Chapter 7

Heaps in statistical mechanics (1)

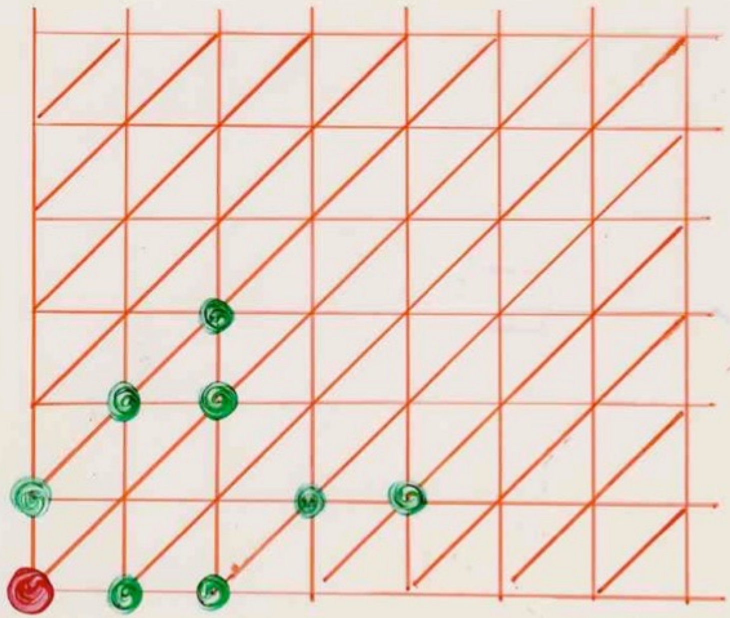
slides: second part of Ch7a

IMSc, Chennai

2 March 2017

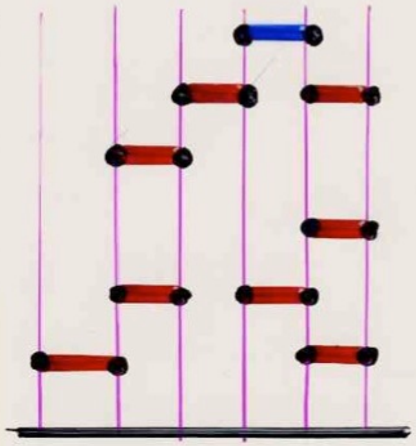
directed animals

on a triangular lattice



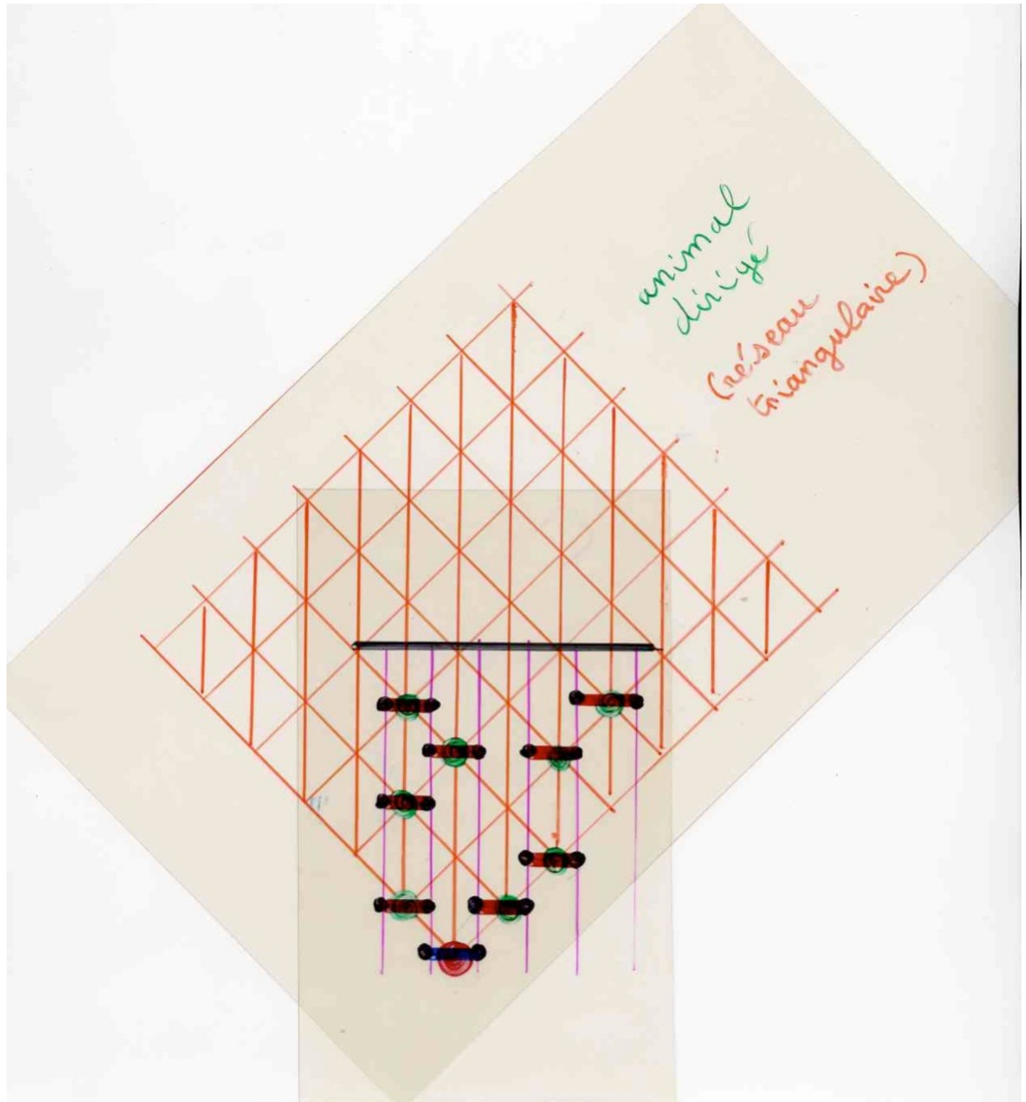
animal
dirigé
(réseau
triangulaire)

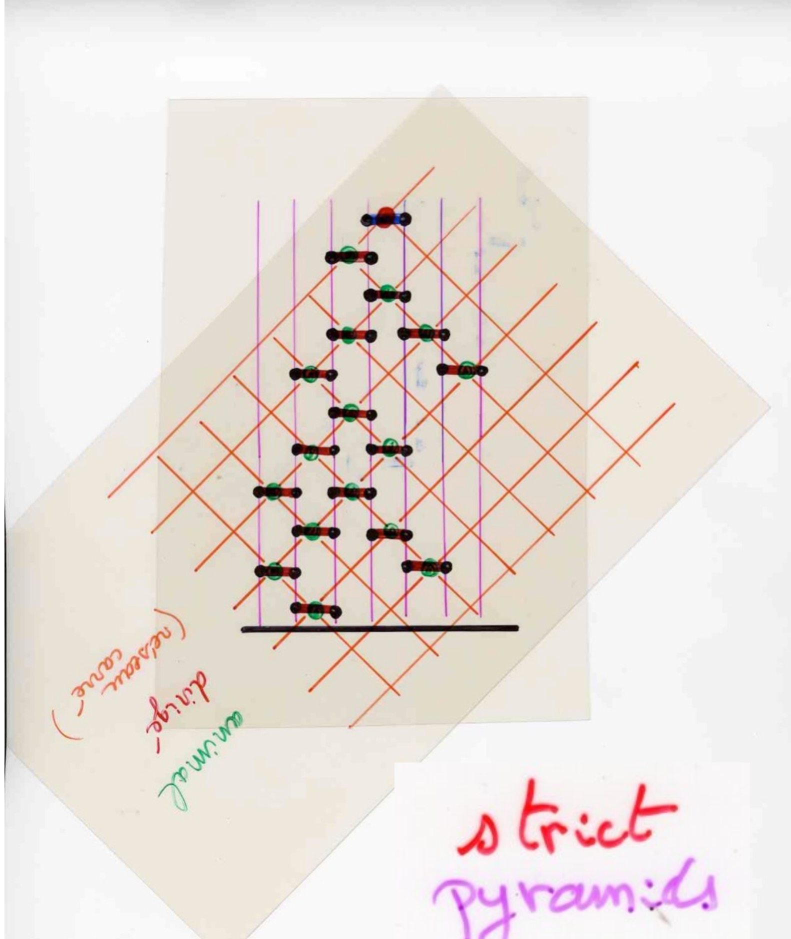
bijection



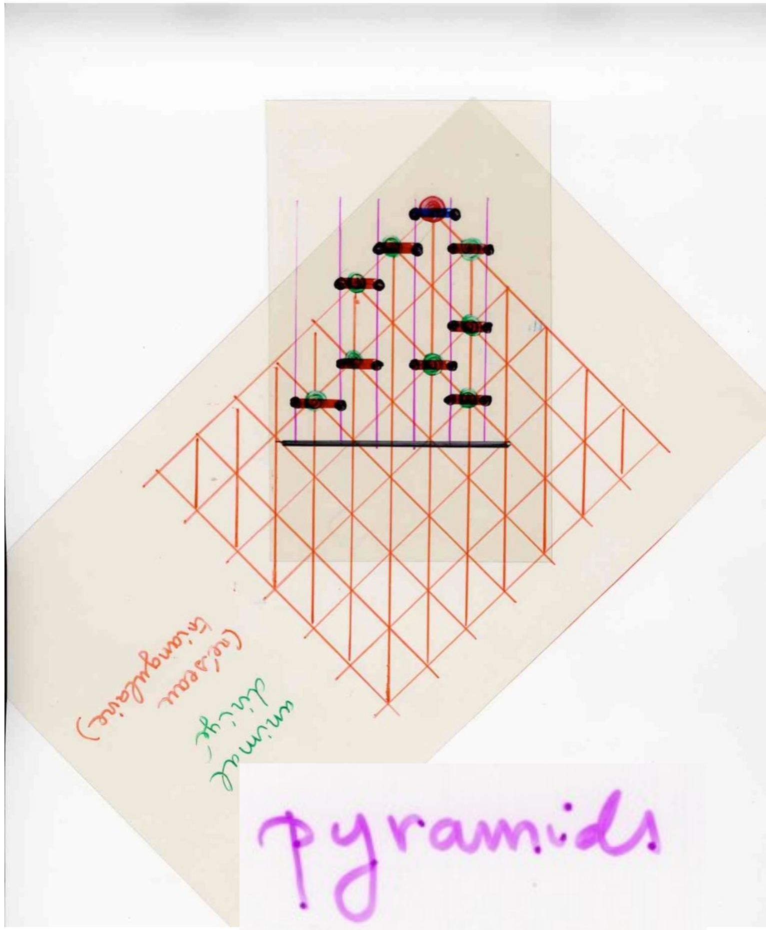
pyramids

directed
animal
(square
lattice)


animal
dirigé
(réseau
triangulaire)



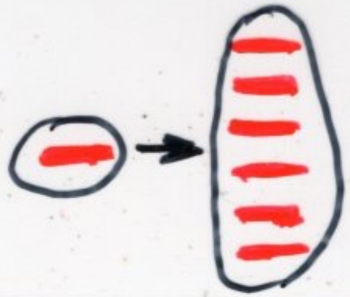
strict
pyramids



pyramids

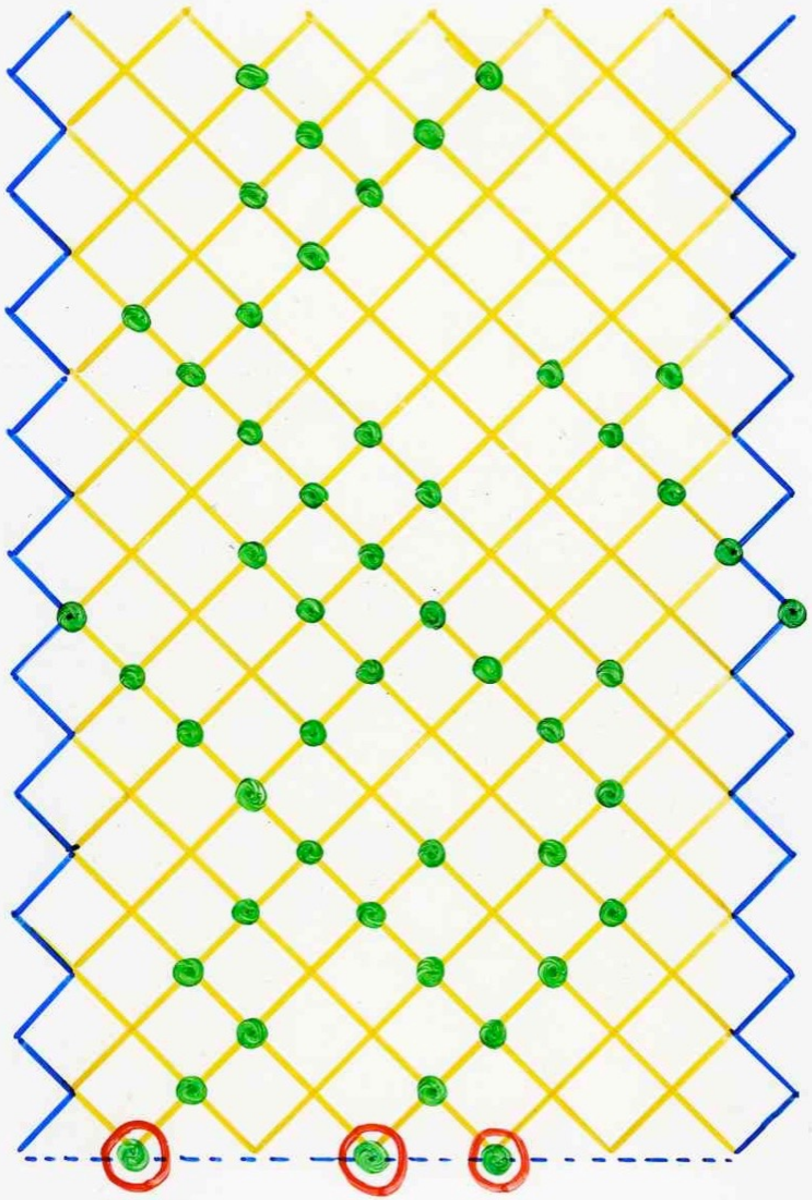
strict
heap $\longrightarrow$ heap

$$t \longrightarrow \frac{1}{1-t}$$



directed animals

on bounded strip



directed animal
on a circular strip

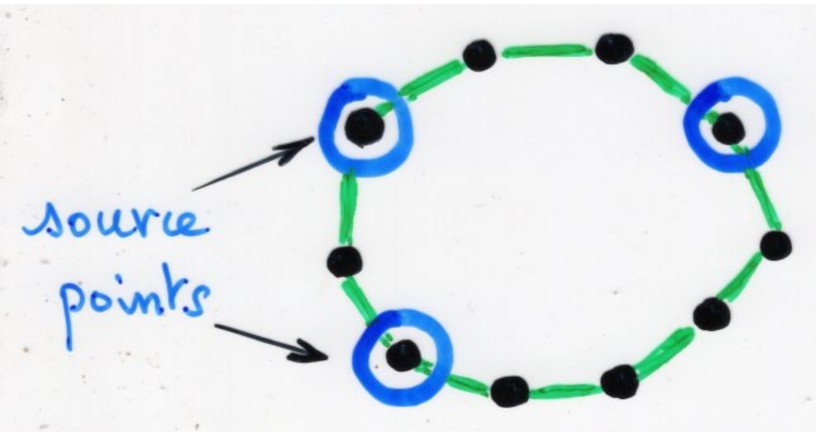
Nadal, Derrida, Vannimenus (1982)

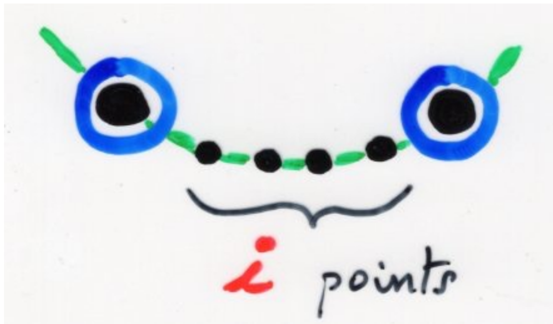
$$\tilde{b}_n^{\leq k} = \frac{1}{k} \sum_{p=0}^{k-1} (-1)^p \sin\alpha_p \prod_{i=1}^{k-1} \left(\frac{\sin(i+\frac{1}{2})\alpha_p}{\sin\frac{\alpha_p}{2}} \right)^{N_i} (1+2\cos\alpha_p)^{n-1}$$

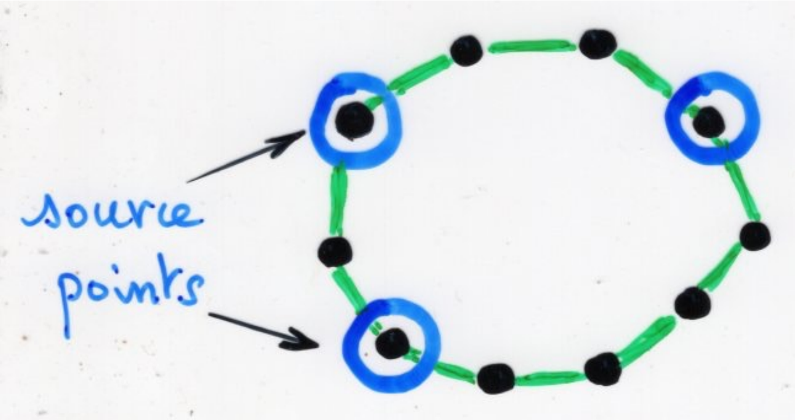
animals
circular strip
width k

$$\alpha_p = \frac{2p+1}{2k}\pi$$

N_i = number of i-holes







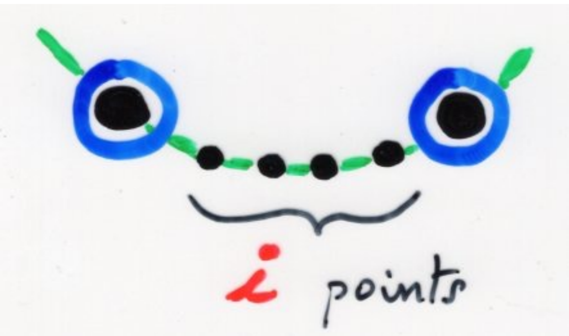
$$\frac{N}{D}$$

$$D = L_n(x)$$

Lucas polynomial

$$N = \prod_i F_i(x)$$

Fibonacci polynomial



N_i = number of i-holes

Nadal, Derrida, Vannimenus (1982)

$$\tilde{b}_n^{\leq k} = \frac{1}{k} \sum_{p=0}^{k-1} (-1)^p \sin\alpha_p \prod_{i=1}^{k-1} \left(\frac{\sin (i+\frac{1}{2})\,\alpha_p}{\sin \frac{\alpha_p}{2}} \right)^{N_i} (1+2\cos\alpha_p)^{n-1}$$

animals

circular strip

width k

$$\alpha_p = \frac{2p+1}{2k}\pi$$

$$\frac{N}{D} \qquad N = \prod_i F_i(x) \qquad D = L_n(x)$$

$$T_n(x) = \frac{1}{2} C_n(2x) \qquad C_n^* = L_n(x^2) \qquad \cos(n\theta) = T_n(\cos\theta)$$

zeros of $T_n(x)$: $\left\{ \cos\left(\frac{(2k-1)}{2n}\pi\right), \quad k = 1, .., n \right\}$

combinatorial understanding

of the thermodynamic limit

the case a 1D gas model

$$Z_n(t) = F_n(-t)$$

Fibonacci polynomials

thermodynamic limit

$$\lim_{n \to \infty} \left(Z_n(t) \right)^{1/n}$$

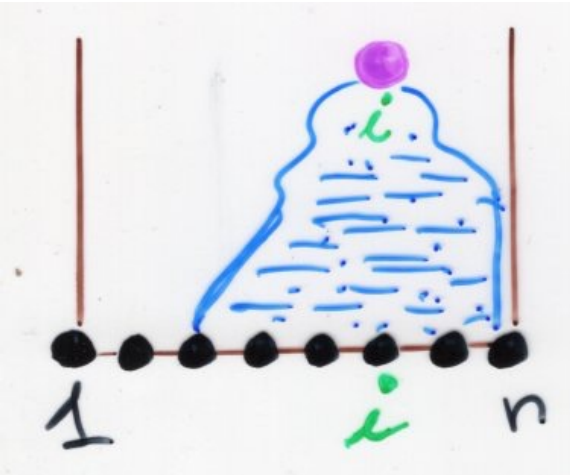
$$\log Z_n^{1/n}(t) = \frac{1}{n} \log Z_n(t)$$

$$= -\frac{1}{n} \log \frac{1}{Z_n(t)}$$

$$-t \frac{d}{dt} \log Z_n^{1/n}(-t) = \underbrace{\frac{1}{n} t \frac{d}{dt} \log \frac{1}{Z_n(-t)}}_{\frac{1}{n} \mathrm{Pyr}_n(t)}$$

$\mathrm{Pyr}_n(t)$ generating function of pyramids on $[1,n]$

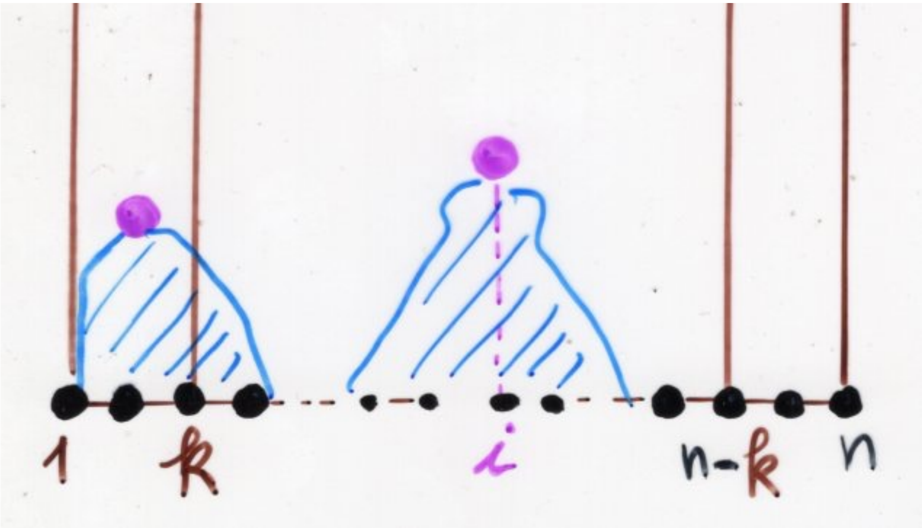
$$\mathrm{Pyr}_n(t) = \sum_{1\leq i\leq n} \mathrm{Pyr}_n^i(t)$$



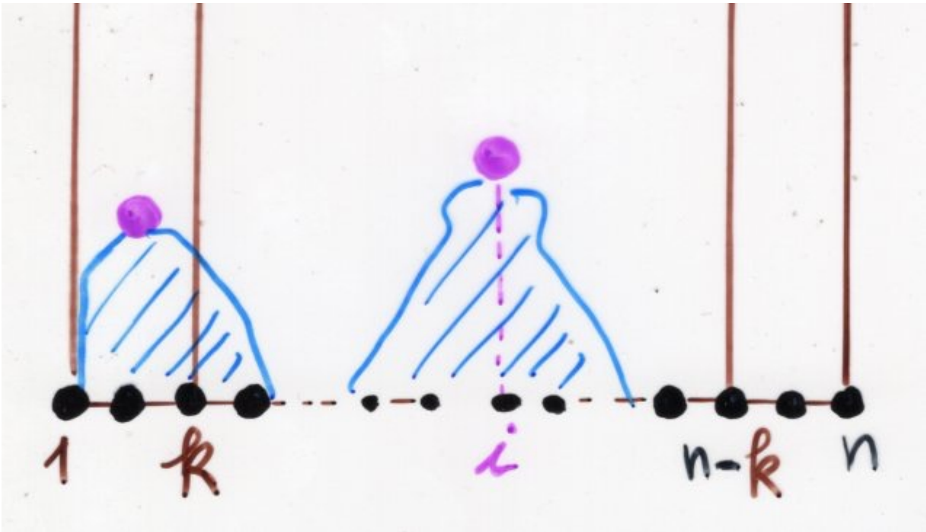
generating function of pyramids over $[1,n]$ with $\pi(\max) = i$

Let $k \geq 1$

$$\mathrm{Pyr}^{i}_{n,k}(t) = t + a^{i}_{n,2}t^2 + \cdots + a^{i}_{n,k}t^k + \cdots$$



for $k \leq i \leq n-k$ $\quad \mathrm{Pyr}^{i}_{n}(t)$ does not depend of i



$$\mathrm{Pyr}_{n,k}(t) = [n-2(k-1)]\,\mathrm{Pyr}^{*}_{n,k}(t) + \sum_{\substack{1\le i<k \\ \text{or } k<i\le n}} \mathrm{Pyr}^{i}_{n,k}$$

$$\frac{1}{n}\mathrm{Pyr}_{n,k}(t) = \left(1-\frac{2(k-1)}{n}\right)\mathrm{Pyr}^{*}_{n,k}(t) + \frac{1}{n}\sum_{\substack{1\le i<k \\ \text{or} \\ k<i\le n}} \mathrm{Pyr}^{i}_{n,k}(t)$$

k fixed
$n \to \infty$

$\longrightarrow 1$

$\longrightarrow 0$

define

$$\rho_n(t) = t\frac{d}{dt}\log Z_n(t)$$

$$\rho(t) = t\frac{d}{dt}\log Z(t)$$

density of the gas
t activity

$$\rho_n(t) \longrightarrow \rho(t)$$

means: for any k, the coefficients of
the first k terms of $\rho_n(t) \longrightarrow$ coeff. of $\rho(t)$

density of the gas

t activity

$$\rho(t) = t \frac{d}{dt} \log Z(t)$$

$-\rho(-t)$ is the generating function of pyramids on $\mathbb{Z}$ (up to translation)

$$\frac{1}{2}\binom{2n}{n} \qquad \frac{1}{2}\frac{1}{\sqrt{1-4t}}$$

directed animals on the triangular lattice lattice

D.Dhar

equivalence
directed animal model
with
hard gas model

relation with
crystal growth model
stochastic lattice gas

$$Z(t) = \sum_{n \geq 0} b_n \frac{t^n}{n!}$$

b_n = nb of "assemblée" of signed labeled pyramids with (m)

(up to translation)

minimum label on the top piece

b_n divisible by $n!$?

research problem

$$\rho(t) = t\frac{d}{dt}\log Z(t)$$

$$Z(t) = 1 + t - 2t^2 + 5t^3 - 14t^4 + \dots + (-1)^{n+1} C_n + \dots$$

"assembléé" of labeled pyramids on $\mathbb{Z}$ (up to translation) with condition (m)

Catalan number

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

(the label of the maximal piece is the smallest) research problem

 1

assemblée

labeled

(m)

 $_3$ +  $_1$ $= 4 = 2 \times 2!$

$3 + 1 = 4 = 2 \times 2$

C_2 Catalan $2!$

$10 = \frac{1}{2}\binom{6}{2}$ pyramids 3 dimers

10×2 labeling

3 pyr 3 labeling

$20 + 9 + 1 = 30 = 5 \times 6$

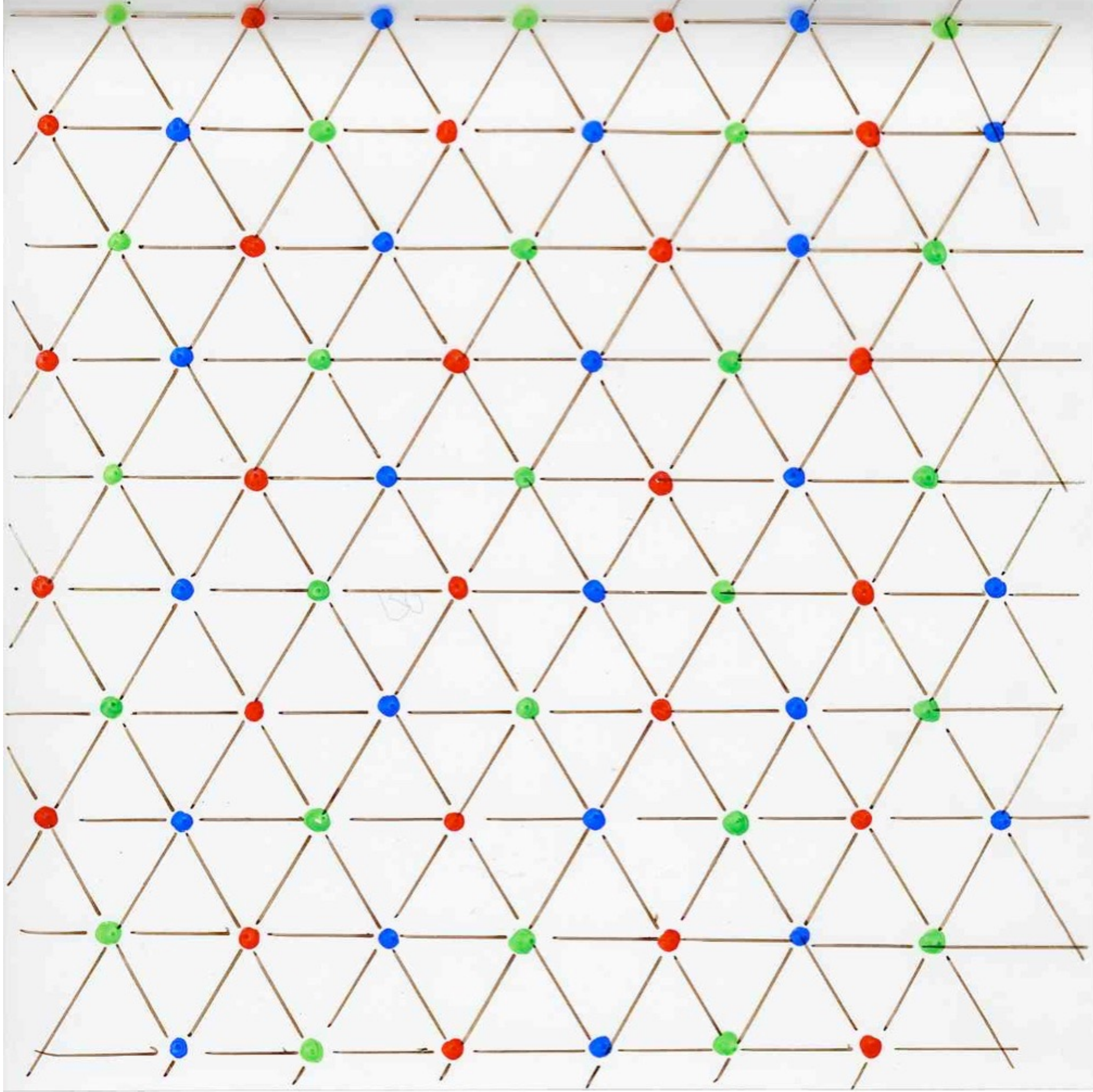
C_3 Catalan $3!$

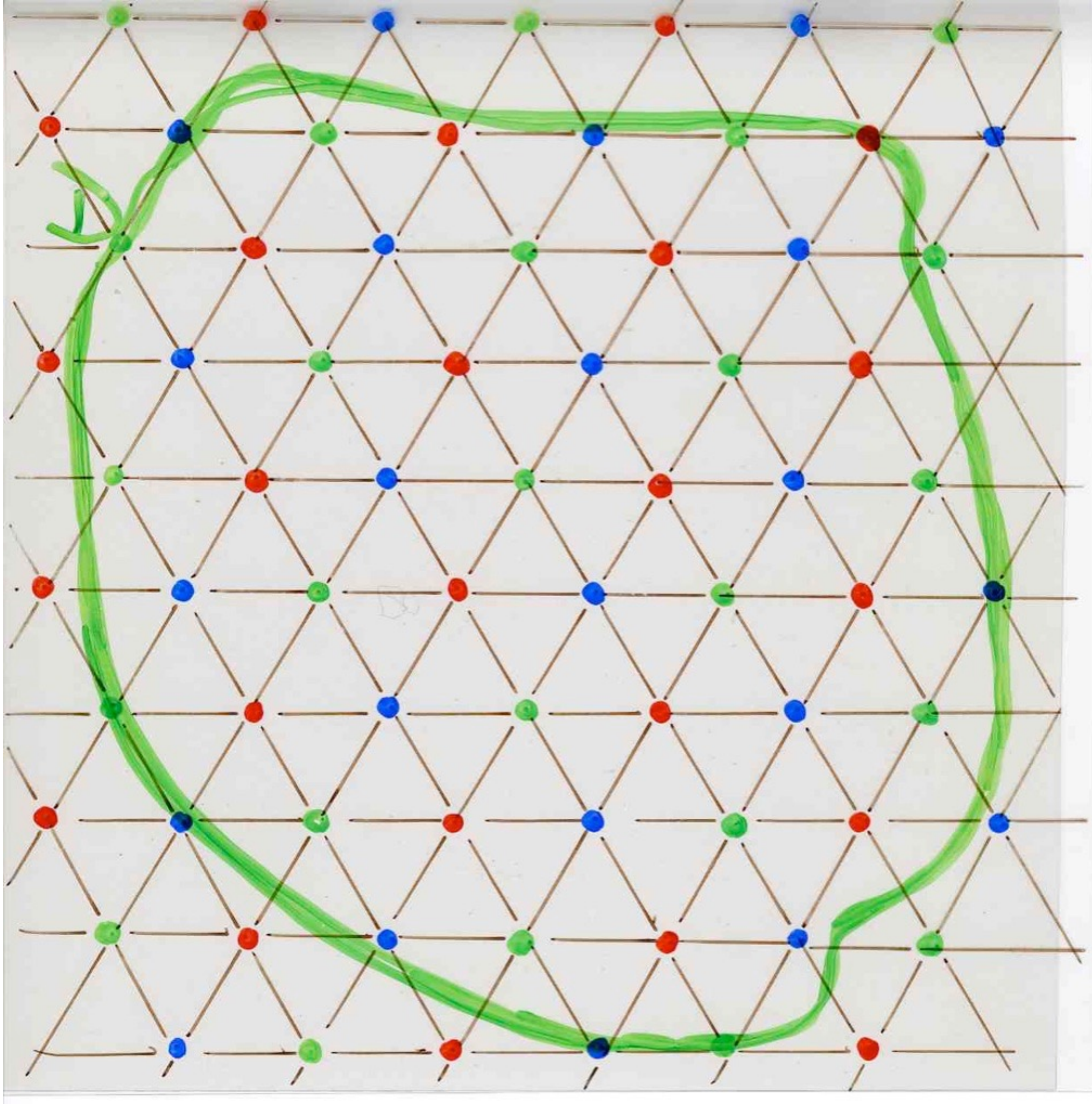
Hard Hexagons

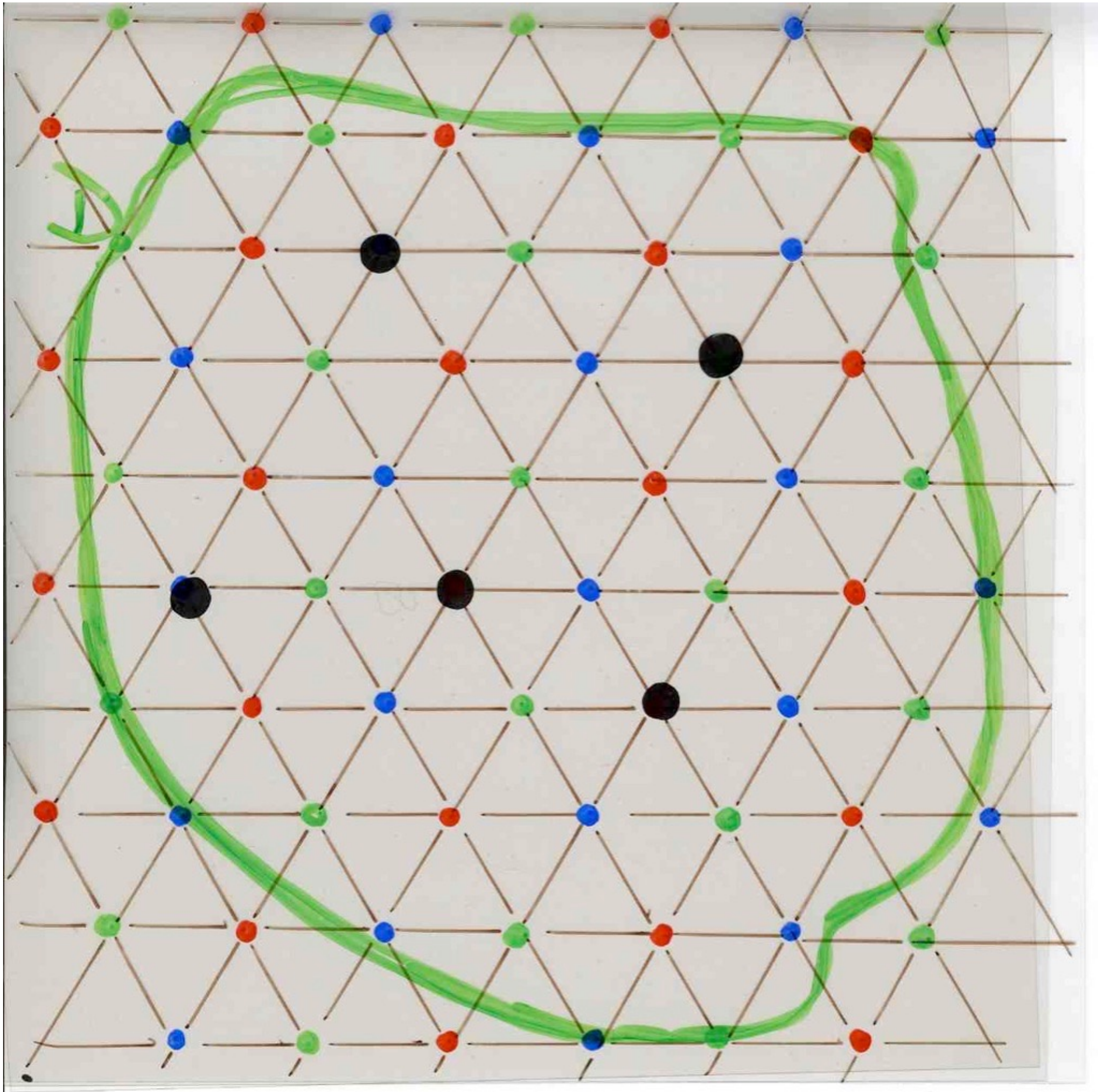
Baxter

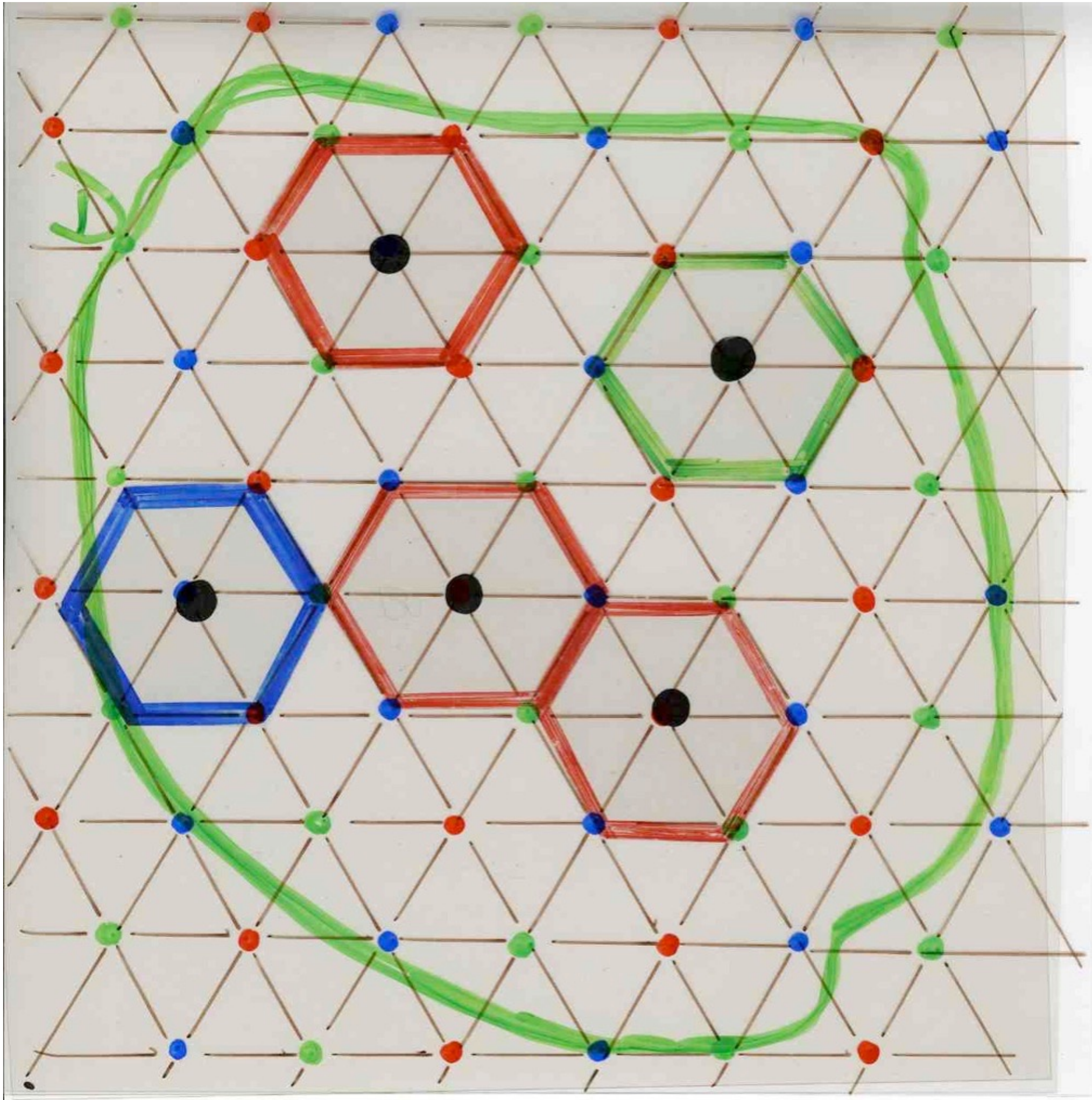
1980
1982

gas model
with "hardcore interactions"









partition function

$$Z_D(t) = \sum_{n \geqslant 0} a_{n,D}\, t^n$$

$$Z(t) = \lim_{\text{"}D \to \infty\text{"}} \left(Z_D(t) \right)^{1/|D|}$$

thermodynamic limit

$$\rho(t) = t \frac{d}{dt} \log Z(t)$$

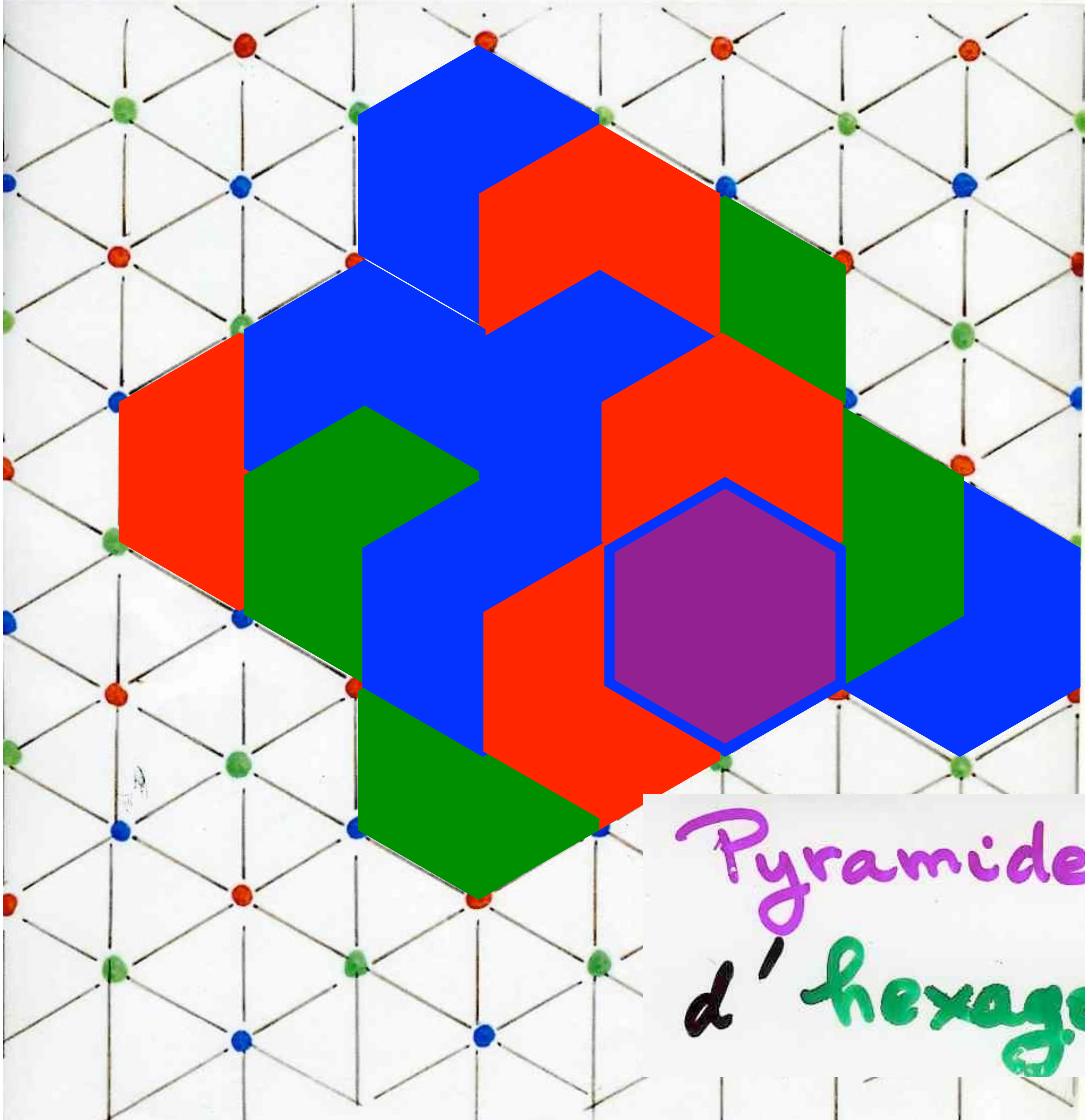
Proposition

$$-\rho(-t) = \sum_{n \geq 1} a_n t^n$$

ρ density of the gas

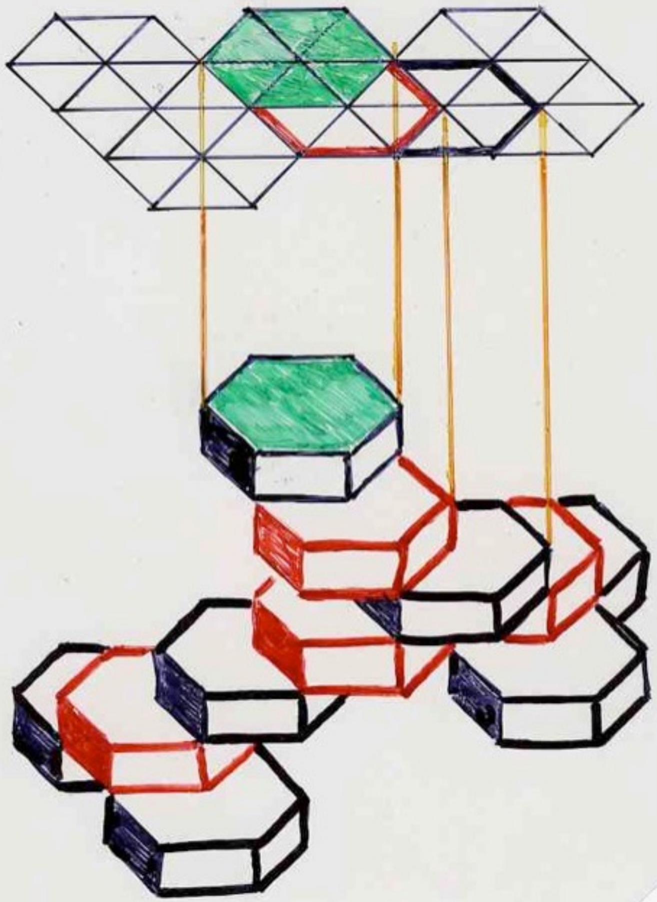
t activity

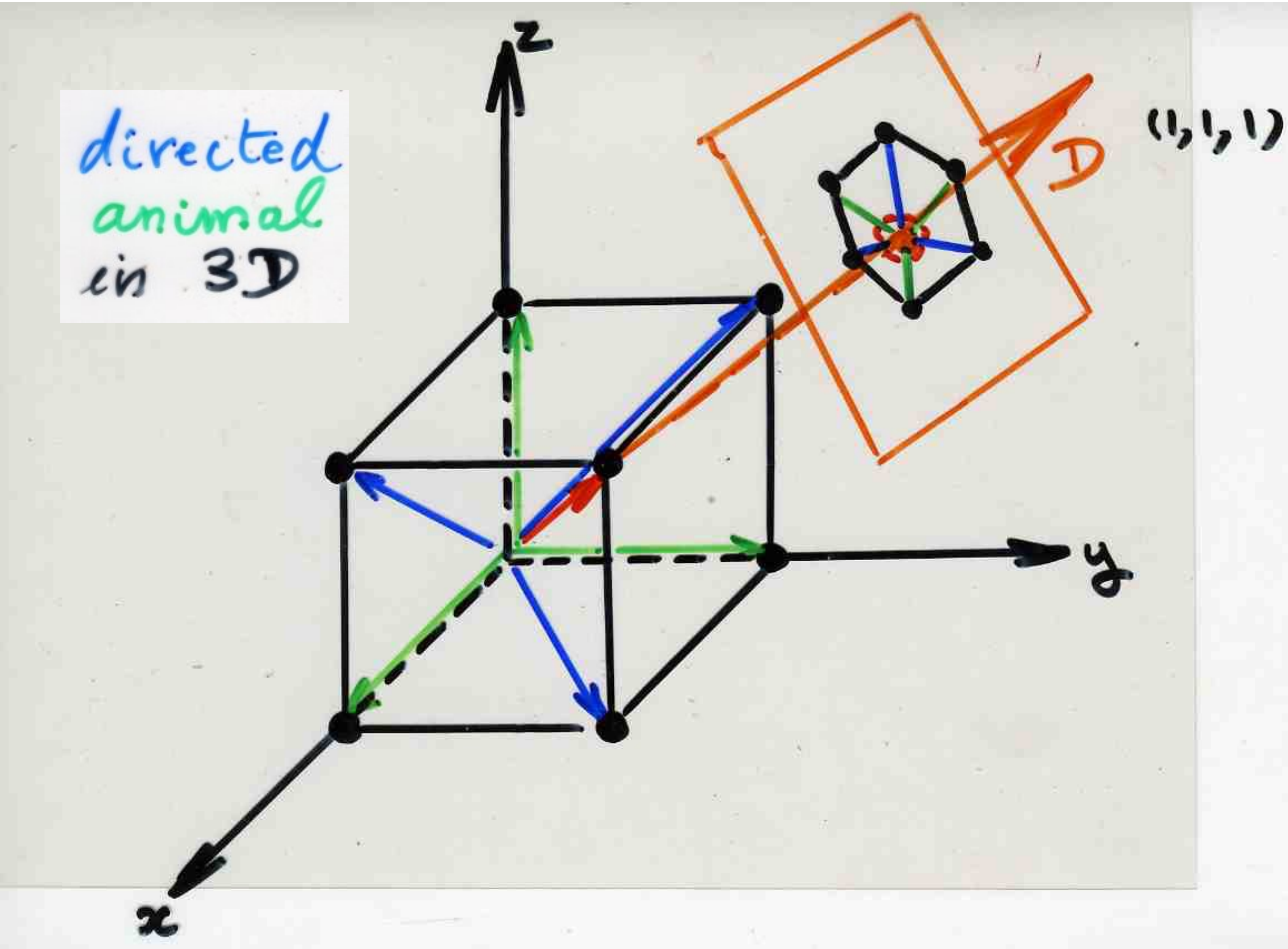
a_n number of pyramids on the triangular lattice with n hexagons (up to translation)



Pyramide d'hexagones

$$-p(-t) = y$$




directed
animal
in 3D
z
y
x
D
(1,1,1)

$$\rho(t) = t - 7t^2 + 58t^3 - 519t^4 + 4856t^5 - \cdots$$

combinatorial understanding

of the thermodynamic limit

the case of a 2D gas model

partition function

$$Z_D(t) = \sum_{n \geqslant 0} a_{n,D}\, t^n$$

$$Z(t) = \lim_{\text{"}D \to \infty\text{"}} \left(Z_D(t) \right)^{1/|D|}$$

thermodynamic limit

$$\rho(t) = t\,\frac{d}{dt}\log \mathbf{Z}(t)$$

Proposition

$$-\rho(-t) = \sum_{n\geq 1} a_n t^n$$

number of pyramids
on the triangular lattice
with n hexagons
(up to translation)

proof

D finite domain of the triangular lattice

$s \in D$ $\qquad P_{D,s}(t)$ generating function for pyramid

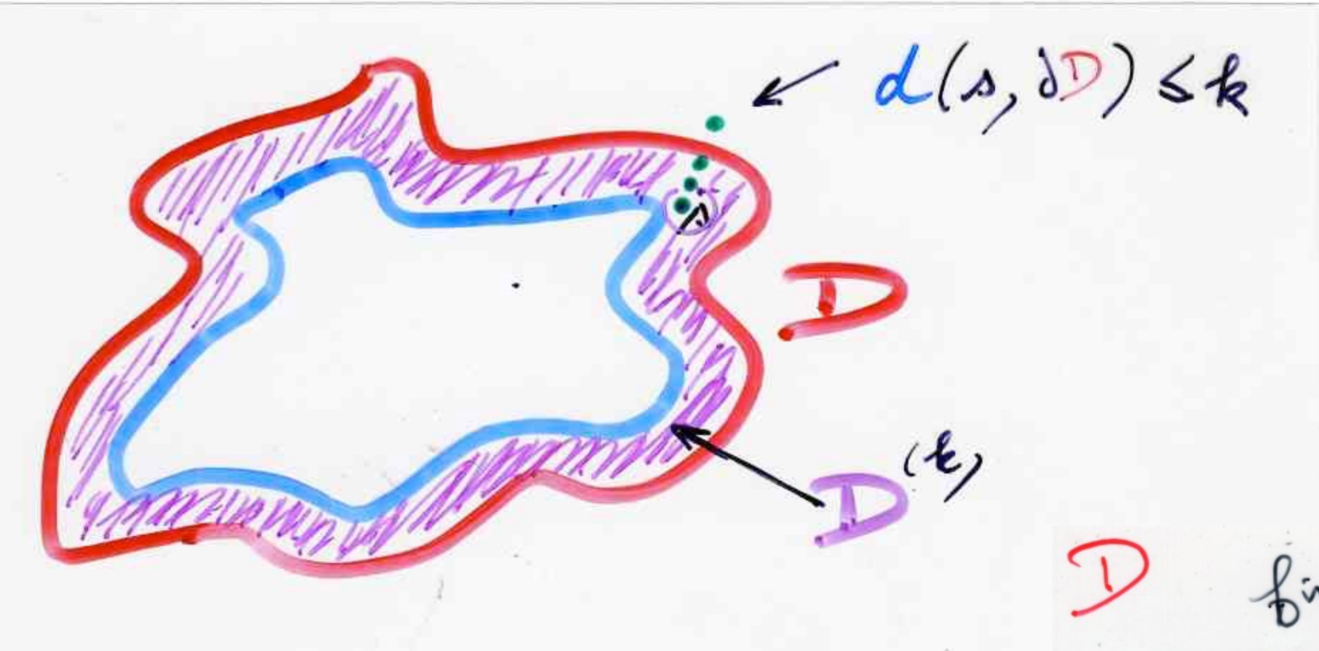
- projection in D
- maximal piece s

Compute: $\displaystyle \frac{1}{|D|} \sum_{s \in D} P_{D,s}(t) = \frac{1}{|D|} P_D(t)$

$\underbrace{\qquad\qquad}$

Hexagon Pyramids on a tube of base D

$$P_D(t) = (-t) \frac{d}{dt} \log Z_D^{-1}(-t)$$



D finite domain of the triangular lattice

Def. $d(s, \partial D)$
smallest length of paths (on Hex) to go from s to the outside of D

$$D^{(k)} = \{ s \in D \,,\; d(s, \partial D) \leq k \}$$

Proposition

Sequence $D_1 \subseteq D_2 \subseteq \cdots D_n \subseteq$
such that for every k

$$\frac{|D_n^{(k)}|}{|D_n|} \to 0$$

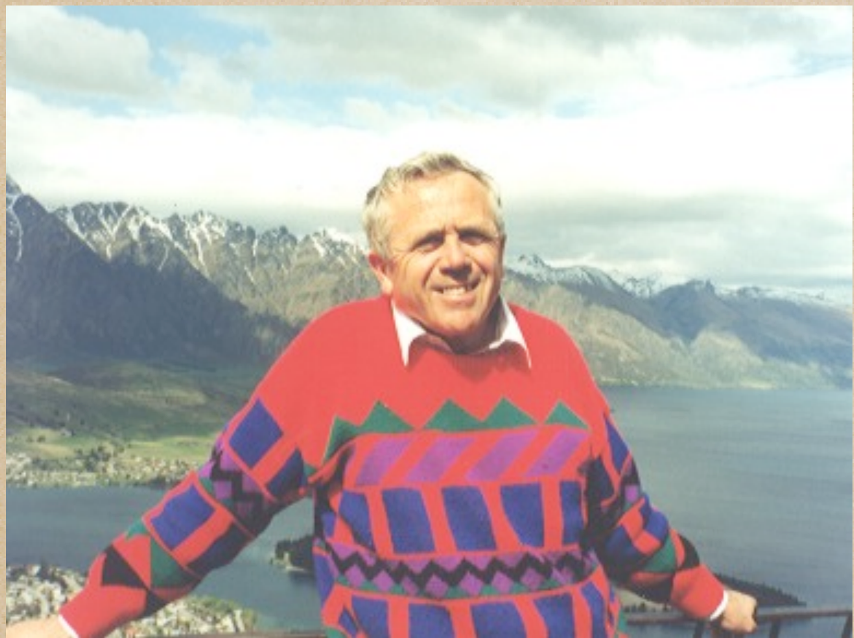
Then: $\frac{1}{|D_n|} \mathbf{P}_{D_n}(t) \to \mathbf{P}(t)$ generating function for Pyramid on Hex (up to translation)

means: $f_n(t) = \sum_{i \geq 0} a_{n,i} t^i$; $f(t) = \sum_{i \geq 0} a_i t^i$

then for every i, $a_{n,i} \to a_i$.

solution of the hard hexagons model

(R. Baxter, 1980)



Rogers - Ramanujan identities

$$R_{I} \quad \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

$$R_{II} \quad \sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

"La fraction continue" de Ramanujan

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots\cdots}}}}}} = \frac{\displaystyle\sum_{n\geqslant 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\displaystyle\sum_{n\geqslant 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}$$

$$R(q) = \prod_{n \geqslant 0} \frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+3})(1-q^{5n+2})} = \frac{R_{II}}{R_{I}}$$

$$R(q) = \prod_{n\geq 0} \frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+3})(1-q^{5n+2})} = \frac{R_{II}}{R_{I}}$$

$$t = -q\left[R(q)\right]^5$$

$$R(q) = \prod_{n\geq 0} \frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+3})(1-q^{5n+2})} = \frac{R_{II}}{R_{I}}$$

$$t = -q\left[R(q)\right]^5$$

$$Y(q) = \prod_{n\geq 0} \frac{(1-q^{6n+2})(1-q^{6n+3})^2(1-q^{6n+4})(1-q^{5n+1})^2(1-q^{5n+4})^2(1-q^{5n})^2}{(1-q^{6n+1})(1-q^{6n+5})(1-q^{6n})^2(1-q^{5n+2})^3(1-q^{5n+3})^3}$$

$$R(q) = \prod_{n \geq 0} \frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+3})(1-q^{5n+2})} = \frac{R_{II}}{R_{I}}$$

$$t = -q\left[R(q)\right]^5$$

$$Y(q) = \prod_{n \geq 0} \frac{(1-q^{6n+2})(1-q^{6n+3})^2(1-q^{6n+4})(1-q^{5n+1})^2(1-q^{5n+4})^2(1-q^{5n})^2}{(1-q^{6n+1})(1-q^{6n+5})(1-q^{6n})^2(1-q^{5n+2})^3(1-q^{5n+3})^3}$$

$$Z(t) = Y(q(t))$$

- critical temperature

$$T_c = \frac{11 + 5\sqrt{5}}{2}$$

- critical exponent $\frac{5}{6}$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{5}$$

$\frac{5}{6}$ critical exponent

Baxter

(1980)

$t_c = 11.09017..$

$\frac{1}{2}(11+5\sqrt{5})$

Gaunt 1967

critical temperature

for hexagons :

$(\phi)^5$

golden ratio

$$= \left(\frac{1+\sqrt{5}}{2}\right)^5$$

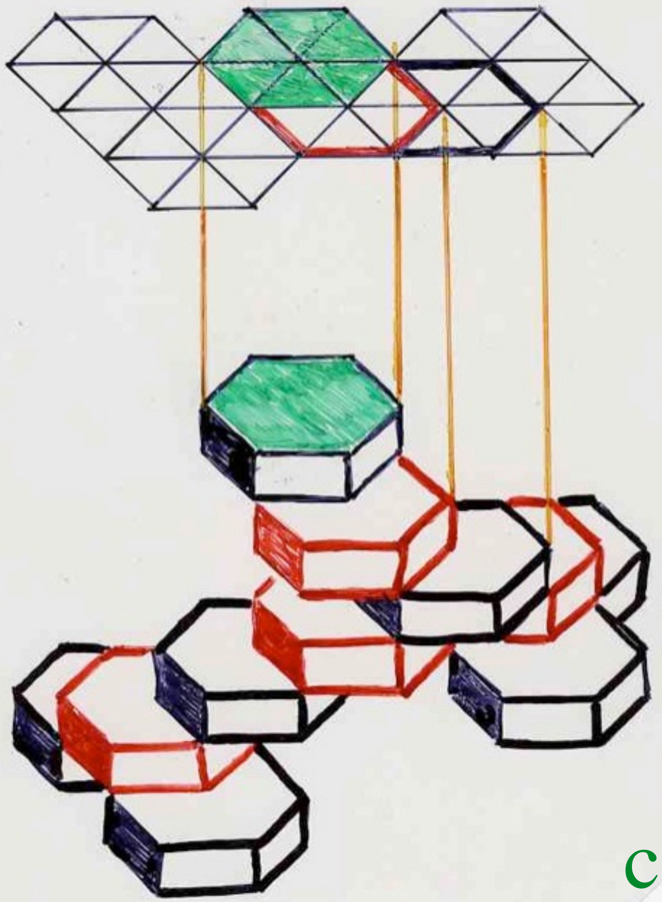
helium monolayer
absorbed onto
a graphite surface

(Riedel 1981)

research problem

$$-p(-t) = y$$

algebraic
generating
function



direct
combinatorial
explanation ?

?

$$Z(t) = \sum_{n \geq 0} b_n \frac{t^n}{n!}$$

b_n = nb of "assemblée" of signed labeled pyramids with (*)

(up to translation)

minimum label on the top piece

b_n divisible by $n!$?

$$Z(t) = 1 + t - 3t^2 + 16t^3 - 106t^4 + 789t^5 - 6318t^6 + \ldots$$

Hard core
lattice gas models

interpretation of the density

hard square ?

algebraic
generating
function ?