Quadratic algebras, combinatorial physics and planar automaton

CMI, Chennai
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Planar automaton

Def. planar automaton $\mathcal{P}$

- 3 finite sets
  - $\mathcal{B}$ horizontal alphabet
  - $\mathcal{A}$ vertical alphabet
  - S planar labels (state)

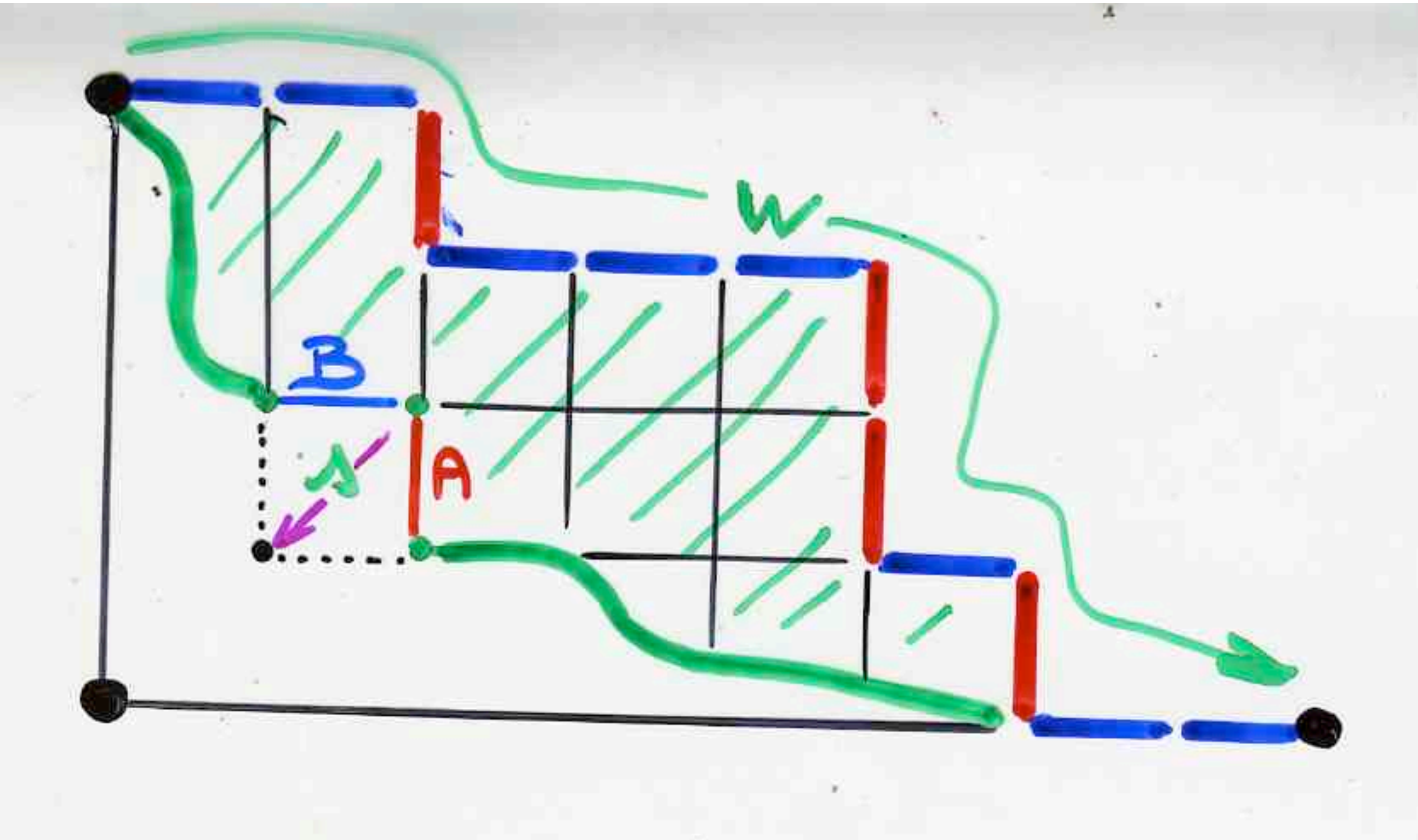
- θ (partial) transition function

$$(s, B, A) \xrightarrow{\theta} (B', A') \quad \text{or} \quad \emptyset$$

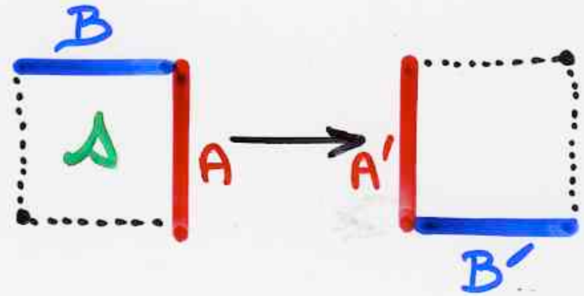
$s \in S$; $B, B' \in \mathcal{B}$; $A, A' \in \mathcal{A}$

- $w \in (\mathcal{A} \cup \mathcal{B})^*$ initial word
- uv, $u \in \mathcal{A}^*$, $v \in \mathcal{B}^*$ final word

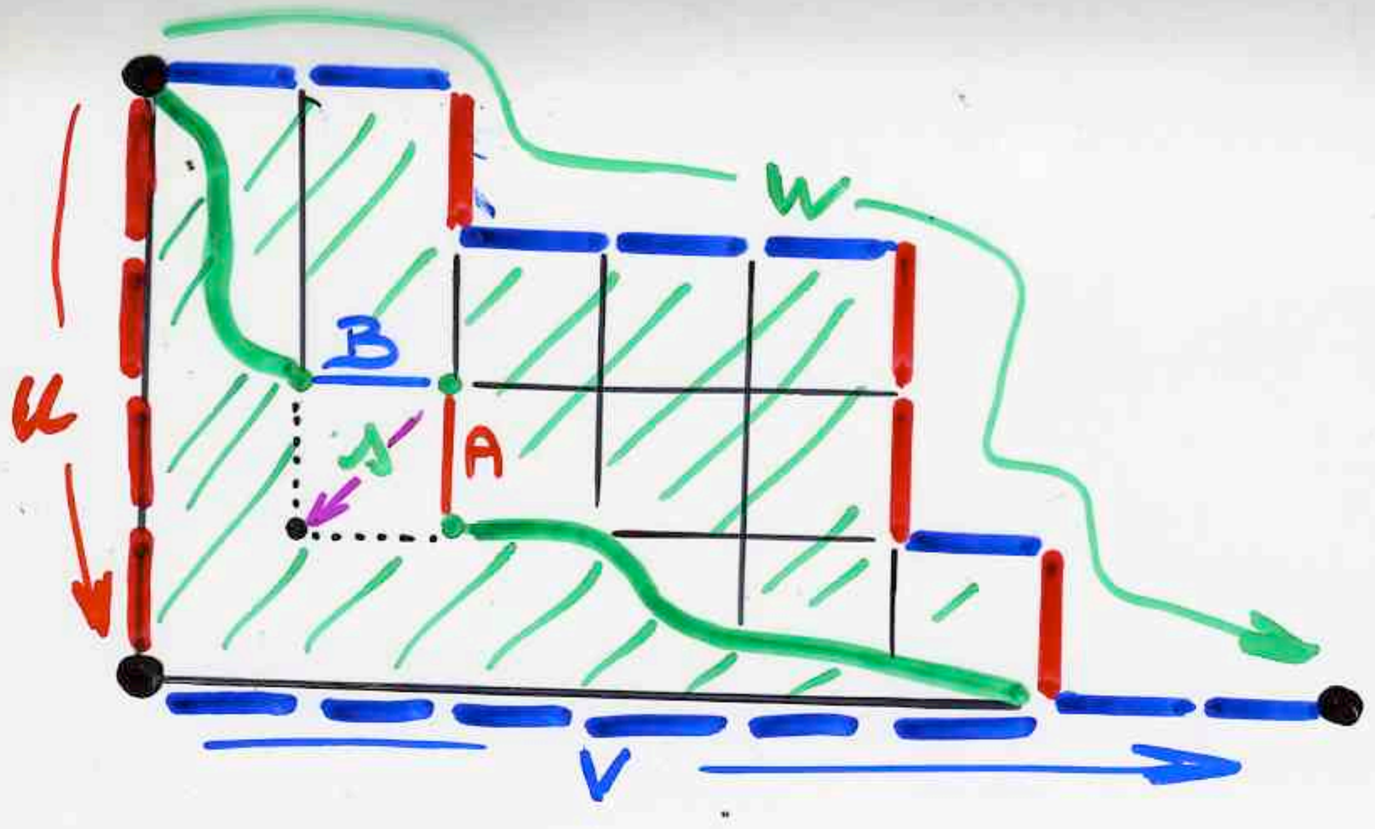
Def. tableau T accepted by a planar automaton $\mathcal{P}=(S, \mathcal{B}, \mathcal{A}, \theta, w, uv)$



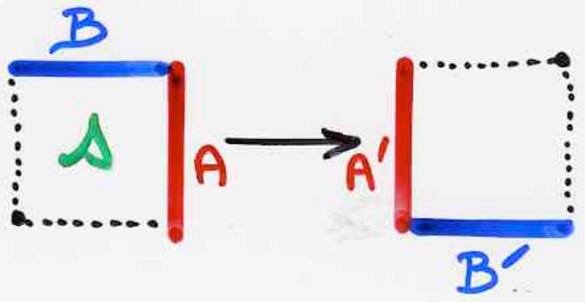
planar rewriting



Def. tableau T accepted by a planar automaton $P = (S, \mathcal{B}, \alpha, \theta, w, uv)$

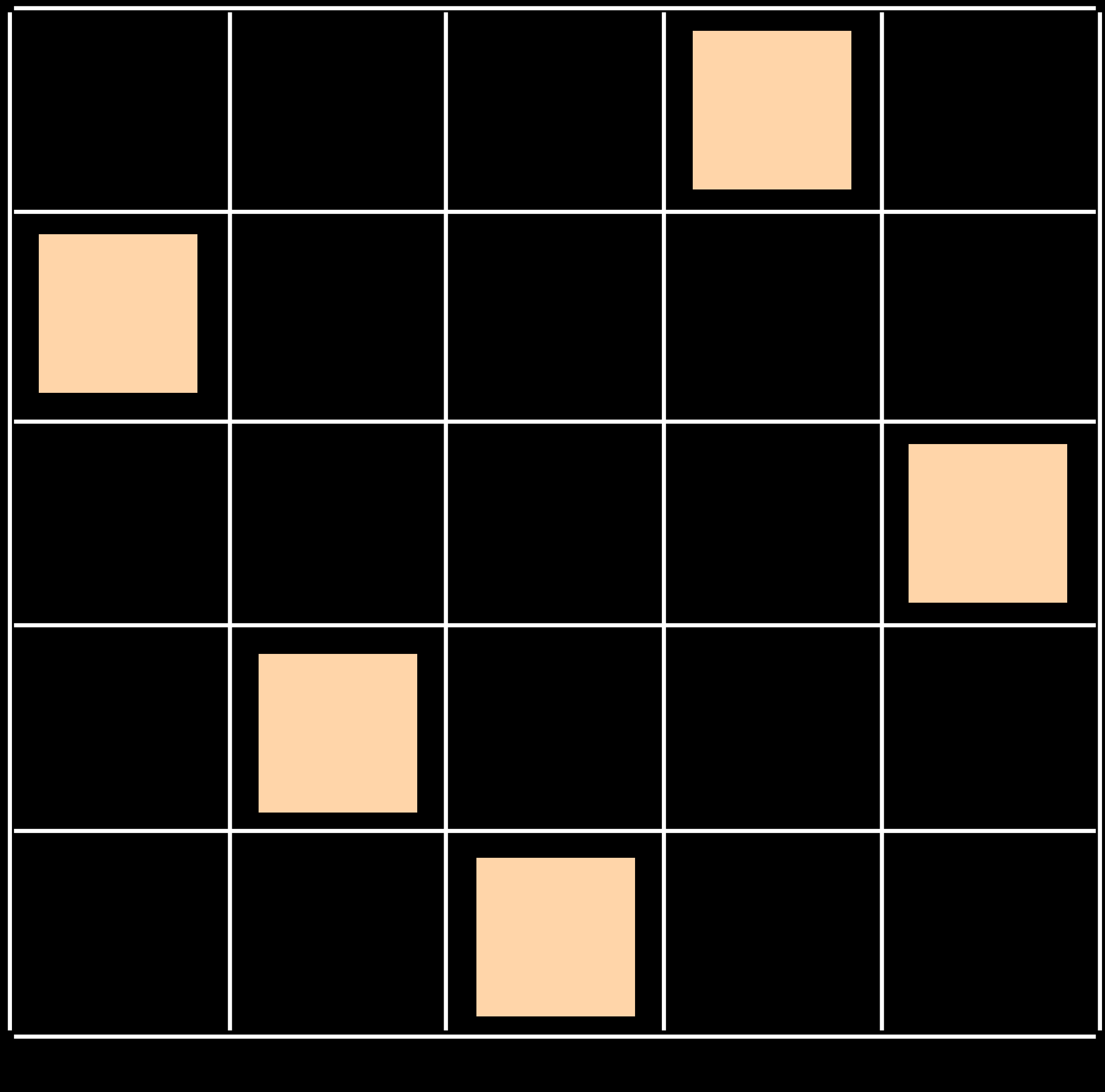


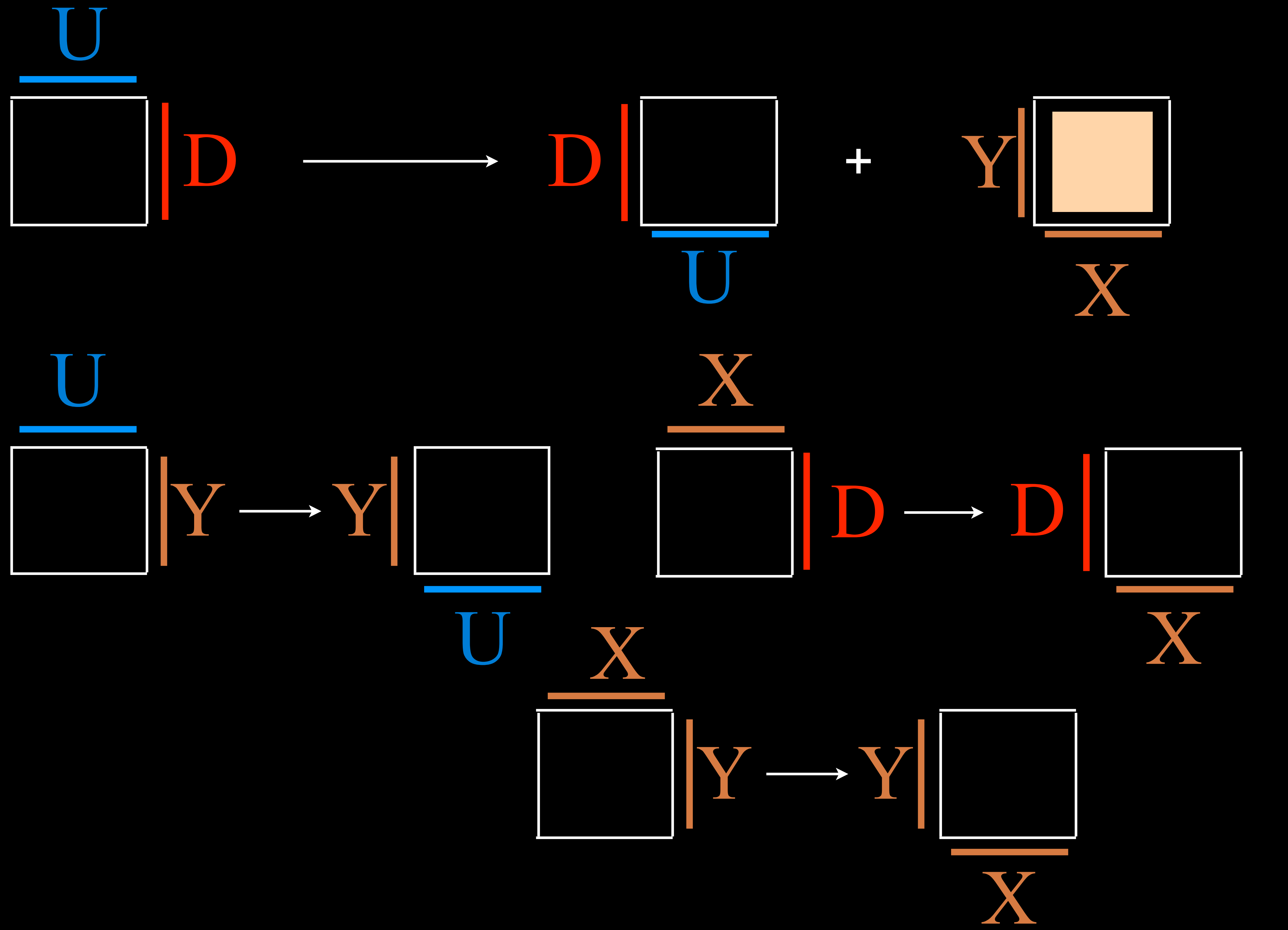
planar rewriting



Planar automata

example 1: permutations

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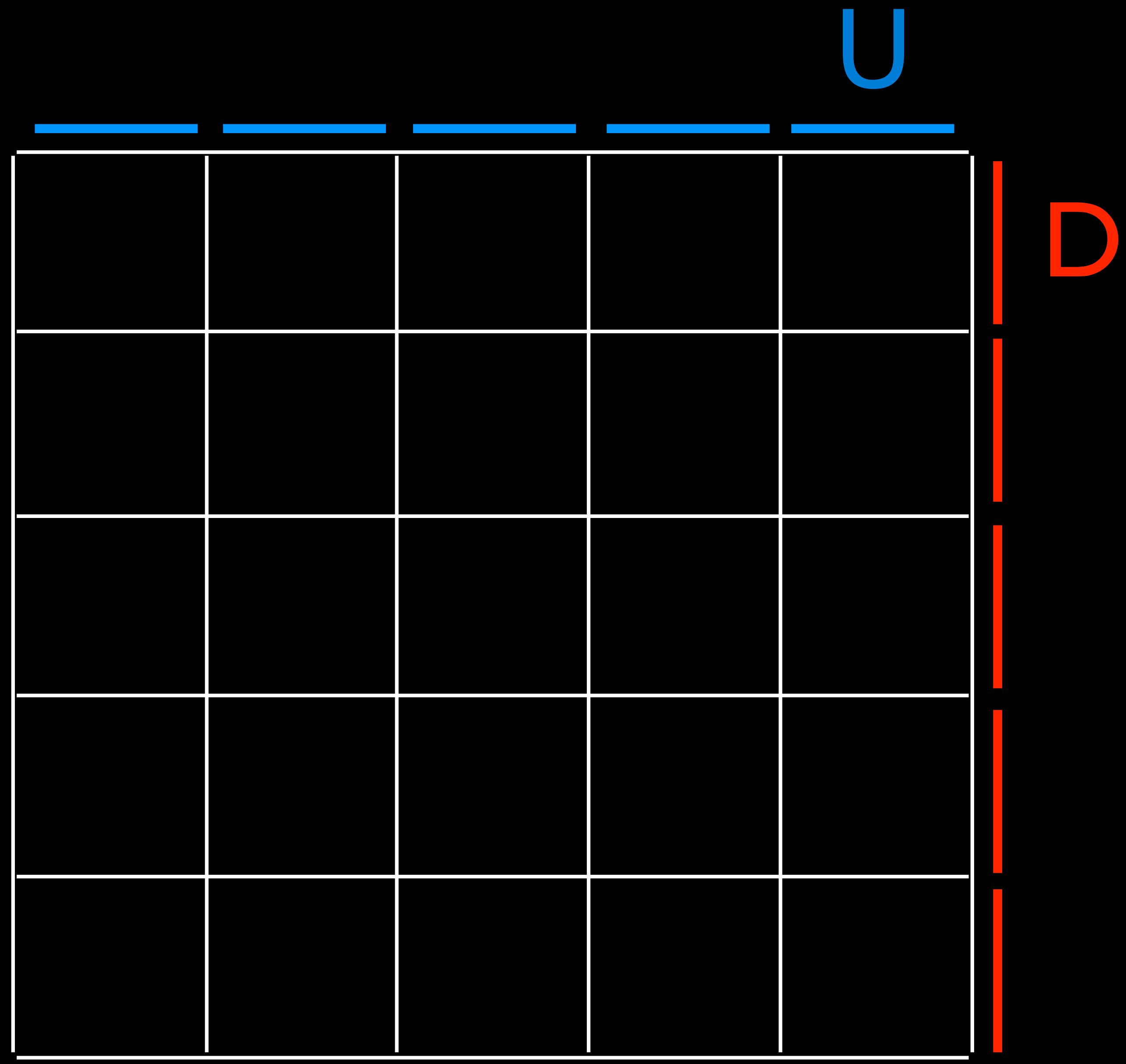
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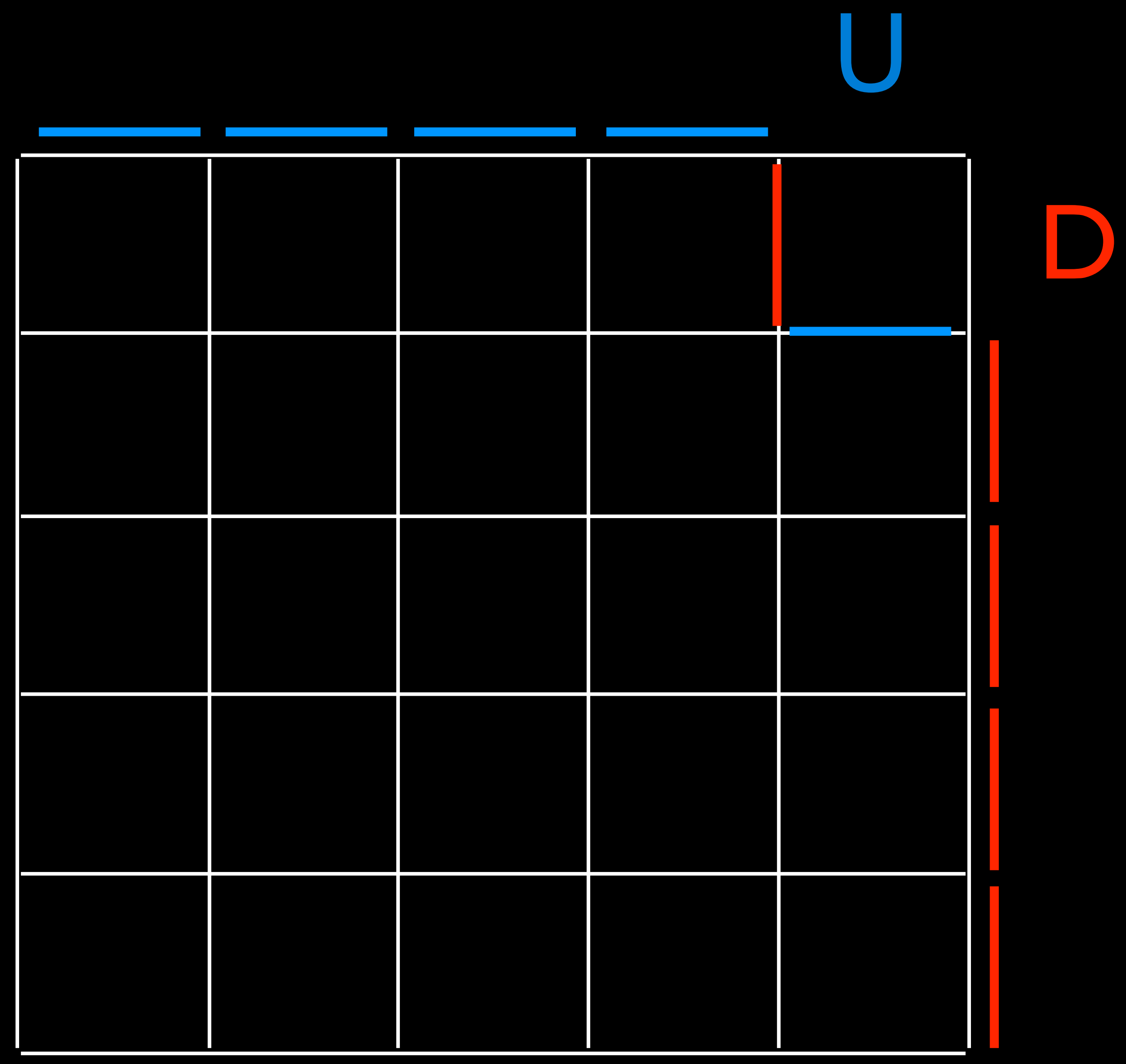
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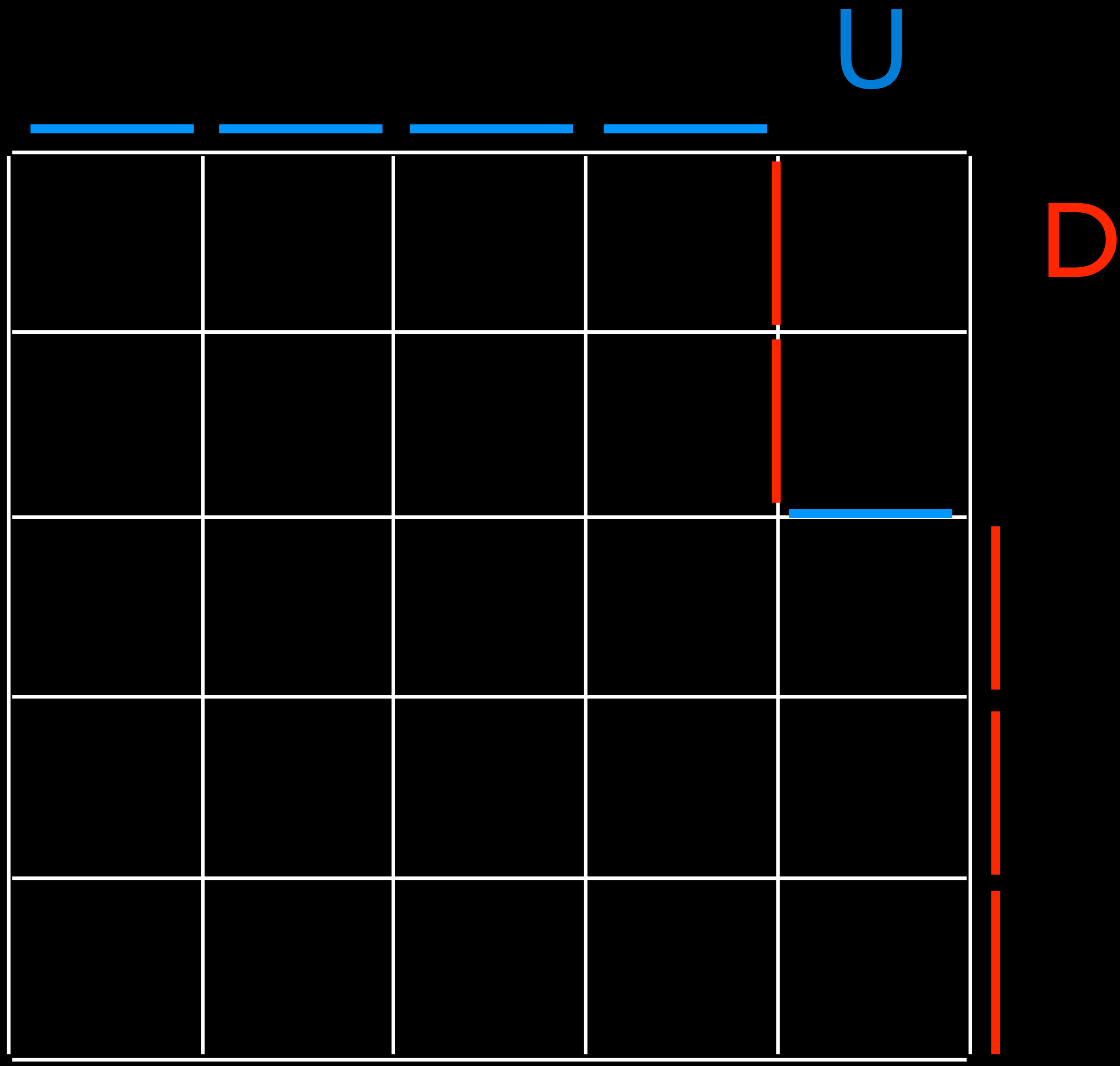
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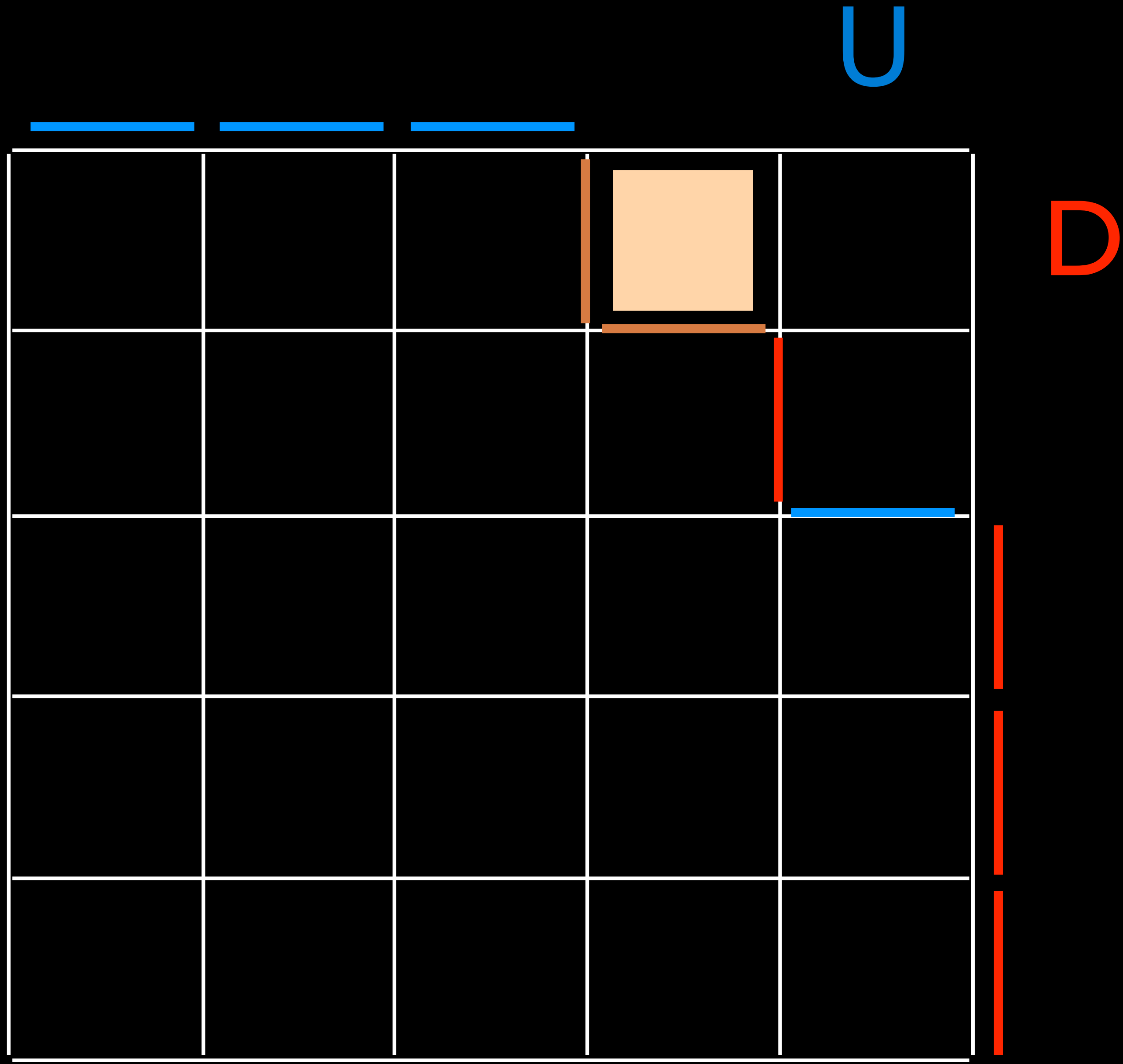
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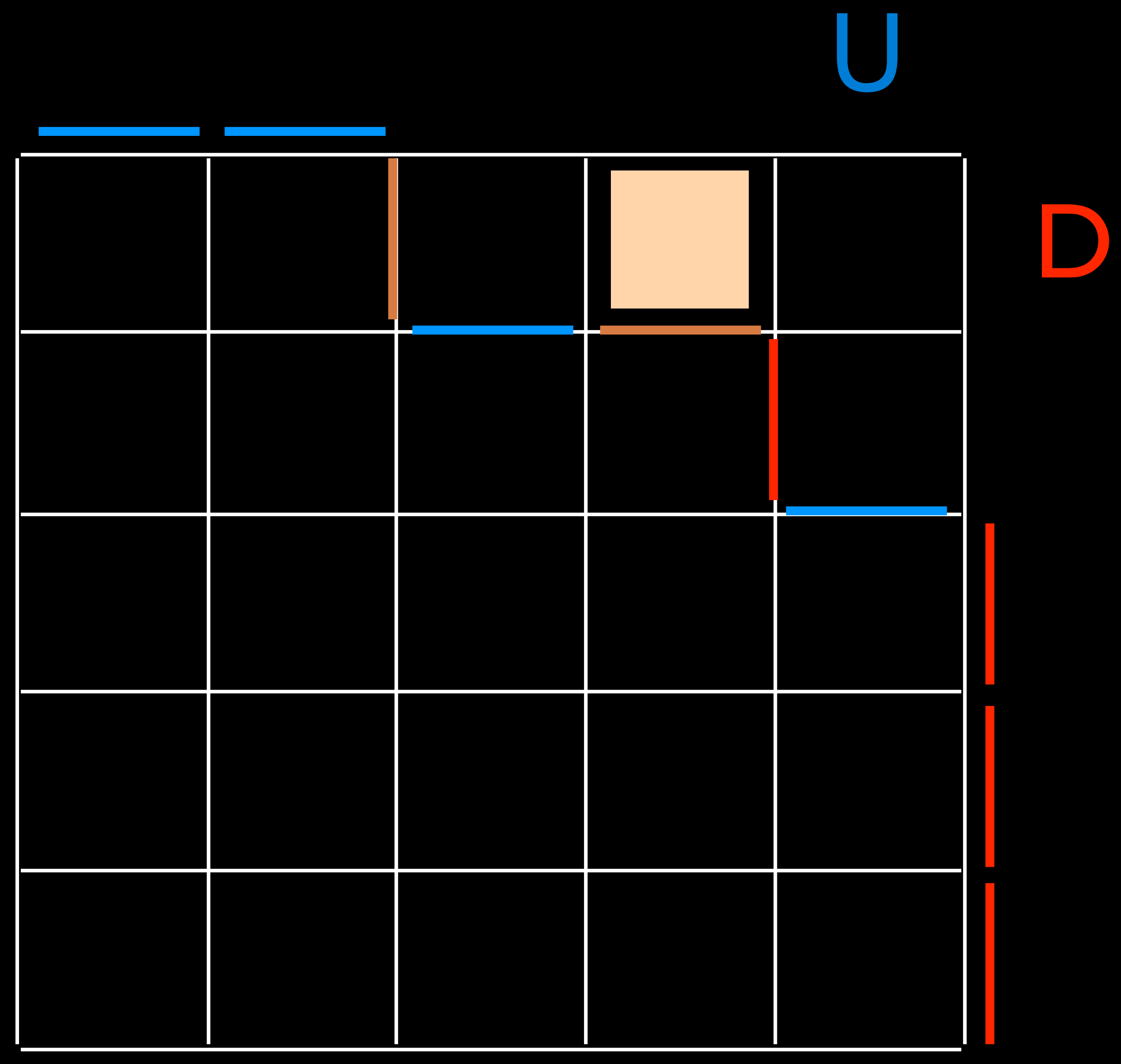
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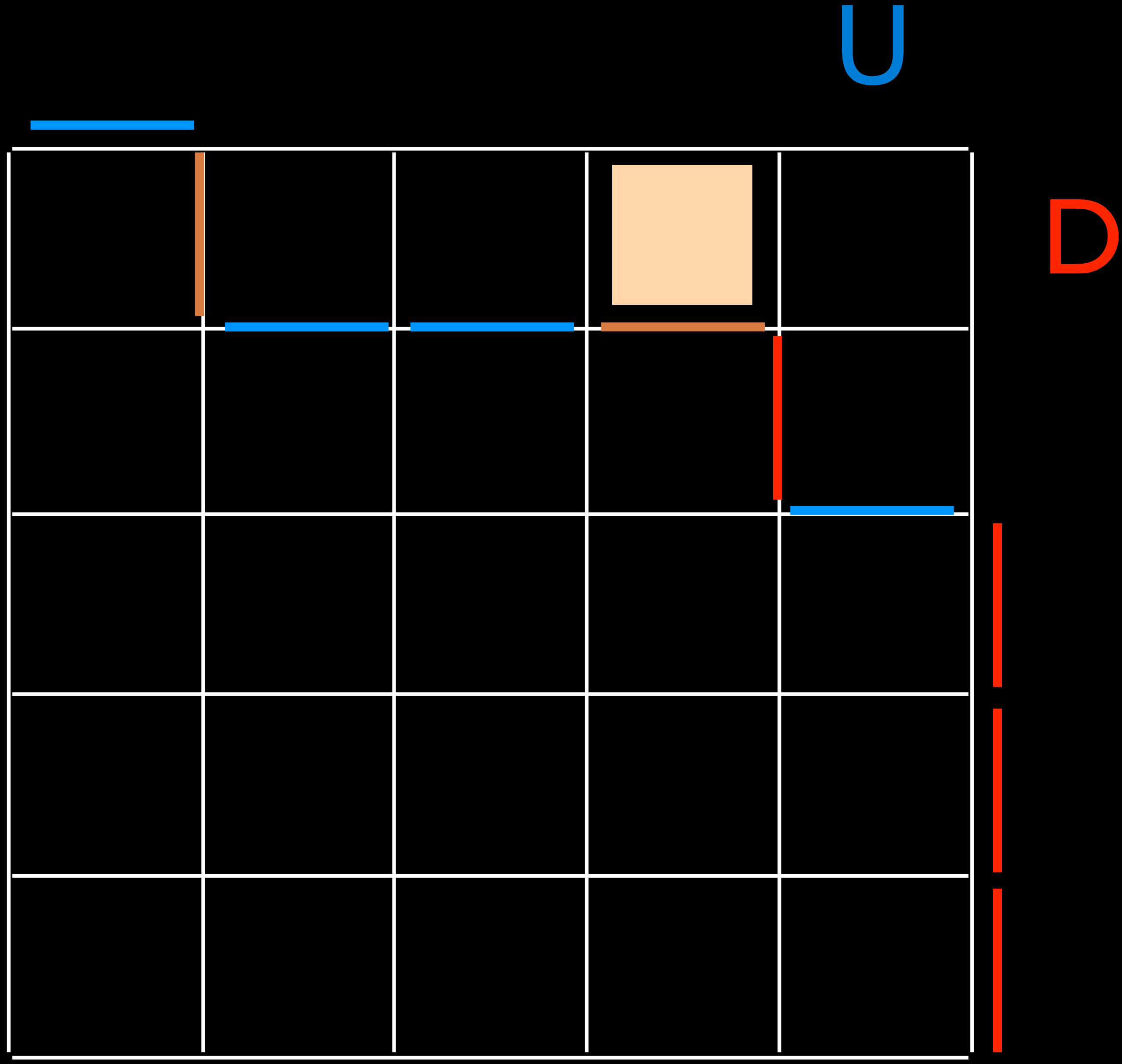
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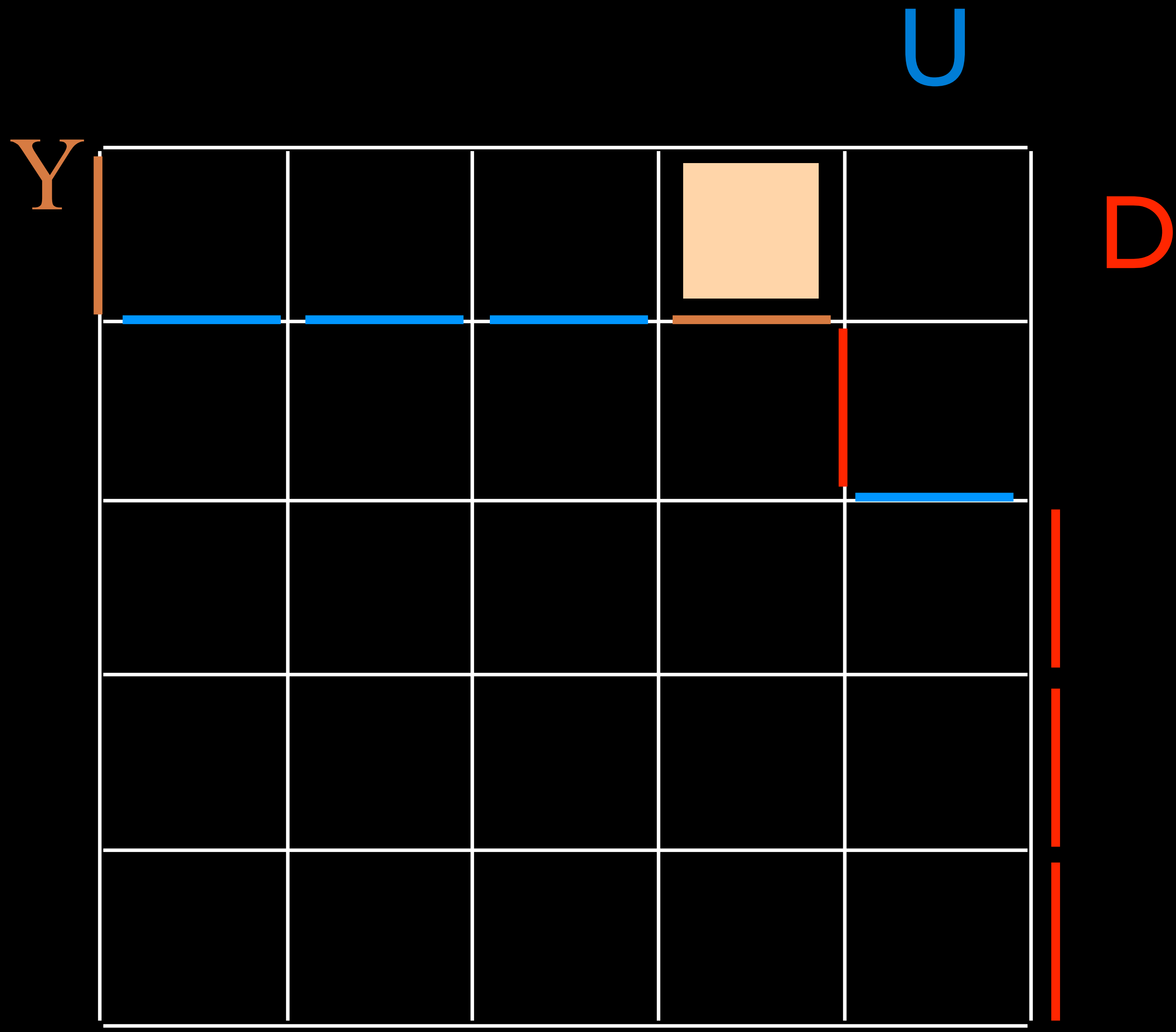
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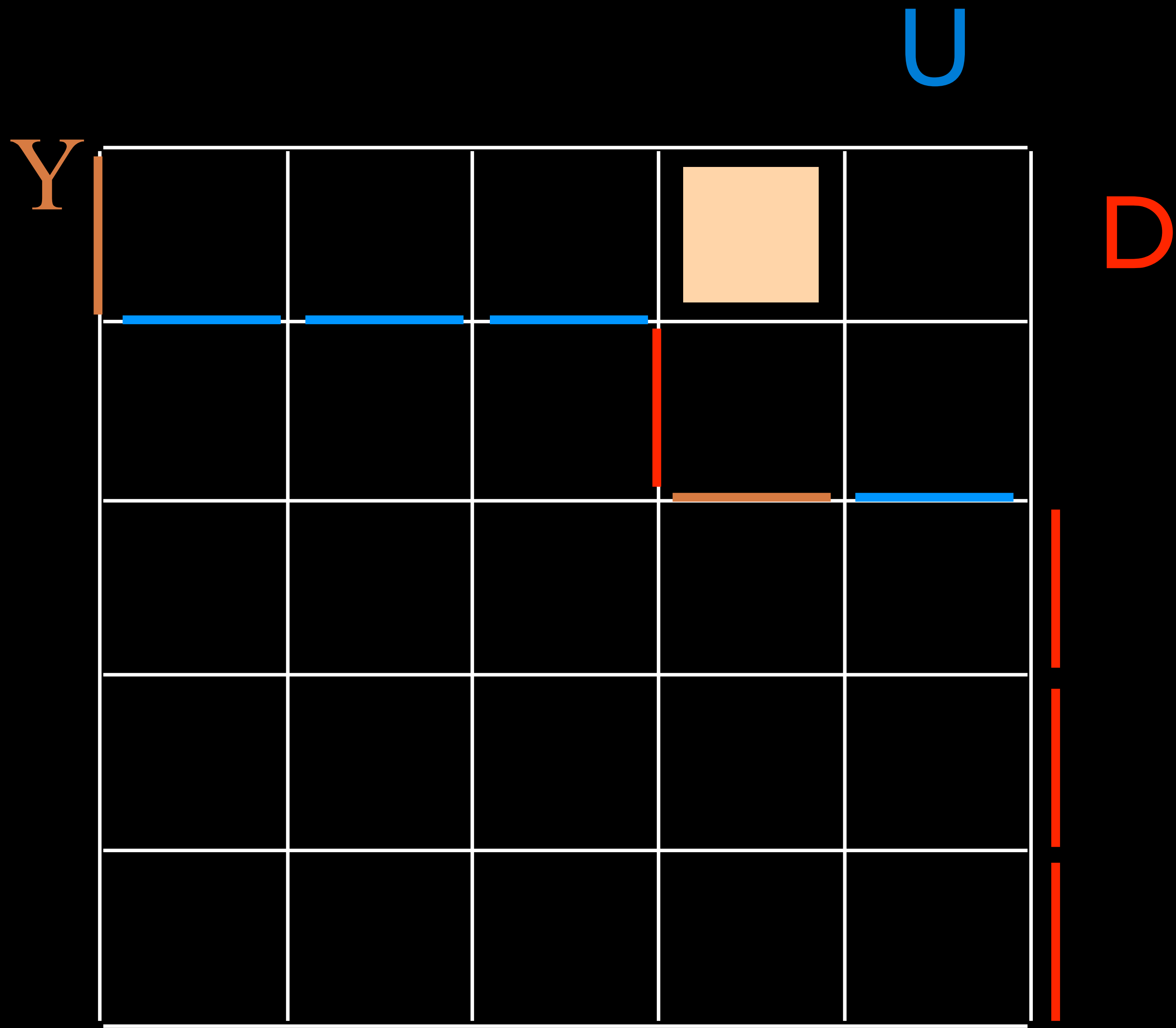
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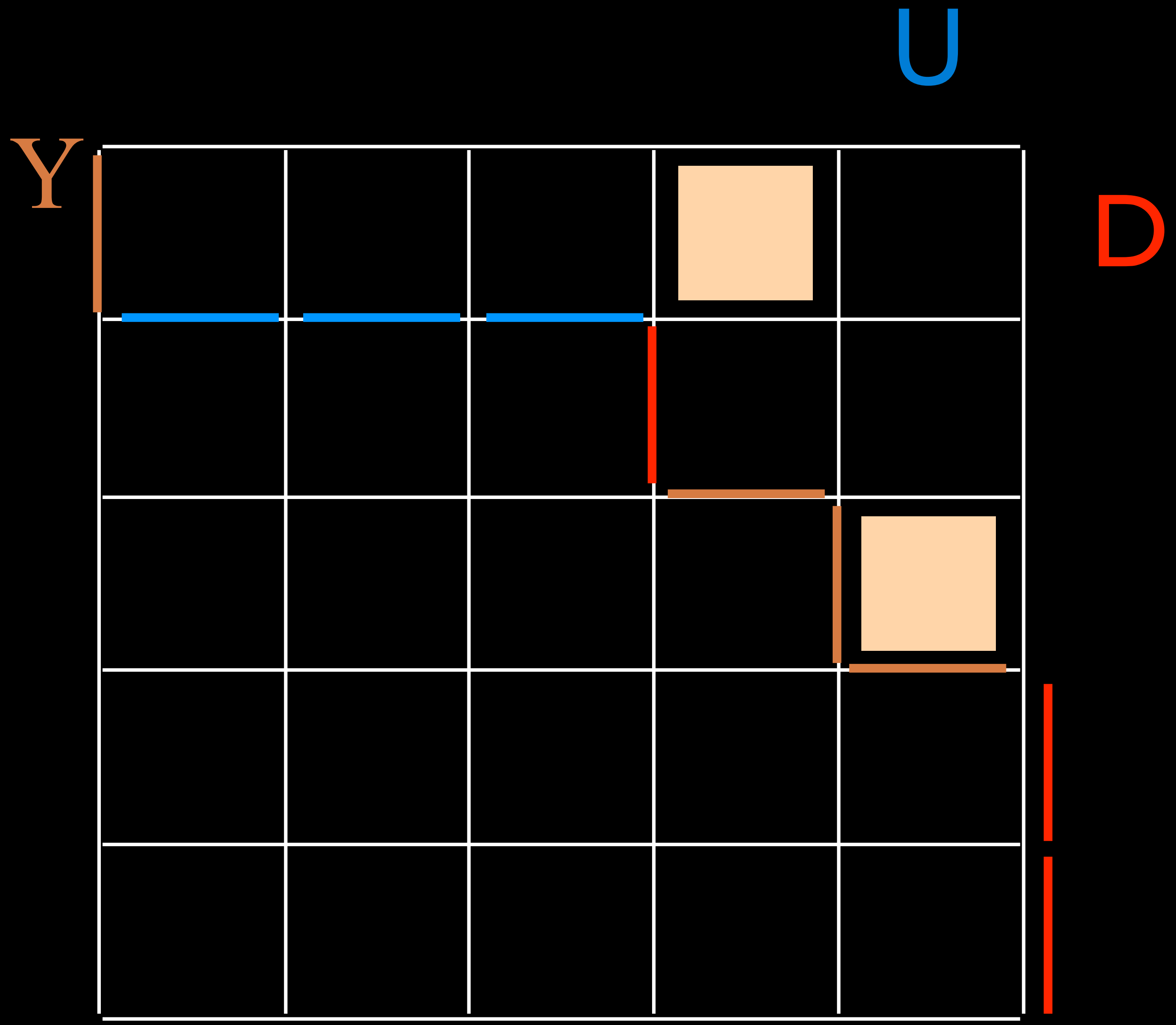
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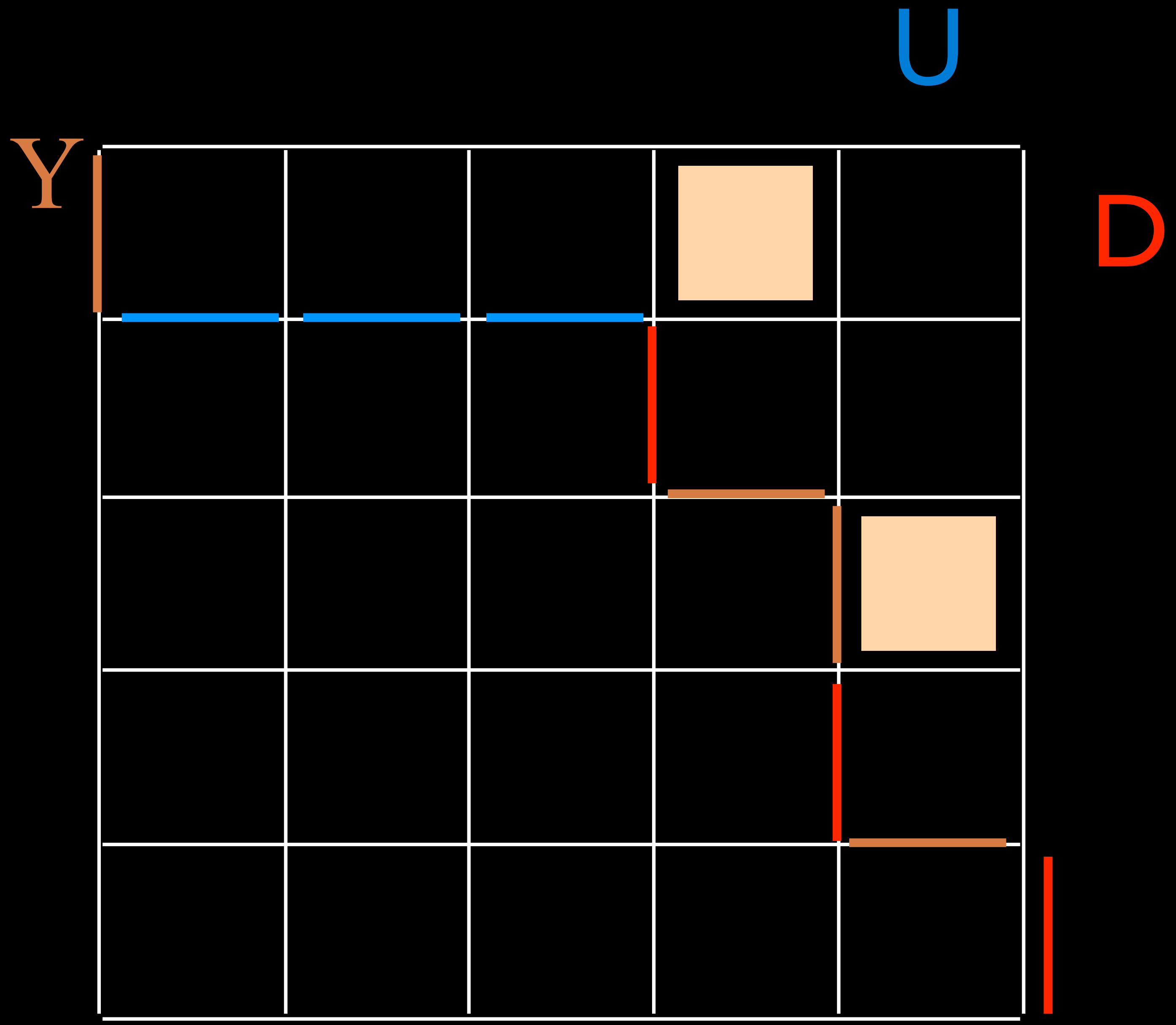
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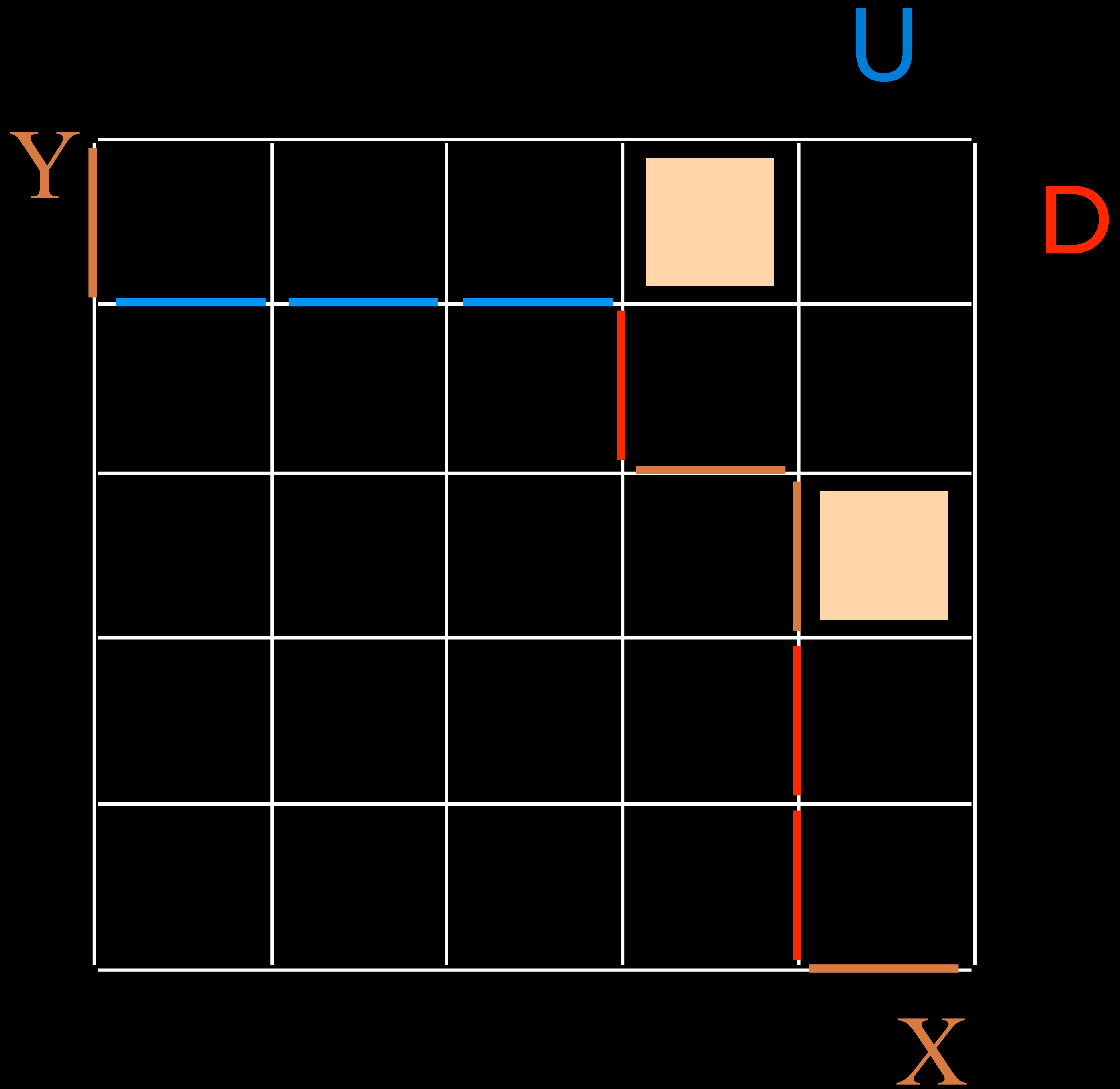

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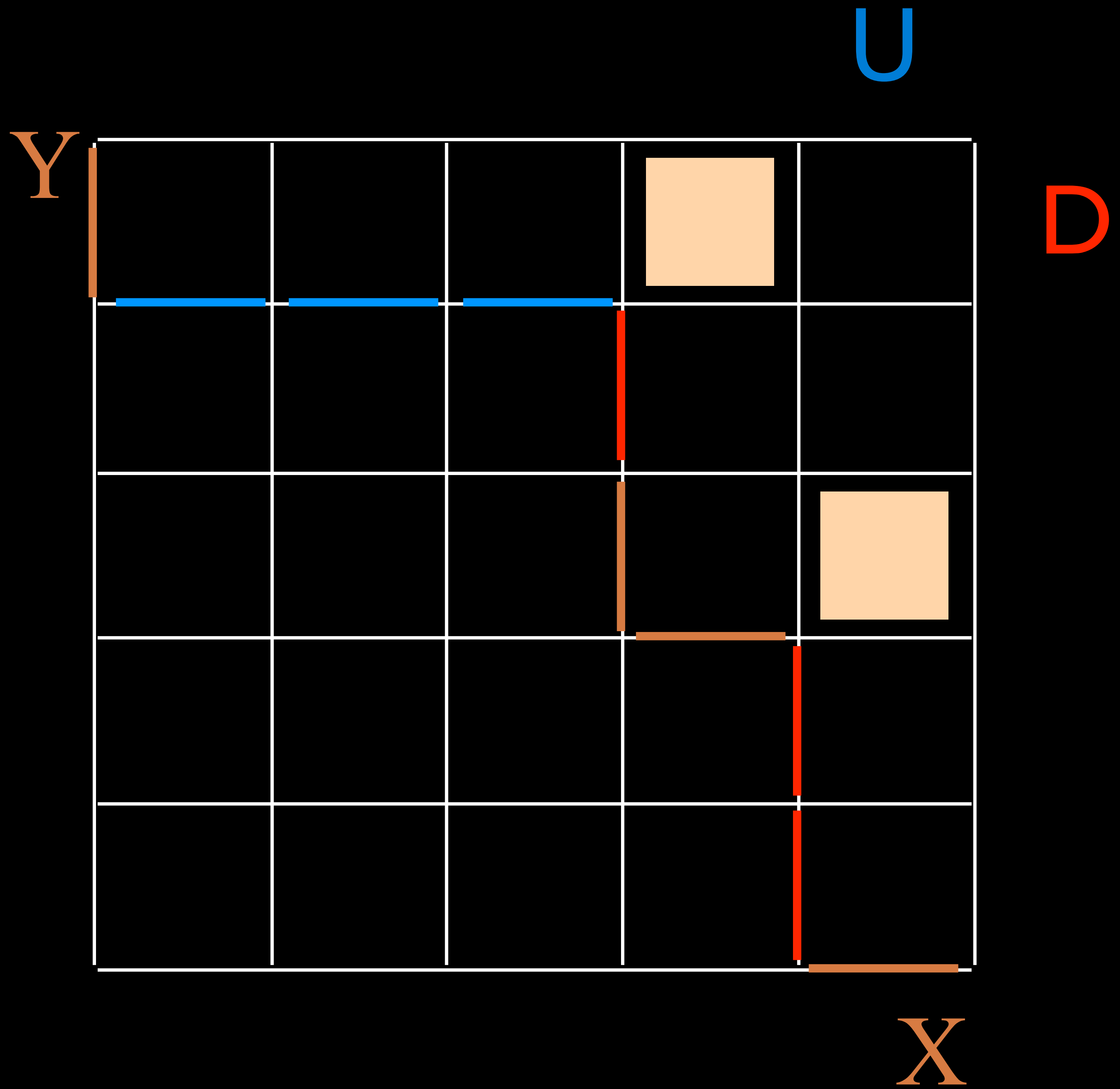

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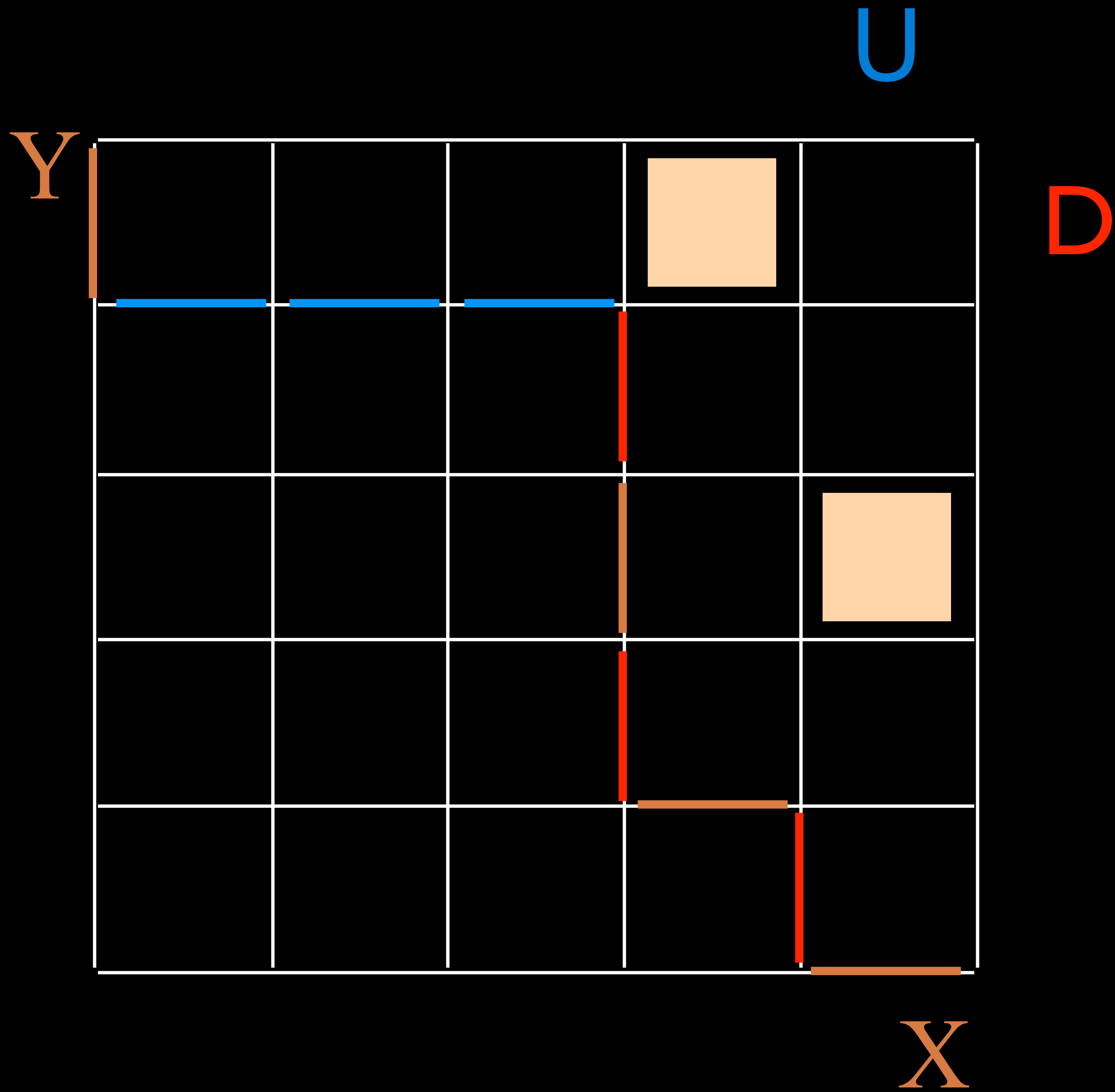
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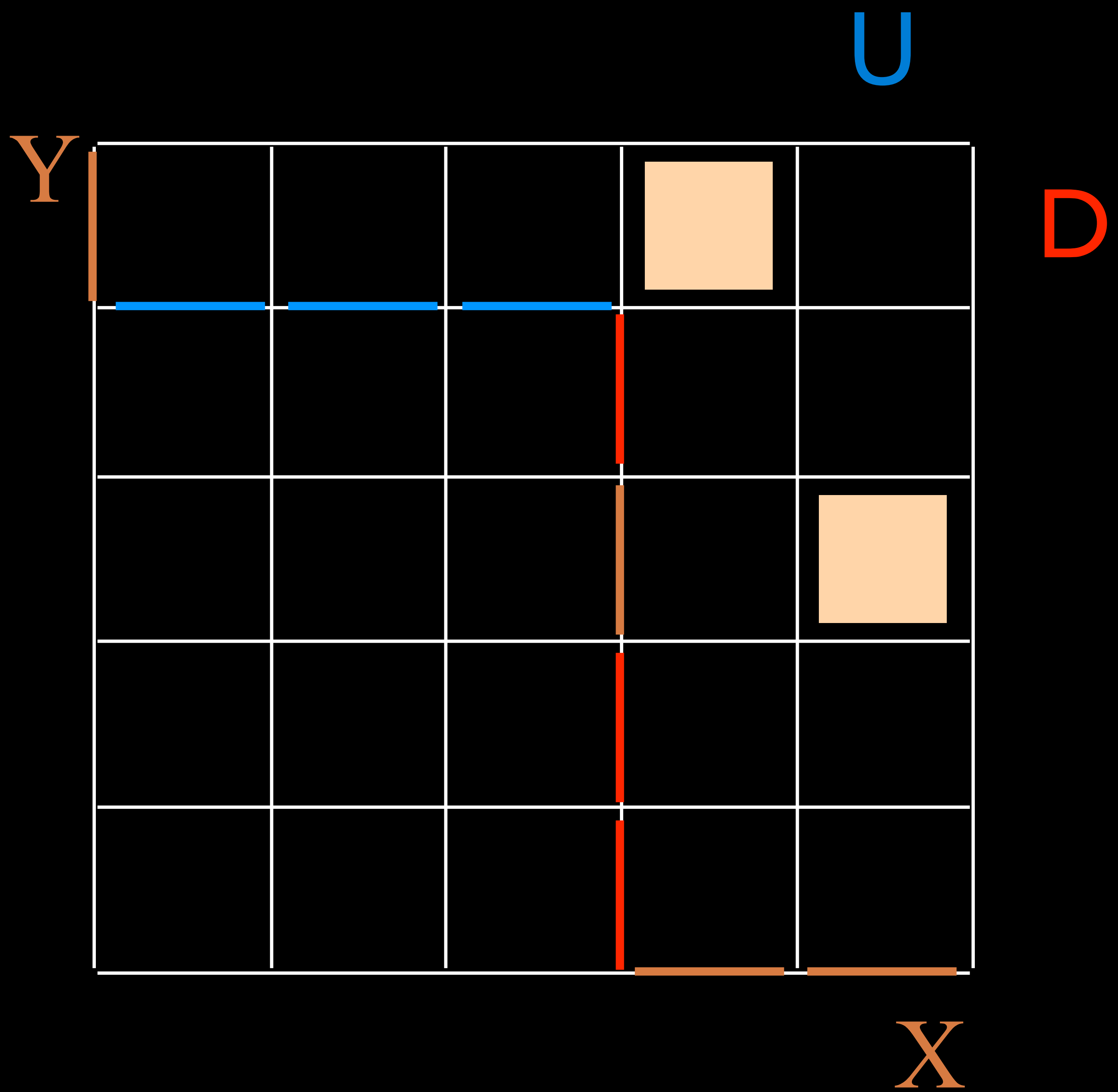
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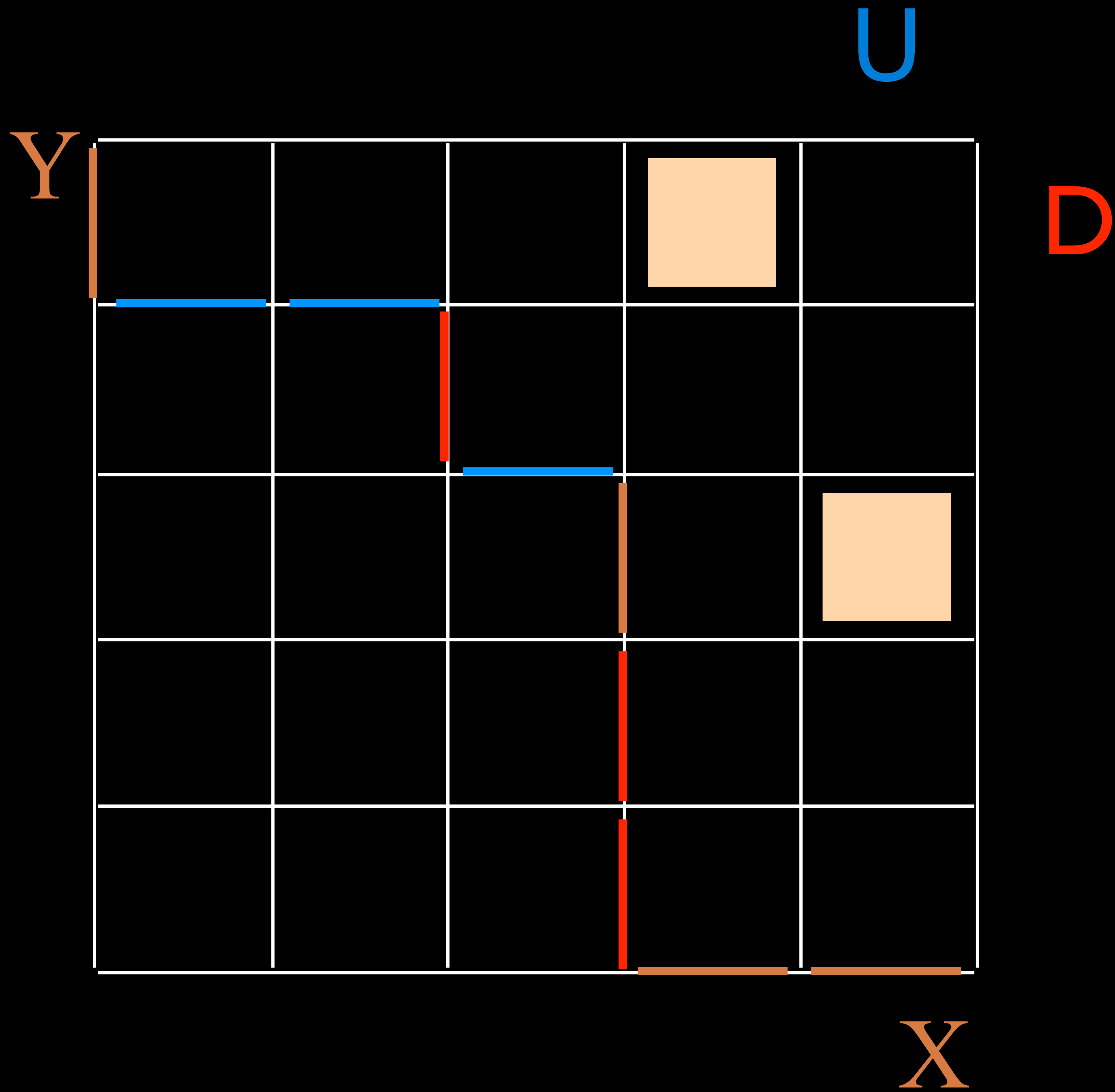
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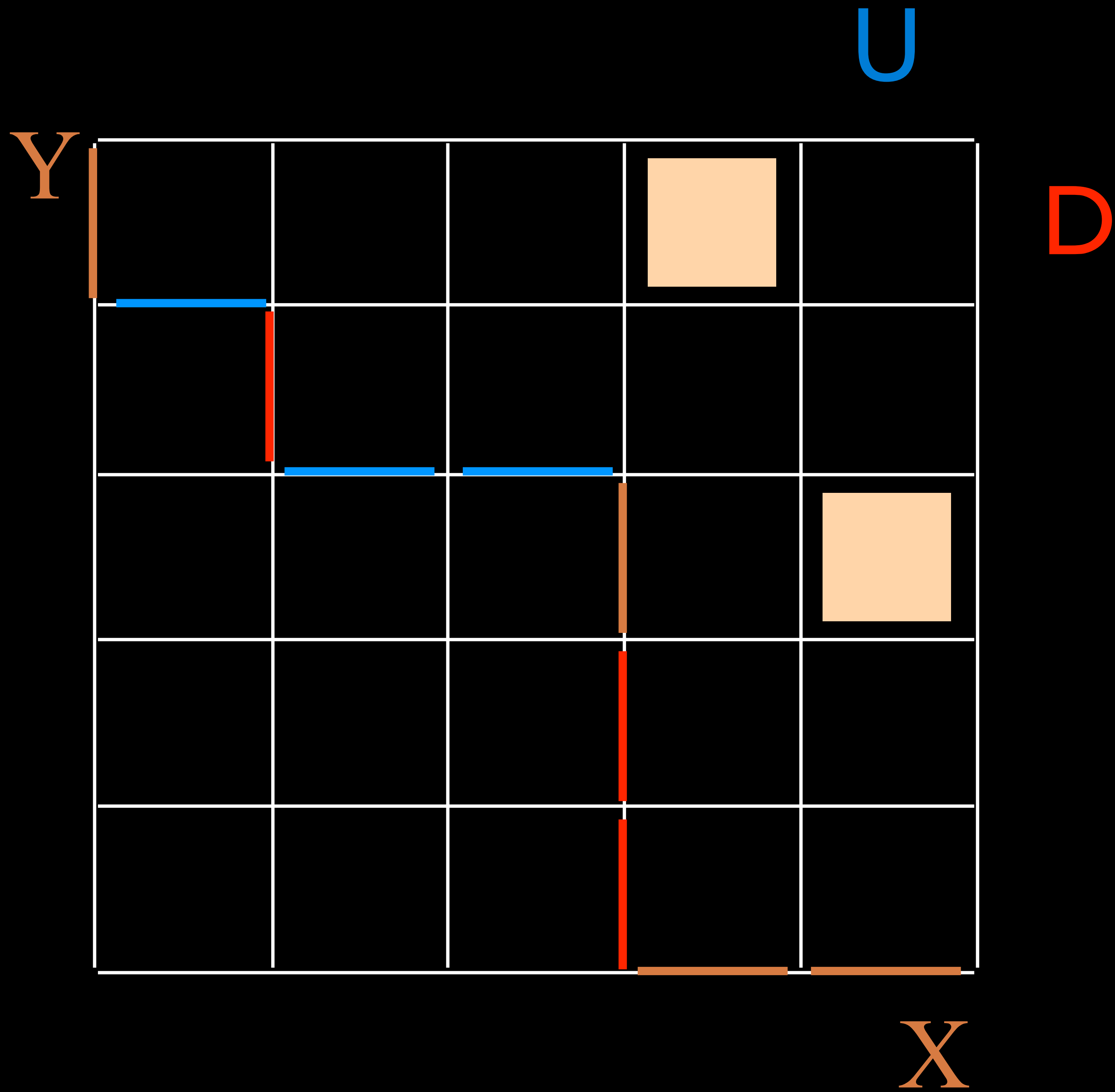
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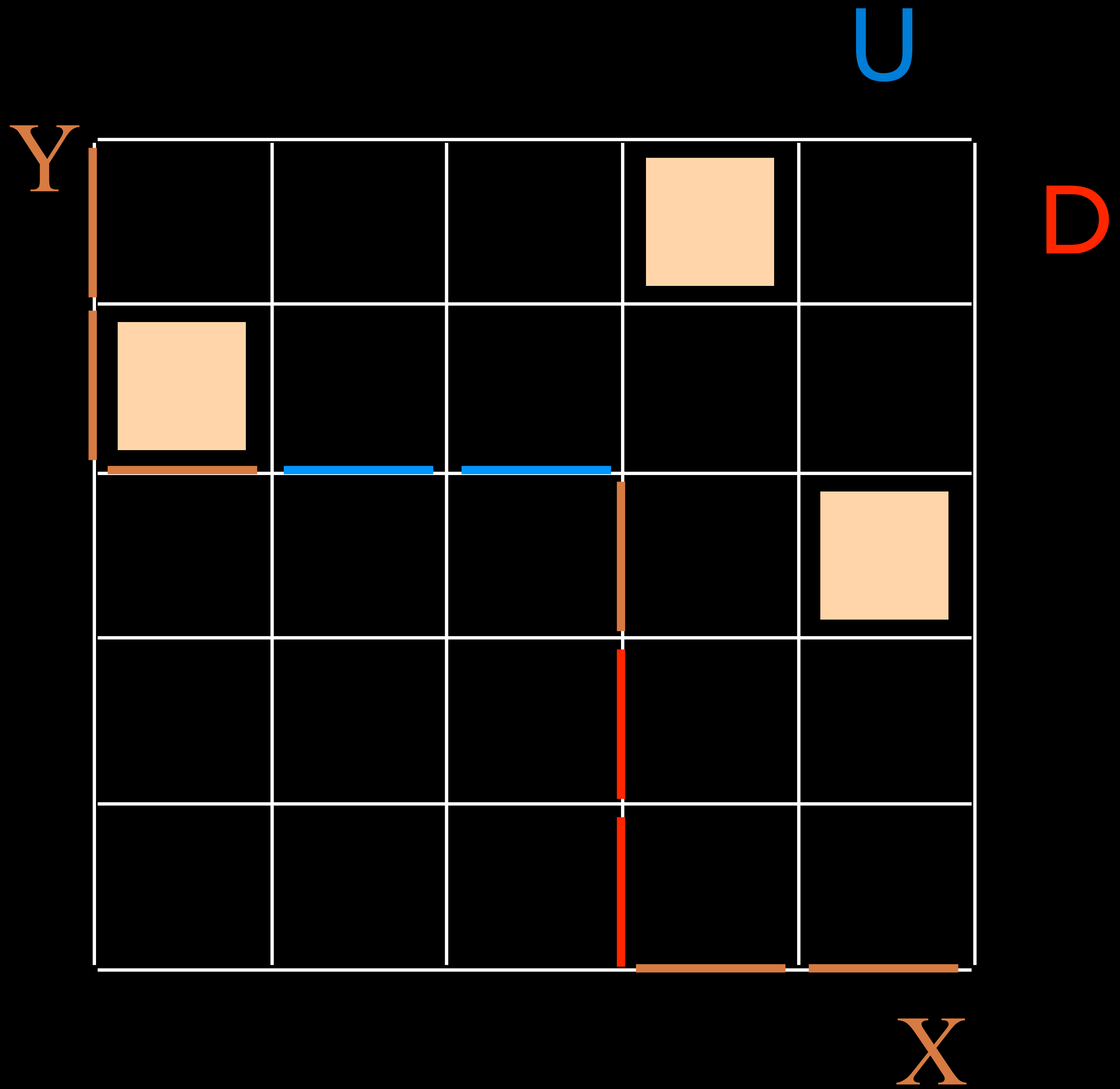
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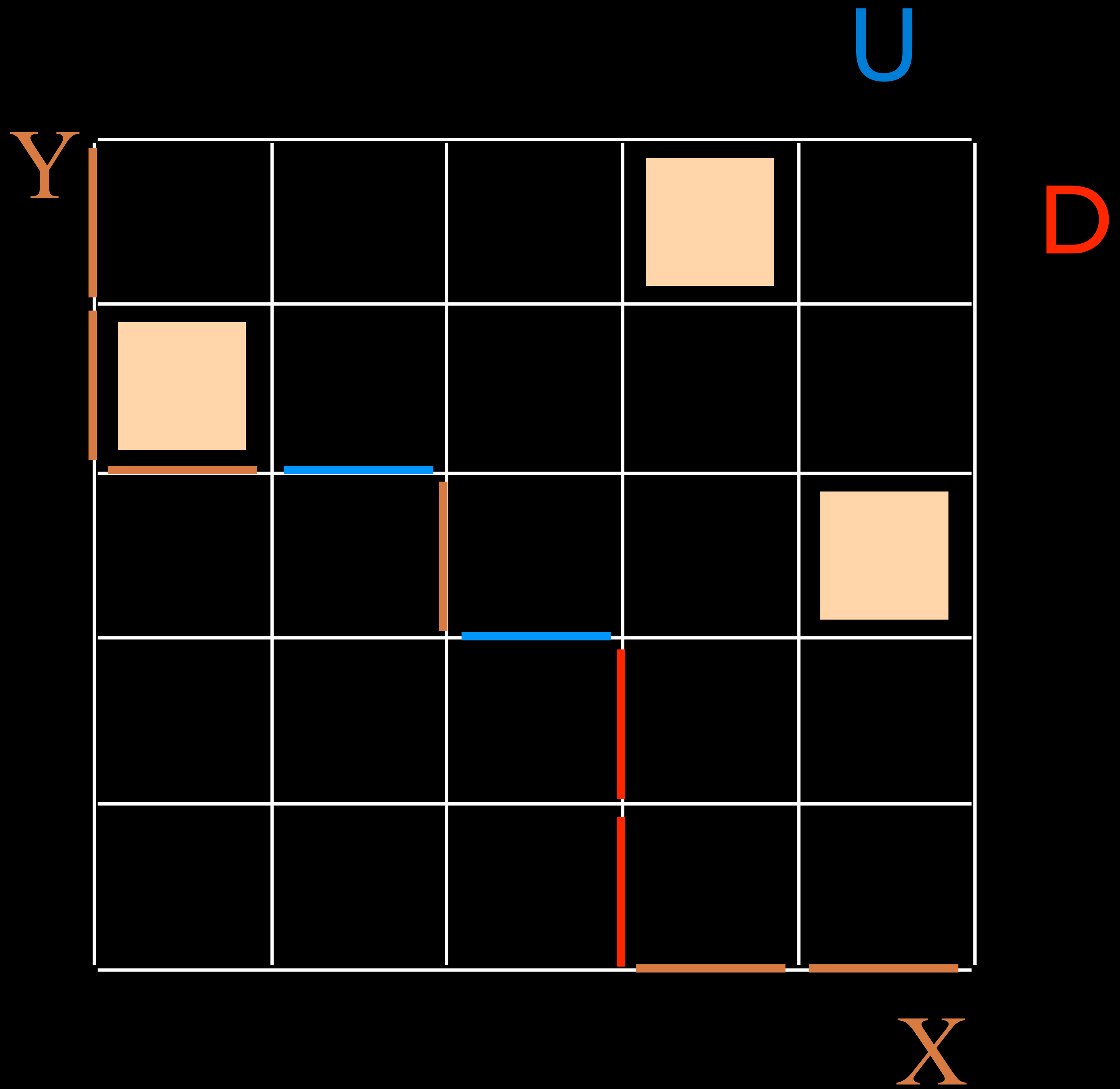
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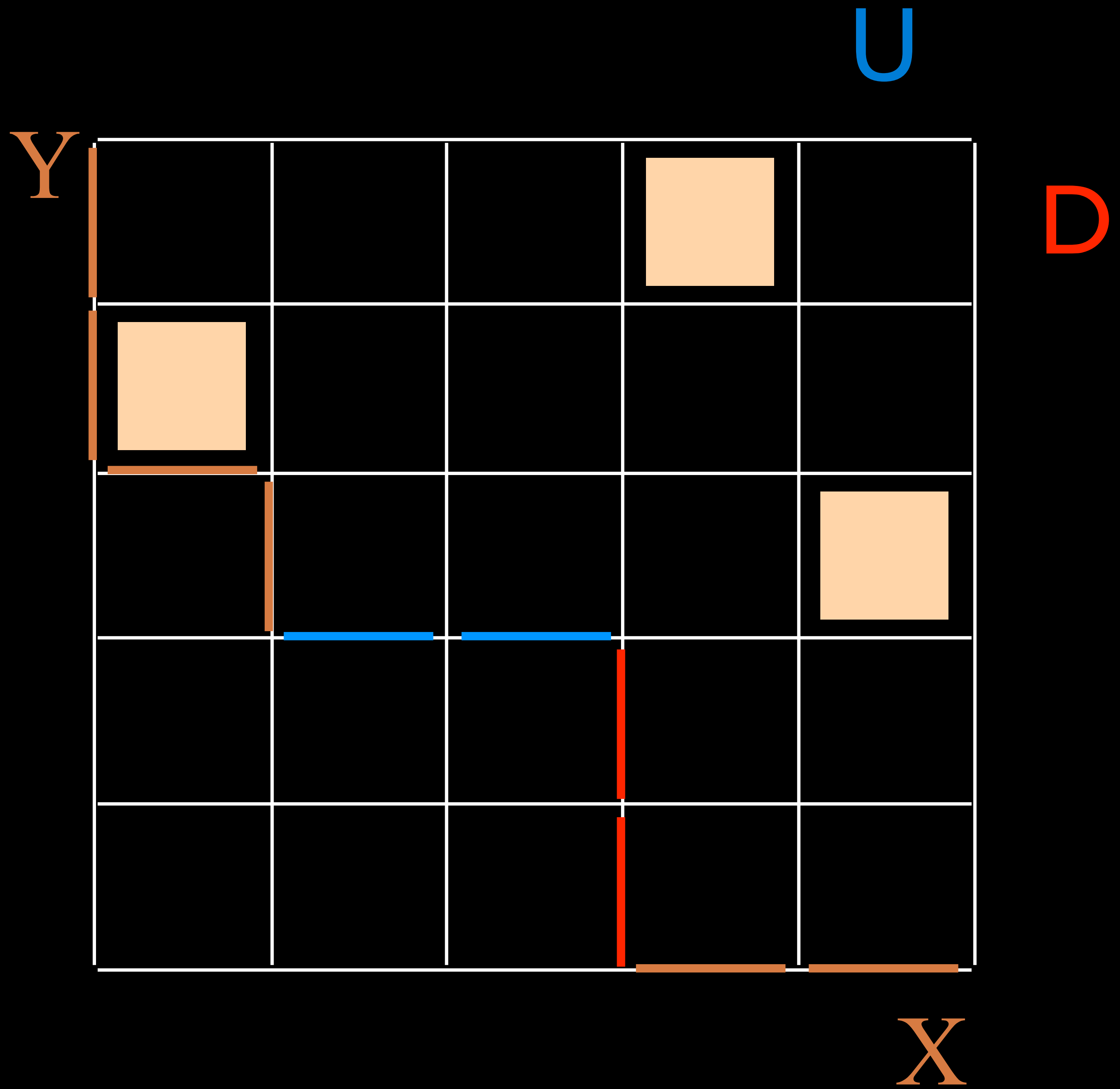

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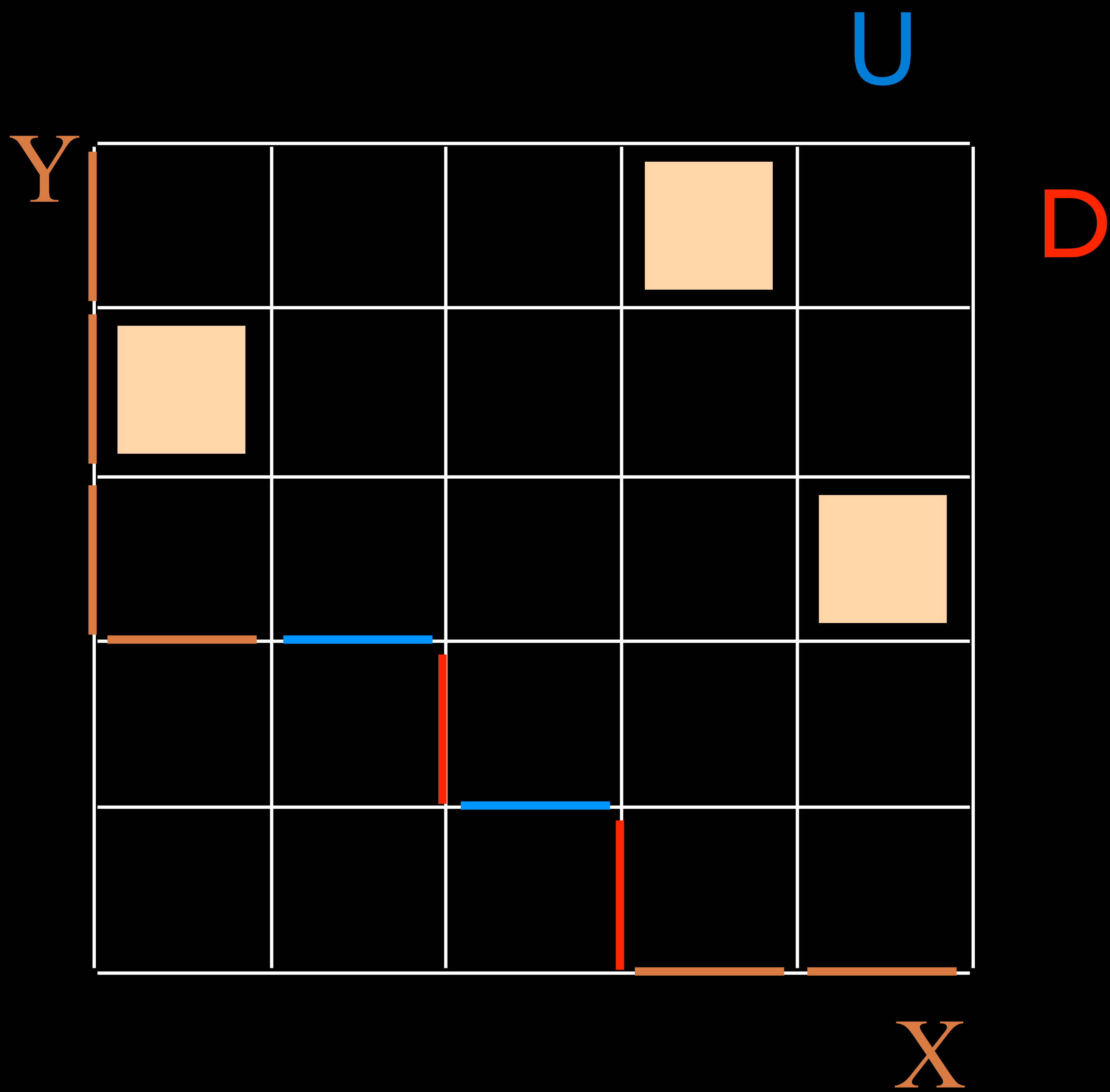
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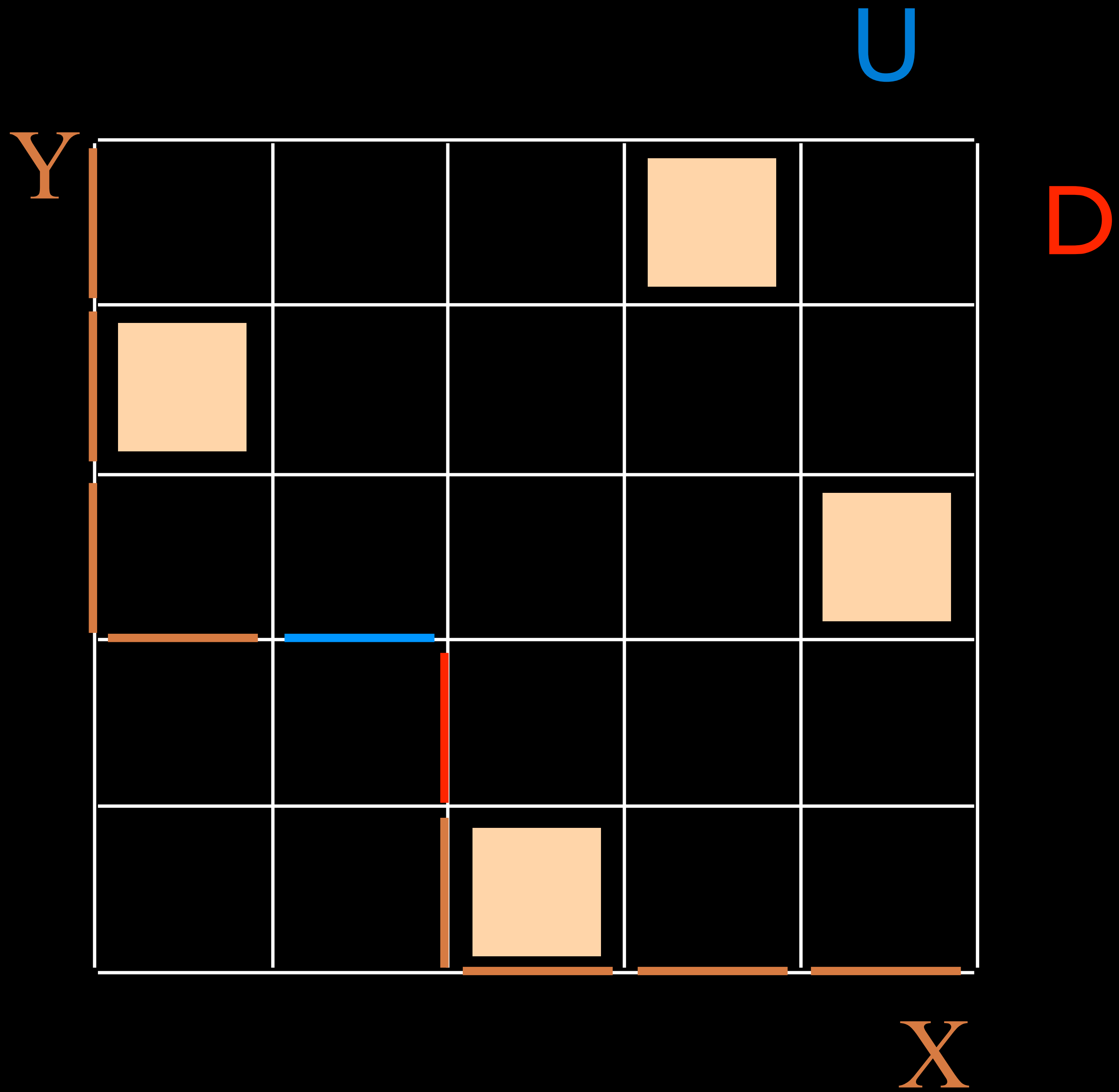
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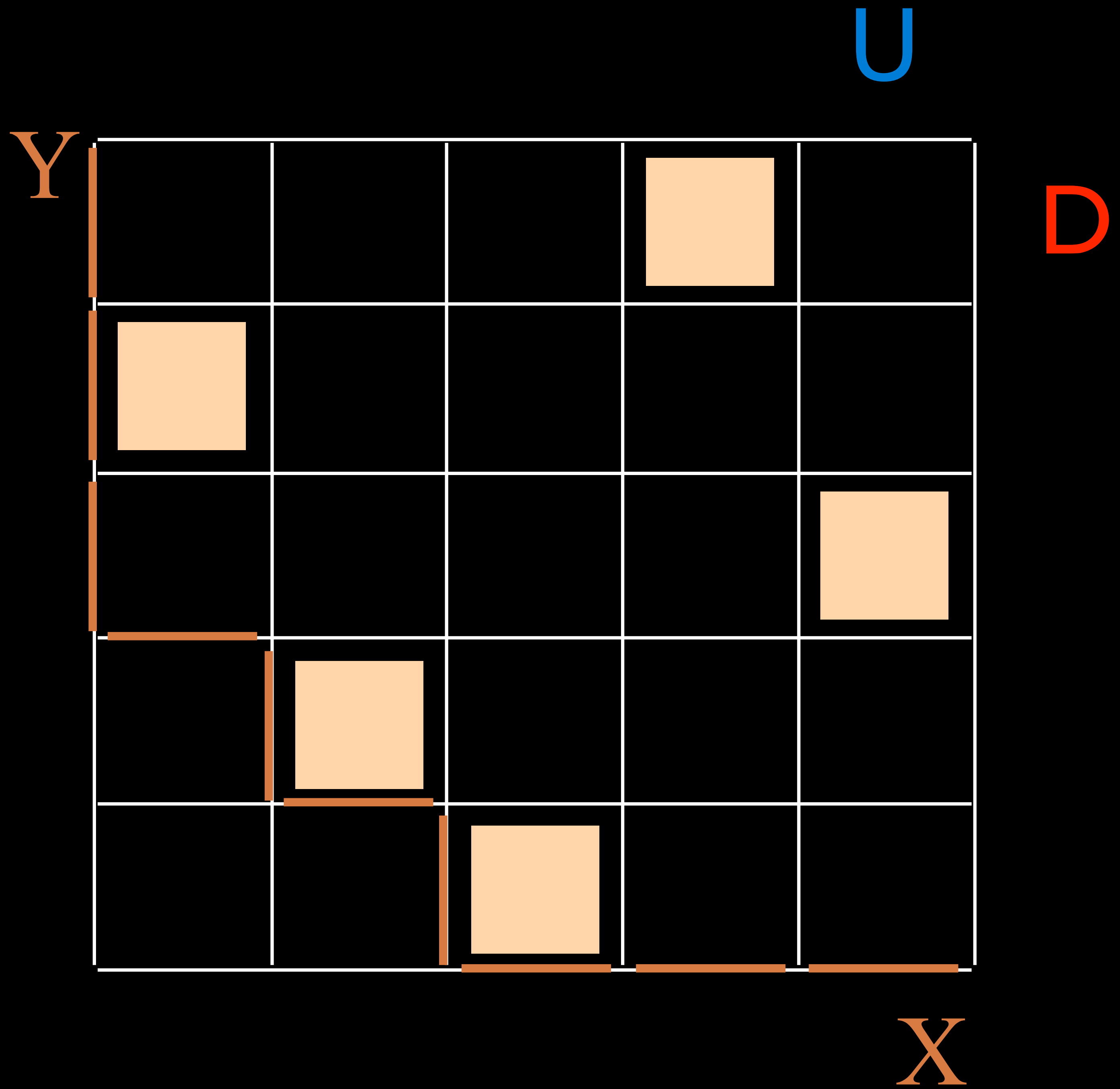
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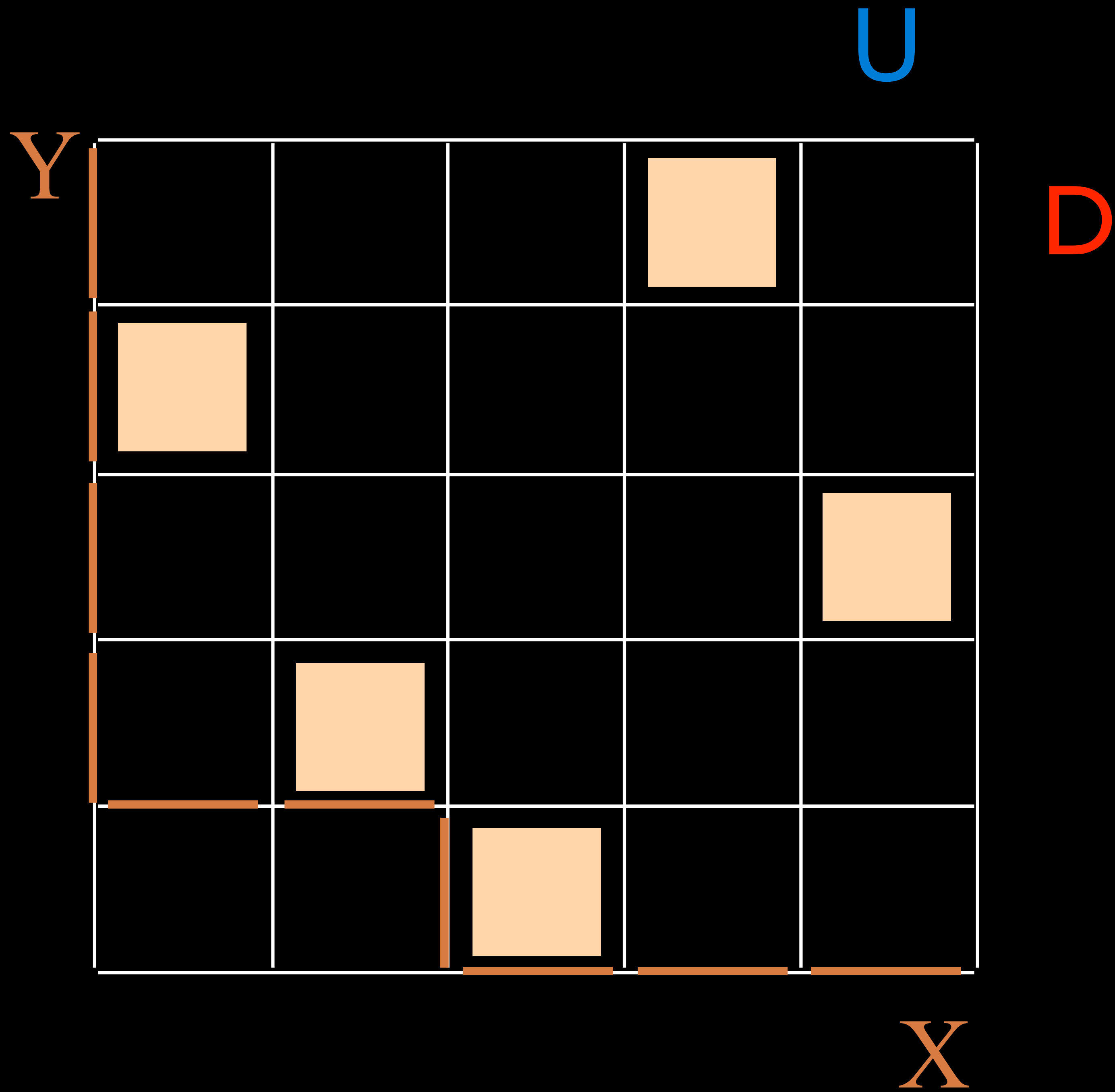

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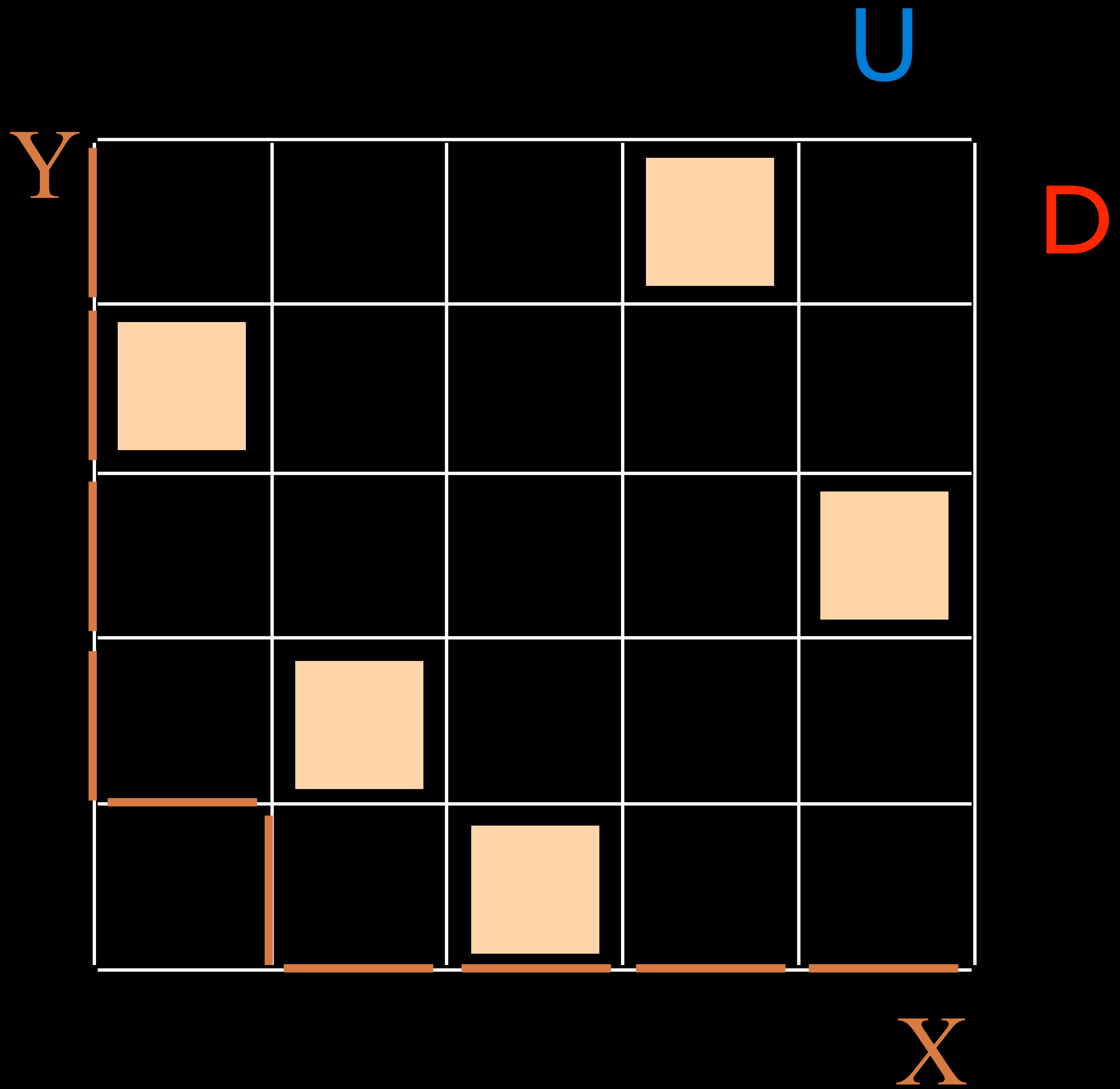
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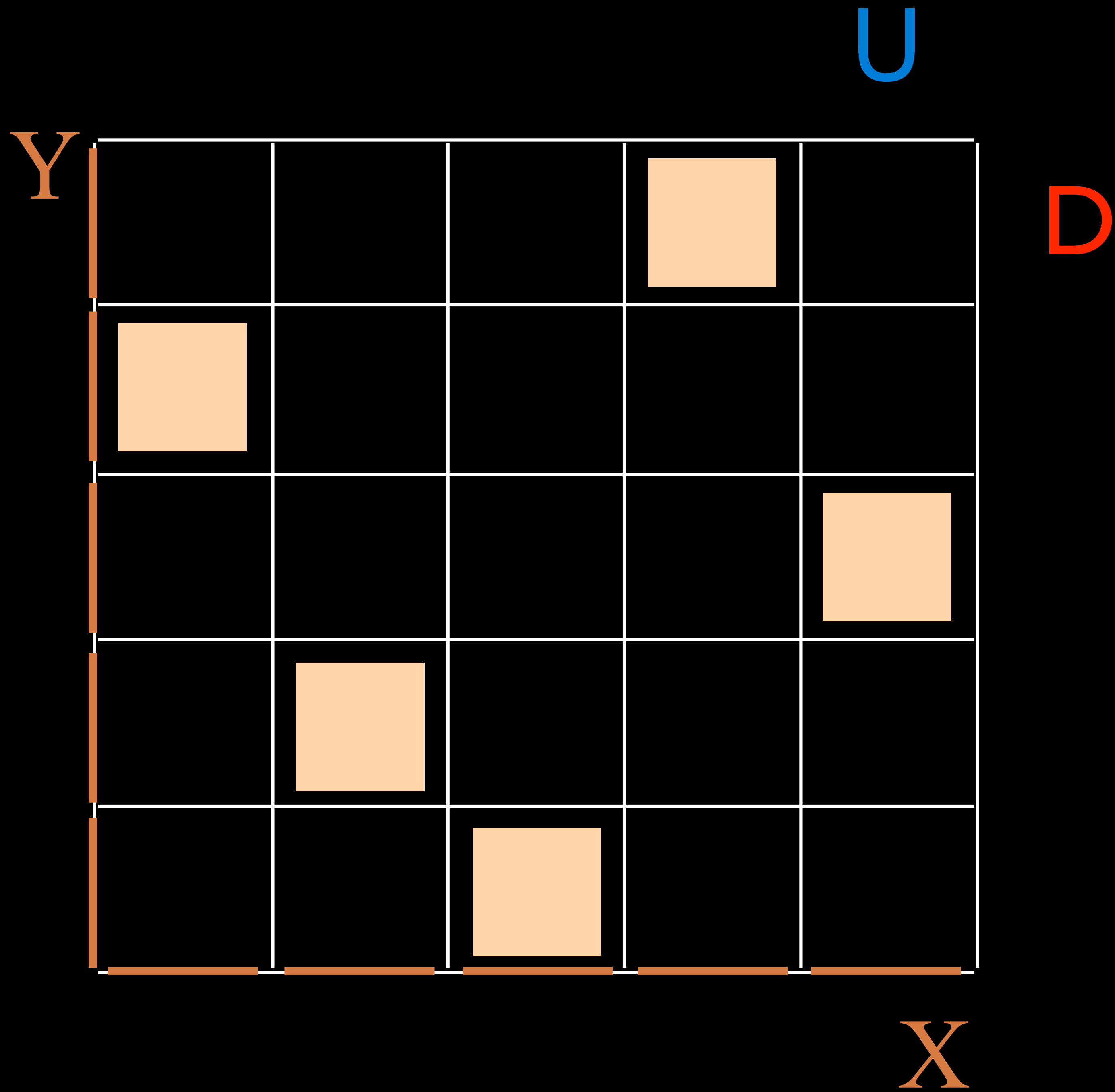
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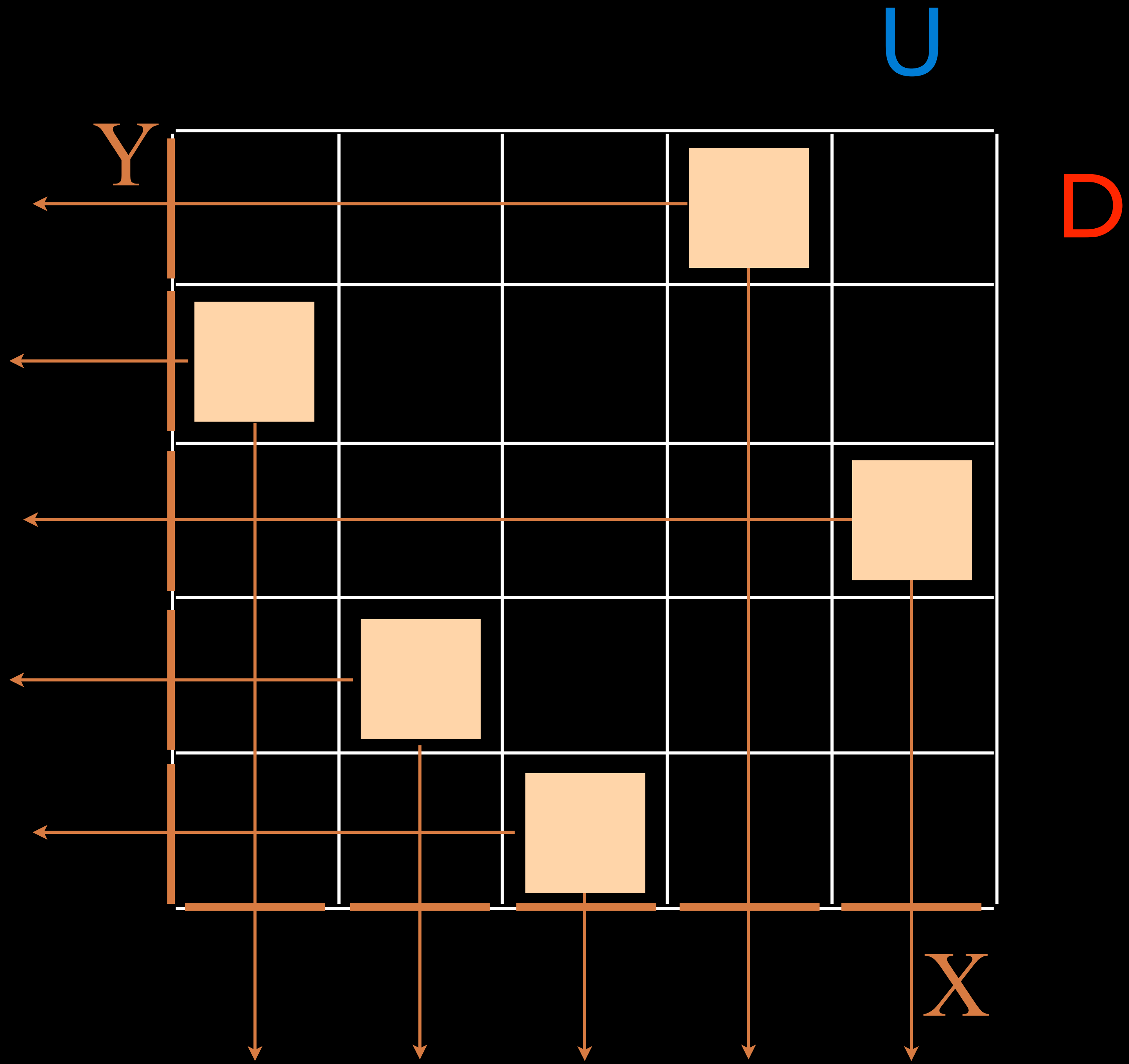
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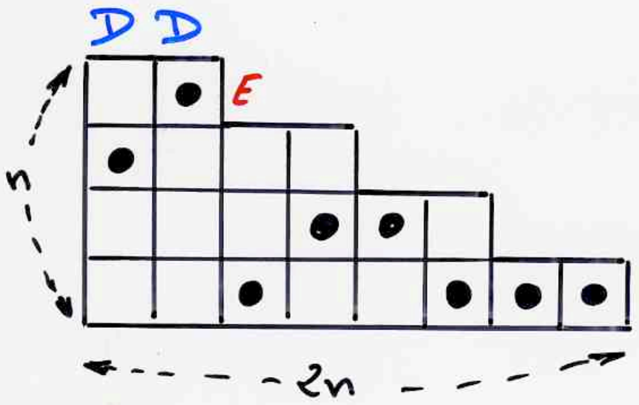
Planar automata

example 2: Genocchi numbers

example surjective pistol



Genocchi numbers G_{2n+2}

nombres de Genocchi

$$G_{2n} = 2(2^{2n}-1)\,B_{2n}$$

Bernoulli

$$2^{2n} G_{2n+2} = (n+1)\,T_{2n+1}$$

Angelo Genocchi
1817 - 1889

Hinc igitur calculo inſtituto reperi

$$
\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= 3 \\
D &= 17 \\
E &= 155 = 5.31 \\
F &= 2073 = 691.3 \\
G &= 38227 = 7.5461 = 7.\frac{127.129}{3}. \\
H &= 929569 = 3617.257 \\
I &= 28820619 = 43867.9.73 \quad \&c.
\end{aligned}
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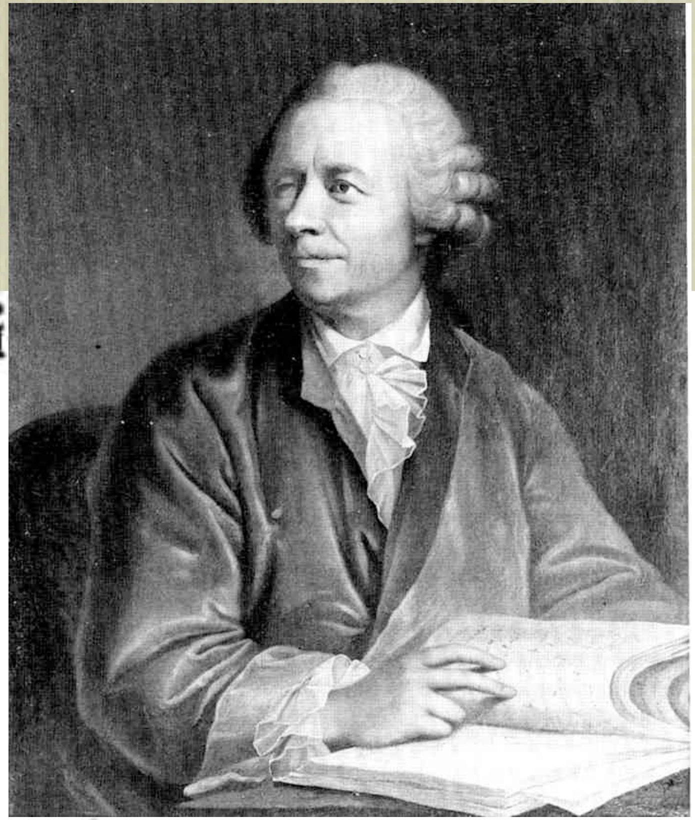


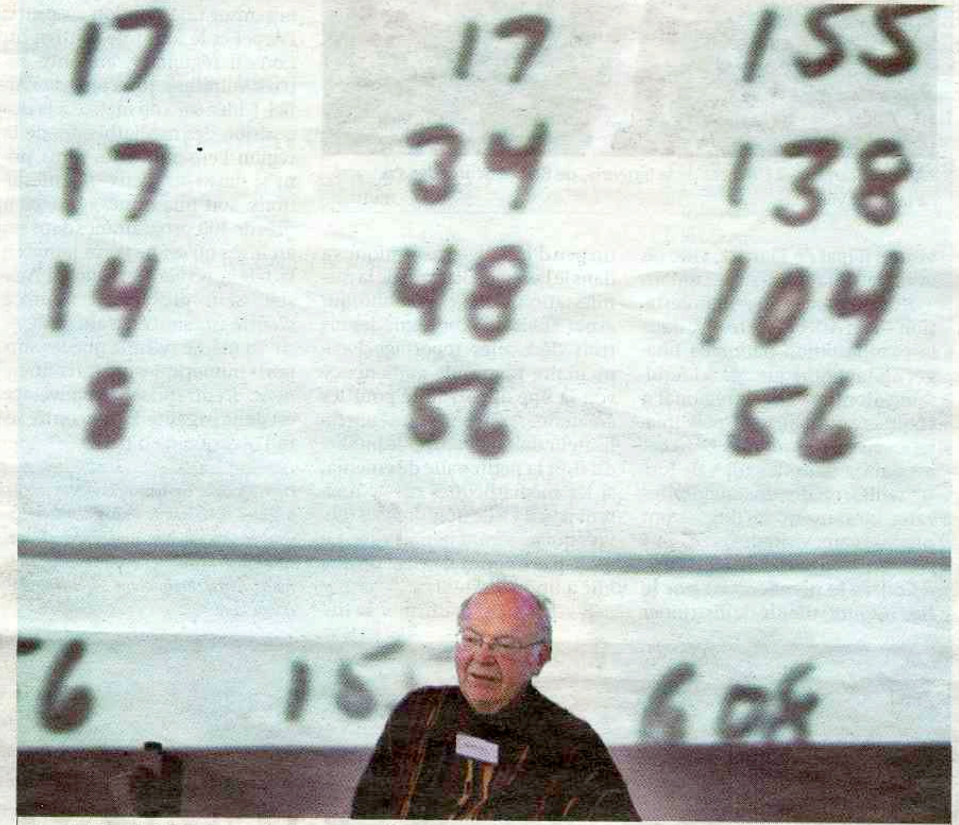
BORDEAUX 1. Le professeur Donald Knuth consacre sa vie à la programmation informatique, considérée comme un art. Il vient d'être sacré docteur honoris causa à Bordeaux

L'ermite de l'informatique

: Bernard Broustet

Une sommité de l'informatique mondiale a séjourné en Gironde ces derniers jours. Donald Knuth, 69 ans, a été sacré mardi docteur honoris causa de l'université Bordeaux 1, après avoir été lundi au centre d'une journée d'échanges qui réunissait une bonne partie du gratin français et européen de la recherche en informatique (1).

Depuis son premier contact, il y a un demi-siècle, avec un monumental et dinosaurien IBM 650, Donald Knuth n'a cessé d'être habité par la passion de l'informatique. Physicien, puis mathématicien de formation, ce géant affable et modeste a voué sa vie à ce qu'il appelle « l'art de la programmation informatique ». Car, à ses yeux, plus qu'une technique, c'est une forme d'activité qui requiert à la fois rigueur, intuition et sens esthétique. Les programmes informatiques réussis ont une sorte de beauté à laquelle même les non-spécialistes peuvent être sensibles.



Donald Knuth, à Bordeaux, le 29 octobre. À 69 ans, il animait une journée d'échanges avec le gratin européen de la recherche en informatique PHOTO LAURENT THEILLET

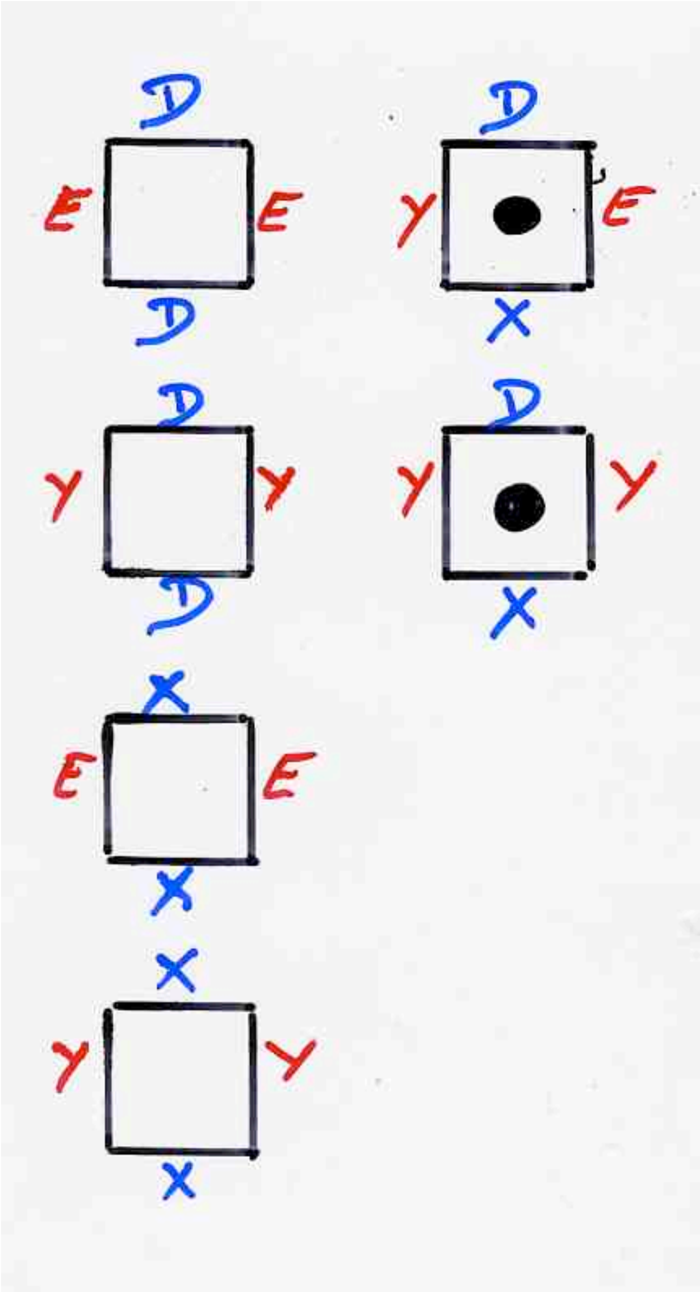
Une encyclopédie. Au long de sa carrière académique (pour l'essentiel à l'université californienne de Stanford), Donald Knuth a fait preuve d'une grande fécondité, en jouant notamment un rôle essentiel dans le développement de langages toujours utilisés par la communauté des mathématiciens. Mais, à 55 ans, le professeur Knuth a décidé de prendre sa retraite de Stanford. Il trouve que les fonctions administratives sont trop absorbantes pour lui permettre de mener à bien l'œuvre entamée à la fin des années 60 sous le titre de « Art of computer programming », sorte d'encyclopédie de l'algorithmique et de la programmation informatique. Donald Knuth a publié, il y a quelque temps déjà, les trois premiers volumes de cette gigantesque somme, traduite en russe, en japonais, en polonais, etc. mais pas en français. Le quatrième tome est pour bientôt. Et Donald Knuth se dit décidé à poursuivre sa tâche tant qu'il en aura la force. Ses ouvrages, dont les ventes cumulées au fil des ans approchent le million d'exemplaires, visent essentiellement les informaticiens et créateurs de programmes. Une communauté certes minoritaire à travers le monde, mais qui se trouve investie d'une mission considérable. En quelques décennies, l'écriture informatique a aidé à résoudre d'innombrables problèmes. « Mais il y en a tant d'autres qui attendent des solutions, notamment dans le domaine médical », affirme le professeur émérite de Stanford.

Un chèque de 2,56 dollars. Pour mener à bien sa tâche, Donald Knuth s'est imposé une vie d'ermite. D'ordinaire, sa journée débute par la bibliothèque ou la piscine. Après quoi, il passe tout le reste de son temps à sa table de travail, dimanche compris. Il n'a plus d'e-mail depuis le début des années 90, considérant que le courrier électronique représente une perte de temps, dès lors qu'on veut aller au fond des choses et non pas rester à leur surface. Une secrétaire lui fait passer les messages considérés comme les plus urgents. Pour le reste, Donald Knuth demande qu'on lui écrive par courrier ordinaire ou par fax, dont il prend parfois connaissance avec des mois de retard. Il s'oblige, en revanche, à tenir aussi scrupuleusement que possible sa promesse d'envoyer un chèque de 2,56 dollars à tout lecteur ayant détecté une erreur dans un de ses livres. Par ailleurs, pour se détendre, il pratique l'orgue, appris dans sa prime jeunesse auprès de son père qui partagea sa vie entre la musique et l'enseignement.

L'orgue de Sainte-Croix. Donald Knuth n'est pas fermé aux choses de ce monde. Sur son site Internet, à la rubrique « Questions qui ne me sont pas fréquemment posées », il demande entre autres : « Pourquoi mon pays a-t-il le droit d'occuper l'Irak ? », « Pourquoi mon pays ne soutient-il pas une Cour internationale de justice ? » Mais cet homme de conscience ne se veut pas militant, pas plus qu'il n'aspire au vedettariat et à la richesse. « Beaucoup de gens, dit-il, ont tendance à considérer que l'informatique, c'est surtout des histoires de business, d'entreprise. Ce n'est pas mon cas. » Sortant de sa semi-réclusion, Donald Knuth s'est donc laissé convaincre d'accepter les hommages de l'université de Bordeaux, après celles de Harvard, d'Oxford, de Tübingen. Il a eu le coup de foudre pour la beauté et l'agrément de la ville. Et il n'oubliera sans doute pas de sitôt l'orgue illustre de l'église Sainte-Croix (2), sur lequel il a eu le bonheur d'exercer son talent.

(1) Ces journées étaient organisées par le Laboratoire bordelais de recherche en informatique (Labri).

(2) Thierry Semenoux, professeur d'orgue au conservatoire de Bordeaux, a joué dans ce domaine un rôle de cicérone auprès de Donald Knuth.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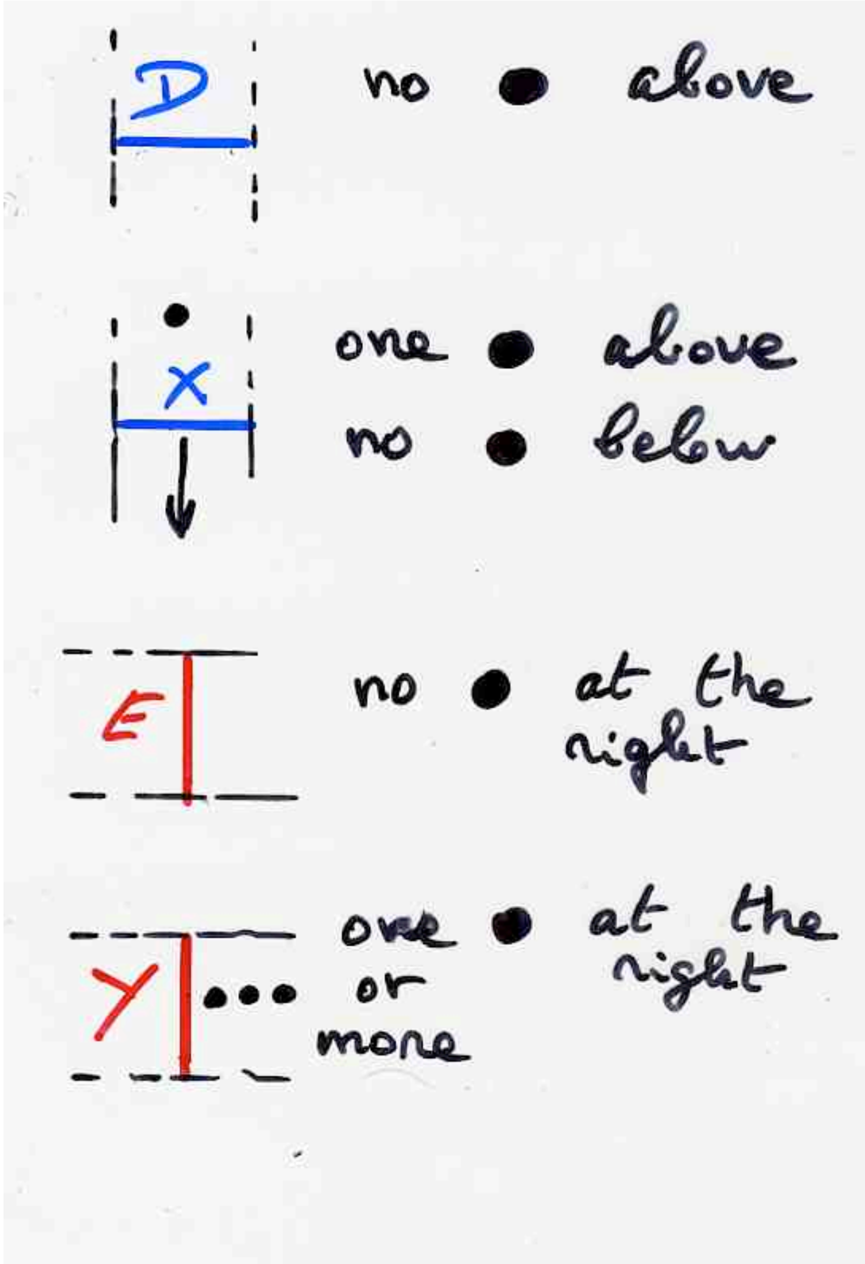
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no • above
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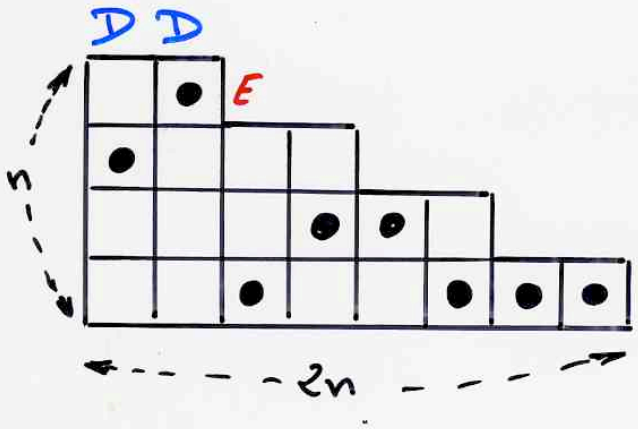
or more

example surjective pistol



Genocchi numbers G_{2n+2}

$$c\,(\,Y^n,\, X^{2n};\, (D^2E)^n\,) = G_{2n+2}$$

$u \qquad v \qquad\qquad w$

Planar automata

example 3:
alternating sign matrices (ASM)

Def - **ASM** alternating sign matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(i) entries: $0, 1, -1$

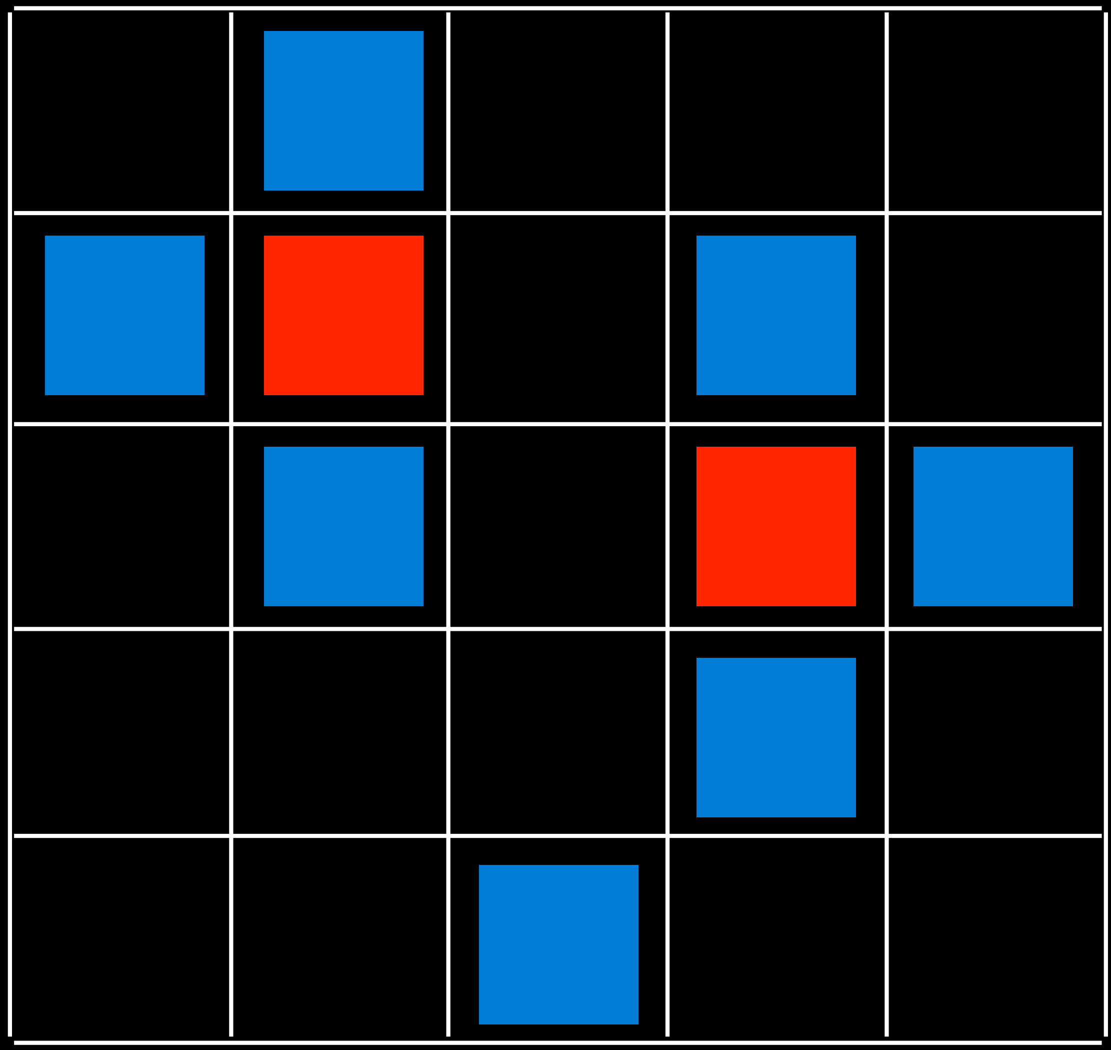
(ii) sum of entries in each $\left\{\begin{array}{l}\text{row}\\ \text{column}\end{array}\right. = 1$

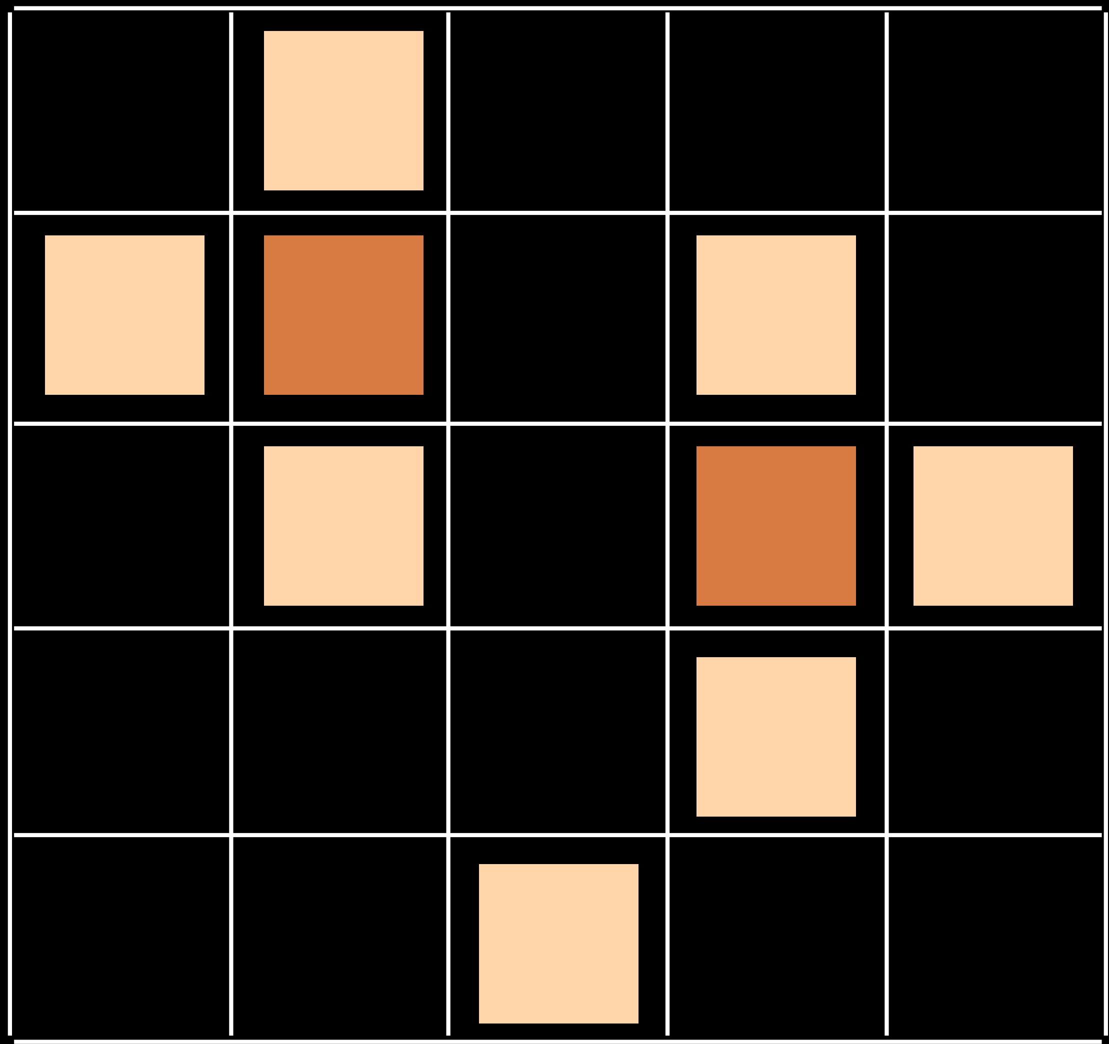
(iii) non-zero entries alternate in each $\left\{\begin{array}{l}\text{row}\\ \text{column}\end{array}\right.$

ASM

Alternating sign matrices

$$
\begin{matrix}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & -1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & 1 & -1 & 1 \\
\cdot & \cdot & 1 & -1 & 1 & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot
\end{matrix}
$$





Permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad + \underset{\text{permutations}}{6}$$

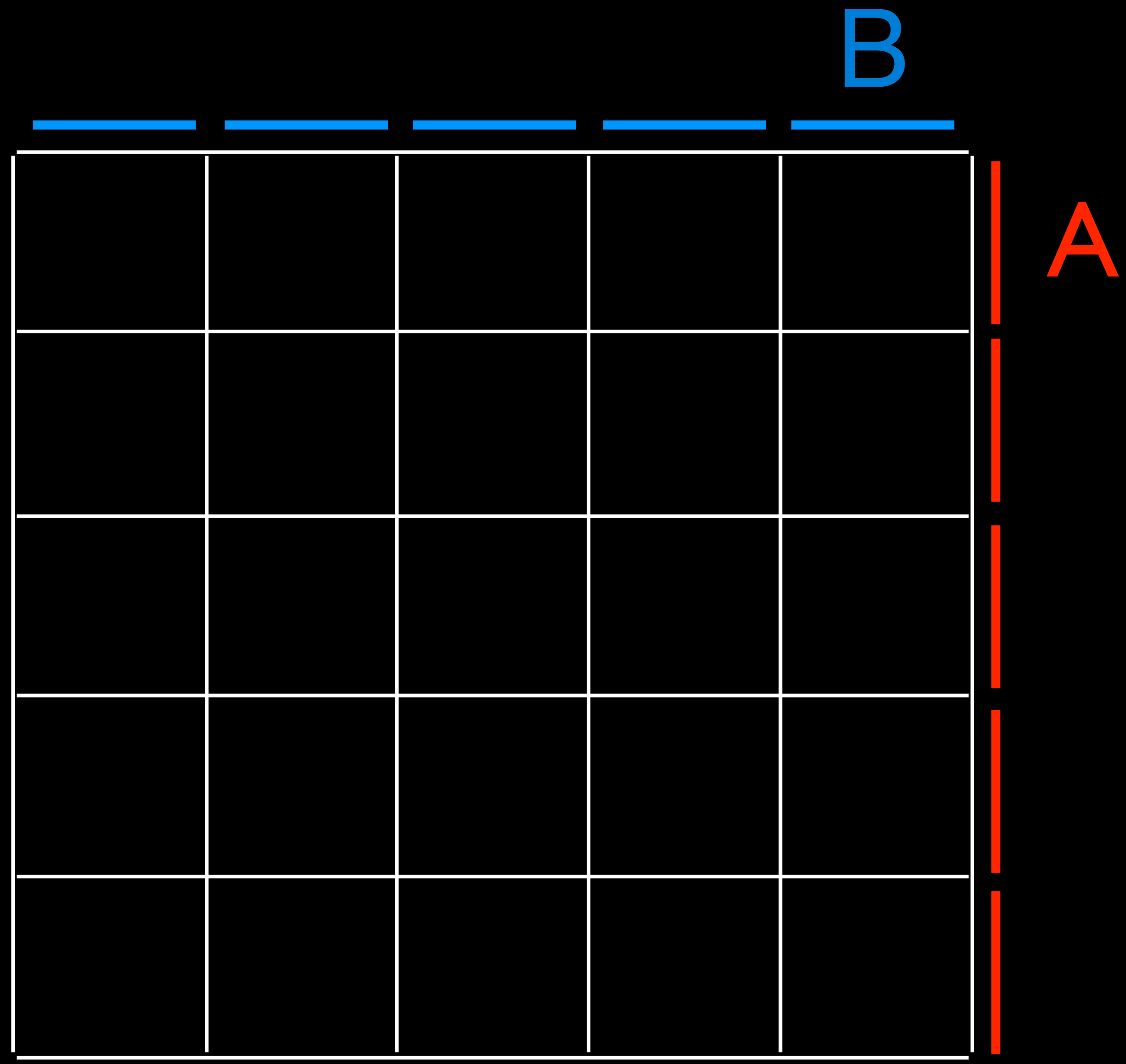
$1, 2, 7, 42, 429, \ldots.$

$$1,\ 2,\ 7,\ 42,\ 429,\ \ldots$$

$$\frac{1!\ 4!\ \cdots\ (3n-2)!}{n!\ (n+1)!\ \cdots\ (n+n-1)!}$$

alternating sign matrices (ex-) conjecture

Mills, Robbins, Rumsey (1982)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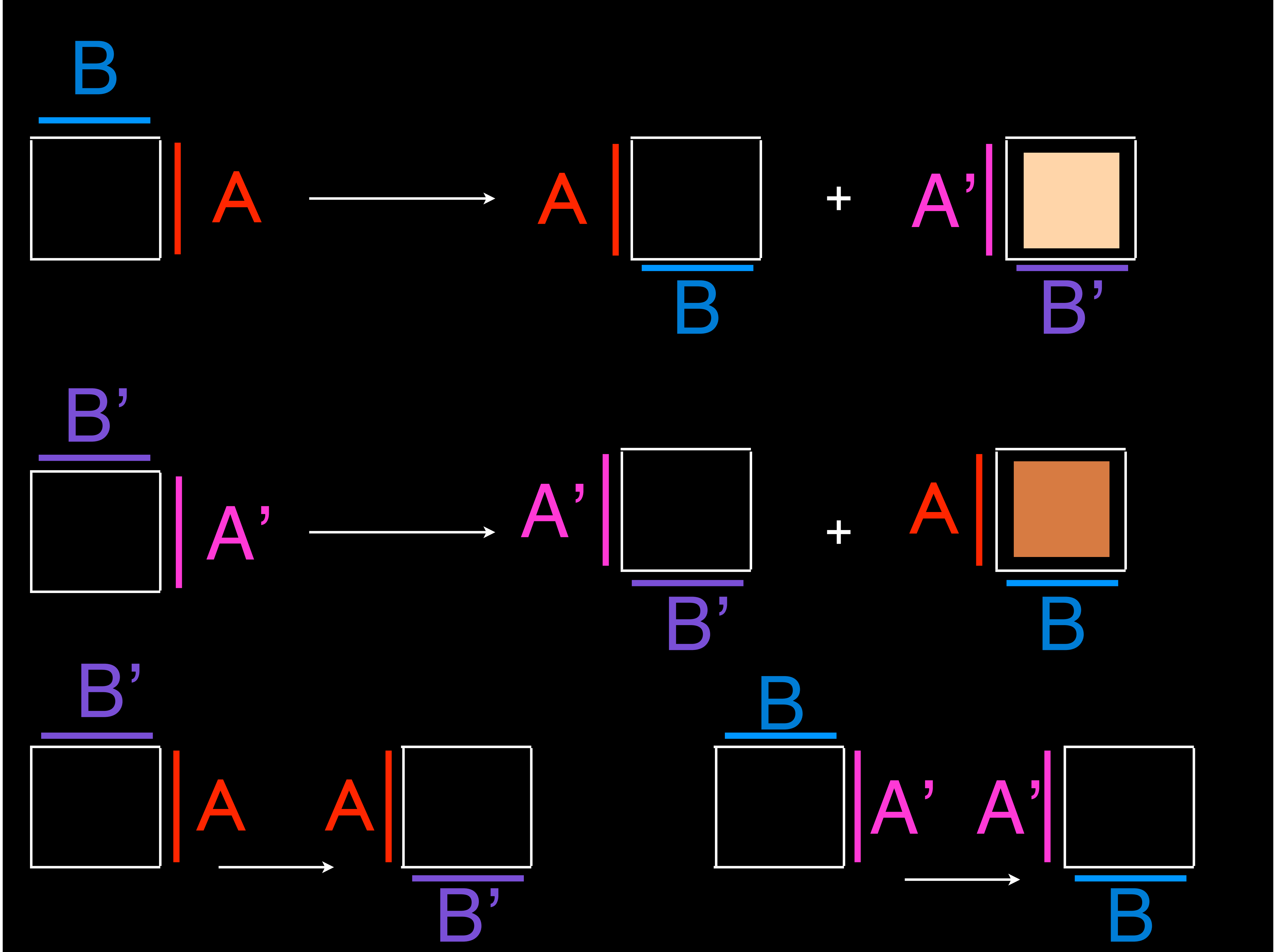
A



B
A
A
B
+
A'
B'
B'
A'
A'
B'
+
A
B
B'
A
A
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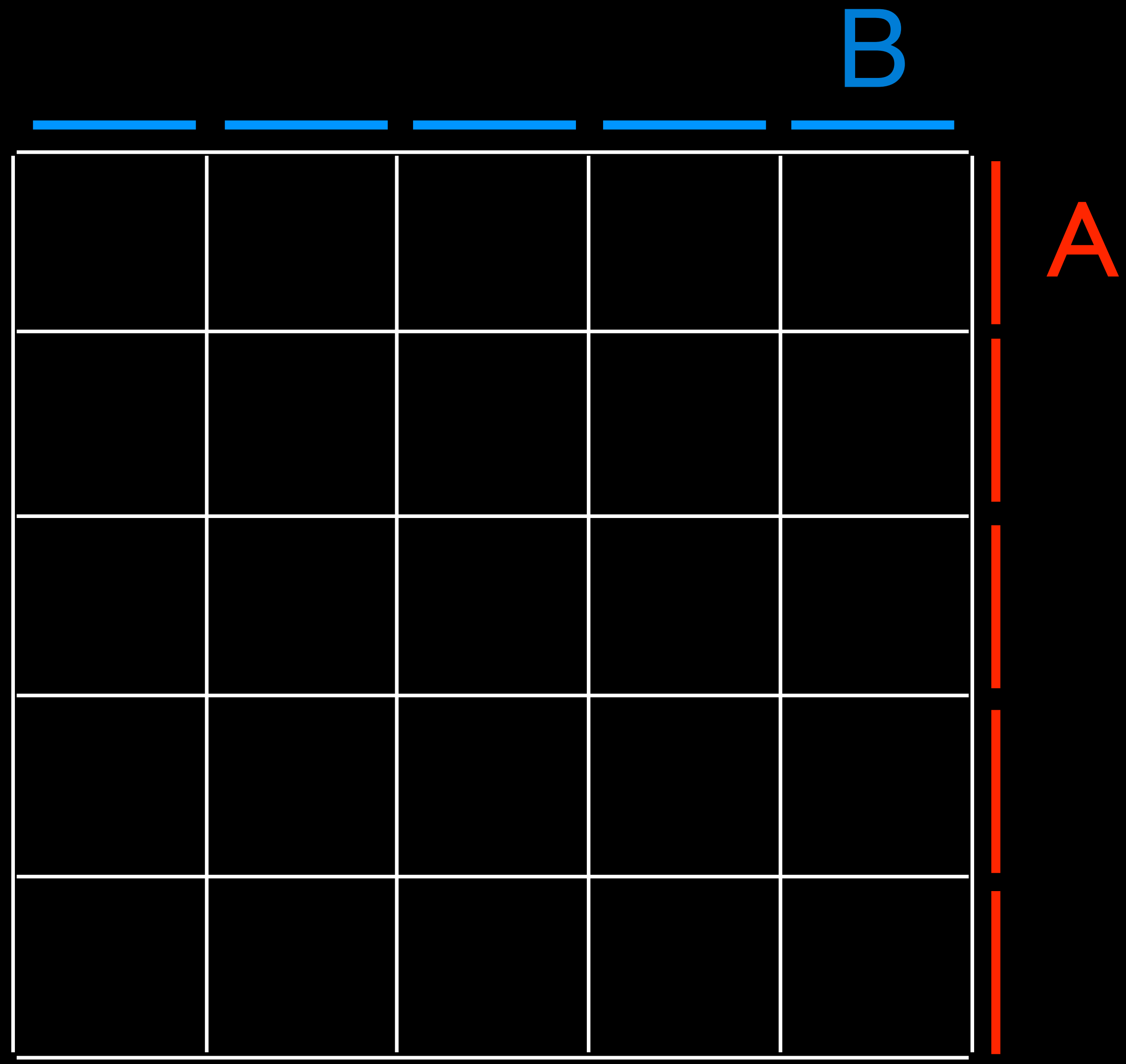
B
A'
A'
B


B
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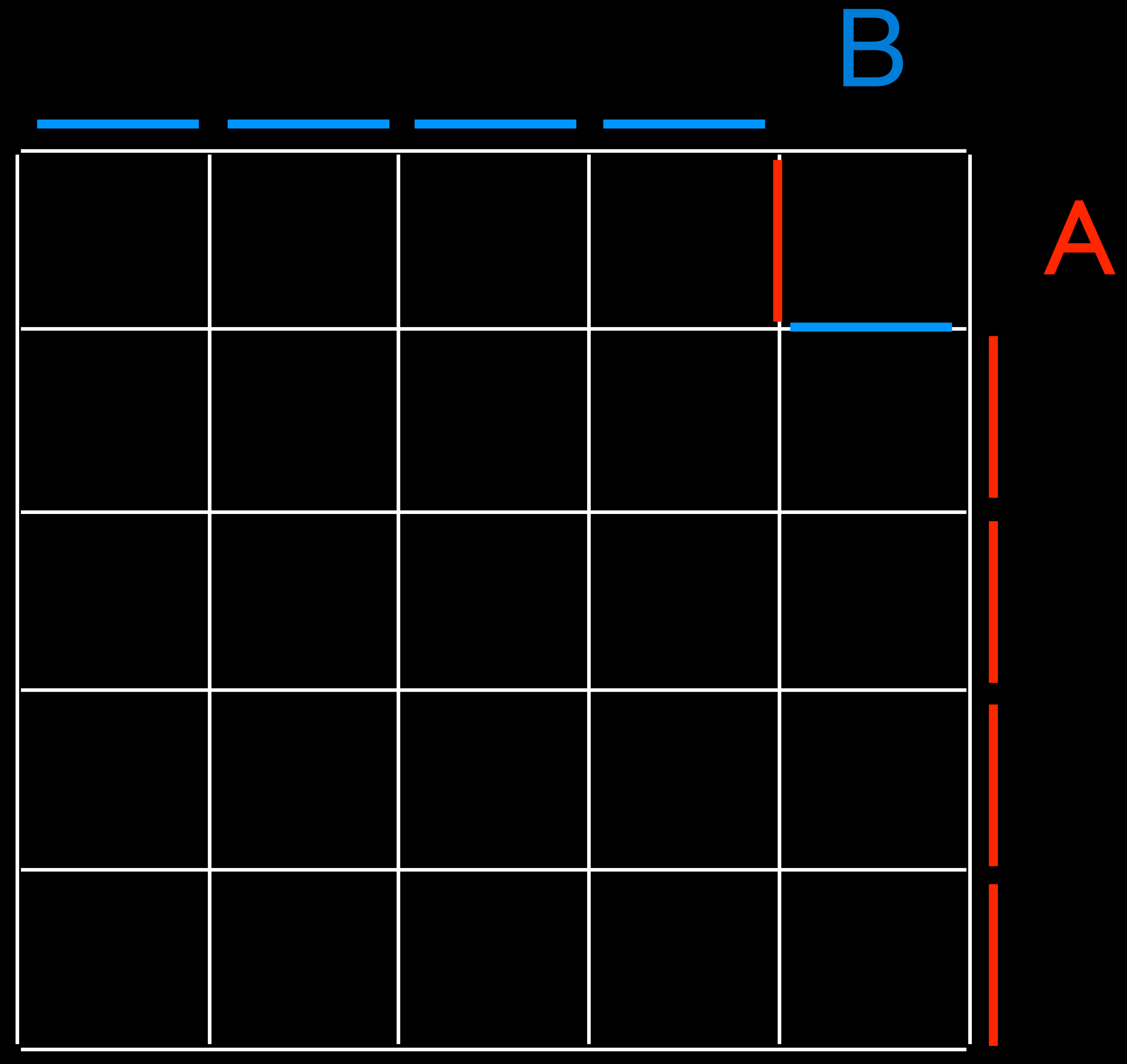
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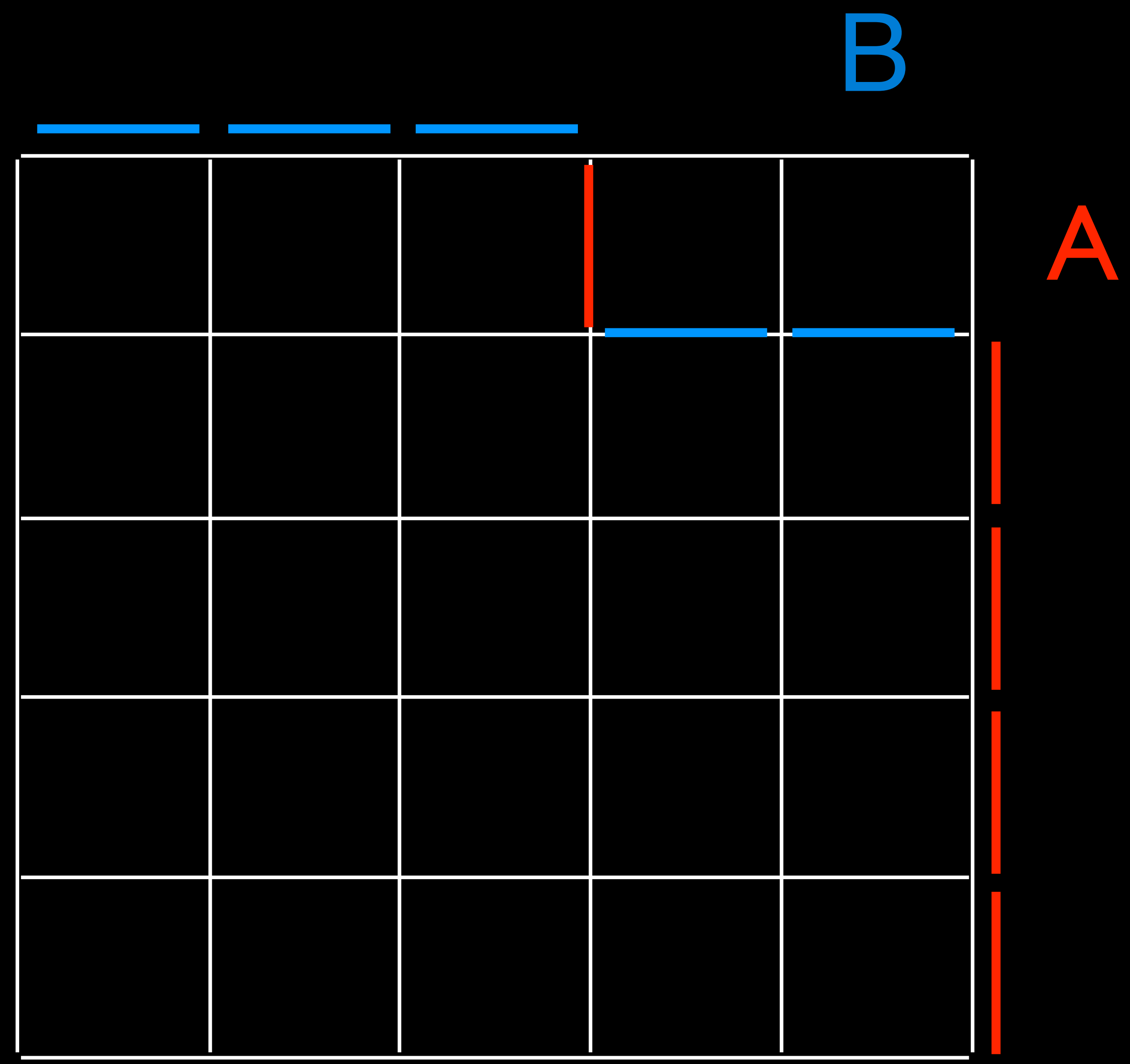
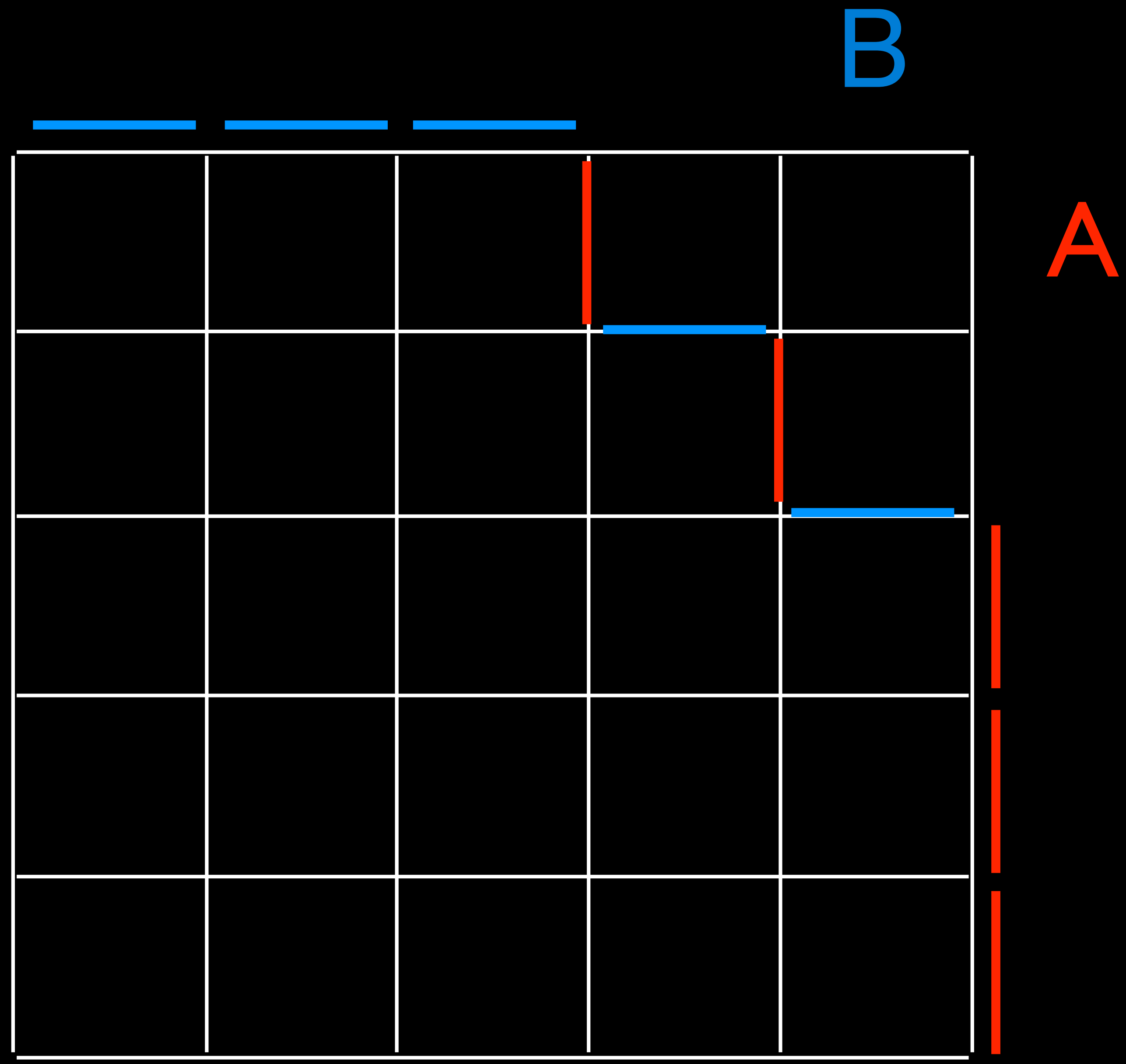

B
A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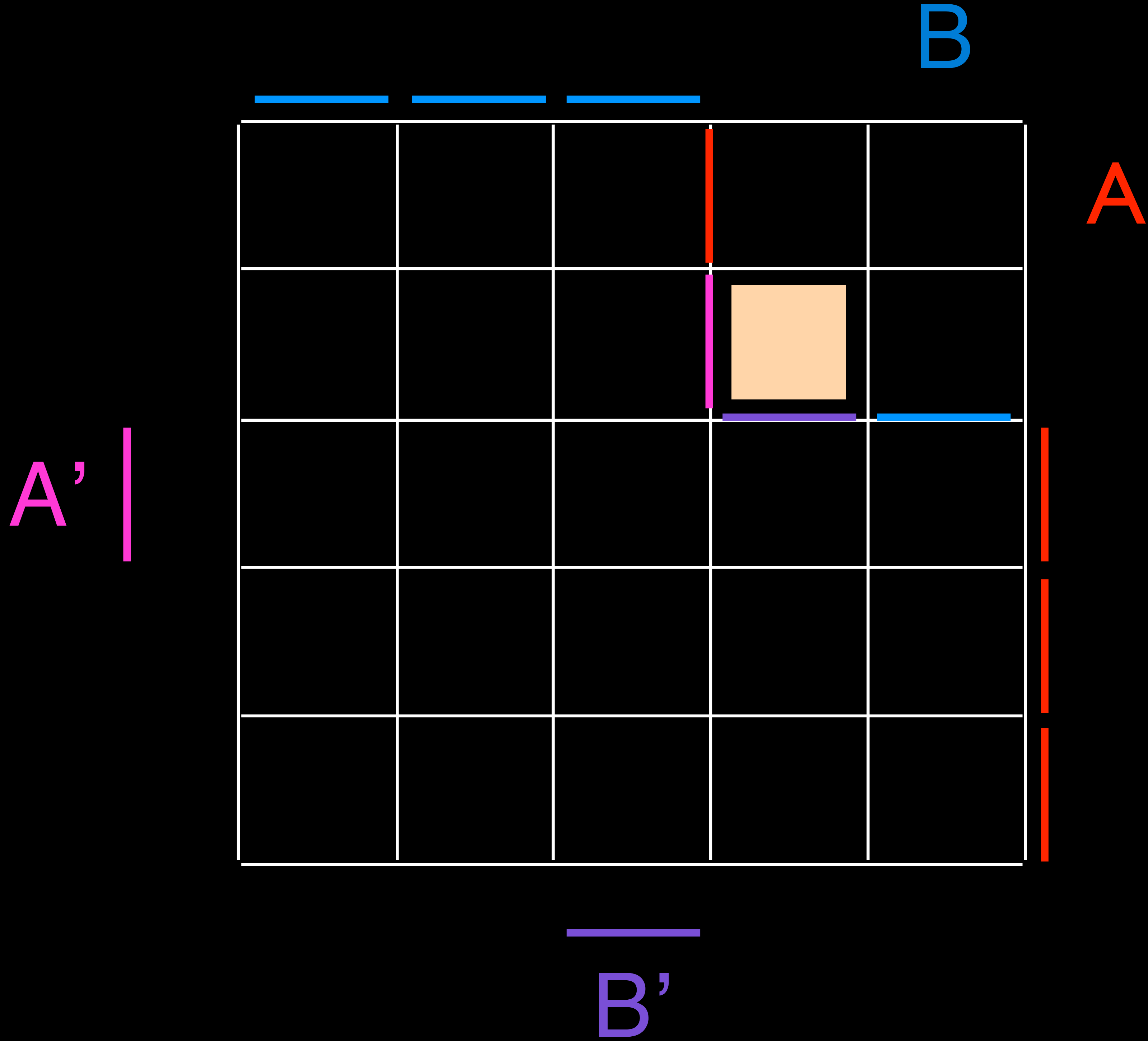
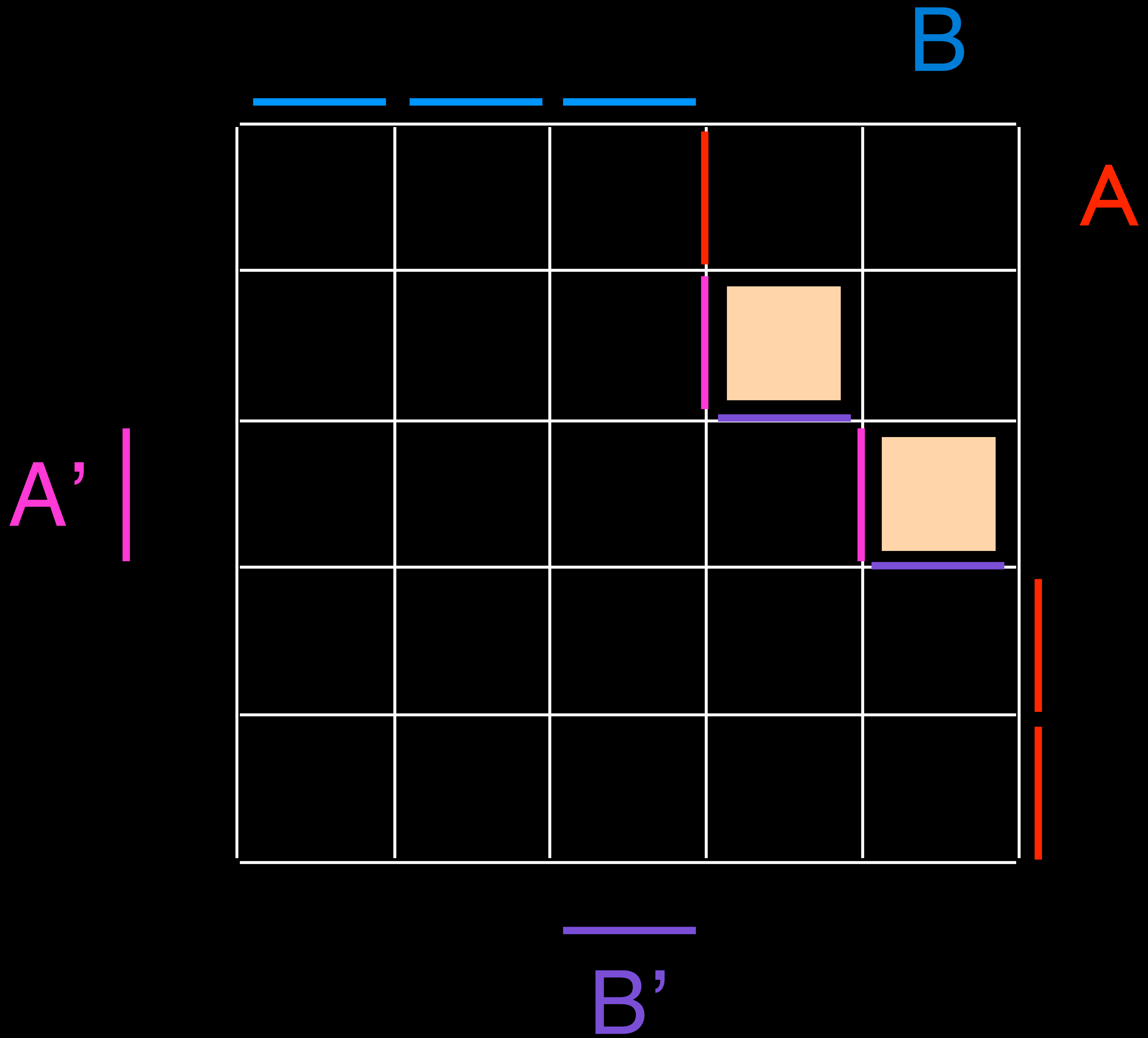

B
A
A'
B'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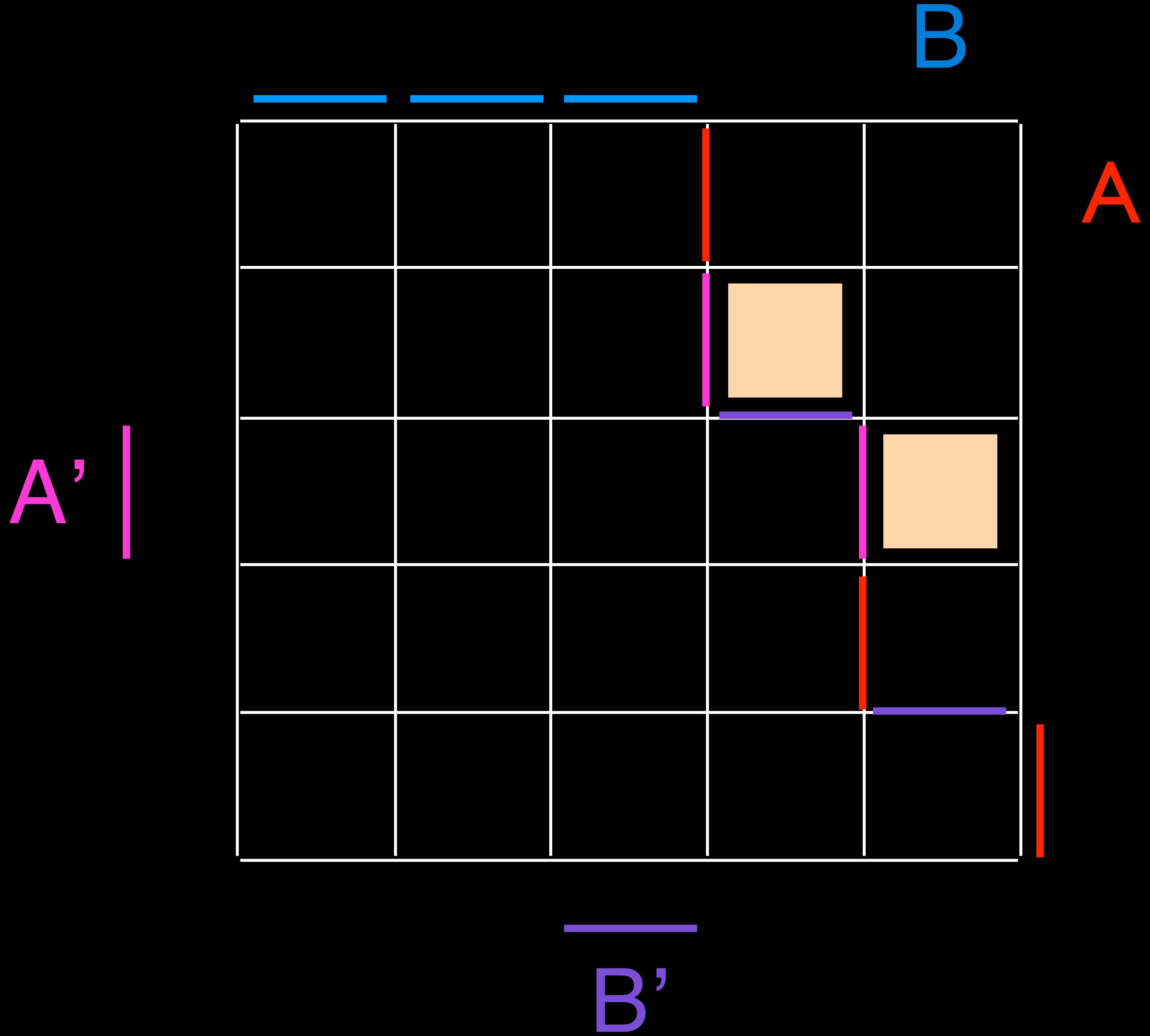
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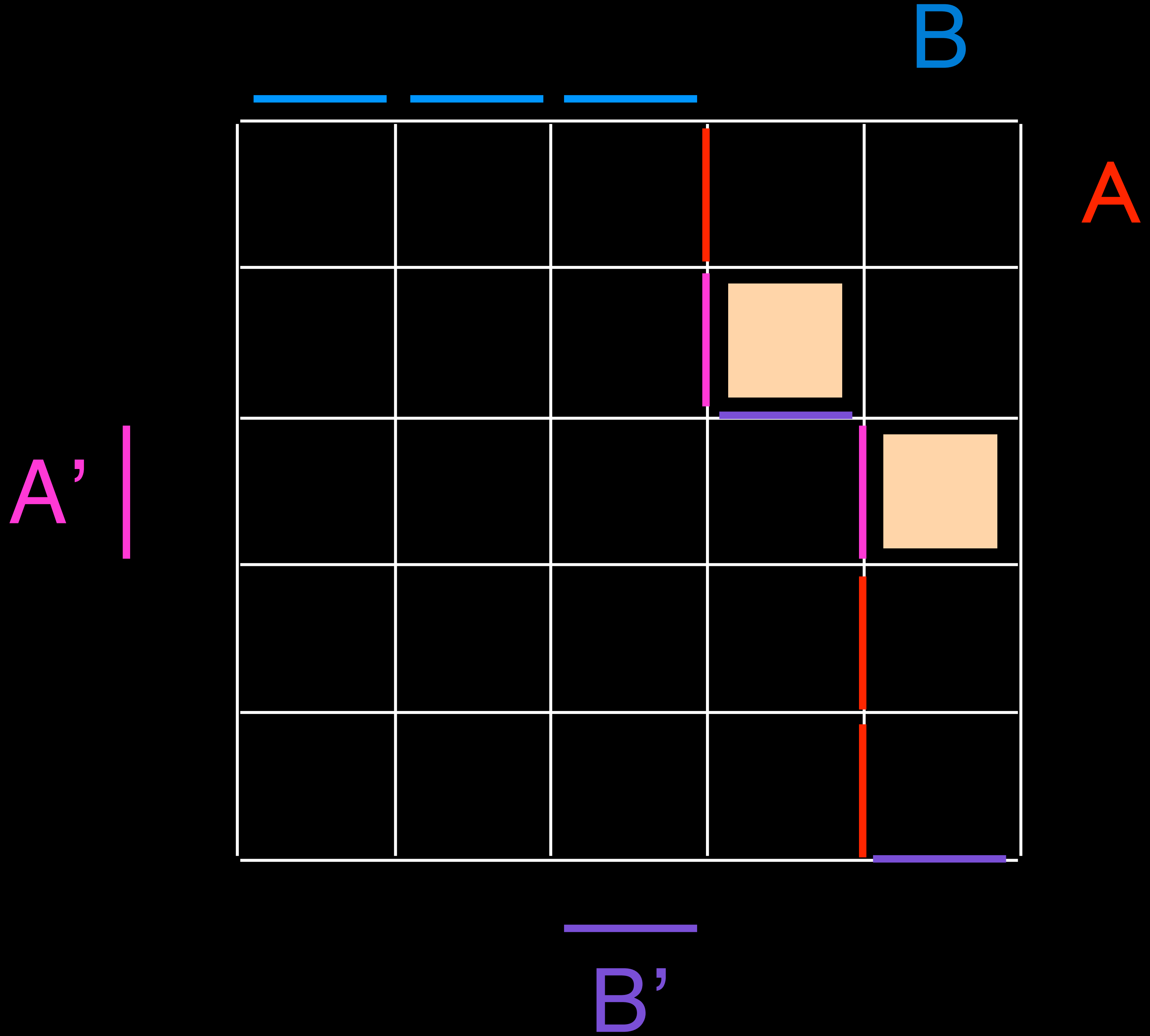
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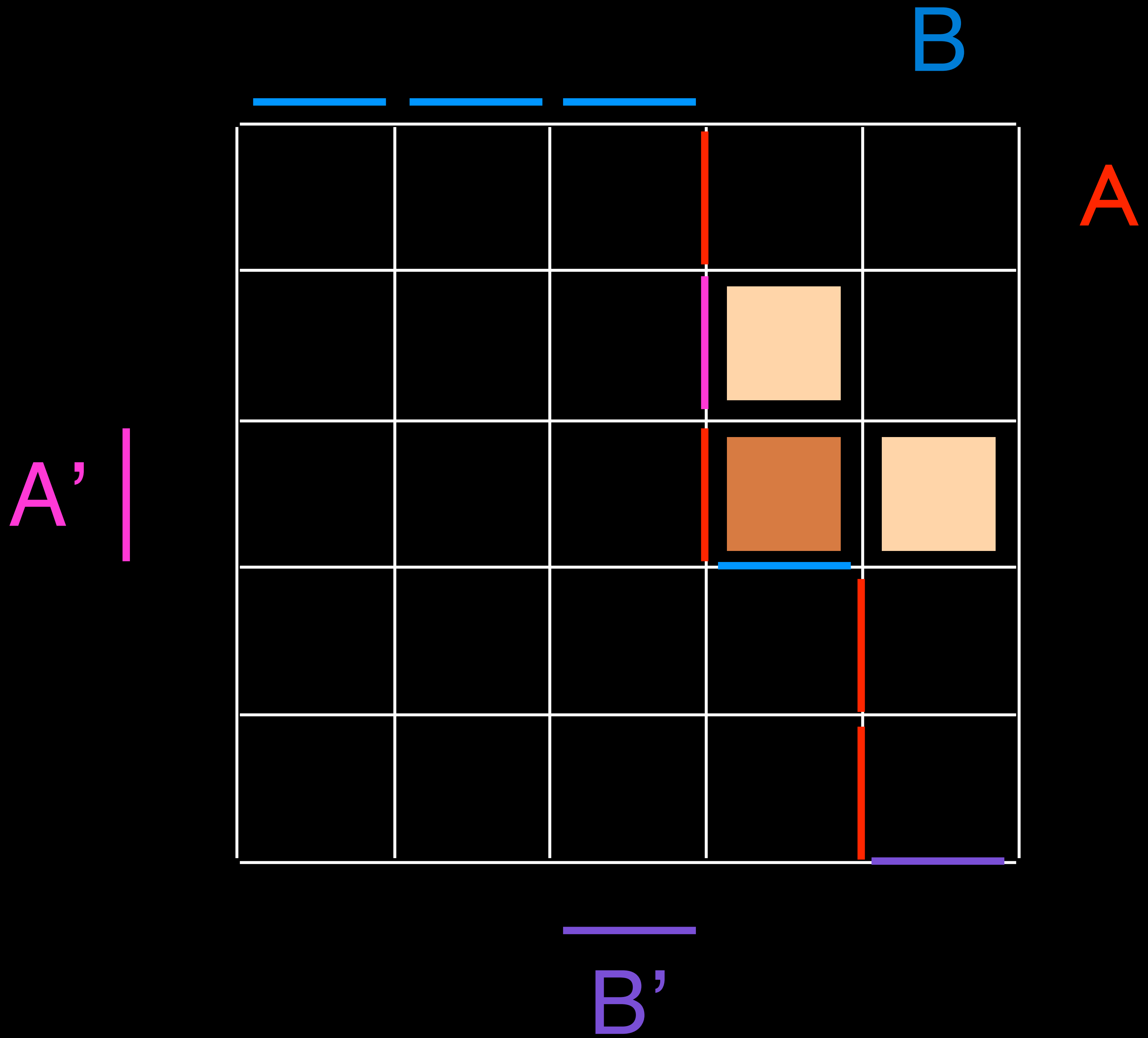
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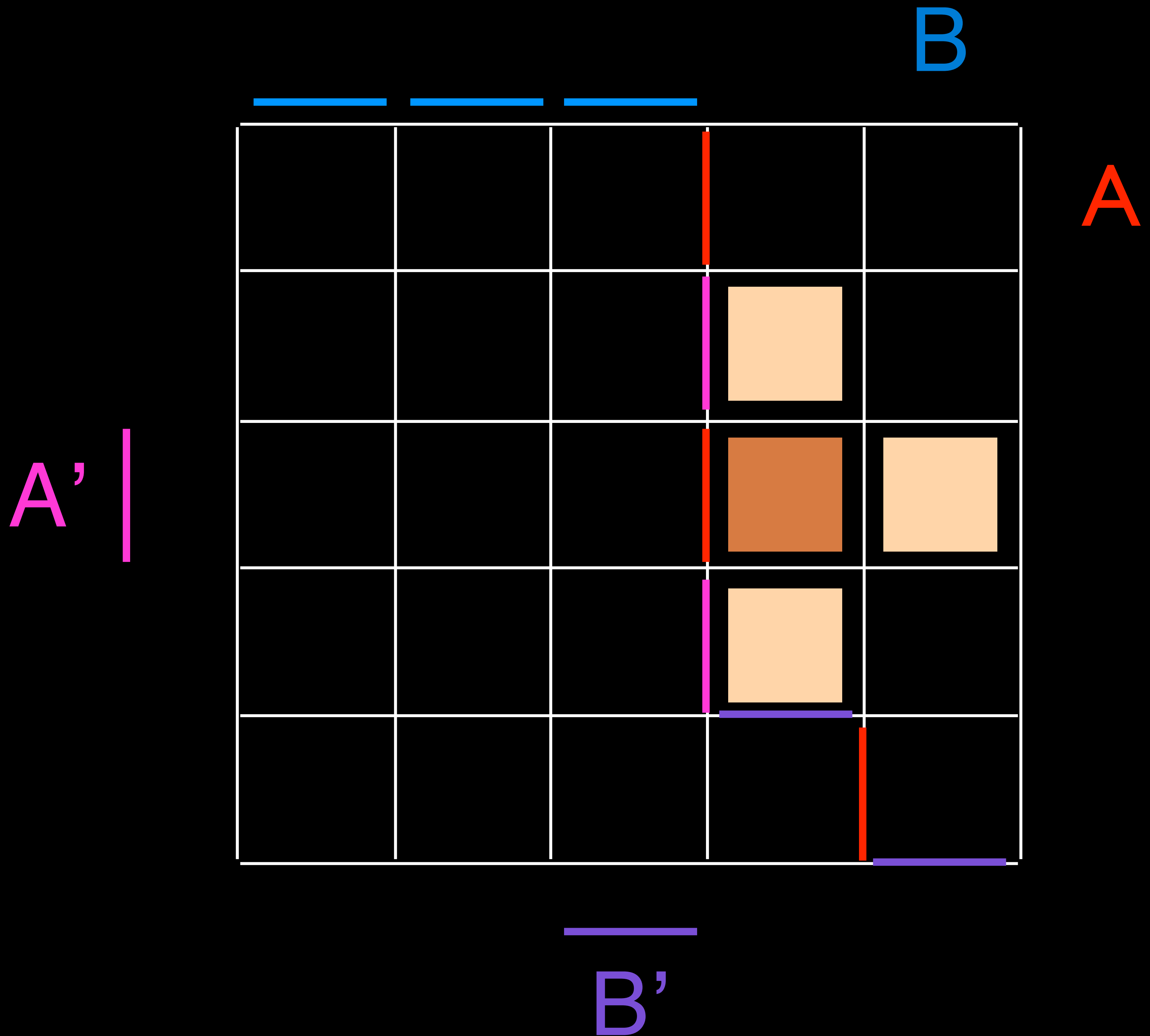
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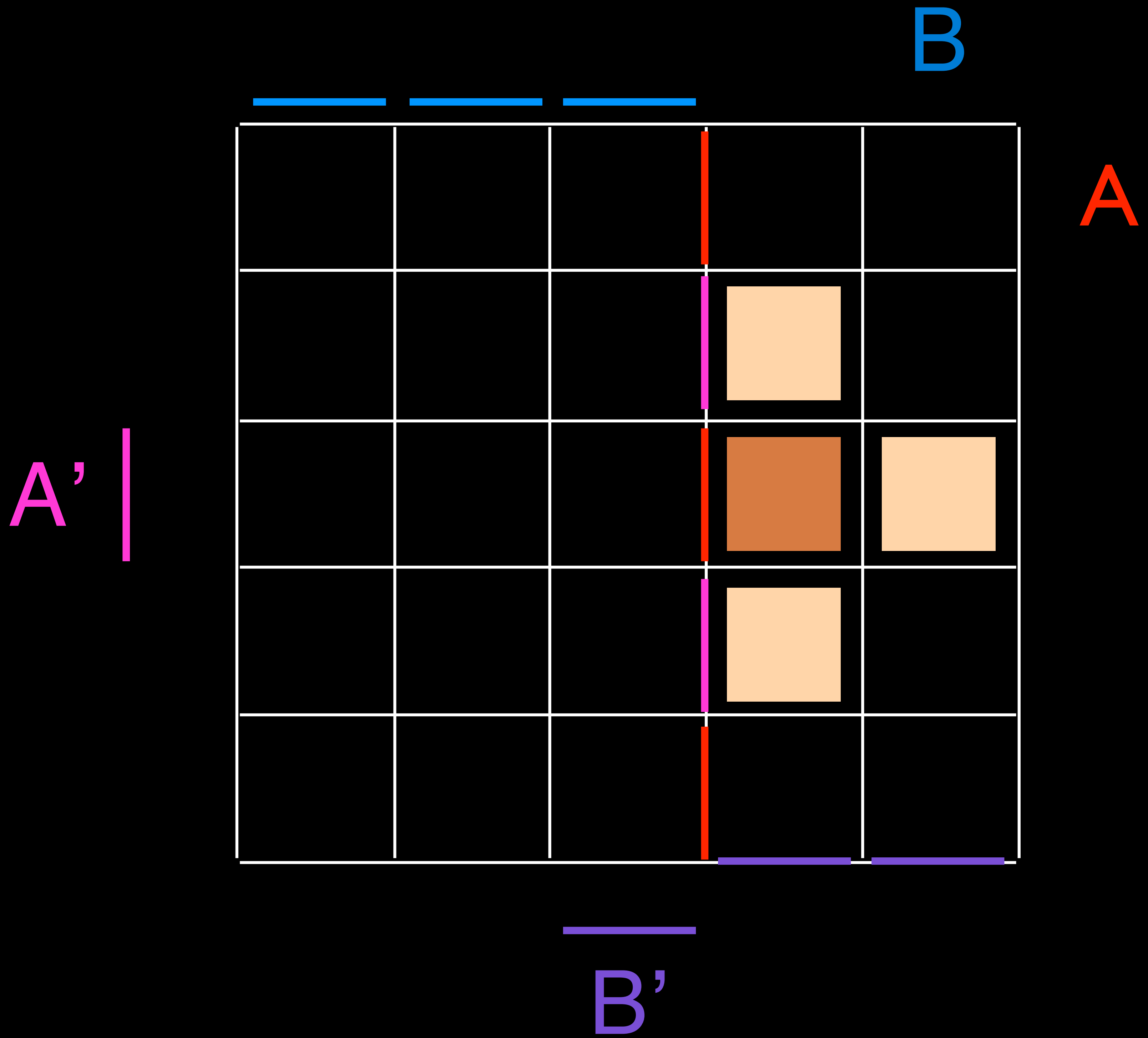

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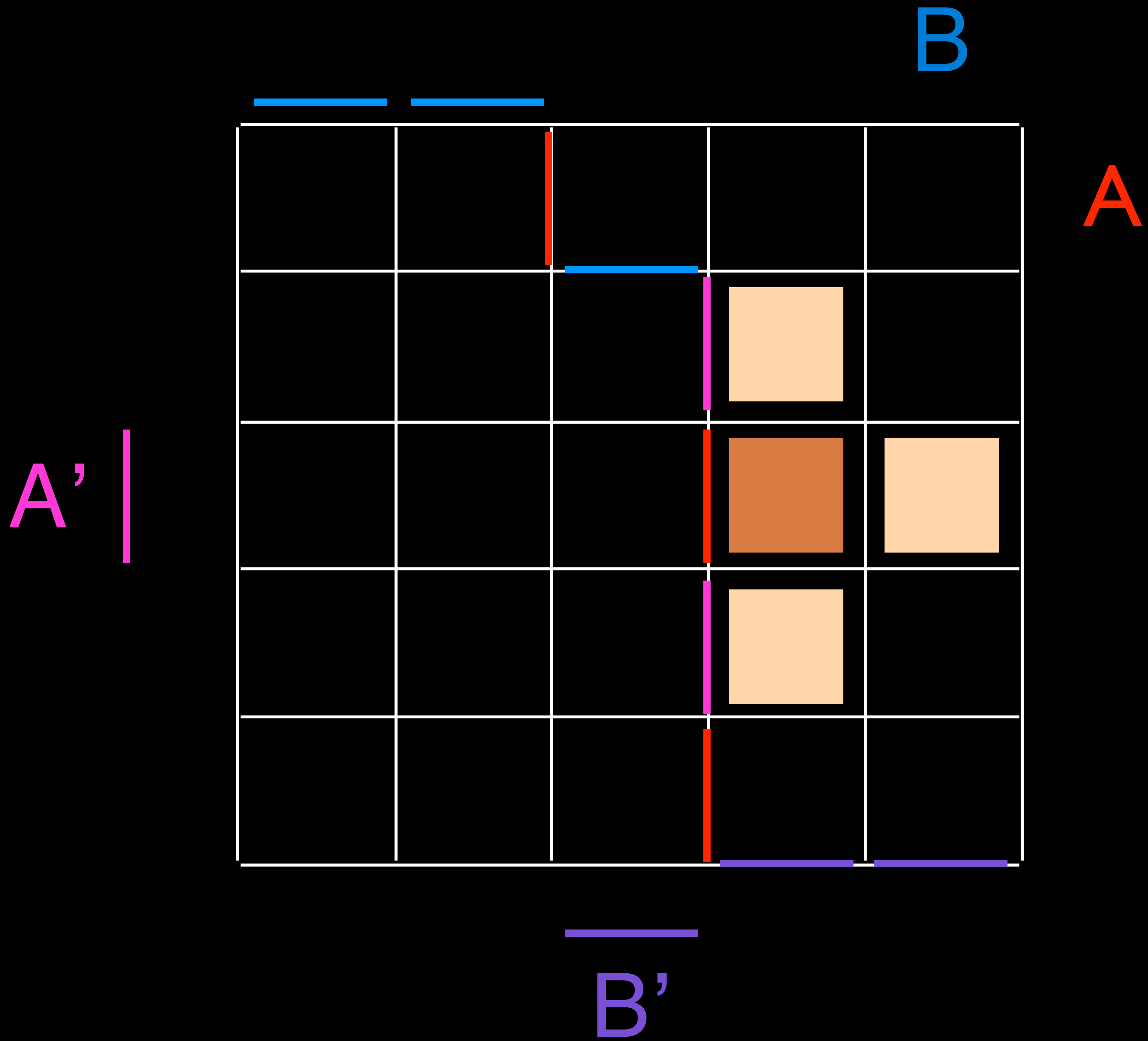

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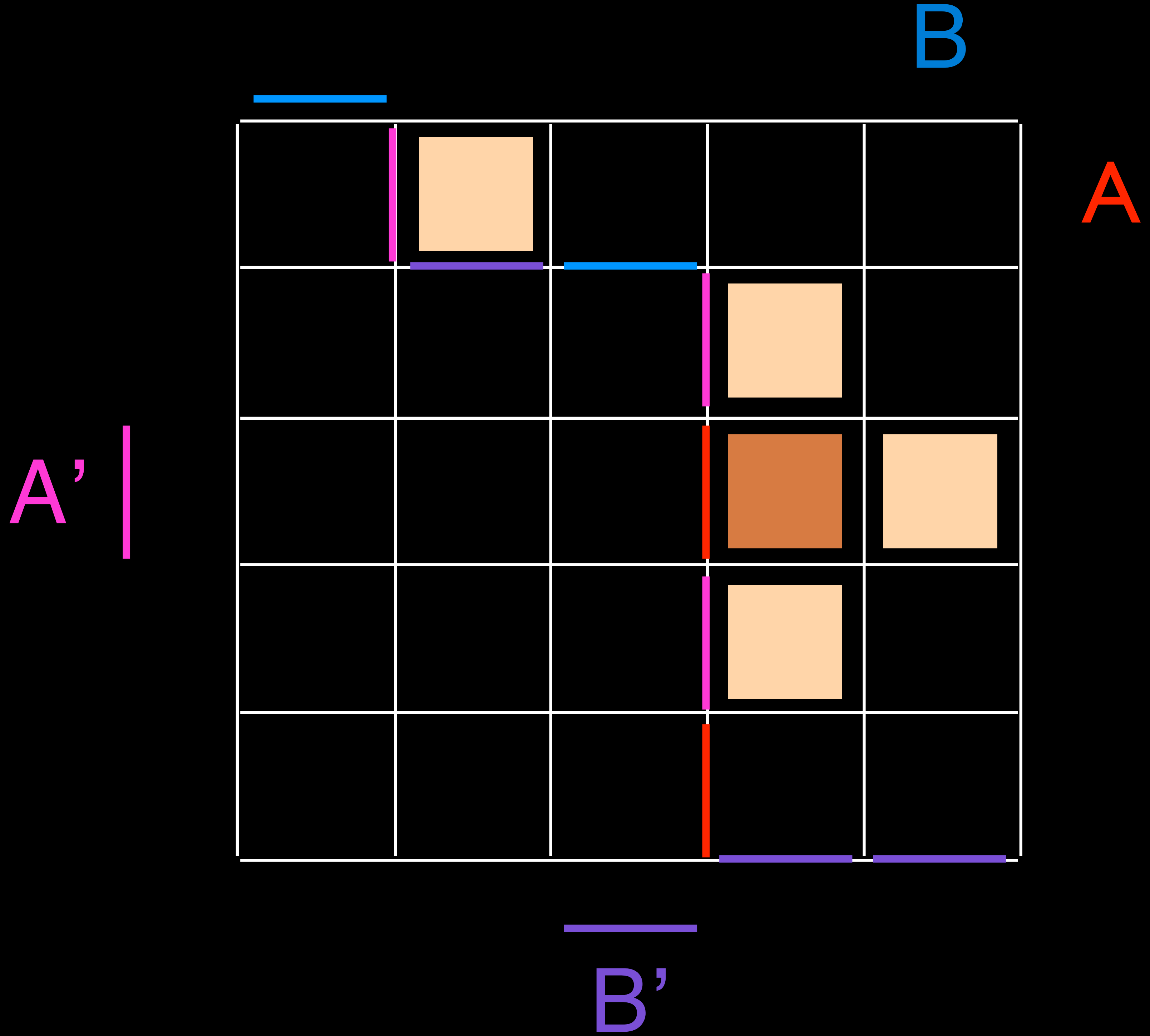
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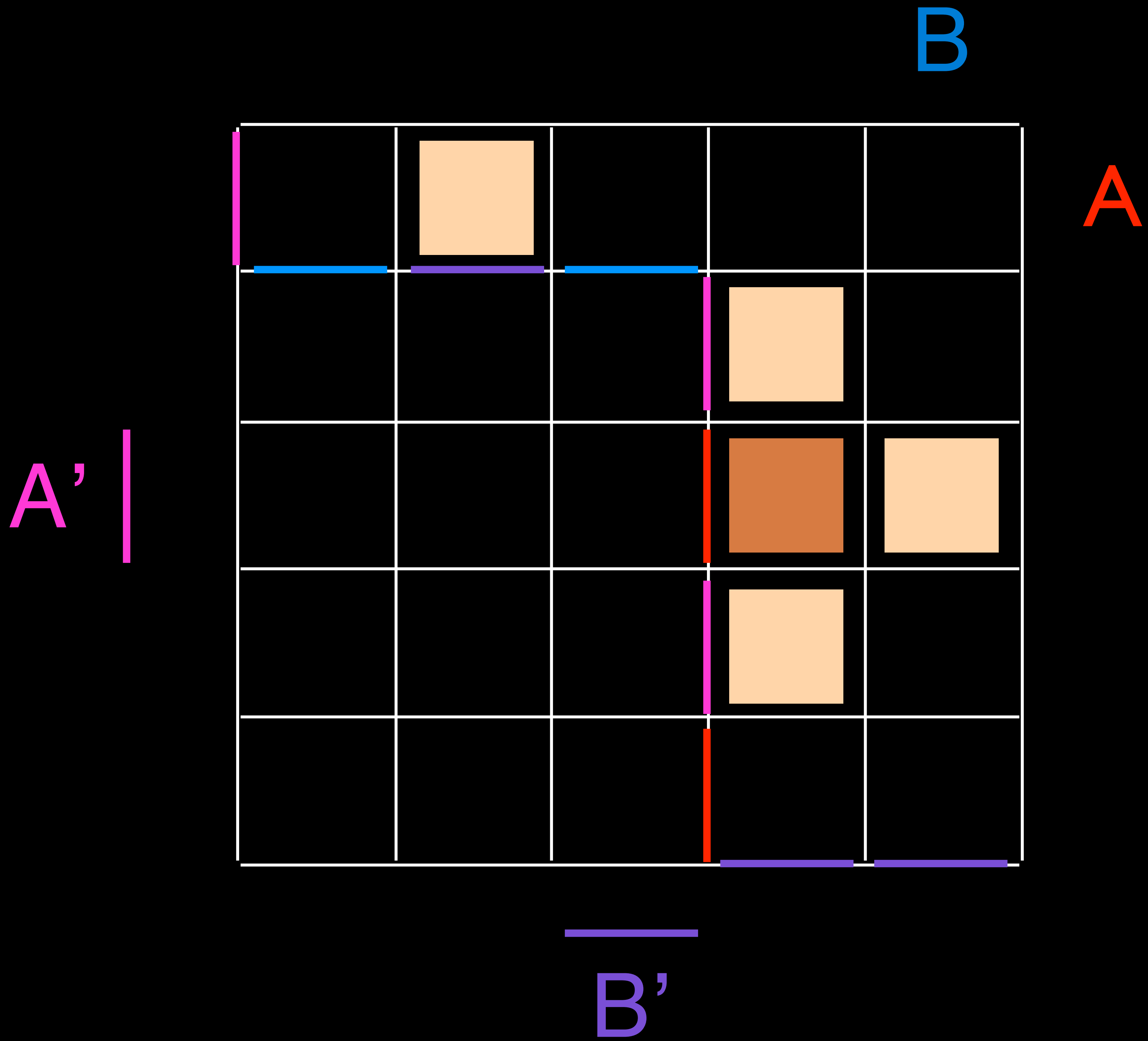
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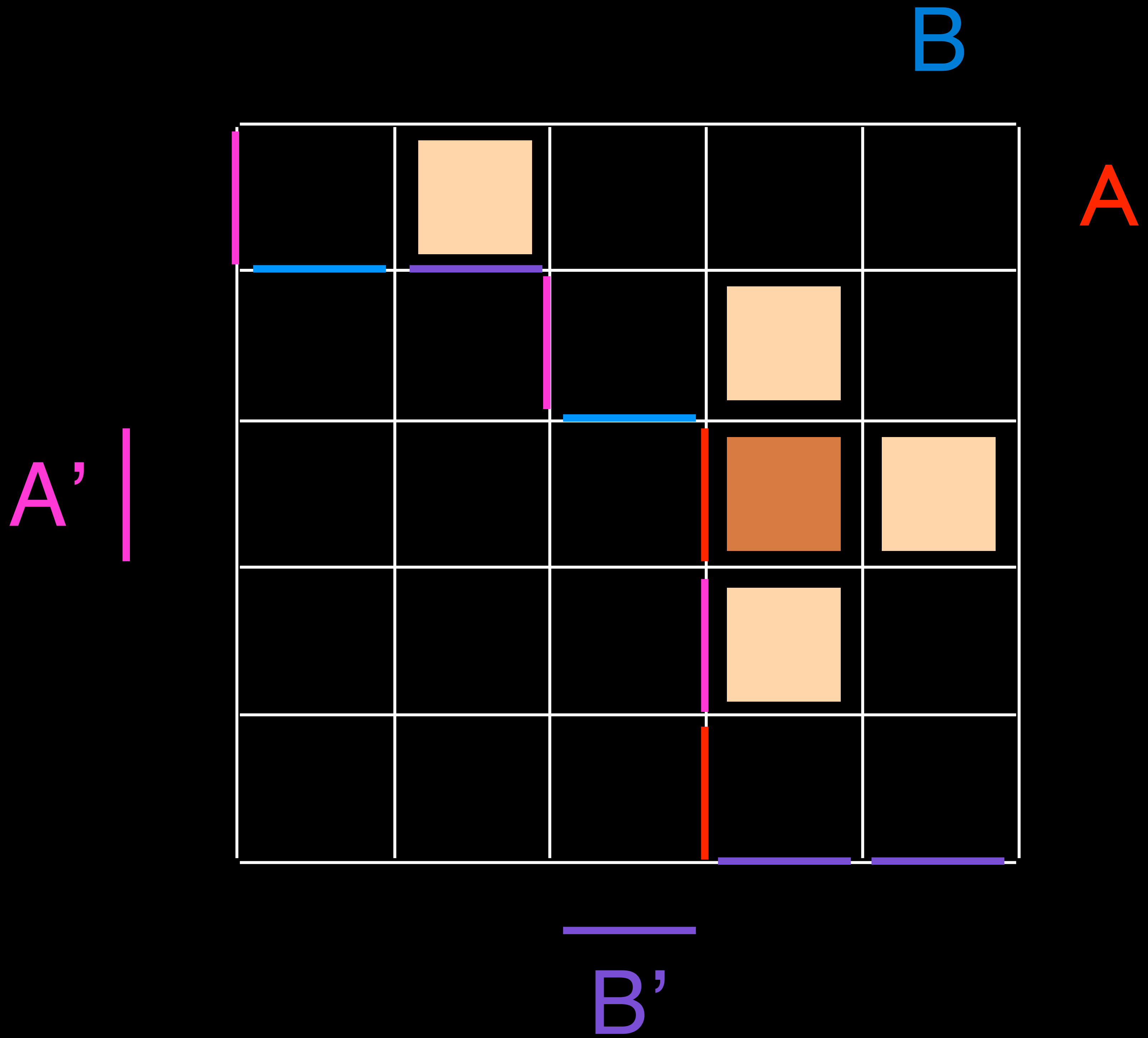
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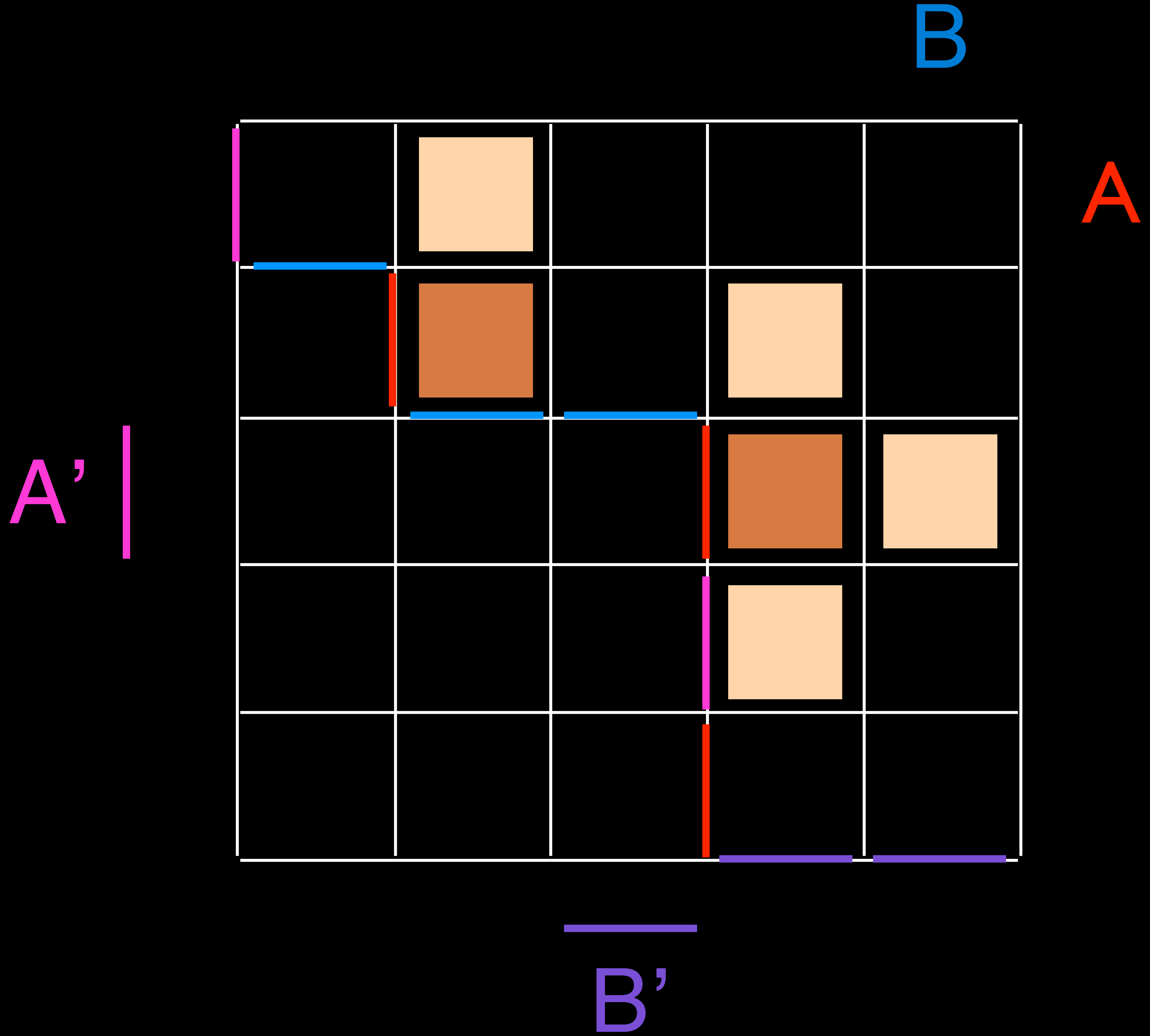

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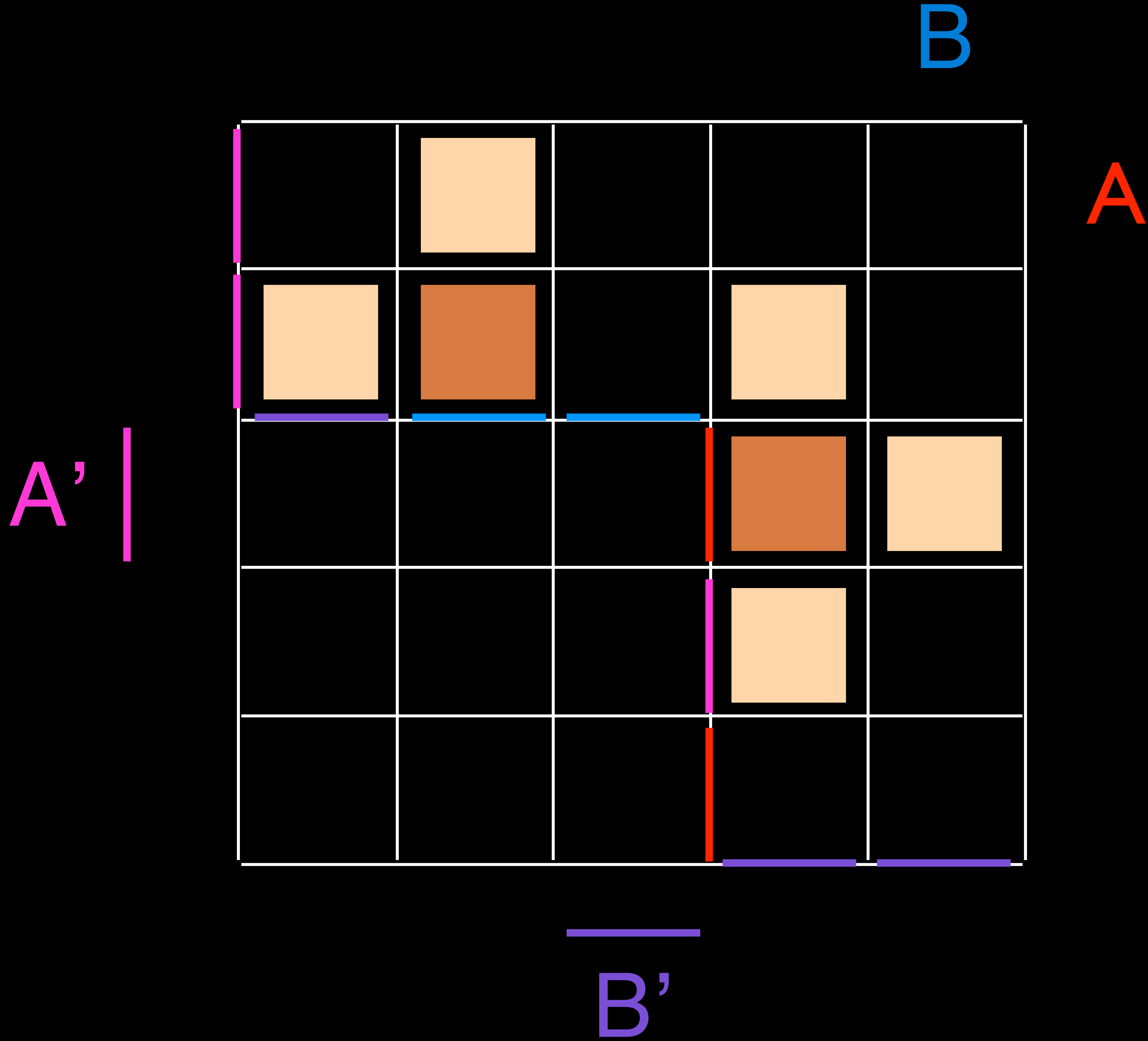
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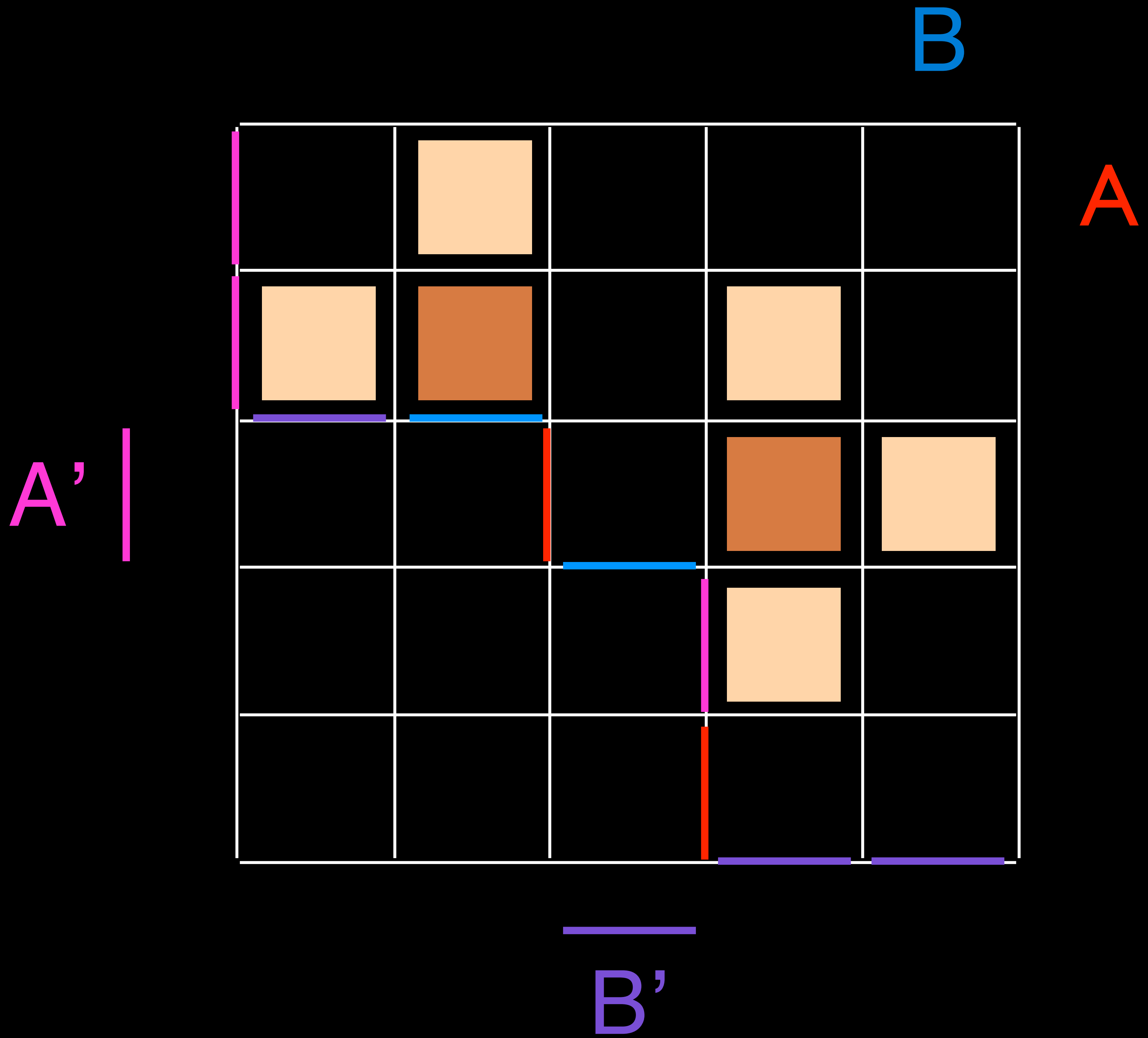
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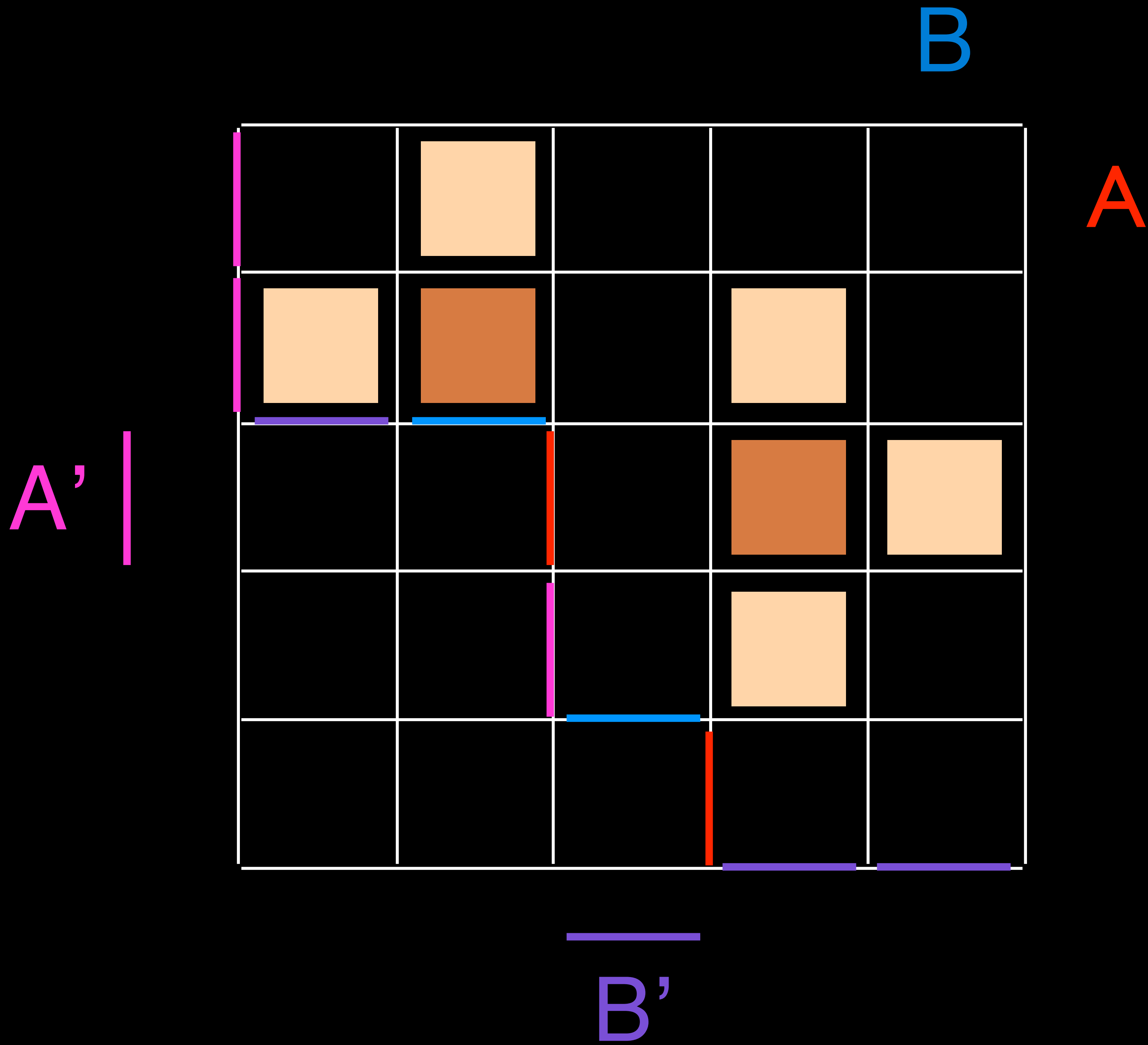
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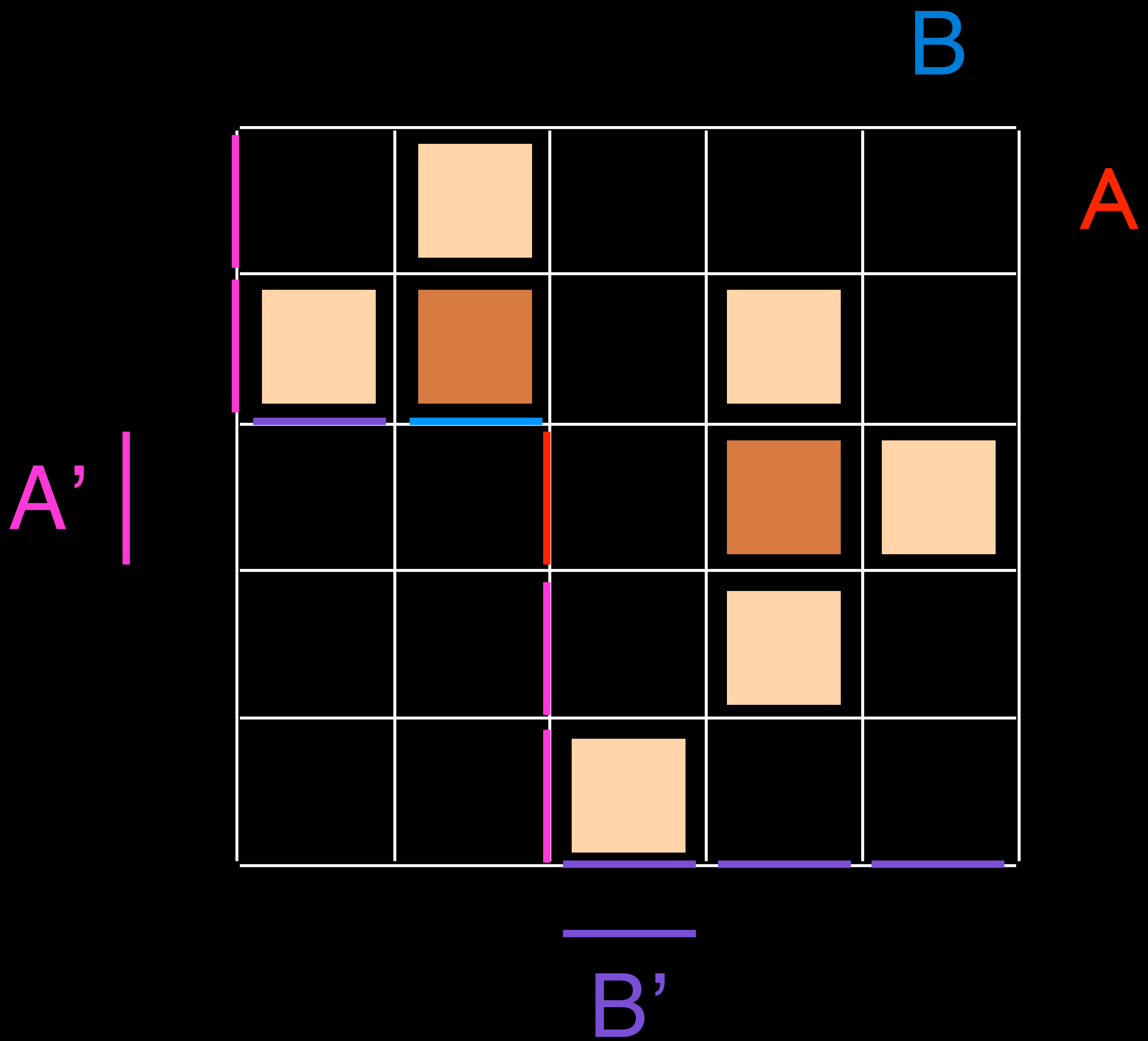
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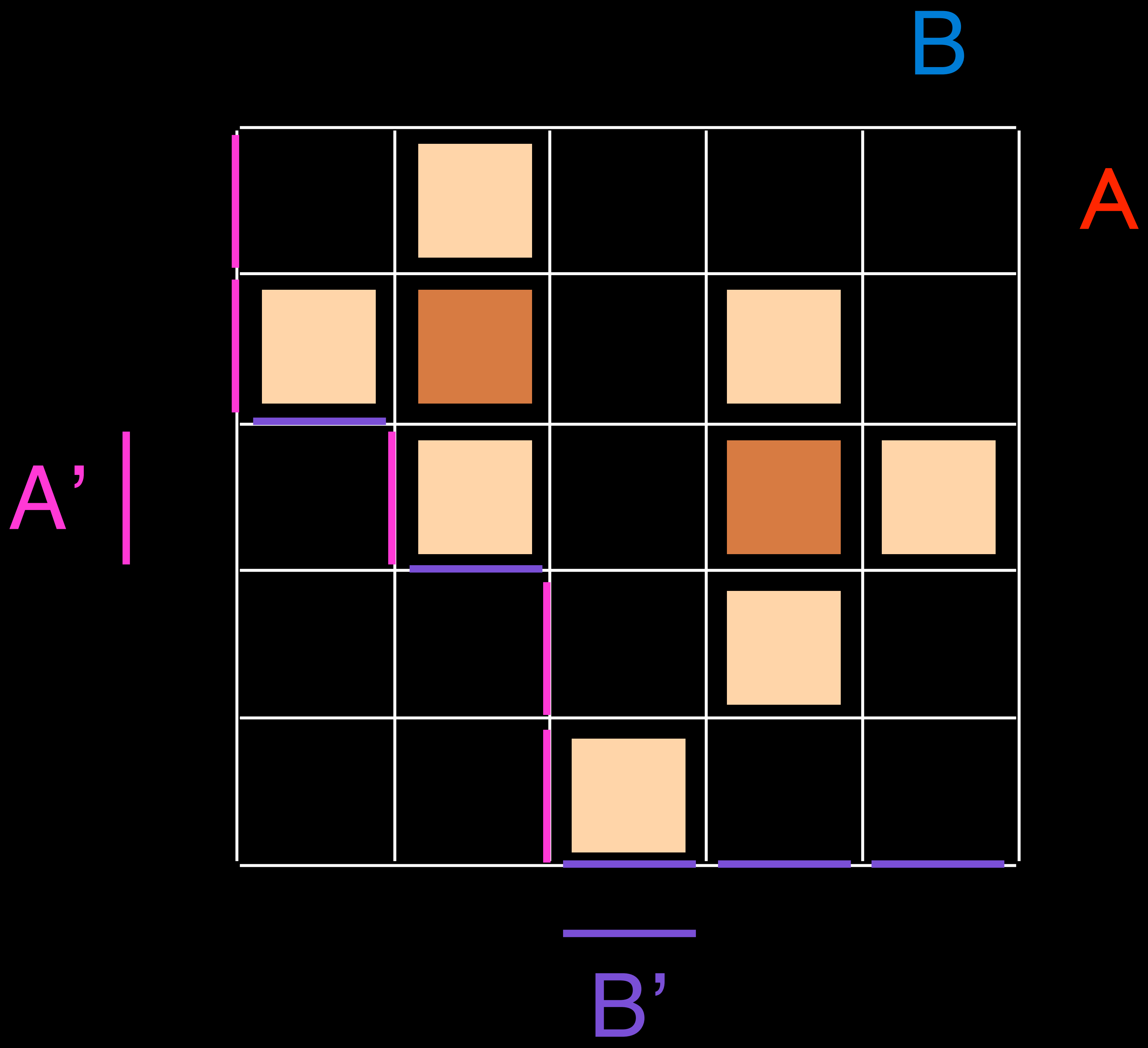
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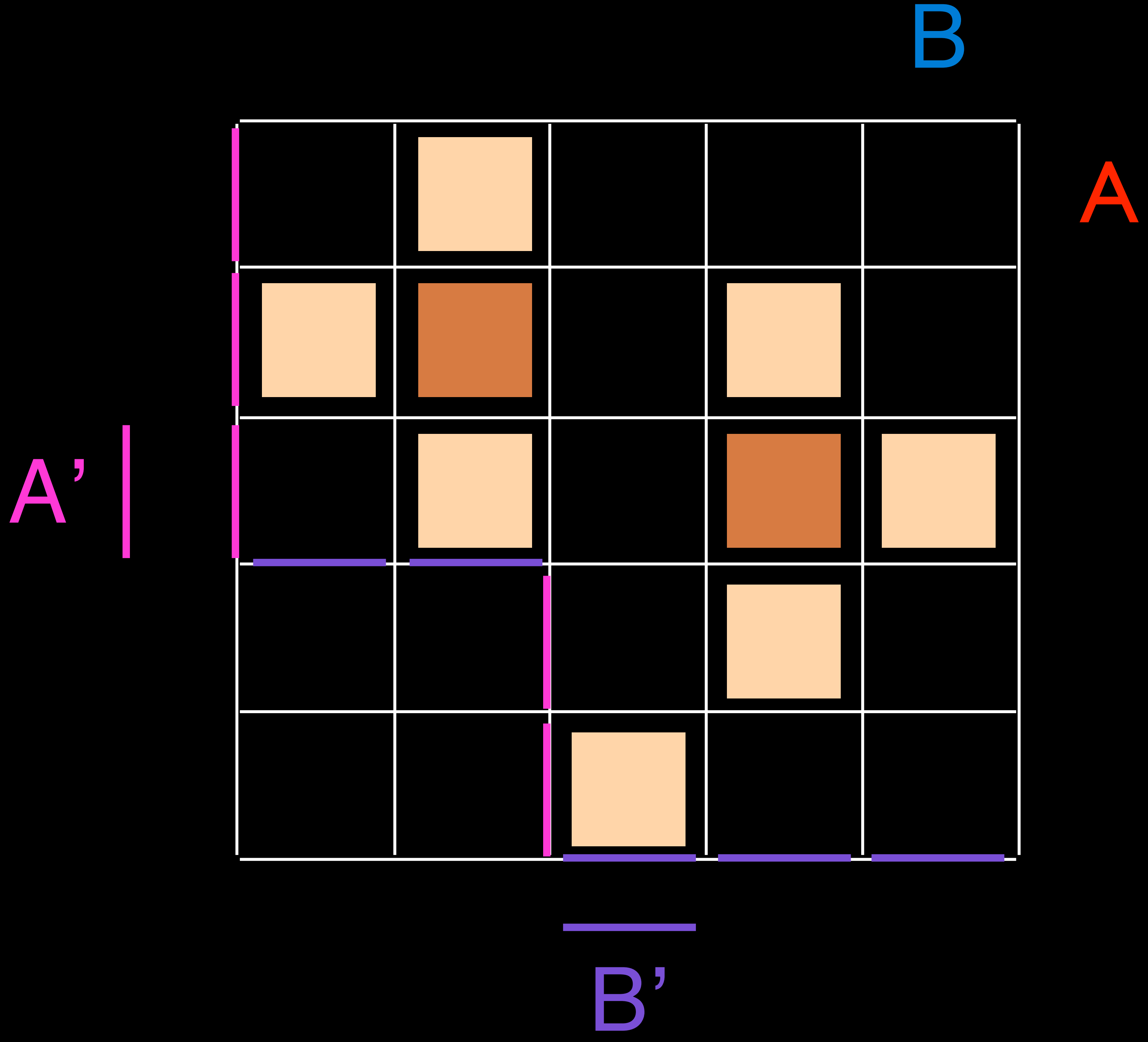
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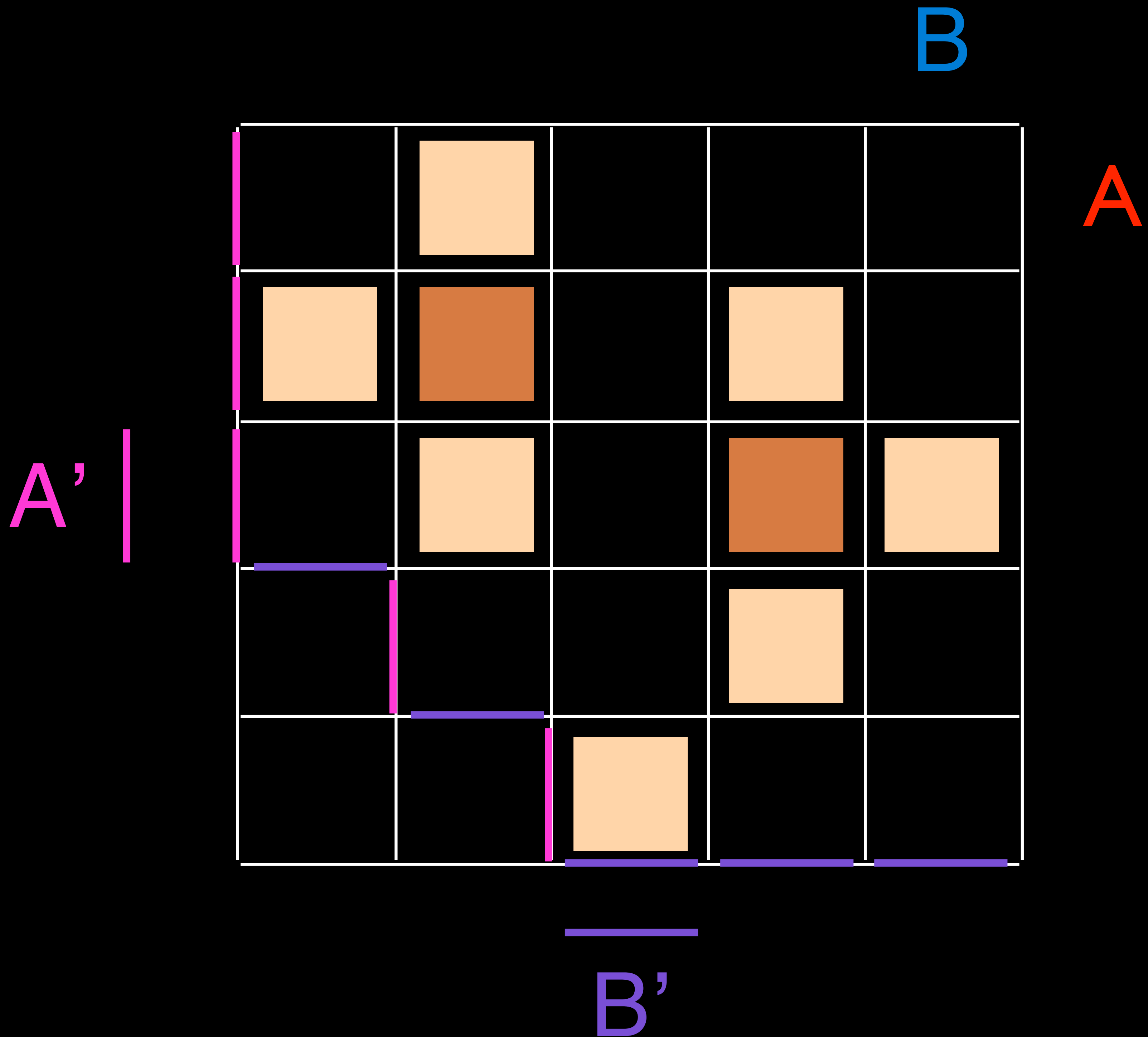
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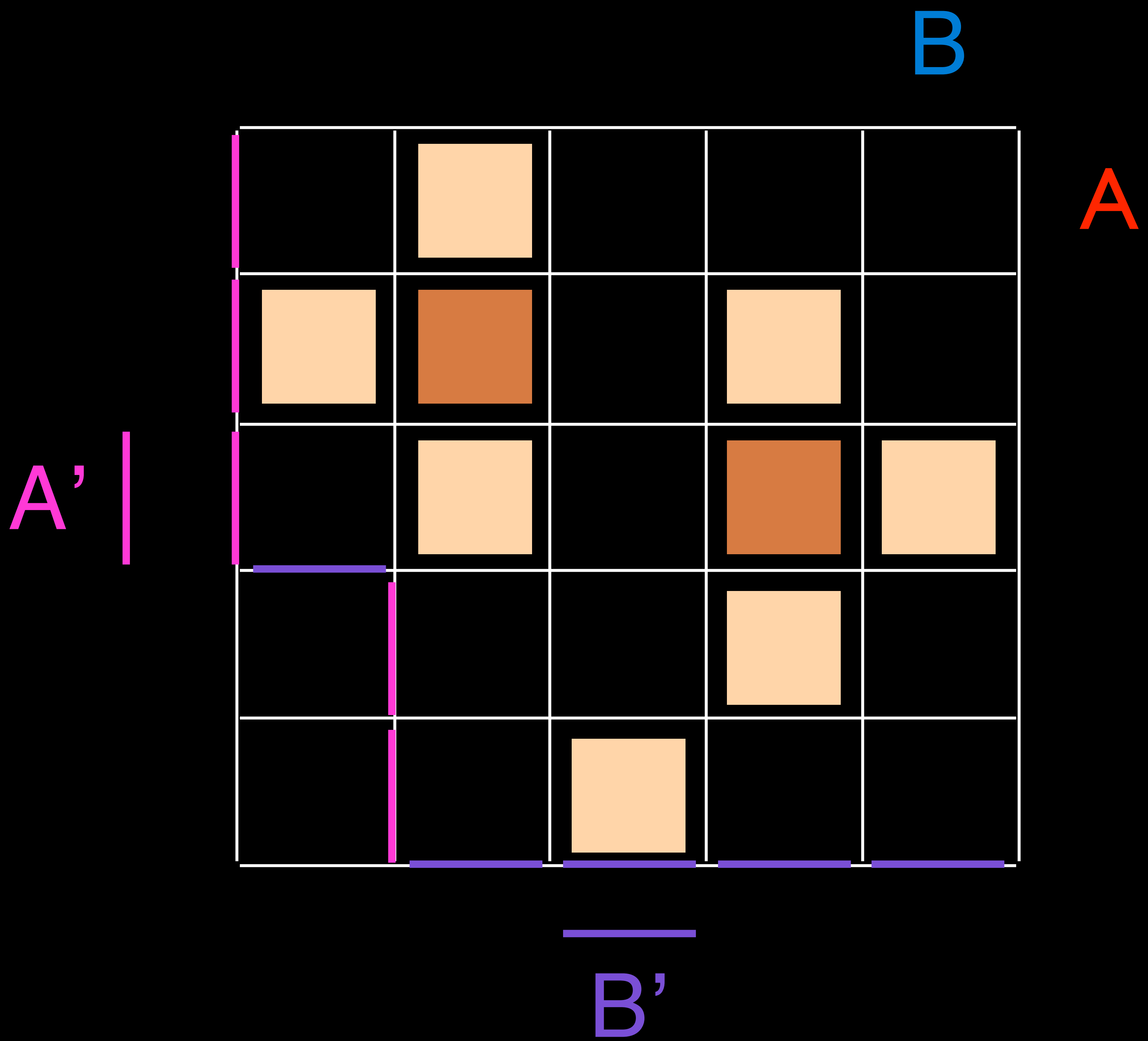

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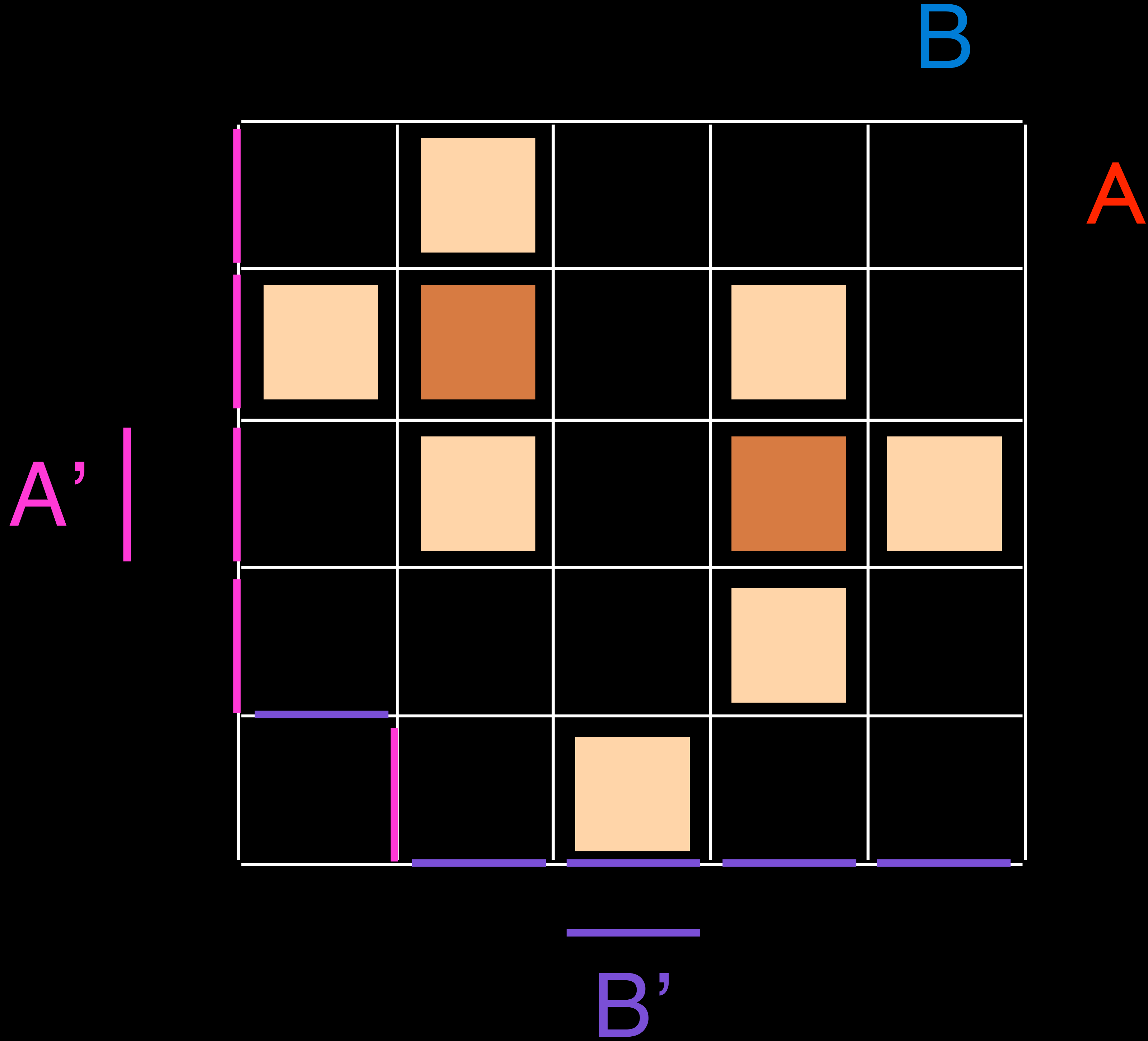
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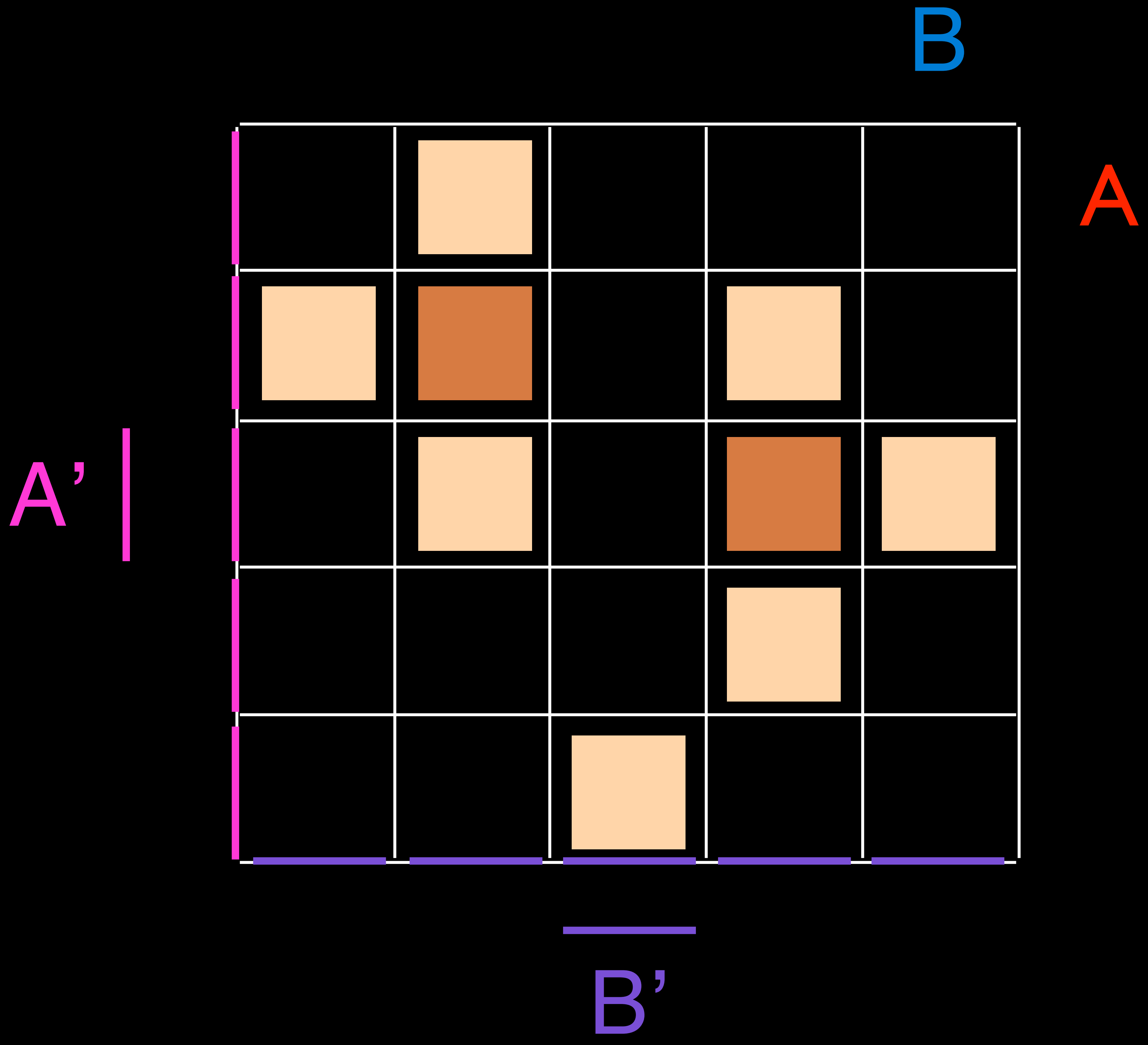
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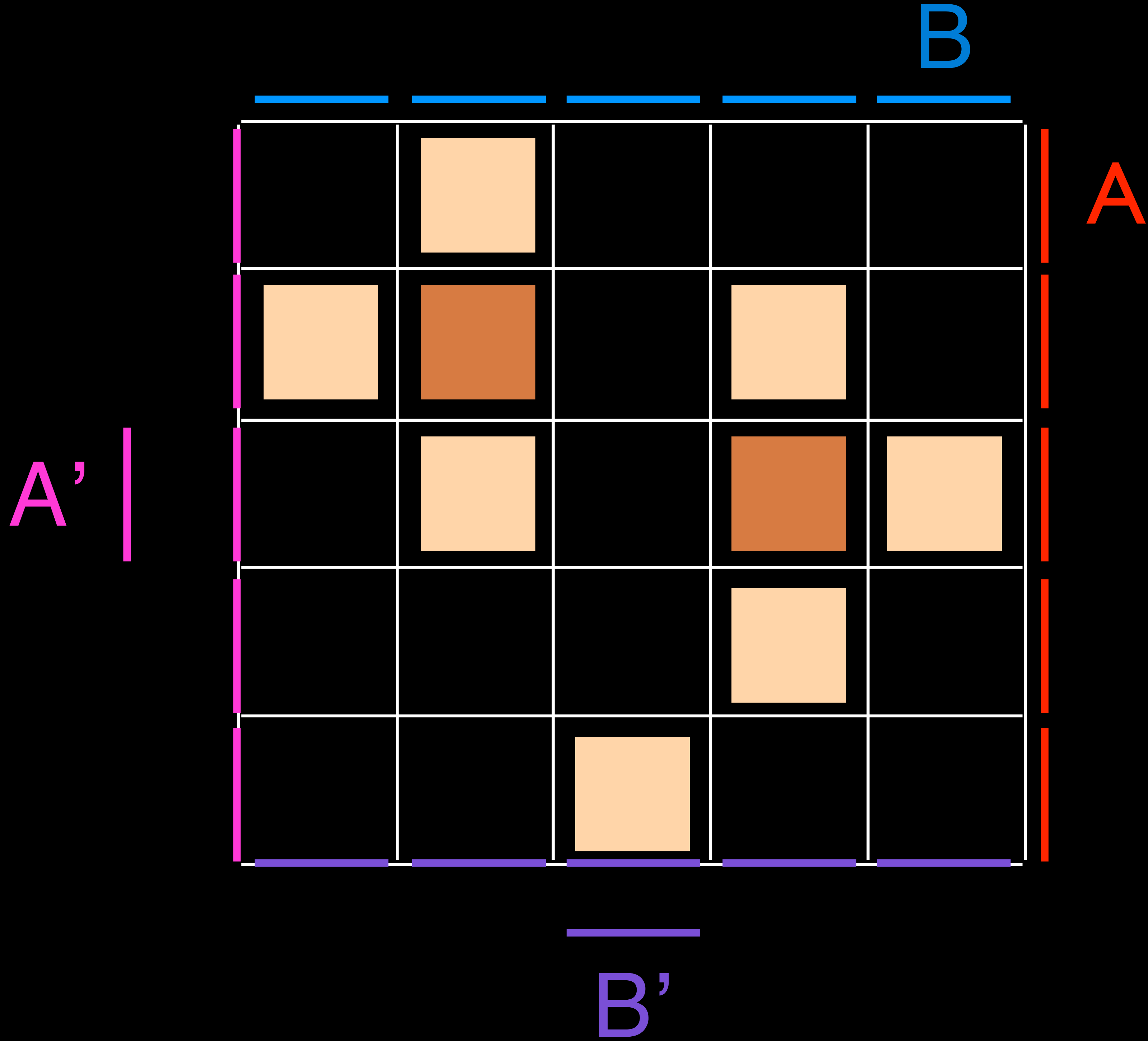

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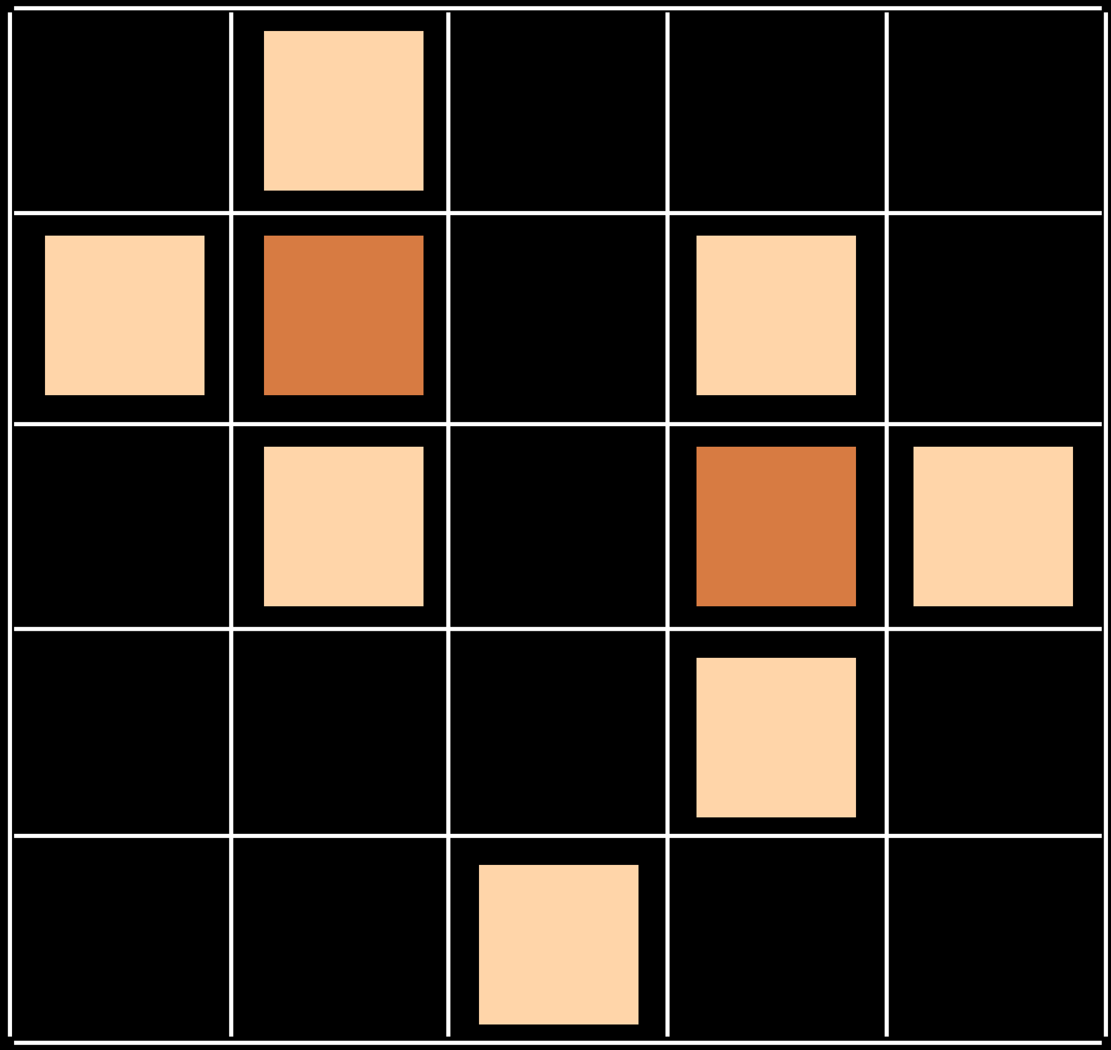
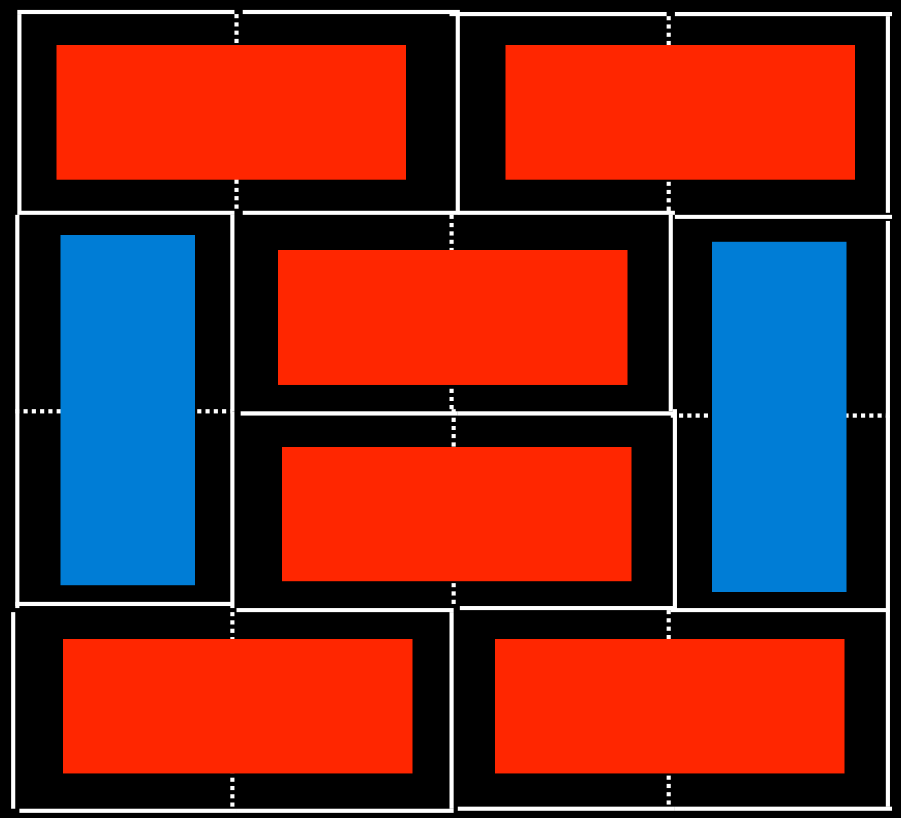


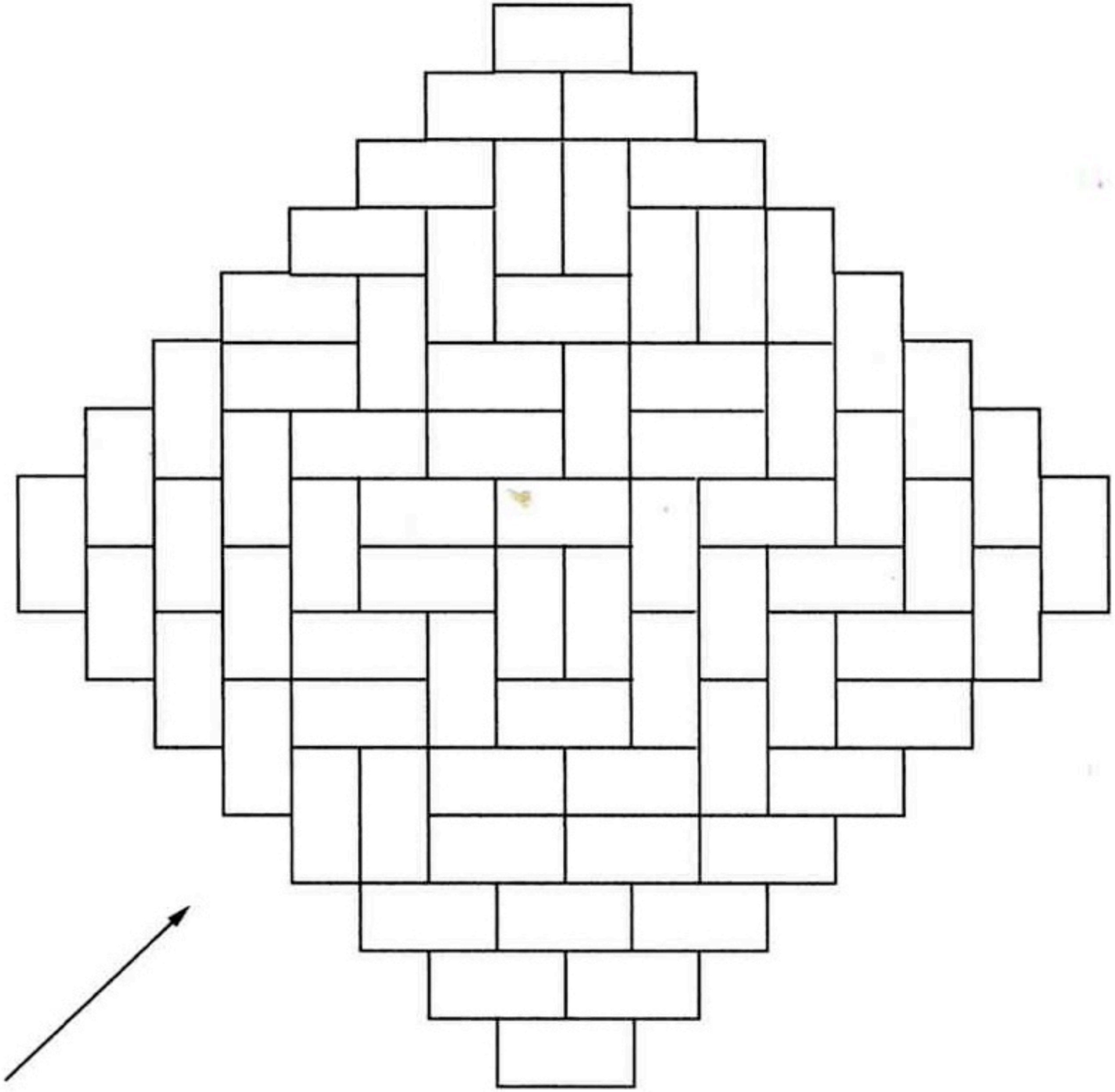
Planar automata

other examples



a tiling
on the
square lattice

$$2^{n(n-1)/2}$$



Elkies,
Kuperberg,
Larsen,
Propp
(1992)

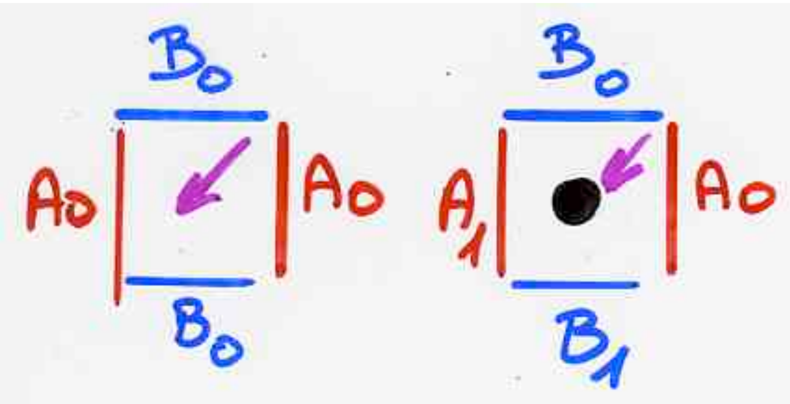
The "RSK planar automaton"

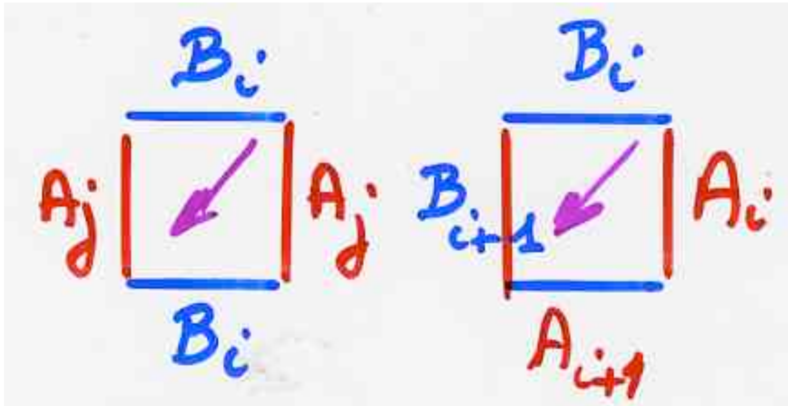
$\mathcal{B} = \{B_0, B_1, \ldots B_k\}$

$\mathcal{A} = \{A_0 \quad A_1 \quad A_k\}$

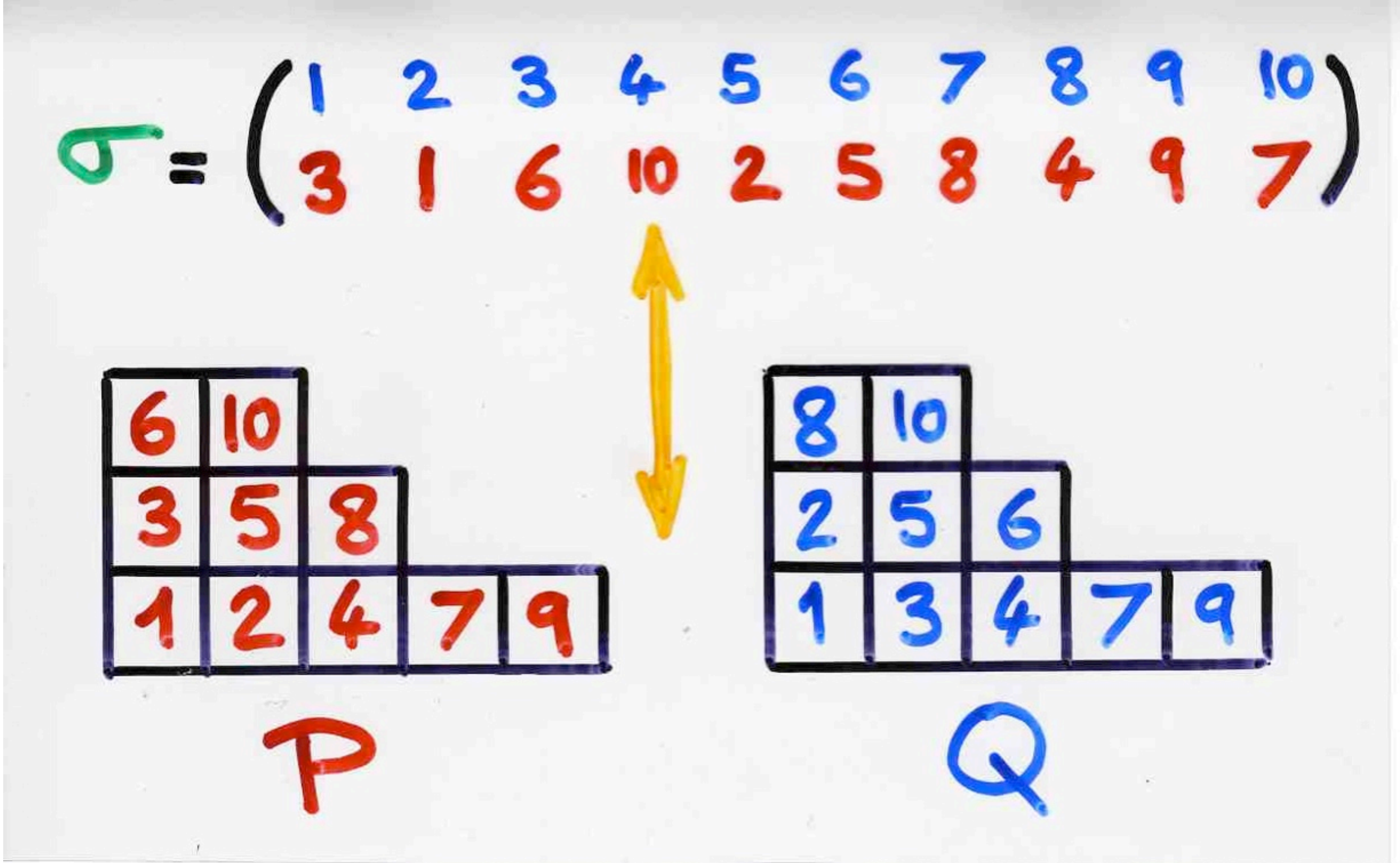
$w \in \{B_0, A_0\}^*$

$S = \{\square, \boxdot\}$



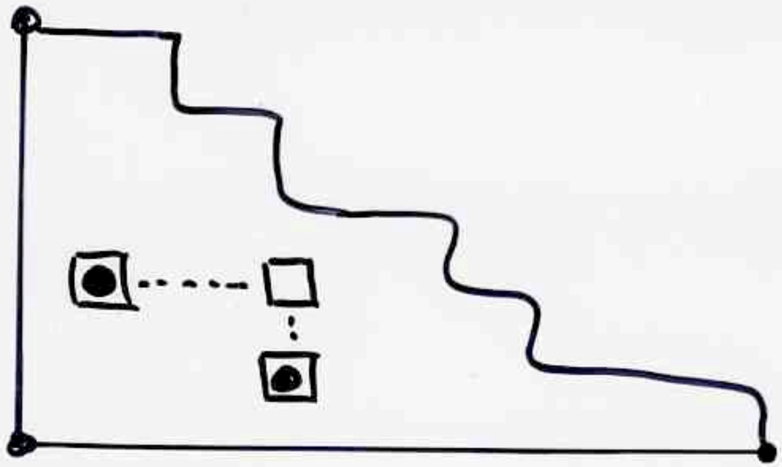


$i \neq j$



The Robinson-Schensted correspondence (RSK)

«Figures»
accepted by planar automata ?

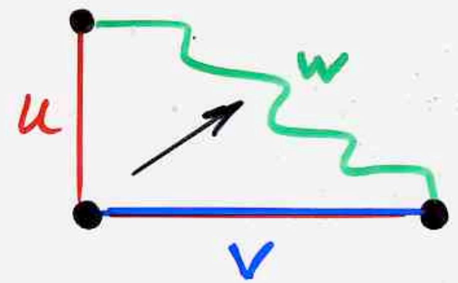


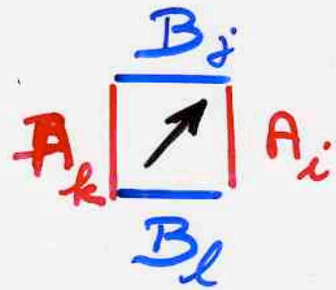
Postnikov

⅃-diagrams

Le-...

reverse planar automata





relation with quadratic algebras

Heisenberg
operators
U, D

$$UD = DU + 1$$

$$UD = DU + I$$

$$UD \to DU \qquad UD \to I$$

$$UD = DU + I$$

<u>Lemme</u> - Tout mot $w \in \{U, D\}^*$ s'écrit

$$w = \sum_{i,j \geq 0} c_{i,j}(w) D^i U^j$$

unicité

$$UD = DU + Id$$

$$U^n D^n = ?$$

$$\begin{aligned} UUUDDD &= UU(DU + Id)DD \\ &= UUDUDD + UUDD \\ &= UDUUDD + 2\,UUDD \\ &= DUUUDD + 3\,UUDD \end{aligned}$$

$$
\begin{aligned}
UUDD &= UDUD + UD \\
&= DUUD + 2\,UD \\
&= DUDU + DU + 2\,(DU + \mathrm{Id}) \\
&= DDUU + 2\,DU \\
&= DDUU + 4\,DU + 2\,\mathrm{Id}
\end{aligned}
$$

$$
\begin{aligned}
U^3D^3 &= DU\,(DDUU + 4\,DU + 2\,\mathrm{Id}) + \\
&\qquad 3\,(DDUU + 4\,DU + 2\,\mathrm{Id}) \\
&= DDUDUU + DDUU \\
&\quad + 4\,(DDUU + DU) + 2\,DU \\
&\quad + 3\,DDUU + 12\,DU + 6\,\mathrm{Id} \\
&= D^3U^3 + 9\,D^2U^2 + 18\,DU + 6\,\mathrm{Id}
\end{aligned}
$$

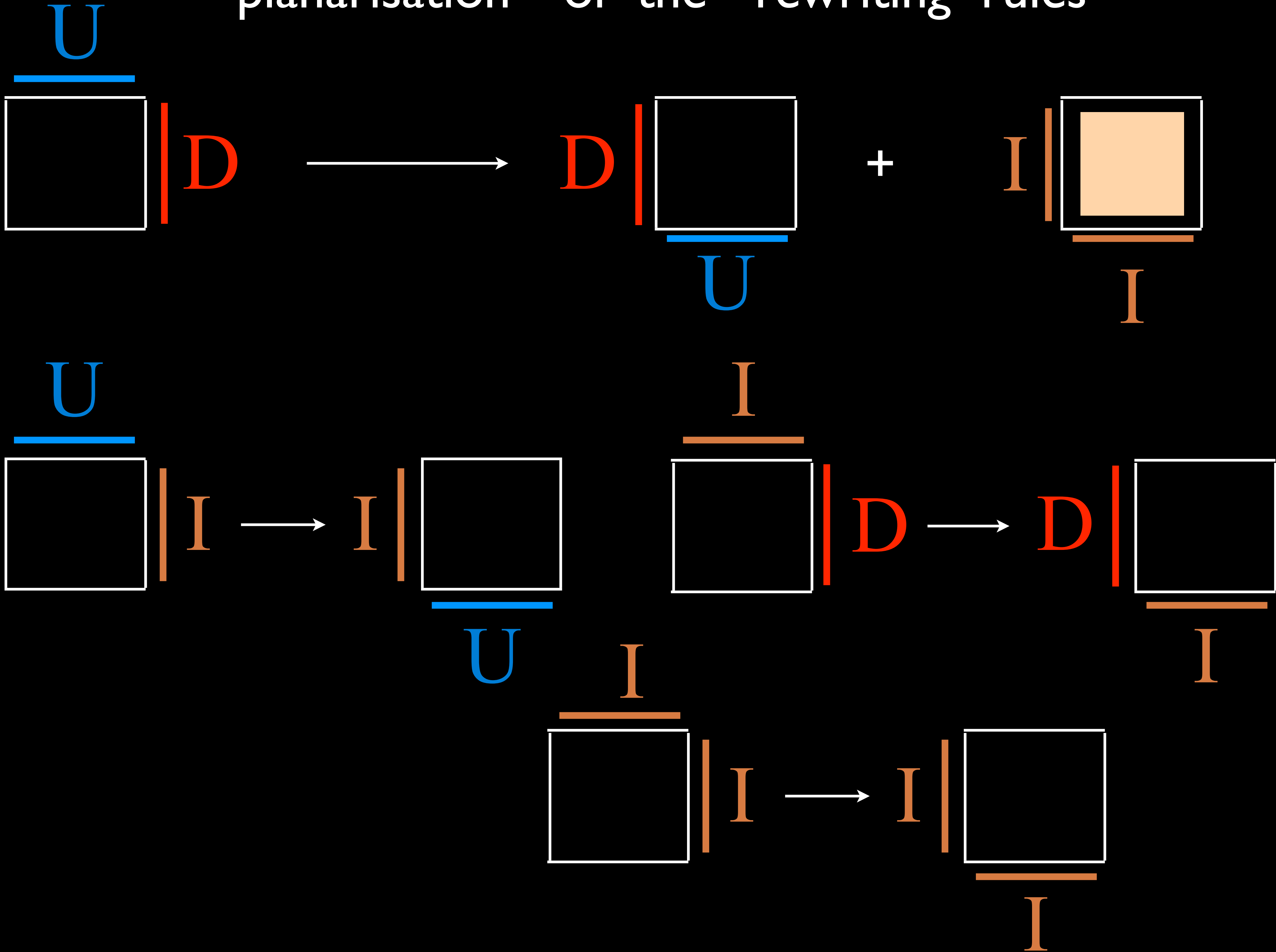
$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} \, D^i U^i$$

normal ordering

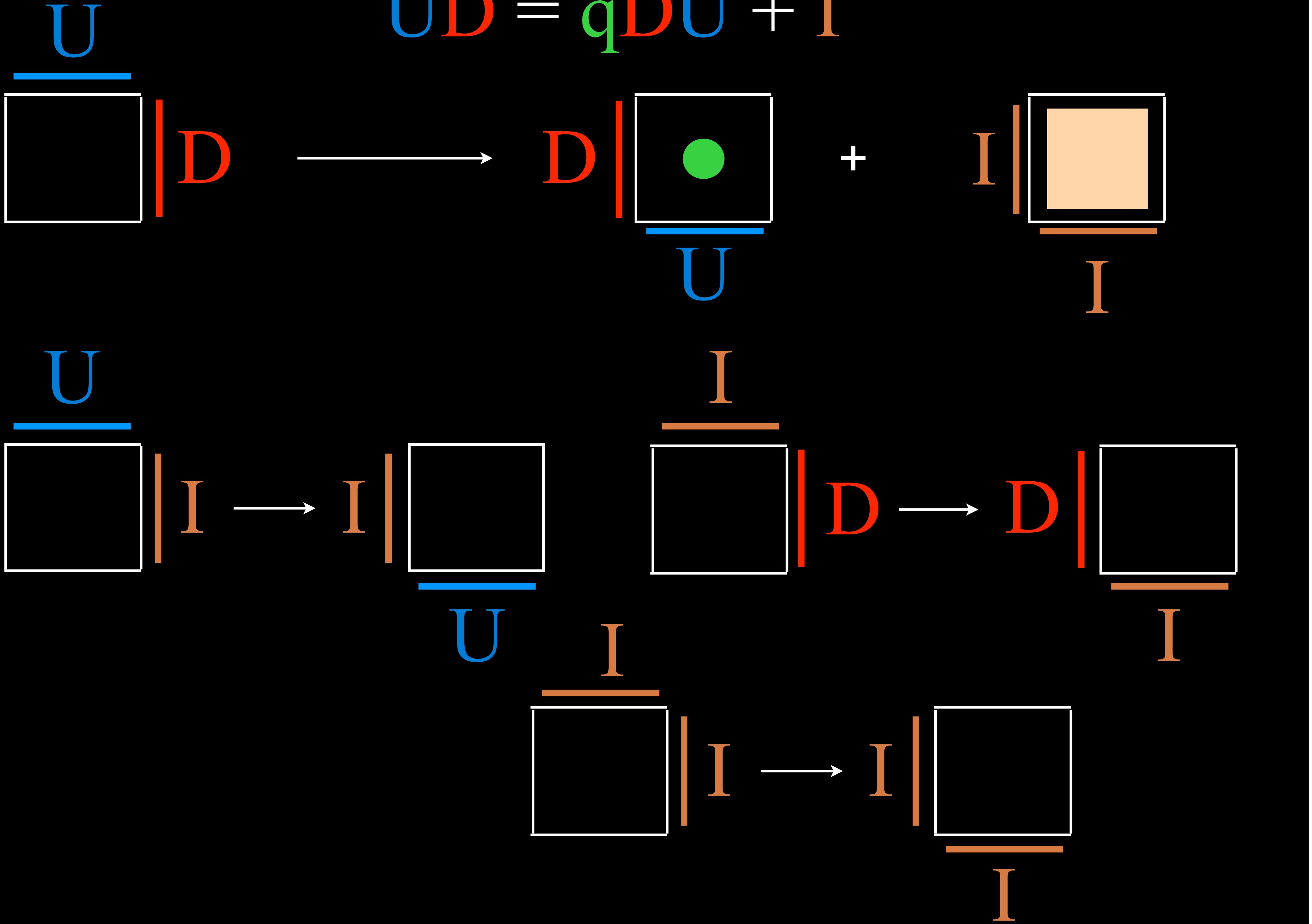
$$c_{n,0} = n!$$

$$c_{n,i} = \binom{n}{i}^2 (n-i)!$$

“planarisation” of the “rewriting rules”



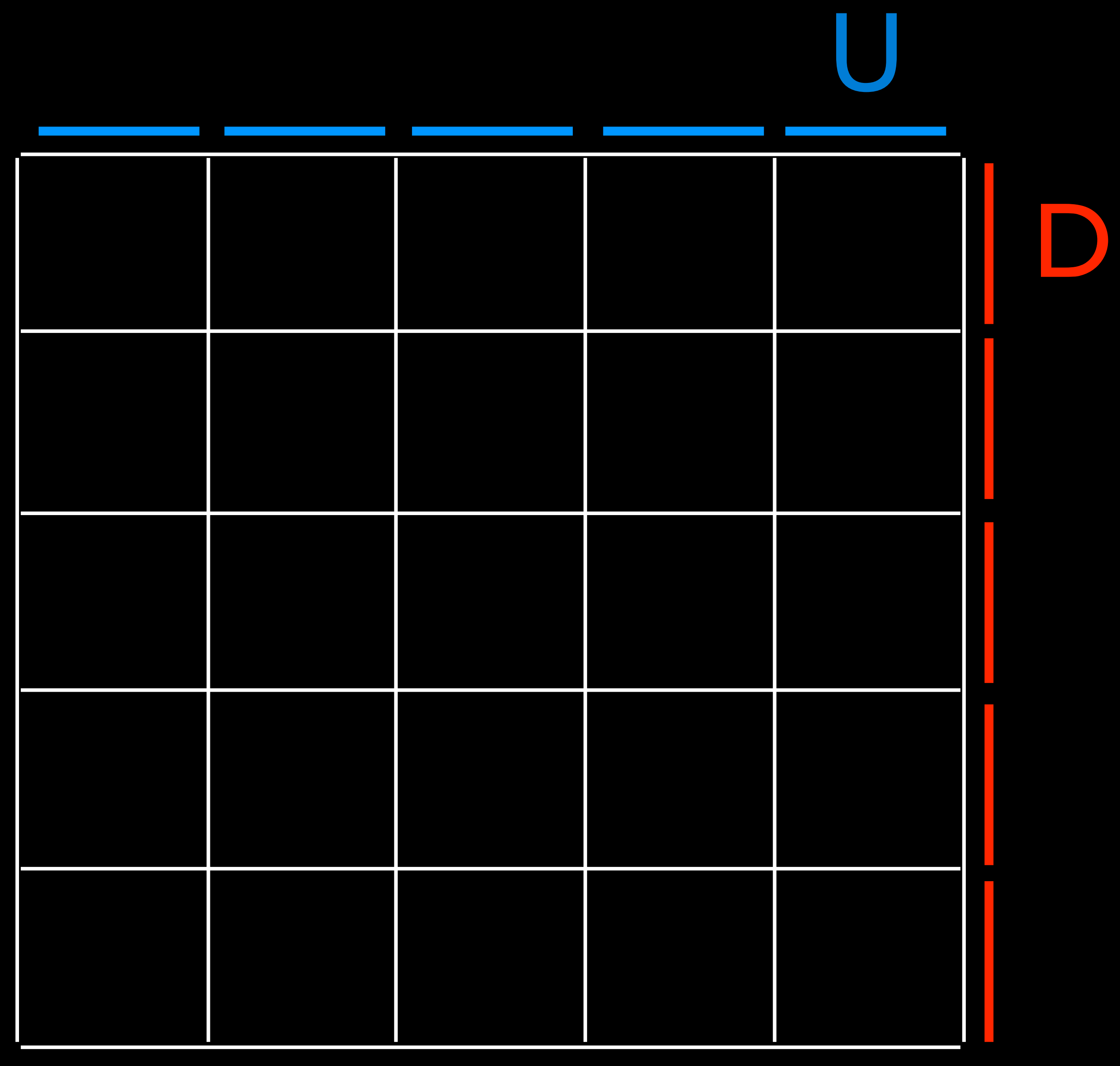
$$UD = qDU + I$$

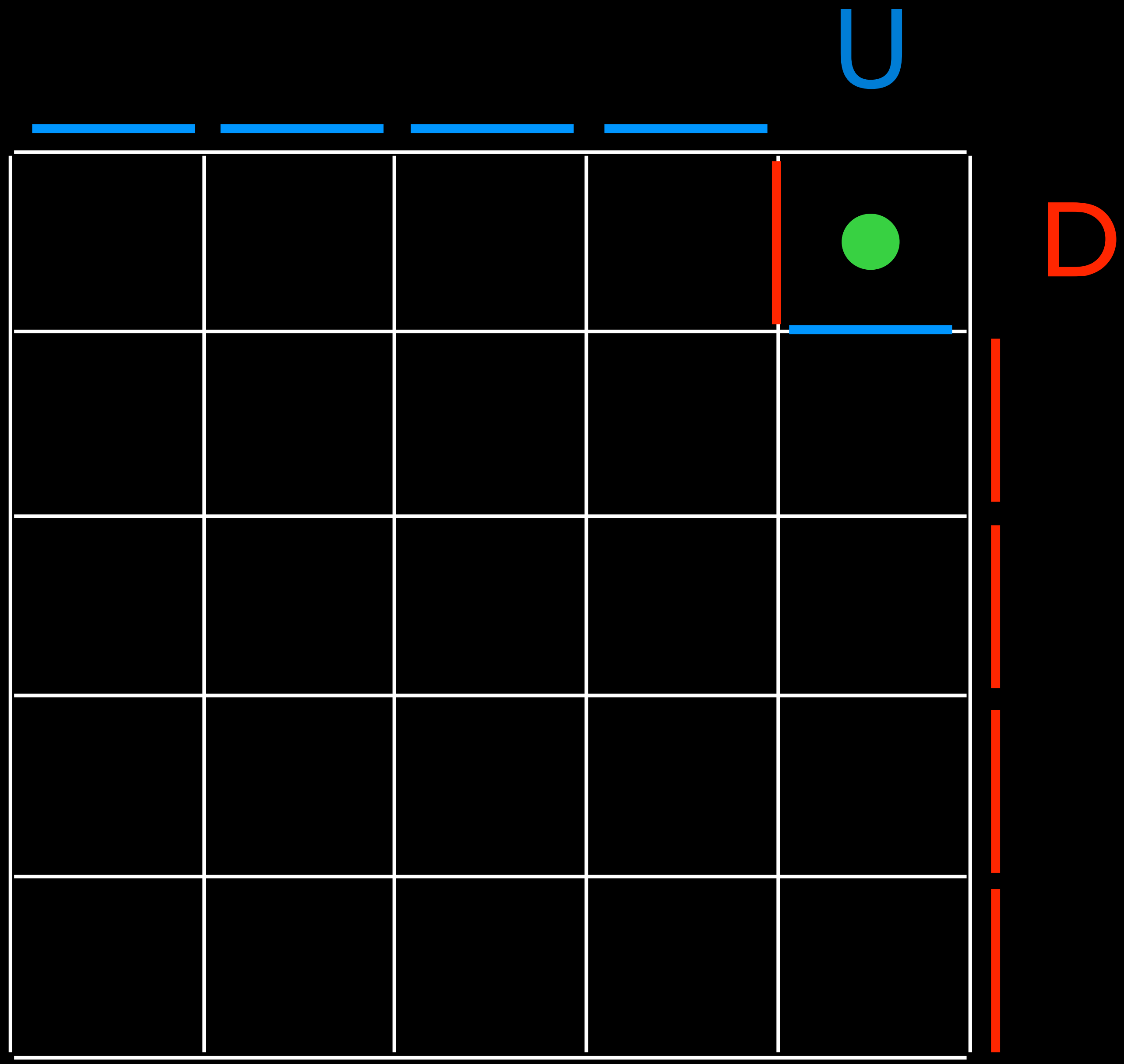


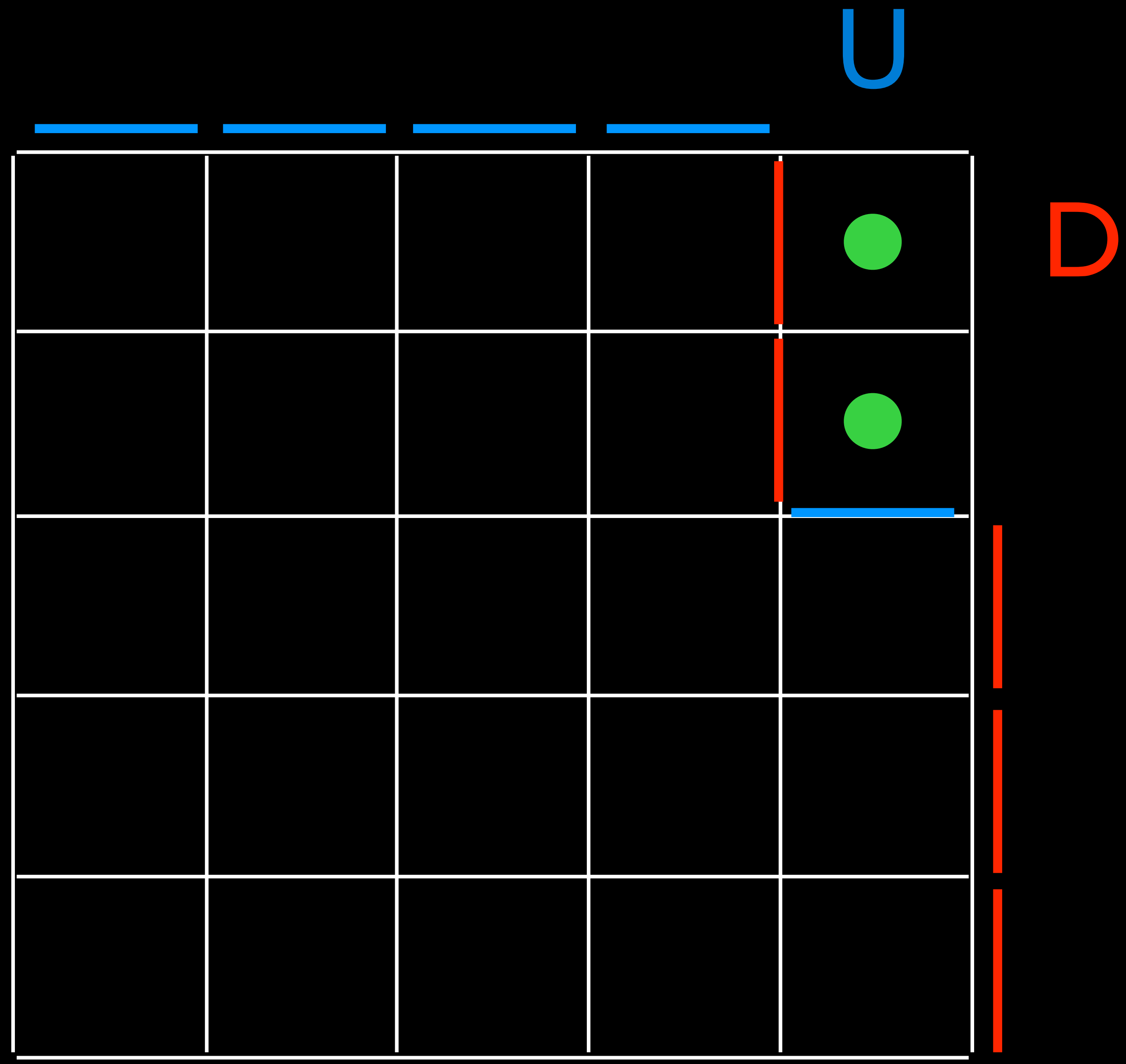
$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

$$\begin{cases} UD \to DU \\ U I_v \to I_v U \\ I_h D \to D I_h \\ I_h I_v \to I_v I_h \end{cases} \qquad UD \to I_v I_h$$

rewriting rules


U
D


U
D


U
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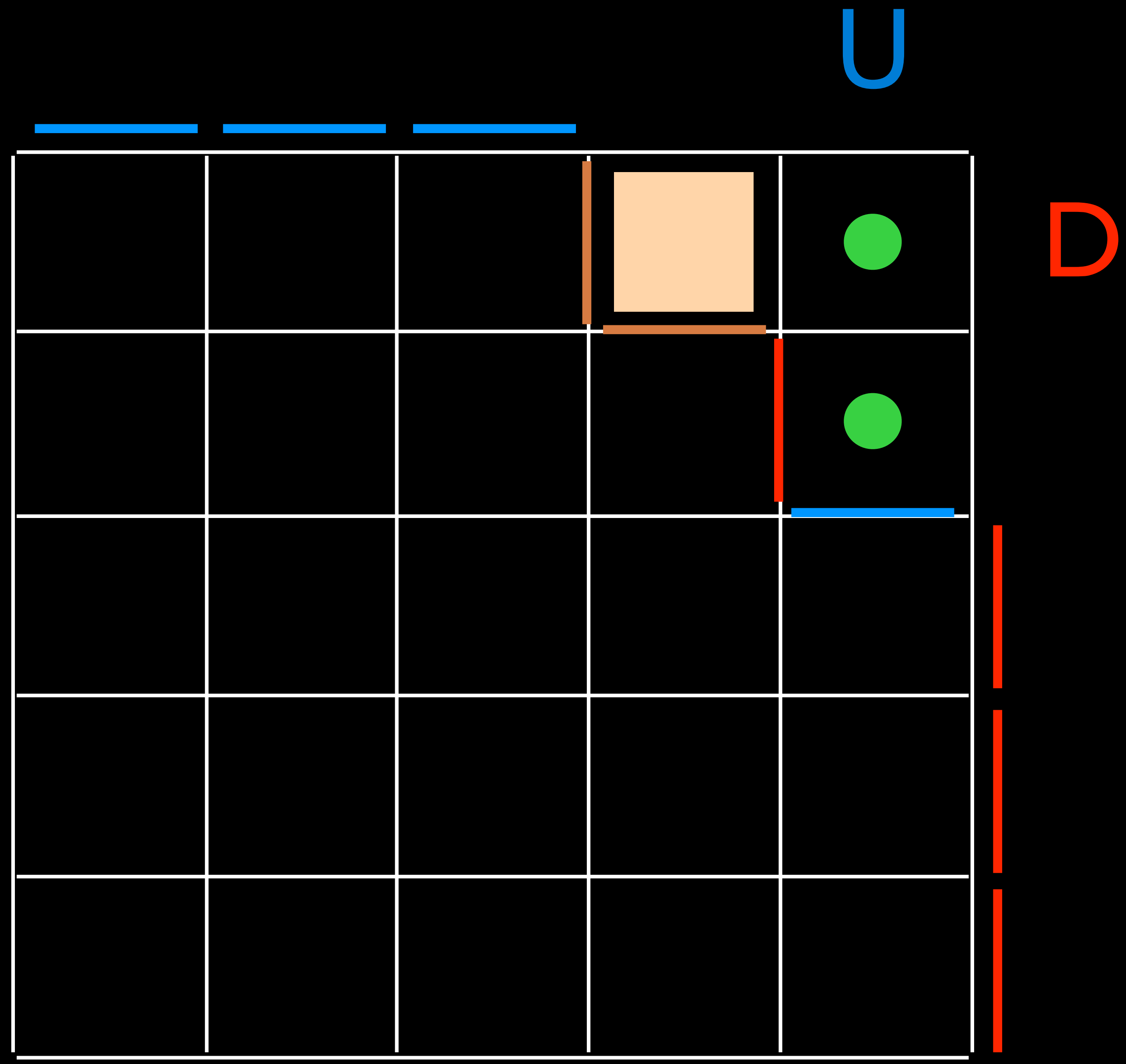
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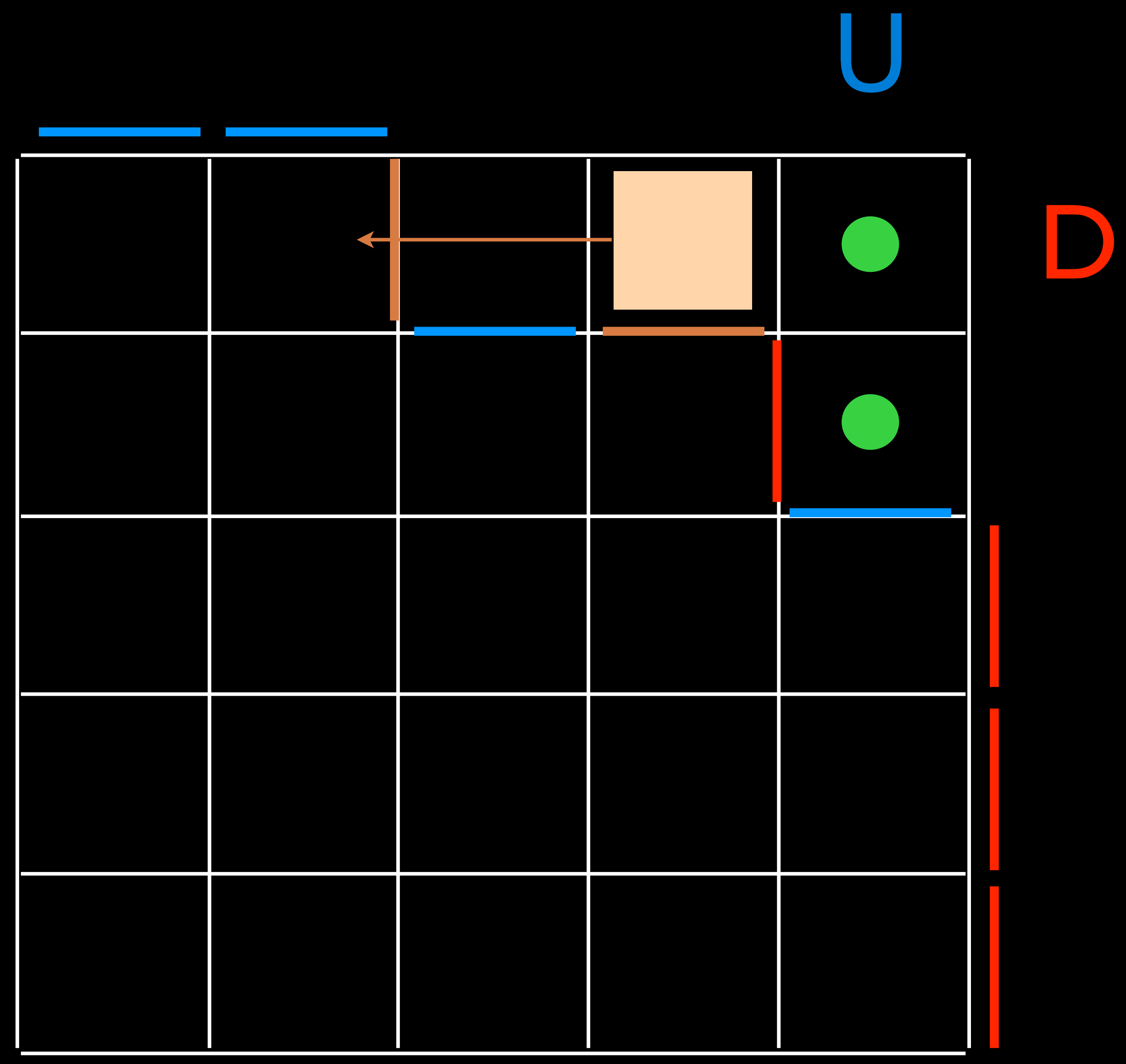
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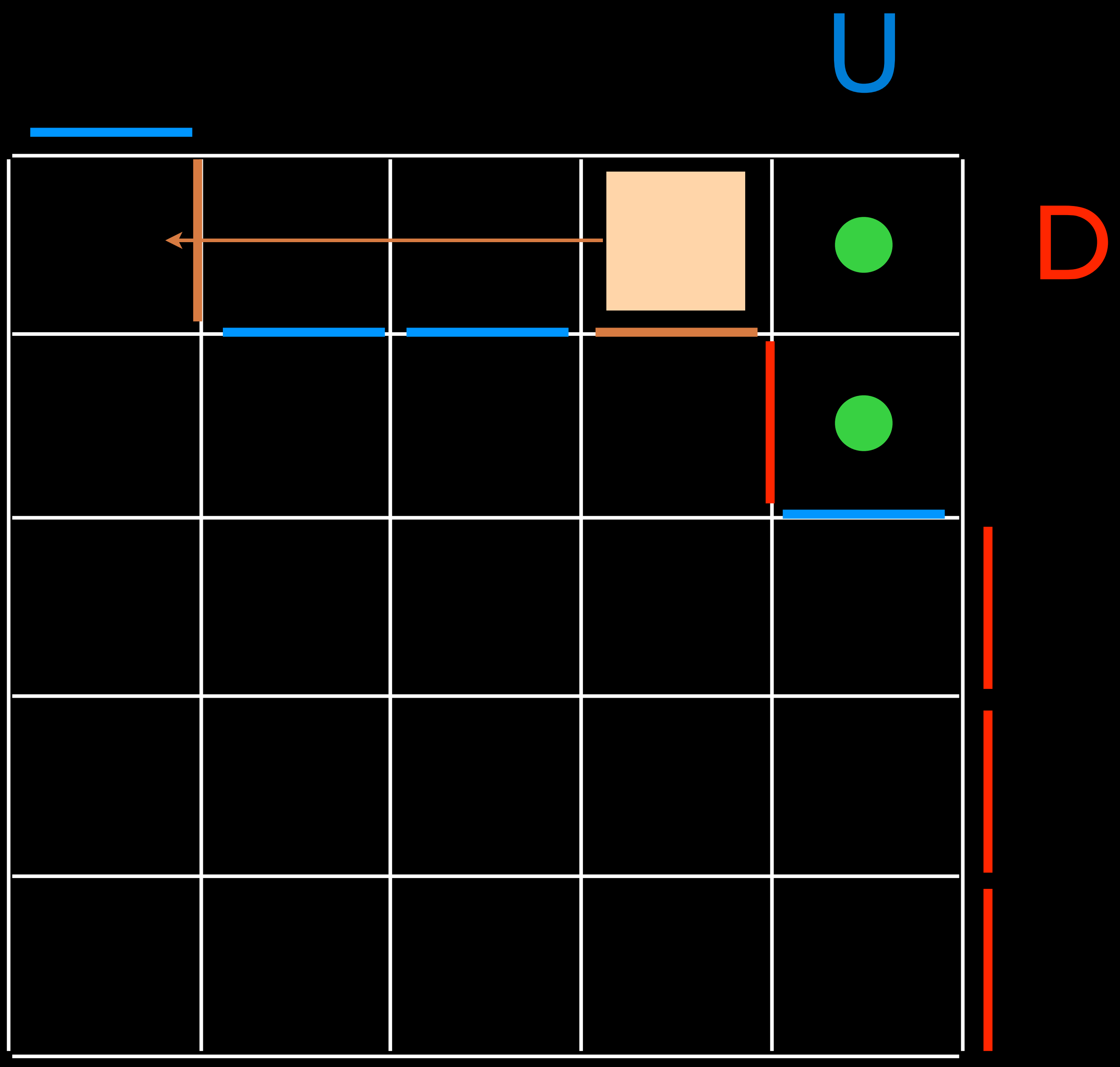
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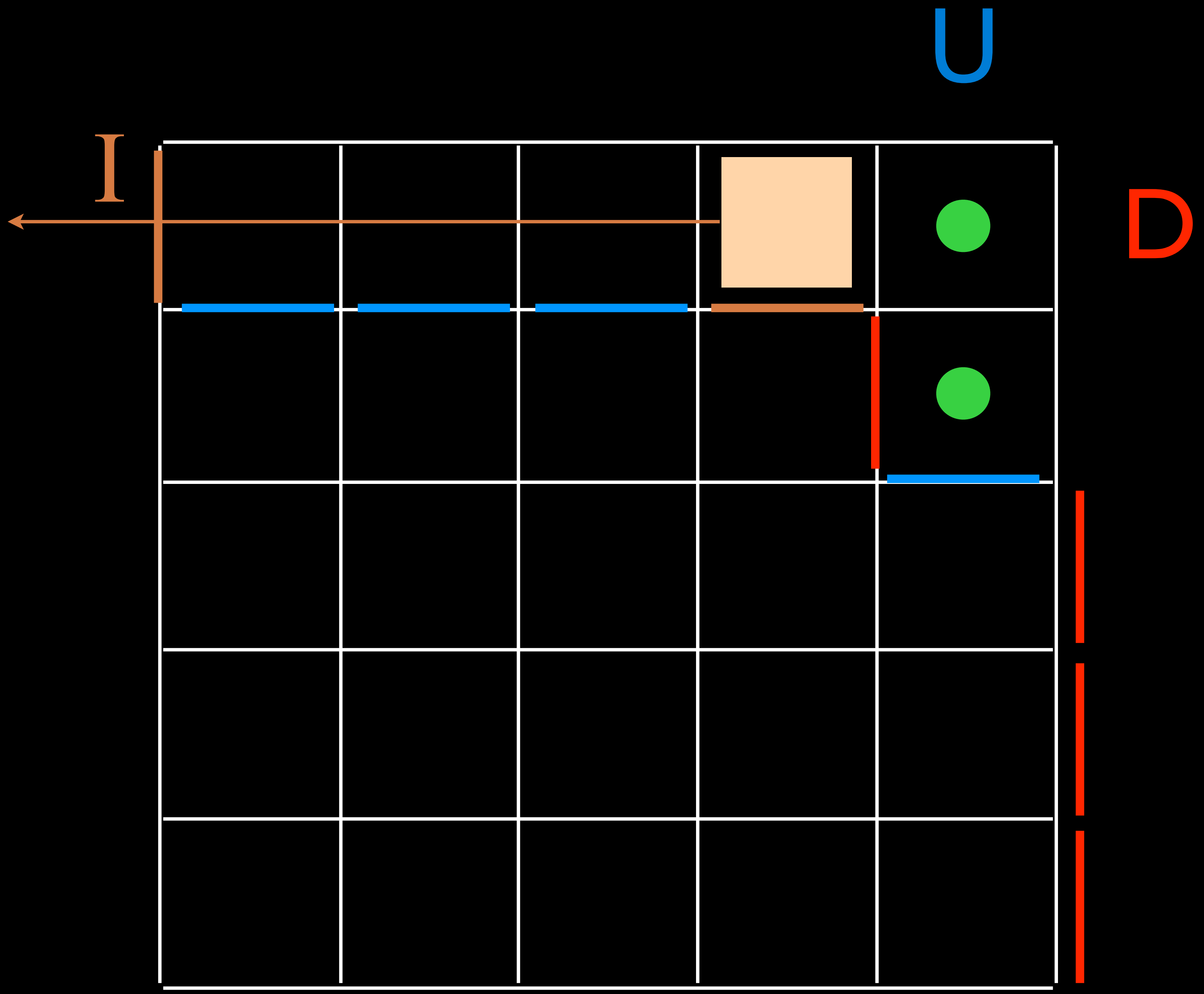
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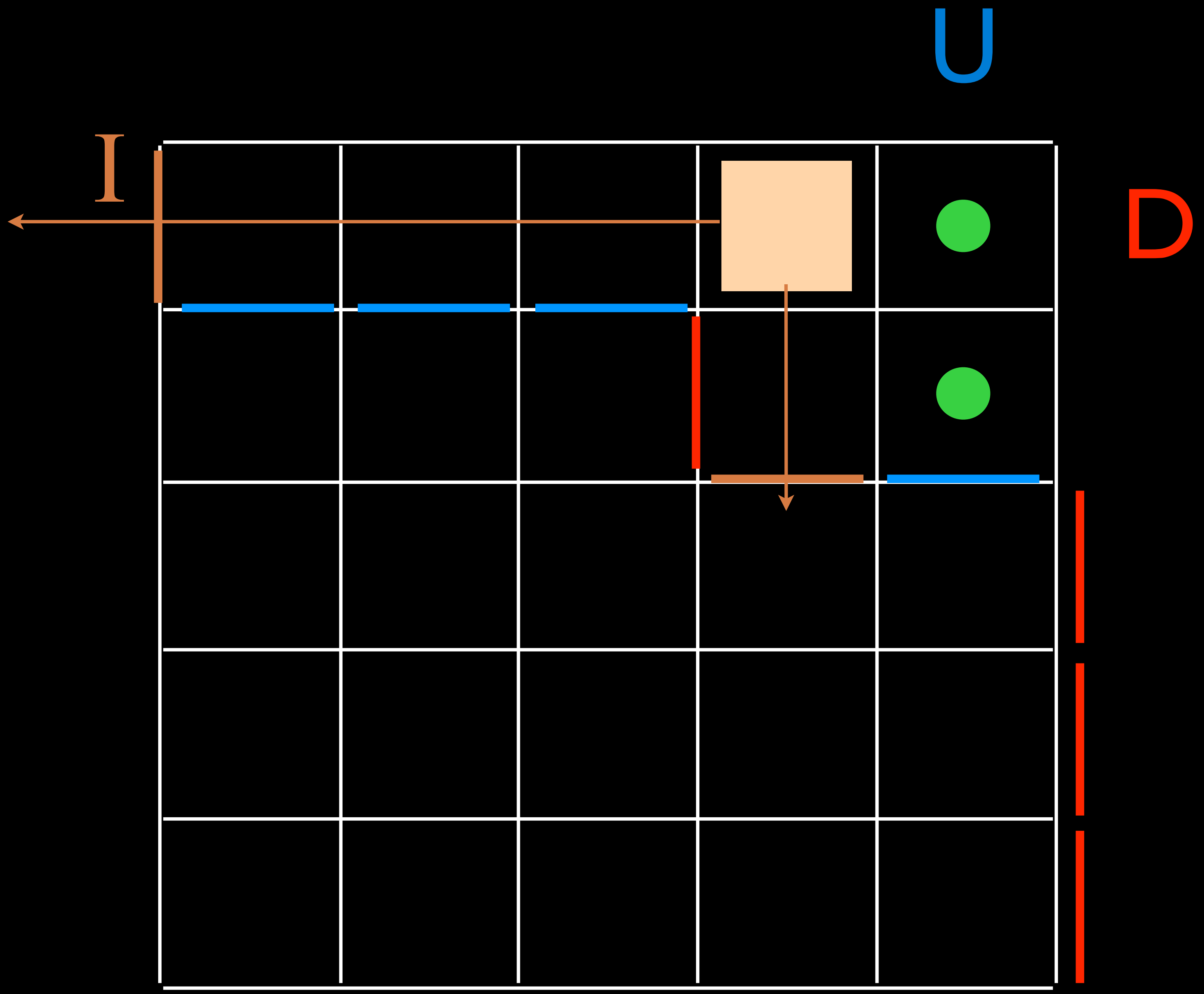
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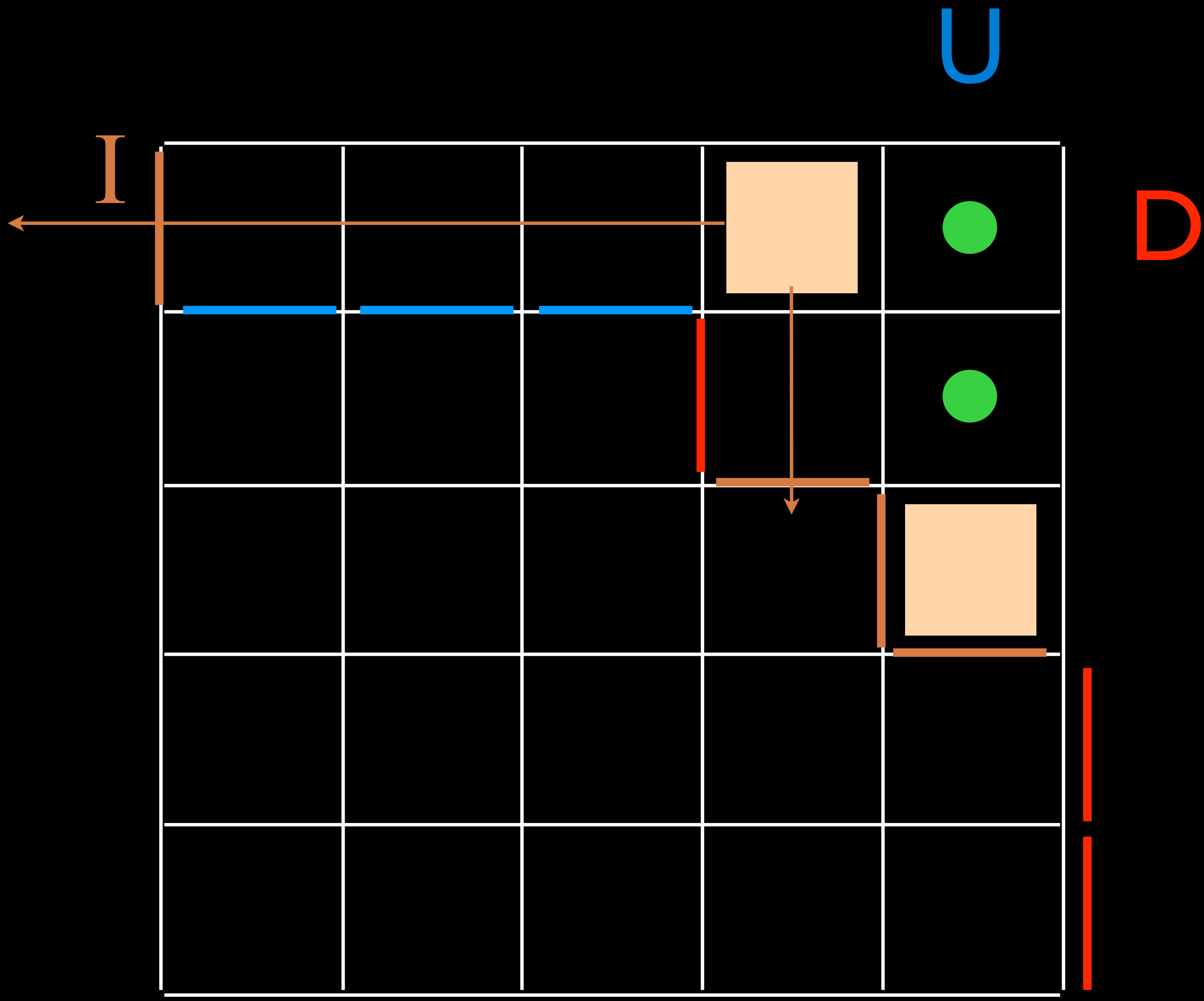
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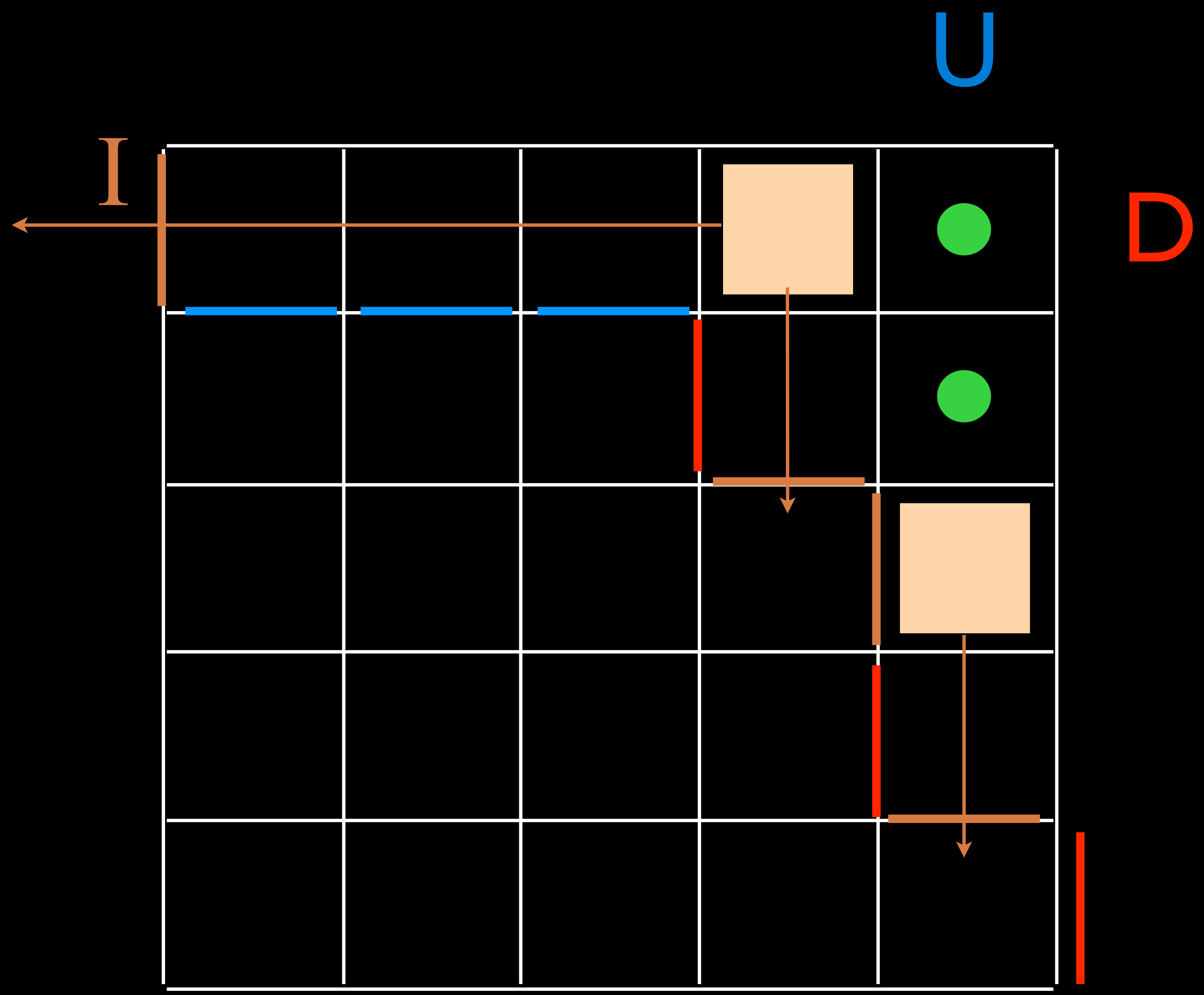
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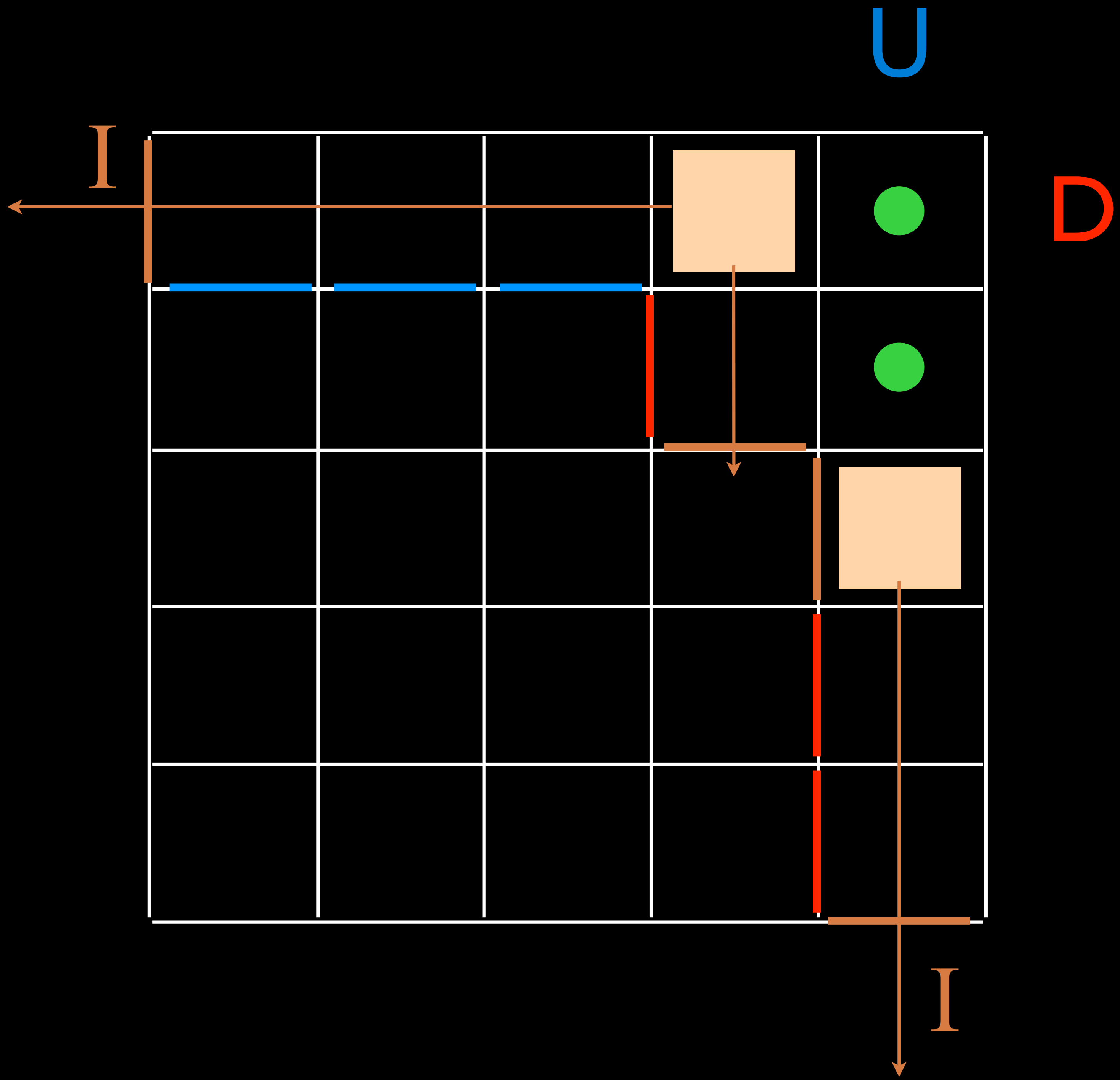

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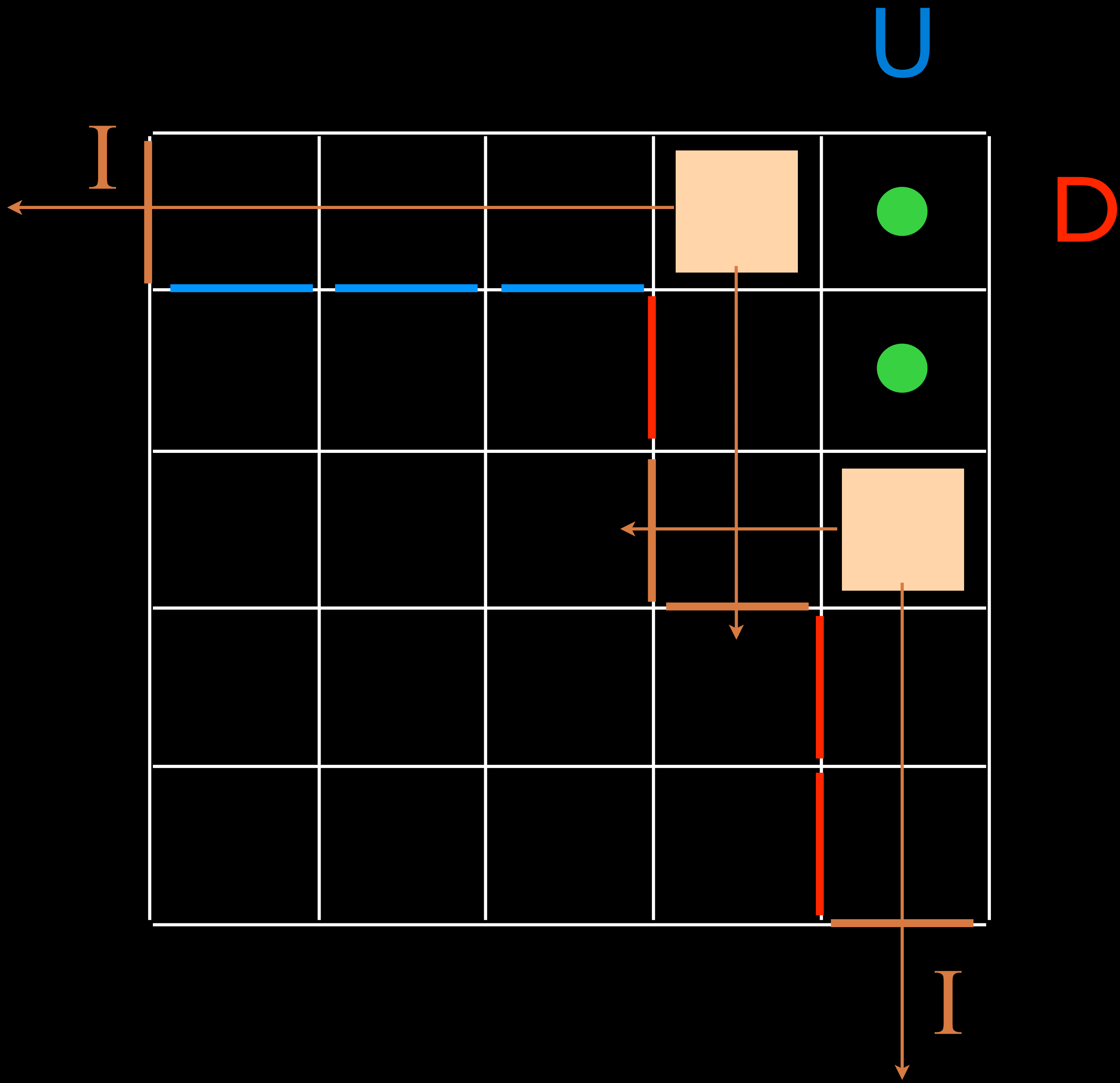
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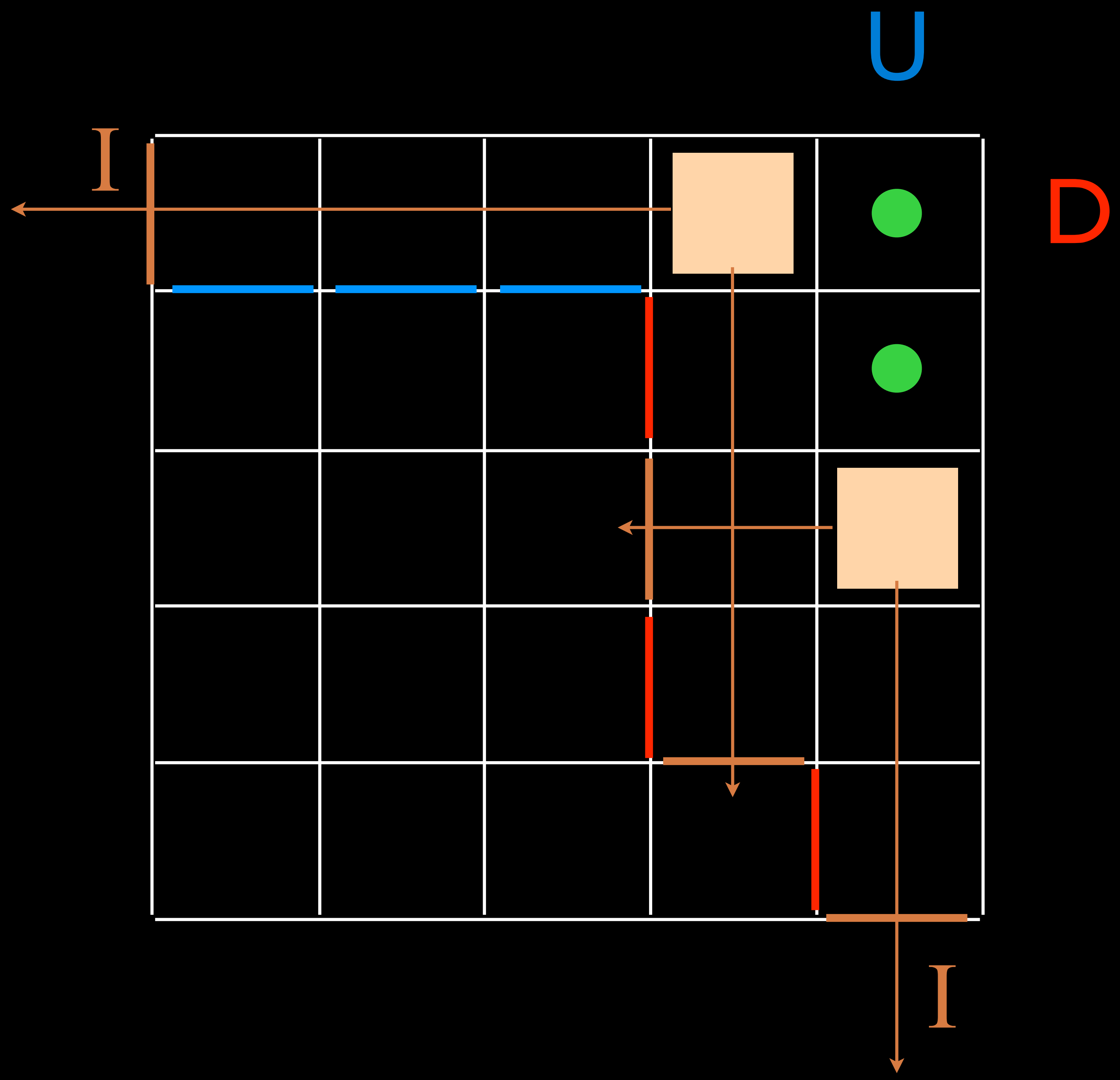

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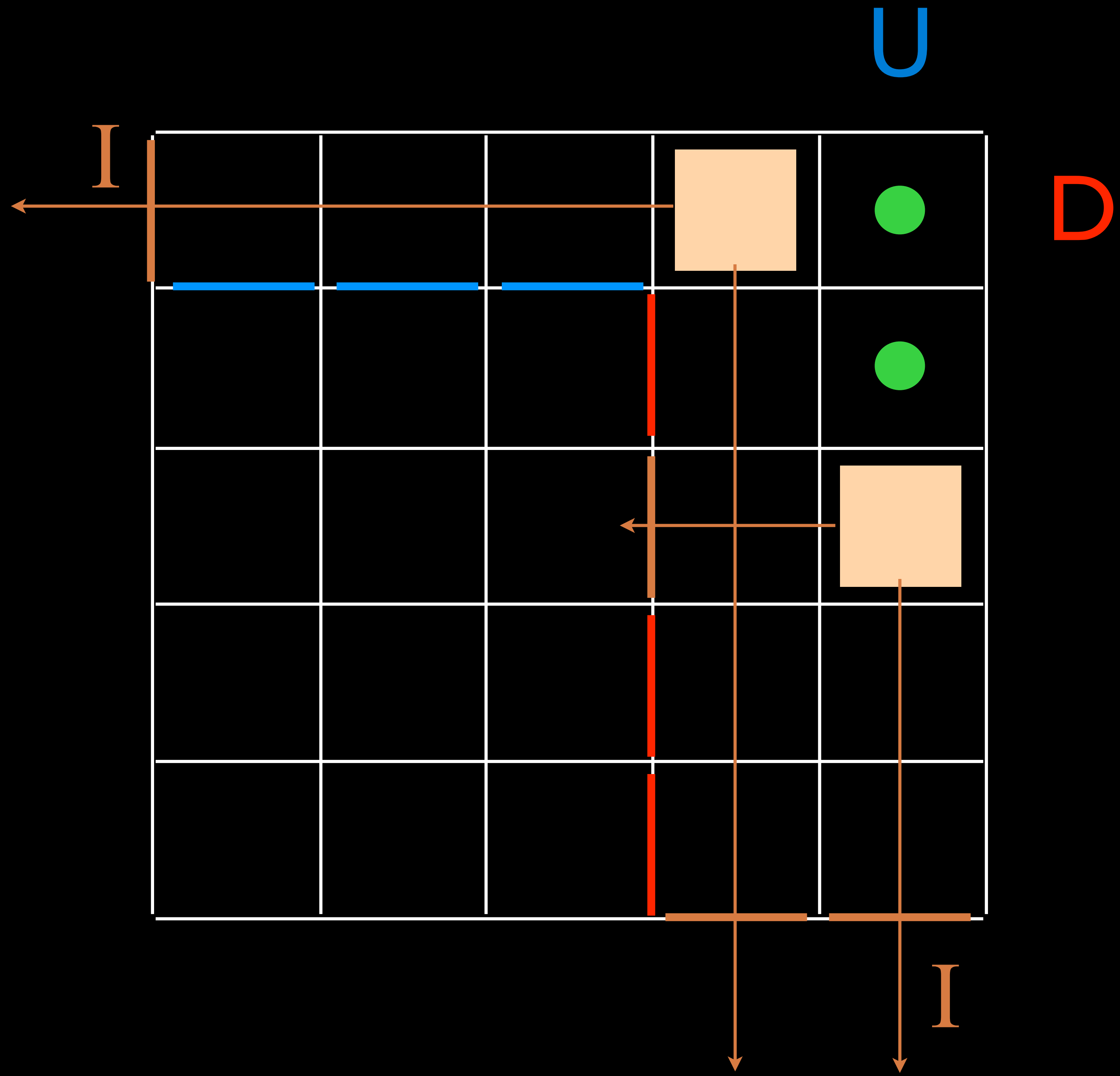
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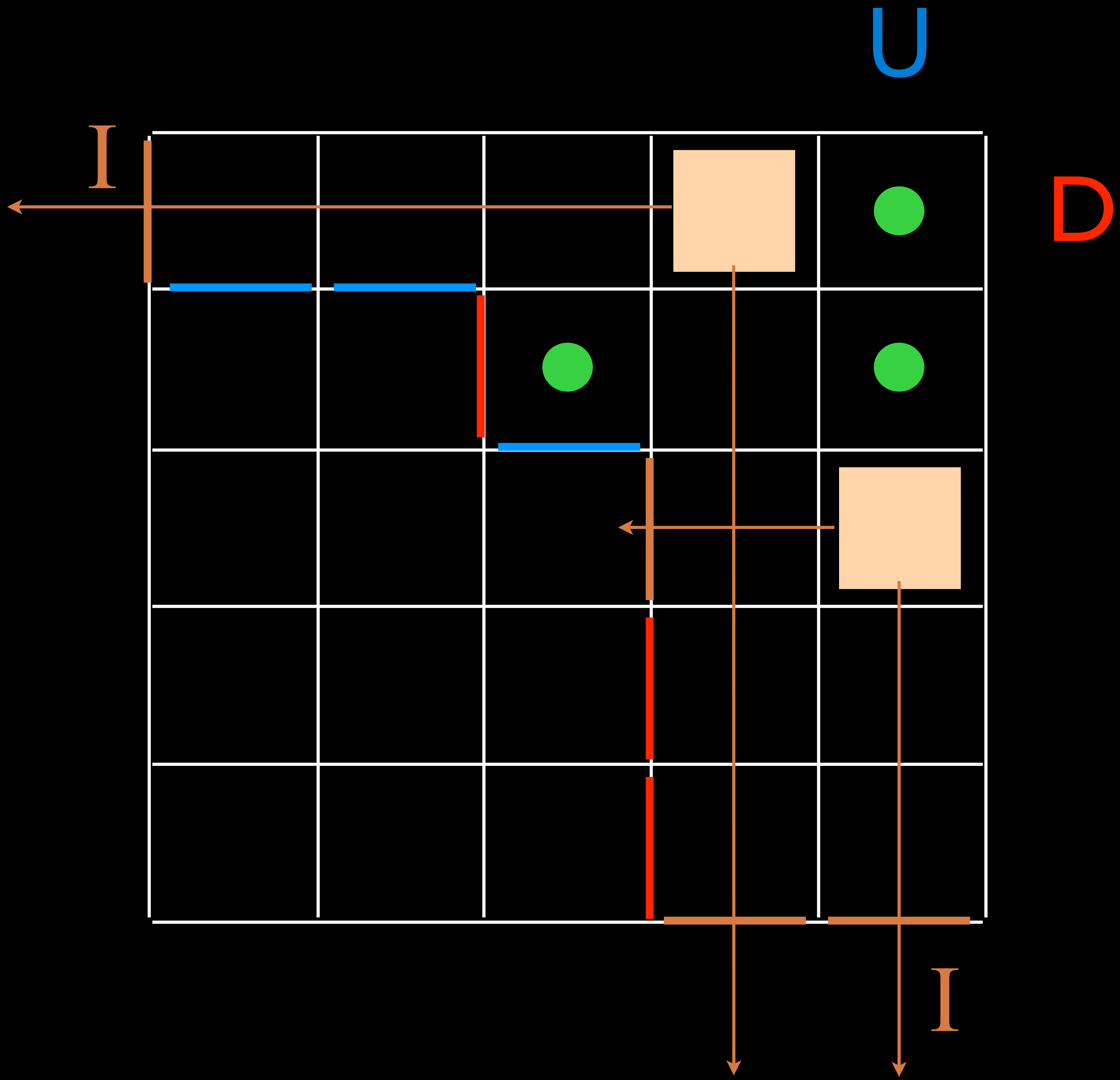
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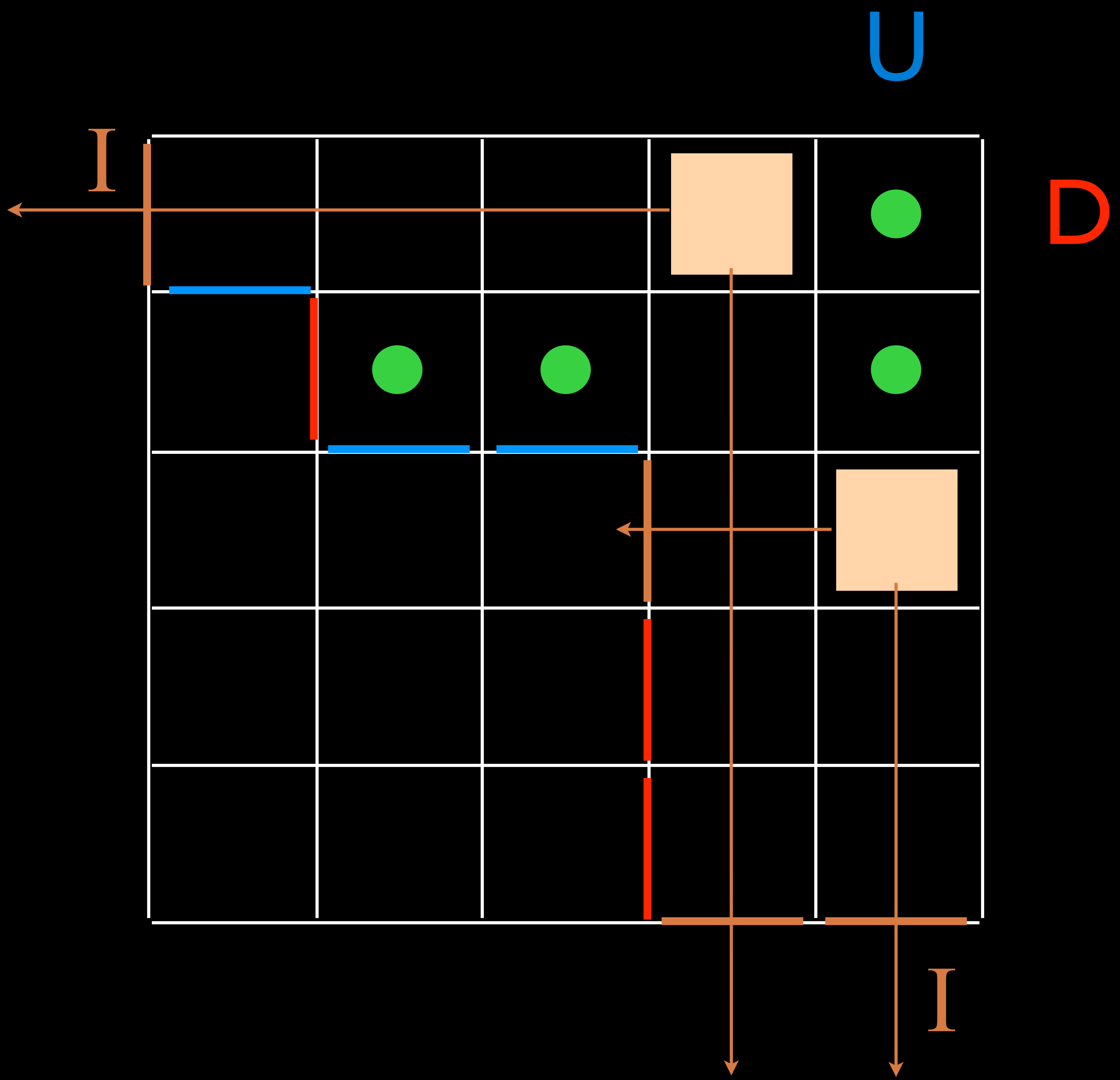
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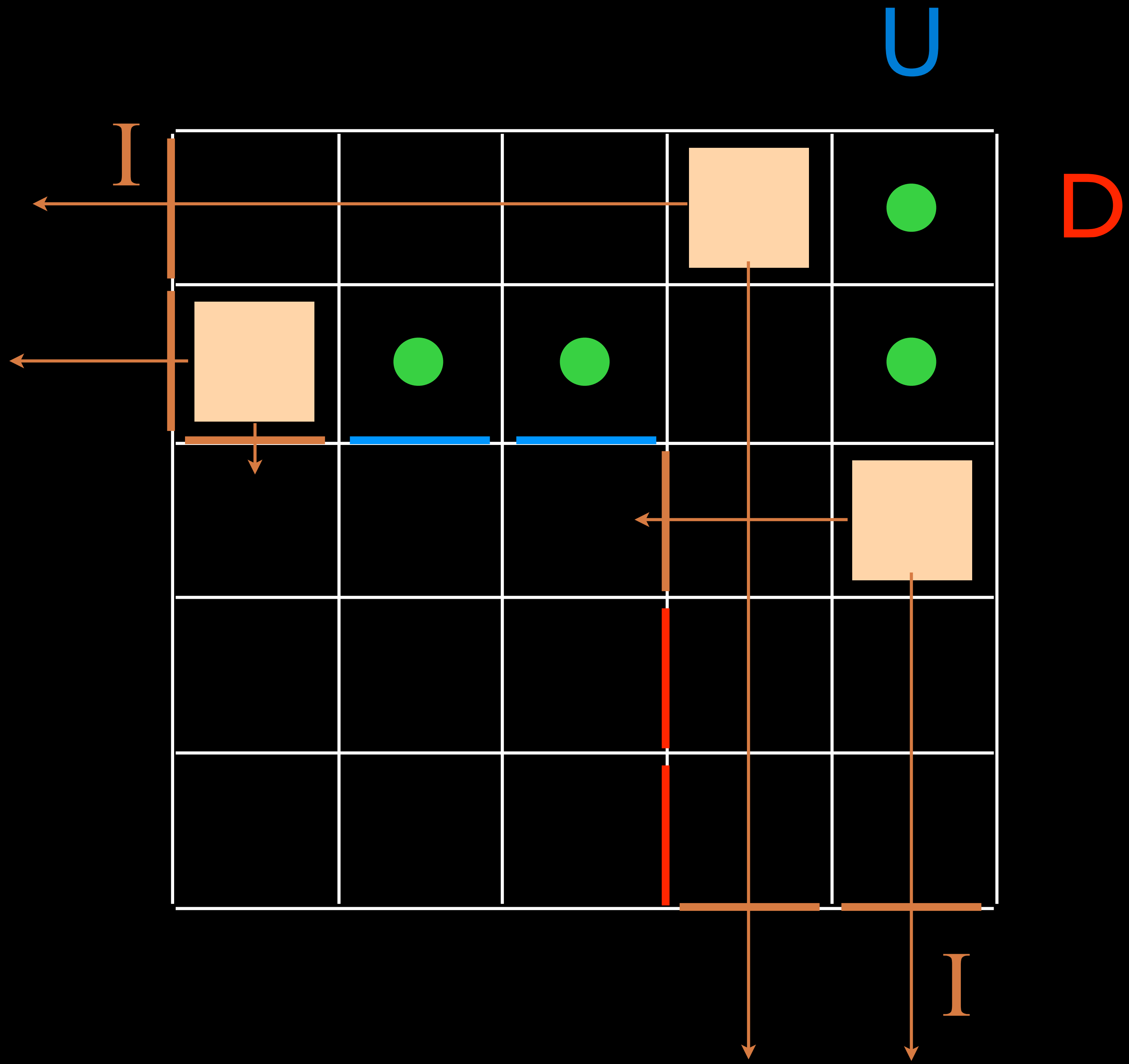

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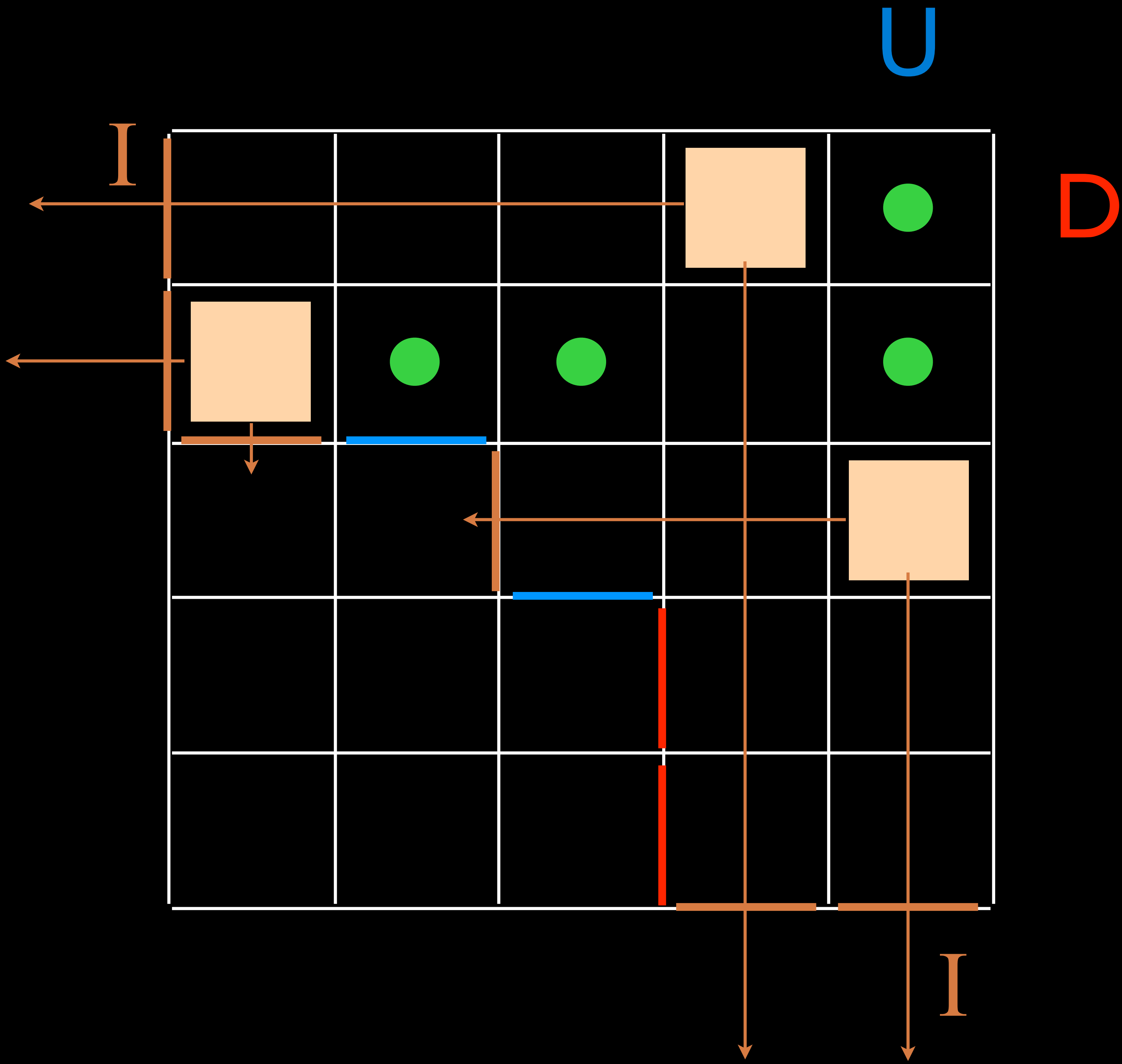
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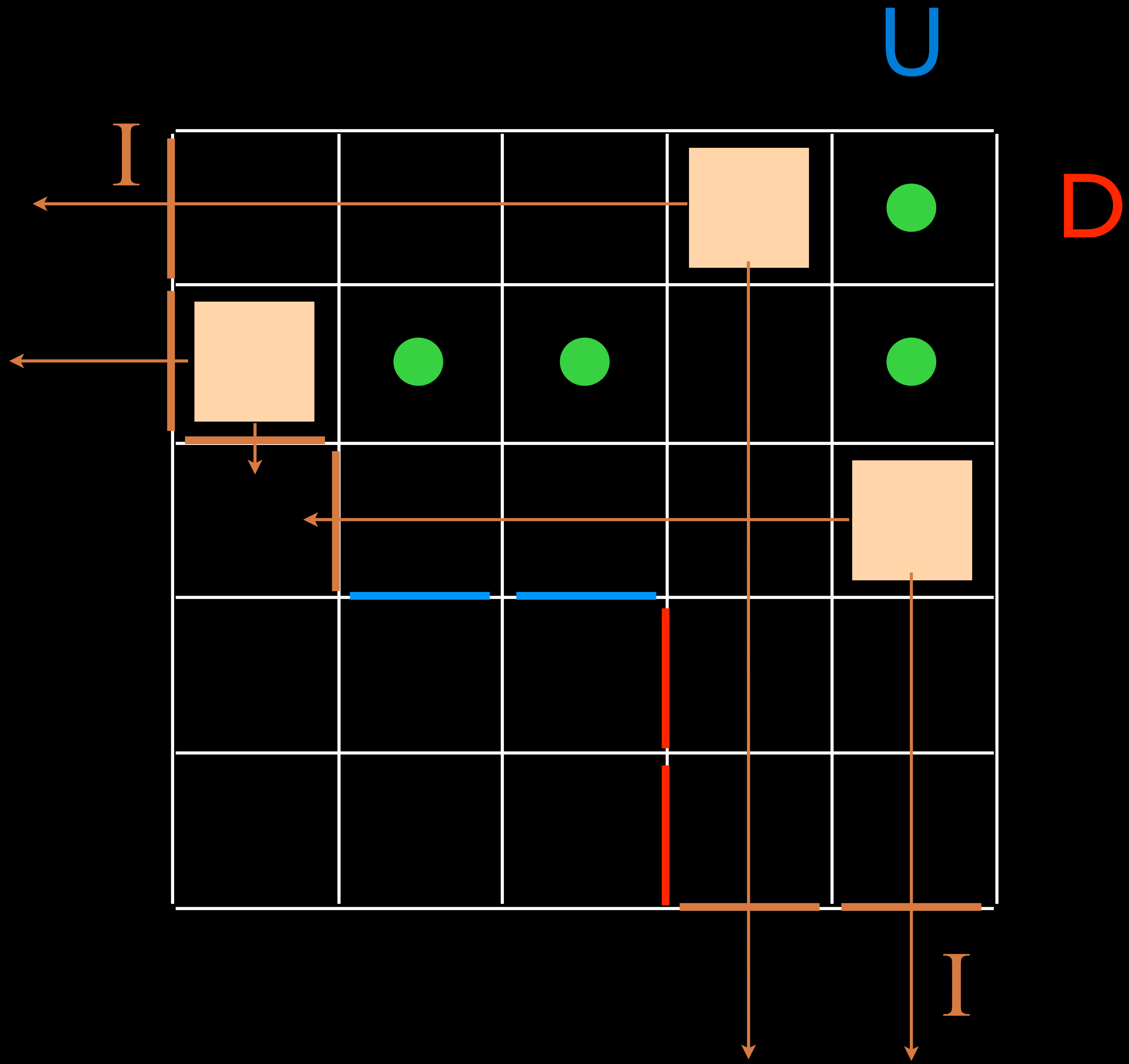
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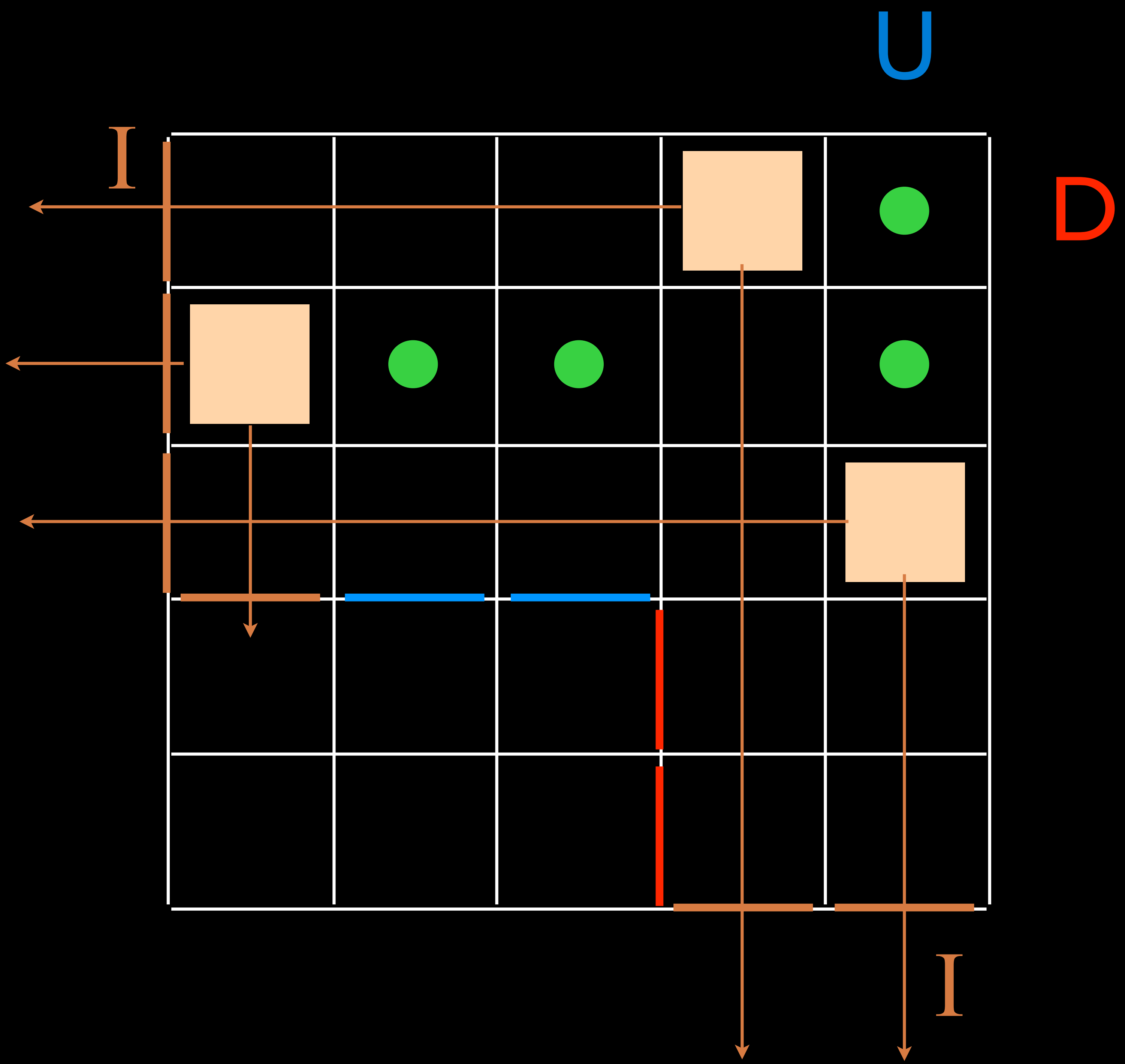
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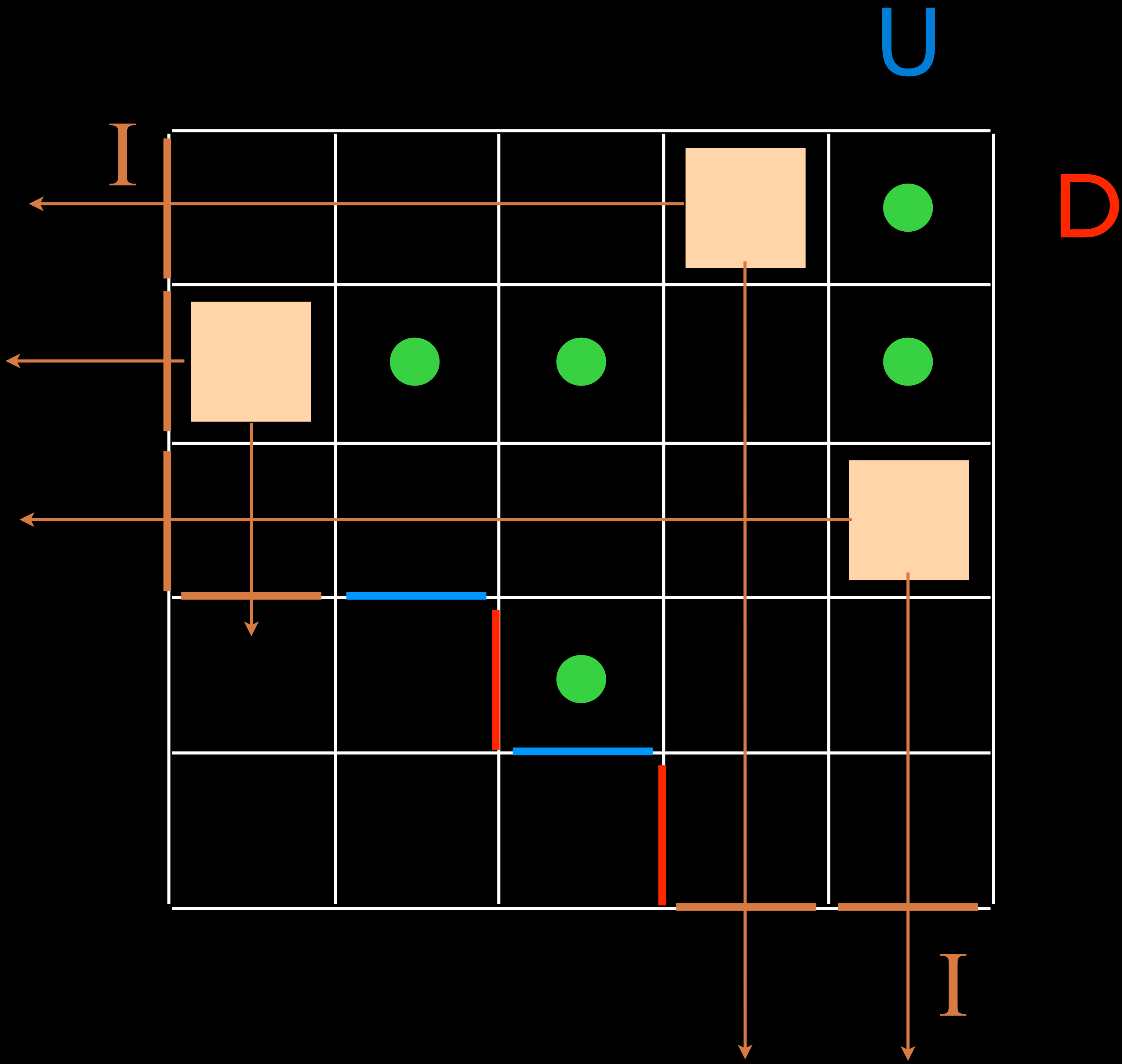
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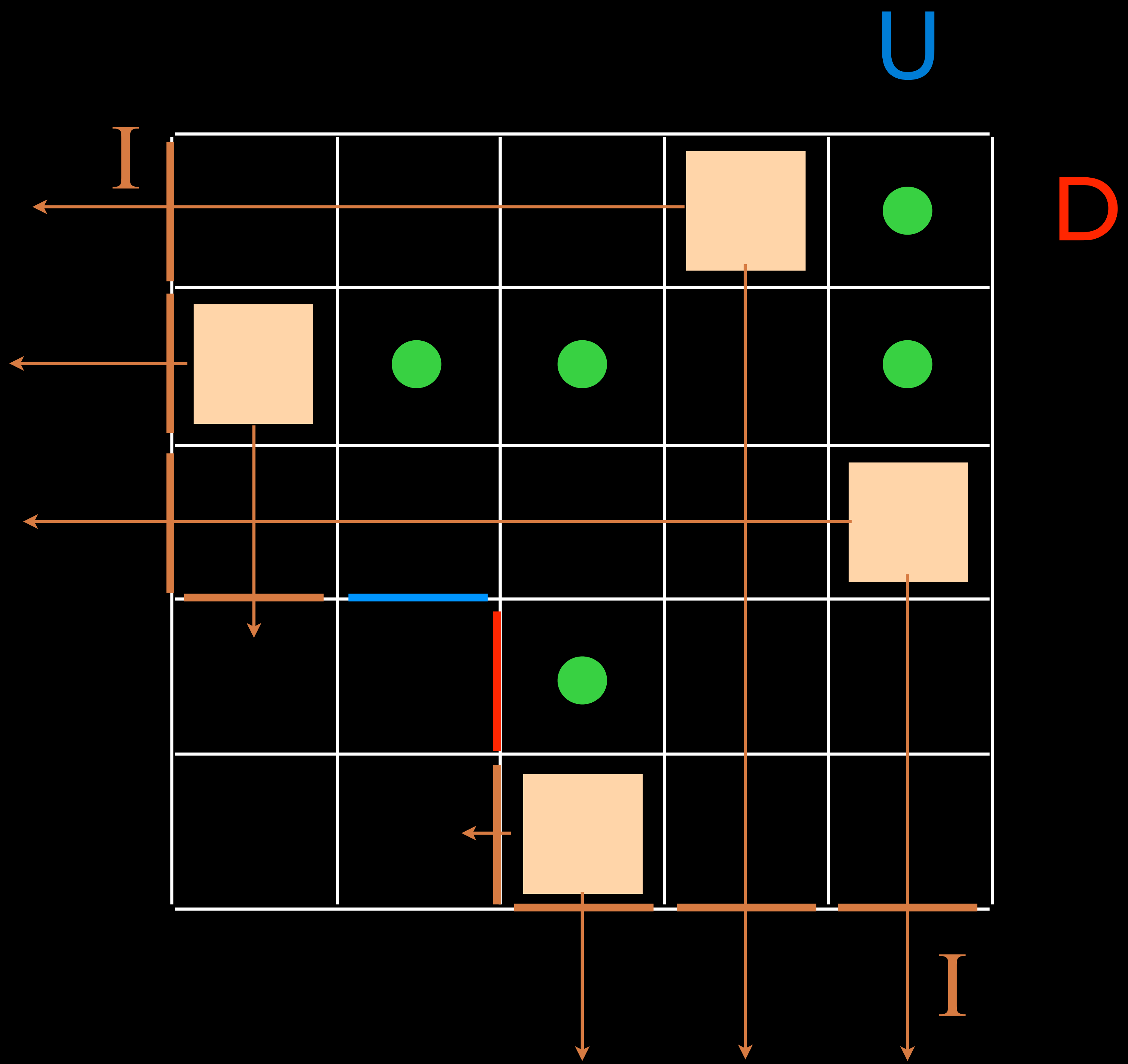
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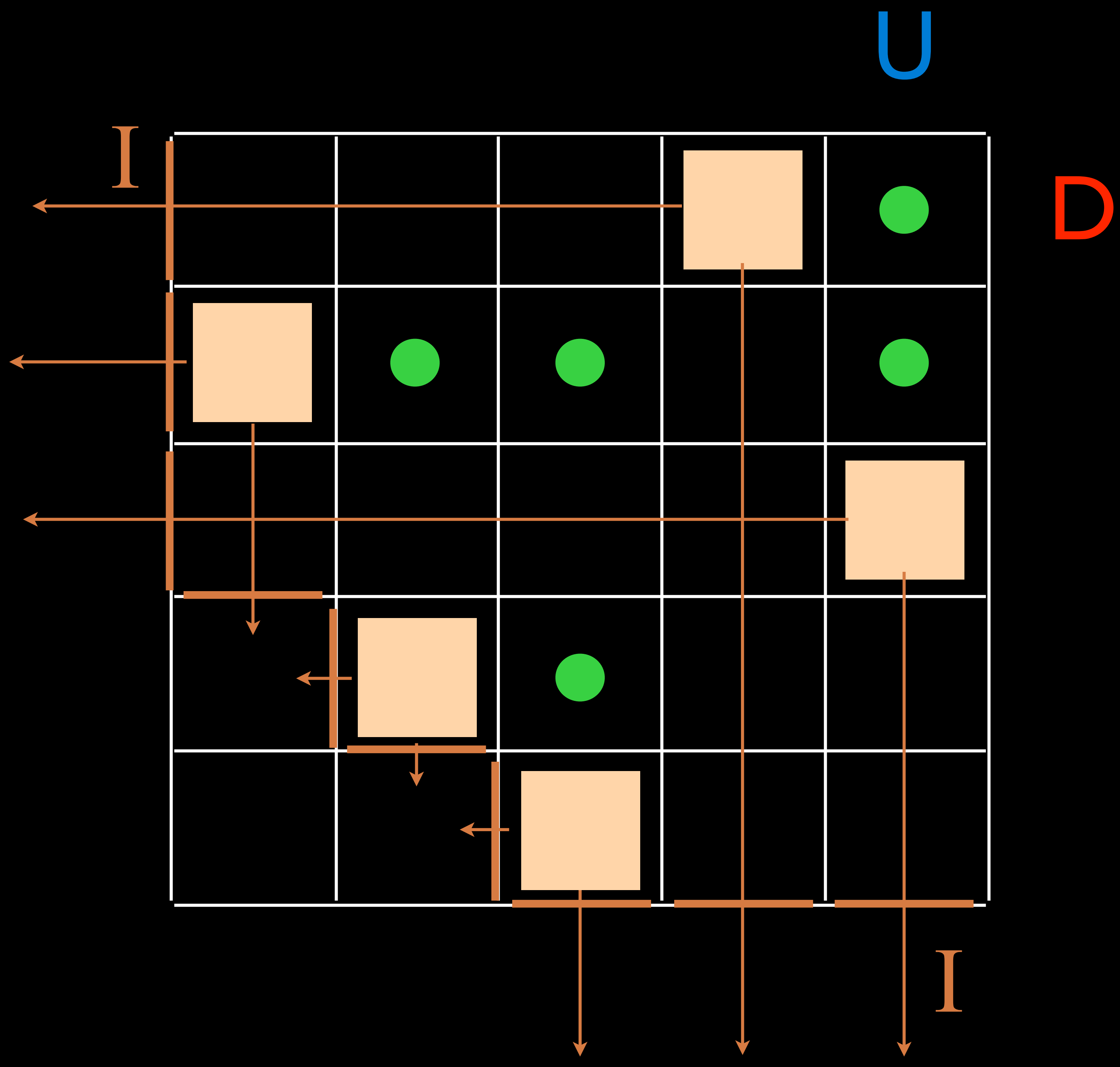
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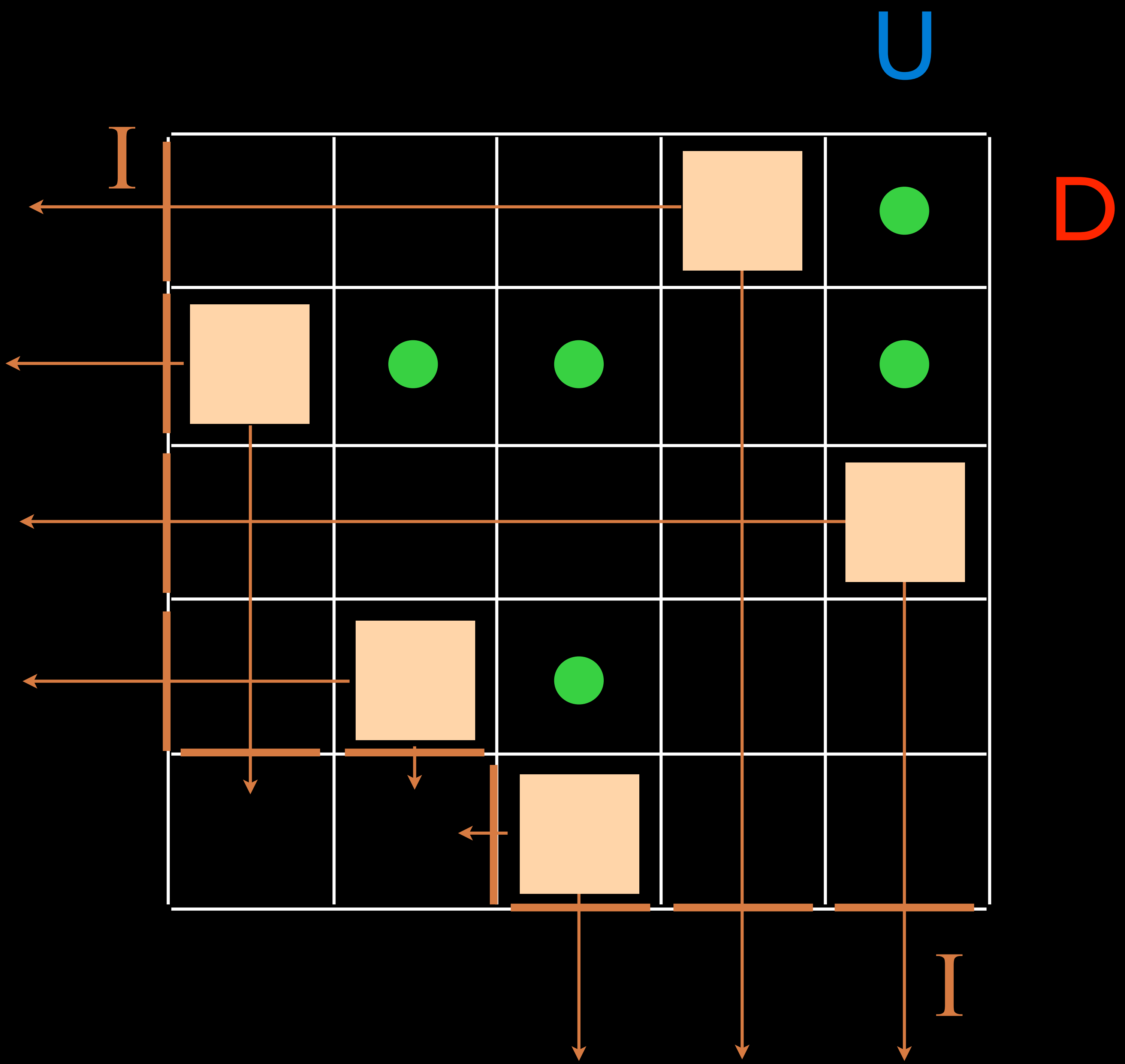
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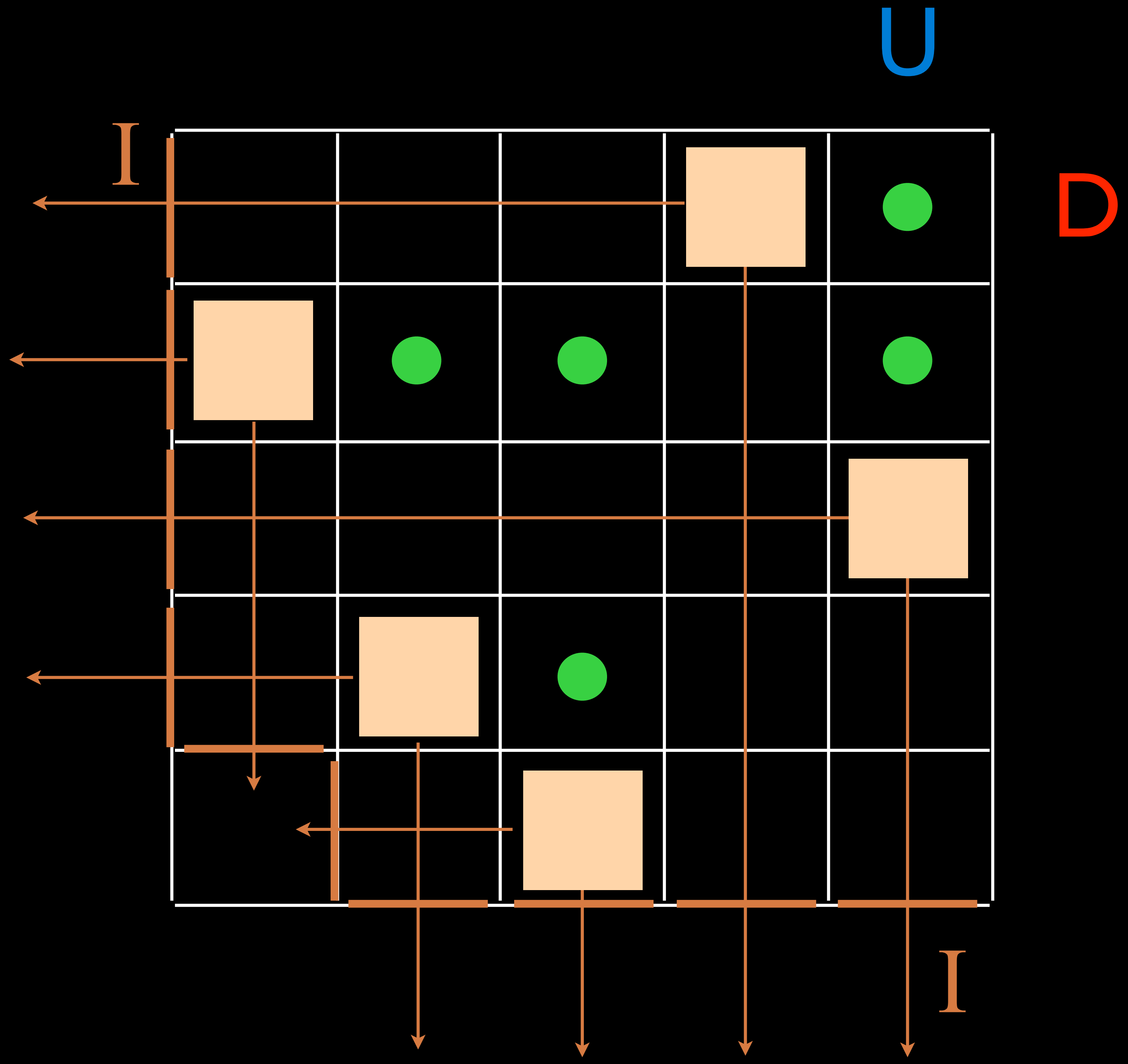
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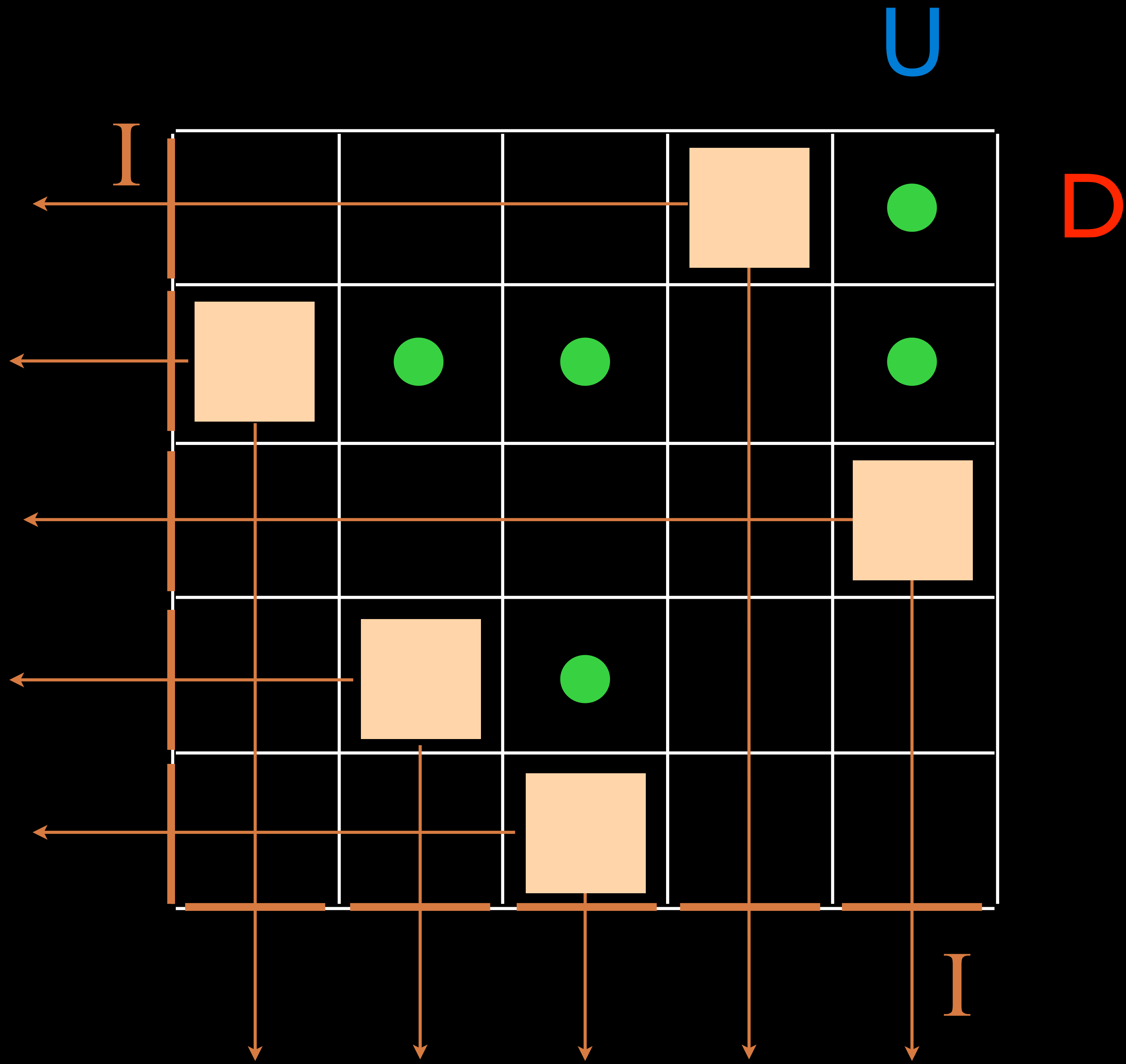


U
D
I
I

$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

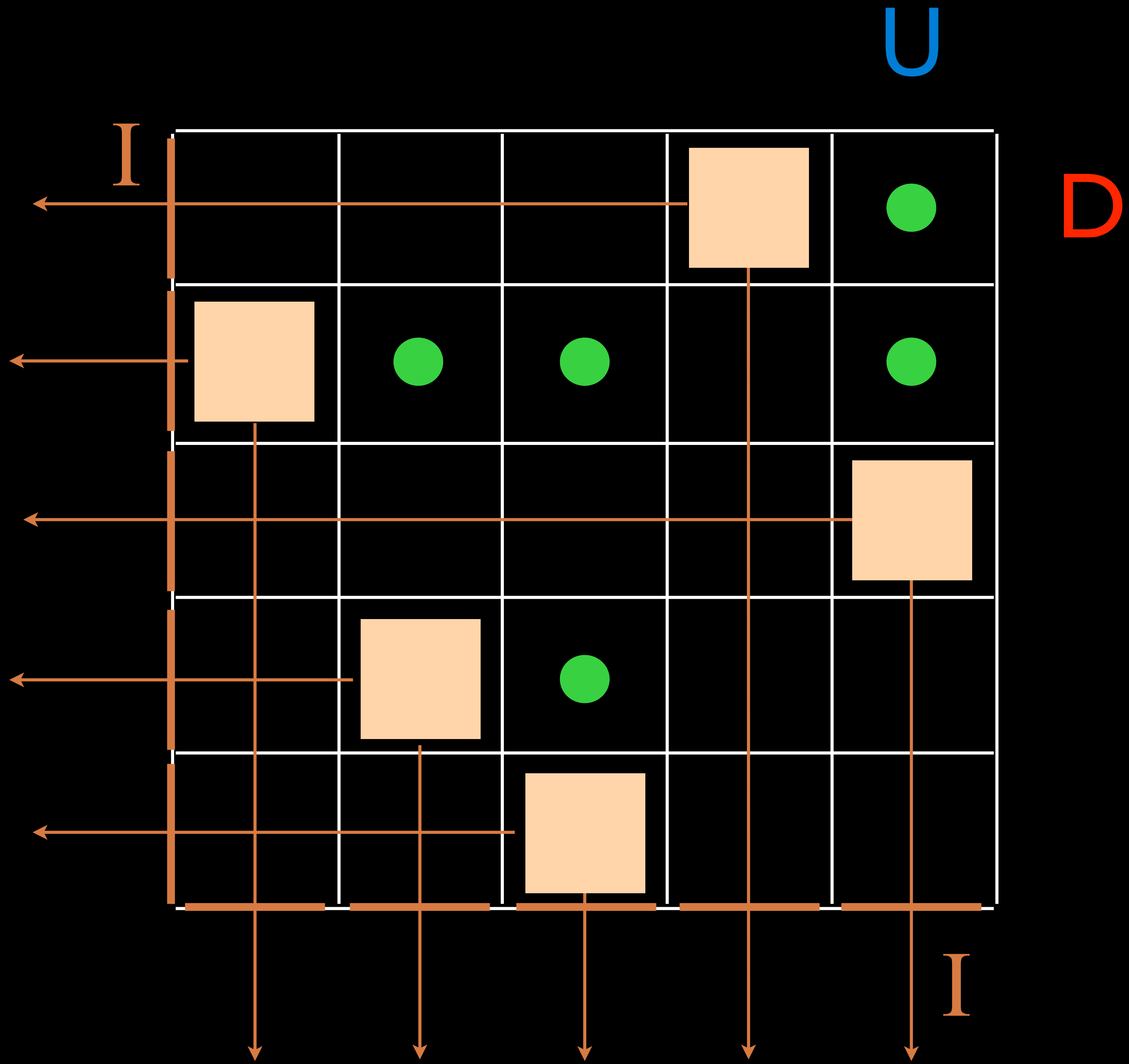
Quadratic algebra Q

5 rewriting rules

"complete" Q-tableau (5 labels)

$\downarrow$

Q-tableau (2 labels)


U
I
D
I

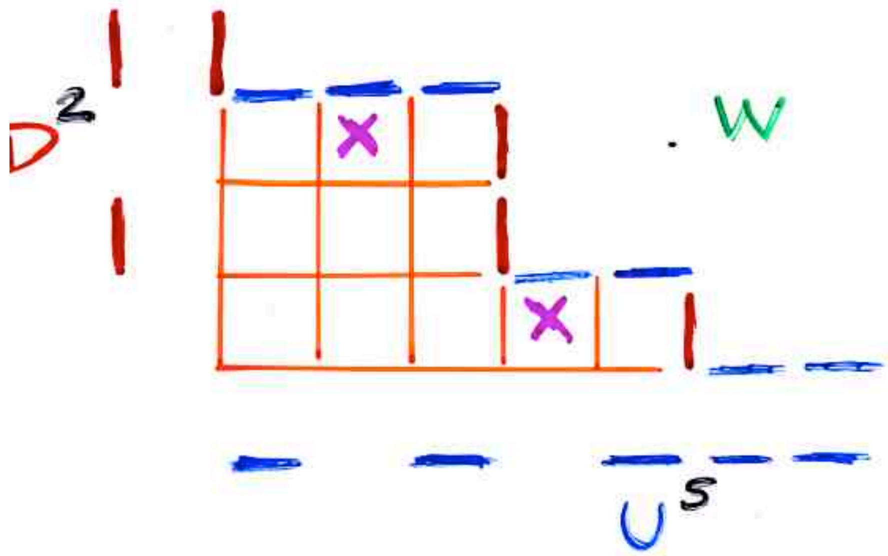
$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} \, D^i U^i$$

normal ordering

$$c_{n,0} = n!$$

$$c_{n,i} = \binom{n}{i}^2 (n-i)!$$

notation $w \to F_w$ diagram Ferrers



<u>Prop.</u> $c_{i,j}(w)$ = nb de "placement" de k tours sur F

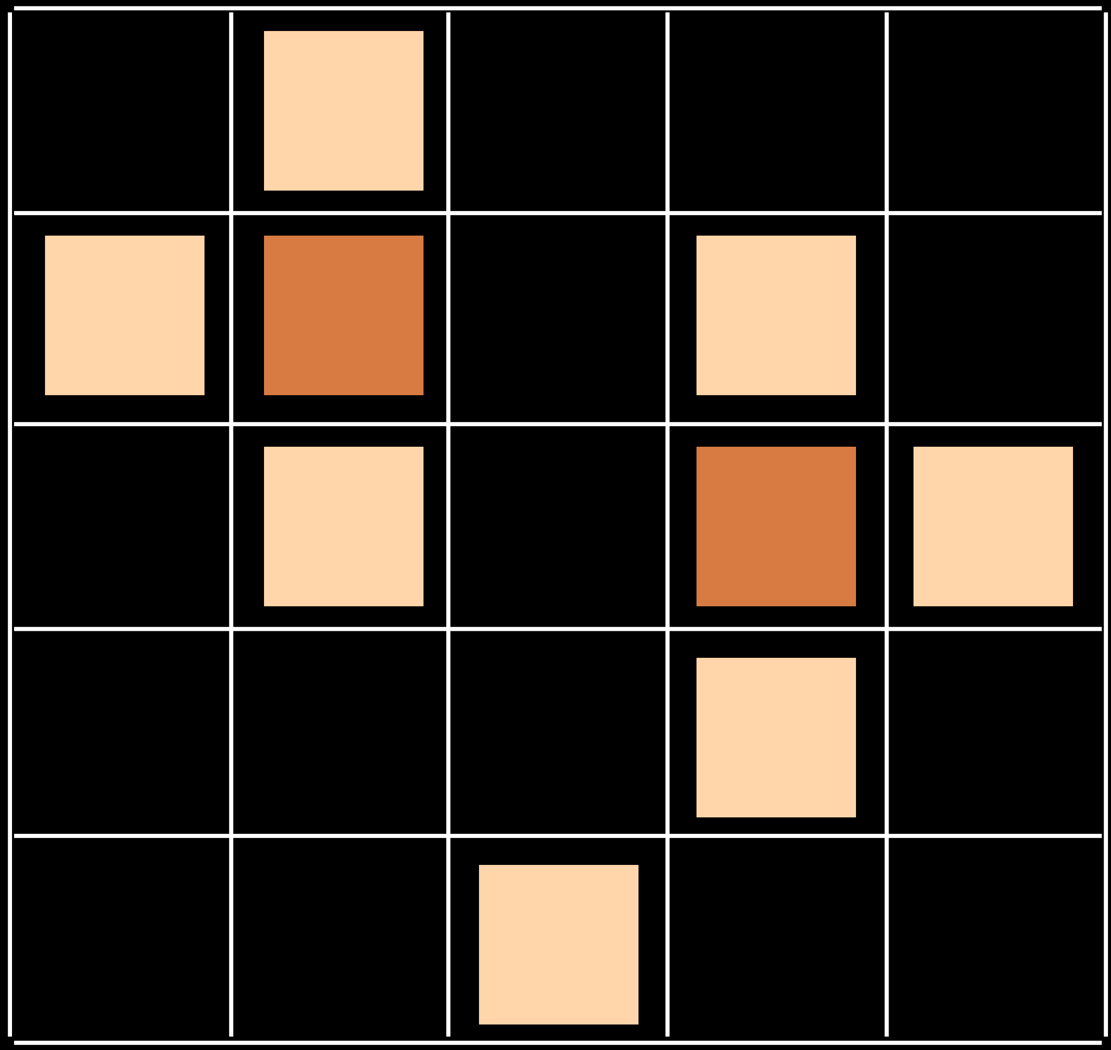
avec $i = |w|_D - k$
$j = |w|_U - k$

quadratic algebra for
alternating sign matrices (ASM)

ASM

$$\begin{matrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & -1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & 1 \\ \cdot & \cdot & 1 & -1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{matrix}$$

Alternating sign matrices



A, A', B, B',

commutations

$$\begin{cases} B A = A B + A' B' \\ B' A' = A' B' + A B \end{cases}$$

$$\begin{cases} B' A = A B' \\ B A' = A' B \end{cases}$$

Lemma. Any word $w(A, A', B, B')$ in letters A, A', B, B', can be uniquely written

$$\sum c(u, v; w)\ \underbrace{u(A, A')}_{\substack{\text{word} \\ \text{in } A,\, A'}}\ \underbrace{v(B, B')}_{\substack{\text{word} \\ \text{in } B,\, B'}}$$

Prop. For $w = B^n A^n$

$u = A'^n$, $v = B'^n$

$c(u, v; w) =$ the number of $n \times n$ ASM (alternating sign matrices)

complete Q-tableaux

Quadratic algebra $\mathbb{Q}$

generators $\mathcal{B} = \{B_j\}_{j \in J}$

$\mathcal{A} = \{A_i\}_{i \in I}$

commutation relations

$$B_j A_i = \sum_{k,l} c_{ij}^{kl} A_k B_l \quad \text{for every } \begin{matrix} i \in I \\ j \in J \end{matrix}$$

Lemma. In $\mathbb{Q}$ every word $w \in (\mathcal{A} \cup \mathcal{B})^*$ can be written in a unique way

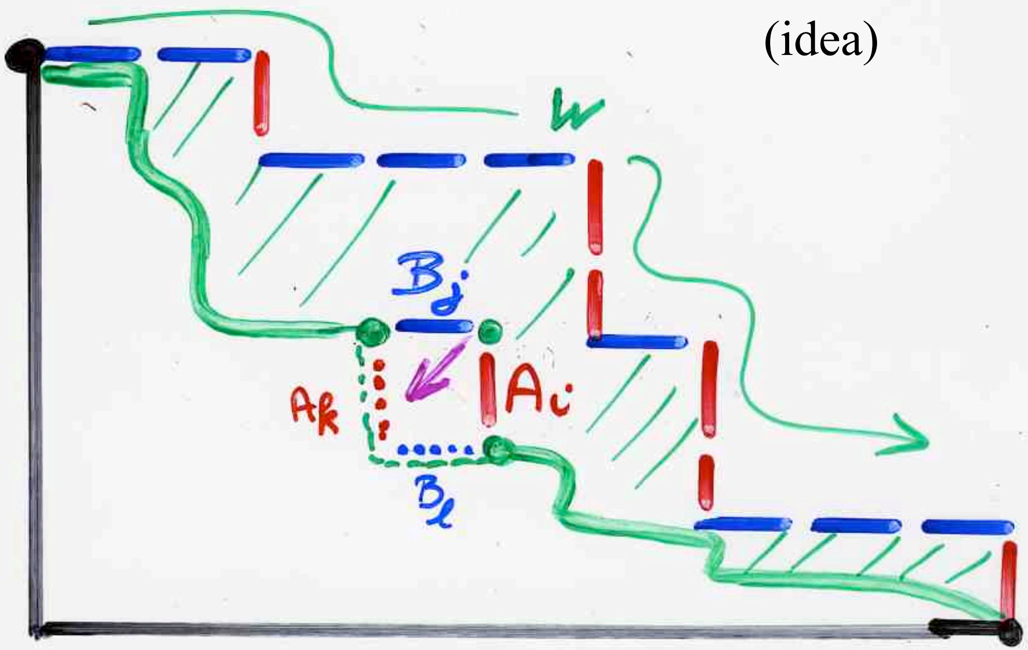
$$w = \sum_{\substack{u \in \mathcal{A}^* \\ v \in \mathcal{B}^*}} c(u, v; w)\, uv$$

This polynomial can be obtained by successive rewriting rules: any occurrence $B_j A_i \longrightarrow \sum c_{ij}^{kl} A_k B_l$ until no more such occurrence.

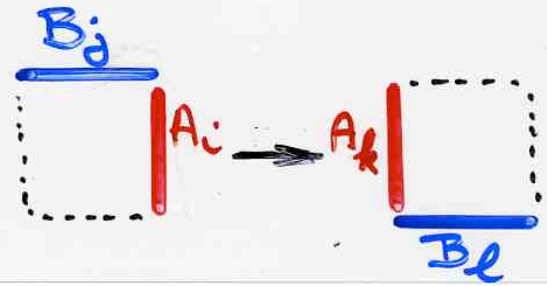
(Lemma) independent of the order of rewriting

Proof:

(idea)



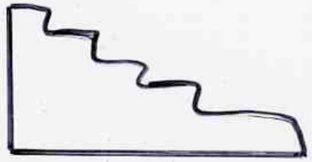
"planar" rewriting



complete

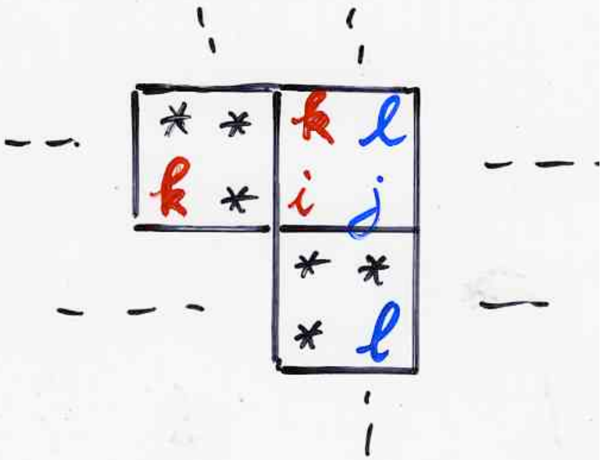
Def. Q-tableau

Ferrers diagram F



each cell $\alpha \in F$ labeled $\begin{array}{|cc|} \hline k & l \\ i & j \\ \hline \end{array}$ $\quad i, k \in I$, $\; j, l \in J$

with "compatibility" condition:



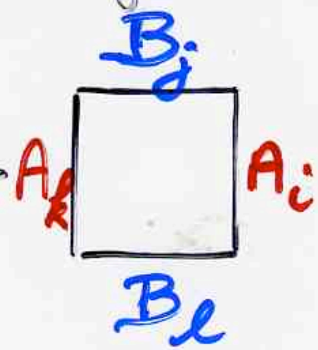
commutation relations

$$B_j A_i = \sum_{k,l} c_{ij}^{kl} A_k B_l \qquad \begin{array}{l} i \in I \\ j \in J \end{array}$$

complete

Def. edge-labeling of a Q-tableau T

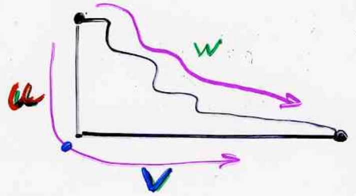
each cell α $\begin{array}{|cc|} \hline k & l \\ i & j \\ \hline \end{array}$ $\longrightarrow$



Def. For T a complete Q-tableau

$uwb(T) \in (\mathcal{A} \cup \mathcal{B})^*$ upper word border

$lwb(T)$ lower word border



Def. weight of a complete Q-tableau T

$$p(T) = \prod_{\substack{\text{cells}\\ \alpha \in F}} c_{i\,j}^{k\,l} \qquad \alpha = \boxed{\begin{matrix} k & l \\ i & j \end{matrix}}$$

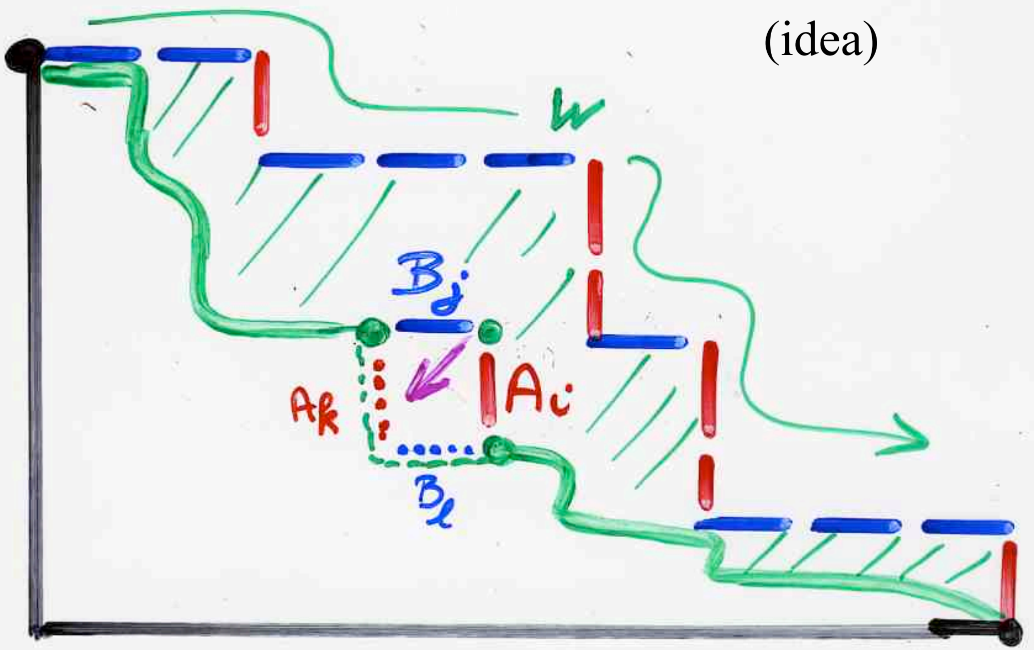
Prop For any $w \in (\mathcal{A} \cup \mathcal{B})^*$, $u \in \mathcal{A}^*$, $v \in \mathcal{B}^*$

$$c(u, v; w) = \sum_{T} p(T)$$

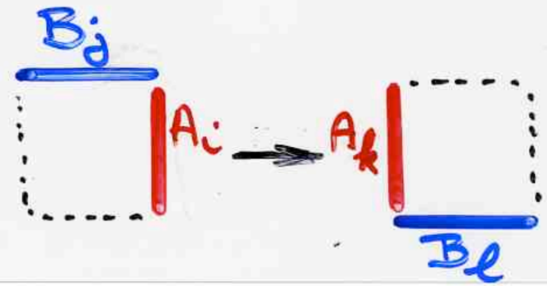
T complete Q-tableau

$$\begin{cases} uwb(T) = w \\ lwb(T) = uv \end{cases}$$

Proof: (idea)



"planar" rewriting



Q-tableaux

S set of labels

$$\varphi : \left\{ \begin{bmatrix} k & l \\ i & j \end{bmatrix} \right\} = R \longrightarrow S$$

set of rewriting rules

$$B_j A_i \to C_{ij}^{kl} A_k B_l$$

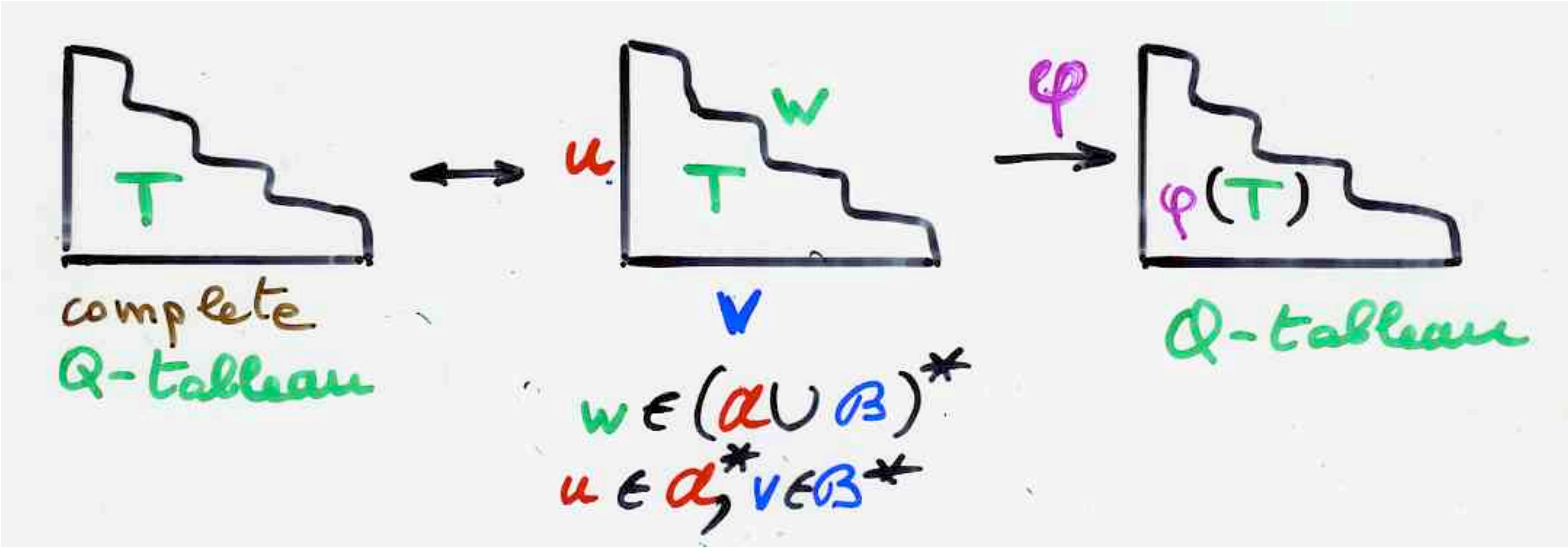
such that:

if $\begin{pmatrix} k & l \\ i & j \end{pmatrix} \neq \begin{pmatrix} k' & l' \\ i' & j' \end{pmatrix}$ and $\varphi\begin{pmatrix} k & l \\ i & j \end{pmatrix} = \varphi\begin{pmatrix} k' & l' \\ i' & j' \end{pmatrix}$

then $(i, j) \neq (i', j')$

<u>Def</u> Q-tableau

"image" by φ of a "complete Q-tableau"



w-compatible

w fixed

$\{$ set of Q-tableaux w-compatible $\}$

$\Updownarrow$ bijection

$\{$ set of complete Q-tableaux T with $uwb(T) = w$ $\}$

Q-tableaux: examples

$$UD = DU + I$$

Weyl-Heisenberg algebra

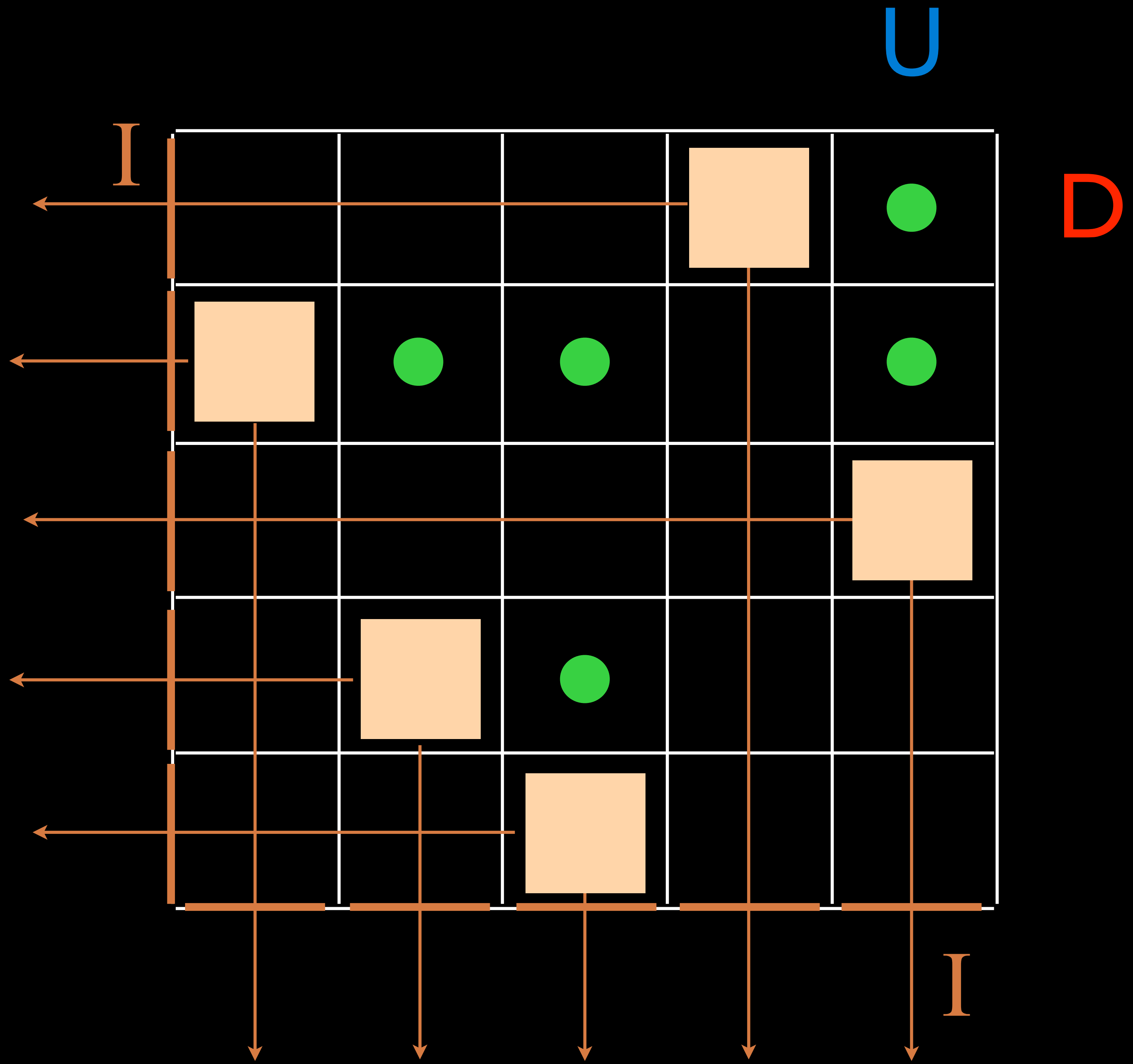
$$\begin{cases} UD = qDU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

$$w = U^n D^n \qquad c(u, v; w) = n!$$
$$uv = I_v^n I_h^n$$

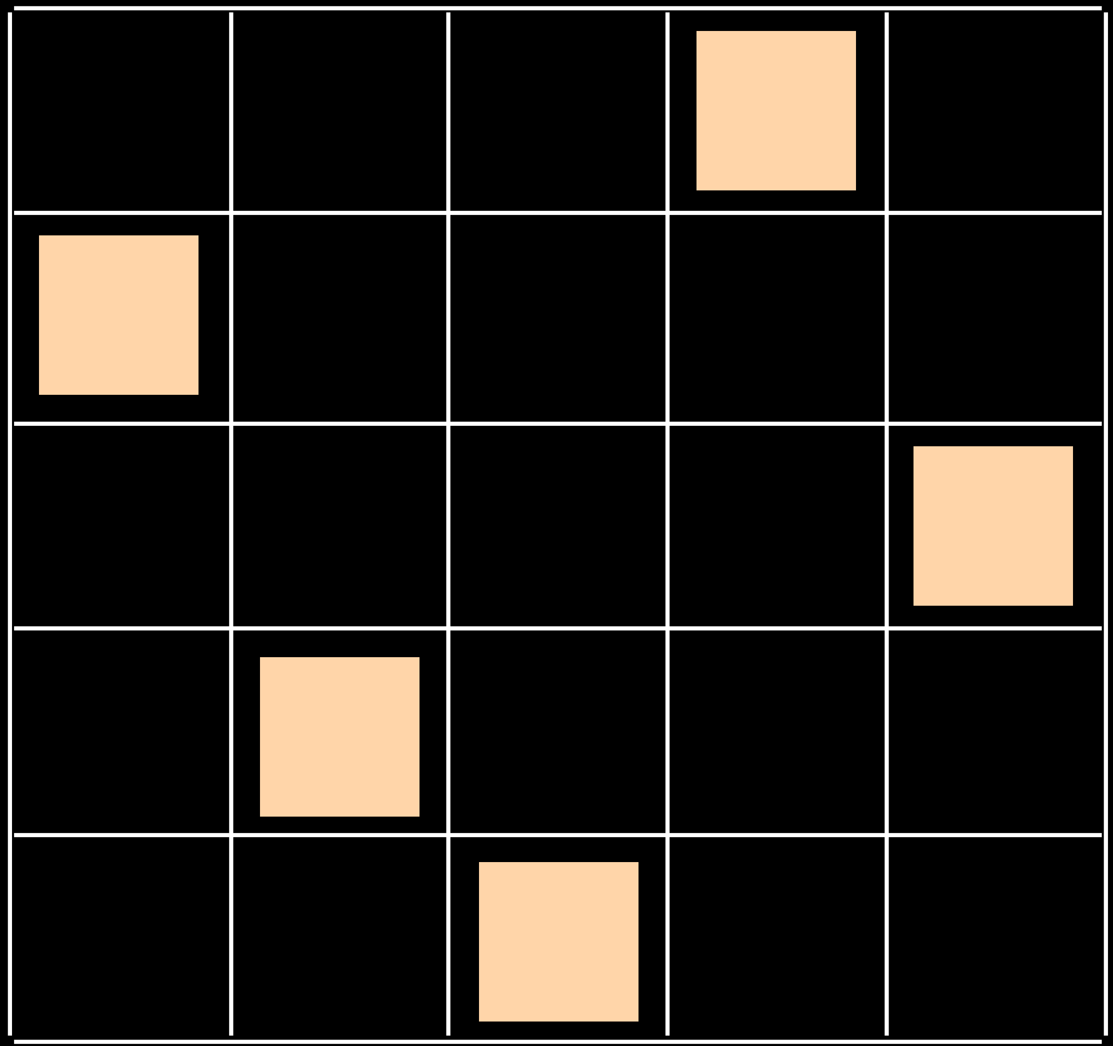
complete

Q-tableau $\longleftrightarrow$ Permutations $\mathfrak{S}_n$

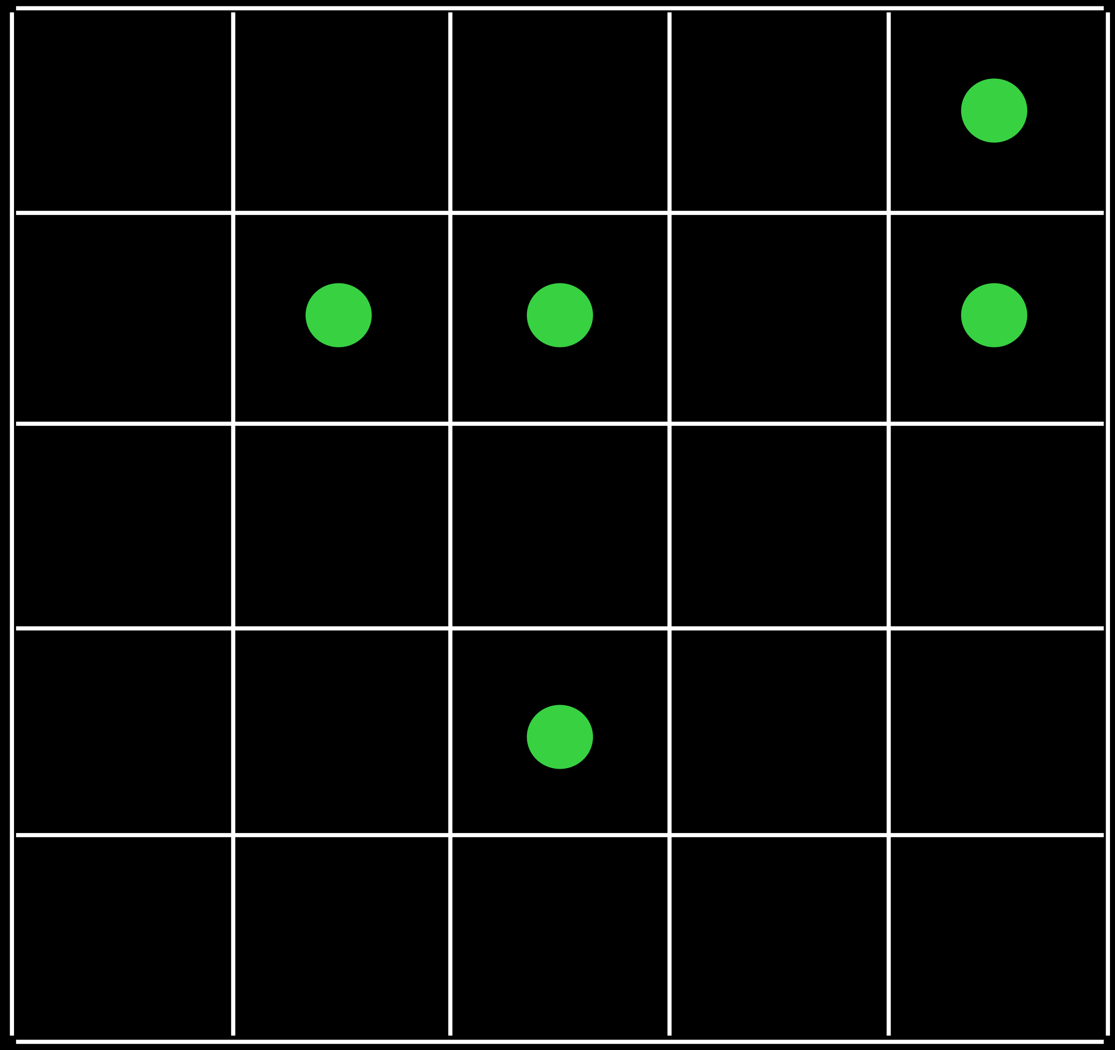
$$\begin{cases} uwb(T) = U^n D^n \\ lwb(T) = I_v^n I_h^n \end{cases}$$



permutation as a complete Q- tableau



permutation as a Q- tableau



another Q-tableau:
Rothe diagram of a permutation

equivalence
Q-tableaux -- planar automaton

equivalence

Q-tableaux $\longleftrightarrow$ tableaux accepted by a planar automaton

Q quadratic algebra

$\mathcal{P}=(S, \mathcal{B}, \alpha, \theta, w, uv)$

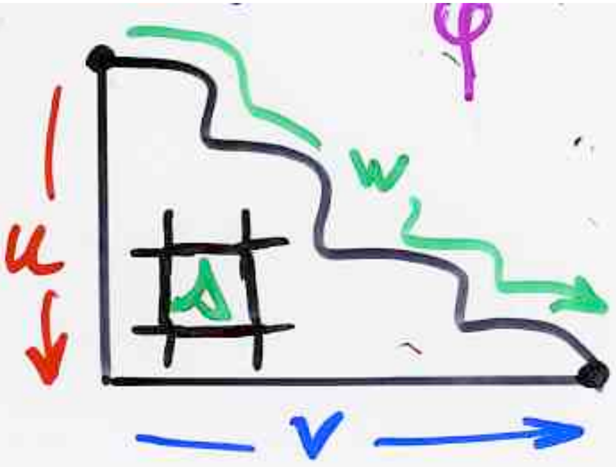
φ

with $\mathcal{P}$ satisfying

$\theta(s, B, A) = \theta(t, B, A)$

$\Rightarrow \quad s = t$

φ



$$BA = \sum_{\substack{s \in S \\ (B', A') = \theta(s, B, A)}} A' B'$$

The 8-vertex algebra
(or XYZ - algebra)
(or Z - algebra)

The quadratic algebra $\mathbf{Z}$

4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\cdots}$, $t_{\cdots}$

$$\begin{cases} B A = q_{00} \, A B + t_{00} \, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet} \, A_\bullet B_\bullet + t_{\bullet\bullet} \, A B \\ B_\bullet A = q_{\bullet 0} \, A B_\bullet + t_{\bullet 0} \, A_\bullet B \\ B A_\bullet = q_{0\bullet} \, A_\bullet B + t_{0\bullet} \, A B_\bullet \end{cases}$$

alternating sign matrices (ASM)

(as a Q-tableau)

$$t_{\bullet\circ} = t_{\circ\bullet} = 0$$

The quadratic algebra Z

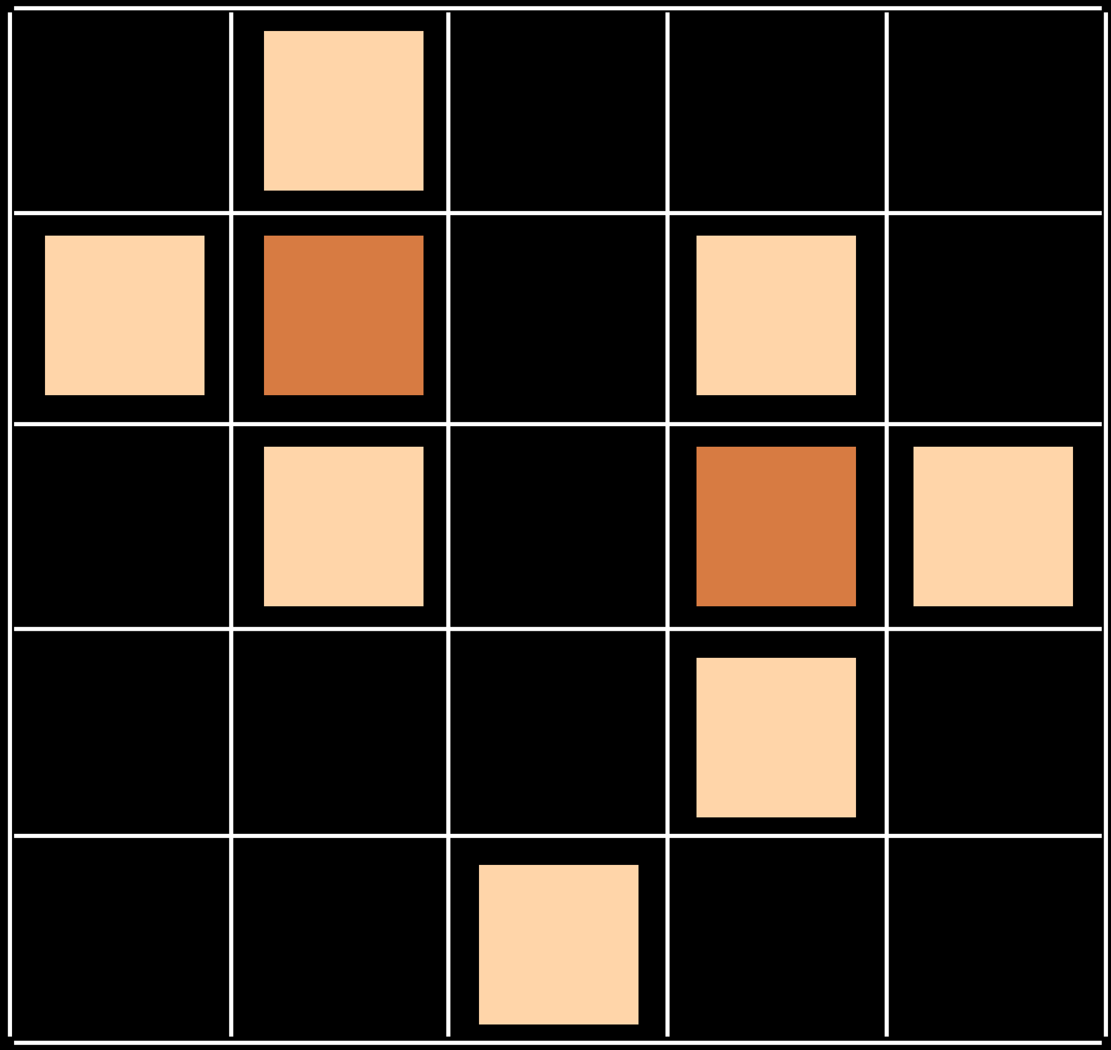
4 generators $B_\bullet \; A_\bullet \; B \; A$

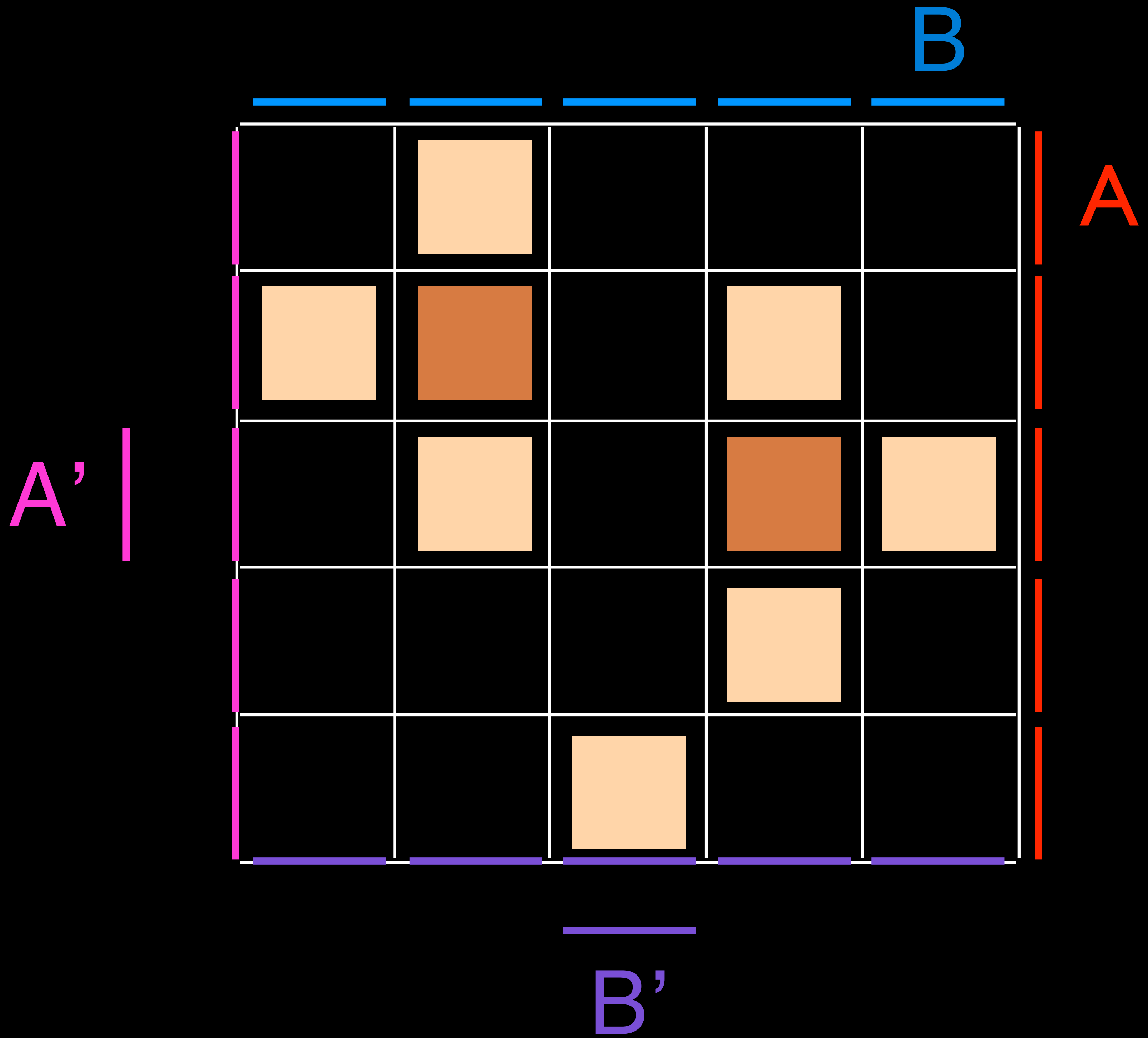
8 parameters $q_{\dots}$, $t_{\dots}$

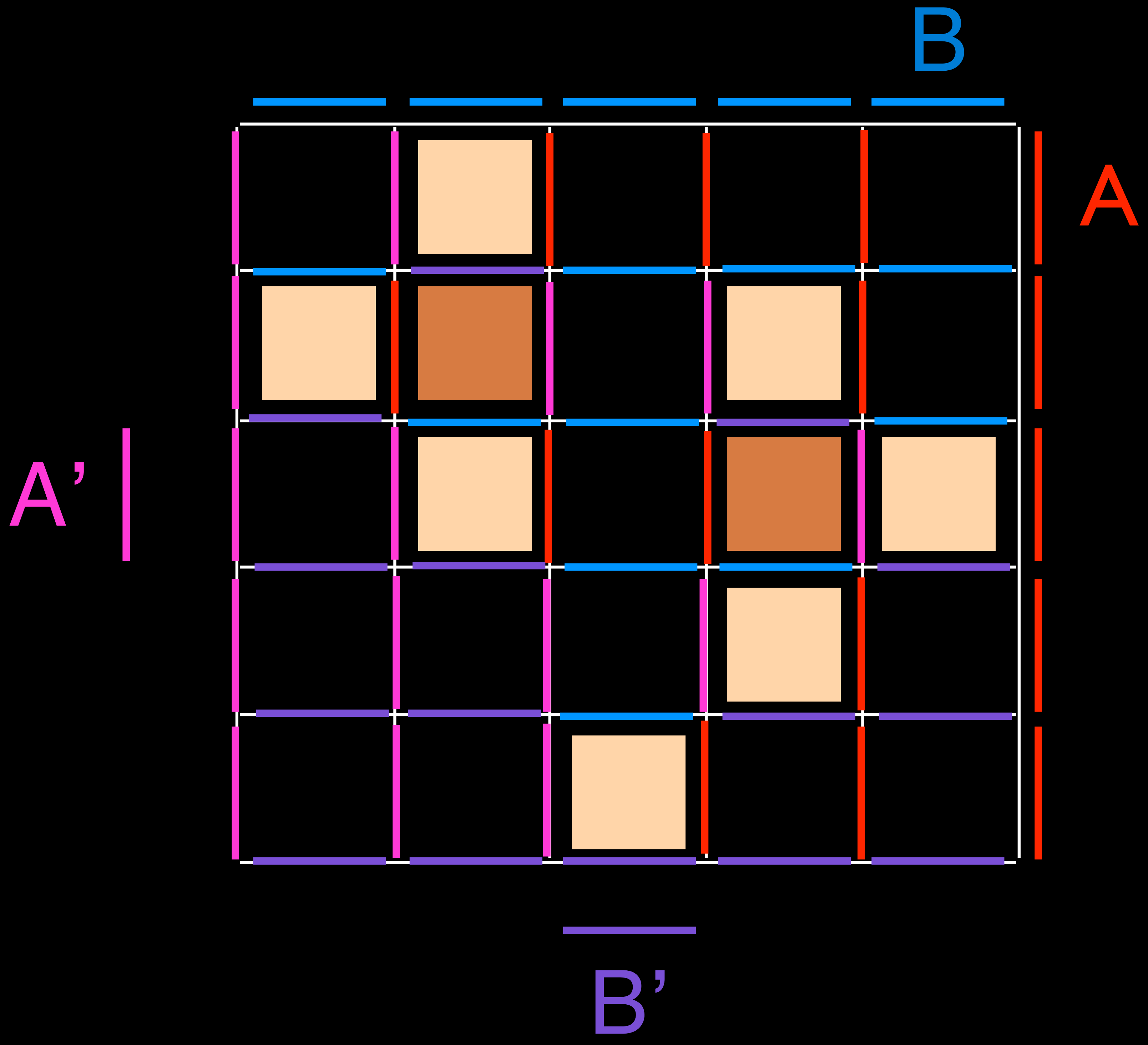
$$\left\{\begin{array}{l}
BA = q_{\circ\circ}\, AB + t_{\circ\circ}\, A_\bullet B_\bullet \\
B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, AB \\
B_\bullet A = q_{\bullet\circ}\, A B_\bullet + 0\, A_\bullet B \\
B A_\bullet = q_{\circ\bullet}\, A_\bullet B + 0\, A B_\bullet
\end{array}\right.$$

$w = B^n A^n \qquad uv = A_\bullet^n \, B_\bullet^n$

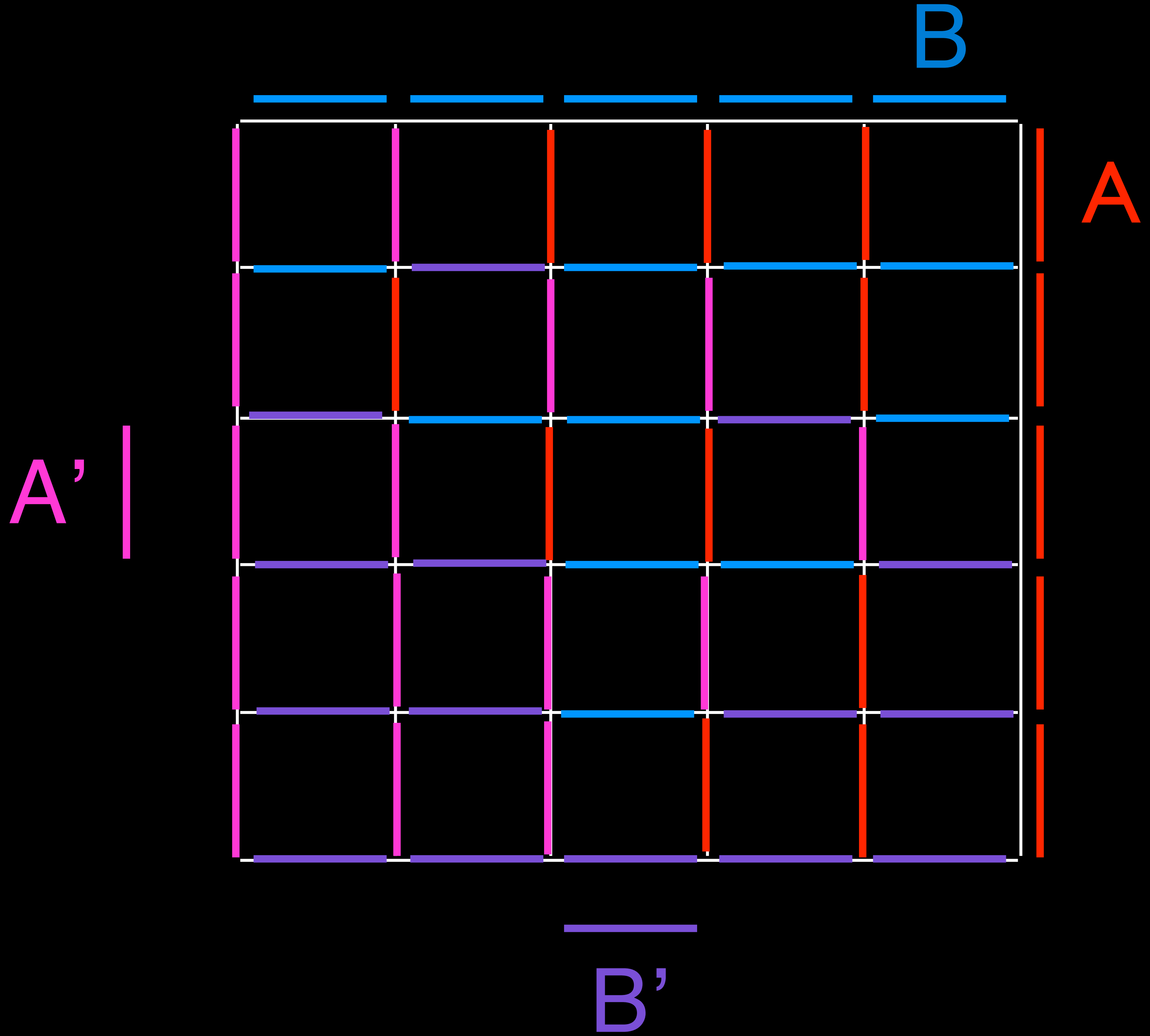
$c(u, v; w) =$ nb of ASM $n \times n$

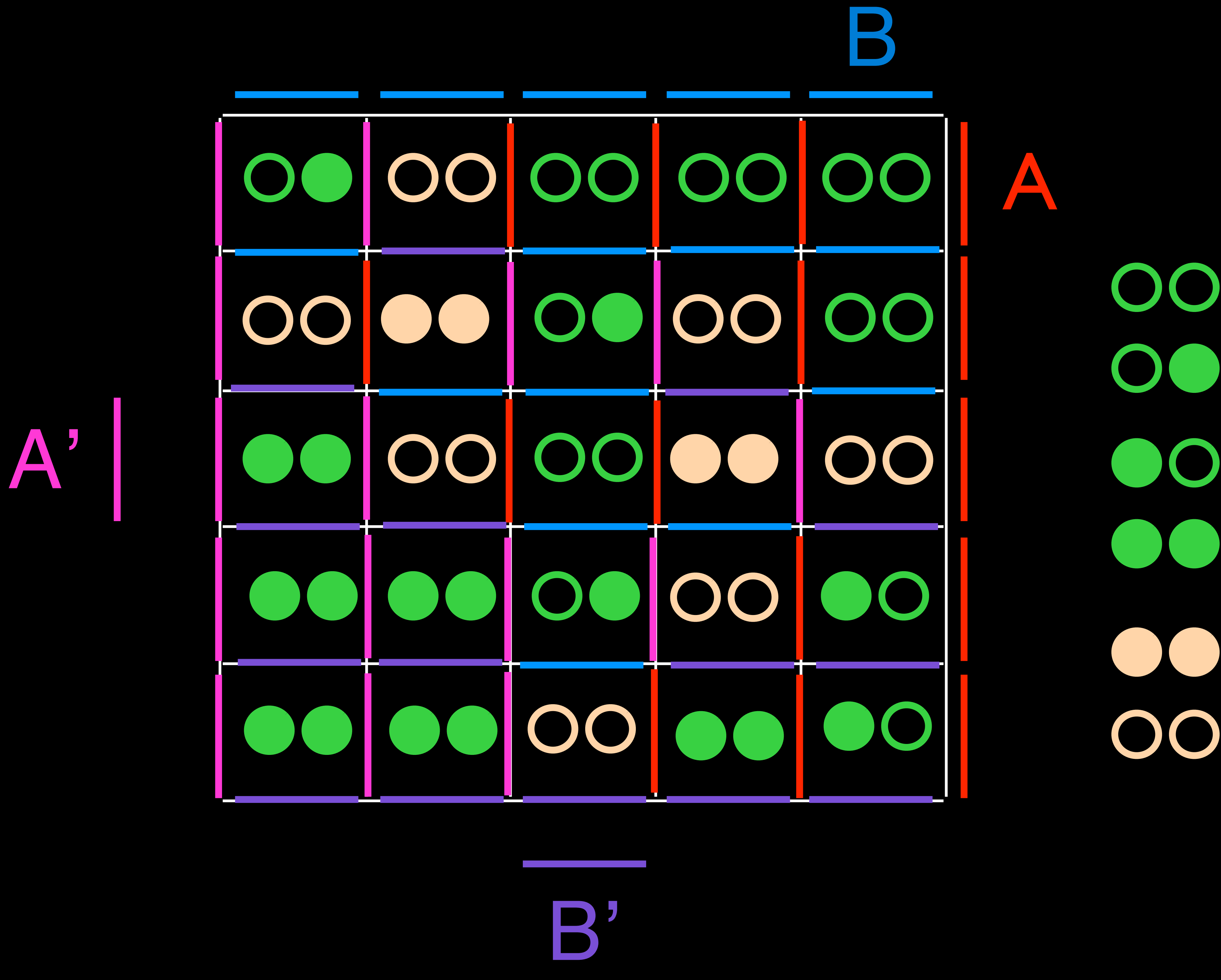



B
A
A'
B'

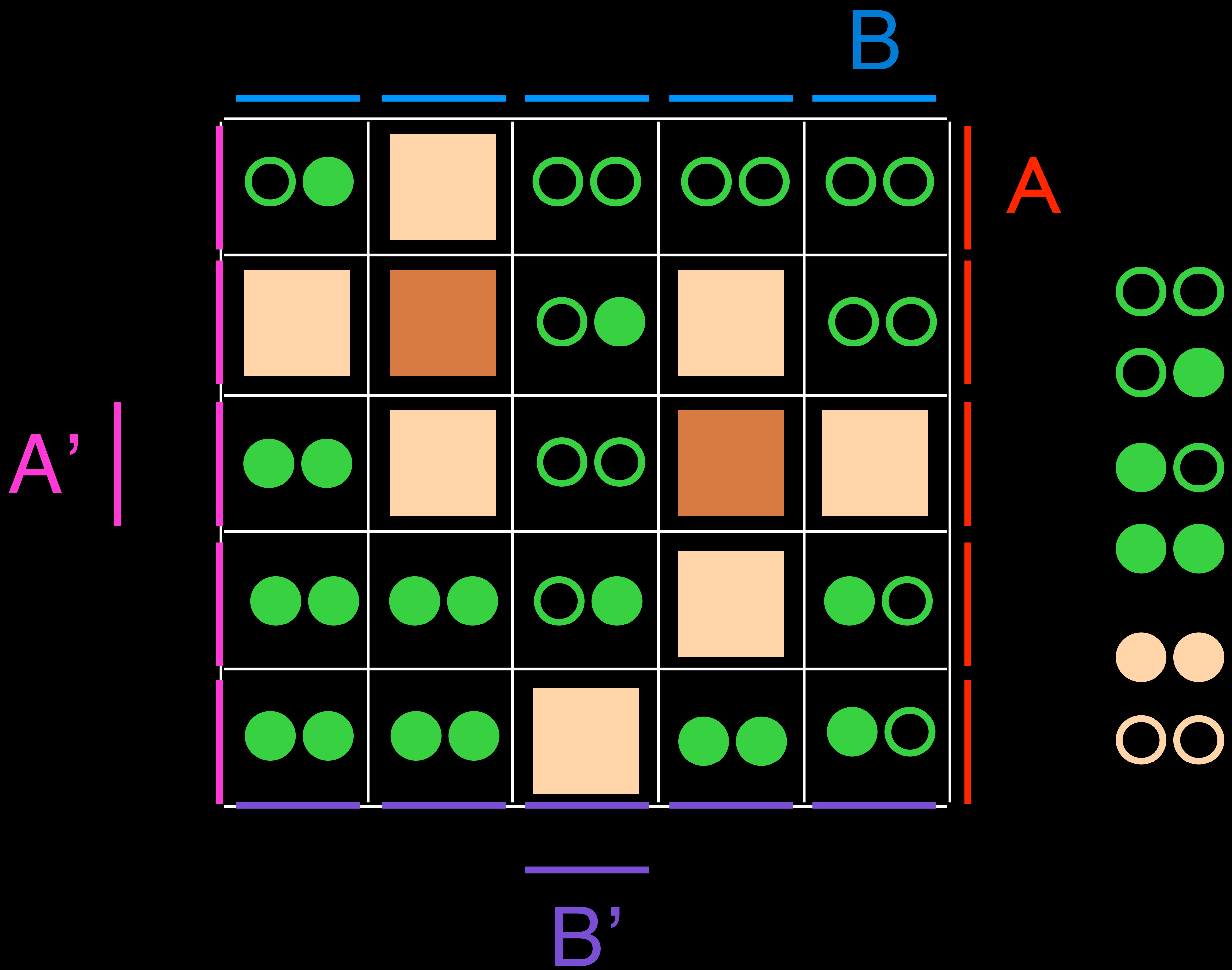


B
A
A’
B’


B
A
A'
B'

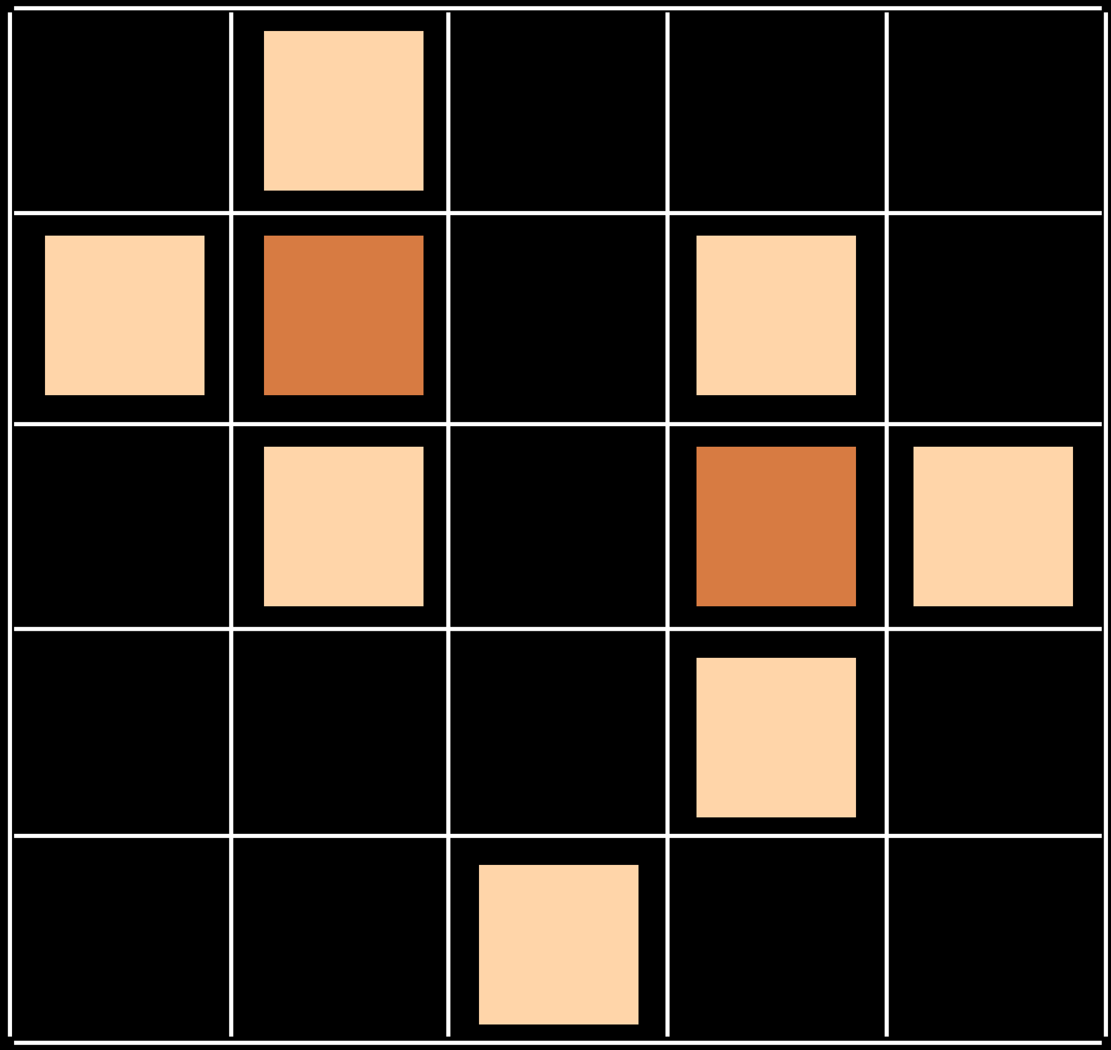


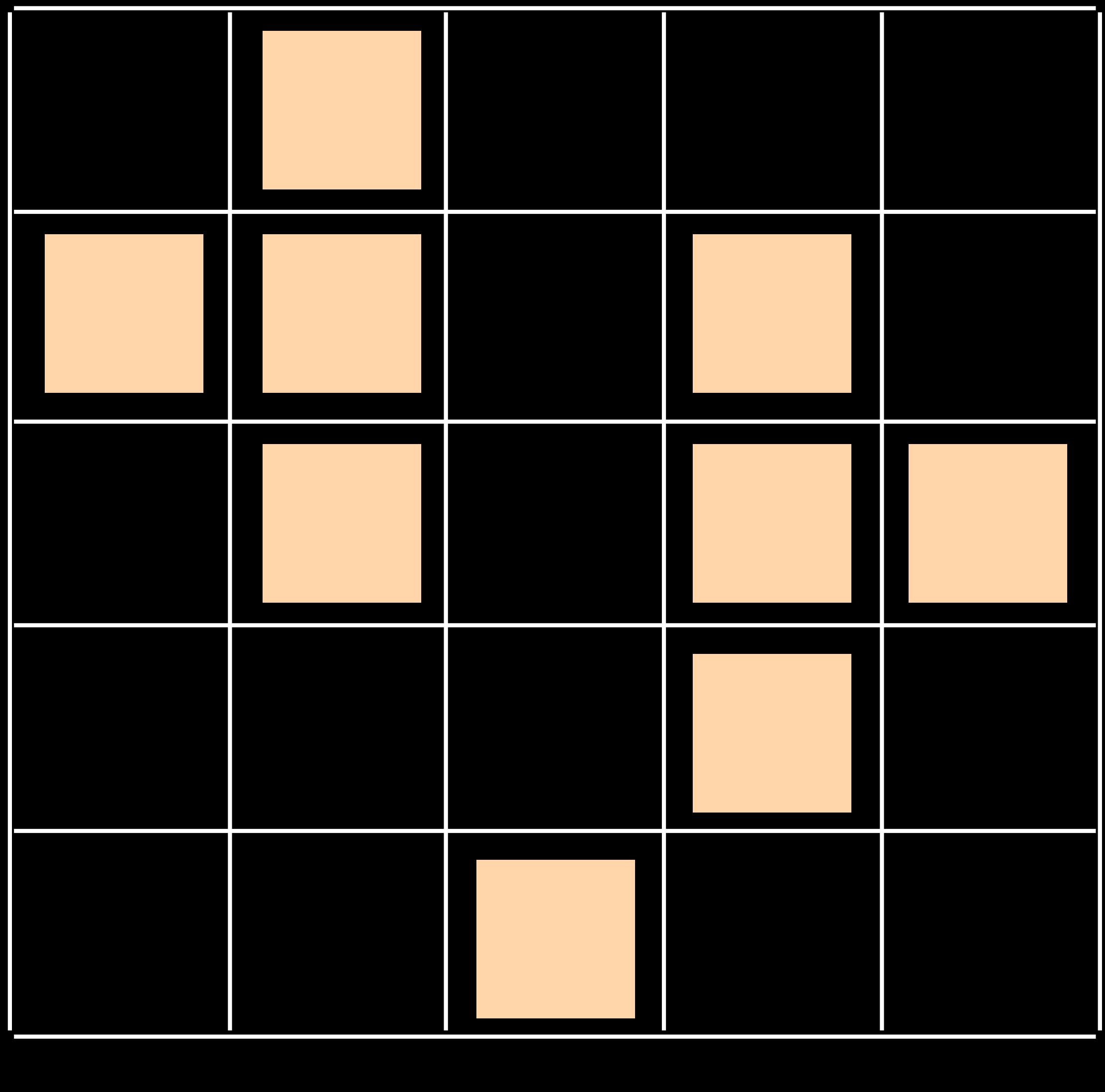
B
A
A'
B'


B
A
A’
B’

A alternating sign matrix $\longrightarrow$ $\varphi(A)$ B.A.BA configuration

$\left.\begin{matrix} 1 \\ -1 \end{matrix}\right\} \longrightarrow \bullet$

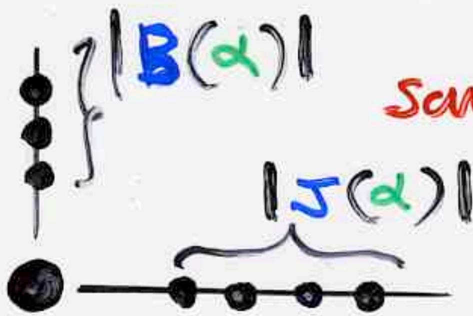




A alternating sign matrix $\longrightarrow$ $\varphi(A)$ B.A.BA configuration

$\left.\begin{matrix} 1 \\ -1 \end{matrix}\right\} \longrightarrow \bullet$

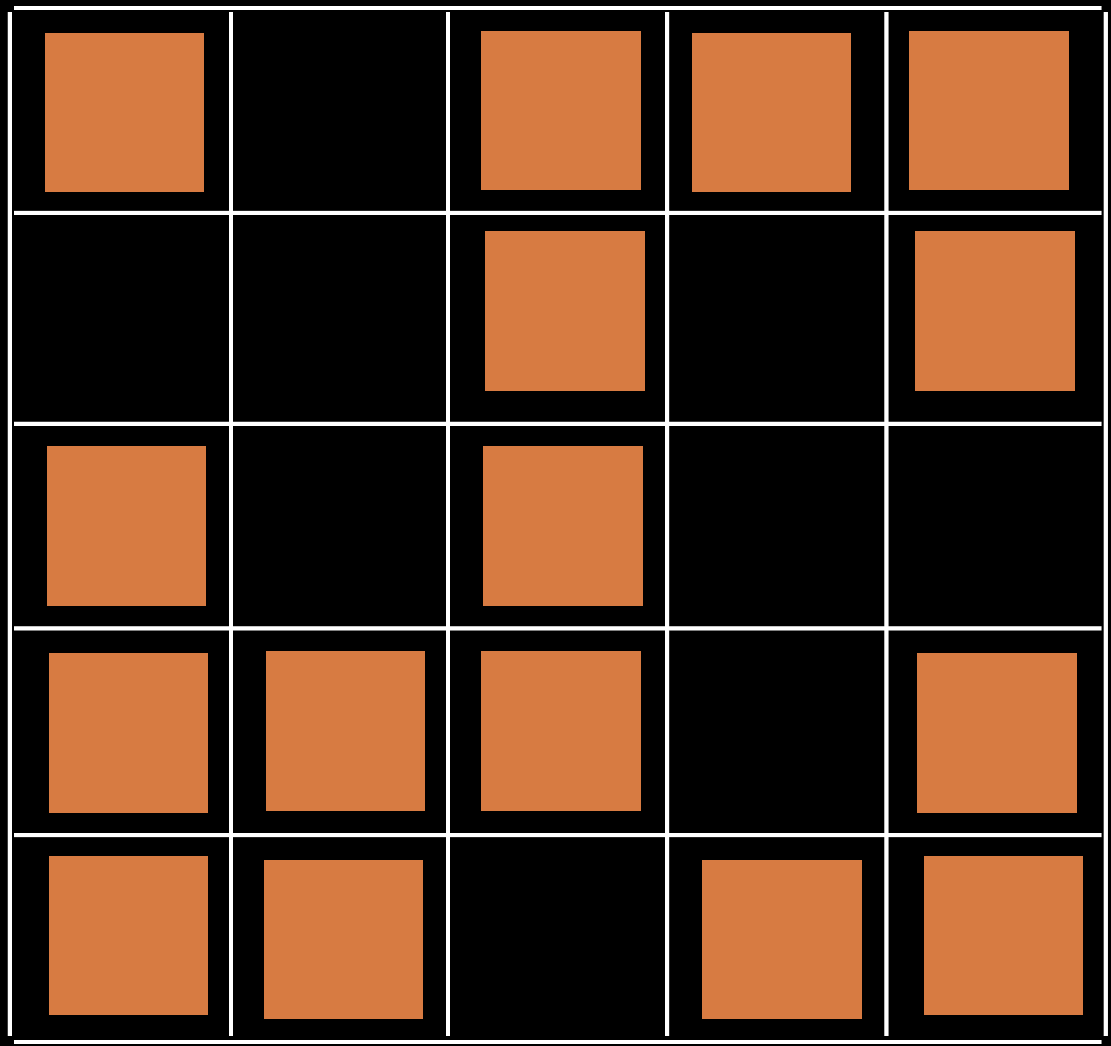
<u>characterisation</u> :



same parity

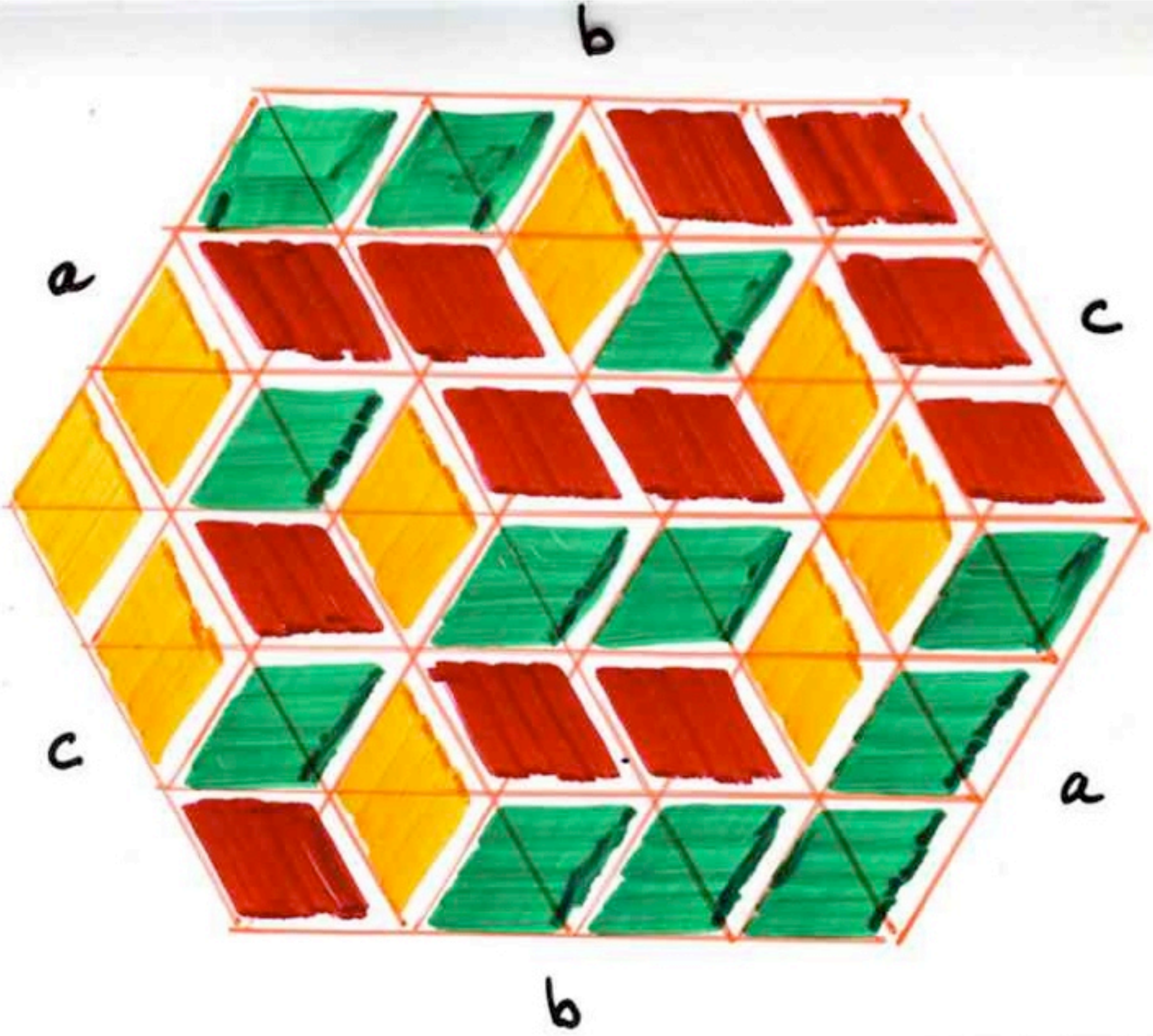
cell $\alpha \in C$

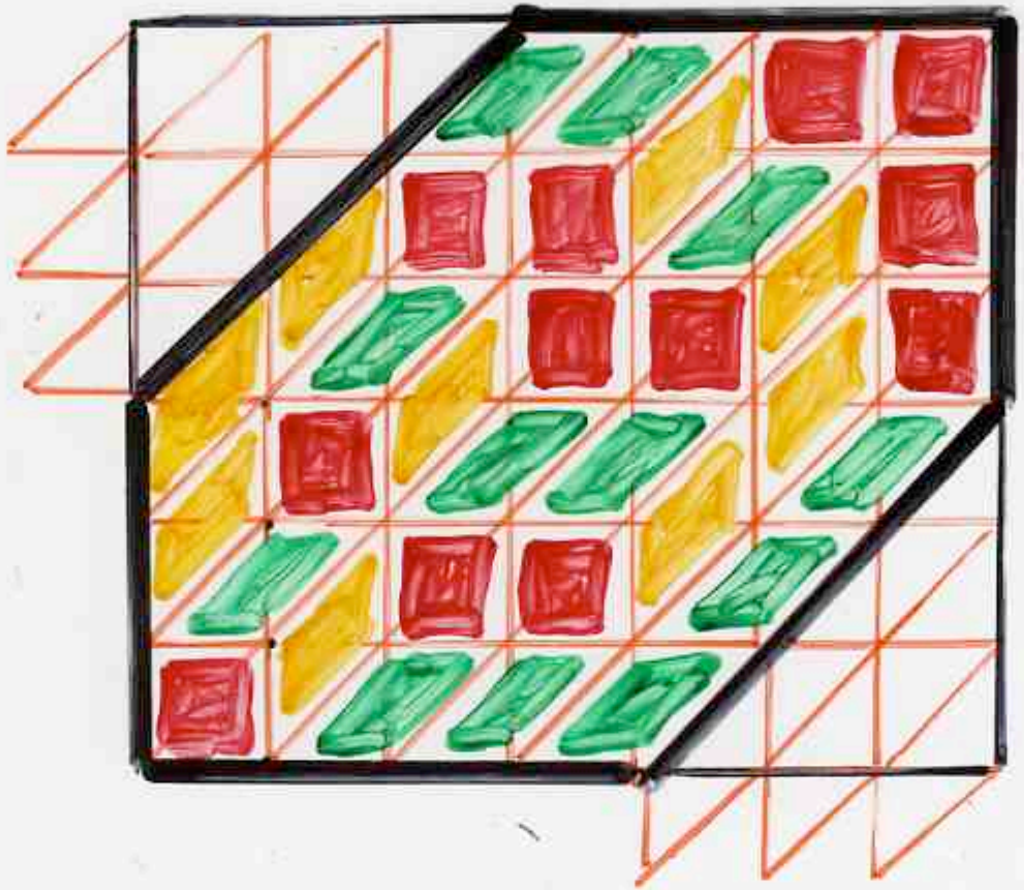
$+$ in each row and column odd number of cells in C

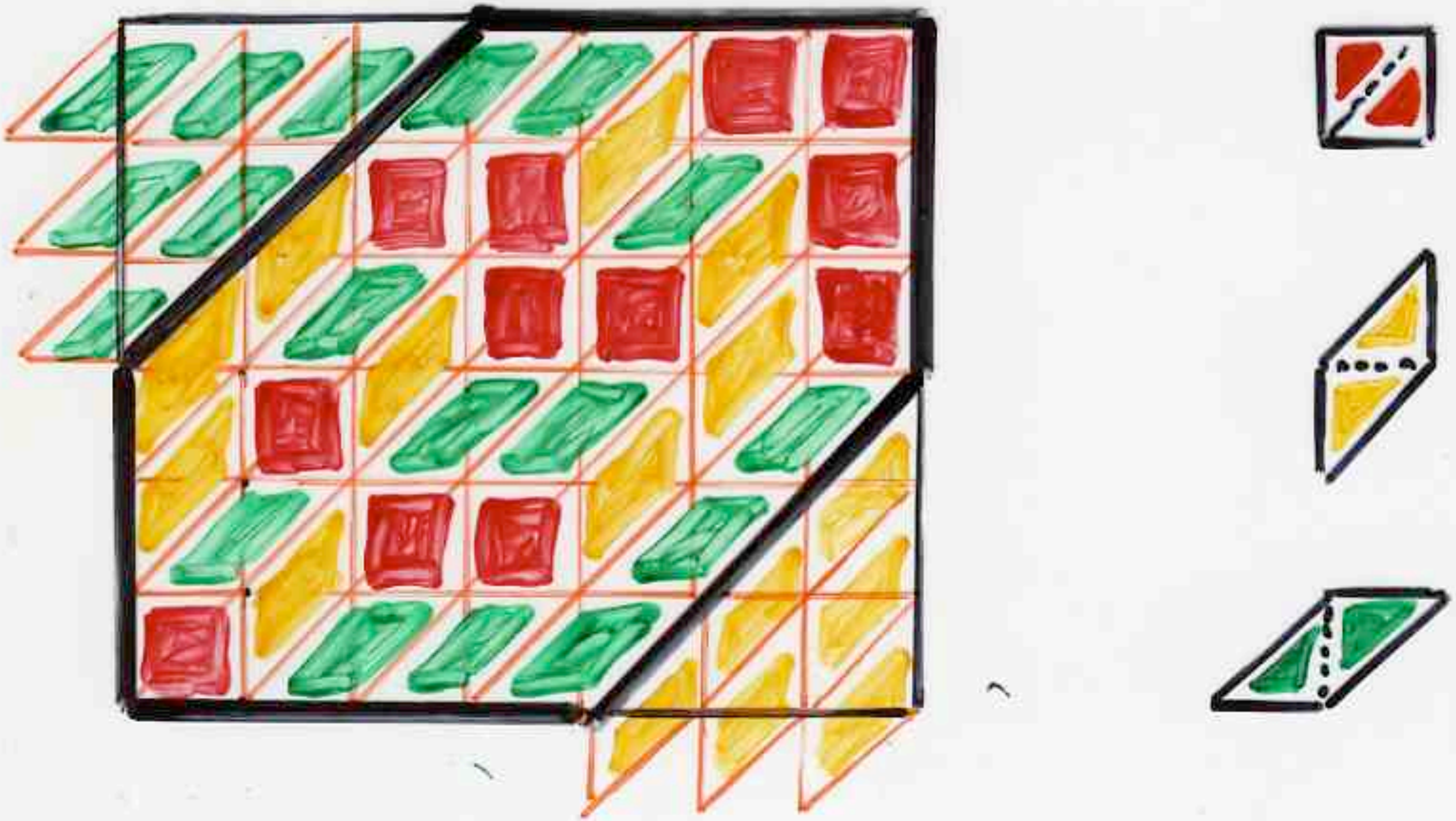


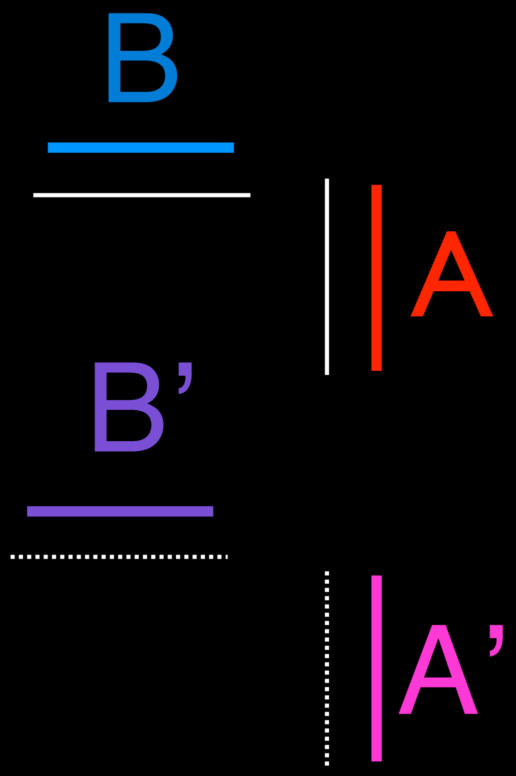
Prop. The number of configurations B.A. BA on $n \times n$ is $2^{(n^2)}$

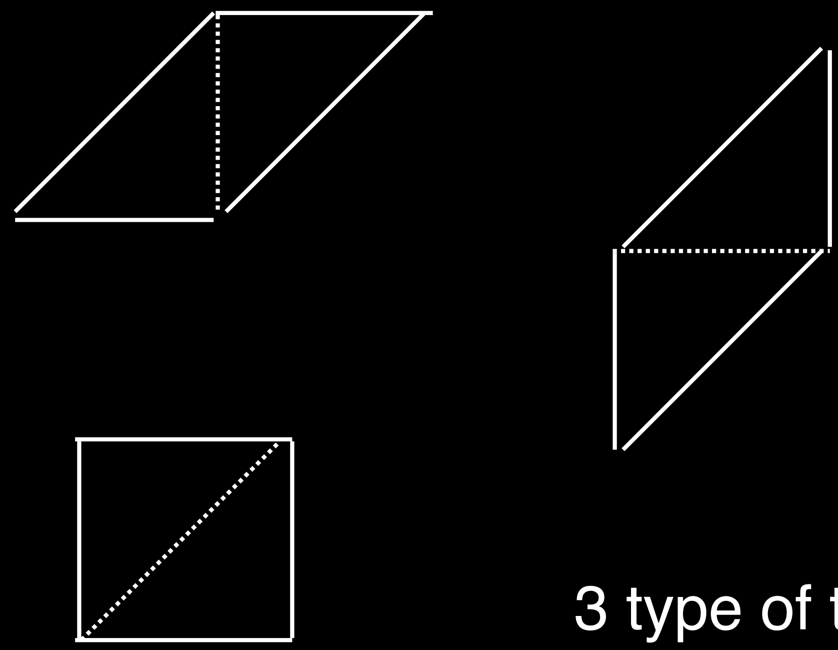
rhombus tilings


b
a
c
c
a
b



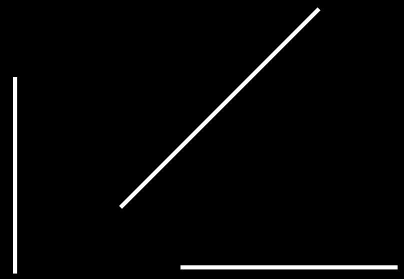




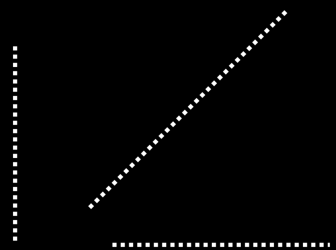


3 type of tiles

border of a tile

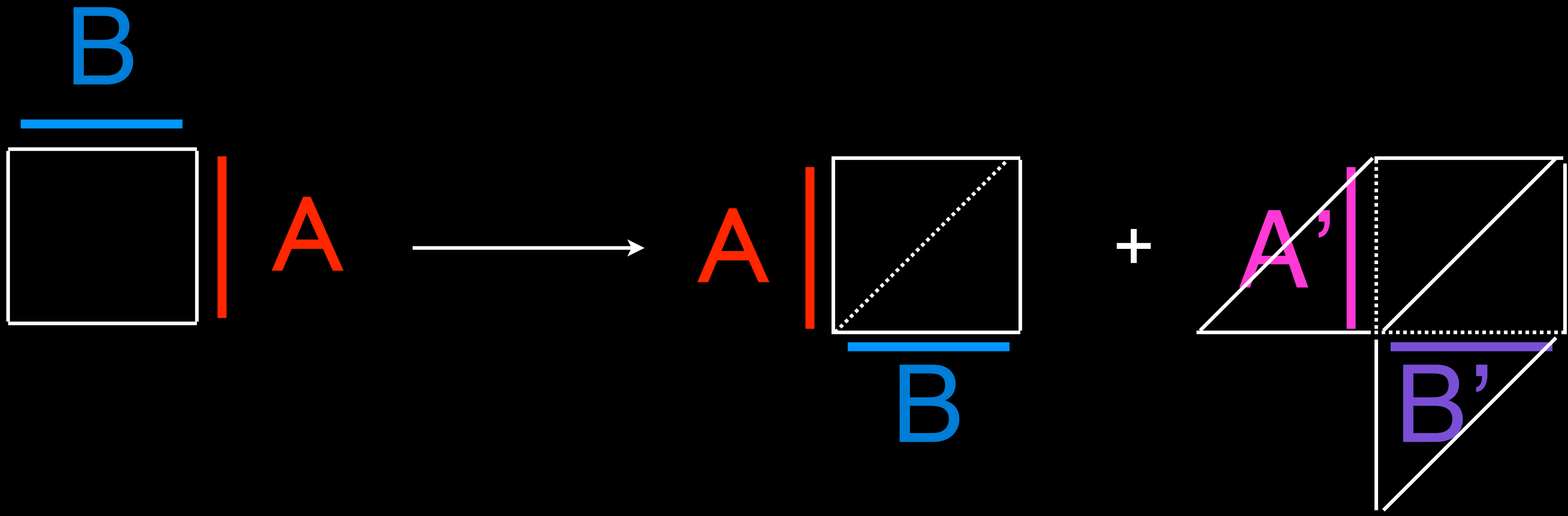


inside a tile

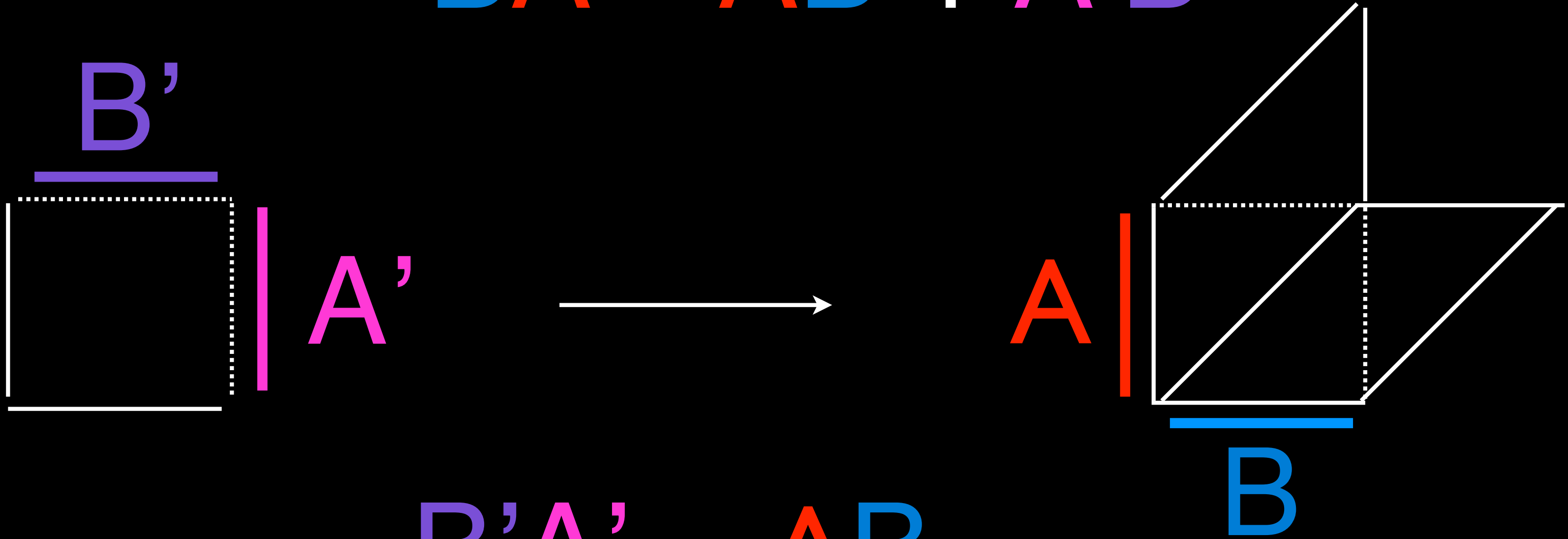


coding of the edges
for tilings
of the triangular lattice

“rewriting rules” for tilings of the triangular lattice

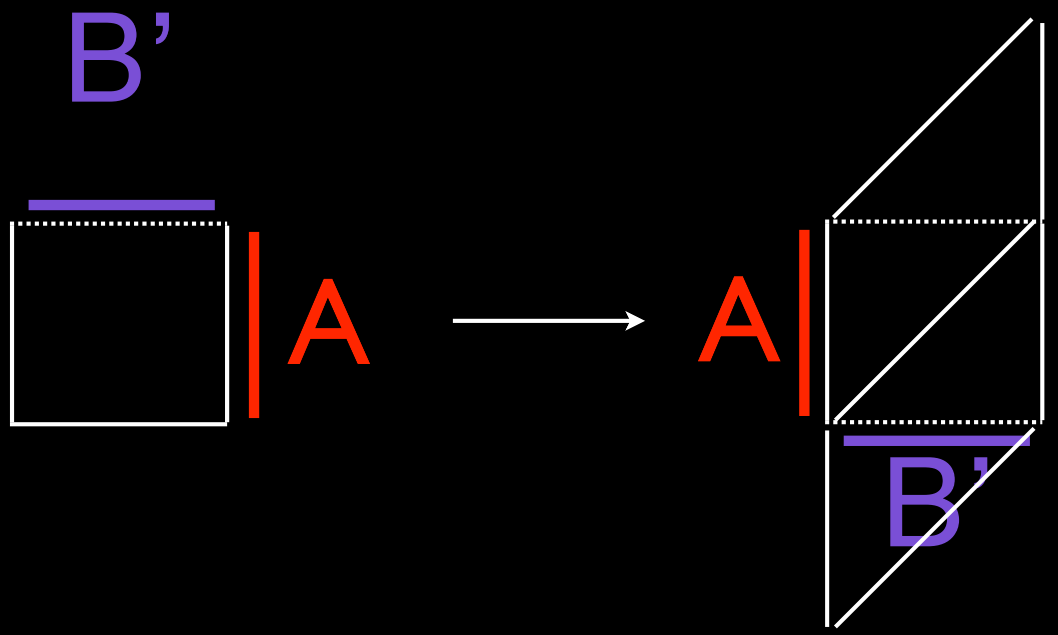


$$BA = AB + A'B'$$

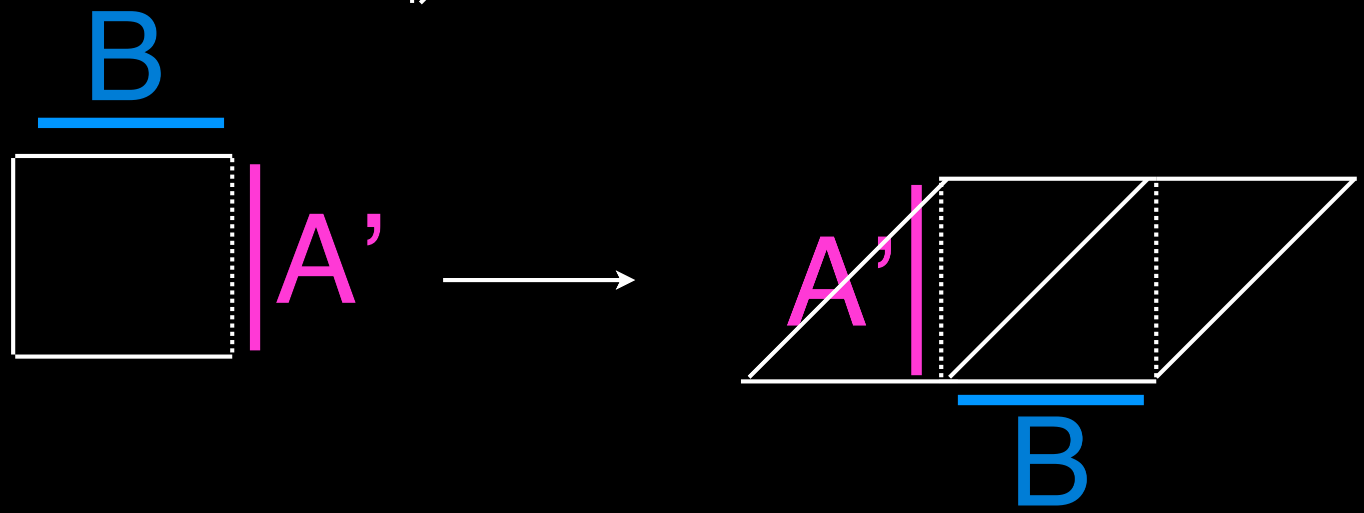


$$B'A' = AB$$

“rewriting rules” for tilings of the triangular lattice



$$B' A = A B'$$



$$B A' = A' B$$

"rewriting rules" for tilings of the triangular lattice

$$BA = AB + A'B'$$

$$B'A' = \qquad AB$$

$$B'A = AB'$$

$$BA' = A'B$$

same as for ASM , except the rewriting rule

$B'A' \longrightarrow A'B'$ is forbidden

$$\begin{cases} t_{\bullet\circ} = t_{\circ\bullet} = 0 \\ q_{\bullet\bullet} = 0 \end{cases} \qquad \text{(ASM)}$$

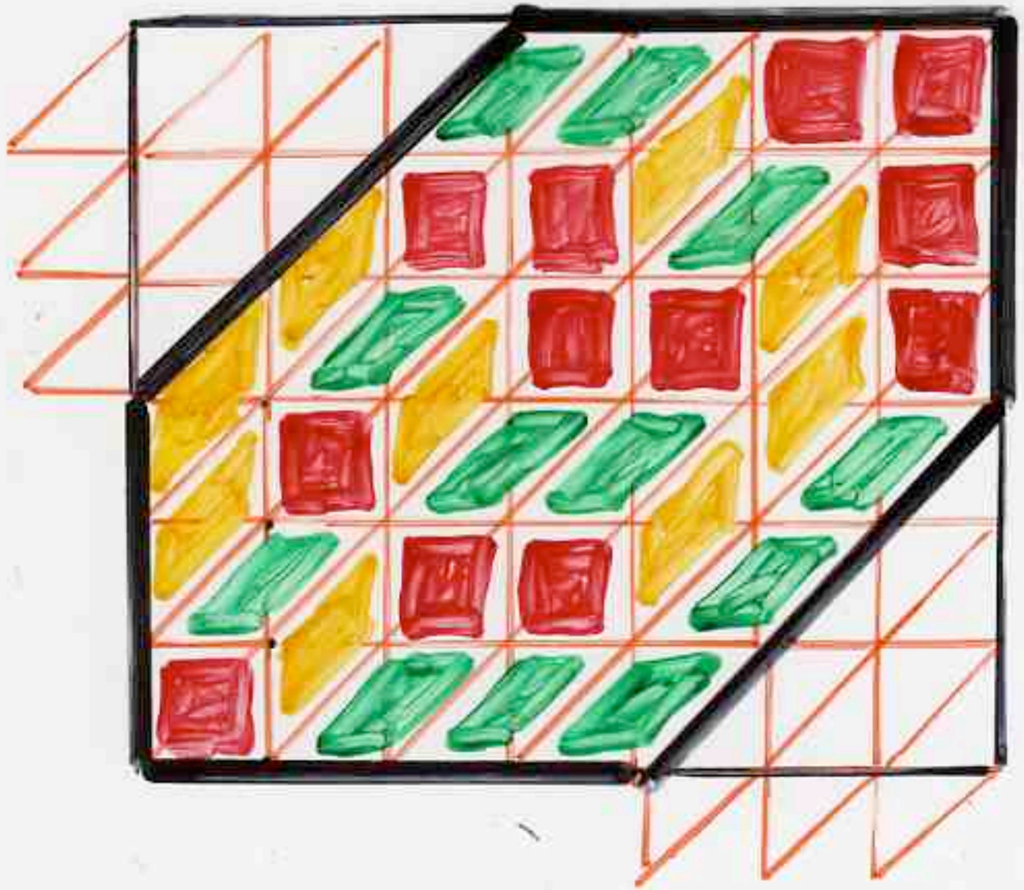
Rhombus tilings

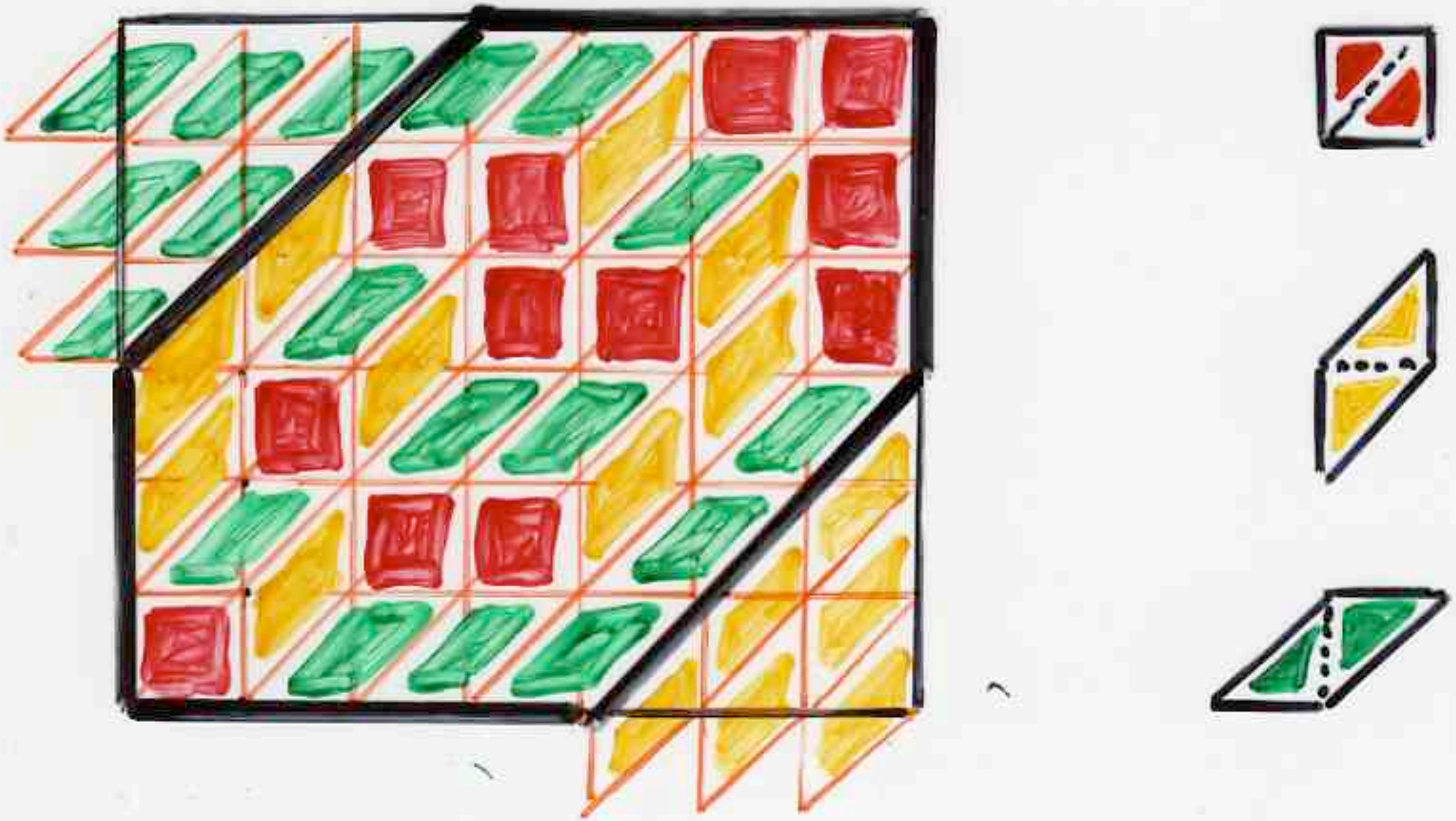
The quadratic algebra $\mathbb{Z}$

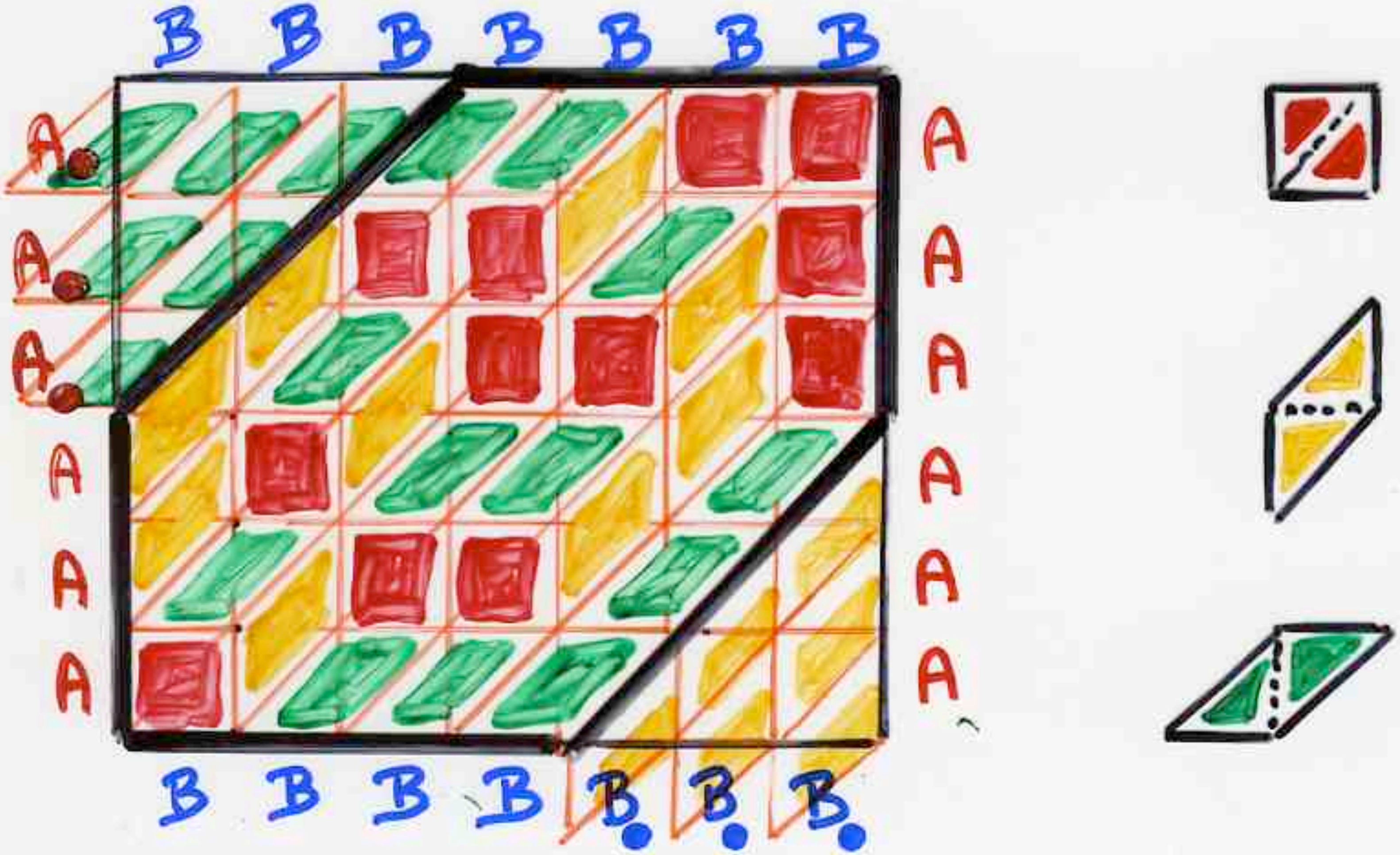
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\cdots}$, $t_{\cdots}$

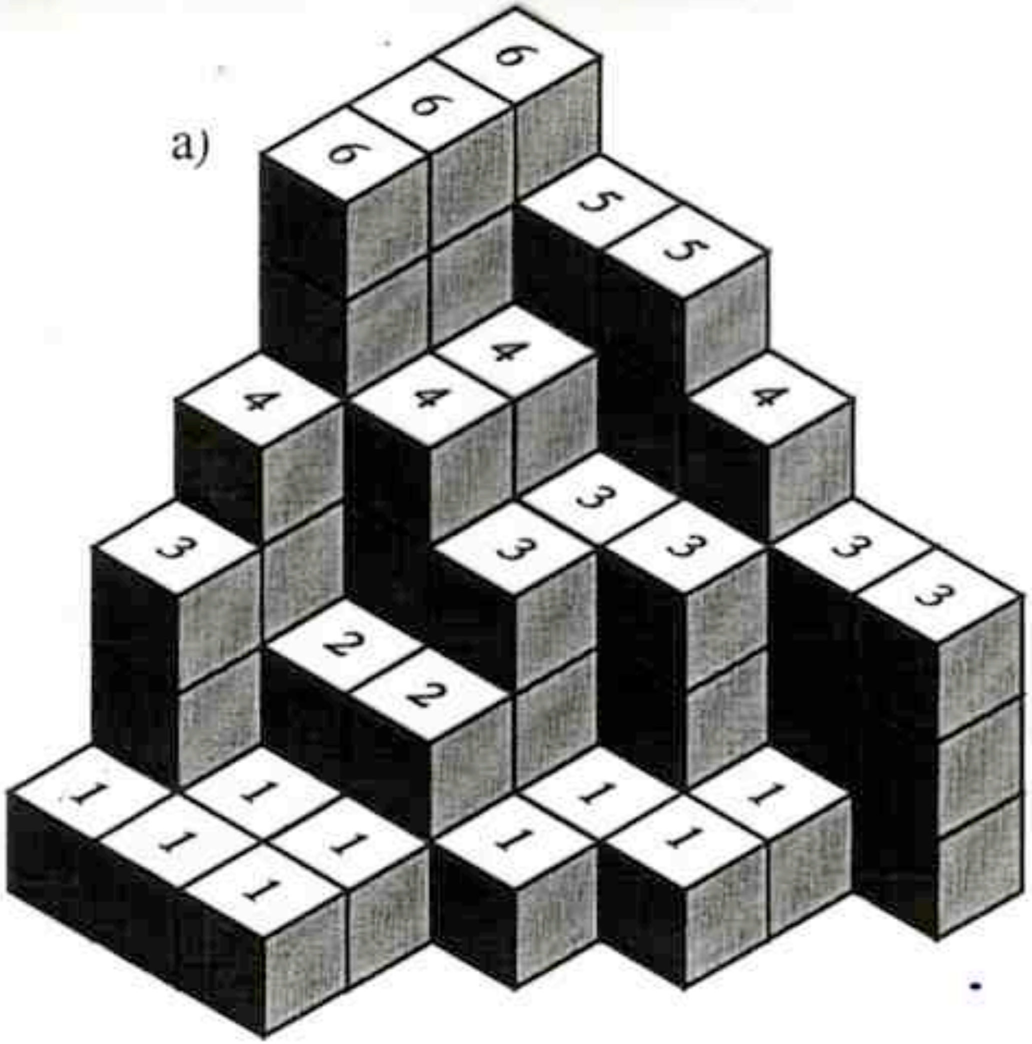
$$\begin{cases} B\,A = q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet\, B_\bullet \\ B_\bullet\, A_\bullet = \bigcirc\, A_\bullet\, B_\bullet + t_{\bullet\bullet}\, A\,B \\ B_\bullet\, A = q_{\bullet\circ}\, A\, B_\bullet + \bigcirc\, A_\bullet\, B \\ B\, A_\bullet = q_{\circ\bullet}\, A_\bullet\, B + \bigcirc\, A\, B_\bullet \end{cases}$$





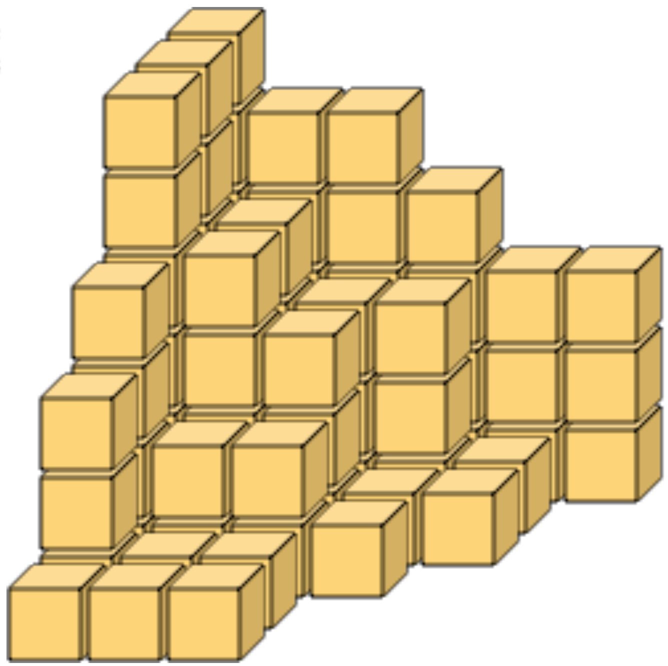

B B B B B B B
A A A A A A
A A A A A A
B B B B B B B

a)



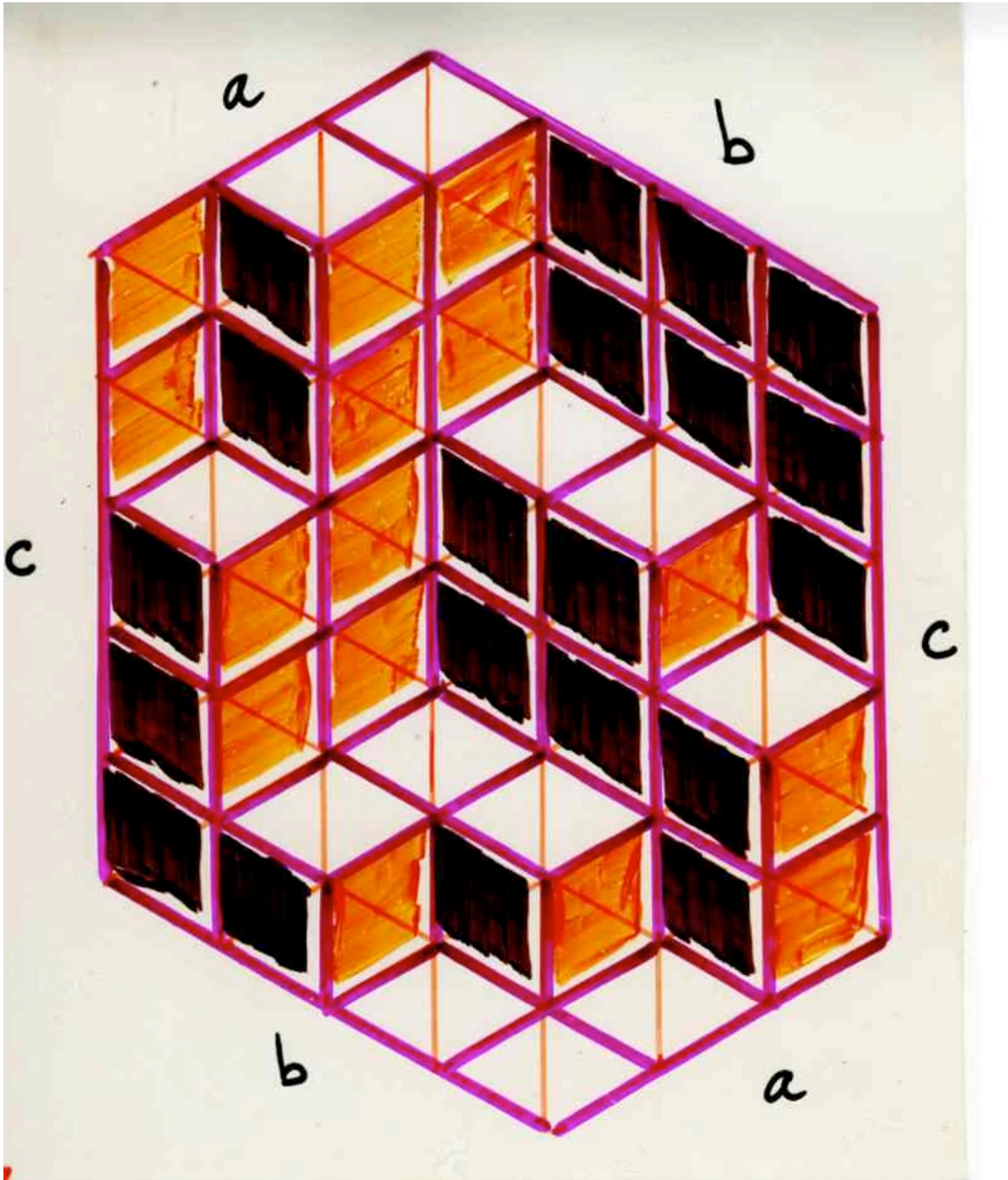
b)

6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			



example:
plane
partitions
in a box

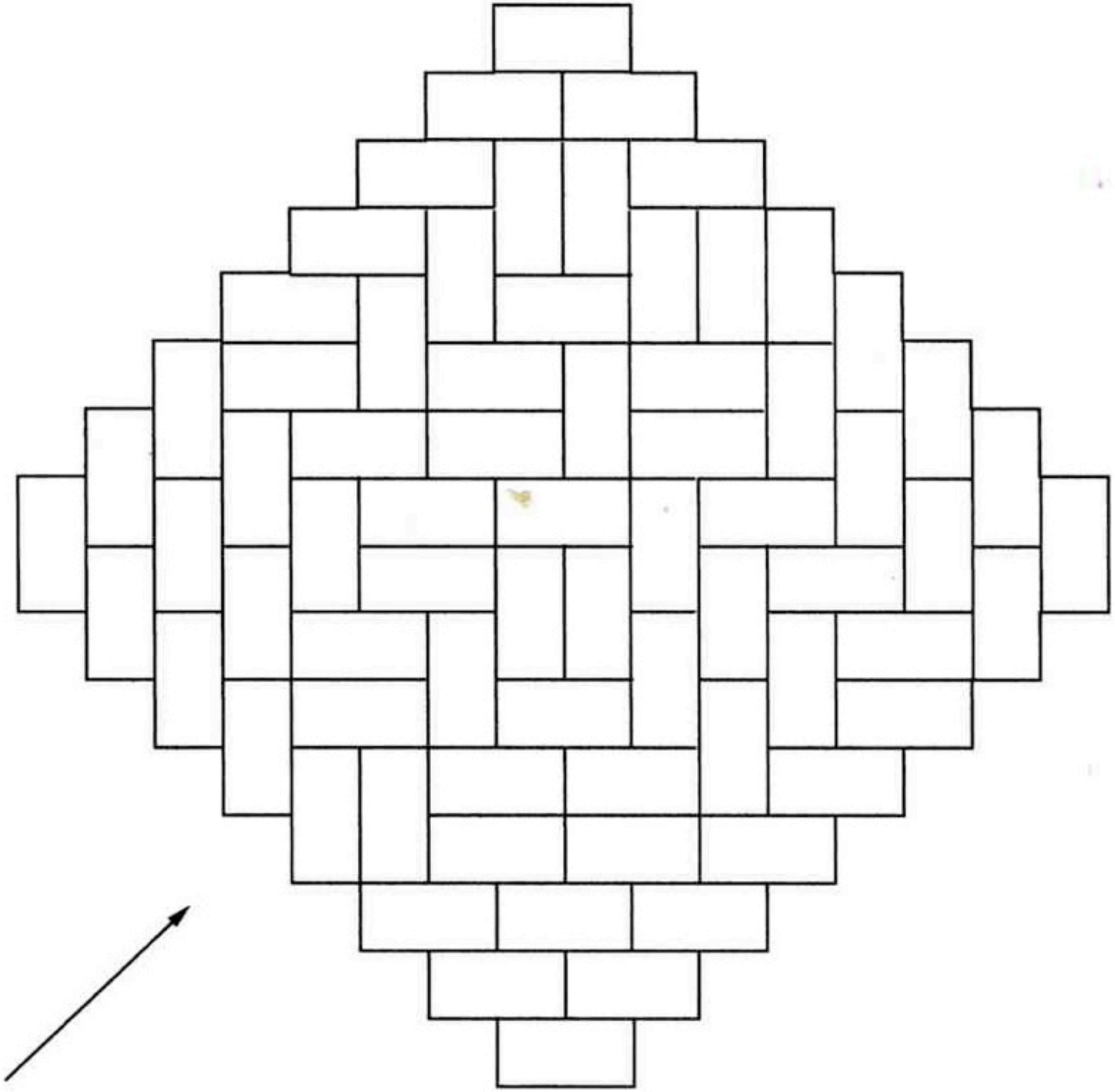
(MacMahon
formula)



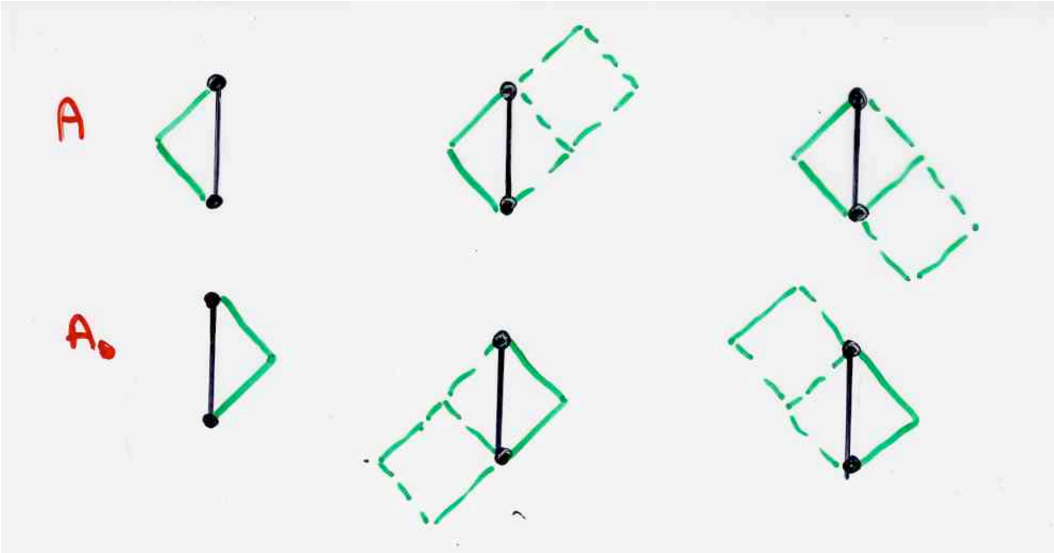
$$\prod_{\substack{1 \leq i \leq a \\ 1 \leq j \leq b \\ 1 \leq k \leq c}} \frac{i+j+k-1}{i+j+k-2}$$

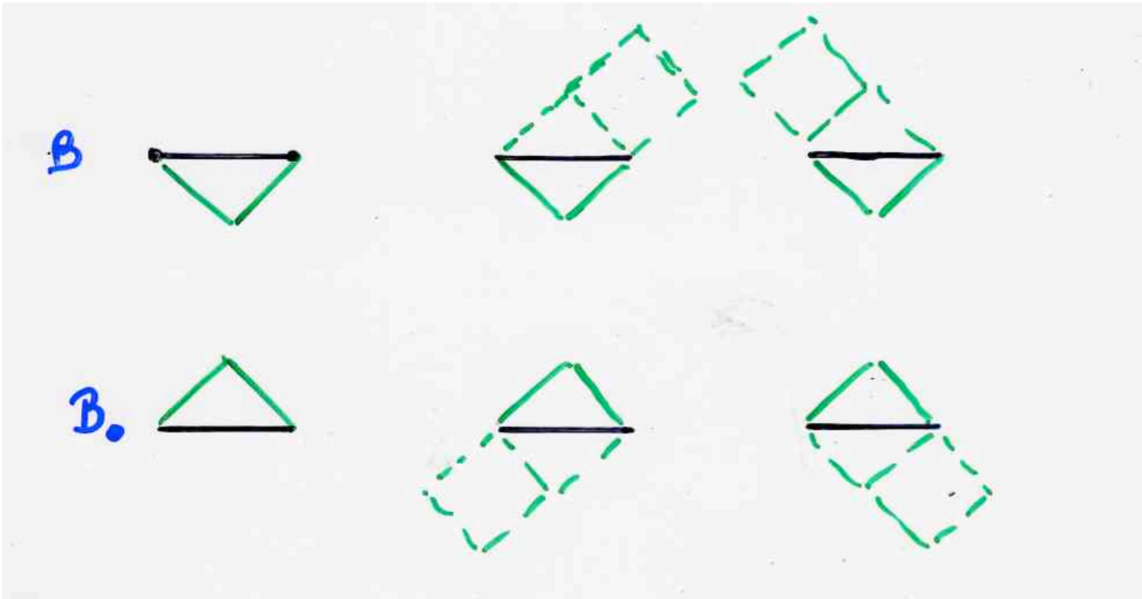
Aztec tilings

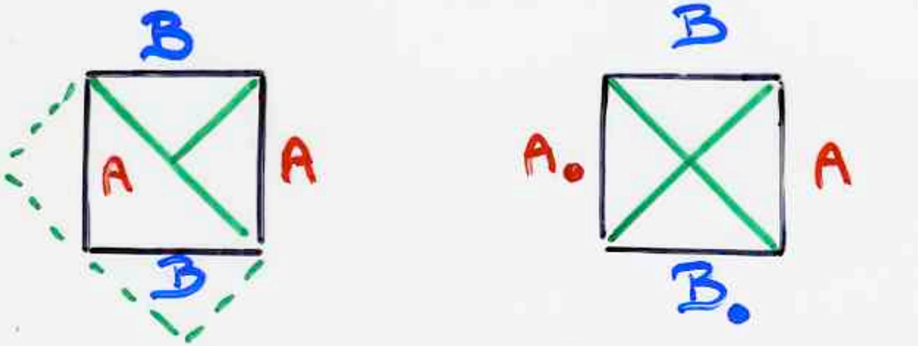
$2^{n(n-1)/2}$



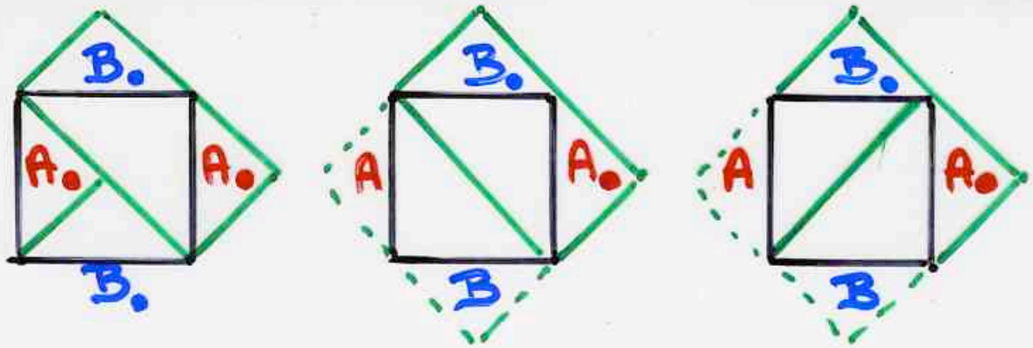
Elkies,
Kuperberg,
Larsen,
Propp
(1992)


A
A•


B
B•



$$BA = AB + A_\bullet B_\bullet$$



$$B_\bullet A_\bullet = A_\bullet B_\bullet + 2\,AB$$

$$B_\bullet A = A B_\bullet \qquad B A_\bullet = A_\bullet B$$

Aztec tilings

$$t_{\bullet\circ} = t_{\circ\bullet} = 0 \qquad \text{(ASM)}$$

$$t_{\bullet\bullet} = 2 \qquad \text{(nb of } -1 \text{ in ASM)}$$

The quadratic algebra Z

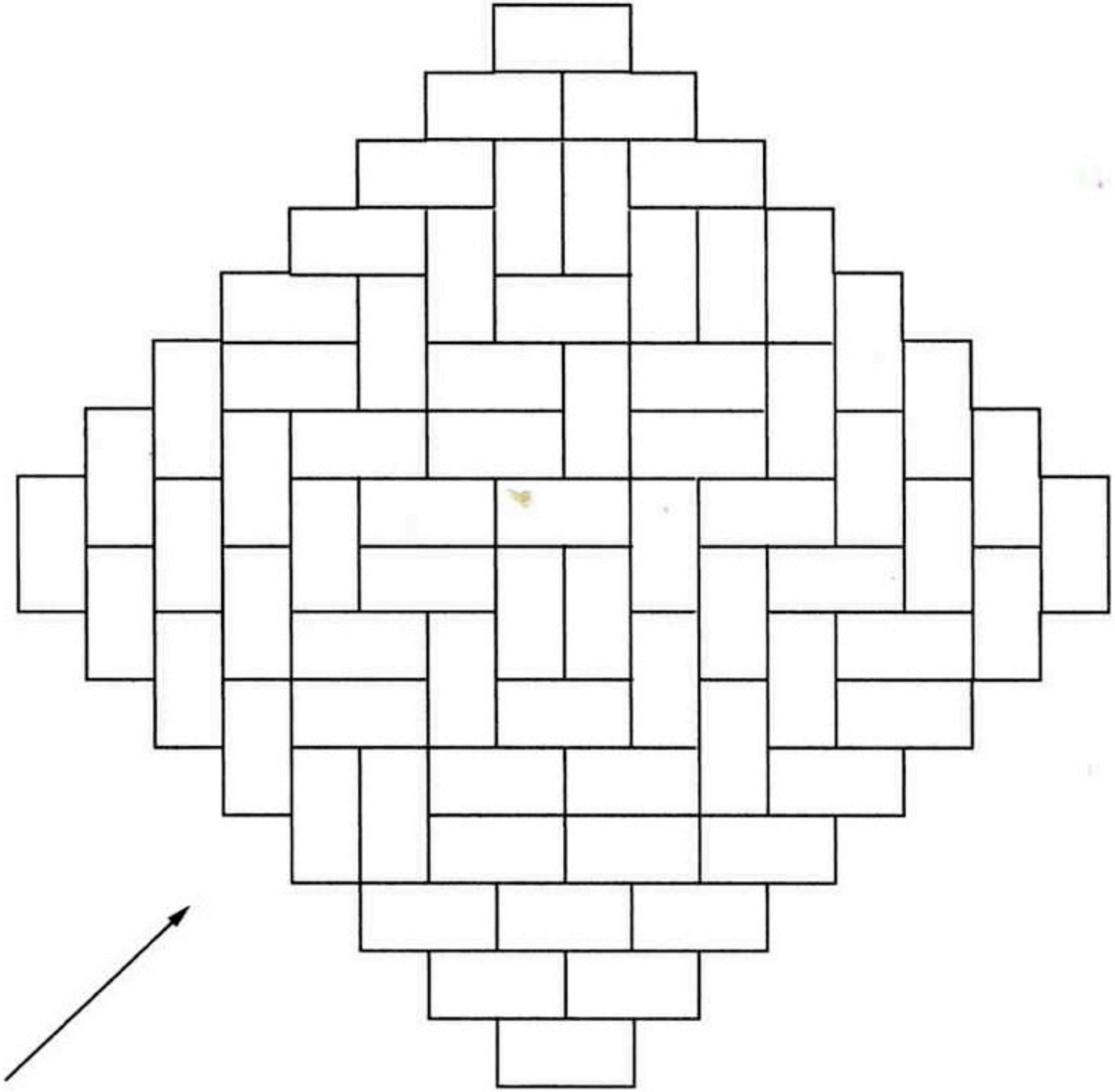
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} BA = q_{\circ\circ} AB + t_{\circ\circ} A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet} A_\bullet B_\bullet + 2\, AB \\ B_\bullet A = q_{\bullet\circ} A B_\bullet + 0\, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} A_\bullet B + 0\, A B_\bullet \end{cases}$$

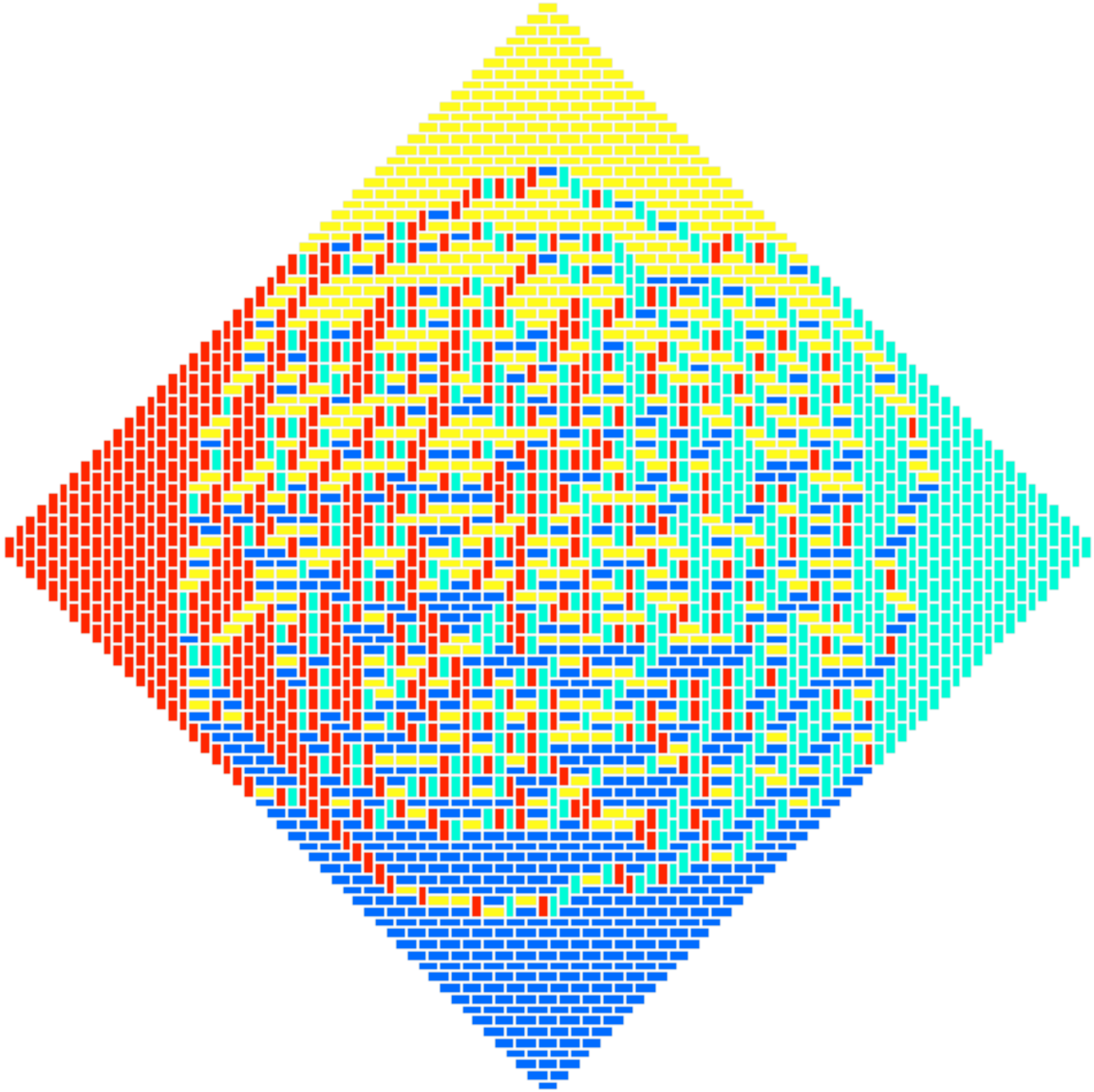
$2^{n(n-1)/2}$

$A_n(2)$

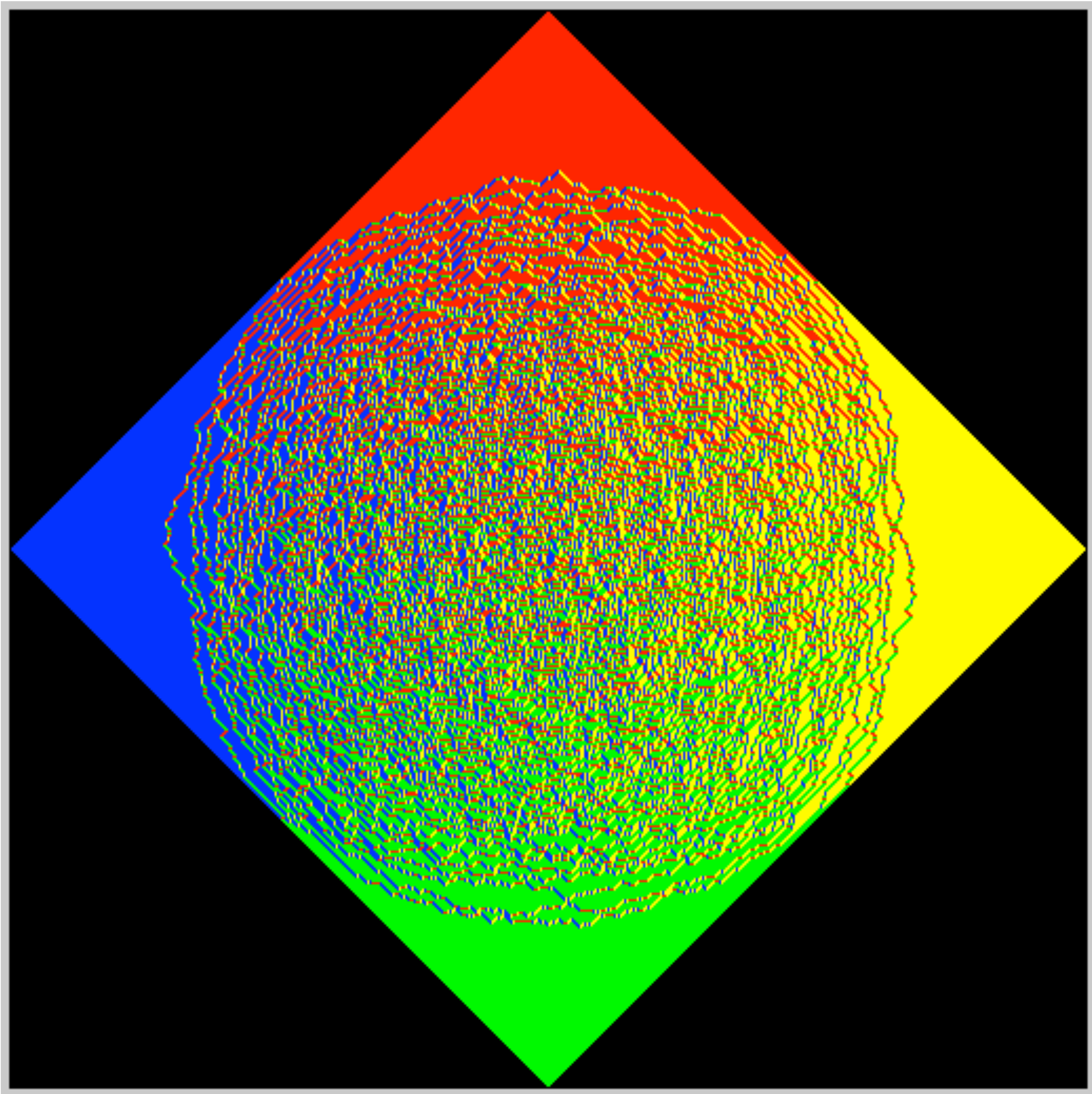


Elkies,
Kuperberg,
Larsen,
Propp
(1992)

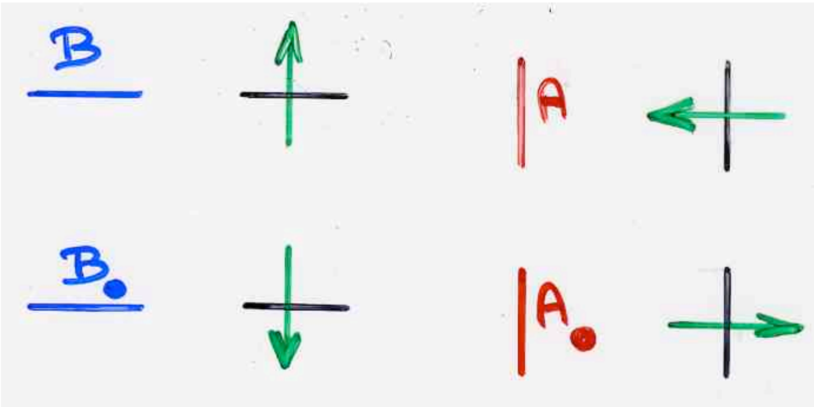
random
Aztec
tilings



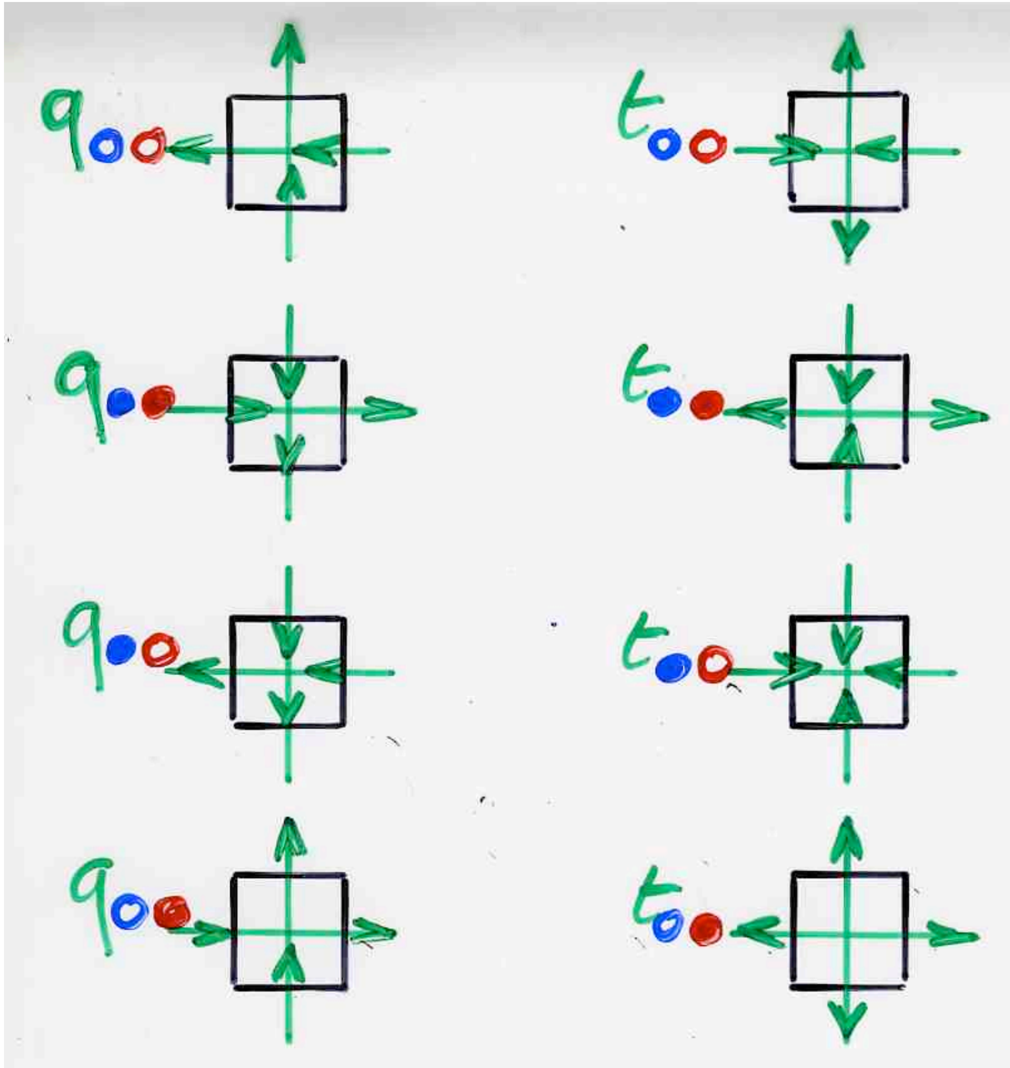
the
«artic
circle»
theorem



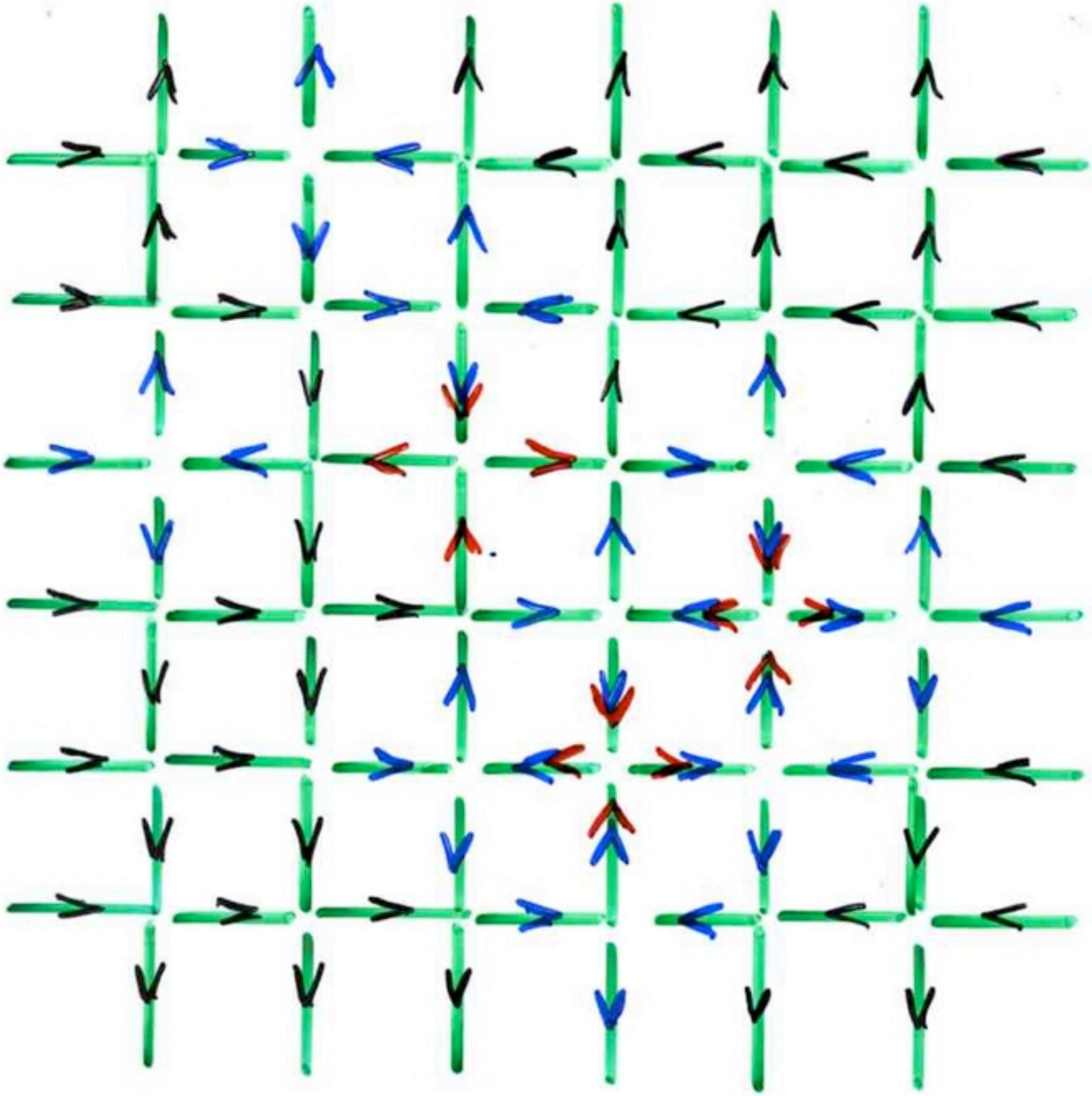
geometric interpretation of XYZ-tableaux



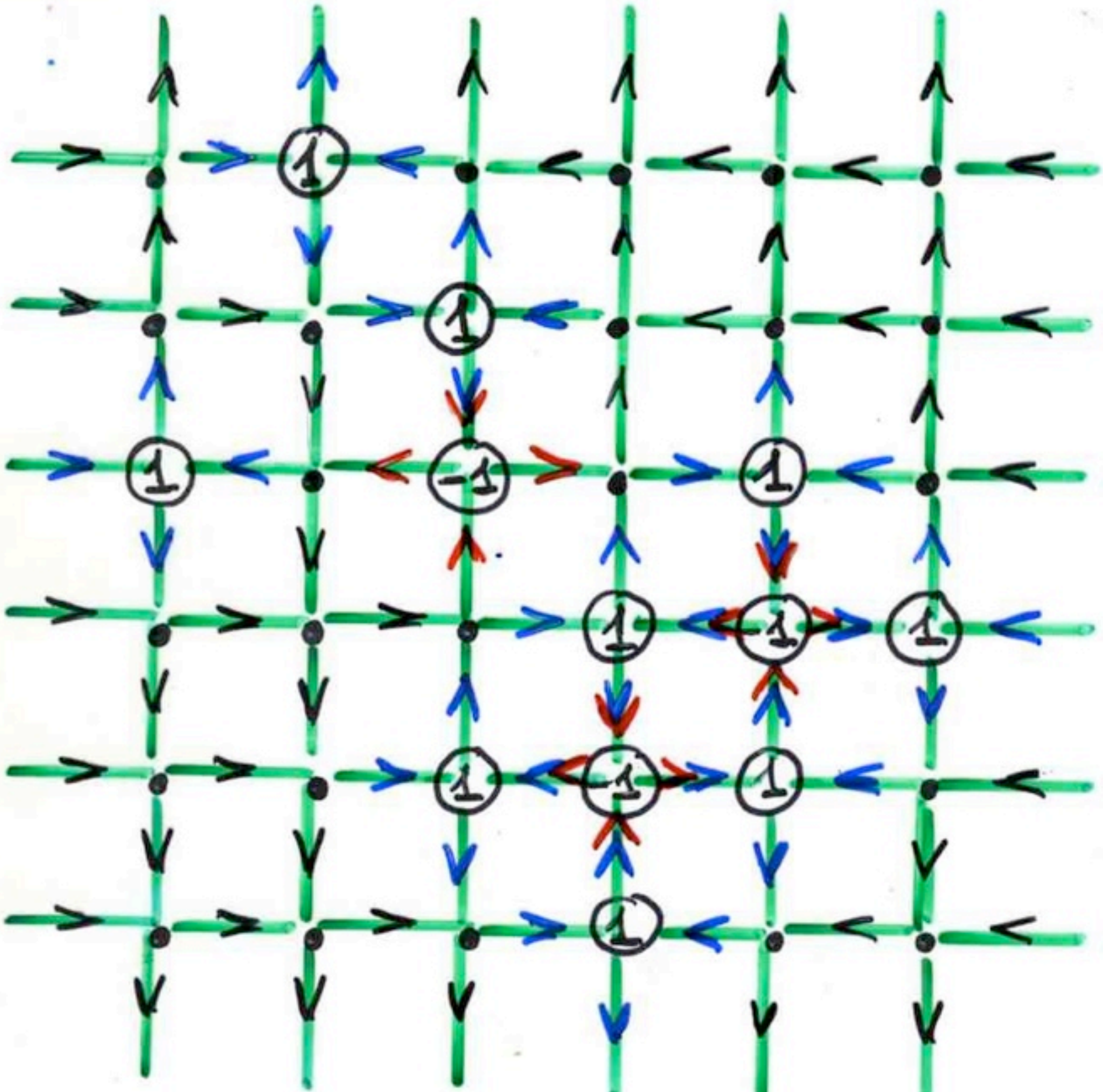
8 - vertex model



The
6-vertex
model



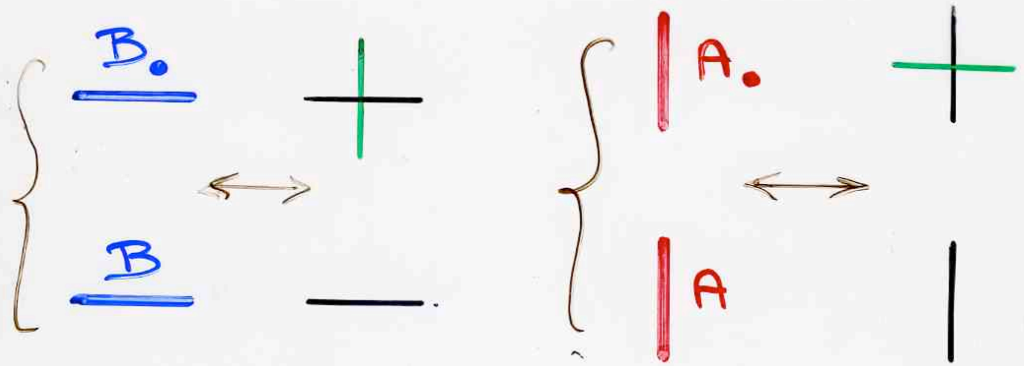
The 6-vertex model

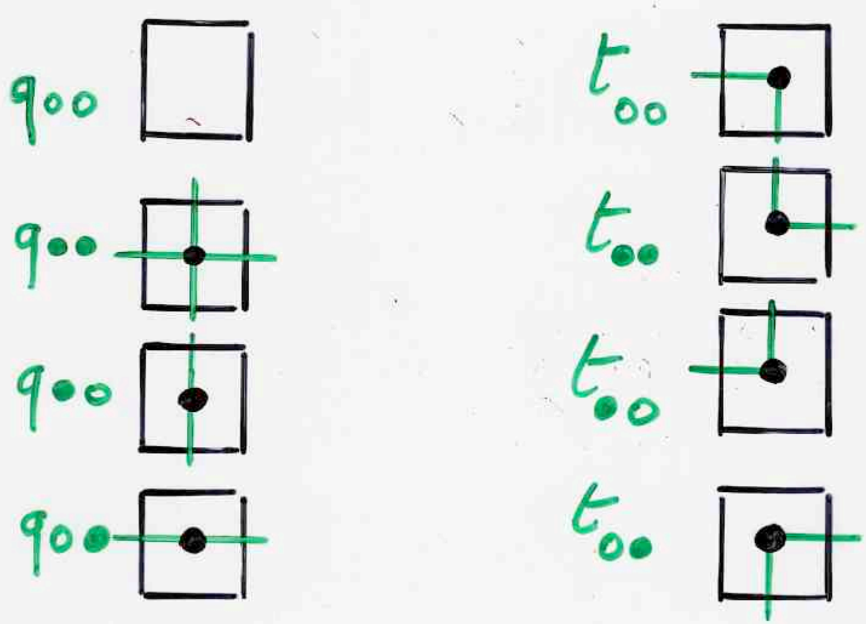


Razumov - Stroganov (ex)- conjecture 2000-2001
on quantum spin chains XXZ

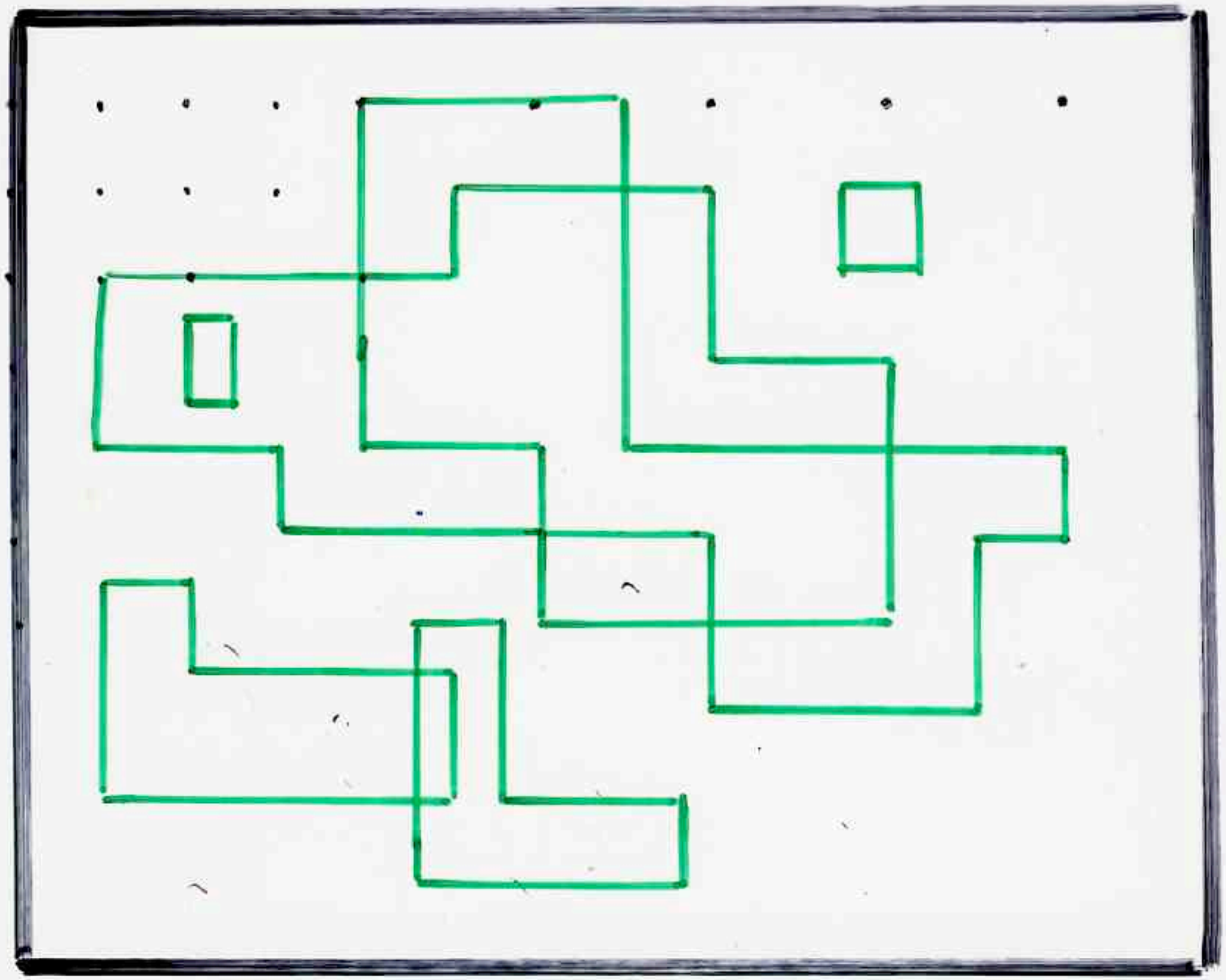
proof by :
L. Cantini and A.Sportiello (2010)
completely combinatorial proof

geometric interpretations of Z-tableaux





8-vertex model



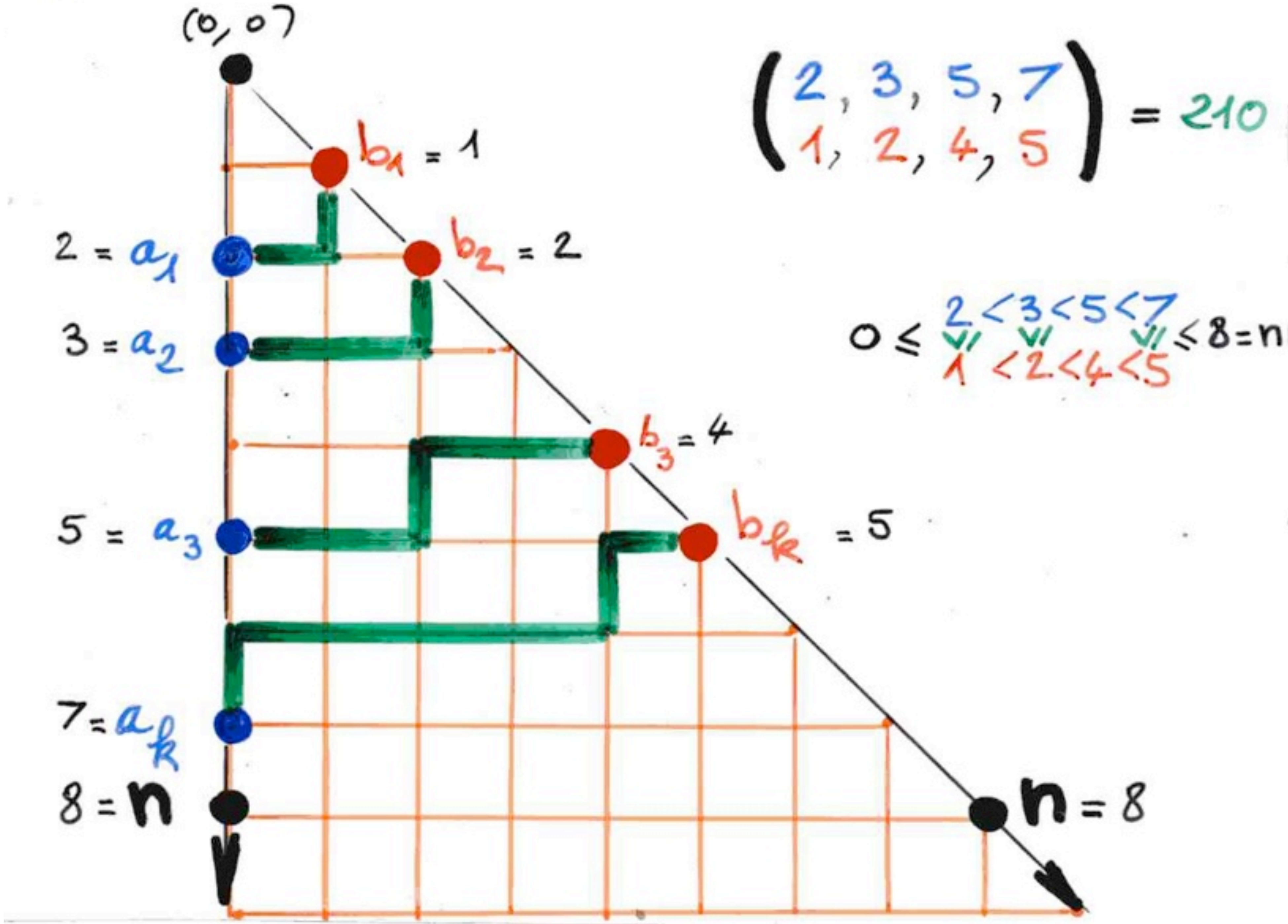
"closed" graph

Ising model

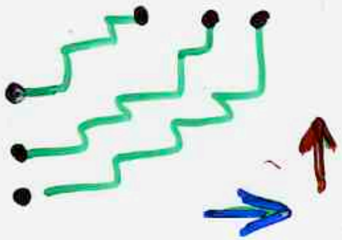
$$w = B^m A^n$$

$$uv = A^n B^m$$

non-intersecting paths



example: binomial determinant



$$\begin{cases} t_{\circ\circ} = 0 \\ q_{\bullet\bullet} = t_{\bullet\bullet} = 0 \end{cases}$$

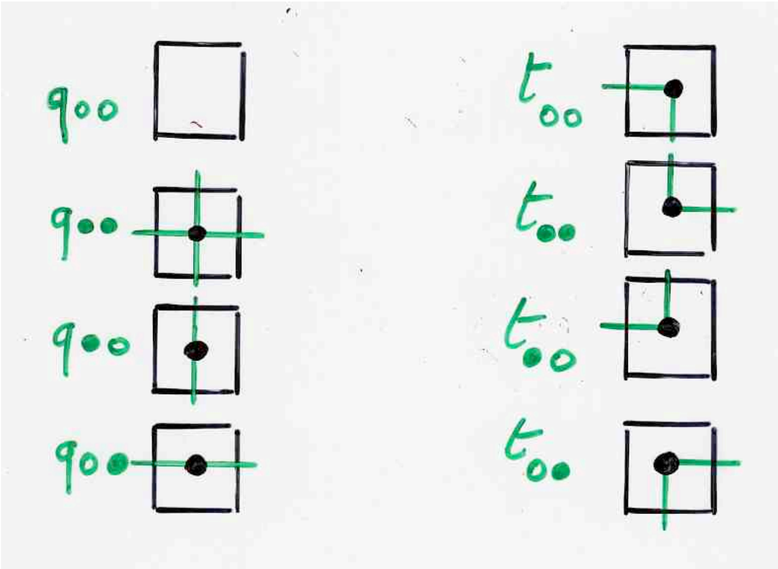
$A \leftrightarrow A_\bullet$
exchanging

$$\begin{cases} t_{\circ\bullet} = 0 \\ q_{\bullet\circ} = t_{\bullet\circ} = 0 \end{cases}$$

The quadratic algebra $\mathcal{Z}$

4 generators $B_\bullet \; A_\bullet \; B \; A$
8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} B A = q_{\circ\circ} \, A B + \bigcirc \, A_\bullet B_\bullet \\ B_\bullet A_\bullet = \bigcirc \, A_\bullet B_\bullet + \bigcirc \, A B \\ B_\bullet A = q_{\bullet\circ} \, A B_\bullet + t_{\bullet\circ} \, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} \, A_\bullet B + t_{\circ\bullet} \, A B_\bullet \end{cases}$$



The quadratic algebra $\mathcal{Z}$

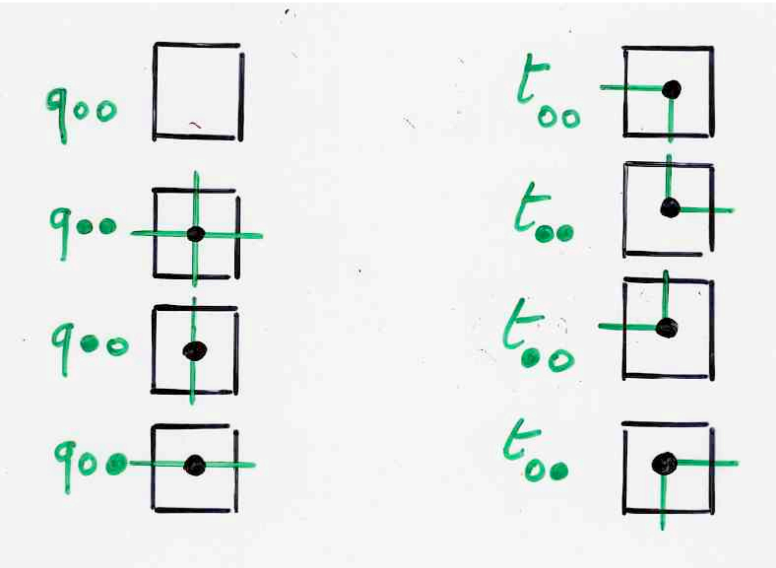
4 generators $B_\bullet \; A_\bullet \; B \; A$
8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} B A = q_{\circ\circ} \, A B + t_{\circ\circ} \, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet} \, A_\bullet B_\bullet + t_{\bullet\bullet} \, A B \\ B_\bullet A = \bigcirc \, A B_\bullet + \bigcirc \, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} \, A_\bullet B + \bigcirc \, A B_\bullet \end{cases}$$

non intersecting paths

$$\begin{cases} q_{\bullet\bullet} = 0 \\ t_{\bullet\circ} = t_{\circ\bullet} = 0 \end{cases}$$

(ASM)
(osc. paths)



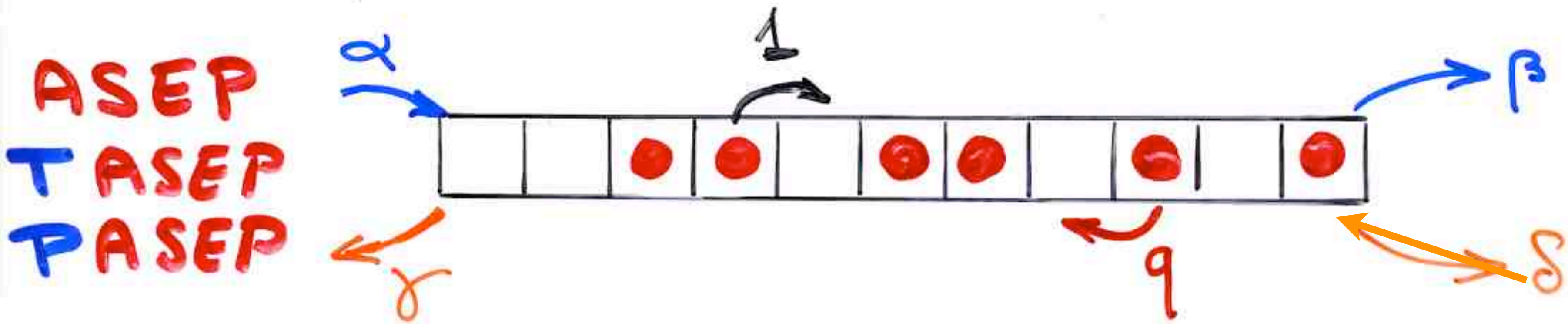
The quadratic algebra Z

4 generators $B_\bullet \; A_\bullet \; B \; A$
8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} B A = q_{\circ\circ} A B + t_{\circ\circ} A_\bullet B_\bullet \\ B_\bullet A_\bullet = \bigcirc A_\bullet B_\bullet + t_{\bullet\bullet} A B \\ B_\bullet A = q_{\bullet\circ} A B_\bullet + \bigcirc A_\bullet B \\ B A_\bullet = q_{\circ\bullet} A_\bullet B + \bigcirc A B_\bullet \end{cases}$$

The PASEP algebra

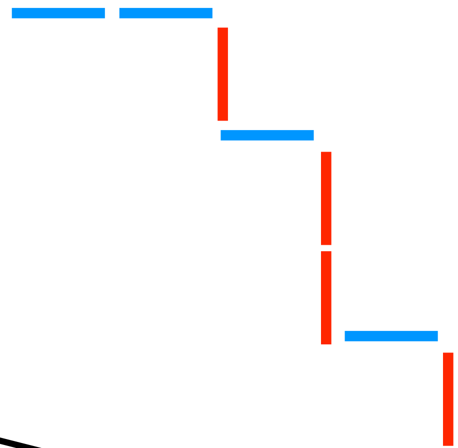
$$DE = qED + E + D$$


ASEP
TASEP
PASEP
α
1
β
γ
q
δ

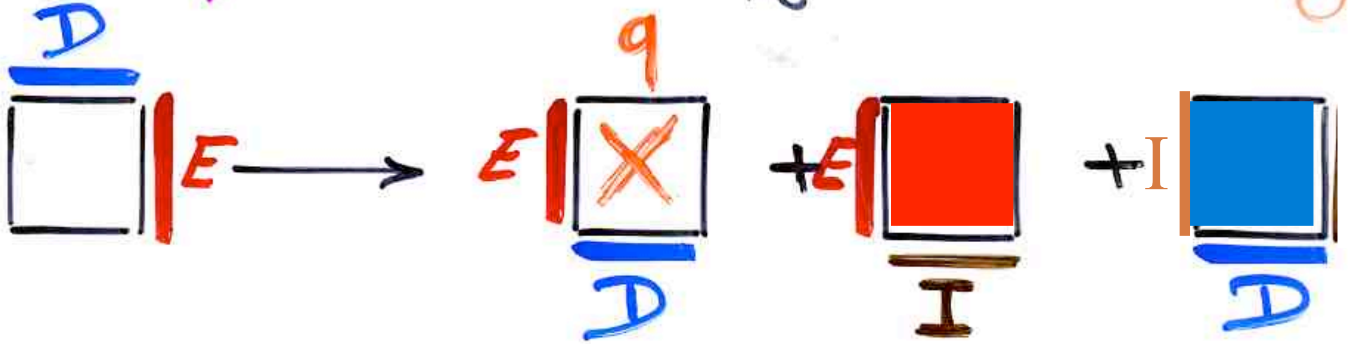
D D E D E E D E

D DE (D E) E D E

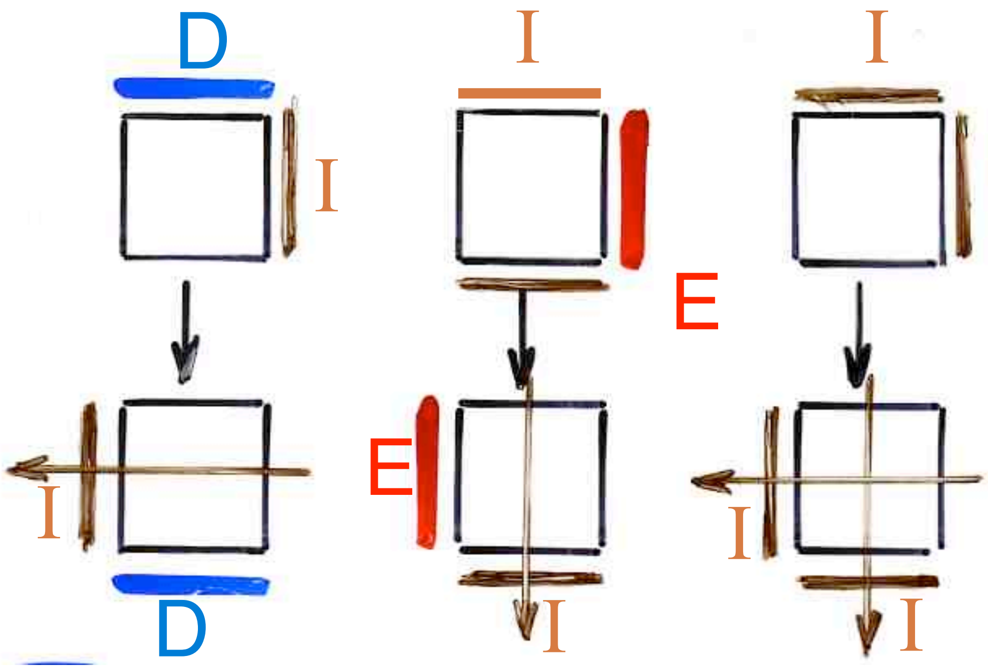
DDE(E)EDE + DDE(ED)EDE + DDE(D)EDE



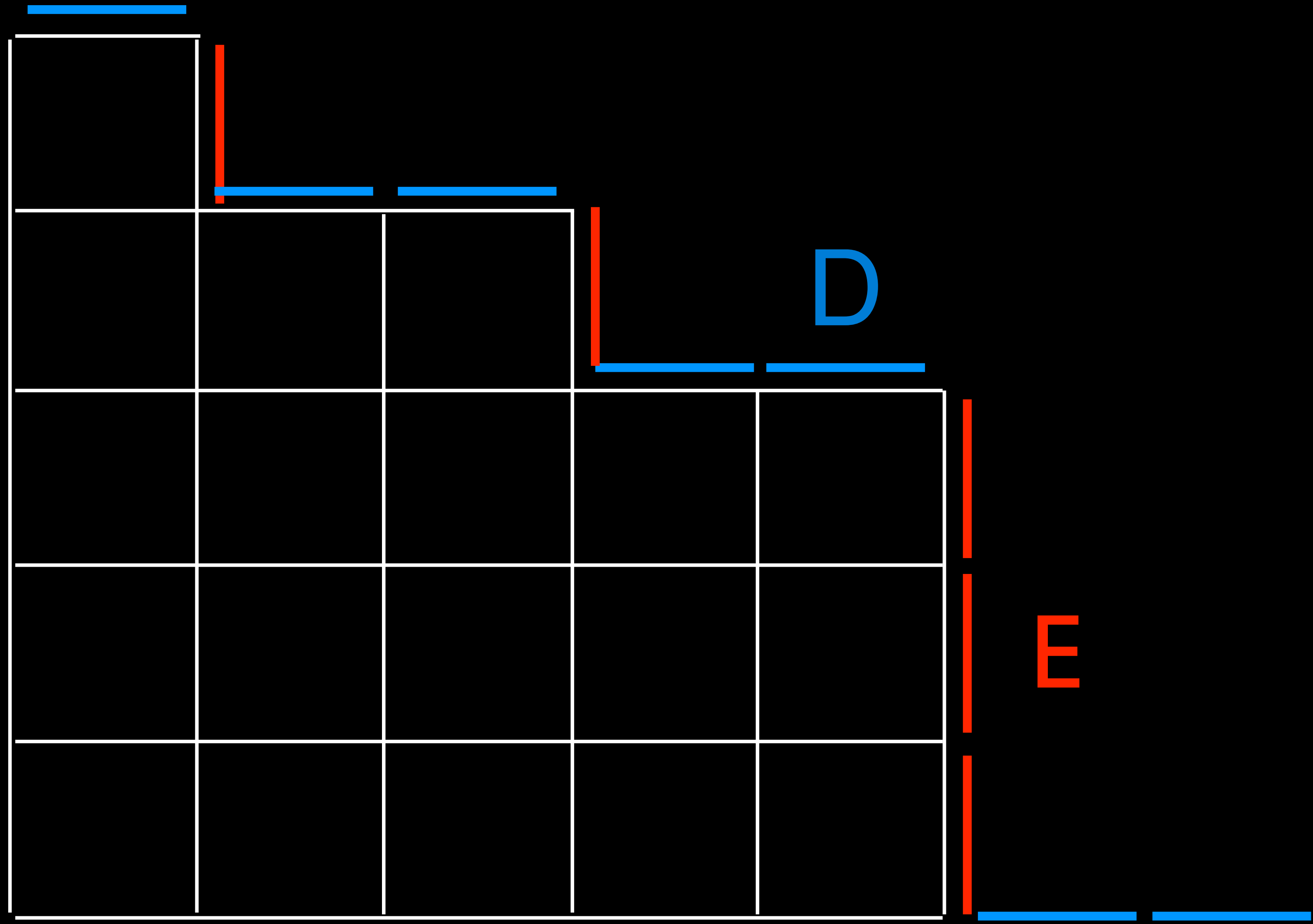
Proof: "planarization" of the rewriting rules

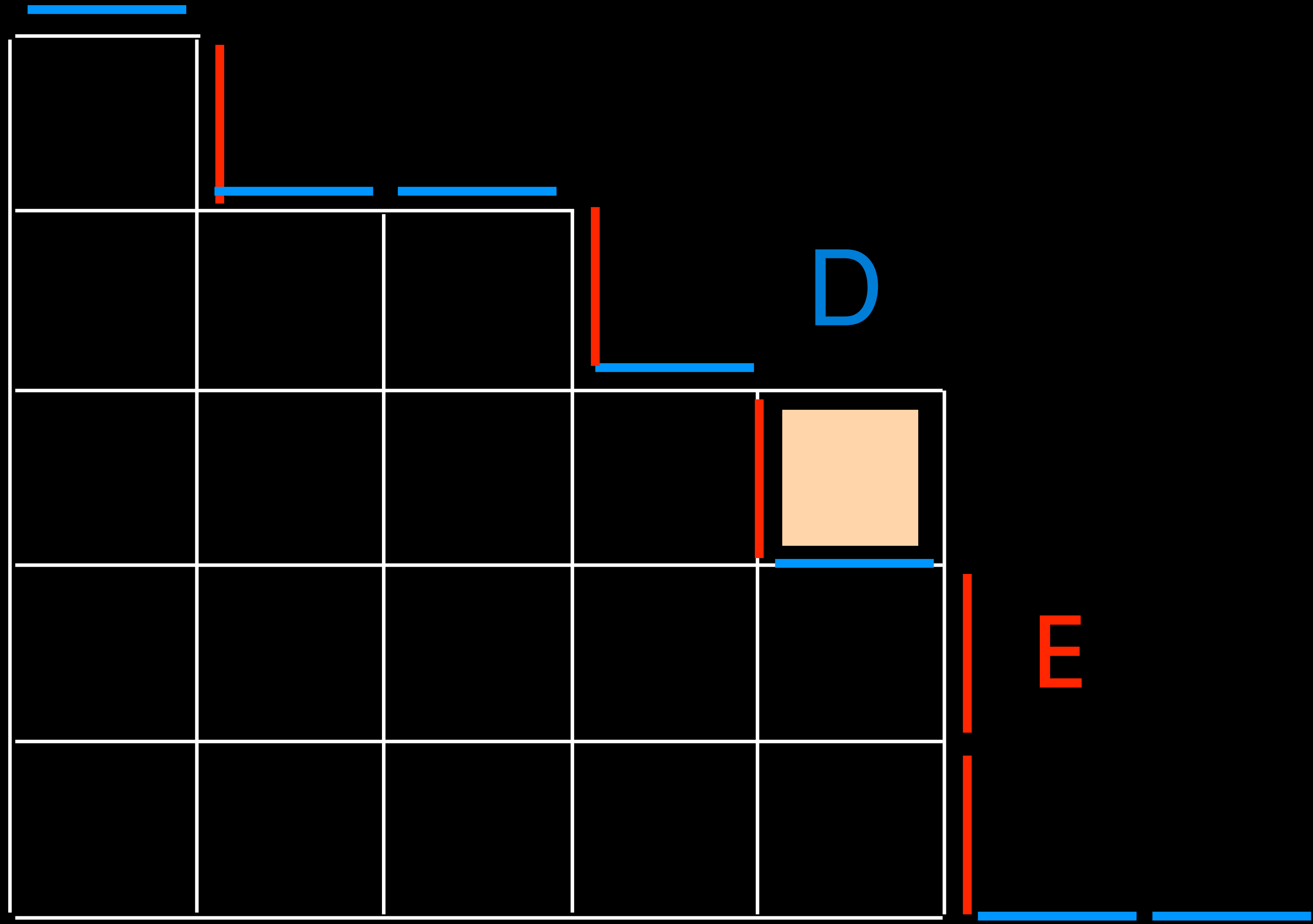


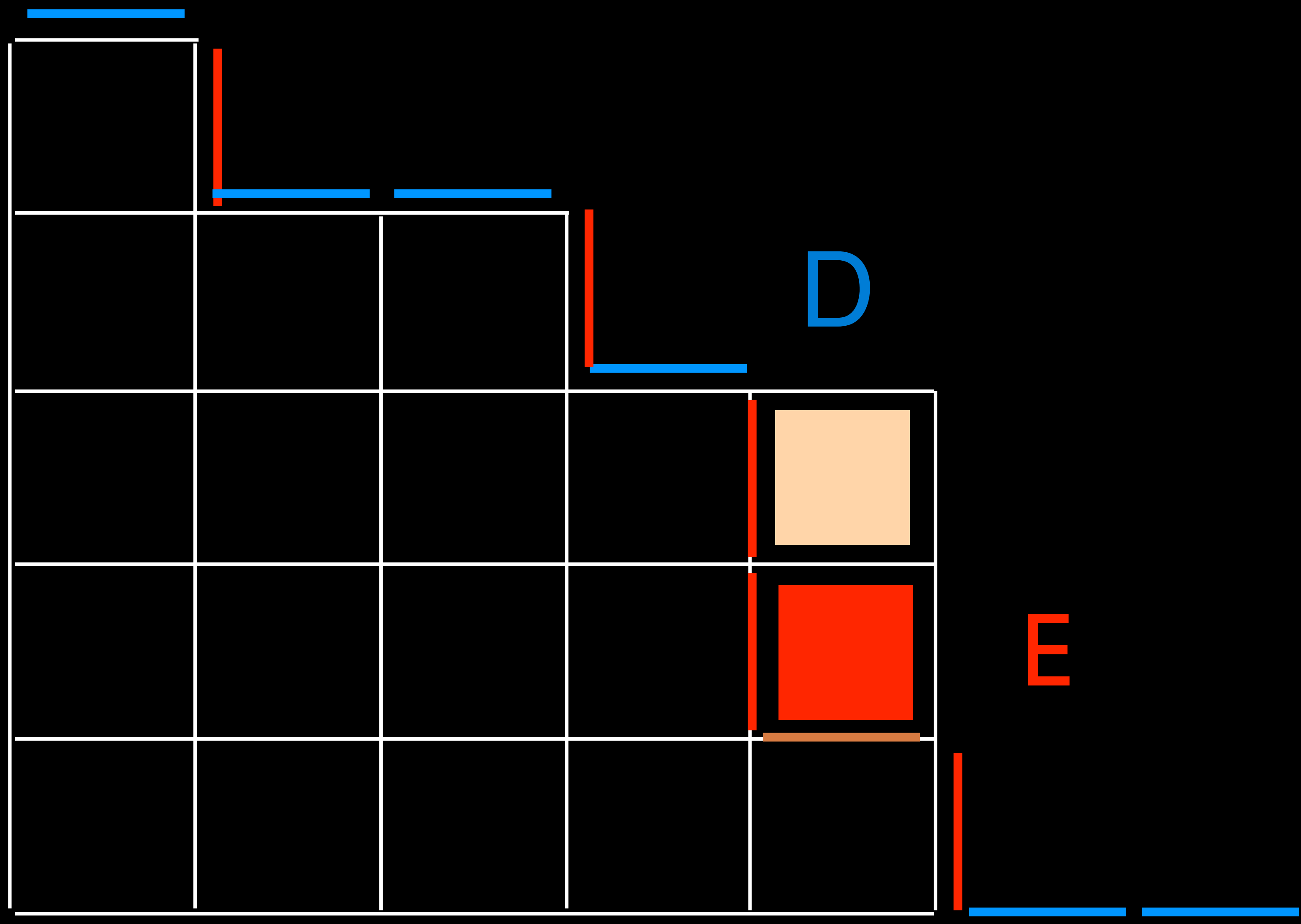
I identity



$$DE = qED + EI_h + I_vD$$
$$DI_v = I_vD$$
$$I_hE = EI_h$$
$$I_hI_v = I_vI_h$$


D
E


D
E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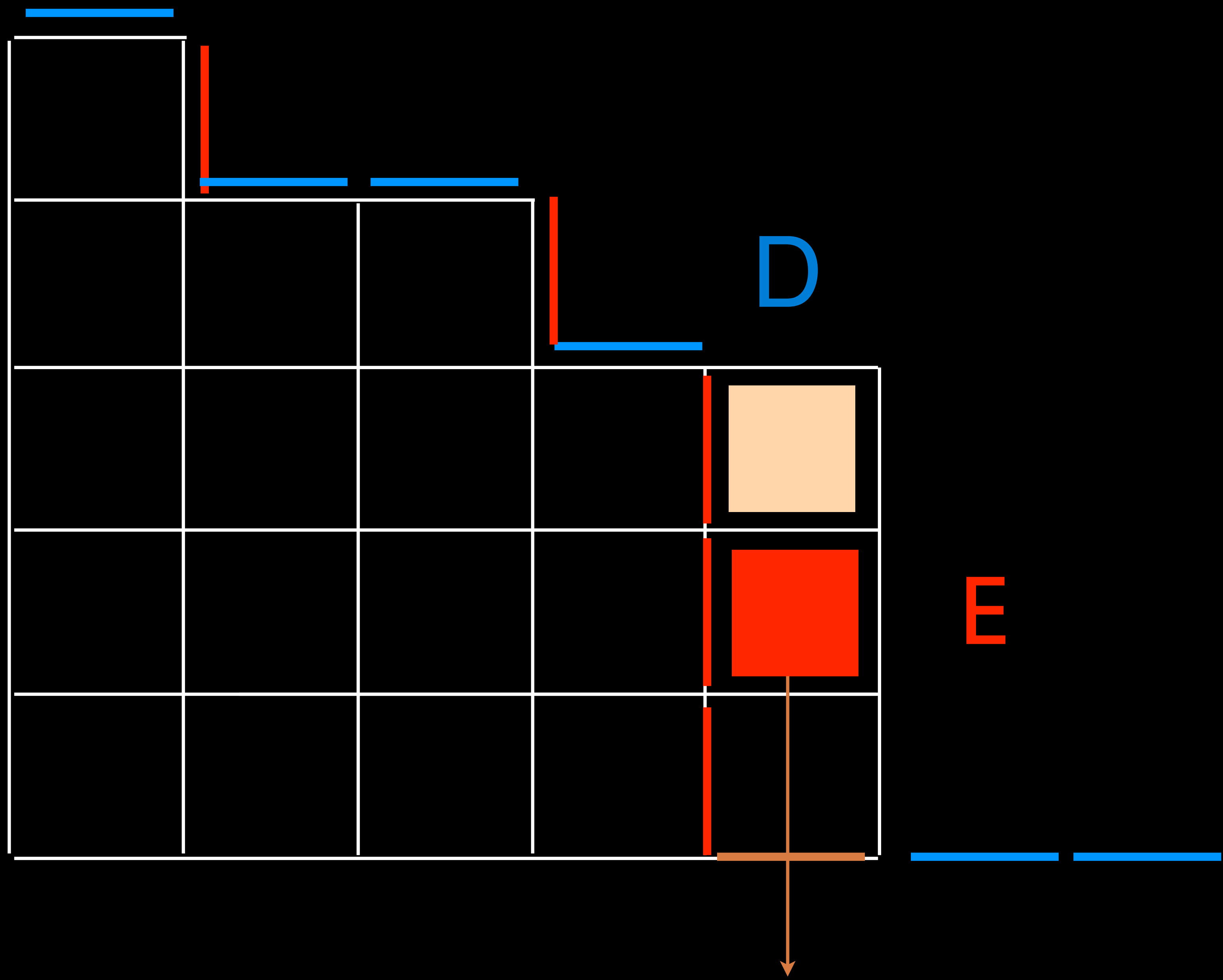
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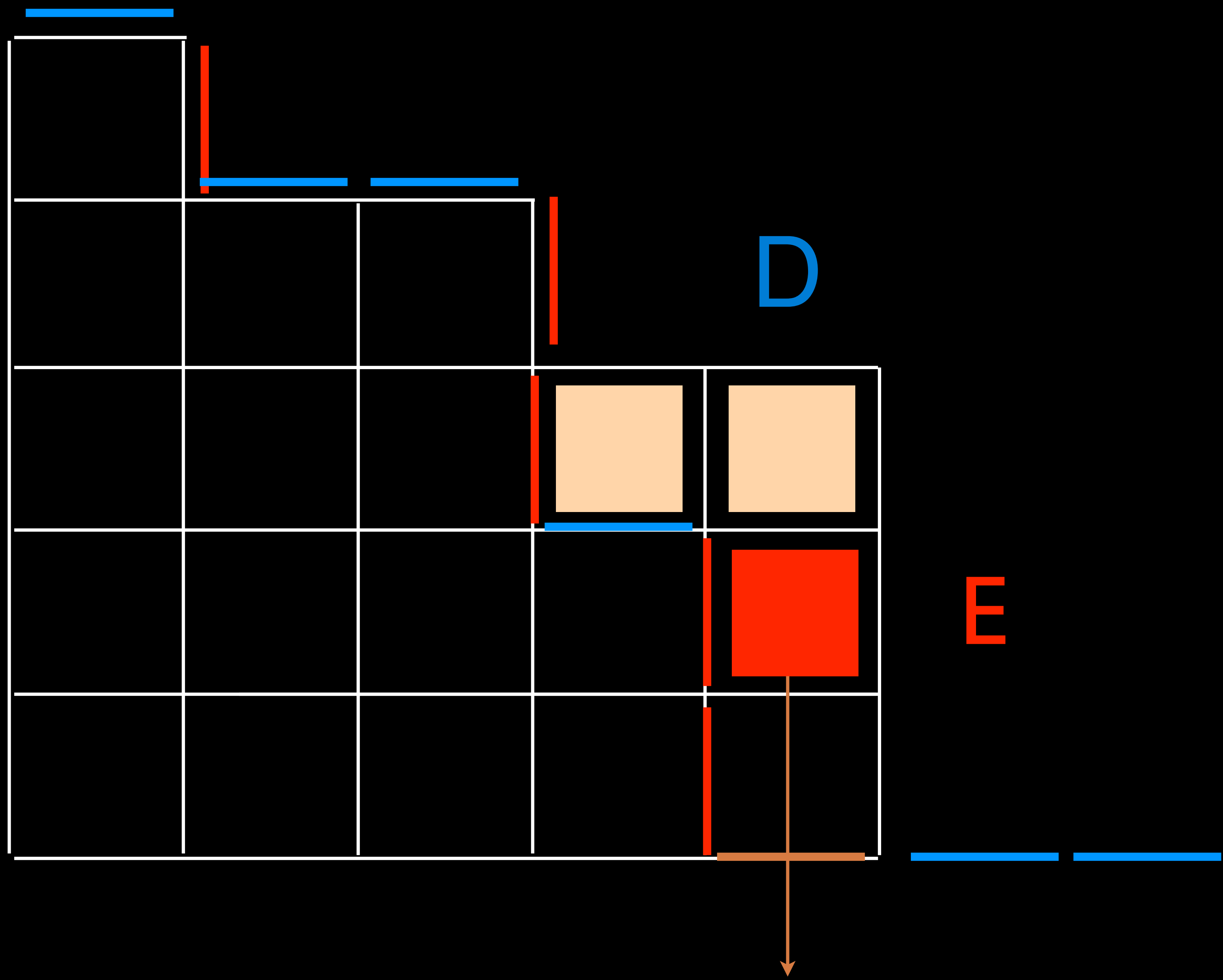
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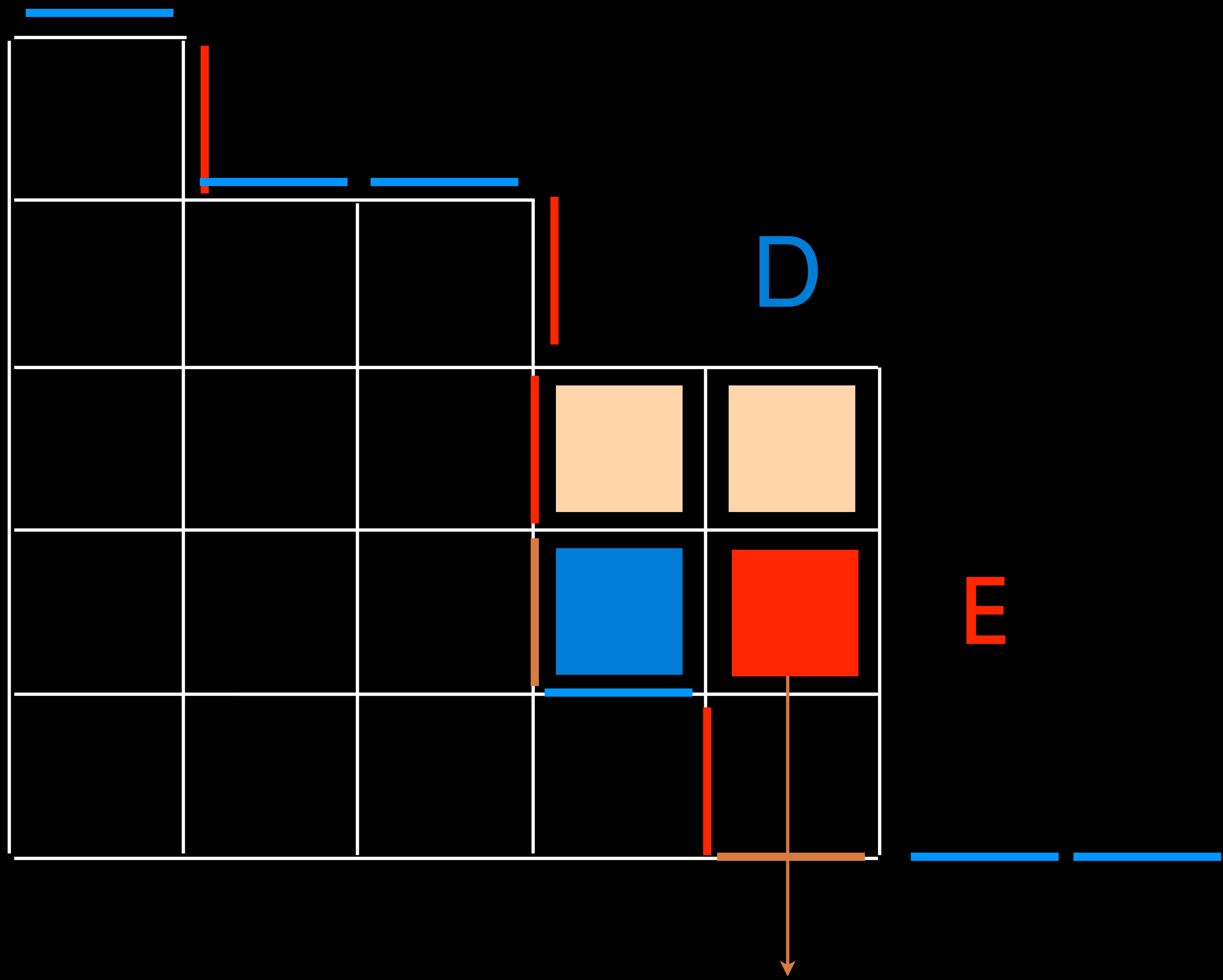
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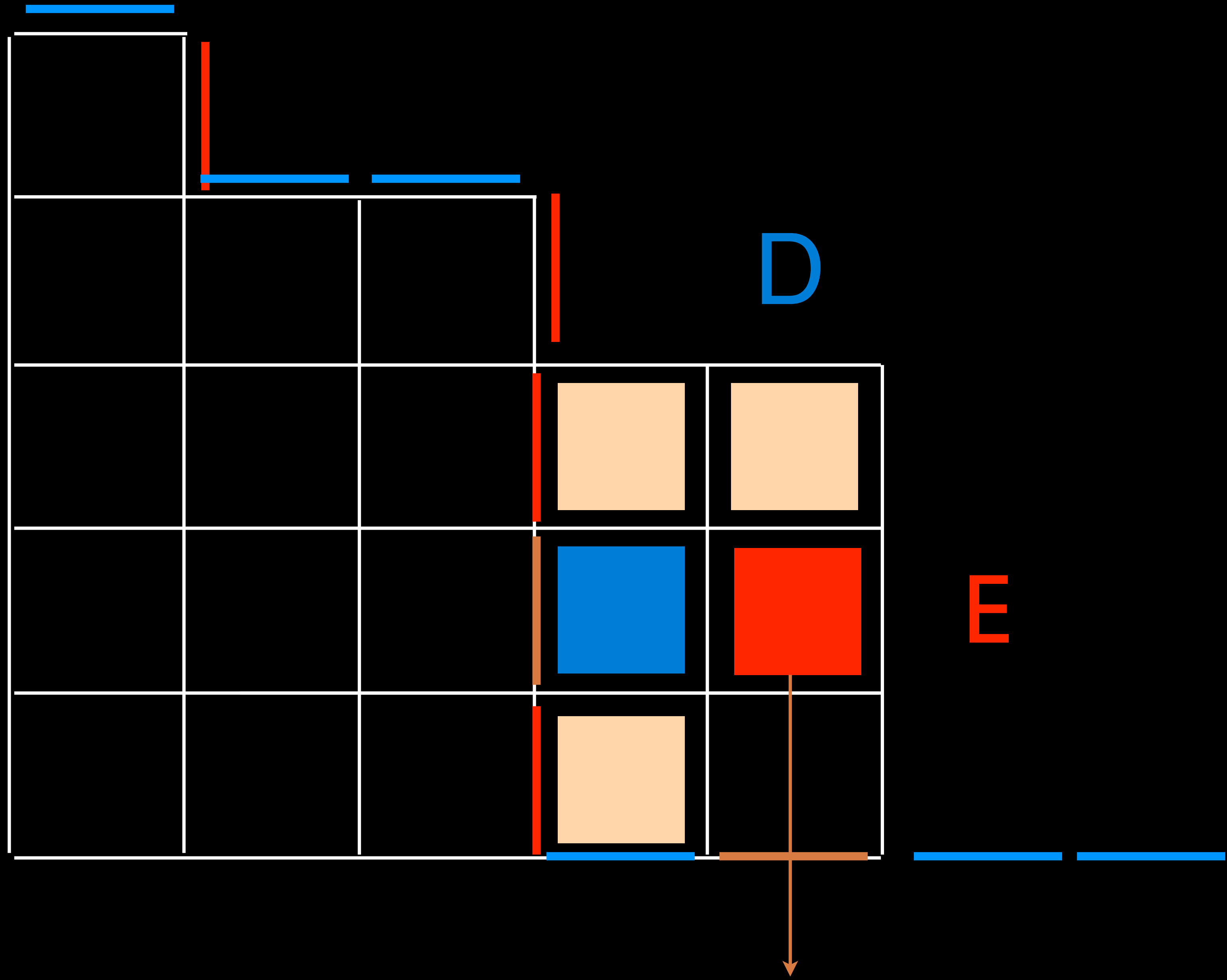
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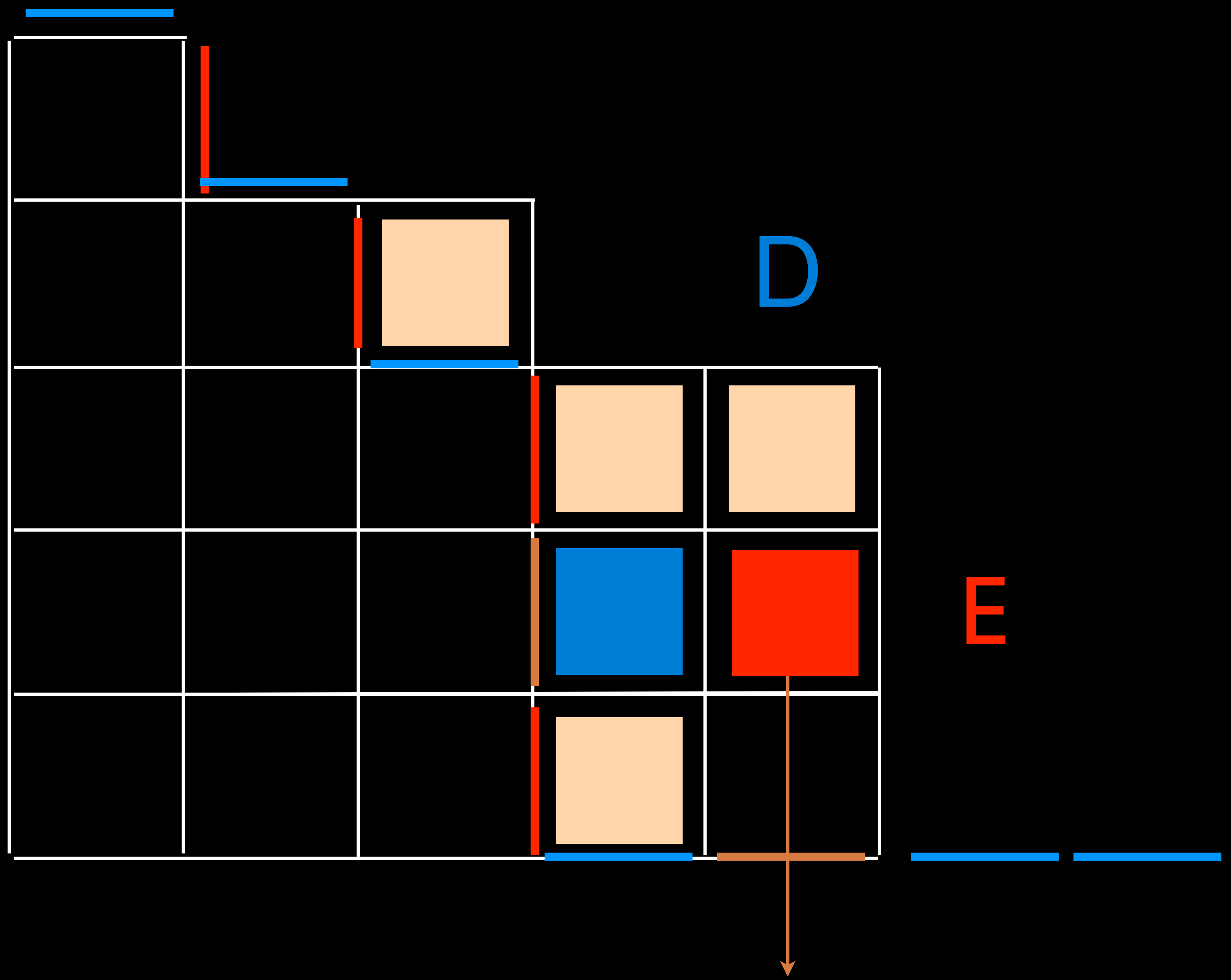
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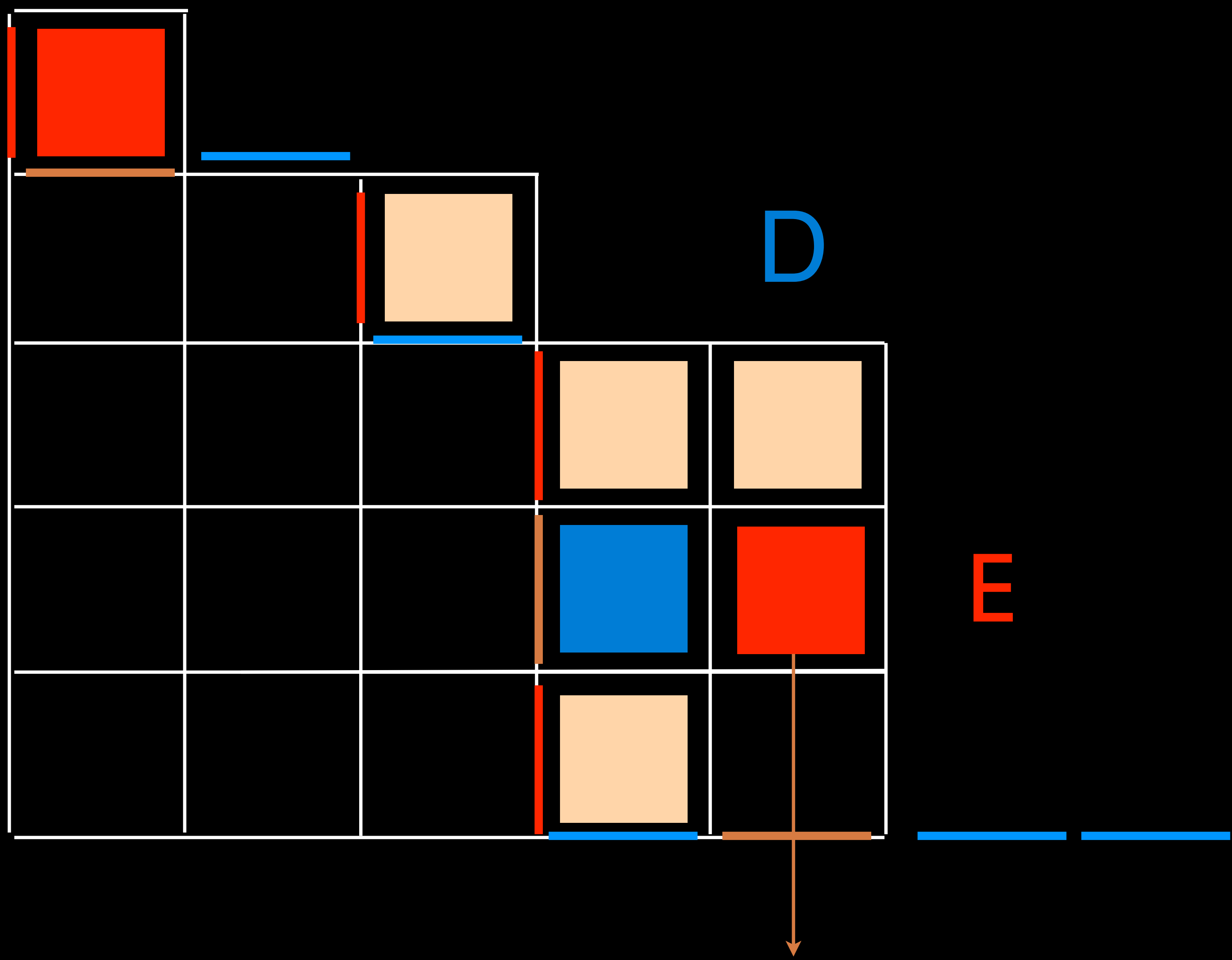
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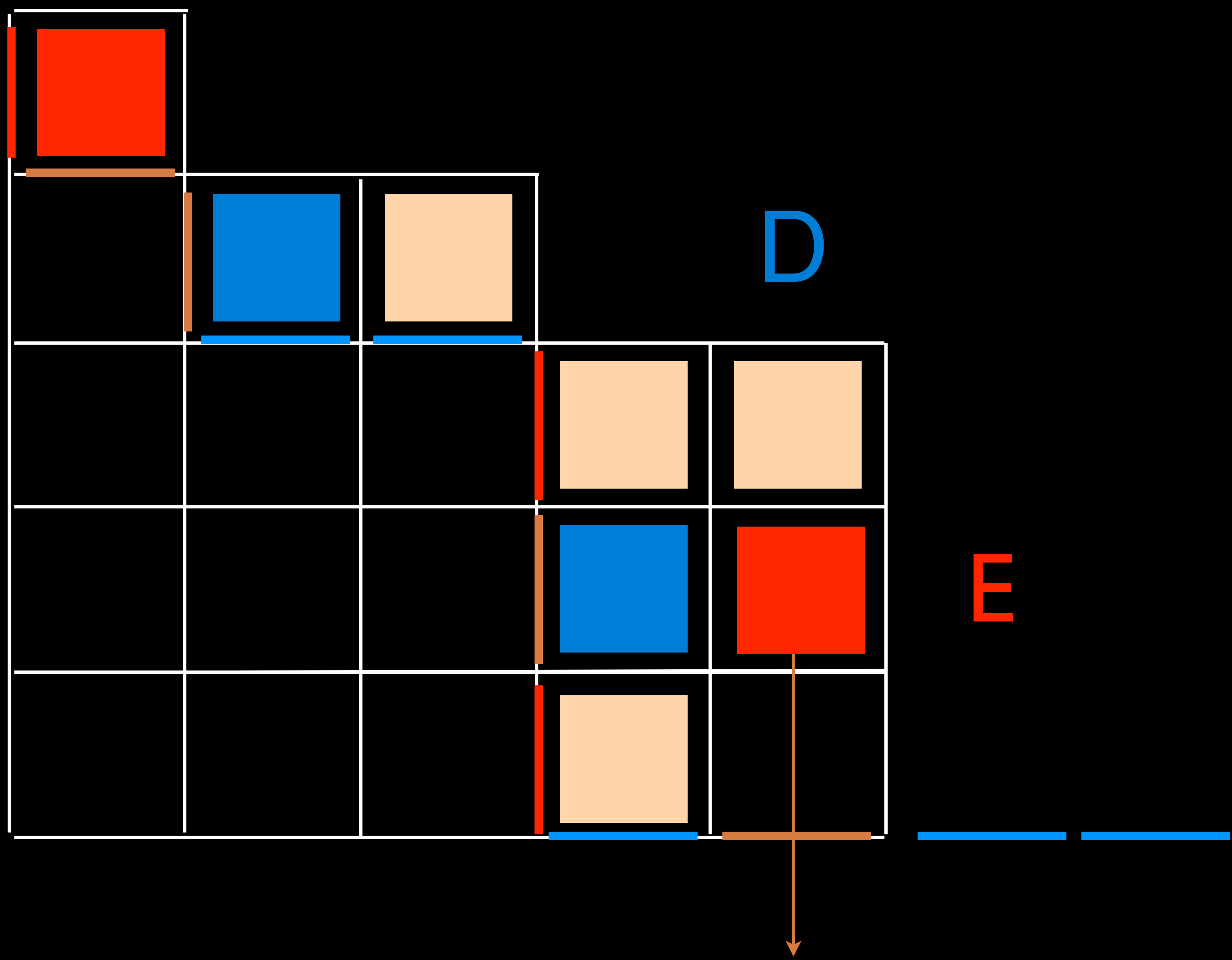
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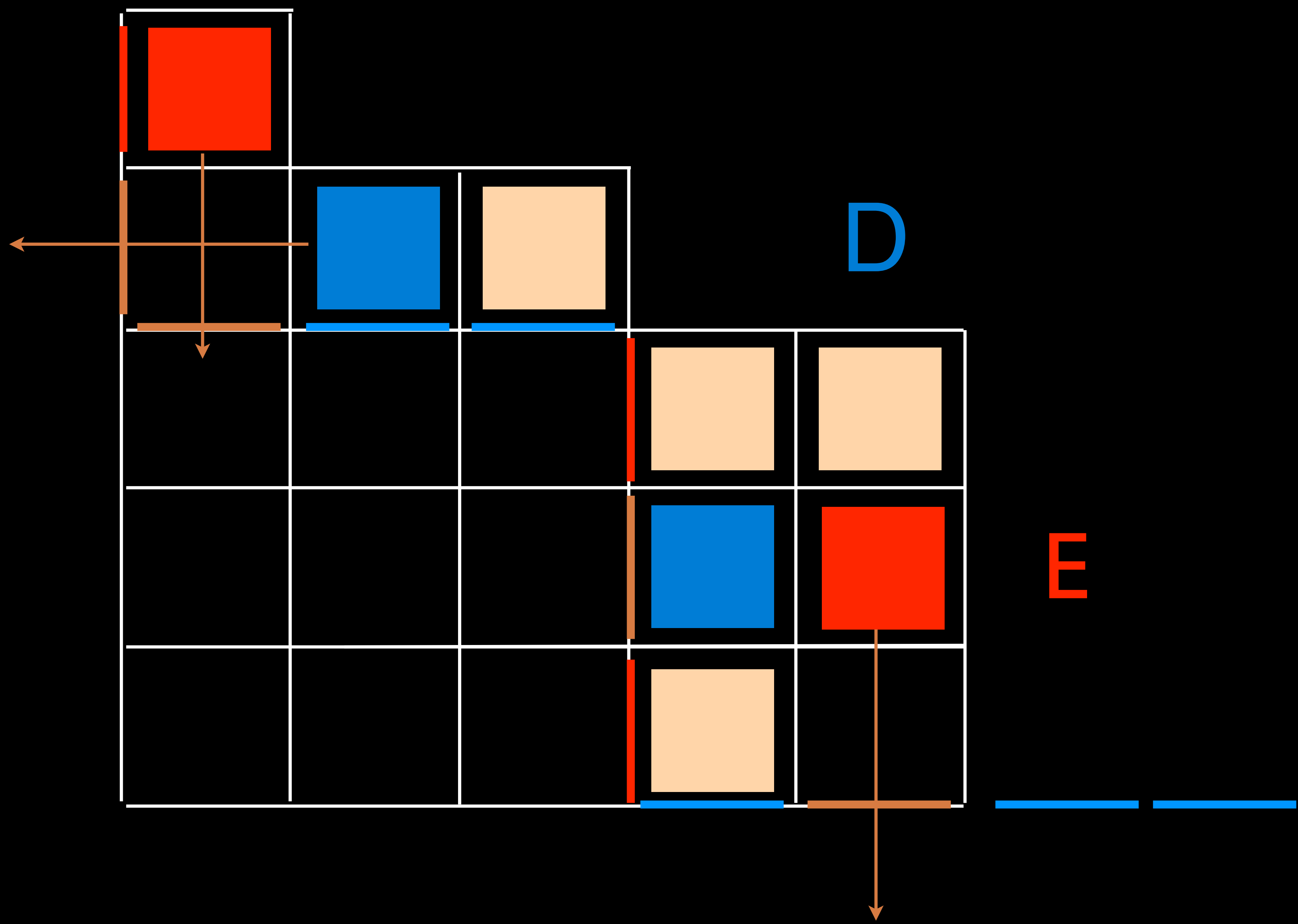

D
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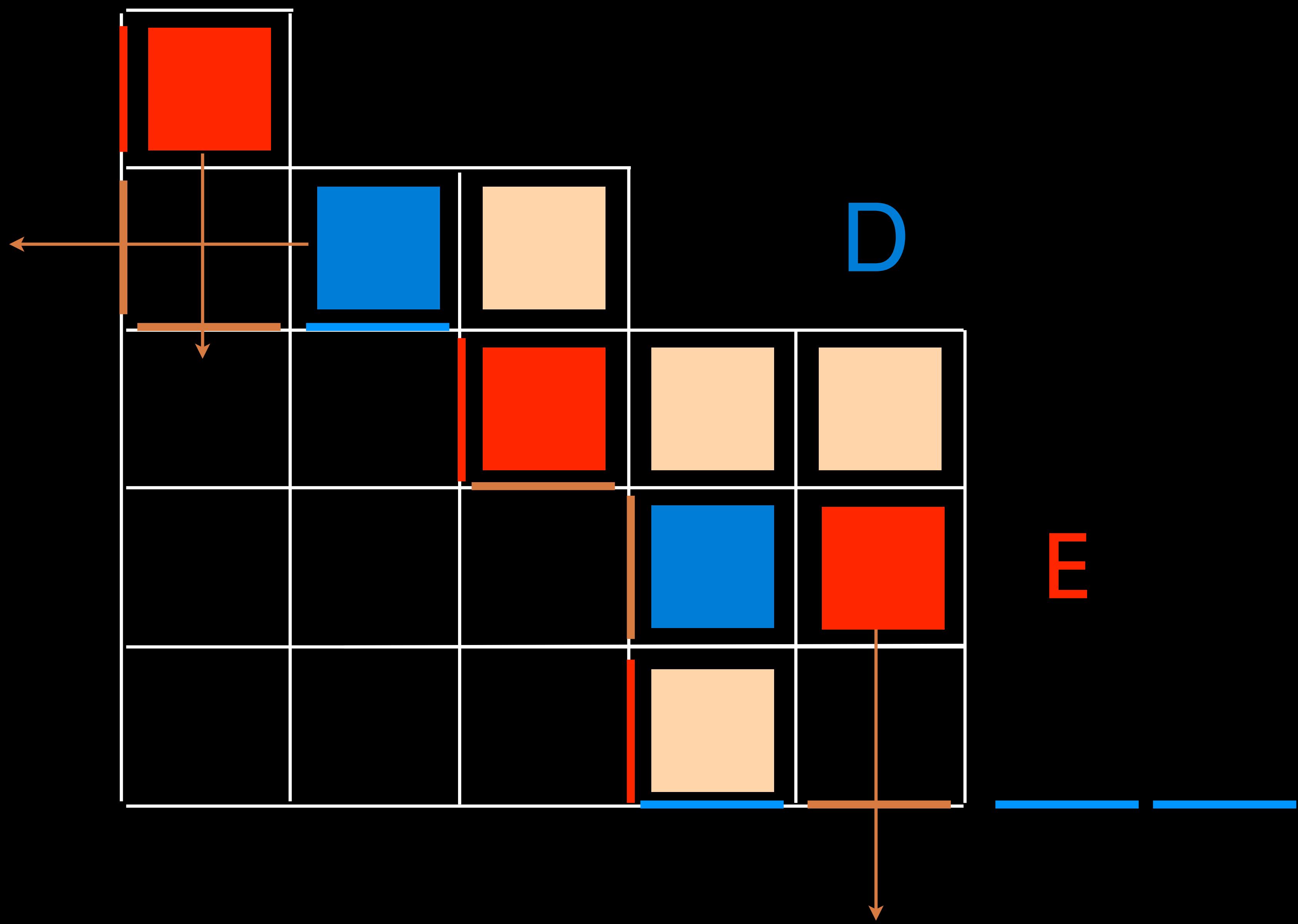
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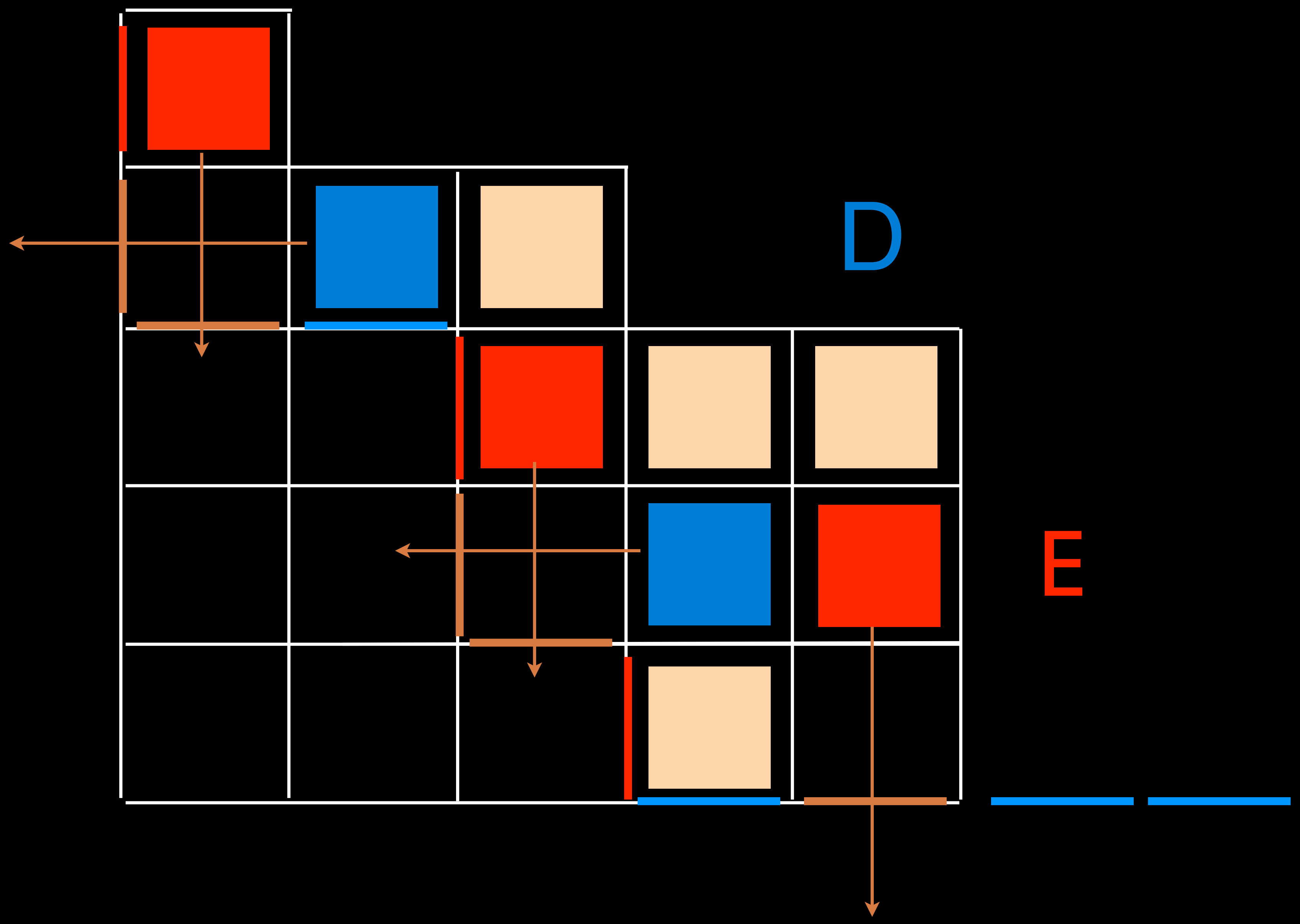
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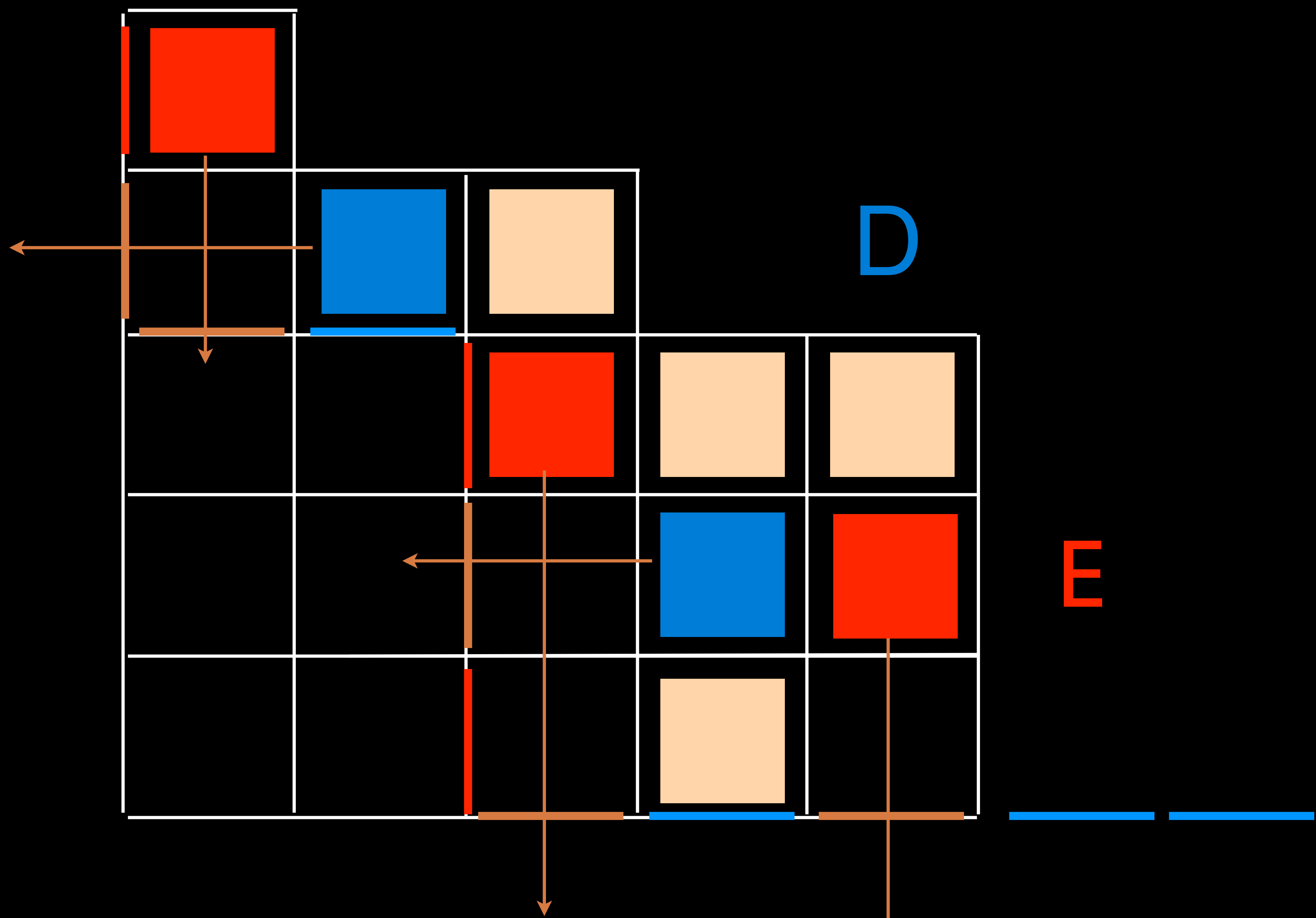
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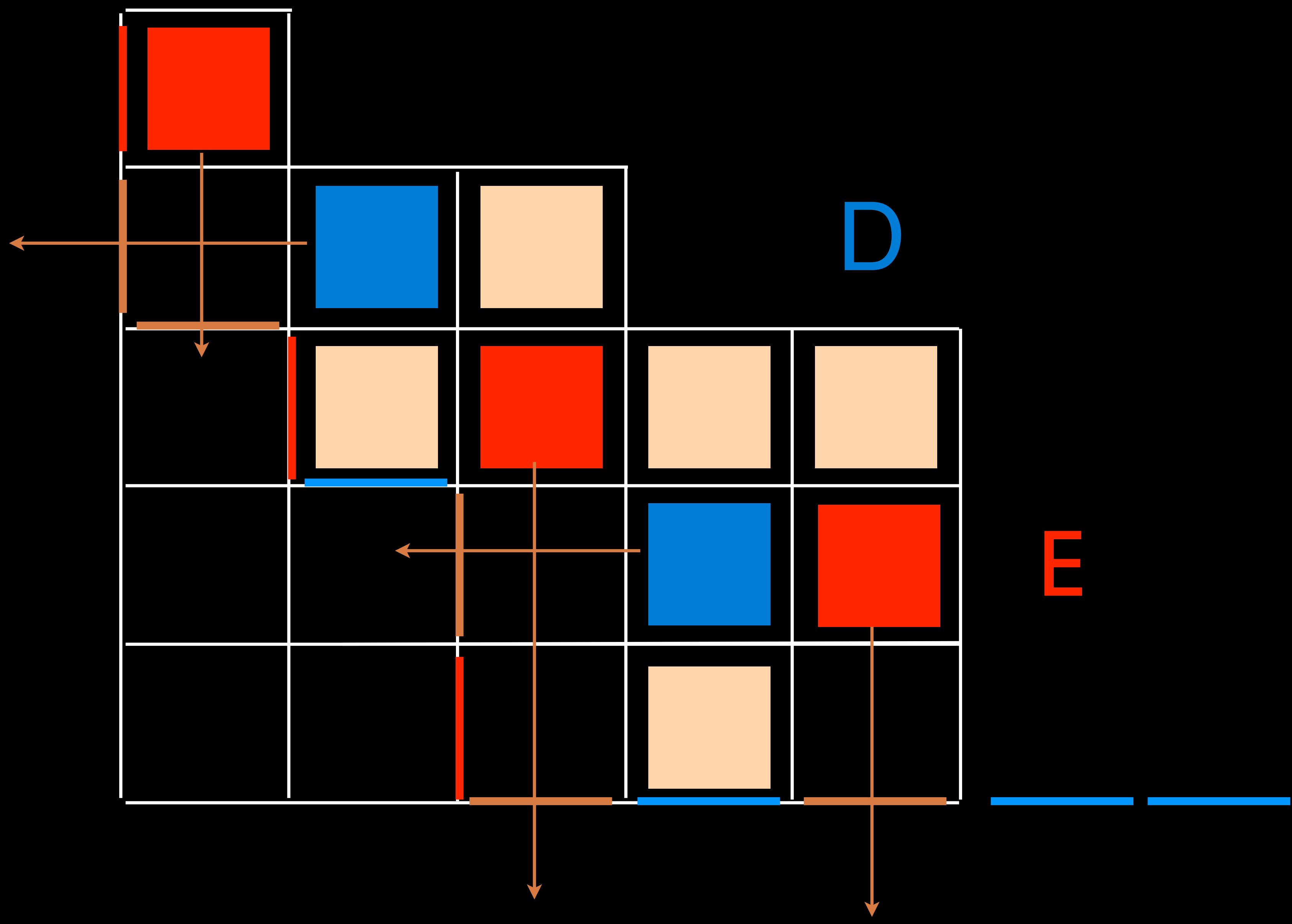
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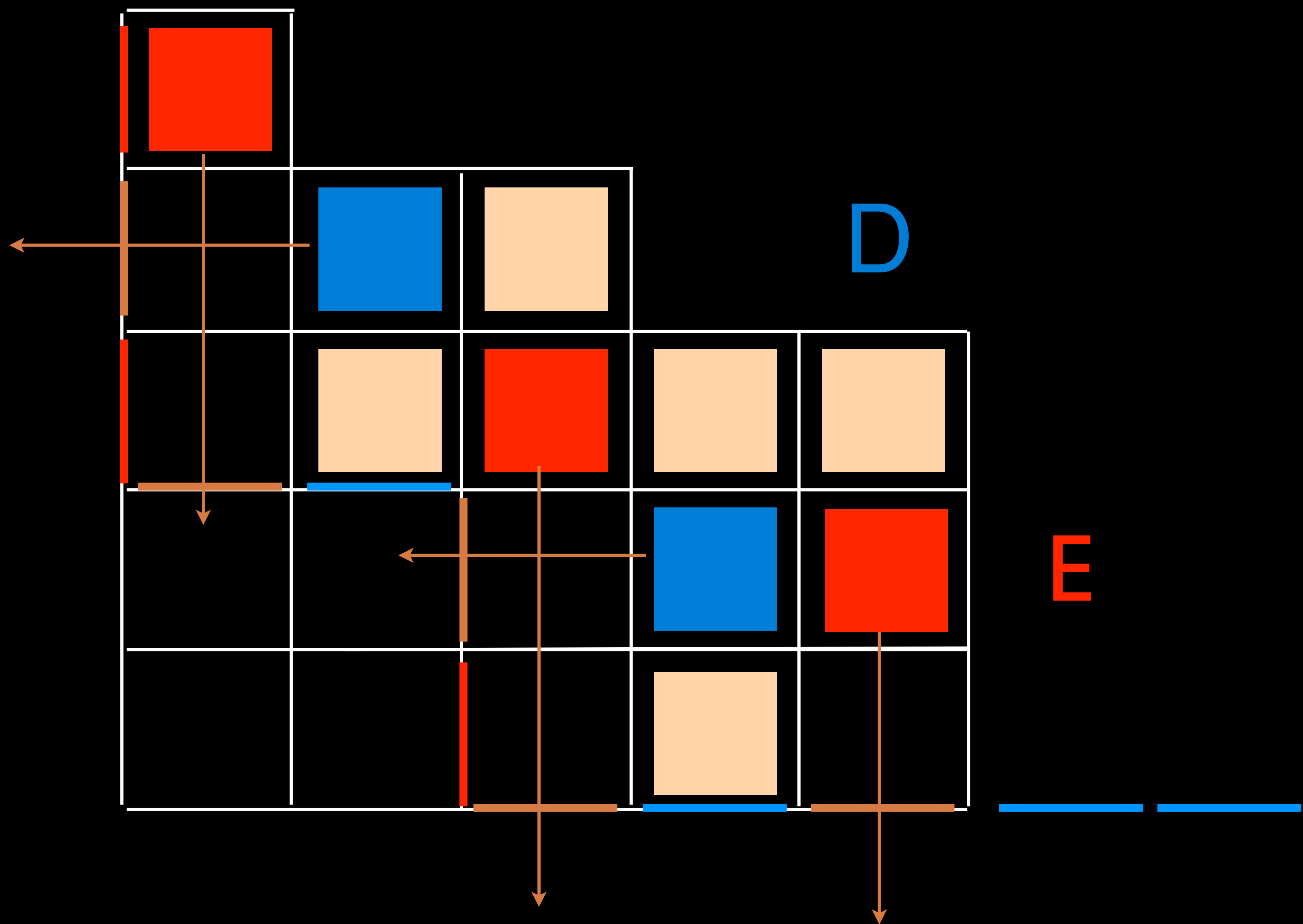
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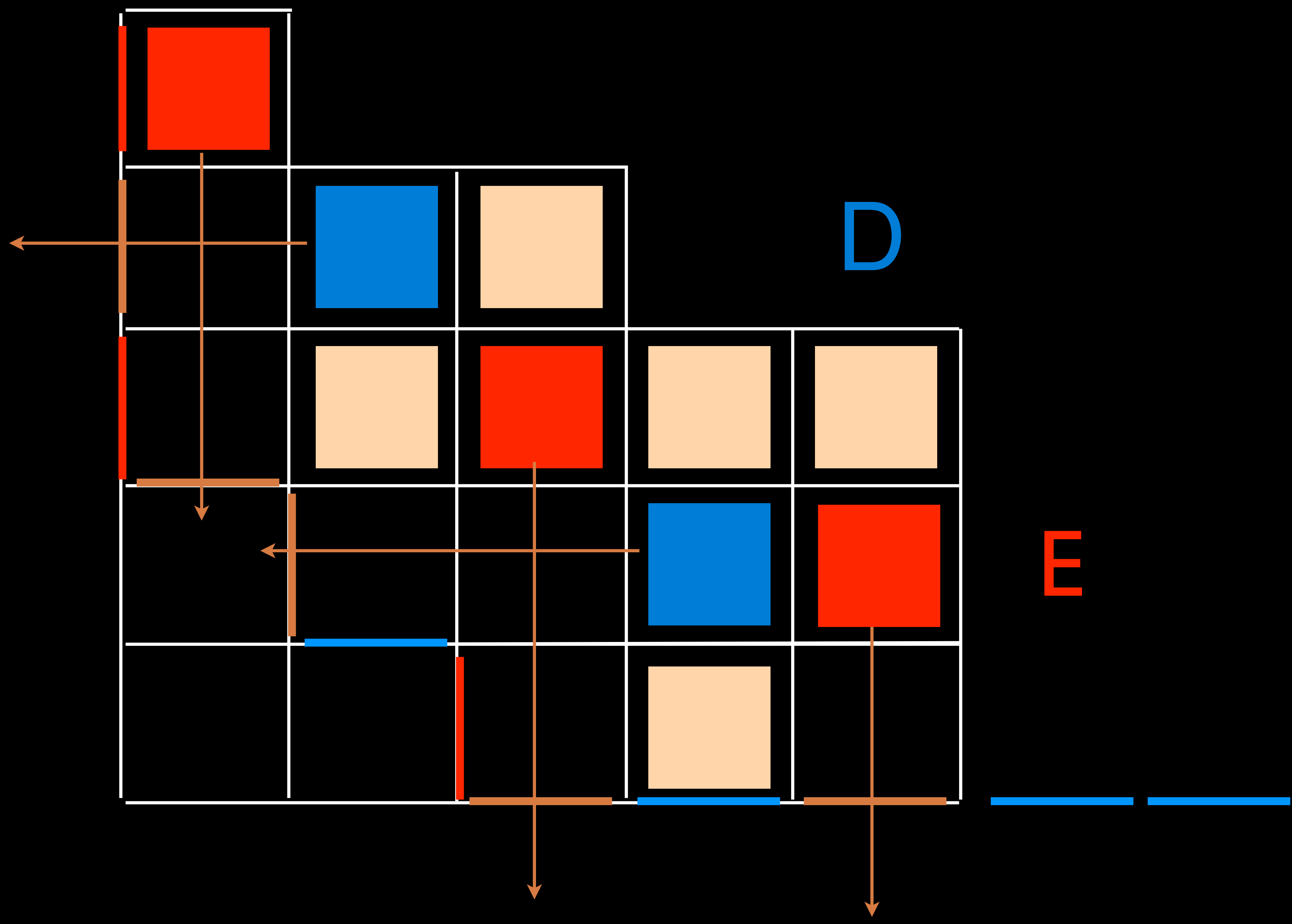

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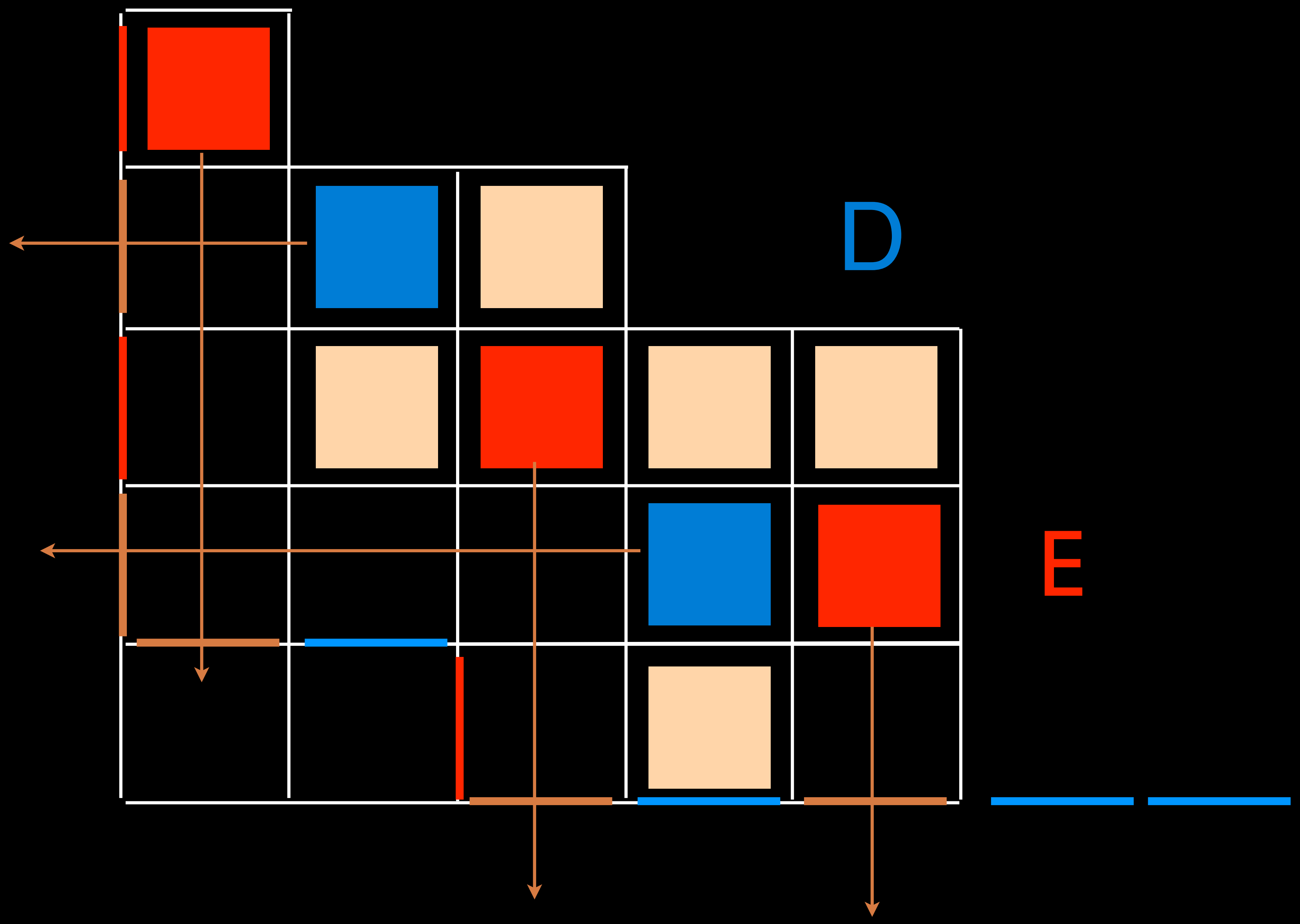
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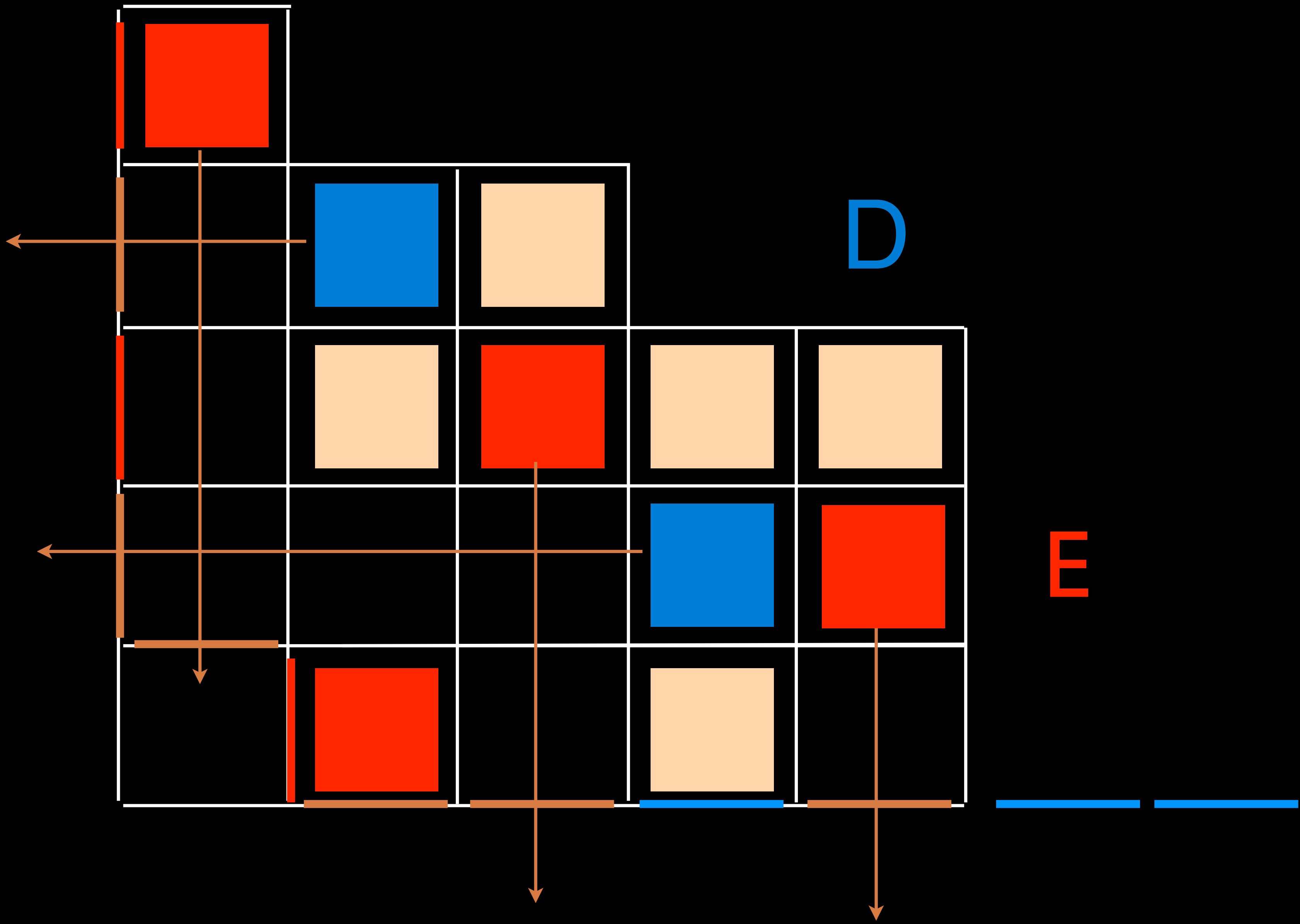
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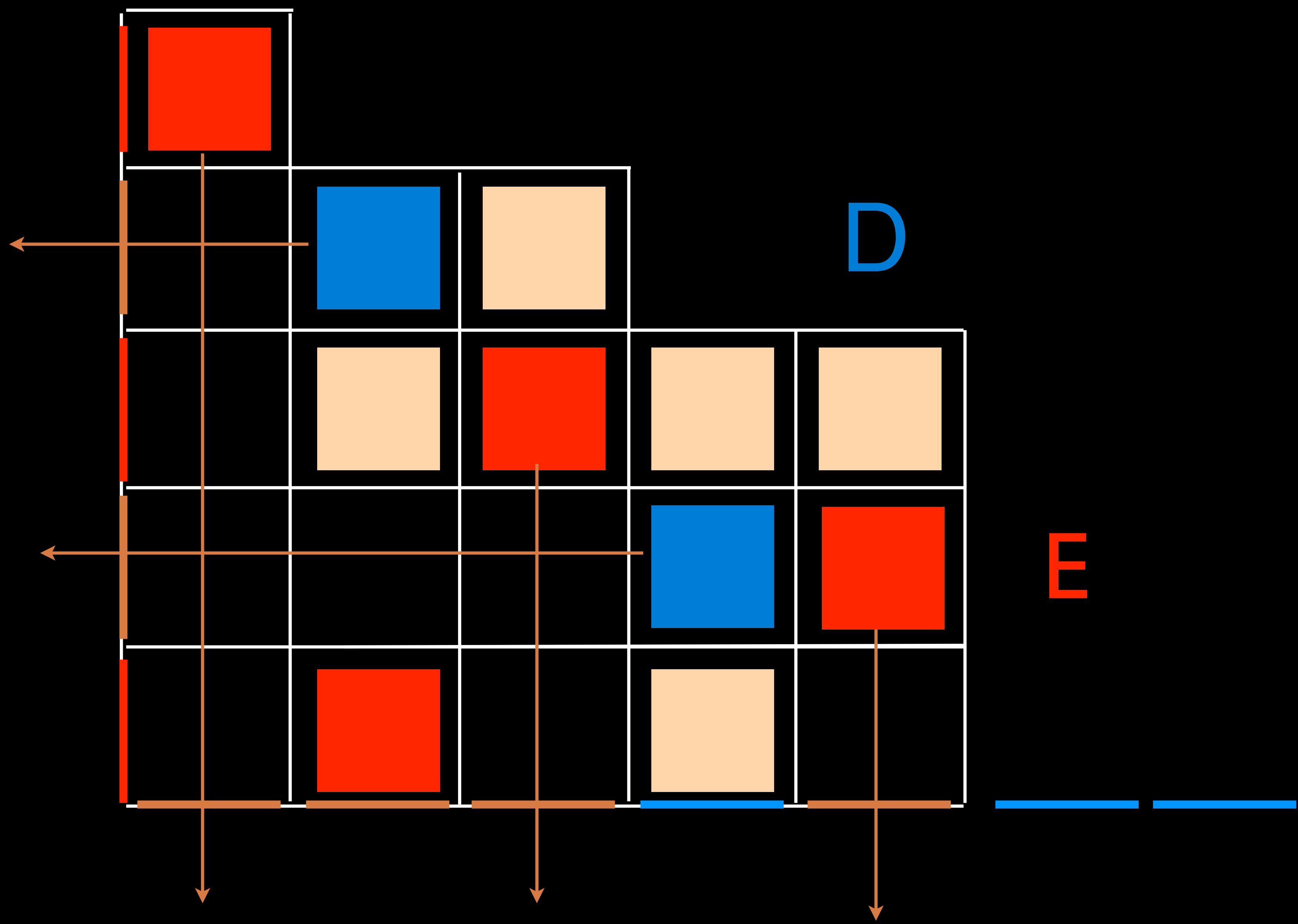
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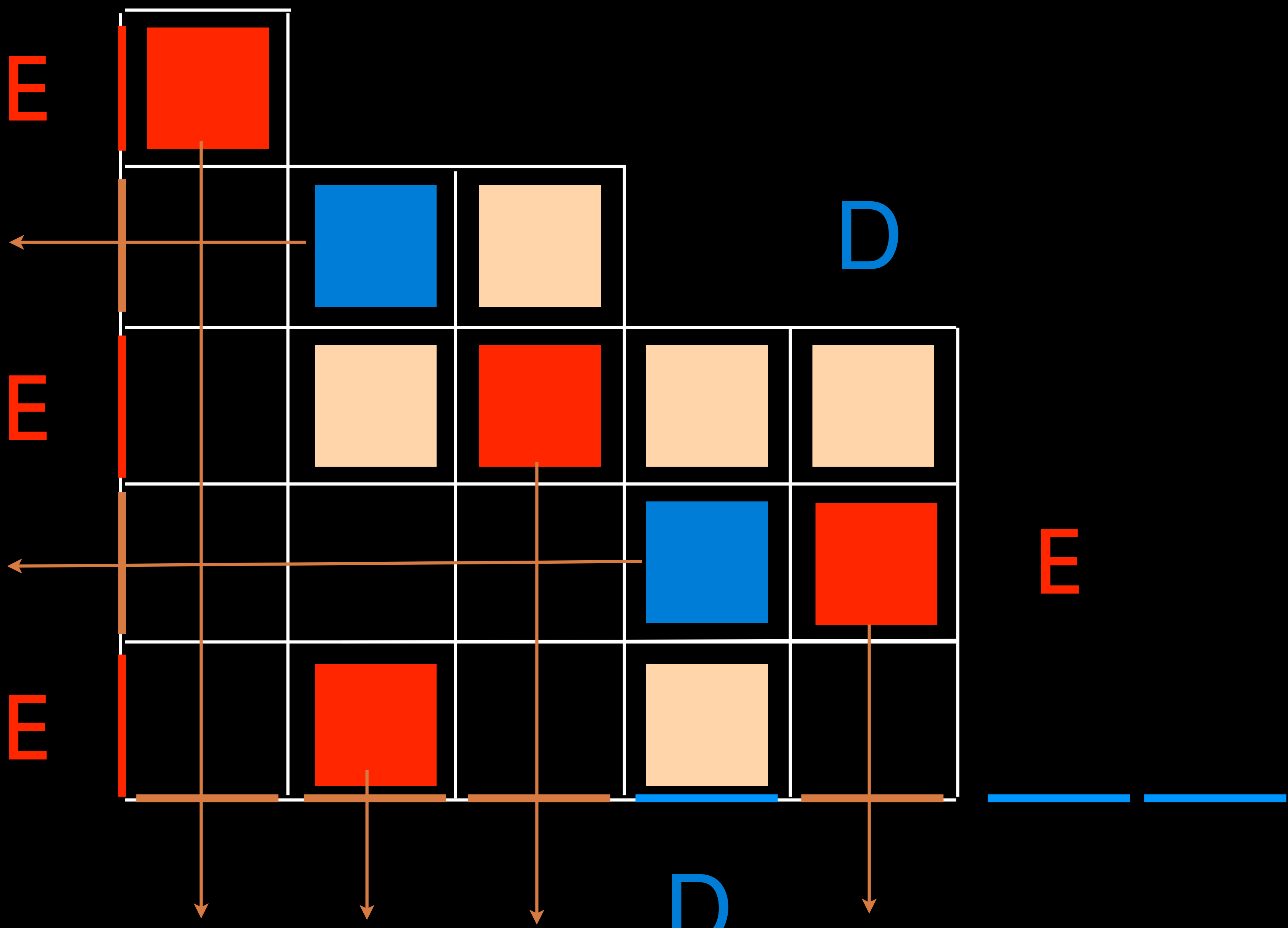
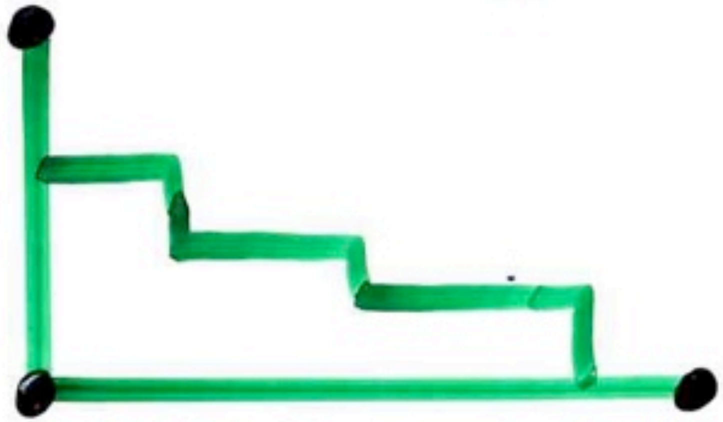

E
D
E
E
E
D

alternative tableaux

alternative tableau

- Ferrers diagram F

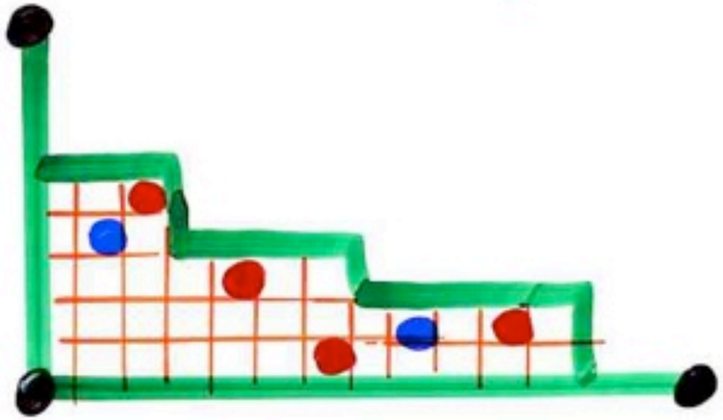


(possibly empty rows or column)

(nb of rows) + (nb of columns) $= n$

alternative tableau

- Ferrers diagram F (possibly empty rows or column)

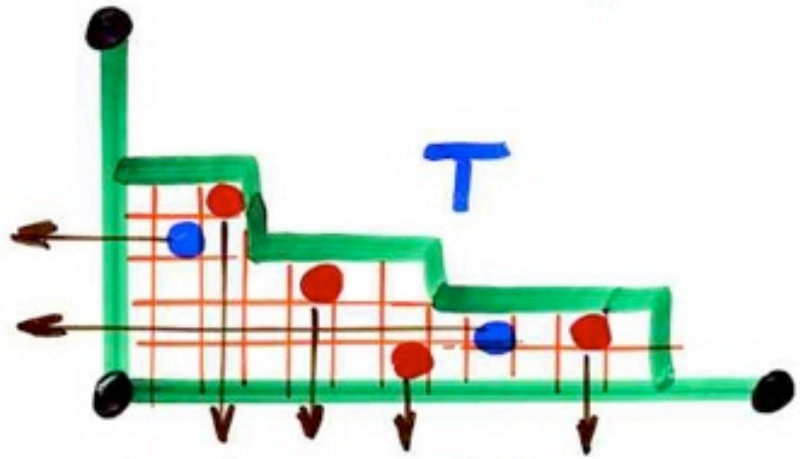


(nb of rows) + (nb of columns) $= n$

- some cells are coloured **red** or **blue**

alternative tableau T

- Ferrers diagram **F** (possibly empty rows or column)



(nb of rows) + (nb of columns) = n

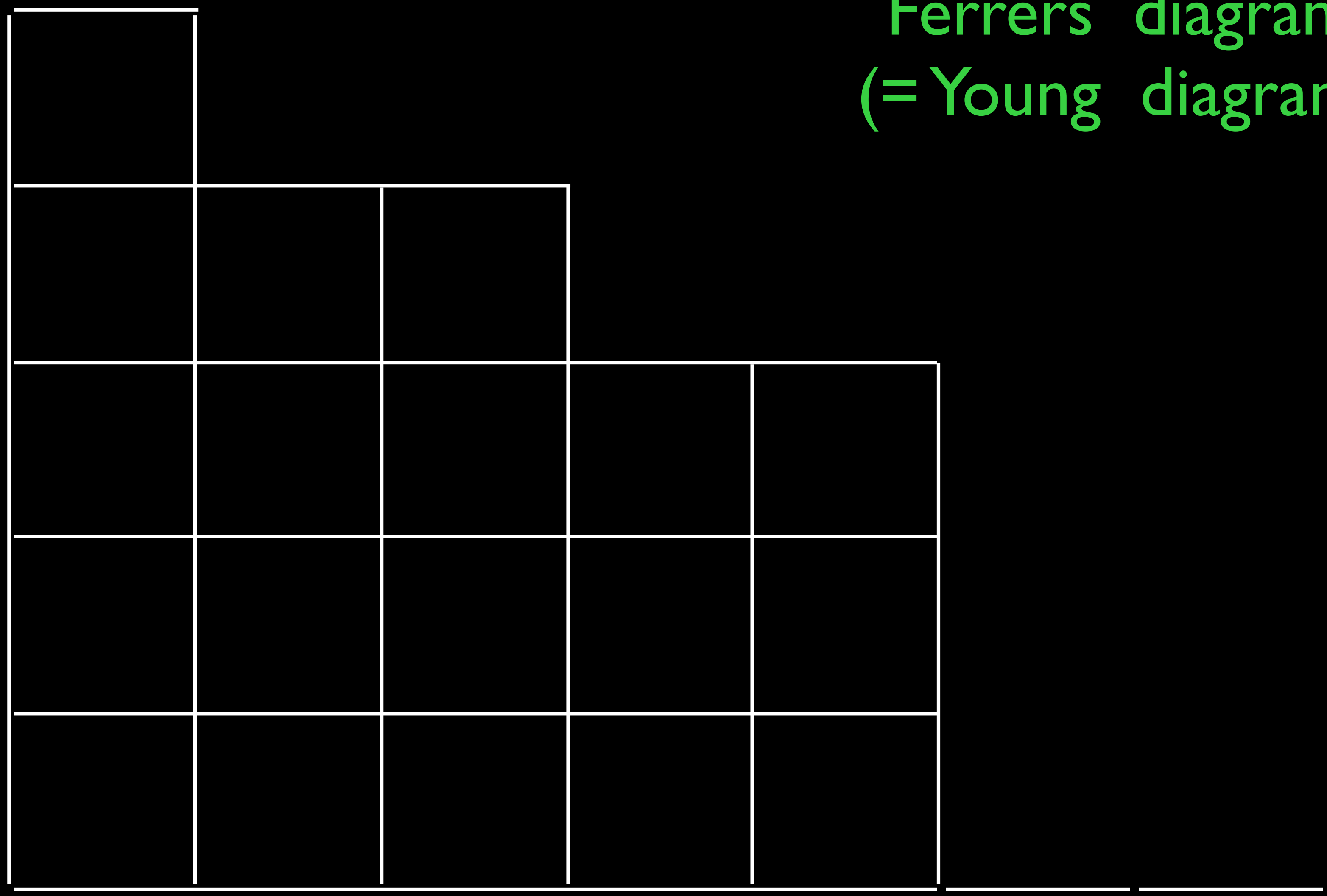
- some cells are coloured **red** or **blue**

- $\begin{cases} \text{no coloured cell at the left of} \\ \text{no coloured cell below} \end{cases}$

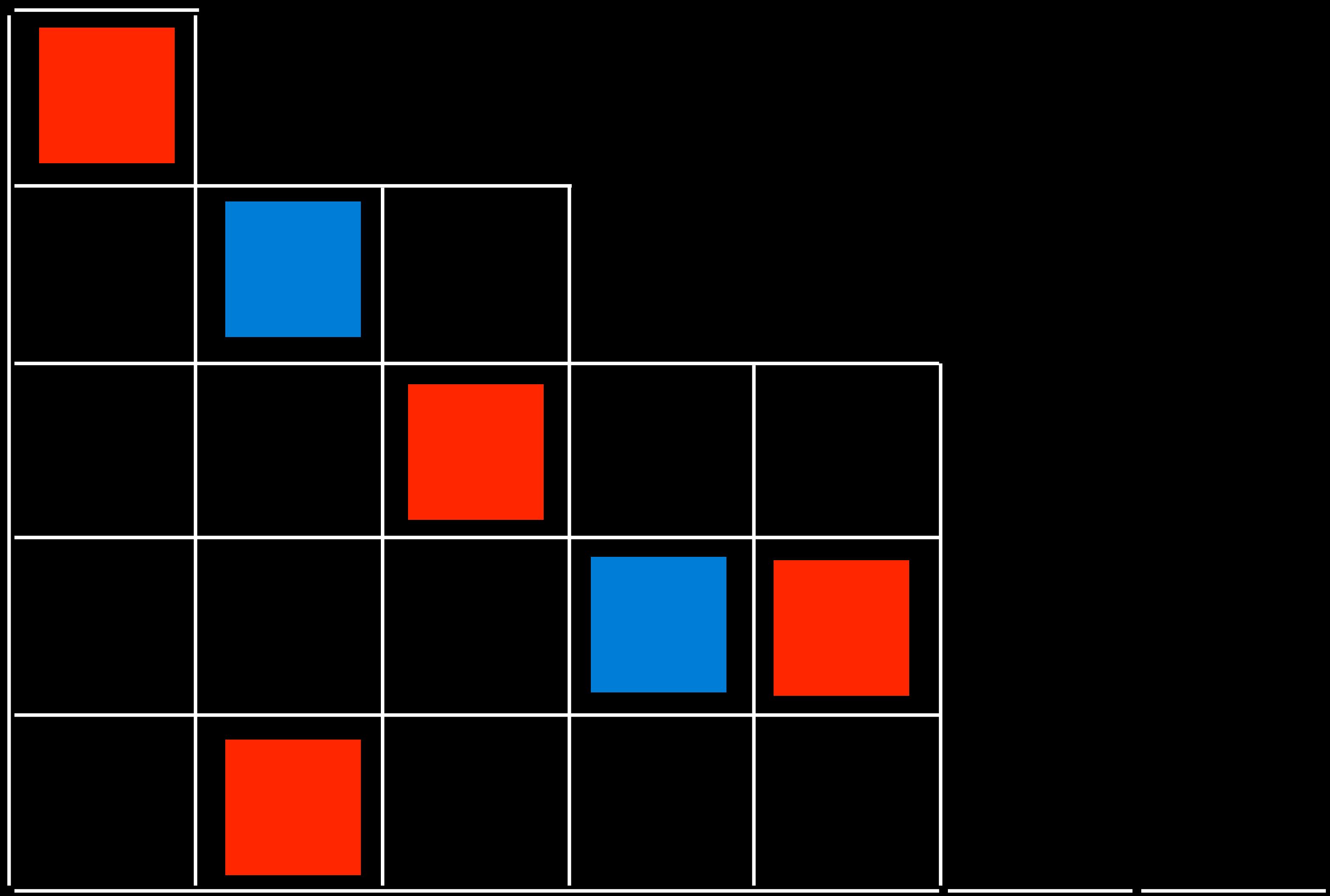
n size of T

alternative tableau

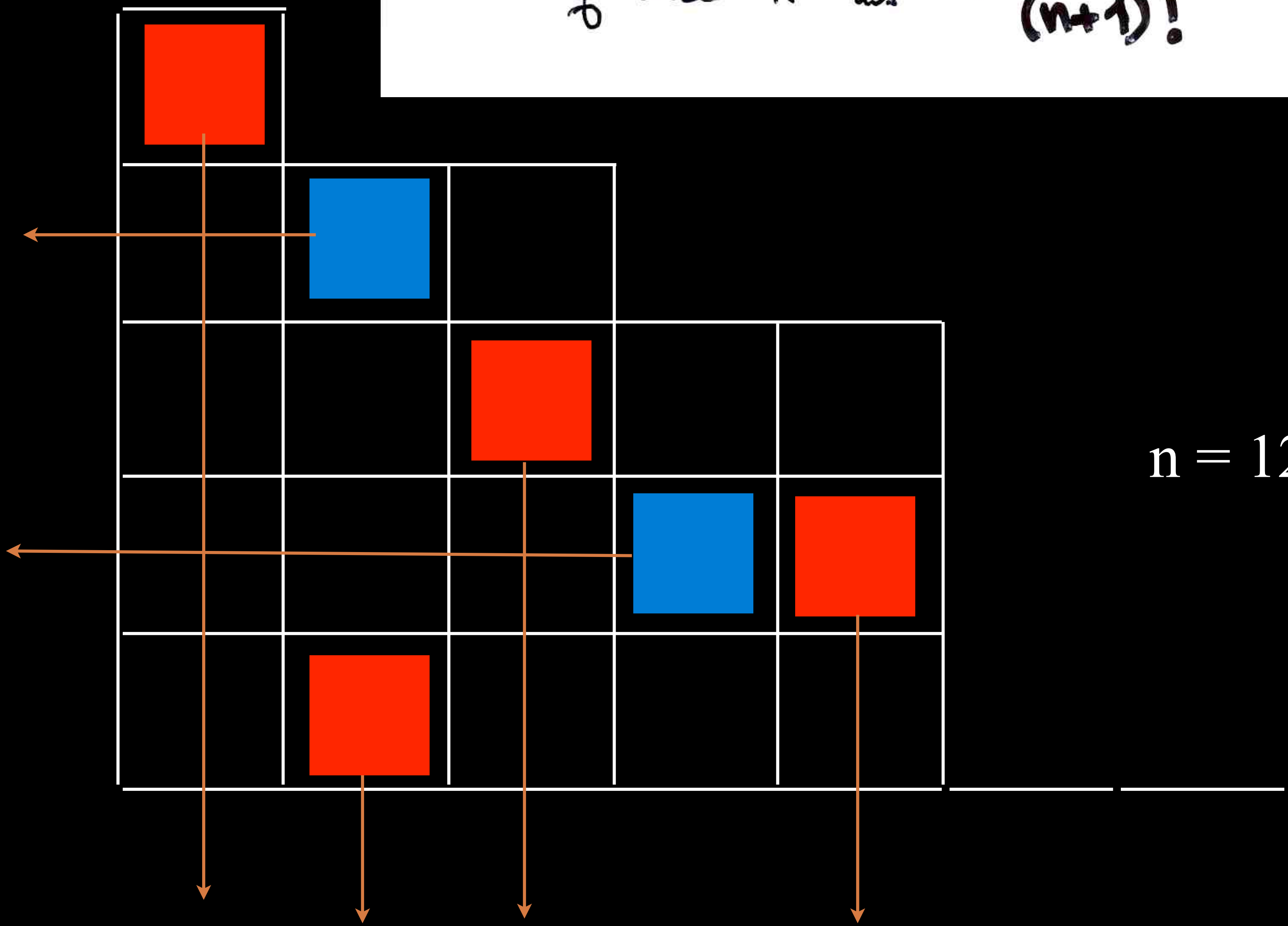
Ferrers diagram
(= Young diagram)



alternative tableau



Prop. The number of alternative tableaux of size n is $(n+1)!$



$n = 12$

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

unicity

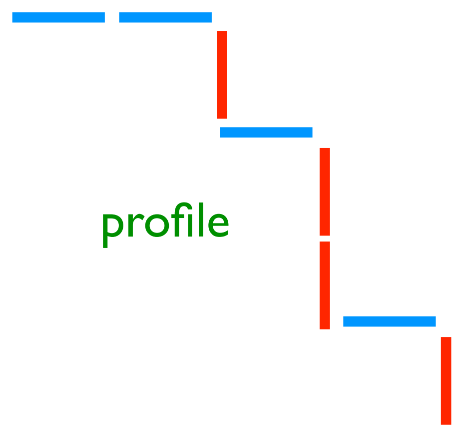
T alternative tableau with profile w

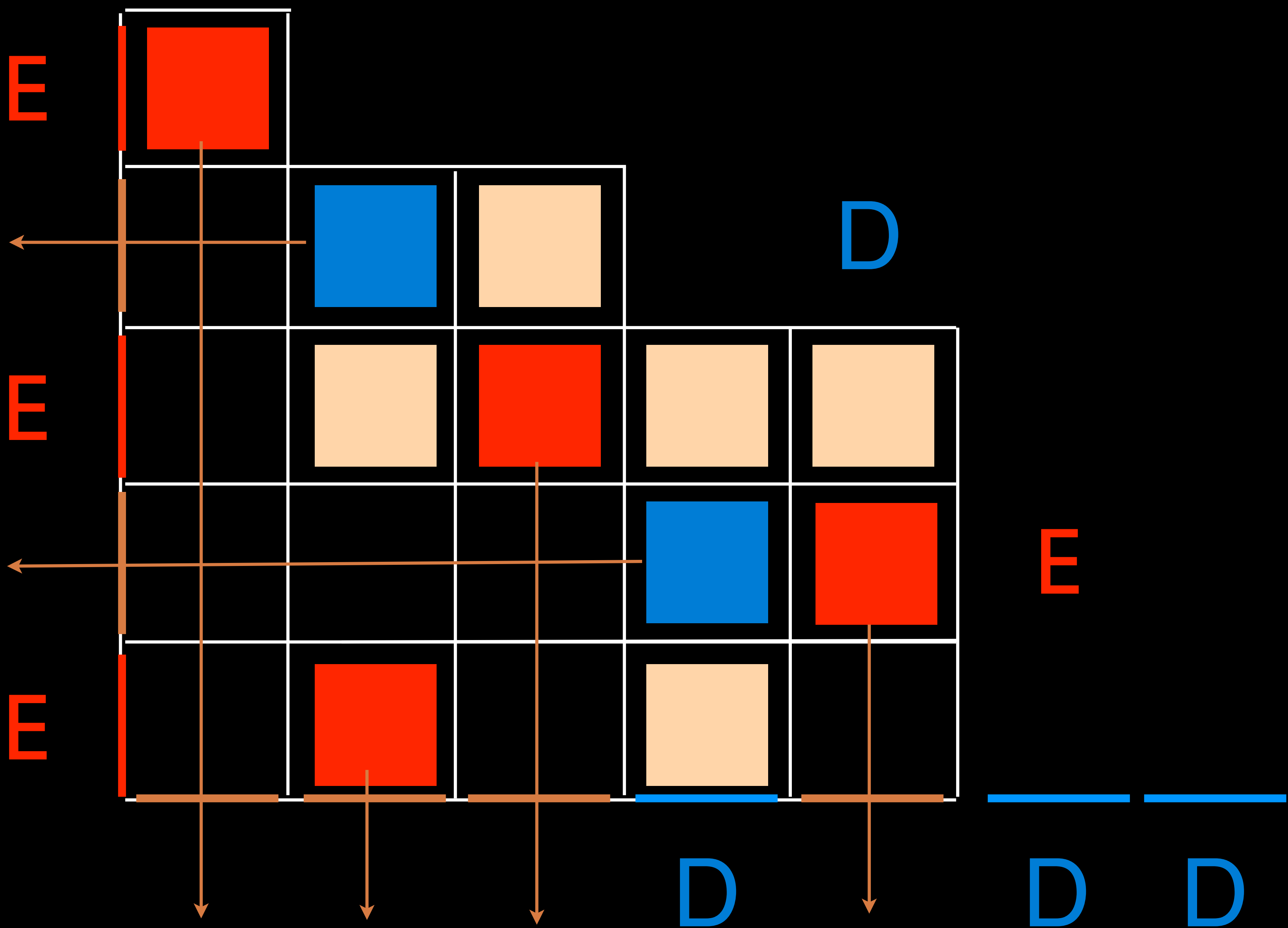
$k(T)$ = nb of $\square$

$i(T)$ = nb of rows without blue cell

$j(T)$ = nb of columns without red cell

w = D D E D E E D E $\longrightarrow$



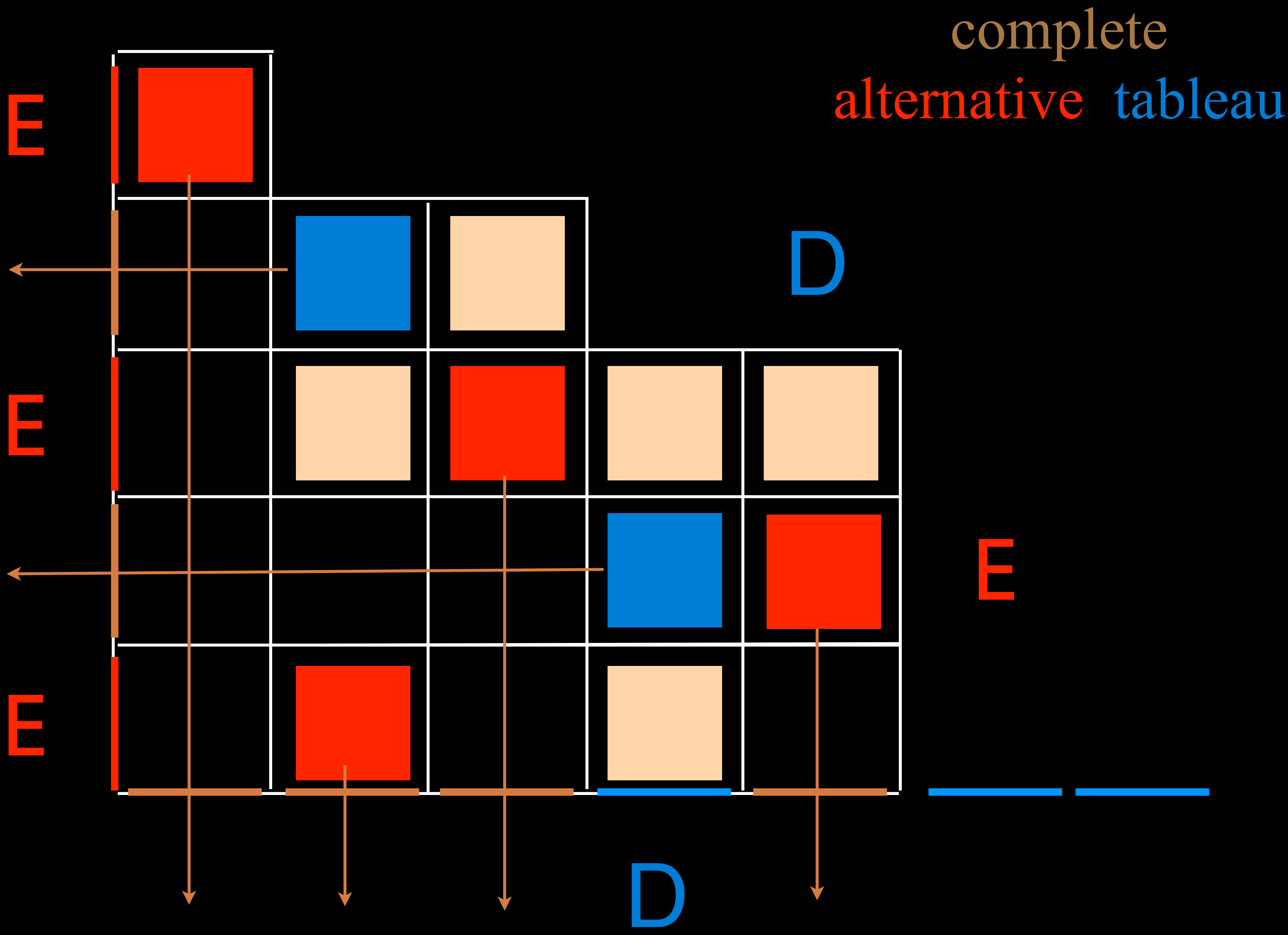

E
D
E
E
E
D
D
D

PASEP -tableaux:
complete tableaux
alternative tableaux

$$DE = qED + E + D$$

PASEP algebra

$$DE = qED + EI_h + I_vD$$
$$DI_v = I_vD$$
$$I_hE = EI_h$$
$$I_hI_v = I_vI_h$$

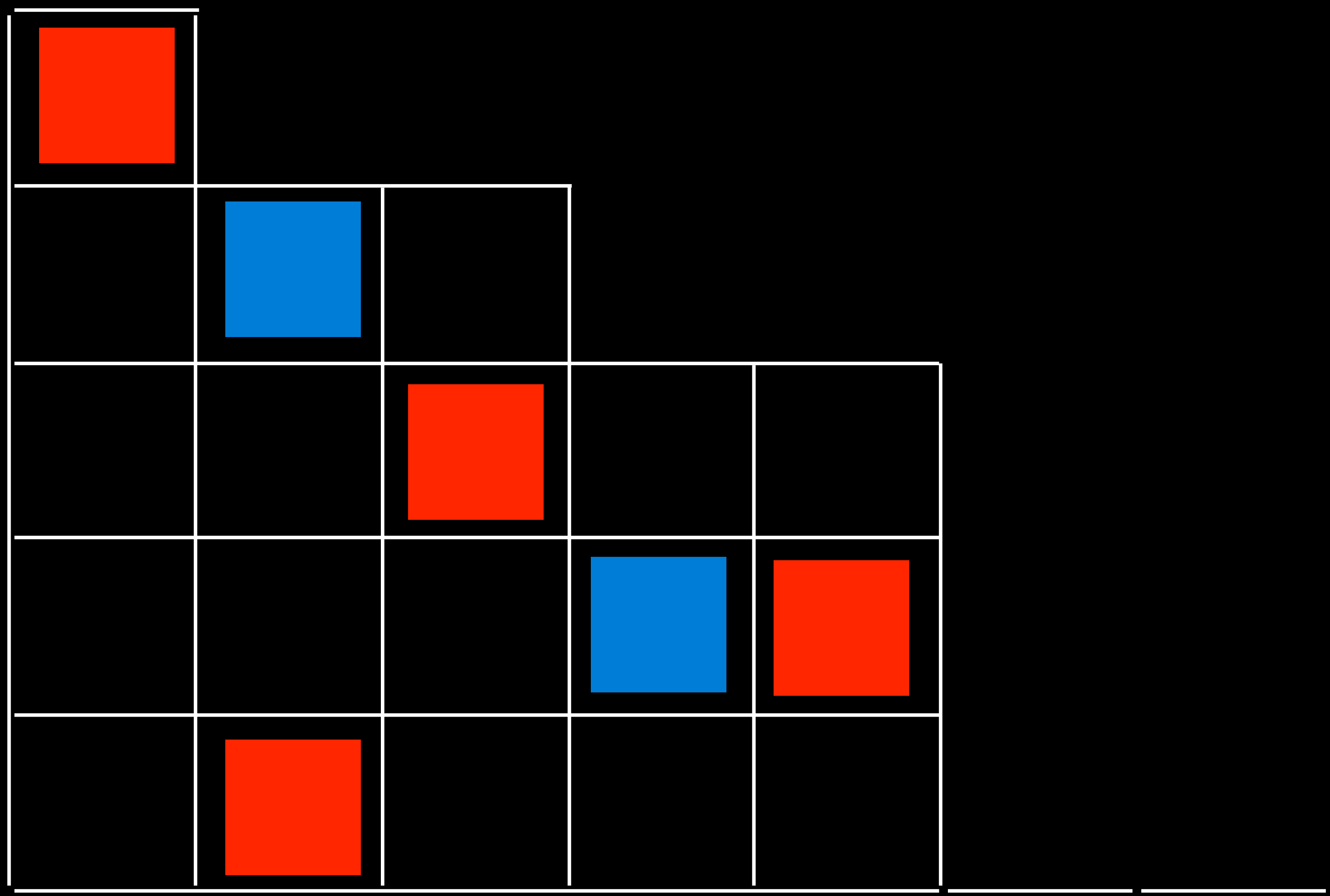

complete
alternative tableau
E
D
E
E
E
D

complete

Q-tableau $\longleftrightarrow$ alternative tableaux

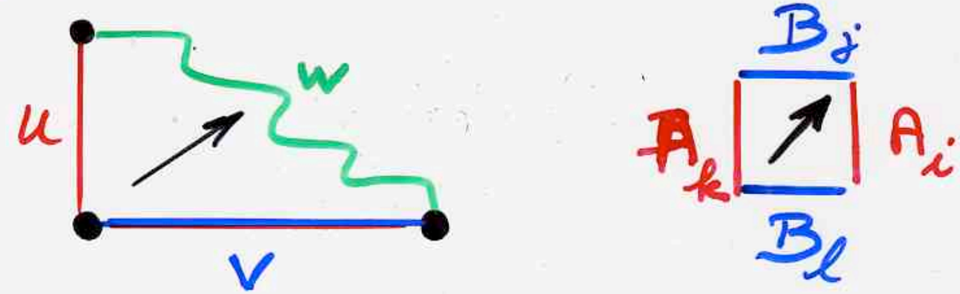
Q PASEP algebra

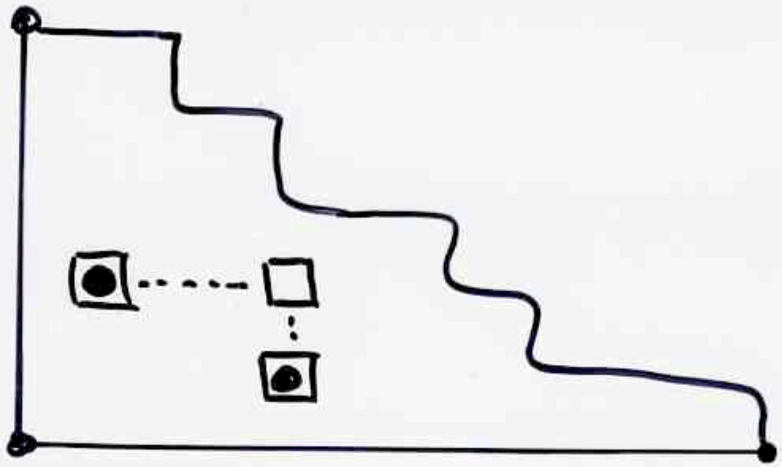
alternative tableau



reverse quadratic algebra
reverse planar automata

reverse planar automata





Postnikov

⅃-diagrams

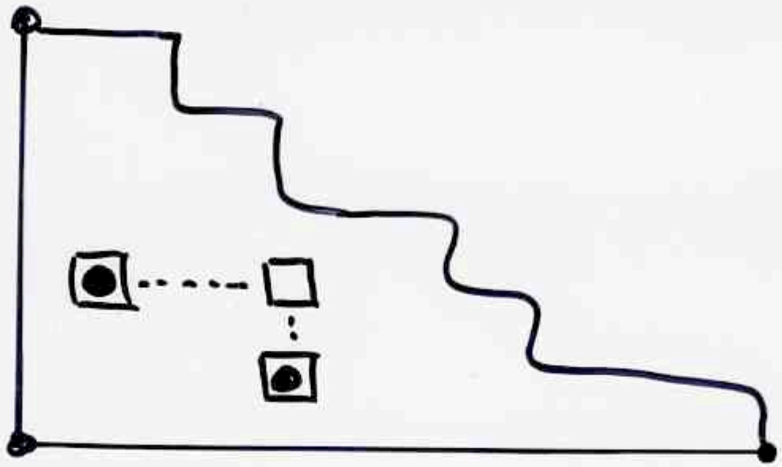
Le-...

$$AB = BA + XY$$

$$YX = XY$$

$$AX = XA + XY$$

$$YB = BY + XY$$



Postnikov

⅃-diagrams

Le-...

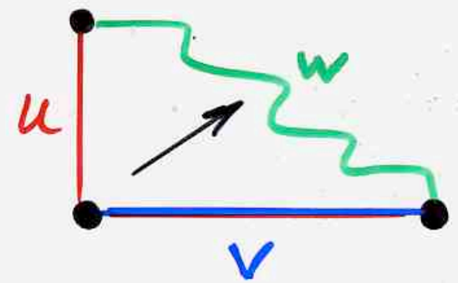
$$AB = \square\, BA + \boxdot\, XY$$

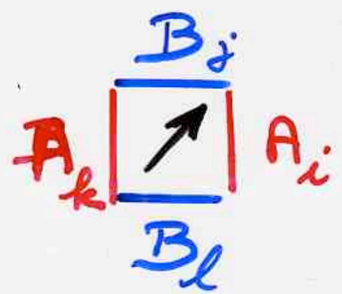
$$YX = \boxdot\, XY$$

$$AX = \square\, XA + \boxdot\, XY$$

$$YB = \square\, BY + \boxdot\, XY$$

reverse planar automata





reverse
(or dual ?)

quadratic algebra

$$Q : \left\{ B_j A_i = \sum_{k,l} c_{ij}^{kl} A_k B_l \right. \quad \forall\, i \in I,\ \forall\, j \in J$$

$$Q^{\dagger} : \left\{ A_k B_l = \sum_{i,j} c_{ij}^{kl} B_j A_i \right. \quad \forall\, k \in I,\ l \in J$$

(possibly 0)

ex

TASEP

$$\mathcal{Q}\quad \begin{cases} DE = q\,ED + EX + YD \\ DY = YD \\ XE = EX \\ XY = YX \end{cases} \qquad \underline{D}_{|E} \qquad \underline{X}_{|Y}$$

$$\mathcal{Q}^{\dagger}\quad \begin{cases} ED = q\,DE \\ EX = DE + XE \\ YD = DE + DY \\ YX = XY \end{cases}$$

ex Weyl-Heisenberg

$\underline{U}$ $|D$ $\quad$ $\underline{X}$ $|Y$

$$Q \begin{cases} UD = q_1 DU + tYX \\ UY = YU \\ XD = DX \\ XY = q_2 YX \end{cases}$$

$$Q^{\dagger} \begin{cases} YX = tUD + q_2 XY \\ DU = q_1 UD \\ YU = UY \\ DX = XD \end{cases}$$

The quadratic algebra $\mathbb{Z}$

4 generators $B_\bullet \; A_\bullet \; B \; A$

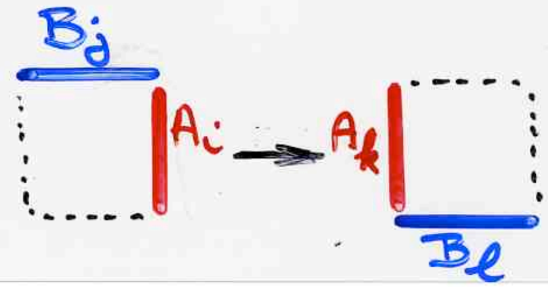
8 parameters $q_{\cdots}\,,\; t_{\cdots}$

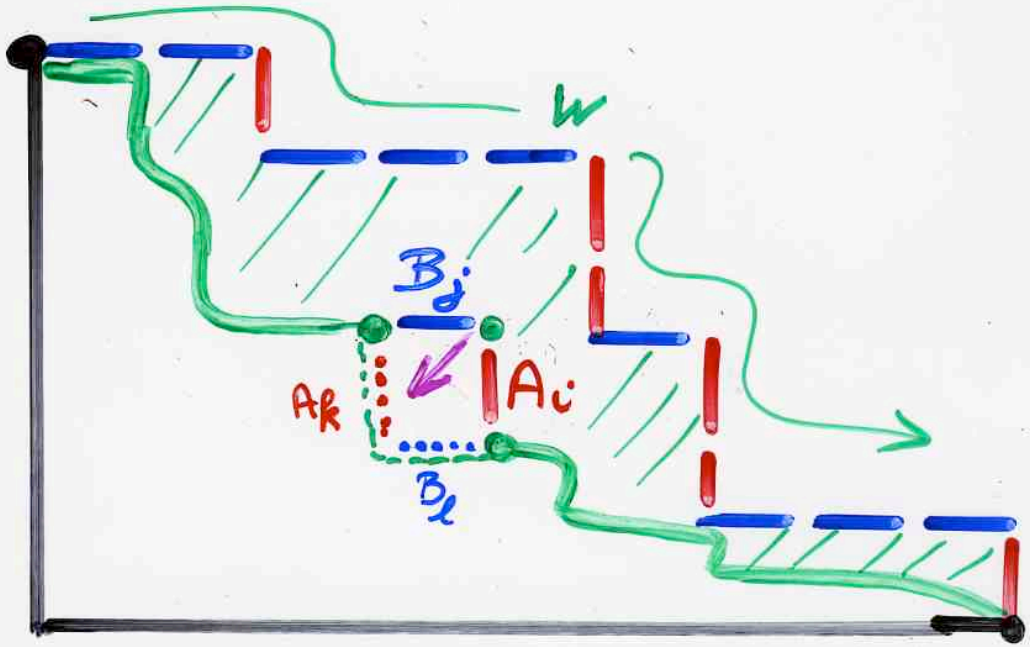
$$
\left\{
\begin{aligned}
B\,A &= q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet B_\bullet \\
B_\bullet A_\bullet &= q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, A\,B \\
B_\bullet A &= q_{\bullet\circ}\, A\,B_\bullet + t_{\bullet\circ}\, A_\bullet B \\
B\,A_\bullet &= q_{\circ\bullet}\, A_\bullet B + t_{\circ\bullet}\, A\,B_\bullet
\end{aligned}
\right.
$$

$$
\mathbb{Z}^t
\left\{
\begin{aligned}
A\,B &= q_{\circ\circ}\, B\,A + t_{\bullet\bullet}\, B_\bullet A_\bullet \\
A_\bullet B_\bullet &= q_{\bullet\bullet}\, B_\bullet A_\bullet + t_{\circ\circ}\, B\,A \\
A_\bullet B &= q_{\circ\bullet}\, B\,A_\bullet + t_{\bullet\circ}\, B_\bullet A \\
A\,B_\bullet &= q_{\bullet\circ}\, B_\bullet A + t_{\circ\bullet}\, B\,A_\bullet
\end{aligned}
\right.
$$

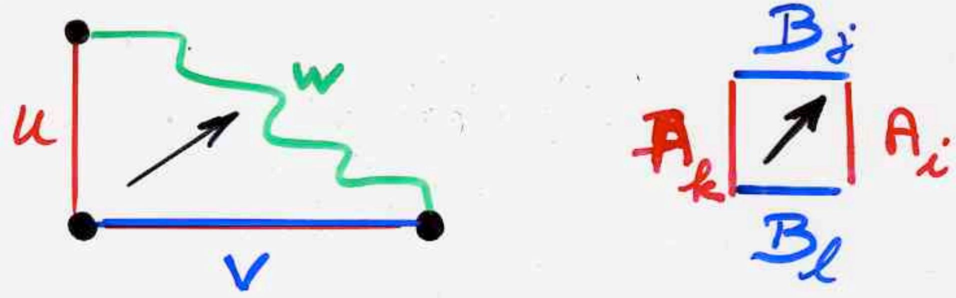
reverse Q-tableaux

"planar" rewriting





reverse planar automata

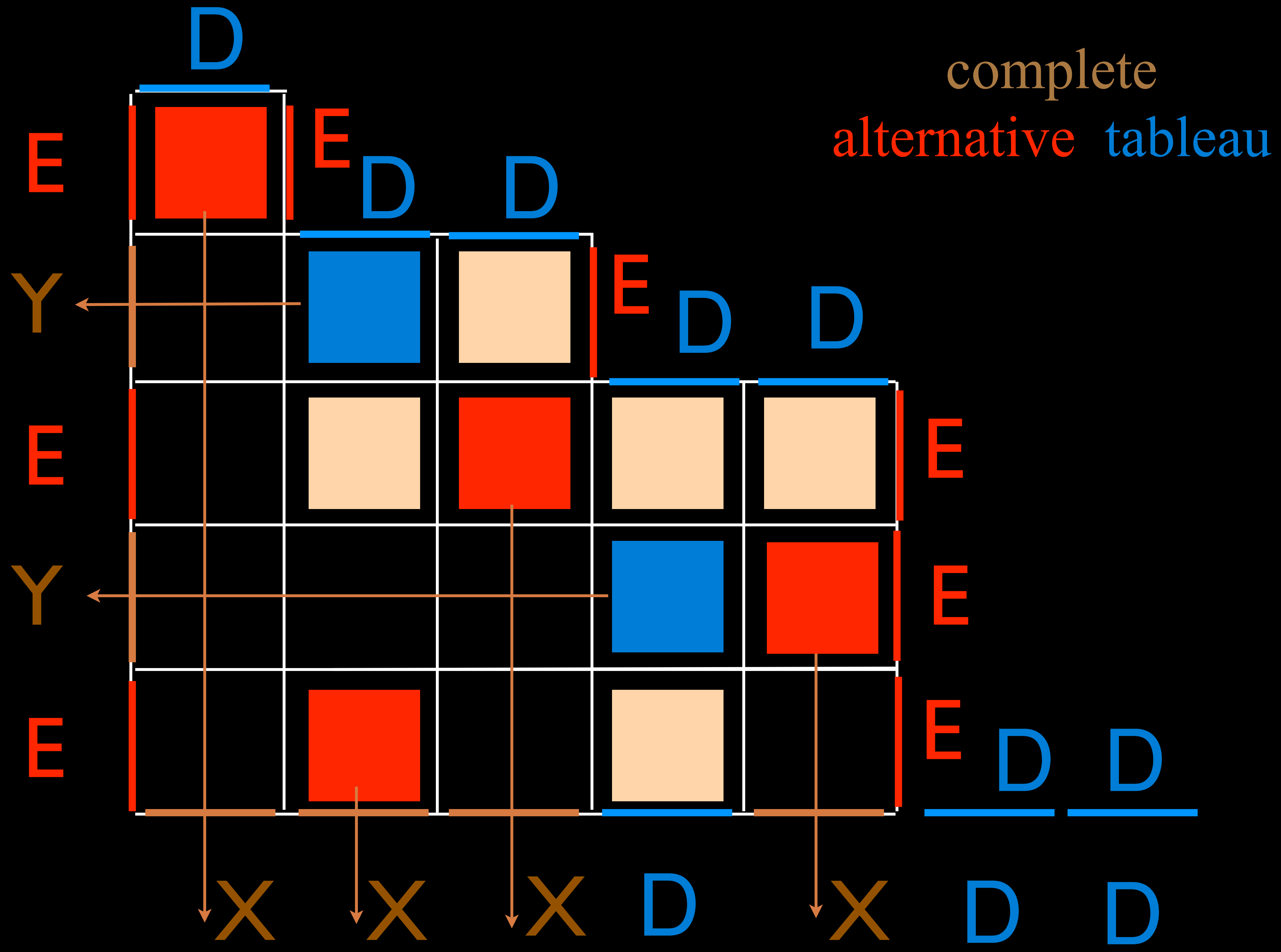


PASEP algebra

$$
Q\begin{cases}
DE = qED + EX + YD \\
XE = EX \\
DY = YD \\
XY = YX
\end{cases}
$$

reverse PASEP algebra
(dual)

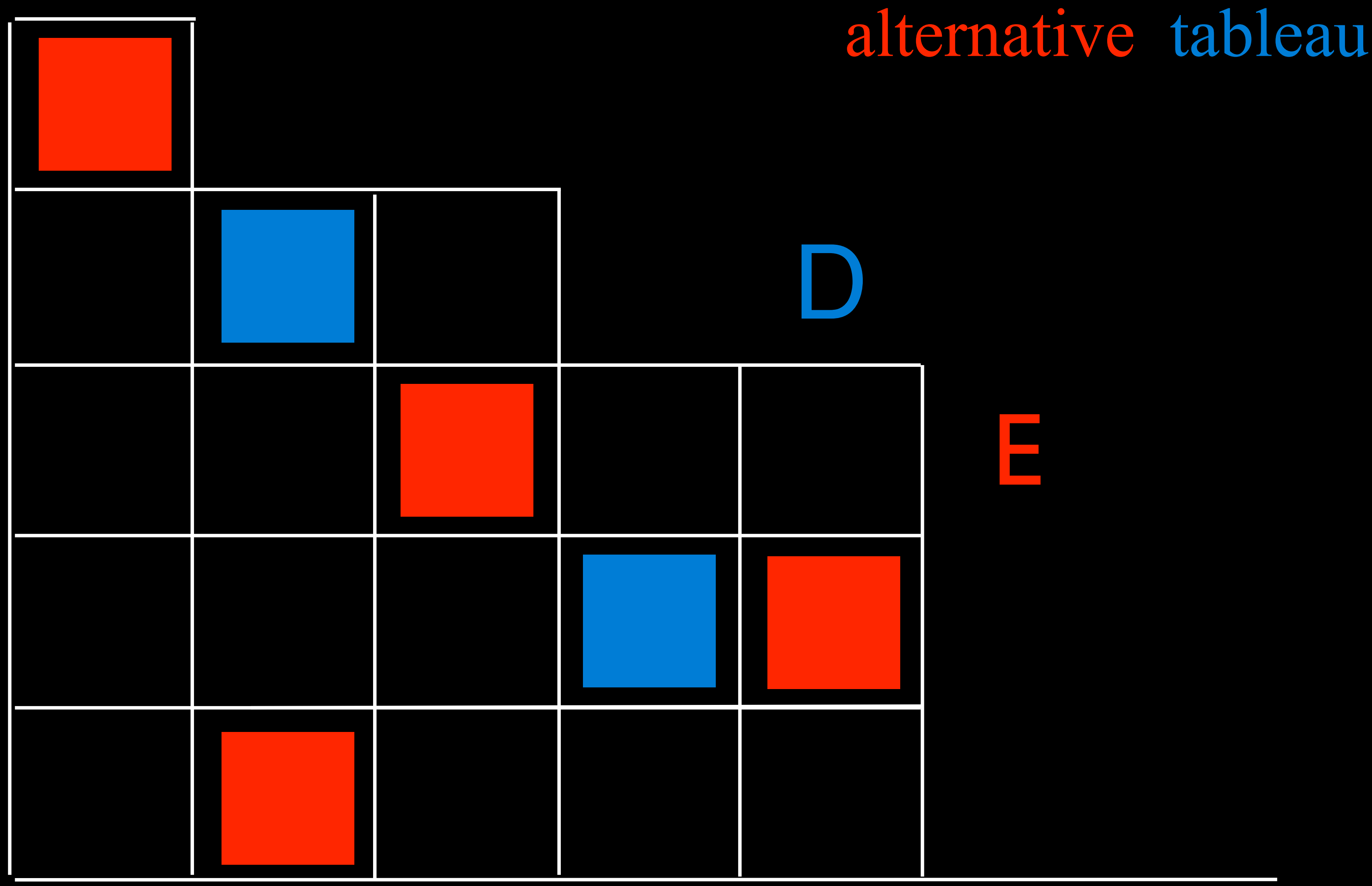
$$
Q^{\dagger}\begin{cases}
ED = qDE \\
EX = XE + DE \\
YD = DY \qquad DE \\
YX = XY
\end{cases}
$$


complete
alternative tableau
D
E
E D D
Y
E D D
E
E
Y
E
E D D
X X X D X D D

$$
Q\begin{cases}
DE = \square\, ED + \blacksquare\, EX + \blacksquare\, YD \\
XE = \square\, EX \\
DY = \square\, YD \\
XY = \square\, YX
\end{cases}
$$

alternative tableaux

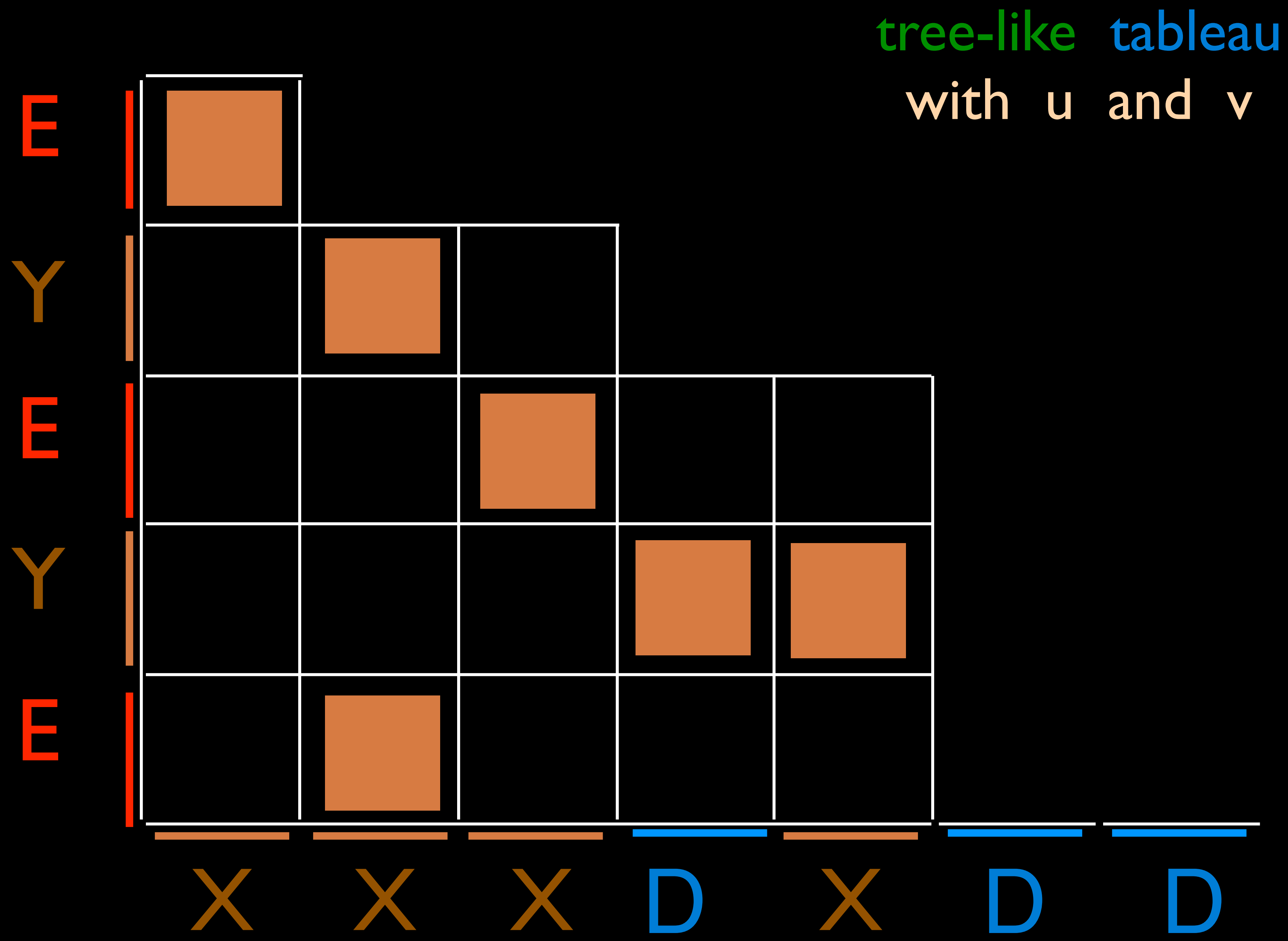
$$
Q^{+}\begin{cases}
ED = \square\, DE \\
EX = \square\, XE + \blacksquare\, DE \\
YD = \square\, DY \quad \blacksquare\, DE \\
YX = \square\, XY
\end{cases}
$$


alternative tableau
D
E

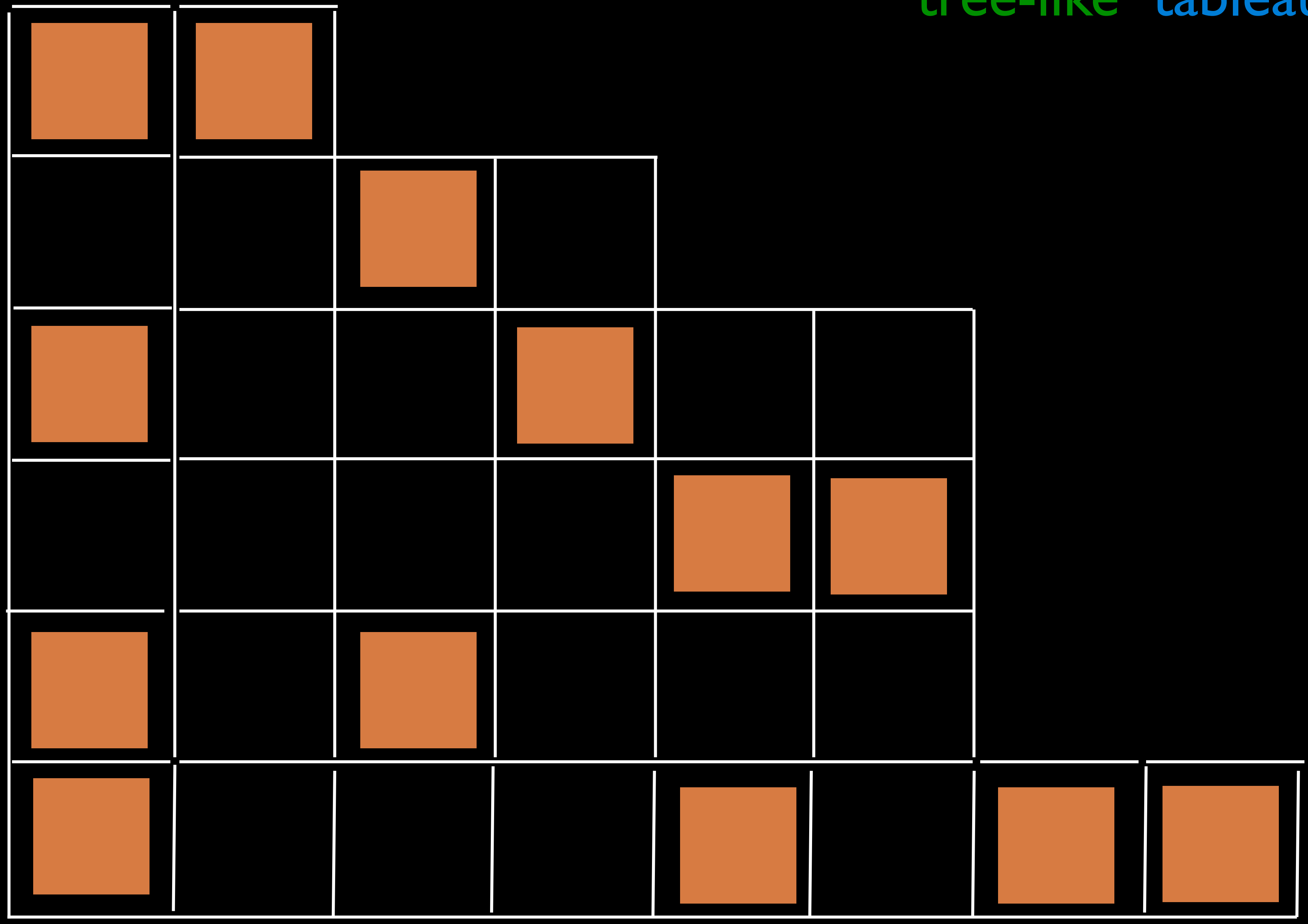
$$Q \begin{cases} DE = \square\, ED + \blacksquare\, EX + \blacksquare\, YD \\ XE = \square\, EX \\ DY = \square\, YD \\ XY = \square\, YX \end{cases}$$

tree-like tableaux

$$Q^{+} \begin{cases} ED = \square\, DE \\ EX = \square\, XE + \blacksquare\, DE \\ YD = \square\, DY \quad \blacksquare\, DE \\ YX = \square\, XY \end{cases}$$


tree-like tableau
with u and v
E
Y
E
Y
E
X X X D X D D

tree-like tableau



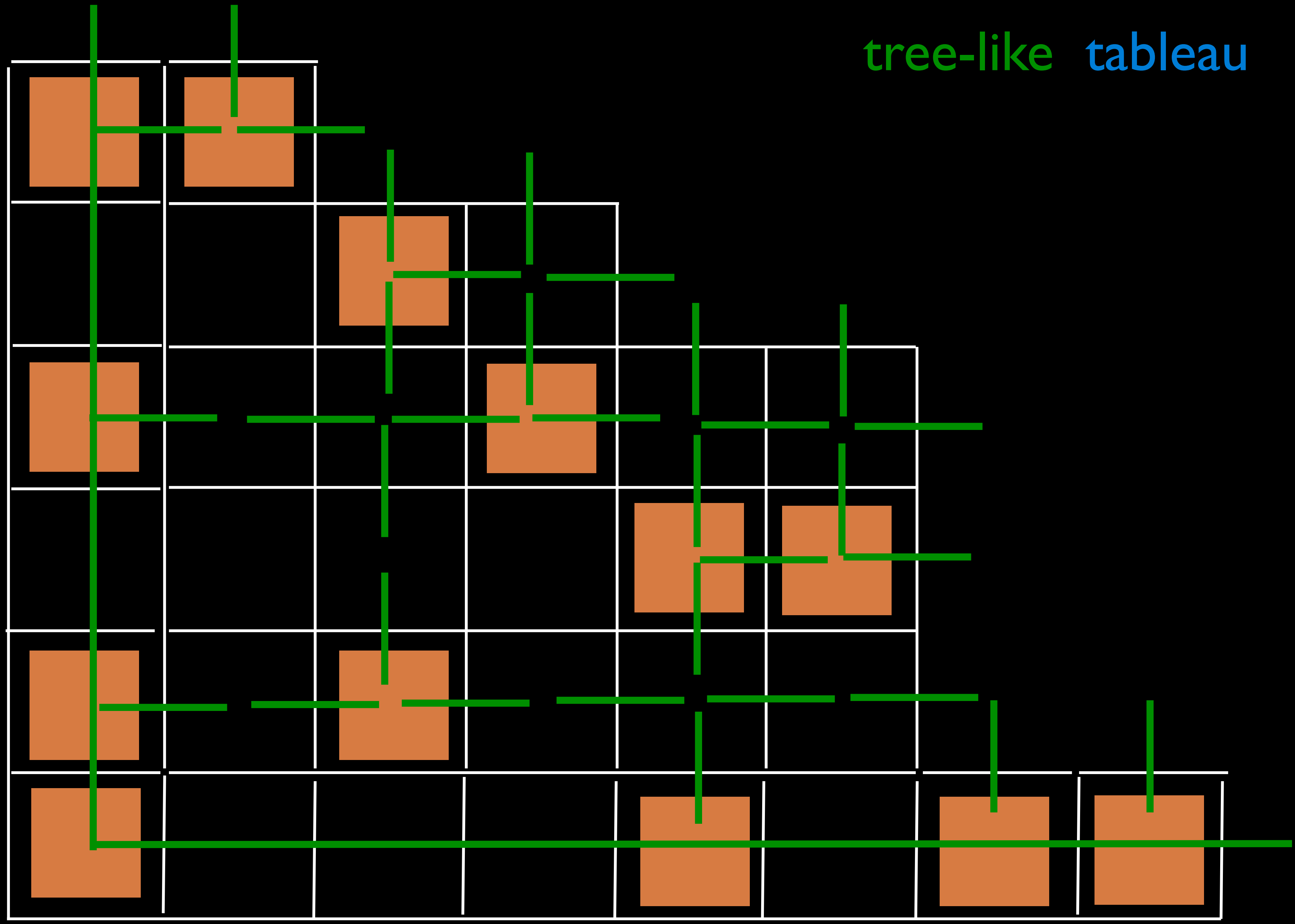
Def- Tree-like tableaux

(Aval, Boussicault, Nadeau) 2011

(i) botom left cell possesses a point $\boxed{\bullet}$
called root point

(ii) for every non-root pointed cell c,
$\exists$ either a pointed cell below c in same column
or a pointed cell to its left in same row
but no both

(iii) every column and every row possesses
at least one pointed cell

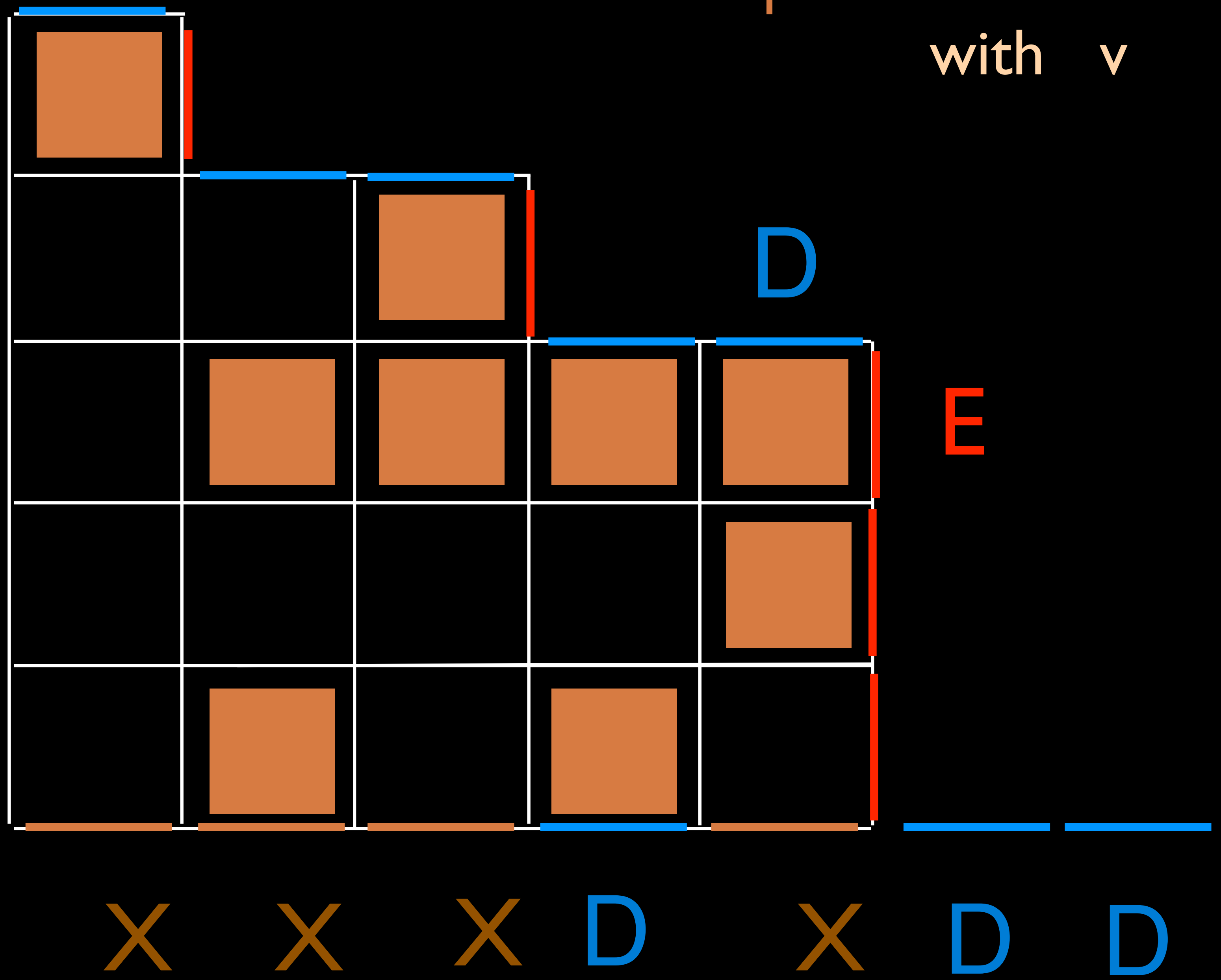

tree-like tableau

$$
Q \begin{cases}
DE = \blacksquare\, ED + \blacksquare\, EX + \square\, YD \\
XE = \square\, EX \\
DY = \square\, YD \\
XY = \square\, YX
\end{cases}
$$

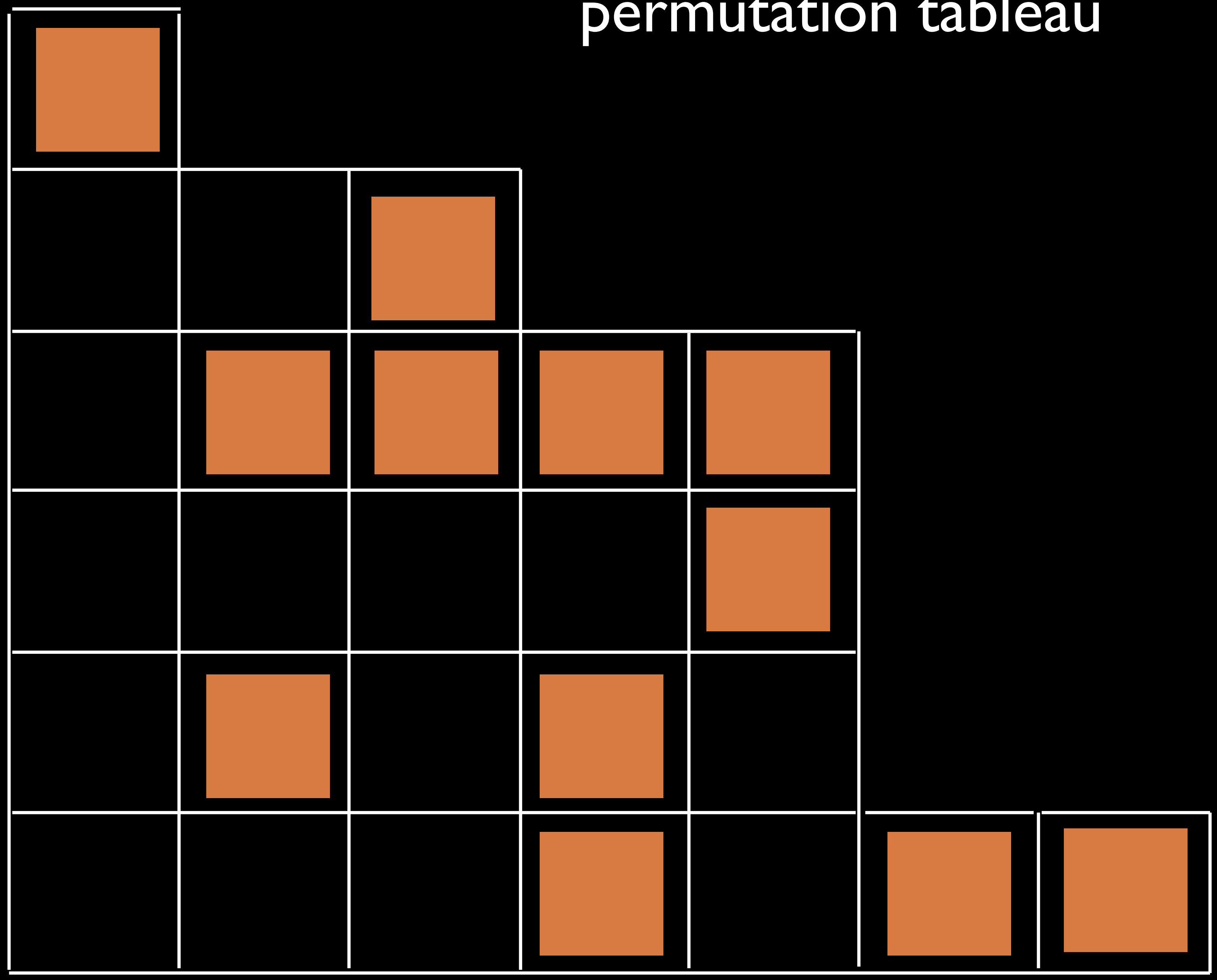
permutation tableaux

$$
Q^{\dagger} \begin{cases}
ED = \blacksquare\, DE \\
EX = \square\, XE + \blacksquare\, DE \\
YD = \square\, DY \quad \square\, DE \\
YX = \square\, XY
\end{cases}
$$

permutation tableau
with v



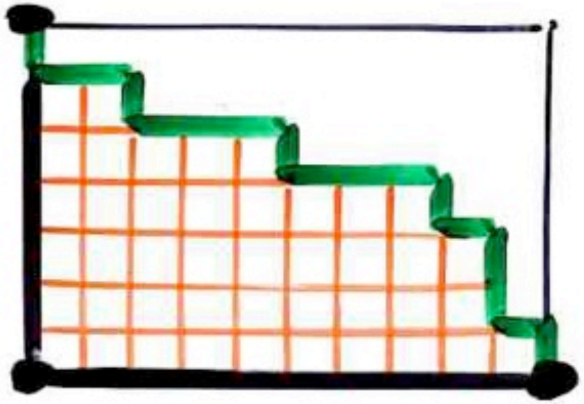
permutation tableau



permutation tableaux

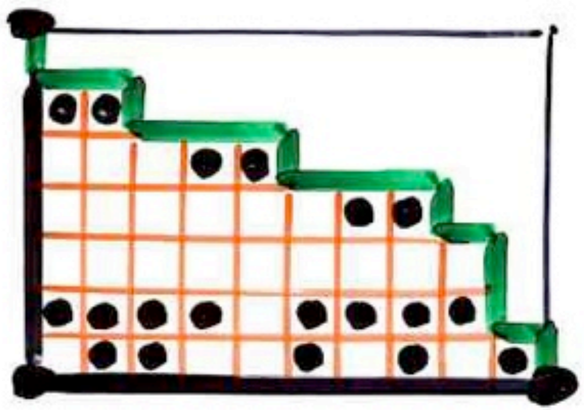
Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

(i)

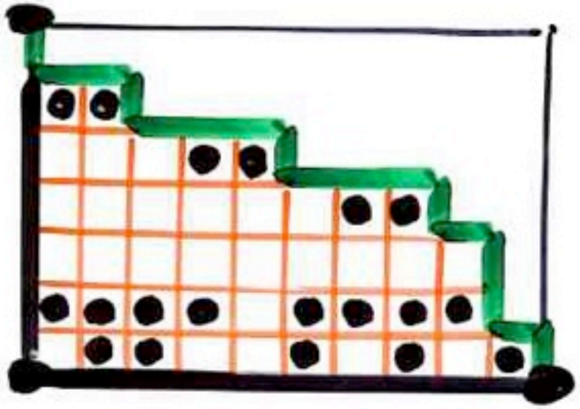
(ii)

□ = 0 ⊡ = 1

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$

rectangle



filling of the cells with 0 and 1

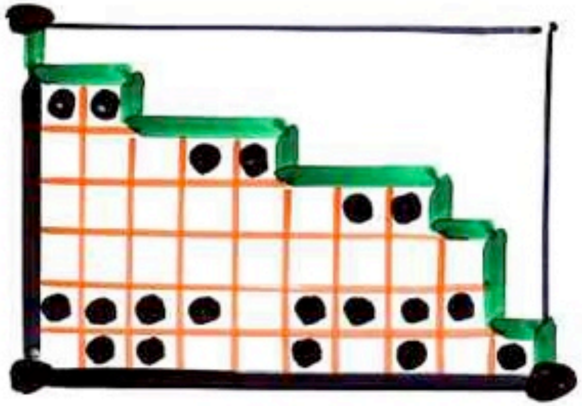
(i) in each column : at least one 1

□ = 0 ⊡ = 1

(ii)

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



filling of the cells with 0 and 1

(i) in each column : at least one 1

(ii) $\begin{matrix} 1 & \cdots & 0 \\ & & \vdots \\ & & 1 \end{matrix}$ forbidden

□ = 0 ■ = 1

permutation tableau

A. Postnikov (2001, ...)
totally nonnegative part of the Grassmannian
E. Steingrímsson, L. Williams (2005)

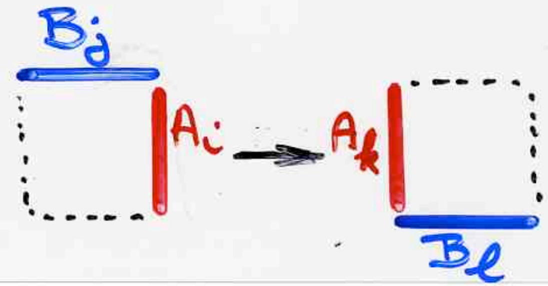
combinatorial interpretations of the
stationary probabilities for the PASEP

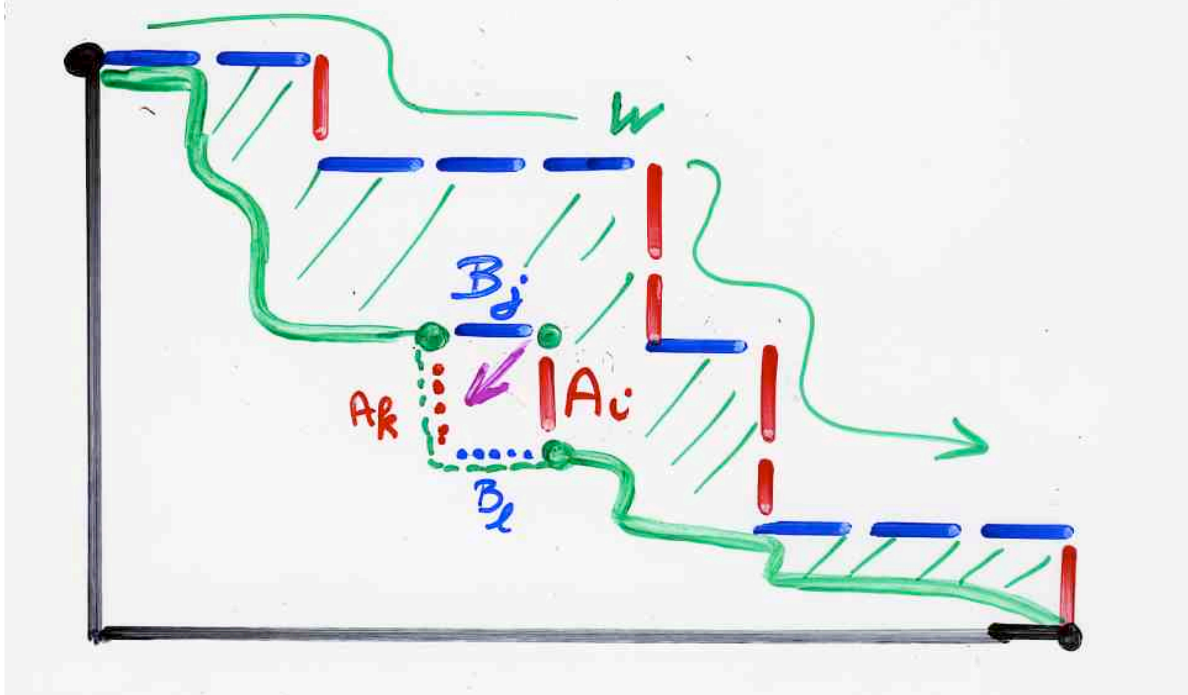
Corteel, Williams (2006) PASEP
Partially Asymmetric Exclusion Process

M. Josuat-Vergès (2007)

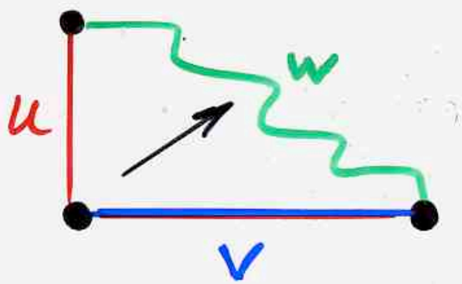
rewriting planar automata

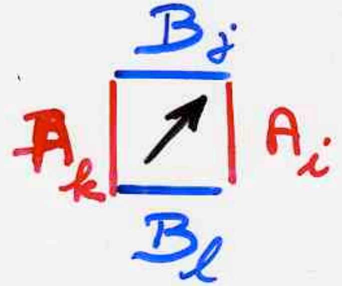
"planar" rewriting





reverse planar automata





«Figures»
accepted by planar automata ?

Bijections between pattern-avoiding fillings of Young diagrams

Josuat-Vergès (2008)

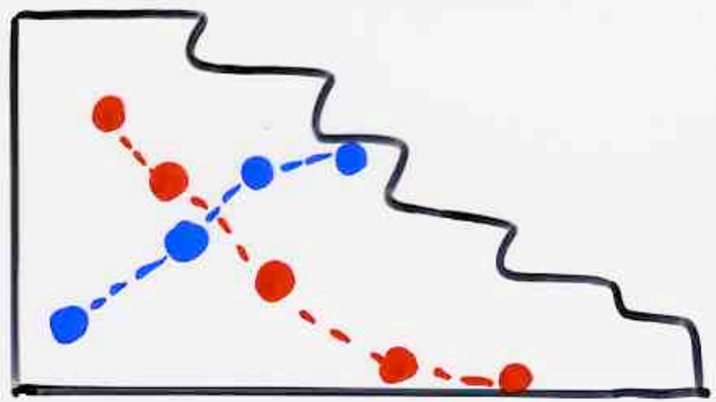
⅂-diagrams

X-diagrams

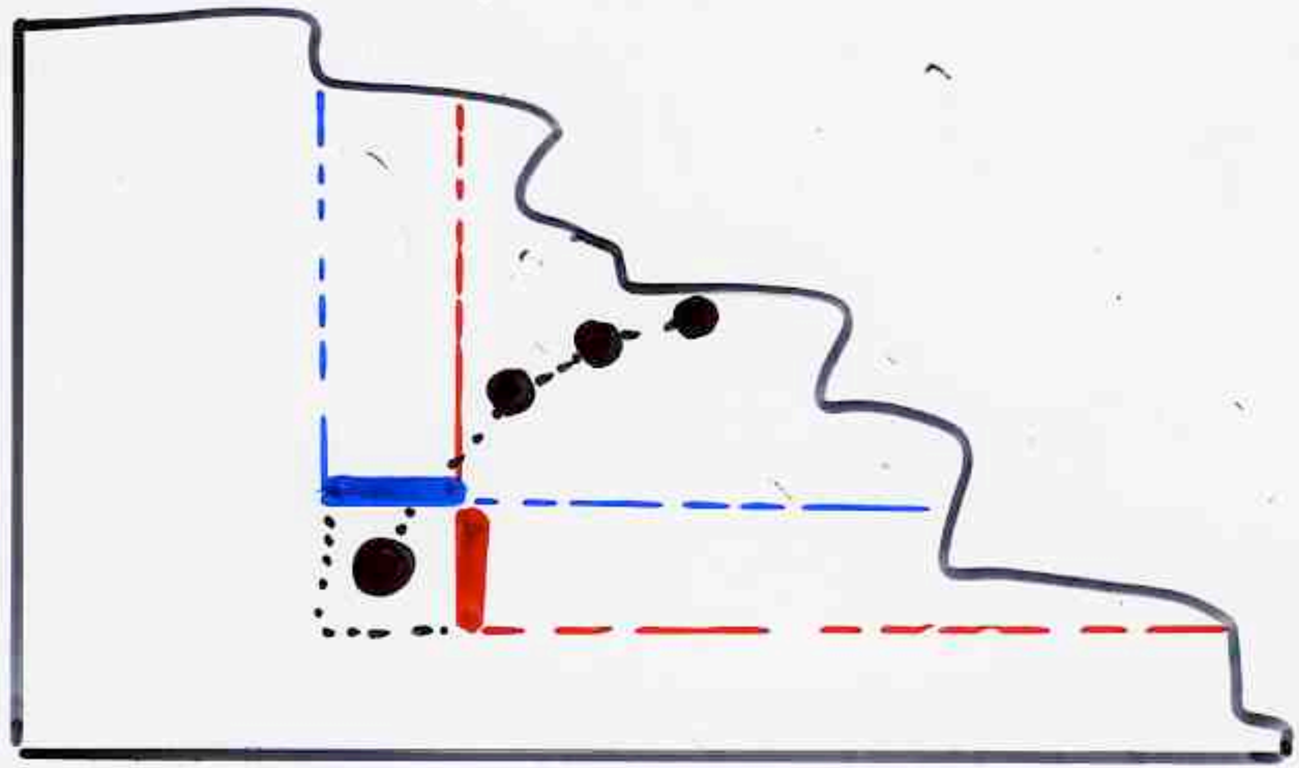
?

increasing
decreasing
chains in fillings of Ferrers shapes

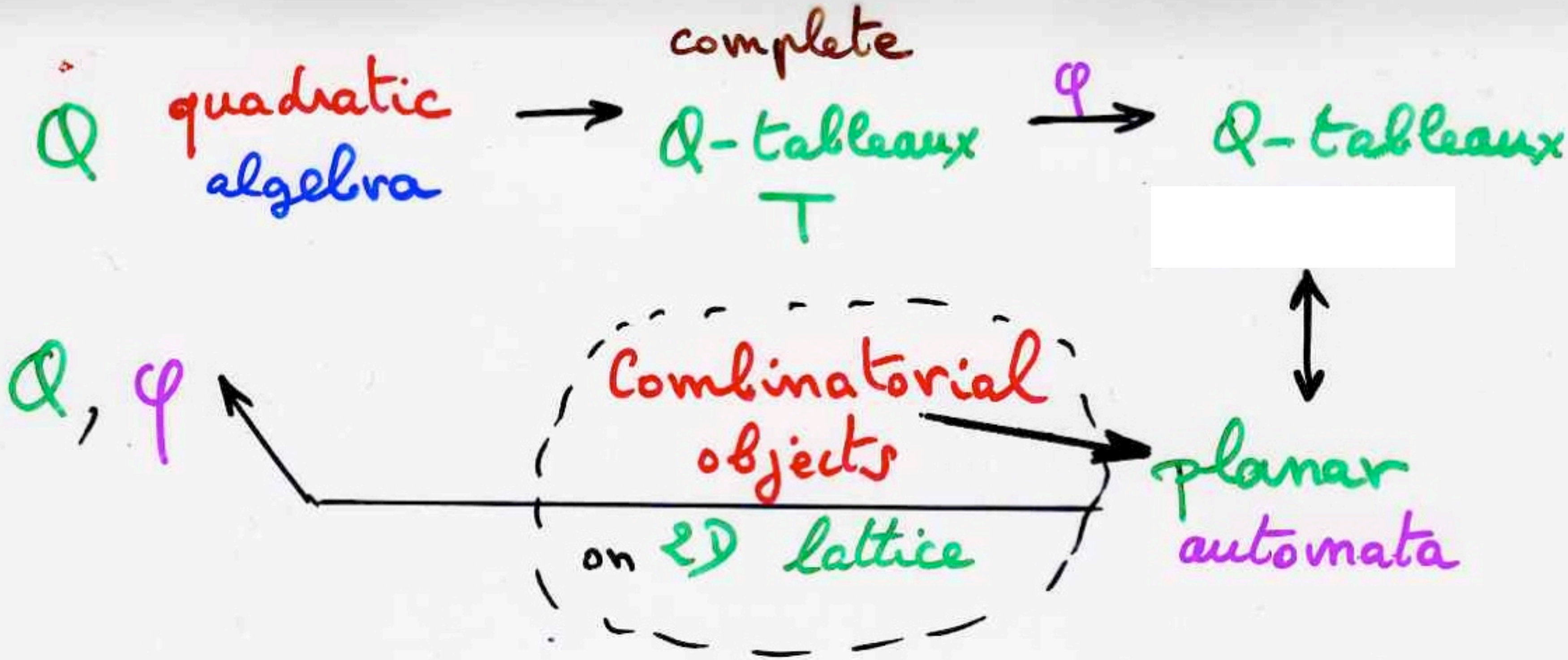
(Jonsson, 2005) (Krattenthaler, 2006)
(Backelin, West, Xin, 2005) (Bousquet-Mélou, Steingrímsson, 2005) ....



increasing
decreasing
subsequences
(chains)



conclusion
problems
more ....



Q quadratic algebra
complete
Q-tableaux
T
φ
Q-tableaux
Q, φ
Combinatorial objects
on 2D lattice
planar automata

- formula for $c(u, v; w)$?

  Q determinant ?

- or at least efficient procedure for computing $c(u, v; w)$?

- generating function ?

"The cellular Ansatz"

combinatorial
objects
on a 2d lattice

representation
by operators

Physics

data structures
"histories"

bijections

orthogonal
polynomials

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

towers placements

permutations

RSK

$\longleftrightarrow$ pairs of Tableaux Young

tableaux alternatifs $\longleftrightarrow$ permutations

Laguerre histories

$DE = qED + E + D$

PASEP

Q-tableaux

ex: ASM,
(alternating sign matrices)
FPL(fully packed loops)
tilings, 8-vertex

quadratic algebra Q

?

commutations
rewriting rules
planarisation

planar
automata

Koszul algebras
duality
J.L.Loday

March 1, IMSc

"Combinatorial operators and quadratic algebras"
(RSK and PASEP)

March 2, IIT Madras

"The combinatorics of some exclusion models
in physics" (PASEP)

Thank you !

planar automata ex1: permutatations

ex2: Genocchi ex3: ASM Fig accepted

UD=DU+I Q algebras for ASM

complete Q-tableaux Q-tableaux ex Q-tableaux

the equivalence 8-vertex algebra

ASM rhombus tilings Aztec tilings

geometric interp XYZ tableaux non-intersec paths

PASEP algebra AT more PASEP tableaux

reverse automata reverse Q-tableaux perm tableaux

rewriting auto Fig accepted conclusion,pb, more