Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics,
a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc
January-March 2017

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Chapter 7

Heaps in statistical mechanics (3)

(slides: first part)

q-Bessel functions in physics

IMSc, Chennai
16 March 2017

Bessel functions

Bessel functions

$$J_\alpha(x) = \sum_m \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

$$\Gamma(m) = (m-1)!$$

canonical solutions

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha)y = 0$$

modified Bessel functions

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix)$$

q-analog

$$n! \rightarrow 1(1+q)\cdots(1+q+\cdots+q^{n-1})$$

$$\frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}$$

$$J_0 = \sum_{n\geq 0} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n (yq)_n}$$

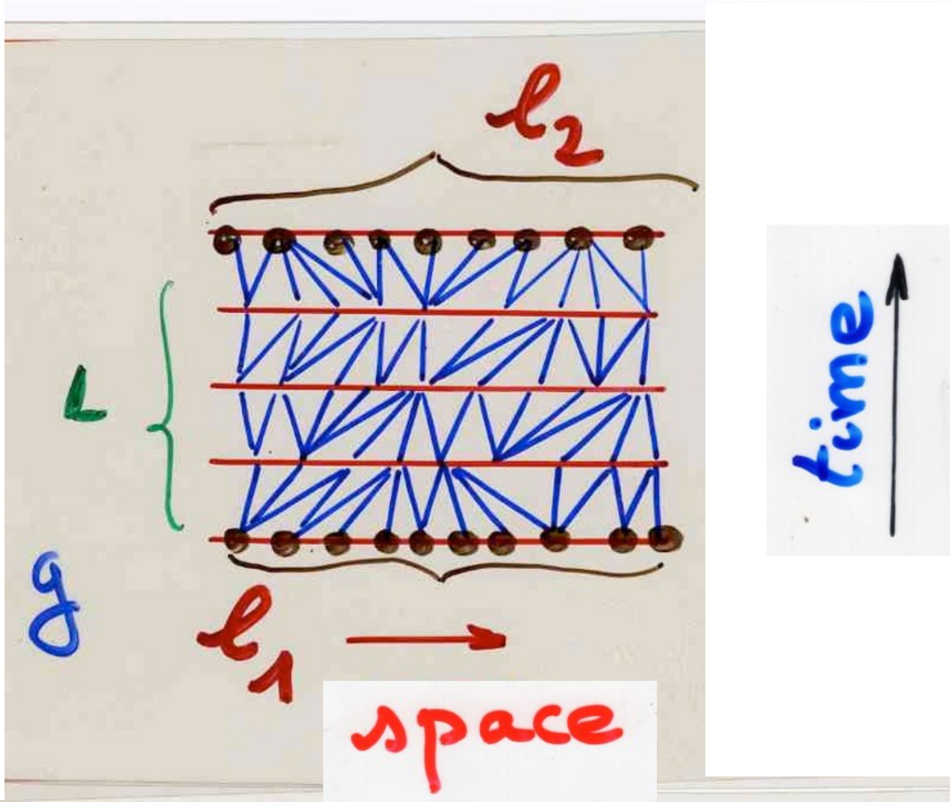
$$J_1 = \sum_{n\geq 1} \frac{(-1)^{n-1} x^n q^{\binom{n+1}{2}}}{(q)_{n-1} (yq)_n}$$

notation $\quad (a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$

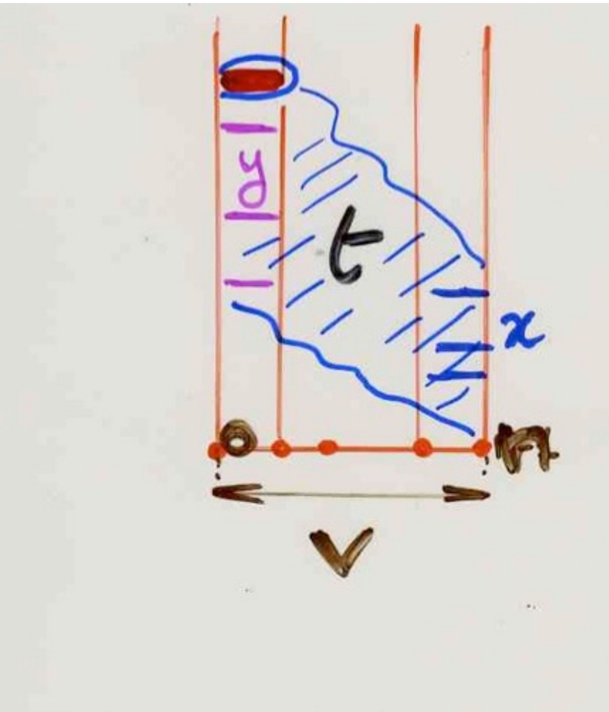
from the previous lecture

Lorentzian triangulations

in 2D qantum gravity



Path integral amplitude for the propagation from geometry ℓ_1 to ℓ_2

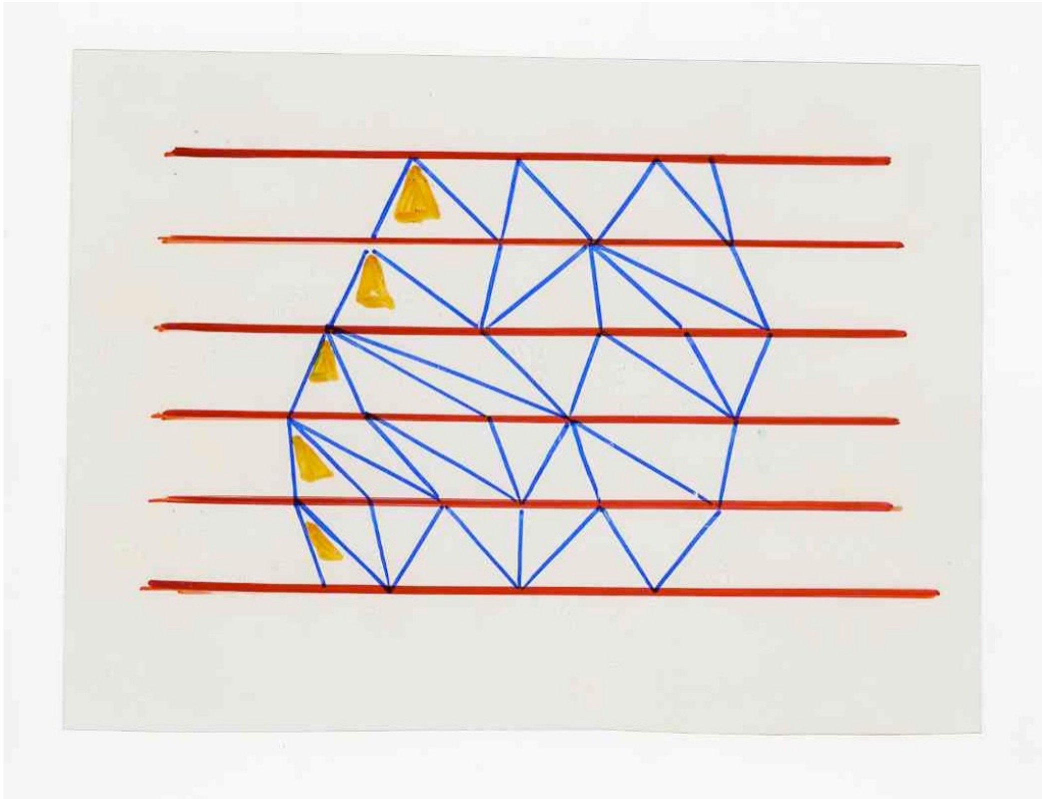


generating function
for pyramids of
dimers with 4
parameters

- t, v, y
- x number of dimers in the last column

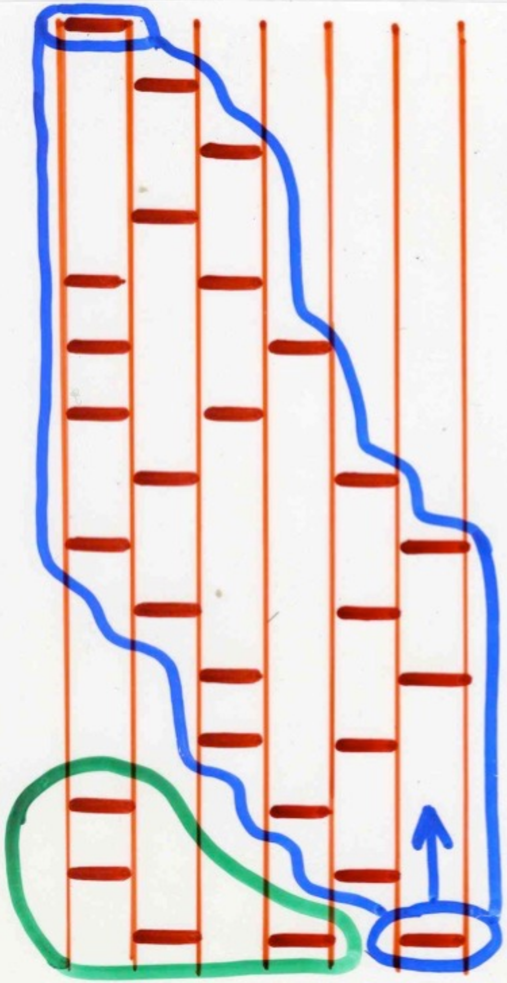
Catalan number !

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

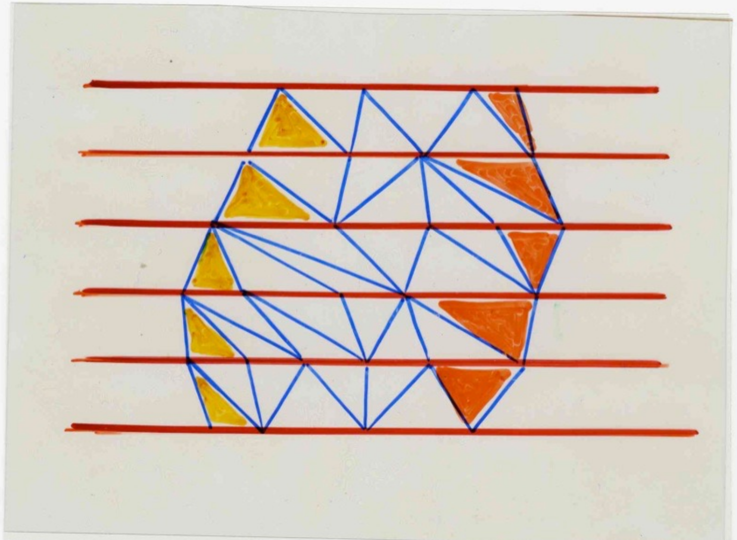


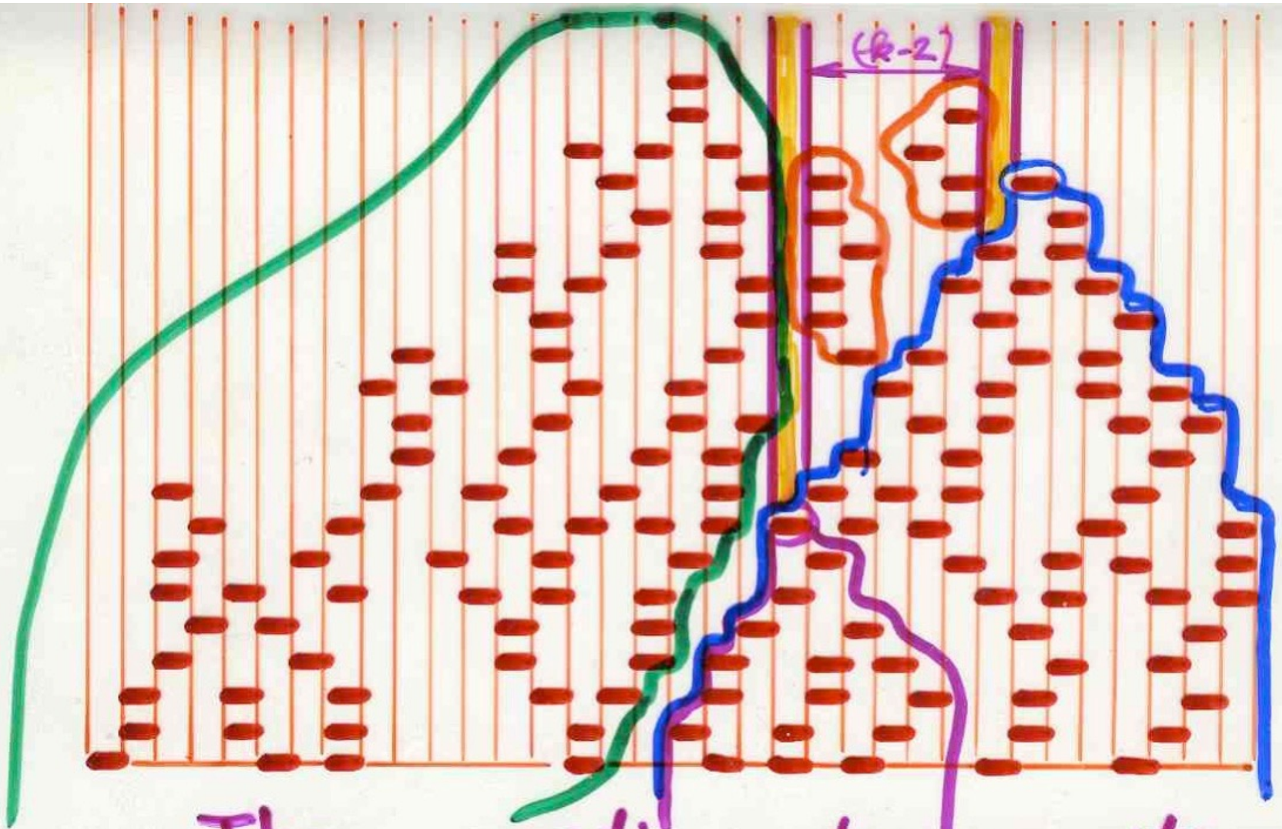
Proposition

generating function for pyramids of dimers with 4 parameters
- t, v, y
- x number of dimers in the last column



$$\frac{y\,t^n v^n}{\tilde{F}_n(t,y,1)\,\tilde{F}_{n+1}(t,y,x)}$$





The nordic decomposition

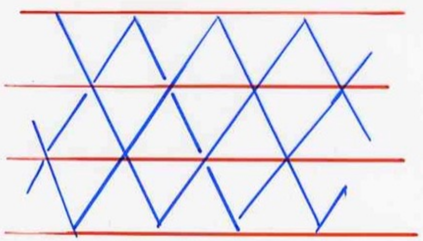
$$C = \frac{Q}{1-Q} + C\sum_{k\geq 1} \frac{Q}{1-Q}\times Q^{k} \times \frac{1}{F_{k-1}}$$

$$F_n = \frac{(1-Q^{n+1})}{(1-Q)(1+Q)^n}$$

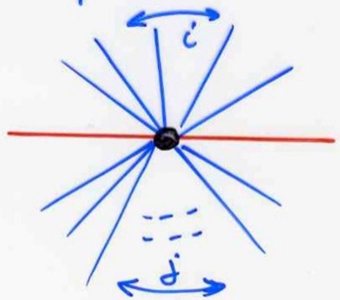
$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n}\times(1+Q+\dots+Q^n)$$

curvature

of the space-time



flat



$$a^{|i-3|+|j-3|}$$

$$\text{total curvature} = \prod_{\text{all points}} a^{(\cdots)}$$

continuum limit

I_1 modified Bessel function

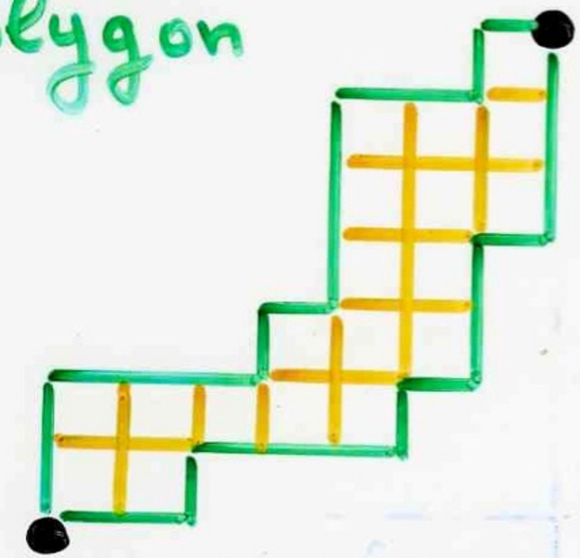
$$G_\Lambda(L_1, L_2; T) = \frac{e^{-(\coth\sqrt{\Lambda}T)\sqrt{\Lambda}(L_1+L_2)}}{\operatorname{sh}\sqrt{\Lambda}\,T} \; \frac{\sqrt{\Lambda L_1 L_2}}{L_2} \, I_1\!\left(\frac{2\sqrt{\Lambda L_1 L_2}}{\operatorname{sh}\sqrt{\Lambda}\,T}\right)$$

Parallelogram polyominoes
(staircase polygons)

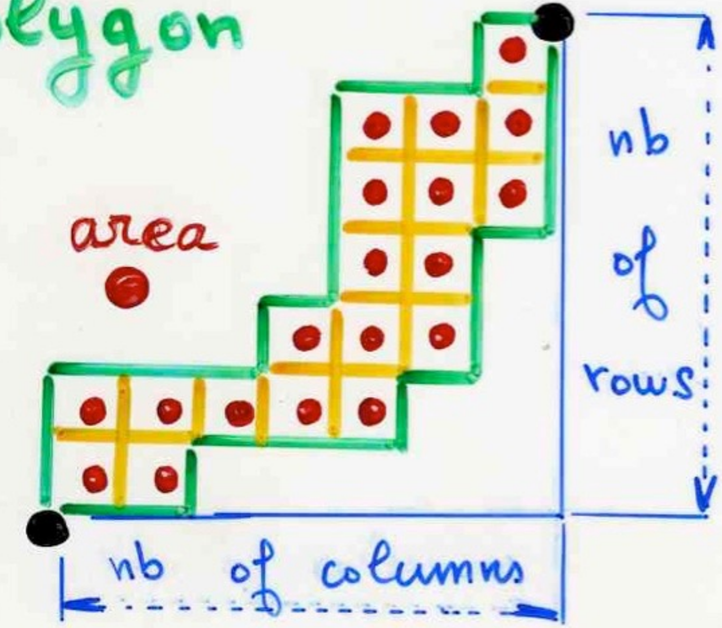
and q-Bessel functions

M.Bousquet-Mélou, X.V. (1992)

staircase polygon



staircase polygon



$a_{m,n,p}$

generating function

$$f(x,y;q) = \sum_{m,n,p} a_{m,n,p}\, x^m y^n q^p$$

$$= \sum_{P} x^{c(P)}\, y^{r(P)}\, q^{\alpha(P)}$$

P: staircase polygons

$c(P)$: nb of columns

$r(P)$: nb of rows

$\alpha(P)$: area

parallelogram polyominoes

$$\begin{cases} x & \text{length} \quad (\text{nb of columns}) \\ y & \text{height} \quad (\text{" rows "}) \\ q & \text{area} \end{cases}$$

Klarner, Rivest (1974)
Bender

Delest, Fedou (1989)
Brak, Guttmann (1990)
Bousquet-Mélou, X.V. (1990)

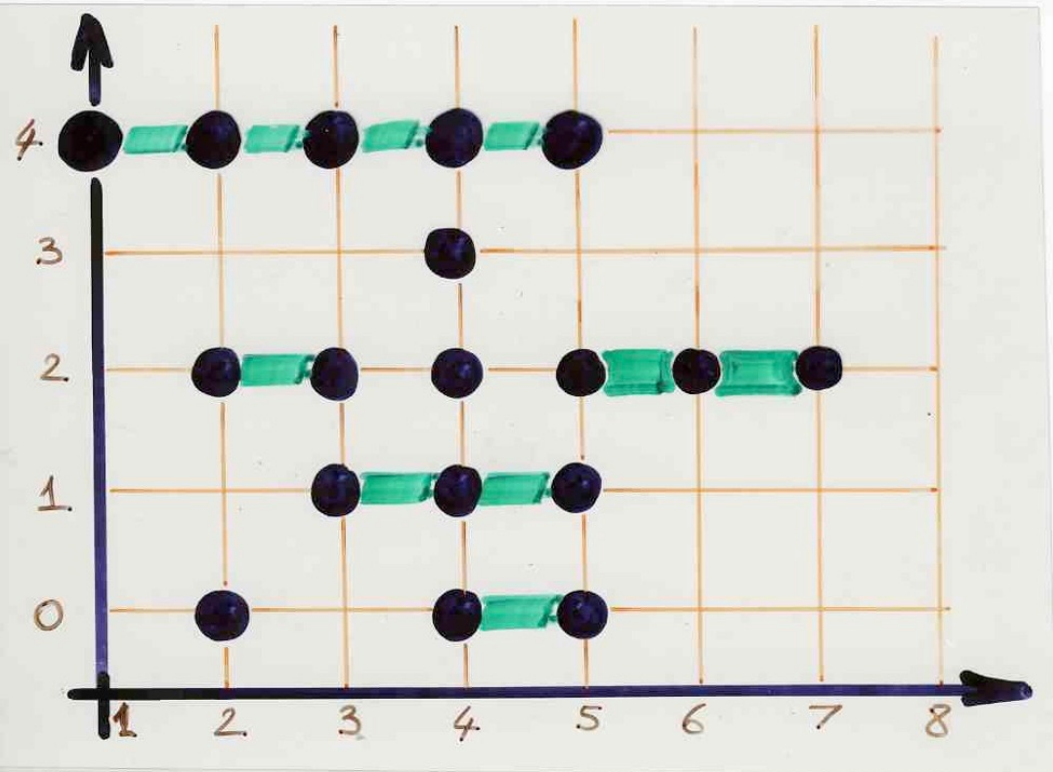
$$y \frac{J_1(x,y,q)}{J_0(x,y,q)}$$

$$J_0 = \sum_{n \geq 0} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n (yq)_n}$$

$$J_1 = \sum_{n \geq 1} \frac{(-1)^{n-1} x^n q^{\binom{n+1}{2}}}{(q)_{n-1} (yq)_n}$$

notation $\quad (a)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$

bijetion parallelogram polyominoes
semi-pyramids of segments



poids $q^{j} t u^{(j-i)}$

bijection

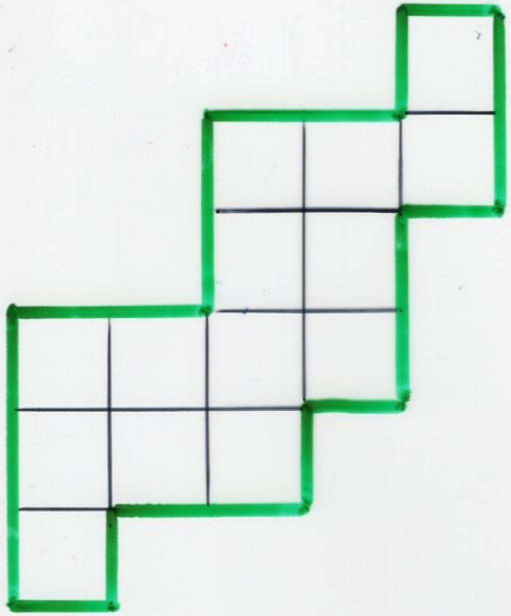
- pyramids of segments E on $\mathbb{N}^+$

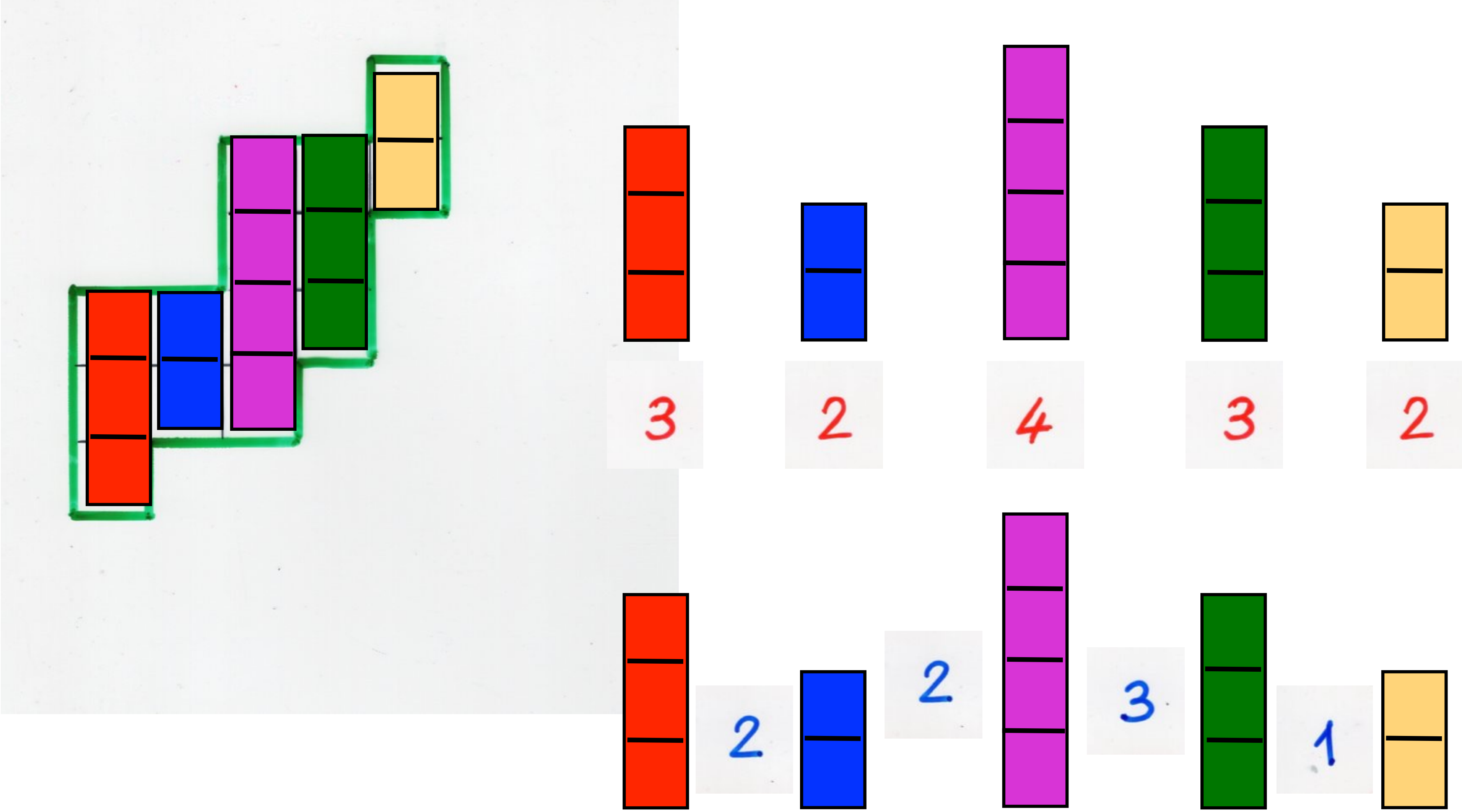
$$\pi\left(\begin{matrix}\text{unique}\\ \text{maximal}\\ \text{piece}\end{matrix}\right) = [1, k] \quad , k \geq 0$$

- parallelogram polyominoes Λ

$$\underbrace{q}_{\text{area}}{}^{a(\Lambda)} \; \underbrace{t}_{\text{number of columns}}{}^{c(\Lambda)} \; \underbrace{u}_{\text{number of rows}}{}^{r(\Lambda)-1} = v(E)$$

$$v(\underbrace{[i;j]}_{\text{segment}}) = q^{j} t u^{(j-i)}$$

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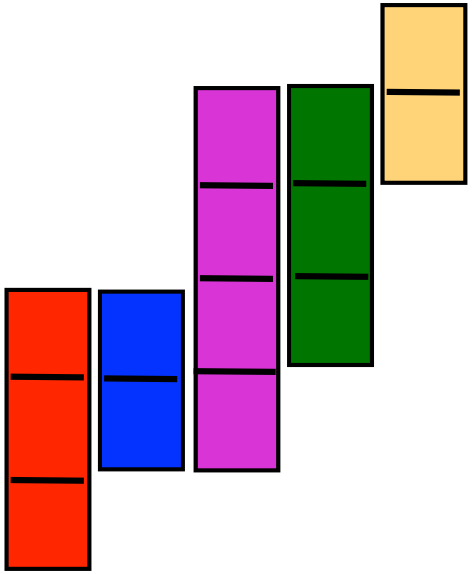

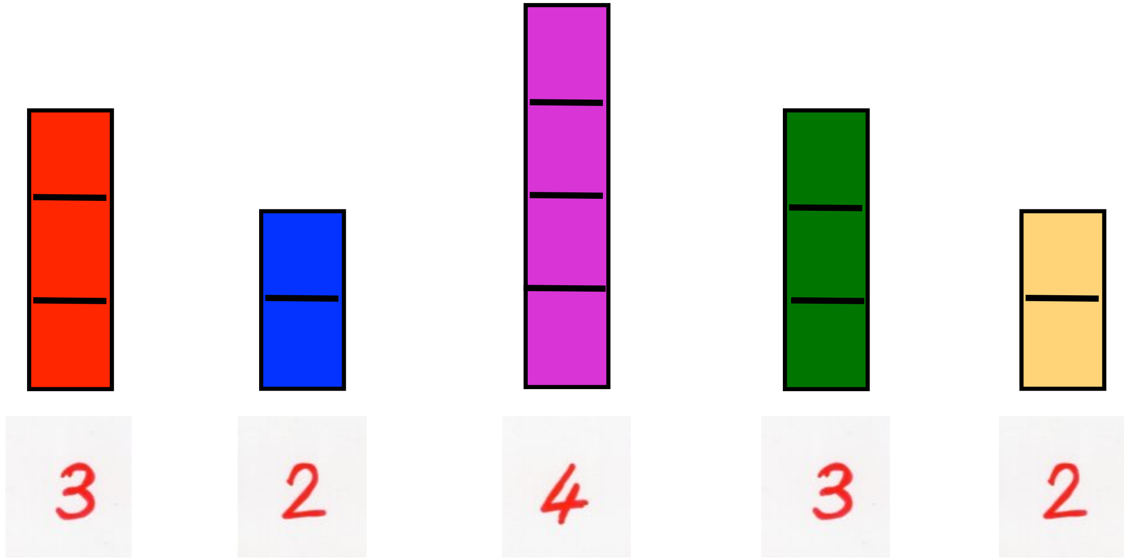
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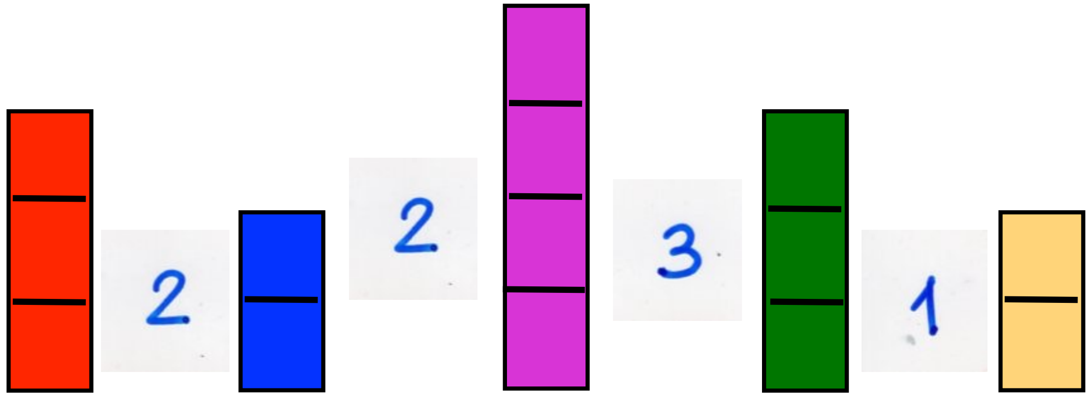
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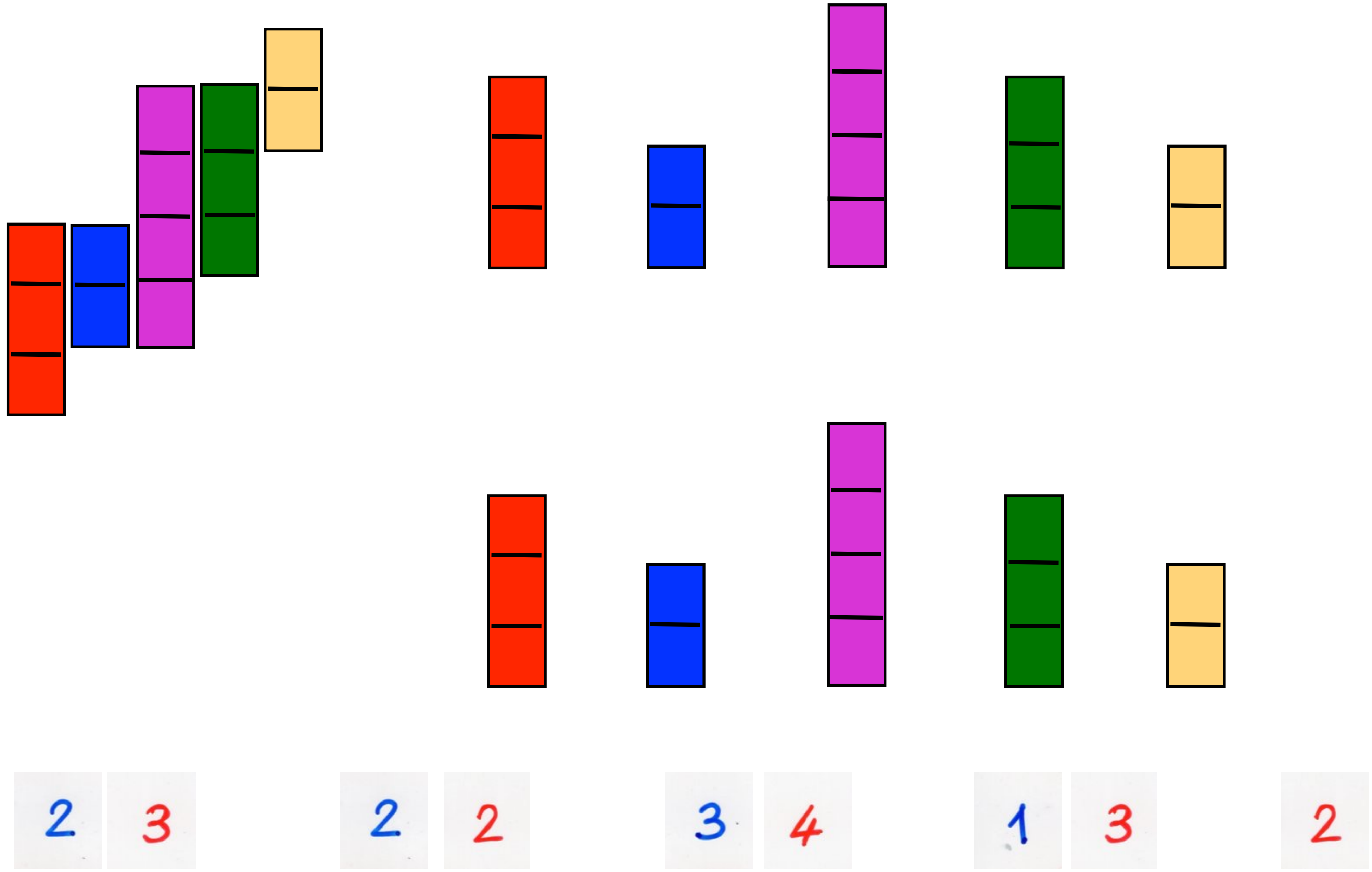
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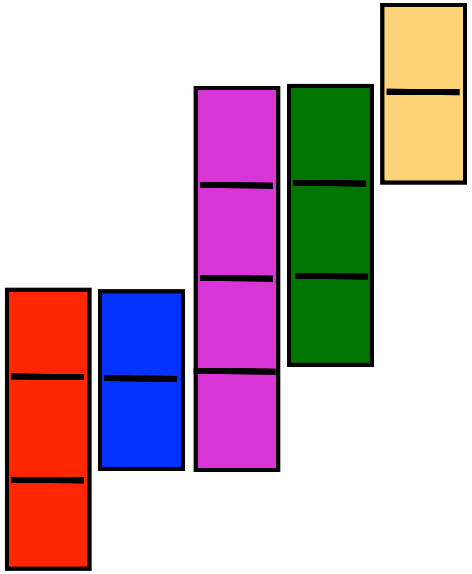

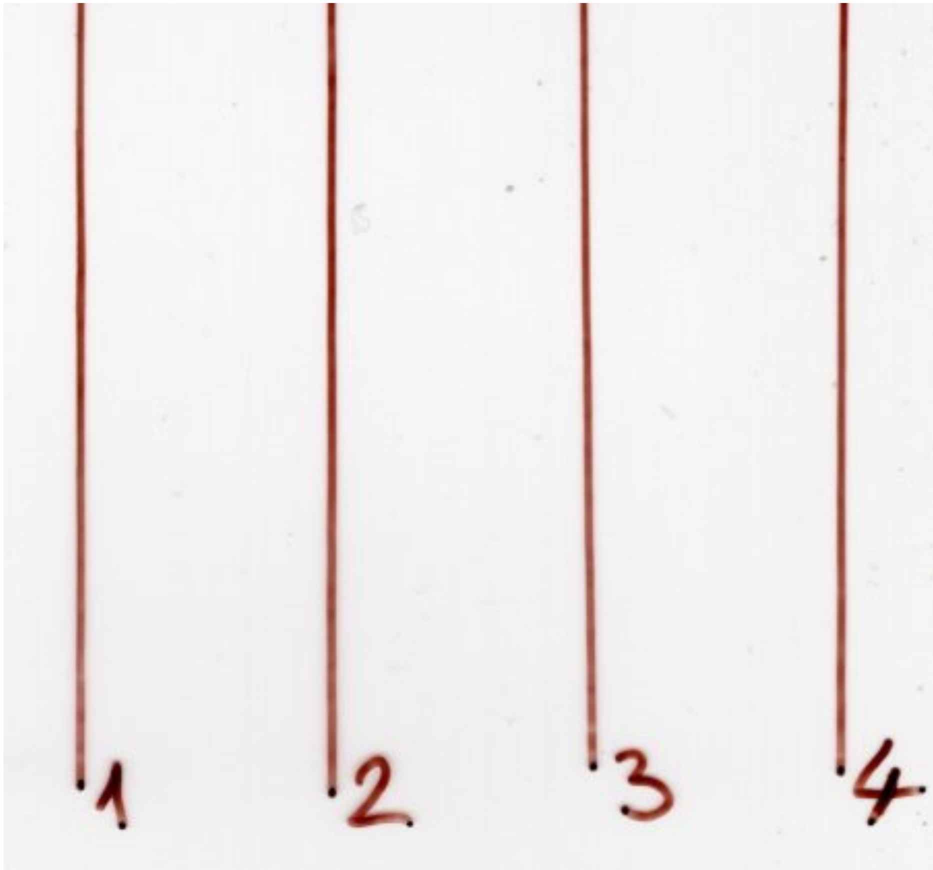
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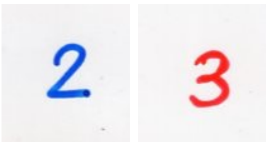
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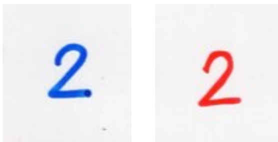
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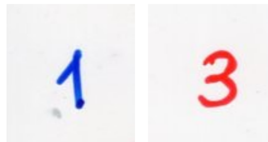
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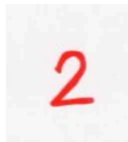
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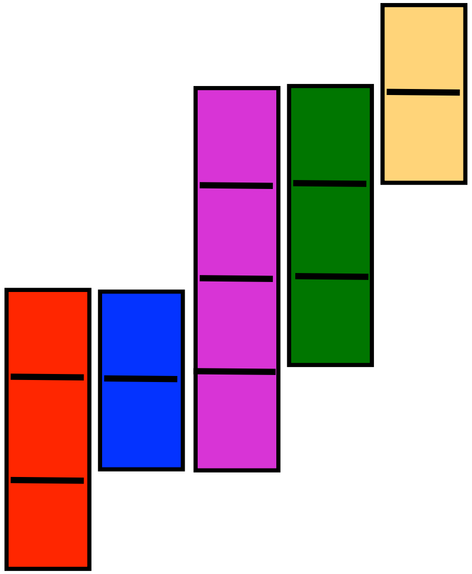

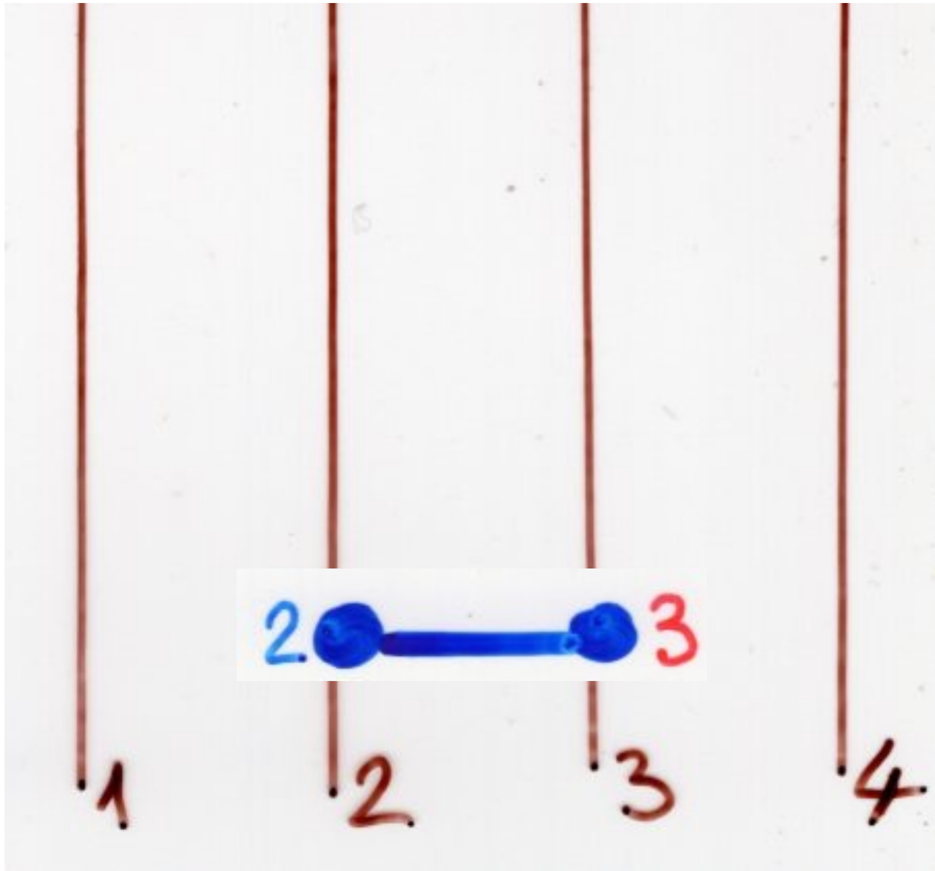
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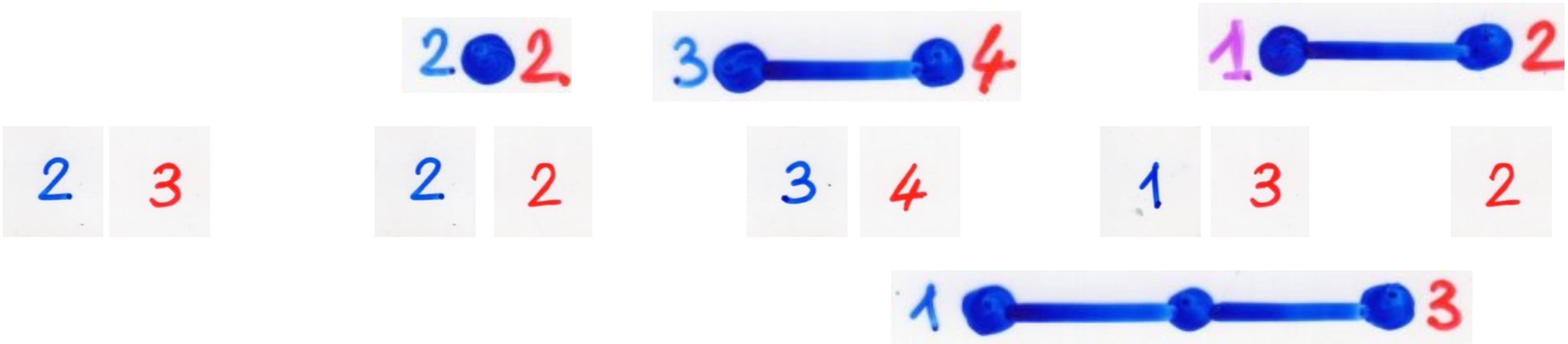
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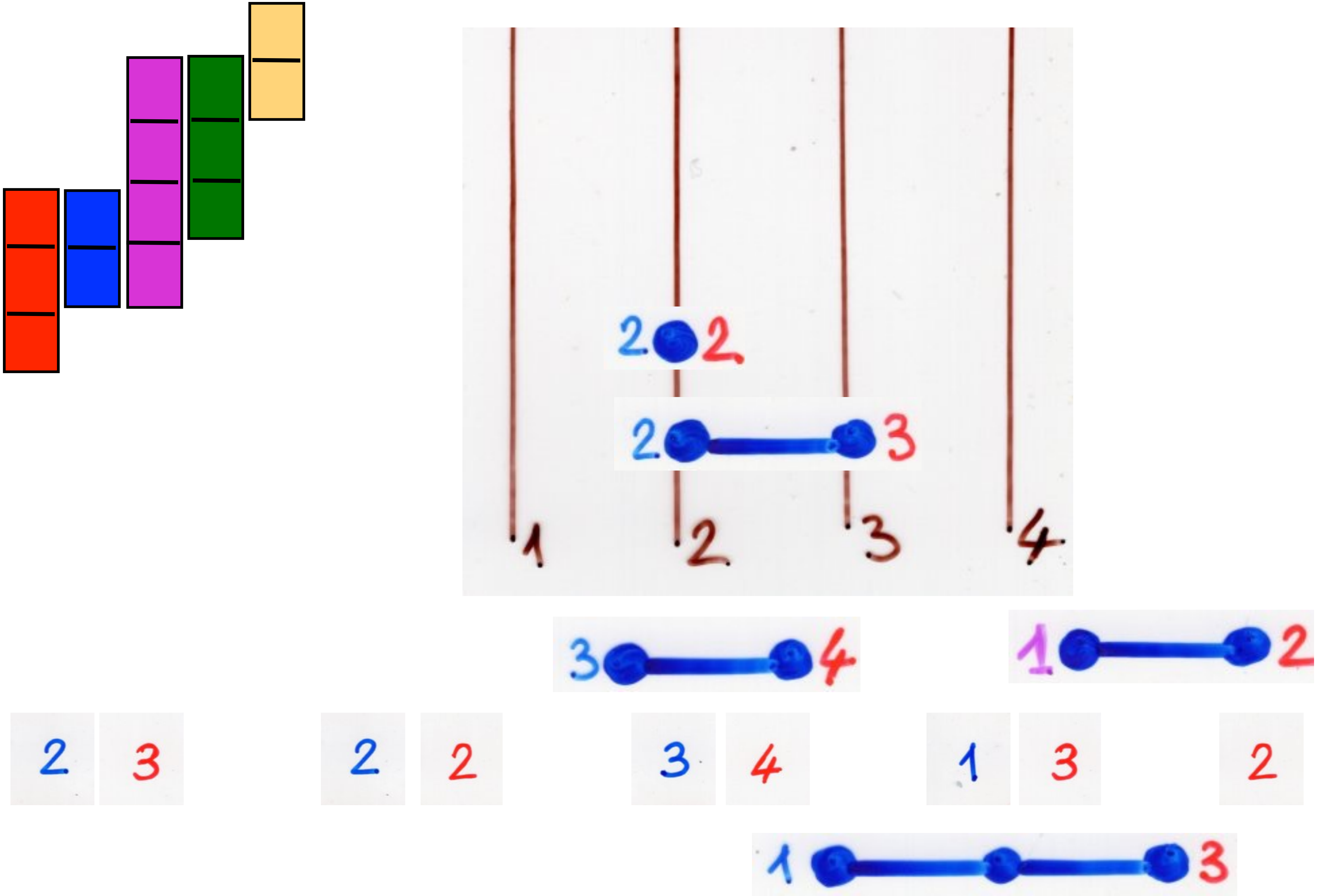

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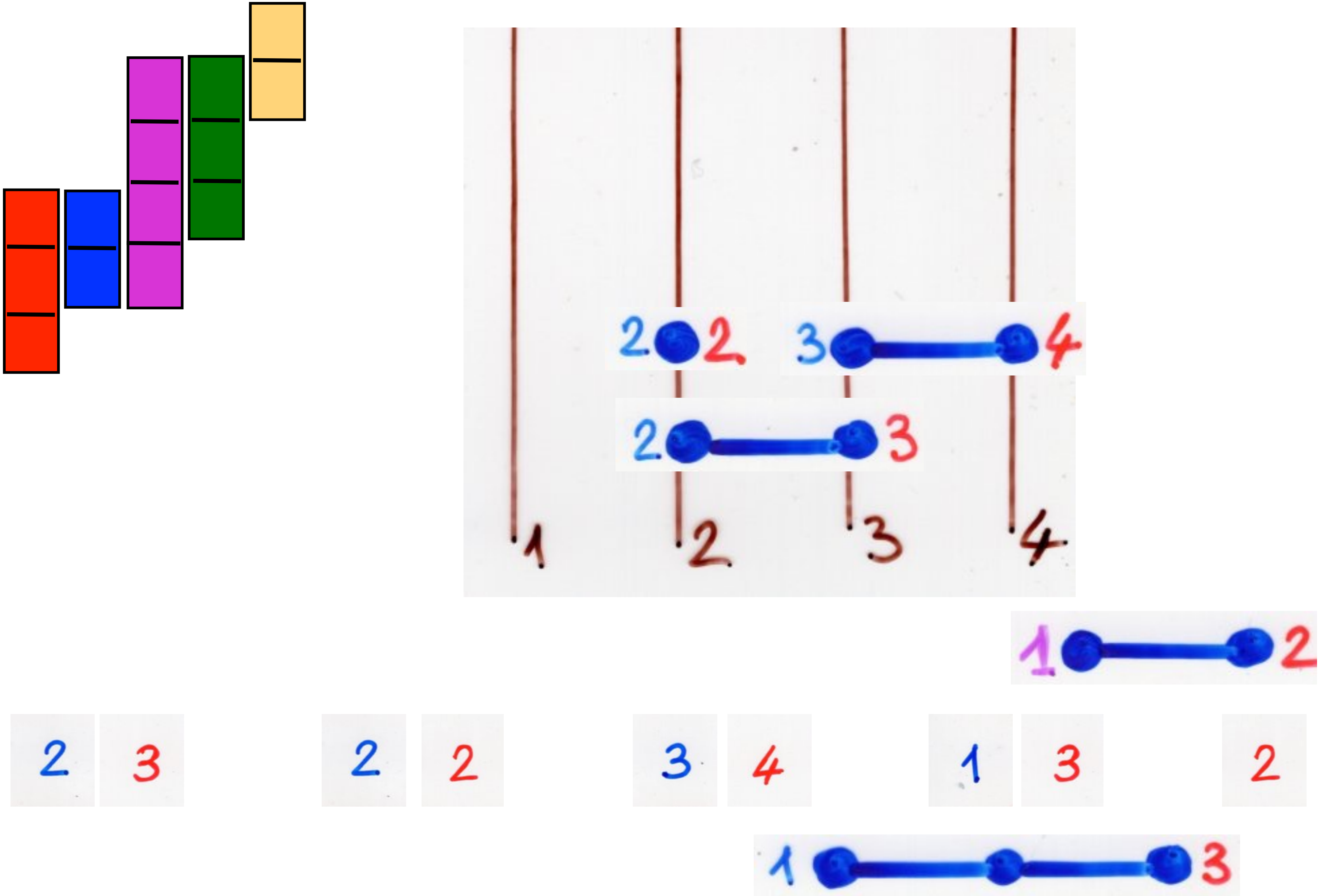

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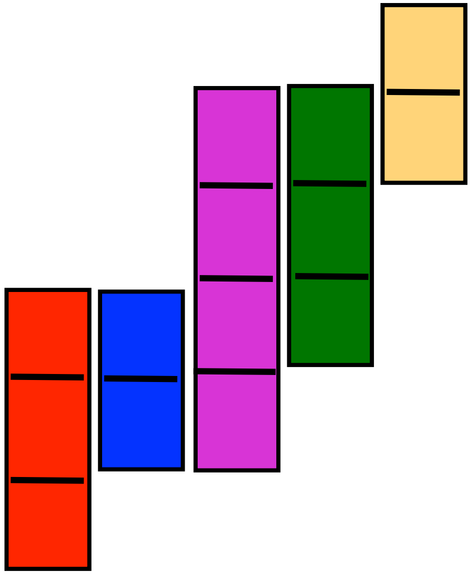

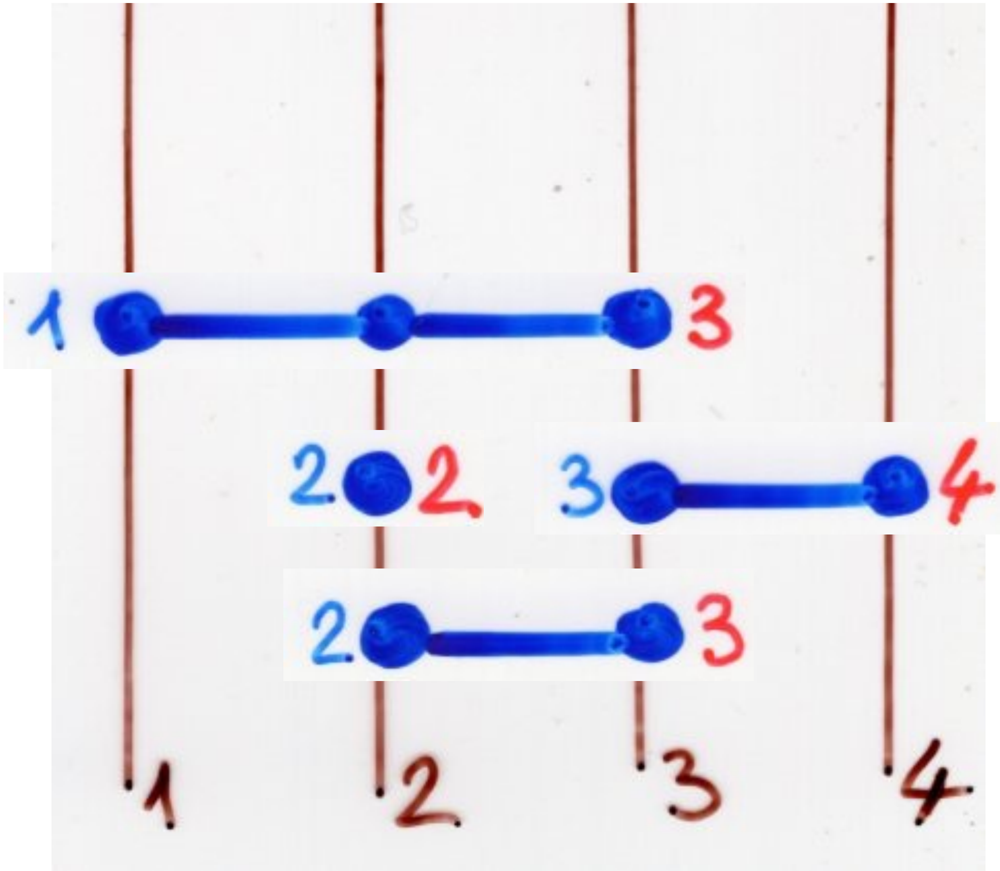
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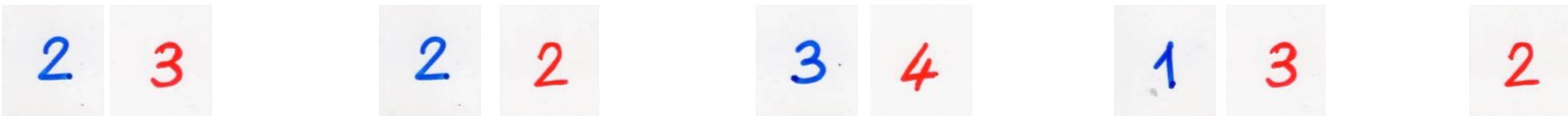
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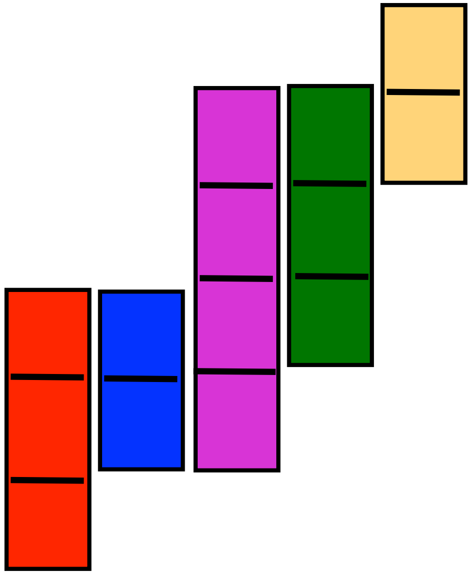

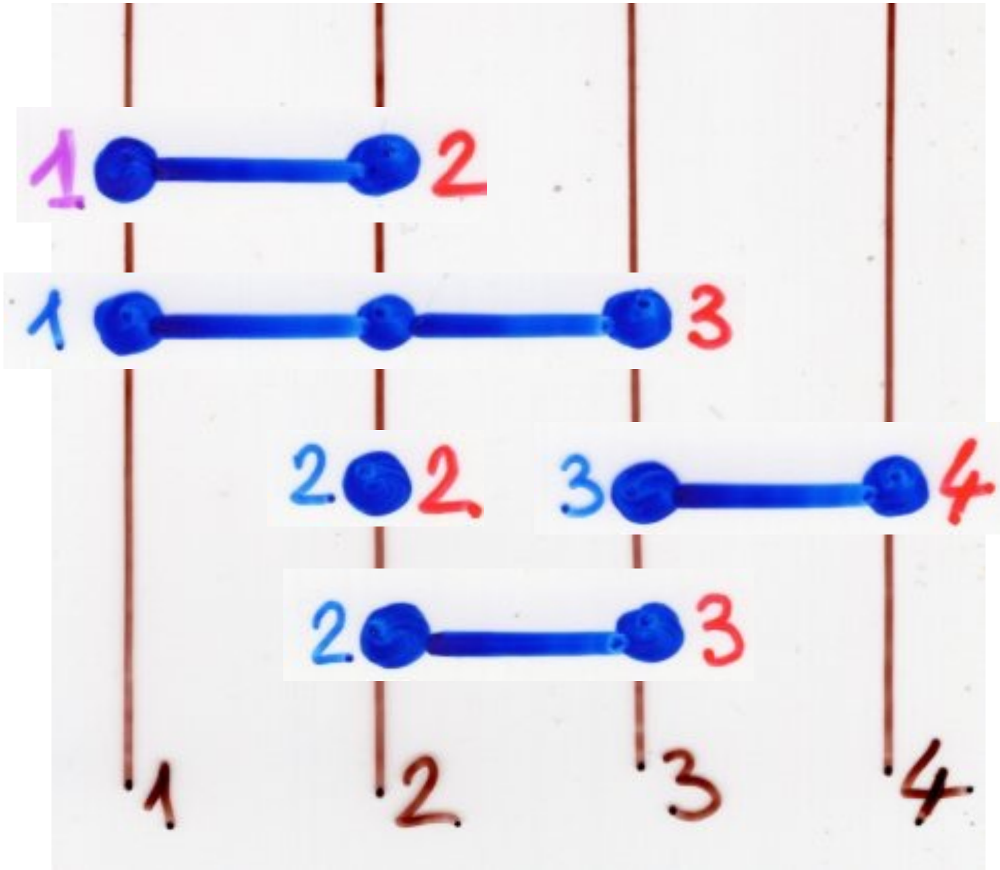
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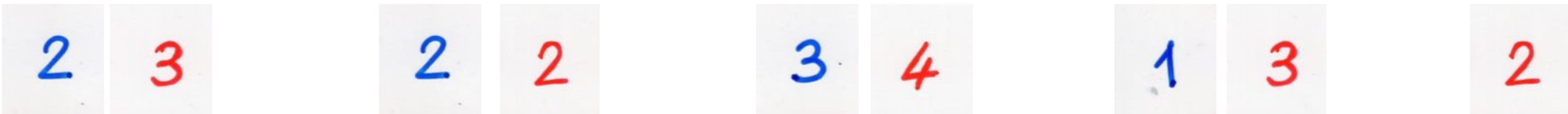
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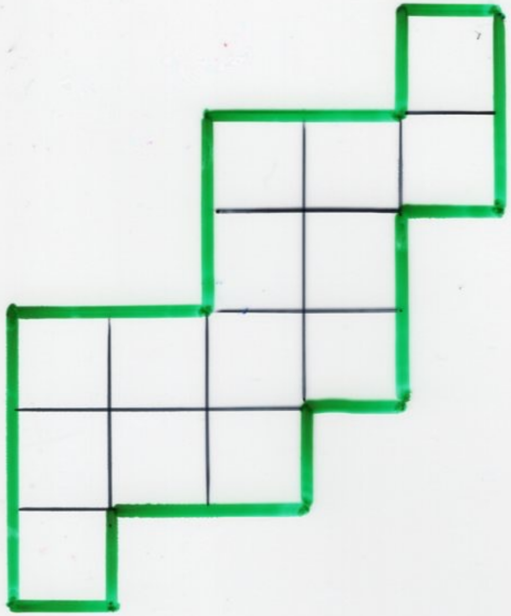

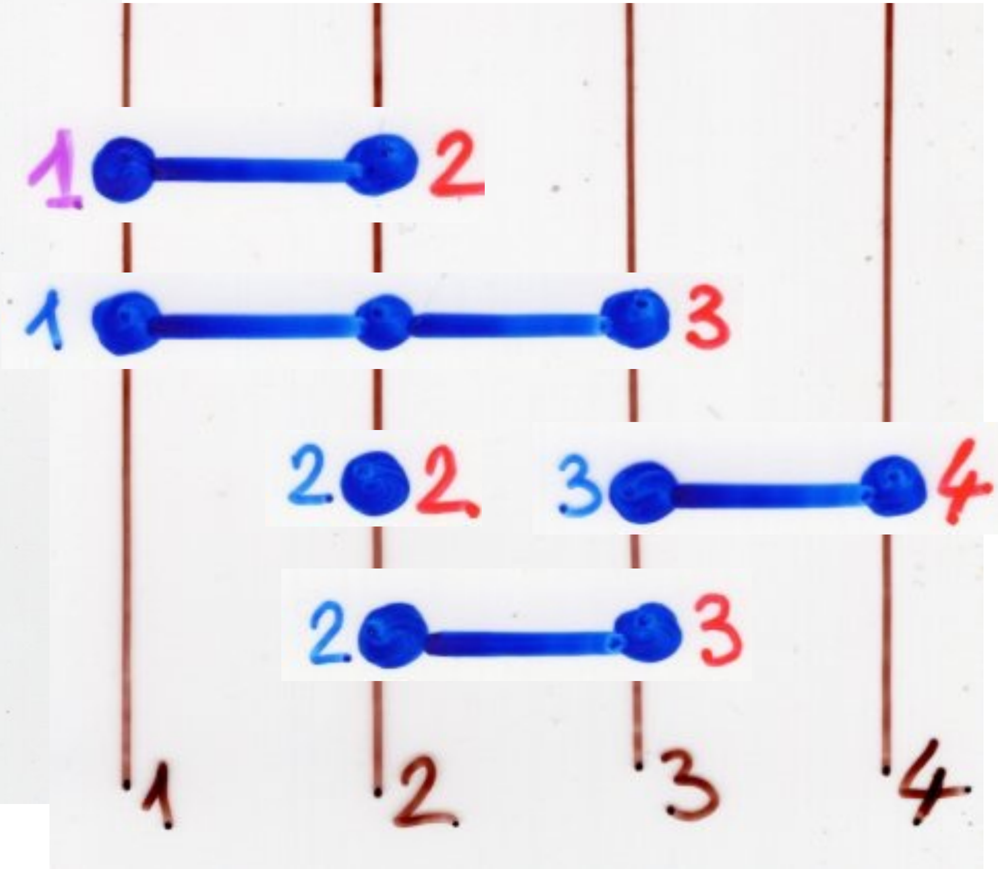

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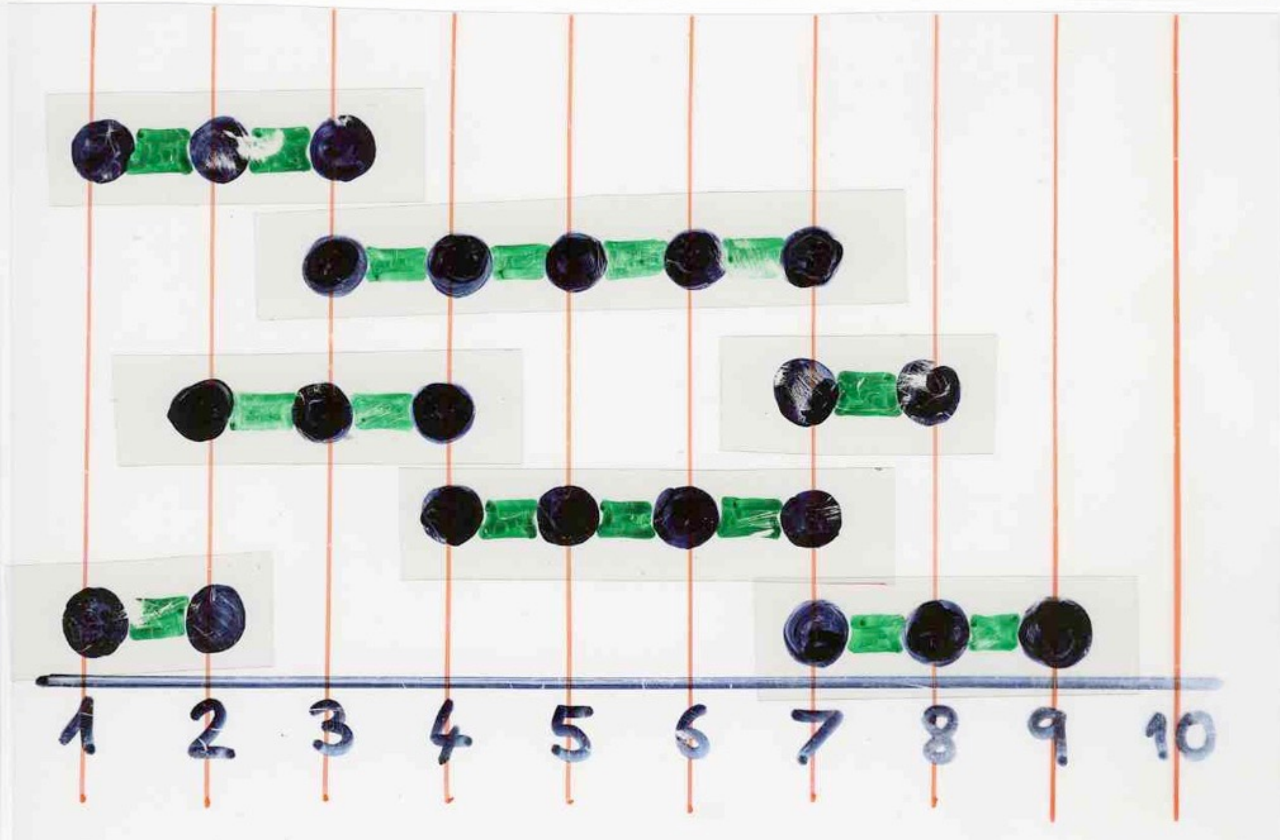
generating function

$$f(t,u;q) = \frac{N}{D}$$



extension of the inversion lemma

$M \subseteq P$

$$\sum_{\substack{E \\ \Pi(\text{maximal pieces}) \in M}} v(E) = \frac{N}{D}$$

$$D = \sum_{\substack{F \\ \text{trivial heaps}}} (-1)^{|F|} v(F)$$

$$N = \sum_{\substack{F \\ \text{trivial heaps} \\ \text{pieces} \notin M}} (-1)^{|F|} v(F)$$

Segments $\quad v([i,j]) = q^{j}\, t\, u^{(j-i)}$

$$D = \sum_{n\geq 0} \frac{(-1)^n t^n q^n q^{\binom{n}{2}}}{(1-q)\cdots(1-q^n)(1-uq)\cdots(1-uq^n)}$$

$$D = \sum_{\mathcal{C}} (-1)^{|\mathcal{C}|}\, v(\mathcal{C})$$

(q-Bessel)

$\mathcal{C}$ configuration of 2 by 2 disjoint segments

$$v(\mathcal{C}) = \prod v(\text{each segment})$$

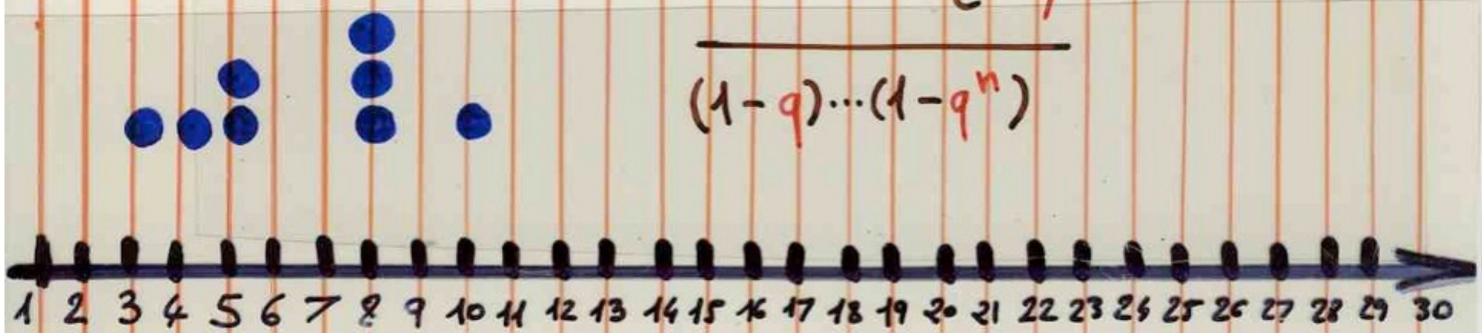
from integers partitions

to q-Bessel functions

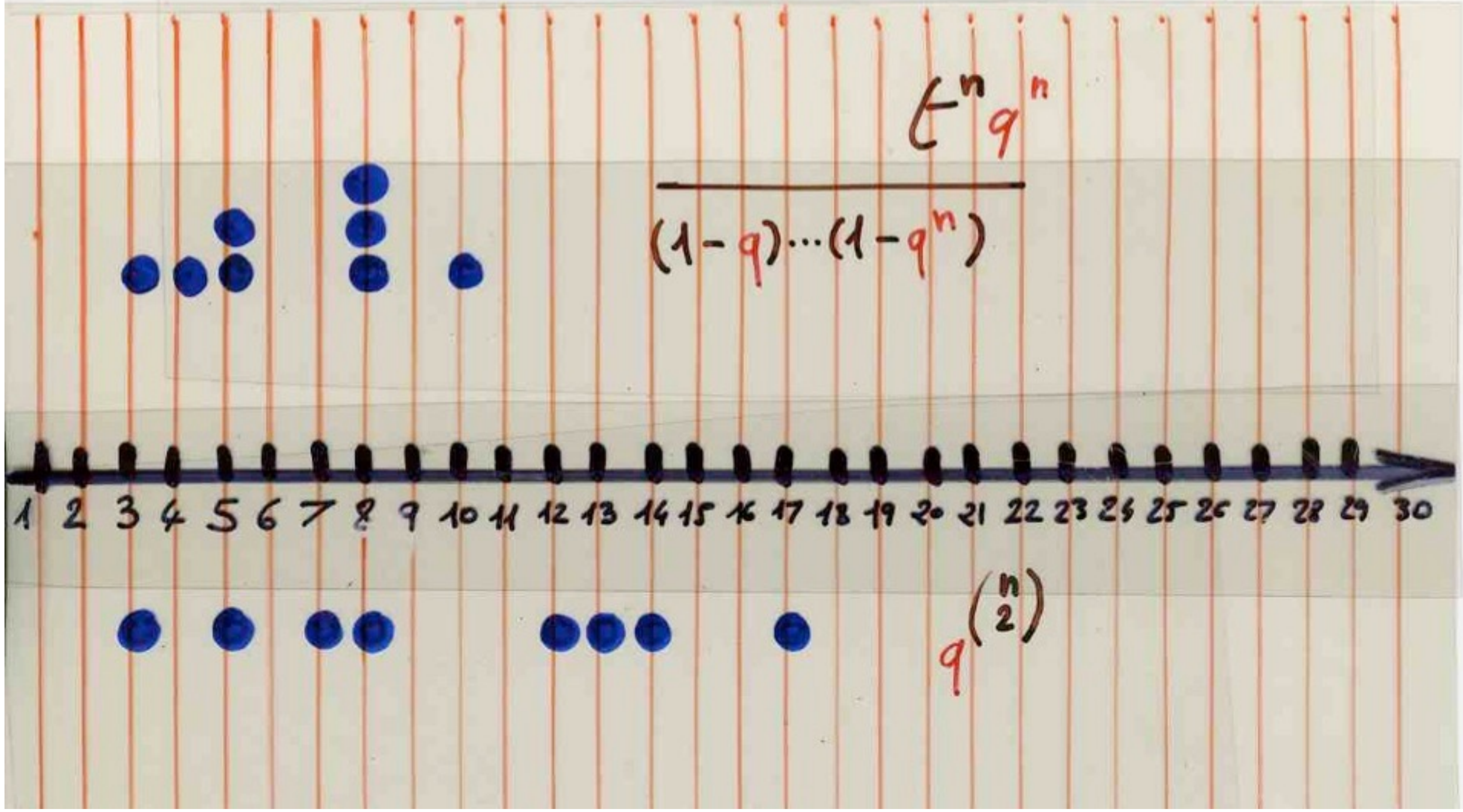
$$D = \sum_{n \geq 0} \frac{(-1)^n \, t^n q^n \, q^{\binom{n}{2}}}{(1-q)\cdots(1-q^n)(1-uq)\cdots(1-uq^n)}$$

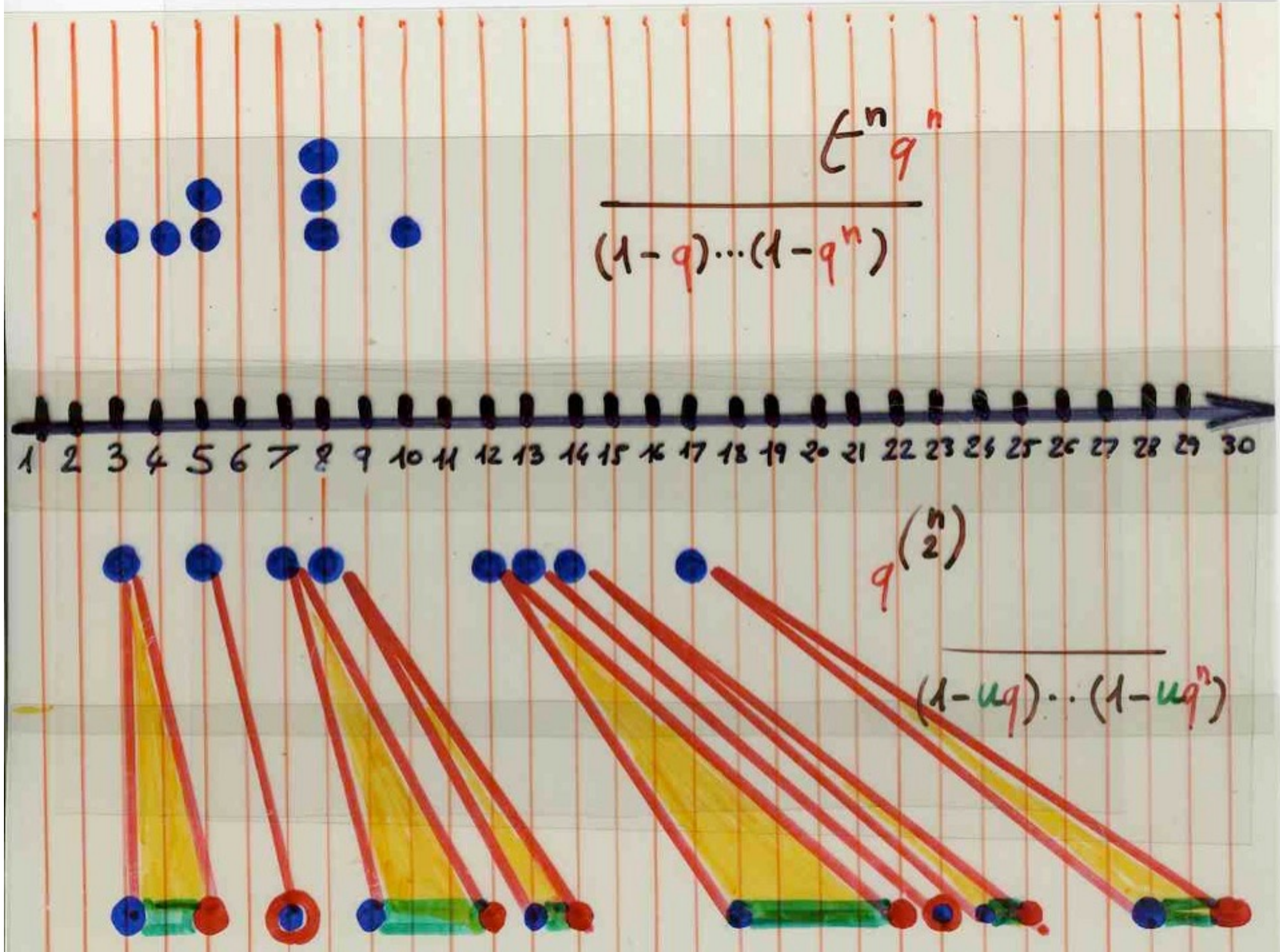
$$D = \sum_{n \geq 0} (-1)^n \frac{q^{\binom{n}{2}}}{(1-uq)\cdots(1-uq^n)}$$

$$\frac{t^n q^n}{(1-q)\cdots(1-q^n)}$$



$$D = \sum_{n \geq 0} \frac{(-1)^n}{(1-uq)\cdots(1-uq^n)}$$

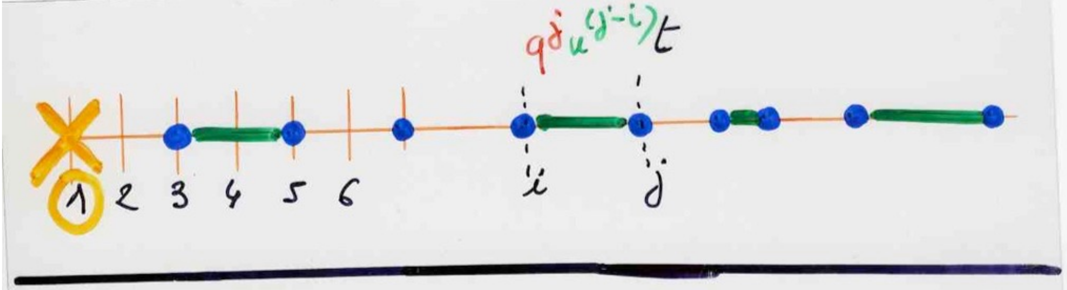


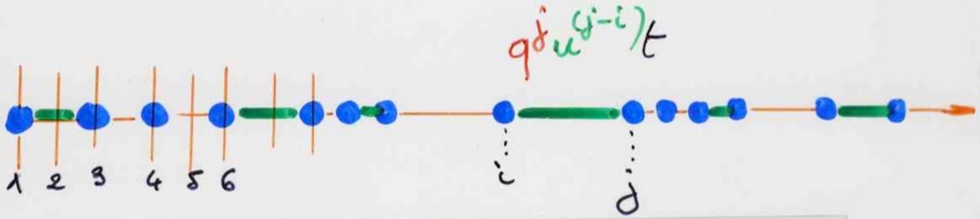


$$\frac{t^n q^n}{(1-q)\cdots(1-q^n)}$$

$$\frac{q^{\binom{n}{2}}}{(1-uq)\cdots(1-uq^n)}$$

$$N = u \sum_{n\geq 1} \frac{(-1)^{n-1}\, t^n\, q^n\, q^{\binom{n}{2}}}{(1-q)\cdots(1-q^{n-1})(1-uq)\cdots(1-uq^n)}$$





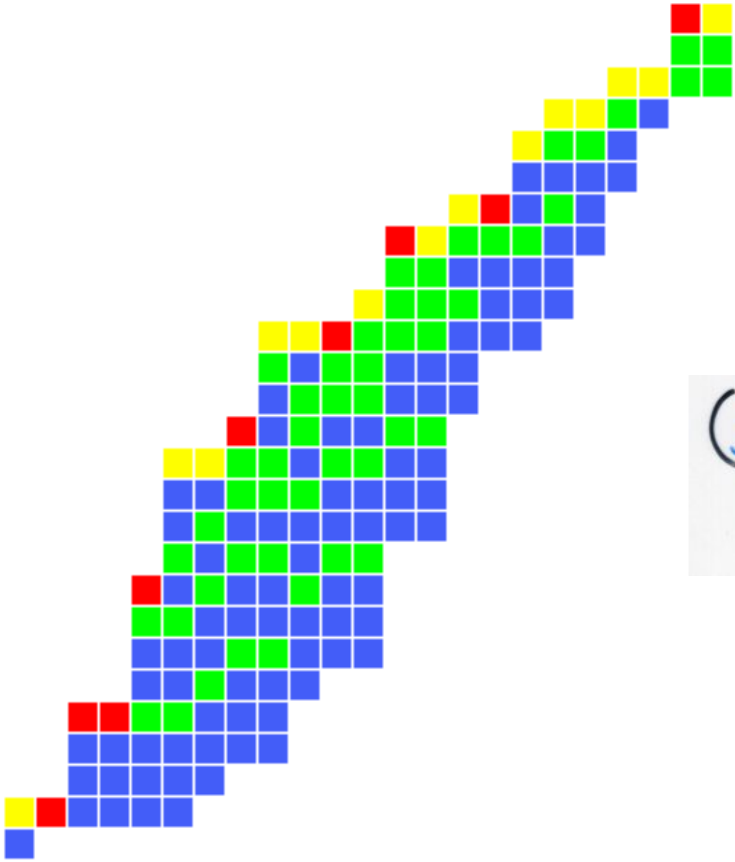
Segments $v([i;j]) = q^j t u^{(j-i)}$

$$D = \sum_{n\geq 0} \frac{(-1)^n\, t^n\, q^n\, q^{\binom{n}{2}}}{(1-q)\cdots(1-q^n)(1-uq)\cdots(1-uq^n)}$$

random parallelogram polyominoes

random parallelogram polyominoe

(staircase polygon)



fixed perimeter

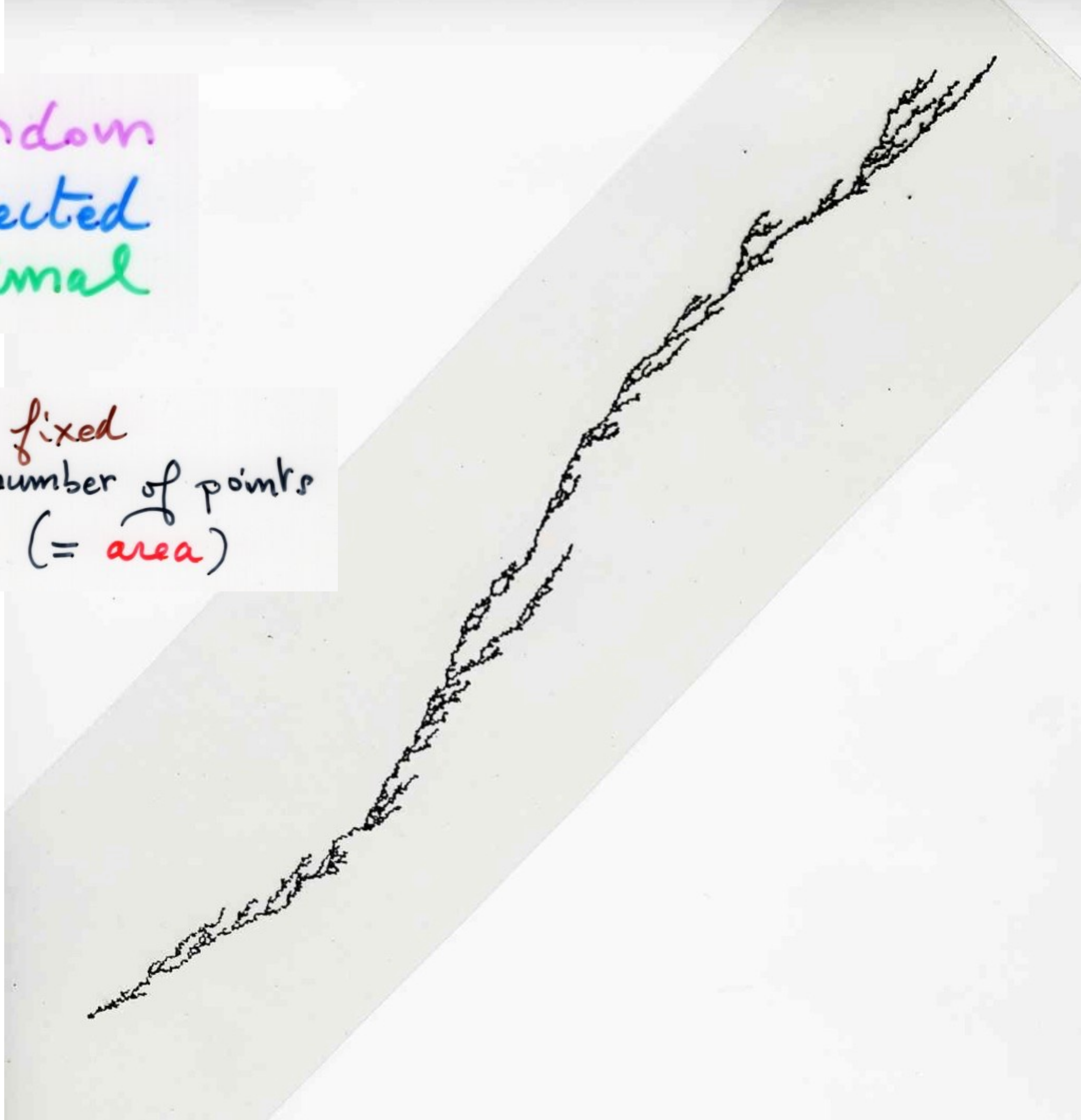
fixed area

(I. Dutour, J.M. Fedou)

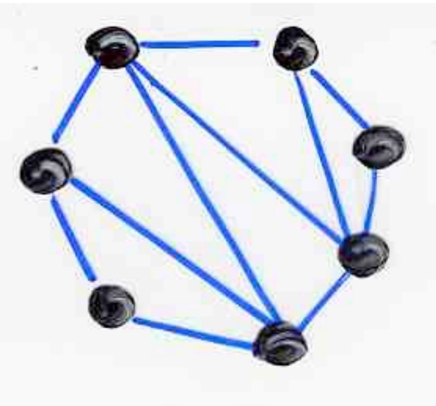
q- algebraic grammar

random
directed
animal

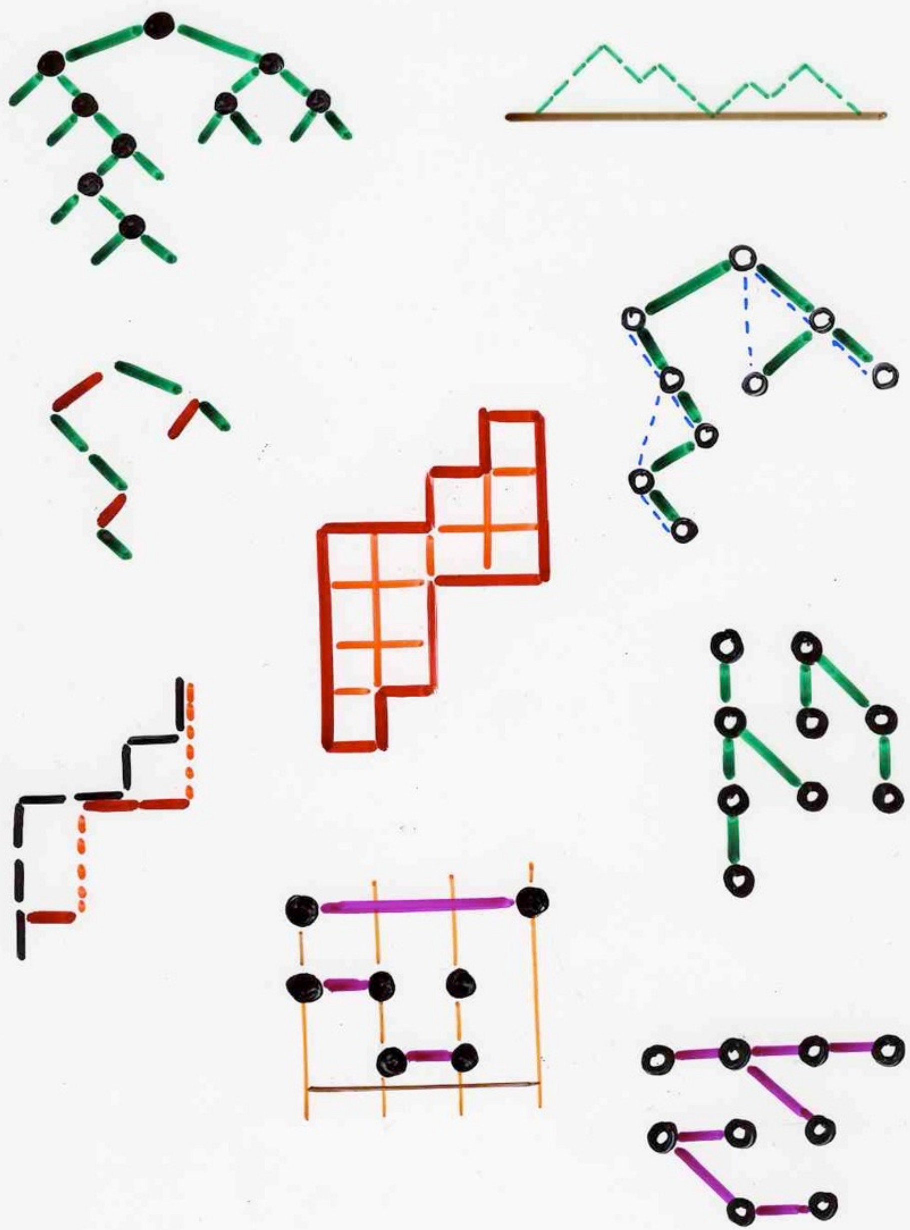
fixed
number of points
(= area)

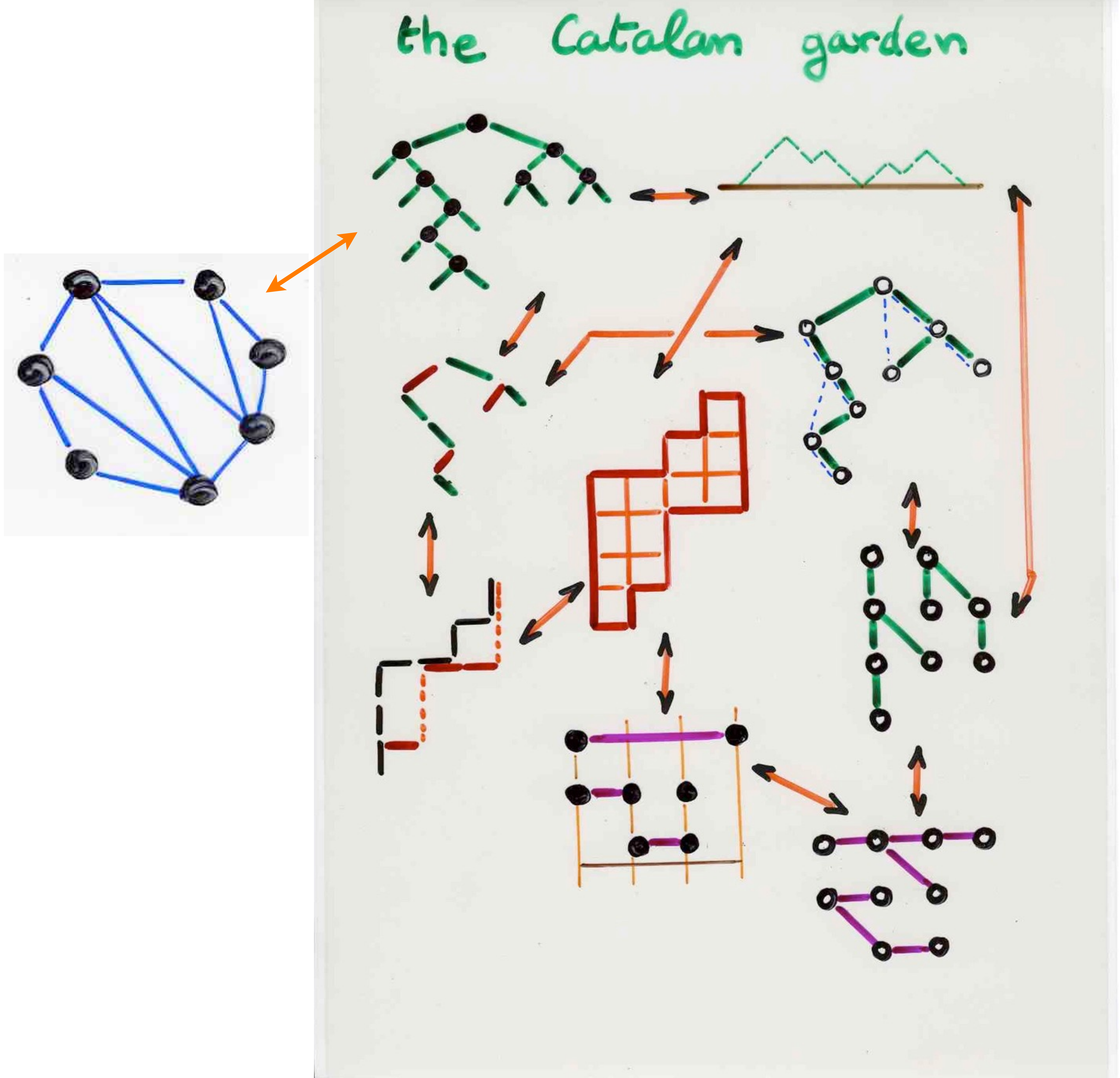


The Catalan garden



the Catalan garden




the Catalan garden

A festival of bijections ....

other description of the bijection:

1. with the stairs decomposition of a heap of dimers

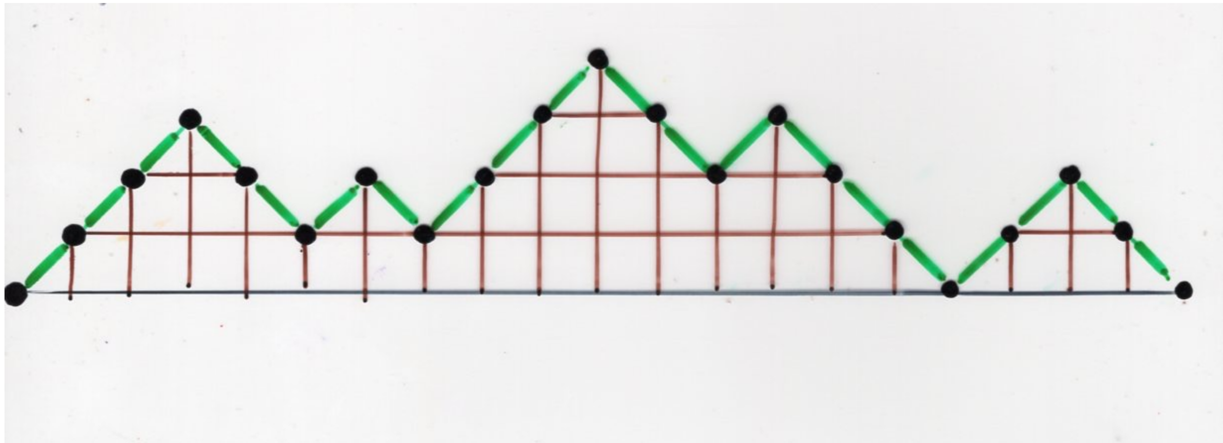
bijection

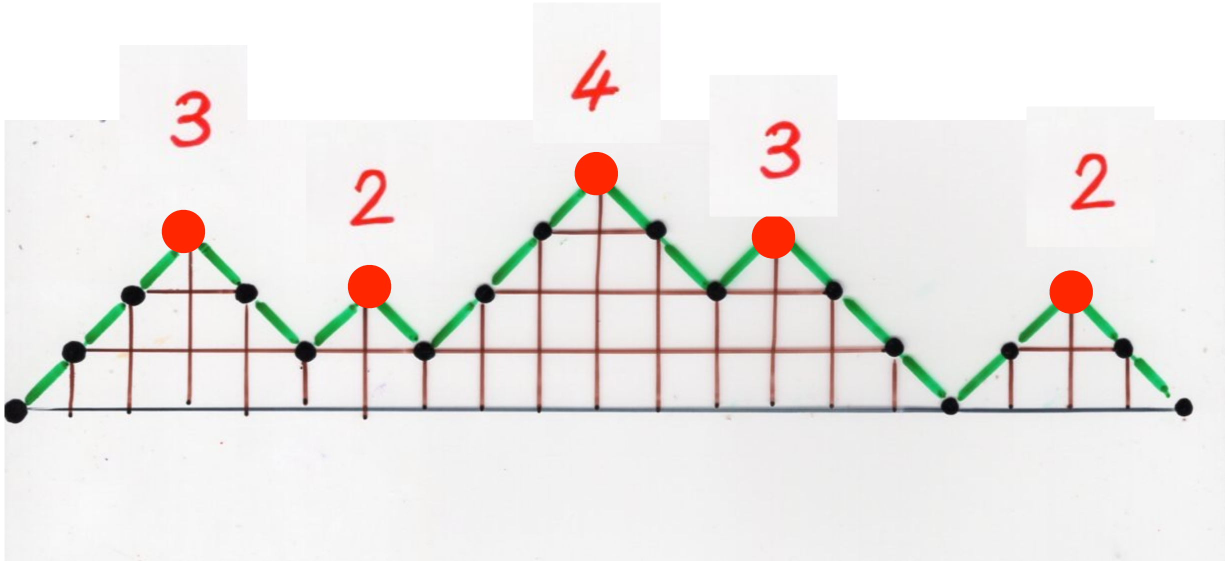
staircase polygons

Dyck paths

Ch 2a (IMSc 2016)
p 110-116

The Catalan garden

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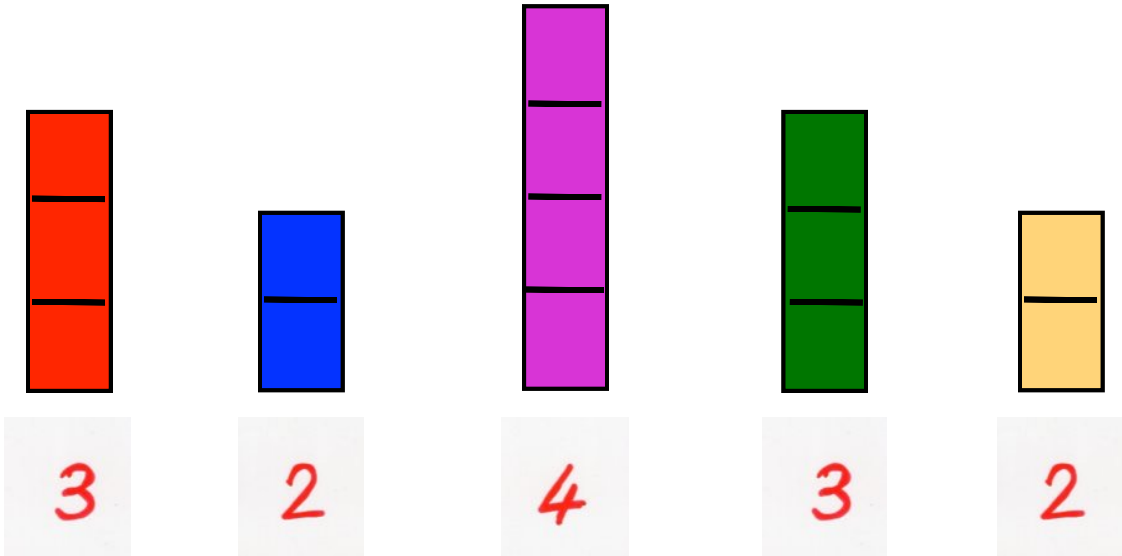
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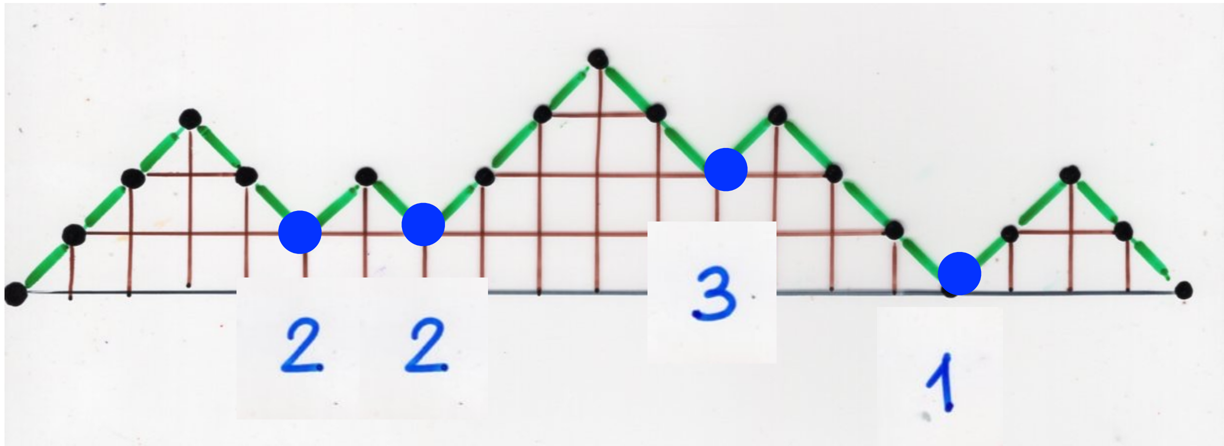
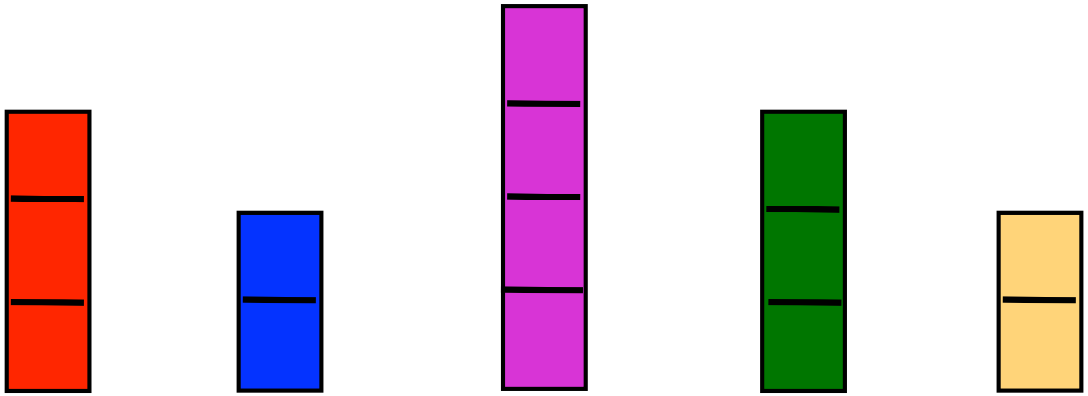
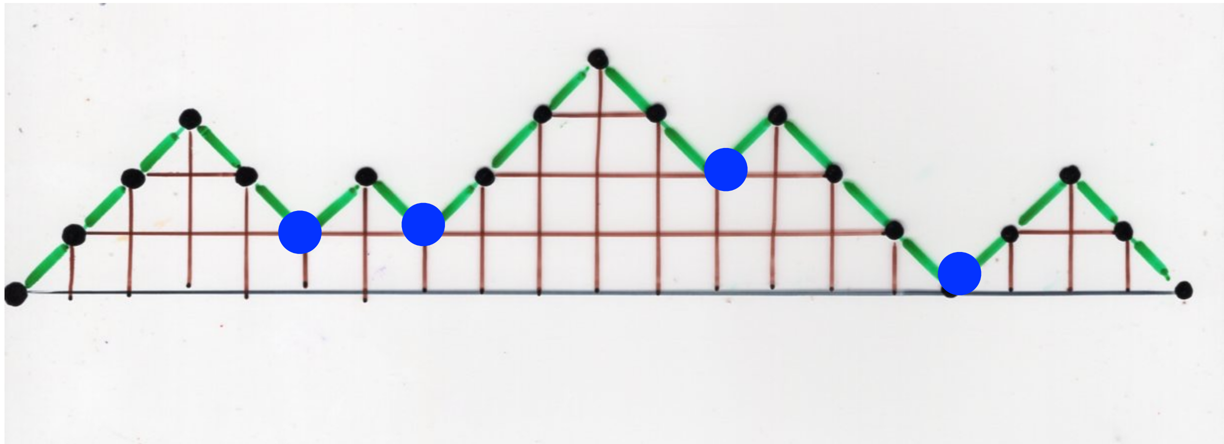
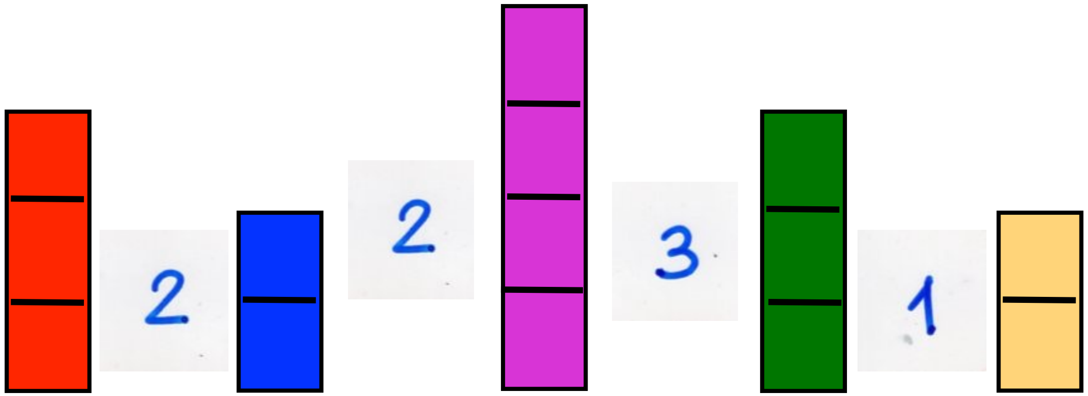
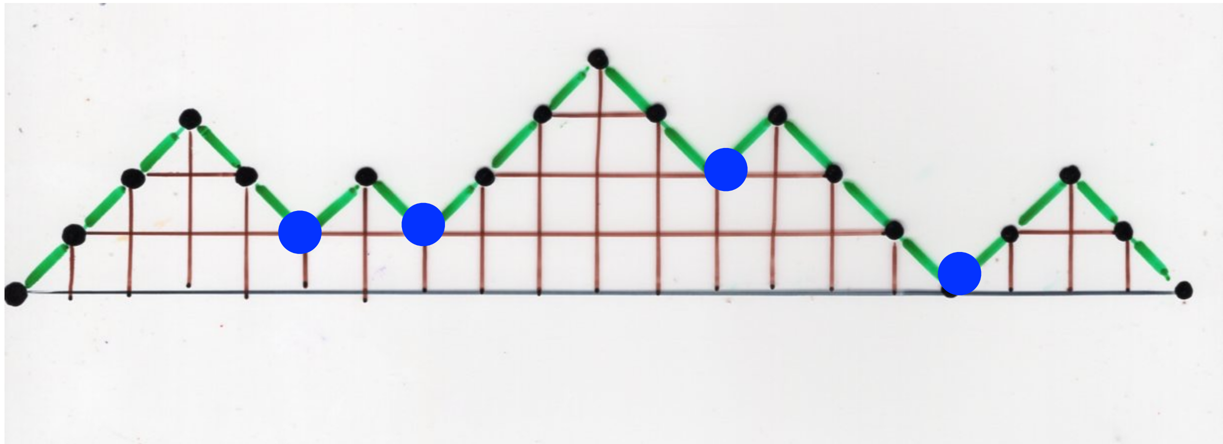
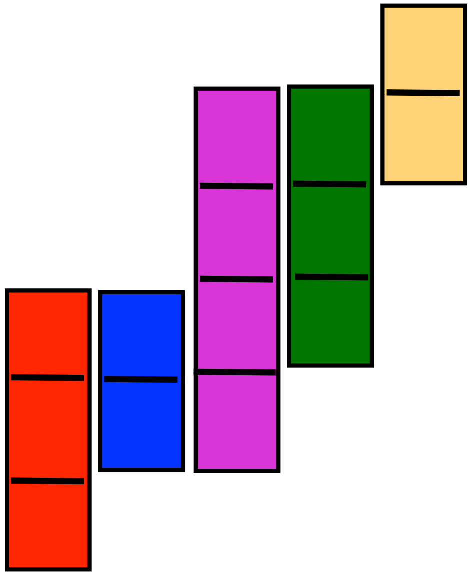

2
2
3
1

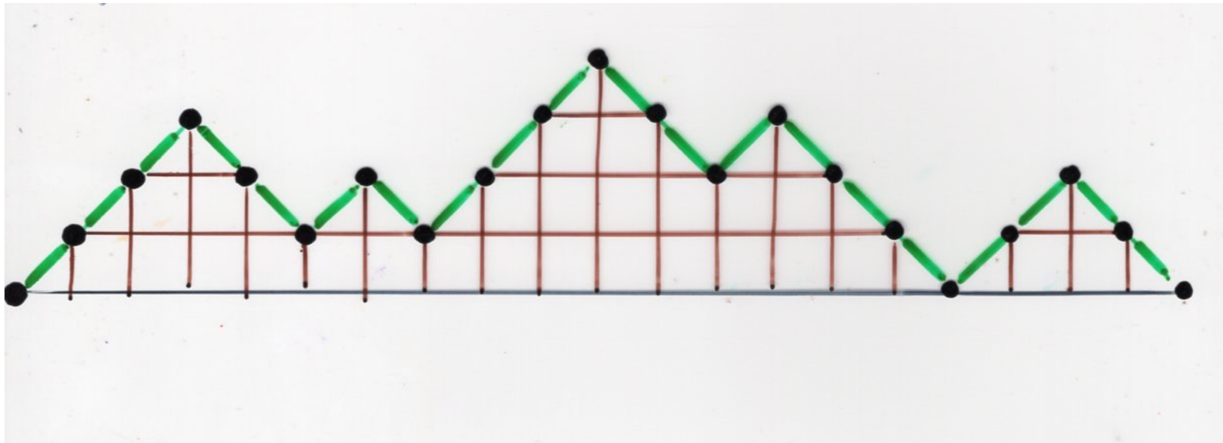


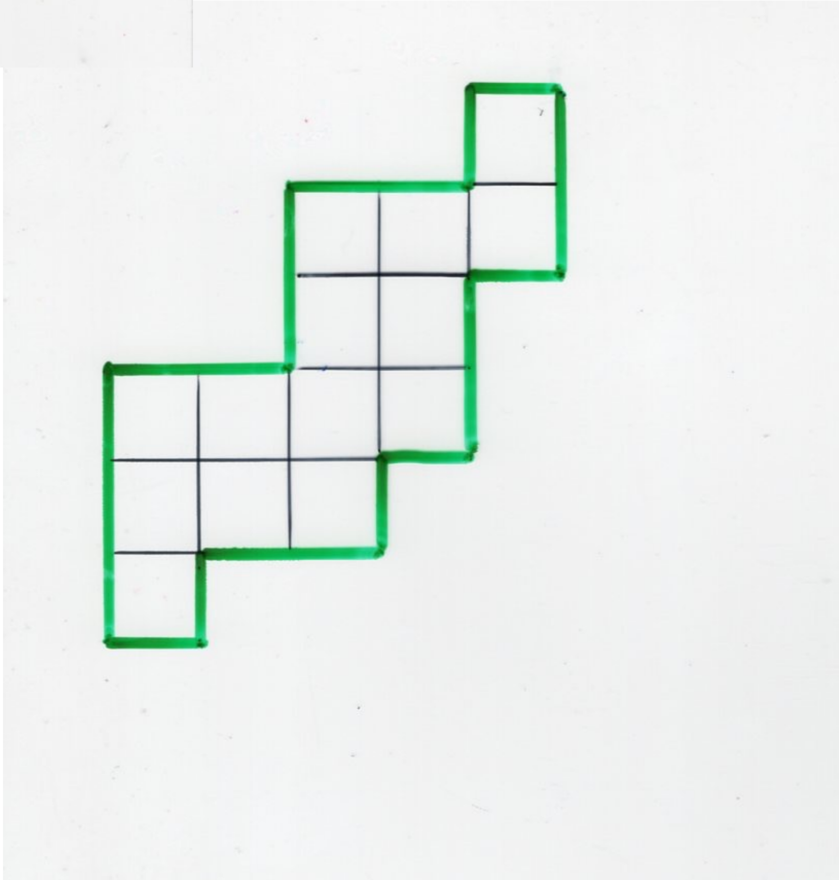



2
2
3
1









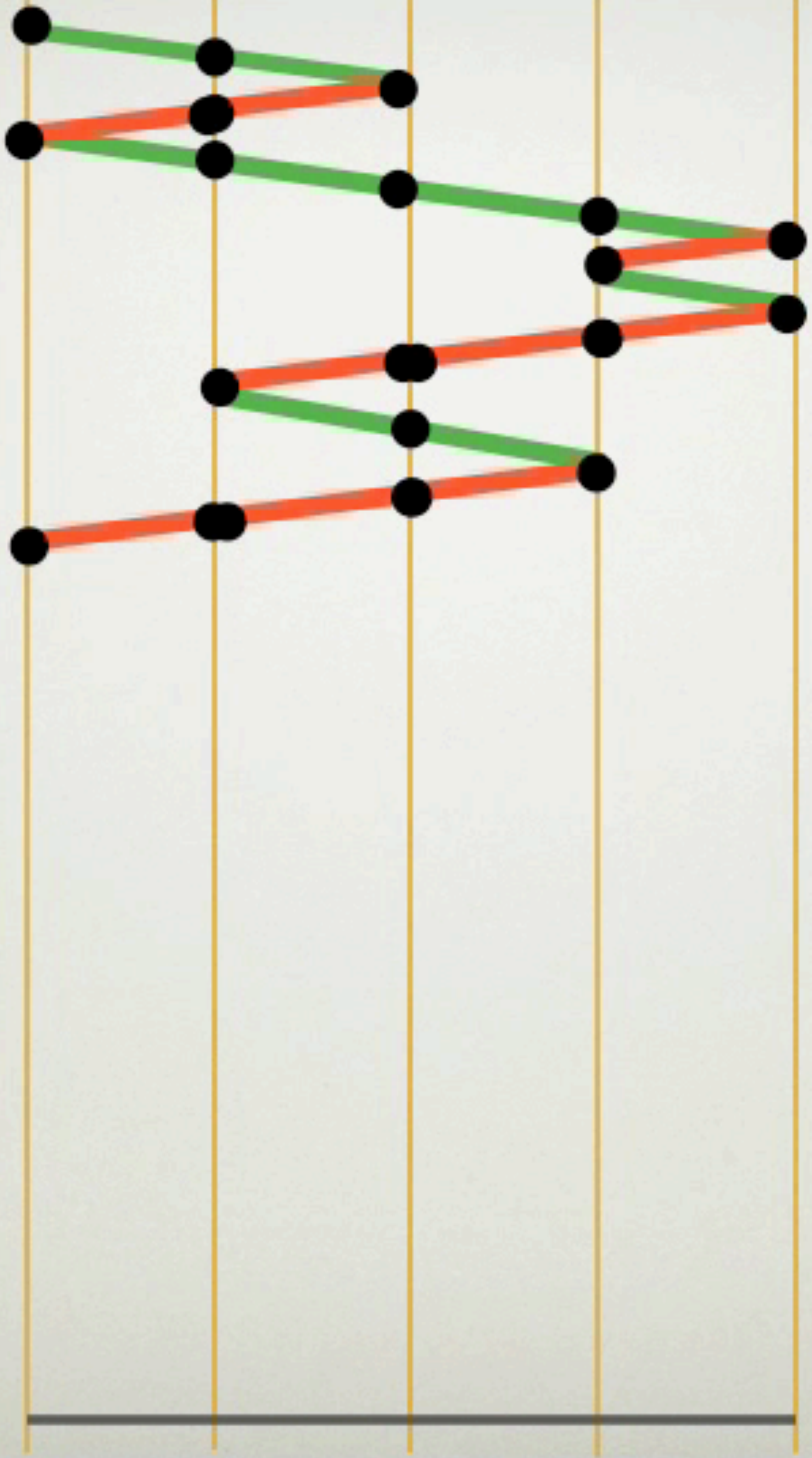
bijections

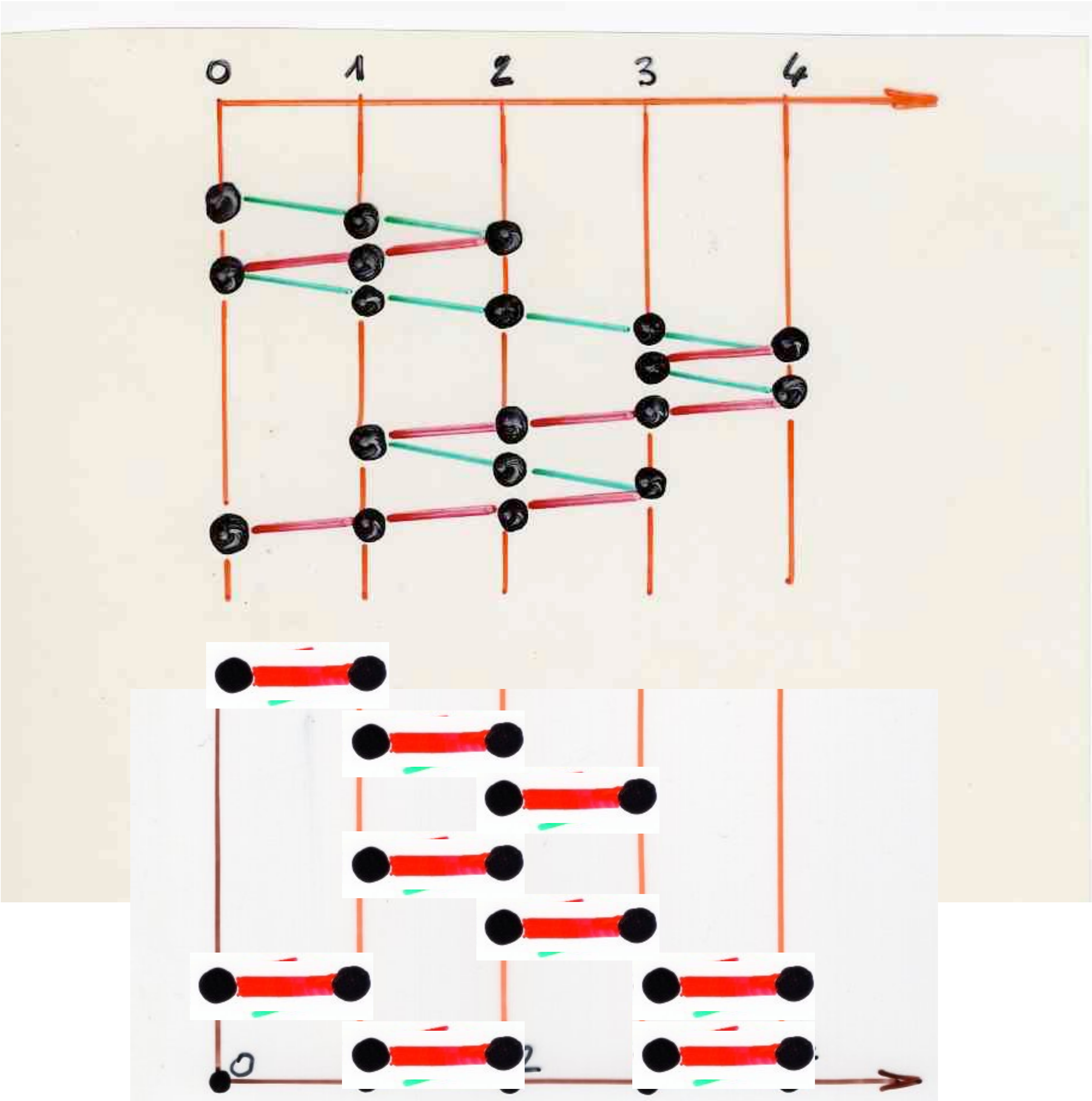
staircase polygons

Dyck paths

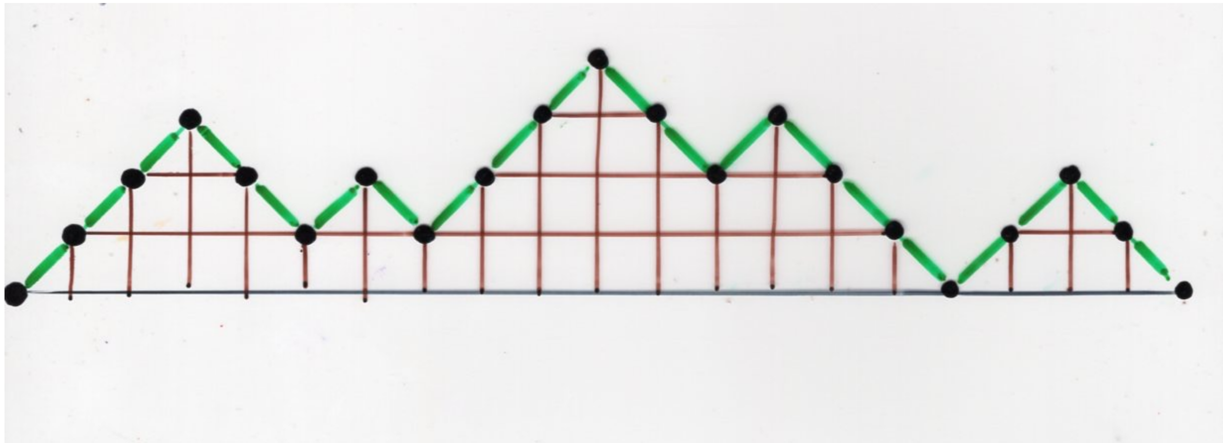
path ω on X $\xleftrightarrow{\chi}$ (η, E)

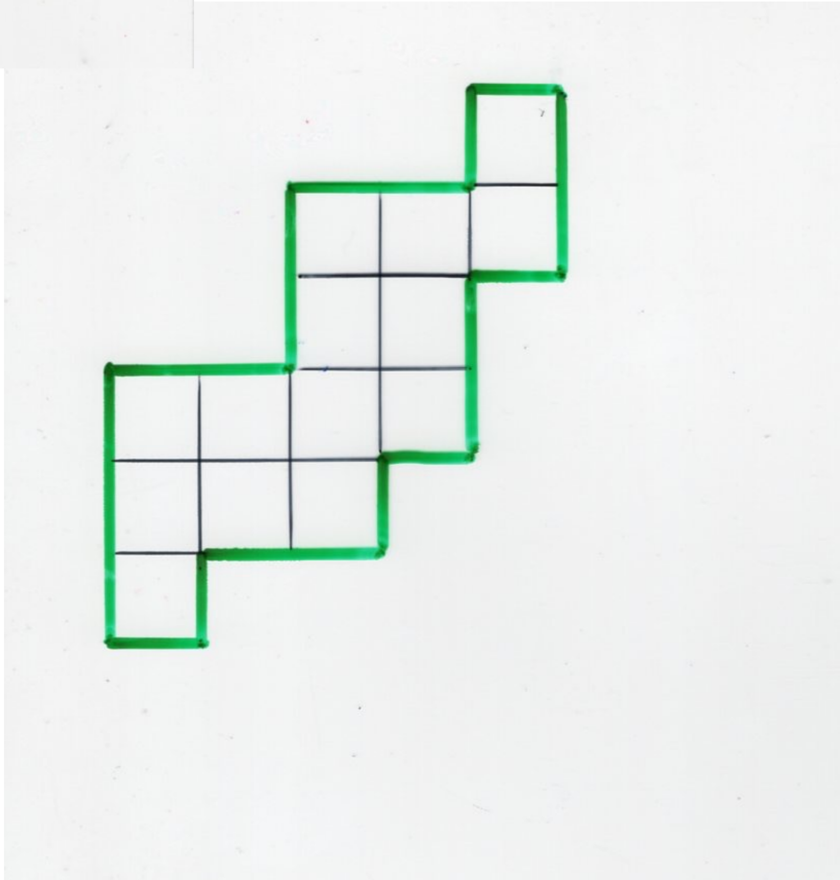
semi-pyramids of dimers

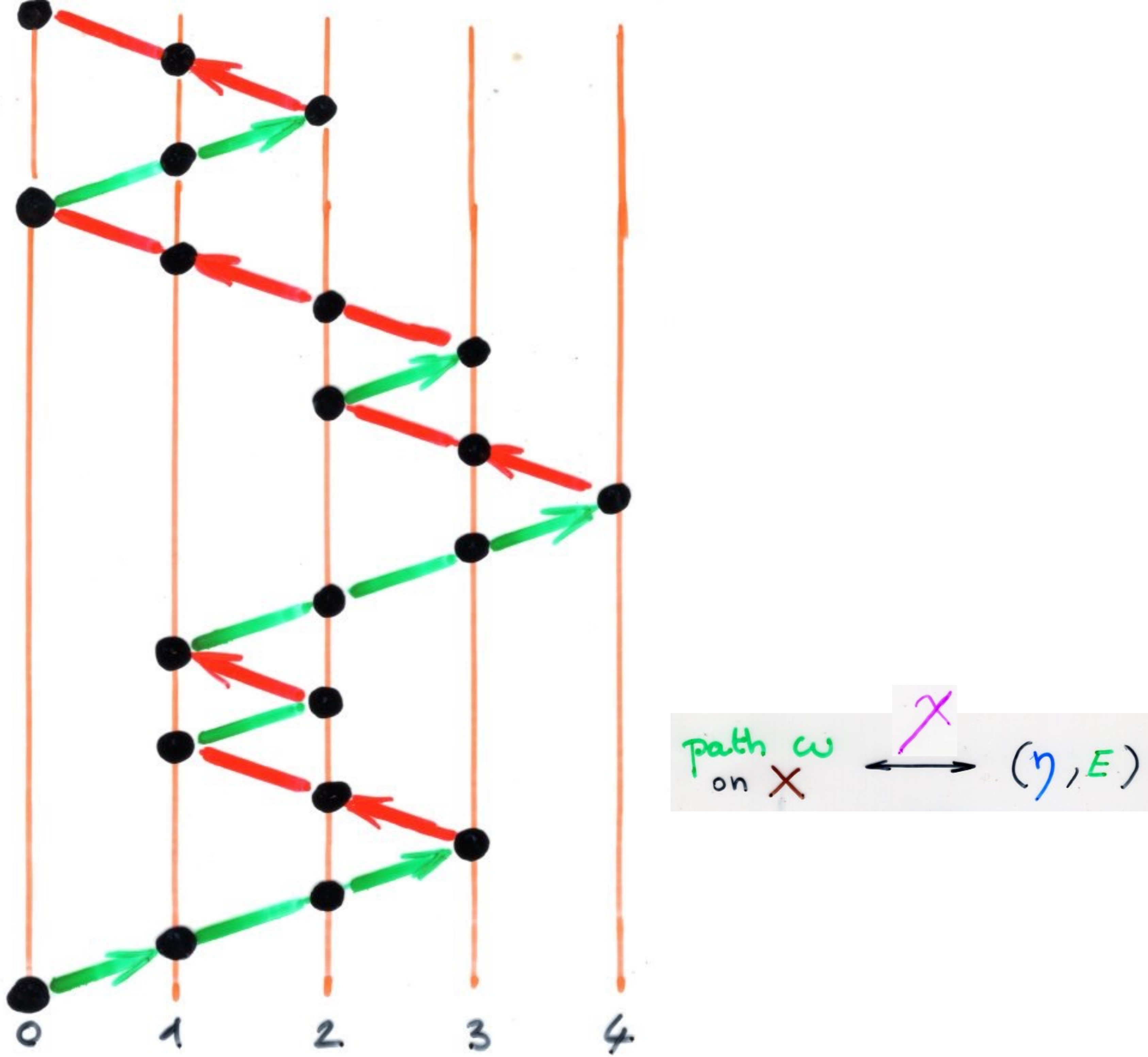


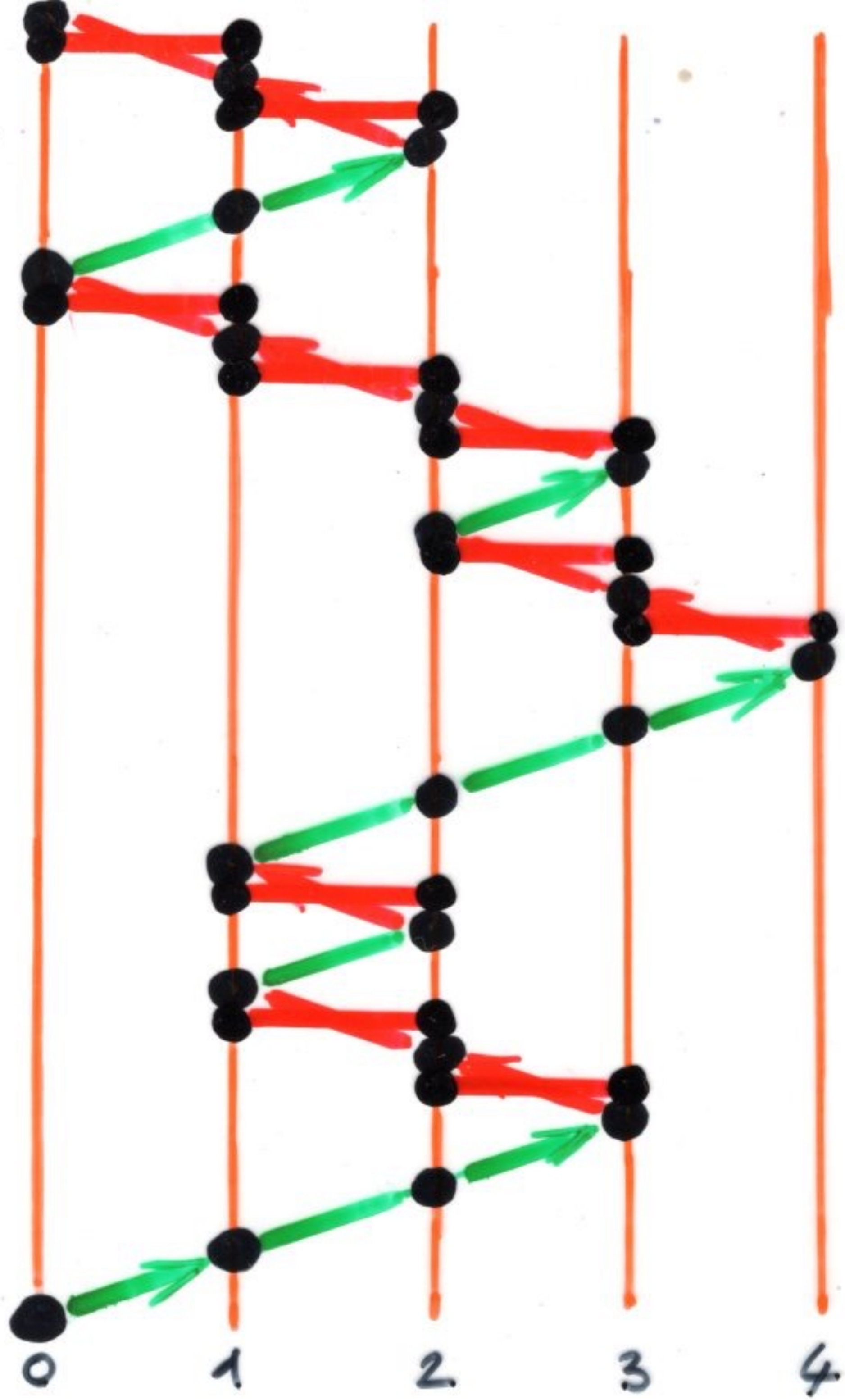


violin:
G. Duchamp

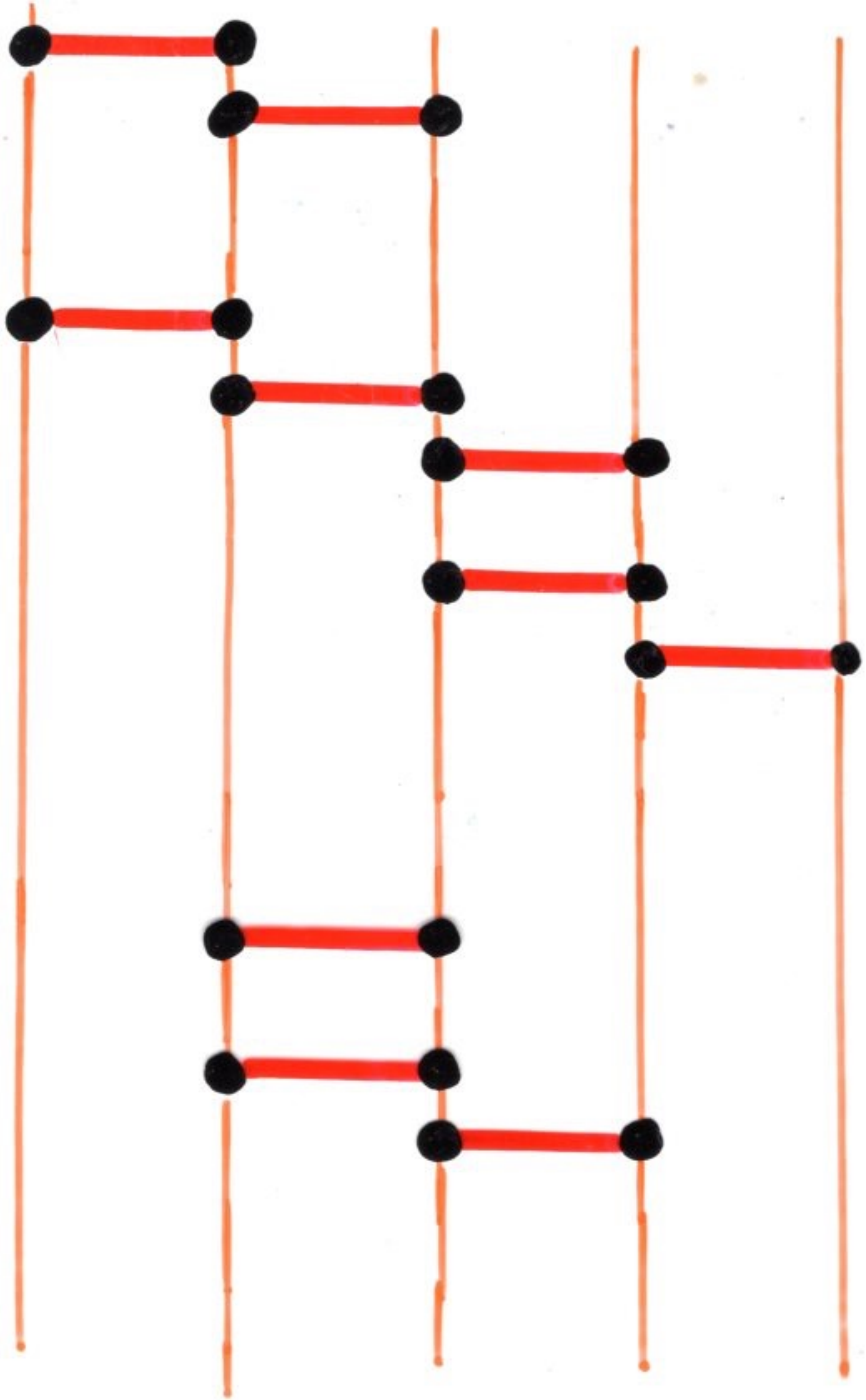





0
1
2
3
4
path ω
on X
χ
(η, E)


0
1
2
3
4


path ω
on X
χ
(η, E)



path ω on X $\xleftrightarrow{\chi}$ (η, E)

bijections

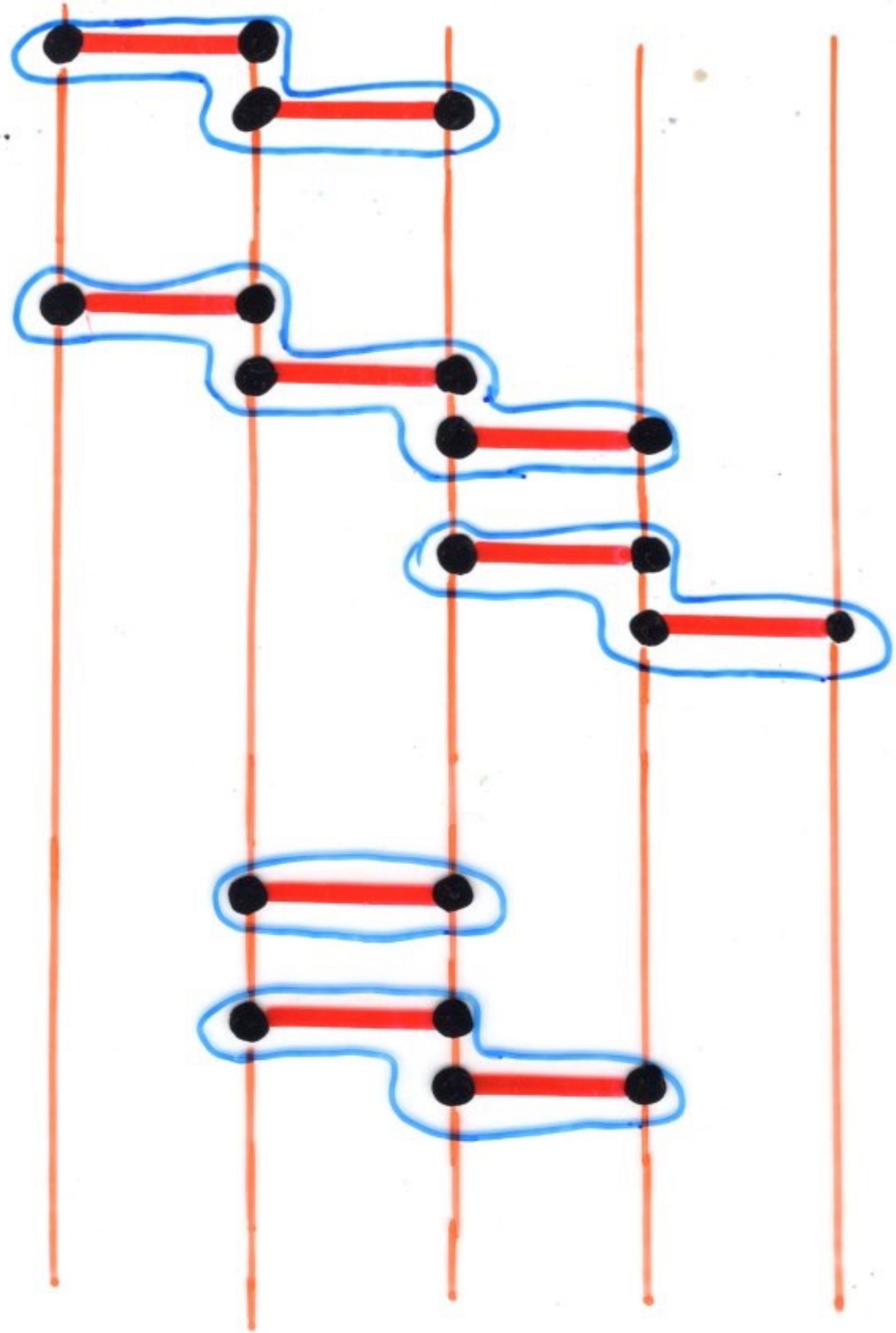
staircase polygons

Dyck paths

semi-pyramids of dimers

stair decomposition

Ch6a, p 50



bijections

staircase polygons

Dyck paths

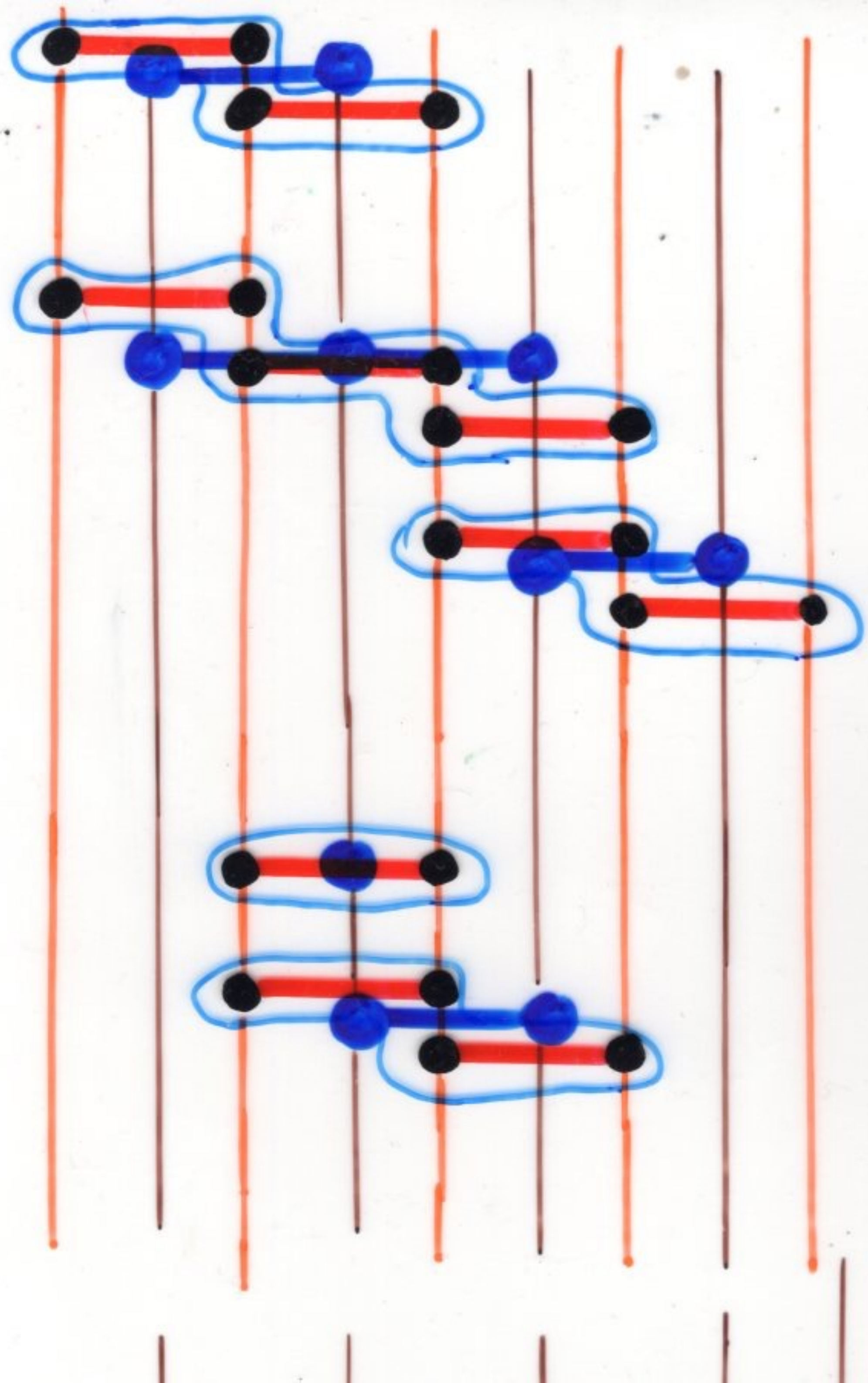
semi-pyramids of dimers

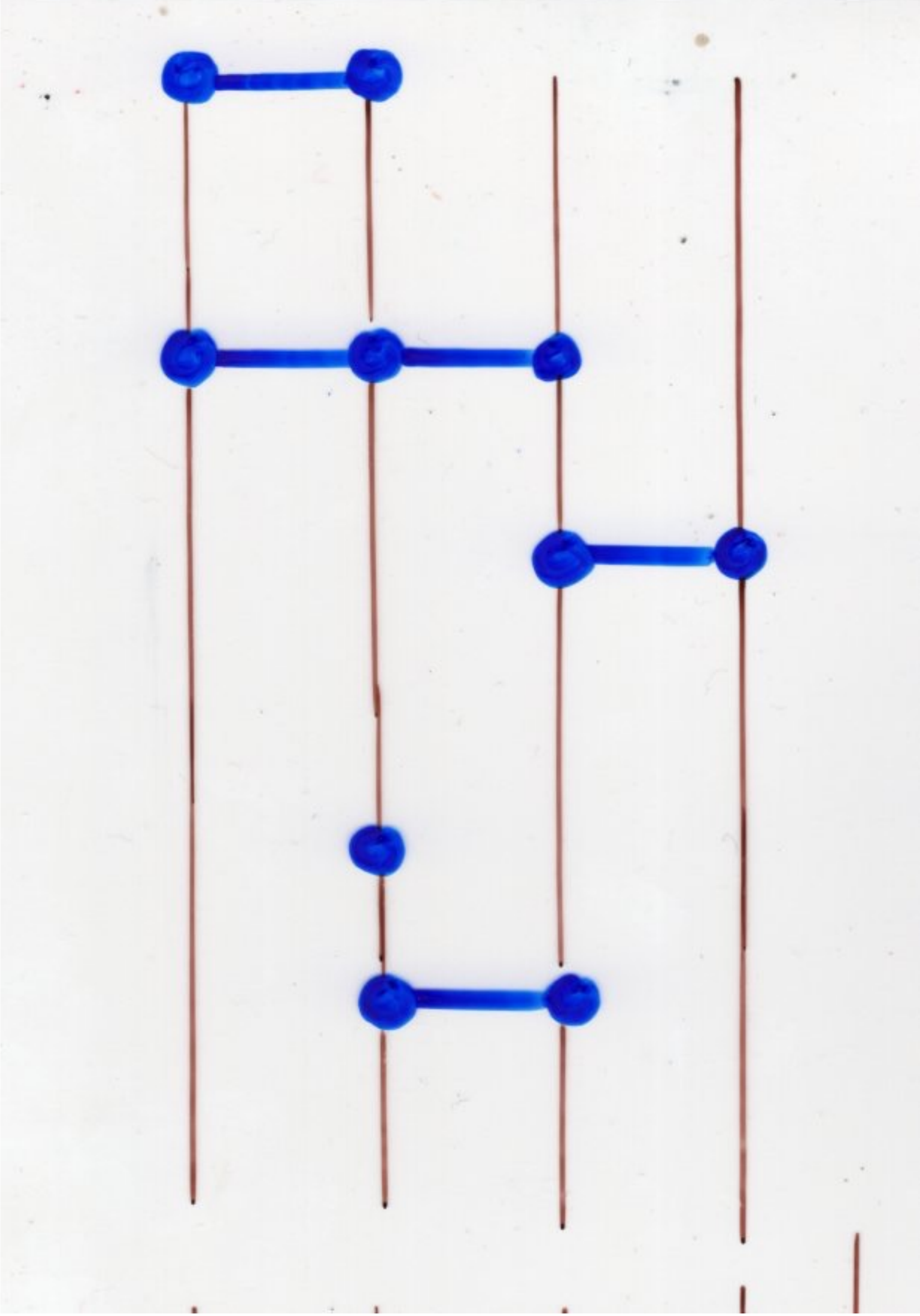
stair decomposition

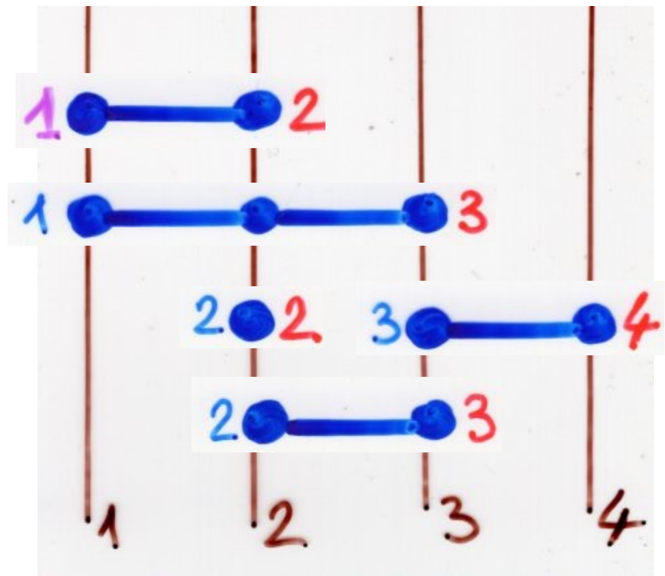
Ch6a, p 50

semi-pyramids of segments

Ch6a, p 55






1 2
1 3
2 2 3 4
2 3
1 2 3 4

a festival of bijections

parallelogram polyominoes

(staircase polygons)

Dyck paths

semi-pyramids of dimers (on $\mathbb{N}$)

stairs decomposition

semi-pyramids of segments (on $\mathbb{N}$)

other description of the bijection:

2. with Lukasiewicz paths

Lukasiewicz path

$\omega = (s_0, \dots, s_n)$

$s_0 = (0,0)$, $s_n = (n, 0)$

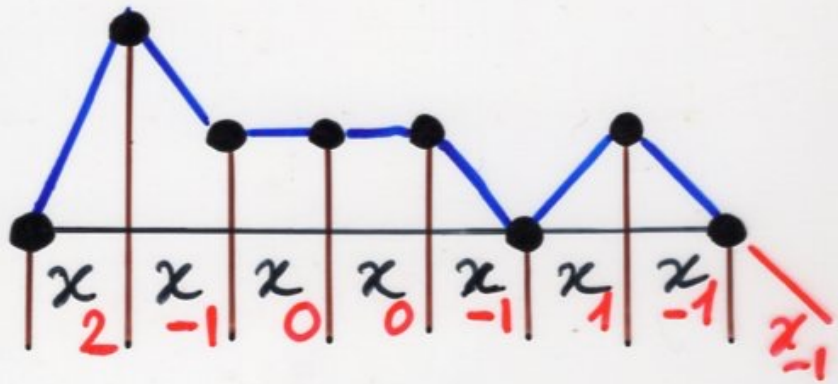
elementary step $s_i = (x_i, y_i)$ $s_{i+1} = (x_{i+1}, y_{i+1})$

with

$x_{i+1} = 1 + x_i$ $y_{i+1} \geqslant y_i - 1$

Ch 2a (IMSc 2016)

p 60



Ch2a, course 2016, p 60-63

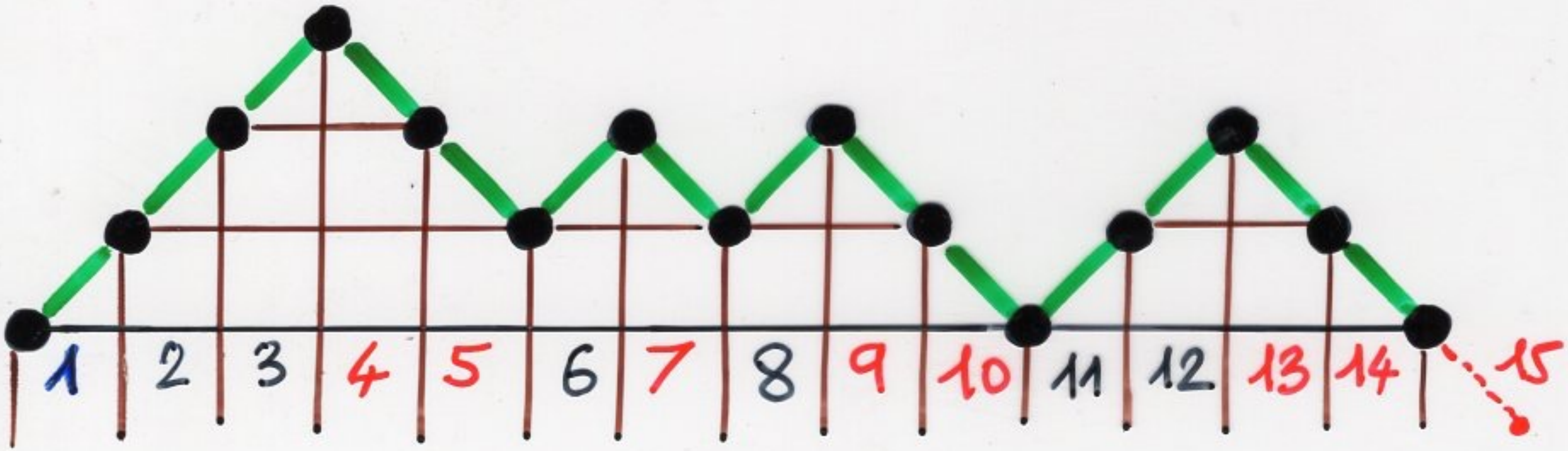
bijection

Dyck paths

Ch2a (IMSc 2016)
p 60

Lukasiewicz paths

The Catalan
garden


1
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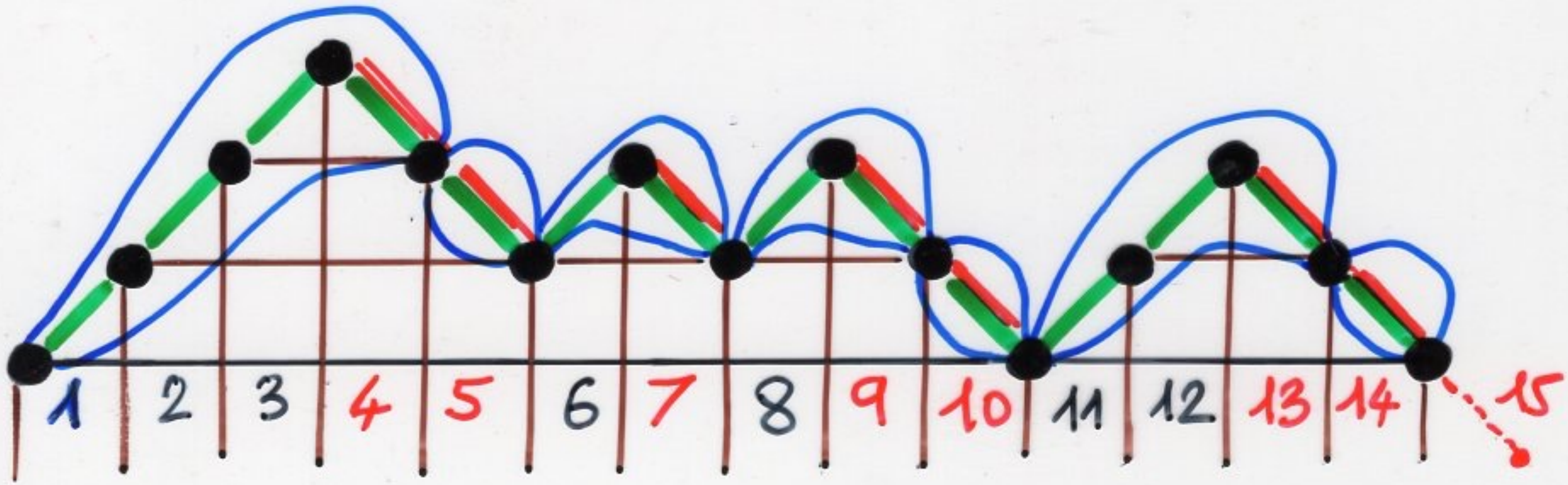
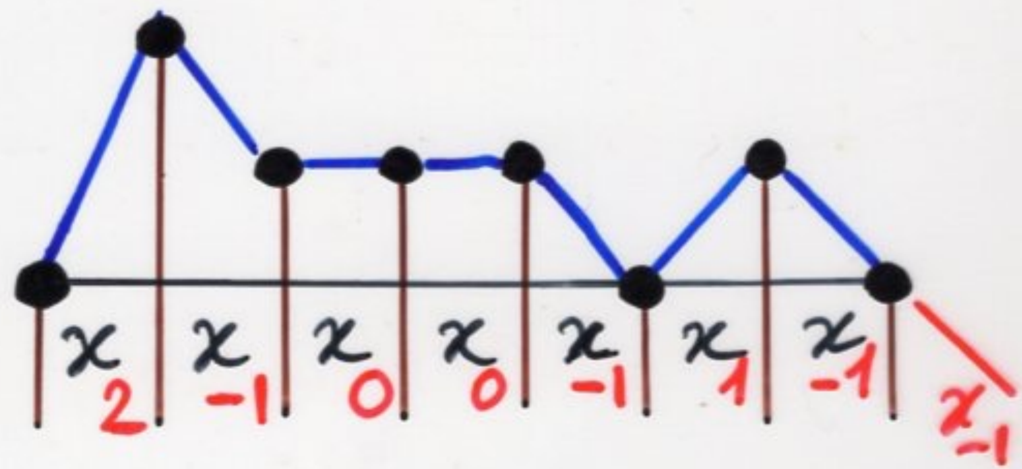
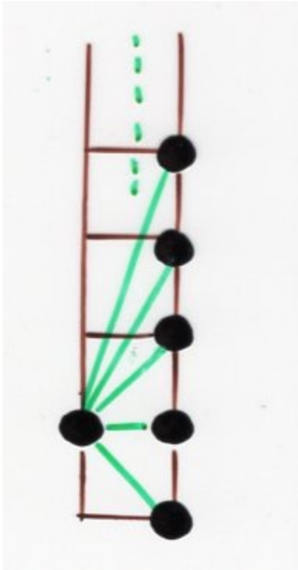
10
11
12
13
14
15

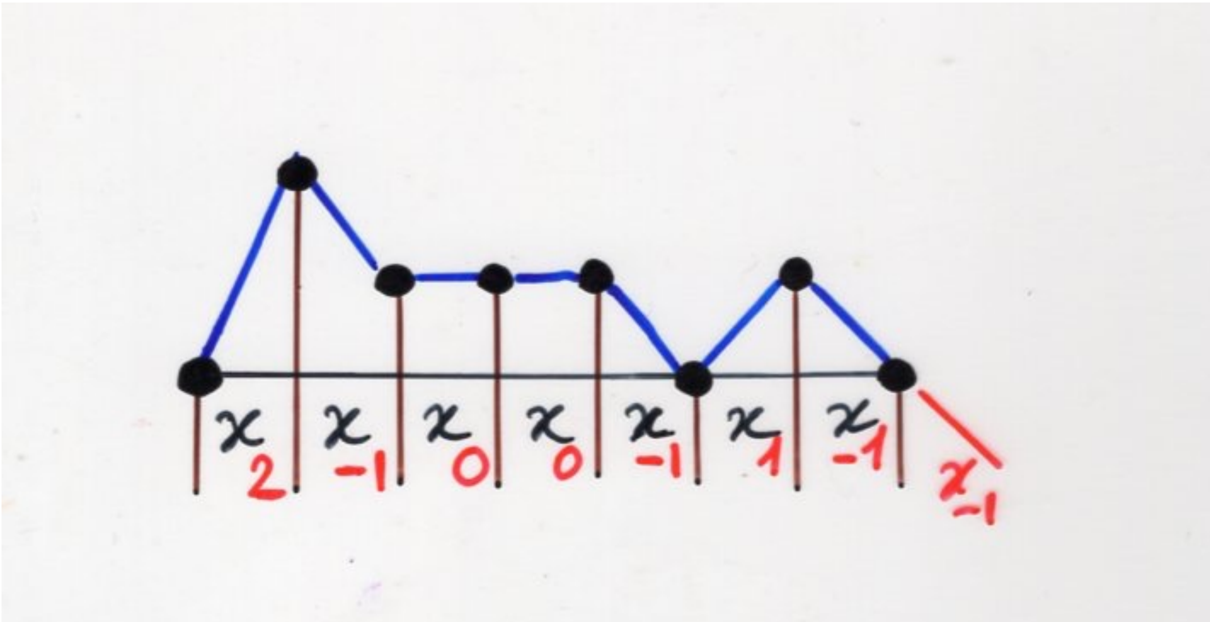


1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

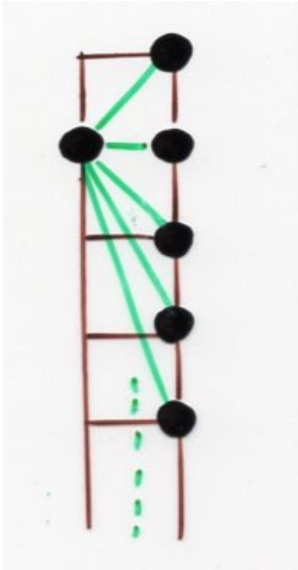


x_2
x_{-1}
x_0
x_0
x_{-1}
x_1
x_{-1}
x_{-1}

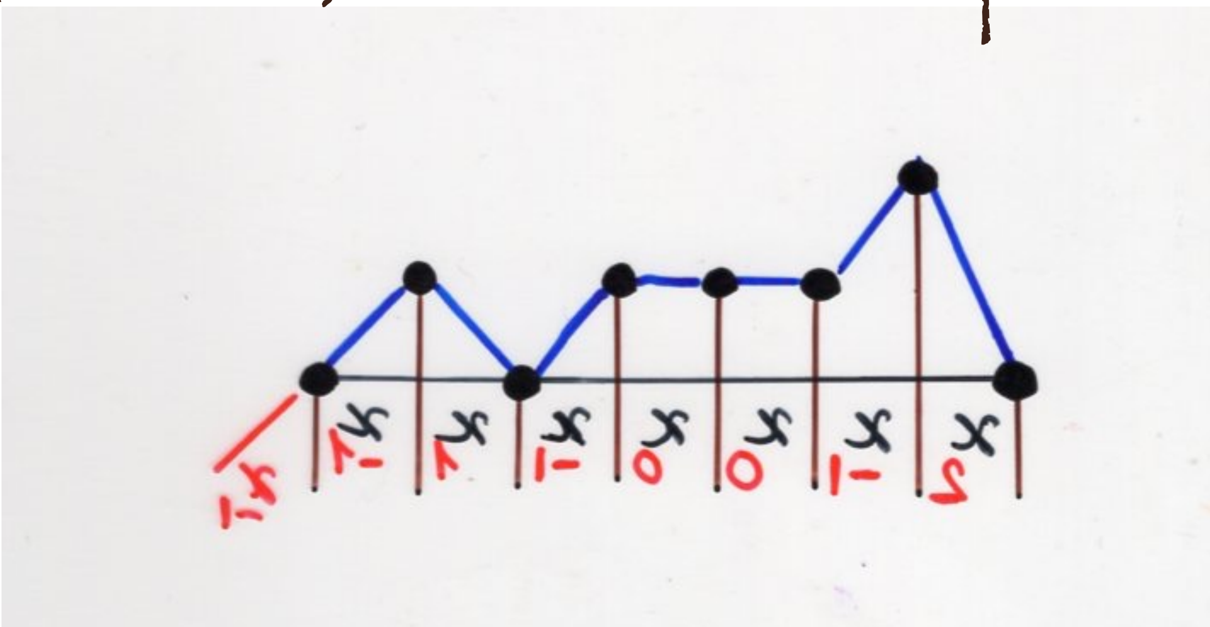


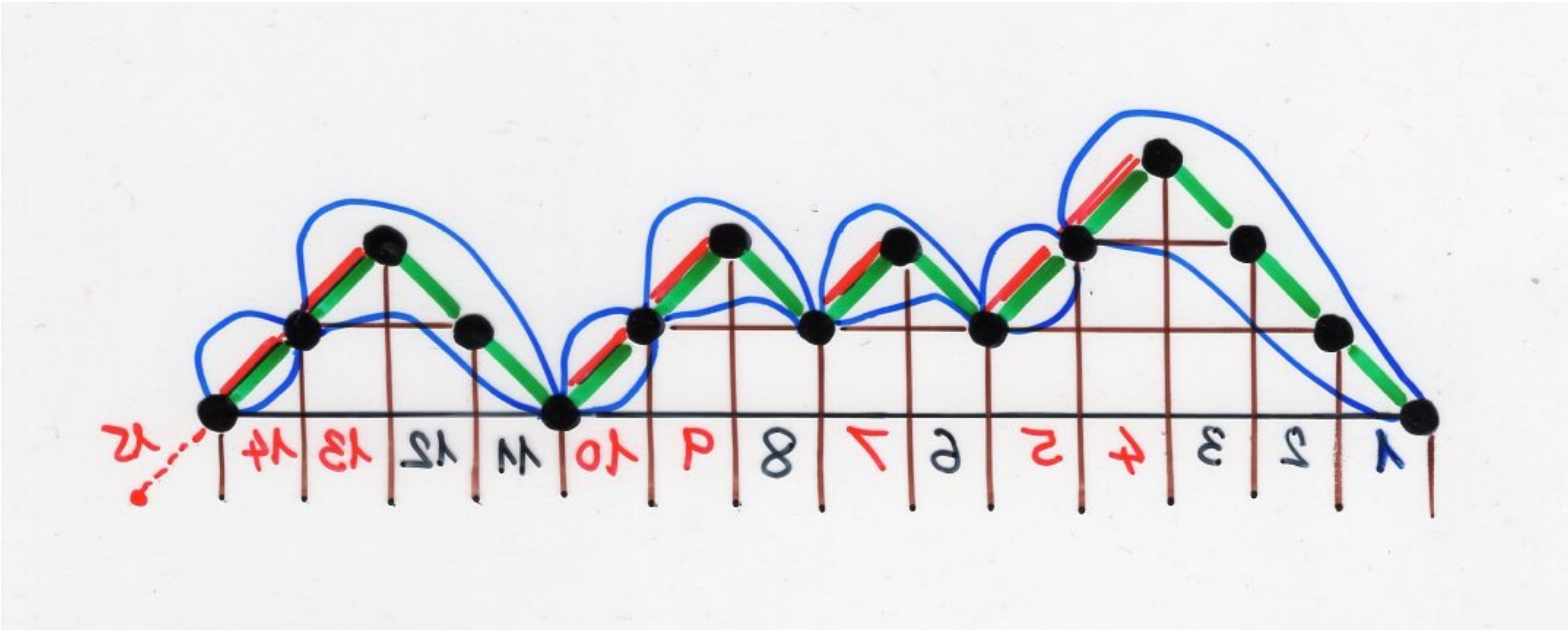


x_2 x_{-1} x_0 x_0 x_{-1} x_1 x_{-1} x_{-1}



(reverse) Lukasiewicz paths





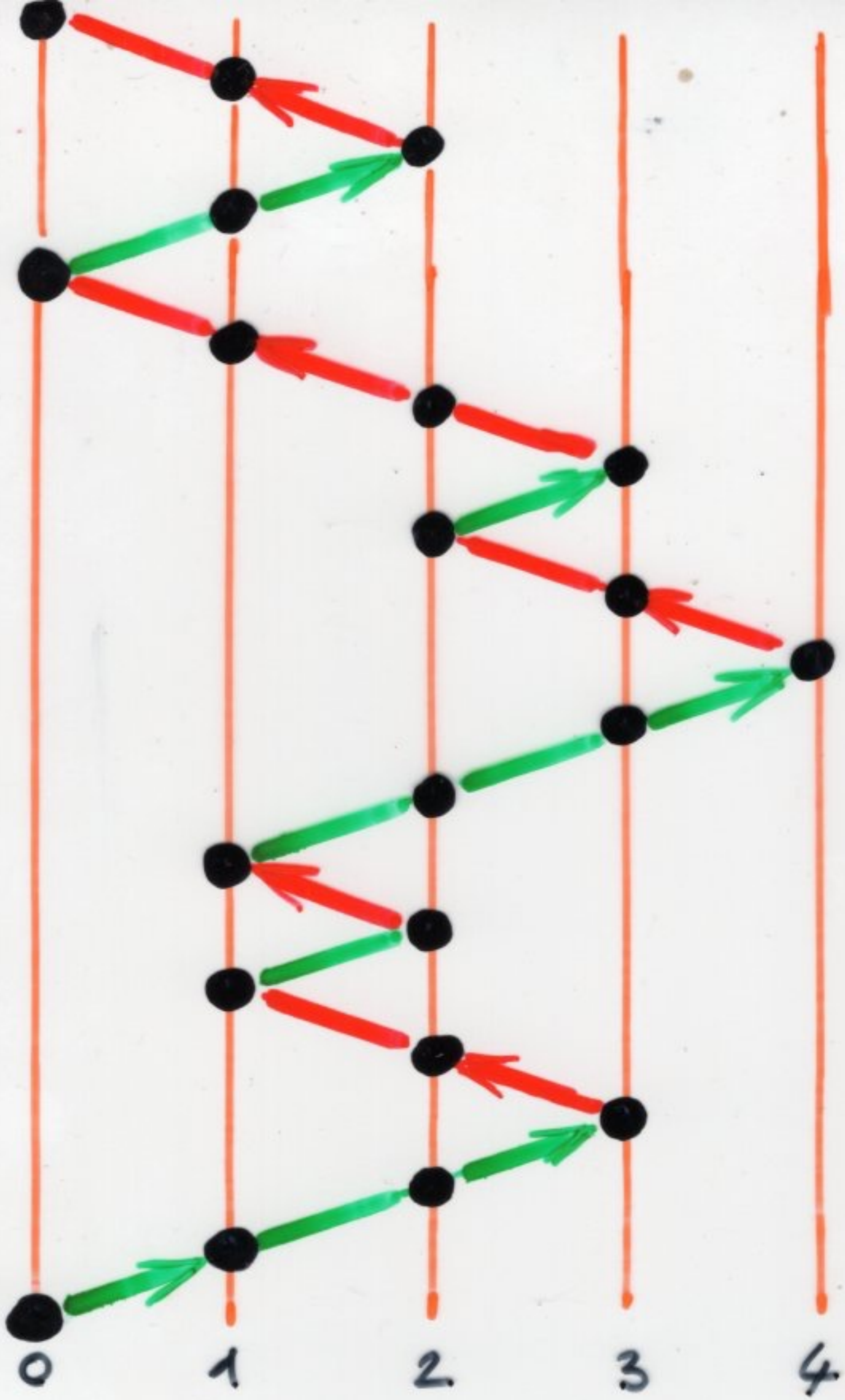
(reverse) Lukasiewicz paths

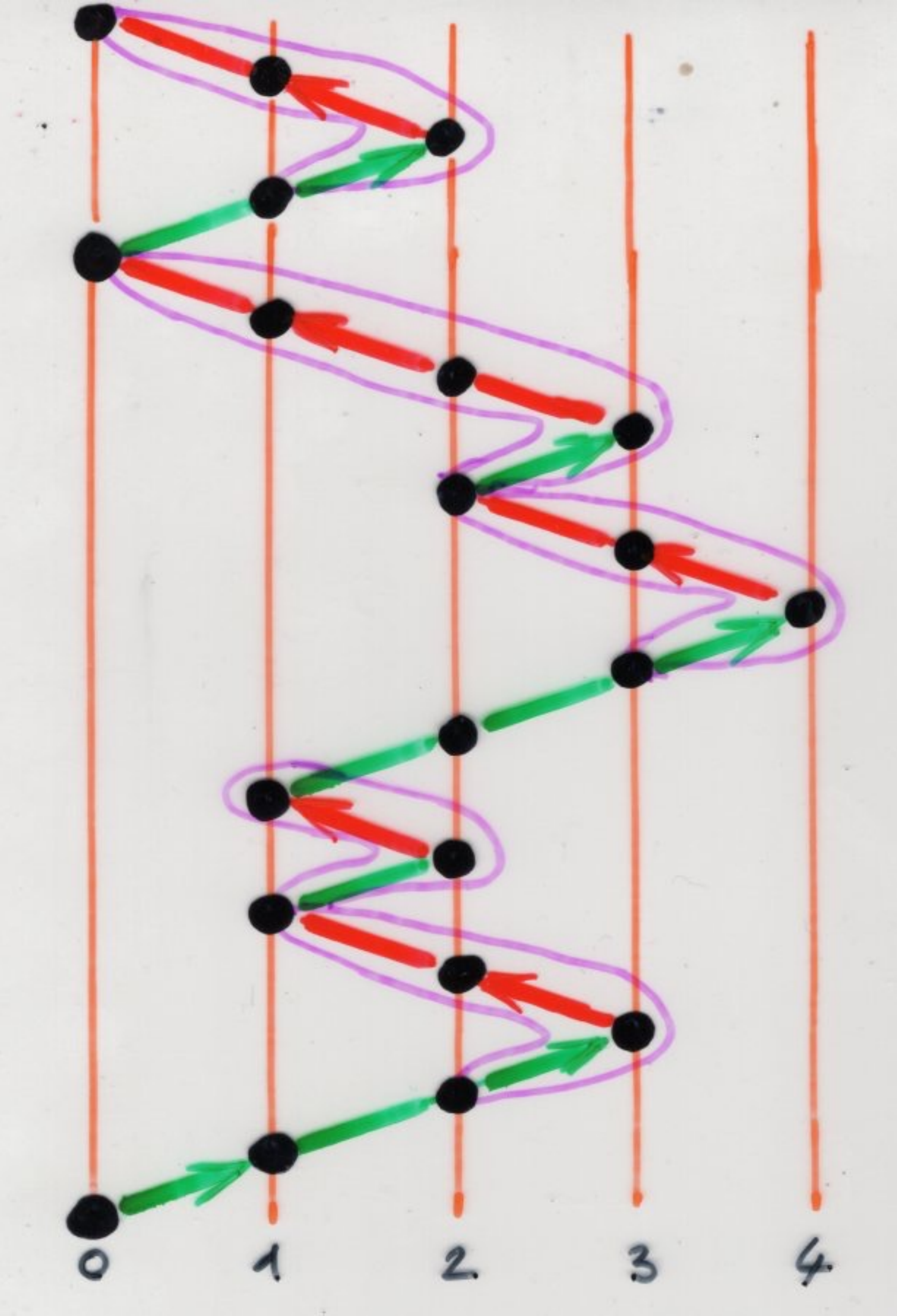
bijections

staircase polygons

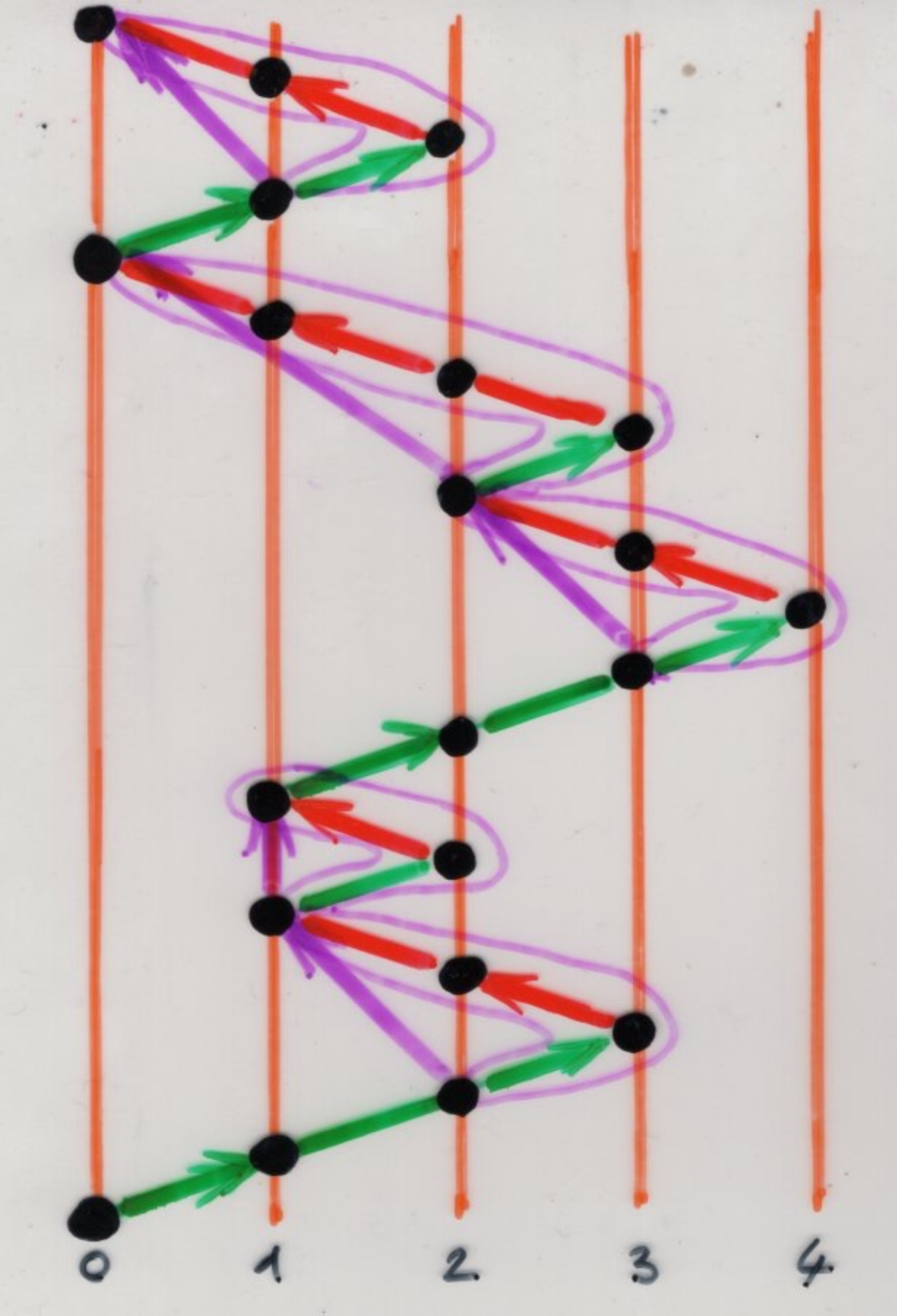
Dyck paths

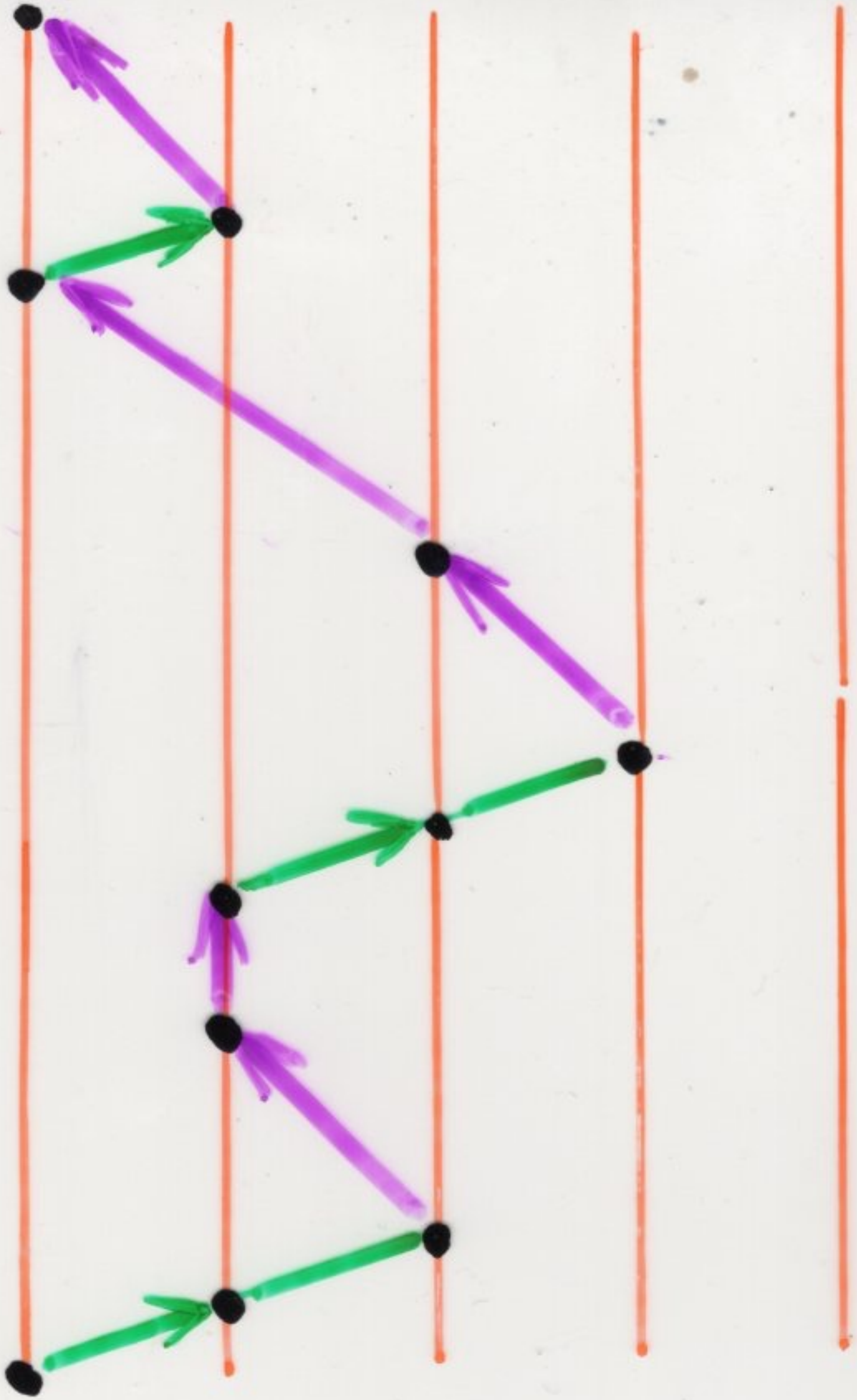
(reverse) Lukasiewicz paths


0
1
2
3
4



0
1
2
3
4


0
1
2
3
4



bijections

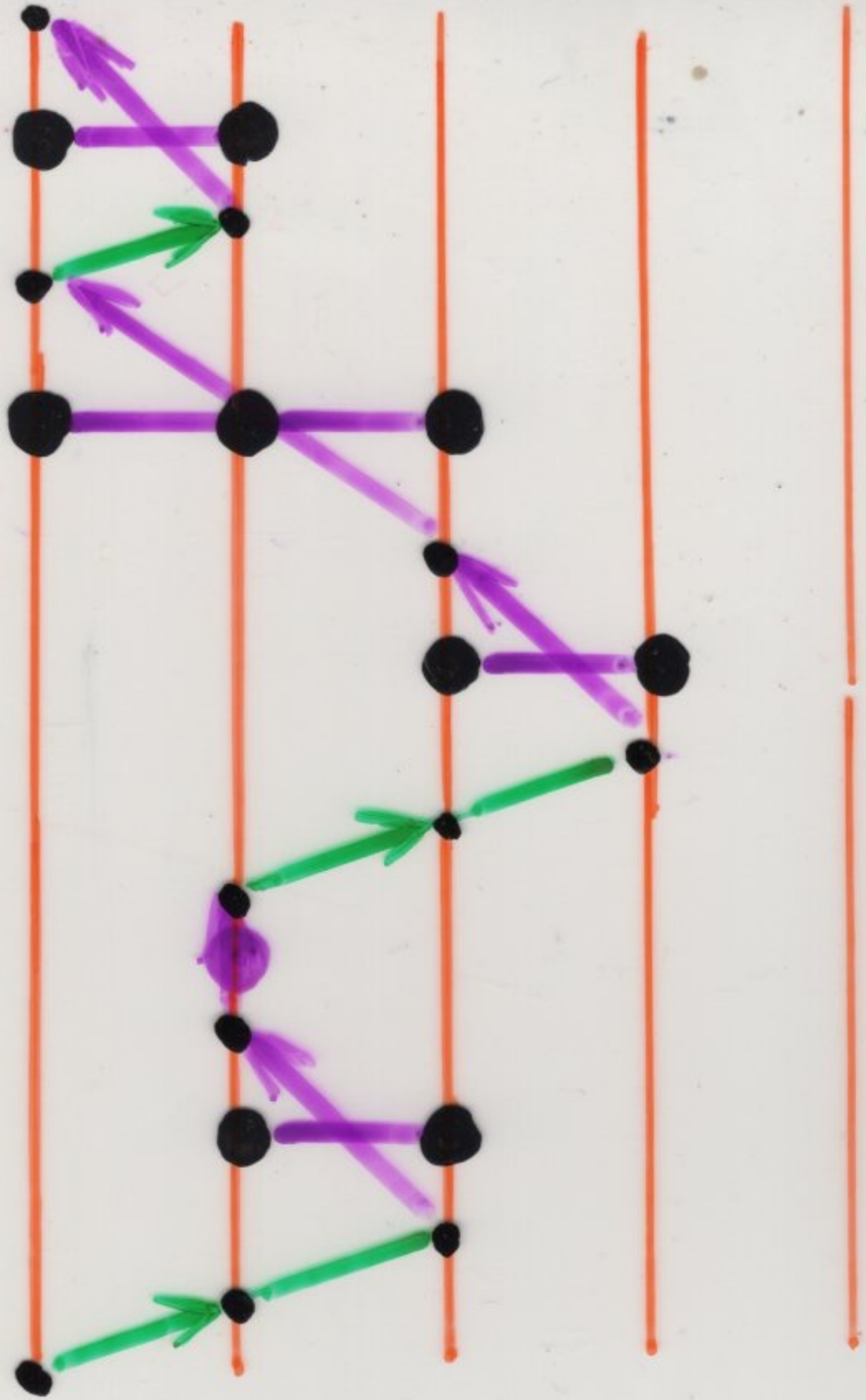
staircase polygons

Dyck paths

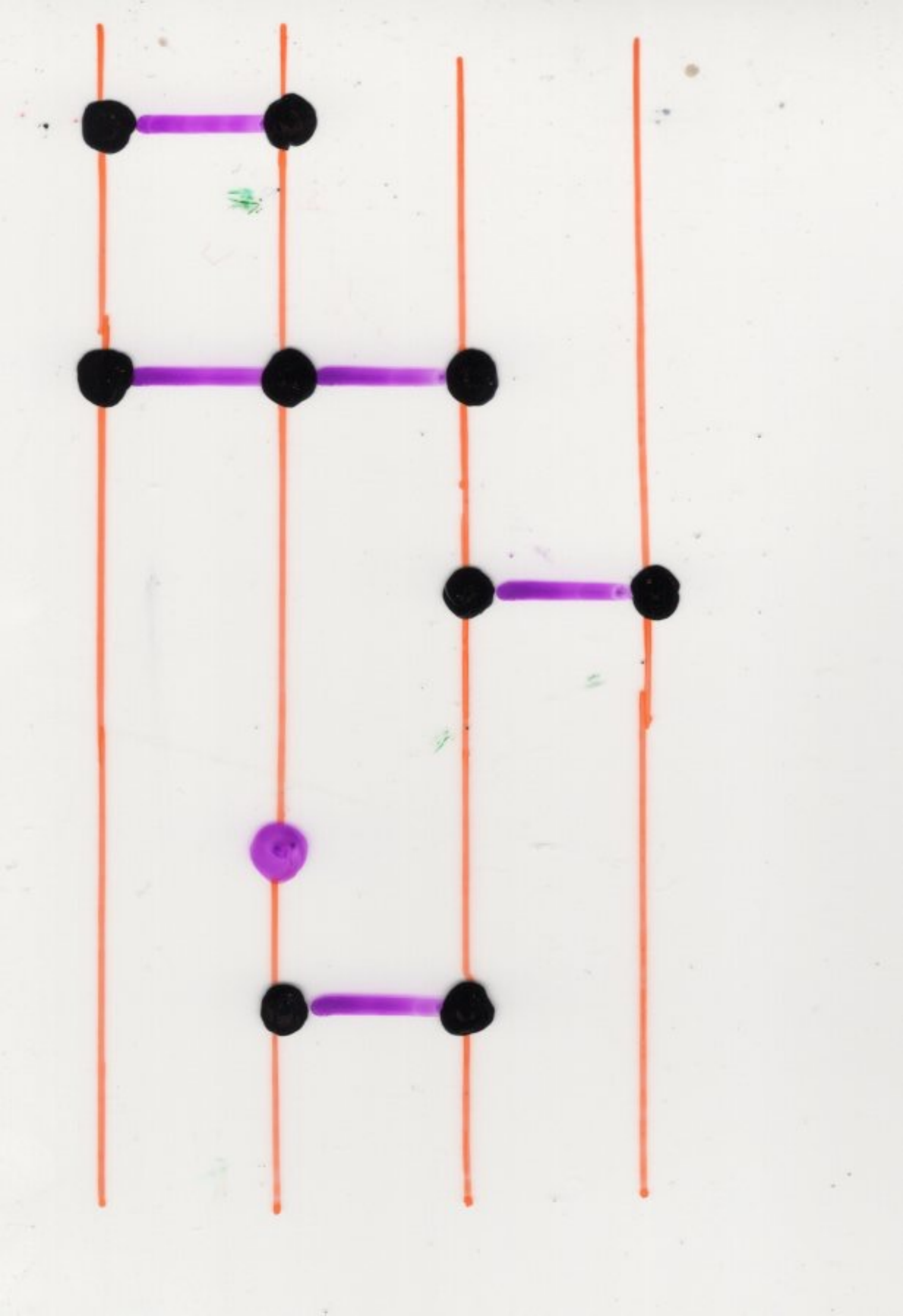
Lukasiewicz paths

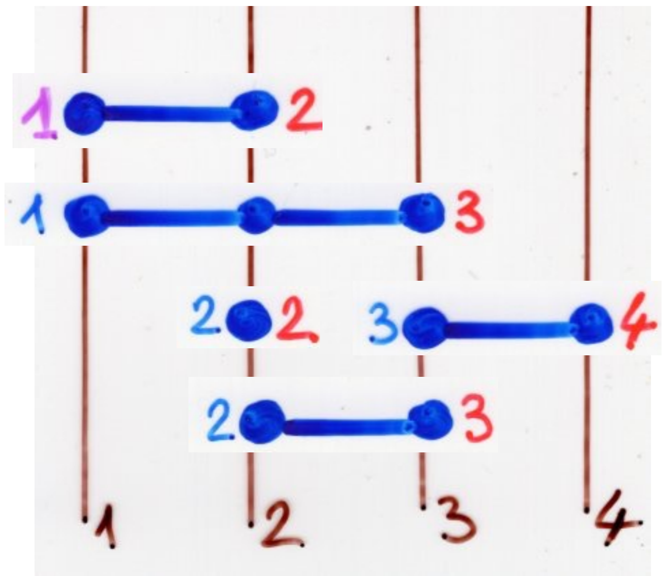
path ω on X $\xleftrightarrow{\chi}$ (η, E)

semi-pyramids of segments



path ω on X $\overset{\chi}{\longleftrightarrow}$ (η, E)




1 2
1 3
2 2
3 4
2 3
1 2 3 4

a festival of bijections

parallelogram polyominoes

(staircase polygons)

Dyck paths

semi-pyramids of dimers (on $\mathbb{N}$)

(reverse) Lukasiewicz paths

stairs decomposition

χ

semi-pyramids of segments (on $\mathbb{N}$)

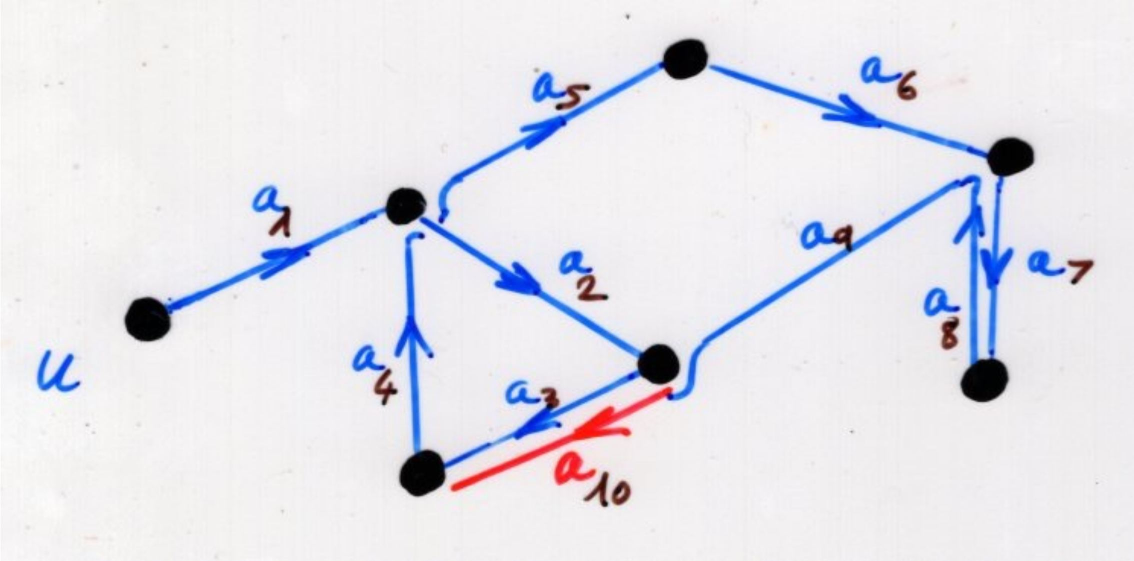
exercise Ch 6a, p 59

other description of the bijection:

3. with the bijection

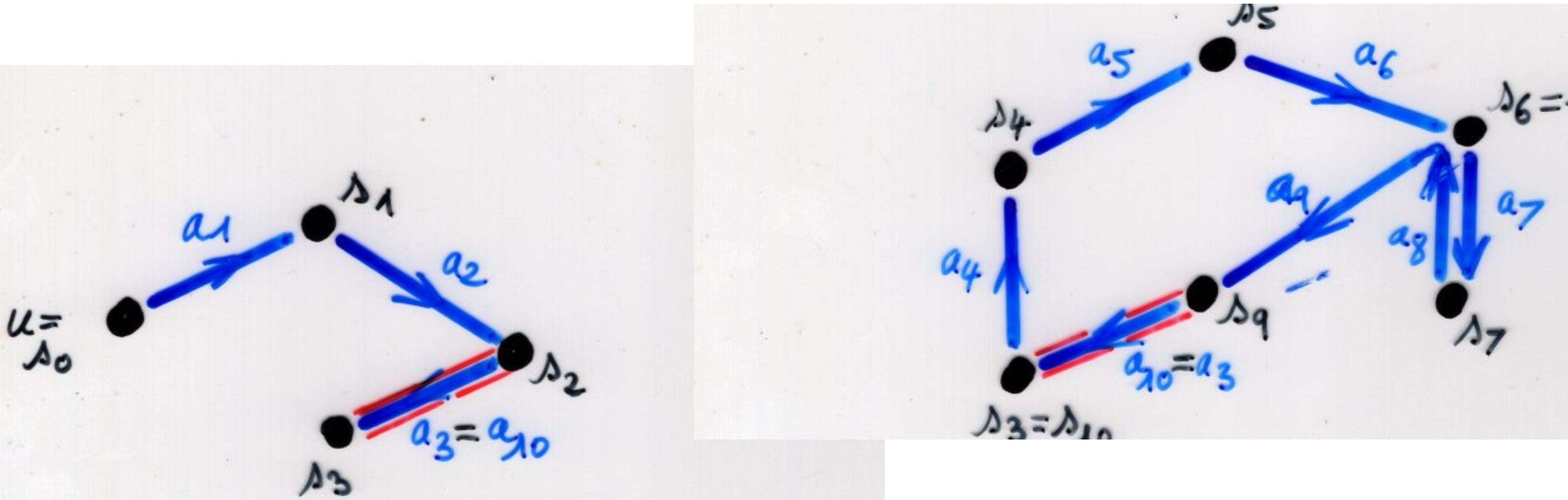
(paths — heaps of oriented loops)

$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$

Ch 5b, p21-29



bijections

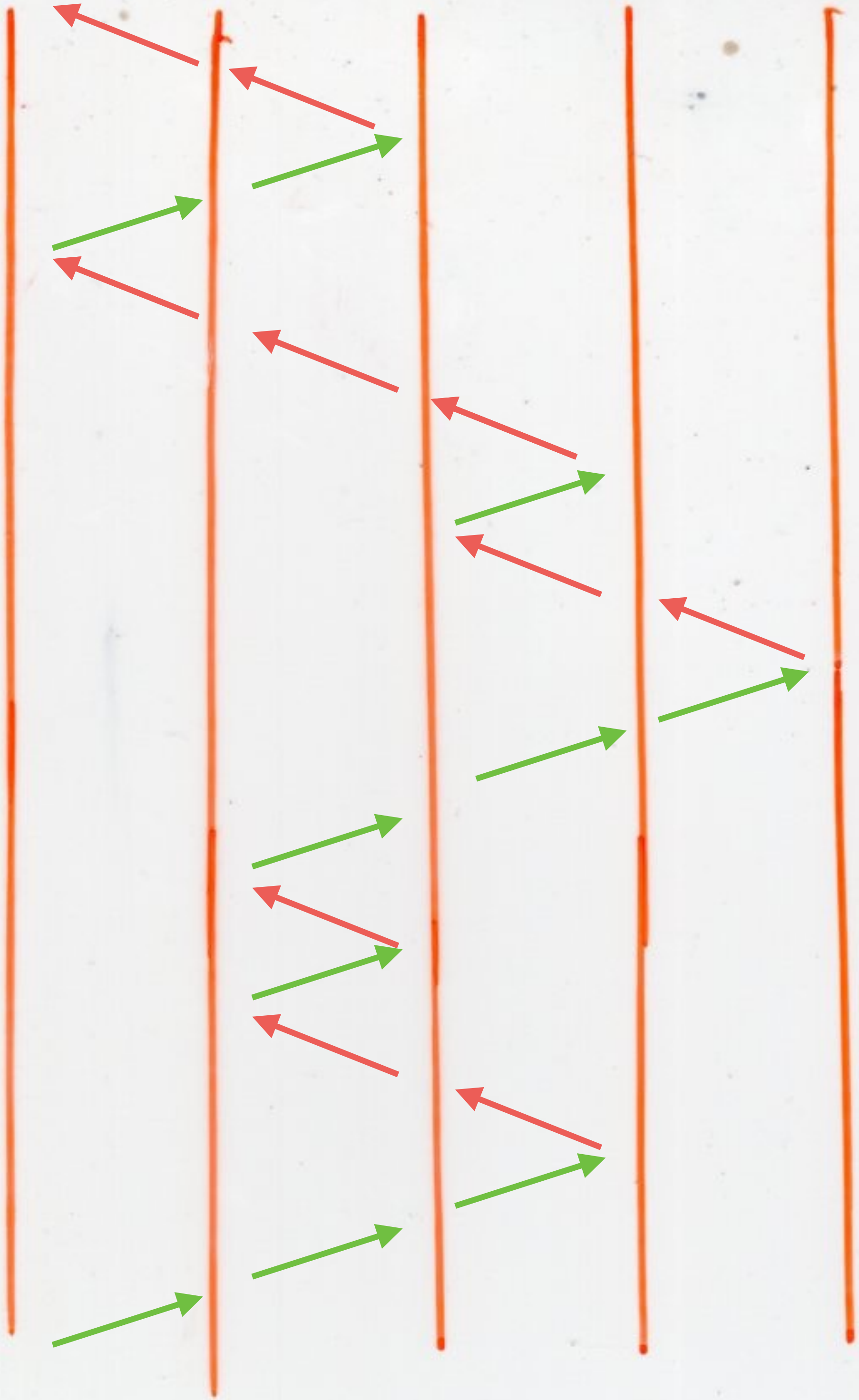
staircase polygons

Dyck paths

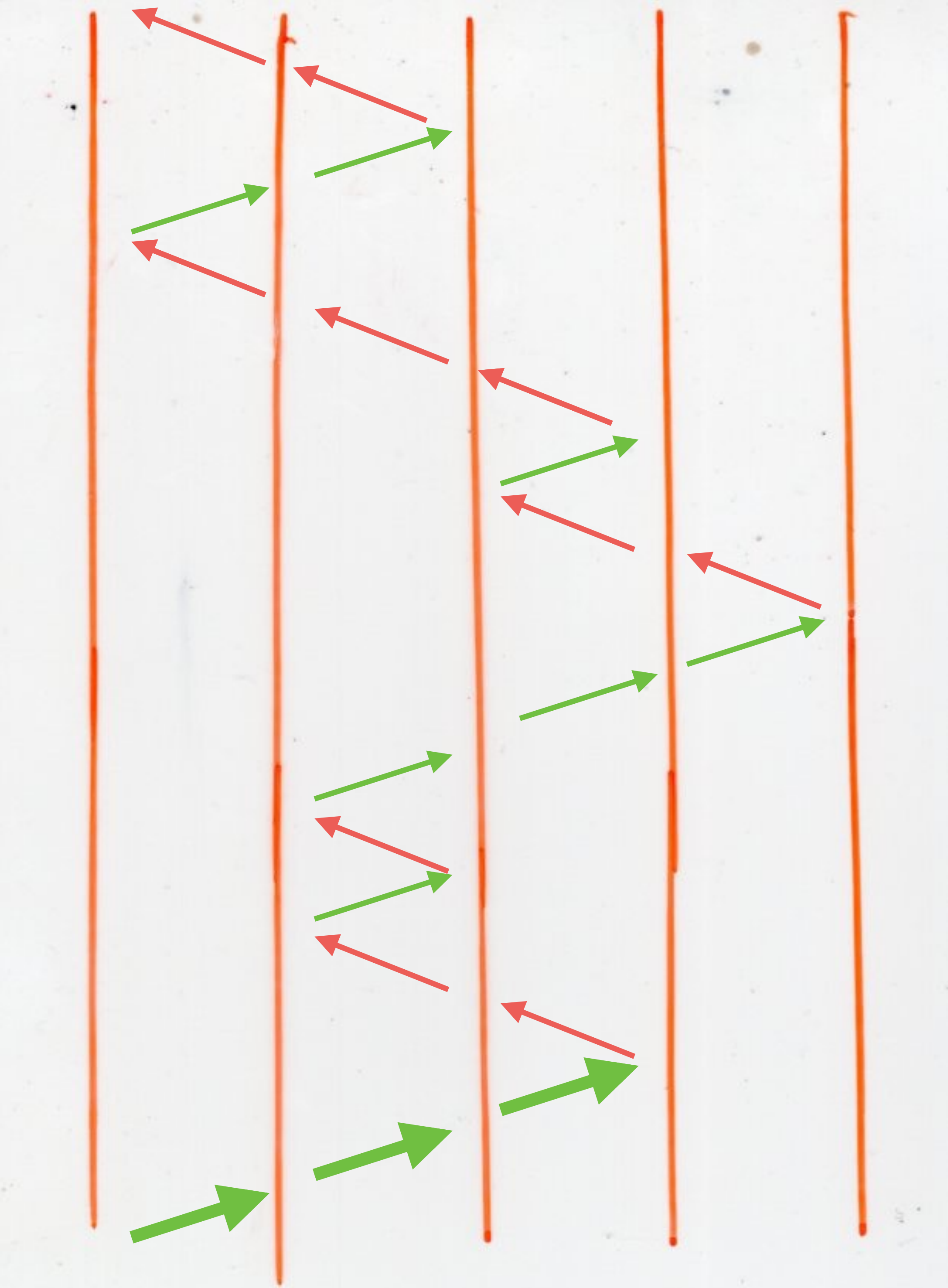
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$

Ch 5b , p 21-29

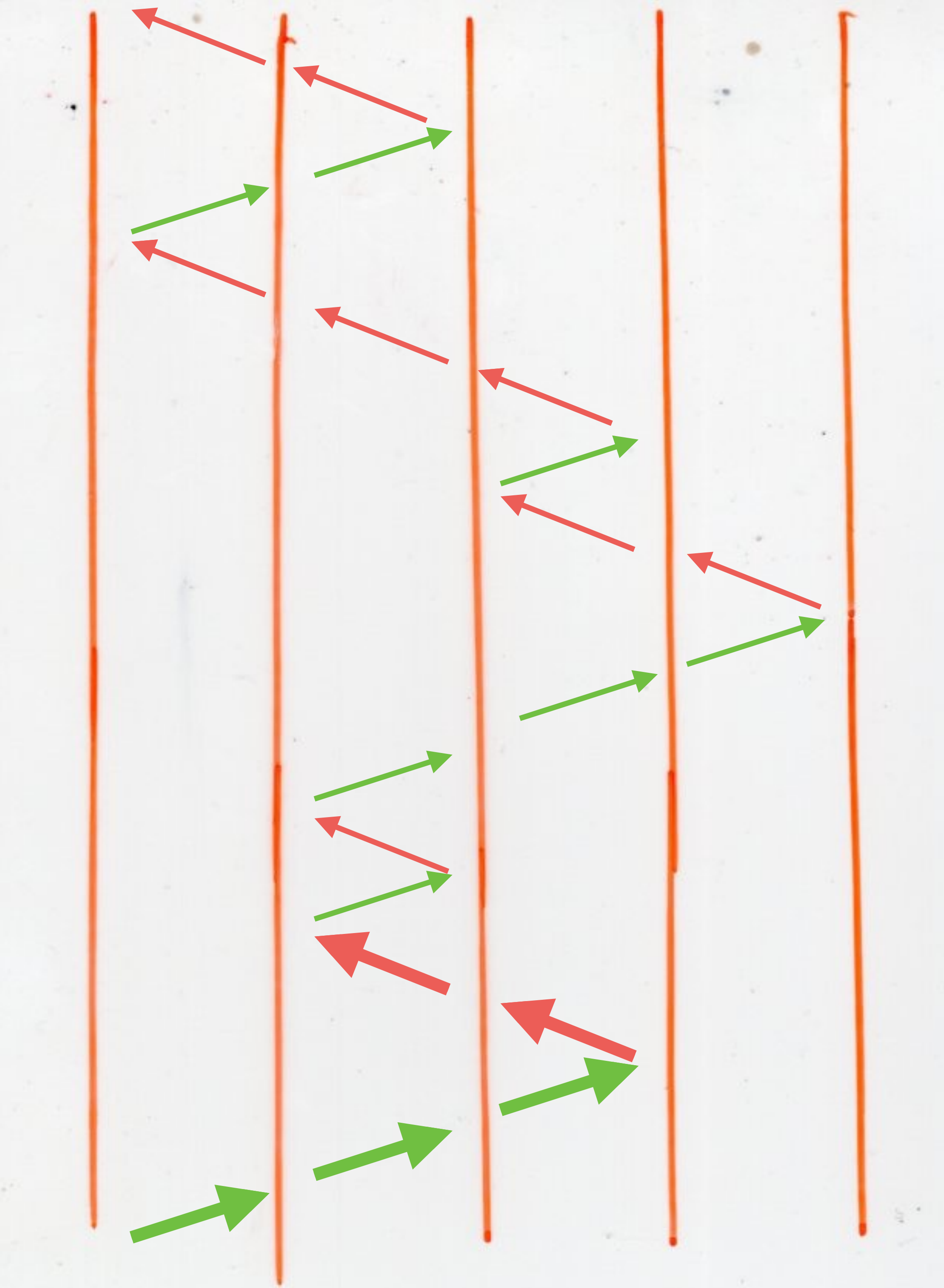
heaps of oriented loops



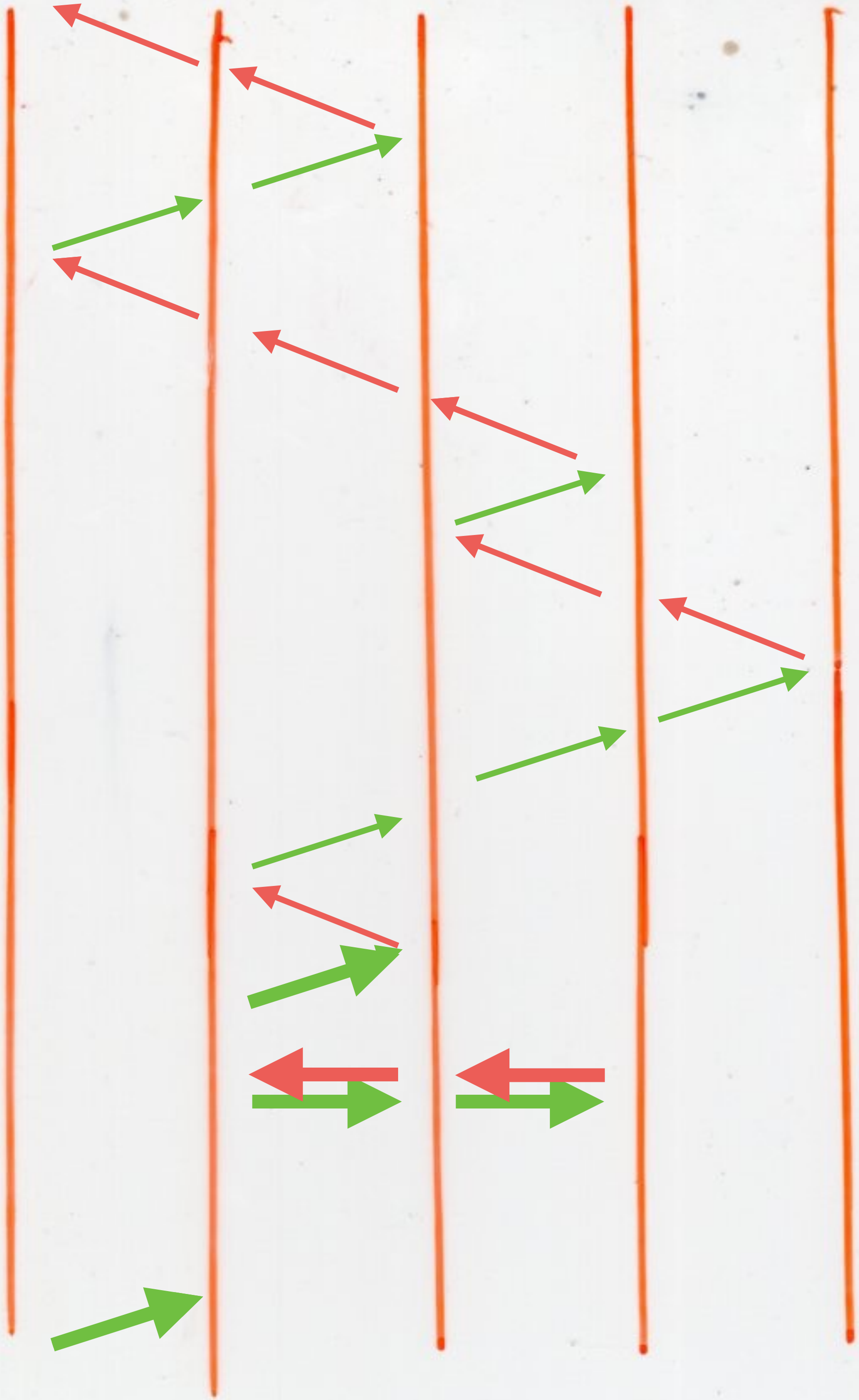
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



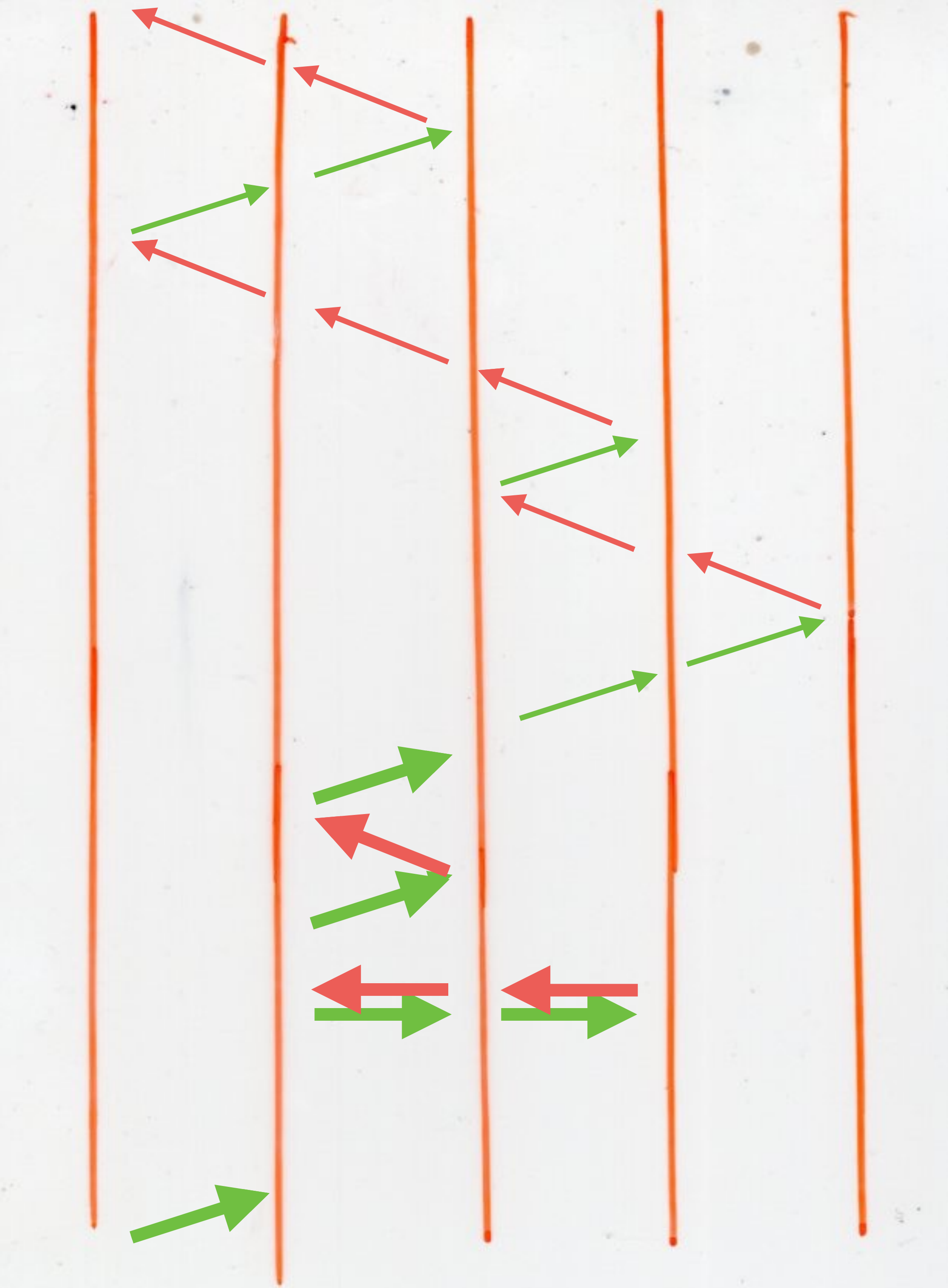
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



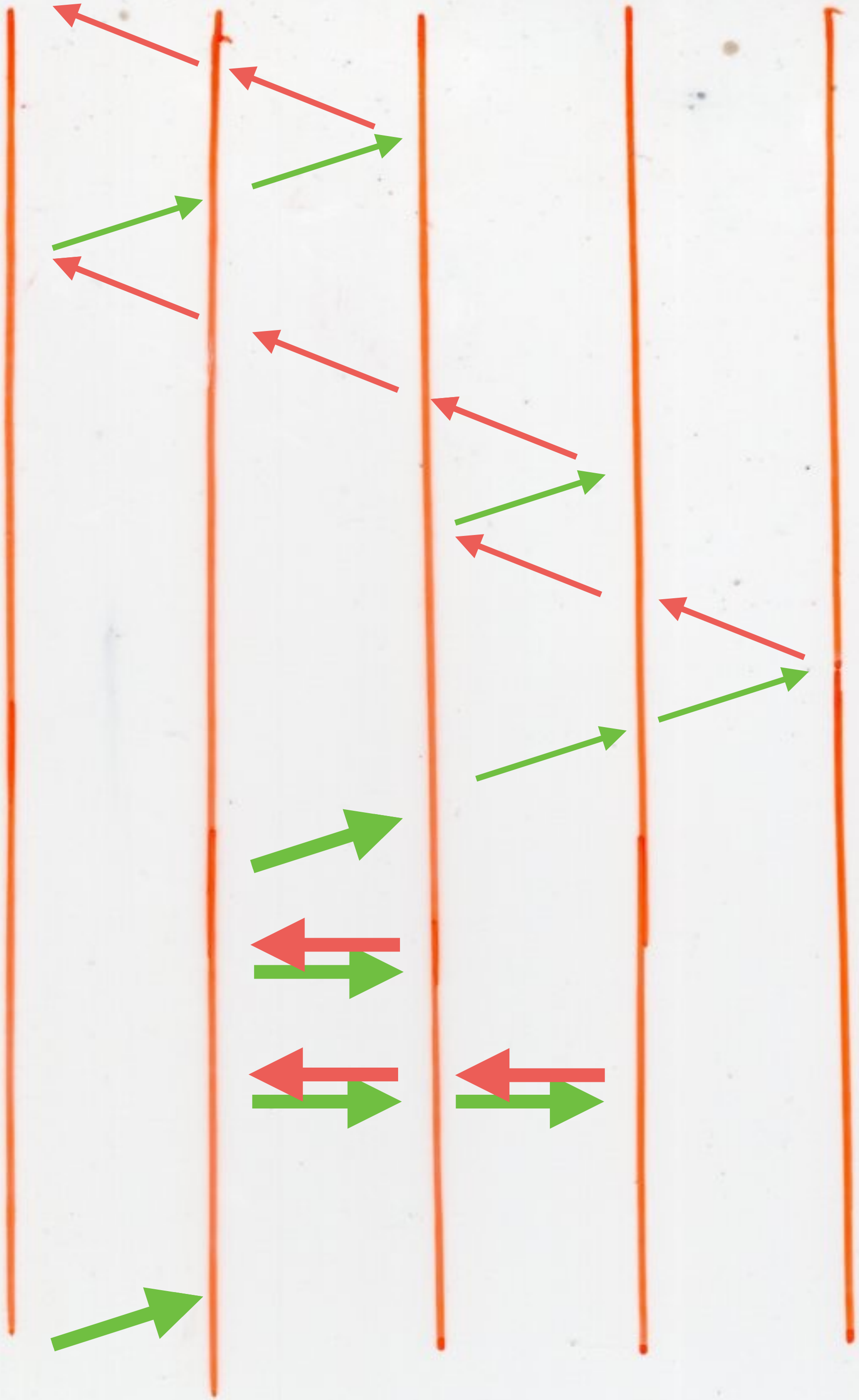
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



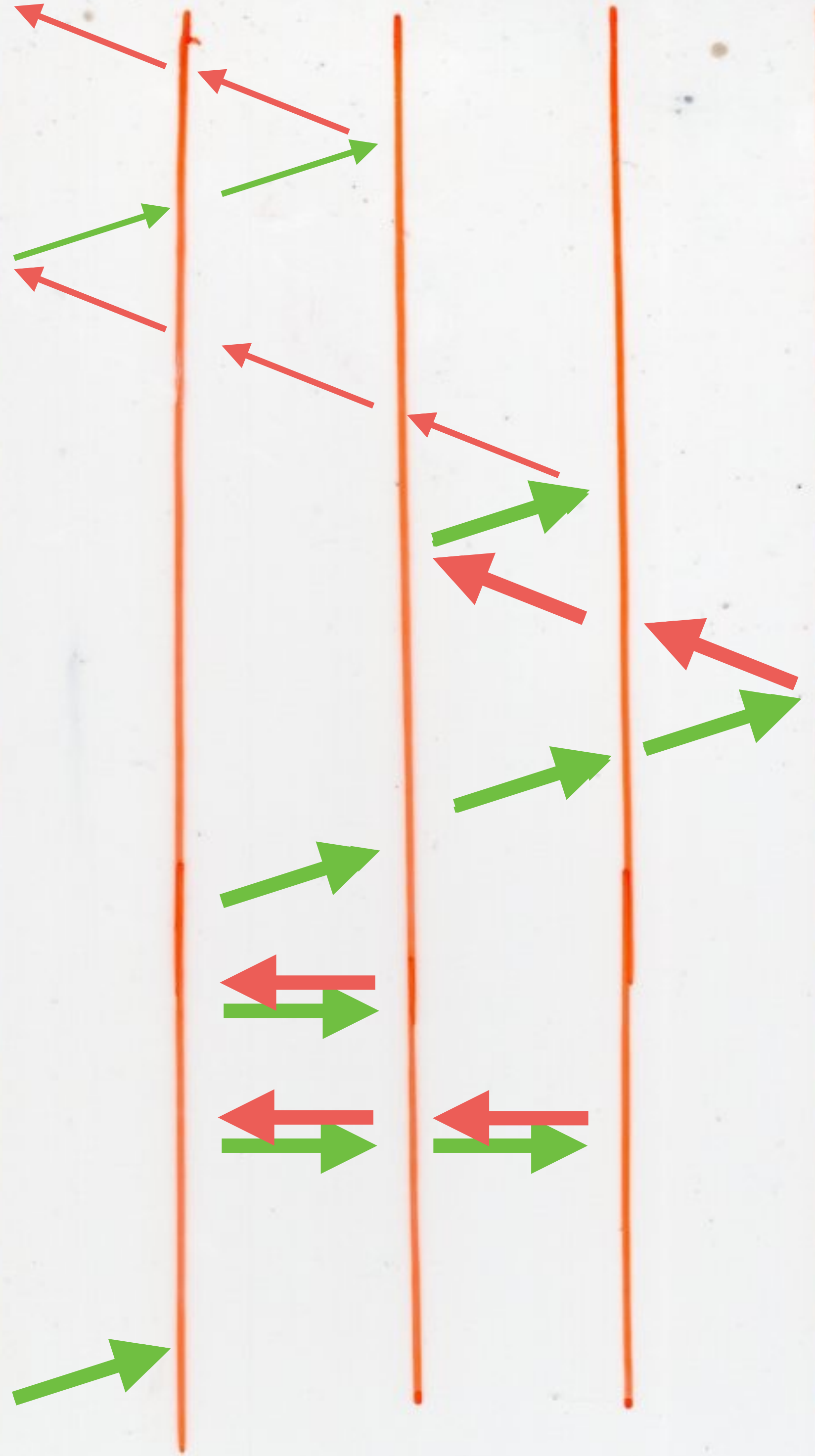
$$\underset{u\rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



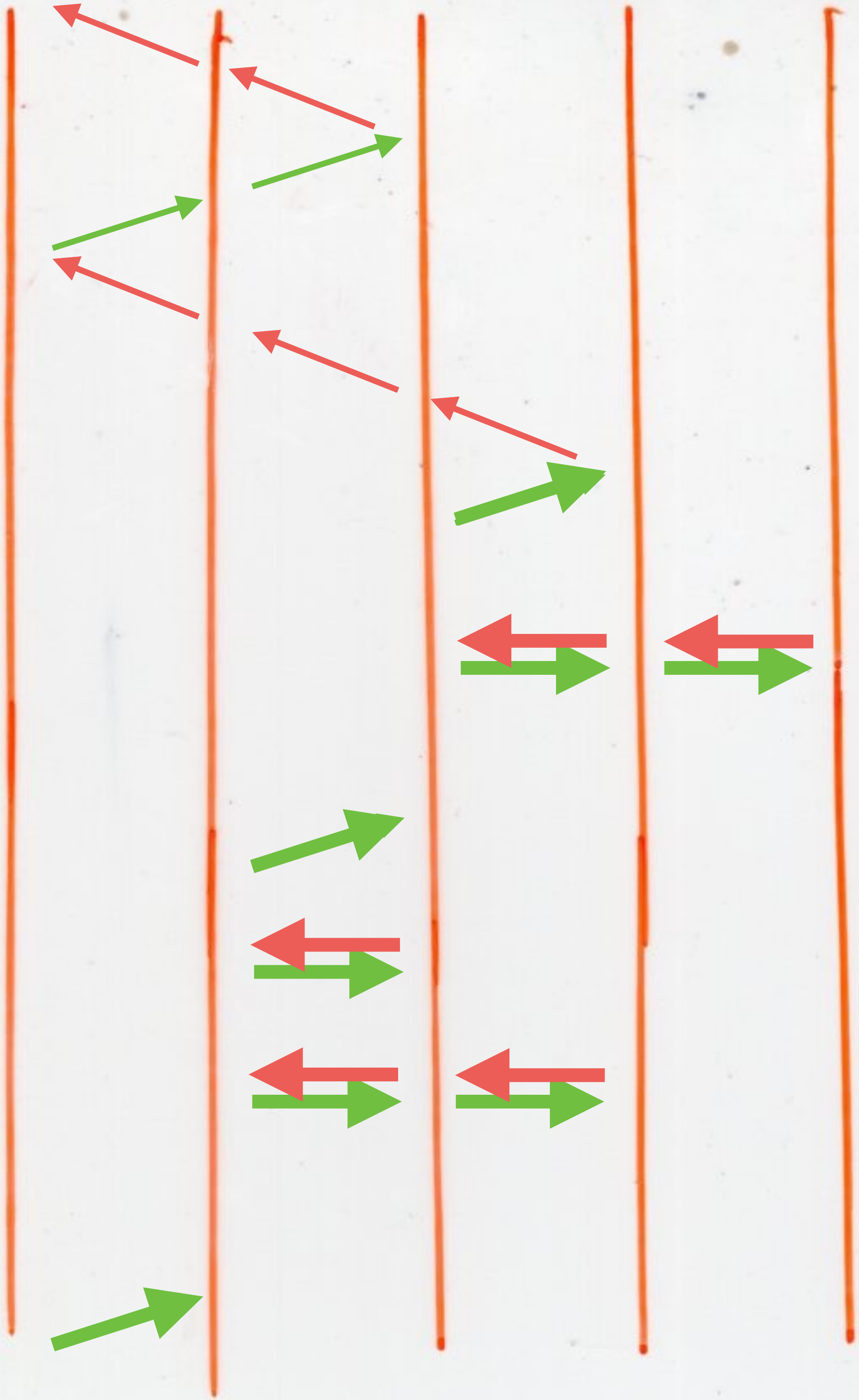
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



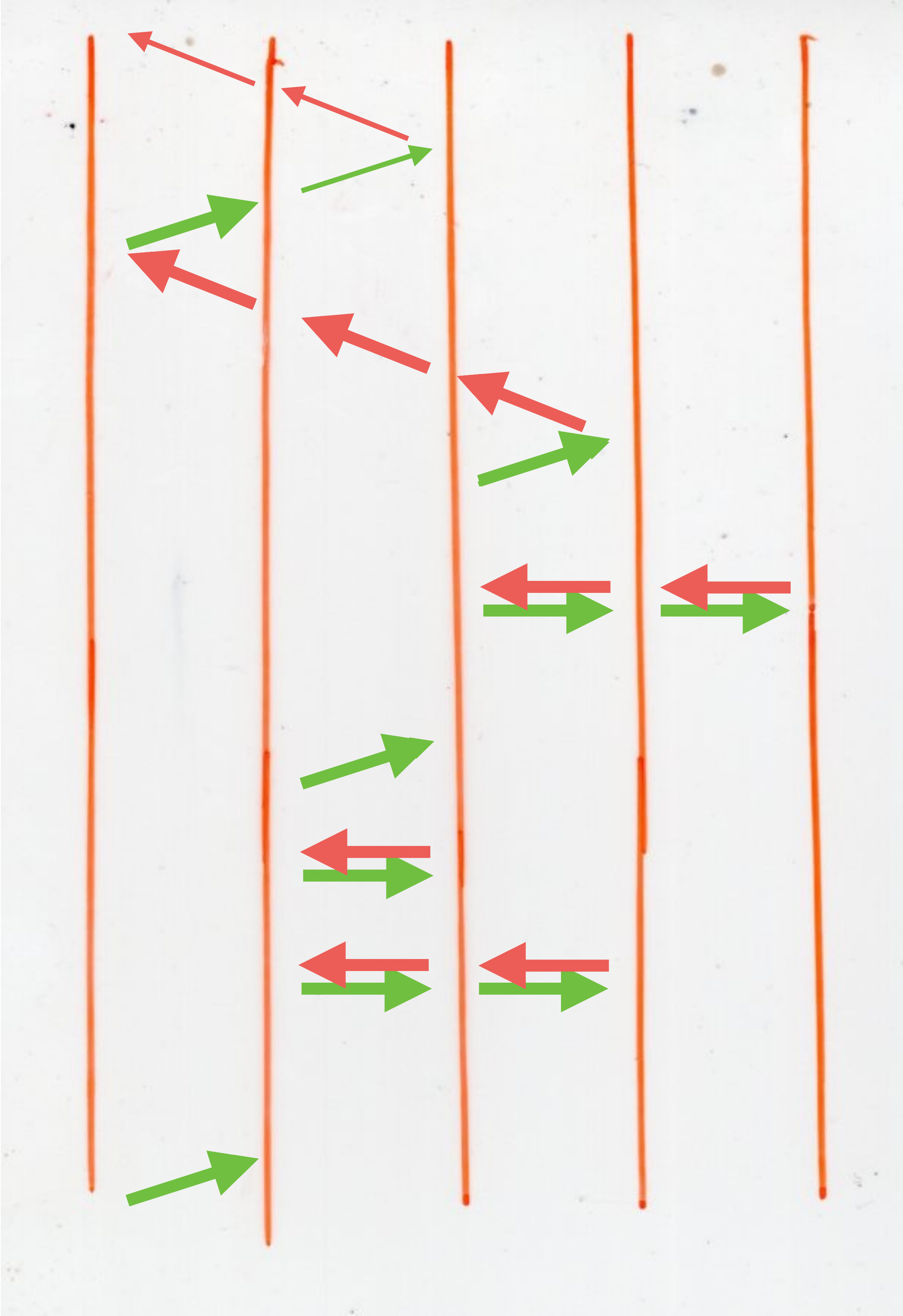
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



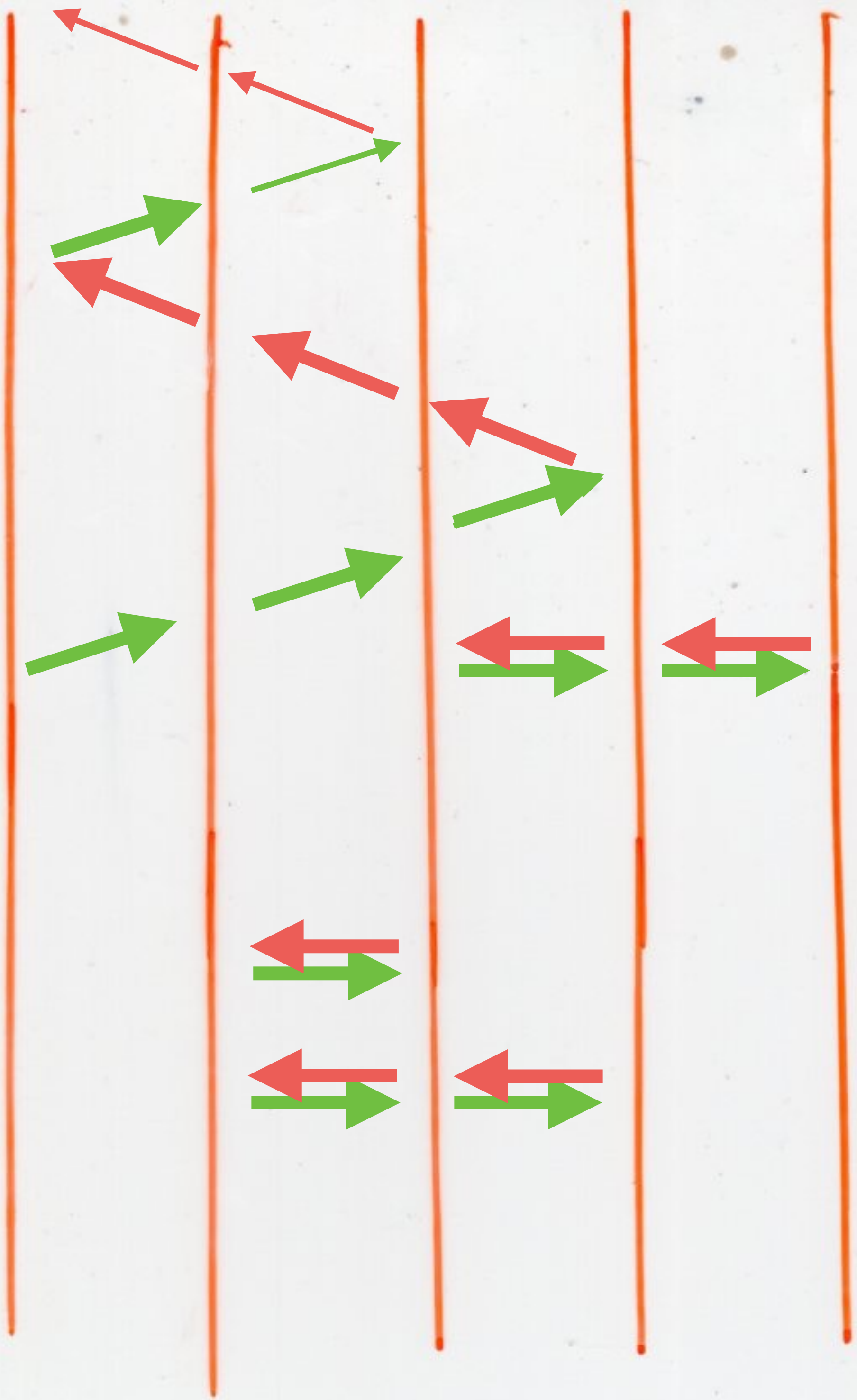
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



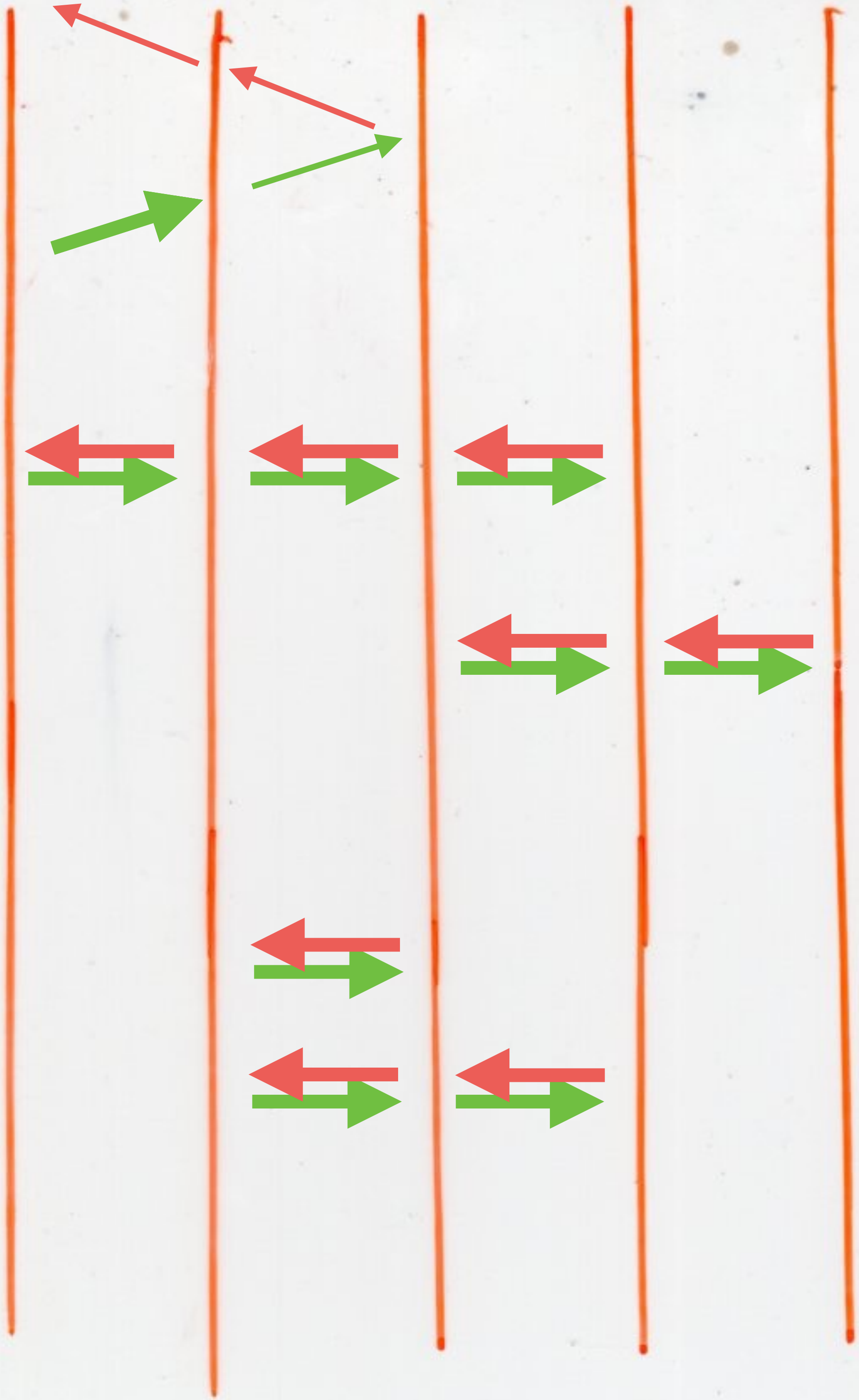
$$\underset{u\rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



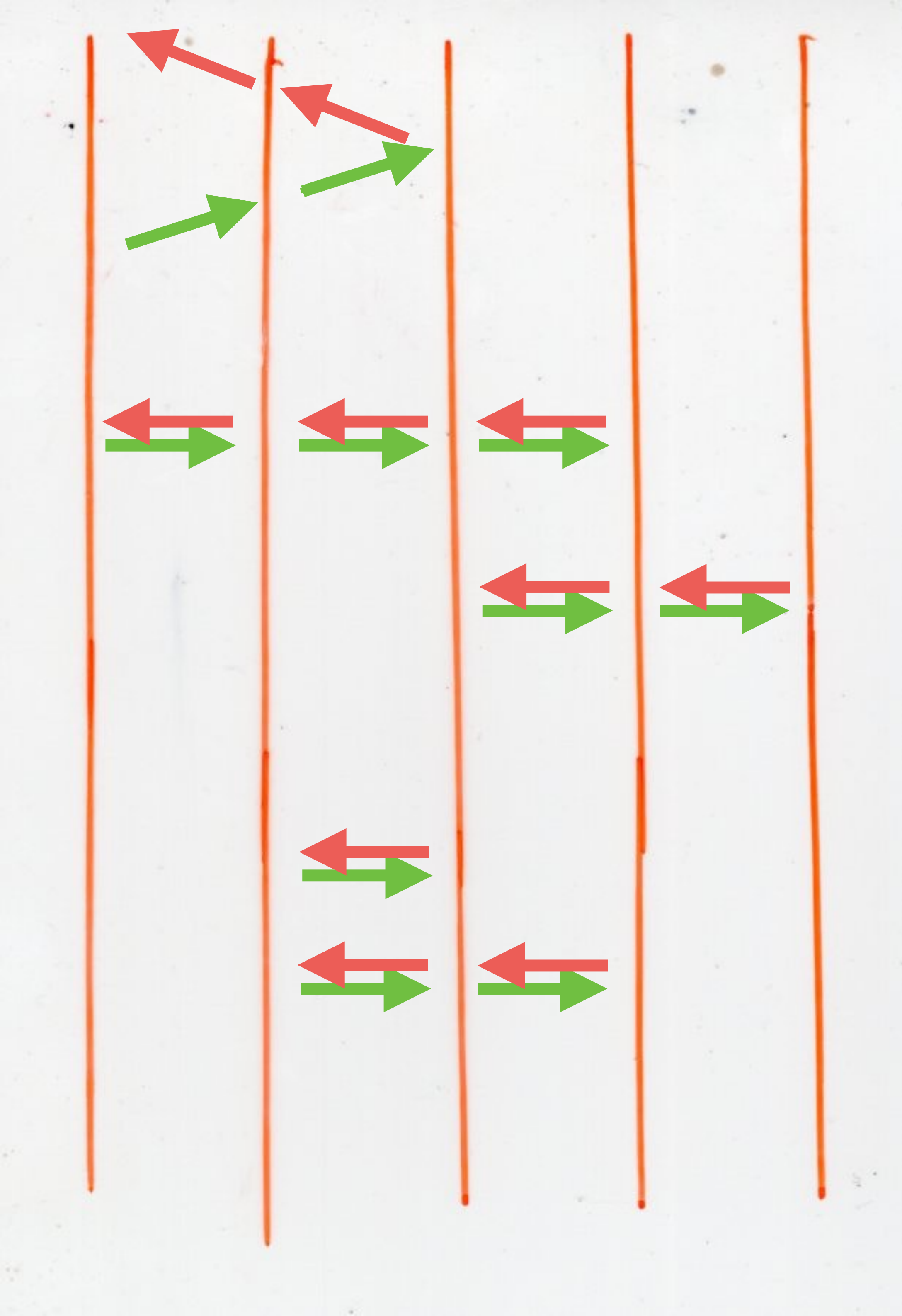
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$

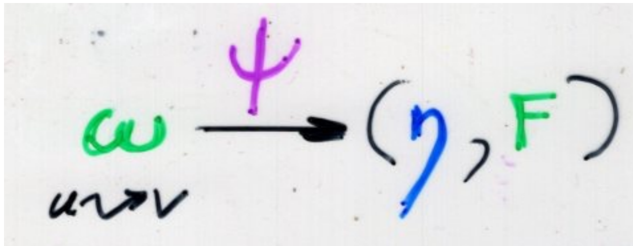


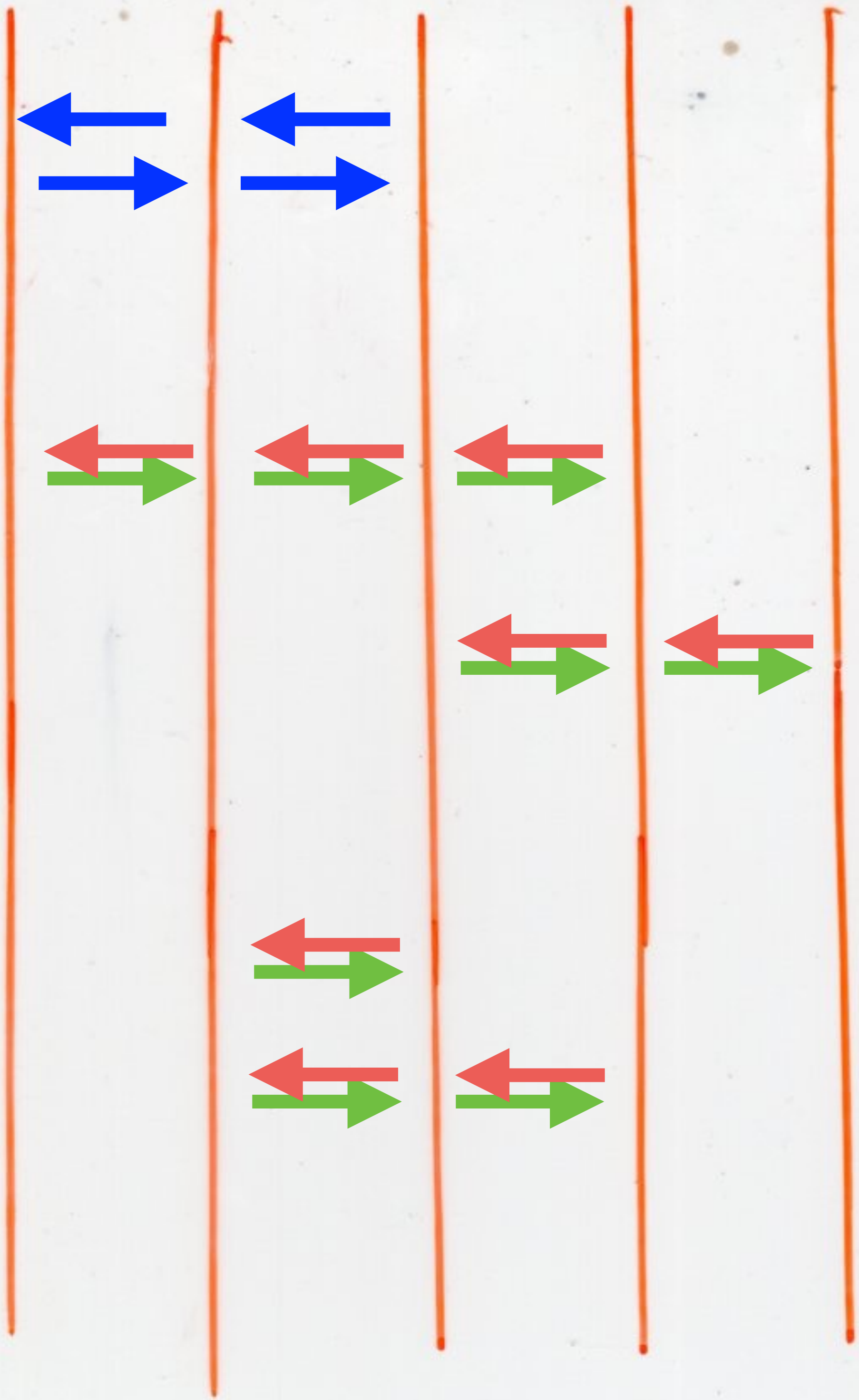
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



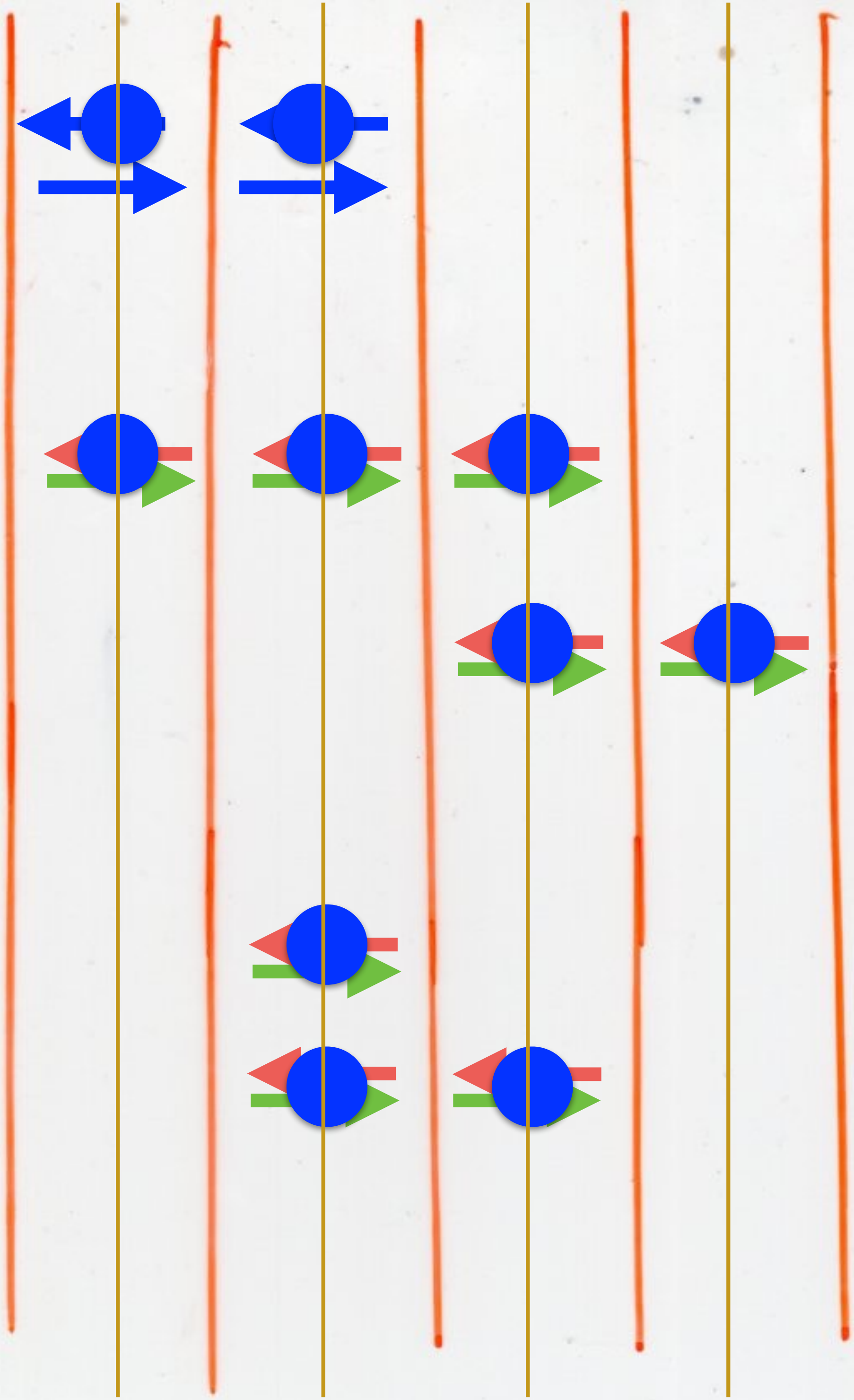
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



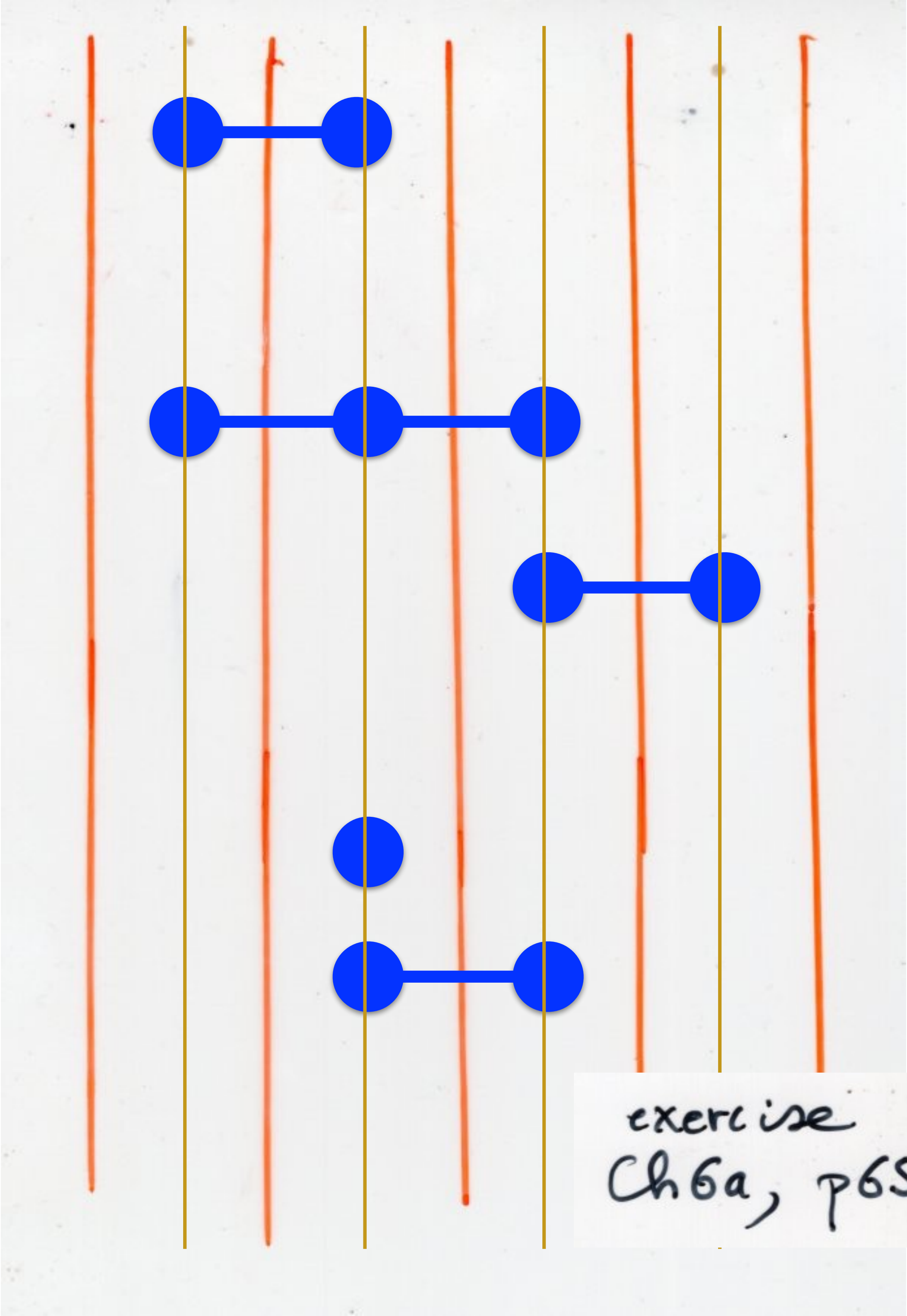

$\omega \xrightarrow{\psi} (\eta, F)$
$u \rightsquigarrow v$



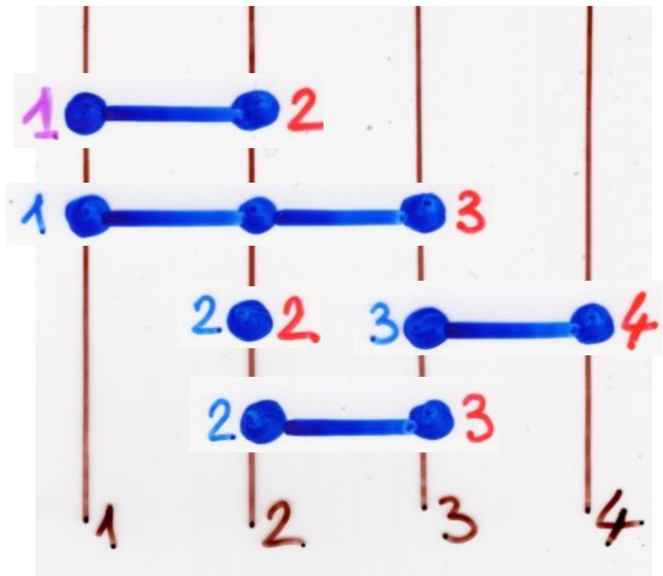
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



exercise
Ch6a, p65



a festival of bijections