Canopée des arbres binaires
et
intervalles genevois de l'associaèdre

(2ème partie)

GT LaBRI
15 novembre 2013

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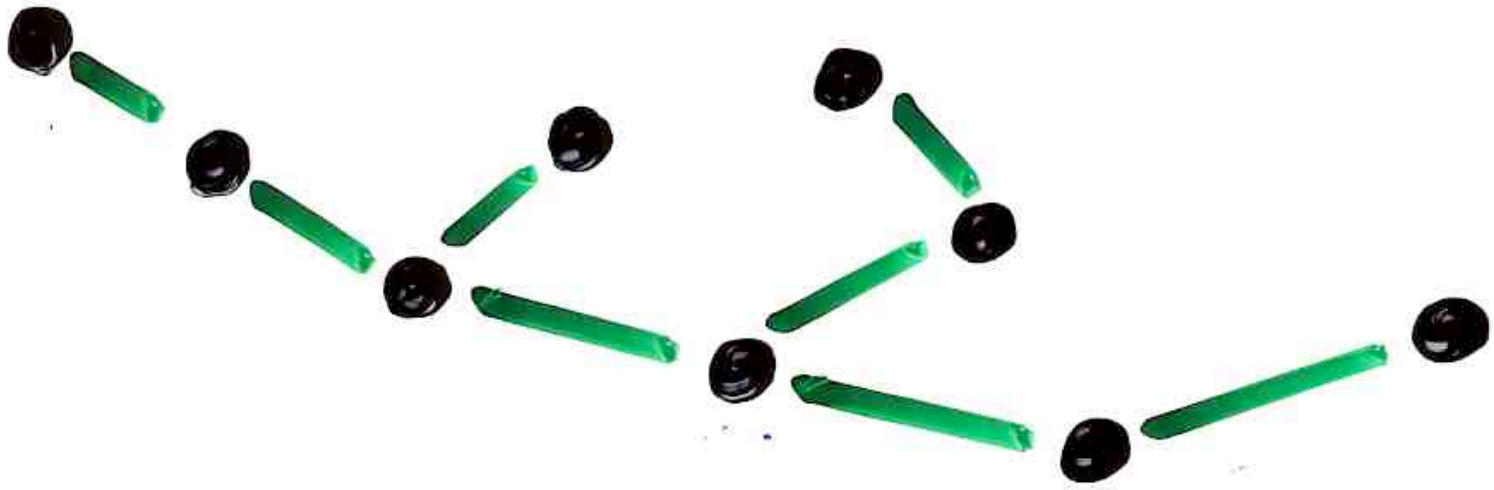
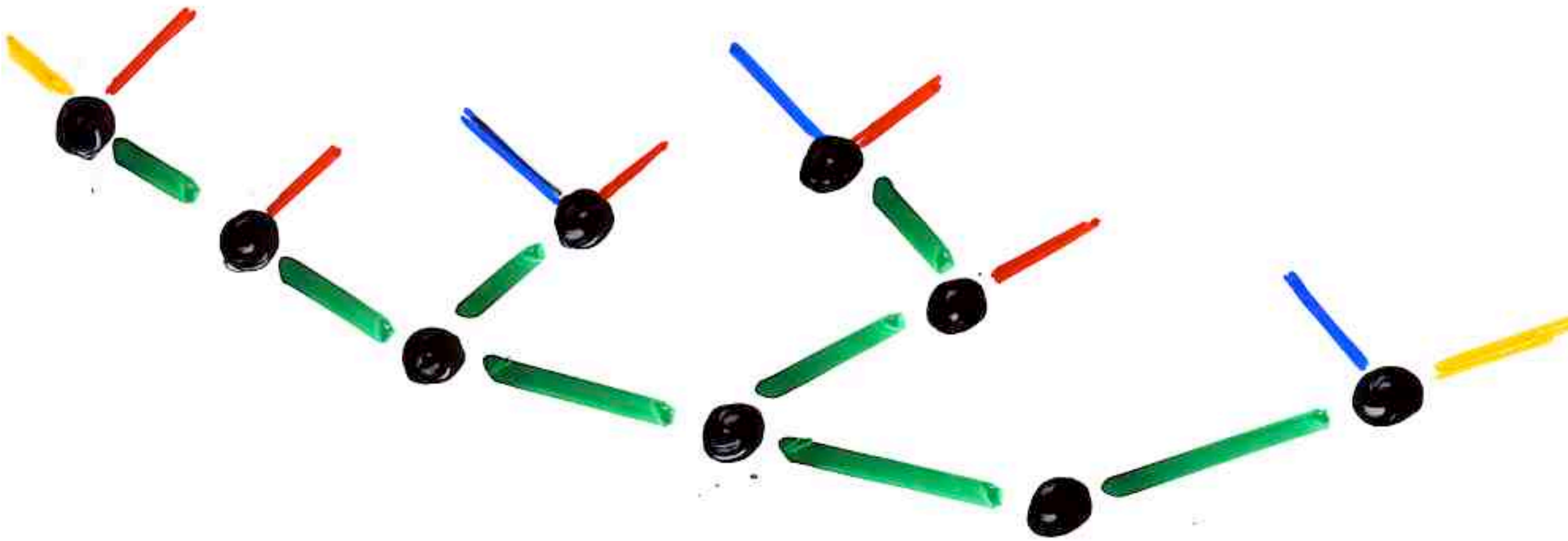
LaBRI, CNRS, Bordeaux

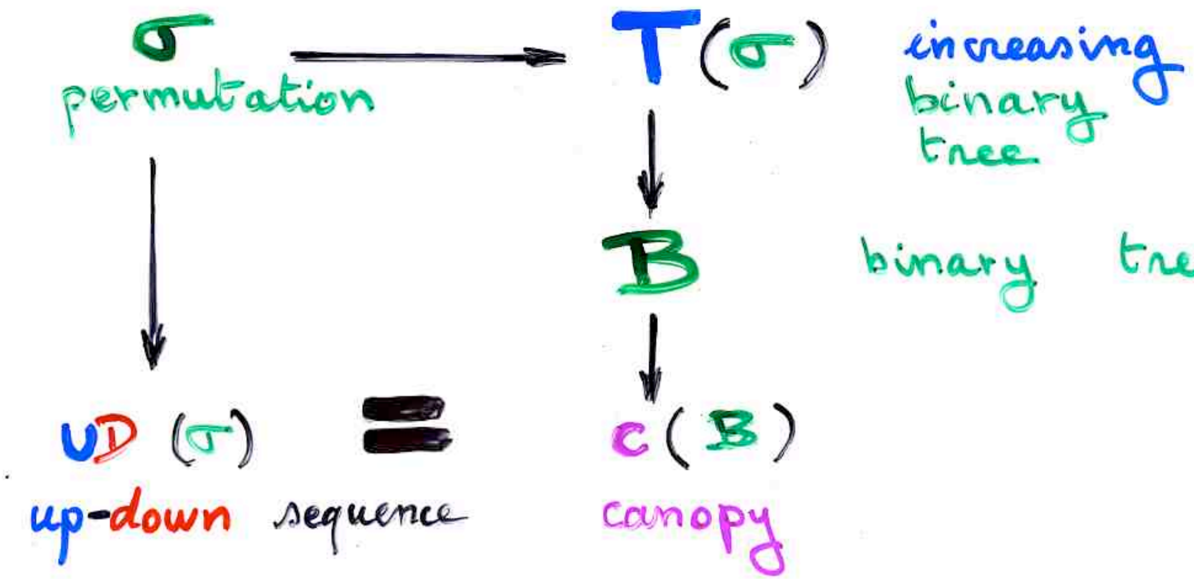
canopy of a binary tree

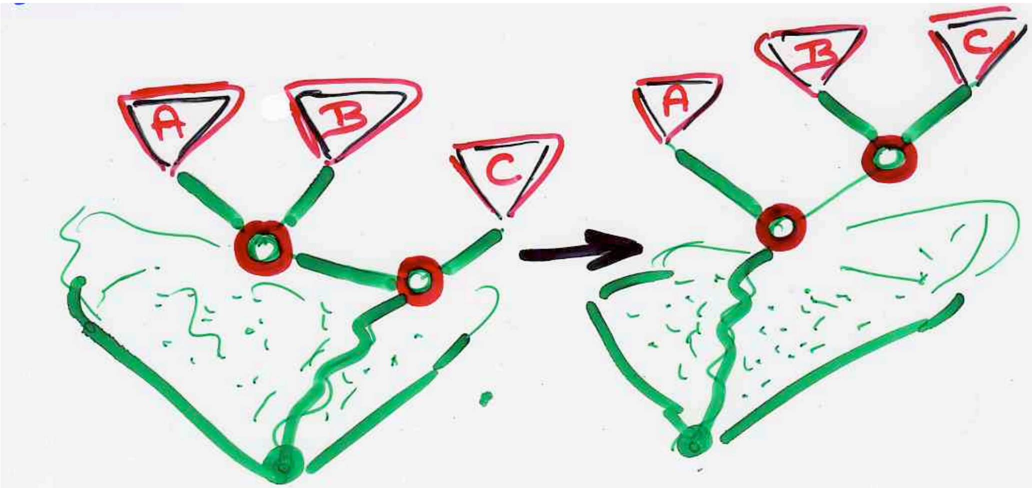


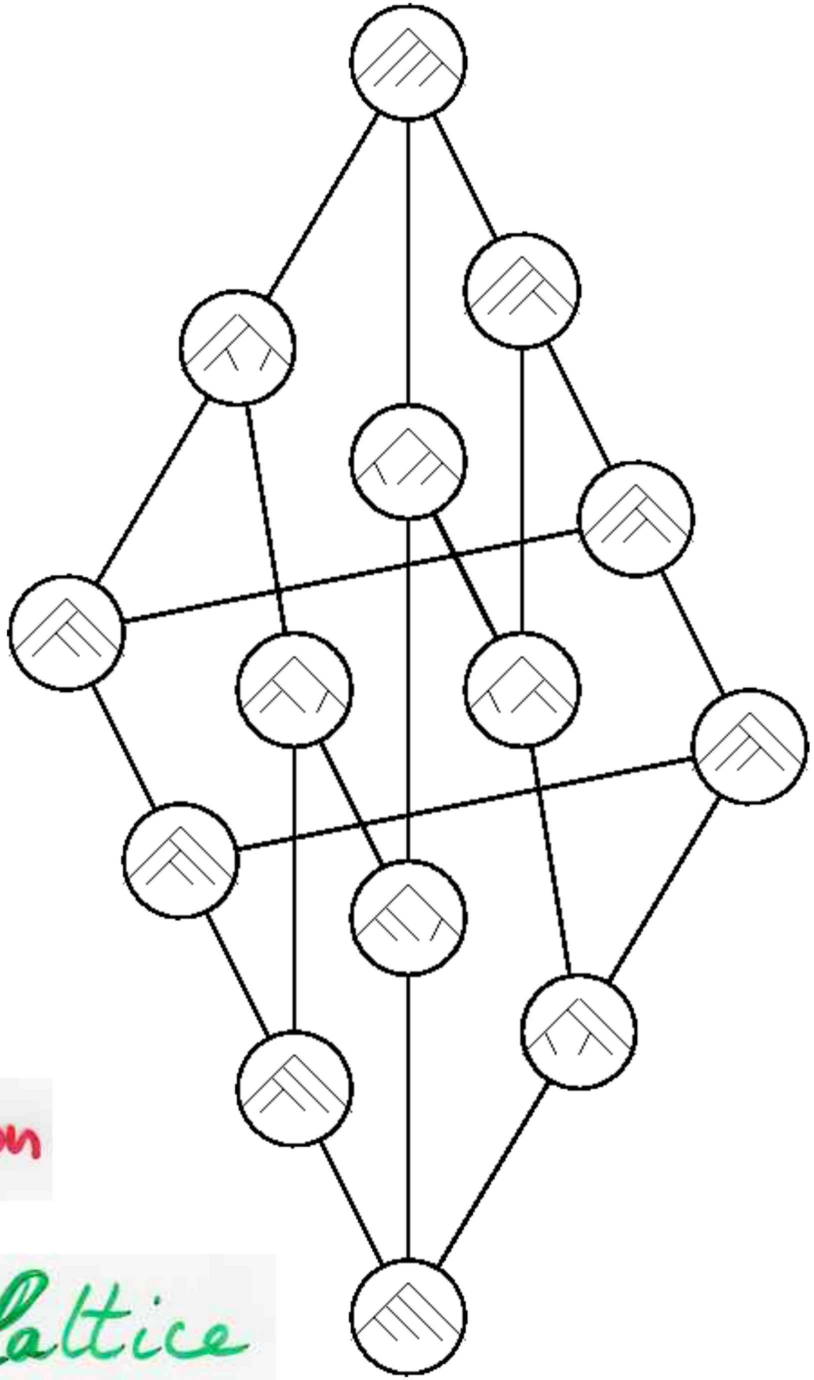


canopy of a binary tree

$C(B) = - - + - + - - +$

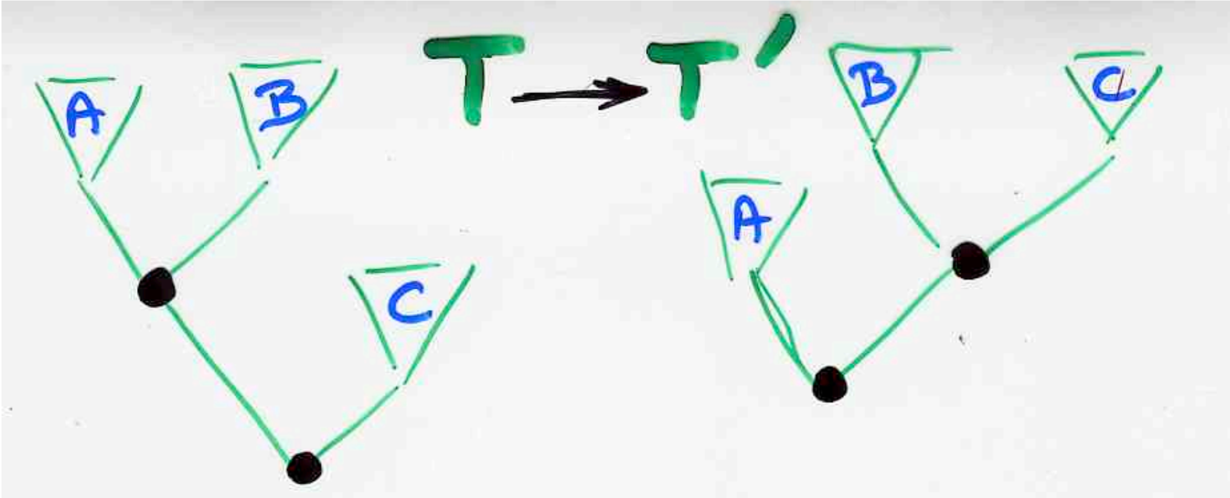

σ
permutation
T(σ)
increasing binary tree
B
binary tree
UD(σ)
up-down sequence
=
c(B)
canopy





order relation

Tamari lattice



if $B \neq \bullet$ canopy is invariant

if $B = \bullet$ canopy $c(T')$ not invariant

$$c(T) = c(A) \overset{+}{/} c(B)\, c(C)$$

$$c(T') = c(A) \underset{-}{\backslash} c(B)\, c(C)$$

$$c(\bullet) = \emptyset$$

the theorem

relating canopy and Tamari lattice

Prop- (i) • The set of binary trees having a given canopy w is an interval $J(w)$ of the Tamari lattice

(ii) • this interval can be extended to an initial interval of the Young lattice

i.e. $\exists$ (integer) partition μ such that $J(w)$ is in bijection with $I(\mu)$, the set of partition $\emptyset \preccurlyeq \lambda \preccurlyeq \mu$ (inclusion of Ferrers diagrams / Young)

with $T \leqslant T' \Rightarrow f(T) \geqslant f(T')$

Tamari lattice — Young lattice

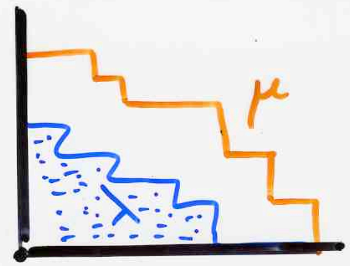
$$J(w) \xrightarrow[\text{bijection}]{f} I(\mu)$$

Young lattice

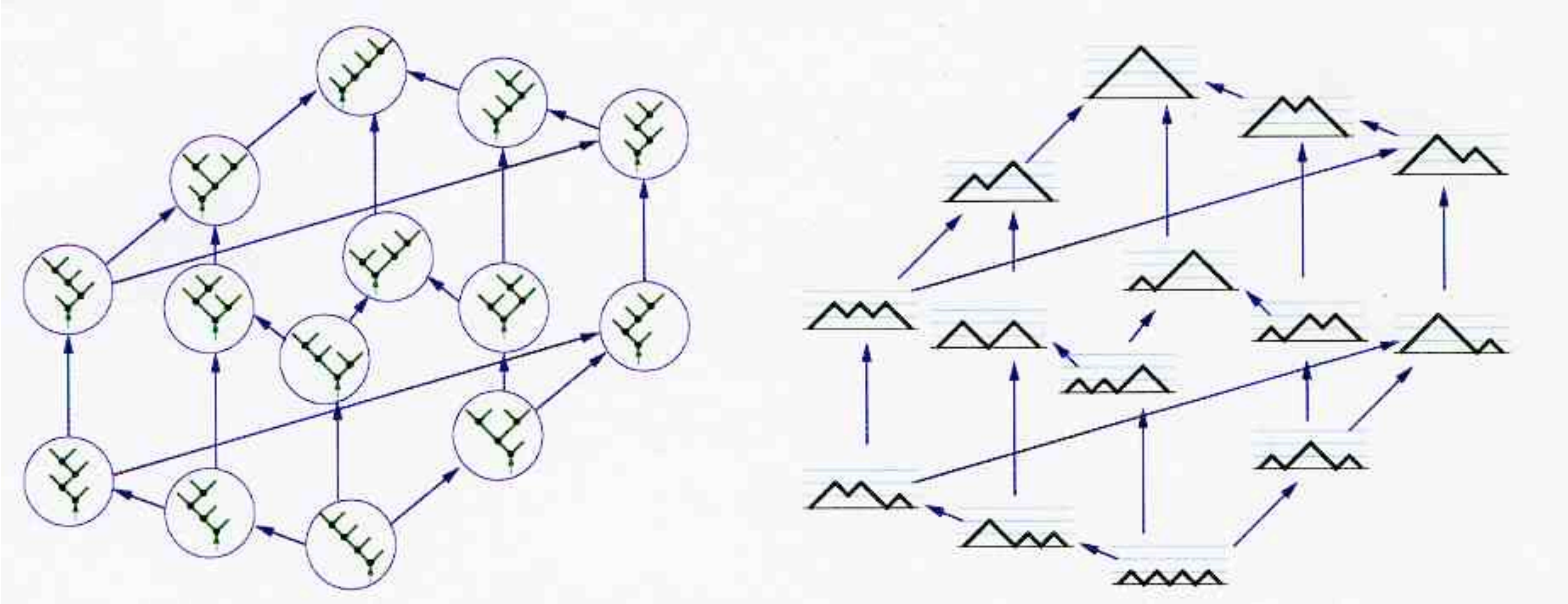
partition μ

$I(\mu)$

$\lambda \preccurlyeq \mu$

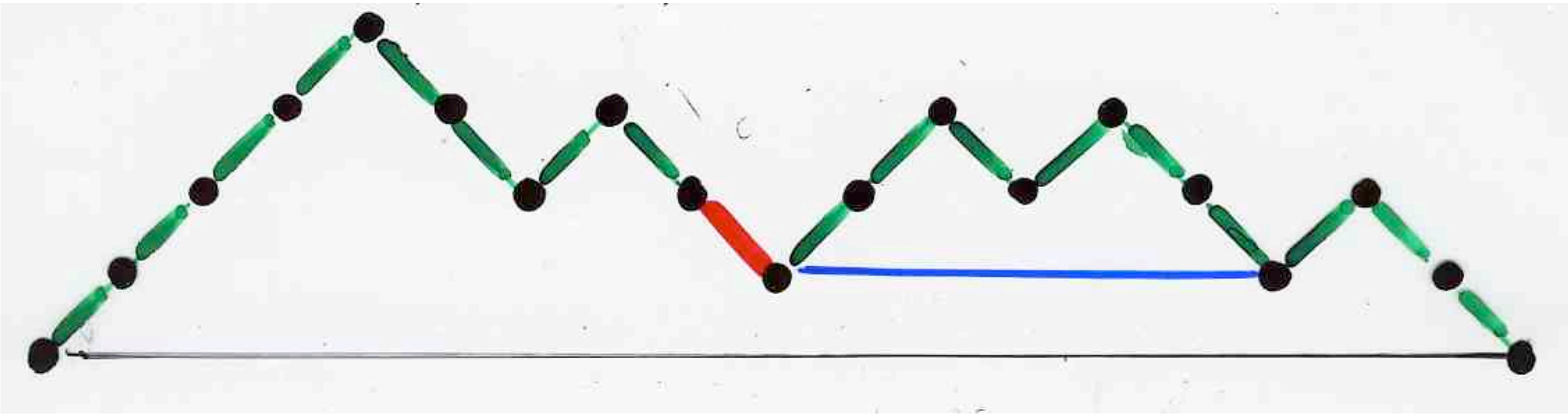


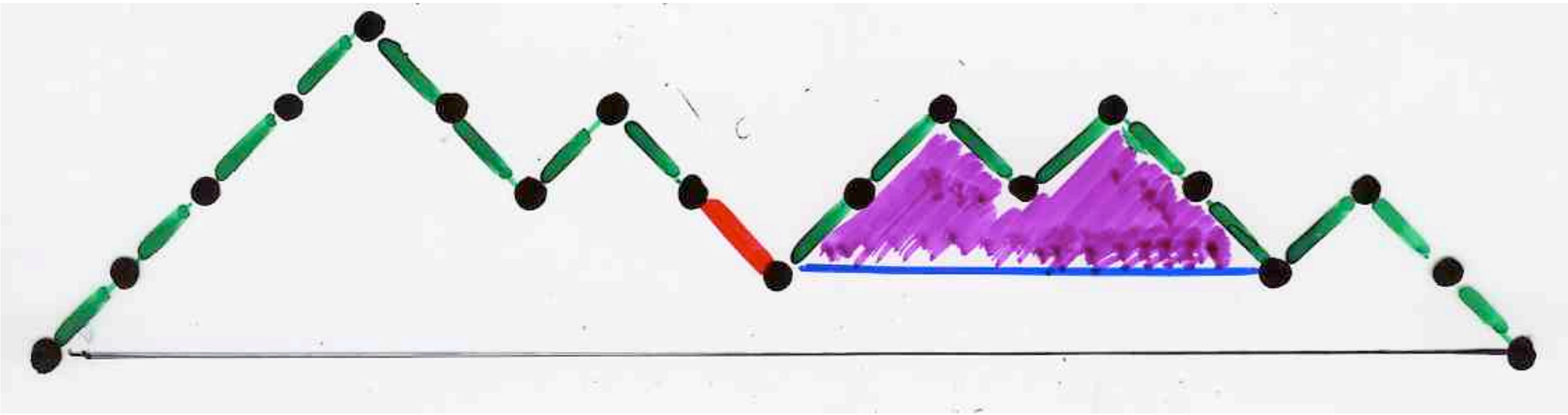
initial segment in the Young lattice (or lower ideal)



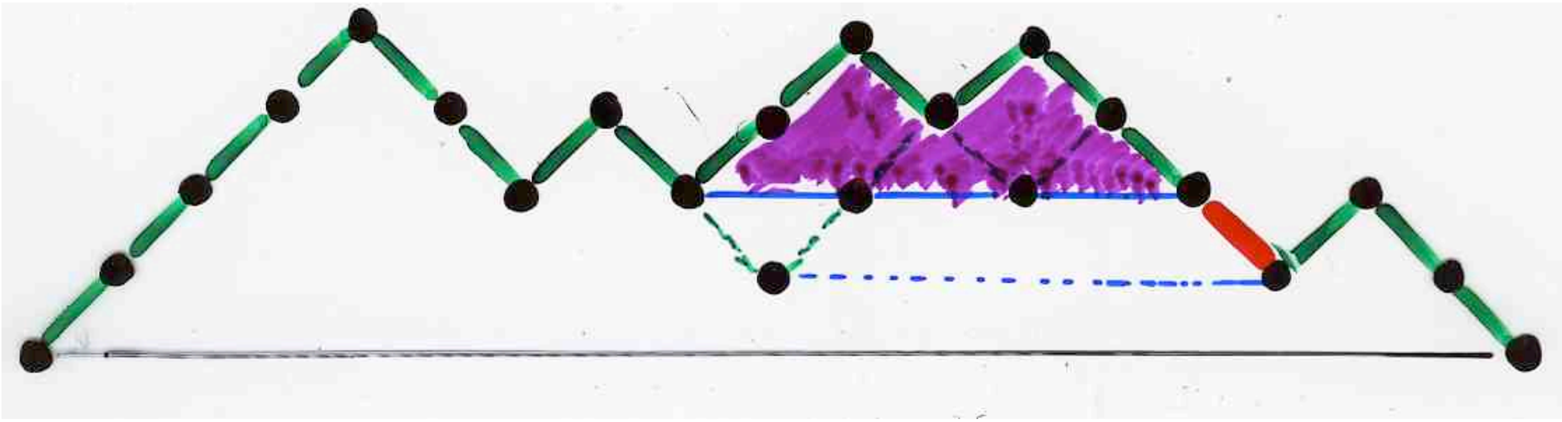
$C_4 = 14$

Catalan





facteur Dyck primitif



facteur Dyck primitif

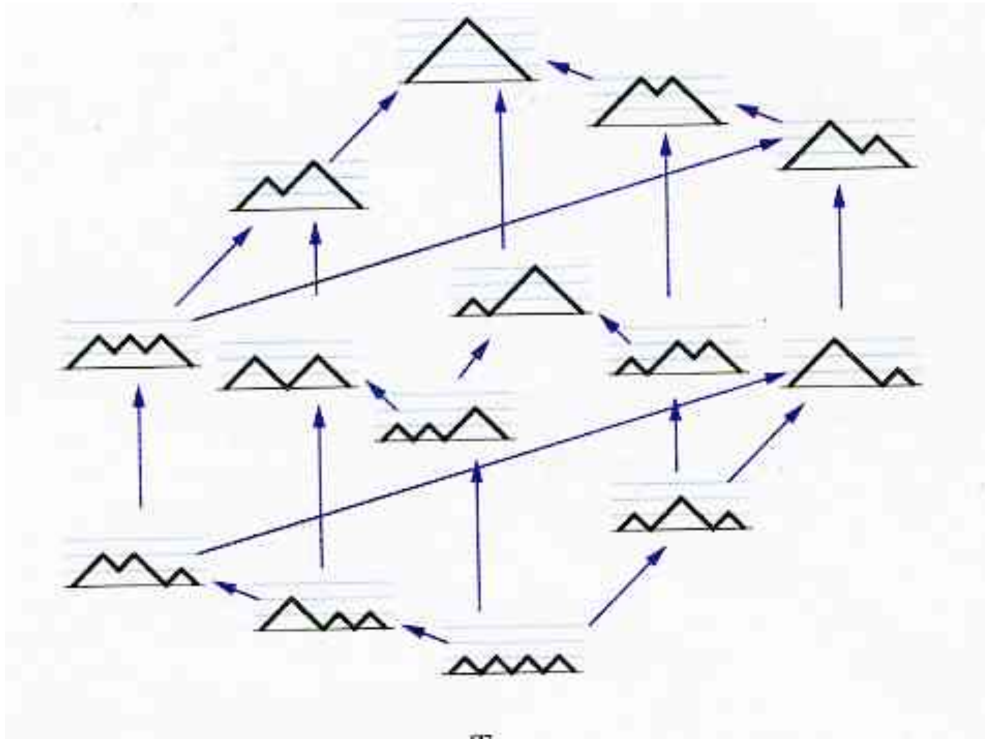
If $T \leqslant T'$ in $(\text{Tamari})_n$ lattice

then $T \leqslant T'$ in $(\text{Dyck})_n$ lattice

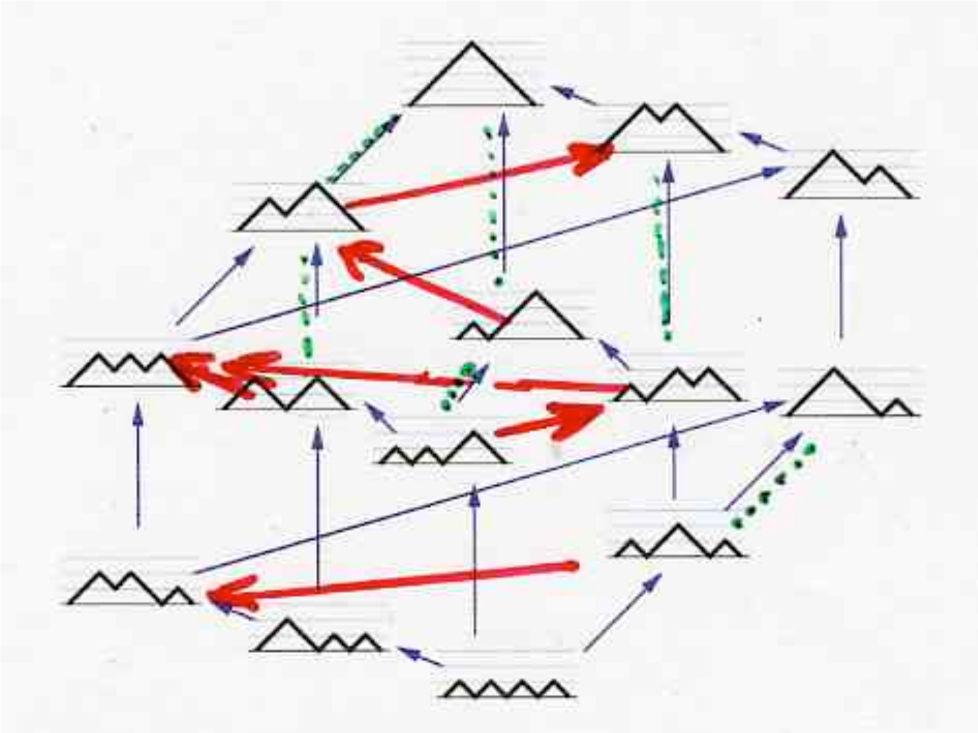
[i.e. T below T']

converse not true

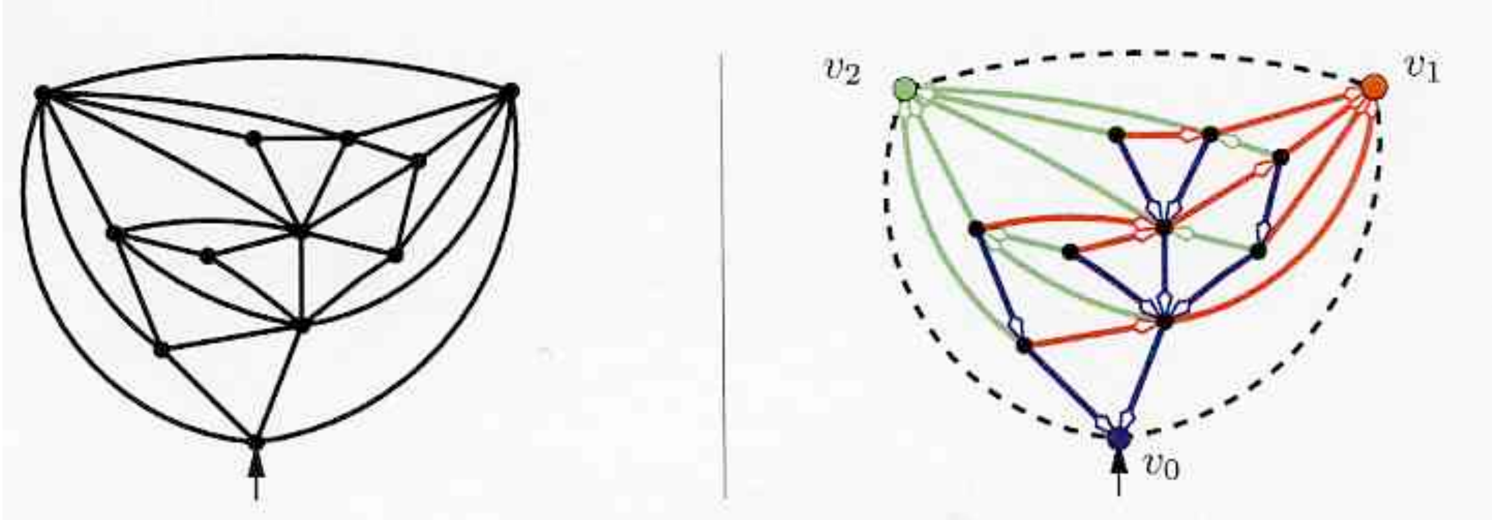
$(\text{Dyck})_n$ extension of $(\text{Tamari})_n$



$\left(\text{Tamari lattice}\right)_4$



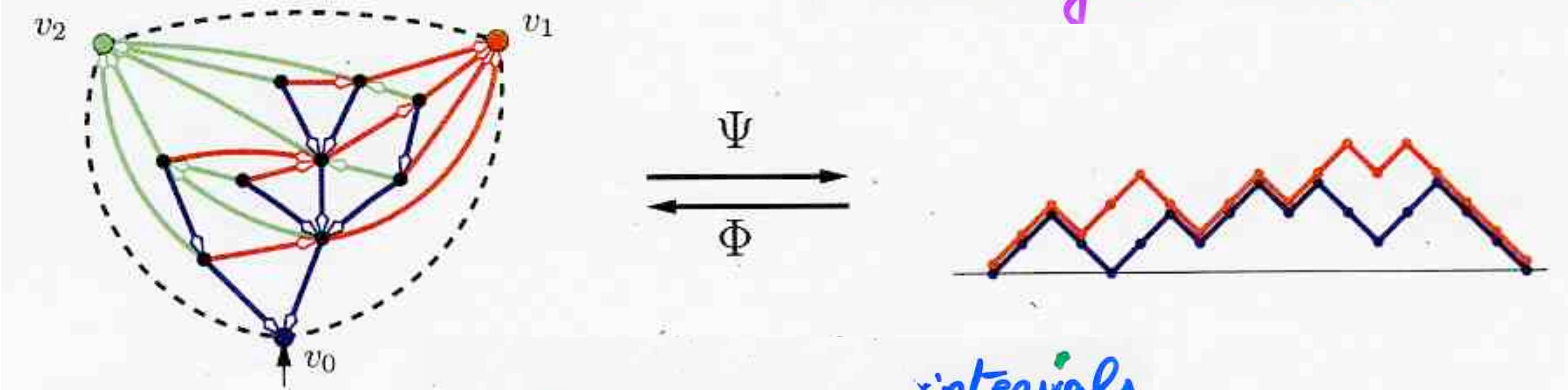
$(\text{Dyck lattice})_4$



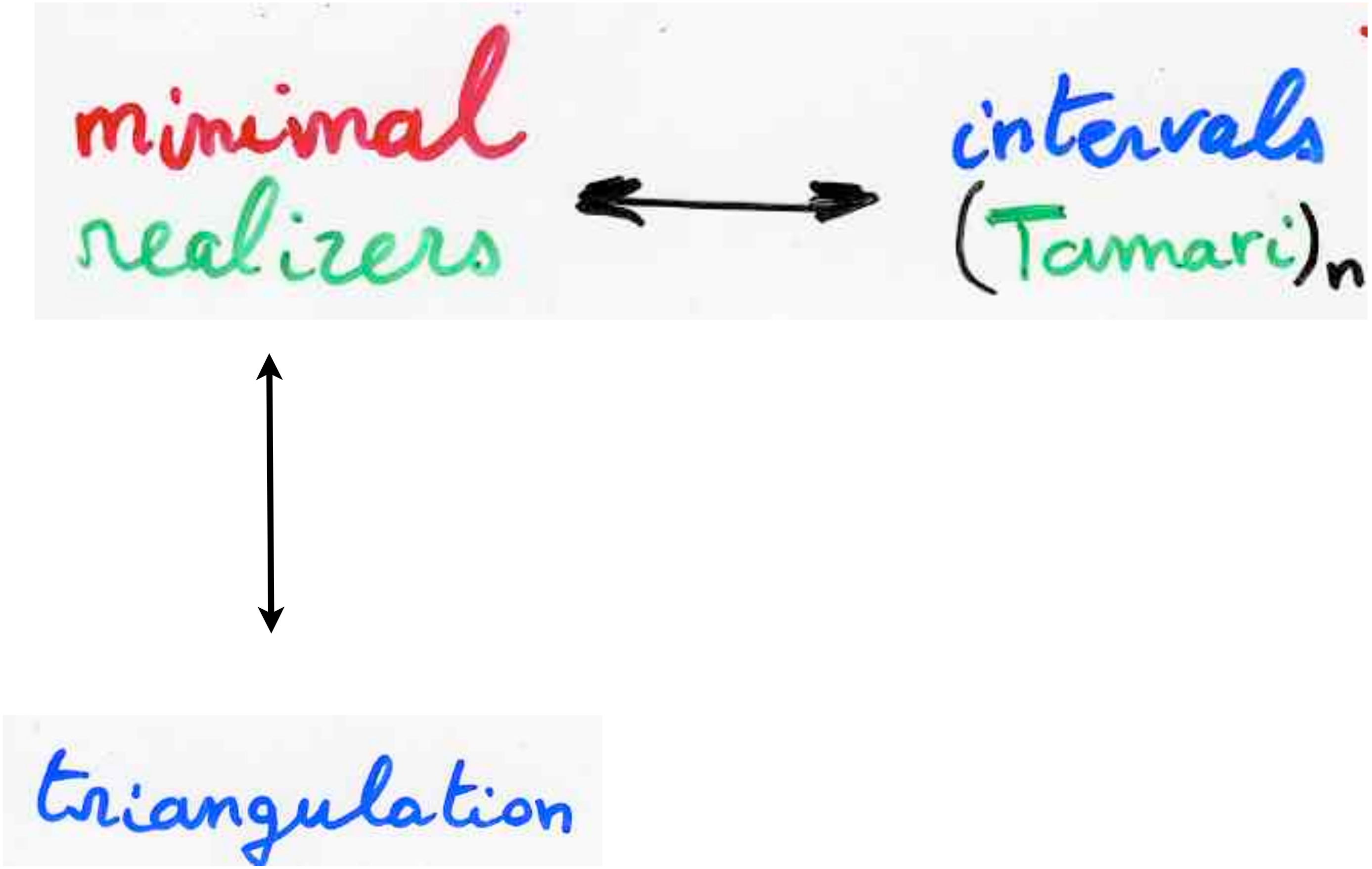
triangulation

realizers

or Schnyder woods



realizers $\longleftrightarrow$ intervals $(\text{Dyck})_n$


minimal realizers
intervals (Tamari)$_n$
triangulation

another Catalan bijection

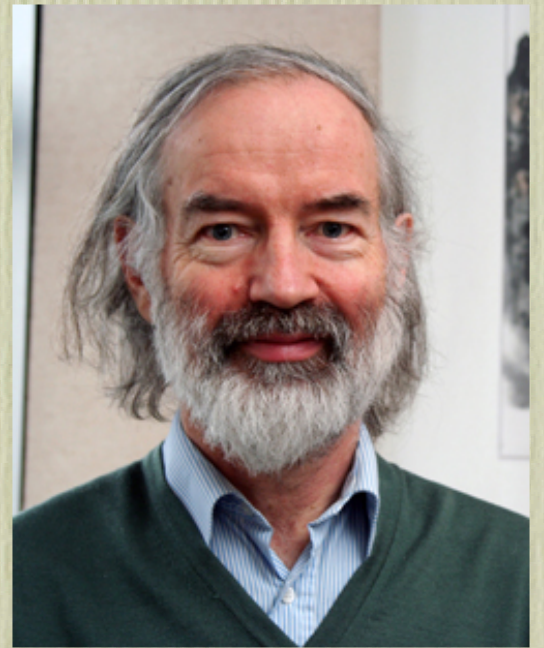
presented for the «LascouxFest 60th»

LascouxFest

52ème SLC

Domaine Saint-Jacques,

Otrrott, Mars 2004

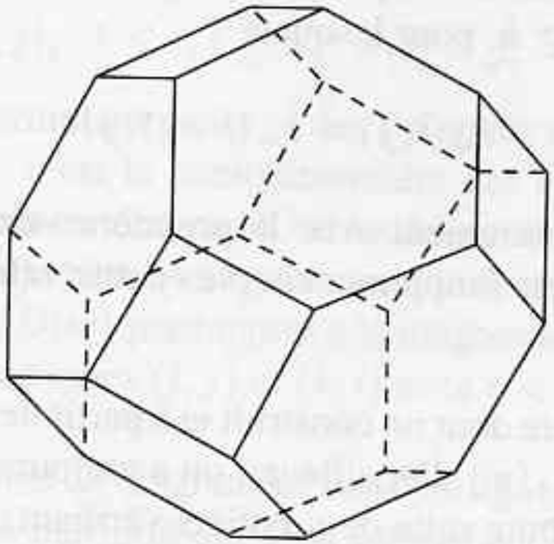


permutohedron

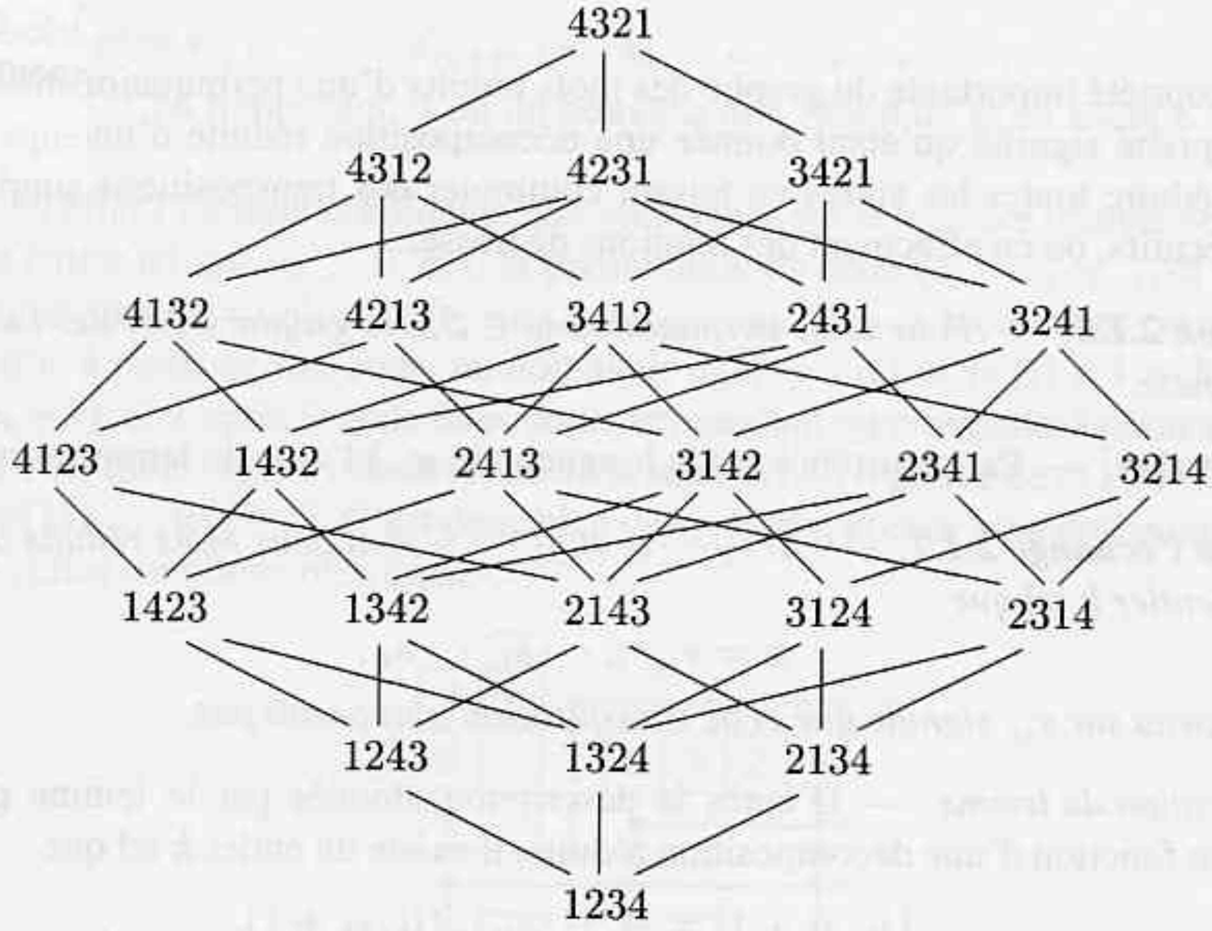
Alain Lascoux

(1944–2013)

permutohedron



2. Le permutoèdre Π_3.



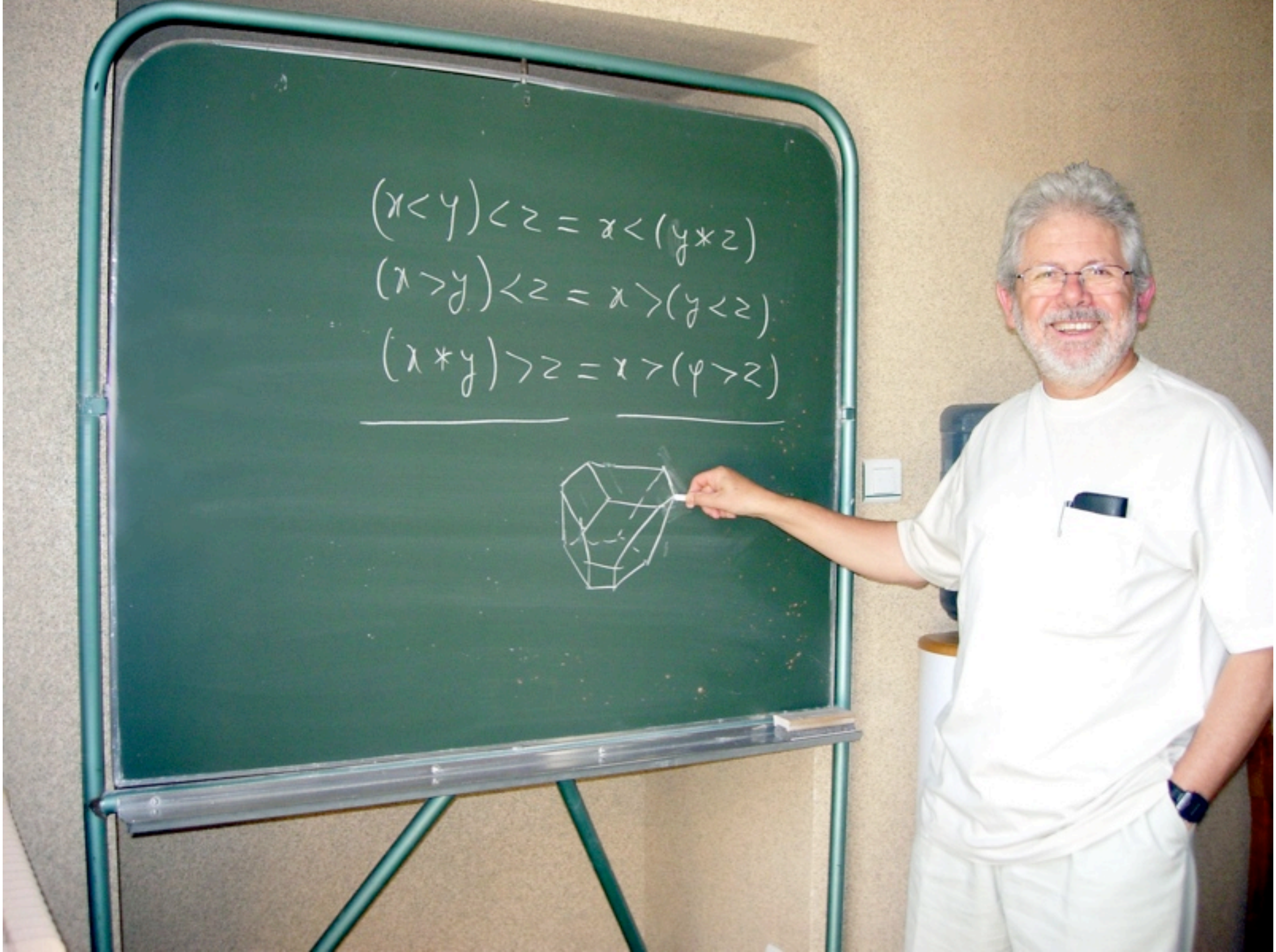
Gil Kalai

LYCEE d'Etat " LOUIS LE GRAND "
- P a r i s -

Année Scolaire
1963-1964

LYCEE d'Etat " LOUIS LE GRAND "
- P a r i s -

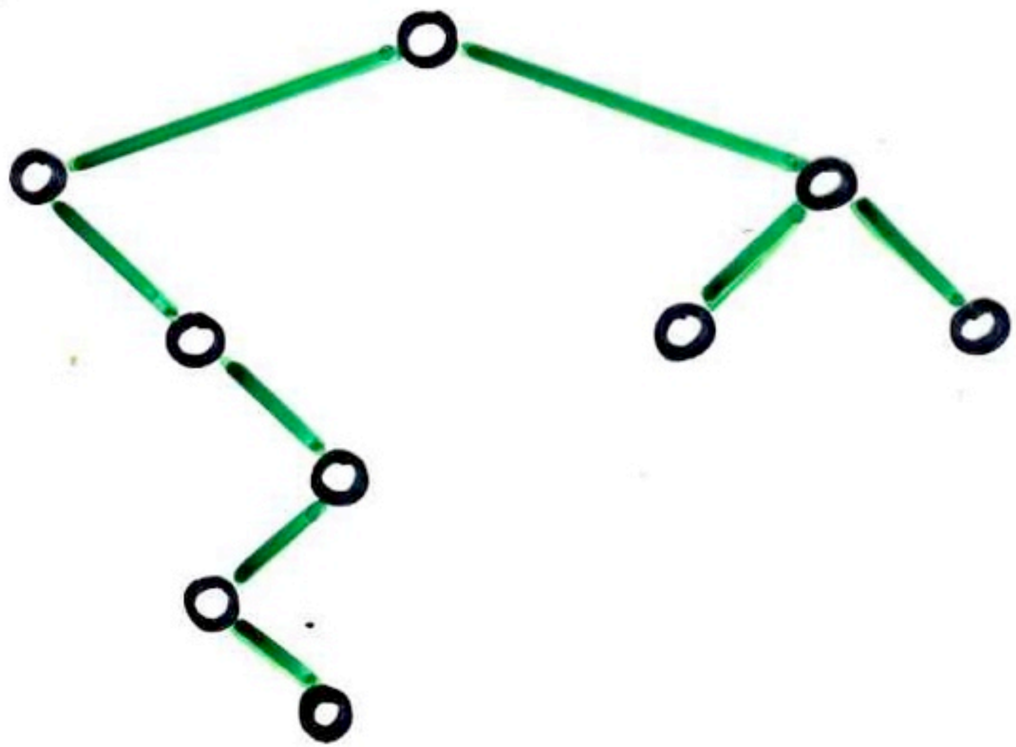
Année Scolaire
1963-1964


(x<y)<z = x<(y*z)
(x>y)<z = x>(y<z)
(x*y)>z = x>(φ>z)

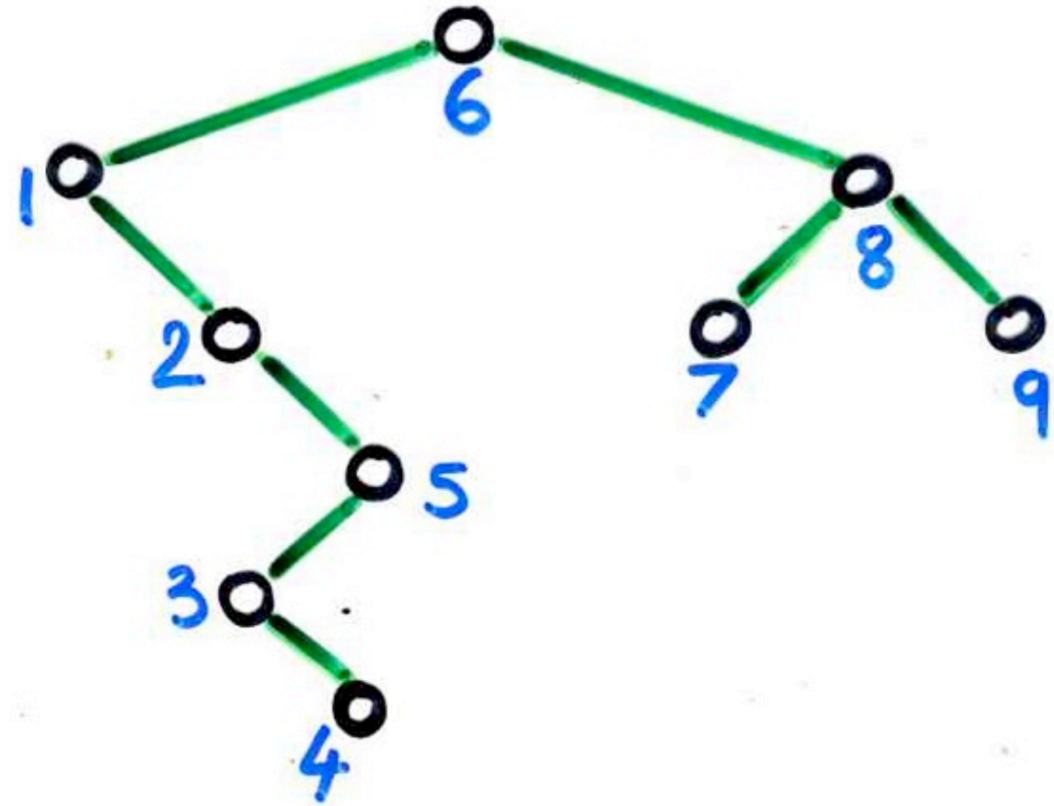
another Catalan bijection

presented for the «LascouxFest 60th»

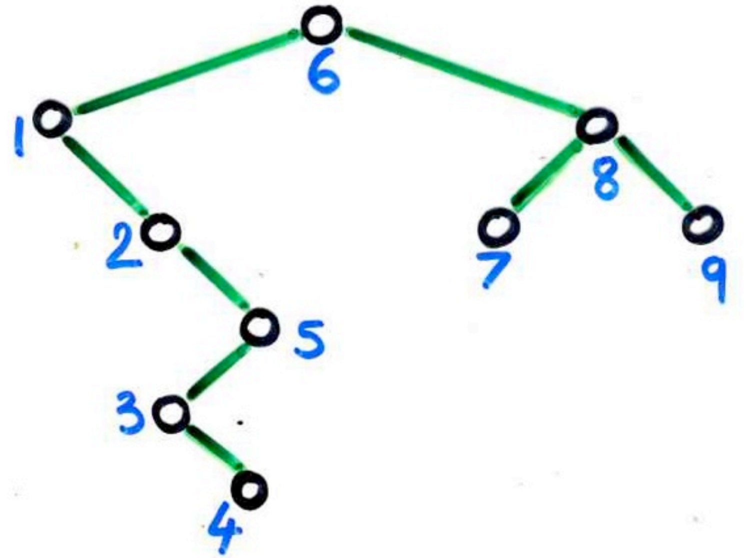
equivalent definition of the canopy

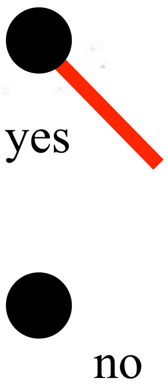


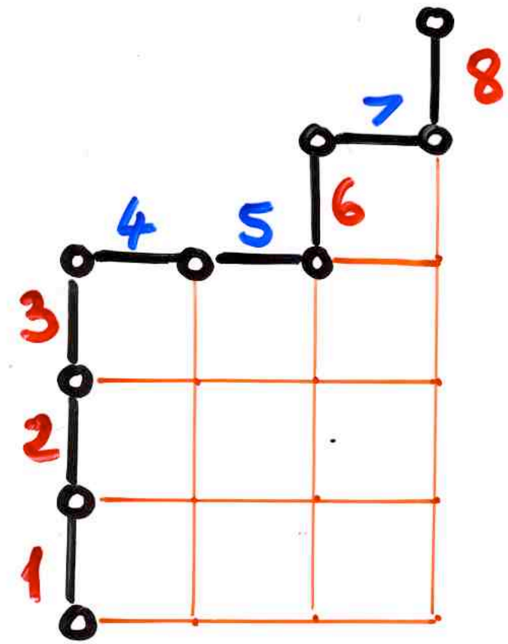
Symmetric order



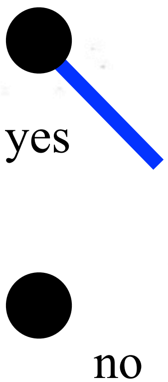
Symmetric order



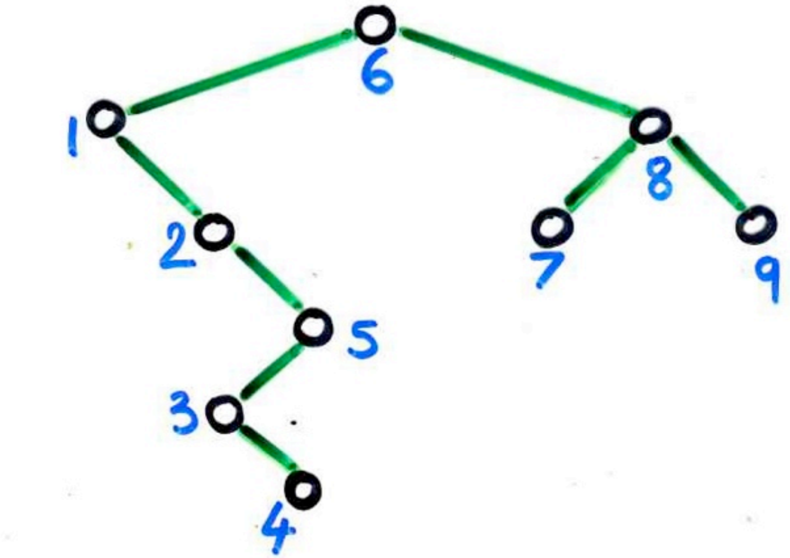




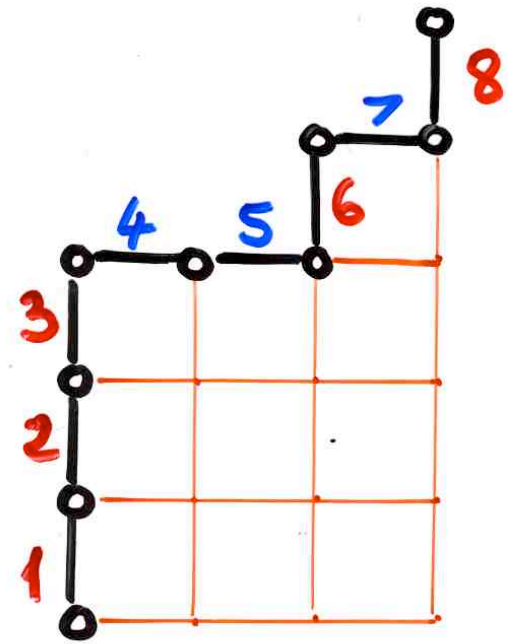
1
2
3
4
5
6
7
8

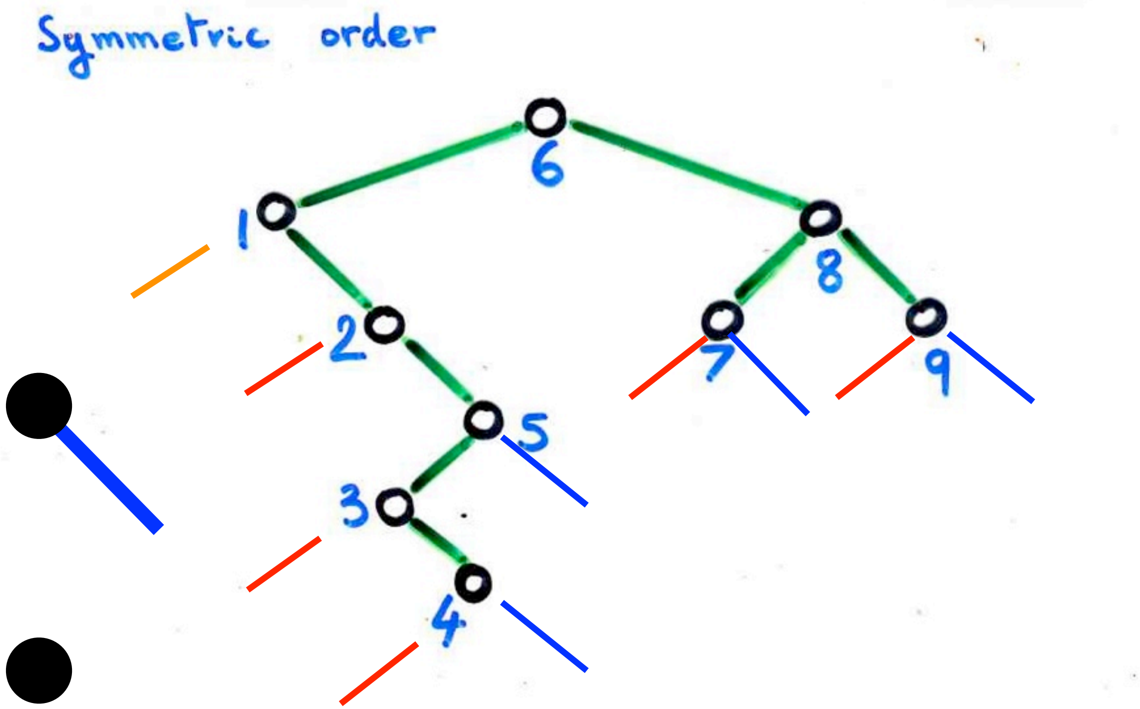


yes
no

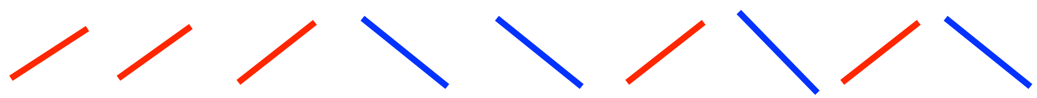


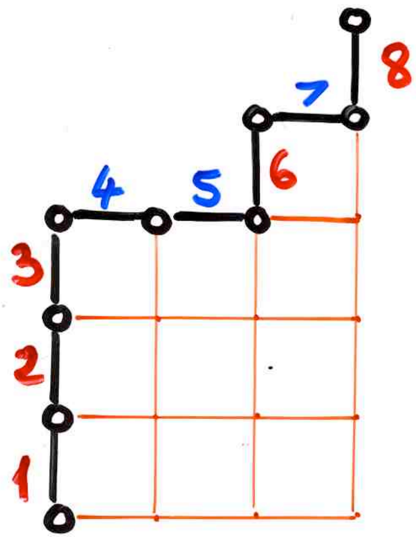
Symmetric order
6
1
8
2
7
9
5
3
4



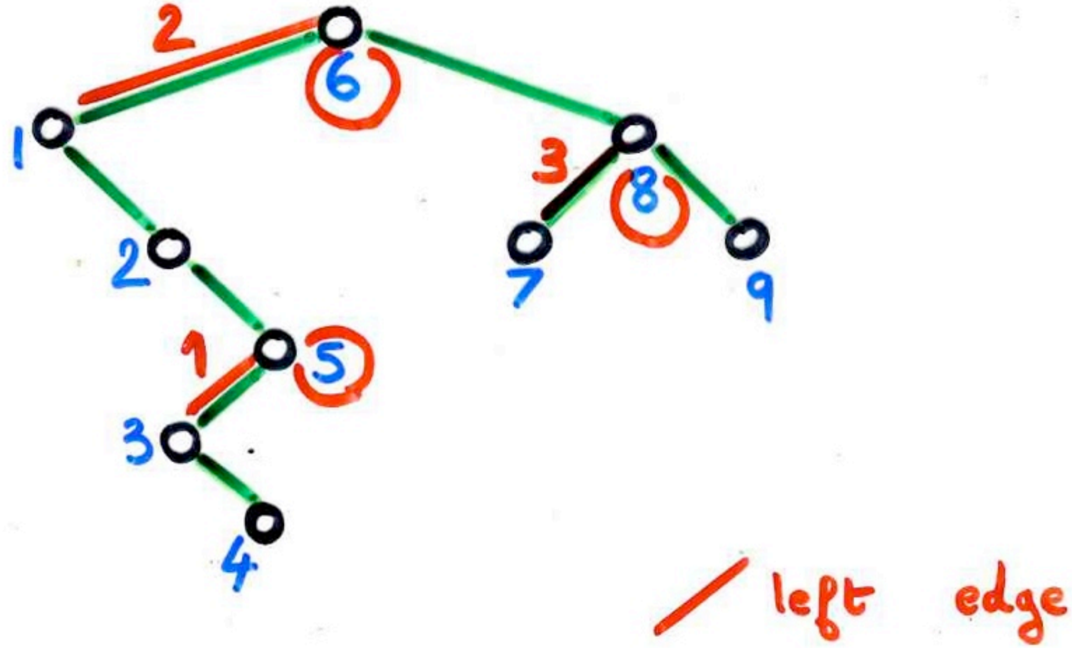


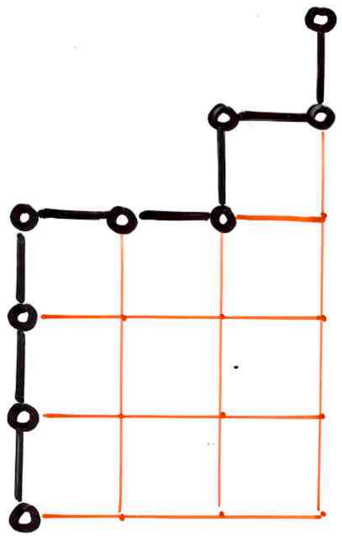
canopy =

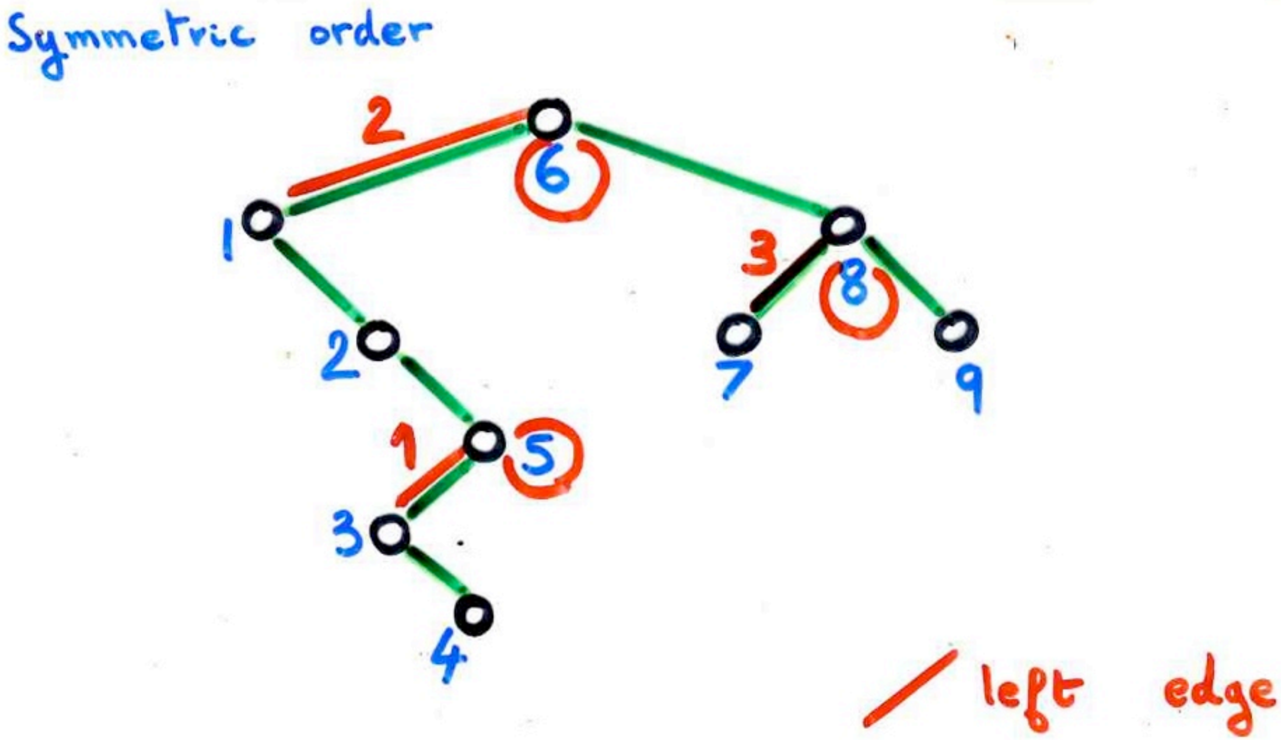




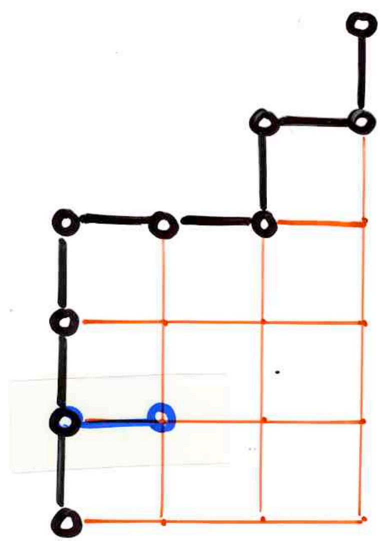
Symmetric order



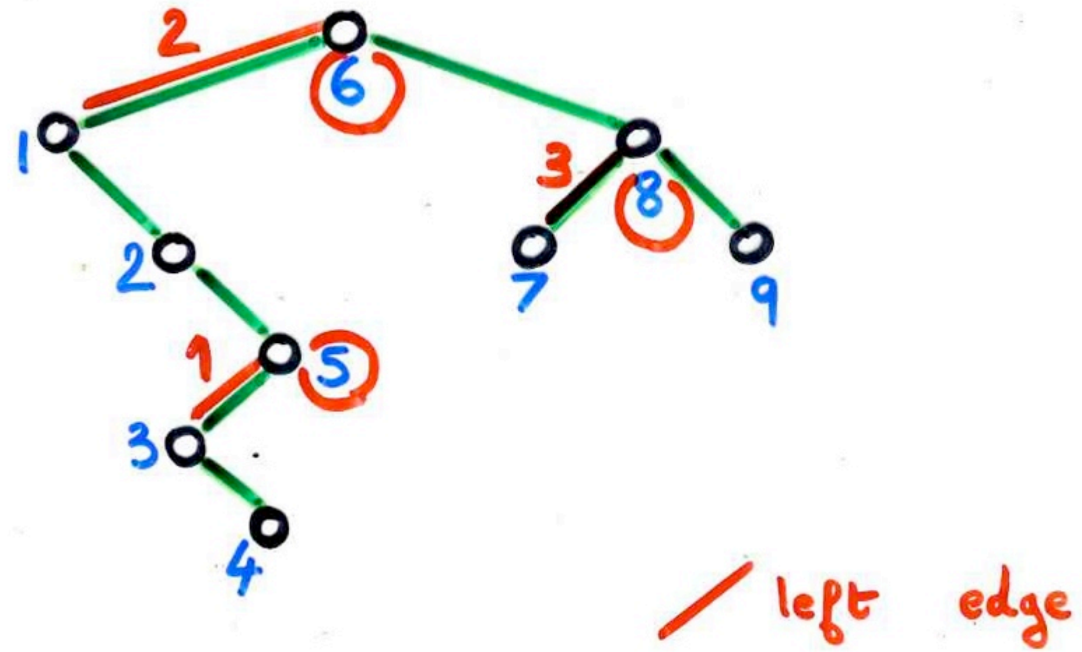




hauteur
à droite

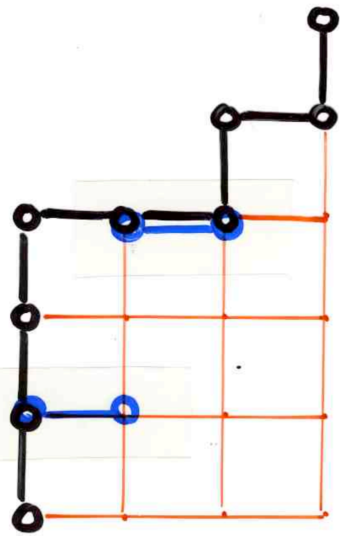


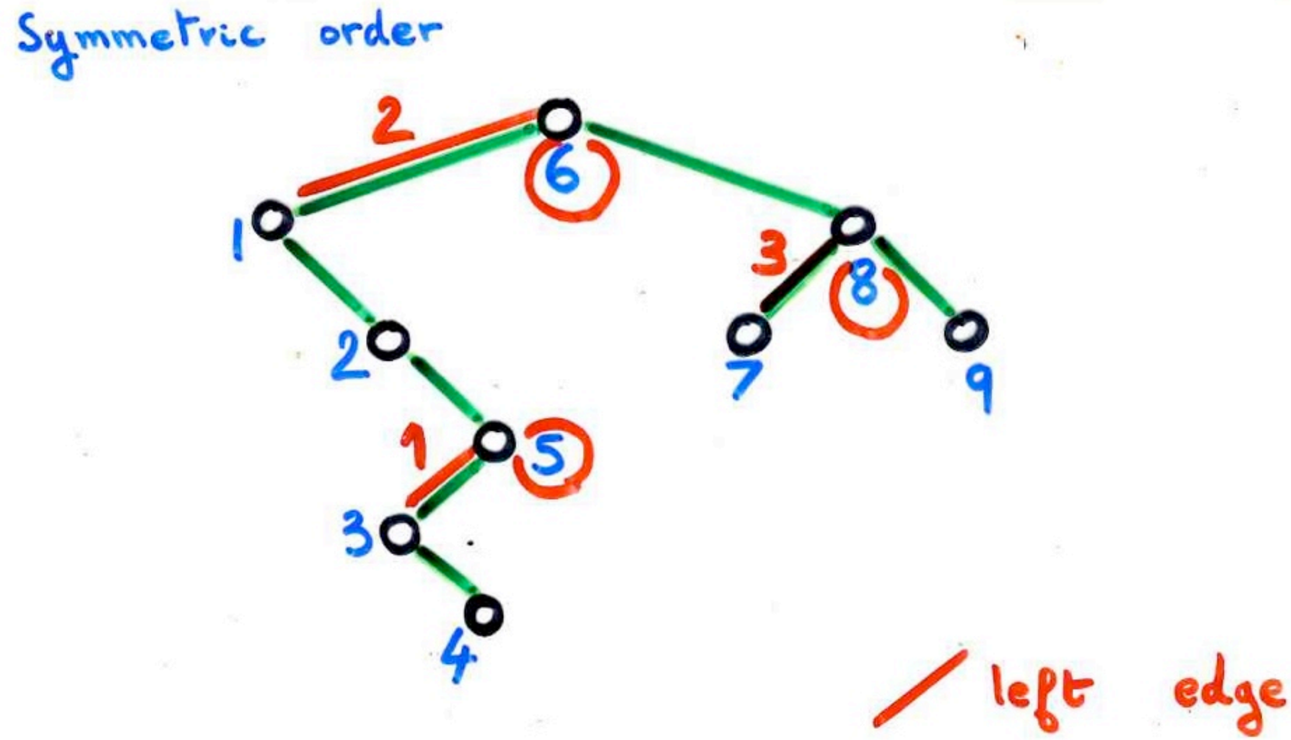
Symmetric order

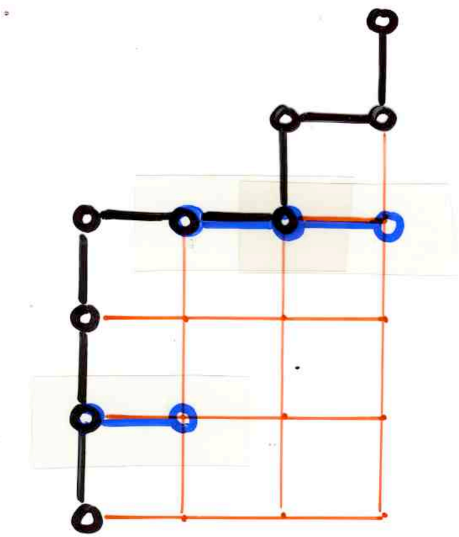


hauteur

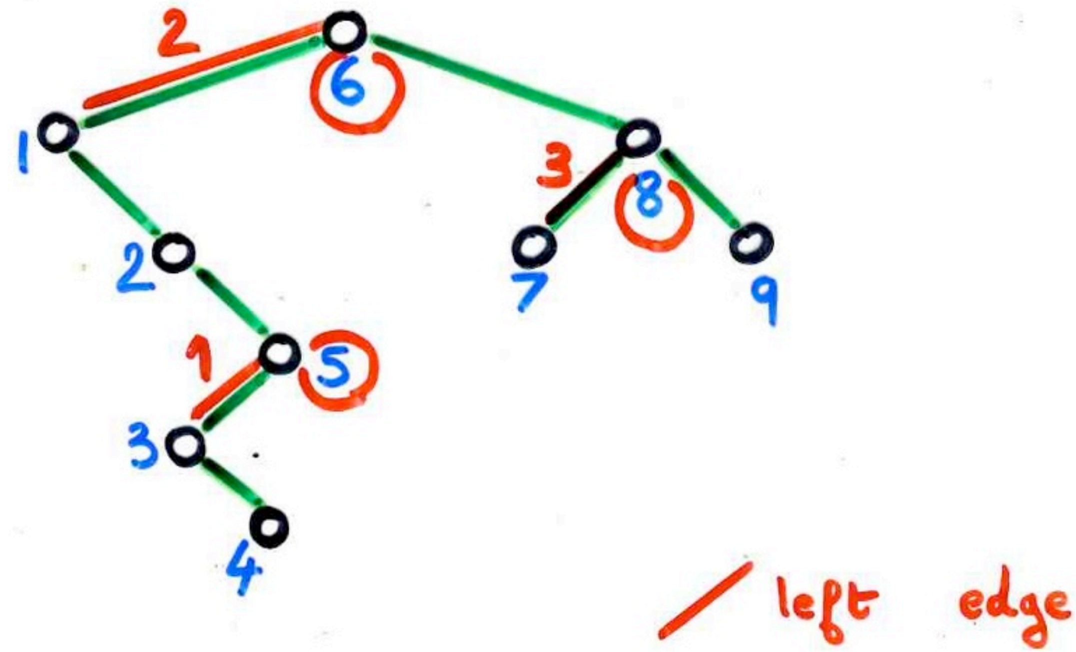
à droite

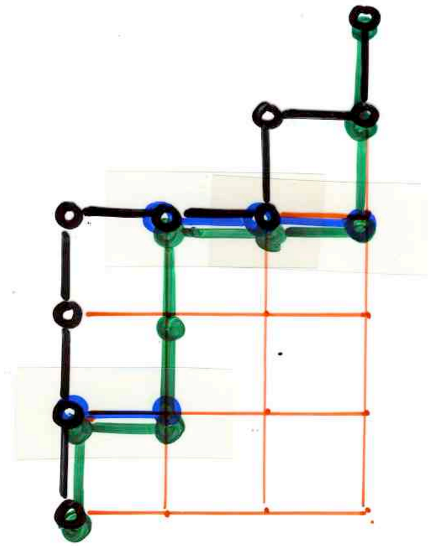



Symmetric order
1
2
3
4
5
6
7
8
9
1
2
3
left edge

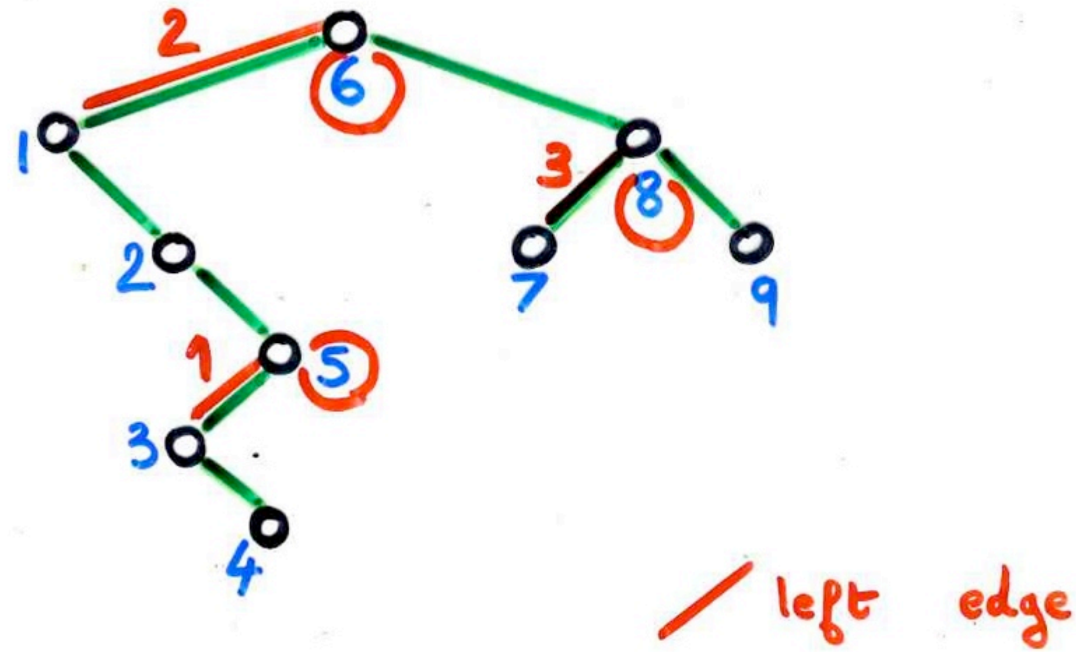


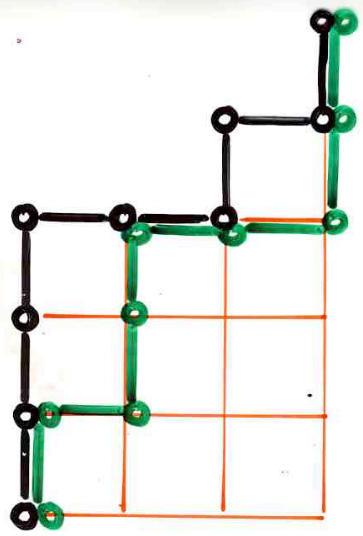
Symmetric order

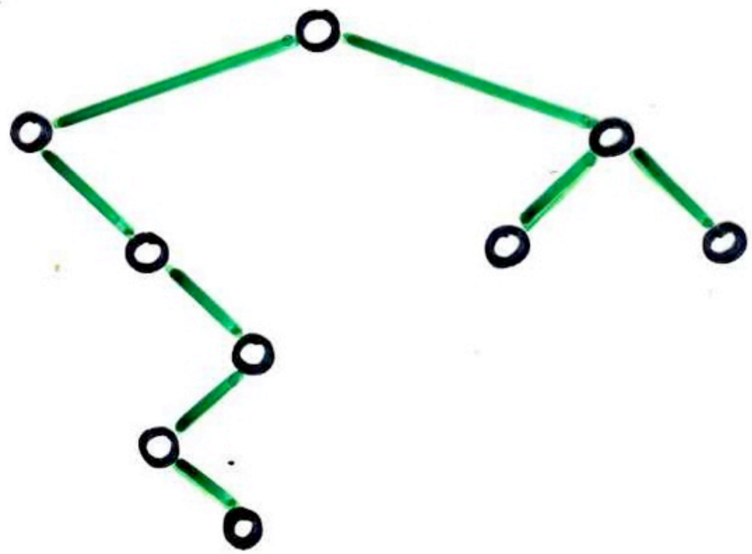




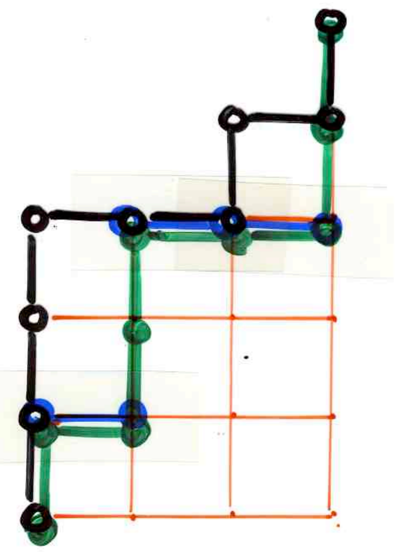
Symmetric order







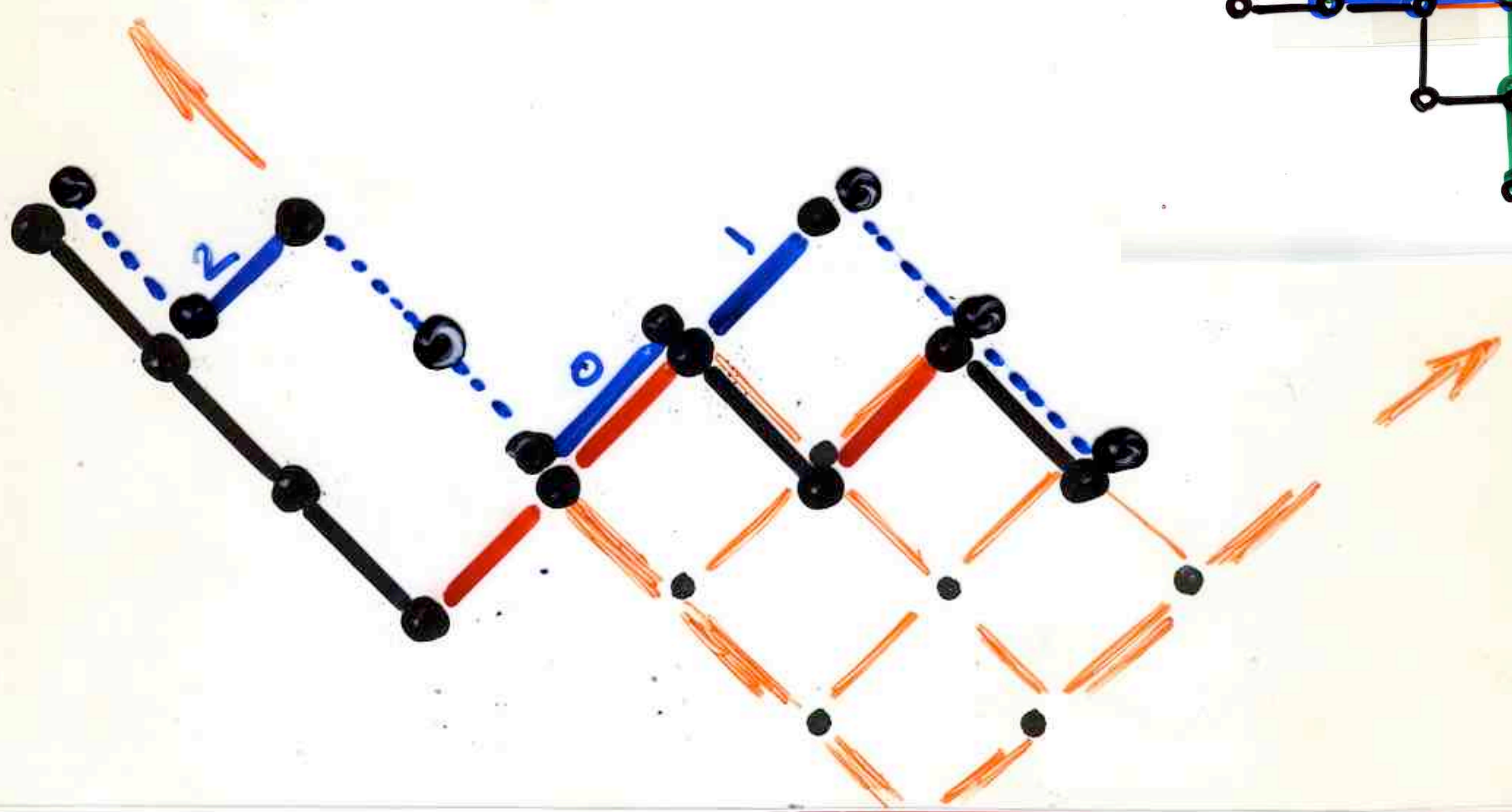
bijection inverse

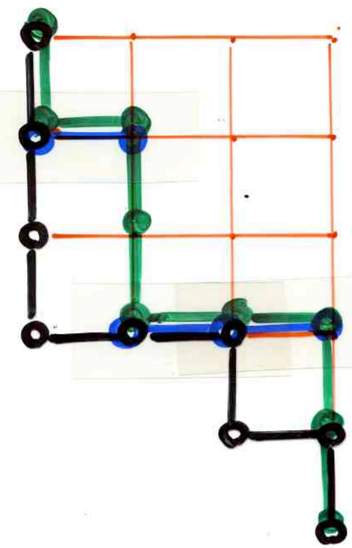


l' algorithme

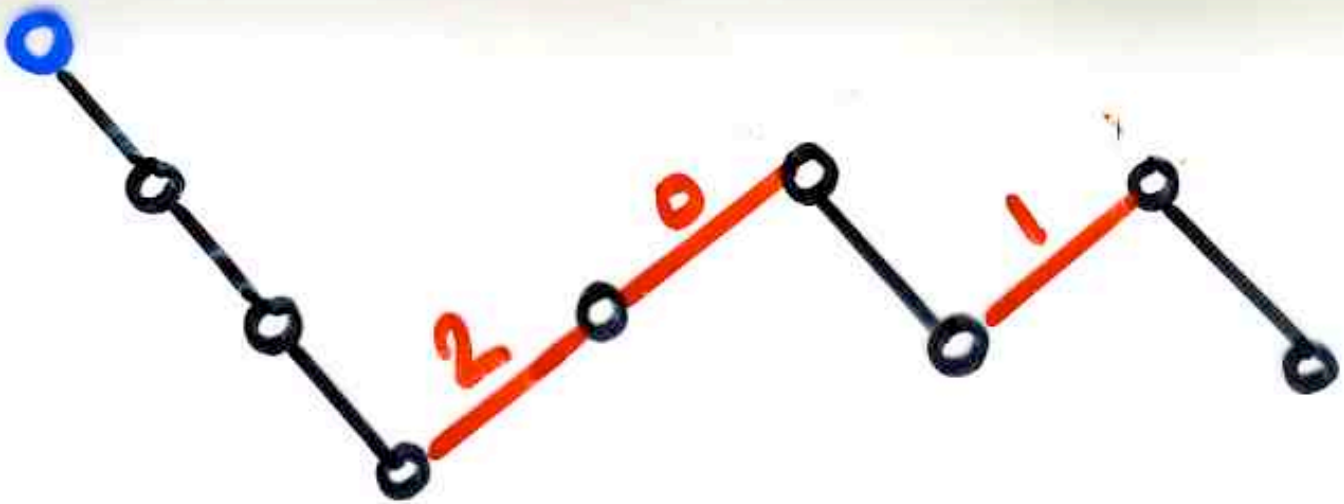
"glisser - pousser"

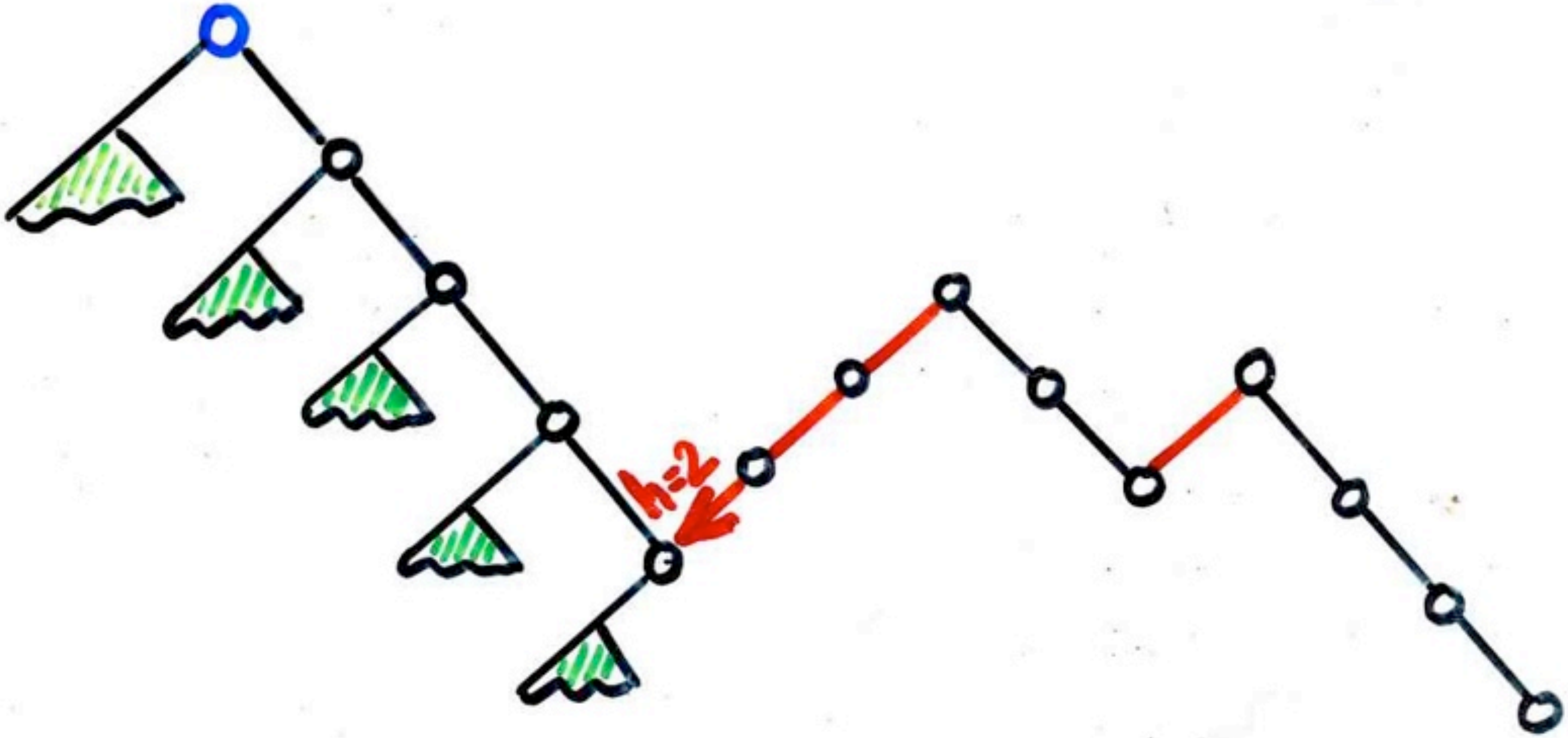
l'algorithme
"glisser - pousser"

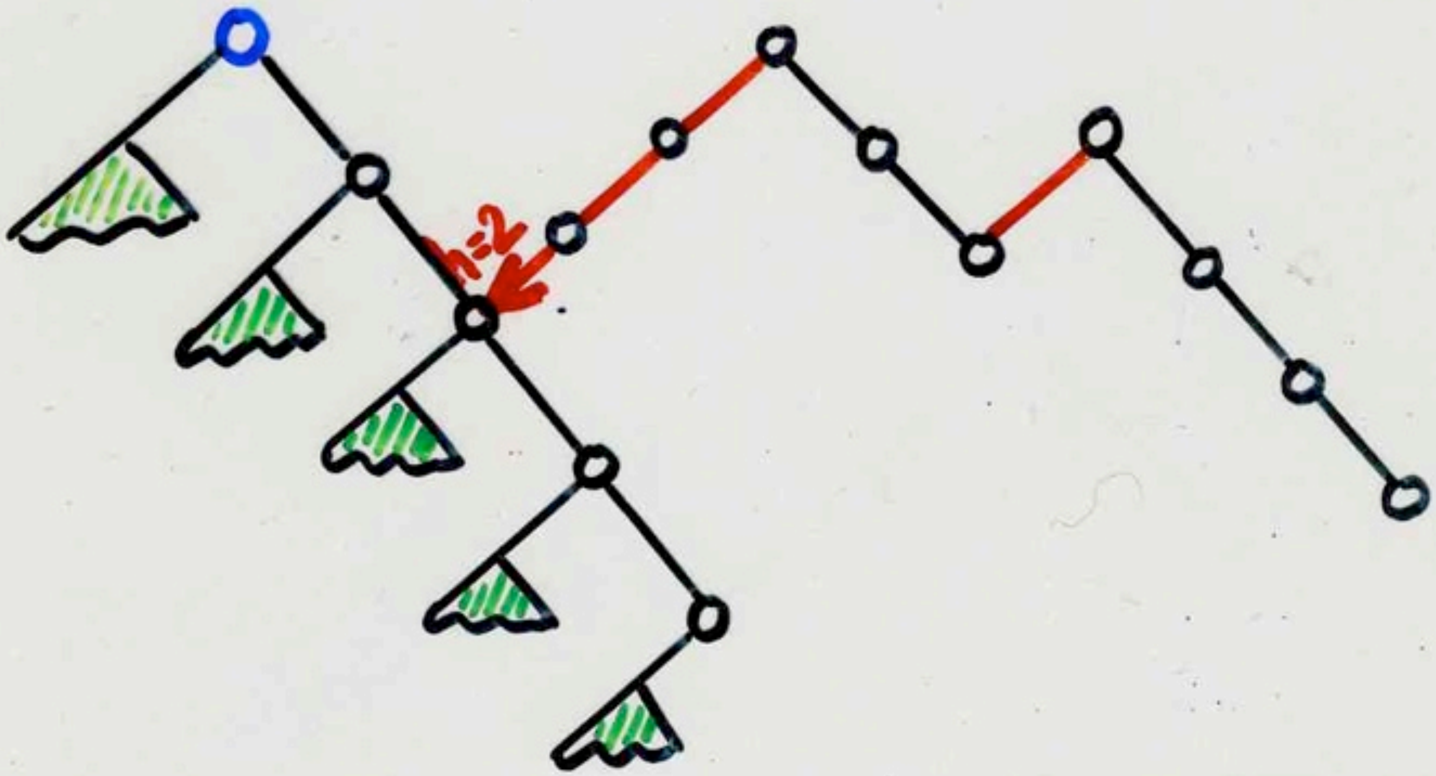




l'algorithme
"glisser - pousser"




h=2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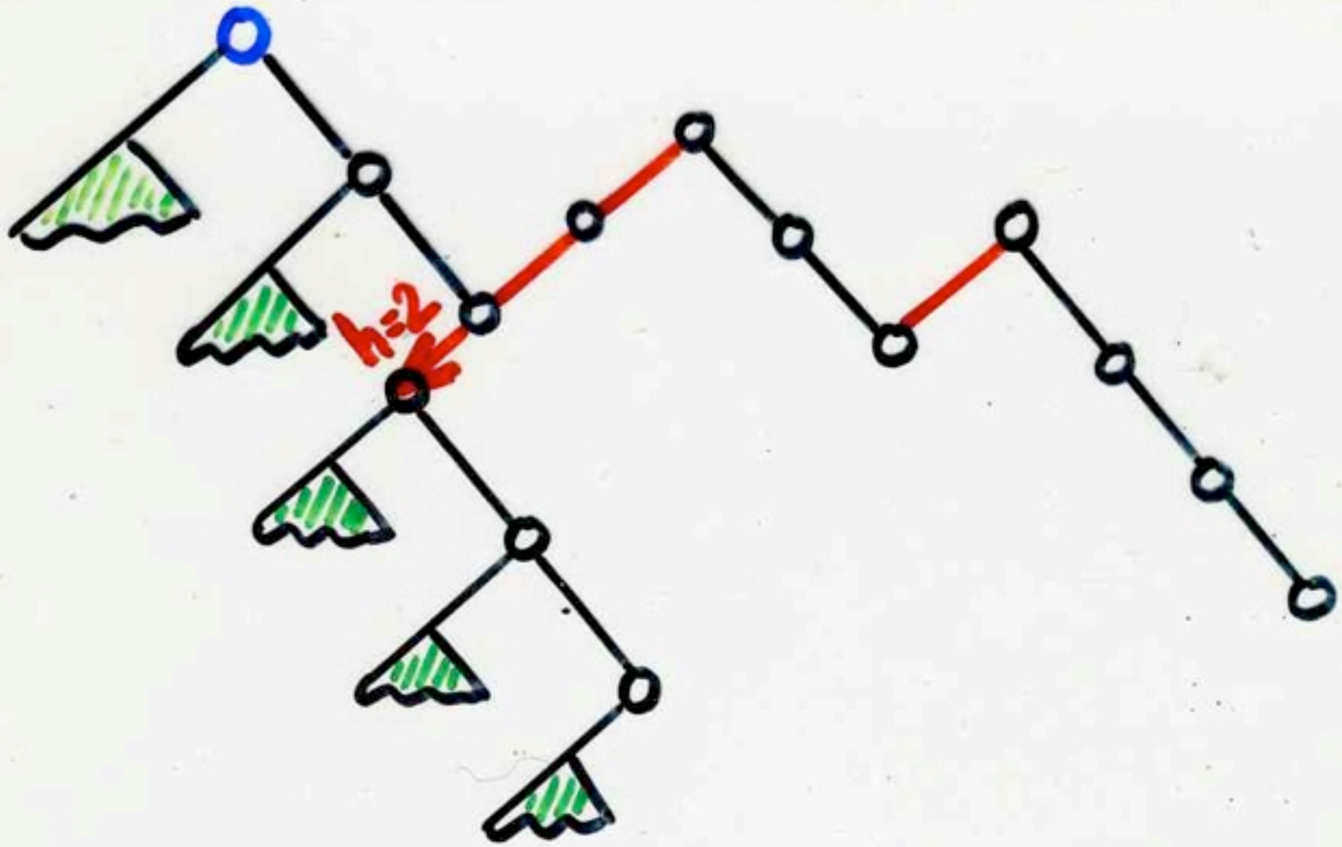
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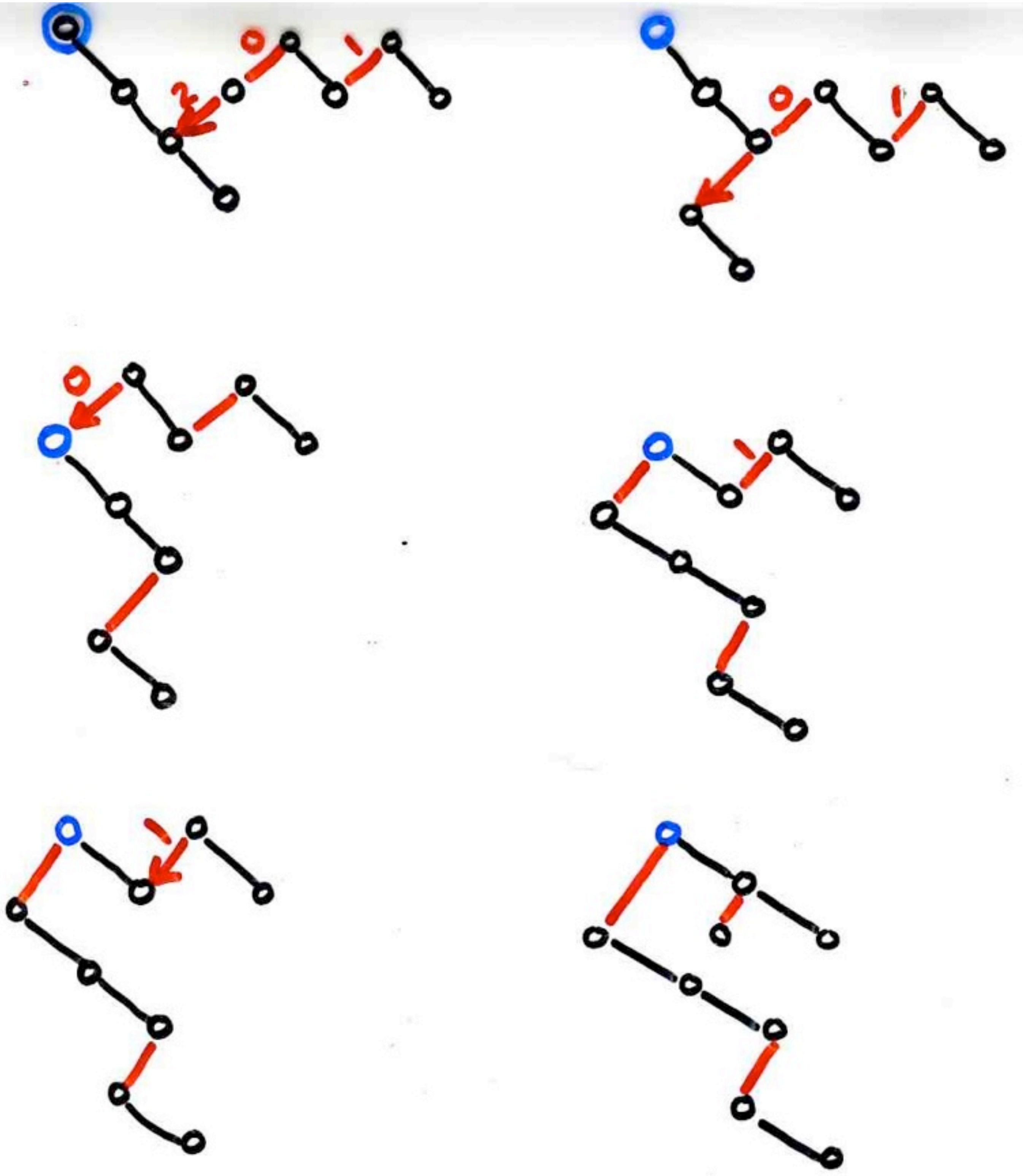
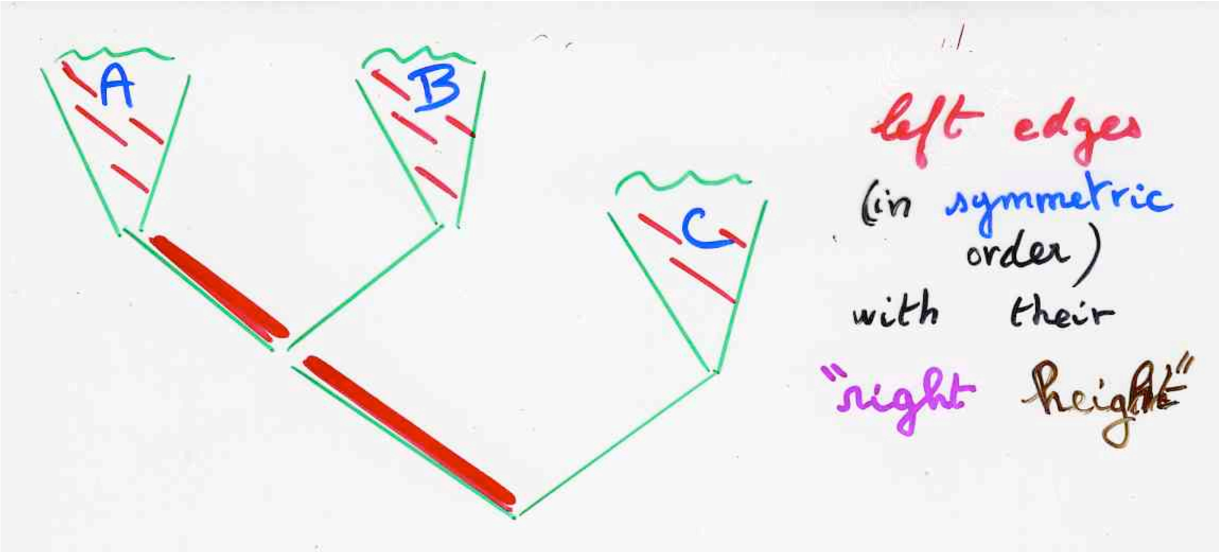
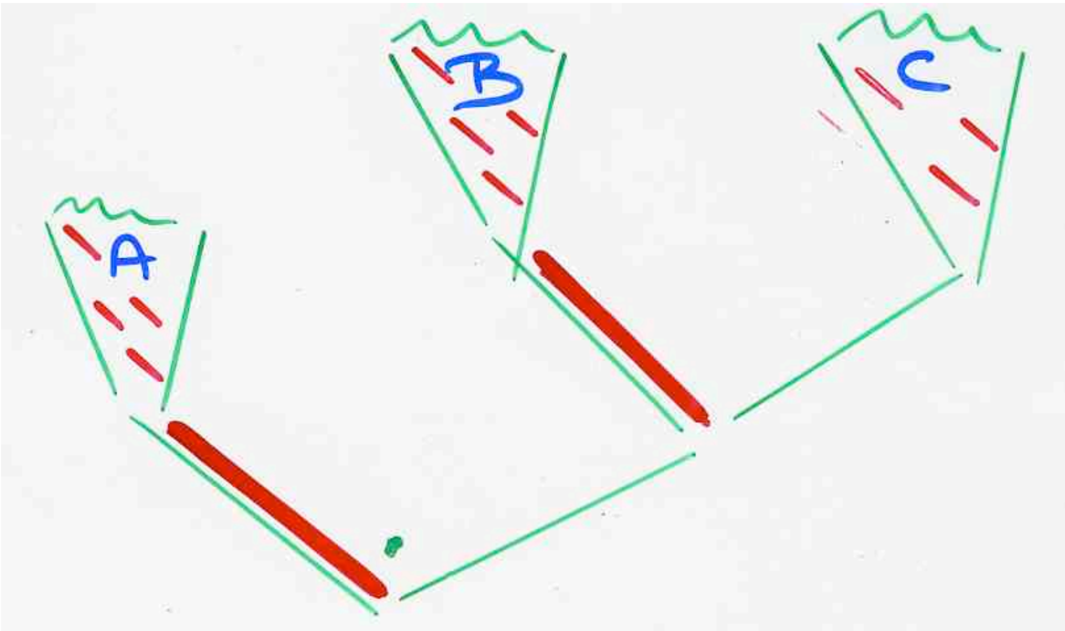
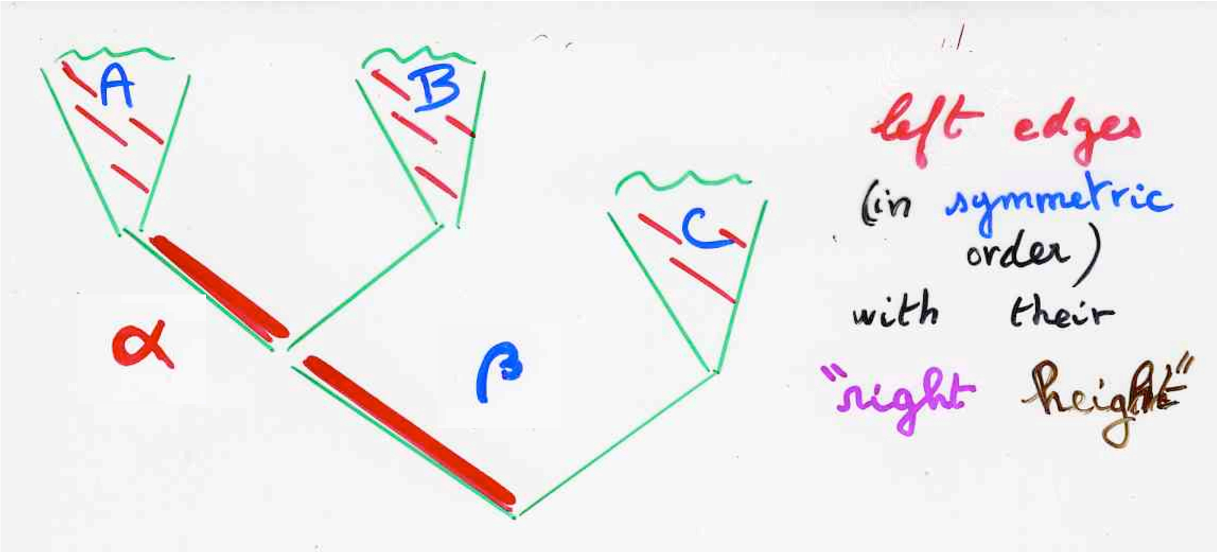
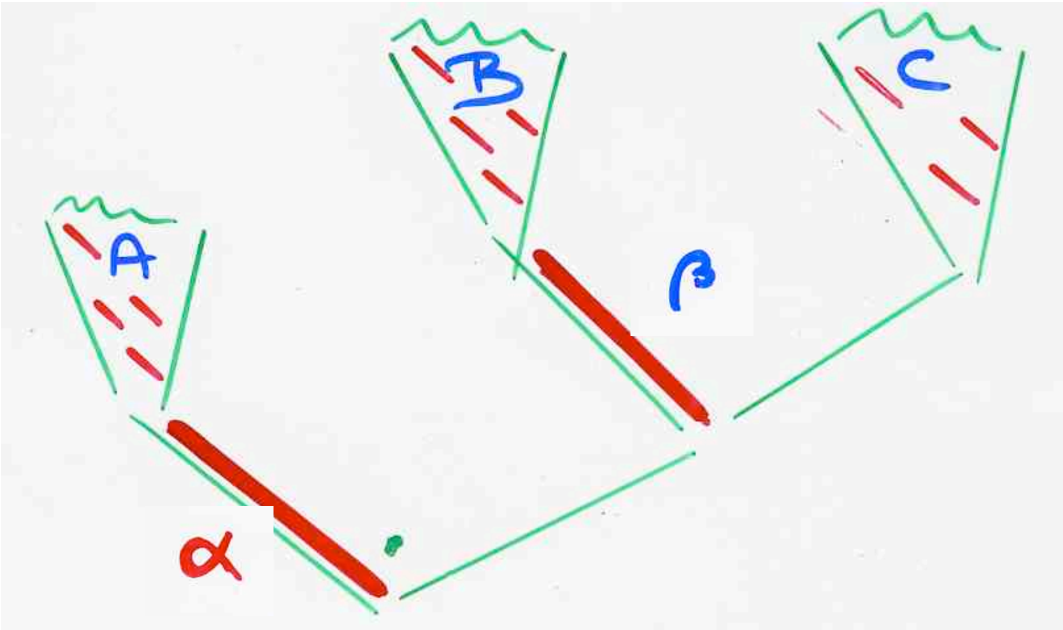


2
0
1


A
B
C
left edges
(in symmetric order)
with their
"right height"

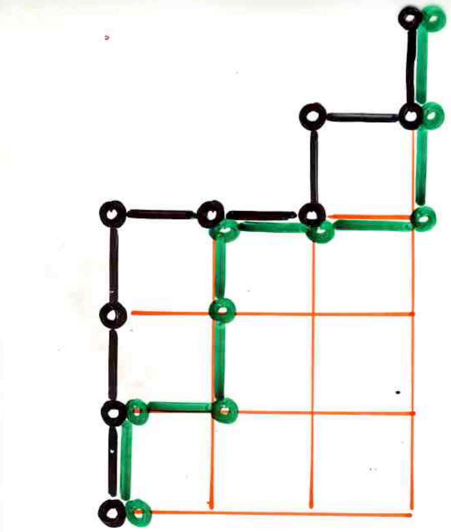

B
C
A





preservation of the symmetric order for left edges

right height:
+1 in C
and for β

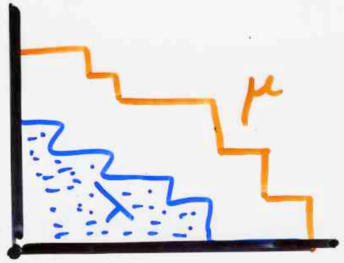


Young lattice

partition μ

$I(\mu)$

$\lambda \leq \mu$



i.e. $\exists$ (integer) partition μ such that $J(w)$ is in bijection with $I(\mu)$, the set of partition $\emptyset \leq \lambda \leq \mu$

(inclusion of Ferrers / Young diagrams)

with $T \leq T' \Rightarrow f(T) \geq f(T')$

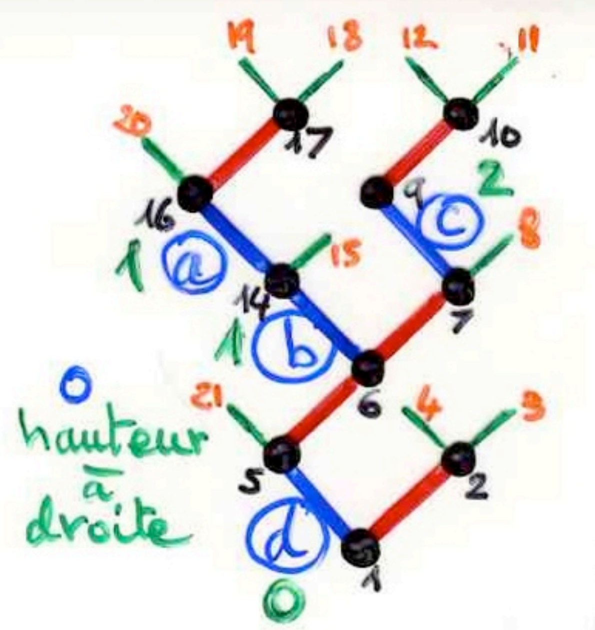
Tamari lattice — Young lattice

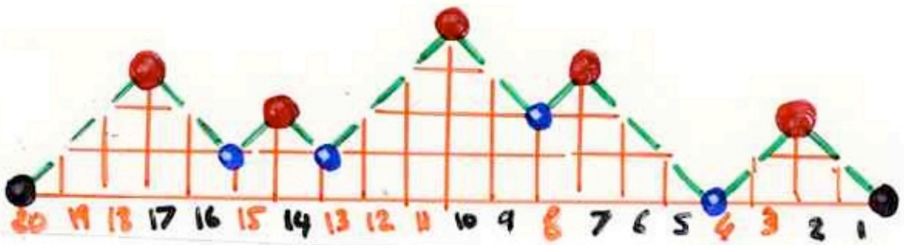
$$J(w) \xrightarrow[\text{bijection}]{f} I(\mu)$$

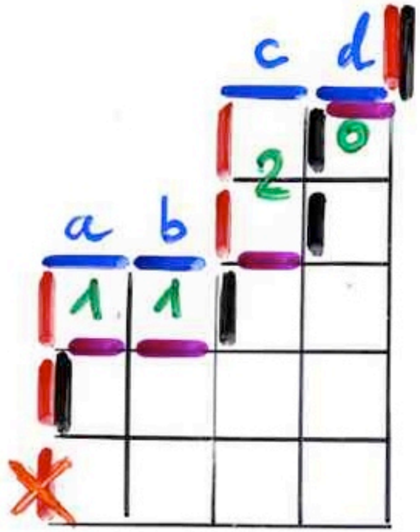
Prop. (i) • The set of binary trees having a given canopy w is an interval $I(w)$ of the Tamari lattice

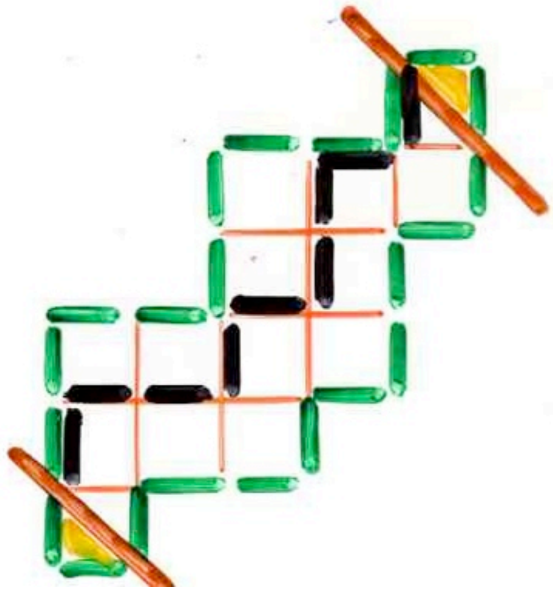
Proof:

with another
description
of the Catalan
bijection


19 18 12 11
17 10
20
16 9 2 c 8
1 a 15
14 7
1 b 6
0
hauteur
à
droite
21 4 3
5 2
d 1
0


20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1


c d
0
2
a b
1 1



Prop. (i) • The set of binary trees having a given canopy w is an interval of the Tamari lattice $J(w)$

intervalles genevois

(ii) • this interval can be extended to an initial interval of the Young lattice

i.e. $\exists$ (integer) partition μ such that $J(w)$ is in bijection with $I(\mu)$, the set of partition $\emptyset \leqslant \lambda \leqslant \mu$

(inclusion of Ferrers diagrams)
Young

with $T \leqslant T' \Rightarrow f(T) \geqslant f(T')$

Tamari lattice — Young lattice

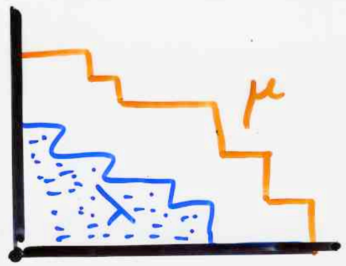
$$J(w) \xrightarrow[\text{bijection}]{f} I(\mu)$$

Young lattice

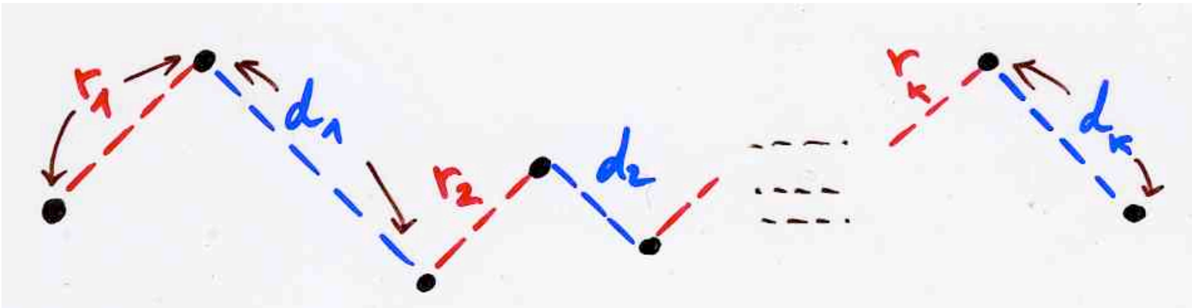
partition μ

$I(\mu)$

$\lambda \leqslant \mu$



initial segment in the Young lattice (or lower ideal)



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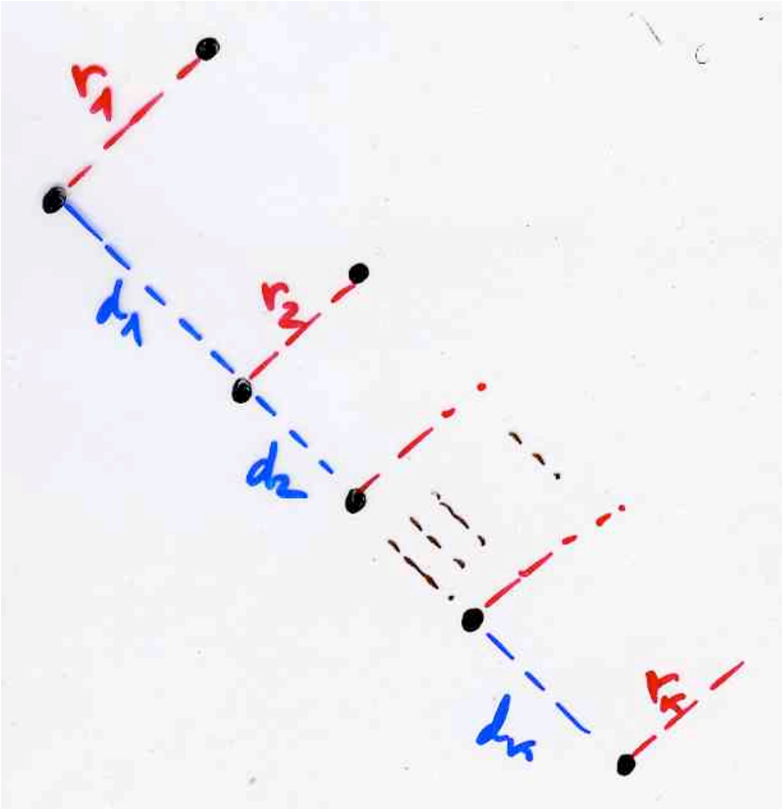

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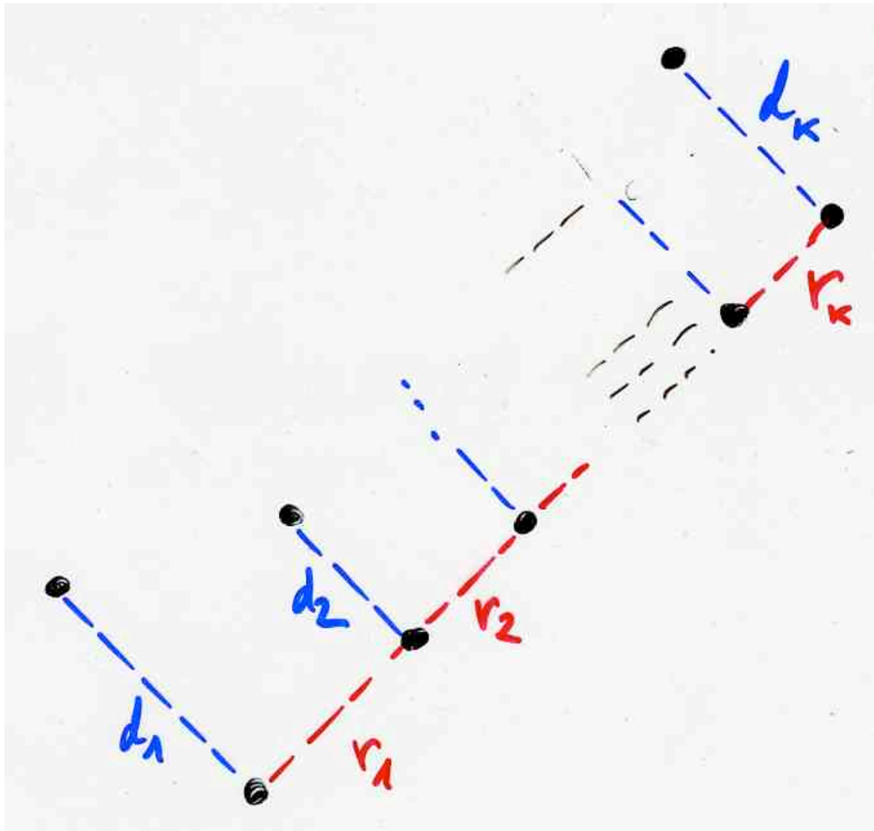


dk
rk
d2
r2
d1
r1

enumeration
of the size of these intervals

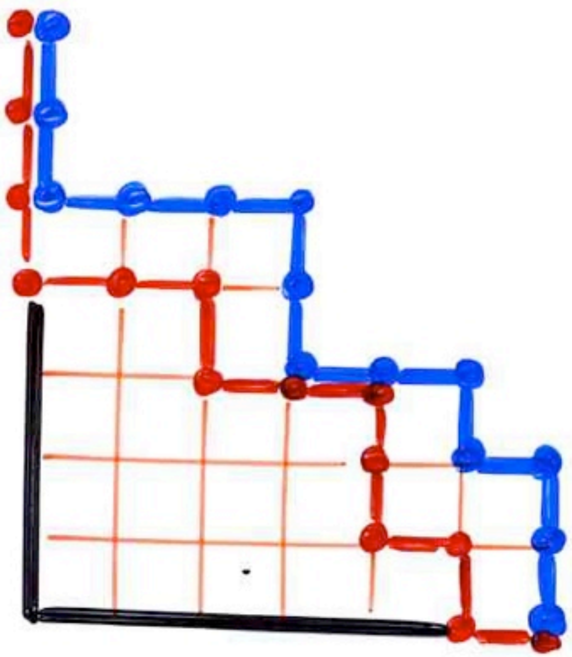
Kreweras's determinant
Narayana (1955)

$$\det \binom{\lambda_i + 1}{j - i + 1}_{1 \le i, j \le k}$$

k = nb of 0's in s

λ_i = nb of 1's to the left of the i^{th} zero

déterminant
Kreweras
Narayana



$\lambda = (0, 0, 3, 3, 5, 6, 6)$

ex: $s = (10100)$ $\qquad \lambda = (1,2,2)$

$\omega =$ 

$$\det\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = \begin{vmatrix} \binom{2}{1} & \binom{2}{2} & \binom{2}{3} \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} \\ \binom{3}{-1} & \binom{3}{0} & \binom{3}{1} \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 3 \\ 0 & 1 & 3 \end{vmatrix} = 9$$

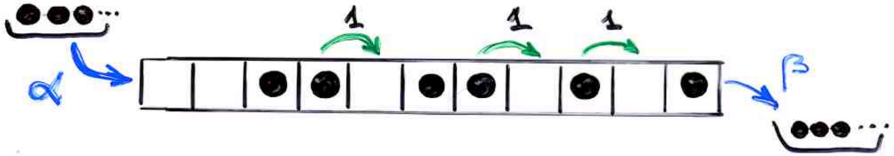
$$P(10100) = \frac{9}{132}$$

Geneva intervals .... ?

intervalles genevois

TASEP

"totally asymmetric exclusion process"



stationary probabilities

$$\frac{1}{C_{n+1}} \sum_{\substack{\text{binary trees } T \\ c(T) = w}} \alpha^{lb(T)} \beta^{rb(T)}$$

canopy

$$Z_n = \sum_{i=1}^{n} \frac{i}{2n-i} \binom{2n-i}{n} \frac{\alpha^{-(i+1)} - \beta^{-(i+1)}}{\alpha^{-1} - \beta^{-1}}$$

Olya Mandelstam
(2013)

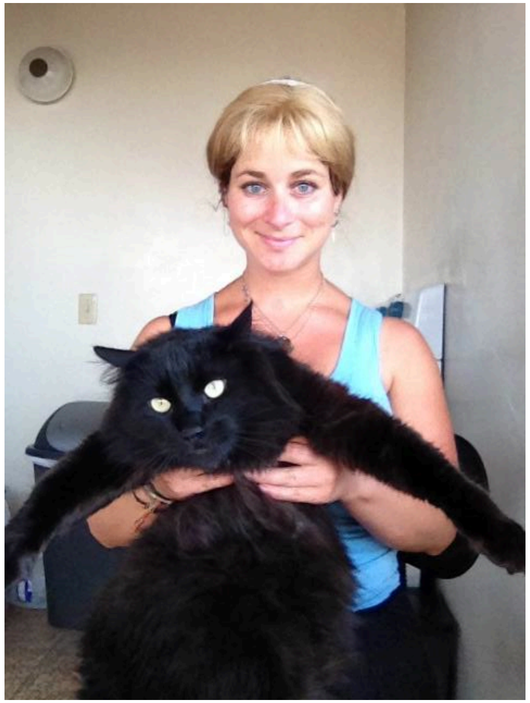
(α, β) - analog of Narayana's determinant

TASEP with 2 parameters

$$\mathcal{P}_{\{\lambda_1, \dots, \lambda_k\}}(\alpha, \beta) = \det A_\lambda^{\alpha, \beta}$$

$$A_\lambda^{\alpha, \beta} = (A_{i,j})$$

$$A_{i,j} = \begin{cases} 0 & \text{for } j < i-1 \\ 1 & \text{for } j = i-1 \\ \beta^{j-i} \alpha^{\lambda_i - \lambda_{j+1}} \left(\binom{\lambda_{j+1}}{j-i} + \binom{\lambda_{j+1}}{j-i+1} \right) \\ + \beta^{j-i} \alpha^{\lambda_i - \lambda_j} \sum_{\ell=0}^{\lambda_j - \lambda_{j+1} - 1} \alpha^{\ell} \left(\binom{\lambda_j - \ell - 1}{j-i-1} + \binom{\lambda_j - \ell - 1}{j-i} \right) & \text{for } j \geq i \end{cases}$$



$s = (\tau_1, \dots, \tau_n)$

$$P_n(s; \alpha, \beta) = \frac{1}{Z_n} \sum_{B} \bar{\alpha}^{\,lb(B)} \, \bar{\beta}^{\,rb(B)}$$

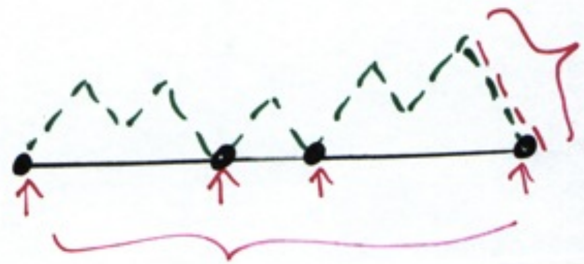
binary trees

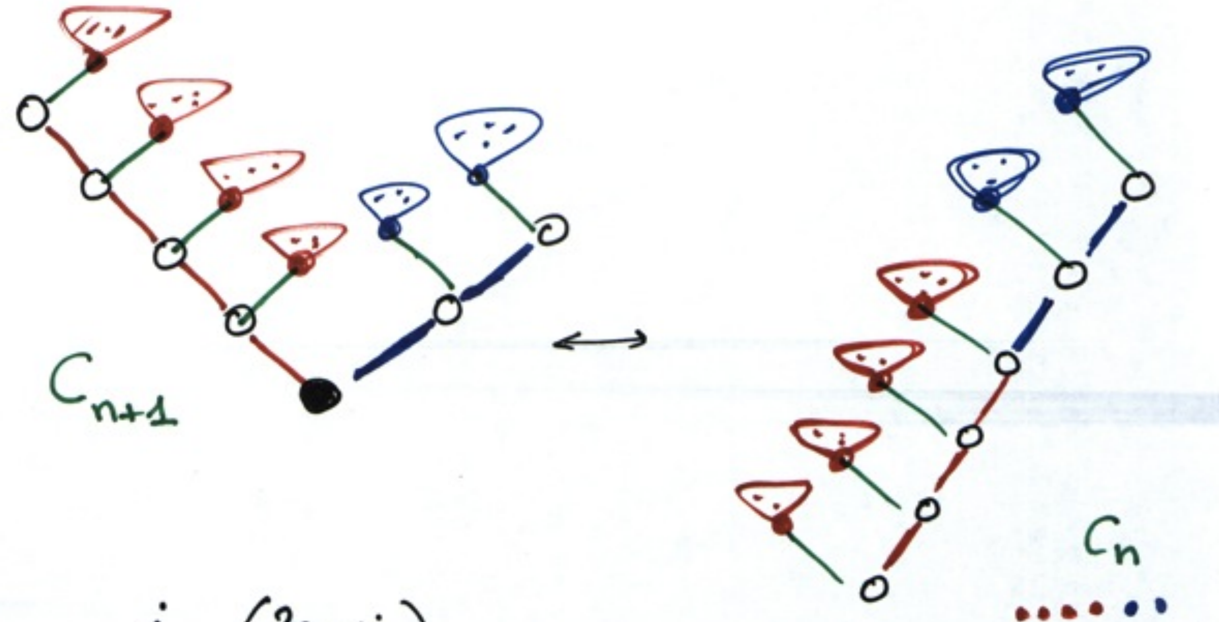
canopy s

$$Z_n = \sum_{i=1}^{n} \frac{i}{2n-i} \binom{2n-i}{n} \frac{\bar{\alpha}^{(i+1)} - \bar{\beta}^{(i+1)}}{\bar{\alpha} - \bar{\beta}}$$

partition function

"ballot" numbers





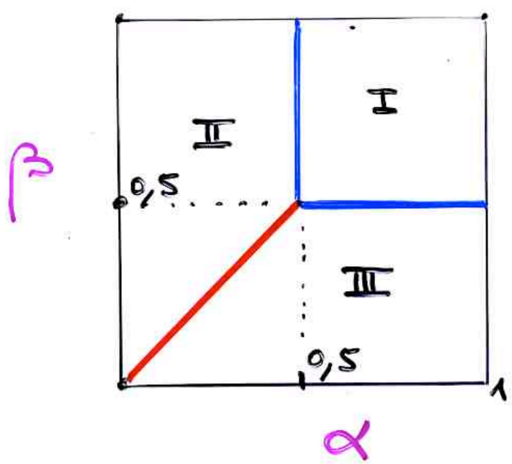
$$\frac{i}{2n-i}\binom{2n-i}{n}$$

$$\left[\bar{\alpha}^{(i)} + \bar{\alpha}^{(i-1)}\bar{\beta} + \dots + \bar{\alpha}\,\bar{\beta}^{(i-1)} + \bar{\beta}^{(i)}\right]$$

$$\sum_n Z_n t^n = \frac{1}{\left(1-\bar{\alpha} f(t)\right)} \times \frac{1}{\left(1-\bar{\beta} f(t)\right)}$$

$f(t) = t$ (generating function of Catalan numbers)

$$= \frac{1-\sqrt{1-4t}}{2}$$



$n \to \infty$

$\rho = \langle \tau_i \rangle$ = taux moyen d'occupation

i loin des bords

(I) $\rho = 1/2$

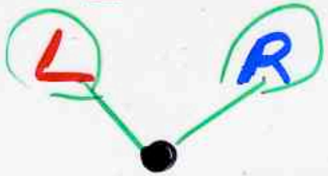
(II) $\rho = \alpha$

(III) $\rho = 1-\beta$

Tamari polynomials

$$B_{\emptyset} = 1$$

$$B_T(x) = x\, B_L(x)\, \frac{x B_R(x) - B_R(1)}{x-1}$$

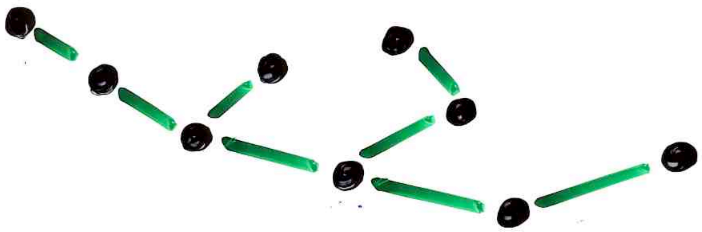
$T =$ 

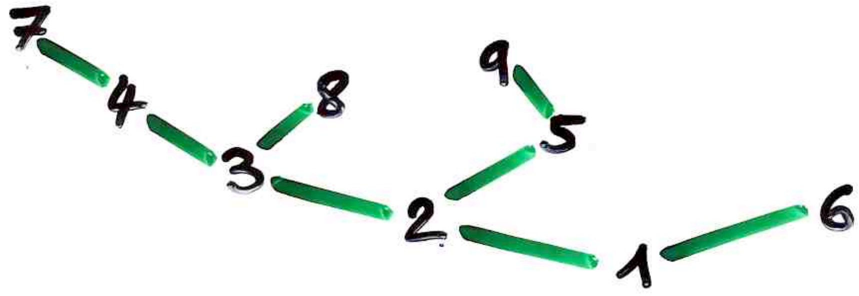
Tamari polynomials

combinatorial structures

	hypercube	associahedron	permutahedron
	lexicographic order (boolean lattice inclusion) ⟷	Tamari order ⟷	weak Bruhat order
dim	2^n	C_n Catalan	$n!$

A. Björner, M. Wachs (1991)





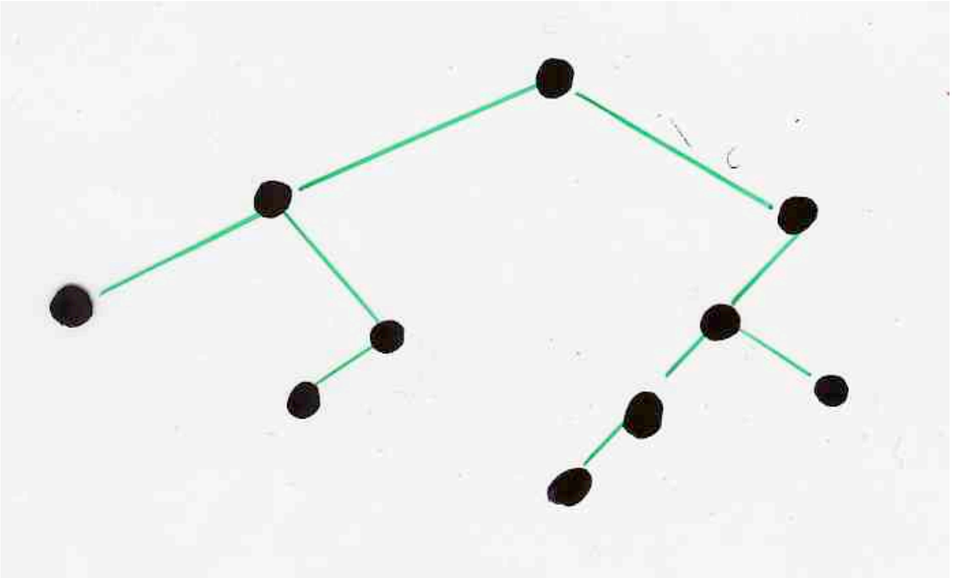
$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$

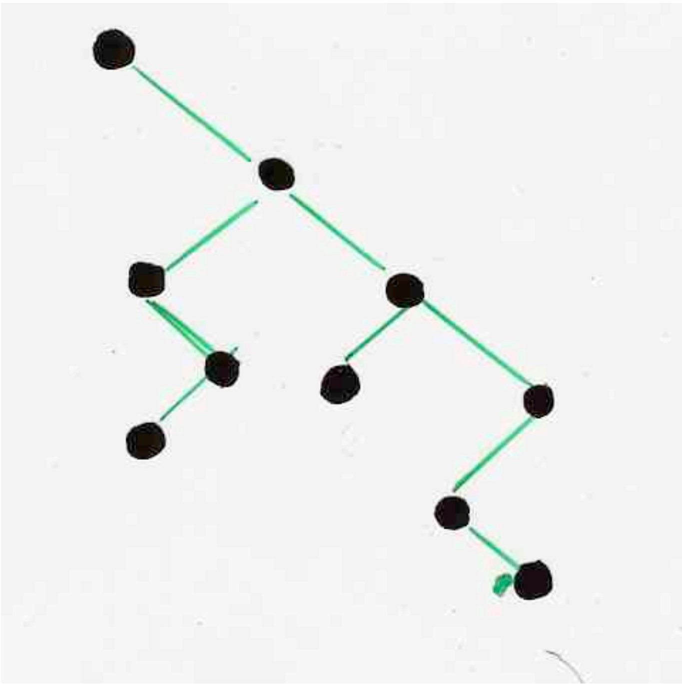
V. Pons

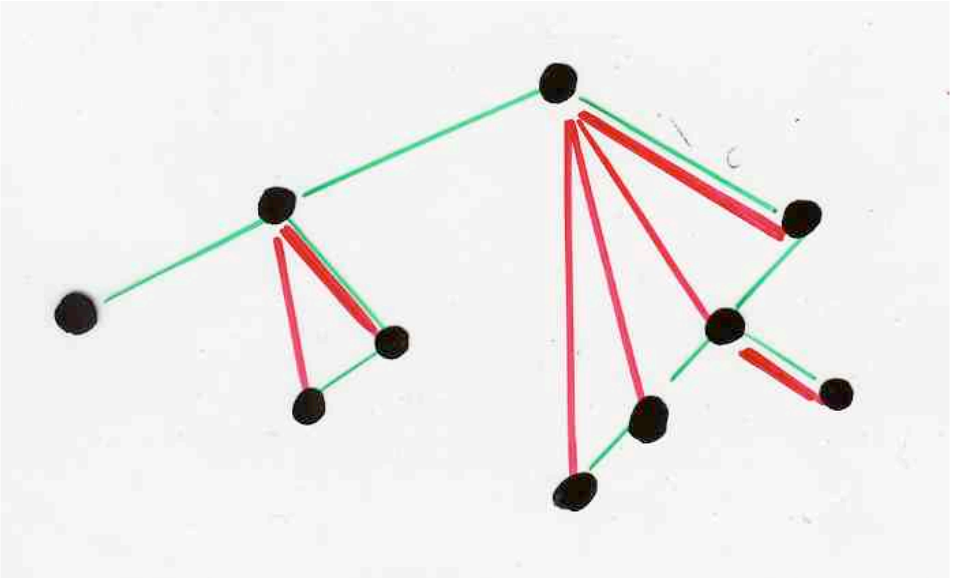
Tamari interval-posets $\Leftrightarrow$ Tamari intervals

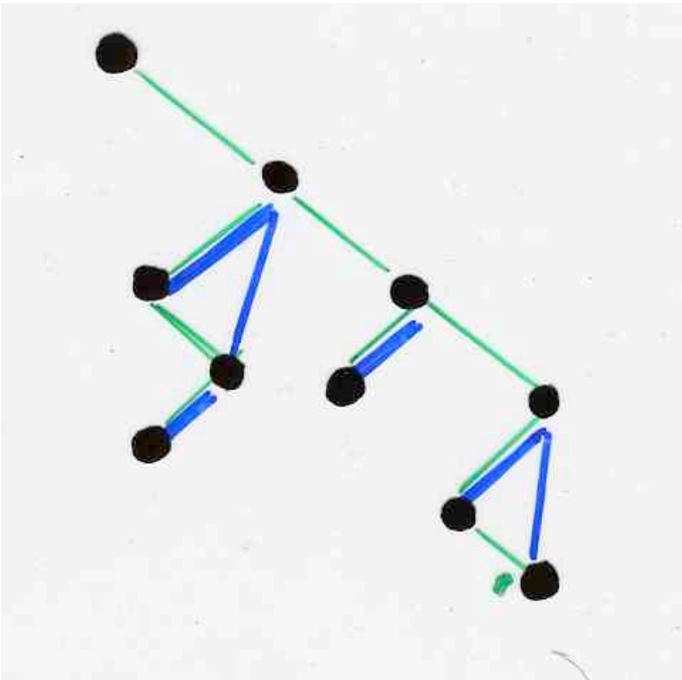
(FPSAC, 2013) thesis (Oct. 2013)

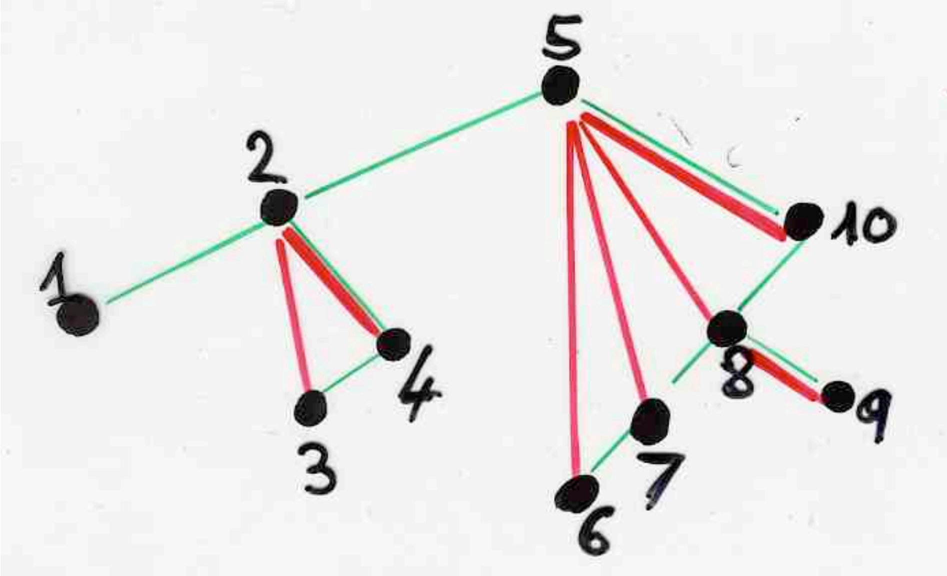
G. Chatel











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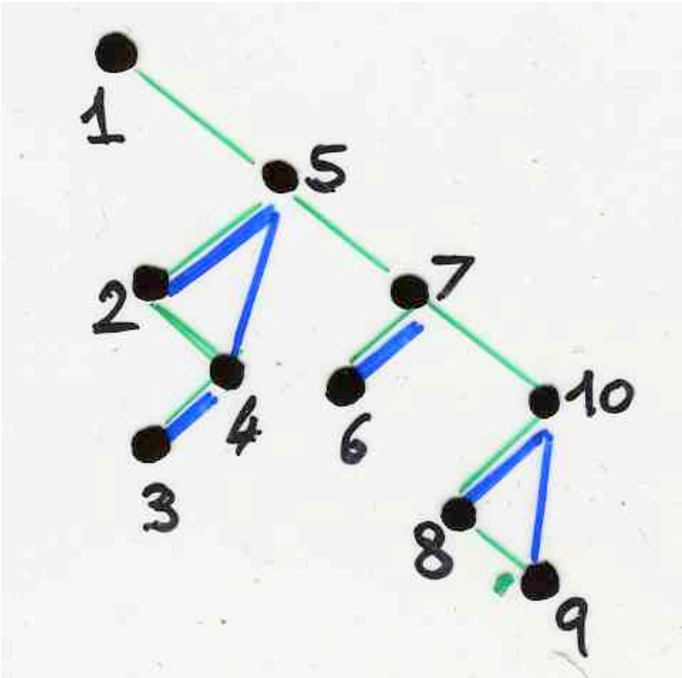

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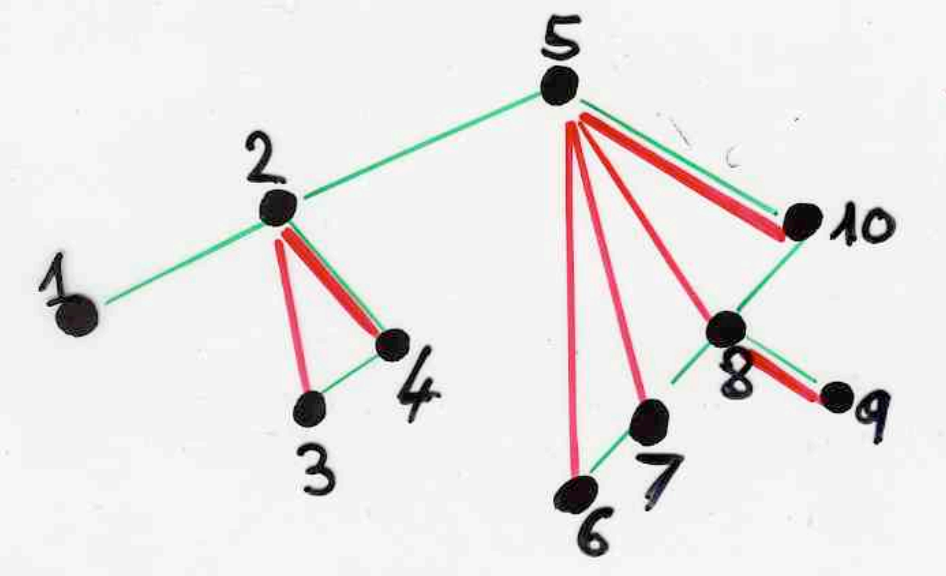
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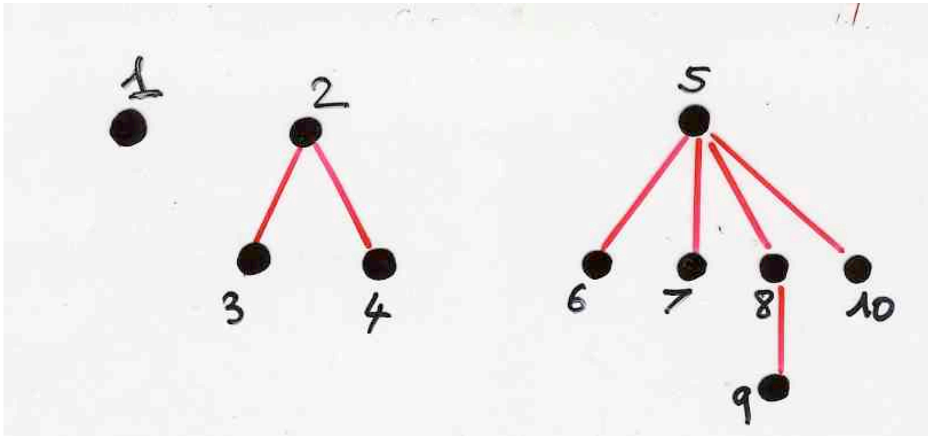
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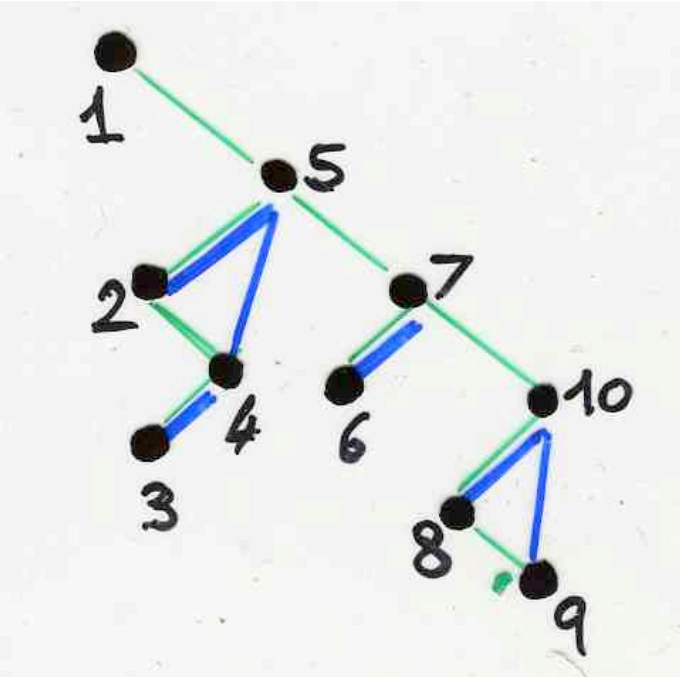
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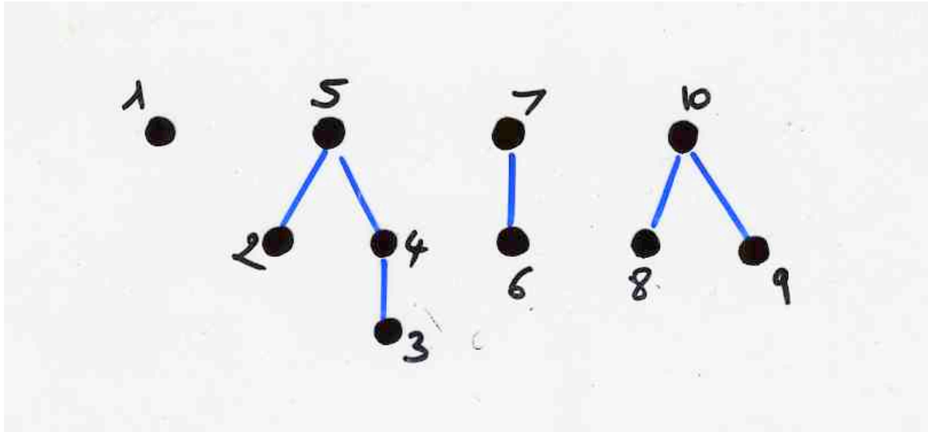
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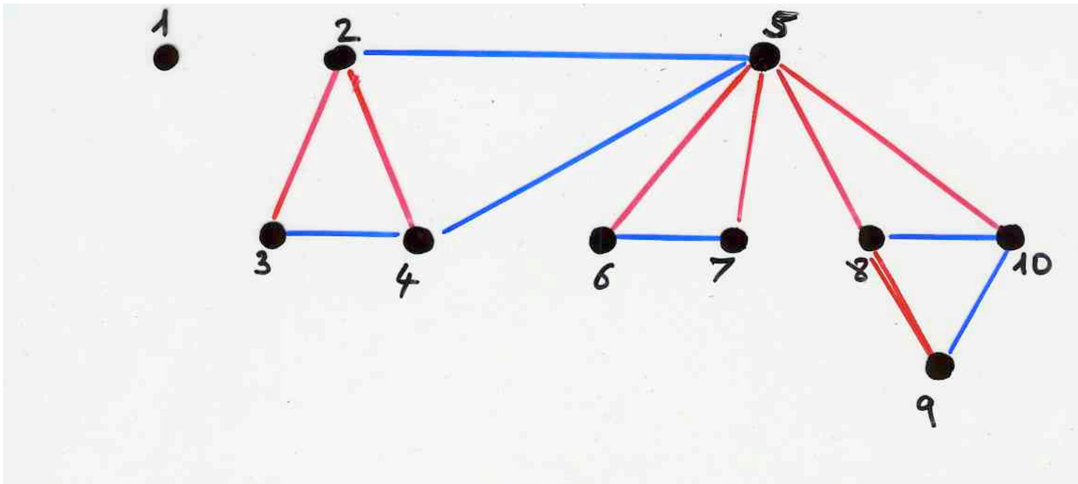
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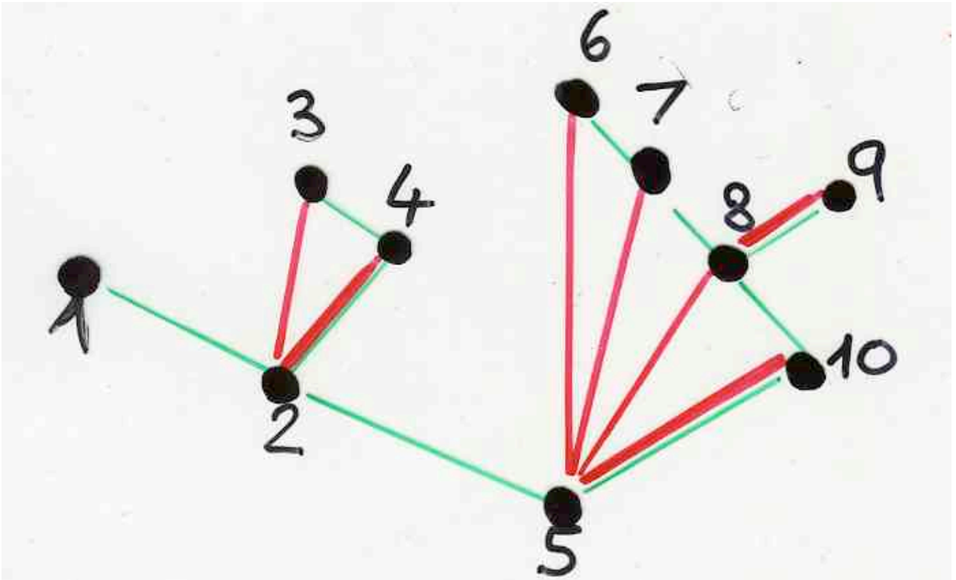
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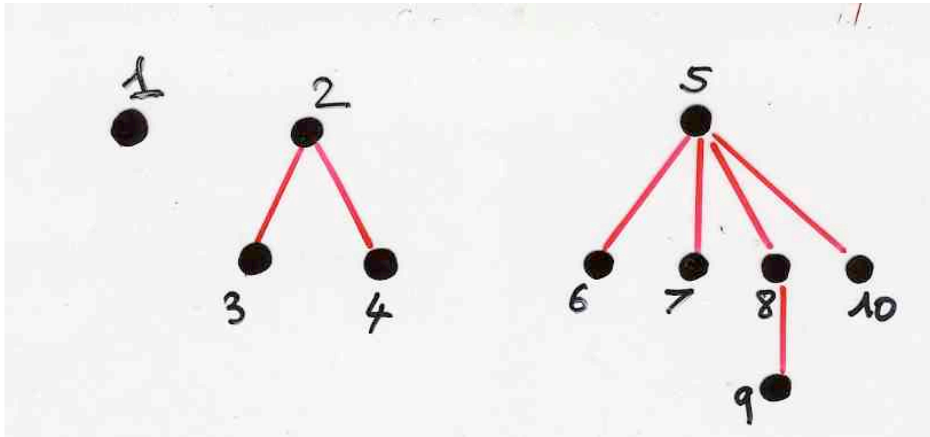
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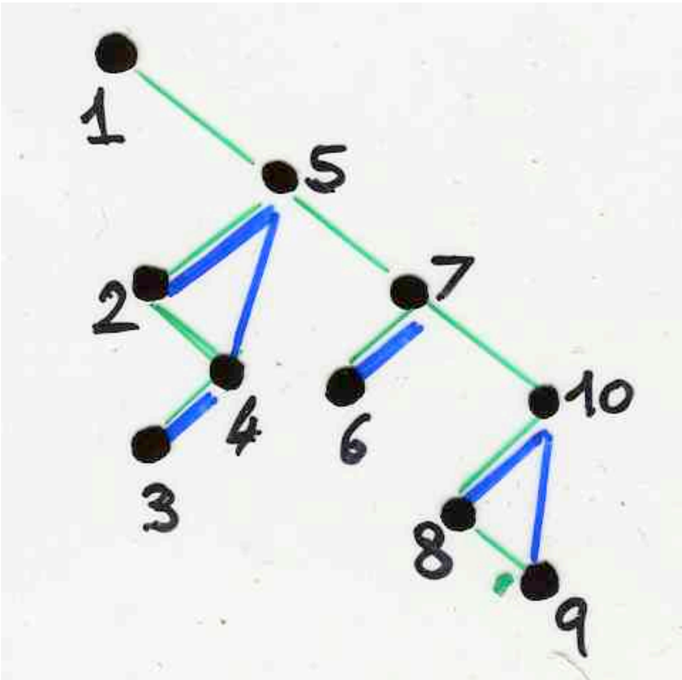
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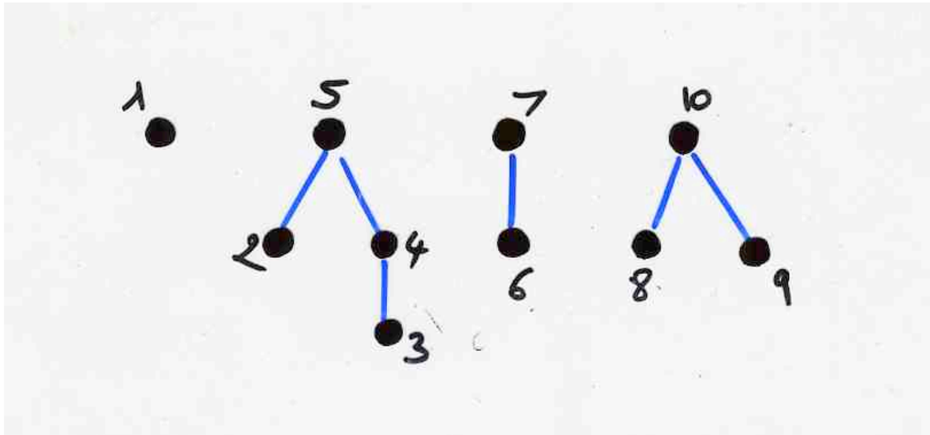
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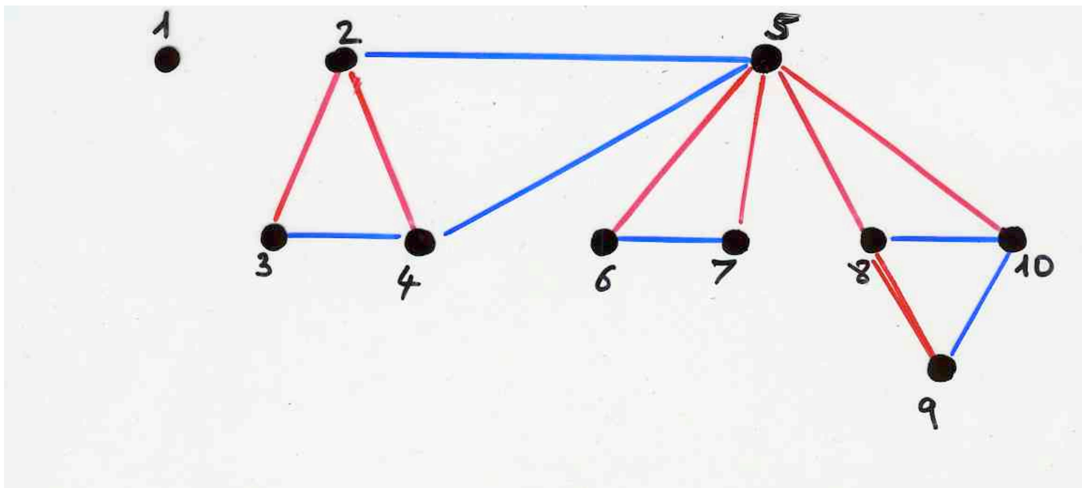
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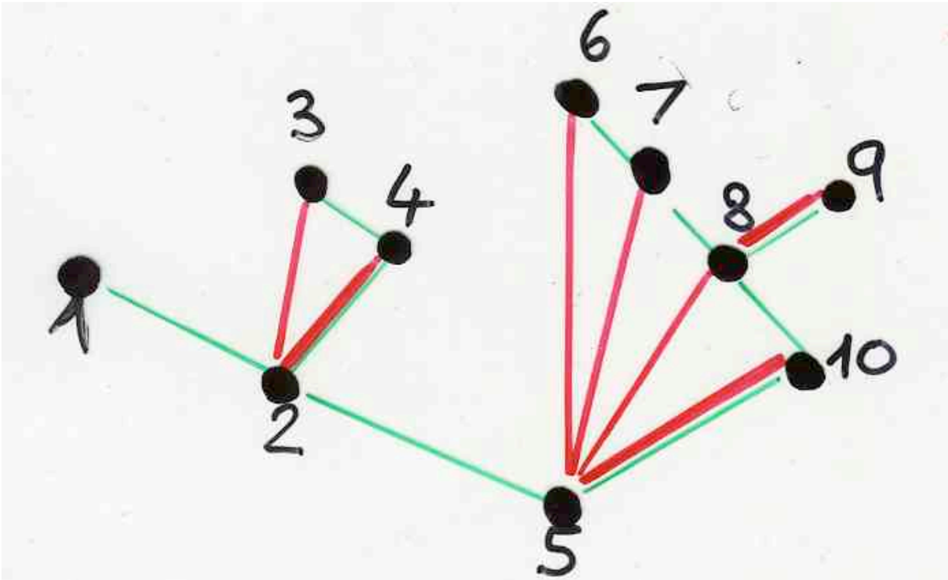
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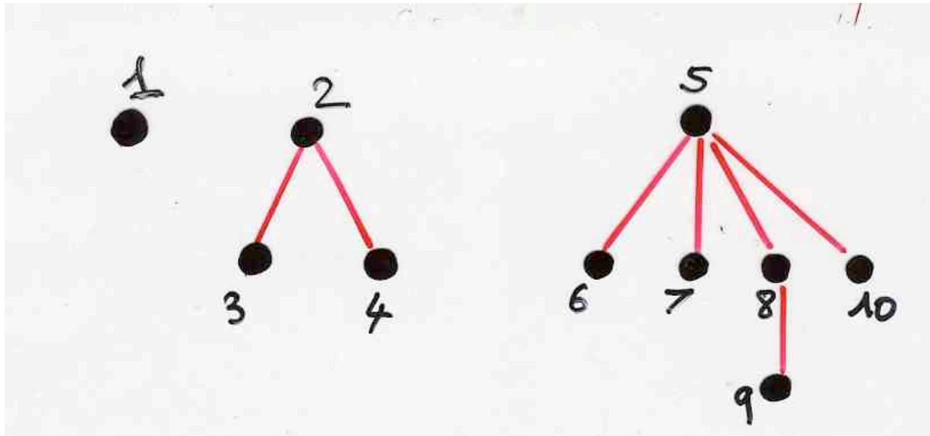
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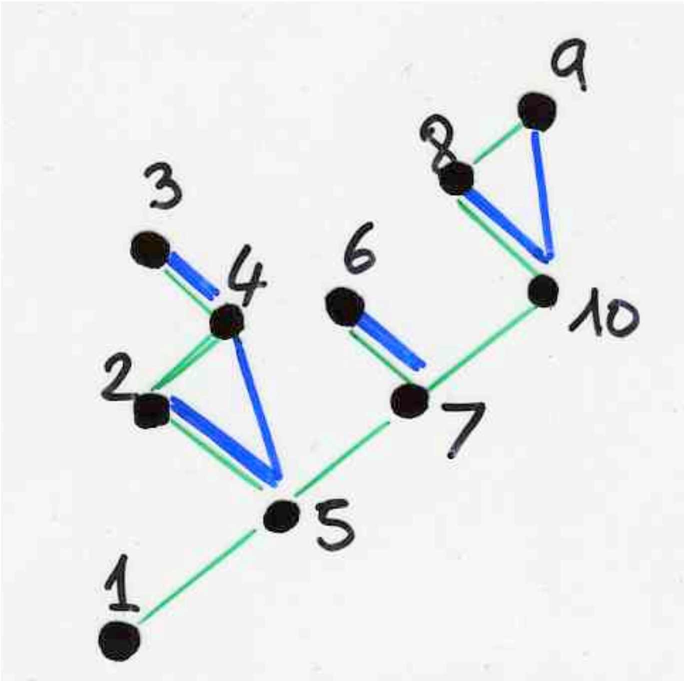
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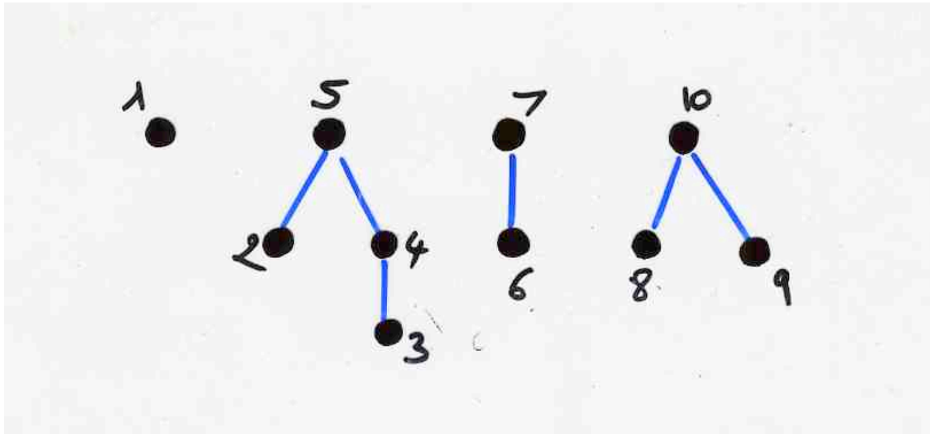
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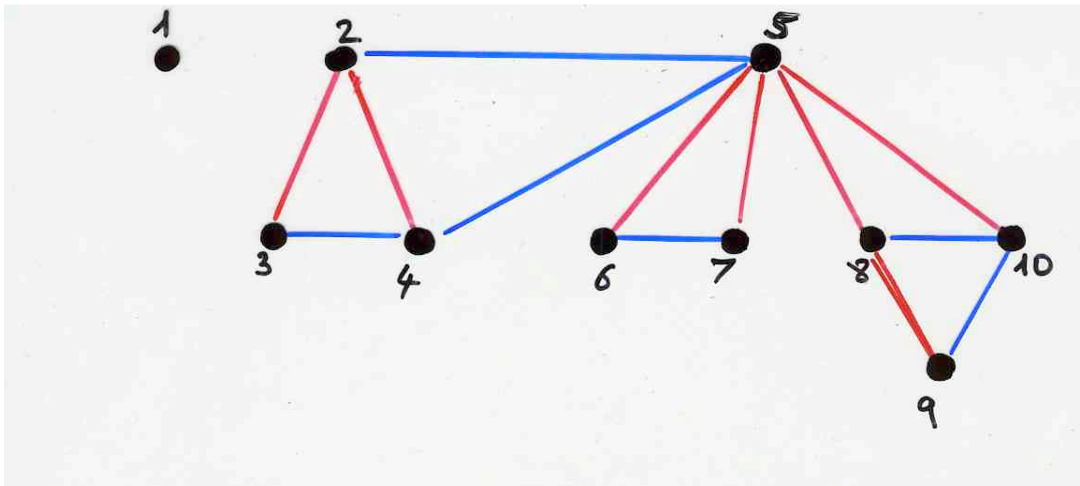
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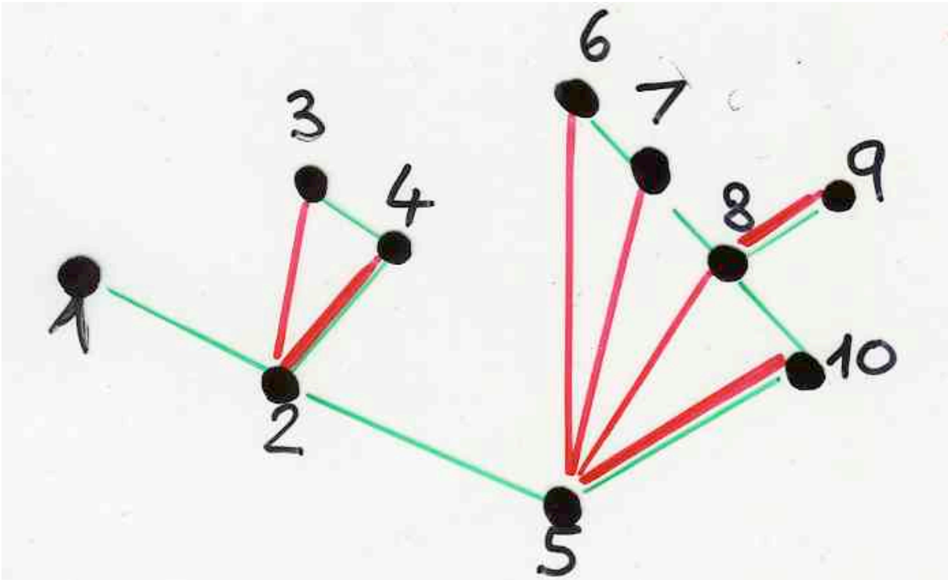
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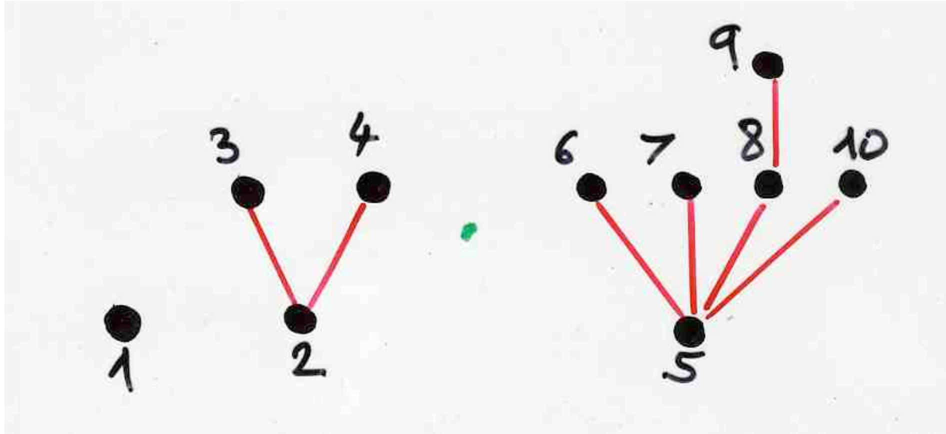
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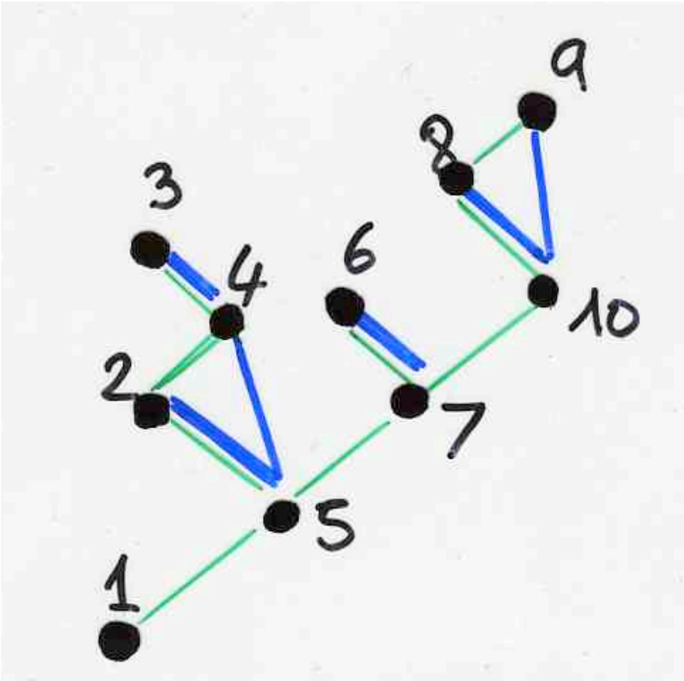
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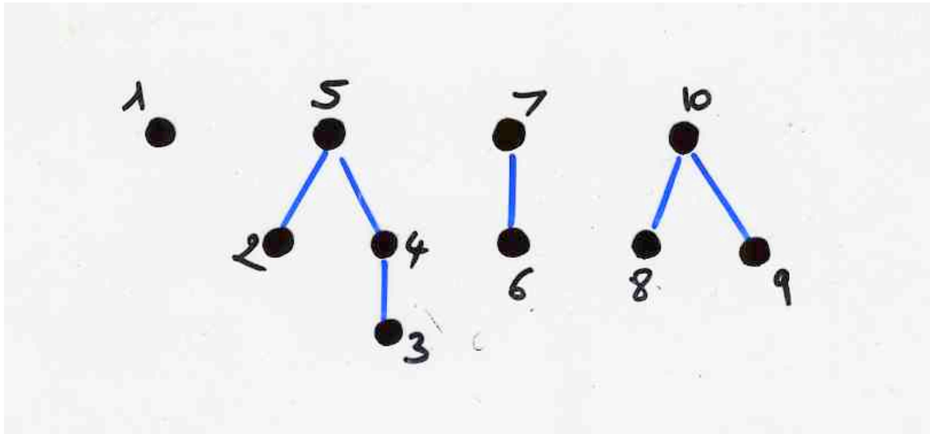
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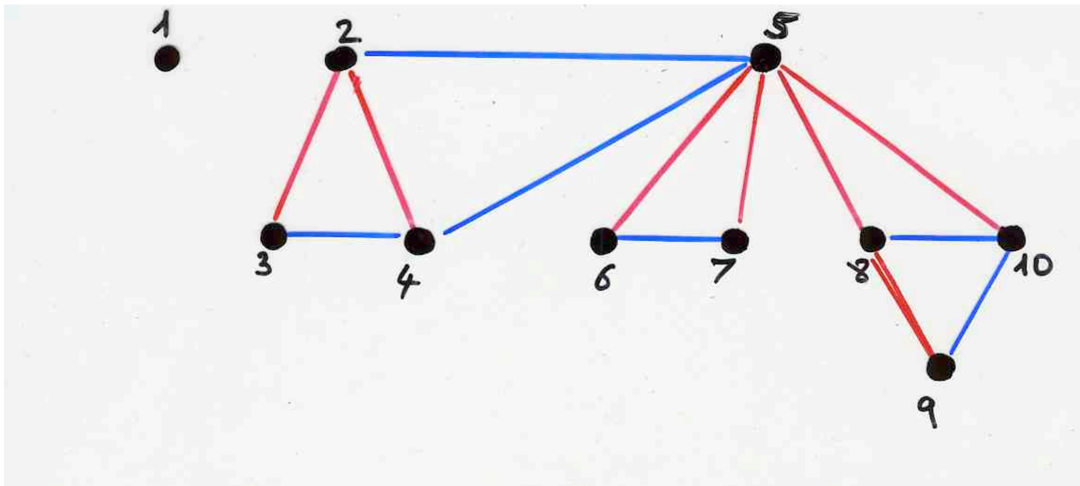
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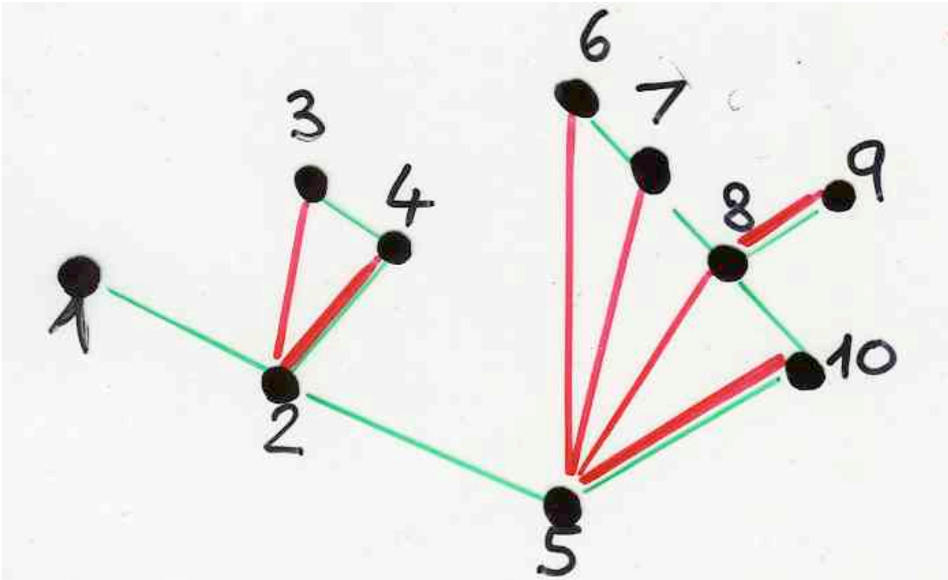
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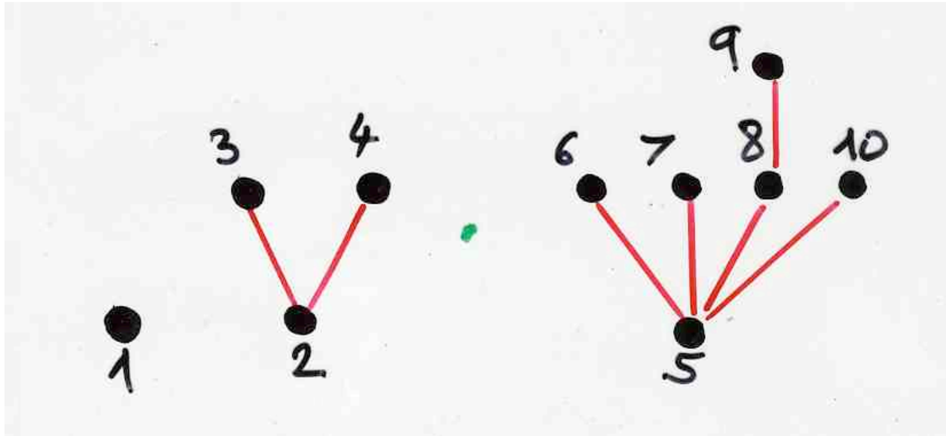
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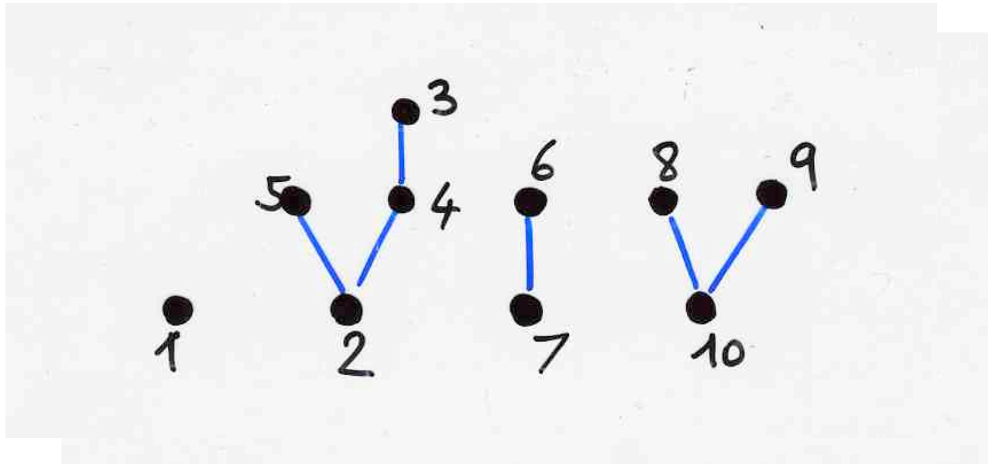
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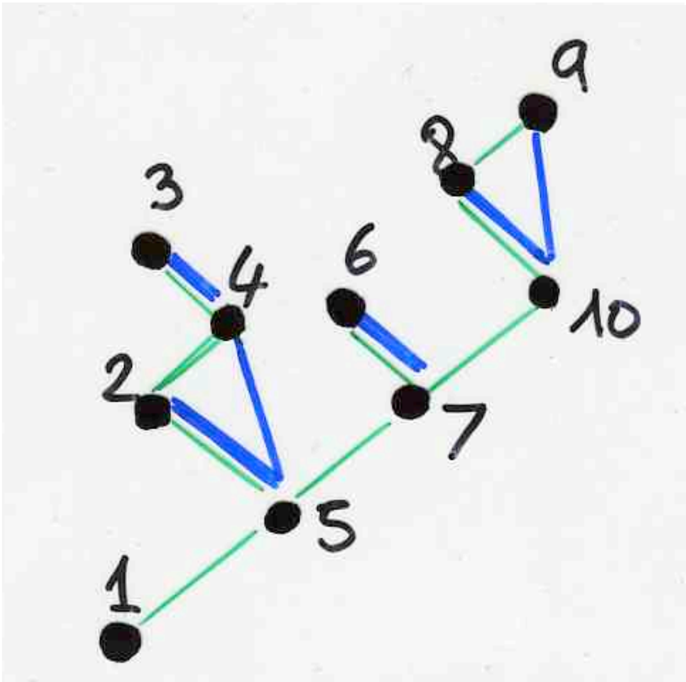
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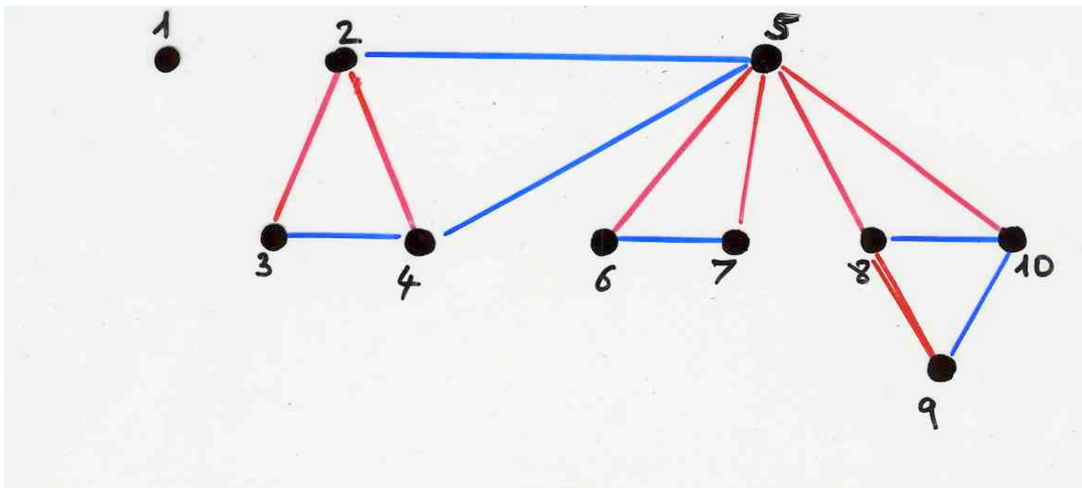
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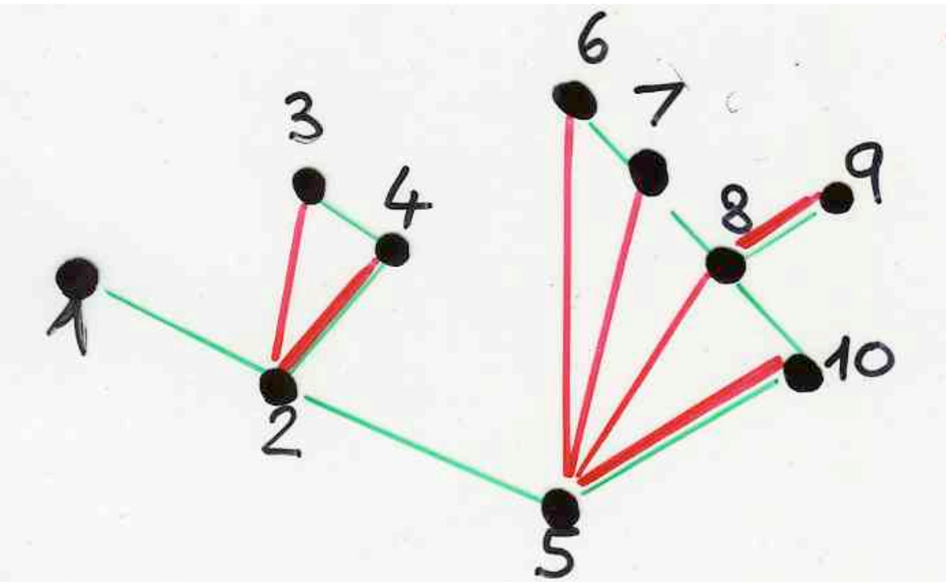
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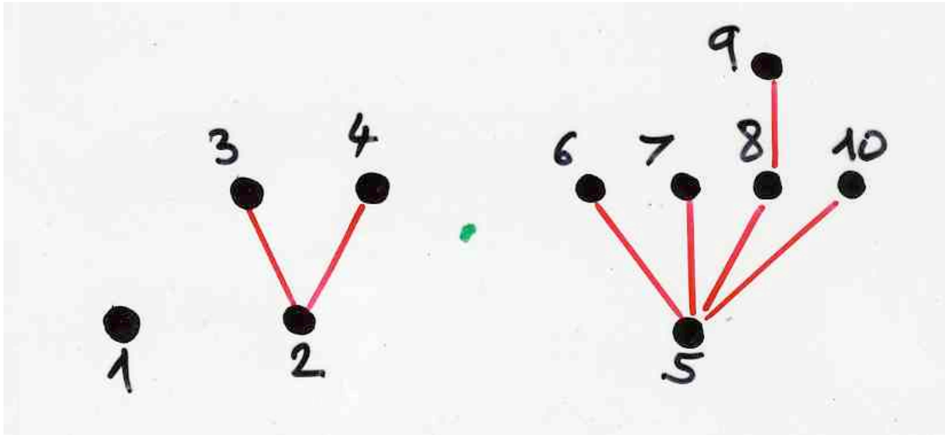
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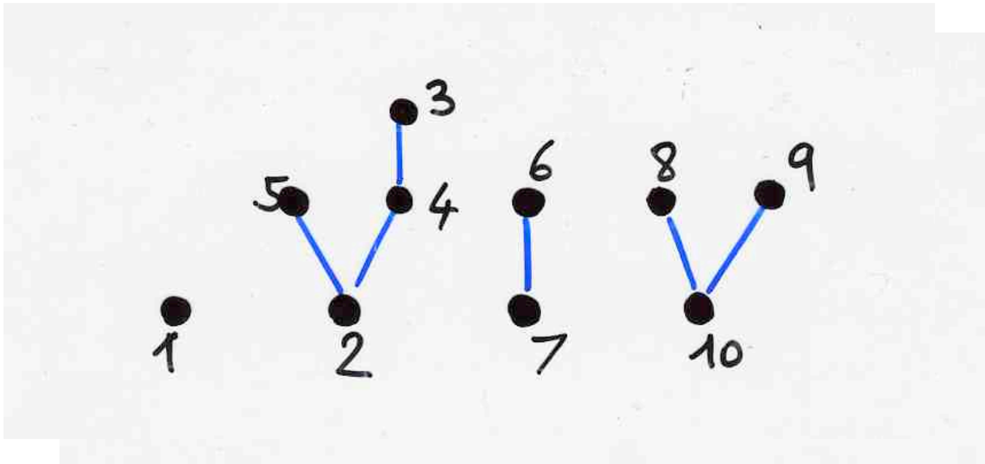
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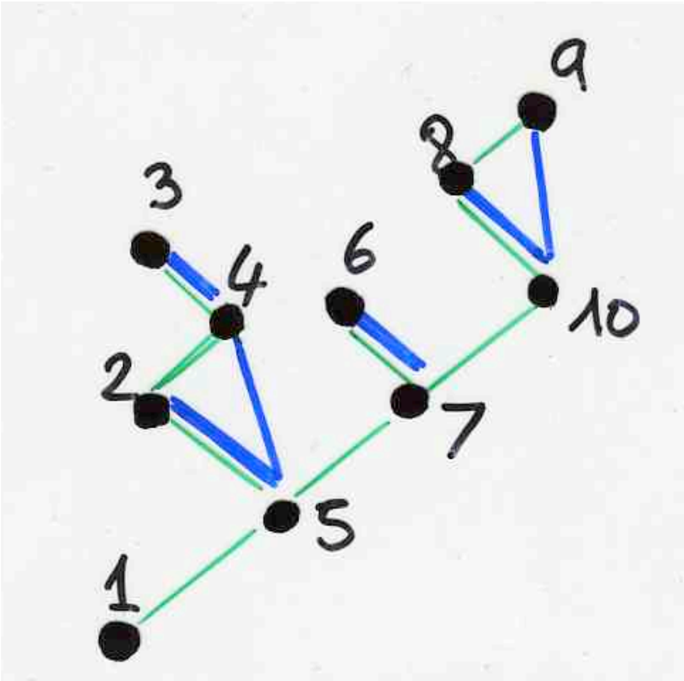
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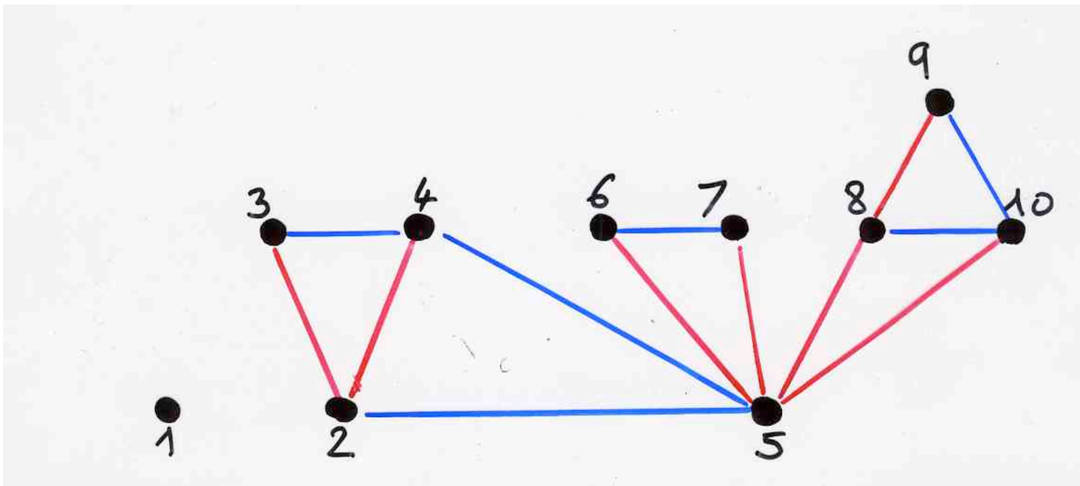
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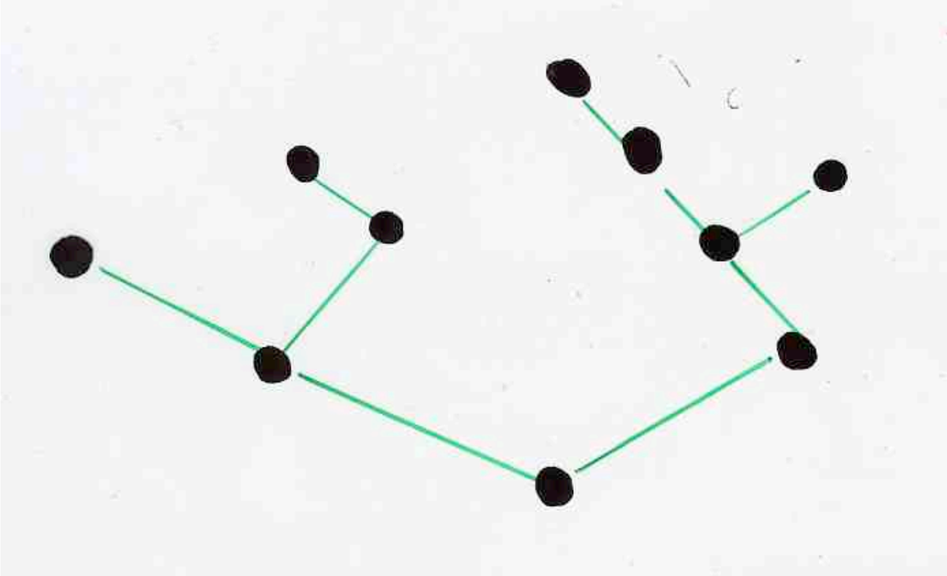

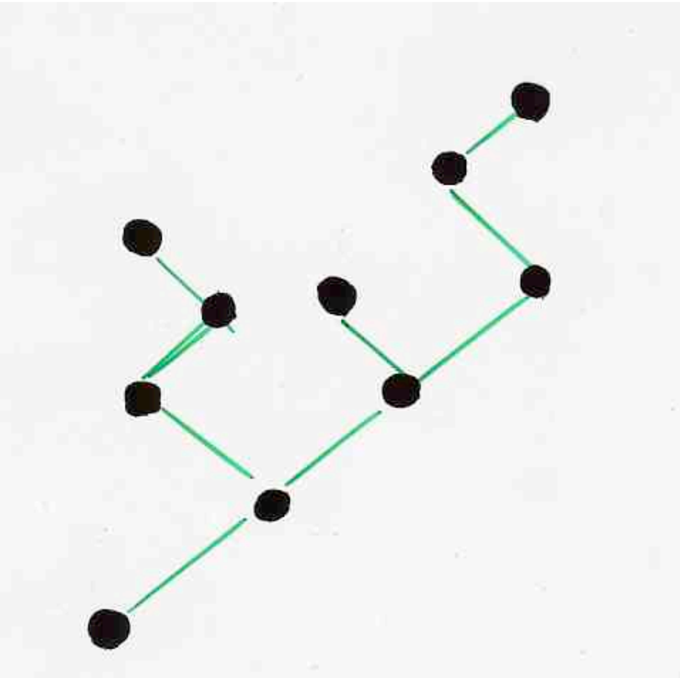

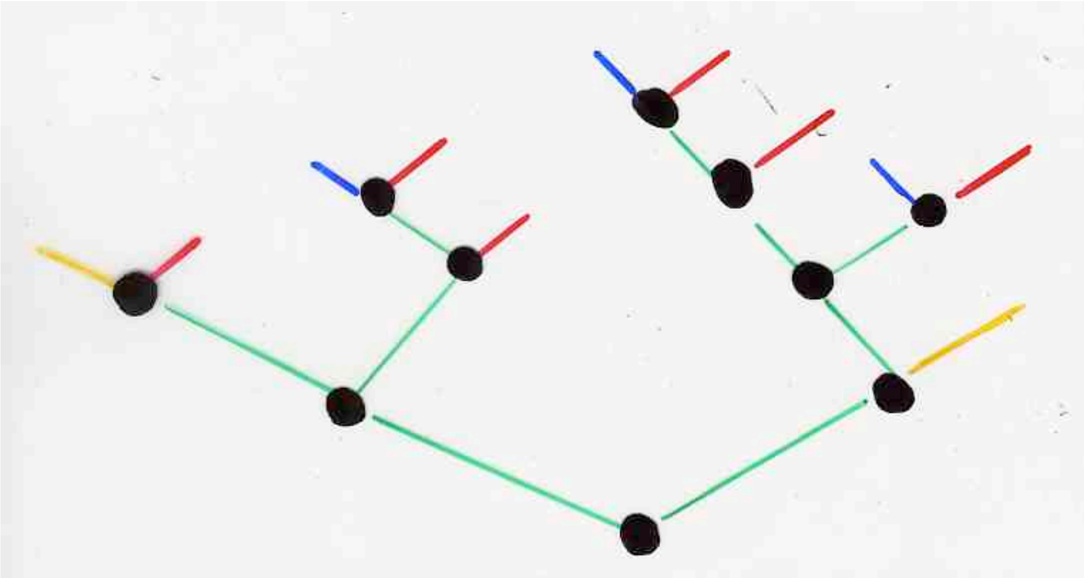

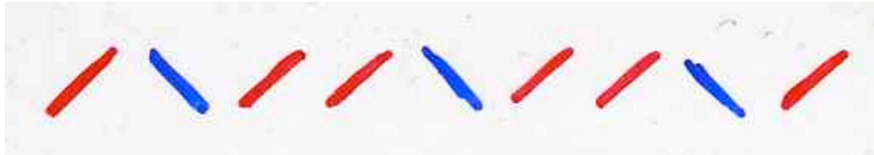

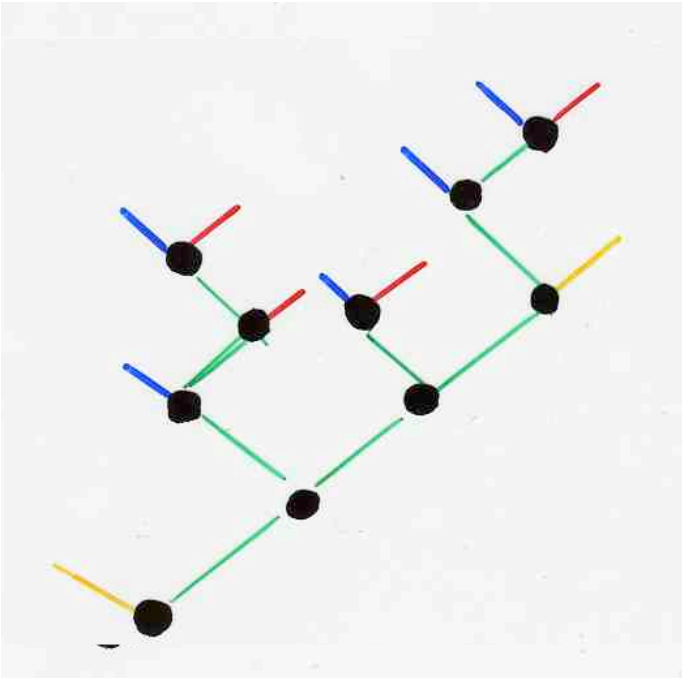

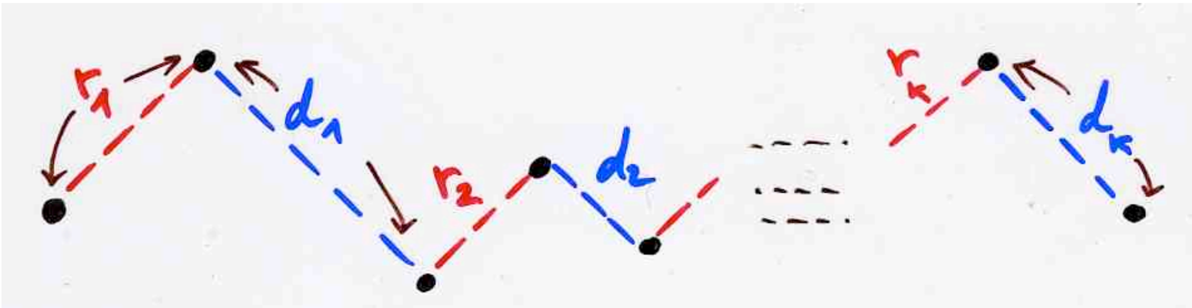

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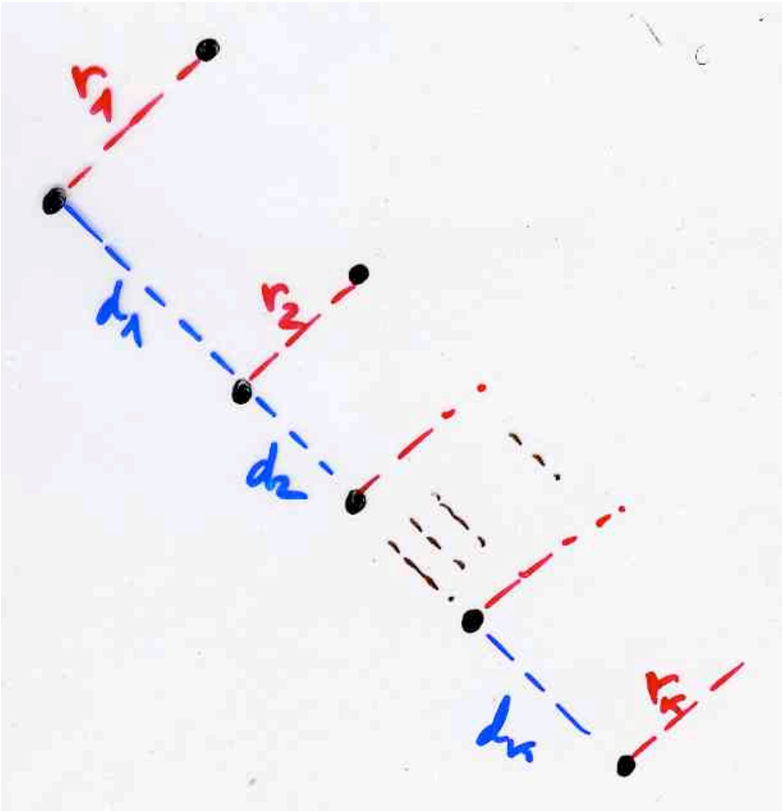
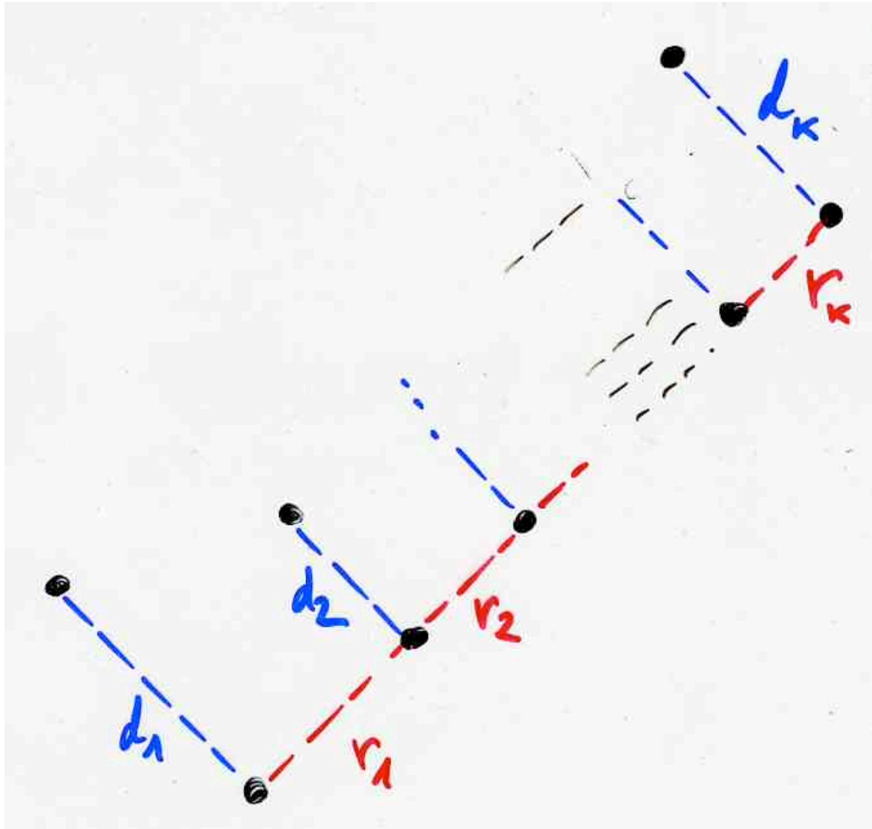
rk
dk



r1
d1
r2
d2
dk
rk



dk
rk
d2
r2
d1
r1

merci !