An extension of Tamari lattices

joint work with
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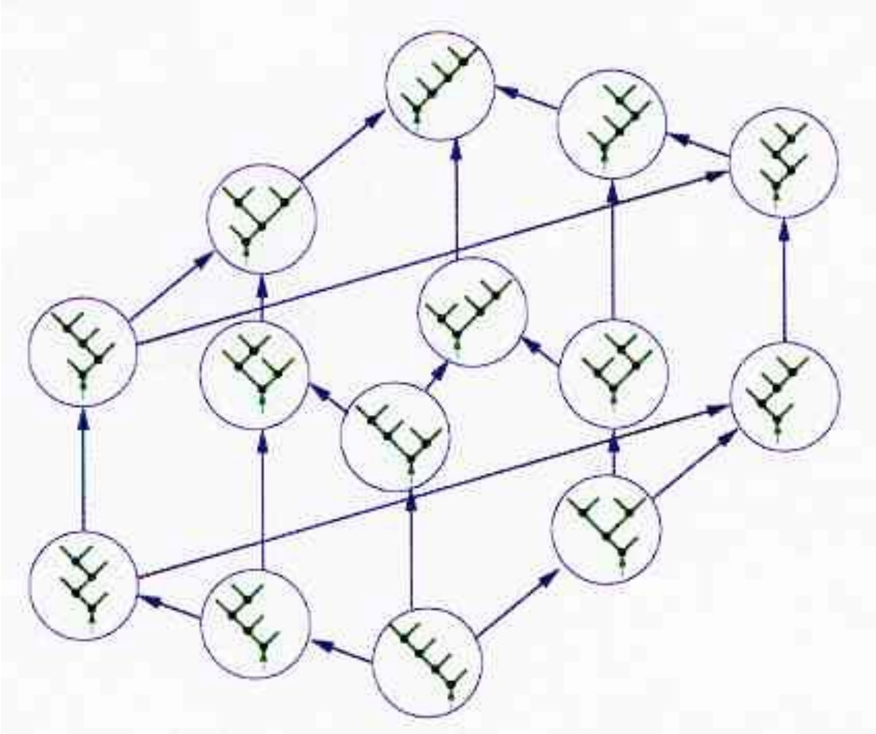
Introduction

Tamari

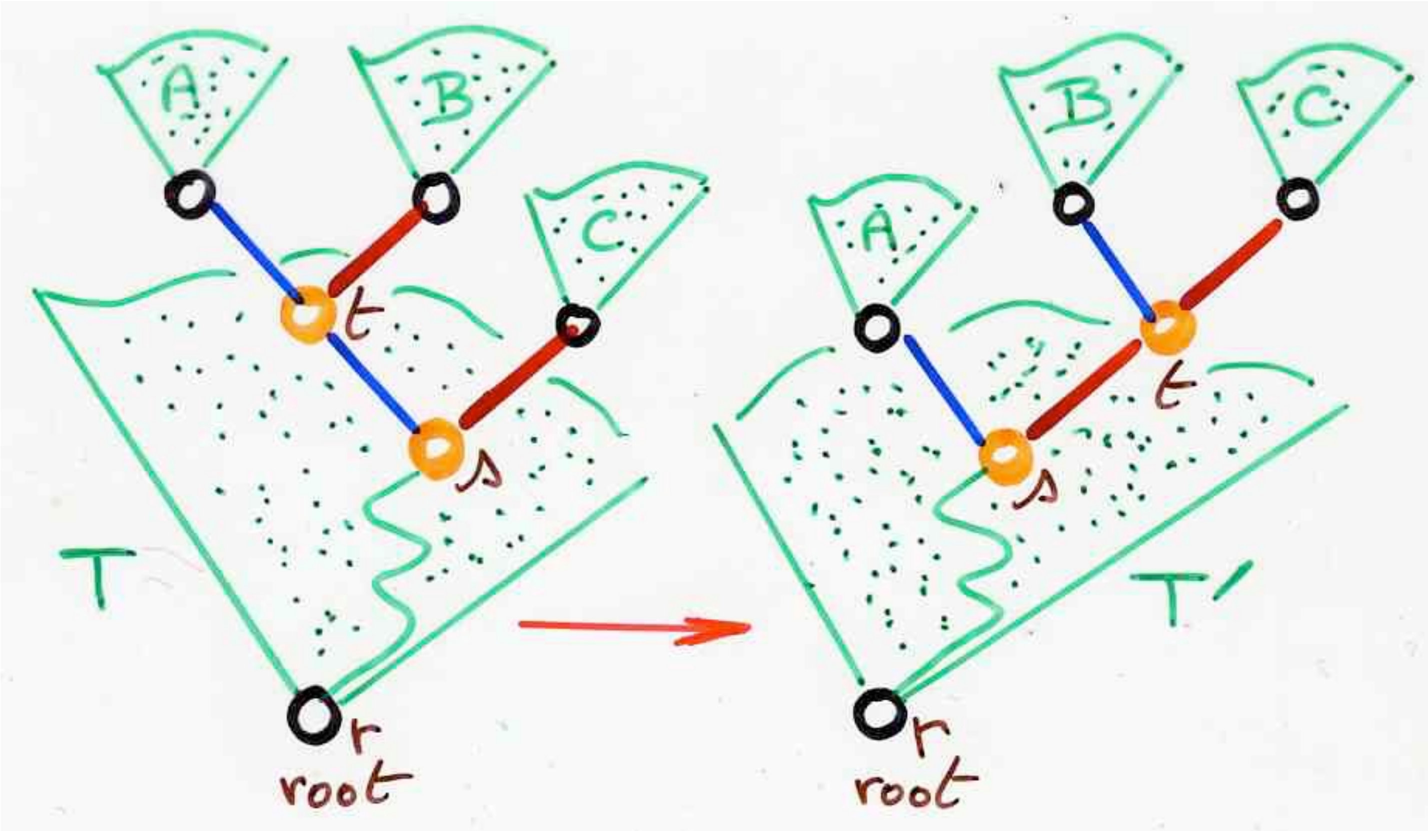
m-Tamari



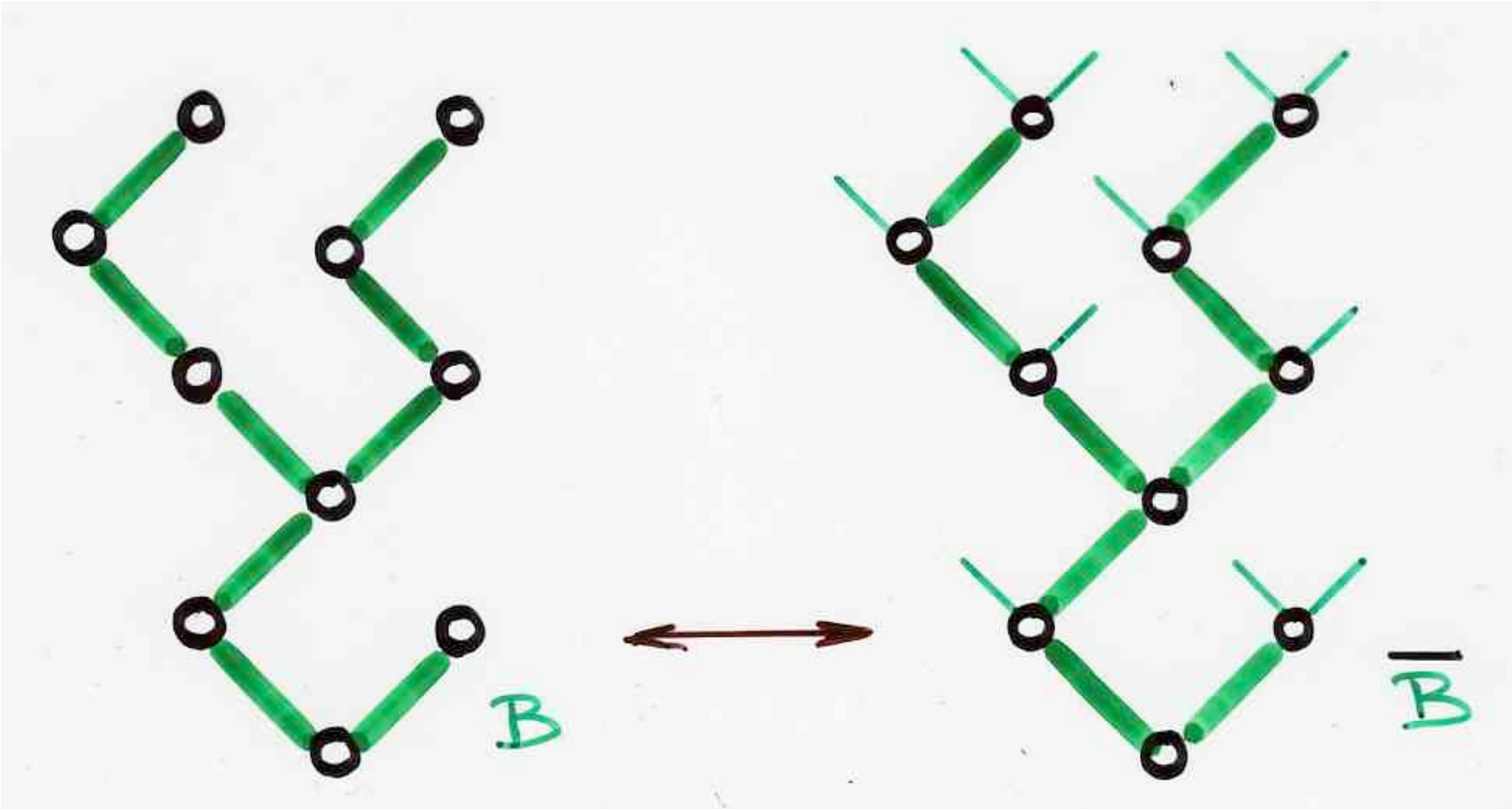
Tamari lattice

$C_4 = 14$
Catalan

Dov Tamari (1951) thèse Sorbone
"Monoides préordonnés et chaînes de Malcev"

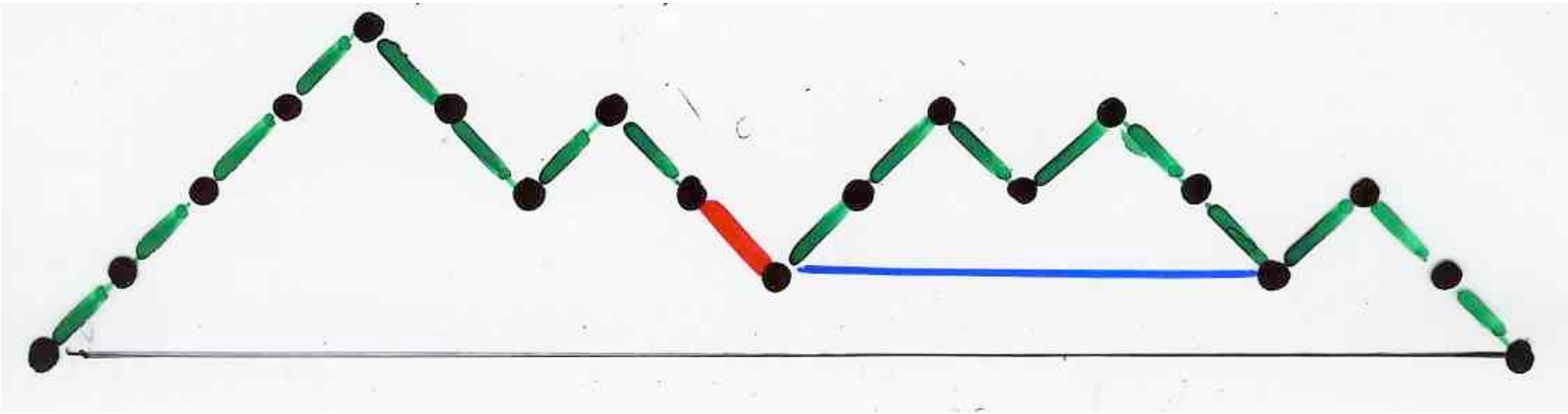


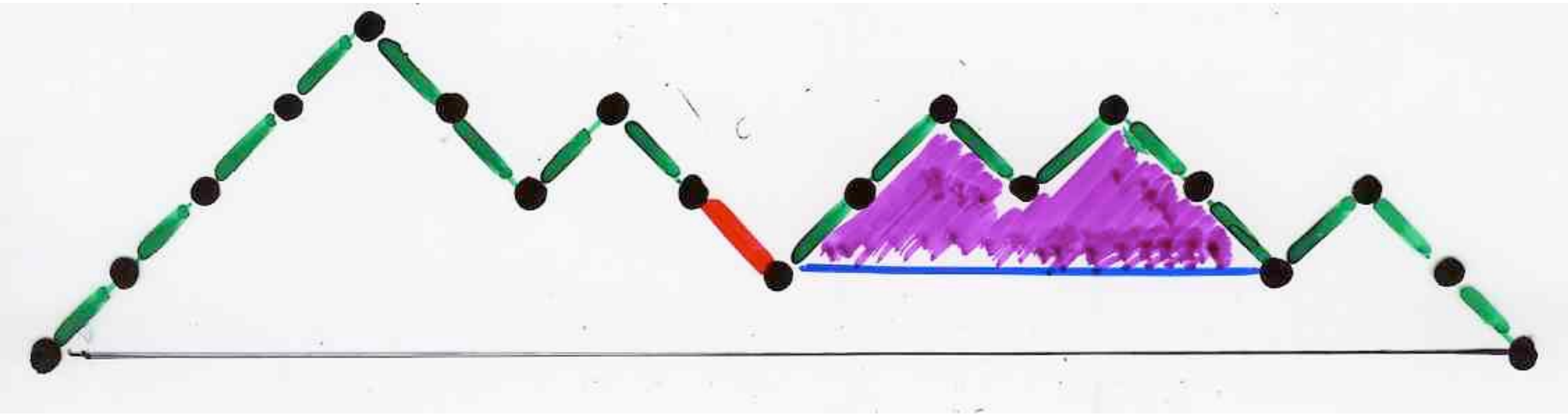
Rotation in a binary tree:
the covering relation in the
Tamari lattice



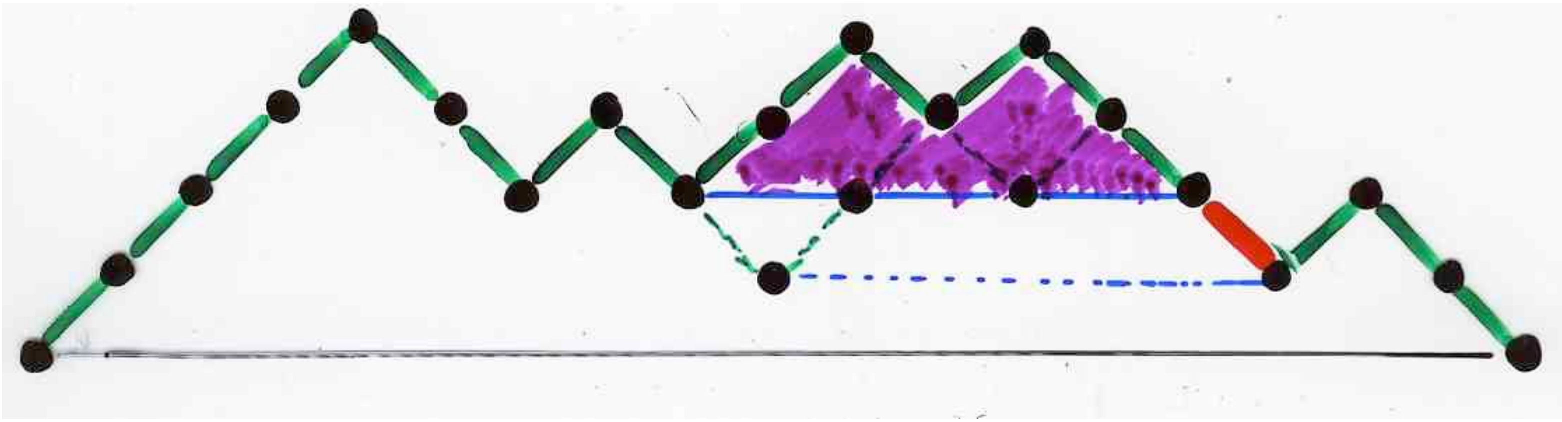
a binary tree B
and its associated complete binary tree $\overline{B}$
(full)

the Tamari lattice
in term
of Dyck paths

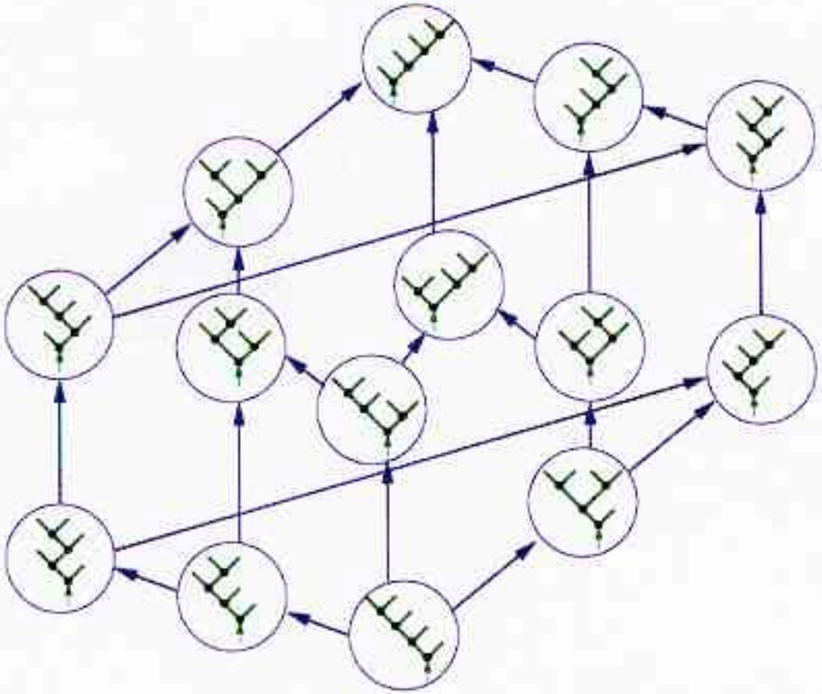


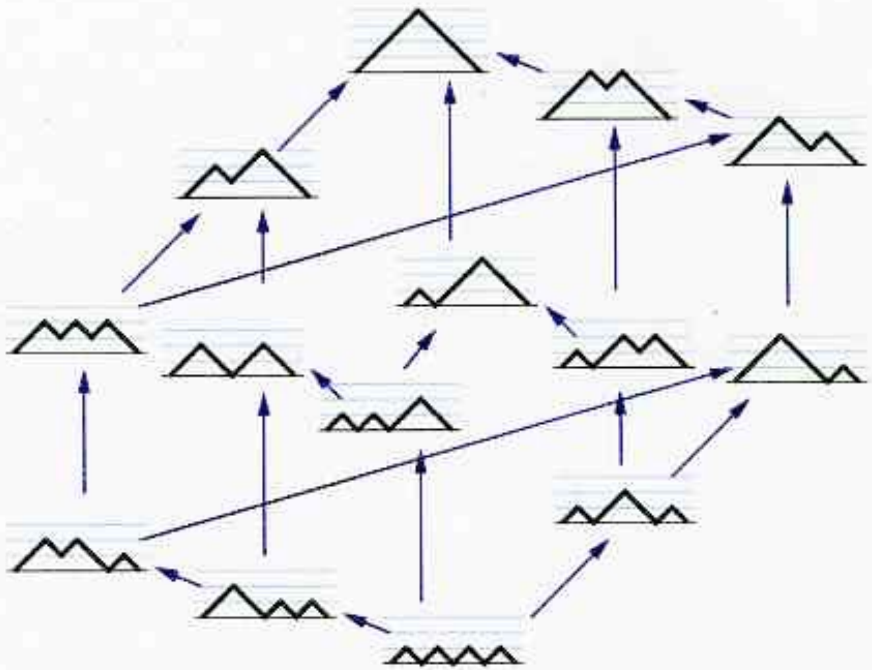


factor Dyck primitif



factor Dyck primitif





$C_4 = 14$

Catalan

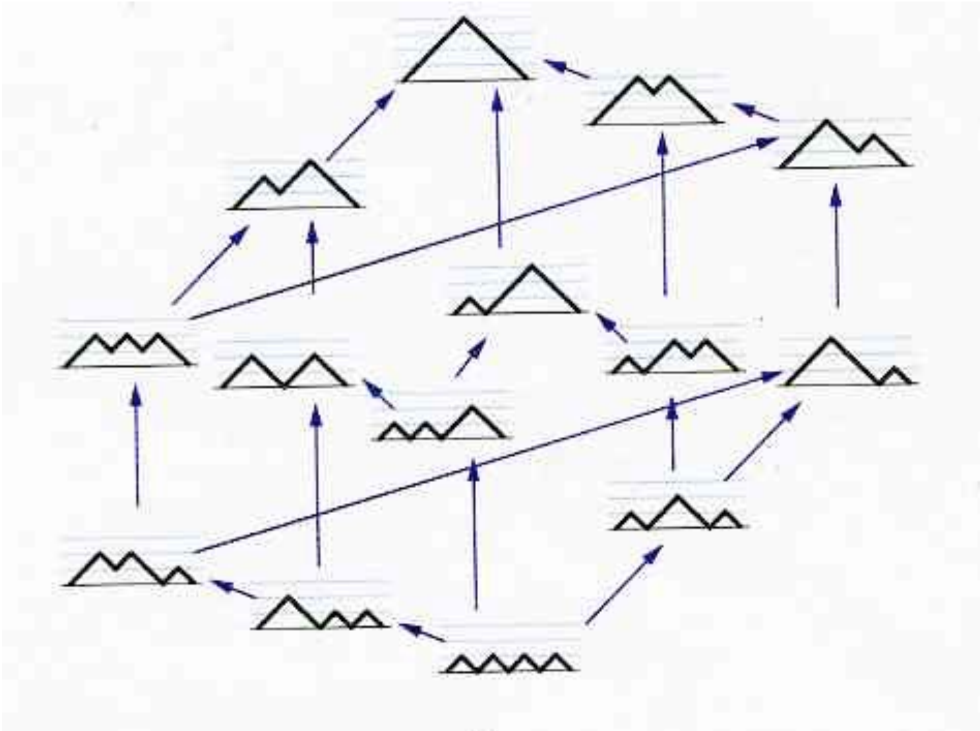
If $T \leqslant T'$ in $(\text{Tamari})_n$ lattice

then $T \leqslant T'$ in $(\text{Dyck})_n$ lattice

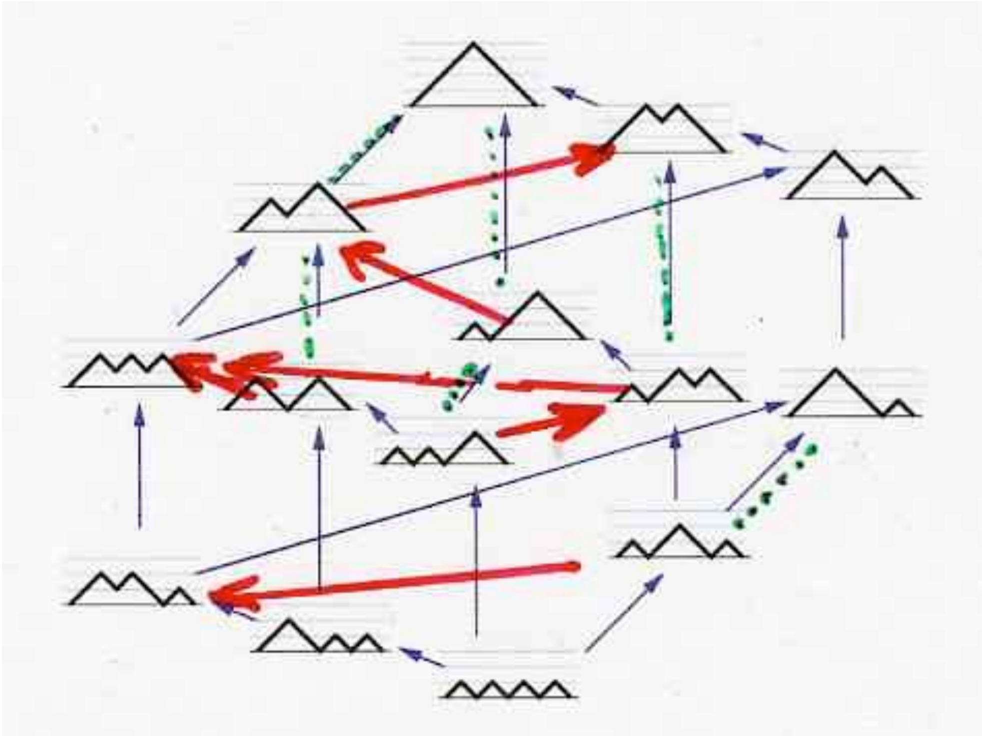
[i.e. T below T']

converse not true

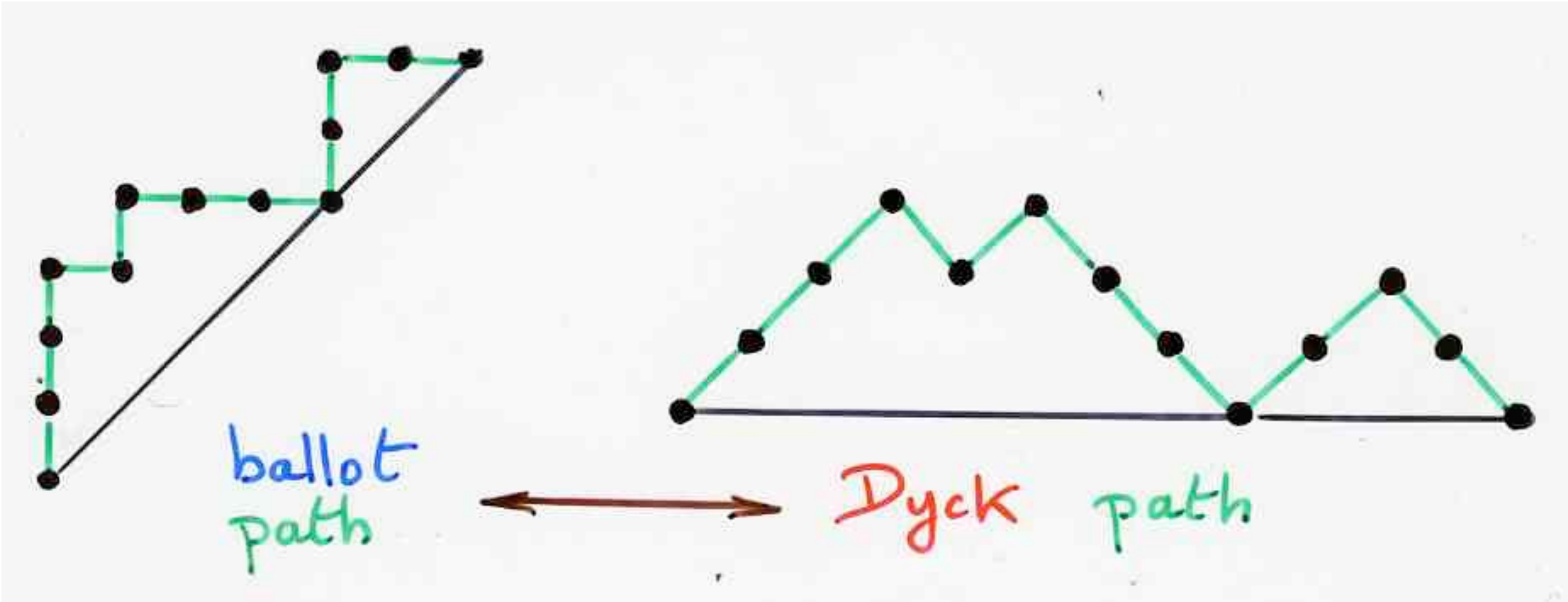
$(\text{Dyck})_n$ extension of $(\text{Tamari})_n$



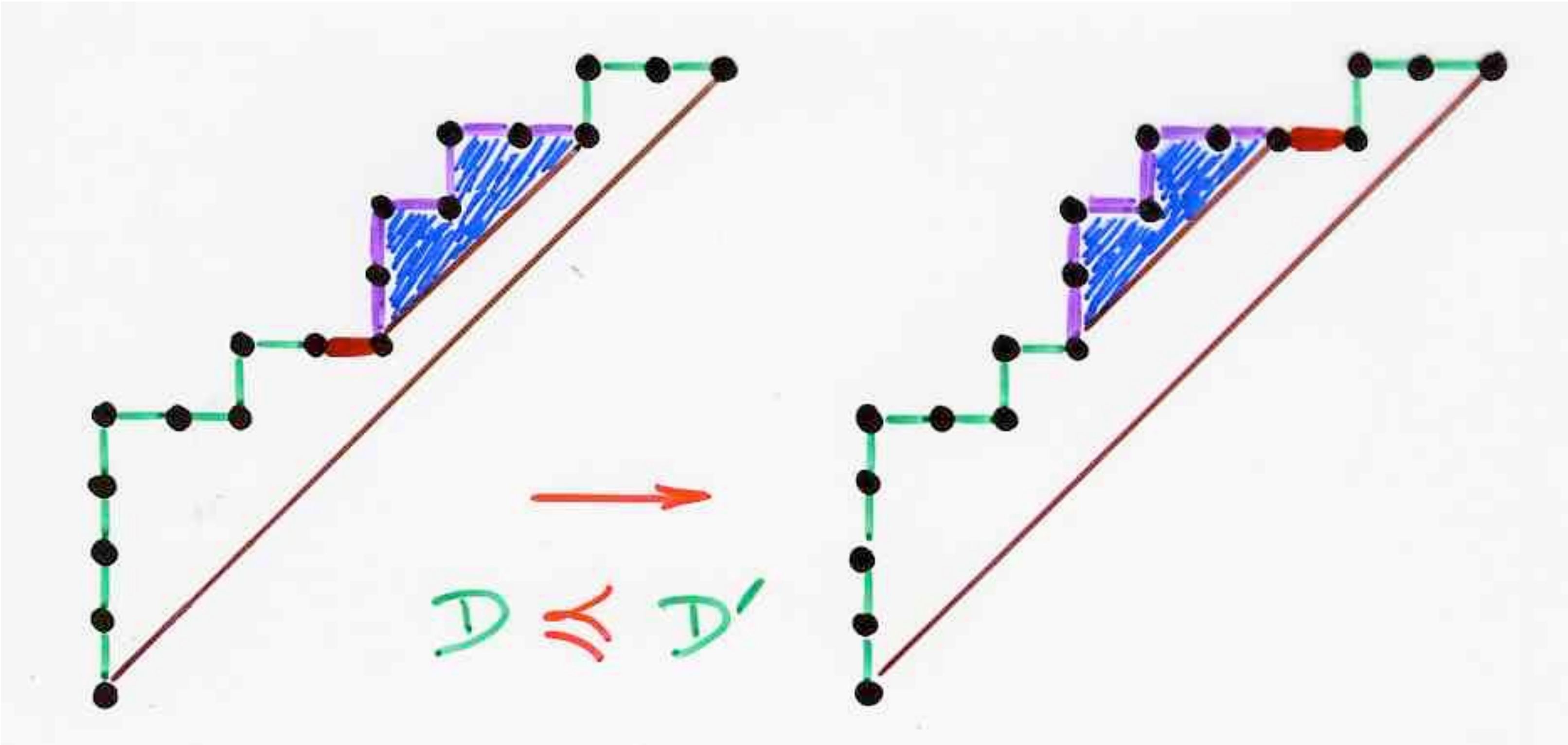
(Tamari lattice)$_4$



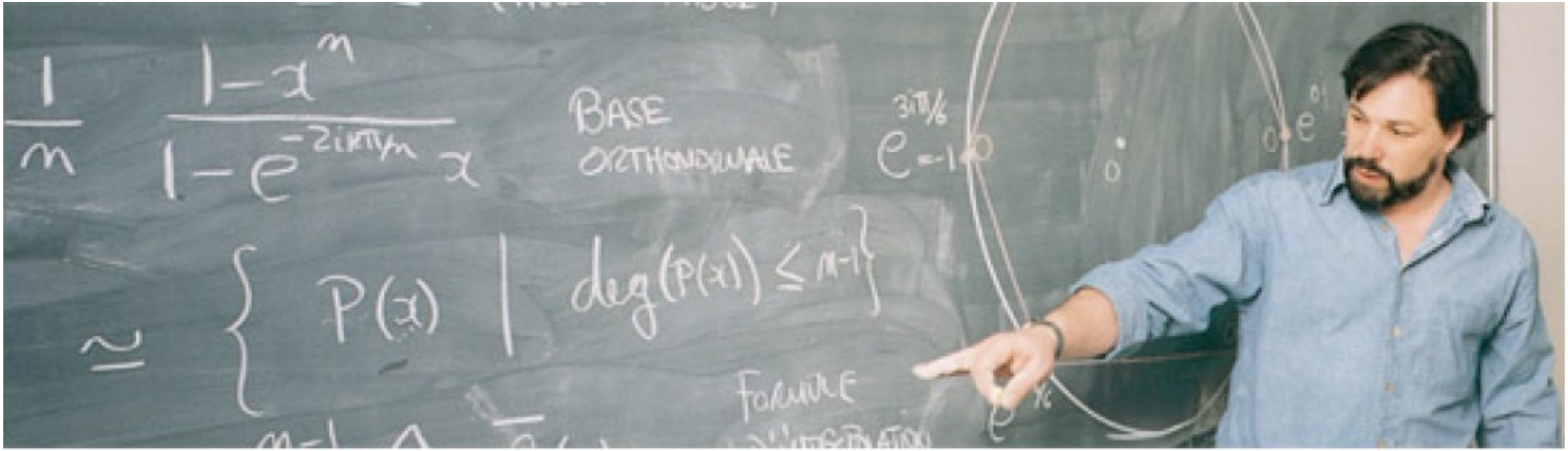

(Dyck)4
lattice



vocabulary: ballot path
Dyck path



the Tamari covering relation
for ballot (Dyck) path



François Bergeron

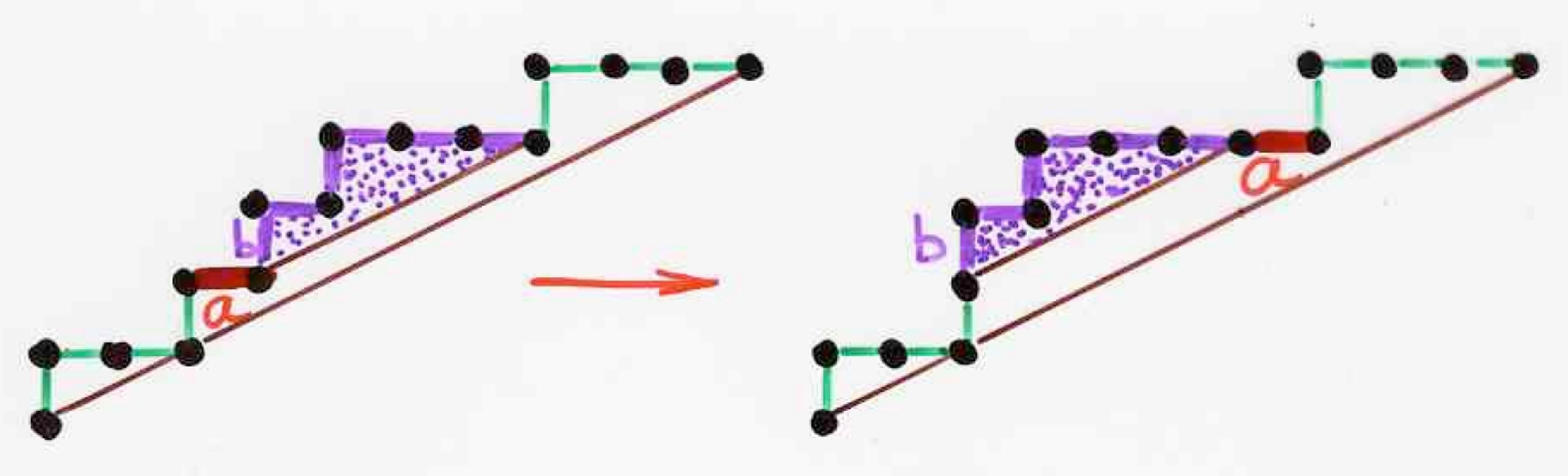
diagonal
coinvariant
spaces

Adriano Garsia

F. Bergeron (2008) introduced the m-Tamari lattice

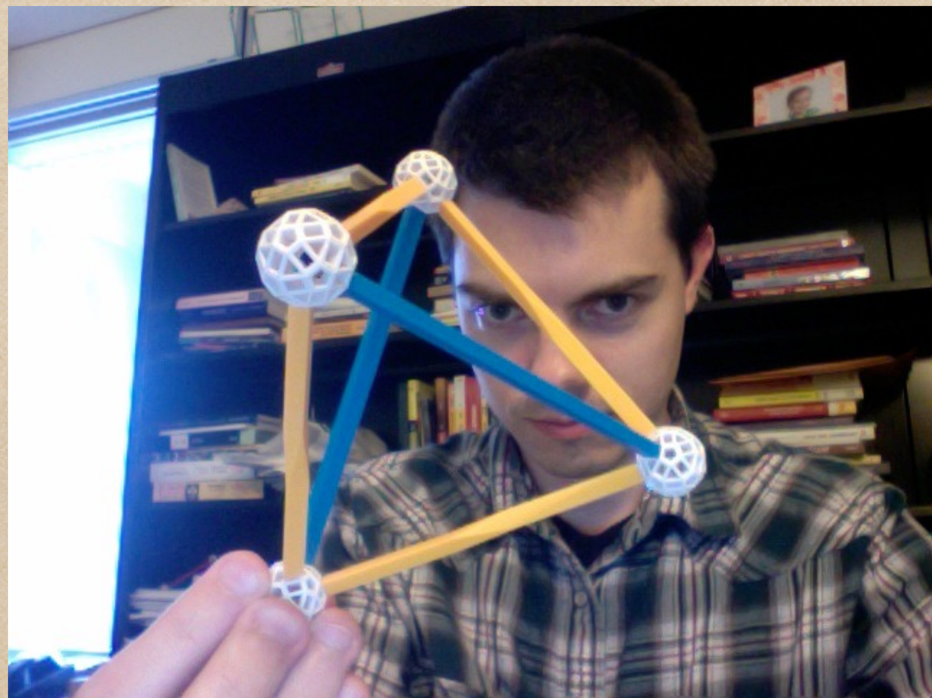
dimensión $\frac{1}{(m+1)n+1}\binom{(m+1)n+1}{mn}$

m-ballot paths



the covering relation in the
m-Tamari lattice
(m = 2)

Rational Catalan Combinatorics



Rational Catalan Combinatorics

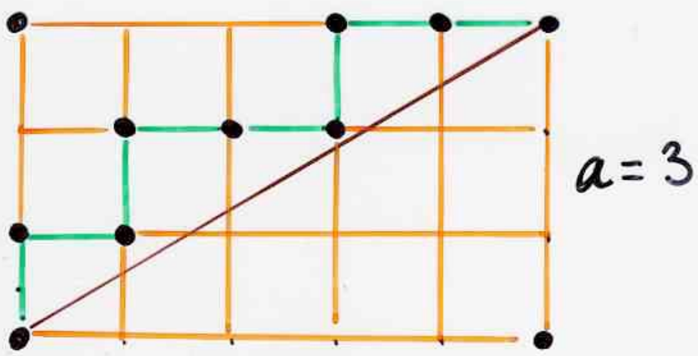
D. Armstrong

$$Cat(a,b) = \frac{1}{a+b}\binom{a+b}{a,\,b}$$

number of (a,b)- ballot paths $= Cat(a,b)$

Grossman (1950)
Bizley (1954)

rational ballot (Dyck) paths

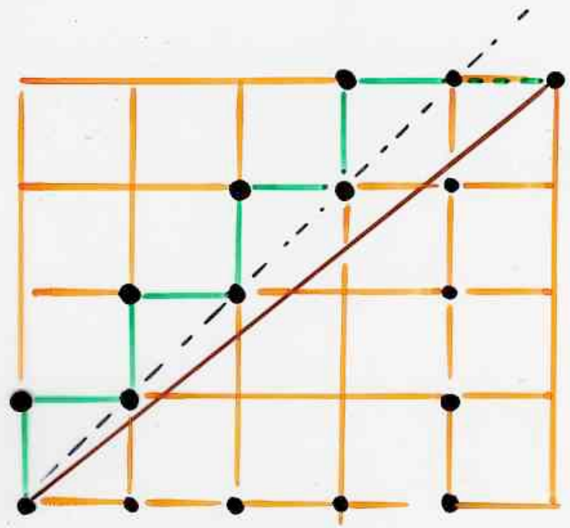


$(a,b) = (3,5)$

$(a, b) = (n, n+1) \longrightarrow C_n$ Catalan nb

$(a, b) = (n, mn+1) \longrightarrow \frac{1}{(m+1)n+1} \binom{(m+1)n+1}{n}$

Fuss-Catalan nb

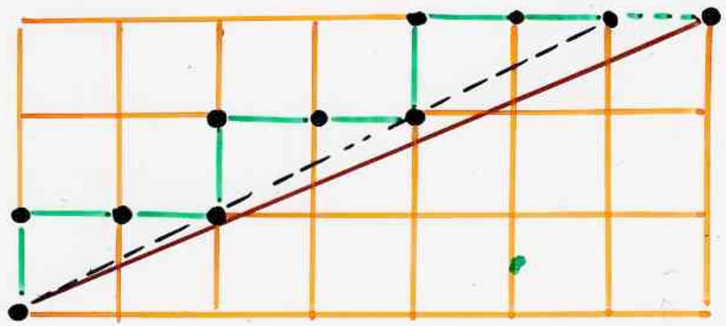


$n = 4$

$n+1 = 5$

$(a, b) = (4, 5)$

Catalan



$n = 3$

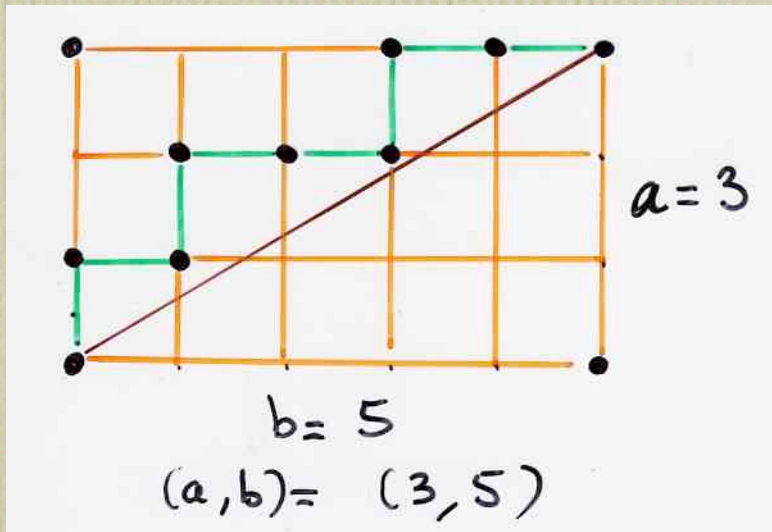
$m = 2$ $mn+1 = 7$

$(a, b) = (3, 7)$

Fuss-Catalan

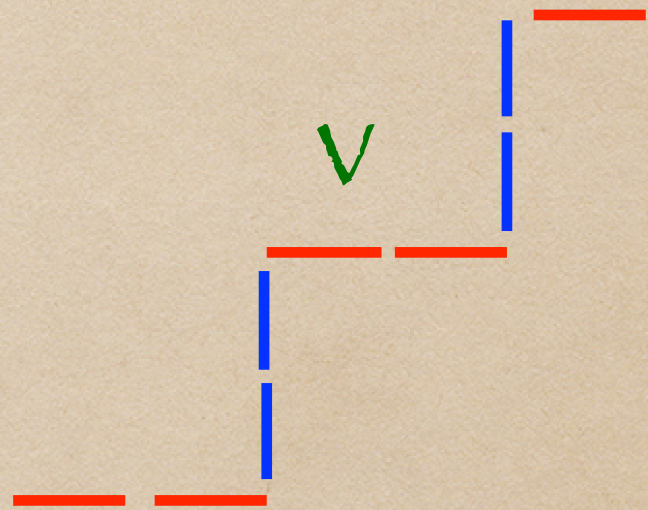
question:

define an (a,b)- Tamari lattice ?

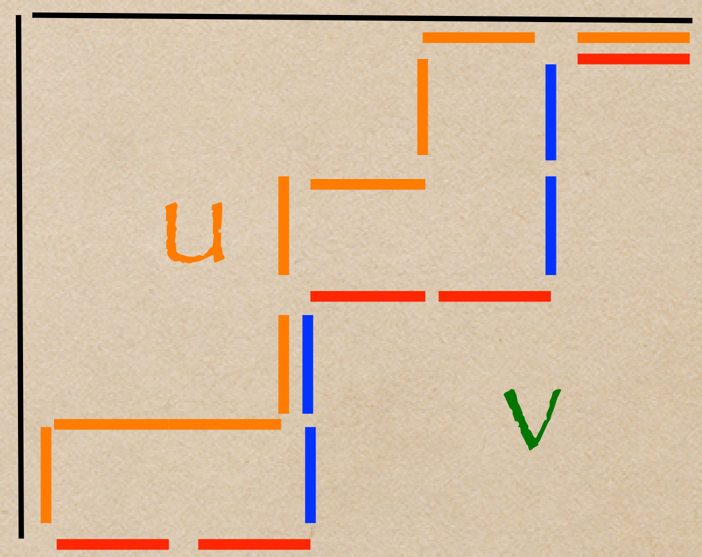


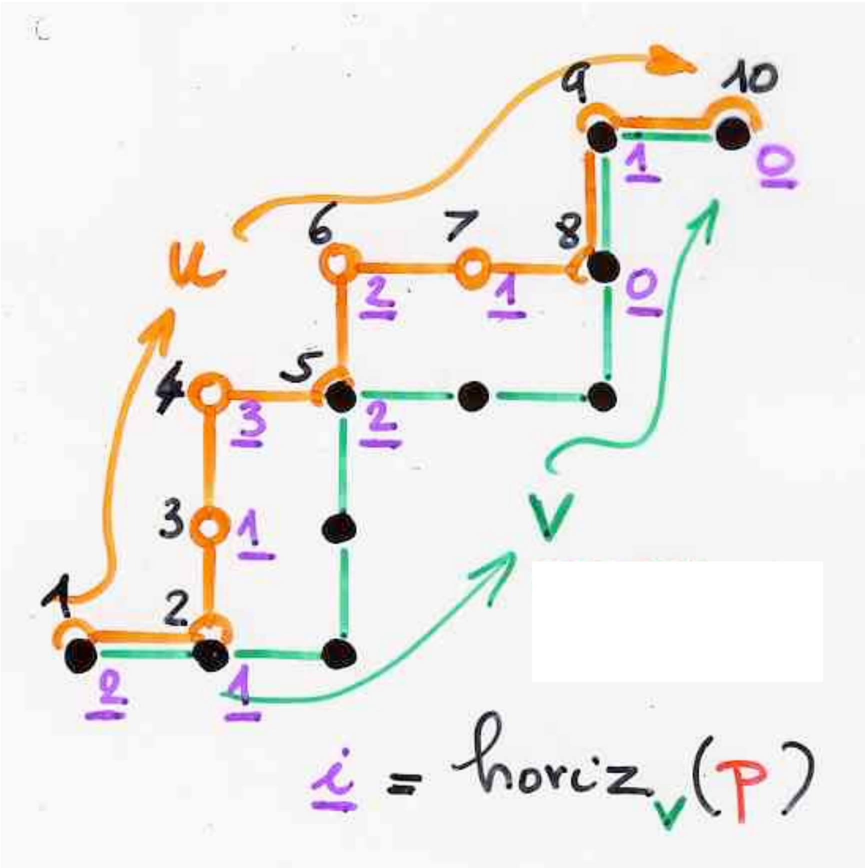
extension: Tamari T_v

transcental
Catalan
combinatorics ?

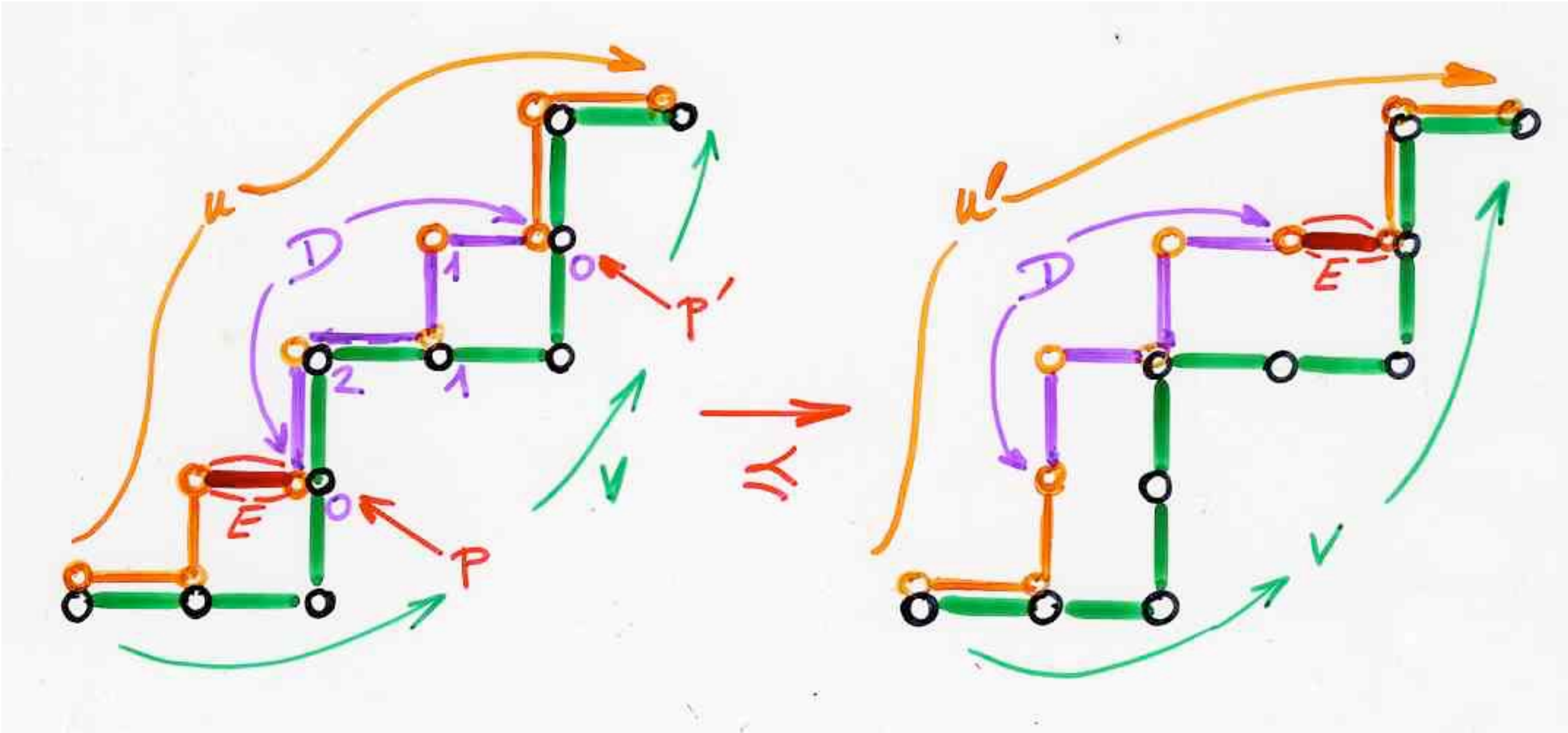


Ferrers diagrams F(u) included in F(v)





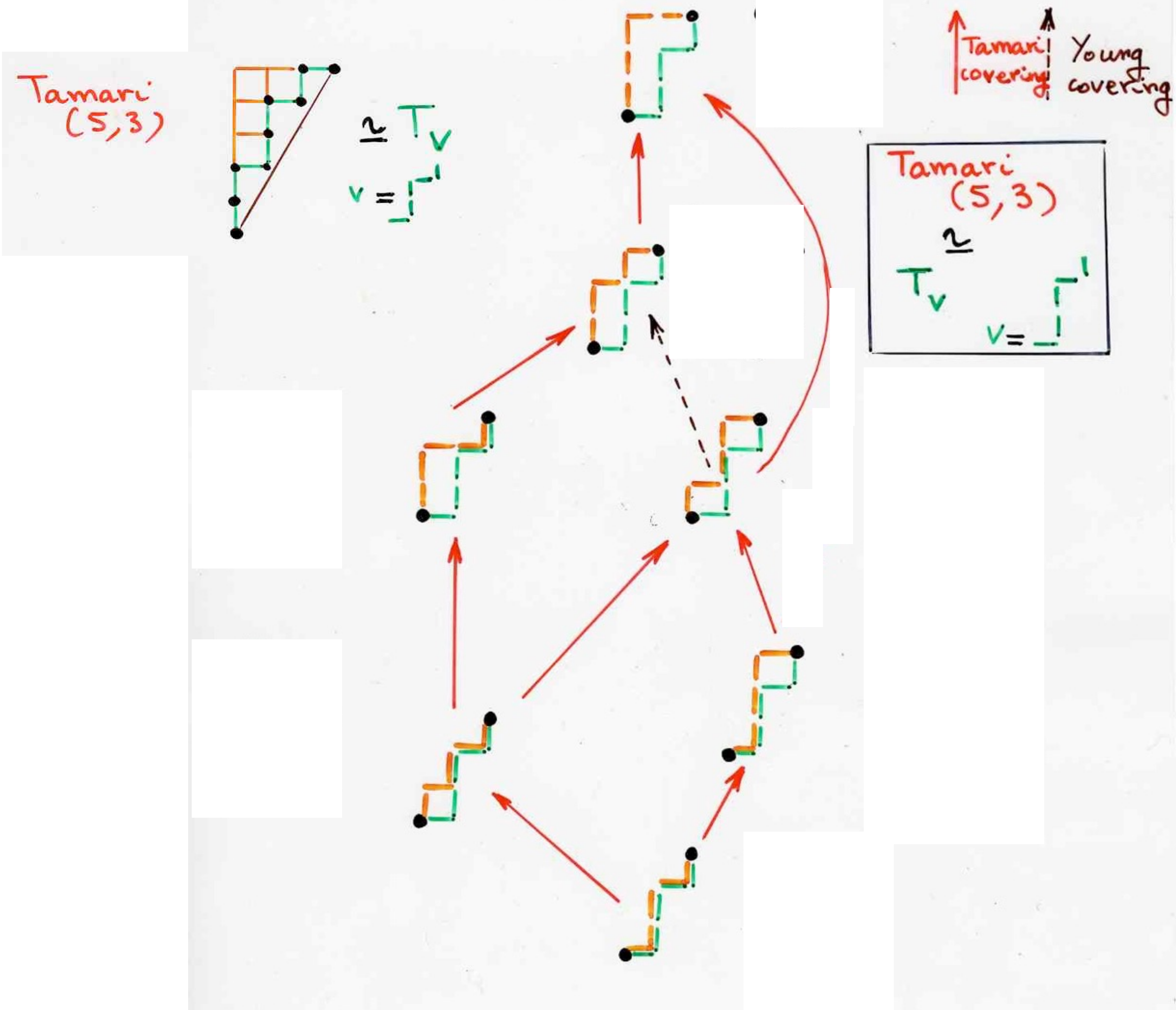
a pair (u, v) of paths
with the "horizontal distance"
$\text{horiz}_v(p)$


u
D
1
1
2
1
0
p'
E
0
p
v
≼
u'
D
E
v

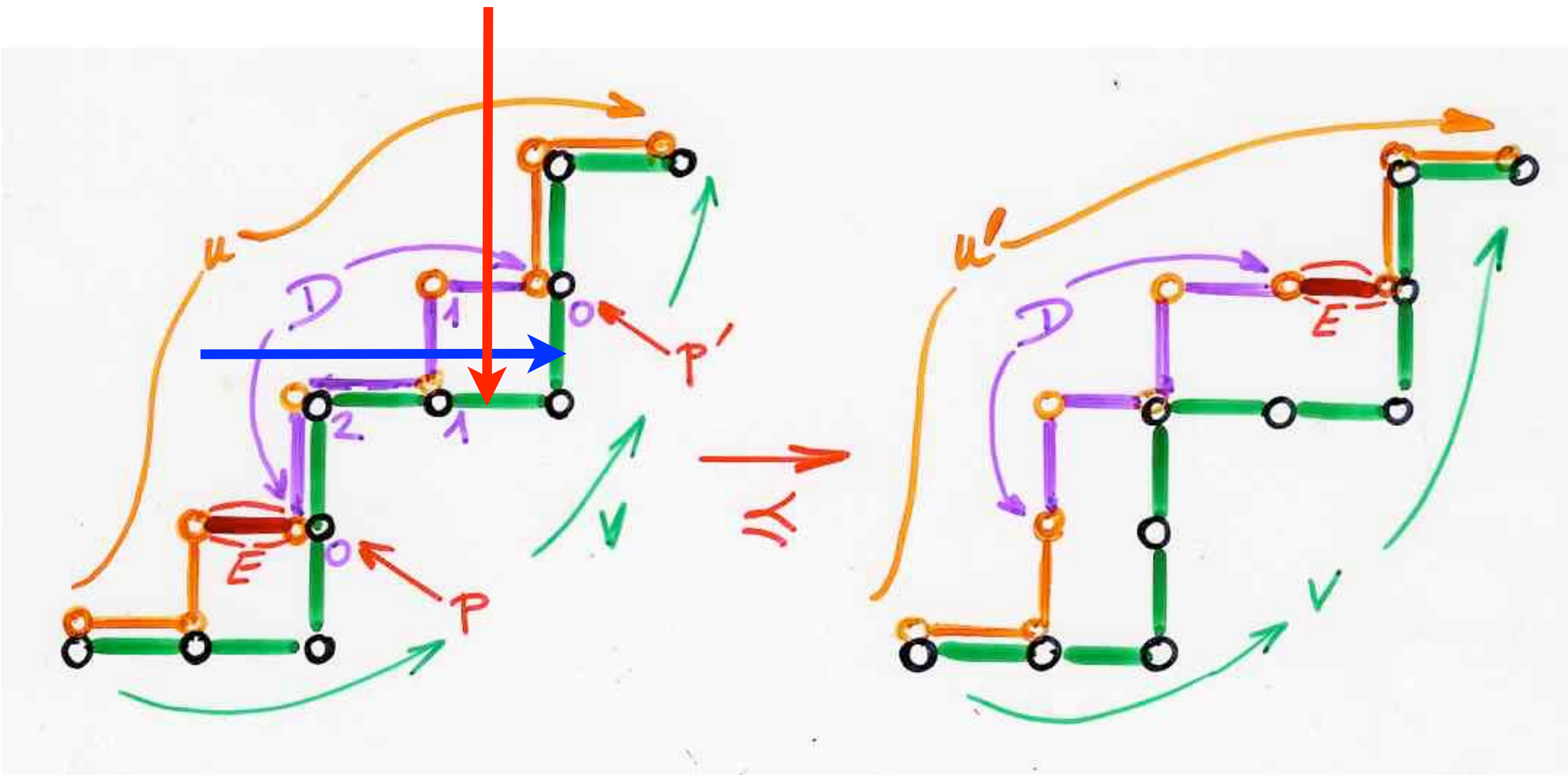

the covering relation
in the poset T_v

Thm 1. For any path v

T_v is a lattice



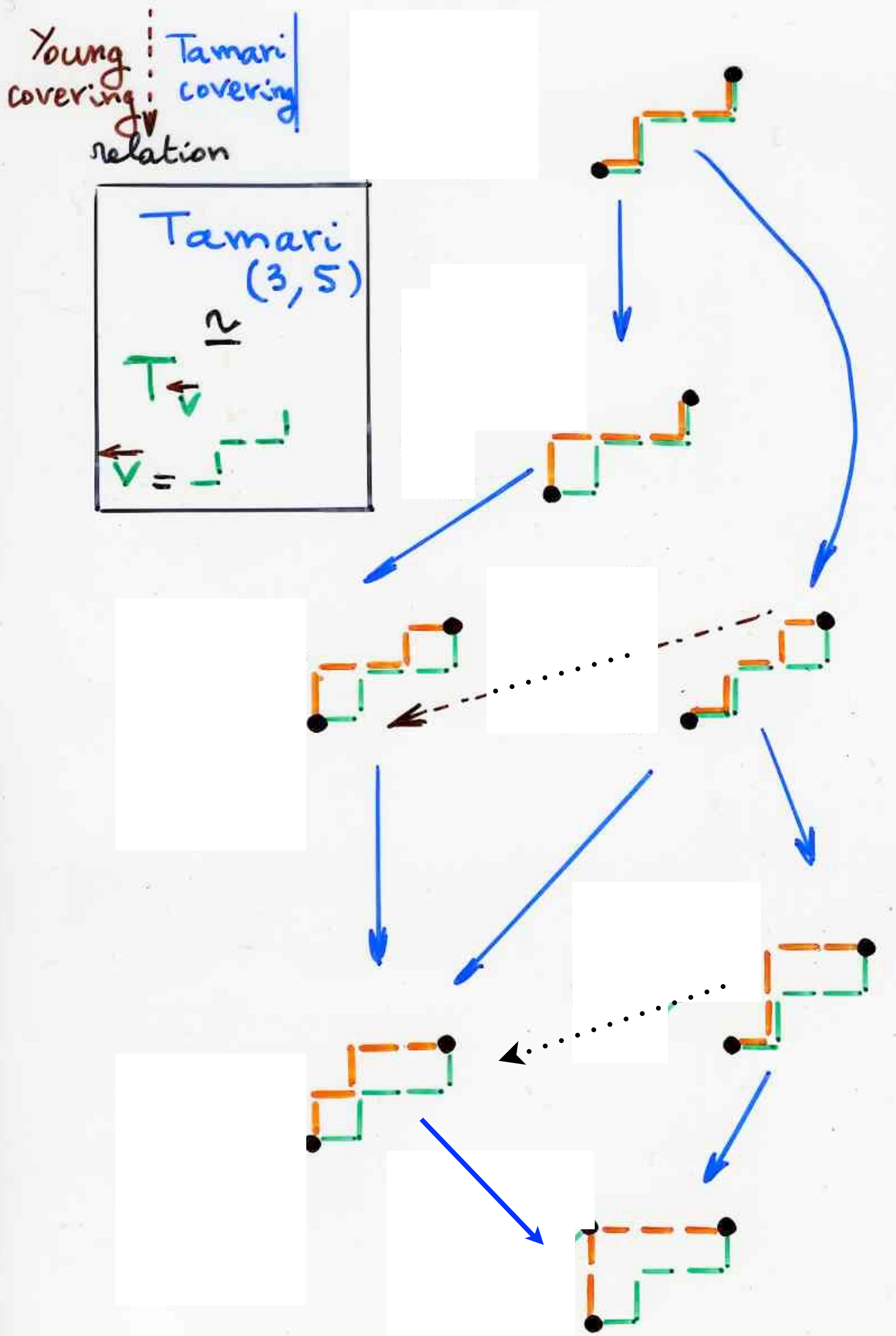
Tamari (5,3)
≃ T_v
v =
Tamari covering
Young covering
Tamari (5,3)
≃
T_v
v =

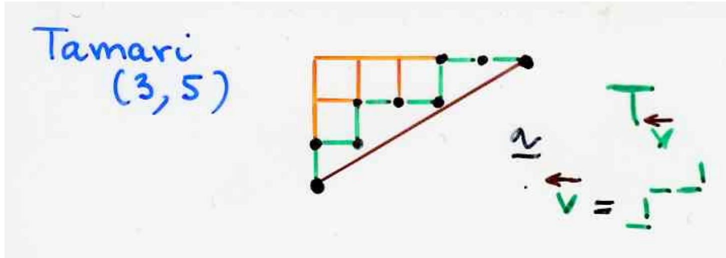


«row covering relation»

«column covering relation»

mirror image, exchange N and E

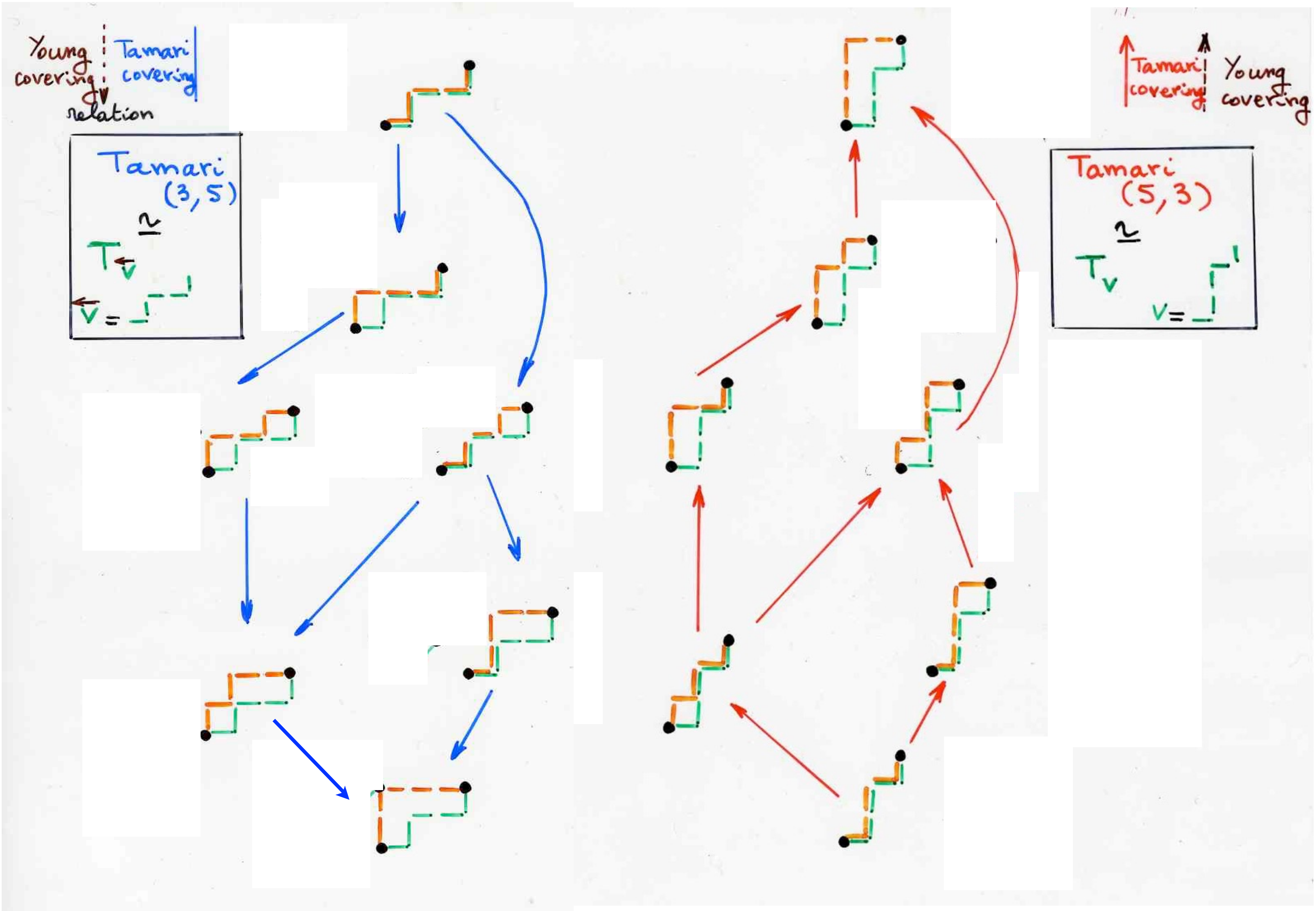

Young covering relation
Tamari covering
Tamari (3,5)
≃
T_v
v =


Tamari (3,5)
≃
T_v
v =

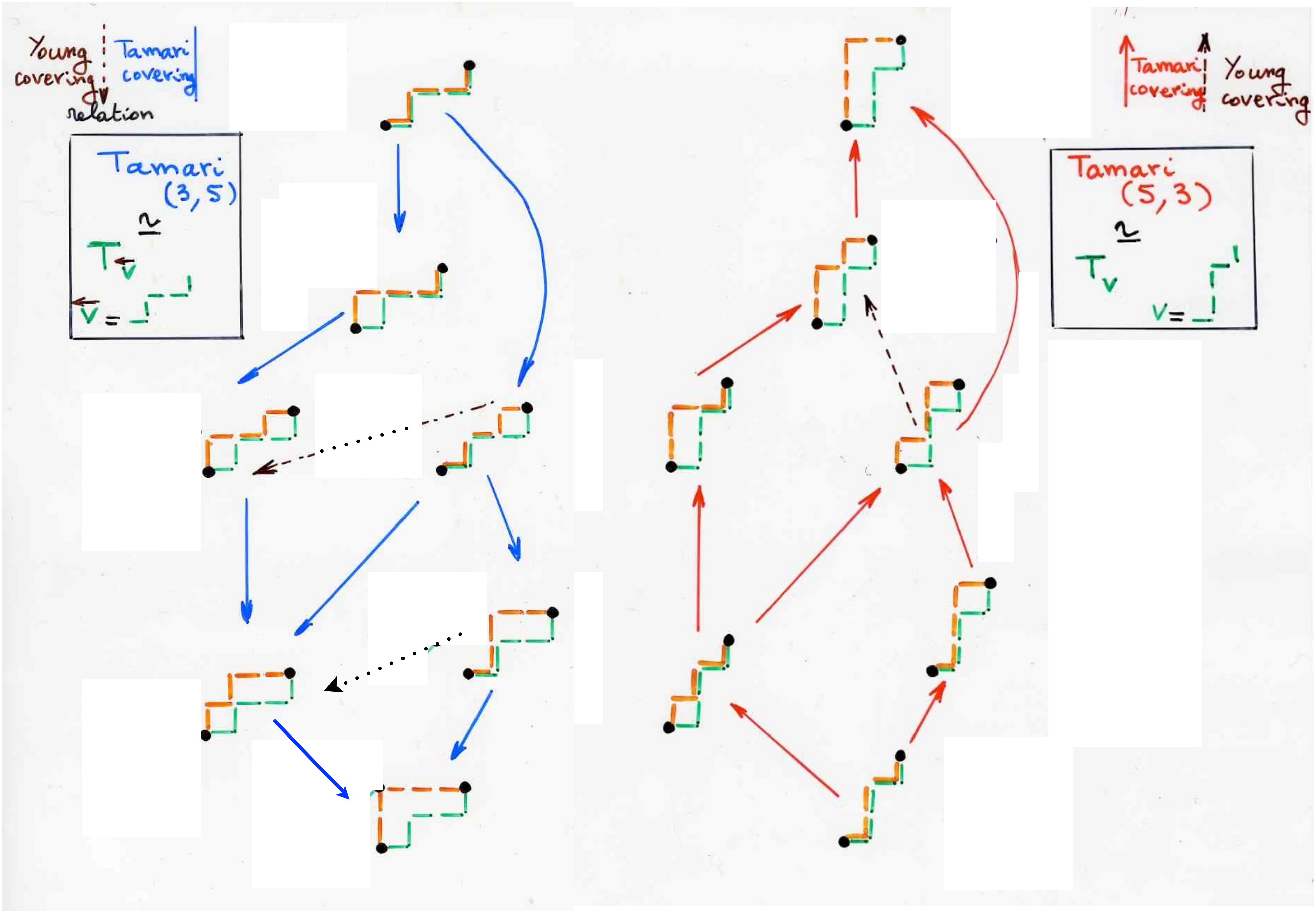
Thm 1. For any path v
T_v is a *lattice*

Thm 2. The *lattice* T_v
is isomorphic to the *dual* of $T_{\overleftarrow{v}}$

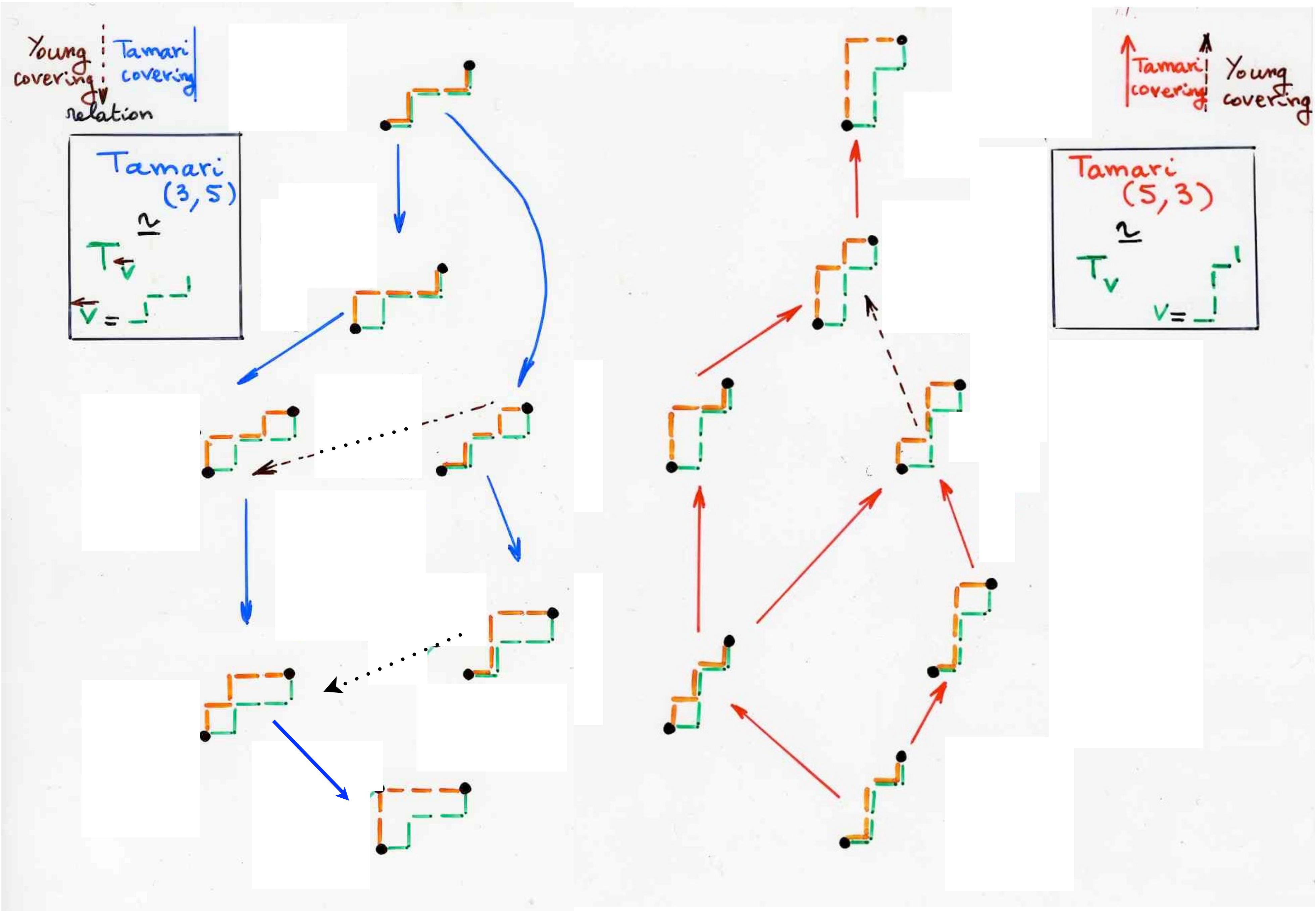
Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$

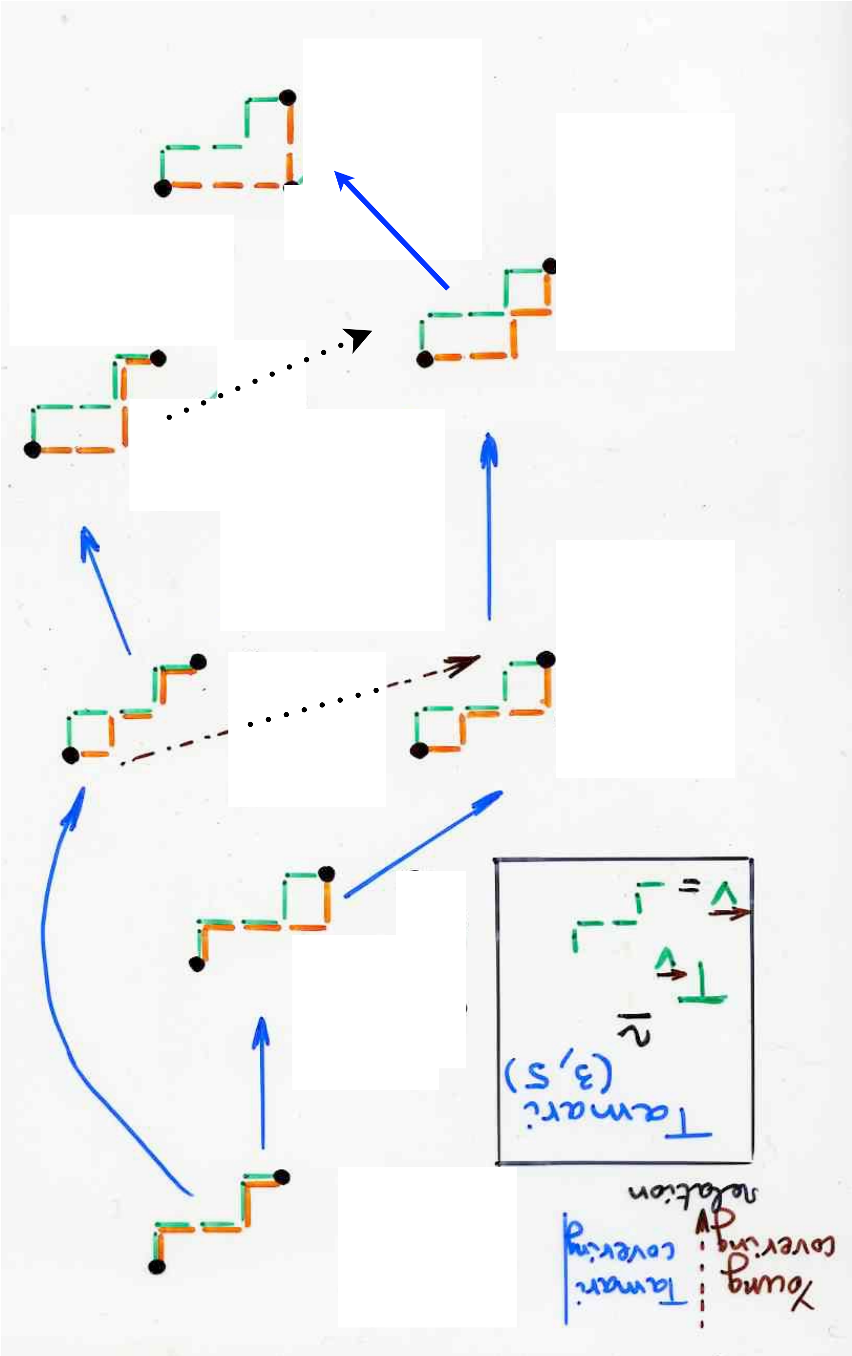


Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$

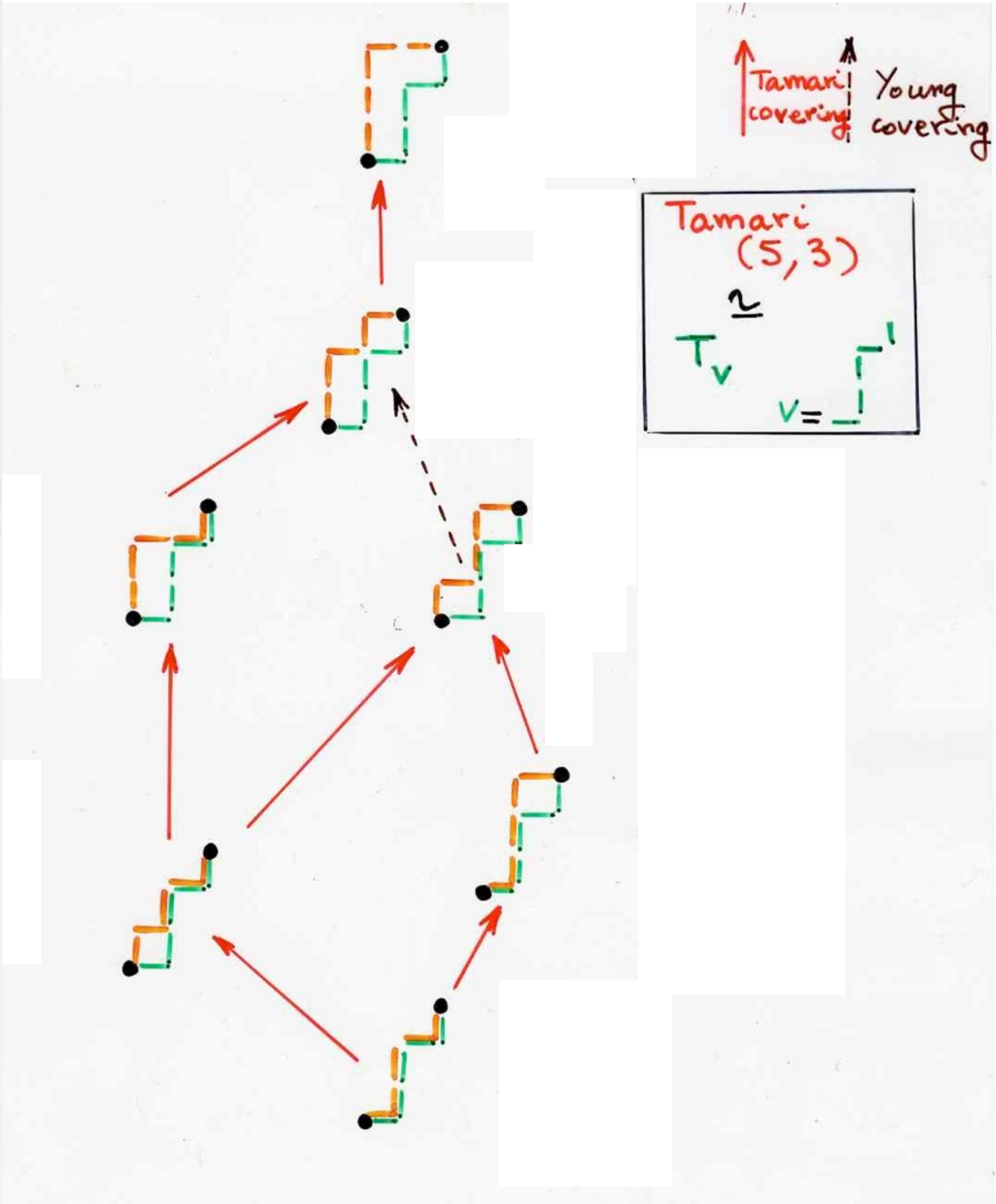


Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$



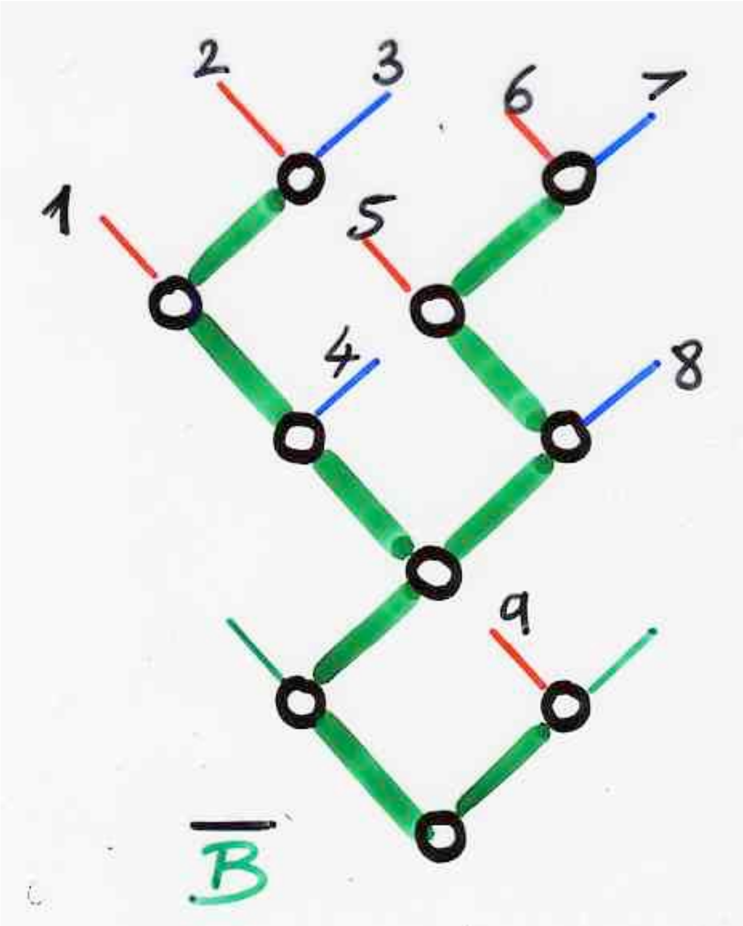


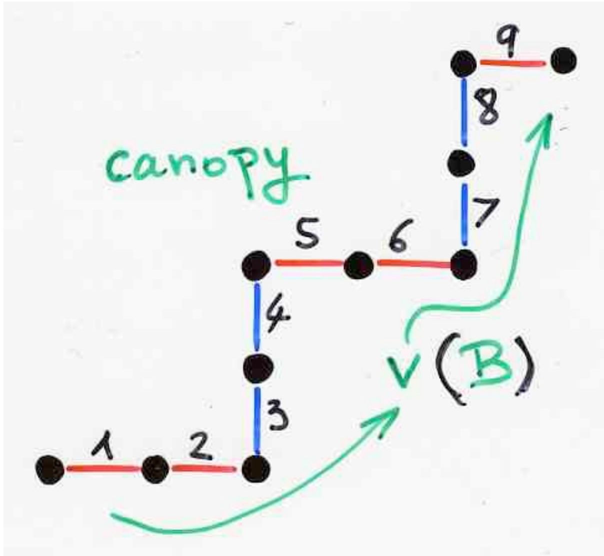
Tamari (3,5)
≃
T_v
v =
Tamari covering
Young covering relation



Tamari covering
Young covering
Tamari (5,3)
≃
T_v
v=

The canopy of a binary tree





definition of the canopy

J.-L.Loday, M. Ronco (1998, 2012)

algebraic structures

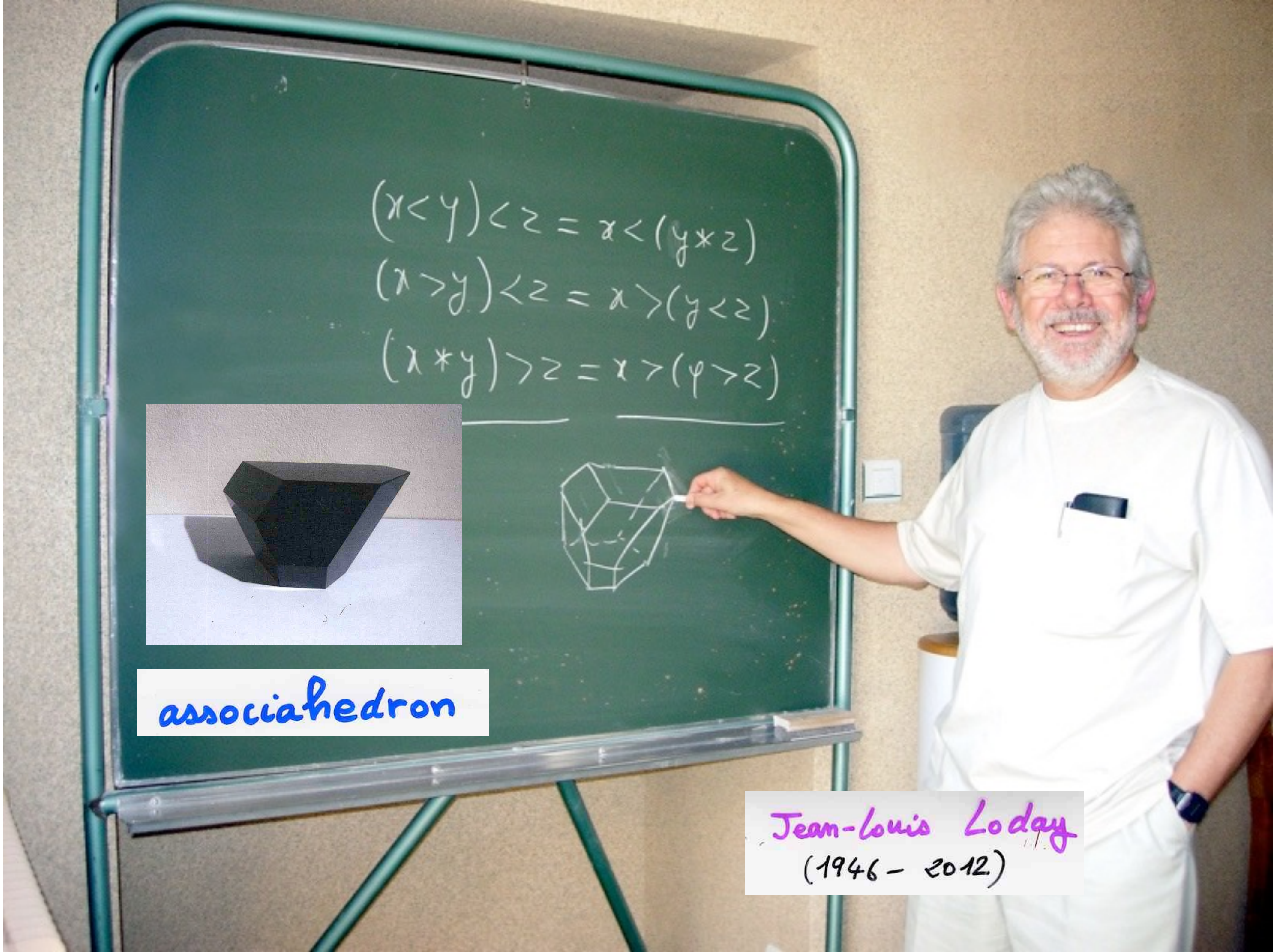
Hopf algebra

dim 2^{n-1} C_n $n!$

Catalan

Boolean lattice inclusion ⟷ Tamari order ⟷ weak Bruhat order

J.-L.Loday, M. Ronco (1998, 2012)


(x<y)<z = x<(y*z)
(x>y)<z = x>(y<z)
(x*y)>z = x>(y>z)
associahedron
Jean-Louis Loday
(1946 – 2012)

C. Hohlweg, C. Lange (2007)
F. Chapoton, S. Fomin, A. Zelevinsky (2002)

extensions :

C. Ceballos	J.-P. Labbé	C. Stump
V. Pilaud	N. Bergeron	F. Santos
N. Reading	H. Thomas	A. Postnikov
R. Marsh	M. Reineke	C. Athanasiadis
D. Speyer	S. Stella	G. Ziegler

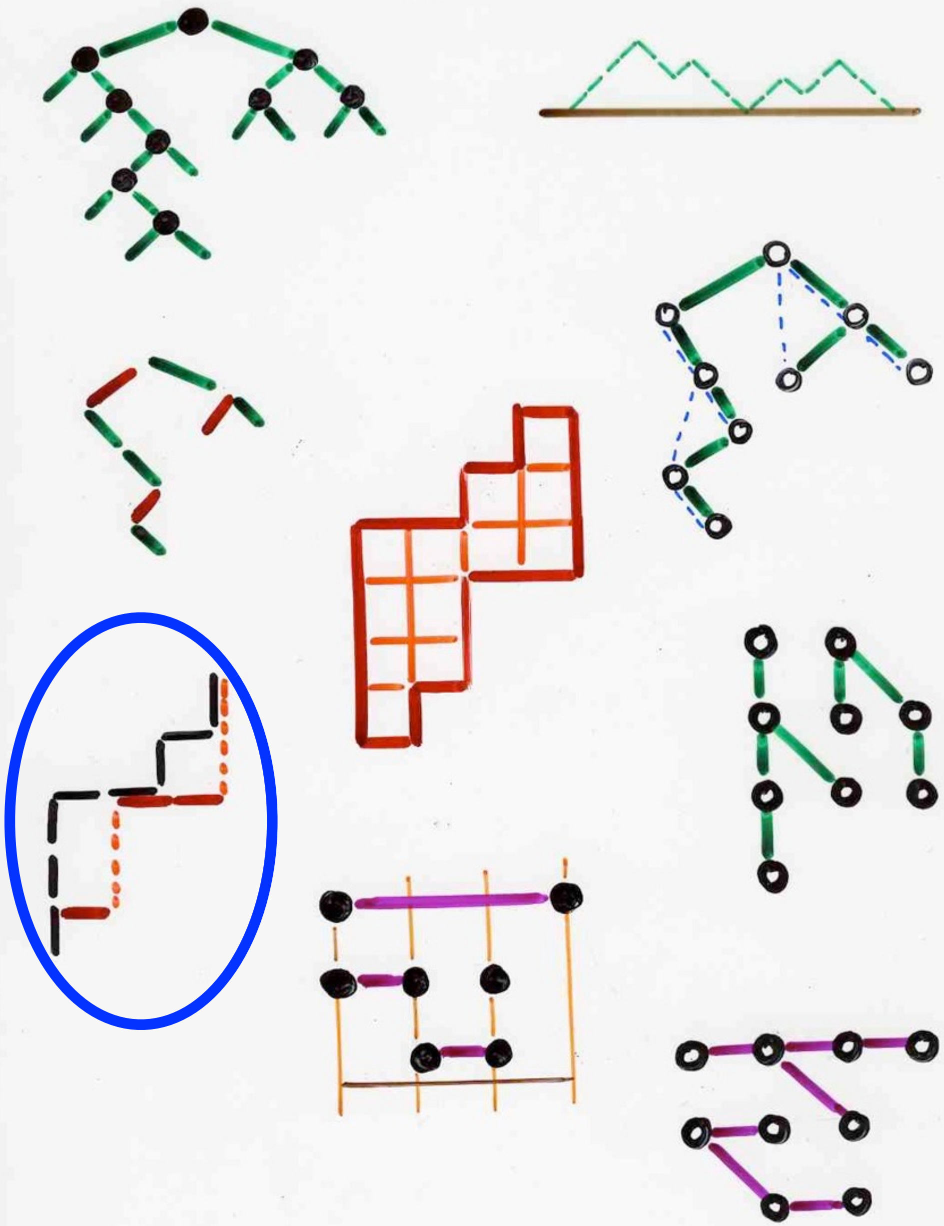
Gil Kalai

and many others ...

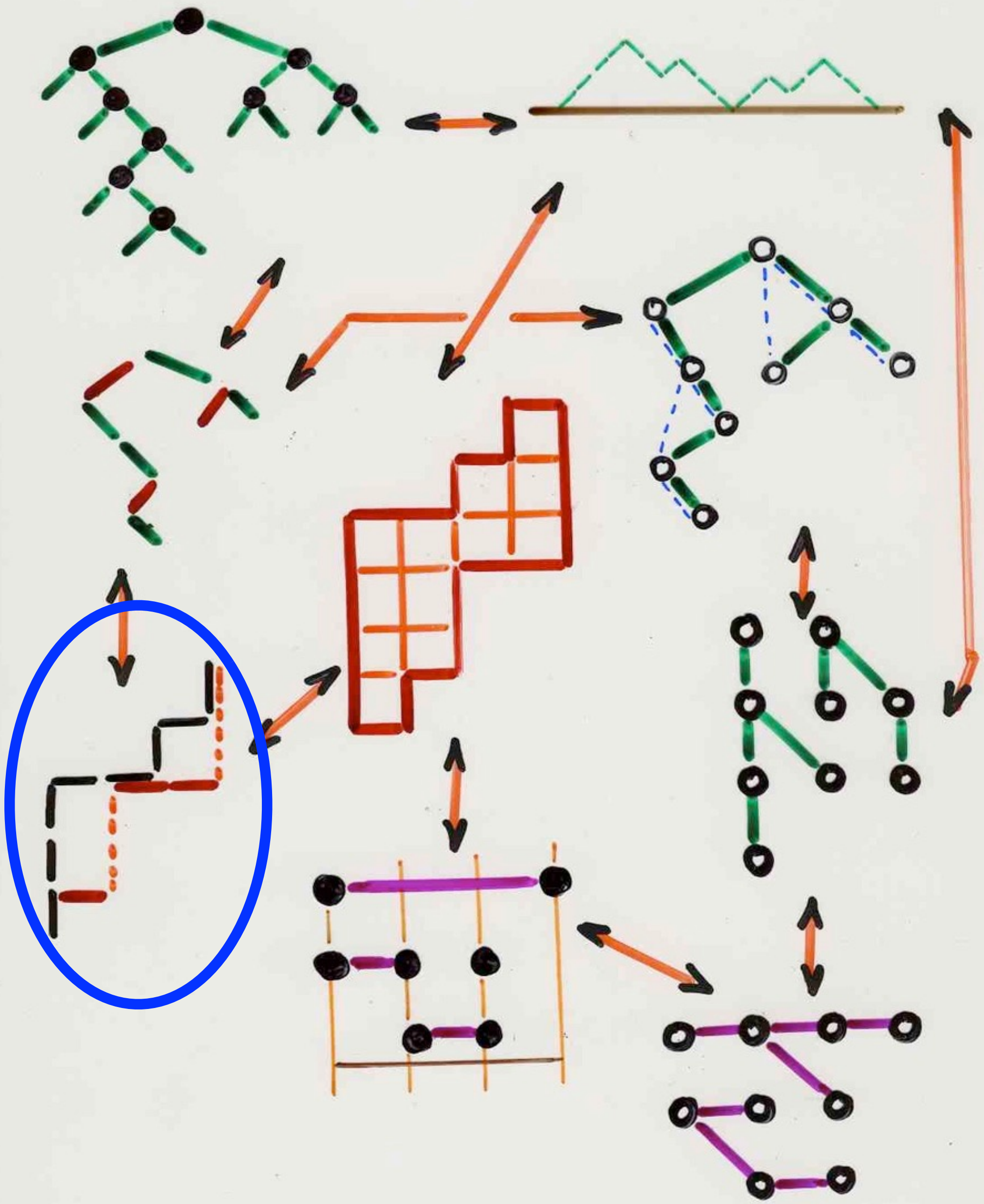
A bijection

of the Catalan garden

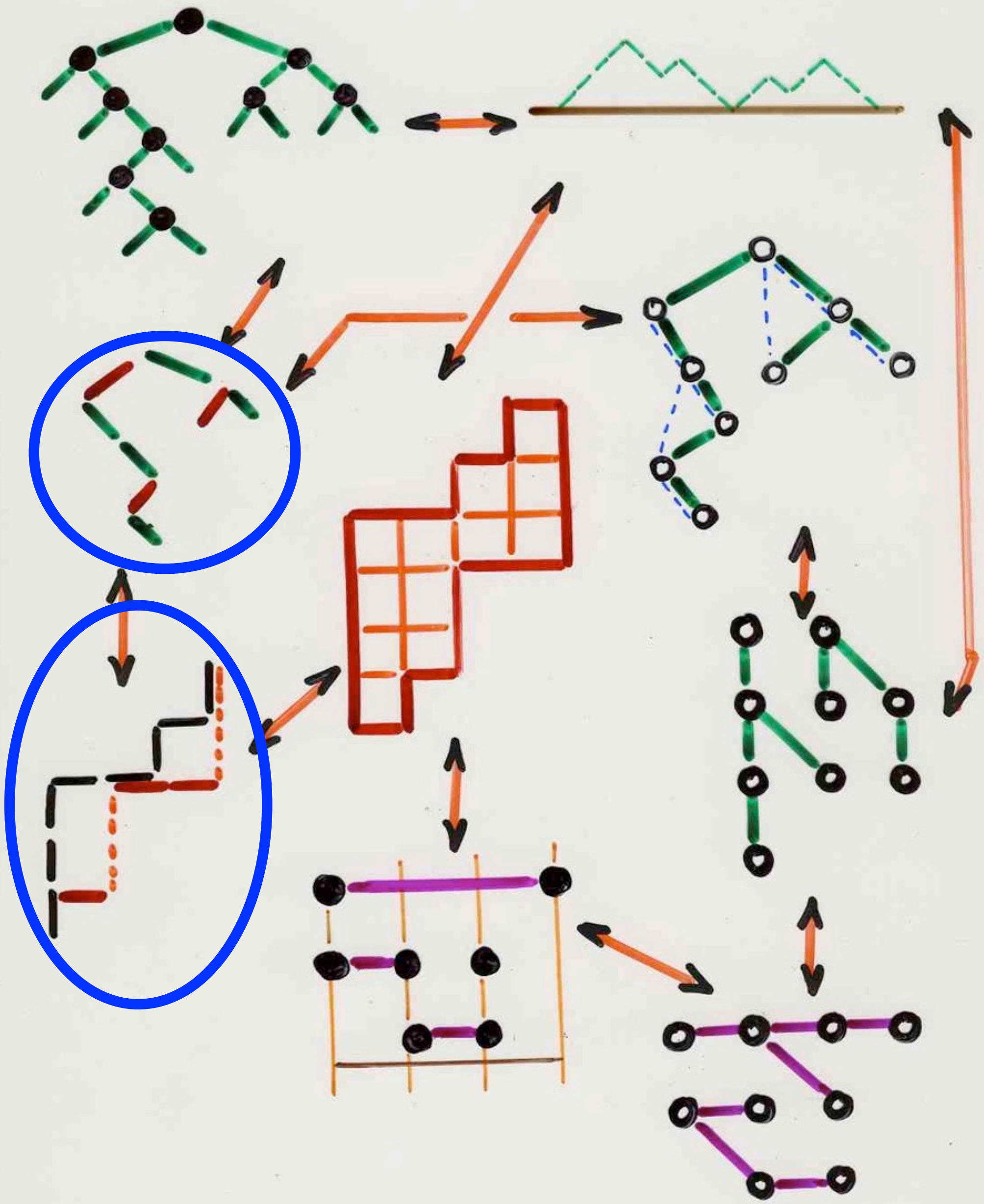
the Catalan garden



the Catalan garden

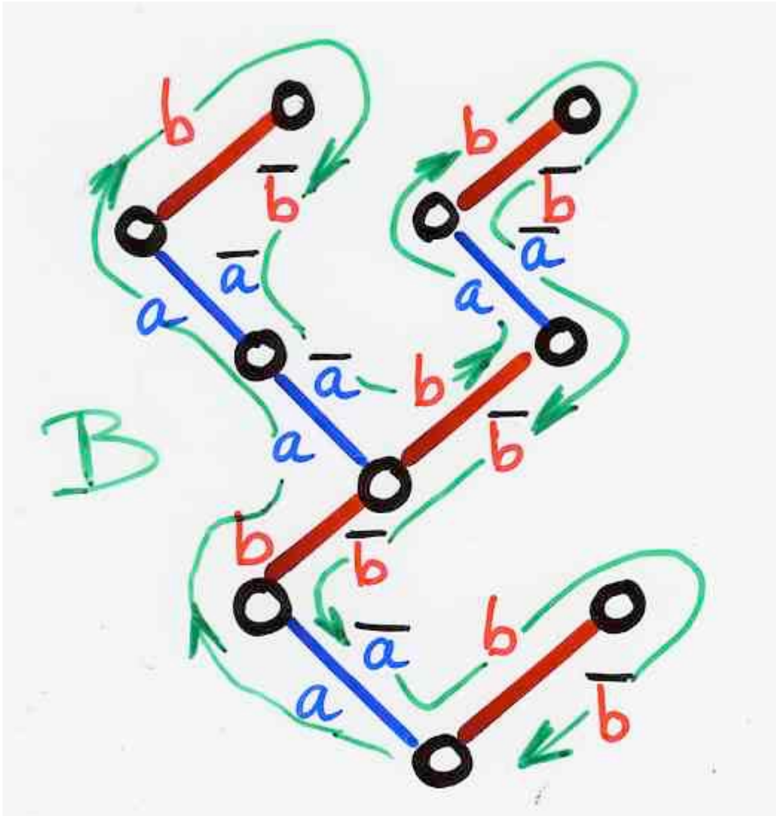


the Catalan garden



A bijection

binary tree B $\longrightarrow$ pair of paths (u,v)



walk around a binary tree B

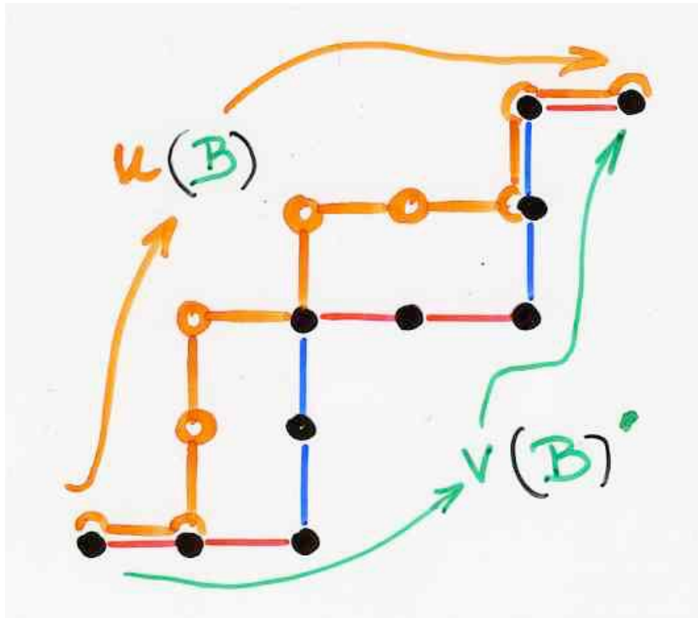
the words $\left\{ \begin{array}{l} w(B) \\ u(B) \\ v(B) \end{array} \right.$

$$w(B) = a\, b\, a\, a\, b\, \bar{b}\, \bar{a}\, \bar{a}\, b\, a\, b\, \bar{b}\, \bar{a}\, \bar{b}\, \bar{b}\, \bar{a}\, b\, \bar{b}$$

$$u(B) = \bar{b}\, \bar{a}\, \bar{a}\, \bar{b}\, \bar{a}\, \bar{b}\, \bar{b}\, \bar{a}\, \bar{b}$$

$$v(B) = b\, b\, \bar{a}\, \bar{a}\, b\, b\, \bar{a}\, \bar{a}\, b$$

$$\bar{a} \rightarrow N \qquad \left.\begin{array}{l} b \\ \bar{b} \end{array}\right\} \rightarrow E$$



the pair (u, v) of paths associated to a binary tree B

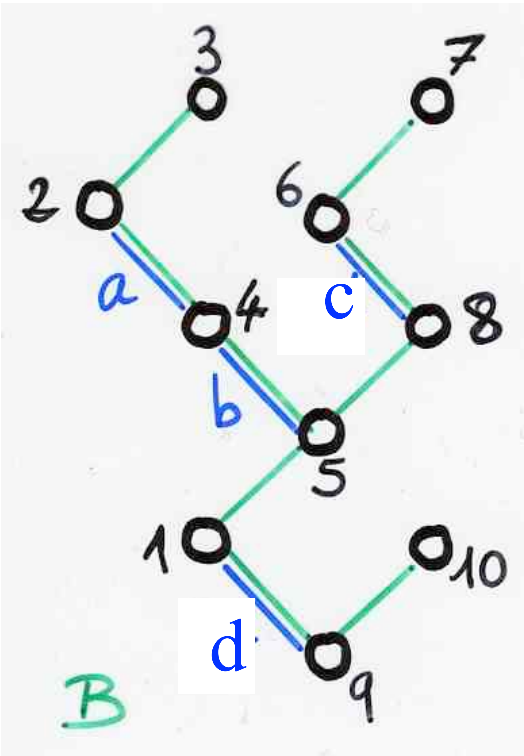
v (B) is the Canopy

reverse bijection

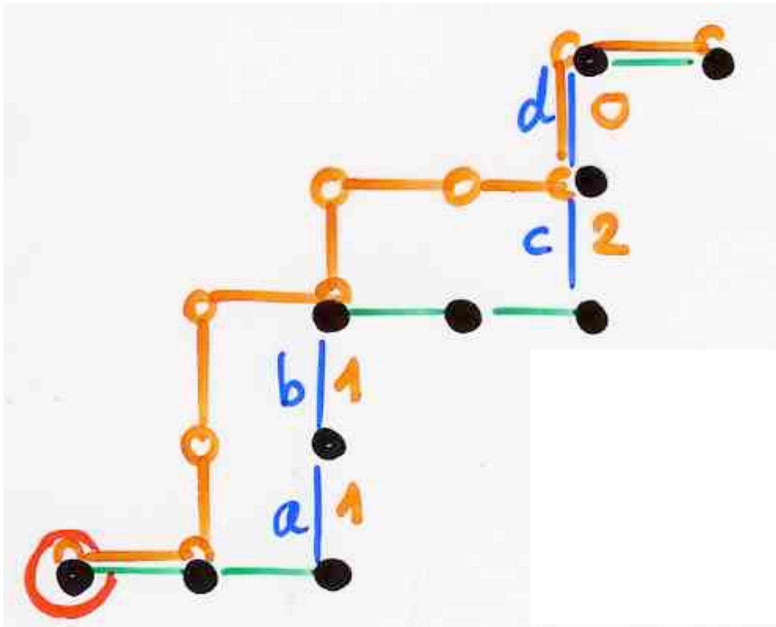
binary tree B ⟵ pair of paths (u,v)

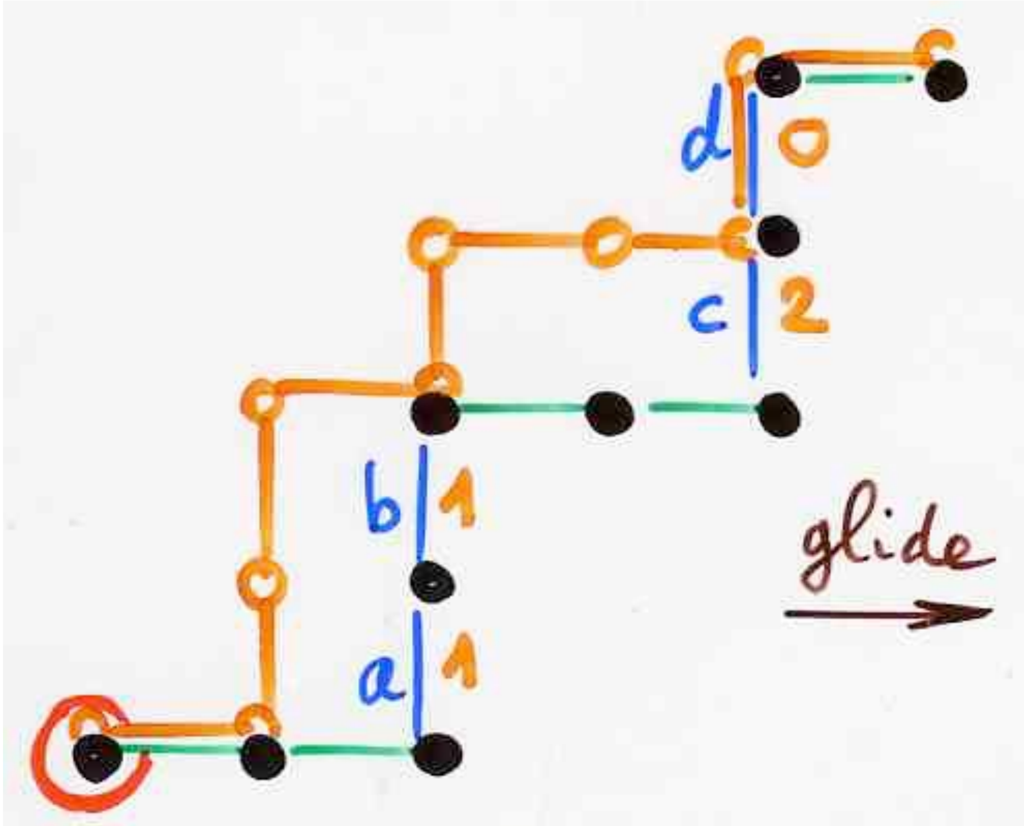
the «push-gliding» algorithm

a lemma



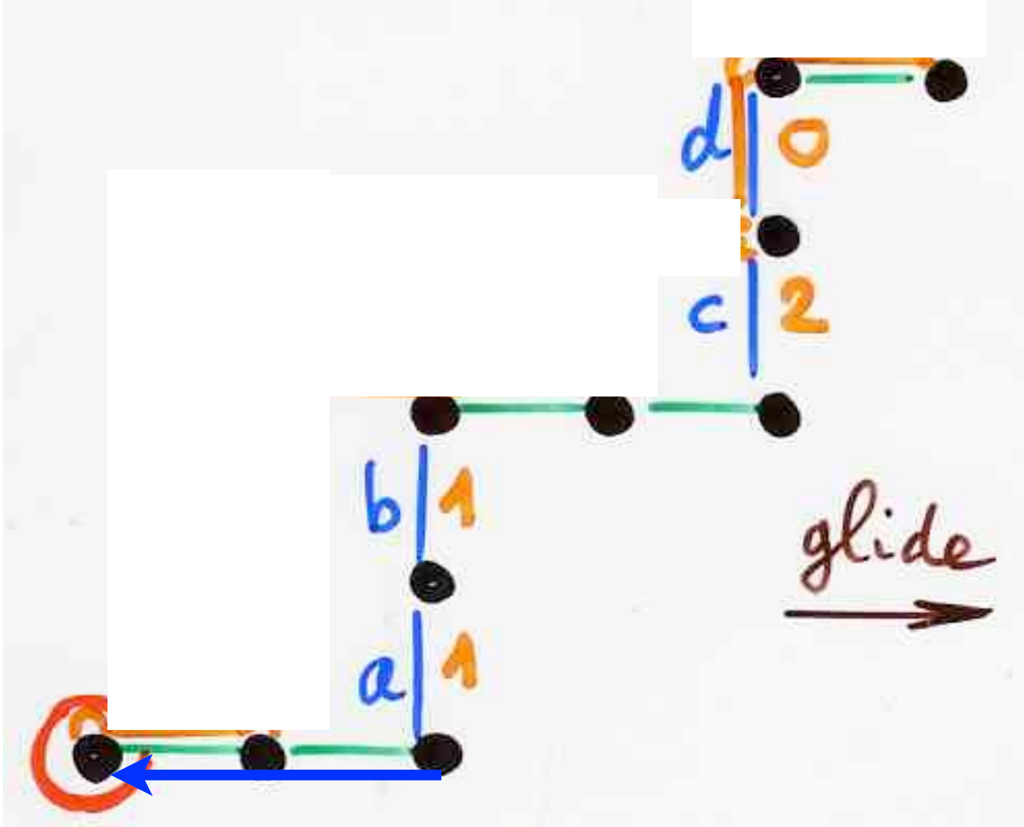
symmetric order





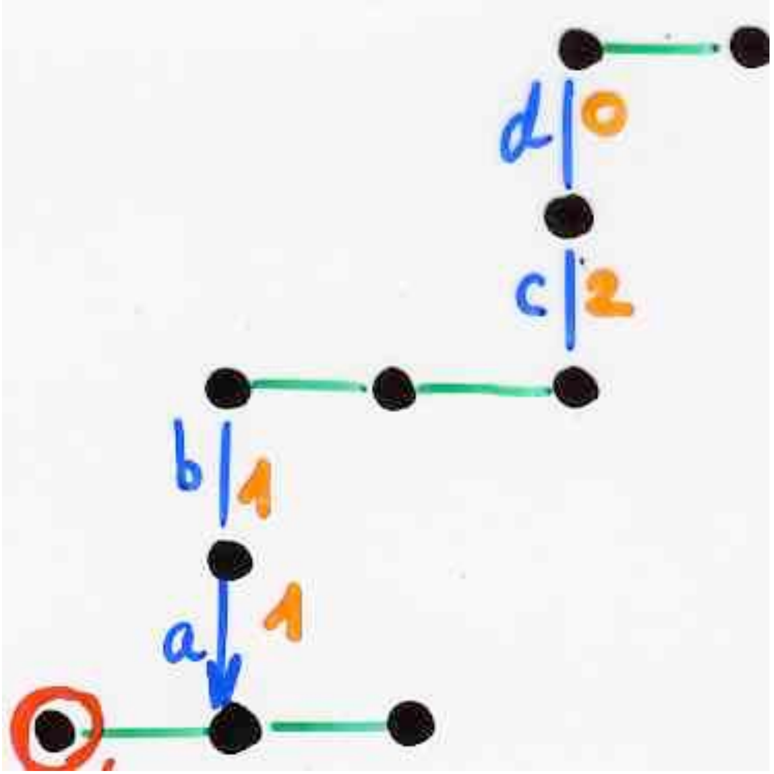
reverse bijection

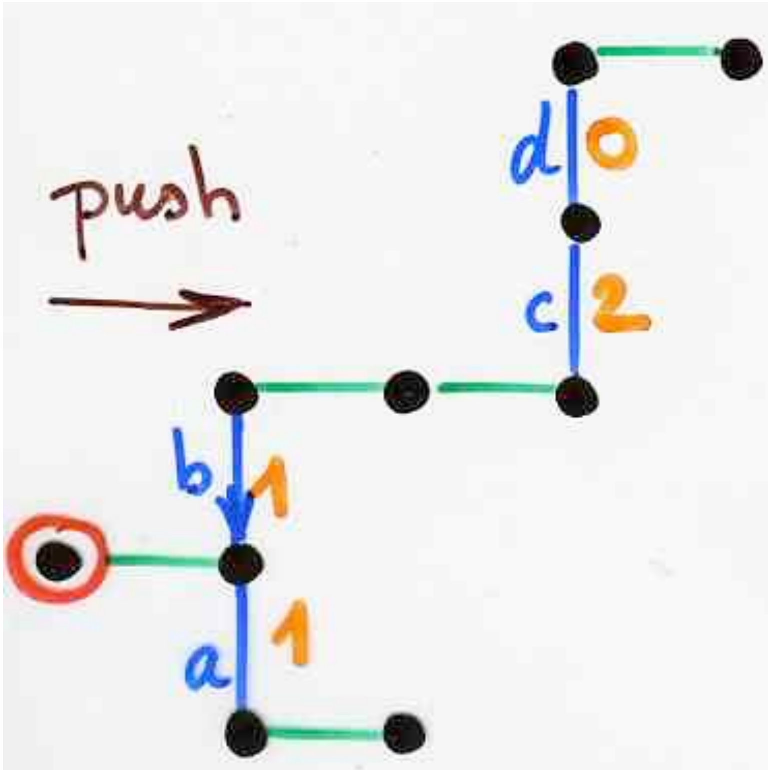
the "push-gliding" algorithm

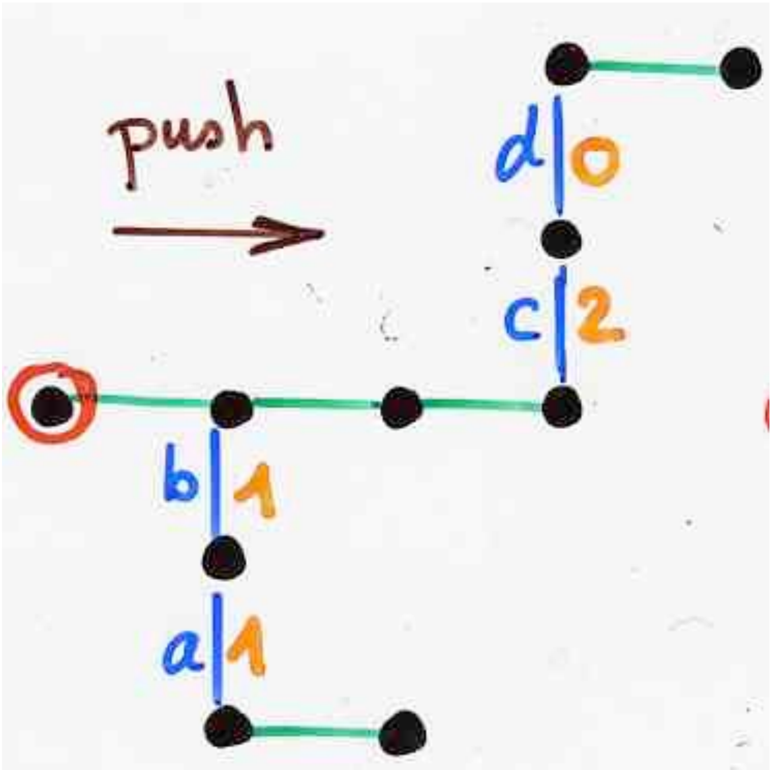


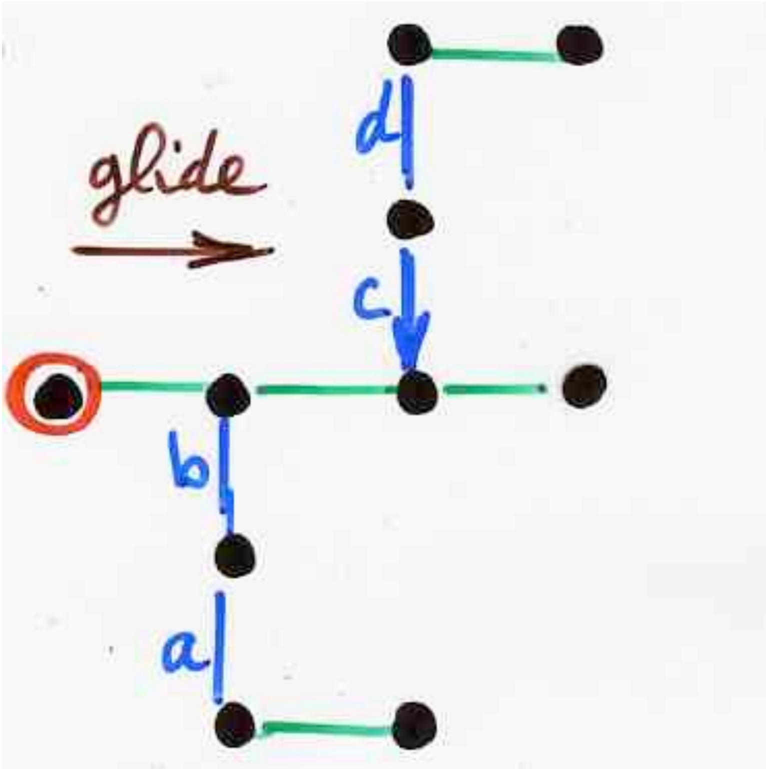
reverse bijection

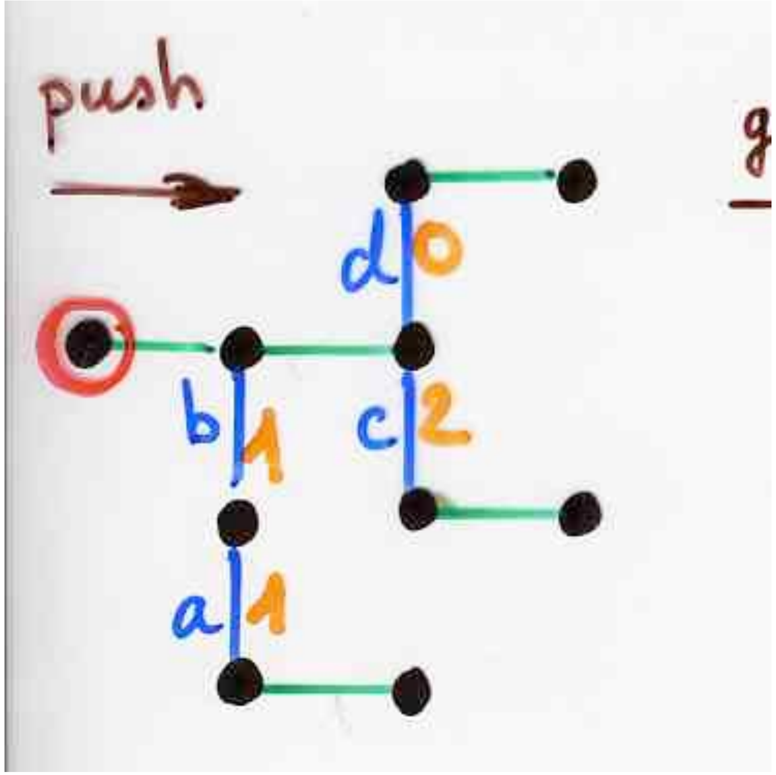
the "push-gliding" algorithm

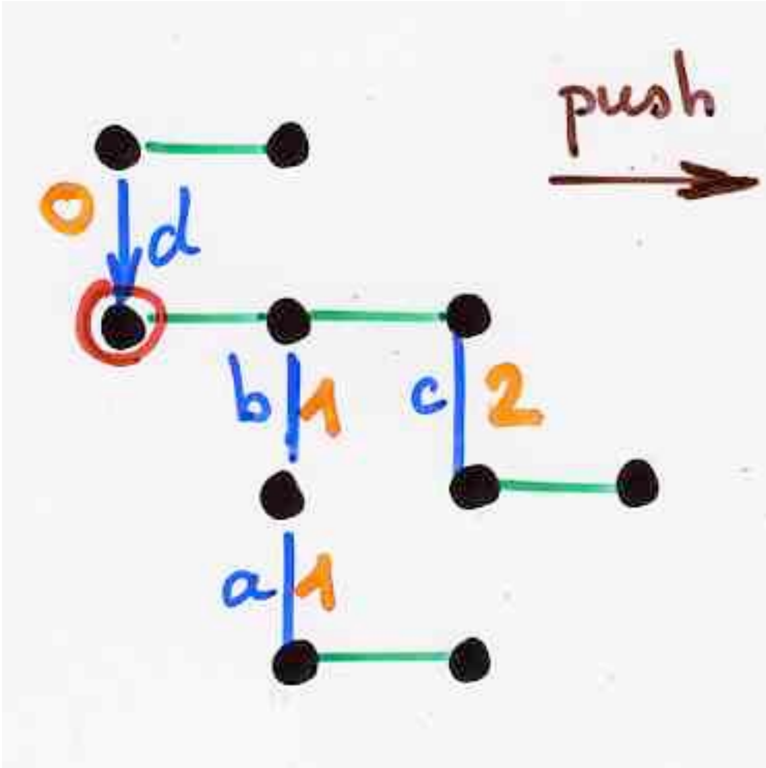

d
0
c
2
b
1
a
1

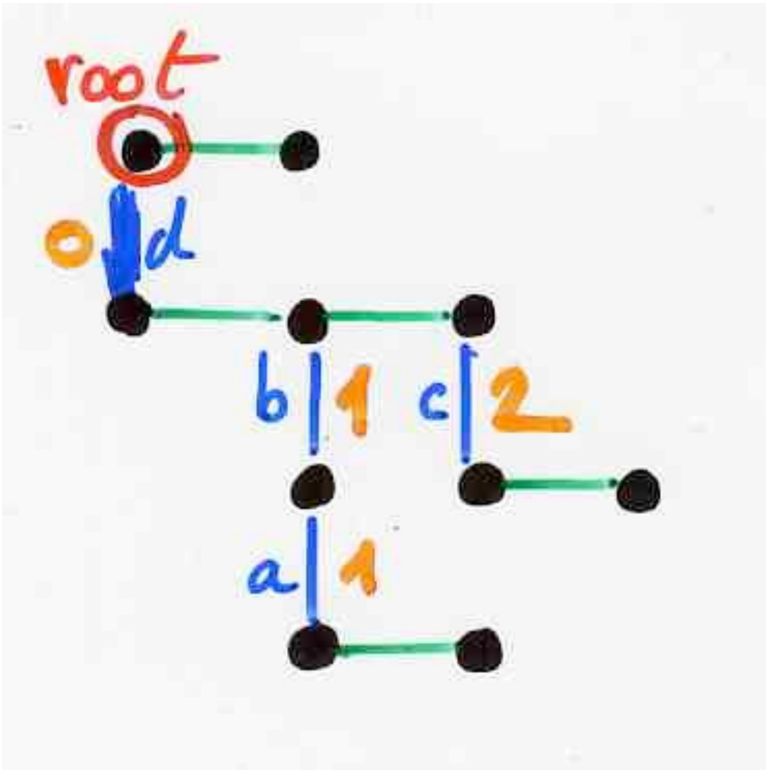

push
d
0
c
2
b
1
a
1

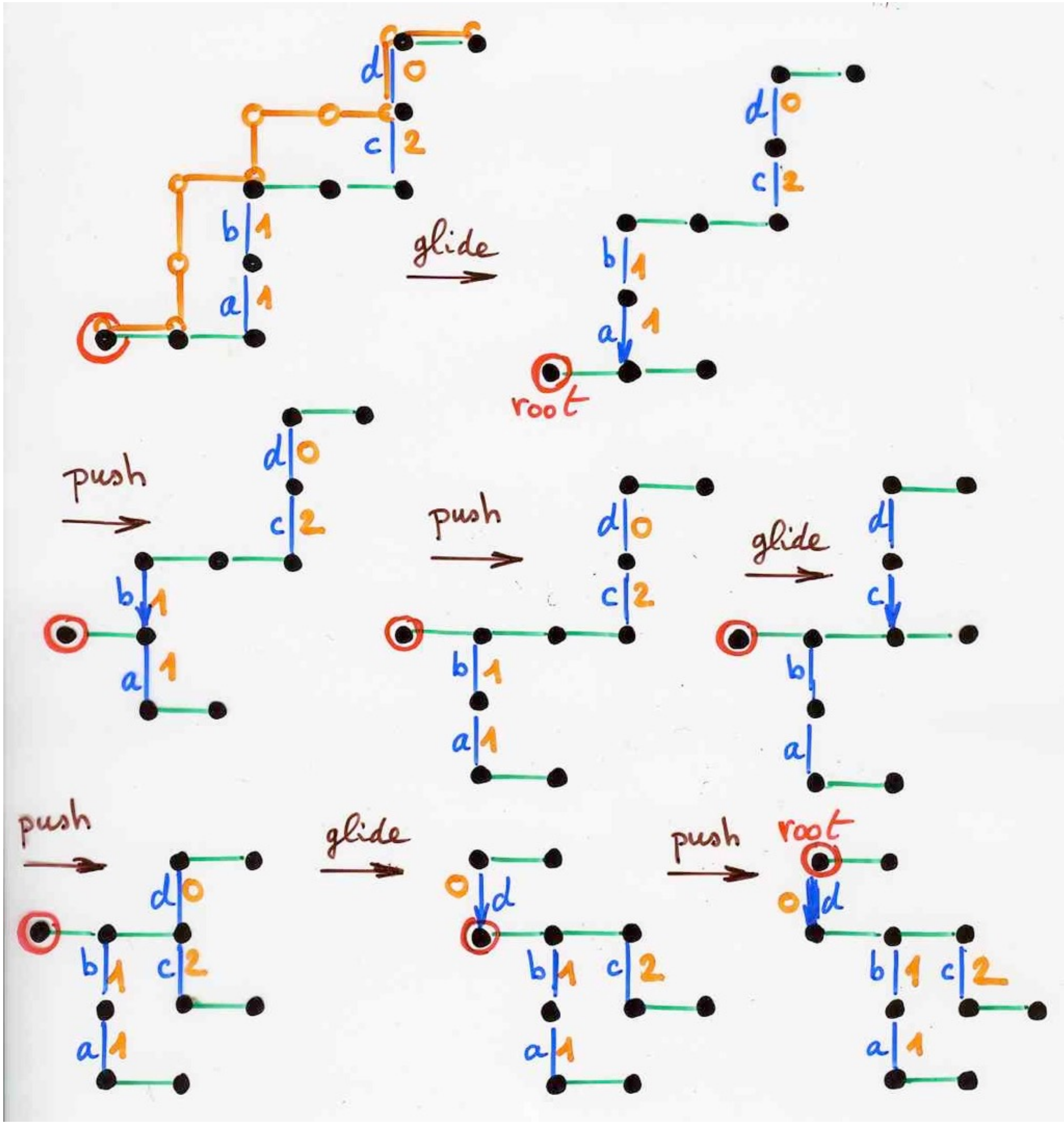

push
d 0
c 2
b 1
a 1


glide
d
c
b
a


push
d
0
b
1
c
2
a
1
g


push
d
b
1
c
2
a
1


root
0
d
b
1
c
2
a
1



d 0
c 2
b 1
a 1
glide
root
push
push
glide
push
glide
push
root

canopy and Tamari lattice

Prop.(i) • The set of binary trees having a given canopy V is an interval of the Tamari lattice I(V)

idea of proof

algebraic structures

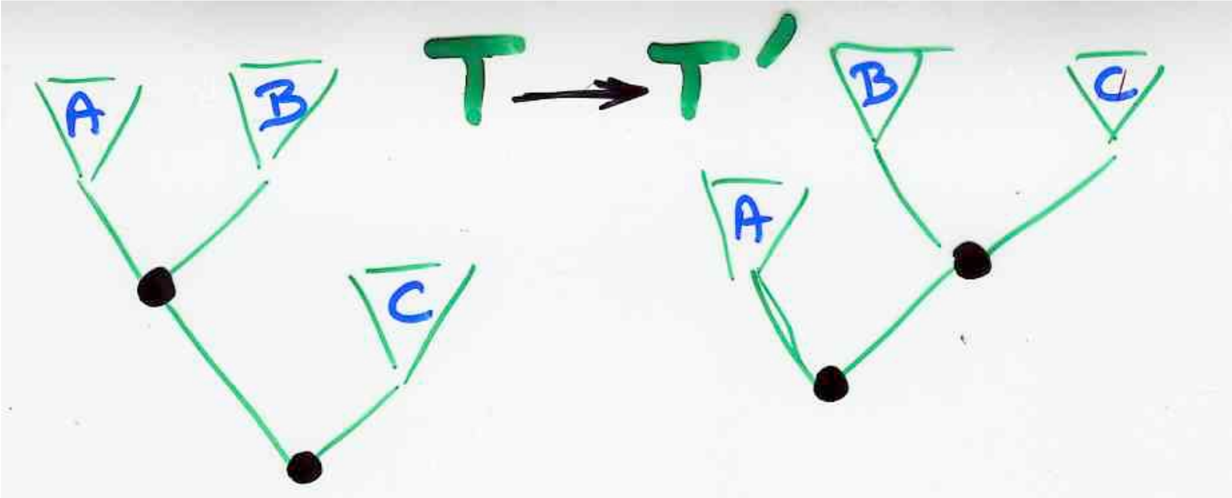
Hopf algebra

dim 2^{n-1} C_n $n!$

Catalan

Boolean lattice inclusion ↔ Tamari order ↔ weak Bruhat order

J.-L.Loday, M. Ronco (1998, 2012)



if $B \neq \bullet$ canopy is invariant

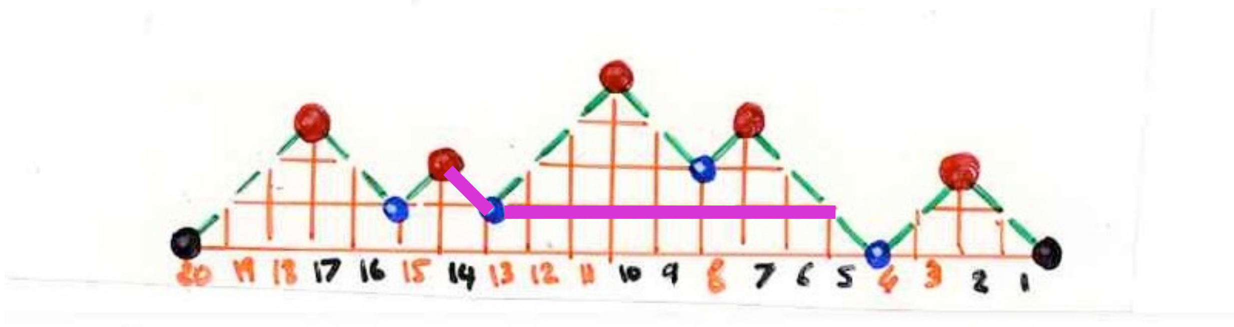
if $B = \bullet$ canopy $c(T')$ not invariant

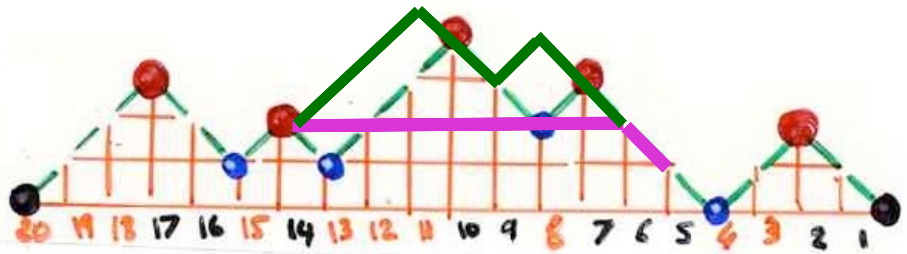
$$c(T) = c(A) \overset{+}{/} c(B)\, c(C)$$

$$c(T') = c(A) \underset{-}{\backslash} c(B)\, c(C)$$

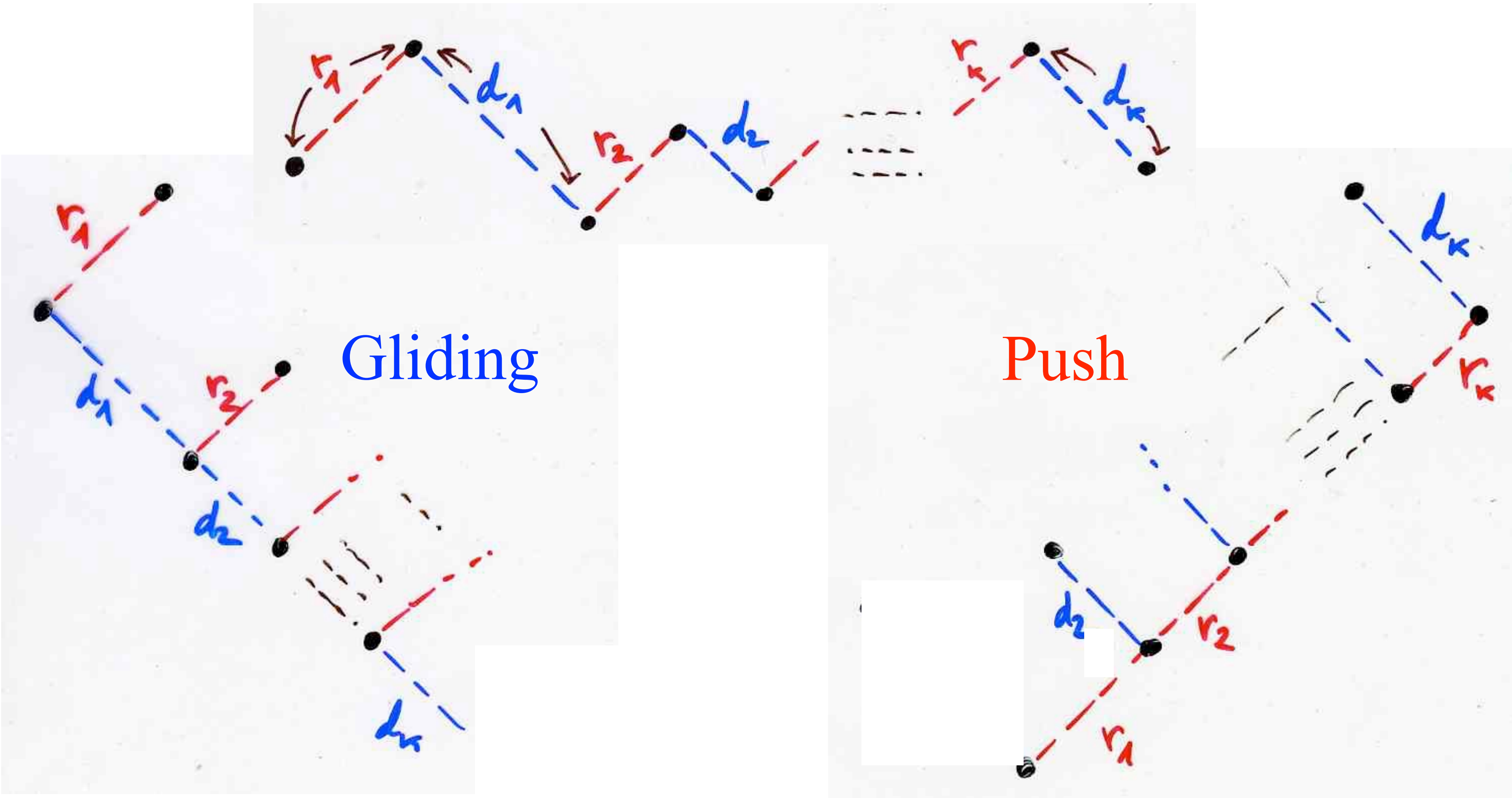
$$c(B) = \varnothing$$

forbidden move



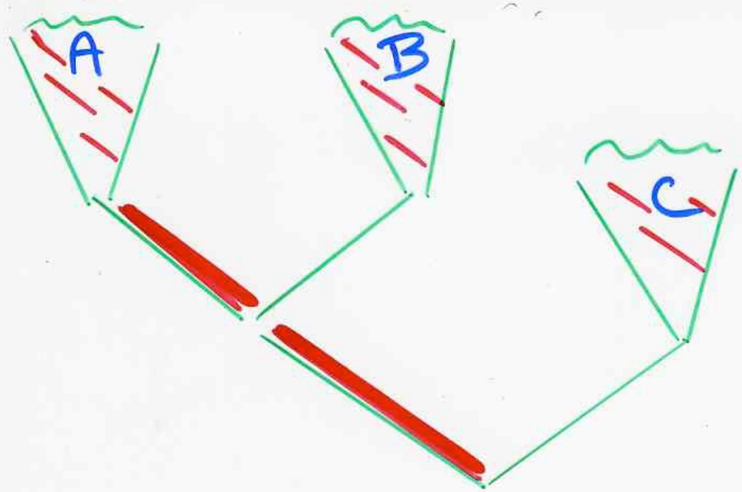

20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

Prop.(i) The set of binary trees having a given canopy w is an interval I(w) of the Tamari lattice

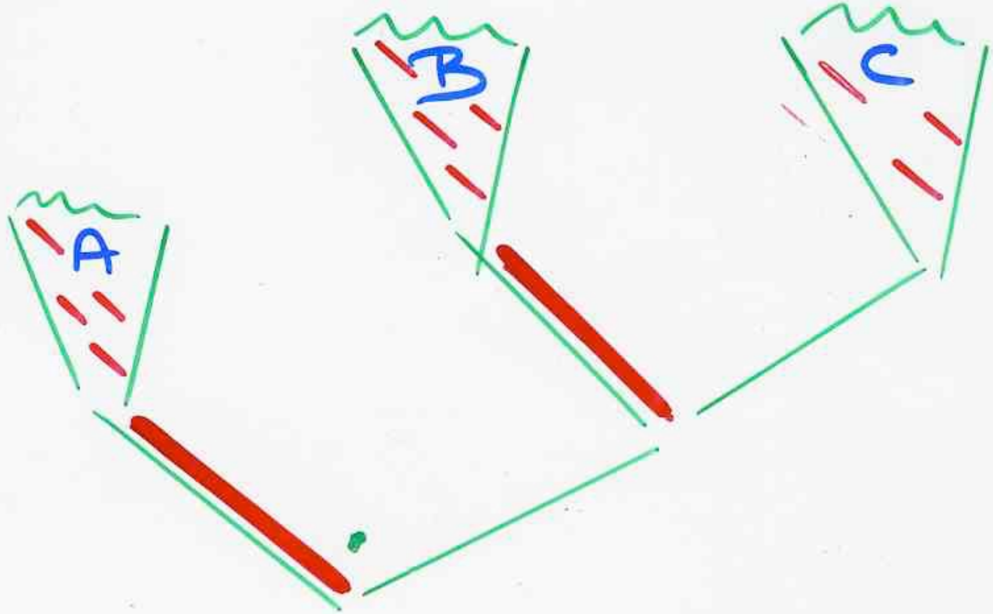


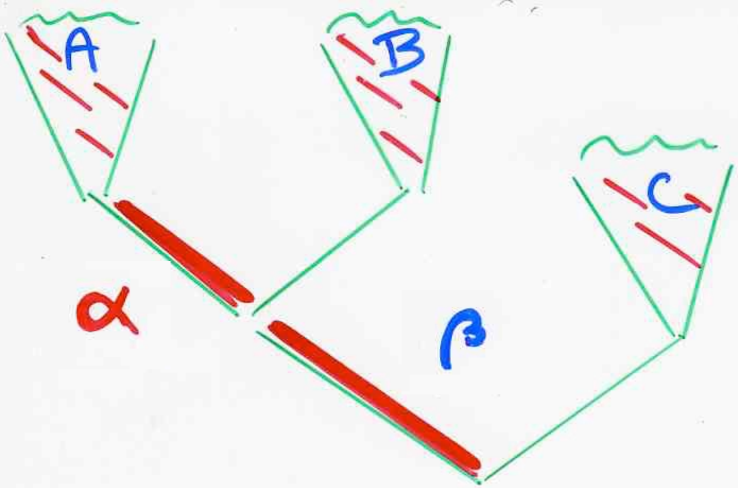
Prop.-(i) • The set of binary trees having a given canopy v is an interval $\mathcal{I}$(v) of the Tamari lattice

(ii) This interval I(v) is isomorphic to T_v

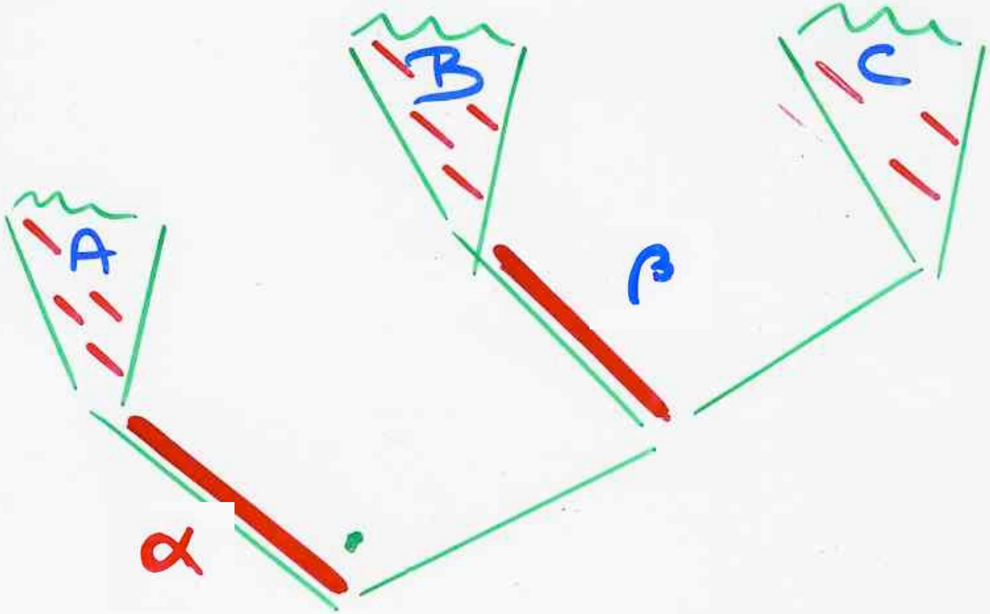


left edges
(in symmetric
order)
with their
"right height"



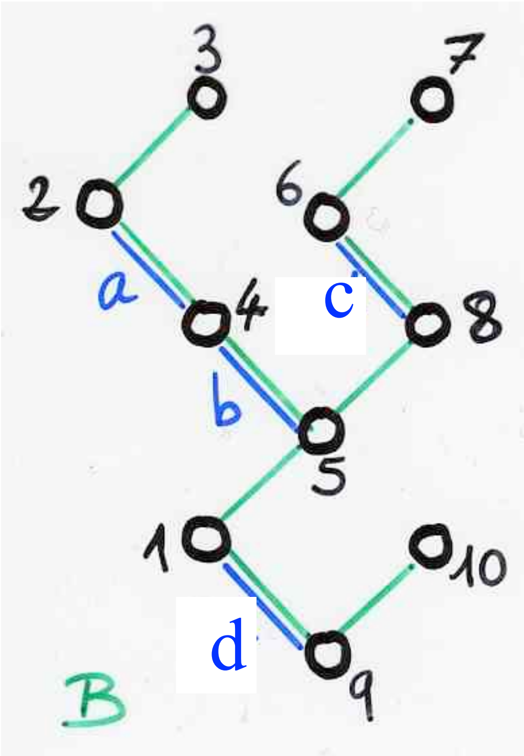


left edges
(in symmetric order)
with their
"right height"

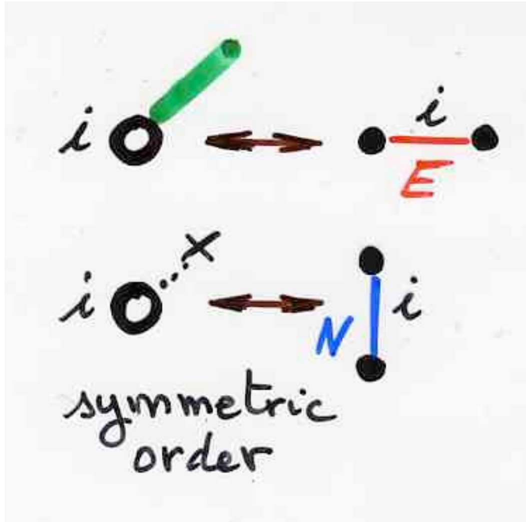


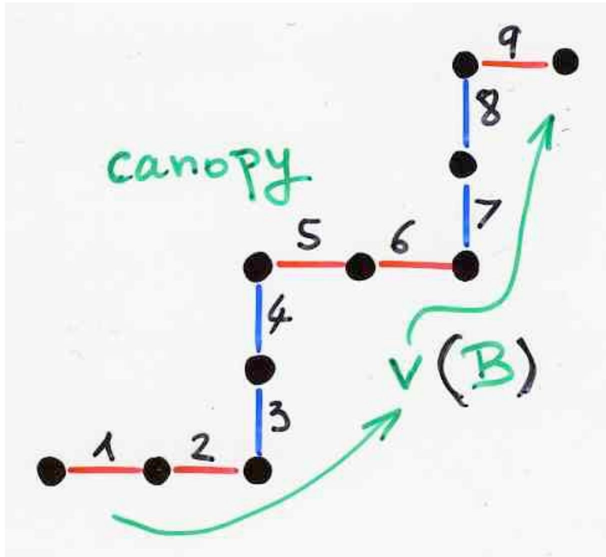
preservation
of the
symmetric order
for left edges

right height:
+1 in C
and for β

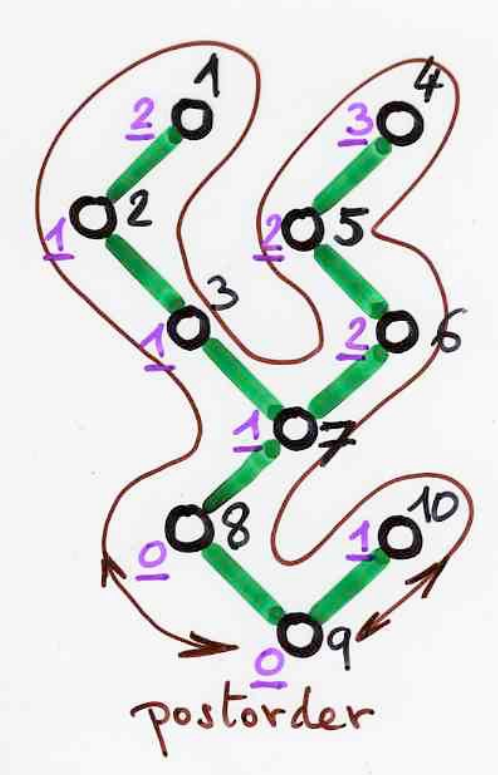


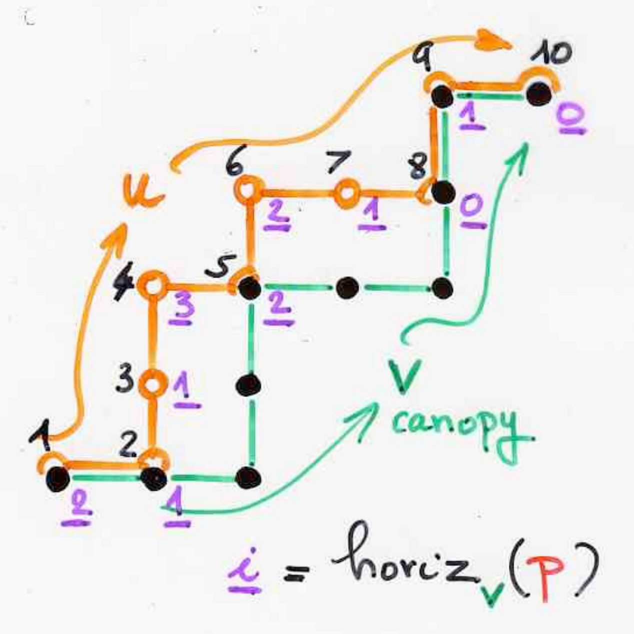
symmetric order



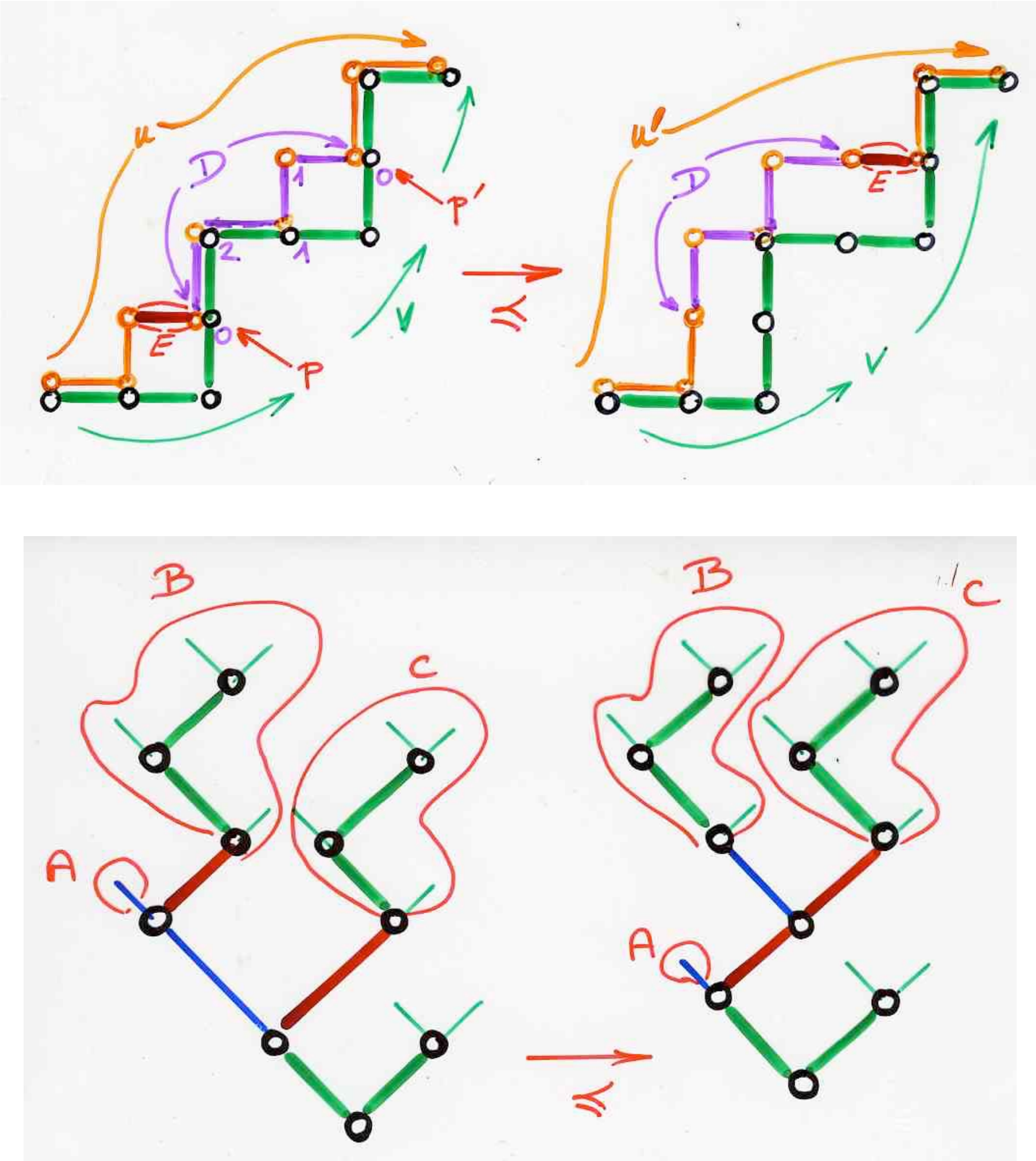


third definition of the canopy





another lemma



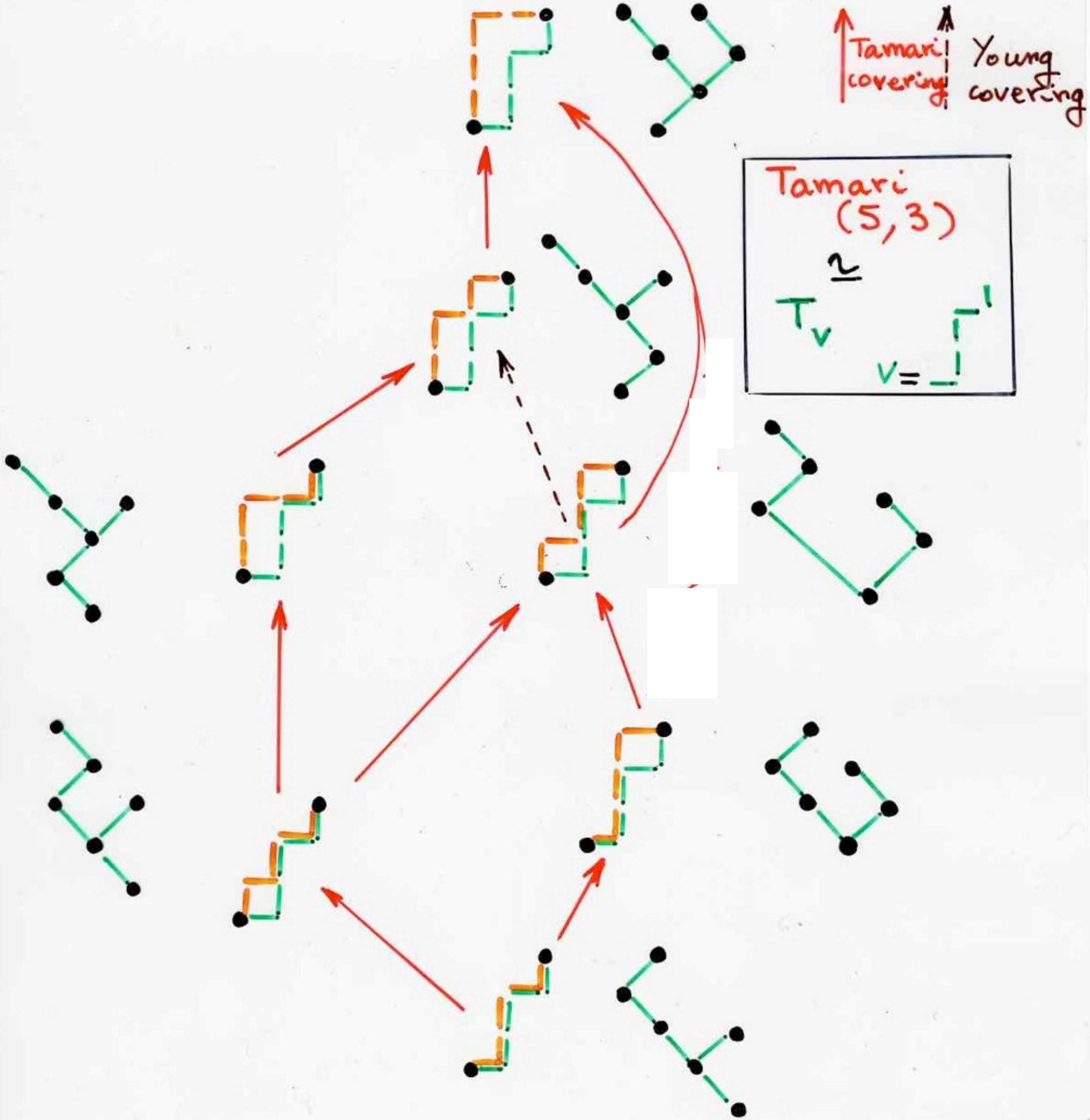
the covering relation in T_V
and the corresponding rotation
in (ordinary) T

Thm 1. For any path v

T_v is a lattice

is a consequence of (ii)

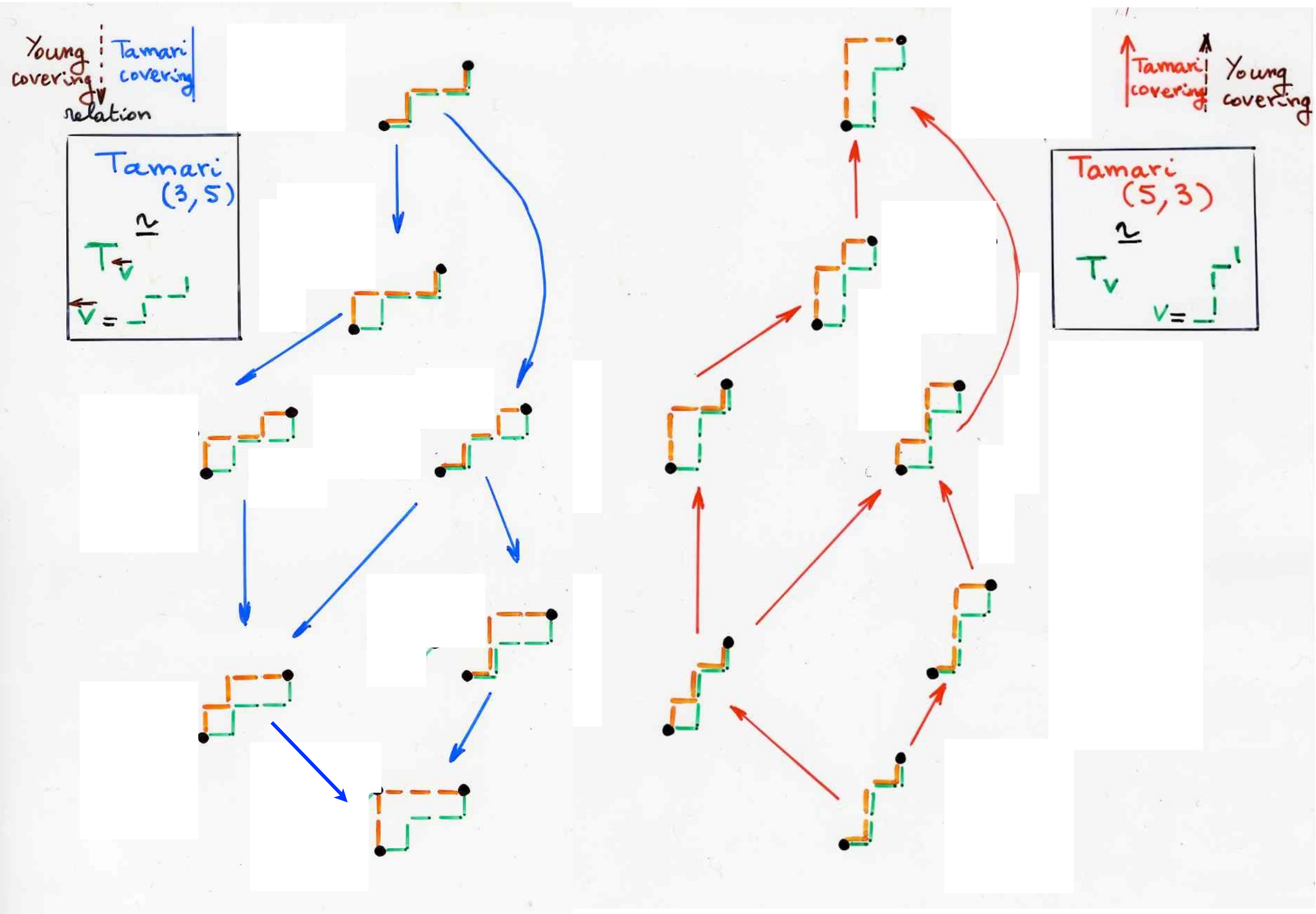
(ii) This interval $I(v)$ is isomorphic to T_v

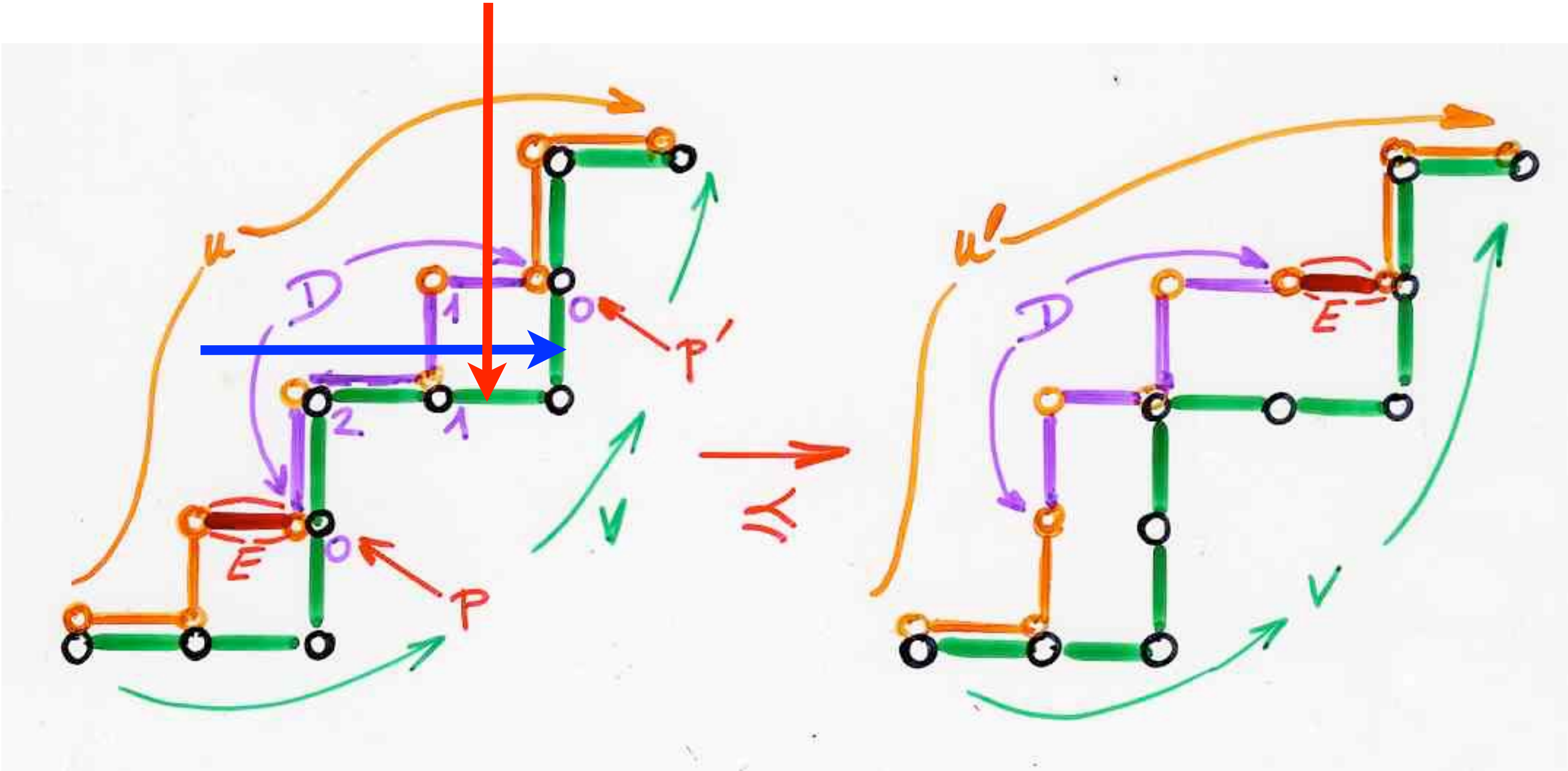


Tamari covering
Young covering
Tamari (5,3)
≃
T_v
v =

proof of the duality

Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$

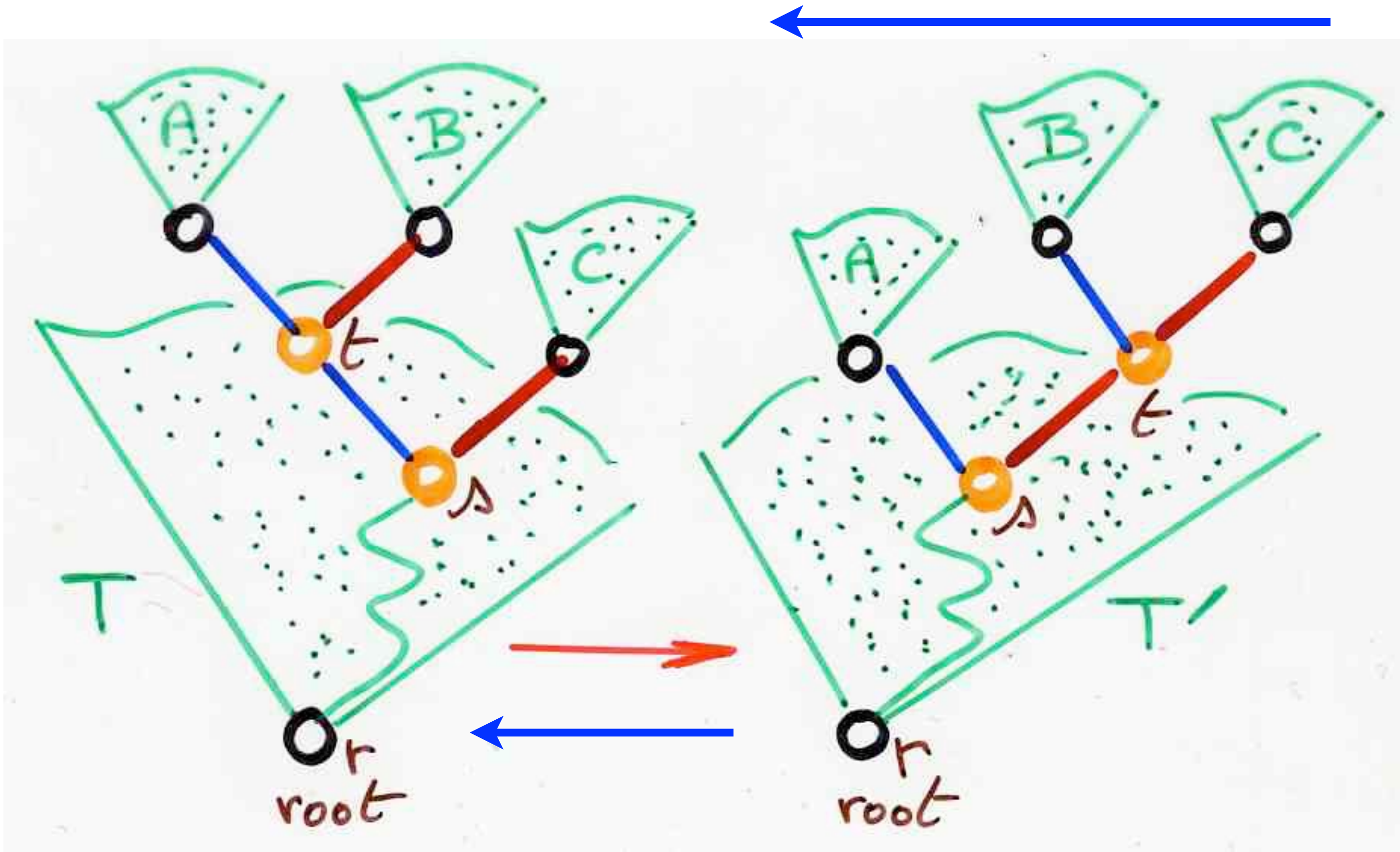




«row covering relation»

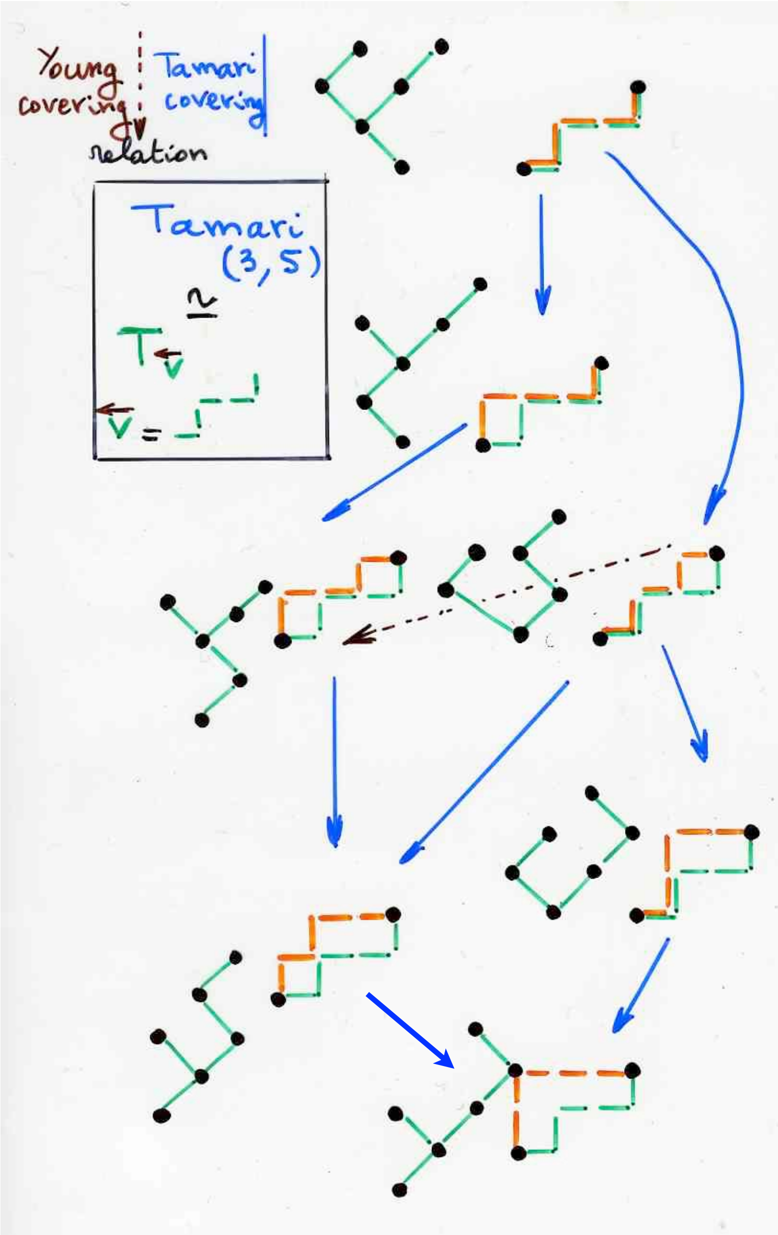
«column covering relation»

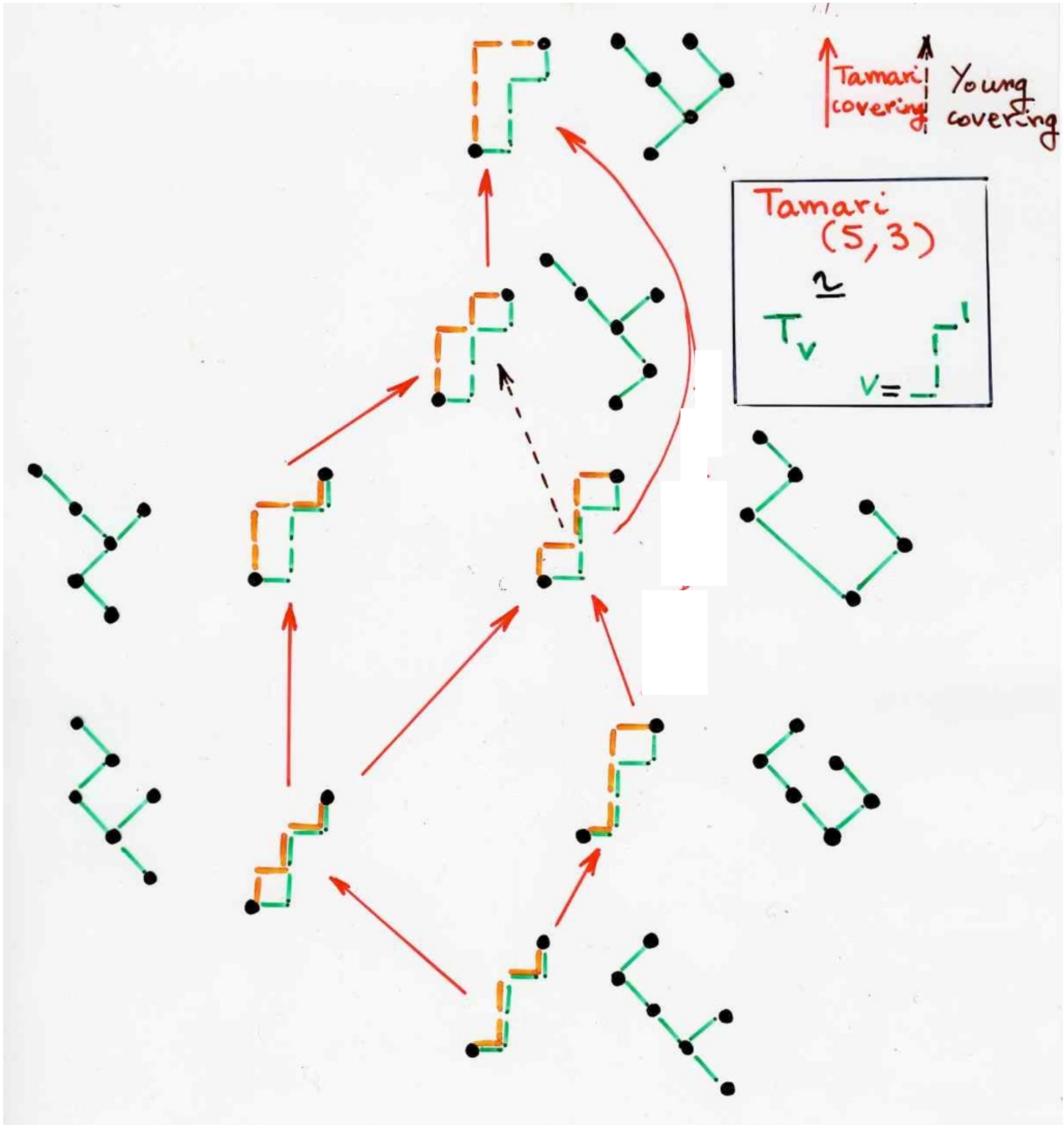
mirror image, exchange N and E

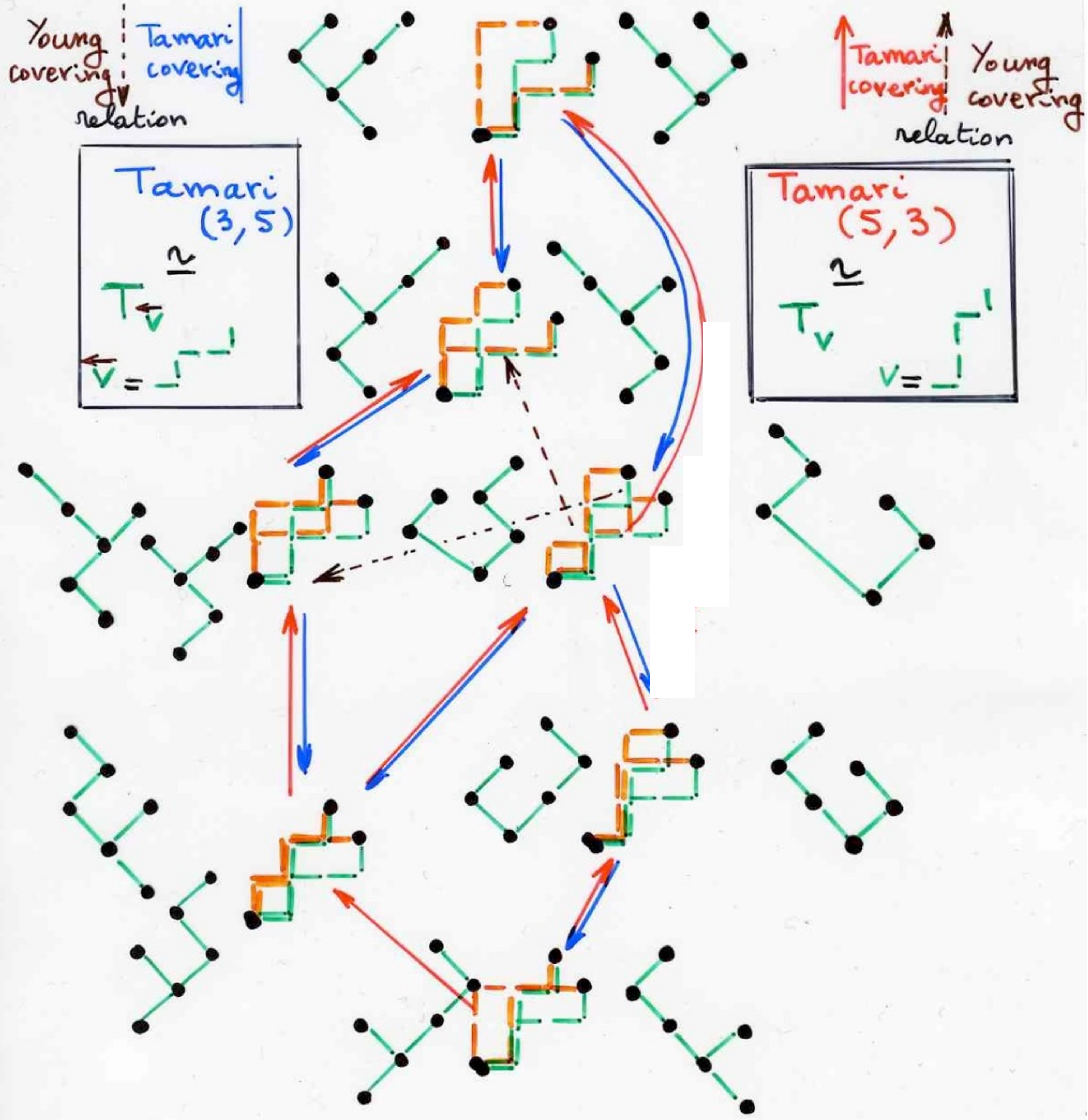


Rotation in a binary tree:
the covering relation in the
Tamari lattice

Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$






Young covering relation
Tamari covering
Tamari covering
Young covering relation
Tamari (3, 5)
≃
Tamari (5, 3)
≃

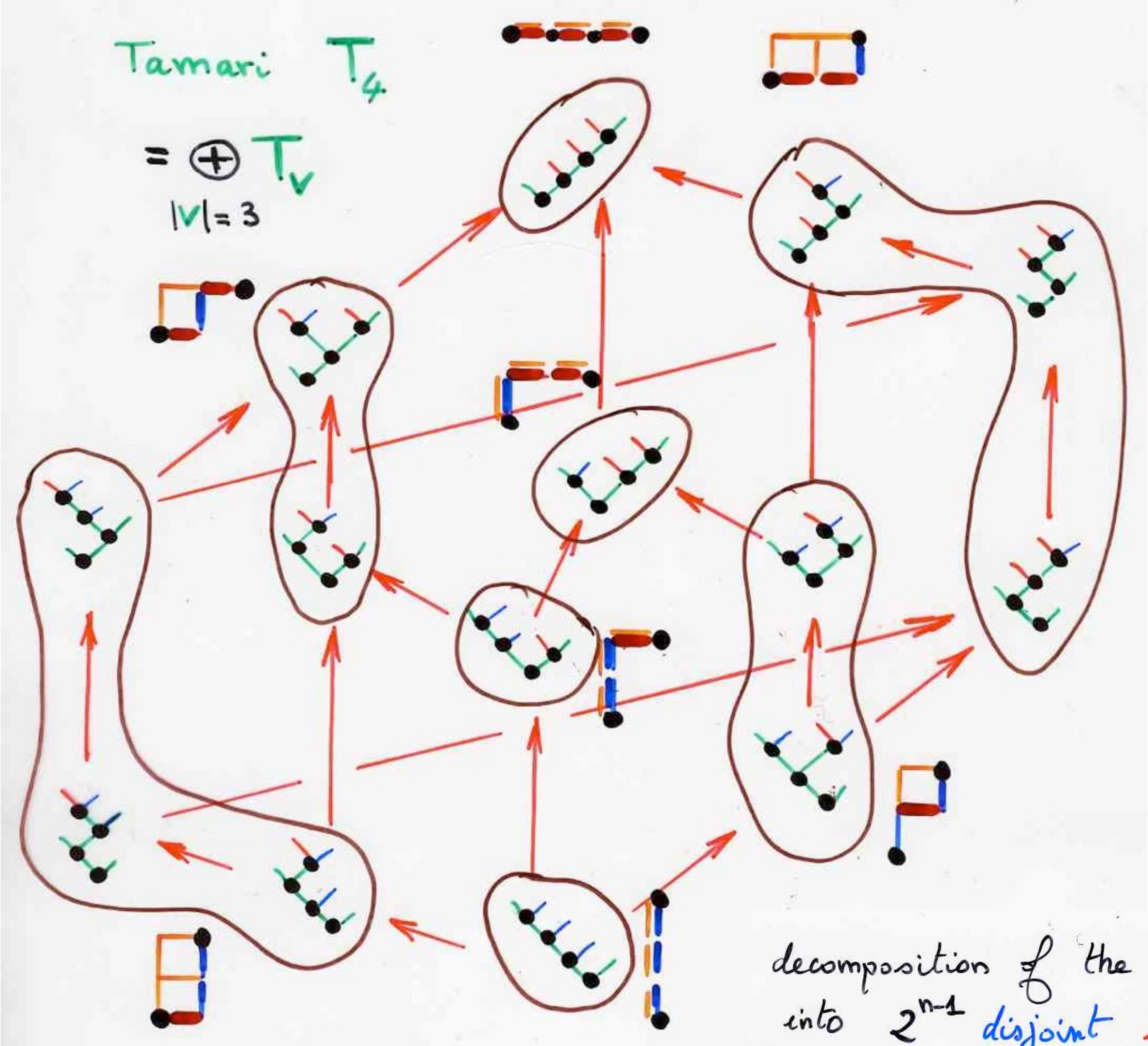
Thm 1. For any path v, T_v is a lattice

Thm 2. The lattice T_v is isomorphic to the dual of $T_{\overleftarrow{v}}$

Thm 3. The usual Tamari lattice T_n can be partitioned into intervals indexed by the 2^{n-1} paths v of length $(n-1)$ with $\{E, N\}$ steps,

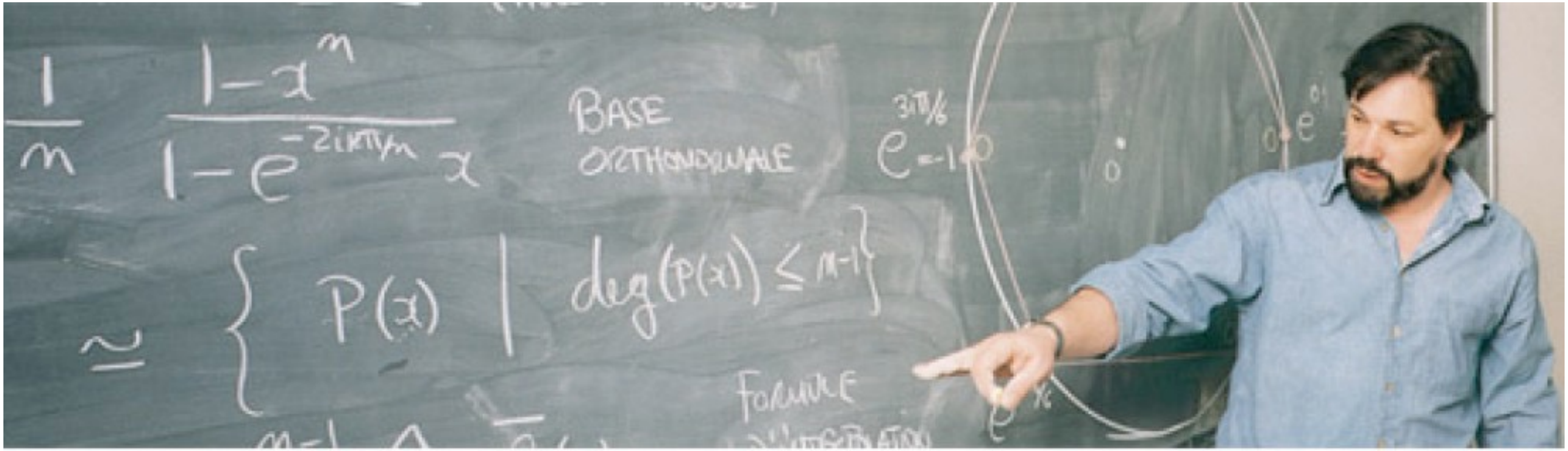
$$T_n \cong \biguplus_{|v|=n-1} I_v ,$$

where each $I_v \cong T_v$.



decomposition of the lattice T_n into 2^{n-1} disjoint intervals

relation with
diagonal coinvariant spaces



François Bergeron

diagonal
coinvariant
spaces

Adriano Garsia

Armstrong, Garsia, Haglund, Haiman, Hicks

Lee, Li, Loehr, Morse, Remmel, Rhoades,

Stout, Xin, Warrington, Zabrocki, ...

+ ......

$X = (x_{i,j})_{\substack{1 \le i \le k \\ 1 \le j \le n}}$ matrix of variables

$\sigma \in \mathfrak{S}_n$ symmetric group

$\sigma \cdot (X) = (x_{i,\sigma(j)})_{\substack{1 \le i \le k \\ 1 \le j \le n}}$ action on $\mathbb{C}[X]$

diagonal coinvariant spaces

$DR_{k,n} = \mathbb{C}[X] / \mathcal{J}$

higher diagonal coinvariant spaces

$DR^{m}_{k,n} = \varepsilon^{m-1} \otimes A^{m-1} / \mathcal{J} A^{m-1}$ alternants

$DR^{m\,\varepsilon}_{k,n}$ subspace of alternants

$k=1$ classical

$k=2$ Garsia, Haiman

→ Macdonald polynomials

$DR^{m}_{2,n}$ ε $DR^{m}_{2,n}$

dimension $\frac{1}{(m+1)n+1}\binom{(m+1)n+1}{mn}$ $(mn+1)^{n-1}$

m-ballot paths m-parking functions

$DR^{m}_{2,n}$ 20 years of studies

m-shuffle conjecture

Frobenius series (q,t) sum on m-parking

(area, dinv)

$k=3$ Haiman (conjecture) 1990

$DR^{\varepsilon}_{3,n}$ $DR_{3,n}$

dimension $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$ $2^n (n+1)^{n-2}$

↑

Chapoton (2006)

number of interval Tamari$_n$

Bijective proof FPSAC 2007

Bernardi, N. Bonichon

F. Bergeron (2008) introduced the m-Tamari lattice

conjecture

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1} \qquad (m+1)^n(mn+1)^{n-2}$$

nb of intervals — nb of labelled intervals

M. Bousquet-Mélou, E. Fusy, L.-F. Préville-Ratelle (2011)

nb of intervals of m-Tamari lattices

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1}$$ F. Bergeron

M. Bousquet-Mélou, G. Chapuy, L.-F. Préville-Ratelle (2011)

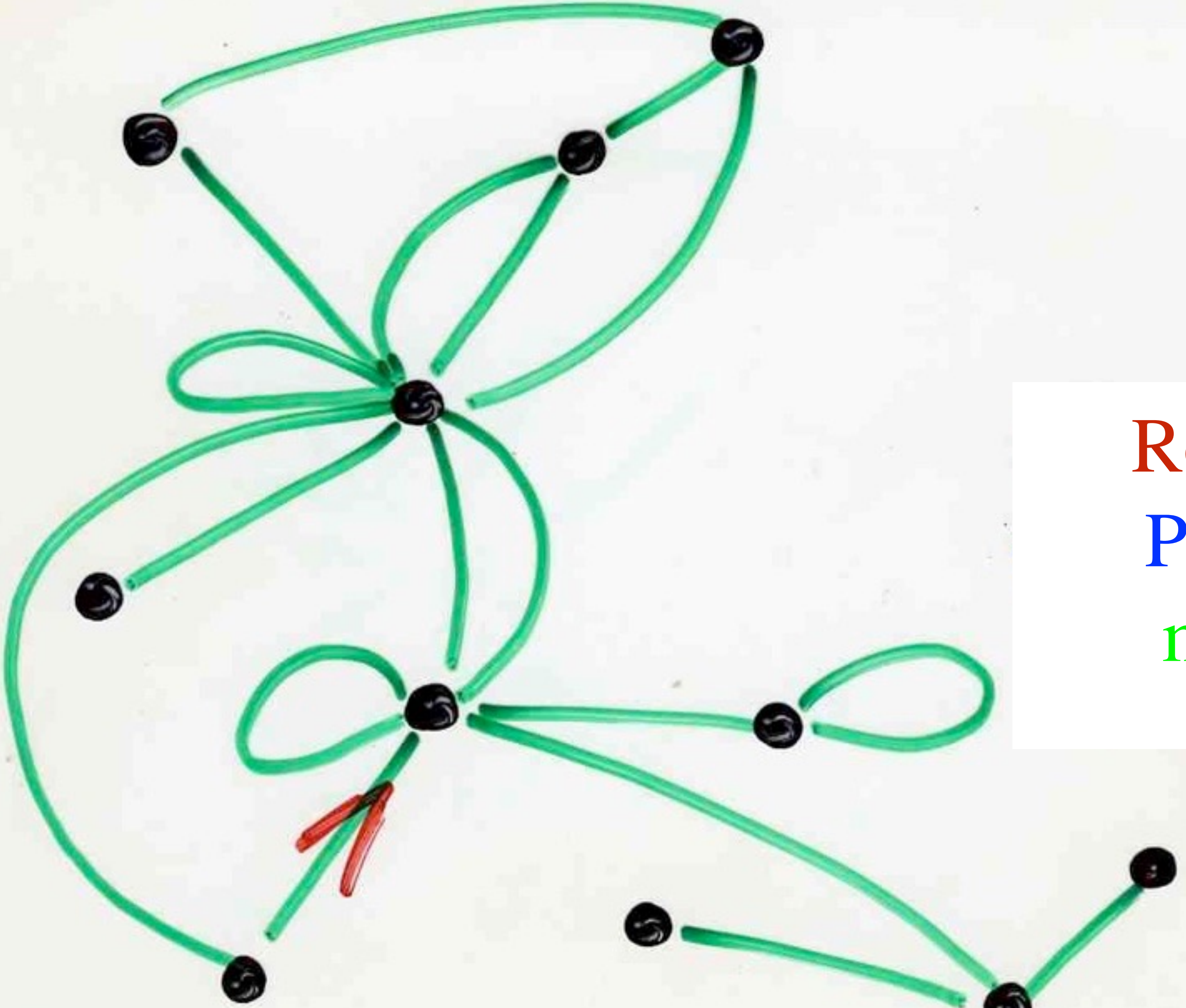
nb of labelled intervals $(m+1)^n(mn+1)^{n-2}$

Proposition (L.-F. Préville-Ratelle)

The total number of intervals in all T_v $|v|=n$

is the number of non-separable planar maps

$$\frac{2(3n+3)!}{(n+2)!(n+3)!}$$



Rooted
Planar
maps

This work was done at Universidad de Talca, Chile
at the invitation of Luc Lapointe,
and in the various following places

Constitution
Isla Negra
Viña del Mar
Valparaiso

Luc Lapointe

Universidad de Talca

La Universidad de Los mejores

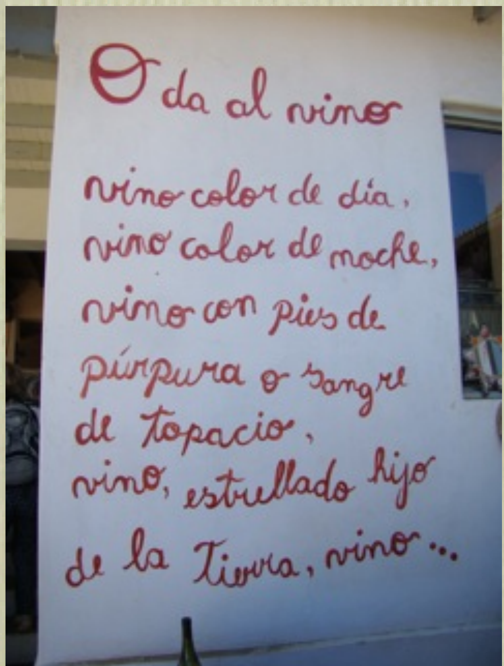
www.utalca.cl

From Talca to Constitution

Maule valley

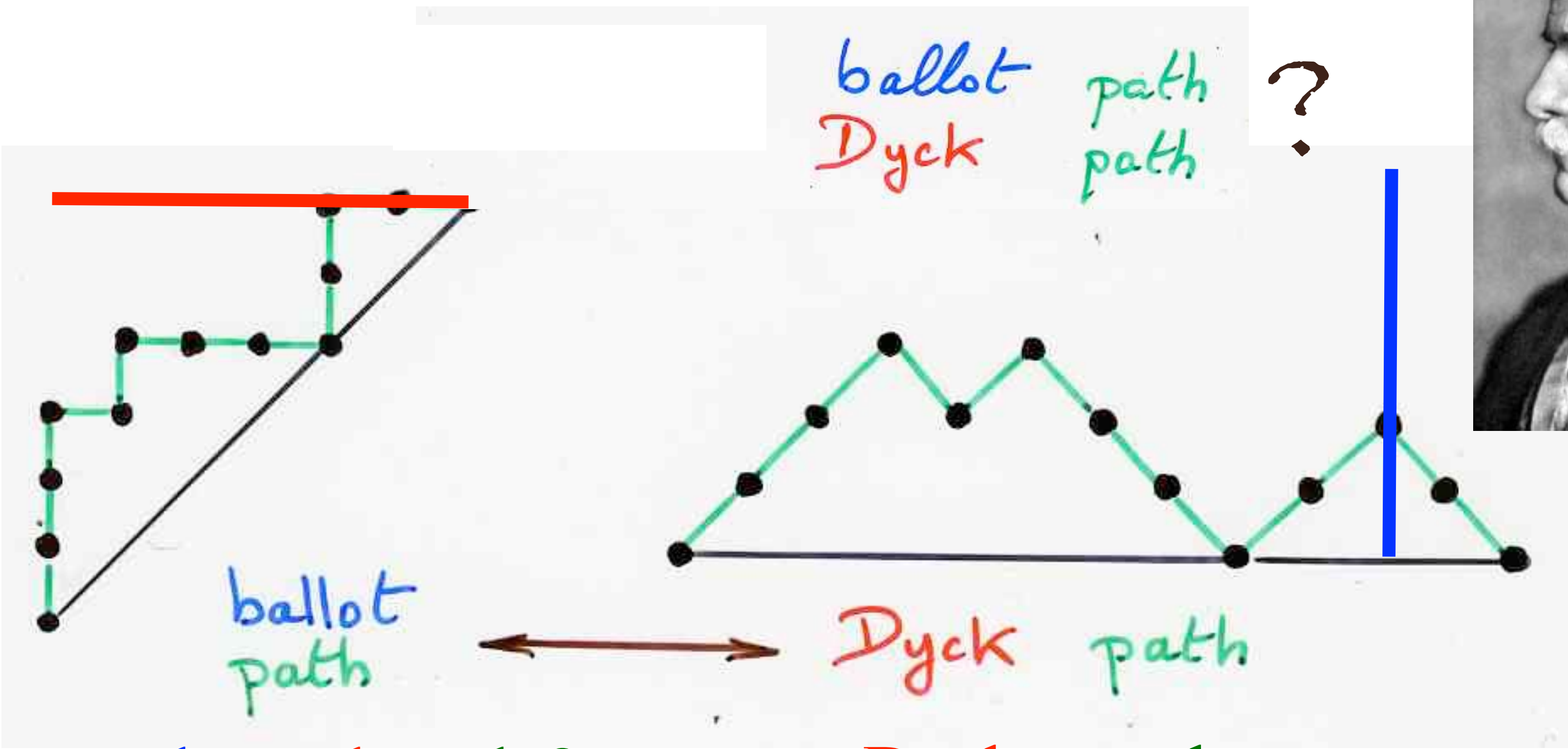
workshop in Valparaiso

Isla Negra
Pablo Neruda



Thank you !

Question of vocabulary



Von Dyck

rotated Dyck path ?

Dyck words

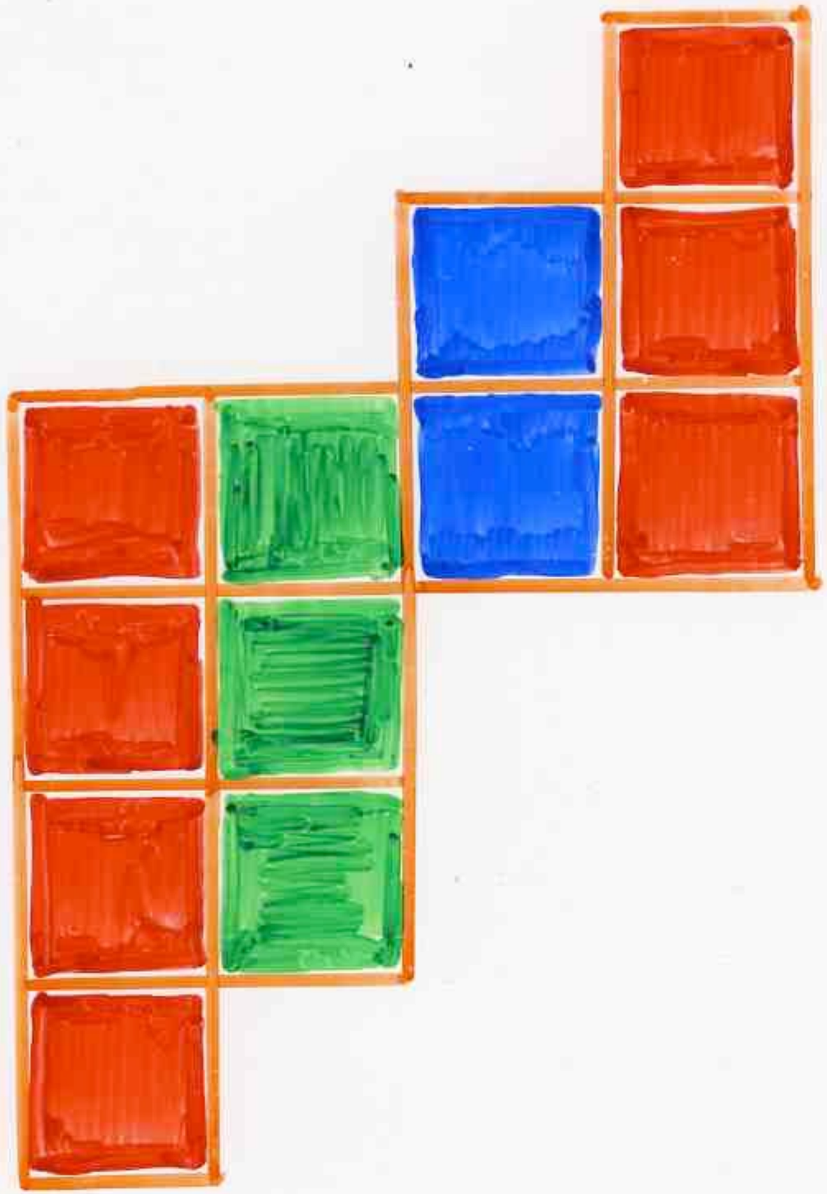
Bertrand
D. André
problème du scrutin
(188x)

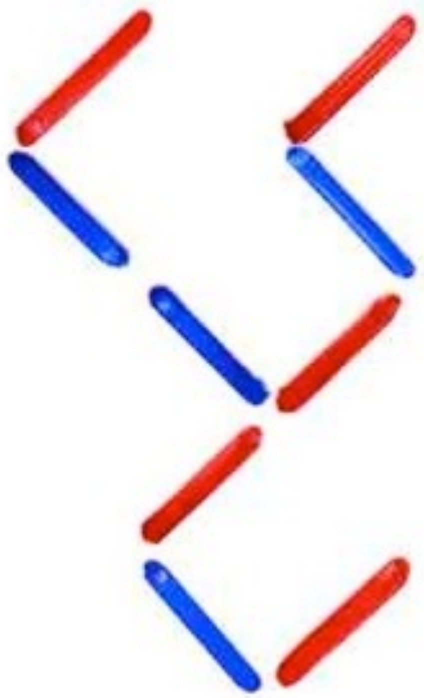
ballot numbers

G.Kerweras «bridge»
D.Dumont «contraction»

Complements

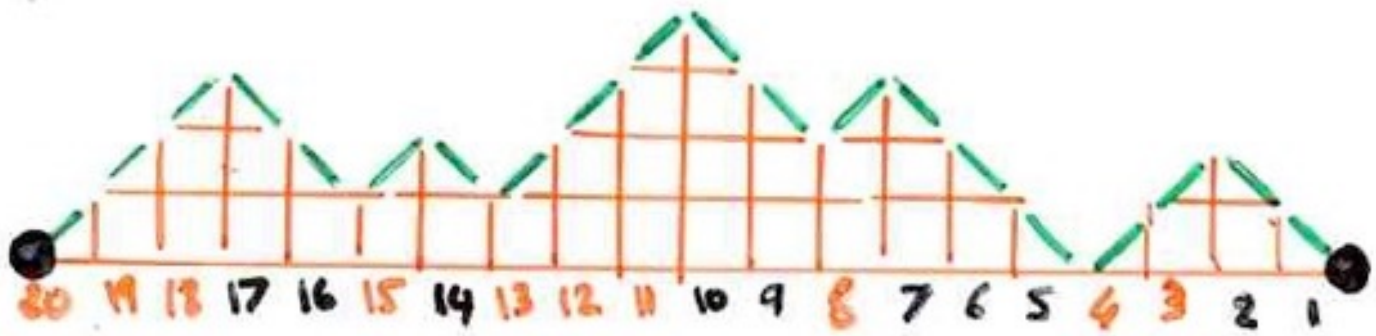
another proof
with staircase polygons

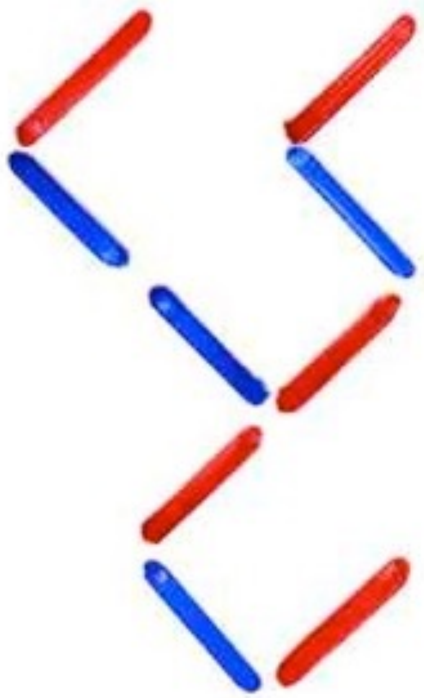




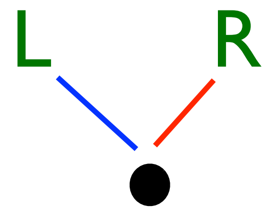
binary tree

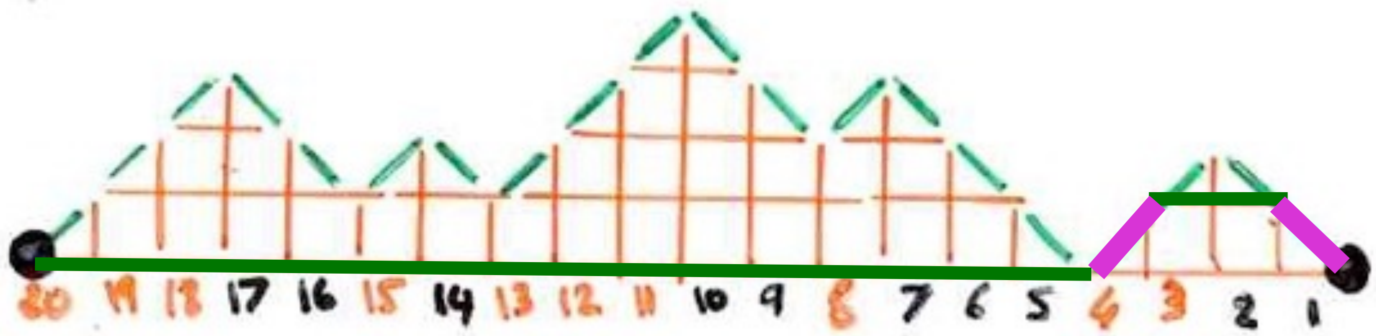
Dyck path

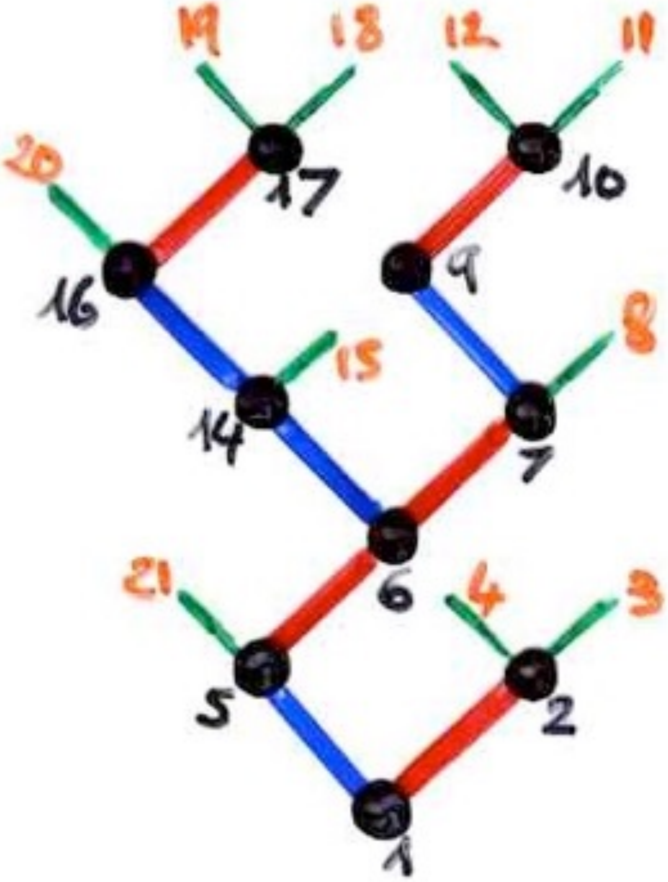




binary tree



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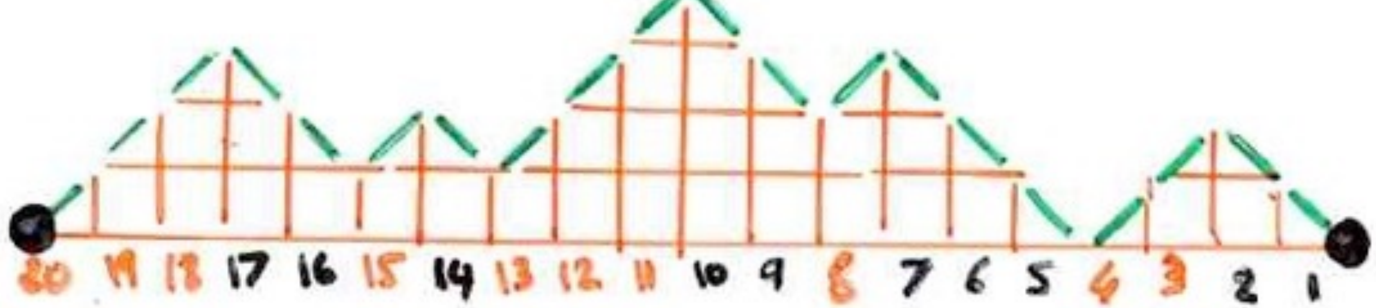
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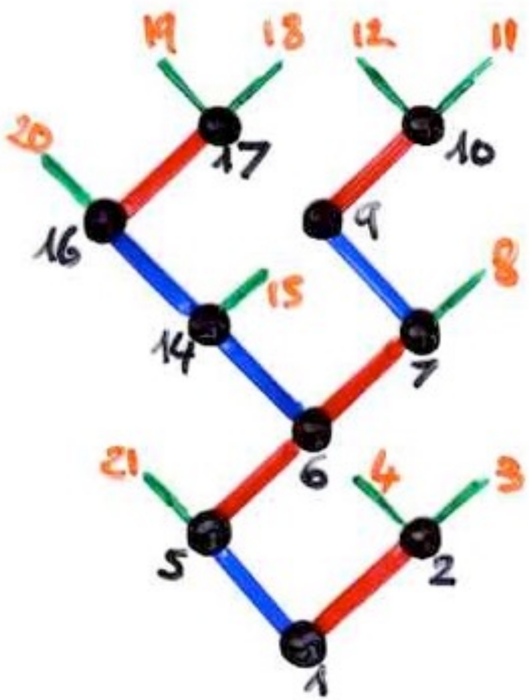

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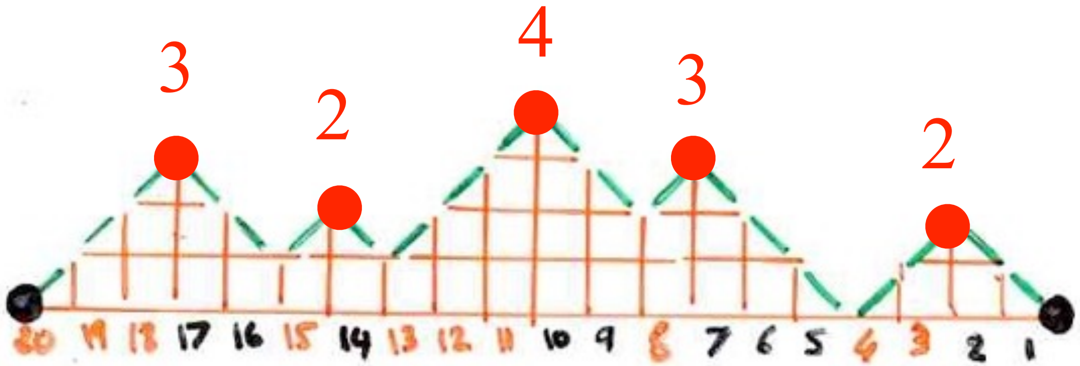

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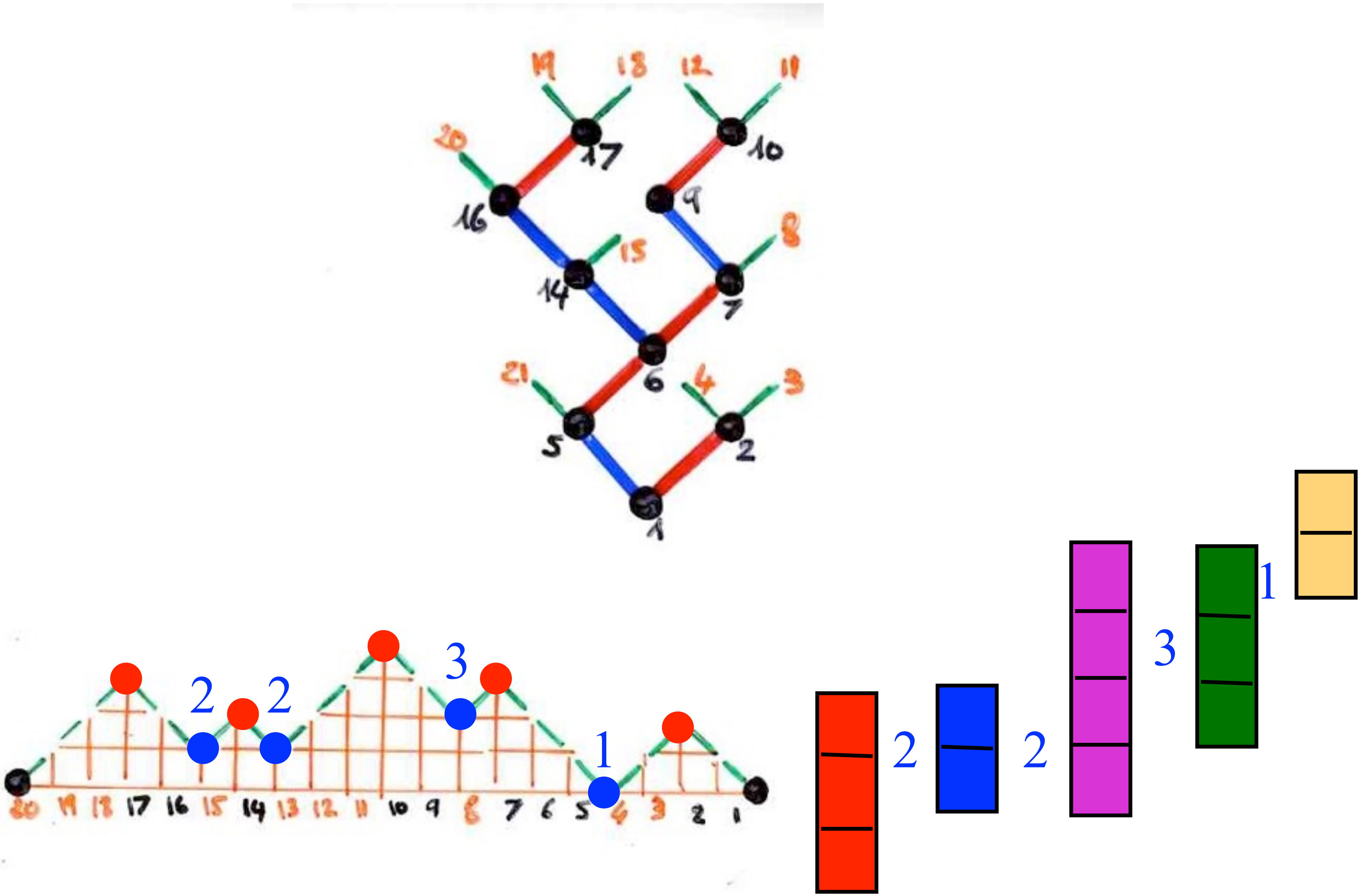
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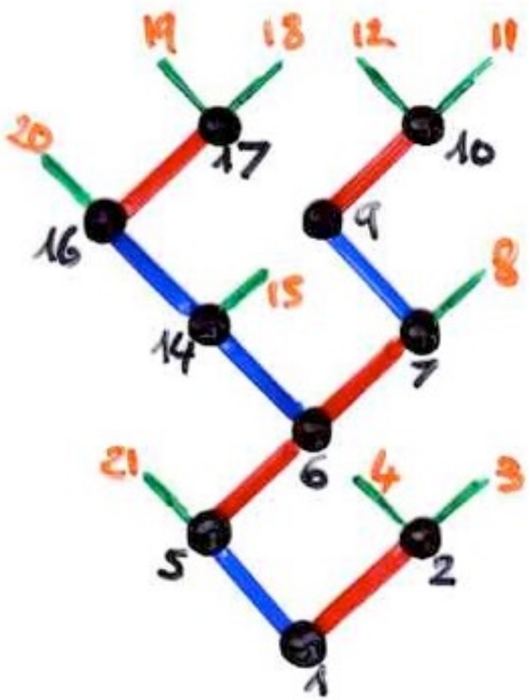
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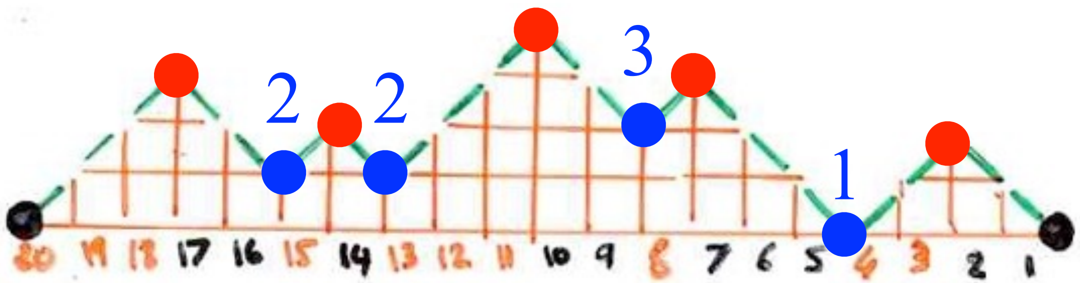
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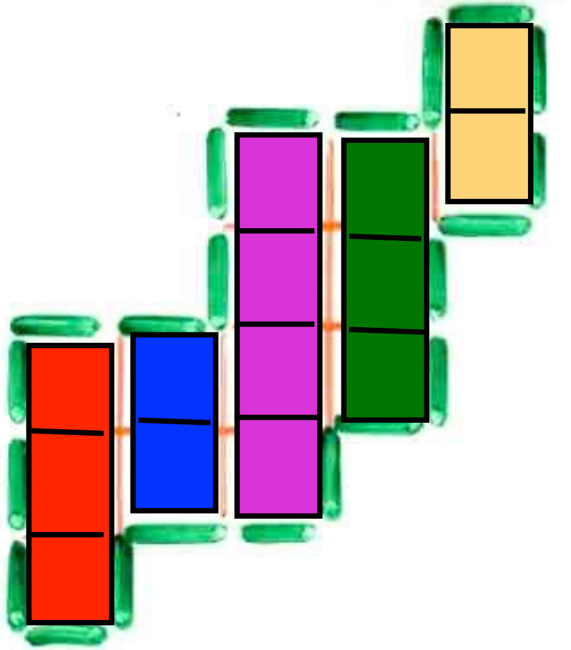

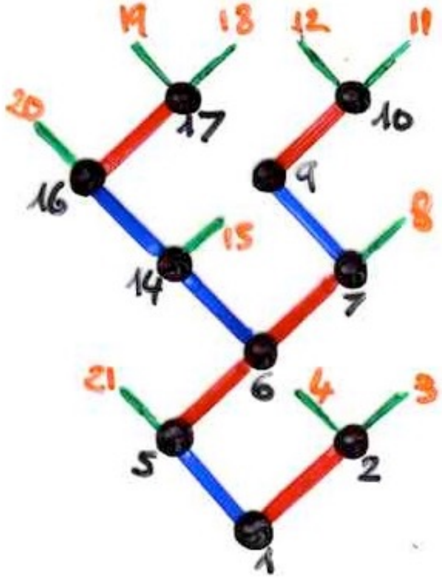
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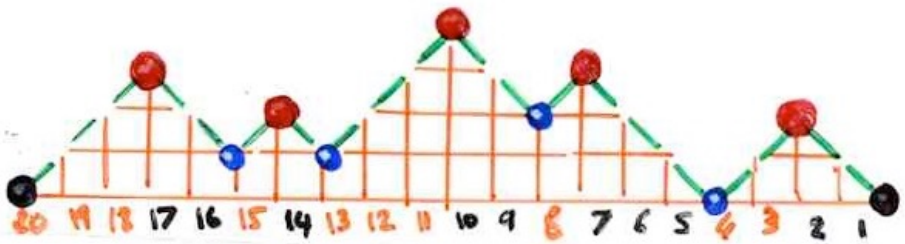
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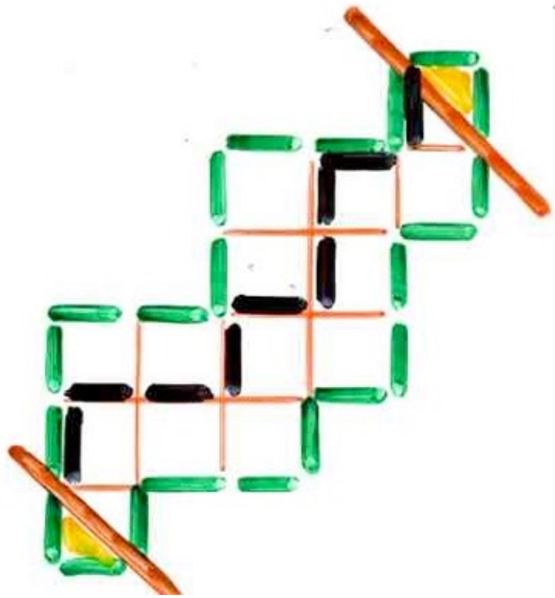
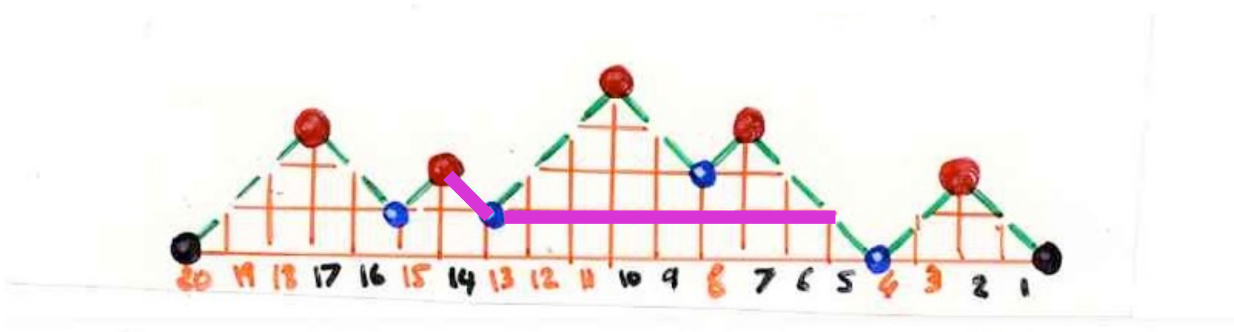
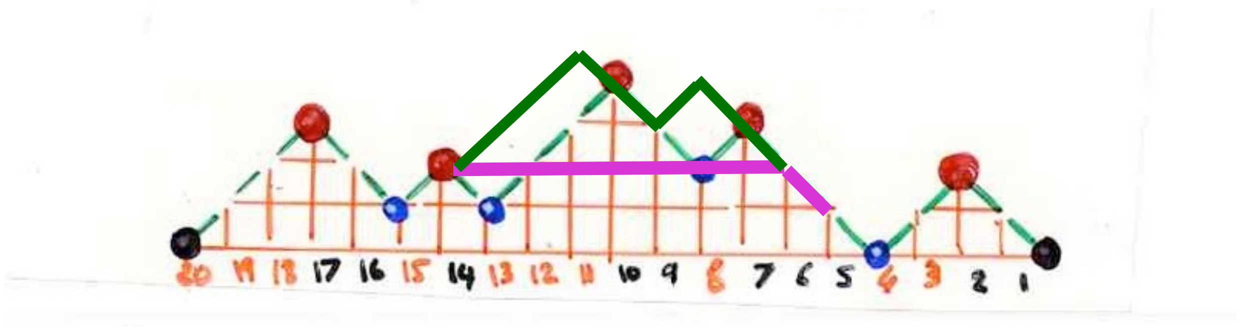


forbidden
move

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