Canopée des arbres binaires
et
intervalles genevois de l'associaèdre

(1ère partie)

GT LaBRI
15 novembre 2013

Xavier Viennot
LaBRI, CNRS, Bordeaux

Dov Tamari (1951) thèse Sorbone
"Monoïdes préordonnés et chaînes de Malcev"

((a,(b, c)),(d, e))
(a (b c))(d e)

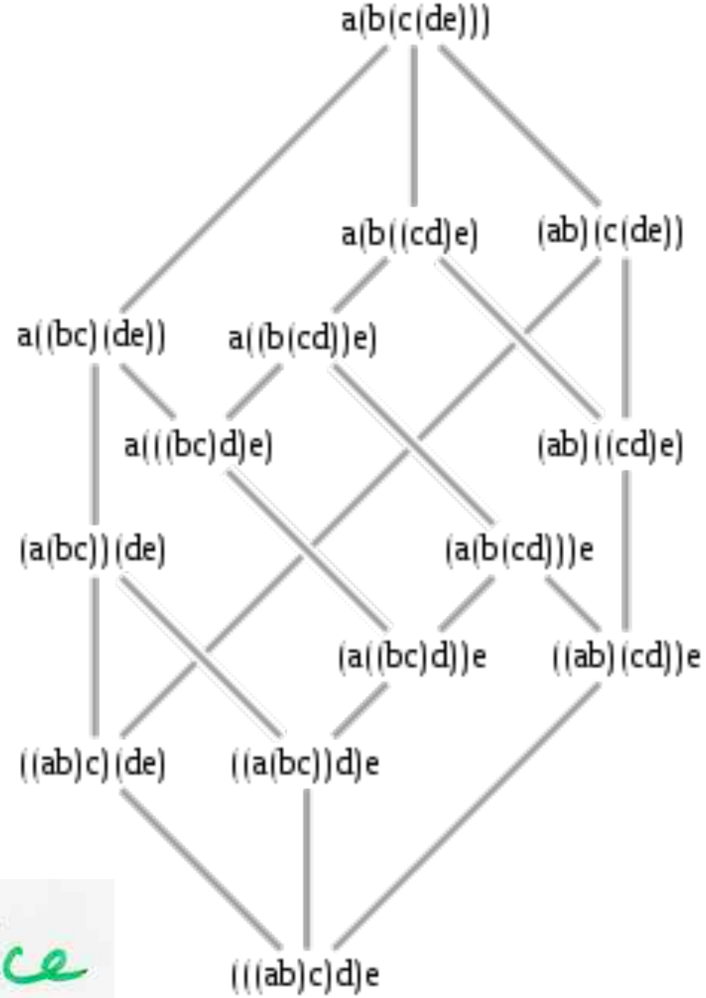
well parenthesis expression

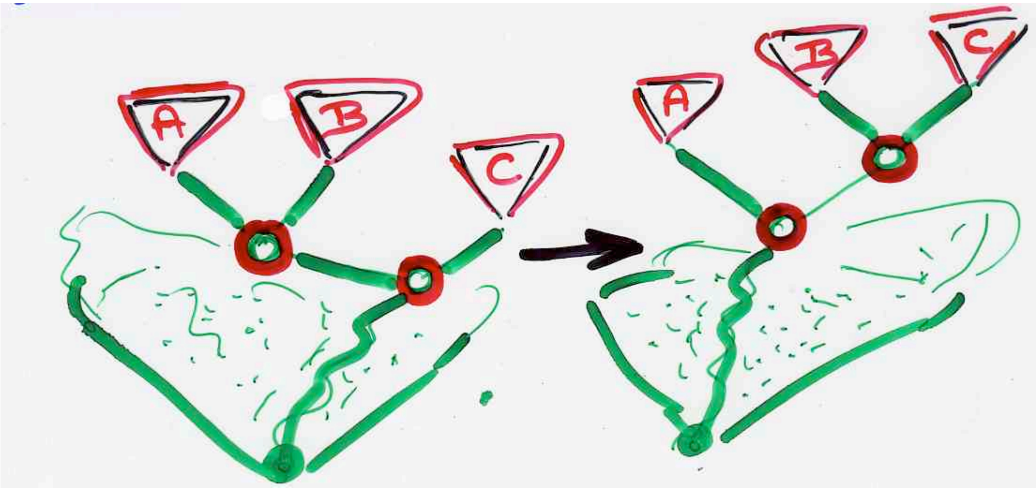
associativity

… ((u v) w) ….
…. (u (v w)) …

order relation

Tamari lattice



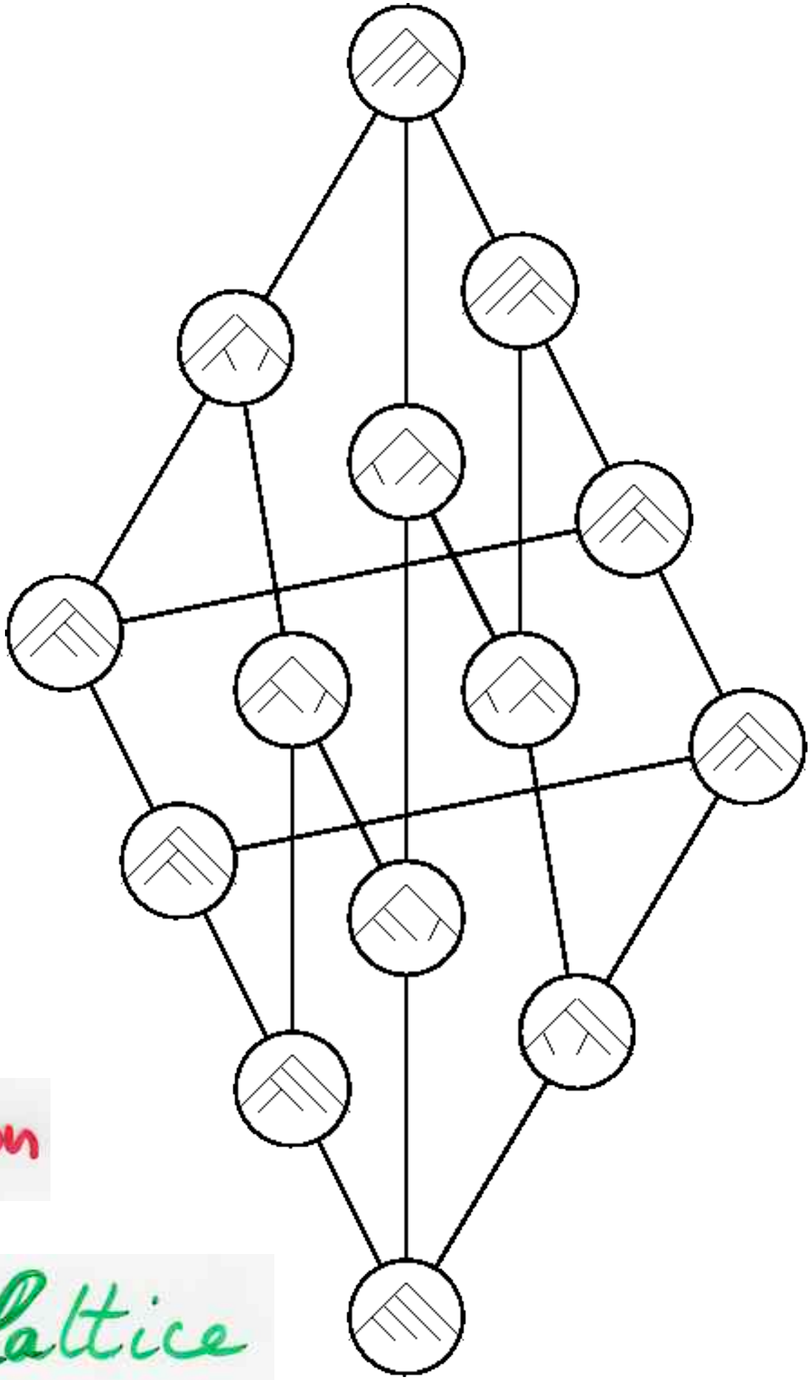


$\cdots((u \; v) \; w)\cdots$

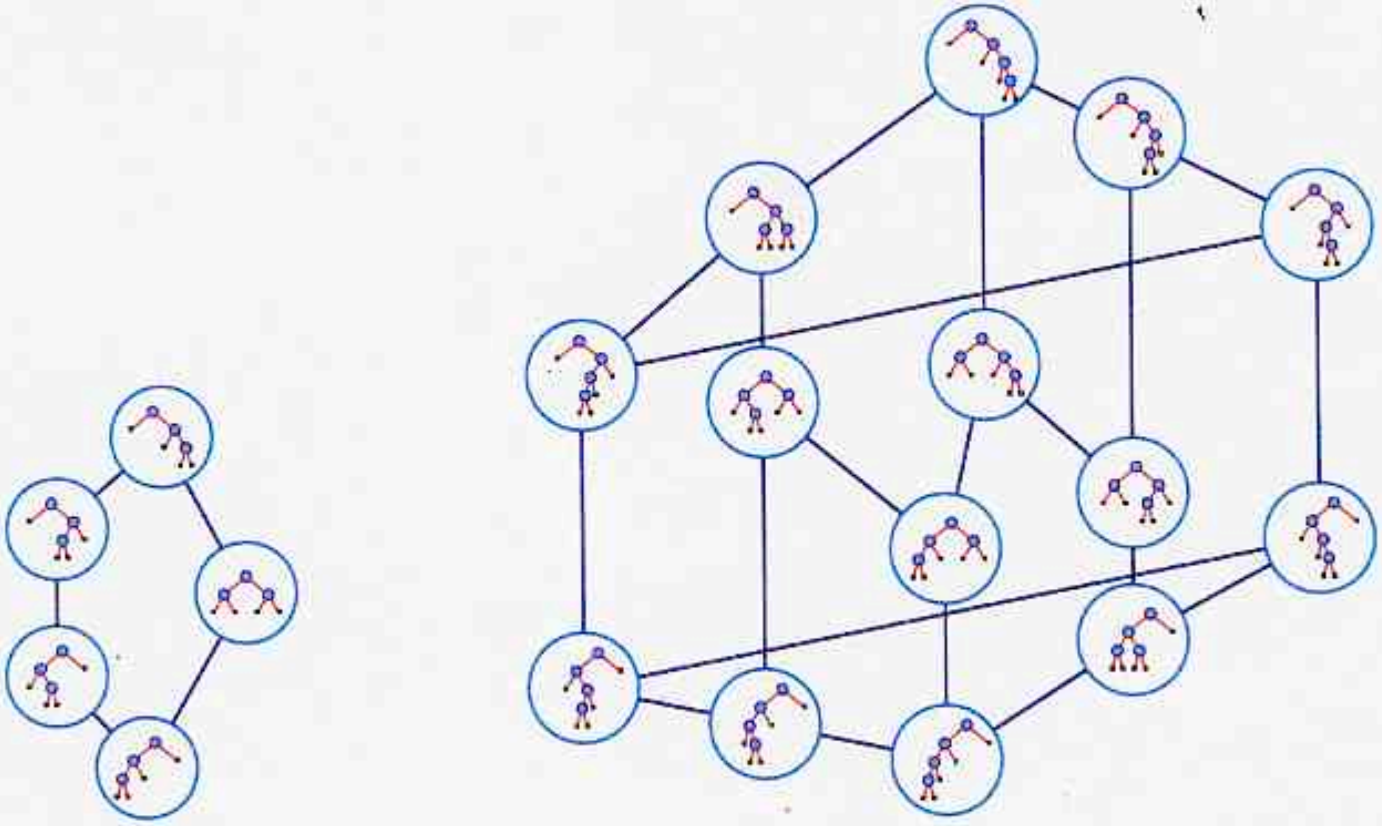
$\cdots(u \; (v \; w))\cdots$

associativity

order relation



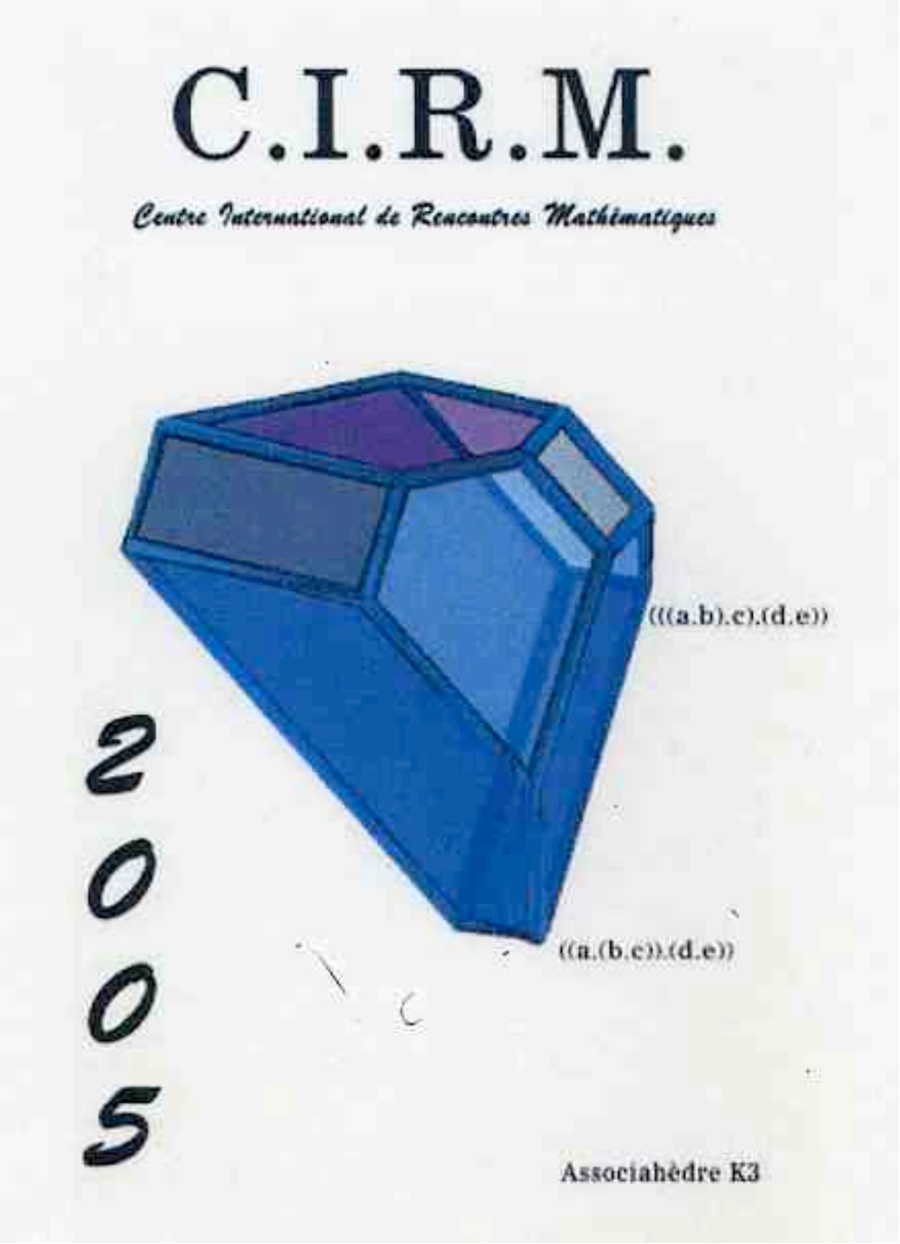
Tamari lattice



(a) $\mathbb{T}_3$ (b) $\mathbb{T}_4$

Fig. 3: The Tamari lattices $\mathbb{T}_3$ and $\mathbb{T}_4$.

associahedron



C.I.R.M.

Centre International de Rencontres Mathématiques

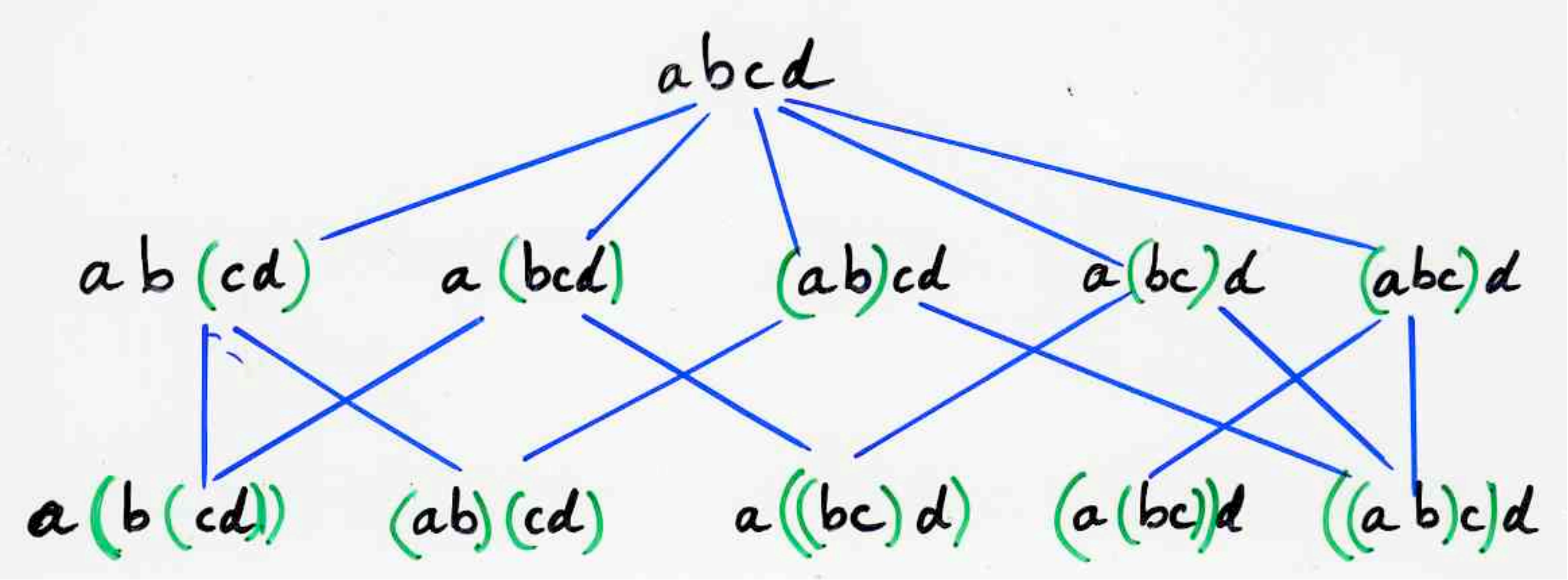
2005

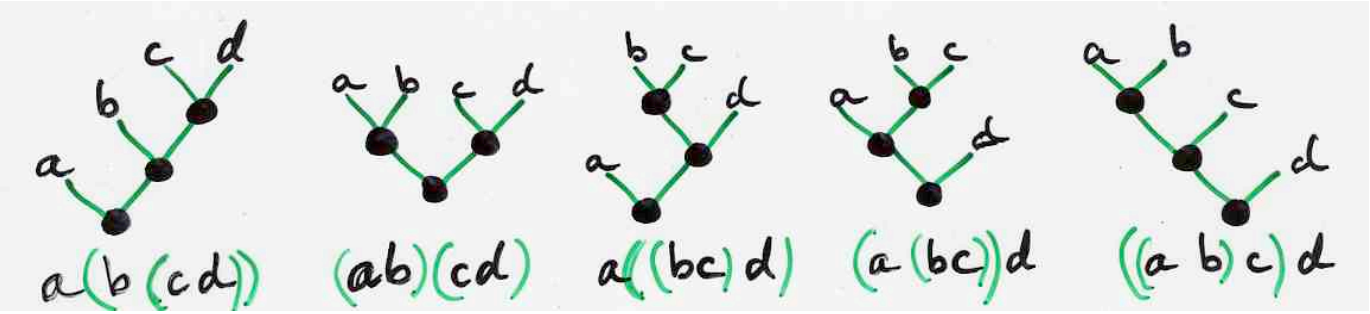
Associahèdre K3

cells structure

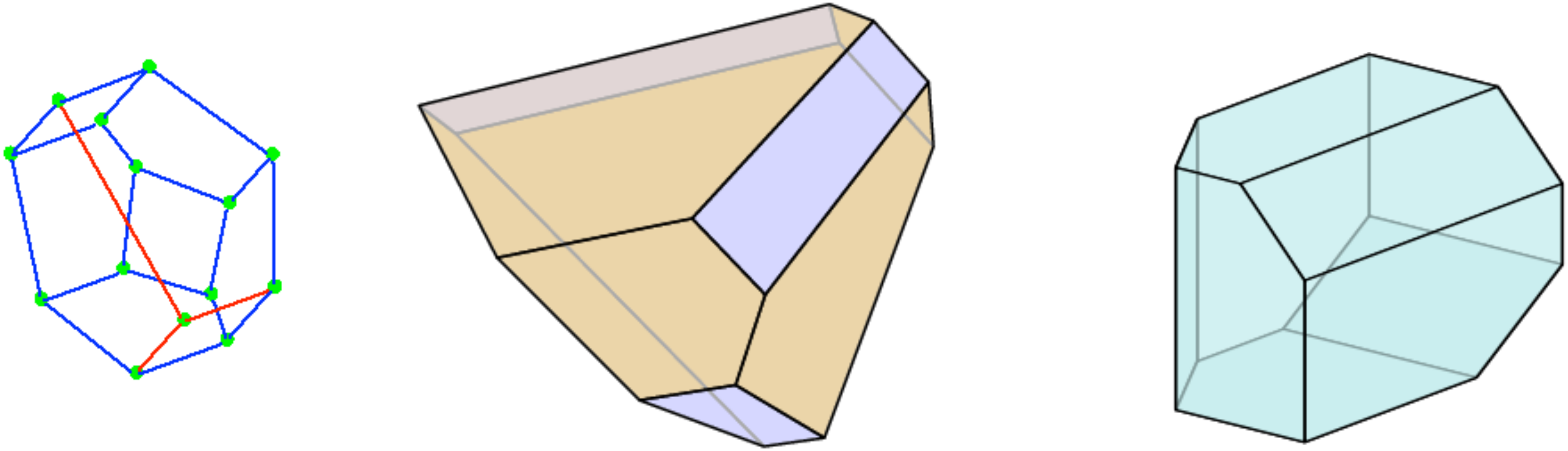
simplicial
complex

polytope



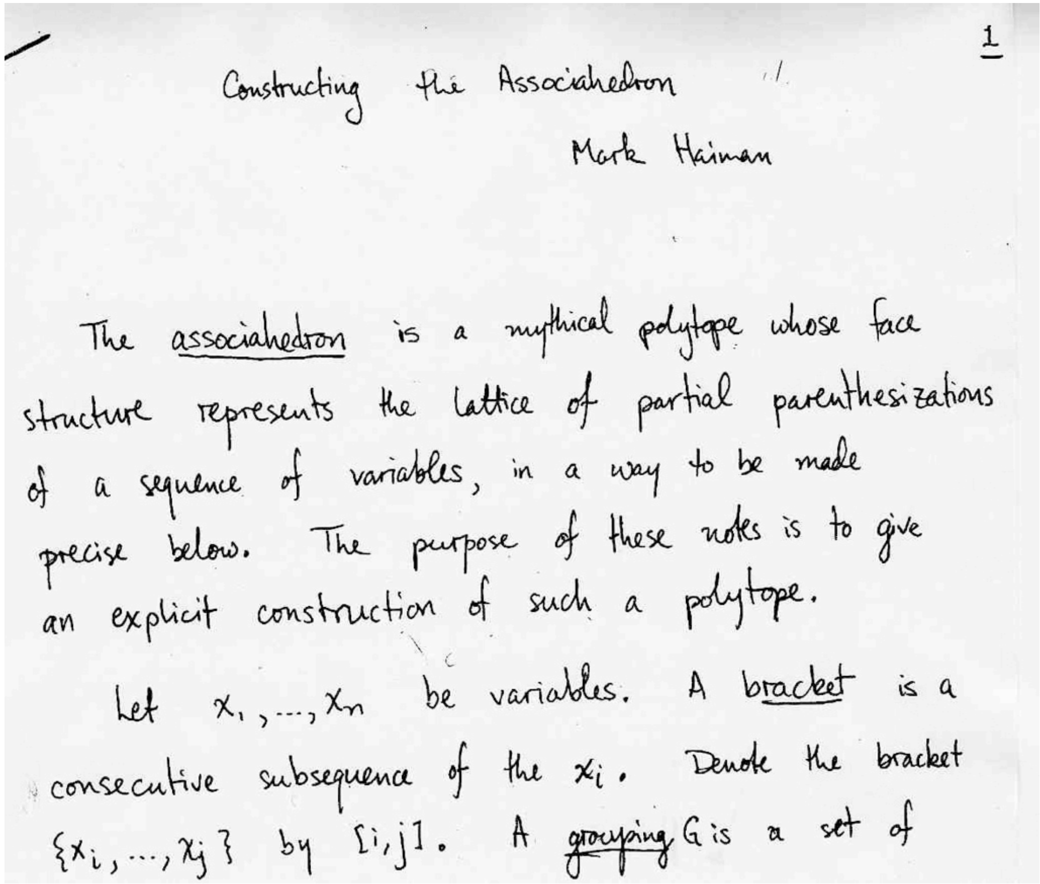


Is it possible to realize the cells structure of the associahedron as the cells of a convex polytope ?



S. Huang, D. Tamari JCTA (1972)
"a simple proof of the lattice property"
D. Huguet, D. Tamari (1978)
face-graph of a polytope ? no proof

M. Haiman (1984) manuscript
C. Lee (1989) first published proof
I. Gelfand, M. Kapranov, A. Zelevinsky (1994)
"secondary polytope" ← fiber polytope



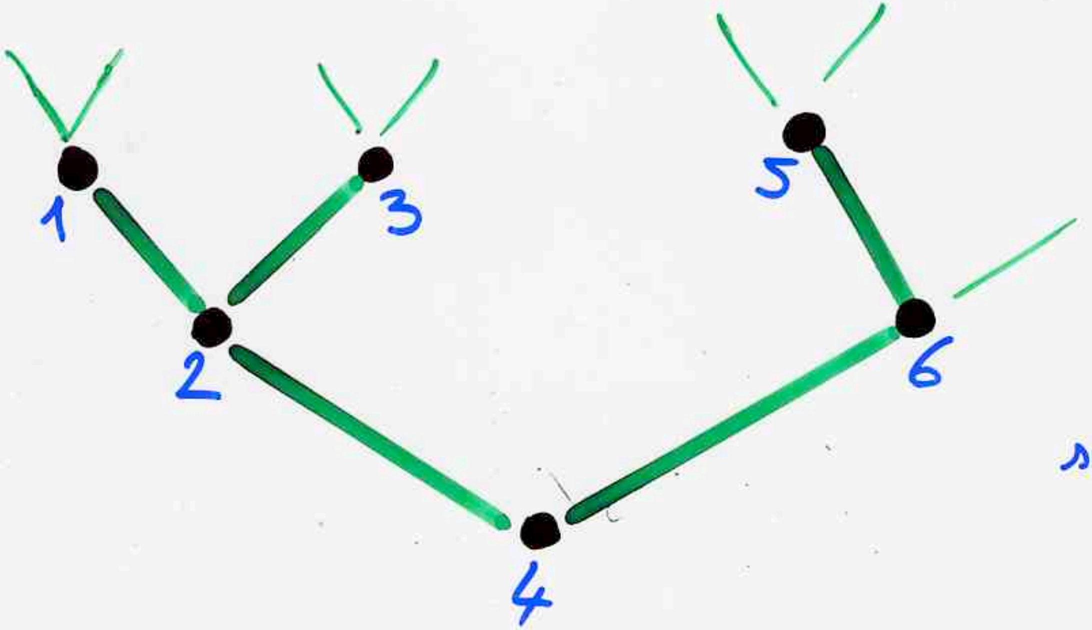
Constructing the Associahedron

Mark Haiman

The *associahedron* is a mythical polytope whose face structure represents the lattice of partial parenthesizations of a sequence of variables, in a way to be made precise below. The purpose of these notes is to give an explicit construction of such a polytope.

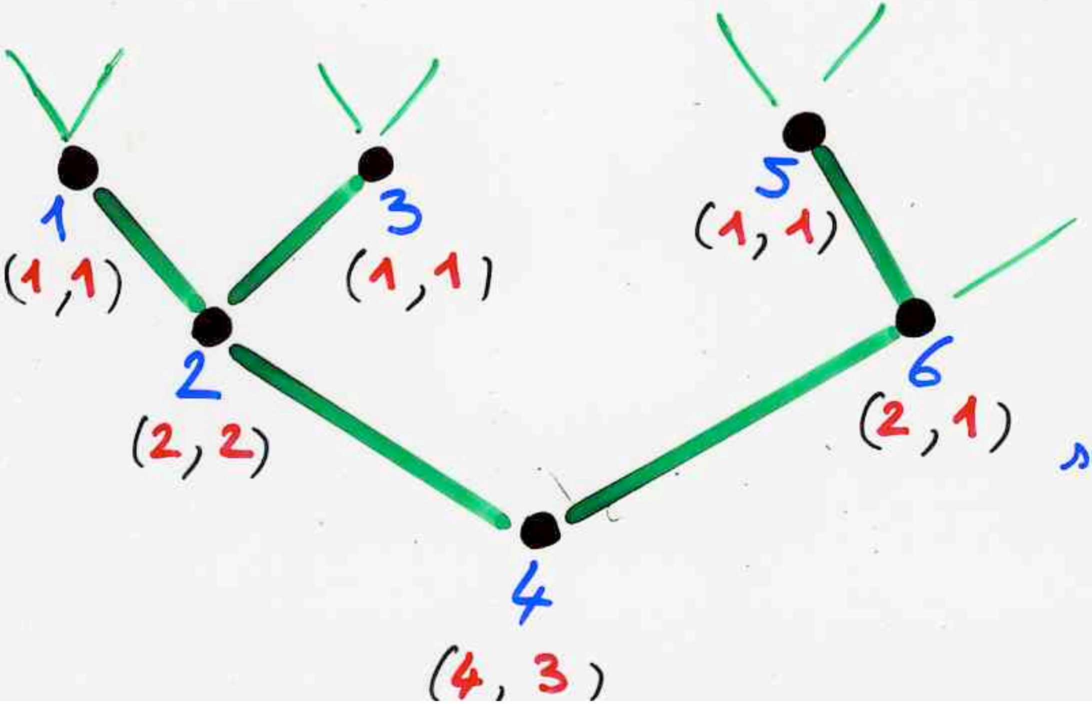
Let $x_1, \dots, x_n$ be variables. A *bracket* is a consecutive subsequence of the x_i. Denote the bracket $\{x_i, \dots, x_j\}$ by $[i,j]$. A *grouping* G is a set of

J.-L. Loday (2004) arXiv : déc 2002
"Realization of the Stasheff polytope"



symmetric order

J.-L. Loday (2004) arXiv: de'c 2002
"Realization of the Stasheff polytope"



symmetric order

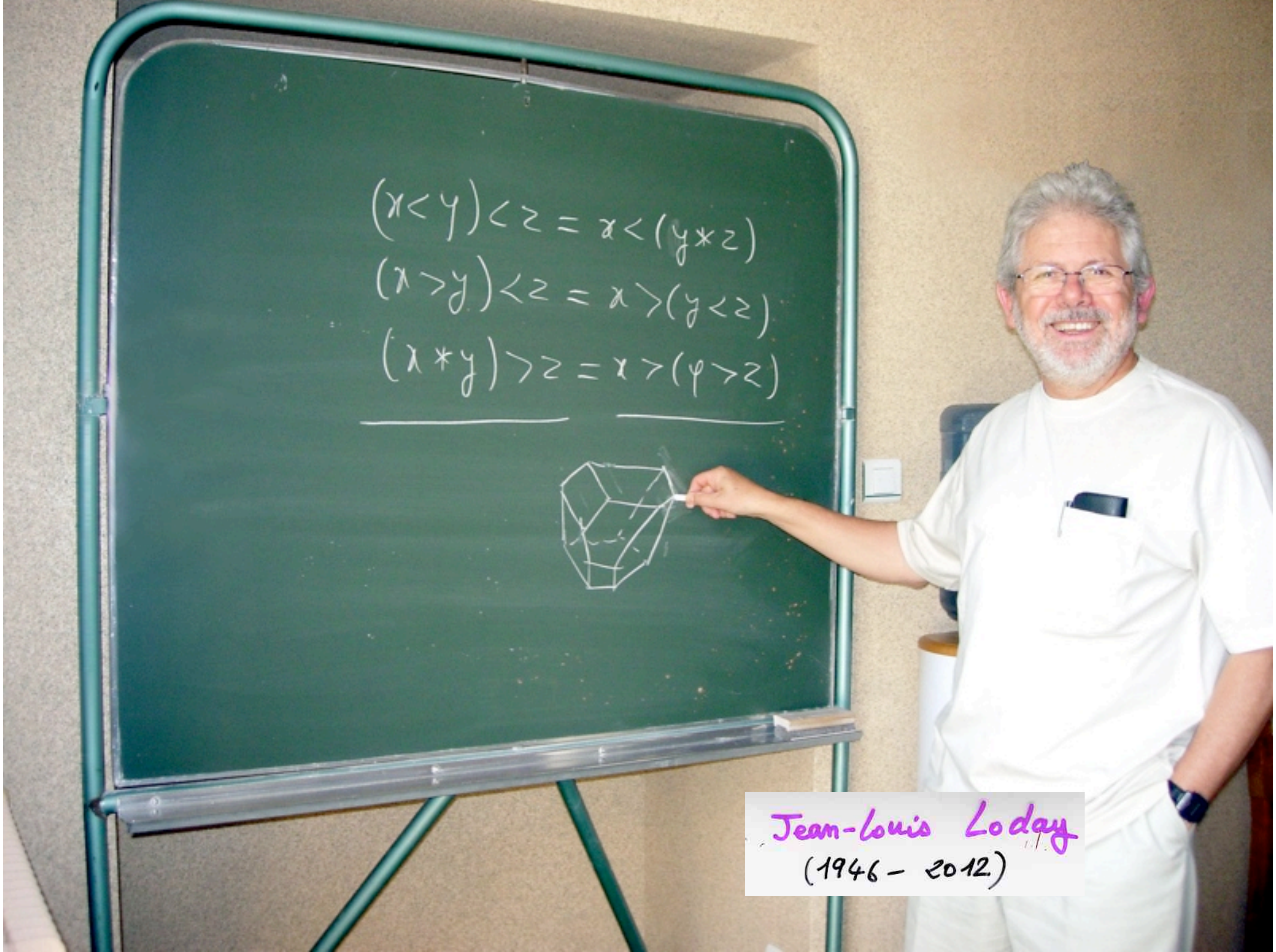
$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ (1, & 4, & 1, & 12, & 1, & 2) \end{array}$$

$\xrightarrow[\text{sum}]{} 21$

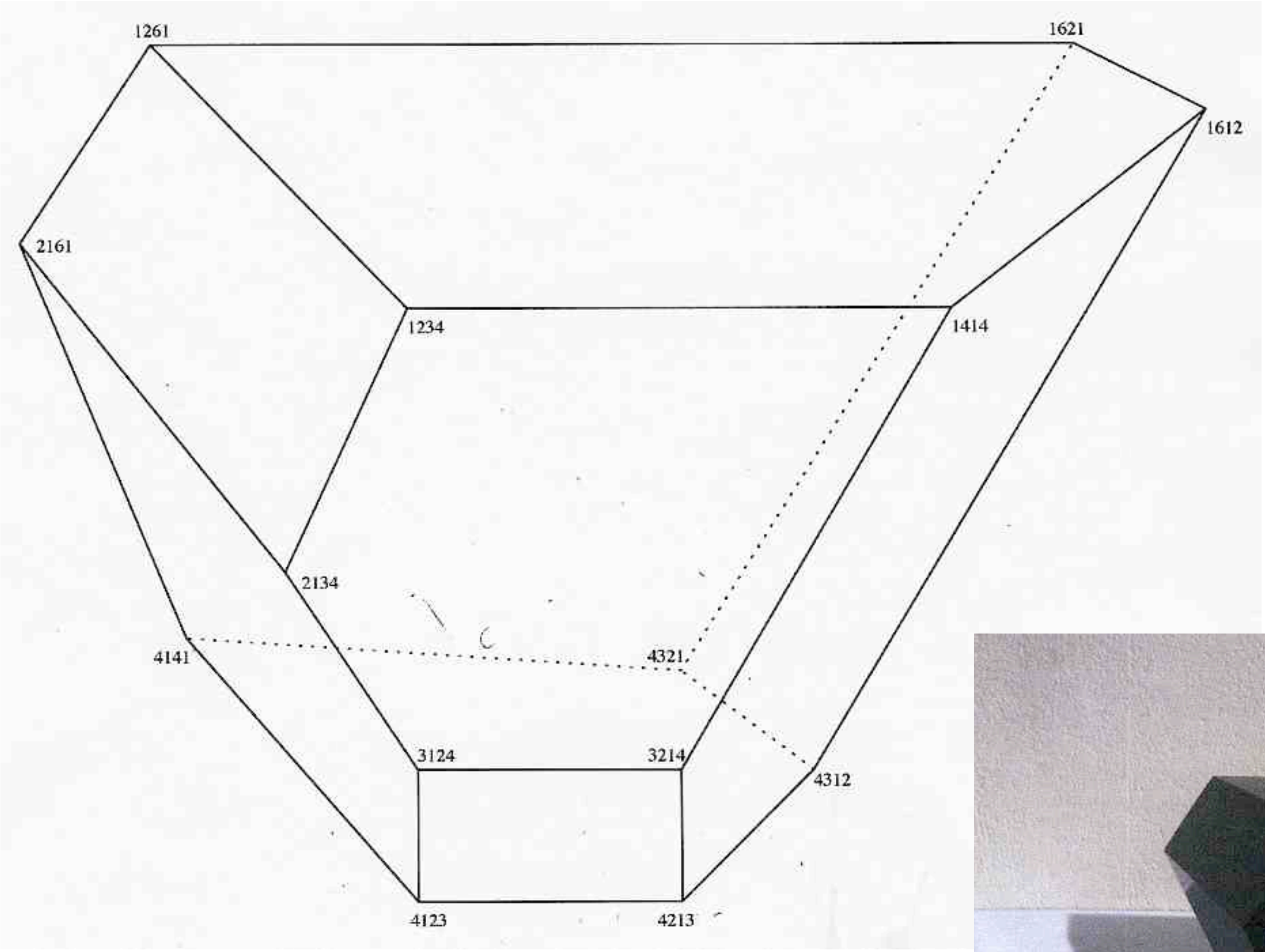
$$\frac{n(n+1)}{2}$$

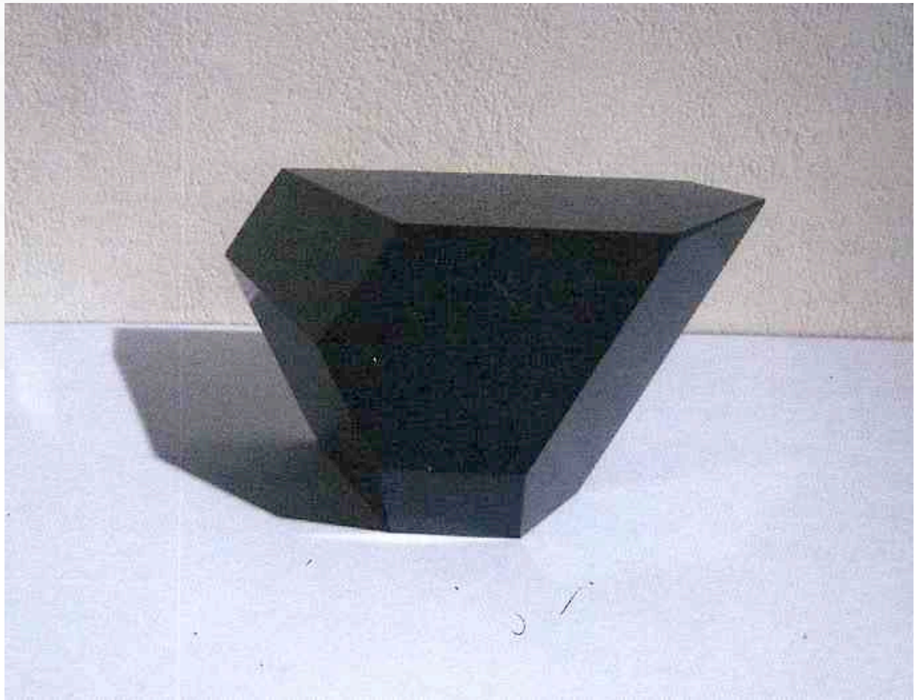
convex hull of the points

hyperplan
$x_1 + \dots + x_n = \frac{n(n+1)}{2}$



Jean-Louis Loday
(1946 – 2012)


1261
1621
1612
2161
1234
1414
2134
4141
4321
3124
3214
4312
4123
4213



C. Hohlweg, C. Lange (2007)
F. Chapoton, S. Fomin, A. Zelevinsky (2002)

extensions :

C. Ceballos
V. Pilaud
N. Reading
R. Marsh
D. Speyer

J.-P. Labbé
N. Bergeron
H. Thomas
M. Reineke
S. Stella

C. Stump
F. Santos
A. Postnikov
C. Athanasiadis
G. Ziegler

Gil Kalai

and many others ...

combinatorial structures

hypercube associahedron permutahedron

lexicographic order ↔ Tamari order ↔ weak Bruhat order

(boolean lattice inclusion)

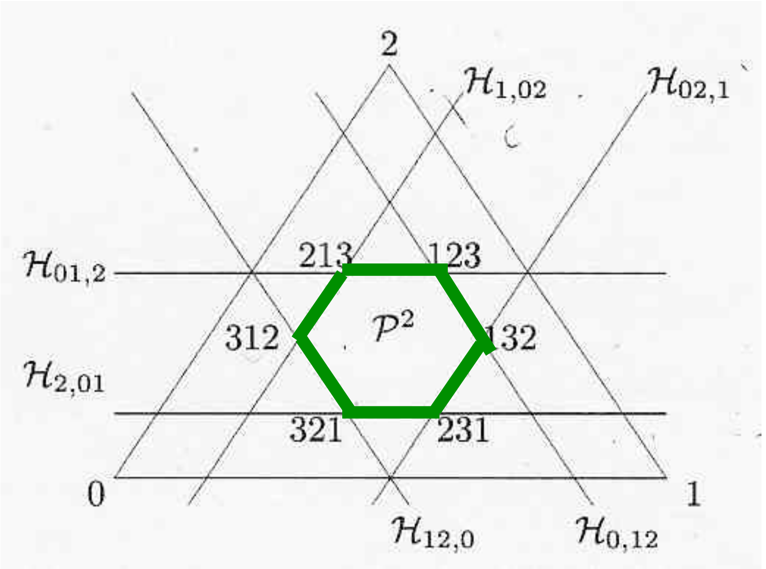
dim 2^n C_n $n!$

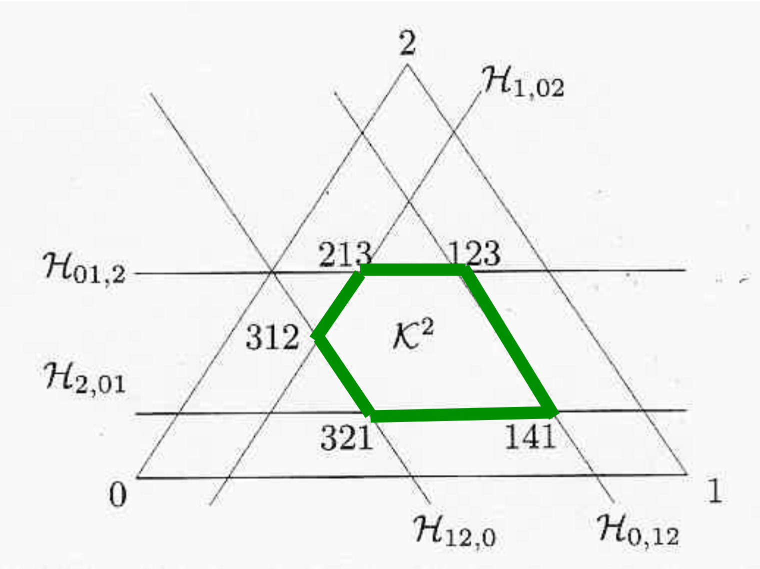
Catalan

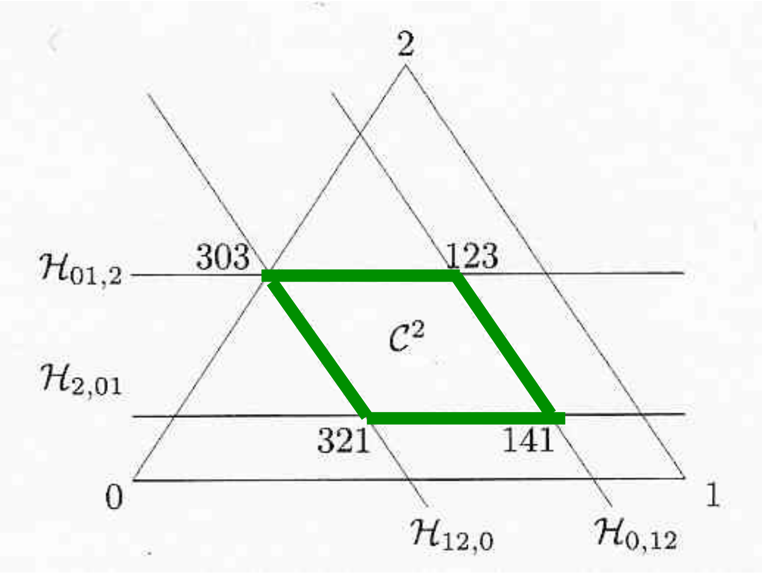
algebraic structures
Hopf algebra

	descent algebra	Loday-Ronco algebra	Reutenauer Malvenuto algebra
dim	2^n	C_n Catalan	$n!$

hypercube		associahedron		permutahedron
lexicographic order (boolean lattice inclusion)	↔	Tamari order	↔	weak Bruhat order

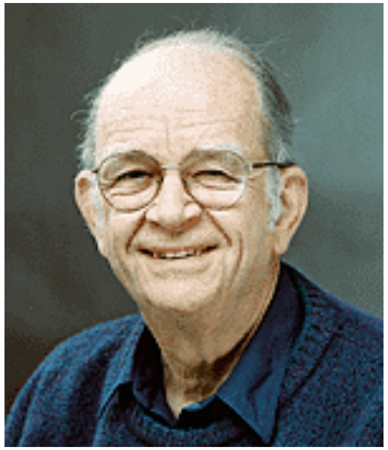

2
$\mathcal{H}_{1,02}$
$\mathcal{H}_{02,1}$
$\mathcal{H}_{01,2}$
213
123
312
$\mathcal{P}^2$
132
$\mathcal{H}_{2,01}$
321
231
0
1
$\mathcal{H}_{12,0}$
$\mathcal{H}_{0,12}$


2
$\mathcal{H}_{1,02}$
$\mathcal{H}_{01,2}$
213
123
312
$\mathcal{K}^2$
$\mathcal{H}_{2,01}$
321
141
0
1
$\mathcal{H}_{12,0}$
$\mathcal{H}_{0,12}$


2
$\mathcal{H}_{01,2}$
303
123
$\mathcal{C}^2$
$\mathcal{H}_{2,01}$
321
141
0
1
$\mathcal{H}_{12,0}$
$\mathcal{H}_{0,12}$

Stasheff polytope (1963)

thesis (1961)



J. Milnor

Homotopy theory

loops

continuous

$[0,1] \xrightarrow{f} X$

topological space

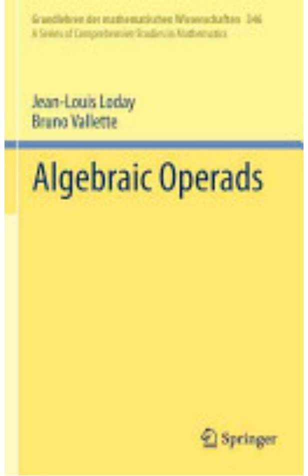
$[0,\frac{1}{2}] \xrightarrow{f} X$

$[\frac{1}{2},1] \xrightarrow{g} X$

Operads

book J.-L. Loday
B. Vallette
(xxiv, 634 p)

Algebraic Operads
(Springer, 2012)



Grundlehren der mathematischen Wissenschaften 346
A Series of Comprehensive Studies in Mathematics

Jean-Louis Loday
Bruno Vallette

Algebraic Operads

Springer

Physics

A. Dimakis, F. Müller-Hoissen

soliton KP-equation
waves in shallow water
⟷ maximal chains in the Tamari lattice


Progress in Mathematics 299

Folkert Müller-Hoissen
Jean Marcel Pallo
Jim Stasheff
Editors

Associahedra, Tamari Lattices and Related Structures

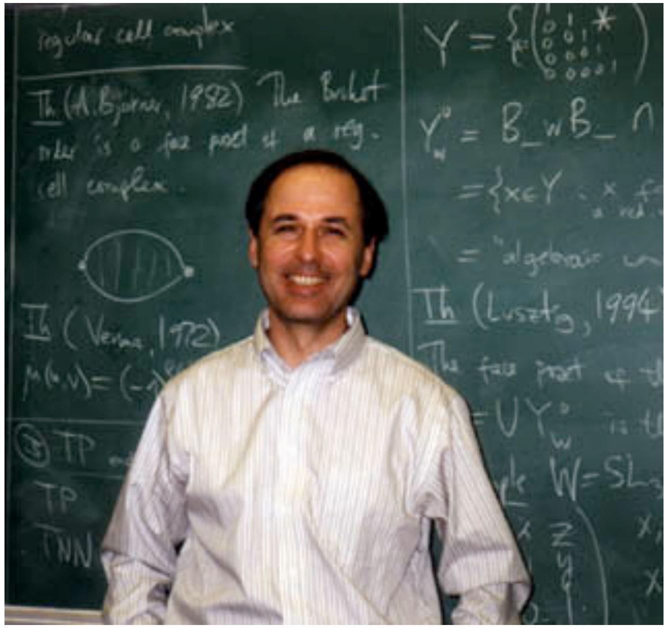
Tamari Memorial Festschrift

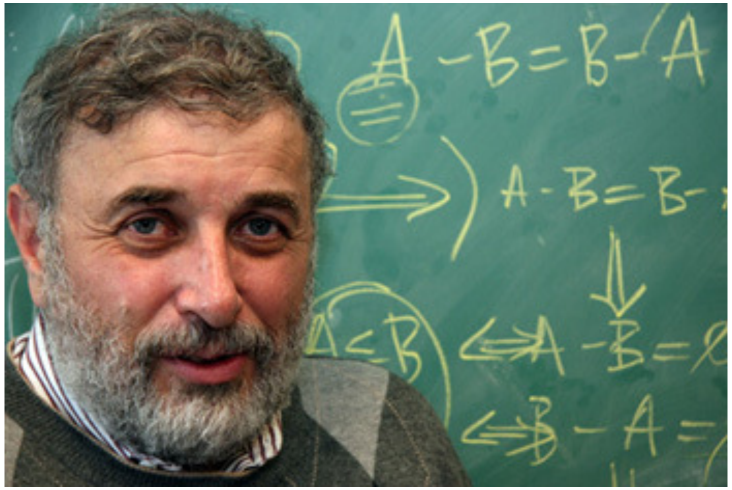
Birkhäuser

"Associahedra, Tamari lattice, and related structure" Progress in Math vol 299
Birkhauser (2012)

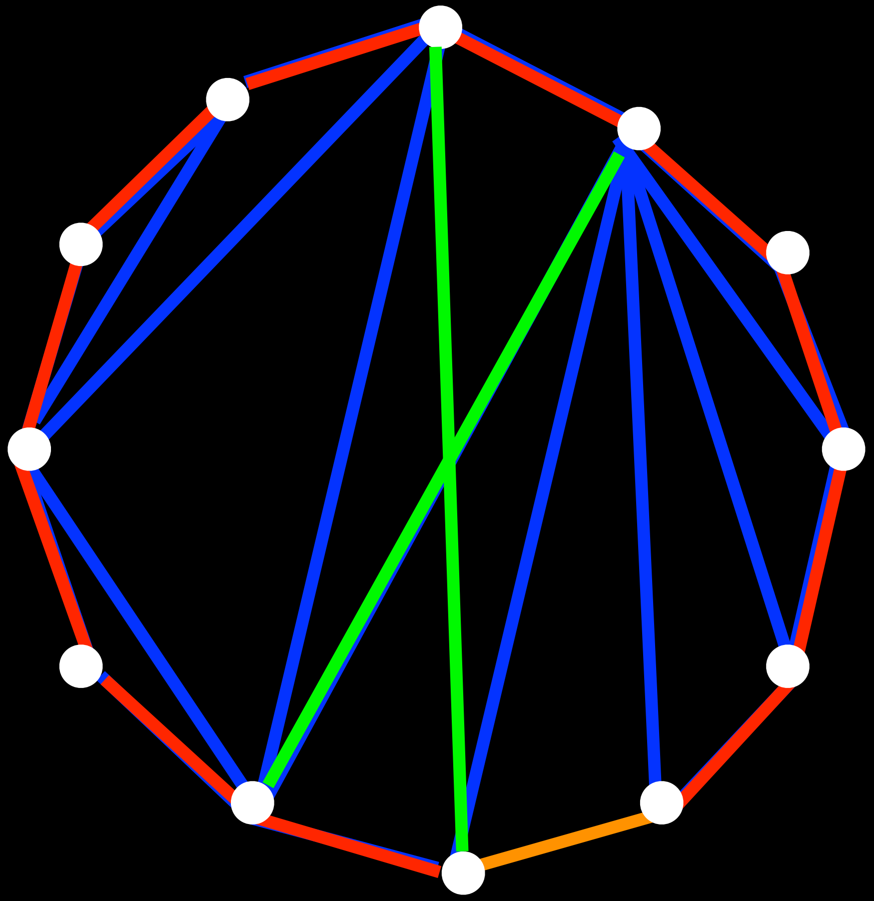
P. Dehornoy
symmetric Thompson monoid
F Thompson group flip distance, diameter

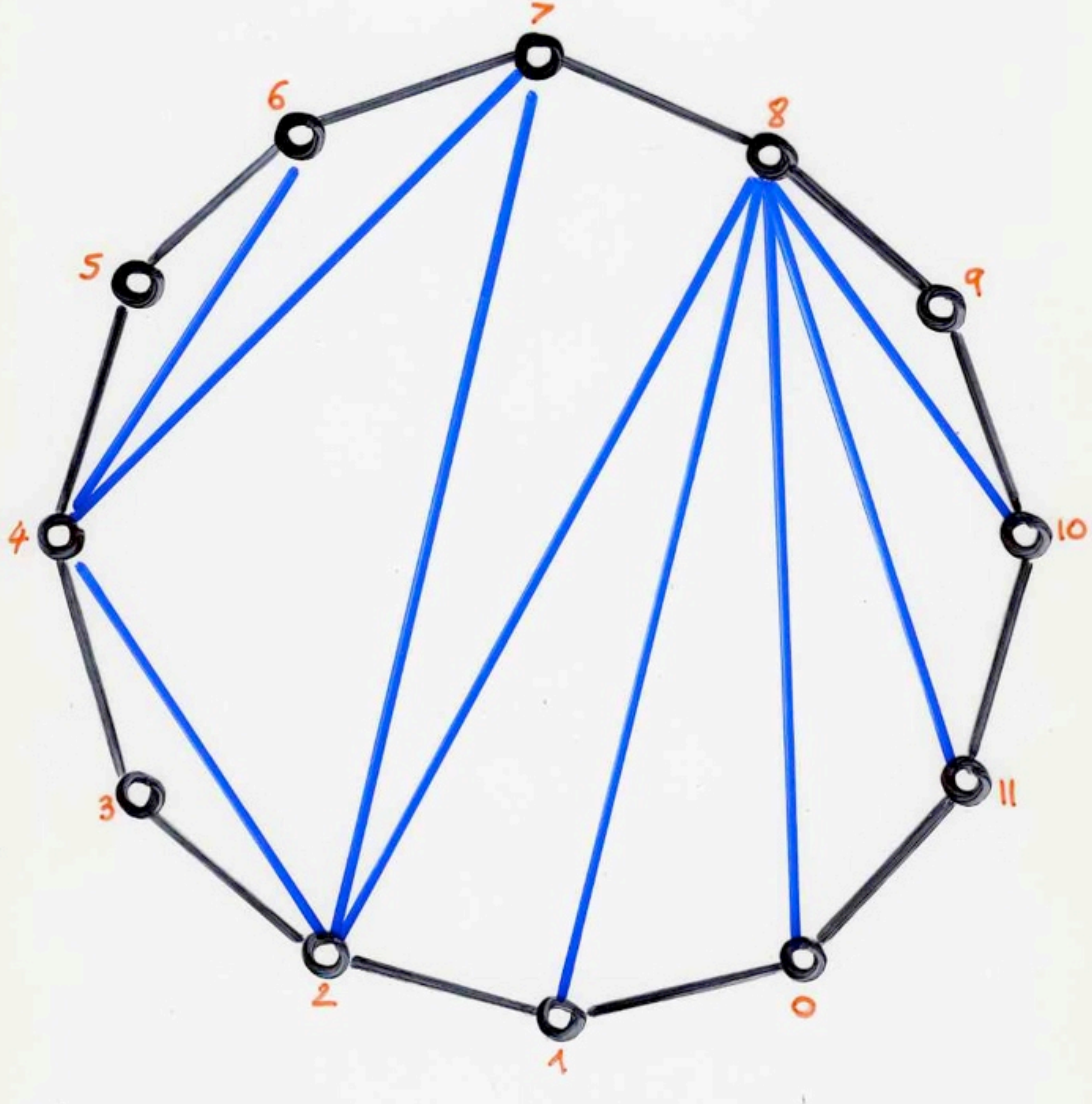
root systems cluster algebras

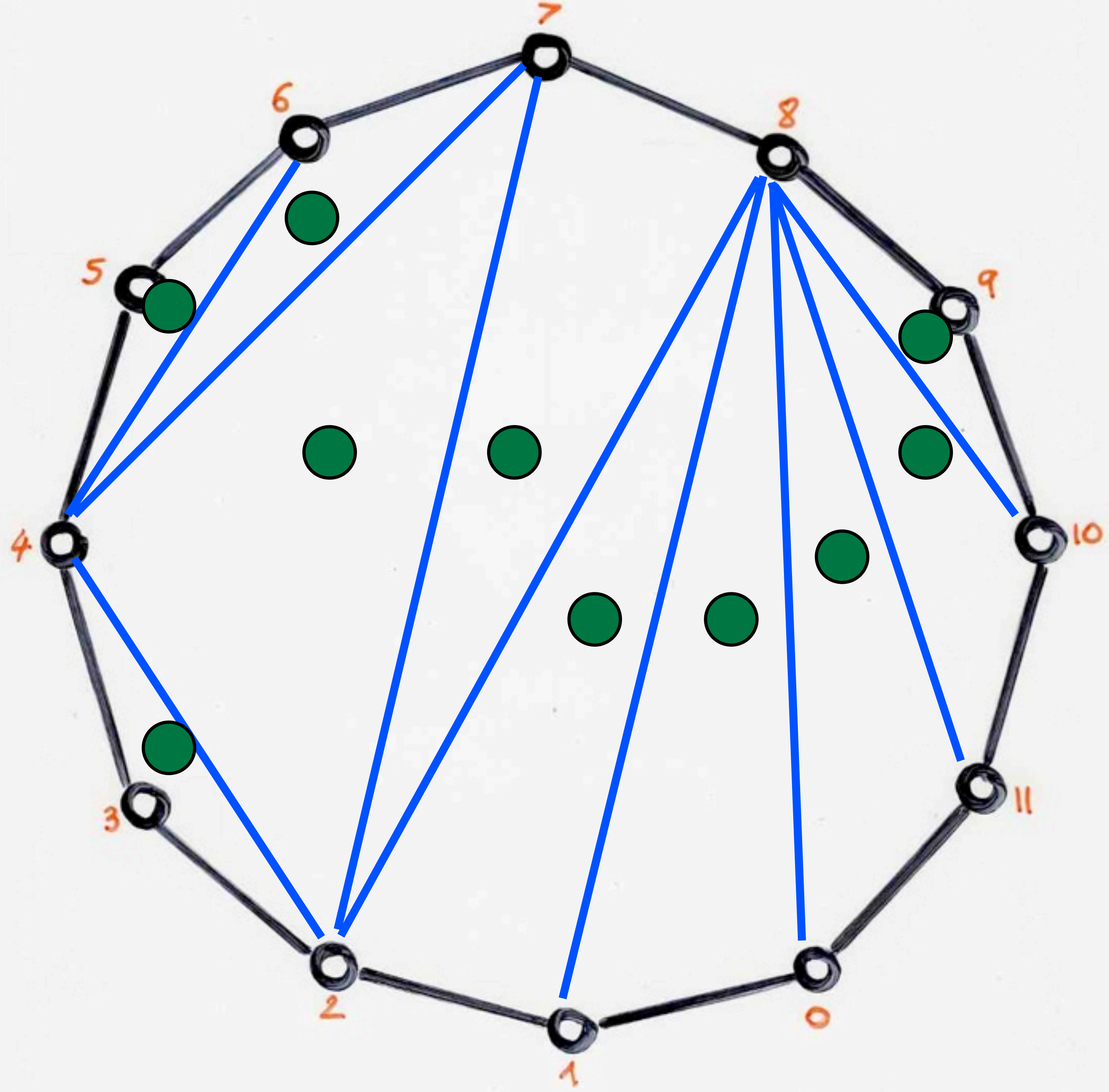


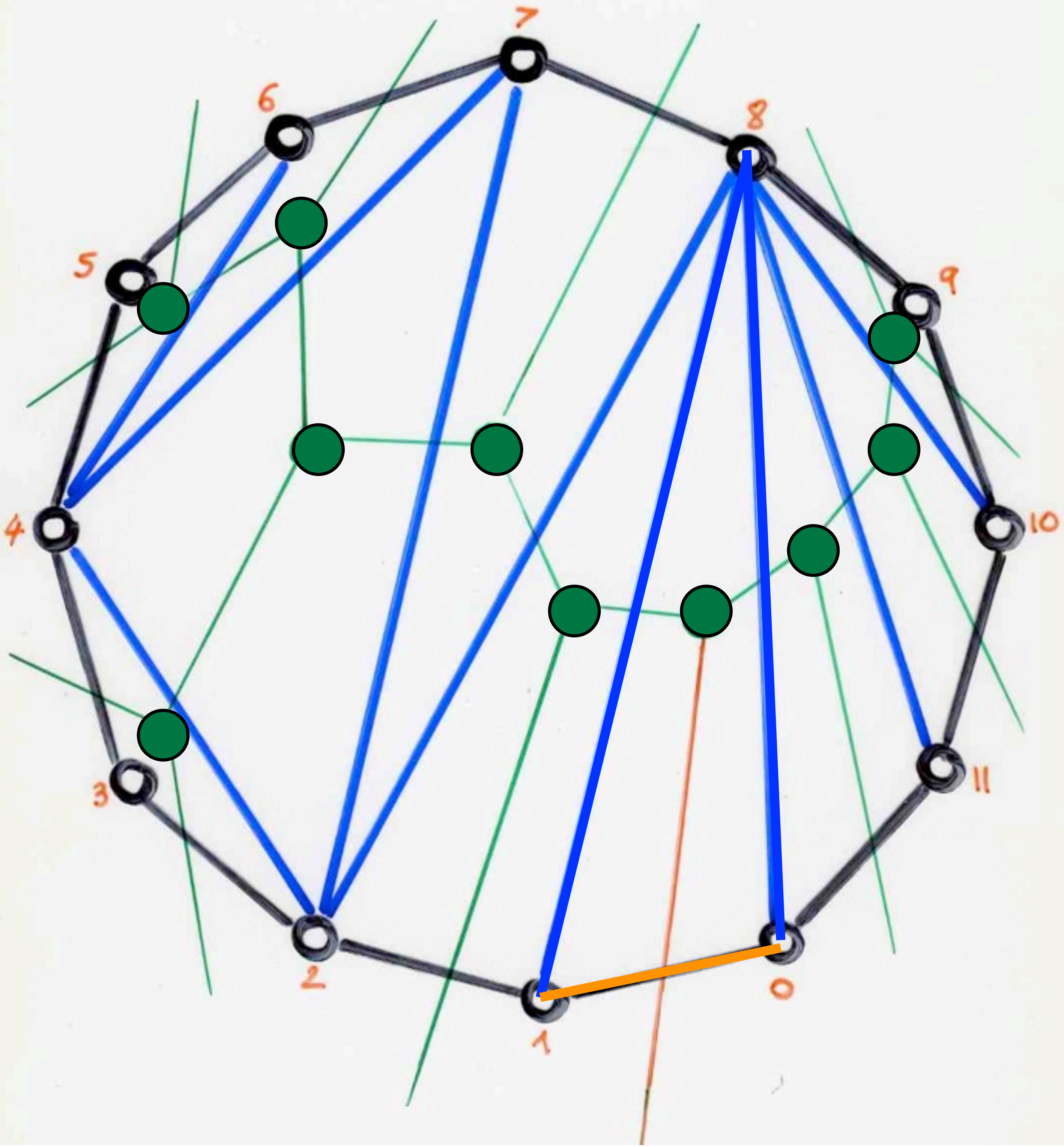


associahedron




7
6
8
5
9
4
10
3
11
2
0
1


7
6
8
5
9
4
10
3
11
2
0
1


7
8
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10
3
11
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1
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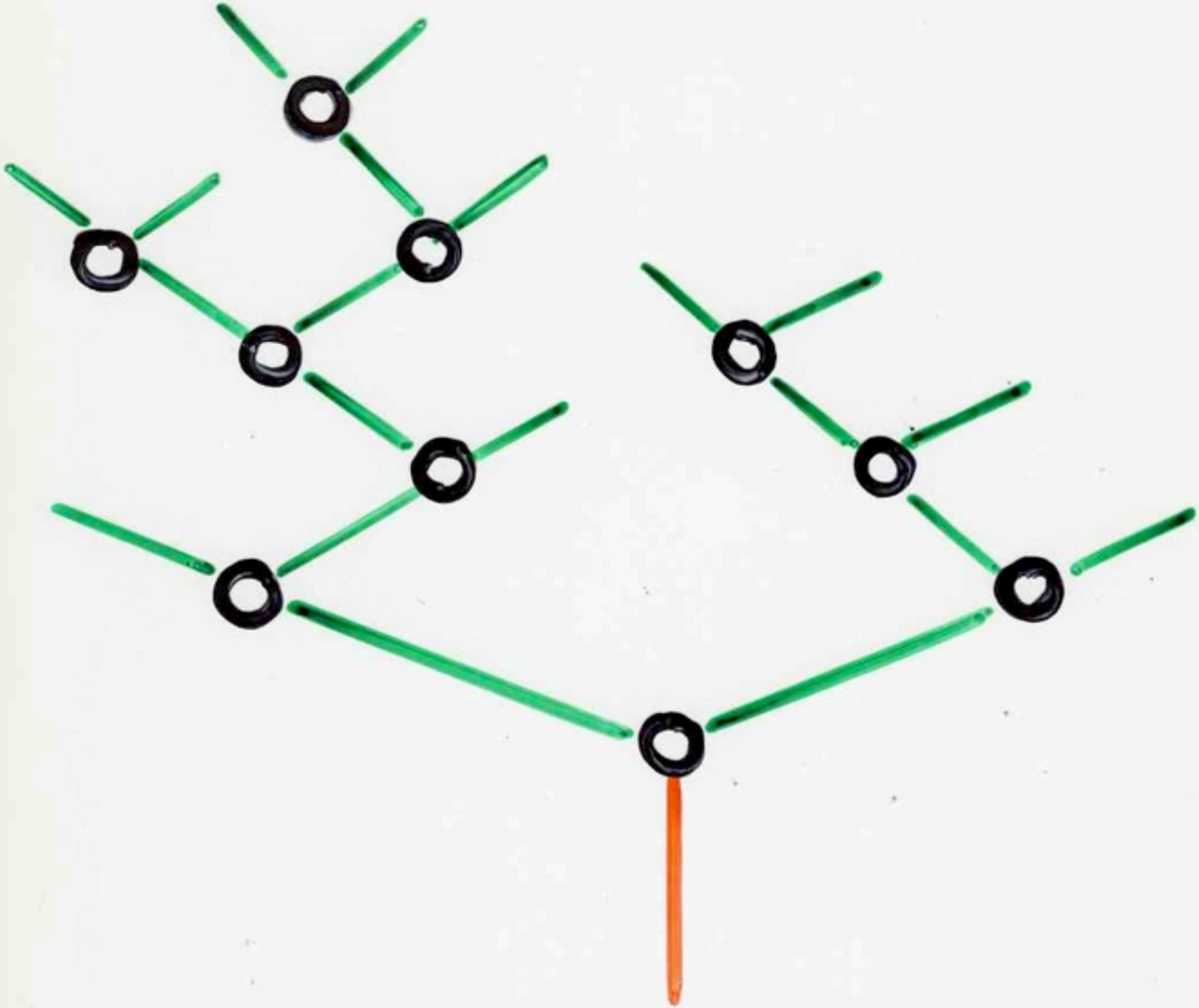
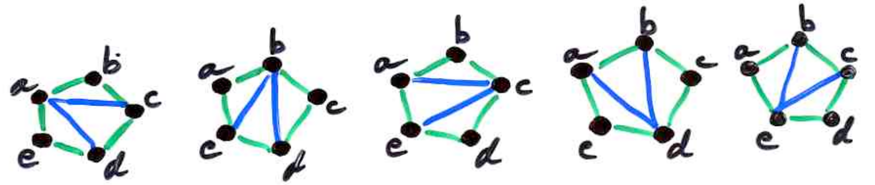


theilet, und solches kan auf 5 verschiedenen Arten geschehen, nehmlich durch die Diagonales I. $\frac{ac}{ad}$; II. $\frac{be}{bd}$; III. $\frac{ca}{ce}$; IV. $\frac{da}{db}$; V. $\frac{ec}{eb}$



Setze ich nun die Anzahl dieser verschiedenen Arten $= x$
so habe ich per inductionem gefunden

wann $n = 3.\ 4.\ 5.\ 6.\ 7.\ 8.\ 9.\ 10$

so ist $x = 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430$

Hieraus habe ich nun den Schluss gemacht, dass generaliter
sey

$$x = \frac{2.6.10.14.18.22.\ldots\ldots(4n-10)}{2.3.4.5.6.7.\ldots\ldots(n-1)} = \frac{(2n)!}{(n+1)!\,n!}$$

$1 = \frac{2}{2}$; $2 = 1.\frac{6}{3}$; $5 = 2.\frac{10}{4}$; $14 = 5.\frac{14}{5}$, $42 =$ [illegible] leicht gefunden, dass also auch eine jede Zahl die folgende [illegible] Die Induction aber, so ich gebraucht, war ziemlich mühsam, doch zweifle ich nicht, dass diese Sach nicht solte weit leichter entwickelt werden können. Ueber die Progression der Zahlen 1, 2, 5, 14, 42, 132, etc. habe ich auch diese Eigenschaft angemerket, dass

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc.} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$$

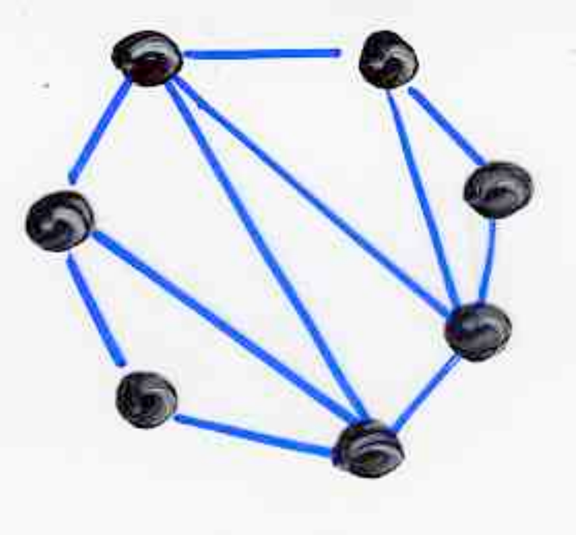
also wann $a = \frac{1}{4}$, so ist $1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} + \text{etc.} = 4$.

Ich bitte mir ferner Ew. Hochwohlgeb. beständige Gewogenheit gehorsamst aus, und habe die Ehre mit der schuldigsten Hochachtung Lebenslang zu beharren

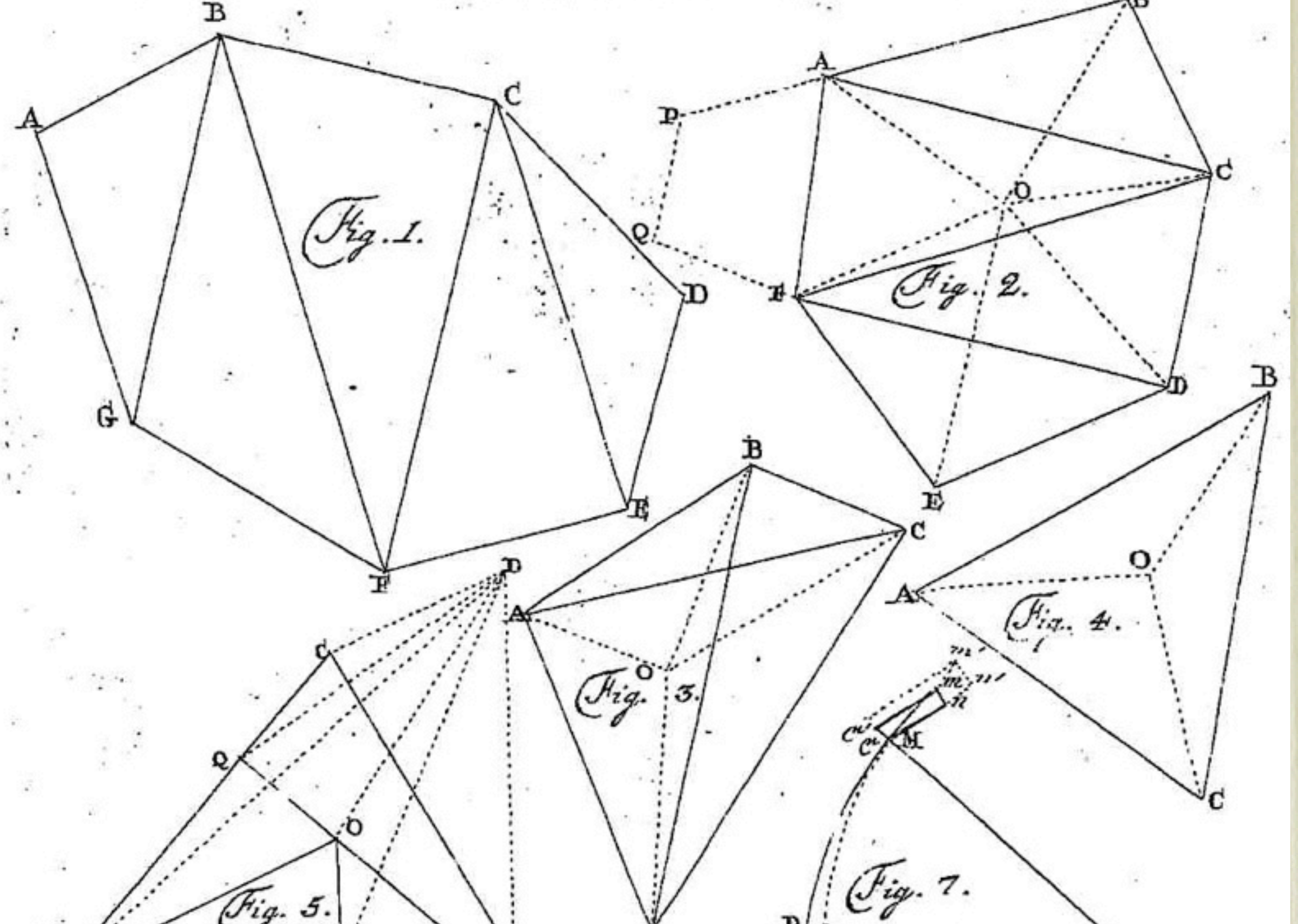
Ew. Hochwohlgebohren

Berlin d. 4ten Sept.
1751.

gehorsamster Diener
L. Euler



Comment. Nov Ac. Imp. Sc. Petrop Tom. IV. Tab III.




4000 BASEL
e-k+f=2
Leonhard Euler
1707-1783
AUSGABETAG 6. MÄRZ 2007
LEONHARD EULER
1707–1783
130
HELVETIA
Leonh. Euler
300
300. Geburtstag
300ème anniversaire
300° anniversario
300th anniversary

Combinatorial problems in lattices

Combinatorial
enumerative
problems
in lattices

poset

Partially
Ordered
Set

lattice

inf

sup

nb of intervals

nb of maximal chains

nb of intervals

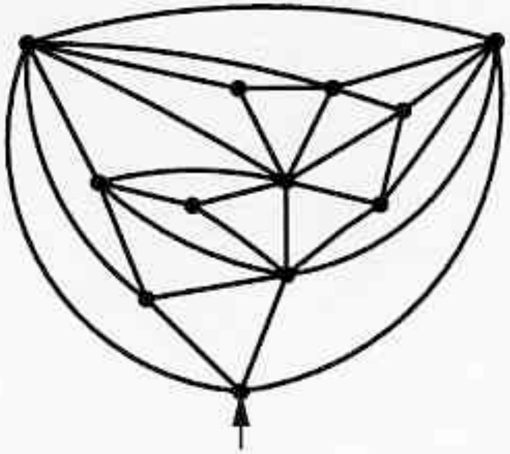
F. Chapoton (2006) $\dfrac{2(4n+1)!}{(n+1)!\,(3n+2)!}$

1, 3, 13, 68, 399, ...

triangulation

Bijective proof FPSAC 2007

Bernardi, N. Bonichon



M. Bousquet-Mélou, E. Fusy, L.-F. Préville-Ratelle (2011)
nb of intervals of m-Tamari lattices

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1}$$

F. Bergeron

M. Bousquet-Mélou, G. Chapuy, L.-F. Préville-Ratelle (2011)
nb of labelled intervals $(m+1)^n(mn+1)^{n-2}$

lattice as a polytope
nb of faces
nb of simplex in a triangulation

for the associahedron
$(n+1)^{n-1}$ parking functions
bijection J.-L. Loday

S. Giraudo FPSAC 2010

set of balanced binary tree

- closed by interval
- $[T_1, T_2]$ interval $\simeq$ $\mathbb{H}_k$ hypercube

product of two binary trees

- in the Loday-Ronco algebra "is" an interval
- in the # algebra "is" an interval

Aval, X.V. SLC

Aval, Novelli, Thibon FPSAC 2011

V. Pons

Tamari interval-posets ⟺ Tamari intervals

(FPSAC, 2013) thesis (Oct. 2013)

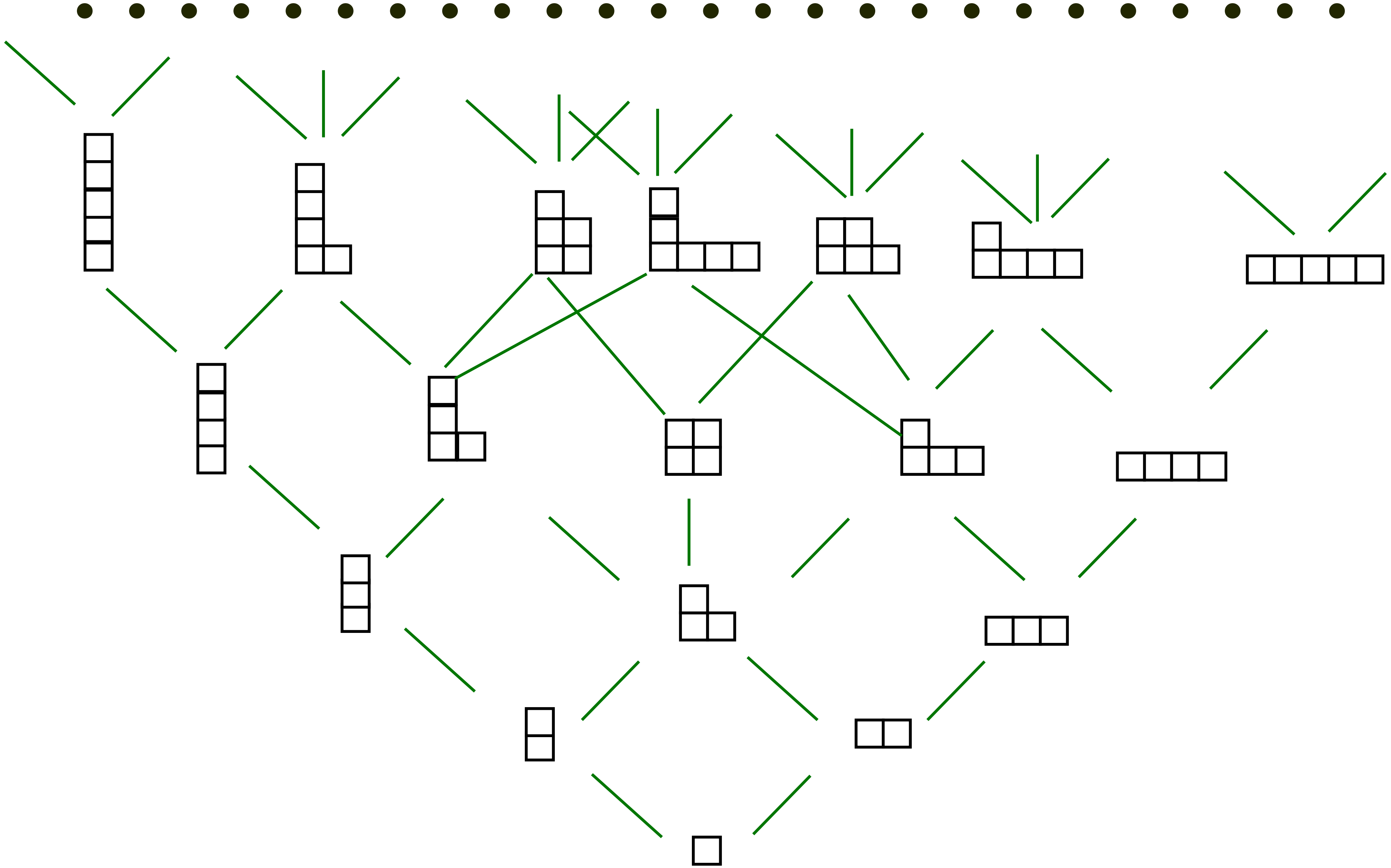
G. Chatel

bijections on Tamari intervals

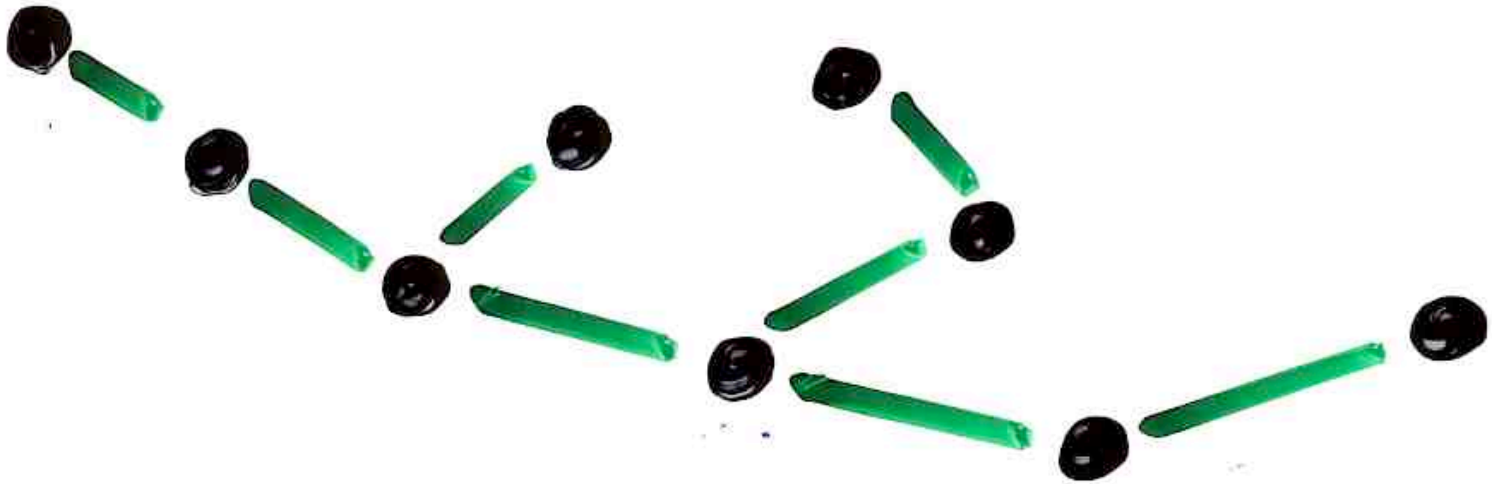
↔ closed flow of an ordered forest

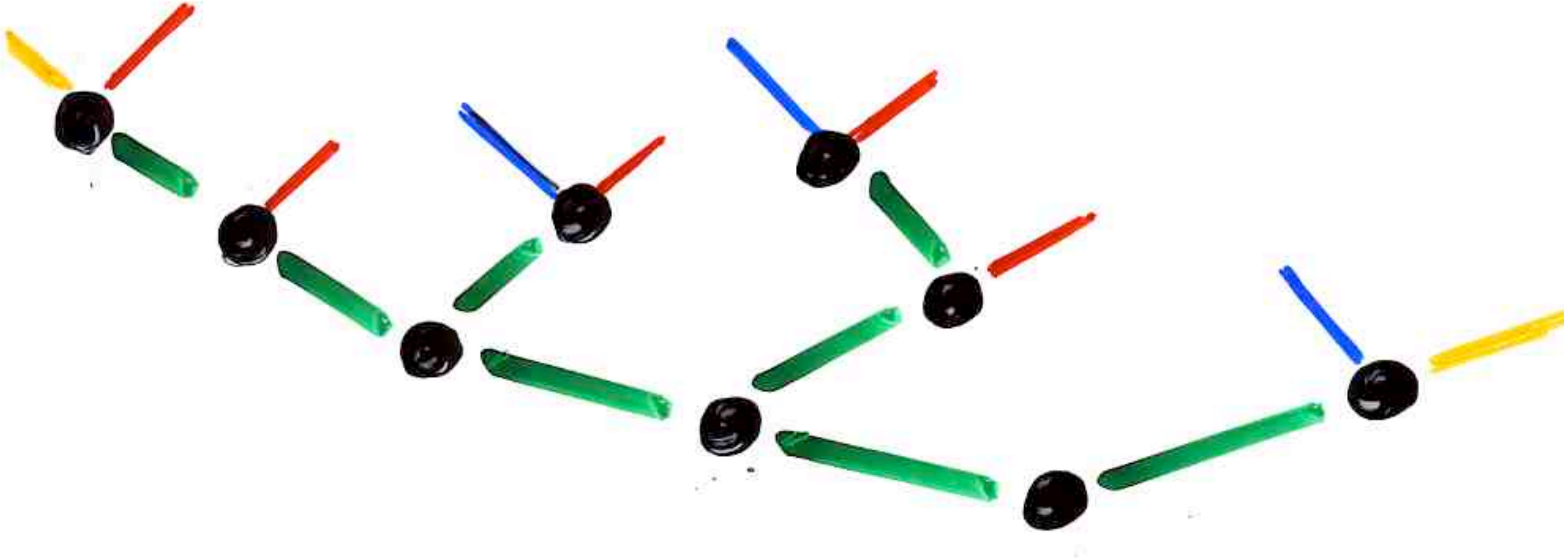
→ Pre-Lie operad F. Chapoton

Young lattice



canopy of a binary tree





canopy of a binary tree

$c(B) = - - + - + - - +$

montée $\sigma(i) < \sigma(i+1)$

$1 \leq i < n$

descente $\sigma(i) > \sigma(i+1)$

$\sigma \in \mathfrak{S}_n$

forme$(\sigma) = w_1 w_2 \cdots w_{n-1}$, $w_i = \begin{cases} + & \text{montée} \\ - & \text{descente} \end{cases}$ en i

UD(σ) mot $w \in \{+, -\}^*$

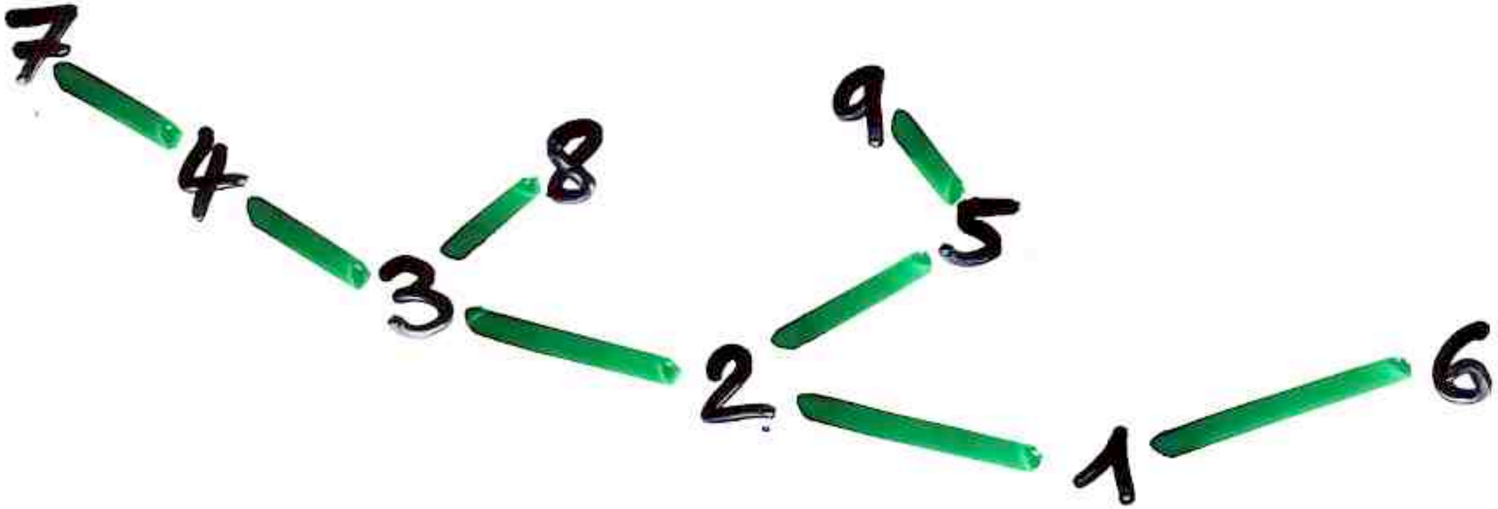
$\sigma = 5\,8\,2\,9\,6\,1\,4\,3$

forme$(\sigma) = + - + - - + -$

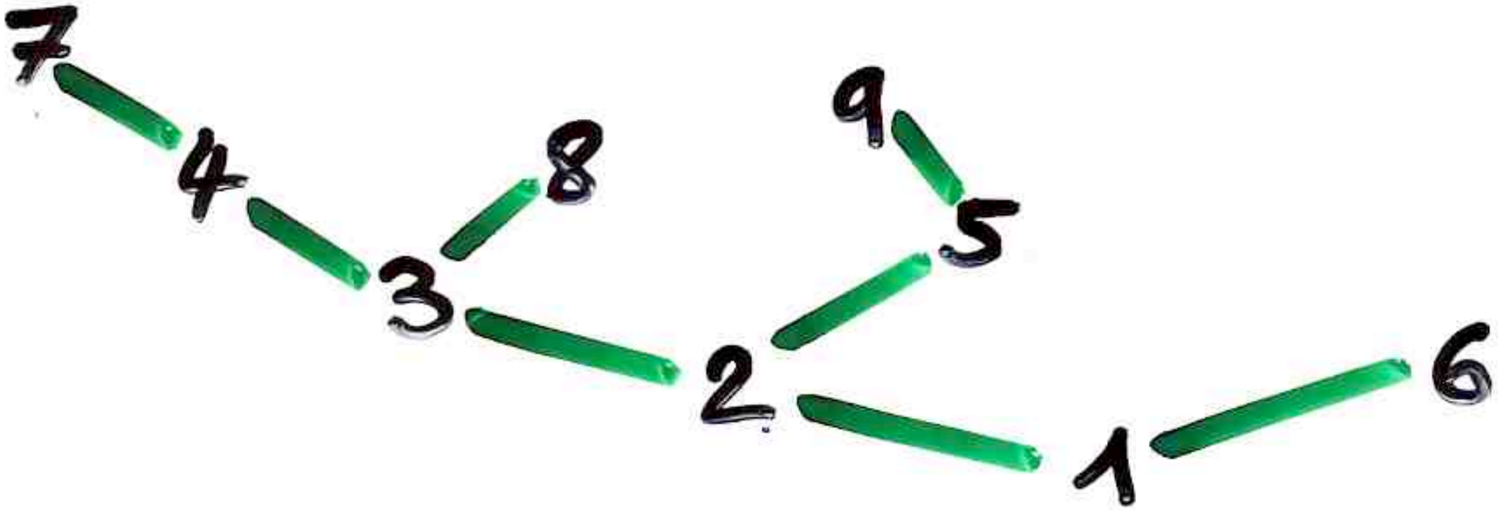
Bijection

increasing binary tree ⟷ permutation

T σ



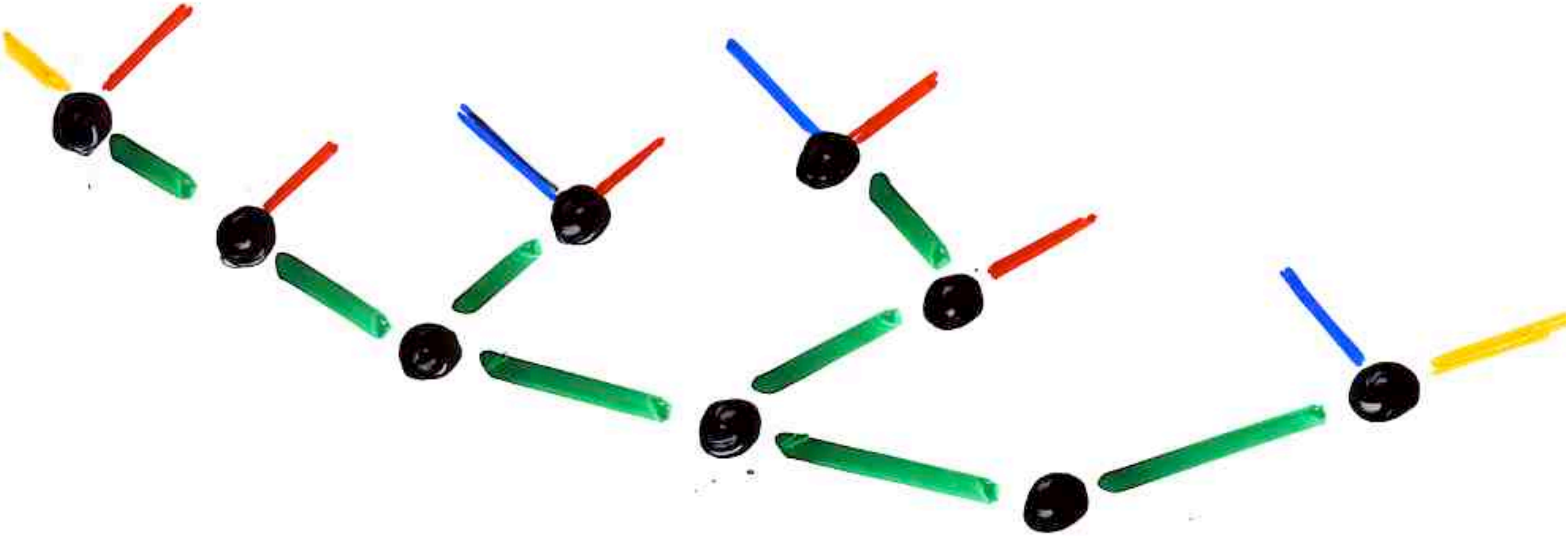
$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$



$\sigma = 7\;4\;3\;8\;2\;9\;5\;1\;6 \;\dots$

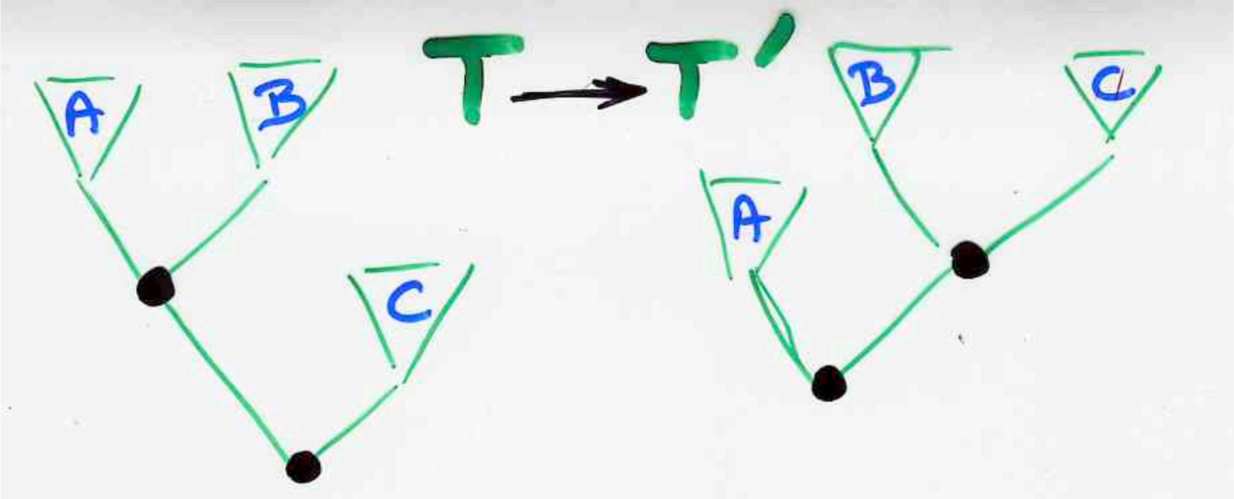
up-down sequence $\quad - \quad - \quad + \quad - \quad + \quad - \quad - \quad +$

$$\begin{tikzcd}
\underset{\text{permutation}}{\sigma} \arrow[r] \arrow[dd] & T(\sigma) \arrow[d] & \text{increasing binary tree} \\
 & B \arrow[d] & \text{binary tree} \\
\underset{\text{up-down sequence}}{UD(\sigma)} \arrow[r, phantom, "="] & \underset{\text{canopy}}{c(B)} &
\end{tikzcd}$$



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6 \ldots$

up-down sequence $-\ -\ +\ -\ +\ -\ -\ +$



if B ≠ ● canopy is invariant

if B = ● canopy $c(T')$

not invariant

$$\overset{*}{c}(T) = c(A) \overset{+}{/} c(B)\, c(C)$$

$$c(T') = c(A) \underset{-}{\backslash} c(B)\, c(C)$$

the theorem

relating canopy and Tamari lattice

Prop. (i) • The set of binary trees having a given canopy w is an interval $J(w)$ of the Tamari lattice

(ii) • this interval can be extended to an initial interval of the Young lattice

i.e. $\exists$ (integer) partition μ such that $J(w)$ is in bijection with $I(\mu)$, the set of partition $\emptyset \leqslant \lambda \leqslant \mu$
(inclusion of Ferrers Young diagrams)

with $T \leqslant T' \Rightarrow f(T) \geqslant f(T')$

Tamari lattice

Young lattice

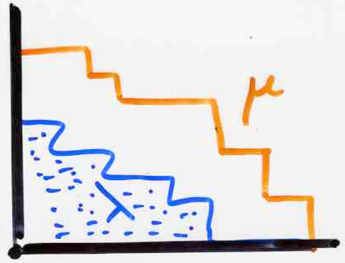
$$J(w) \xrightarrow[\text{bijection}]{f} I(\mu)$$

Young lattice

partition μ

$I(\mu)$

$\lambda \leqslant \mu$



initial segment in the Young lattice (or lower ideal) merci !