Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics, a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

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Chapter 7

Heaps in statistical mechanics
(1)

slides: first part of Ch7a

IMSc, Chennai
2 March 2017

a few words about

statistical mechanics .....

phase transition
critical phenomena

from local interactions
→ global behaviour

exactly
solved
models

Ising
model

Baxter
book (1982)

Onsager (1944)

Statistical physics

critical exponent

$$F(T) \cong \frac{1}{(T - T_c)^{\alpha}}$$

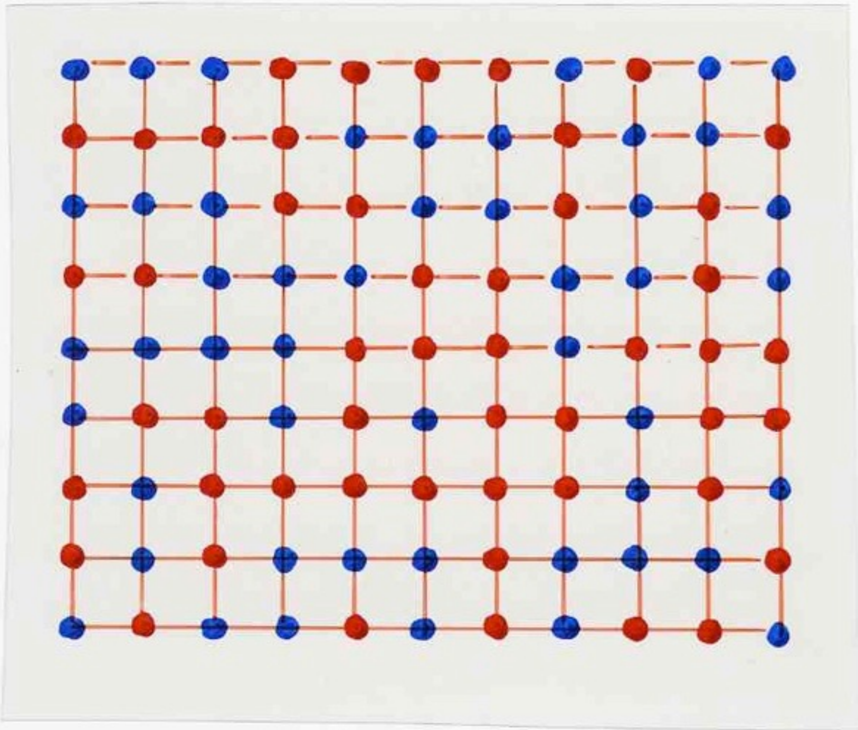
temperature

critical temperature

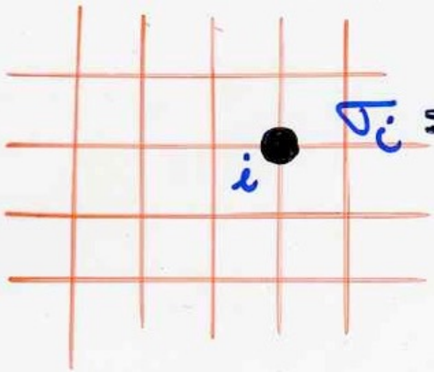
thermodynamic function

example 1:

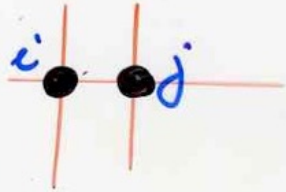
the Ising model



Ising model



$\sigma_i = \pm 1$



interaction

$$E_{ij} = J_{ij}\,\sigma_i\sigma_j$$

total energy

σ configuration

$$E_\sigma = -J\sum_{\substack{i,j\\ \text{adjacent}}} \sigma_i\sigma_j - I\sum_i \sigma_i$$

external field

$J > 0$ ferromagnétisme

$J < 0$ (anti- - - - - - .)

Partition function

$$Z_L = \sum_{\sigma} \exp\left(-E_\sigma / kT\right)$$

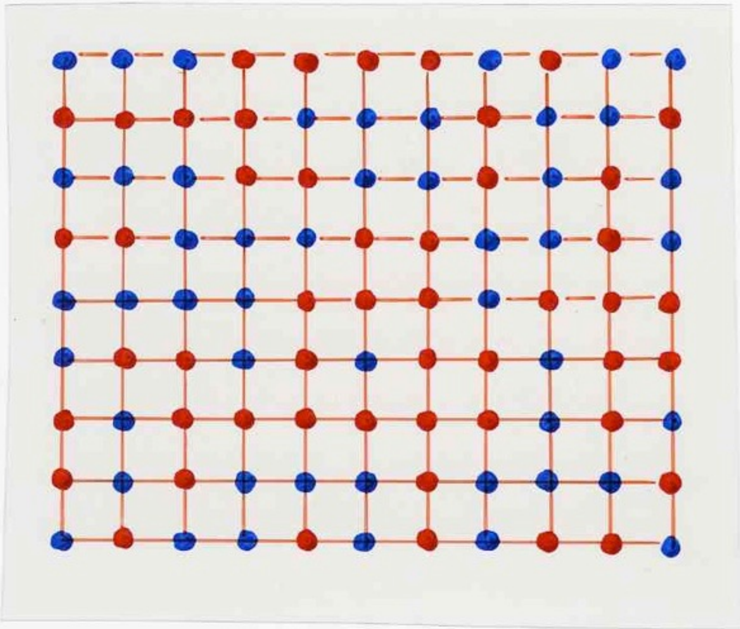
k Boltzmann constant

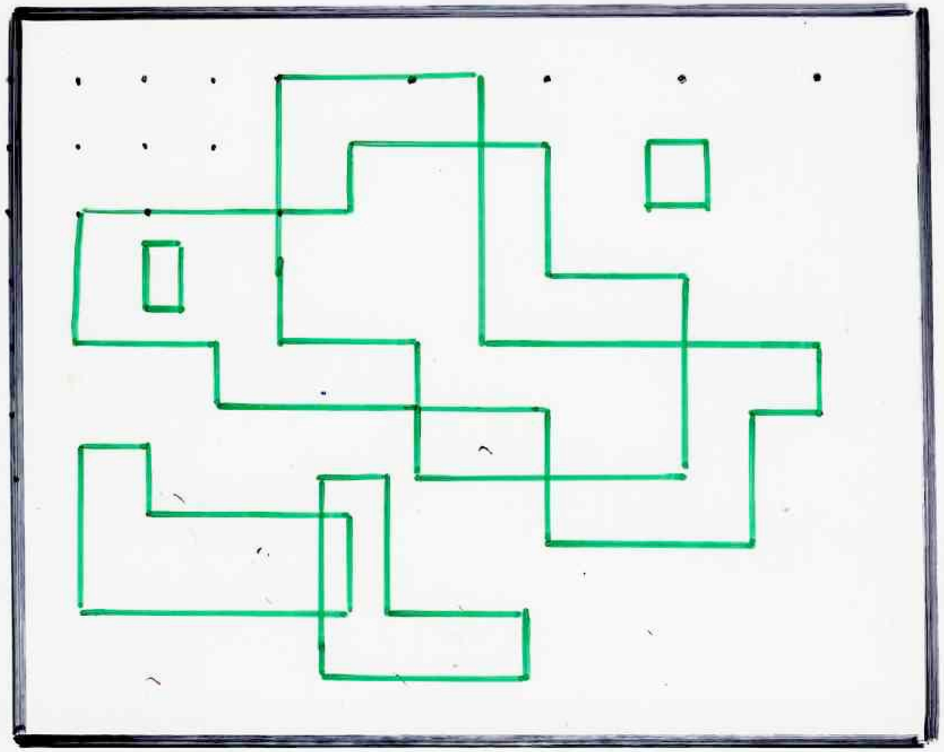
T temperature

$$Z_L = (\operatorname{ch} H)^N \, 2^N \, \mathcal{C}\ell_L(\operatorname{th} H)$$

generating function for closed subgraphs

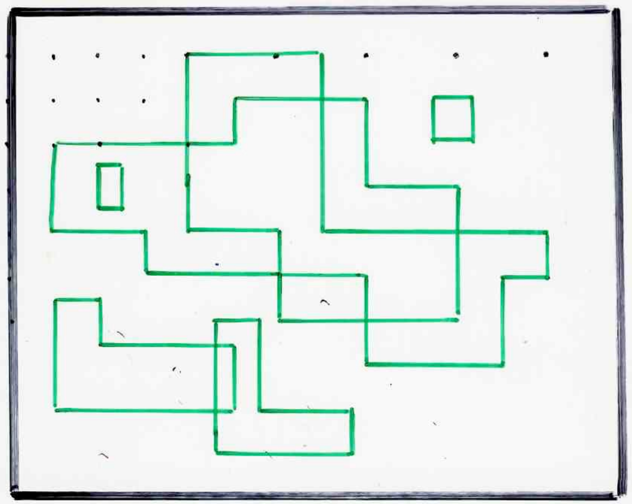
(enumerated by total number of edges)





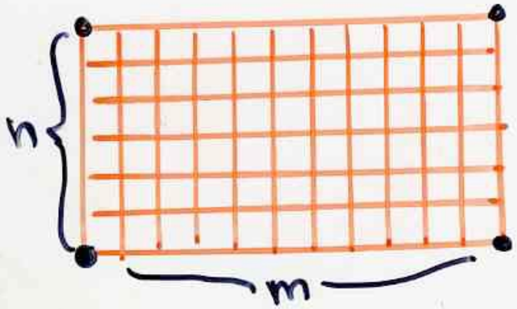
"closed" graph

Ising model



"closed" graph

Ising model

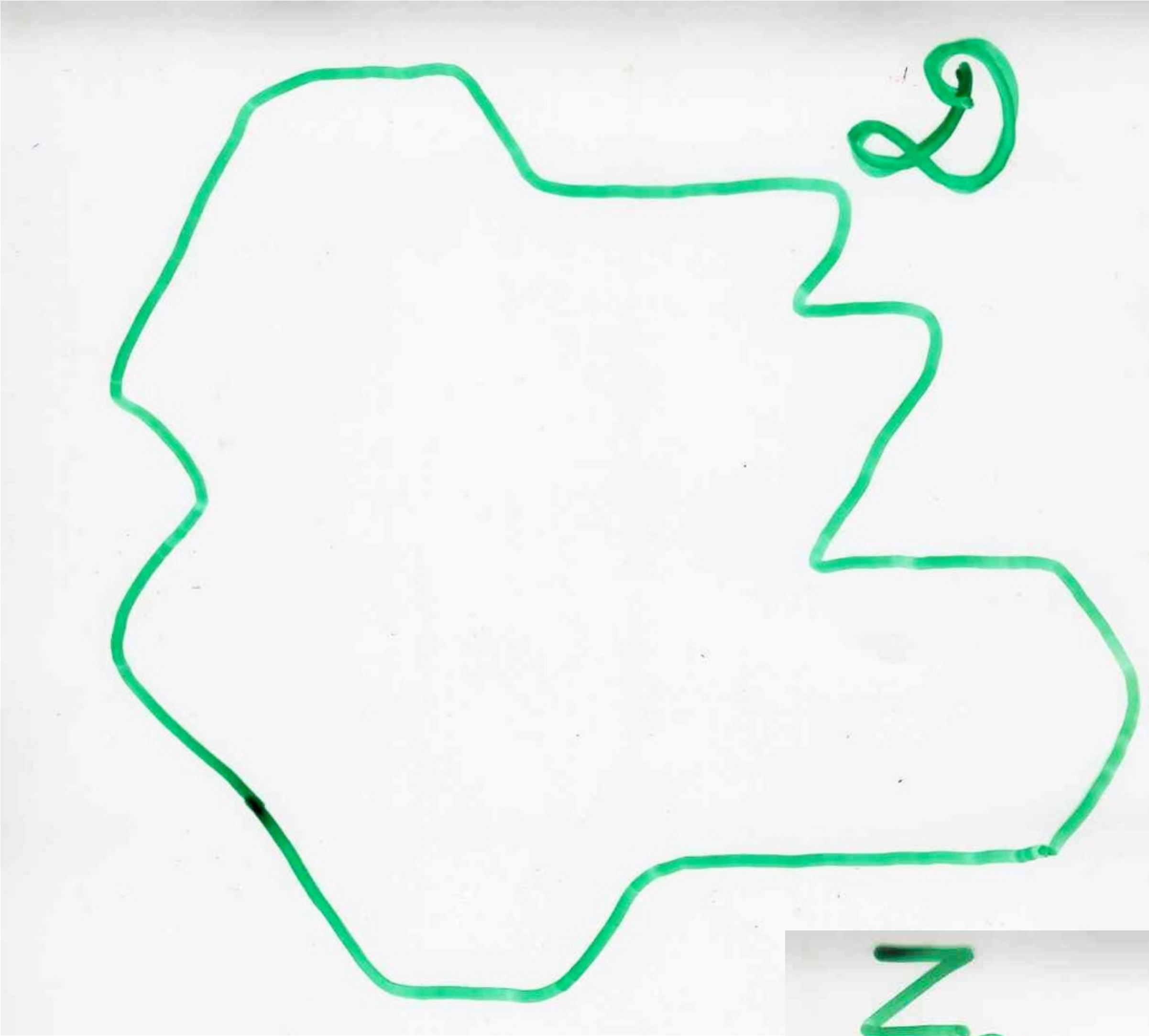


thermodynamic limit

"$N \to \infty$"

$$N = nm$$

$$Z = \lim_{\text{"}N\to\infty\text{"}} Z_{n,m}^{1/N} .$$

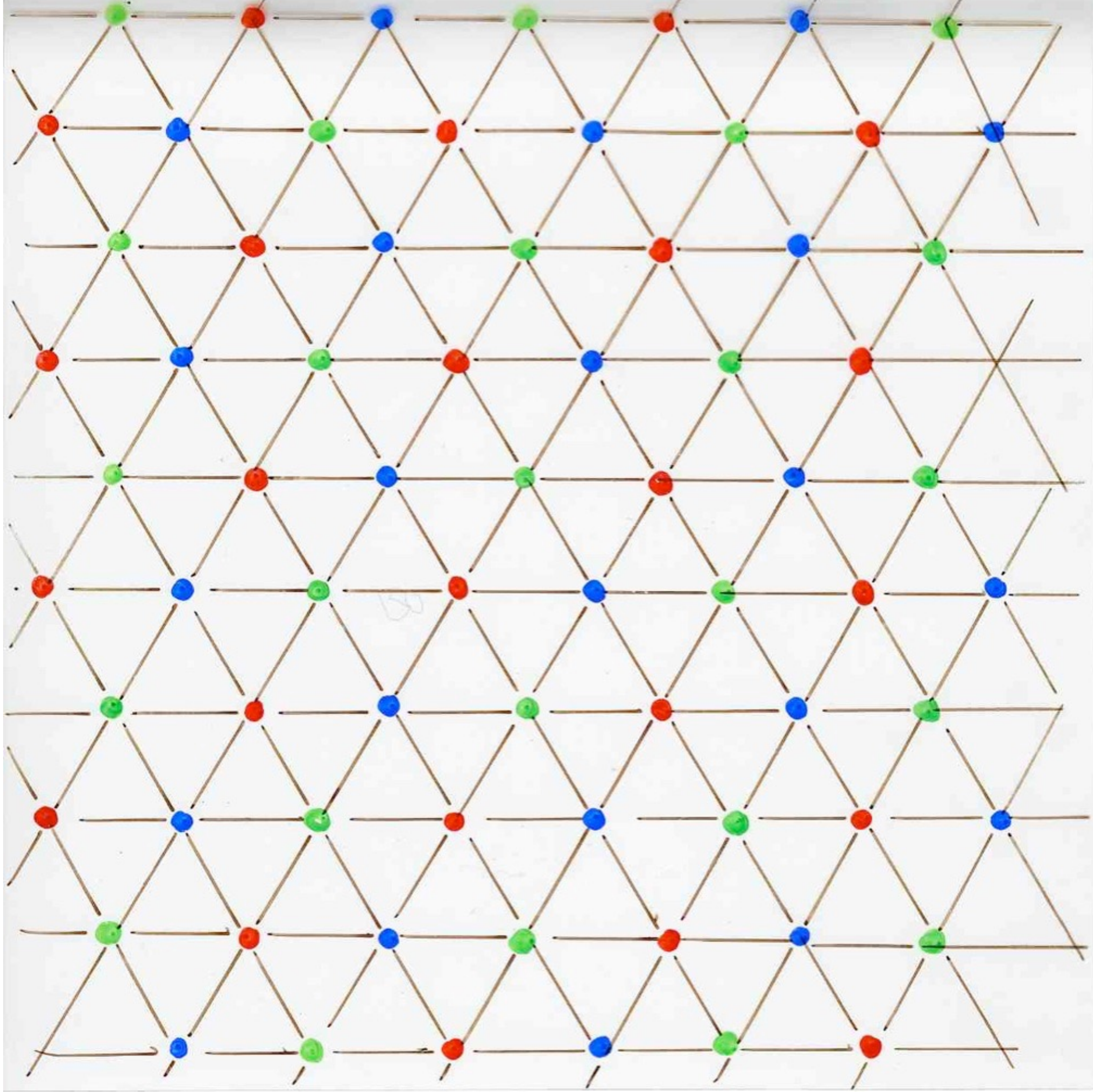


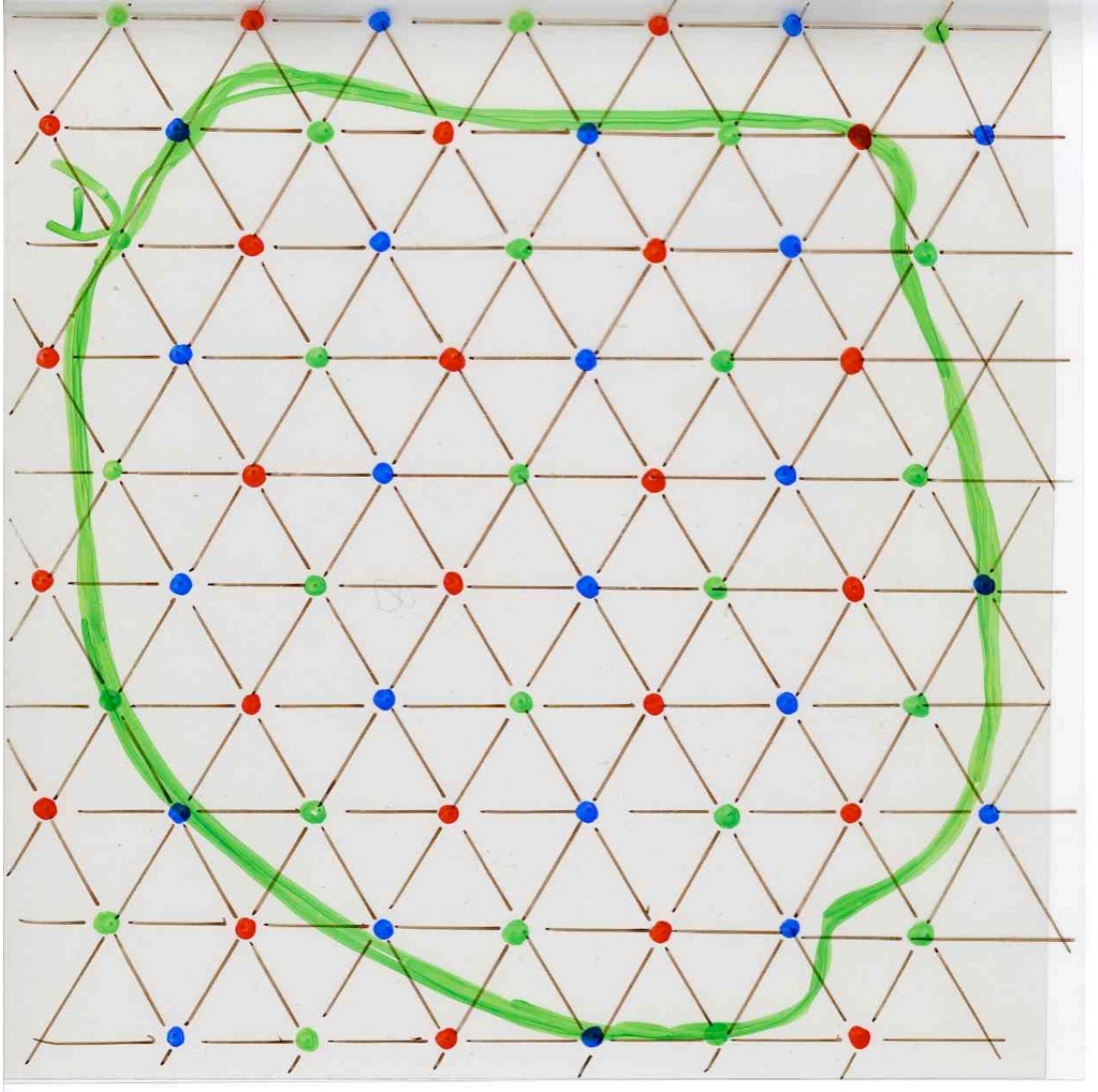
$Z_{\mathcal{D}}$ partition function

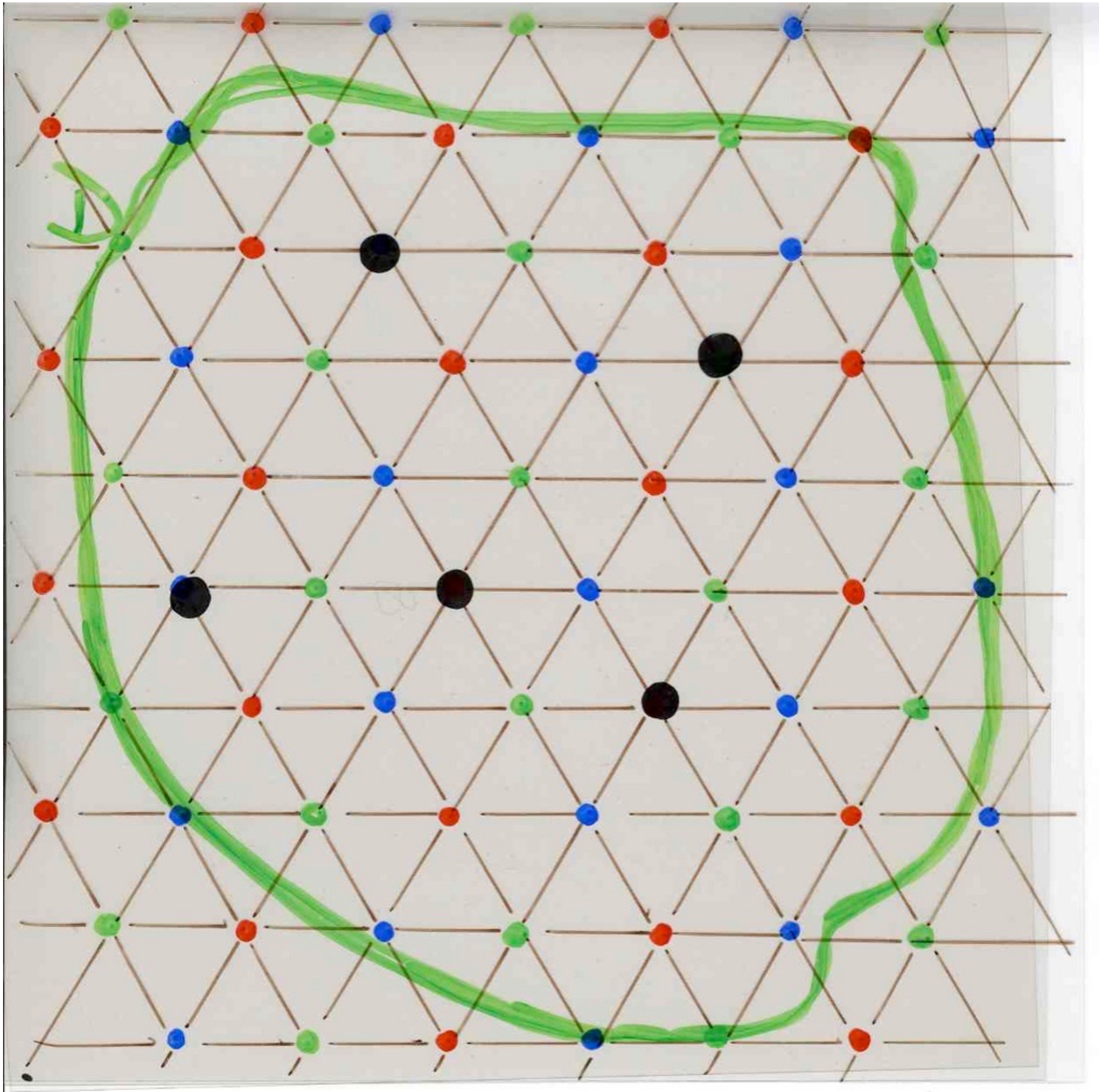
$$Z = \lim_{``\mathcal{D} \to \infty"} Z_{\mathcal{D}}^{1/N}$$

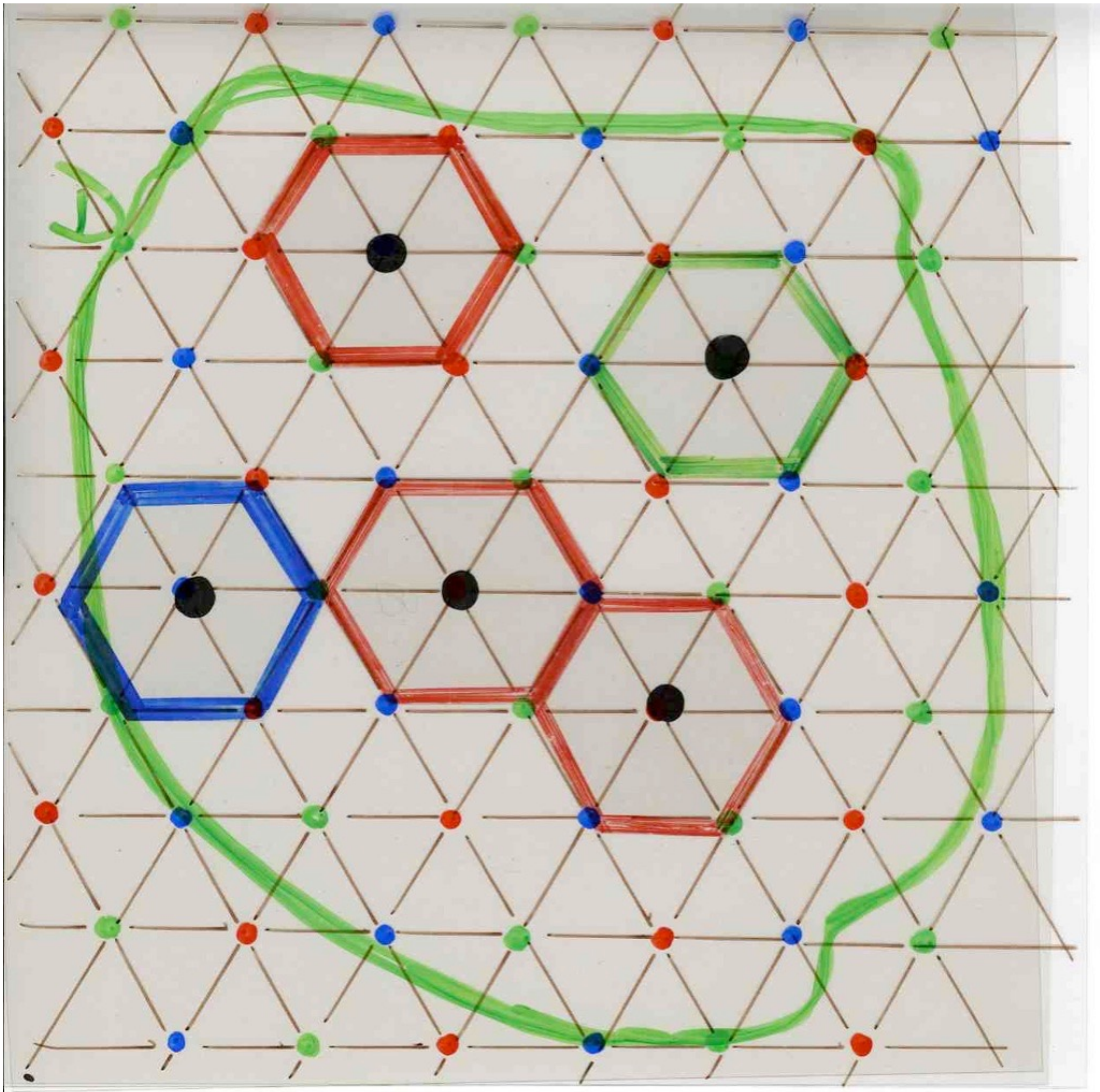
example 2:

gas model









partition function

$$Z_D(t) = \sum_{n \geq 0} a_{n,D}\, t^n$$

$$Z(t) = \lim_{\text{"}D \to \infty\text{"}} \left(Z_D(t) \right)^{1/|D|}$$

thermodynamic limit

Hard Hexagons

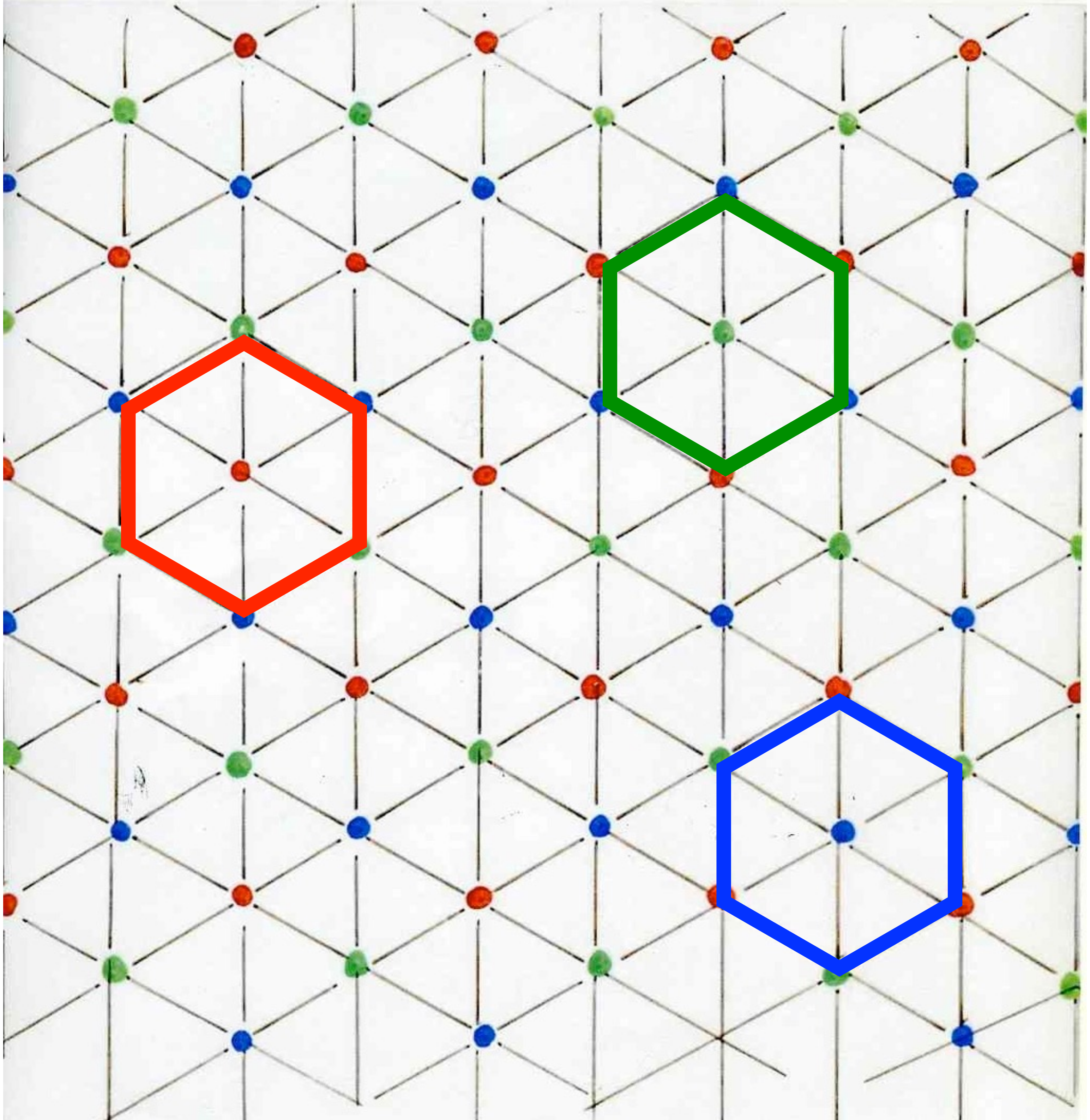
Baxter

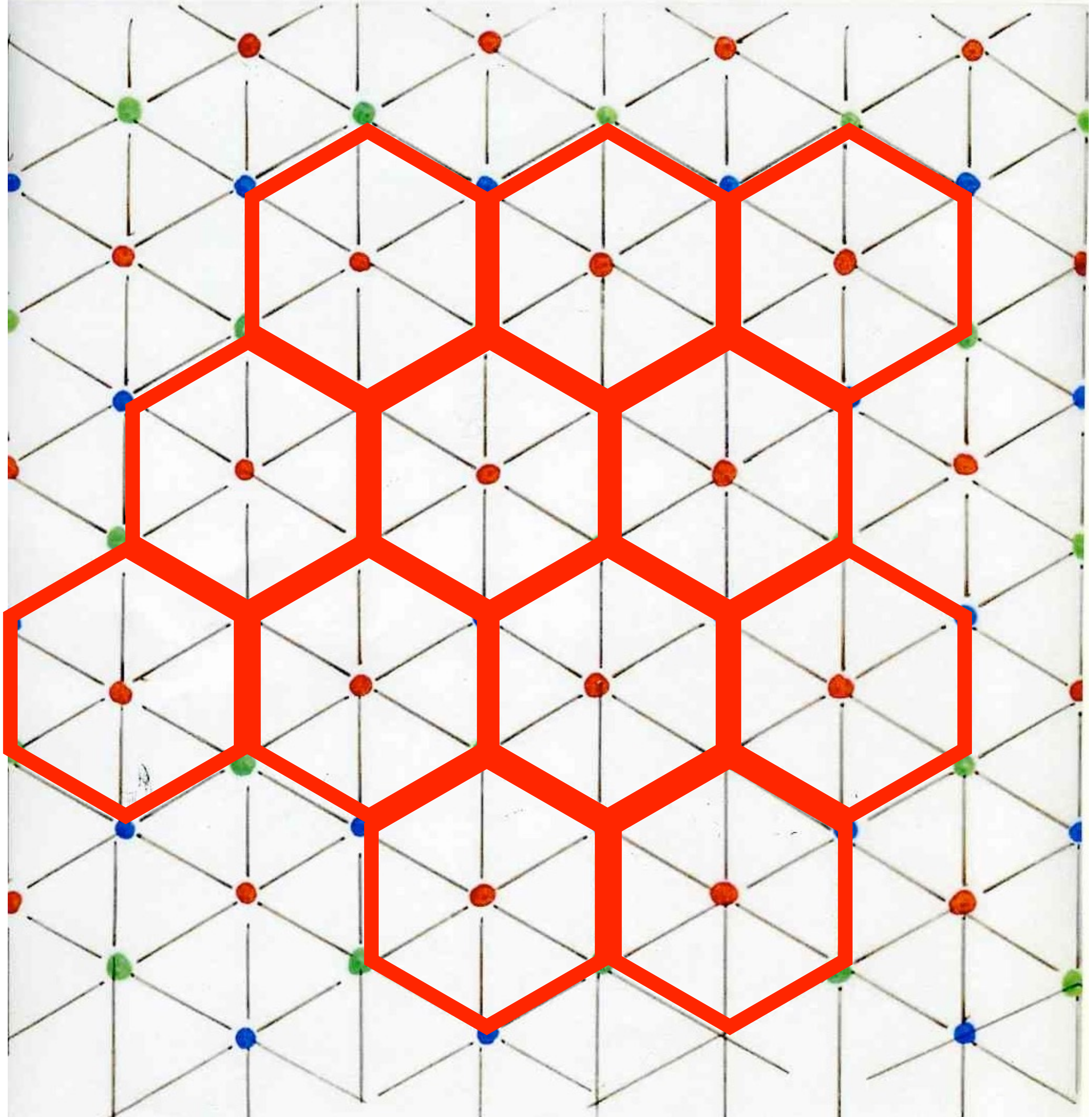
1980

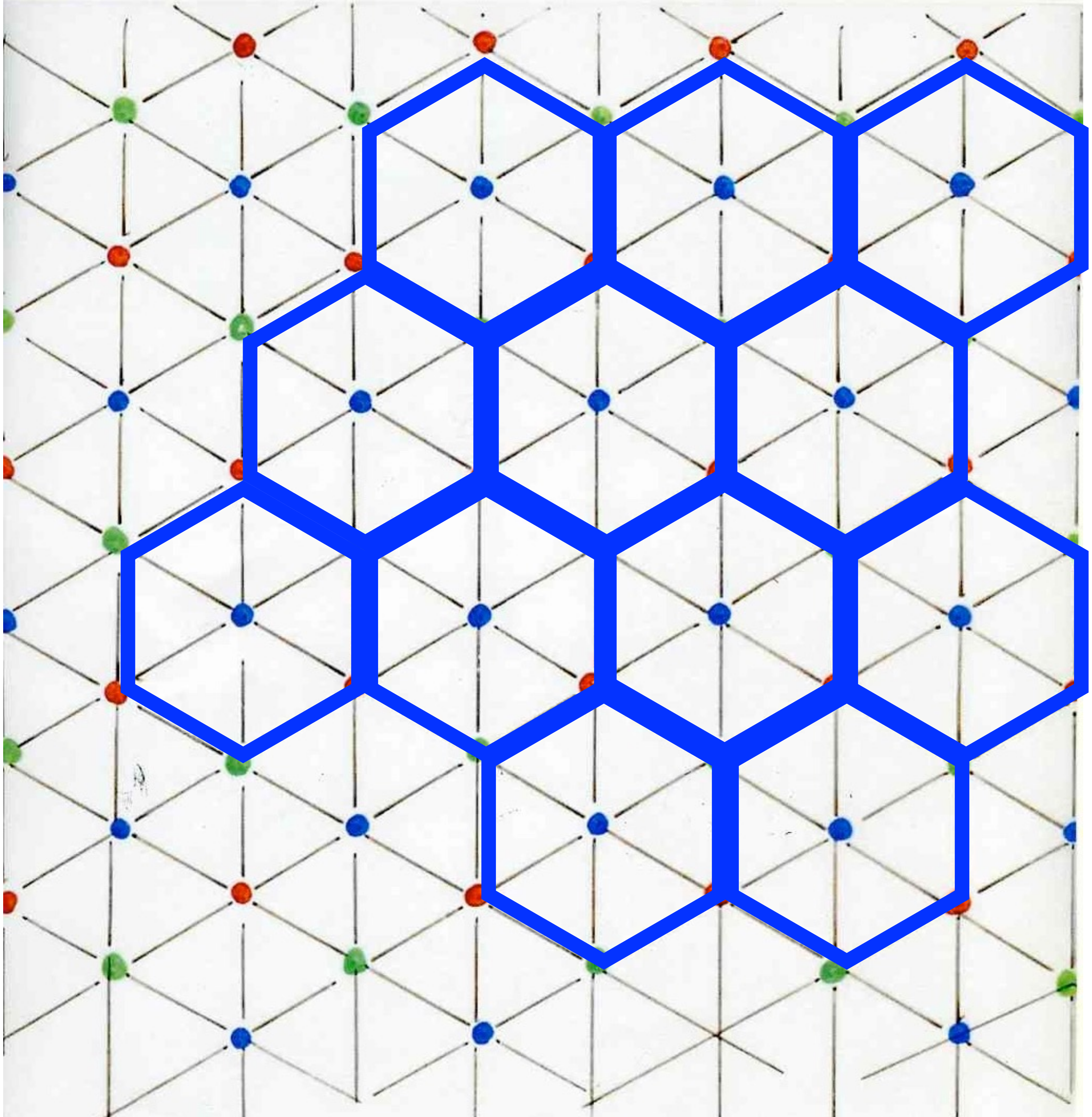
1982

gas model

with "hardcore interactions"



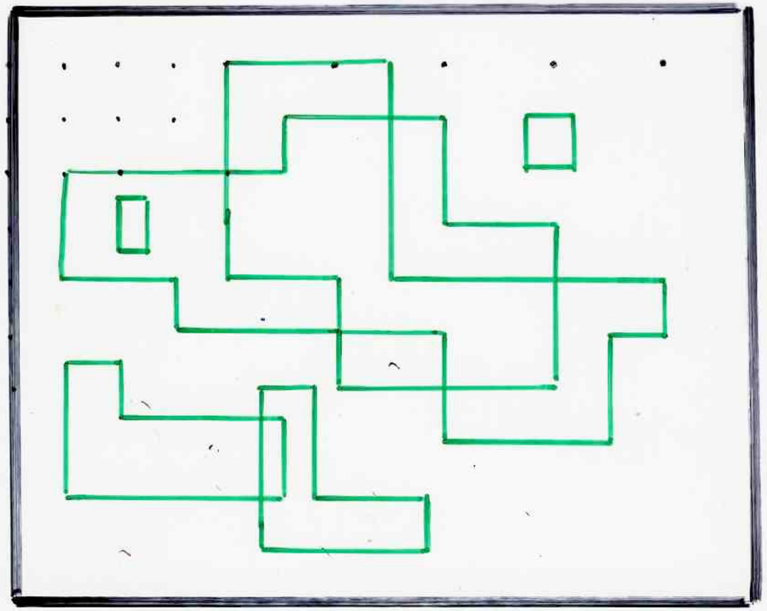




statistical mechanics

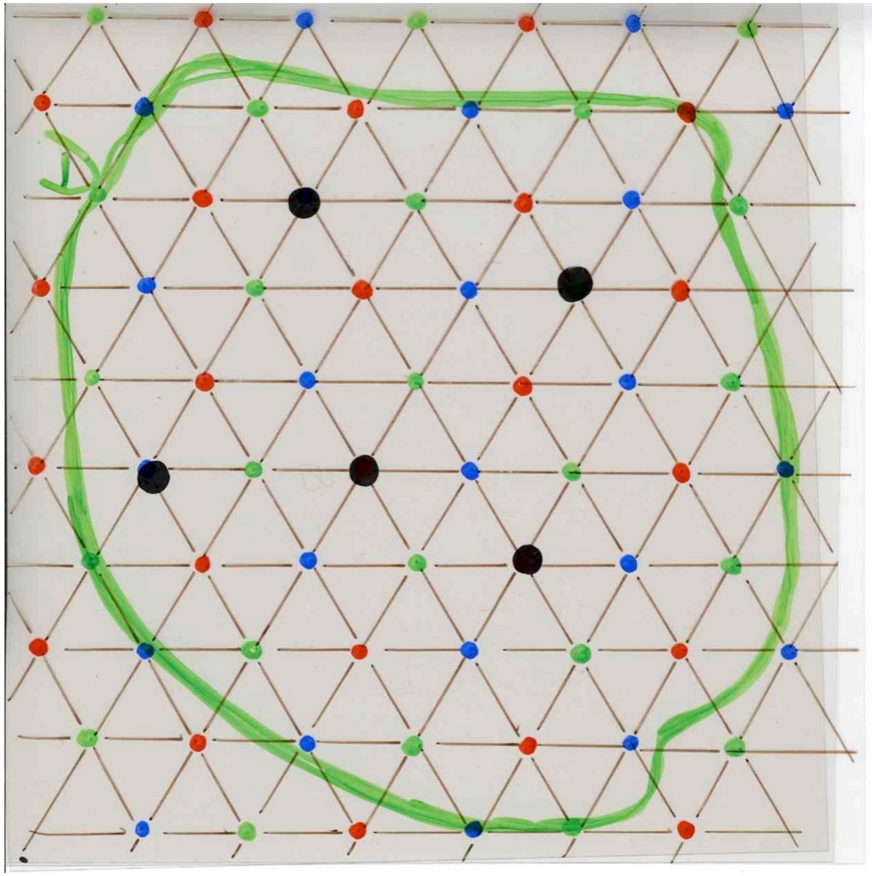
and

combinatorics



"closed" graph Ising model

before the thermodynamic limit



Statistical physics

$$F(T) \cong \frac{1}{(T-T_c)^{\alpha}}$$

critical exponent

temperature

thermodynamic function

critical temperature

Polyominoes
animals
heaps

and physics

$$F(T) = \sum_{n \geq 0} a_n T^n$$

partition function

$$F(T) \simeq \frac{1}{(T-T_c)^{\alpha}}$$

critical exponents

$$a_n \sim \mu^n \, n^{-\theta}$$

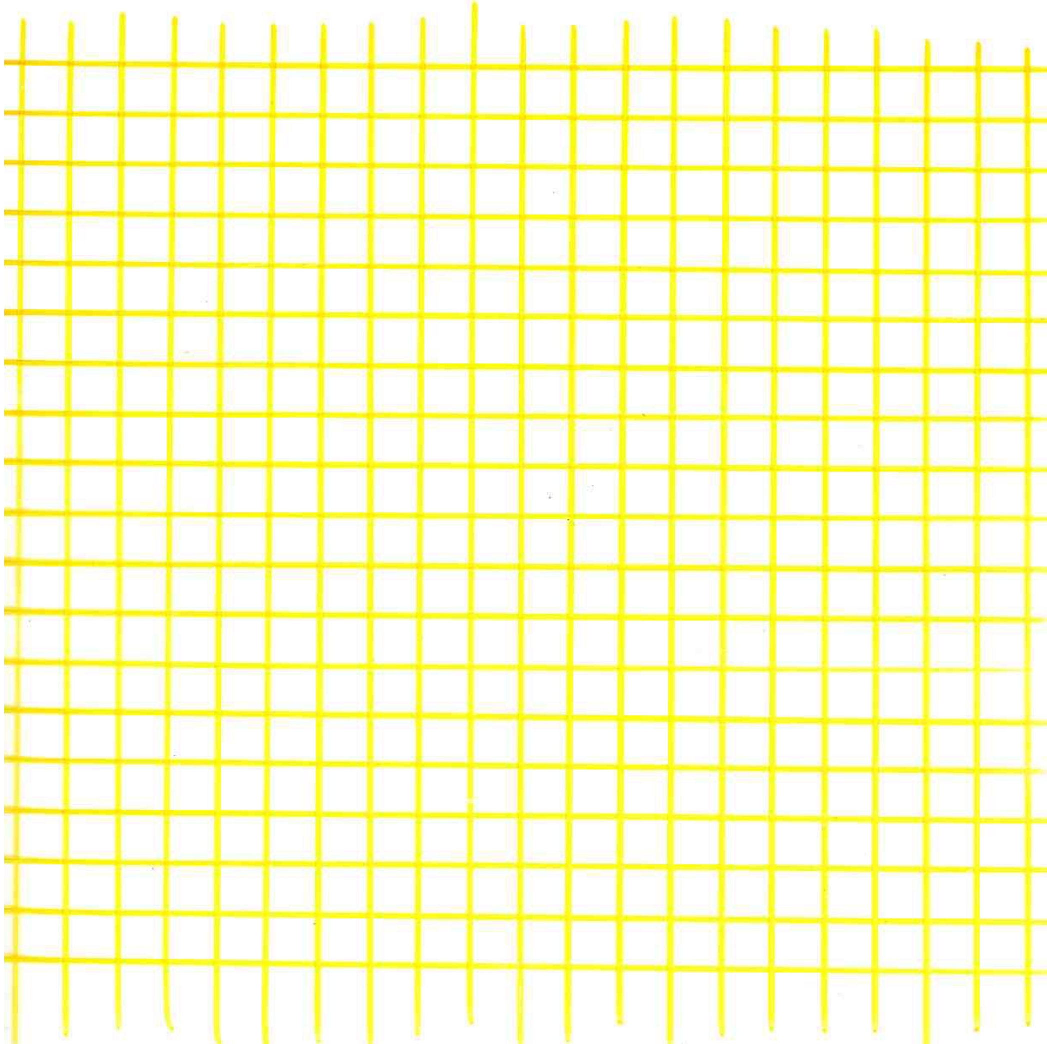
after the

thermodynamic limit

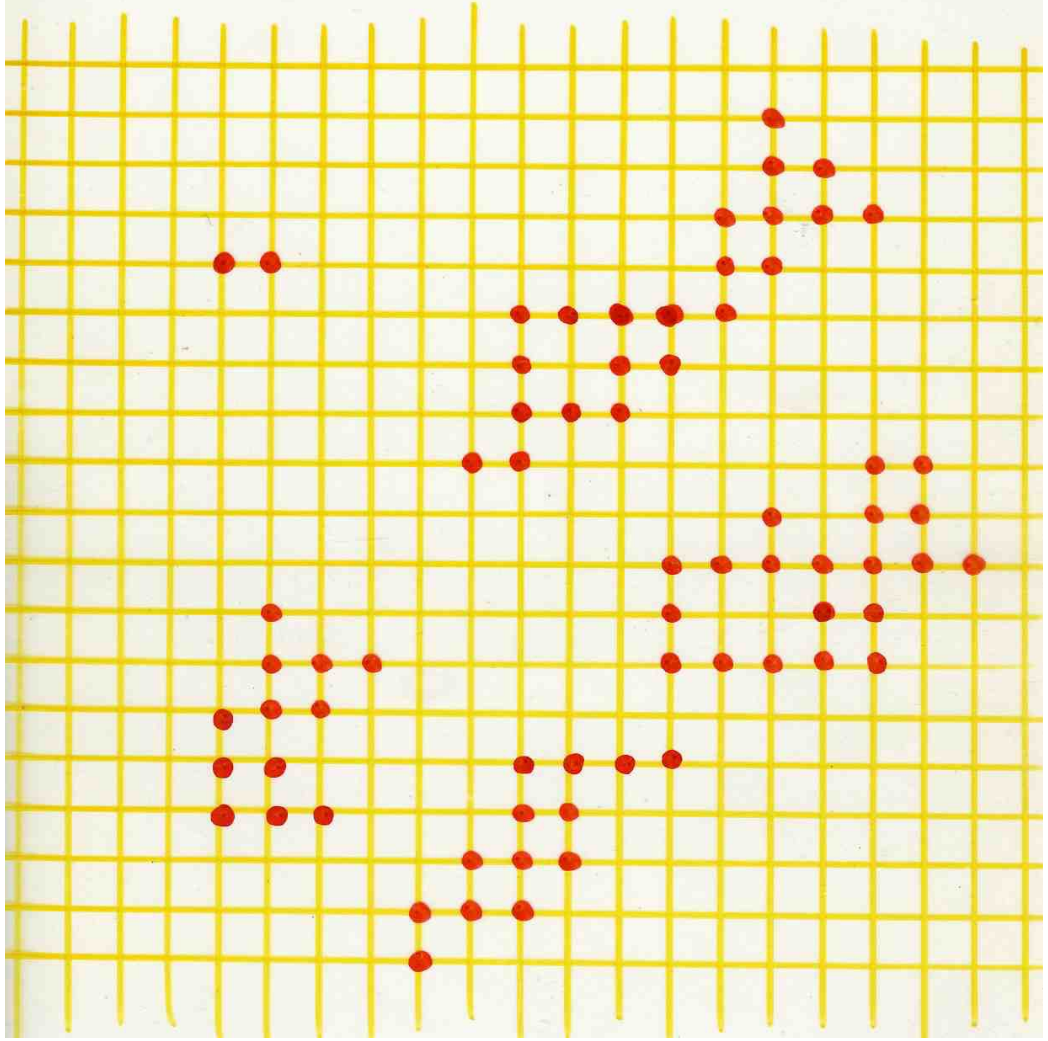
example 3

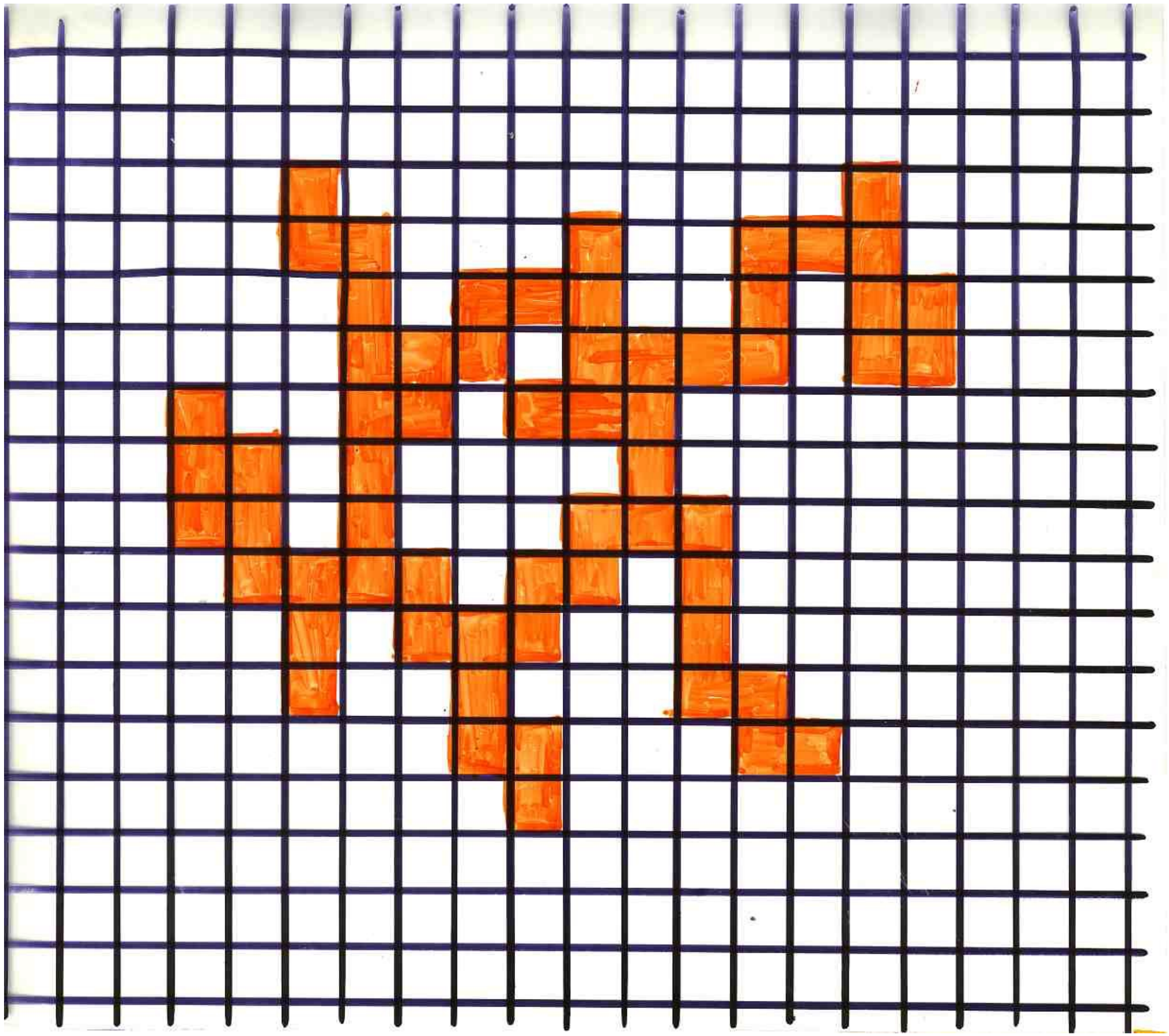
percolation

Percolation

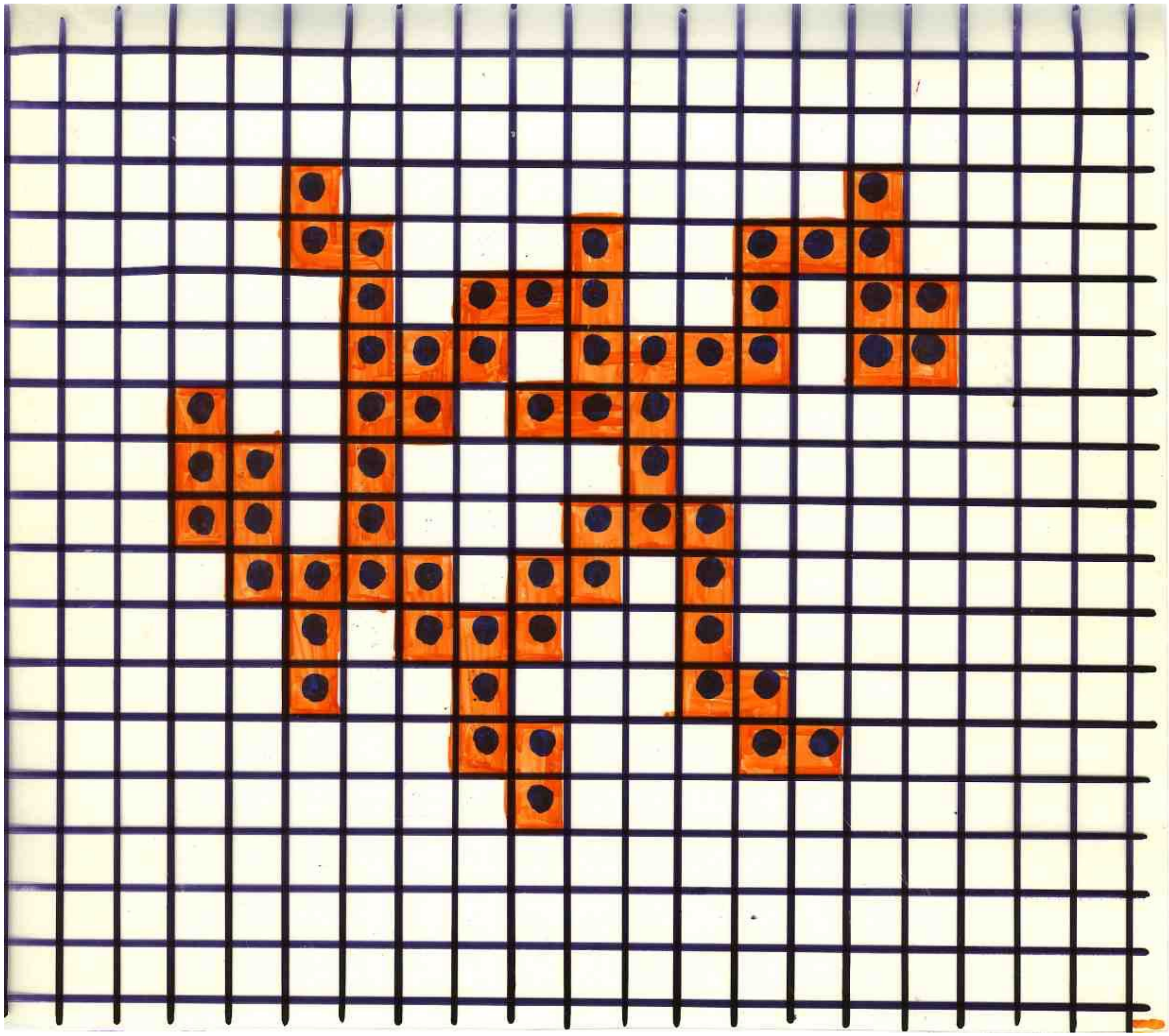


Percolation



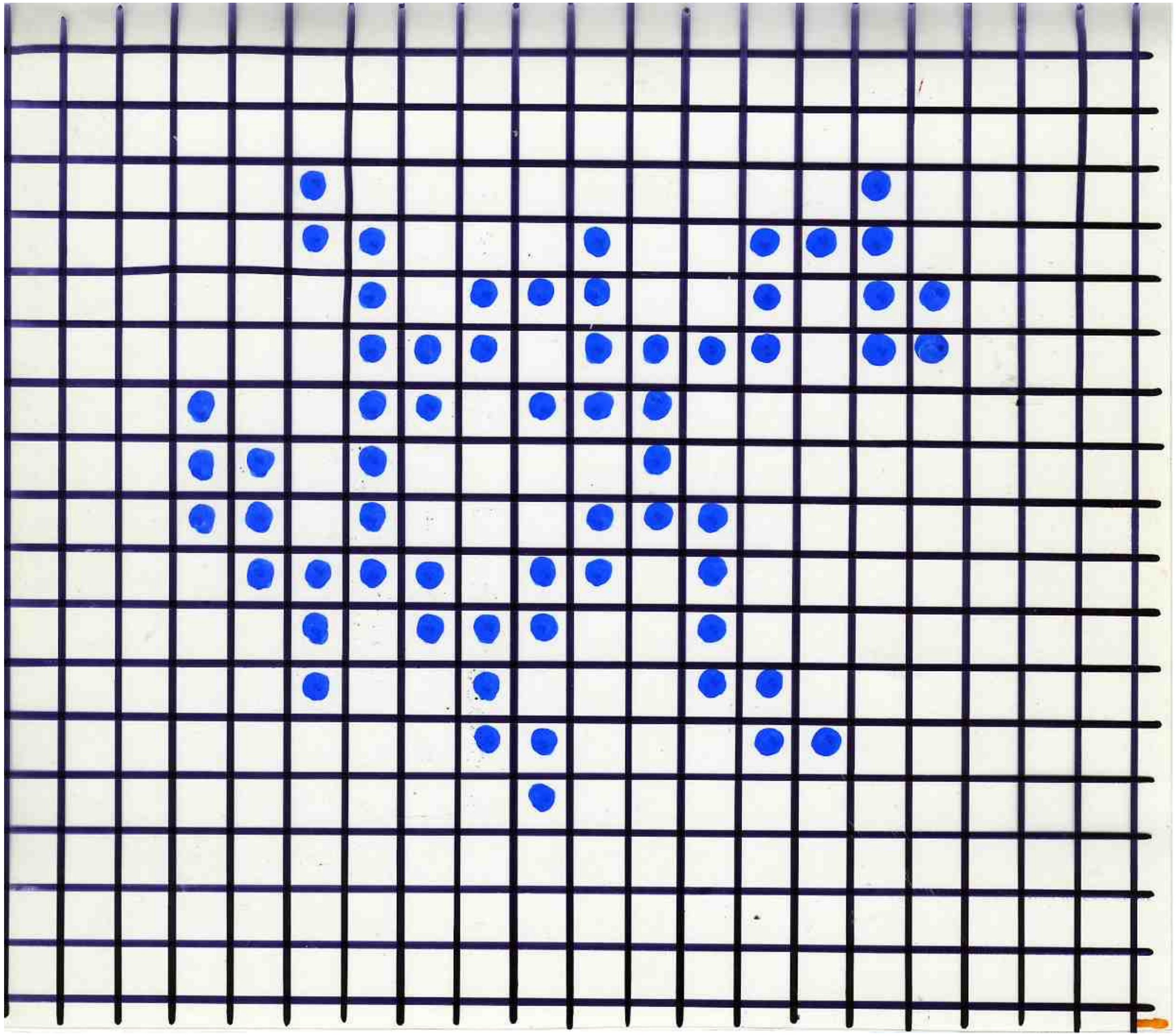


polyomino

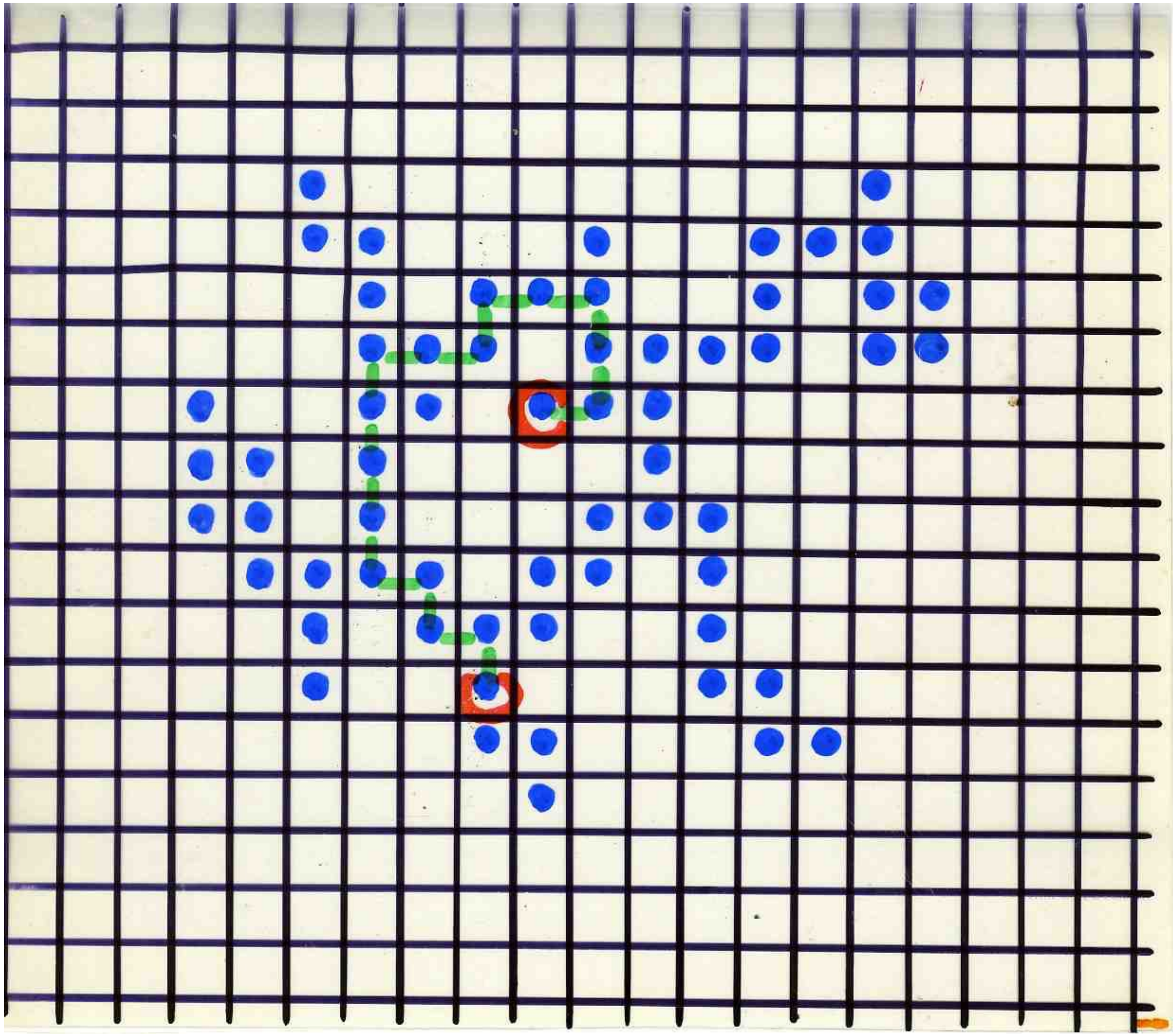


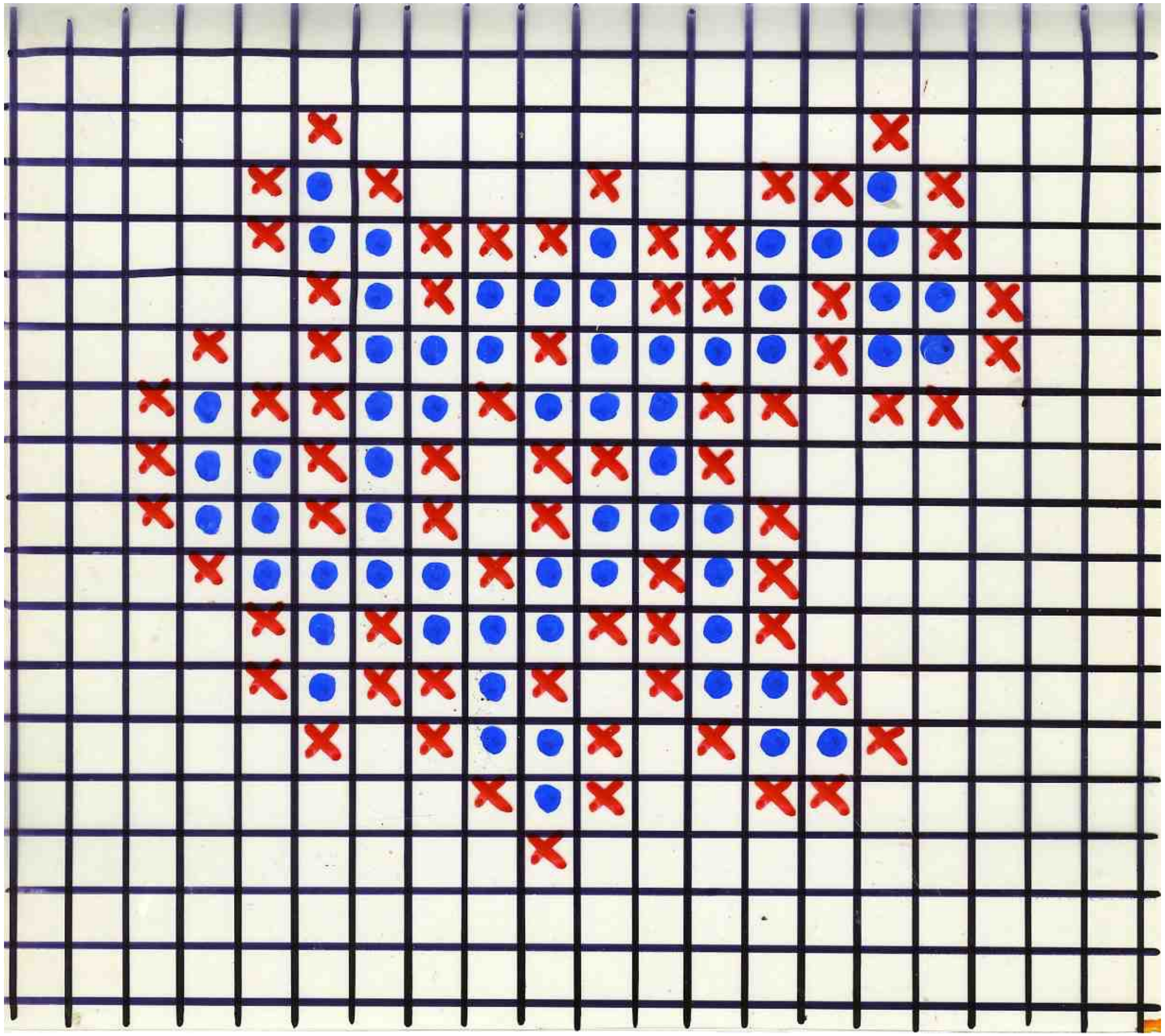
area

perimeter

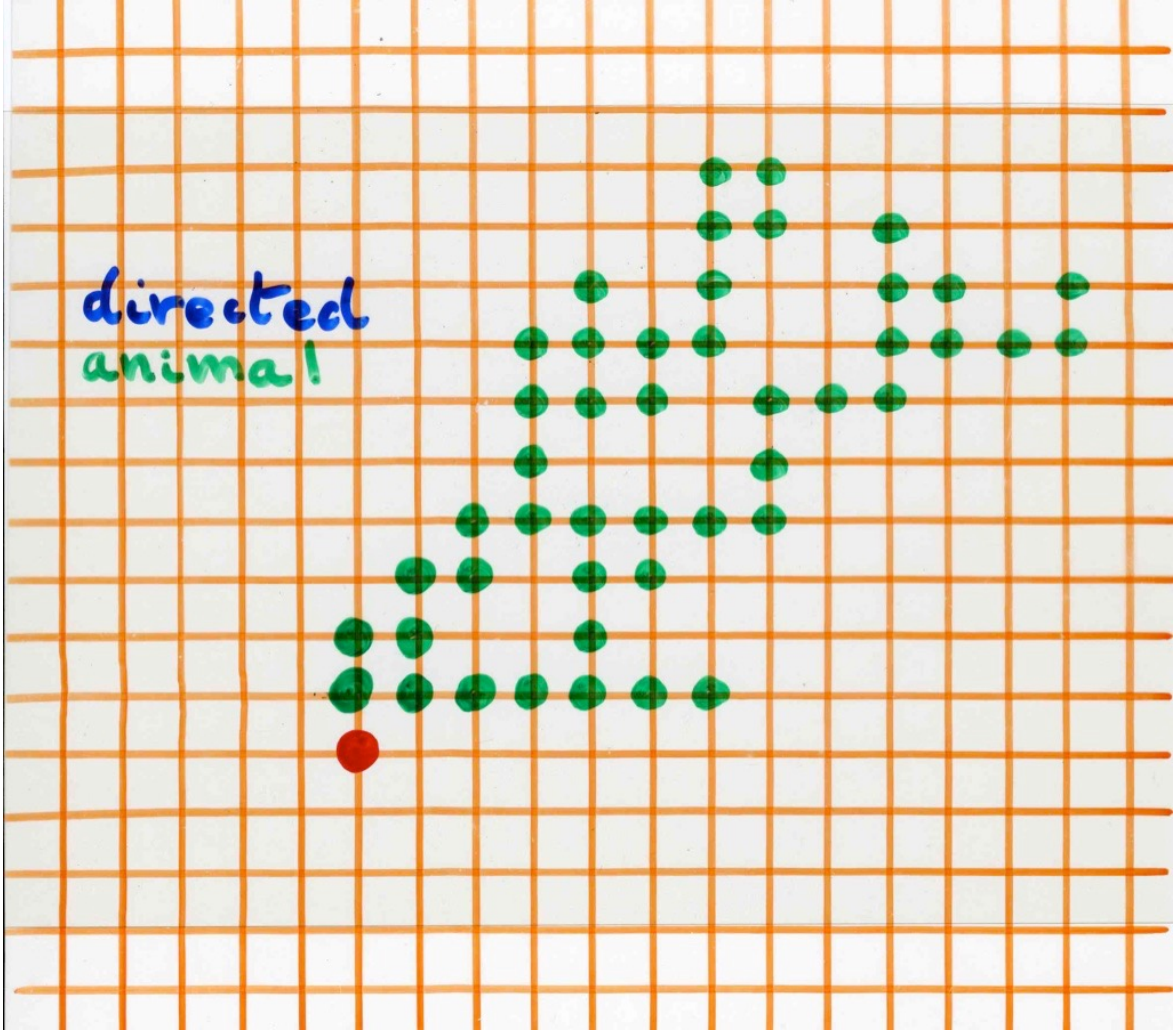


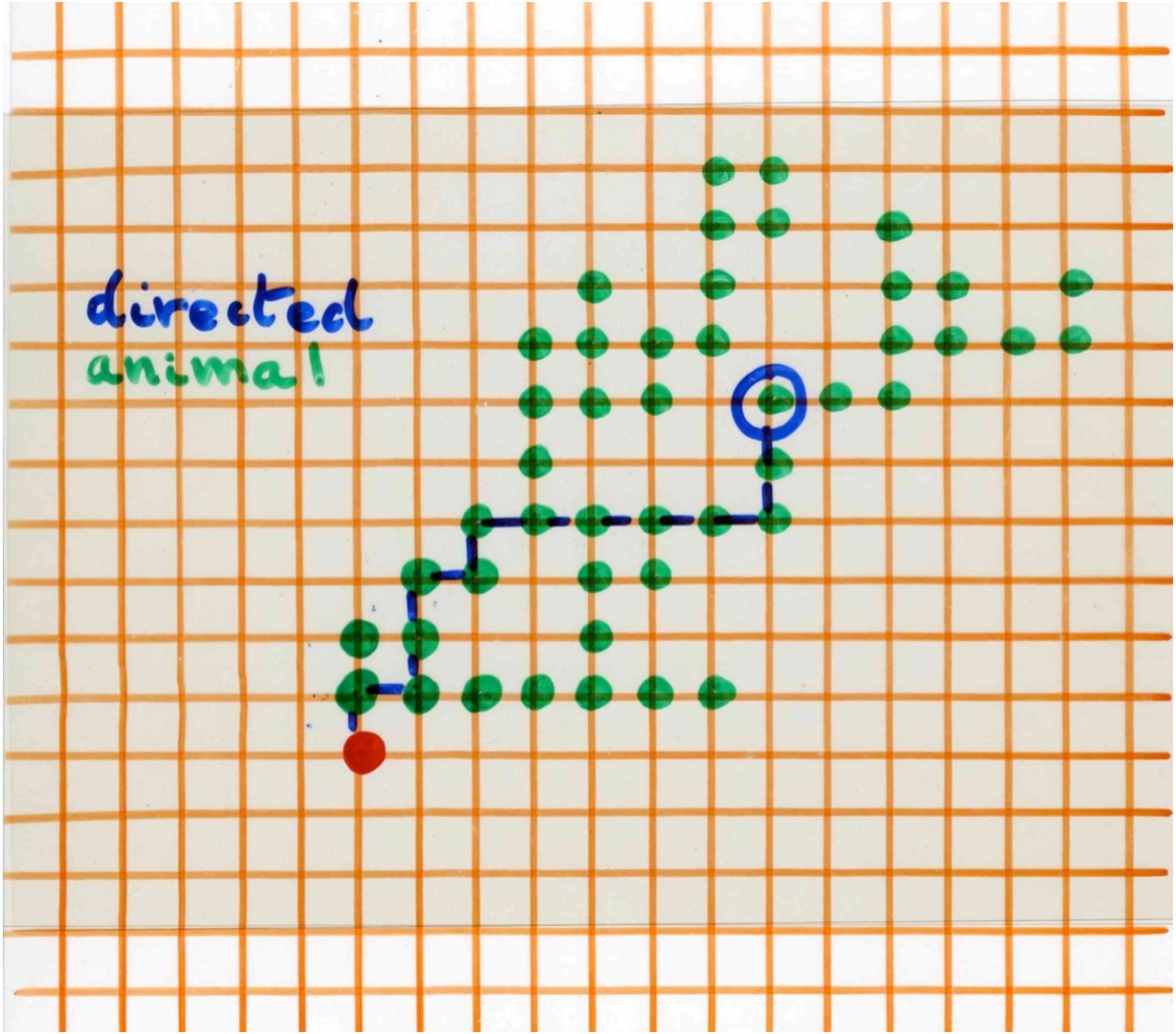
animal





The directed animal model

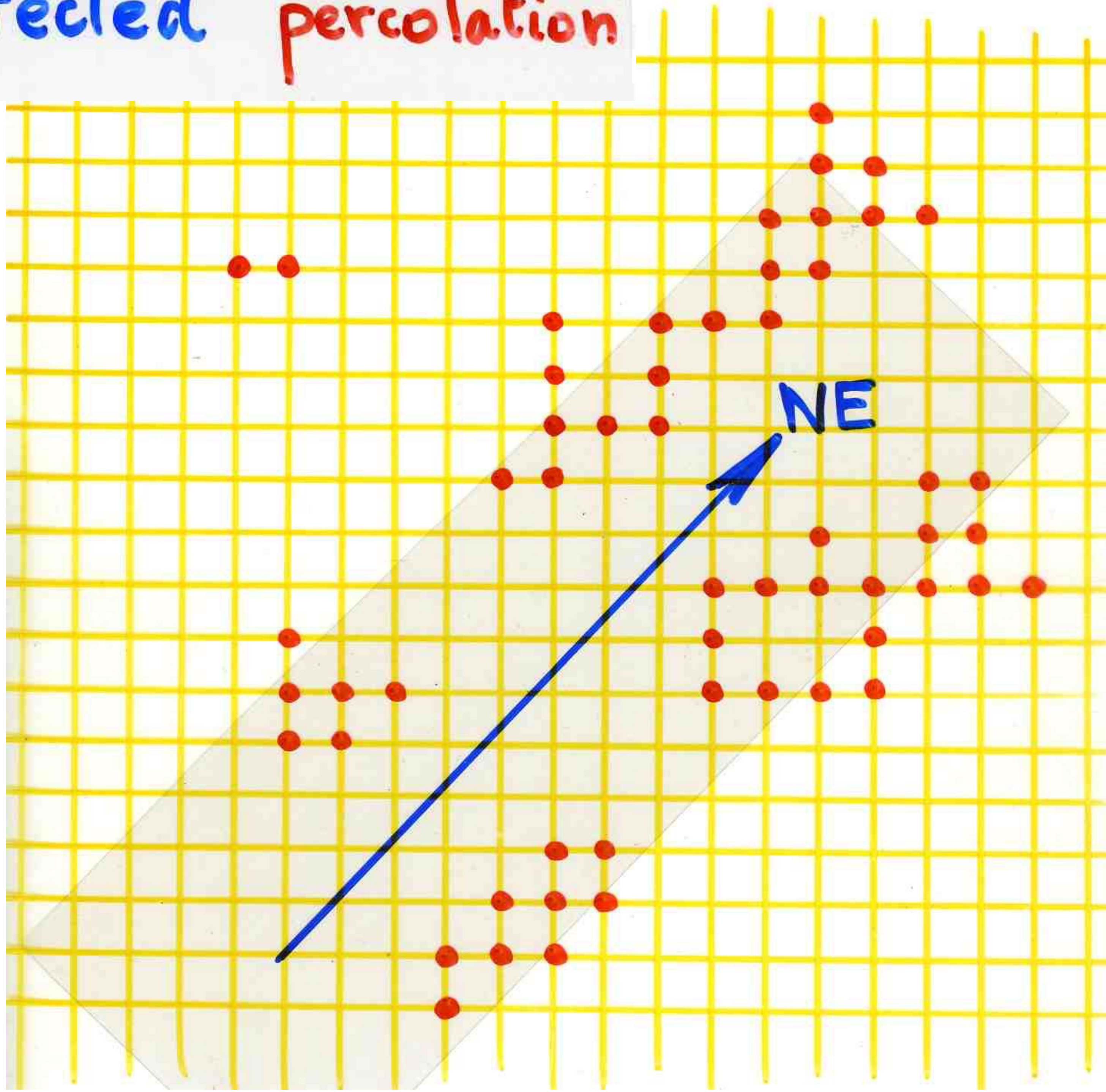

directed
animal

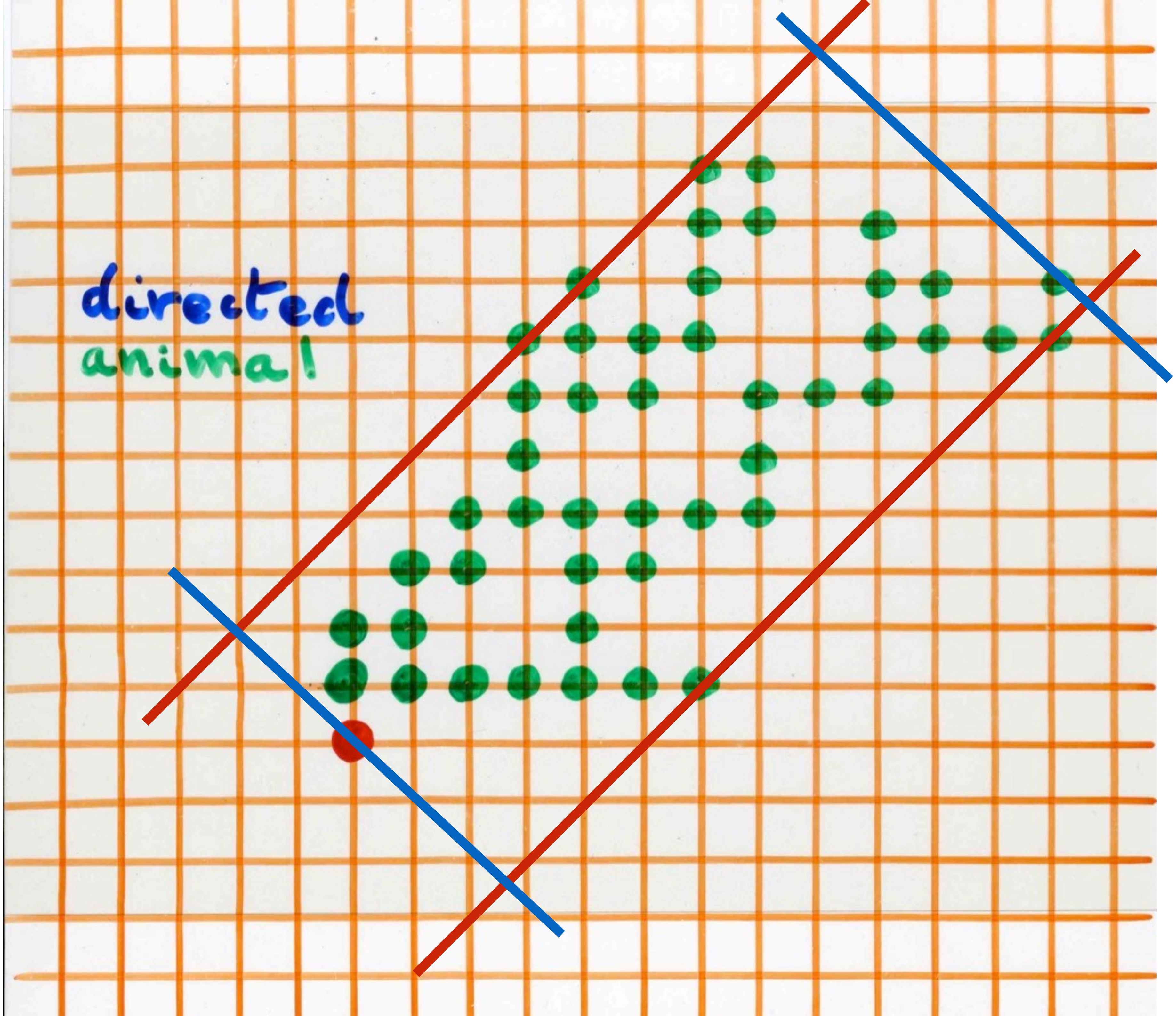

directed
animal

$a_n = \{$ number of directed animals having n points

$$F(t) = \sum_{n \geq 1} a_n t^n$$

directed percolation




directed
animal

$$a_n \sim \mu^n \, n^{-\theta}$$ number of directed animals n points

$$\ell_n \sim n^{\nu_\perp}$$ average width

$$L_n \sim n^{\nu_\parallel}$$ average length

$$a_n \sim \mu^n \, n^{-\theta}$$

number of directed animals n points

$$l_n \sim n^{\nu_\perp}$$

average width

$$L_n \sim n^{\nu_\parallel}$$

average length

Critical exponents

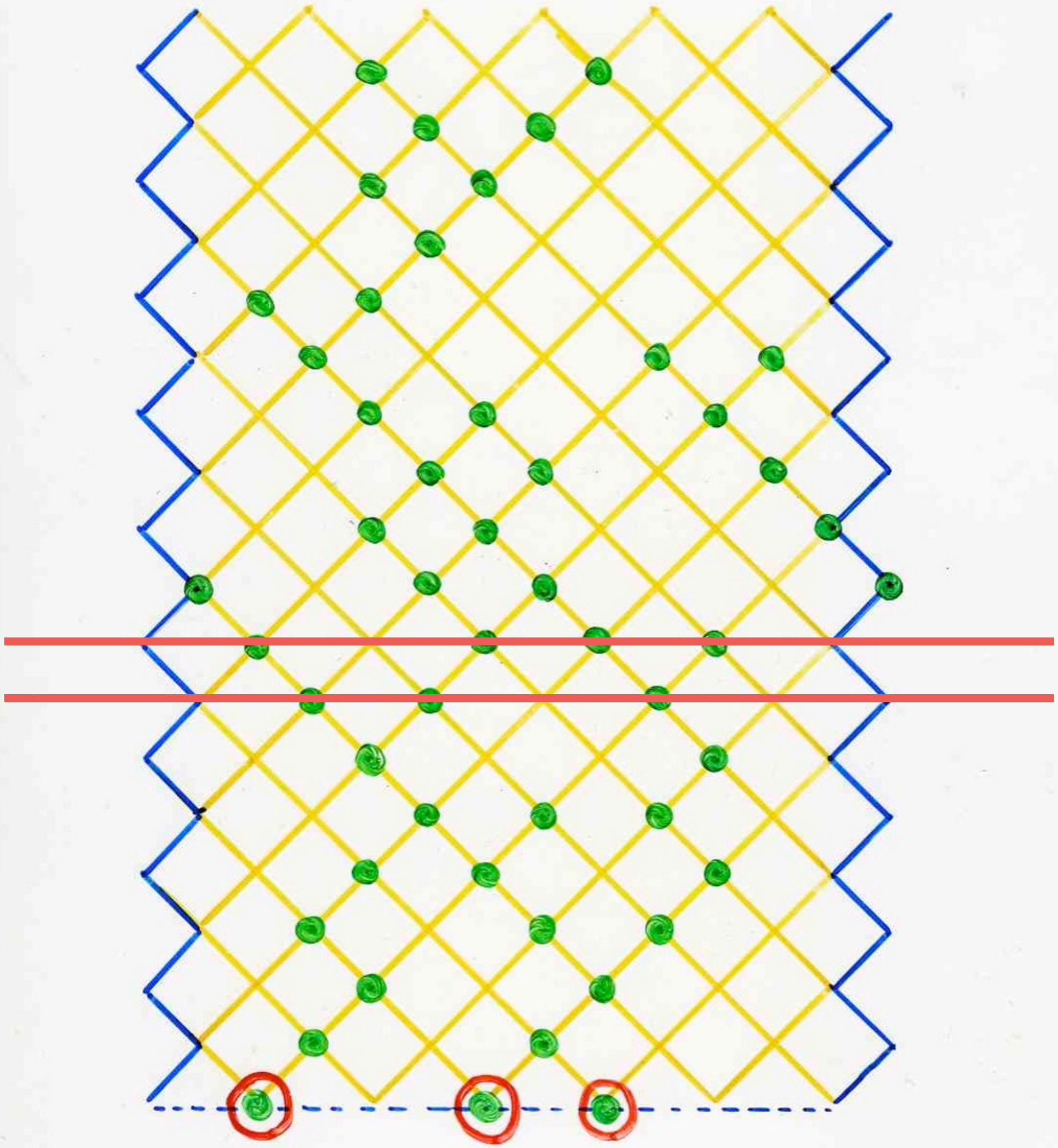
Nadal, Derrida, Vannimenus (1982)

B. Derrida

J. Vannimenus

J.P. Nadal

(1982, 1983)



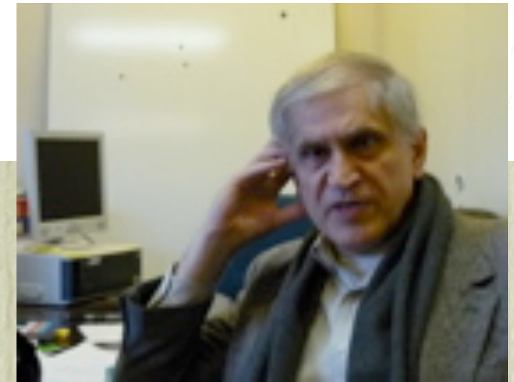
source points

directed animal
on a circular strip

$$\tilde{b}_n^{\leq k} = \frac{1}{k} \sum_{p=0}^{k-1} (-1)^p \sin \alpha_p \prod_{i=1}^{k-1} \left(\frac{\sin(i+\frac{1}{2})\,\alpha_p}{\sin \frac{\alpha_p}{2}} \right)^{N_i} (1 + 2\cos \alpha_p)^{n-1}$$

animals
circular strip
width k

$$\alpha_p = \frac{2p+1}{2k}\,\pi$$



J. Vannimenus

J.P. Nadal

(1982, 1983)

B. Derrida

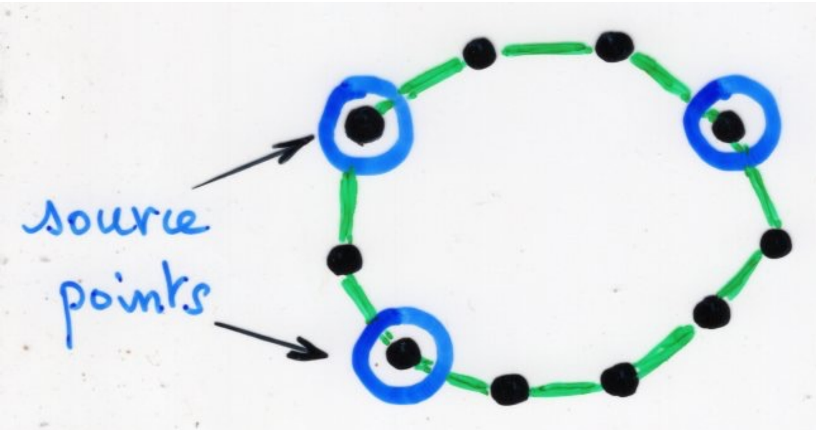
Nadal, Derrida, Vannimenus (1982)

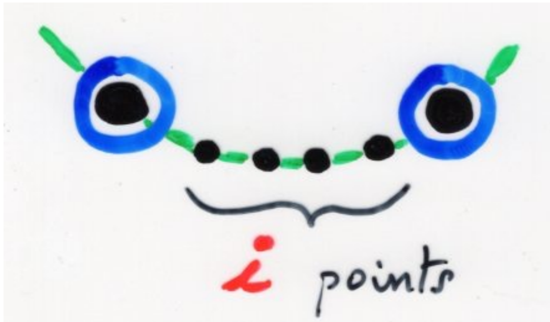
$$\tilde{b}_n^{\leq k} = \frac{1}{k} \sum_{p=0}^{k-1} (-1)^p \sin \alpha_p \prod_{i=1}^{k-1} \left(\frac{\sin (i+\frac{1}{2}) \alpha_p}{\sin \frac{\alpha_p}{2}} \right)^{N_i} (1 + 2 \cos \alpha_p)^{n-1}$$

animals
circular strip
width k

$$\alpha_p = \frac{2p+1}{2k} \pi$$

N_i = number of i-holes





Nadal, Derrida, Vannimenus (1982)

$$a_n^k \sim c(\mu_k)^n$$

$$\mu_k = 1 + 2\cos \pi/2k$$

$$a_n \sim \mu^n n^{-\theta}$$

$$\mu = 3 \qquad \theta = 1/2$$

Hakim, Nadal (1982)

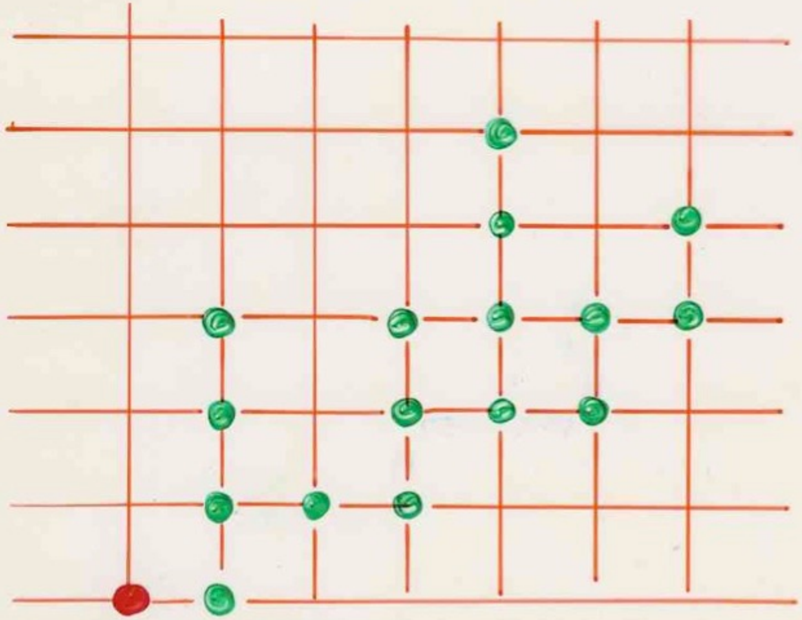
D. Dhar (1982)

$$\nu_\perp = 1/2$$

$$\nu_\parallel = \cancel{9/11} \ ?$$

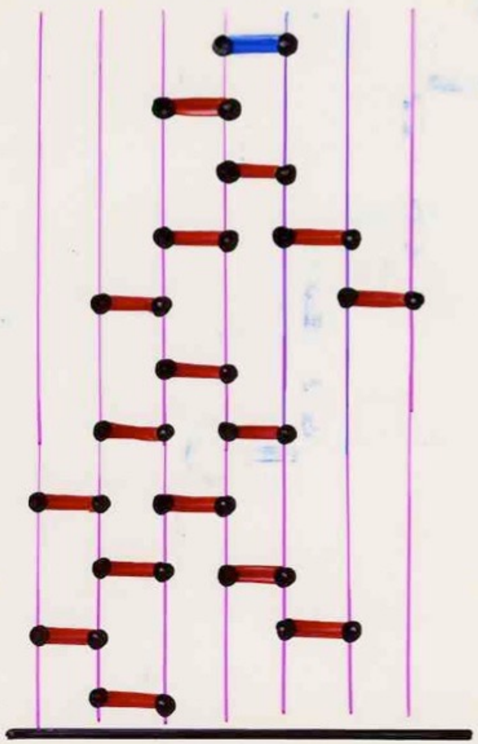
directed animals

and heaps of dimers

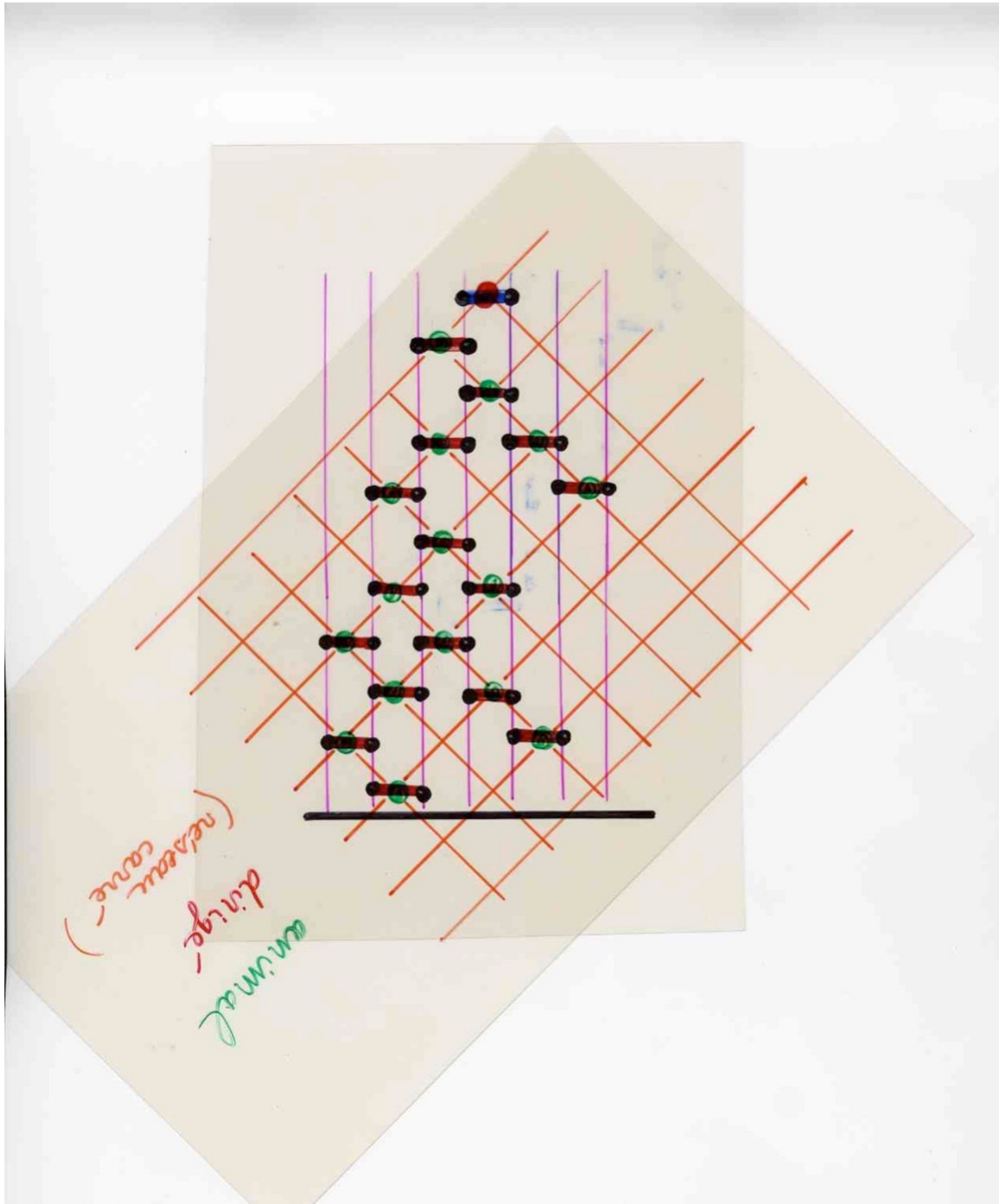


directed
animal
(square
lattice)

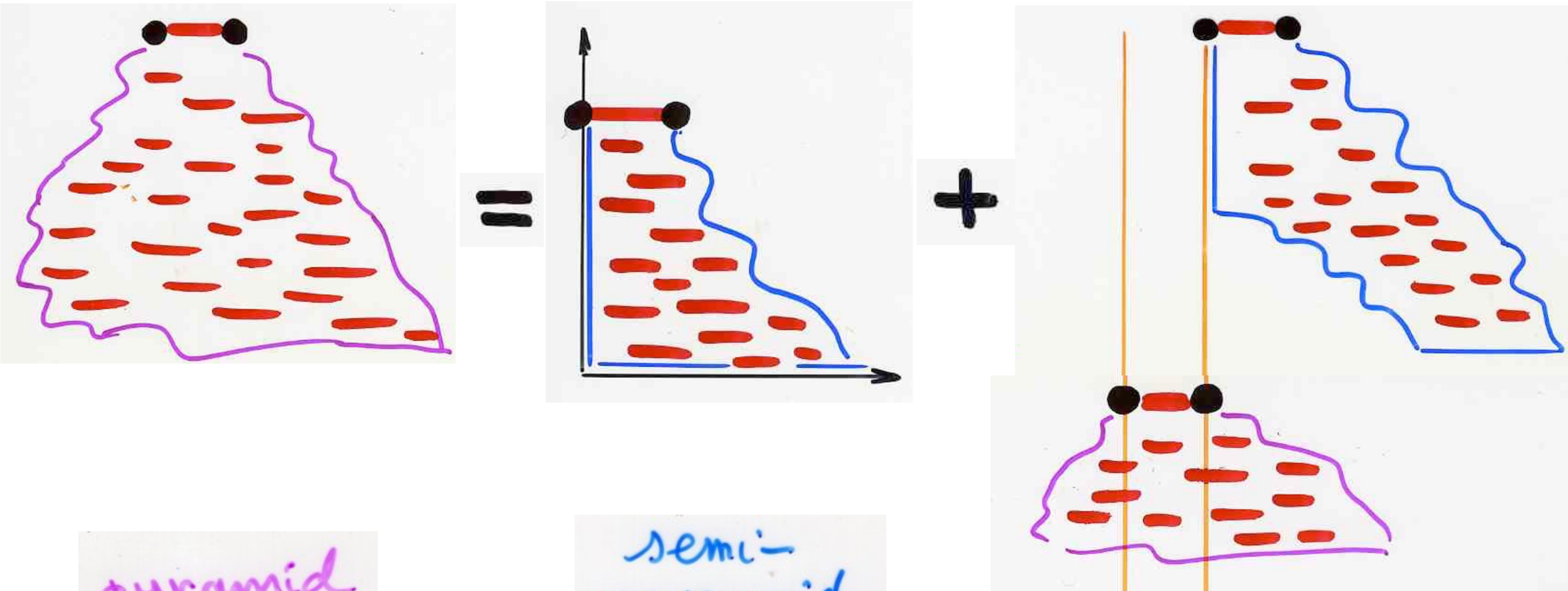
bijection



strict
pyramids



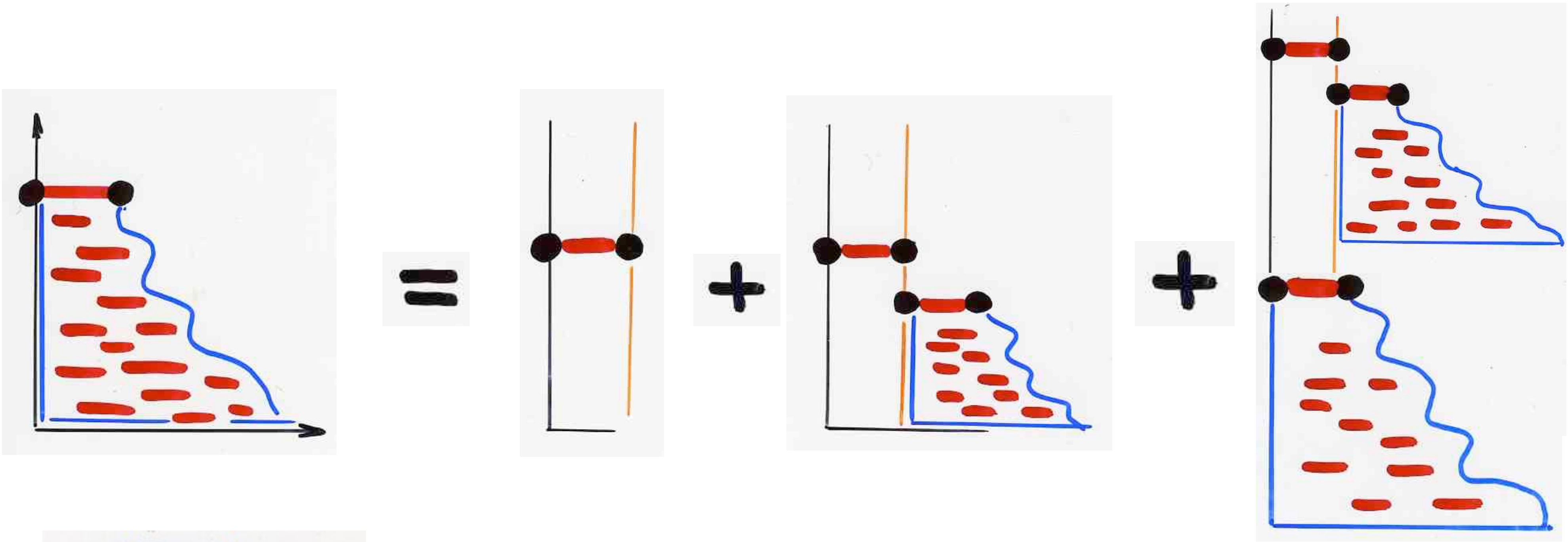
strict
pyramids



pyramid

semi-pyramid

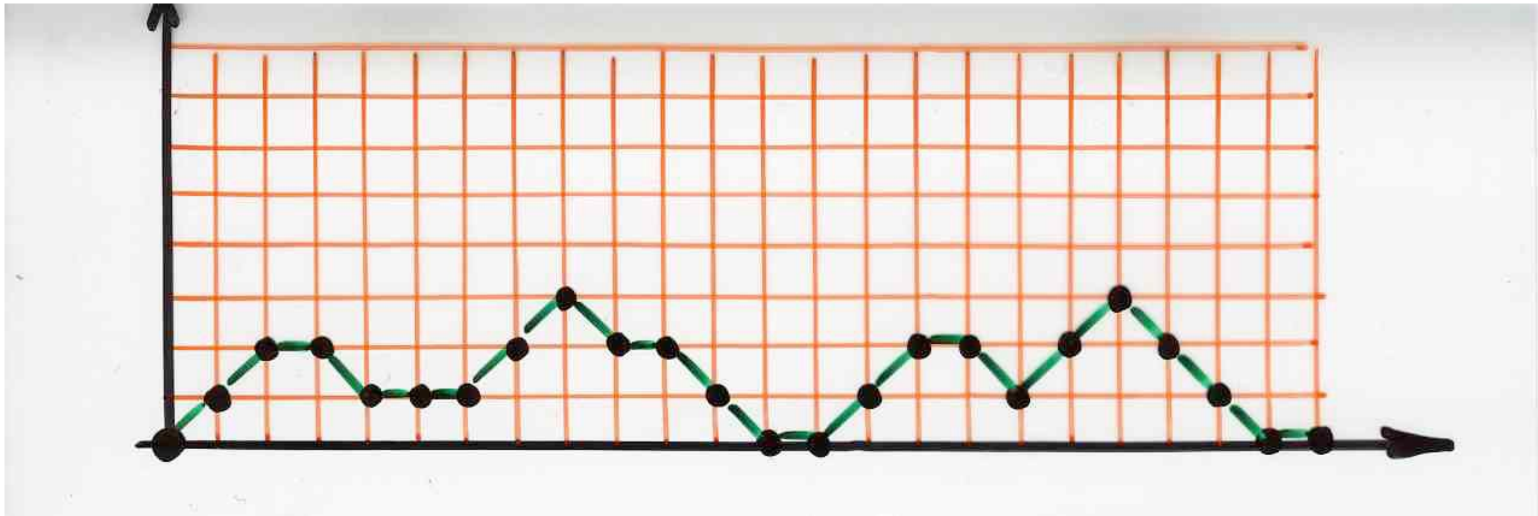
$$P = H + PH$$



semi-
pyramid

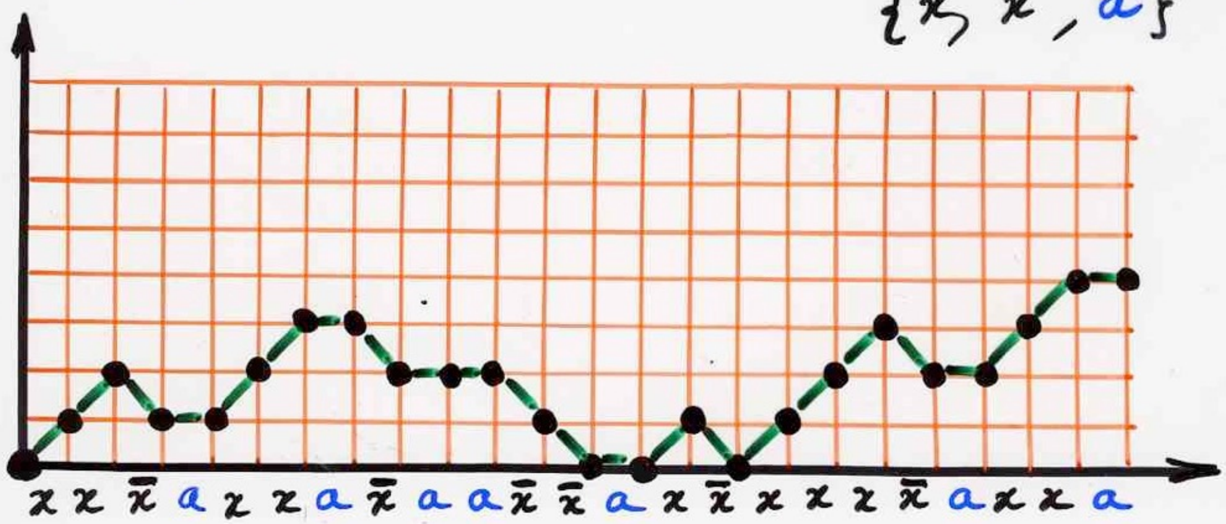
$$H = z + zH + zH^2$$

Motzkin paths



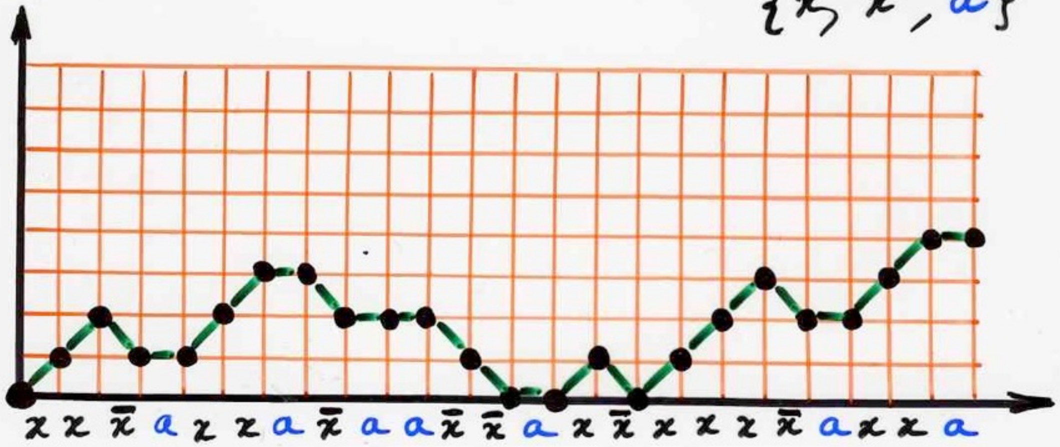
prefix (left factor) of a Motzkin path (word)

$\{x, \bar{x}, a\}$



prefix (left factor) of a Motzkin path
(word)

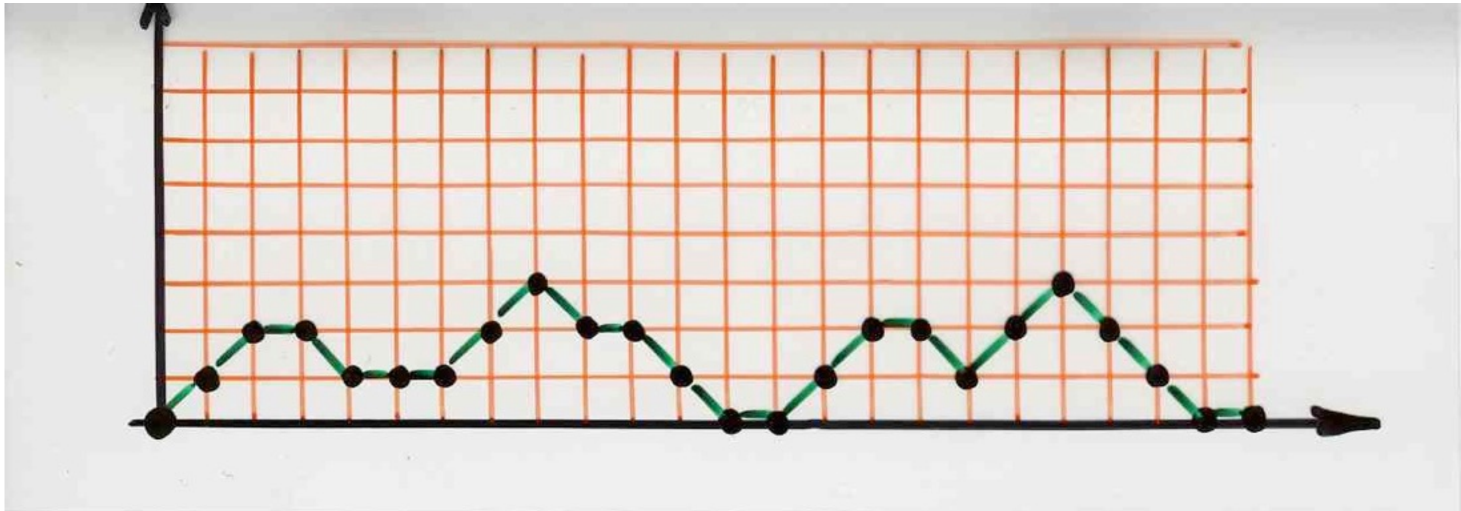
$\{x, \bar{x}, a\}$



- prefix Motzkin path = $\left\{ \begin{array}{l} \bullet \text{ Motzkin path} \\ \bullet \text{ (Motzkin path)} \times \end{array} \right.$

() × (prefix Motzkin path)

$$p = m + t\,p\,m$$



- Motzkin path =
$$\begin{cases} \bullet \ \emptyset \\ \bullet \ (\bullet\!-\!\bullet) \times (\text{Motzkin path}) \\ \bullet \ (\nearrow) \times (\text{Motzkin path}) \times (\searrow) \times (\text{Motzkin path}) \end{cases}$$

$$m = 1 + t\,m + t^2 m^2$$

$$P = H + PH$$

$$H = z + zH + zH^2$$

$$P = t_P$$
$$H = t_m$$

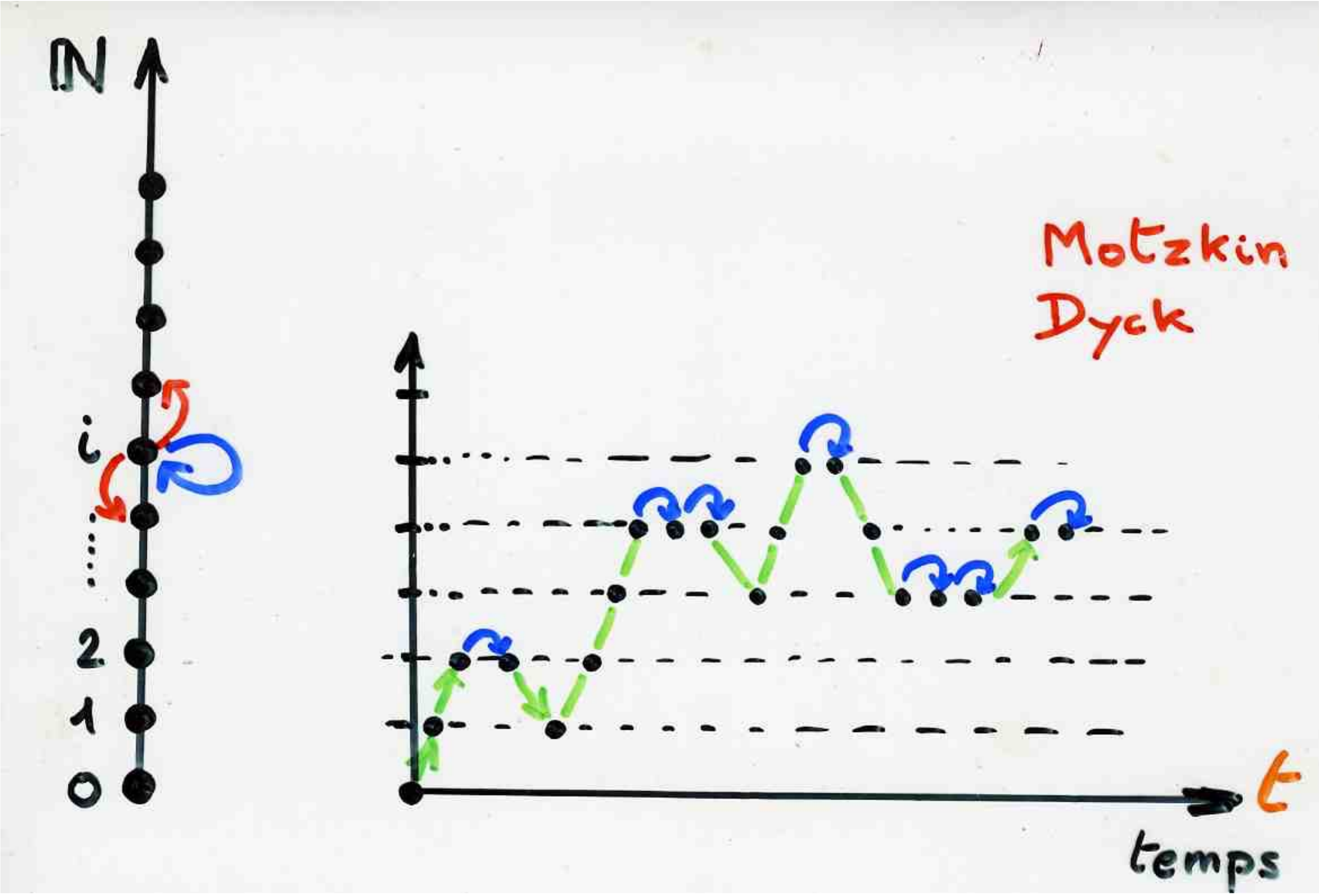
number of directed animals n points = number of prefix of Motzkin paths length $(n-1)$

(Brute force) bijection

2d animals → 1d paths

D. Gouyou-Beauchamps

X. V. (1984)



$$\sum_{n\geq 1} a_n t^n = \frac{1}{2}\left[\left(\frac{1+t}{1-3t}\right)^{1/2} - 1\right]$$

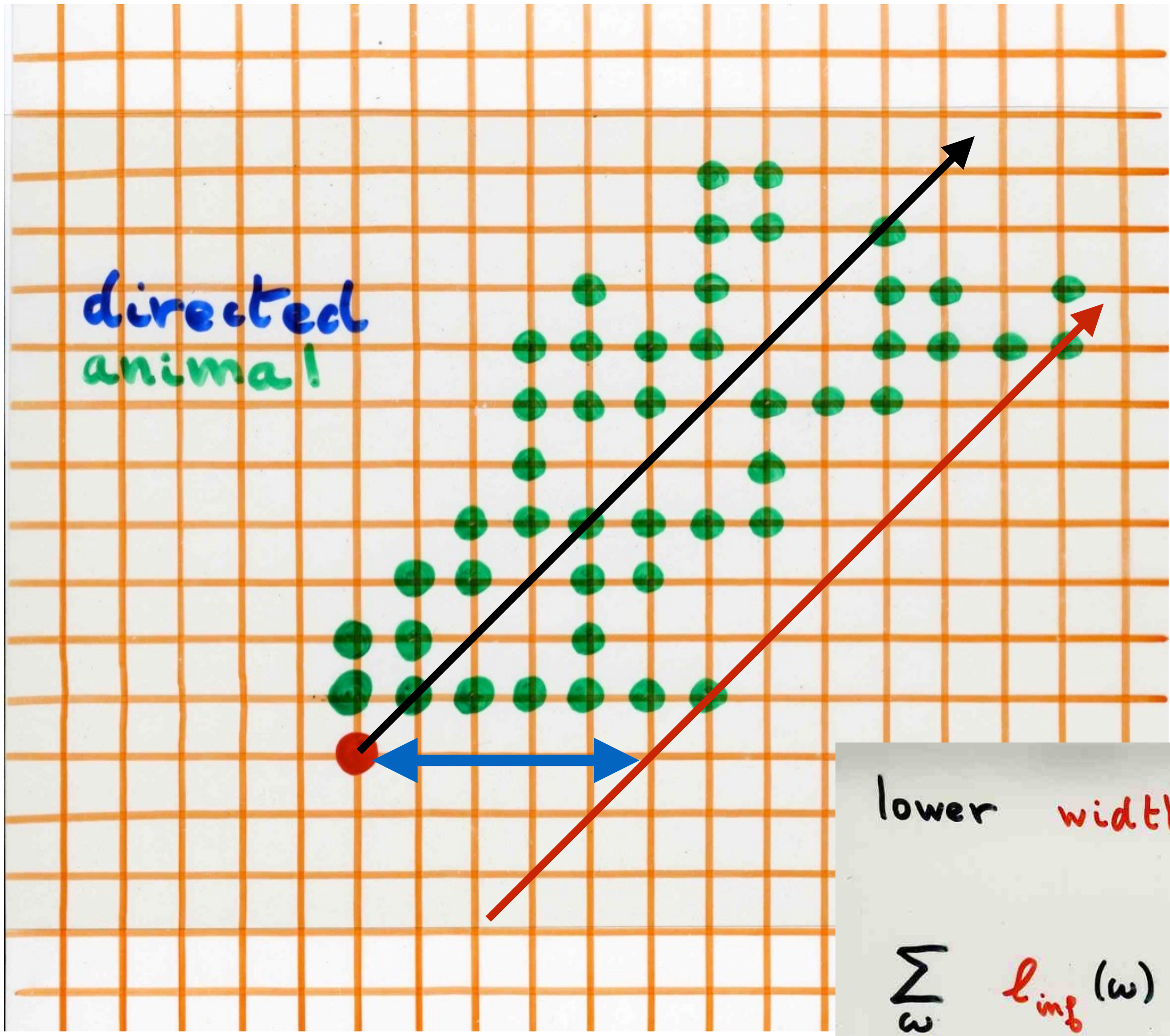
$$a_{n+1} = \sum_{0\leq i\leq n} \binom{n}{i}\binom{i}{\lfloor i/2\rfloor}$$

lower width → level of the final point of the path

$$\sum_{\substack{\omega \\ \text{path} \\ |\omega|=n}} \ell_{\inf}(\omega) = 3^n$$

$$\ell_{n+1} = \frac{2.\,3^n}{a_{n+1}} - 2$$ (Dhar conjecture) 1982

$\ell_n \sim n^{1/2}$ $\nu_\perp = \frac{1}{2}$



lower width → level of the final point of the path

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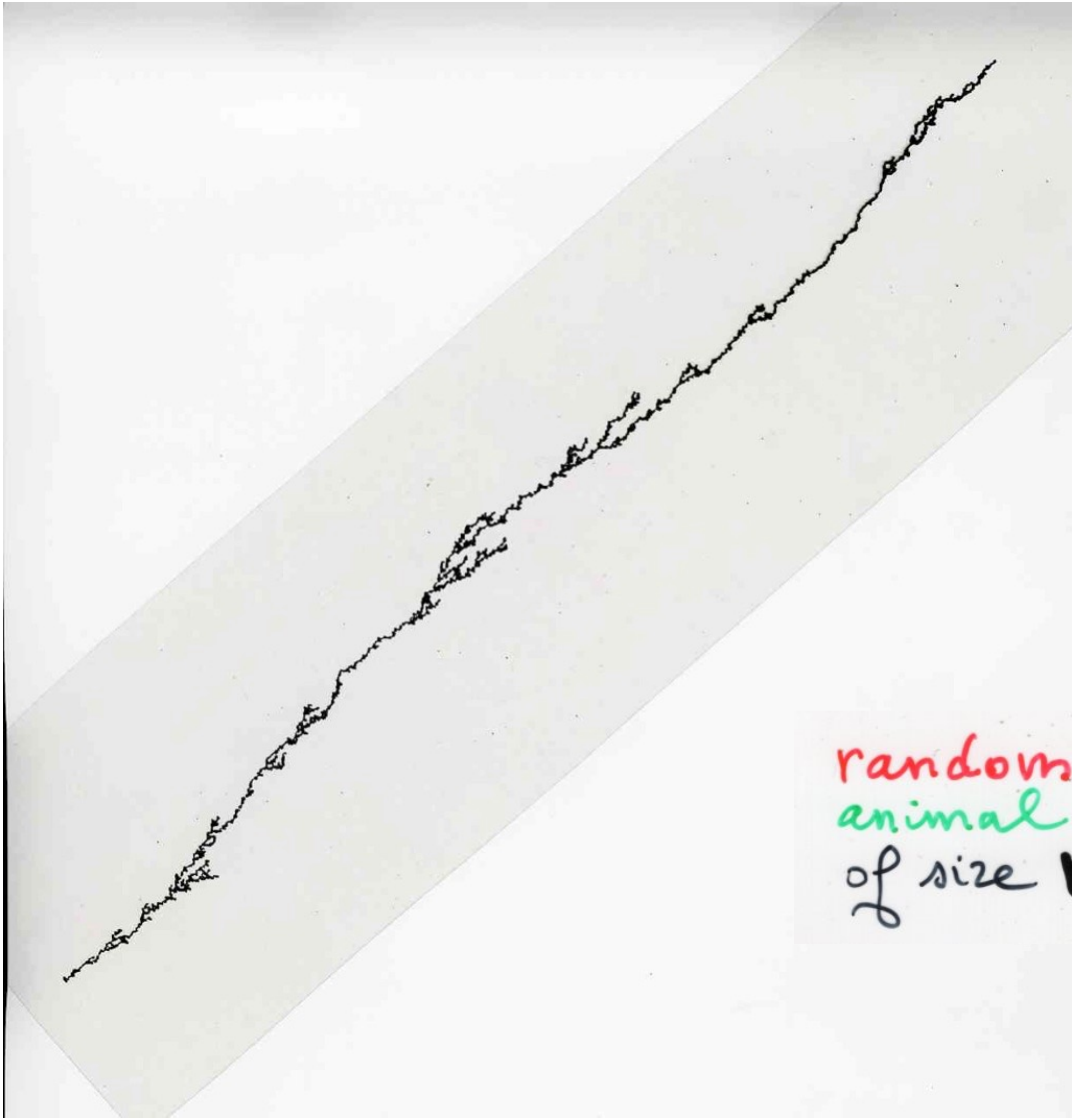
(Dhar conjecture)
1982

$$\ell_n \sim n^{1/2} \qquad \nu_\perp = \frac{1}{2}$$



random
animal
of size n

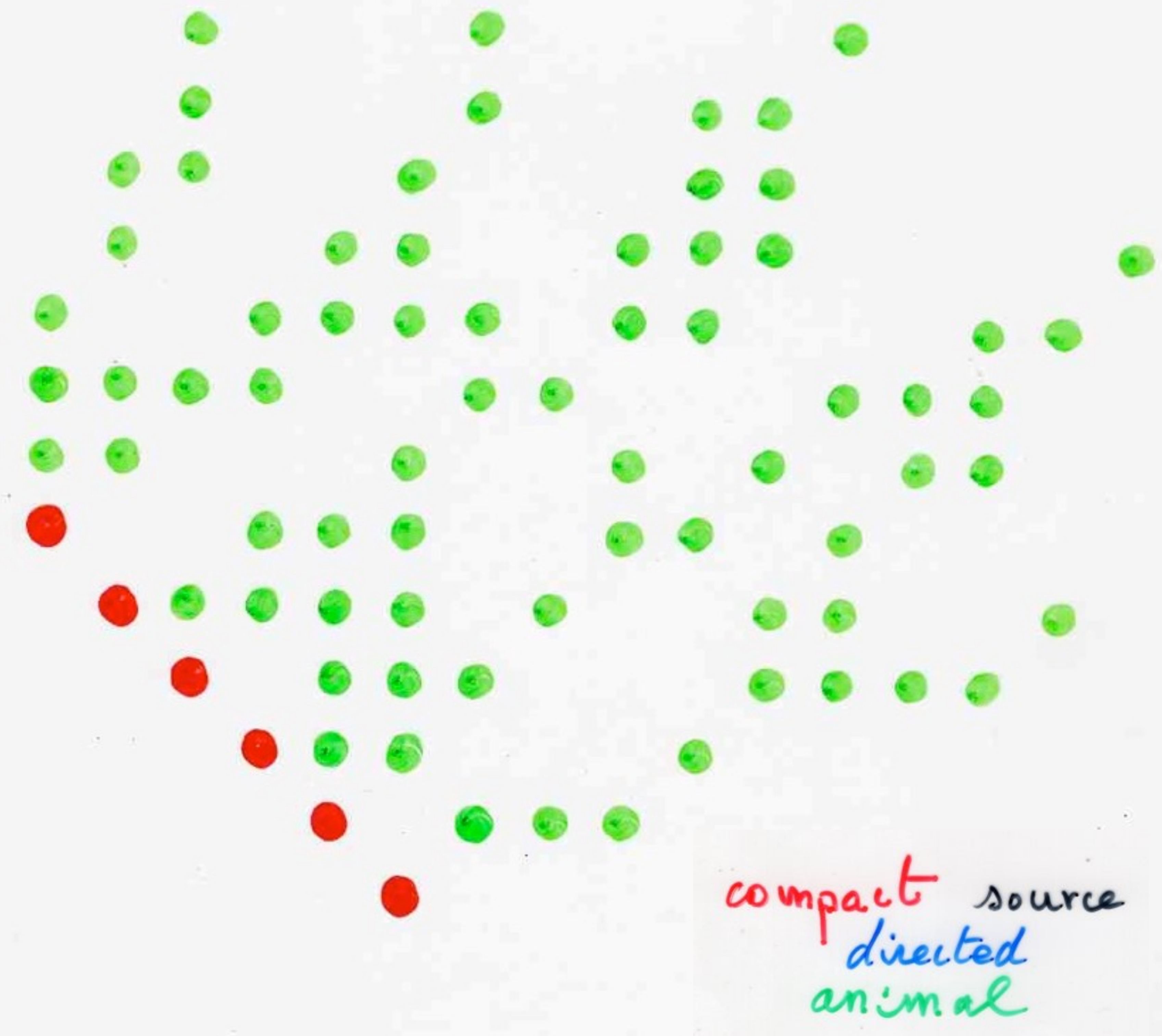
random
animal
of size n

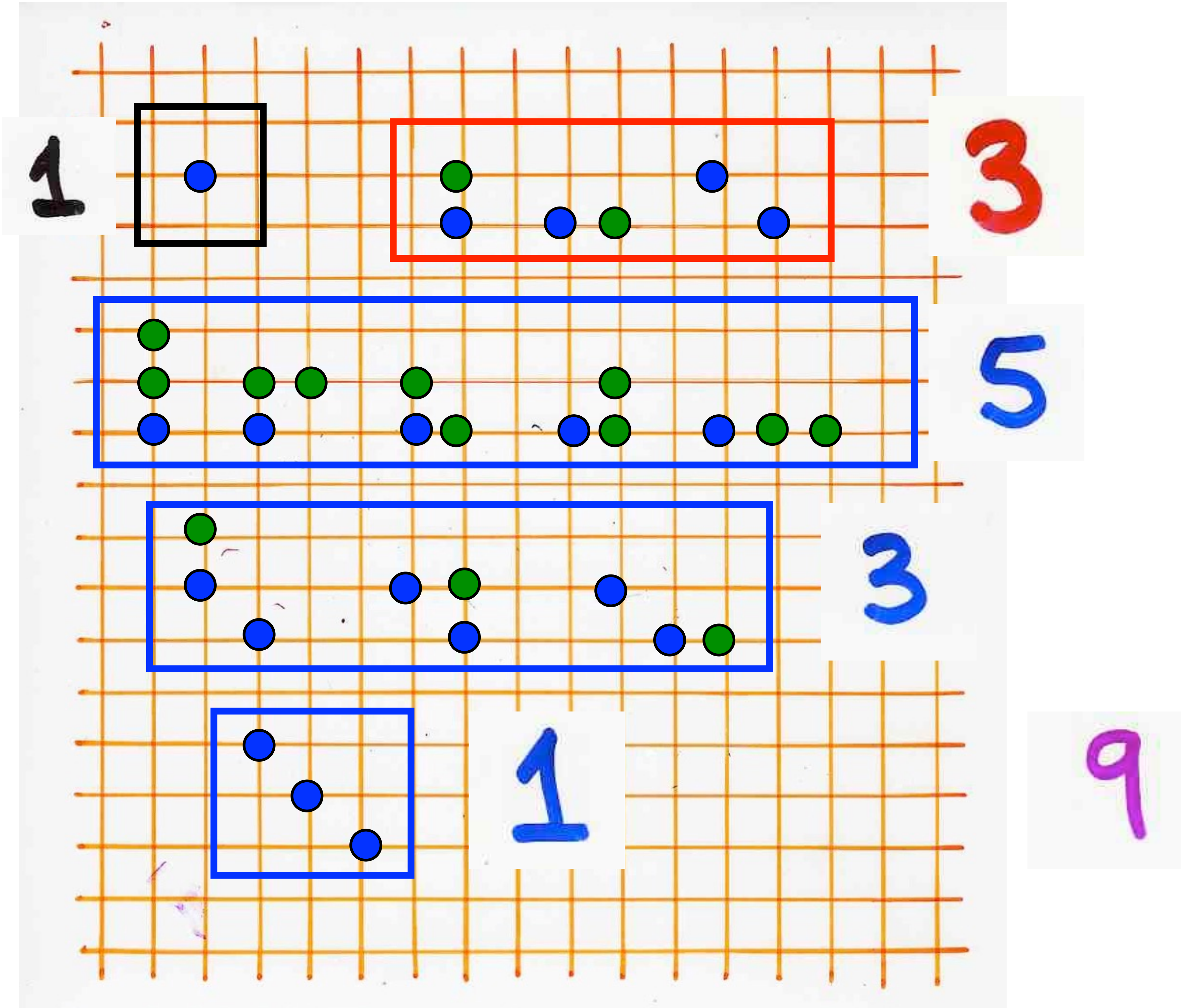
complements

exercise ?

compact source

directed animals


compact source
directed
animal


1
3
5
3
1
9

$1, 3, 9, 27, 81, \ldots$

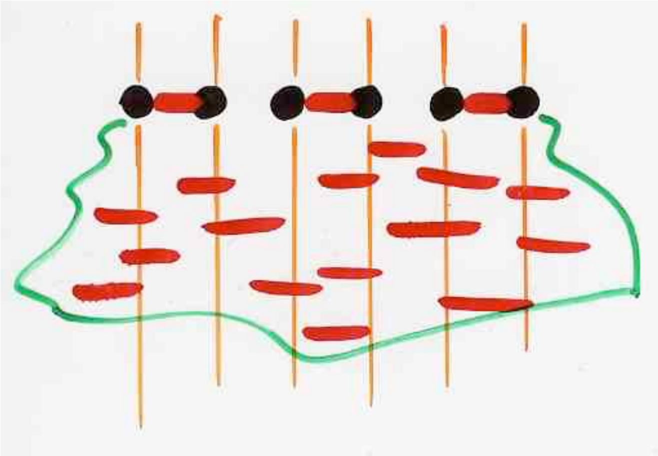
$1, 3, 3^2, 3^3, 3^4, 3^5, 3^6,$!

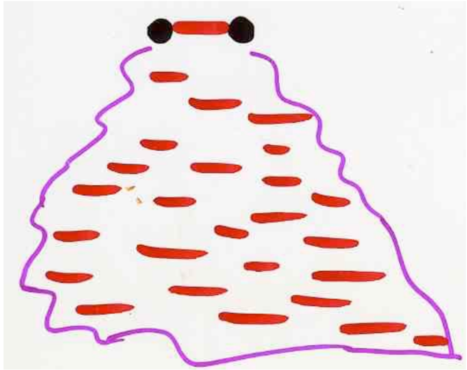
The number of directed animals size $n+1$, with compact source is

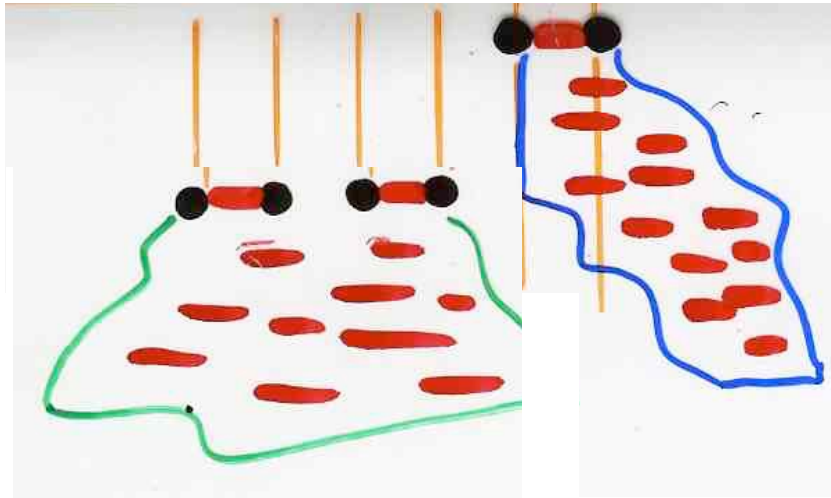
The number of directed animals size $n+1$, with compact source is

$$3^n \; !$$

D. Gouyon-Beauchamps
X. V. (1984)







compact source
directed
animal

pyramid

semi-pyramid

$$X = P + XH$$

$$H = z + zH + zH^2$$

$$P = H + PH$$

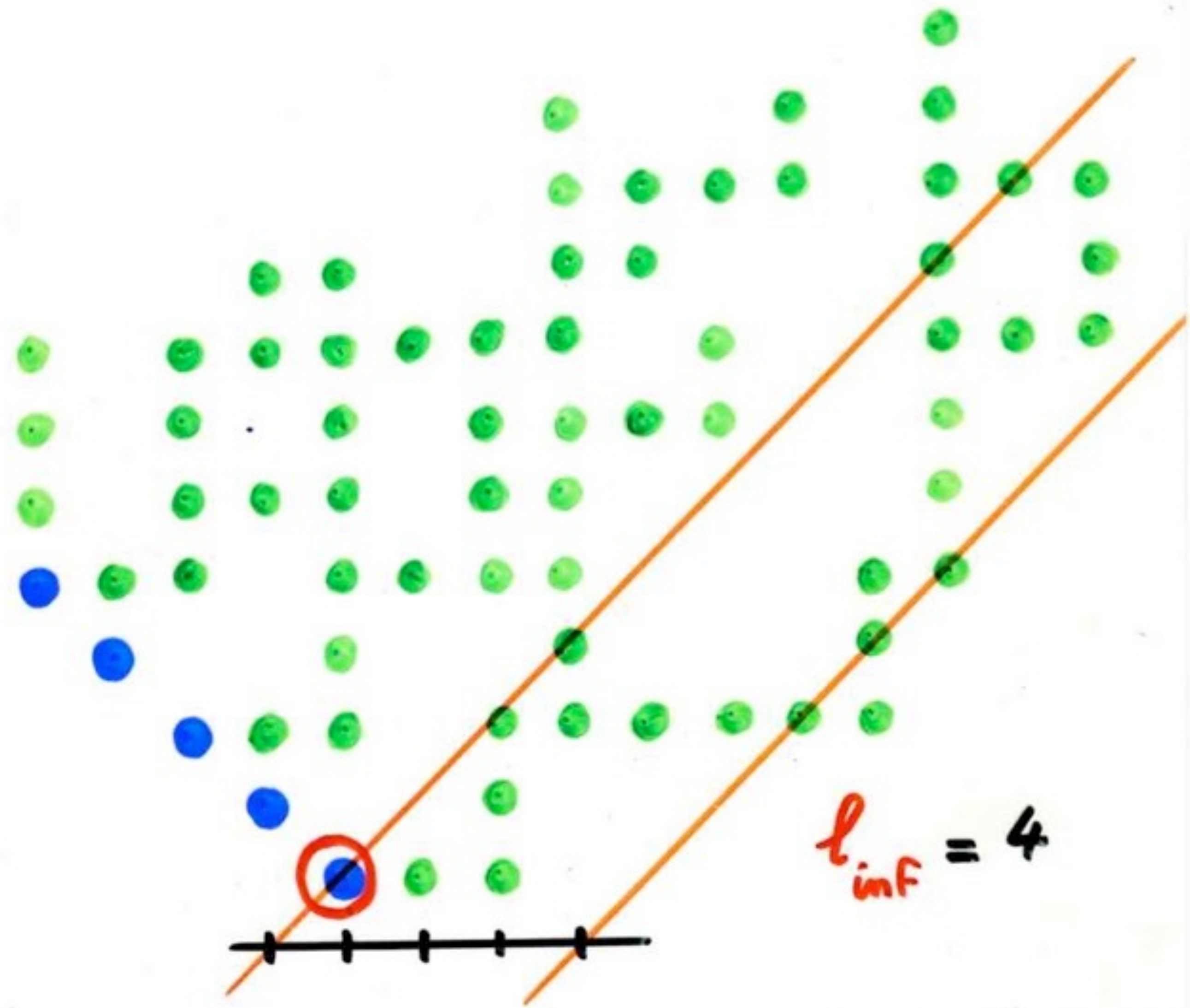
$$X = P + XH$$

$$X = \frac{z}{1-3z}$$

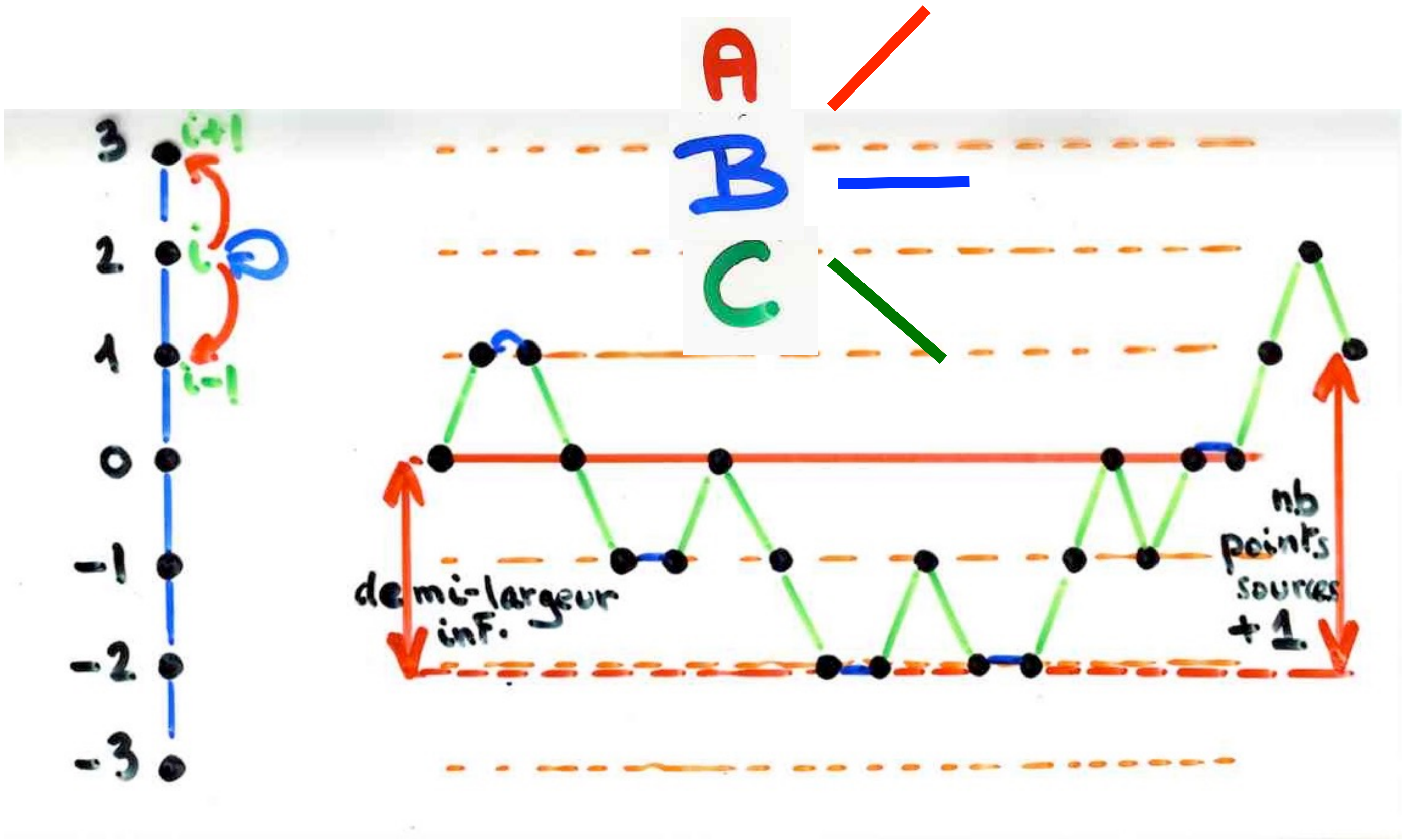
J. Betrema

J.G. Penaud

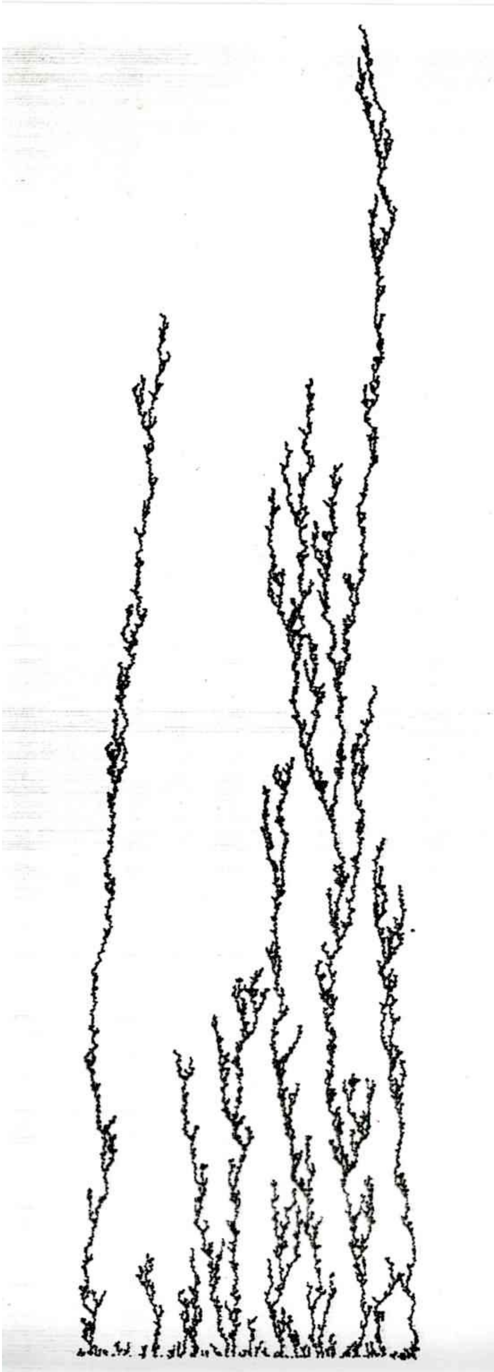
$$= z + 3z^2 + 3^2 z^3 + \ldots + 3^n z^{n+1} + \ldots$$

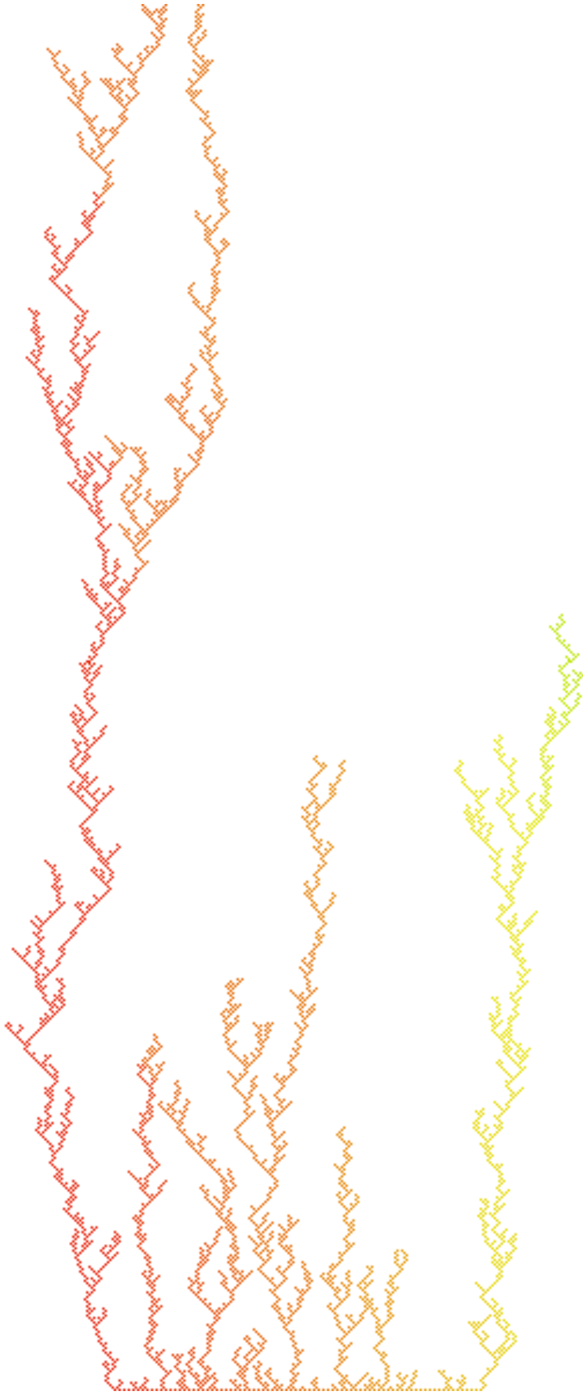


$\ell_{inf} = 4$



D. Gouyou-Beauchamps
X. V.
(1984)





random

compact source
directed
animal