Algèbre de Loday-Ronco
et
tableaux alternatifs de Catalan

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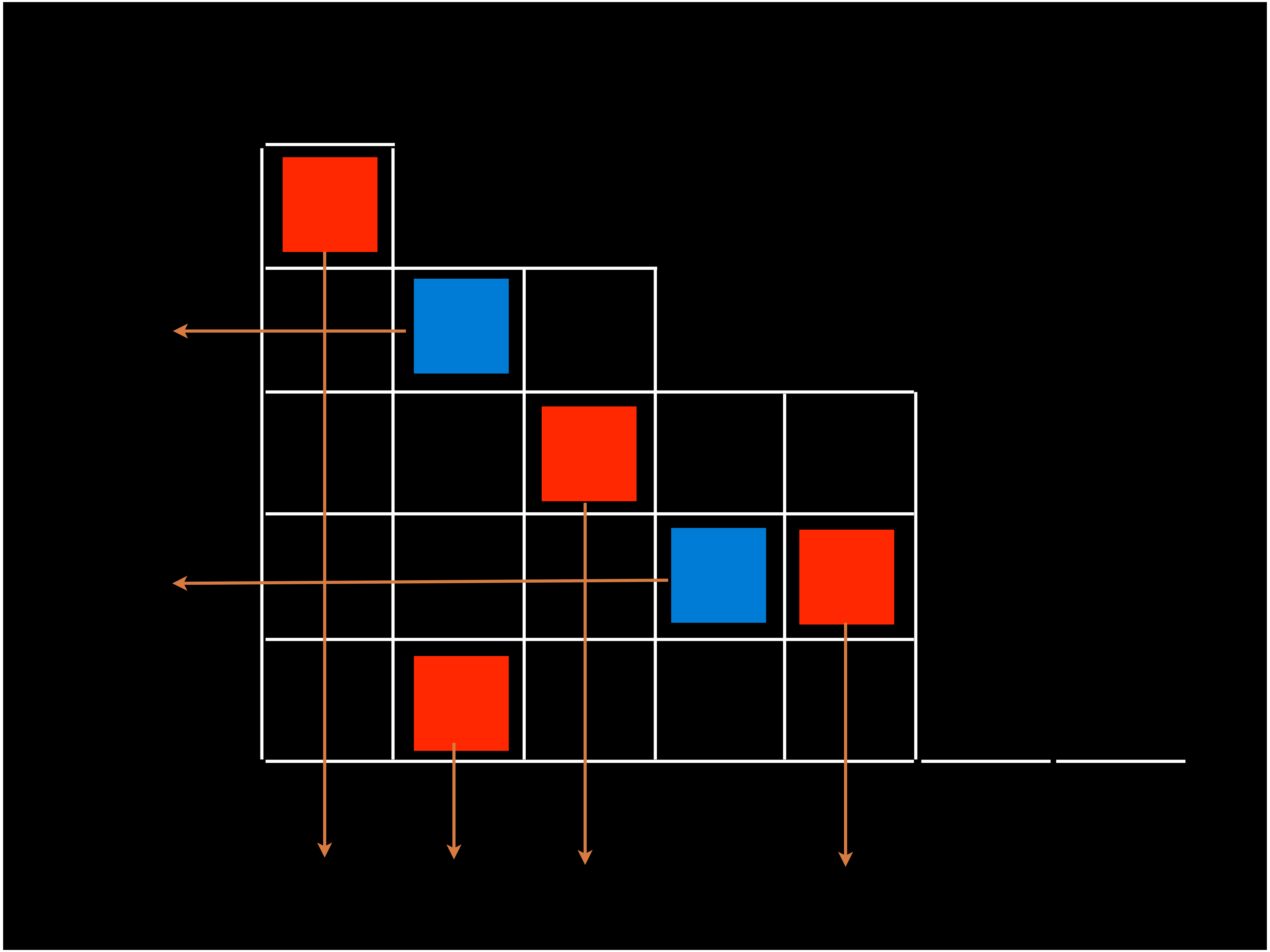
et xgv

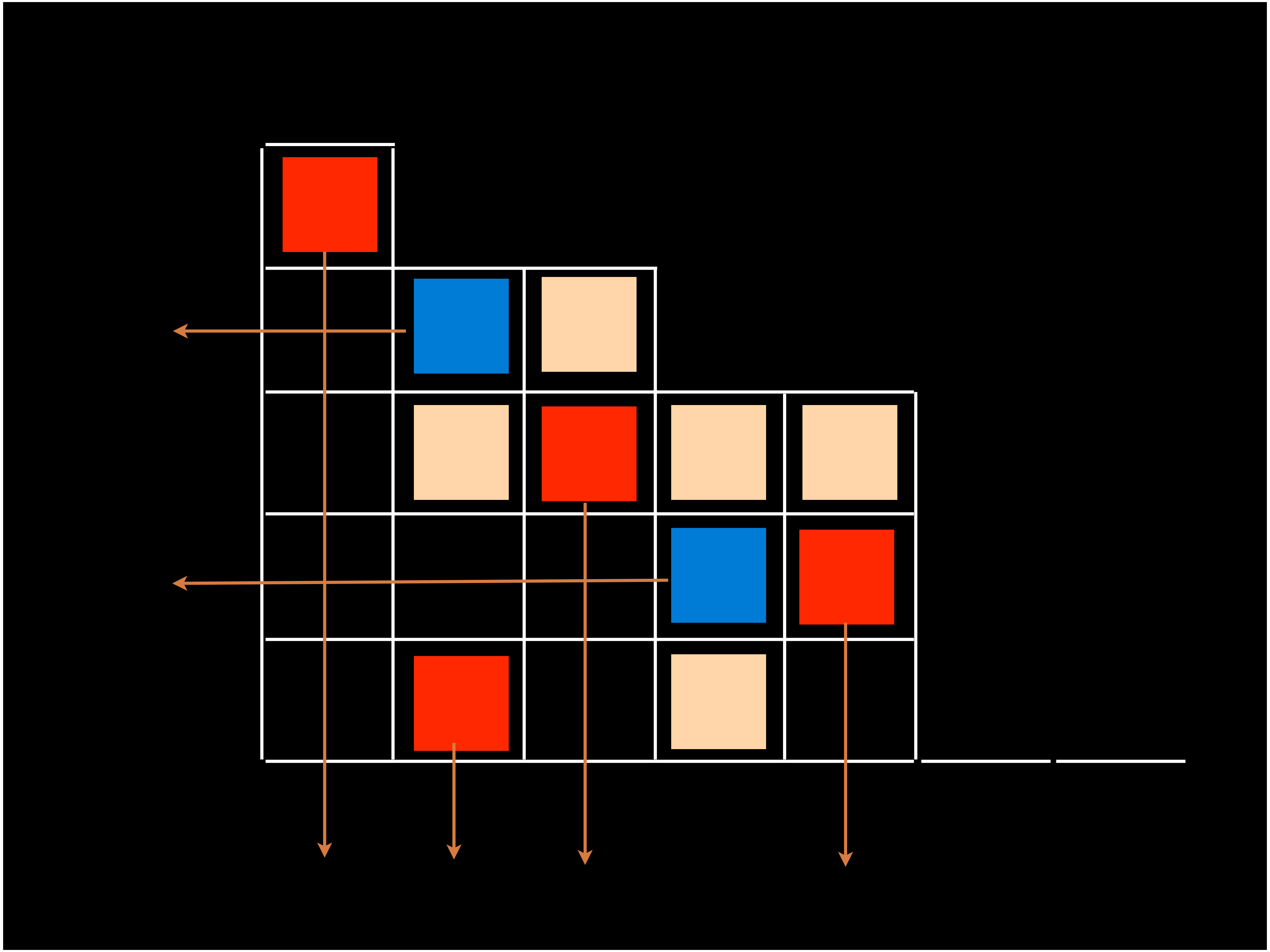
GT, LaBRI

23 Janvier 2009

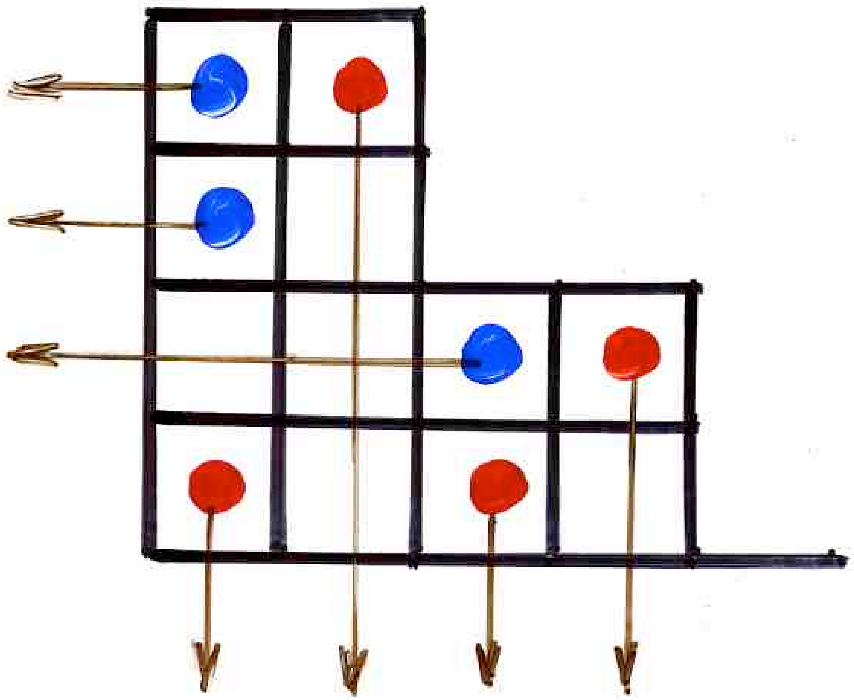
§1 algèbres de Hopf: Malvenuto-Reutenauer, Loday-Ronco, algèbre des descentes

§2 tableaux alternatifs de Catalan



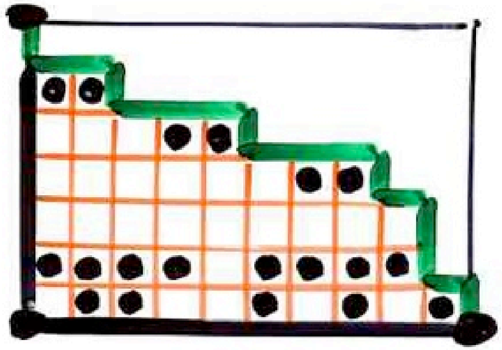


Def Catalan alternative tableau T
alt. tab. without cells $\boxed{\times}$
i.e.: every empty cell is below a red cell or on the left of a blue cell



Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

(i) in each column : at least one 1

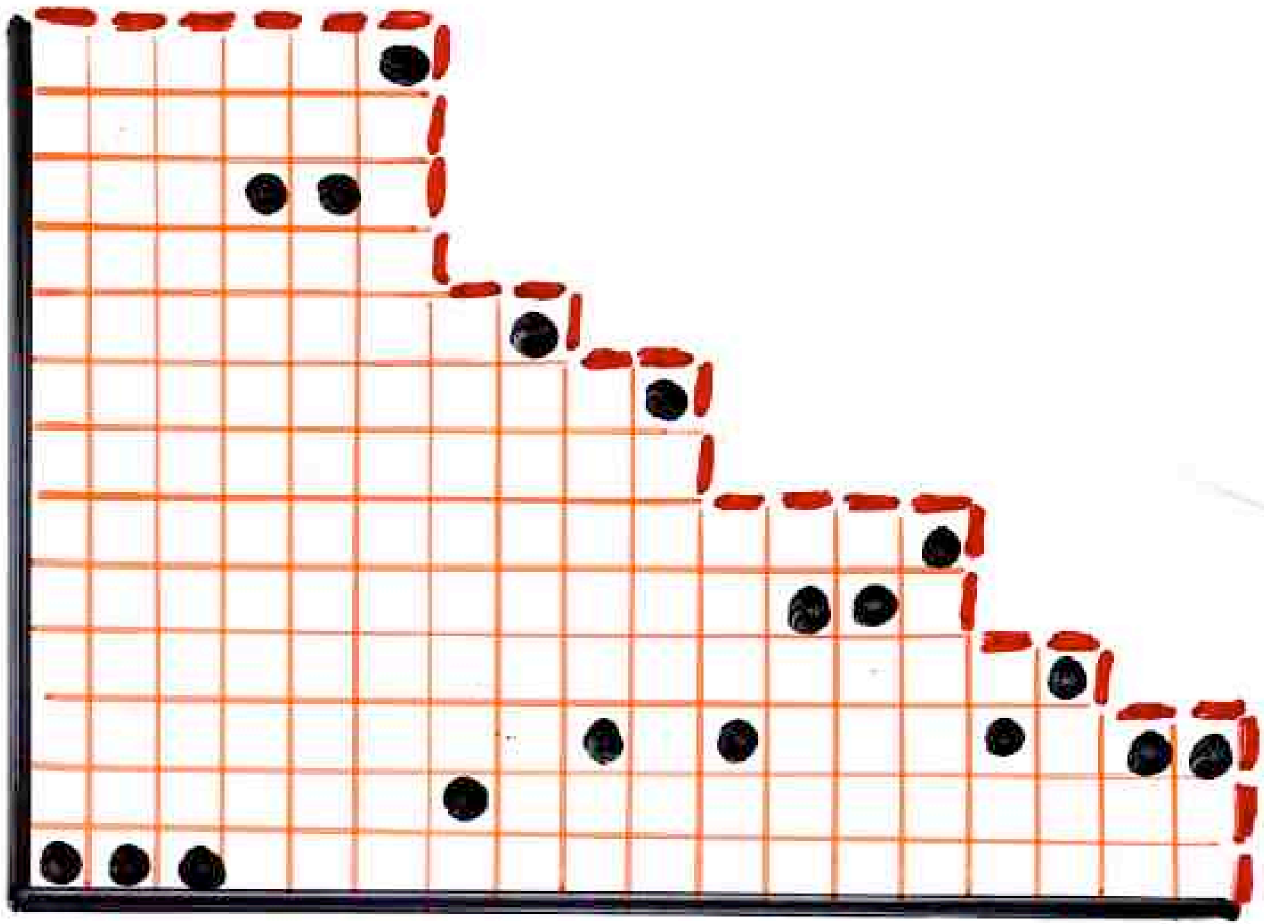
$\square = 0$ $\boxed{\bullet} = 1$

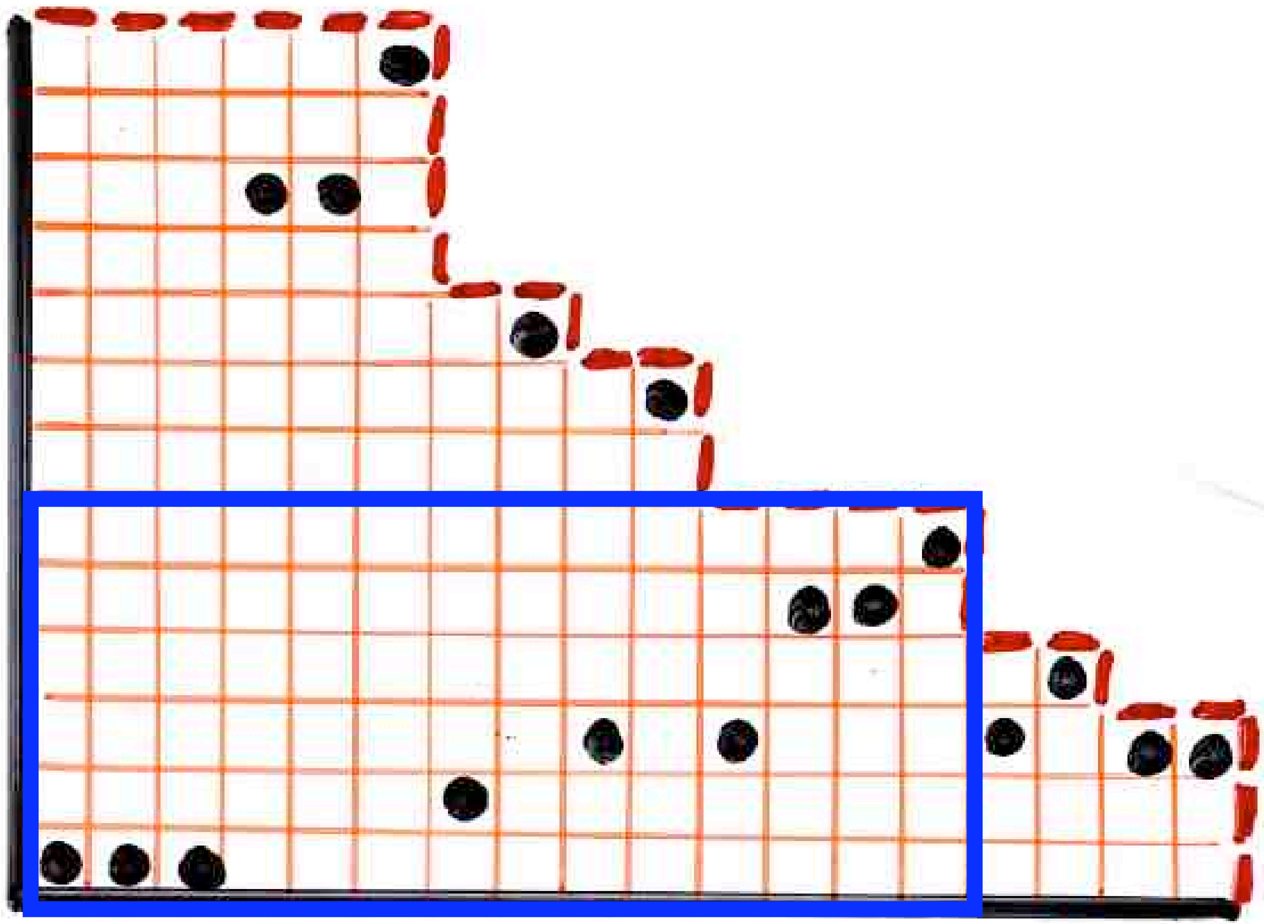
(ii) 

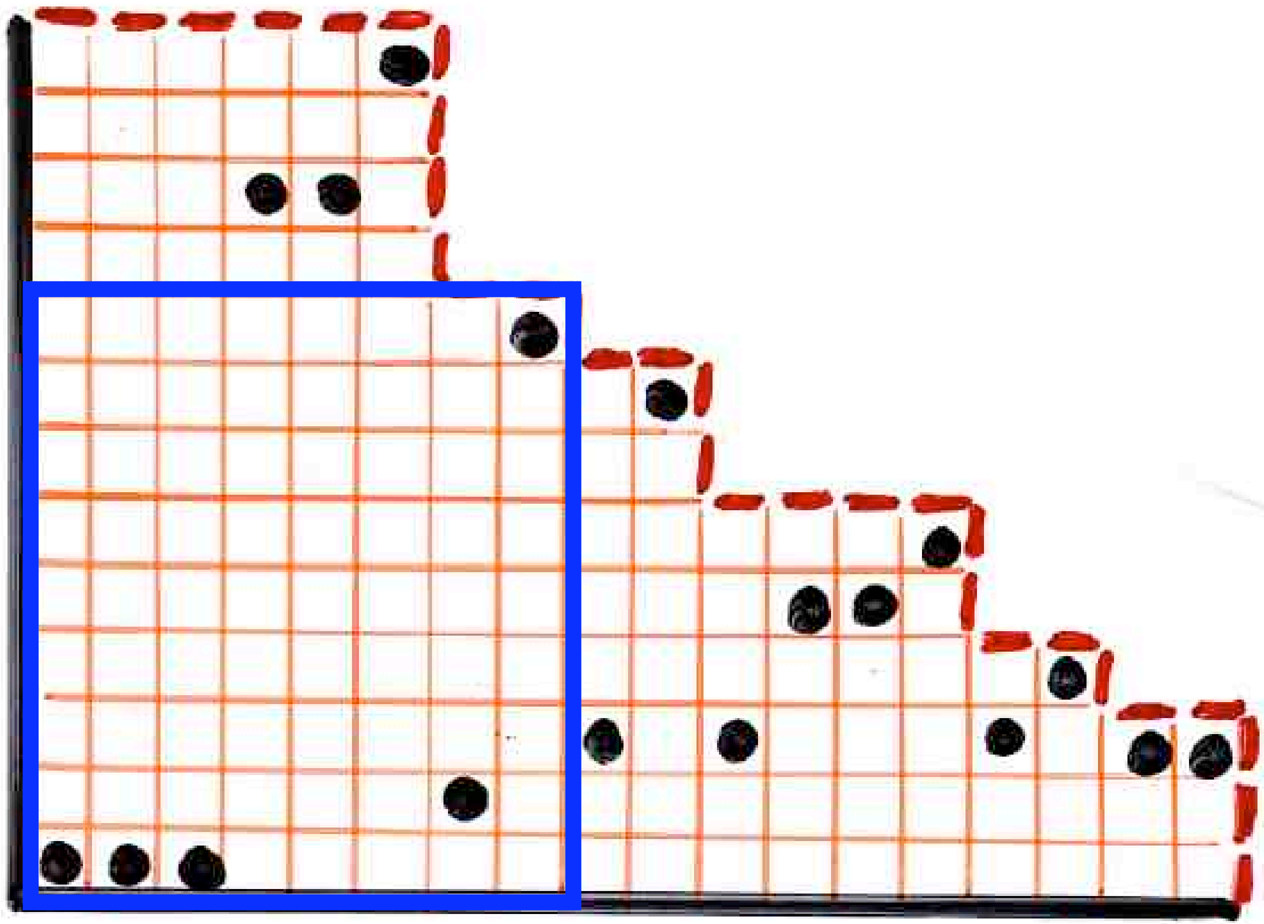
forbidden

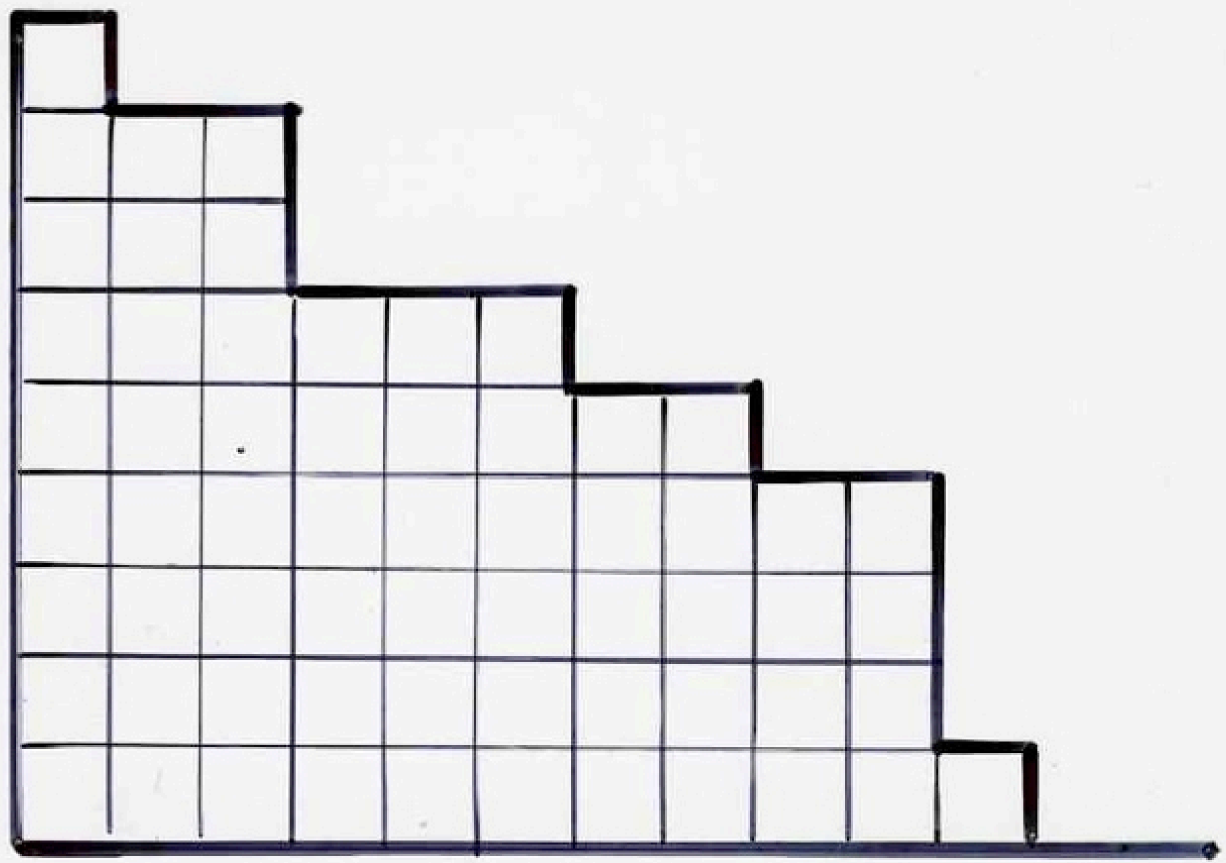
Catalan permutation tableau

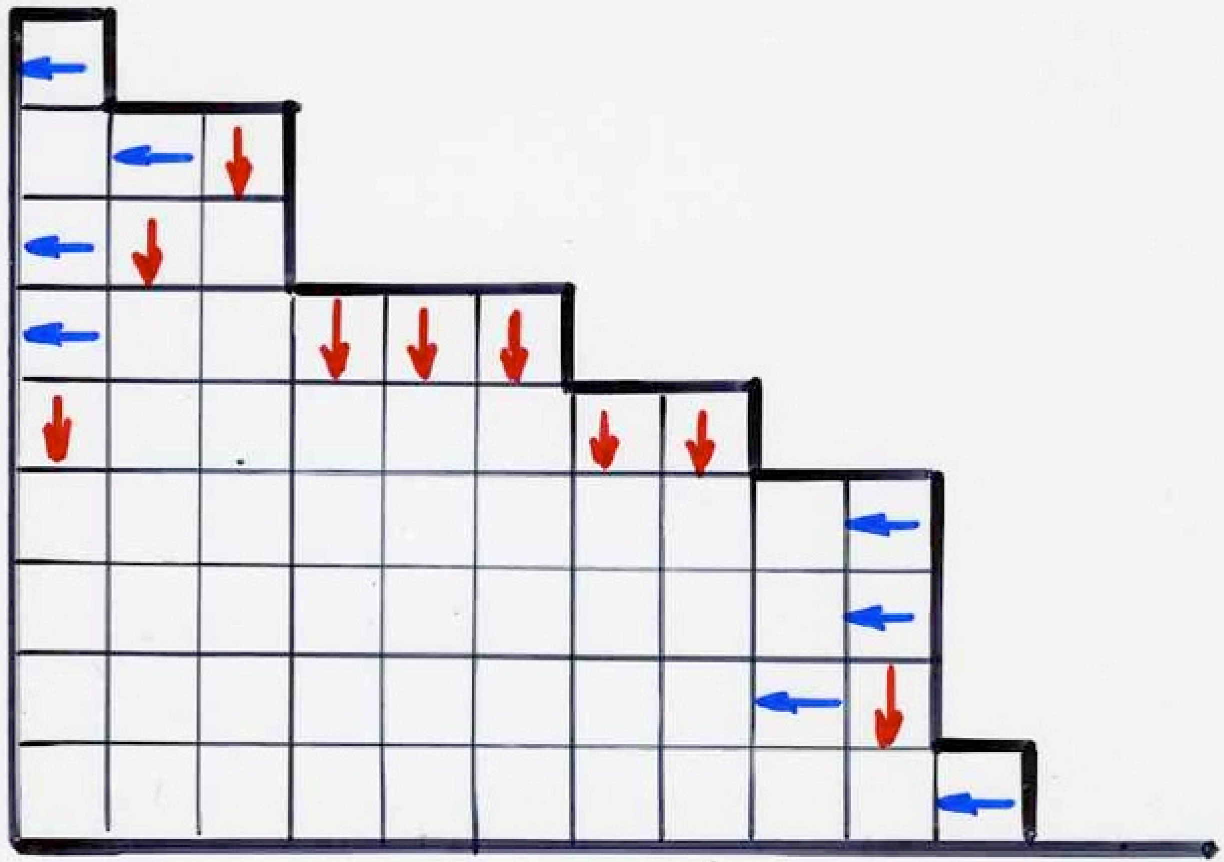
(iii) only one 1 in each column

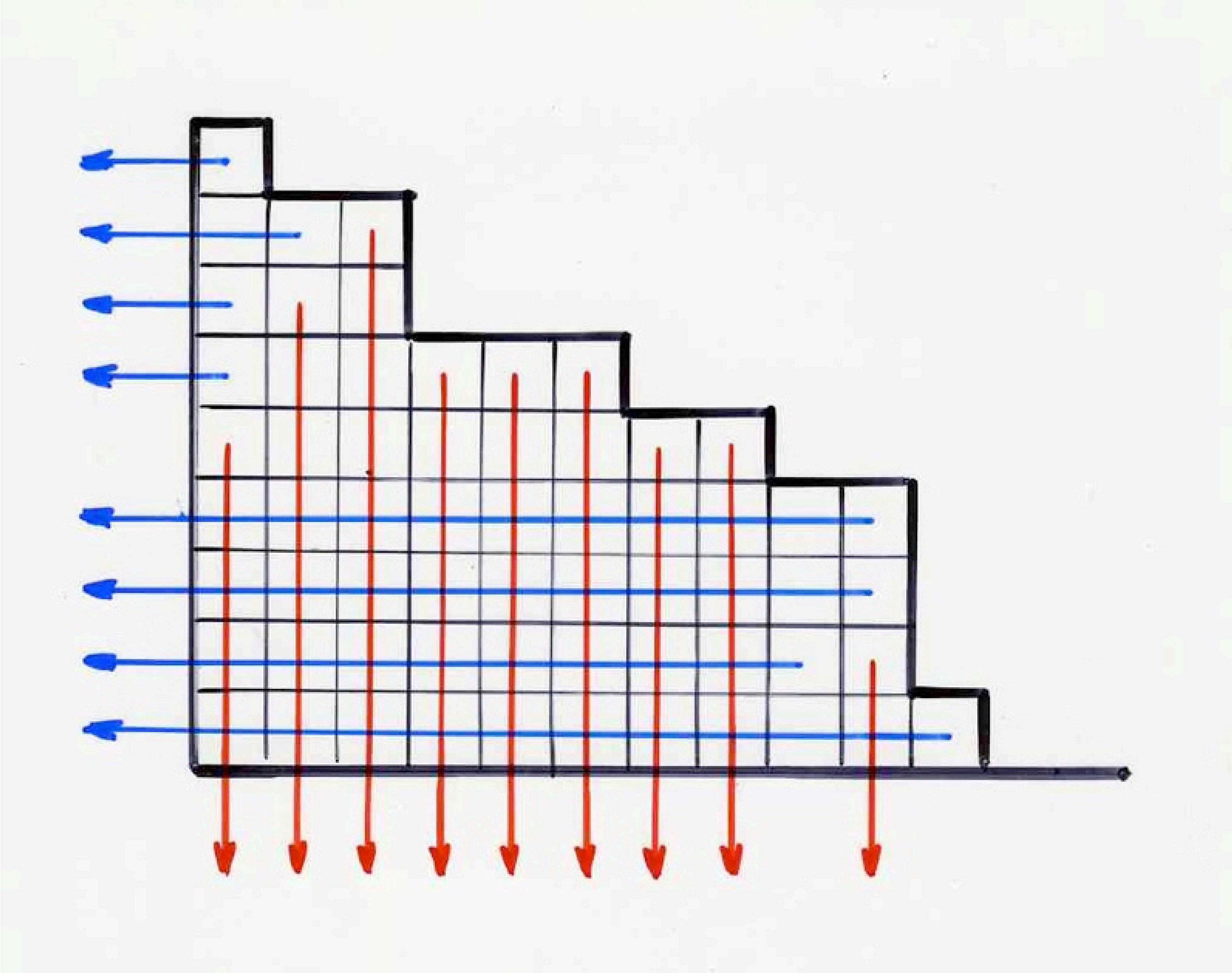


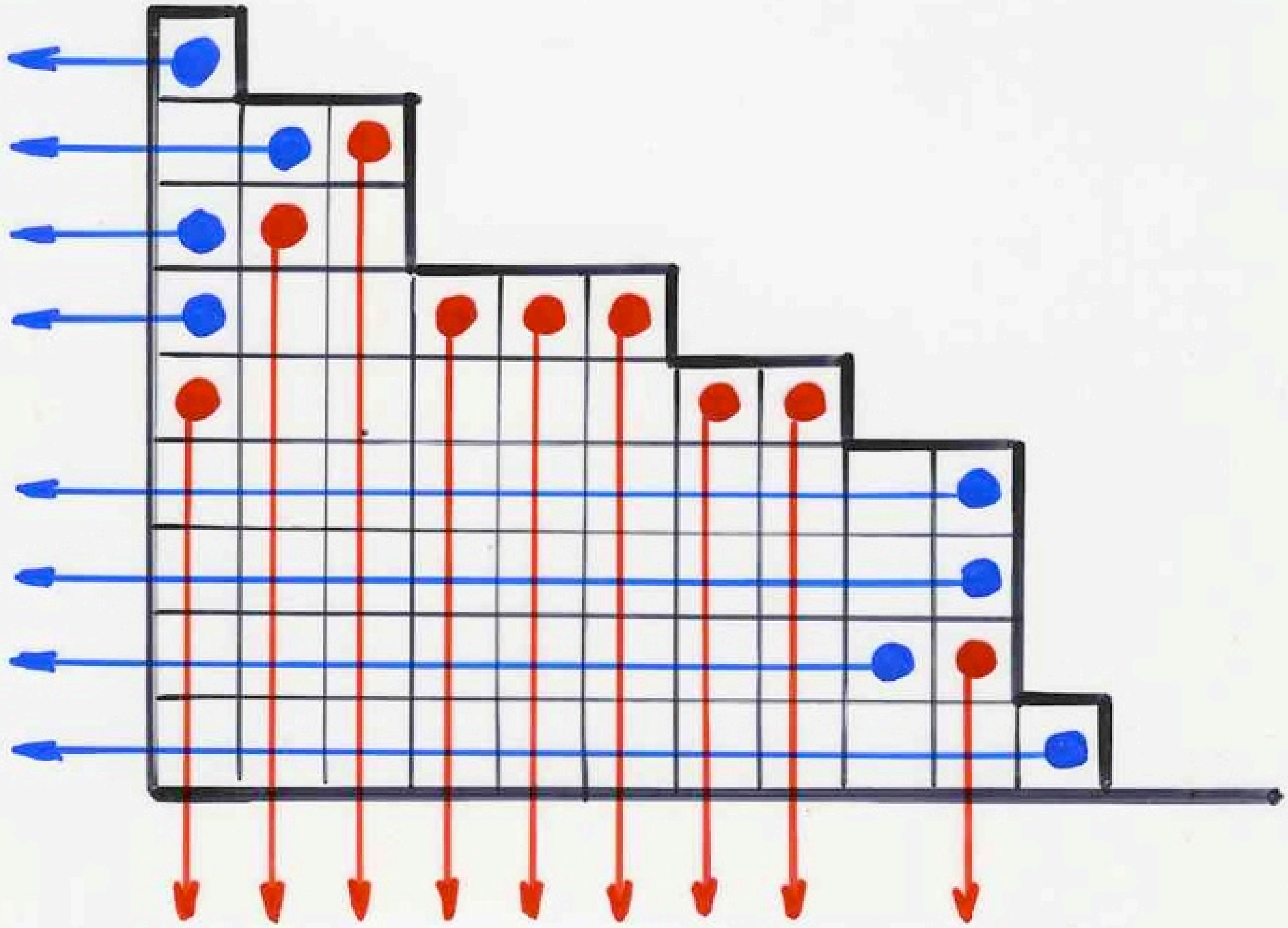


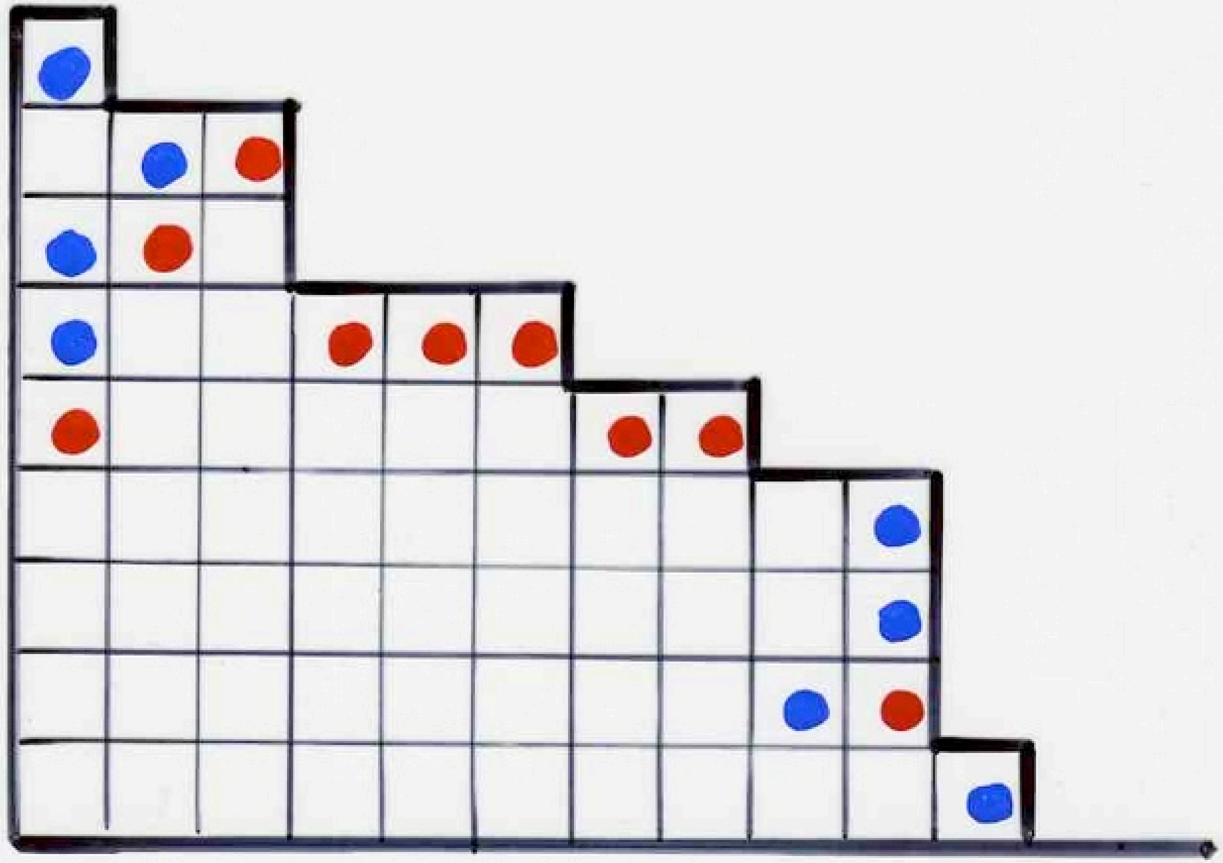


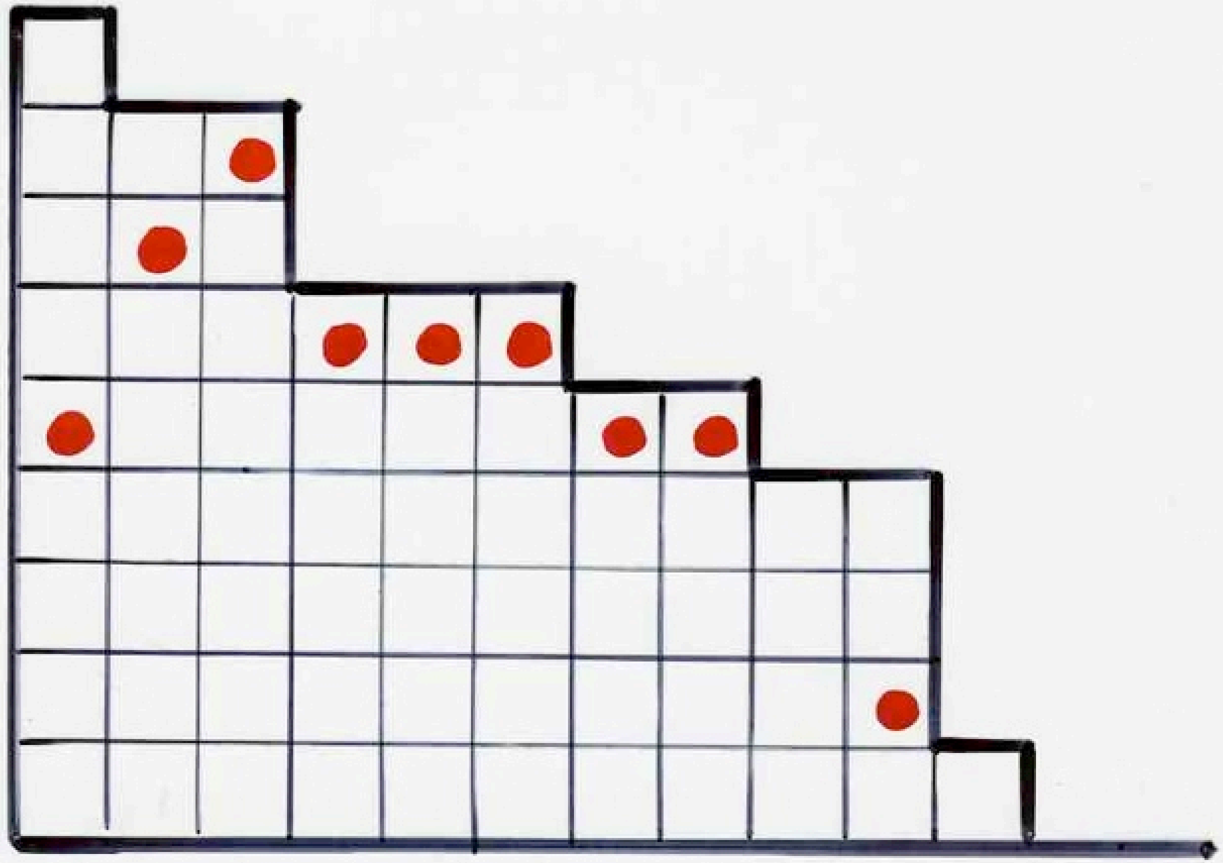




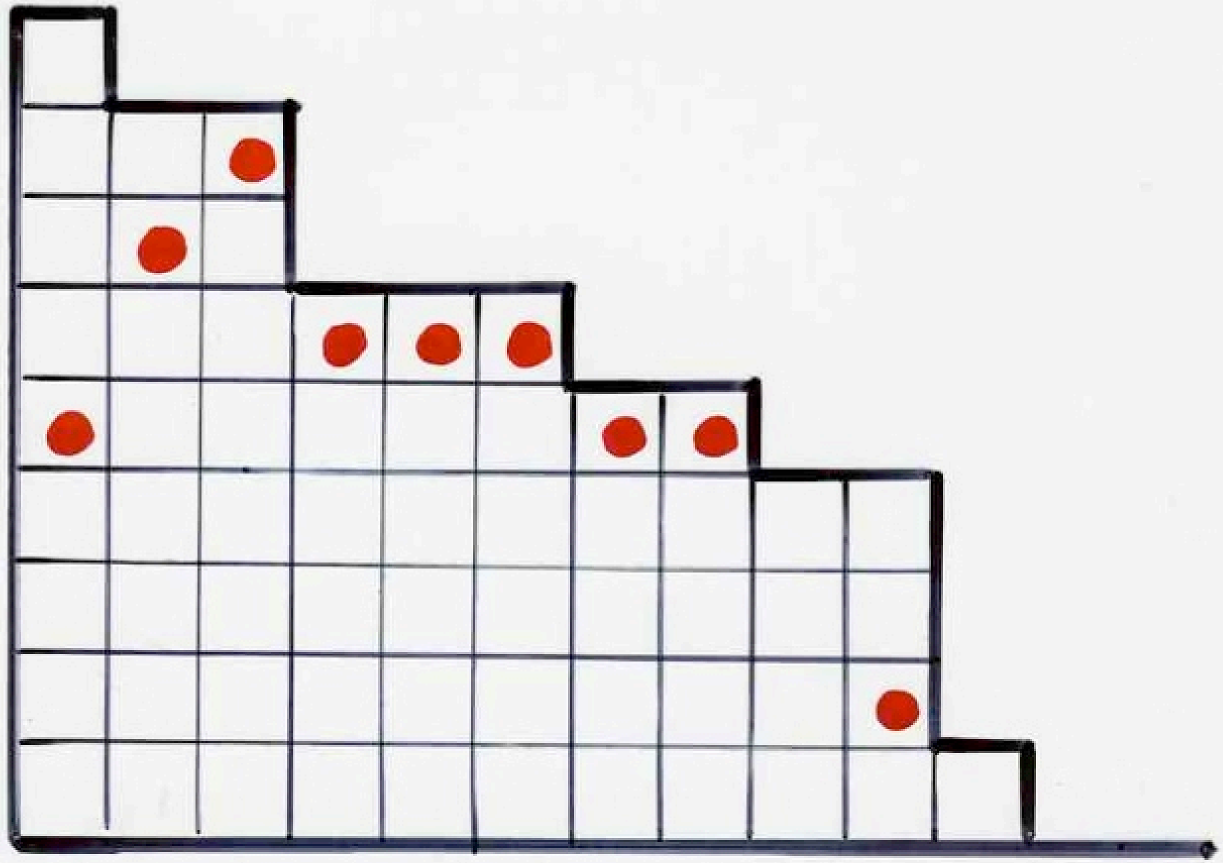




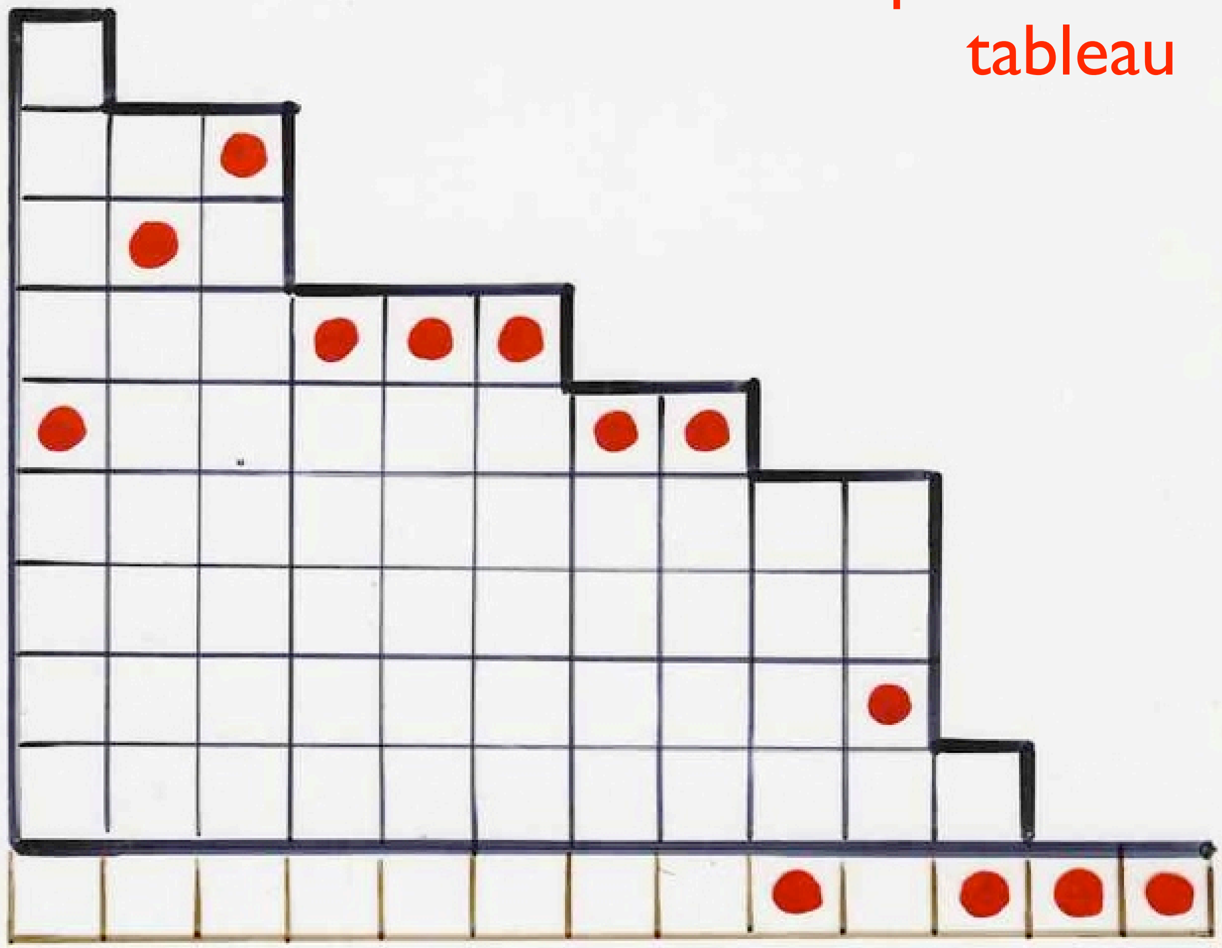




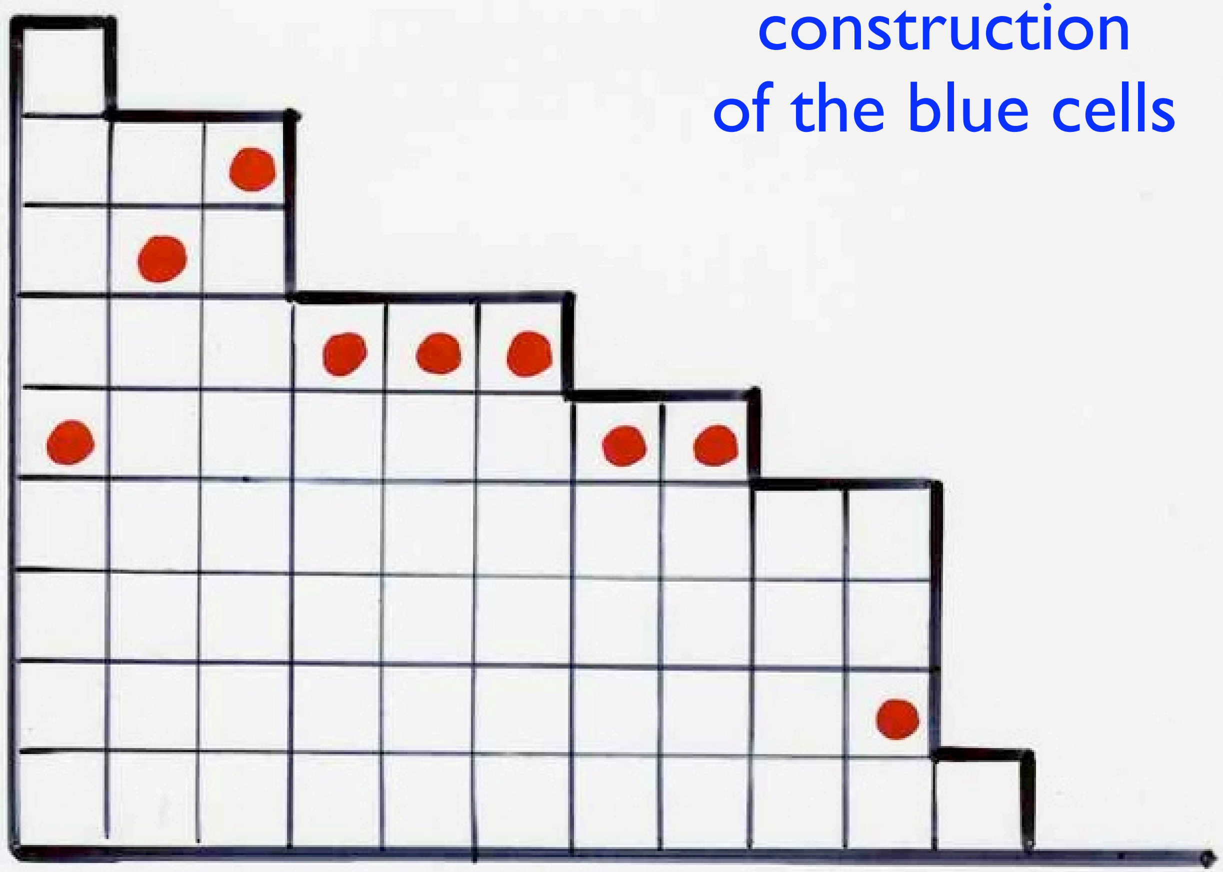
Caractérisation

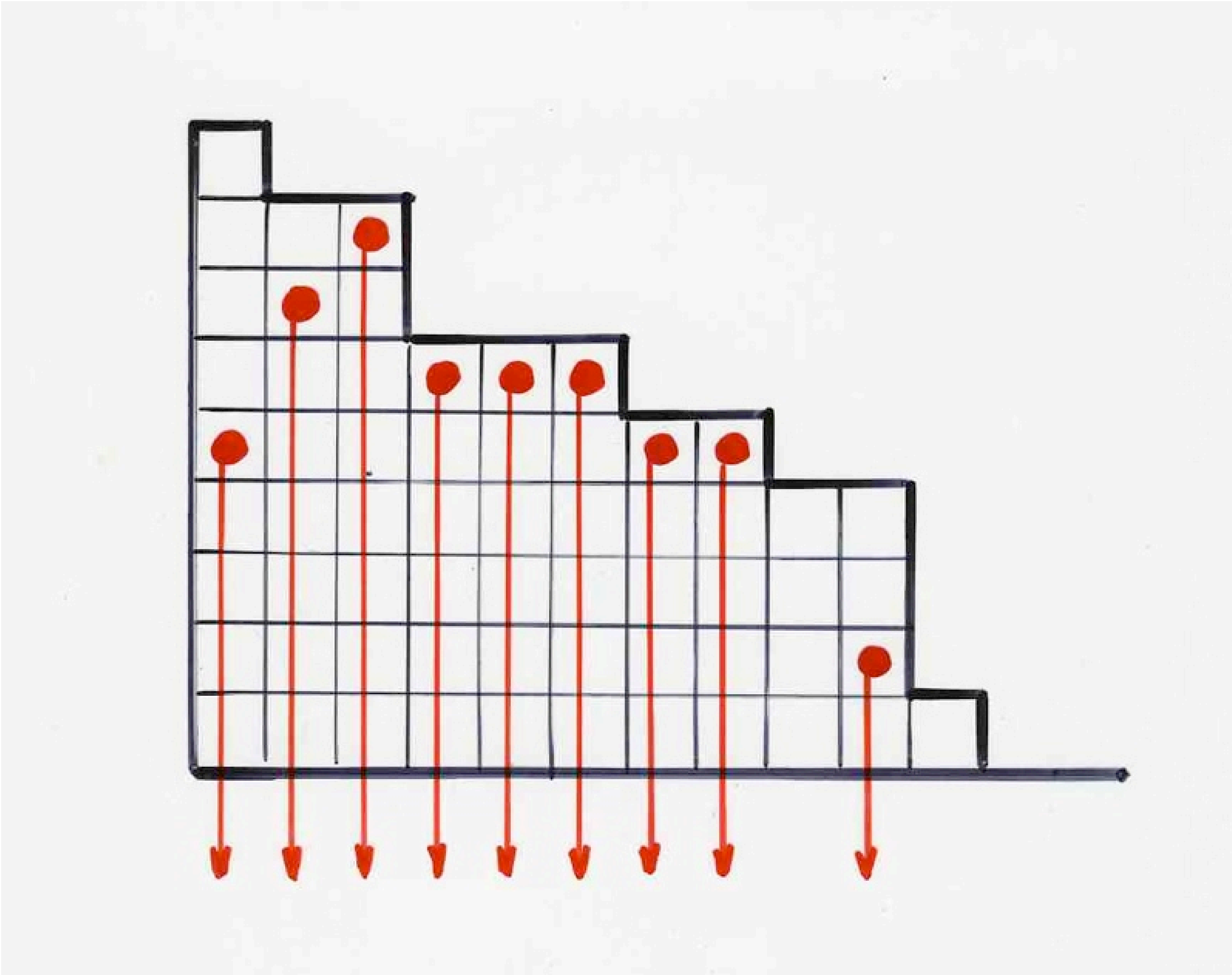


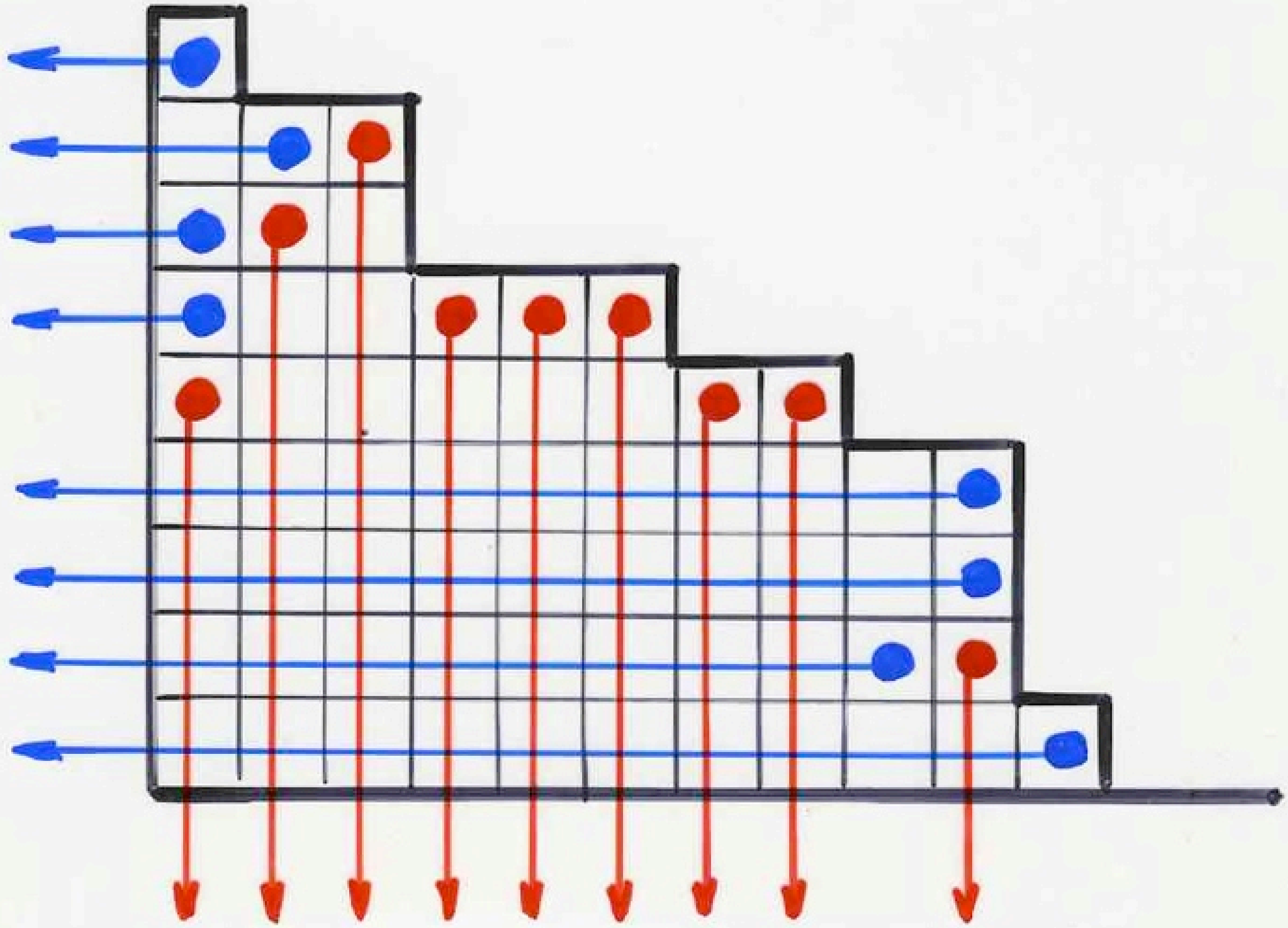
permutation
tableau

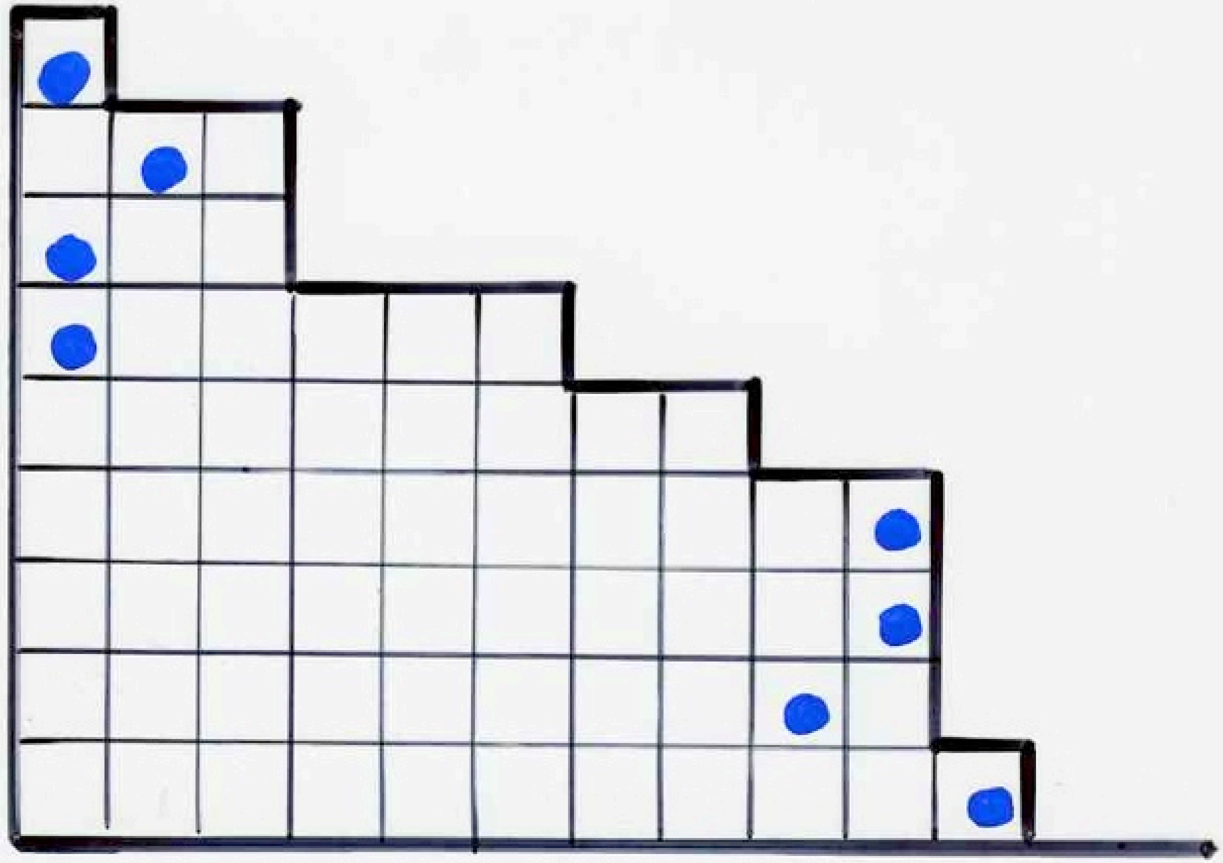


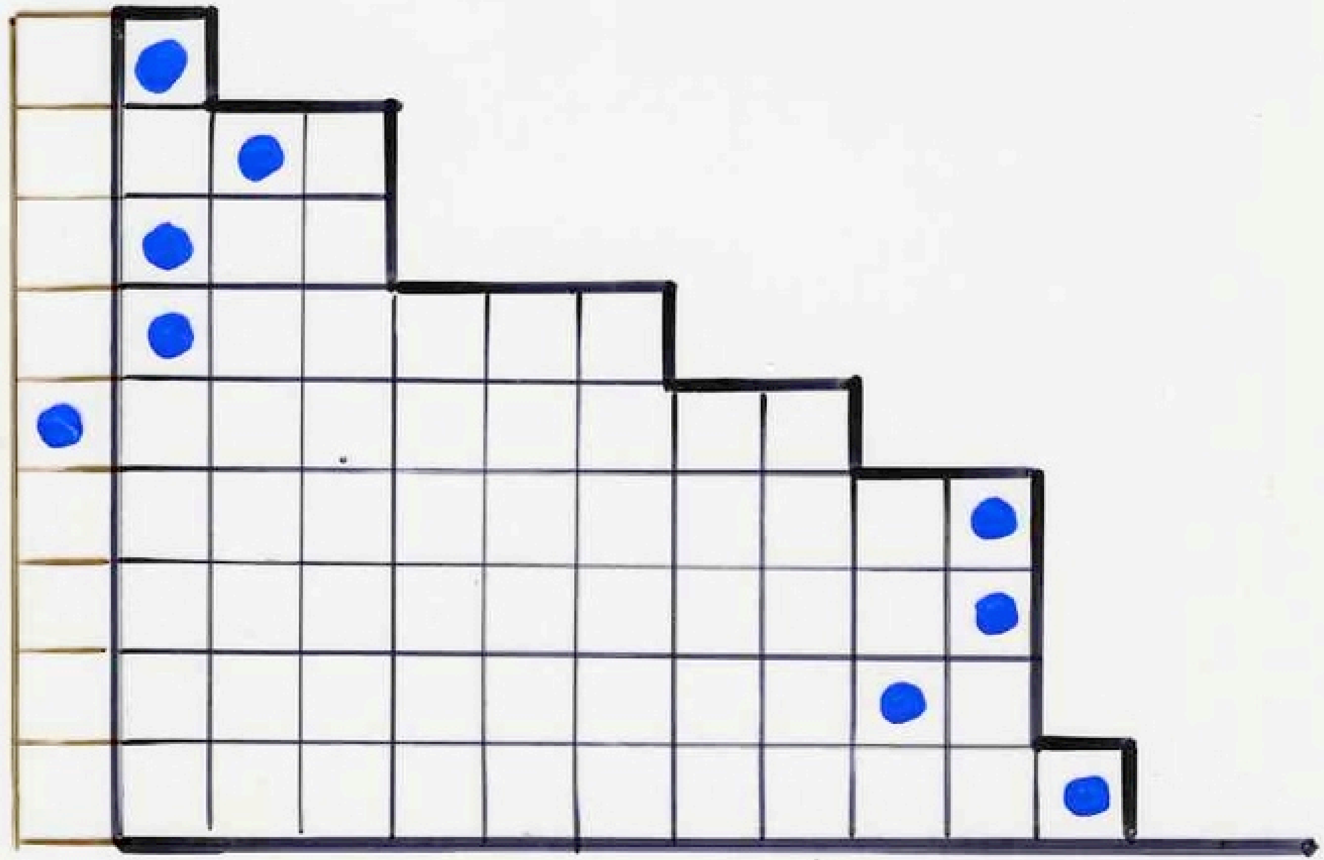
construction
of the blue cells

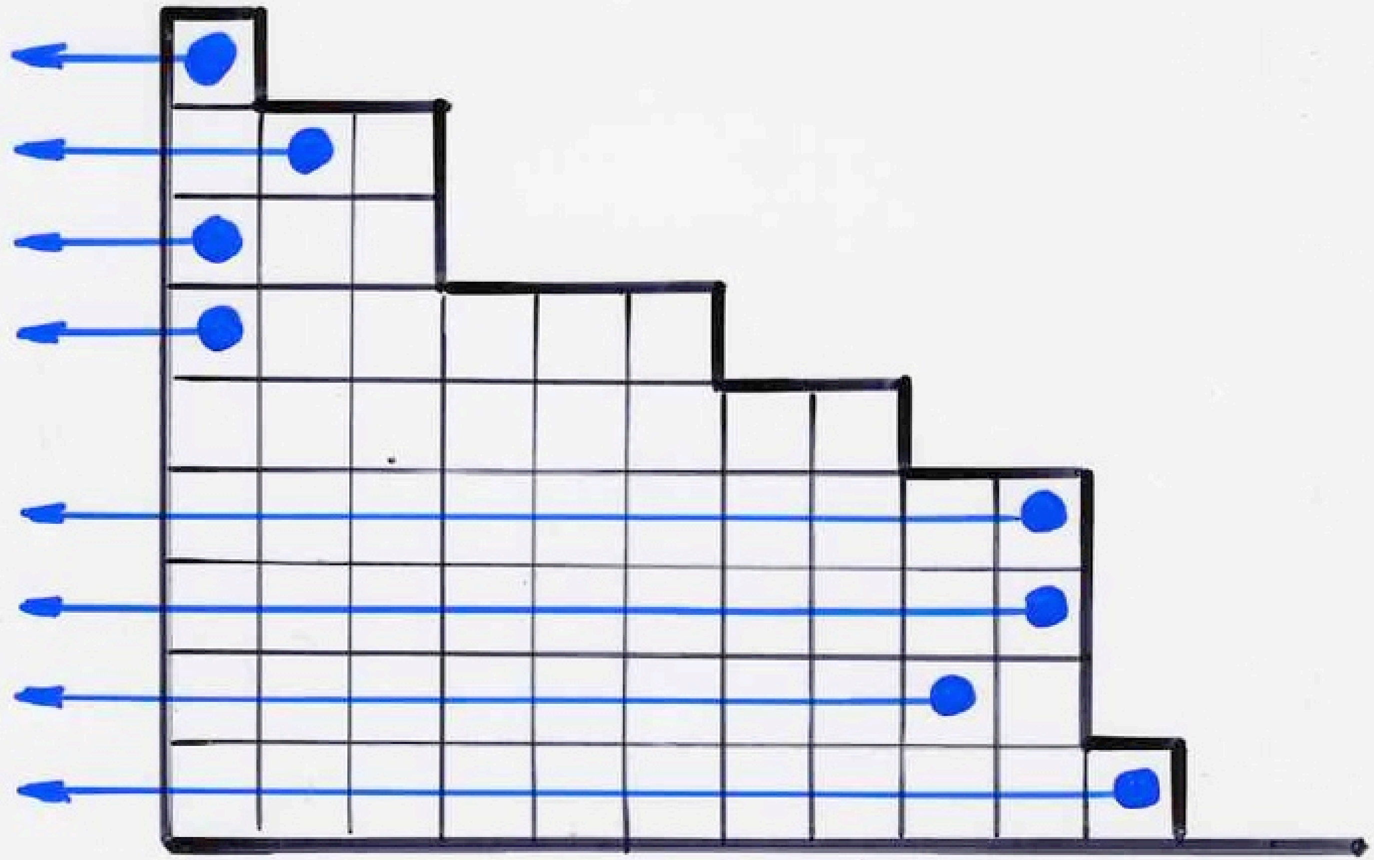


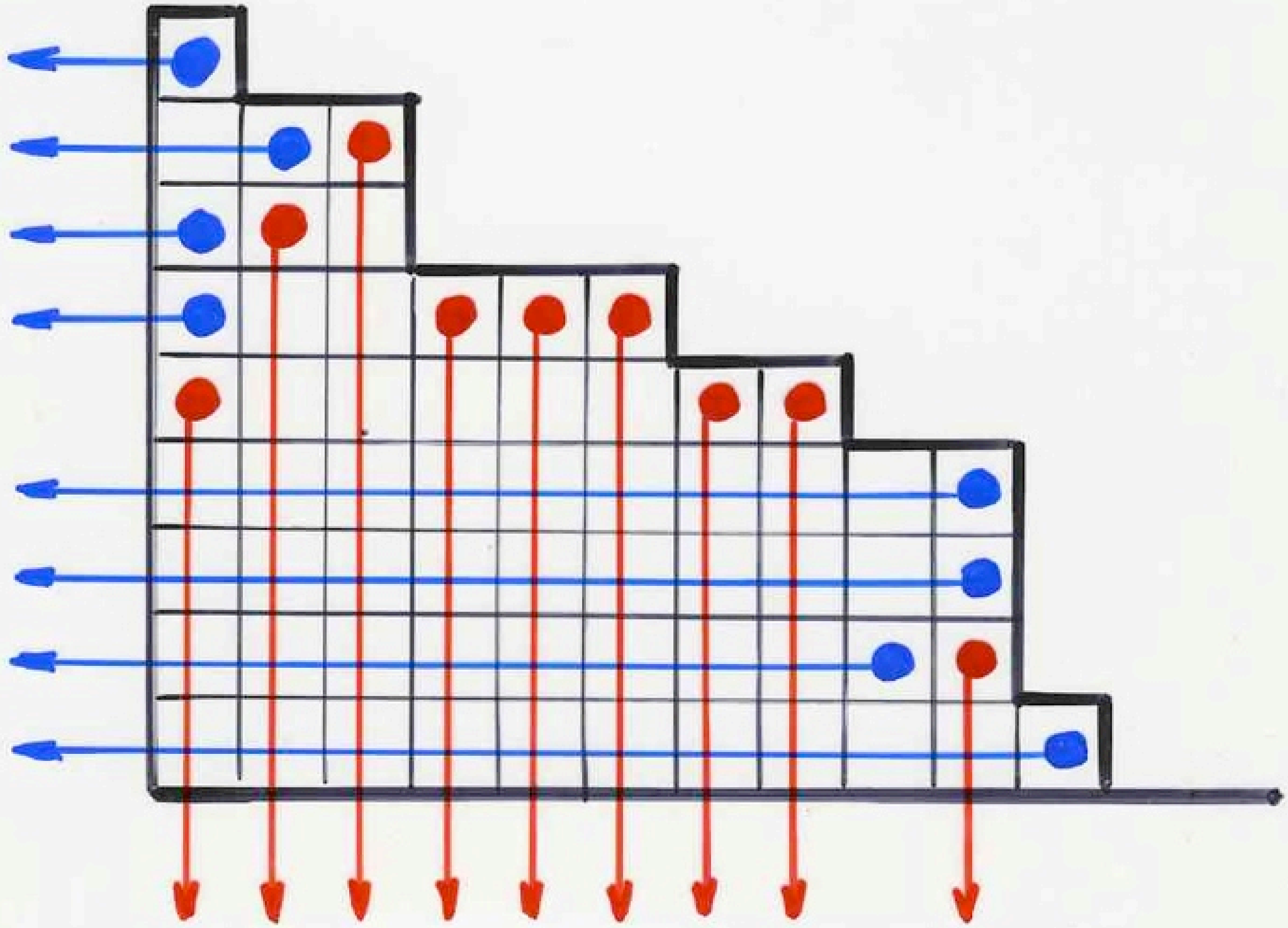


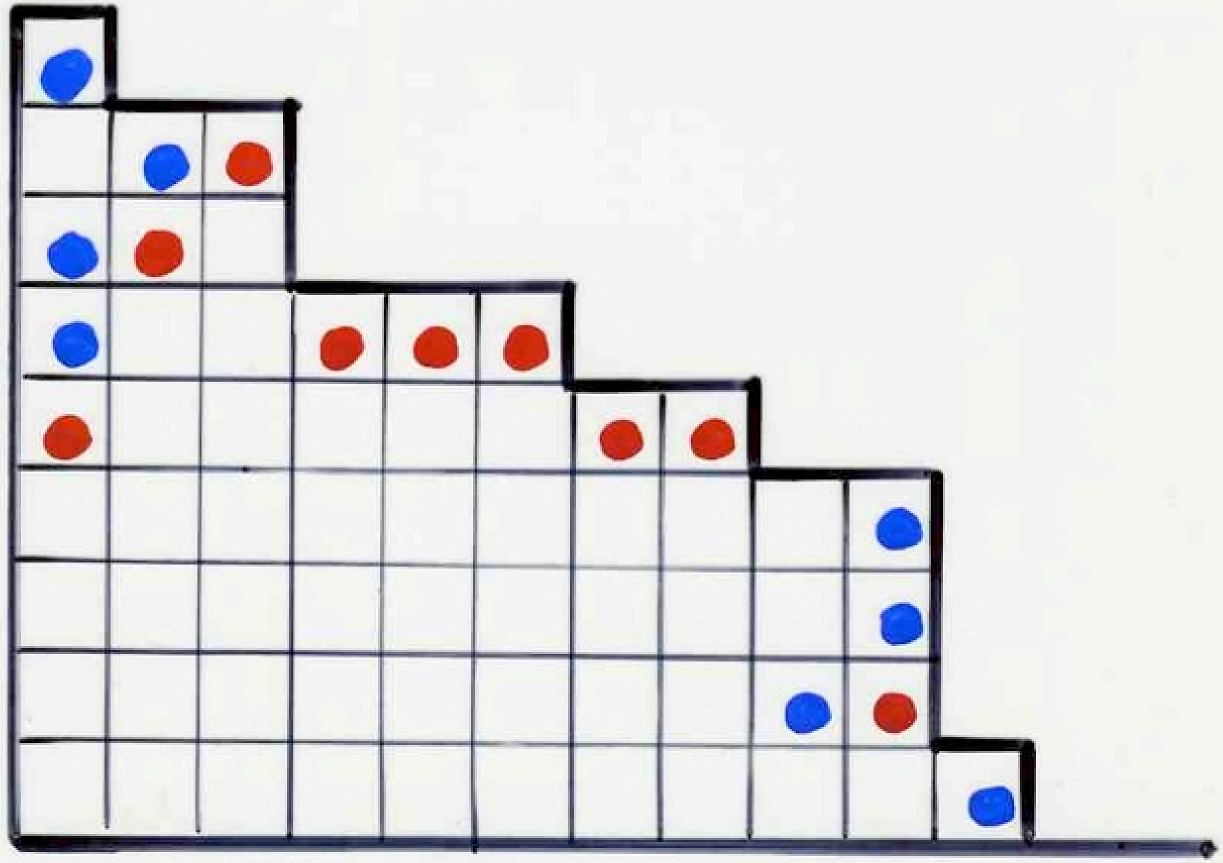




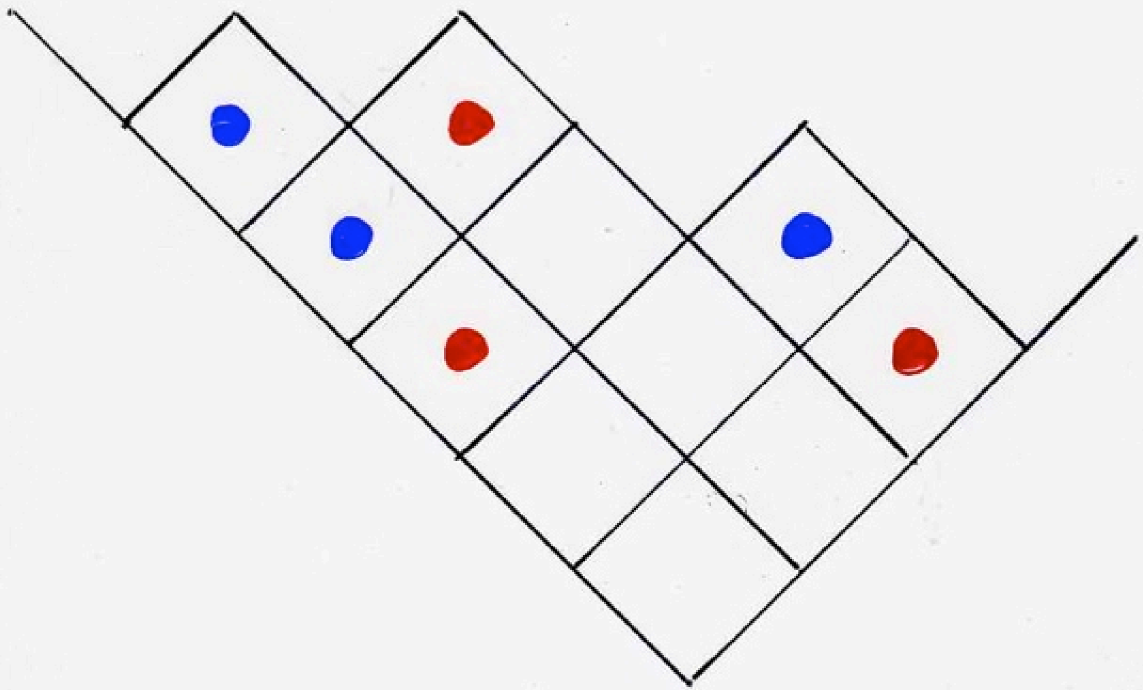


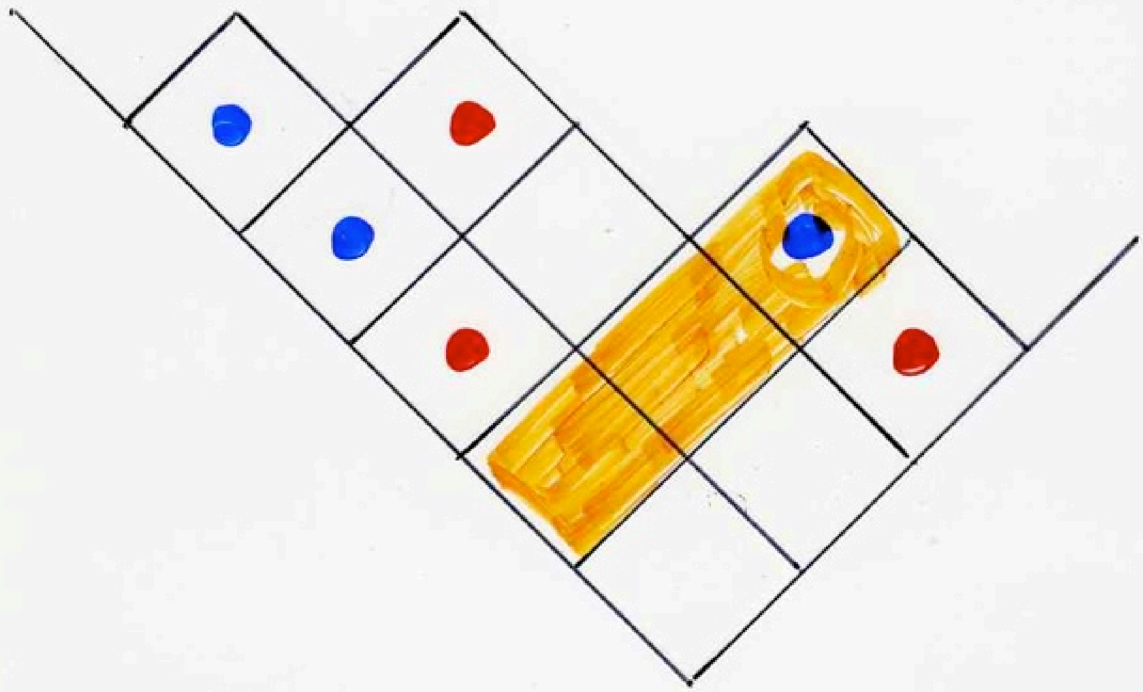


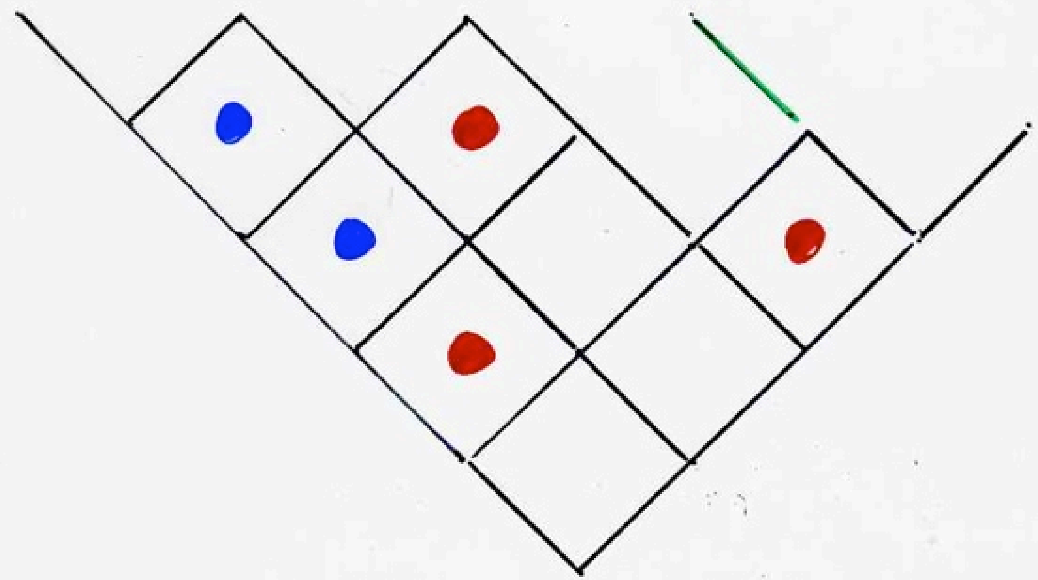


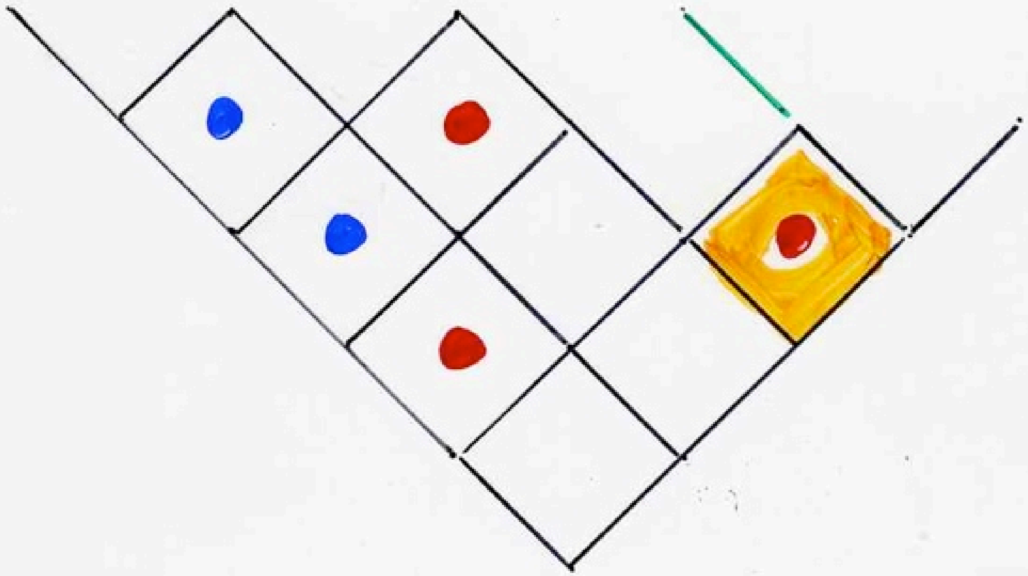


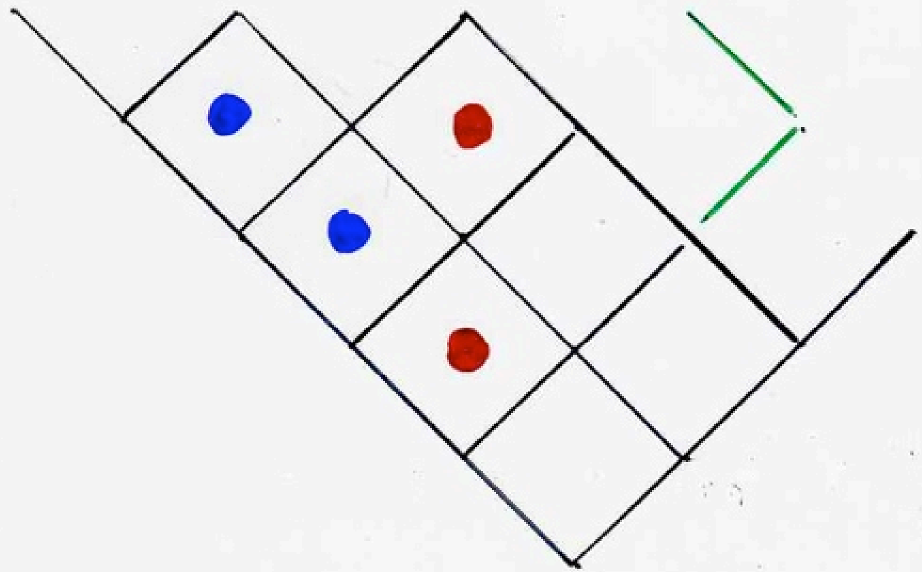
Bijection
tableaux alternatifs de Catalan
arbres binaires

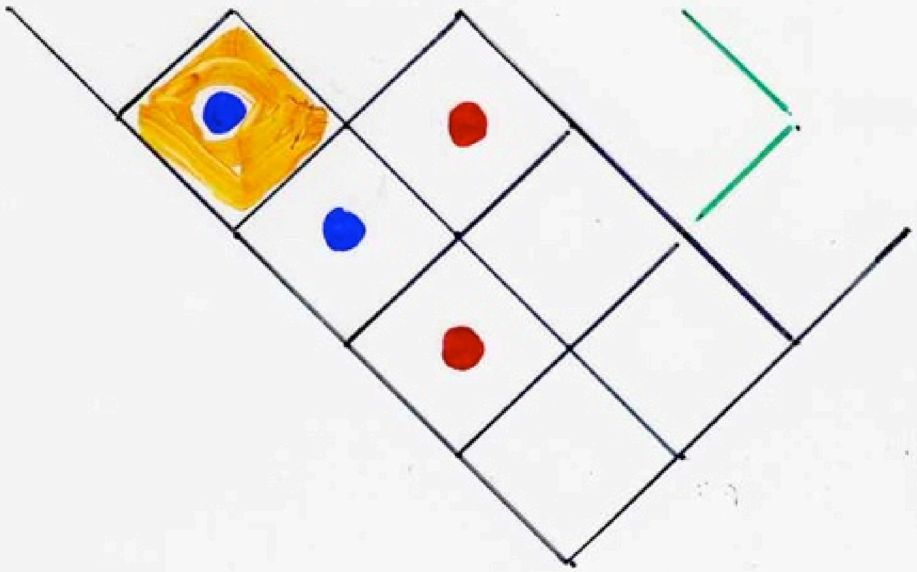


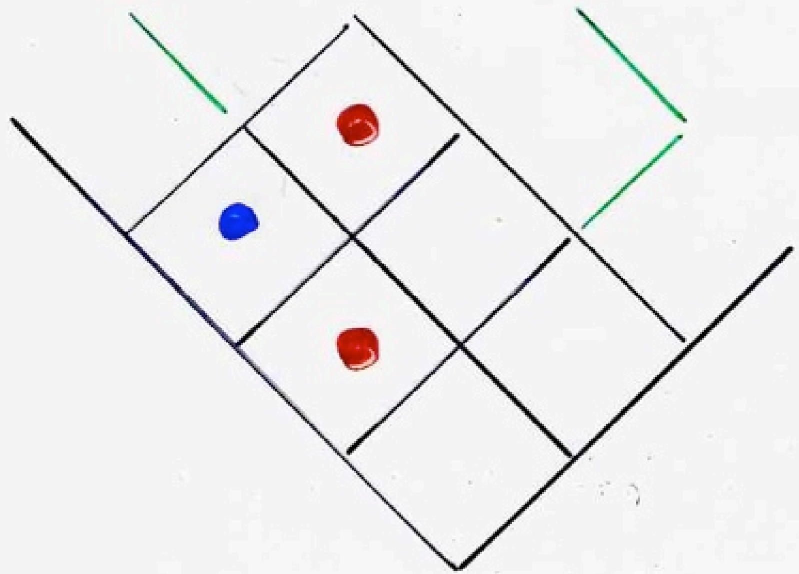


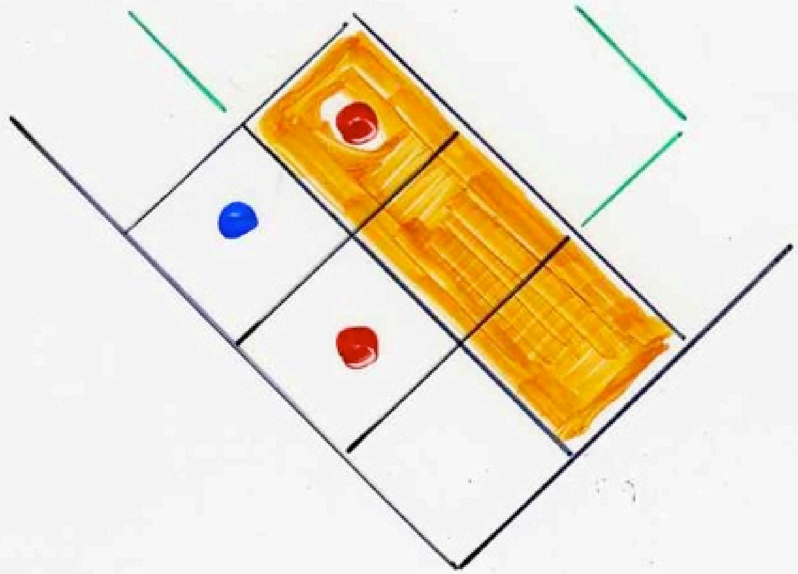


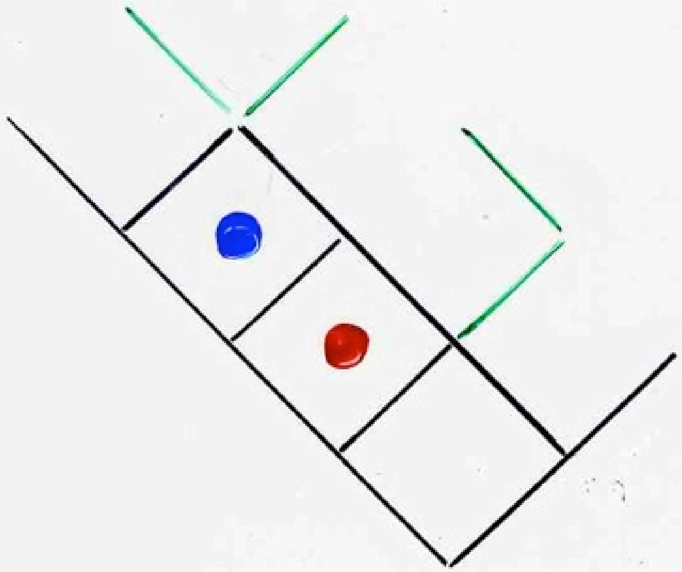


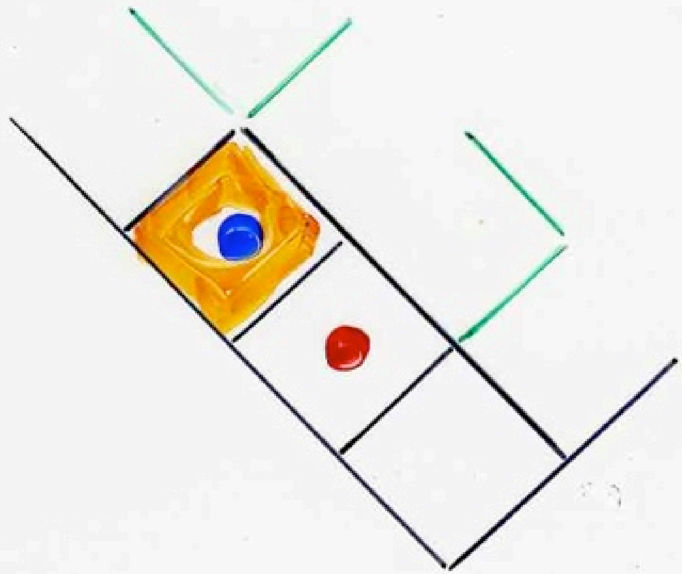


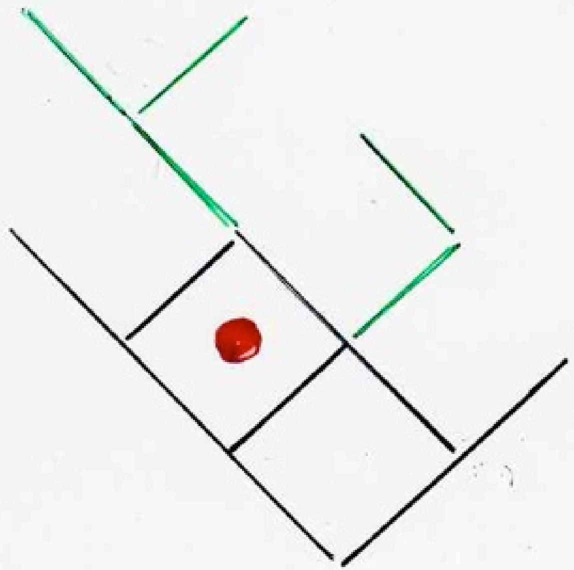


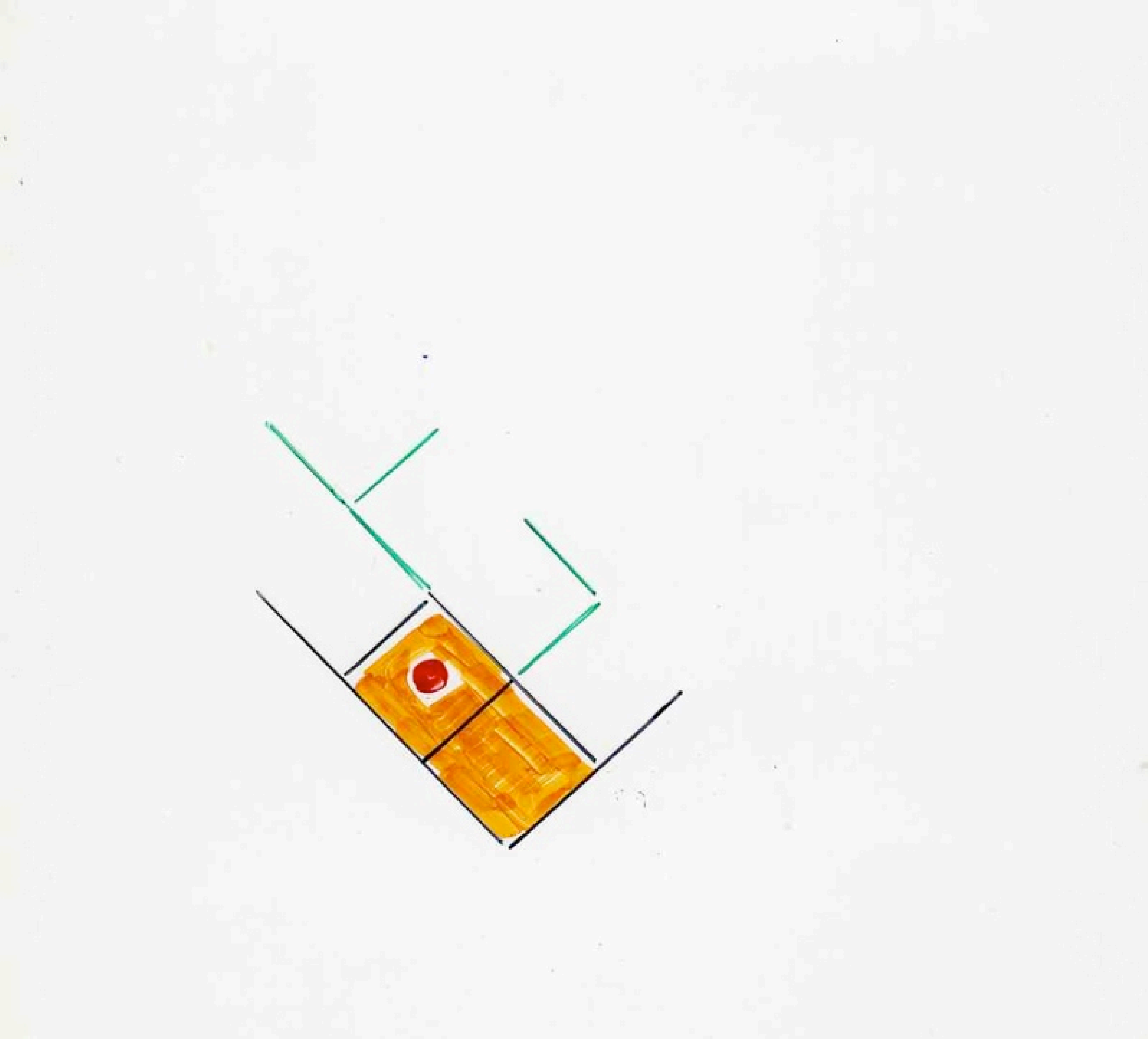


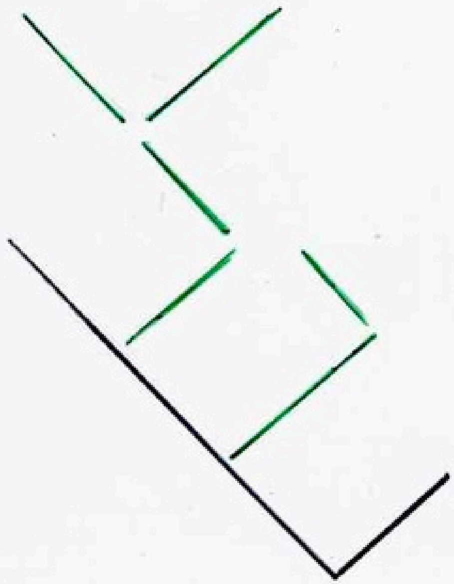






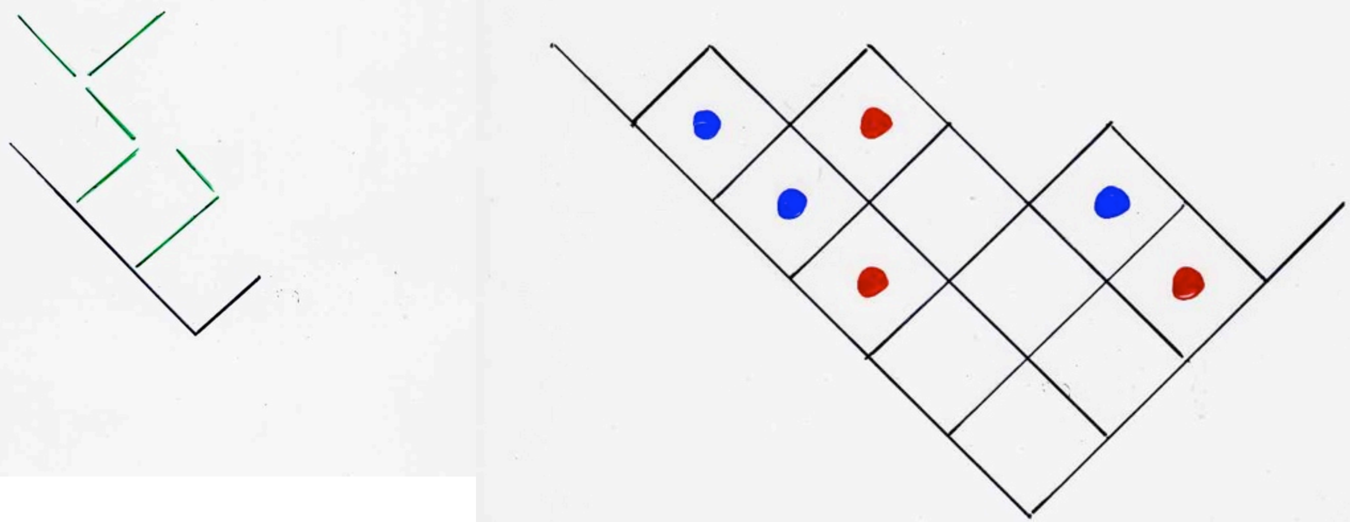






Bijection

tableaux alternatifs de Catalan taille n ⟷ arbres binaires n arêtes



profil (bord) du diagramme de Ferrers ⟵---⟶ canopée

Bijection

tableaux **alternatifs** de Catalan taille n ⟷ arbres binaires n arêtes

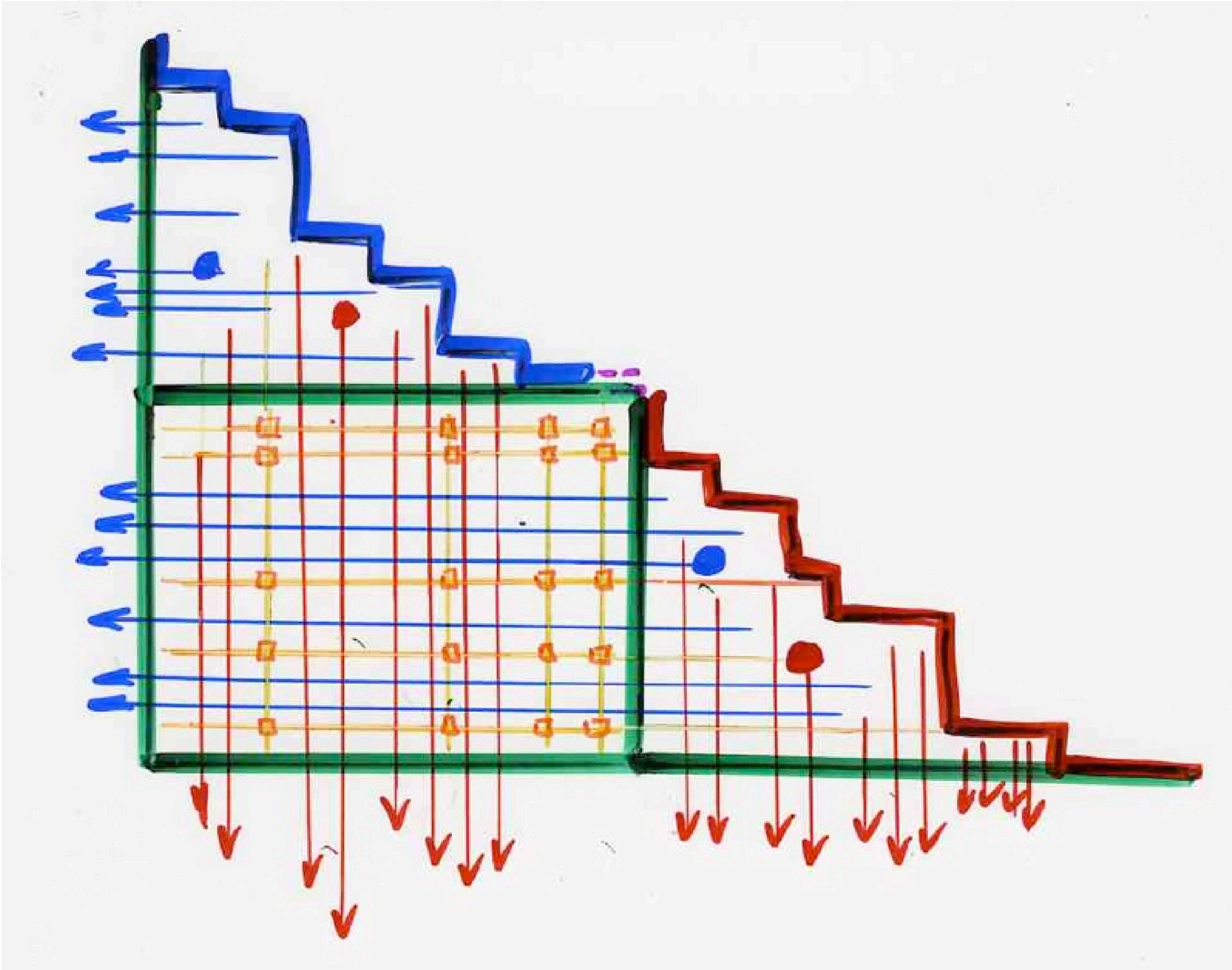
profil (bord) du diagramme de Ferrers ⟵ - - - ⟶ canopée

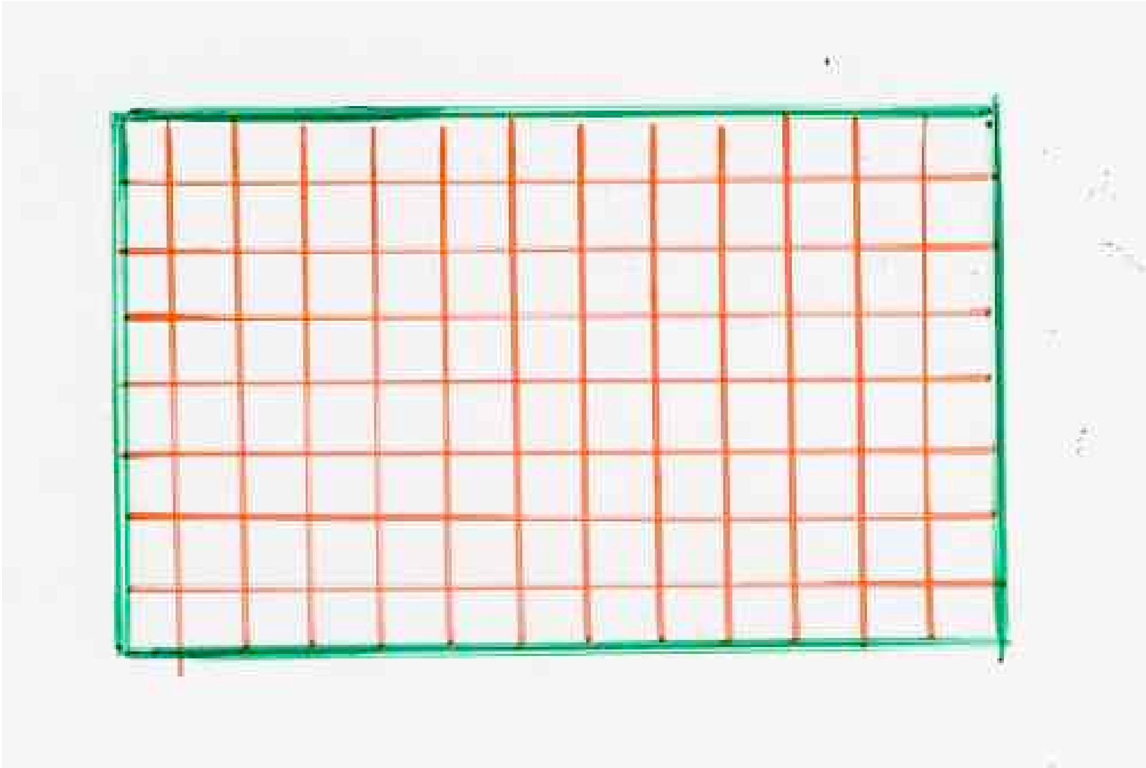
nb de lignes libres (sans ←●)

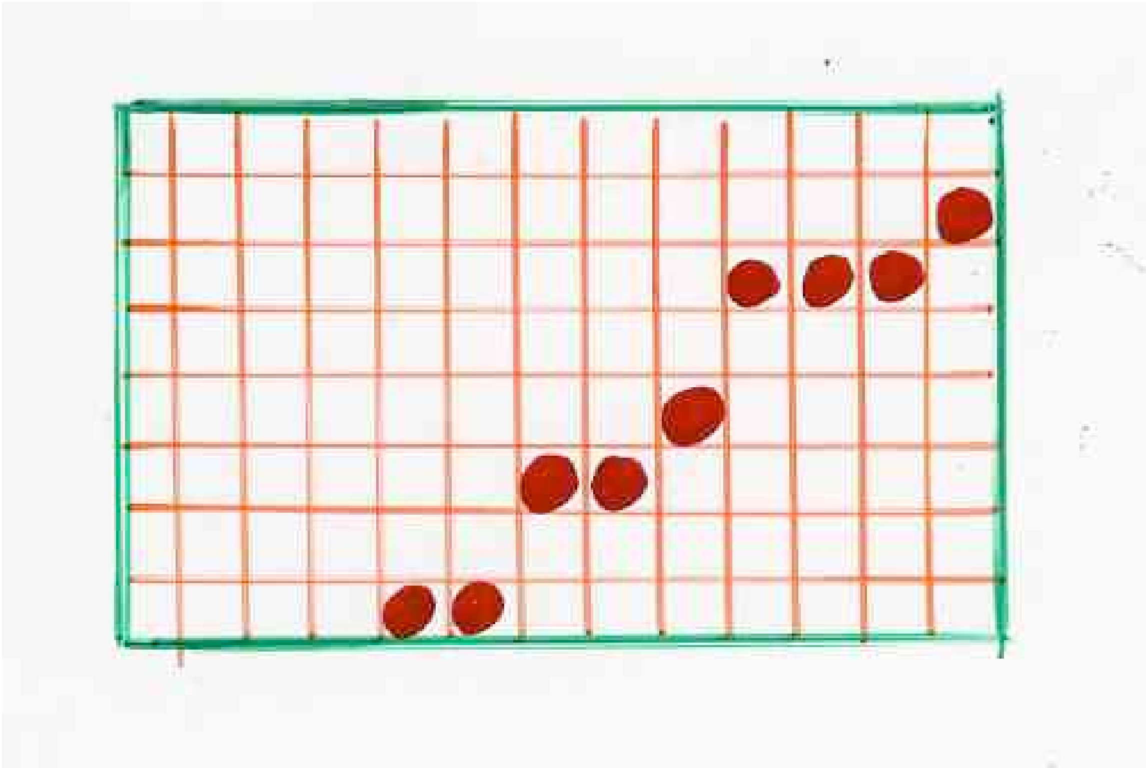
nb de colonnes libres (sans ●↓)

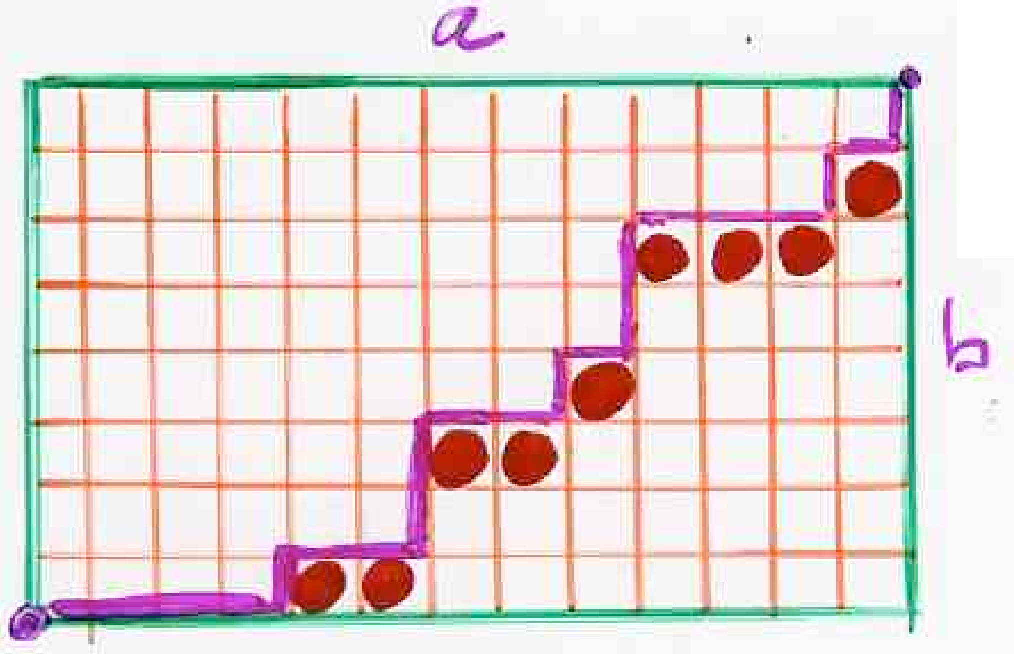
La Question

§3 Le produit de Loday-Ronco
avec
les tableaux alternatifs de Catalan

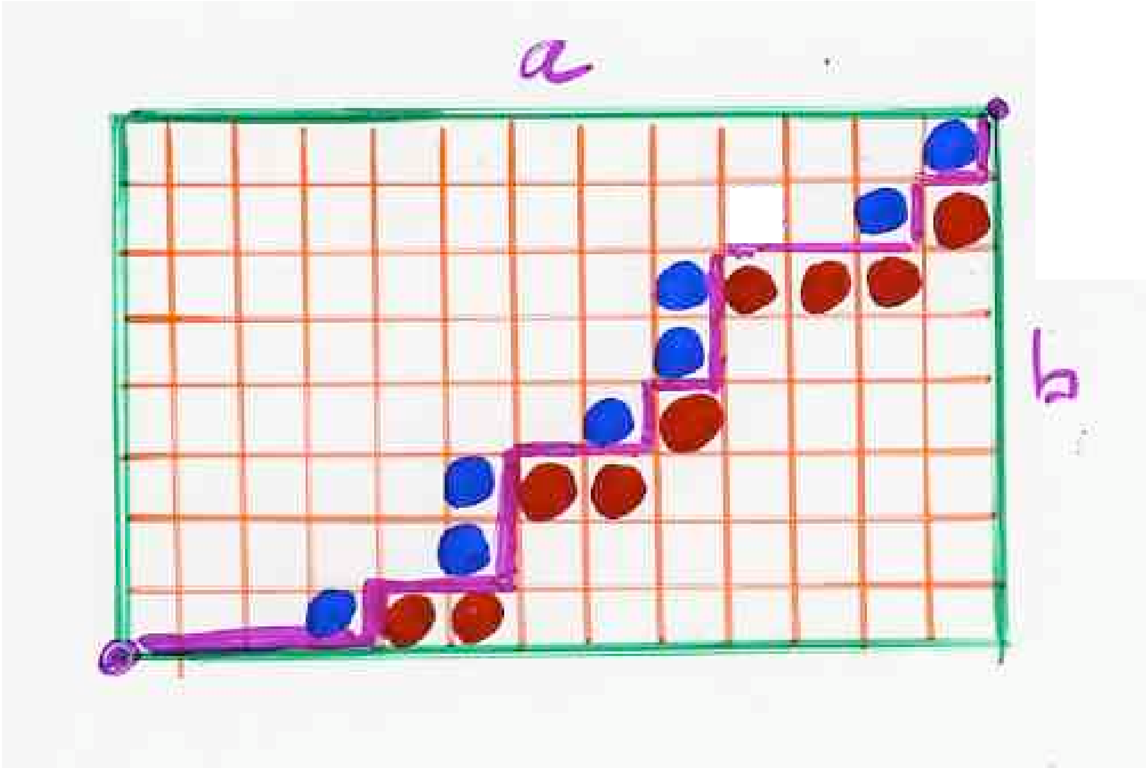








$$\binom{a+b}{a}$$


a
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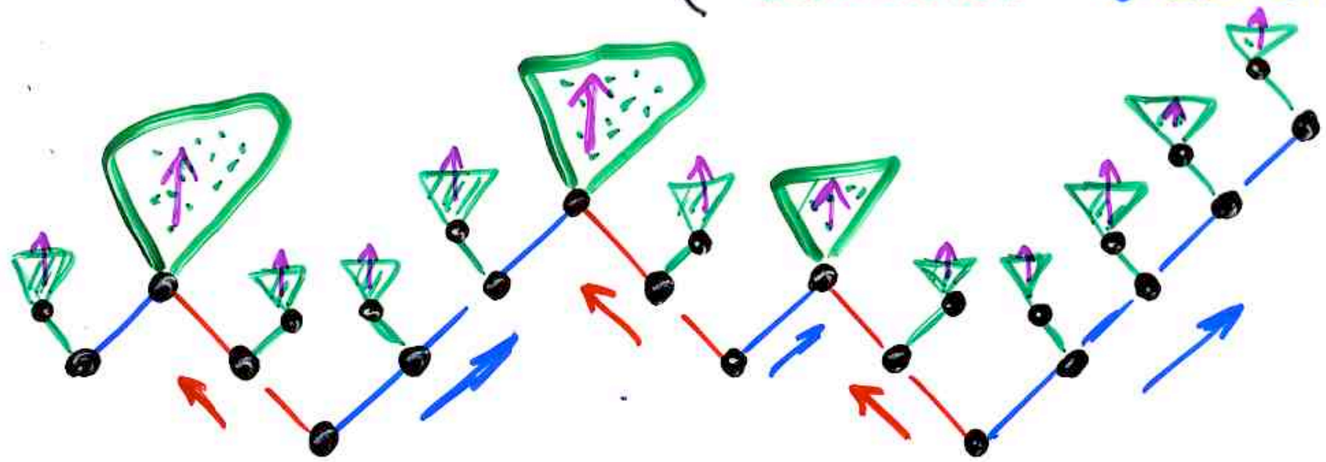
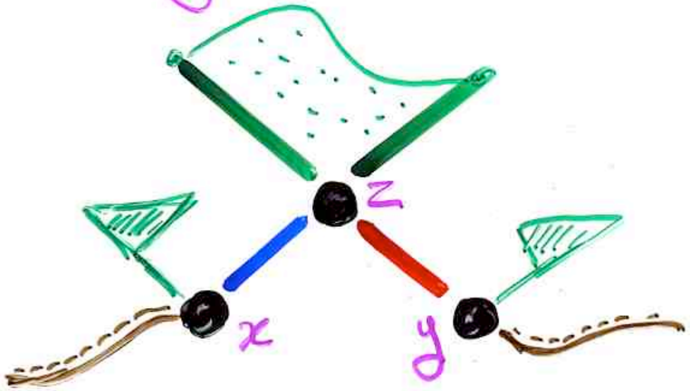
§4 Relation avec le "jeu de taquin" pour les arbres binaires croissants

Def. Increasing woods
("buissons" croissants)

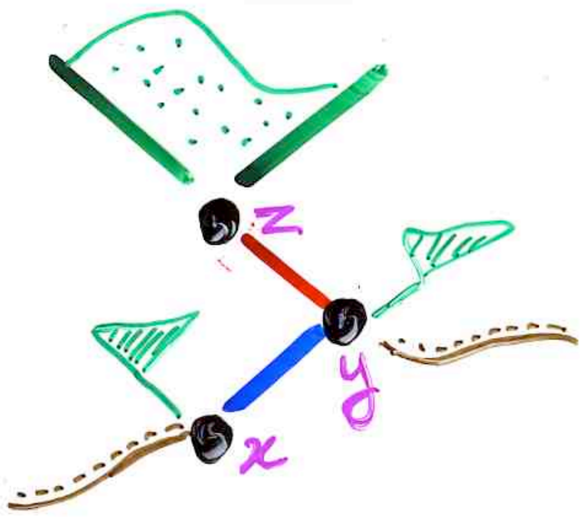


Def - Sliding in an increasing woods

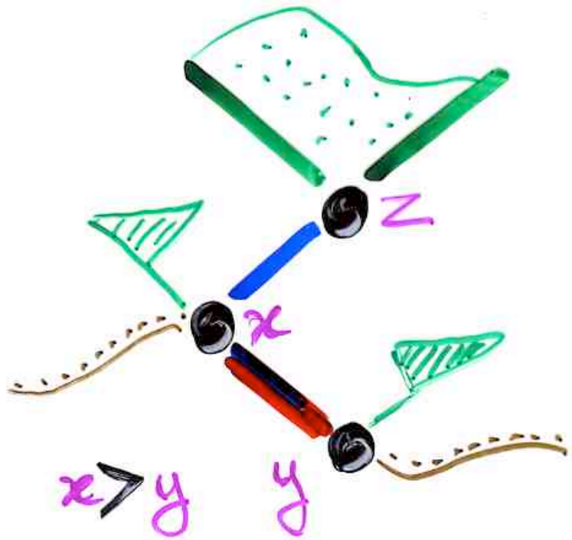


$z > x$

$z > y$



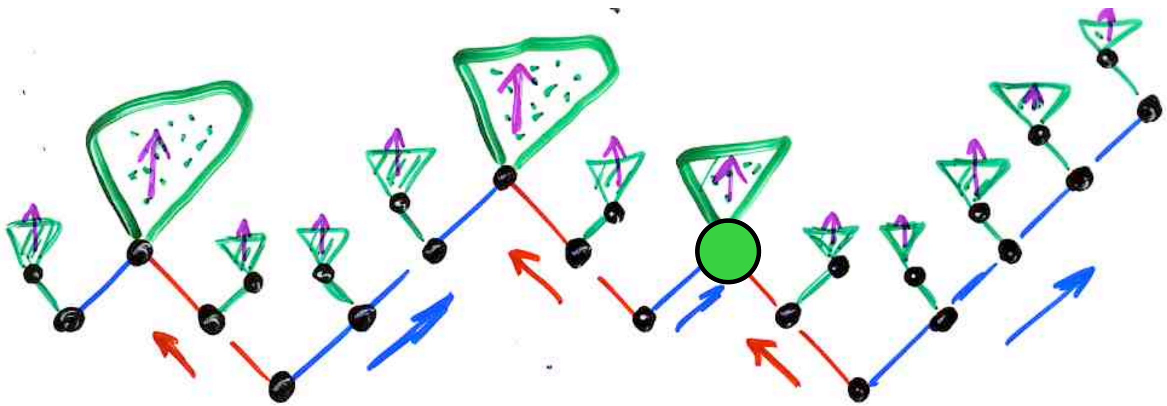
$x < y$

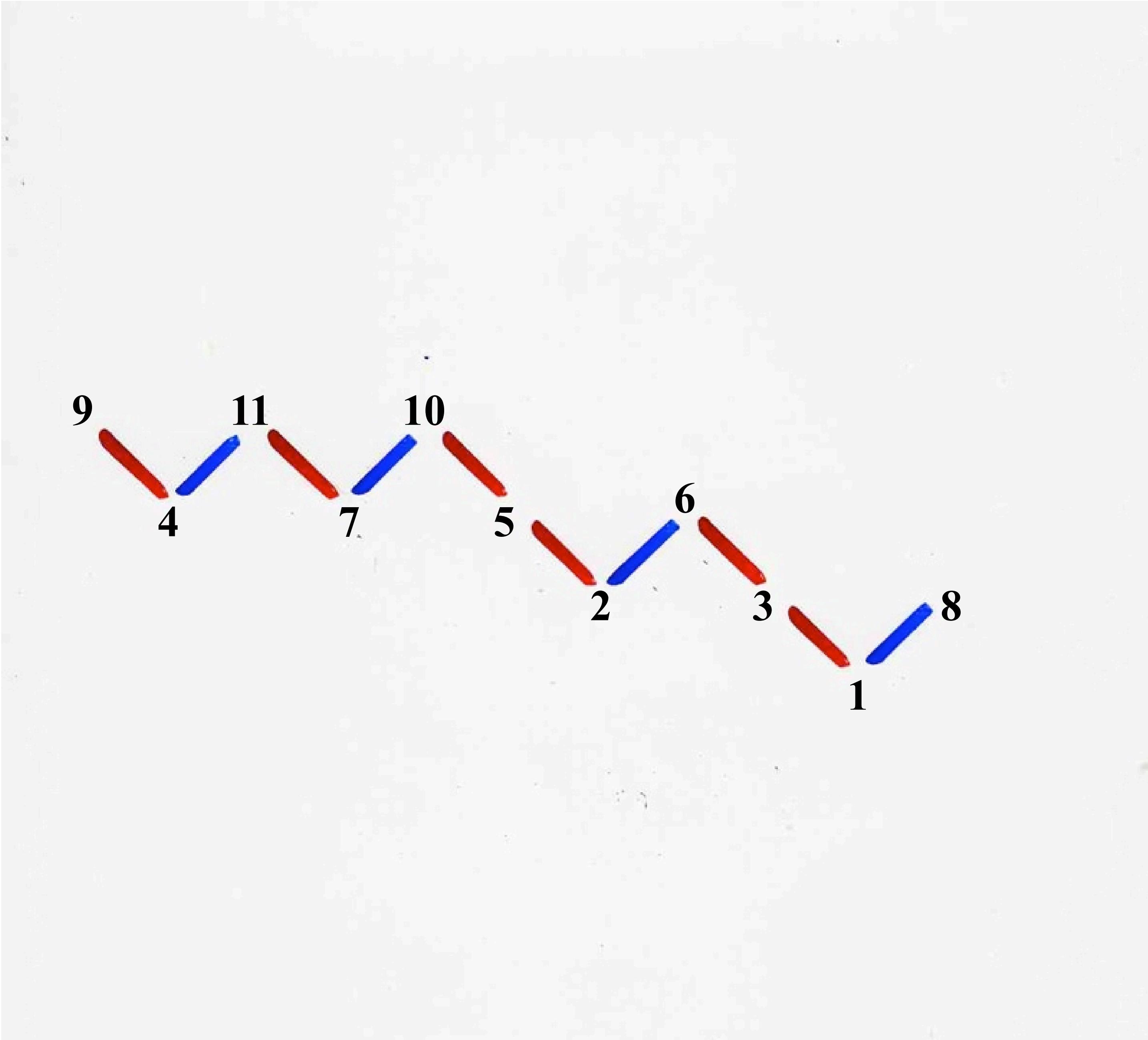


$x > y$

"jeu de taquin"

for increasing woods

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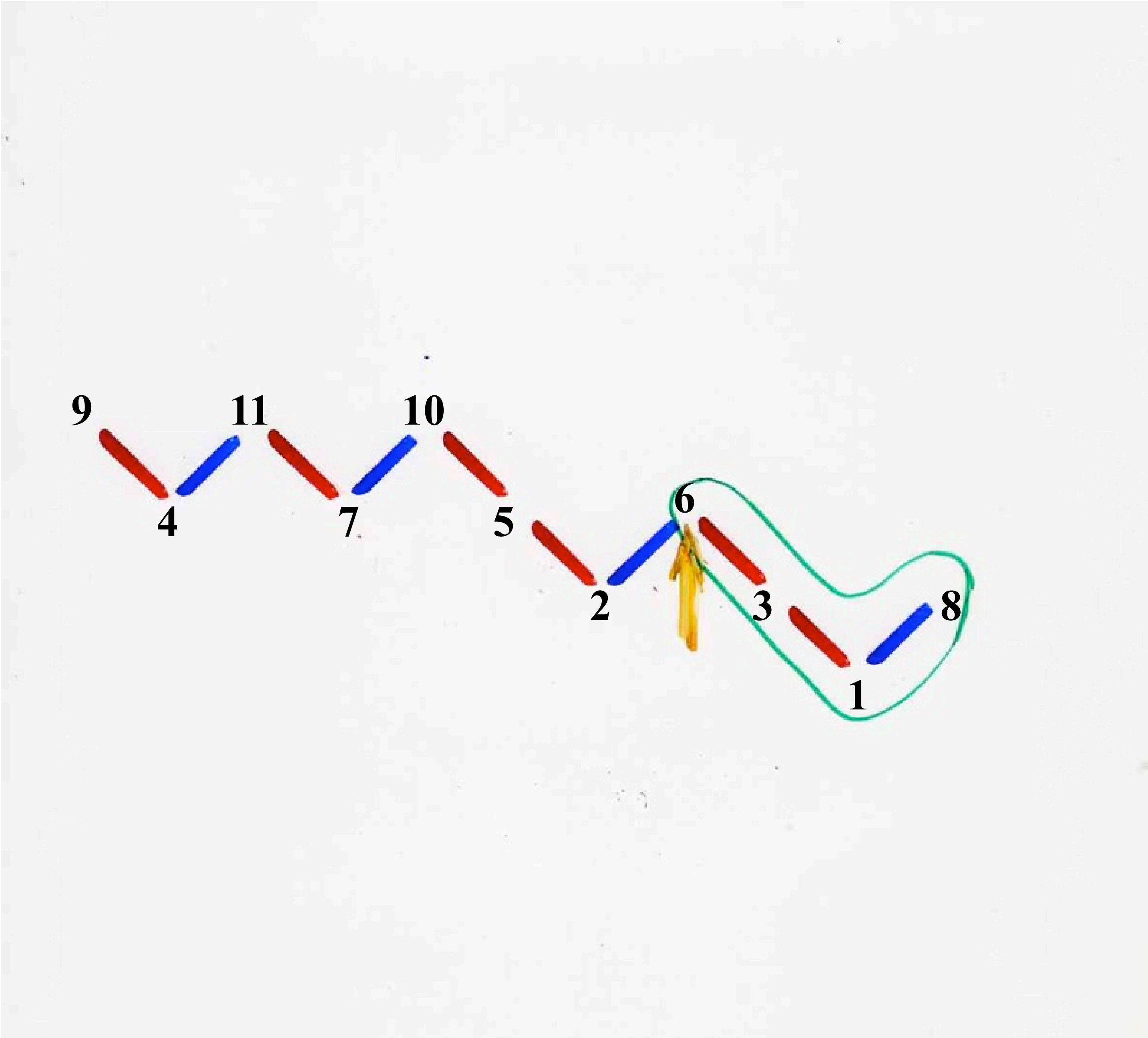
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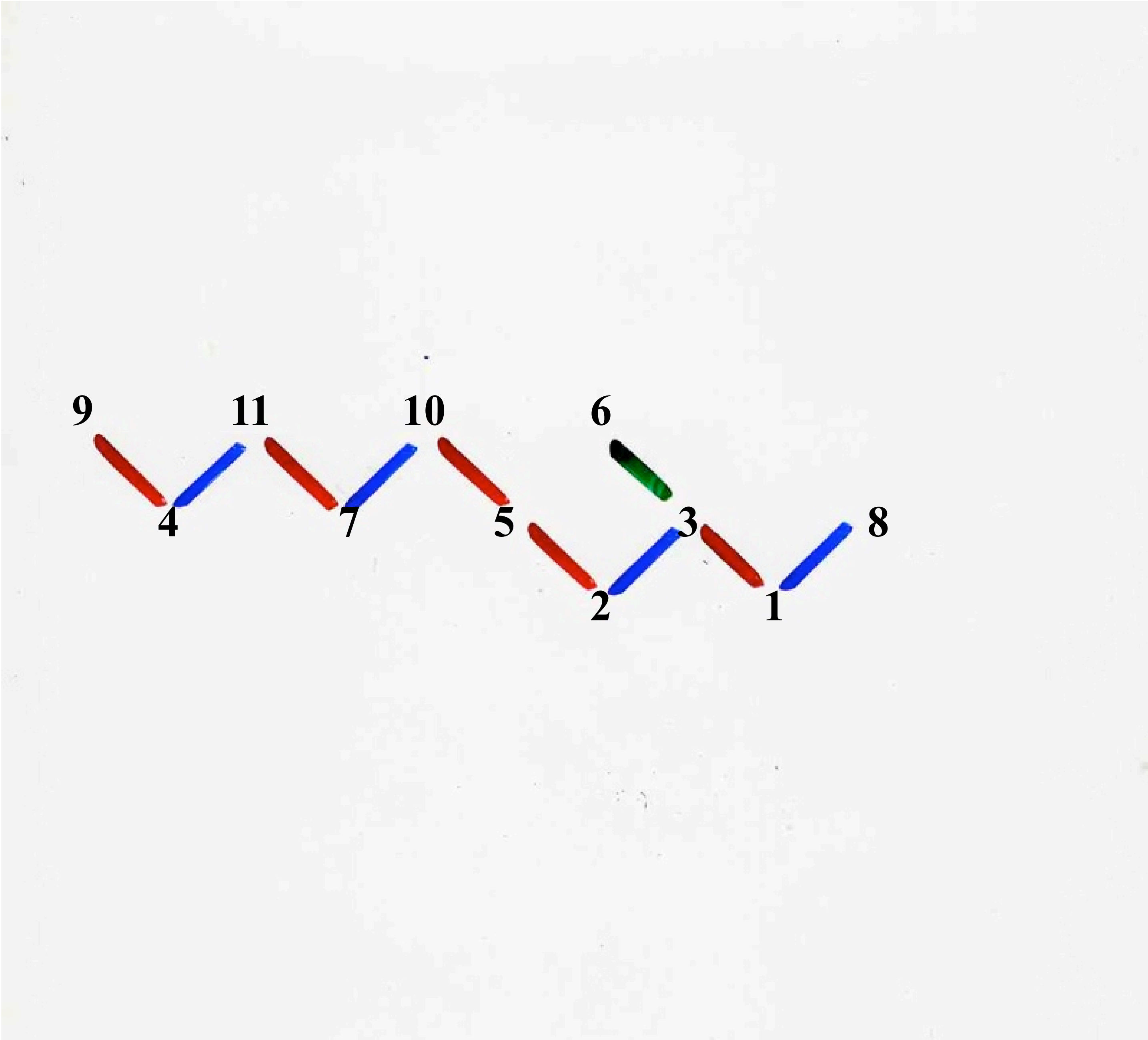
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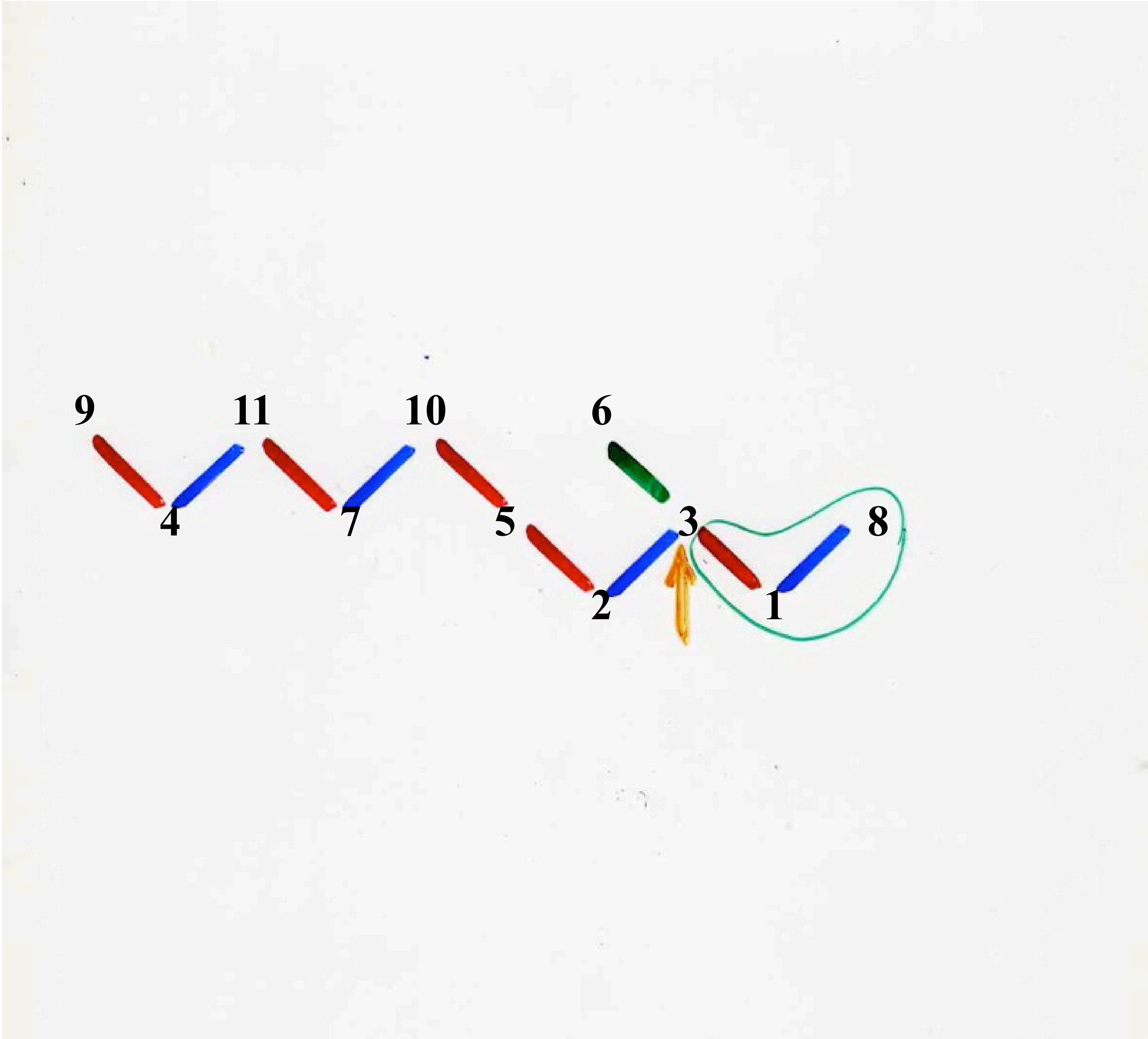

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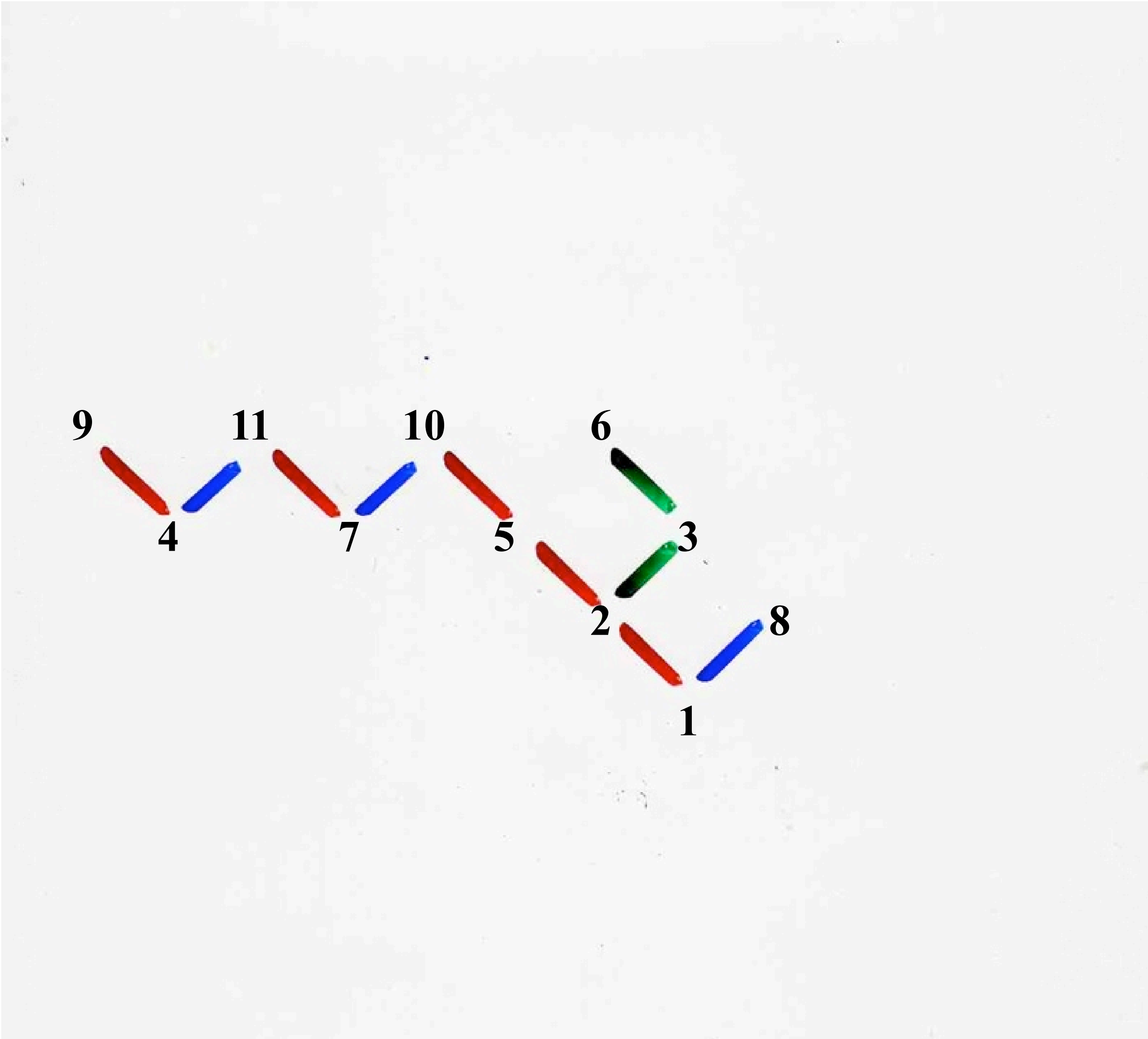
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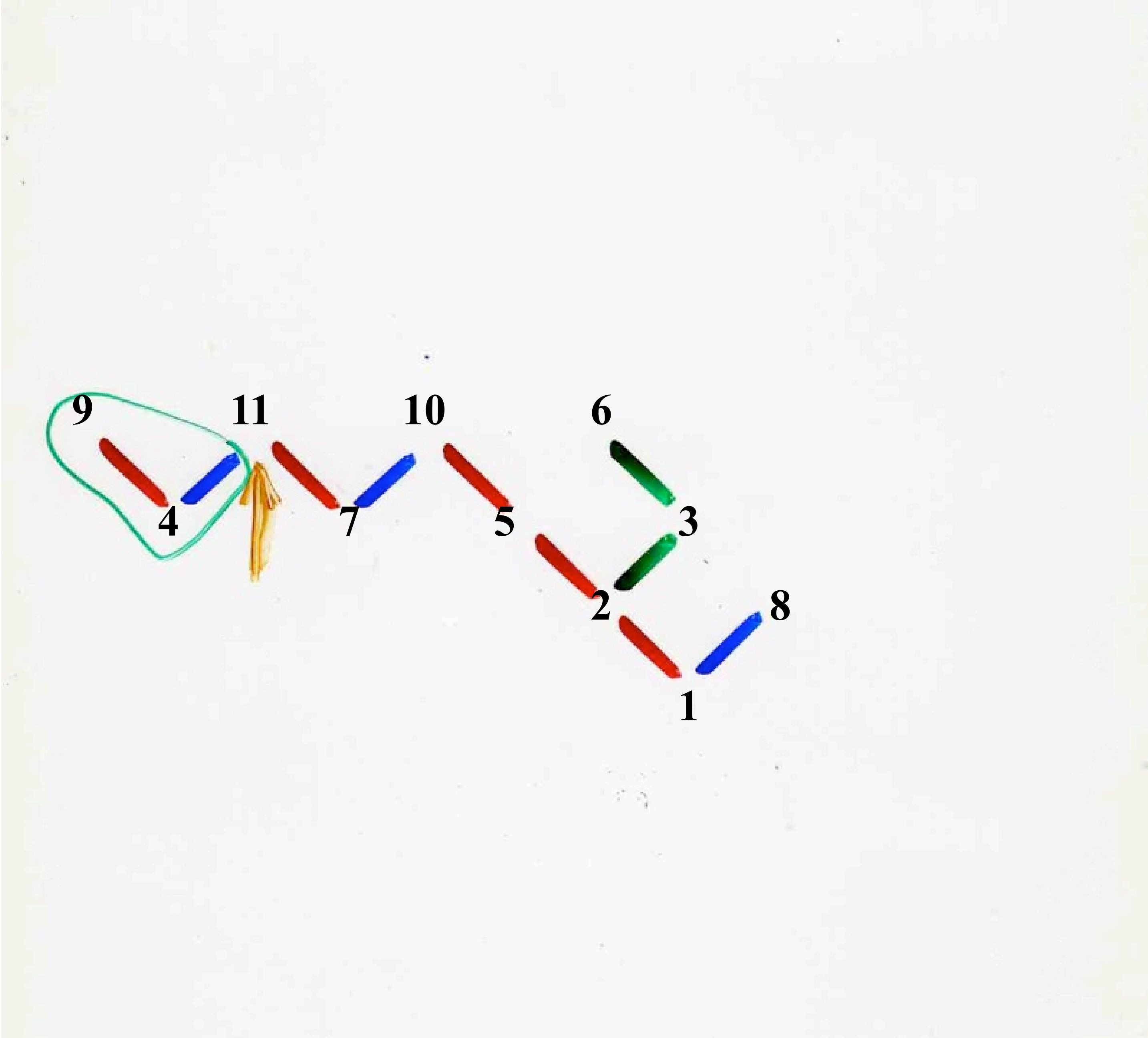
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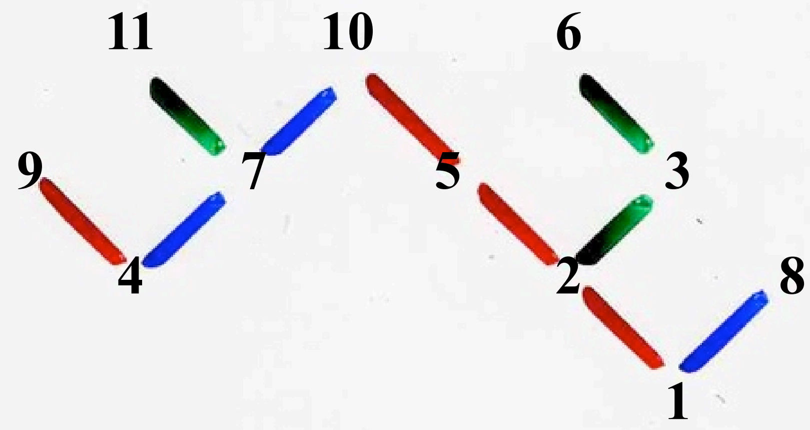
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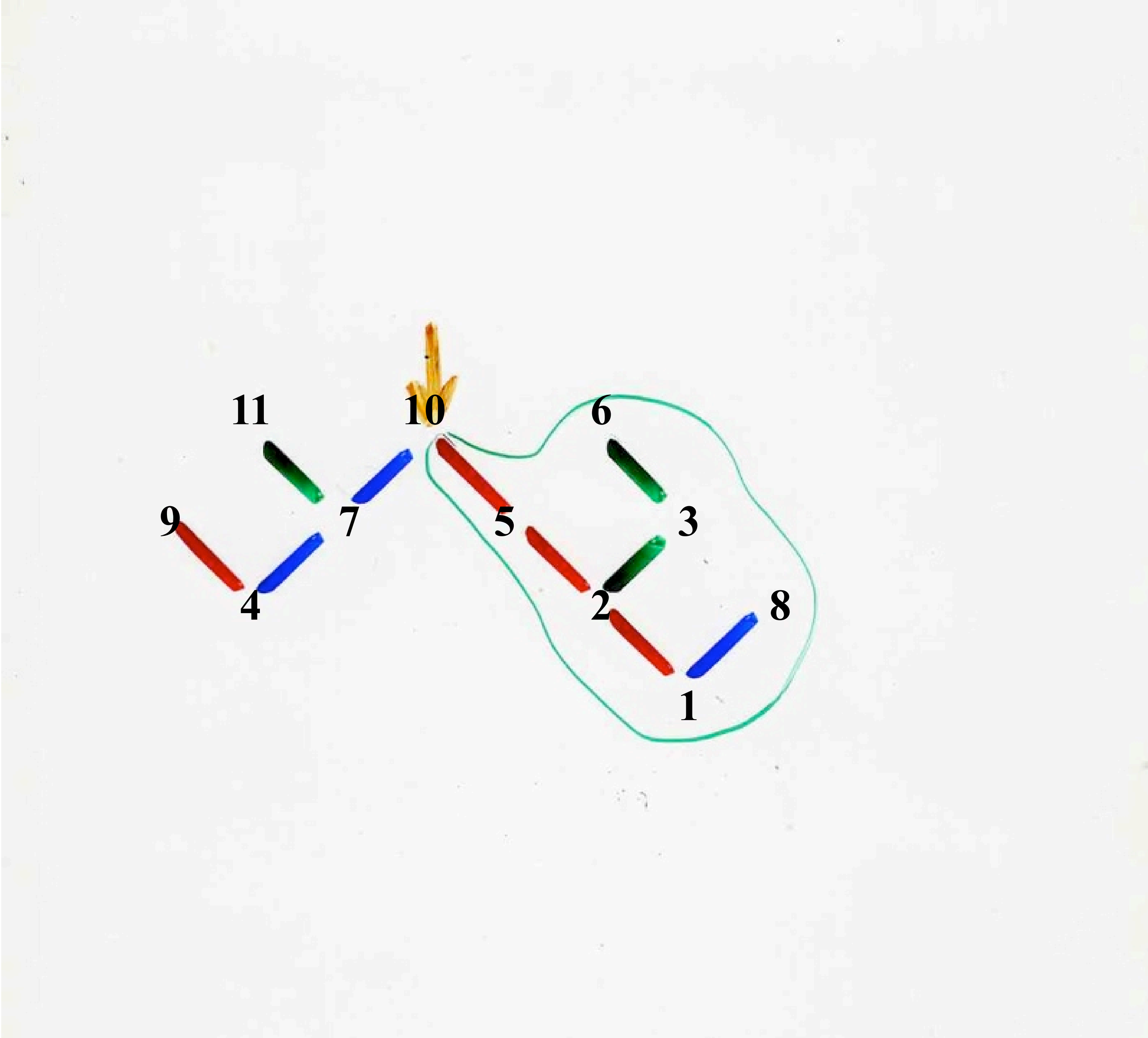
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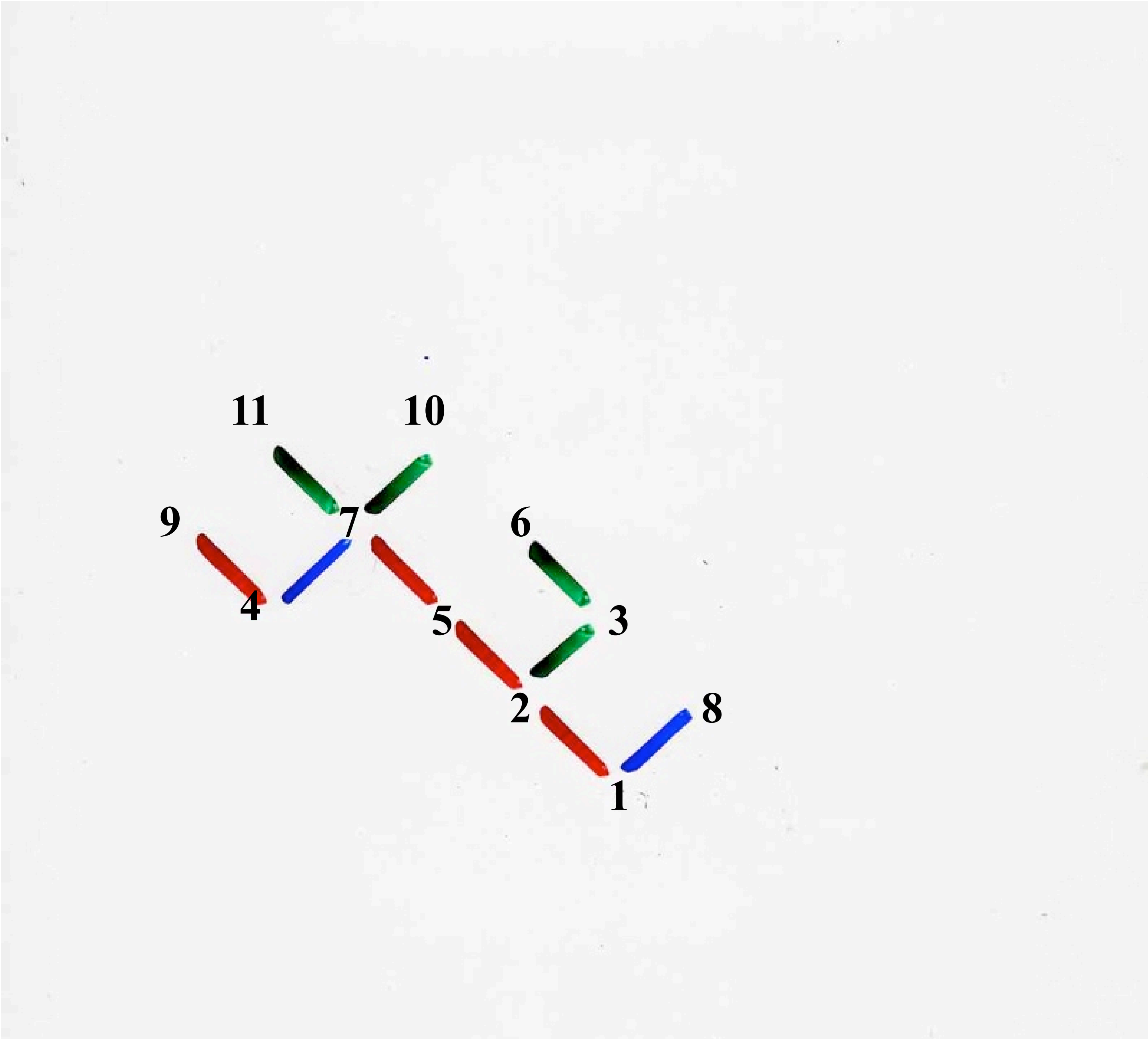

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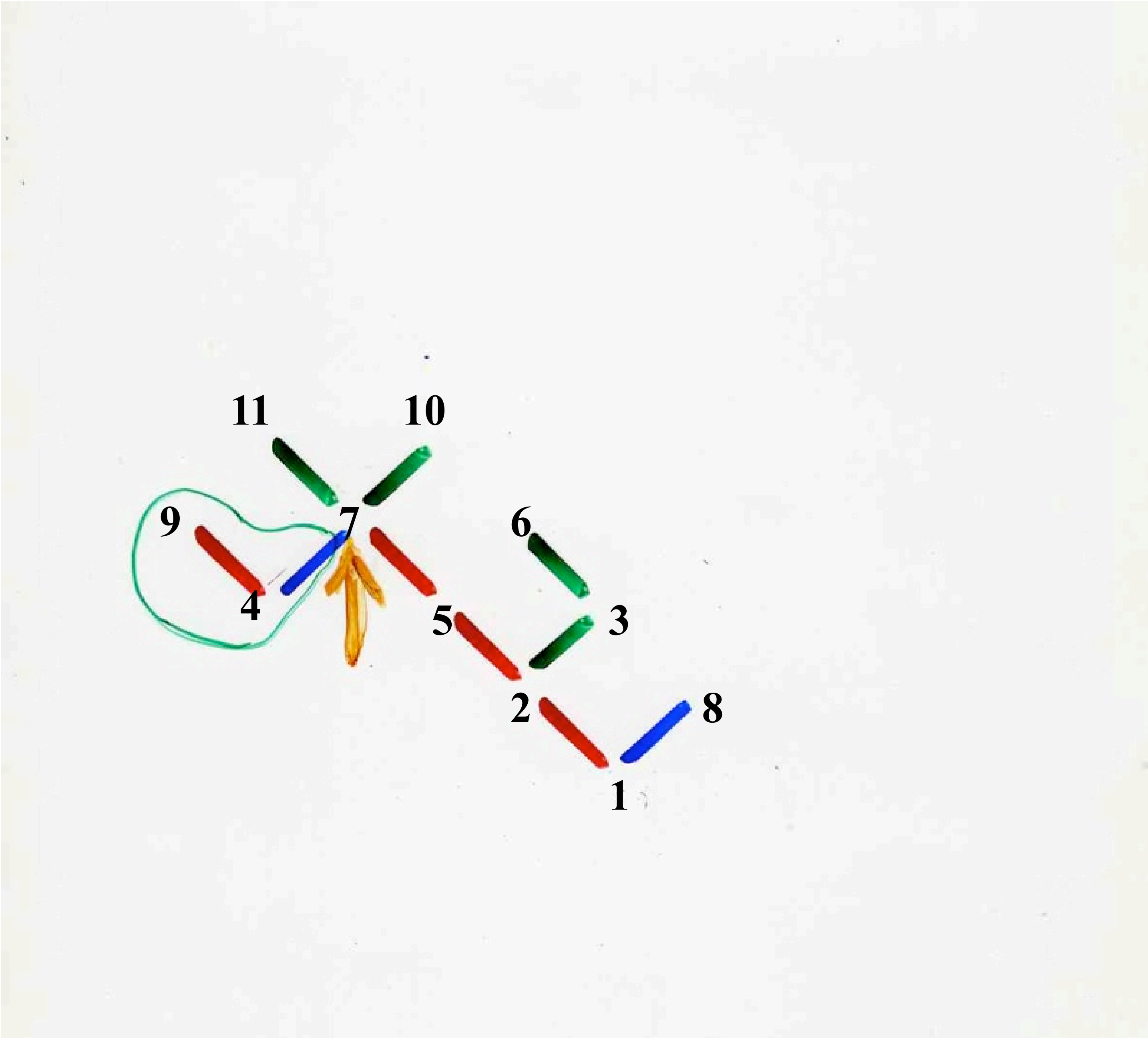
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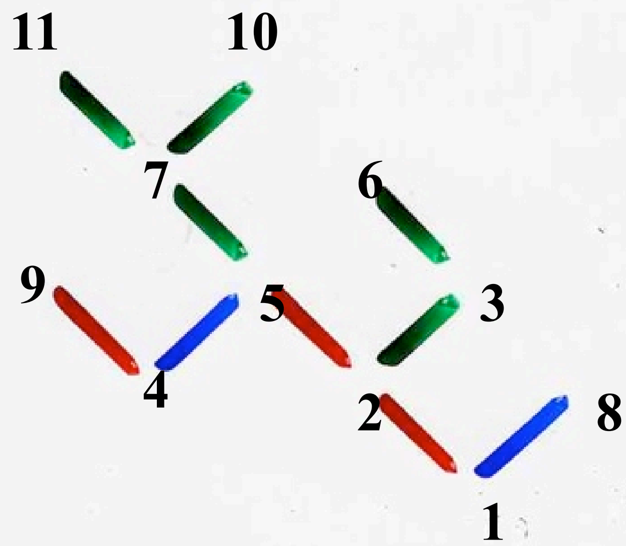
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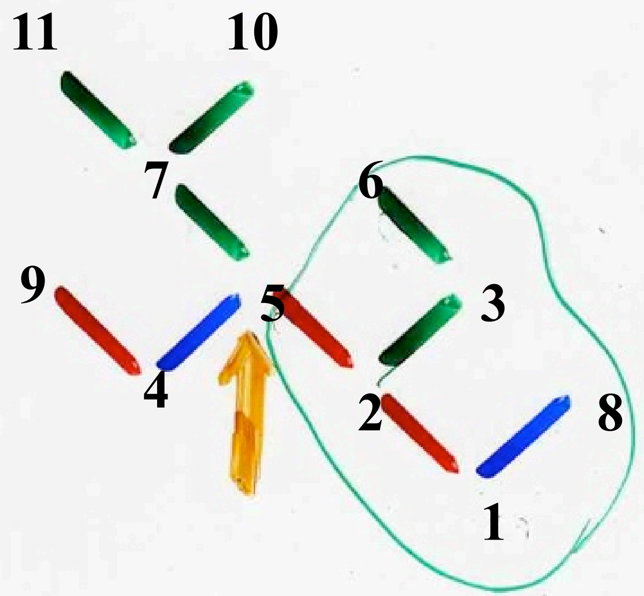
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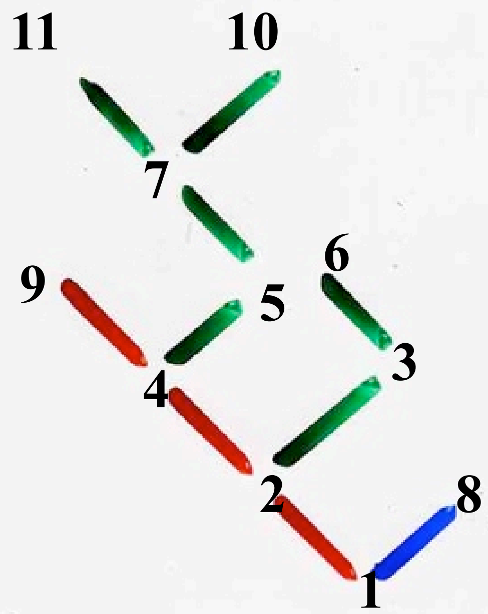
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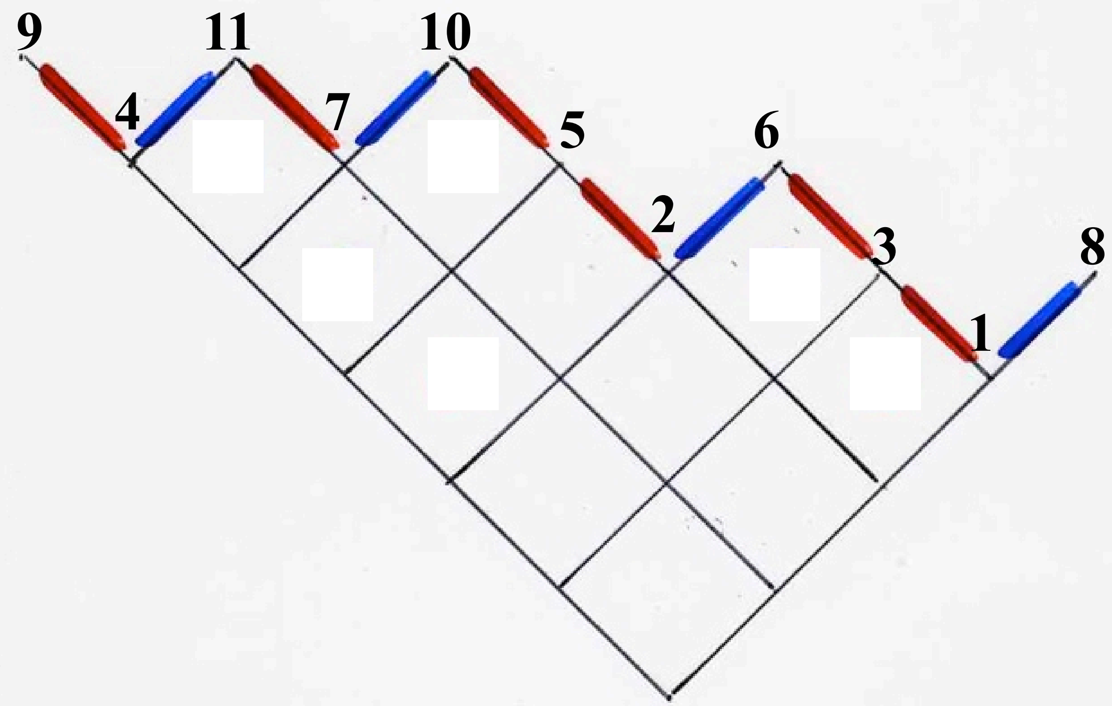
“jeu de taquin”

pour un arbre binaire croissant

$\longrightarrow$

tableau alternatif de Catalan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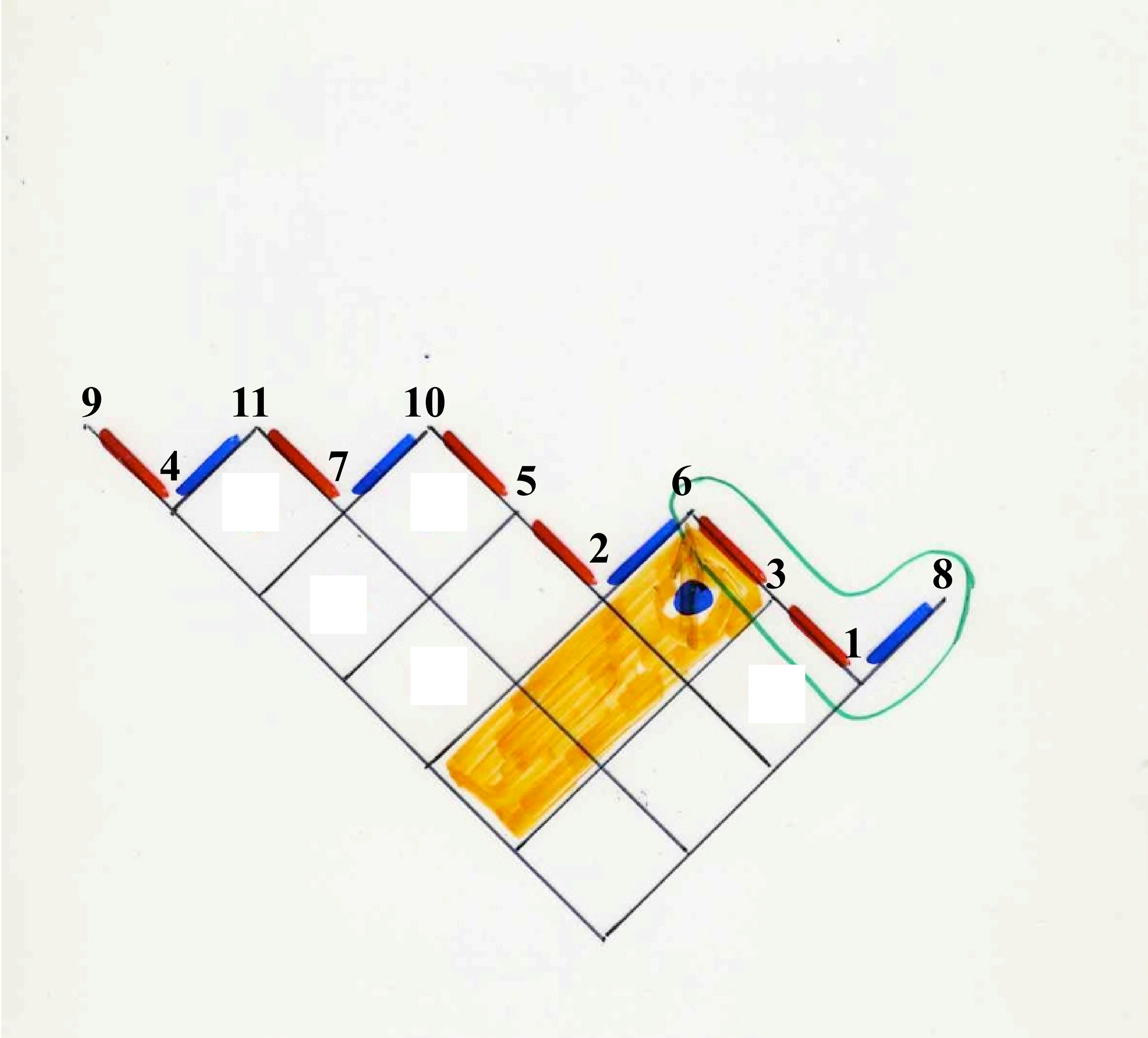
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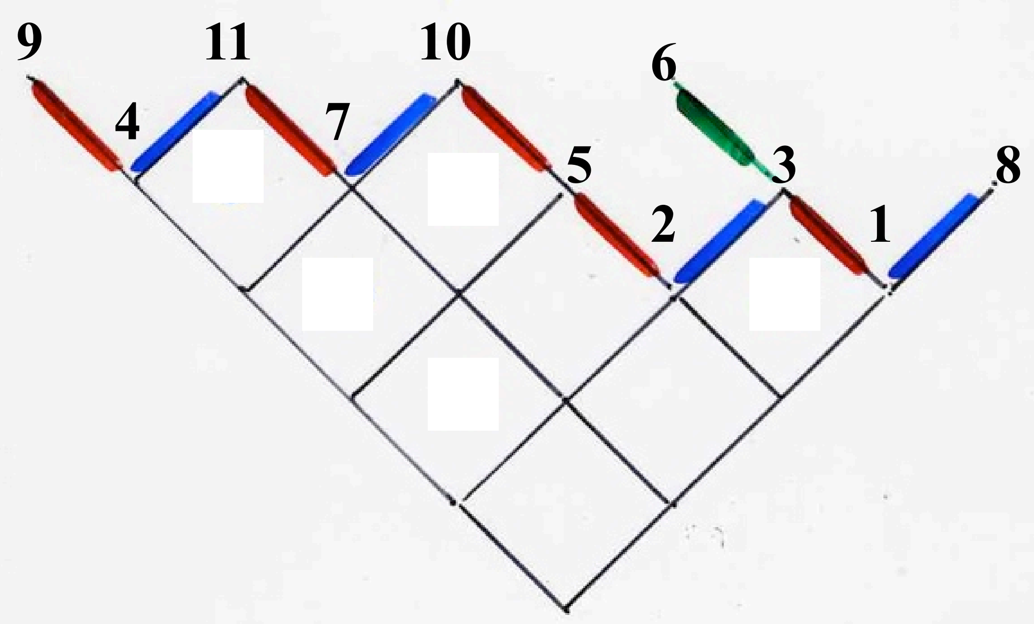
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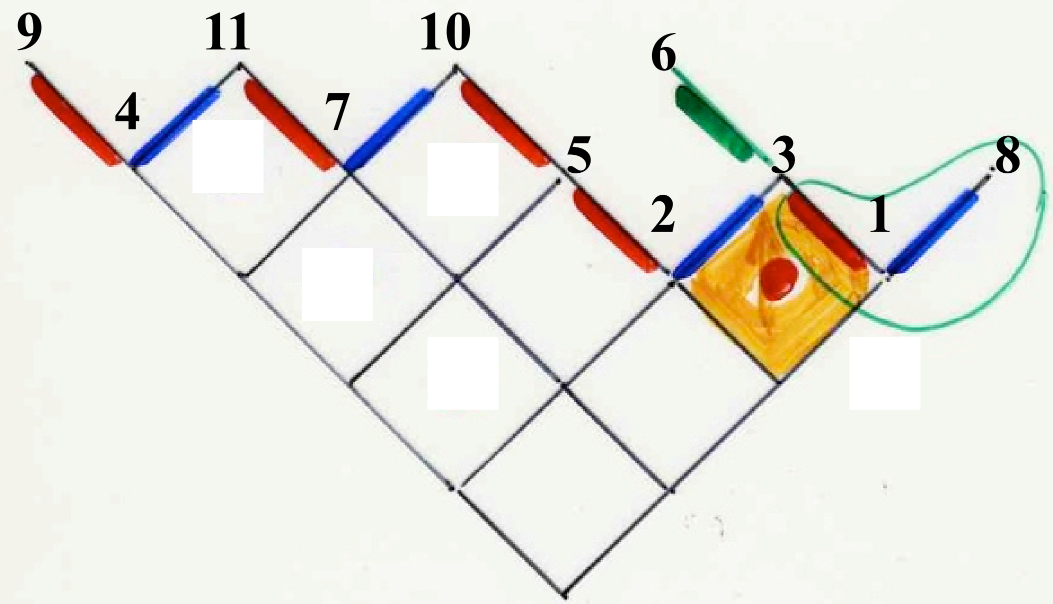

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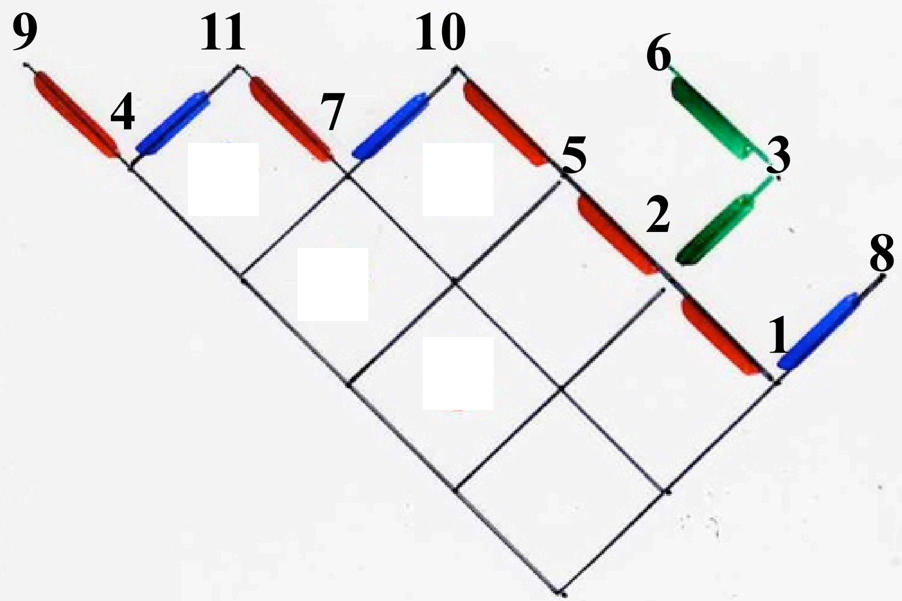
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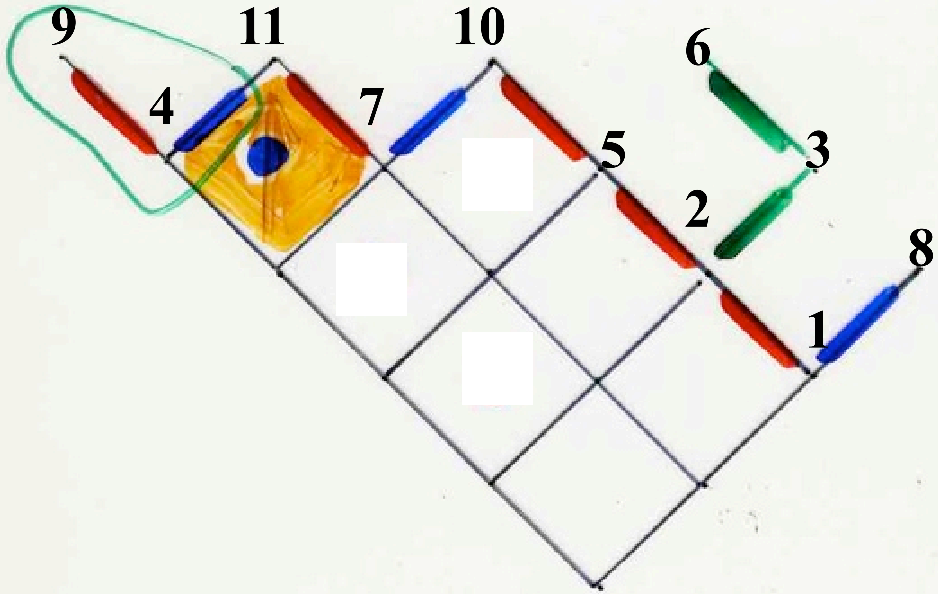
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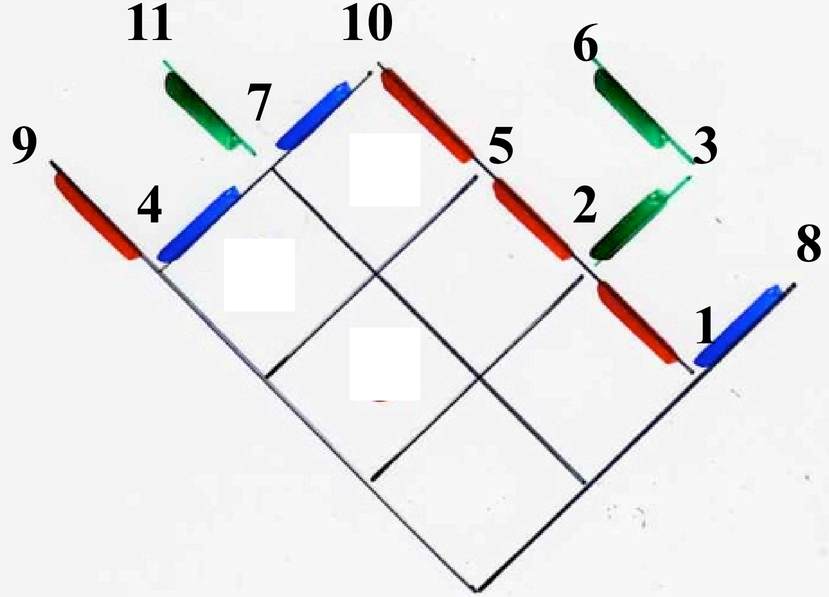

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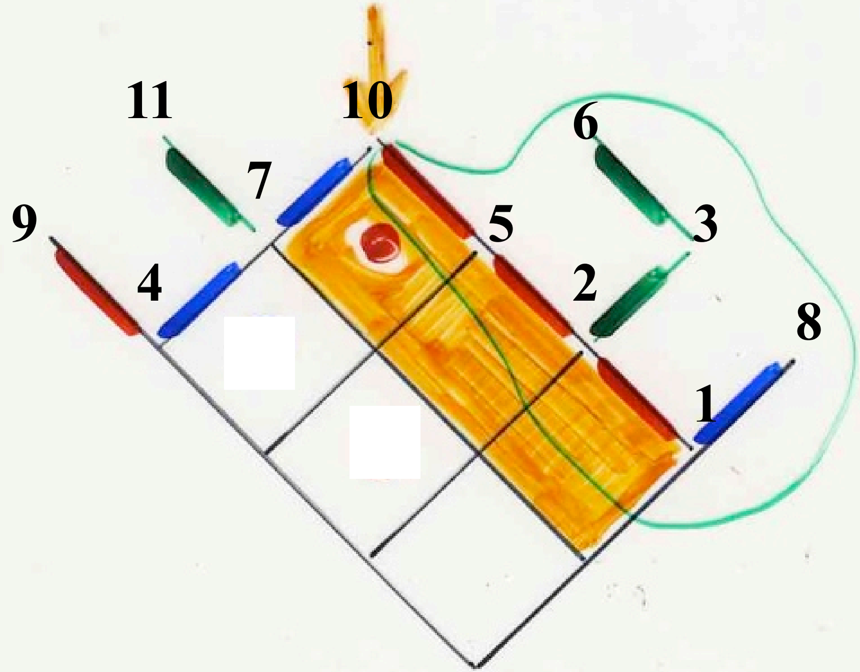
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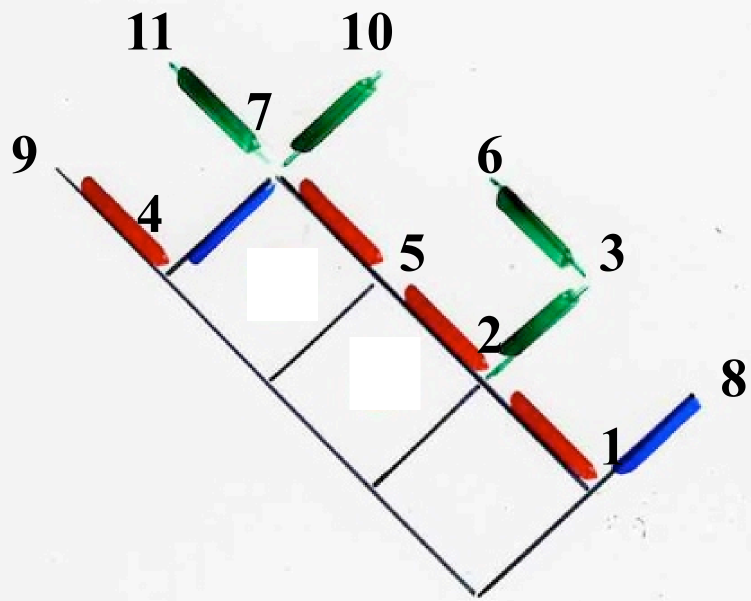

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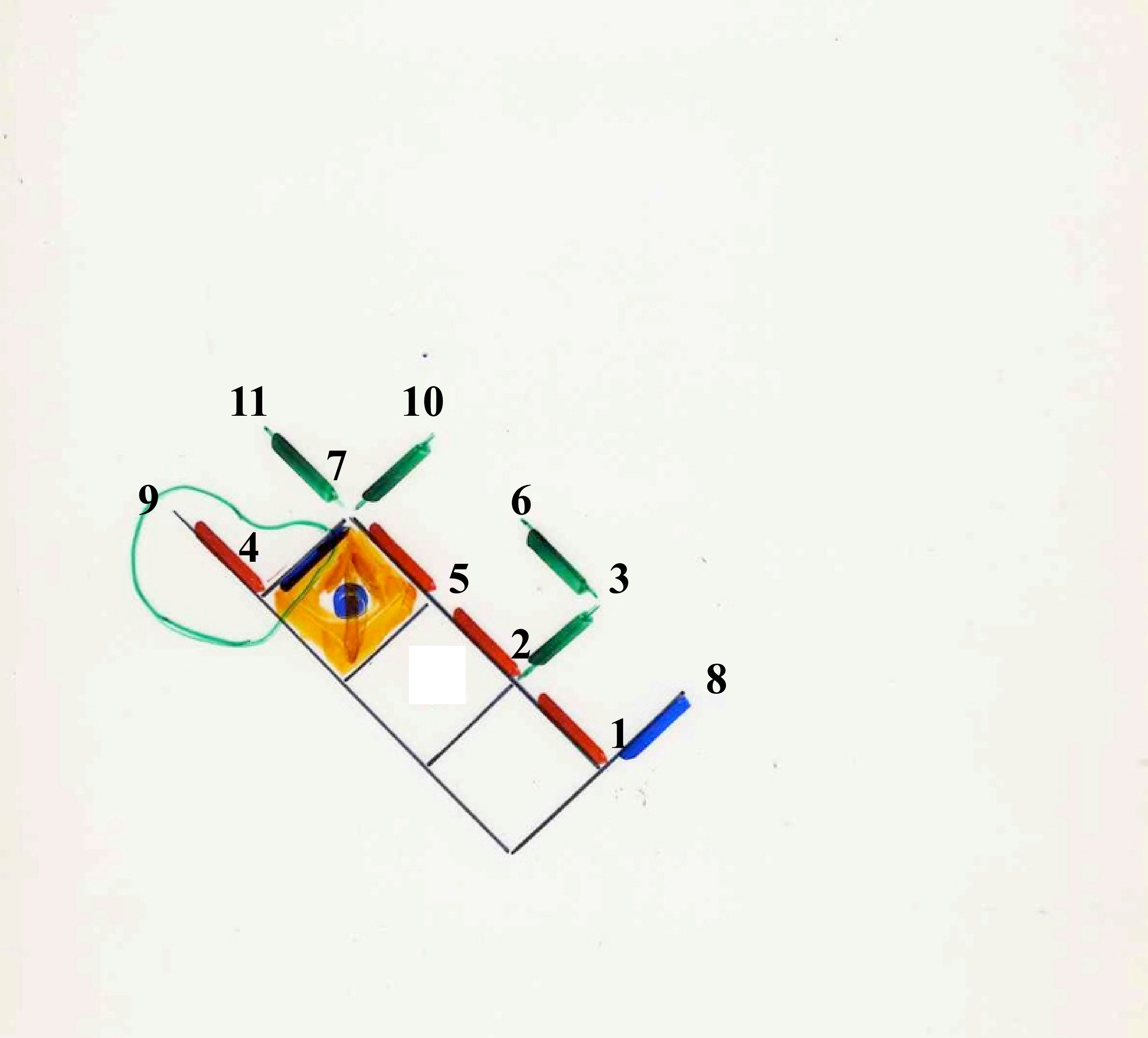

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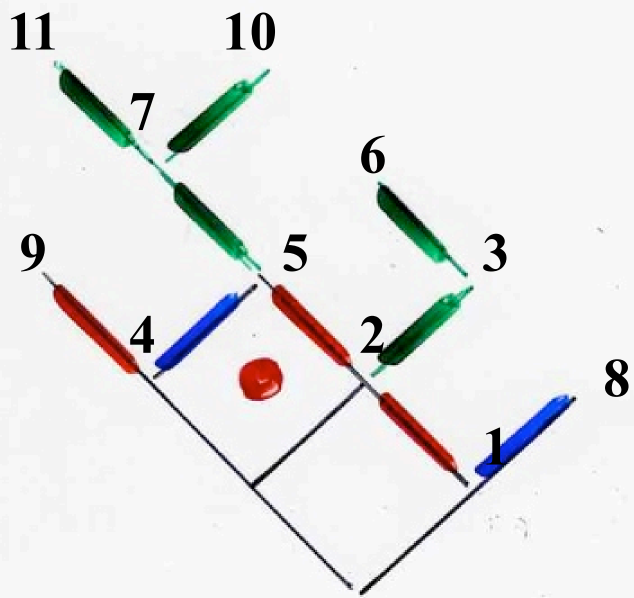
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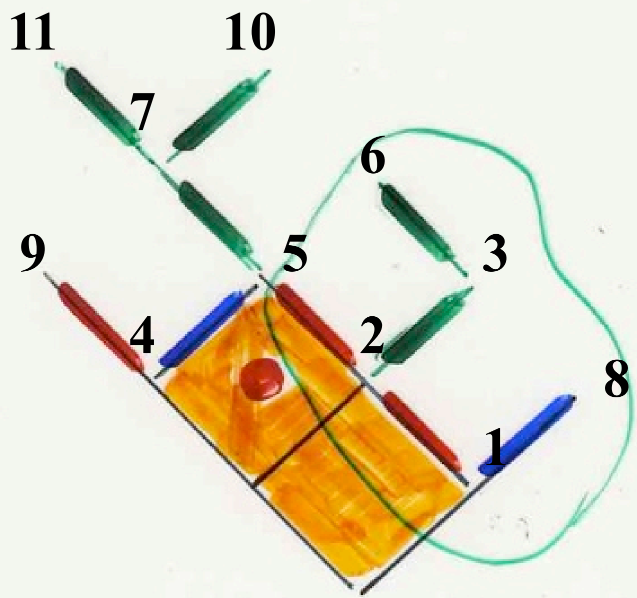
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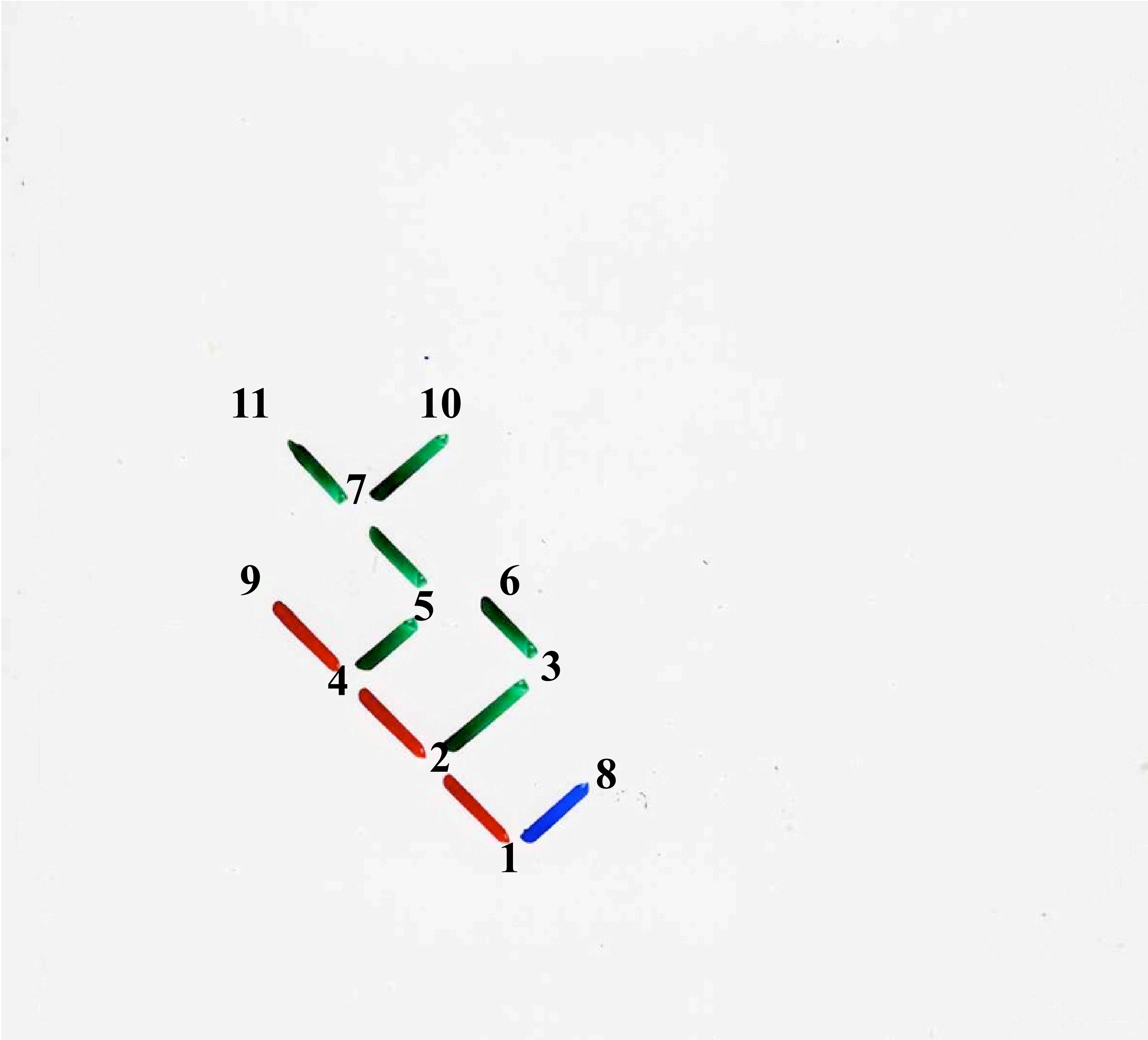
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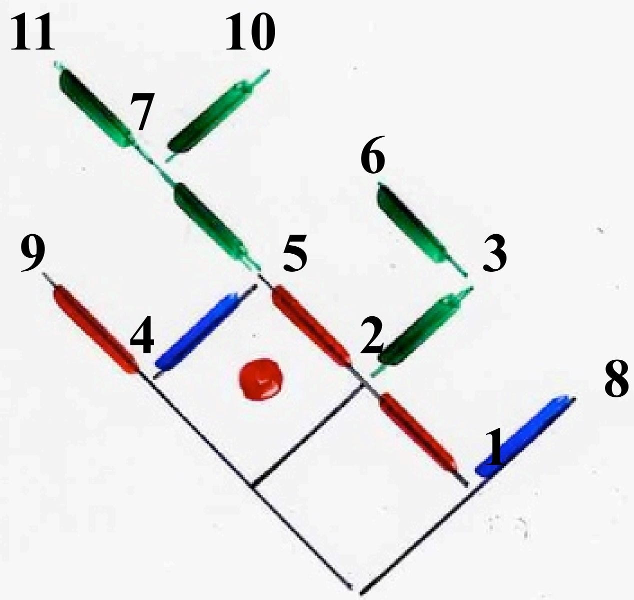
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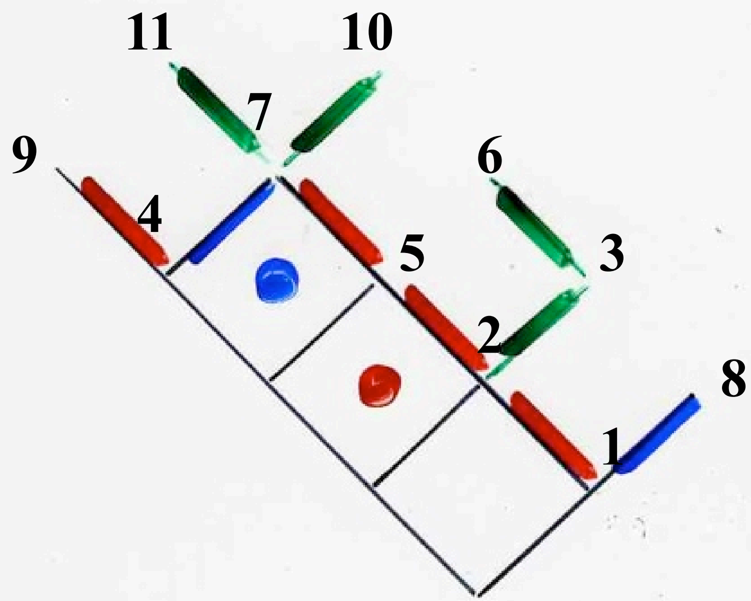

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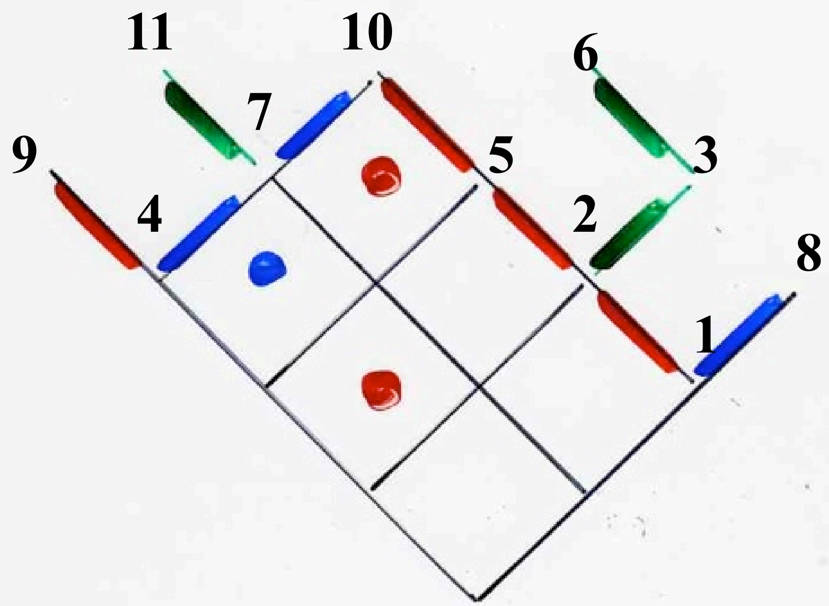
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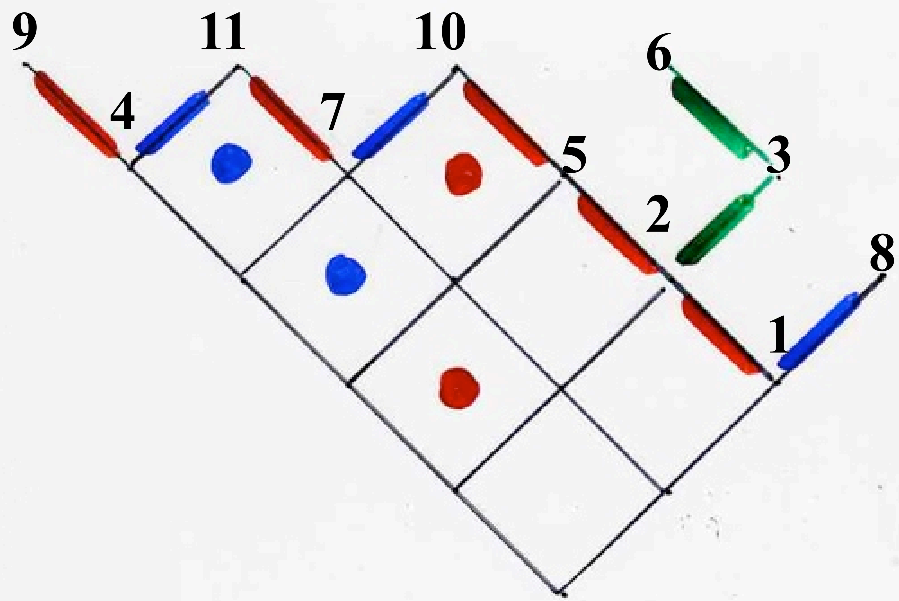

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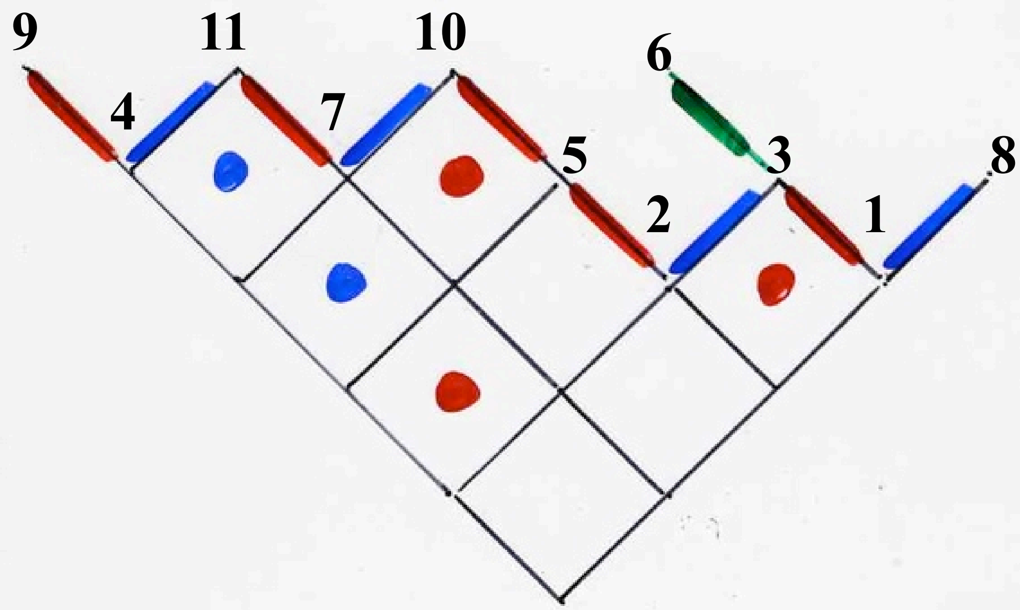
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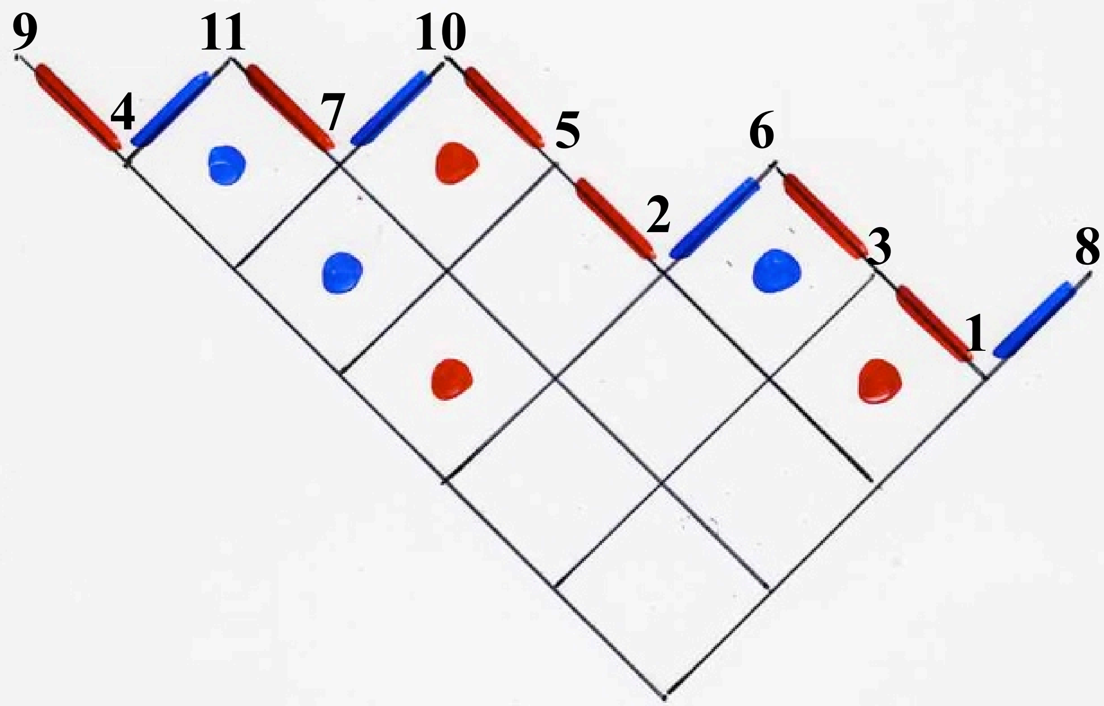
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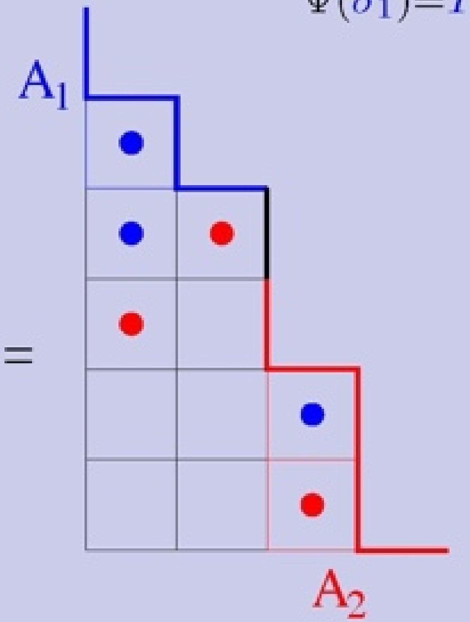
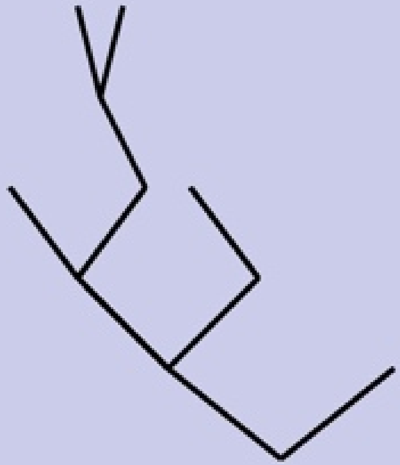
Une autre preuve du produit des tableaux alternatifs de Catalan

La preuve ($\supset$)

$$\overline{\Psi}(T_1) * \overline{\Psi}(T_2) = \sum_{\Psi(\sigma_1)=T_1} \sigma_1 * \sum_{\Psi(\sigma_2)=T_2} \sigma_2 = \sum \sigma \overset{?}{=} \sum_{T\ldots} \sum_{\Psi(\sigma)=T} \sigma$$

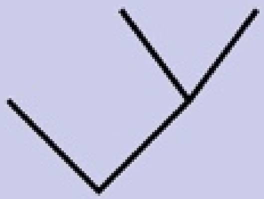
A =


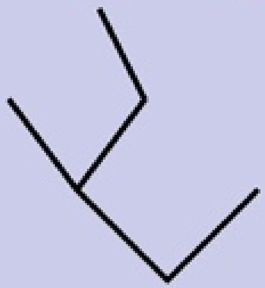


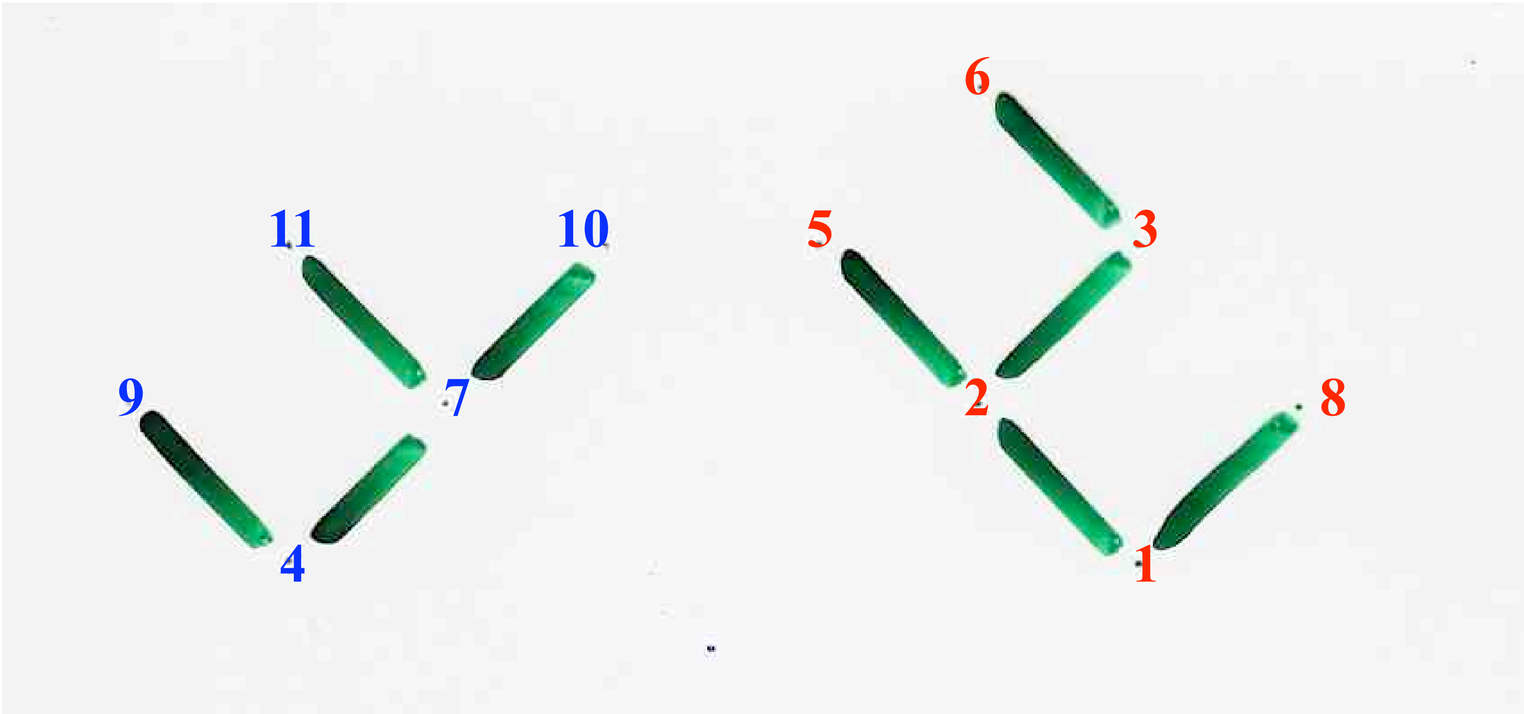
$\sigma = 9\ 4\ 11\ 7\ 10\ 5\ 2\ 6\ 3\ 1\ 8$

$\sigma_1 = 9\ 4\ 11\ 7\ 10$ $\qquad$ $\sigma_2 = 5\ 2\ 6\ 3\ 1\ 8$

$\Psi(\sigma_1) = T_1$ $\qquad$ $\Psi(\sigma_2) = T_2$



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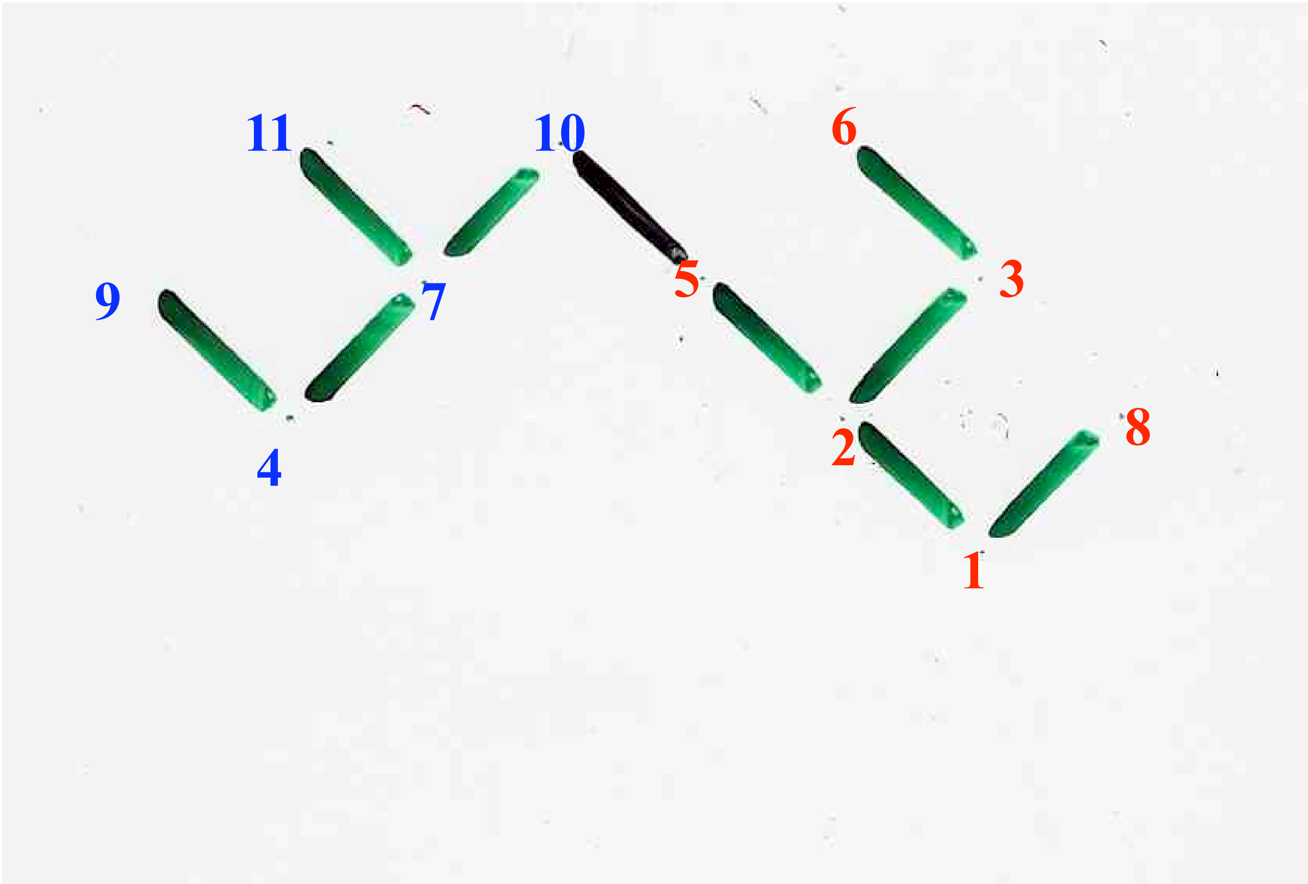
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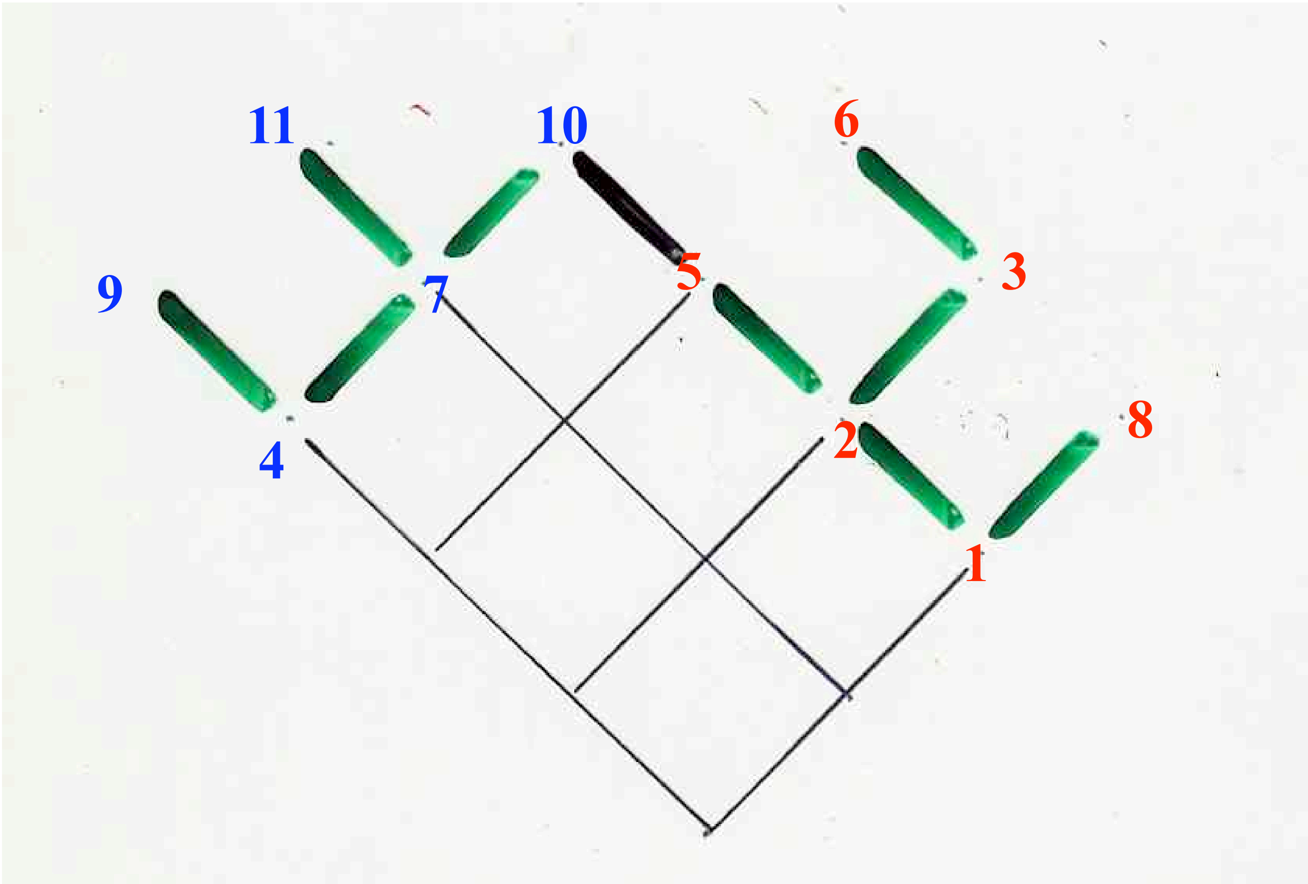

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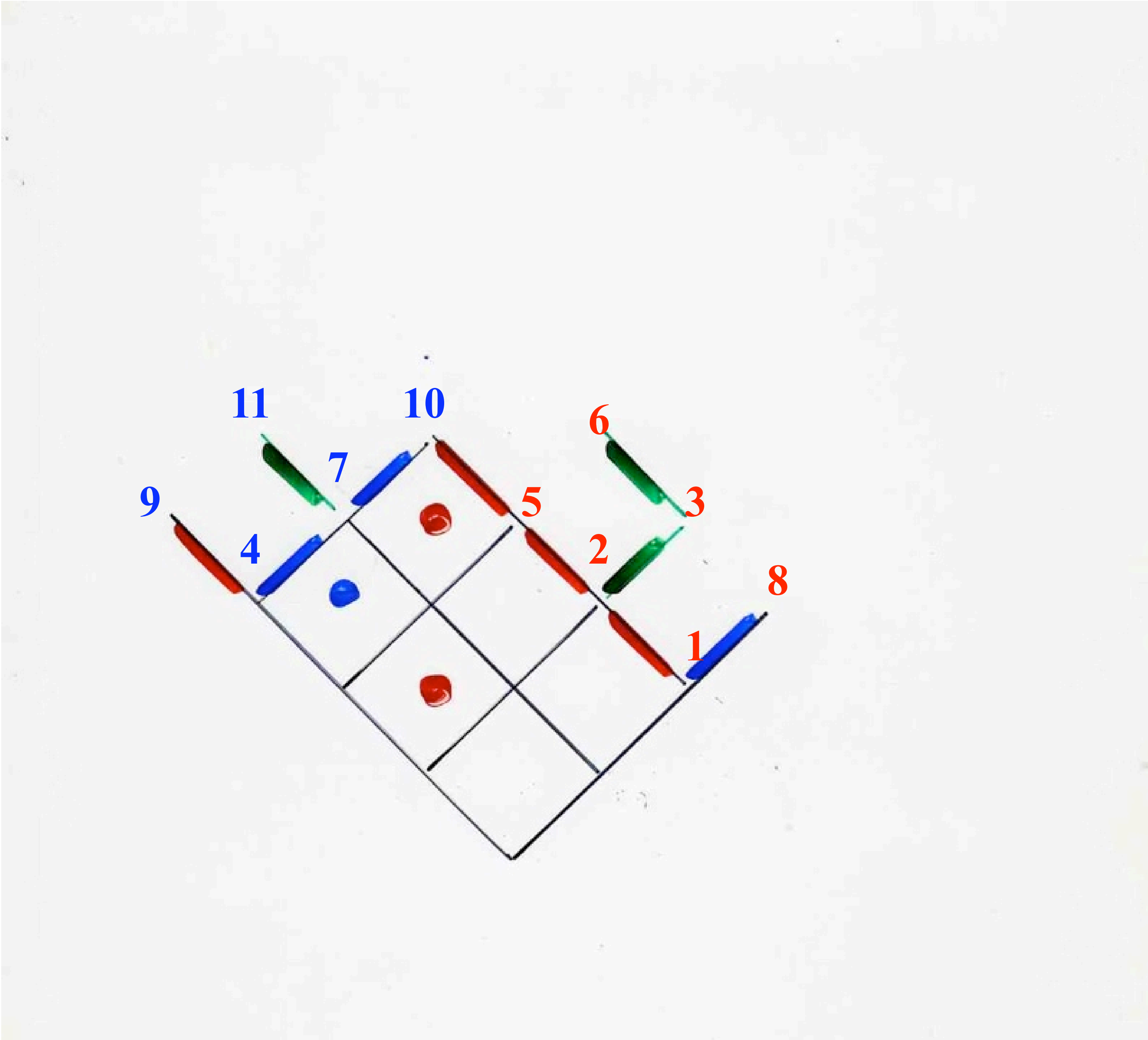
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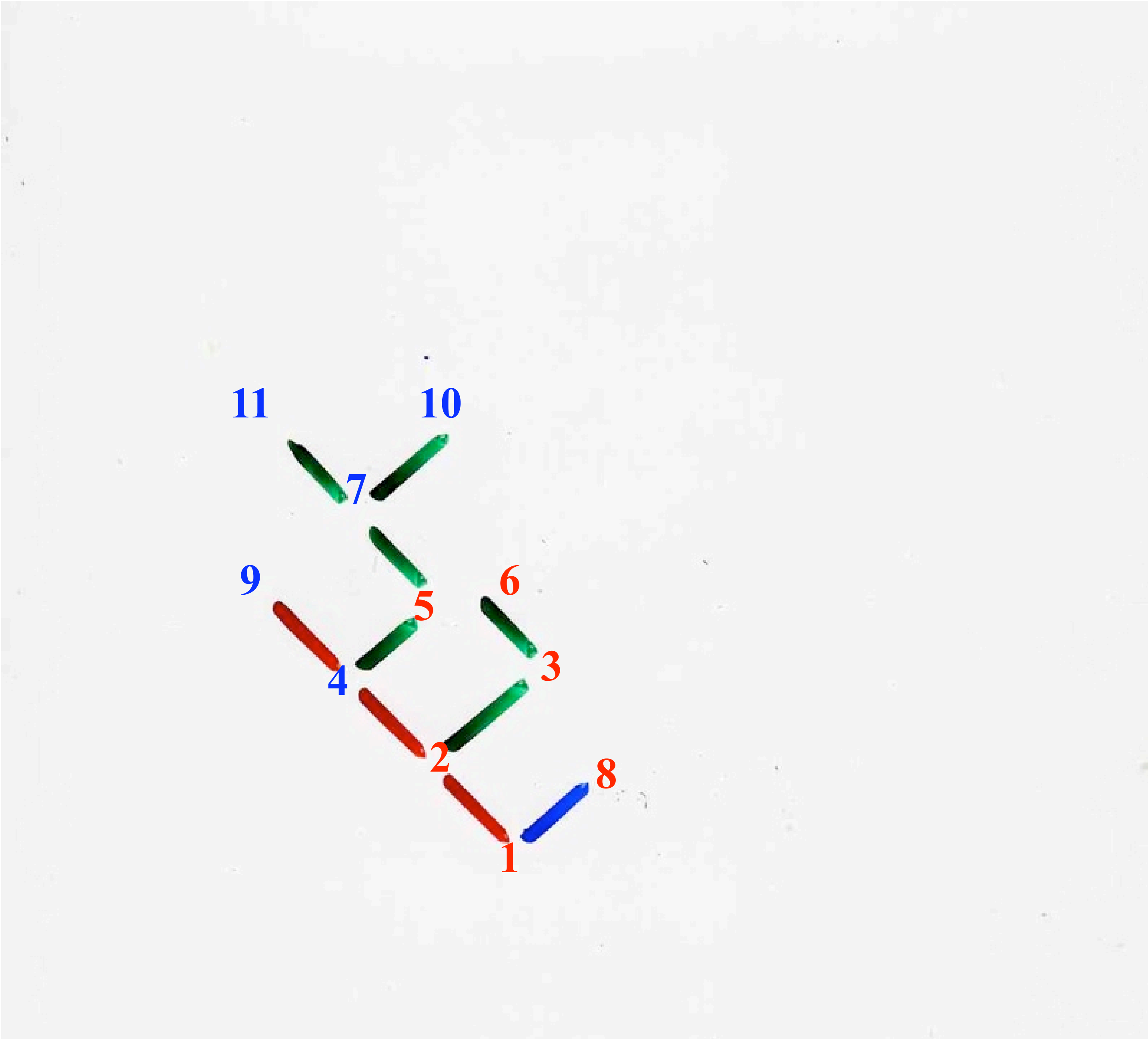
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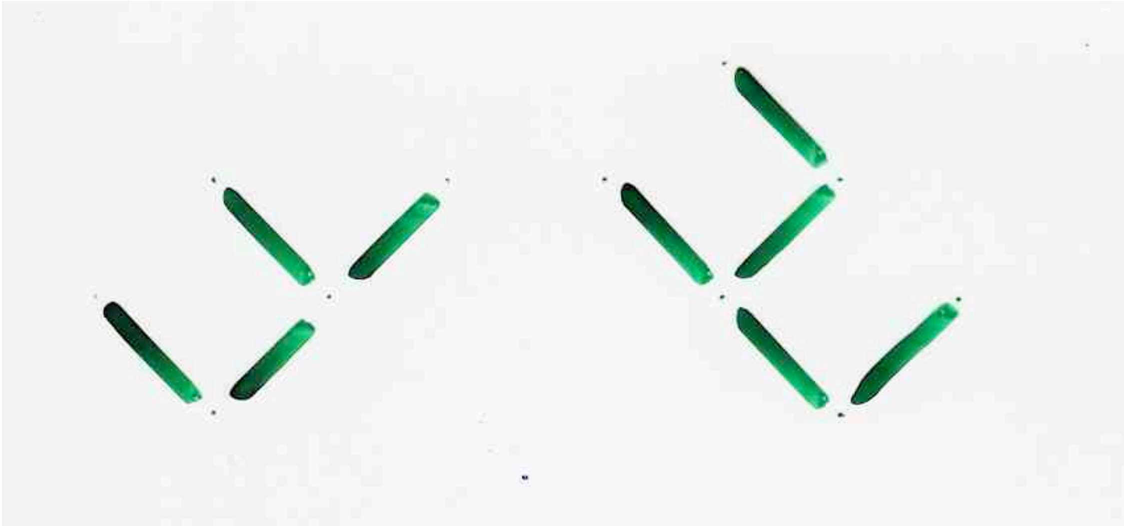
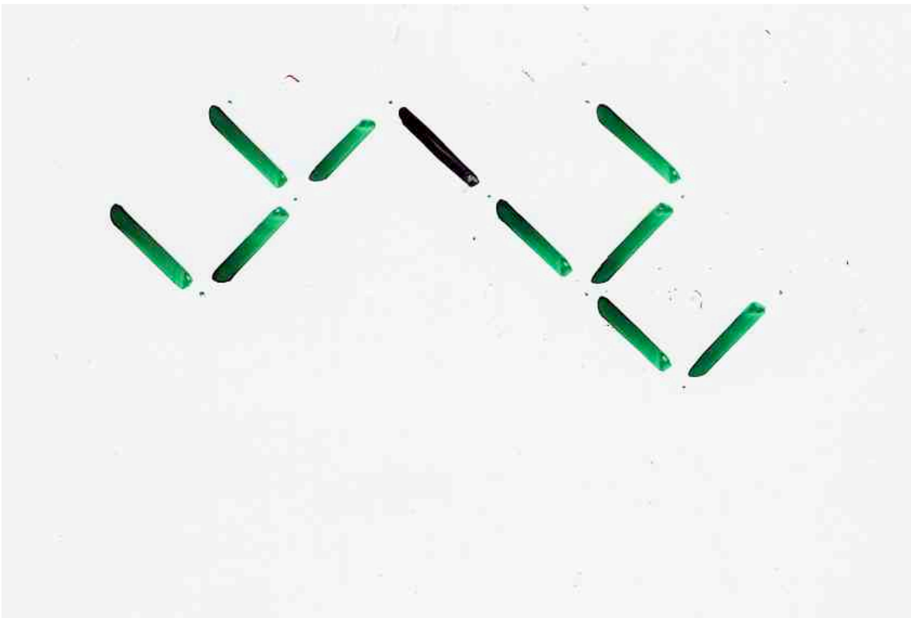
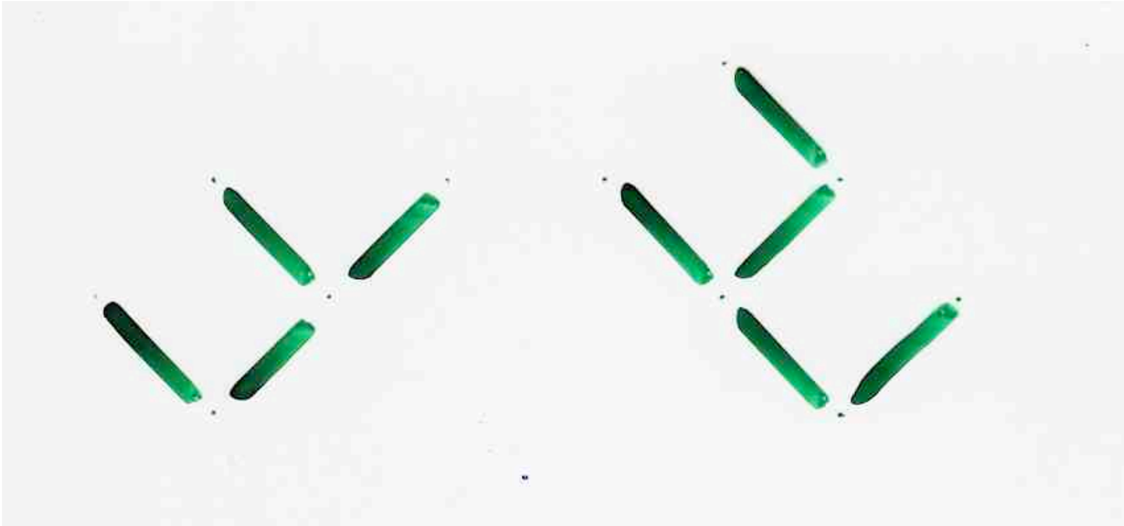
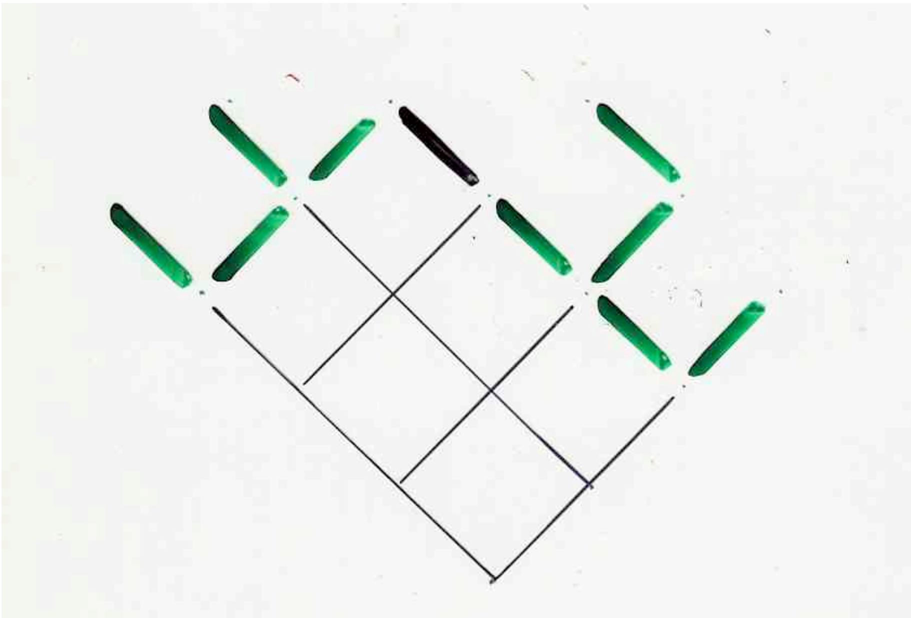
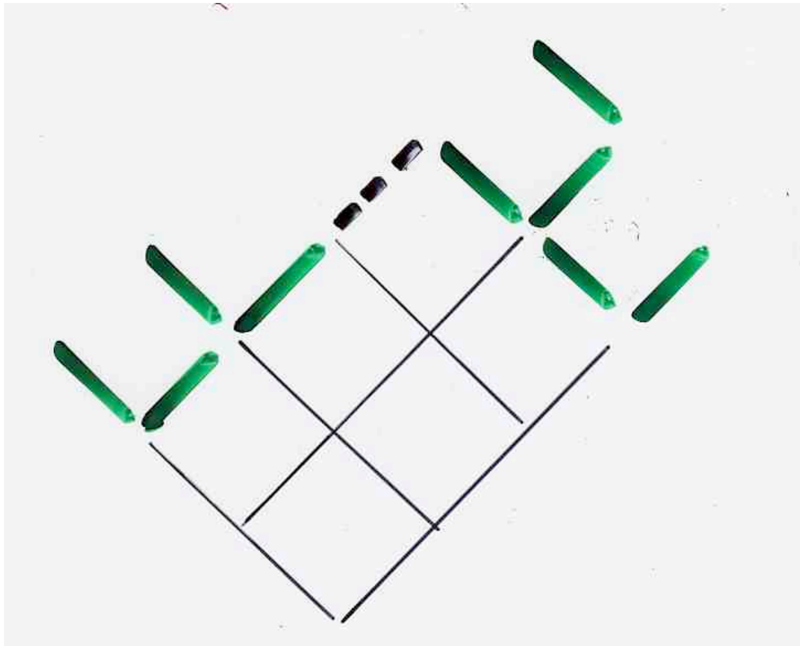


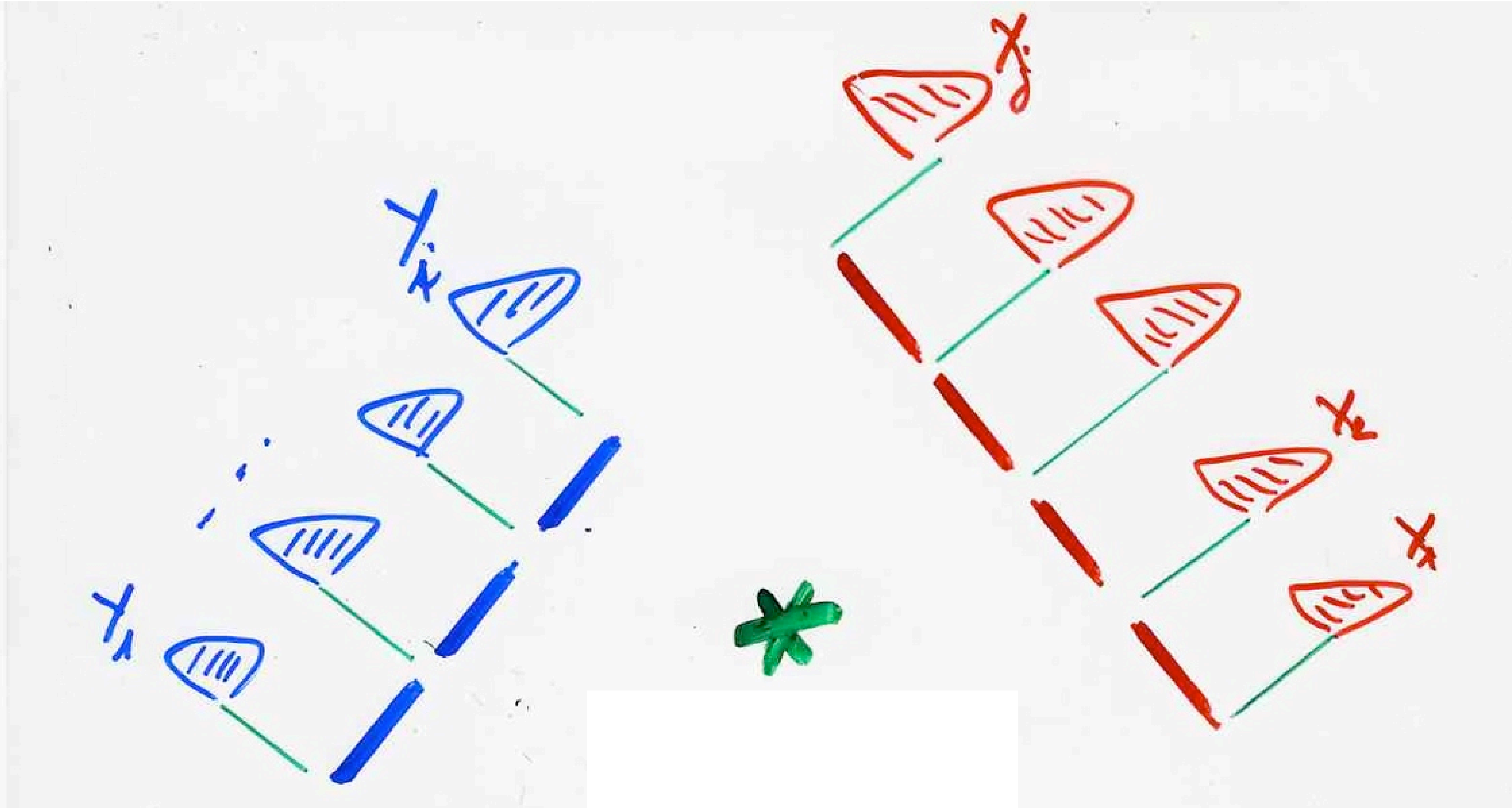




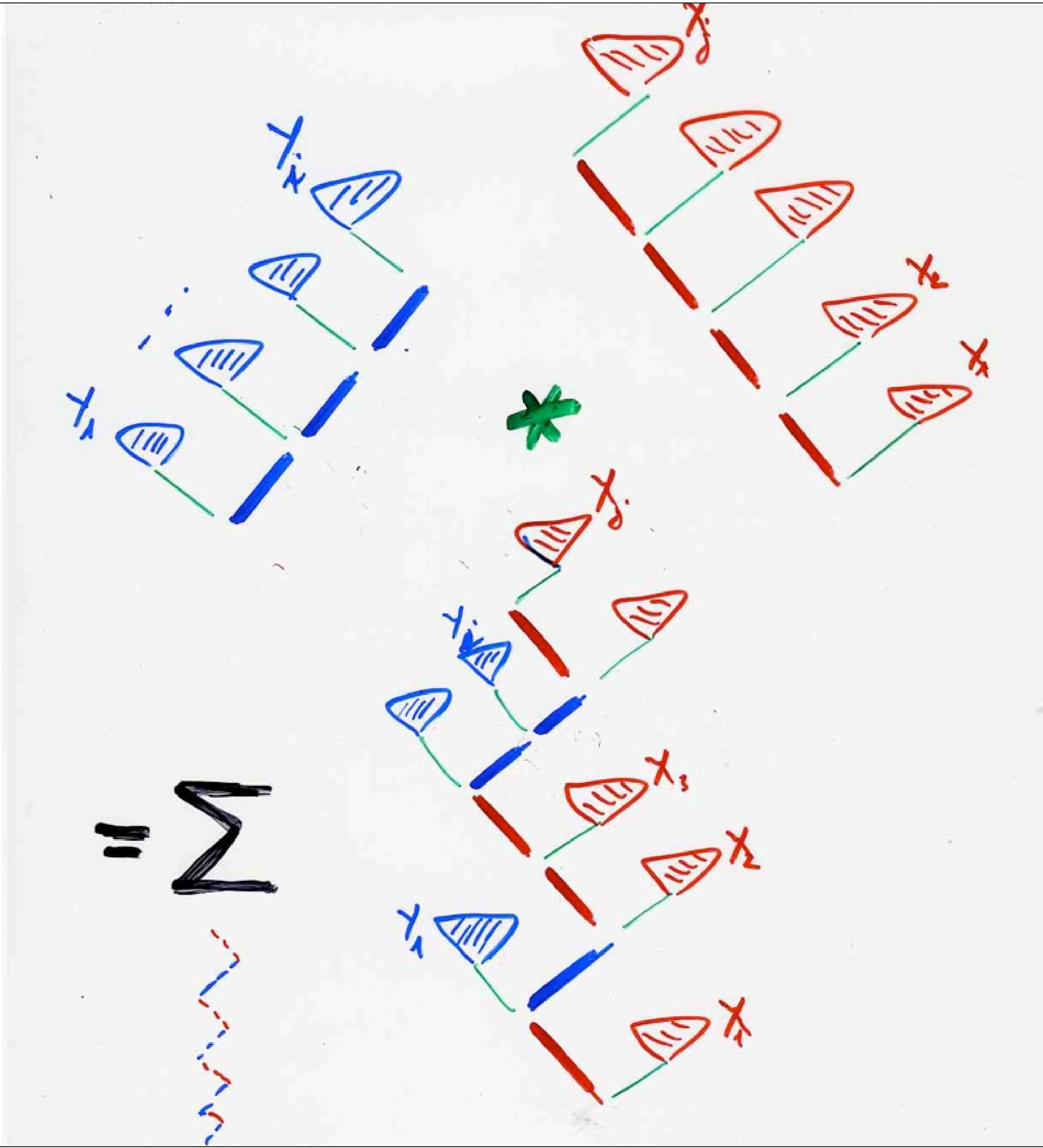




“jeu de taquin”
pour les arbres binaires



y_n
y_1
x_j
x_2
x_1



y_k
y_1
x_j
x_2
x_1
x_j
y_k
x_3
x_2
y_1
x_1
$= \sum$

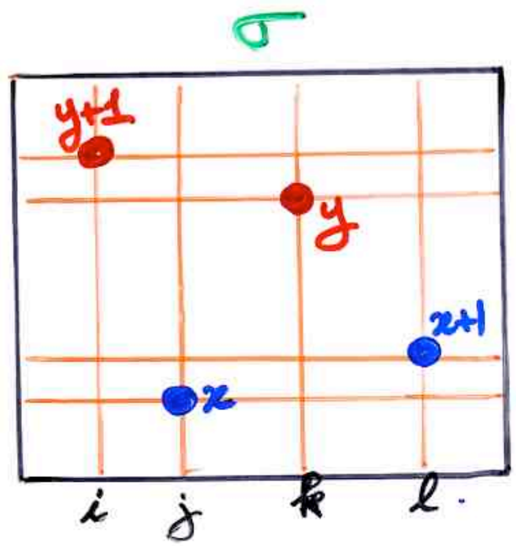
§5 permutations de Bernardi

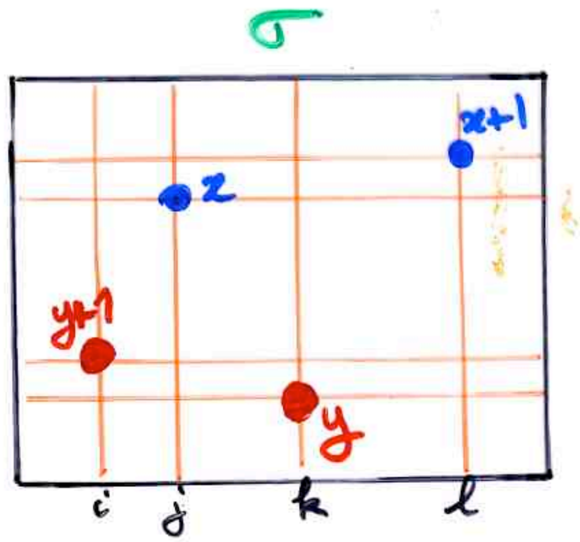
Permutations

with no subsequence of the type

$$\cdots\ (y+1)\ \cdots\ x\ \cdots\ y\ \cdots\ (x+1)\ \cdots$$

ex: $\sigma = 6\ 4\ 5\ 3\ 9\ 7\ 8\ (10)\ 1\ 2$





Permutations

with no subsequence of the type

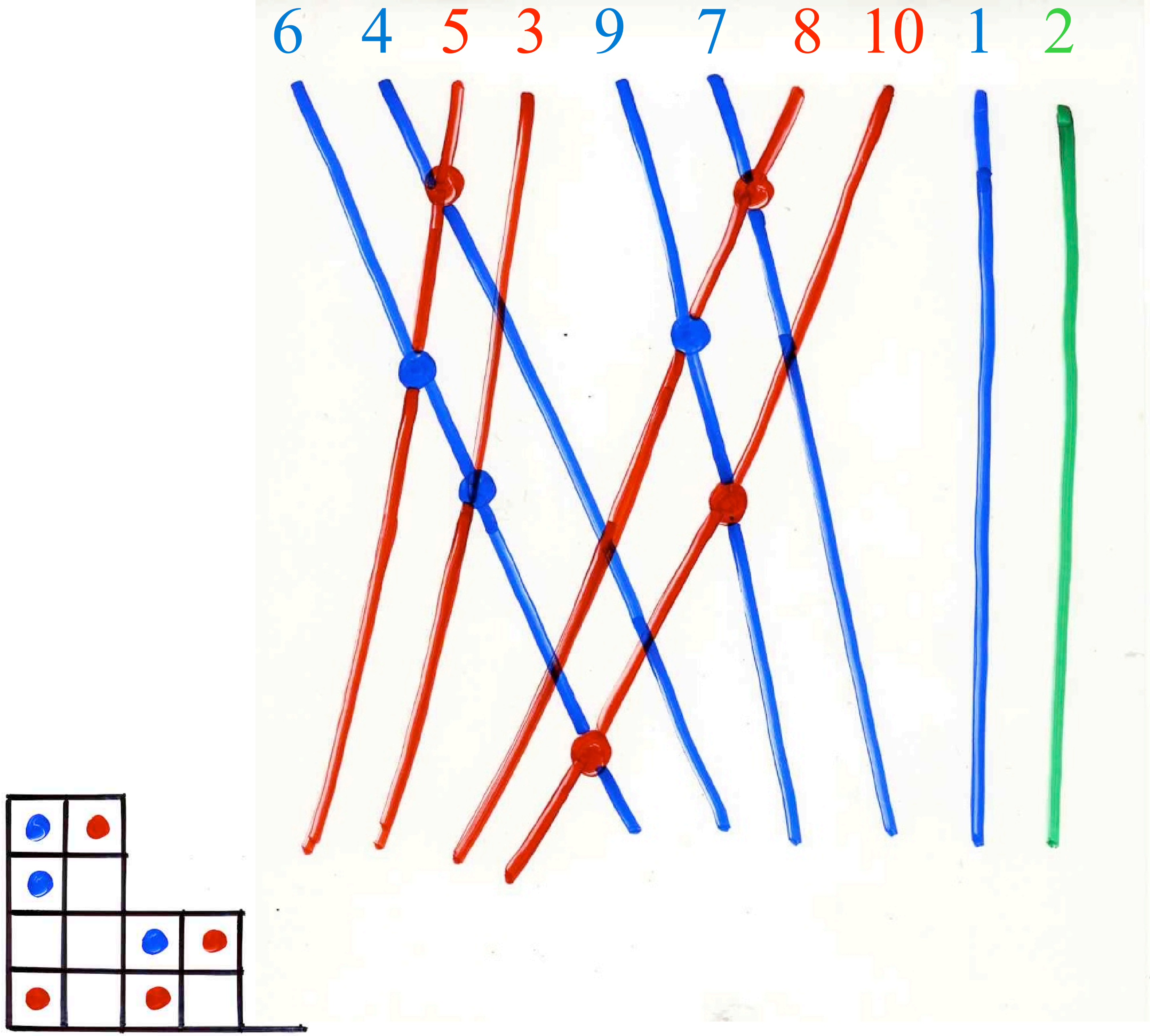
$$\cdots (y+1) \cdots x \cdots y \cdots (x+1) \cdots$$

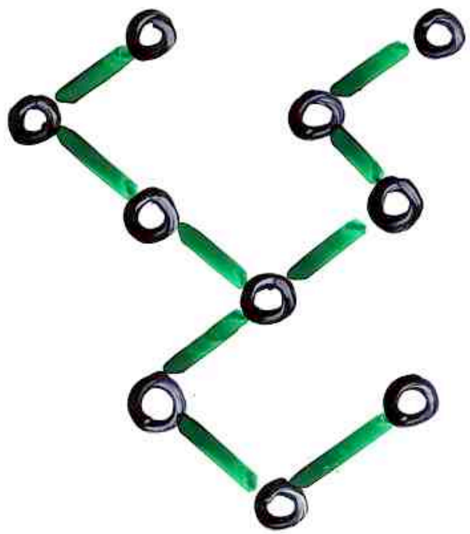
ex: $\sigma = 6\ 4\ 5\ 3\ 9\ 7\ 8\ (10)\ 1\ 2$

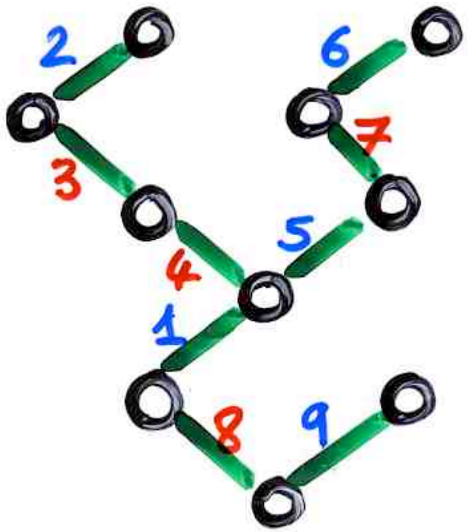
Prop. (O. Bernardi, 2008)
The number of such permutations on n elements is C_n Catalan number

Lemma. $\underset{\text{permutation}}{\sigma} \leftrightarrow T$ alternating tableau

T has no crossing
$\Leftrightarrow$ σ has no subsequence of type
$(y+1) \cdots x \cdots y \cdots (x+1)$


6 4 5 3 9 7 8 10 1 2





2
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4
1
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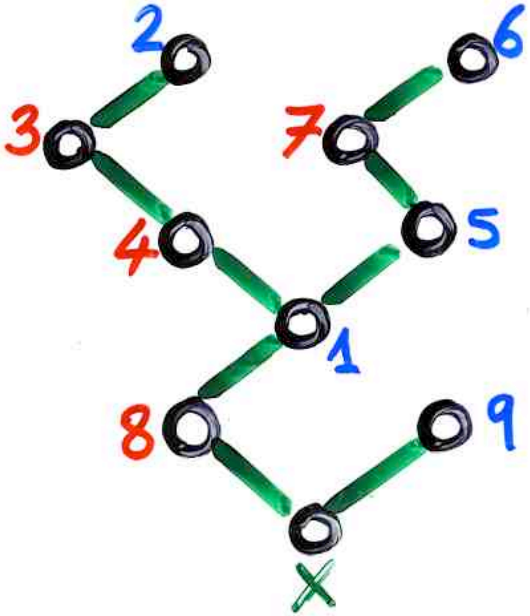
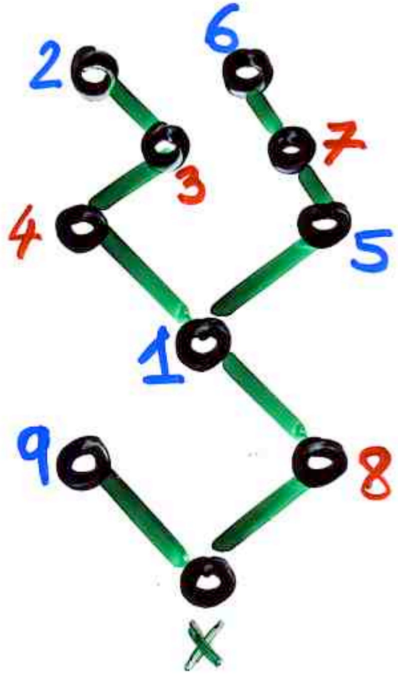
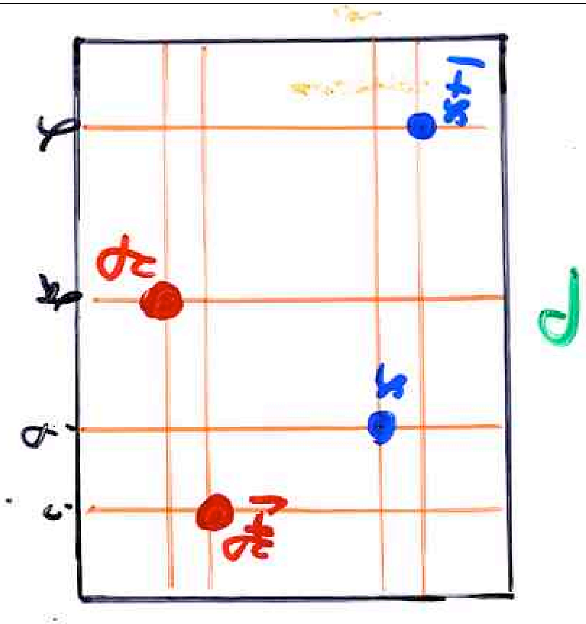
9

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & X \\ 6 & 4 & 5 & 3 & 9 & 7 & 8 & X & 1 & 2 \end{pmatrix}$$

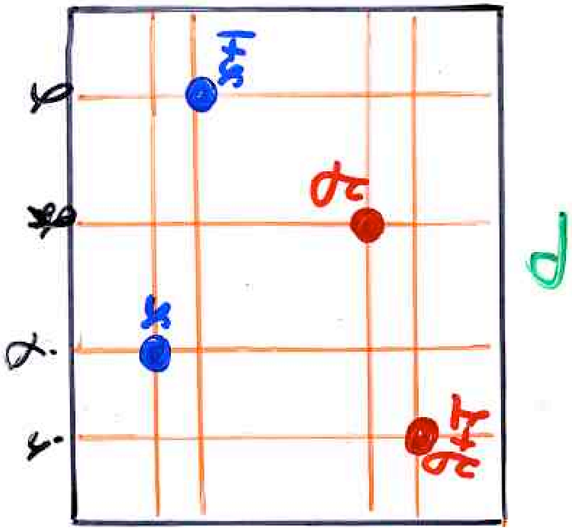




$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & X \\ 9 & X & 4 & 2 & 3 & 1 & 6 & 7 & 5 & 8 \end{pmatrix}$$



31- 24

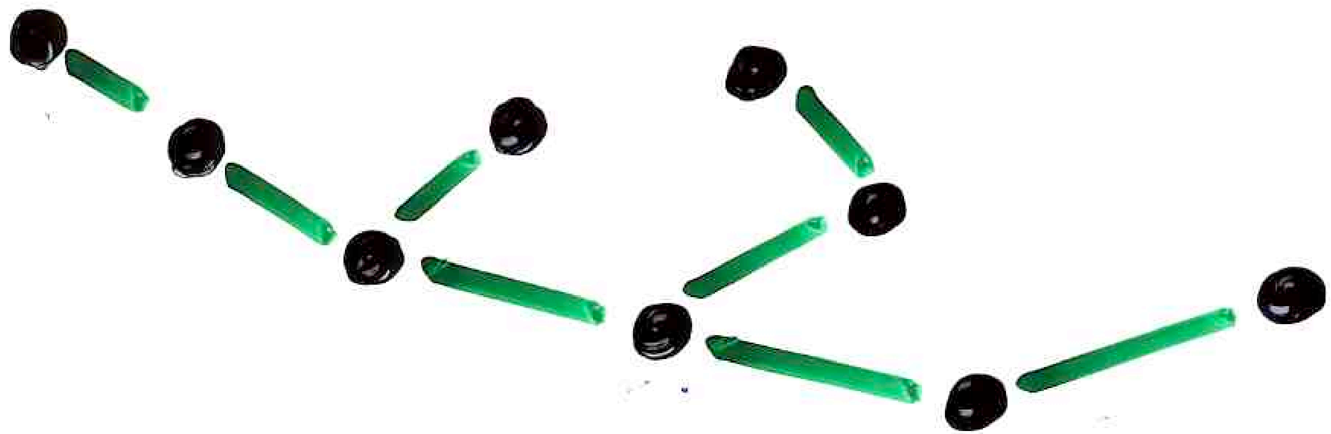


24 - 31

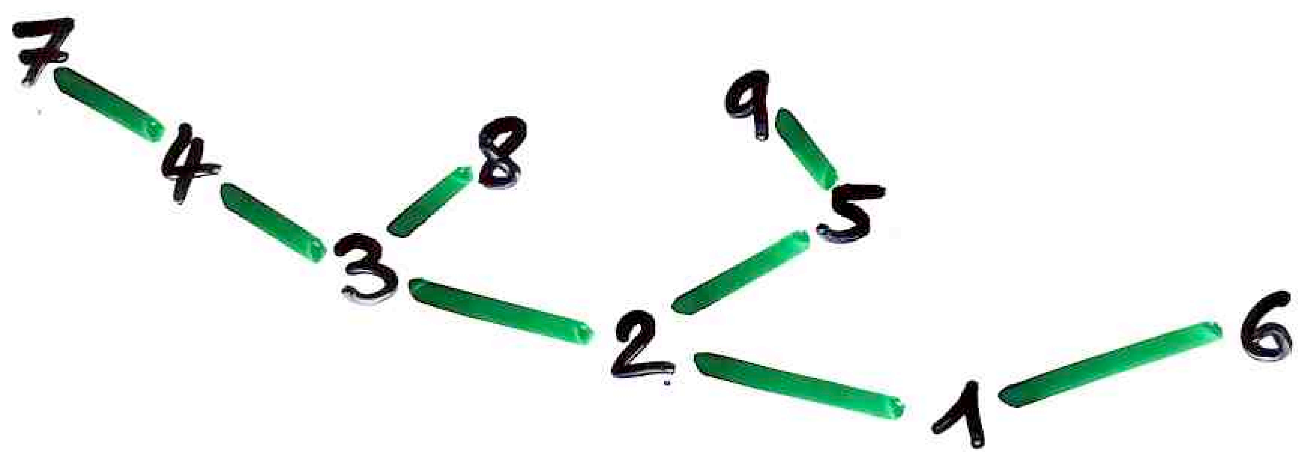
Compléments

Increasing binary trees

Def- Increasing binary tree

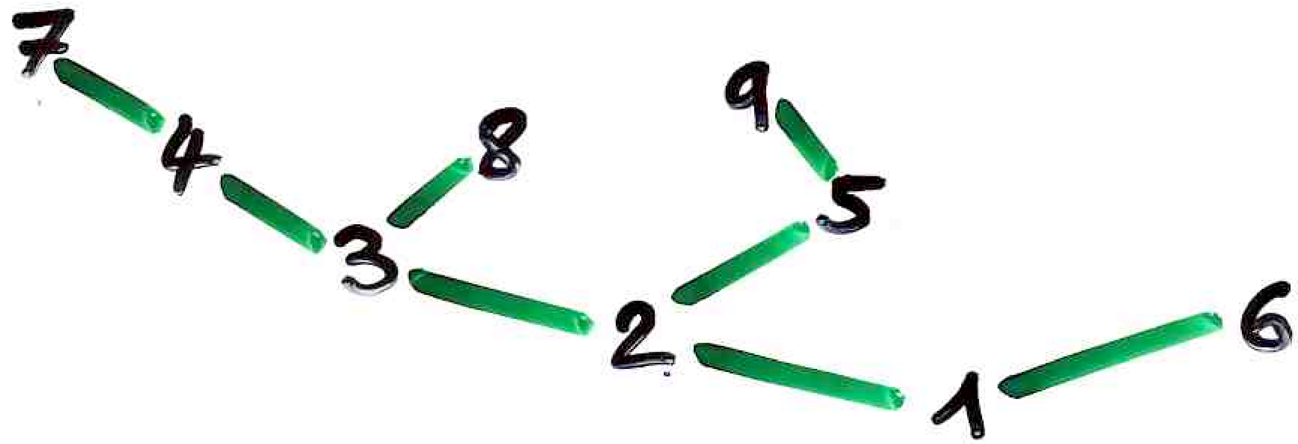


Def- Increasing binary tree



Bijection increasing binary tree $\longleftrightarrow$ permutation

T σ



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$

Bijection increasing binary tree $\longleftrightarrow$ permutation

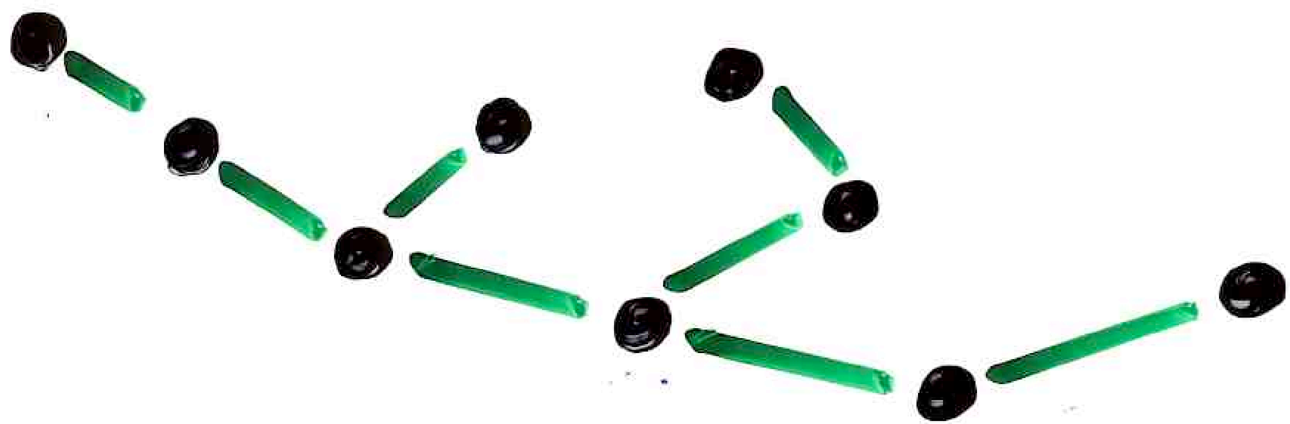
T $\qquad\qquad$ σ

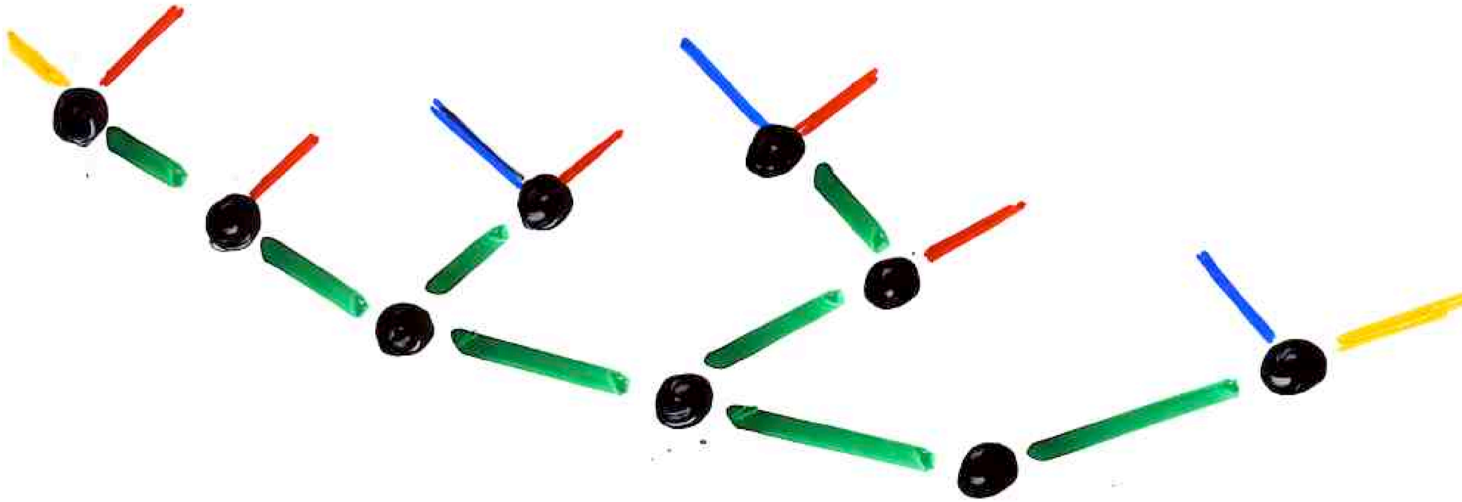
$T \xrightarrow{\pi} \sigma$ $\qquad$ symmetric order of vertices (or "projection")

$\sigma \xrightarrow{\delta} T$ $\qquad$ "déployé"

word $w = u\, m\, v$ $\qquad$ m (unique) minimum letter

$\delta(w) =$ $\delta(u)$ $\delta(v)$
m

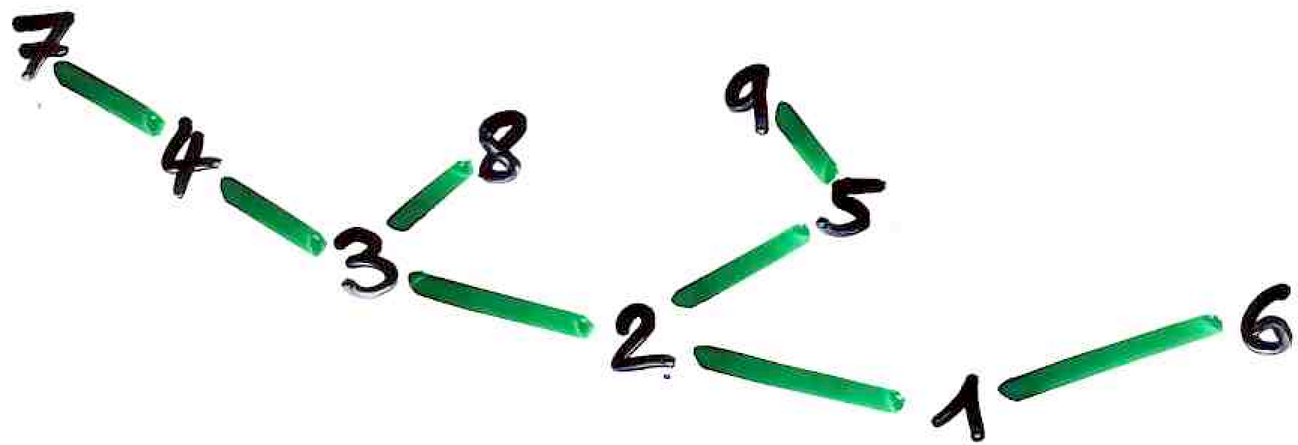




canopy of a binary tree

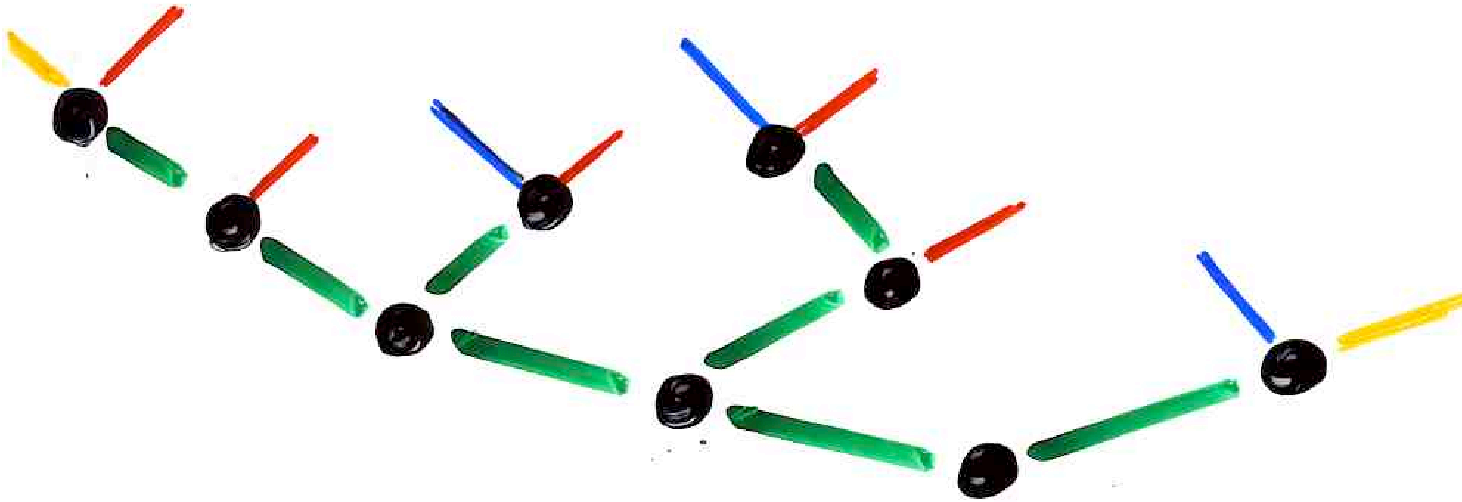
$C(B) = - - + - + - - +$

$$
\begin{tikzcd}
\underset{\text{permutation}}{\sigma} \arrow[rr] \arrow[dd] & & T(\sigma) \arrow[d] & \begin{array}{l}\text{increasing}\\ \text{binary}\\ \text{tree}\end{array} \\
 & & B \arrow[d] & \text{binary tree} \\
\underset{\text{up-down}}{UD(\sigma)} & \text{sequence} \; = & \underset{\text{canopy}}{c(B)} &
\end{tikzcd}
$$



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6 \ldots$

up-down sequence $\quad - \ - \ + \ - \ + \ - \ - \ +$



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6 \ldots$

up-down sequence $\quad - \quad - \quad + \quad - \quad + \quad - \quad - \quad +$

"hook-length formula"

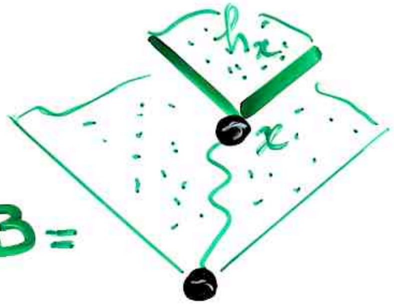
$$\frac{n!}{\prod_x h_x}$$

n nb of vertices

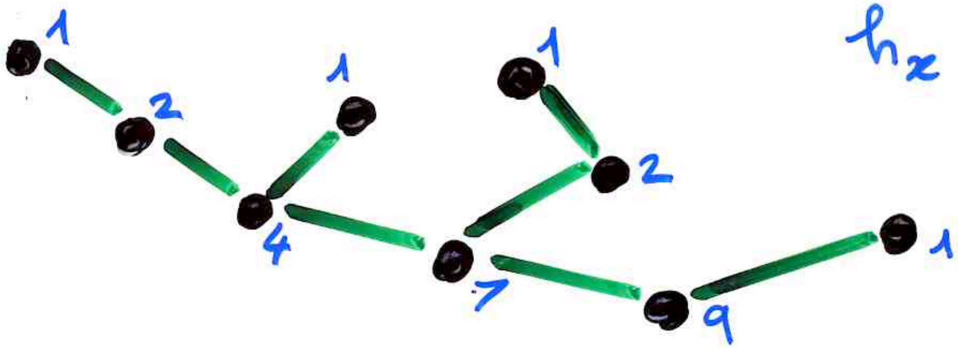
product of size of sub-trees

nb of increasing binary tree for a binary tree B

$B =$



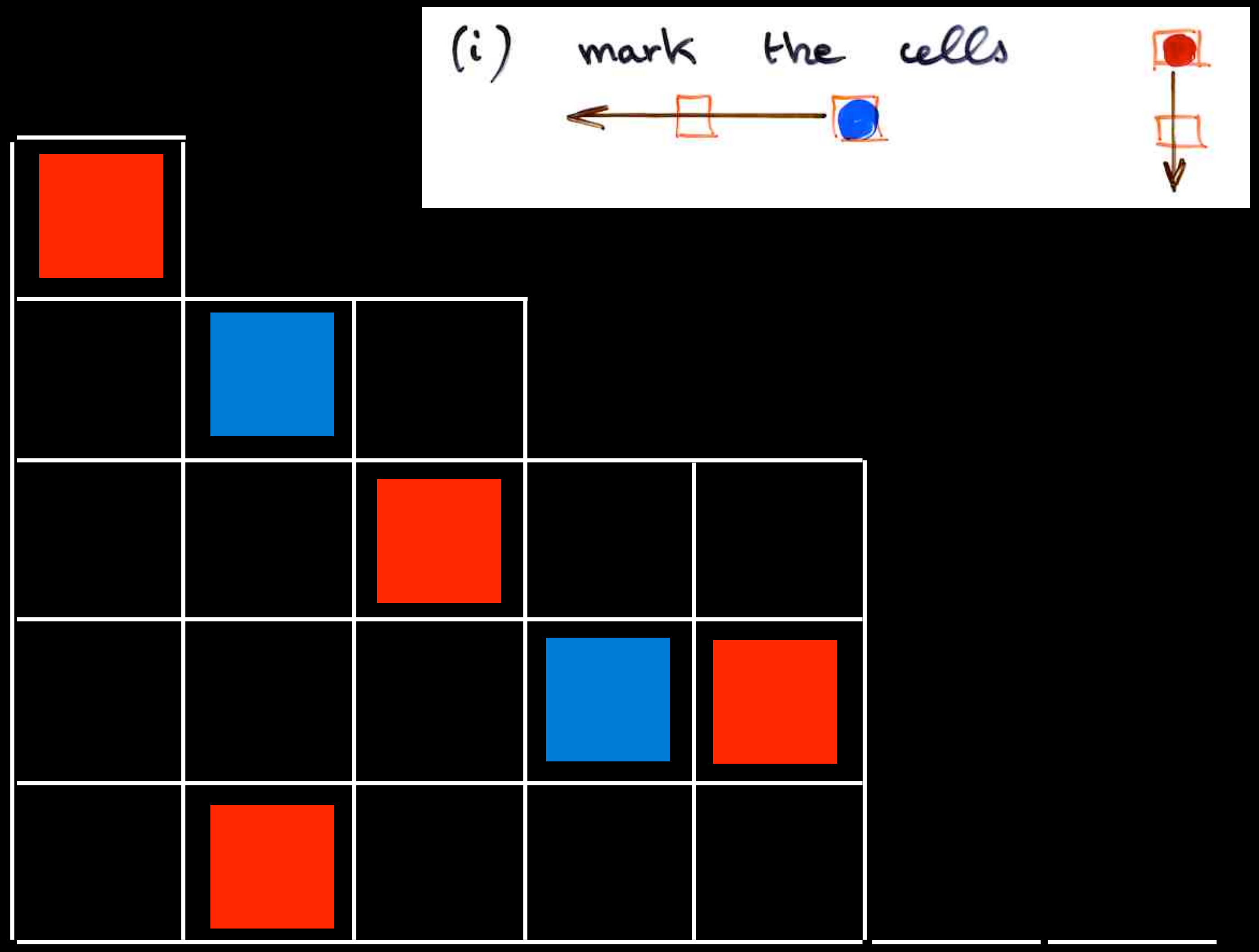
ex:



"hook-length" h_x

$$\frac{9!}{2^2.4.7.9} = 360$$

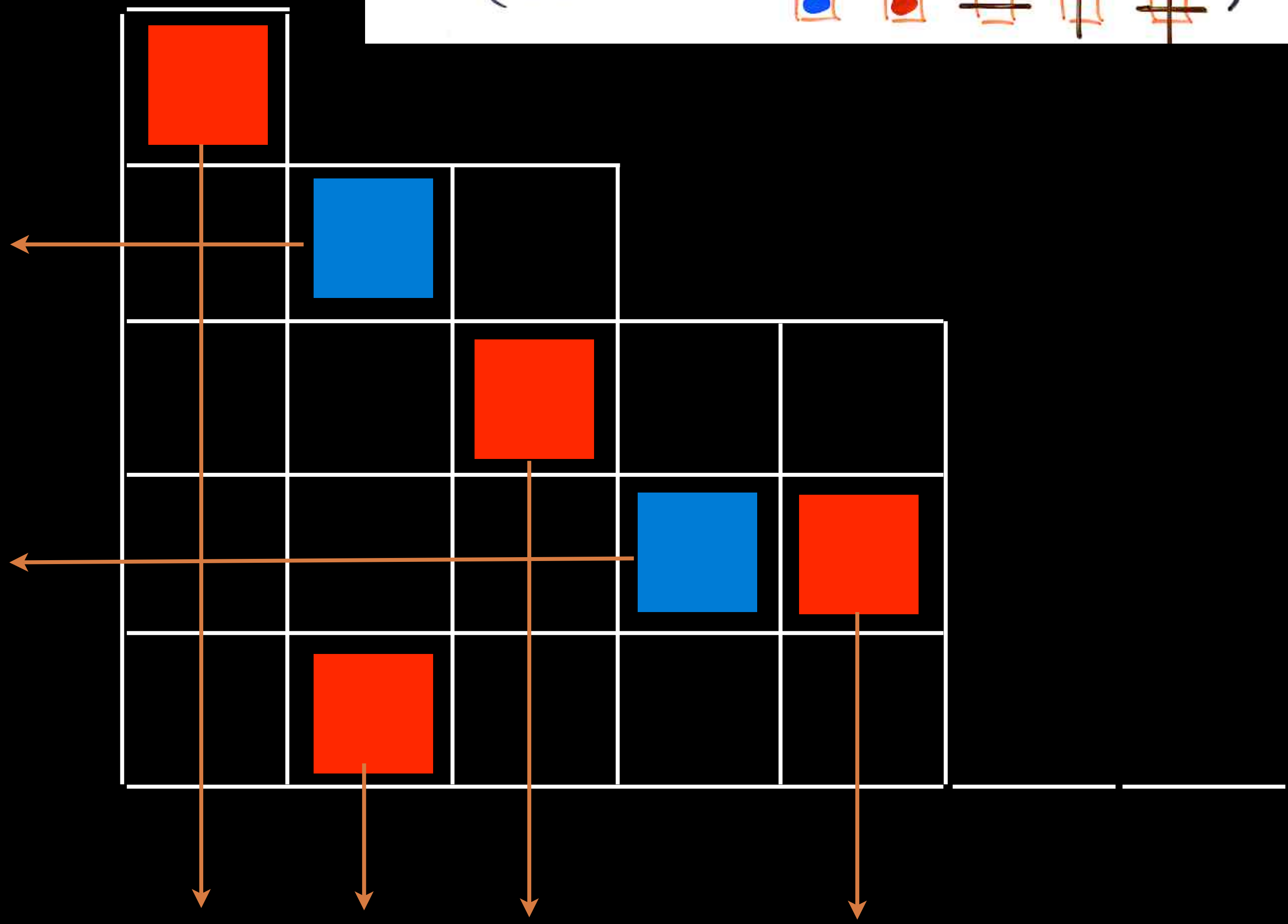
Bijection
alternative tableaux
permutation tableaux

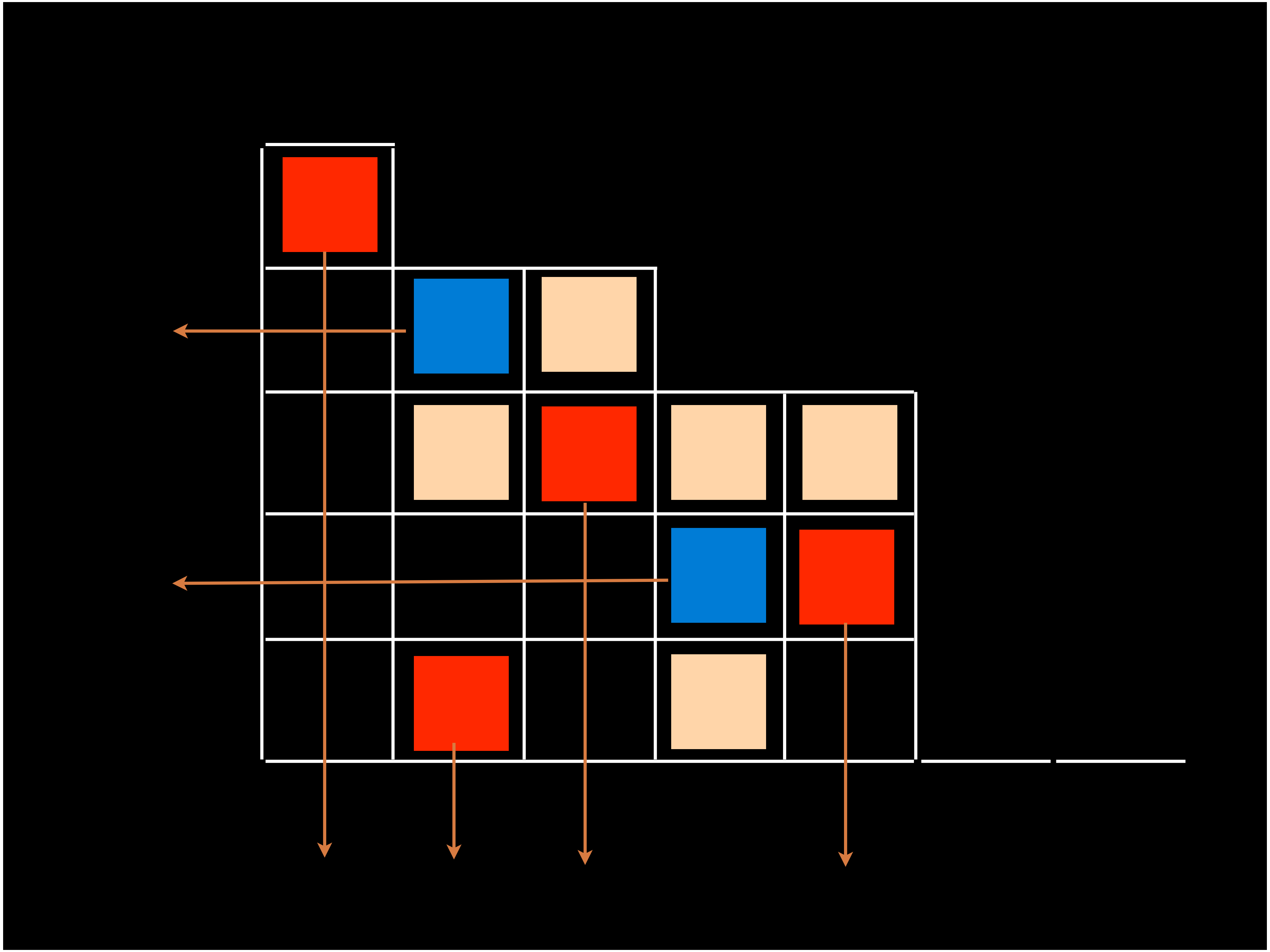


alternative tableau

(ii) mark the empty cells by ✗

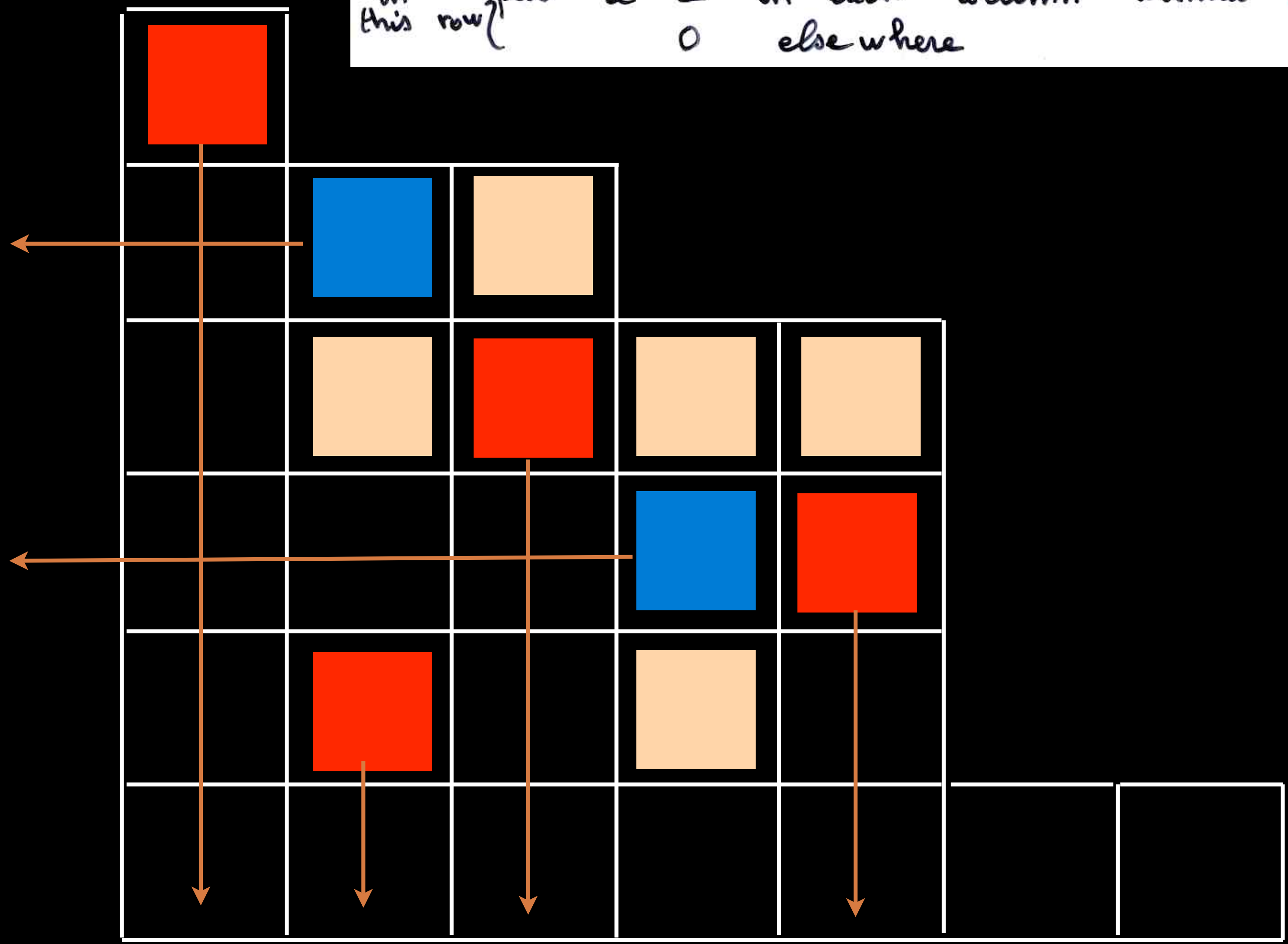
(other than)



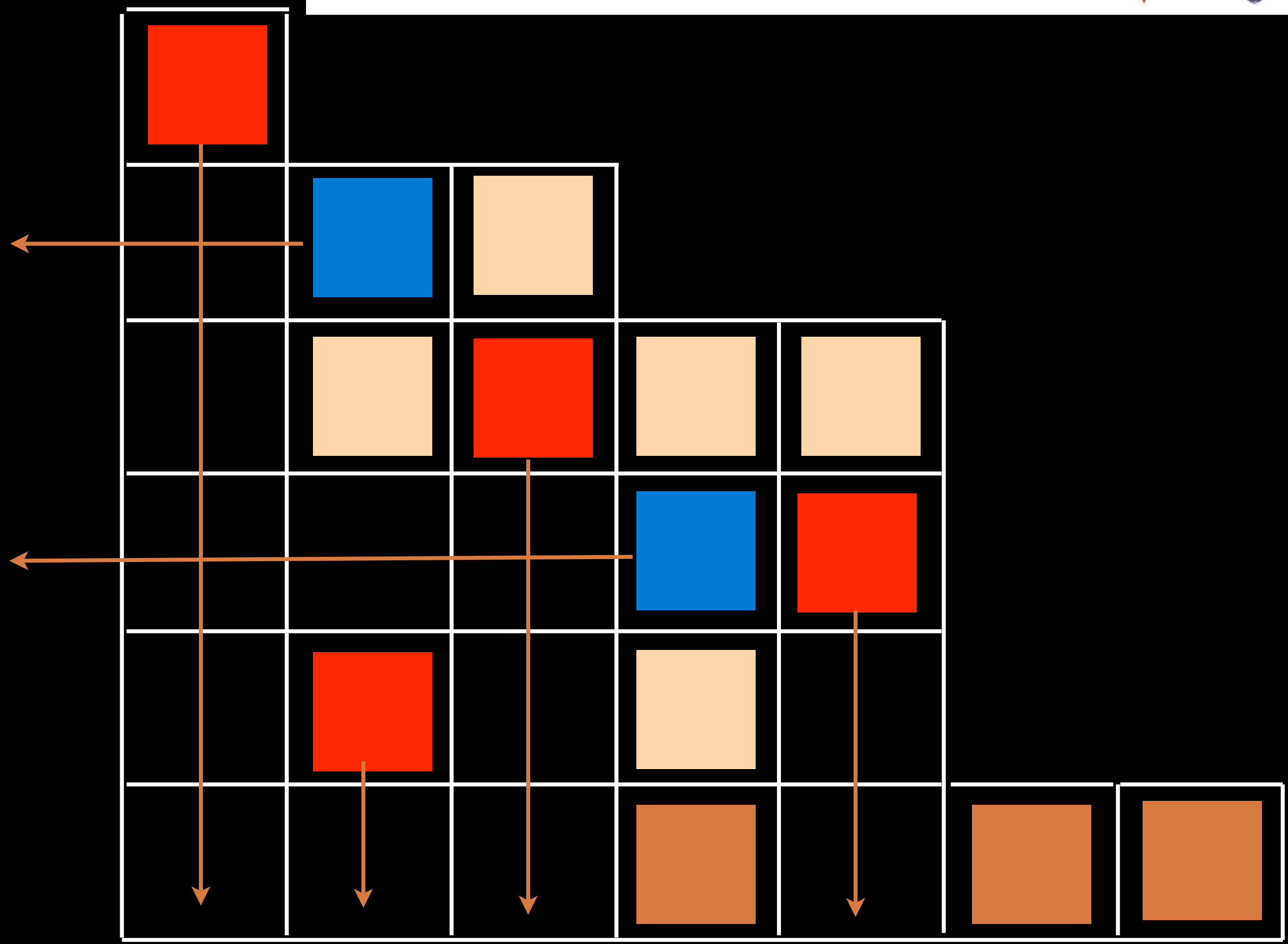


(iv) add a new row below F

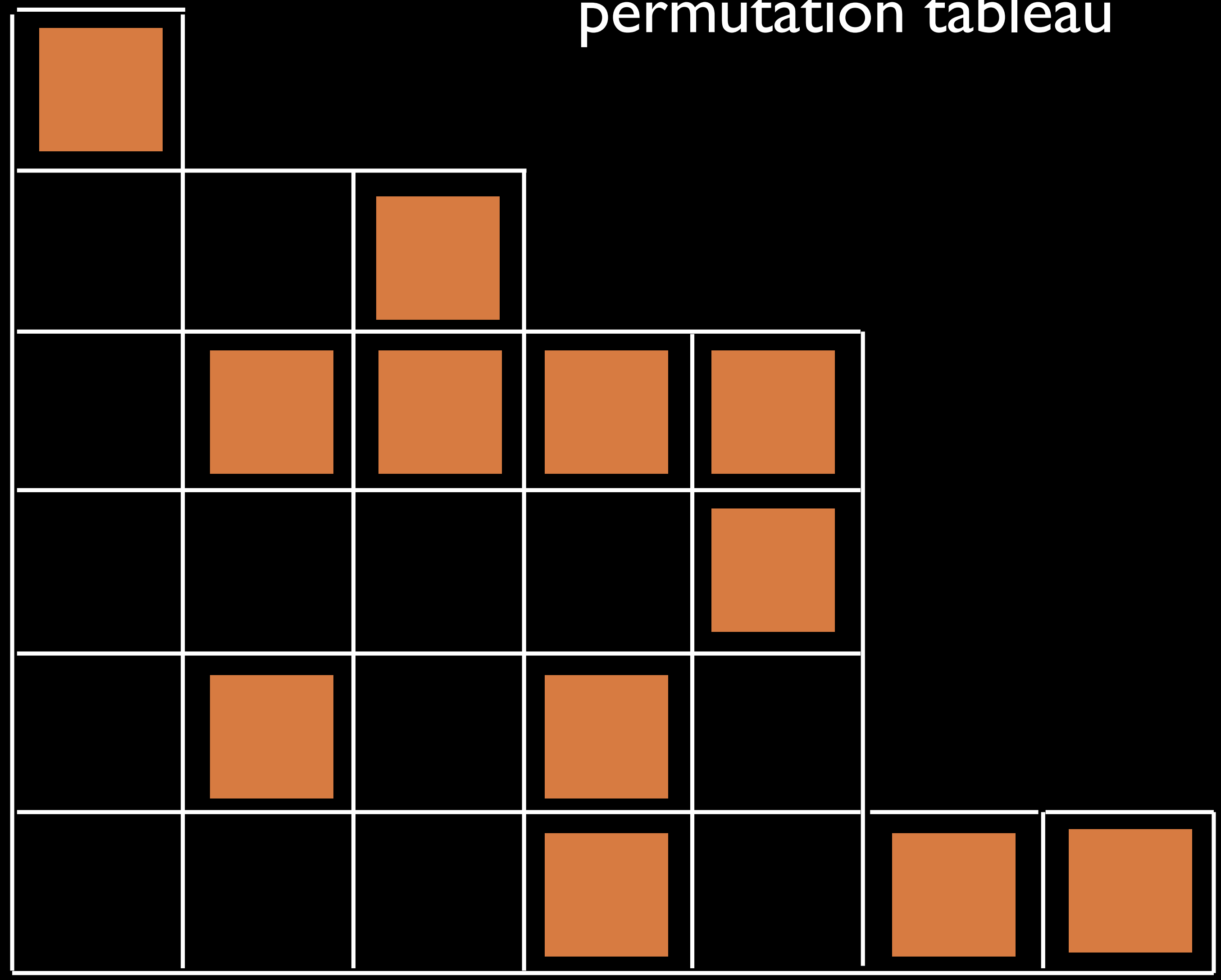
in this row { put a 1 in each column without ↓
0 elsewhere



(iii) • replace the cells or by 1
• replace the cells by 0



permutation tableau



check: AT $\xrightarrow{\varphi}$ PT size $(n+1)$

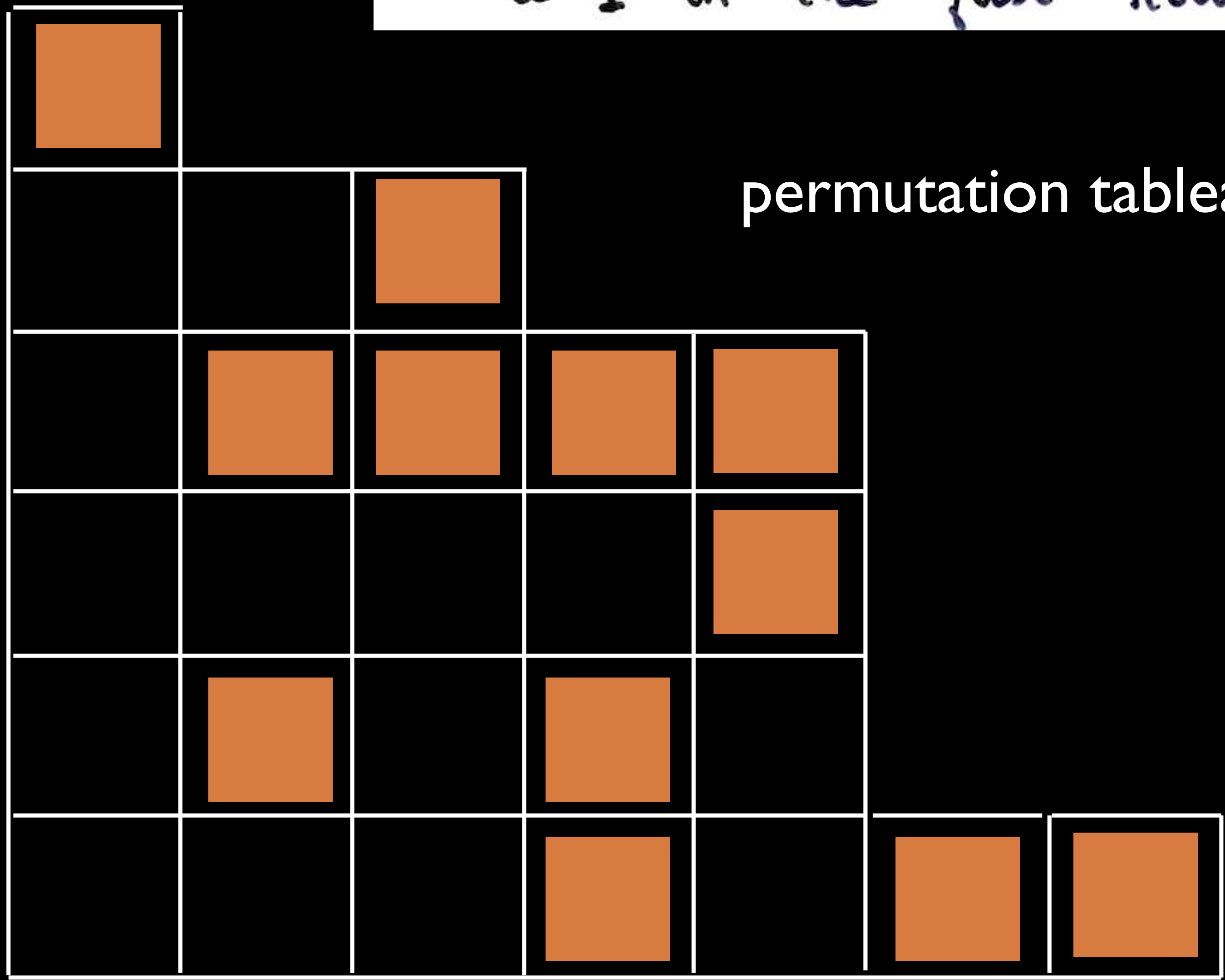
- the exist at least a 1 in each column of PT $= \varphi($AT$)$

impossible

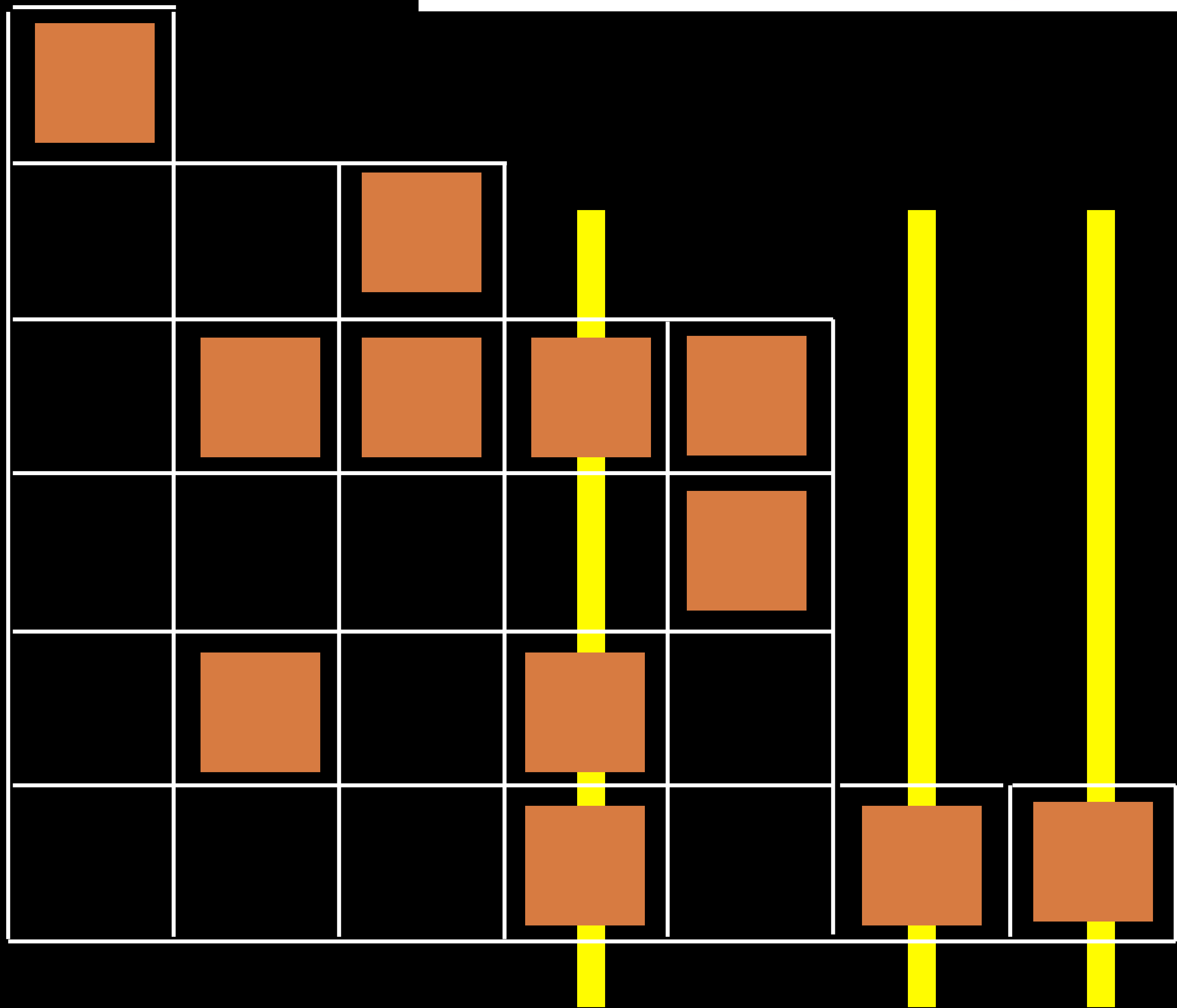
inverse bijection $\psi = \varphi^{-1}$

(i) mark the columns with a 1 in the first row

permutation tableau

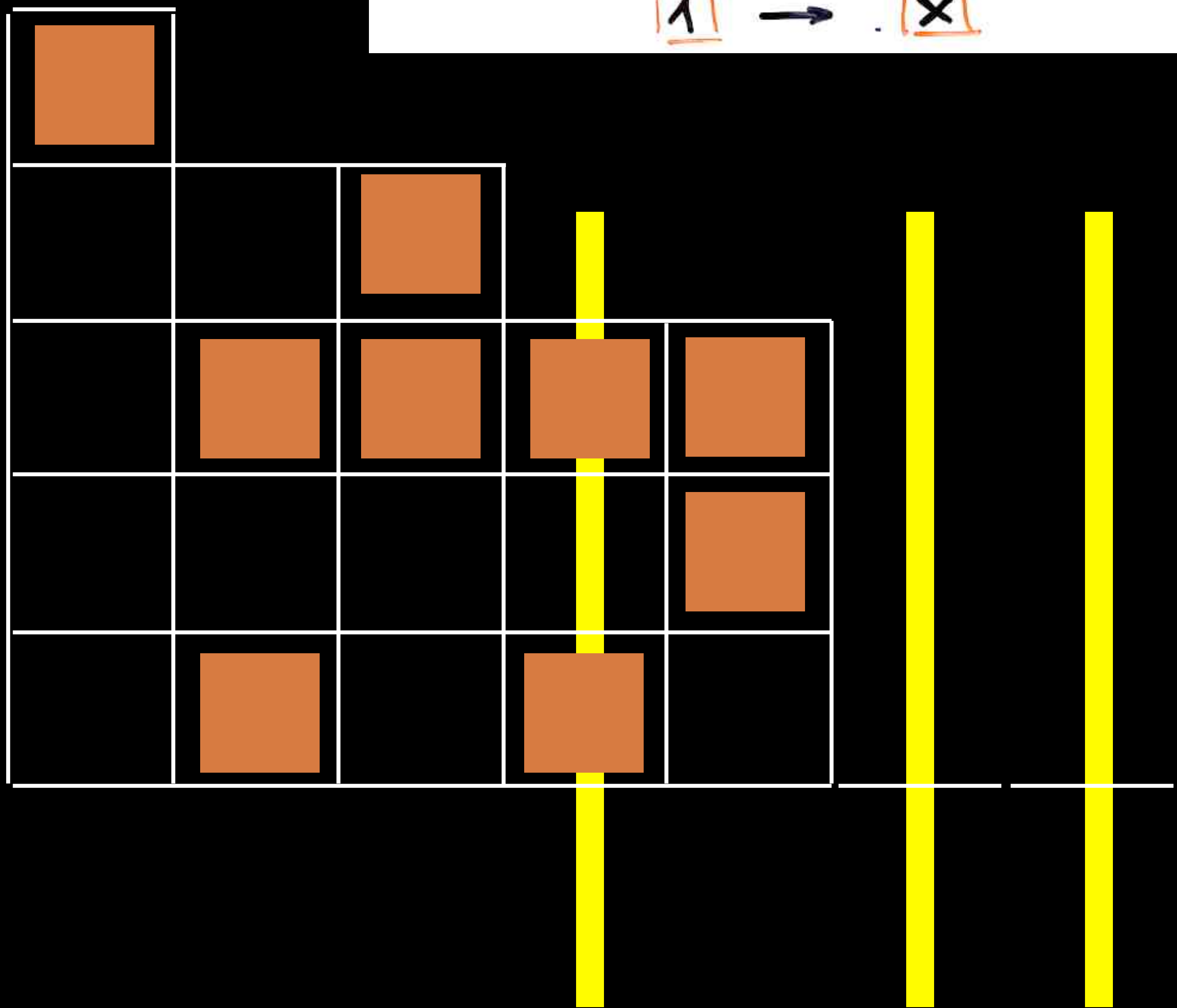


(ii) delete the first row

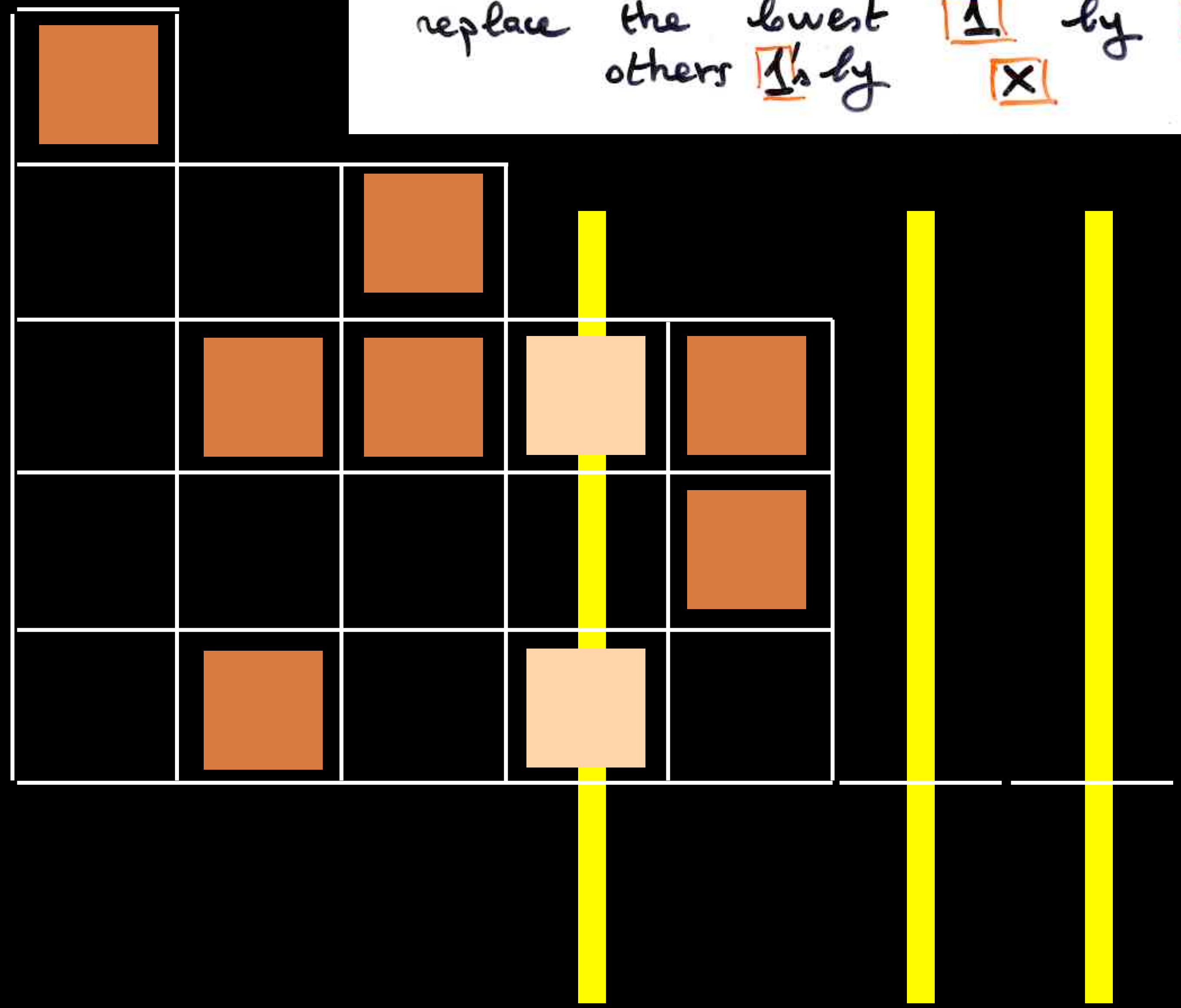


(iii) in each marked column

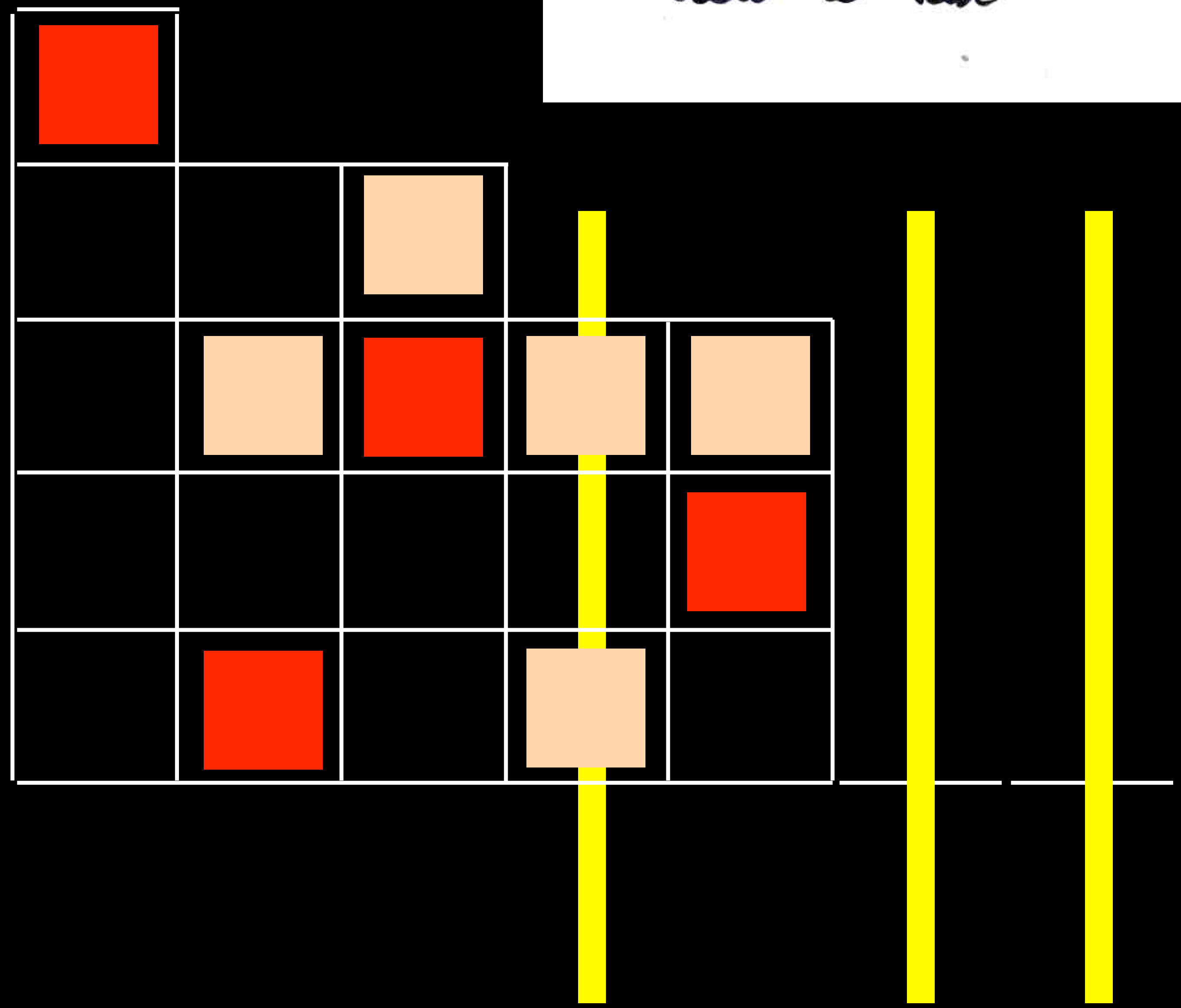
$\boxed{1} \longrightarrow \boxed{\times}$



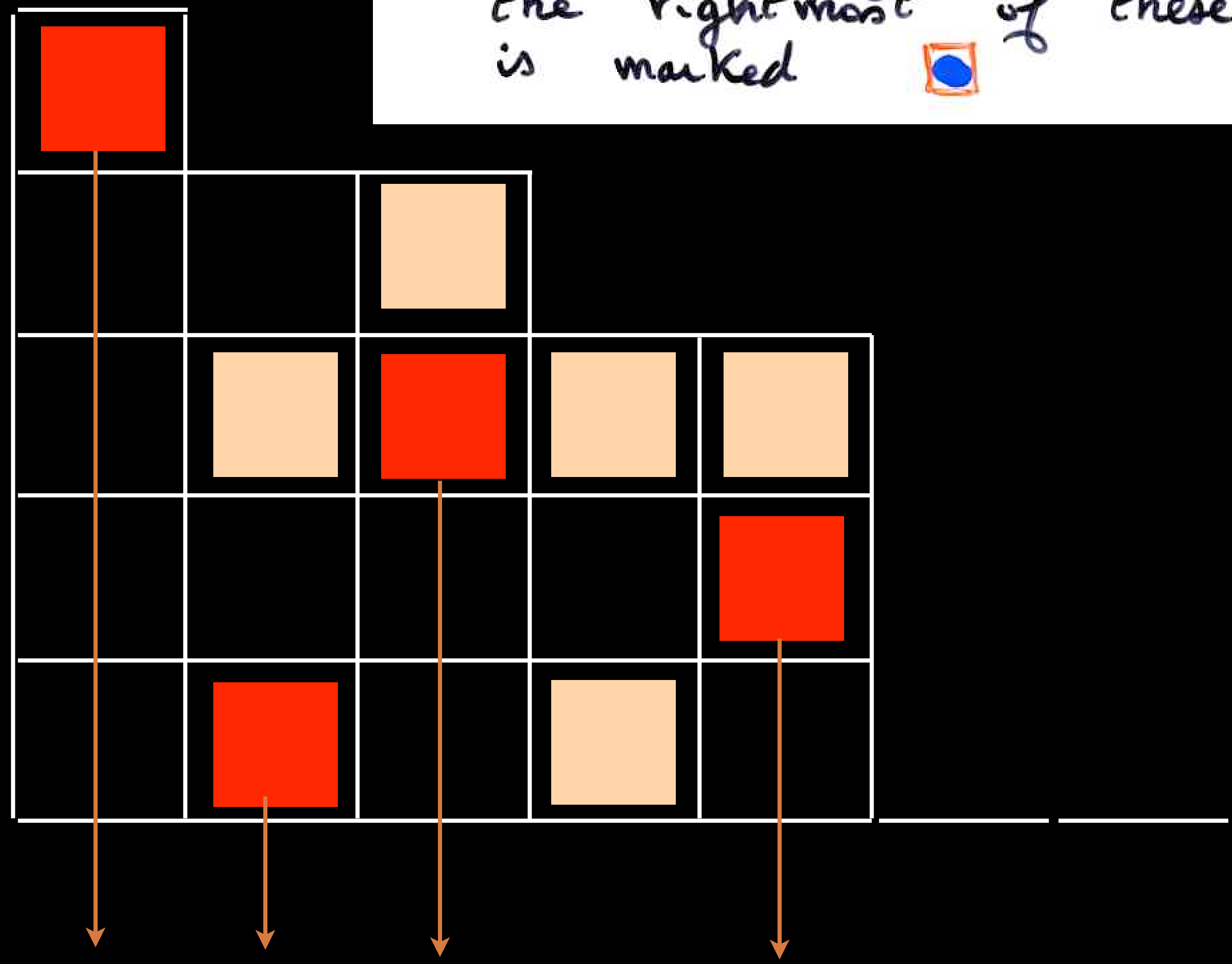
(iv) in each non marked column
(∃ some cells with 1)
replace the lowest $\boxed{1}$ by $\boxed{\bullet}$
others $\boxed{1}$s by $\boxed{\times}$



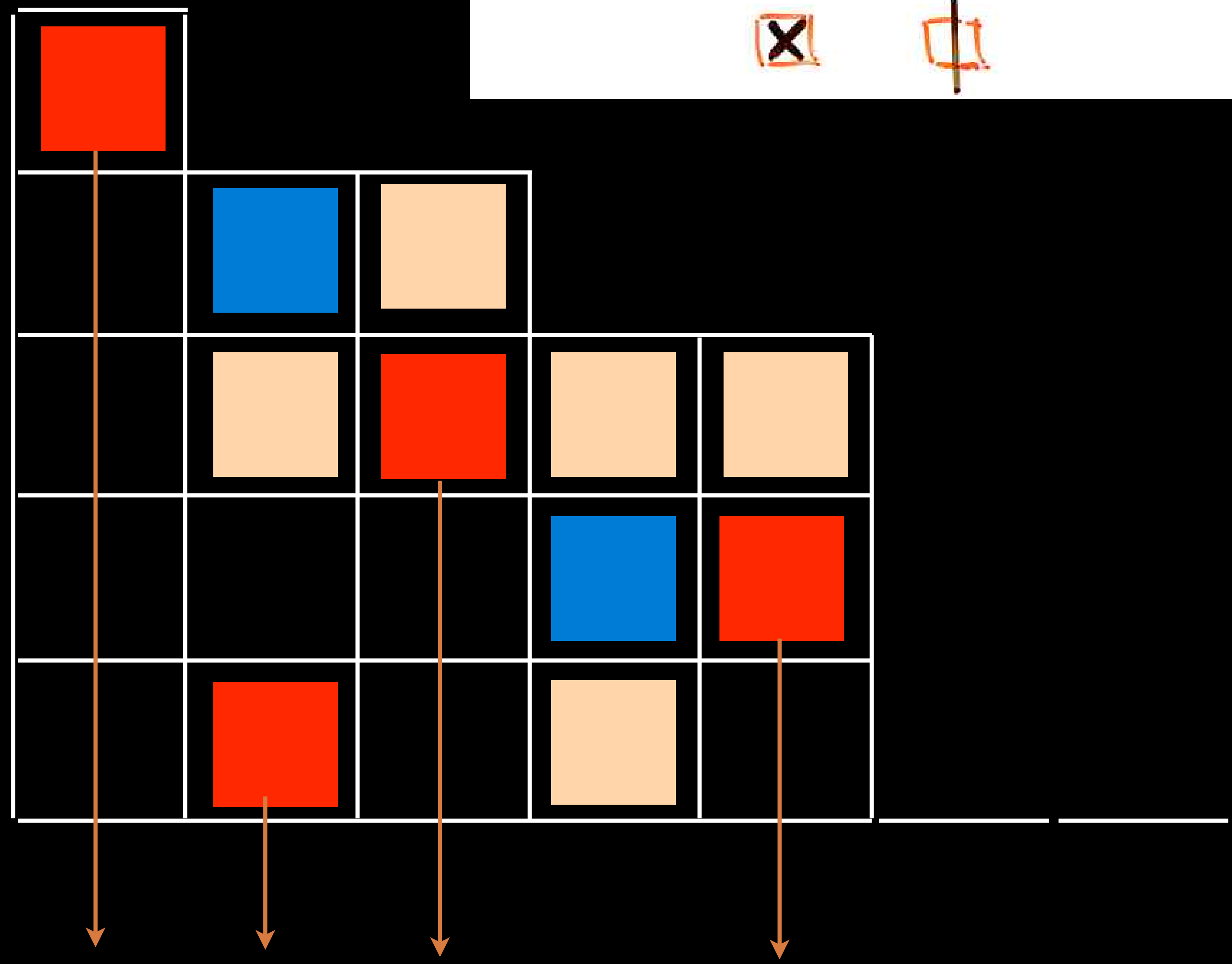
(v) mark the cells below a red



(vi) in each row where there exist empty cells, the rightmost of these cells is marked



(vii) delete the marks $\boxed{\times}$



alternative tableau