From a letter of Leonhard Euler to modern researches at the crossroad of algebra, geometry, combinatorics and physics

K.Madhava Sarma Memorial Distinguished Lecture

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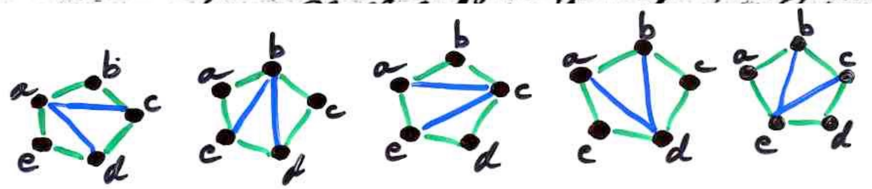
A letter from Leonhard Euler to Christian Goldbach ....

Berlin, 4 September 1751

Leonhard
Euler
1707 - 1783



wenn $n = 3.\ 4.\ 5.\ 6.\ 7.\ 8.\ 9.\ 10$

so ist $x = 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430$

$$x = \frac{2.6.10.14.18.22.\ \ldots\ (4n-10)}{2.3.4.5.6.7.\ \ldots\ (n-1)} = \frac{(2n)!}{(n+1)!\,n!}$$

$1 = \frac{2}{2};\ 2 = 1.\frac{6}{3};\ 5 = 2.\frac{10}{4};\ 14 = 5.\frac{14}{5},\ 42 =$

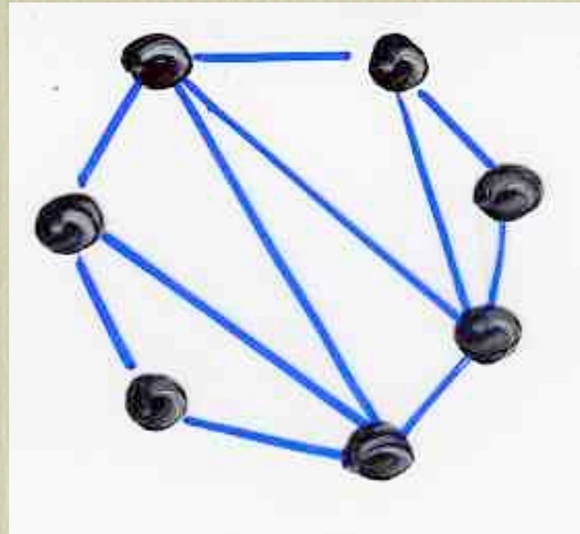
$1,\ 2,\ 5,\ 14,\ 42,\ 132,$ etc.

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$$

also wenn $a = \frac{1}{4}$ so ist $1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} + \text{etc} = 4.$

Berlin d. 4. Sept 1751.

L. Euler



theilet, und solches kann auf 5 verschiedene Arten geschehen vermittelst durch die Diagonales I. $\frac{ac}{ad}$; II. $\frac{bd}{be}$; III. $\frac{ca}{ce}$; IV. $\frac{db}{da}$; V. $\frac{ec}{eb}$.
Ferner kann ein Sechseck durch 3 Diagonales in 4 Triangula zerschnitten, und dieses kann auf 14 verschiedene Arten geschehen.
Nun ist die Frage generaliter, da ein Polygonum von n Seiten durch $n-3$ Diagonales in $n-2$ Triangula zerschnitten wird, auf wie vielerley verschiedene Arten solches geschehen könne.
Setze ich nun die Anzahl dieser verschiedenen Arten $= x$ ... gefunden

so habe ich ...

wenn $n =$ ...

$$1, 2, 5, 14, 42, 132, 429, 1430, \ldots$$

so ist $x = 1, 2, 5, 14, 42, 132, 429, 1430$
Hieraus habe ich nun den Schluss gemacht, dass generaliter sey

$$x = \frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \ldots (4n-10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \ldots (n-1)} = \frac{(2n)!}{(n+1)!\, n!}$$

... $2 = 1 \cdot \frac{6}{3}$, $5 = 2 \cdot \frac{10}{4}$; $14 = 5 \cdot \frac{14}{5}$; $42 = 14 \cdot \frac{18}{6}$; ...

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$n! = 1 \times 2 \times 3 \times \ldots \times n$$

$$\frac{1-2a-\sqrt{1-4a}}{2a^2}$$

$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + etc$

[illegible]

$1+2a+5a^2+14a^3+42a^4+132a^5+etc = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$

also wenn $a = \frac{1}{4}$ so ist $1+\frac{2}{4}+\frac{5}{4^2}+\frac{14}{4^3}+\frac{42}{4^4}+etc = 4.$

[illegible]

Berlin d. 4. Sept 1751.

4 Sept 1751
Berlin

gehorsamster Diener
L. Euler

... summa 5 ...

unter welchen summa $aa+bb+cc+dd$ so beschaffen werden kan.

Dass aber quatuor quadrata

Denn es ist $a+b+c+d=2$; so ist

$$aa+bb+cc+dd = (a+b-1)^2 + (a+c-1)^2 + (b+c-1)^2 + 1$$

folglich ist $aa+bb+cc+dd-1$ in 3 quadrata resolubel. Wann nun ...

Ist nun $8n+3$ in 3 quadrata resolubel, so wird

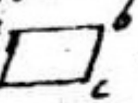
$8n+3 = (a+b-1)^2 + (a+c-1)^2 + (b+c-1)^2$ dergestalt dass $a+b+c+d=2$

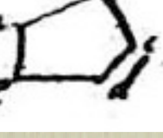
$8n+4 = aa+bb+cc+dd$ dergestalt dass ... Zweifelsfall

Aus diesem ist das schöne Theorema welches ... herzuleiten ...

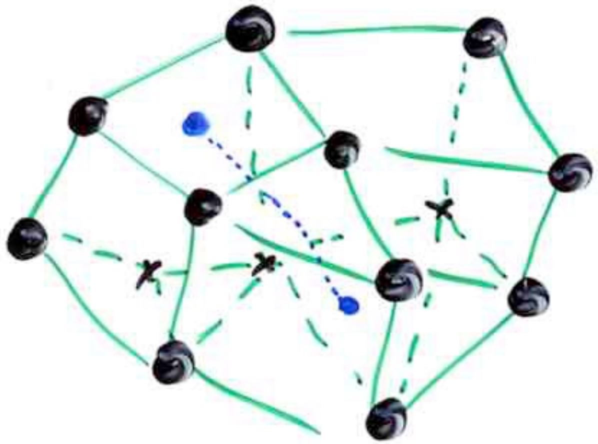
aus dem Theoremate Fermatiano herzuleiten ...

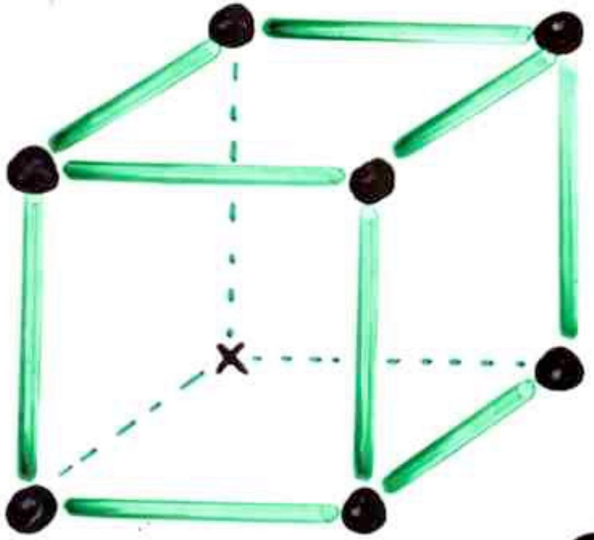
Ich bin neulich auf eine Betrachtung gefallen, welche mir nicht wenig merkwürdig vorkommen. Dieselbe betrifft, auf wie vielerley Arten ein gegebenes Polygonum durch Diagonal Linien in Triangula zerschnitten werden können.

Also ein quadrilaterum  kann entweder durch die Diagonalen ac oder durch bd, und also auf 2erley Art in zwey Triangula resolvirt werden.

Ein Fünfeck  wird durch 2 Diagonales in 3 Triangula zerschnitten werden ...

convex polyhedron



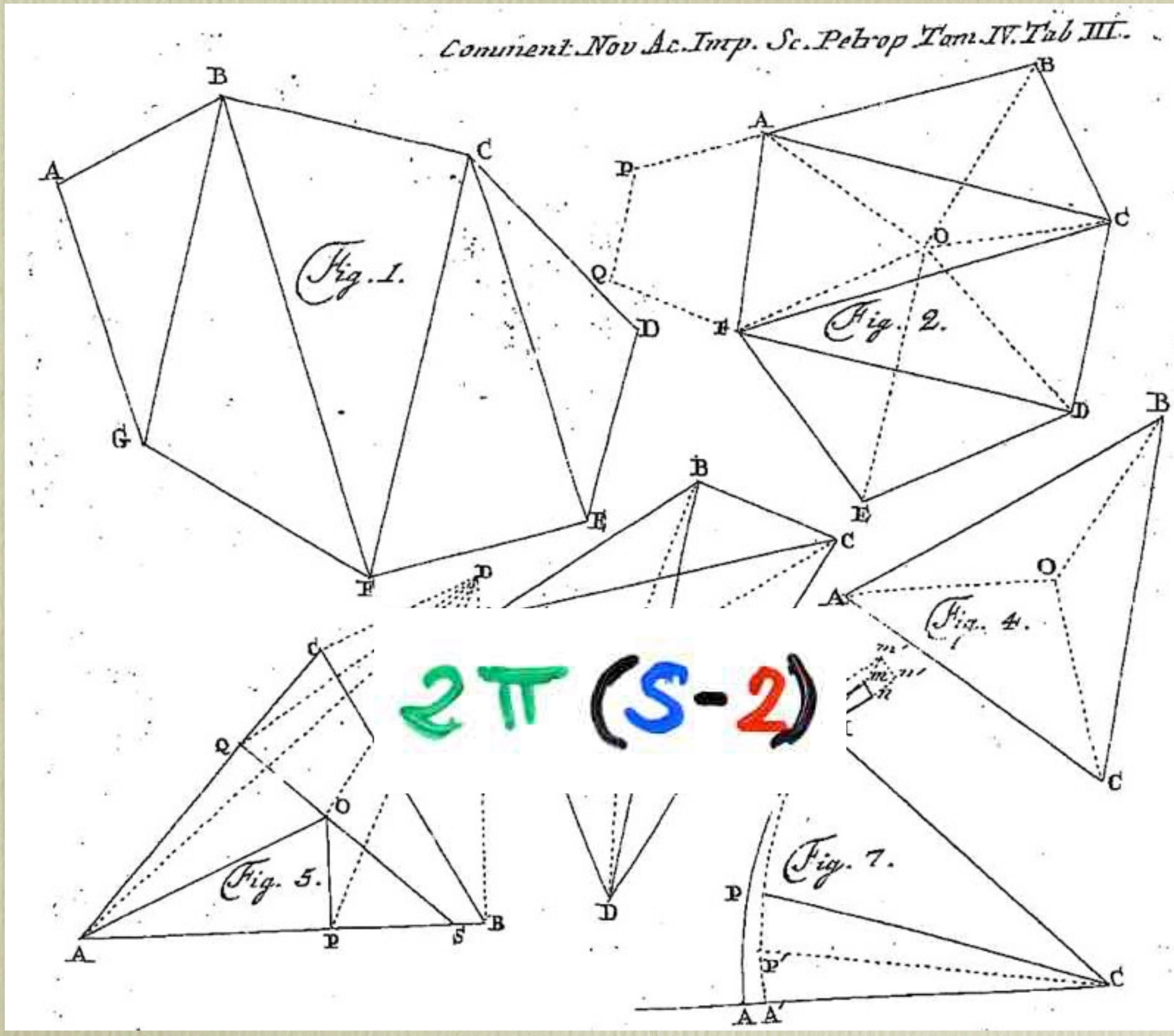


Descartes formula

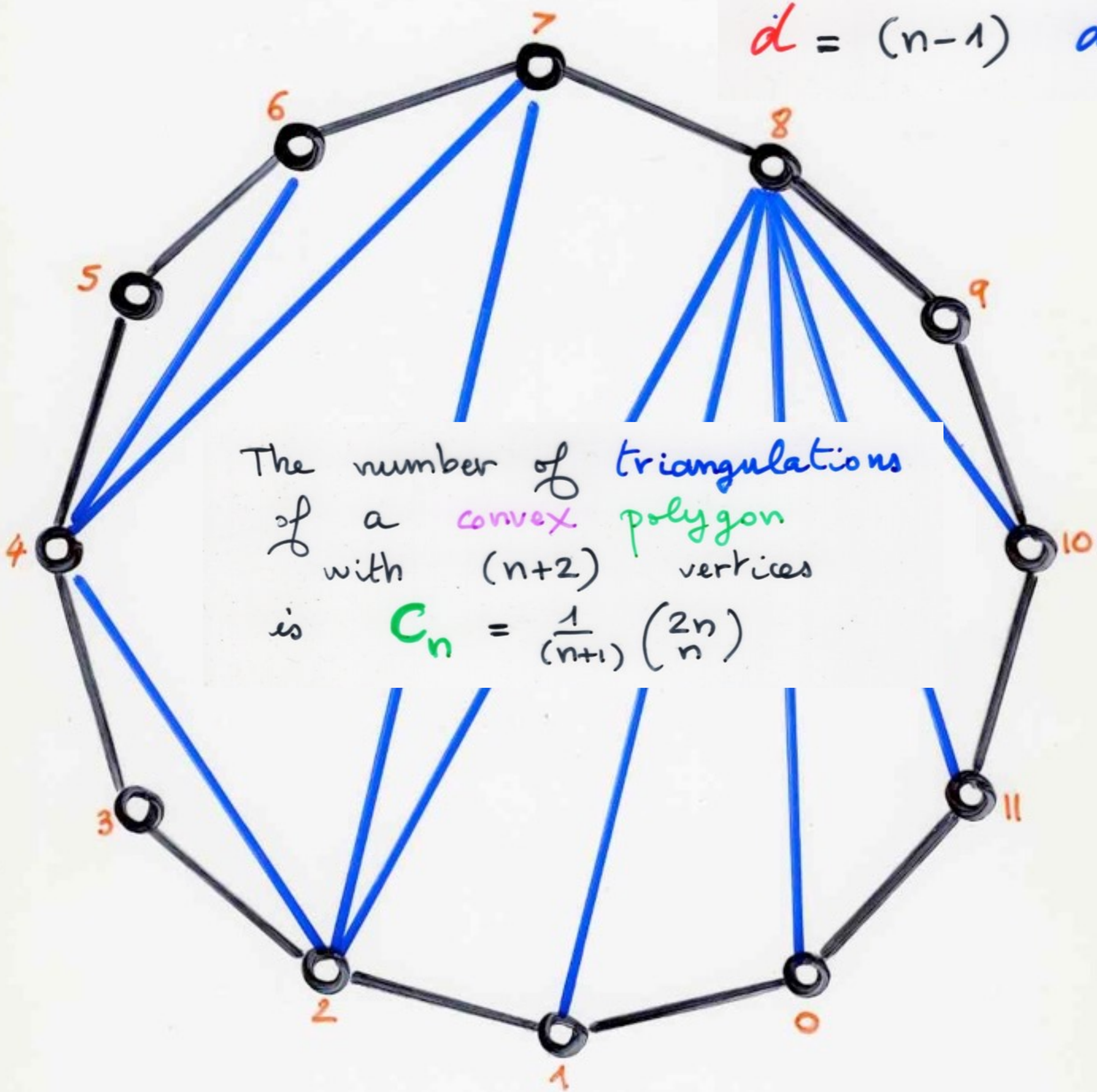
$$\sum \text{defects} = (2\pi) \times (S - A + F)$$

$$S - A + F =$$

$$8 - 12 + 6 = 2$$


Comment. Nov Ac. Imp. Sc. Petrop Tom. IV. Tab III.
Fig. 1.
Fig. 2.
Fig. 4.
Fig. 5.
Fig. 7.
2π (S-2)

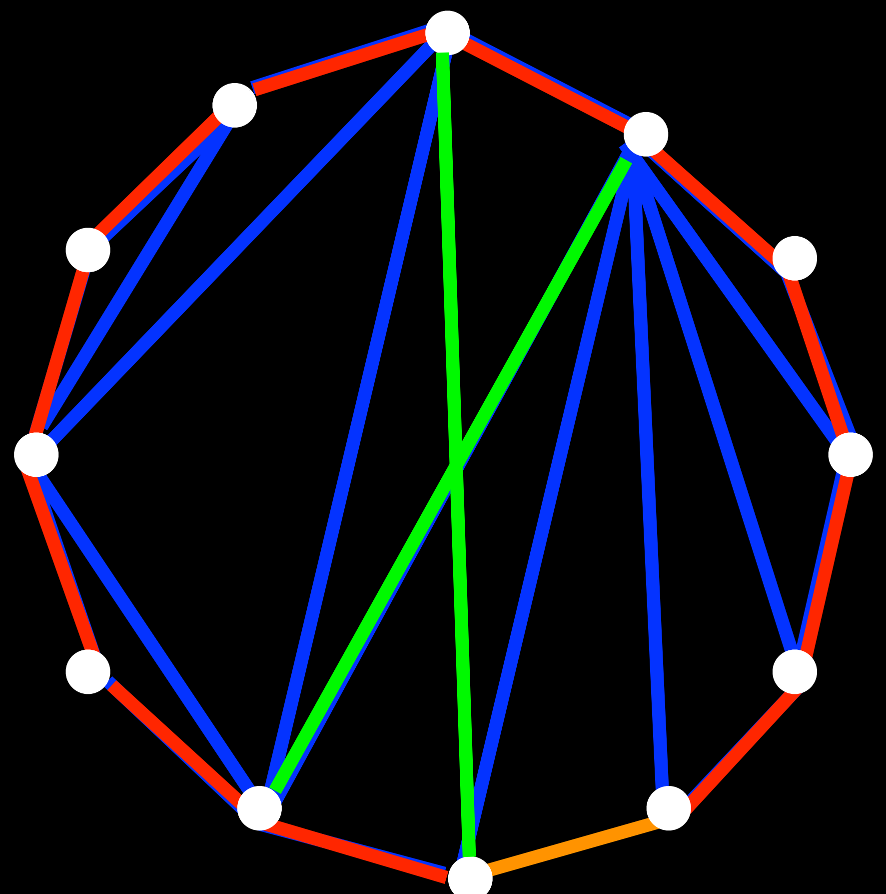
$d = (n-1)$ diagonals

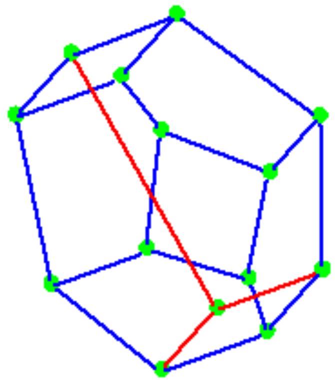


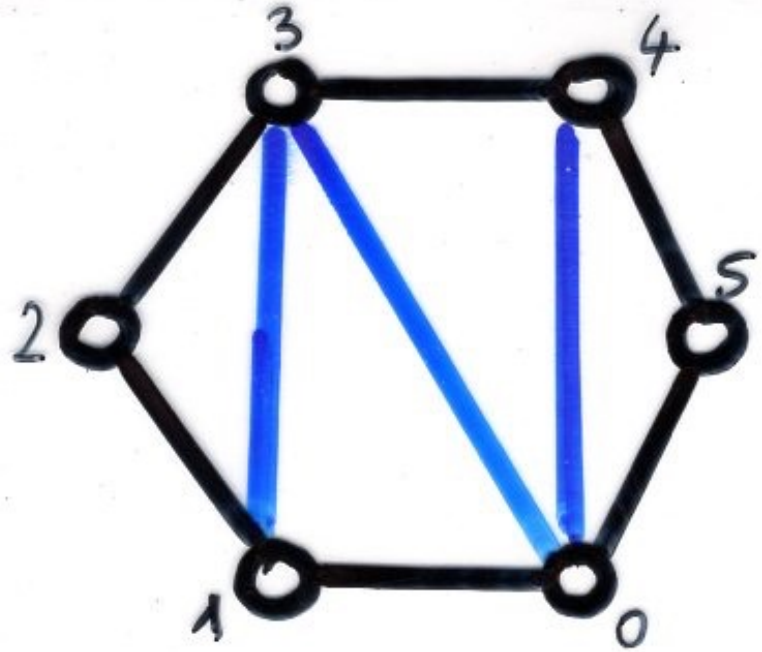
The number of triangulations of a convex polygon with $(n+2)$ vertices is $C_n = \frac{1}{(n+1)}\binom{2n}{n}$

flips for triangulations

associahedron



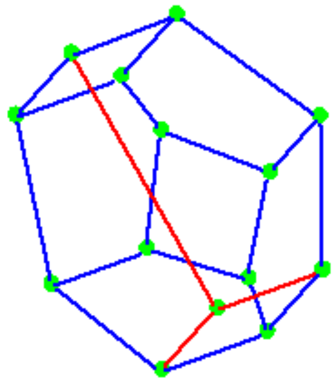


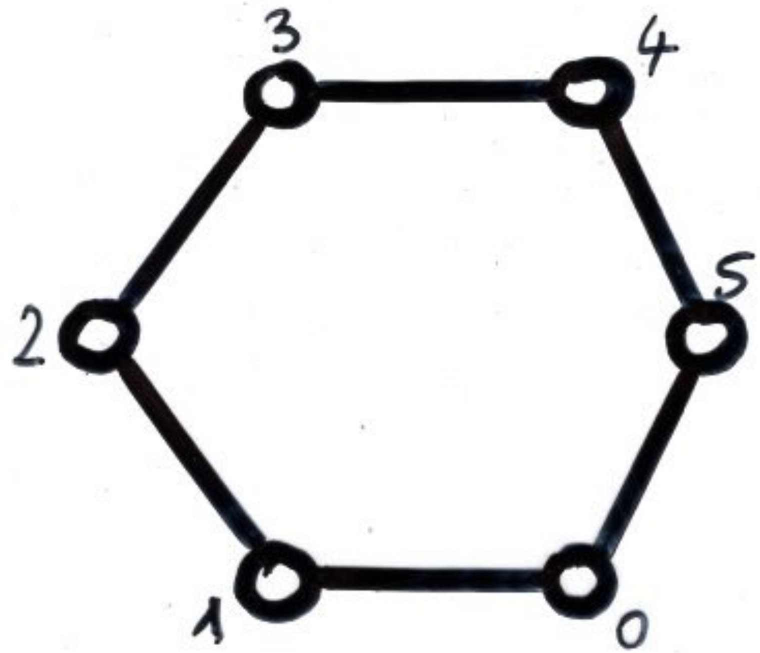


14 vertices

21 edges

Is it possible to realize the cells structure of the associahedron as the cells of a convex polytope ?

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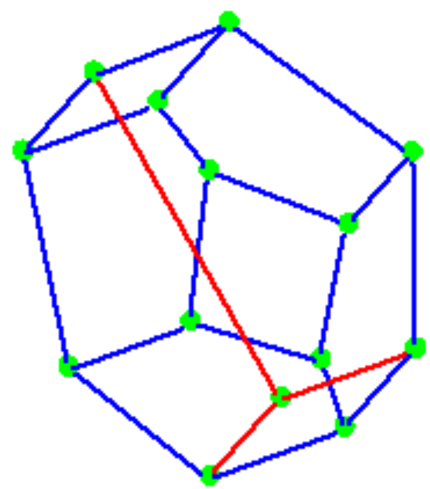

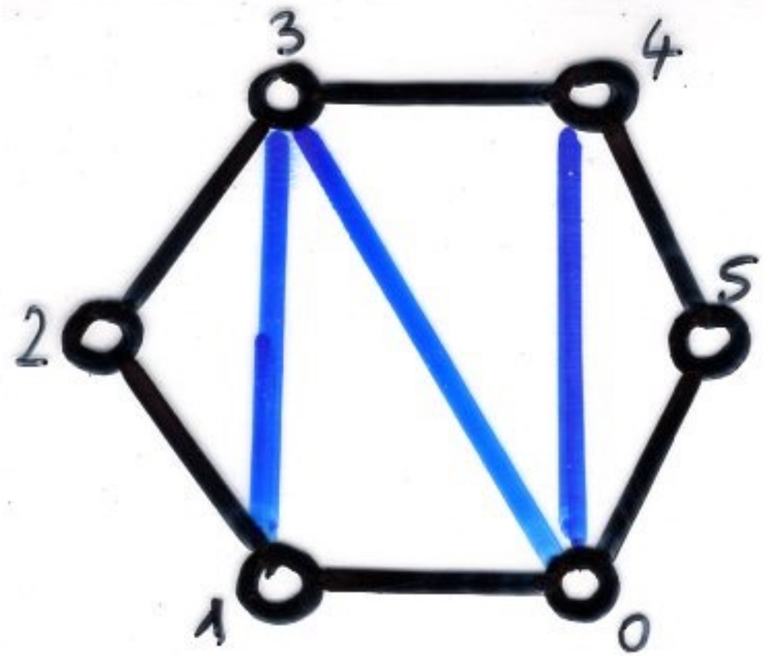
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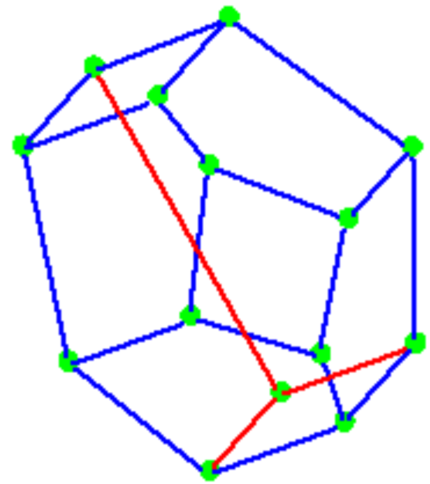

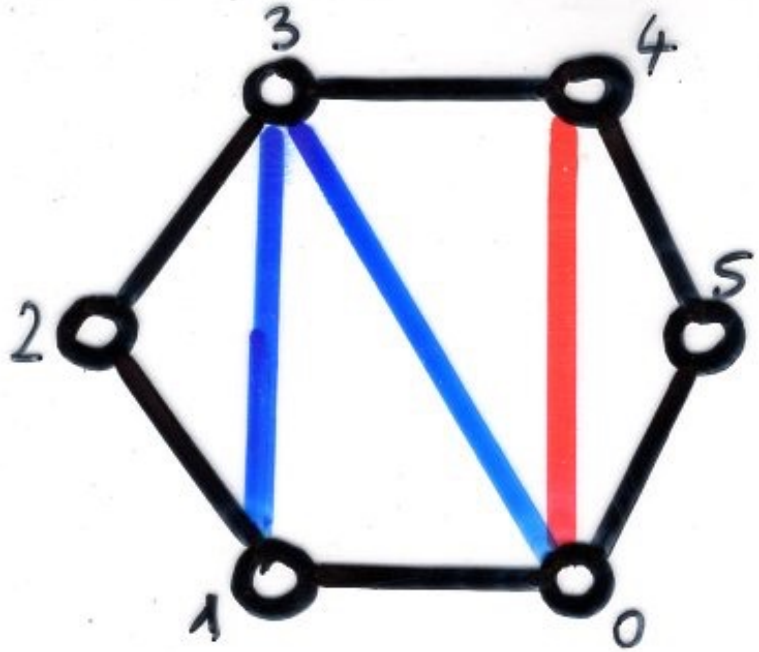
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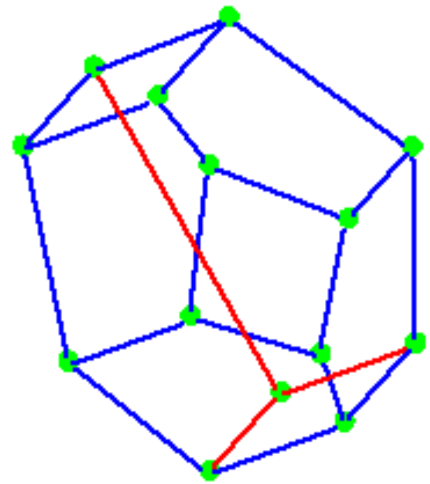

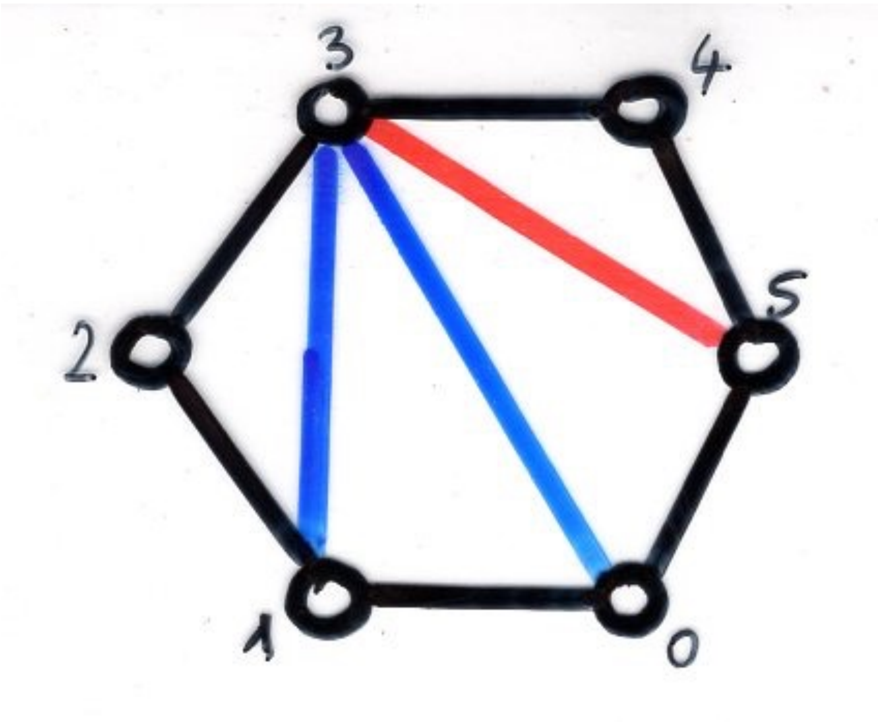
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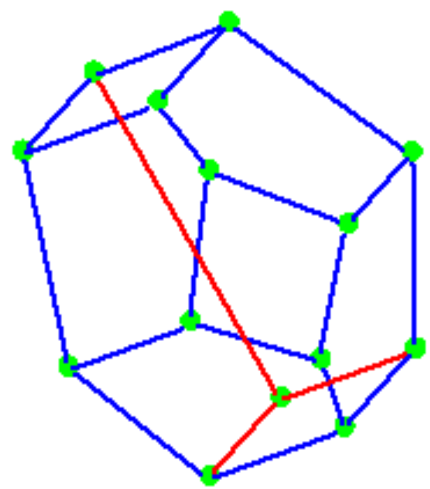
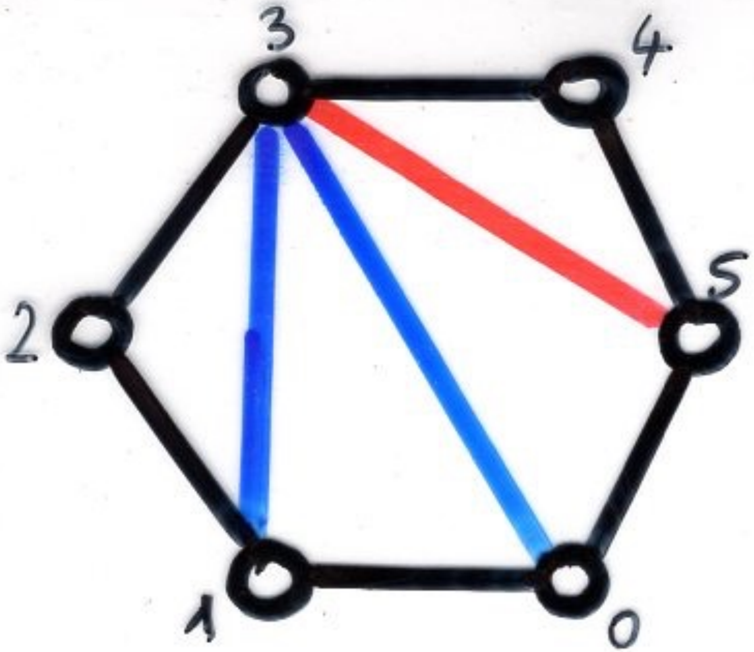
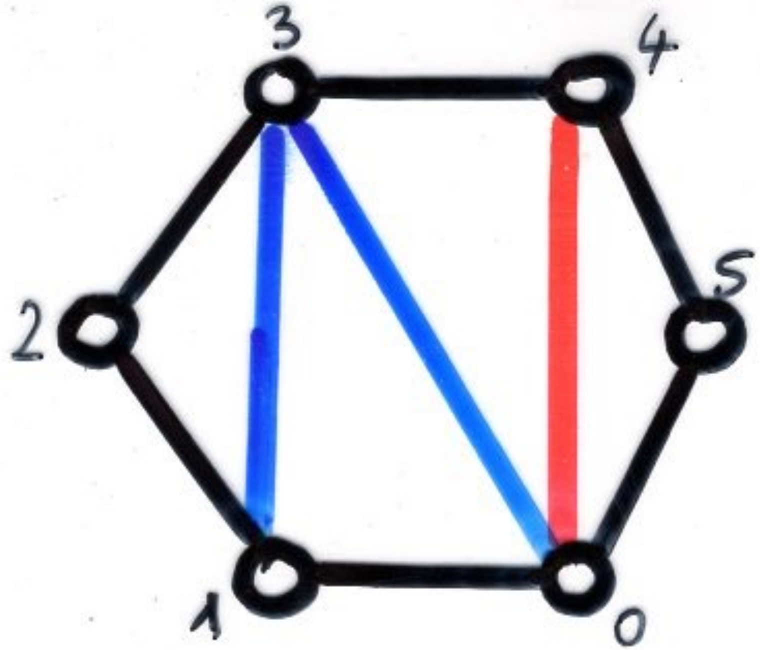
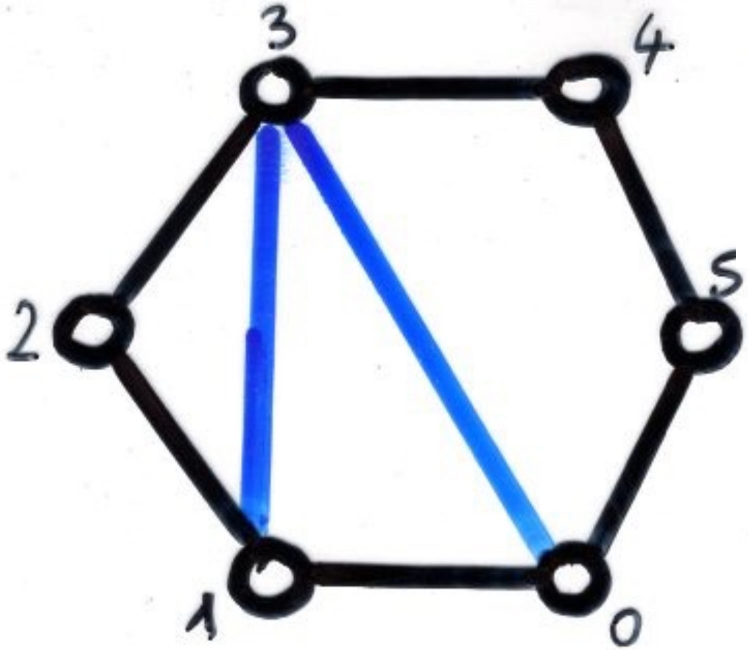


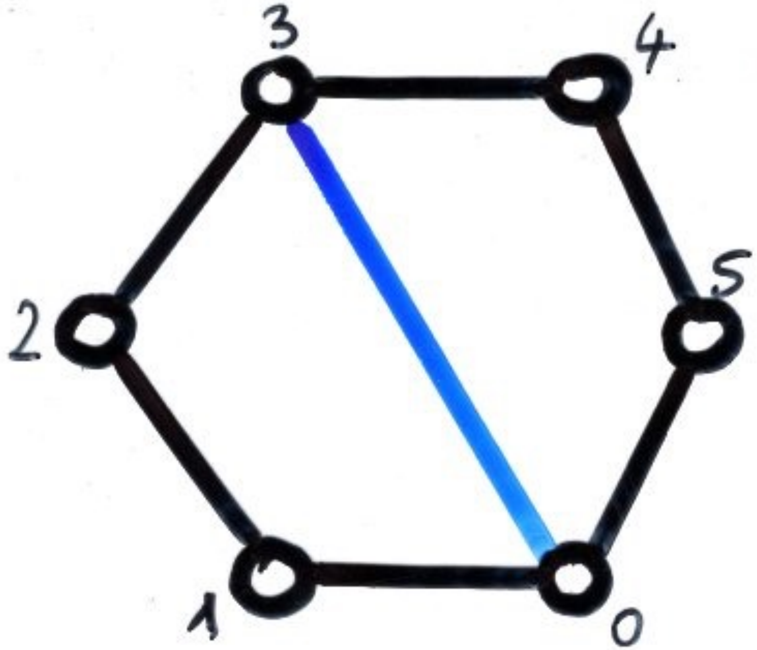




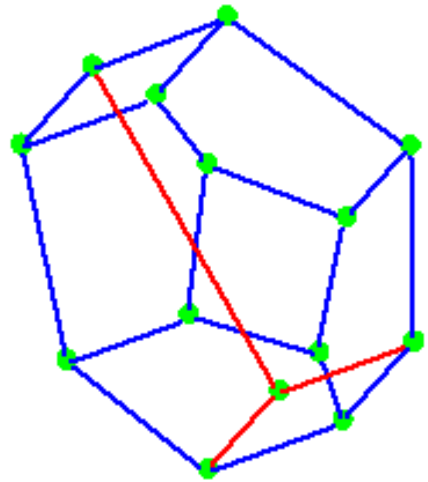


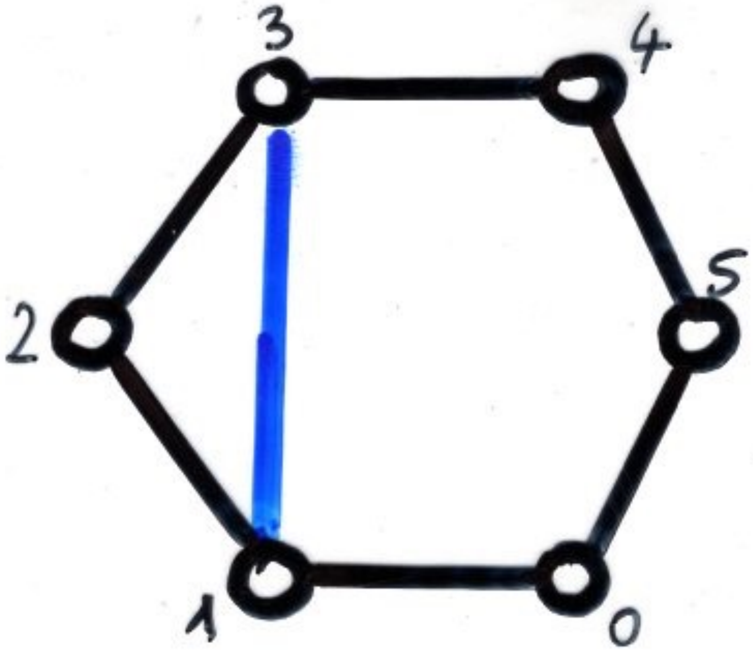
$\longleftrightarrow$ edges

21 edges



faces



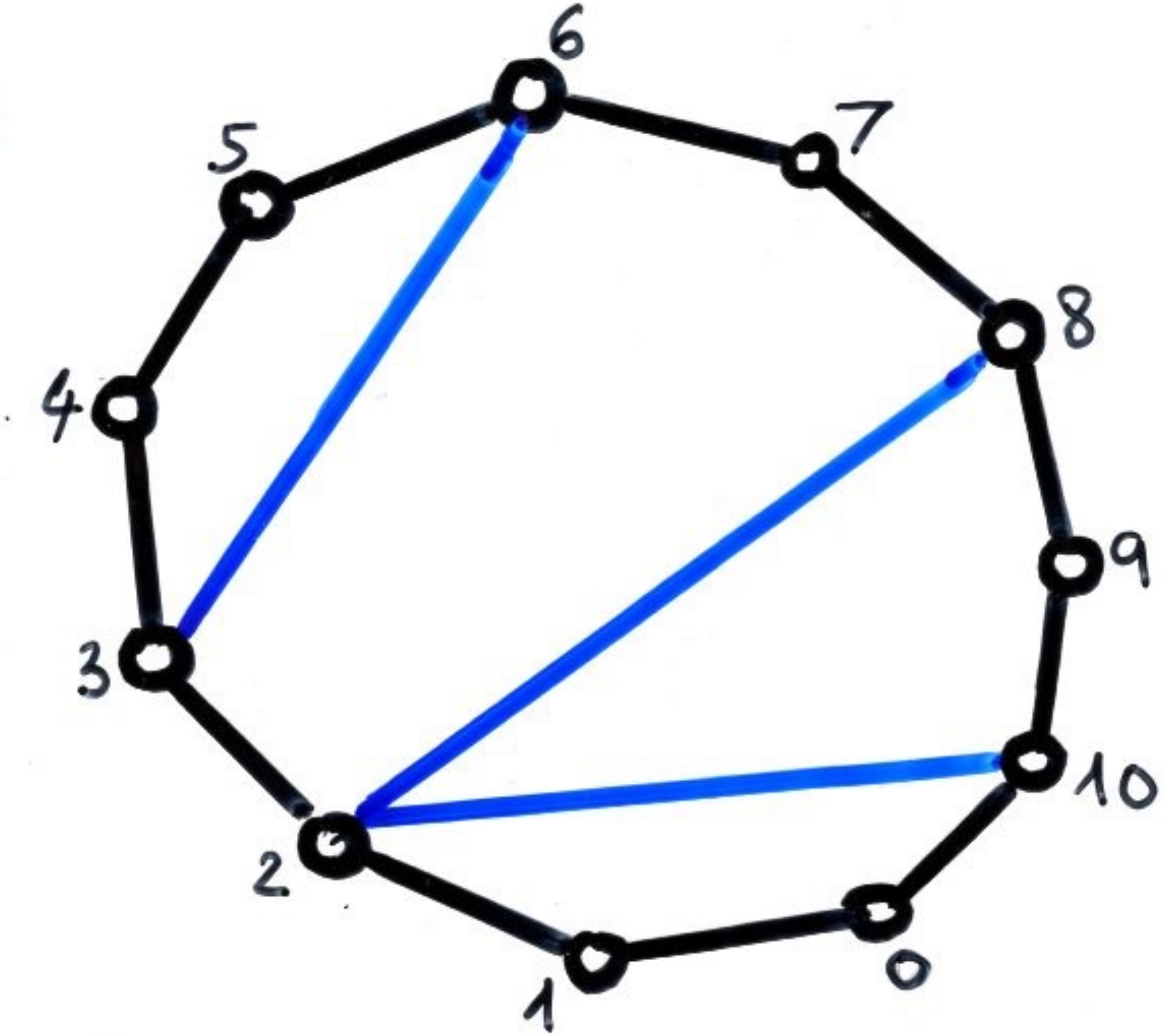


9 faces

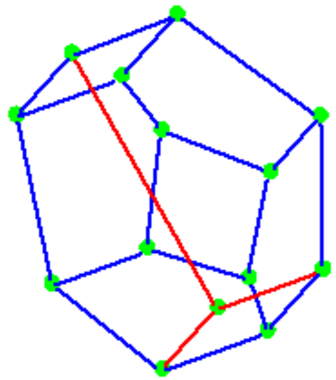
$\begin{cases} 6 & \text{pentagons} \\ 3 & \text{rectangles} \end{cases}$

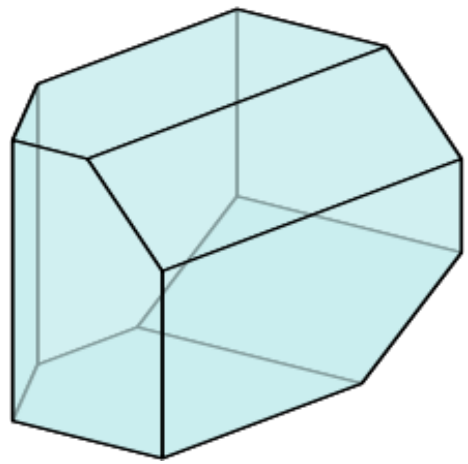
partial triangulations

cells




4000 BASEL
e-k+f=2
Leonhard Euler
1707-1783
AUSGABETAG 6. MÄRZ 2007
LEONHARD EULER
1707–1783
130
300
300. Geburtstag
300ème anniversaire
300° anniversario
300th anniversary



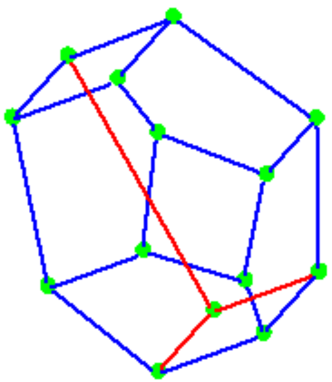


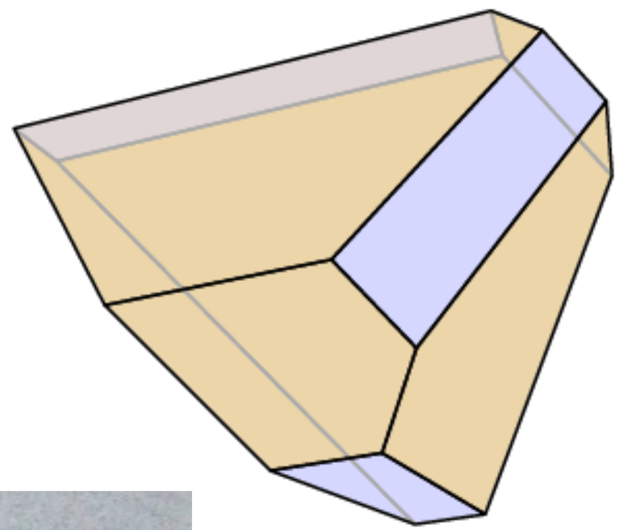
14 vertices
21 edges
9 faces

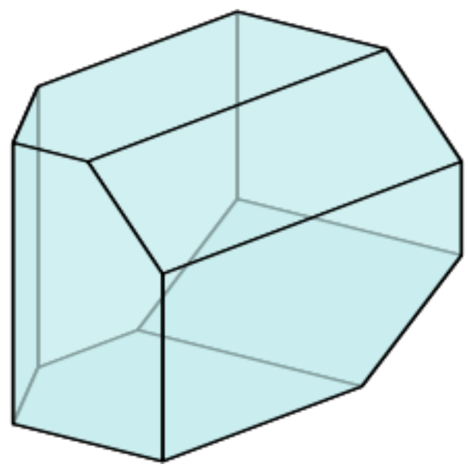
$$S - A + F$$
$$14 - 21 + 9 = 2$$

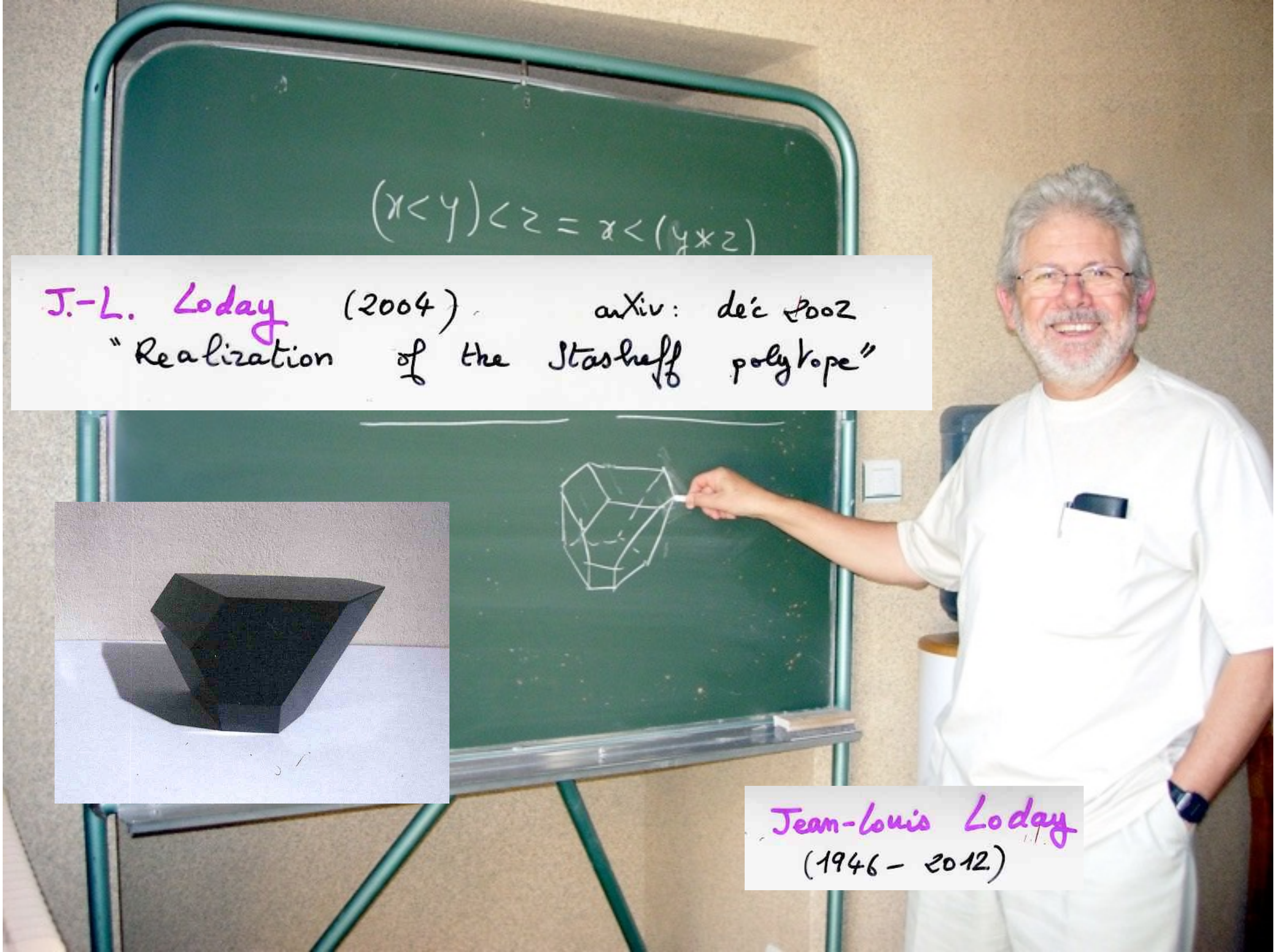
$\begin{cases} 6 \\ 3 \end{cases}$ pentagons
rectangles

Is it possible to realize the cells structure of the associahedron as the cells of a convex polytope ?









J.-L. Loday (2004) arXiv: déc 2002

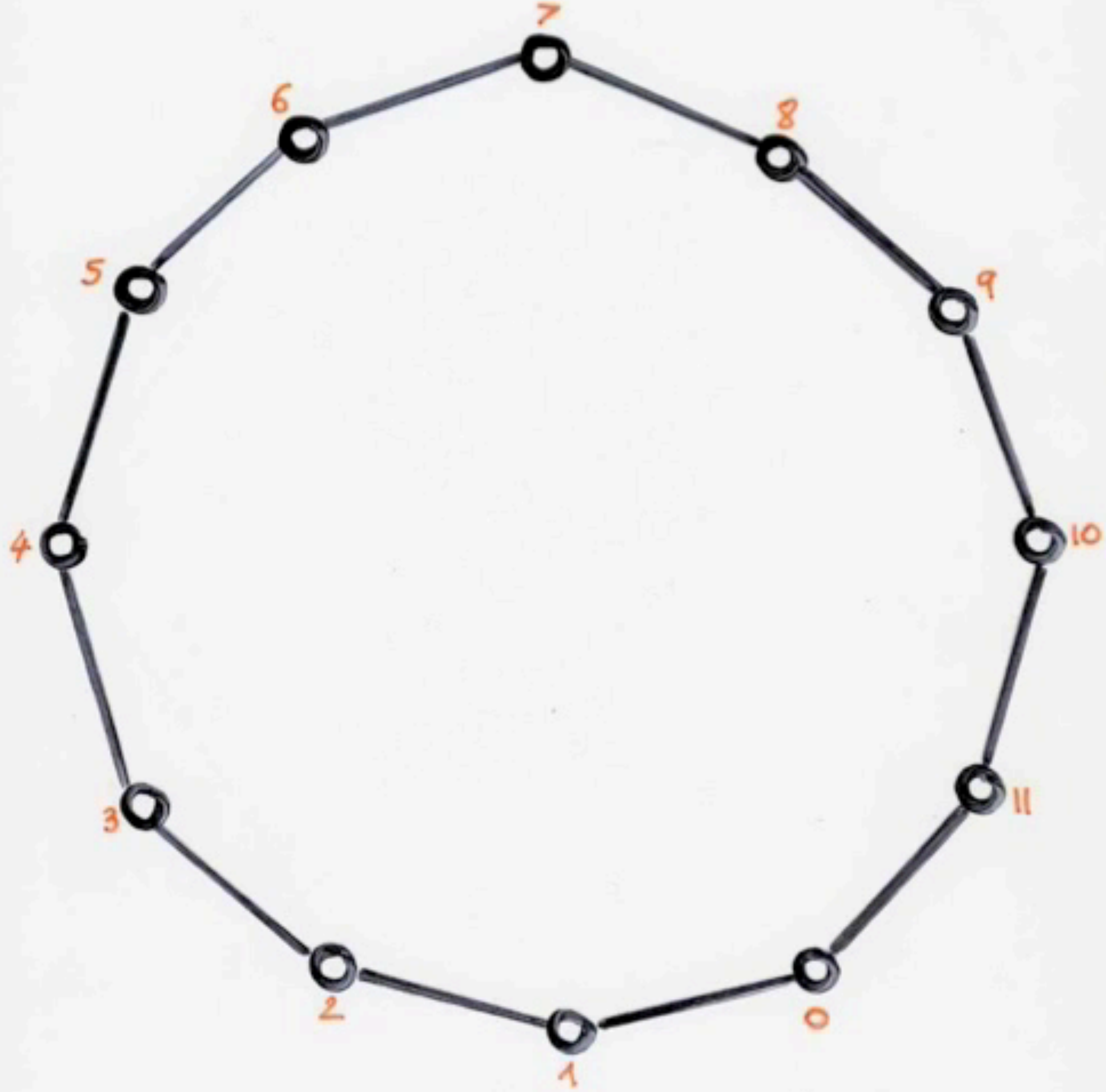
"Realization of the Stasheff polytope"

Jean-Louis Loday

(1946 – 2012)

from triangulations

to binary trees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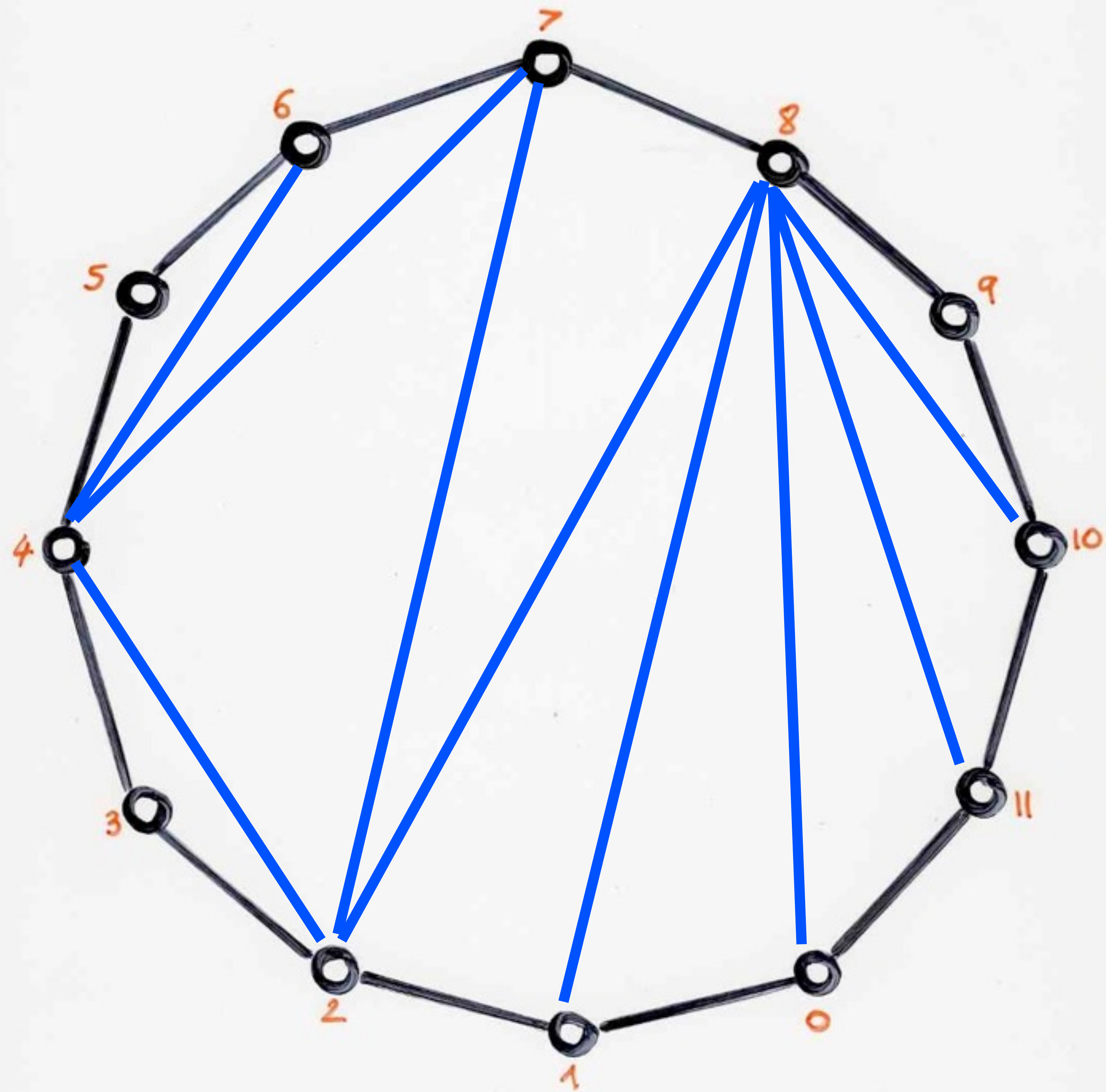

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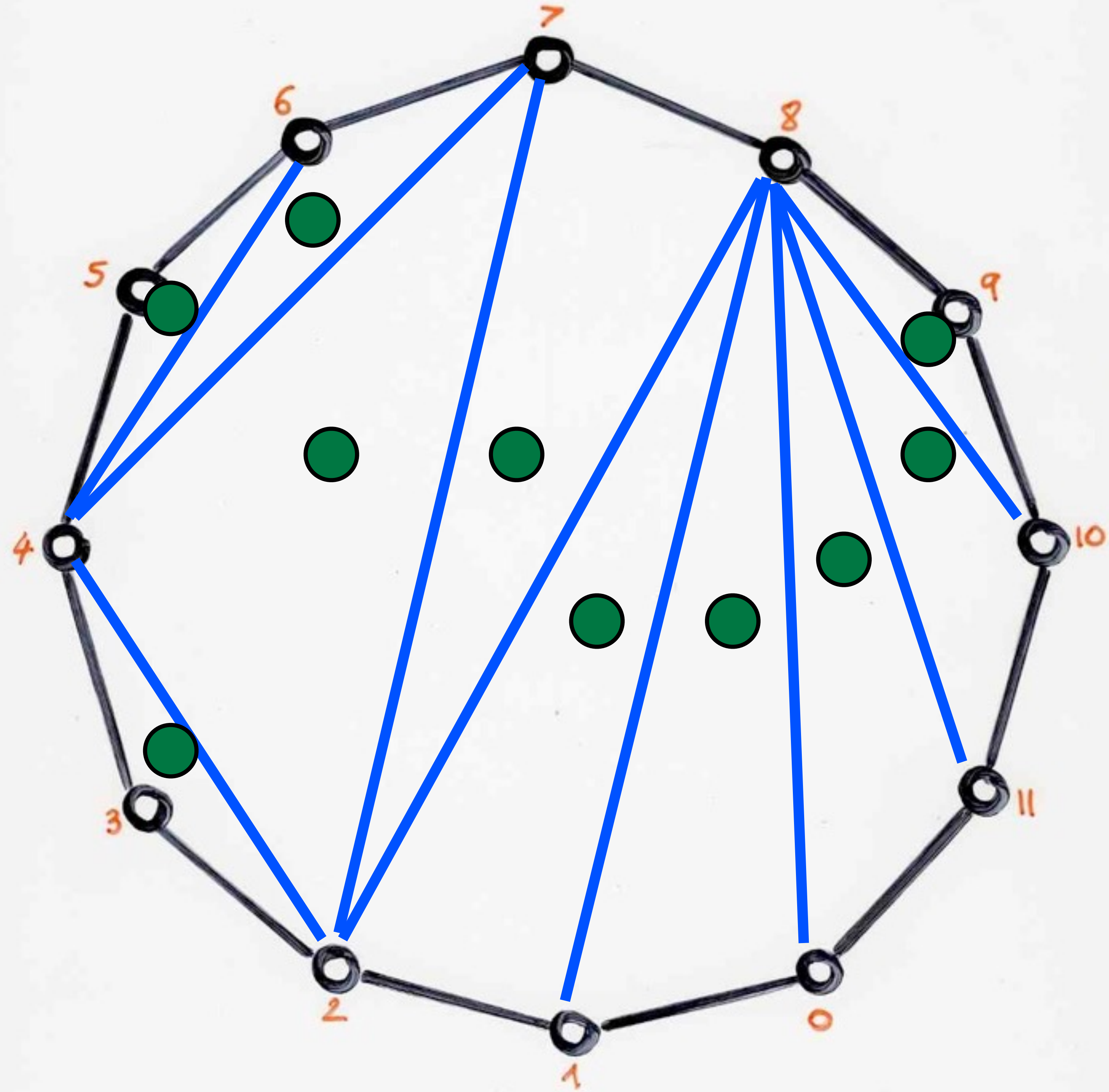

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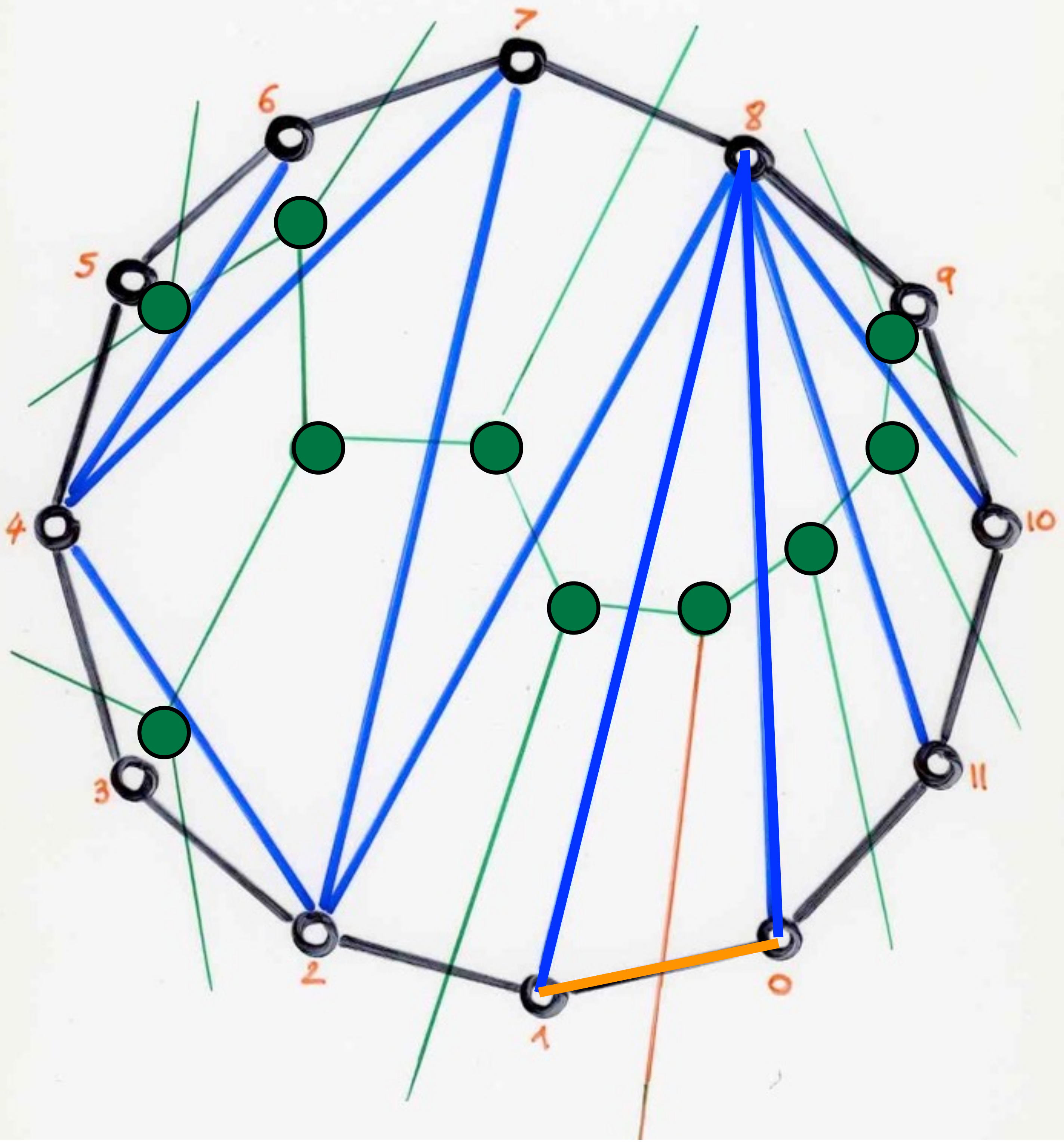

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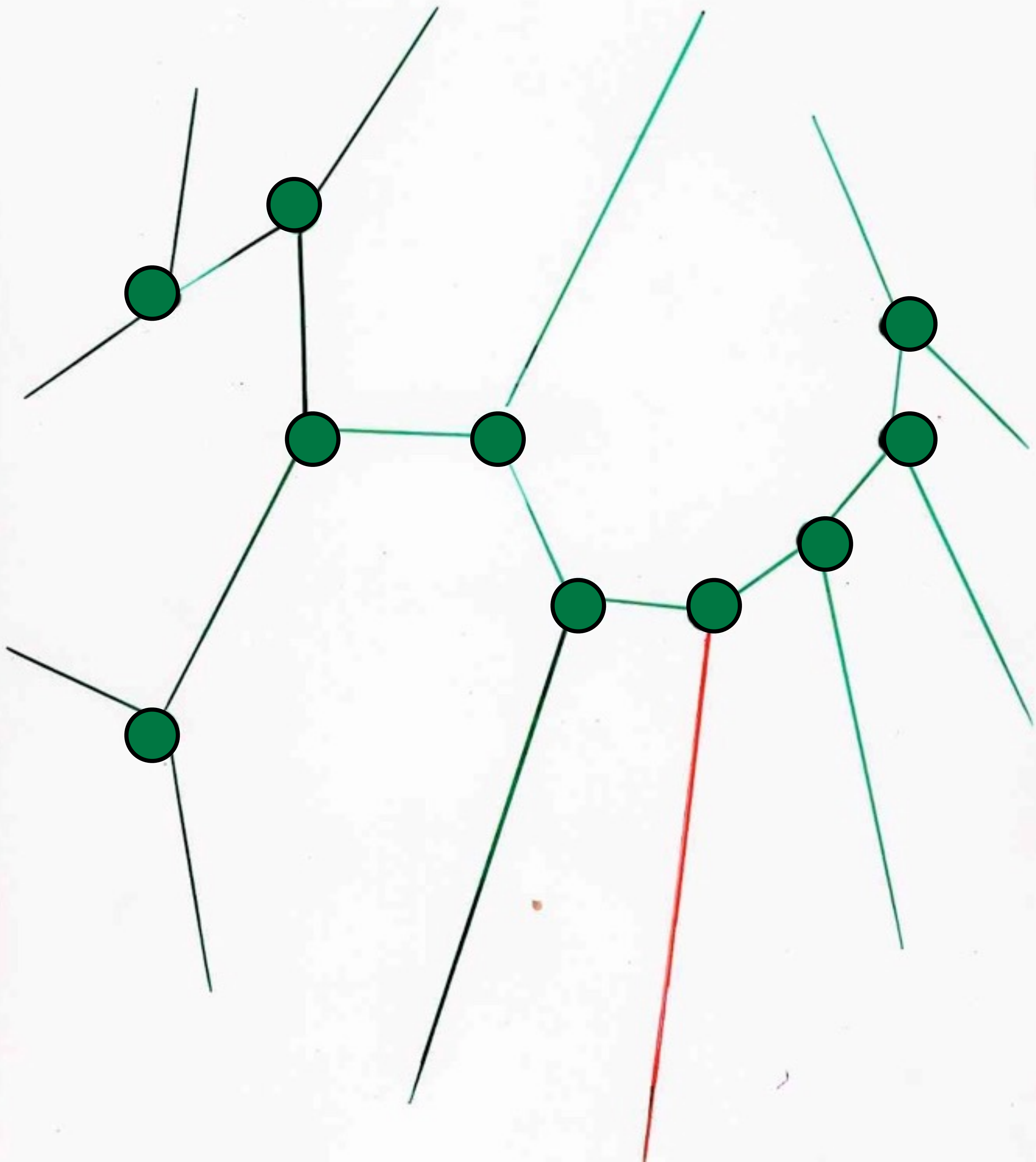
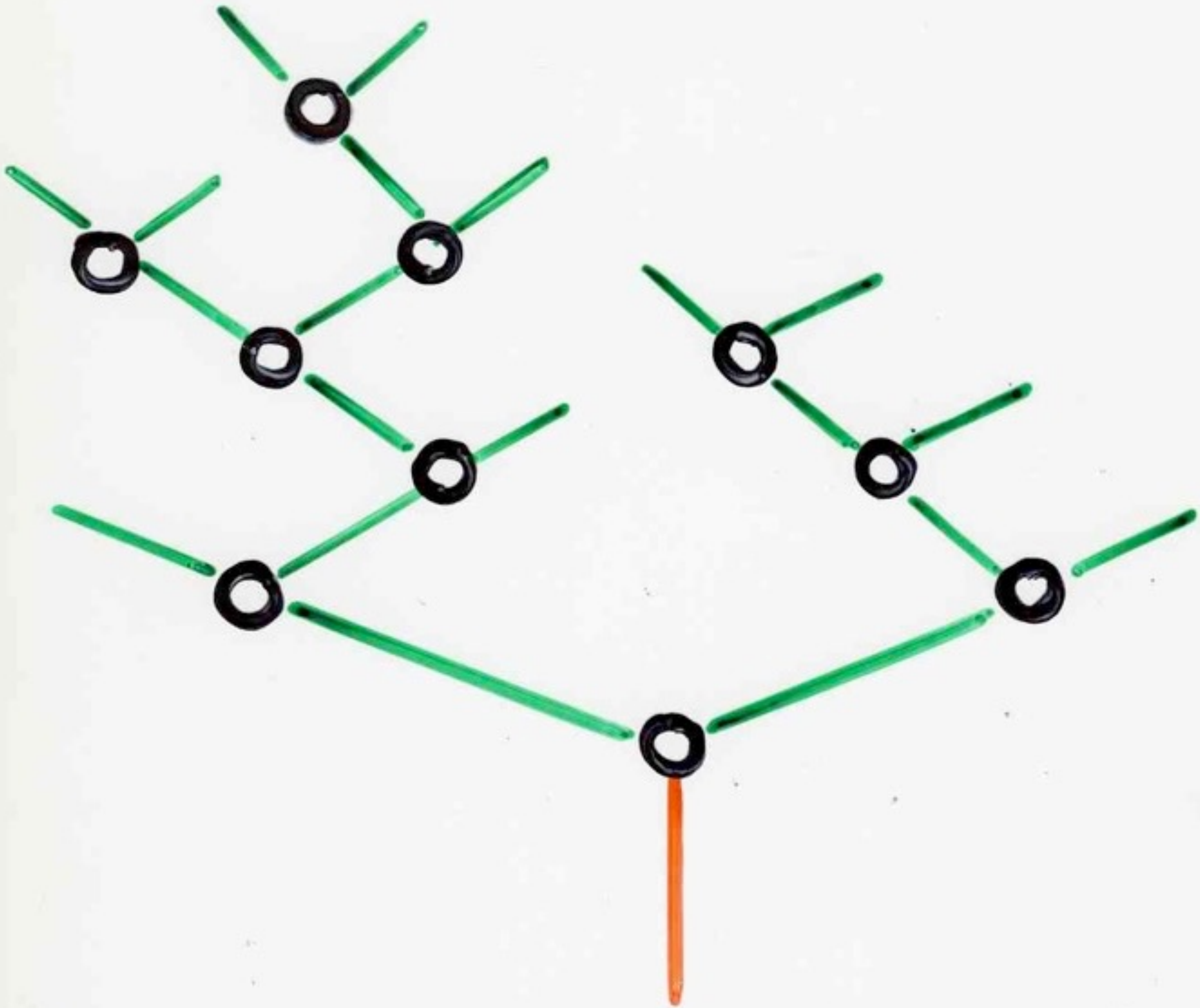
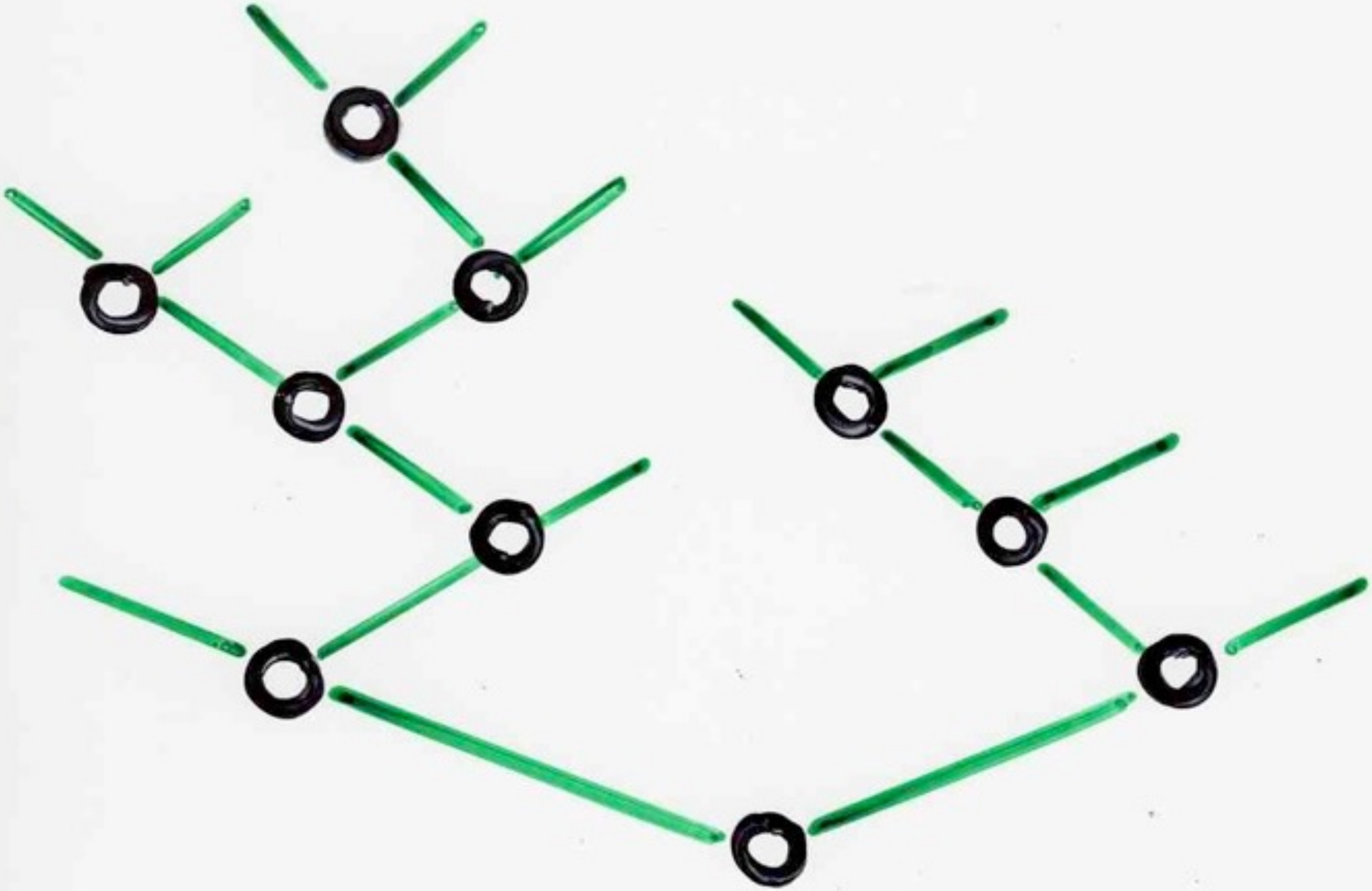
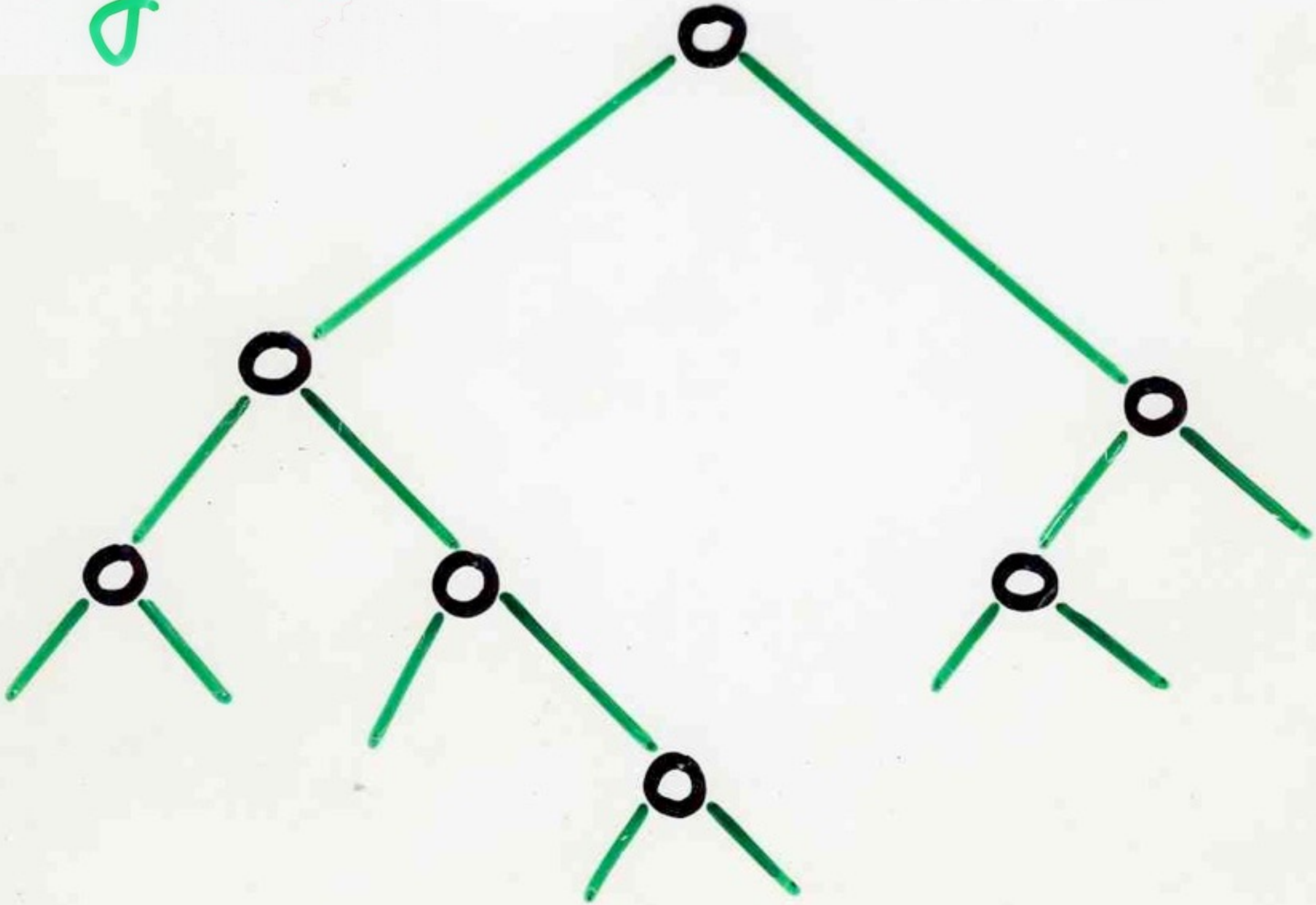




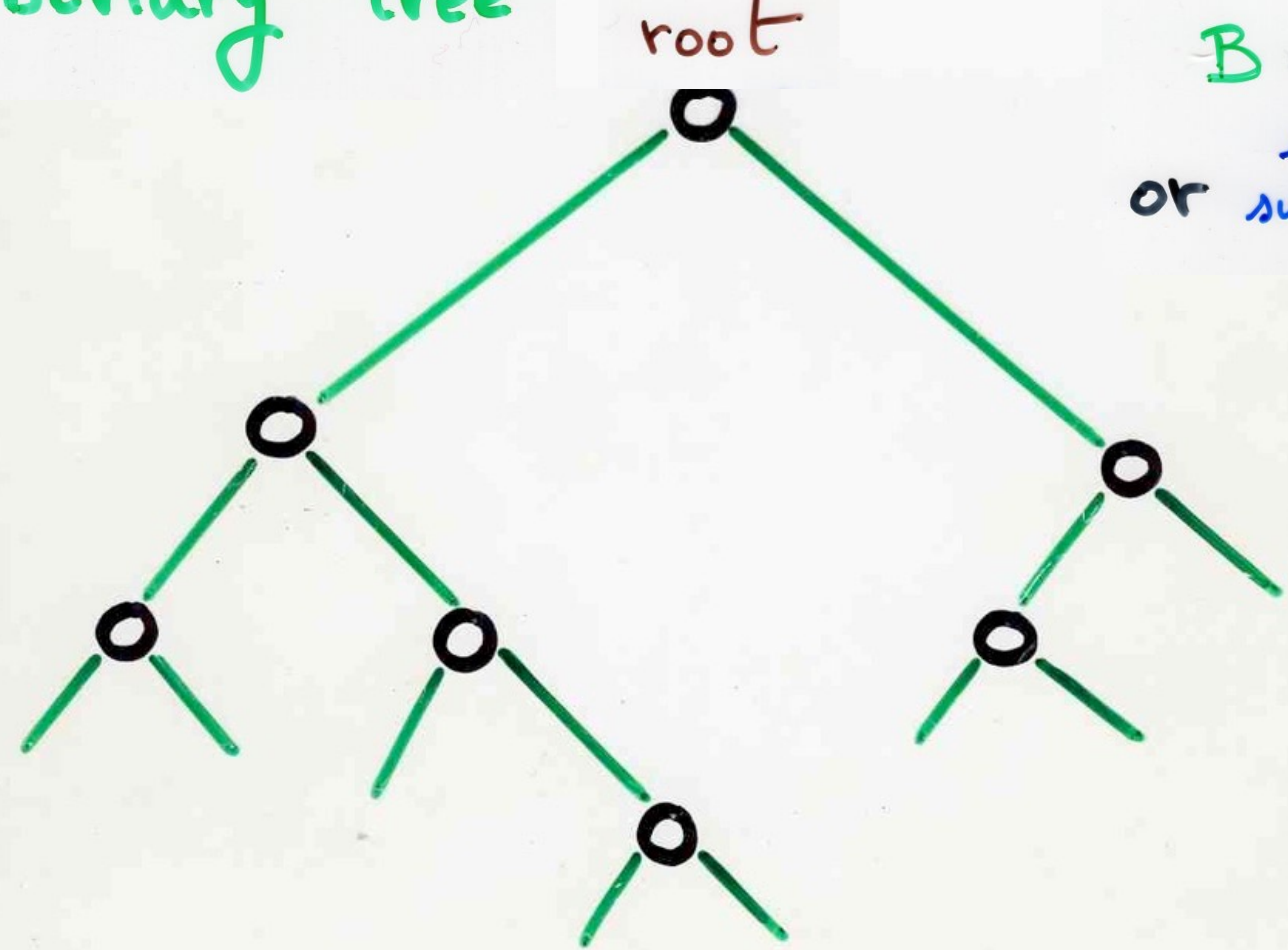


binary trees

binary tree



binary tree

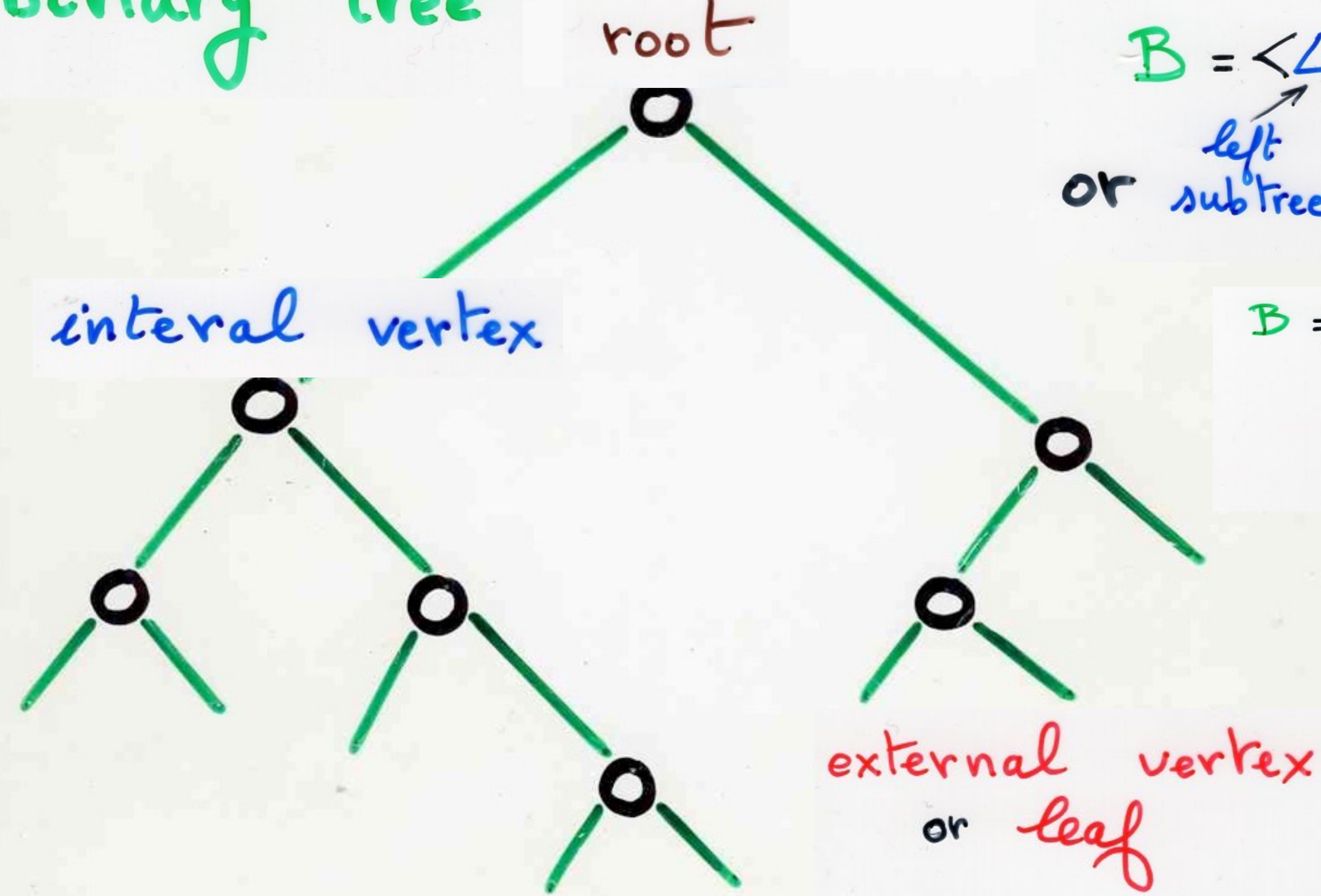


$$B = \langle \underset{\text{left subtree}}{L}, \underset{\text{root}}{r}, \underset{\text{right subtree}}{R} \rangle$$

or

$$B = \langle \underset{\text{leaf or external vertex}}{v} \rangle$$

binary tree



$$B = \langle L, r, R \rangle$$

left subtree, root, right subtree

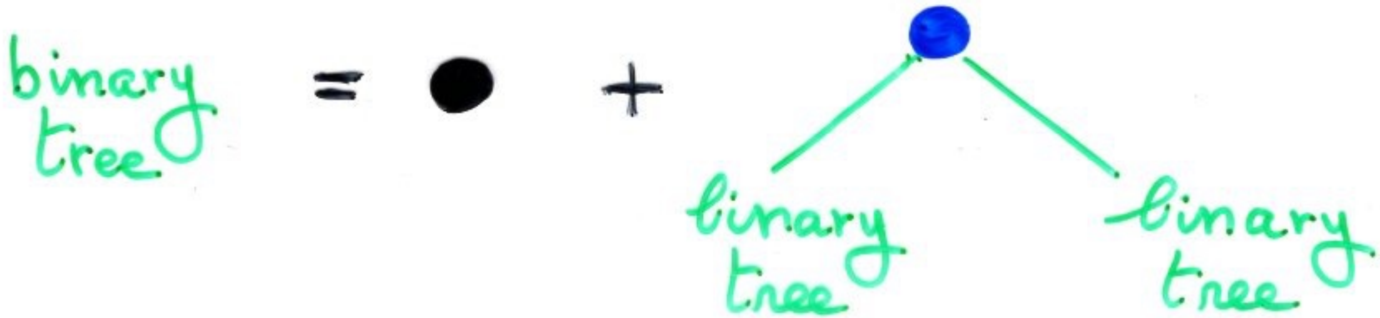
or

$$B = \langle v \rangle$$

leaf or external vertex

C_n = number of binary trees having n internal vertices

(or $n+1$ leaves = external vertices)



$$\underset{\text{binary tree}}{B} = \{\bullet\} + \left(B \times \underset{\text{root}}{\bullet} \times B\right)$$

$$y = 1 + t\,y^2$$

algebraic equation

$$y = 1 + t y^2$$

$$y = 1 + 2t + 5t^2 + 14t^3 + 42t^4 + \cdots + C_n t^n + \cdots$$

$$y = \frac{1 - (1-4t)^{1/2}}{2t}$$

[illegible]

doch zweifle ich nicht, daß diese durch [illegible] Progressionen dergestalt

vertheilt werden können. [illegible] bey der Progression

1, 2, 5, 14, 42, 132, etc. habe ich auch diese Eigenschaft

gemerket, daß

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc.} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$$

also wenn $a=\frac{1}{4}$ so ist $1+\frac{2}{4}+\frac{5}{4^2}+\frac{14}{4^3}+\frac{42}{4^4}+\text{etc.} = 4$.

[illegible] Ew. Hochwohlgeb.

[illegible]

beständiger Ergebenheit [illegible] und [illegible]

habe die Ehre mit der schuldigsten Hochachtung

lebenslang zu beharren

Ew. Hochwohlgebohrnen

[illegible] Berlin d. 4t Sept 1751.

gehorsamster Diener

L. Euler

$$(1+u)^m =$$

$$1 + \frac{m}{1!}u + \frac{m(m-1)}{2!}u^2 + \frac{m(m-1)(m-2)}{3!}u^3 +$$

$$+ \dots$$

$$m = 1/2$$

$$u = -4t$$

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\,n!}$$

$$n! = 1 \times 2 \times \dots \times n$$

3 (1838)

Note sur une Équation aux différences finies;

PAR E. CATALAN.

M. *Lamé* a démontré que l'équation

$$P_{n+1} = P_n + P_{n-1}P_3 + P_{n-2}P_4 + \dots + P_4P_{n-2} + P_3P_{n-1} + P_n, \qquad (1)$$

se ramène à l'équation linéaire très simple,

$$P_{n+1} = \frac{4n-6}{n} P_n. \qquad (2)$$

Admettant donc la concordance de ces deux formules, je vais chercher à en déduire quelques conséquences.

L'intégral

$y = 1 + t y^2$

et comme, dans la question de géométrie qui conduit à ces deux équations, on a $P_3 = 1$, nous prendrons simplement

$$P_{n+1} = \frac{2.6.10.14\dots(4n-6)}{2.3.\ 4.\ 5\dots\ \ n}. \qquad (3)$$

Le numérateur

$$2.6.10.14\dots(4n-6) = 2^{n-1}.1.3.5.7\dots(2n-3) = \frac{2^{n-1}.1.2.3.4.5\dots(2n-2)}{2.4.6.8\dots(2n-2)} = \frac{1.2.3.4\dots(2n-2)}{1.2.3\dots(n-1)}.$$

Donc

$$P_{n+1} = \frac{n(n+1)(n+2)\dots(2n-2)}{2.3.4\dots n}. \qquad (4)$$

Si l'on désigne généralement par $C_{m,p}$ le nombre des combinaisons de m lettres, prises p à p; et si l'on change n en $n+1$, on aura

$$P_{n+2} = \frac{1}{n+1} C_{2n,n}, \qquad (5)$$

ou bien

$$P_{n+2} = C_{2n,n} - C_{2n,n-1}. \qquad (6)$$

II.

Les équations (1) et (5) donnent ce théorème sur les combinaisons :

$$\left.\begin{aligned} \frac{1}{n+1} C_{2n,n} = \frac{1}{n} C_{2n-2,n-1} + \frac{1}{n-1} C_{2n-4,n-2} \times \frac{1}{2} C_{2,1} \\ + \frac{1}{n-2} C_{2n-6,n-3} \times \frac{1}{3} C_{4,2} + \dots + \frac{1}{n} C_{2n-2,n-1}. \end{aligned}\right\} \qquad (7)$$

III.

On sait que le $(n+1)^e$ nombre figuré de l'ordre $n+1$, a pour expression, $C_{2n,n}$: si donc, dans la table des nombres figurés, on prend ceux qui occupent la diagonale; savoir :

$$1,\ 2,\ 6,\ 20,\ 70,\ 252,\ 924\dots;$$

qu'on les divise respectivement par

on obtiendr

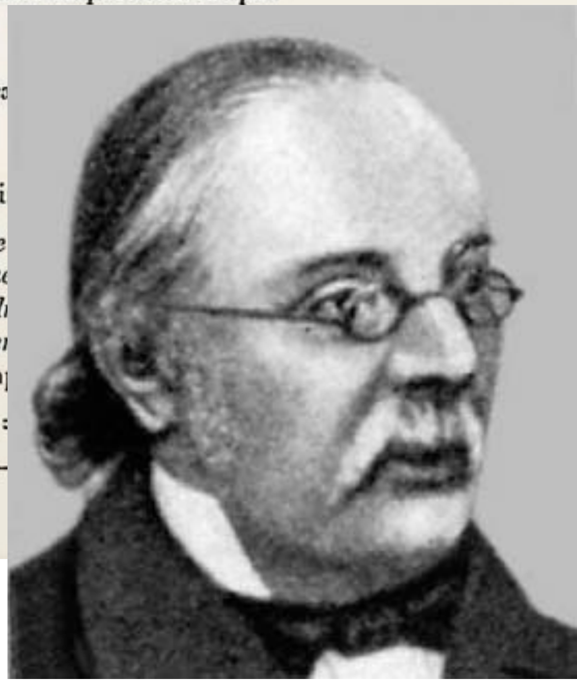
(A)

lesquels joui

Un terme *nme des produits qu* *ême, et dans un ordr* *n multi-pliant les te*

Par exem

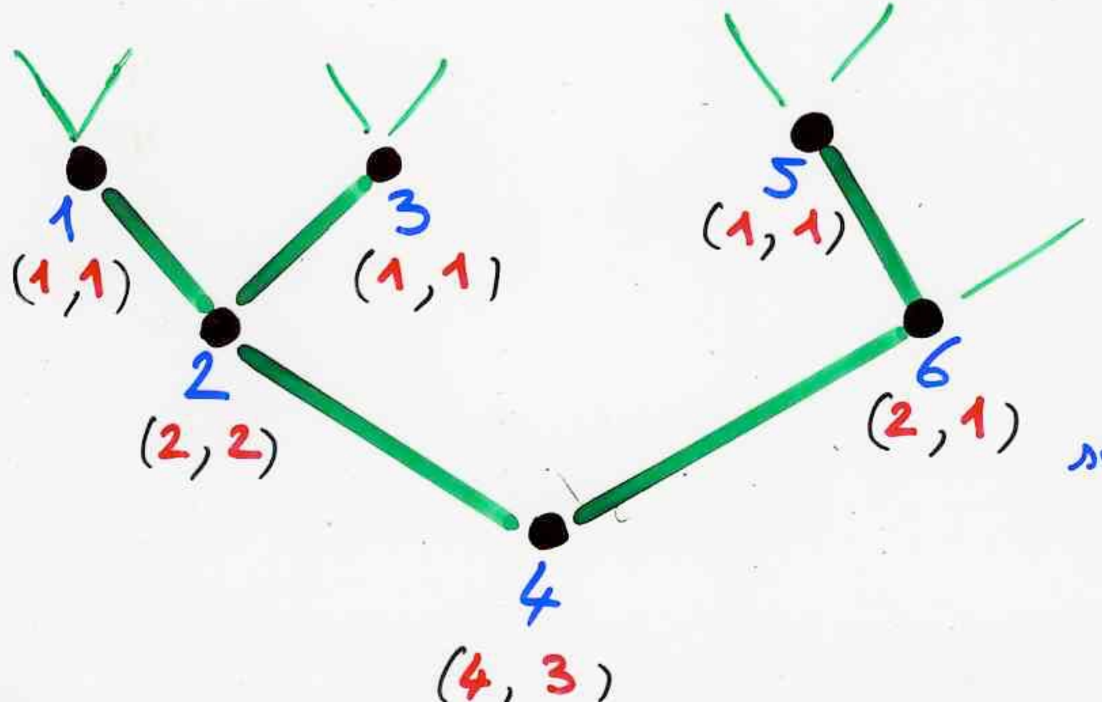
132 =



Eugène Catalan (1814-1894)

realisation of the associahedron

J.-L. Loday (2004) arXiv: déc 2002
"Realization of the Stasheff polytope"



symmetric order

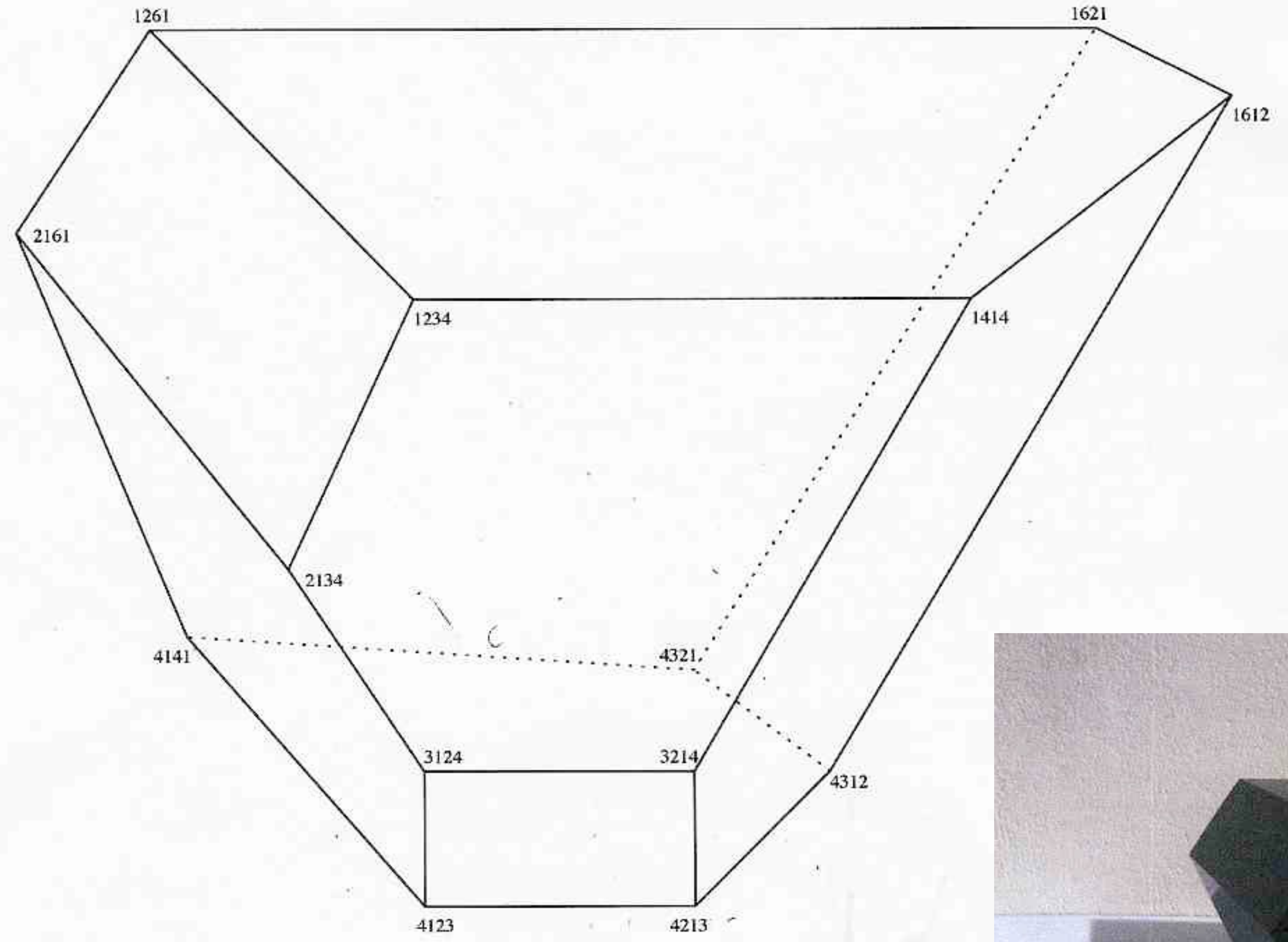
1	2	3	4	5	6
(1	4	1	12	1	2)

$\xrightarrow[\text{sum}]{} 21$

$\dfrac{n(n+1)}{2}$

convex hull of the points

hyperplan
$x_1 + \dots + x_n = \dfrac{n(n+1)}{2}$


1261
1621
1612
2161
1234
1414
2134
4141
4321
3124
3214
4312
4123
4213


Sir Hypparcus,

you said 103049 ?

103049

Plutarch:

Chrysippus says that the number of compound propositions that can be made from only ten simple propositions exceeds a million. Hipparchus, to be sure, refuted this by showing that this number is 103 049.

D. Hough (1994)

M2819 1, 3, 9, 35, 178
Van der Waerden numbers. Ref Loth83 49. [1,2; A5346]

M2820 1, 3, 9, 35, 201, 1827
Coefficients of Bell's formula. Ref NMT 10 65 62. [2,2; A2575, N1134]

M2821 1, 3, 9, 37, 153, 951, 5473, 42729, 353937, 3455083, 30071001, 426685293, 4707929449, 59350096287, 882391484913, 15177204356401, 205119866263713
Sums of logarithmic numbers. Ref TMS 31 79 63. jos. [0,2; A2751, N1135]

M2822 1, 1, 1, 3, 9, 37, 177, 959, 6097, 41641, 325249, 2693691, 24807321, 241586893, 2558036145, 28607094455, 342232522657, 4315903789009, 57569080467073
Expansion of $e^{\tan x}$. Ref JO61 150. [0,4; A6229]

M2823 1, 3, 9, 42, 206, 1352, 10168
Regular semigroups of order n. Ref PL65. MAL 2 2 67. SGF 14 71 77. [1,2; A1427, N1136]

M2824 0, 1, 1, 3, 9, 45, 225, 1575, 11025, 99225, 893025, 9823275, 108056025, 1404728325, 18261468225, 273922023375, 4108830350625, 69850115960625
Expansion of $1 / (1-x)(1-x^2)^{1/2}$. Ref R1 87. [1,4; A0246, N1137]

M2825 1, 1, 1, 3, 9, 48, 504, 14188, 1351563
Threshold functions of n variables. Ref PGEC 19 821 70. MU71 38. [0,4; A1530, N1138]

M2826 3, 9, 54, 450, 4725, 59535, 873180, 14594580
Expansion of an integral. Ref C1 167. [2,1; A1194, N1139]

M2827 1, 3, 9, 89, 1705, 67774
Superpositions of cycles. Ref AMA 131 143 73. [3,2; A3225]

M2828 1, 3, 9, 93, 315, 3855, 13797, 182361, 9256395, 34636833, 1857283155, 26817356775, 102280151421, 1497207322929, 84973577874915, 4885260612740877
Fermat quotients: $(2^{p-1}-1)/p$. Ref Well86 70. [0,2; A7663]

M2829 3, 10, 4, 5, 10, 2, 5, 3, 2, 3, 6, 6, 6, 3, 5, 6, 10, 5, 5, 10, 6, 6, 6, 2, 5, 8, 2, 6, 8, 4, 6, 6, 4, 5, 10, 2, 4, 7, 11, 5, 7, 9, 10, 7, 1, 6, 7, 11, 7, 10, 0, 6, 8, 9, 6, 4, 11, 7, 13, 2, 6, 4, 4
Iterations until $3n$ reaches 153 under x goes to sum of cubes of digits map. Ref Robe92 13. [1,1; A3620]

M2830 1, 3, 10, [illegible] 13361, 73011, 597449, 1865358,

M2819 1, 3, 9, 35, 178
Van der Waerden numbers. Ref Loth83 49. [1,2; A5346]

M2820 1, 3, 9, 35, 201, 1827
Coefficients of Bell's formula. Ref NMT 10 65 62. [2,2; A2575, N1134]

M2821 1, 3, 9, 37, 153, 951, 5473, 42729, 353937, 3455083, 30071001, 426685293, 4707929449, 59350096287, 882391484913, 15177204356401, 205119866263713
Sums of logarithmic numbers. Ref TMS 31 79 63. jos. [0,2; A2751, N1135]

M2822 1, 1, 1, 3, 9, 37, 177, 959, 6097, 41641, 325249, 2693691, 24807321, 241586893, 2558036145, 28607094455, 342232522657, 4315903789009, 57569080467073
Expansion of $e^{\tan x}$. Ref JO61 150. [0,4; A6229]

M2823 1, 3, 9, 42, 206, 1352, 10168
Regular semigroups of order n. Ref PL65. MAL 2 2 67. SGF 14 71 77. [1,2; A1427, N1136]

M2824 0, 1, 1, 3, 9, 45, 225, 1575, 11025, 99225, 893025, 9823275, 108056025, 1404728325, 18261468225, 273922023375, 4108830350625, 69850115960625
Expansion of $1/(1-x)(1-x^2)^{1/2}$. Ref R1 87. [1,4; A0246, N1137]

M2825 1, 1, 1, 3, 9, 48, 504, 14188, 1351563
Threshold functions of n variables. Ref PGEC 19 821 70. MU71 38. [0,4; A1530, N1138]

M2826 3, 9, 54, 450, 4725, 59535, 873180, 14594580
Expansion of an integral. Ref C1 167. [2,1; A1194, N1139]

M2827 1, 3, 9, 89, 1705, 67774
Superpositions of cycles. Ref AMA 131 143 73. [3,2; A3225]

M2828 1, 3, 9, 93, 315, 3855, 13797, 182361, 9256395, 34636833, 1857283155, 26817356775, 102280151421, 1497207322929, 84973577874915, 488526061274087
Fermat quotients: $(2^{p-1}-1)/p$. Ref Well86 70. [0,2; A7663]

M2829 3, 10, 4, 5, 10, 2, 5, 3, 2, 3, 6, 6, 6, 3, 5, 6, 10, 5, 5, 10, 6, 6, 6, 2, 5, 8, 2, 6, 8, 4, 6, 4, 5, 10, 2, 4, 7, 11, 5, 7, 9, 10, 7, 1, 6, 7, 11, 7, 10, 0, 6, 8, 9, 6, [illegible] 7, 13, 2, 6, [illegible]
Iterations until $3n$ reaches 153 under x goes to sum of cubes of dig[illegible]
[1,1; A3620]

M2830 1, 3, 10, 13, 62, 75, 437, 512, 949, 6206, 13361, 7301[illegible]
6193523, 26639450, 59472423, 383473988, 1593368375, [illegible]
Convergents to cube root of 3. Ref AMP 46 105 1866. L1 67[illegible]

M2844
5097243, 16835050, 55602393, 183642229, 606529080, 2003229469, 6616217487
$a(n)=3a(n-1)+a(n-2)$. Ref FQ 15 292 77. ARS 6 168 78. [0,3; A6190]

M2845 1, 3, 10, 33, 111, 379, 1312, 4596, 16266, 58082, 209010, 757259, 2760123, 10114131, 37239072, 137698584, 511140558, 1904038986, 7115422212, 26668376994
A simple recurrence. Ref IFC 16 351 70. [0,2; A1558, N1143]

M2846 1, 1, 3, 10, 33, 147
One-sided hexagonal polyominoes with n cells. Ref jm. [1,3; A6535]

M2847 1, 3, 10, 34, 116, 396, 1352, 4616, 15760, 53808, 183712, 627232, 2141504, 7311552, 24963200, 85229696, 290992384, 993510144, 3392055808, 11581202944
Order-consecutive partitions. Ref HM94. [0,2; A7052]

$$\text{G.f.:}\ \ (1-x)\ /\ (1-4x+2x^2).$$

M2848 1, 3, 10, 35, 126, 462, 1716, 6435, 24310, 92378, 352716, 1352078, 5200300, 20058300, 77558760, 300540195, 1166803110, 4537567650, 17672631900
$C(2n+1,n+1)$. Ref RS3. [0,2; A1700, N1144]

M2849 1, 3, 10, 36, 136, 528, 2080, 8256, 32896, 131328, 524800, 2098176, 8390656, 33558528, 134225920, 536887296, 2147516416, 8590000128, 34359869440
$2^{n-1}(1+2^n)$. Ref JGT 17 625 93. [0,2; A7582]

M2850 1, 3, 10, 36, 137, 543, 2219, 9285, 39587, 171369, 751236, 3328218, 14878455, 67030785, 304036170, 1387247580, 6363044315, 29323149825, 135700543190
Restricted hexagonal polyominoes with n cells: reversion of M2741. Ref PEMS 17 11 70. rcr. [1,2; A2212, N1145]

M2851 1, 3, 10, 37, 151, 674, 3263, 17007, 94828, 562595, 3535027, 23430840, 163254885, 1192059223, 9097183602, 72384727657, 599211936355, 5150665398898
Expansion of $e \uparrow (e \uparrow x + 2x - 1)$. Ref JCT A24 316 78. SIAD 5 498 92. [0,2; A5493]

M2852 1, 1, 3, 10, 38, 154, 654, 2871, 12925, 59345, 276835, 1308320, 6250832, 30142360, 146510216, 717061938, 3530808798, 17478955570, 86941210950
Dissections of a polygon. Ref EDMN 32 6 40. BAMS 54 359 48. [0,3; A1002, N1146]

$$\text{Reversion of}\ \ x(1-x-x^2).$$

M2853 1, 1, 3, 10, 38, 156, 692, 3256, 16200, 84496, 460592, 2611104, 15355232, 93376960, 585989952, 3786534784, 25152768128, 171474649344, 1198143415040
Symmetric permutations. Ref LU91 1 222. LNM 560 201 76. [0,3; A0902, N1147]

$$a(n)\ =\ 2.a(n-1)\ +\ (2n-4).a(n-2).$$

Quadrinomial coefficients. Ref CT 78. [0,3; A5725]

M2844 0, 1, 3, 10, 33, 109, 360, 1189, 3927, 12970, 42837, 141481, 467280, 1543321, 5097243, 16835050, 55602393, 183642229, 606529080, 2003229469, 6616217487
$a(n)=3a(n-1)+a(n-2)$. Ref FQ 15 292 77. ARS 6 168 78. [0,3; A6190]

M2845 1, 3, 10, 33, 111, 379, 1312, 4596, 16266, 58082, 209010, 757259, 2760123, 10114131, 37239072, 137698584, 511140558, 1904038986, 7115422212, 26668376994
A simple recurrence. Ref IFC 16 351 70. [0,2; A1558, N1143]

M2846 1, 1, 3, 10, 33, 147
One-sided hexagonal polyominoes with n cells. Ref jm. [1,3; A6535]

M2847 1, 3, 10, 34, 116, 396, 1352, 4616, 15760, 53808, 183712, 627232, 2141504, 7311552, 24963200, 85229696, 290992384, 993510144, 3392055808, 11581202944
Order-consecutive partitions. Ref HM94. [0,2; A7052]

$$\text{G.f.:}\ (1-x)\ /\ (1-4x+2x^2).$$

M2848 1, 3, 10, 35, 126, 462, 1716, 6435, 24310, 92378, 352716, 1352078, 5200300, 20058300, 77558760, 300540195, 1166803110, 4537567650, 17672631900
$C(2n+1,n+1)$. Ref RS3. [0,2; A1700, N1144]

M2849 1, 3, 10, 36, 136, 528, 2080, 8256, 32896, 131328, 524800, 2098176, 8390656, 33558528, 134225920, 536887296, 2147516416, 8590000128, 34359869440
$2^{n-1}(1+2^n)$. Ref JGT 17 625 93. [0,2; A7582]

M2850 1, 3, 10, 36, 137, 543, 2219, 9285, 39587, 171369, 751236, 3328218, 14878455, 67030785, 304036170, 1387247580, 6363044315, 29323149825, 135700543190
Restricted hexagonal polyominoes with n cells; reversion of M2741. Ref PEMS 17 11 70. rcr. [1,2; A2212, N1145]

M2851 1, 3, 10, 37, 151, 674, 3263, 1[illegible]2595, 3535027, 23430840, 163254885, 1192059223, 909718360[illegible]599211936355, 5150665398898
Expansion of $e \uparrow (e \uparrow x + 2x - 1)$. Re[illegible]8. SIAD 5 498 92. [0,2; A5493]

M2852 1, 1, 3, 10, 38, 154, 654, 2871, 129[illegible]5, 1308320, 6250832, 30142360, 146510216, 717061938, 35308[illegible]570, 86941210950
Dissections of a polygon. Ref EDMN 32 6 40. [illegible]. [0,3; A1002, N1146]

Reversion of x ([illegible]

M2853 1, 1, 3, 10, 38, 156, 692, 3256, 16200[illegible]1104, 15355232, 93376960, 585989952, 3786534784, 251527[illegible], 1198143415040
Symmetric permutations. Ref LU91 1 222. LNM [illegible]902, N1147]

$a(n) = 2.a(n-1) + ($ [illegible]

M2845 1, 3, 10, 33, 111, 379, 1312, 4596, 16266, 58082, 209010, 757259, 2760123, 10114131, 37239072, 137698584, 511140558, 1904038986, 7115422212, 26668376994
A simple recurrence. Ref IFC 16 351 70. [0,2; A1558, N1143]

M2846 1, 1, 3, 10, 33, 147
One-sided hexagonal polyominoes with n cells. Ref jm. [1,3; A6535]

M2847 1, 3, 10, 34, 116, 396, 1352, 4616, 15760, 53808, 183712, 627232, 2141504, 7311552, 24963200, 85229696, 290992384, 993510144, 3392055808, 11581202944
Order-consecutive partitions. Ref HM94. [0,2; A7052]

$$\text{G.f.: } (1-x)\ /\ (1-4x+2x^2).$$

M2848 1, 3, 10, 35, 126, 462, 1716, 6435, 24310, 92378, 352716, 1352078, 5200300, 20058300, 77558760, 300540195, 1166803110, 4537567650, 17672631900
$C(2n+1,n+1)$. Ref RS3. [0,2; A1700, N1144]

M2849 1, 3, 10, 36, 136, 528, 2080, 8256, 32896, 131328, 524800, 2098176, 8390656, 33558528, 134225920, 536887296, 2147516416, 8590000128, 34359869440
$2^{n-1}(1+2^n)$. Ref JGT 17 625 93. [0,2; A7582]

M2850 1, 3, 10, 36, 137, 543, 2219, 9285, 39587, 171369, 751236, 3328218, 14878455, 67030785, 304036170, 1387247580, 6363044315, 29323149825, 135700543190
Restricted hexagonal polyominoes with n cells: reversion of M2741. Ref PEMS 17 11 70. rcr. [1,2; A2212, N1145]

M2851 1, 3, 10, 37, 151, 674, 3263, 17007, 94828, 562595, 3535027, 23430840, 163254885, 1192059223, 9097183602, 72384727657, 599211936355, 5150665398898
Expansion of $e \uparrow (e \uparrow x + 2x - 1)$. Ref JCT A24 316 78. SIAD 5 498 92. [0,2; A5493]

M2852 1, 1, 3, 10, 38, 154, 654, 2871, 12925, 59345, 276835, 1308320, 6250832, 30142360, 146510216, 717061938, 3530808798, 17478955570, 86941210950
Dissections of a polygon. Ref EDMN 32 6 40. BAMS 54 359 48. [0,3; A1002, N1146]

$$\text{Reversion of } x(1-x-x^2).$$

M2853 1, 1, 3, 10, 38, 156, 692, 3256, 16200, 84496, 460592, 2611104 [illegible] 32, 93376960, 585989952, 3786534784, 25152768128, 171474649344, 1 [illegible] 40
Symmetric permutations. Ref LU91 1 222. LNM 560 201 76. [0,3; A090 [illegible]

$$a(n) = 2.a(n-1) + (2n-4).a(n-2).$$

M2866 1, 1, 3, 10, 45, 251, 1638, 12300, 104877, 1000135
From descending subsequences of permutations. Ref JCT A53 99 90. [1,3; A6220]

M2867 1, 1, 3, 10, 45, 256, 1743, 13840, 125625, 1282816, 14554683, 181649920, 2473184805, 36478744576, 579439207623, 9861412096000, 179018972217585
Expansion of ln(1+sinh x). [0,3; A3704]

M2868 1, 3, 10, 45, 272, 2548, 39632, 1104306, 56871880, 5463113568, 978181717680, 326167542296048, 202701136710498400, 235284321080559981952
Symmetric reflexive relations on n nodes: ½ M1650. See Fig M3032. Ref MIT 17 21 55. MAN 174 70 67. JGT 1 295 77. [1,2; A0250, N1153]

M2869 1, 1, 3, 10, 45, 274
Sub-Eulerian graphs with n nodes. Ref ST90. [2,3; A5143]

M2870 1, 1, 3, 10, 47, 246, 1602, 11481, 95503, 871030, 8879558, 98329551, 1191578522, 15543026747, 218668538441, 3285749117475, 52700813279423
Sums of multinomial coefficients. Ref C1 126. [0,3; A5651]

$$\text{G.f.:}\quad 1 \,/\, \Pi\,(1-x^k/k!).$$

M2871 1, 3, 10, 48, 312, 2520, 24480, 277200, 3588480, 52254720
From solution to a difference equation. Ref FQ 25 363 87. [0,2; A5921]

M2872 1, 1, 3, 10, 53, 265, 1700
Sorting numbers. Ref PSPM 19 173 71. [0,3; A2873, N1154]

M2873 0, 1, 1, 3, 10, 56, 468, 7123, 194066, 9743542, 900969091, 153620333545, 48432939150704, 28361824488394169, 30995890806033380784
Nonseparable graphs with n nodes. Ref JCT 9 352 70. CCC 2 199 77. JCT B57 294 93. [1,4; A2218, N1155]

M2874 1, 3, 10, 66, 792, 25506, 2302938, 591901884, 420784762014, 819833163057369, 4382639993148435207, 64588133532185722290294, 2638572375815762804156666529
Signed graphs with n nodes. Ref CCC 2 31 77. rwr. JGT 1 295 77. [1,2; A4102]

M2875 1, 3, 10, 70, 708, 15224, 544152, 39576432, 5074417616, 1296033011648, 604178966756320, 556052774253161600, 954895322019762585664
Self-converse digraphs with n nodes. Ref MAT 13 157 66. rwr. [1,2; A2499, N1156]

M2876 3, 10, 84, 10989, 363883, 82620, 137550709
Coefficients of period polynomials. Ref LNM 899 292 81. [3,1; A6311]

M2867 1, 1, 3, 10, 45, 256, 1743, [illegible]
2473184805, 36478744576, 579439207623, 9861412096000, 179018972217585
Expansion of ln(1+sinh x). [0,3; A3704]

M2868 1, 3, 10, 45, 272, 2548, 39632, 1104306, 56871880, 5463113568, 978181717680,
326167542296048, 202701136710498400, 235284321080559981952
Symmetric reflexive relations on n nodes: ½ M1650. See Fig M3032. Ref MIT 17 21 55. MAN 174 70 67. JGT 1 295 77. [1,2; A0250, N1153]

M2869 1, 1, 3, 10, 45, 274
Sub-Eulerian graphs with n nodes. Ref ST90. [2,3; A5143]

M2870 1, 1, 3, 10, 47, 246, 1602, 11481, 95503, 871030, 8879558, 98329551,
1191578522, 15543026747, 218668538441, 3285749117475, 52700813279423
Sums of multinomial coefficients. Ref C1 126. [0,3; A5651]

$$\text{G.f.:}\quad 1 \,/\, \Pi\,(1-x^k/k!).$$

M2871 1, 3, 10, 48, 312, 2520, 24480, 277200, 3588480, 52254720
From solution to a difference equation. Ref FQ 25 363 87. [0,2; A5921]

M2872 1, 1, 3, 10, 53, 265, 1700
Sorting numbers. Ref PSPM 19 173 71. [0,3; A2873, N1154]

M2873 0, 1, 1, 3, 10, 56, 468, 7123, 194066, 9743542, 900969091, 153620333545,
48432939150704, 28361824488394169, 30995890806033380784
Nonseparable graphs with n nodes. Ref JCT 9 352 70. CCC 2 199 77. JCT B57 294 93. [1,4; A2218, N1155]

M2874 1, 3, 10, 66, 792, 25506, 2302938, 591901884, 420784762014, 819833163057369,
4382639993148435207, 6458813353218572229029 4, 26385723758157628041566665 29
Signed graphs with n nodes. Ref CCC 2 31 77. rwr. JGT 1 295 77. [1,2; A4102]

M2875 1, 3, 10, 70, 708, 15224, 544152, 39576432, 5074417616, 1296033011648,
604178966756320, 556052774253161600, 954895322019762585664
Self-converse digraphs with n nodes. Ref MAT 13 157 66. rwr. [1,2; A2499, N1156]

M2876 3, 10, 84, 10989, 363883, 82620, 137550709
Coefficients of period polynomials. Ref LNM 899 292 81. [3,1; A6311]

M2890 3, 11, 37, 101, 333667, 9091, 9901, 909091, 1111111111111111111,
1111111111111111111111111, 99990001, 999999000001, 909090909090909091
Primes with unique period length. Ref JRM 18 24 85. [1,1; A7615]

M2891 1, 3, 11, 38, 126, 415, 1369, 4521
Paths on square lattice. Ref ARS 6 168 78. [3,2; A6189]

M2892 1, 3, 11, 39, 131, 423, 1331, 4119, 12611, 38343, 116051, 350199, 1054691,
3172263, 9533171, 28632279, 85962371, 258018183, 774316691, 2323474359
$2(3^n - 2^n) + 1$. Ref IJ1 11 162 69. [0,2; A2783, N1159]

M2893 1, 3, 11, 39, 139, 495, 1763, 6279, 22363, 79647, 283667, 1010295, 3598219,
12815247, 45642179, 162557031, 578955451, 2061980415, 7343852147, 26155517271
Subsequences of $[1,...,2n]$ in which each odd number has an even neighbor. Ref GuMo94.
[0,2; A7482]

$$a(n) = 3\,a(n-1) + 2\,a(n-2).$$

M2894 1, 1, 3, 11, 41, 153, 571, 2131, 7953, 29681, 110771, 413403, 1542841, 5757961,
21489003, 80198051, 299303201, 1117014753, 4168755811, 15558008491
$a(n) = 4a(n-1) - a(n-2)$. Ref EUL (1) 1 375 11. MMAG 40 78 67. [0,3; A1835, N1160]

M2895 1, 3, 11, 43, 171, 683, 2731, 10923, 43691, 174763, 699051, 2796203, 11184811,
44739243, 178956971, 715827883, 2863311531, 11453246123, 45812984491
$(2^{2n+1} + 1)/3$. Ref JGT 17 625 93. [0,2; A7583]

M2896 3, 11, 43, 683, 2731, 43691, 174763, 2796203, 715827883, 2932031007403,
768614336404564651, 201487636602438195784363
Primes of form $(2^p + 1)/3$. Ref MMAG 27 157 54. [1,1; A0979, N1161]

M2897 1, 3, 11, 44, 186, 814, 3652, 16689, 77359, 362671, 1716033, 8182213, 39267086,
189492795, 918837374, 4474080844, 21866153748, 107217298977, 527266673134
Fixed hexagonal polyominoes with n cells. Ref RE72 97. dhr. [1,2; A1207, N1162]

M2898 1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, 2646723, 13648869,
71039373, 372693519, 1968801519, 10463578353, 55909013009, 300159426963
Schroeder's second problem: $(n+1)a(n+1) = 3(2n-1)a(n) - (n-2)a(n-1)$. Ref
EDMN 32 6 40. BAMS 54 359 48. RCI 168. C1 57. VA91 198. [1,3; A1003, N1163]

M2912 3, 11, 171, 43691, 2863311531, 12297829382473034411, 226854911280625642308916404954512140971
$(2\uparrow(2^n+1)+1)/3$. Ref dsk. [1,1; A6485]

M2913 1, 3, 11, 173, 2757, 176275, 11278843, 2887207533, 739113849605, 756849694787987, 775013348349049083, 3174453917988010255981
$a(n+2)=(4^{n+1}-5)a(n)-4a(n-2)$. Ref dhl. hpr. [1,2; A3115]

M2914 3, 11, 197, 129615, 430904428717
Spectrum of a certain 3-element algebra. Ref Berm83. [0,1; A7156]

M2915 3, 12, 15, 36, 138, 276, 4326, 21204, 65274, 126204, 204246, 1267356, 10235538, 54791316, 212311746, 678889380, 4946455134, 20113372464
Specific heat for crystobalite lattice. Ref CJP 48 310 70. [0,1; A5392]

M2916 1, 3, 12, 28, 66, 126, 236, 396, 651, 1001
Paraffins. Ref BER 30 1919 1897. [1,2; A5995]

M2917 3, 12, 29, 57, 99, 157, 234, 333, 456, 606, 786, 998, 1245
Series-reduced planted trees with n nodes, $n-4$ endpoints. Ref jr. [9,1; A1860, N1171]

M2918 3, 12, 31, 65, 120, 203, 322, 486, 705, 990, 1353, 1807, 2366, 3045, 3860, 4828, 5967, 7296, 8835, 10605, 12628, 14927, 17526, 20450, 23725, 27378, 31437, 35931
Quadrinomial coefficients. Ref C1 78. [2,1; A5718]

M2919 0, 3, 12, 45, 168, 627, 2340, 8733, 32592, 121635, 453948, 1694157, 6322680, 23596563, 88063572, 328657725, 1226567328, 4577611587, 17083879020
$a(n)=4a(n-1)-a(n-2)$. [0,2; A5320]

M2920 0, 0, 1, 3, 12, 45, 170, 651, 2520, 97502, 37854, 147070
Necklaces with n red, 1 pink and $n-3$ blue beads. Ref MMAG 60 90 87. [1,4; A5656]

M2921 1, 3, 12, 50, 27, 1323, 928, 1080, 48525, 3237113, 7587864, 23361540993, 770720657, 698808195, 179731134720, 542023437008852, 3212744374395
Cotesian numbers. Ref QJMA 46 63 14. [2,2; A2179, N1172]

M2922 1, 3, 12, 52, 238, 1125, 5438, 26715, 132871, 667312, 3377906, 17210522, 88169685, 453810095, 2345209383, 12162367228, 63270384303
n-node animals on f.c.c. lattice. Ref DU92 42. [1,2; A7198]

31013, 38083, 38183, 1003001, 1008001, 1180811, 1183811, 1300031, 1303031
Palindromic reflectable primes. Ref JRM 15 252 83. [1,1; A7616]

M2912 3, 11, 171, 43691, 2863311531, 12297829382473034411, 226854911280625642308916404954512140971
$(2\uparrow(2^n+1)+1)/3$. Ref dsk. [1,1; A6485]

M2913 1, 3, 11, 173, 2757, 176275, 11278843, 2887207533, 739113849605, 756849694787987, 775013348349049083, 3174453917988010255981
$a(n+2)=(4^{n+1}-5)a(n)-4a(n-2)$. Ref dhl. hpr. [1,2; A3115]

M2914 3, 11, 197, 129615, 430904428717
Spectrum of a certain 3-element algebra. Ref Berm83. [0,1; A7156]

M2915 3, 12, 15, 36, 138, 276, 4326, 21204, 65274, 126204, 204246, 1267356, 10235538, 54791316, 212311746, 678889380, 4946455134, 20113372464
Specific heat for crystobalite lattice. Ref CJP 48 310 70. [0,1; A5392]

M2916 1, 3, 12, 28, 66, 126, 236, 396, 651, 1001
Paraffins. Ref BER 30 1919 1897. [1,2; A5995]

M2917 3, 12, 29, 57, 99, 157, 234, 333, 456, 606, 786, 998, 1245
Series-reduced planted trees with n nodes, $n-4$ endpoints. Ref jr. [9,1; A1860, N1171]

M2918 3, 12, 31, 65, 120, 203, 322, 486, 705, 990, 1353, 1807, 2366, 3045, 3860, 4828, 5967, 7296, 8835, 10605, 12628, 14927, 17526, 20450, 23725, 27378, 31437, 35931
Quadrinomial coefficients. Ref C1 78. [2,1; A5718]

M2919 0, 3, 12, 45, 168, 627, 2340, 8733, 32592, 121635, 453948, 1694157, 6322680, 23596563, 88063572, 328657725, 1226567328, 4577611587, 17083879020
$a(n)=4a(n-1)-a(n-2)$. [0,2; A5320]

M2920 0, 0, 1, 3, 12, 45, 170, 651, 2520, 97502, 37854, 147070
Necklaces with n red, 1 pink and $n-3$ blue beads. Ref MMAG 60 90 87. [1,4; A5656]

M2921 1, 3, 12, 50, 27, 1323, 928, 1080, 48525, 3237113, [illegible] 61540993, 770720657, 698808195, 179731134720, 5420234370088[illegible] 4395
Cotesian numbers. Ref QJMA 46 63 14. [2,2; A2179, N117[illegible]

M2922 1, 3, 12, 52, 238, 1125, 5438, 26715, 132871, 66731[illegible] 210522, 88169685, 453810095, 2345209383, 12162367228, 6327[illegible]
n-node animals on f.c.c. lattice. Ref DU92 42. [1,2; A7198]

21489003, 80198051, 299303201, 1117014753, 4168755811, 15558008491
$a(n)=4a(n-1)-a(n-2)$. Ref EUL (1) 1 375 11. MMAG 40 78 67. [0,3; A1835, N1160]

M2895 1, 3, 11, 43, 171, 683, 2731, 10923, 43691, 174763, 699051, 2796203, 11184811, 44739243, 178956971, 715827883, 2863311531, 11453246123, 45812984491
$(2^{2n+1}+1)/3$. Ref JGT 17 625 93. [0,2; A7583]

M2896 3, 11, 43, 683, 2731, 43691, 174763, 2796203, 715827883, 2932031007403, 768614336404564651, 201487636602438195784363
Primes of form $(2^p+1)/3$. Ref MMAG 27 157 54. [1,1; A0979, N1161]

M2897 1, 3, 11, 44, 186, 814, 3652, 16689, 77359, 362671, 1716033, 8182213, 39267086, 189492795, 918837374, 4474080844, 21866153748, 107217298977, 527266673134
Fixed hexagonal polyominoes with n cells. Ref RE72 97. dhr. [1,2; A1207, N1162]

M2898 1, 1, 3, 11, 45, 197, 903, 4279, 2079[illegible], 103049, [illegible]18859, 2646723, 13648869, 71039373, 372693519, 1968801519, 10463[illegible]5909013009, 300159426963
Schroeder's second problem: $(n+1)a(n+$[illegible]$-1)a(n)-(n-2)a(n-1)$. Ref EDMN 32 6 40. BAMS 54 359 48. RCI 168. C[illegible] 198. [1,3; A1003, N1163]

$a(n) = 3\,a(n-1) + 2\,a(n-2)$.

M2894 1, 1, 3, 11, 41, 153, 571, 2131, 7953, 29681, 110771, 413403, 1542841, 5757961, 21489003, 80198051, 299303201, 1117014753, 4168755811, 15558008491
$a(n)=4a(n-1)-a(n-2)$. Ref EUL (1) 1 375 11. MMAG 40 78 67. [0,3; A1835, N1160]

M2895 1, 3, 11, 43, 171, 683, 2731, 10923, 43691, 174763, 699051, 2796203, 11184811, 44739243, 178956971, 715827883, 2863311531, 11453246123, 45812984491
$(2^{2n+1}+1)/3$. Ref JGT 17 625 93. [0,2; A7583]

M2896 3, 11, 43, 683, 2731, 43691, 174763, 2796203, 715827883, 2932031007403, 768614336404564651, 201487636602438195784363
Primes of form $(2^p+1)/3$. Ref MMAG 27 157 54. [1,1; A0979, N1161]

M2897 1, 3, 11, 44, 186, 814, 3652, 16689, 77359, 362671, 1716033, 8182213, 39267086, 189492795, 918837374, 4474080844, 21866153748, 107217298977, 527266673134
Fixed hexagonal polyominoes with n cells. Ref RE72 97. dhr. [1,2; A1207, N1162]

M2898 1, 1, 3, 11, 45, 197, 903, 4279, 2079[illegible], 103049, [illegible]18859, 2646723, 13648869, 71039373, 372693519, 1968801519, 10463[illegible]353, [illegible]5909013009, 300159426963
Schroeder's second problem: $(n+1)a(n+1)=3(2n-1)a(n)-(n-2)a(n-1)$. Ref EDMN 32 6 40. BAMS 54 359 48. RCI 168. C1 57. VA91 198. [1,3; A1003, N1163]

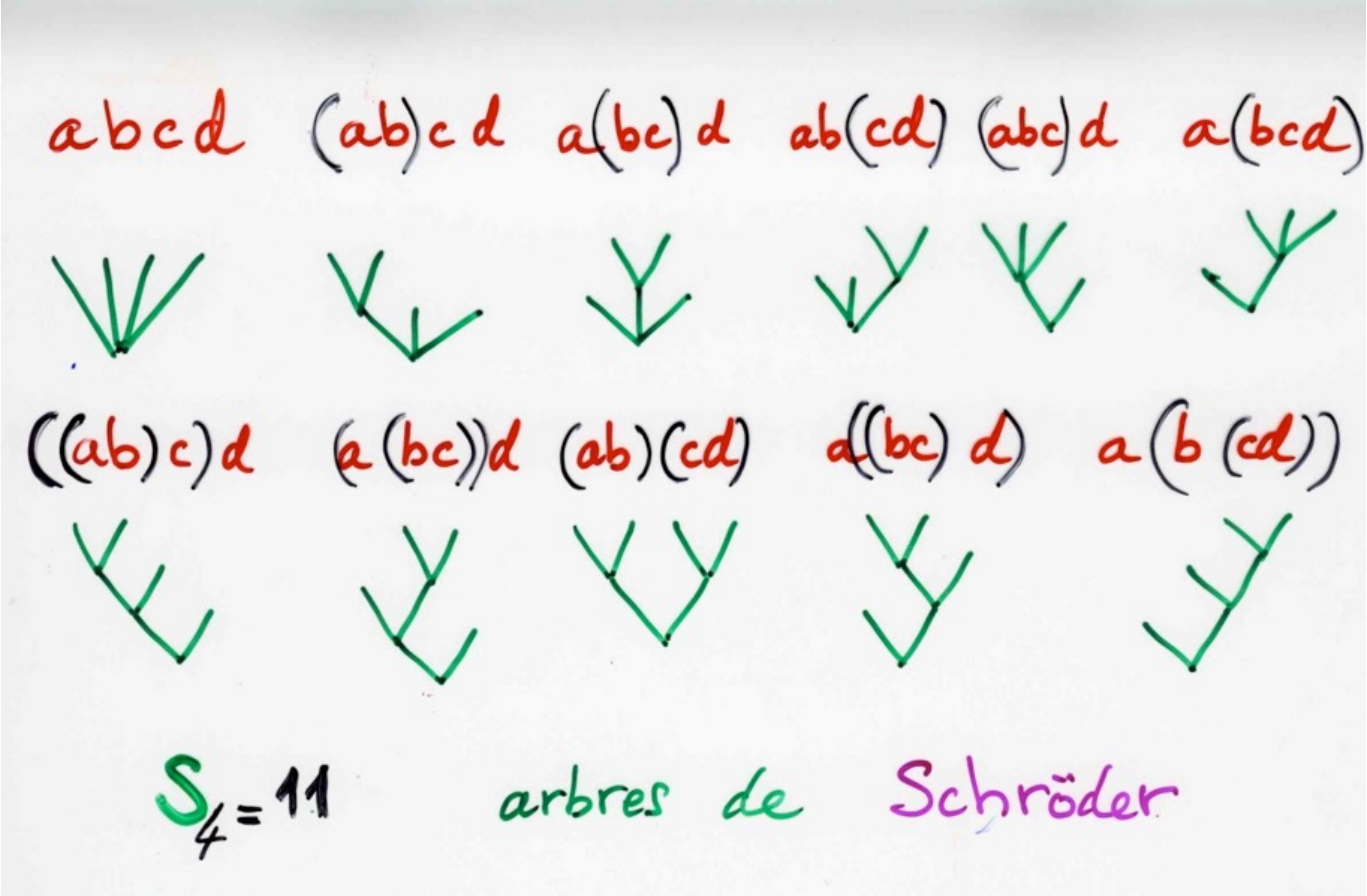
1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, ...

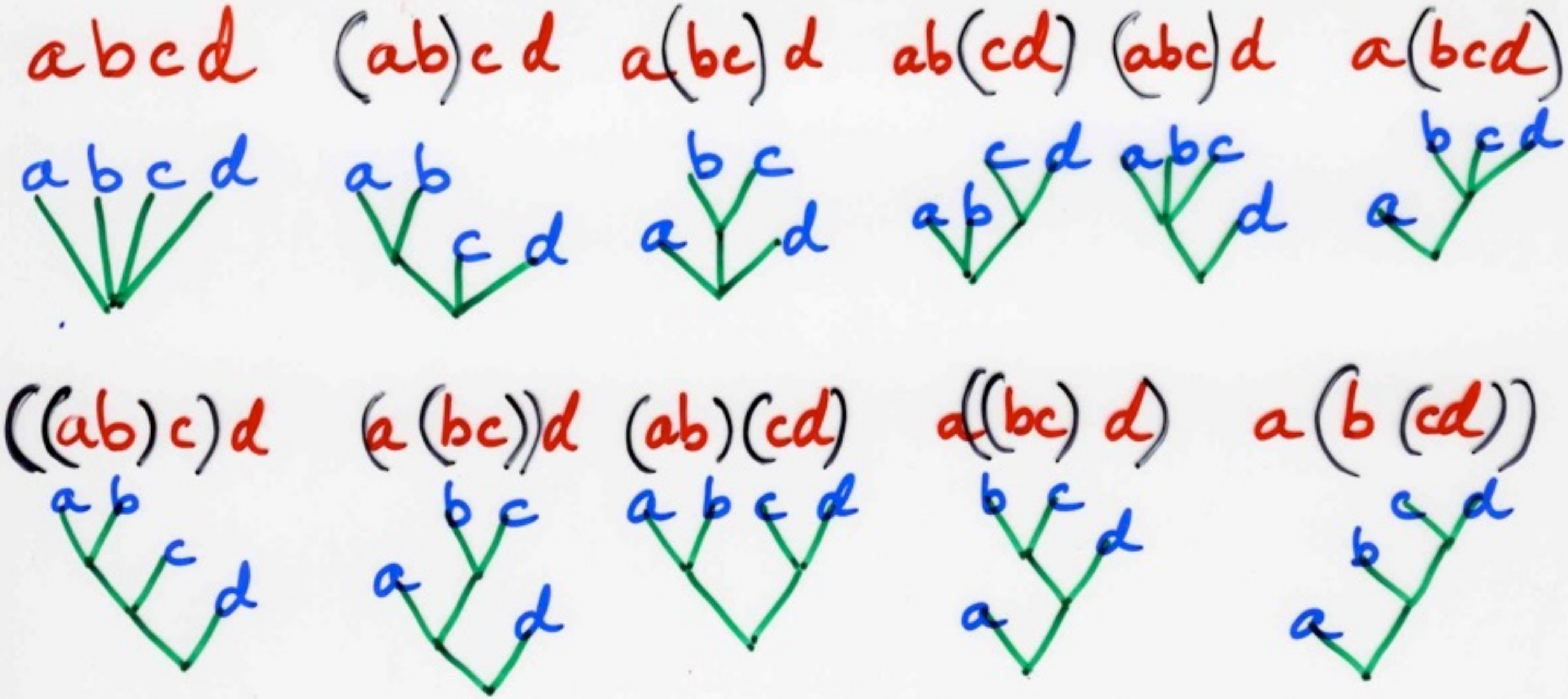
Schröder

1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, ...

$abcd \quad (ab)cd \quad a(bc)d \quad ab(cd) \quad (abc)d \quad a(bcd)$

$((ab)c)d \quad (a(bc))d \quad (ab)(cd) \quad a((bc)d) \quad a(b(cd))$


abcd
(ab)cd
a(bc)d
ab(cd)
(abc)d
a(bcd)
((ab)c)d
(a(bc))d
(ab)(cd)
a((bc)d)
a(b(cd))
S4 = 11
arbres de Schröder

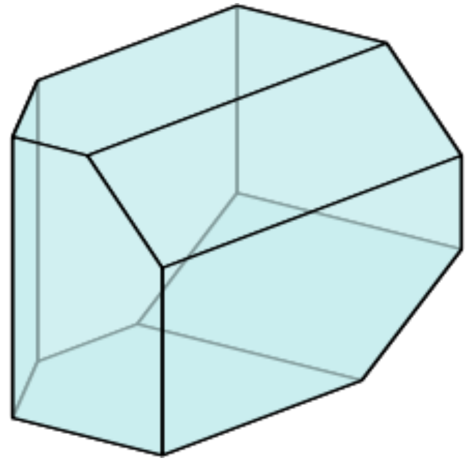


$S_4 = 11$ arbres de Schröder

Total number of cells

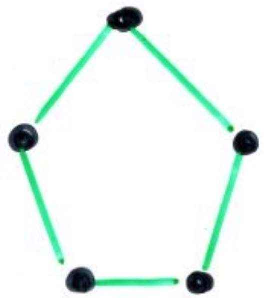
$14 + 21 + 9 + 1 = 45$

vertices edges faces associahedron



$5 + 5 + 1 = 11$

vertices edges



$2 + 1 = 3$

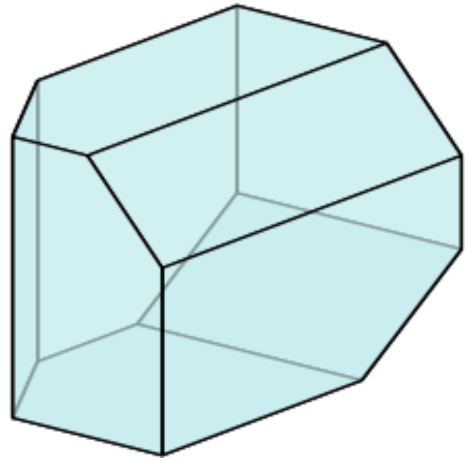
1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, ...

Schröder numbers

Total number of cells

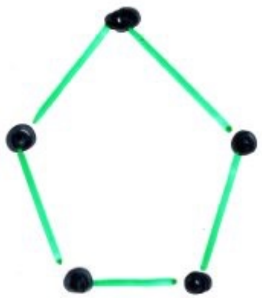
$$14 + 21 + 9 + 1 = 45$$

vertices edges faces associahedron



$$5 + 5 + 1 = 11$$

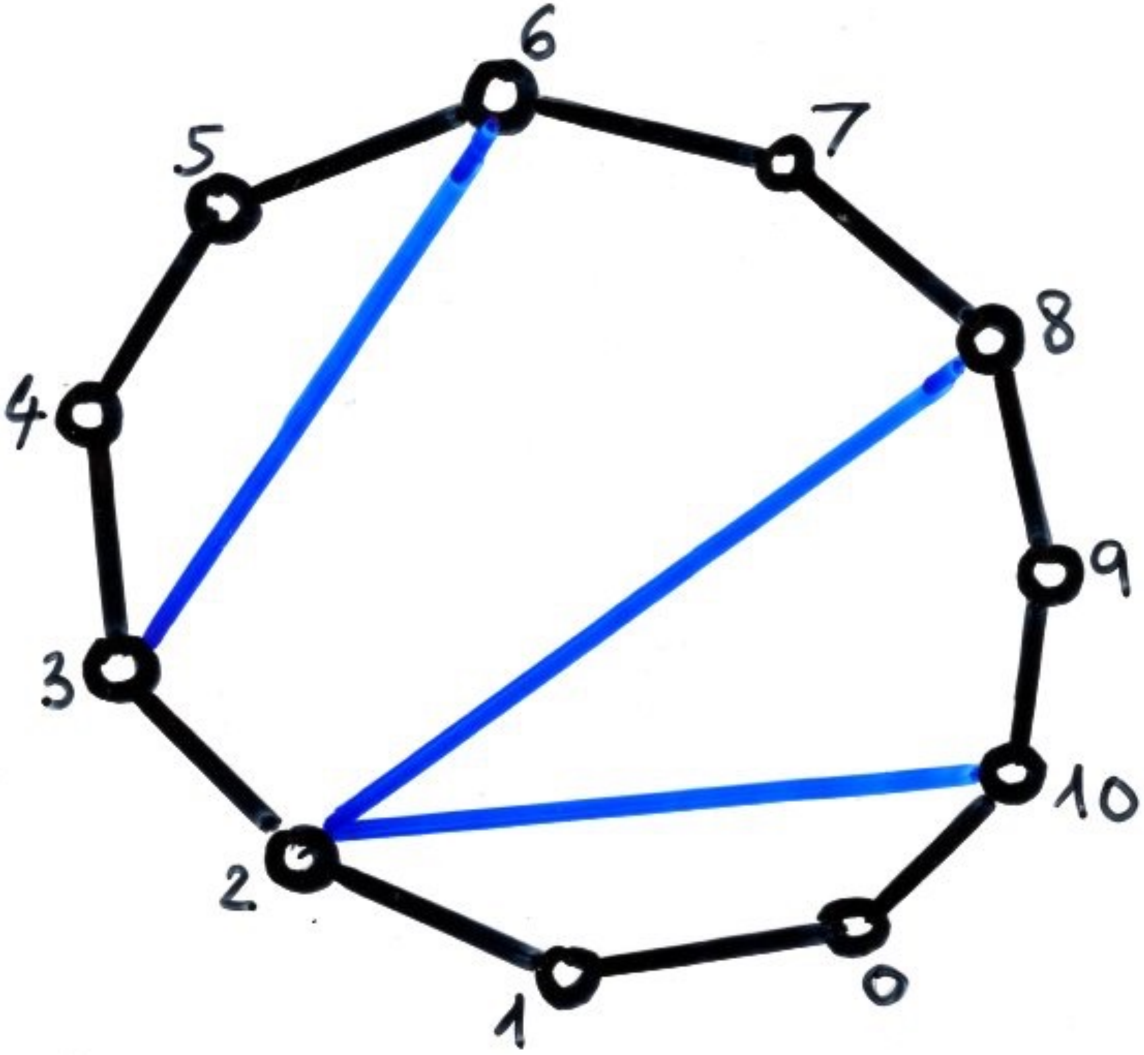
vertices edges



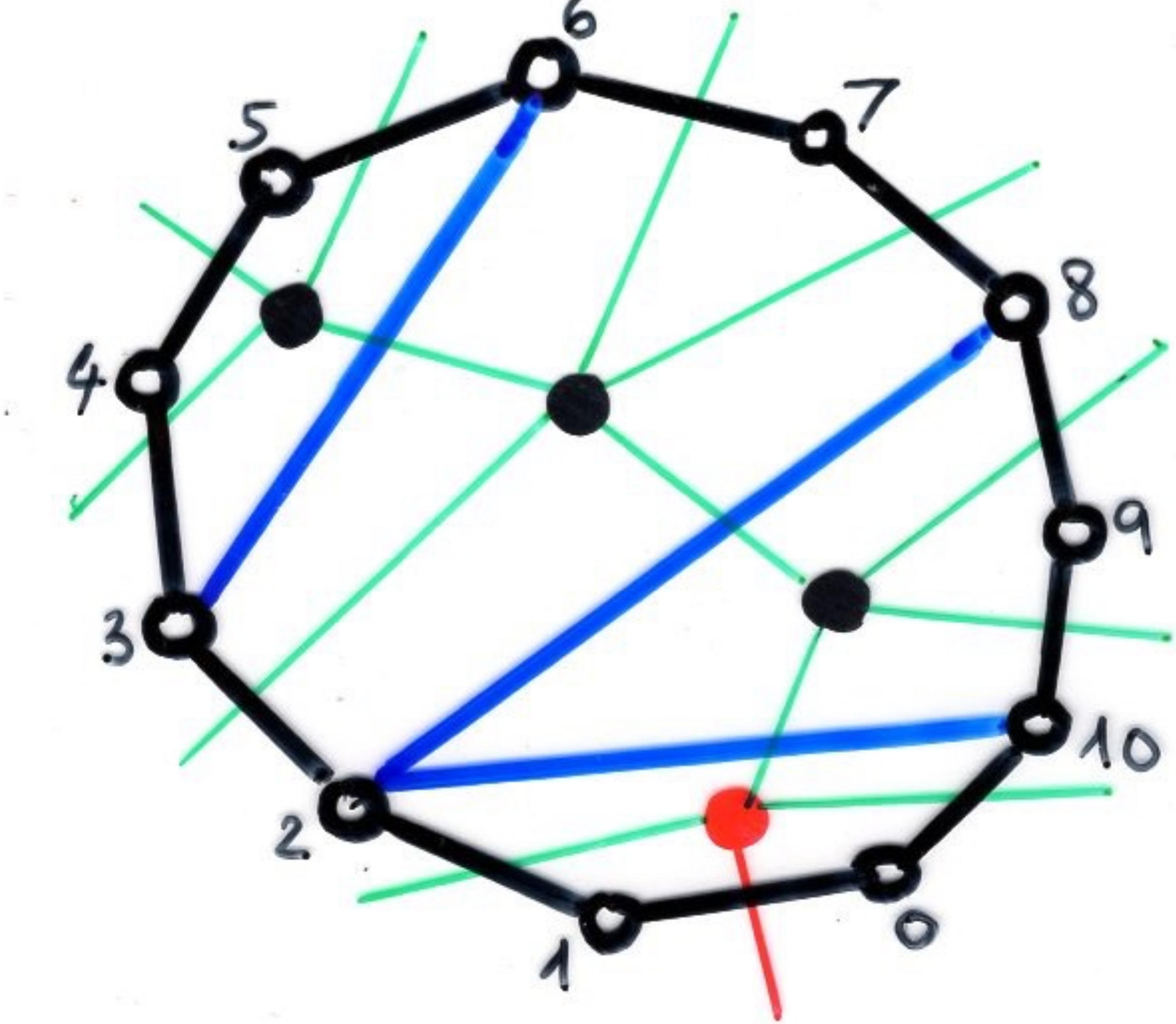
$$2 + 1 = 3$$

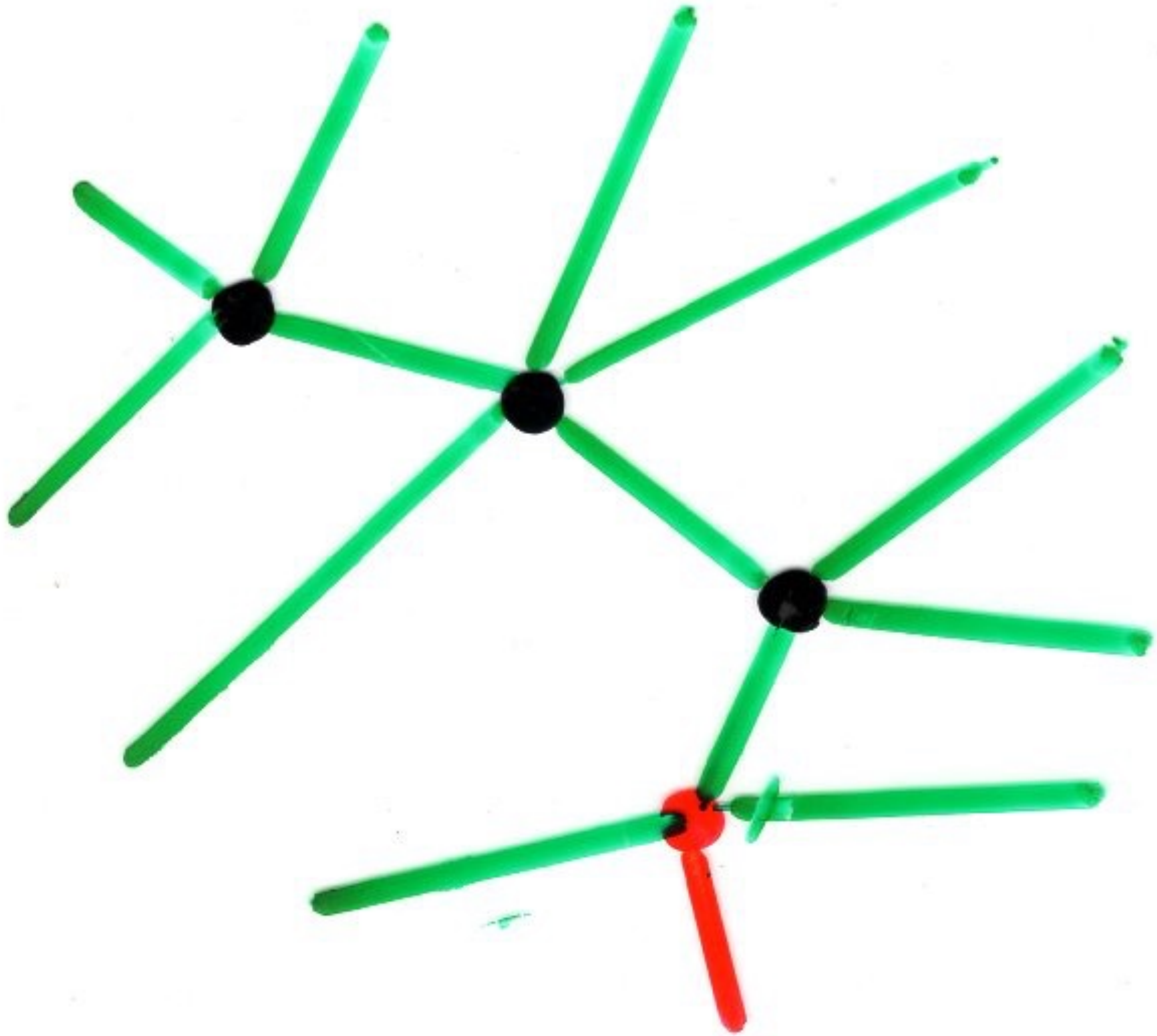
1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, ..

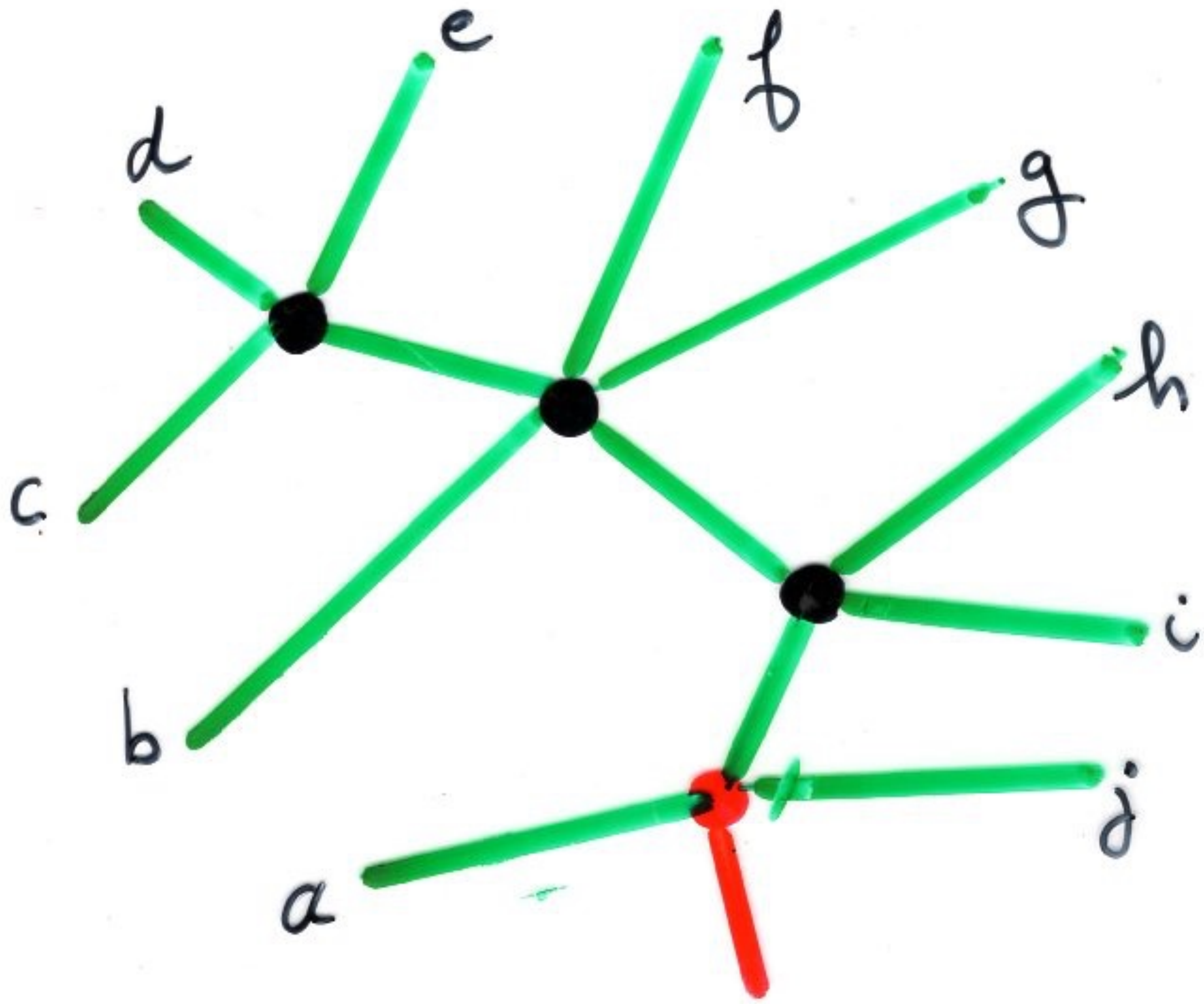
Schröder - Hipparchus numbers

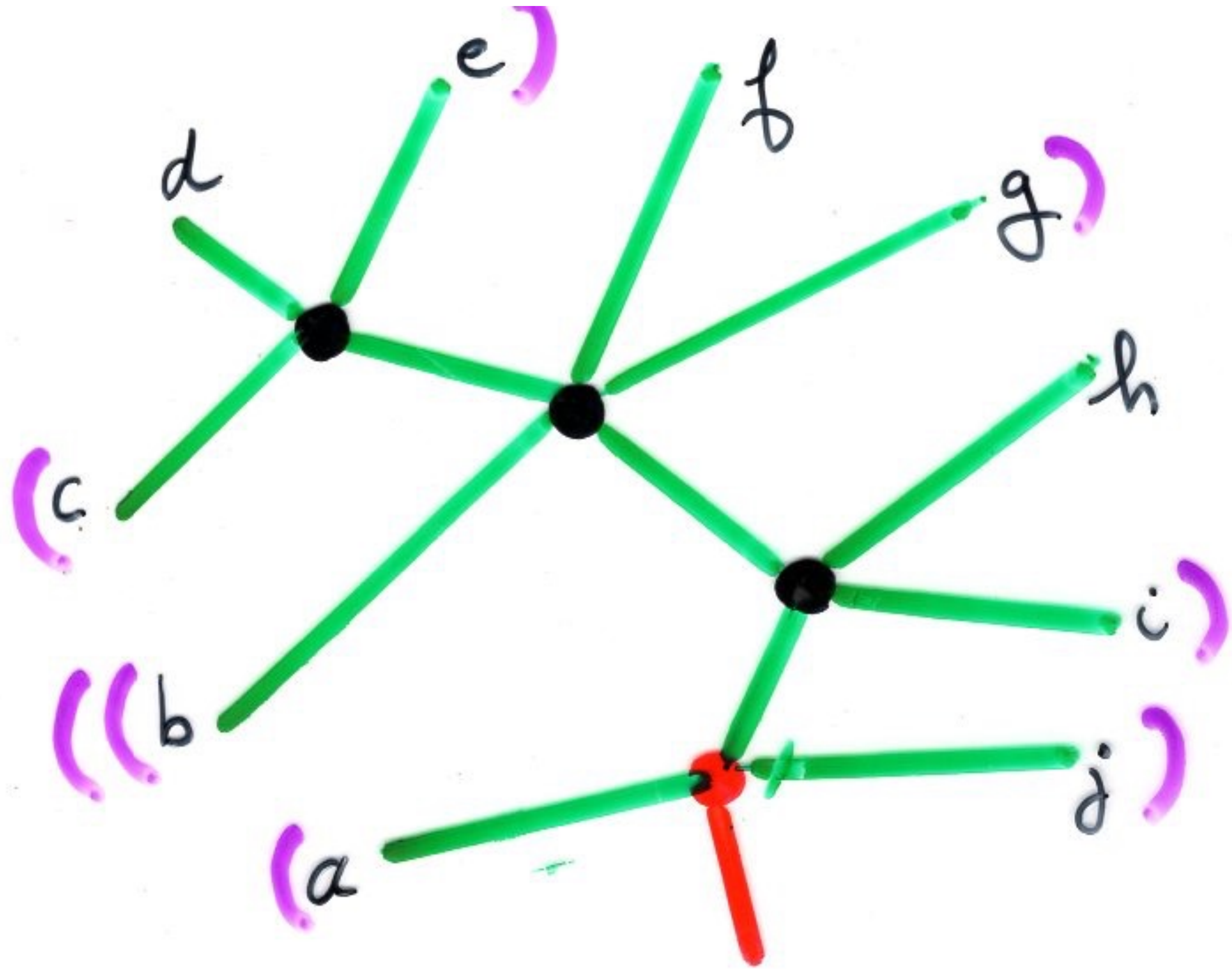


6
7
5
8
4
9
3
10
2
0
1


6
7
5
8
4
9
3
10
2
0
1

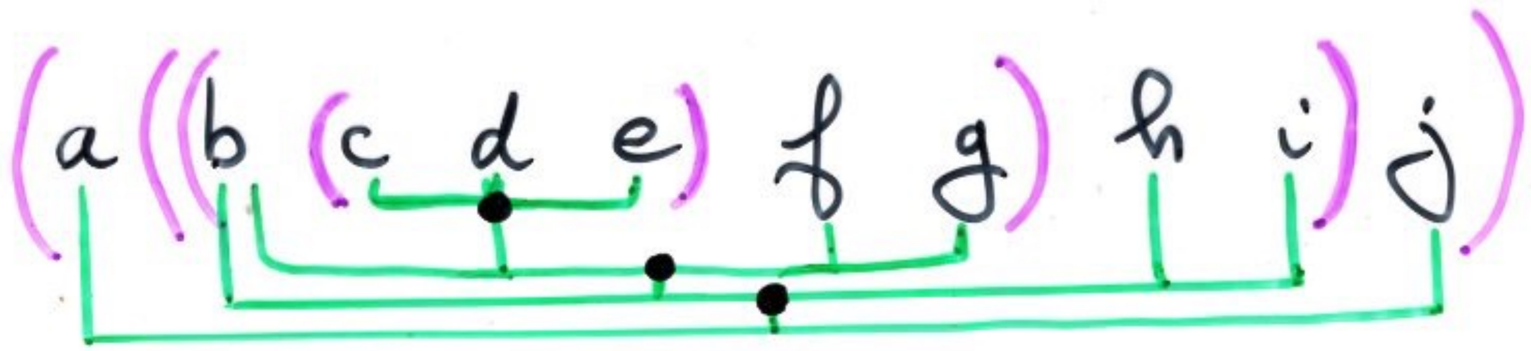



e
f
d
g
h
c
i
b
j
a

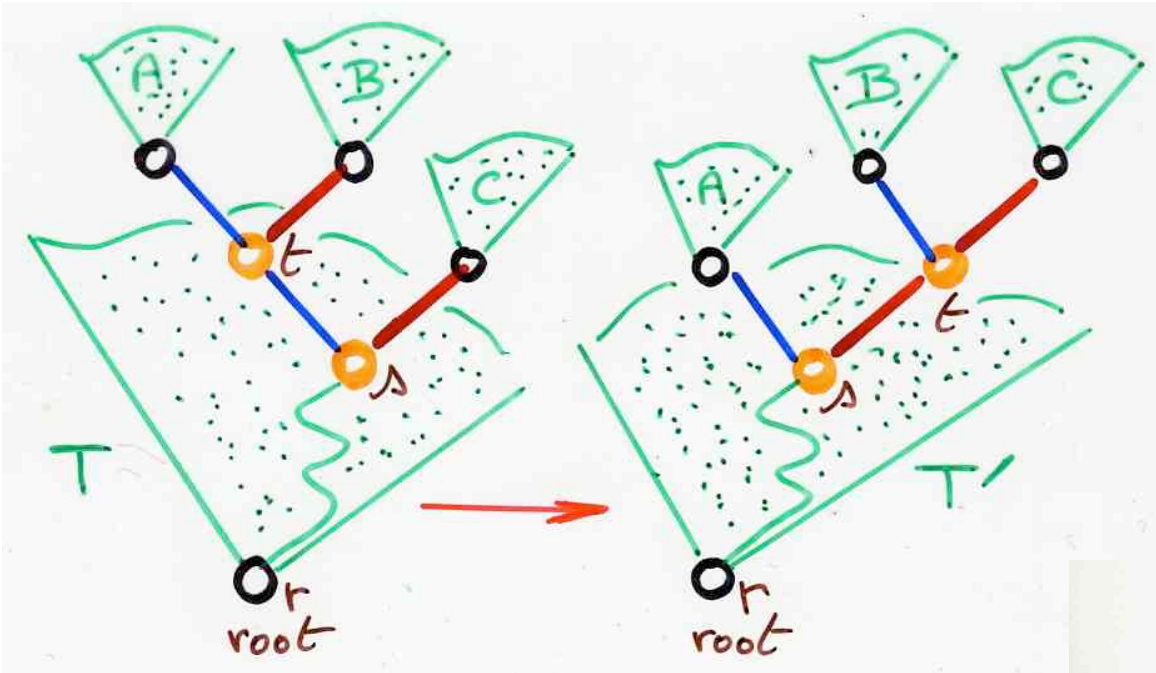


a
b
c
d
e
f
g
h
i
j

(a ((b (c d e) f g) h i) j)



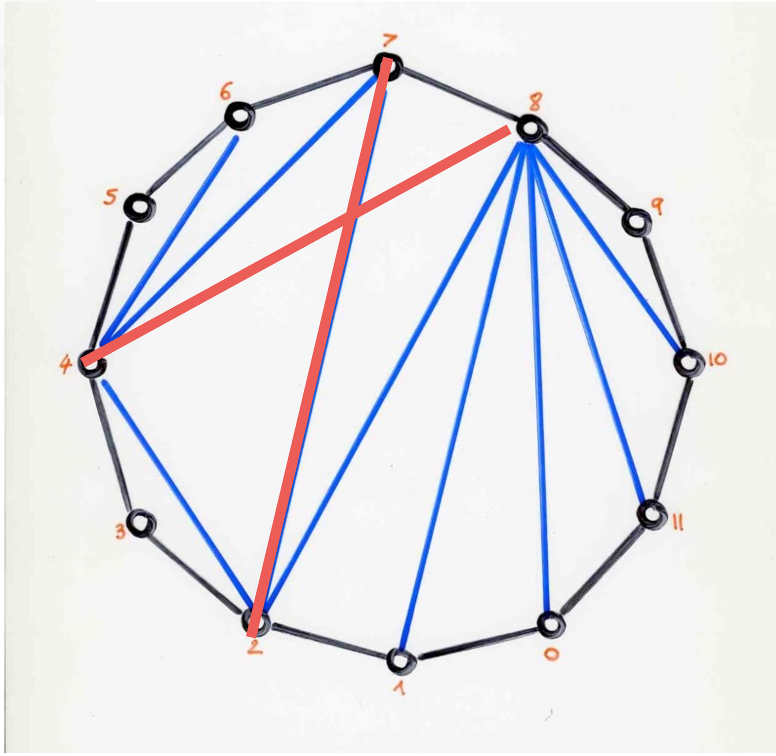
Tamari lattice



Rotation in a binary tree: the covering relation in the Tamari lattice

order relation

Tamari lattice



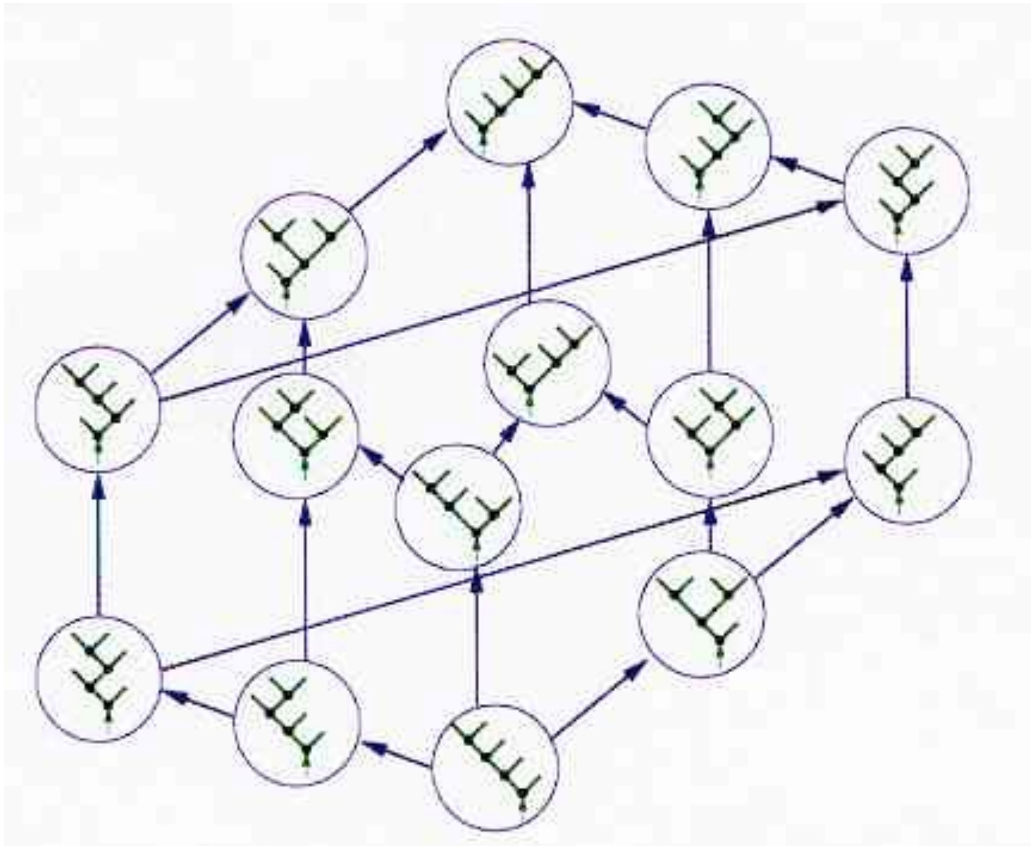
poset $\preccurlyeq$

partially ordered set

covering relation

$\alpha \preccurlyeq \beta$
no γ between
α and β

Hasse diagram



lattice

every two elements have a unique **least upper bound** (join)

and a unique **greatest lower bound** (meet)

Boolean lattice
inclusion

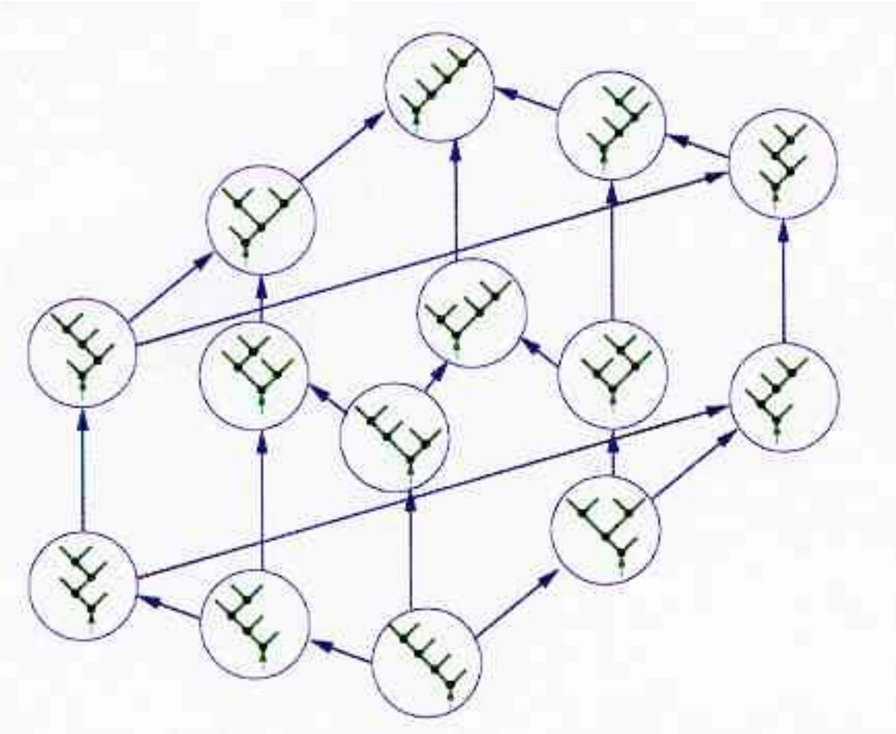
set $\mathcal{P}(X)$ subsets of X

$A \subseteq B$
order relation

$A, B \subseteq X$

$\sup(A, B) = A \cup B$

$\inf(A, B) = A \cap B$



Tamari lattice

$C_4 = 14$

Catalan

Dov Tamari (1951) thèse Sorbone
"Monoides préordonnés et chaînes de Malcev"

$((a,(b,c)),(d,e))$

$(a\,(b\;\;c))(d\;\;e)$

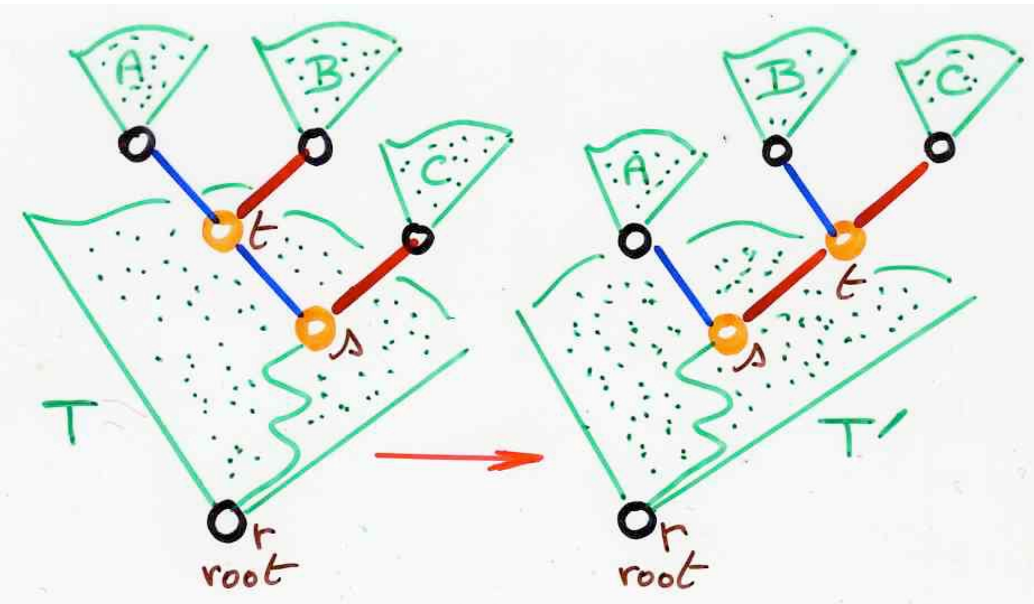
well parenthesis expression

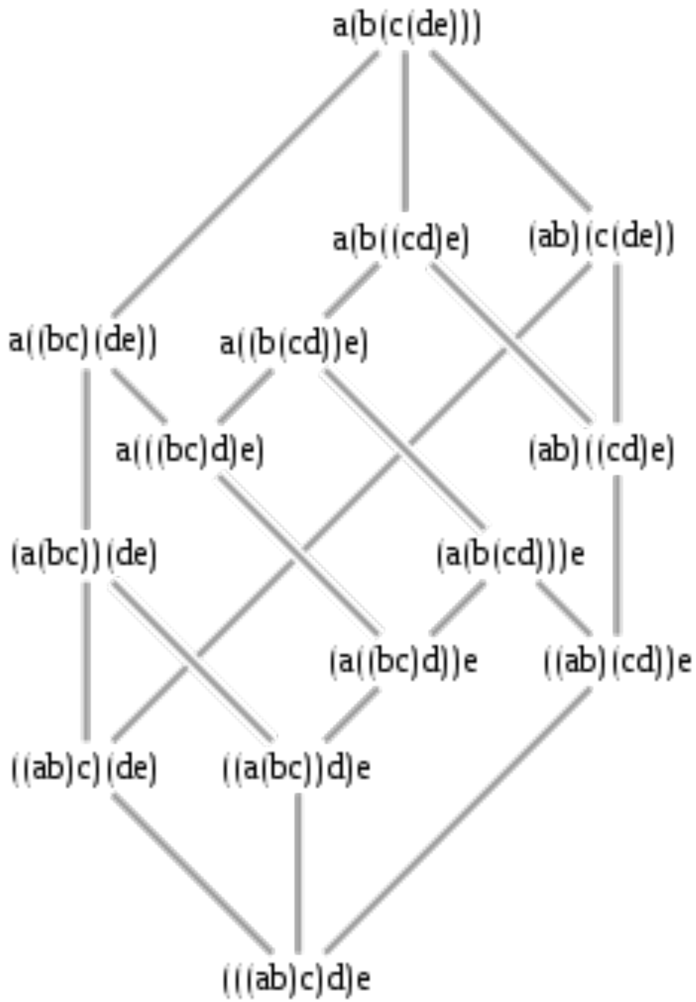
Tamari lattice

associativity

$\cdots((u\;\;v)\;\;w)\cdots$

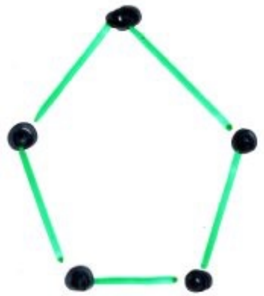
$\cdots(u\;\;(v\;\;w))\cdots$

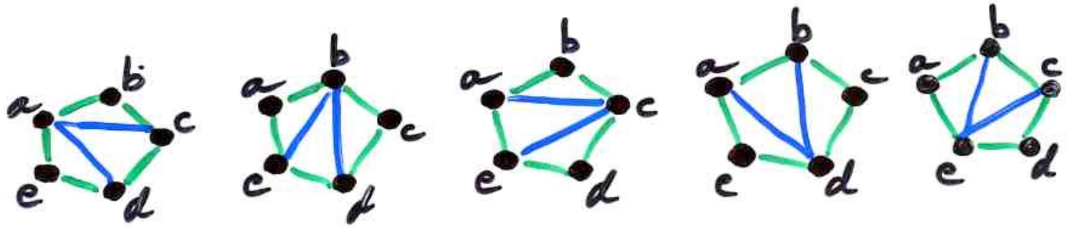


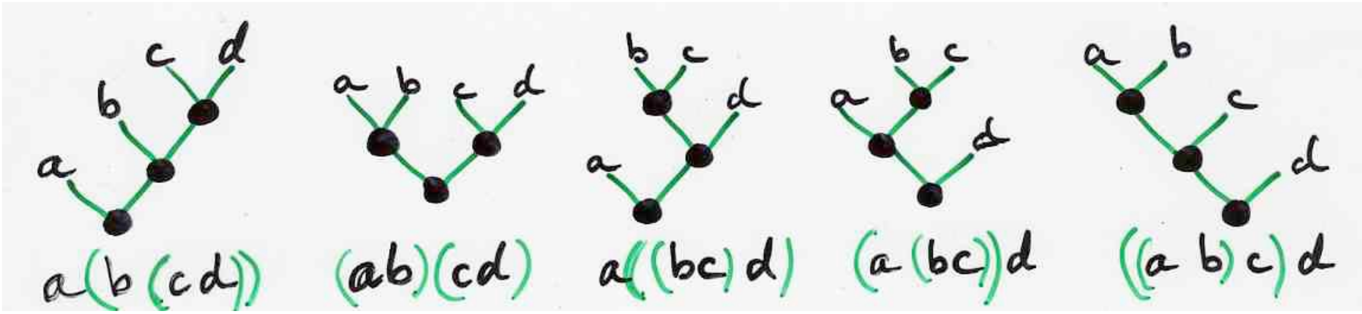


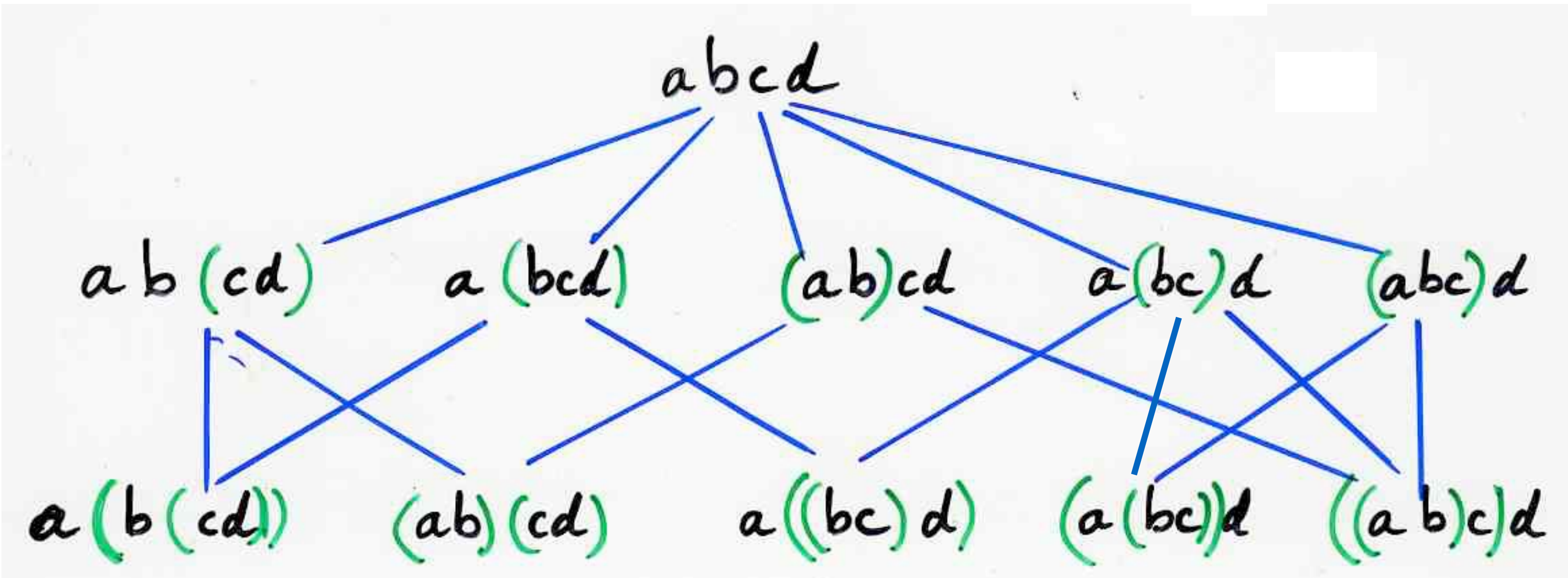
$$5 + 5 + 1 = 11$$

vertices edges



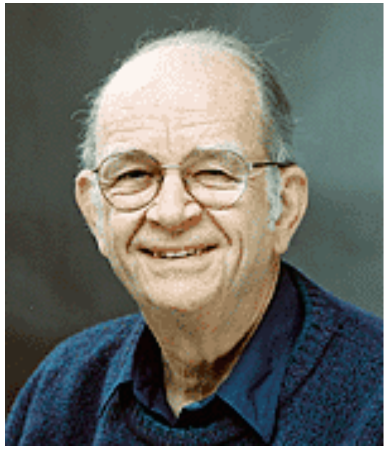






Stasheff polytope (1963)

thesis (1961)



Homotopy theory

loops

continuous

$[0,1] \xrightarrow{f} X$

topological space

$[0,\tfrac{1}{2}] \xrightarrow{f} X$

$[\tfrac{1}{2},1] \xrightarrow{g} X$

C. Hohlweg, C. Lange (2007)
F. Chapoton, S. Fomin, A. Zelevinsky (2002)

C. Ceballos
V. Pilaud
N. Reading
R. Marsh
D. Speyer

J.-P. Labbé
N. Bergeron
H. Thomas
M. Reineke
S. Stella

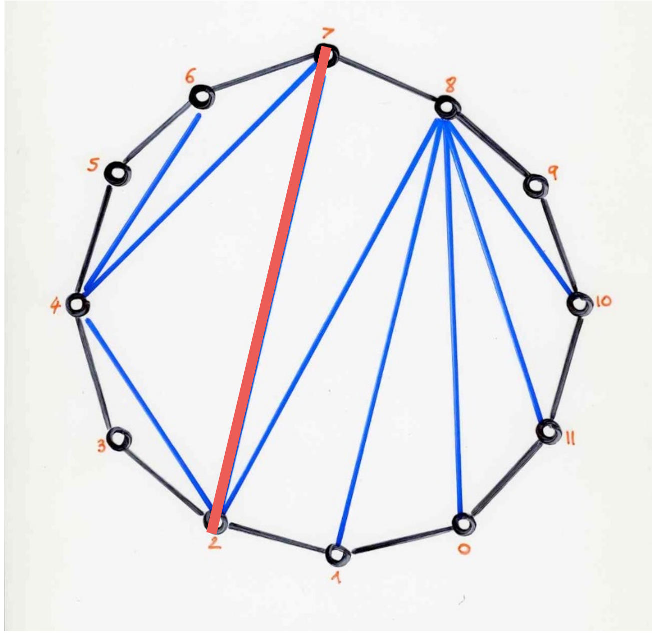
C. Stump
F. Santos
A. Postnikov

G. Ziegler
Gil Kalai

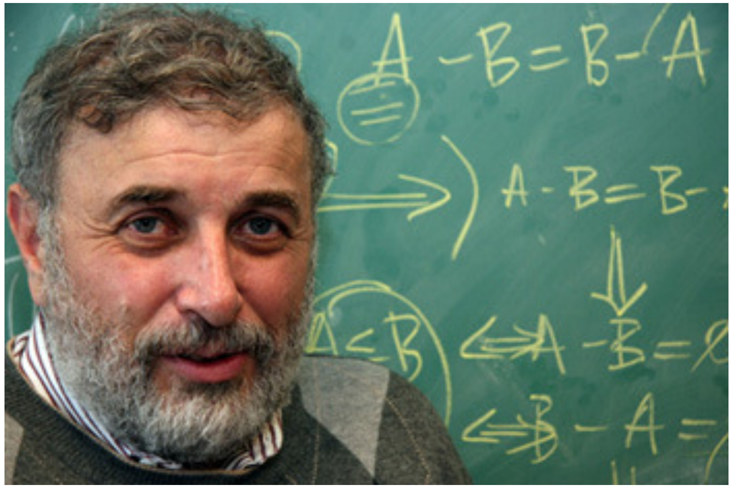
soliton KP- equation
waves in shallow water
⟷ maximal chains in the Tamari lattice

"Associahedra, Tamari lattice, and related structure" Progress in Math vol 299
Birkhauser (2012)

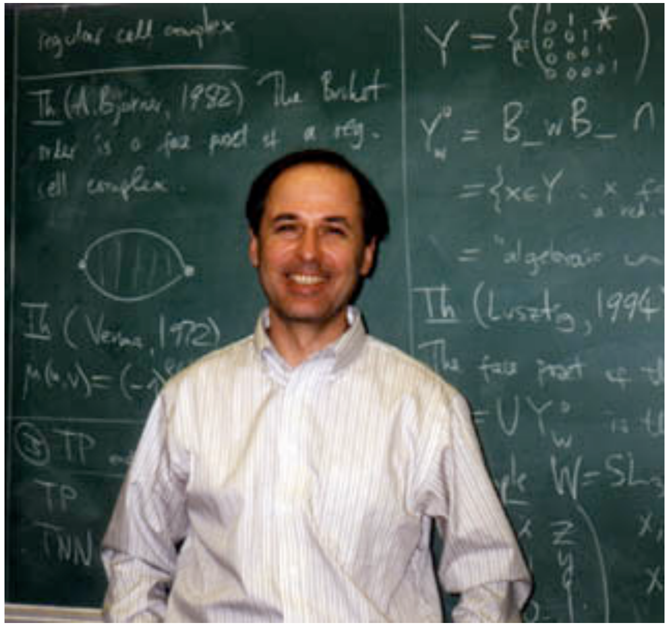
associahedron



root systems cluster algebras



Zelevinsky



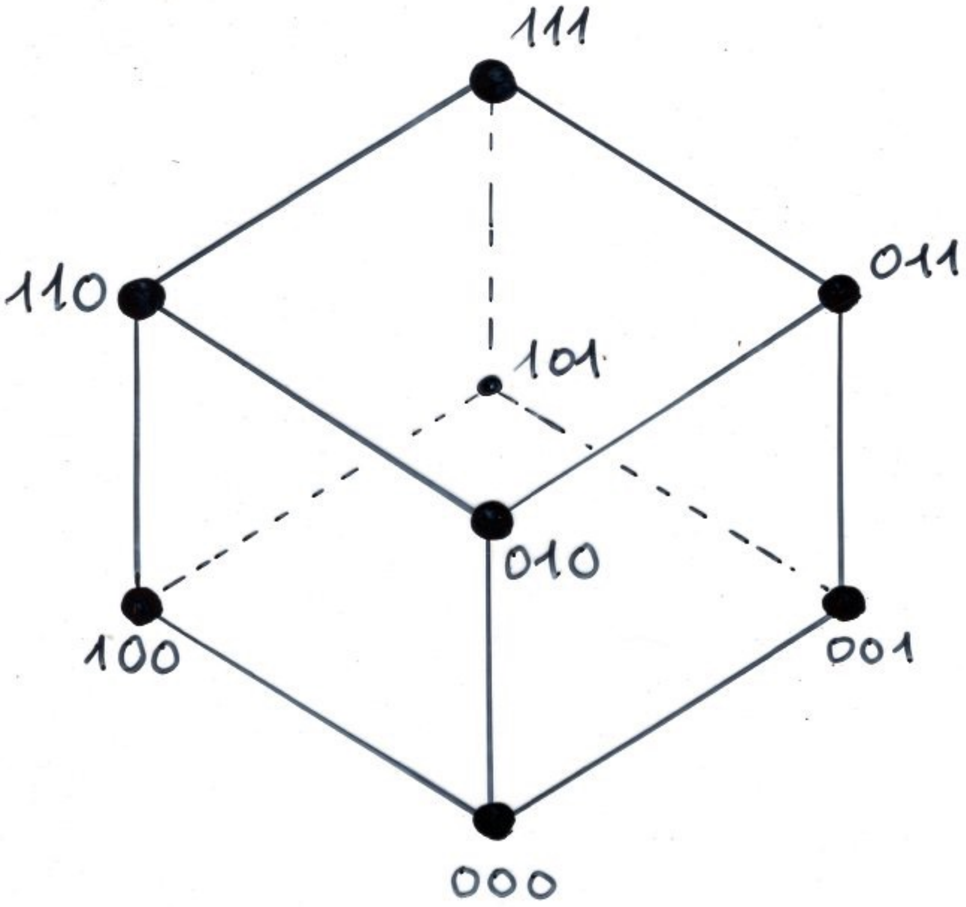
Fomin

3 lattices

words in 0, 1

binary trees

permutations



Boolean lattice
inclusion

$A \subseteq B$
order relation

$|X| = n \qquad X = \{1, 2, \ldots, n\}$

$A = \{2, 3, 6\} \subseteq \{1, 2, \ldots, 8\}$

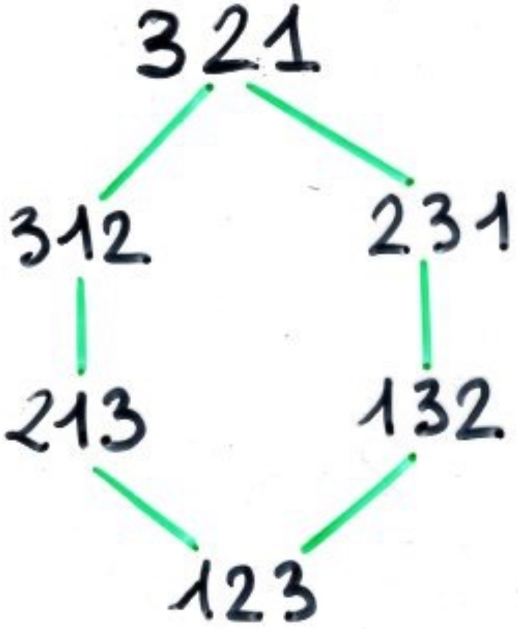
$$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline w = & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}$$

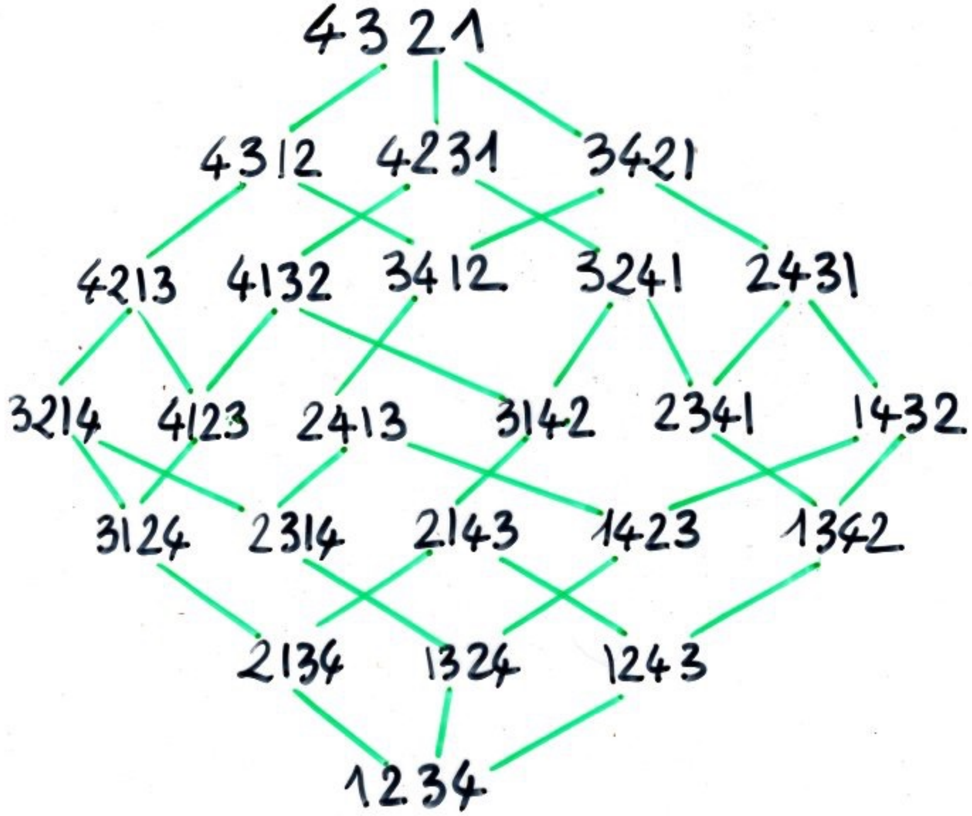
right

weak Bruhat order

$$\sigma = \sigma(1)\cdots\sigma(i)\,\sigma(i+1)\cdots\sigma(n)$$

$$\sigma \quad \cdots\sigma(i+1)\,\sigma(i)\cdots\sigma$$





left

weak Bruhat order

$$\sigma = \sigma(1)\cdots(x)\cdots\cdots(x+1)\cdots\sigma(n)$$

$$\sigma(1)\cdots(x+1)\cdots\cdots(x)\cdots\sigma(n)$$

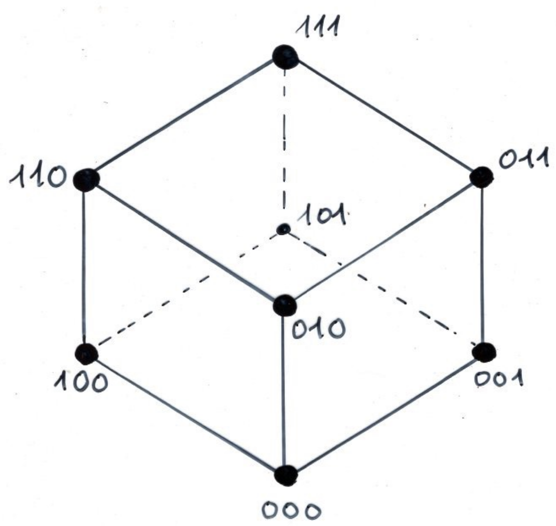
combinatorial structures

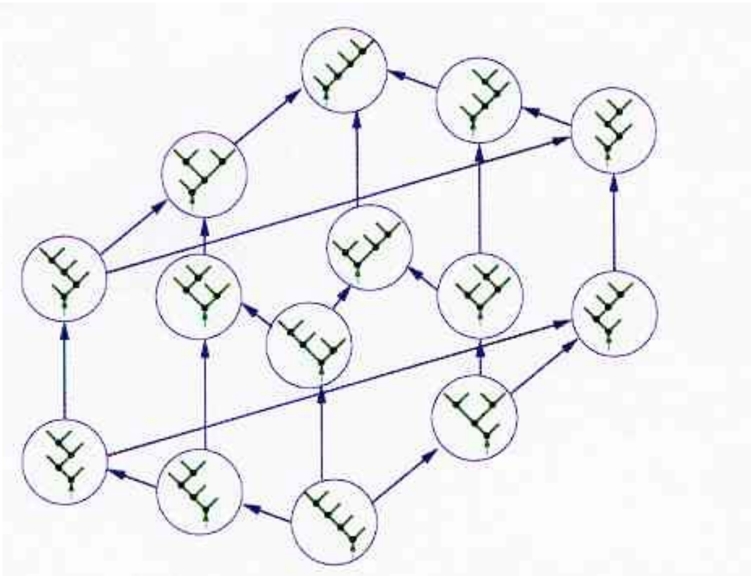
hypercube ⟷ associahedron ⟷ permutahedron

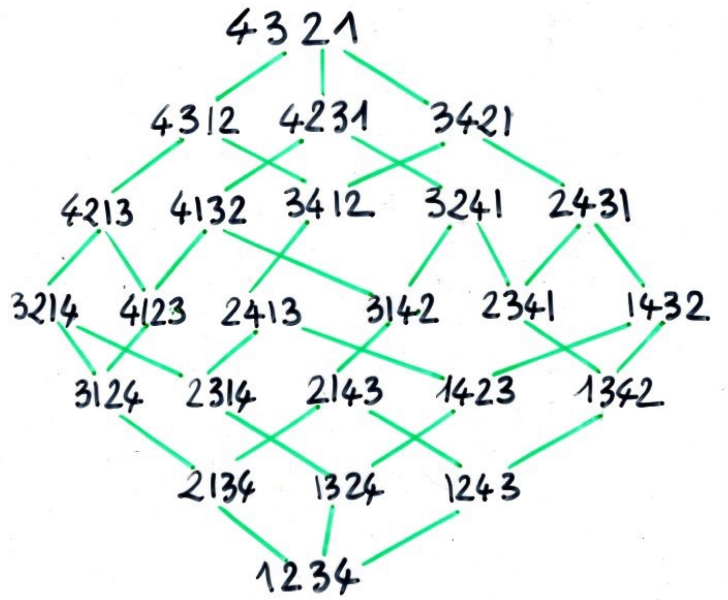
Boolean lattice
inclusion ⟷ Tamari order ⟷ weak Bruhat order

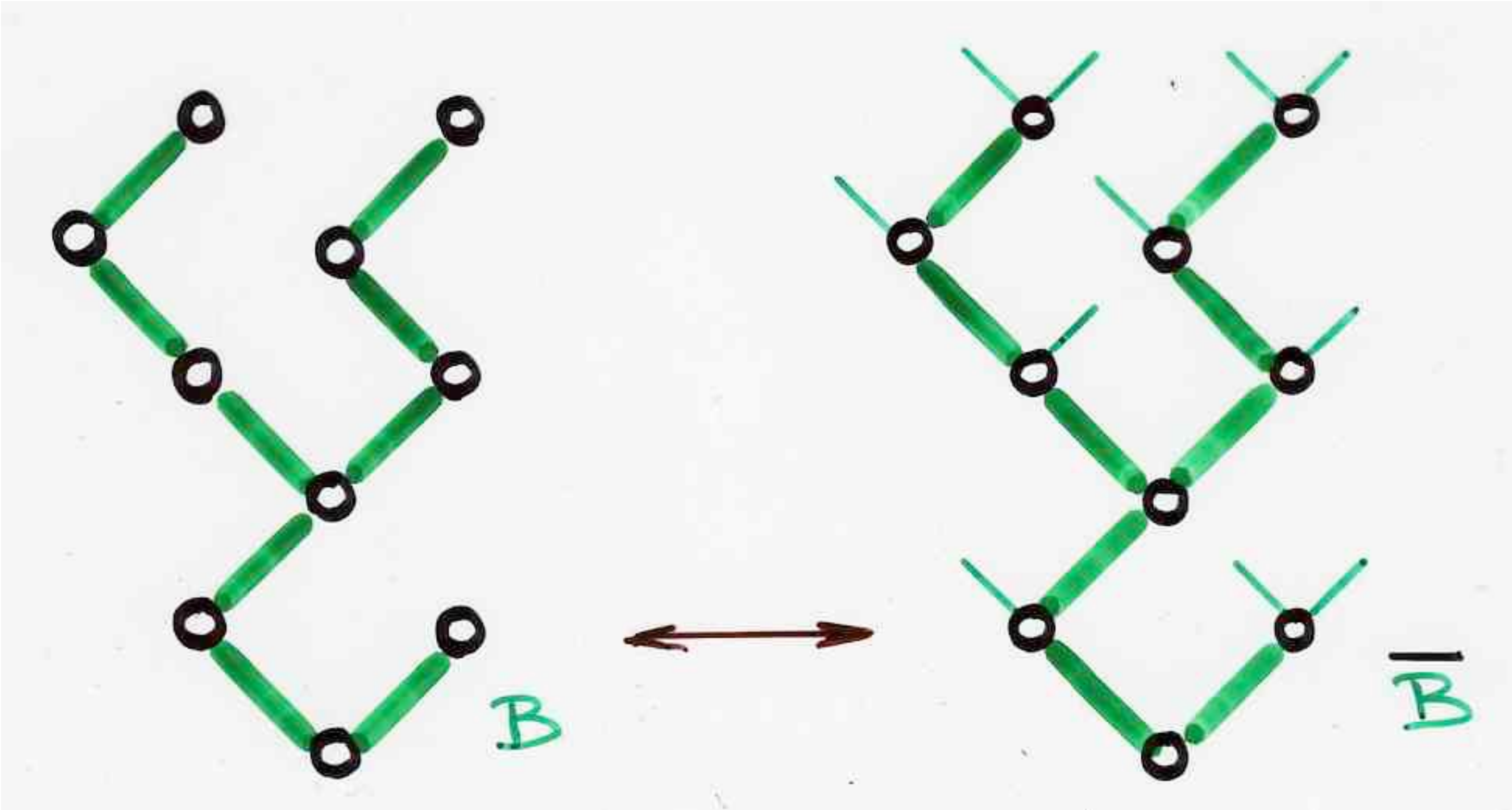
dim 2^{n-1} C_n $n!$

Catalan

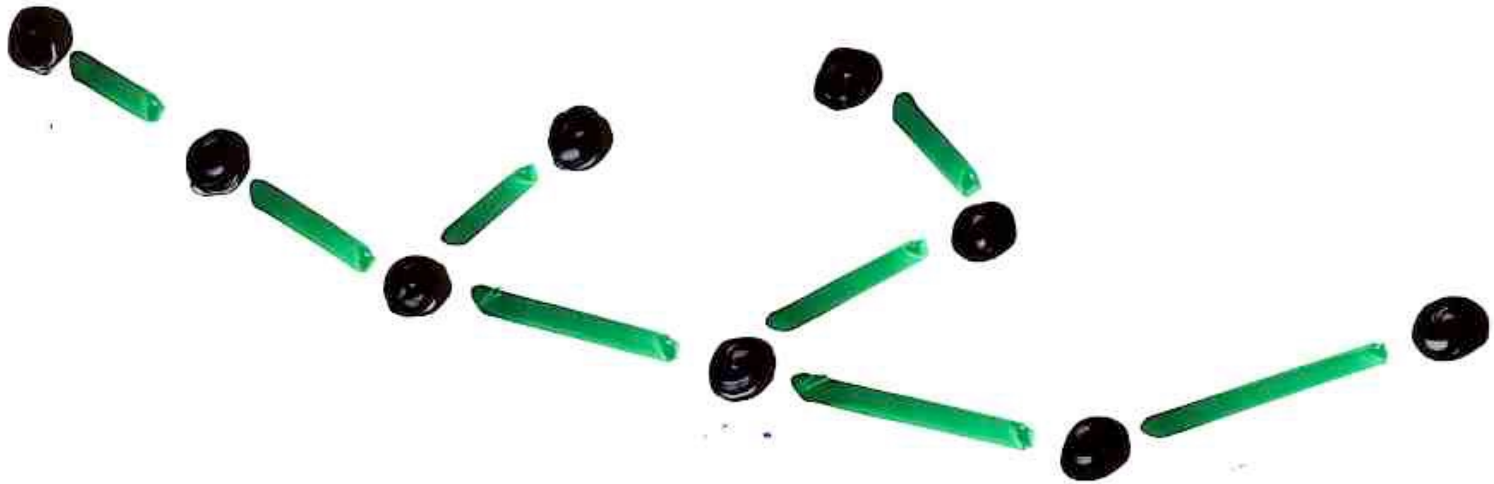




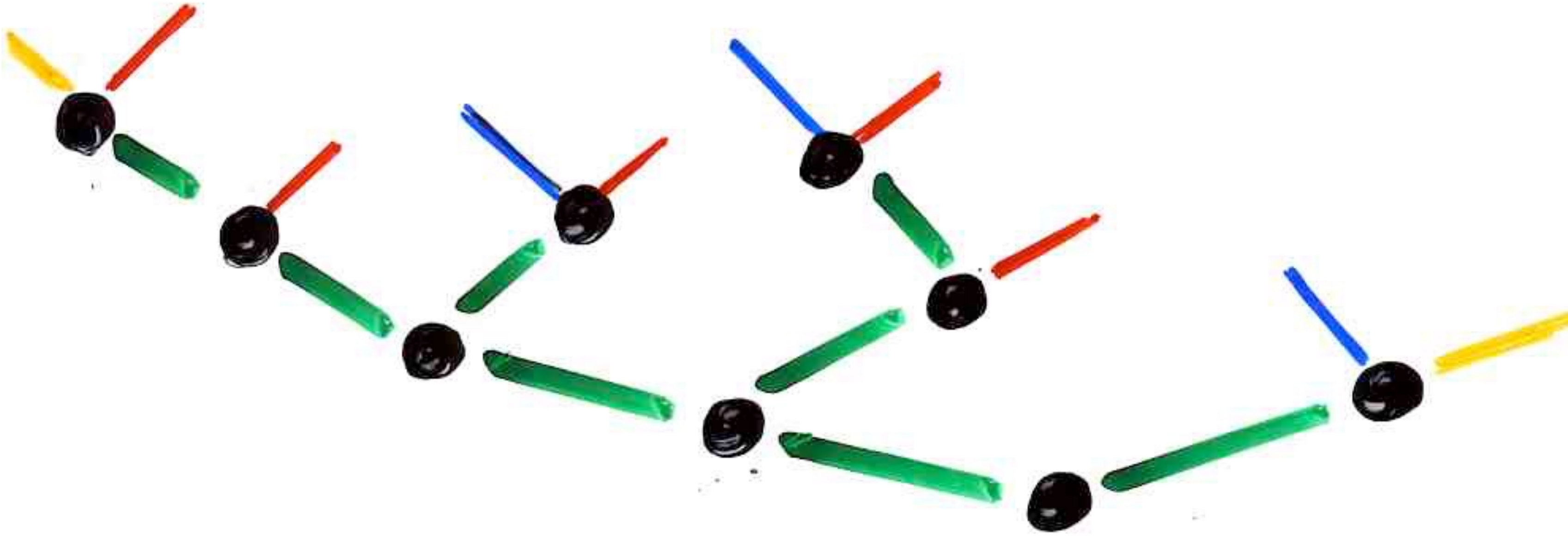




a binary tree B
and its associated complete binary tree $\overline{B}$
(full)



canopy of a binary tree



canopy of a binary tree

$$c(B) = / \; / \; \backslash \; / \; \backslash \; / \; / \; \backslash$$

Loday, Ronco (1998, 2012)

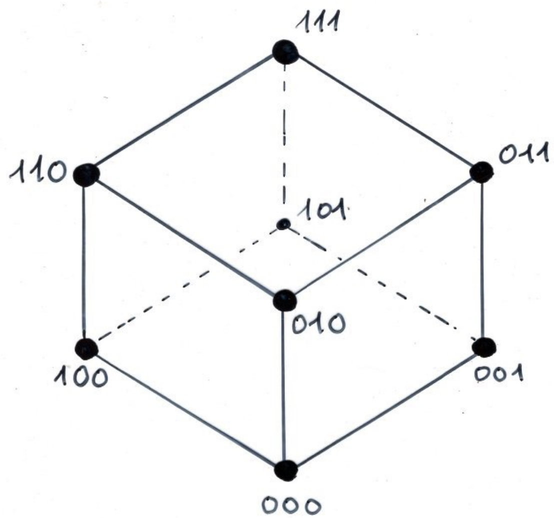
combinatorial structures

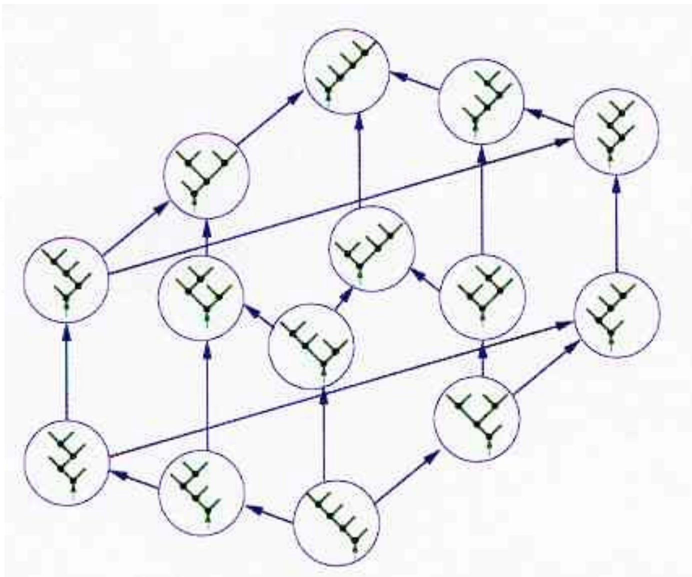
hypercube — associahedron

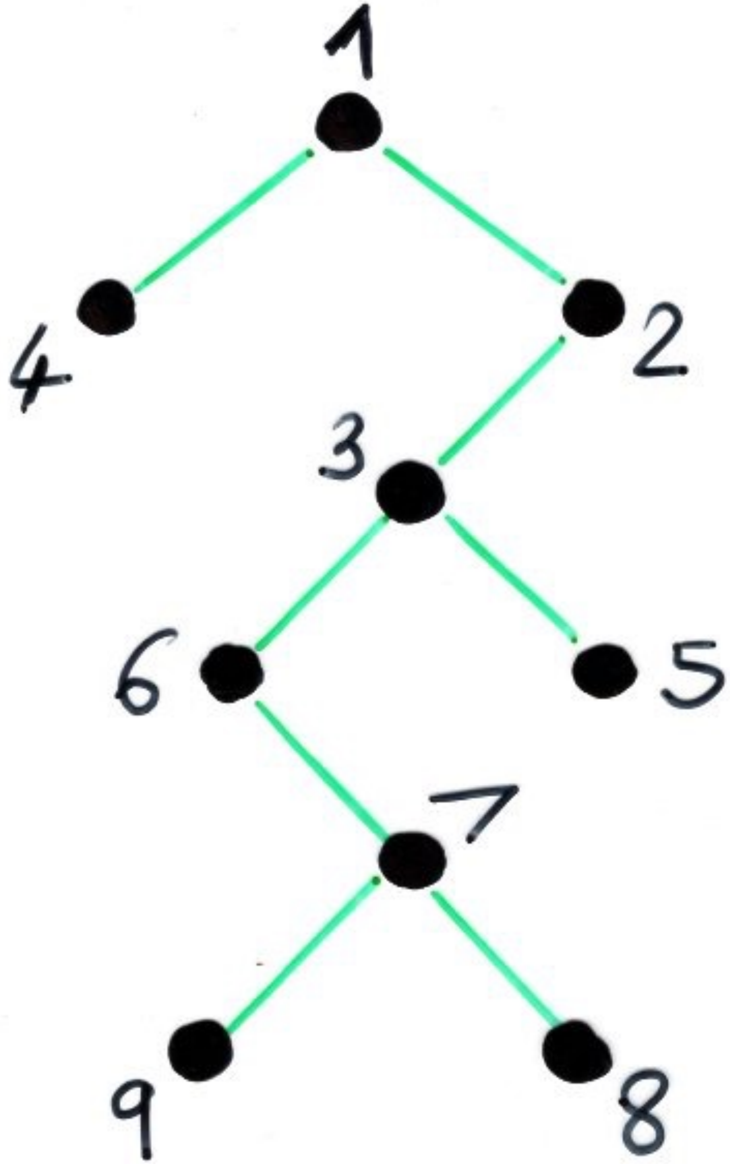
Boolean lattice inclusion ⟷ Tamari order

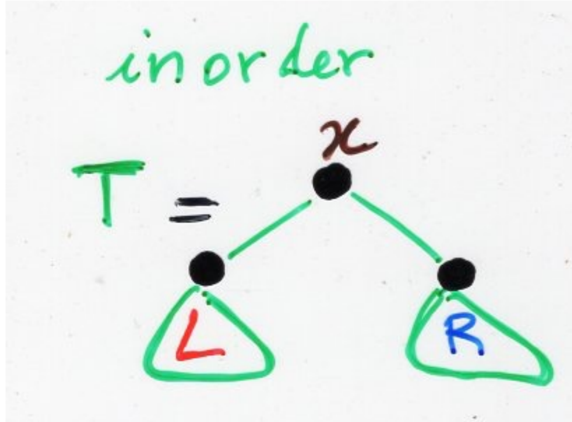
dim $2^{n}-1$ — C_n

Catalan









$$\Pi(T) = \underbrace{\Pi(L)}\, x\, \underbrace{\Pi(R)}$$

projection of $T \in \mathcal{T}_n$

$$\Pi(T) = 4\,1\,6\,9\,7\,8\,3\,5\,2$$

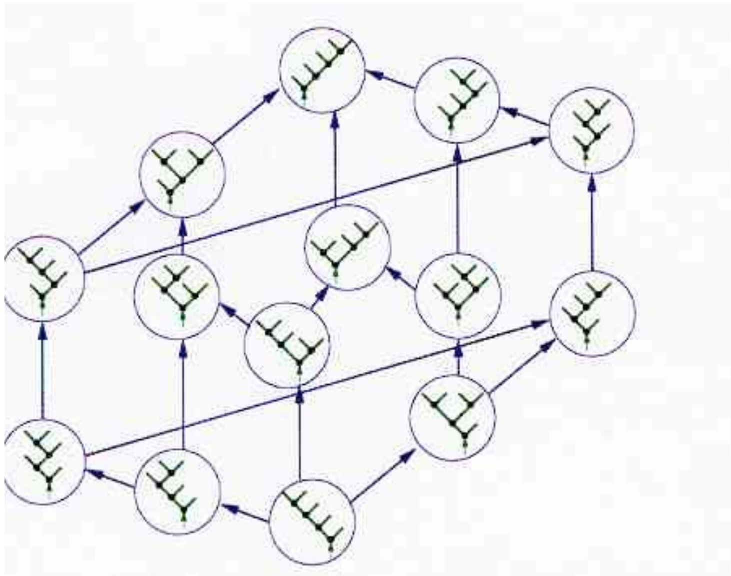
combinatorial structures

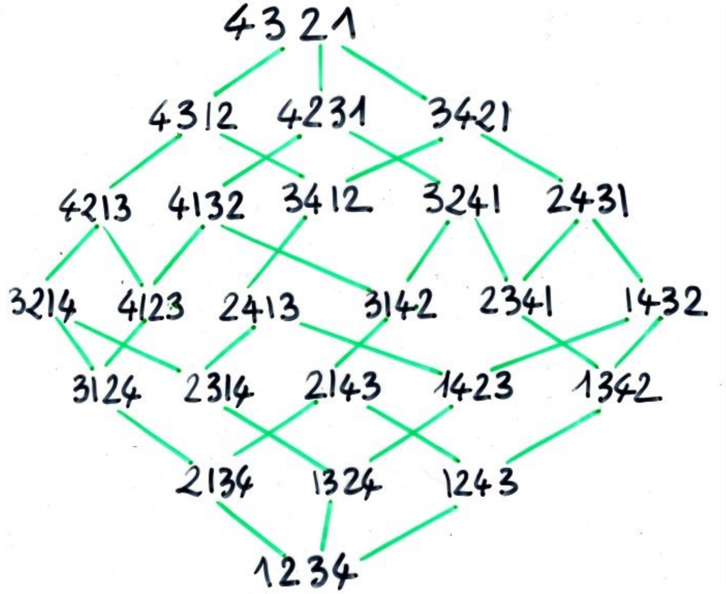
associahedron permutahedron

Tamari order ⟷ weak Bruhat order

C_n $n!$

Catalan





up-down sequence of a permutation

4 7 1 9 2 3 5 8 6

$$
\begin{tikzcd}
\sigma \arrow[r] \arrow[dd] & T(\sigma) \arrow[d] \\
 & B \arrow[d] \\
UD(\sigma) \arrow[r, phantom, "="] & c(B)
\end{tikzcd}
$$

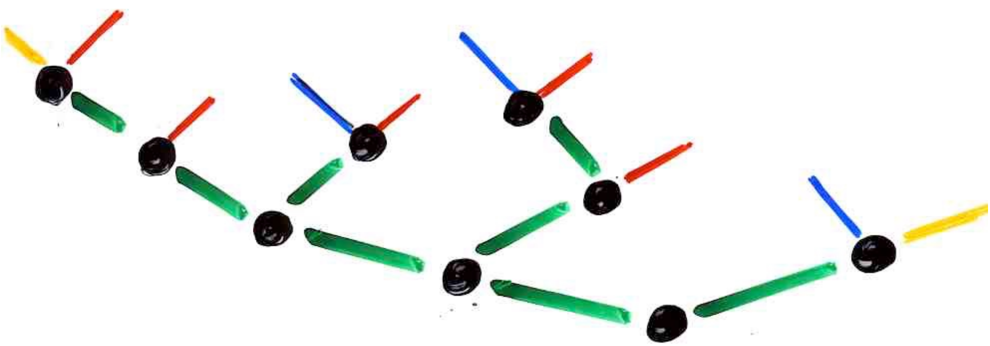
σ permutation

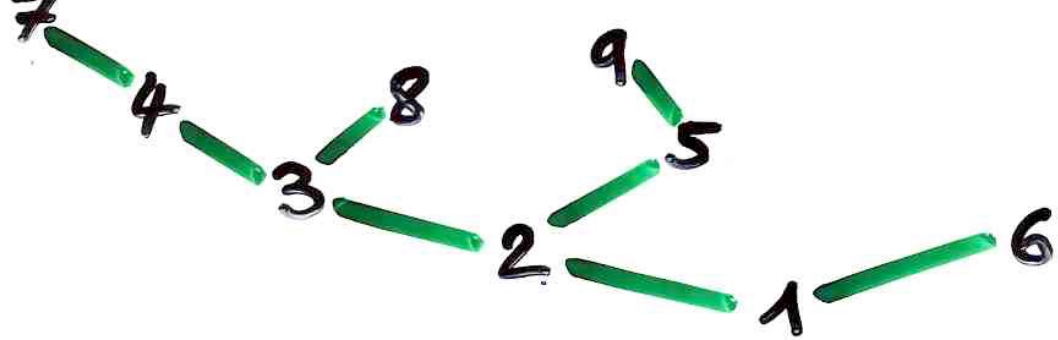
$T(\sigma)$ increasing binary tree

B binary tree

$UD(\sigma)$ up-down sequence

$c(B)$ canopy





$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6 \ldots$

up-down sequence $-\ -\ +\ -\ +\ -\ -\ +$

A. Björner, M. Wachs (1991)

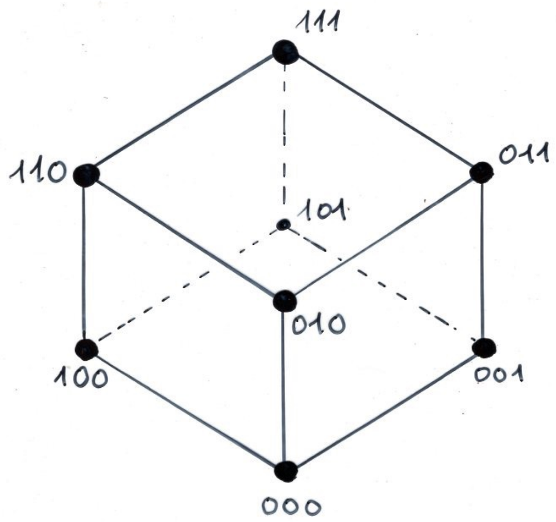
Loday, Ronco (1998, 2012)

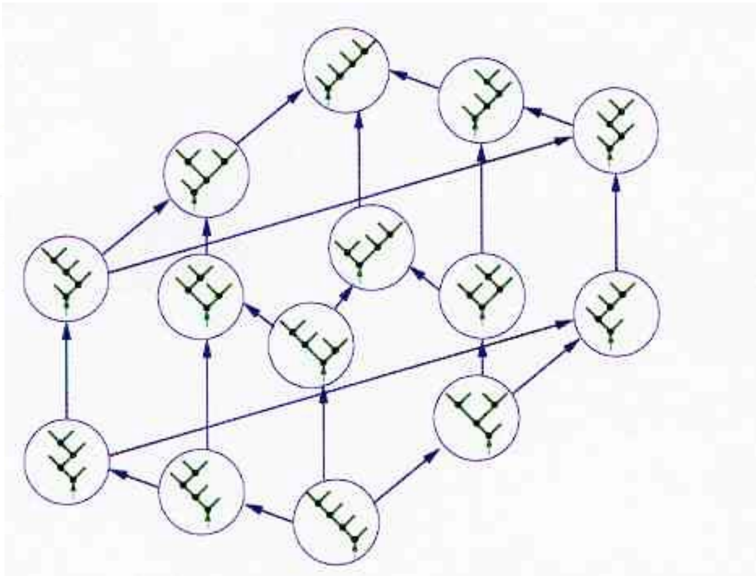
combinatorial structures

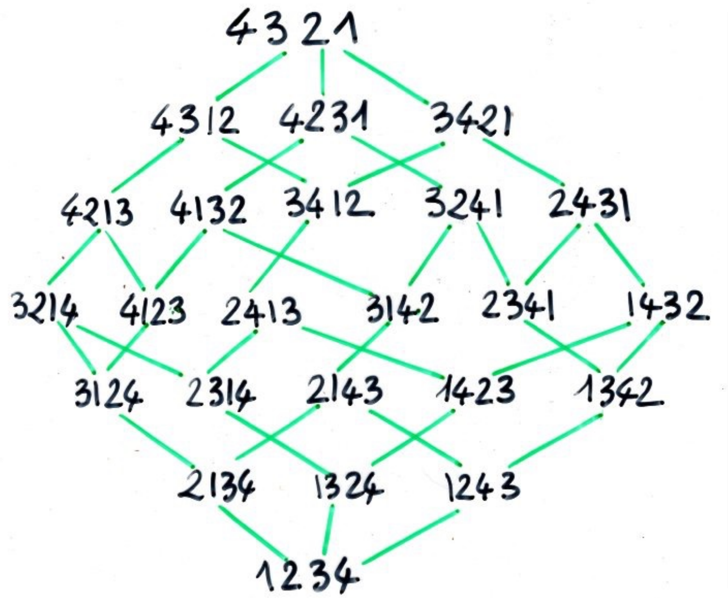
hypercube — associahedron — permutahedron

Boolean lattice inclusion ⟷ Tamari order ⟷ weak Bruhat order

dim 2^{n-1} — C_n Catalan — $n!$





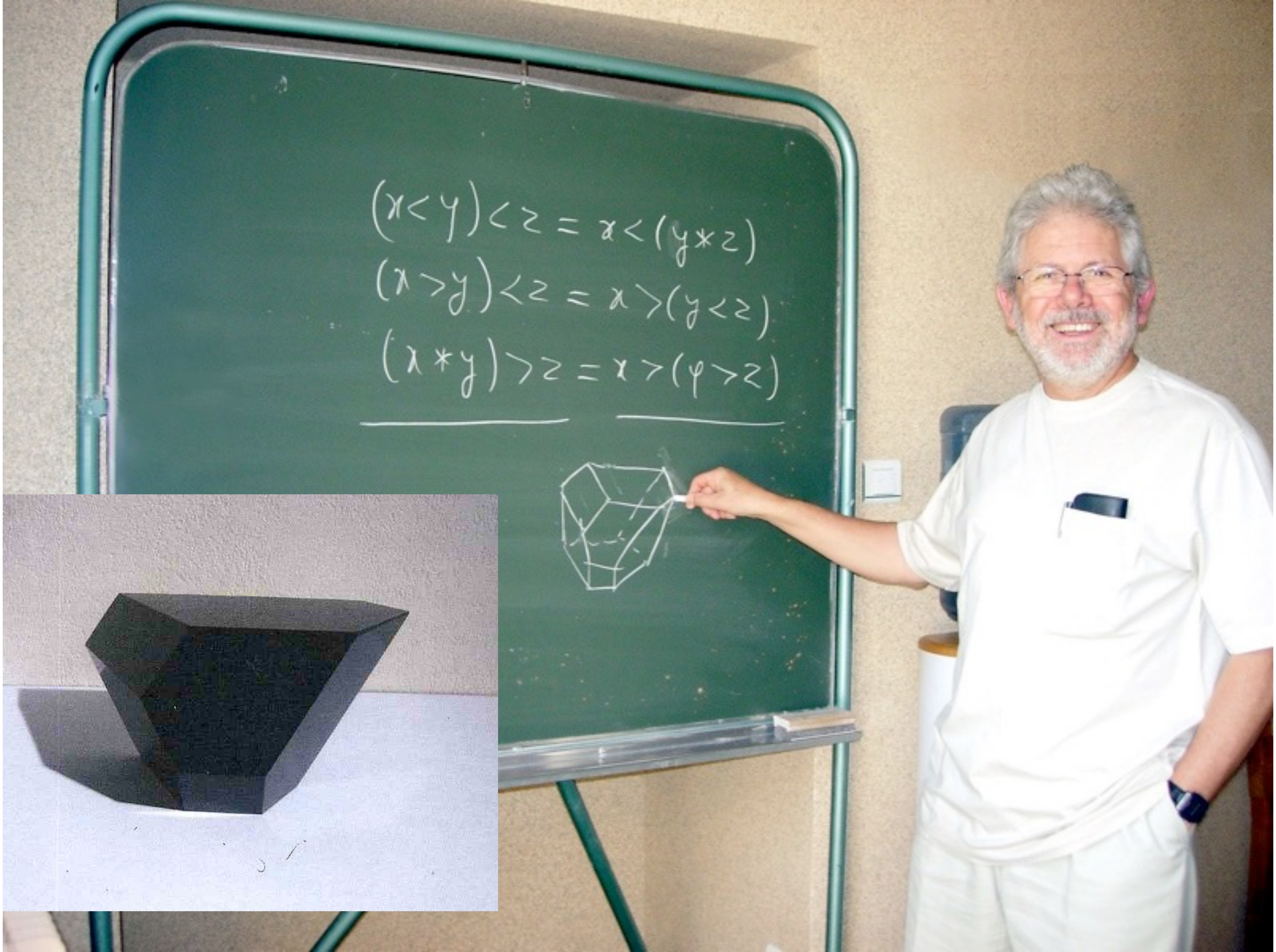


3 geometric structures

hypercube

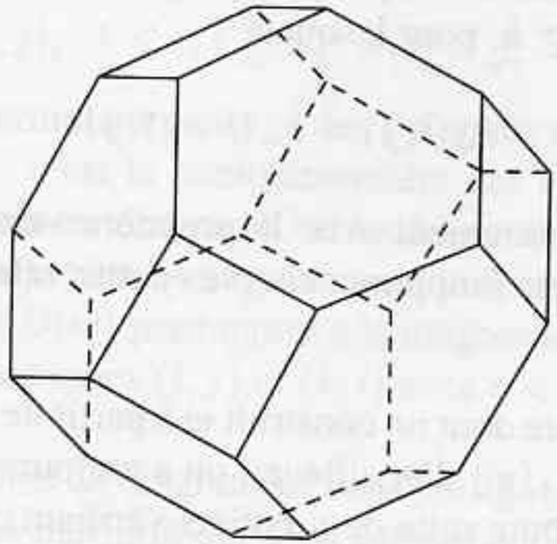
associahedron

permutohedron

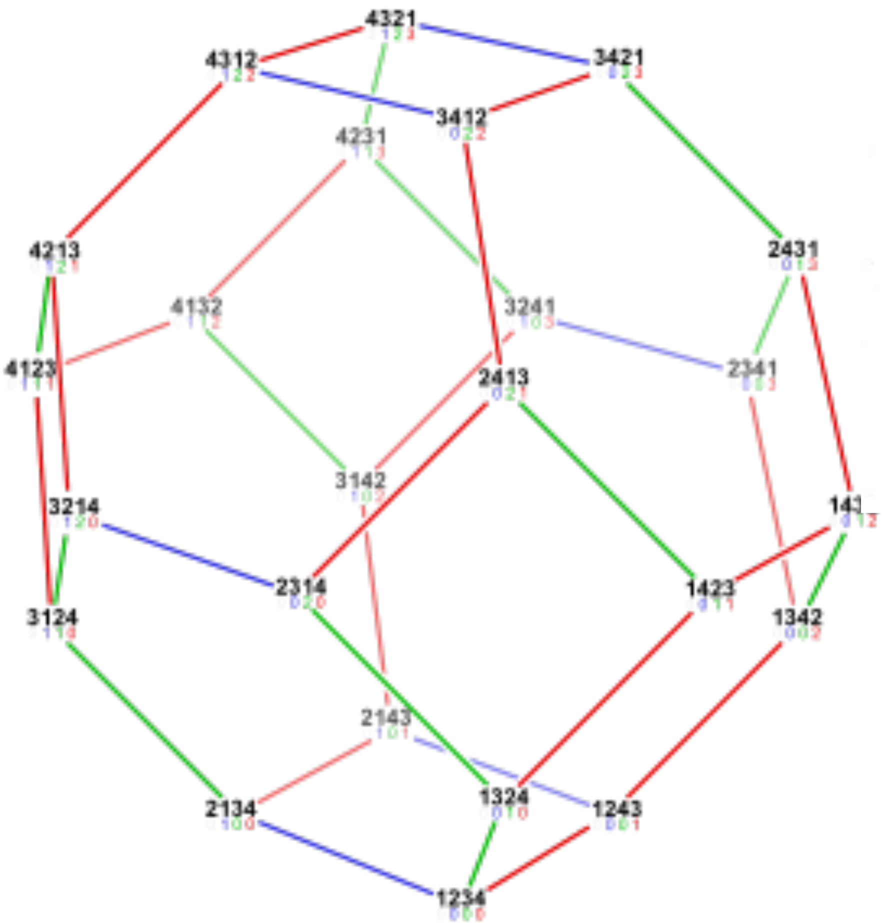

$(x < y) < z = x < (y * z)$
$(x > y) < z = x > (y < z)$
$(x * y) > z = x > (y > z)$

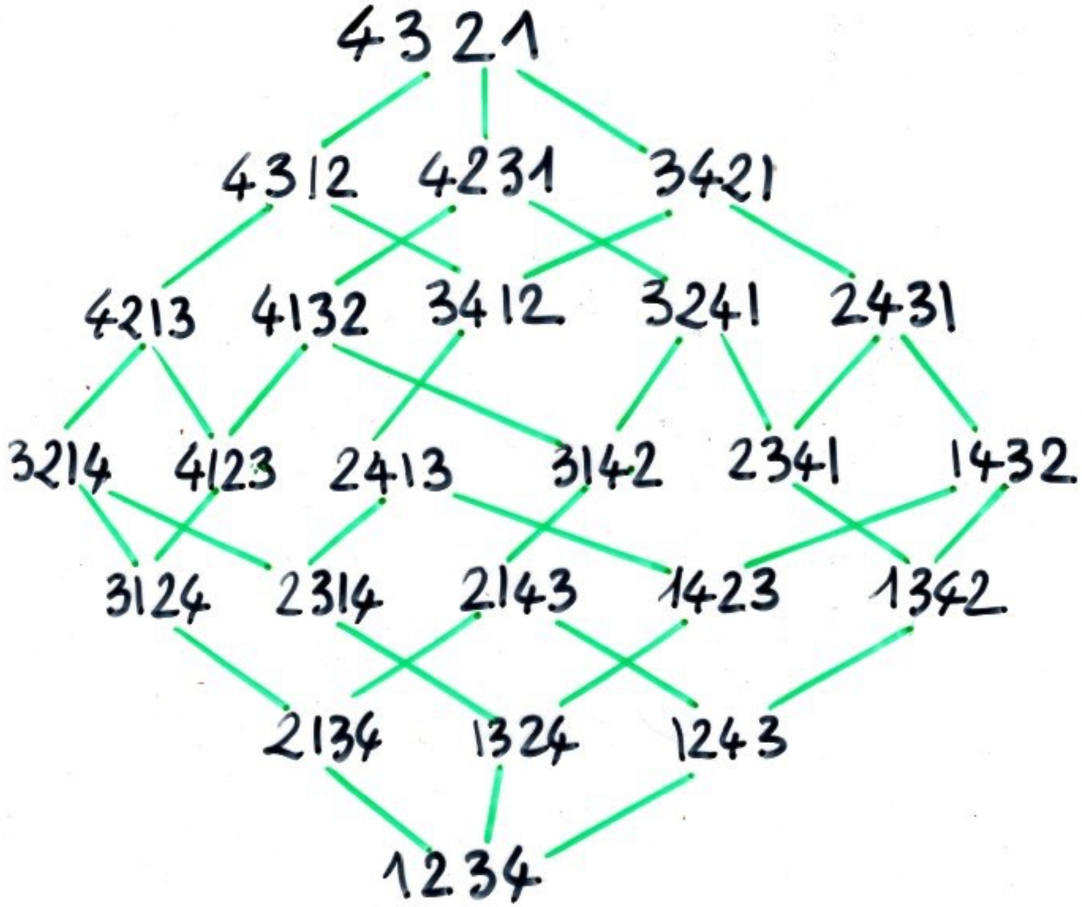
Alain Lascoux
(1944–2013)

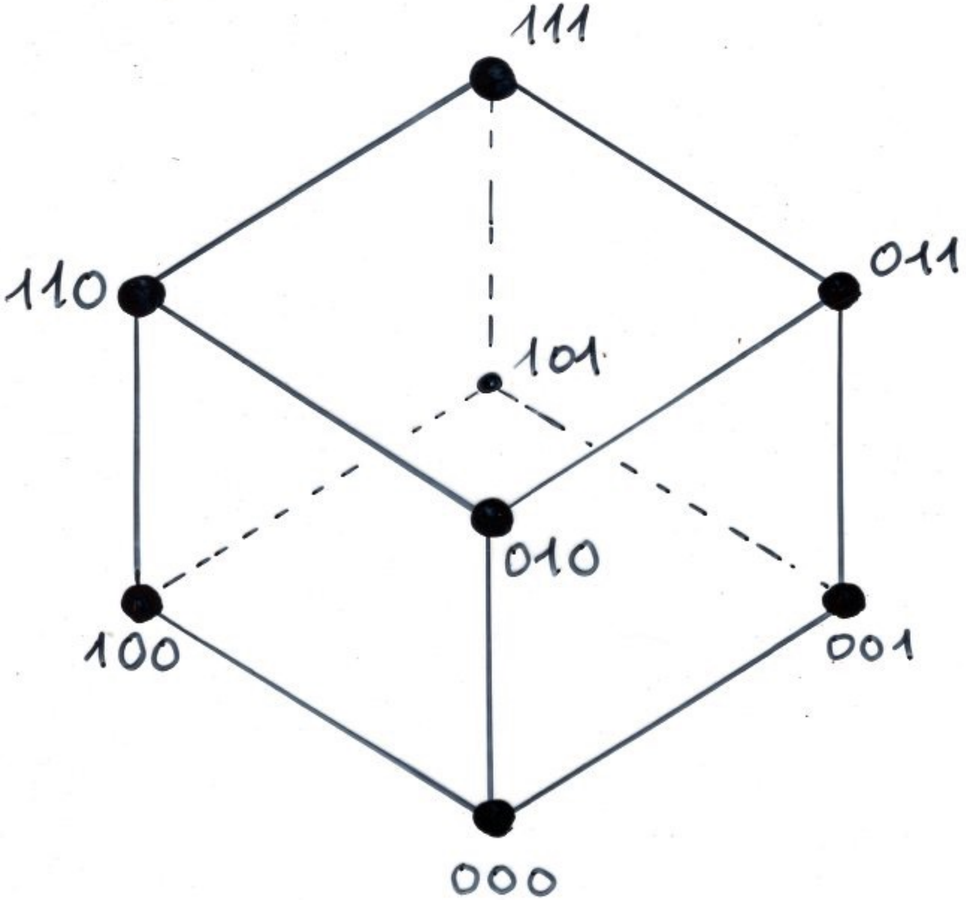
permutohedron



2. Le permutoèdre Π_3.


4321
4312
3421
3412
4231
4213
2431
4132
3241
4123
2413
2341
3142
3214
143
2314
1423
3124
1342
2143
2134
1324
1243
1234


4321
4312 4231 3421
4213 4132 3412 3241 2431
3214 4123 2413 3142 2341 1432
3124 2314 2143 1423 1342
2134 1324 1243
1234



Boolean lattice
inclusion

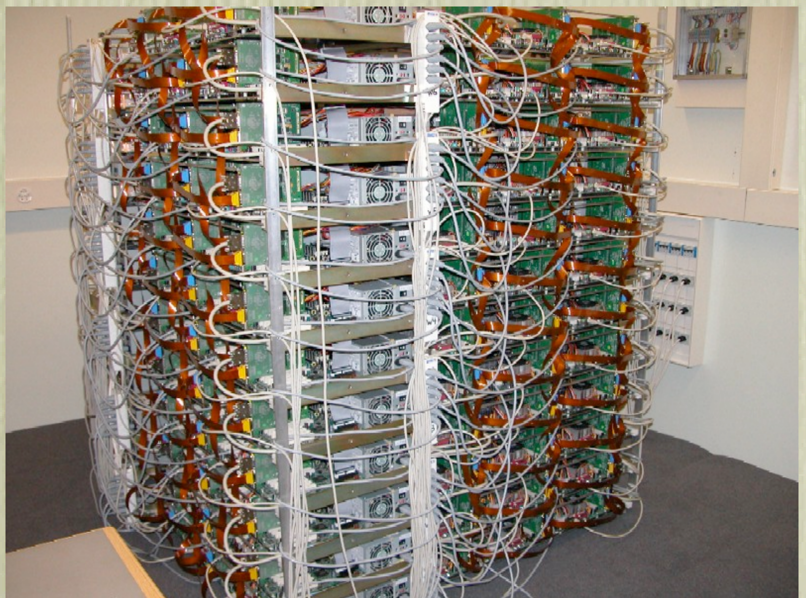
$A \subseteq B$
order relation

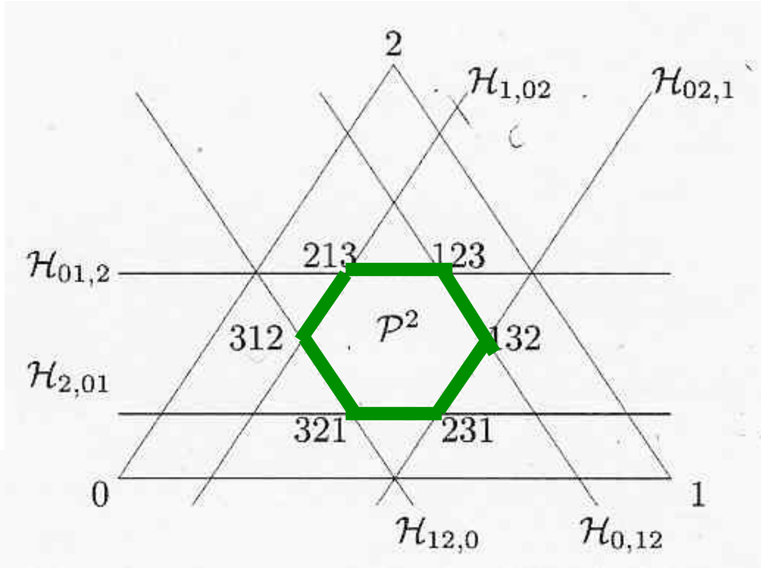
$|X| = n \qquad X = \{1, 2, \ldots, n\}$

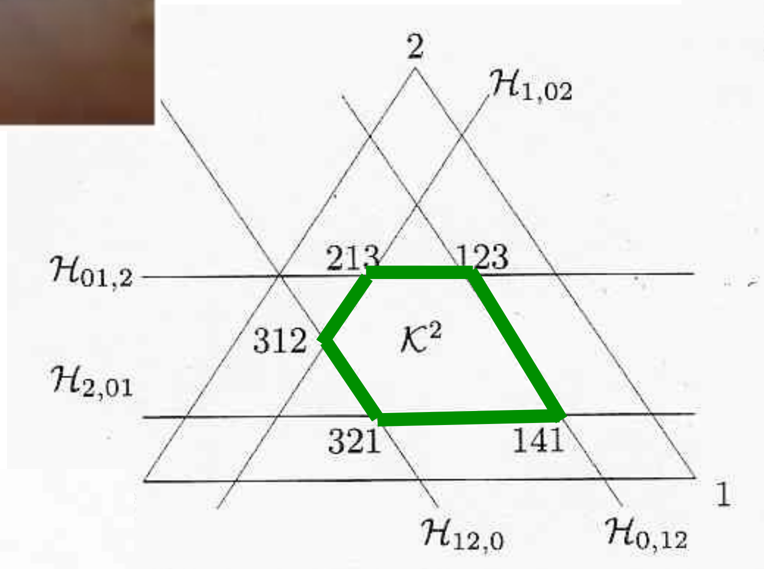
$A = \{2, 3, 6\} \subseteq \{1, 2, \ldots, 8\}$

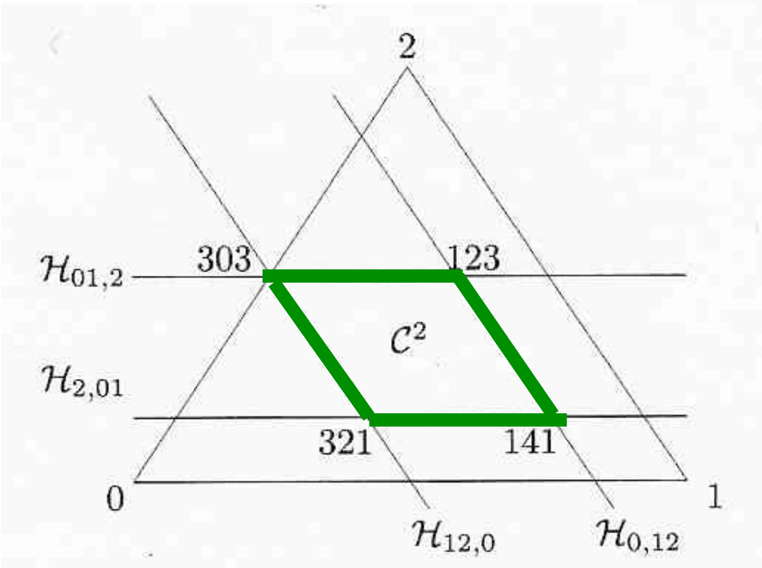
$$\begin{array}{c|cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline w = & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}$$

	Binohedron	associahedron	permutahedron
	00110100110001100000 $\longleftrightarrow$	Tamari order $\longleftrightarrow$	weak Bruhat order
dim	2^{n-1}	C_n	$n!$
		Catalan	




2
$\mathcal{H}_{1,02}$
$\mathcal{H}_{02,1}$
$\mathcal{H}_{01,2}$
213
123
312
$\mathcal{P}^2$
132
$\mathcal{H}_{2,01}$
321
231
0
1
$\mathcal{H}_{12,0}$
$\mathcal{H}_{0,12}$


2
$\mathcal{H}_{1,02}$
$\mathcal{H}_{01,2}$
213
123
312
$\mathcal{K}^2$
$\mathcal{H}_{2,01}$
321
141
1
$\mathcal{H}_{12,0}$
$\mathcal{H}_{0,12}$


2
$\mathcal{H}_{01,2}$
303
123
$\mathcal{C}^2$
$\mathcal{H}_{2,01}$
321
141
0
1
$\mathcal{H}_{12,0}$
$\mathcal{H}_{0,12}$

3 algebraic structures

descent algebra

Loday-Ronco
algebra

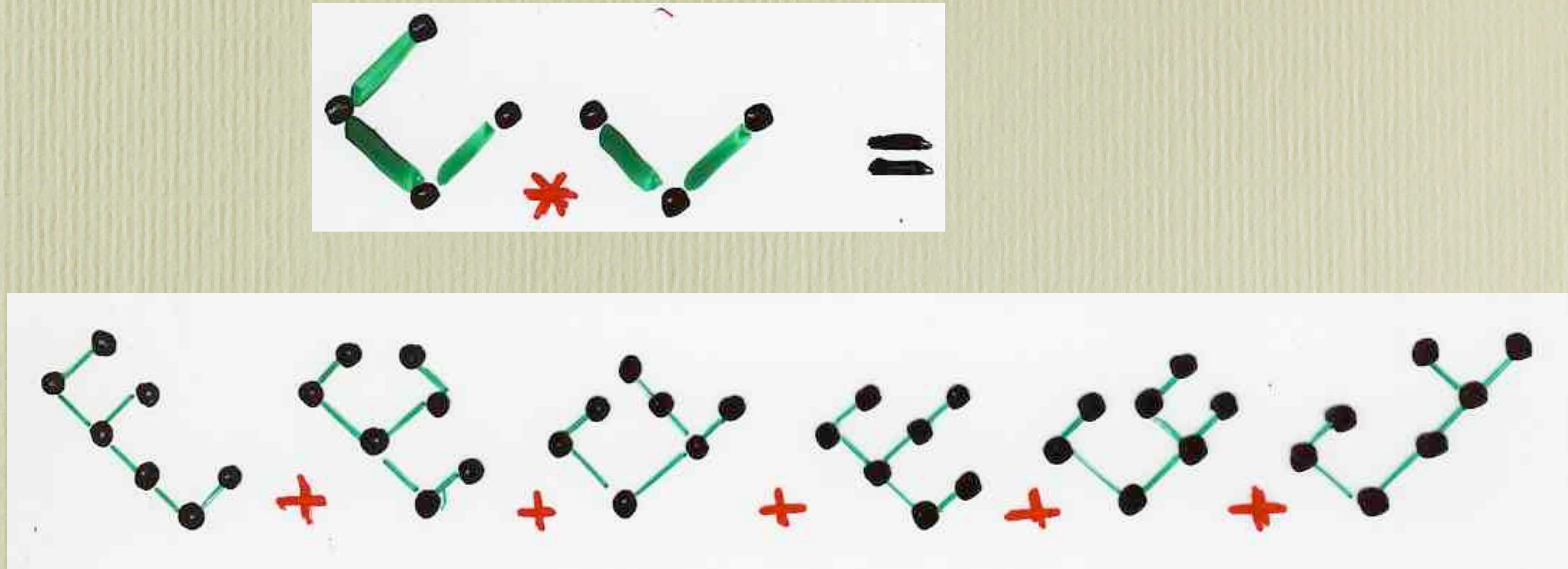
Reutenauer-
Malvenuto algebra

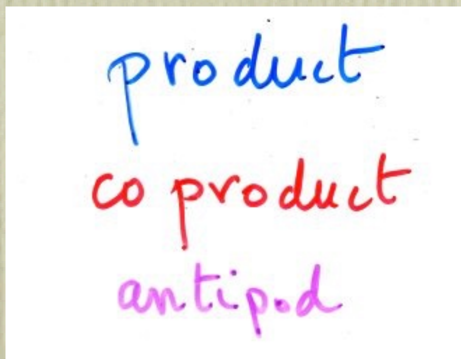
algebraic structures
Hopf algebra

	descent algebra	Loday-Ronco algebra	Reutenauer Malvenuto algebra
dim	2^{n-1}	C_n Catalan	$n!$

hyperacube ⟷ associahedron ⟷ permutahedron

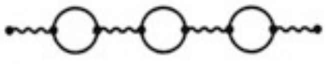
Boolean lattice inclusion ⟷ Tamari order ⟷ weak Bruhat order product of two binary trees



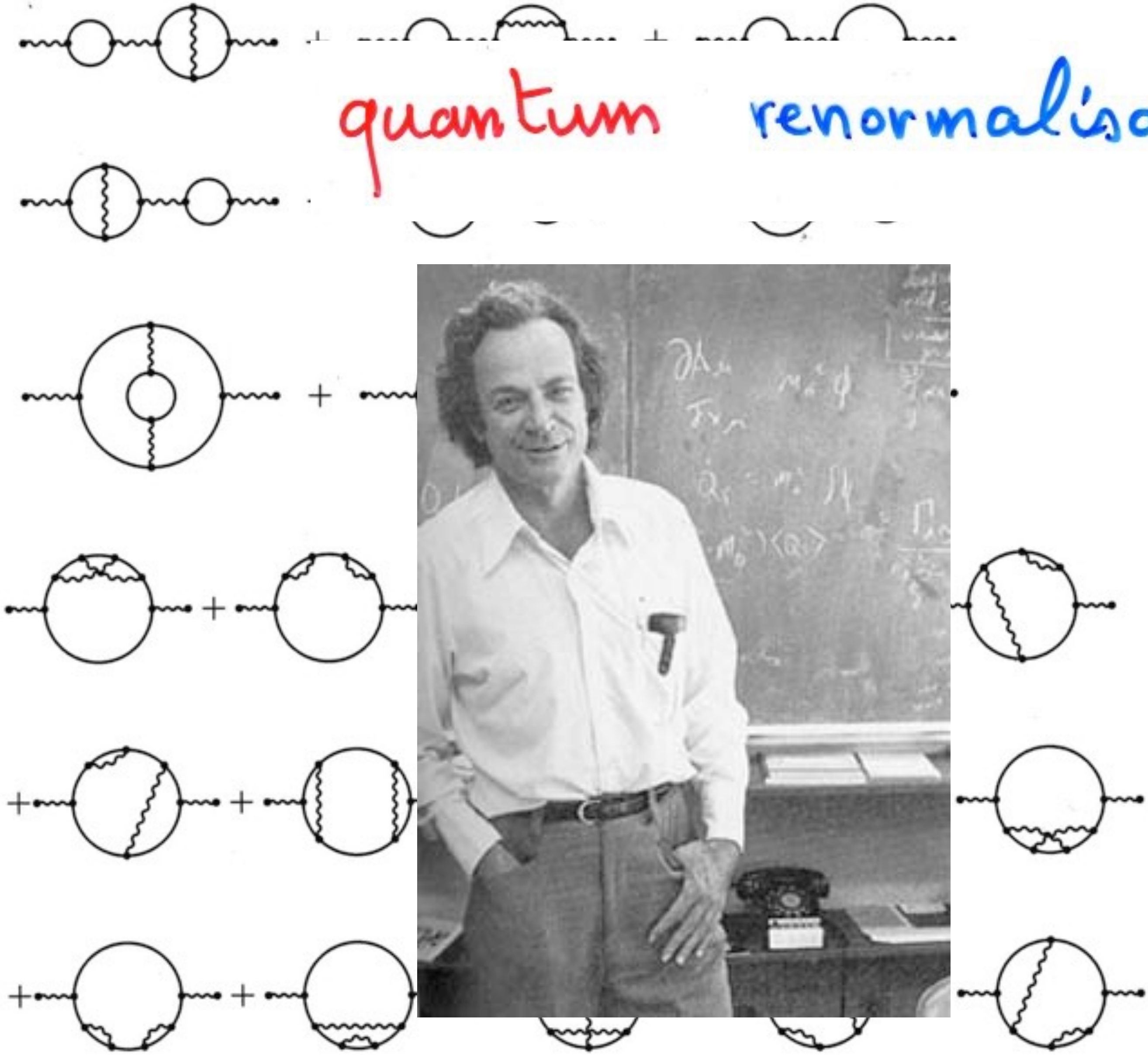


relation with Feynman diagrams

quantum physics

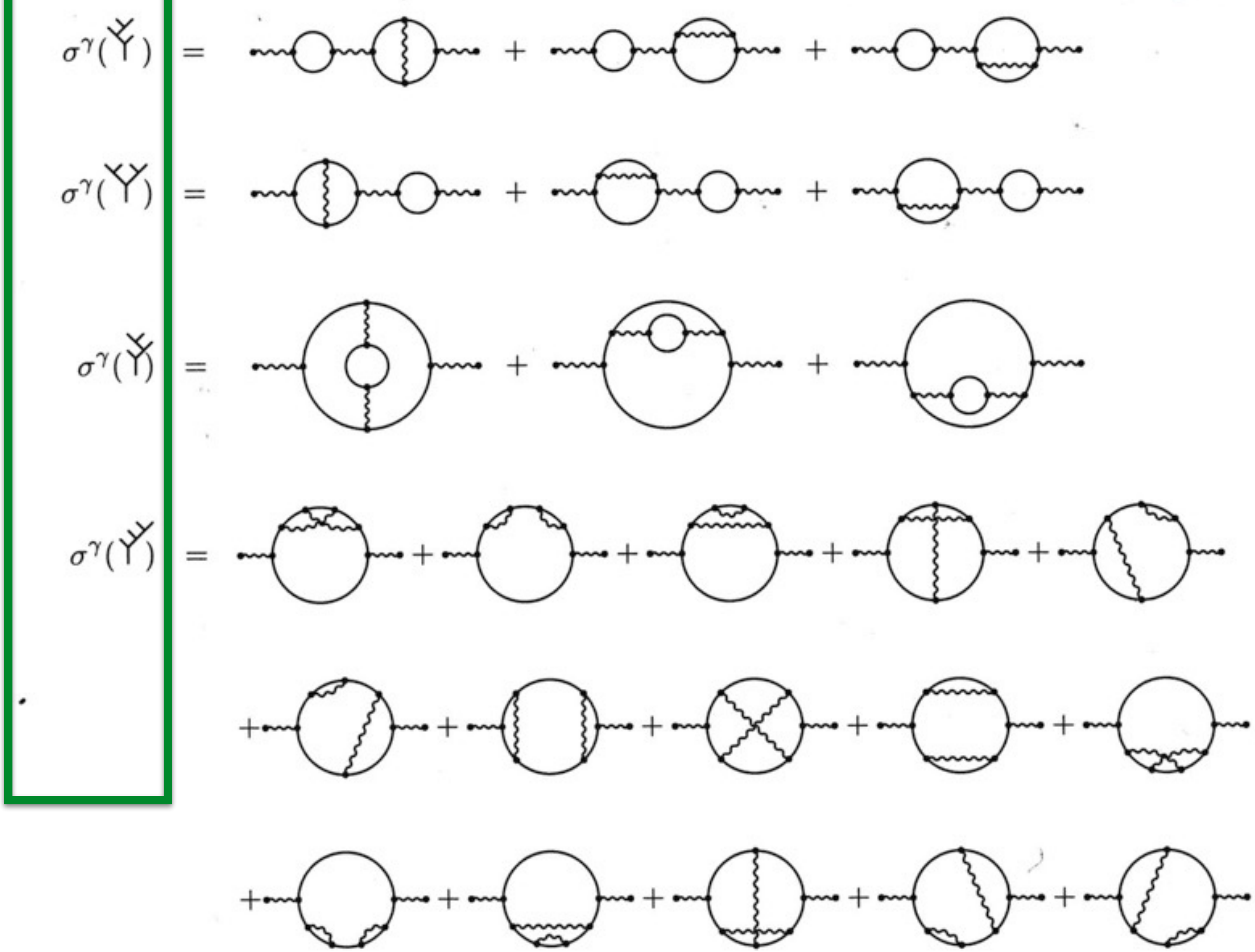


Feynman diagrams



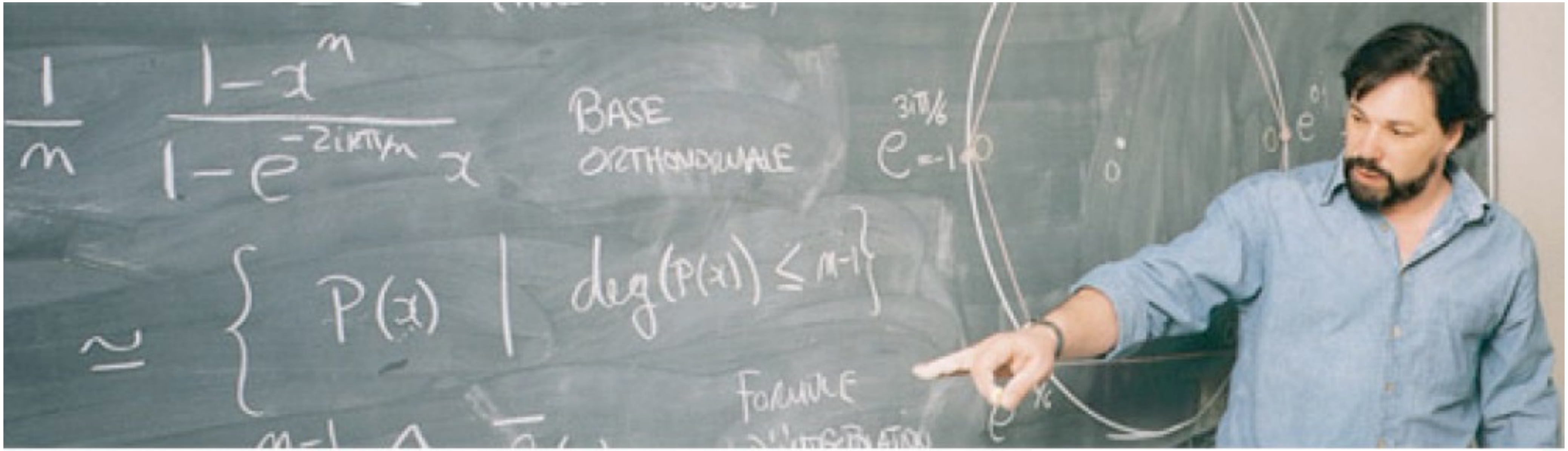
quantum renormalisation

Feynman diagrams



Connes-Kreimer
Hopf algebra of trees

relation with diagonal coinvariant spaces



F. Bergeron

diagonal
coinvariant
spaces

A. Garsia

$X = (x_{i,j})_{\substack{1\le i\le k\\ 1\le j\le n}}$ matrix of variables

$\sigma \in \mathfrak{S}_n$ symmetric group

$\sigma(X) = (x_{i,\sigma(j)})_{\substack{1\le i\le k\\ 1\le j\le n}}$ action on $\mathbb{C}[X]$

diagonal coinvariant spaces

$DR_{k,n} = \mathbb{C}[X]/J$ $\quad DR^{\varepsilon}_{k,n}$

Armstrong, Garsia, Haglund, Haiman, Hicks
Lee, Li, Loehr, Morse, Remmel, Rhoades,
Stout, Xin, Warrington, Zabrocki, ...

\+ ......

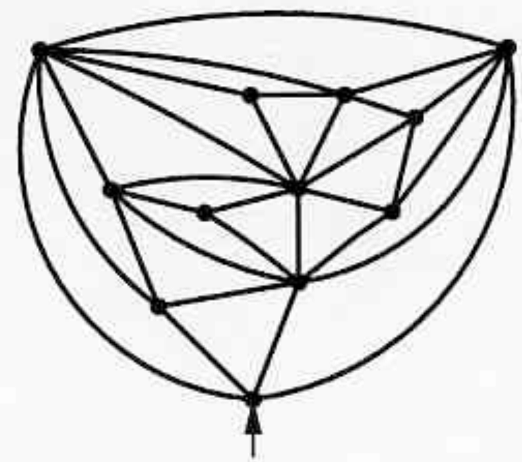
$k=3$ $\qquad DR^{\varepsilon}_{3,n}$

dimension $\dfrac{2}{n(n+1)}\dbinom{4n+1}{n-1}$

Haiman (conjecture) (1990)

number of intervals of Tamari$_n$

Chapoton (2006)



triangulation

Bijective proof

FPSAC 2007

Bernardi, N. Bonichon

higher diagonal coinvariant spaces

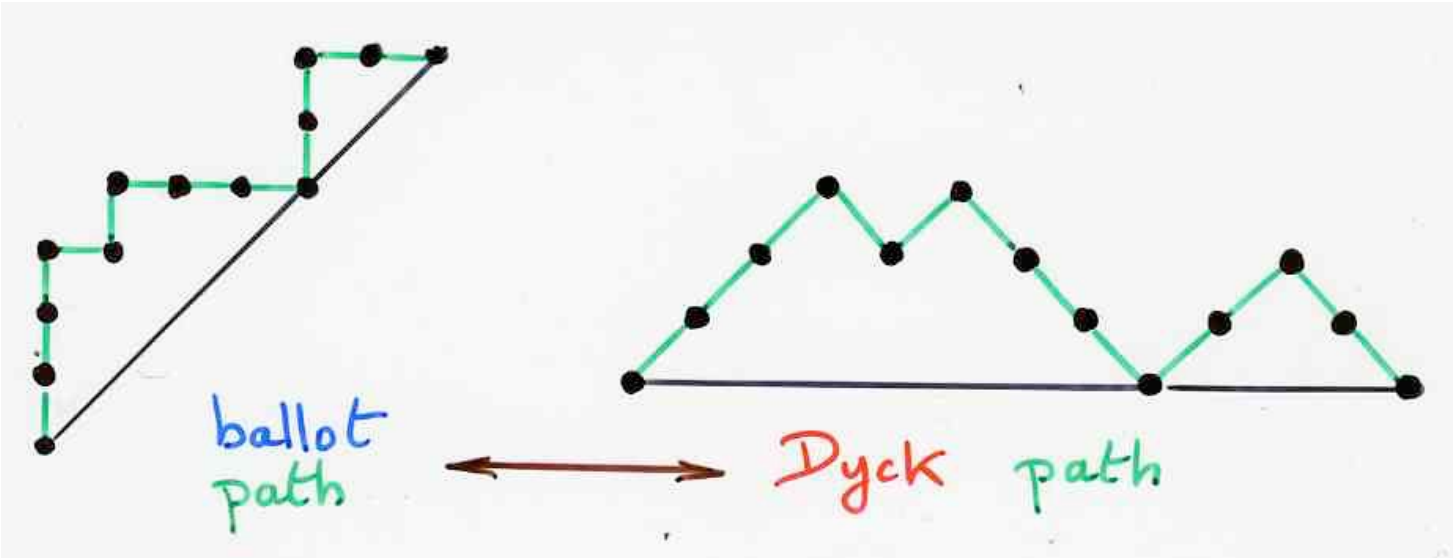
$$DR^{m\,\varepsilon}_{k,n} \qquad DR^{m}_{k,n}$$

$k=2$ Garsia, Haiman

$$DR^{m\,\varepsilon}_{2,n}$$

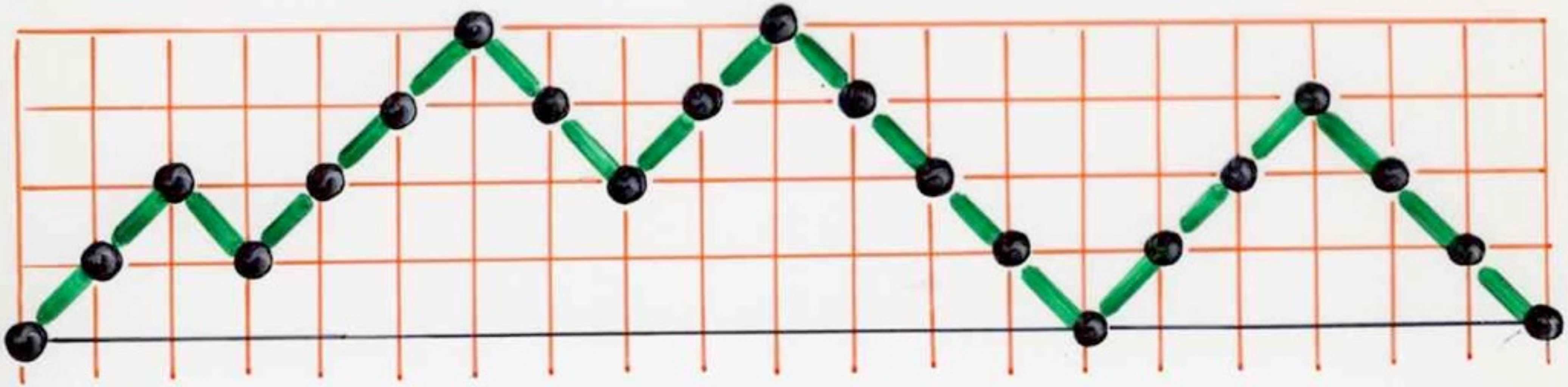
dimension $\dfrac{1}{(m+1)n+1}\dbinom{(m+1)n+1}{mn}$

m - ballot paths



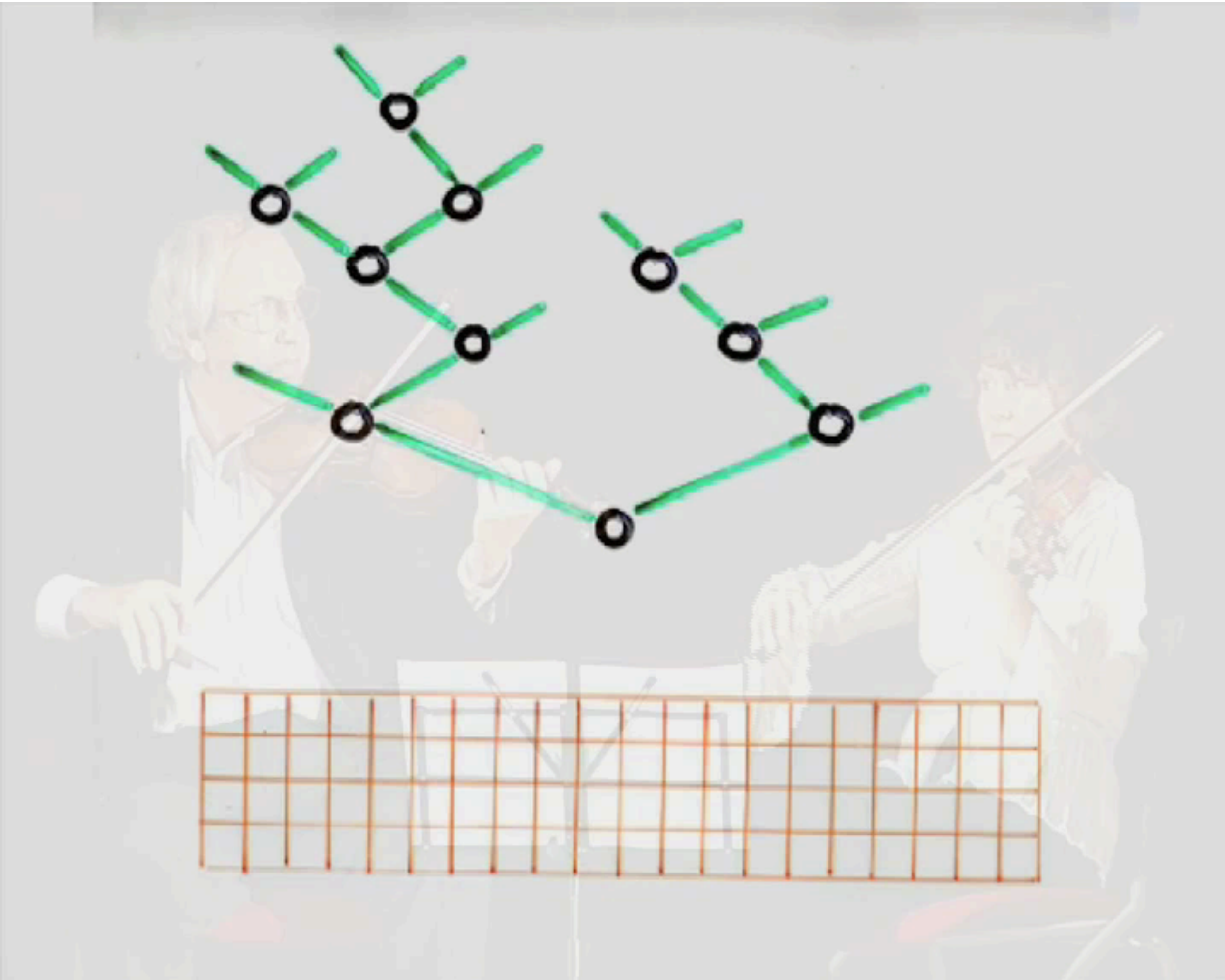
Dyck paths

Dyck path



from binary trees

to Dyck paths ....



violins:

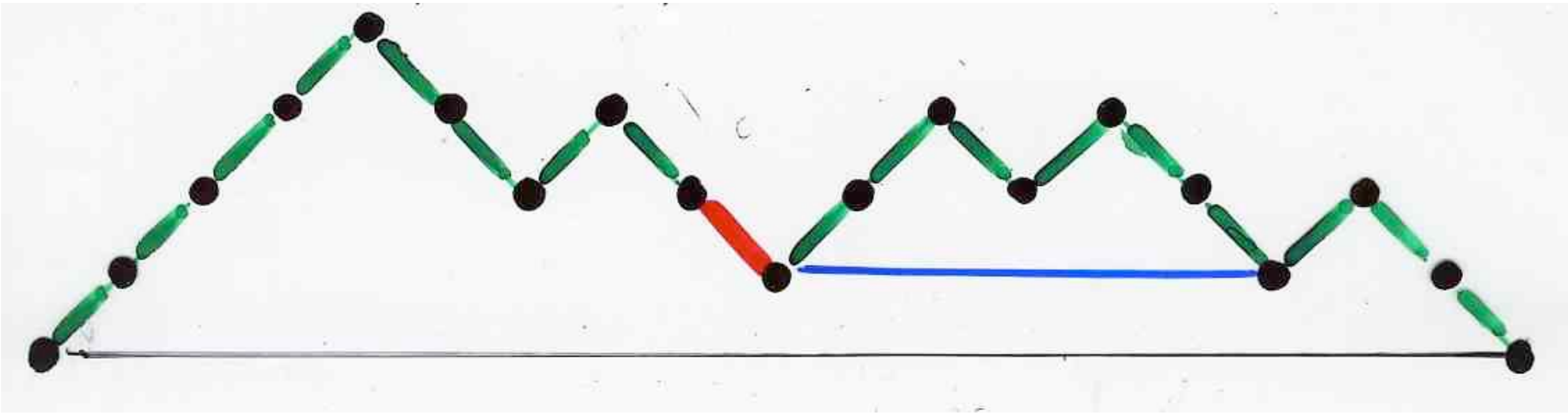
Mariette Freudentheil

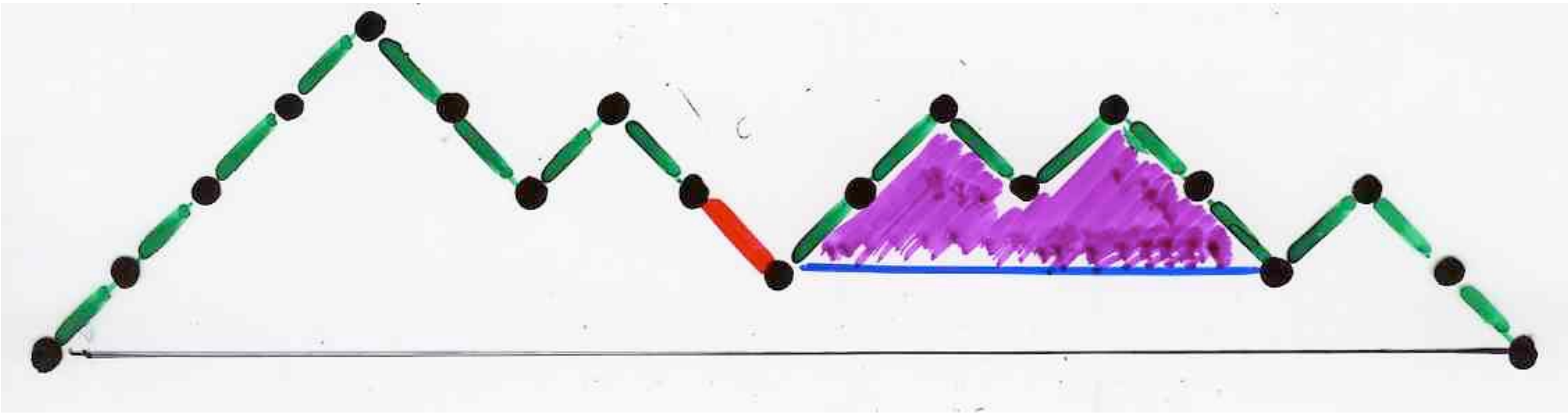
Gérard H.E. Duchamp

Association
Cont'Science

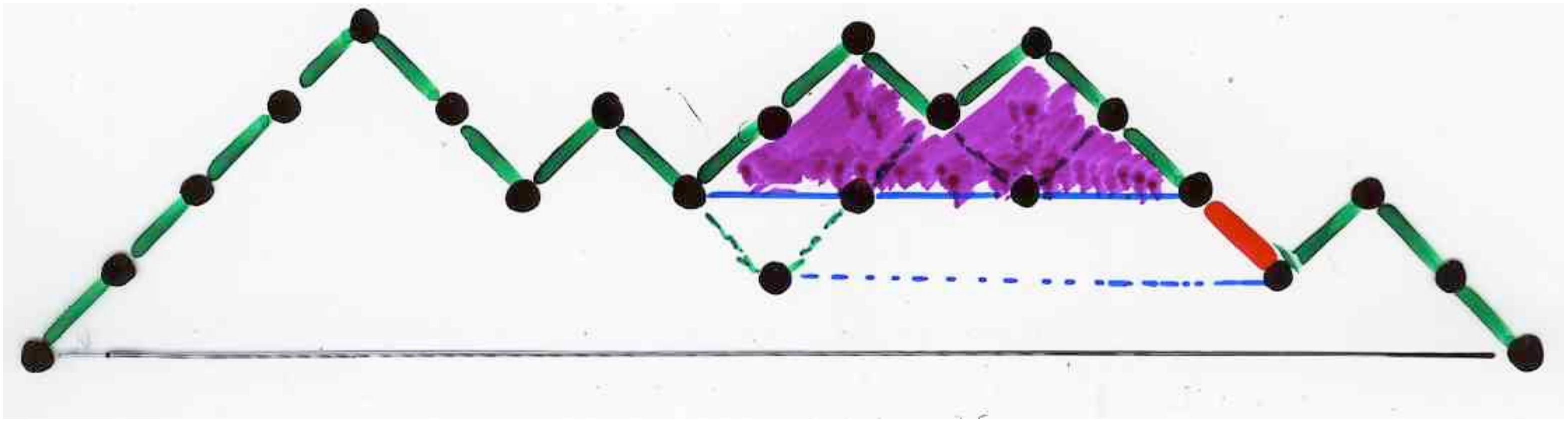
Atelier audiovisuel
Université Bordeaux 1
Yves Descubes
Franck Marmisse

the Tamari lattice in terms of Dyck paths



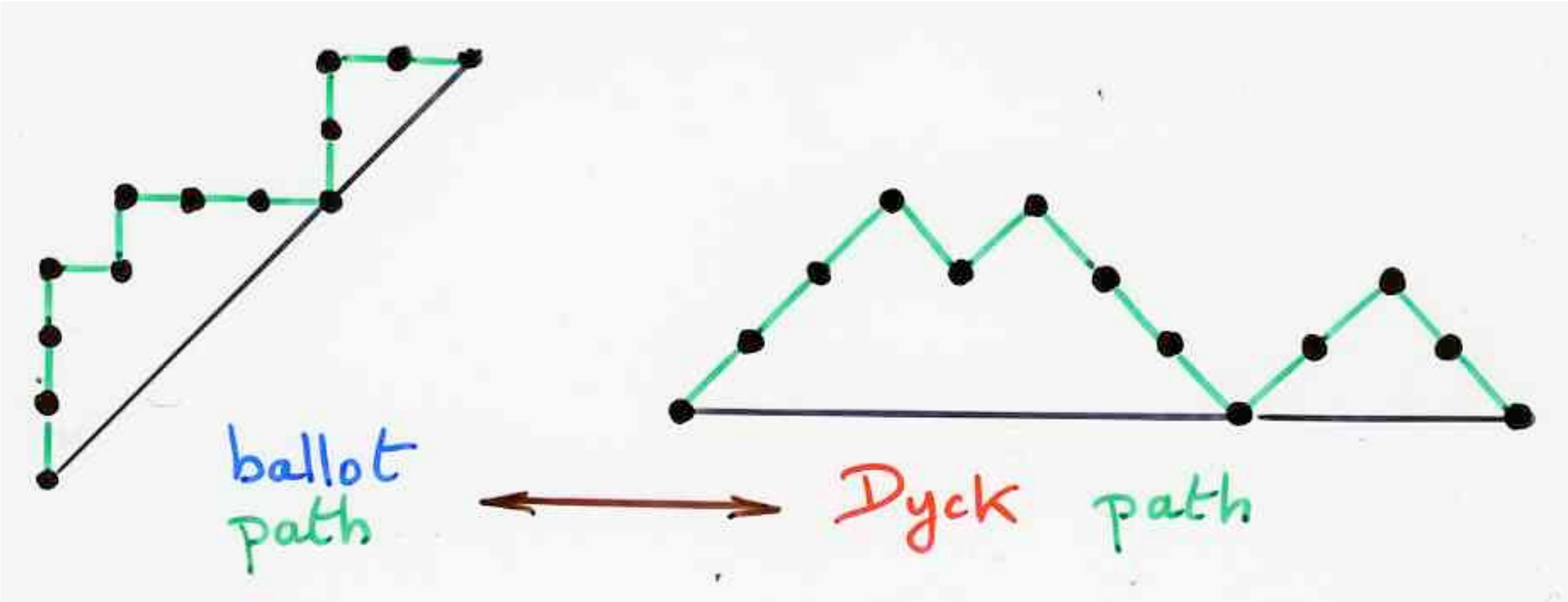


factor Dyck primitif

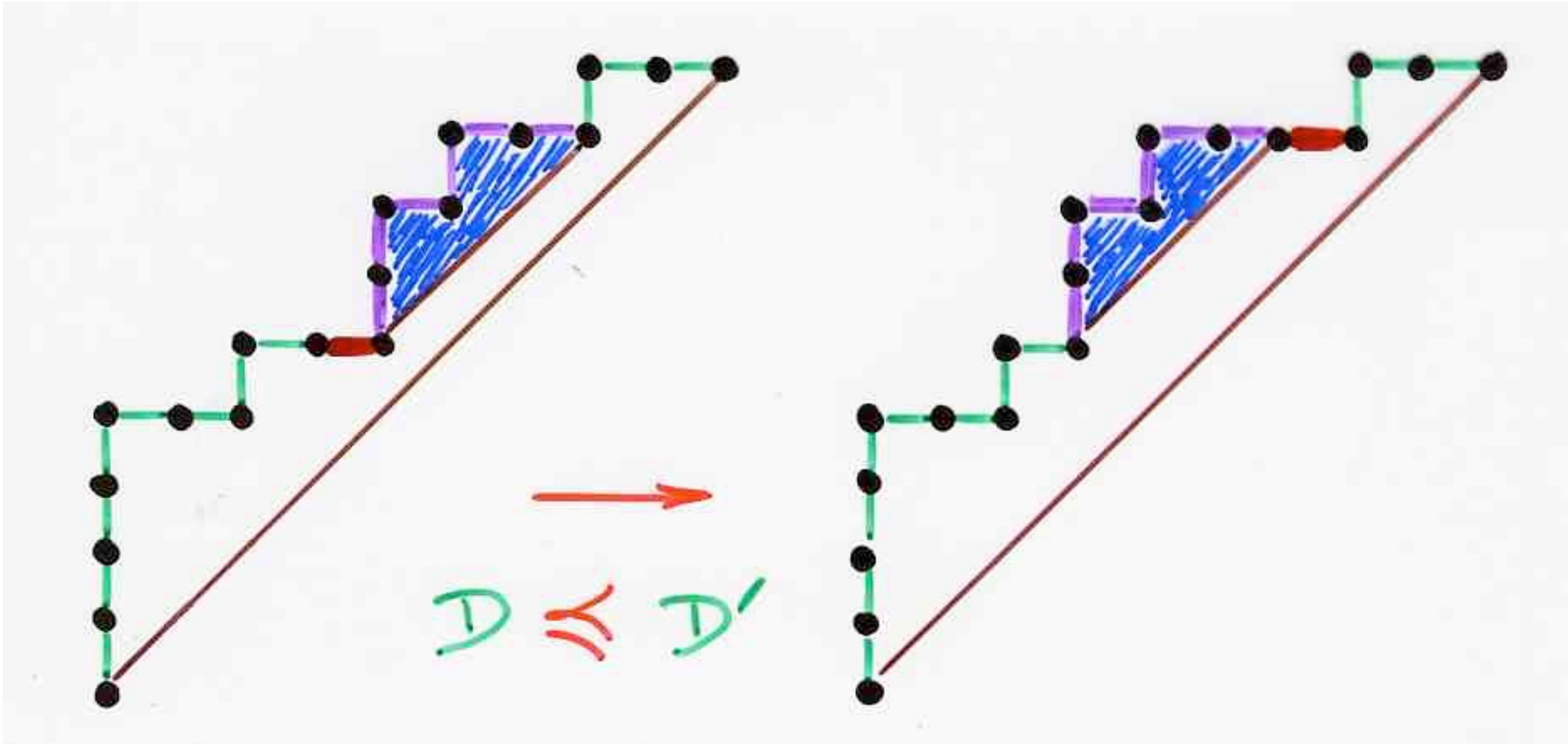


factor Dyck primitif

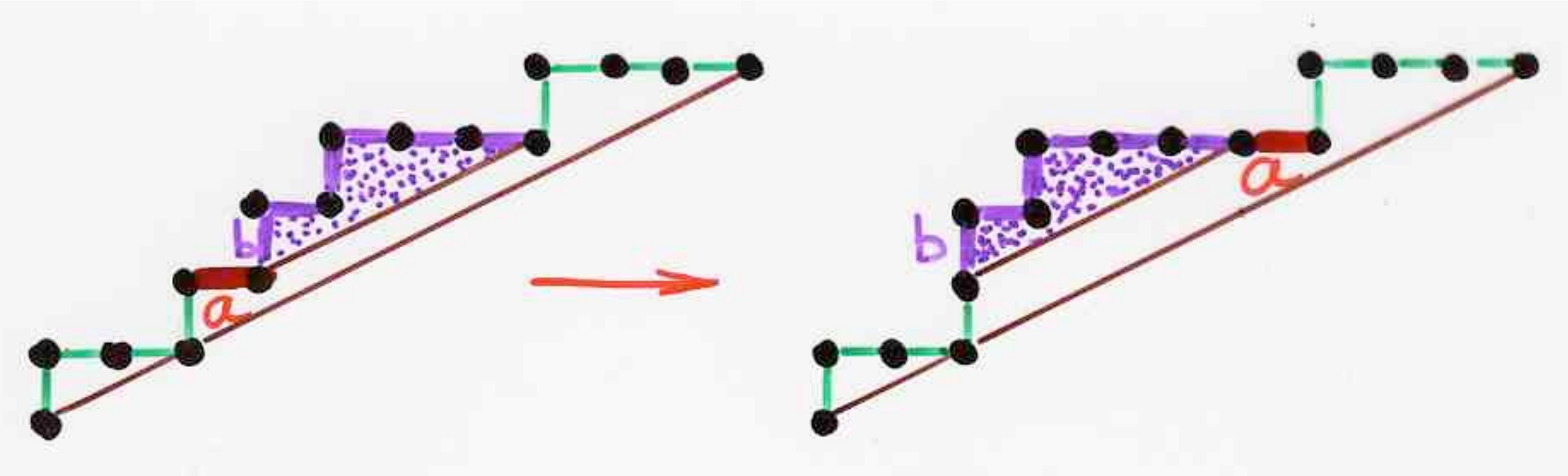
m-Tamari lattice



vocabulary: ballot path
Dyck path



the Tamari covering relation
for ballot (Dyck) path



the covering relation in the
m-Tamari lattice
($m = 2$)

higher diagonal coinvariant spaces

$$DR^{m\,\varepsilon}_{k,n} \qquad DR^{m}_{k,n}$$

$k=2$ Garsia, Haiman

$$DR^{m\,\varepsilon}_{2,n}$$

dimension $\dfrac{1}{(m+1)n+1}\dbinom{(m+1)n+1}{mn}$

m - ballot paths

F. Bergeron (2008) introduced the m-Tamari lattice

conjecture $\dfrac{m+1}{n\,(mn+1)}\dbinom{(m+1)^2 n+m}{n-1}$

nb of intervals

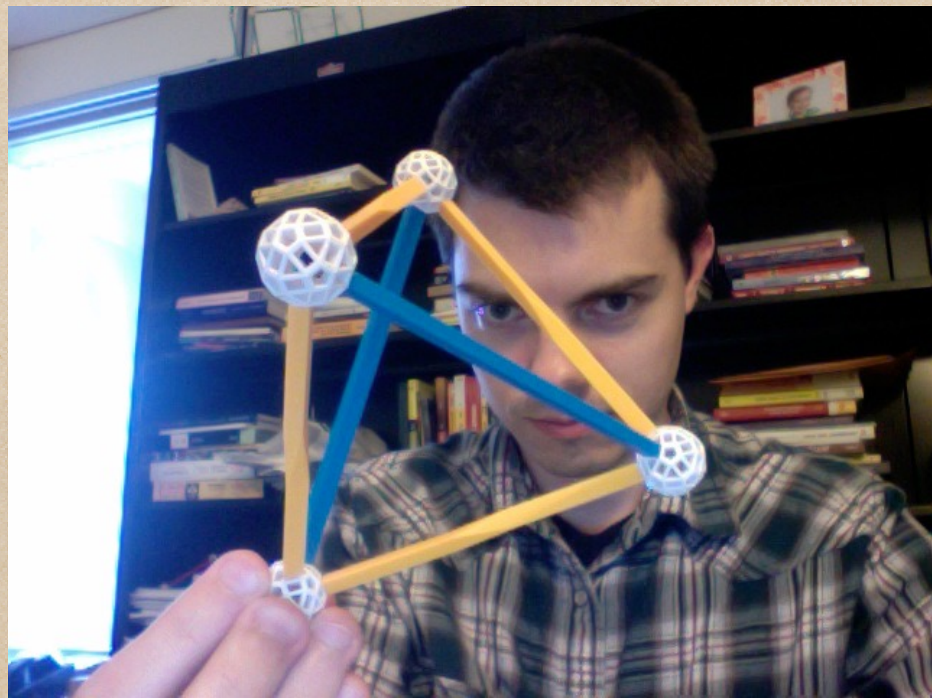
M. Bousquet-Mélou, E. Fusy, L.-F. Préville-Ratelle (2011)

nb of intervals of m-Tamari lattices

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1}$$

F. Bergeron

Rational Catalan Combinatorics



Rational Catalan Combinatorics

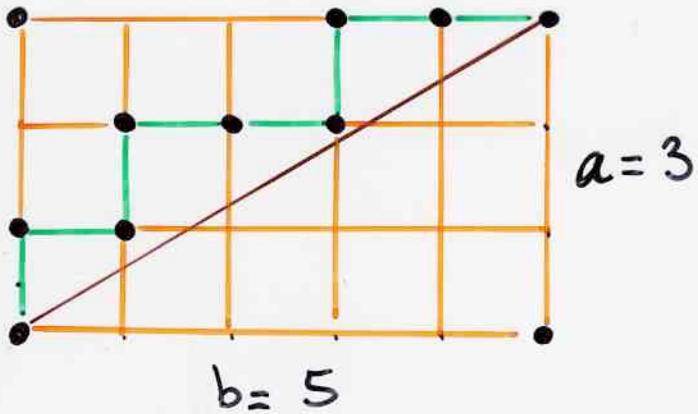
D. Armstrong

$$Cat(a,b) = \frac{1}{a+b}\binom{a+b}{a,b}$$

number of (a,b)- ballot paths $= Cat(a,b)$

Grossman (1950)
Bizley (1954)

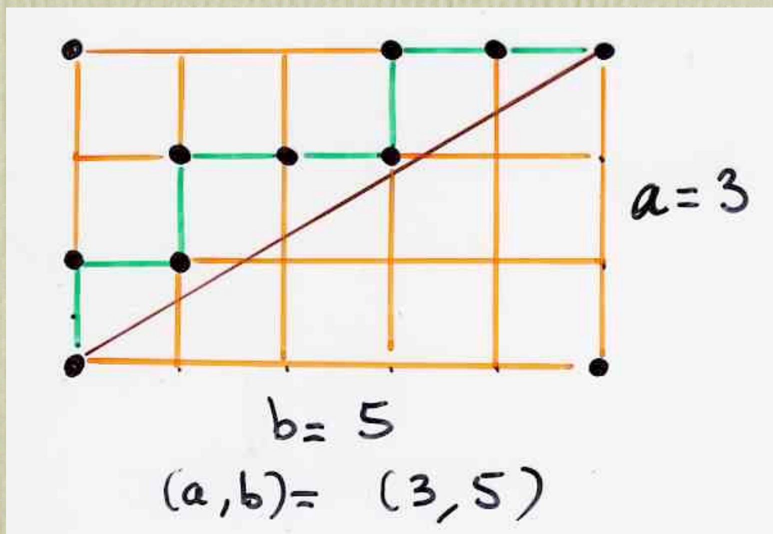
rational ballot (Dyck) paths



$(a,b) = (3,5)$

question:

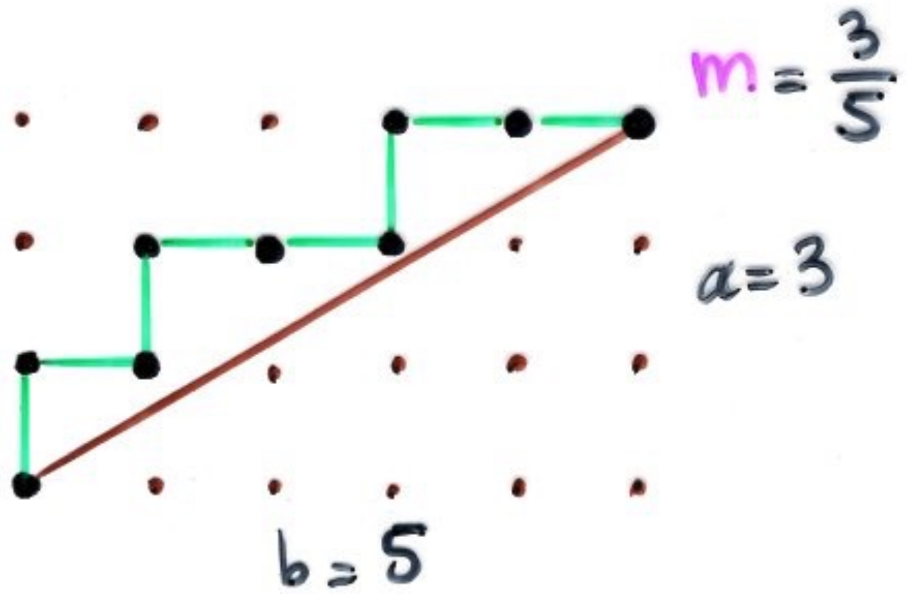
define an (a,b)- Tamari lattice ?



joint work with
Louis-François Préville-Ratelle
U. Talca, Chile

FPSAC'15, Daejeon

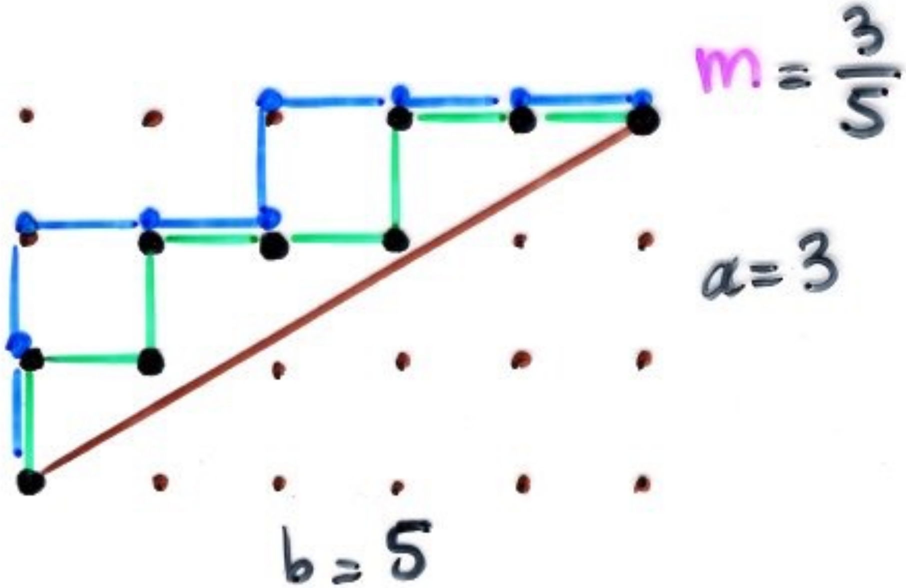
to be published in
Transactions A.M.S.



Rational Catalan Combinatorics

D. Armstrong

$$Cat(a,b) = \frac{1}{a+b}\binom{a+b}{a,\,b}$$

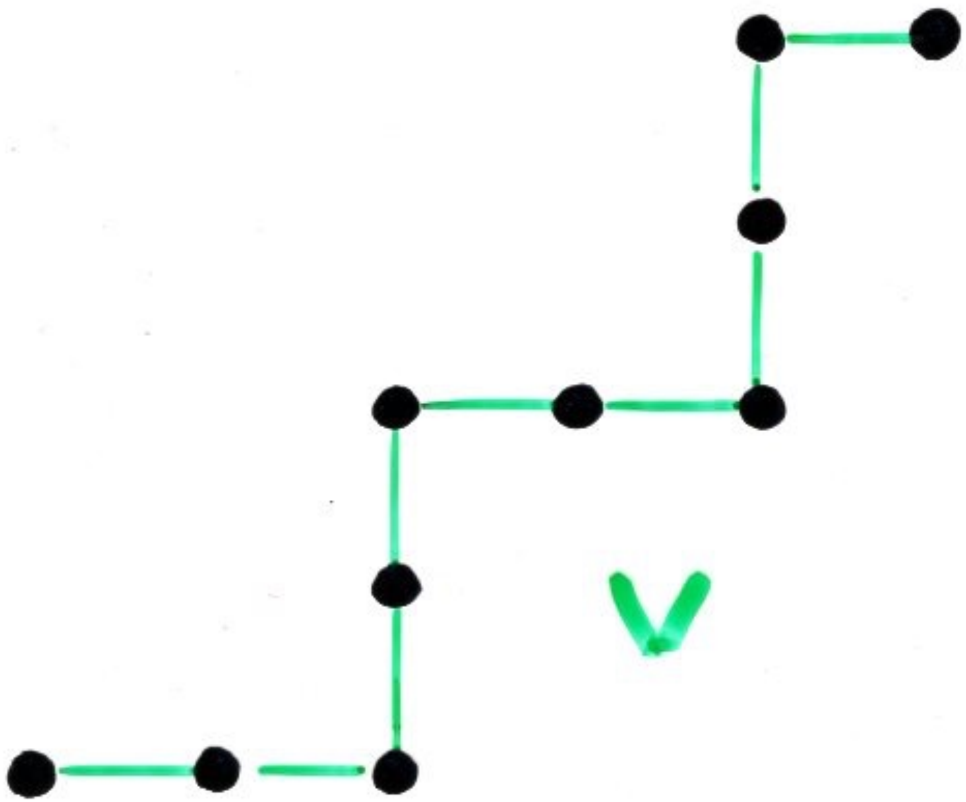


Rational Catalan Combinatorics

D. Armstrong

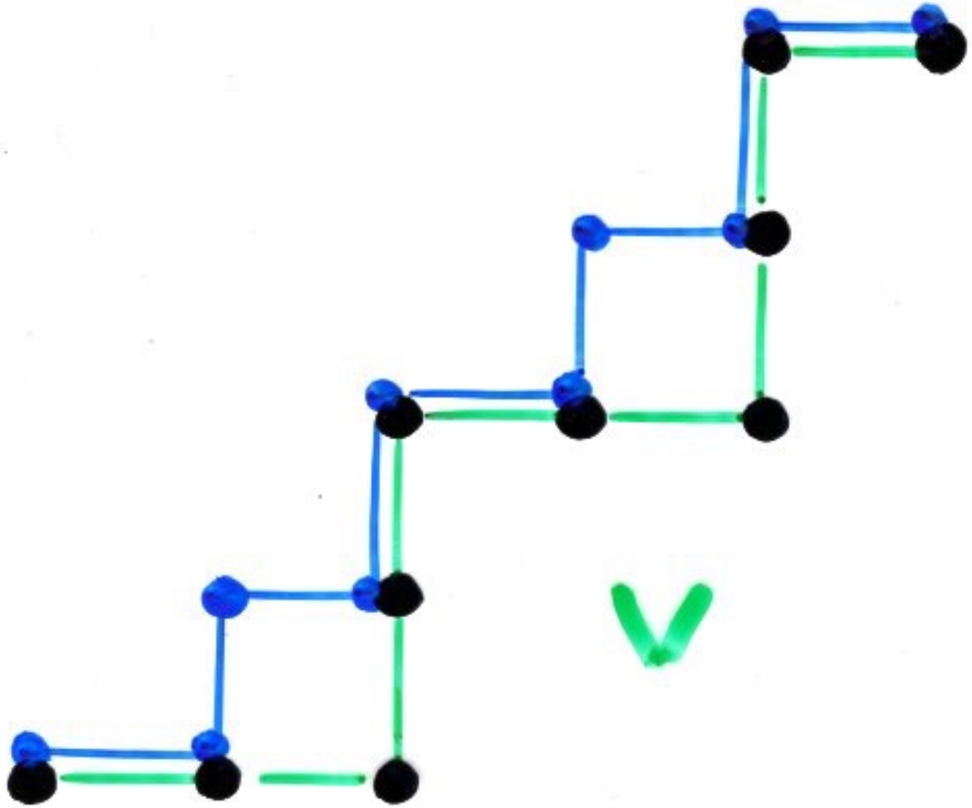
$$\mathrm{Cat}(a,b) = \frac{1}{a+b}\binom{a+b}{a,\,b}$$

Tamari T_v

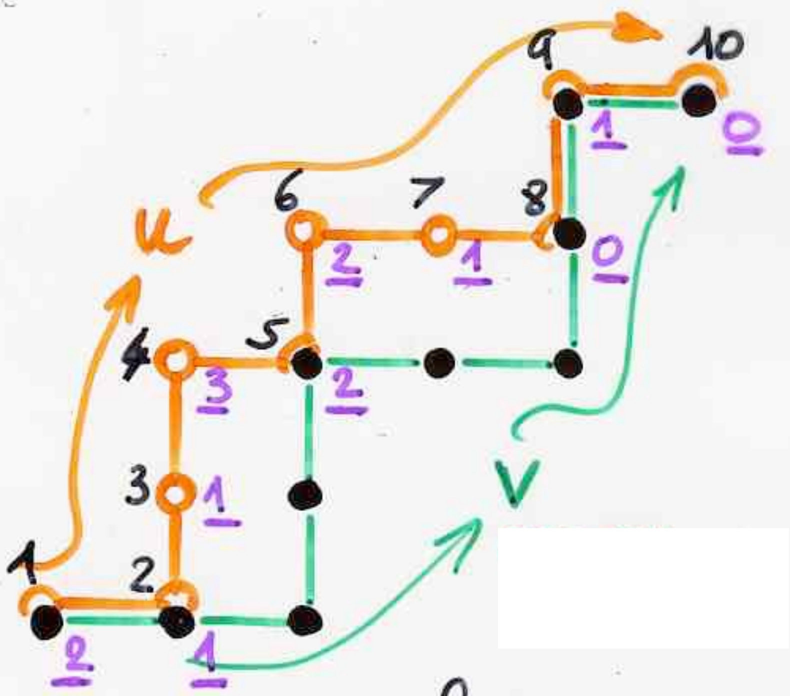


Transcendantal
Catalan
combinatorics ?

Tamari T_v

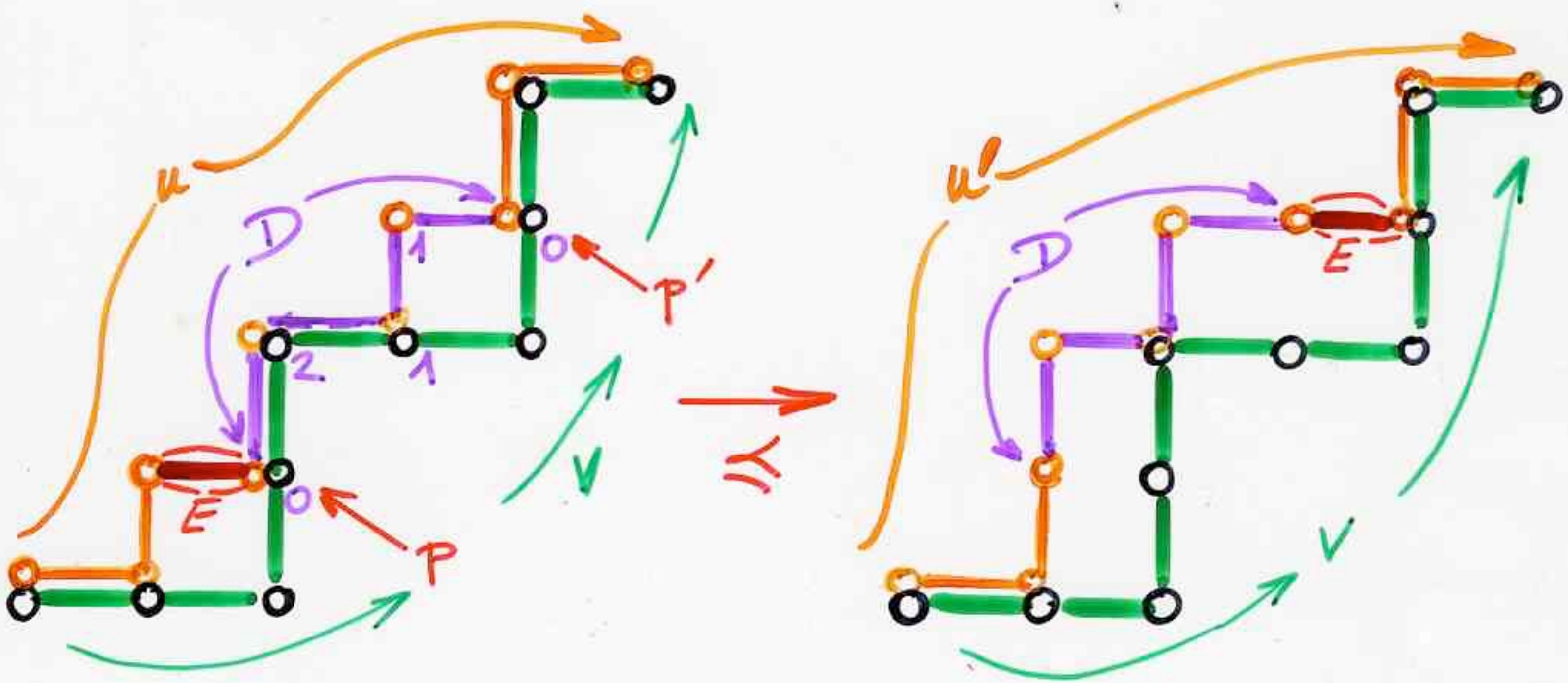


Transcendantal
Catalan
combinatorics?



$\underline{i} = \mathrm{horiz}_{v}(p)$

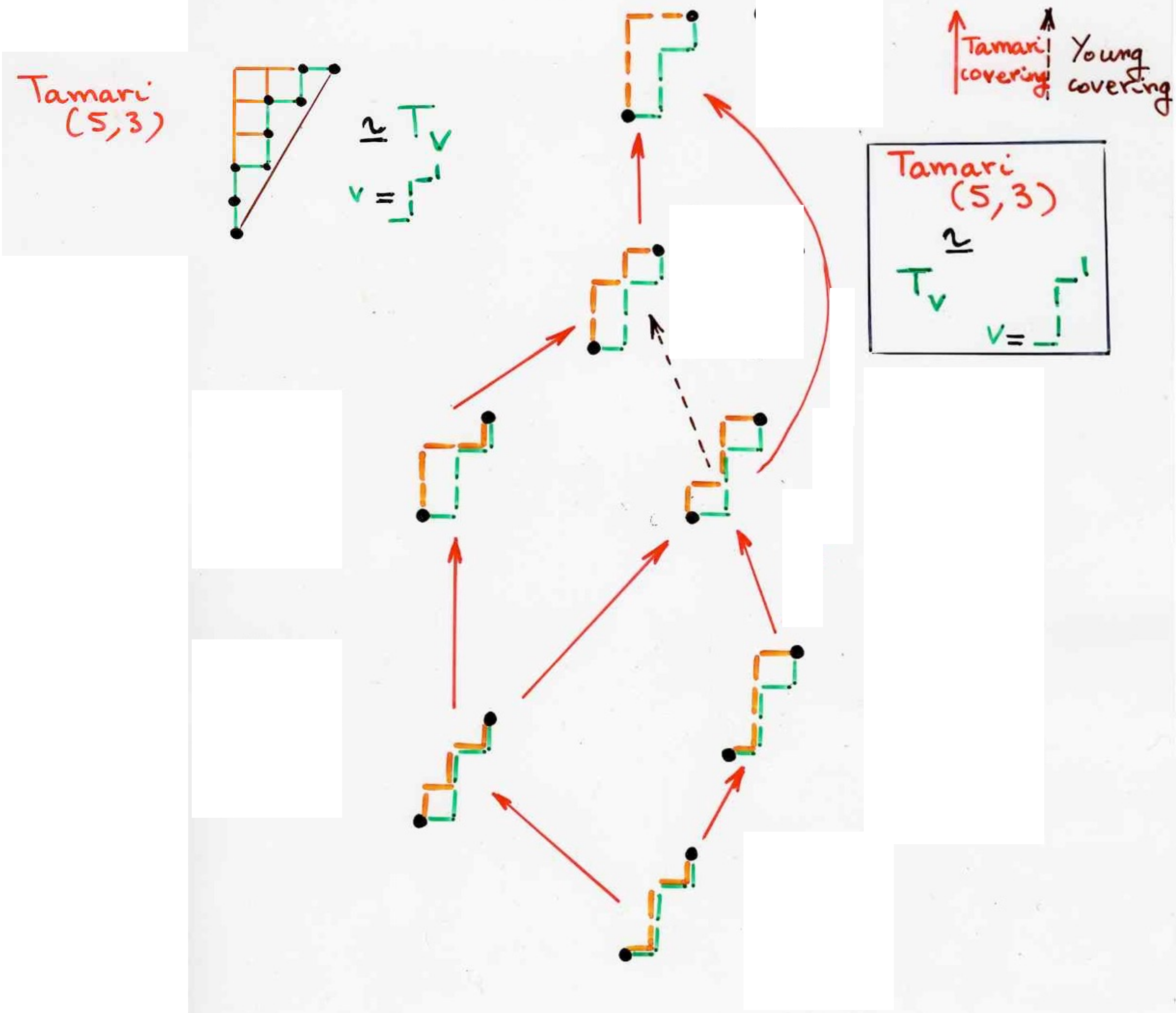
a pair (u, v) of paths
with the "horizontal distance"
$\mathrm{horiz}_{v}(p)$



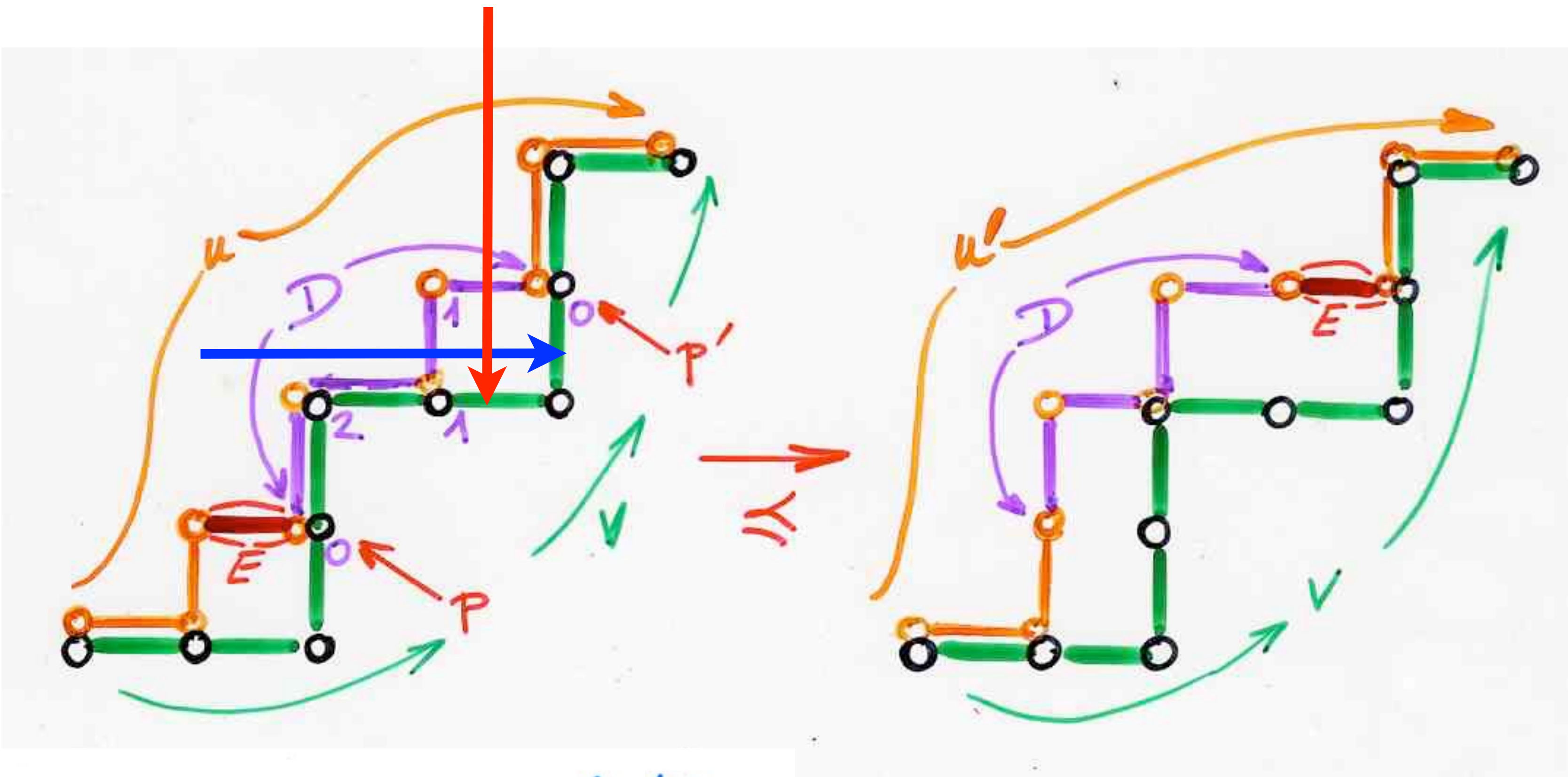
the covering relation
in the poset T_v

Thm 1. For any path v

T_v is a lattice



Tamari (5,3)
≃ T_v
v =
Tamari covering
Young covering
Tamari (5,3)
≃
T_v
v =



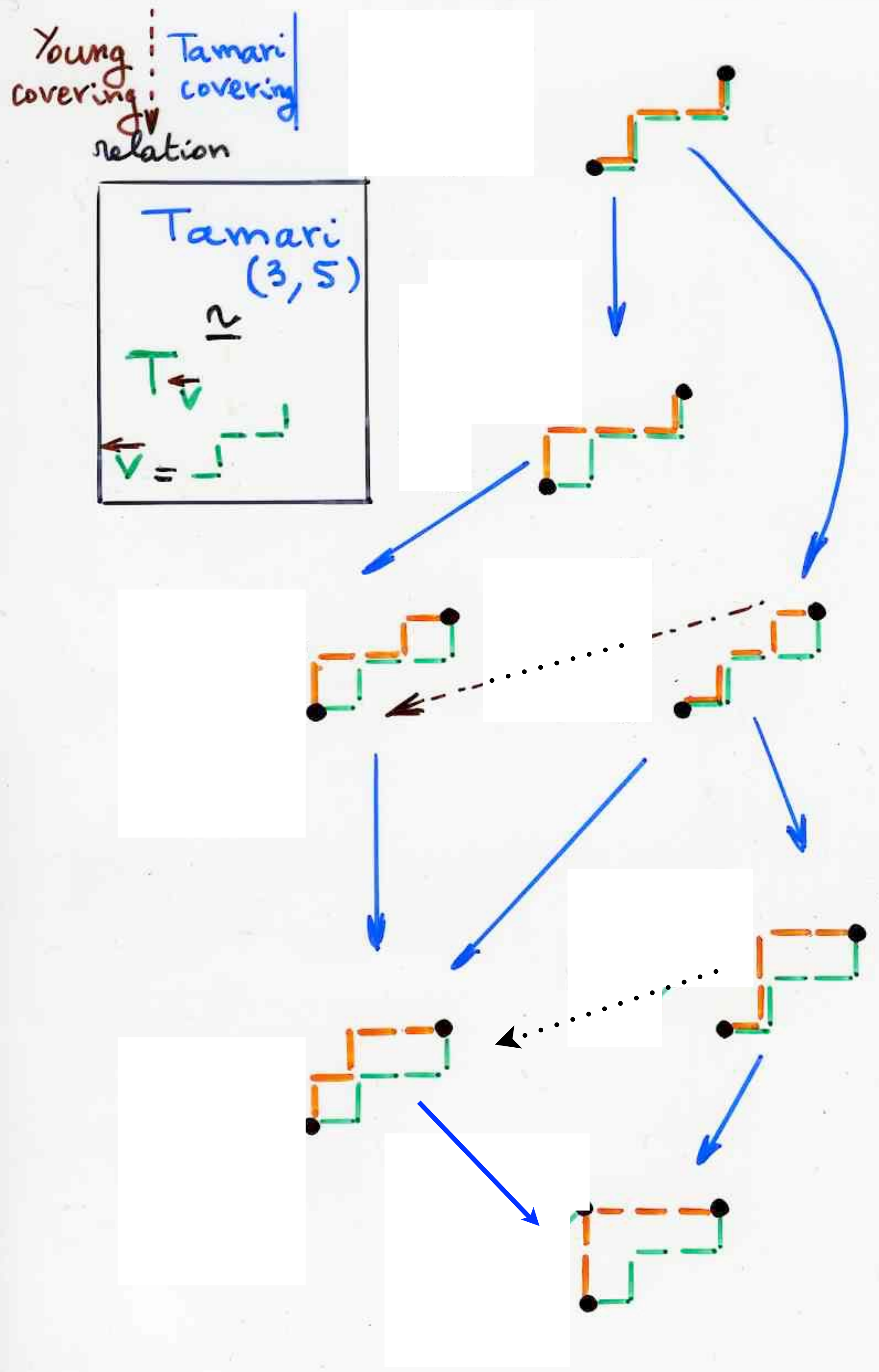
row covering relation

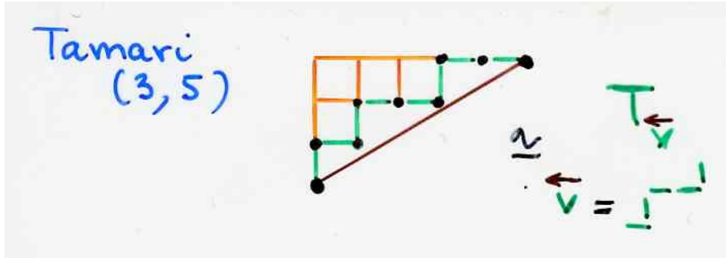
column covering relation

$T_{\overleftarrow{V}}$

$\overleftarrow{V}$ mirror image

exchange N and E

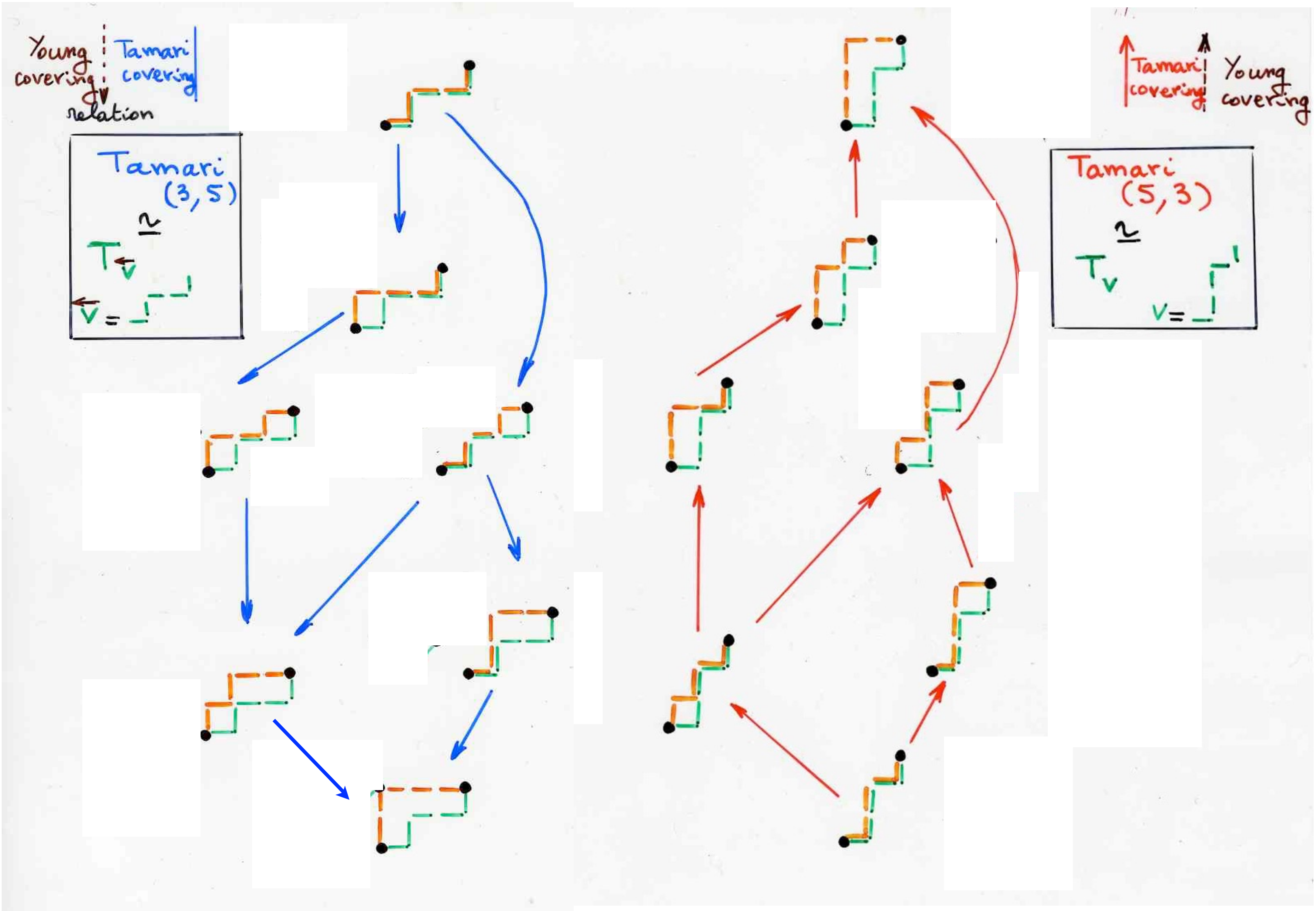

Young covering relation
Tamari covering
Tamari (3,5)
≃


Tamari (3,5)
≃

Thm 1. For any path v
T_v is a lattice

Thm 2. The lattice T_v
is isomorphic to the dual of $T_{\overleftarrow{v}}$

Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$



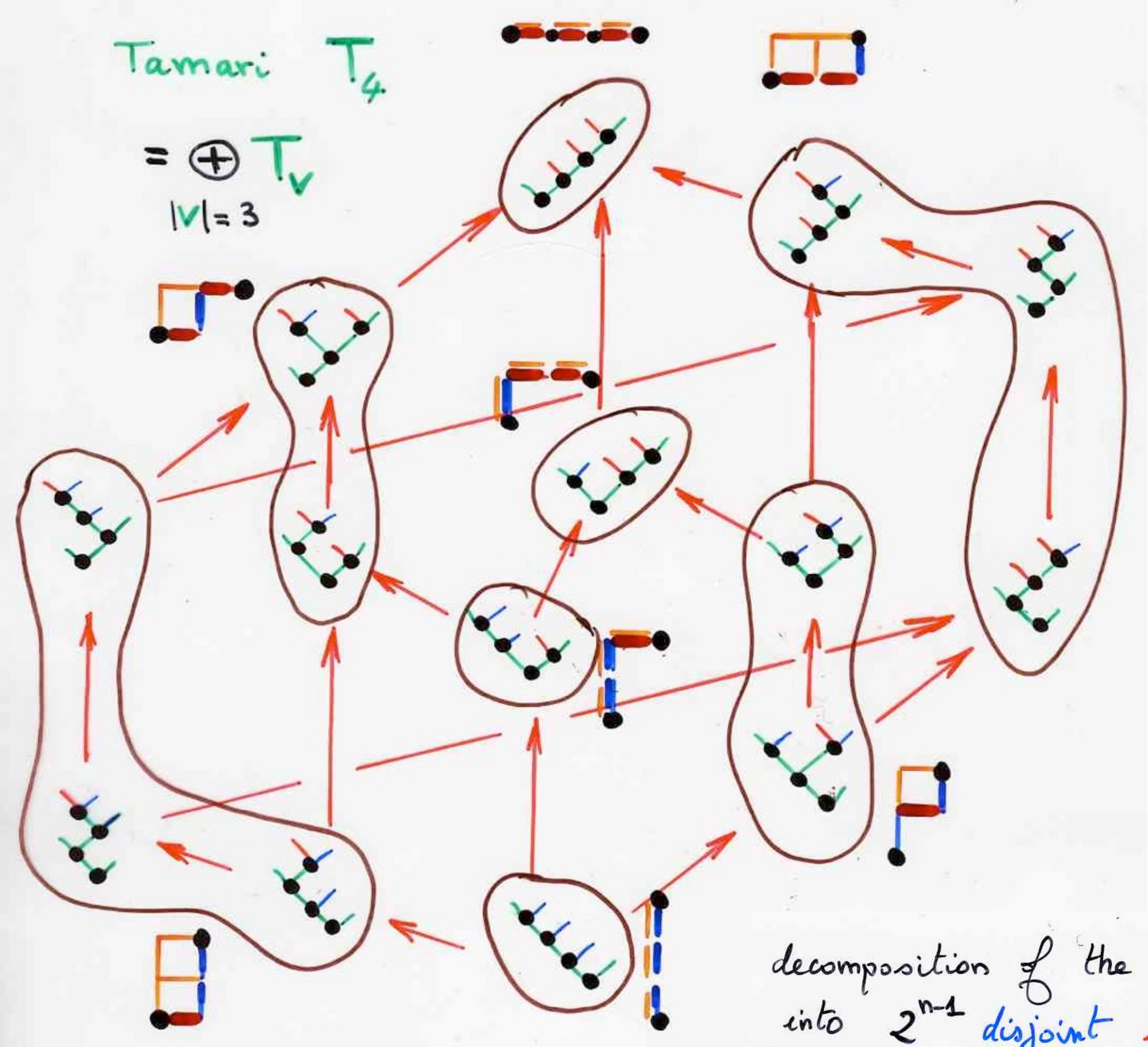
Prop • The set of binary trees having a given canopy v is an interval $I(v)$ of the Tamari lattice.

• This interval $I(v)$ is isomorphic to T_v

Thm3. The usual Tamari lattice T_n can be partitioned into intervals indexed by the 2^{n-1} paths v of length $(n-1)$ with $\{E, N\}$ steps,

$$T_n \cong \biguplus_{|v|=n-1} I_v ,$$

where each $I_v \cong T_v$.



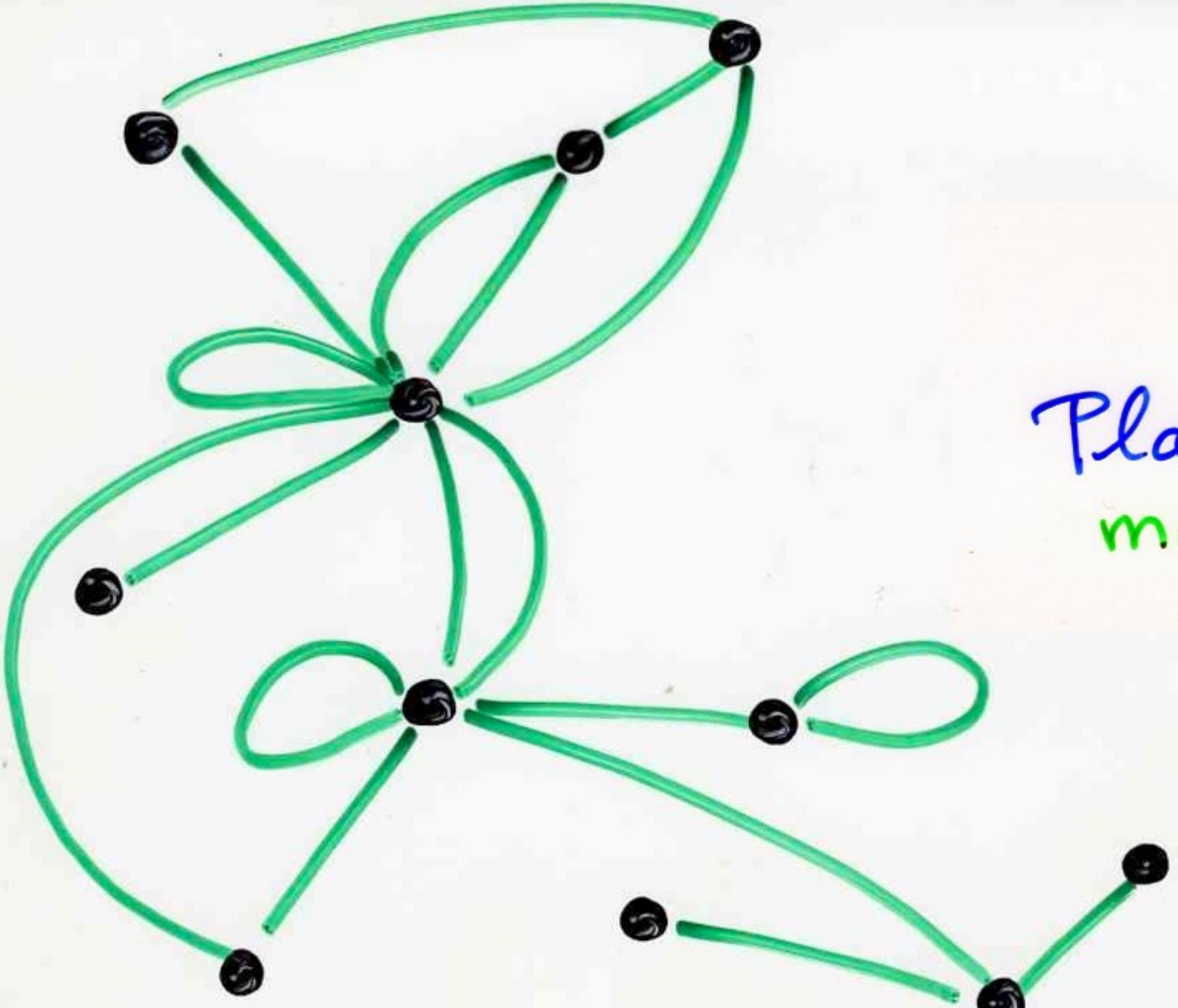
decomposition of the lattice T_n into 2^{n-1} disjoint intervals

Prop (L.-F. Préville-Ratelle)

The total number of intervals in all T_v with $|v| = n$ is the number of non-separable planar maps.

bijective proof: L.-F. Préville-Ratelle, W. Fang (FPSAC'16)

$$\frac{2\,(3n+3)!}{(n+2)!\,(n+3)!}$$



Planar map

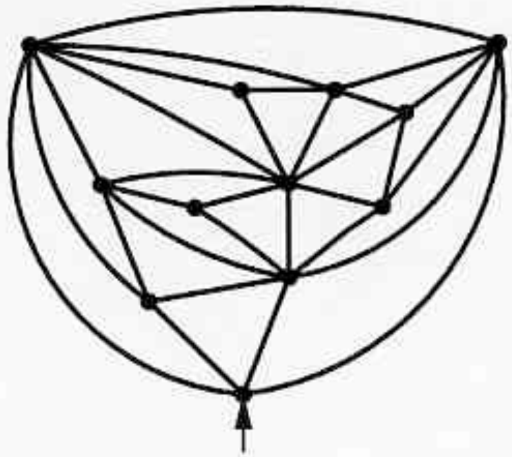
$k=3$ $\quad DR^{\varepsilon}_{3,n}$

dimension $\dfrac{2}{n(n+1)}\dbinom{4n+1}{n-1}$

Haiman (conjecture) (1990)

number of intervals of Tamari_n

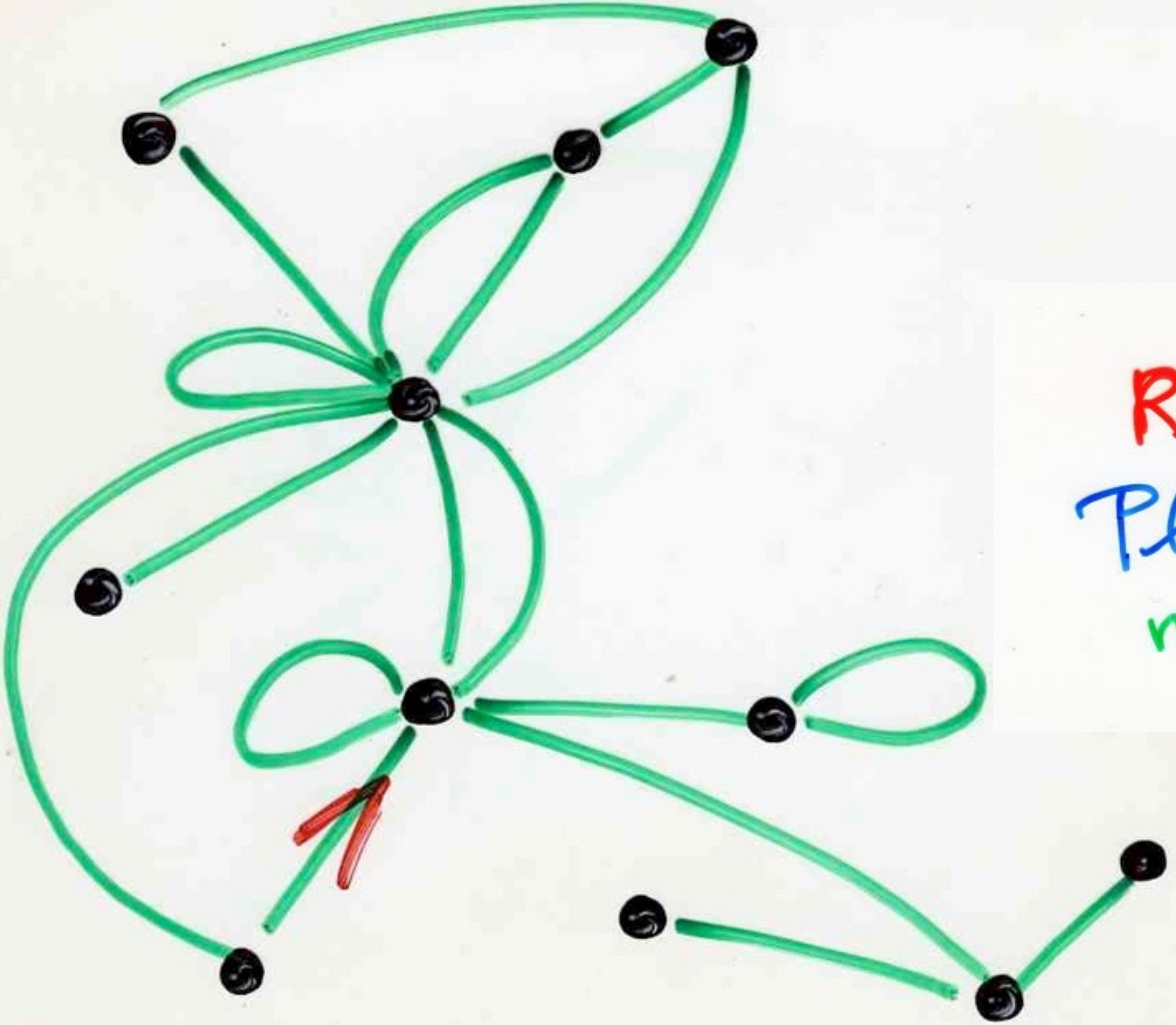
Chapoton (2006)



triangulation

Bijective proof FPSAC 2007

Bernardi, N. Bonichon



Rooted
Planar
map

Tutte (1960)

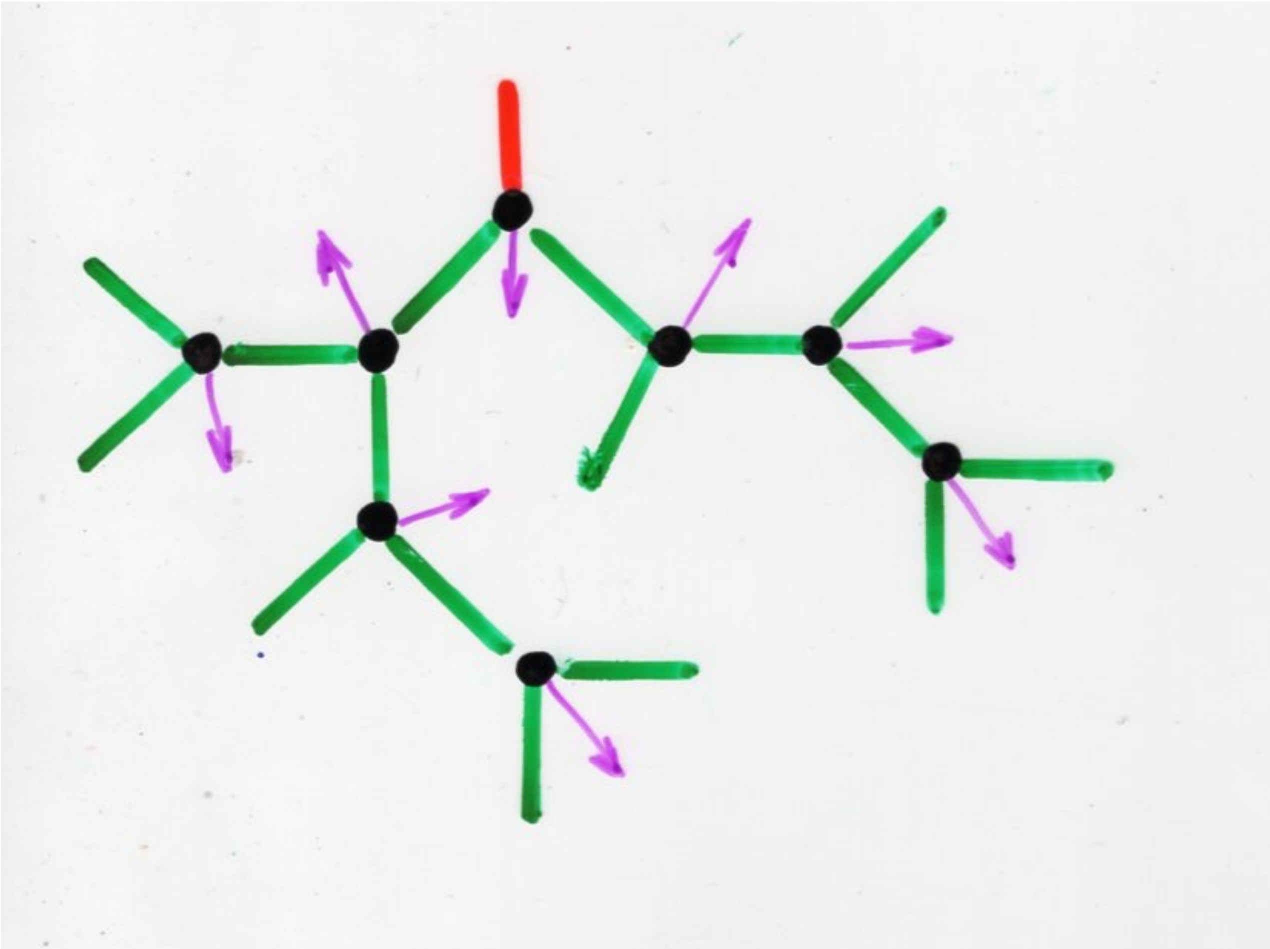
The number of rooted planar maps with m edges is

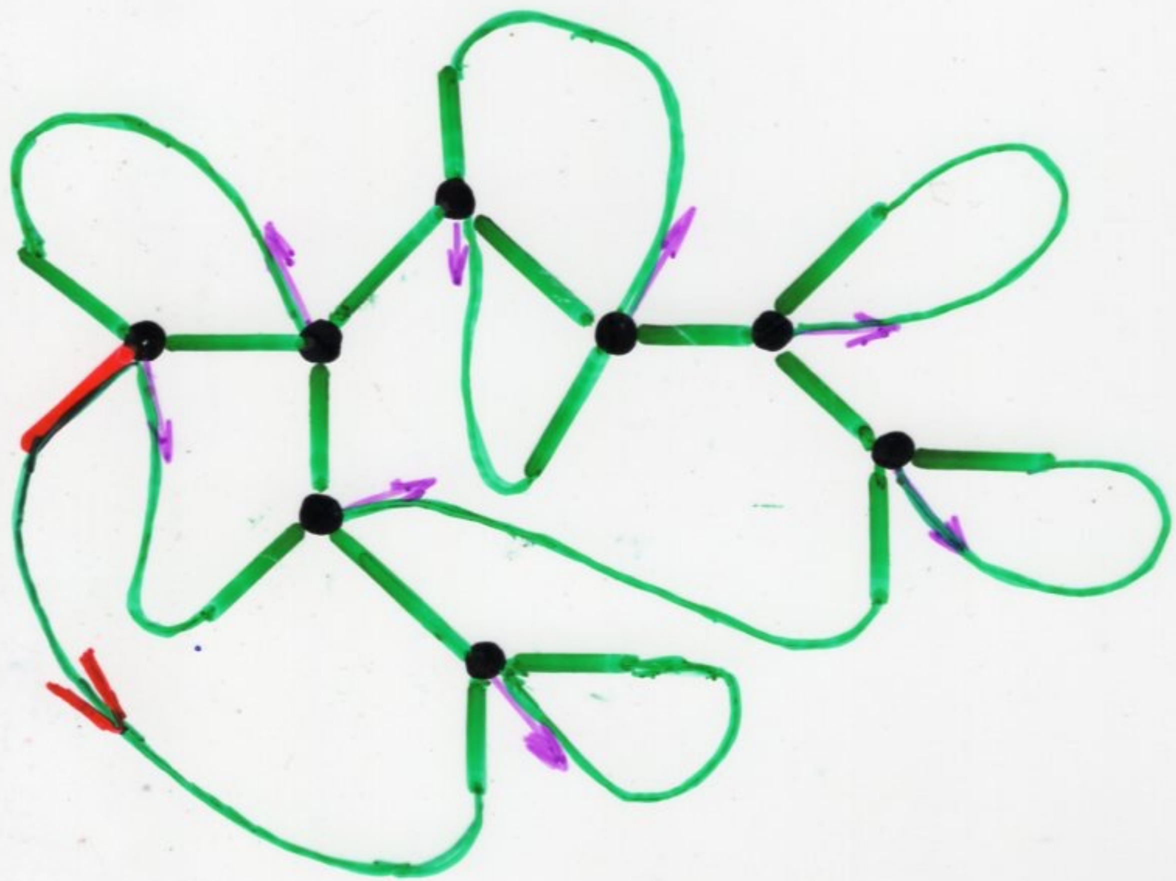
$$\frac{2 \times 3^m}{(m+2)} C_m$$

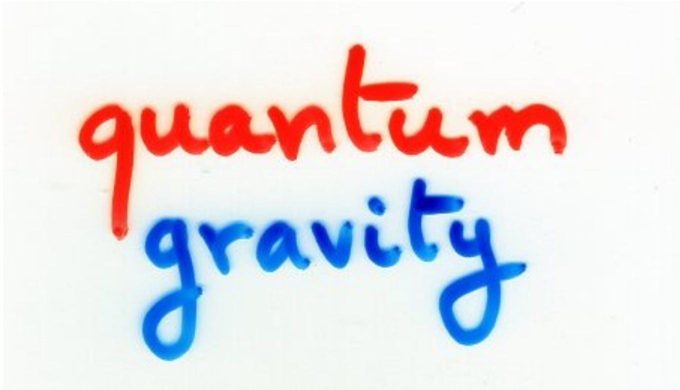
Catalan number

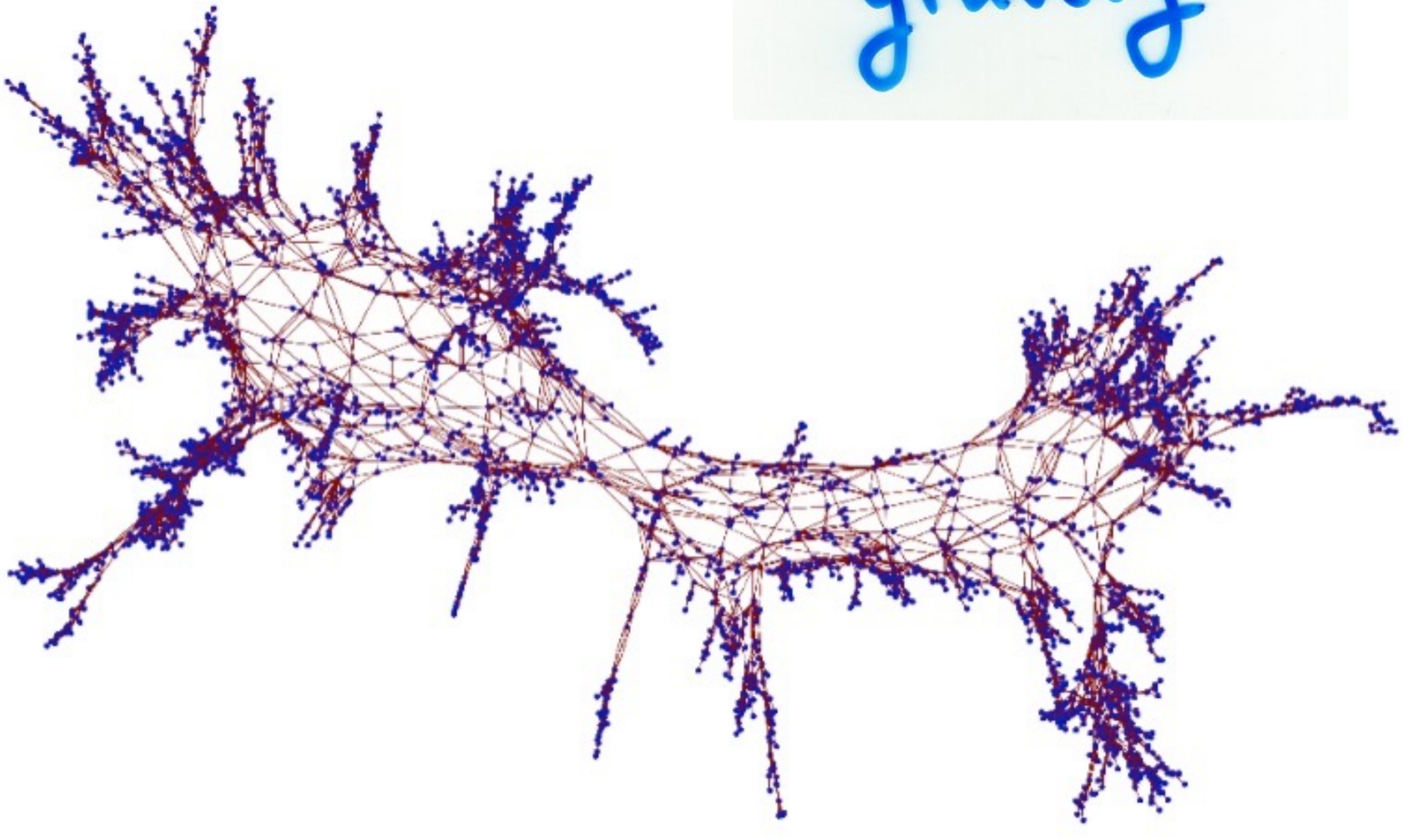
$3^n C_n$

blossoming trees






quantum
gravity



Thank you !

www.xavierviennot.org

ॐ सरस्वत्यै नमः।