Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics,
a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

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CNRS, LaBRI, Bordeaux

www.xavierviennot.org

Chapter 7

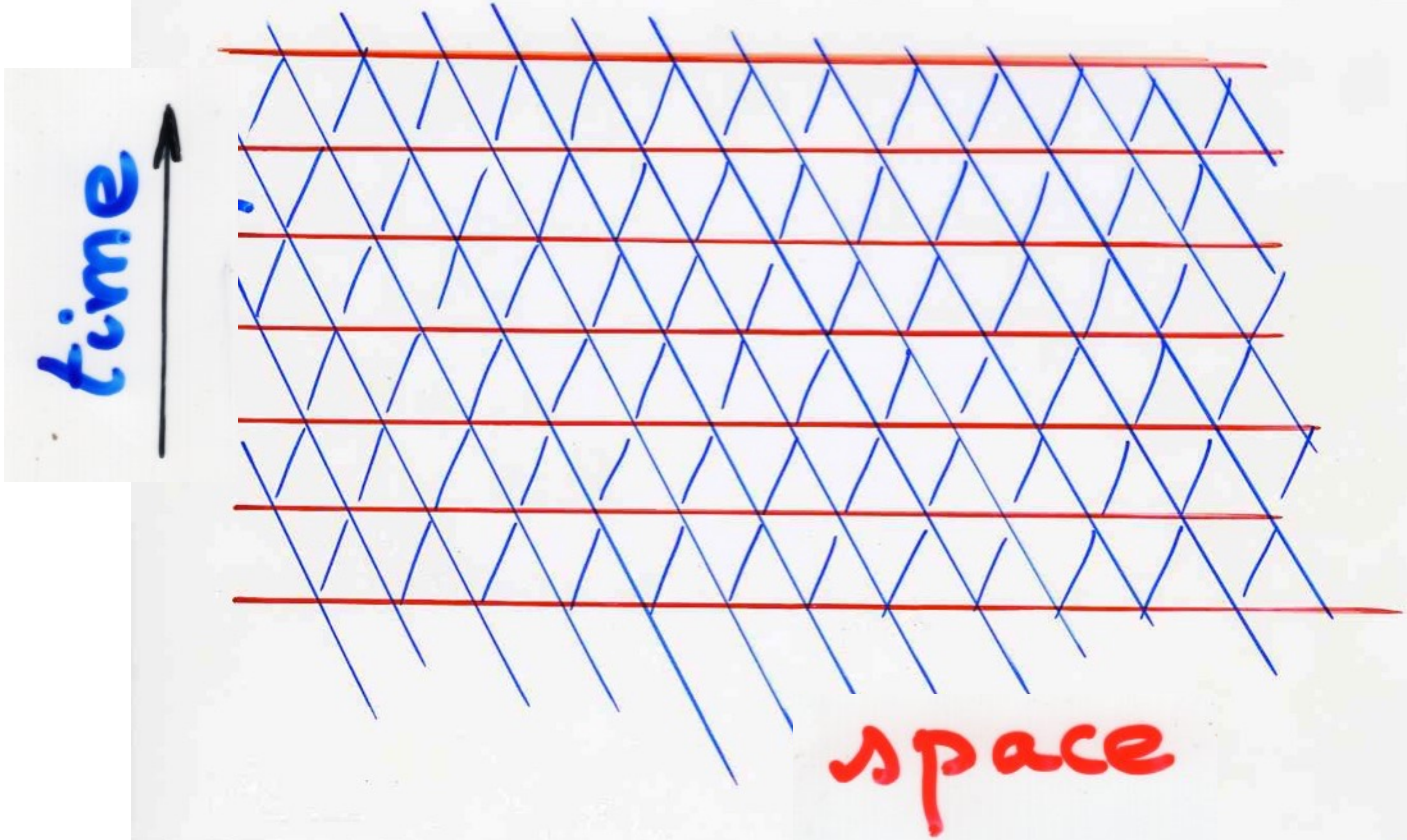
Heaps in statistical mechanics (2)

2nd part of the slides

IMSc, Chennai

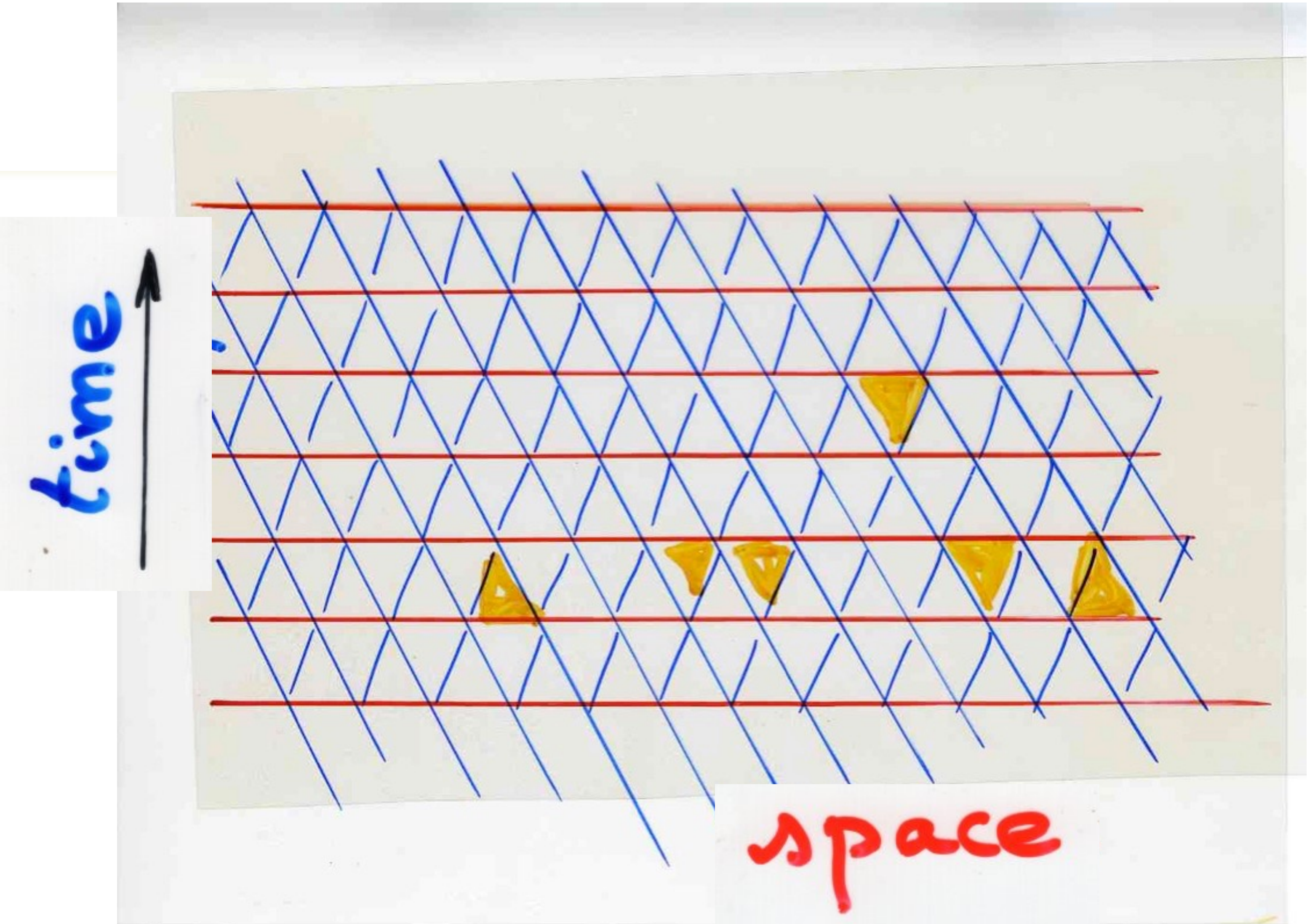
13 March 2017

2D Lorentzian triangulation


time
space

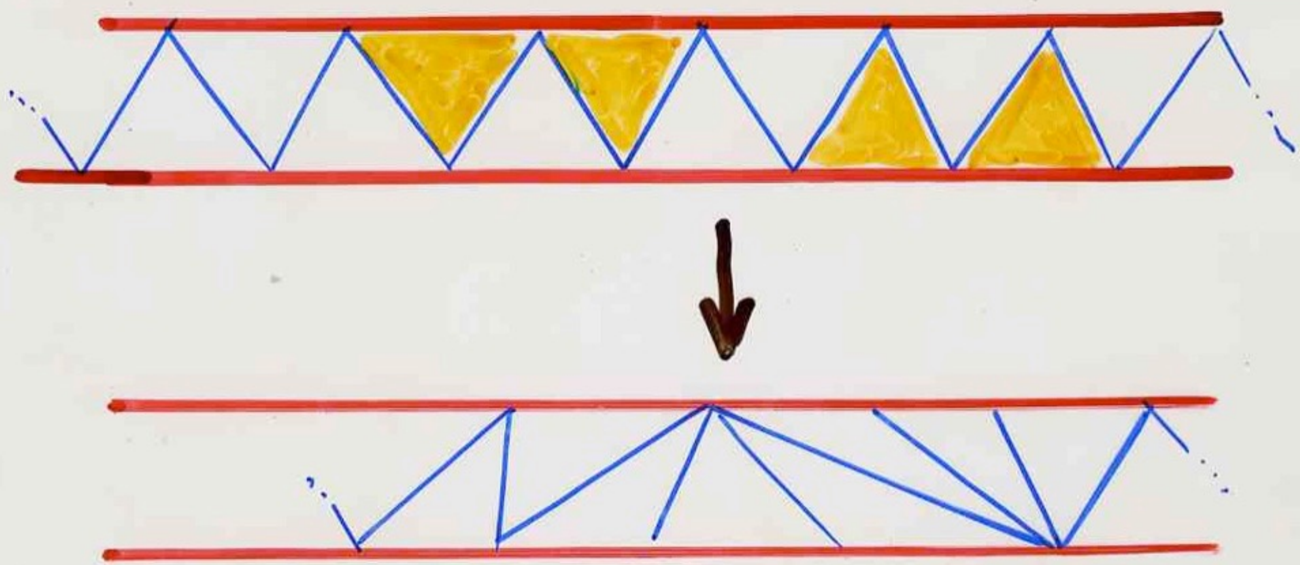
• "quantum fluctuations" of the space-time

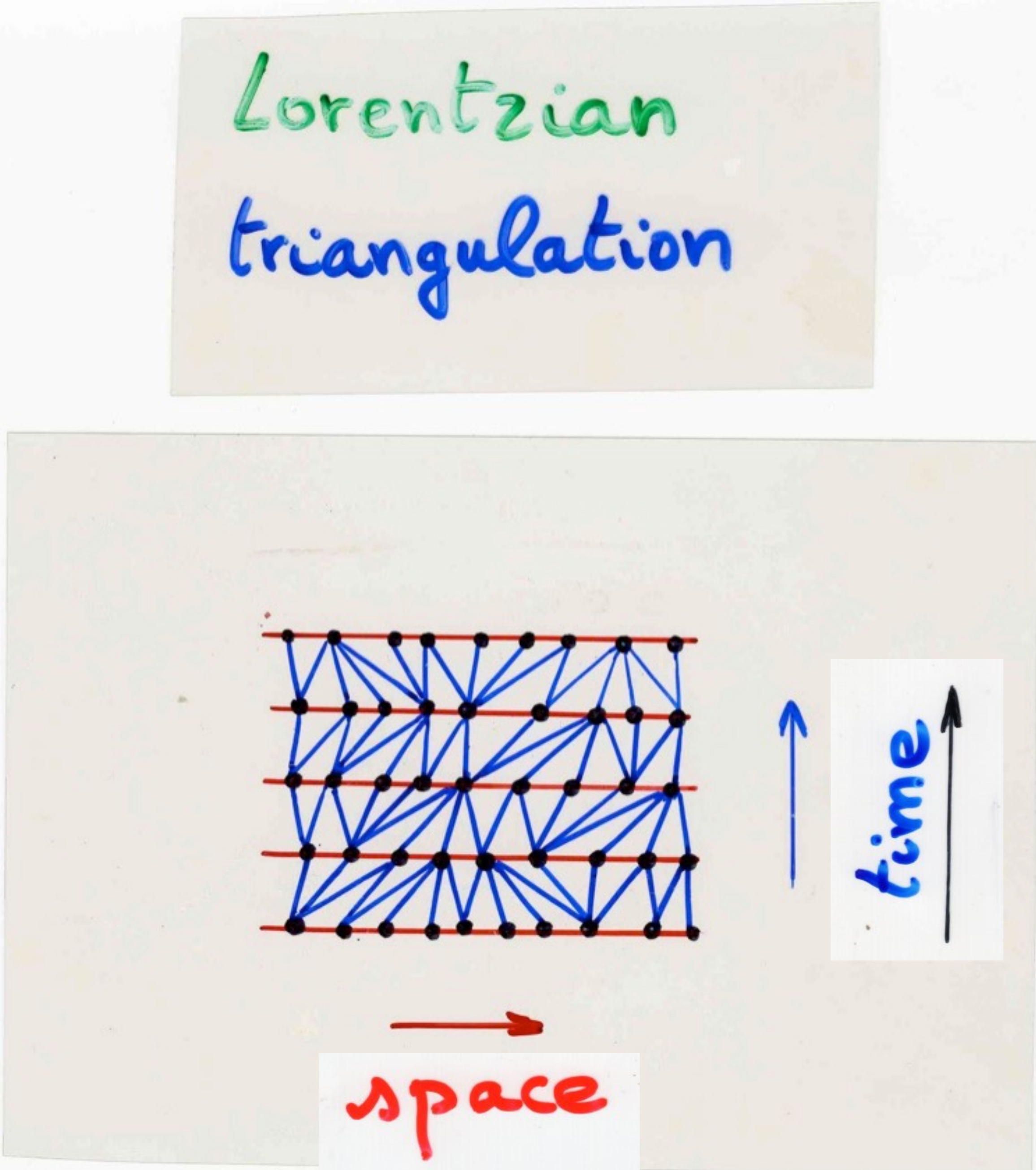
"quantum geometry"



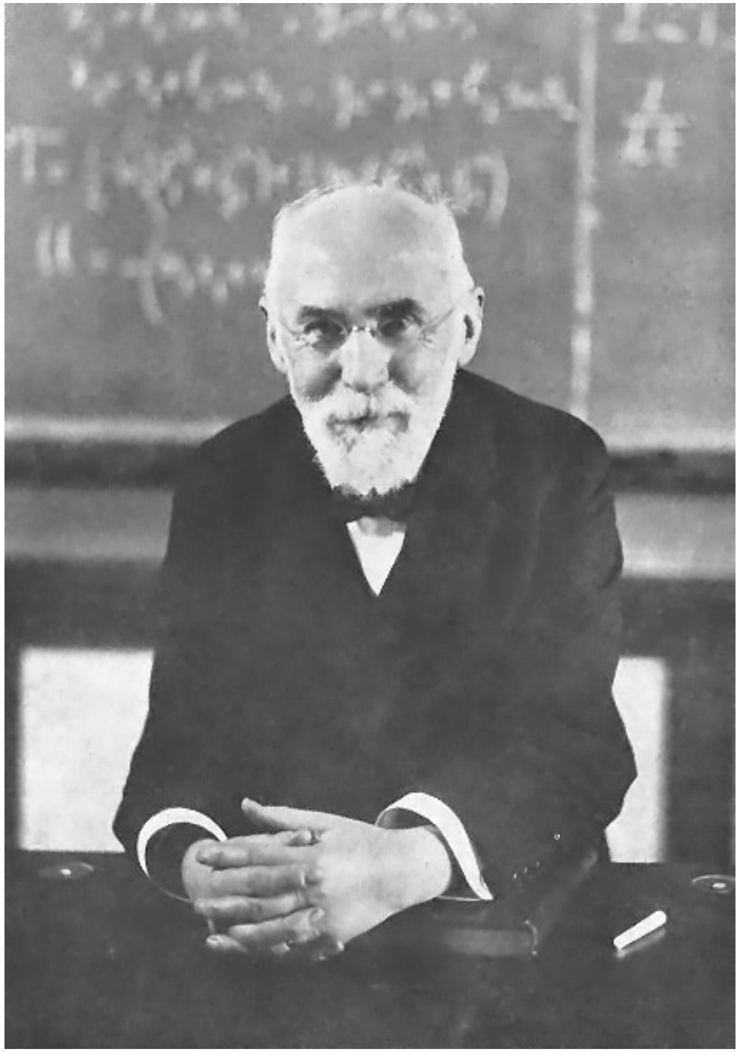
• "quantum fluctuations" of the space-time

"quantum geometry"





2D Lorentzian quantum gravity



Lorentzian
quantum
gravity

$\underbrace{+++}_{\text{space}}\underbrace{-}_{\text{time}}$

2D

$\underbrace{1}_{\text{space}} + \underbrace{1}_{\text{time}}$

euclidian

+ + + +

space time

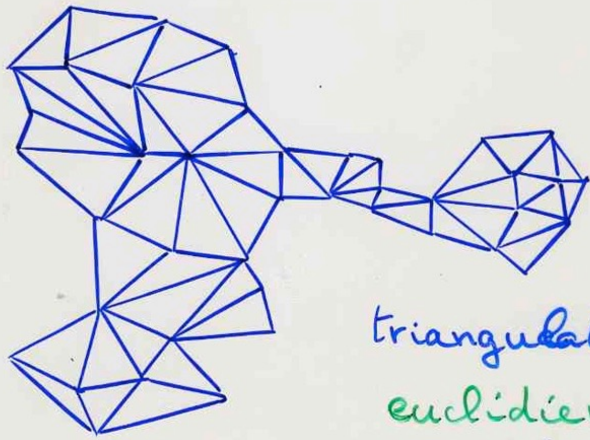
+ + + −

space time

Lorentzian

quantum

gravity



triangulation

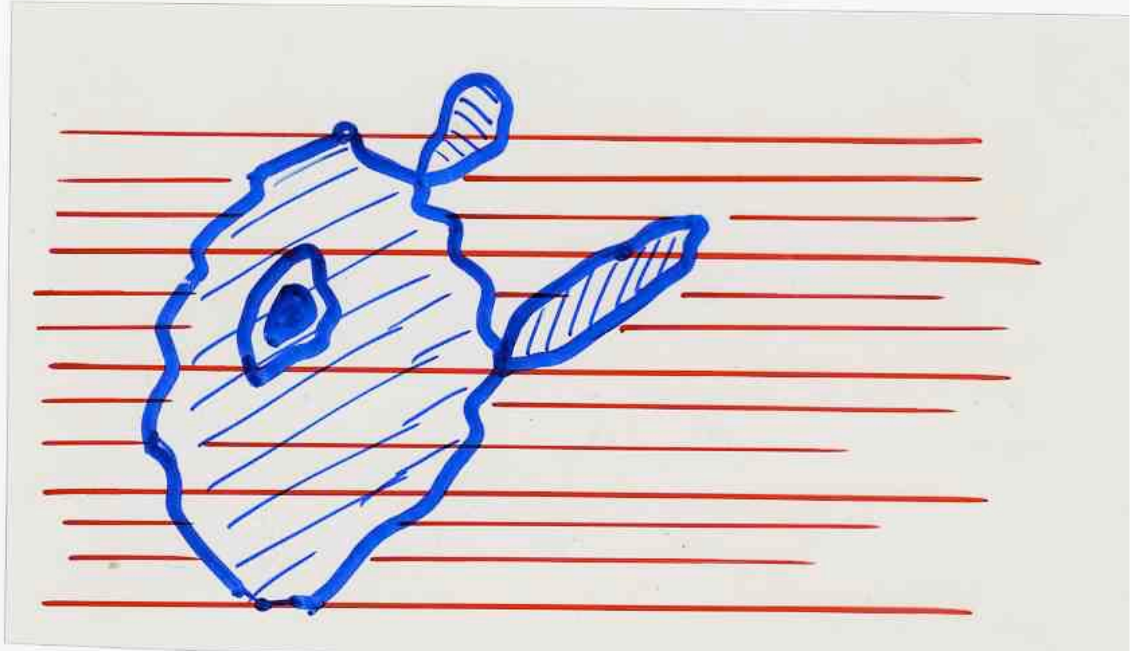
euclidienne

+ +

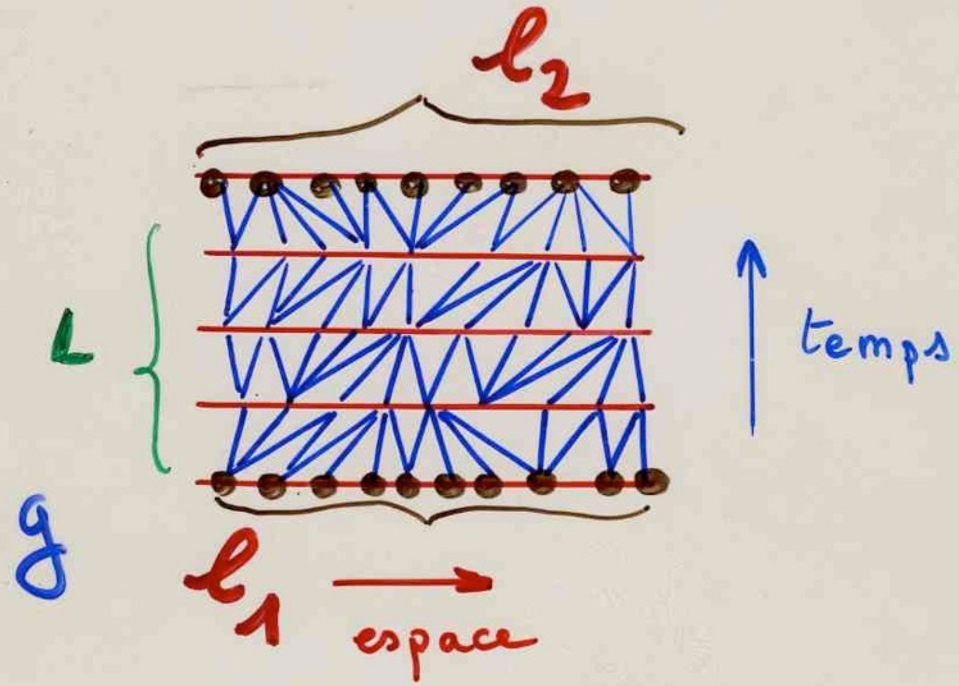
2D

1 + 1

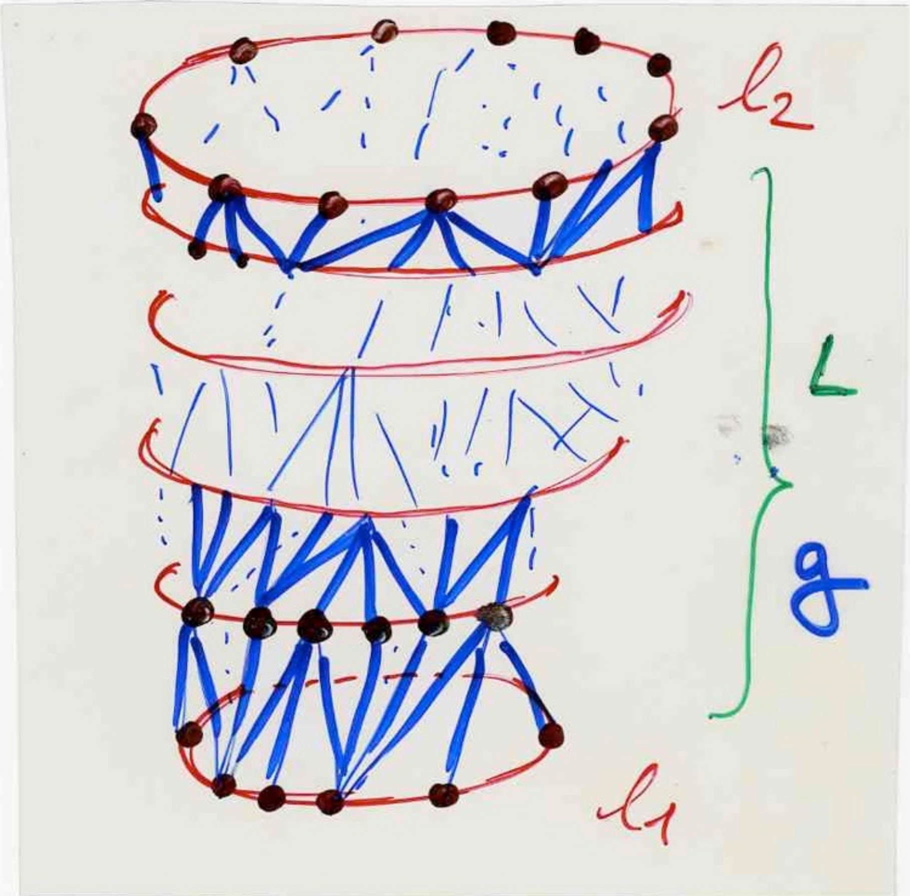
space time



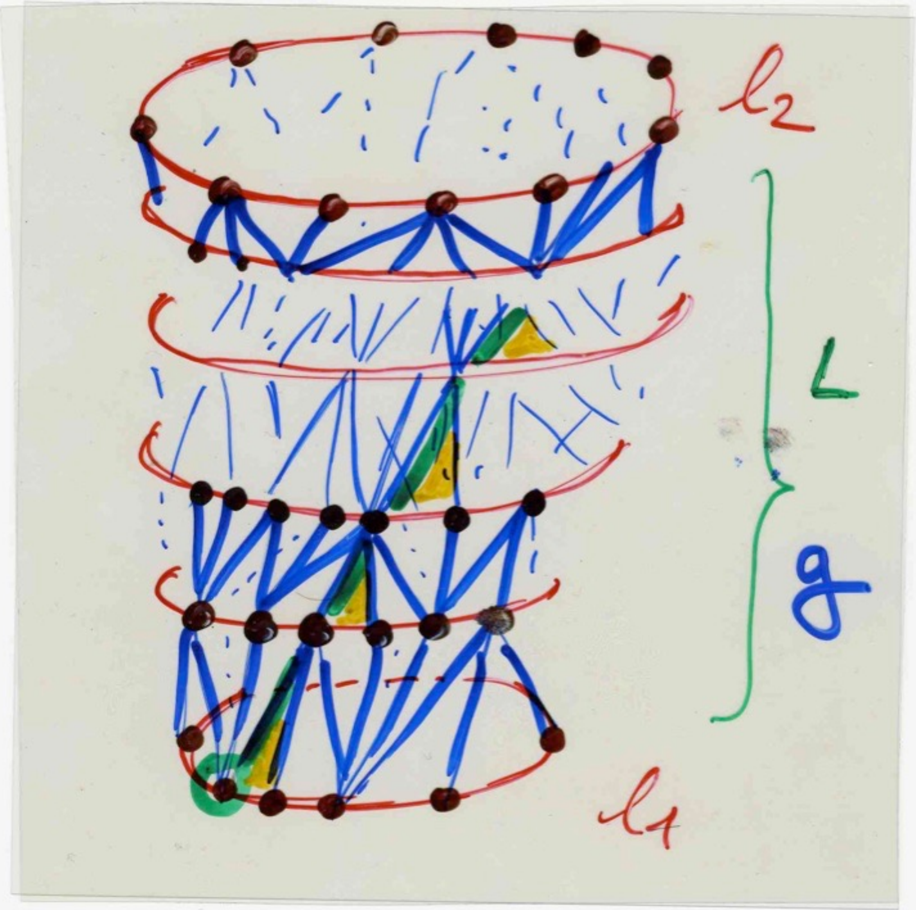
no baby
universe



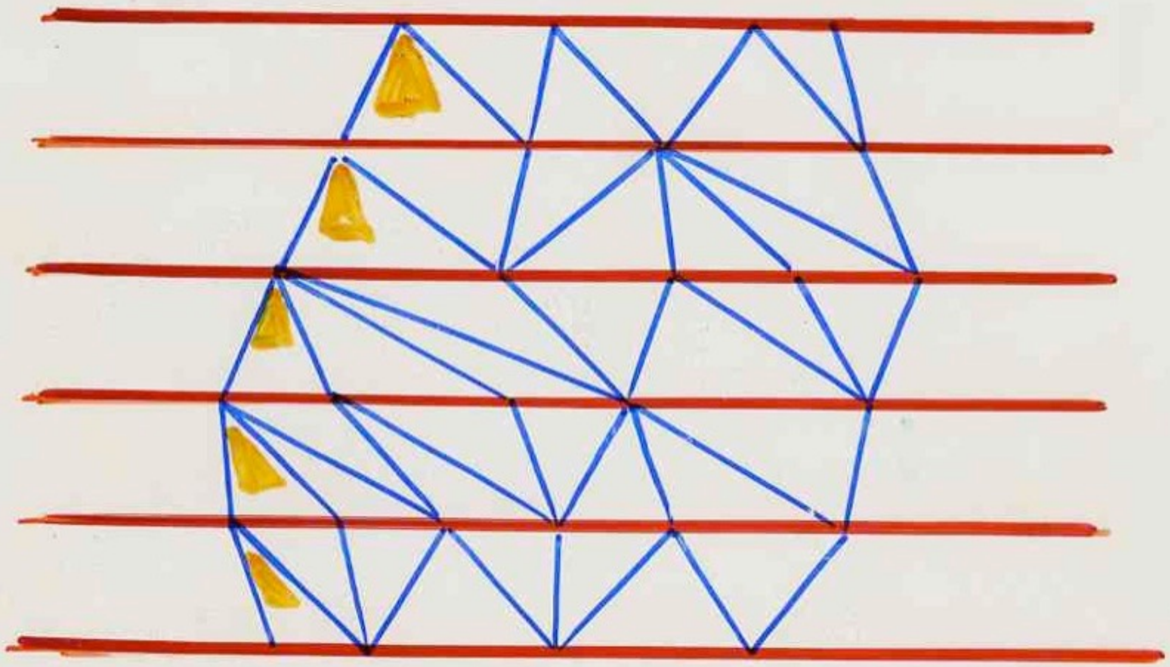
Path integral amplitude
for the propagation from
geometry ℓ_1 to ℓ_2

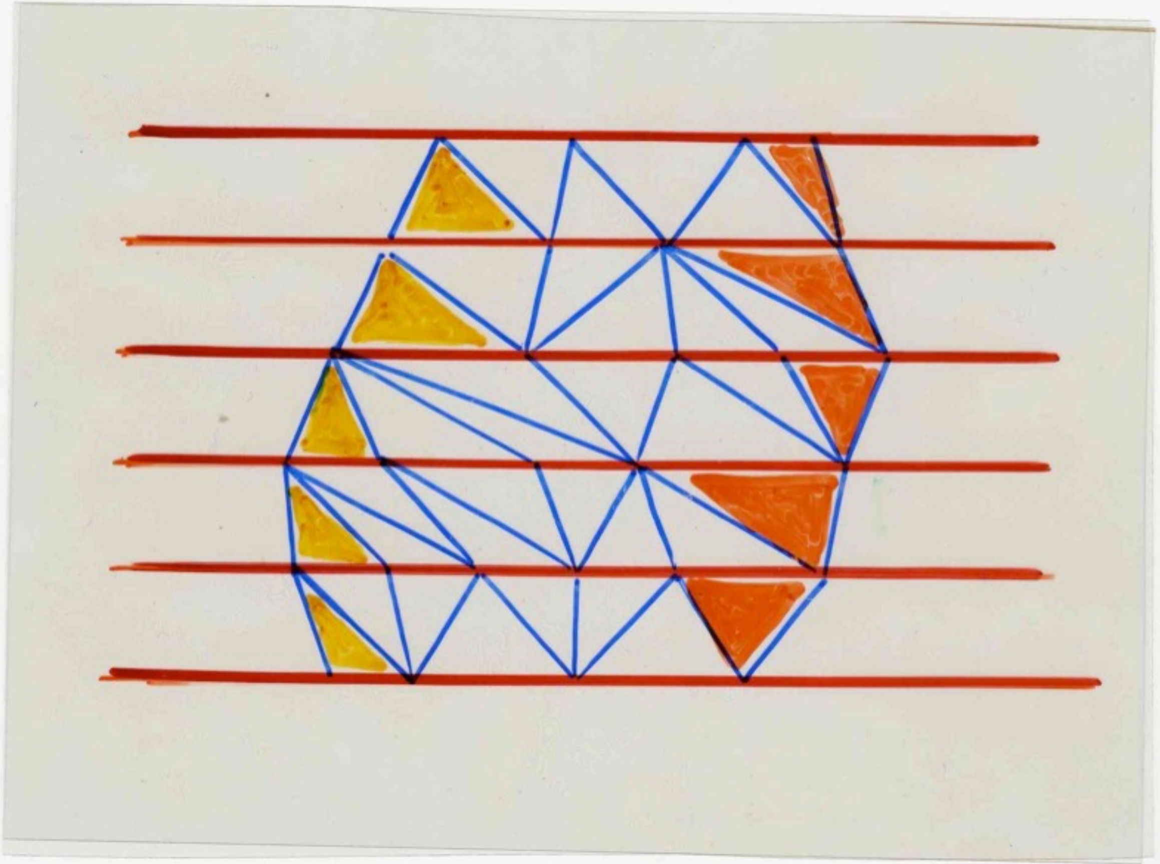


ℓ_2
L
g
ℓ_1



l_2
L
g
l_1

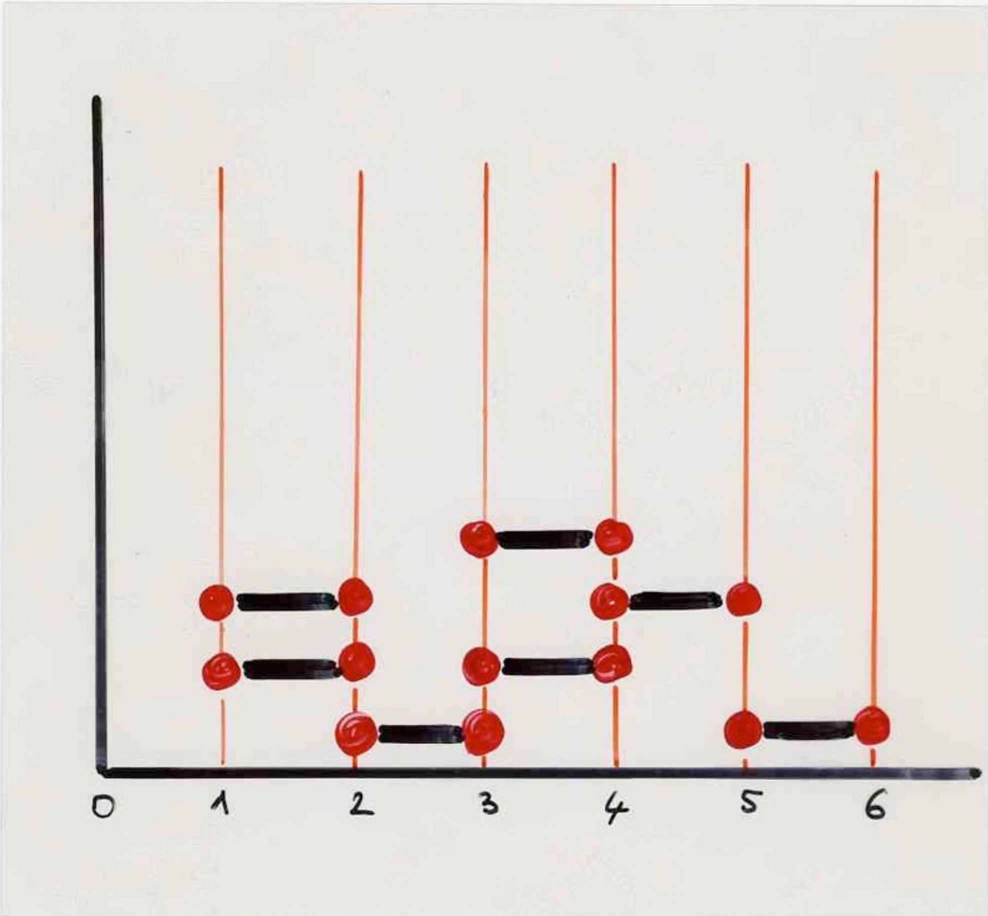




Heaps of dimers

$\updownarrow$

Lorentzian triangulations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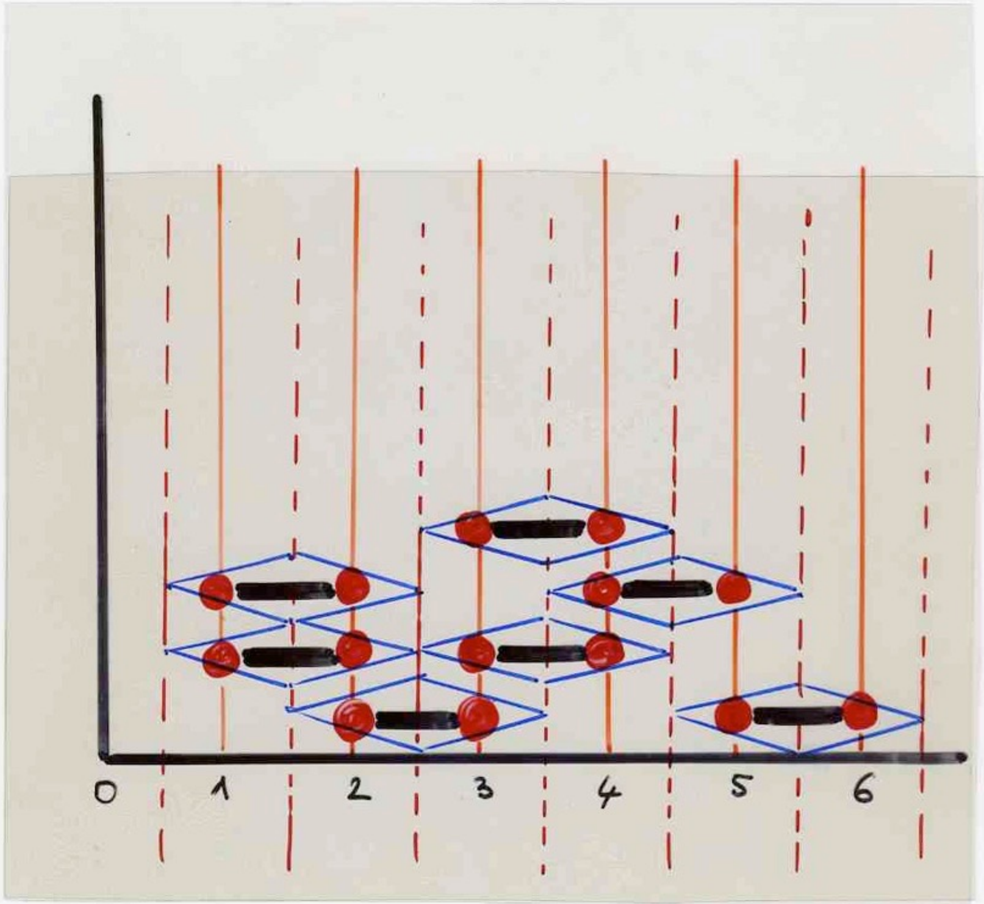

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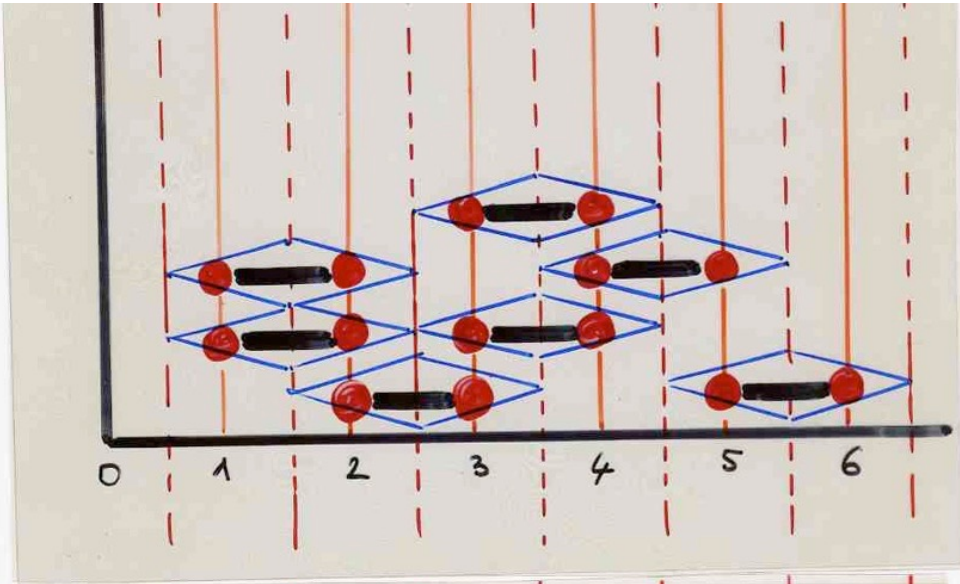
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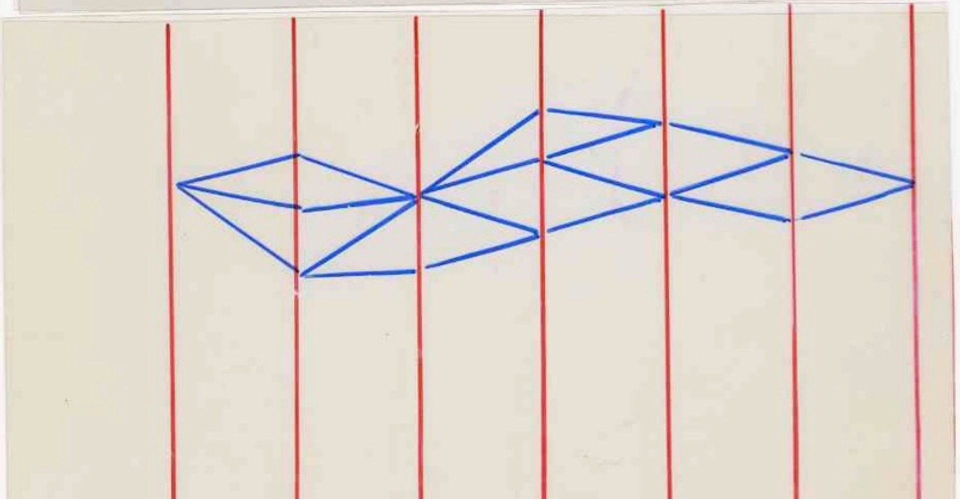

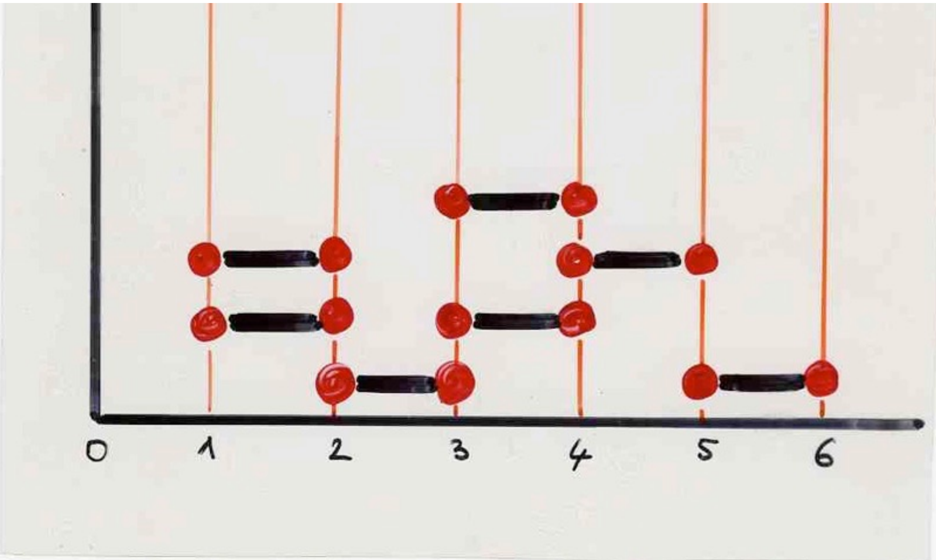
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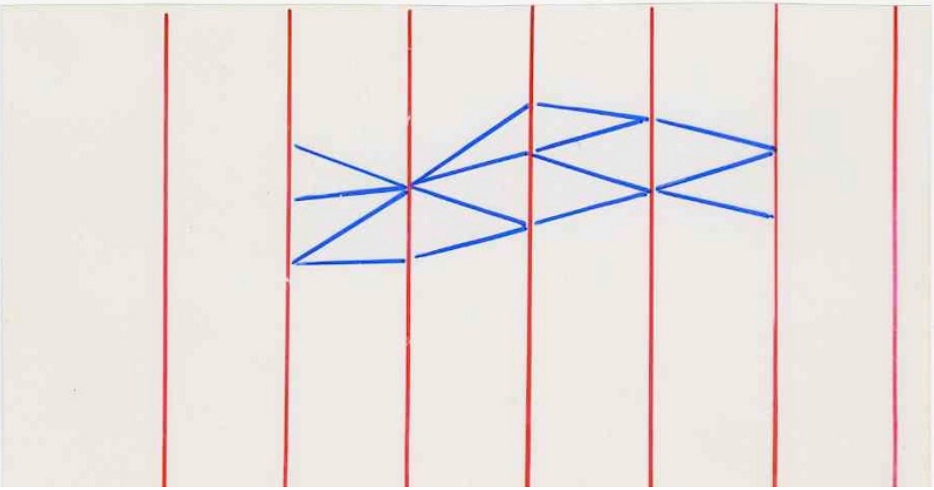

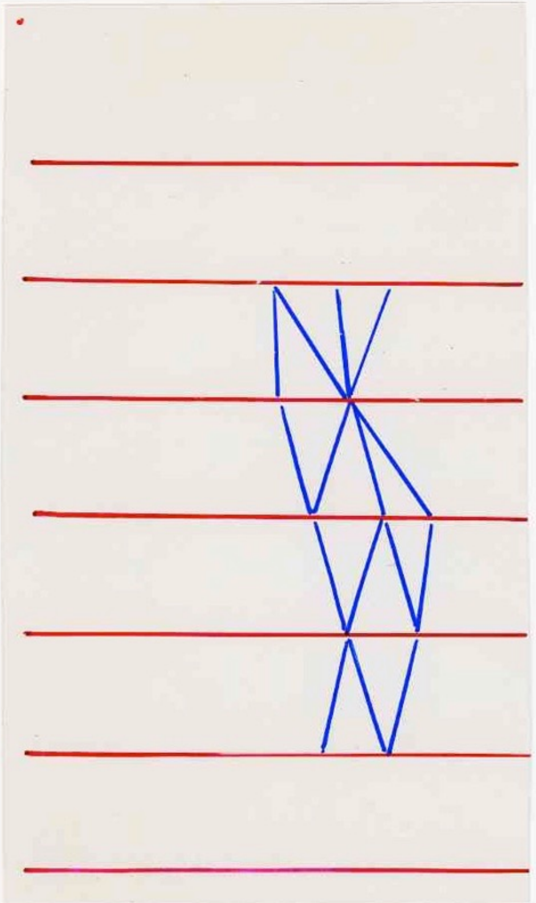

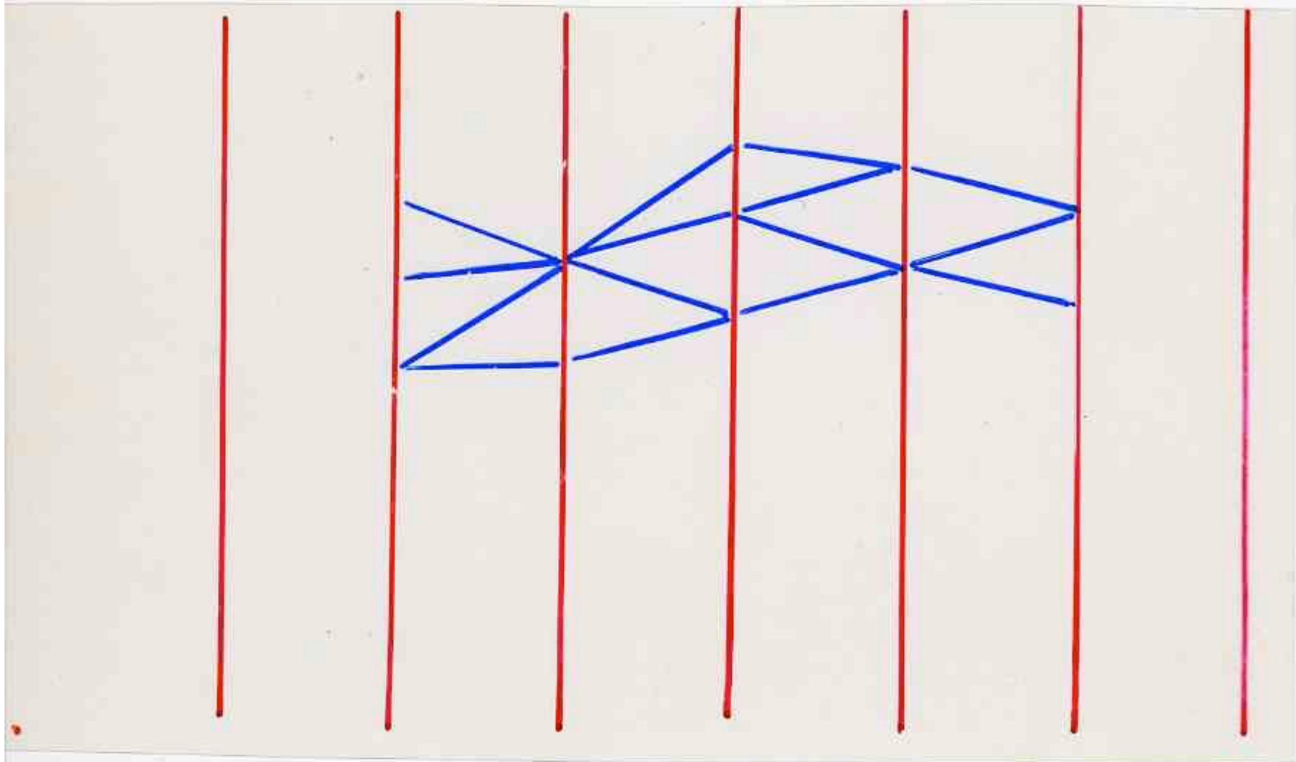

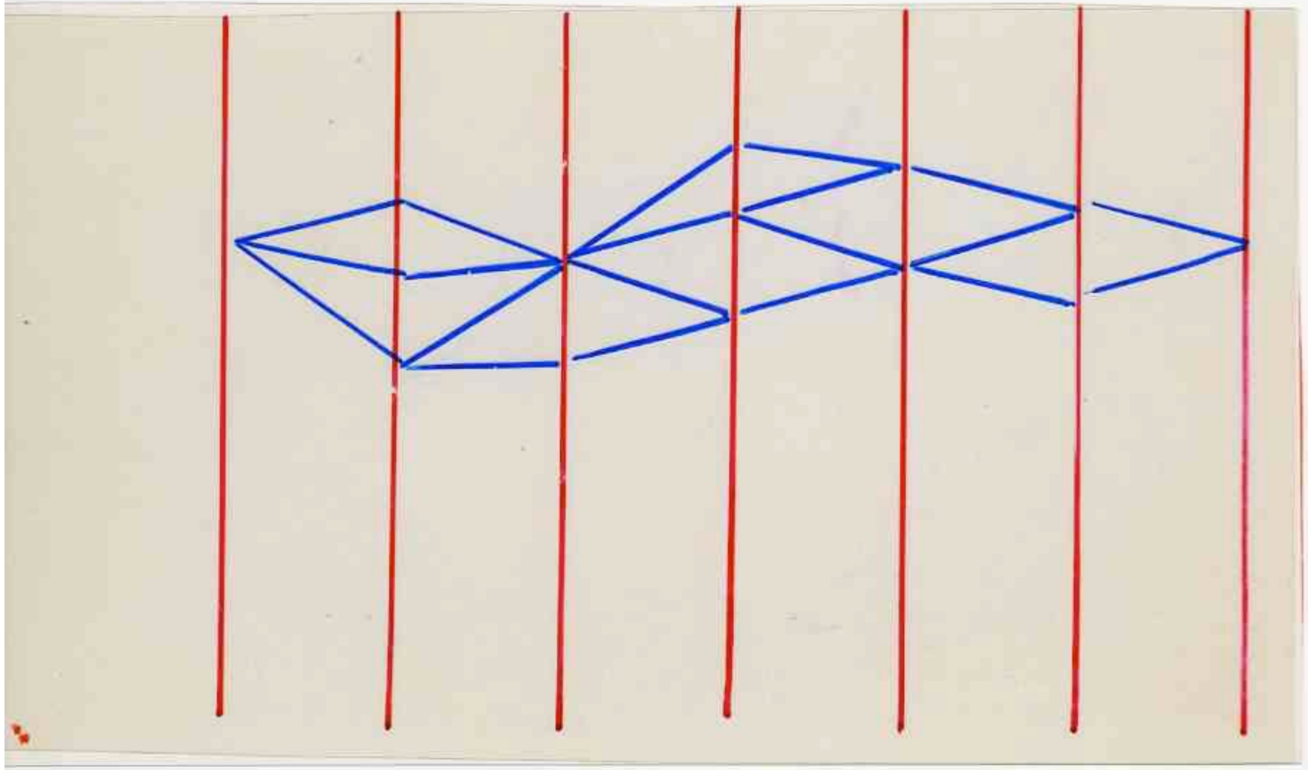

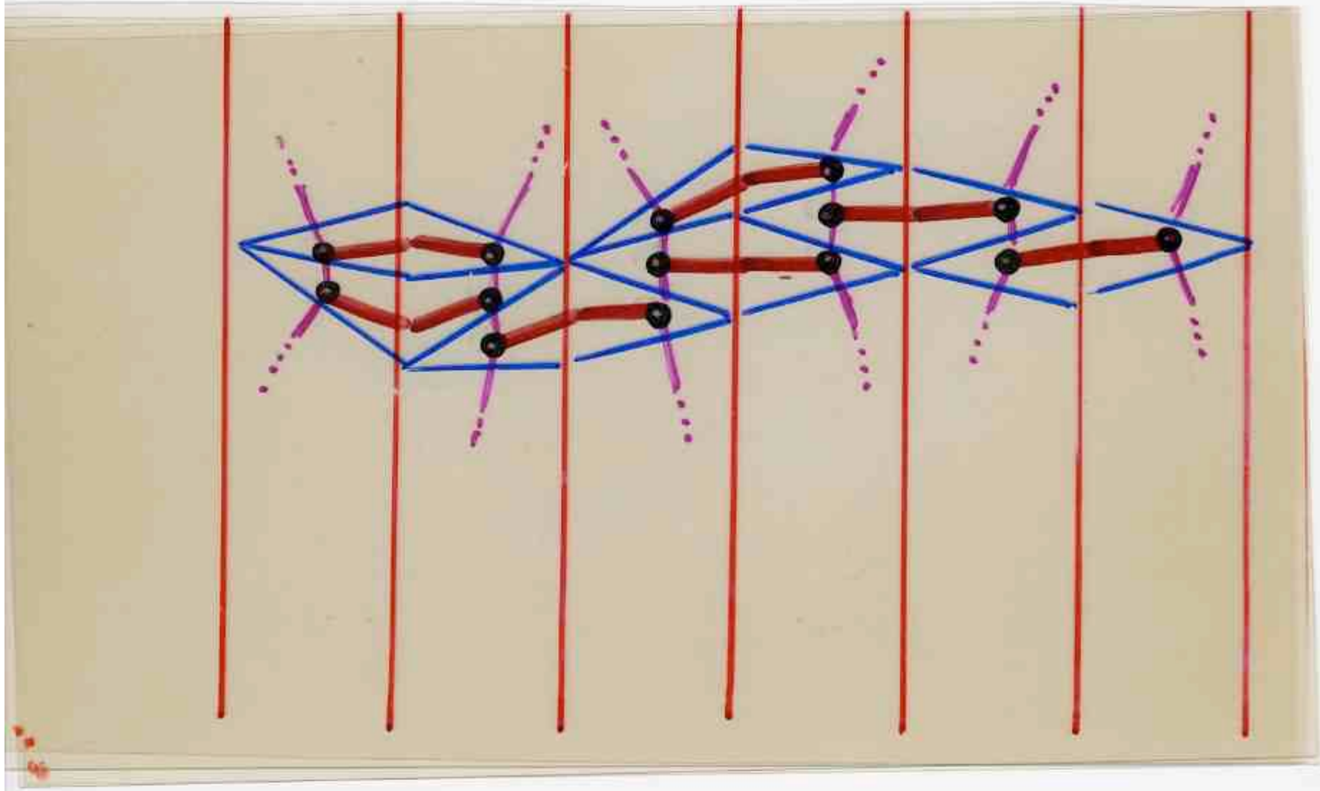

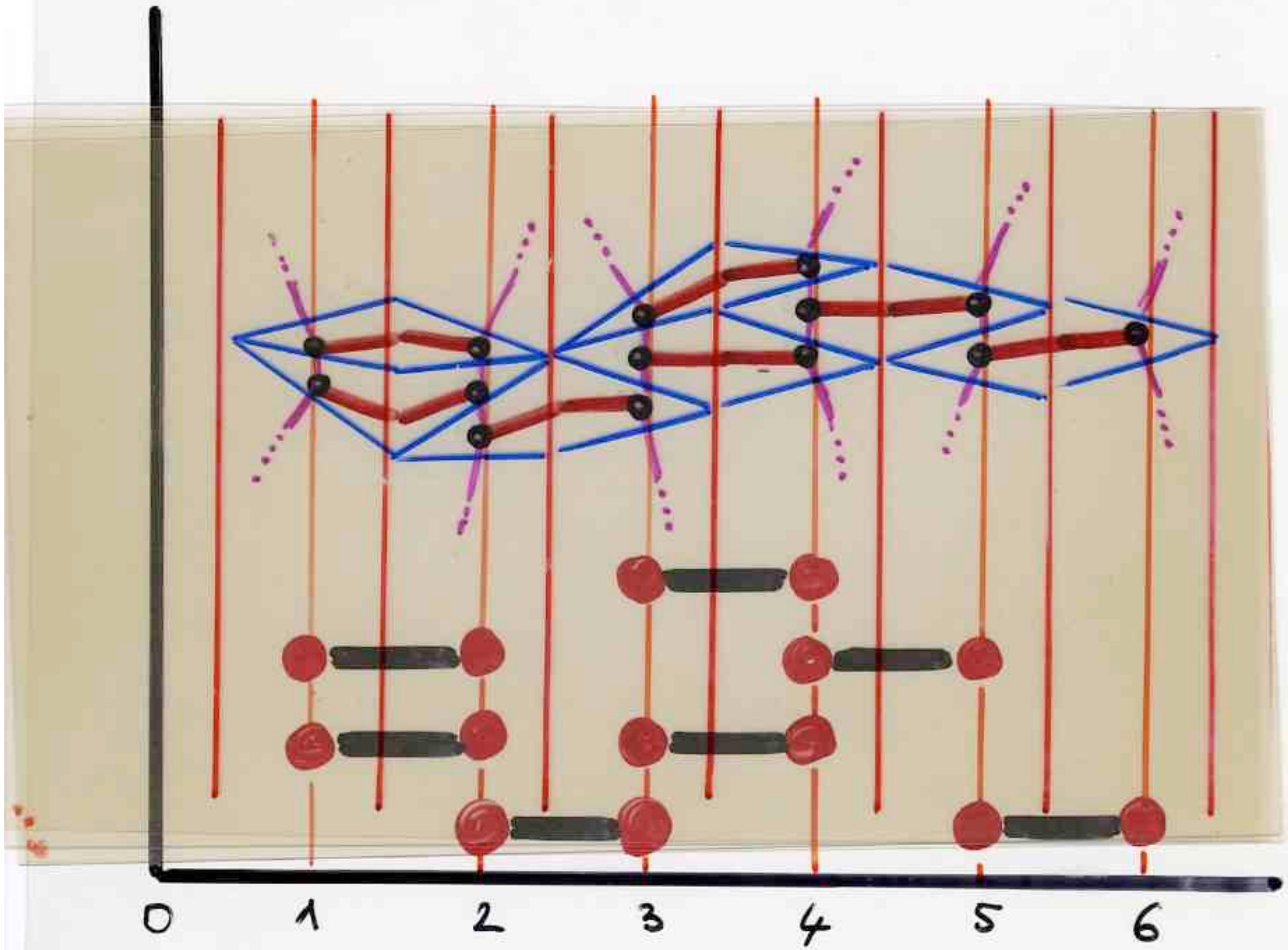
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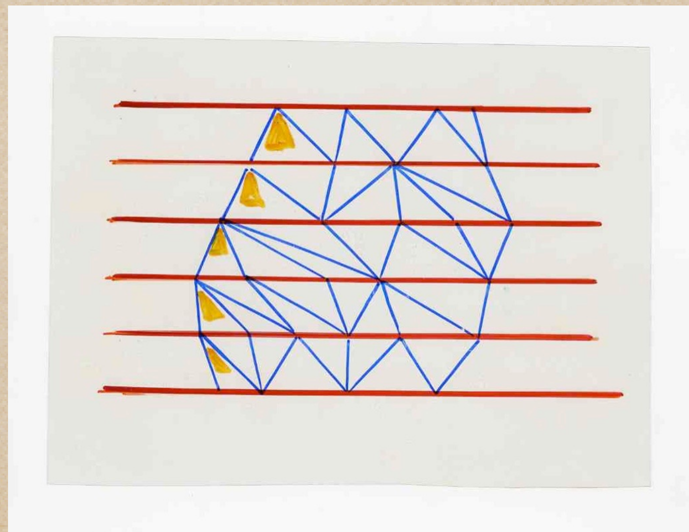
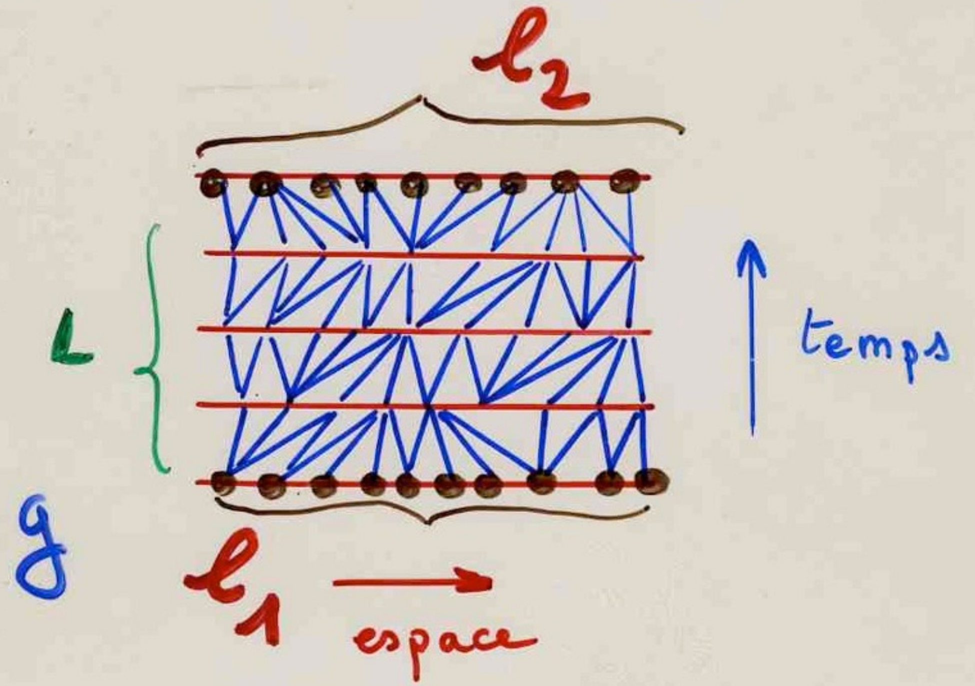
4 parameters

generating functions





Path integral amplitude
for the propagation from
geometry ℓ_1 to ℓ_2

Dyck paths

$\updownarrow$

Heaps of dimers

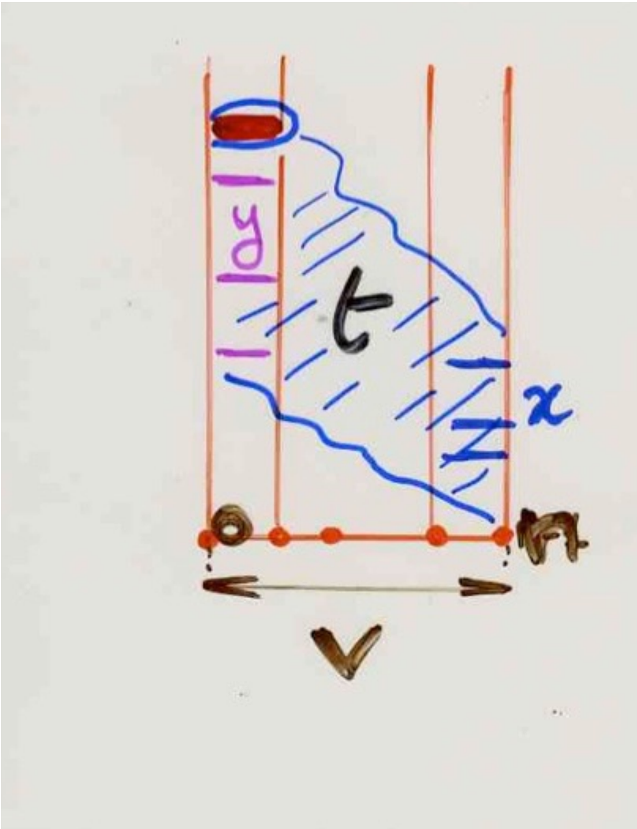
(Pyramids)

$\updownarrow$

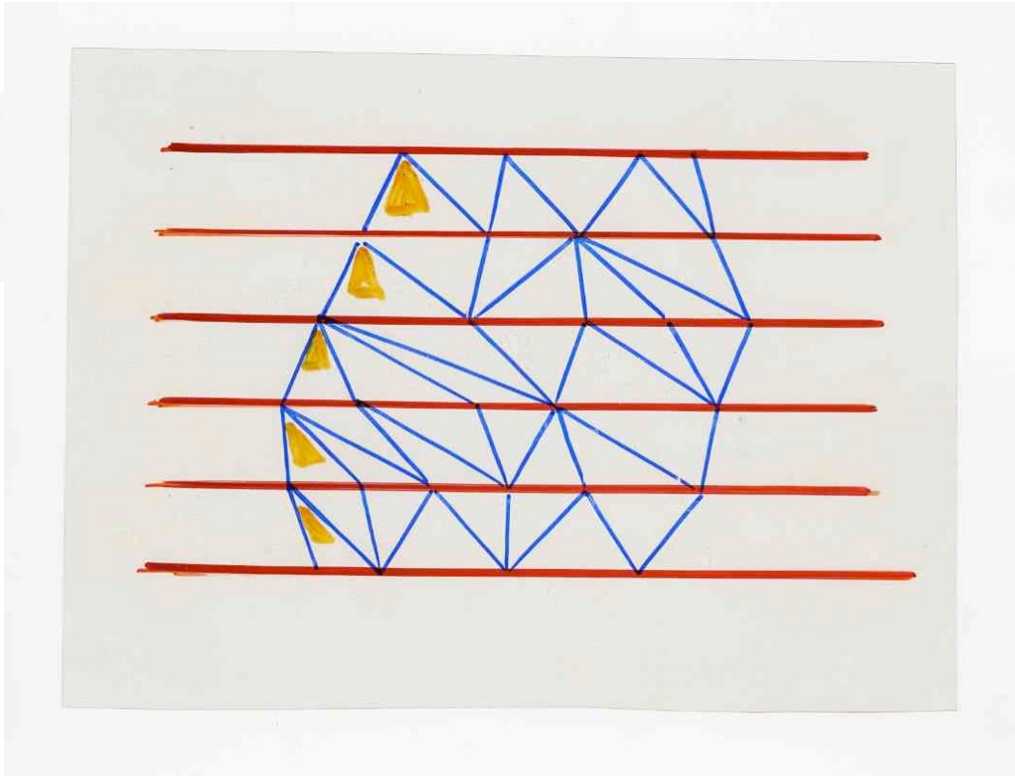
Lorentzian triangulations

(*) border condition

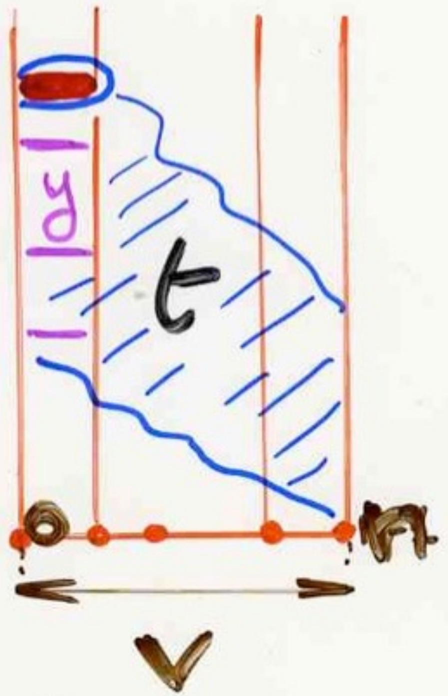
semi-pyramid



bijection



exercise

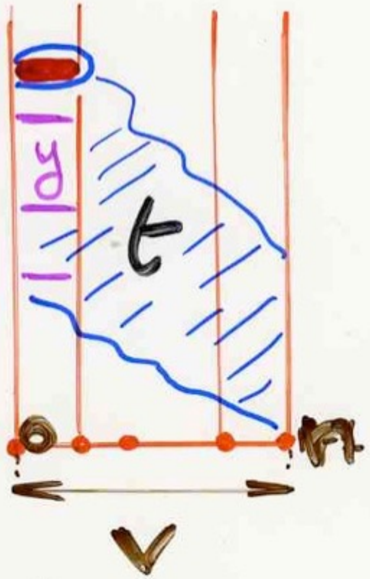


prove that the generating function $Q(t, y, v)$ for pyramids with 3 parameters is given by:

- t number of dimers
- v width
- y number of dimers in the first column

$$Q(t,y,v) = \sum_{n\geq 1} \frac{y\, t^n v^n}{\tilde{F}_n \tilde{F}_{n+1}}$$

(Bousquet-Mélou, Rechnitzer)

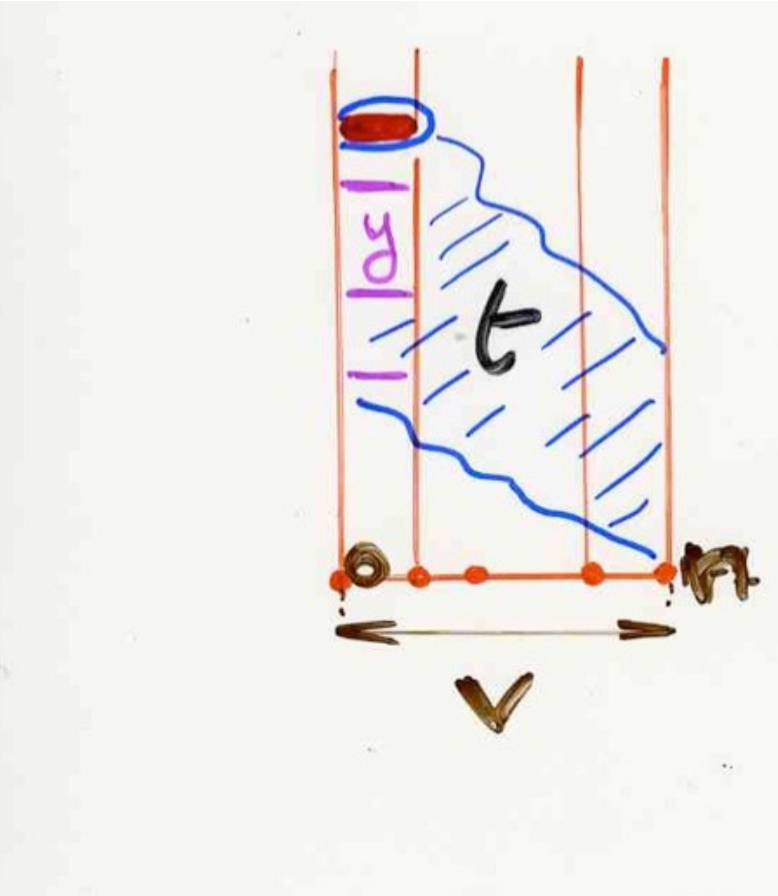


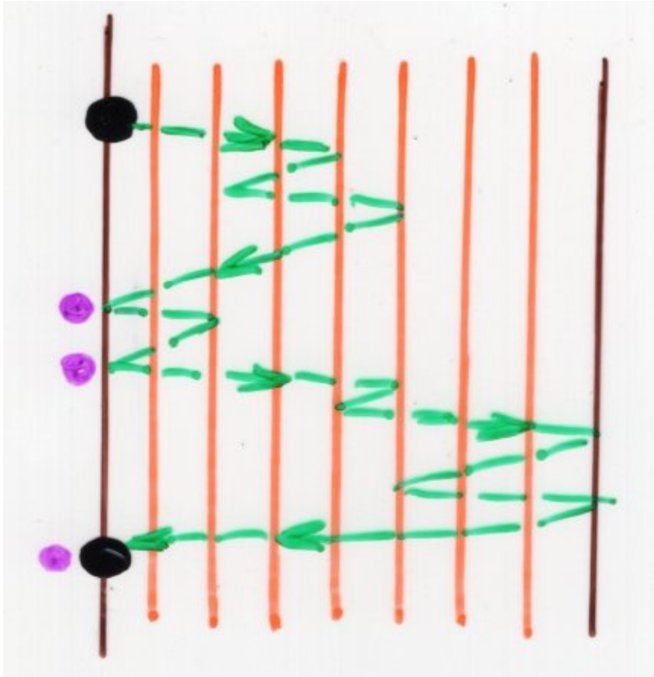
$\tilde{F}_n(t,y)$ defined by:

$$\tilde{F}_n(t,y) = F_{n-1}(t) - y\, F_{n-2}(t)$$

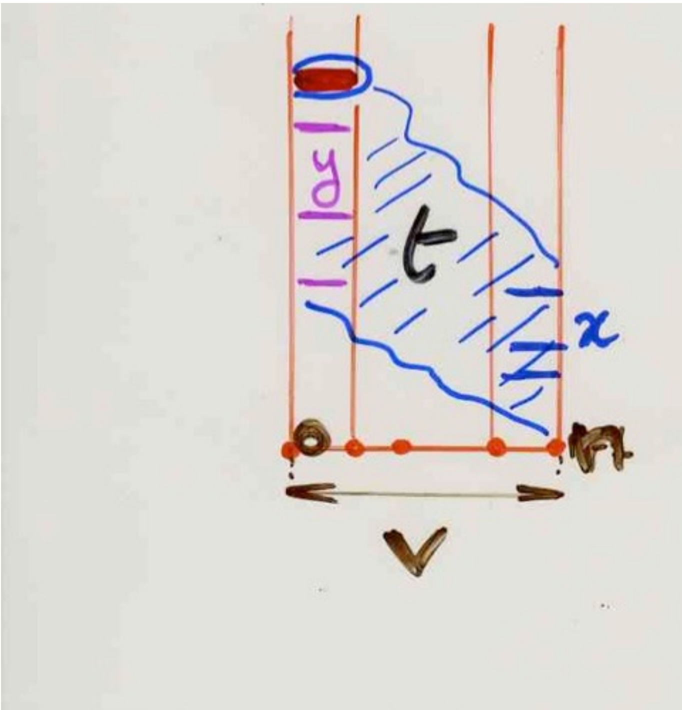
$F_n(t)$ Fibonacci polynomial

hint: use the bijection with Dyck paths





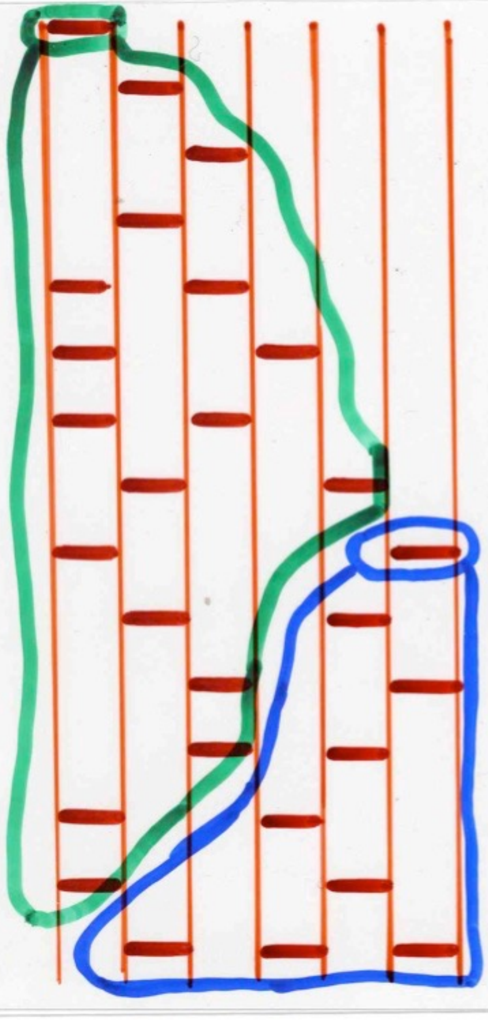
Proposition



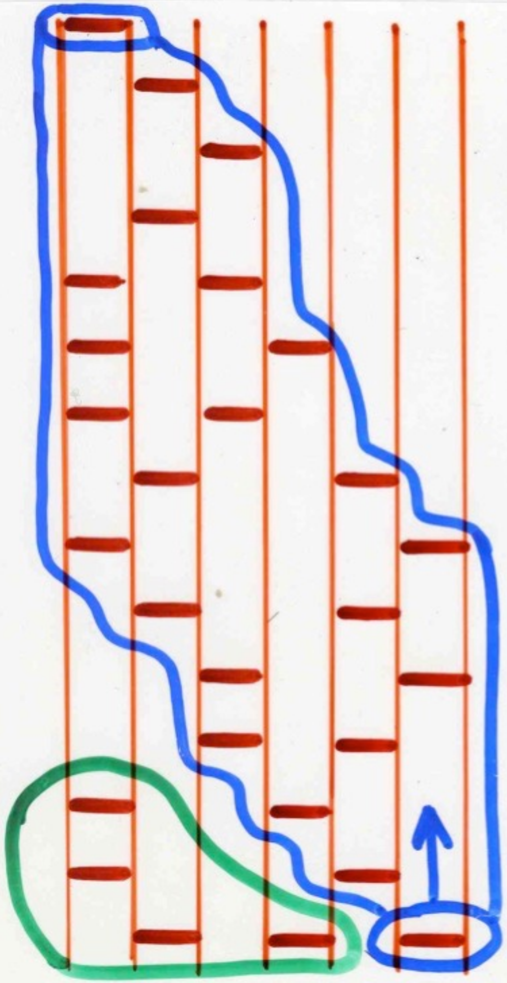
generating function for pyramids of dimers with 4 parameters

- t, v, y
- x number of dimers in the last column

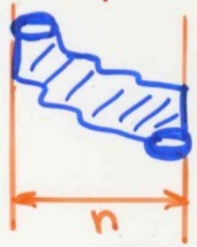
Proposition



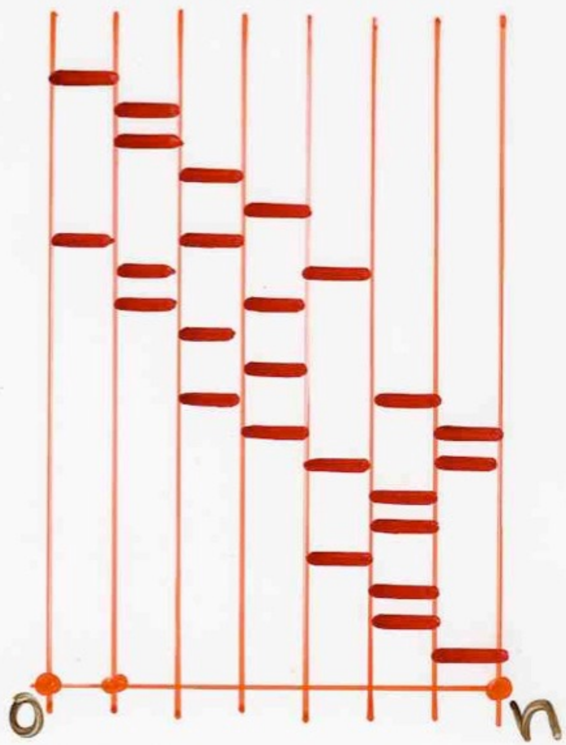
$$\frac{{}_{y}t^{n}v^{n}_{x}}{\tilde{F}_{n}(t,y,1)\,\tilde{F}_{n+1}(t,y,x)} = \frac{{}_{y}t^{n-1}v^{n-1}}{\tilde{F}_{n-1}(t,y,1)\,\tilde{F}_{n}(t,y,1)} \times \frac{x\,t\,v\,\tilde{F}_{n-1}(t,y,1)}{\tilde{F}_{n+1}(t,y,x)}$$



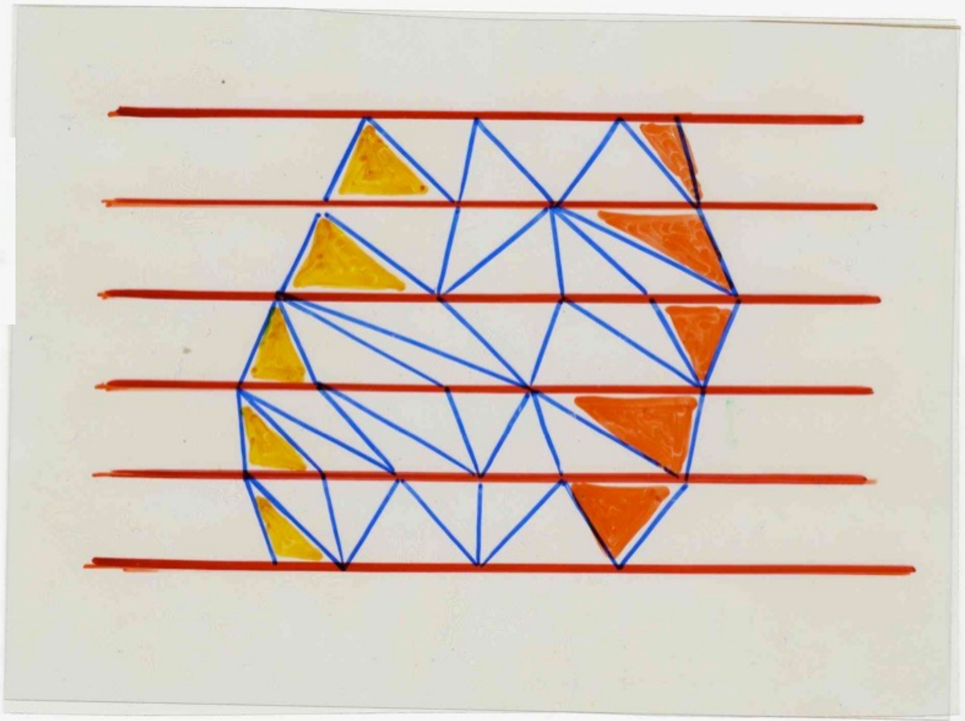
$$\frac{y\,t^{n}v^{n}}{\tilde{F}_{n}(t,y,1)\,\tilde{F}_{n+1}(t,y,x)} = \frac{1}{\tilde{F}_{n}(t,y,1)} \times \frac{y\,t^{n}v^{n}}{\tilde{F}_{n+1}(t,y,x)}$$



double
semi- pyramid

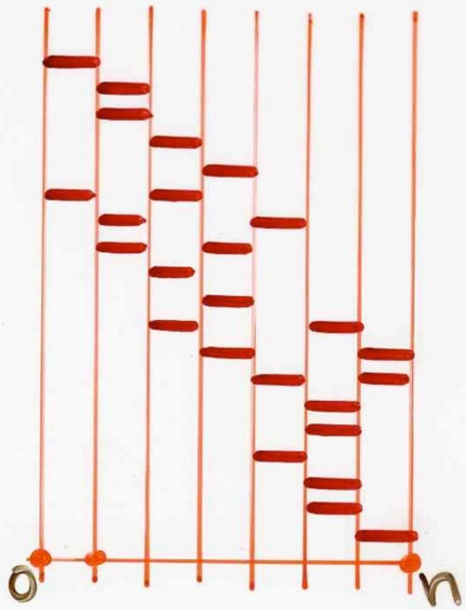


bijection



exercise

semi-double pyramid

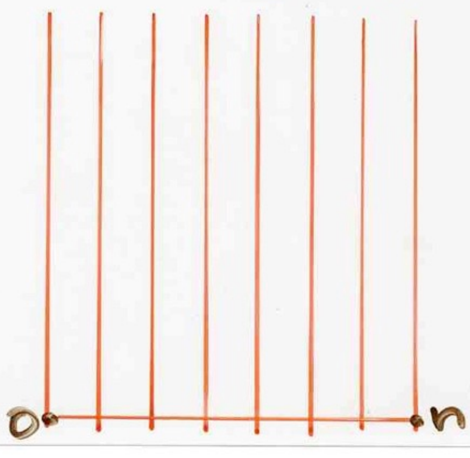


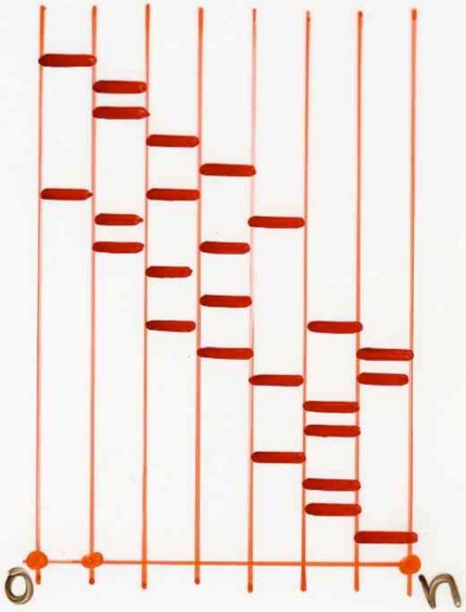
in bijection with (general) heap on $[0,n]$

P_m t^m m dimers

H_{m-n} t^{m-n} $m-n$ dimers

preserve x^{i-1} y^{j-1}





semi- double pyramid

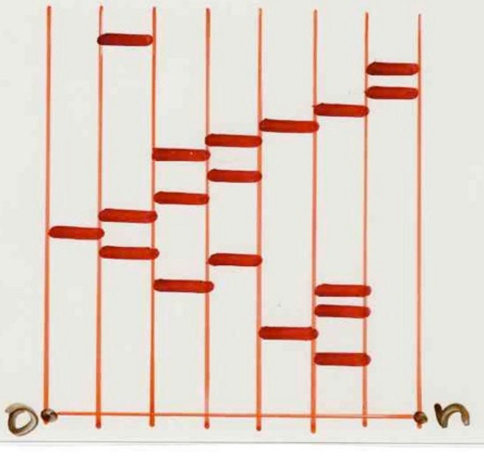
in bijection

with (general) heap on $[0,n]$

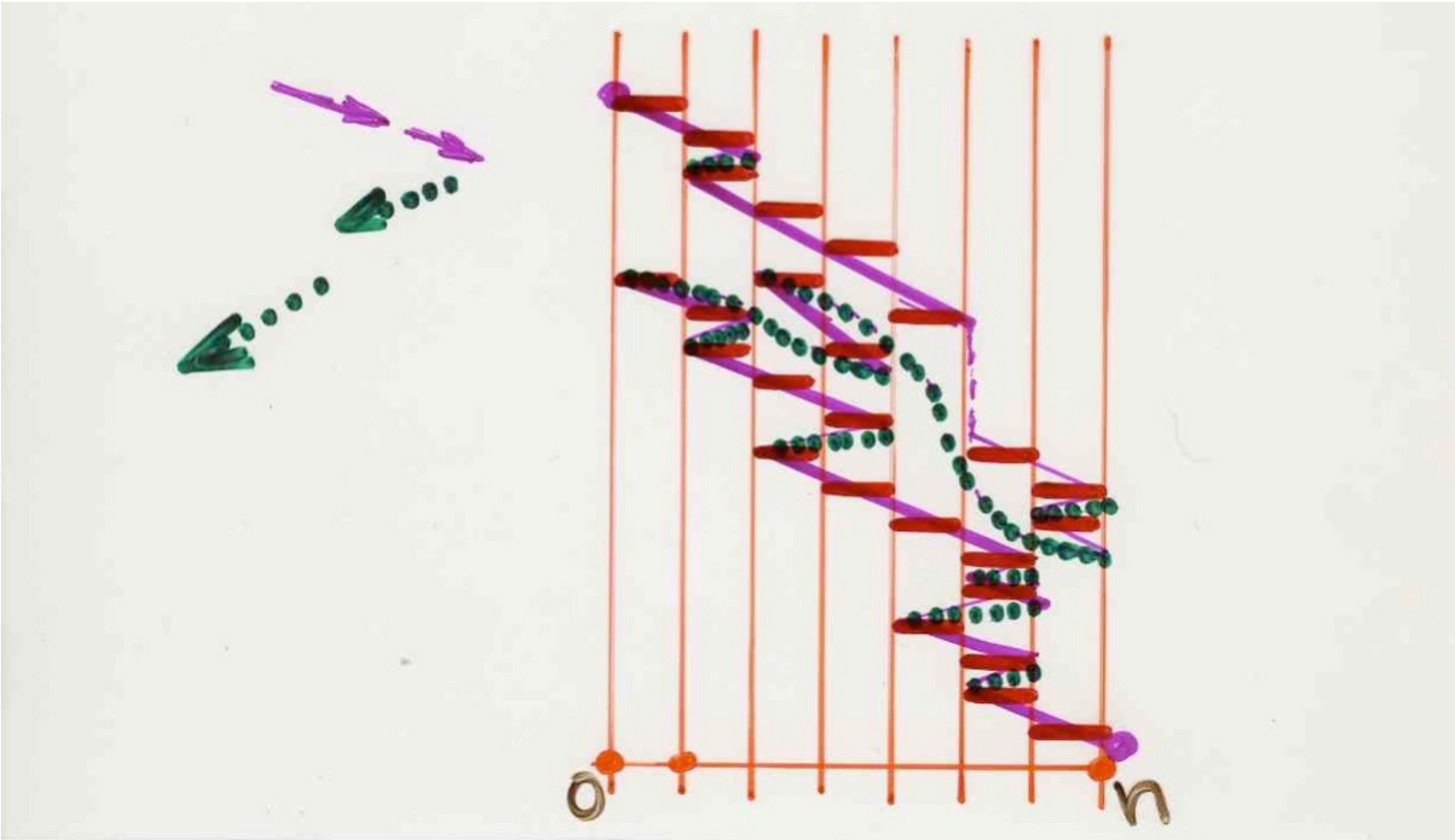
P_m t^m
m dimers

H_{m-n} t^{m-n}
$m-n$ dimers

preserve x^{i-1} y^{j-1}



double
semi- pyramid



in bijection

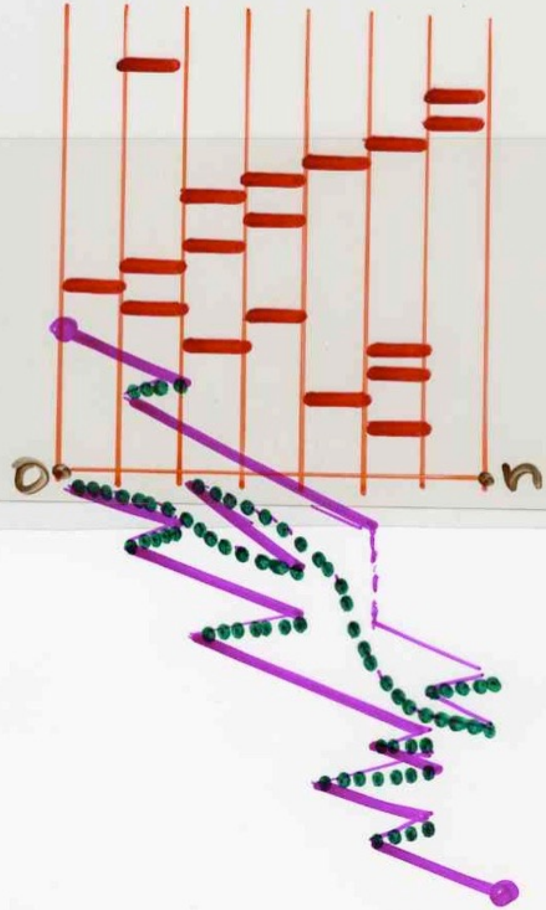
P_m t^m with (general) **heap** on $[0,n]$

m dimers

H_{m-n} t^{m-n}

$m-n$ dimers

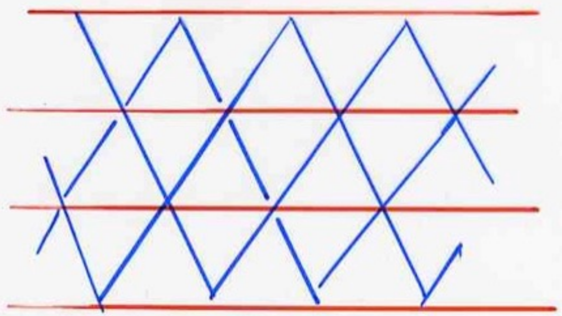
preserve x^{i-1} y^{j-1}

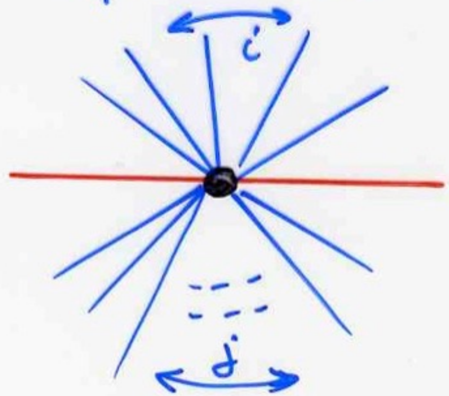


curvature

curvature

of the space-time



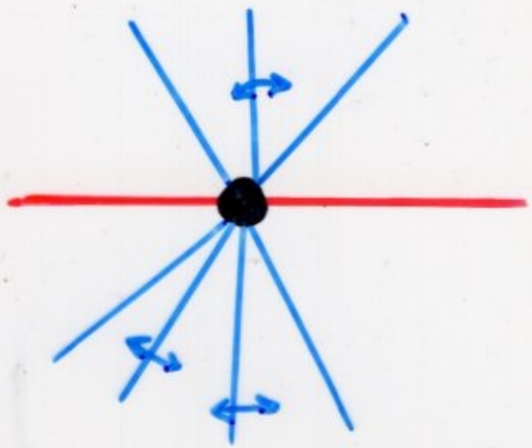


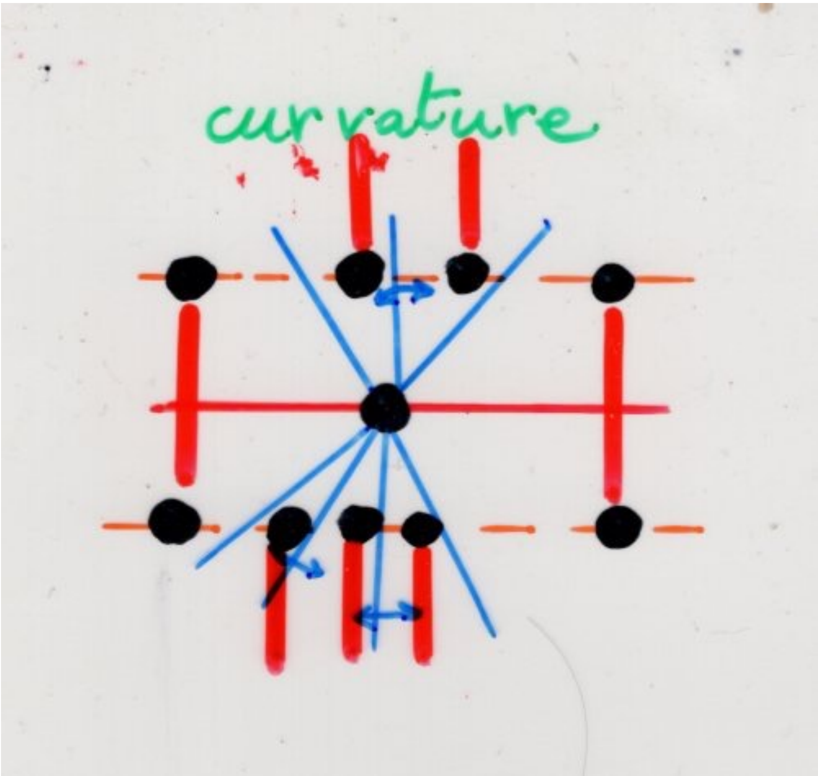
flat

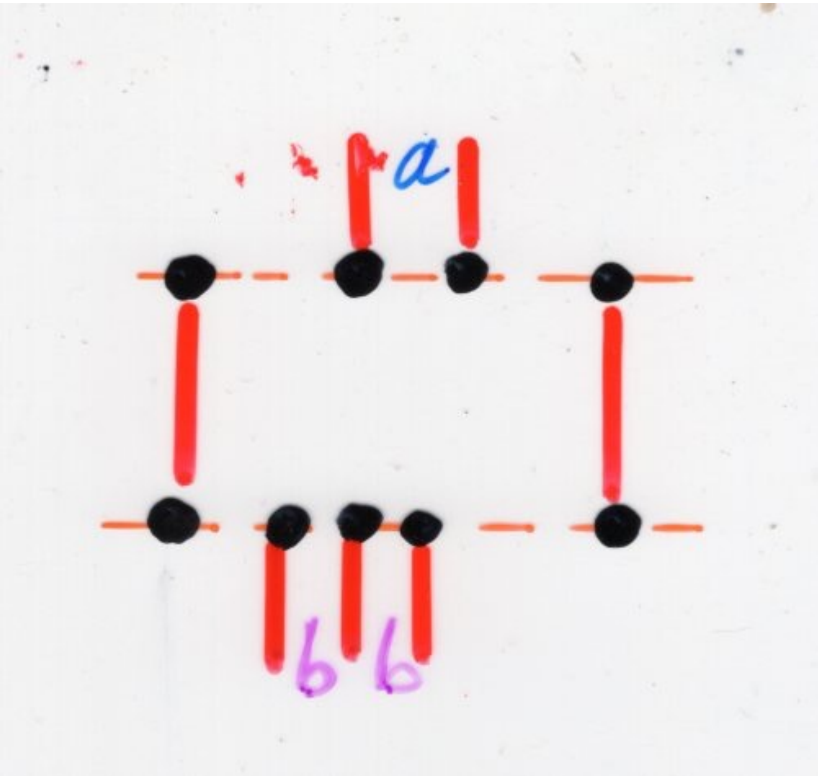
$$a^{|i-3|+|j-3|}$$

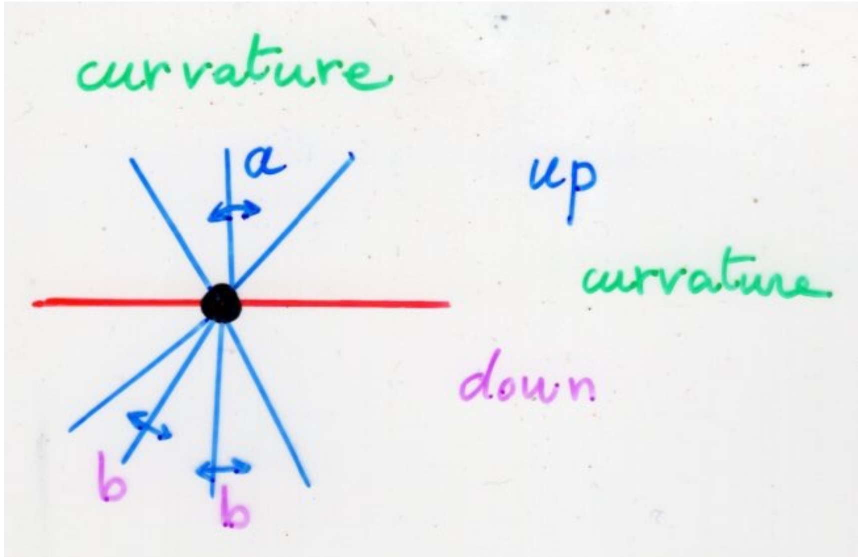
$$\text{total curvature} = \prod_{\text{all points}} a^{(\cdots)}$$

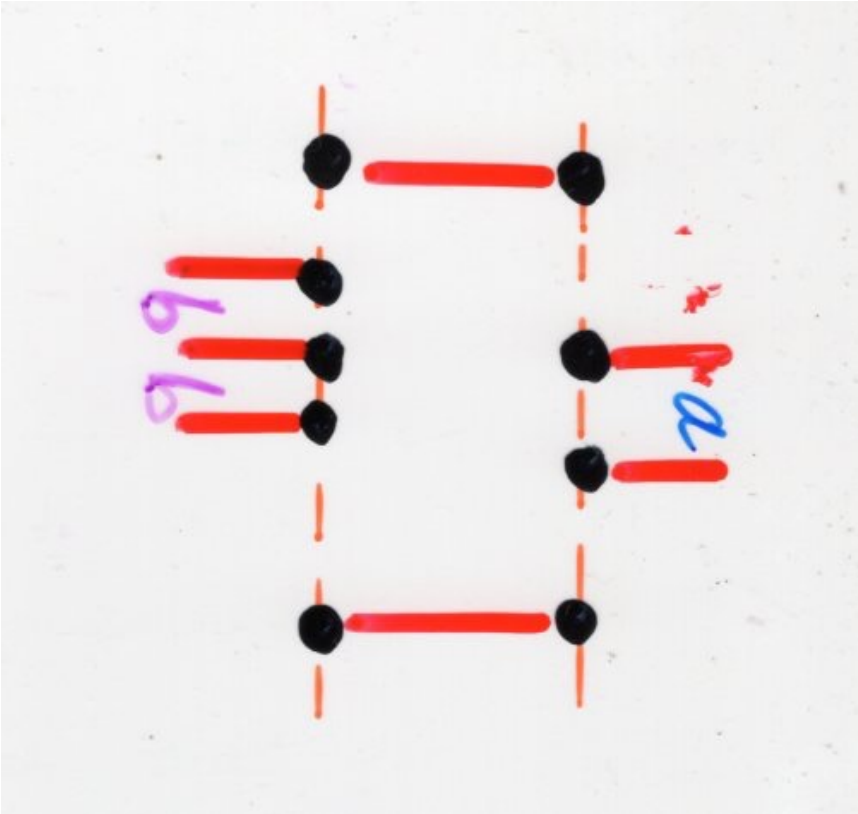
curvature

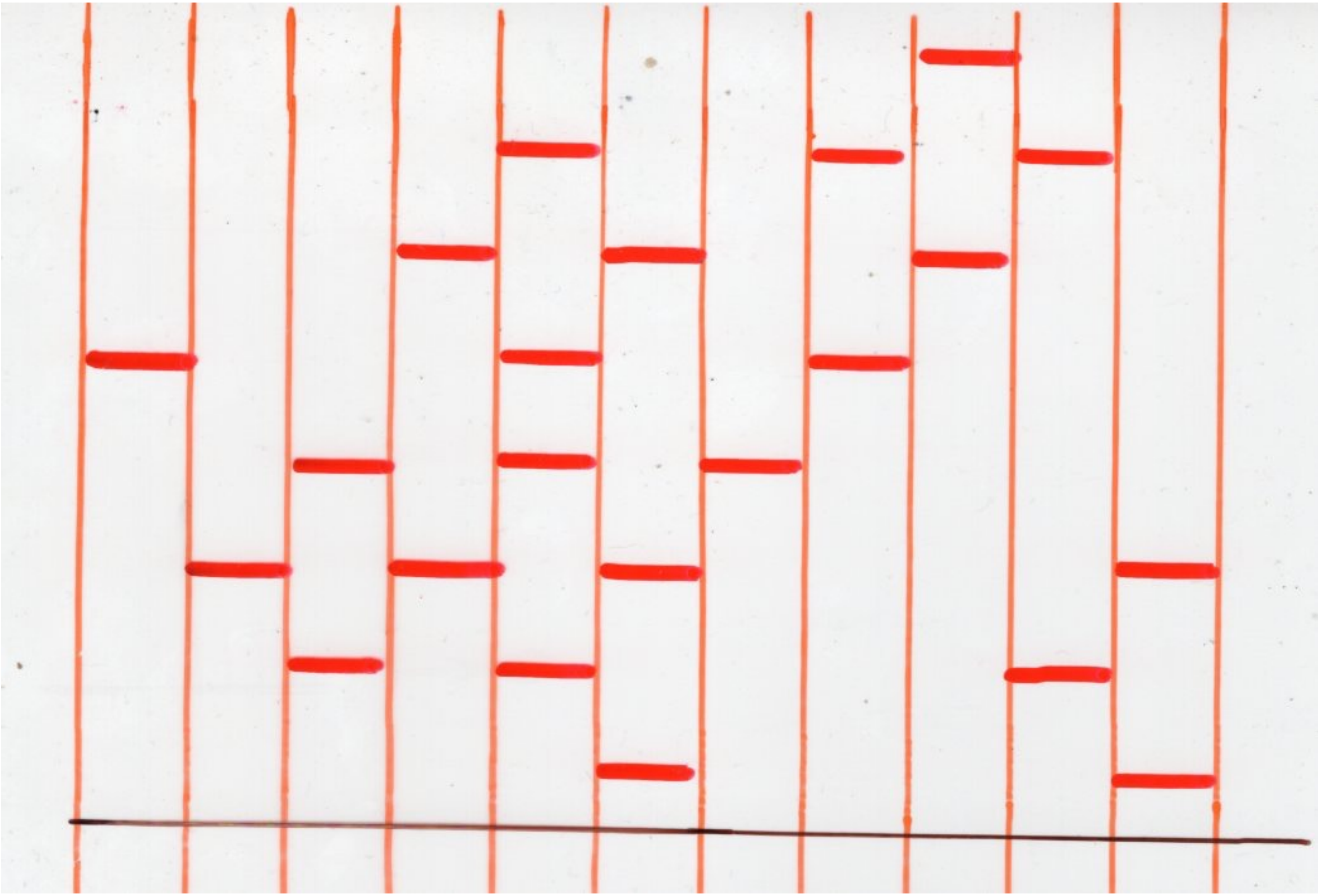


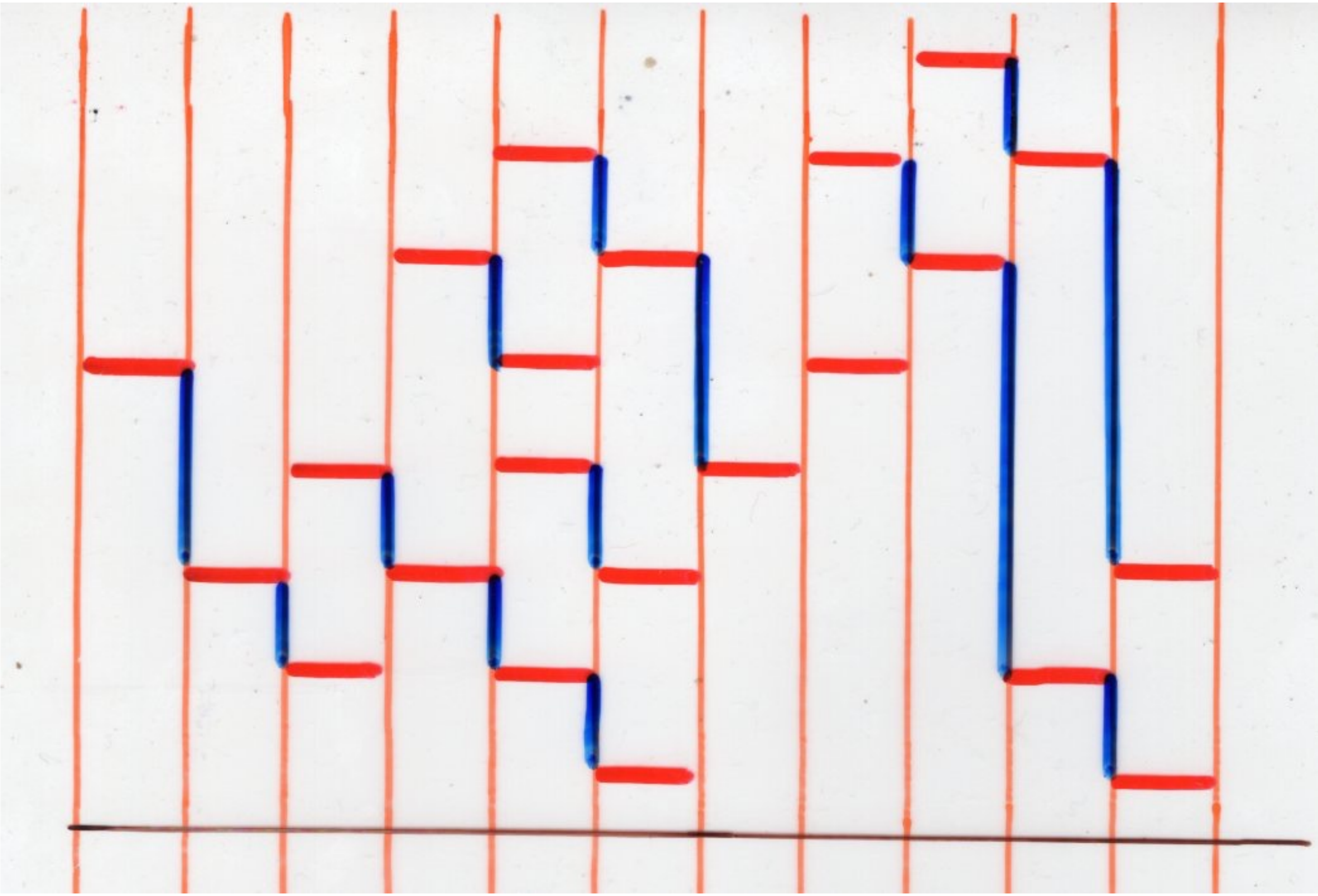

curvature

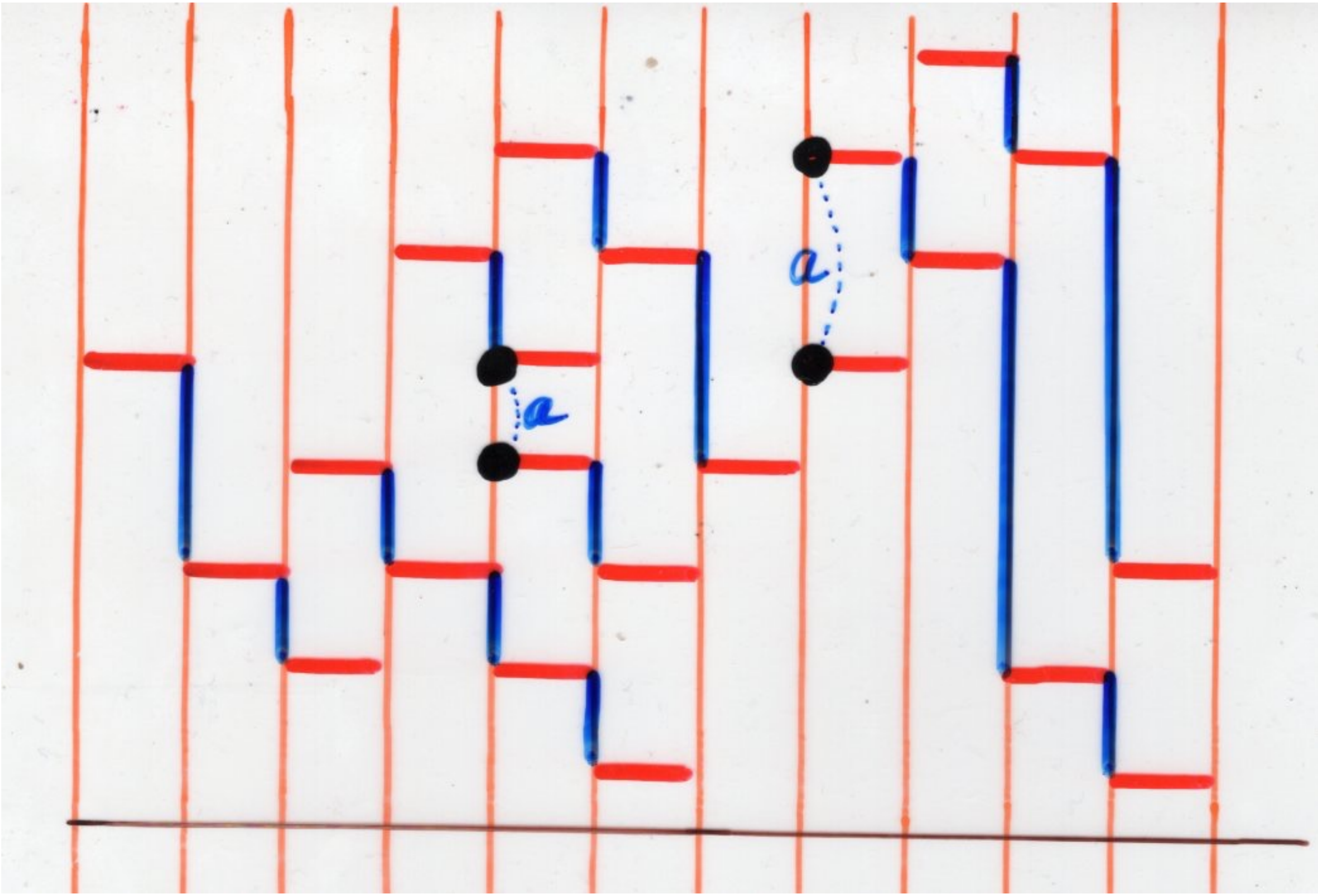

a
b
b

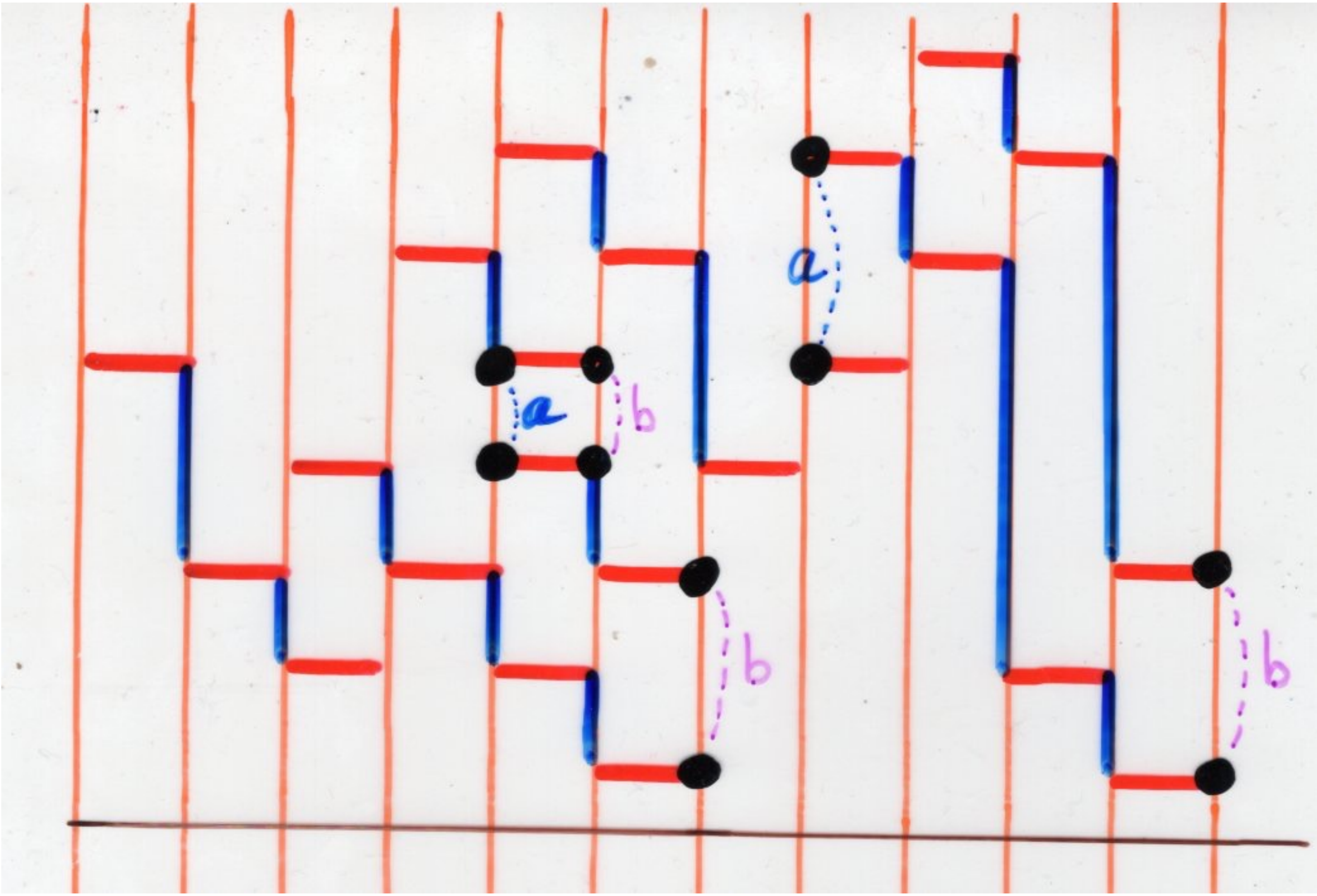

curvature
a
up
curvature
down
b
b

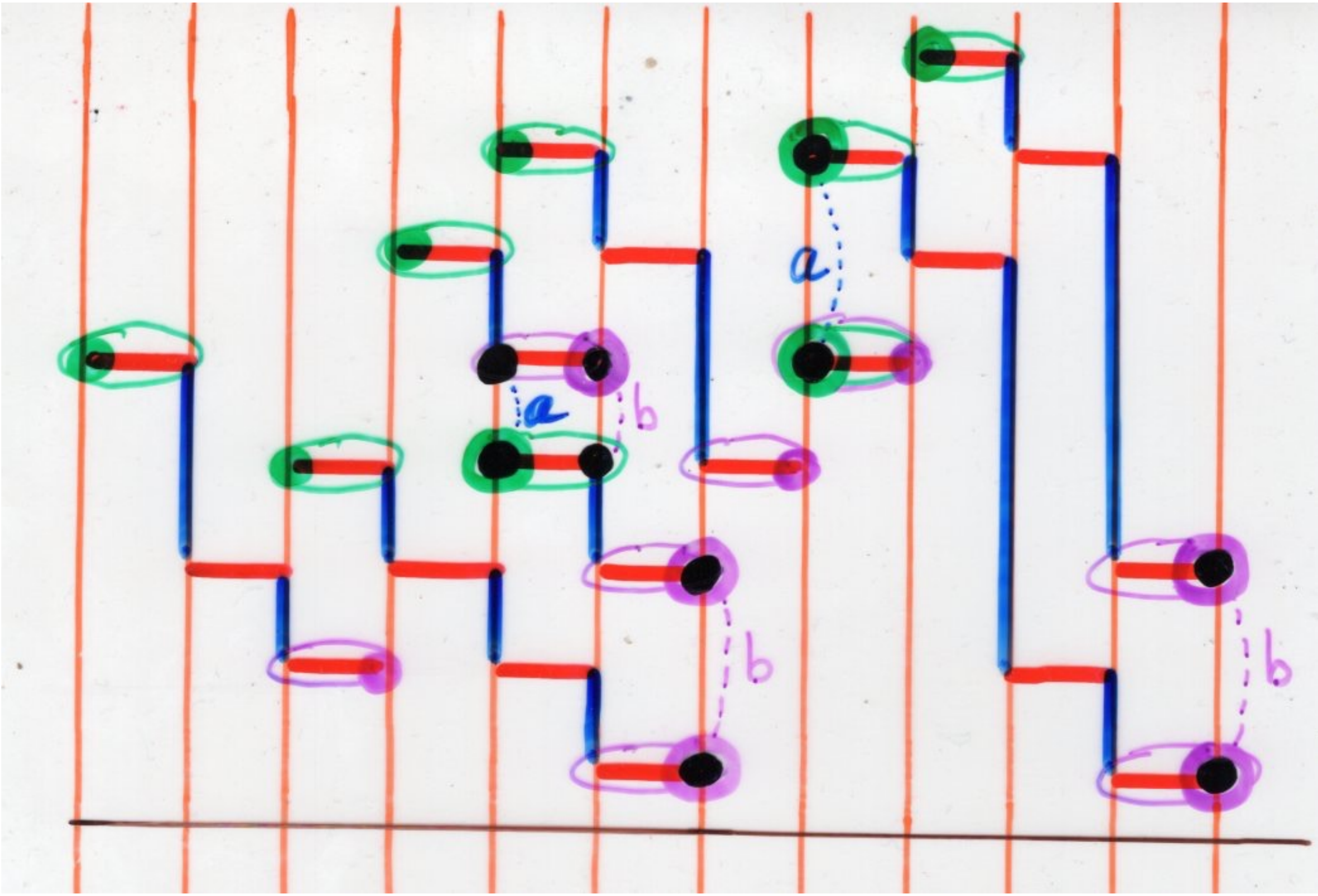

b
b
a

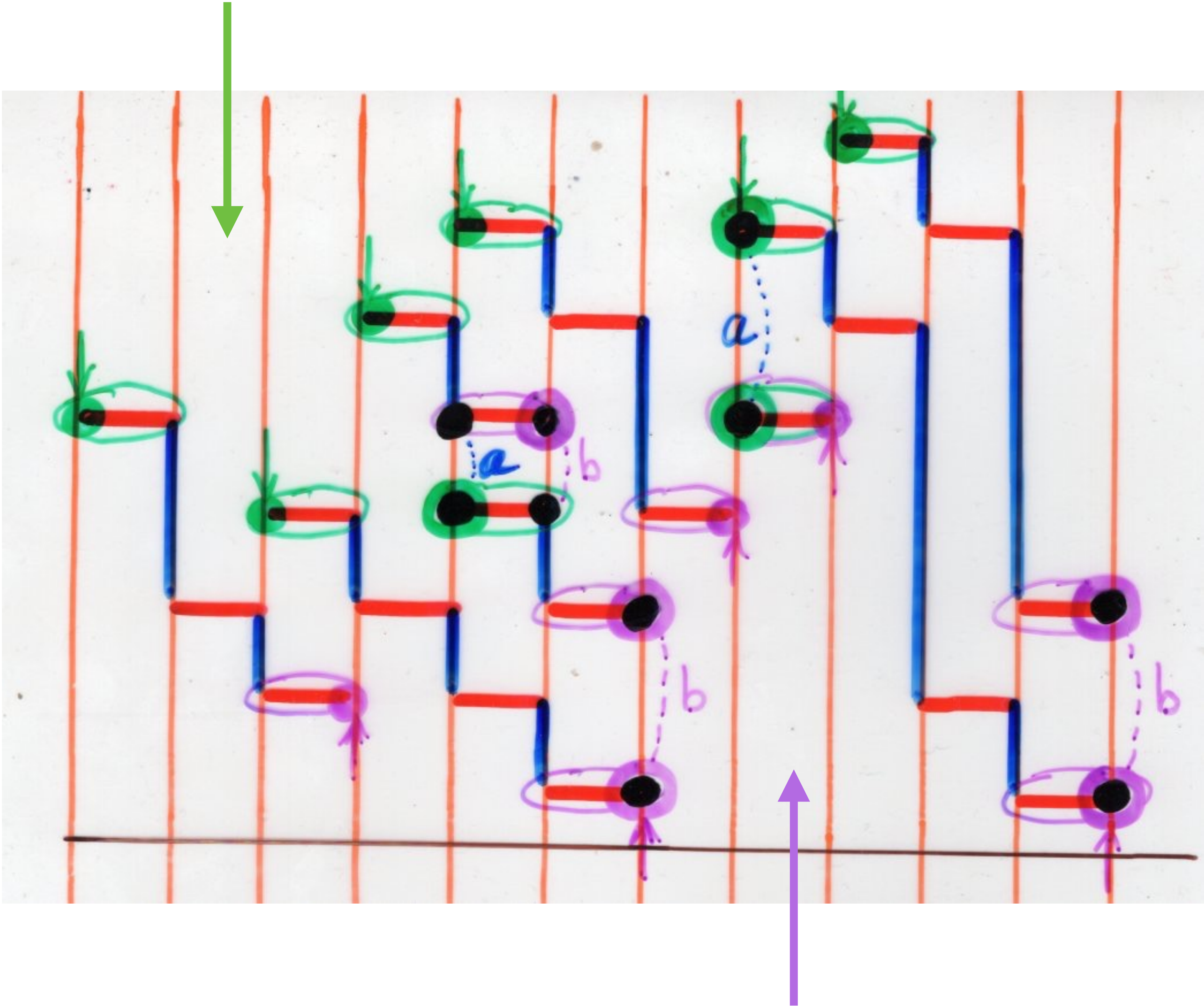





a
a


a
b
b
a
b


a
b
a
b
b


a
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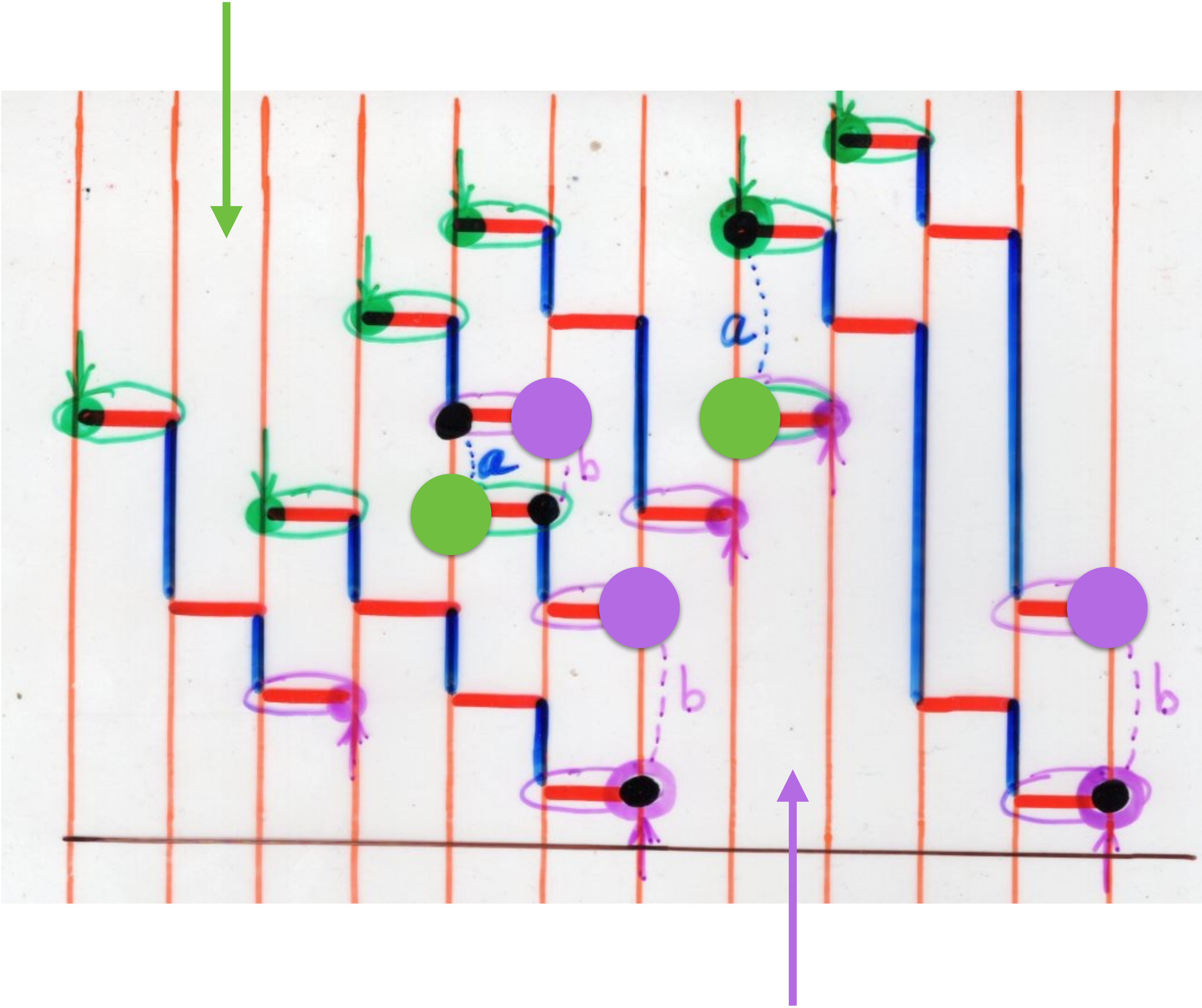
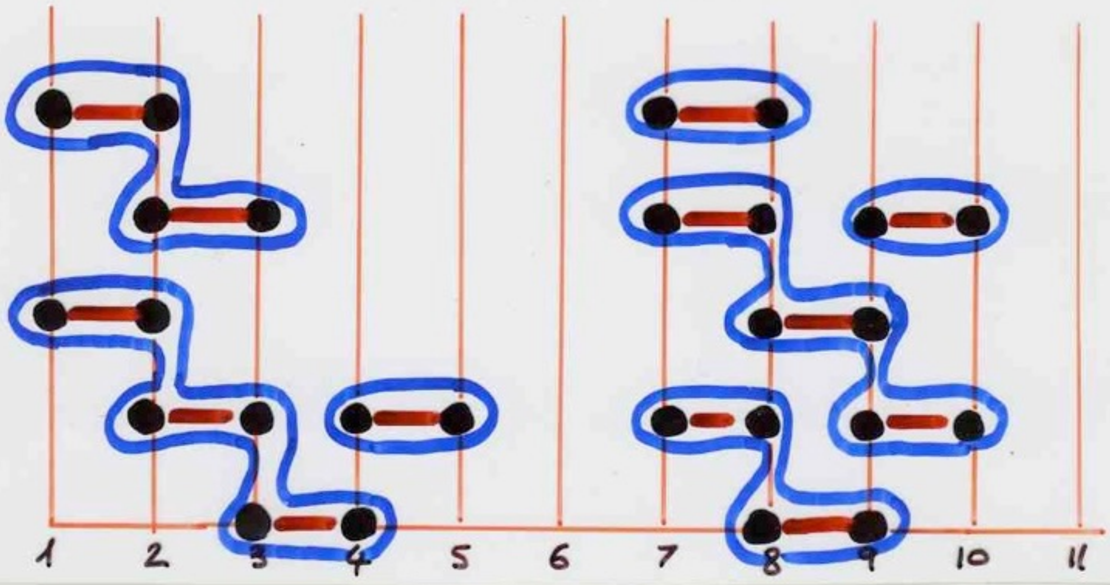
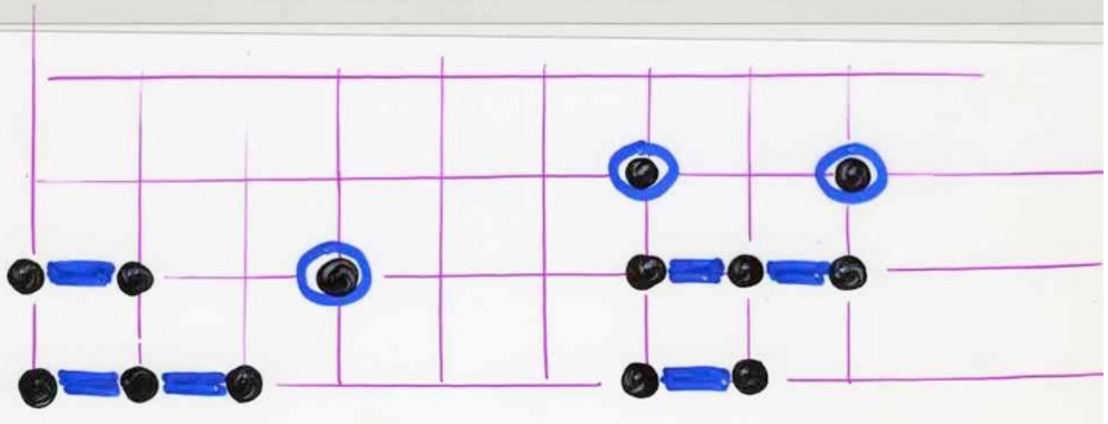
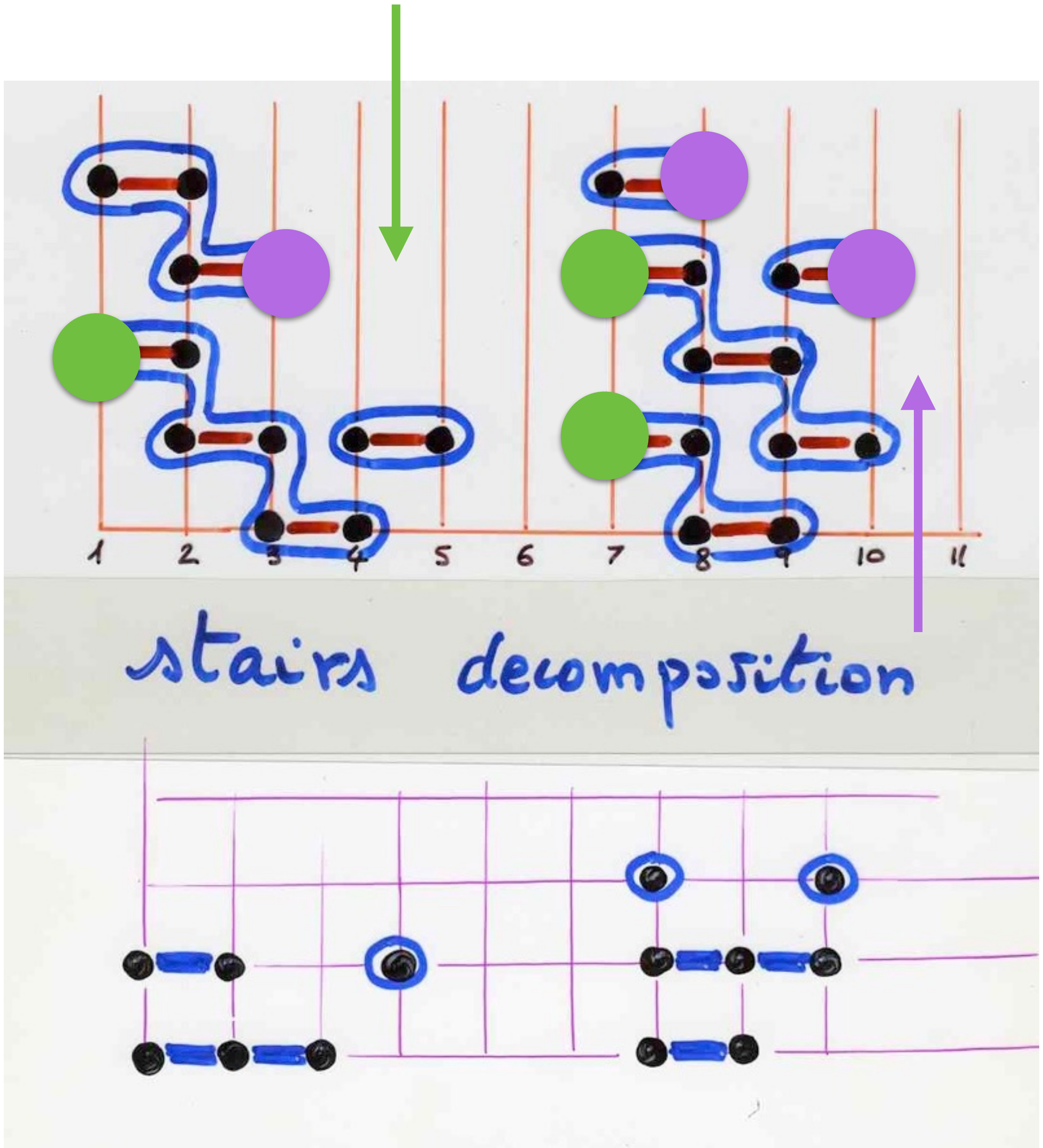
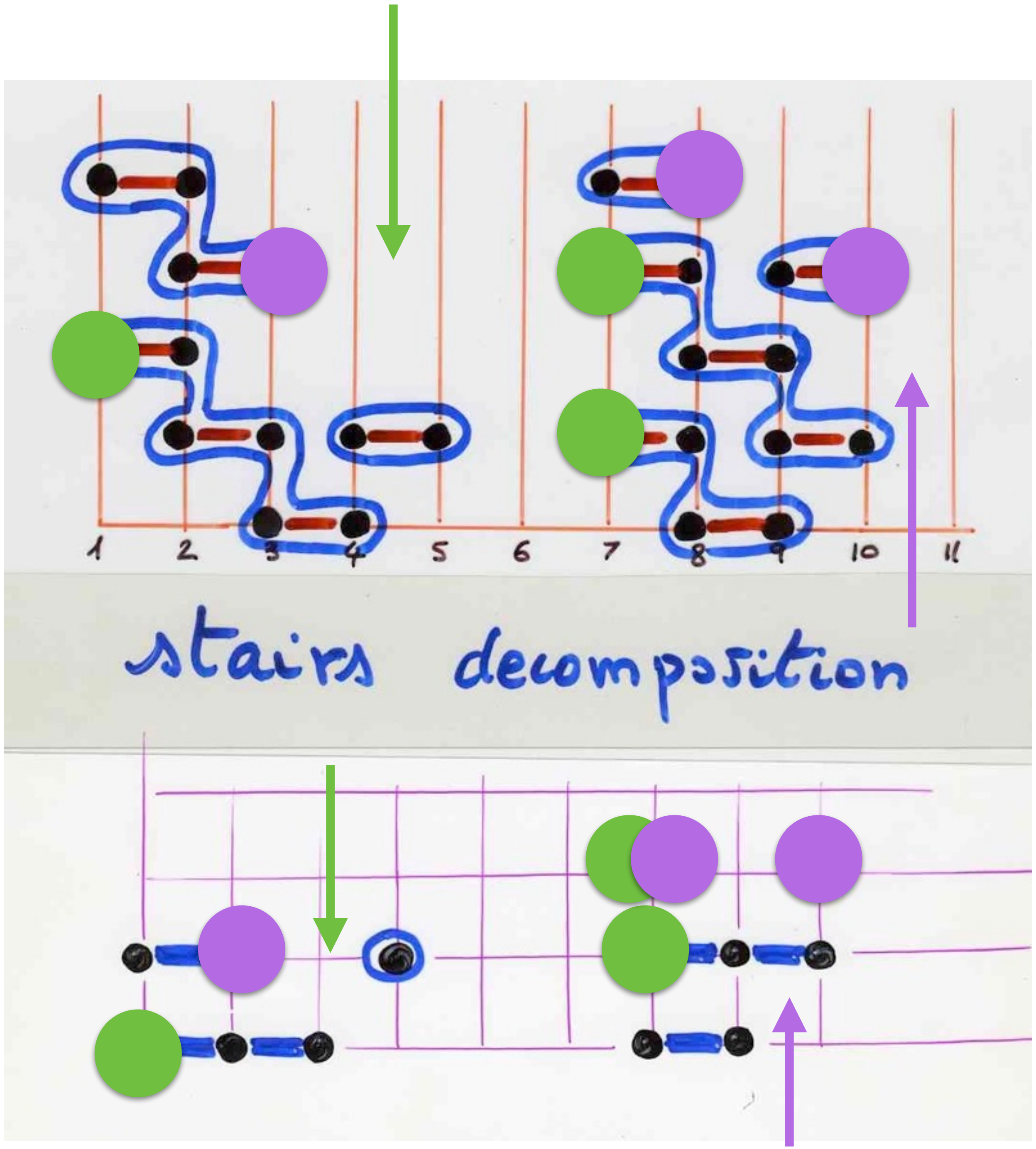

a
a
b
b
b



stairs decomposition




1 2 3 4 5 6 7 8 9 10 11
stairs decomposition


1
2
3
4
5
6
7
8
9
10
11
stairs decomposition

heaps of dimers

on $[0, n]$ with

total curvature $= 0$

up-curvature $= 0$

(or down-)

?

heaps of dimers

on $[0, n]$ with

total curvature $= 0$

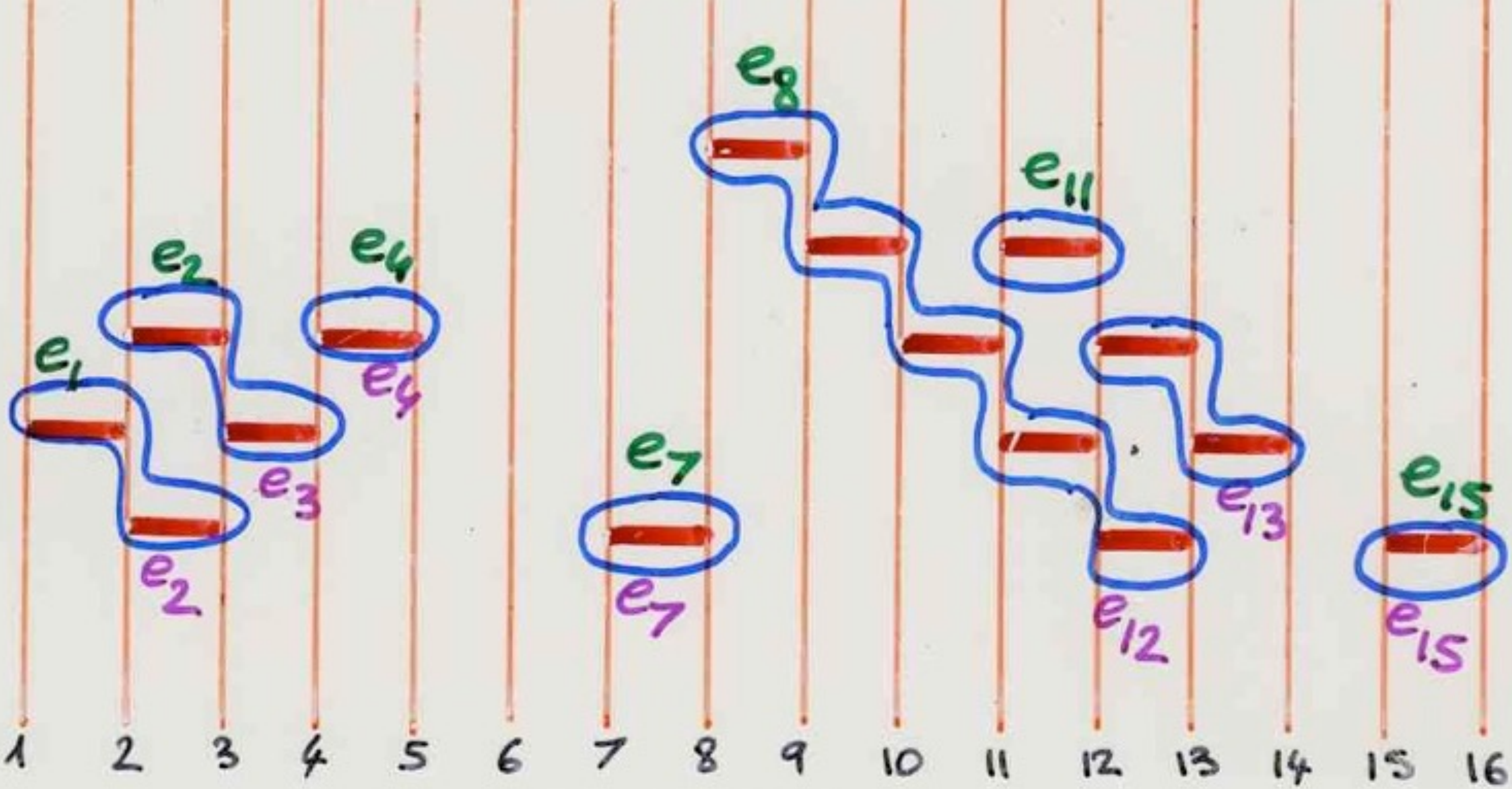
up- curvature $= 0$

(or down-)

number

C_n Catalan number

$n!$



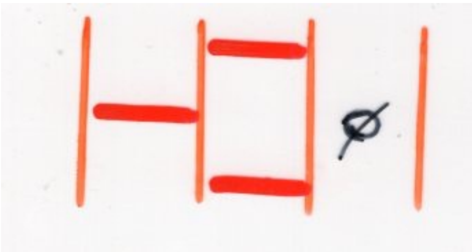
from Chapter 6a

$$\begin{array}{ccccccccccccccccc}
 & & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & & \\
 & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
1 & \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq & n
\end{array}$$

exercise

from Chapter 6a, p86

The number of strict heaps satisfying the condition:



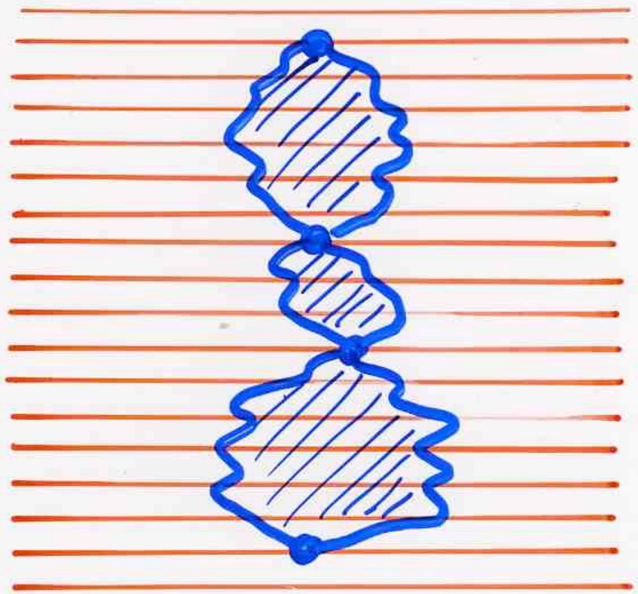
$\Longleftrightarrow$ $\min(S_1) < \cdots < \min(S_k)$

is $n!$

$\Longleftrightarrow$ $\max(S_1) < \cdots < \max(S_k)$

Lorentzian triangulations
in 2D qantum gravity

the nordic decomposition
of a heap of dimers

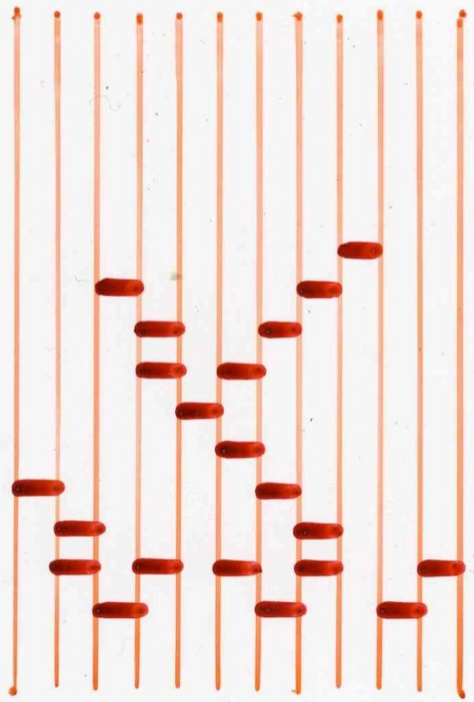


(general)
Lorentzian
triangulation

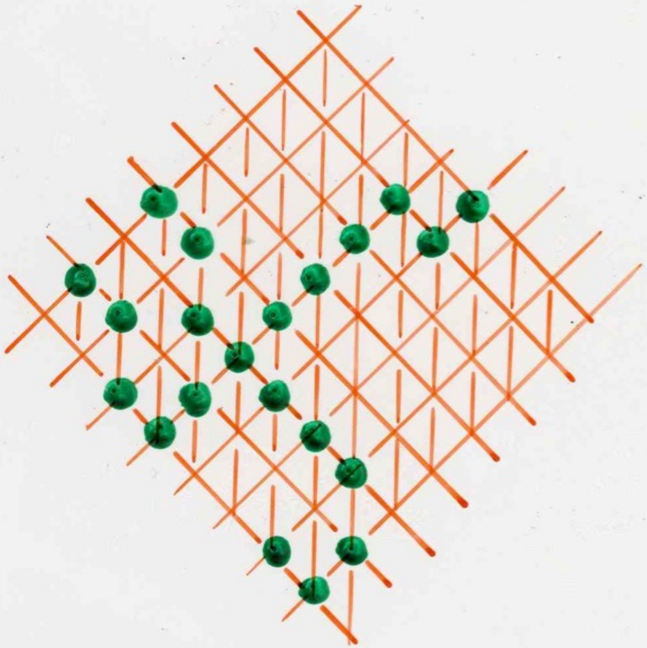
Lorentzian triangulation with no articulation points

$\Longleftrightarrow$

connected heap of dimers



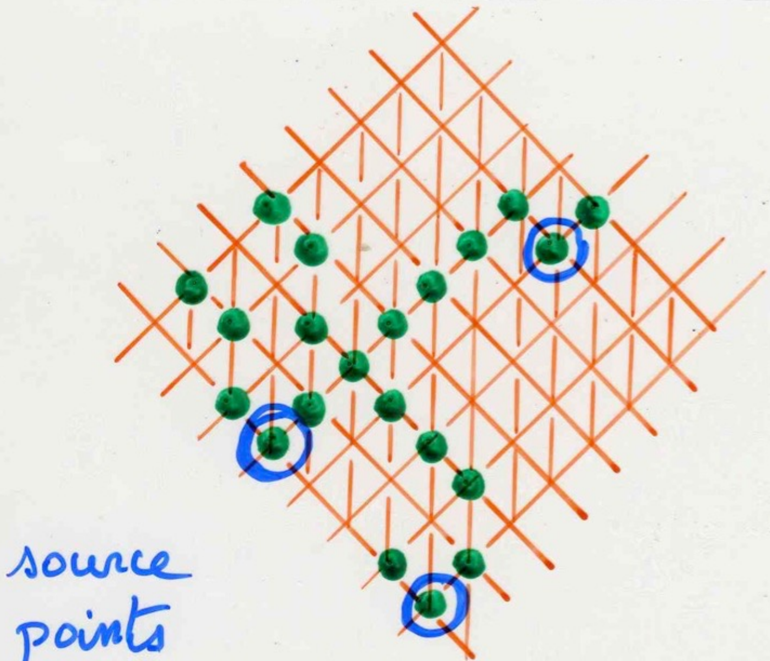
connected heap of dimers



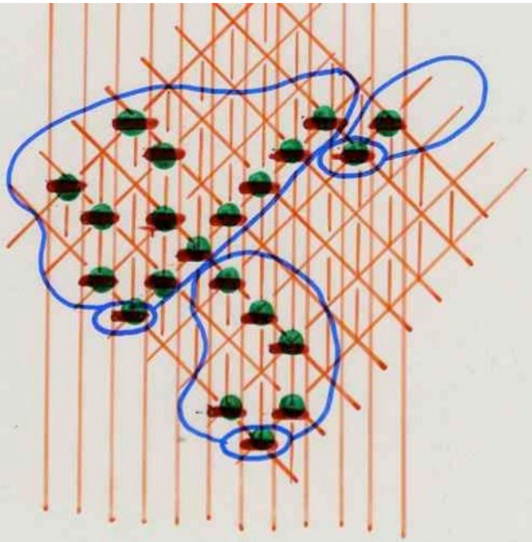
multidirected animal

(triangular lattice)

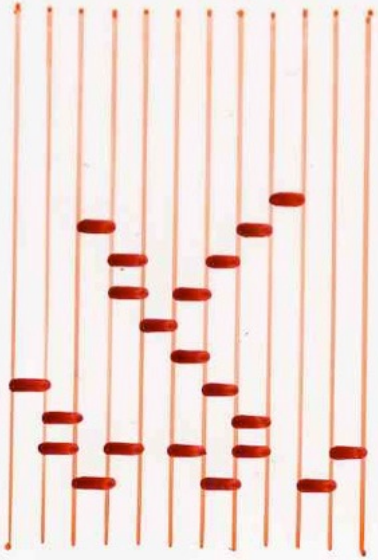
(Bousquet-Mélou, Rechnitzer, 2002)



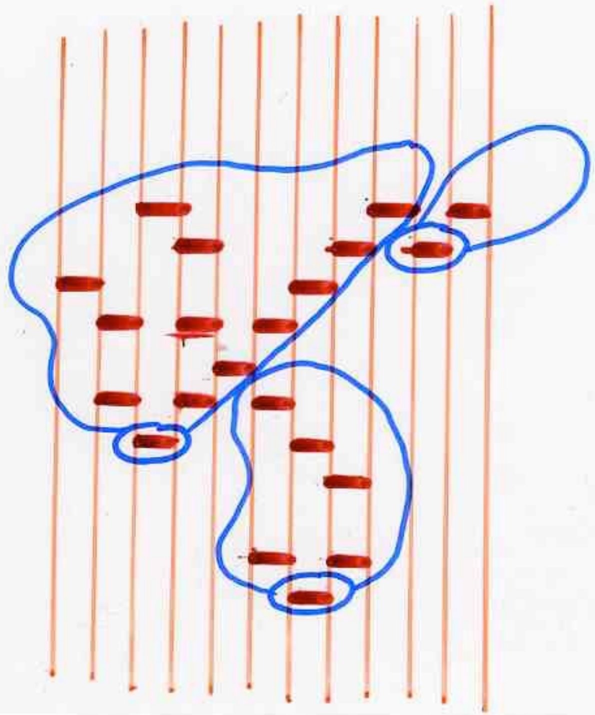
multidirected
animal
(triangular
lattice)
(Bousquet-Mélou,
Rechnitzer, 2002)



multidirected animal (triangular lattice) (Bousquet-Mélou, Rechnitzer, 2002)



connected heap of dimers



$$Q(t) = \frac{1-2t-\sqrt{1-4t}}{2t}$$

generating function for

half-pyramid $(\neq \emptyset)$

$$= \sum_{n \geqslant 1} C_n t^n$$

Catalan

$C(t)$ g.f. connected heap

Bousquet-Mélou, Rechnitzer (2002)

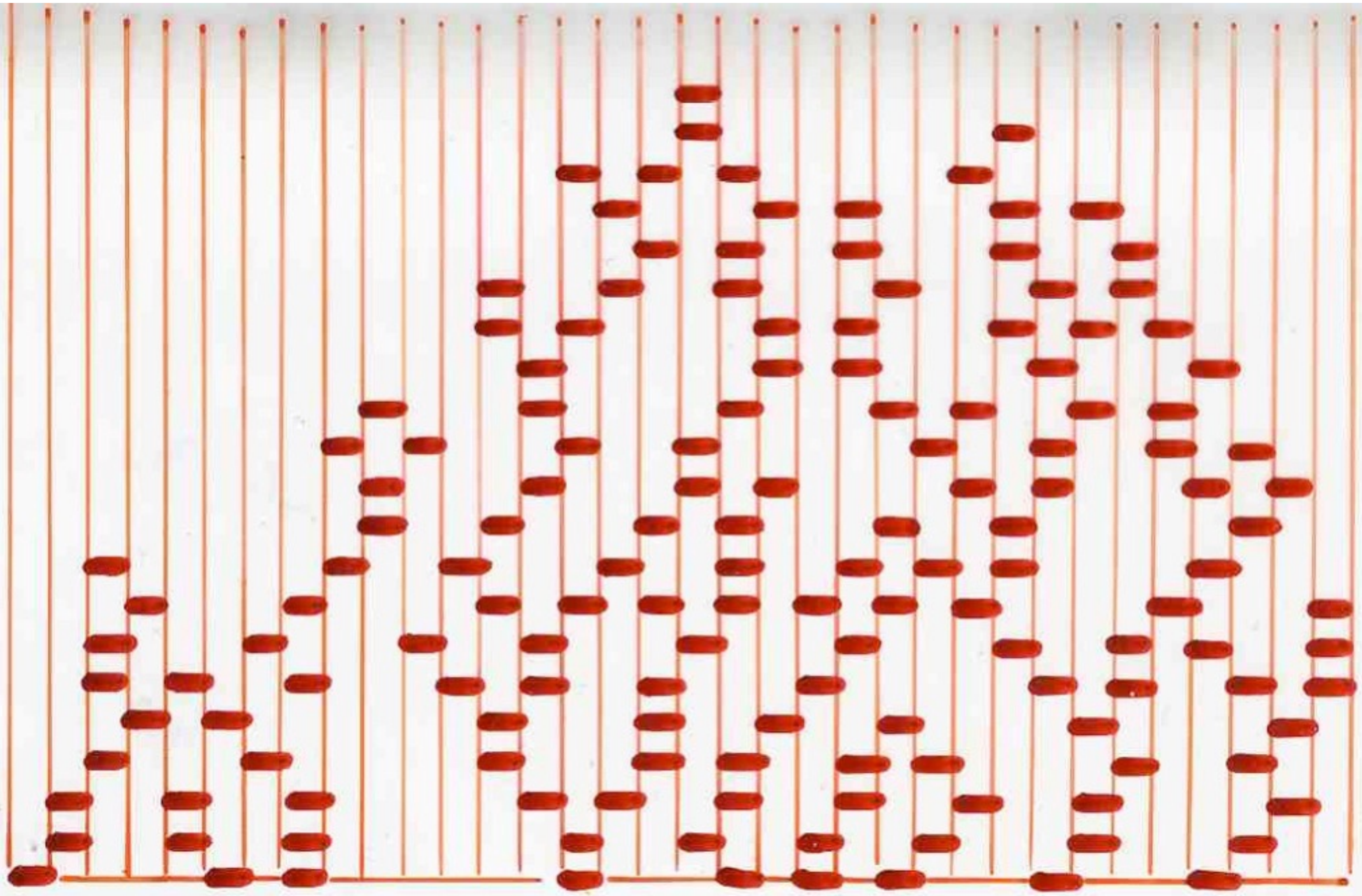
$$C(t) = \frac{Q}{(1-Q)\left[1 - \sum_{k\geq 1} \frac{Q^{k+1}}{1-Q^{k}(1+Q)}\right]}$$

$$C = \frac{Q}{(1-Q)\left[1-\boxed{\sum_{k\geq 1} \frac{Q^{k+1}(1+Q)^{k-1}}{1-Q^{k}}}\right]}$$

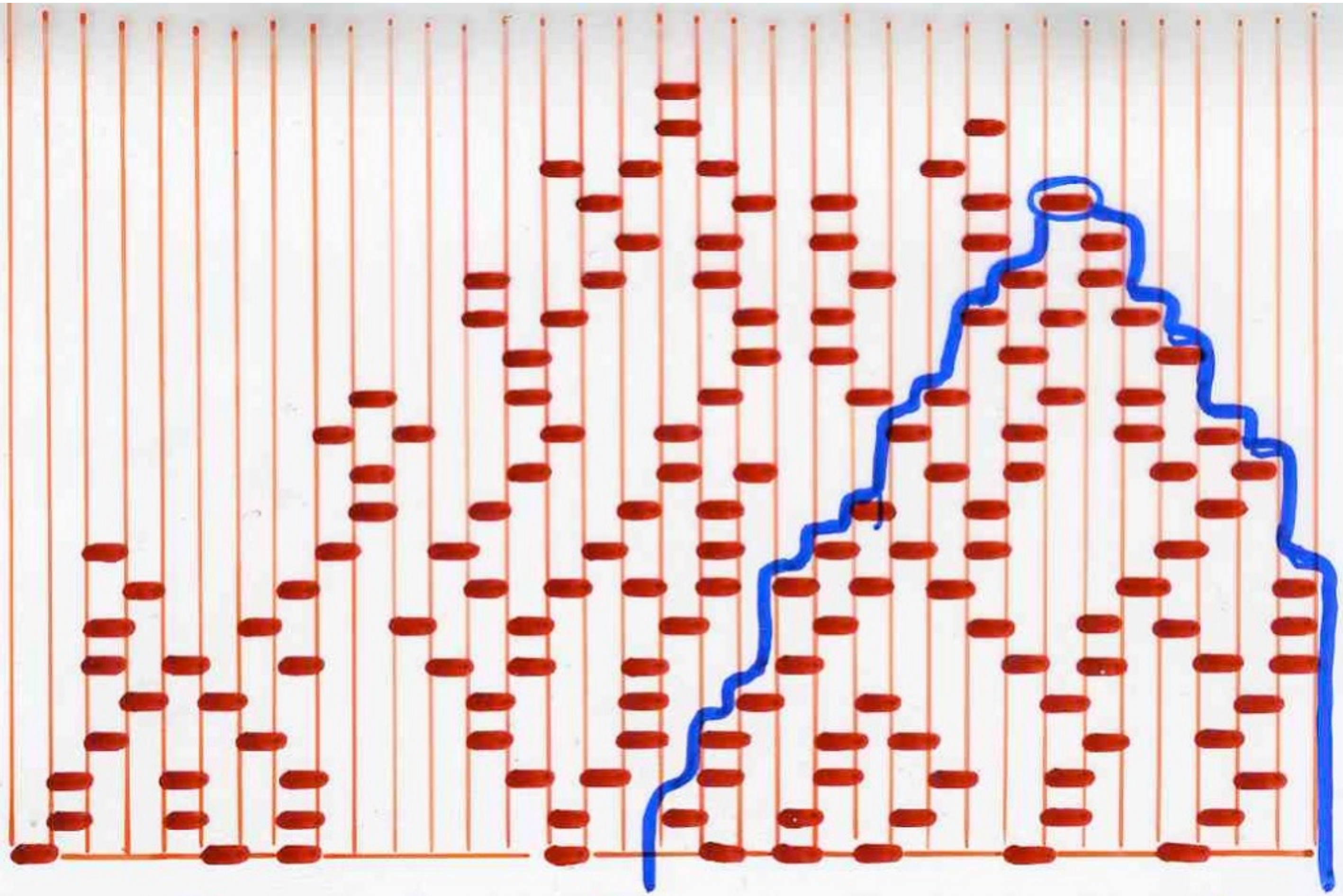
$$= \sum_{k\geq 1} \frac{Q^{k+1}}{1-Q^{k}(1+Q)}$$

bijective proof

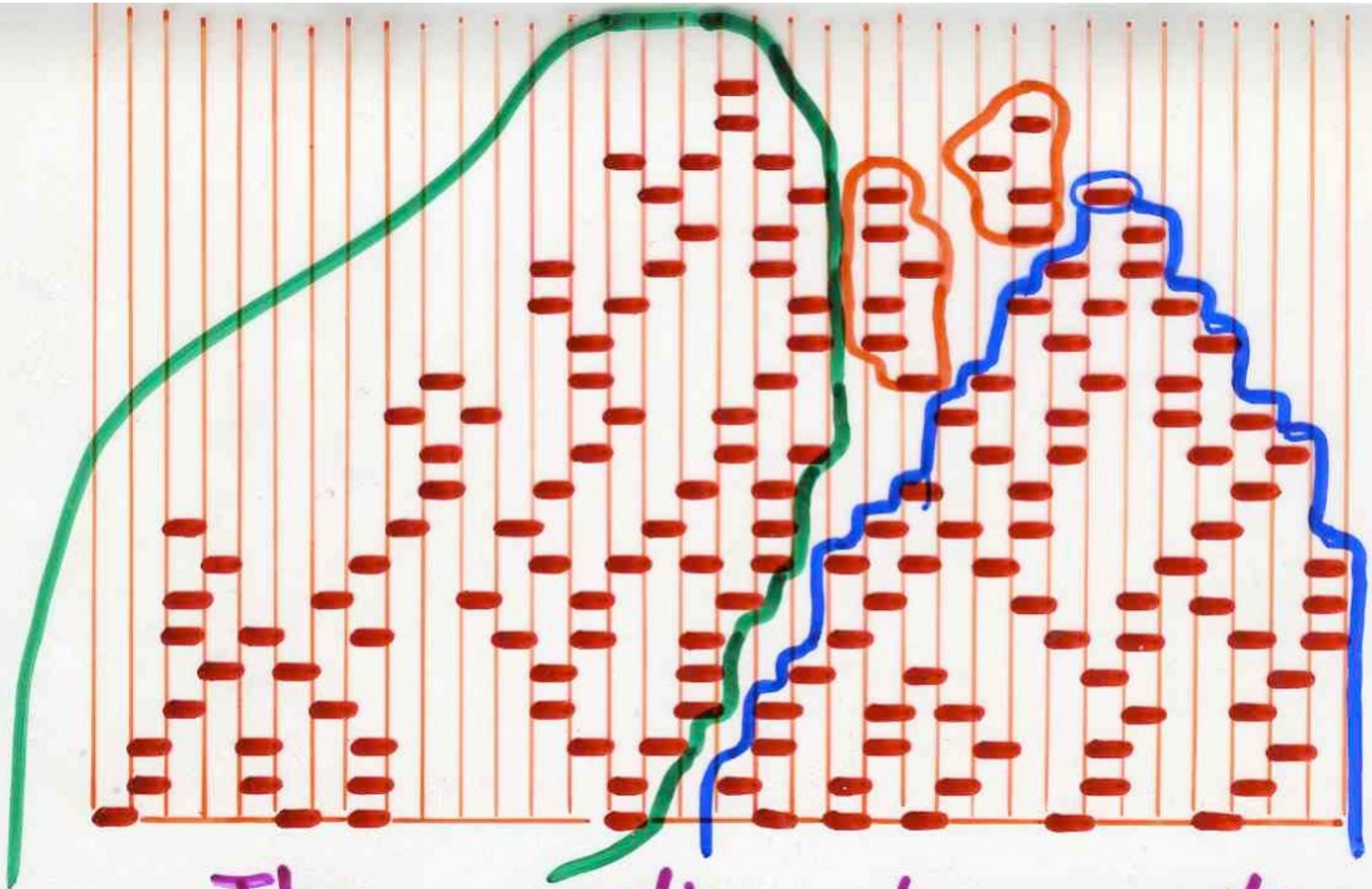
X.V. (2005)



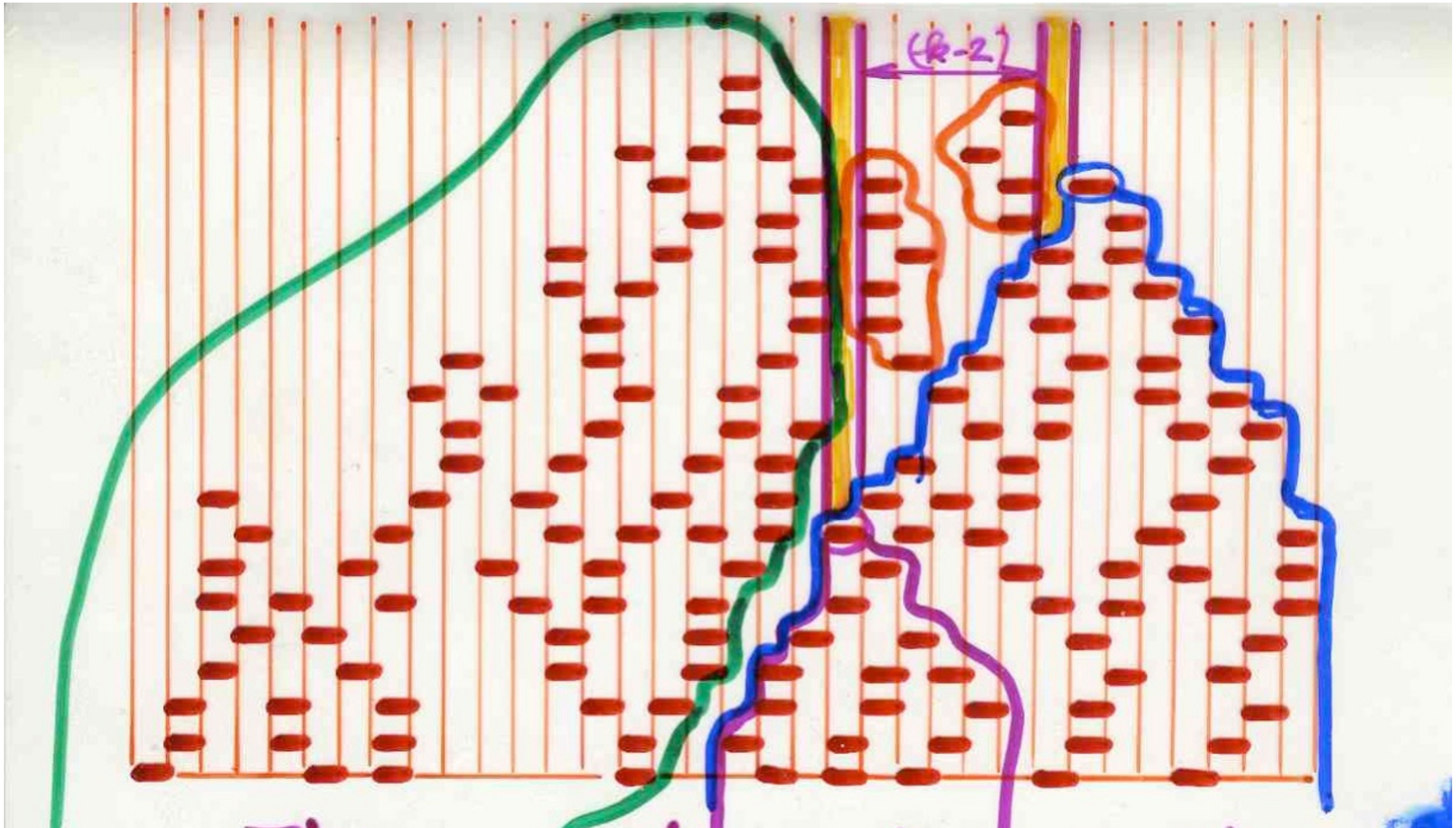
The nordic decomposition



The nordic decomposition

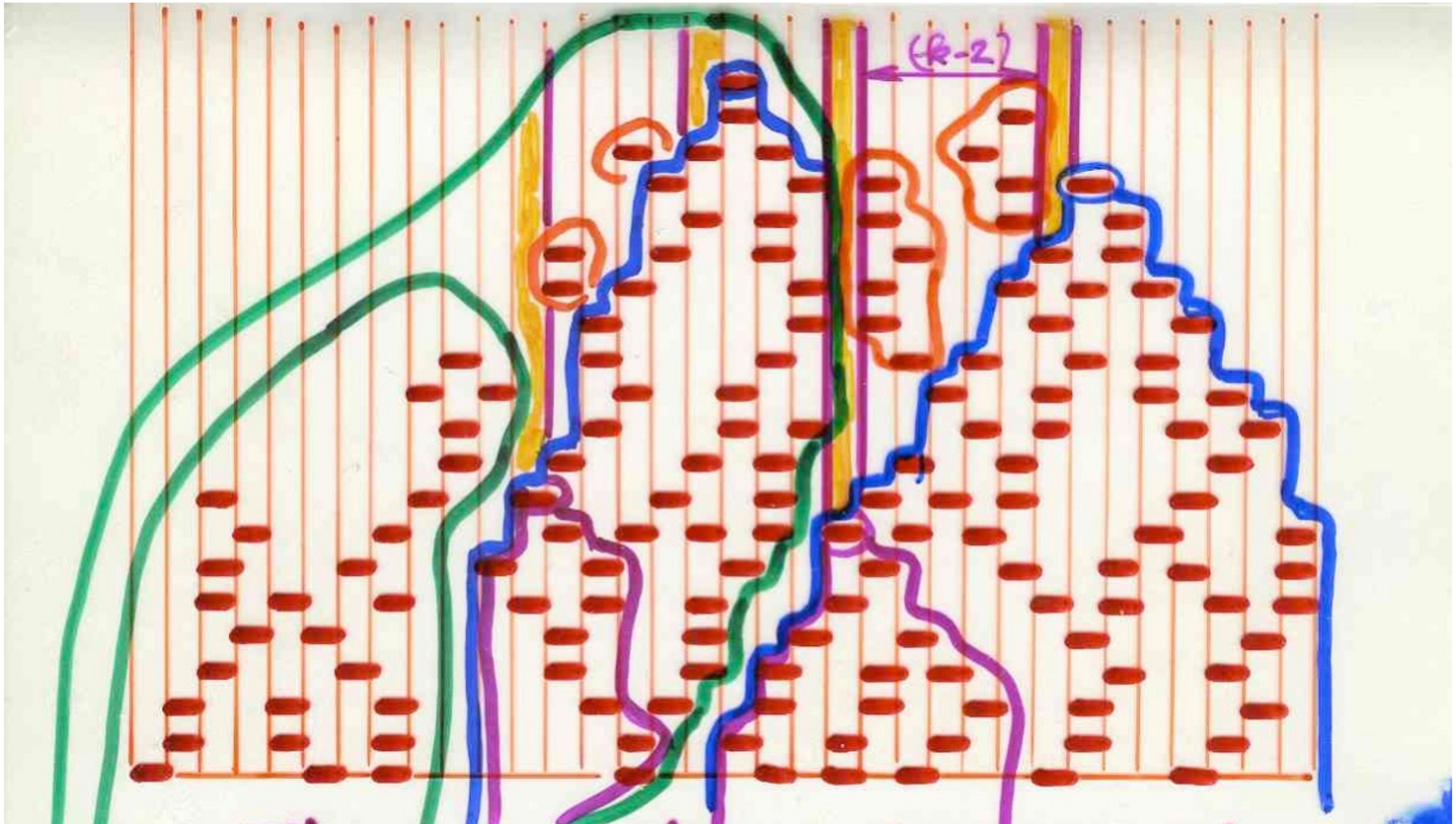


The nordic decomposition

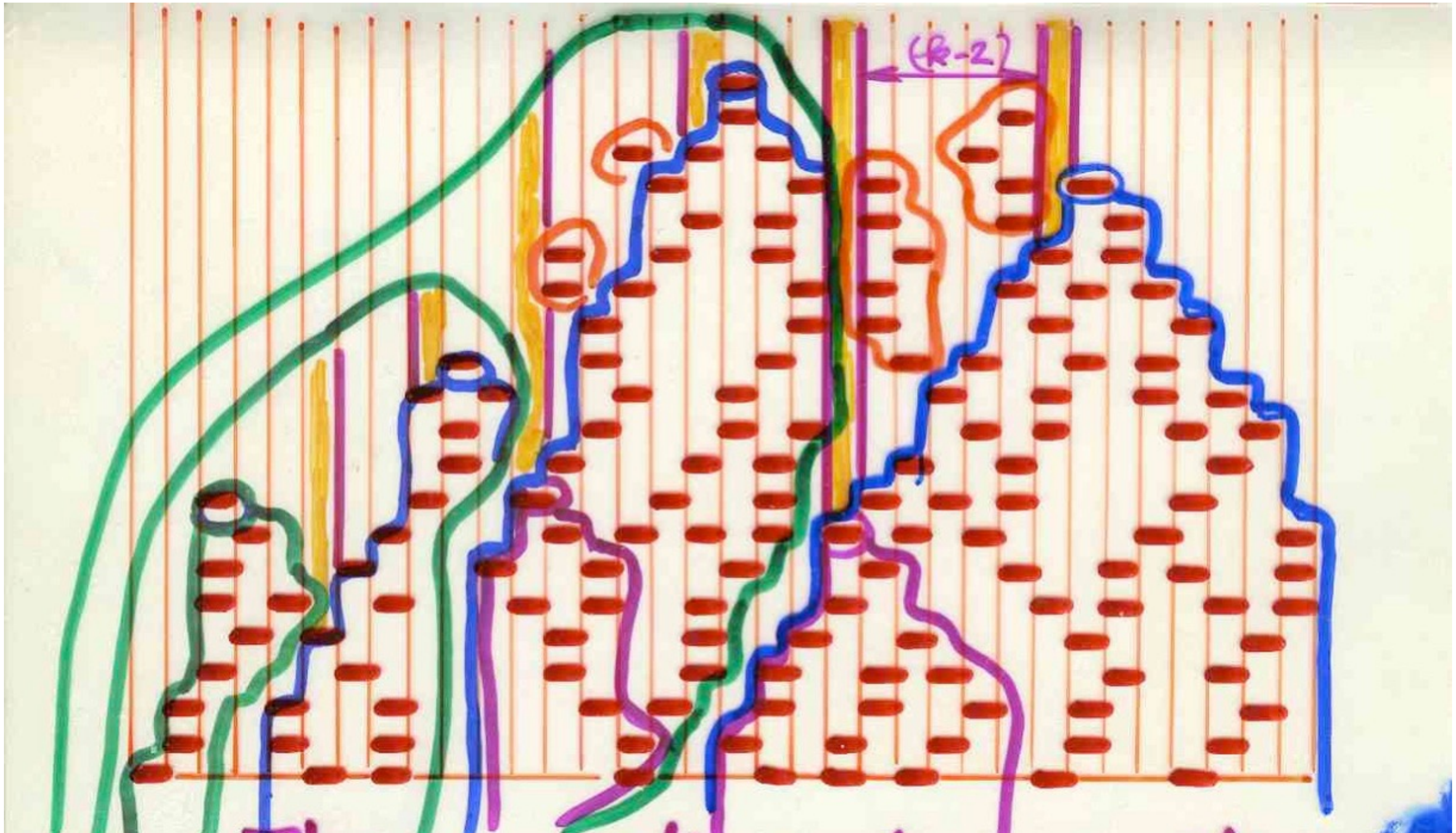


The nordic decomposition

$$C = \frac{Q}{1-Q} + C \sum_{k \geq 1} \frac{Q}{1-Q} \times Q^{k} \times \frac{1}{F_{k-1}}$$



The nordic decomposition



The nordic decomposition

$$C = \frac{Q}{1-Q} + C \sum_{k \geq 1} \frac{Q}{1-Q} \times Q^{k} \times \frac{1}{F_{k-1}}$$

$$C = \frac{Q}{1-Q} \times \frac{1}{\left[1-\left(\sum_{k\geq 1} \frac{Q}{1-Q} \times Q^{k} \times \frac{1}{F_{k-1}}\right)\right]}$$

connected heap

$$F_n = \frac{(1-Q^{n+1})}{(1-Q)(1+Q)^n}$$

$$C = \frac{Q}{(1-Q)\left[1-\sum_{k\geq 1} \frac{Q^{k+1}(1+Q)^{-k-1}}{1-Q^{k}}\right]}$$

$$C = \frac{Q}{(1-Q)\left[1-\boxed{\sum_{k\geq 1}\frac{Q^{k+1}(1+Q)^{-k-1}}{1-Q^{k}}}\right]}$$

$$= \sum_{k\geq 1}\frac{Q^{k+1}}{1-Q^{k}(1+Q)}$$

solution exercise Ch2b, p103

Fibonacci polynomials

and

generating function of Catalan numbers

notations $D = 1 + Q$

generating function of Catalan numbers

$$Q(t) = \frac{1-2t-\sqrt{1-4t}}{2t}$$

generating function for half-pyramid $(\neq \emptyset)$

$$= \sum_{n\geq 1} C_n t^n$$

Catalan

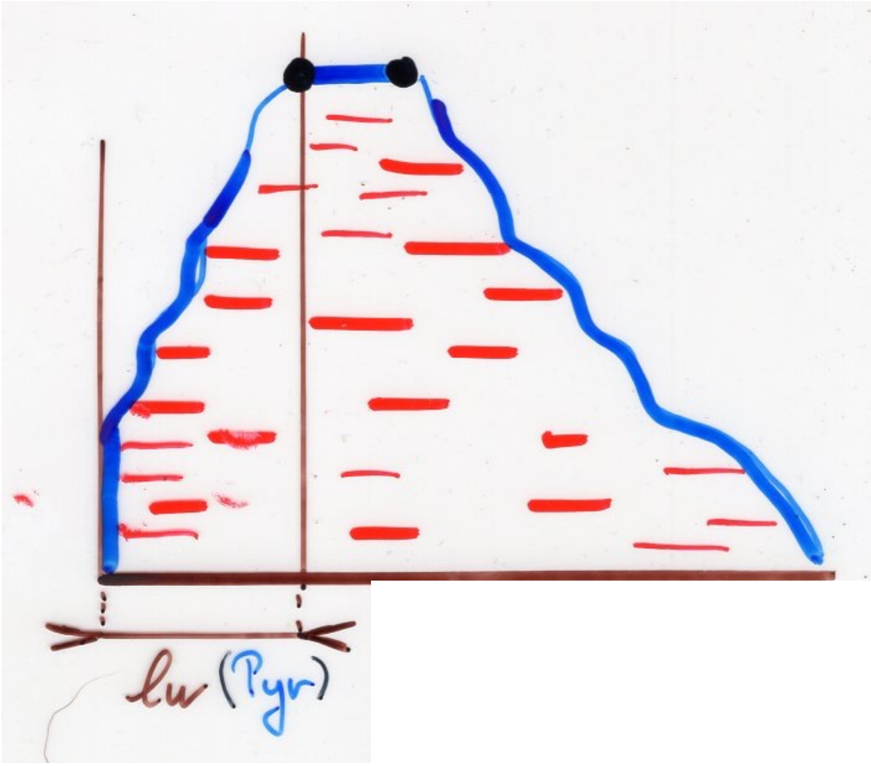
$F_n(t)$ n^{th} Fibonacci polynomial

we want to prove the following identity.

$$F_n = \frac{(1-Q^{n+1})}{(1-Q)(1+Q)^n}$$

$$\Leftrightarrow$$

$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n} \times (1+Q+\ldots+Q^n)$$



semi-pyramid :

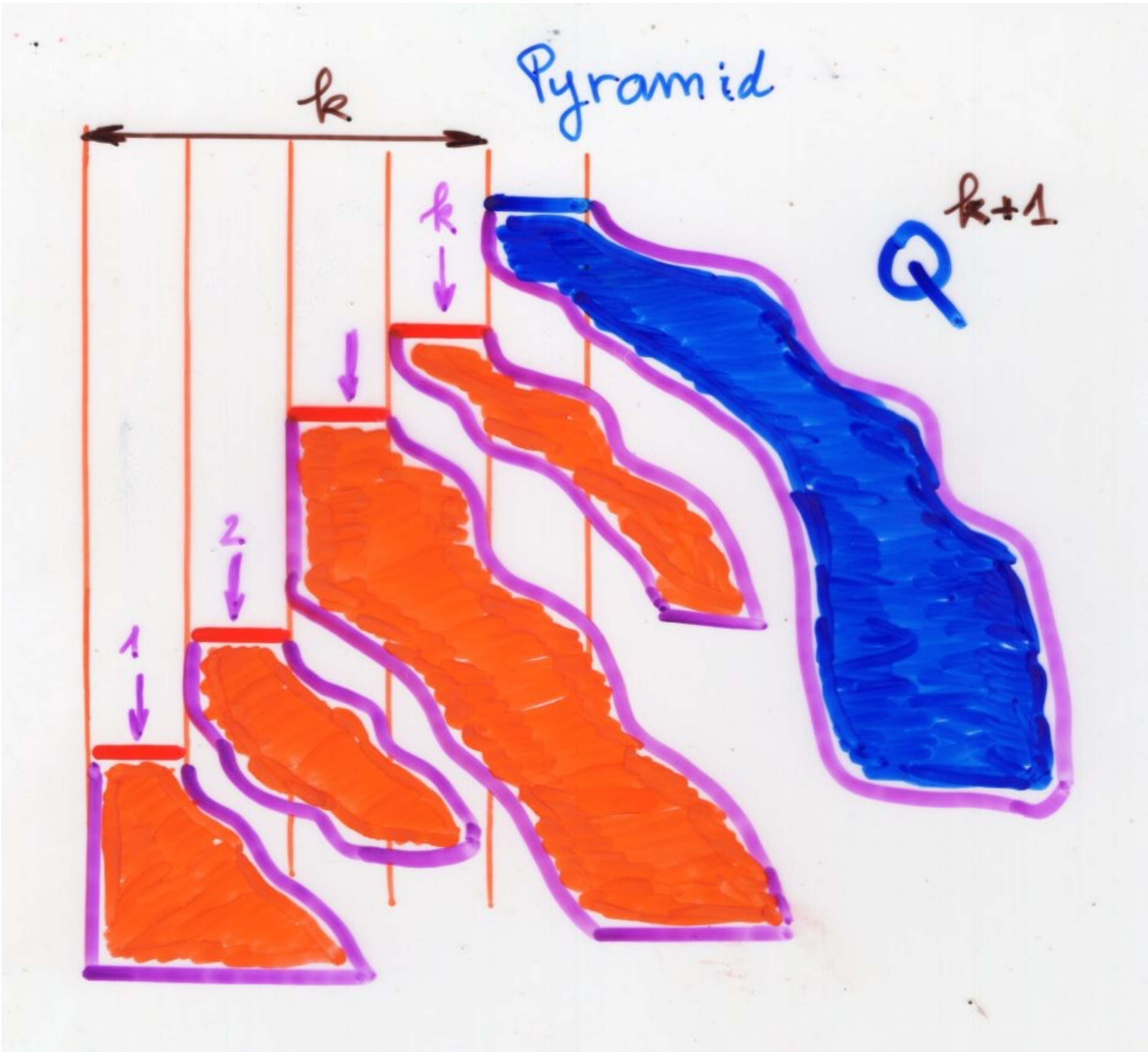
lw (Pyr) = 0

left-width of a pyramid of dimers

lw (Pyr)

a) Prove that the generating function of (non-empty) pyramids of dimers Pyr with left-width lw (Pyr) = k, is equal to

$$Q^{k+1}$$


Pyramid
k
k
2
1
Q k+1

b)

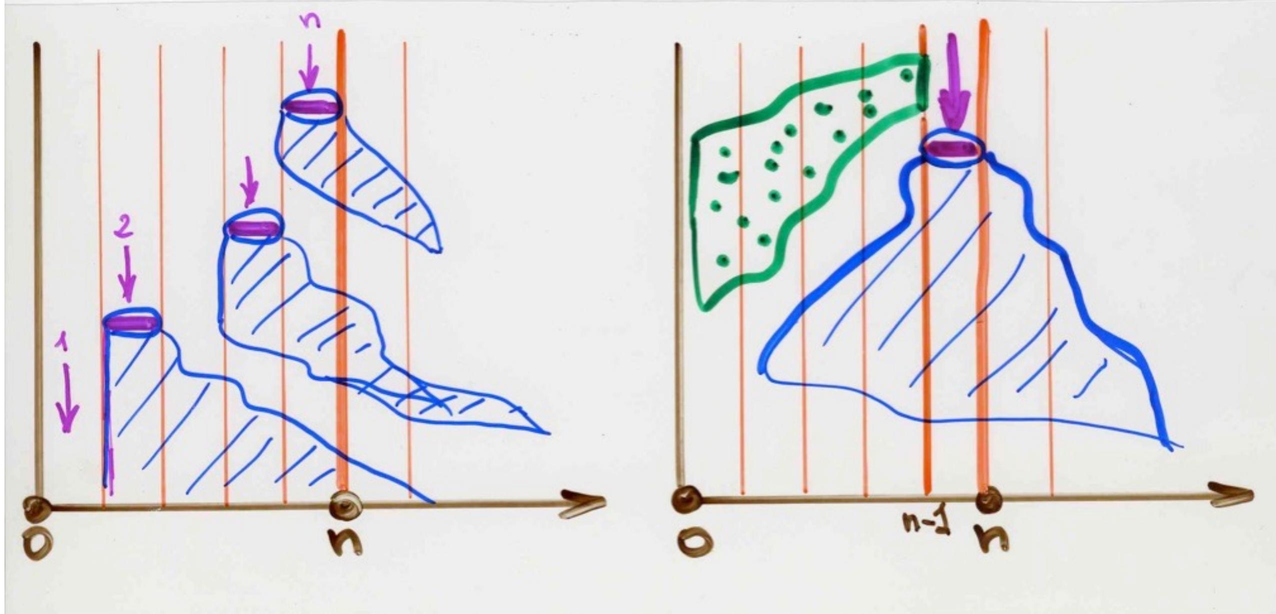
Prove that both sides of the identity are the generating function of:

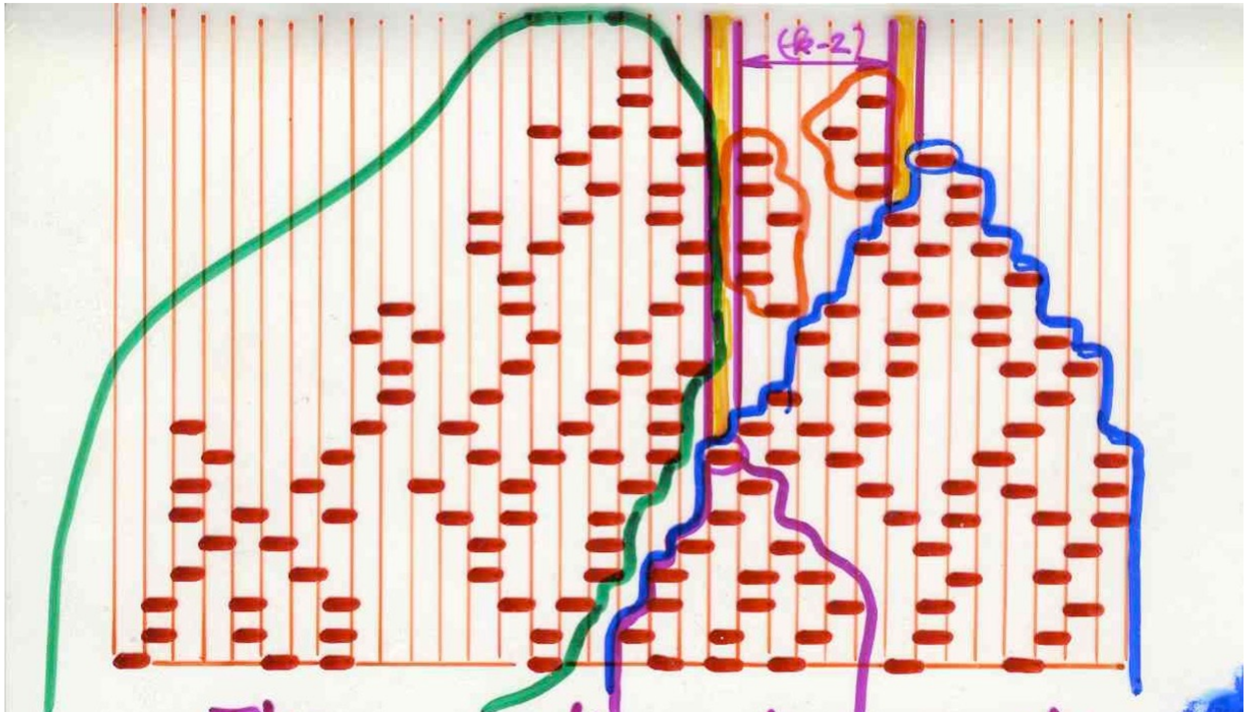
$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n} \times (1+Q+\ldots+Q^n)$$

heaps of dimers on $[0, \infty[$

maximal pieces, projection $\subseteq [0, n]$

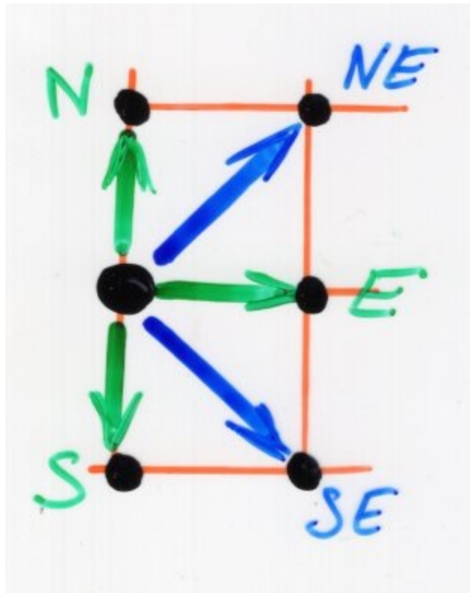
$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n} \times (1+Q+\dots+Q^n)$$





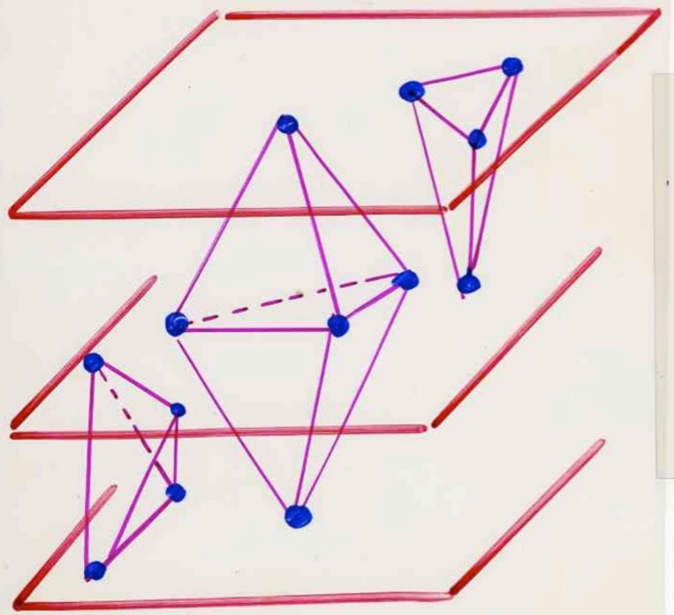
The nordic decomposition

A. Bacher (2016)
partially directed animals



Lorentzian quantum gravity

(1+1) + 1 dimension



3d

4d

?

Benedetti, Loll, Zamponi (2007)
arXiv: 0704.3214

Benedetti, thesis (2007)
arXiv: 0707.3070

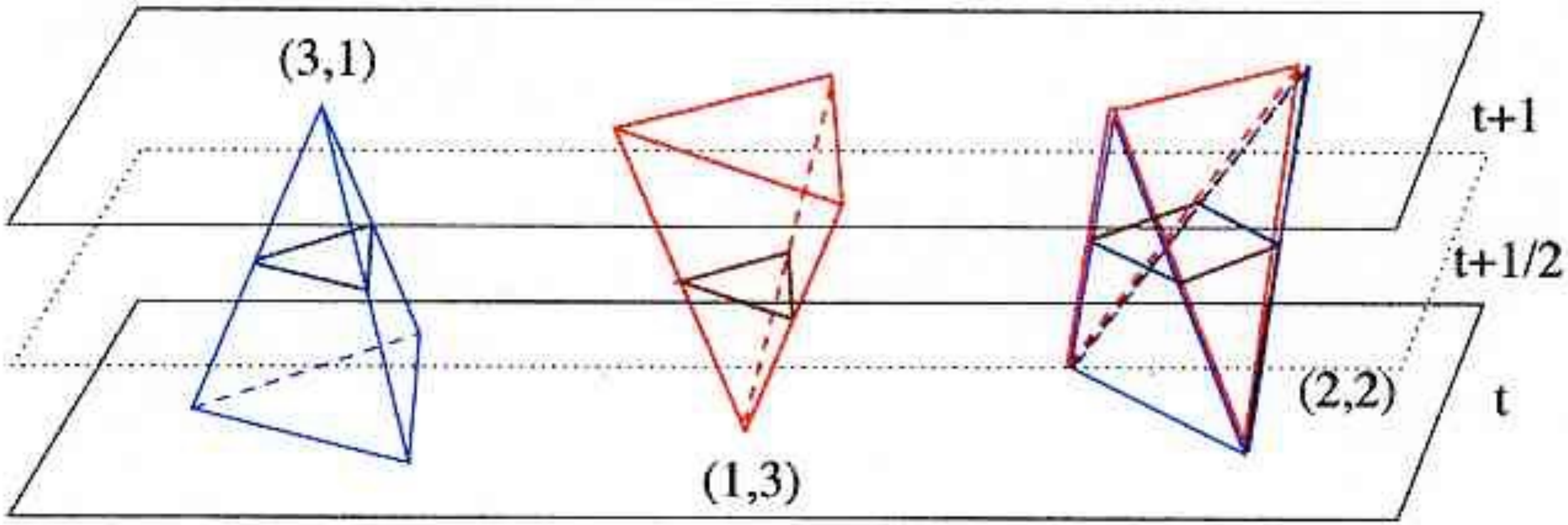
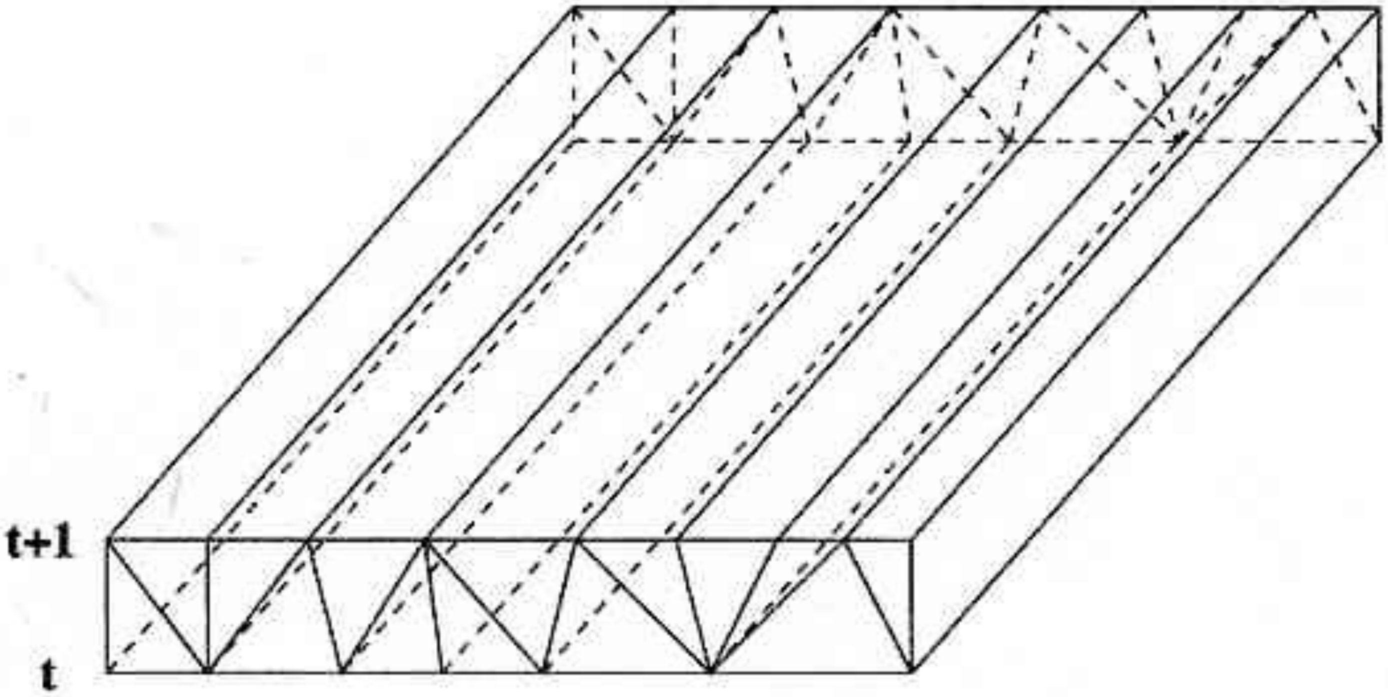


Figure 4: The three types of tetrahedral building blocks and their intersections at time $t + 1/2$.


t+1
t

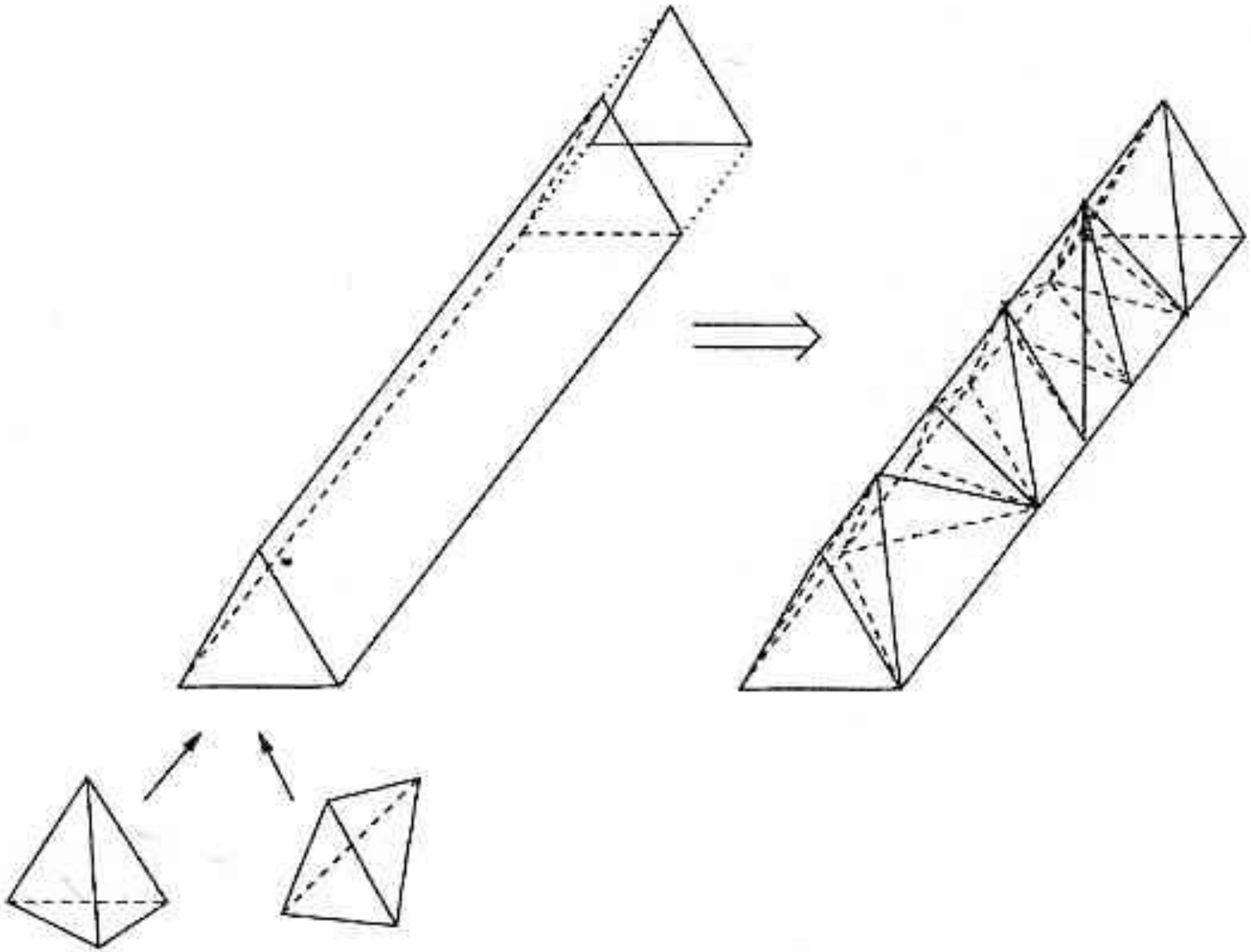
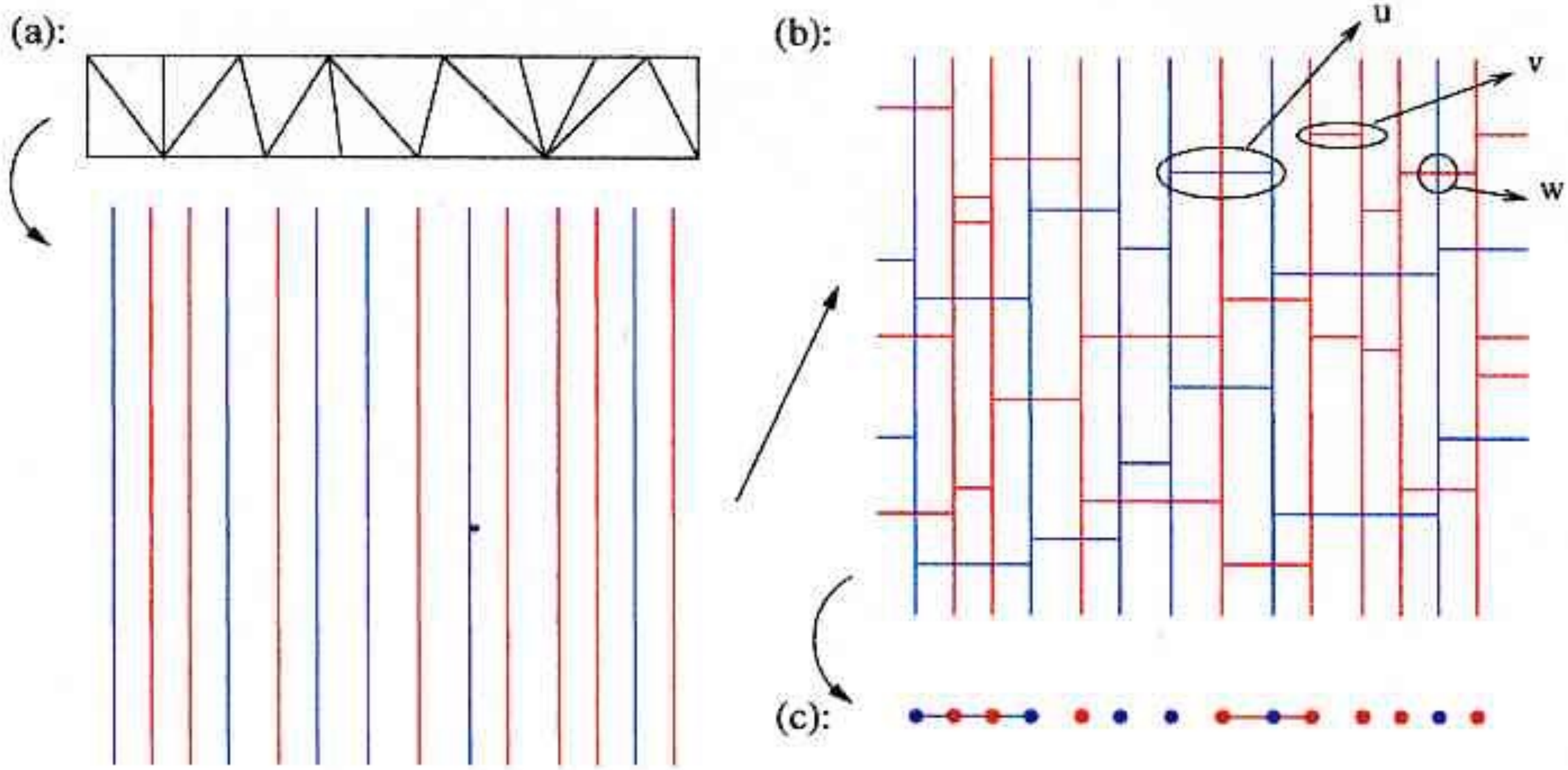


Figure 2: A triangulated prism constructed as a tower over a two-dimensional triangle.


(a):
(b):
u
v
w
(c):