Alternative tableaux
and
pattern in permutations

xgv

GT, LaBRI

14 November 2008

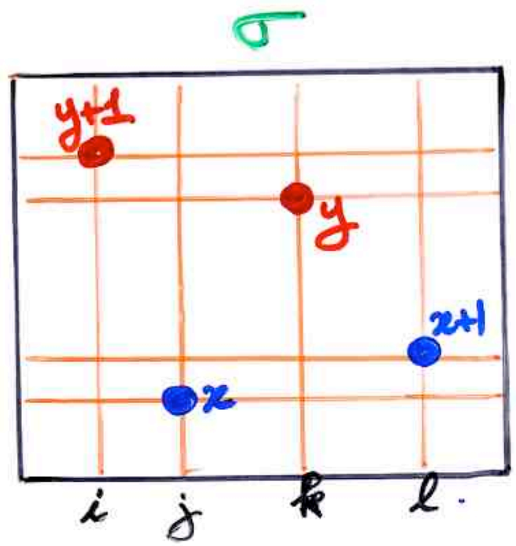
Permutations

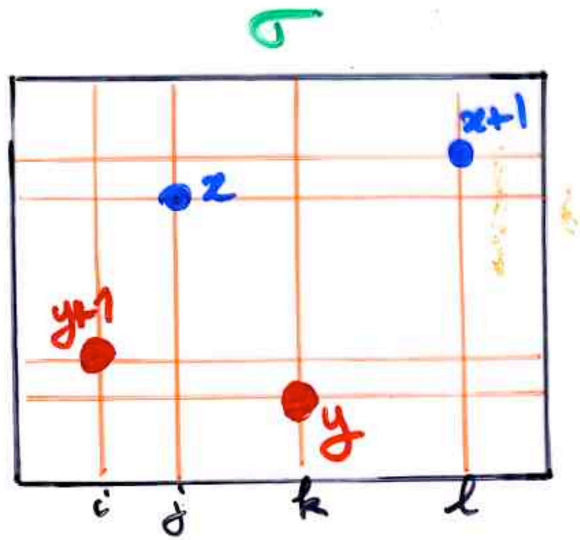
with no subsequence of the type

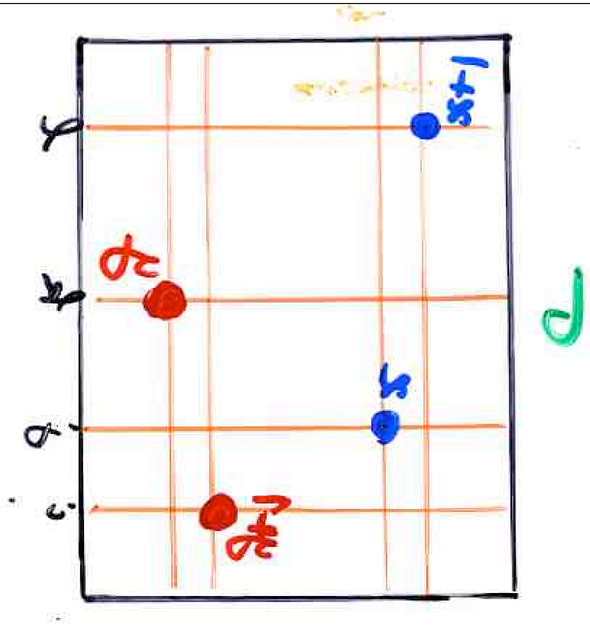
$$\cdots (y+1) \cdots x \cdots y \cdots (x+1) \cdots$$

ex: $\sigma = 6\ 4\ 5\ 3\ 9\ 7\ 8\ (10)\ 1\ 2$

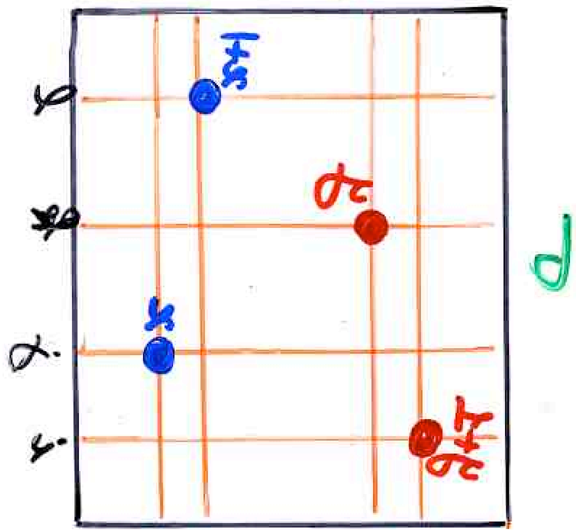
Prop. (O. Bernardi, 2008)
The number of such permutations on n elements is C_n Catalan number


σ
y+1
y
x+1
x
i
j
k
l


σ
x+1
x
y+1
y
i
j
k
l



31- 24

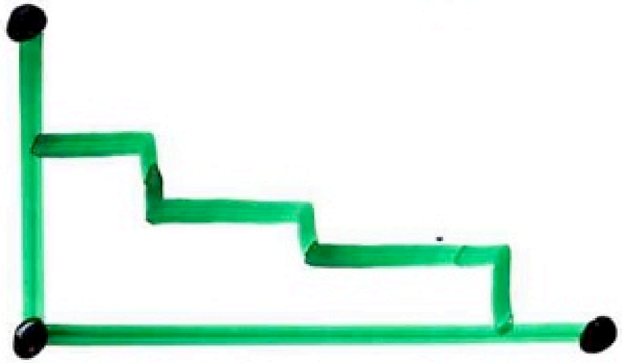


24 - 31

§1
alternative tableau: definition

alternative tableau

– Ferrers diagram **F**

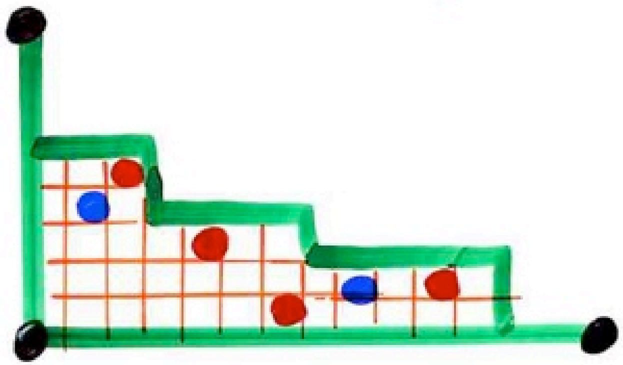


(possibly empty rows or column)

(nb of rows) + (nb of columns) = n

alternative tableau

- Ferrers diagram F



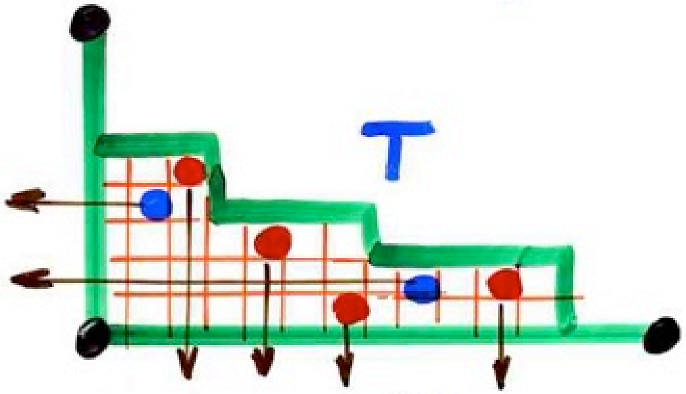
(possibly empty rows or column)

(nb of rows) + (nb of columns) $= n$

- some cells are coloured **red** or **blue**

alternative tableau T

- Ferrers diagram F (possibly empty rows or column)



(nb of rows) + (nb of columns) = n

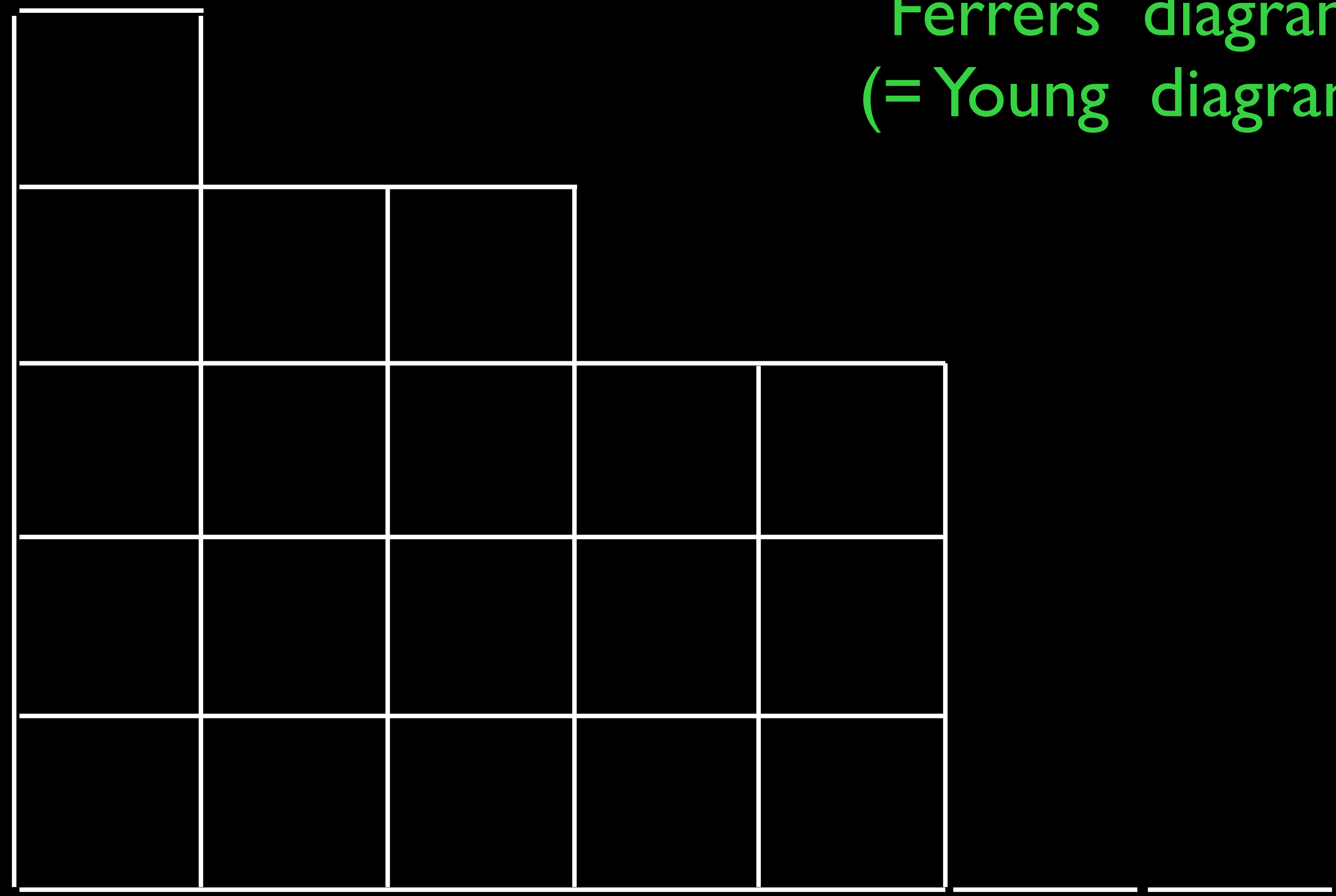
- some cells are coloured **red** or **blue**

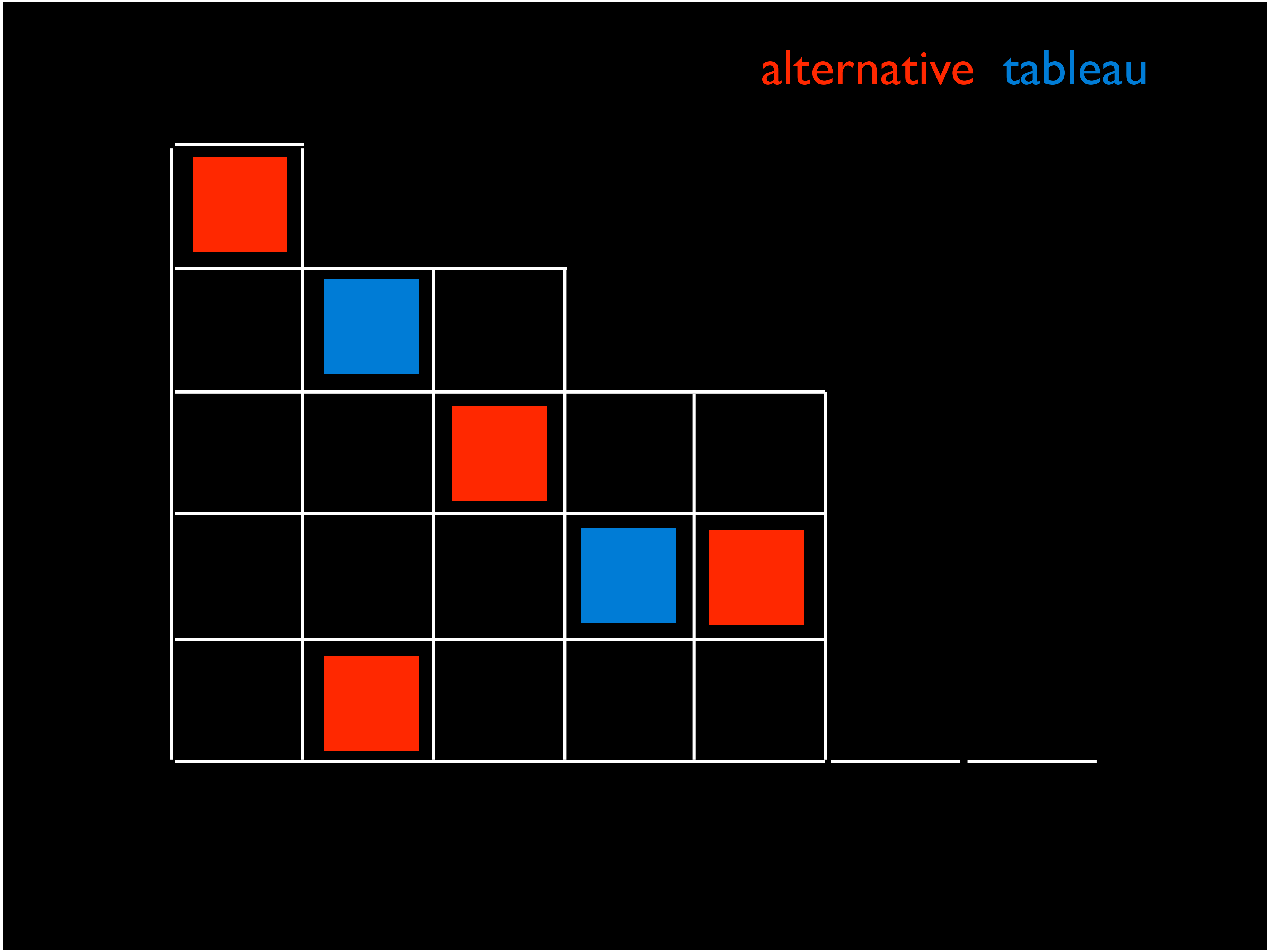
- { no coloured cell at the left of ■ (blue)
  { no coloured cell below ■ (red)

n size of T

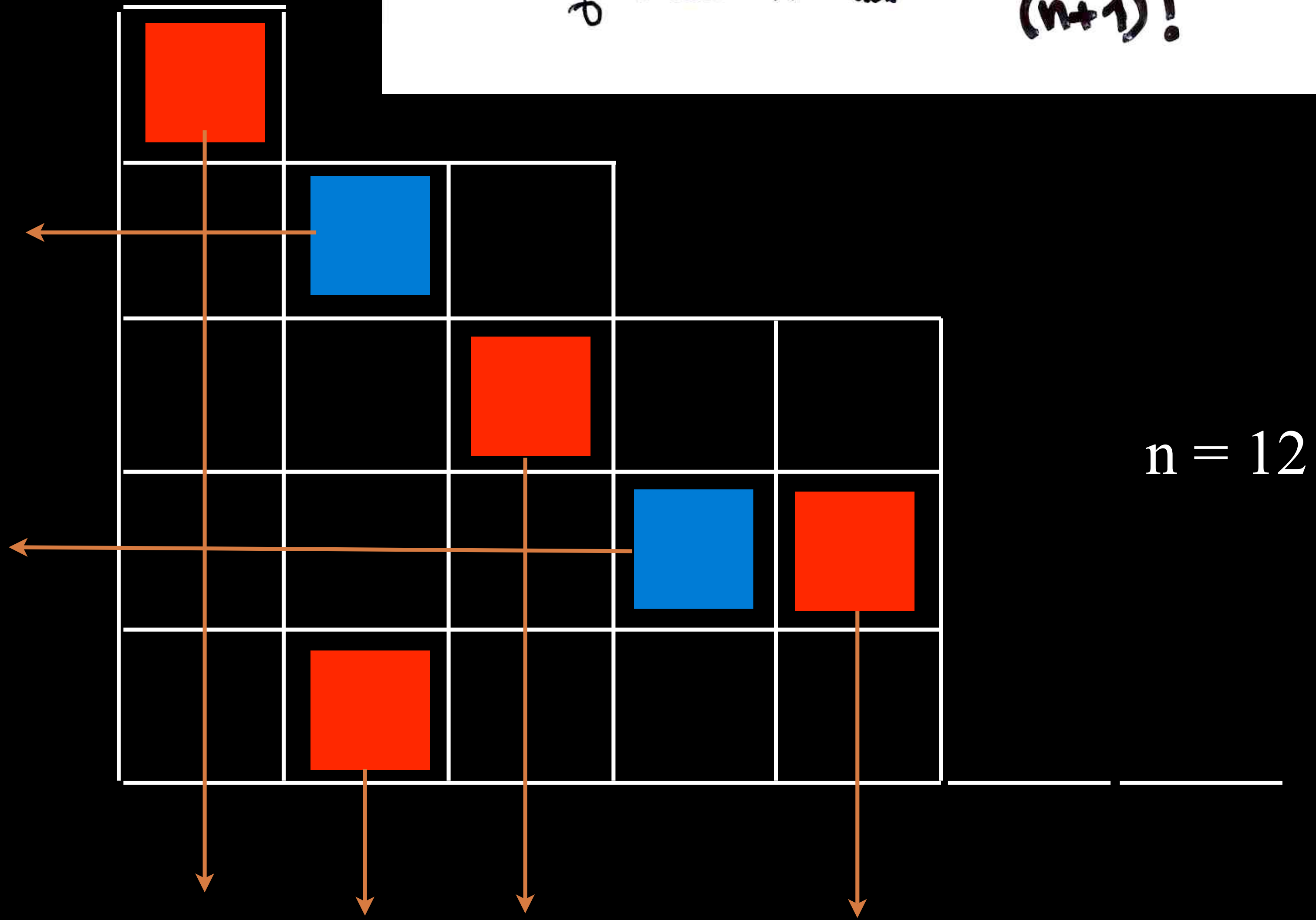
alternative tableau

Ferrers diagram
(= Young diagram)




alternative tableau

Prop. The number of *alternative tableaux* of size n is $(n+1)!$



ex: n=2

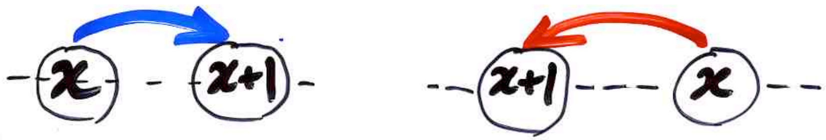
§2 The “exchange-fusion” algorithm

Def – Permutation $\sigma = \sigma(1) \cdots \sigma(n)$

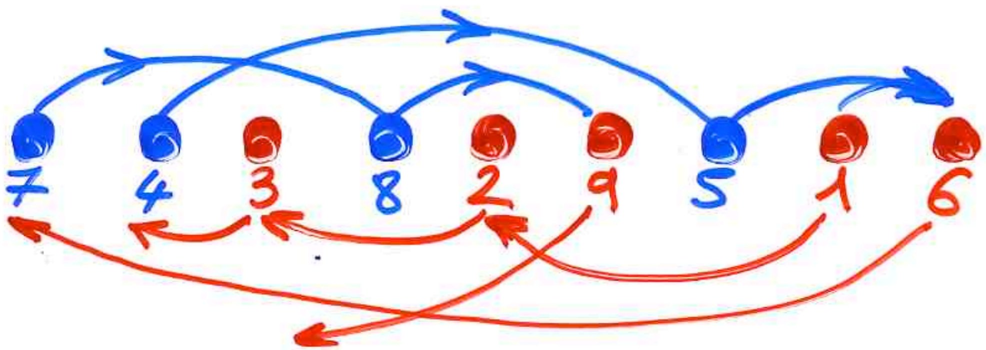
$x = \sigma(i)$, $1 \leq x < n$

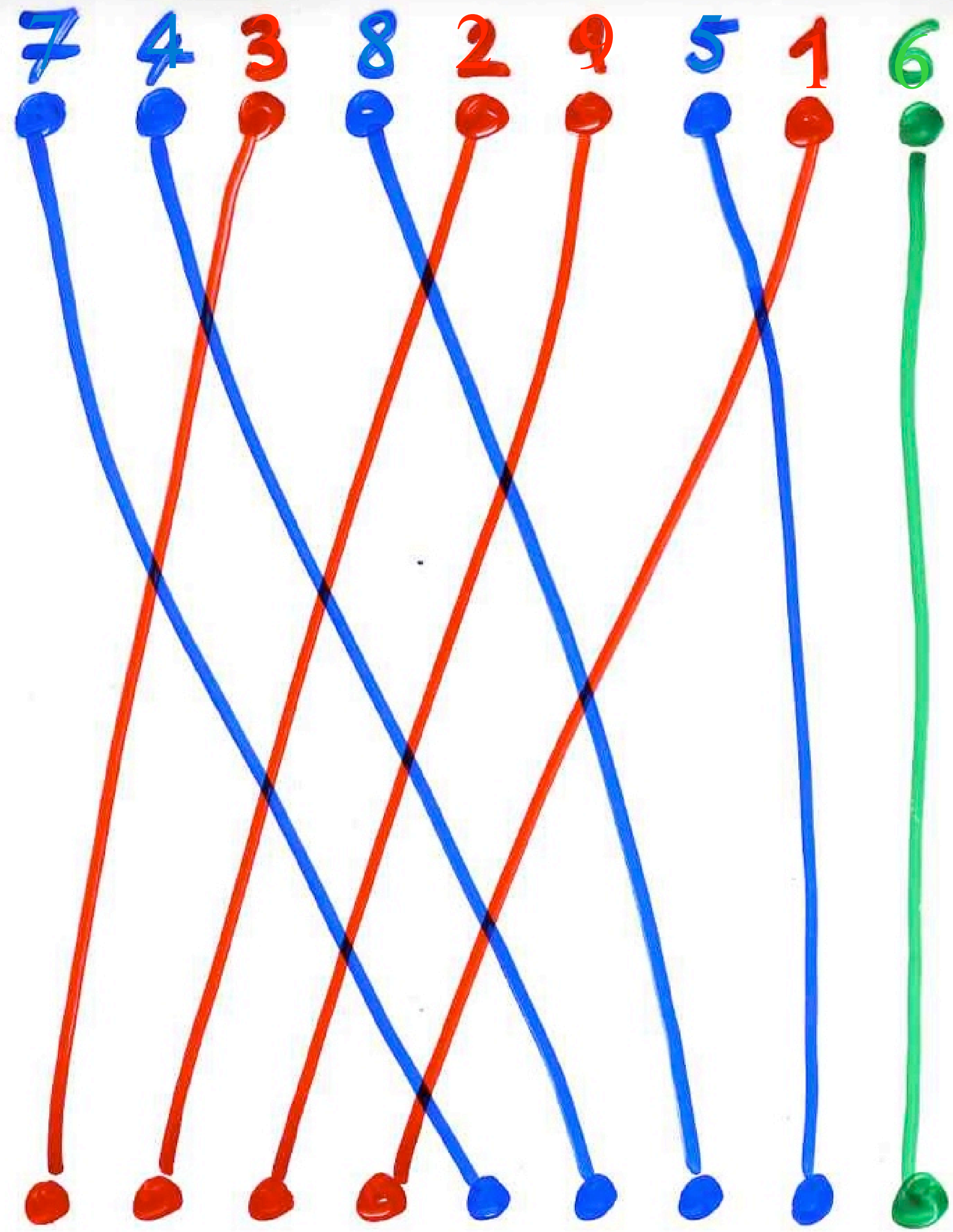
(valeur) x $\begin{cases} \text{avance} \\ \text{recul} \end{cases}$ $x+1 = \sigma(j)$, $\begin{cases} i < j \\ j < i \end{cases}$

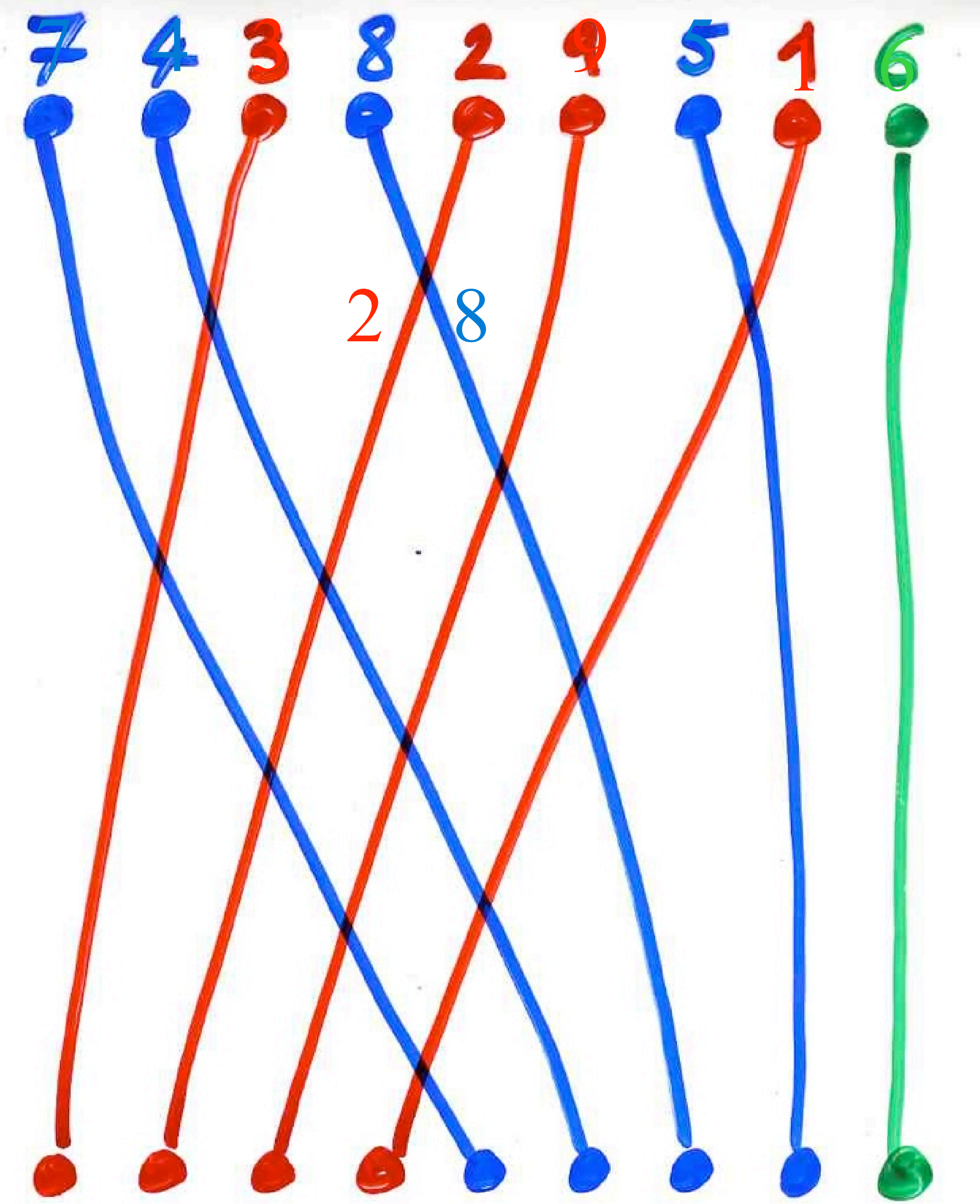
- convention $x = n$ est un recul

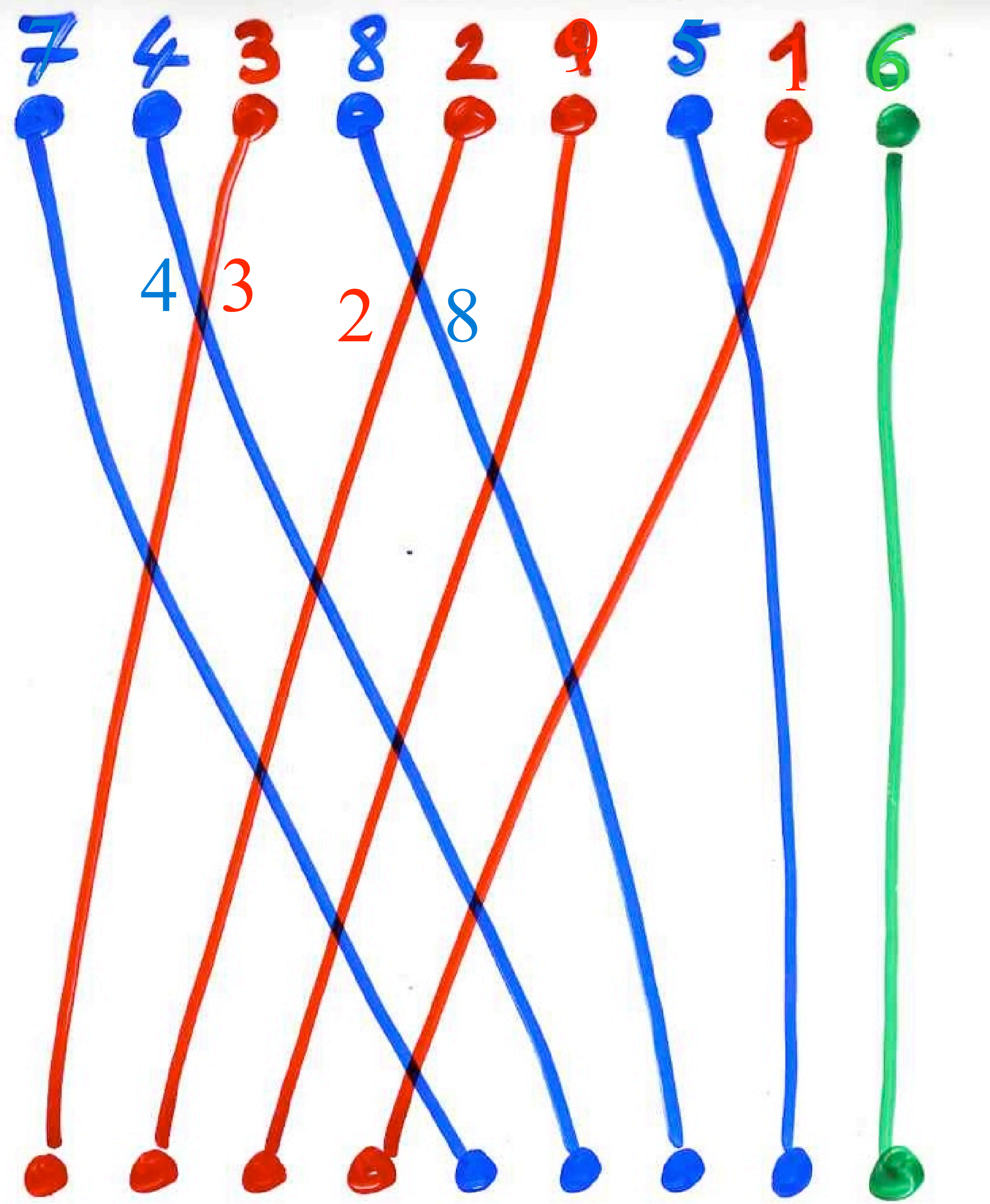


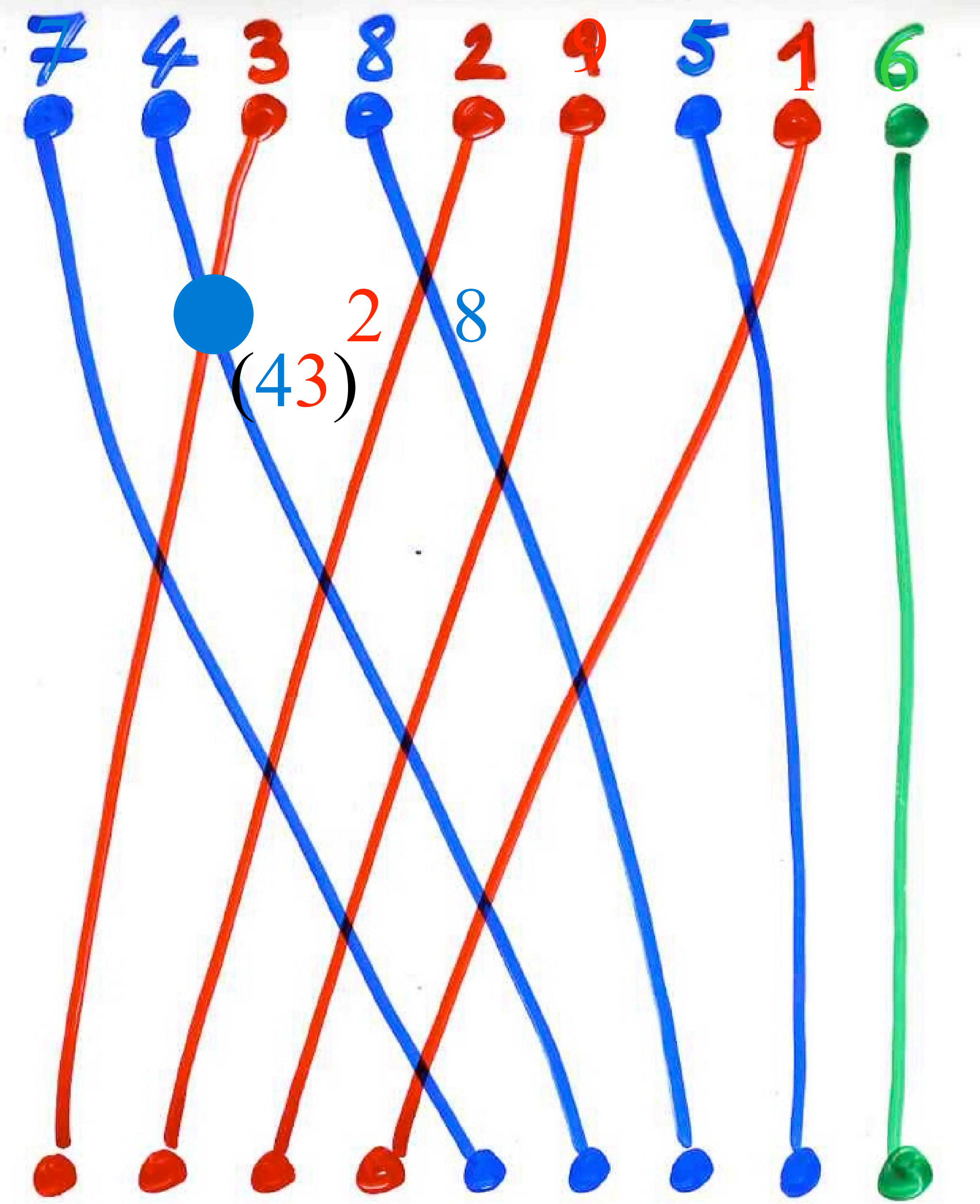
$$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$$

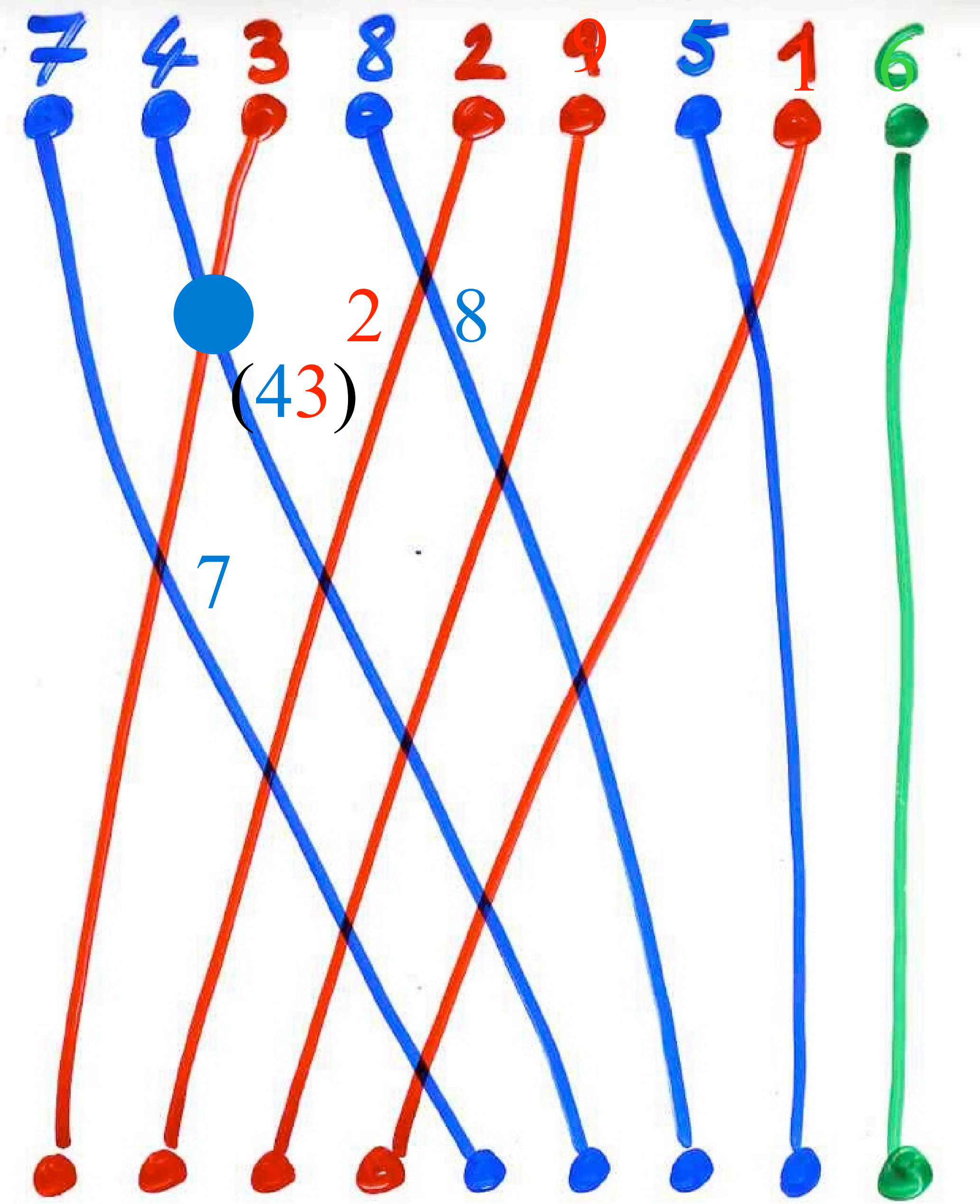


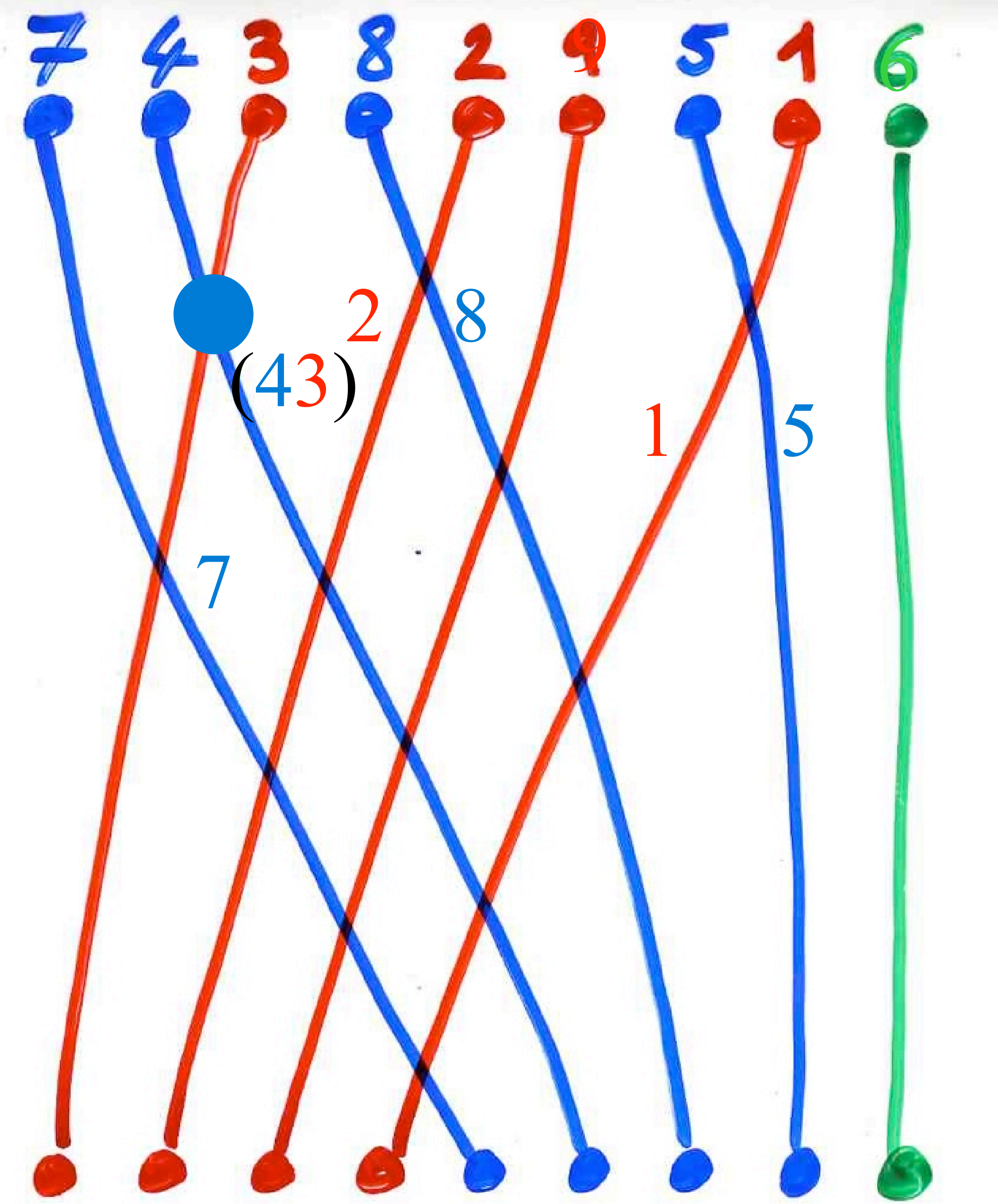

7 4 3 8 2 9 5 1 6


7 4 3 8 2 9 5 1 6
2 8

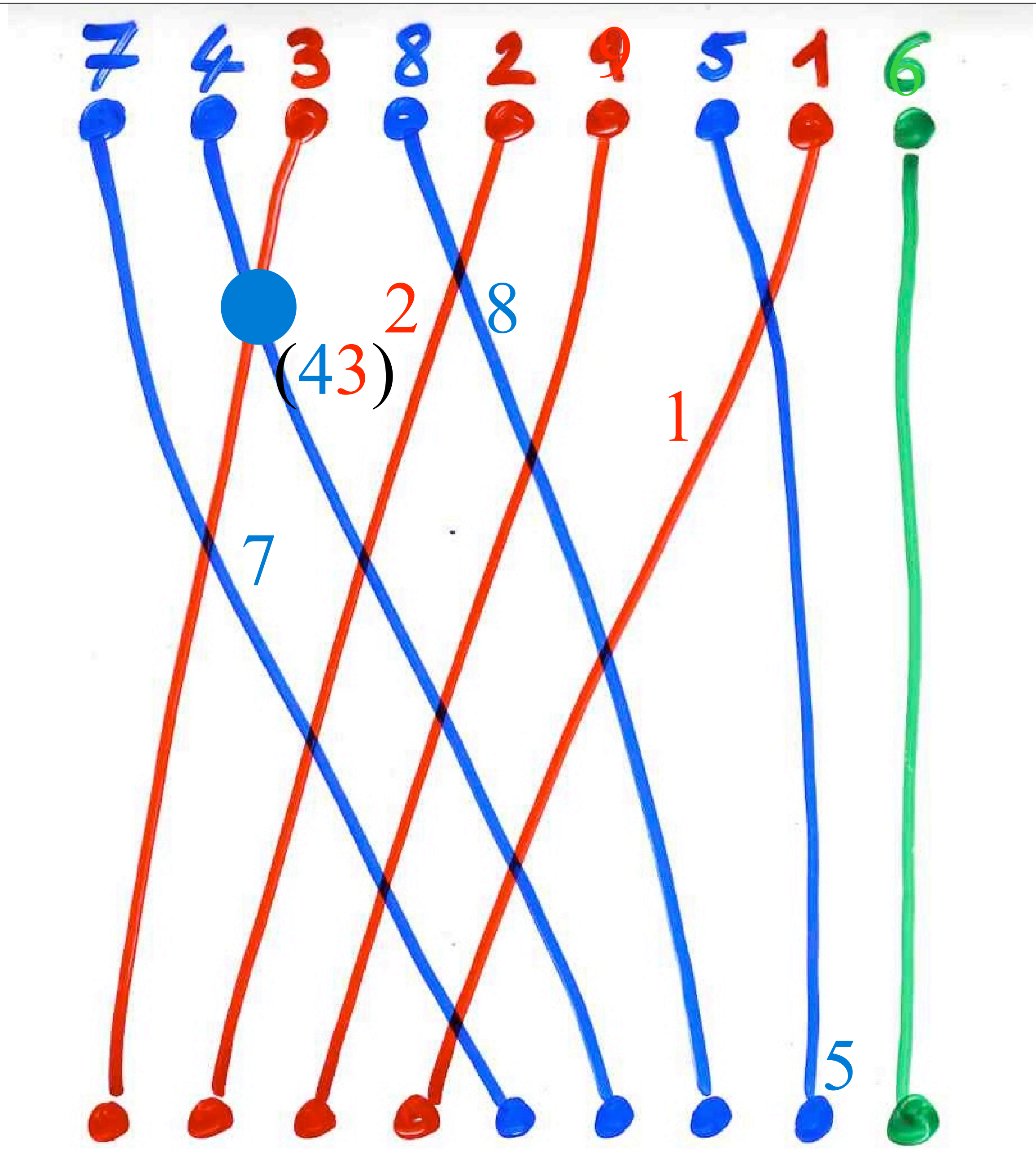

7 4 3 8 2 9 5 1 6
4 3 2 8


7 4 3 8 2 9 5 1 6
2 8
(43)

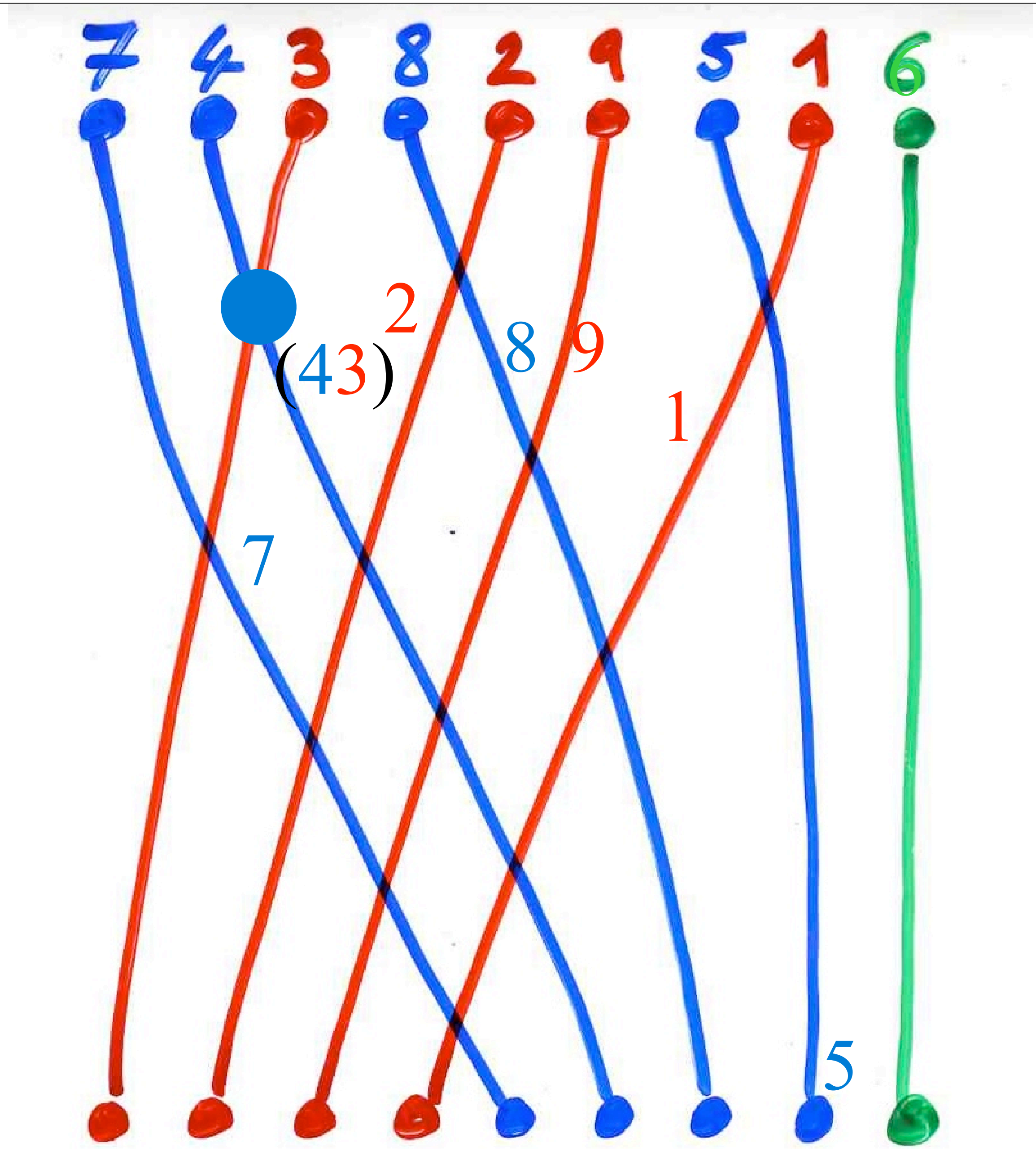

7 4 3 8 2 9 5 1 6
2
8
(43)
7



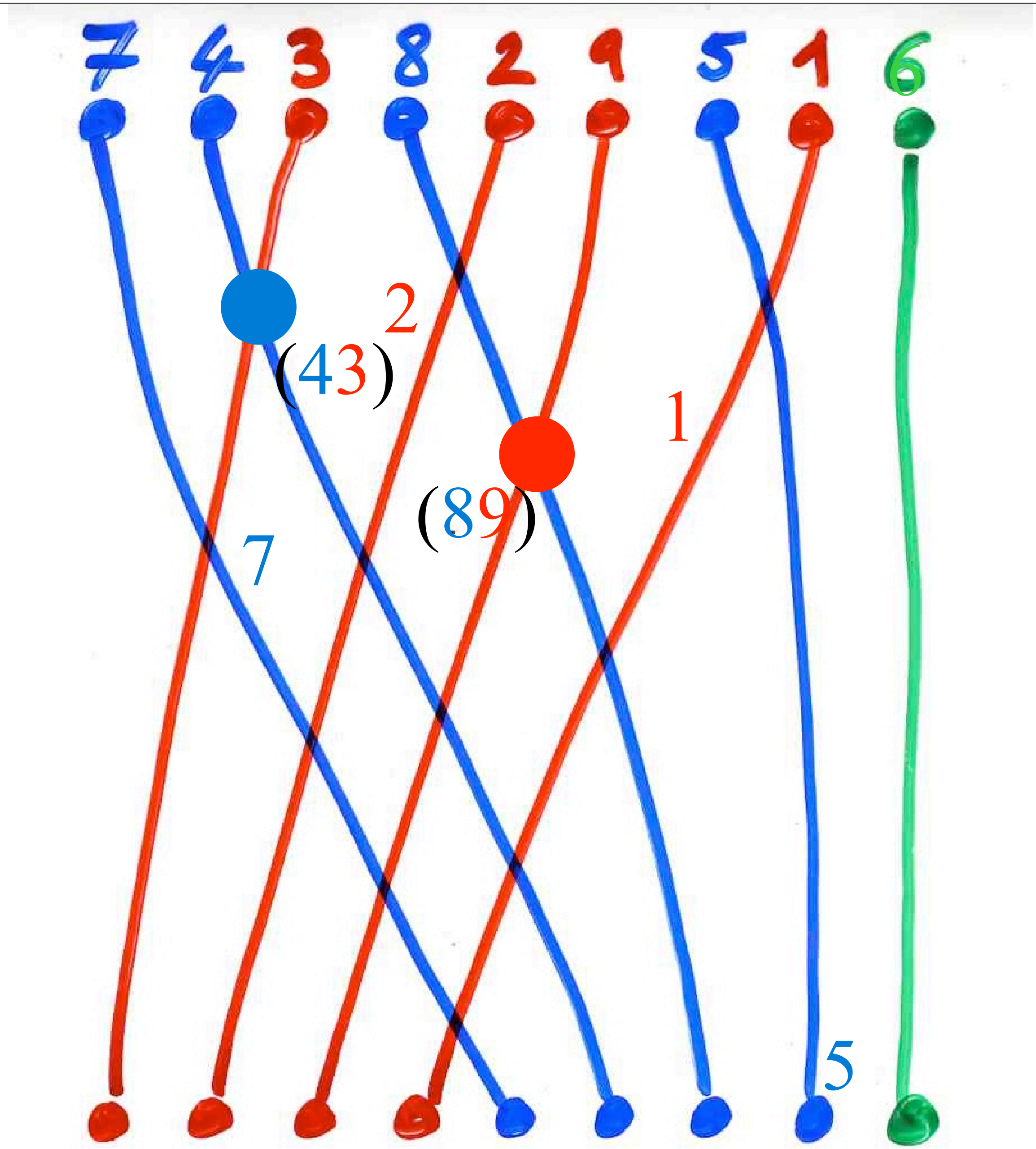
7 4 3 8 2 9 5 1 6
2
8
(43)
1
5
7

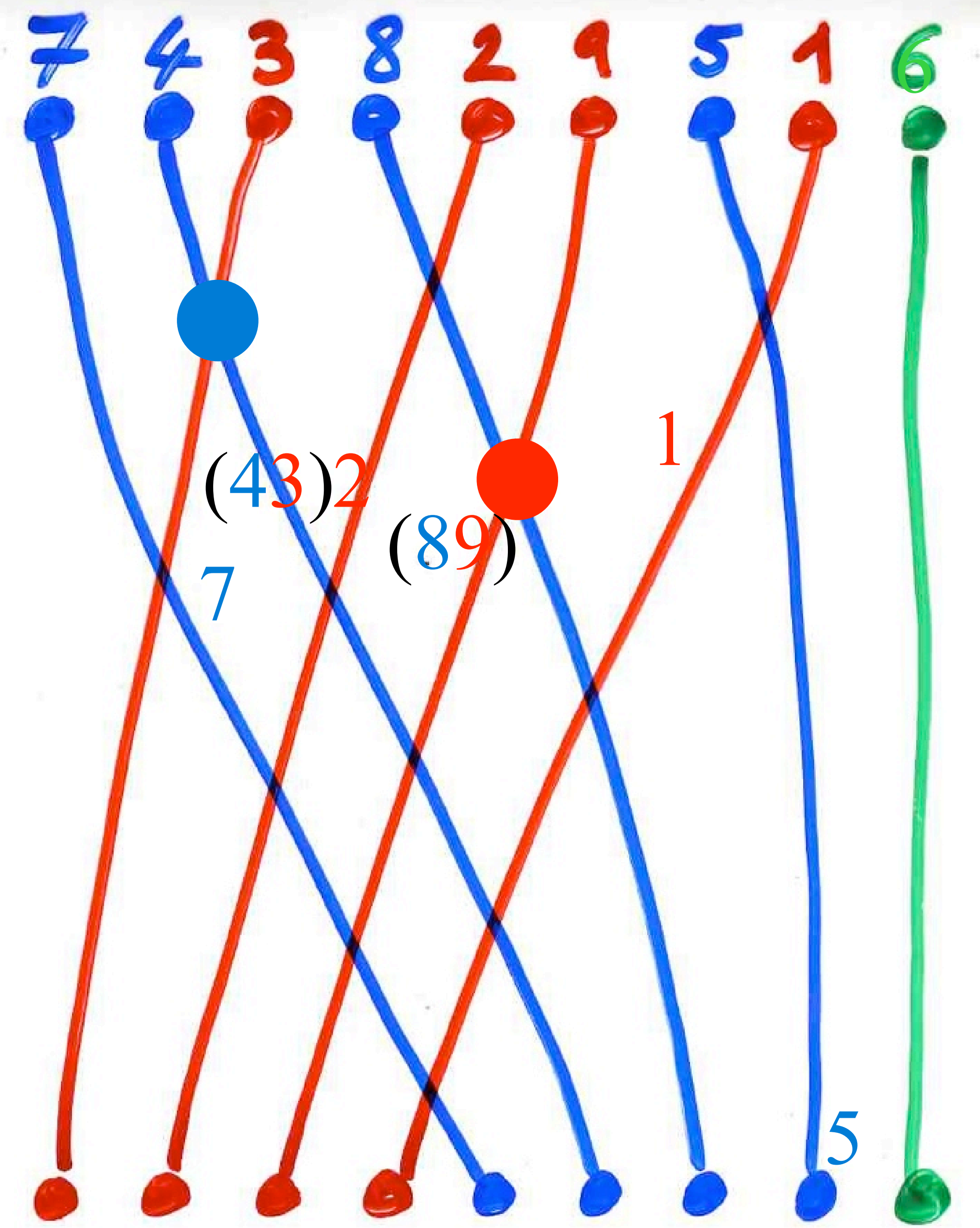


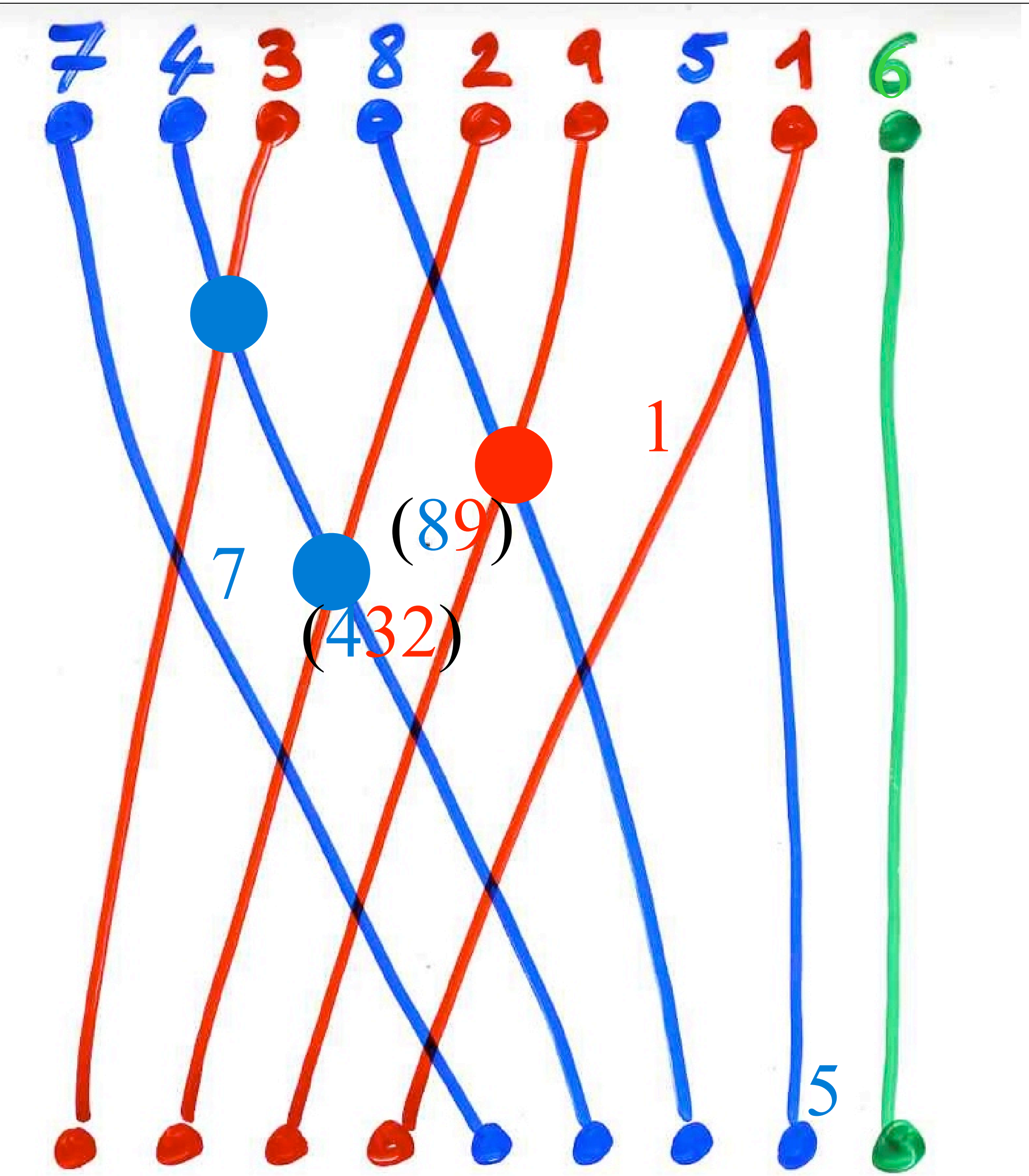
7 4 3 8 2 9 5 1 6
2
8
(43)
1
7
5



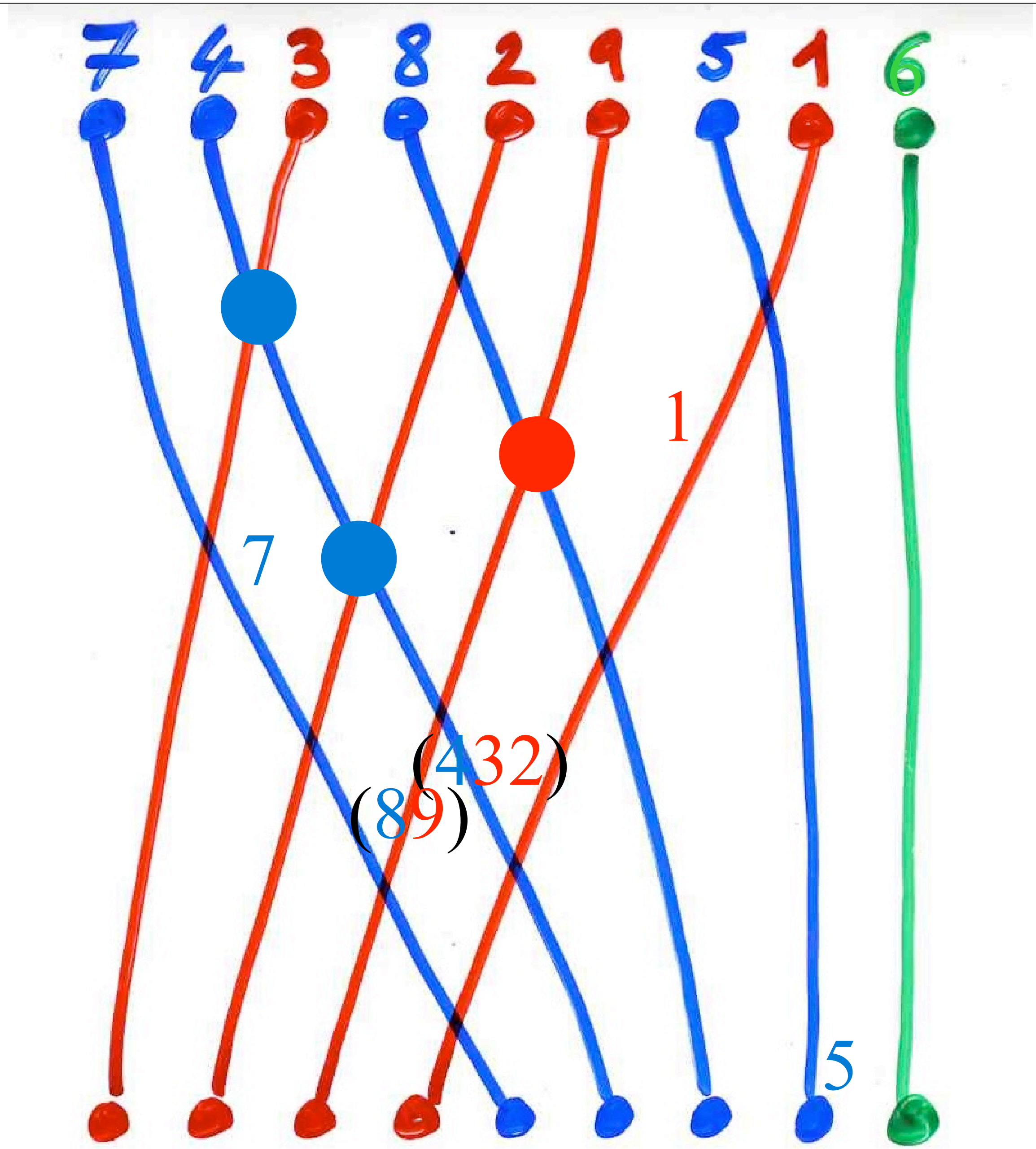
7 4 3 8 2 9 5 1 6
2
(43)
8
9
1
7
5


7 4 3 8 2 9 5 1 6
(43)
2
(89)
1
7
5

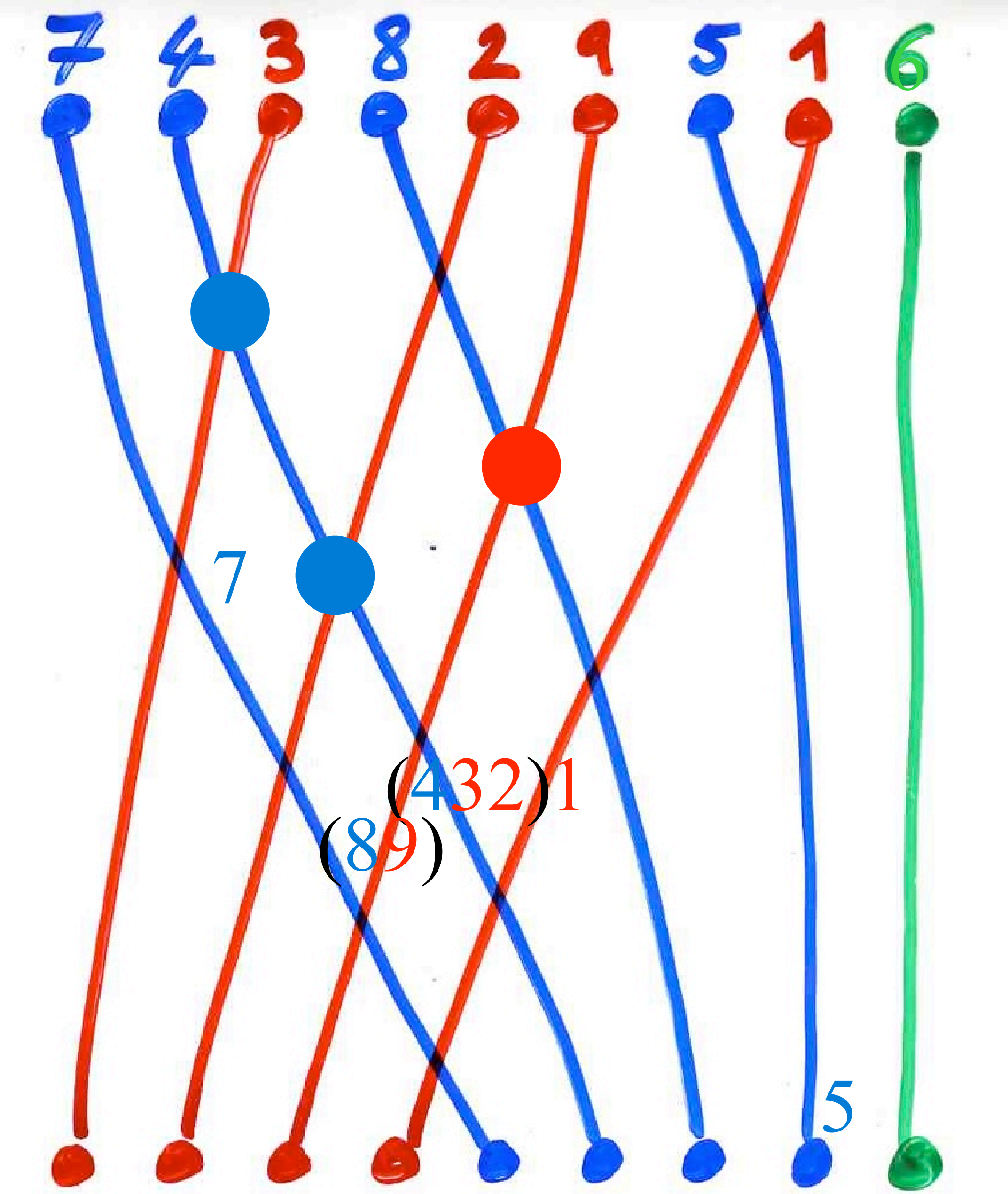

7 4 3 8 2 9 5 1 6
(43)2
(89)
1
7
5



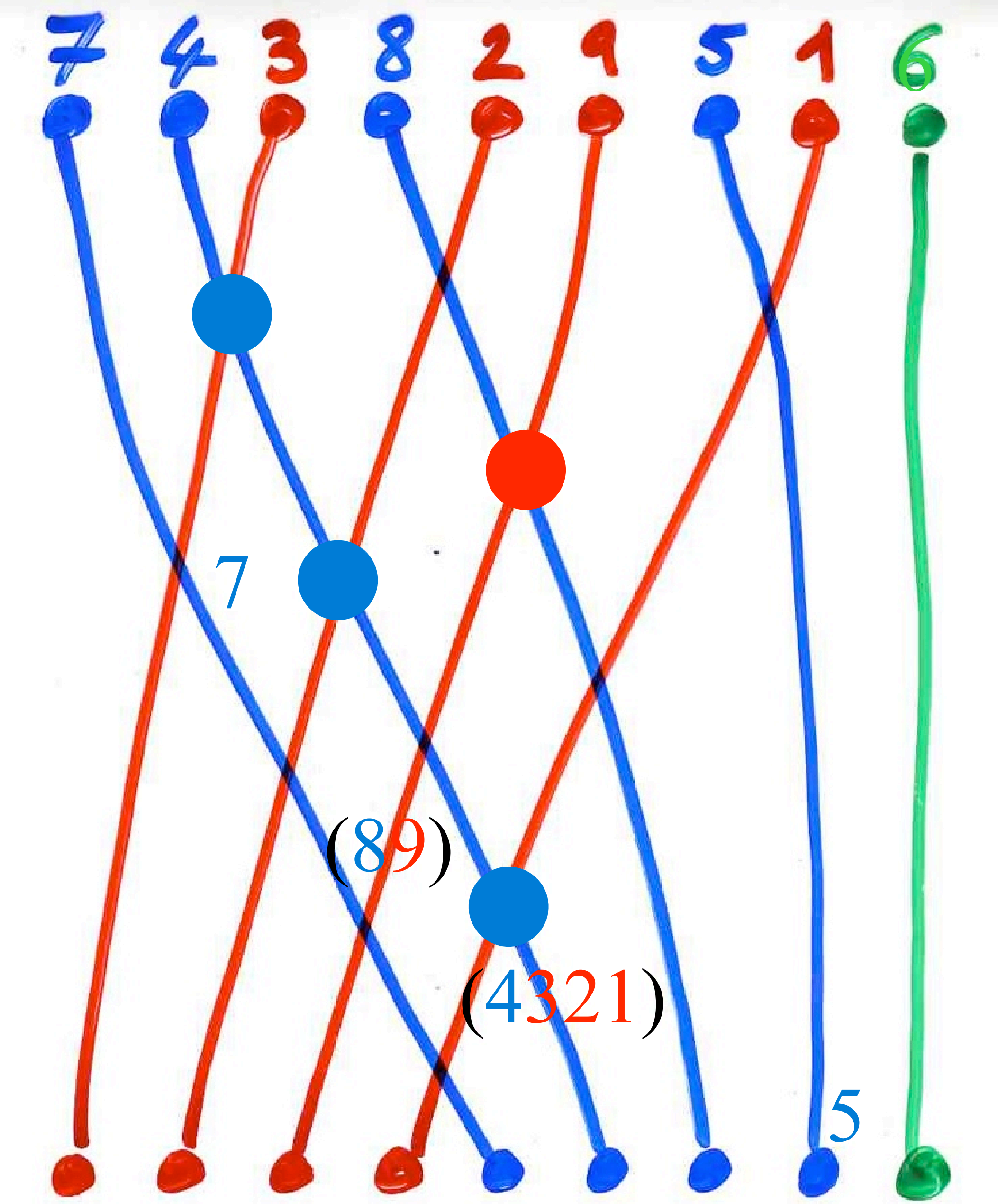
7 4 3 8 2 9 5 1 6
1
(89)
7
(432)
5



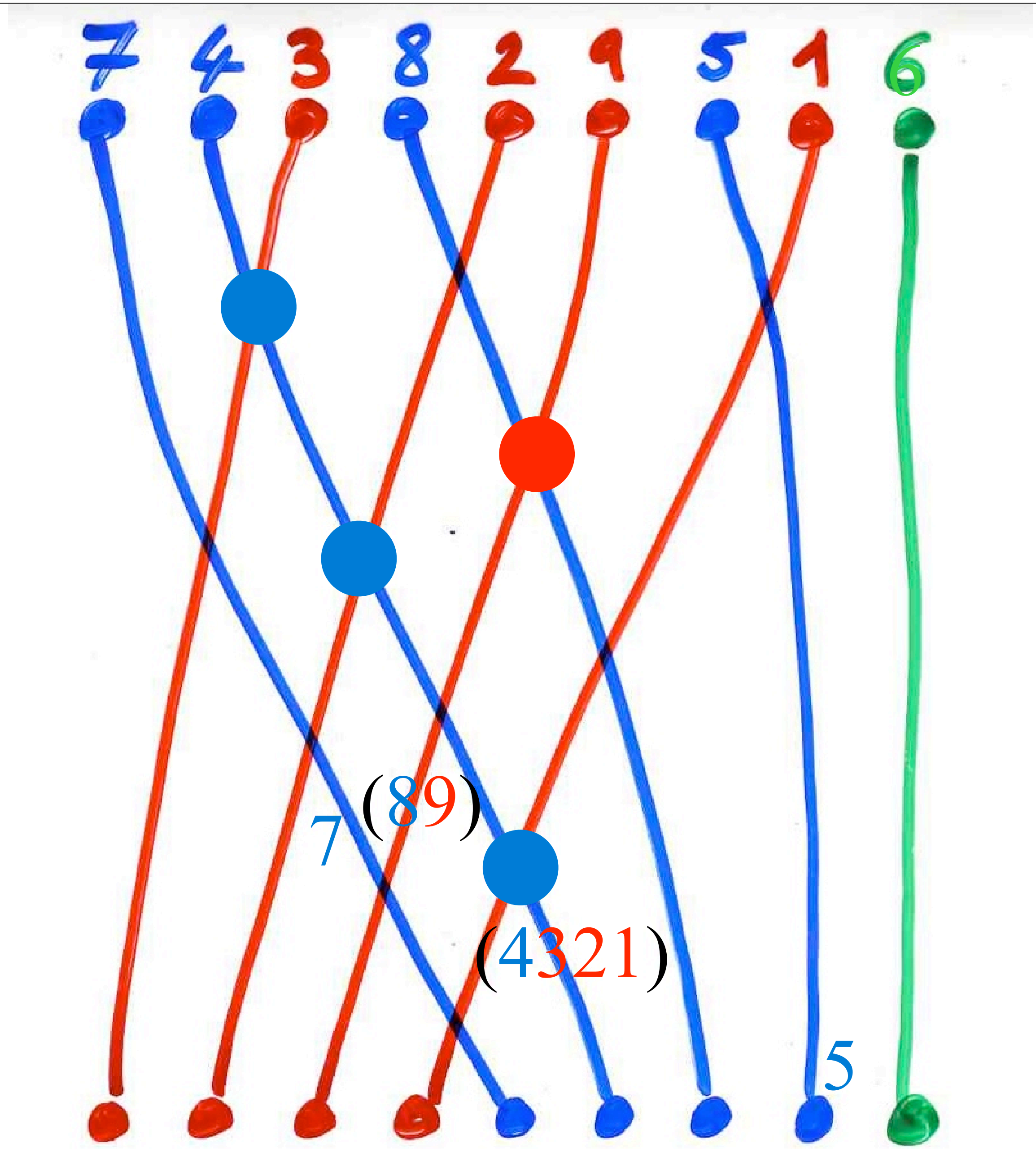
7 4 3 8 2 9 5 1 6
1
7
(432)
(89)
5



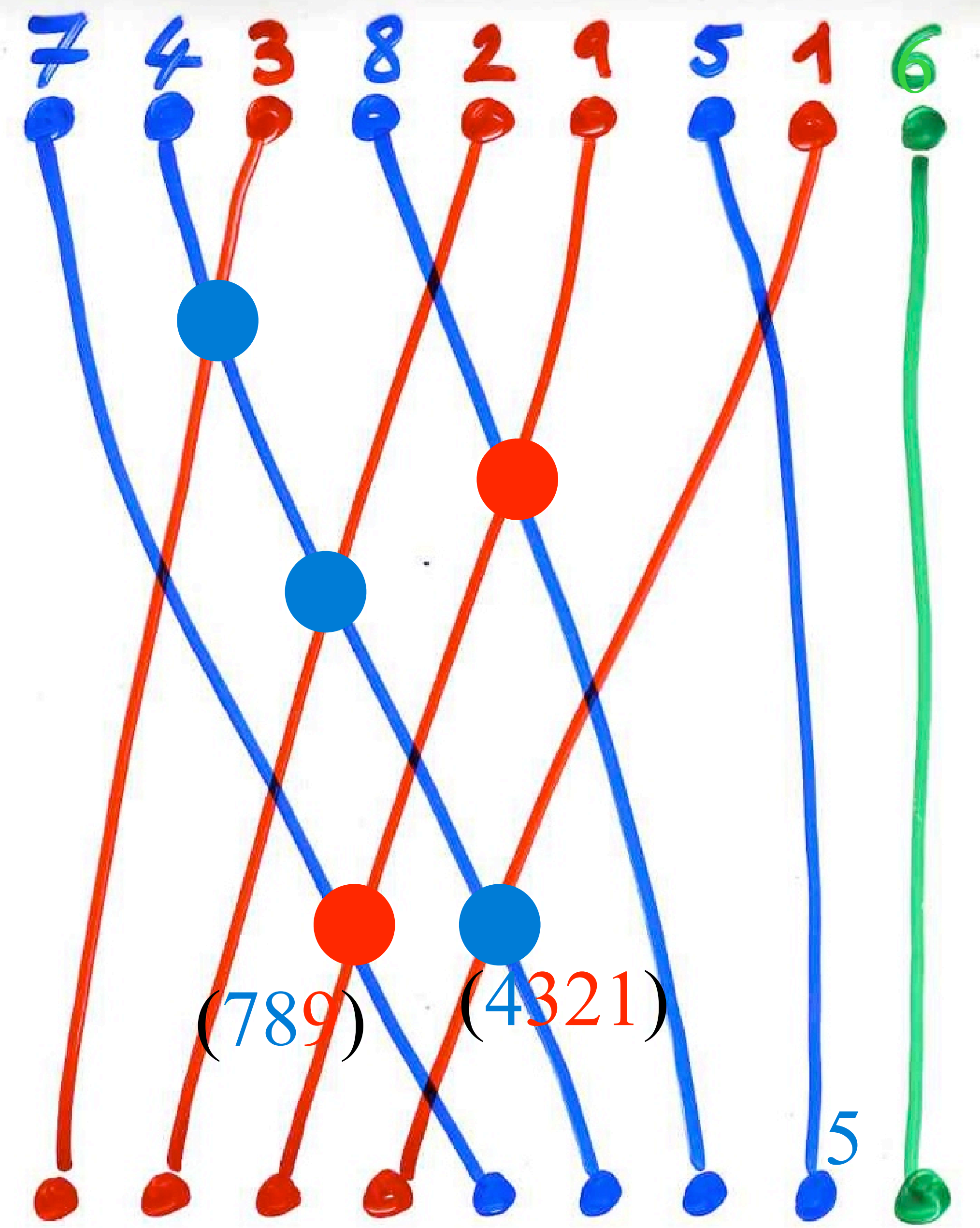
7 4 3 8 2 9 5 1 6
7
(432)1
(89)
5



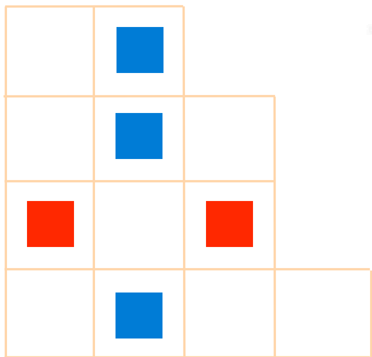
7 4 3 8 2 9 5 1 6
7
(89)
(4321)
5

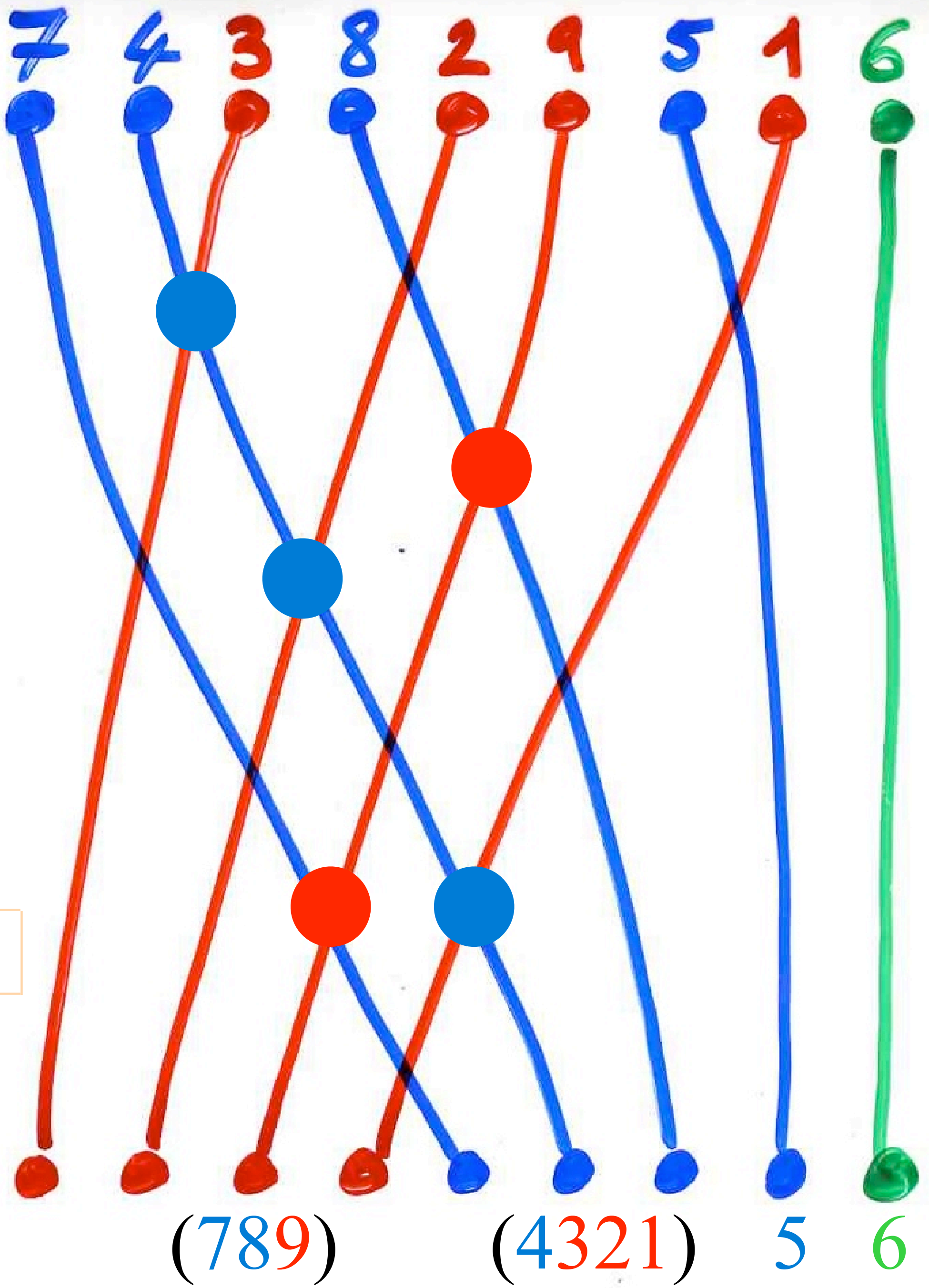


7 4 3 8 2 9 5 1 6
(89)
7
(4321)
5


7 4 3 8 2 9 5 1 6
(789)
(4321)
5

“exchange-fusion” algorithm





Description of the "blocks" (= words) appearing at the end of the "exchange-fusion" algorithm

$\sigma = \sigma(1) \cdots \sigma(n)$

u, v } subword of σ, with letters { $< \sigma(n+1)$, $> \sigma(n+1)$

$u = x_1 u_1 \cdots x_k u_k$; $\{x_1, \ldots, x_k\}$ left-to-right maxima

$v = v_\ell y_\ell \cdots v_1 y_1$; $\{y_1, \ldots, y_\ell\}$ right-to-left maxima

Prop. (Bernardi, Nadeau, xgv)

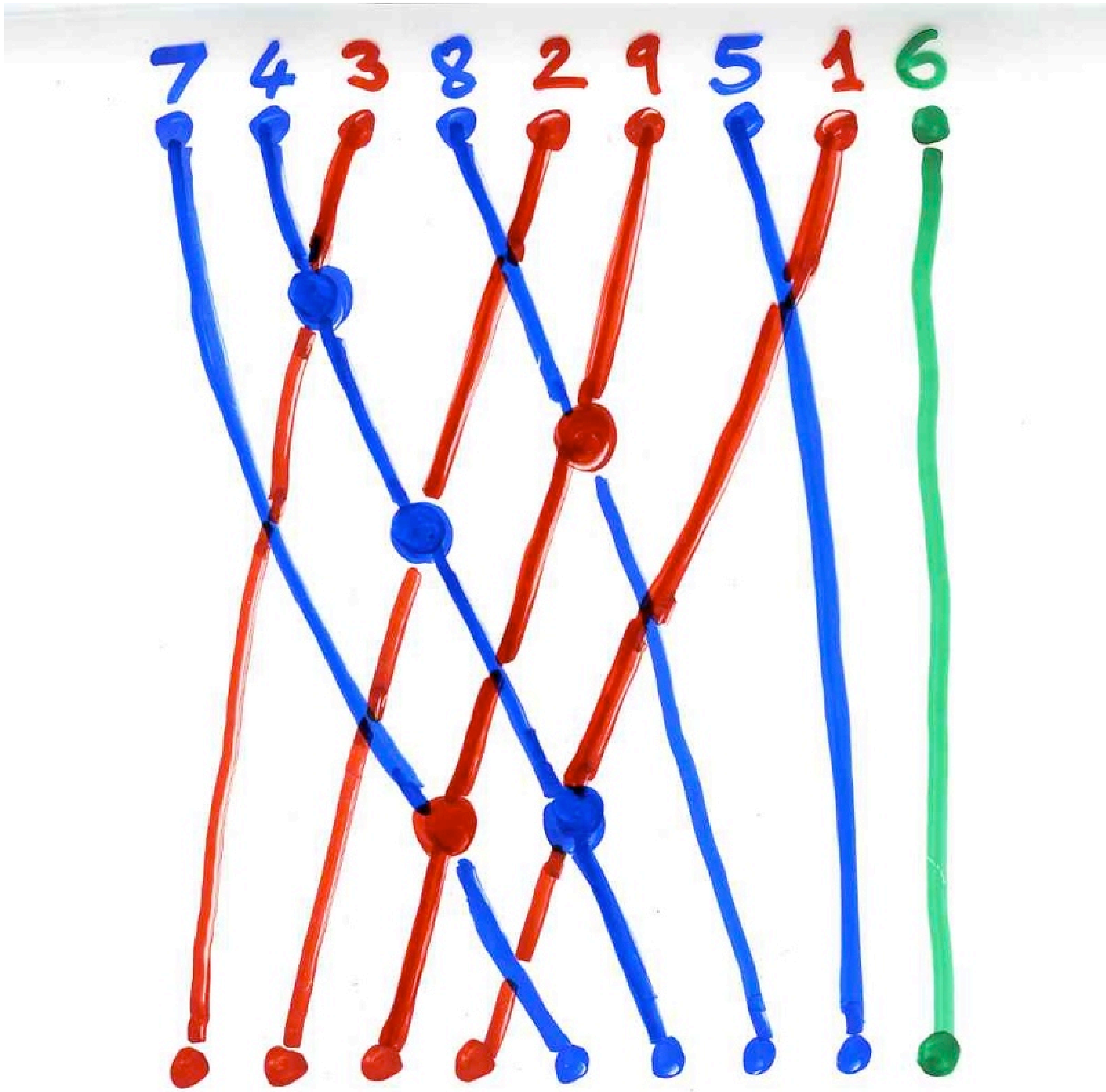
The "final" blue blocks are

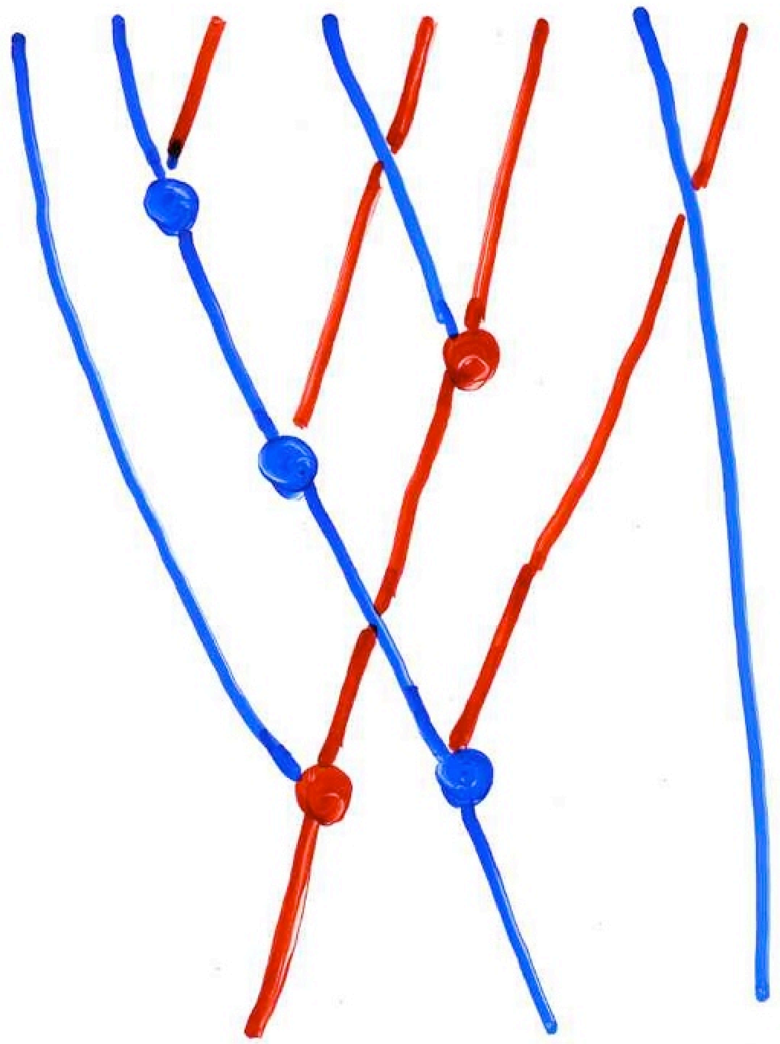
$(x_1 u_1), \ldots, (x_k u_k)$

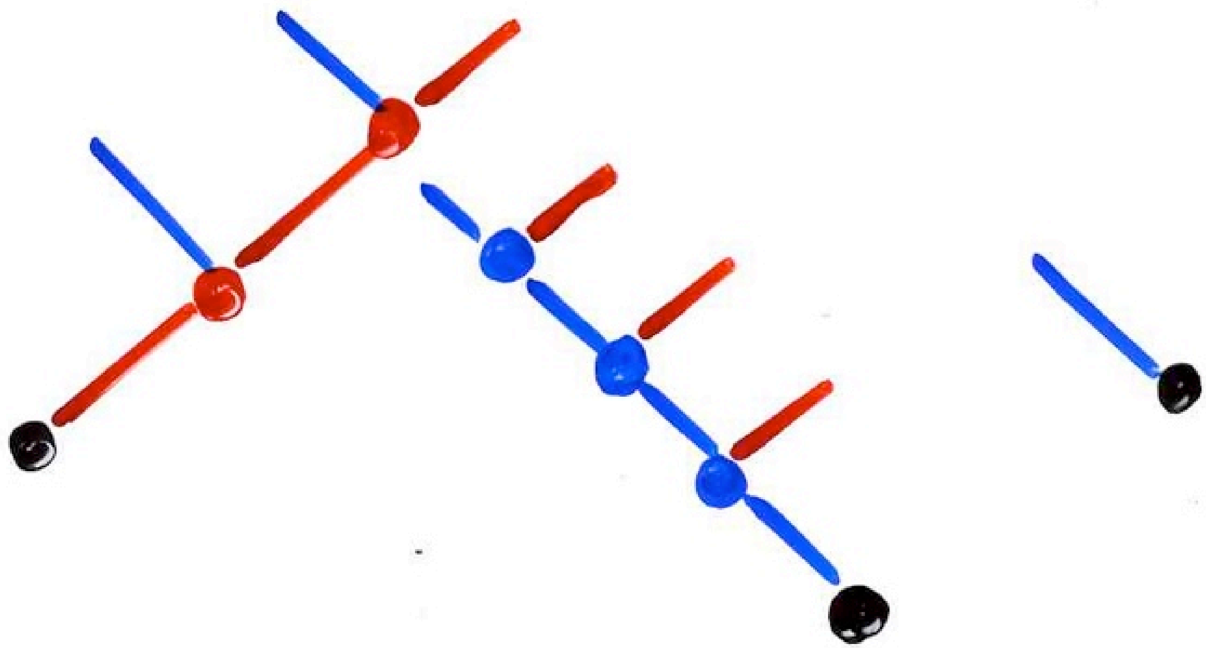
The "final" red blocks are

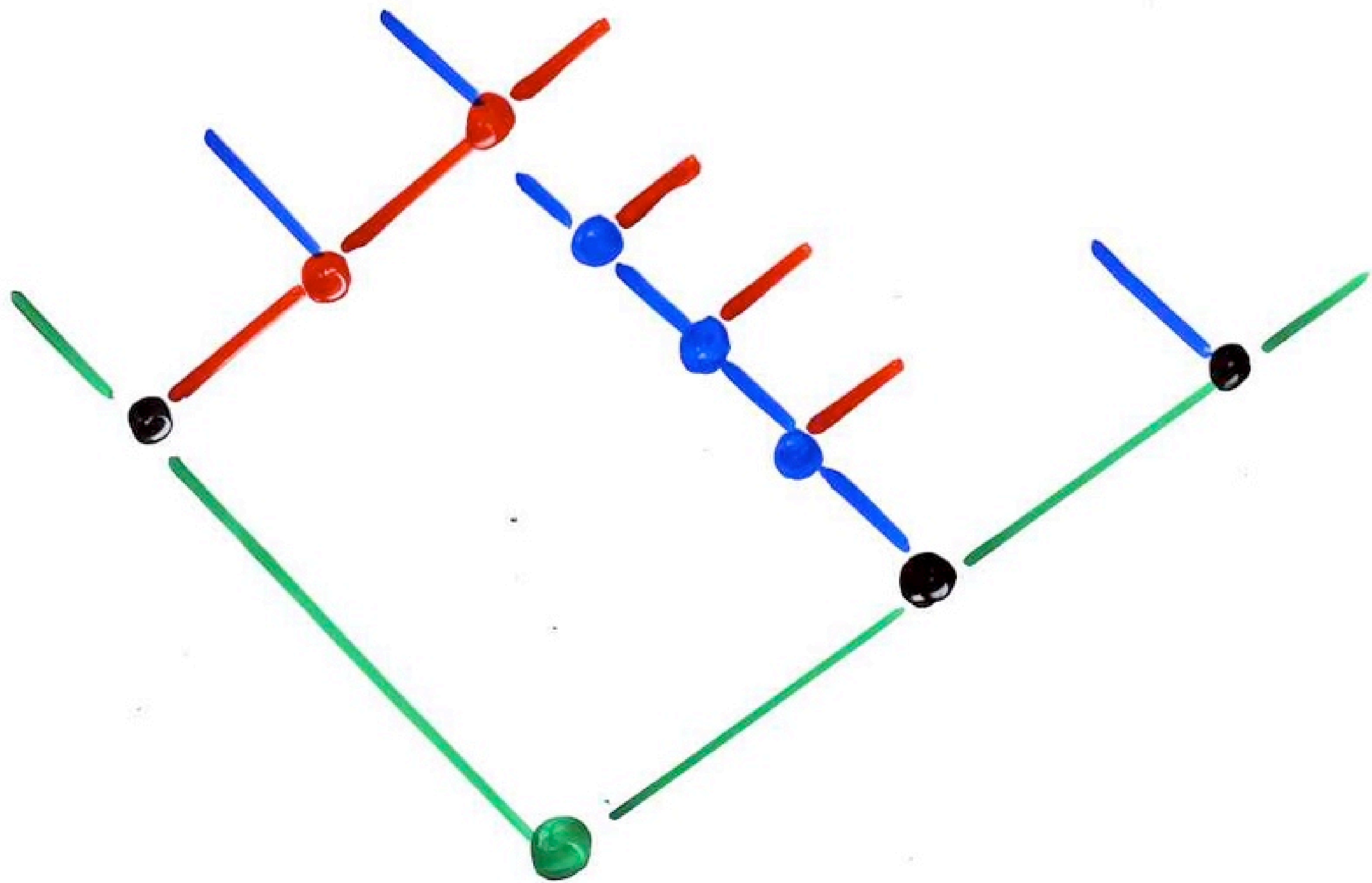
$(v_\ell y_\ell), \ldots, (v_1 y_1)$

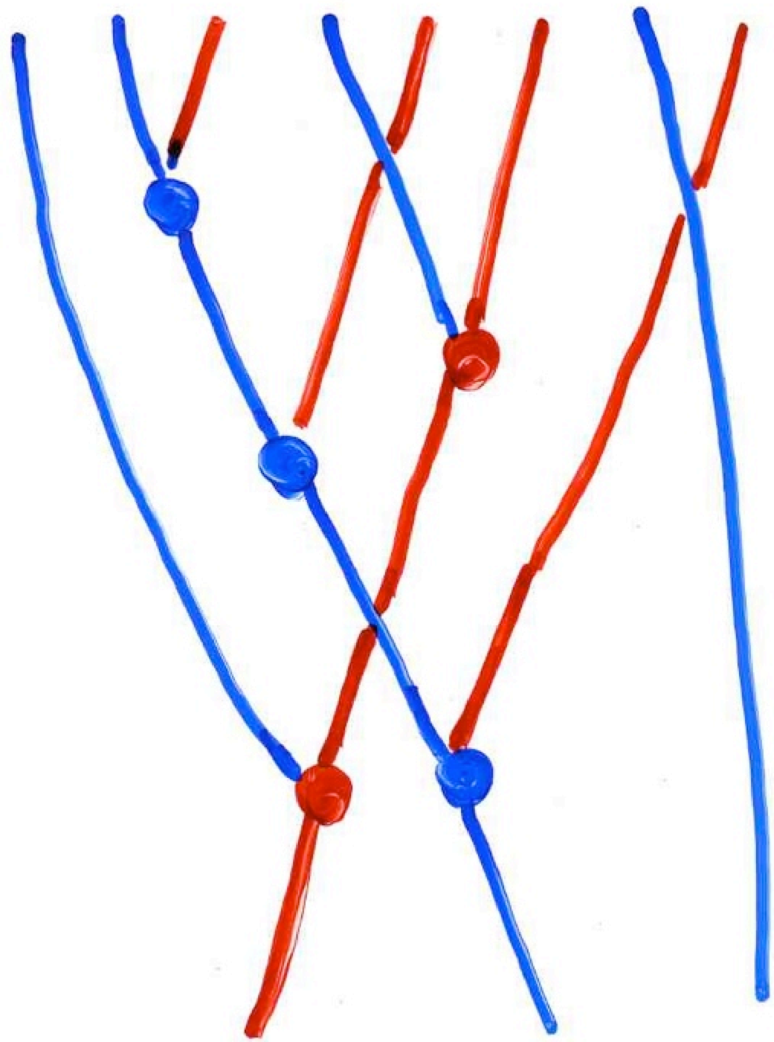
§3 The inverse "exchange-fusion" algorithm

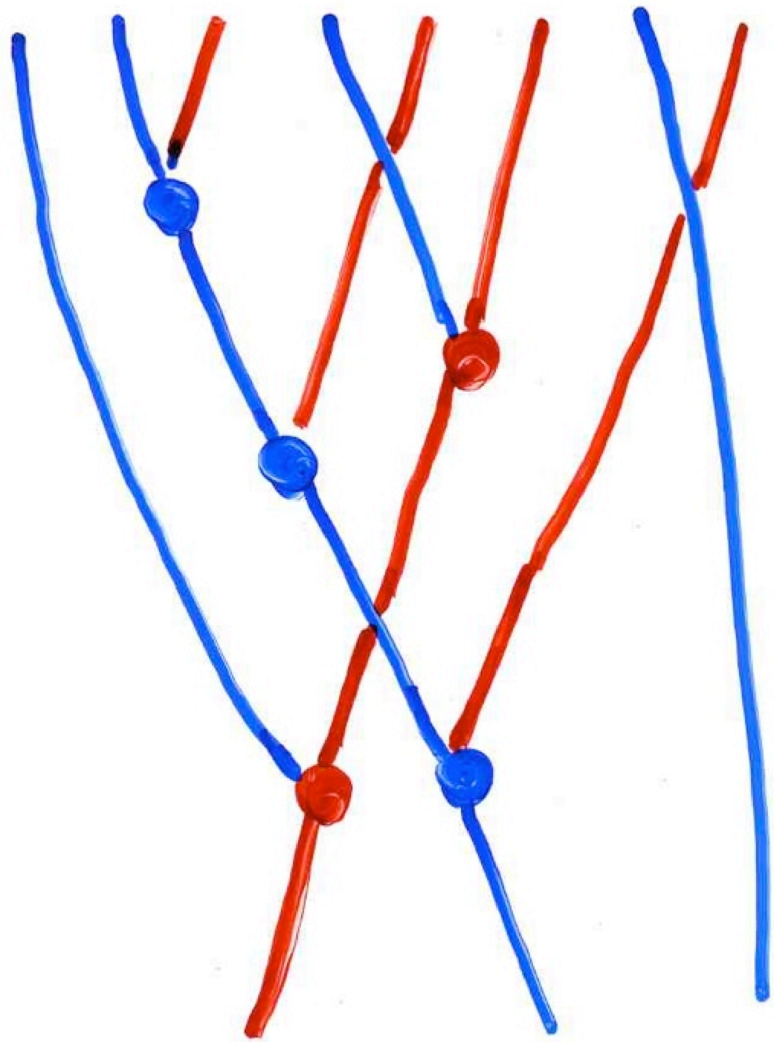

7 4 3 8 2 9 5 1 6



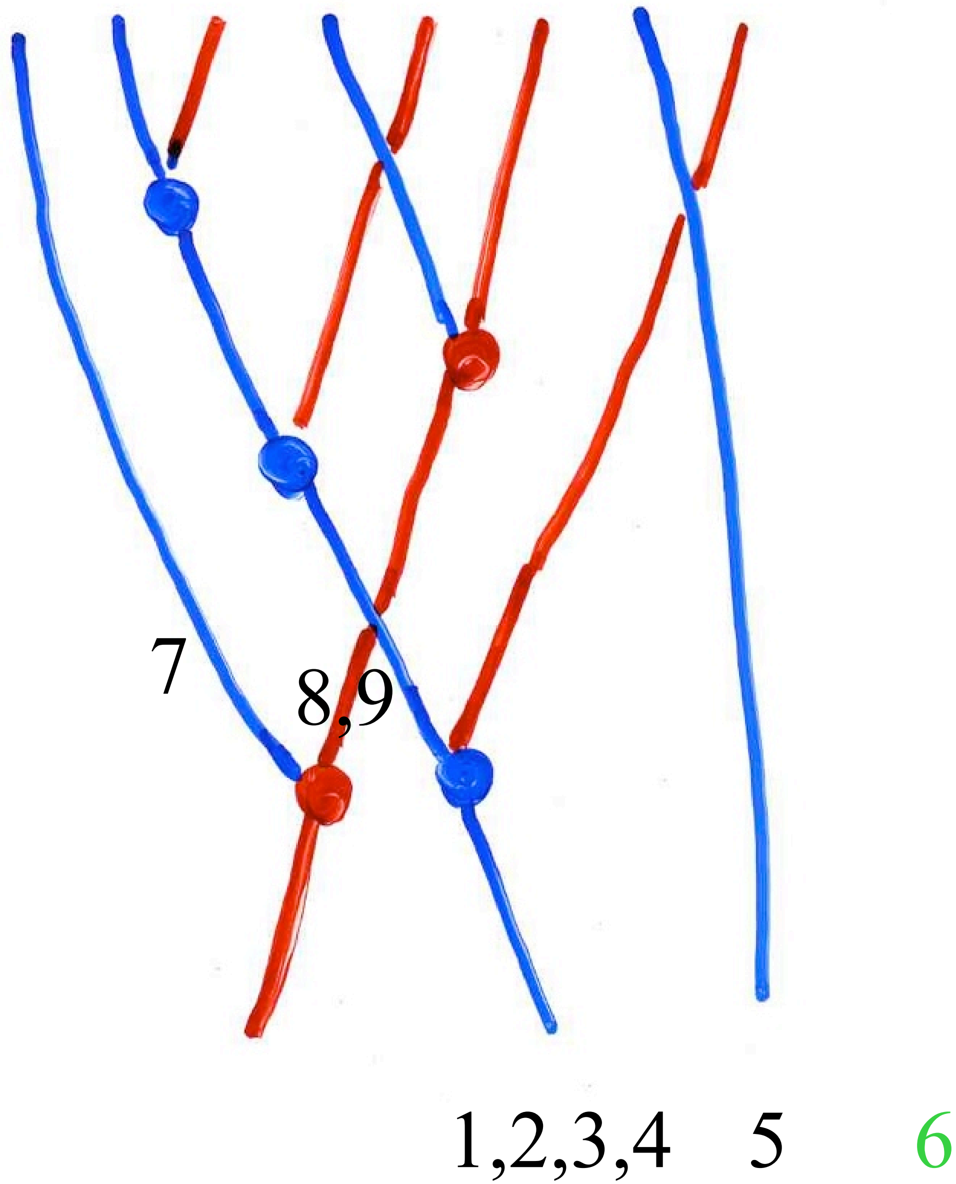


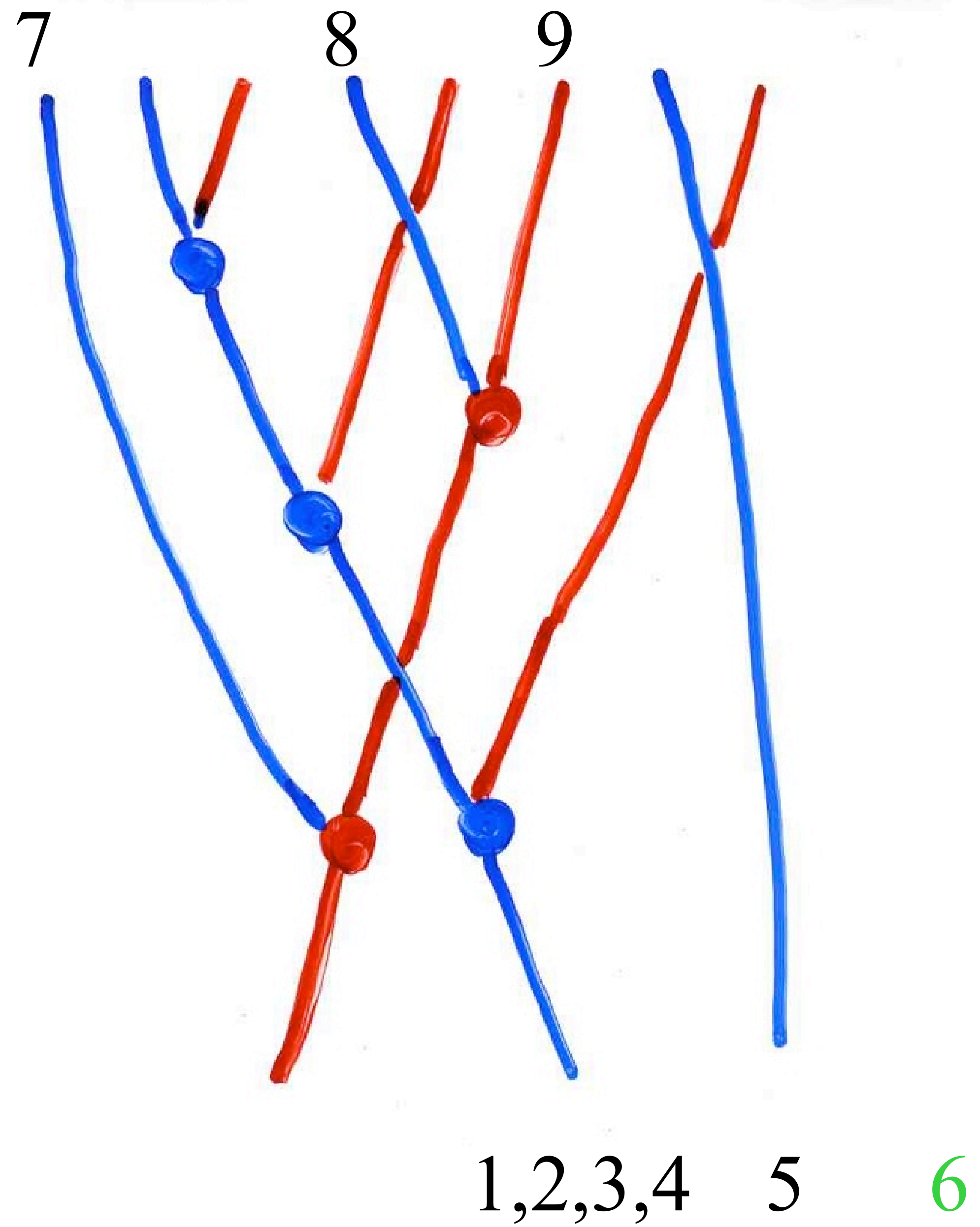


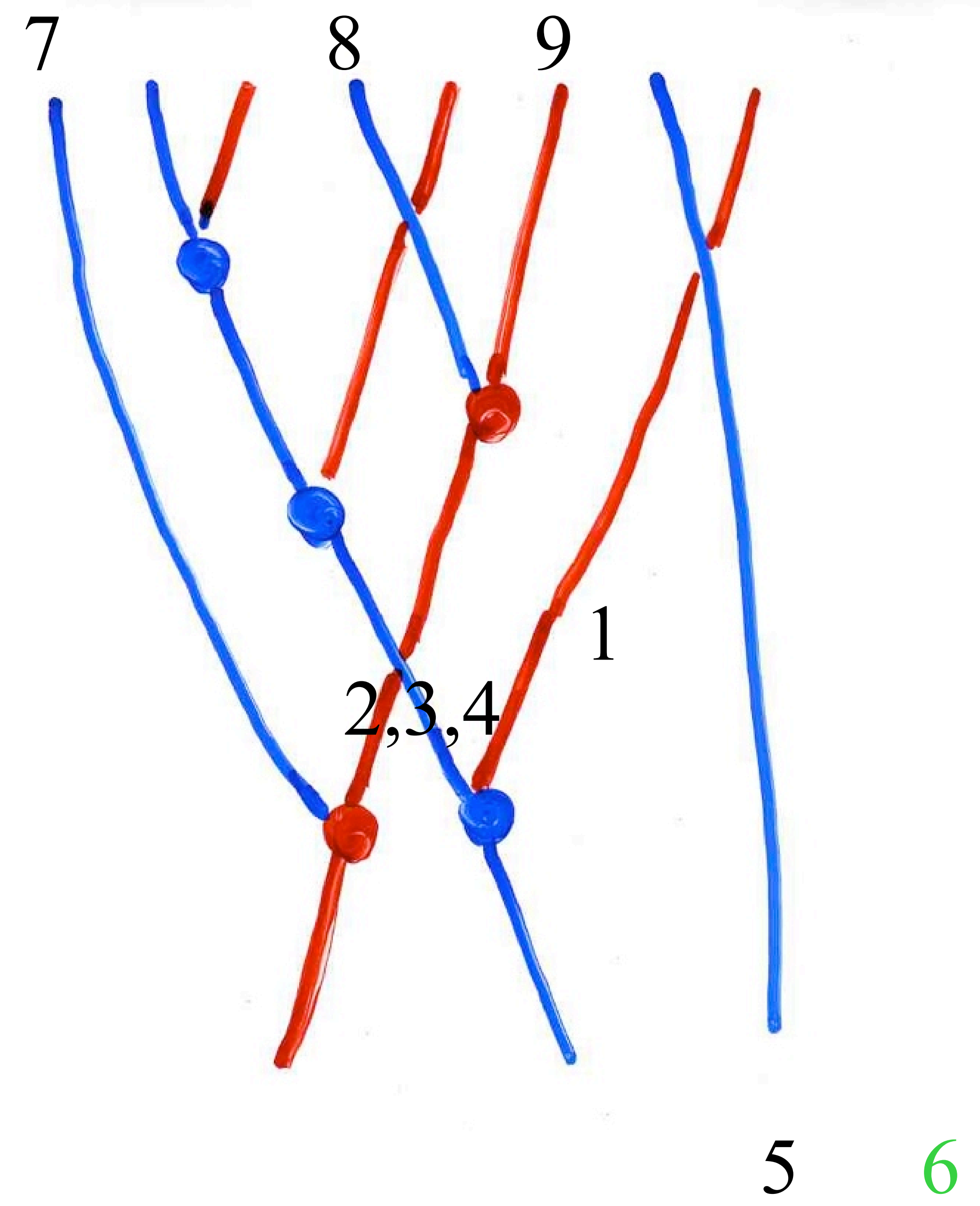


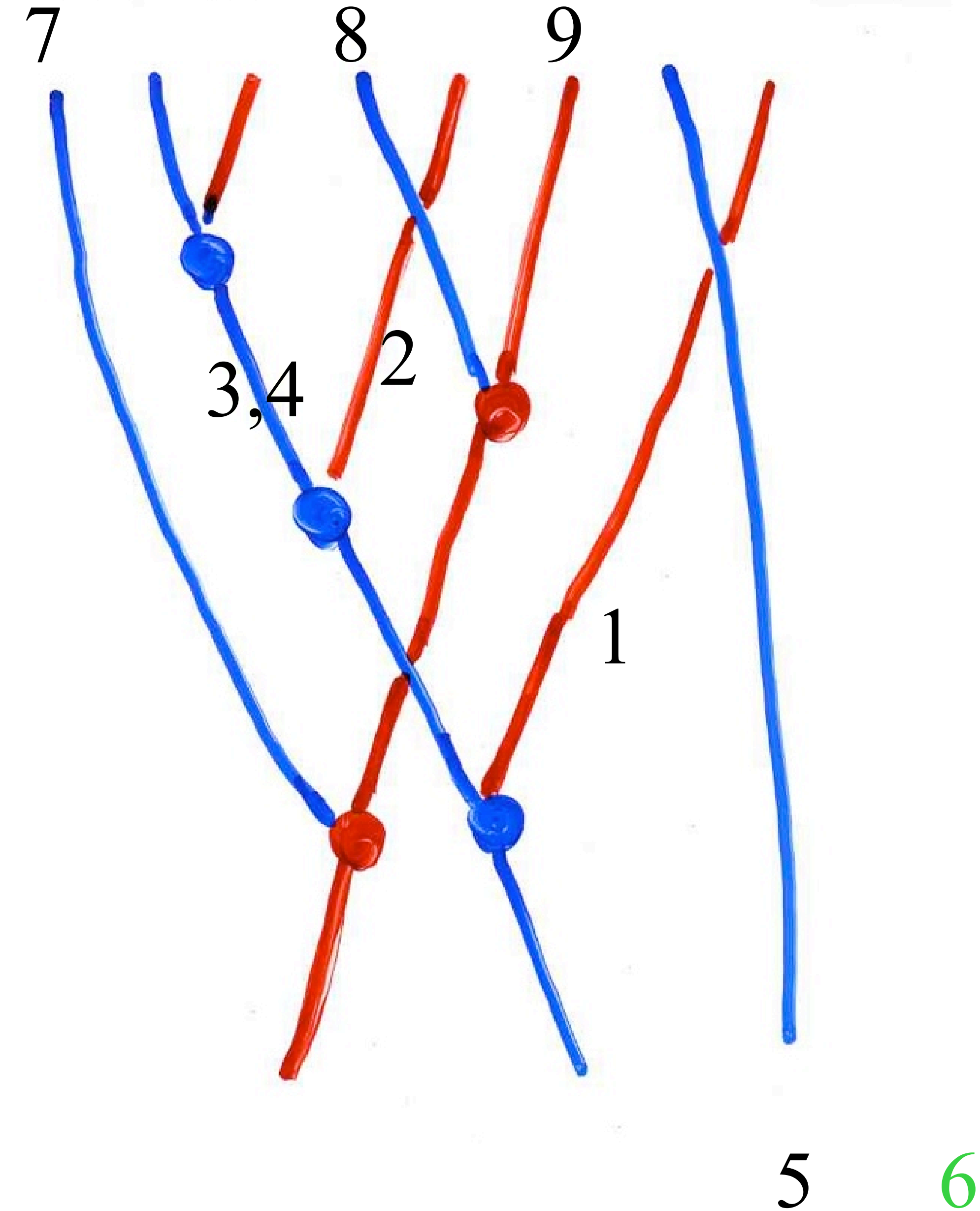


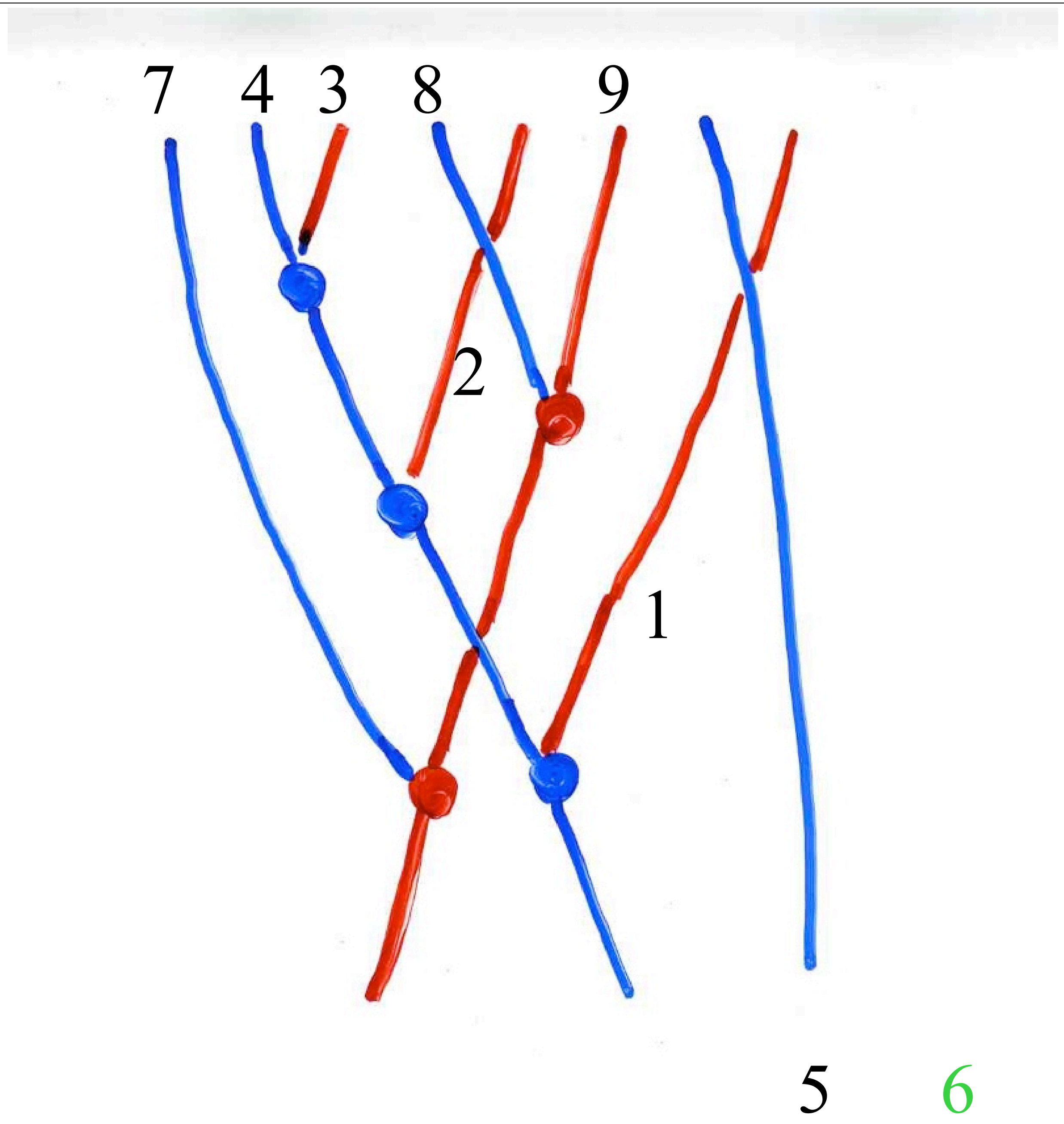
7,8,9 1,2,3,4 5 6

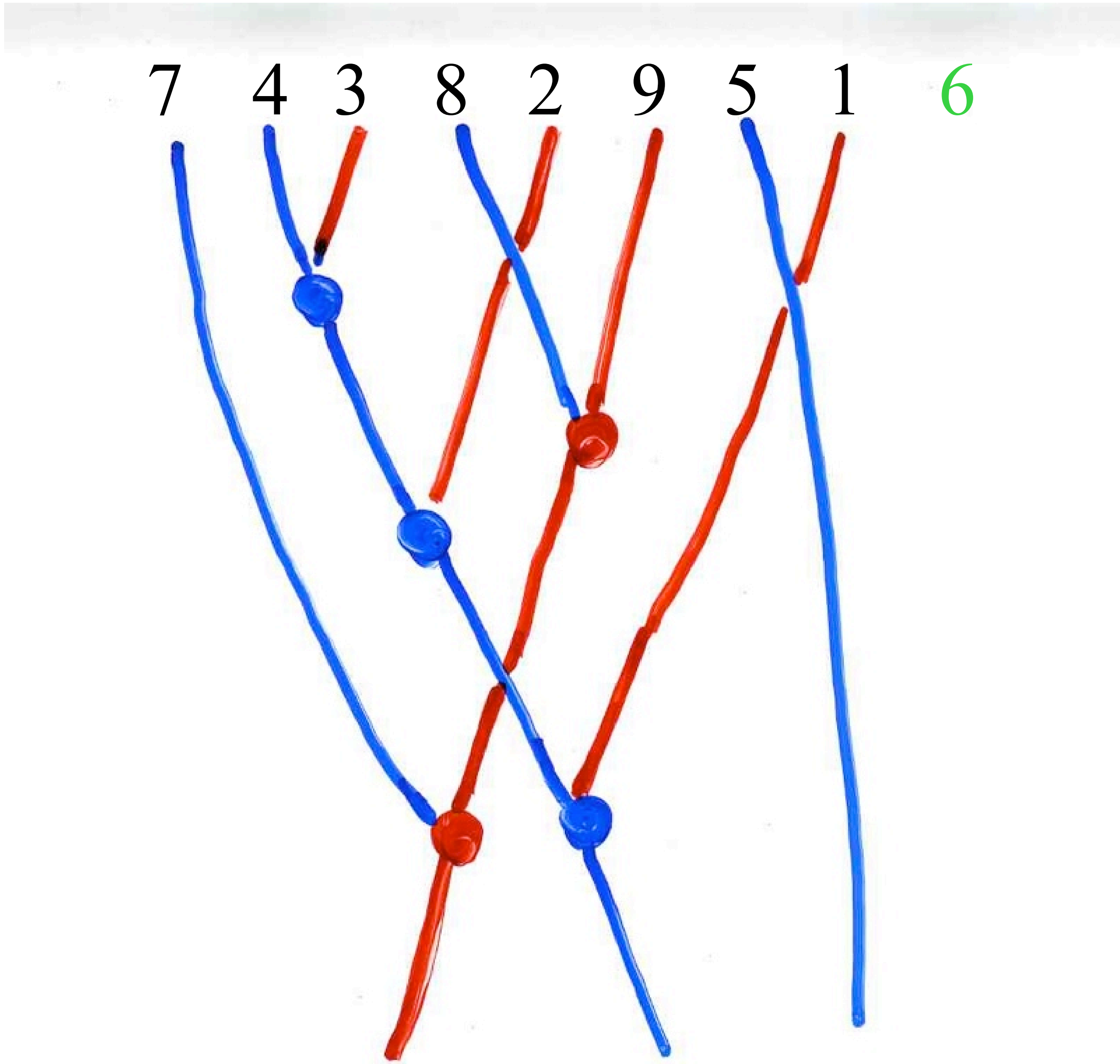

7
8,9
1,2,3,4
5
6


7
8
9
1,2,3,4
5
6


7
8
9
1
2,3,4
5
6


7
8
9
3,4
2
1
5
6


7
4
3
8
9
2
1
5
6


7 4 3 8 2 9 5 1 6

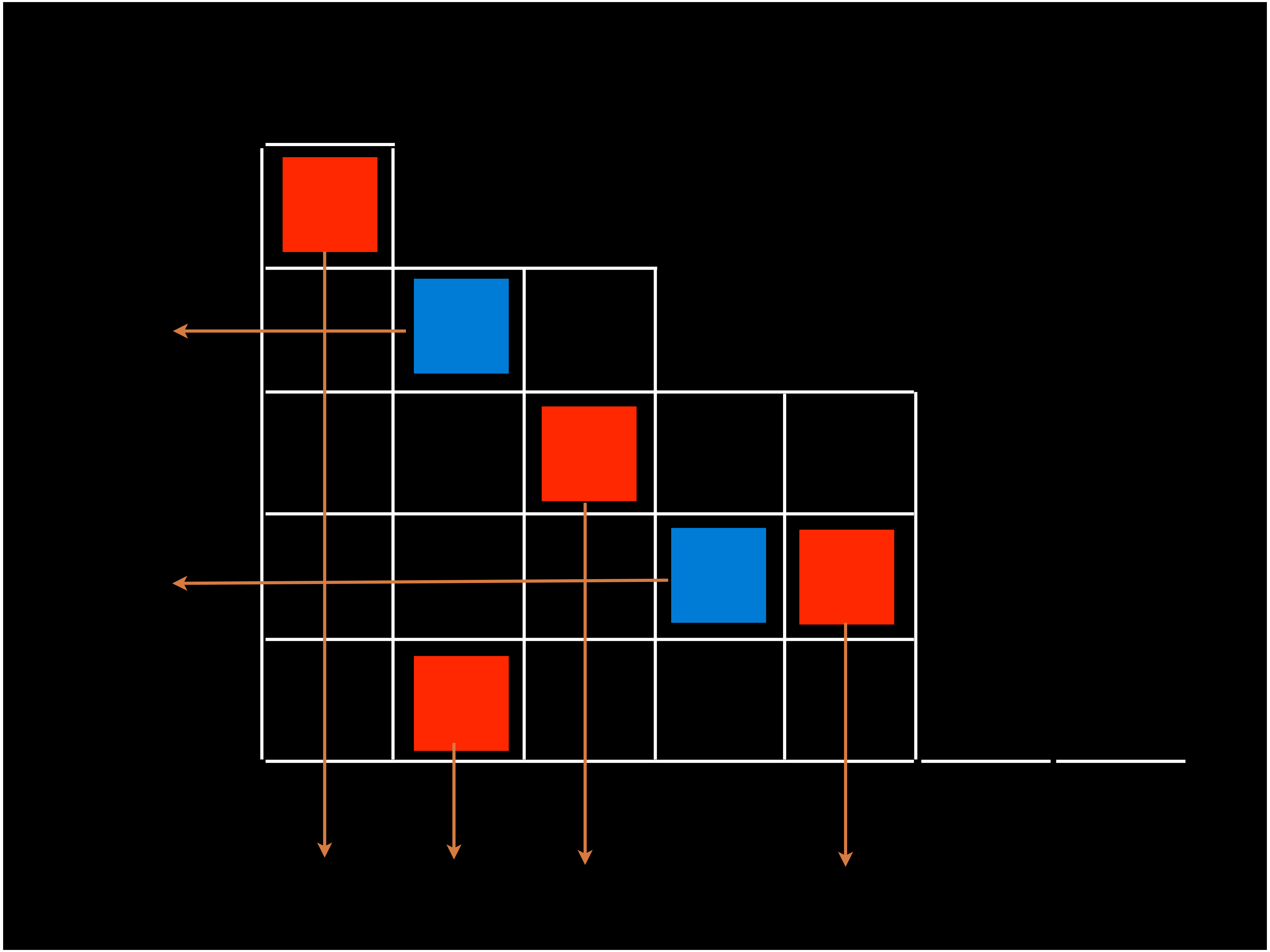
§ 4 The q-Laguerre parameter

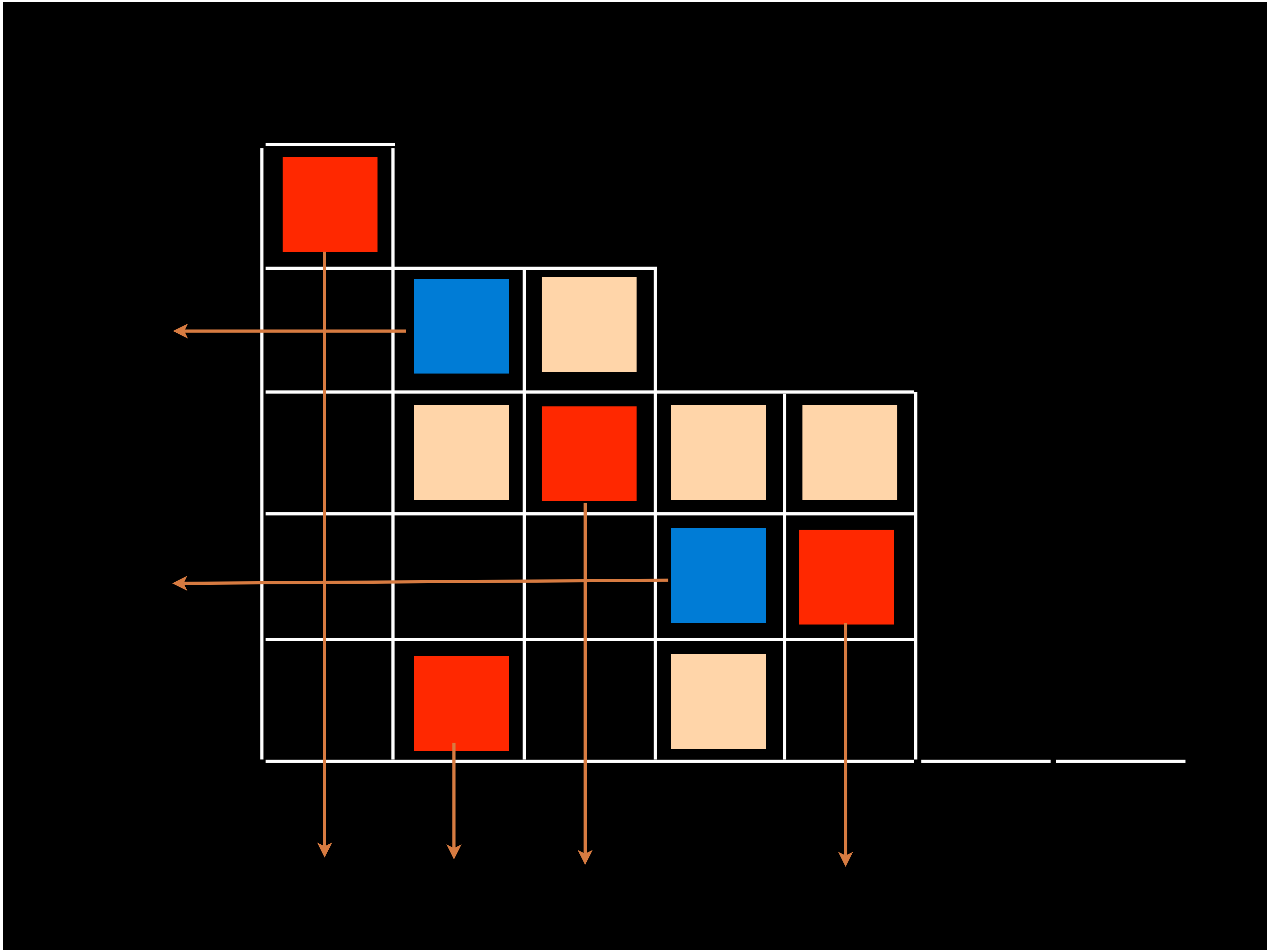
Def - Crossing of an alternative tableau:
a non-colored cell which is
neither at the left of a blue cell
neither below a red cell

crossing of alternating tableau

↕

exchange in the "exchange-fusion"
algorithm





From work of Corteel, Nadeau, Steingrimsson, Williams

we know that parameter

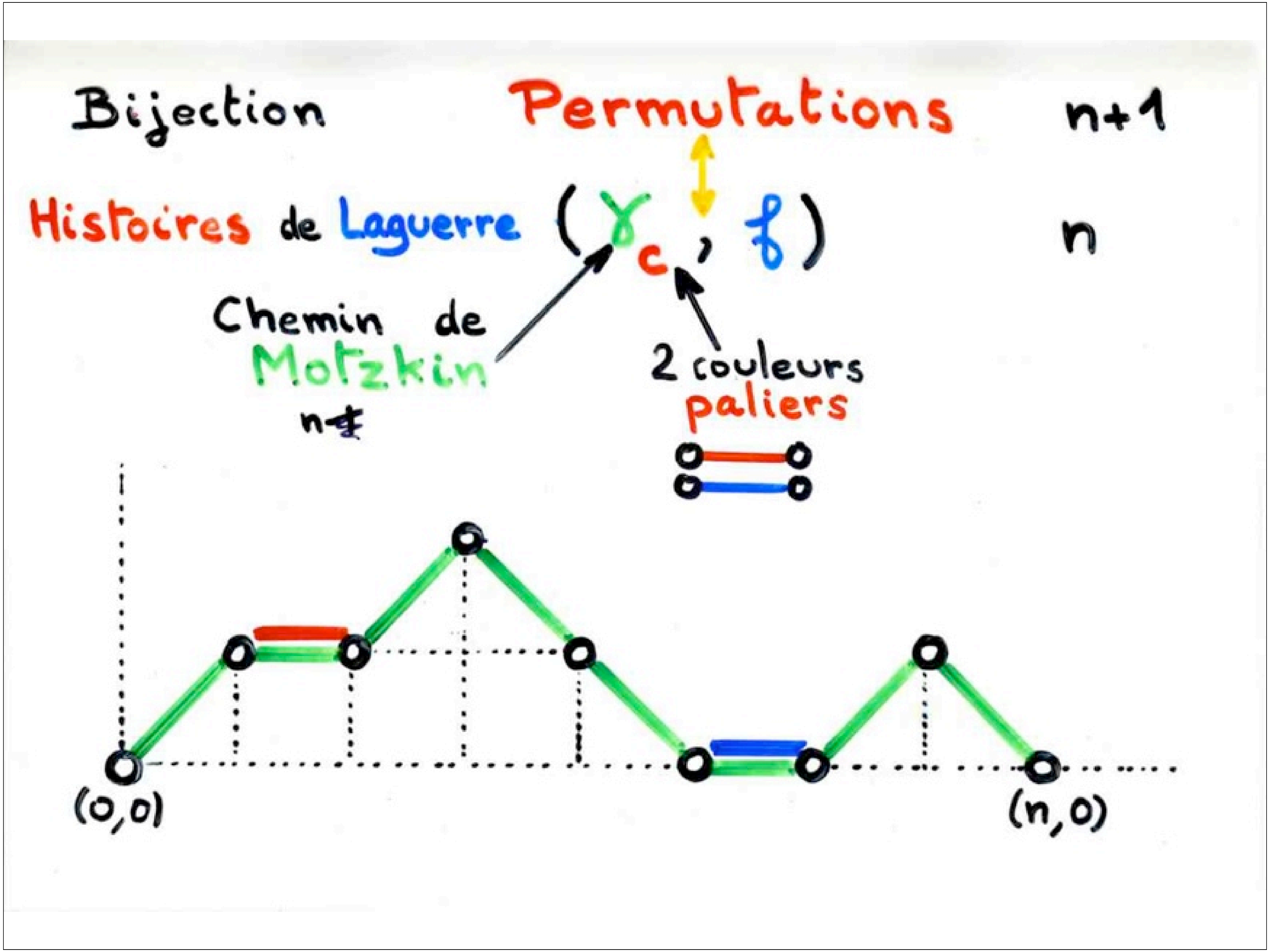
"number of crossing" in alternating tableaux :

same distribution as

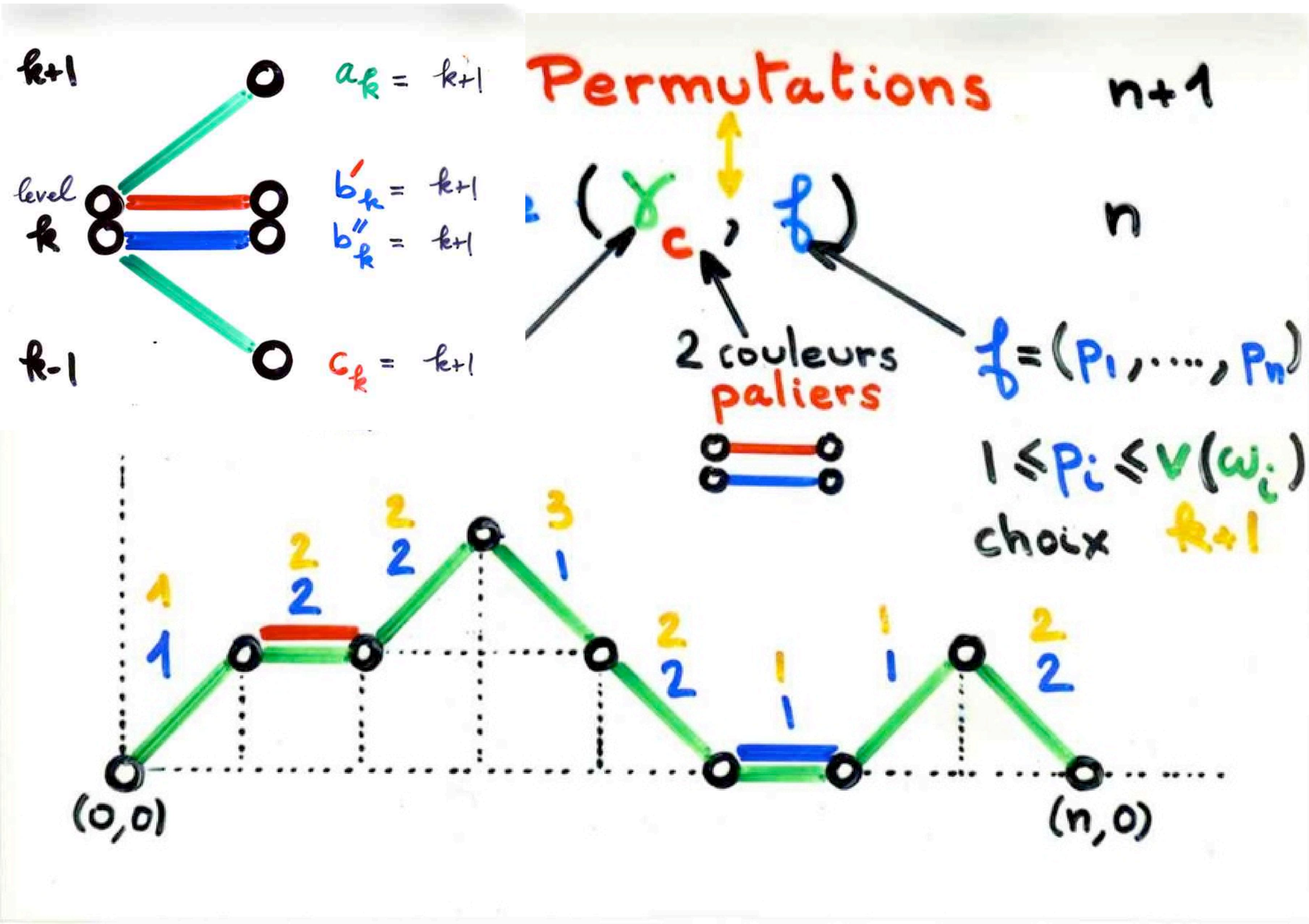
"q-analog of Laguerre histories"

Bijection
Laguerre histories
permutations

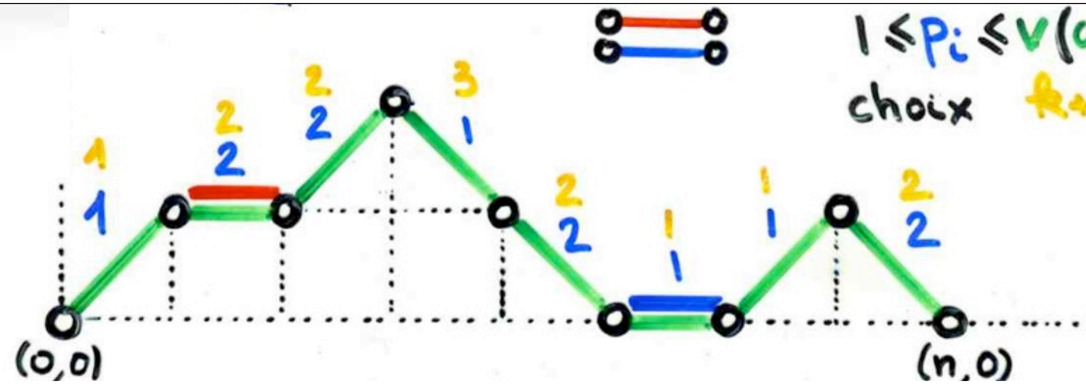
Françon-xgv., 1978



Bijection
Permutations
n+1
Histoires de Laguerre
(γ c ; f)
n
Chemin de Motzkin
2 couleurs
paliers
(0,0)
(n,0)


Permutations
n+1
n
k+1
level
k
k-1
$a_k = k+1$
$b'_k = k+1$
$b''_k = k+1$
$c_k = k+1$
2 couleurs
paliers
$f=(p_1,\dots,p_n)$
$1 \leq p_i \leq v(\omega_i)$
choix k+1
(0,0)
(n,0)

$h = (\omega_c ; (p_1, \ldots, p_n))$



$1 \leq p_i \leq v(\omega_i)$

choix $k+1$

x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
n= 8		2	2
9	•		

␣

␣ 1 ␣

␣ 1 ␣ 2

␣ 1 ␣ 3 ␣ 2

4 1 ␣ 3 ␣ 2

4 1 ␣ 3 5 2

4 1 6 ␣ 3 5 2

4 1 6 ␣ 7 ␣ 3 5 2

4 1 6 ␣ 7 8 3 5 2

4 1 6 9 7 8 3 5 2 $= \sigma \in \mathfrak{S}_{n+1}$

Bijection réciproque

$$\mathcal{L}_n \xrightarrow{\Pi \circ \Theta} \mathfrak{S}_{n+1}$$

- convention

$\sigma \in \mathfrak{S}_n$, $\sigma(0) = \sigma(n+1) = 0$

Def.

$x \in [1, n]$

(x valeur
i indice

pic	$\sigma(i-1) < x = \sigma(i) > \sigma(i+1)$
creux	$\sigma(i-1) > x = \sigma(i) < \sigma(i+1)$
double montée	$\sigma(i-1) < x = \sigma(i) < \sigma(i+1)$
double descente	$\sigma(i-1) > x = \sigma(i) > \sigma(i+1)$

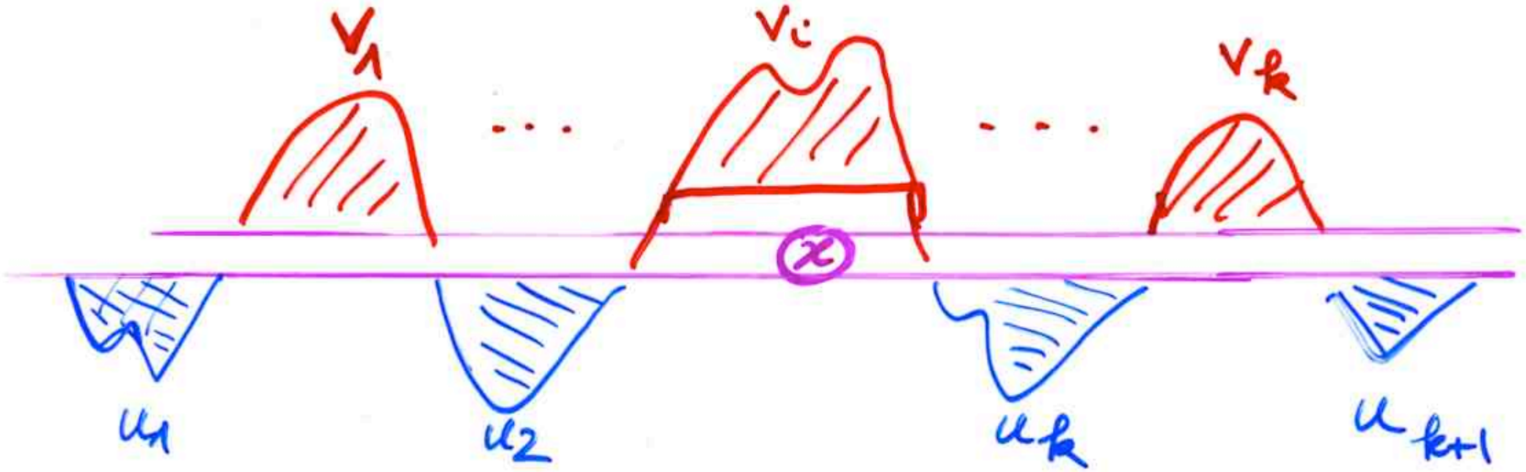
Def - $\sigma \in \mathfrak{S}_n$, $x \in [1, n]$

x- decomposition

- $\sigma = u_1 v_1 \cdots u_k v_k u_{k+1}$
- lettres $(u_i) < x$
- lettres $(v_j) \geq x$
- mots $v_1, u_2, \ldots, u_k, v_k$ non vides

ex. $\sigma = 4\,1\,6\,9\,7\,8\,3\,5\,2$, $x = 3$

$$\underbrace{\;}_{u_1}\ \overbrace{4}^{v_1}\ \underbrace{1}_{u_2}\ \overbrace{6\,9\,7\,8\,\textcircled{3}\,5}^{v_2}\ \underbrace{2}_{u_3}$$



v_1
v_i
v_k
x
u_1
u_2
u_k
u_{k+1}

Bijection réciproque $\quad \mathcal{L}_n \xrightarrow{\pi\circ\theta} \mathfrak{S}_{n+1}$

$$\sigma \in \mathfrak{S}_{n+1} \longrightarrow \left(\omega_c \,;\, (p_1, \dots, p_n)\right)$$

- ω_c

  $i^{\text{ème}}$ pas

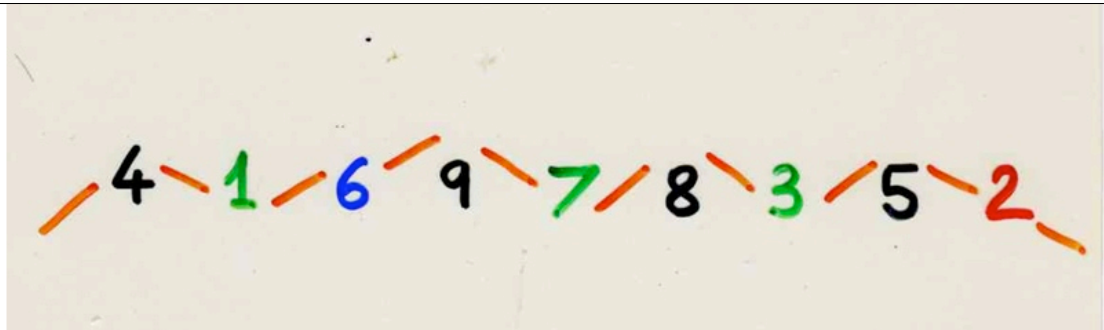
i creux

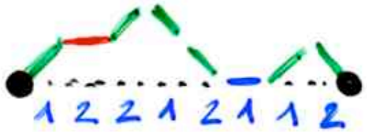
i pic

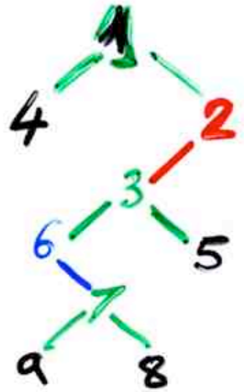
i double montée

i double descente

- $p_i = j$ ssi i lettre de v_j dans la i-decomposition de σ

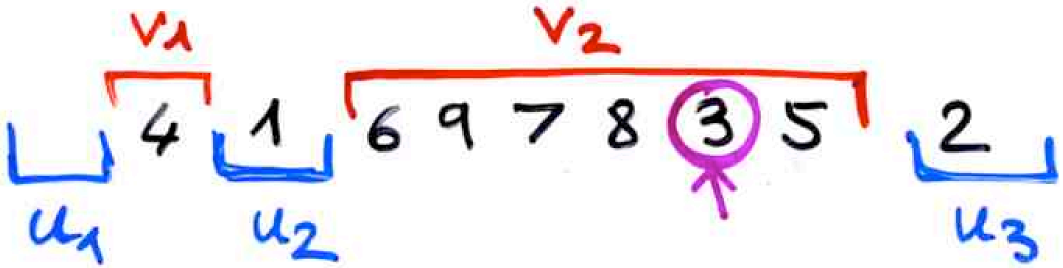






4 1 6 9 7 8 3 5 2

ex. $\sigma = 4\,1\,6\,9\,7\,8\,3\,5\,2$, $x = 3$



Lemme - $h = (\omega_c \,;\, (p_1, \dots, p_n)) \in \mathcal{L}_n$ Laguerre history

$\Pi \circ \Theta_4$ ↓

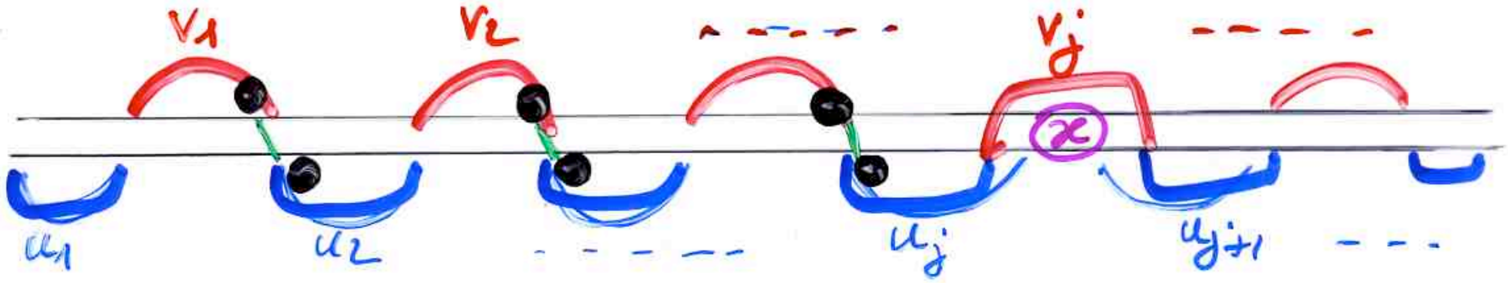
permutation $\sigma \in \mathfrak{S}_{n+1}$

$p_x = j$ est aussi :

$j = 1 +$ nb de triplets (a, b, x) ayant le "motif" (31-2) c.à d :

$a = \sigma(i)$, $b = \sigma(i+1)$, $x = \sigma(\ell)$

$i < i+1 < \ell$ $\quad b < x < a$



"q-analogue" of Laguerre histories

Bernardi interpretation
of
crossing of alernative tableaux
in term of
the corresponding permutation

Prop (O. Bernardi, 2008)

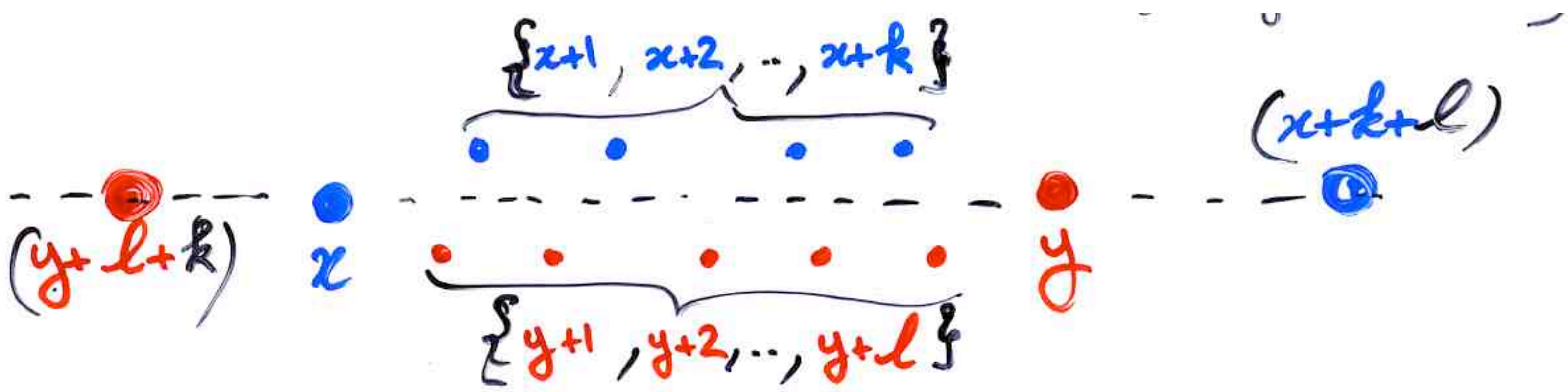
σ permutation ($(n+1)$ elements) $\longleftrightarrow$ T alternating tableau

"exchange-fusion" algorithm

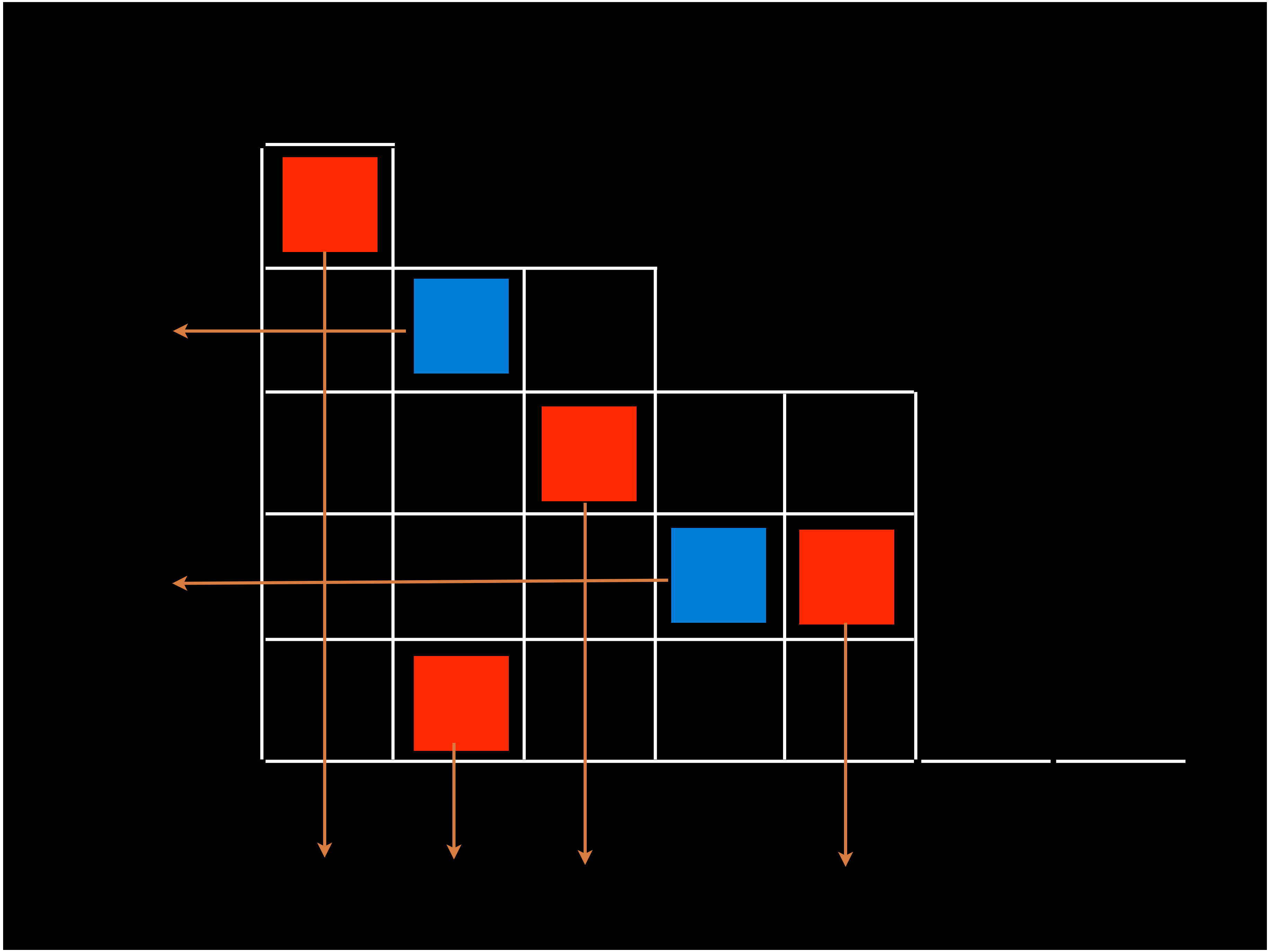
The number of crossings of T is the number of pairs (x, y), $x = \sigma(i)$, $y = \sigma(j)$, $1 \leq i < j \leq n+1$ such that: $\exists$ $k, l \geq 0$, such that:

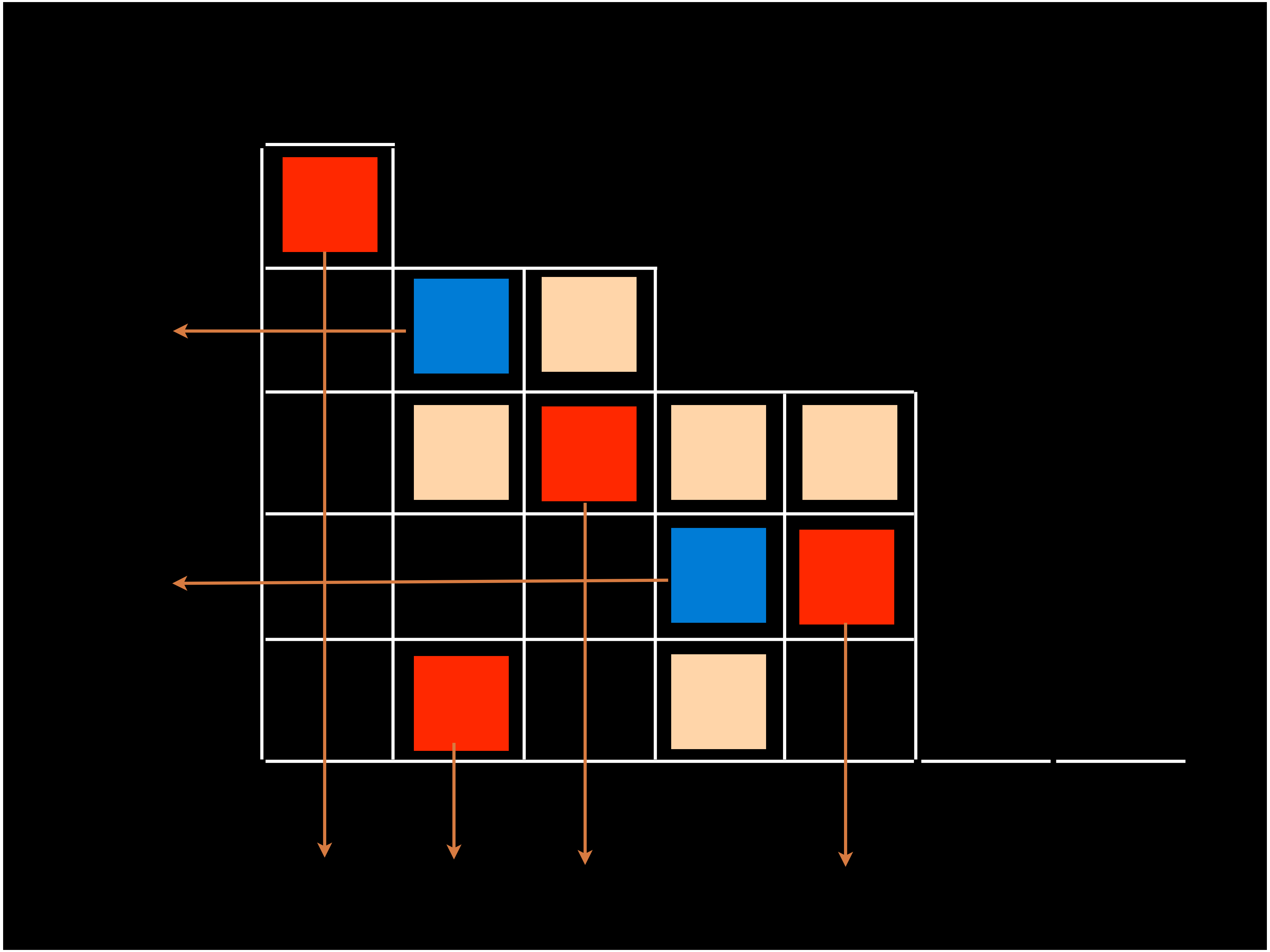
- the set of values $\{x+1, x+2, \ldots, x+k ; y+1, \ldots, y+l, \}$ are located (in the word σ) between x and y
- $x+k+1$ is located at the right of y
- $y+l+1$ is located at the left of x

(convention: $(n+2)$ at the left of everybody)


{x+1, x+2, .., x+k}
(x+k+l)
(y+l+k)
x
y
{y+1, y+2, .., y+l}

§ 5 Catalan alternative tableaux

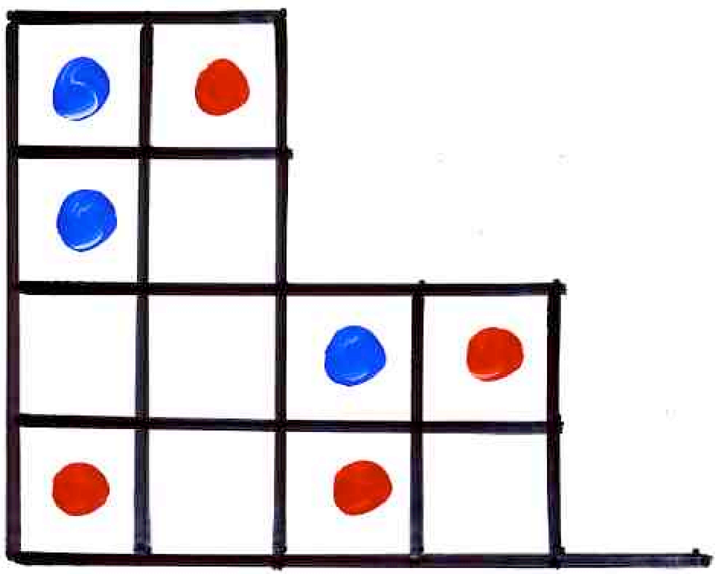




Def Catalan alternative tableau T

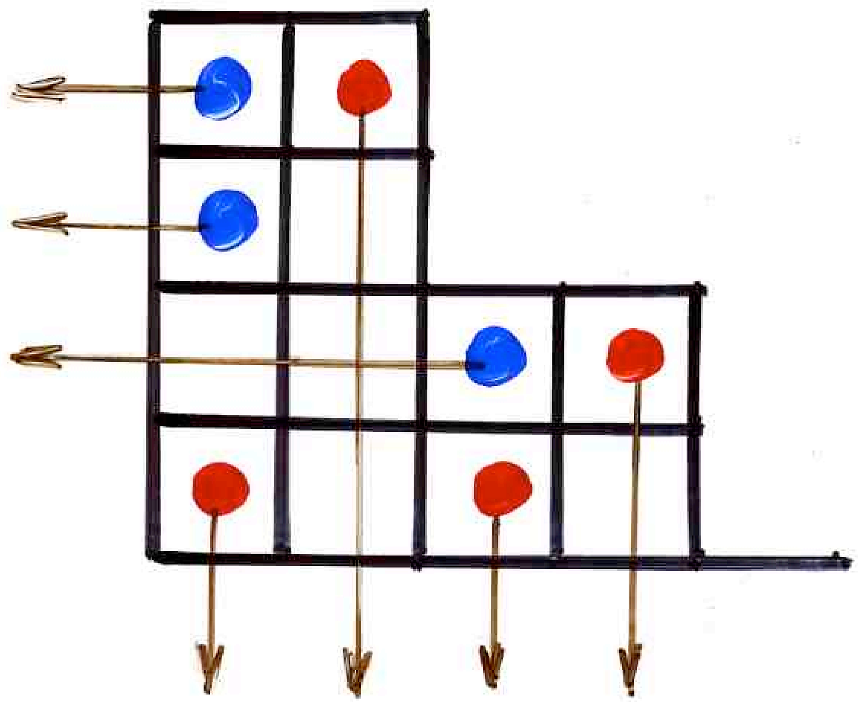
alt. tab. without cells [X]

i.e: every empty cell is below a red cell or on the left of a blue cell



Def Catalan alternative tableau T
alt. tab. without cells ☒

i.e: every empty cell is below a red cell or on the left of a blue cell



Lemma. σ (permutation) $\Longleftrightarrow$ T alternating tableau

T has no crossing

$\Longleftrightarrow$ σ has no subsequence of type

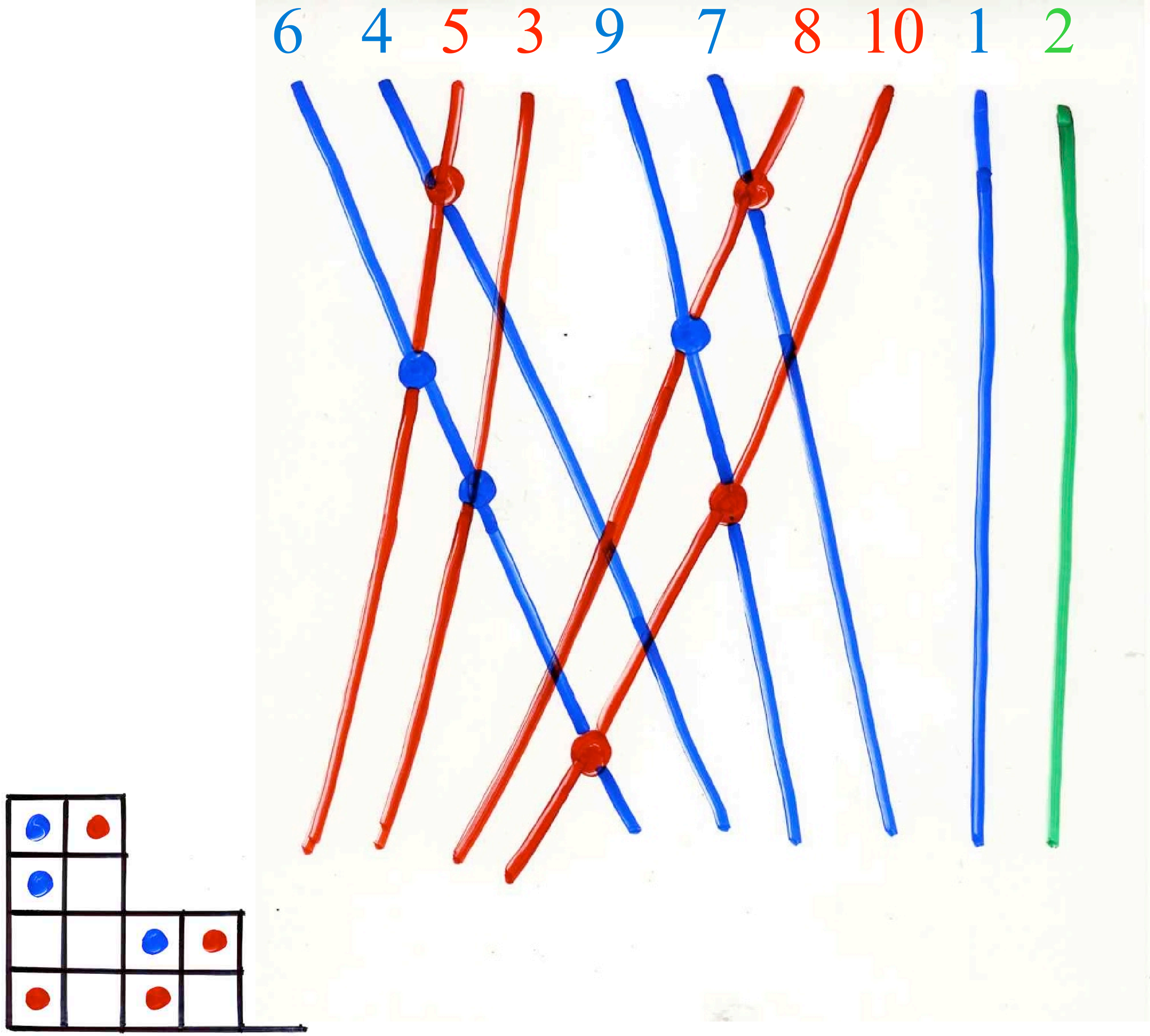
$(y+1) \cdots x \cdots y \cdots (x+1)$

Prop. (O. Bernardi, 2008)

The number of such permutations on n elements is C_n Catalan number

ex: $\sigma = 6\ 4\ 5\ 3\ 9\ 7\ 8\ (10)\ 1\ 2$

Bijection
permutations with no $y+1, x, y, x+1$
and binary trees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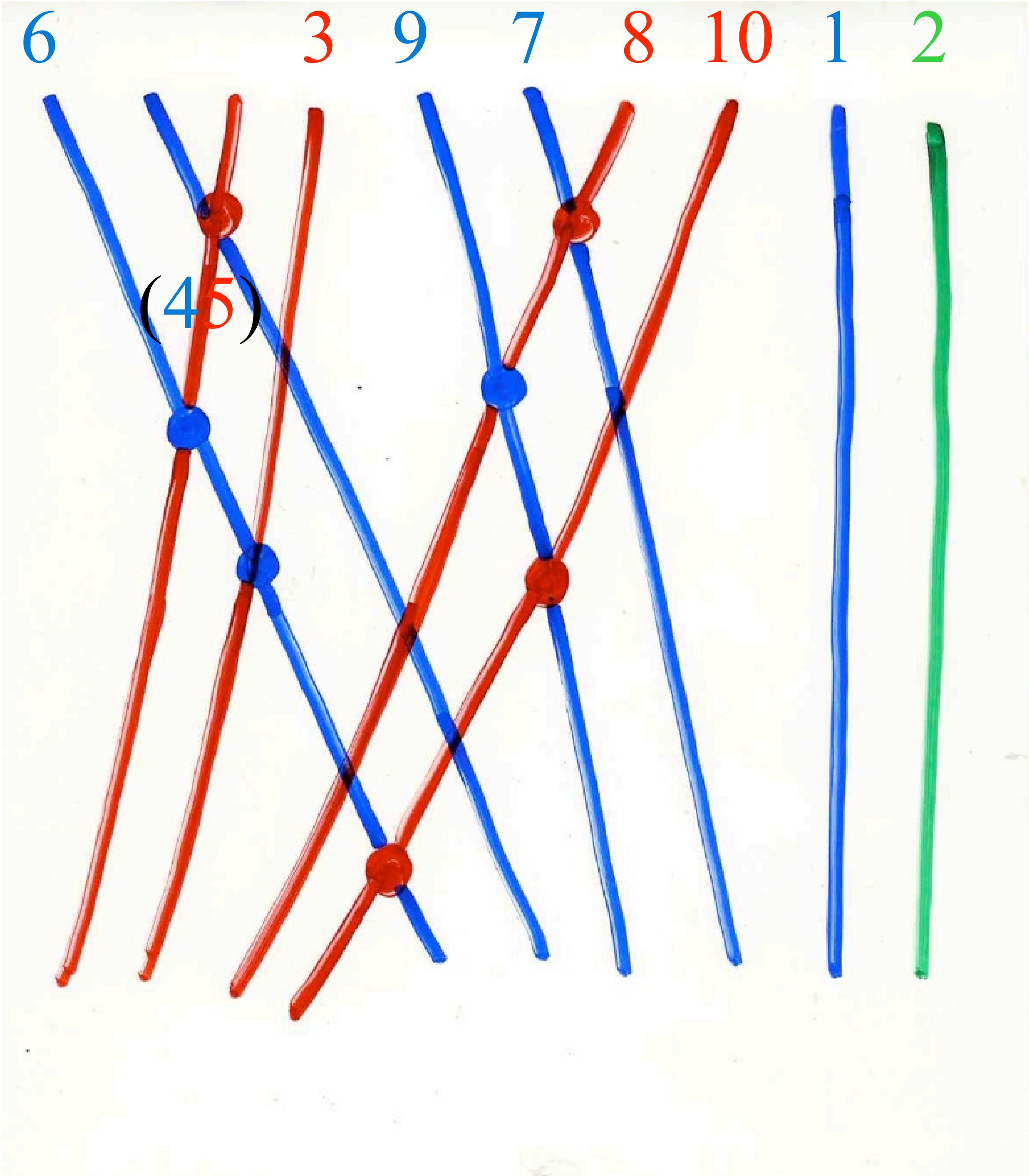
6
3
9
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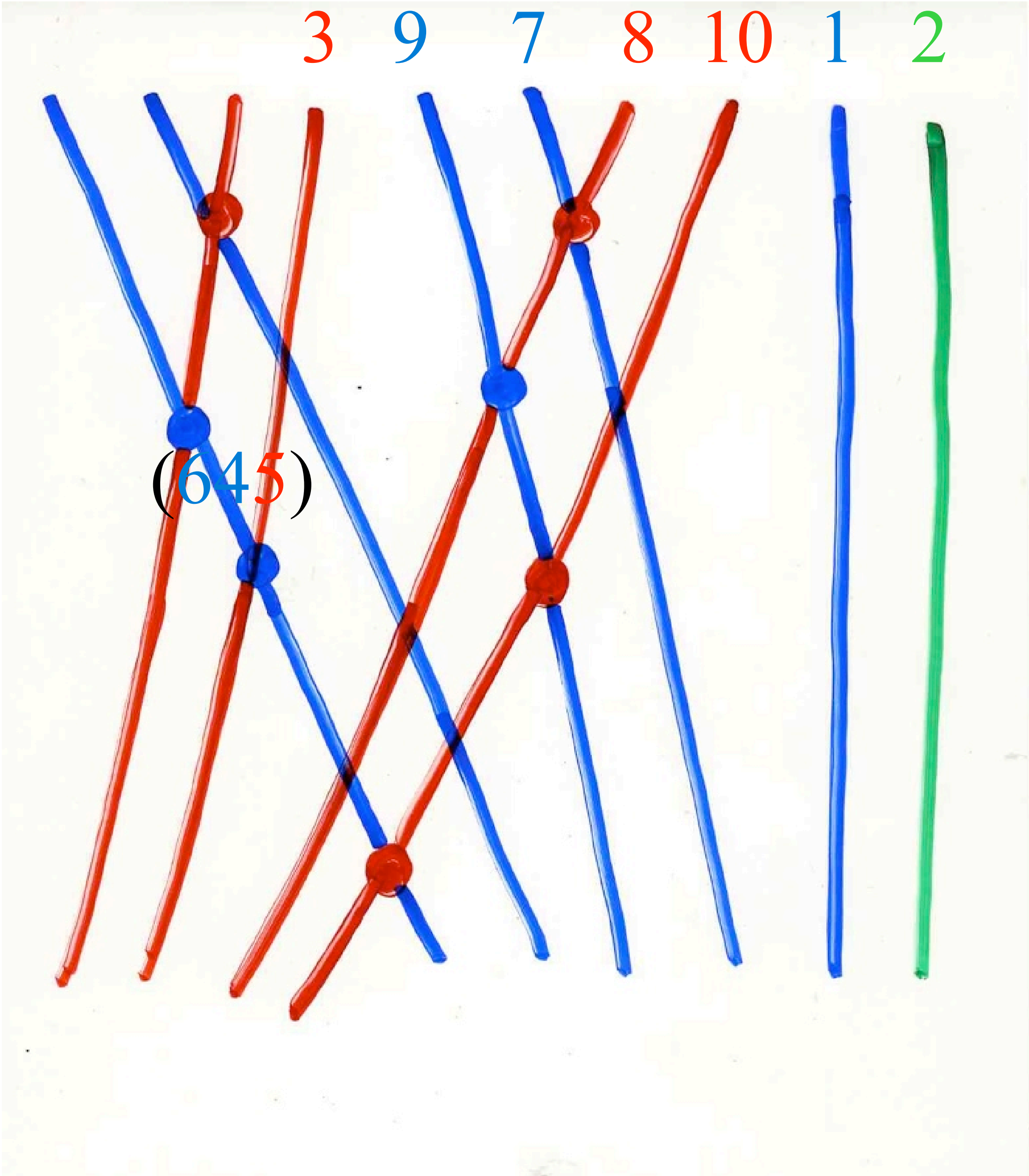
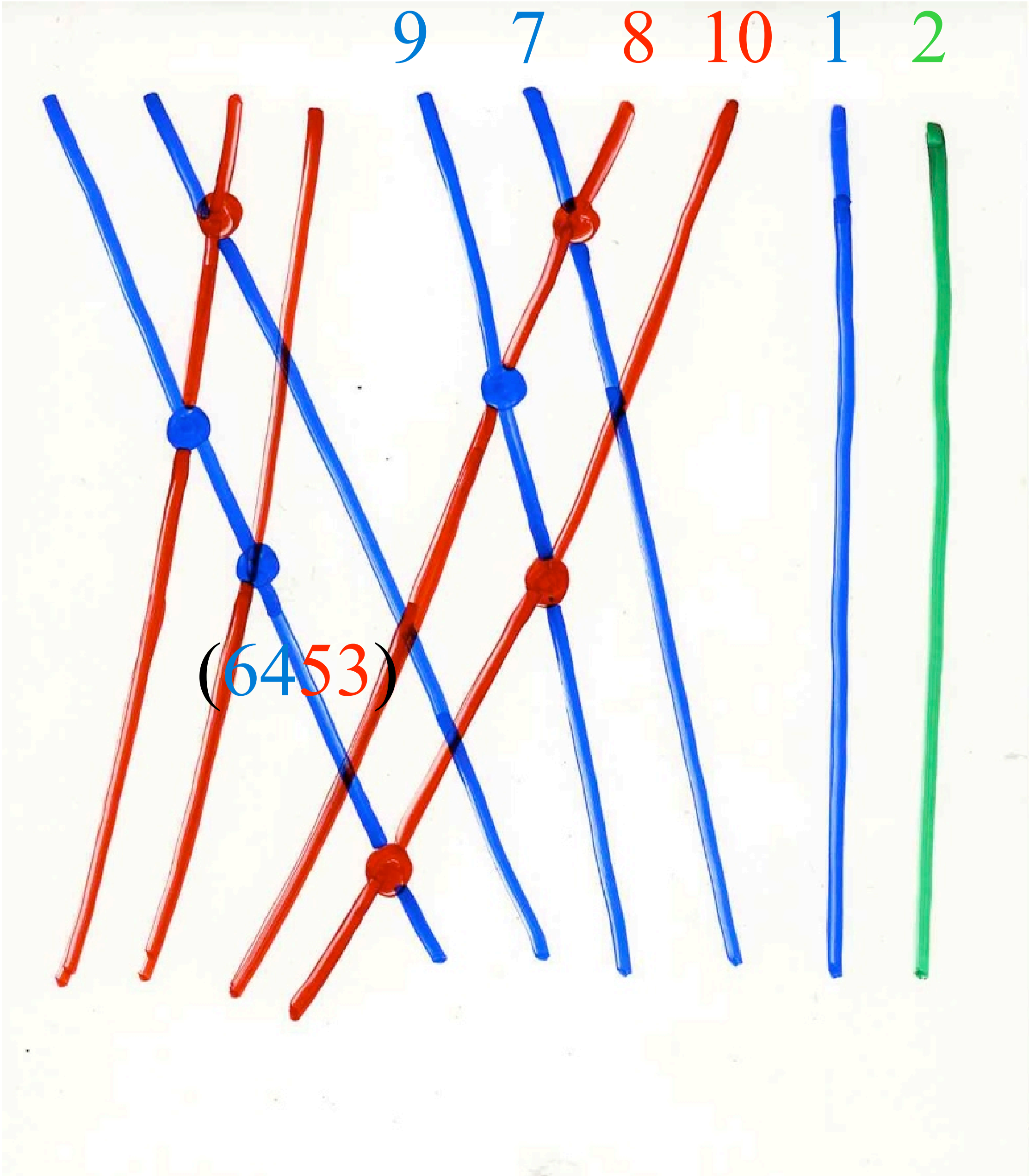
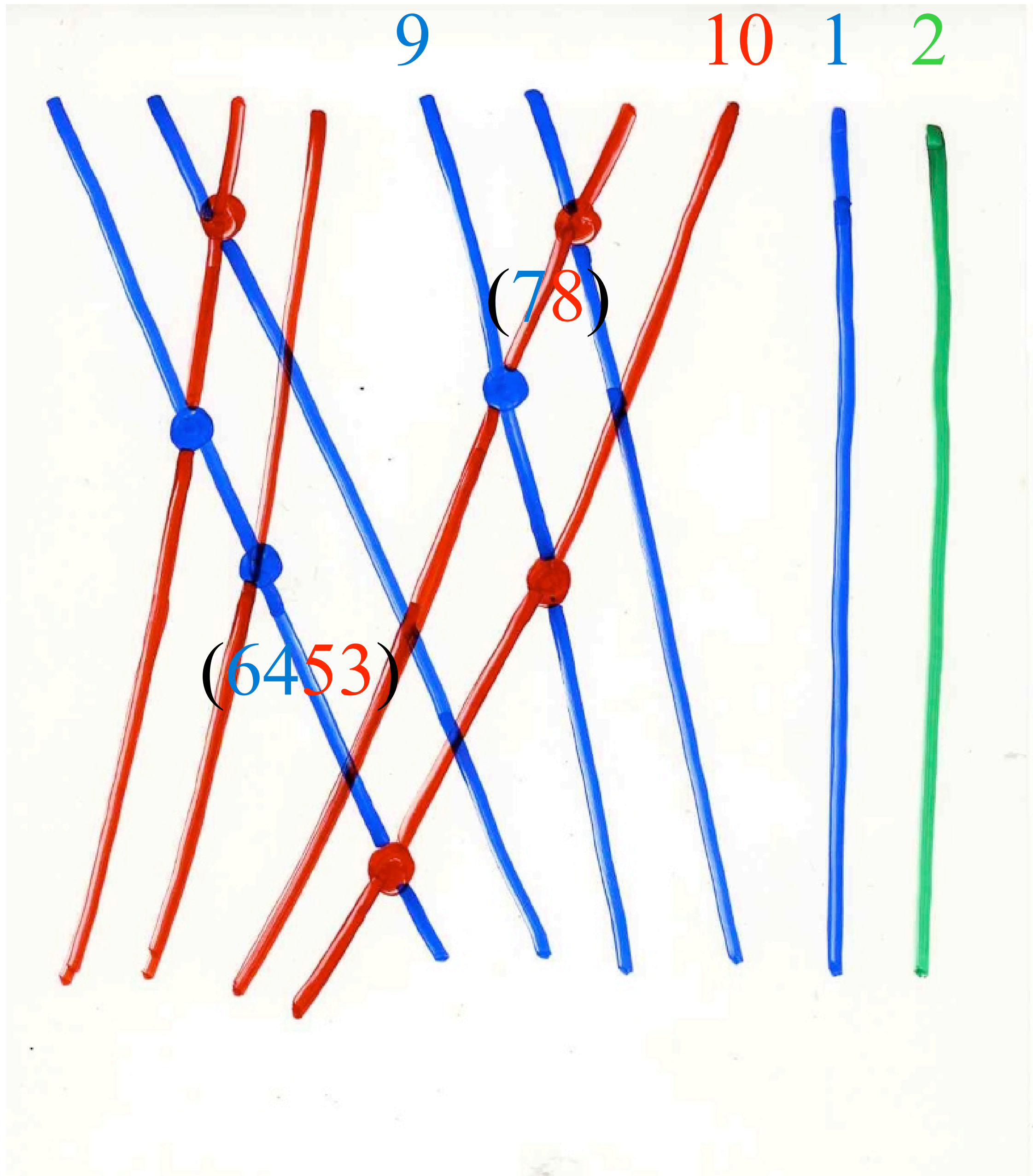
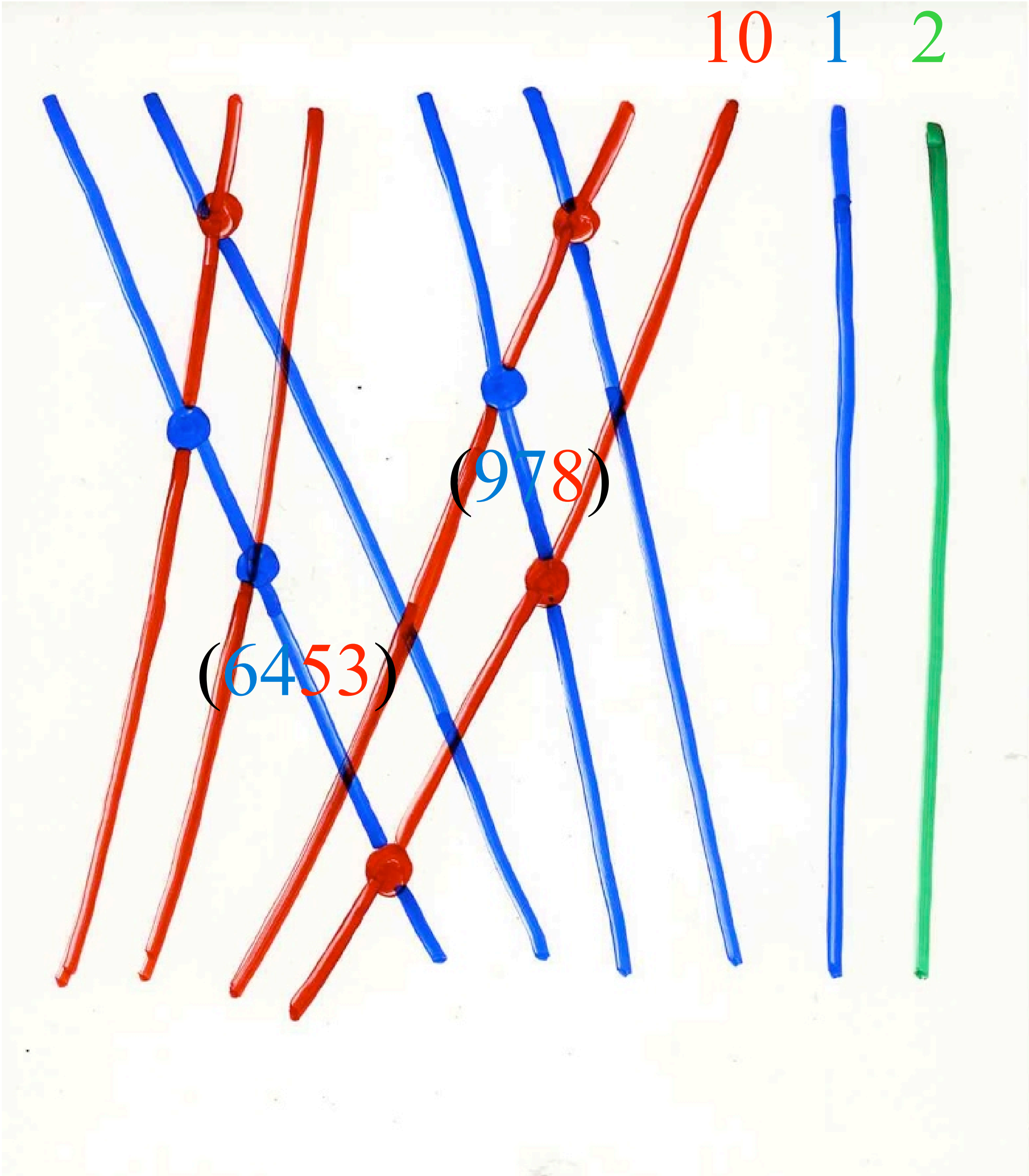
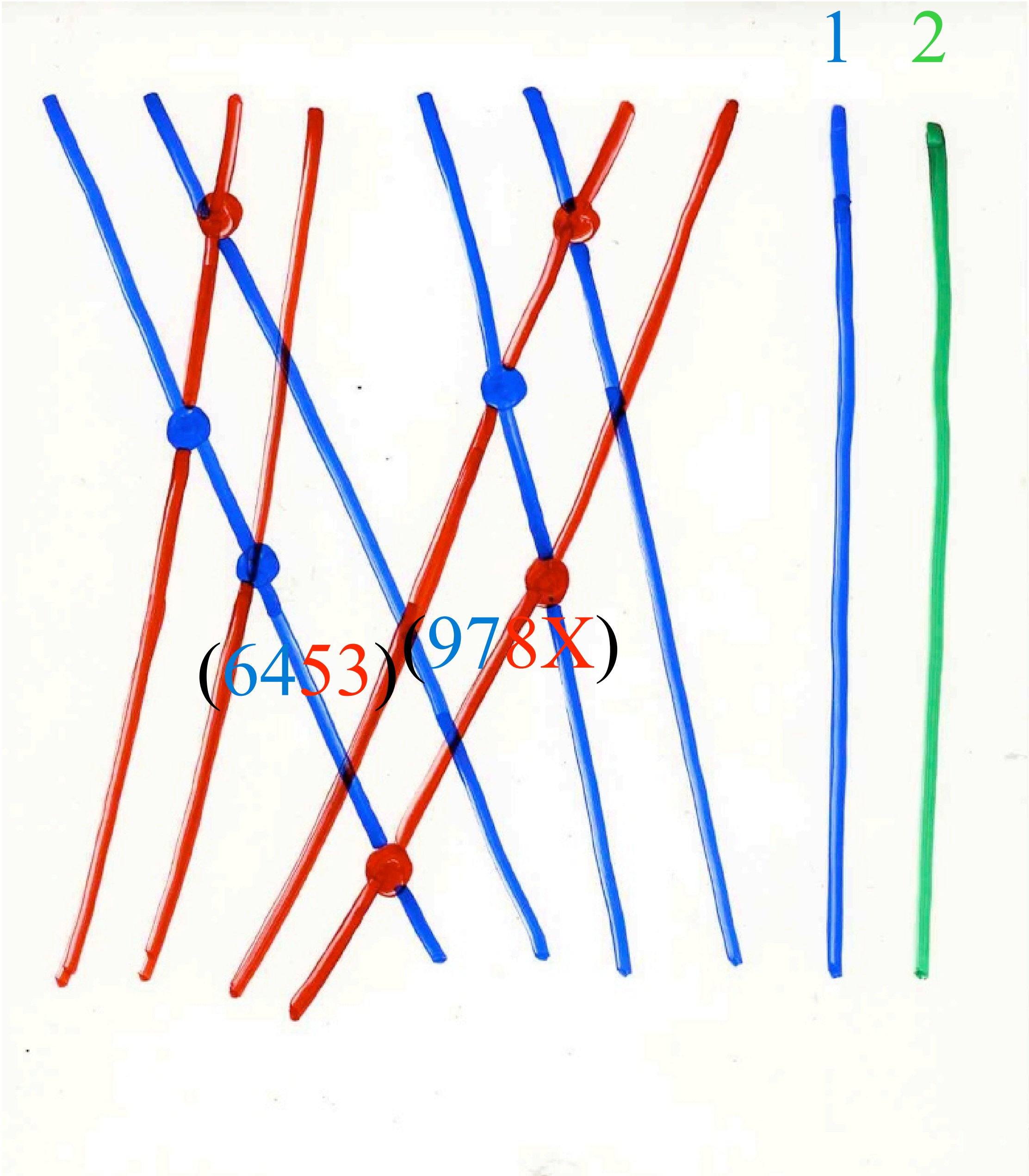
10
1
2
(45)


3 9 7 8 10 1 2
(645)

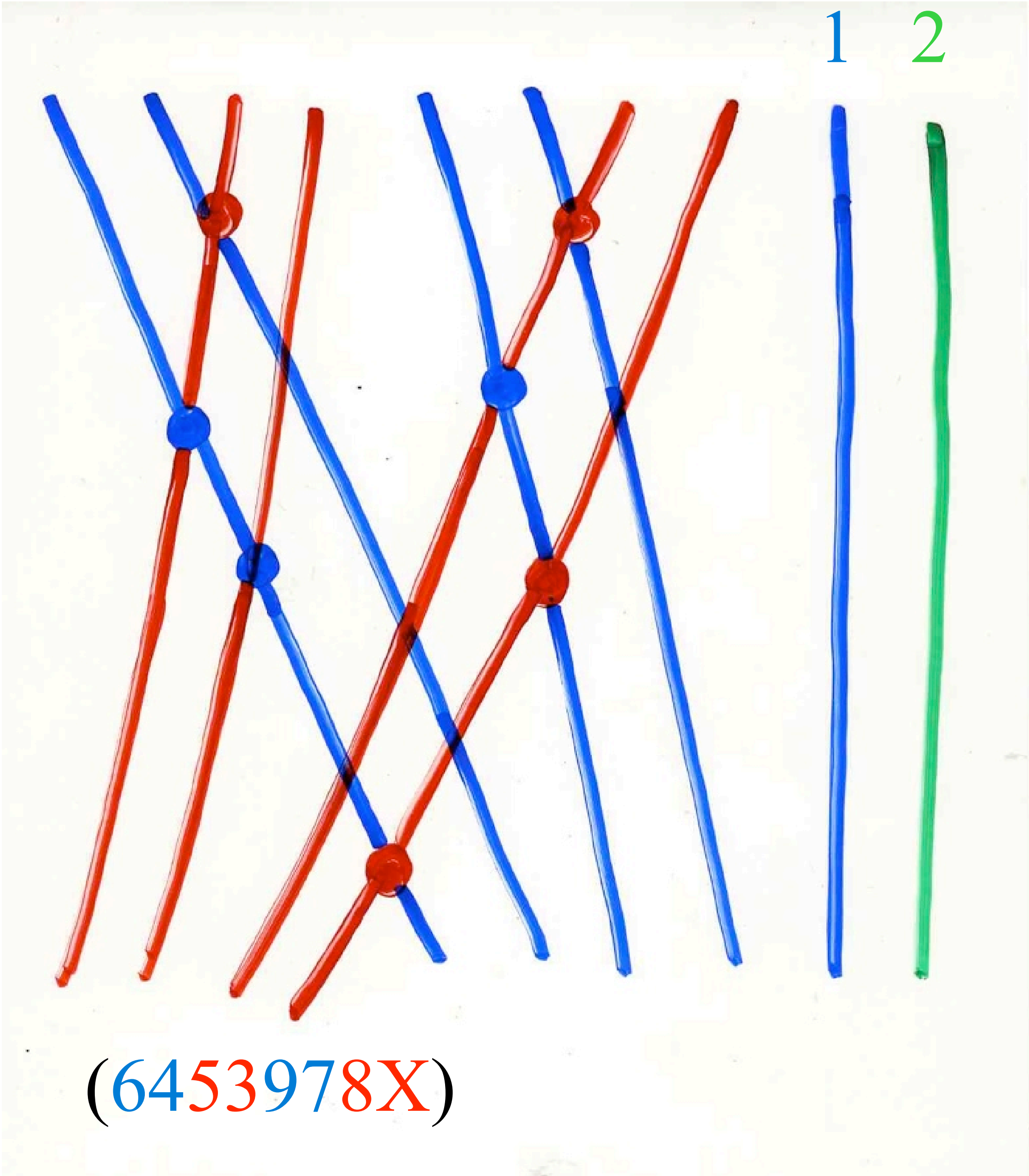

9 7 8 10 1 2
(6453)

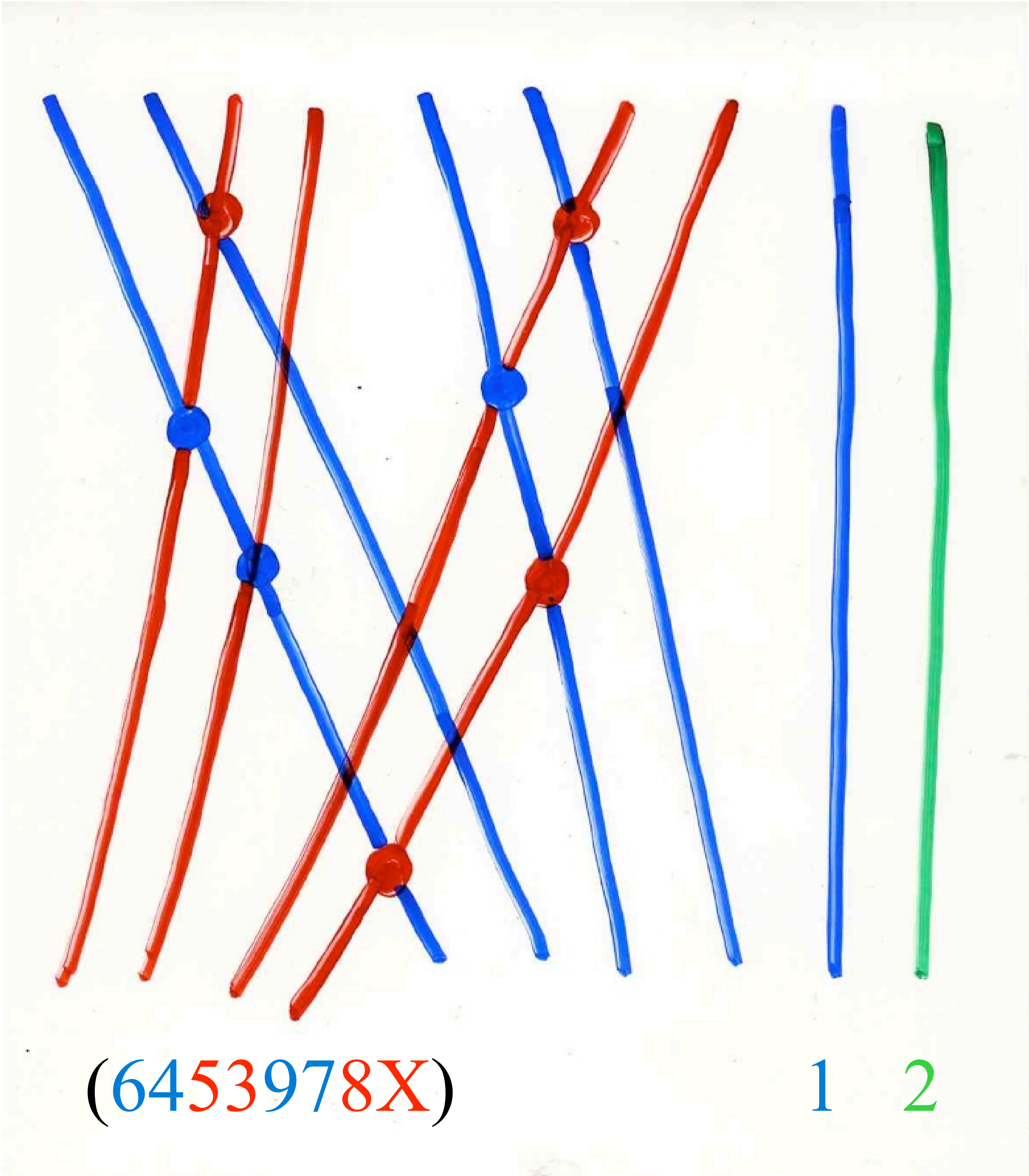

9
10
1
2
(78)
(6453)

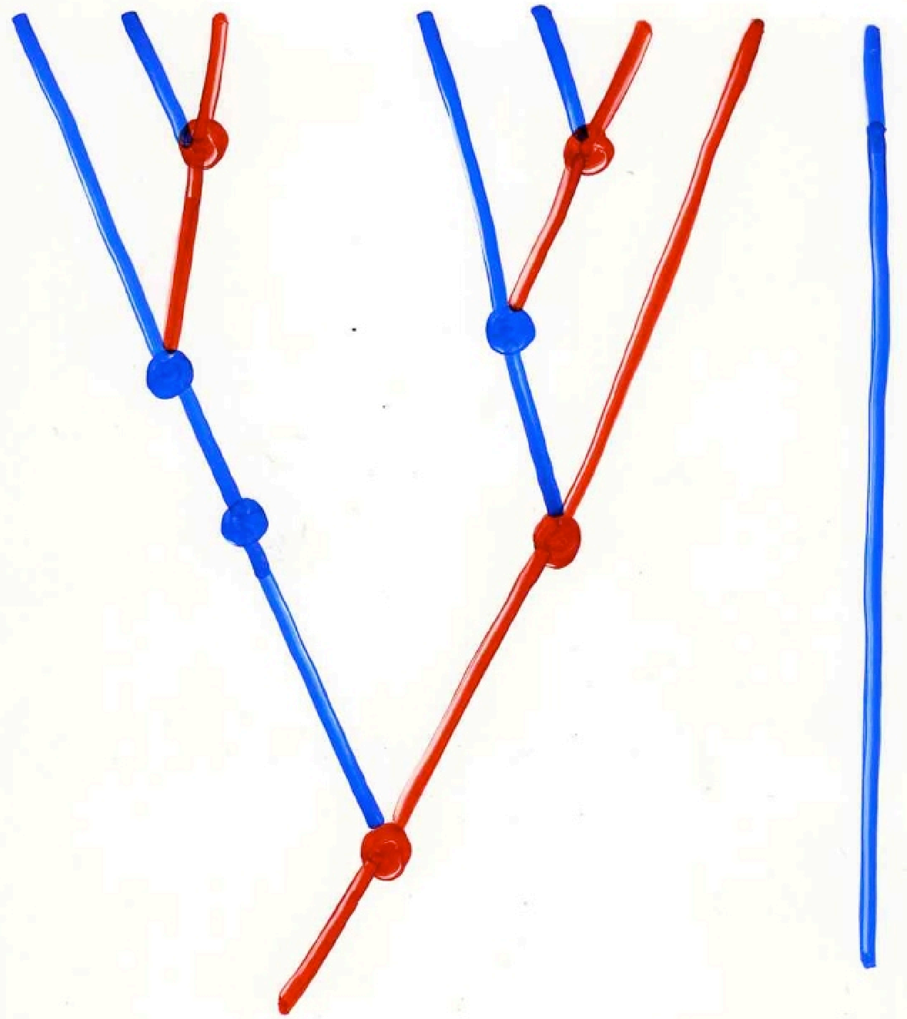

10
1
2
(978)
(6453)

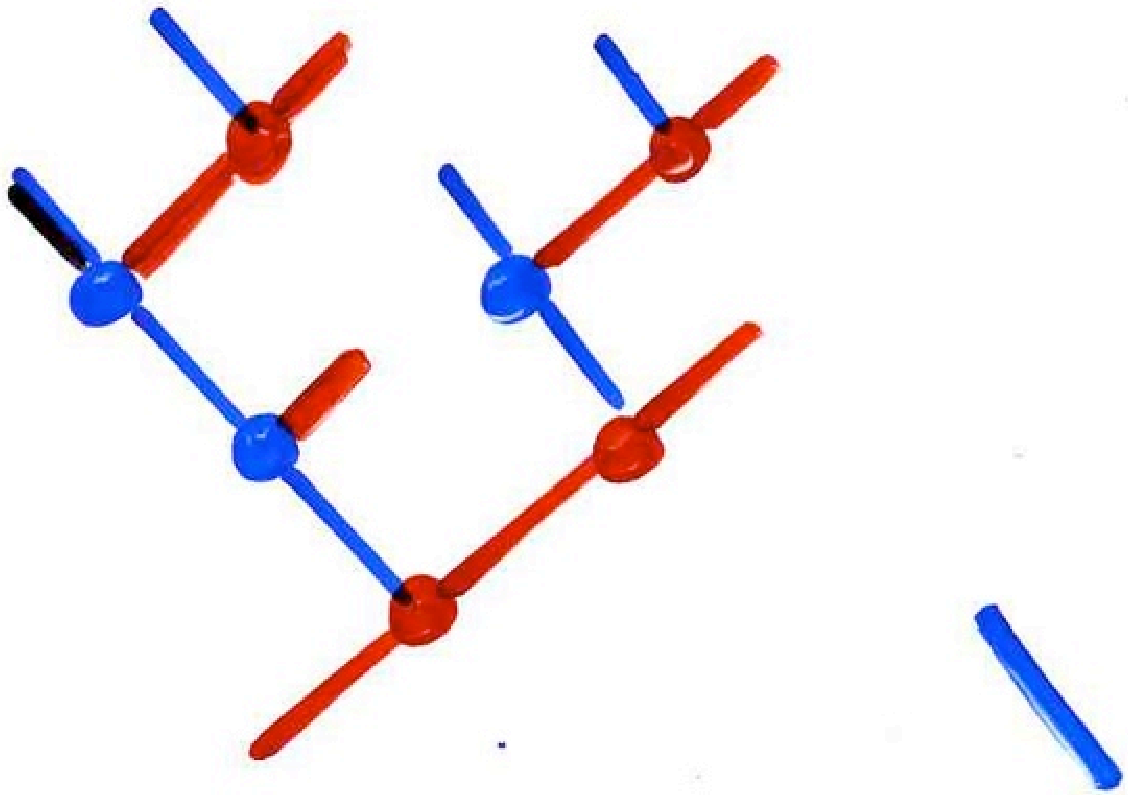


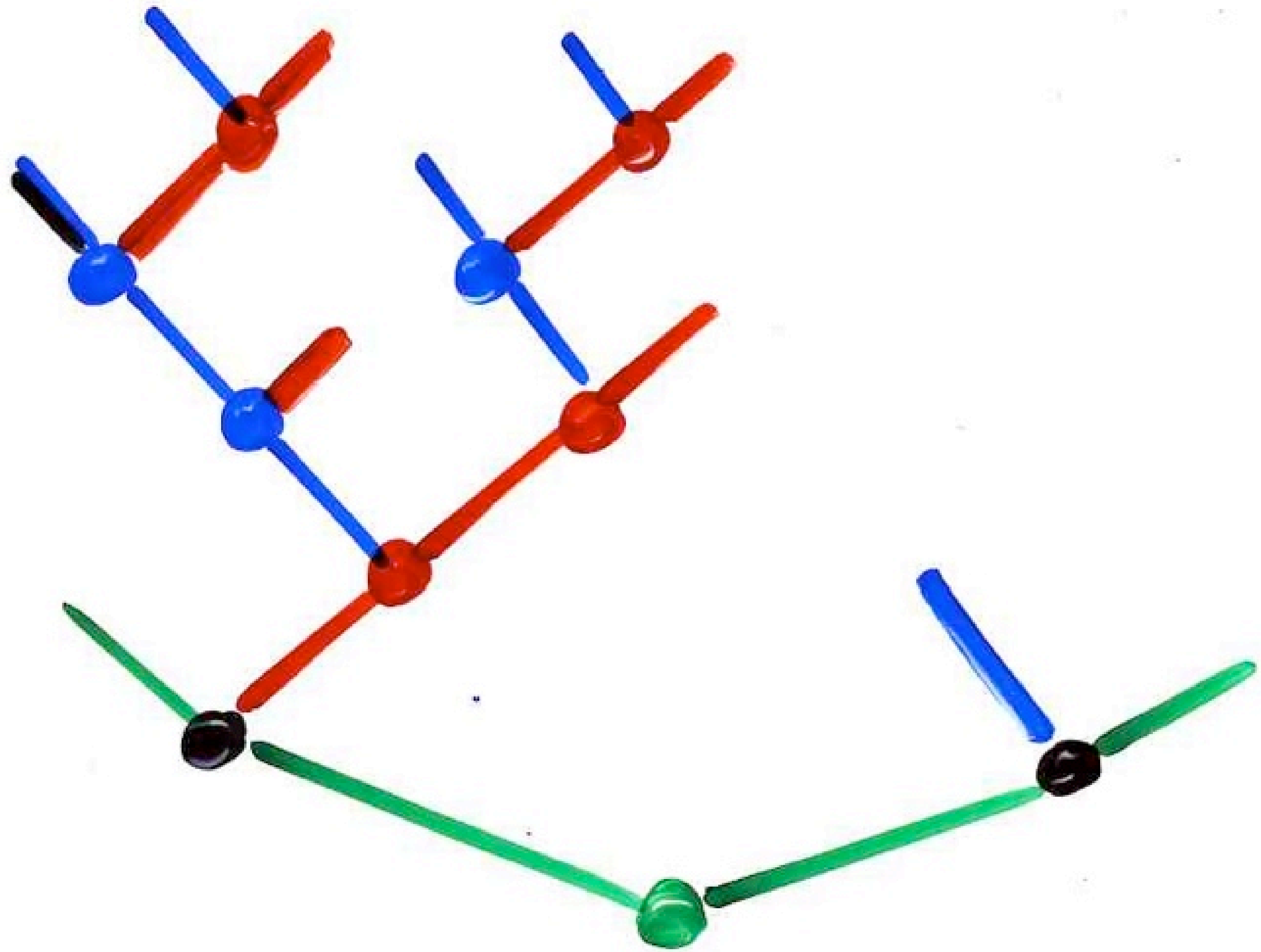
1
2
(6453)(978X)

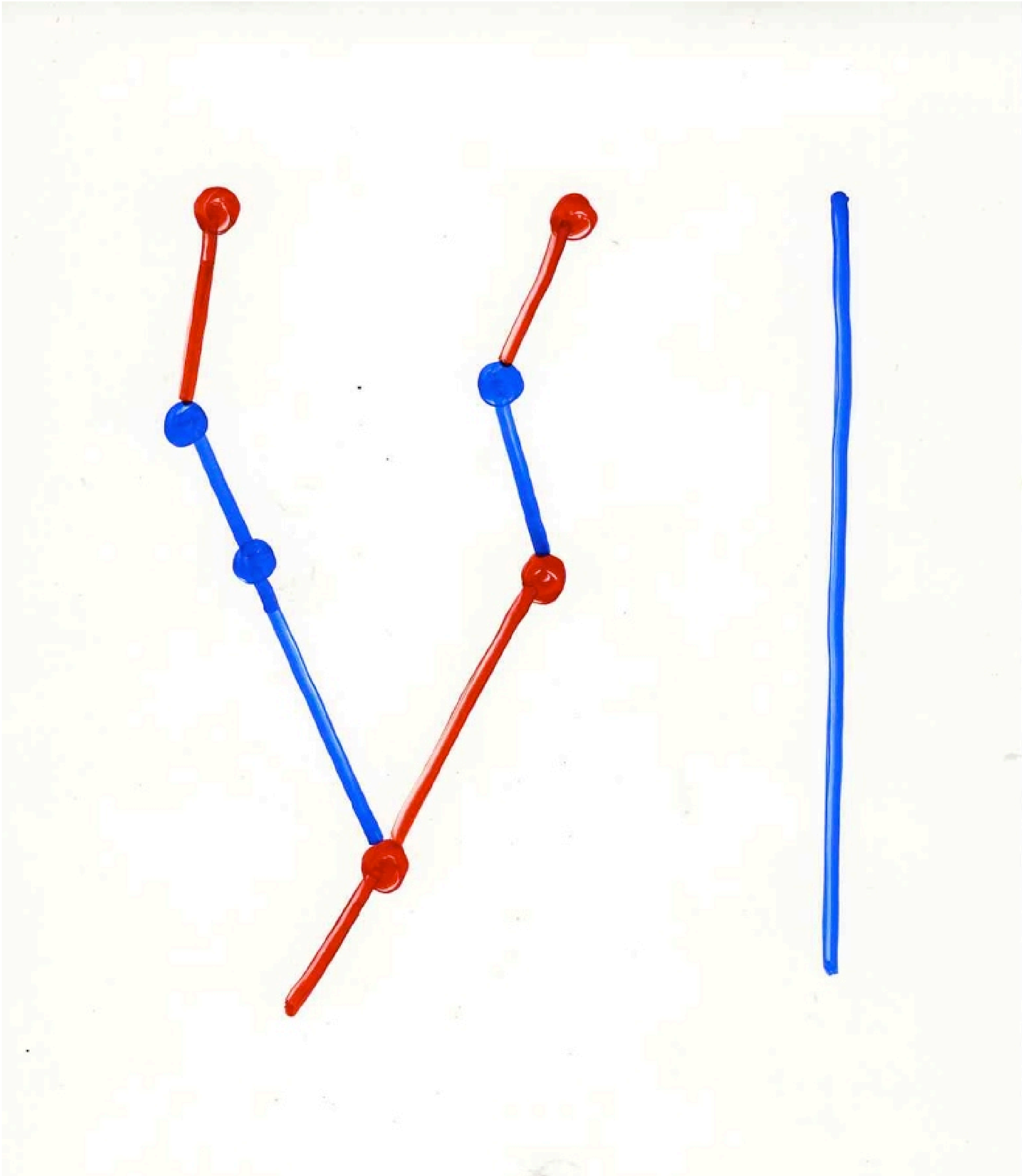

1
2
(6453978X)


(6453978X)
1 2







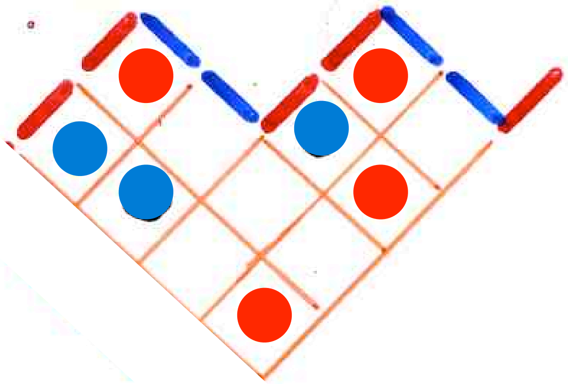


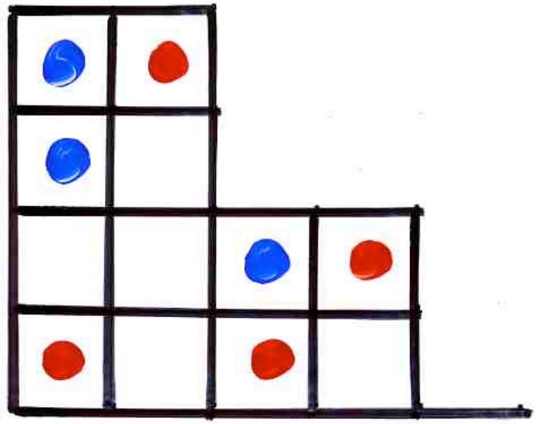
a “taquin-like” bijection

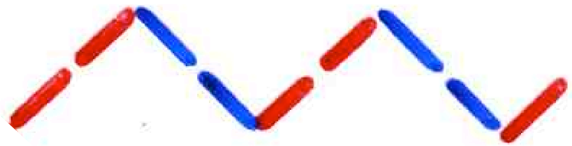
between

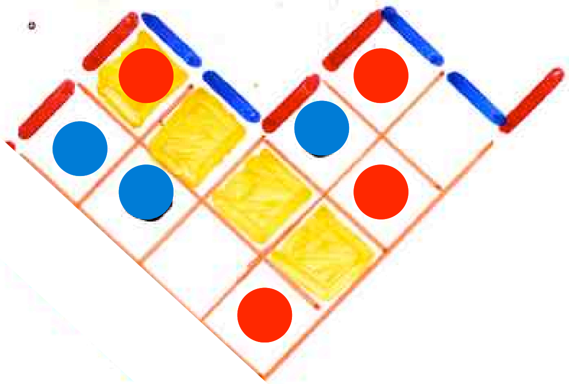
alternative Catalan tableaux

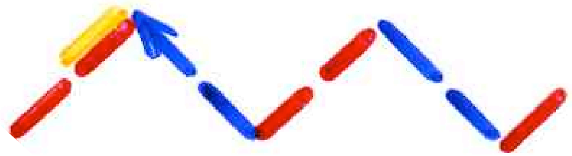
and binary trees

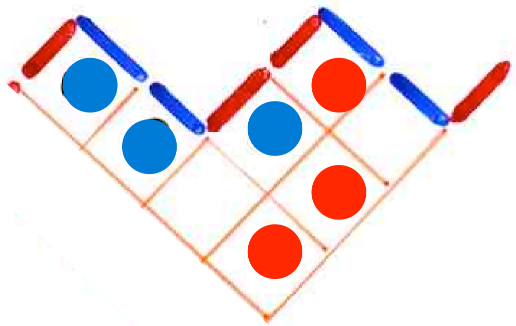


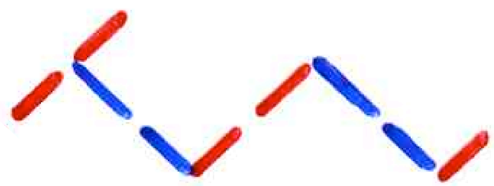


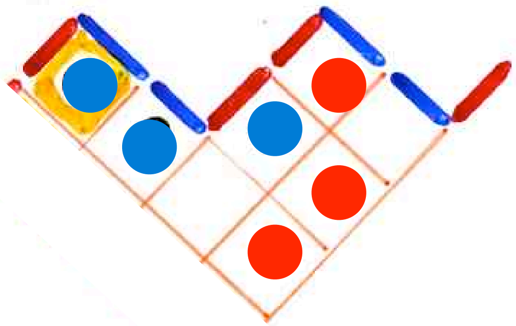


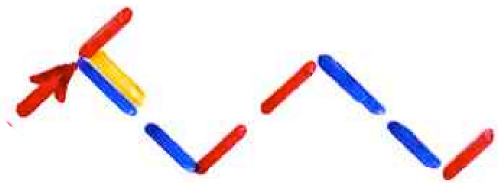


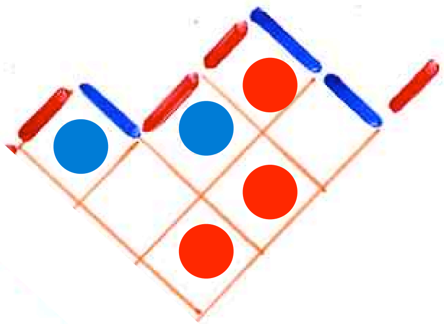


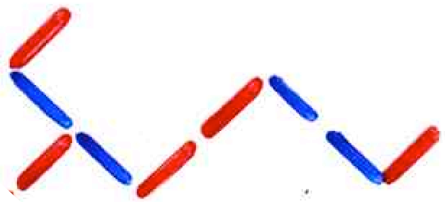


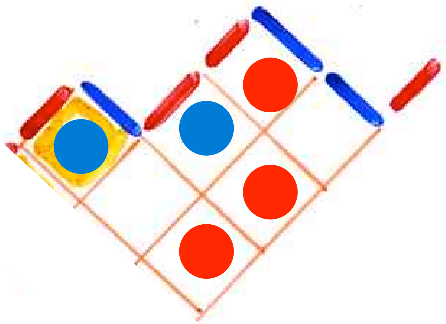


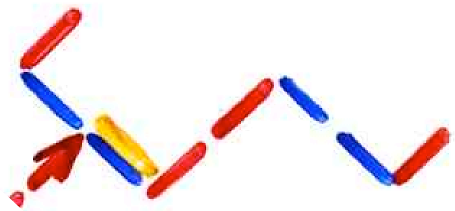


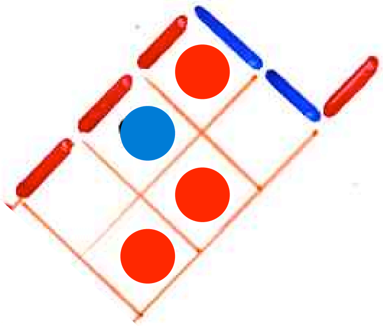


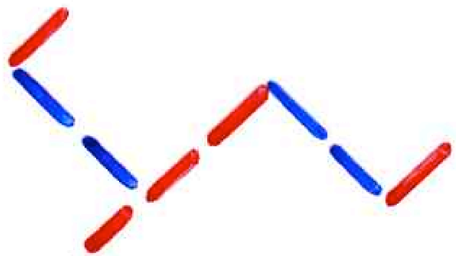


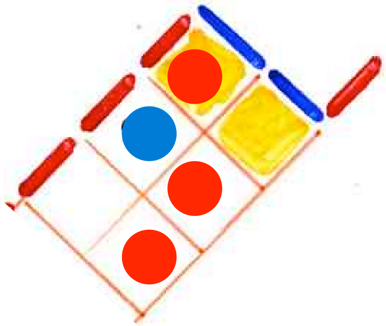


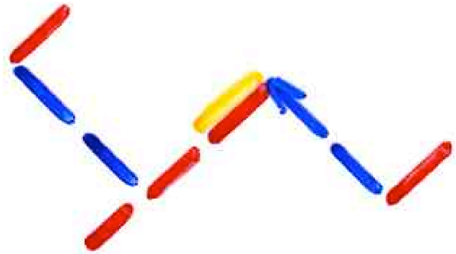


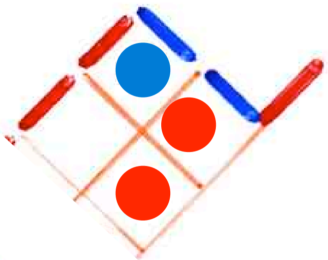


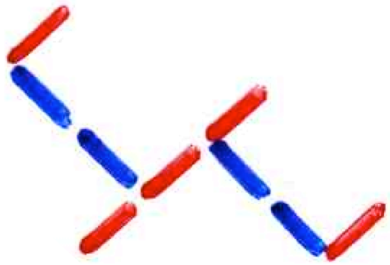


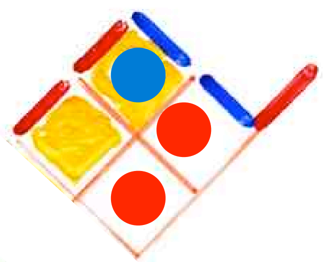


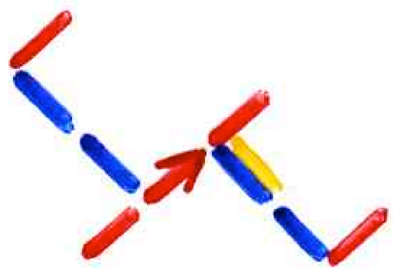


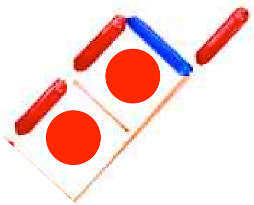


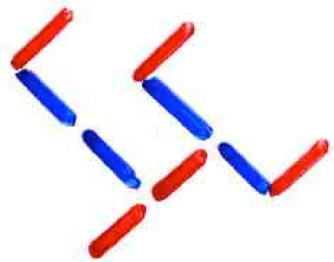


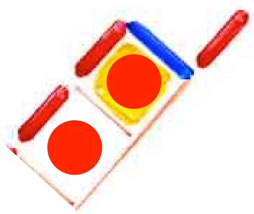


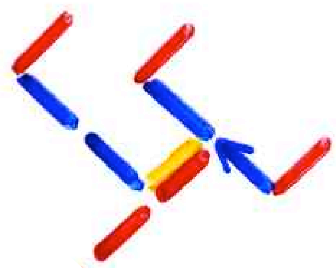


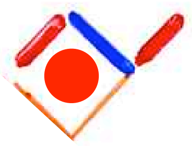


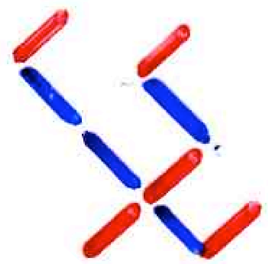


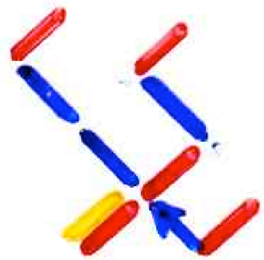


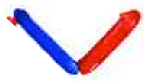


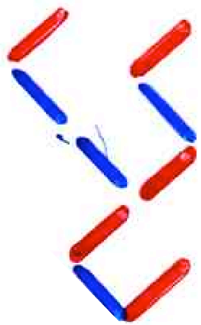


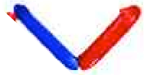
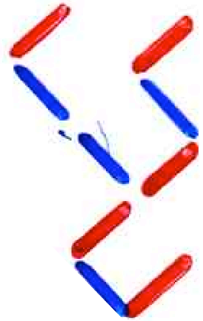
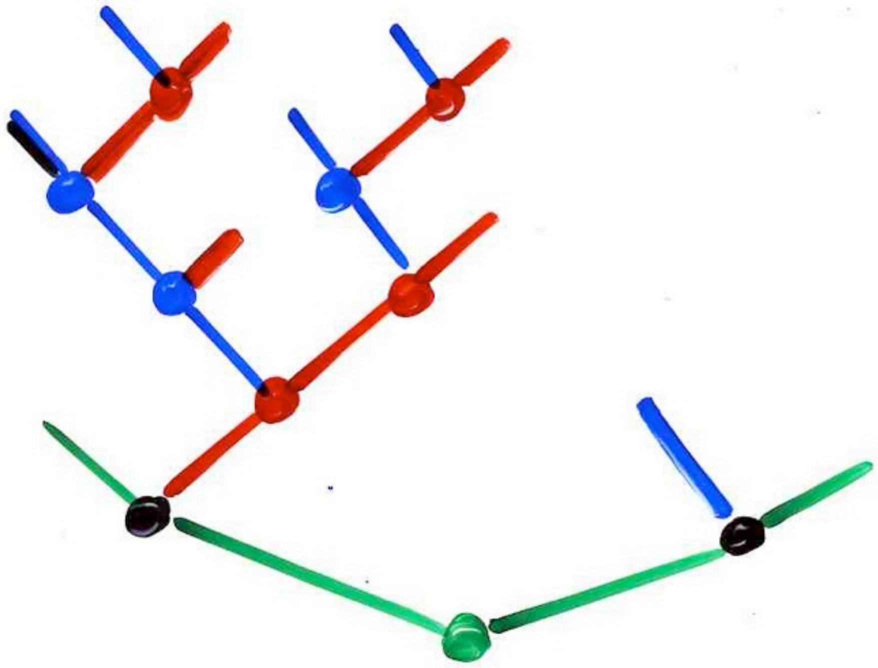




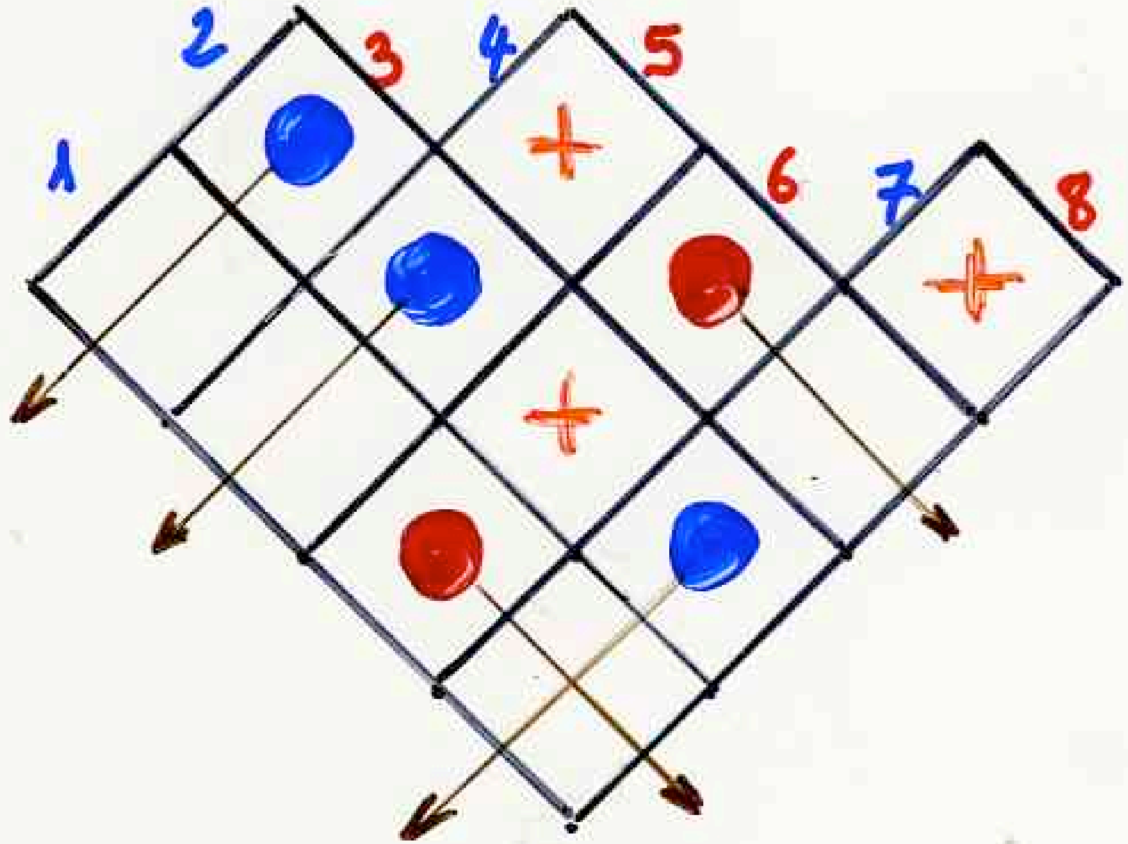






Bijection
alternative tableaux
alternative binary trees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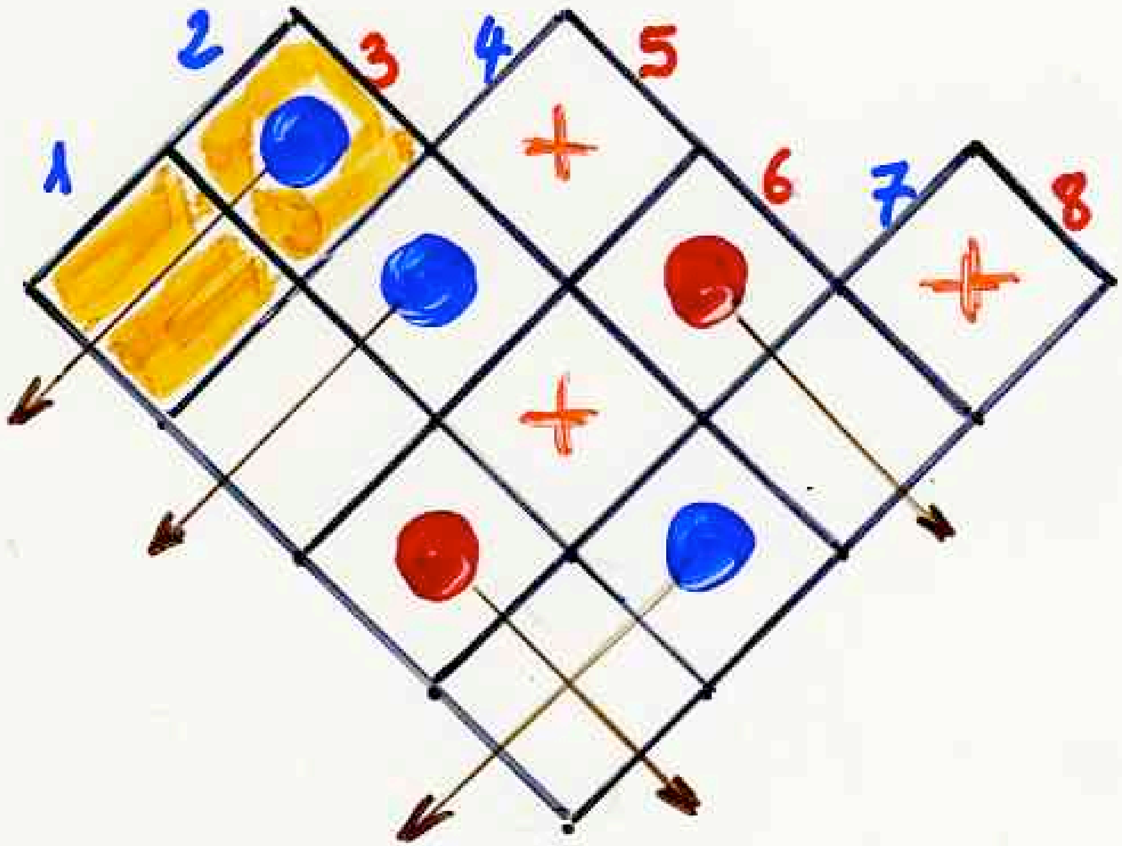
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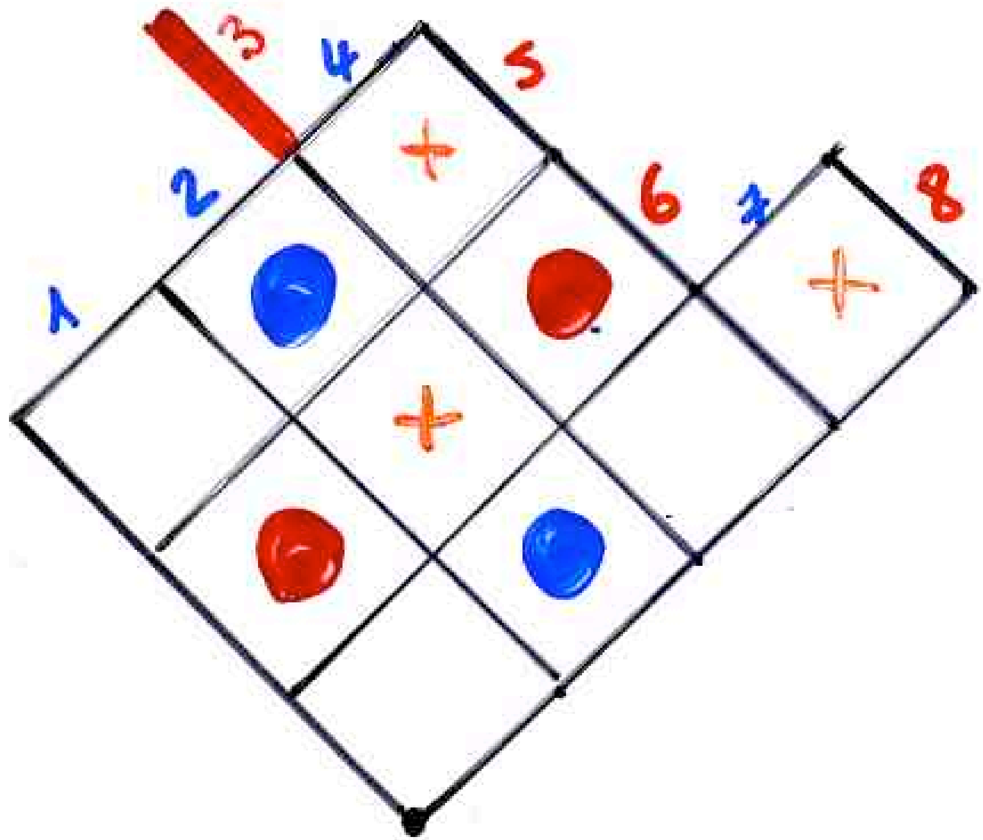

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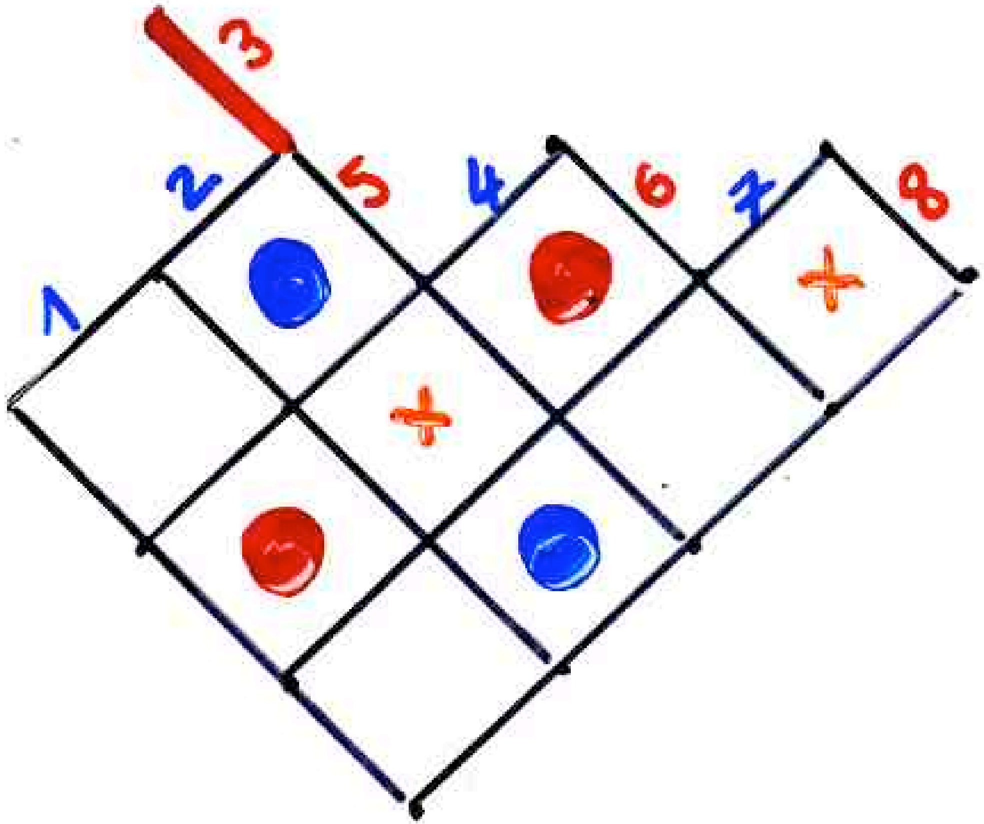

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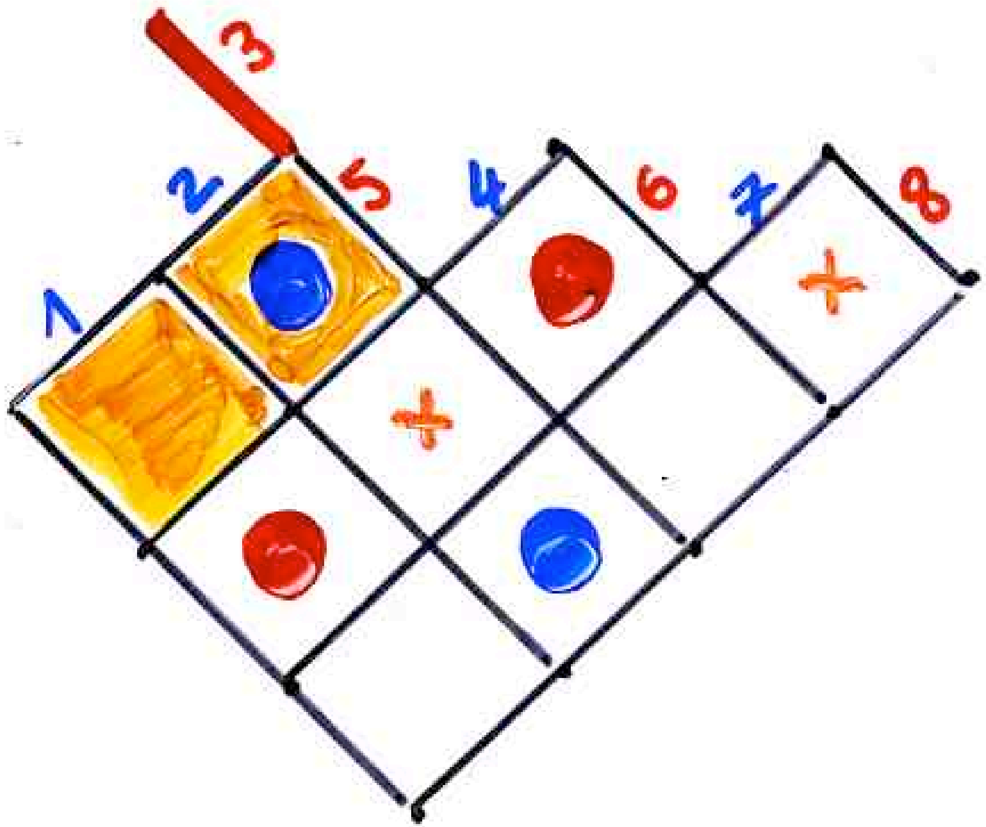
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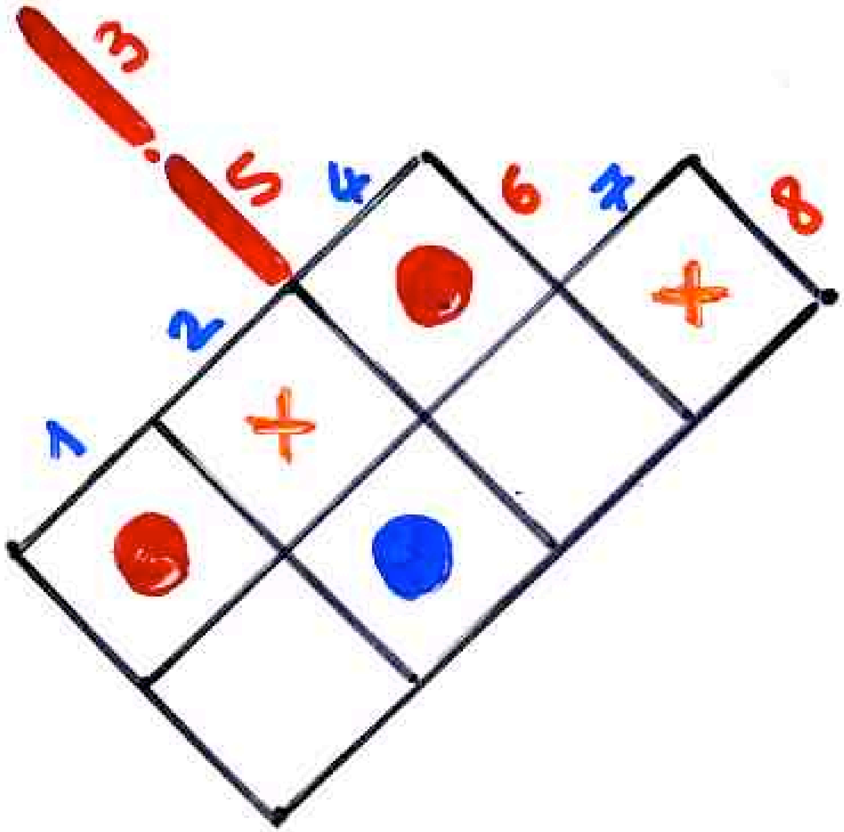

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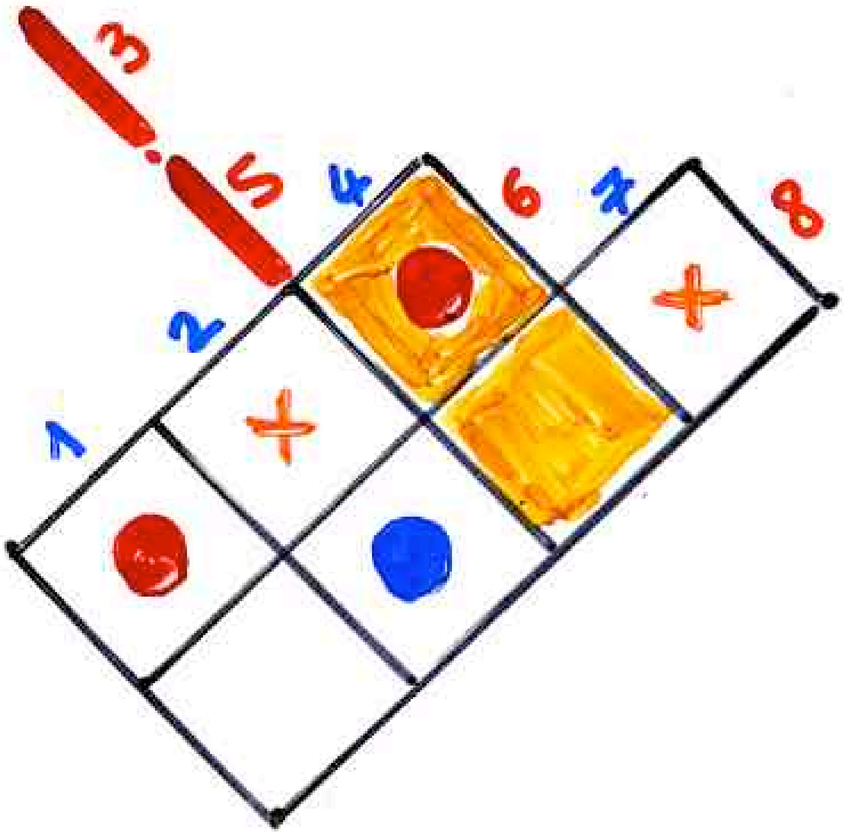
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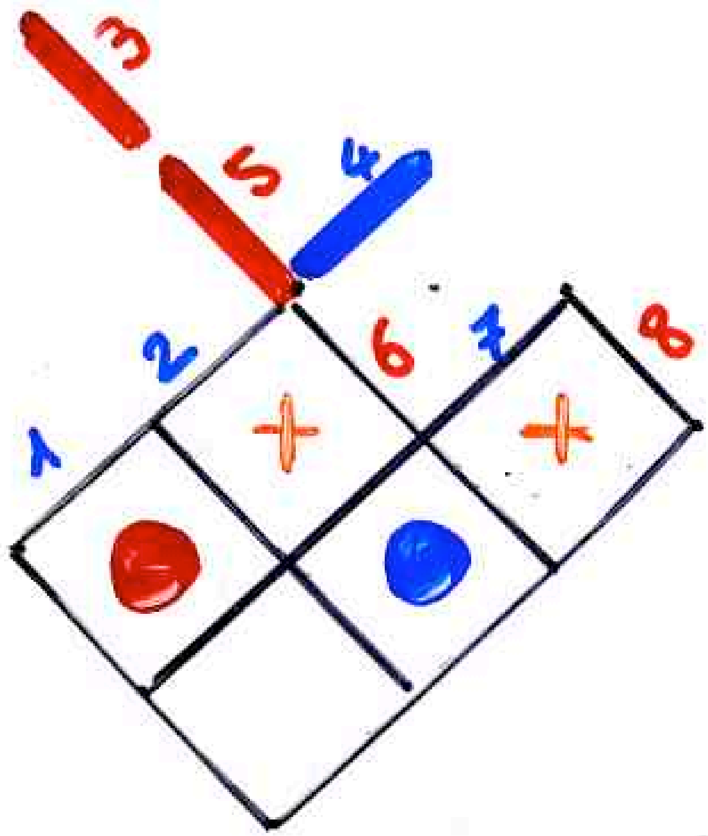

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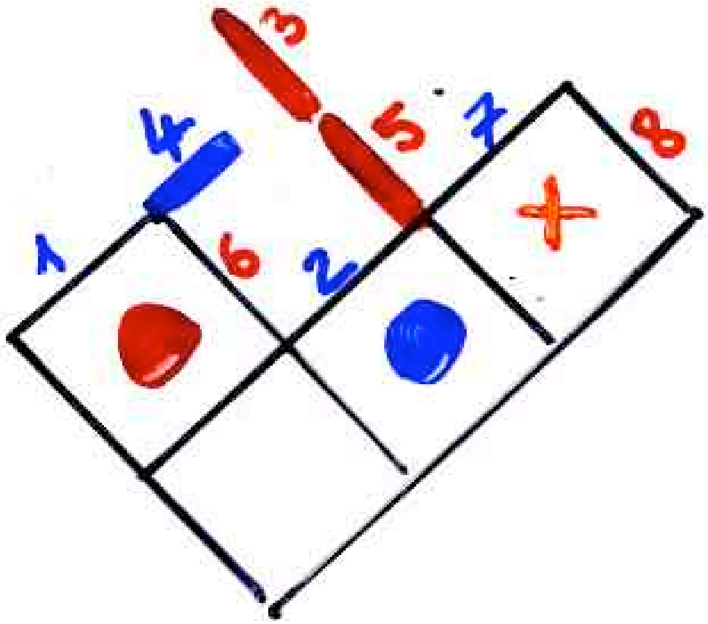
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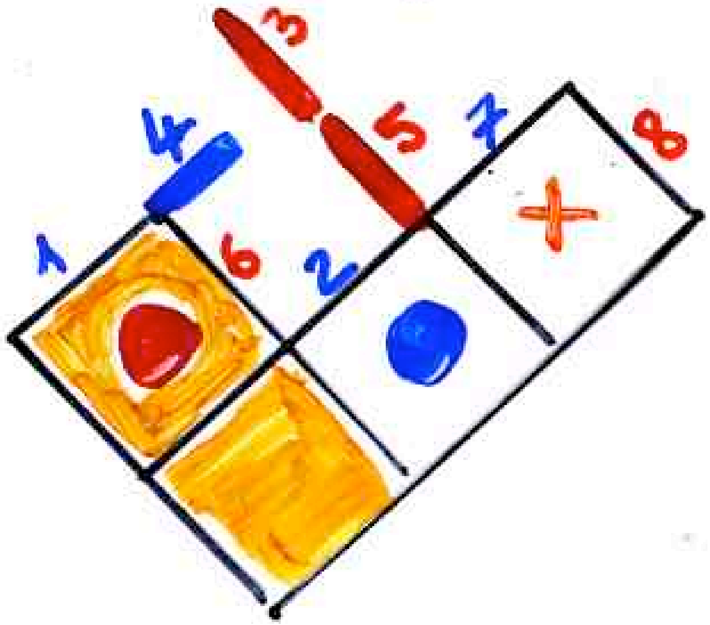

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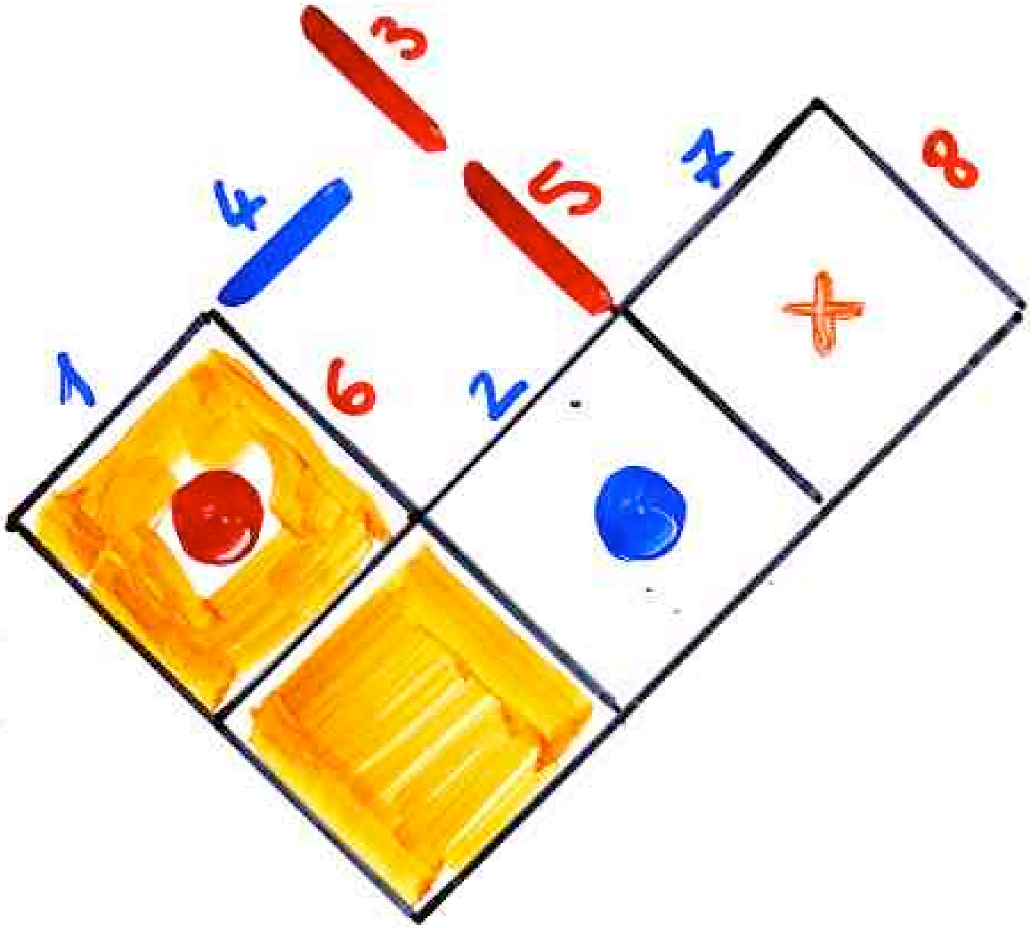

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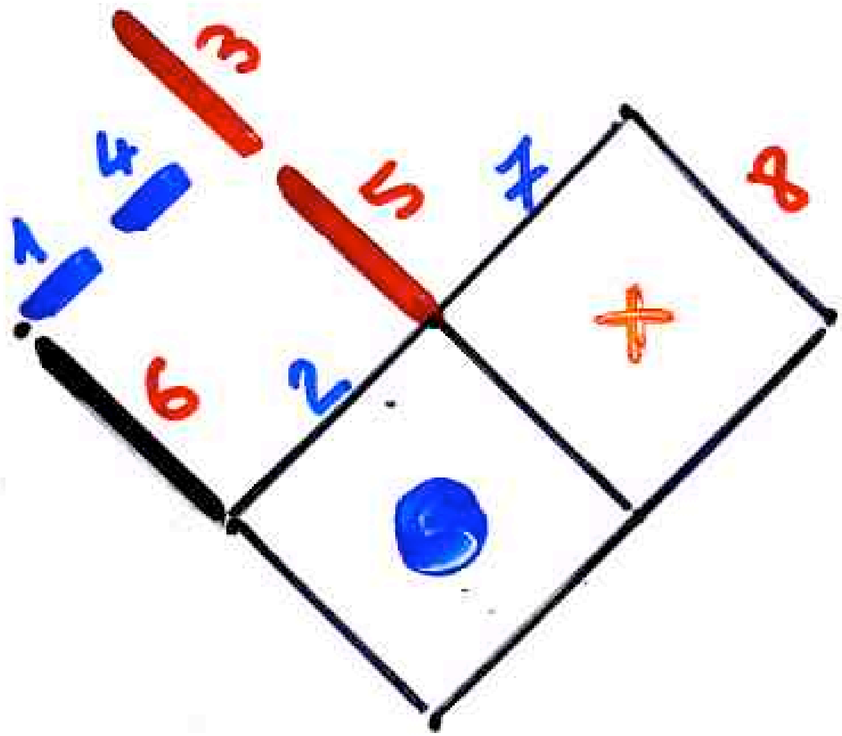
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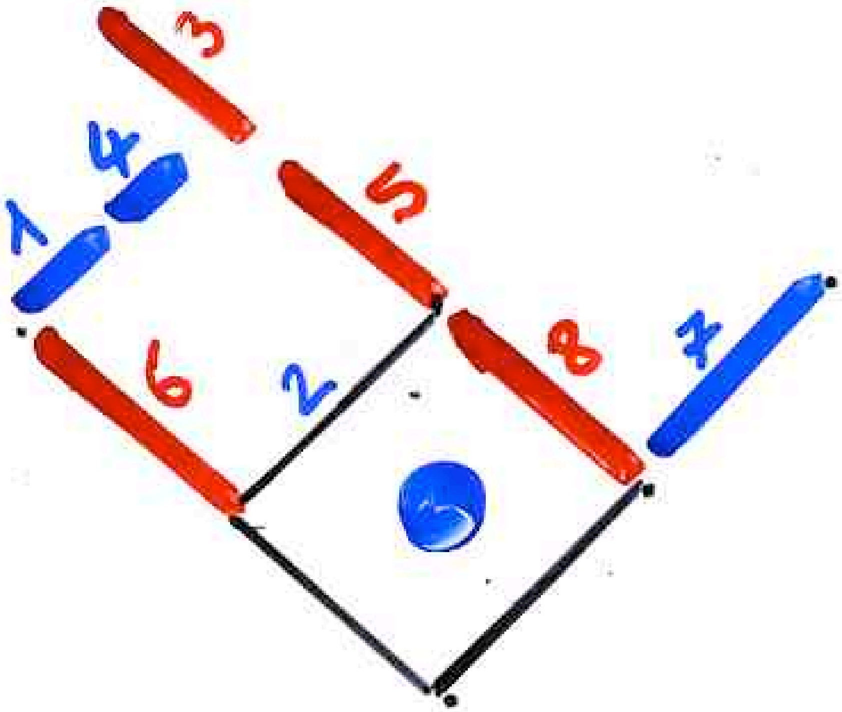

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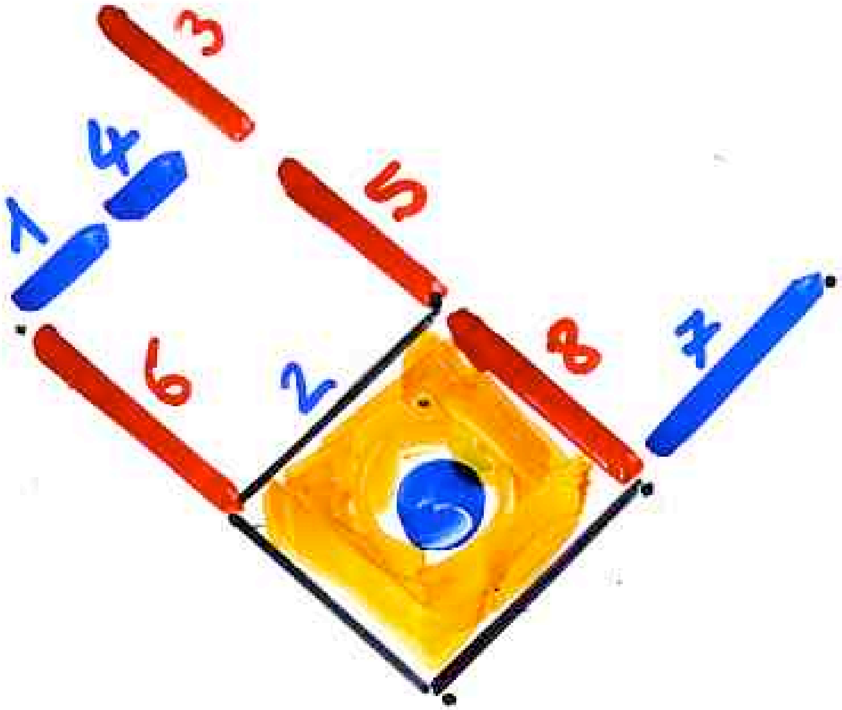
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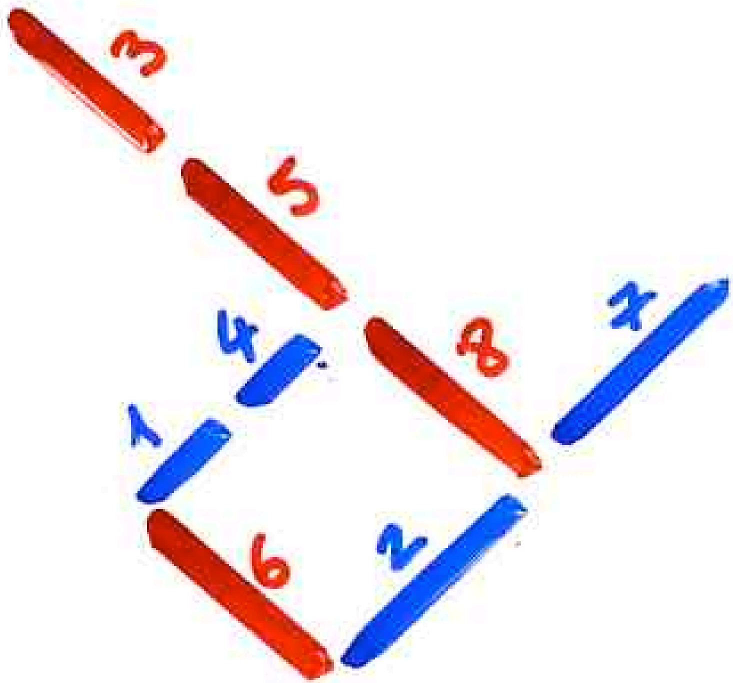

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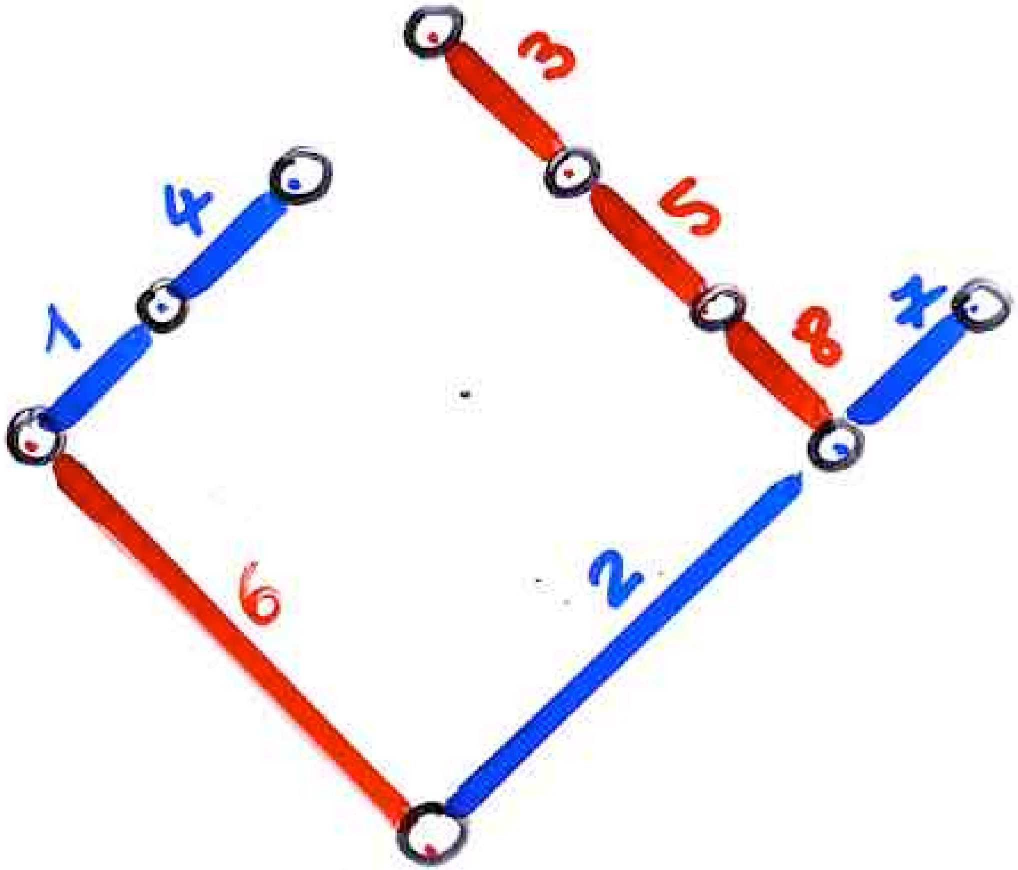
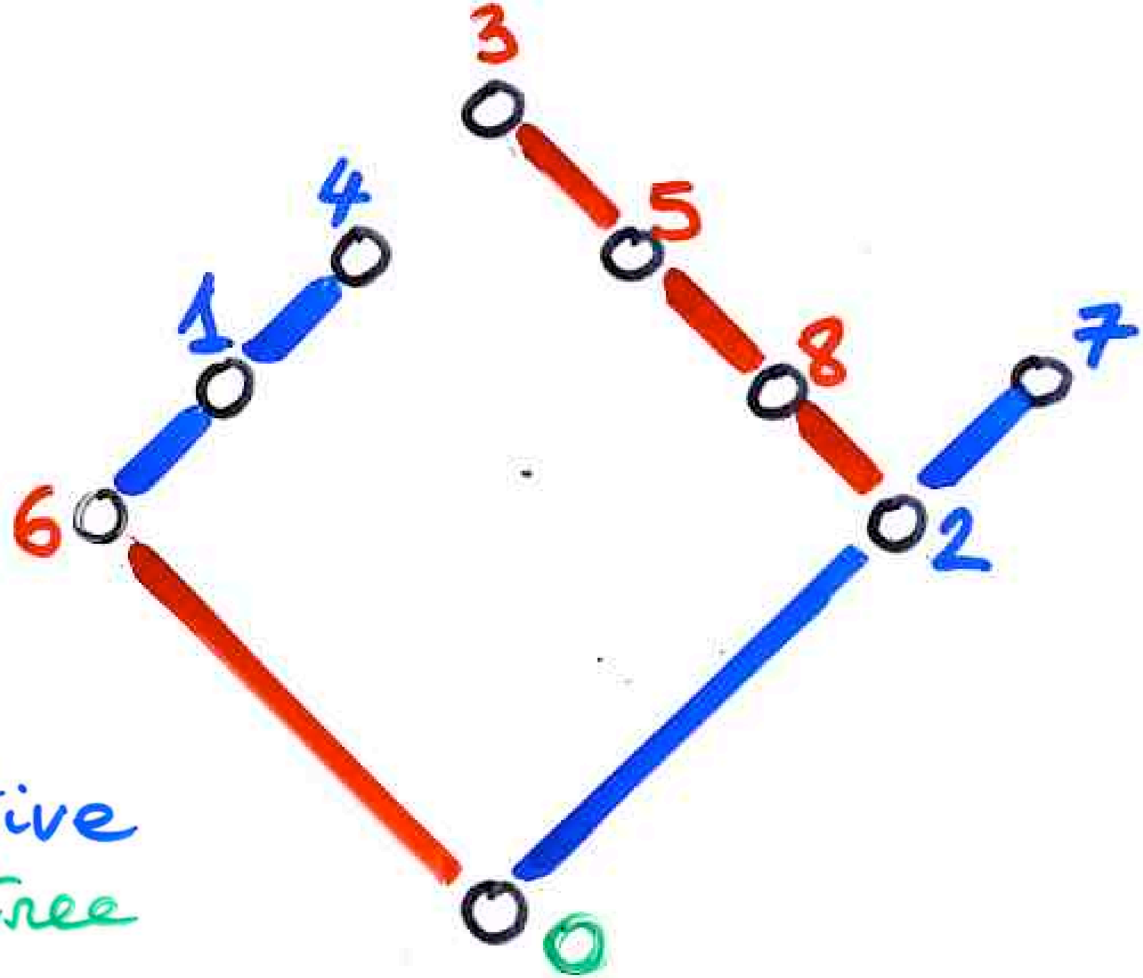
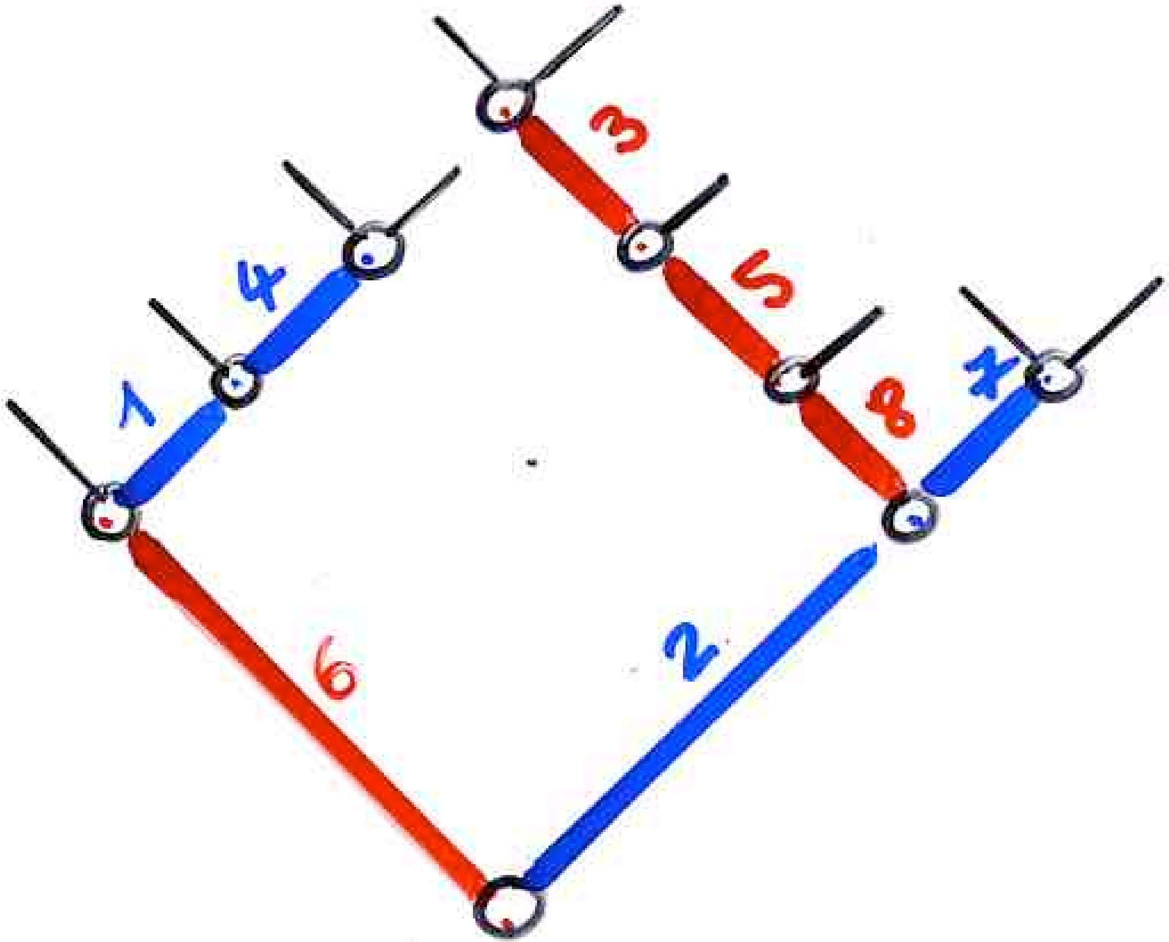
6
2



3
4
5
1
8
7
6
2



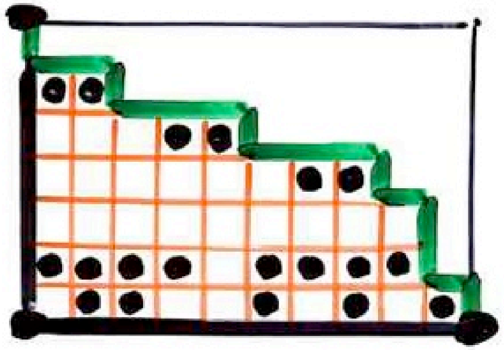
alternative
binary tree
(P. Nadeau)


1
4
3
5
8
7
6
2

Catalan Permutation tableaux

Permutation Tableau

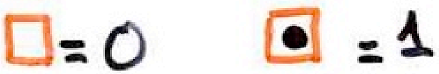
Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

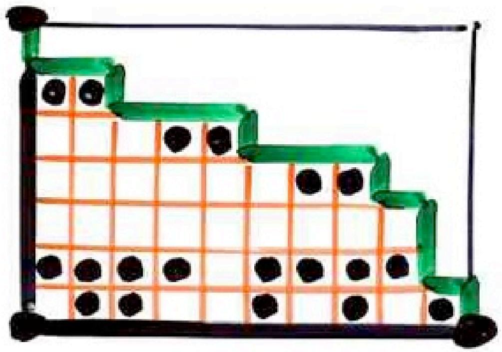
(i)

(ii)



Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



□ = 0   ⊡ = 1

filling of the cells with 0 and 1

(i) in each column : at least one 1

(ii) 1 - - - - 0 / 1 forbidden

permutation tableau

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)

Corteel, Williams (2006) PASEP

Partially Asymmetric Exclusion Process

Catalan tableau

(i), (ii) permutation tableau

+ (iii) only one 1 in each column

Catalan tableau

(i), (ii) permutation tableau

+ (iii) only one 1 in each column

Steingrímsson, Williams (2005, 06)

tableaux $\xrightarrow{\Phi}$ S_n $\xrightarrow{\Psi}$ S_n permutation

Catalan tableau $\longleftrightarrow$ permutation avoiding (2-31)

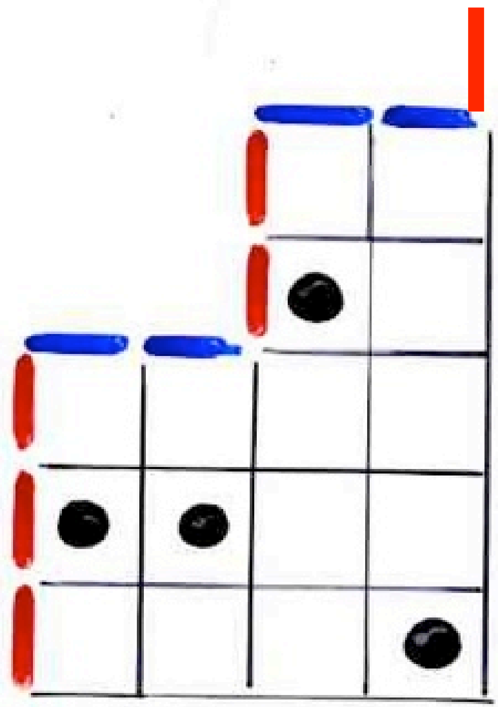
A. Claesson (2001) C_n Catalan nb

Alexander Burstein, 2006

Sylvie Corteel, 2006

Niklas Eriksen, 2006

Astrid Reifegerste, 2006



§1 alternative tableau: definition

§2 The "exchange-fusion" algorithm

§3 The inverse "exchange-fusion" algorithm

§4 The q-Laguerre parameter

Bijection Laguerre histories permutations

§5 Catalan alternative tableaux

Bijection permutations with no $y+1, x, y, x+1$ and binary trees

bijection alternative Catalan tableaux --- binary trees

Bijection alternative tableaux alternative binary trees

Catalan Permutation tableaux