Alternating sign matrices: at the crossroad of algebra, combinatorics and physics

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quantum mechanics:
spin chain model

Spin chains and combinatorics

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The XXZ quantum spin chain model with periodic boundary conditions is one of the most popular integrable models which has been investigating by the Bethe Ansatz method during the last 35 years [3]. It is described by the Hamiltonian

$$H_{XXZ} = -\sum_{j=1}^{N} \left\{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right\}, \qquad \vec{\sigma}_{N+1} = \vec{\sigma}_1. \tag{1}$$

The nonzero wave function components are

$$N=3: \quad \psi_{001}=1;$$
$$N=5: \quad \psi_{00011}=1, \quad \psi_{00101}=2;$$
$$N=7: \quad \psi_{0000111}=1, \quad \psi_{0001101}=\psi_{0001011}=3, \quad \psi_{0010011}=4 \quad \psi_{0010101}=7.$$

All components not included in the list can be obtained by shifting. Notice that the components of the ground state are positive in accordance with the Perron–Frobenius theorem.

Let us continue the list. For $N=9$ the components of the eigenvector with the energy $-27/2$ and $S_z=-1/2$ are

$$\psi_{000001111}=1, \quad \psi_{000010111}=4, \quad \psi_{000011011}=6, \quad \psi_{000100111}=7,$$
$$\psi_{000101011}=17, \quad \psi_{000101101}=14, \quad \psi_{000110011}=12, \quad \psi_{001001011}=21,$$
$$\psi_{001010011}=25, \quad \psi_{001010101}=42.$$

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We omit nonzero components which can be obtained by the reflection of the order of sites since this transformation is a symmetry of our state, as it is for the ground state. For example, we have

$$\psi_{000011101}=\psi_{000010111}=4.$$

1, 2, 7, 42, 429, ...

?

M1791 0, 1, 2, 7, 32, 181, 1214, 9403, 82508, 808393, 8743994, 103459471, 1328953592, 18414450877, 273749755382, 4345634192131, 73362643649444, 1312349454922513
$a(n)=n.a(n-1)+(n-2)a(n-2)$. Ref R1 188. [0,3; A0153, N0706]

$$\text{E.g.f.:}\ (1-x)^{-3}\, e^{-x}.$$

M1792 1, 1, 2, 7, 32, 181, 1232, 9787, 88832, 907081, 10291712, 128445967, 1748805632, 25794366781, 409725396992, 6973071372547, 126585529106432
Expansion of $1/(1-\sinh x)$. Ref ARS 10 138 80. [0,3; A6154]

M1793 0, 1, 1, 2, 7, 32, 184, 1268, 10186, 93356, 960646, 10959452, 137221954, 1870087808, 27548231008, 436081302248, 7380628161076, 132975267434552
Stochastic matrices of integers. Ref DUMJ 35 659 68. [0,4; A0987, N0707]

M1794 1, 2, 7, 33, 192
Permutations of length n with n in second orbit. Ref C1 258. [2,2; A6595]

M1795 1, 2, 7, 34, 209, 1546, 13327, 130922, 1441729, 17572114, 234662231, 3405357682, 53334454417, 896324308634, 16083557845279, 306827170866106
$a(n)=2n.a(n-1)-(n-1)^2a(n-2)$. Ref SE33 78. [0,2; A2720, N0708]

M1796 1, 2, 7, 34, 257, 2606, 32300, 440564, 6384634
Polyhedra with n nodes. Ref GR67 424. UPG B15. Dil92. [4,2; A0944, N0709]

M1797 2, 7, 35, 219, 1594, 12935, 113945, 1070324, 10586856, 109259633, 1168384157, 12877168147, 145656436074, 1685157199175, 19886174611045
Two-rowed truncated monotone triangles. Ref JCT A42 277 86. Zei93. [1,1; A6947]

M1798 1, 1, 2, 7, 35, 228, 1834, 17382, 195866, 2487832, 35499576, 562356672, 9794156448, 186025364016, 3826961710272, 84775065603888, 2011929826983504
Coefficients of iterated exponentials. Ref SMA 11 353 45. [0,3; A0154, N0710]

M1799 1, 2, 7, 35, 228, 1834, 17582, 195866, 2487832, 35499576, 562356672, 9794156448, 186025364016, 3826961710272, 84775065603888, 2011929826983504
Expansion of $\ln(1+\ln(1+x))$. [0,2; A3713]

M1800 1, 0, 1, 2, 7, 36, 300, 3218, 42335, 644808
Circular diagrams with n chords. Ref BarN94. [0,4; A7474]

M1801 1, 2, 7, 36, 317, 5624, 251610, 33642660, 14685630688
$n \times n$ binary matrices. Ref CPM 89 217 64. SLC 19 79 88. [0,2; A2724, N0711]

M1802 2, 7, 37, 216, 1780, 32652
Semigroups of order n with 2 idempotents. Ref MAL 2 2 67. SGF 14 71 77. [2,1; A2787, N0712]

M1803 1, 2, 7, 37, 266, 2431, 27007, 353522, 5329837, 90960751, 1733584106, 36496226977, 841146804577, 21065166341402, 569600638022431
$a(n)=(2n-1)a(n-1)+a(n-2)$. Ref RCI 77. [0,2; A1515, N0713]

M1804 1, 1, 2, 7, 38, 291, 2932, 36961, 561948, 10026505, 205608536, 4767440679, 123373203208, 3525630110107, 110284283006640, 3748357699560961
Forests of labeled trees with n nodes. Ref JCT 5 96 68. SIAD 3 574 90. [0,3; A1858, N0714]

M1805 1, 1, 2, 7, 40, 357, 4824, 96428, 2800472, 116473461
n-element partial orders contained in linear order. Ref nbh. [0,3; A6455]

M1806 1, 2, 7, 41, 346, 3797, 51157, 816356, 15050581, 314726117, 7359554632, 190283748371, 5389914888541, 165983936096162, 5521346346543307
Planted binary phylogenetic trees with n labels. Ref LNM 884 196 81. [1,2; A6677]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \,.\,.\, n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 6453983693872074973935 6
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

M1810 0, 1, 2, 7, 44, 361, 3654, 44207, 622552, 10005041, 180713290, 3624270839, 79914671748, 1921576392793, 50040900884366, 1403066801155039
Modified Bessel function $K_n(1)$. Ref AS1 429. [0,3; A0155, N0716]

M1811 0, 1, 2, 7, 44, 447, 6749, 142176, 3987677, 143698548, 6470422337, 356016927083, 23503587609815, 1833635850492653, 166884365982441238
$a(n)=n(n-1)a(n-1)/2+a(n-2)$. [0,3; A1046, N0717]

M1812 1, 2, 7, 44, 529, 12278, 565723, 51409856, 9371059621, 3387887032202, 2463333456292207, 3557380311703796564, 10339081666350180289849
Sum of Gaussian binomial coefficients $[n,k]$ for $q=4$. Ref TU69 76. GJ83 99. ARS A17 328 84. [0,2; A6118]

M1813 2, 7, 52, 2133, 2590407, 3374951541062, 5695183504479116640376509, 16217557574922386301420514191523784895639577710480
Free binary trees of height n. Ref JCIS 17 180 92. [1,1; A5588]

M1814 1, 1, 2, 7, 56, 2212, 2595782, 3374959180831, 5695183504489239067484387, 16217557574922386301420531277071365103168734284282
Planted 3-trees of height n. Ref RSE 59(2) 159 39. CMB 11 87 68. JCIS 17 180 92. [0,3; A2658, N0718]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \, . \, . \, n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 64539836938720749739356
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \ldots n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 6453983693872074973956
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

ASM

Alternating sign matrices

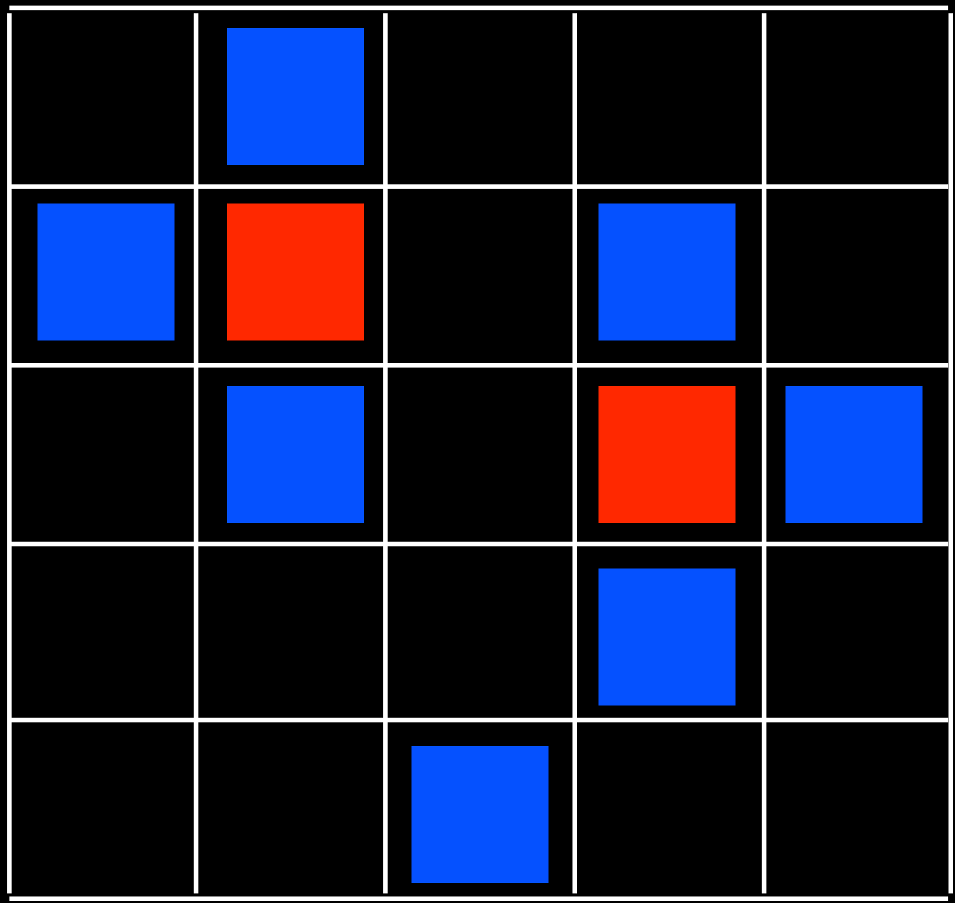
"Matrices à signes alternés"

(alternating sign matrices)

- entrées : 0, 1, −1
- somme des entrées, ligne, colonne = 1
- entrées ≠ 0 alternent en signe ligne, colonne

ex :

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



Permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad + \; 6$$

permutations

$$1, \; 2, \; 7, \; 42, \; 429, \; \ldots$$

“ What else have you got in your pocket ?” he
went on, turning to A
“ Only a thimble,”
“ Hand it over her
Then they all crow
while the Dodo solen

Lewis Carroll

“Alice aux pays des merveilles”

C. I. Dodgson (1866)

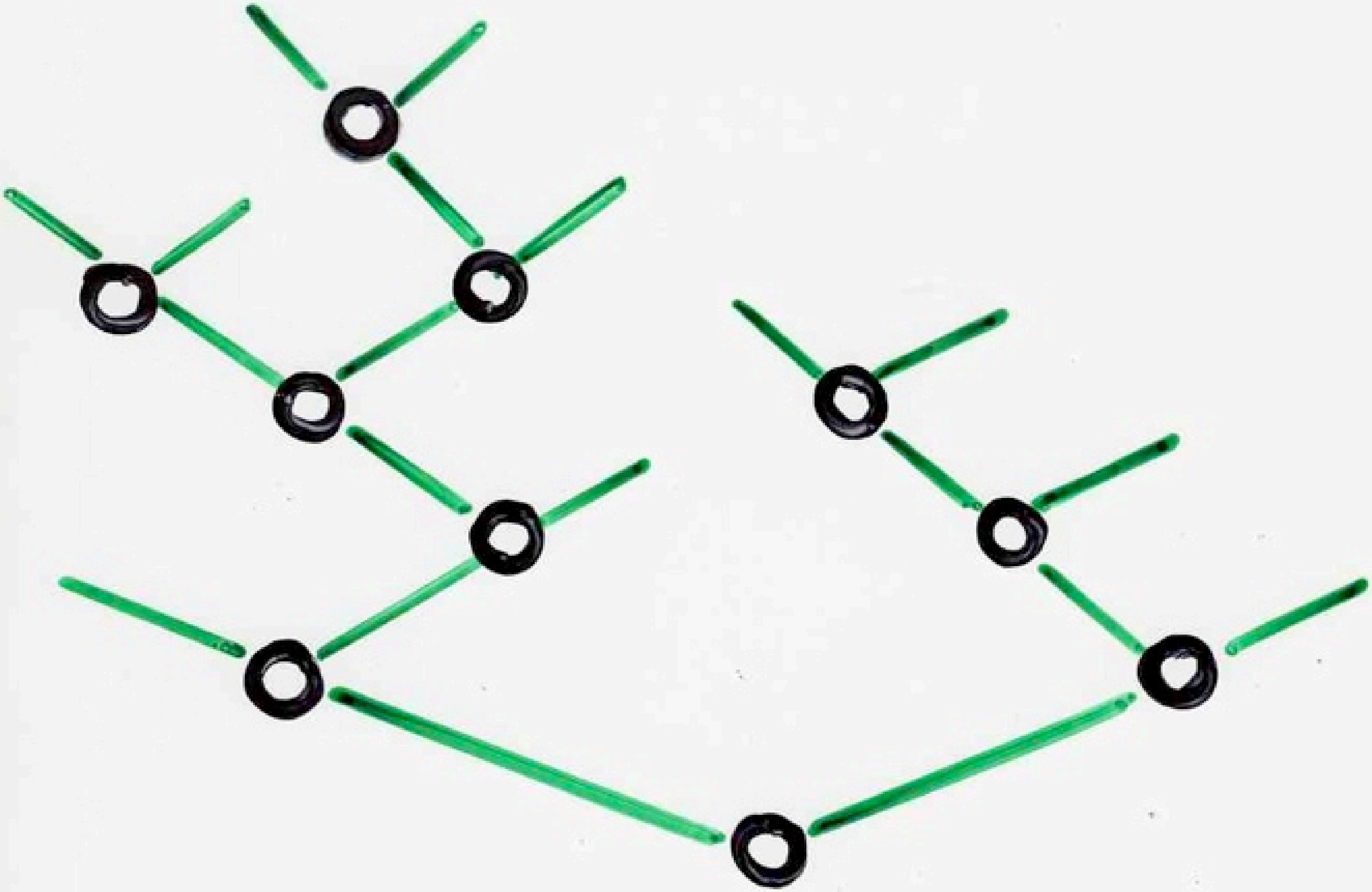
Condensation of determinants

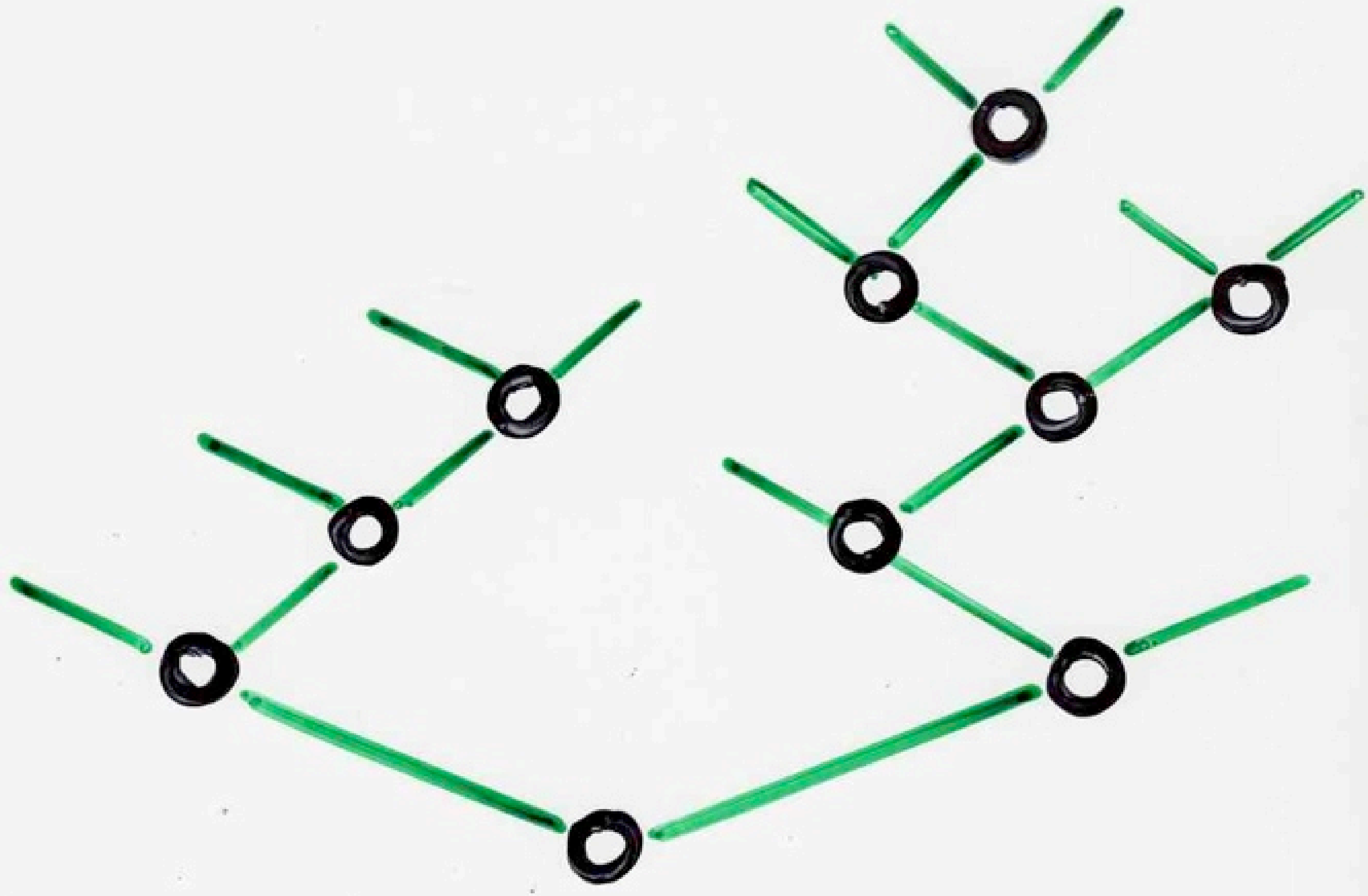
$$det(M) = \frac{M_{NO} M_{SE} - M_{NE} M_{SO}}{M_C}$$

enumerative combinatorics

Combinatoire énumérative

$$a_n = \ ?$$





C_n = nombre d'arbres binaires ayant n sommets internes

(et donc $n+1$ feuilles)

<u>nombre de Catalan</u>

generating function

formal power series

$$y = 1 + 2t + 5t^2 + 14t^3 + 42t^4 + \dots + C_n t^n + \dots$$

Wilf

corde à linge

série génératrice

$$f(t) = a_0 + a_1 t + a_2 t^2 + \ldots\ldots$$

$$\cdots + a_n t^n + \ldots\ldots$$

$$f(t) = \sum_{n \geqslant 0} a_n t^n$$

binary trees

recurrence

$$C_{n+1} = \sum_{i+j=n} C_i C_j$$

$$C_0 = 1$$

$$y = 1 + t y^2$$

equation algébrique

$$y = \frac{1 - (1 - 4t)^{1/2}}{2t}$$

$$(1+u)^m =$$

$$1 + \frac{m}{1!}u + \frac{m(m-1)}{2!}u^2 + \frac{m(m-1)(m-2)}{3!}u^3 + \cdots$$

$$m = \frac{1}{2}$$

$$u = -4t$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

(1838)

Note sur une Équation aux différences finies ;

Par E. CATALAN.

M. *Lamé* a démontré que l'équation

$$P_{n+1} = P_n + P_{n-1}P_3 + P_{n-2}P_4 + \ldots + P_4P_{n-2} + P_3P_{n-1} + P_n, \quad (1)$$

se ramène à l'équation linéaire très simple,

$$P_{n+1} = \frac{4n-6}{n} P_n. \quad (2)$$

Admettant donc la concordance de ces deux formules, je vais chercher à en déduire quelques conséquences.

I.

L'intégrale de l'équation (2) est

$$P_{n+1} = \frac{6}{3} \cdot \frac{10}{4} \cdot \frac{14}{5} \ldots \frac{4n-6}{n}$$

et comme, dans la question de géométrie qui … équations, on a $P_3 = 1$, nous prendrons simplem…

$$P_{n+1} = \frac{2.6.10.14\ldots(4n-6)}{2.3.\,4.\,5\ldots\quad n}$$

Le numérateur

$$2.6.10.14\ldots(4n-6) = 2^{n-1}.1.3.5.7.\ldots$$
$$= \frac{2^{n-1}.1.2.3.4.5\ldots(2n-2)}{2.4.6.8\ldots(2n-2)} = \ldots$$

Donc

$$P_{n+1} = \frac{n(n+1)(n+2)\ldots(2n-\ldots}{2.3.4\ldots n}$$

Si l'on désigne généralement par $C_{m,p}$ le nombre des combinaisons de m lettres, prises p à p ; et si l'on change n en $n+1$, on aura

$$P_{n+2} = \frac{1}{n+1} C_{2n,n}, \quad (5)$$

ou bien

$$P_{n+2} = C_{2n,n} - C_{2n,n-1}. \quad (6)$$

II.

Les équations (1) et (5) donnent ce théorème sur les combinaisons :

$$\left.\begin{aligned} \frac{1}{n+1} C_{2n,n} &= \frac{1}{n} C_{2n-2,n-1} + \frac{1}{n-1} C_{2n-4,n-2} \times \frac{1}{2} C_{2,1} \\ &+ \frac{1}{n-2} C_{2n-6,n-3} \times \frac{1}{3} C_{4,2} + \ldots + \frac{1}{n} C_{2n-2,n-1}. \end{aligned}\right\} \quad (7)$$

III.

On sait que le $(n+1)^e$ nombre figuré de l'ordre $n+1$, a pour expression, $C_{2n,n}$: si donc, dans la table des nombres figurés, on prend ceux qui occupent la diagonale ; savoir :

1, 2, 6, 20, 70, 252, 924… ;

…ise respectivement par

1, 2, 3, 4, 5, 6, 7… ;

…a une nouvelle suite de nombres,

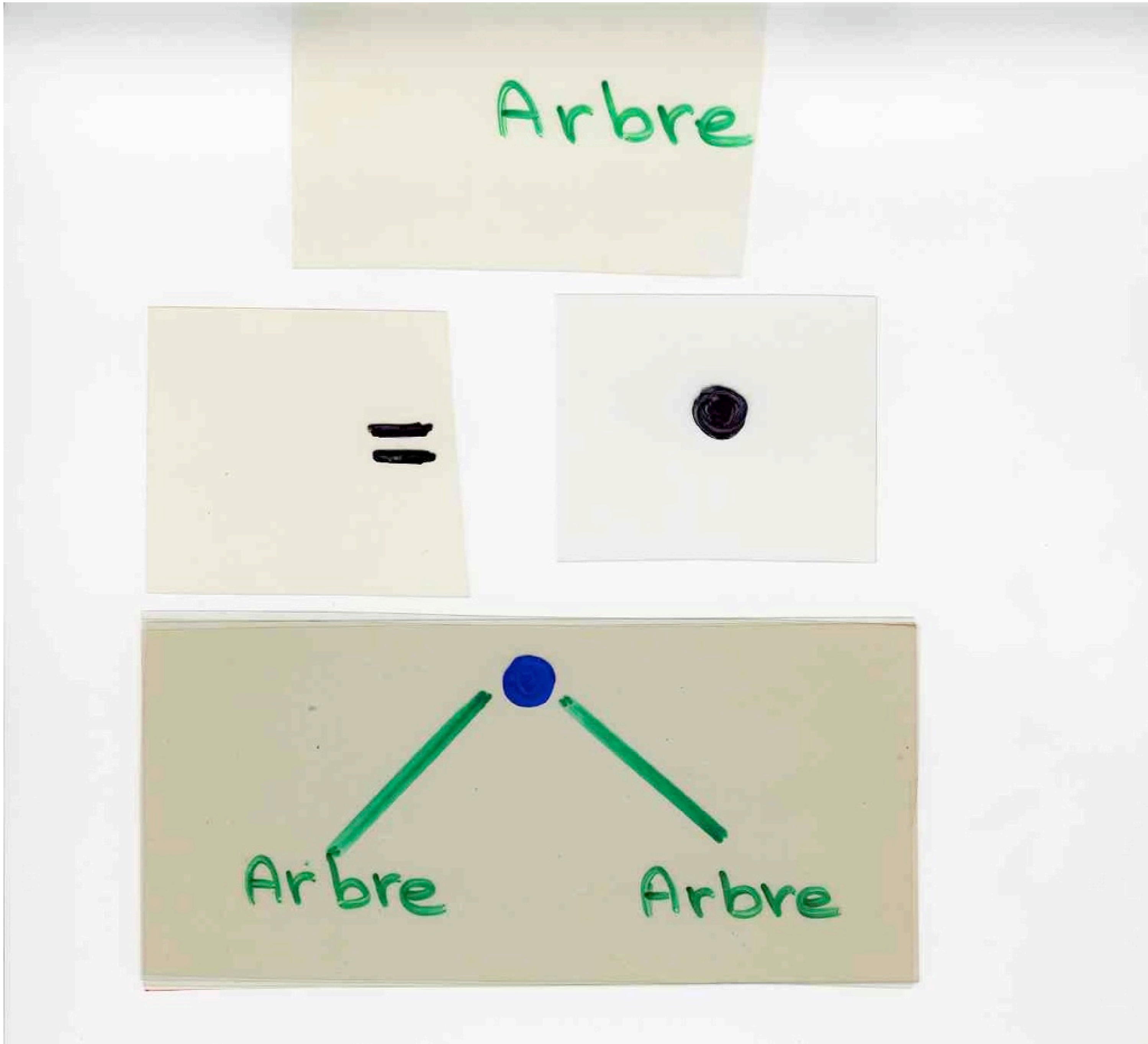
1, 1, 2, 5, 14, 42, 132…, (A)

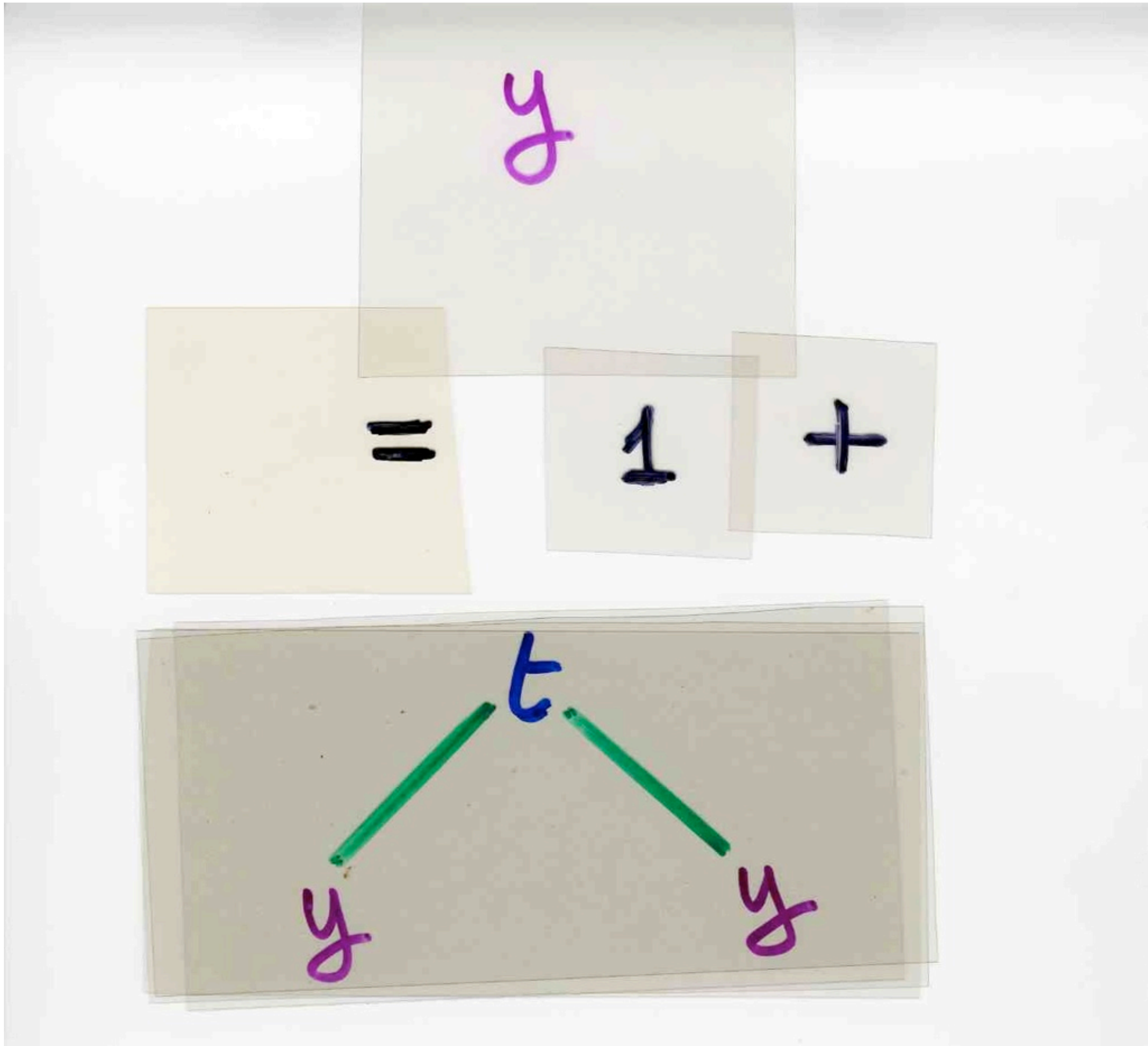
…ront de cette propriété :

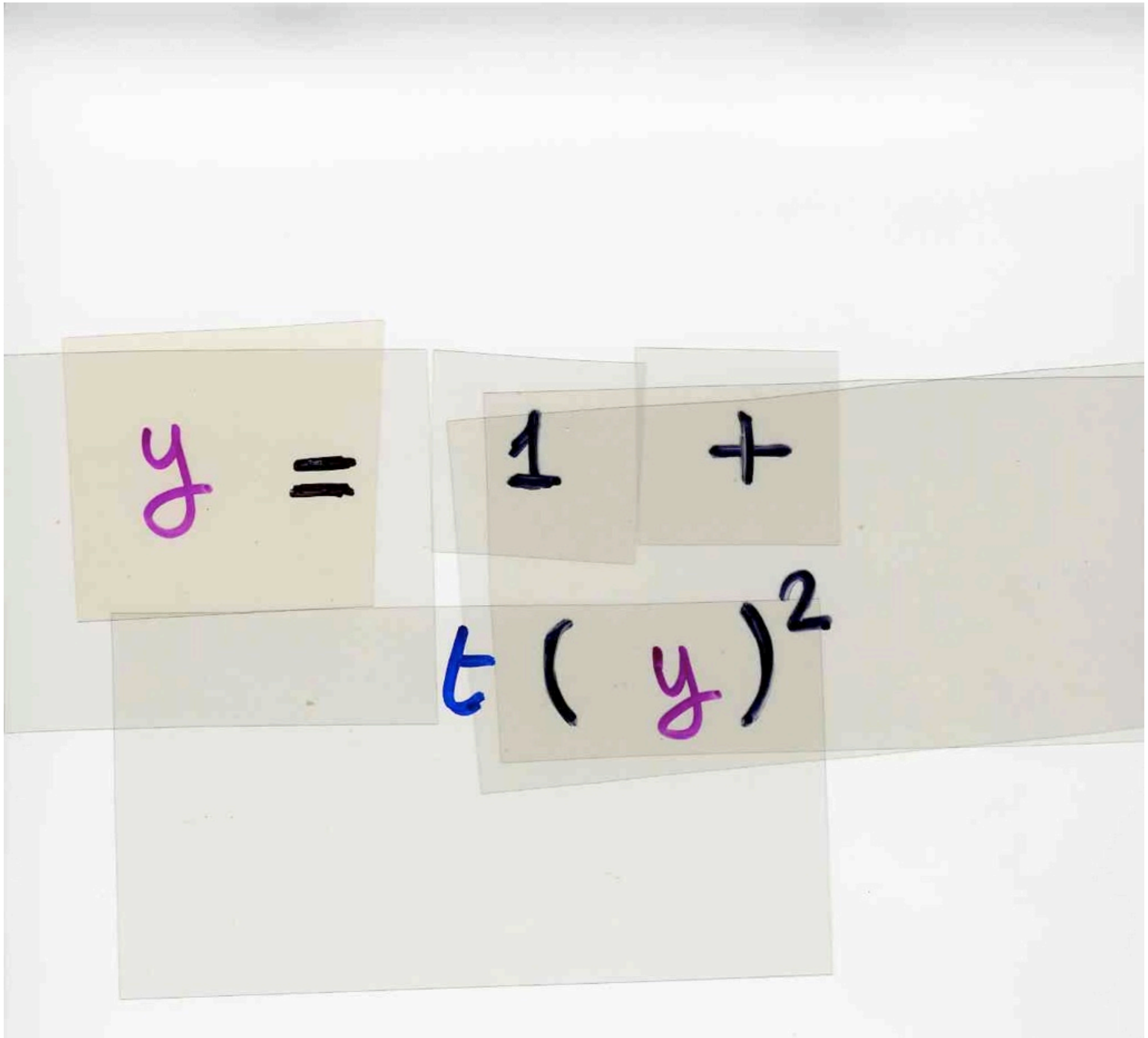
… quelconque de la suite (A) est égal à la somme des … l'on obtient en écrivant au-dessous d'elle-même, et …re inverse, la série des termes précédents, et en multi… …rmes correspondants des deux séries.

…ple,

$$\ldots = 1.42 + 1.14 + 2.5 + 5.2 + 14.1 + 42.1.$$


Arbre
=
Arbre
Arbre

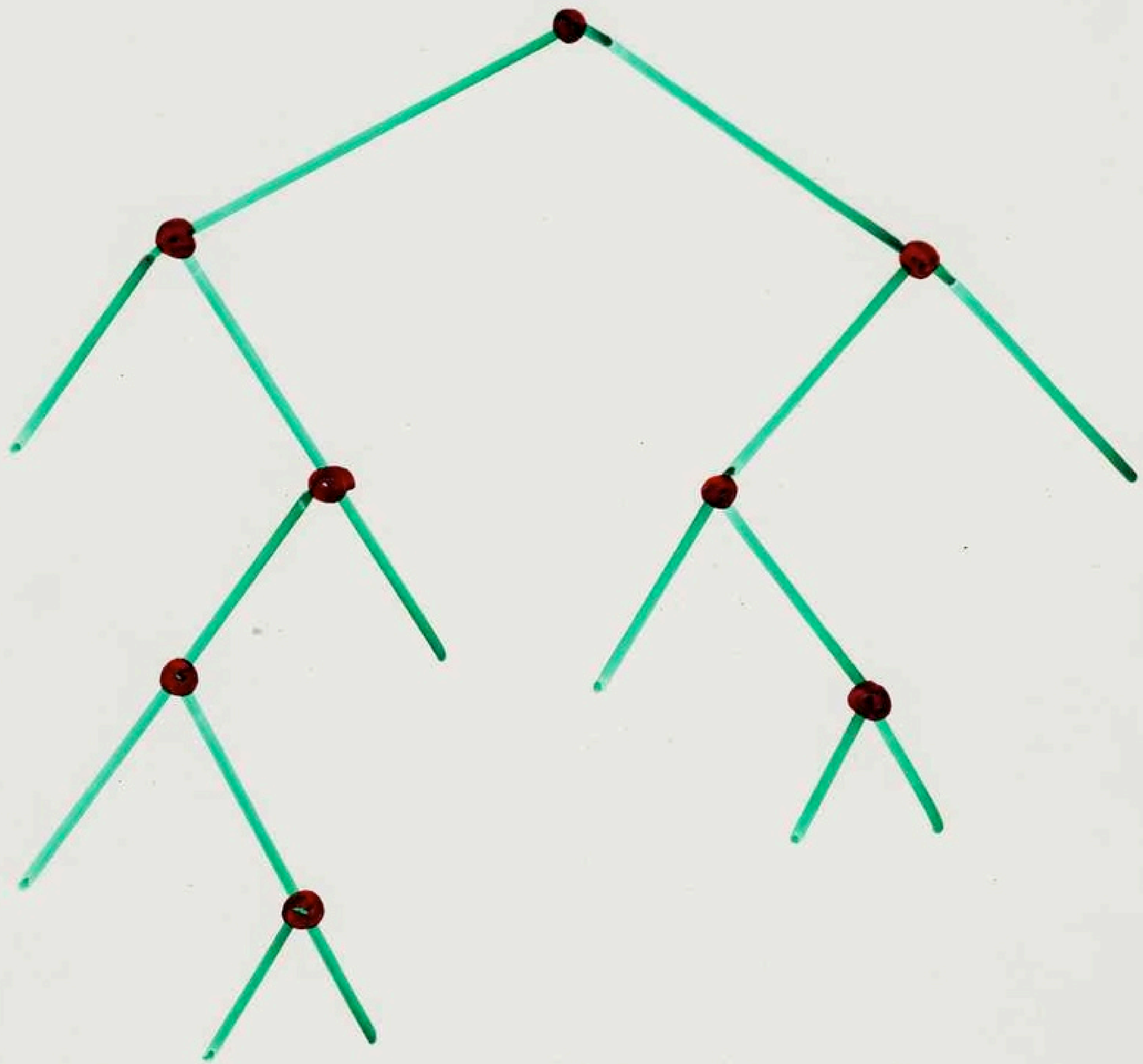

y
=
1
+
t
y
y

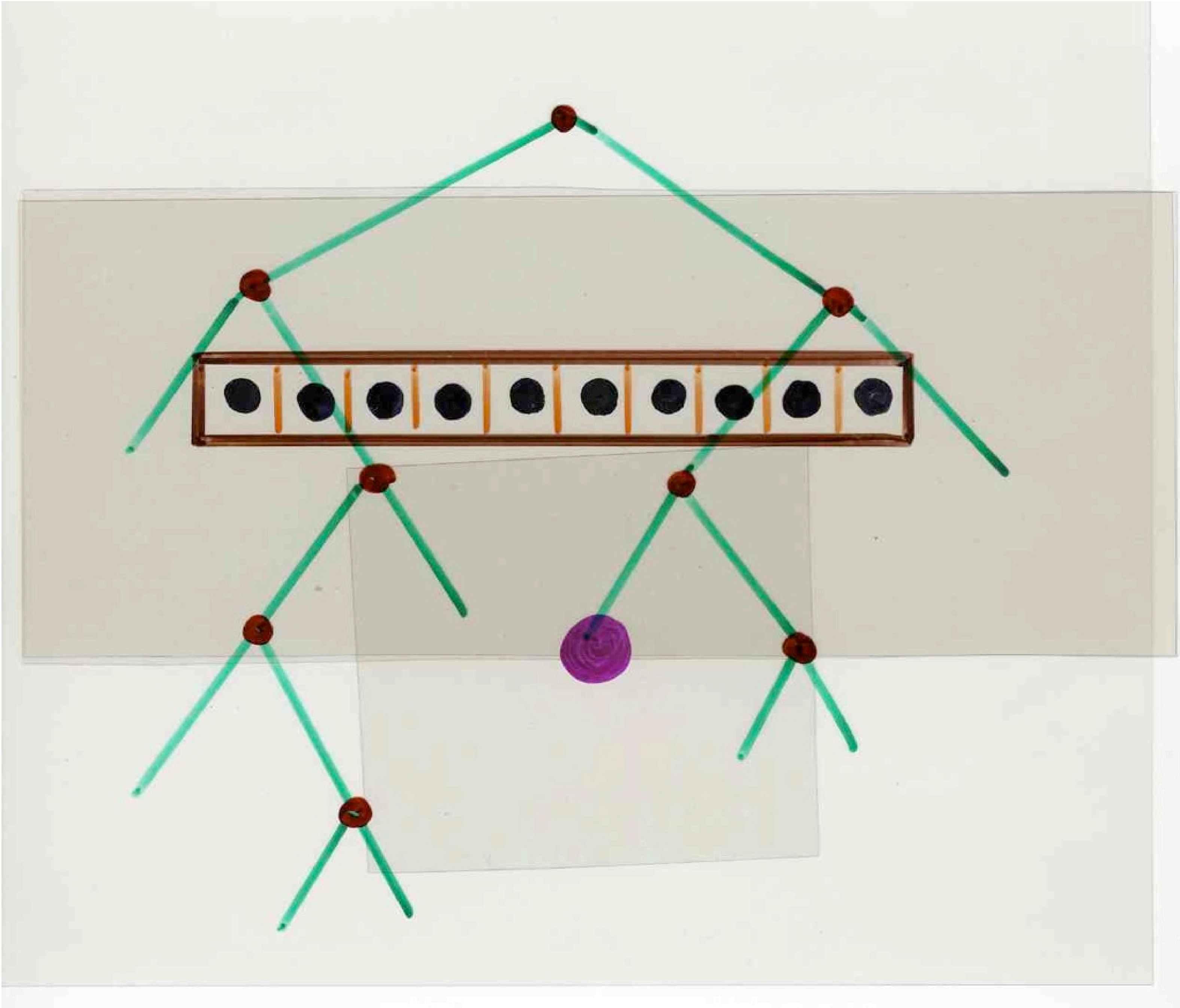

$y = 1 + t(y)^2$

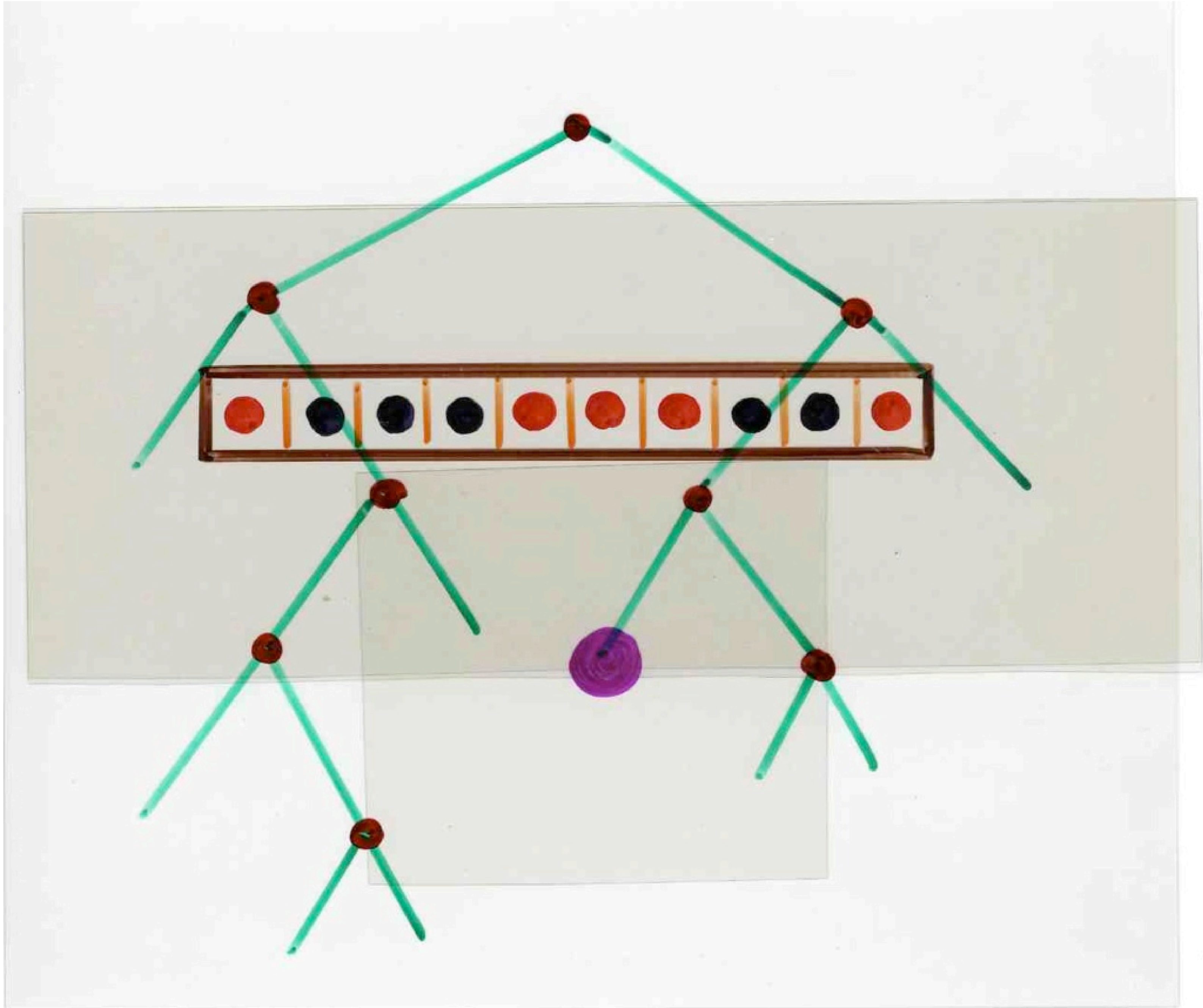
bijective proof

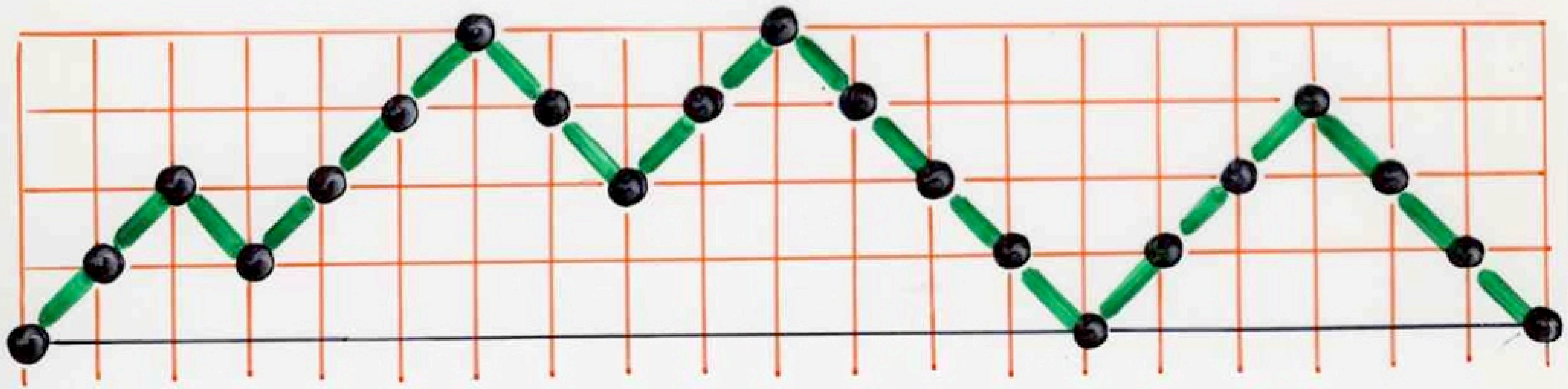
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$(n+1)\,C_n = \binom{2n}{n}$$









Dyck path

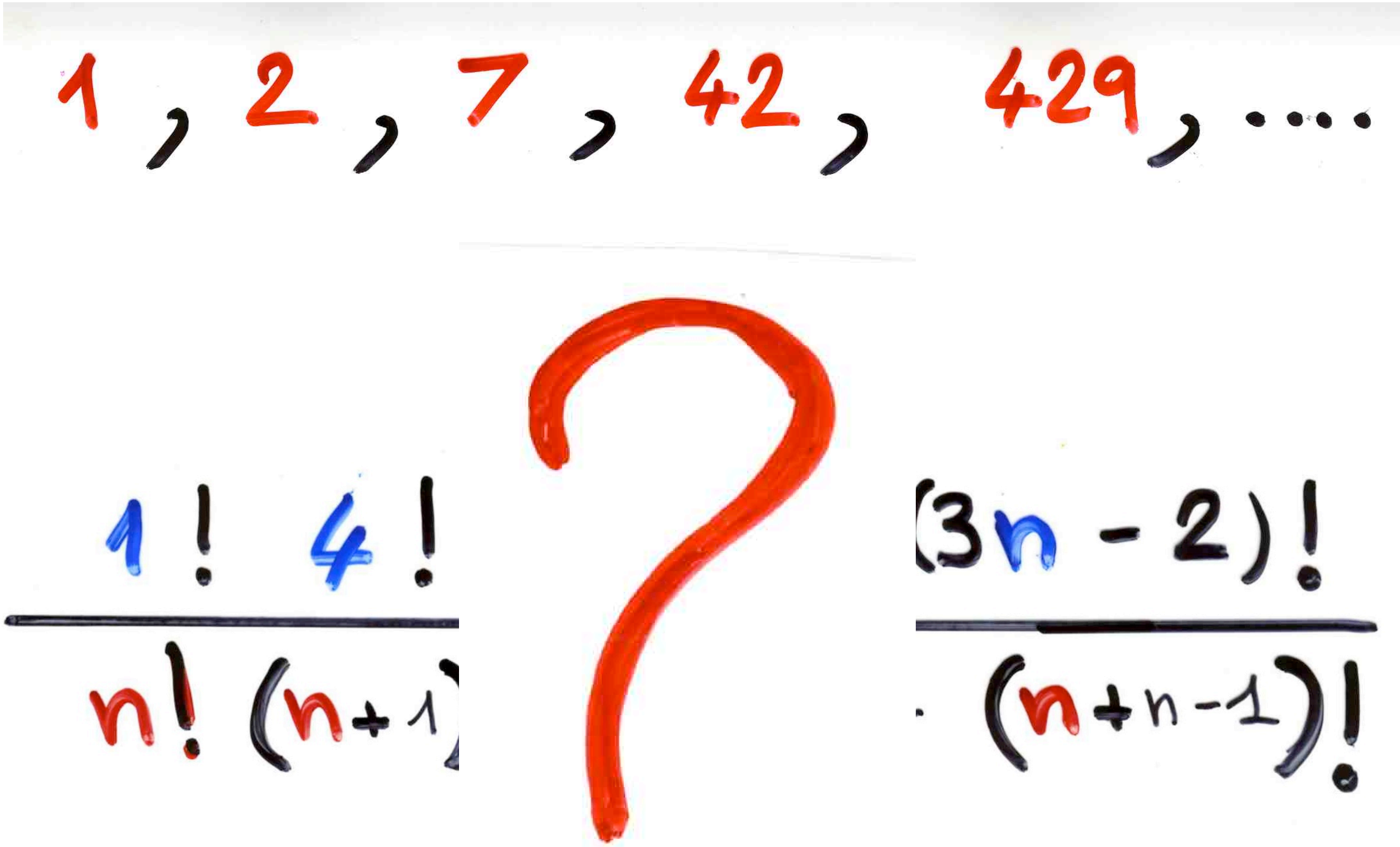
enumeration of ASM

$$1\,,\ 2\,,\ 7\,,\ 42\,,\ 429\,,\ \ldots.$$

$$\frac{1!\ 4!\ 7!\quad (3n-2)!}{n!\,(n+1)!\,(n+2)!\cdots(n+n-1)!}$$

alternating sign matrices conjecture

Mills, Robbins, Rumsey (1982)



alternating sign matrices conjecture

Mills, Robbins, Rumsey (1982)

deviner une formule

$$a_n = \frac{(\text{produit})}{(\text{produit})}$$

RATE

C. Krattenthaler
M. Rubey

$$b_n = \frac{a_{n+1}}{a_n}$$

$rat(n)$

$$\frac{C_{n+1}}{C_n} = \frac{2(2n+1)}{(n+2)}$$

Catalan

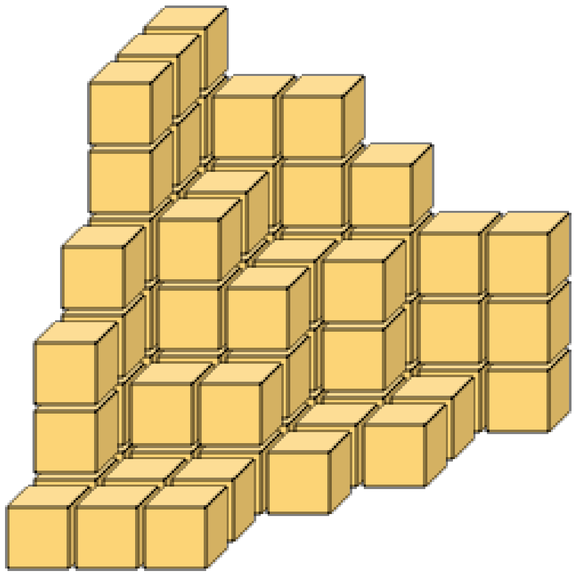
$$c_n = \frac{b_{n+1}}{b_n}$$

Robbins

The Mathematical Intelligencer (1991)

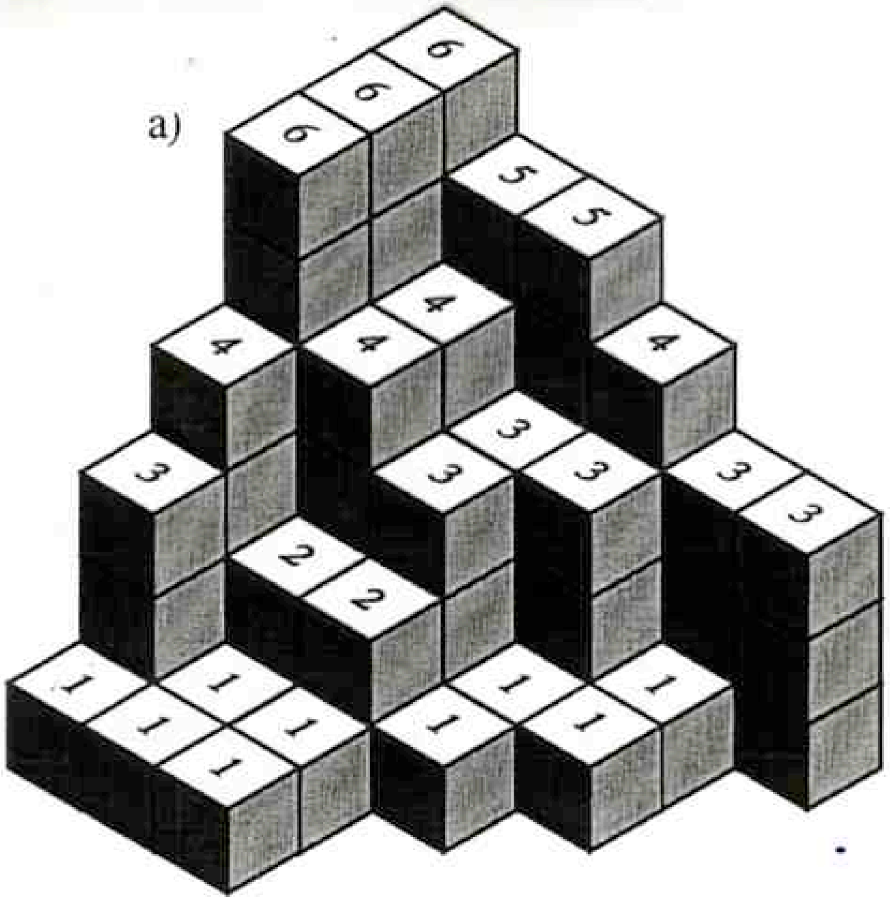
“These conjectures are of such compelling simplicity that it is hard to understand how any mathematician can bear the pain of living without understanding why they are true”

plane partitions



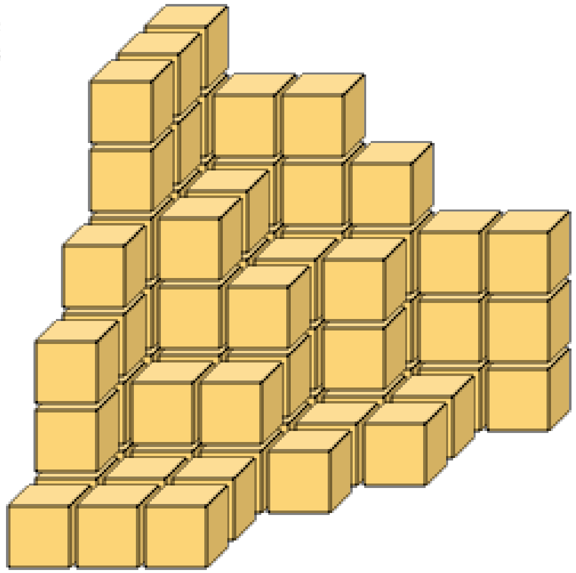
(figure from David Bressoud)

a)



b)

6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

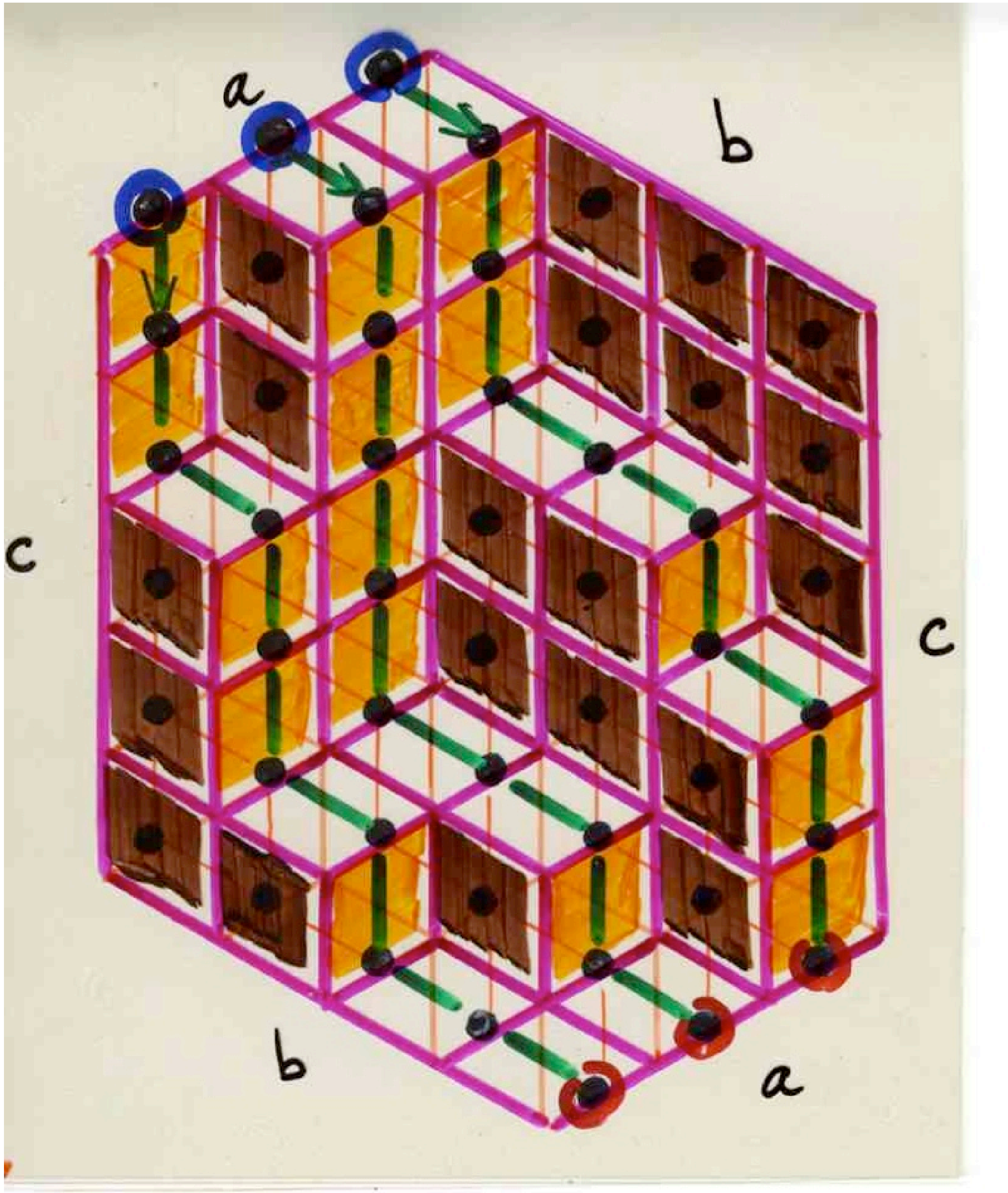


(figure from
David Bressoud)

6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

$$\prod_{\substack{1 \leq i \leq a \\ 1 \leq j \leq b \\ 1 \leq k \leq c}} \frac{i+j+k-1}{i+j+k-2}$$

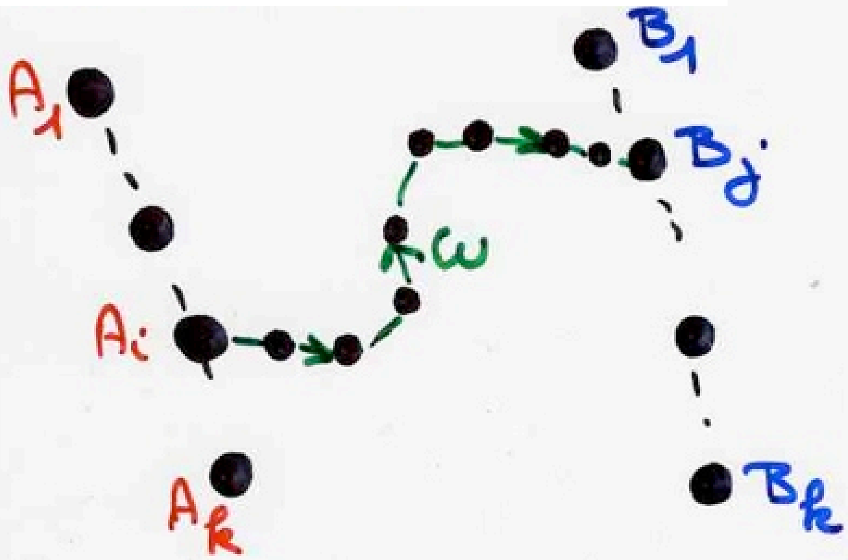
Plane partitions
and
paths



a
b
c
c
b
a

The “LGV Lemma”

LGV methodology



$A_1, \dots, A_k$

$B_1, \dots, B_k$

path

$\omega = (s_0, \dots, s_n) \quad s_i \in \Pi$

valuation

$v : \Pi \times \Pi \to \mathbb{K}$ ring

$v(\omega) = v(s_0, s_1) \cdots v(s_{n-1}, s_n)$

$$a_{ij} = \sum_{A_i \rightsquigarrow B_j} v(\omega)$$

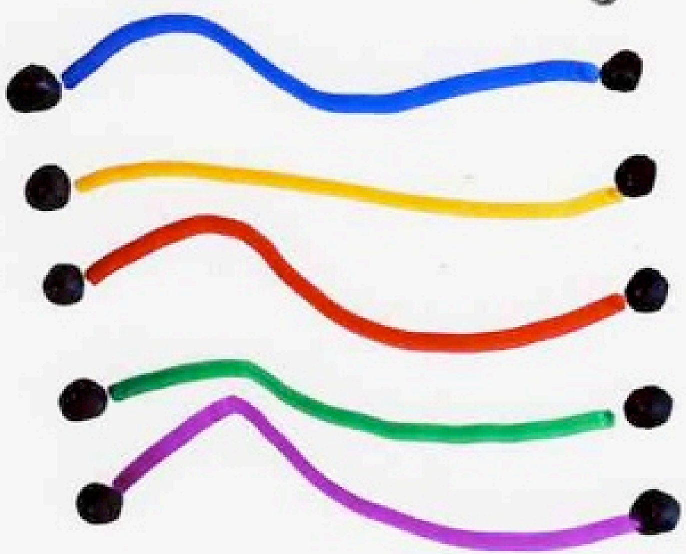
suppose finite sum

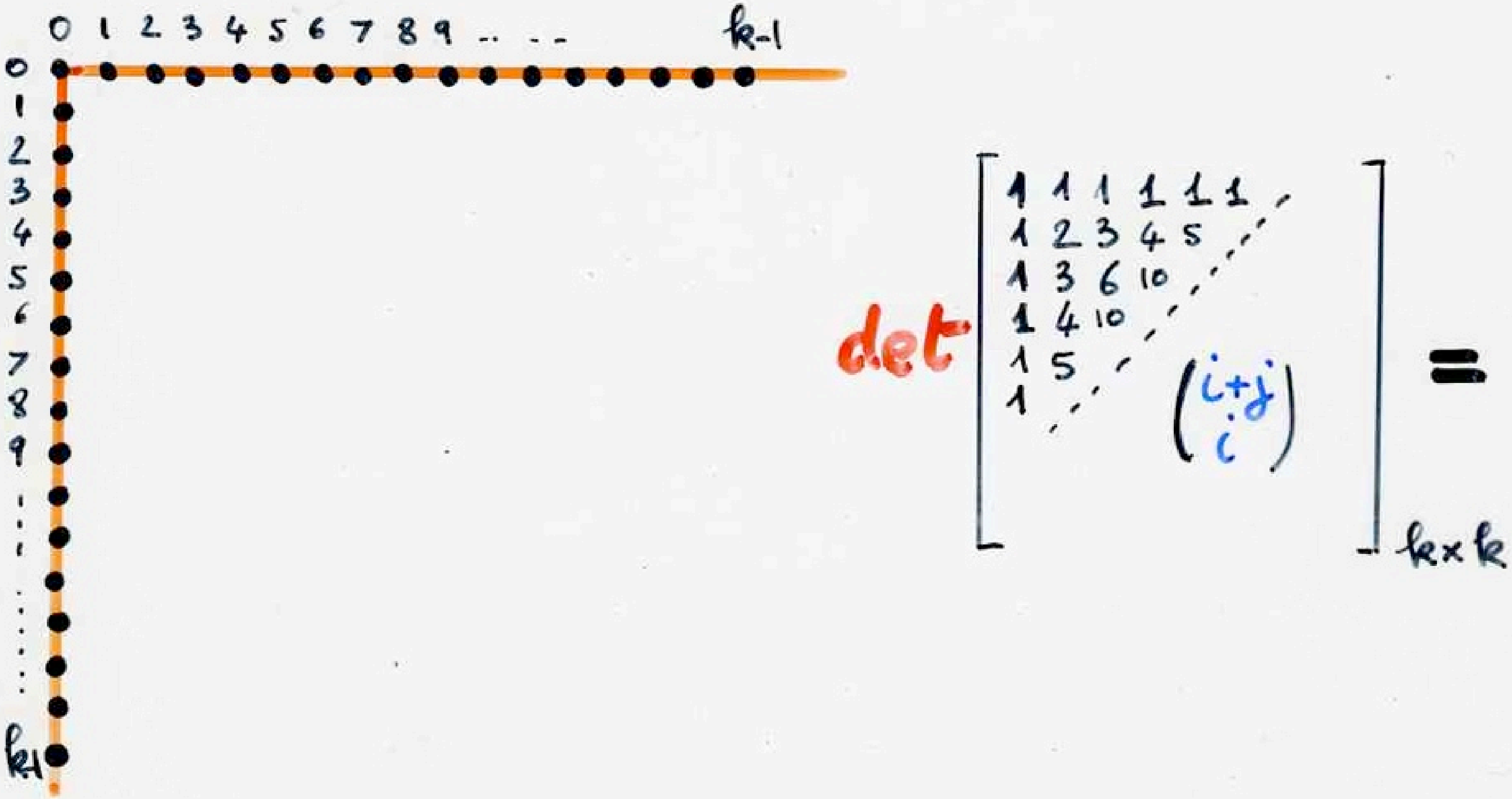
Prop. (C)

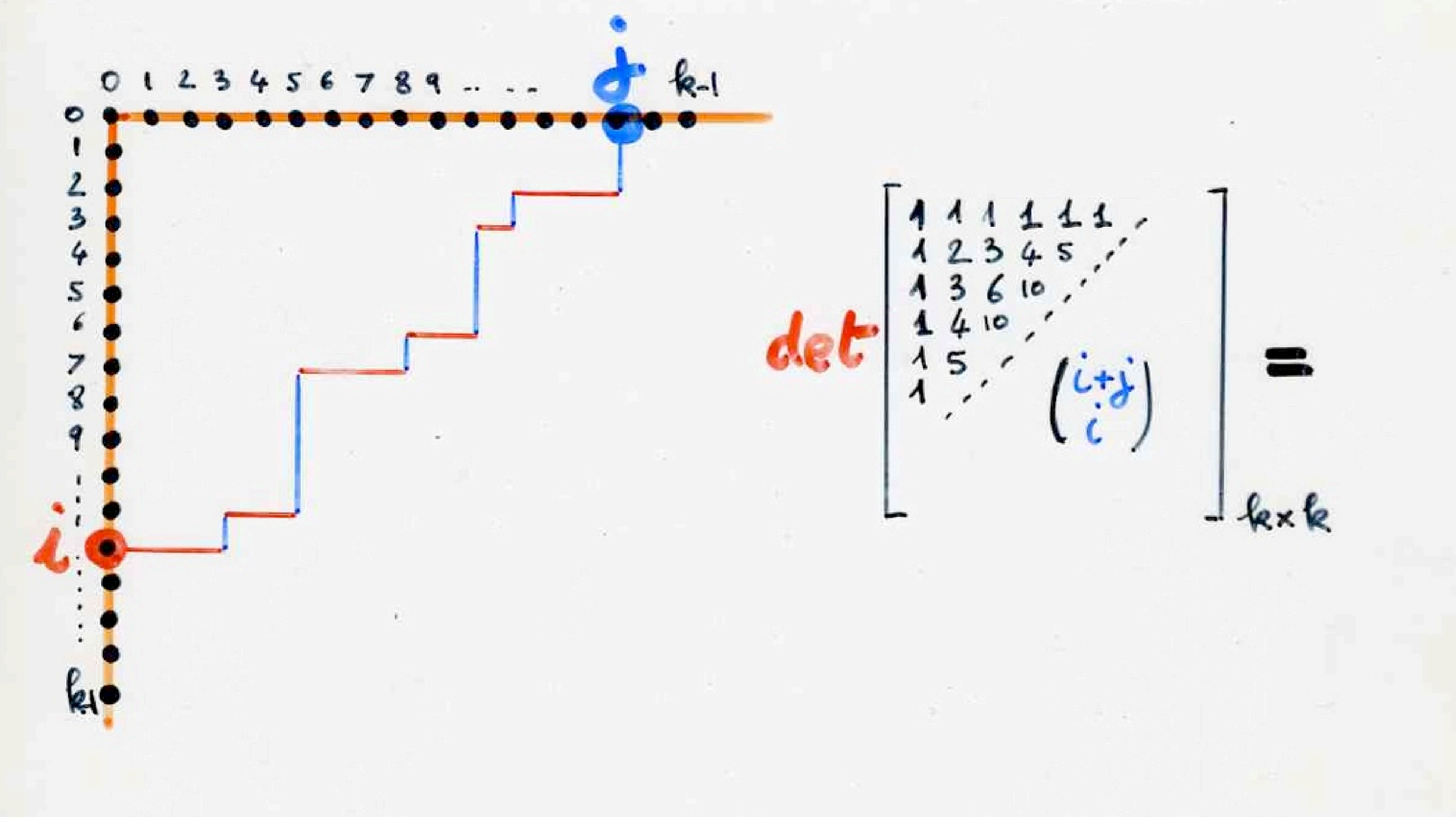
$$\det(a_{ij}) = \sum_{\Omega=(\omega_1,\dots,\omega_k)} v(\omega_1)\dots v(\omega_k)$$

$\omega_i : A_i \rightsquigarrow B_i$

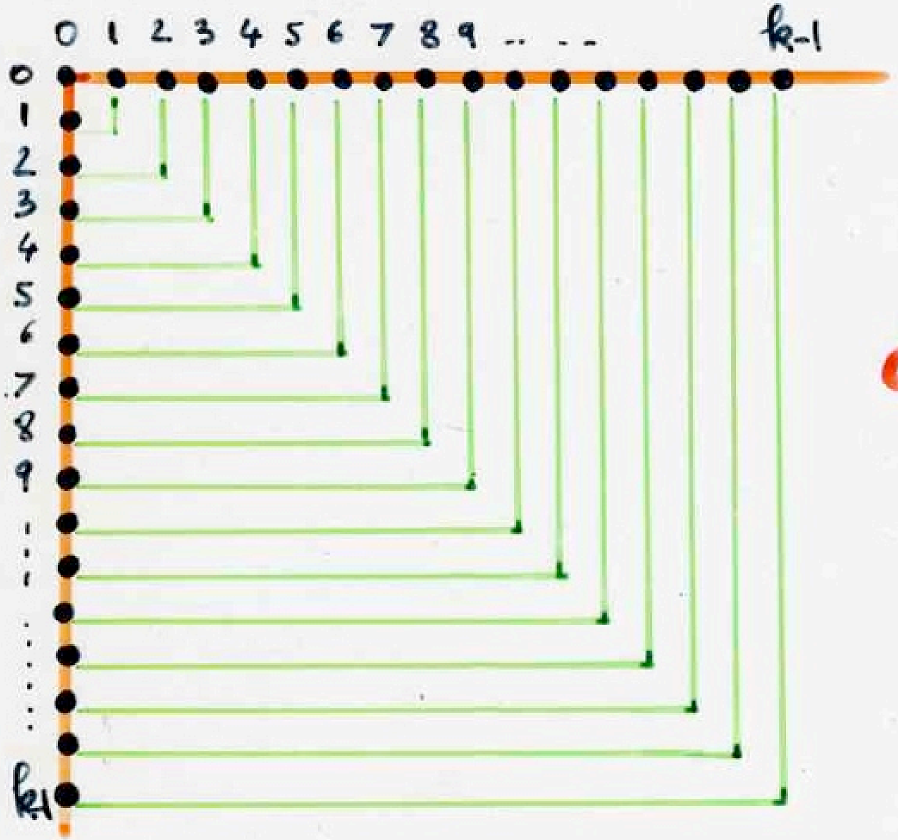
2 by 2 disjoints




0 1 2 3 4 5 6 7 8 9 .. .. k-1
0 1 2 3 4 5 6 7 8 9
k-1
det
1 1 1 1 1 1
1 2 3 4 5
1 3 6 10
1 4 10
1 5
1
(i+j i)
k×k
=



0 1 2 3 4 5 6 7 8 9 j k-1
0 1 2 3 4 5 6 7 8 9 i k-1
det
1 1 1 1 1 1
1 2 3 4 5
1 3 6 10
1 4 10
1 5
1
(i+j over i)
=
k×k



$$\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & & \\ 1 & 3 & 6 & 10 & & & \\ 1 & 4 & 10 & & & \binom{i+j}{i} & \\ 1 & 5 & & & & & \\ 1 & & & & & & \\ \vdots & & & & & & \end{bmatrix}_{k\times k} = 1$$

binomial determinant

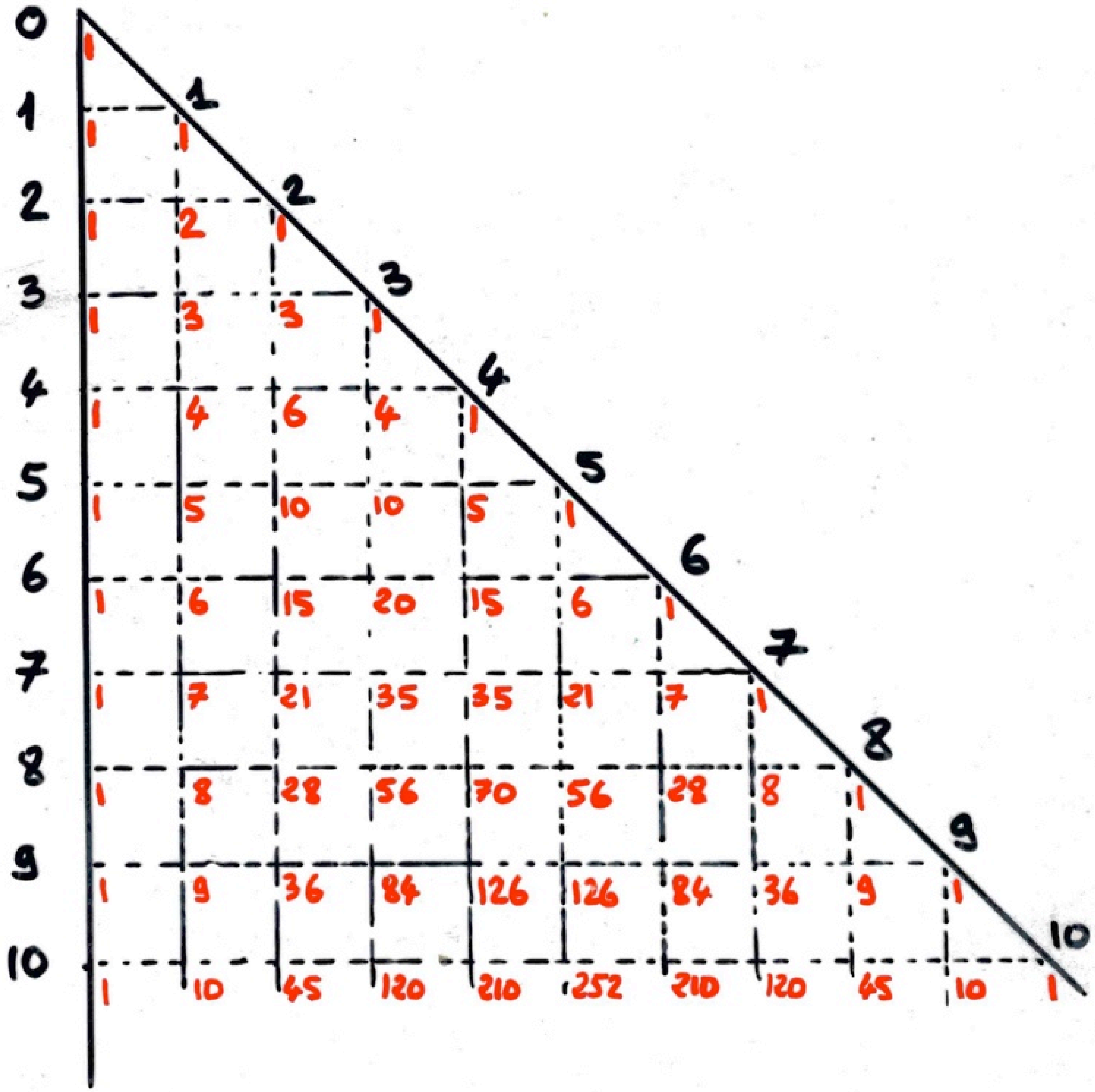
$0 \leq a_1 < \cdots < a_k$

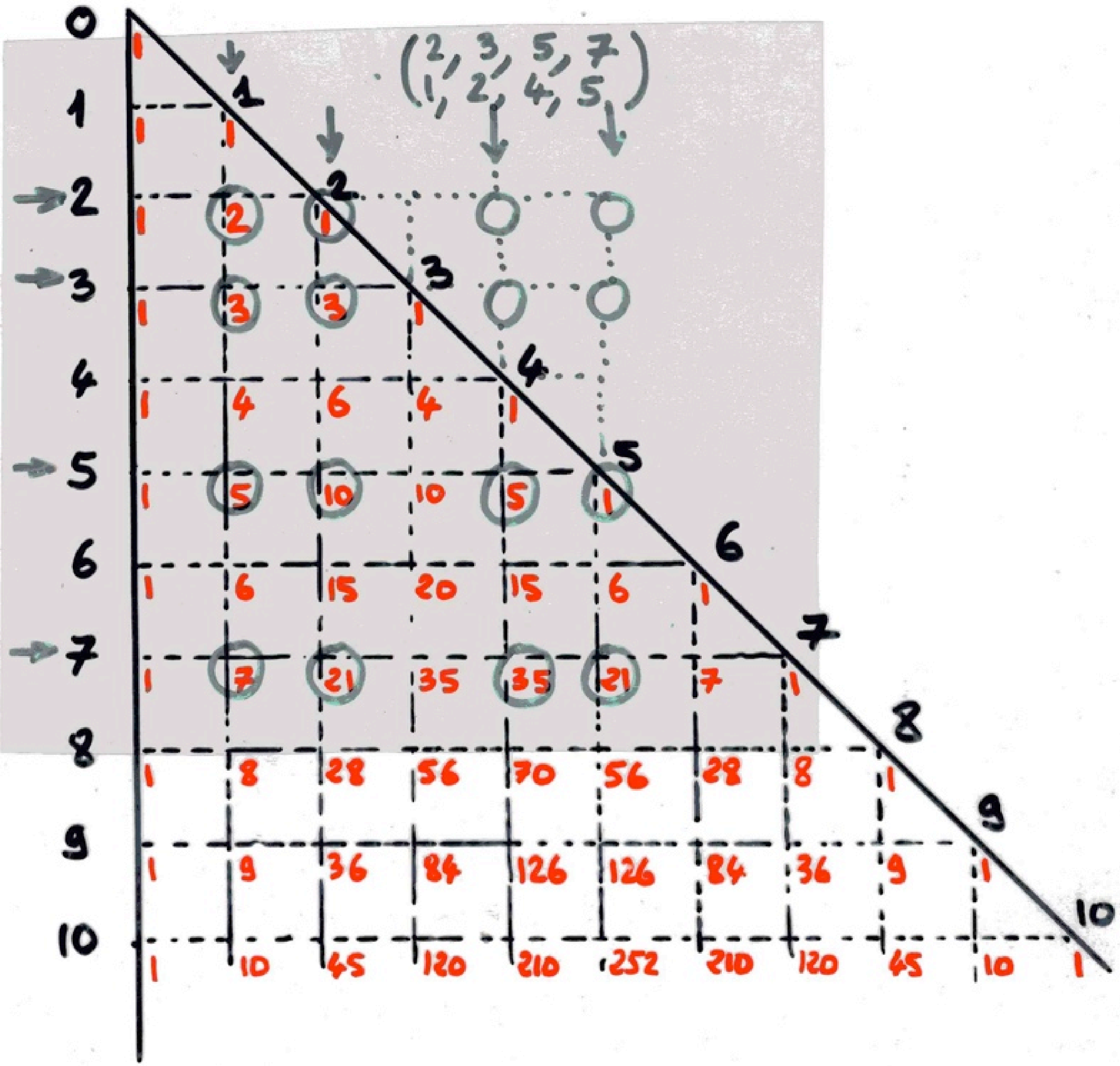
$0 \leq b_1 < \cdots < b_k$

$$\binom{a_1, \ldots, a_k}{b_1, \ldots, b_k} = \text{dét}\left(\binom{a_i}{b_j}\right)_{1 \leq i,j \leq k}$$

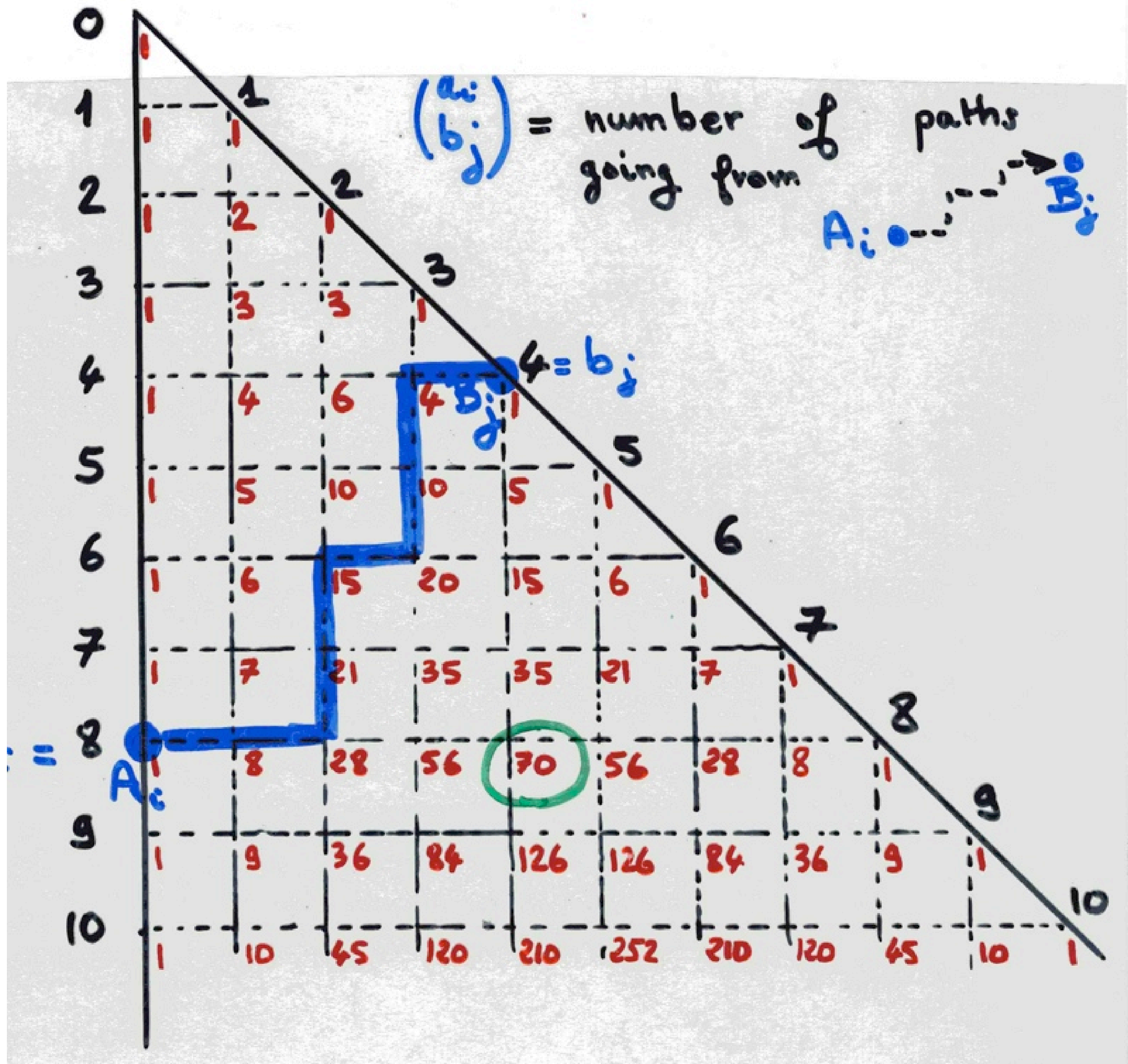
I. Gessel, X.G. Viennot
(Adv. in Maths
58 (1985) 300-321)

Binomial Determinant

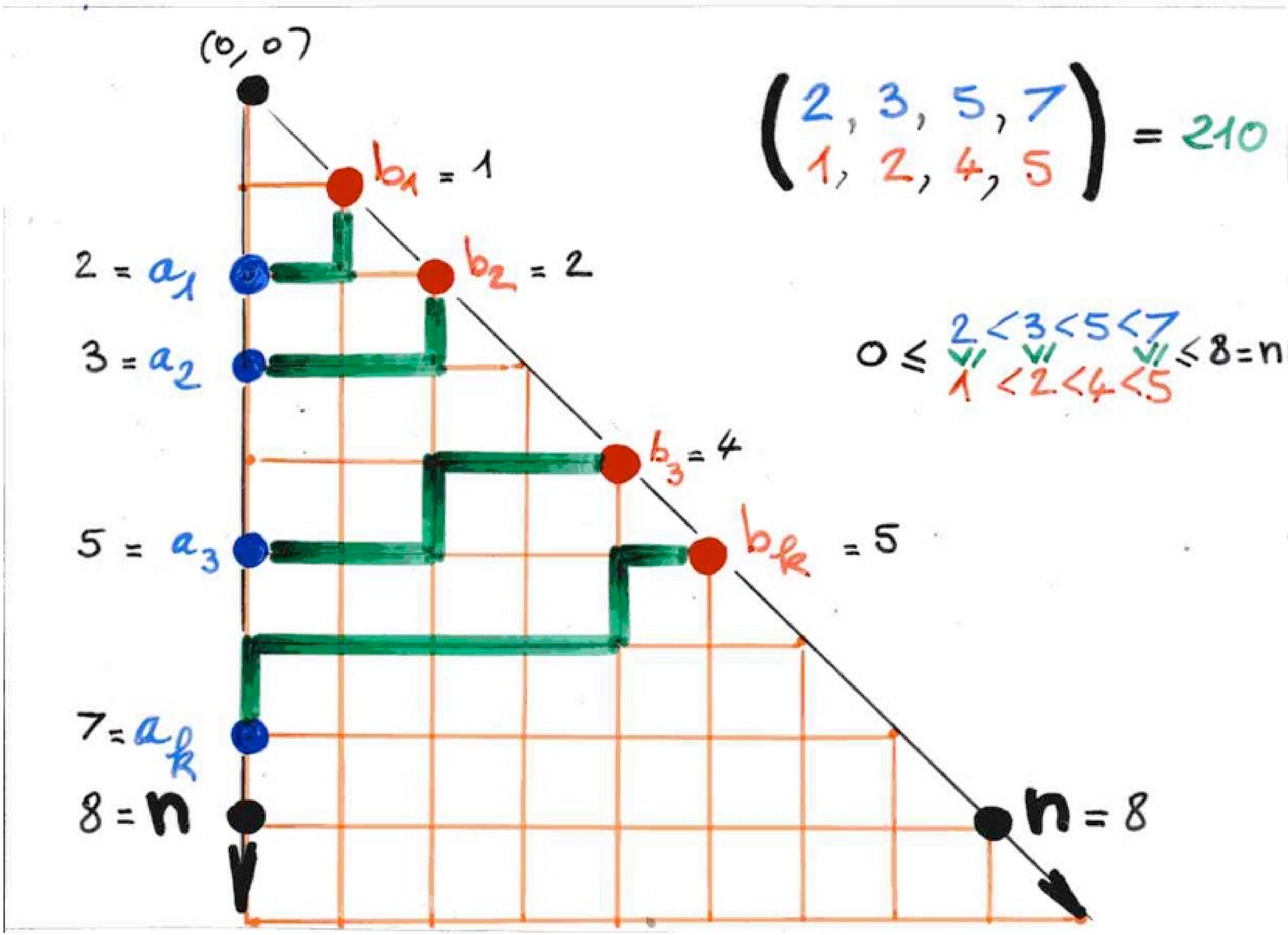

0
1
1 1
2
1 2 1
3
1 3 3 1
4
1 4 6 4 1
5
1 5 10 10 5 1
6
1 6 15 20 15 6 1
7
1 7 21 35 35 21 7 1
8
1 8 28 56 70 56 28 8 1
9
1 9 36 84 126 126 84 36 9 1
10
1 10 45 120 210 252 210 120 45 10 1



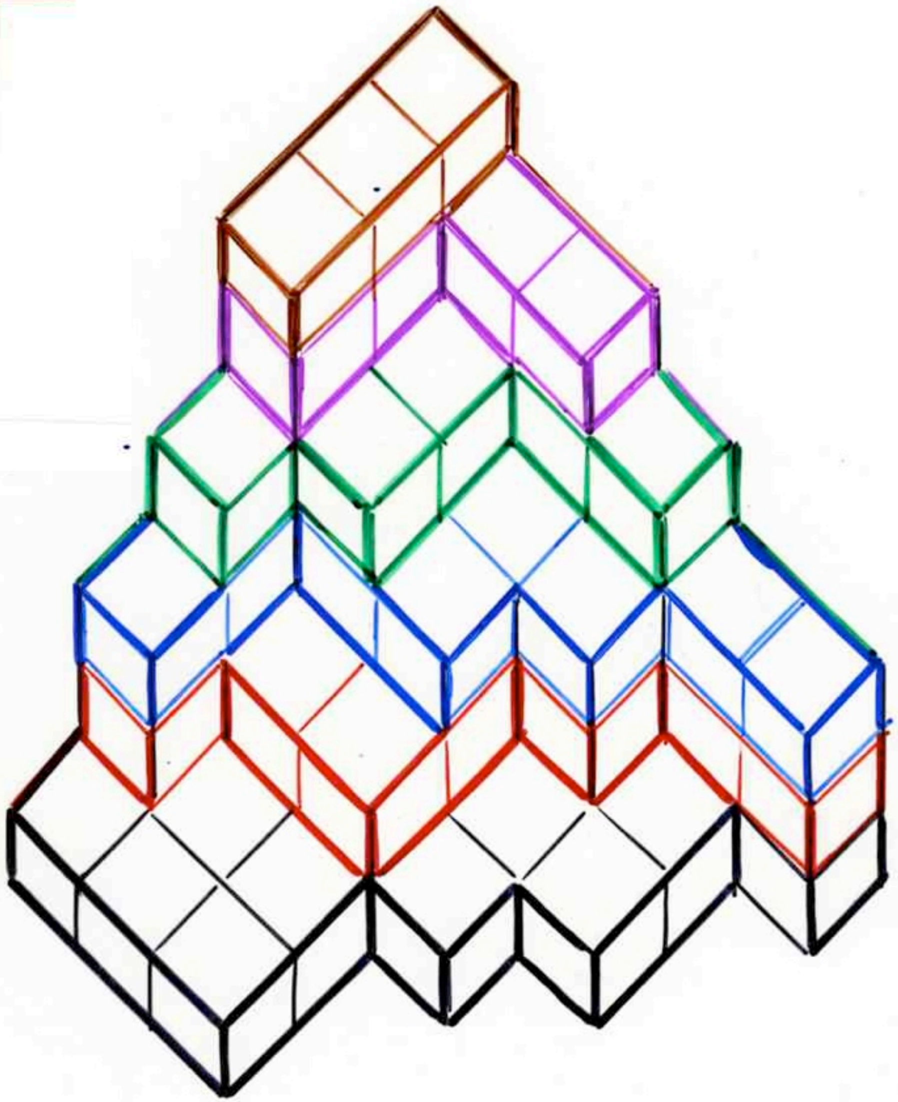
(2, 3, 5, 7
1, 2, 4, 5)
0 1
1 1 1
2 1 2 1
3 1 3 3 1
4 1 4 6 4 1
5 1 5 10 10 5 1
6 1 6 15 20 15 6 1
7 1 7 21 35 35 21 7 1
8 1 8 28 56 70 56 28 8 1
9 1 9 36 84 126 126 84 36 9 1
10 1 10 45 120 210 252 210 120 45 10 1

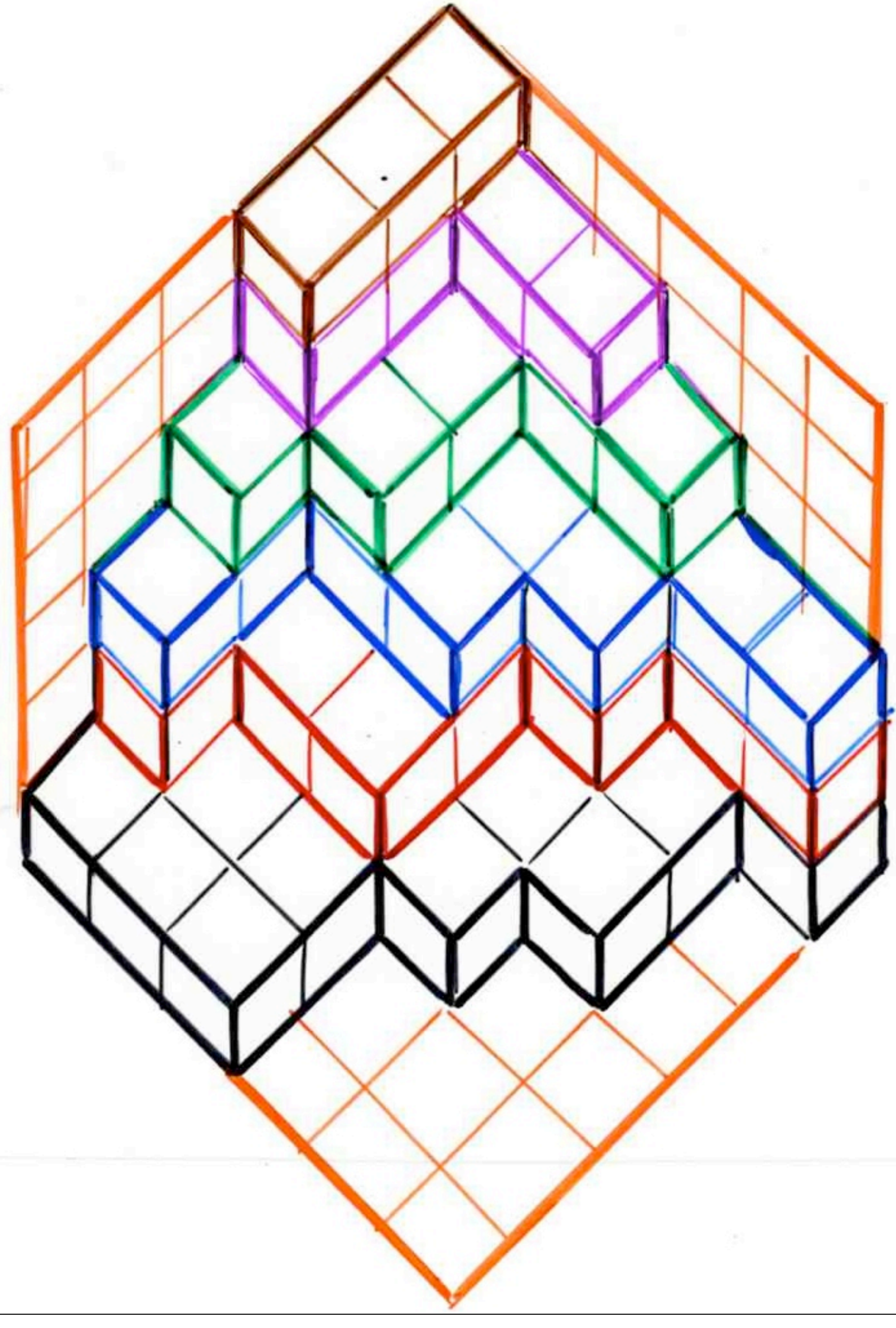


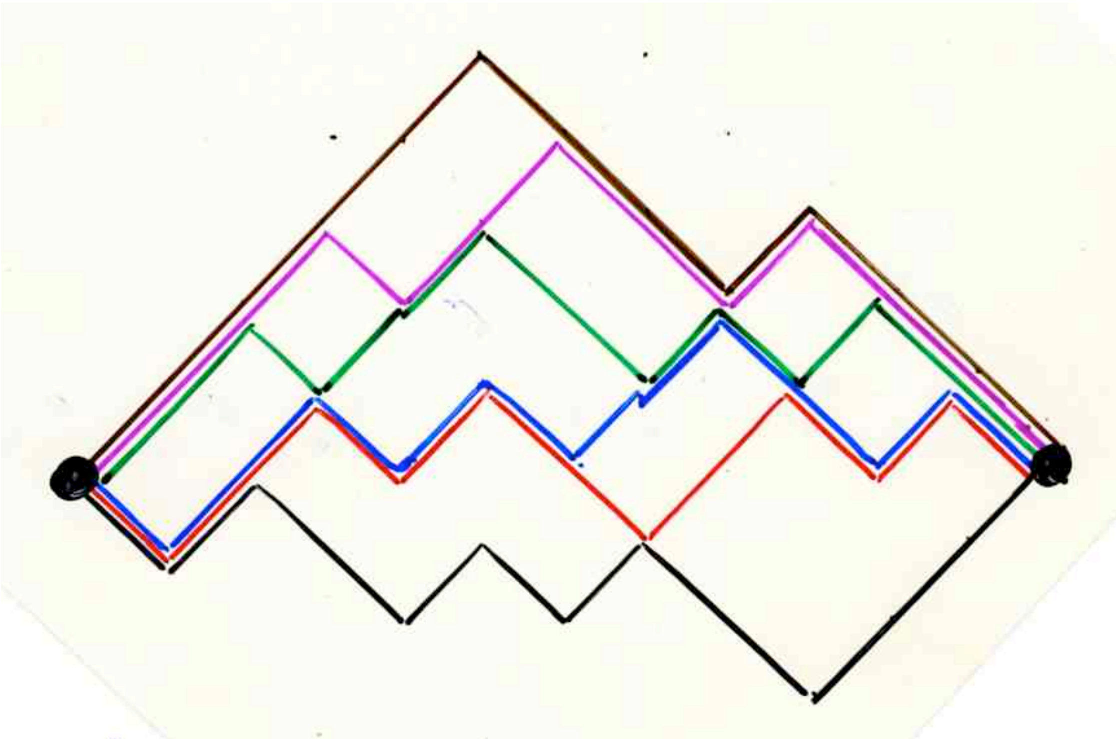
$\binom{a_i}{b_j}$ = number of paths going from A_i to B_j
4 = b_j
B_j
= 8
A_i

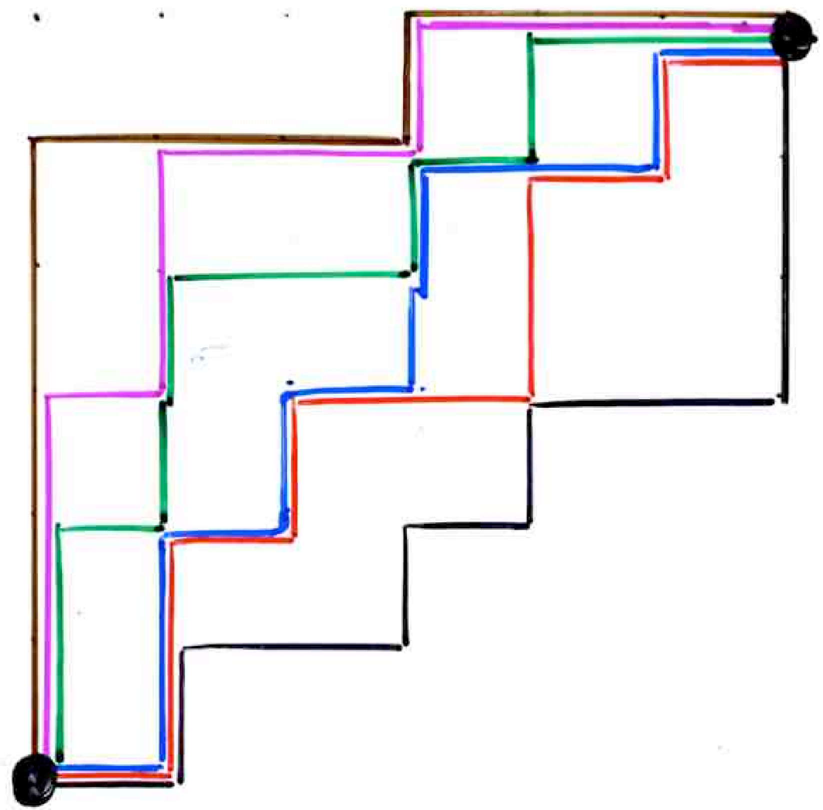

(0, 0)
b1 = 1
2 = a1
b2 = 2
3 = a2
b3 = 4
5 = a3
bk = 5
7 = ak
8 = n
n = 8
(2, 3, 5, 7 / 1, 2, 4, 5) = 210
0 ≤ 2 < 3 < 5 < 7 ≤ 8 = n
1 < 2 < 4 < 5

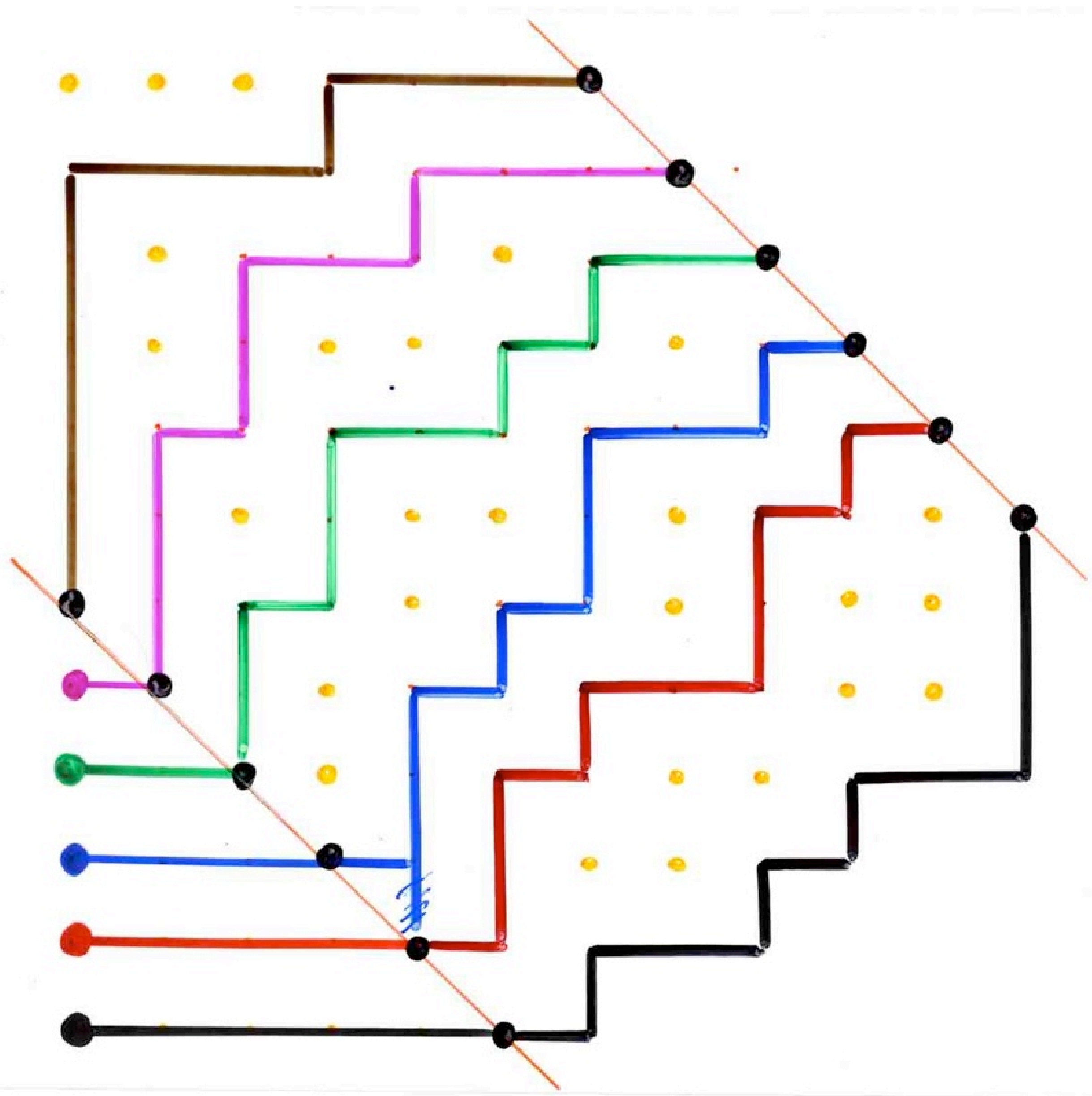
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			



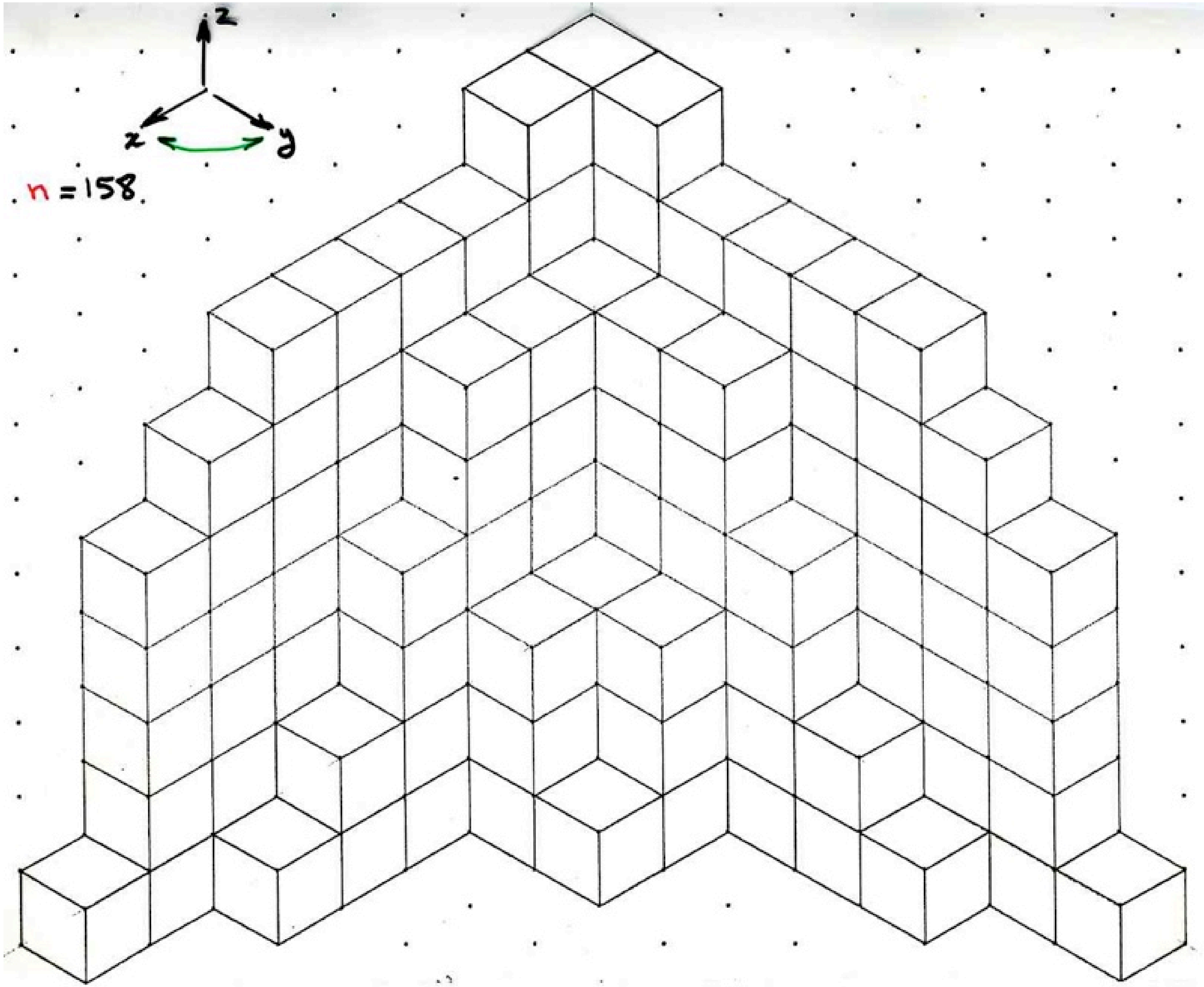








symmetric plane partitions


z
x
y
n = 158

Conjecture de MacMahon

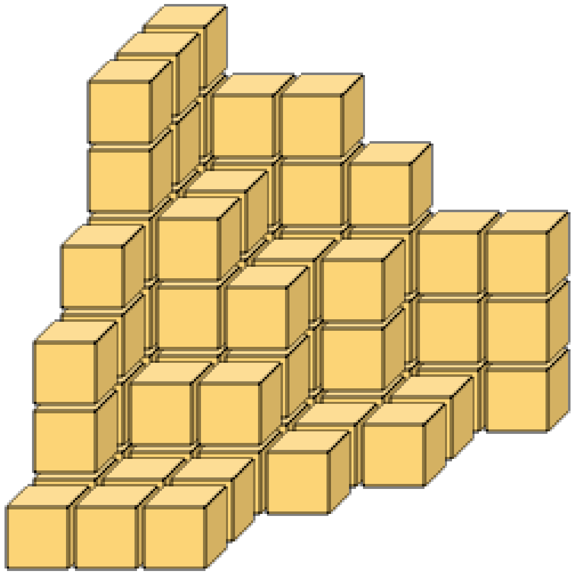
preuve:

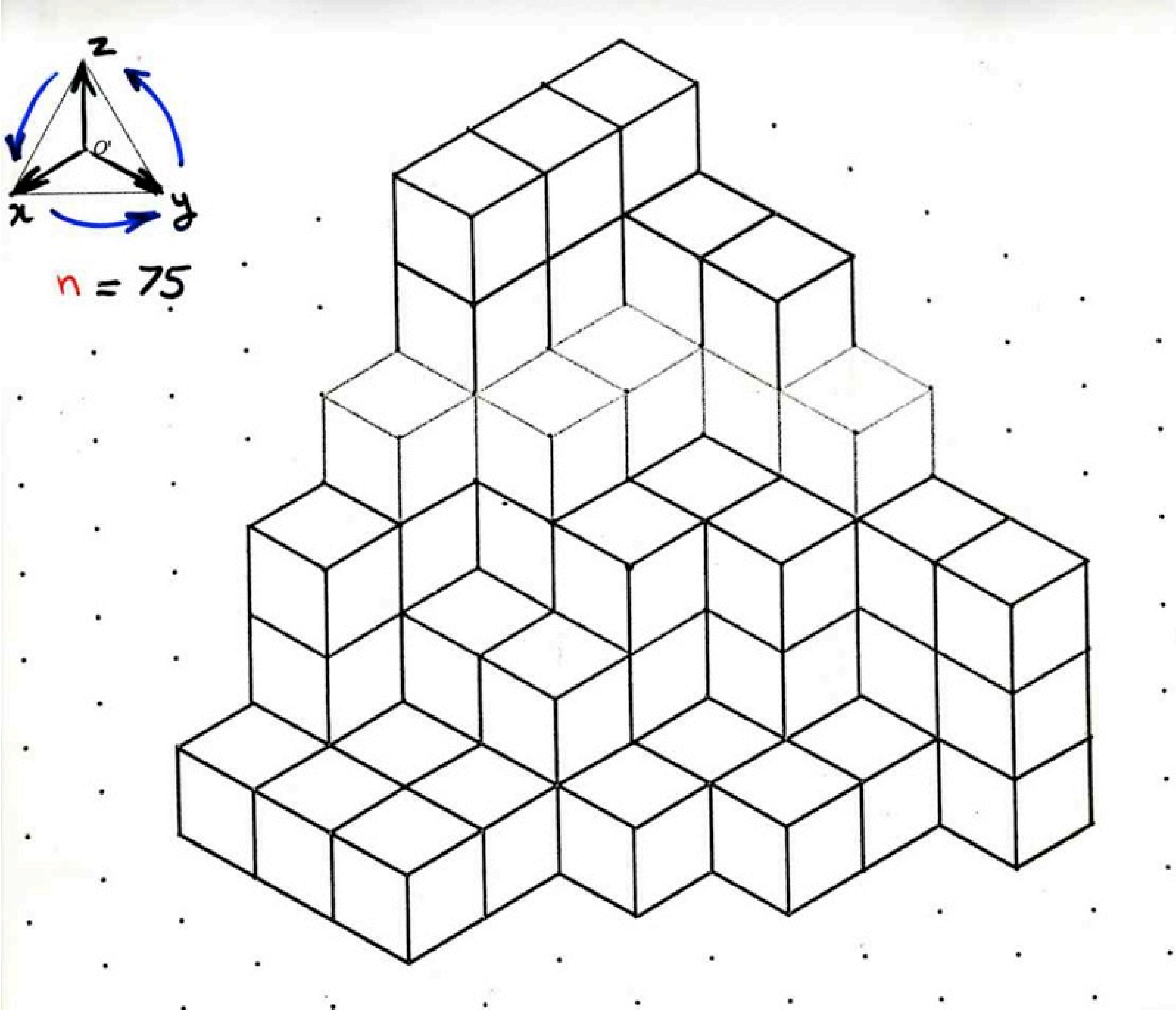
- G. Andrews (1978)
- I. Macdonald (1979)

q-séries

cyclicaly symmetric plane partitions

Partitions planes
cycliquement symétriques



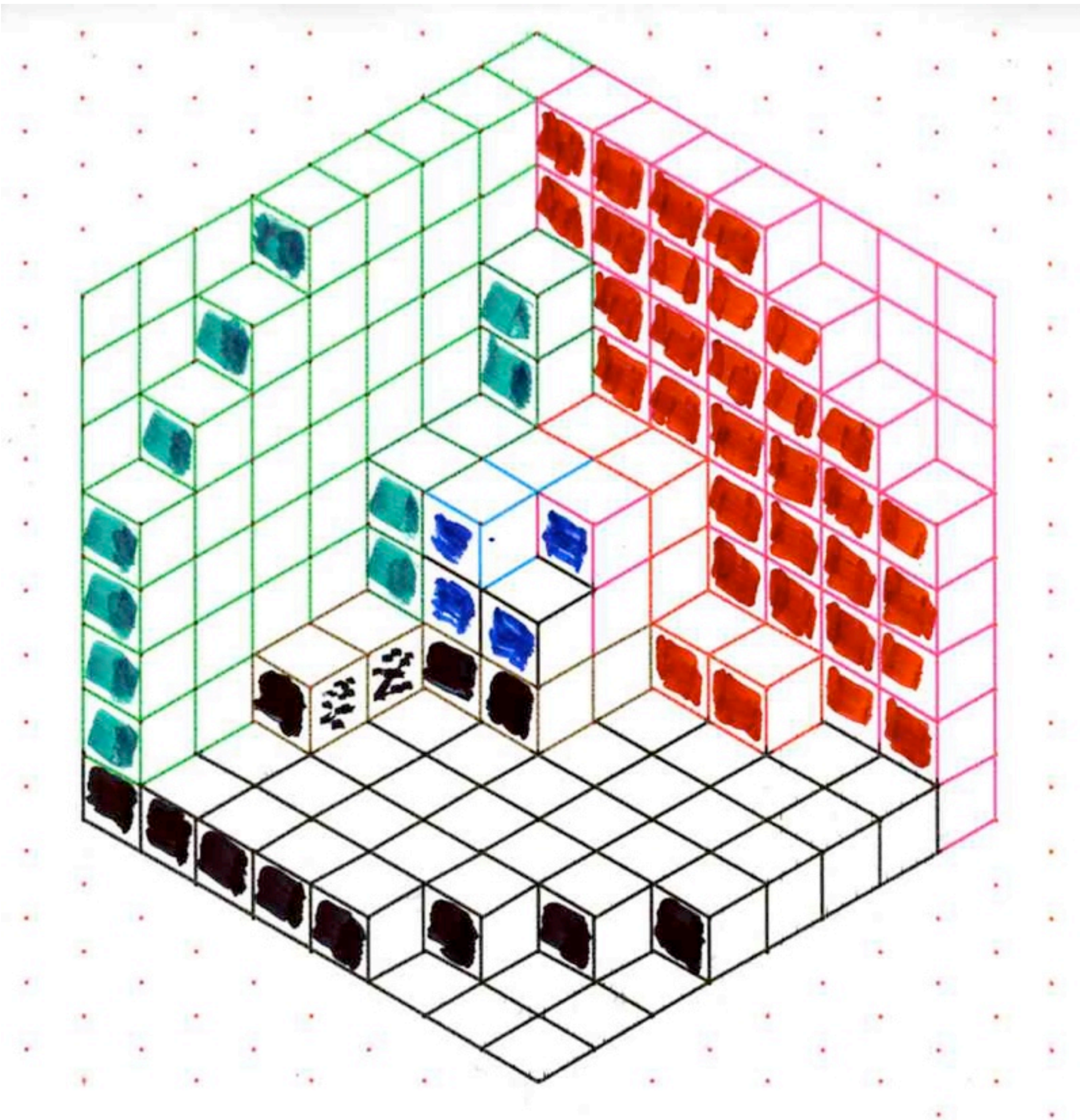

z
O'
x
y
n = 75

conjecture de Macdonald

preuve

- G. Andrews (q=1), (1979) PPD
- W. Mills, D. Robbins, H. Rumsey (1982)

totally symmetric plane partitions



Ten formulae

complémentaire
d'une
partition plane

self-complementary

totally symmetric

3D partitions

G. Andrews

3D Ferrers diagrams up to symmetries

10 formulae

$$\frac{(\text{product})}{(\text{product})}$$

Stanley (1985)

Table 2. Symmetry classes of plane partitions.

1. No restrictions

$$P_1(n) = \prod_{1 \leq i,j,k \leq n} \frac{i + j + k - 1}{i + j + k - 2}$$

2. Symmetric (exchange of x and y axes)

$$P_2(n) = \prod_{\substack{1 \leq i,j,k \leq n \\ i \leq j}} \frac{i + j + k - 1}{i + j + k - 2}$$

3. Cyclically symmetric (cyclic permutations of the three coordinate axes)

$$P_3(n) = \prod_{1 \leq i < j,k \leq n} \frac{i + j + k - 1}{i + j + k - 2} \prod_{1 \leq i \leq j \leq n} \frac{2i + j - 1}{2i + j - 2}$$

4*. Totally symmetric (all six permutations of the coordinate axes)

$$P_4(n) = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i + j + k - 1}{i + j + k - 2}$$

5. Self-complementary

$$P_5(2n) = P_1(n)^2$$

6. Complement = mirror image

$$P_6(2n) = \binom{3n - 1}{2n - 1} \prod_{1 \leq i \leq j \leq 2n-2} \frac{2n + i + j + 1}{i + j + 1}$$

7. Symmetric and self-complementary

$$P_7(2n) = P_1(n)$$

8. Cyclically symmetric and complement = mirror image

$$P_8(2n) = \prod_{i=0}^{n-1} \frac{(3i + 1)(6i)!(2i)!}{(4i + 1)!(4i)!}$$

9*. Cyclically symmetric and self-complementary

$$P_9(2n) = P_{10}(2n)^2$$

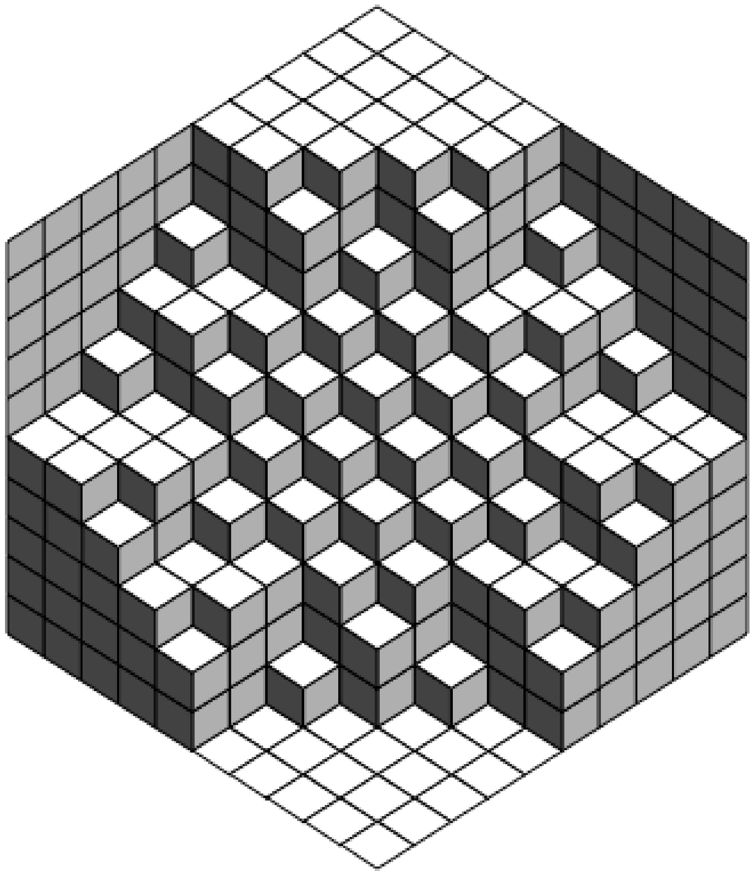
10*. Totally symmetric and self-complementary

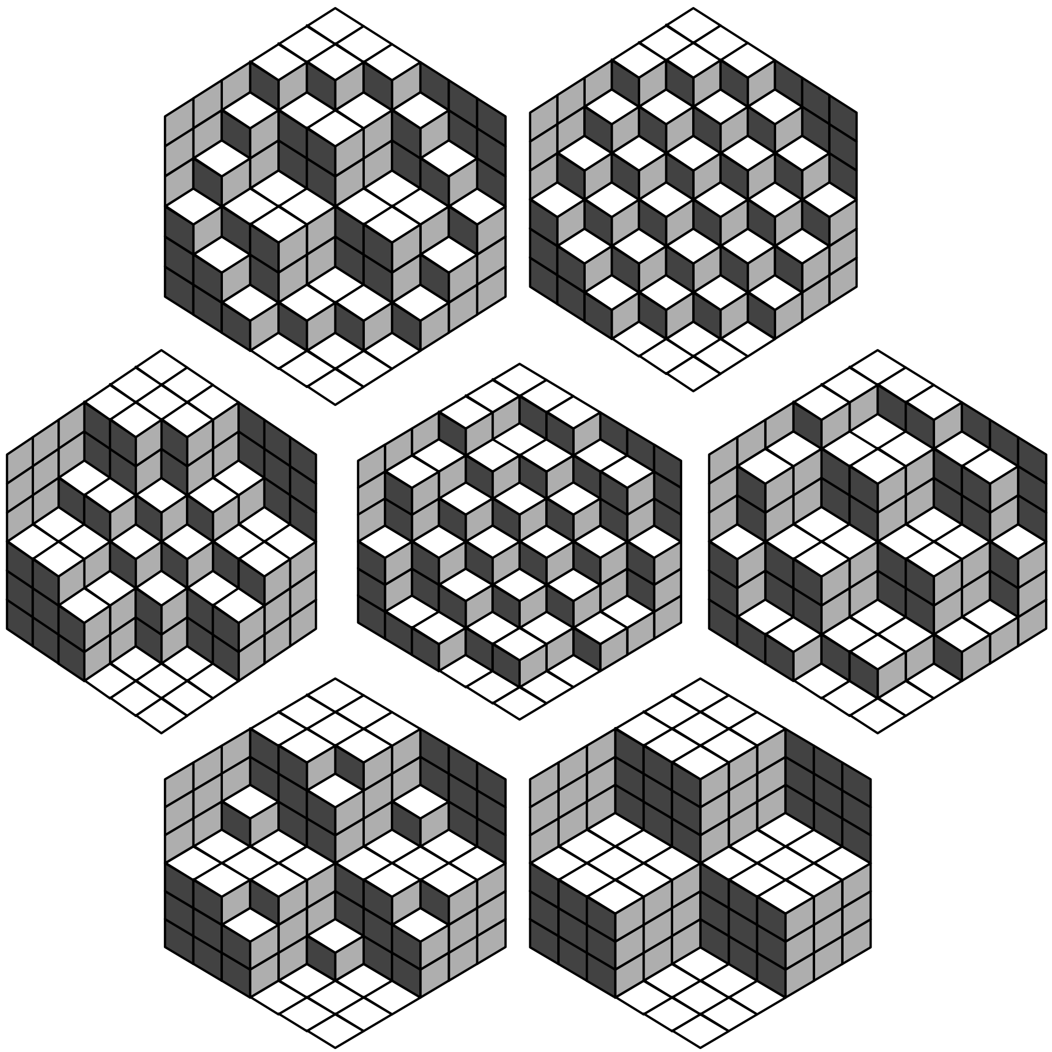
$$P_{10}(2n) = \prod_{i=0}^{n-1} \frac{(3i + 1)!}{(n + i)!}$$

Remarks: $P_k(n)$ gives the formula for the number of plane partitions from the k-th symmetry class whose Ferrers graph fits in an n-by-n-by-n box. There are no plane partitions for odd n in the self-complementary symmetry classes. For those symmetry classes marked with an asterisk the given formula has not been proved.

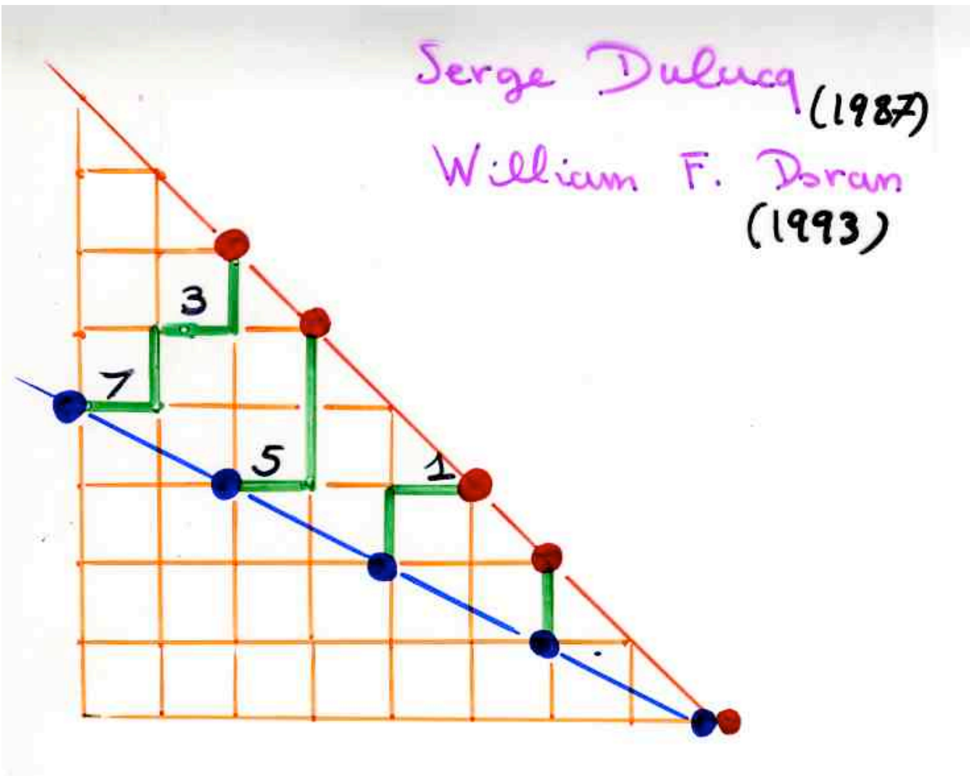
TSSCPP

totaly symmetric
self complementary
plane partitions

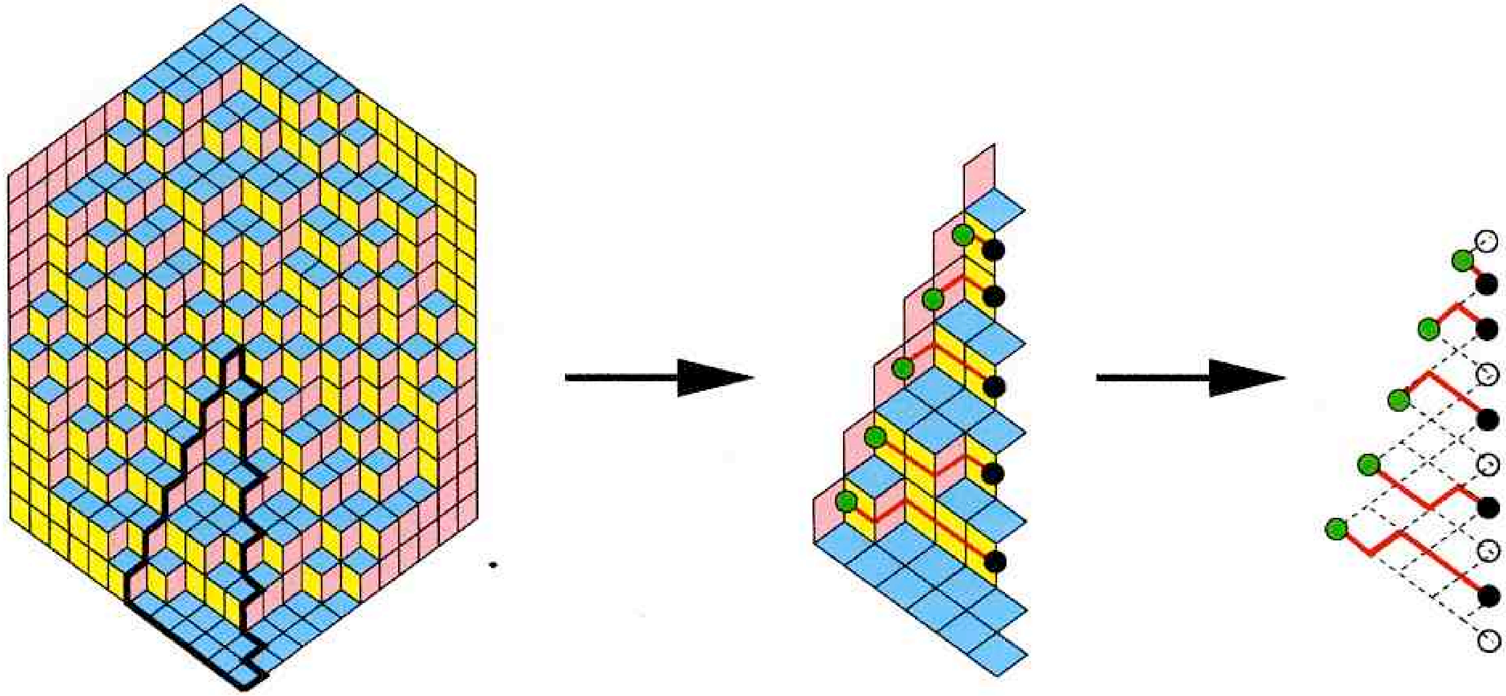




$$M = \begin{cases} \binom{j}{i-j} & 0 \le j \le i \le 2j \\ 0 & \text{sinon} \end{cases}$$



1	.	.	.	.	.	.
.	1	.	.	.	.	.
.	1	1	.	.	.	.
.	.	2	1	.	.	.
.	.	1	3	1	.	.
.	.	.	3	4	1	.
.	.	.	1	6	5	1



Di Francesco (2006)

the last of the ten

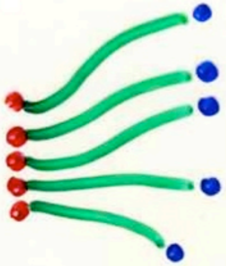
the T.S.S.C.P.P. conjecture

"tour de force" G. Andrews (1994)

- p.p.a.c.t.s

  ↕

- configuration de chemins deux à deux disjoints

  ↓

- Pfaffien

  ↓

- déterminant

  ↓

- calcul

série hypergéométrique

méthode **WZ**

Wilf, Zeilberger

A=B

D. Zeilberger (1992– 1995)

(+ 90 checkers)

Proof of the A. S. M. conj.

PROOF OF THE ALTERNATING SIGN MATRIX CONJECTURE [1]

Doron ZEILBERGER[2]

Checked by[3]: David Bressoud and

Gert Almkvist, Noga Alon, George Andrews, Anonymous, Dror Bar-Natan, Francois Bergeron, Nantel Bergeron, Gaurav Bhatnagar, Anders Björner, Jonathan Borwein, Mireille Bousquest-Mélou, Francesco Brenti, E. Rodney Canfield, William Chen, Chu Wenchang, Shaun Cooper, Kequan Ding, Charles Dunkl, Richard Ehrenborg, Leon Ehrenpreis, Shalosh B. Ekhad, Kimmo Eriksson, Dominique Foata, Omar Foda, Aviezri Fraenkel, Jane Friedman, Frank Garvan, George Gasper, Ron Graham, Andrew Granville, Eric Grinberg, Laurent Habsieger, Jim Haglund, Han Guo-Niu, Roger Howe, Warren Johnson, Gil Kalai, Viggo Kann, Marvin Knopp, Don Knuth, Christian Krattenthaler, Gilbert Labelle, Jacques Labelle, Jane Legrange, Pierre Leroux, Ethan Lewis, Daniel Loeb, John Majewicz, Steve Milne, John Noonan, Kathy O'Hara, Soichi Okada, Craig Orr, Sheldon Parnes, Peter Paule, Bob Proctor, Arun Ram, Marge Readdy, Amitai Regev, Jeff Remmel, Christoph Reutenauer, Bruce Reznick, Dave Robbins, Gian-Carlo Rota, Cecil Rousseau, Bruce Sagan, Bruno Salvy, Isabella Sheftel, Rodica Simion, R. Jamie Simpson, Richard Stanley, Dennis Stanton, Volker Strehl, Walt Stromquist, Bob Sulanke, X.Y. Sun, Sheila Sundaram, Raphaële Supper, Nobuki Takayama, Xavier G. Viennot, Michelle Wachs, Michael Werman, Herb Wilf, Celia Zeilberger, Hadas Zeilberger, Tamar Zeilberger, Li Zhang, Paul Zimmermann .

Dedicated to my Friend, Mentor, and Guru, Dominique Foata.

Two stones build two houses. Three build six houses. Four build four and twenty houses. Five build hundred and twenty houses. Six build Seven hundreds and twenty houses. Seven build five thousands and forty houses. From now on, [exit and] ponder what the mouth cannot speak and the ear cannot hear.

(Sepher Yetsira IV,12)

Abstract: The number of $n \times n$ matrices whose entries are either -1, 0, or 1, whose row- and column- sums are all 1, and such that in every row and every column the non-zero entries alternate in sign, is proved to be $[1!4!\ldots(3n-2)!]/[n!(n+1)!\ldots(2n-1)!]$, as conjectured by Mills, Robbins, and Rumsey.

[1] To appear in Electronic J. of Combinatorics (Foata's 60th Birthday issue). Version of July 31, 1995; original version written December 1992. The Maple package ROBBINS, accompanying this paper, can be downloaded from the www address in footnote 2 below.

[2] Department of Mathematics, Temple University, Philadelphia, PA 19122, USA.
E-mail:`zeilberg@math.temple.edu`. WWW:`http://www.math.temple.edu/~zeilberg`. Anon. ftp: `ftp.math.temple.edu`, directory `/pub/zeilberg`. Supported in part by the NSF.

[3] See the Exodion for affiliations, attribution, and short bios.

Subsublemma 1.1.3:

$$\sum_{\pi\in\mathcal{S}_k} \mathrm{sgn}(\pi)\cdot\pi\left[\frac{x_1x_2^2\dots x_k^k}{(1-x_k)(1-x_kx_{k-1})\cdots(1-x_kx_{k-1}\dots x_1)}\right]=\frac{x_1\cdots x_k\prod_{1\le i<j\le k}(x_j-x_i)}{\prod_{i=1}^k(1-x_i)\prod_{1\le i<j\le k}(1-x_ix_j)} \quad . \tag{Issai}$$

[Type '`S113(k);`' in ROBBINS, for specific k.]

Proof : See [PS], problem VII.47. Alternatively, (Issai) is easily seen to be equivalent to Schur's identity that sums all the Schur functions ([Ma], ex I.5.4, p. 45). This takes care of subsublemma 1.1.3. □

Inserting $(Issai)$ into $(Stanley)$, expanding $\prod_{1\le i<j\le k}(x_j-x_i)$ by Vandermonde's expansion,

$$\sum_{\pi\in\mathcal{S}_k}\mathrm{sgn}(\pi)\cdot\pi(x_1^0x_2^1\cdots x_k^{k-1}) \quad ,$$

using the antisymmetry of Δ_k once again, and employing crucial fact $\aleph_4$, we get the following string of equalities:

$$b_k(n)=\frac{1}{k!}CT_{x_1,\dots,x_k}\left\{\frac{\Delta_k(x_1,\dots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n}x_i^{n+k-1}}\left(\frac{x_1\cdots x_k\prod_{1\le i<j\le k}(x_j-x_i)}{\prod_{i=1}^k(1-x_i)\prod_{1\le i<j\le k}(1-x_ix_j)}\right)\right\}$$

$$=\frac{1}{k!}CT_{x_1,\dots,x_k}\left\{\frac{\Delta_k(x_1,\dots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-2}\prod_{1\le i<j\le k}(1-x_ix_j)}\left(\sum_{\pi\in\mathcal{S}_k}\mathrm{sgn}(\pi)\cdot\pi(x_1^0x_2^1\dots x_k^{k-1})\right)\right\}$$

$$=\frac{1}{k!}\sum_{\pi\in\mathcal{S}_k}CT_{x_1,\dots,x_k}\left\{\pi\left[\frac{\Delta_k(x_1,\dots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-2}\prod_{1\le i<j\le k}(1-x_ix_j)}\left(\prod_{i=1}^k x_i^{i-1}\right)\right]\right\}$$

$$=\frac{1}{k!}\sum_{\pi\in\mathcal{S}_k}CT_{x_1,\dots,x_k}\left\{\pi\left[\frac{\Delta_k(x_1,\dots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-i-1}\prod_{1\le i<j\le k}(1-x_ix_j)}\right]\right\}$$

$$=\frac{1}{k!}\sum_{\pi\in\mathcal{S}_k}CT_{x_1,\dots,x_k}\left\{\frac{\Delta_k(x_1,\dots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-i-1}\prod_{1\le i<j\le k}(1-x_ix_j)}\right\}$$

$$=\frac{1}{k!}\left(\sum_{\pi\in\mathcal{S}_k}1\right)CT_{x_1,\dots,x_k}\left\{\frac{\Delta_k(x_1,\dots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-i-1}\prod_{1\le i<j\le k}(1-x_ix_j)}\right\}$$

$$=CT_{x_1,\dots,x_k}\left\{\frac{\Delta_k(x_1,\dots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-i-1}\prod_{1\le i<j\le k}(1-x_ix_j)}\right\}, \tag{George'''}$$

where in the last equality we have used Levi Ben Gerson's celebrated result that the number of elements in $\mathcal{S}_k$ (the symmetric group on k elements,) equals $k!$. The extreme right of $(George''')$ is exactly the right side of $(MagogTotal)$. This completes the proof of sublemma 1.1. □

"EXTREME UGLYNESS CAN BE BEAUTIFUL"

Doron Zeilberger

Bordeaux, May, 1991

3rd FPSAC

Kuperberg (1995)

6-vertex model

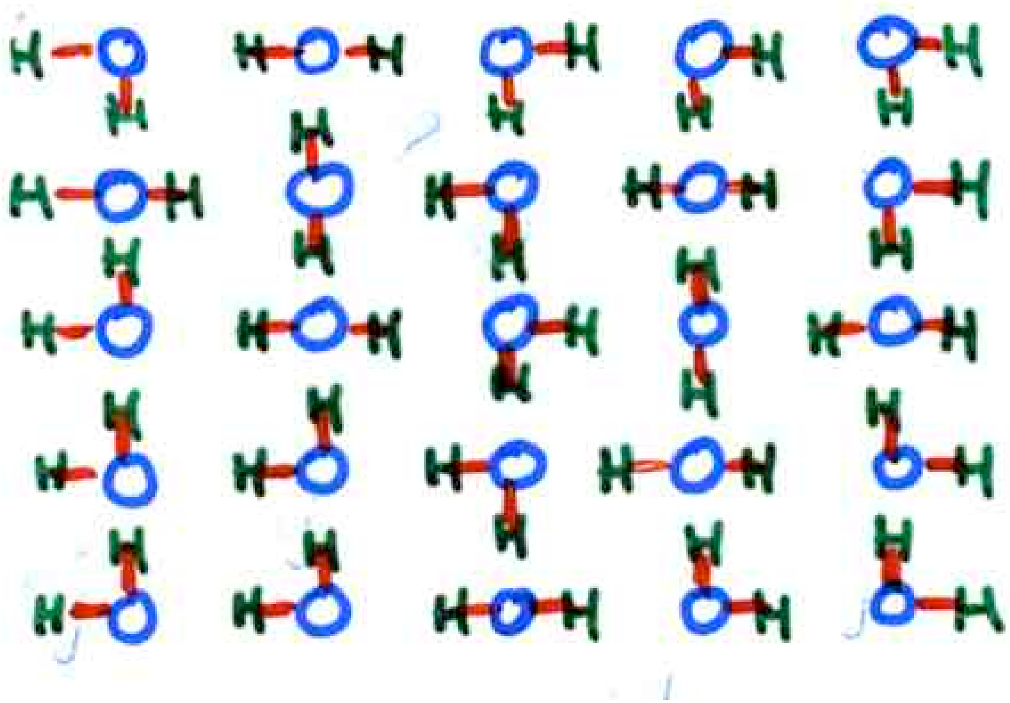
(ice model)

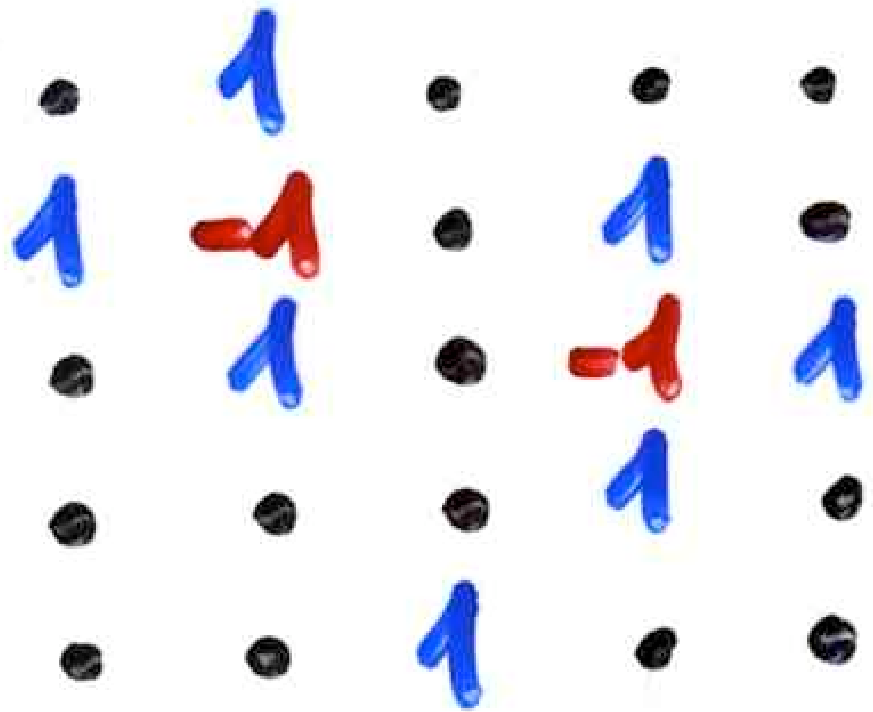
with domain wall boundary condition

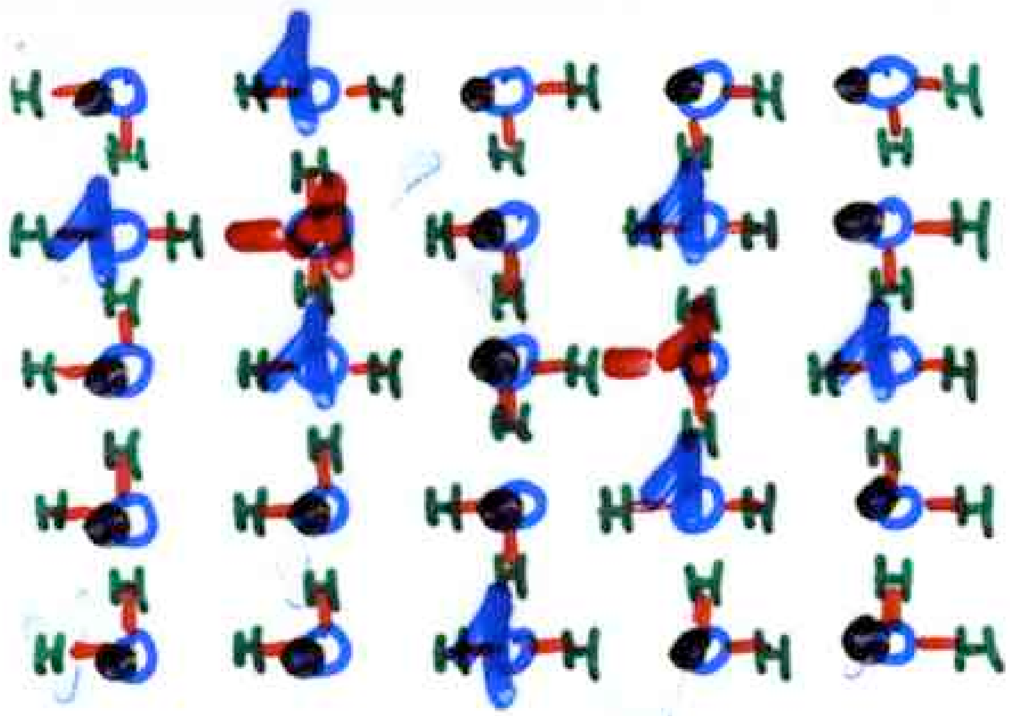
ice model

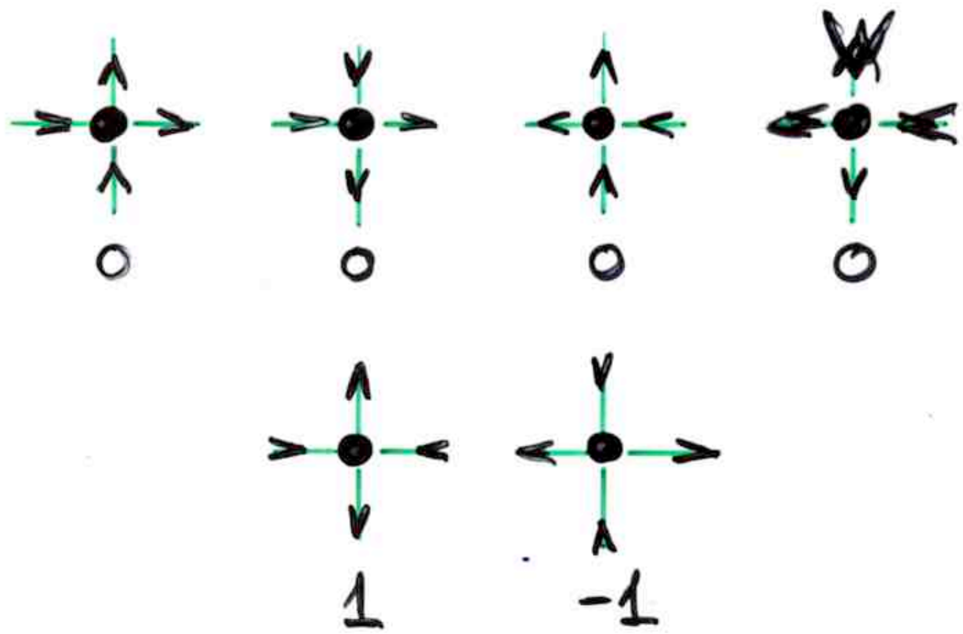
or

six-vertex model

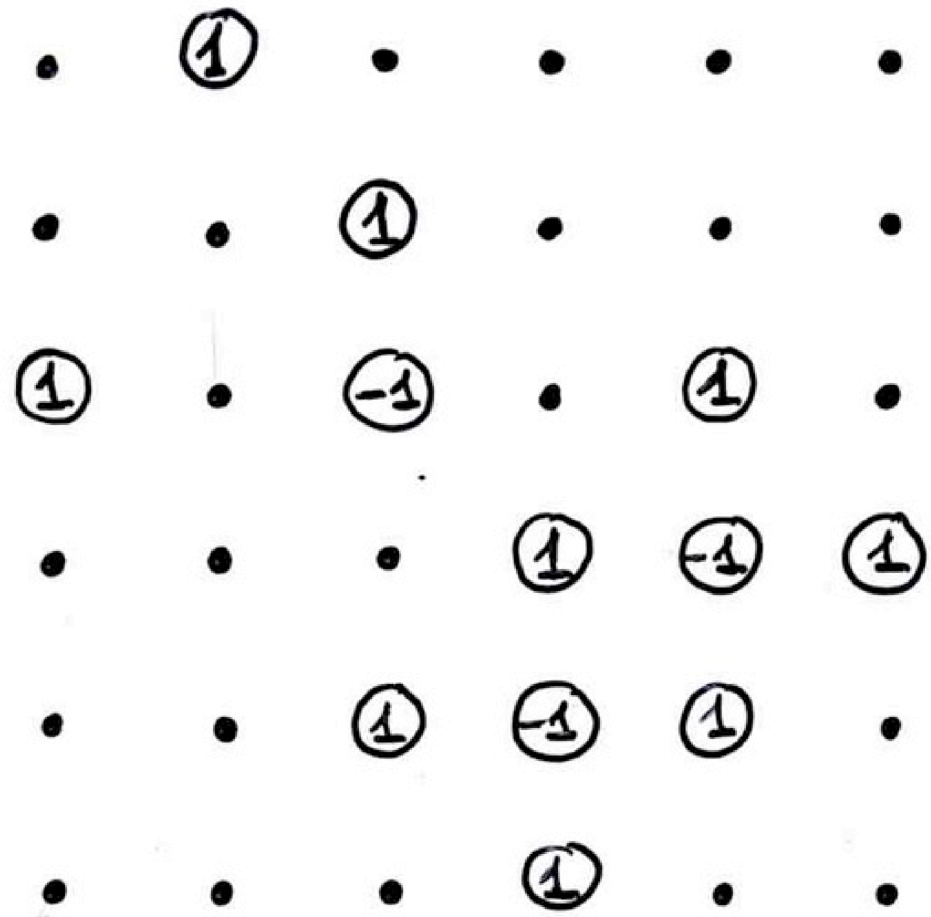


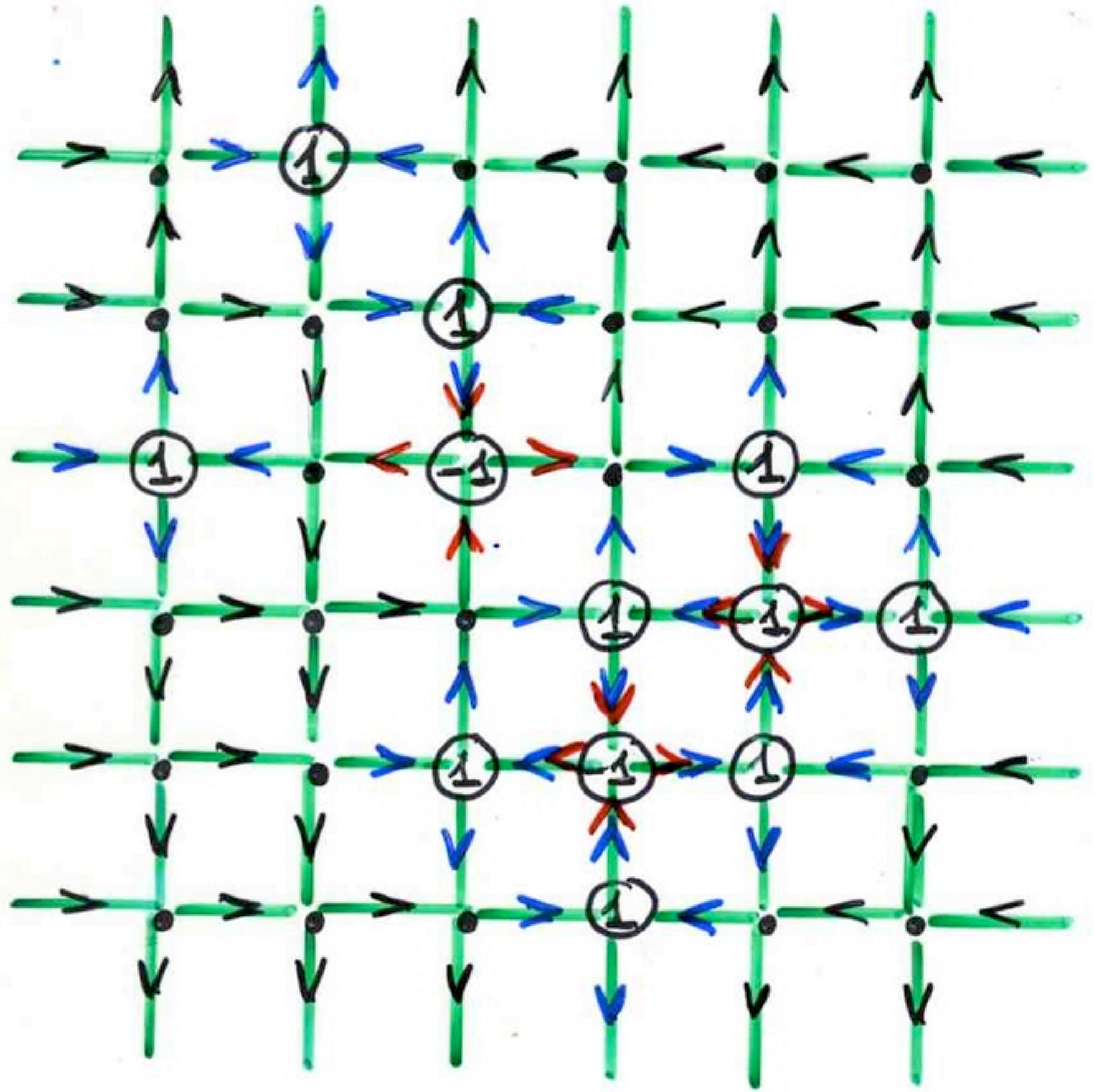






6-vertex model


1
1
1 -1 1
1 -1 1
1 -1 1
1


1
1
1
-1
1
1
1
1
1
1
1
1

fonction de partition

Gaudin

Korepin, Bogoliubov, Izergin

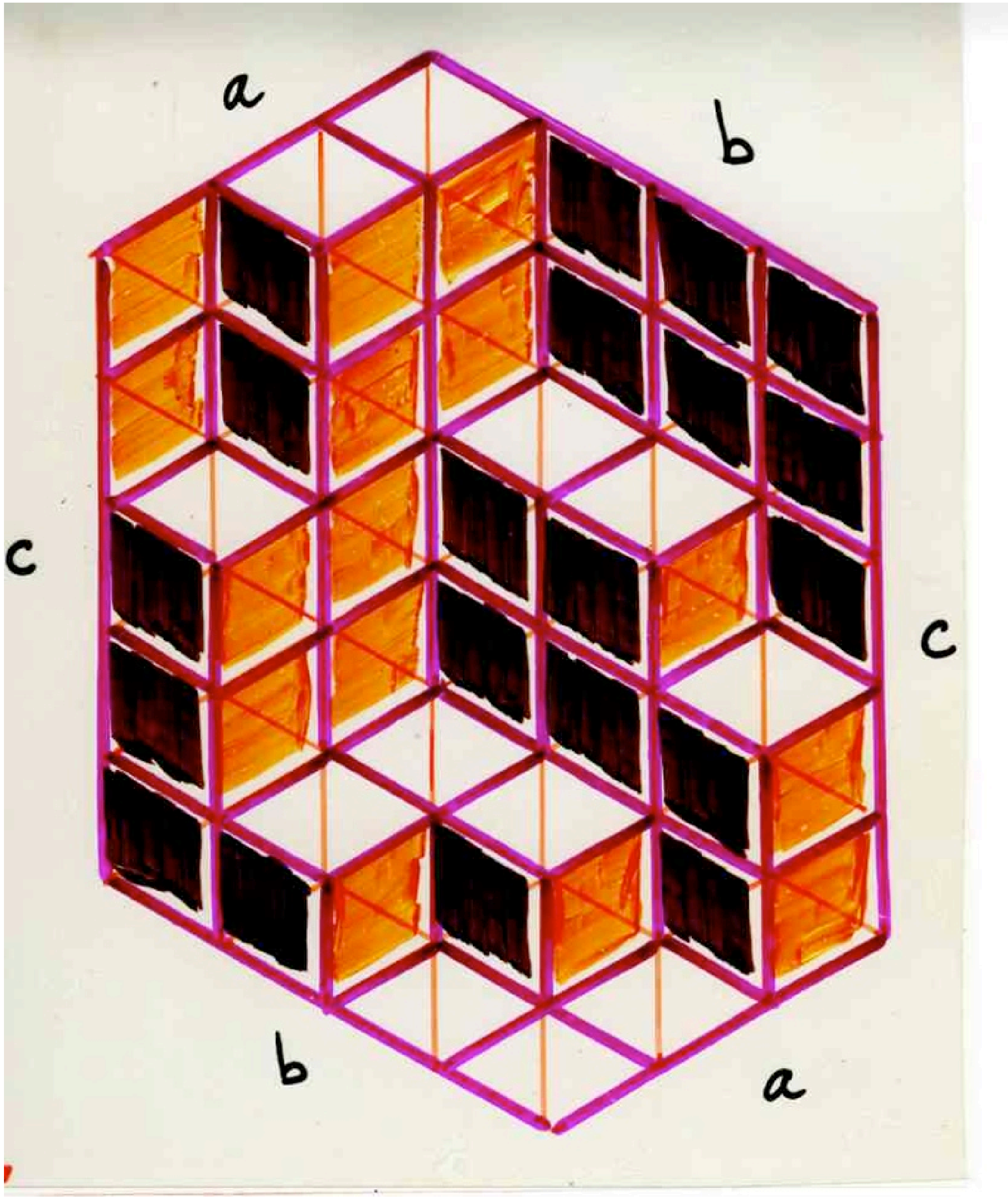
"Quantum Inverse Scattering Method and Correlation Functions" (1993)

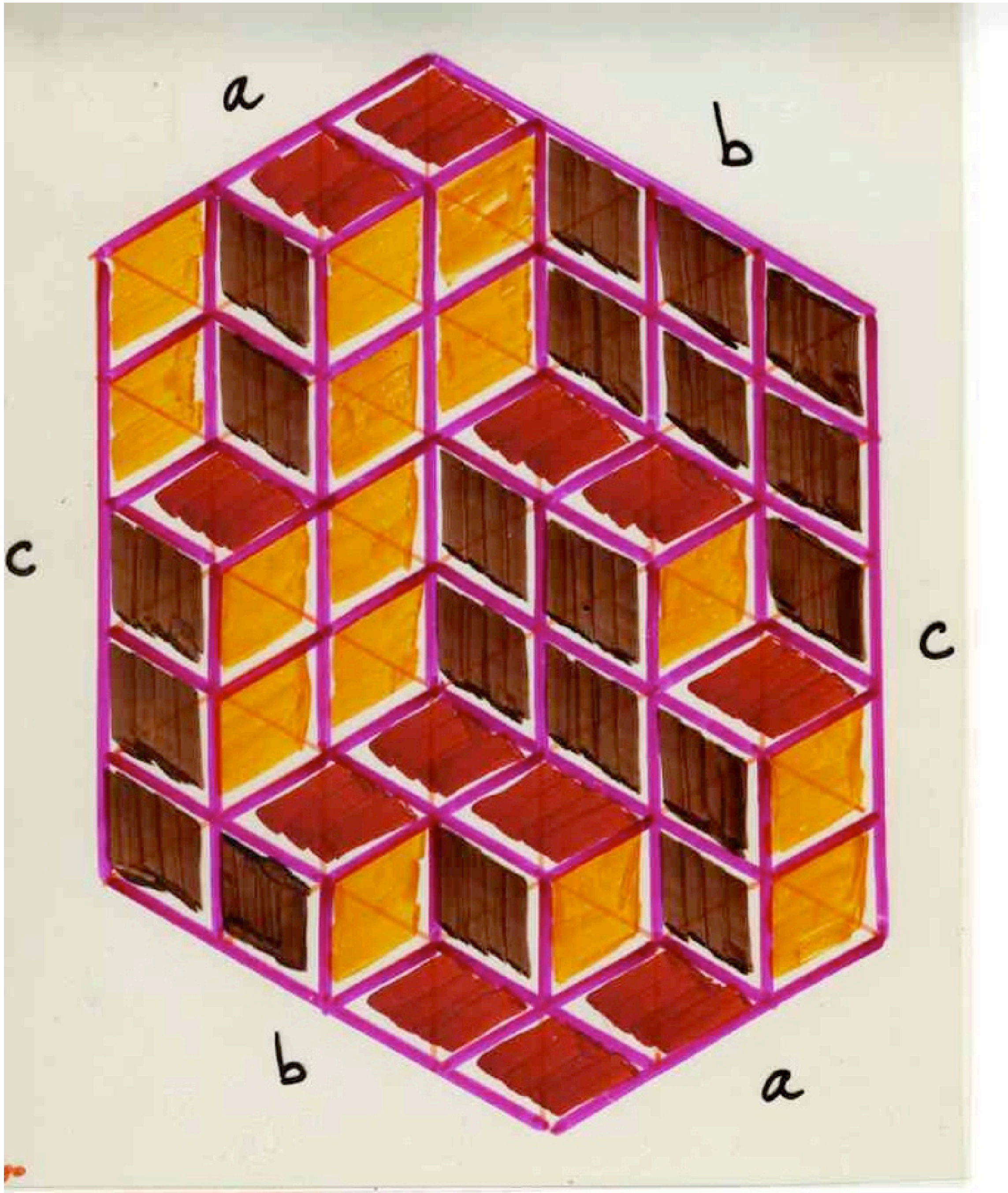
$$Z_n(\vec{x};\vec{y};a) = \frac{\prod_{i=1}^{n} x_i/y_i \prod_{1\le i,j\le n} (x_i/y_j)(ax_i/y_j)}{\prod_{1\le i<j\le n} (x_i/x_j)(y_j/y_i)} \det(M)$$

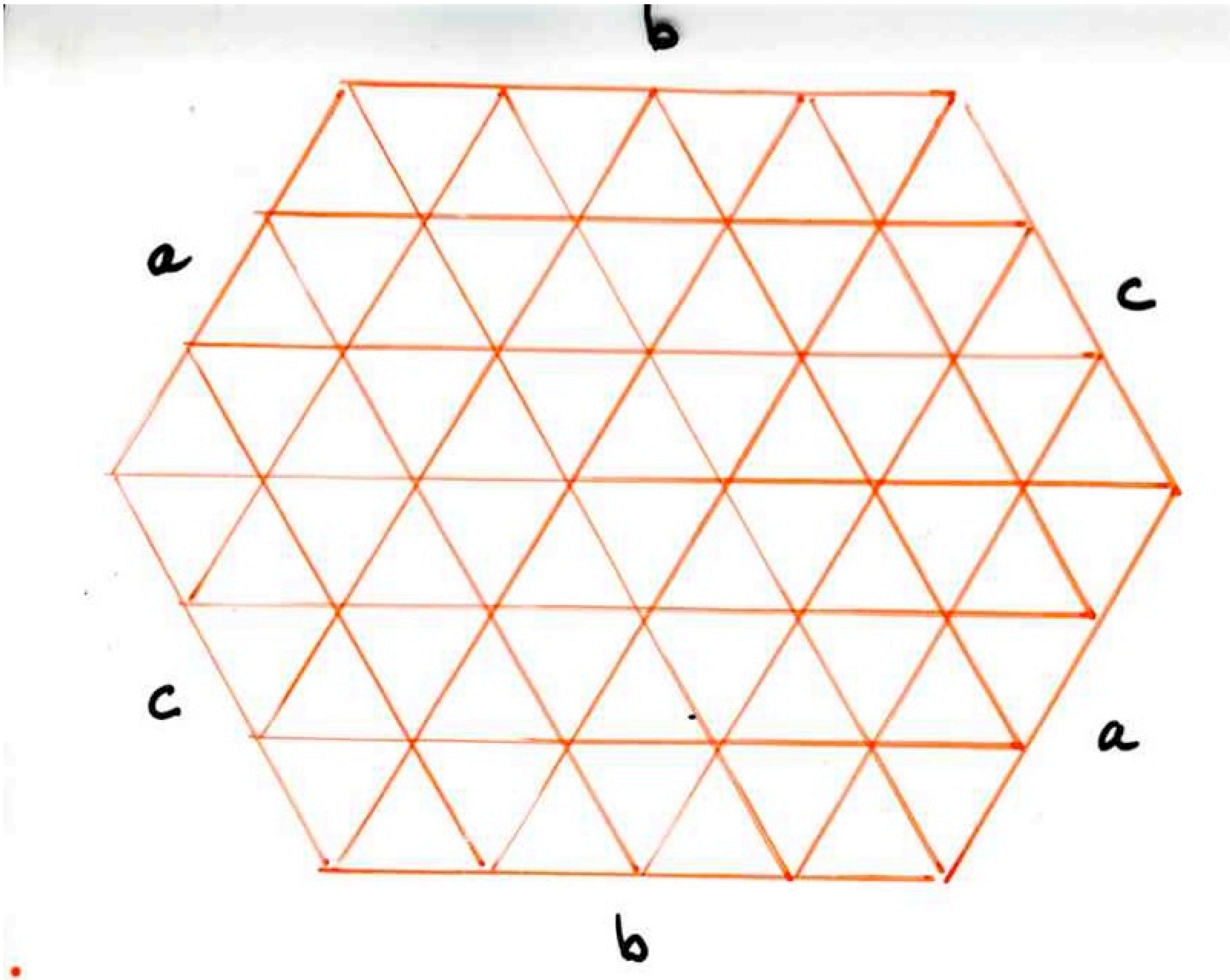
$$M = \frac{1}{(x_i/y_j)(ax_i/y_j)}$$

équation Yang-Baxter

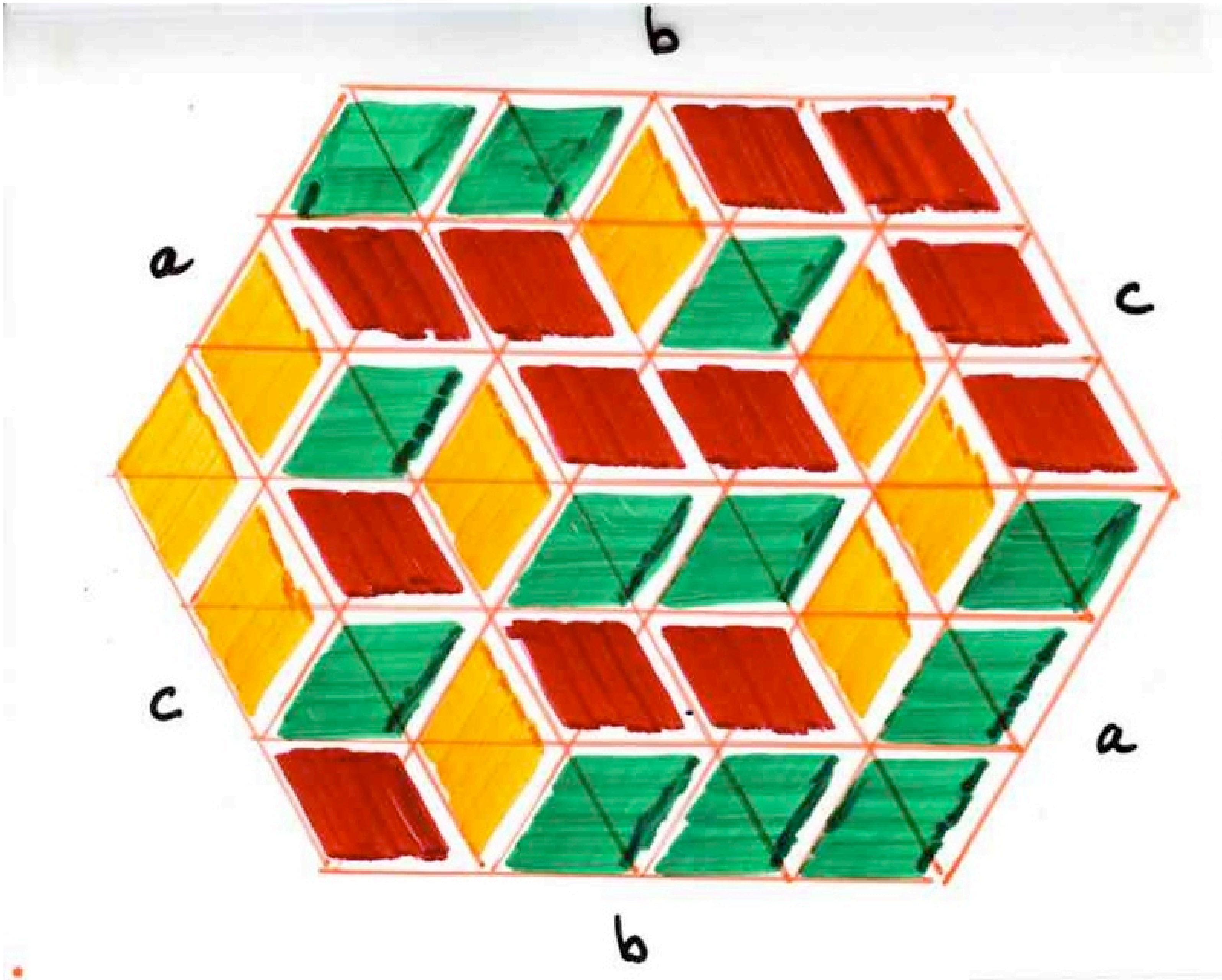
Tilings and matching

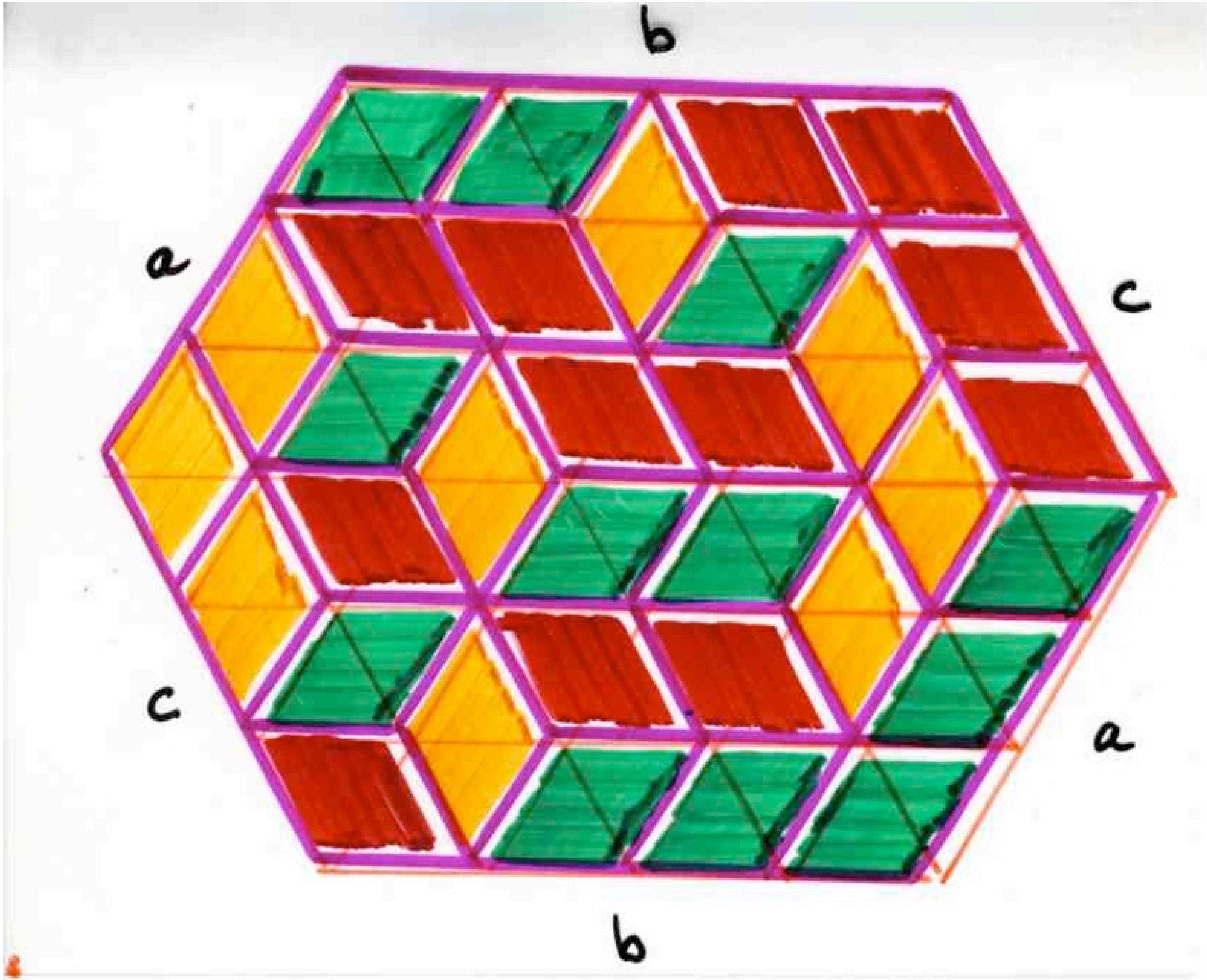

a
b
c
c
b
a


a
b
c
c
b
a



b
a
c
c
a
b

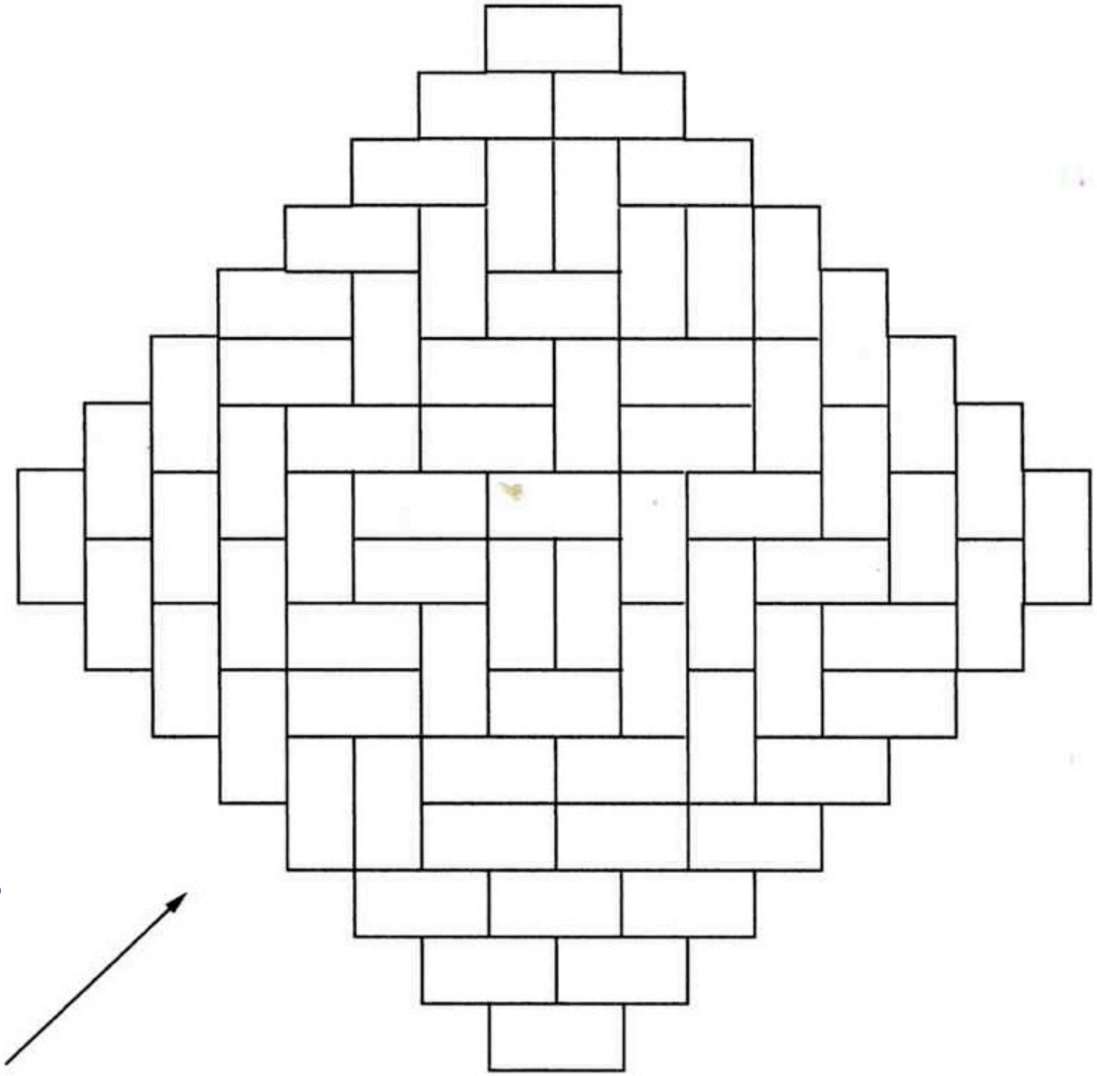

b
a
c
c
a
b


b
a
c
c
a
b

Aztec

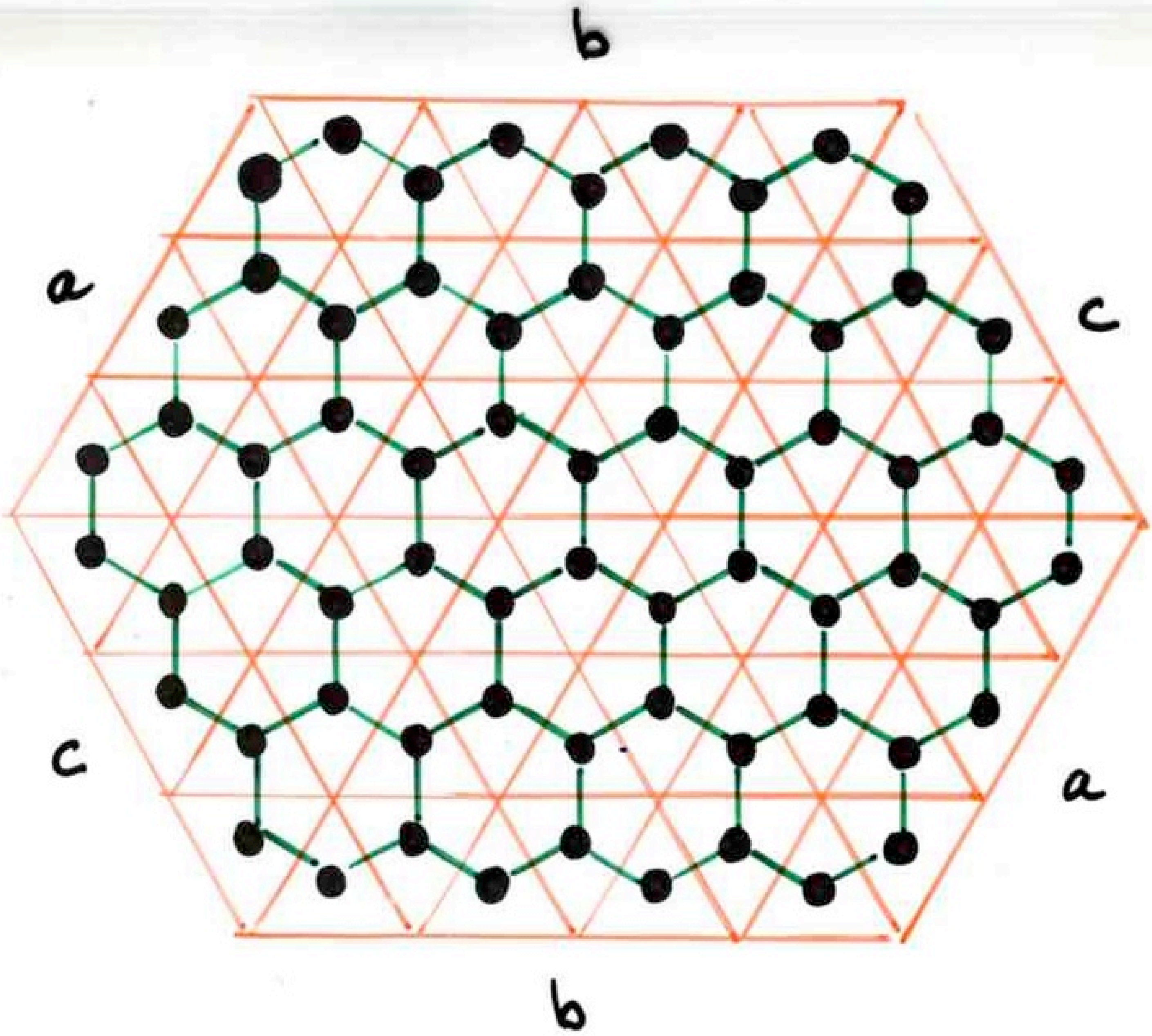
$2^{n(n-1)/2}$

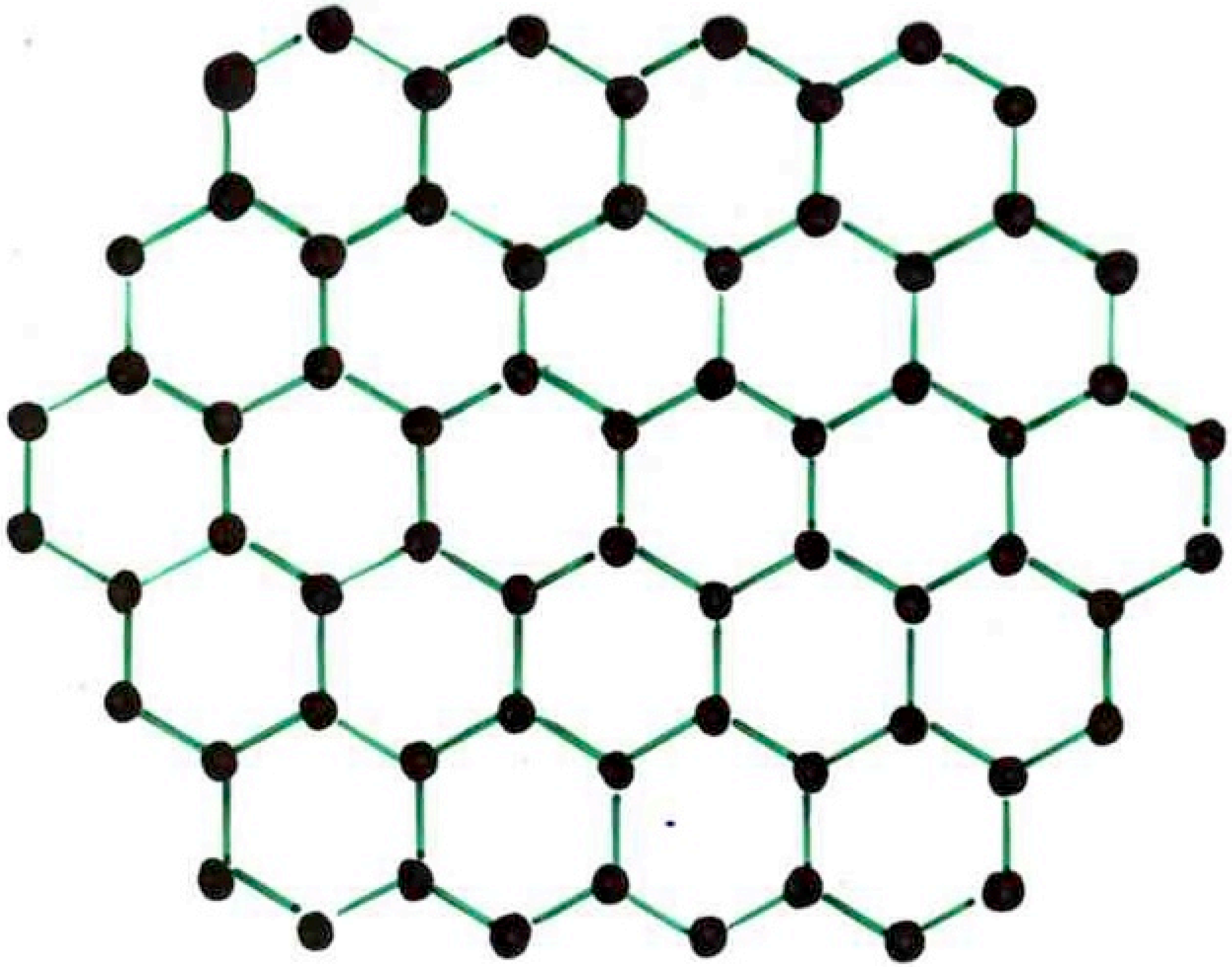
$A_n(2)$

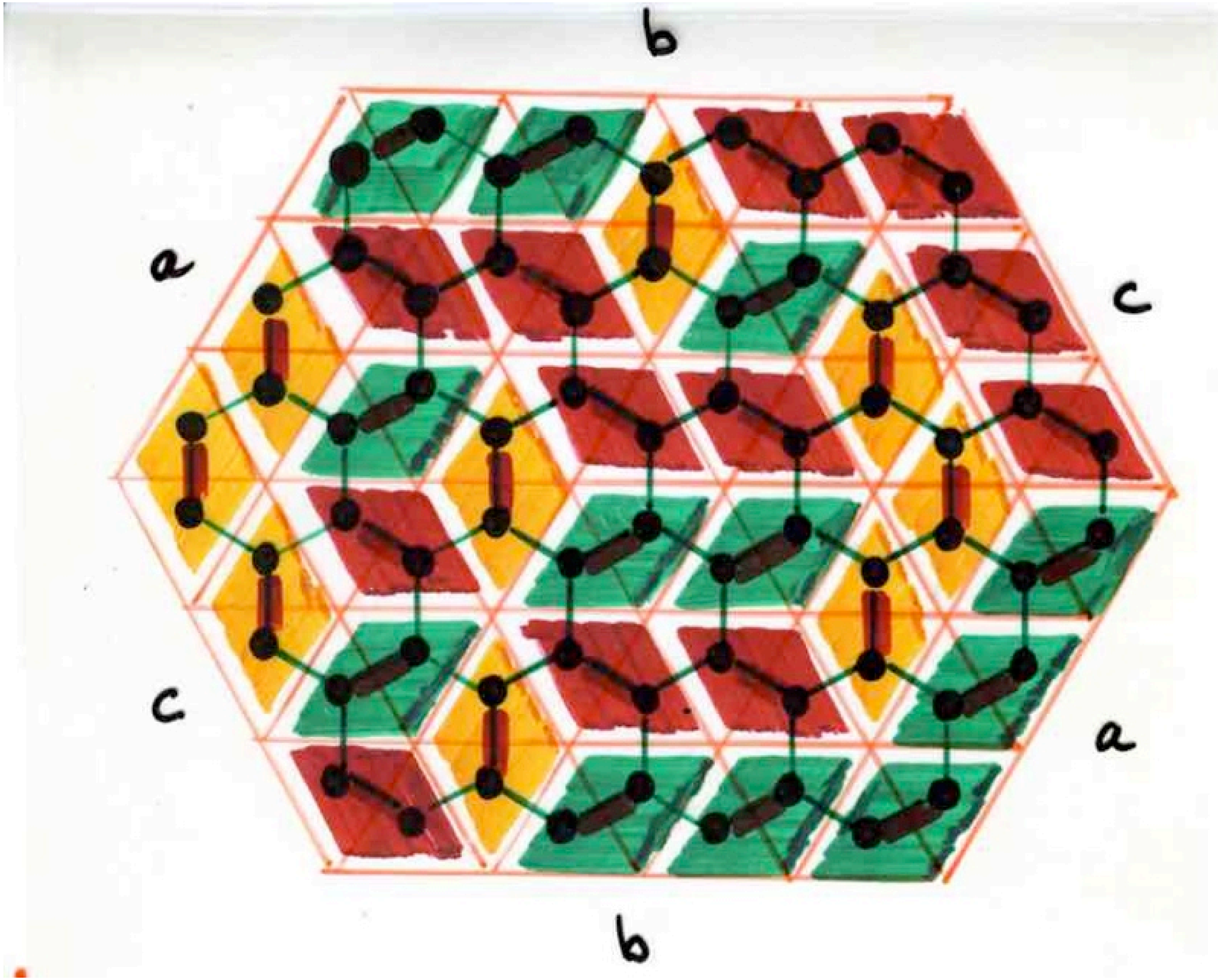


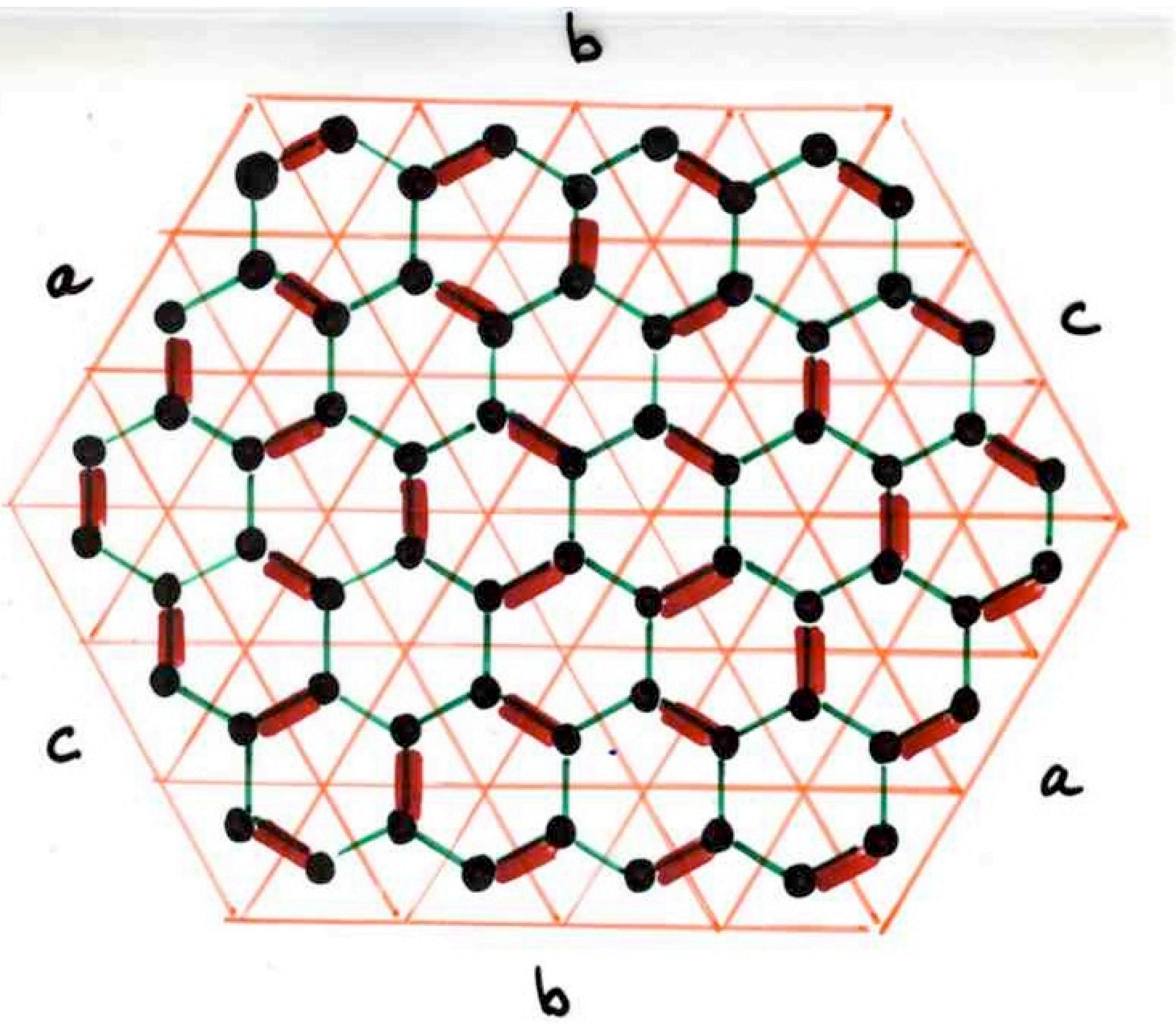
Elkies,
Kuperberg,
Larsen,
Propp
(1992)

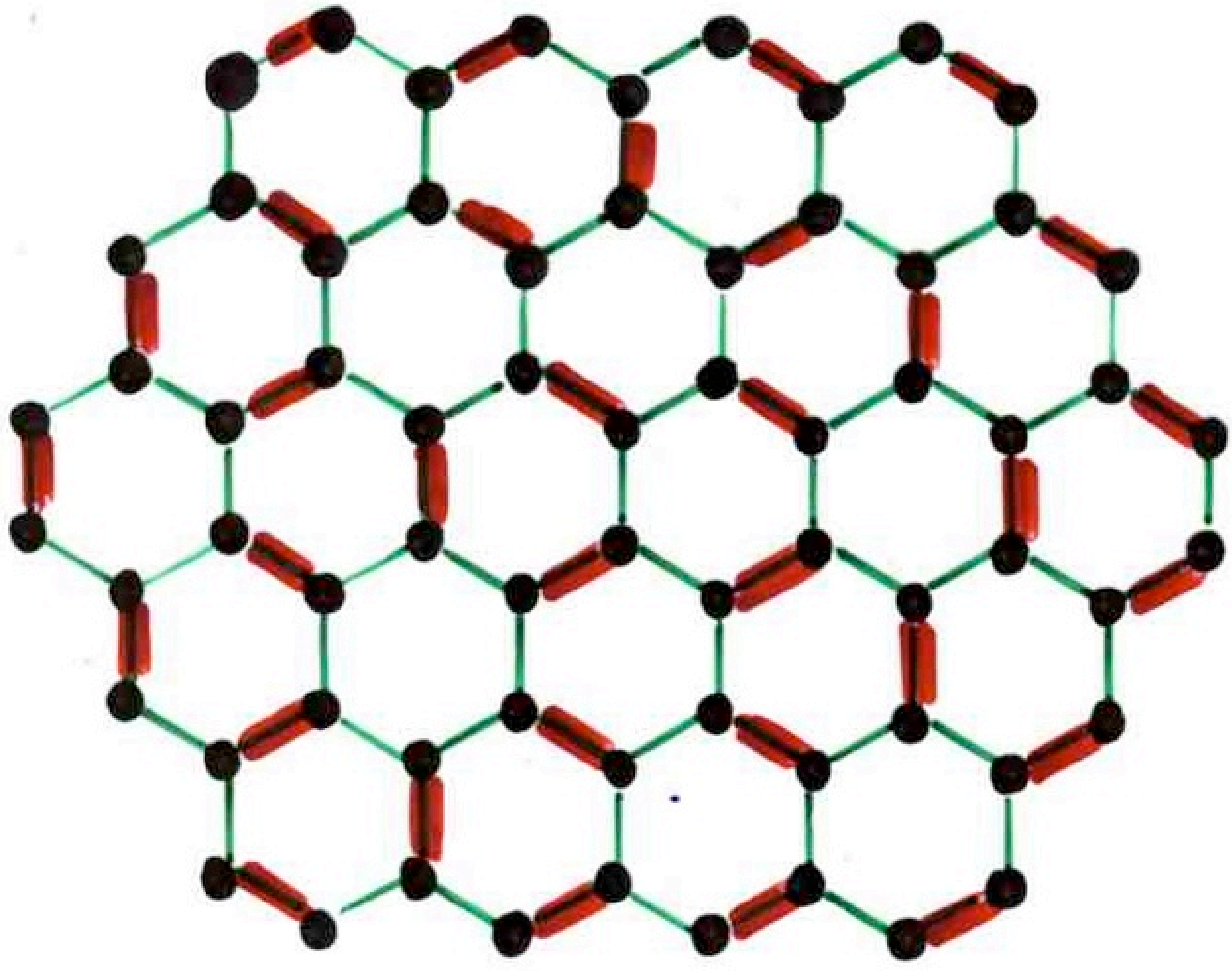
Matchings


b
a
c
c
a
b




b
a
c
c
a
b


b
a
c
c
a
b



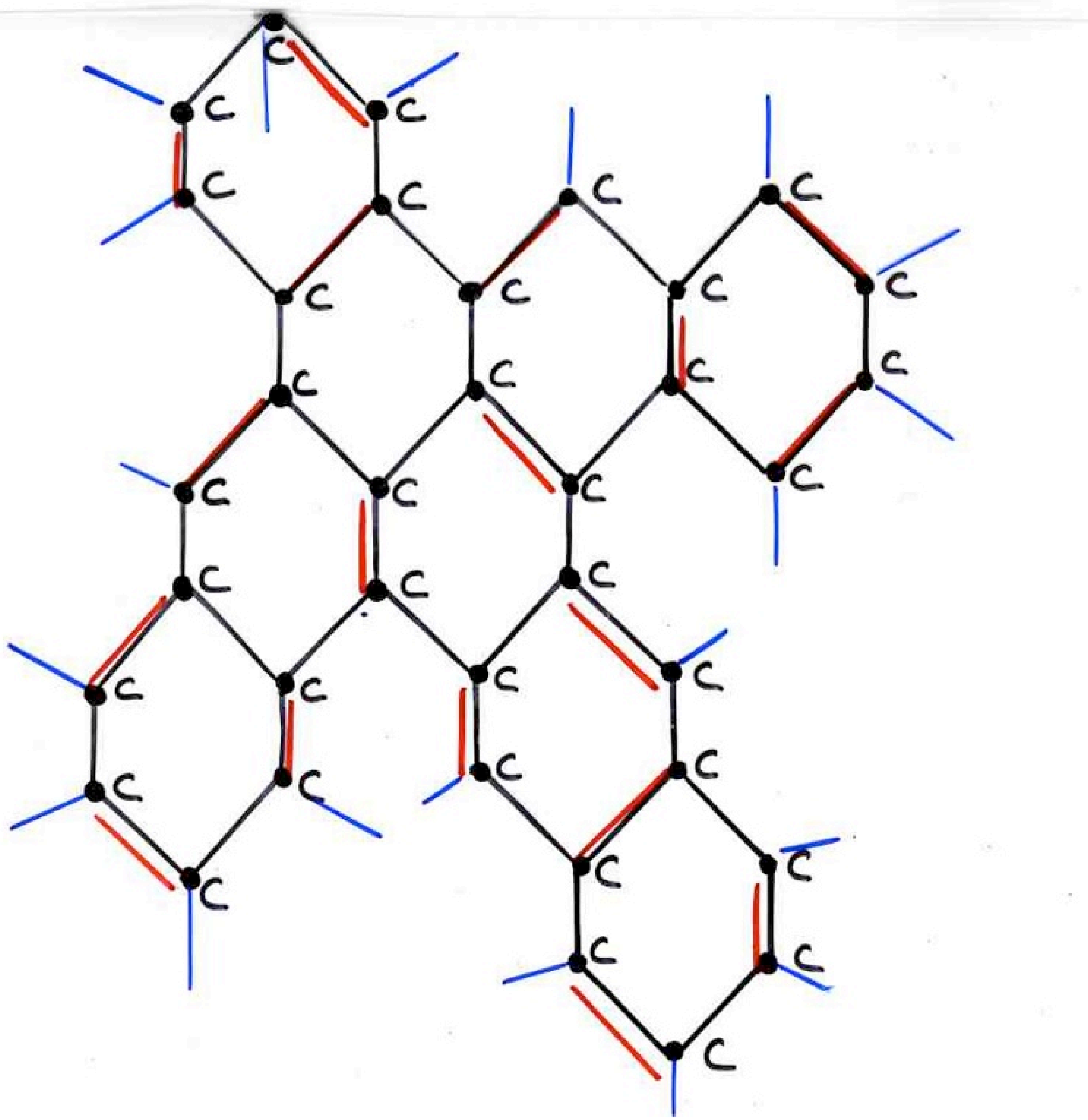
- dénombrement de
  couplages parfaits
- graphe planaire

  méthode du Pfaffien

- modèle d'Ising (1925)

  Kasteleyn, Fisher, Temperley
  (1961, ...)

  Onsager (1944)



Pfaffian

Pfaffians

Soichi Okada (1989)

Basil Gordon (1971)

John Stembridge (1990)

Pfaffian

Pfaf (1815)
Caianiello (1953, 59)
Wick

$T = (a_{ij})$ $\quad 1 \leq i < j \leq 2k$

ex:

$$\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{34} \end{vmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$$

Proofs and Confirmations
The story of the alternating sign matrix conjecture

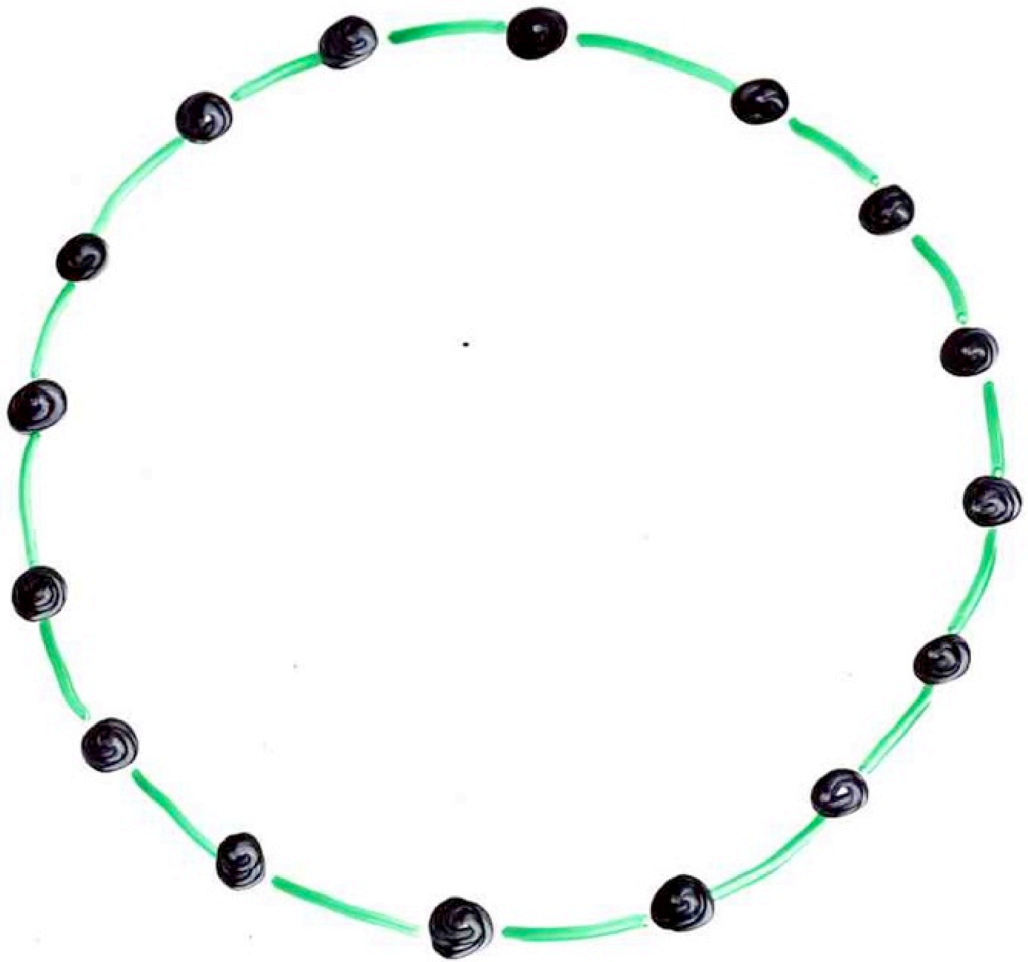
David M. Bressoud
Macalester College
Saint Paul, MN

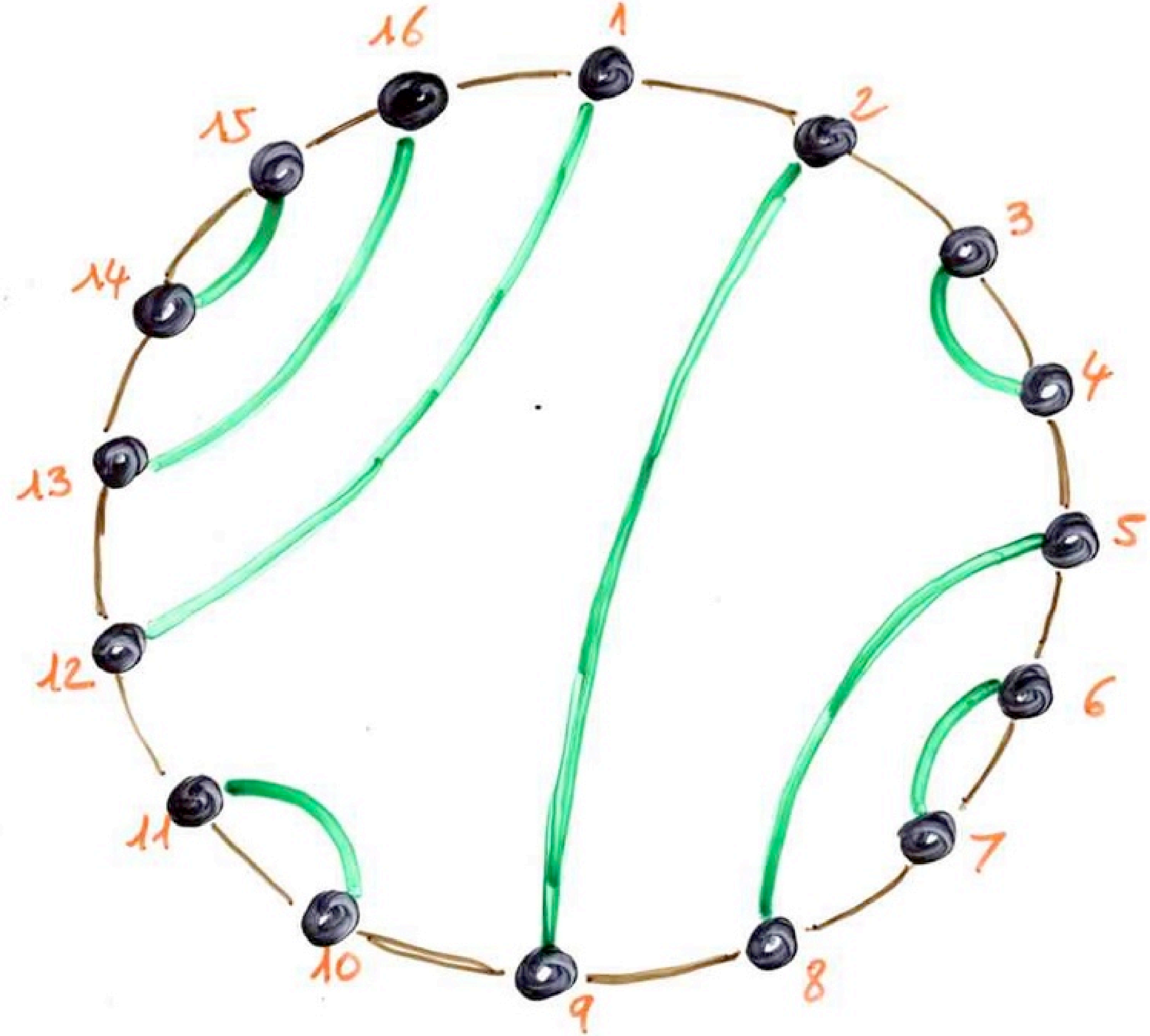
July 28, 1997

Razumov -Stroganov conjecture

(2000, ...)

Markov chain
on
chord diagrams

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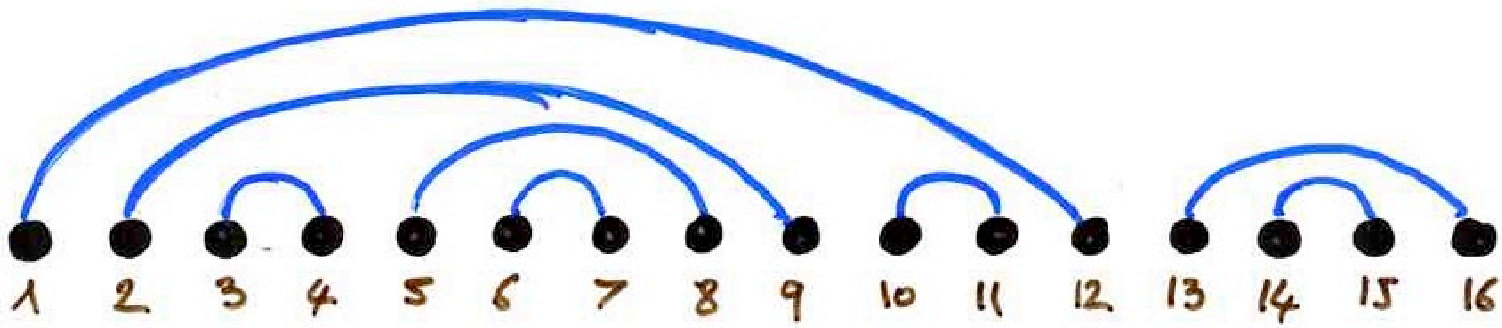
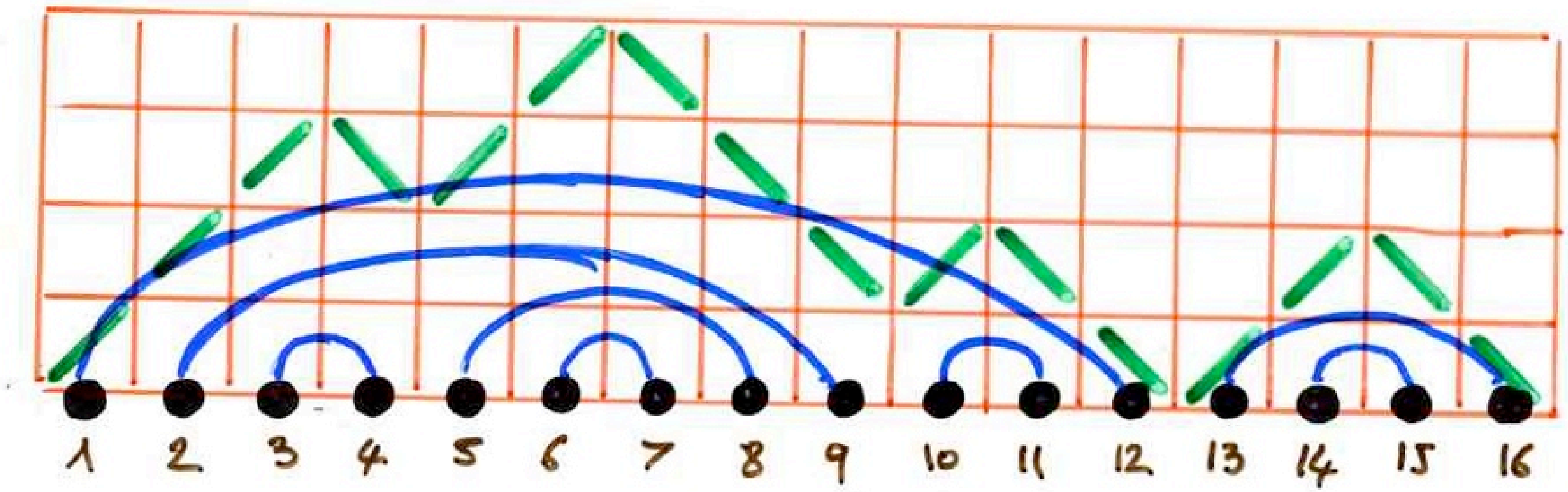
16


1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16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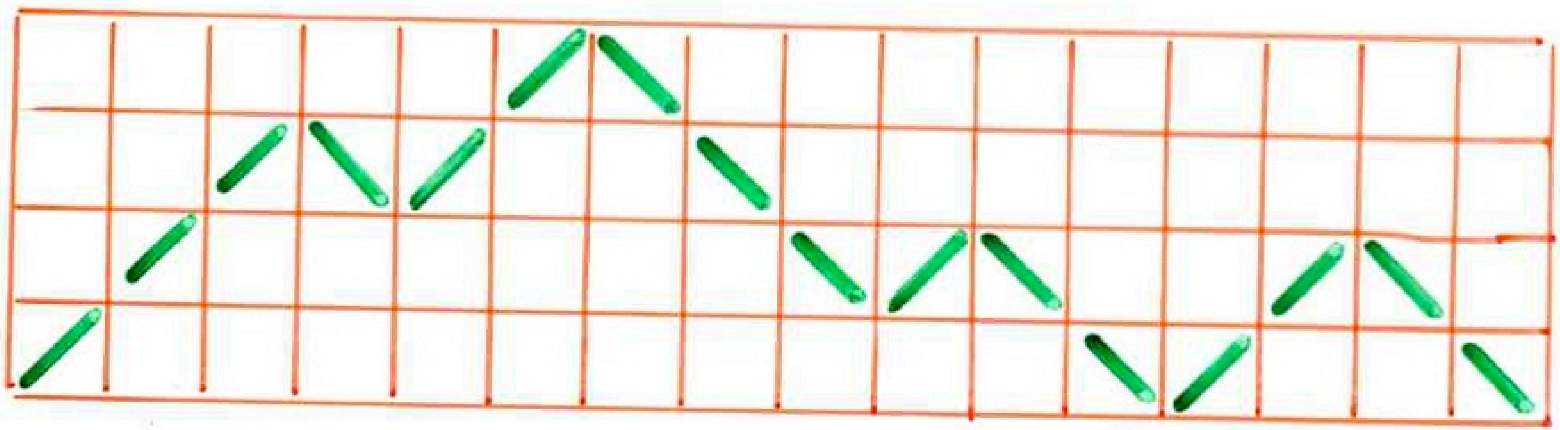

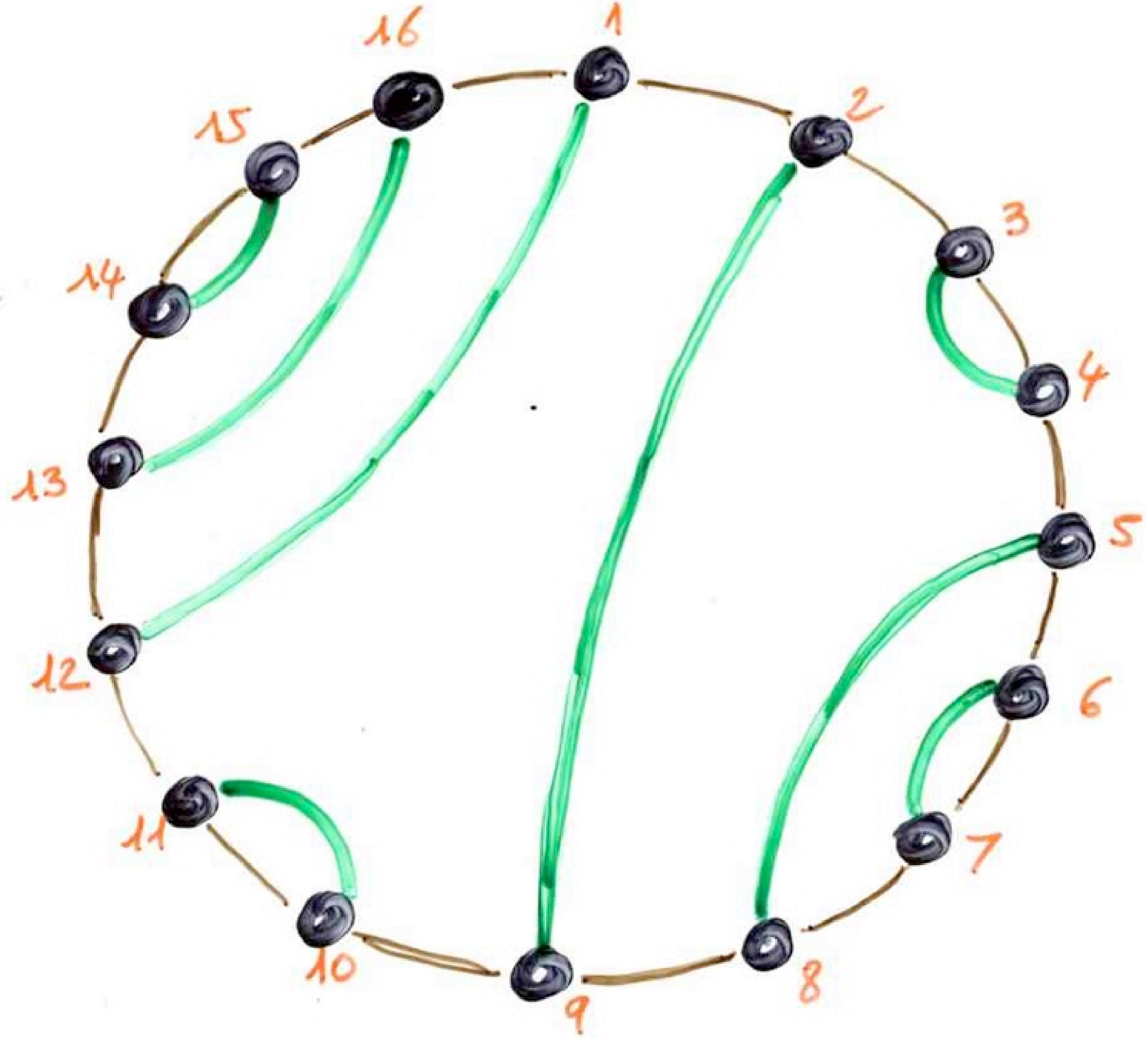
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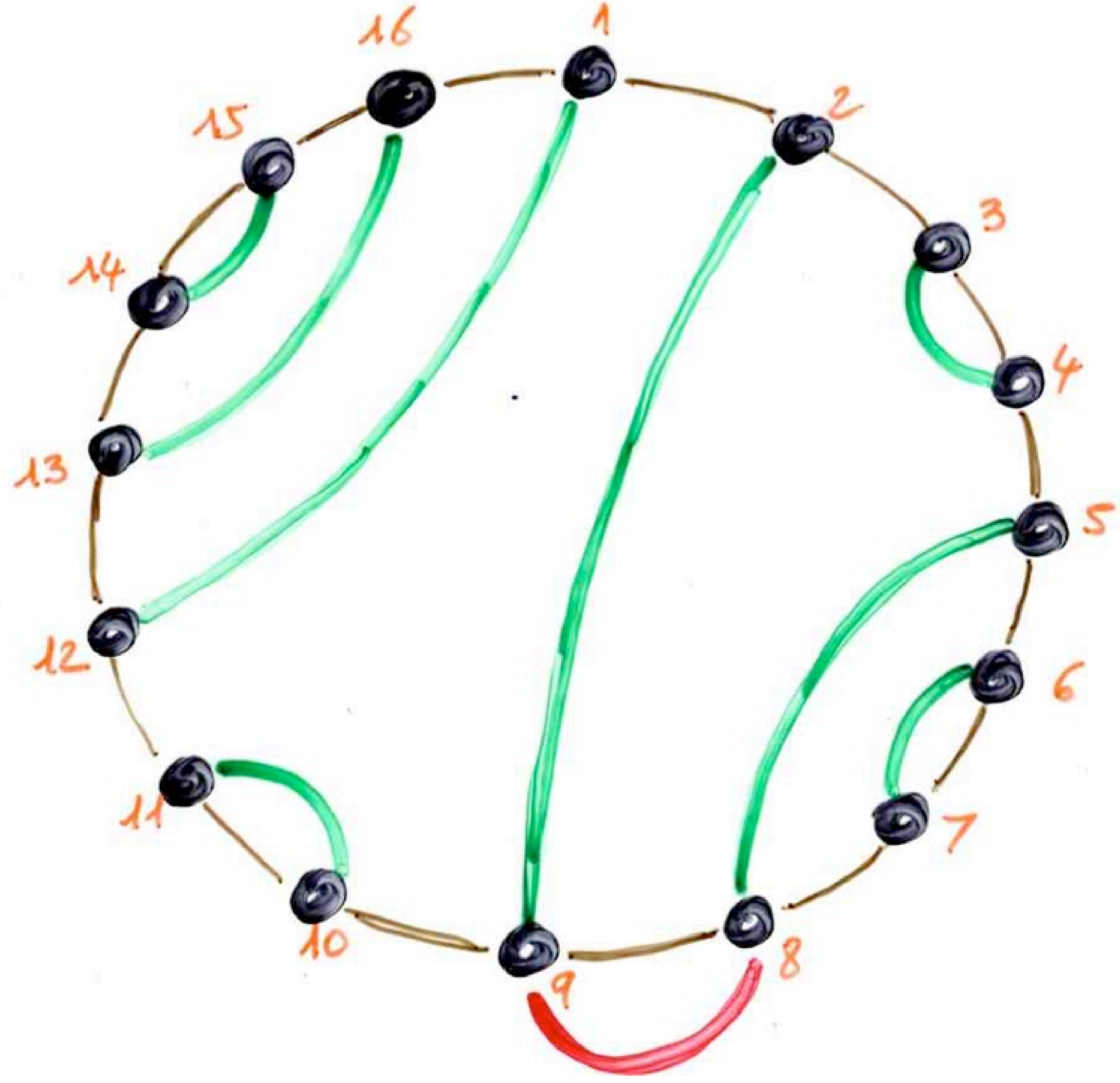
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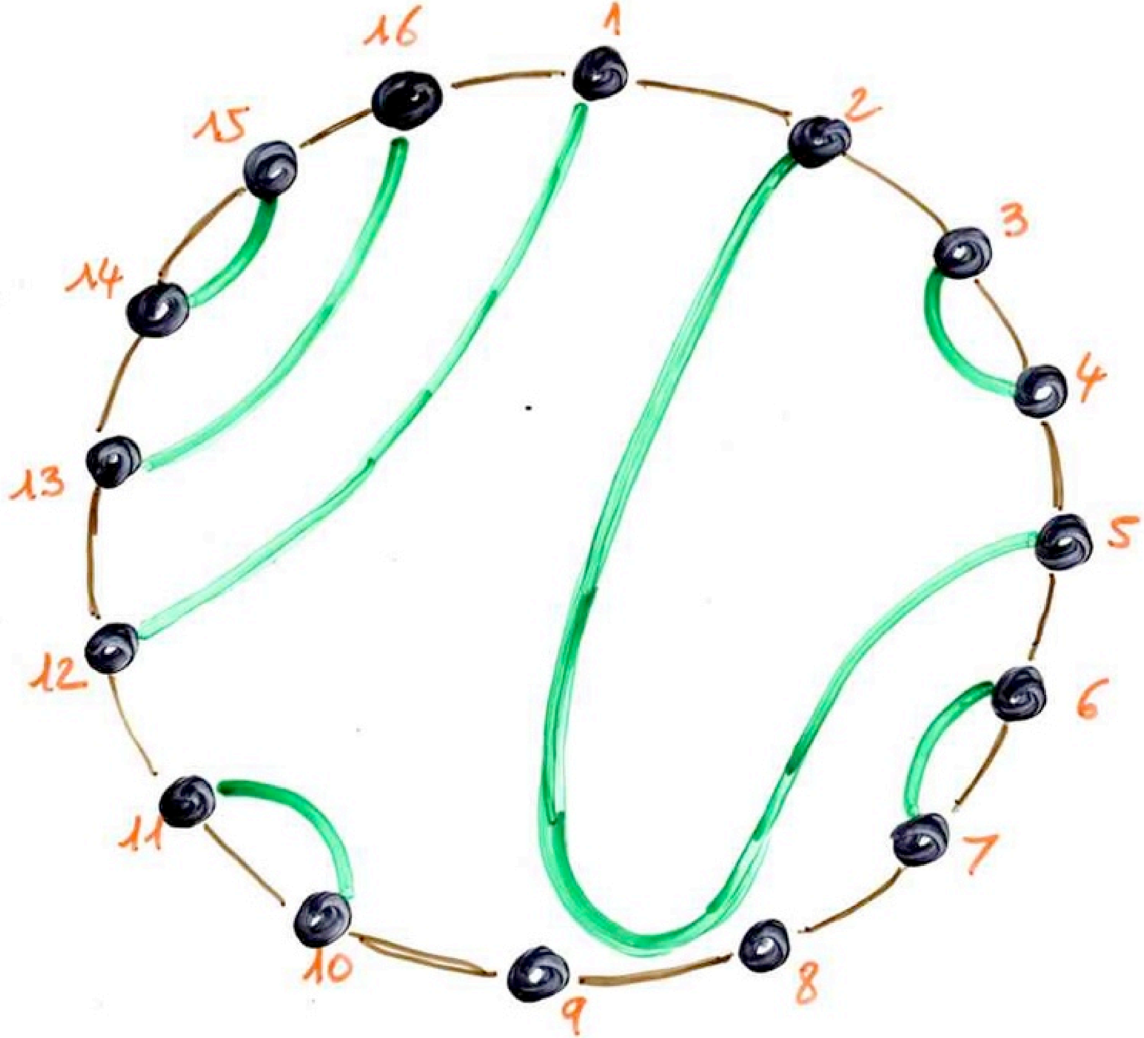
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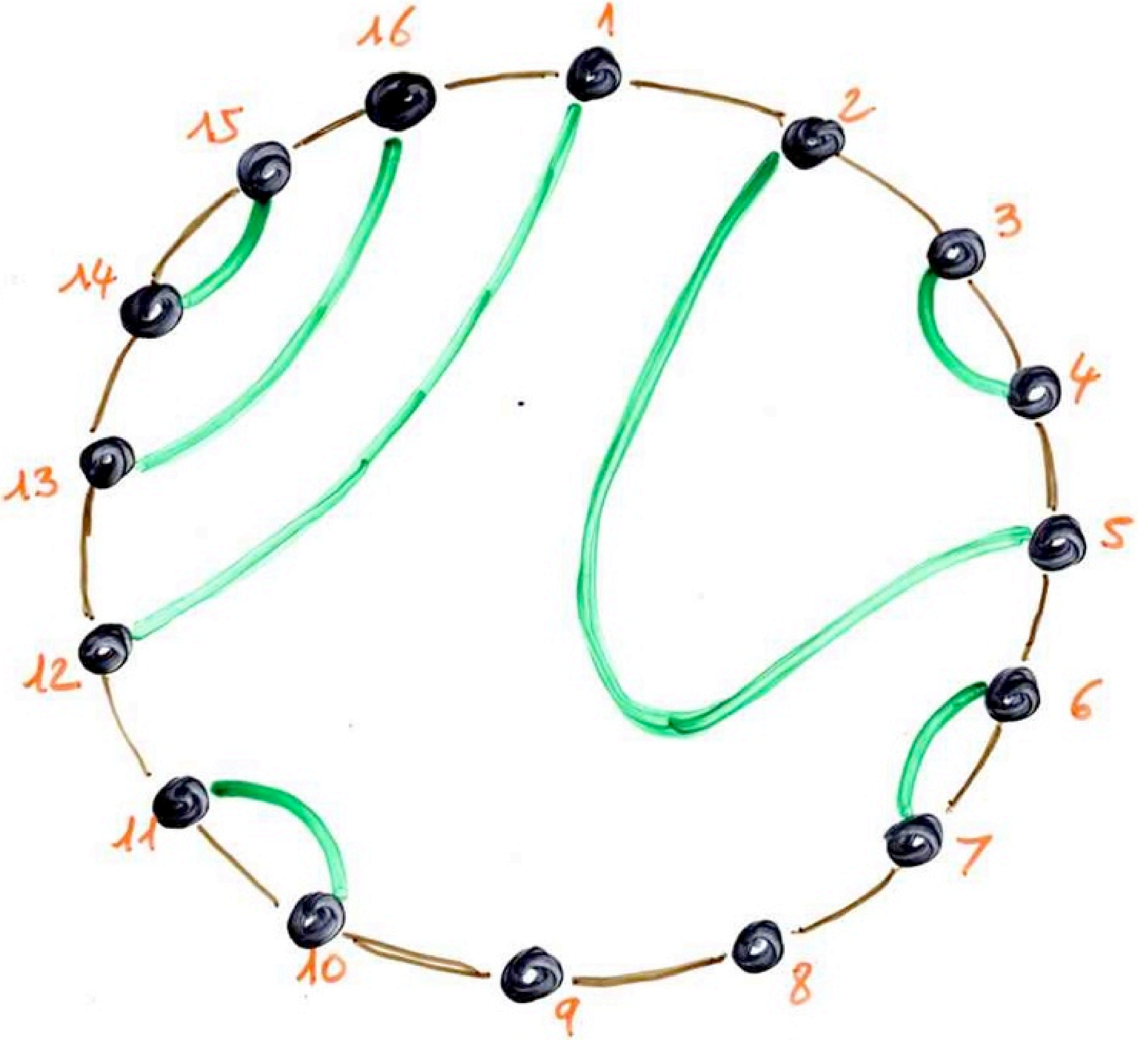
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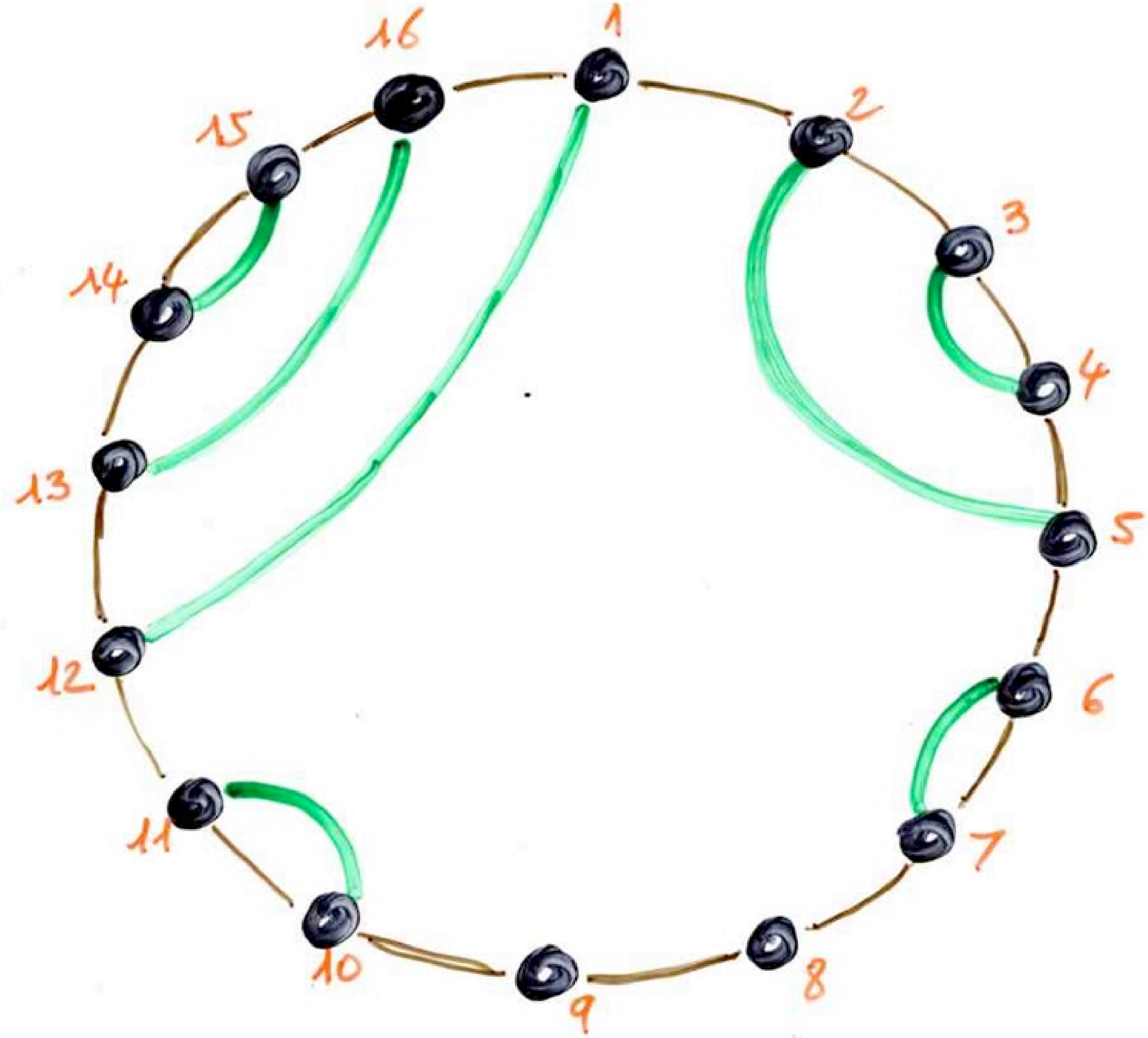
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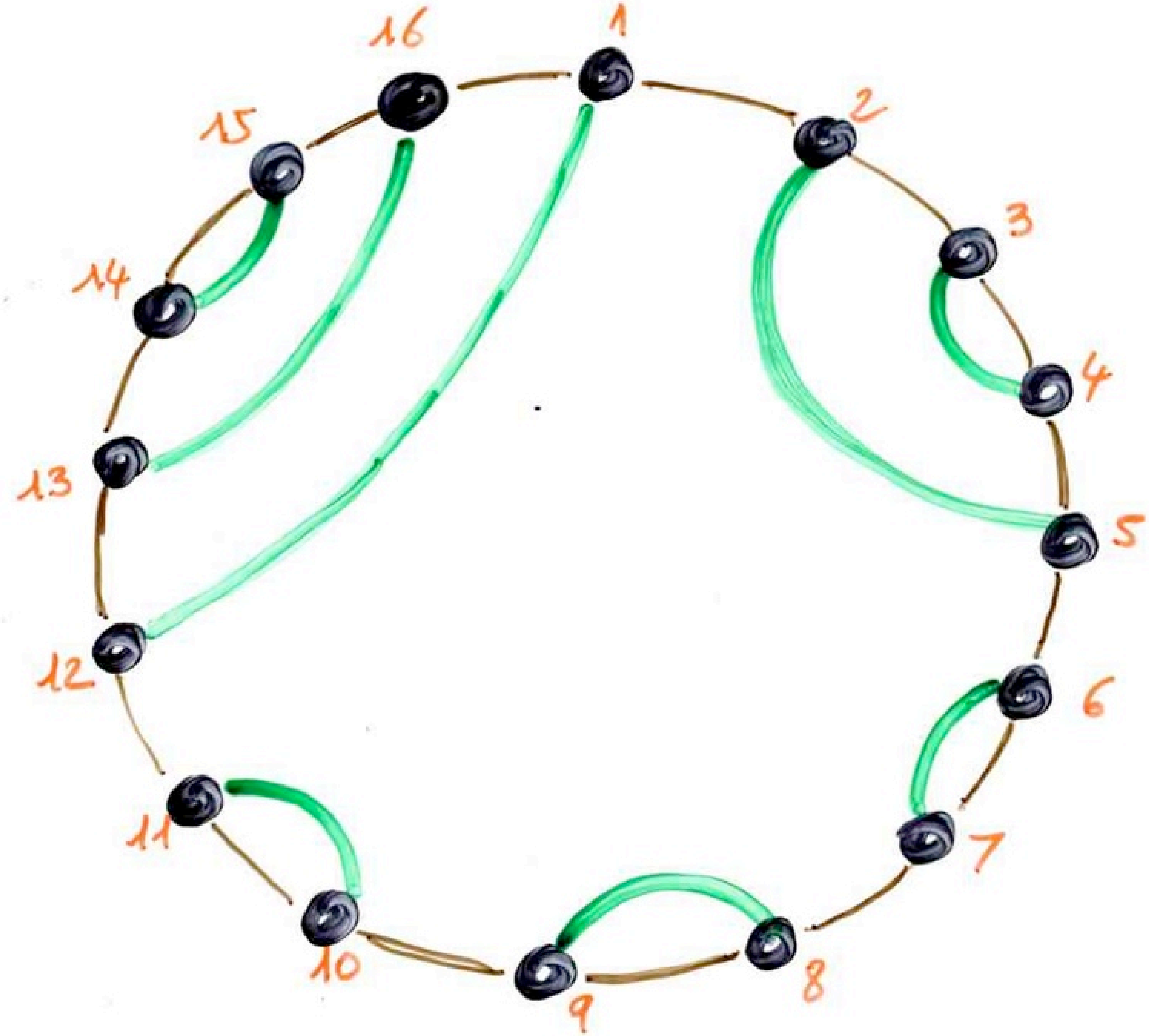
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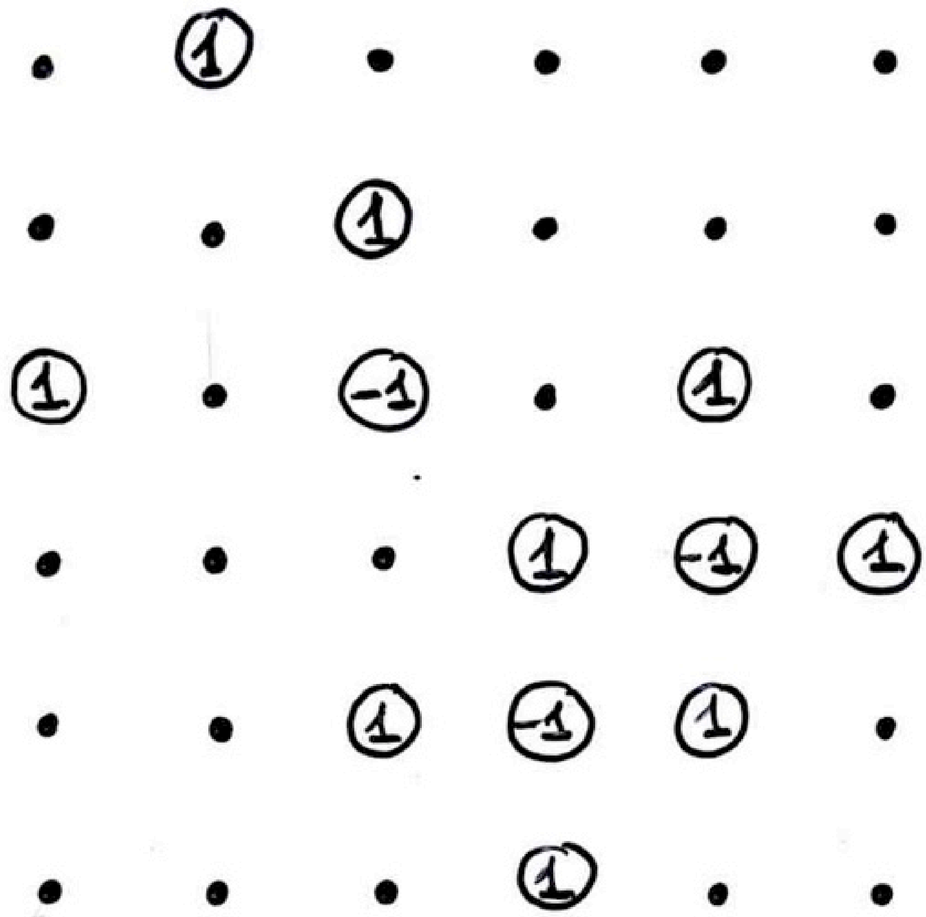
stationary probabilities

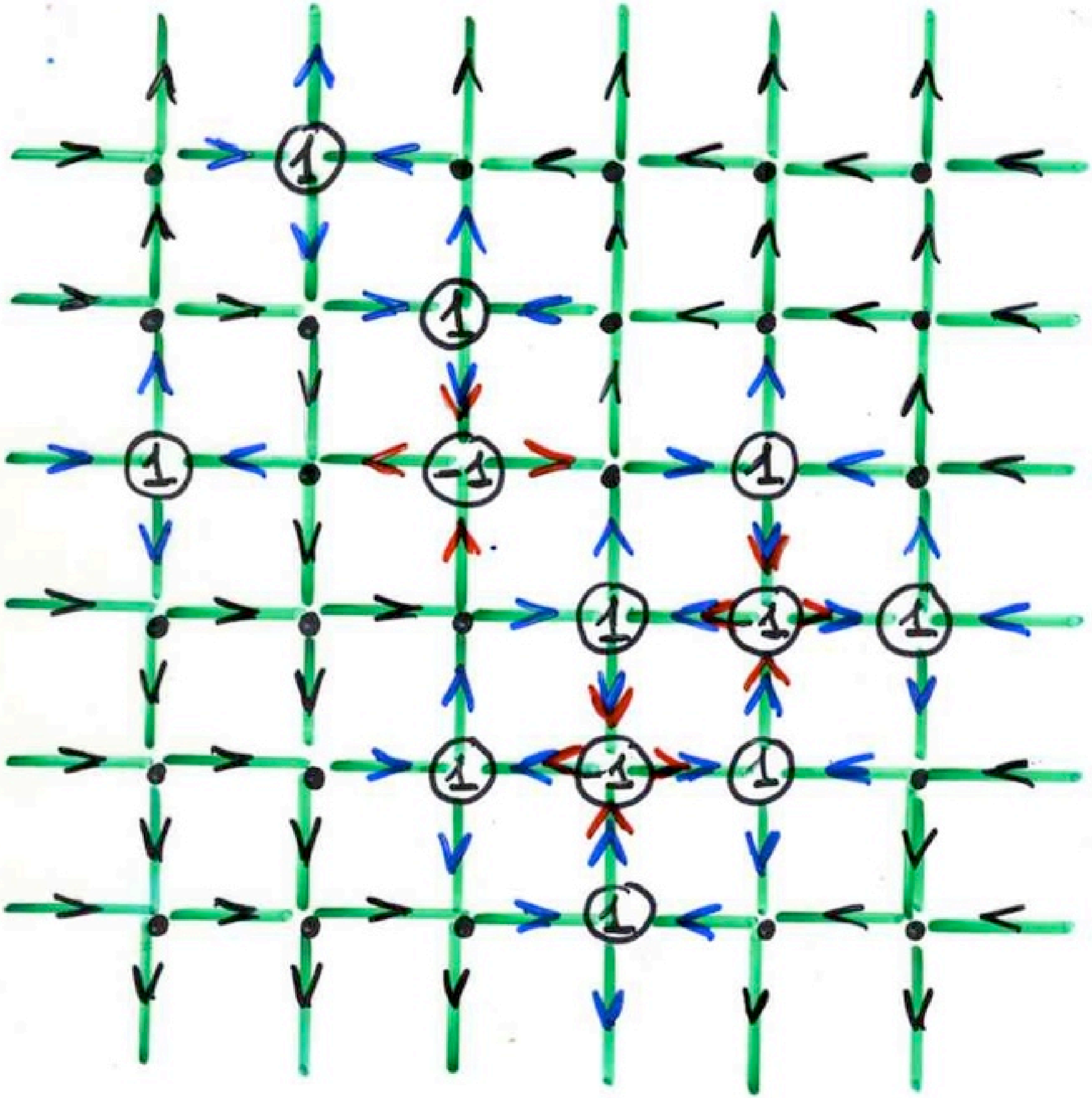
FPL

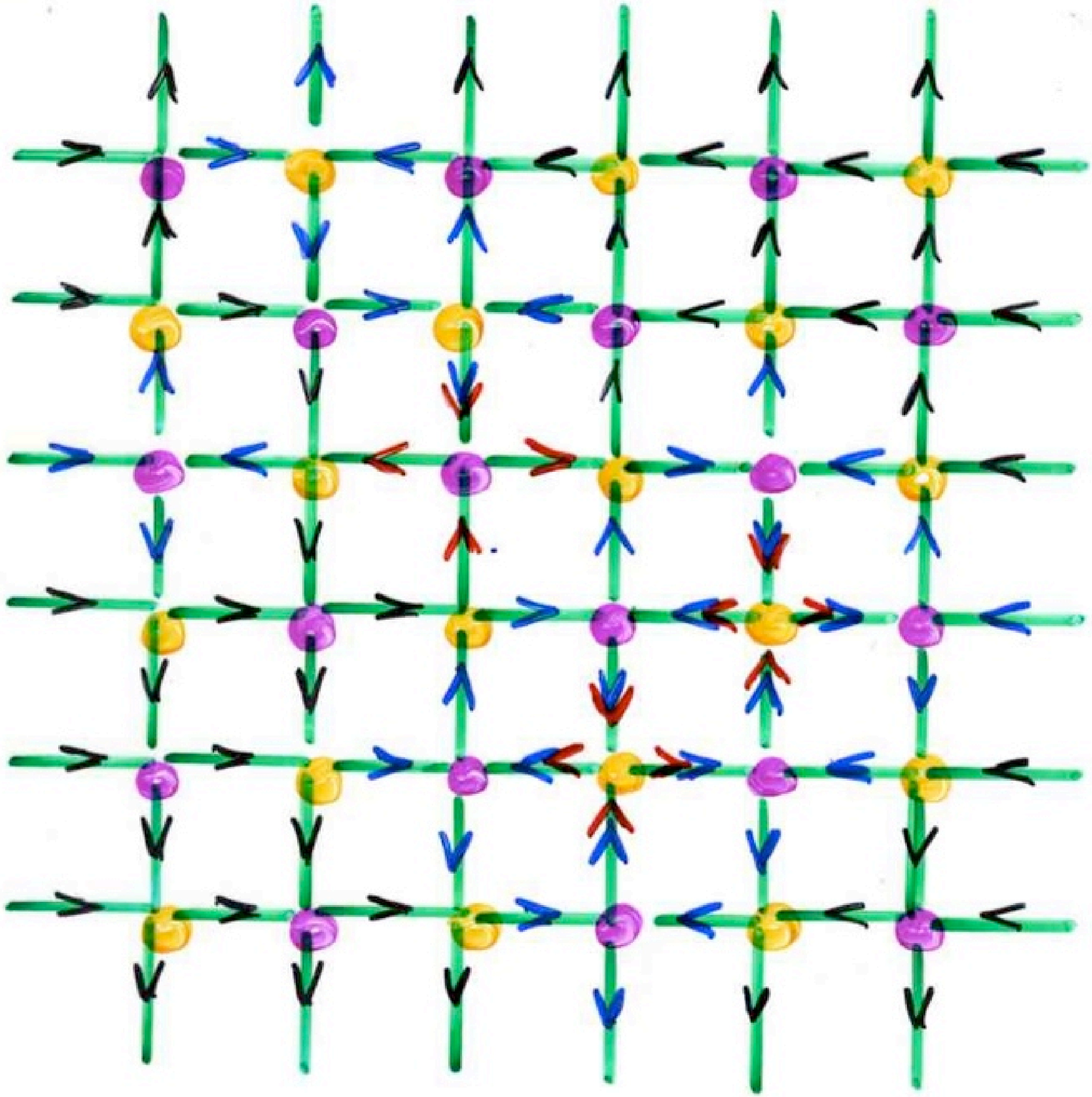
"Fully packed loop configurations"

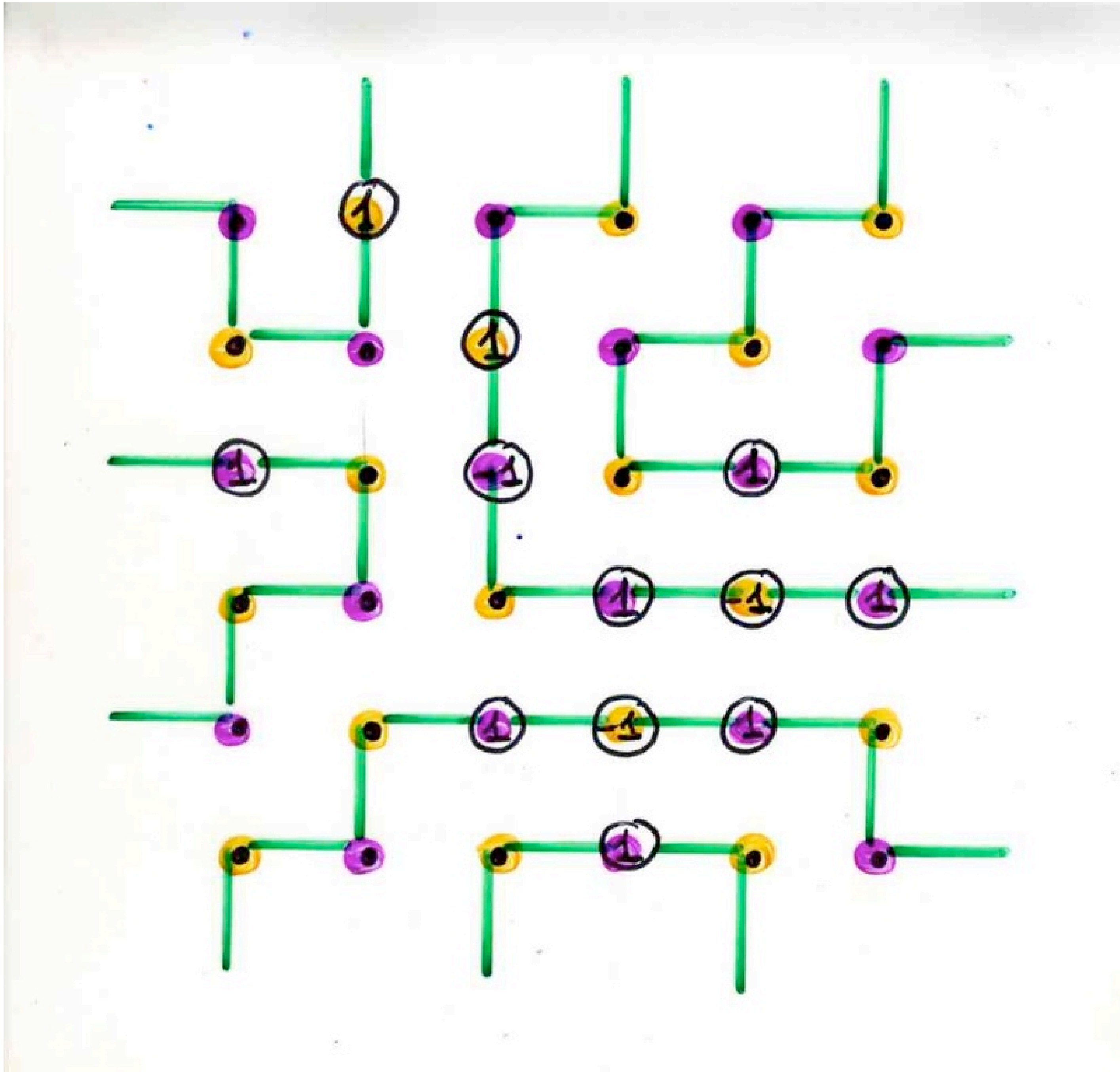
The bijection AMS FPL



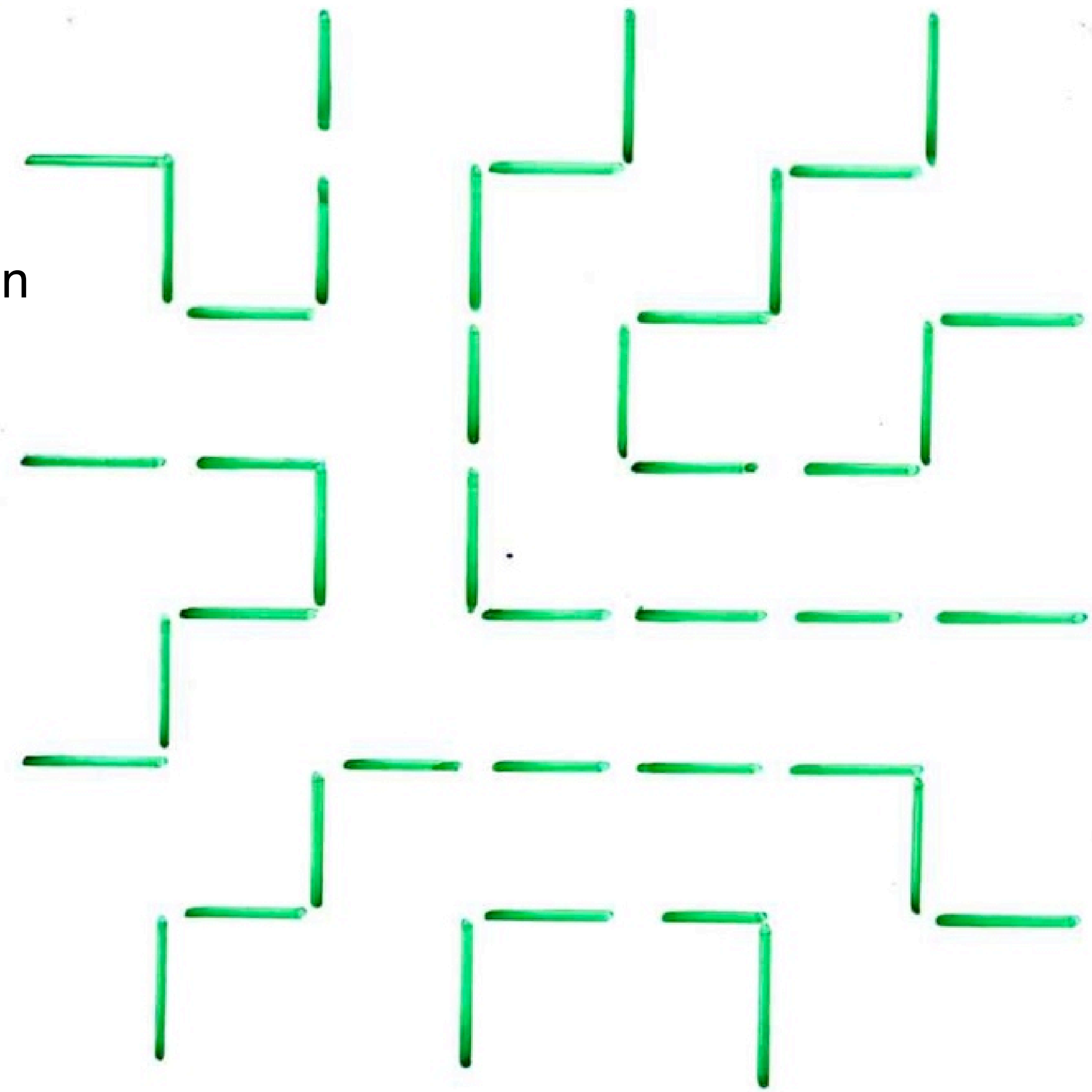
The 6-vertex model



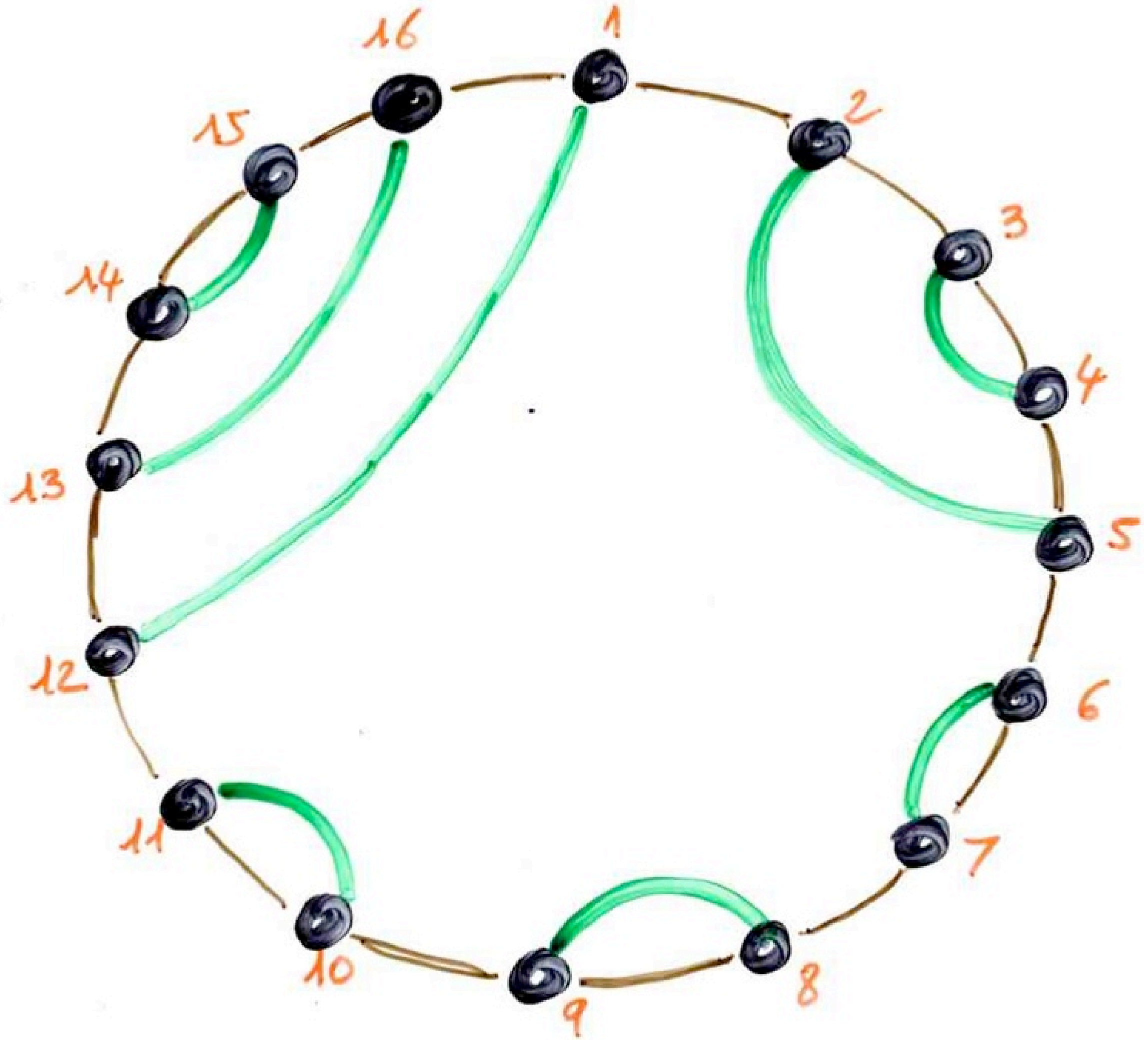



1
1
1
-1
1
1
-1
1
1
-1
1
1

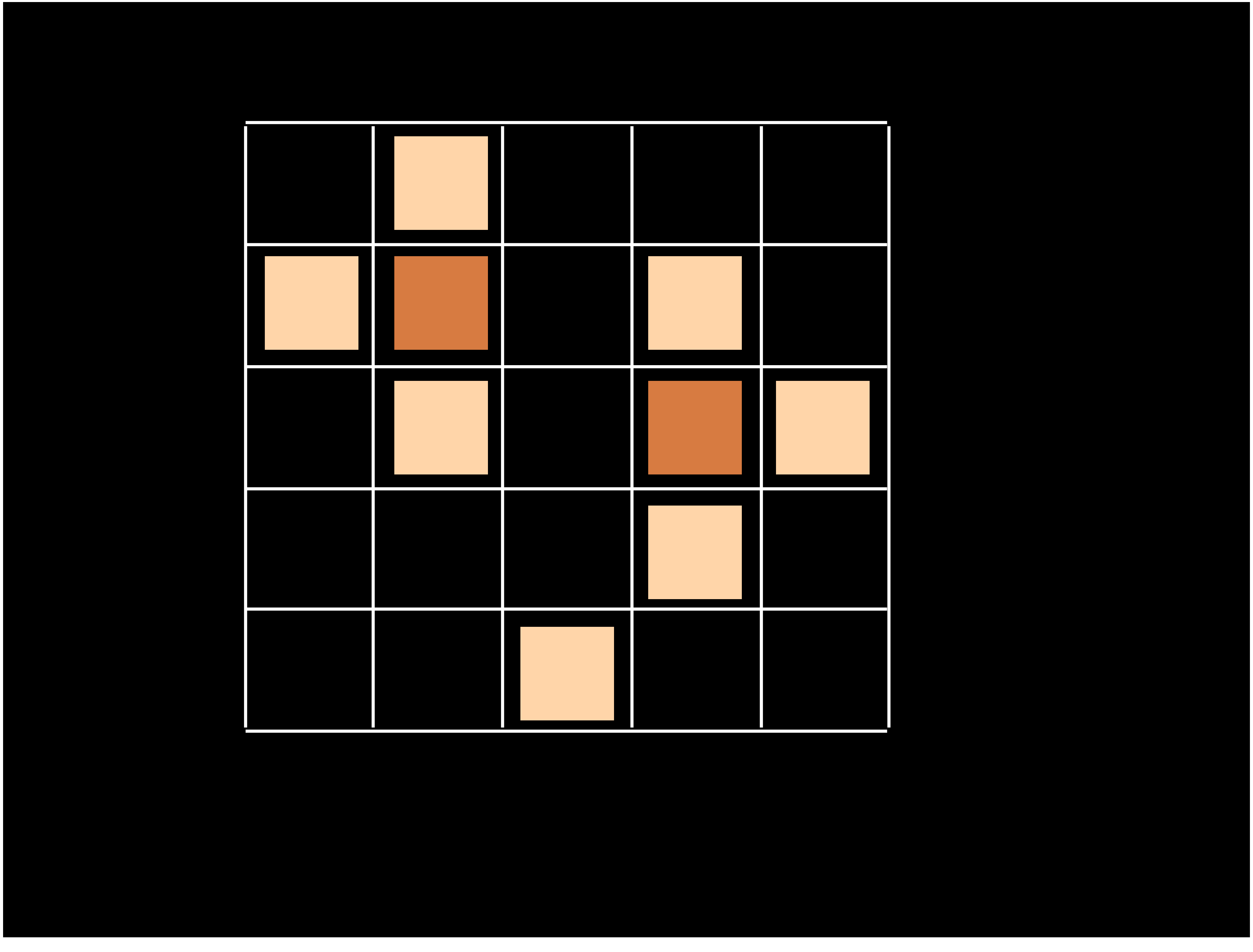
FPL
“Fully
Packed
Loop”
configuration

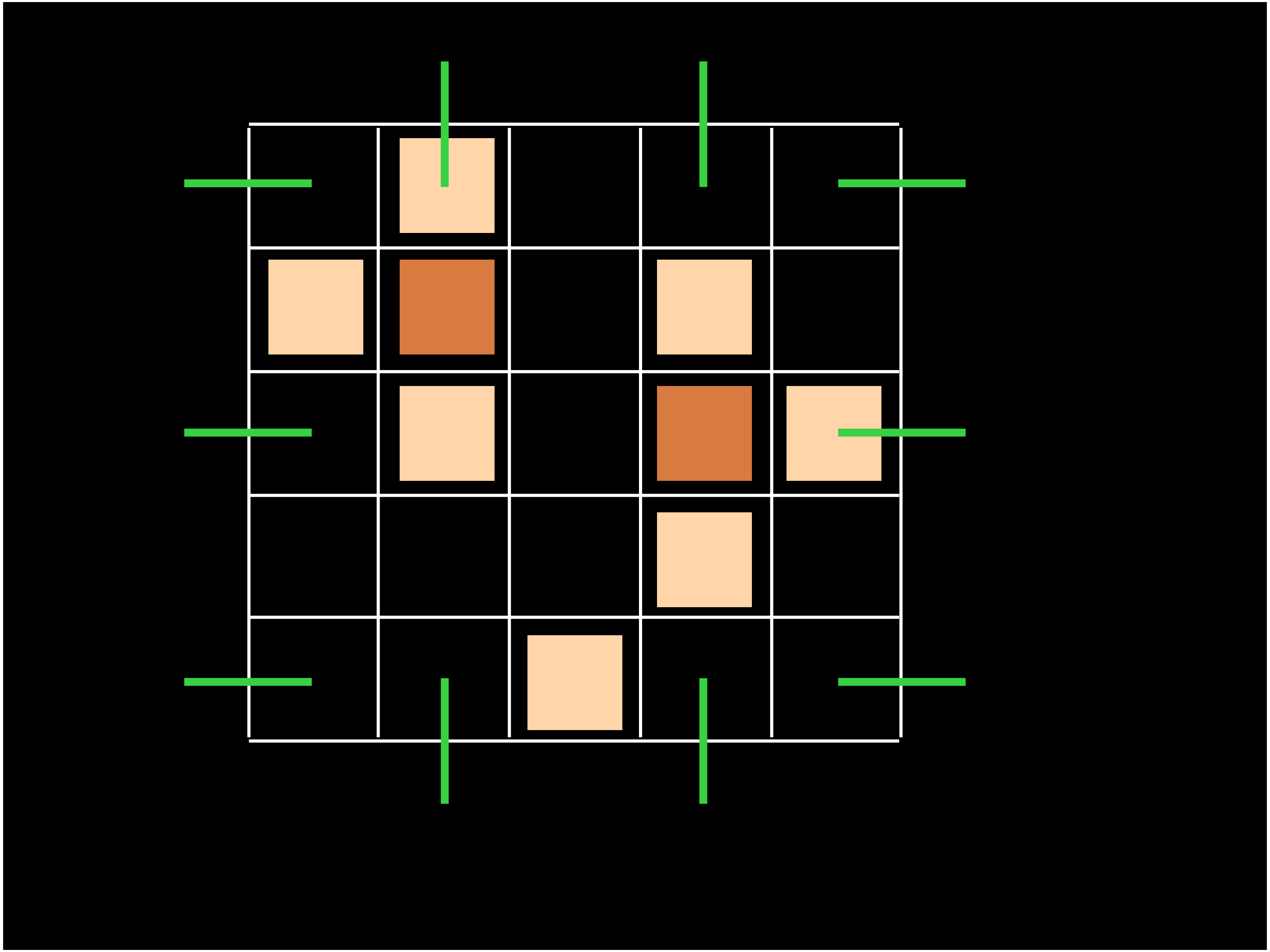


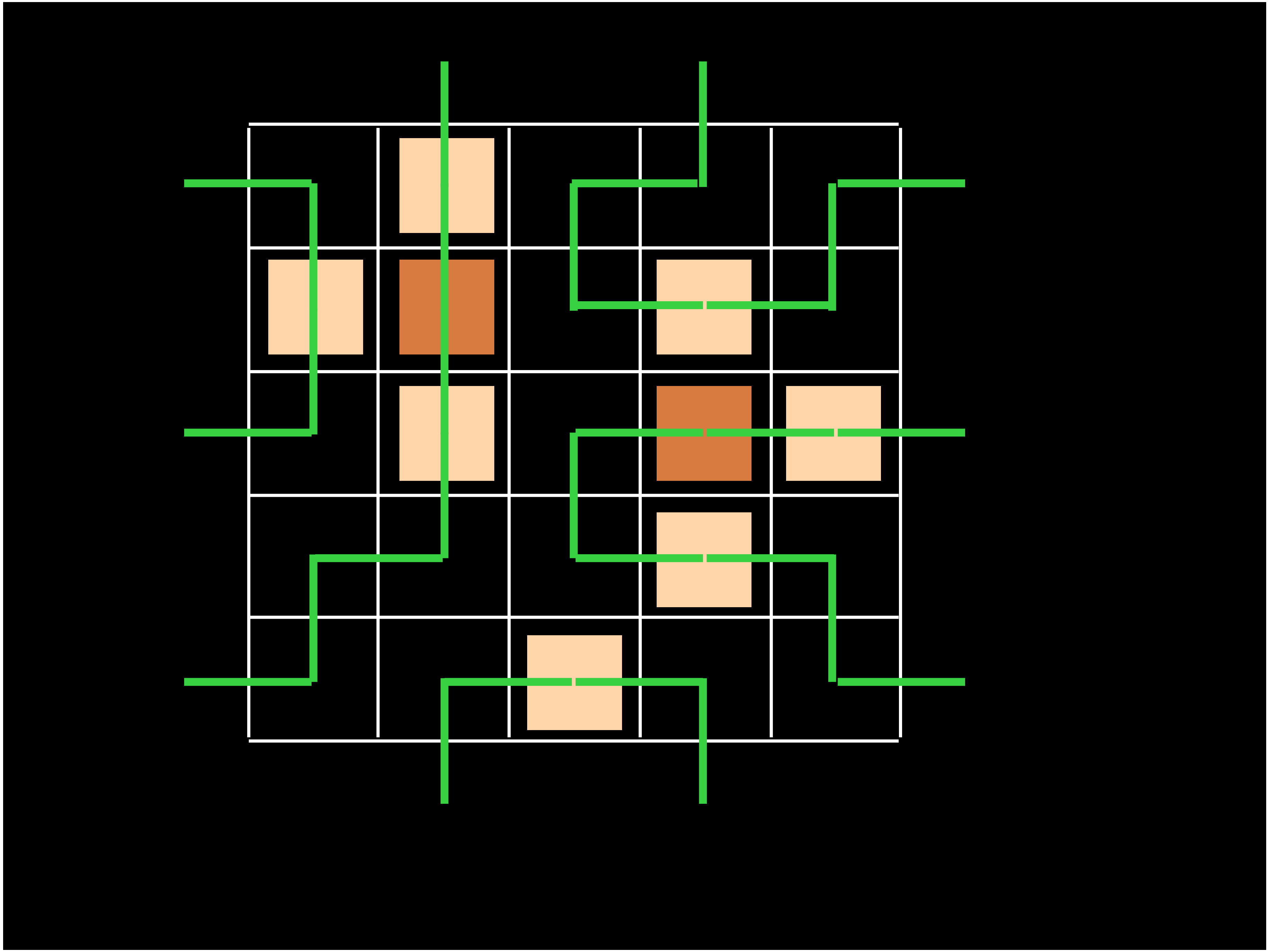
Razumov-Stroganov conjecture



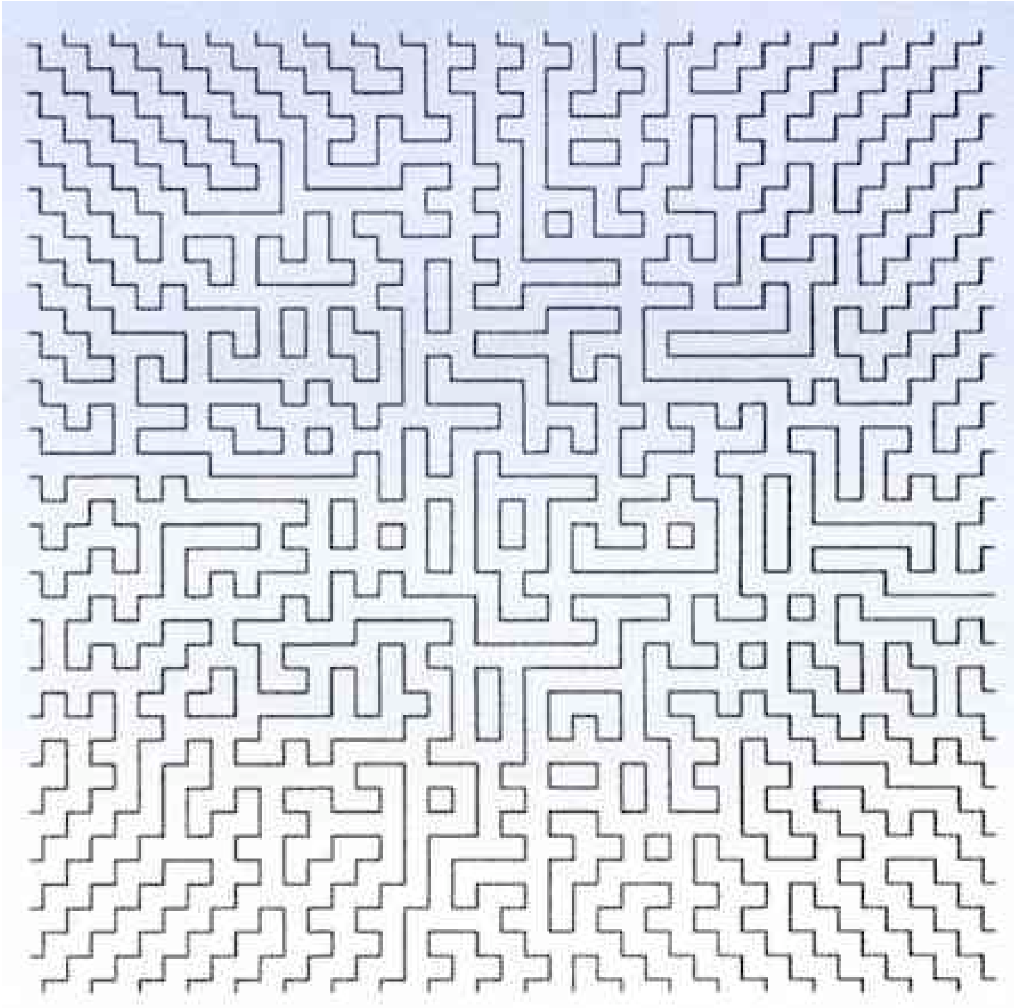
stationary
probabilities

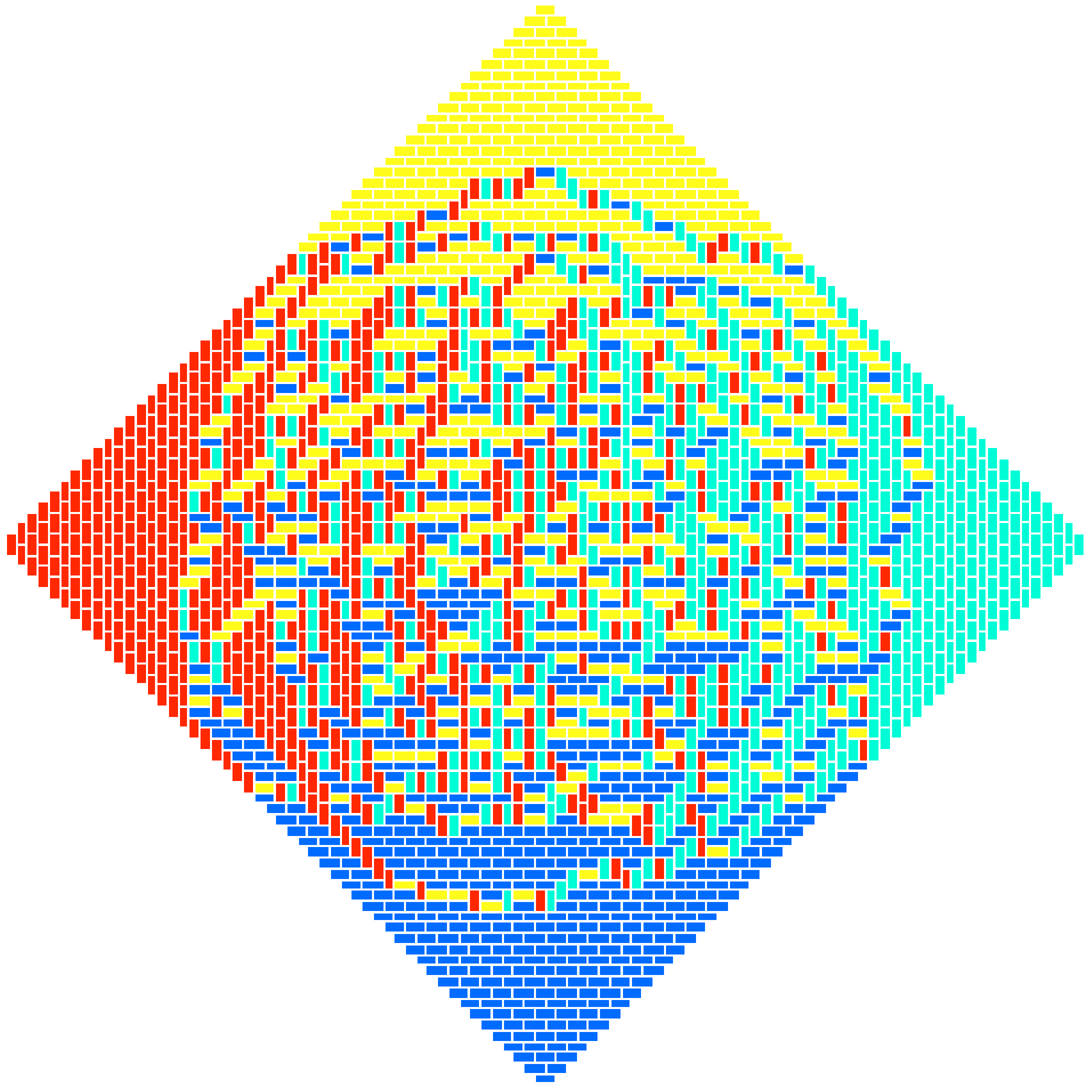


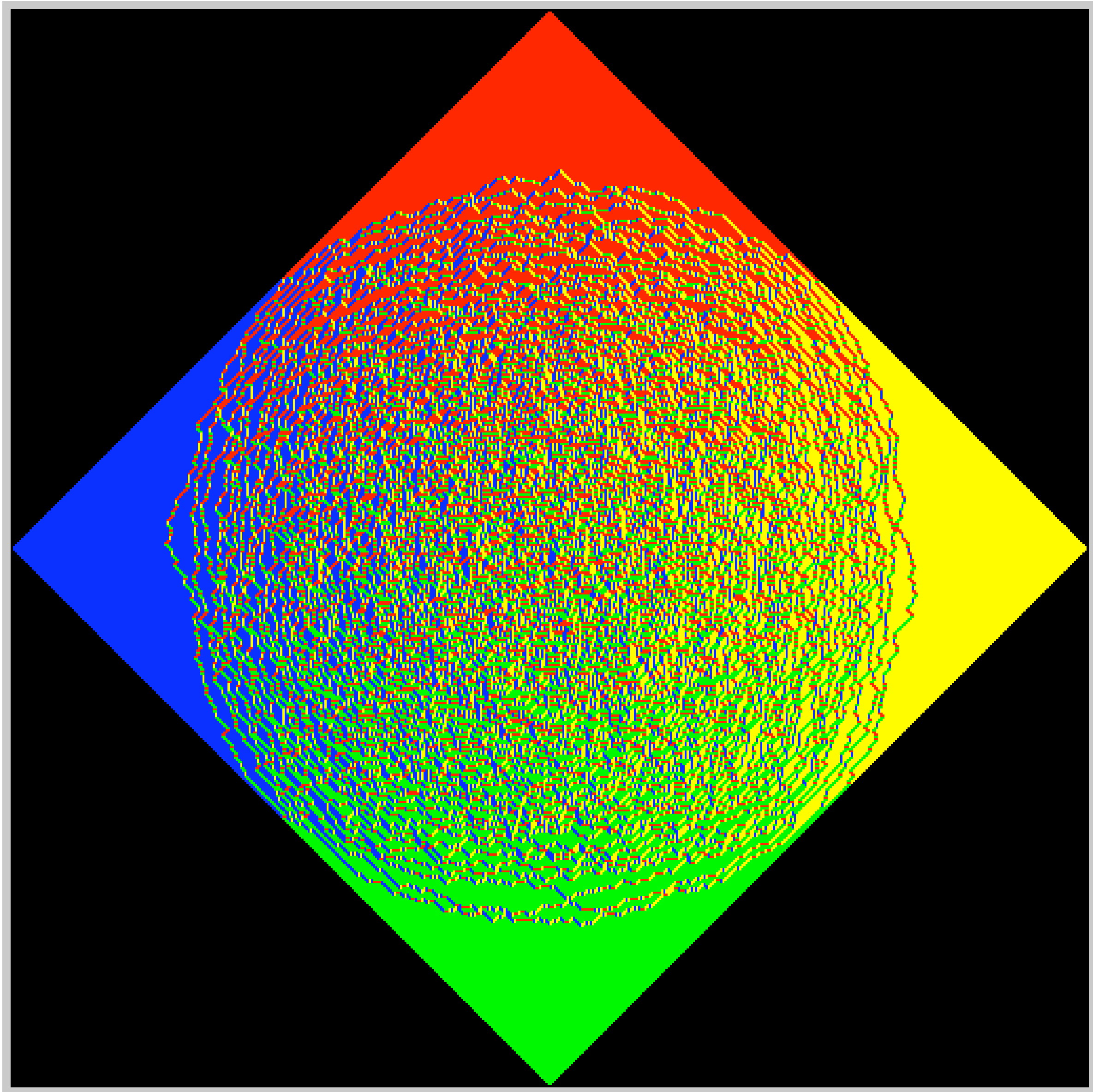




Random combinatorial structures







Andrei Okoukov

avec Richard Kenyon

algebraic approach
with
operators and commutations

$A, A', B, B',$

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

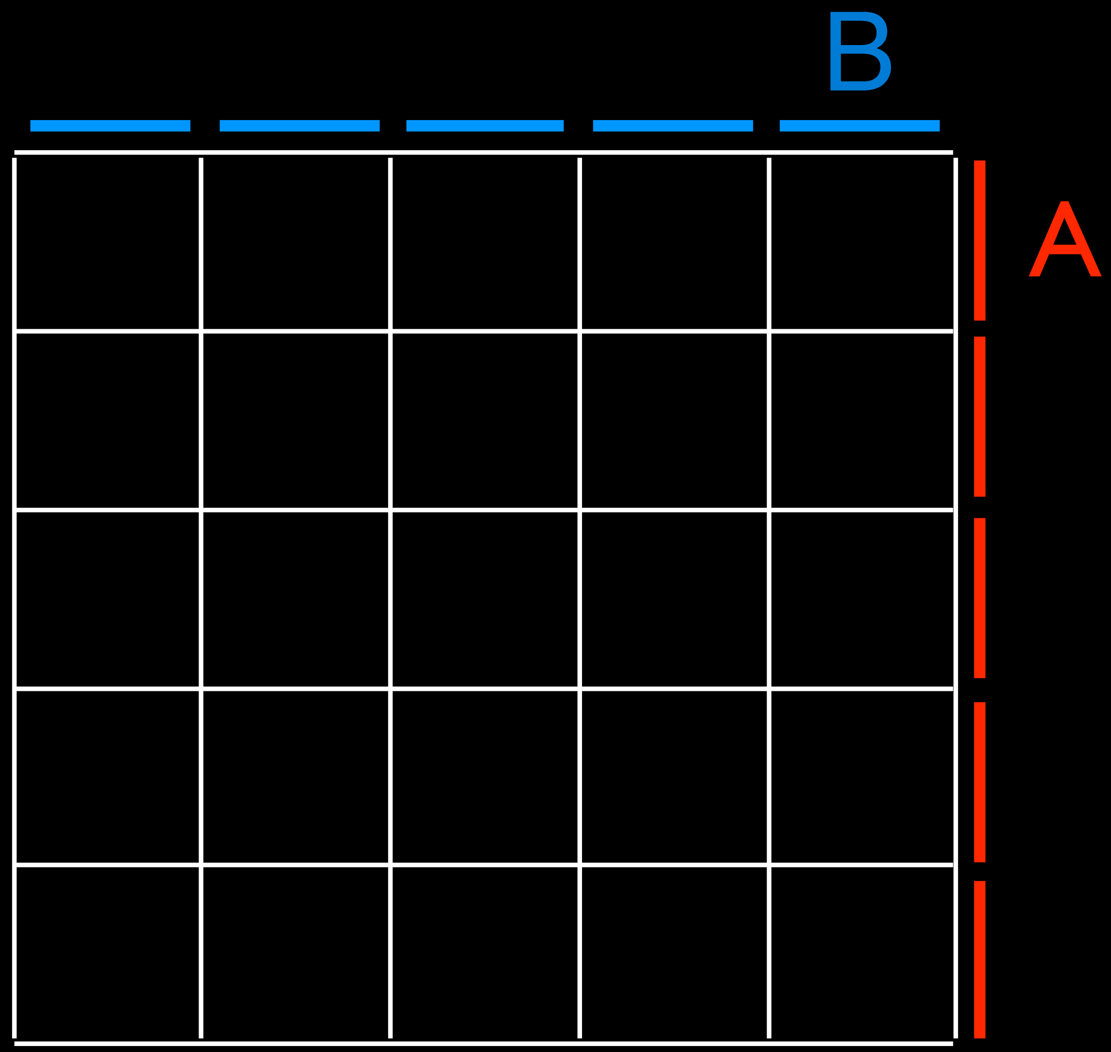
Lemma. Any word $w(A, A', B, B')$ in letters A, A', B, B', can be uniquely written

$$\sum C(u, v; w)\ \underbrace{u(A, A')}_{\text{word in } A,\, A'}\ \underbrace{v(B, B')}_{\text{word in } B,\, B'}$$

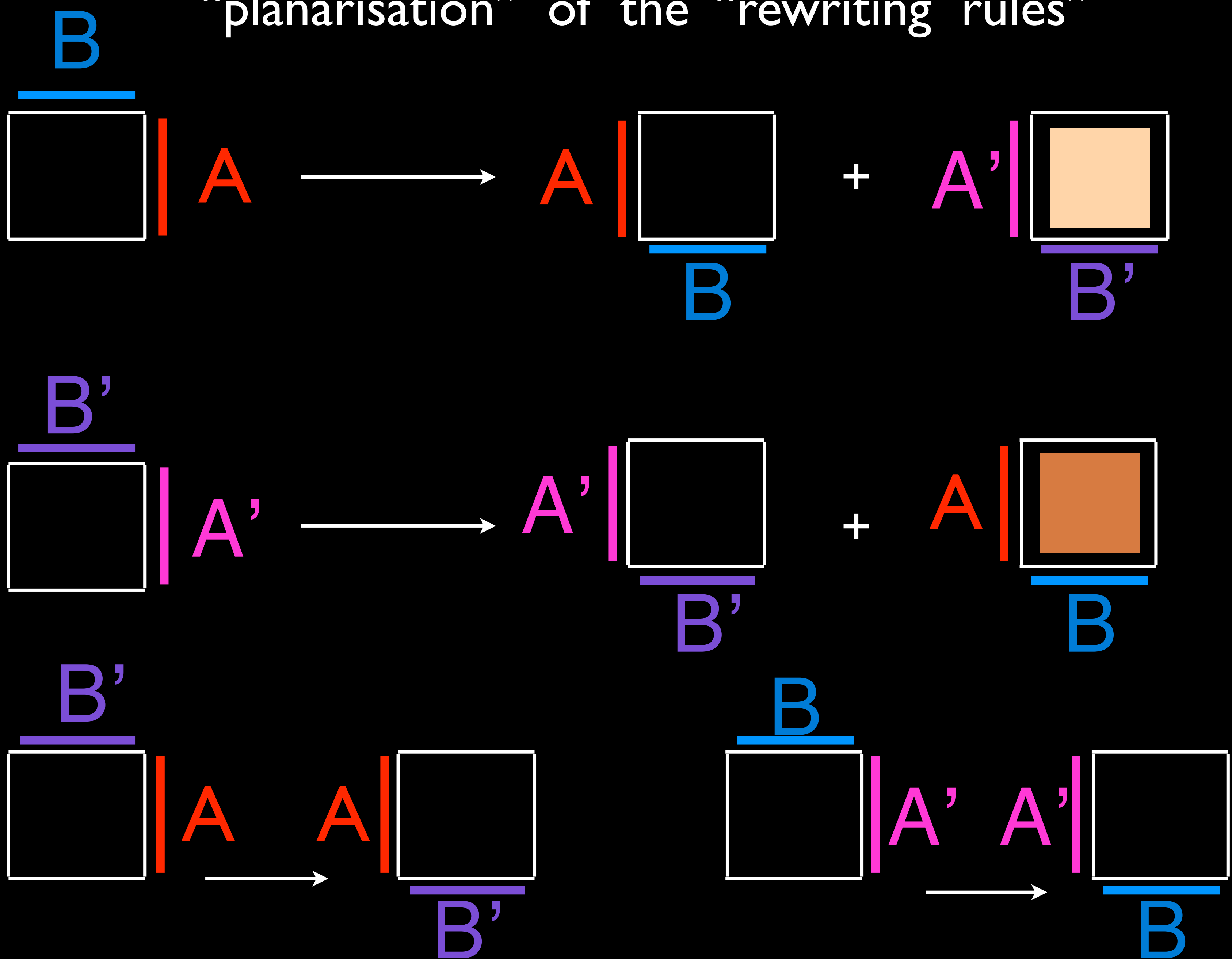
Prop. For $w = B^n A^n$

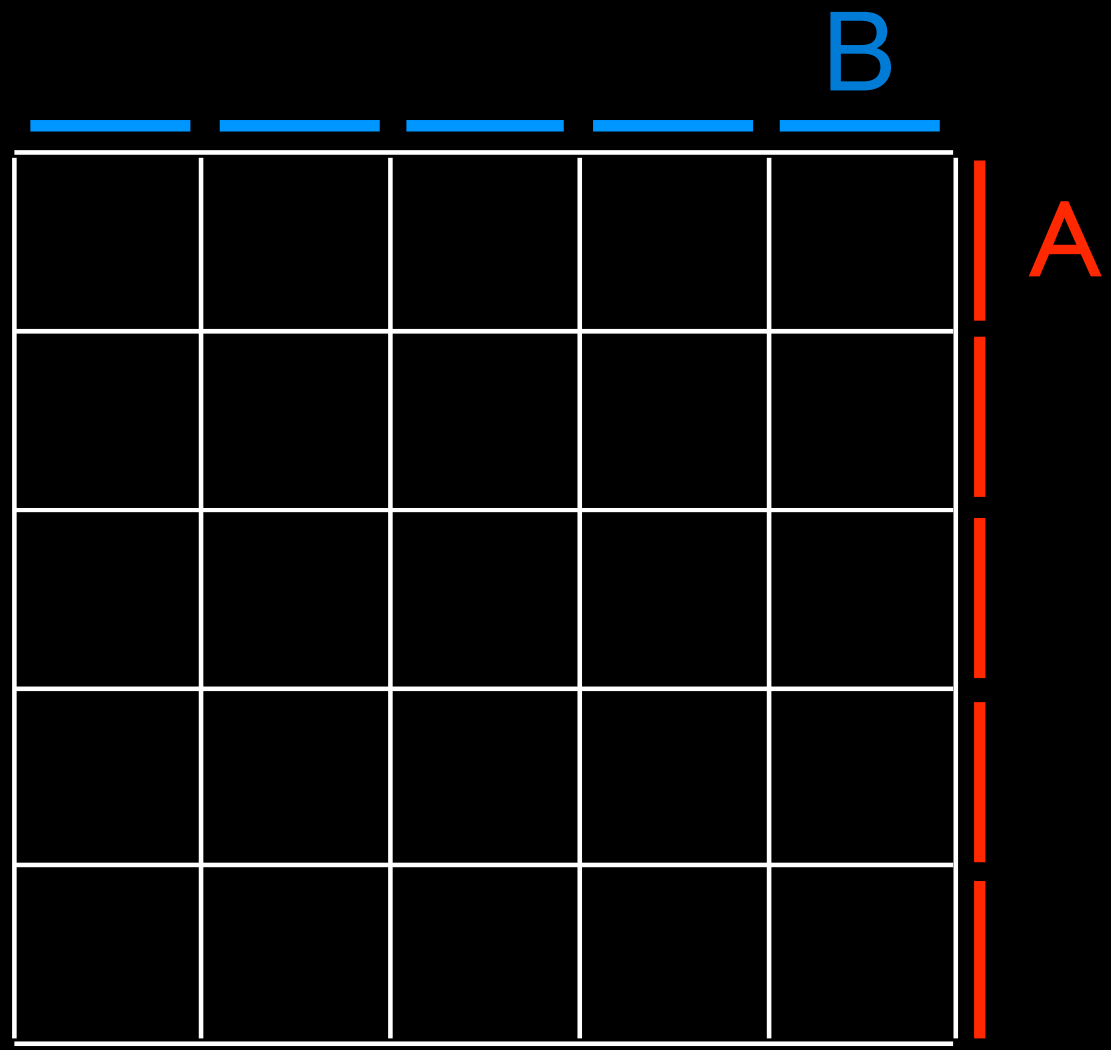
$u = A'^n$, $v = B'^n$

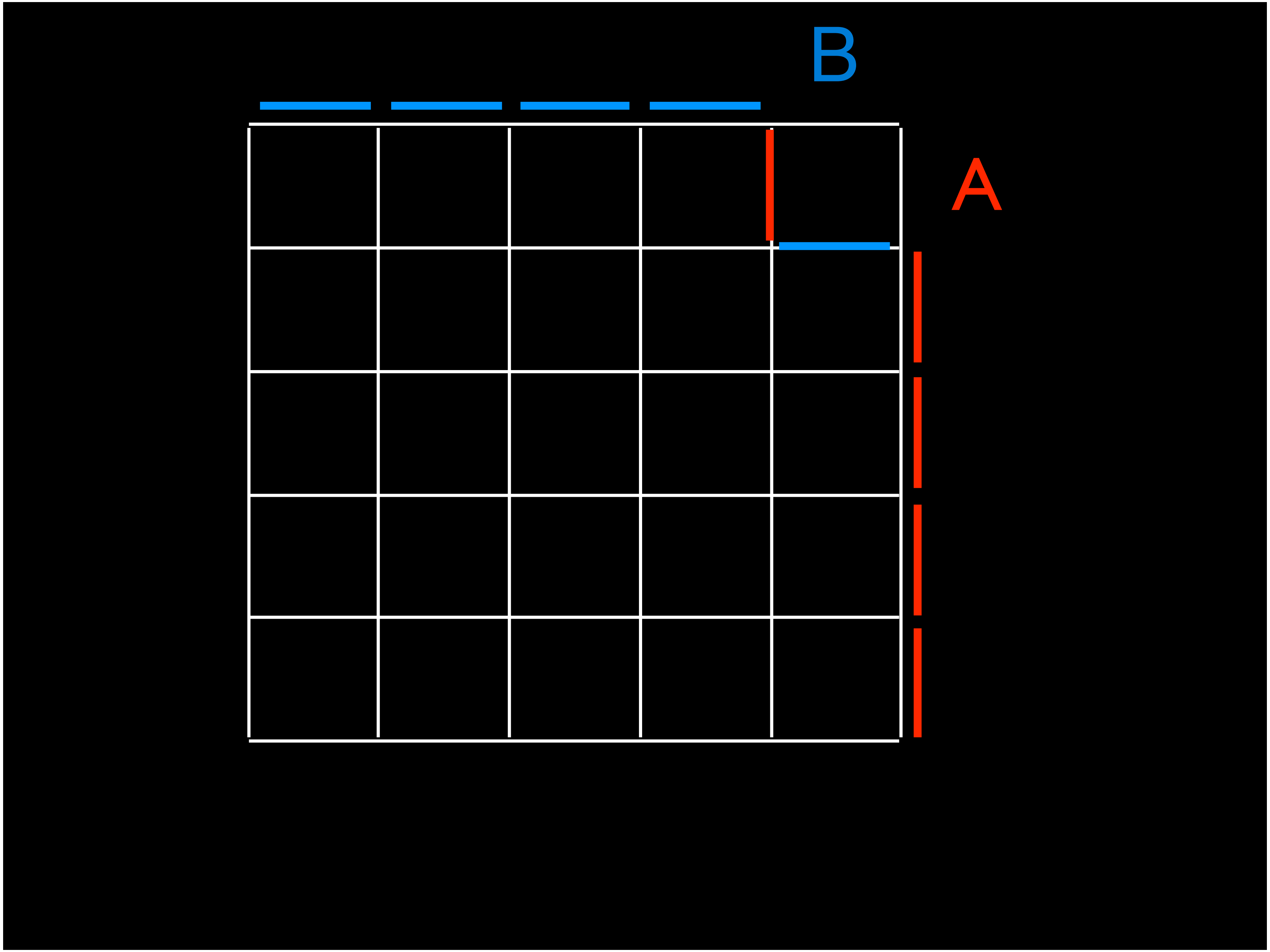
$C(u, v; w)$ = the number of $n \times n$ ASM (alternating sign matrices) “planar” proof:



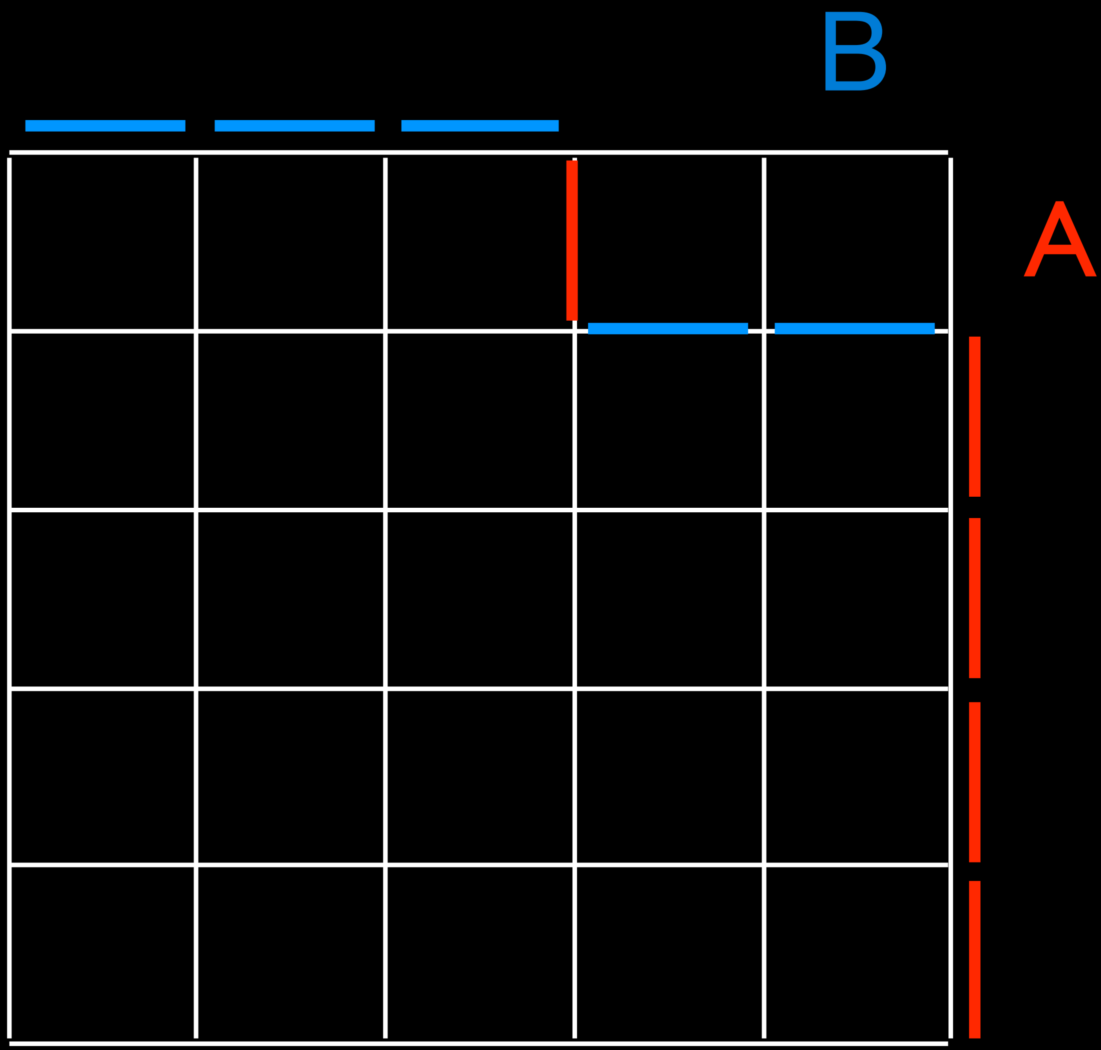
“planarisation” of the “rewriting rules”

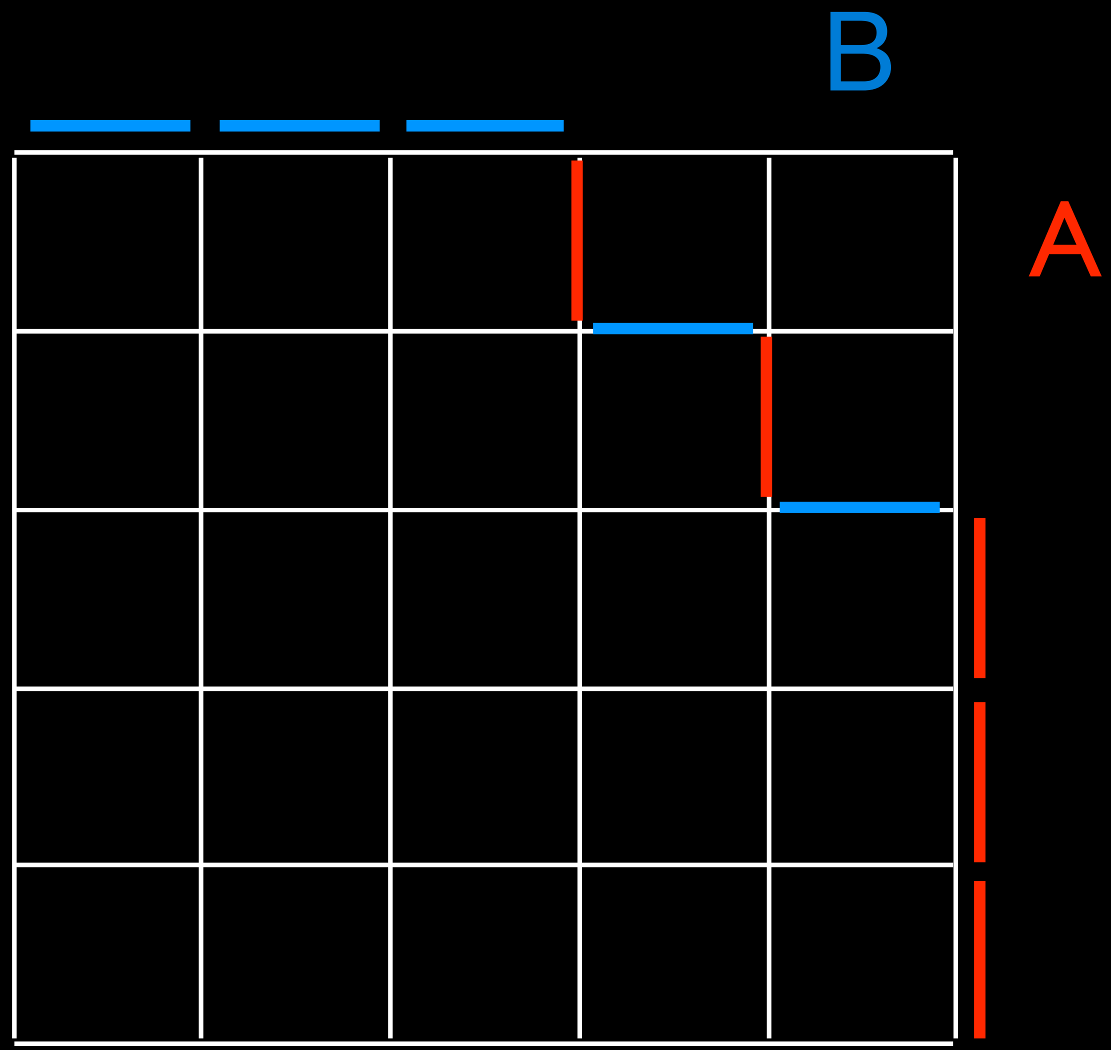


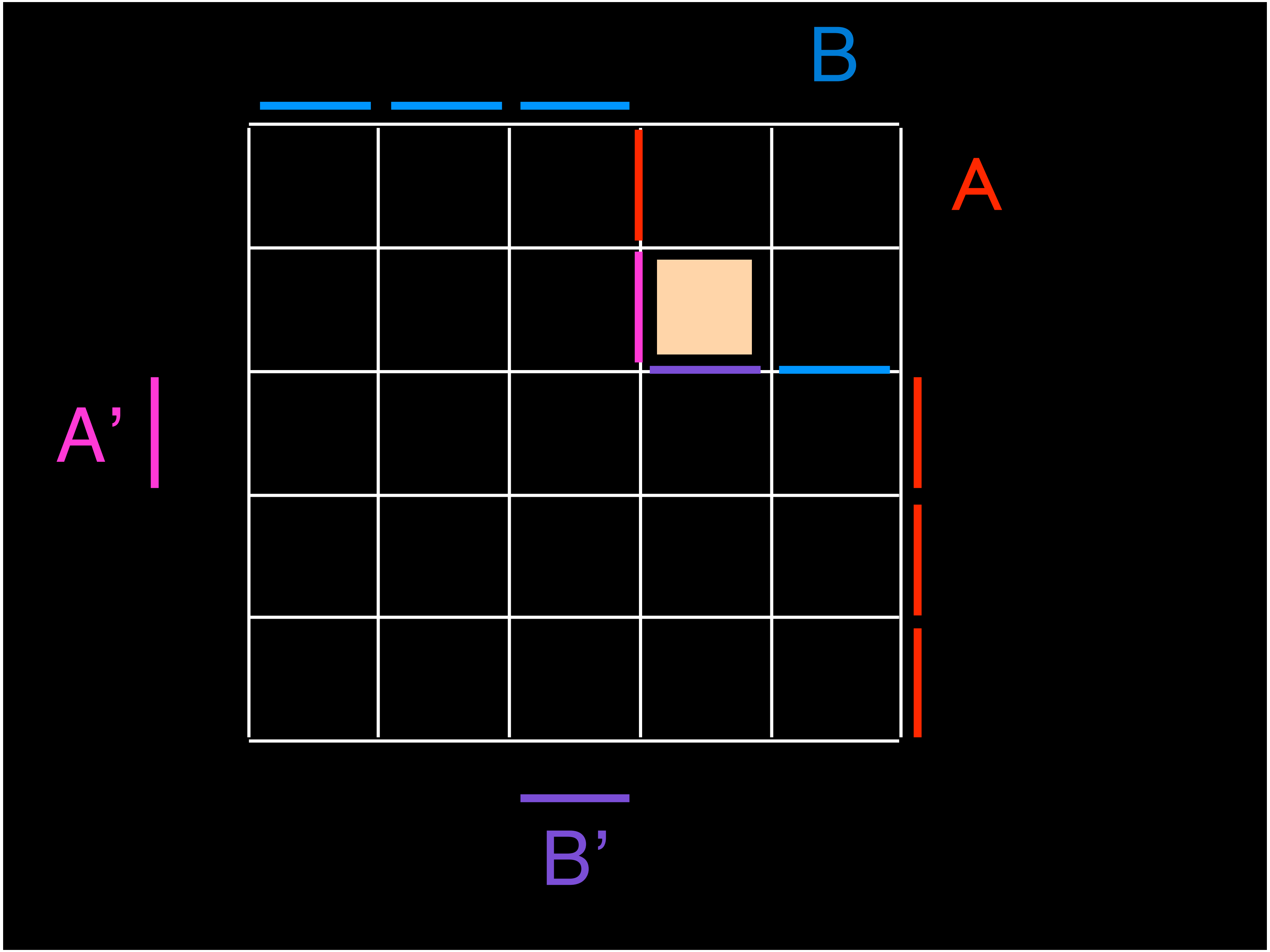

B
A

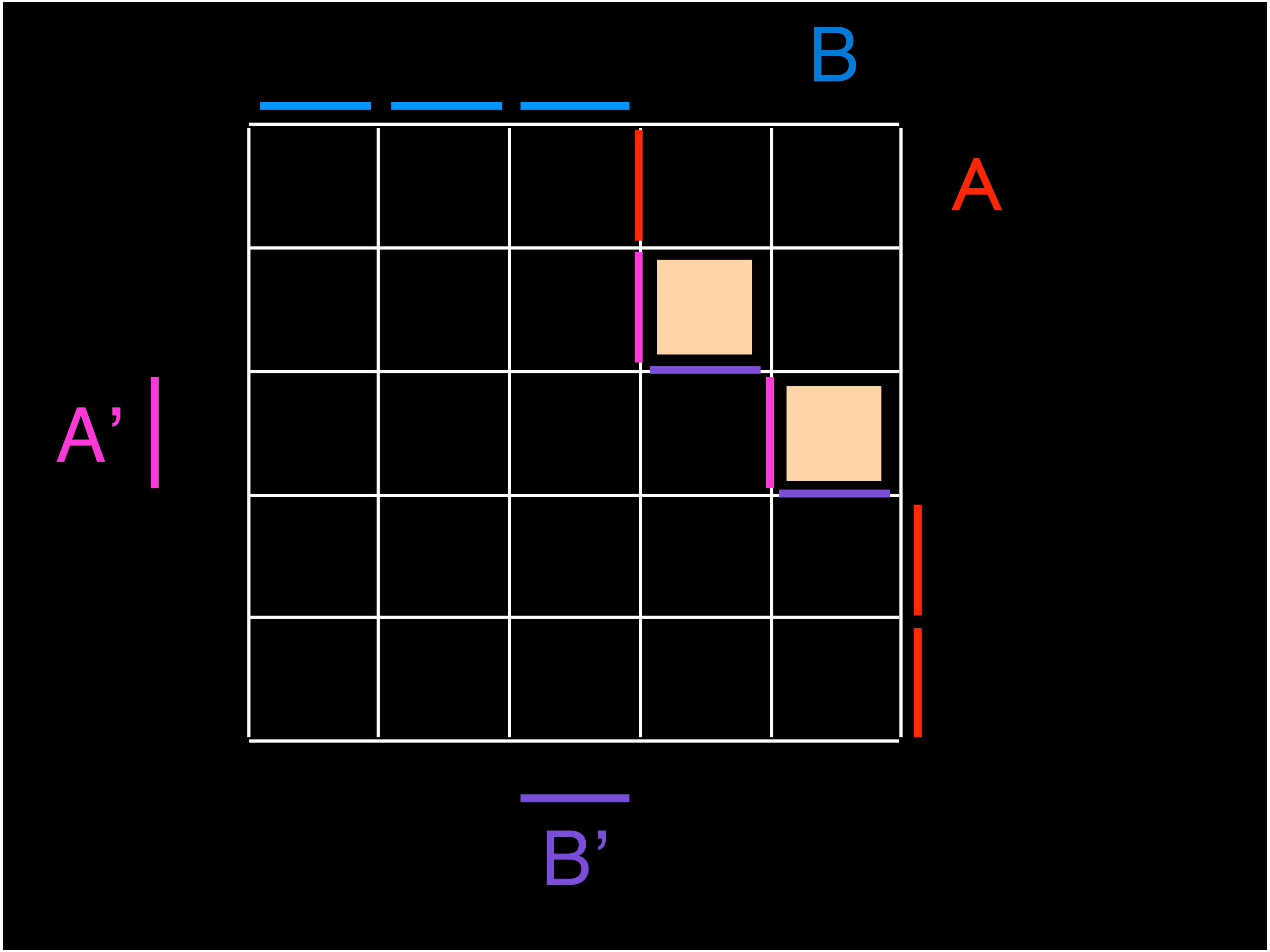


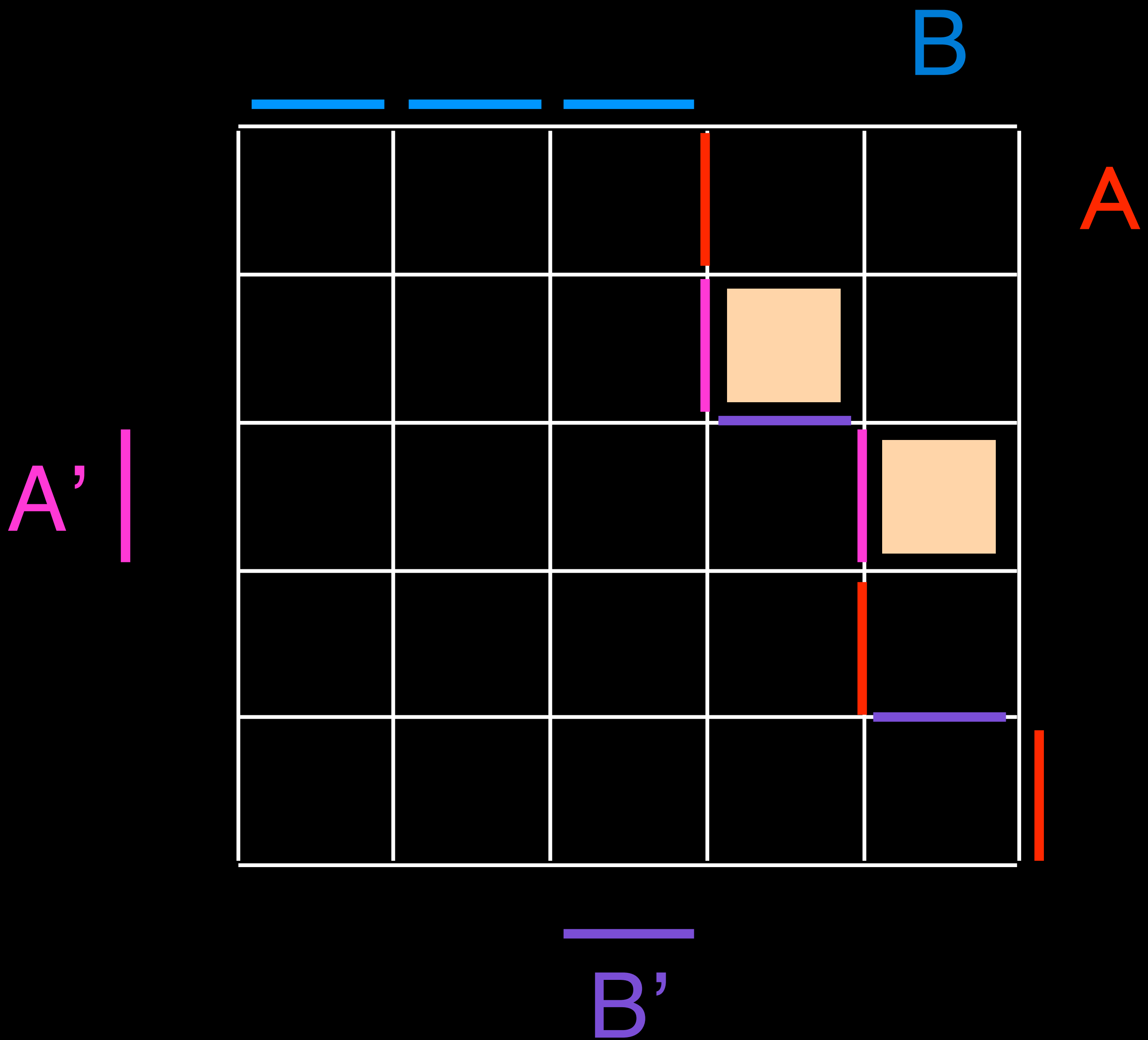
B
A

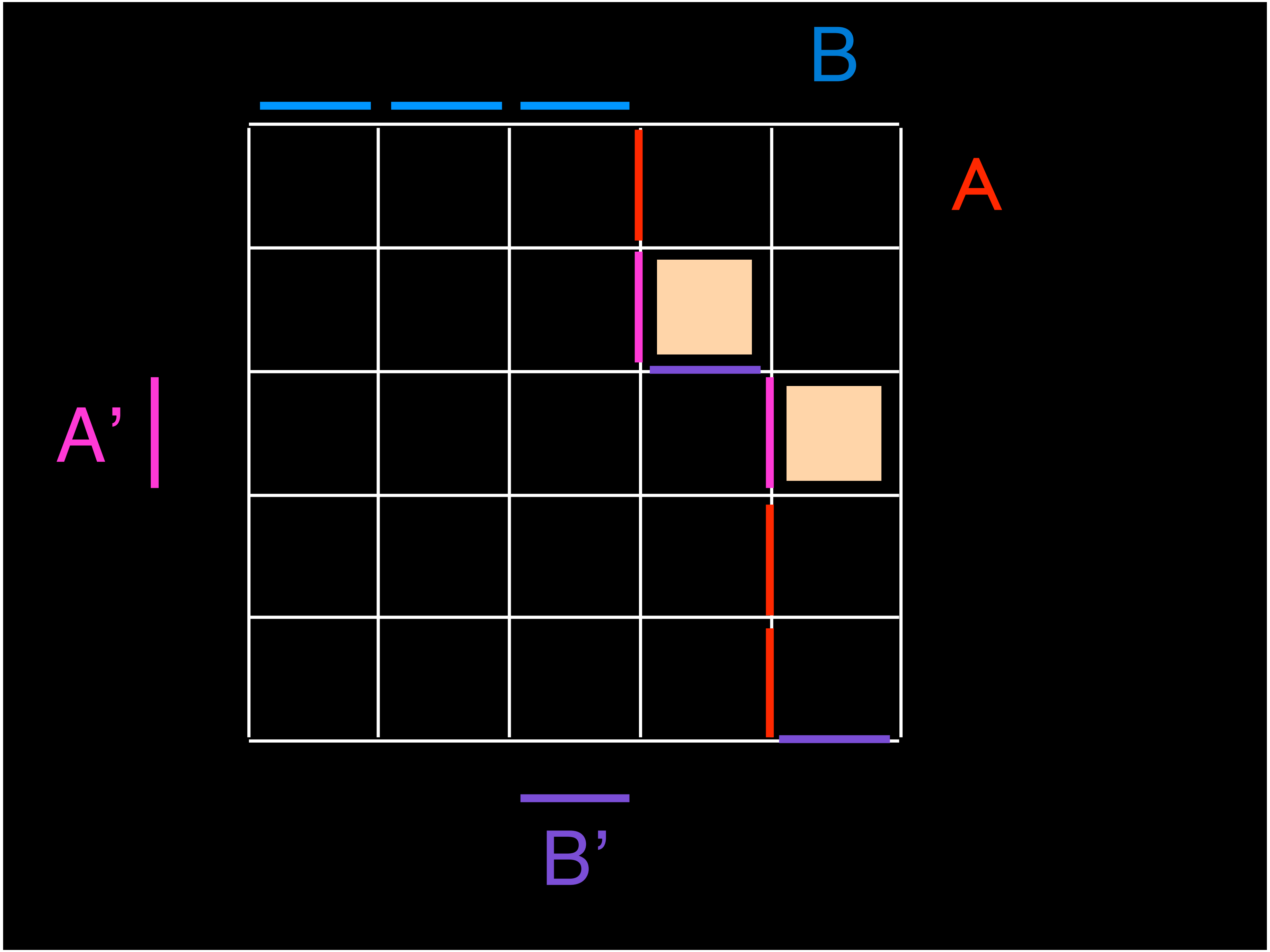

B
A

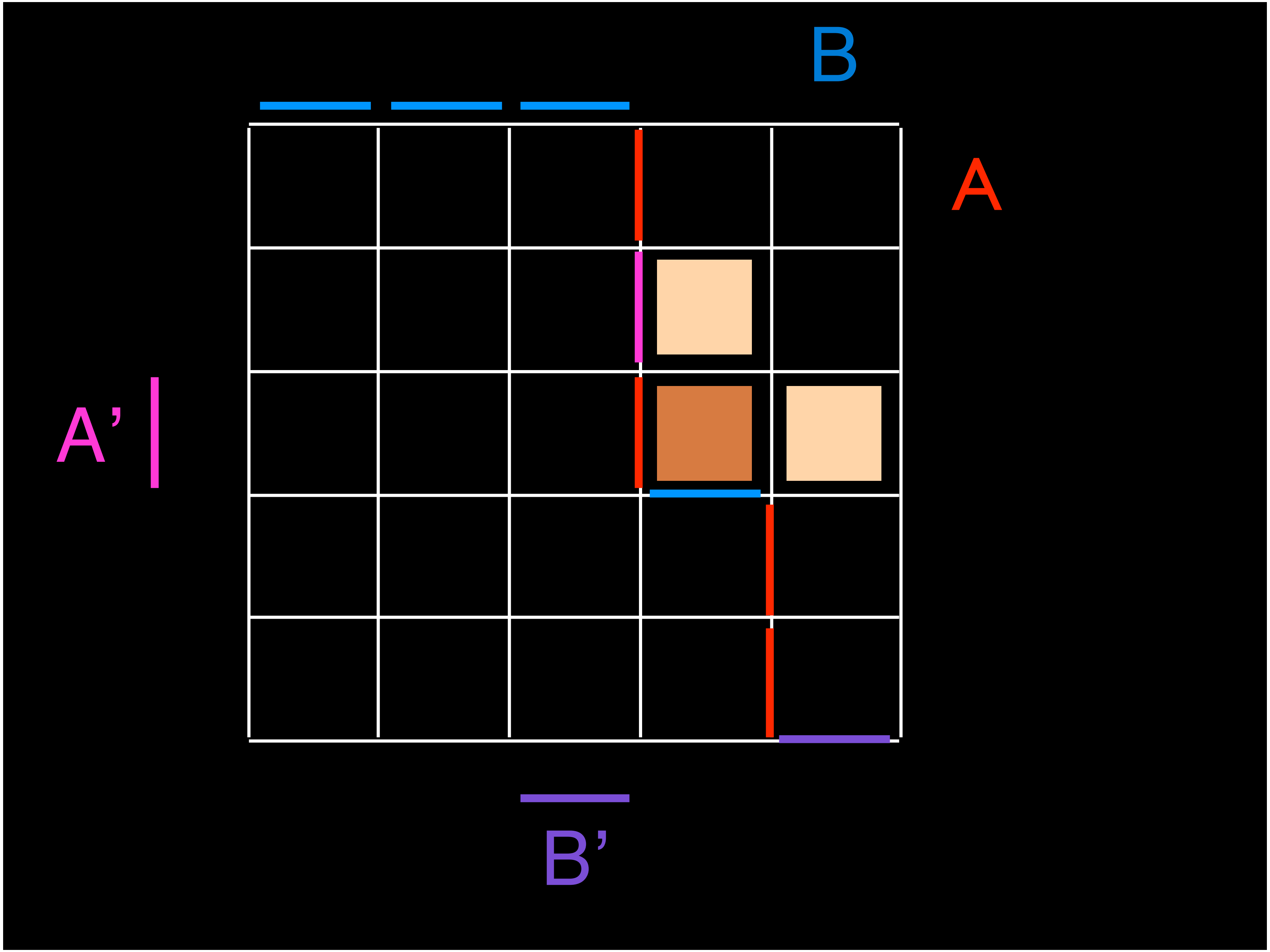

B
A

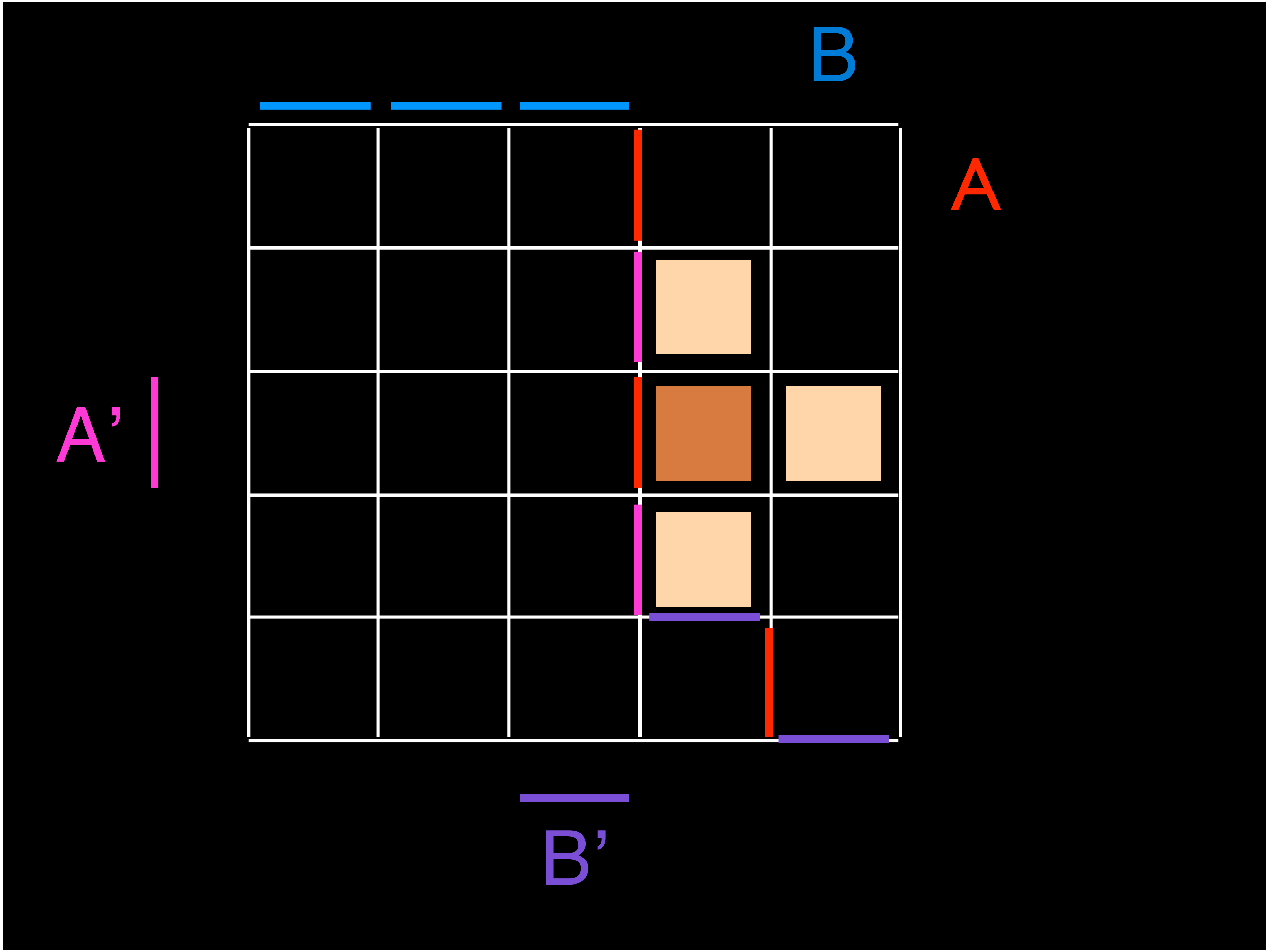

B
A
A'
B'


B
A
A'
B'


B
A
A'
B'


B
A
A'
B'


B
A
A'
B'


B
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A'
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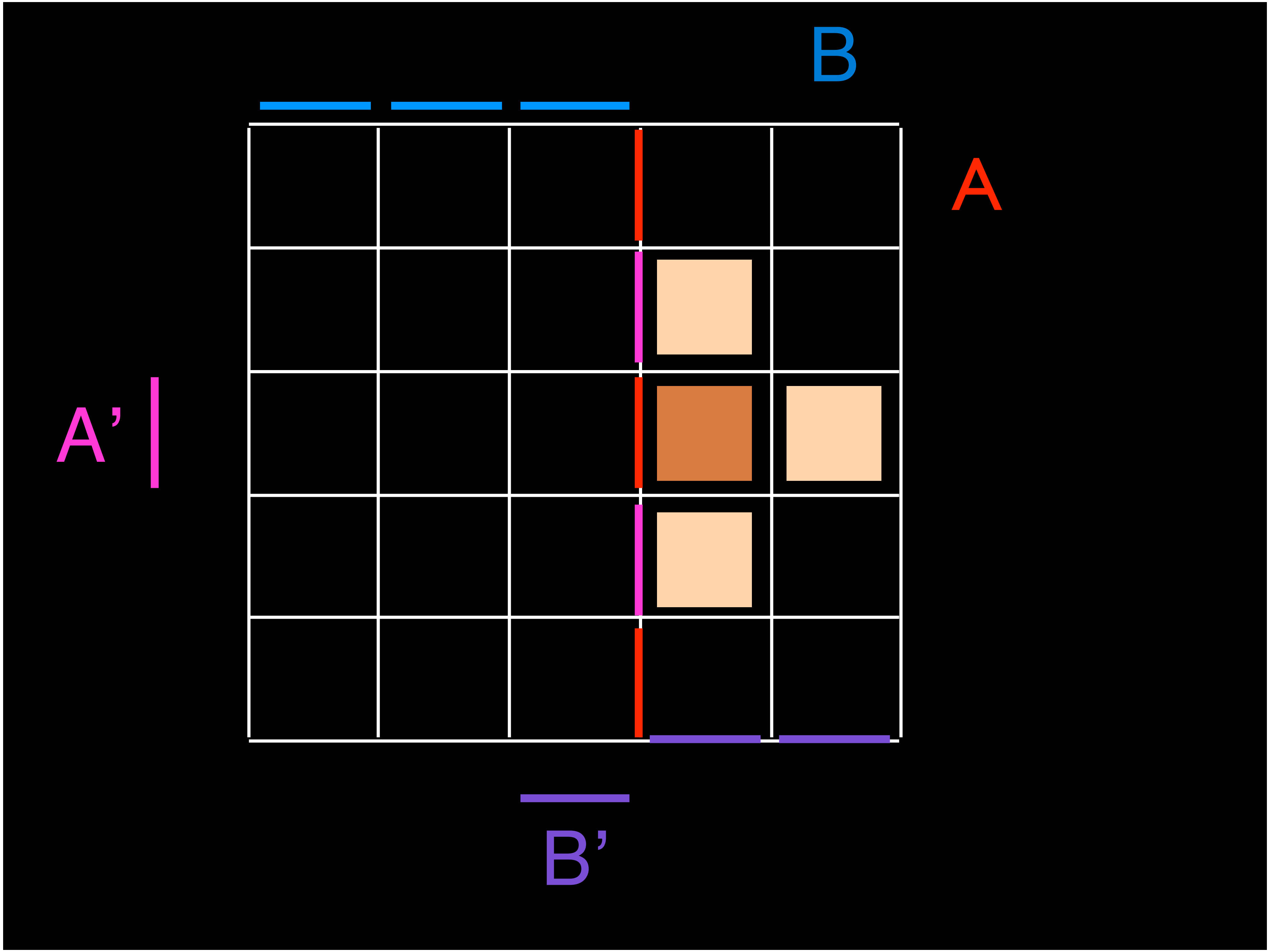
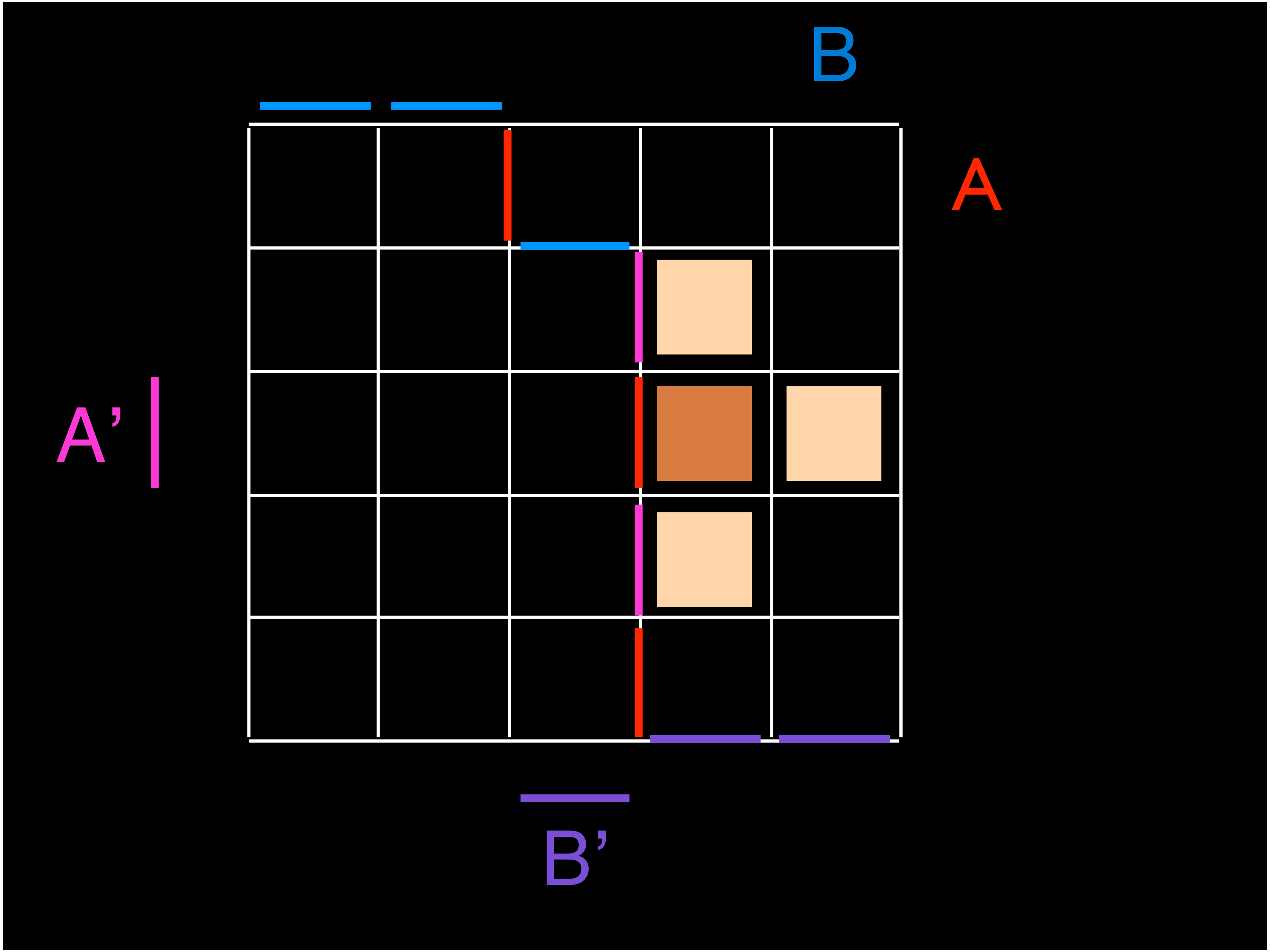


B
A
A'
B'


B
A
A'
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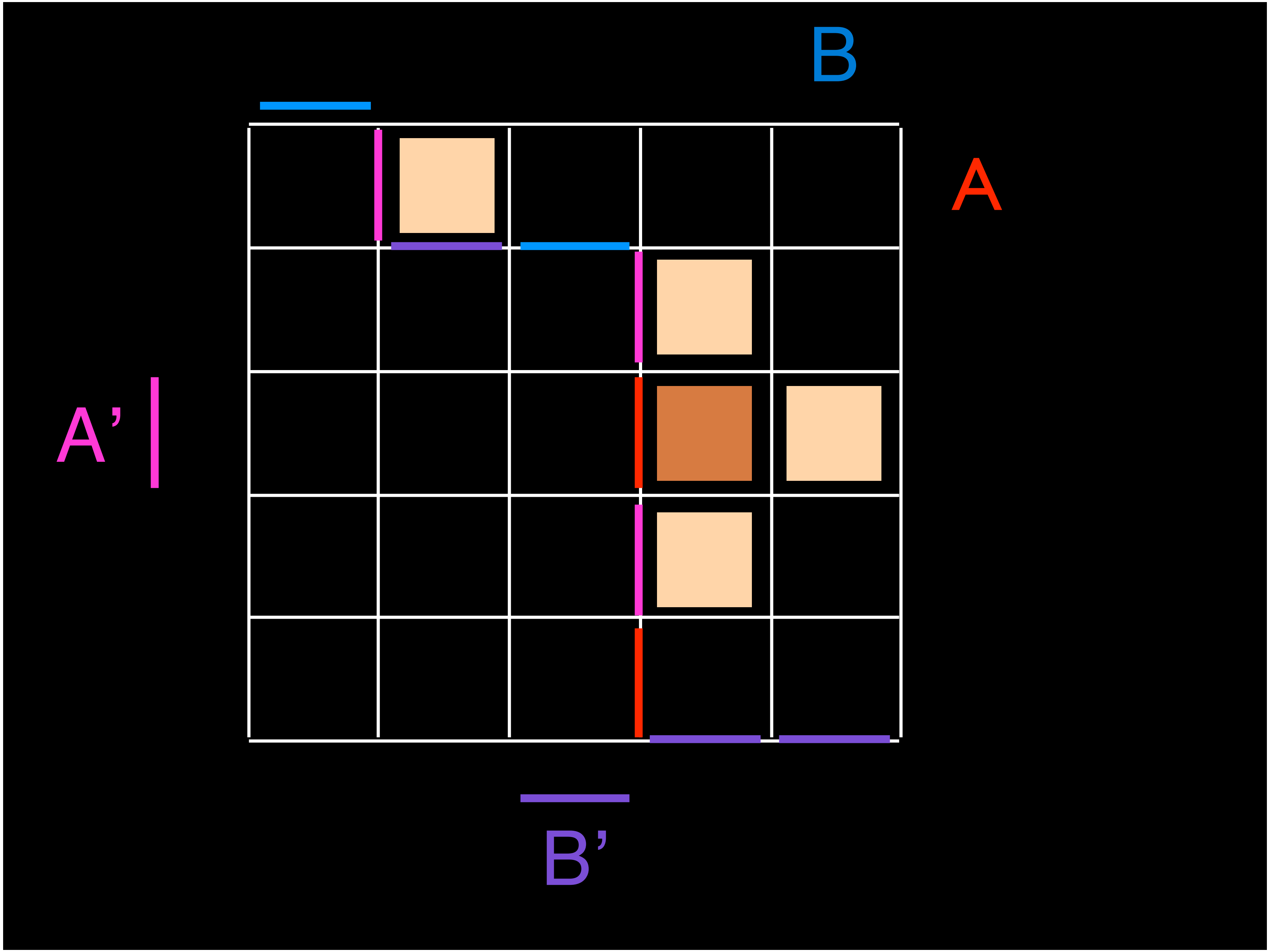
B
A
A'
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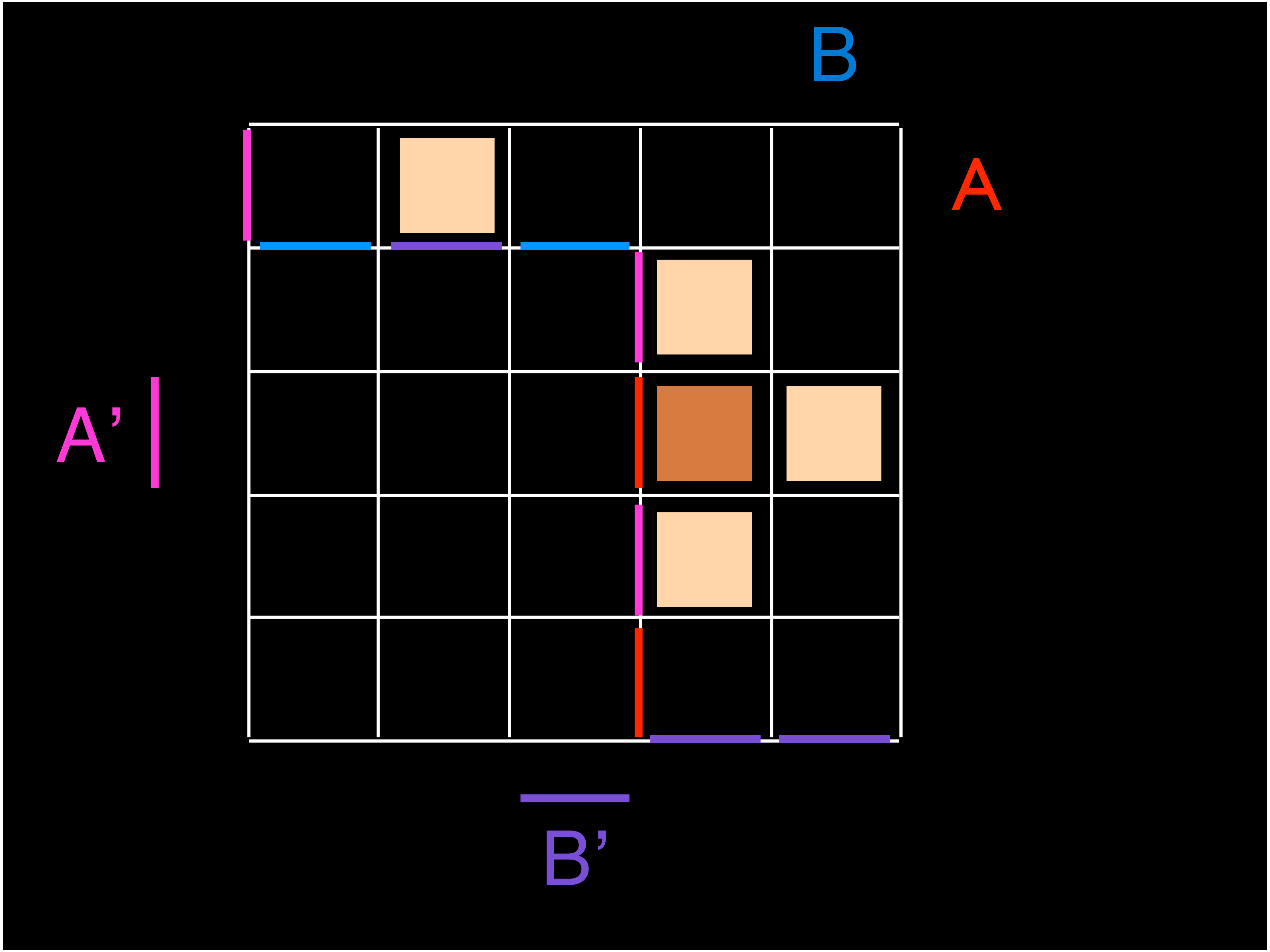
B
A
A'
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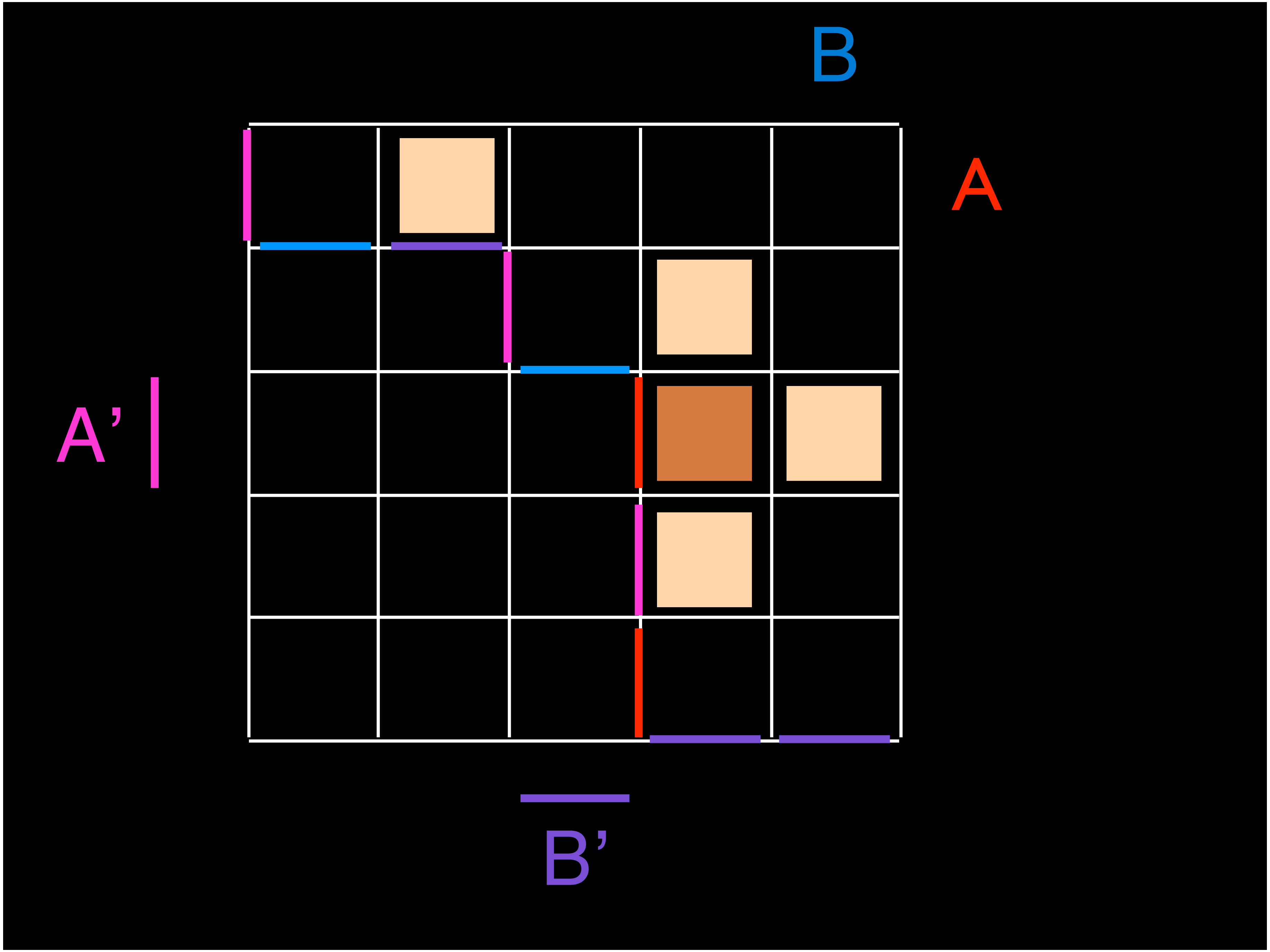
B
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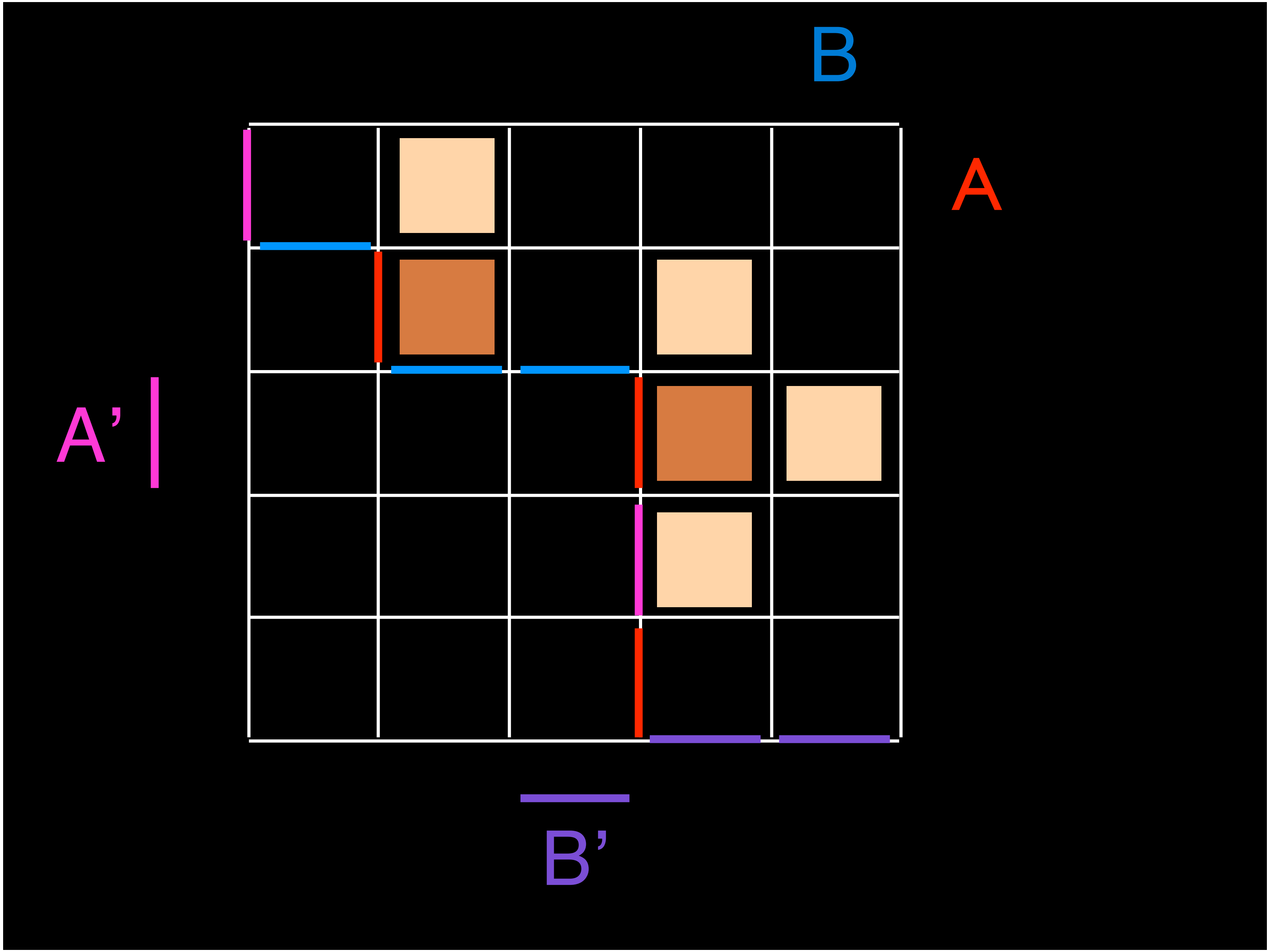
B
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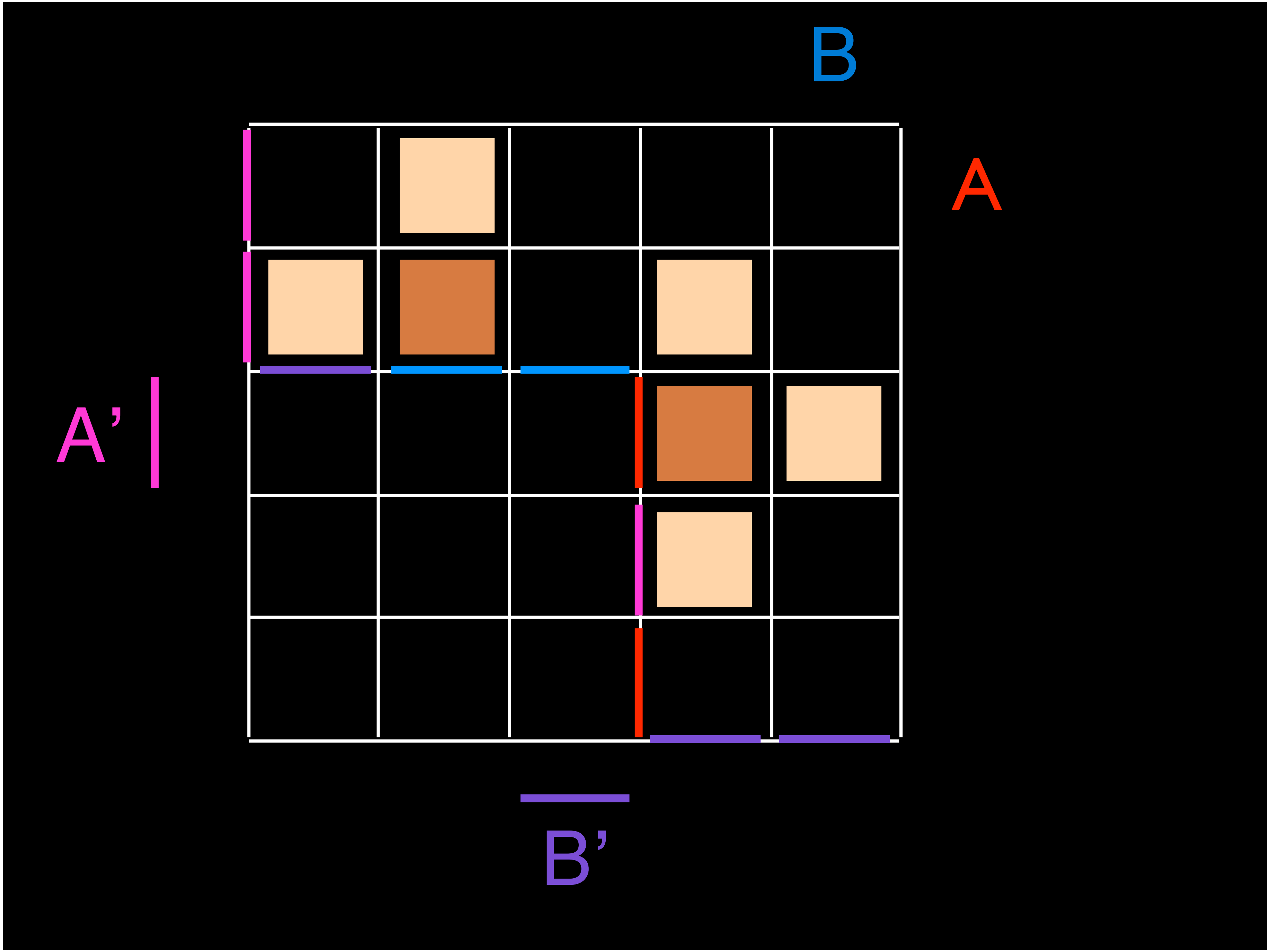
B
A
A'
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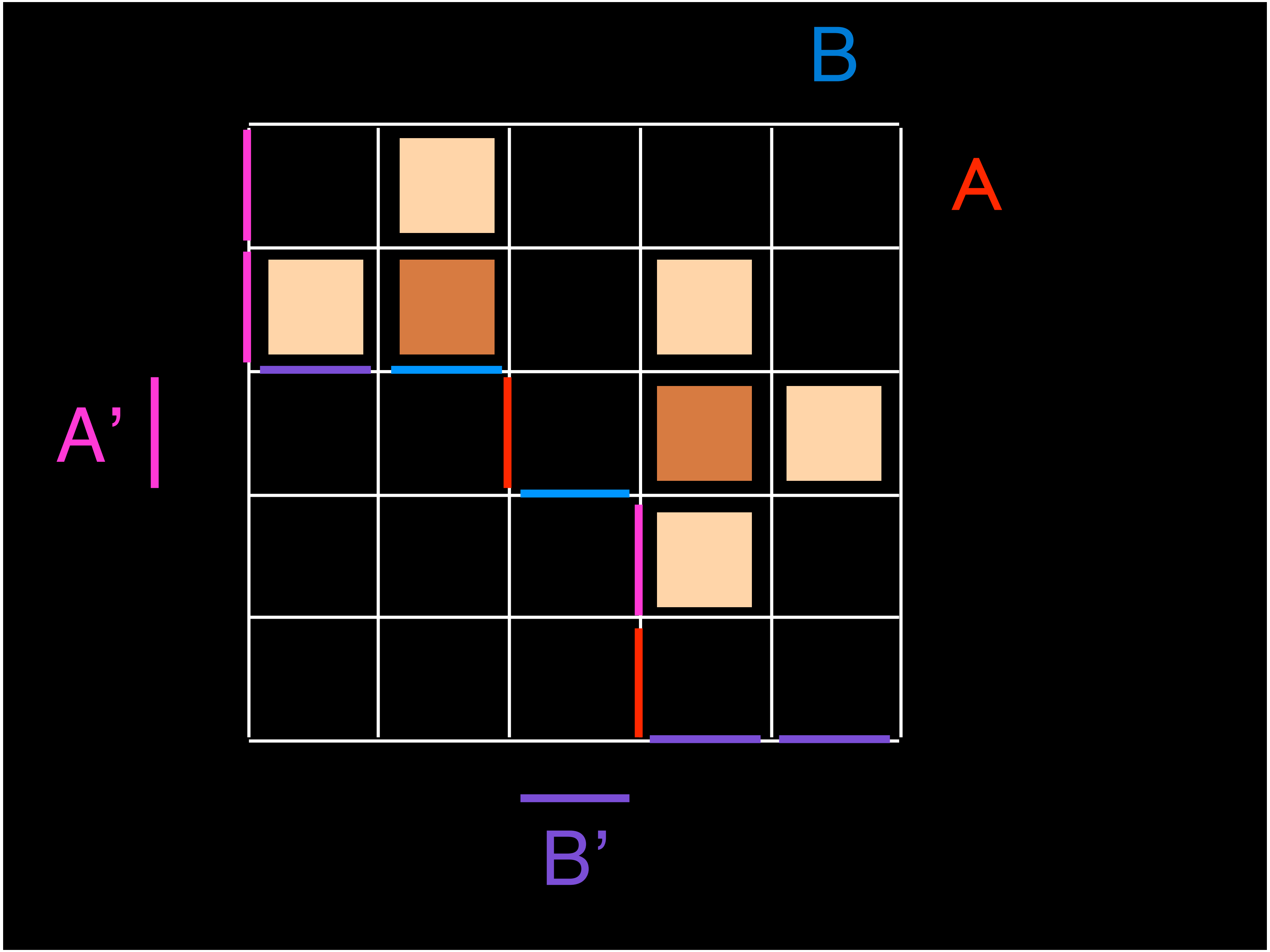
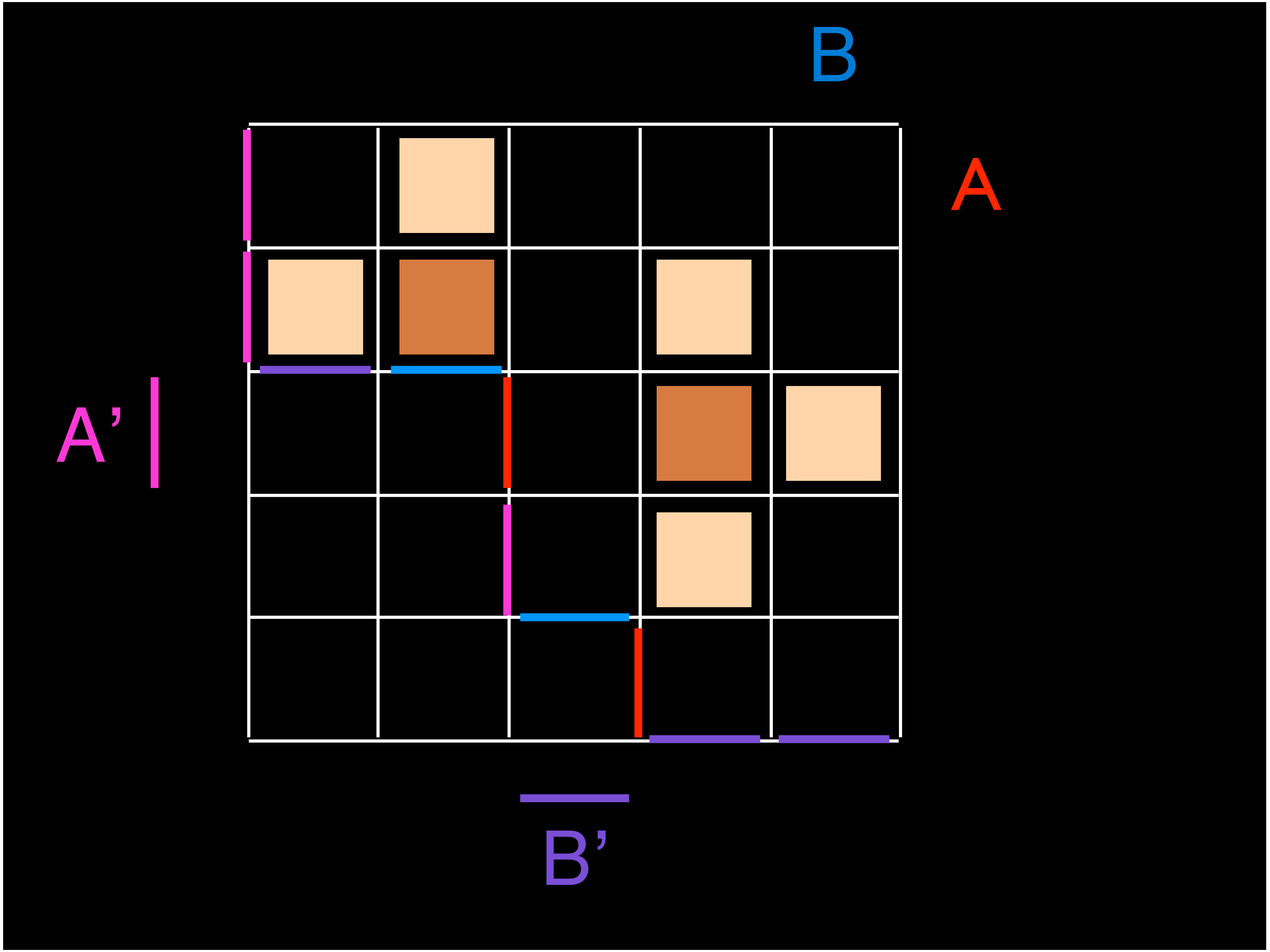
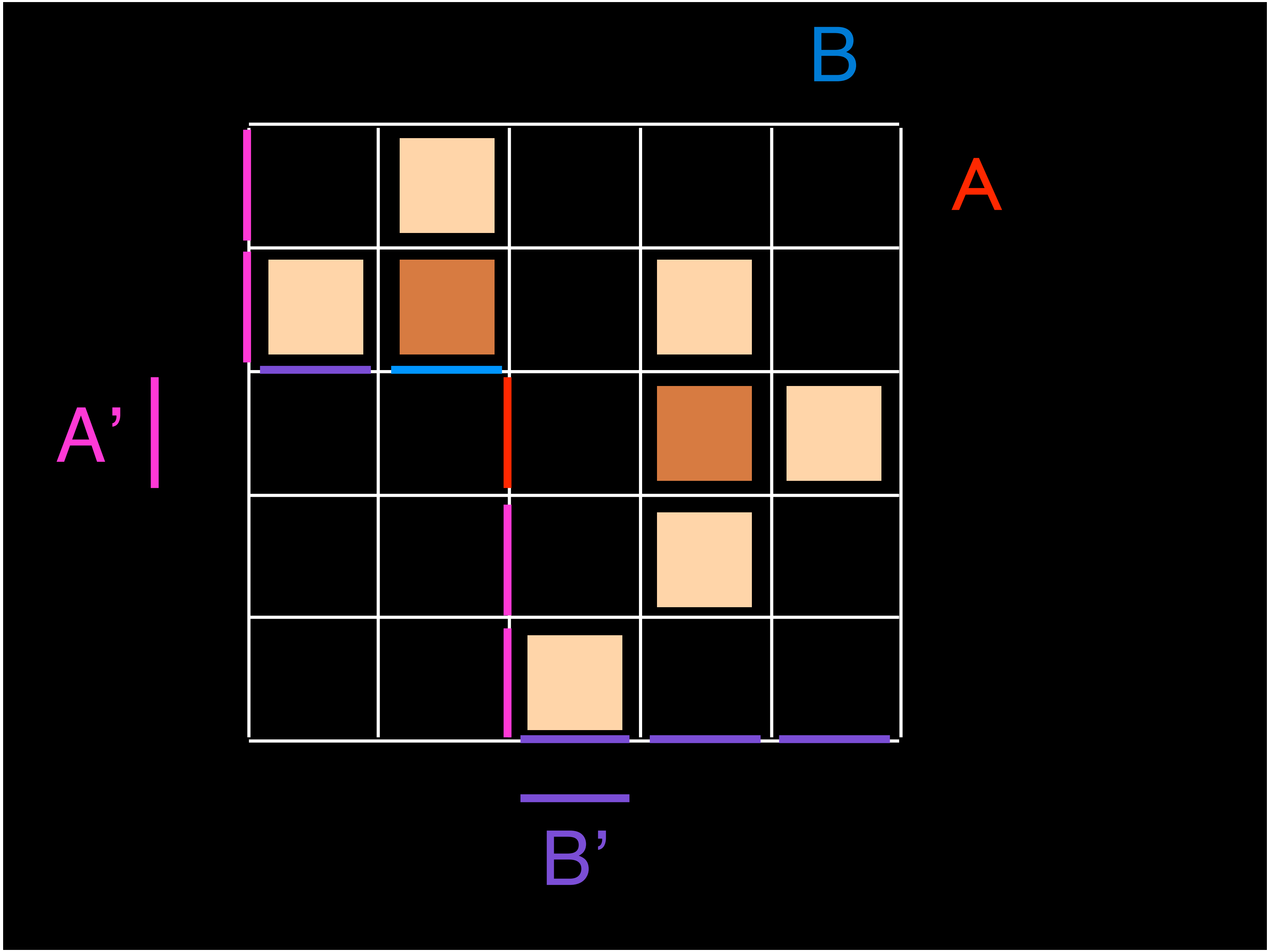
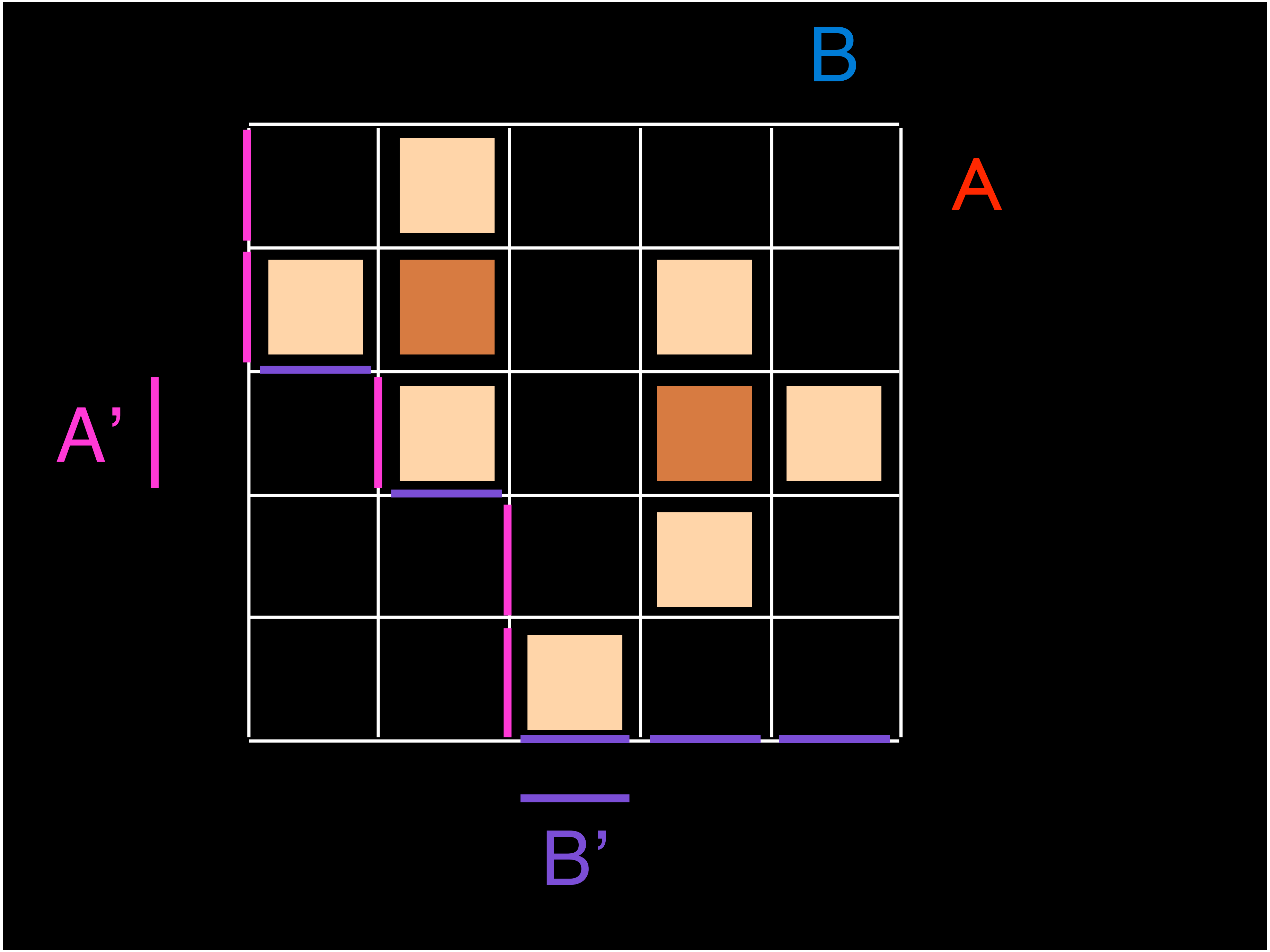

B
A
A'
B'



B
A
A'
B'


B
A
A'
B'


B
A
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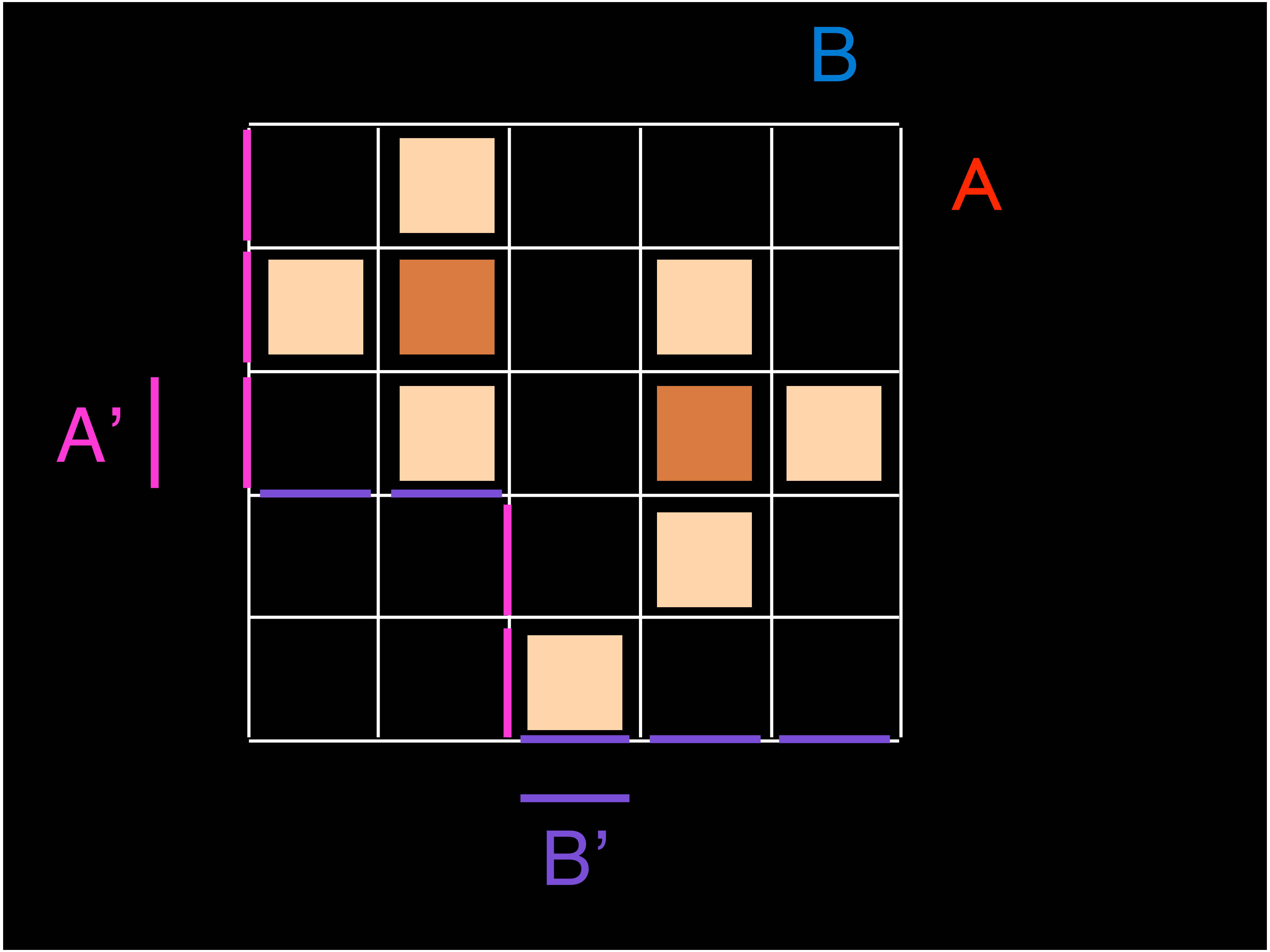
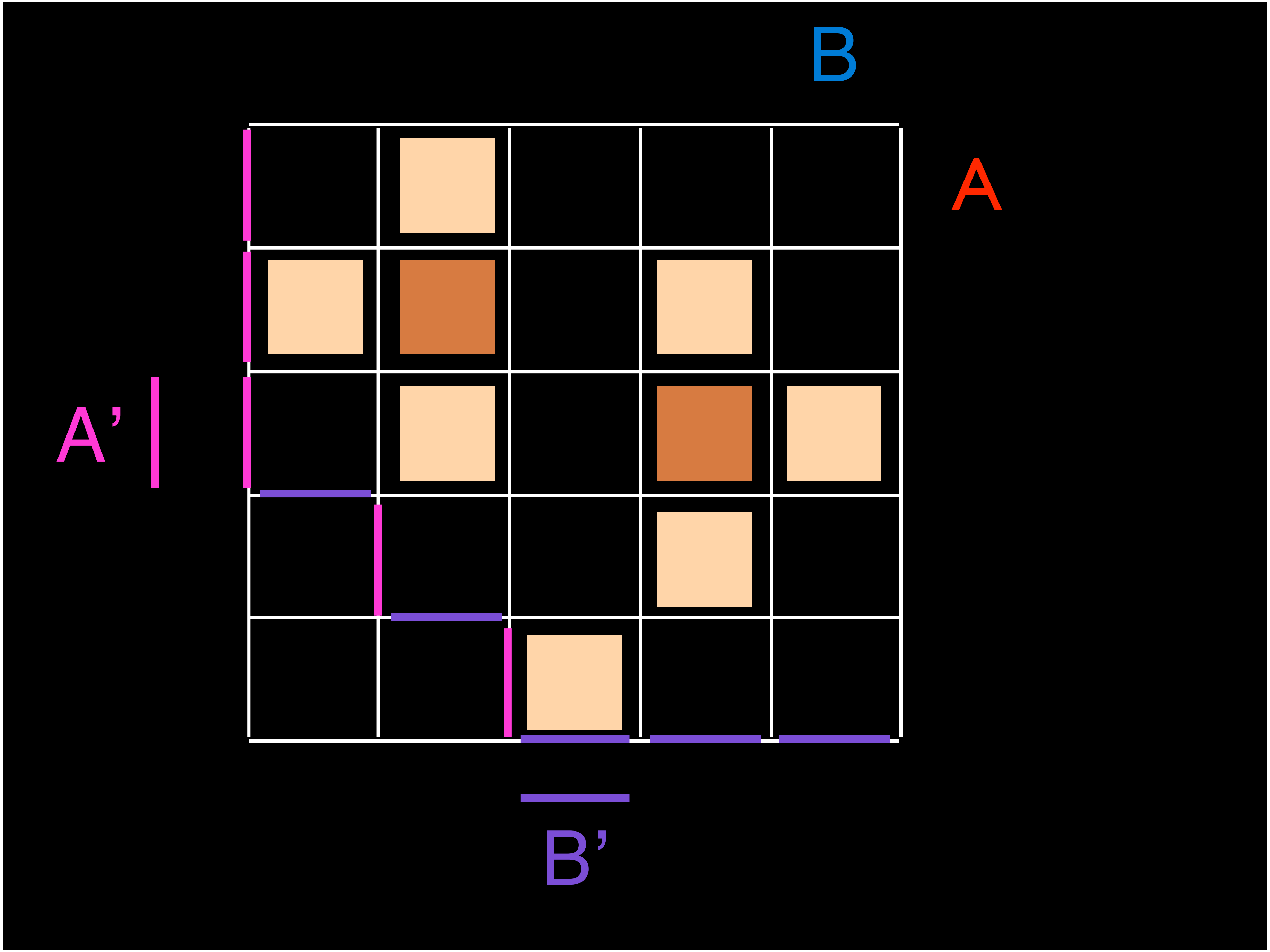
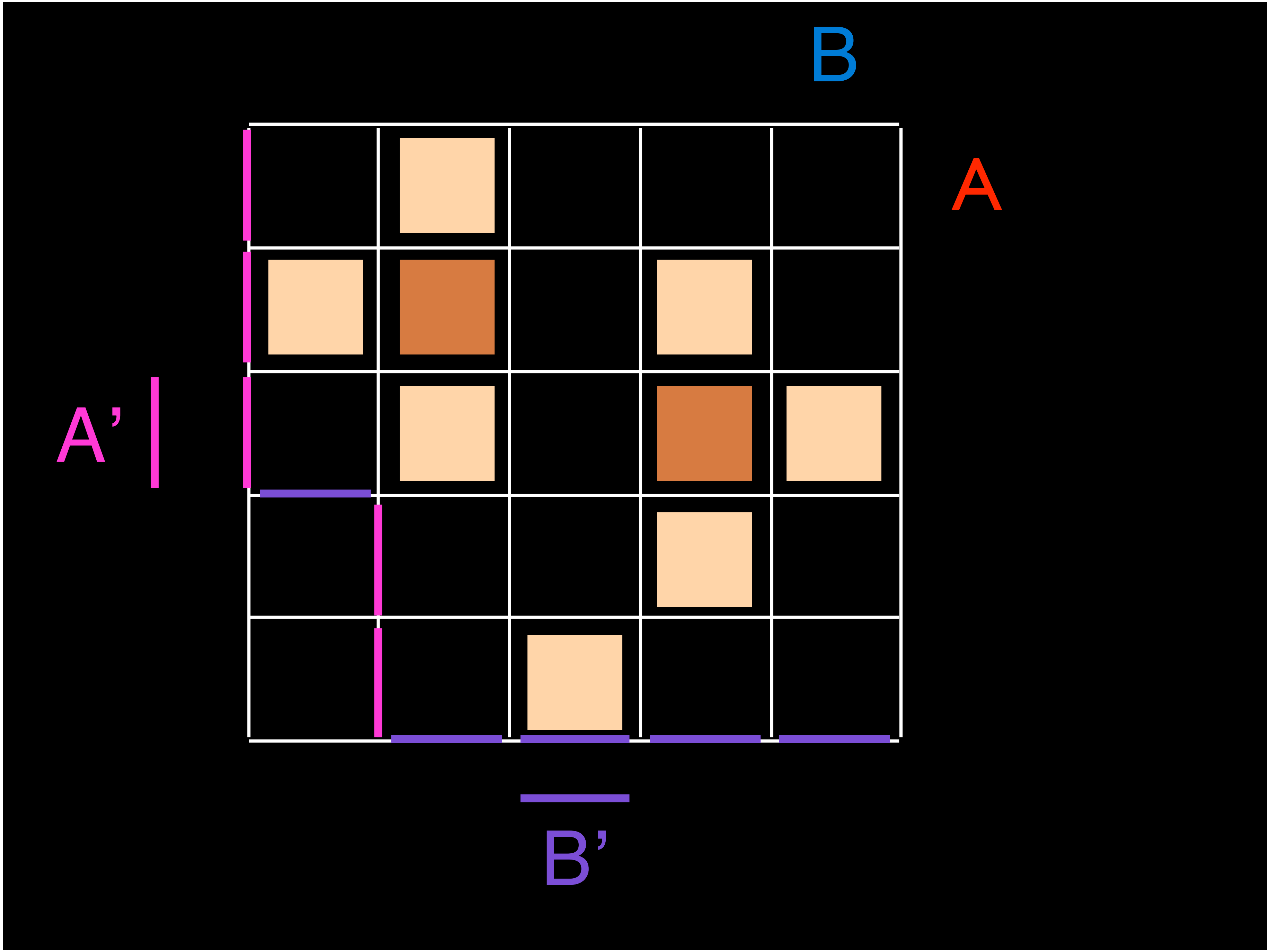
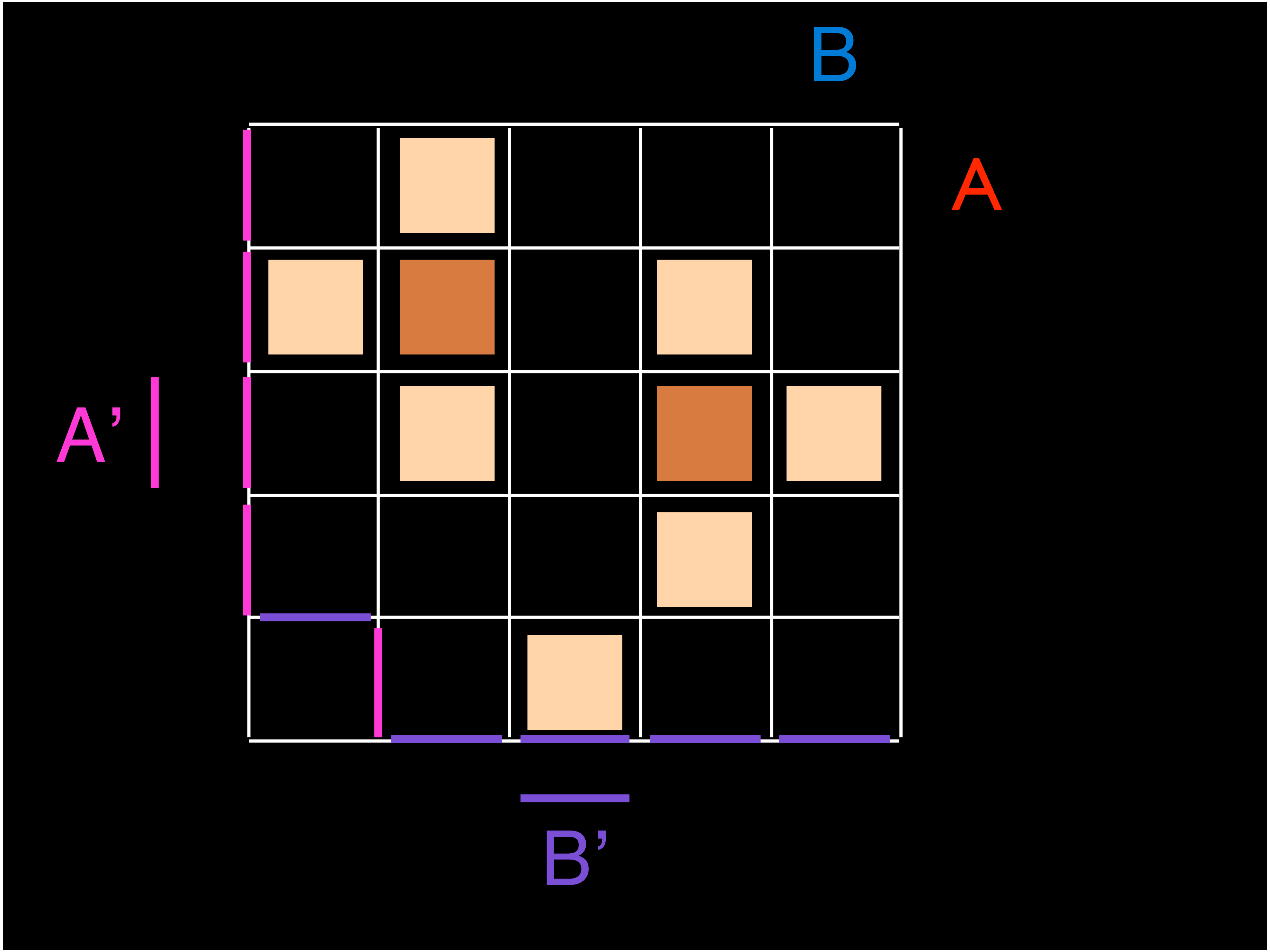
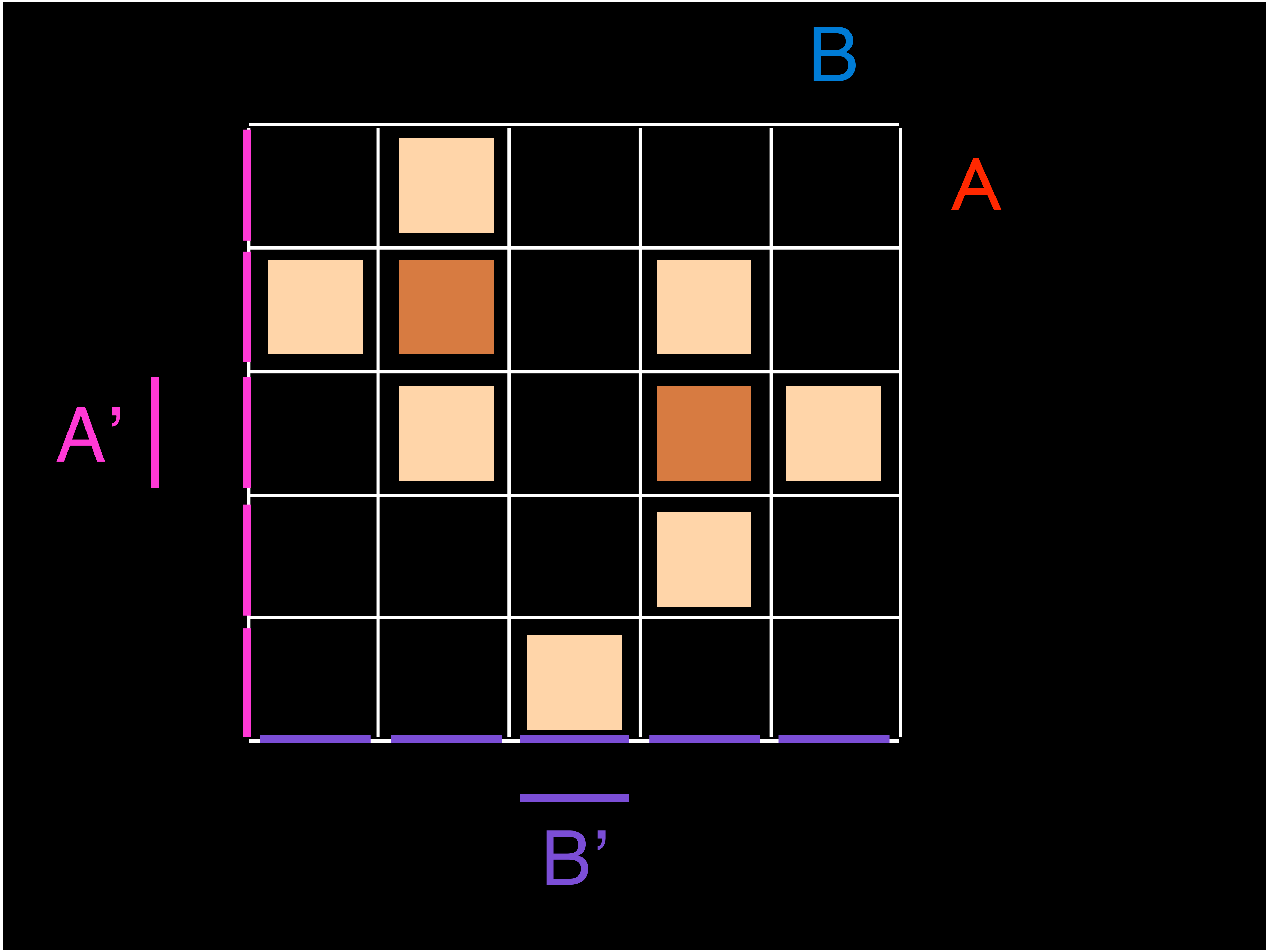

B
A
A’
B’

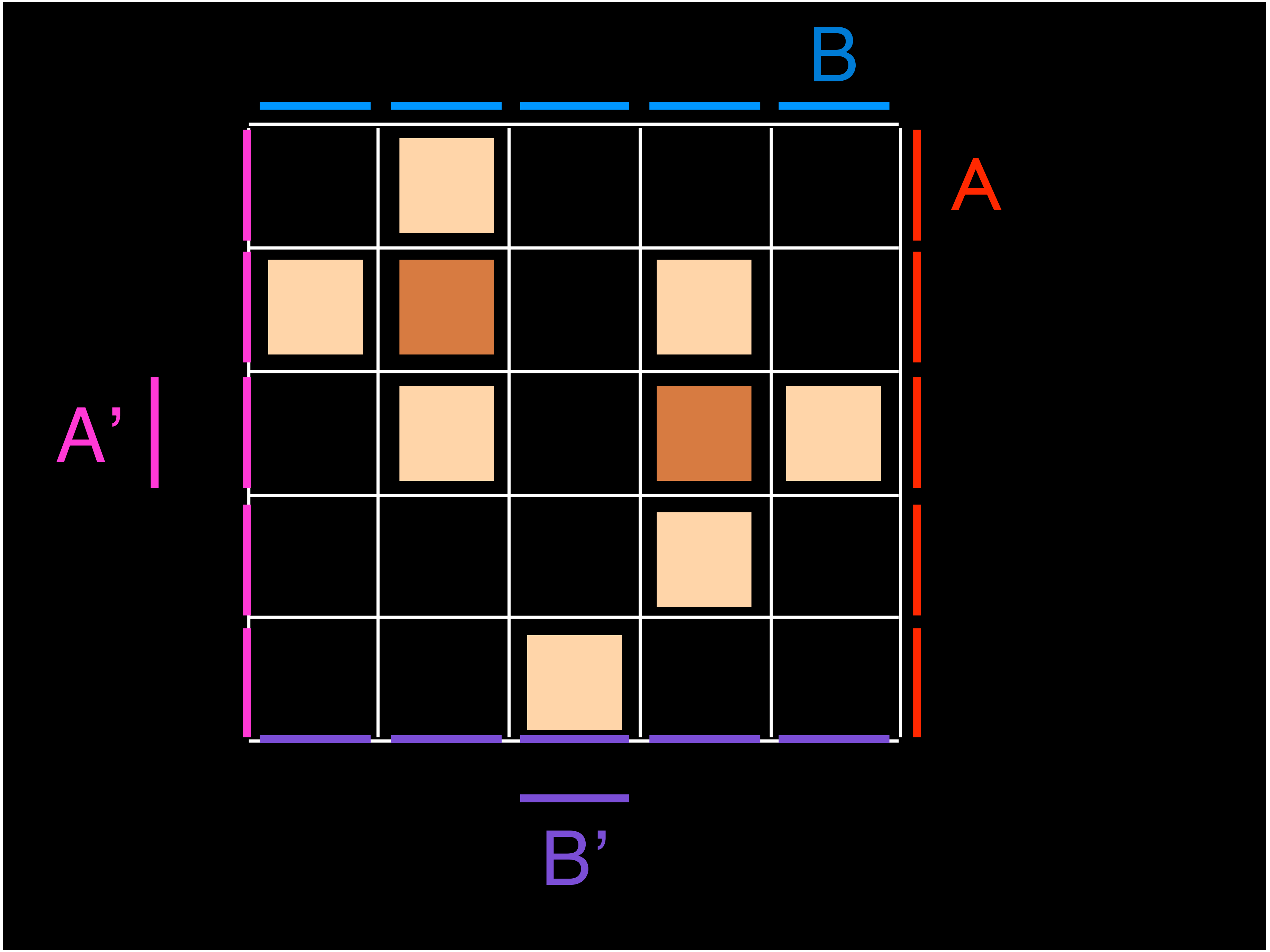


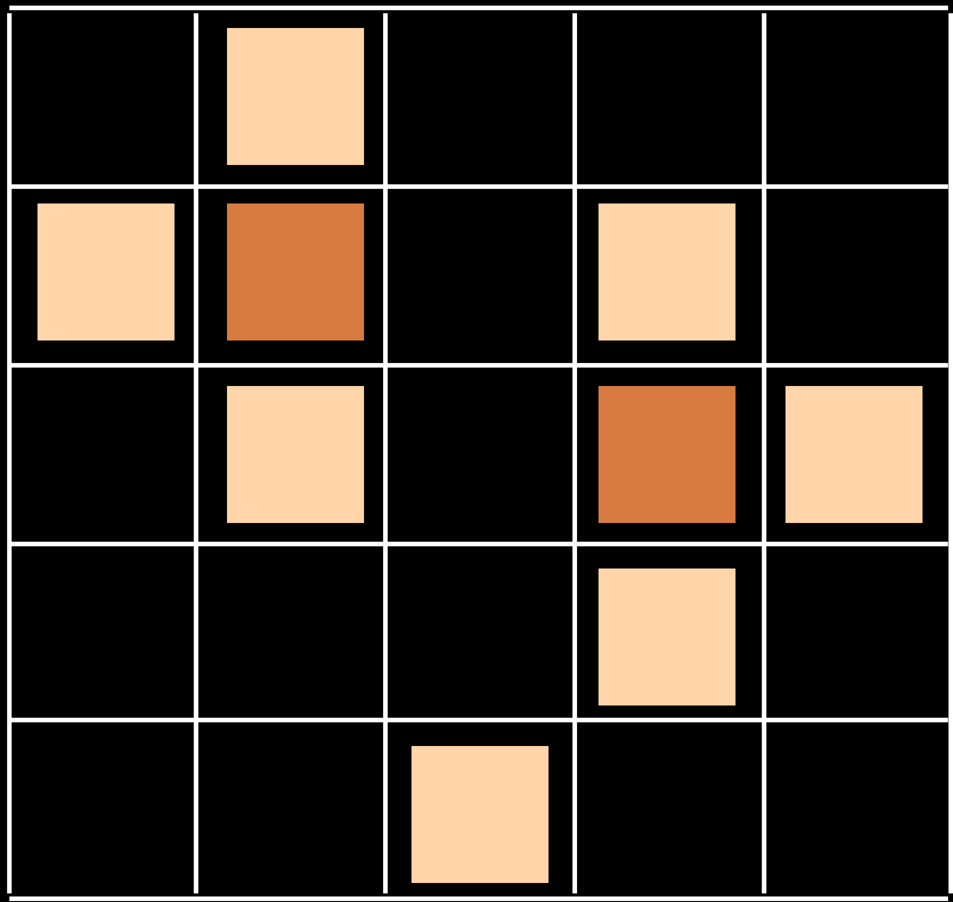
B
A
A’
B’


B
A
A'
B'


B
A
A'
B'


B
A
A'
B'

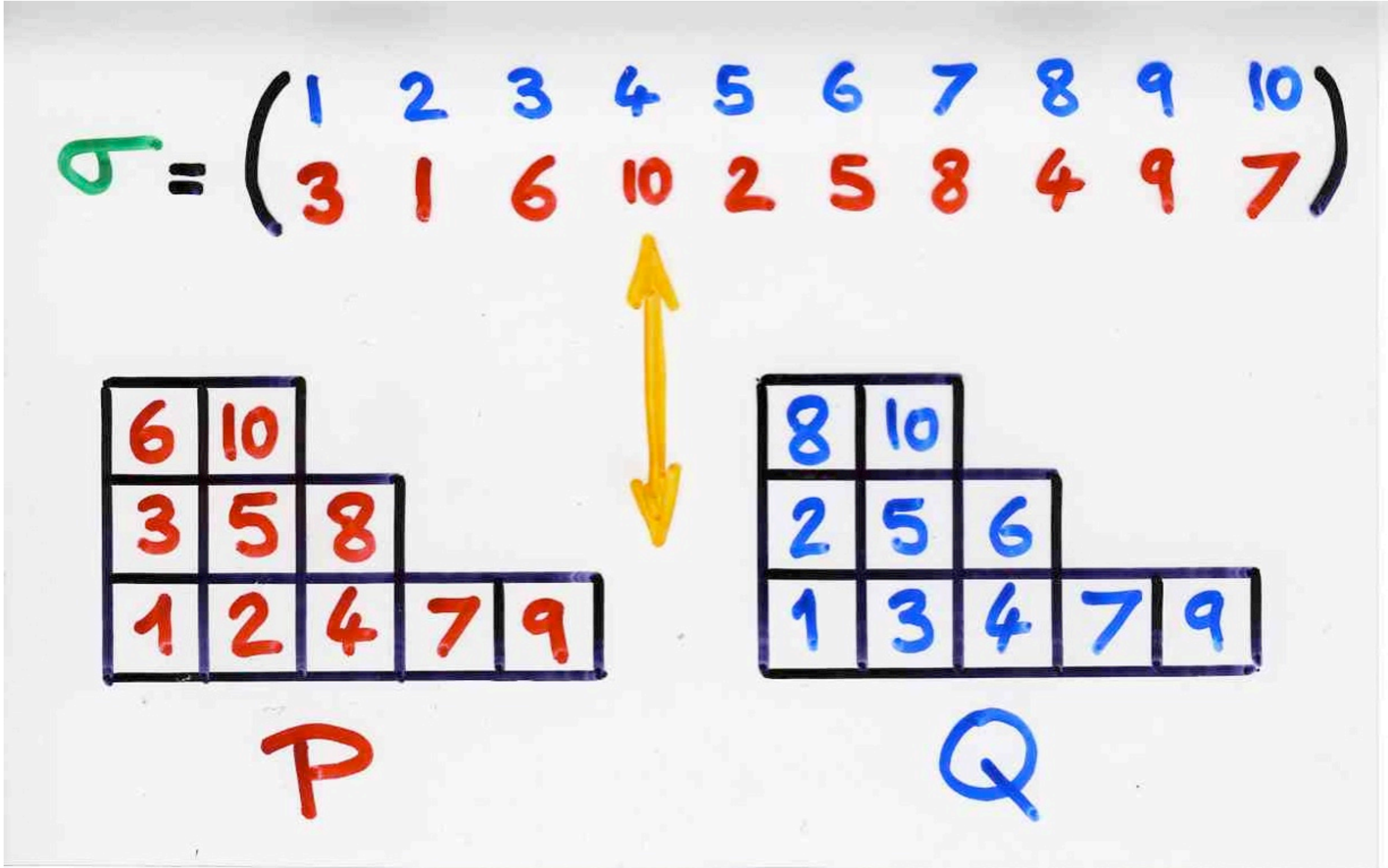

B
A
A'
B'



Heisenberg
operators
U, D

$UD = DU + I$

RSK
Robison-
Schensted-
Knuth
correspondence



σ =
1 2 3 4 5 6 7 8 9 10
3 1 6 10 2 5 8 4 9 7
6 10
3 5 8
1 2 4 7 9
8 10
2 5 6
1 3 4 7 9
P
Q

differential poset

Fomin, Stanley

$$U^n D^n =$$

$$U \cdots U \underbrace{U D}_{(DU + I)} D \cdots D$$

. . . .

Robinson-Schensted-Schützenberger bijection

Operators for tilings of the triangular lattice

Operators and commutations for tilings of the triangular lattice

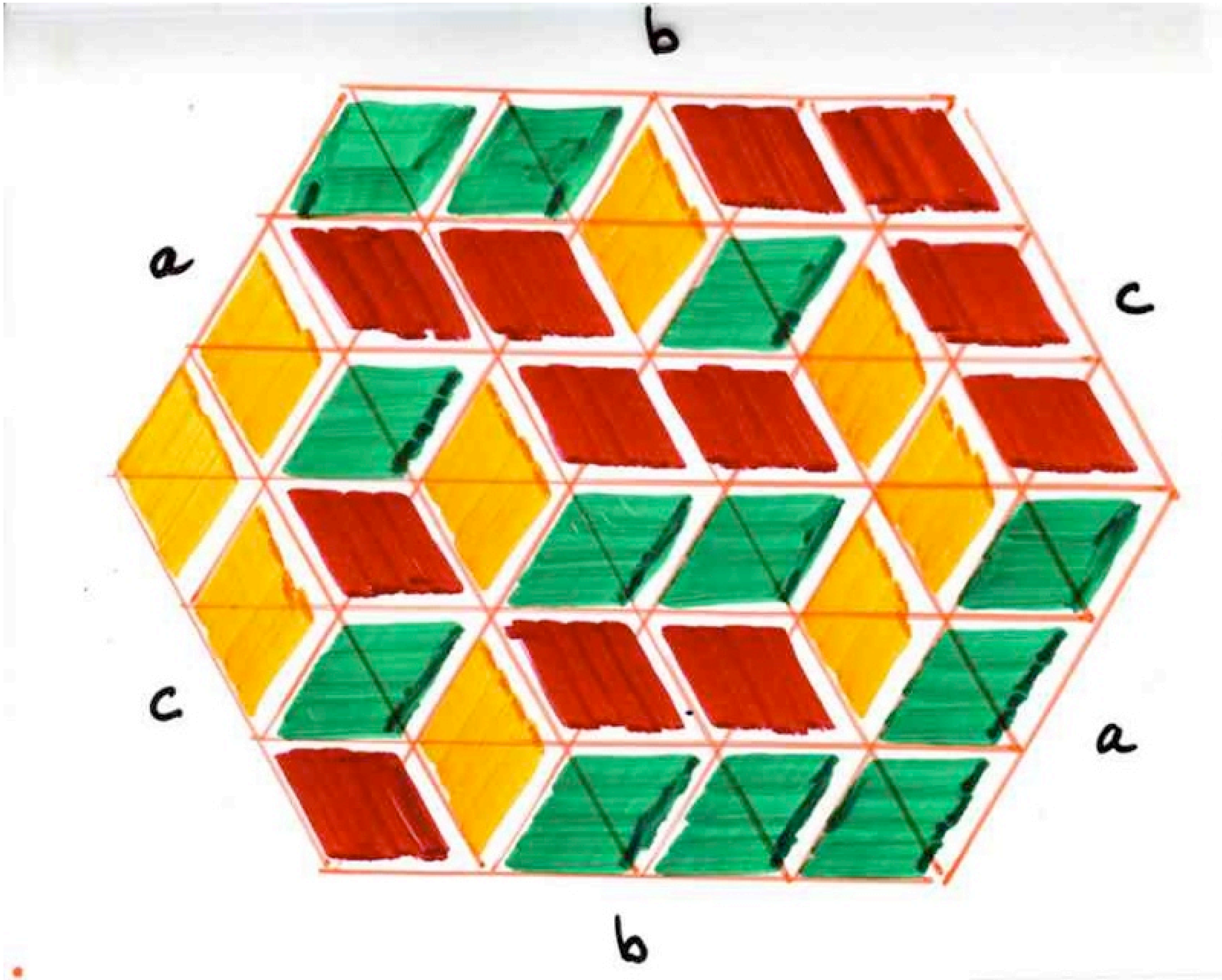
$A, A', B, B',$

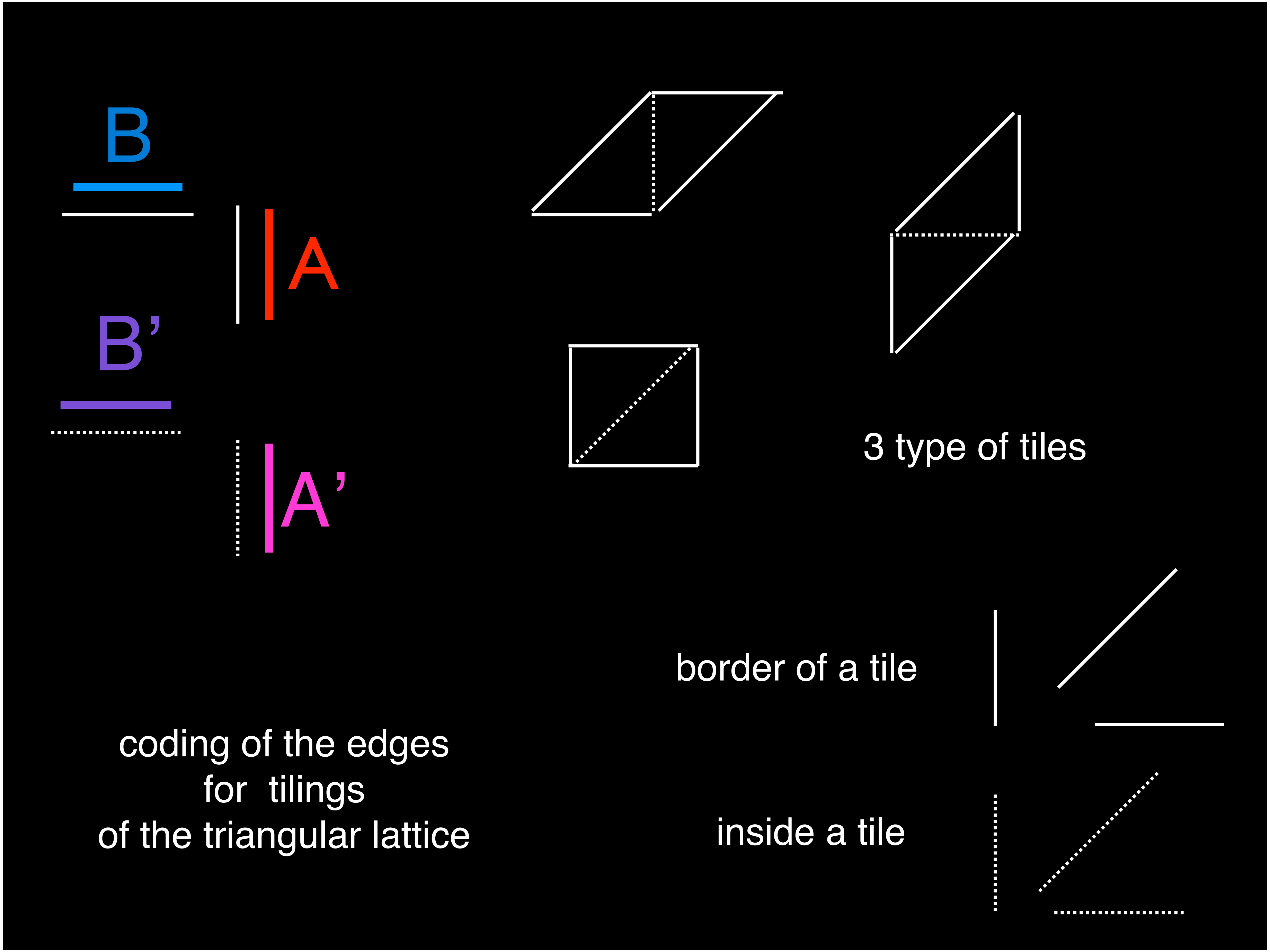
commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = AB \end{cases}$$

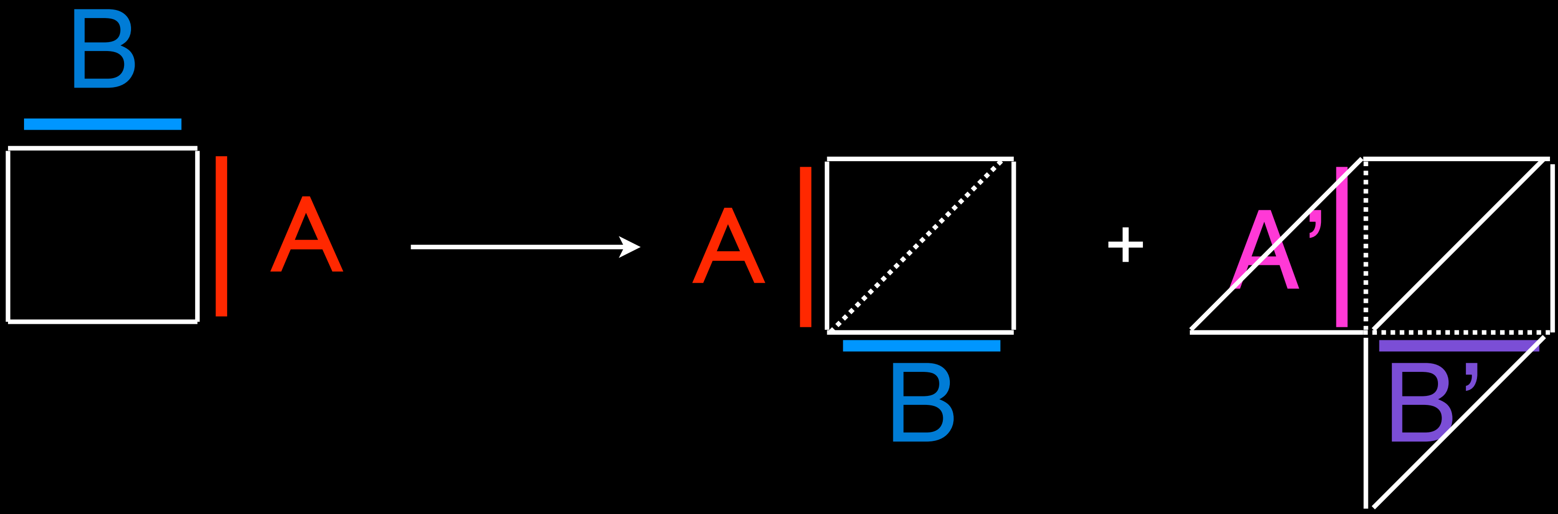
$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

(same as for ASM but with B'A' ⟶ A'B' missing !):


b
a
c
c
a
b


B
A
B'
A'
3 type of tiles
border of a tile
inside a tile
coding of the edges
for tilings
of the triangular lattice

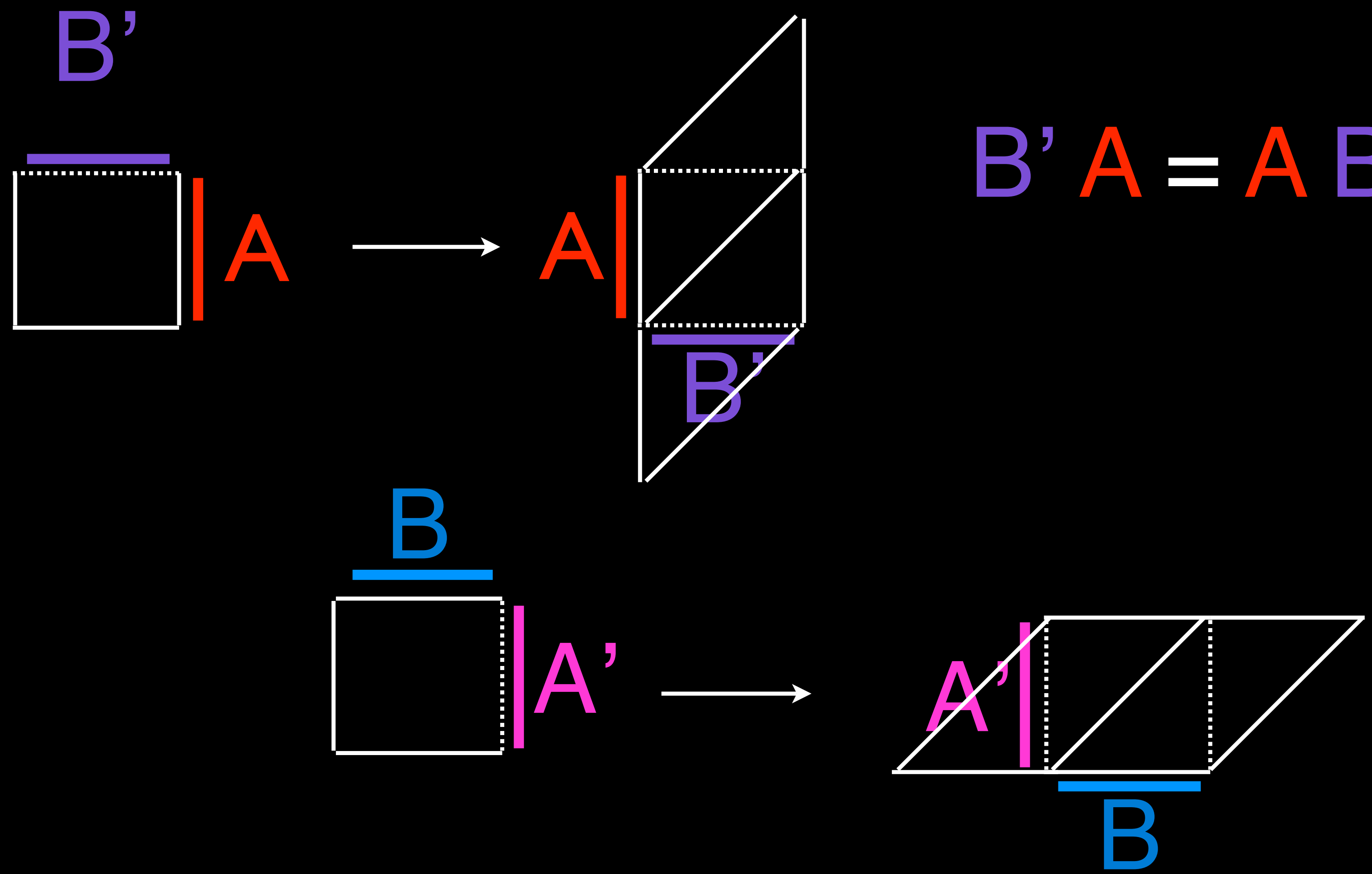
“rewriting rules” for tilings of the triangular lattice



BA = AB + A’B’

B’A’ = AB

“rewriting rules” for tilings of the triangular lattice



B’ A = A B’

B A’ = A’ B

“rewriting rules” for tilings of the triangular lattice

$$BA = AB + A'B'$$

$$B'A' = AB$$

$$B'A = AB'$$

$$BA' = A'B$$

same as for ASM , except the rewriting rule

$B'A' \longrightarrow A'B'$ is forbidden

8- parameters quadratic algebra

commutations

$$\begin{cases} BA = q_1 AB + q_2 A'B' \\ B'A' = q_3 A'B' + q_4 AB \end{cases}$$

$$\begin{cases} B'A = q_5 AB' + q_6 A'B \\ BA' = q_7 A'B + q_8 AB' \end{cases}$$

some perspectives

Questions.

- find a "combinatorial representation" for operators A, A', B, B'.

- analogue of RSK (Robinson-Schensted-Knuth) for ASM ?

- analogue of "local rules" (Fomin)

- direct proof of the formula

$$A_n = \prod_{j=1}^{n} \frac{(3j-2)!}{(n+j-1)!}$$

(nb of ASM of size n)

= 1, 2, 47, 429, ...

?

ASM

1-, 2-, 3- enumeration $A_n(x)$

Colomo, Pronco, (2004)

Hankel determinants

(continuous) Hahn, Meixner-Pollaczek,
(continuous) dual Hahn orthogonal polynomials

Ismail, Lin, Roan (2004)
XXZ spin chains and Askey-Wilson operator

xgv website :

http://www.labri.fr/perso/viennot/

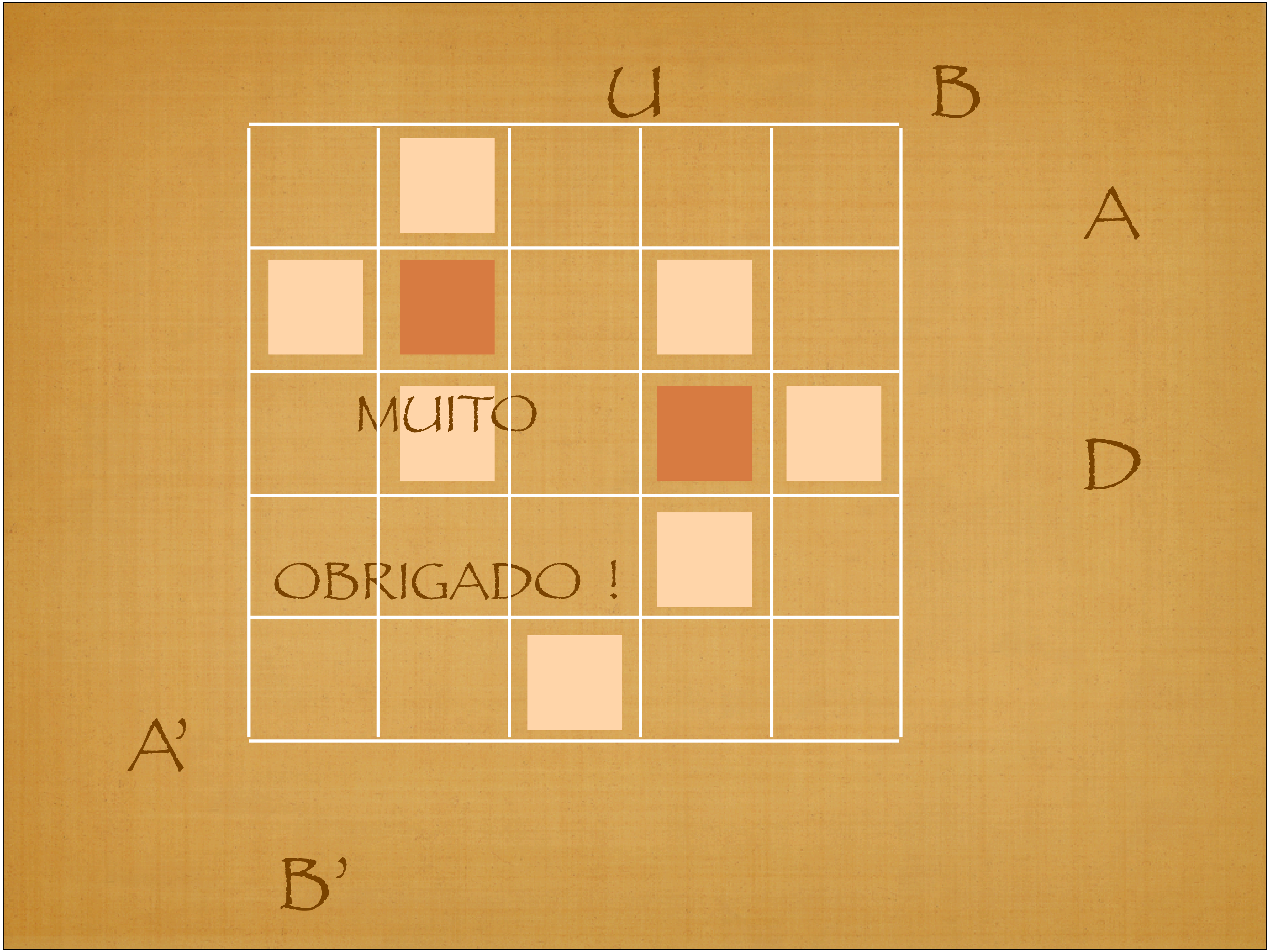
Recherche, cv, publications, exposés, diaporamas, livres, petite école, photos: voir ma page personnelle ici

Vulgarisation scientifique voir la page de l'association Cont'Science

downloadable papers, slides and lecture notes, etc ... here

(the summary on page “recherches” and most slides are in english)

MUITO OBRIGADO


U
B
A
MUITO
D
OBRIGADO !
A'
B'

spin chains   ASM   enum combin   generating function

binary trees   bijective proof   ASM enumeration

Plane Partitions   PP and Paths   LGV   binomial det

Sym PP   CSPP   TSPP   10 formules   TSSCPP

6-vertex

Tilings   Aztec   Matchings   Pfaffiens

RS conj   FPL   Random

Algebraic approach   RSK, U D   Operators for tilings

Perspectives   fin