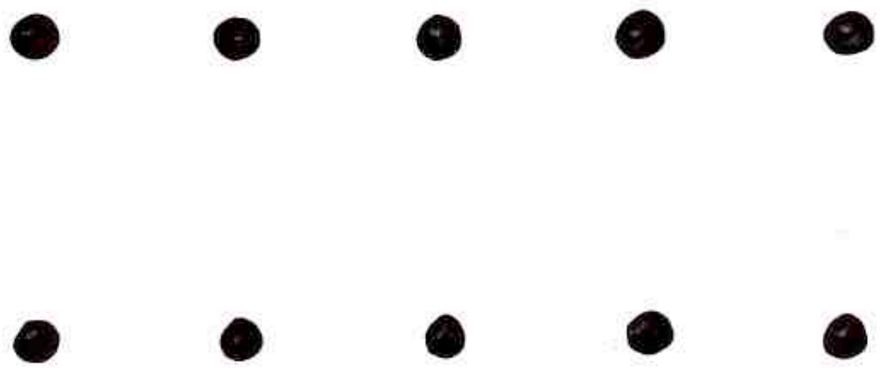
Primitive operations
for “dictionnaries” data structure:
add or delete any elements, asking questions (with positive or negative answer)



number of choices for each
primitive operations

priority queue

Polya urn



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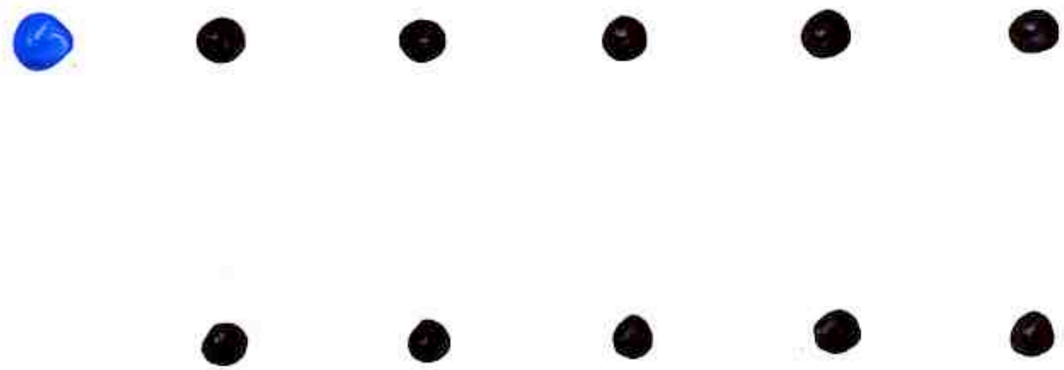


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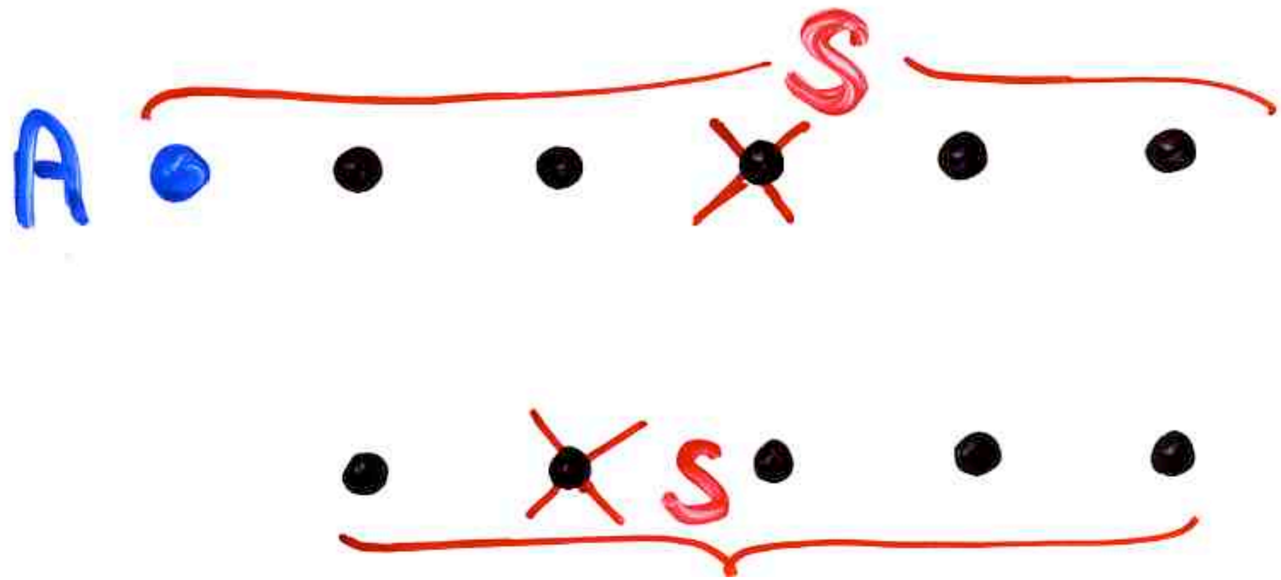
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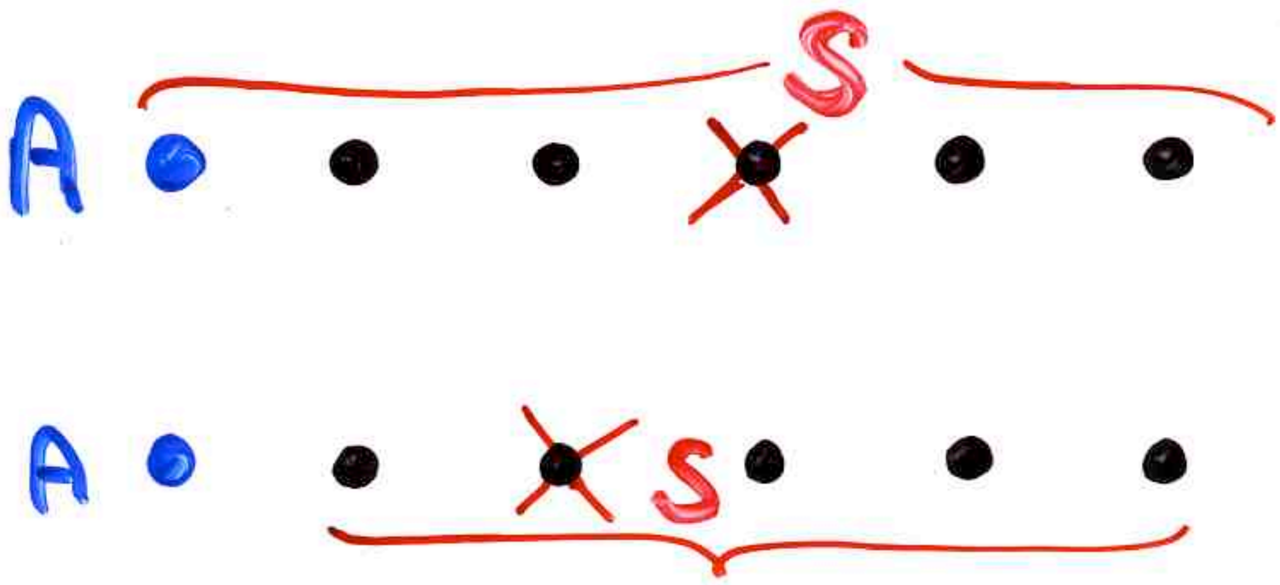
$AS - SA = I$



A produit par x

S $\frac{d}{dx}(\ )$

polynôme d'Hermite $H_n(x)$

$\lambda_k = k$; $b_k = 0$

$(k \geq 1)$ $(k \geq 0)$

$a_k = 1$ $\begin{cases} b'_k = 0 \\ b''_k = 0 \end{cases}$ $c_k = k$

Histoires d'Hermite

Hermite history

$$h = (\omega \; ; \; f)$$

ω: Dyck path

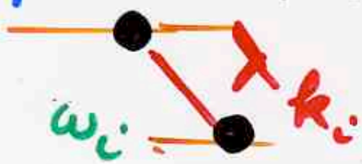
f: choice function

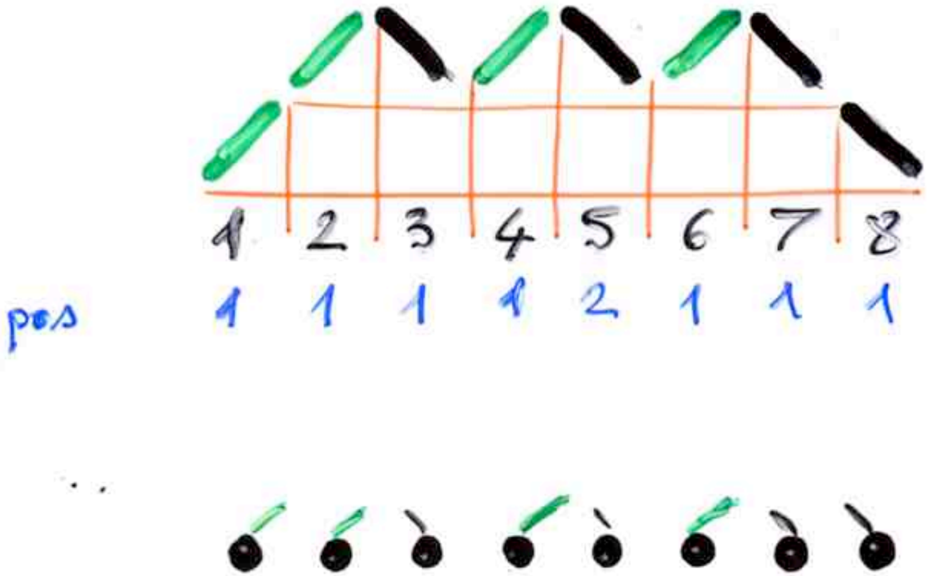
$\omega = \omega_1 \dots \omega_{2n}$

$f = (p_1, \dots, p_{2n})$

$p_i = 1$

$1 \le p_i \le v(\omega_i) = \lambda_{k_i}$





The cellular Ansatz

From quadratic algebra Q
to combinatorial objects (Q-tableaux)
and bijections

"The cellular Ansatz"

combinatorial
objects
on a 2d lattice

representation
by operators

Physics

data structures
"histories"

bijections

orthogonal
polynomials

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

$DE = qED + E + D$

PASEP

towers placements

permutations $\longleftrightarrow$ pairs of Tableaux Young

RSK

tableaux alternatifs $\longleftrightarrow$ permutations

Laguerre histories

Q-tableaux

ex: ASM,
(alternating sign matrices)
FPL(fully packed loops)
tilings, 8-vertex

quadratic algebra Q

?

commutations
rewriting rules

planarisation

planar
automata

Koszul algebras
duality

Thank you!