How to color a map with (-1) color?

(second part)

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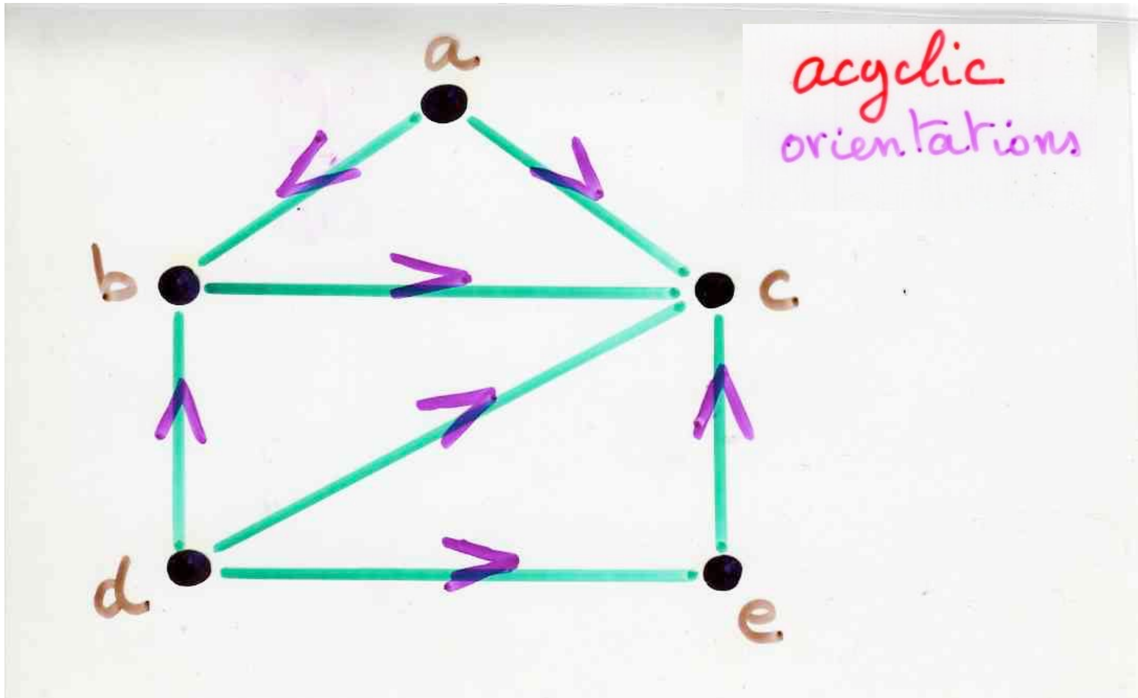
proof of Stanley's theorem

Proposition (Stanley, 1973)

$$a(G) = (-1)^{n(G)} \gamma_G(-1)$$



4 ideas

- (proper) coloring gives a partition of the vertices V of the graph G into trivial heaps
(called in graph theory indepent sets)

sequence of trivial heaps
$\longrightarrow$ a heap on the graph G

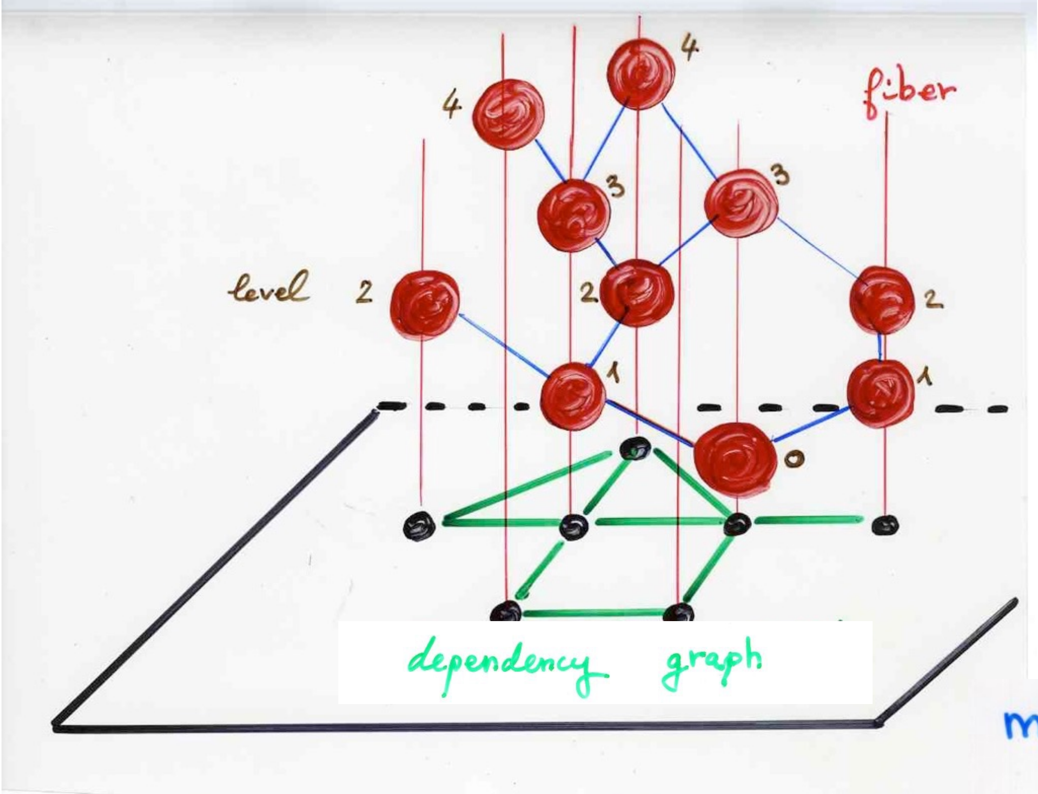
- if f is the generating function of combinatorial objects

$$\frac{1}{1-f}$$ g.f. of sequences of such objects

- Inversion Lemma for heaps (or commutation) monoids

- multilinear heaps

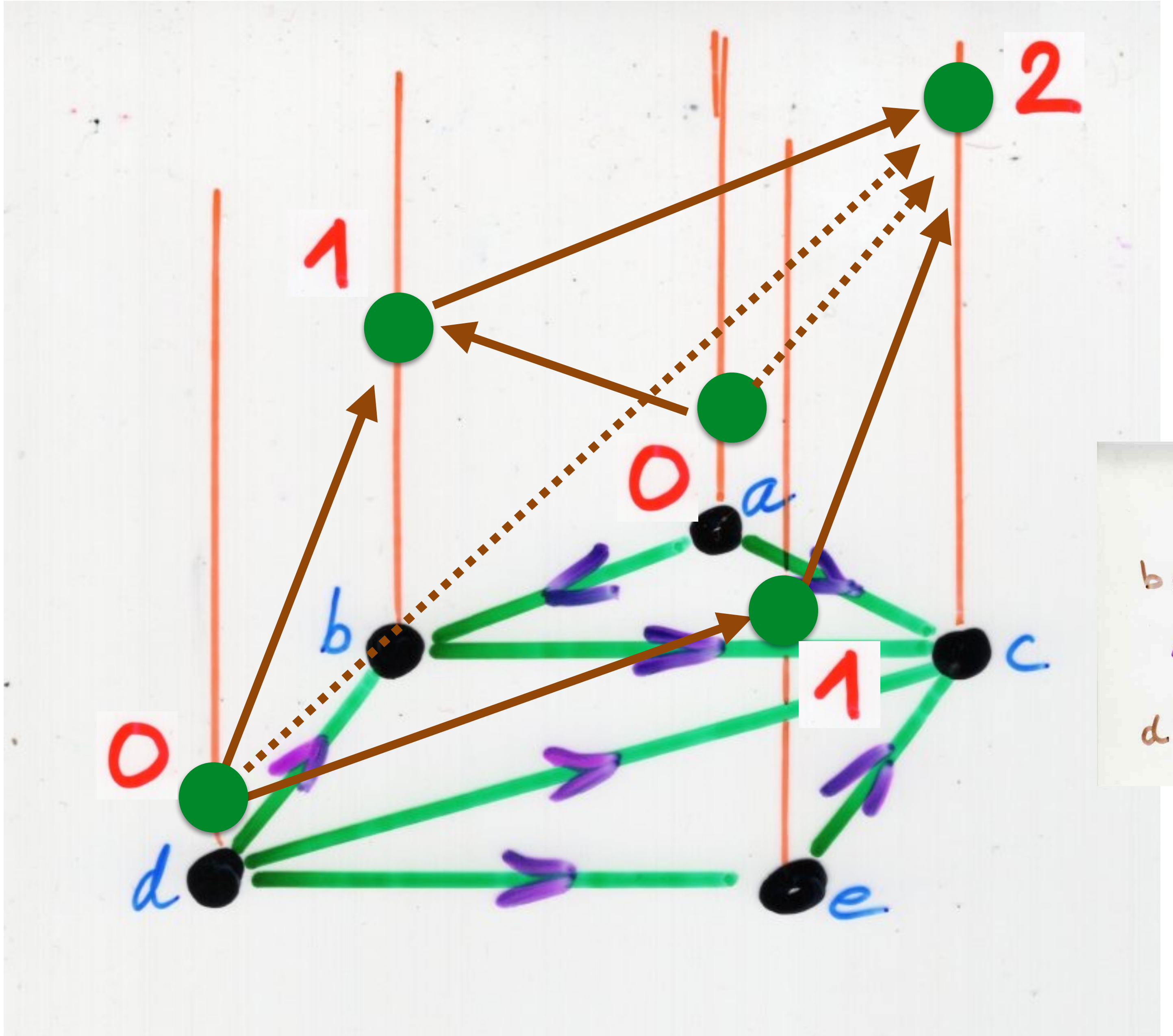
Definition A heap F is multilinear iff in each fiber $\pi^{-1}(v)$, $v \in V$ there is one and only one piece of F

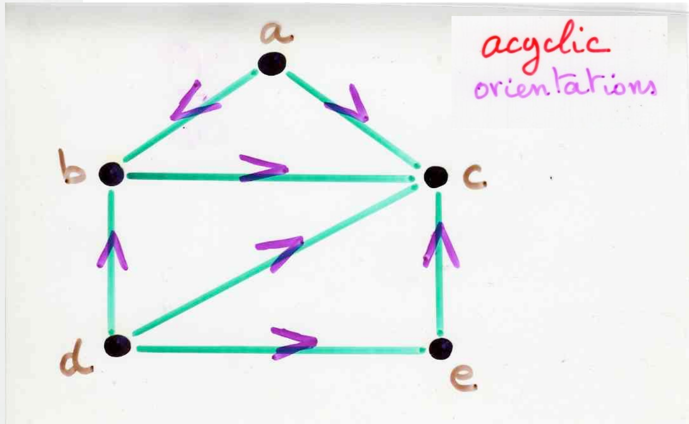


Bijection

multilinear heaps on G $\longleftrightarrow$ acyclic orientations of G

Bijection multilinear heaps on G ⟷ acyclic orientations of G





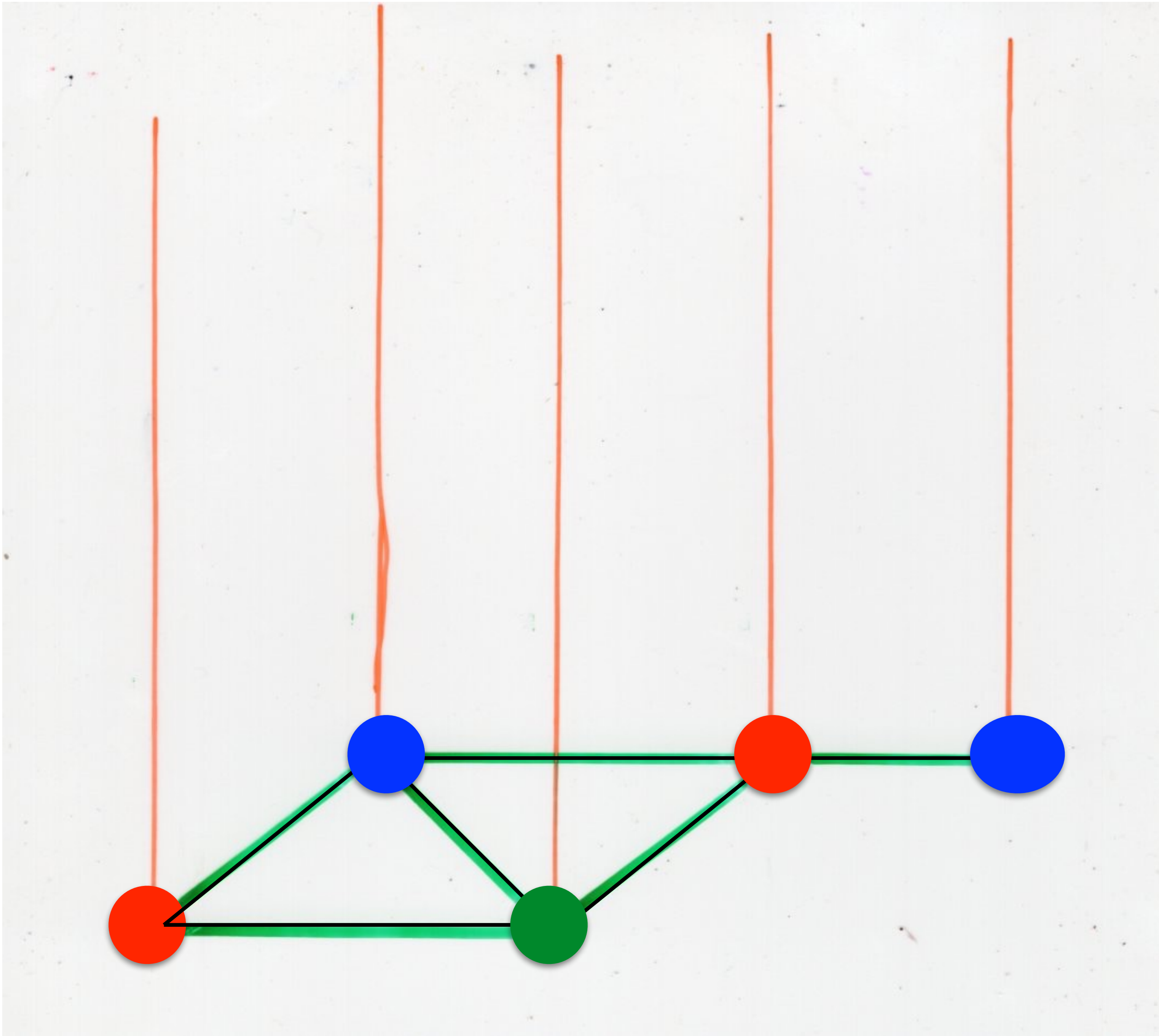
λ possible colors k are used

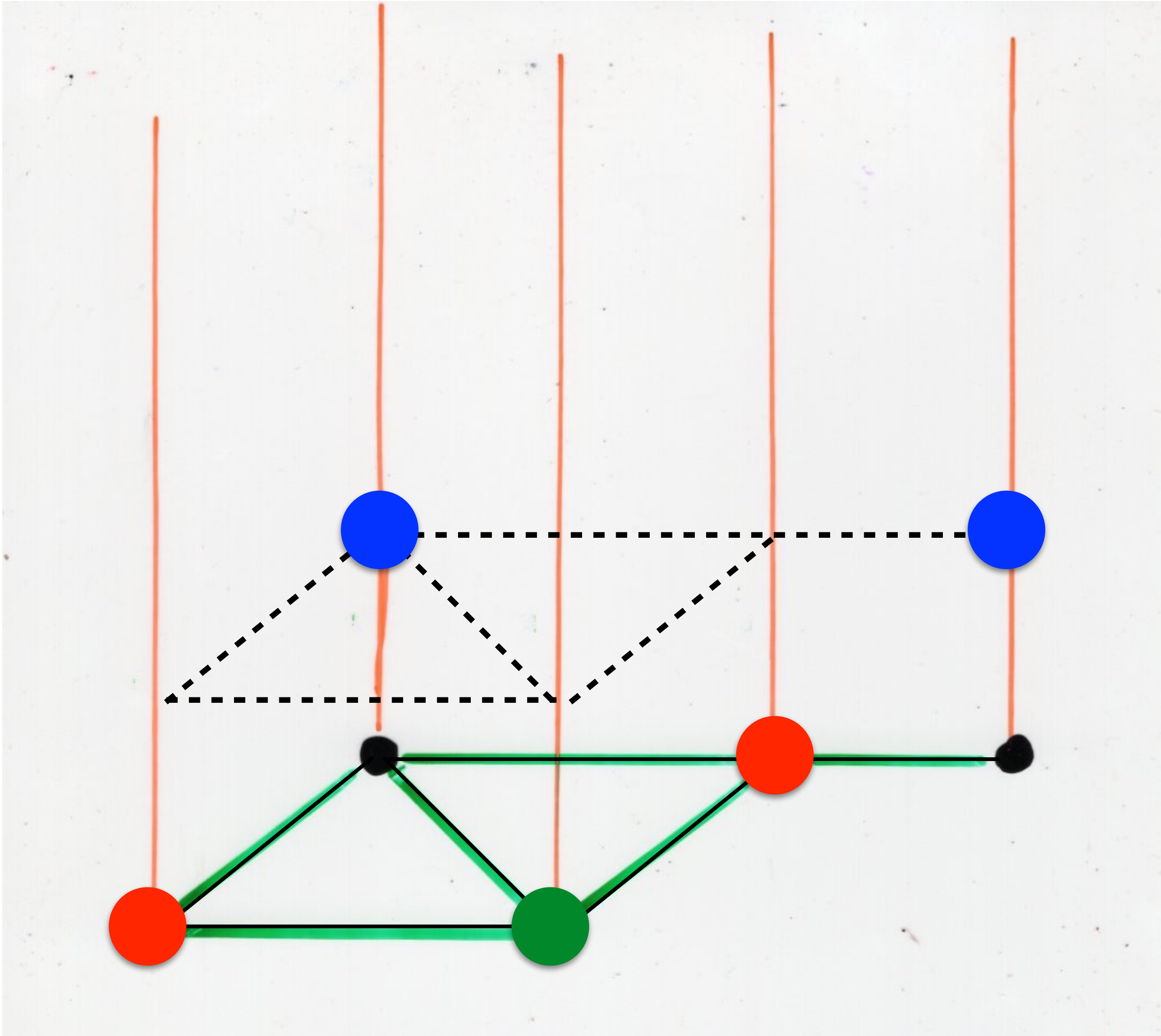
define a total order on the colors $c_1, \dots, c_k$

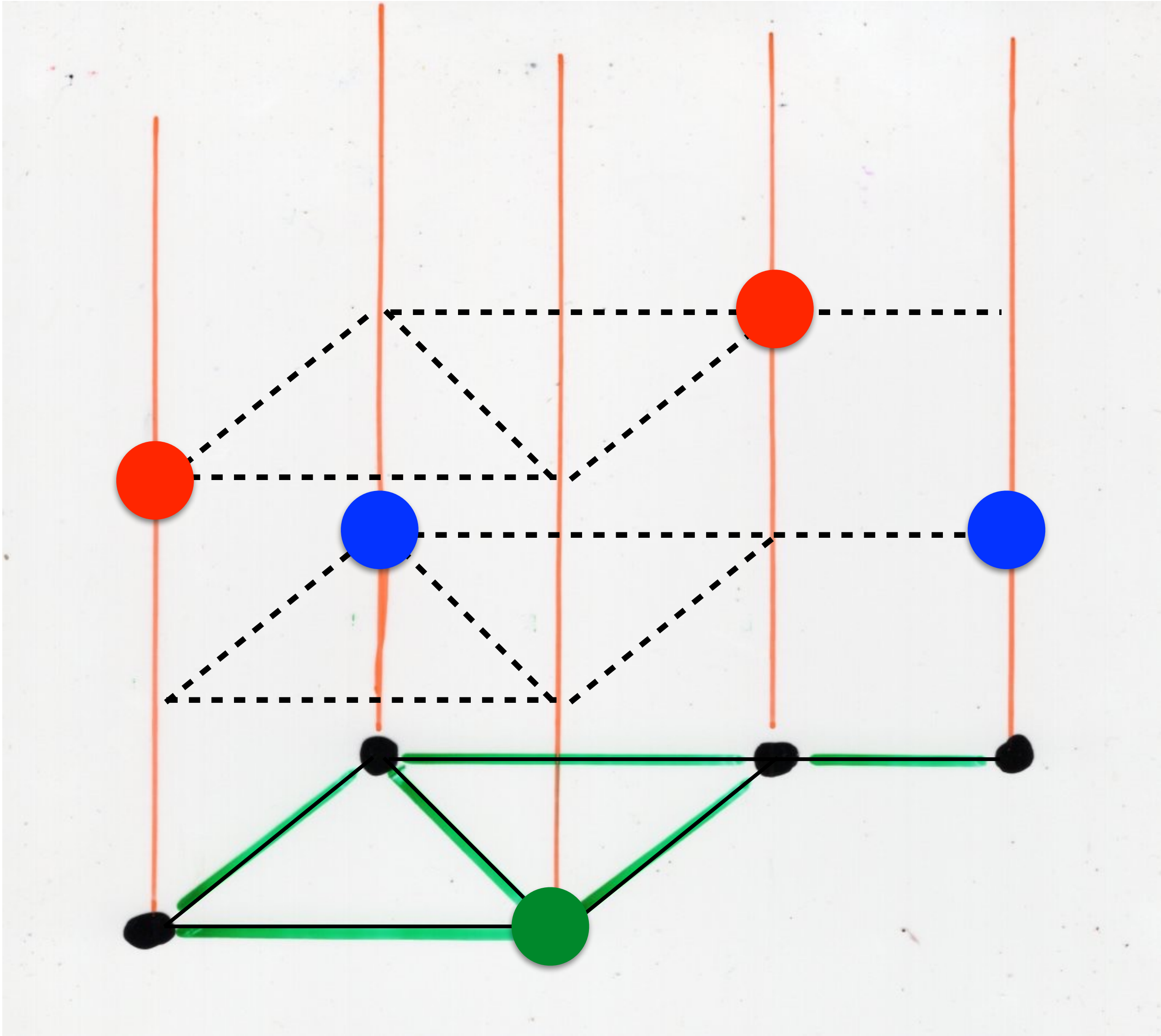
$\longrightarrow$ $(T_1, \dots, T_k)$

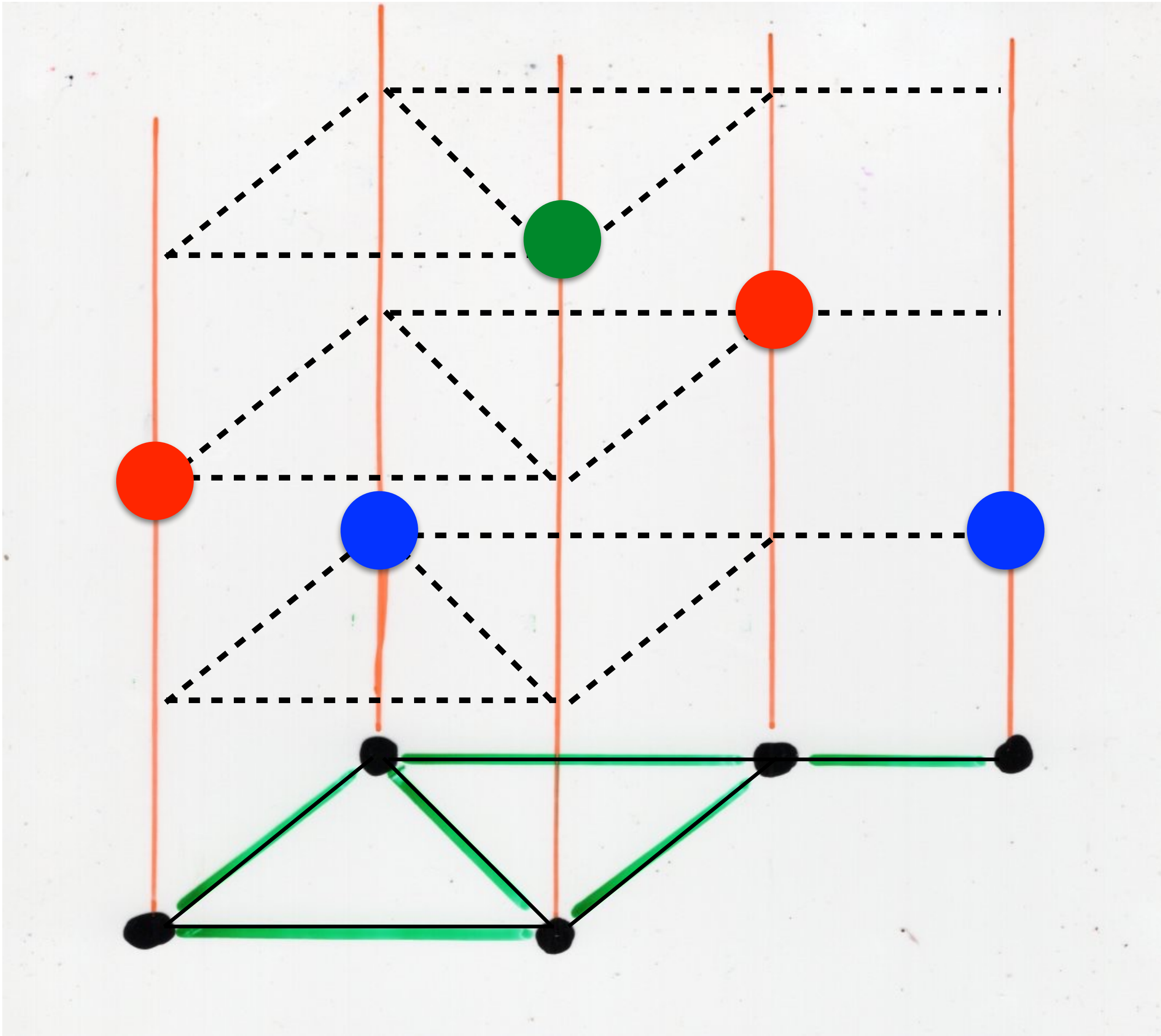
sequence of trivial heaps

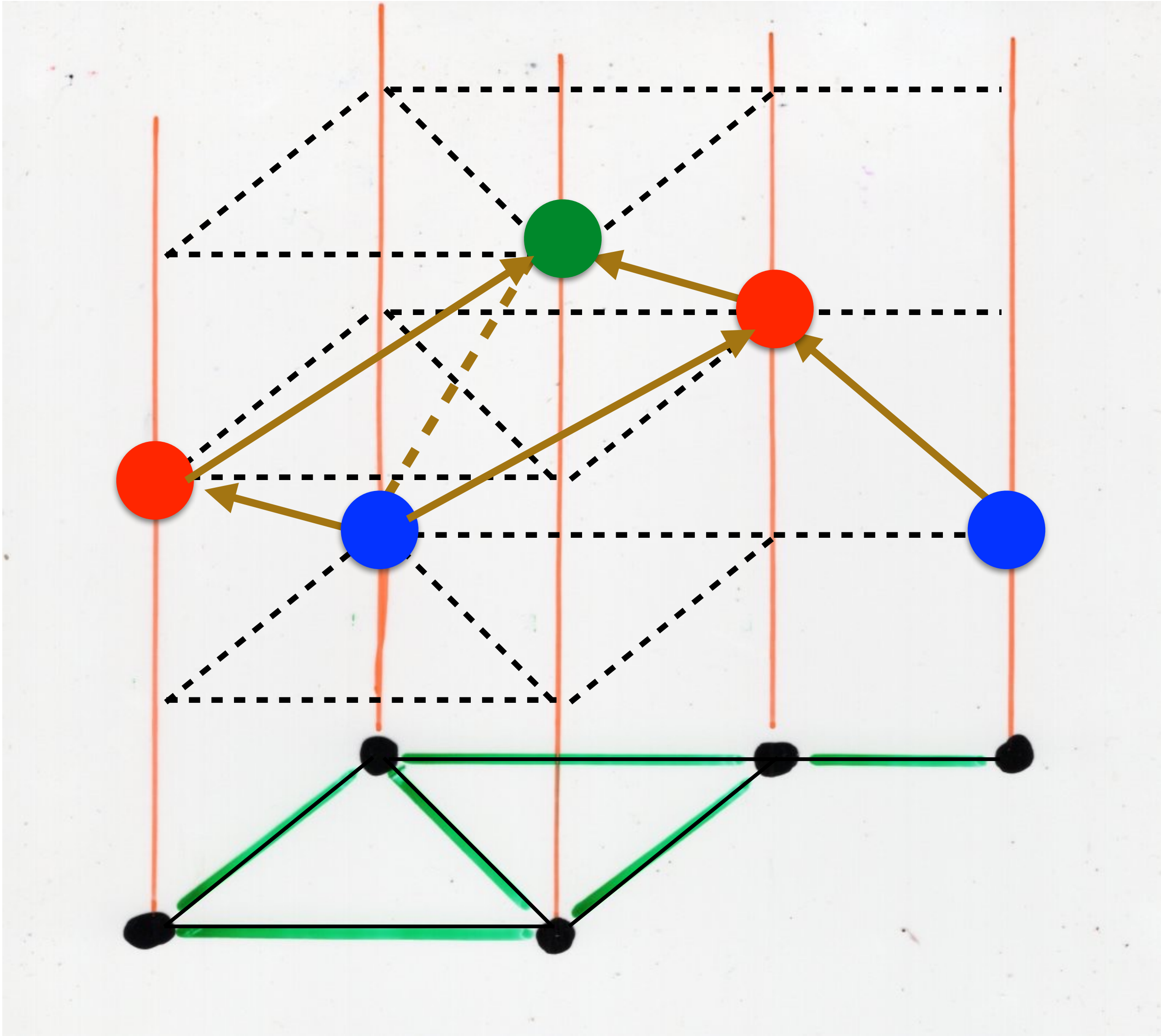
$F = T_1 \odot \cdots \odot T_k$ is a multilinear heap











Definition

F heap of $H(V, E)$

a layer factorization of F is a sequence $(T_1, \ldots, T_k)$ of trivial heaps such that $F = T_1 \odot \cdots \odot T_k$

(product of heaps)

$(F; (T_1, \ldots, T_k))$ is called a layered heap

$\beta_k(F)$ number of layer factorizations of F

Definition colored layered heap is a layered heap $(F; (T_1, \ldots, T_k))$ where each layer T_i is colored (i.e. all the pieces of T_i have the same color) with the condition that all layers have distinct colors

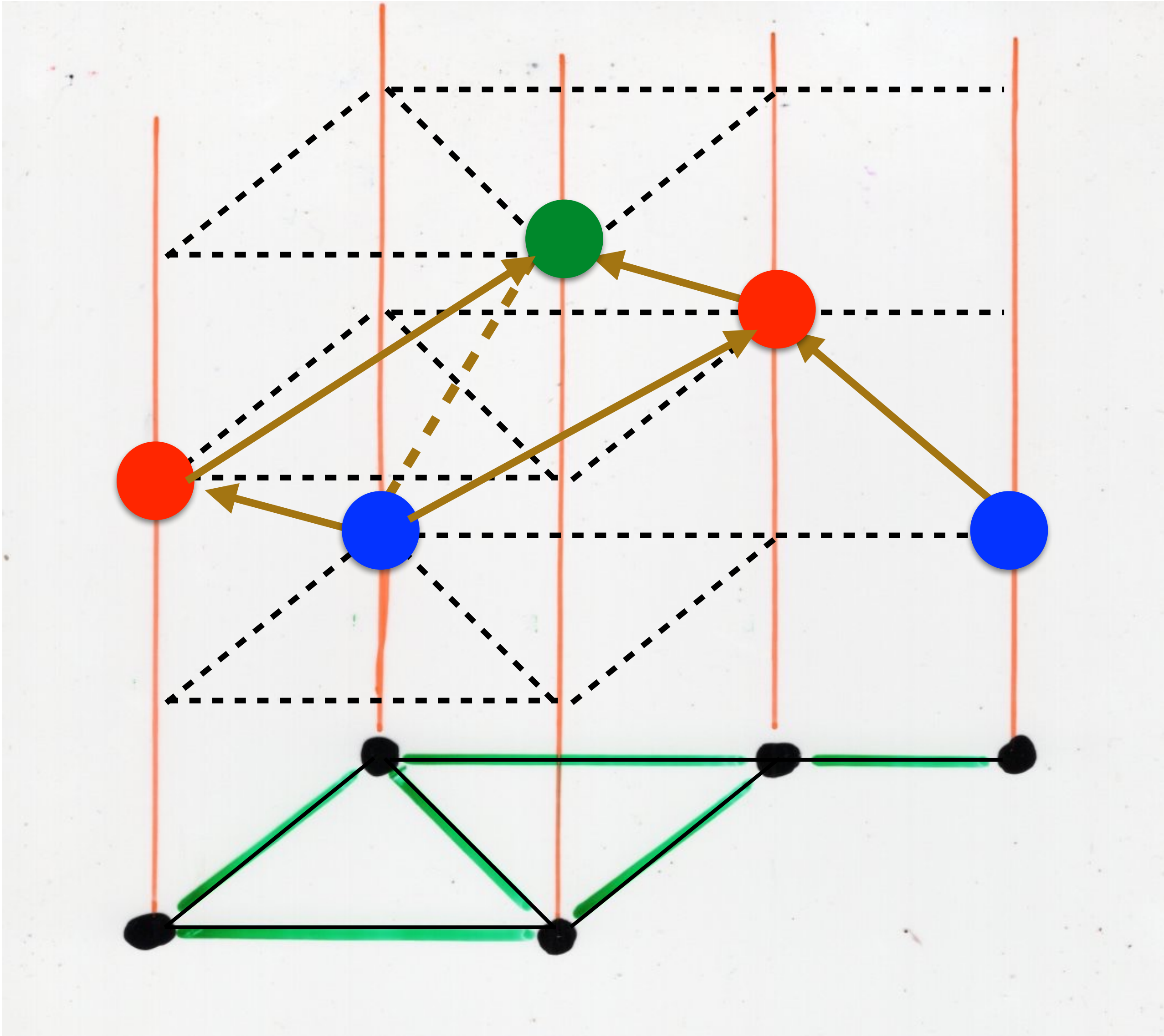
If λ is the number of possible colors, the number of colored layer associated to the heap F is:

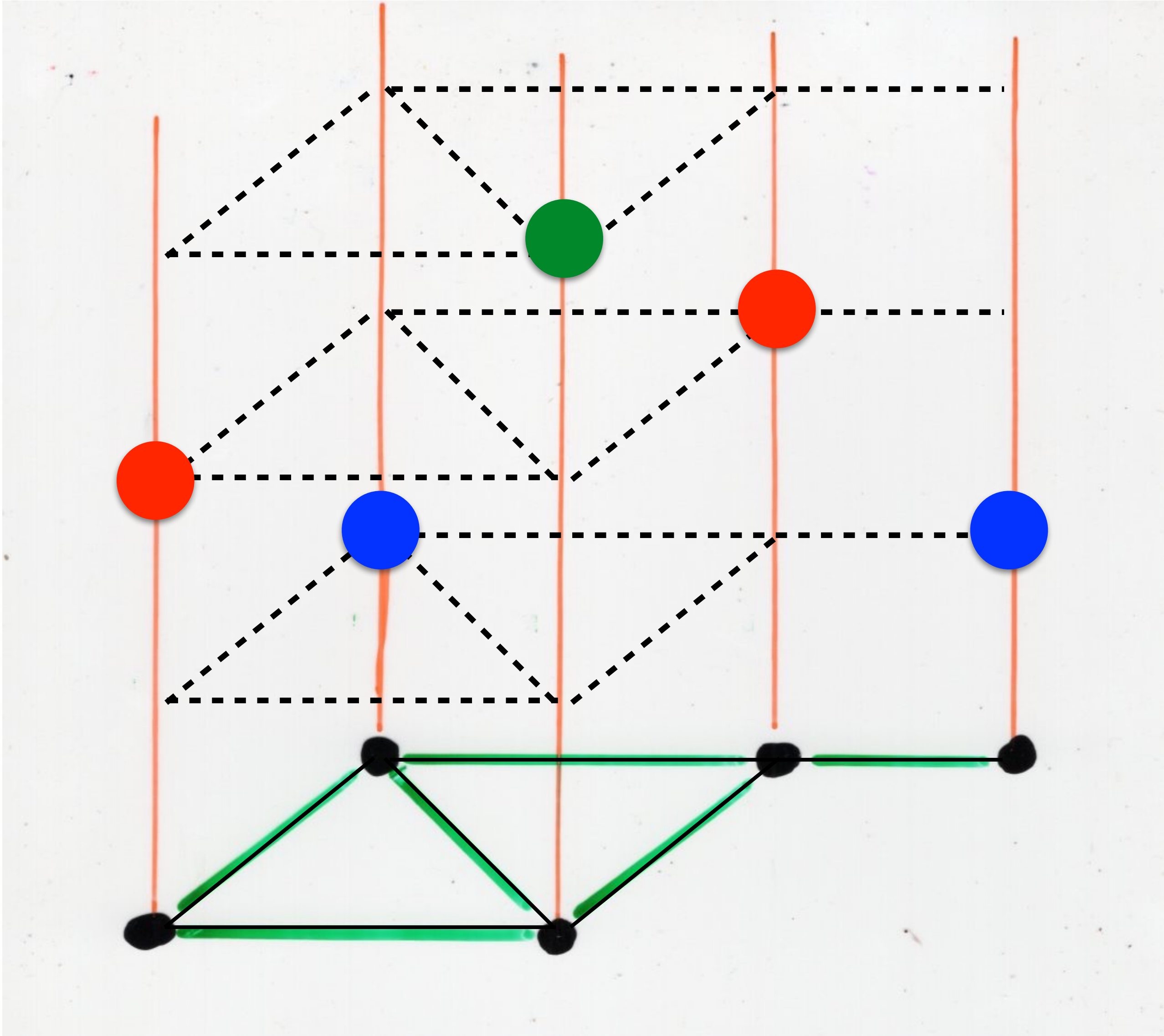
$$\beta_k(F)\, \lambda(\lambda-1) \cdots (\lambda-k+1)$$

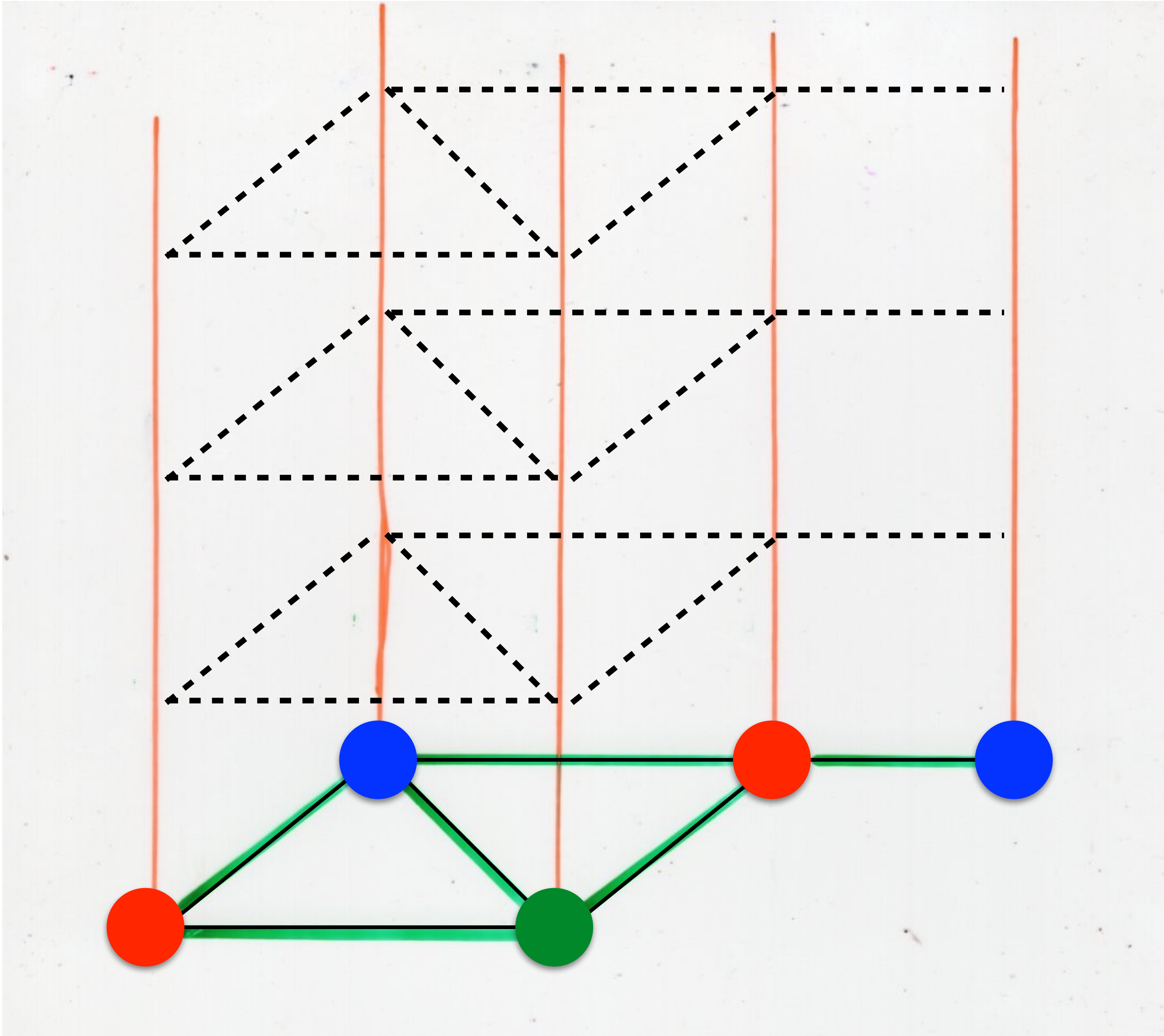
Definition A heap F is covering the graph G iff for any vertex $v \in V$ of G the fiber above v is not empty
(the fiber is the chain of pieces of F with projection on v)

(the fiber above v is the chain $\pi^{-1}(v)$)

multilinear $\longleftrightarrow$ ordered coloring
colored layered
heap







Proposition

$$\gamma_G(\lambda) = \sum_{\substack{F \\ \text{multilinear} \\ \text{heap over } G}} \sum_{k \geq 1} \beta_k(F) \frac{\lambda(\lambda-1)\ldots(\lambda-k+1)}{k!}$$

$$\binom{\lambda}{k}$$

$$\gamma_G(-1) = \sum_{\substack{F \\ \text{multilinear} \\ \text{heap over } G}} \sum_{k \geq 1} \beta_k(F) (-1)^k$$

Definition Chromatic power series of the graph G (with weighted heaps)

$$\Gamma_G^{v}(\lambda) = \sum_{\substack{F \\ \text{heap} \\ \text{covering } G}} \sum_{k \geq 1} \beta_k(F)\, v(F)\, \frac{\lambda(\lambda-1)\cdots(\lambda-k+1)}{k!}$$

$$\gamma_G(\lambda)$$

multilinear

sequence of trivial heaps

$(T_1, \ldots, T_k)$

$$f = \sum_{T} v(T)$$

generating function of trivial heaps

$$\frac{1}{1-f}$$

g.f. of sequence of trivial heaps

add a variable t for taking account of the parameter k

$$\frac{1}{1 - t\left(\sum_{\substack{T \\ \text{trivial} \\ \text{heap}}} v(T)\right)} = \sum_{\substack{F \\ \text{heap} \\ \text{on } G}} \beta_k(F)\, v(F)\, t^k$$

$$t = -1 \qquad \bar{v}(\alpha) = -v(\alpha)$$

α basic piece
= vertex of G

$$\frac{1}{1 + \sum_{\substack{T \\ \text{trivial} \\ \text{heap}}} (-1)^{|T|}\, \bar{v}(T)} = \sum_{\substack{F \\ \text{heap} \\ \text{on } G}} \bar{v}(F)$$

$$\frac{1}{1+\left(\sum_{\substack{T \\ \text{trivial} \\ \text{heap}}} v(T)\right)} = \sum_{\substack{F \\ \text{heap} \\ \text{on } G}} \beta_k(F)\, v(F)\, (-1)^k$$

$$t = -1 \qquad \bar{v}(\alpha) = -v(\alpha)$$

α basic piece
= vertex of G

=

$$\frac{1}{1+\sum_{\substack{T \\ \text{trivial} \\ \text{heap}}} (-1)^{|T|}\,(T)} = \sum_{\substack{F \\ \text{heap} \\ \text{on } G}} (-1)^{|F|}\, v(F)$$

$$\sum_{\substack{F \\ \text{heap} \\ \text{on } G}} \beta_k(F)\, v(F)\, (-1)^k = \sum_{\substack{F \\ \text{heap} \\ \text{on } G}} (-1)^{|F|}\, v(F)$$

$$\Gamma_G^{v}(\lambda) = \sum_{\substack{F \\ \text{heap} \\ \text{covering } G}} \sum_{k \geq 1} \beta_k(F)\, v(F)\, \frac{\lambda(\lambda-1)\ldots(\lambda-k+1)}{k!}$$

$$\sum_{\substack{F \\ \text{heap} \\ \text{on } G \\ \text{covering } G}} \beta_k(F)\, v(F)\, (-1)^k = \sum_{\substack{F \\ \text{heap} \\ \text{on } G \\ \text{covering } G}} (-1)^{|F|}\, v(F)$$

$$\lambda = -1$$

$$\Gamma_G^v(\lambda) = \sum_{\substack{F \\ \text{heap} \\ \text{covering } G}} \sum_{k \geq 1} \beta_k(F)\, v(F)\, \frac{\lambda(\lambda-1)\cdots(\lambda-k+1)}{k!}$$

$$\sum_{\substack{F \\ \text{heap} \\ \text{on } G}} \beta_k(F)\, v(F)\, (-1)^k = \sum_{\substack{F \\ \text{heap} \\ \text{on } G}} (-1)^{|F|}\, v(F)$$

covering G

covering G

$$\Gamma_G^{v}(-1)$$

$$\Gamma_G^{v}(-1) = \sum_{\substack{F \\ \text{heap} \\ \text{covering } G}} (-1)^{|F|}\, v(F)$$

$$\sum_{\substack{F \\ \text{heap} \\ \text{on } G}} \beta_k(F)\, v(F)\, (-1)^k = \sum_{\substack{F \\ \text{heap} \\ \text{on } G}} (-1)^{|F|}\, v(F)$$

covering G

multilinear

covering G

multilinear

$$\gamma_G^{v}(-1)$$

$$\gamma_G^{v}(-1) = \sum_{\substack{F \\ \text{multilinear} \\ \text{heap on } G}} (-1)^{n(G)}\, v(F)$$

$$\gamma_G^{\vee}(-1) = \sum_{\substack{F \\ \text{multilinear} \\ \text{heap on } G}} (-1)^{n(G)} \, v(F)$$

$$\downarrow$$

number of acyclic orientations of G

$v(\alpha) = 1$
$\alpha \in V$

Bijection

multilinear heaps on G $\longleftrightarrow$ acyclic orientations of G

$\square$
end of proof

Greene, Zaslavsky (1983)

- number of acyclic orientations with one sink $= \pm$ linear term of $\gamma_G(\lambda)$

  $\rightarrow$ proved with hyperplane arrangements

Gebhard, Sagan (2000) 3 other proofs

$\rightarrow$ Lass (2001) proof with heaps

$$\Gamma_G^{v}(\lambda) = \sum_{\substack{F \\ \text{heap} \\ \text{covering } G}} \sum_{k \geq 1} \beta_k(F)\, v(F) \frac{\lambda(\lambda-1)\cdots(\lambda-k+1)}{k!}$$

$$\gamma_G(\lambda)$$

multilinear

<u>Definition</u> multicoloring of the graph G associated to $\mathbf{k} = (k_1, \dots, k_n)$

$|V| = n$ $\quad V = (1, 2, \dots, n)$

is an assignement of colors to the vertices of G in vertex $i \in V$ receives k_i colors, such that adjacent vertices receive only disjoint colors.

$$\gamma^{G}_{\mathbf{k}}(\lambda)$$

number of multicoloring associated to $\mathbf{k}$ with λ colors

Bijection

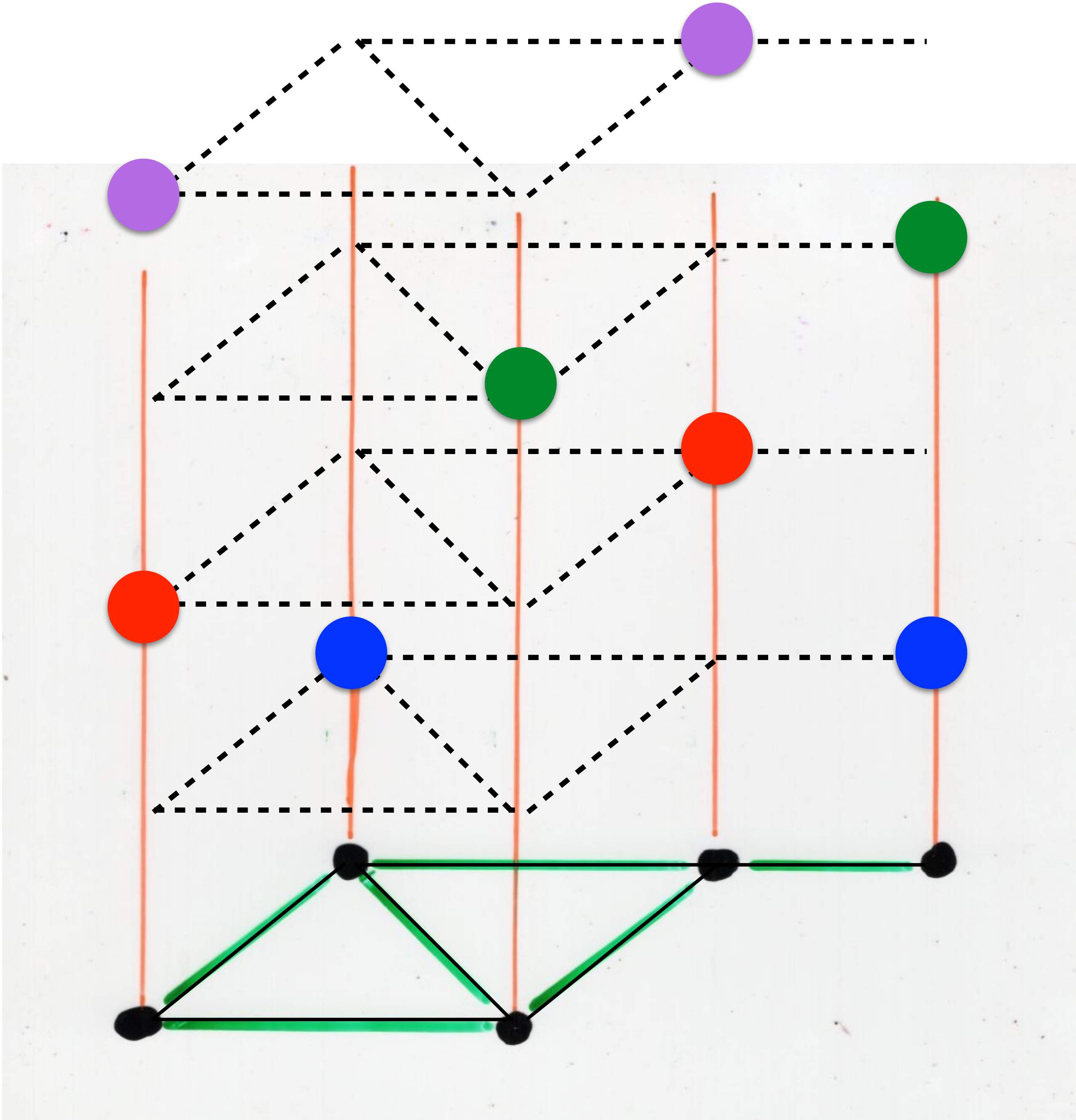
colored layered heap covering G
(having k layers)

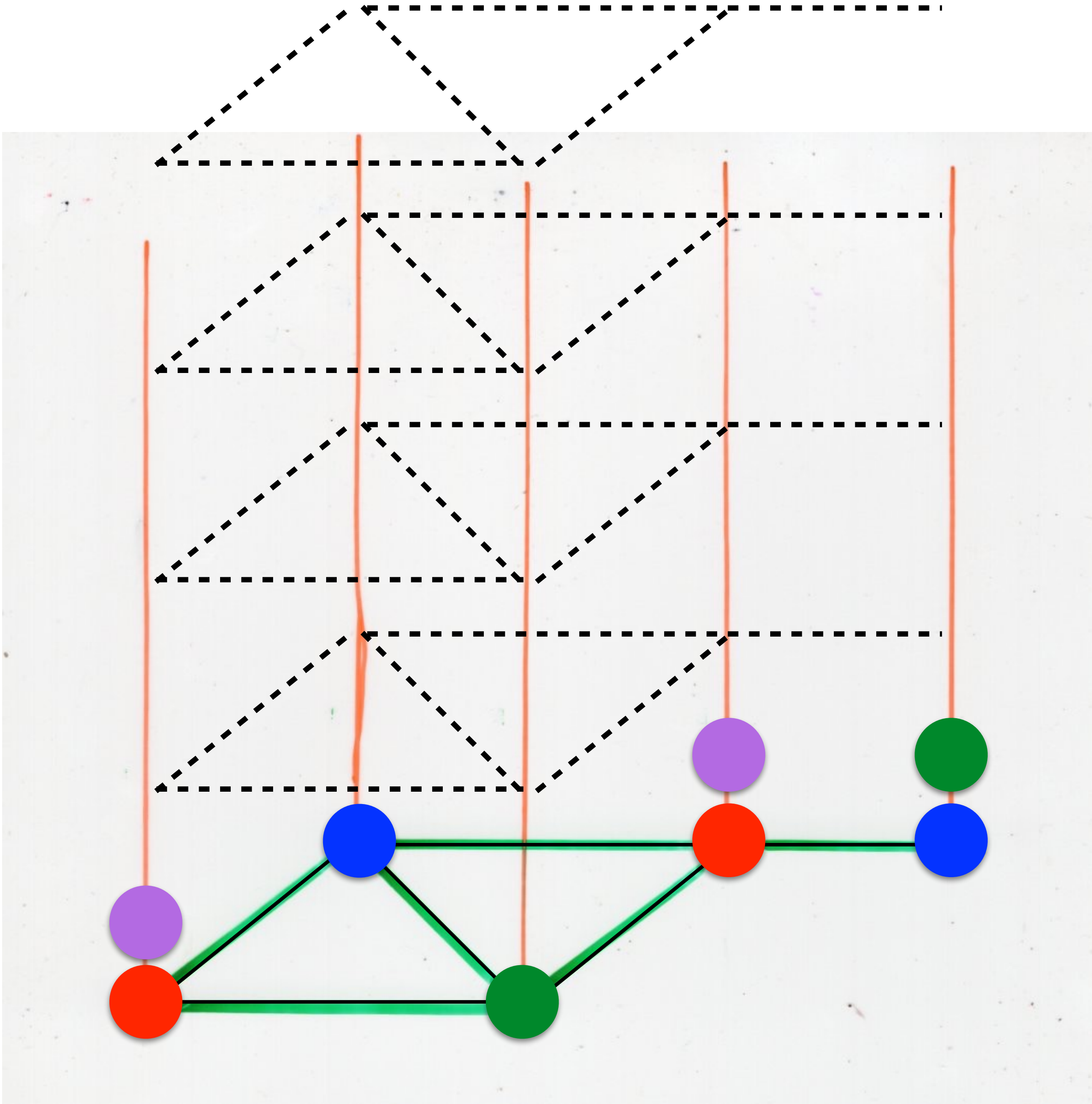
$\updownarrow$

ordered multicoloring
(i.e. the k colors used in the multicoloring are totally ordered)

multilinear colored layered heap $\leftrightarrow$ ordered coloring

$\mathbf{k} = (1, 1, \dots, 1)$





« Behind the walls »
Jean-Pierre Muller 2013

« Behind the walls »
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$$\Gamma_G^{v}(\lambda) = \sum_{\substack{F \\ \text{heap} \\ \text{covering } G}} \sum_{k \geq 1} \beta_k(F)\, v(F) \frac{\lambda(\lambda-1)\cdots(\lambda-k+1)}{k!}$$

$\gamma_G(\lambda)$

multilinear

$\longrightarrow$ interpretation for $\gamma_{\mathbf{k}}^{G}(\lambda)$ $\quad \mathbf{k} = (k_1, \ldots, k_n)$

- chromatic polynomials from Kac-Moody algebras
  Venkatesh, Viswanath (2015)

- multi-chromatic polynomial related to root multiplicities for Borcherds-Kac-Moody algebras

  Arunkumar, Kus, Venkatesh (2016)

- interpretation with free partially commutative Lie algebra in terms of factorization of a heap into Lyndon heaps (2017)

other interactions between heaps and graphs theory

Riemann zeta function

$$\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$$

Ihara-Selberg zeta function of a graph

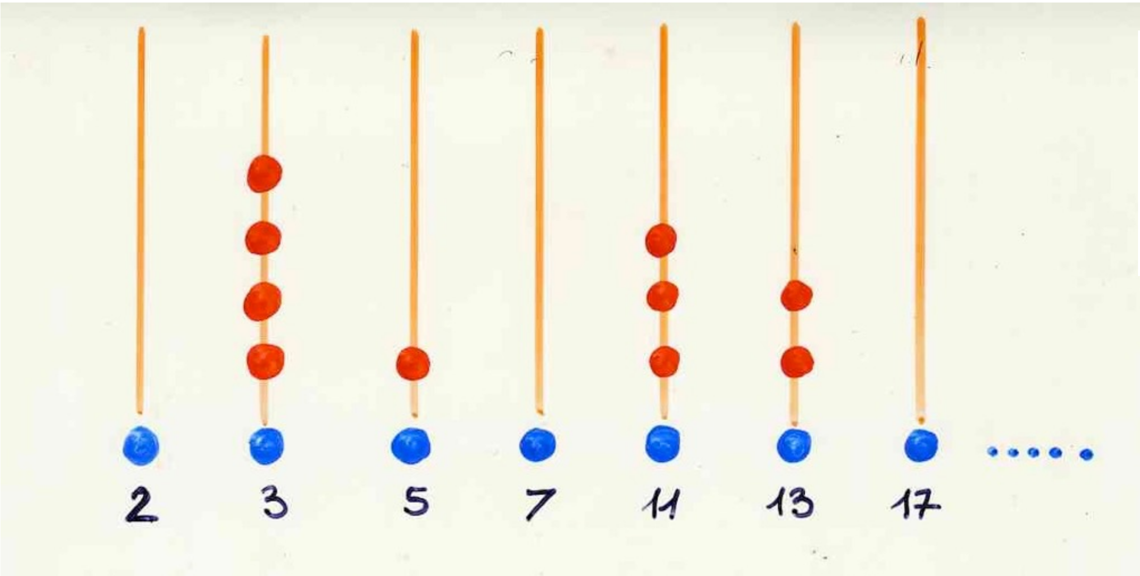
$$Z_G(t)$$

$\mathbb{N}^+ = \mathbb{N} - \{0\}$

$\mathbb{N}^+$ multiplicative monoid

$n \in \mathbb{N}^+ \longrightarrow p_1^{\alpha_1} \cdots p_k^{\alpha_k}$

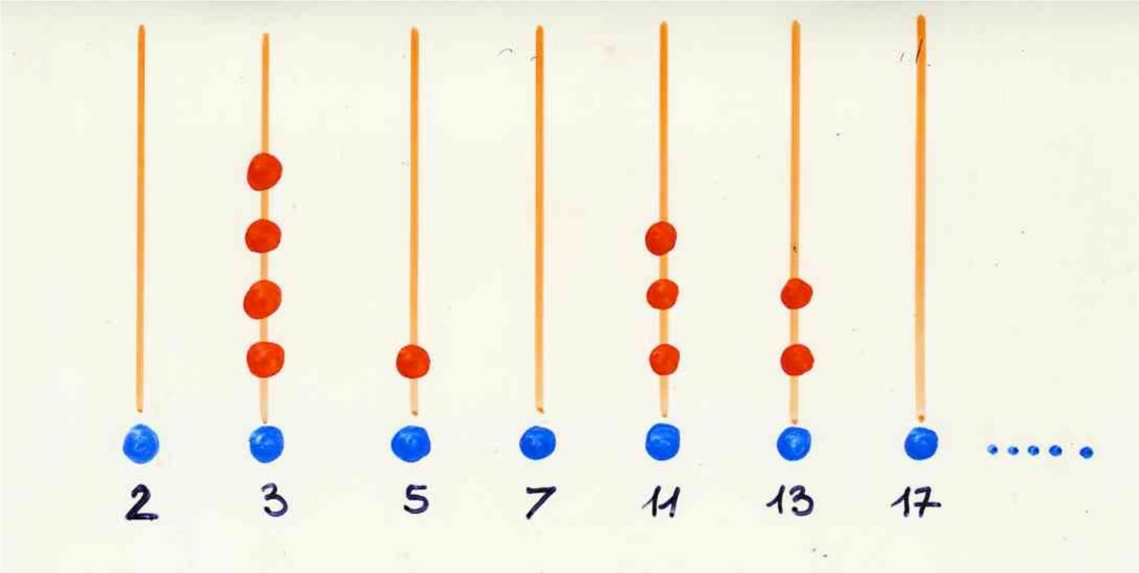
for $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$
prime numbers decomposition



$H(\mathbb{N}^+, \mathcal{C})$

$a \not{\mathcal{C}}\, b$ for any $a, b \in \mathbb{N}^+$
except $a \mathcal{C} a$

$$\sum_{n \geq 1} n^{-s} = \left(\sum_{n \geq 1} \mu(n)\, n^{-s} \right)^{-1}$$



$$n^{-s} = p_1^{-s\alpha_1} \cdots p_k^{-s\alpha_k}$$

Möbius classic in number theory

for $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$
prime numbers decomposition

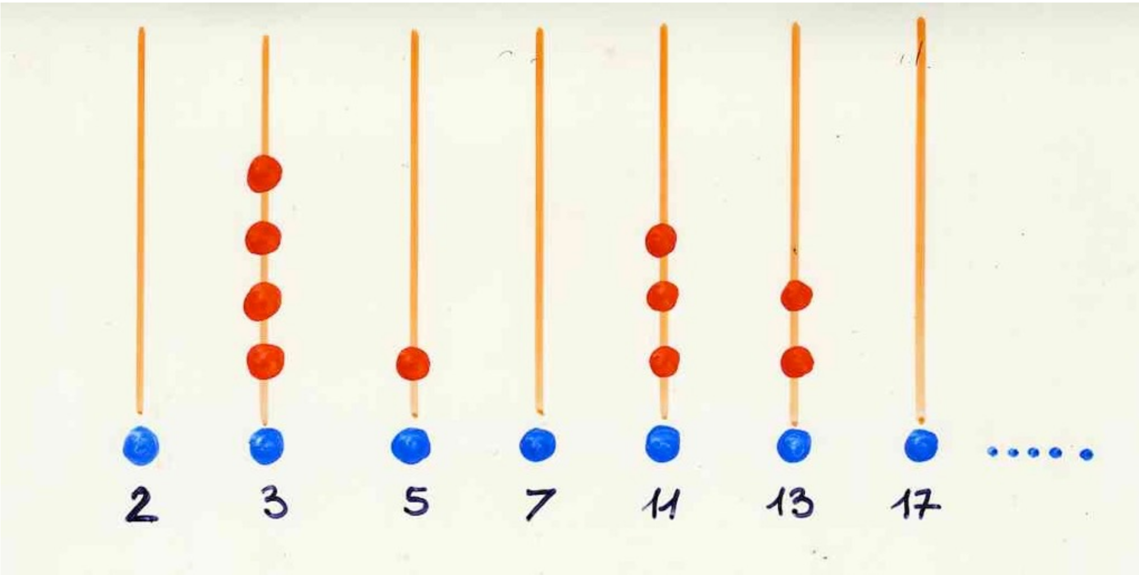
$$\mu(n) = \begin{cases} \bullet\ 0 & \text{if } n \text{ is divisible by a square} \\ \bullet\ (-1)^k & \text{else} \end{cases}$$

$$g(n) = \sum_{d \mid n} f(d)$$

$$\Updownarrow$$

$$f(n) = \sum_{d \mid n} \mu(d)\, g(n/d)$$

$$\sum_{n\geq 1} n^{-s} = \left(\sum_{n \geq 1} \mu(n)\, n^{-s} \right)^{-1}$$



$$n^{-s} = p_1^{-s\alpha_1} \cdots p_k^{-s\alpha_k}$$

Euler identity

$$\zeta(s) = \prod_{p} \left(\frac{1}{1 - p^{-s}} \right)$$

p prime number

$$\zeta(s) = \prod_{p} \left(\frac{1}{1-p^{-s}} \right)$$

prime number

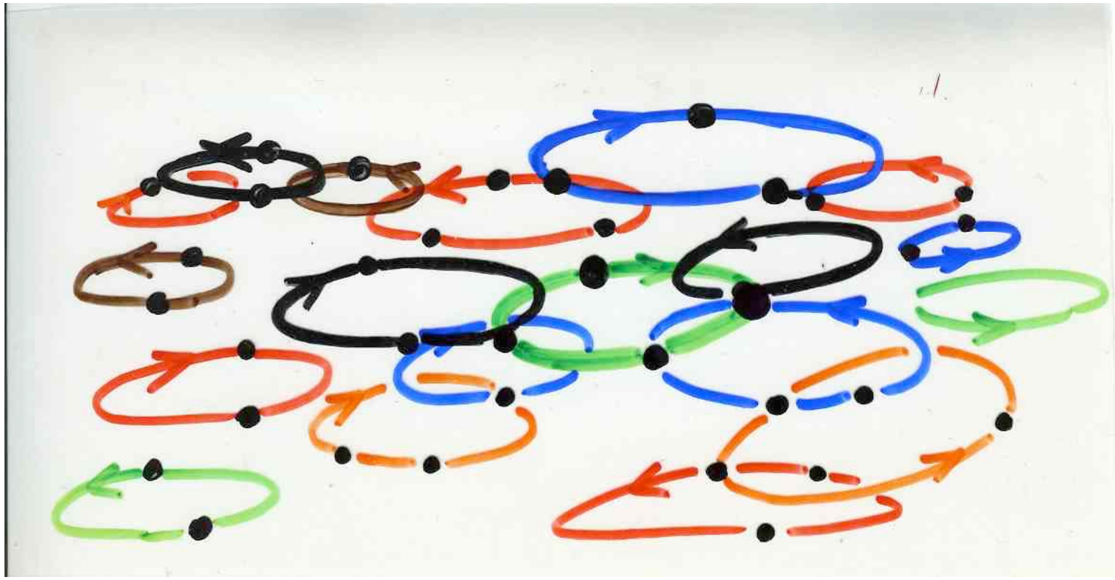
$$Z_G(t) = \prod_{[C]} \frac{1}{(1-t^{|C|})}$$

some "prime" over the graph G

Ihara-Selberg zeta function of a graph $Z_G(t)$

$$\zeta_G(t) = \prod_{[C]} \frac{1}{(1-t^{|C|})}$$

equivalence class
prime
circuit

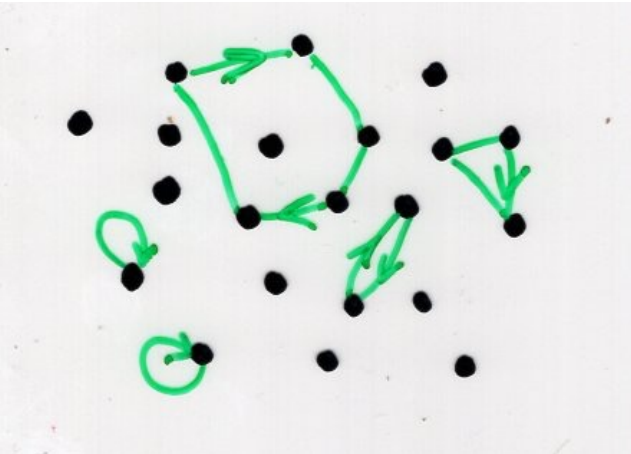


Giscard, Rochet (2016)
extending number theory
to paths on Graphs

$$\zeta_G(t) = \frac{1}{\det(I-A)}$$

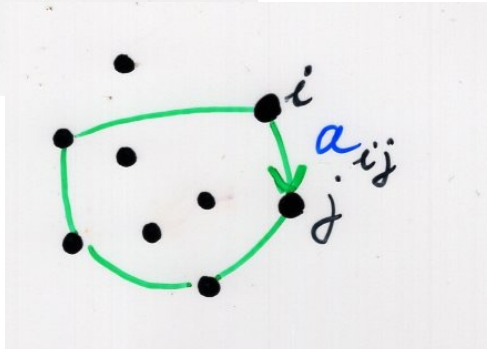
$$A = (a_{ij})_{1 \le i,j \le k}$$

$$\det(I - A) = \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \text{permutation}}} (-1)^{\mathrm{inv}(\sigma)} \, a_{1\sigma(1)} \cdots a_{k\sigma(k)}$$



$$\sum_{\substack{\gamma_1, \ldots, \gamma_r \\ \text{2 by 2 disjoint cycles}}} (-1)^r \, v(\gamma_1) \cdots v(\gamma_r)$$

$$X = [1, k]$$



heaps and linear algebra

<u>Lemma</u> $X = \{1, 2, \dots, k\}$

$A = (a_{i,j})$ $n \times n$ matrix

$$(I - A)^{-1}_{i,j} = \sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} v(\omega)$$

with $v(i,j) = a_{i,j}$

Bijection

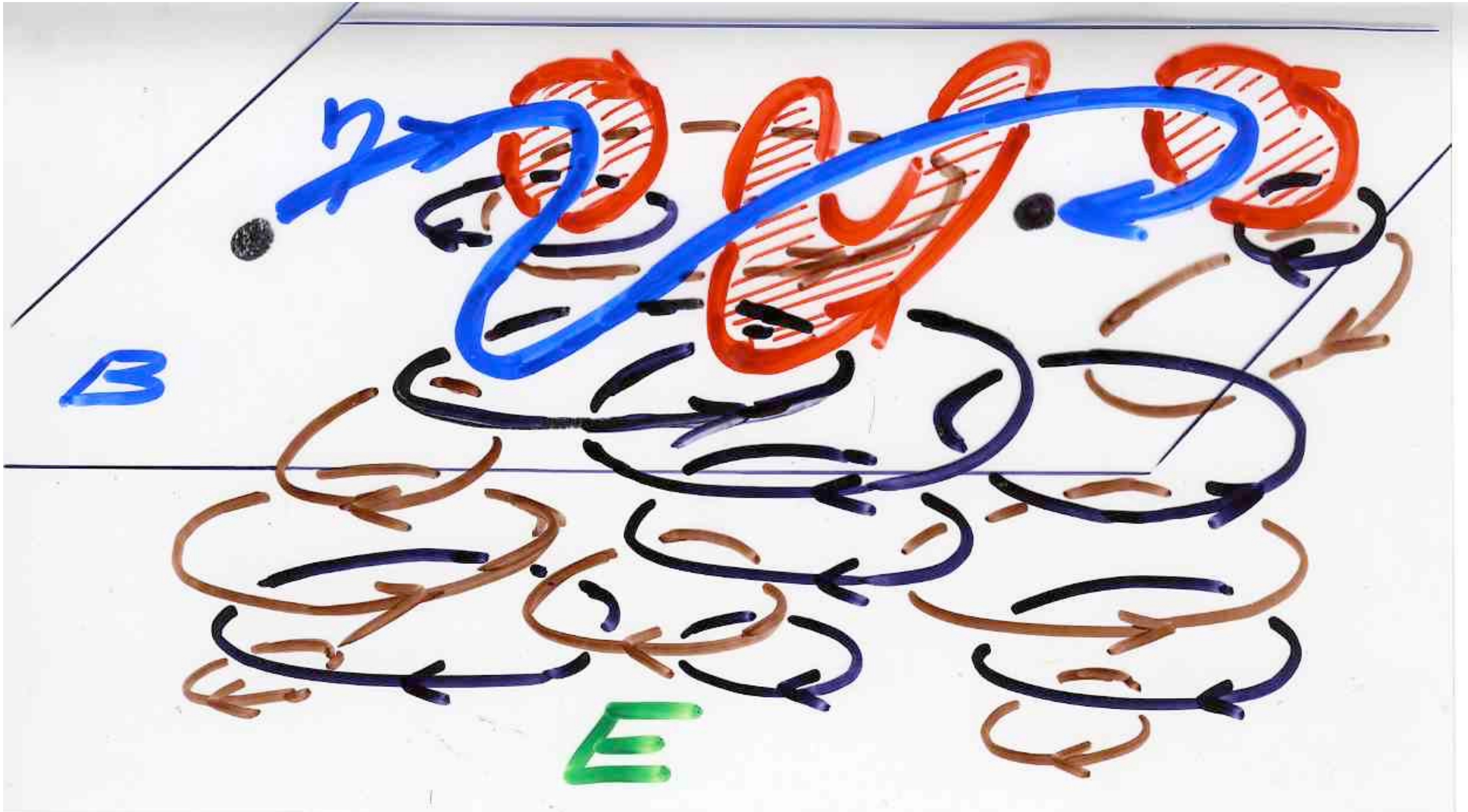
$u, v \in X$

path ω on X going from u to v $\overset{\chi}{\longleftrightarrow}$ (η, E)

- η self-avoiding path going from u to v
- E heap of cycles such that the projections $\alpha = \pi(m)$ of the maximal pieces intersect η ($\underset{\text{cycle}}{\alpha}$ and $\underset{\text{path}}{\eta}$ has a common vertex)

$$v(\omega) = v(\eta)\, v(E)$$

The bijection χ

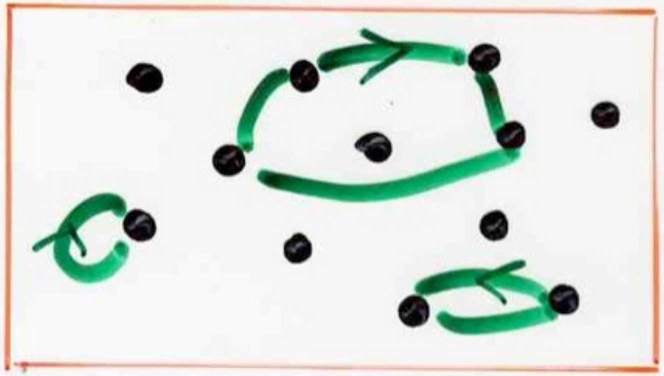


Prop. $$\sum_{\substack{\omega \\ i \rightsquigarrow j}} v(\omega) = \frac{N_{ij}}{D}$$

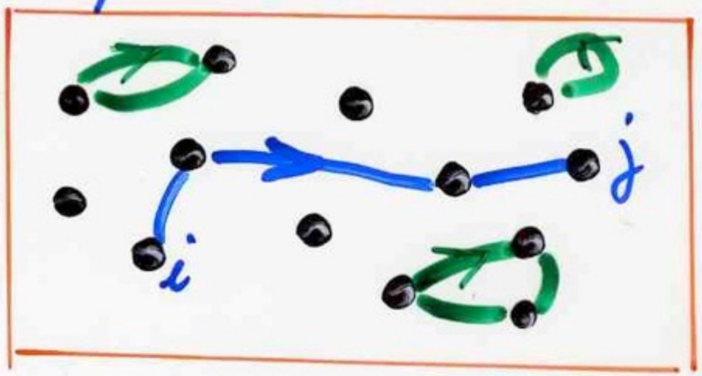
$$N_{ij} = \sum_{\eta} v(\eta) N_\eta$$

η self-avoiding path $i \rightsquigarrow j$

$$D = \sum_{\substack{\{\gamma_1, \dots, \gamma_r\} \\ \text{2 by 2 disjoint cycles}}} (-1)^r v(\gamma_1) \cdots v(\gamma_r)$$



$$N_{ij} = \sum_{\{\eta; \gamma_1, \dots, \gamma_r\}} (-1)^r v(\eta)\, v(\gamma_1) \cdots v(\gamma_r)$$

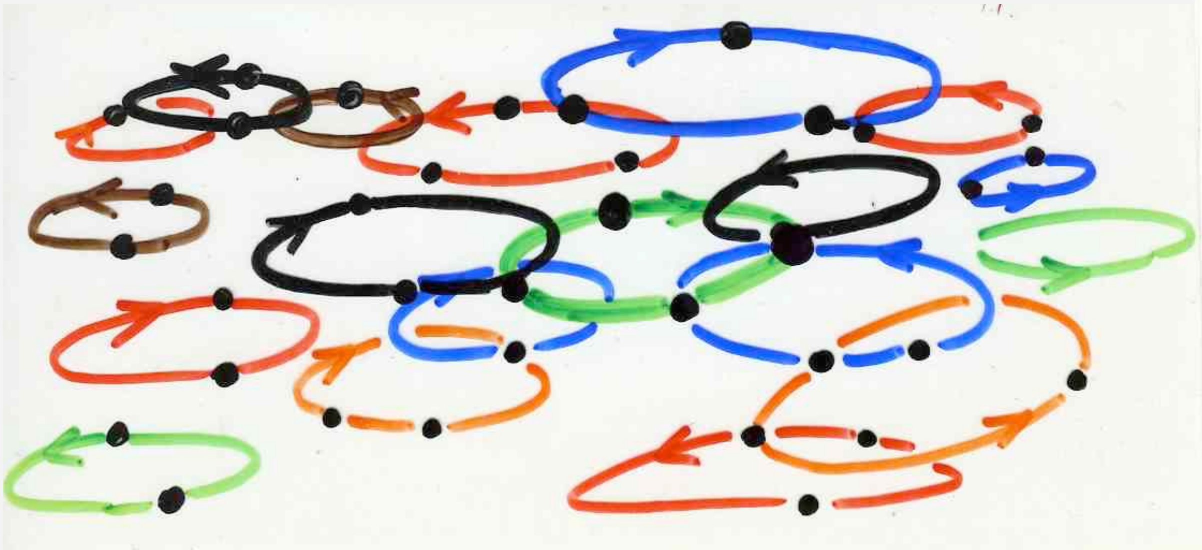


$$\log(\det(B)) = \mathrm{Tr}(\log(B))$$

$$B = (1 - A)^{-1}$$

$$\frac{1}{\det(I - A)} = \sum_{E} v(E)$$

E heap of cycles on $[1, k]$



heaps in

mathematics

theoretical physics

computer science

3 basic lemma

- generating functions for **heaps** $\dfrac{N}{D}$ "trivial" heaps
- log (heaps) = pyramids
- path = heap

Basic definitions and theorems

- commutation monoids and heaps of pieces : basic definitions
- generating functions for heaps
  - $\dfrac{1}{D}$, $\dfrac{N}{D}$, inversion lemma
  - logarithmic lemma
- Heaps and paths, flow monoid, rearrangements

path = heap

rearrangement = heap$_{\text{cycles}}$ of

Some applications to classical mathematics

- heaps and linear algebra:
  bijective proofs of classical theorems
- heaps and combinatorial theory of
  orthogonal polynomials and continued fraction
- heaps and algebraic graph theory

Some applications in theoretical physics

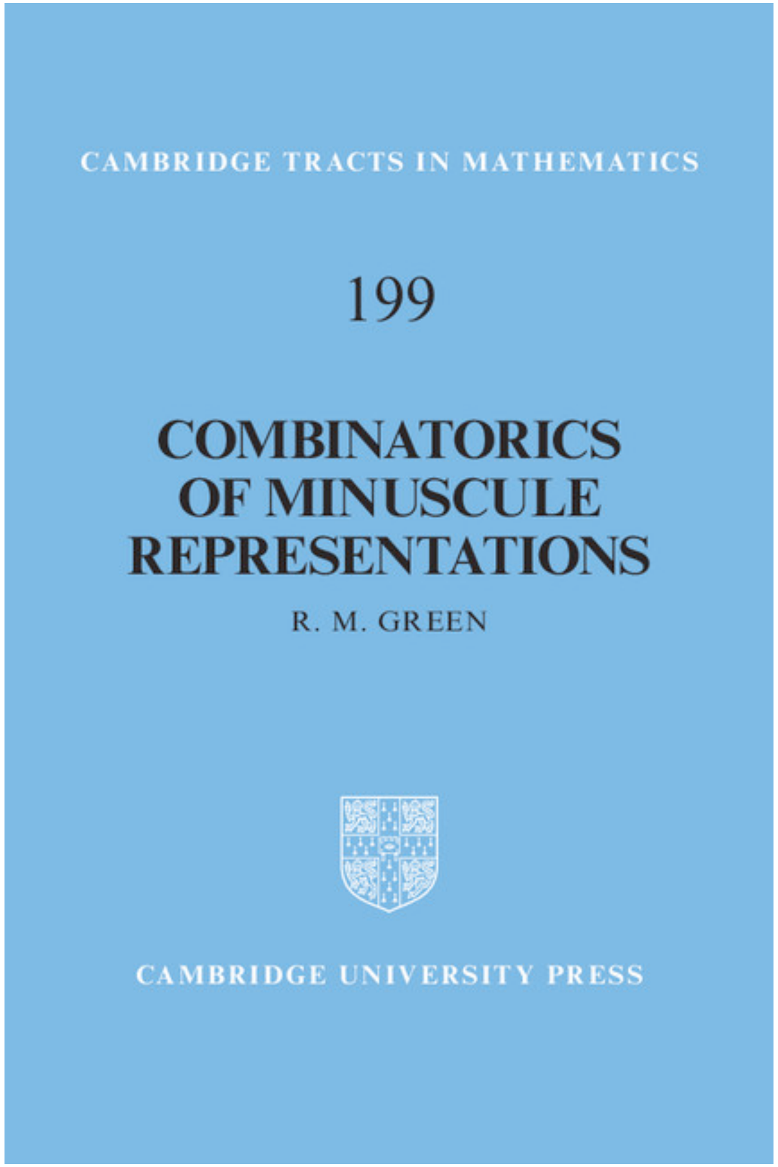
- directed animals and gas model in statistical physics
- Lorentzian triangulations in 2D quantum gravity
- q-Bessel functions in physics: polyominoes and SOS model

Applications to more advanced mathematics

- fully commutative class of words
  in Coxeter groups

  → representation theory of Lie algebras
  with operators on heaps

  Temperley-Lieb algebra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COMBINATORICS OF MINUSCULE REPRESENTATIONS
R. M. GREEN
CAMBRIDGE UNIVERSITY PRESS

R. Green (2013)

Complementary topics

- zeta function on graph and number theory (Giscard, Rochet)
- minuscule representations of Lie algebra (R. Green and students) book
- basis of free partially commutative Lie algebra (Lalonde, Duchamp-Krob, ...)
    - computer science: the SAT problem revisited with heaps (D. Knuth, vol 4, Fascicle 6)
    - computer science: Petri nets, asynchronous automata, Zielonka theorem
- statistical physics: (T. Helmuth) Ising model revisited
- string theory and heaps gauge theory, quivers (Ramgoolam)

Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics,

a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

www.xavierviennot.org/coursIMSc2017

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Thank you !

ॐ सरस्वत्यै नमः।