Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics, a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

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Chapter 6

Heaps and Coxeter groups
(1)

fully commutative elements
and Temperley-Lieb algebra

IMSc, Chennai
23 February 2017

The heap monoid
of a
Coxeter group

finite set S

M square symmetric matrix

indexed by S

$$\begin{cases} m_{ss} = 1 \\ m_{st} = m_{ts} \in \{2, 3, \ldots\} \cup \{\infty\} \end{cases}$$

Coxeter group W

generators S

relations

- $s^2 = 1$ for all $s \in S$
- $\underbrace{sts\ldots}_{m_{st}} = \underbrace{tst\ldots}_{m_{st}}$ if $m_{st} < \infty$

braid relations

- $s^2 = 1$ for all $s \in S$

- $\underbrace{sts\ldots}_{m_{st}} = \underbrace{tst\ldots}_{m_{st}}$ if $m_{st} < \infty$

$$(st)^{m_{st}} = 1$$

$$\begin{cases} m_{ss} = 1 \\ m_{st} = m_{ts} \in \{2, 3, \ldots\} \cup \{\infty\} \end{cases}$$

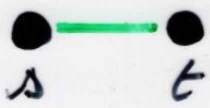
Coxeter matrix M

Coxeter system (W, S)

Coxeter graph Γ

Coxeter graph Γ

vertex set S



edge iff $m_{s,t} \geqslant 4$

Coxeter-Dynkin diagram

labeled graph Γ

$H(W, S)$ heap monoid

associated to the Coxeter group W
(in fact Coxeter system)
(W, S)

set of basic pieces S
dependency relation $\mathcal{C}$

$s \,\mathcal{C}\, t$

iff $m_{s,t} \neq 2$

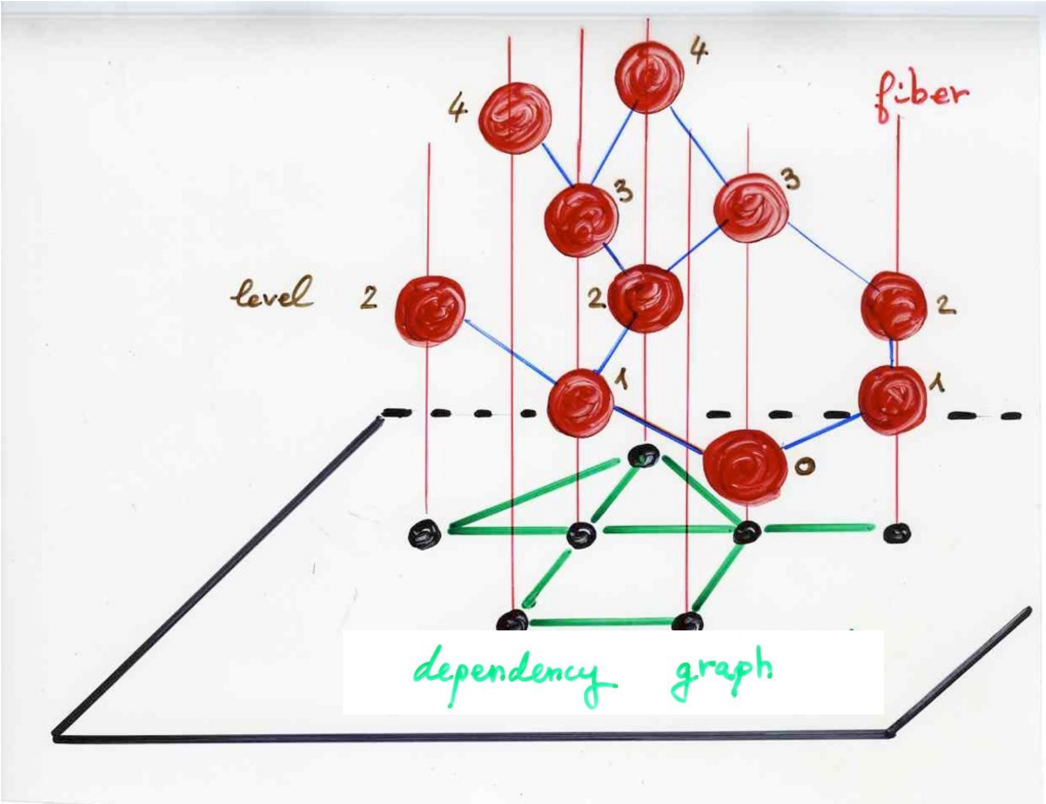
in other words

$$\begin{cases} - \; s \,\mathcal{C}\, s \\ - \; s \,\mathcal{C}\, t, \; (s \neq t) \text{ iff } \end{cases}$$

$H(W,S)$ is the heap monoid associated to the graph Γ as in Ch 5

Γ is the dependency graph

$$H(W,S) = H(\Gamma)$$



$$\Gamma = (S, E)$$

finite poset $(H, \preceq)$

labeling map π

$H \xrightarrow[\pi]{} \Gamma$

(denoted E in the original paper "étiquette")

second definition of heaps, Ch1c, p29, 31

(i) $\alpha, \beta \in E$, $\pi(\alpha)\, \mathcal{C}\, \pi(\beta) \Rightarrow \begin{cases} \alpha \preceq \beta \\ \text{or} \\ \beta \preceq \alpha \end{cases}$

(ii') $\preceq$ is the transitive closure of the relation in (i)

$\alpha \preceq \beta$ and $\pi(\alpha)\, \mathcal{C}\, \pi(\beta)$

can be rewritten

can be rewritten as :

(i)' for every vertex $s \in S$
$H_s = \pi^{-1}(\{s\})$ is a chain

fiber over $s \in S$

for any edges $\{s,t\}$ of Γ
$H_{s,t} = \pi^{-1}(\{s,t\})$ is a chain

fiber over $\{s,t\}$ edge of Γ

chain = totally ordered subset of H

(ii)' The order relation $\preccurlyeq$ is the transitive closure of the relations given by all chains of (i)'

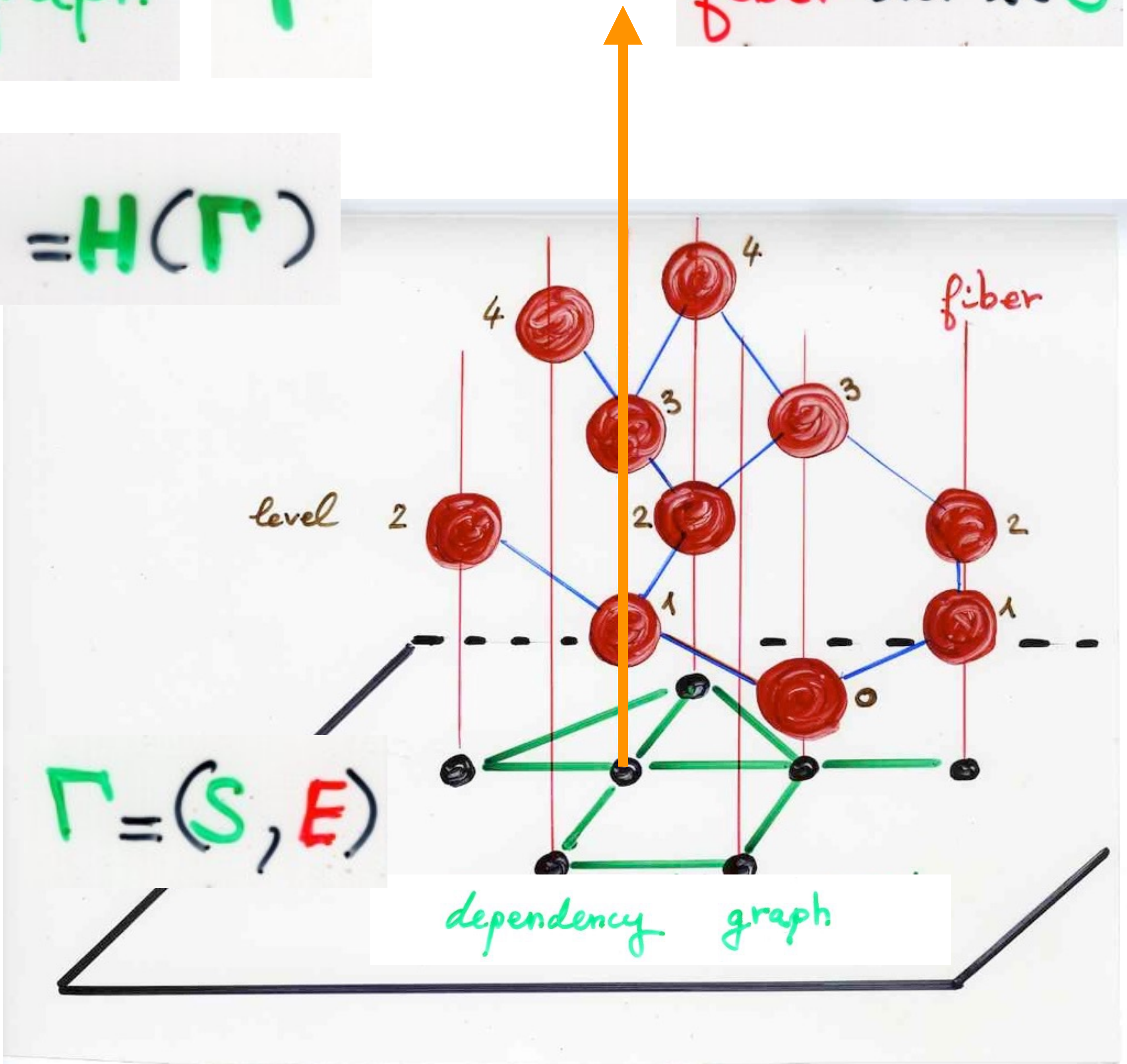
H_s $\quad$ $H_{s,t}$

(i.e. the smallest partial ordering containing these chains)

Coxeter graph Γ

fiber over $s \in S$

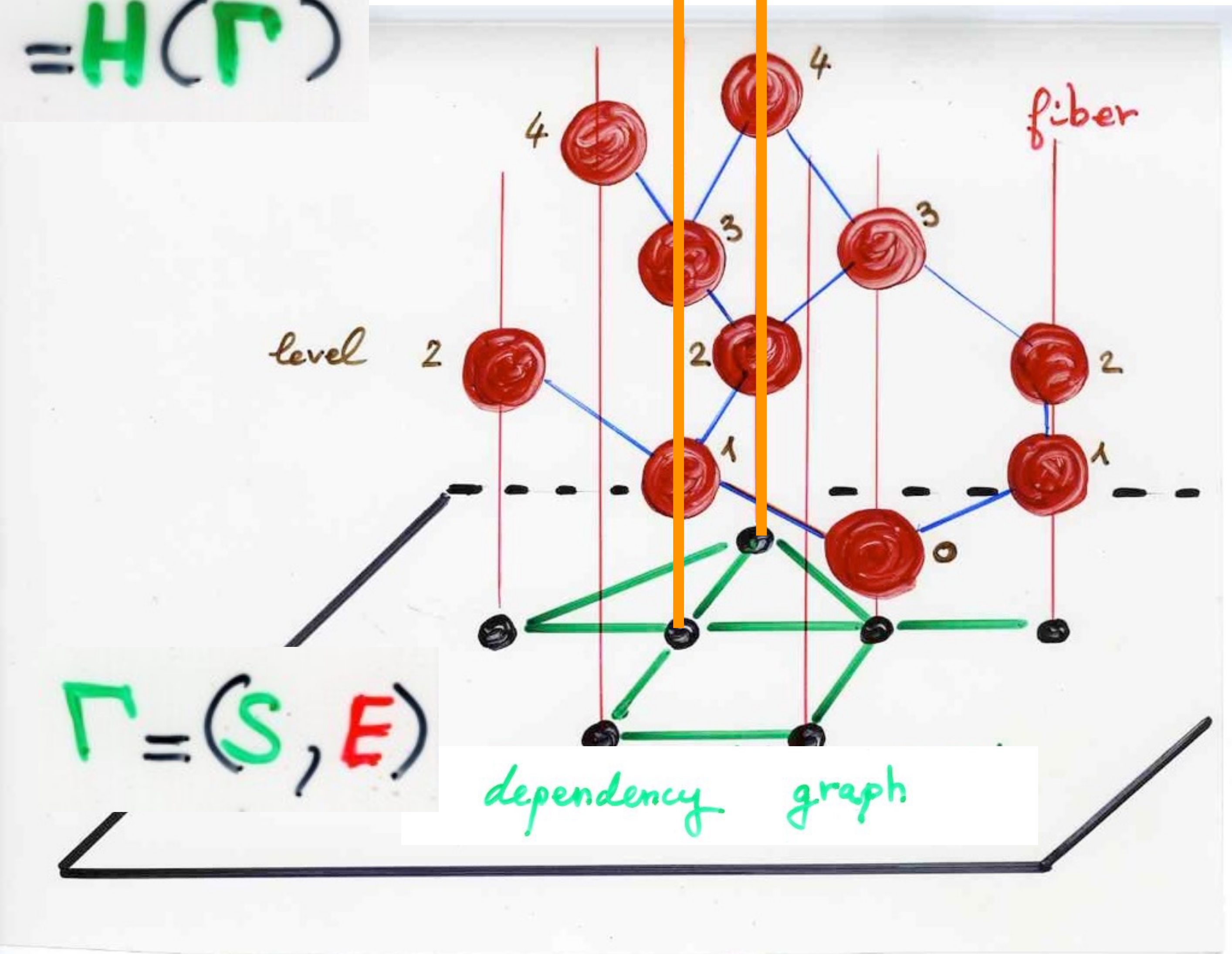
$H(W,S) = H(\Gamma)$



Coxeter graph Γ

fiber over $\{s,t\}$ edge of Γ

$H(W,S) = H(\Gamma)$



reduced decomposition

$w = s_1 \cdots s_n$ $s_i \in S$

length $n = \ell(w)$

reduced decomposition:
factorization $w = s_1 \cdots s_n$
minimal length

$R(w)$ set of reduced
decomposition of w

(Matsumoto property)

Given two reduced decompositions of w, there is a sequence of braid relations which can be applied to transform one into the other.

$[w]$ commutation class C

= set of reduced decompositions obtained by using only the commutation relations

$$st = ts \quad \text{for } s,t \in S \text{ and } m_{s,t} = 2$$

Lemma. The set $R(w)$ of reduced decompositions is a disjoint union of commutation classes.

For each of them, there exist a heap $H(C)$ of $\mathbf{H}(W,S)$ such that C is exactly the set of linear extensions of the poset $H(C)$

Heaps of dimers

and the symmetric group

Symmetric group $\mathfrak{S}_n$

S_n

$n!$ permutations

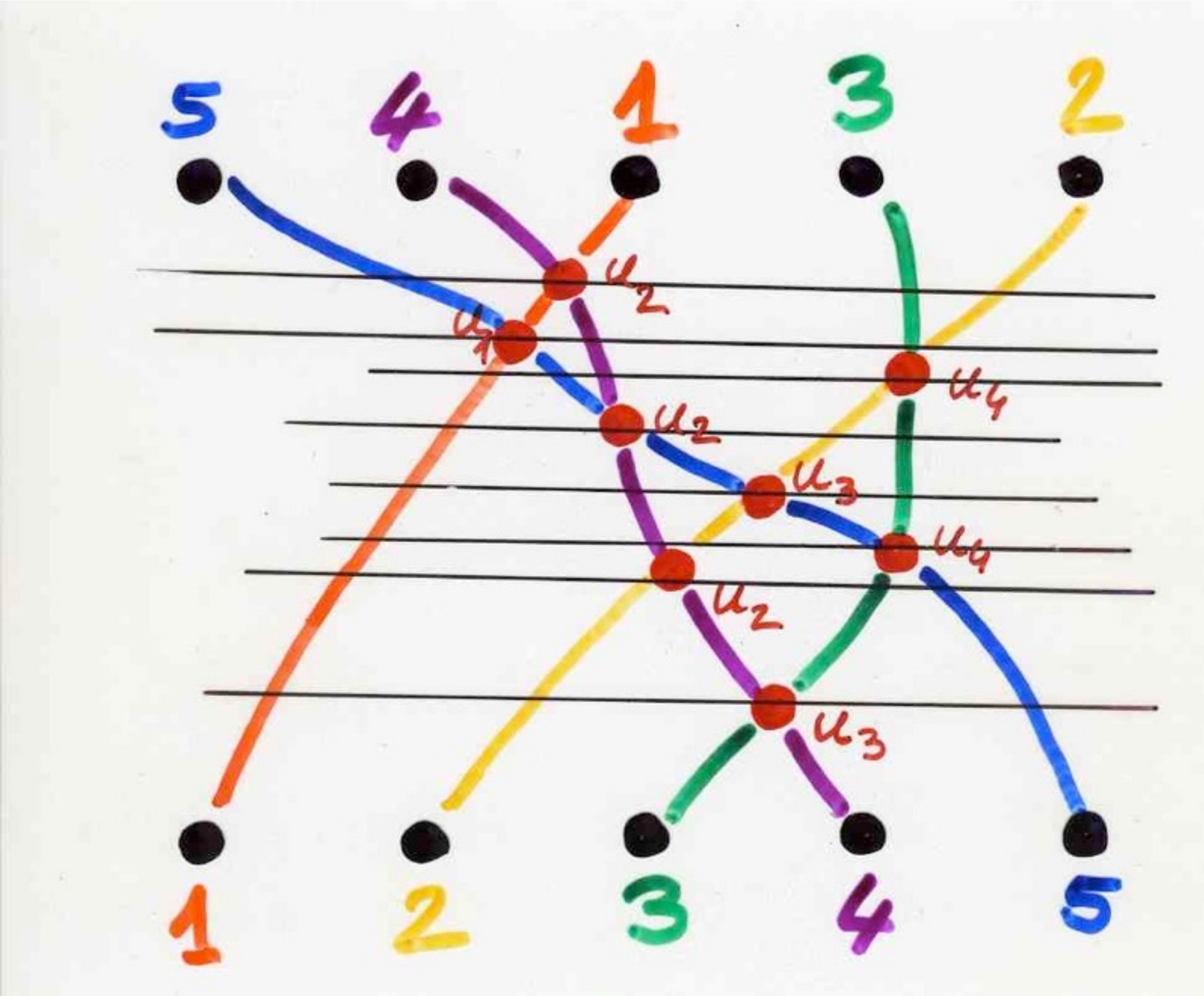
$$\sigma_i = (i, i+1) \qquad i = 1, 2, \ldots, n-1$$

transposition of two consecutive elements

$$\begin{cases} (i) & \sigma_i \sigma_j = \sigma_j \sigma_i , \qquad |i-j| \geq 2 \\ (ii) & \sigma_i^2 = 1 , \\ (iii) & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} . \end{cases}$$

Moore - Coxeter
Yang - Baxter

Coxeter graph


5 4 1 3 2
u2
u1
u4
u2
u3
u4
u2
u3
1 2 3 4 5

$$u_i(a_1 \cdots a_i\, a_{i+1} \cdots a_n)$$
$$= (a_1 \cdots a_{i+1}\, a_i \cdots a_n)$$

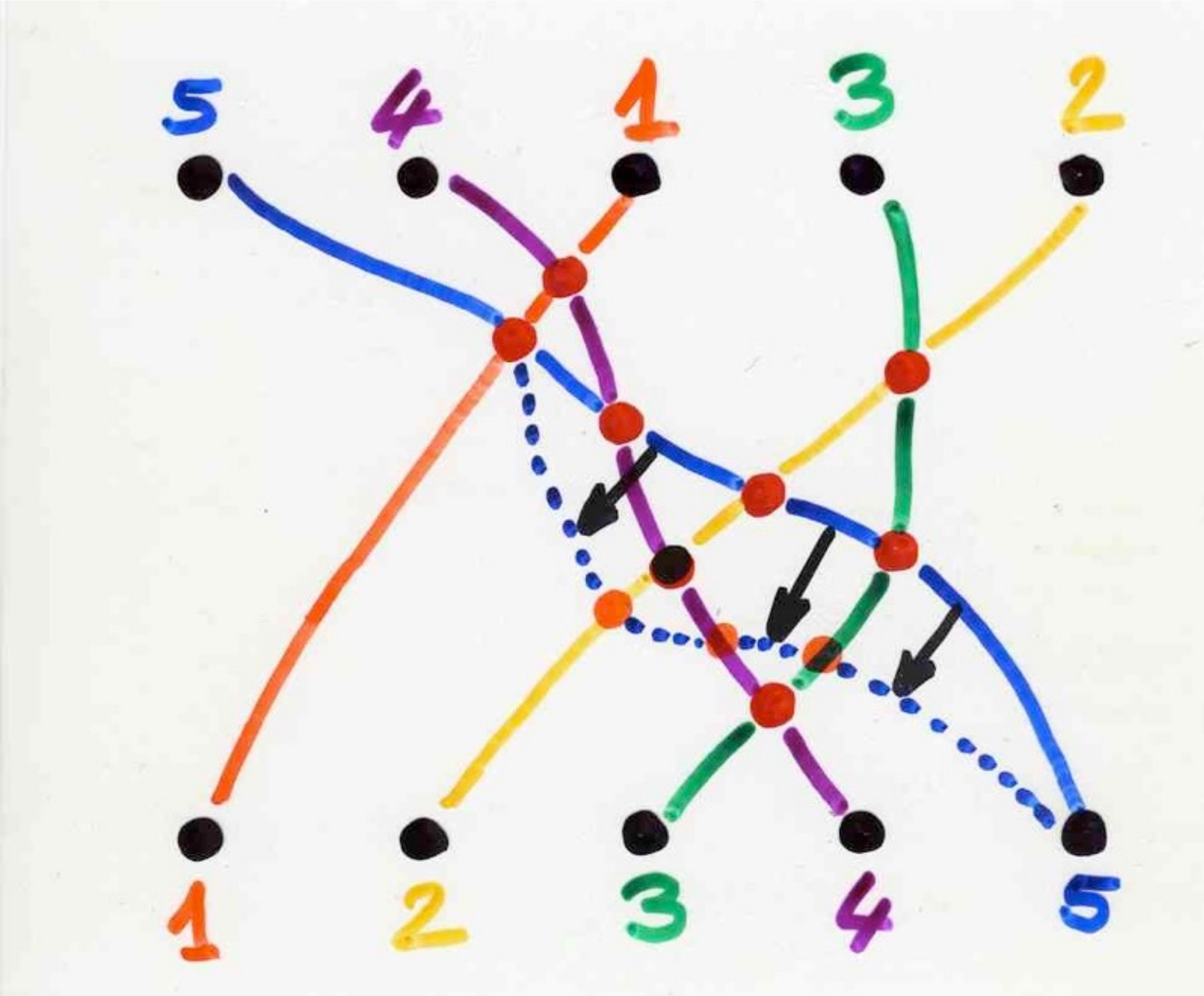
equivalently :

if $\quad \sigma = u_{i_1} \cdots u_{i_k}(12 \cdots n)$

$$\sigma^{-1} = s_{i_1} \cdots s_{i_k}$$

$$s_i = (i, i+1)$$

transposition


5
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3
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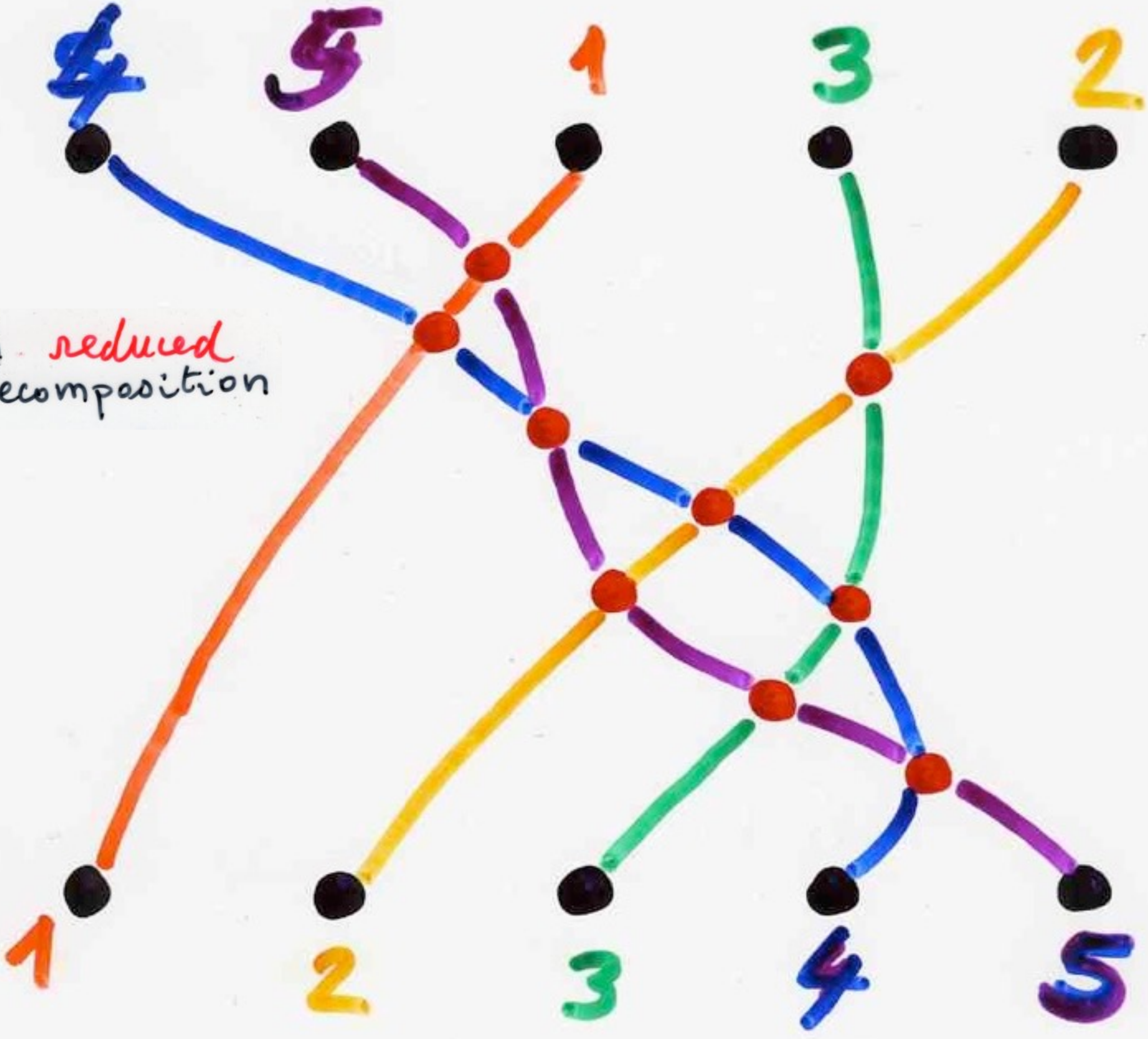
reduced decomposition
of a permutation

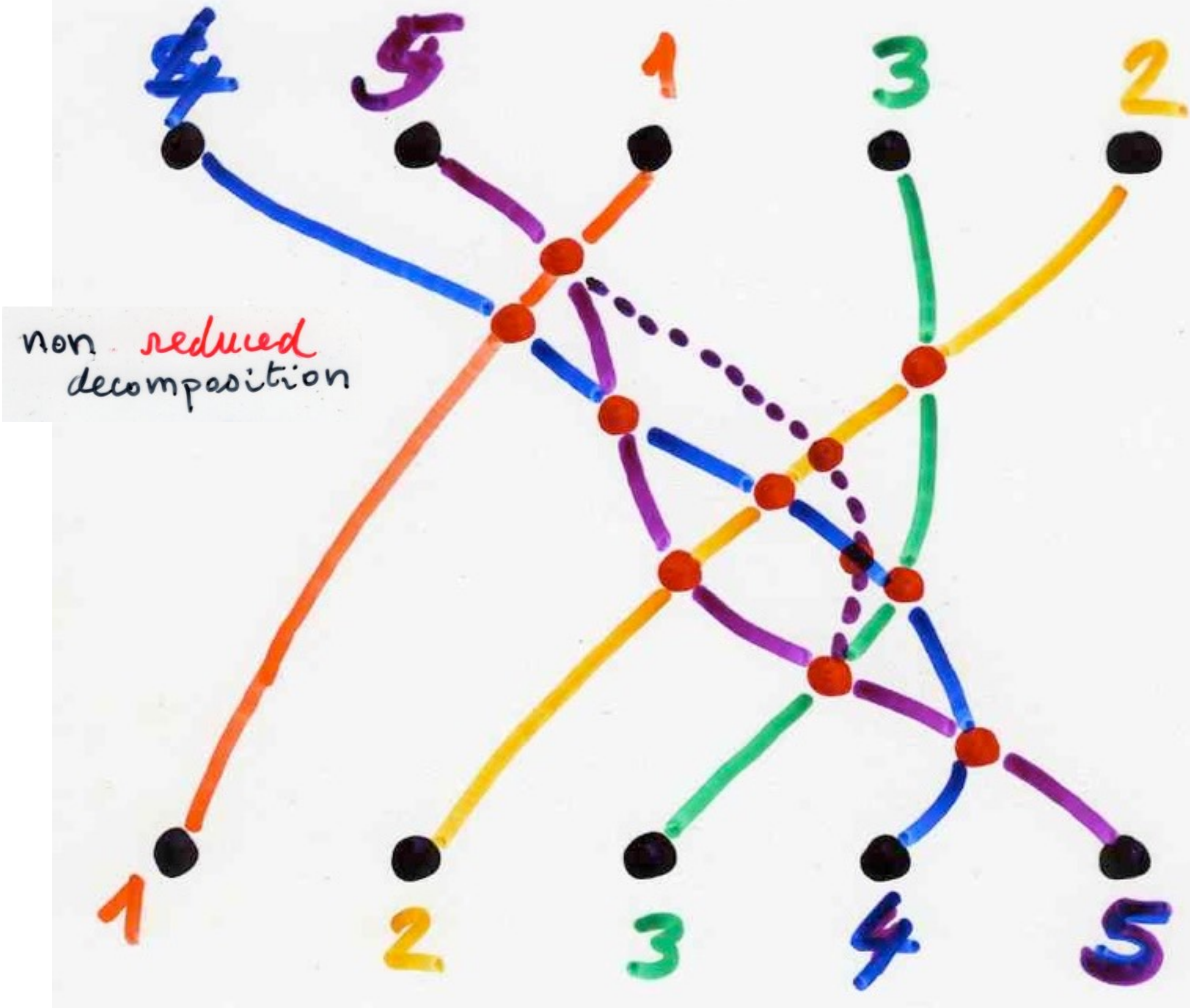
$$\sigma = u_{i_1} \dots u_{i_k}$$

k minimum

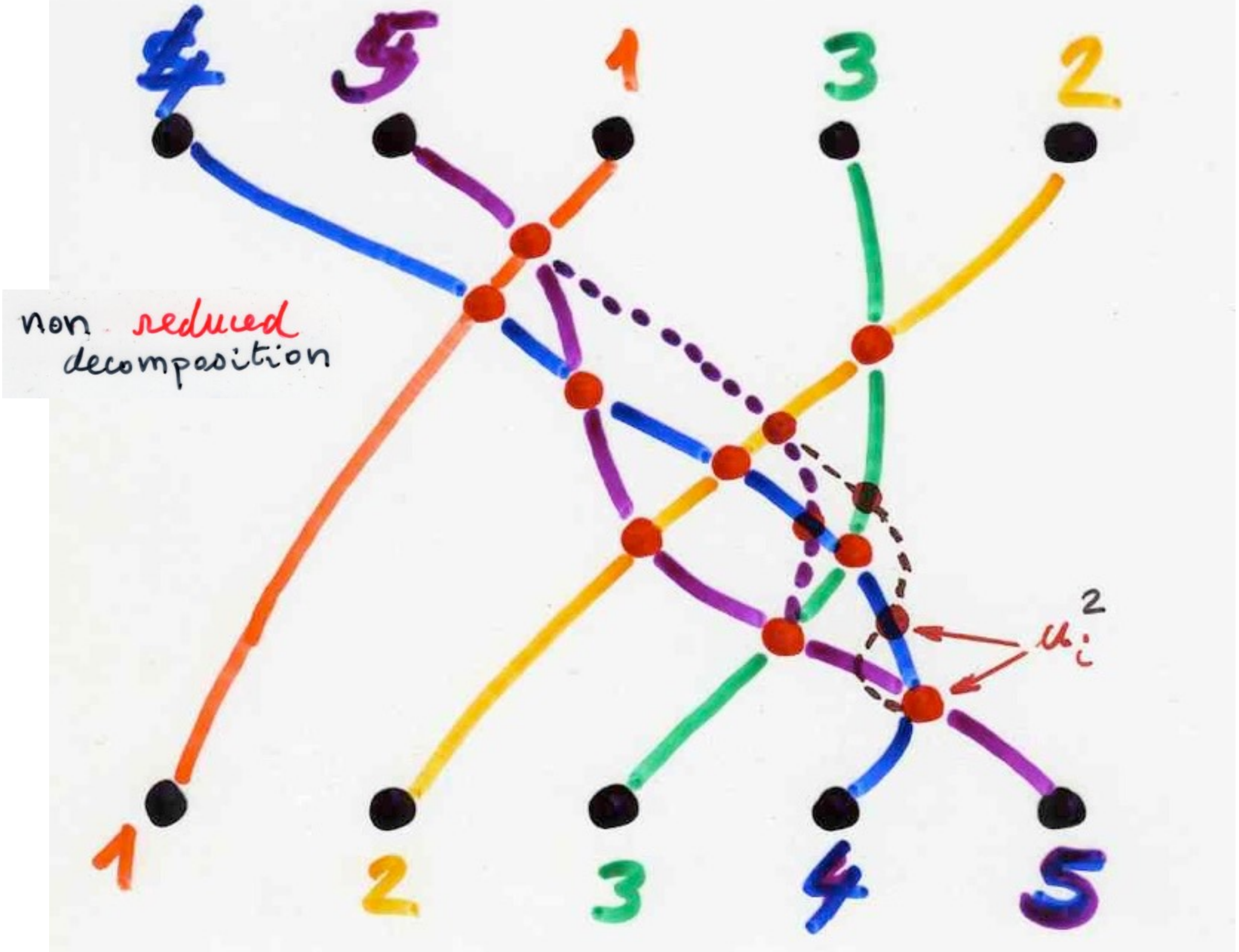
(nb of inversion)



non reduced decomposition



4 5 1 3 2
non reduced
decomposition
1 2 3 4 5


4
5
1
3
2
non reduced
decomposition
u_i^2
1
2
3
4
5

number of reduced decomposition of $\sigma \in S_n$ $= \sum_{\substack{\text{reduced} \\ \text{commutation} \\ \text{class of} \\ \sigma}} \left(\text{linear extensions} \right)$

Elnitsky

<u>Lemma</u>. The set $R(w)$ of reduced decompositions is a disjoint union of commutation classes.

For each of them, there exist a heap $H(C)$ of $\mathbf{H}(W, S)$ such that C is exactly the set of linear extensions of the poset $H(C)$

heaps of dimers

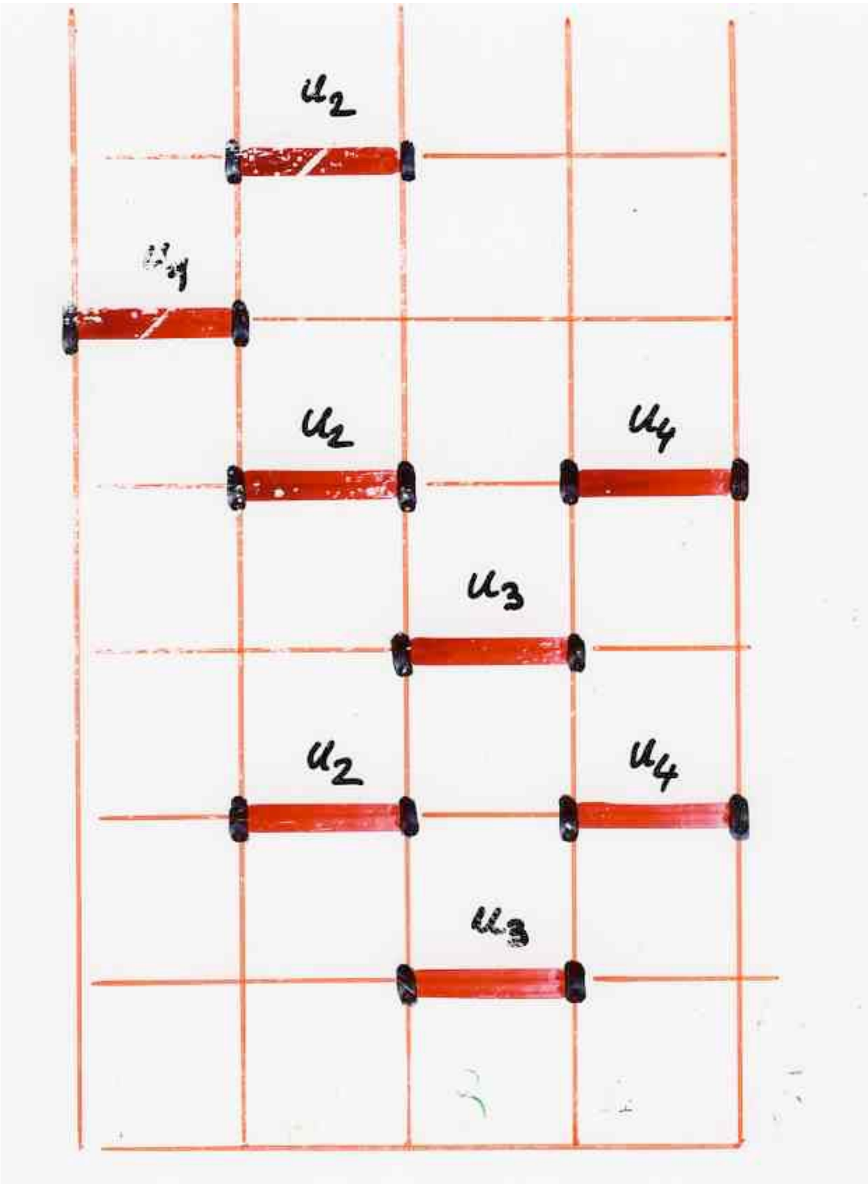
$(i, i+1)$

on $\{0, 1, \ldots, n-1\}$

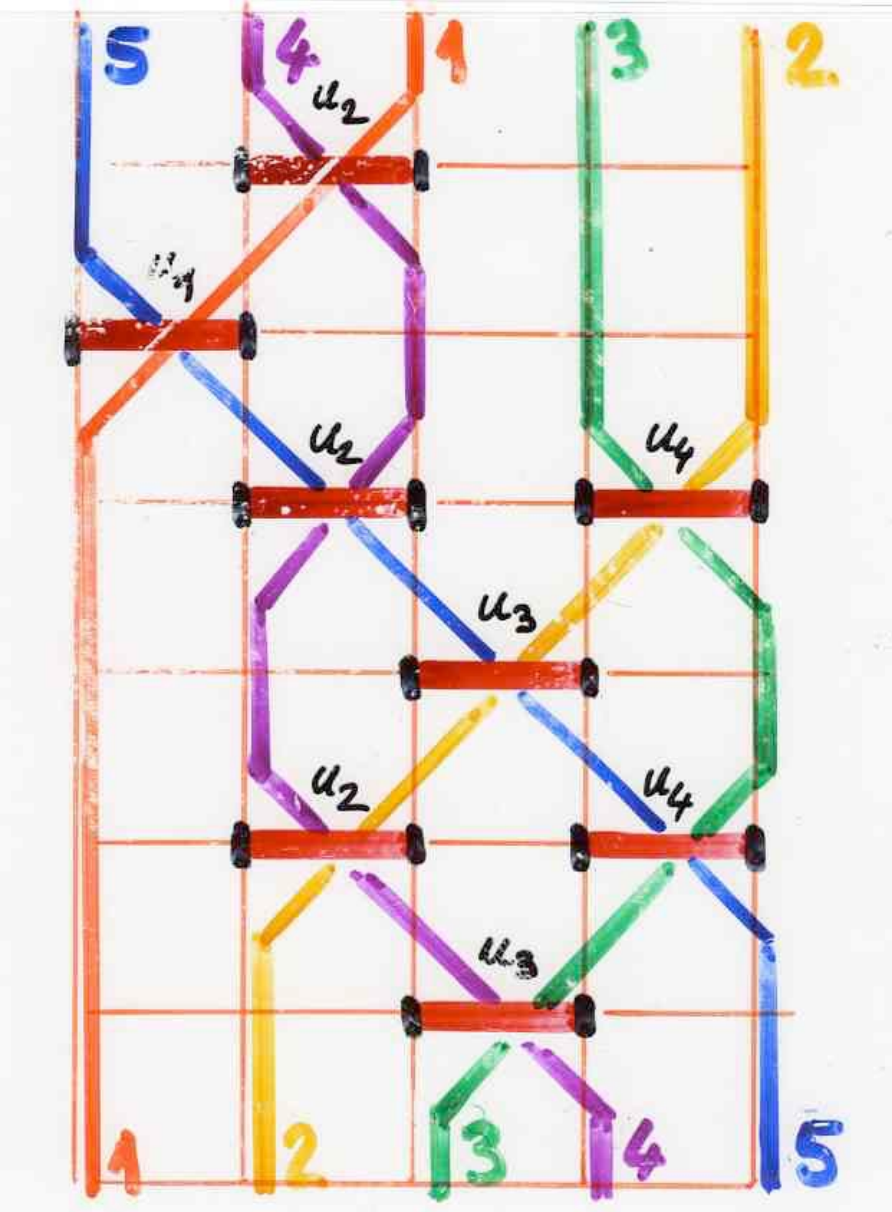
generators $\{\sigma_0, \sigma_1, \ldots, \sigma_{n-1}\}$

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

iff $|i-j| \geq 2$



heap of dimers $[1, n]$ $\longrightarrow$ permutation S_n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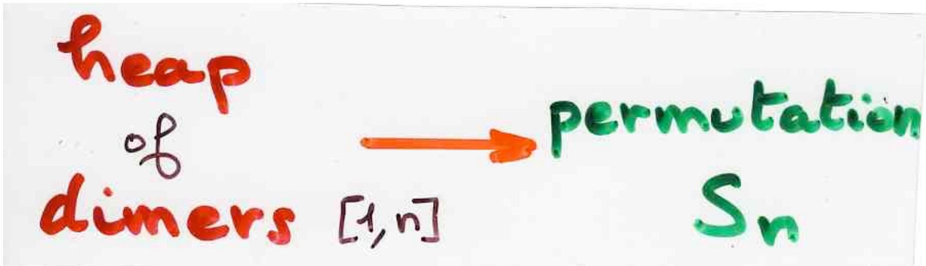
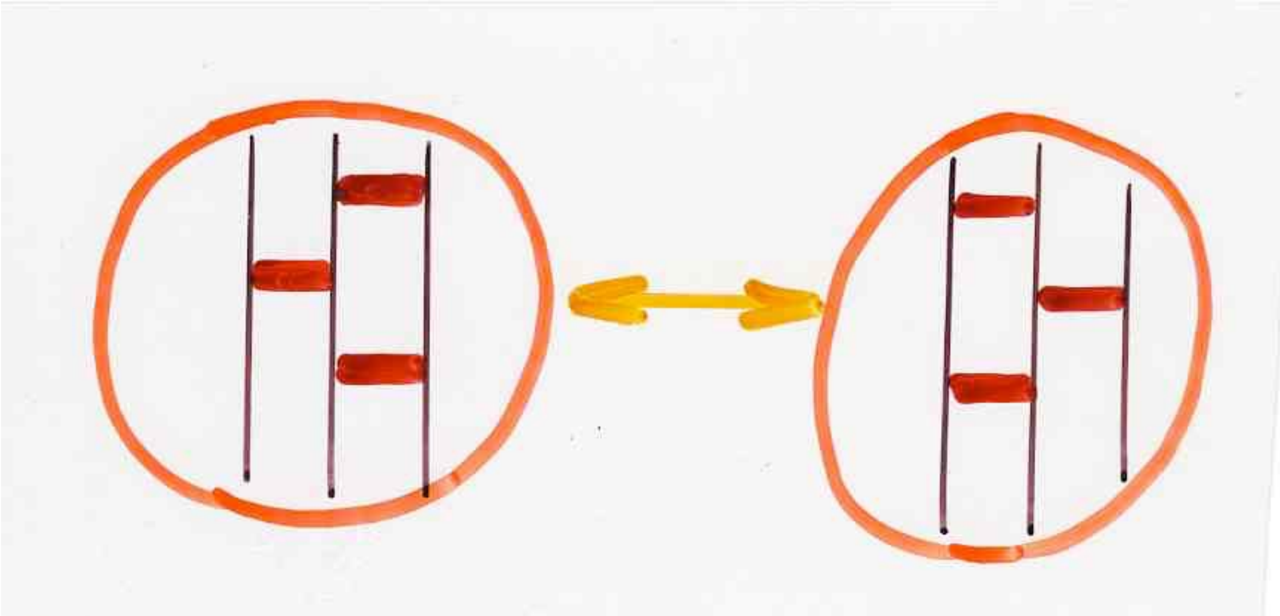
4
5


heap
of
dimers [1, n]
permutation
S_n



fully commutative elements in Coxeter groups

Definition An element w of the Coxeter group W is *fully commutative* iff $R(w)$ is reduced to one *commutation class*.

The corresponding heap $H(w)$ will also be called *fully commutative* (FC)

Lemma (Stembridge)(1995)

A reduced word w represents a FC element iff no elements of its commutation class $[w]$ contains a factor $\underbrace{sts\ldots}_{m_{s,t}}$ for a $m_{s,t} \geq 3$

Proposition (Stembridge, 1995)

A heap $H \in H(W,S)$ is FC iff

(i)

and

(ii)

strict heap

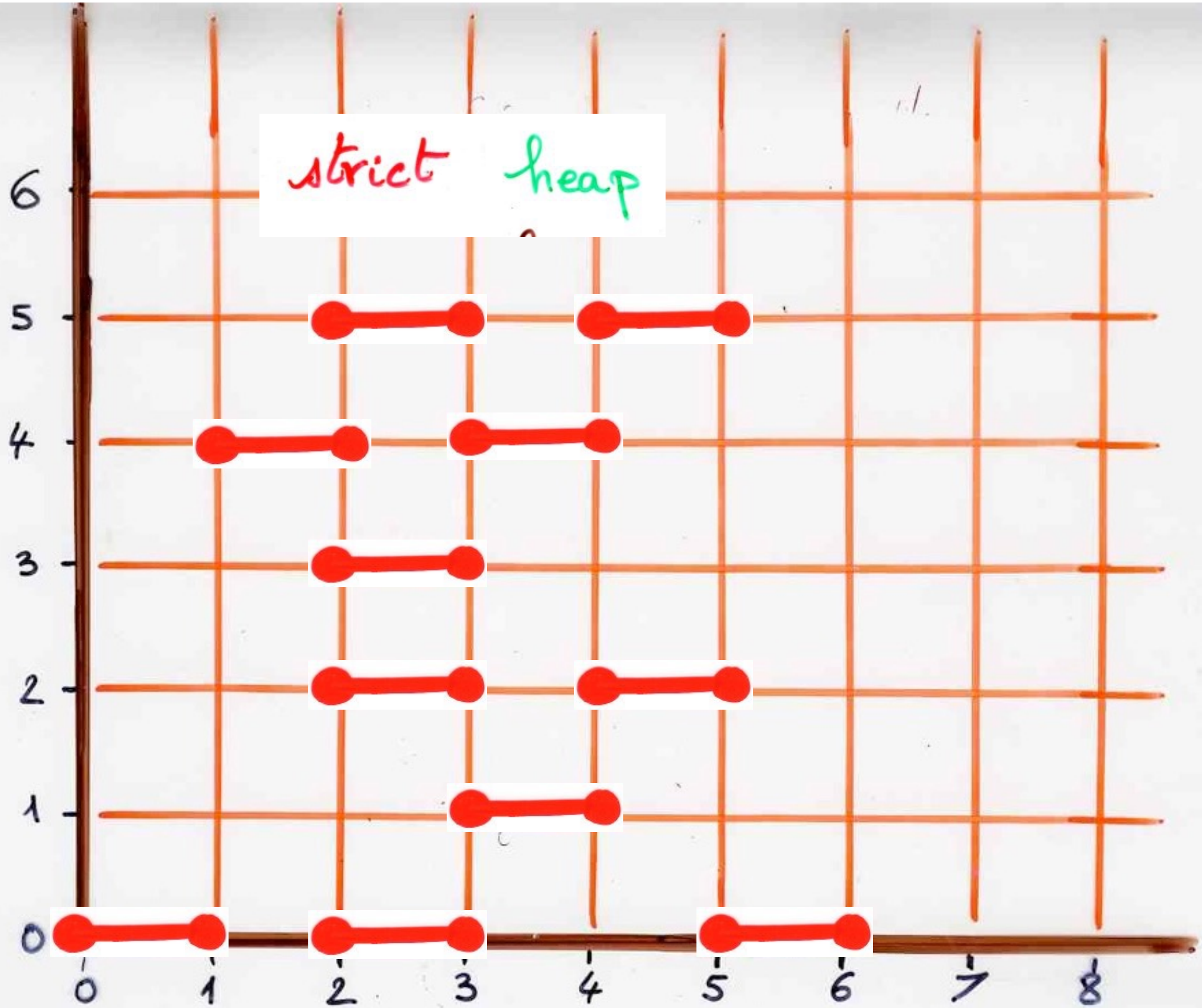
convex chain

strict heap

<u>Definition</u> strict heap H

iff no covering relation $x \prec y$ in H such that $\pi(x) = \pi(y)$

<u>example</u> heaps of dimers on $\mathbb{N}$


strict heap
6
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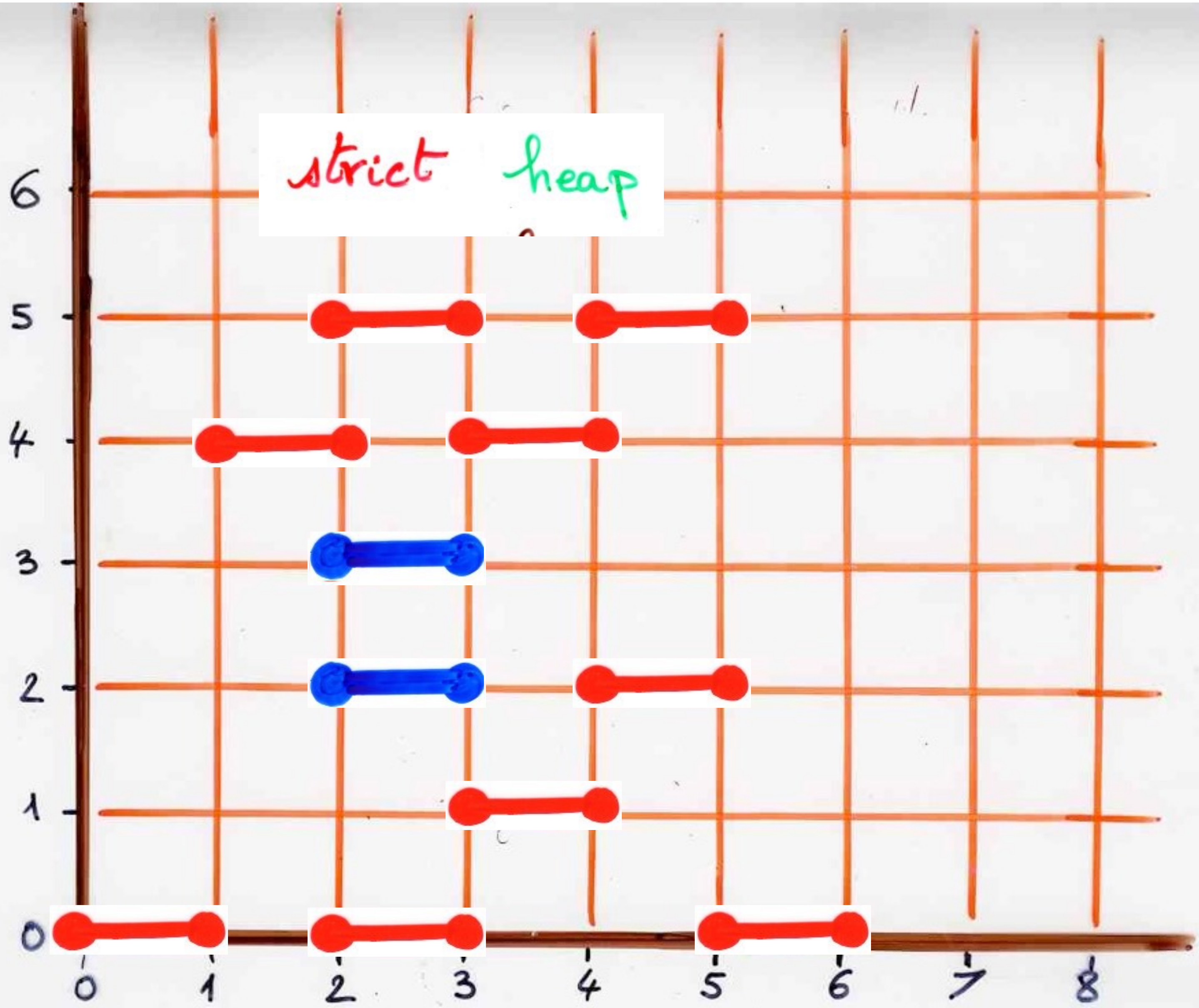
1
0
0 1 2 3 4 5 6 7 8


strict heap
6
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2
1
0
0 1 2 3 4 5 6 7 8

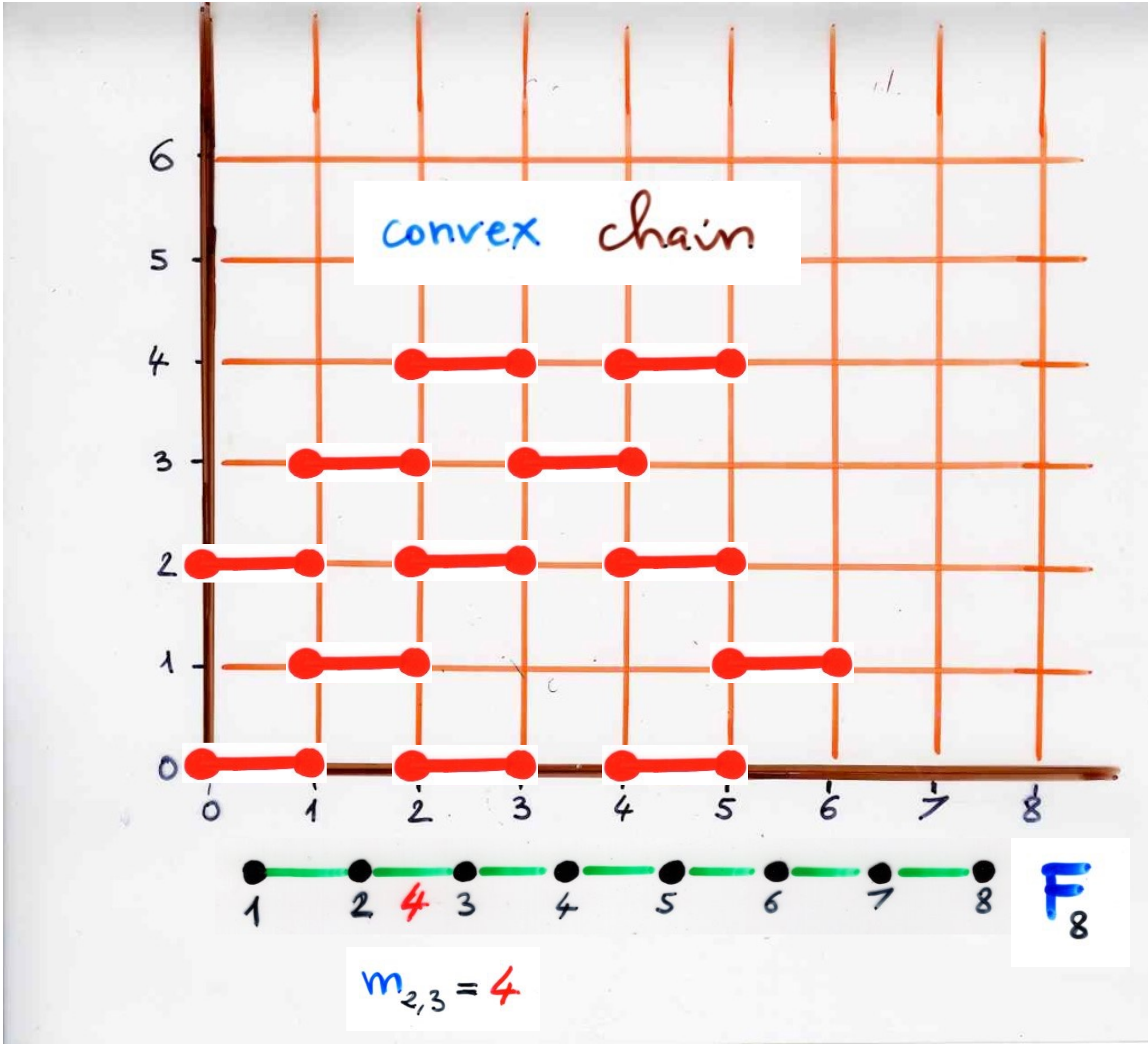
convex chain

Definition convex chain in a poset

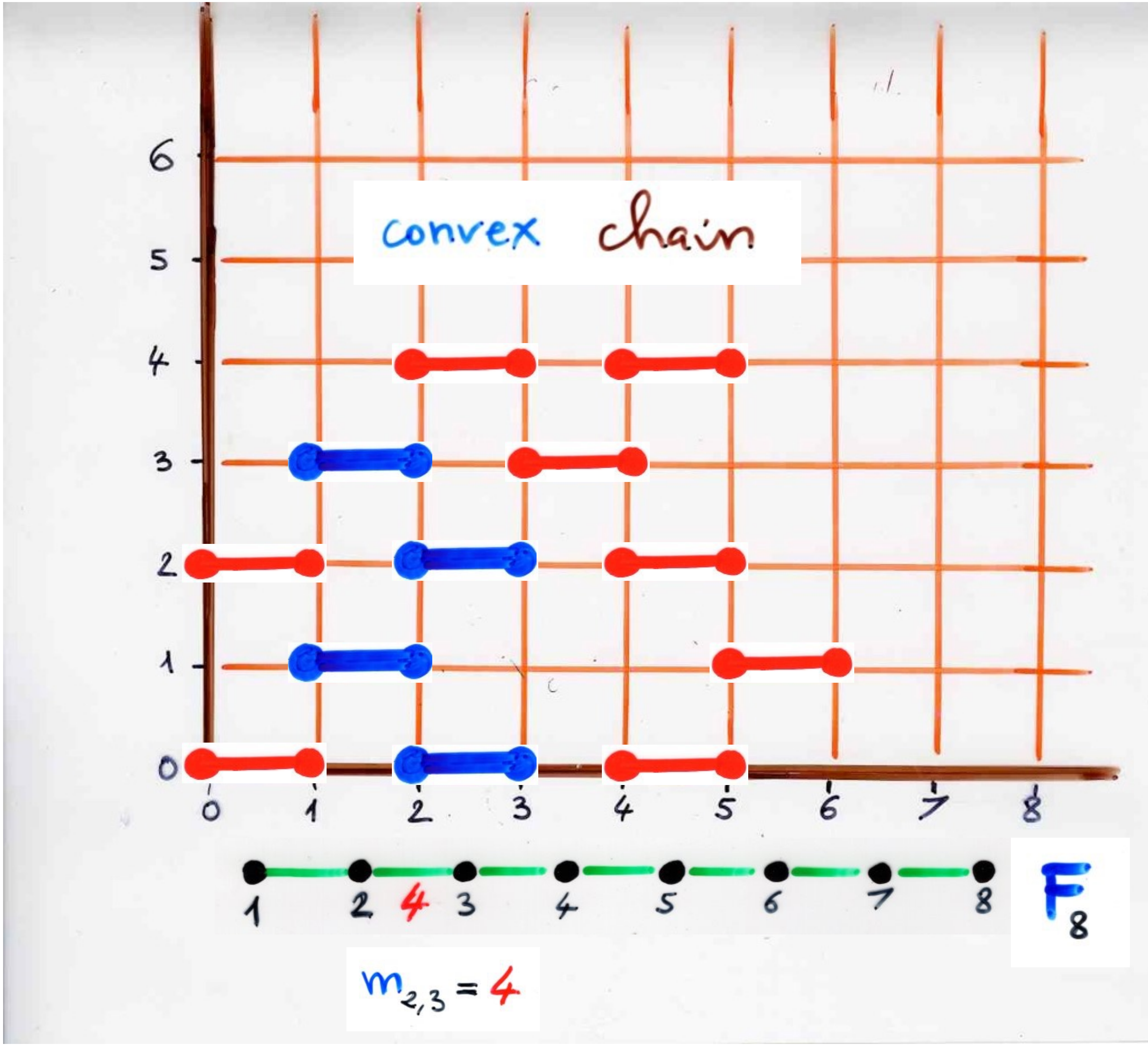
$x_1 < x_2 < \cdots < x_k$ iff

the only elements z with $x_1 < z < x_k$

are the elements of the chain



convex chain
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8
1 2 4 3 4 5 6 7 8
F_8
$m_{2,3} = 4$


convex chain
6
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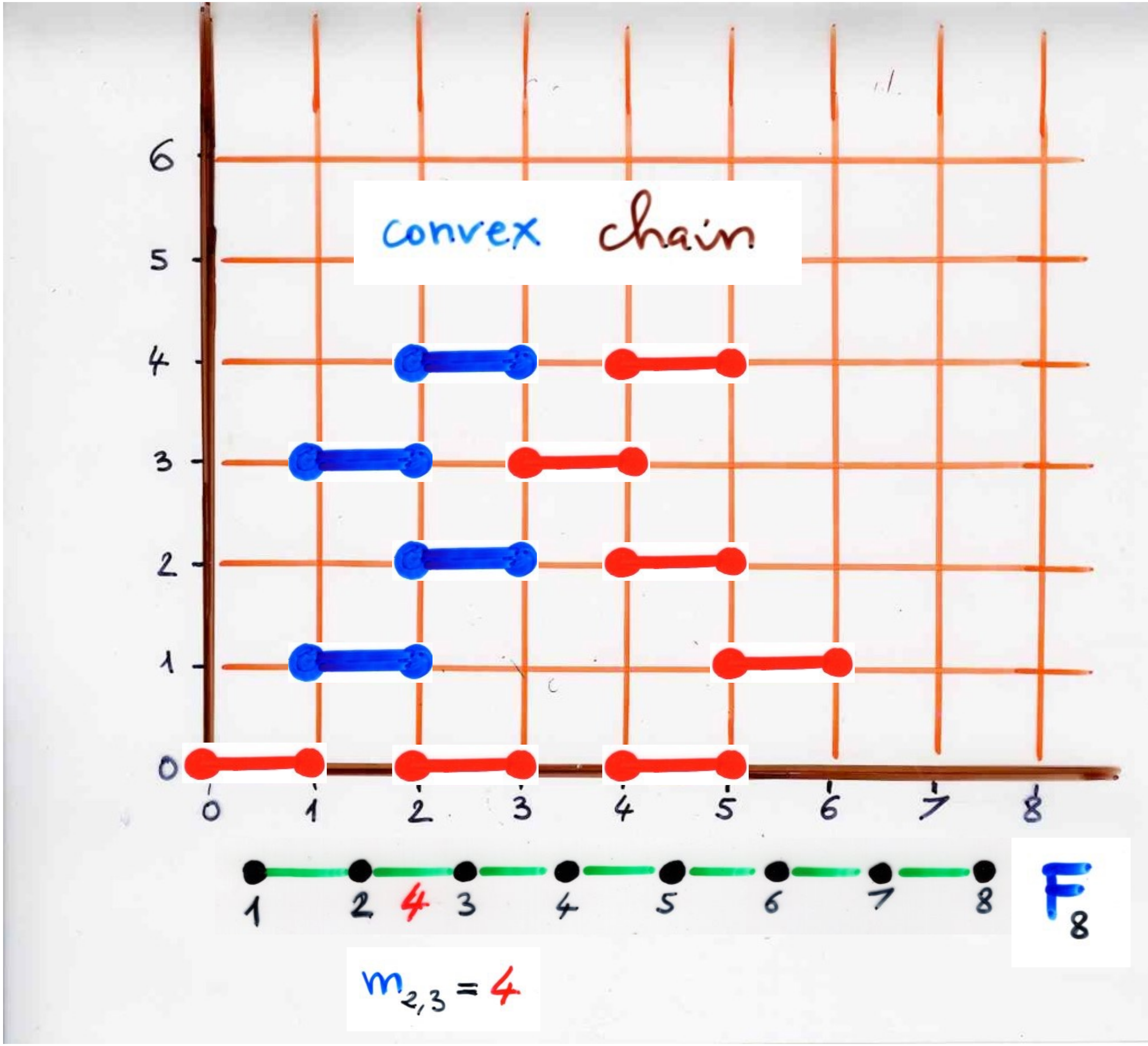
0 1 2 3 4 5 6 7 8
1 2 4 3 4 5 6 7 8
F_8
$m_{2,3} = 4$


convex chain
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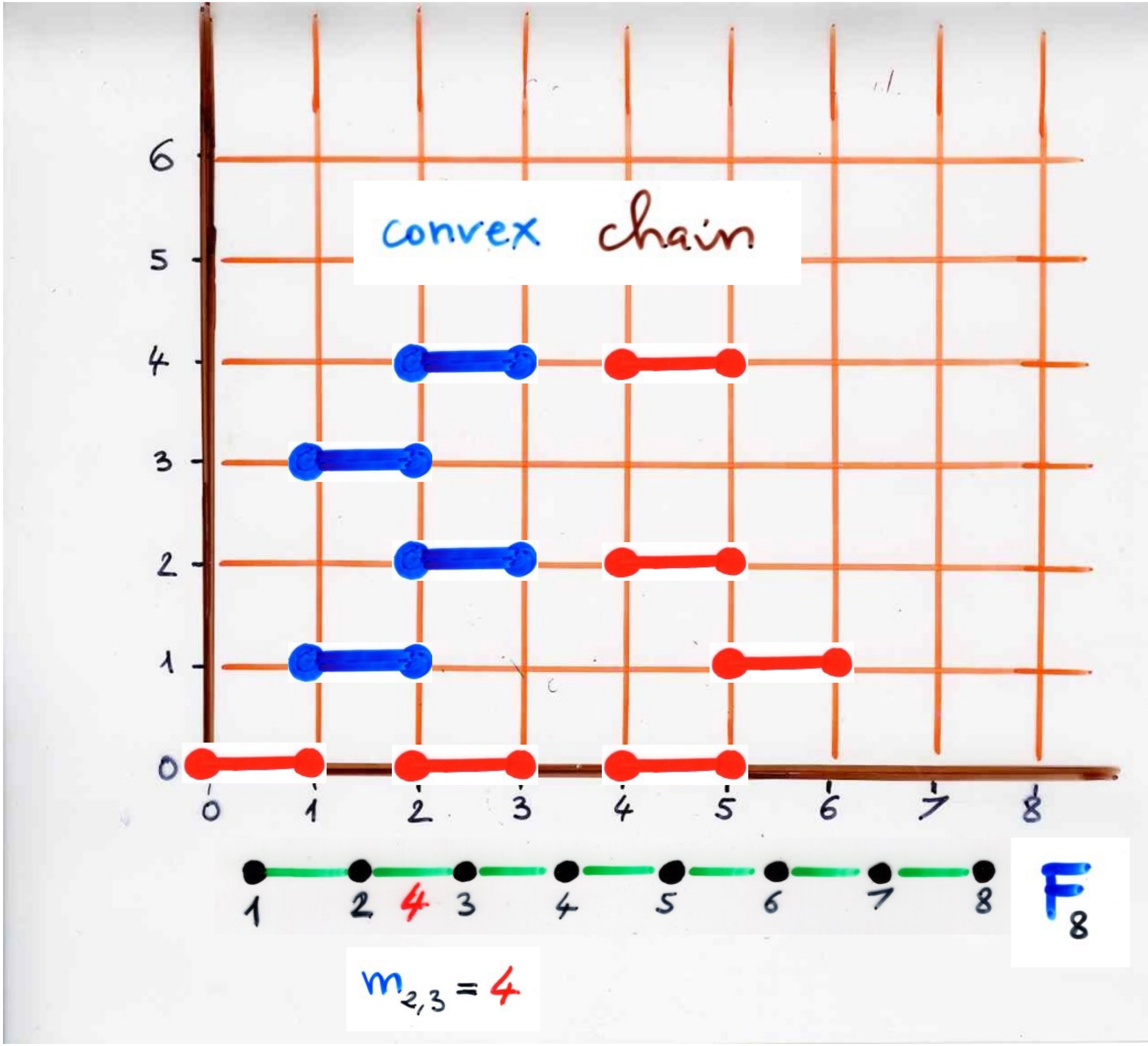
0 1 2 3 4 5 6 7 8
1 2 4 3 4 5 6 7 8
F_8
$m_{2,3} = 4$


convex chain
6
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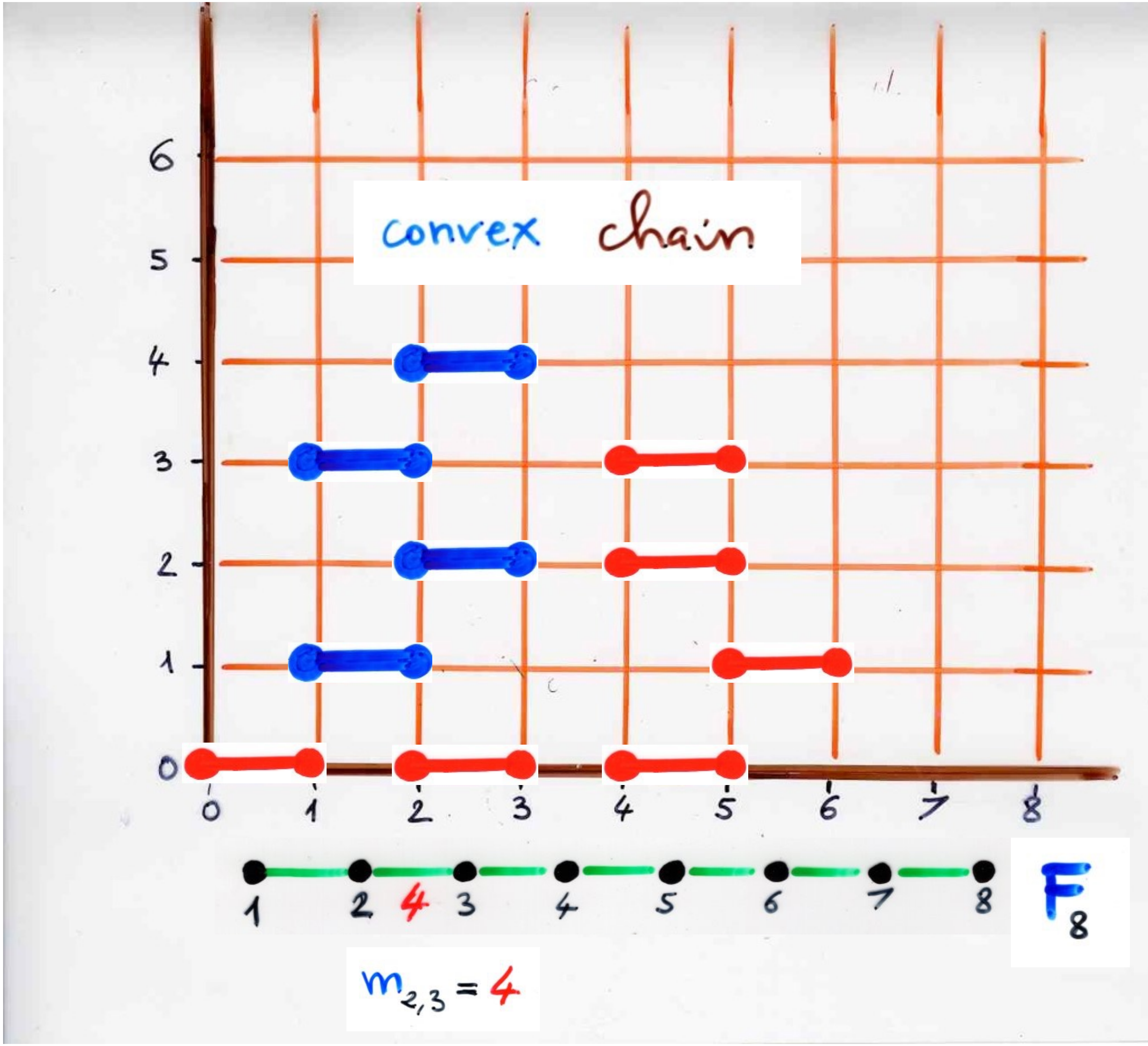
0 1 2 3 4 5 6 7 8
1 2 4 3 4 5 6 7 8
F_8
$m_{2,3} = 4$


convex chain
6
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2
1
0
0 1 2 3 4 5 6 7 8
1 2 4 3 4 5 6 7 8
F8
m2,3 = 4

Proposition (Stembridge, 1995)

A heap $H \in \mathcal{H}(W,S)$ is FC iff

(i) ~~h~~ H is strict

and

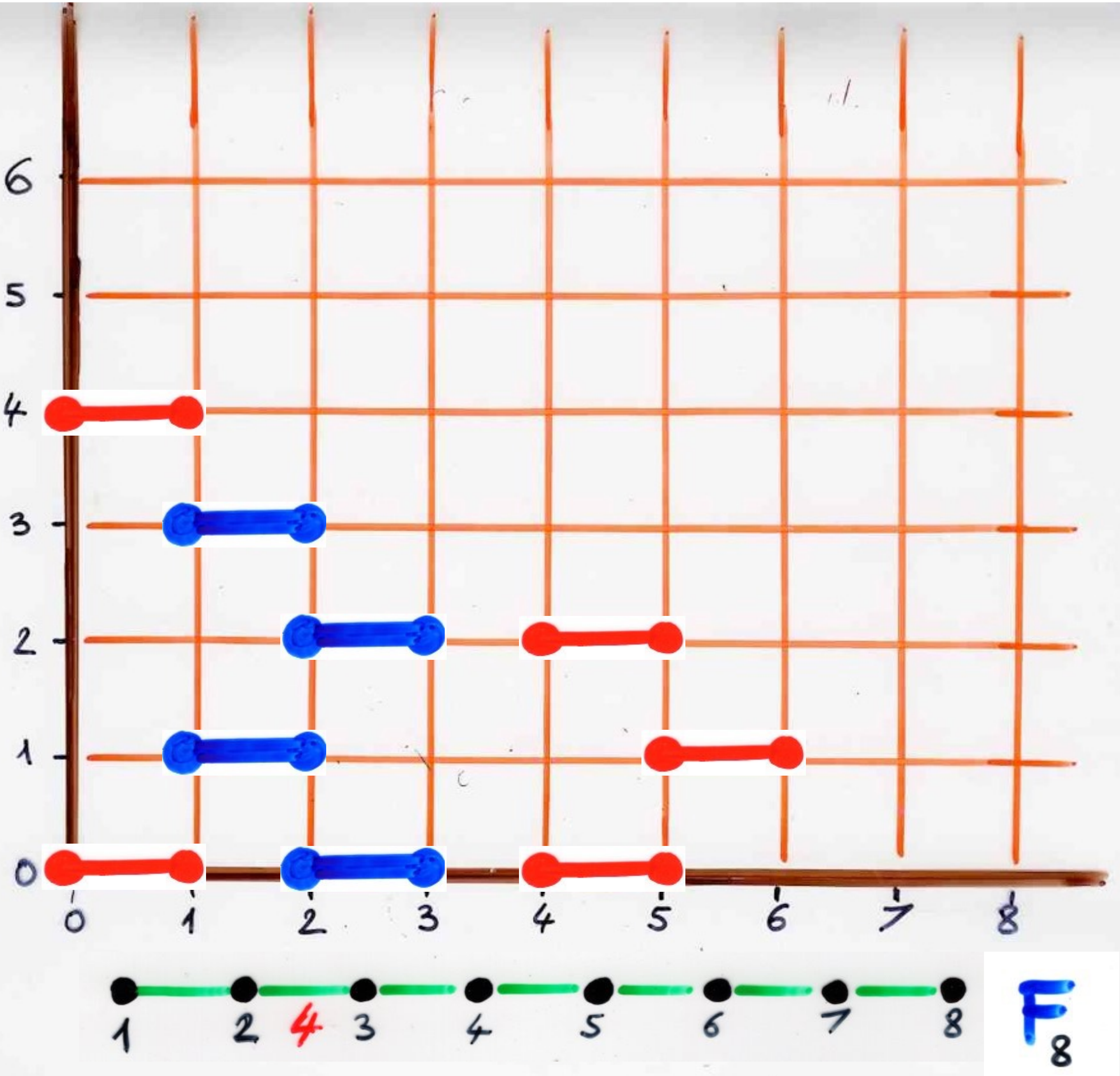
(ii) ~~to~~ H does not contain a convex chain such that

$$x_1 \prec x_2 \prec \cdots \prec x_{m_{s,t}}$$

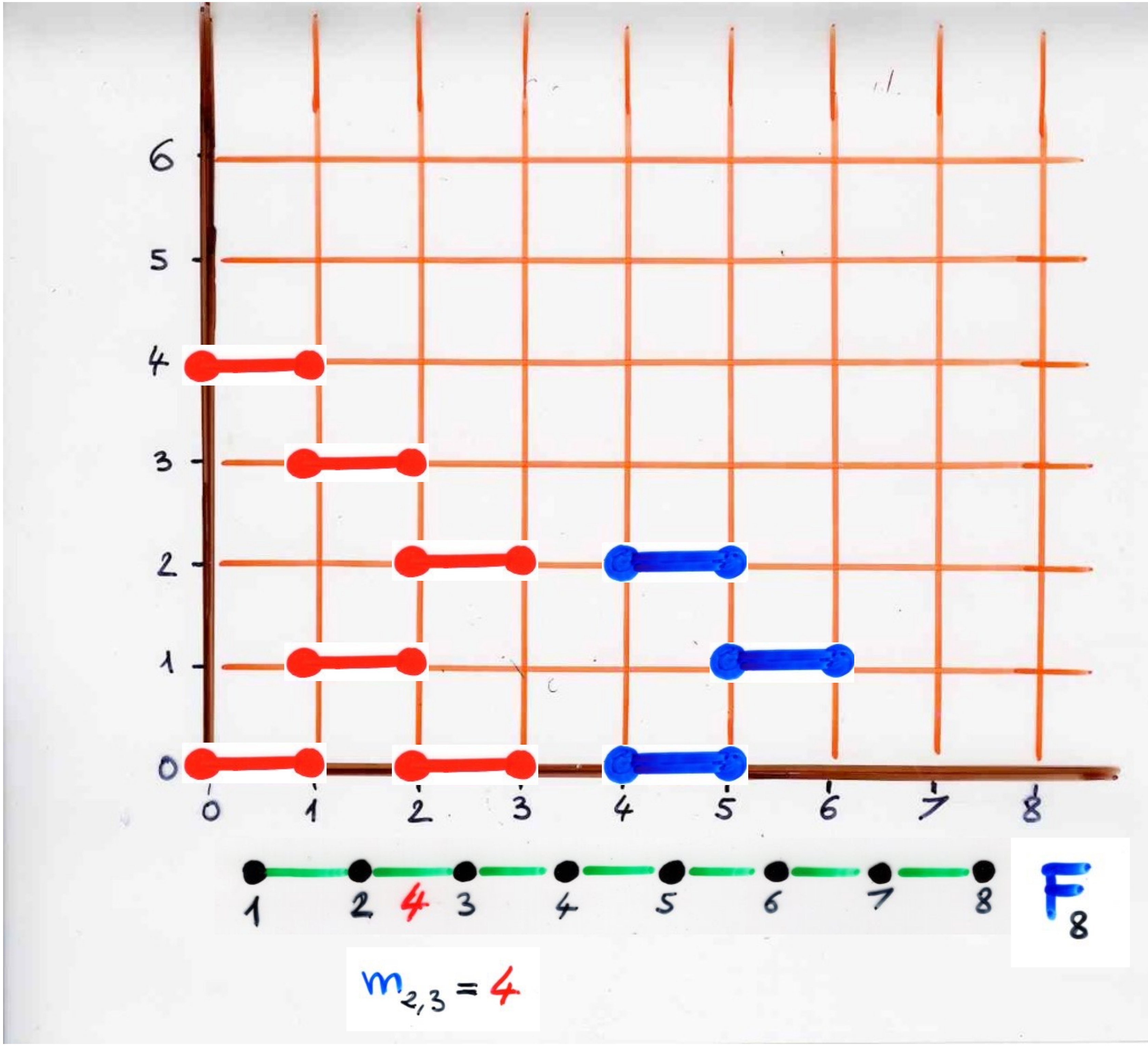
$$x_1, x_3, x_5, \ldots \in \pi^{-1}(\{s\})$$

$$x_2, x_4, \ldots \in \pi^{-1}(\{t\})$$

where $3 \leqslant m_{s,t} < \infty$



$m_{2,3} = 4$



6
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0
0 1 2 3 4 5 6 7 8
1 2 4 3 4 5 6 7 8
F_8
$m_{2,3} = 4$

seminal papers

→ Stembridge (1996, 98)

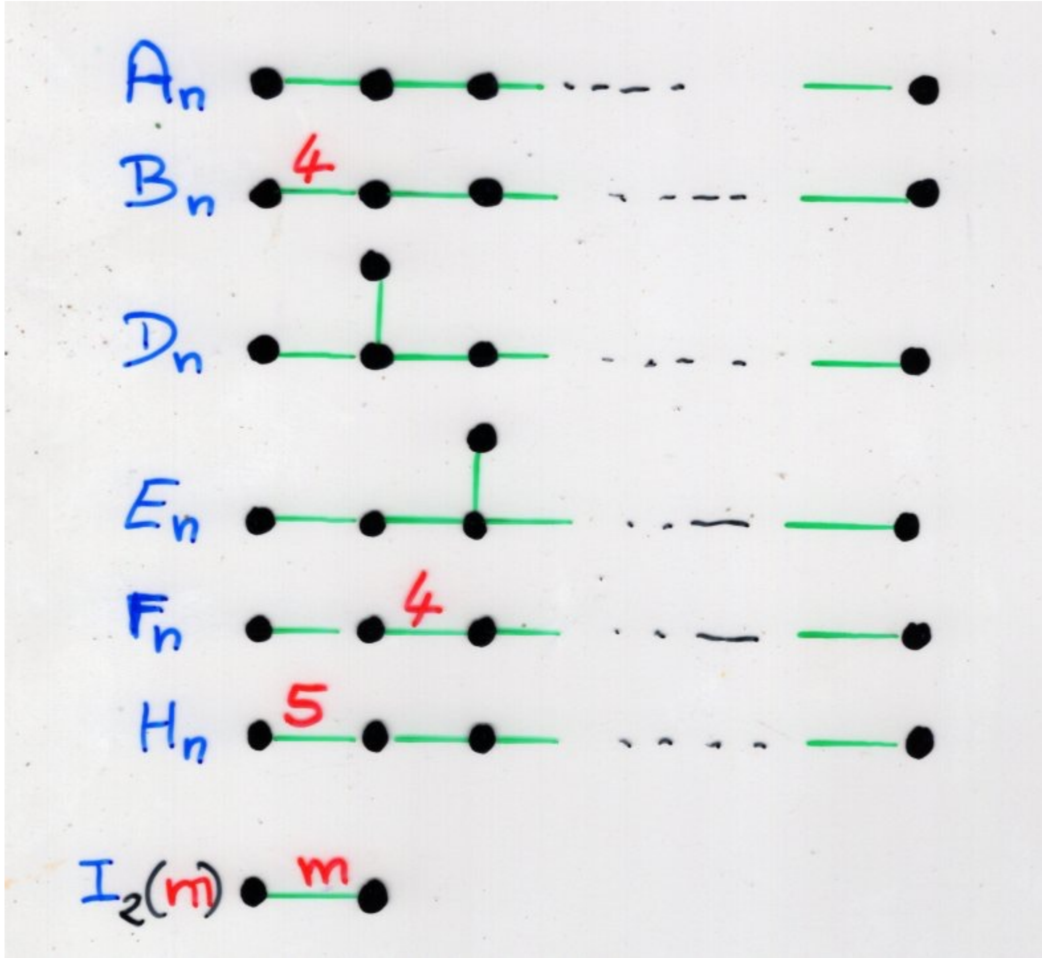
- classification of Coxeter groups with a finite number of FC elements
- enumeration in each of these cases
  → always algebraic generating functions

→ Fan (1995) for $m_{s,t} \leq 3$ (simply laced)

→ Graham (1995)

FC elements in any Coxeter group W naturally index a basis of the generalized Temperley-Lieb algebra of W

finite Coxeter groups



A_n

B_n

D_n

$E_6 \quad E_7 \quad E_8$

F_4

$H_3 \quad H_4$

$I_2(m)$

The list of FC-finite Coxeter groups

$I_2(5) = H_2$

$I_2(6) = G_2$

→ FC elements in relation with
Kazhdan-Lusztig polynomials
Greene, Shi, Cellini, Papi, ..

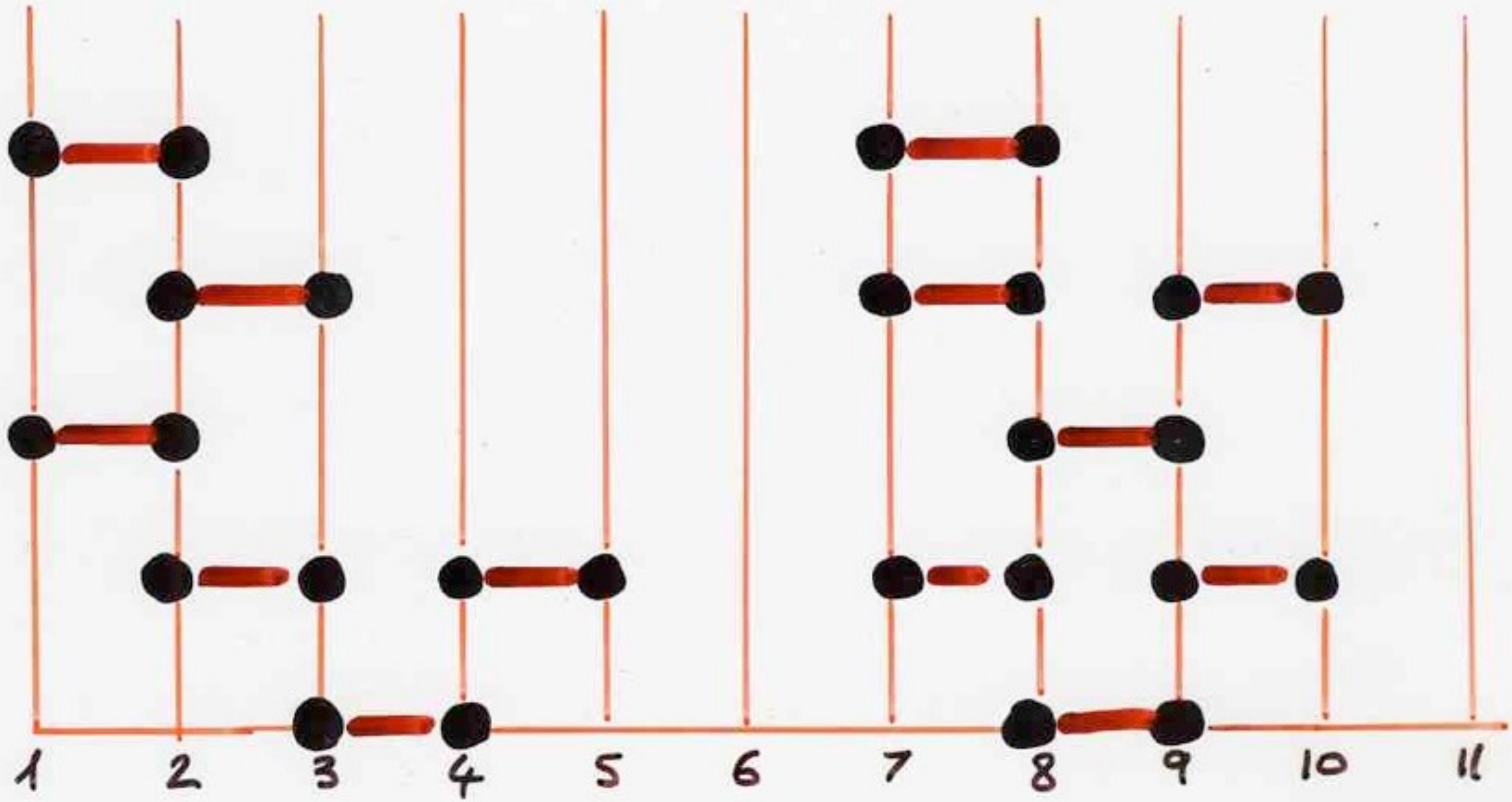
affine Coxeter groups
Biagioli, Jouhet, Nadeau (2014, 2015)
" " " , Bousquet-Mélou (2016)

Hanusa, Jones (2010)

fully commutative elements

(for the symmetric group)

the stairs decomposition of a heap of dimers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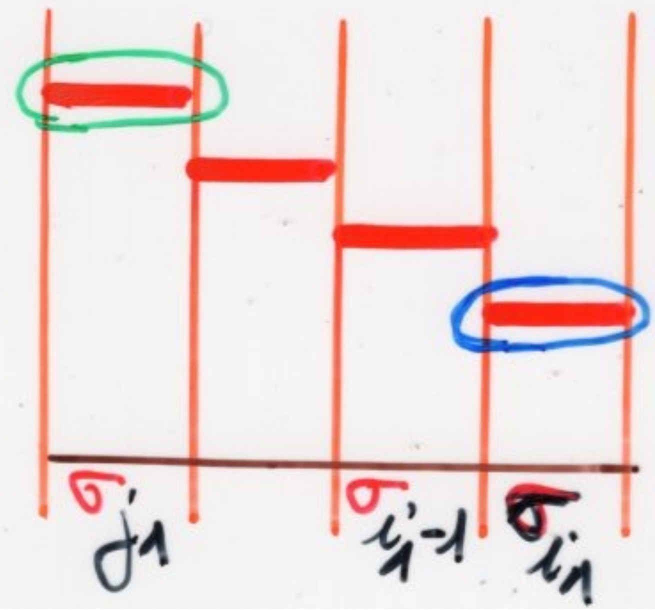
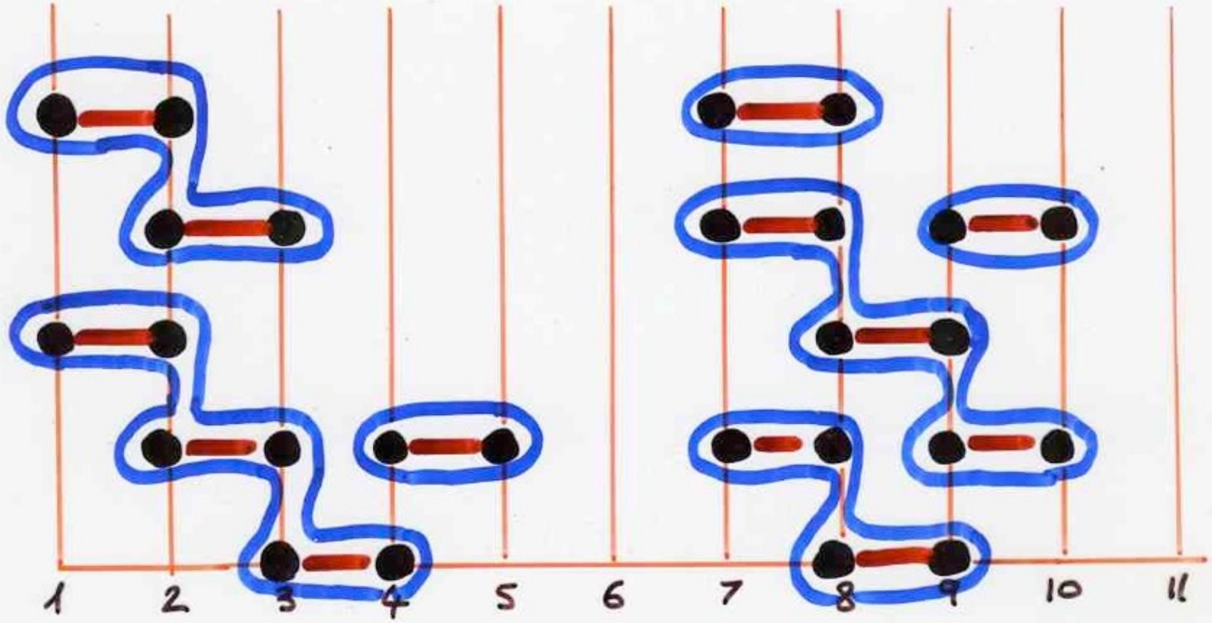
a stair is

a convex chain of dimers

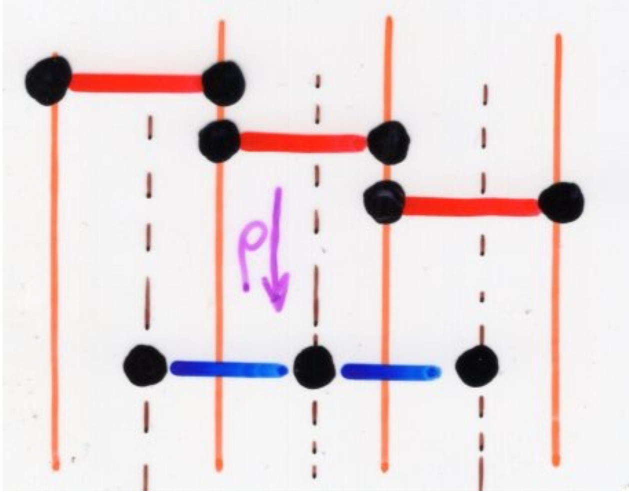
$\sigma_i \prec \sigma_{i-1} \prec \cdots \prec \sigma_k$

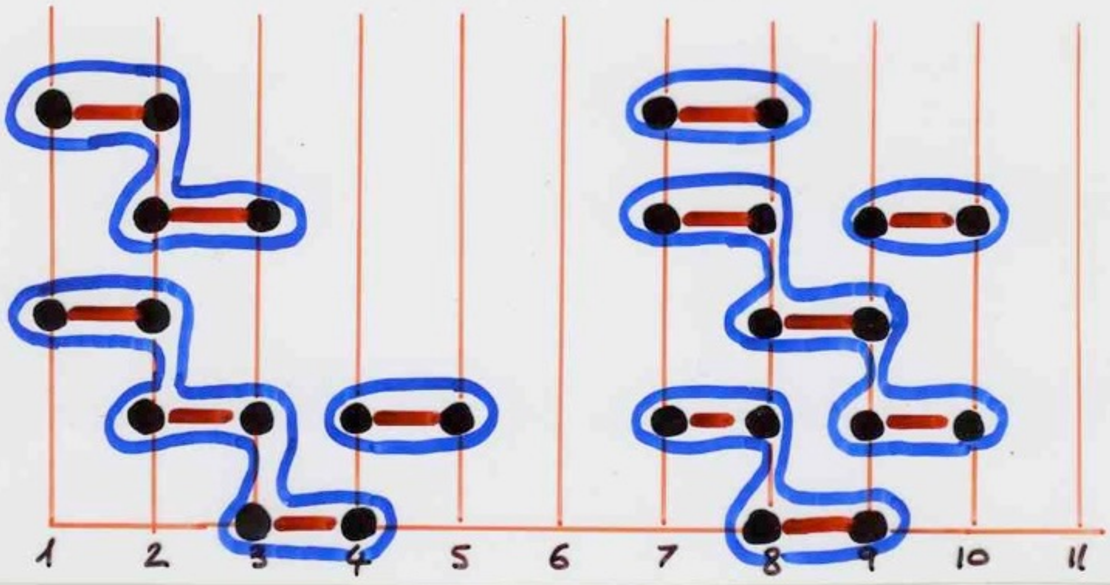




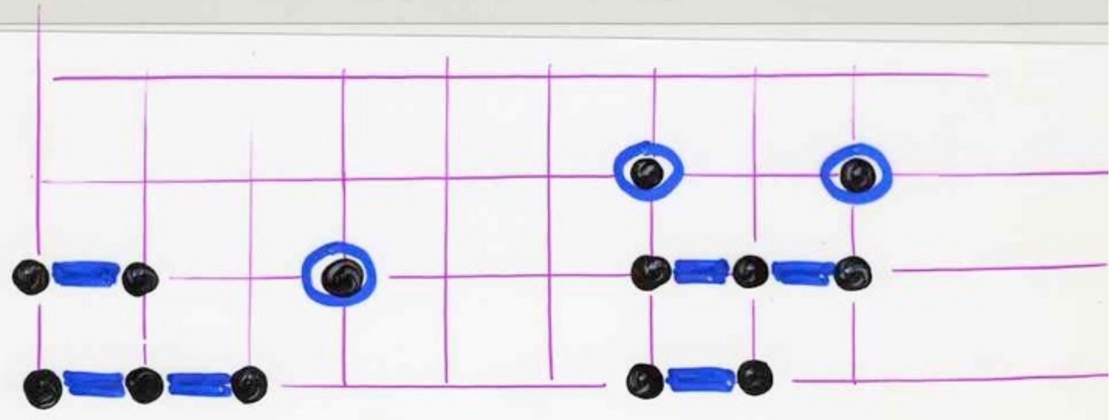
stairs decomposition

substitution





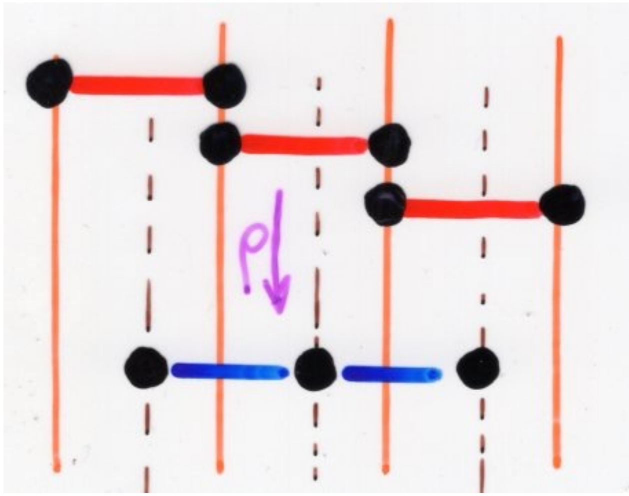
stairs decomposition

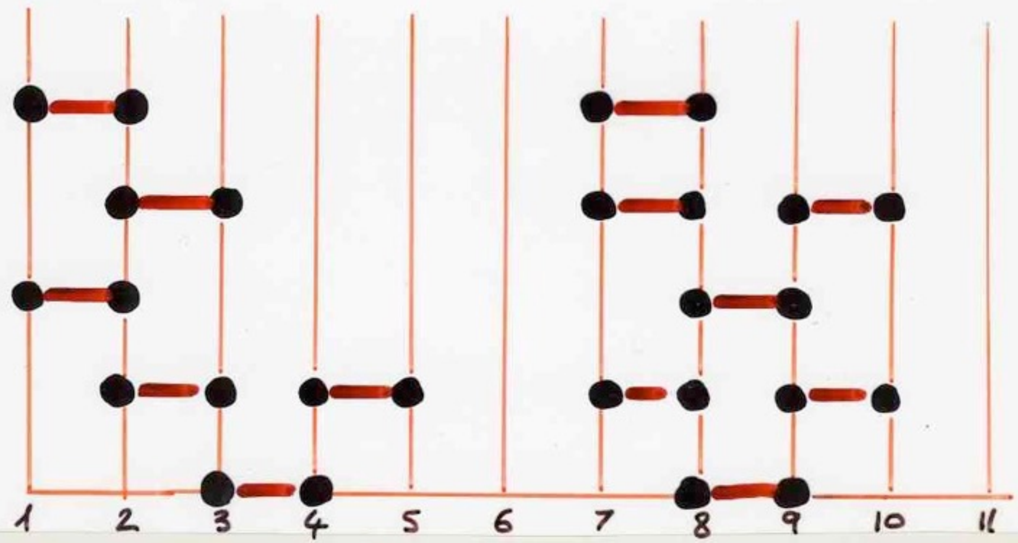


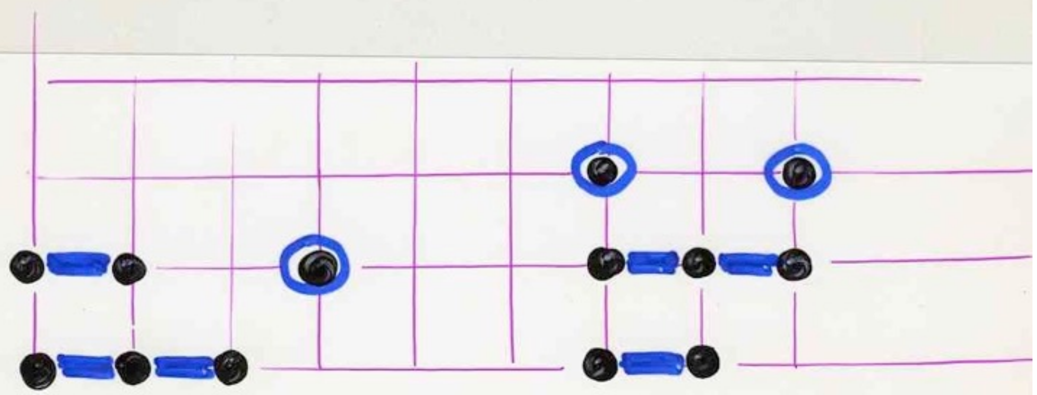
Proposition The stairs decomposition of a heap of dimers on $\mathbb{N}$ gives a bijection ρ

$$\text{heap of dimers on } \mathbb{N} \xrightarrow{\rho} \text{heap of segments on } \mathbb{N}$$

substitution




1 2 3 4 5 6 7 8 9 10 11



exercise

exercise

$$\begin{array}{ccc} \text{Dyck paths} & \xrightarrow{\chi} & \text{pyramids} \\ \updownarrow & & \text{of dimers} \\ \text{Łukasiewicz paths} & \xrightarrow{\chi} & \text{pyramids} \\ & & \text{of segments} \end{array}$$

from course IMSc 2016

p51 and p60-63

bijection

Dyck paths

Lukasiewicz paths

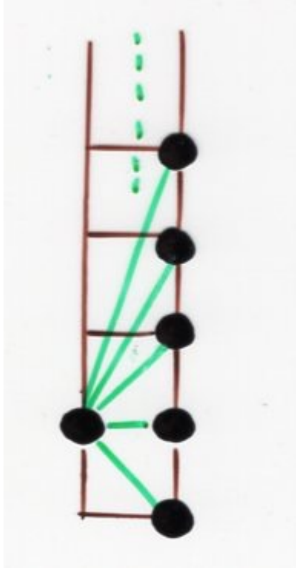
Lukasiewicz path

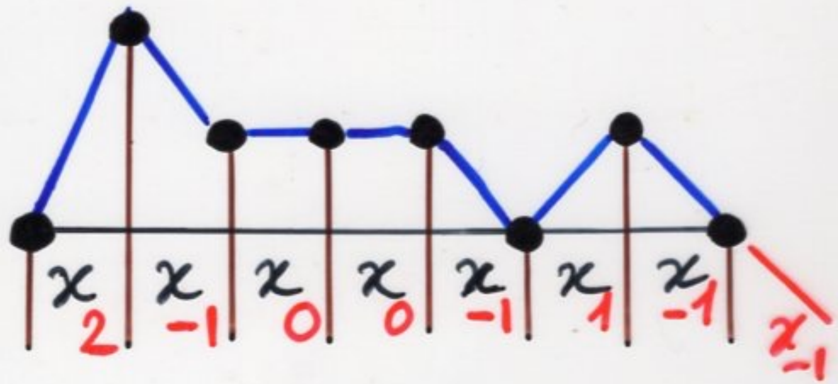
$\omega = (s_0, \dots, s_n)$

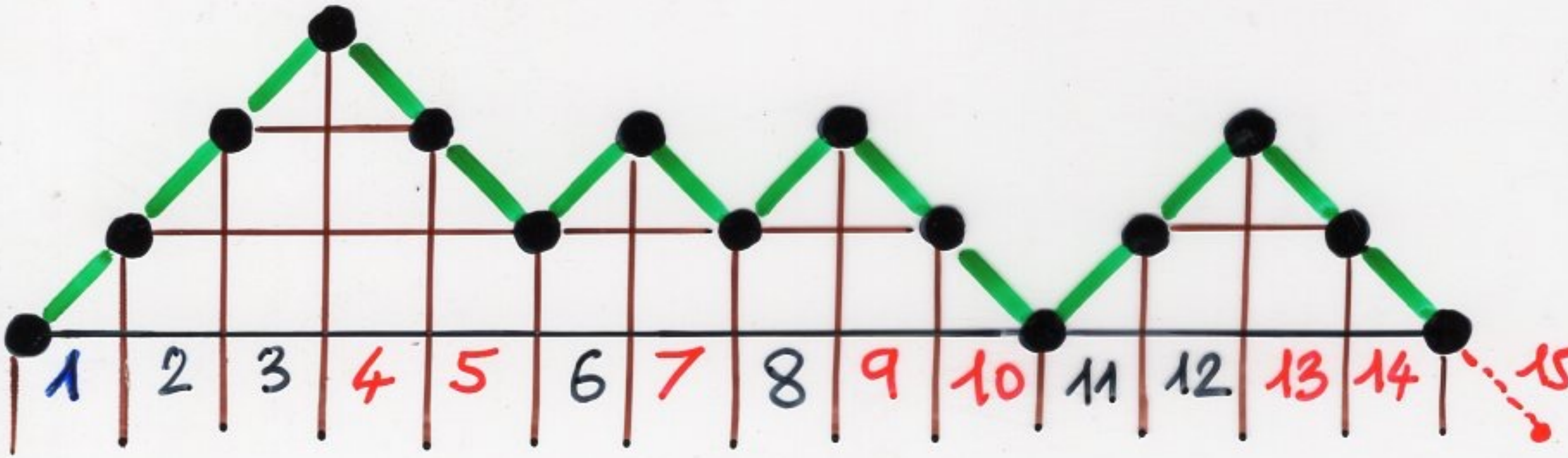
$s_0 = (0,0)$, $s_n = (n,0)$

elementary step $s_i = (x_i, y_i)$ $s_{i+1} = (x_{i+1}, y_{i+1})$

$x_{i+1} = 1 + x_i$ with $y_{i+1} \geqslant y_i - 1$



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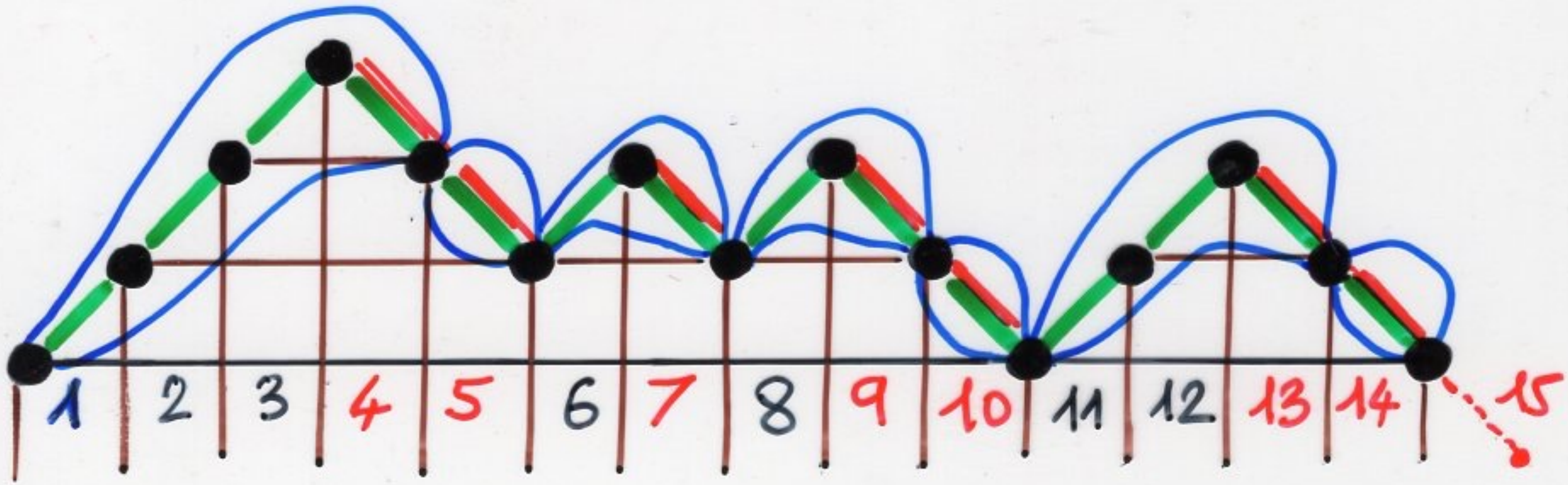

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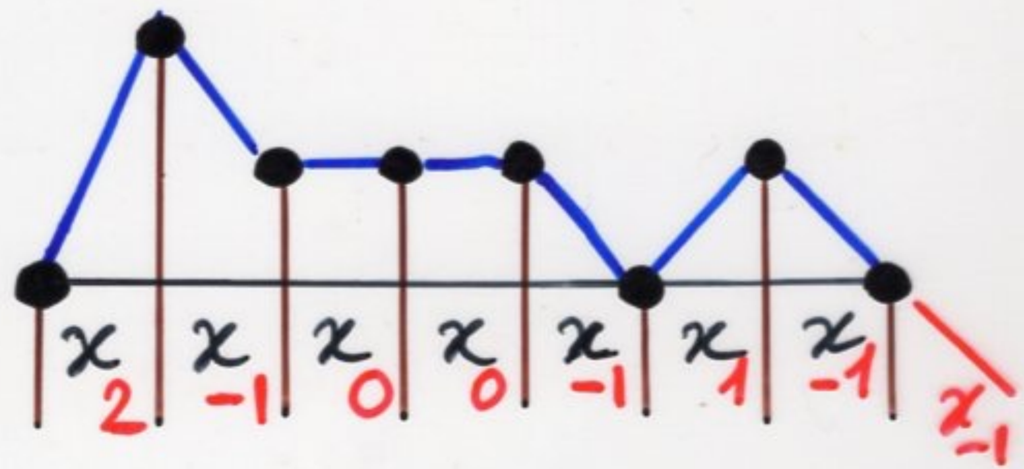

x_2
x_{-1}
x_0
x_0
x_{-1}
x_1
x_{-1}
x_{-1}

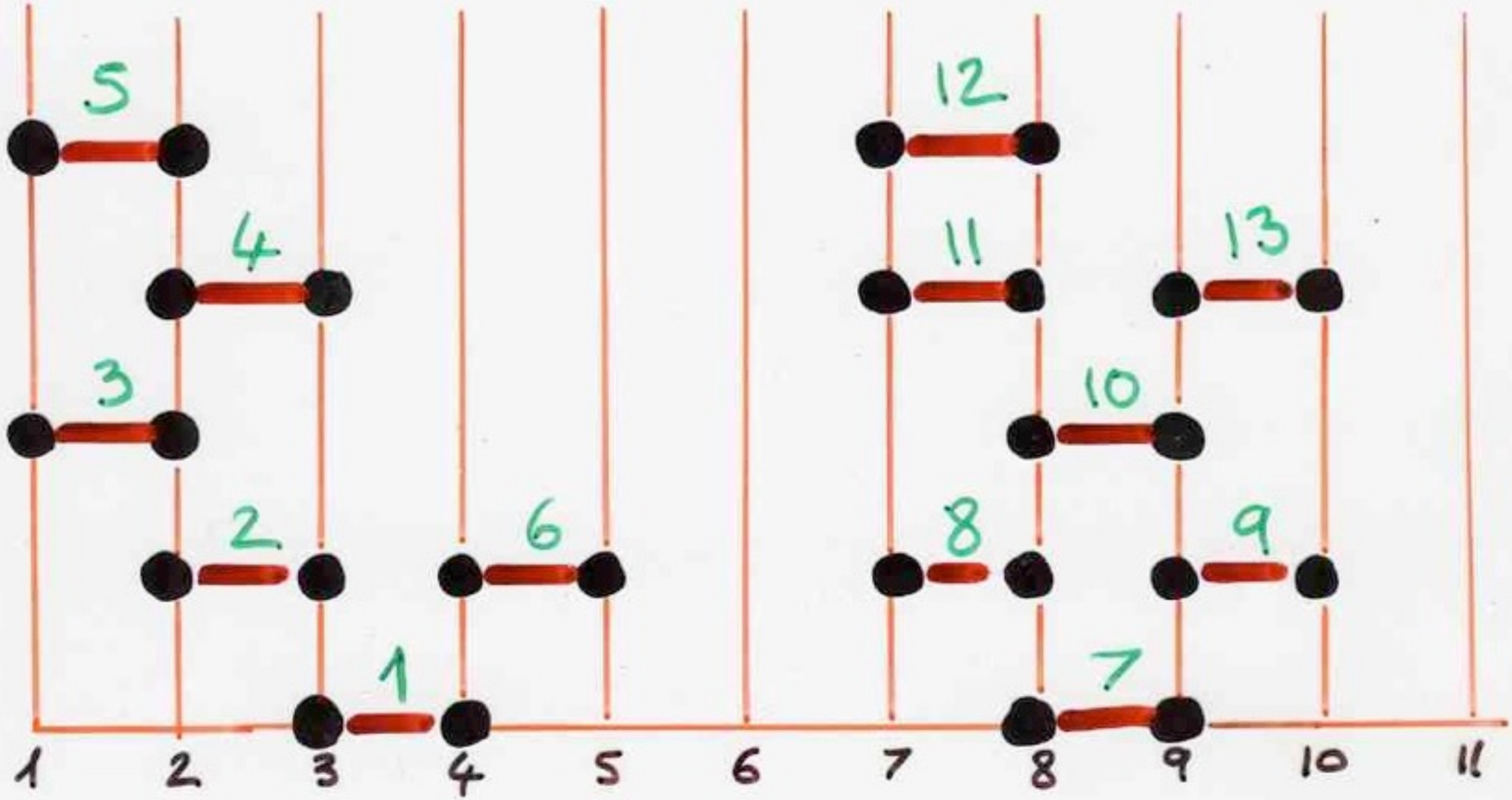
exercise

$$\begin{array}{ccc}
\text{Dyck paths} & \xrightarrow{\chi} & \text{pyramids of dimers} \\
\updownarrow & & \big\downarrow \rho \;\; ? \;\; \text{compare} \\
\text{Łukasiewicz paths} & \xrightarrow{\chi} & \text{pyramids of segments}
\end{array}$$

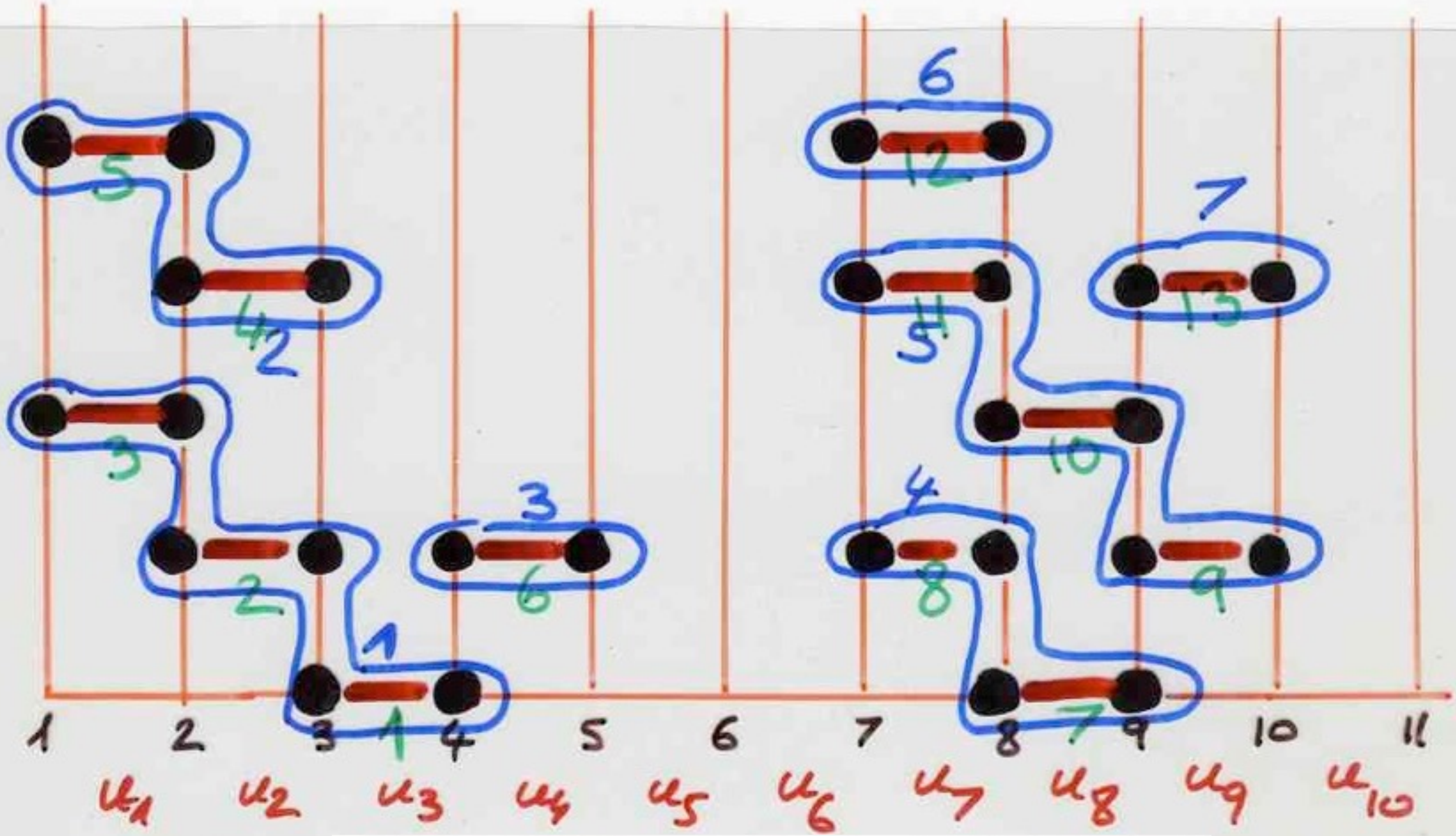
$$\text{Dyck paths} \xrightarrow{\psi} \text{pyramids of oriented loops}$$

relation ?

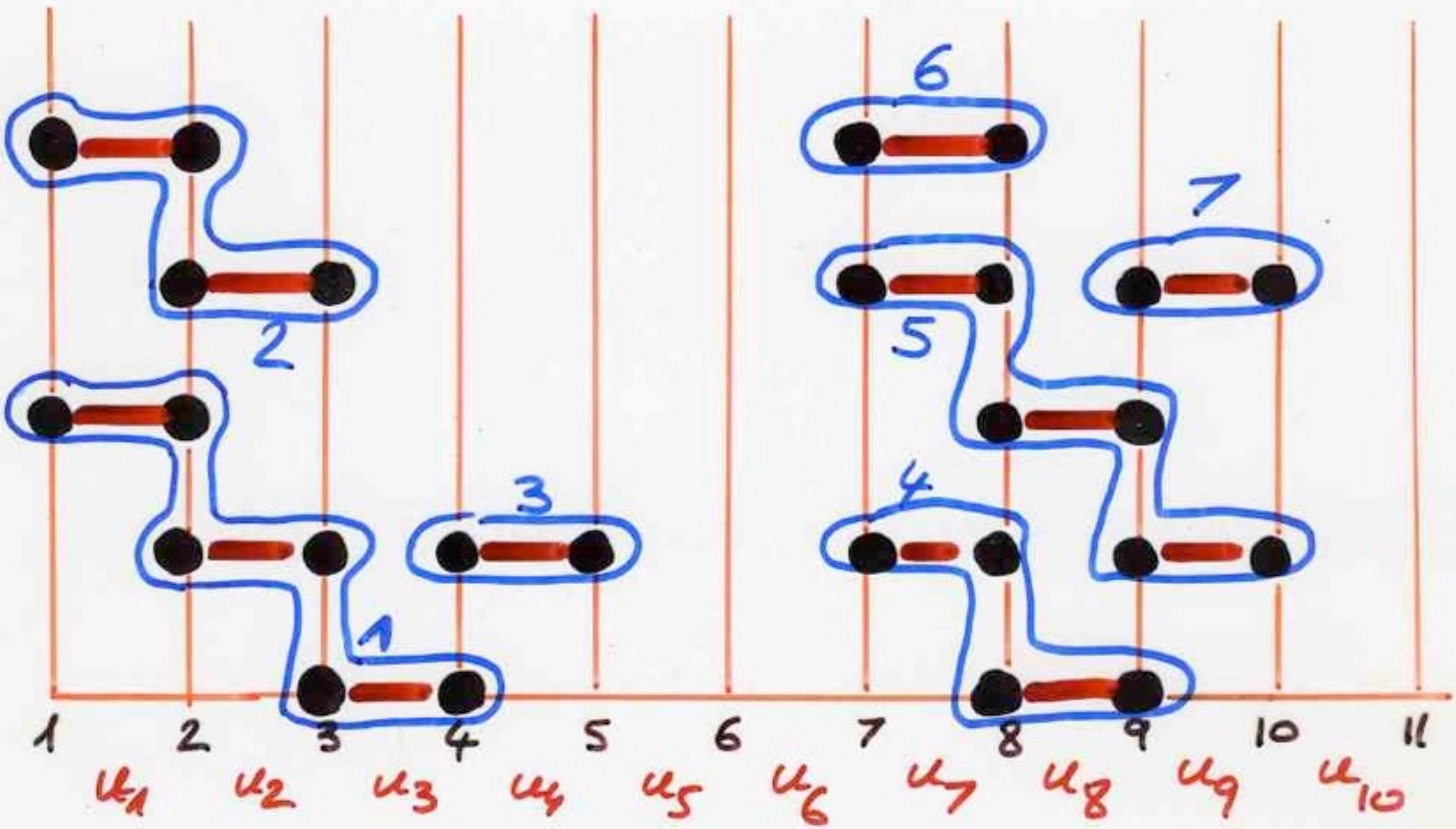
total order
of the stairs
in a heap
of dimers



lexicographic normal form



lexicographic normal form



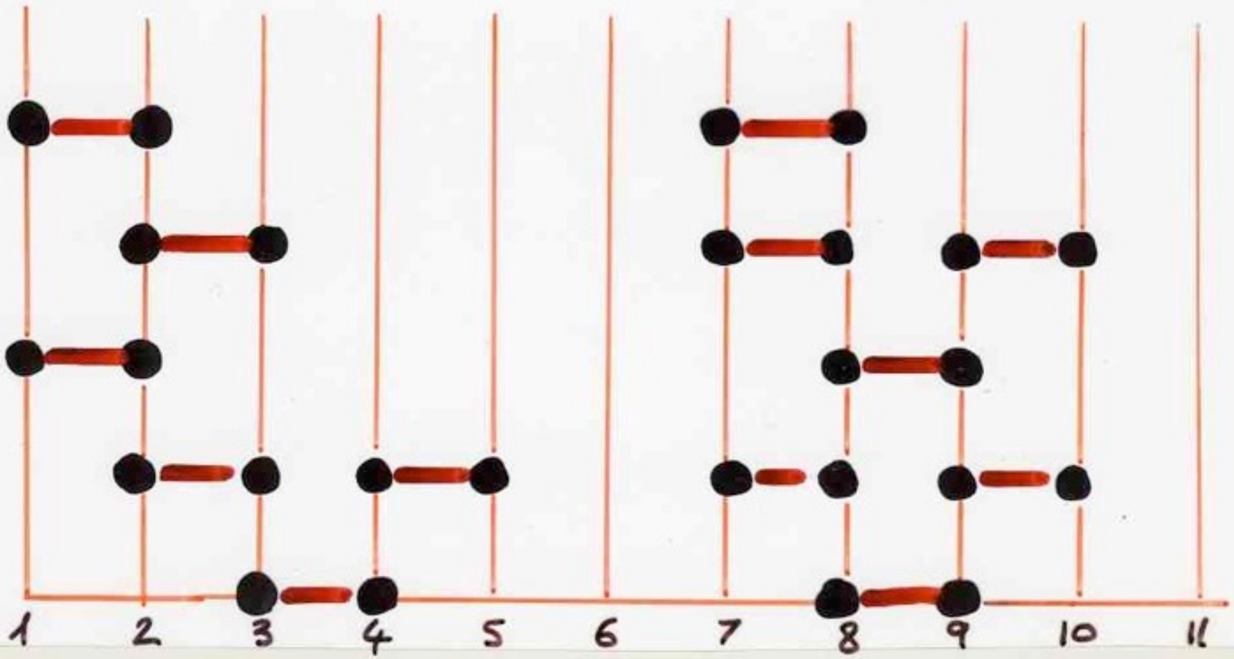
6
7
2
5
3
4
1
1 2 3 4 5 6 7 8 9 10 11
u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8 u_9 u_{10}

ordering the segments

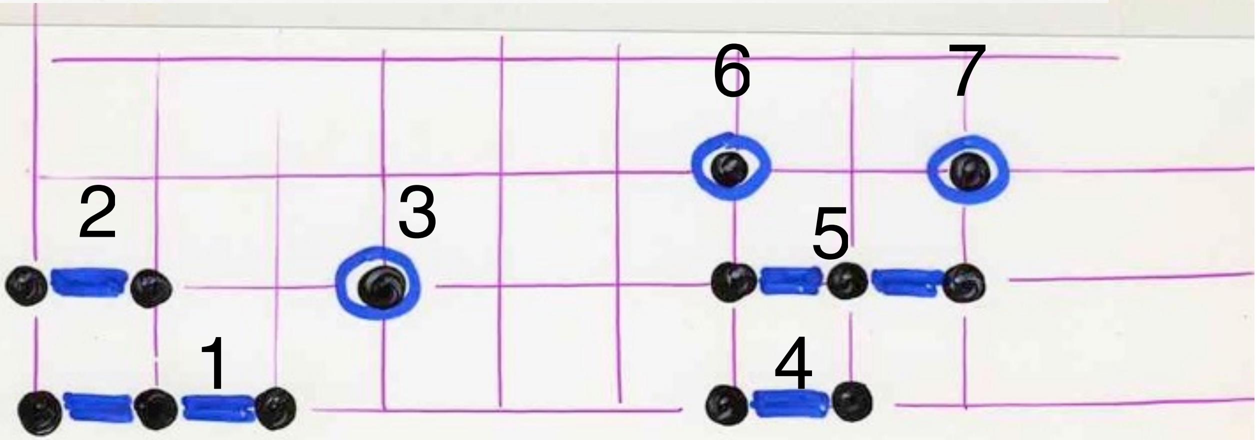
total order of the segments in a heap of segments

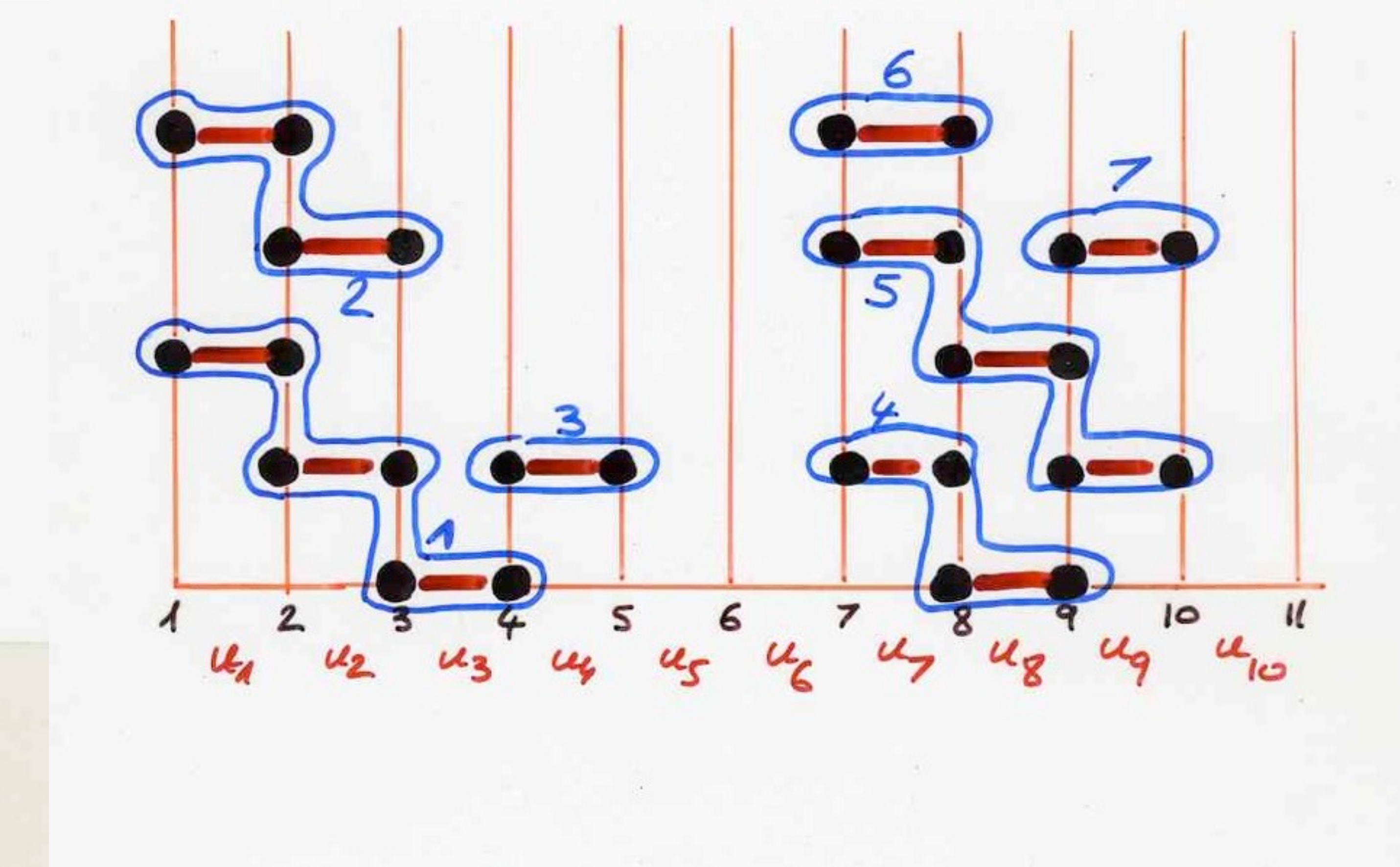
$\longrightarrow$

total order of the stairs in a heap of dimers



lexicographic normal form




1
2
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6
7
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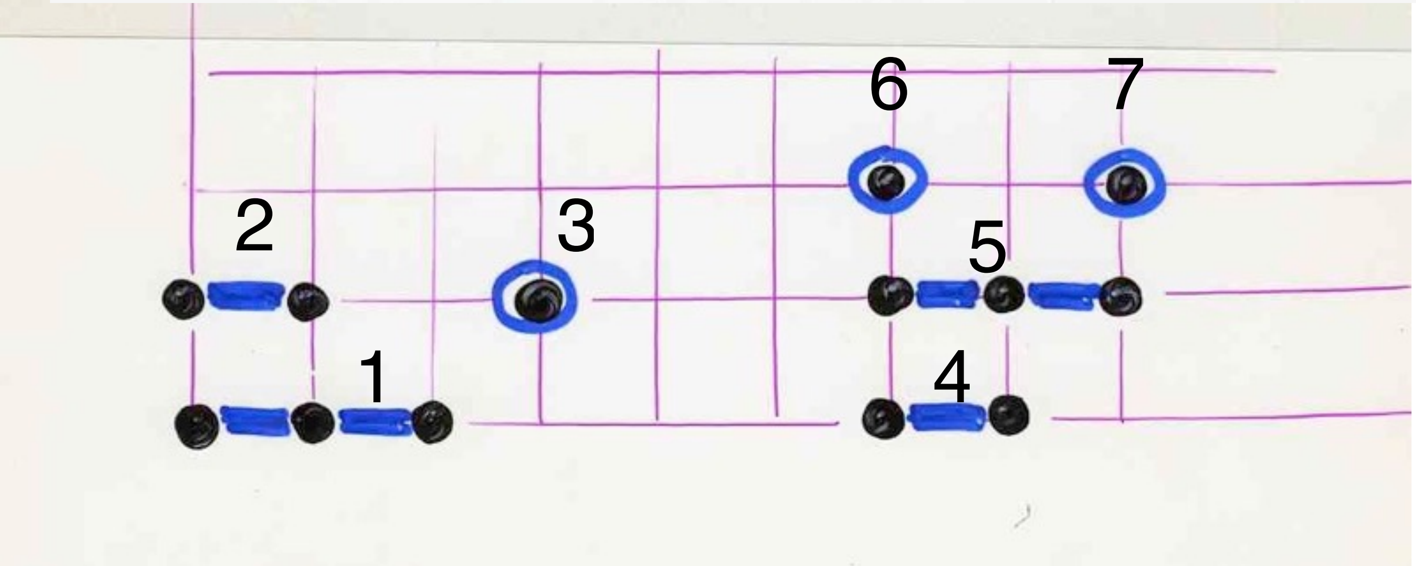
11
u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8 u_9 u_{10}


6
7
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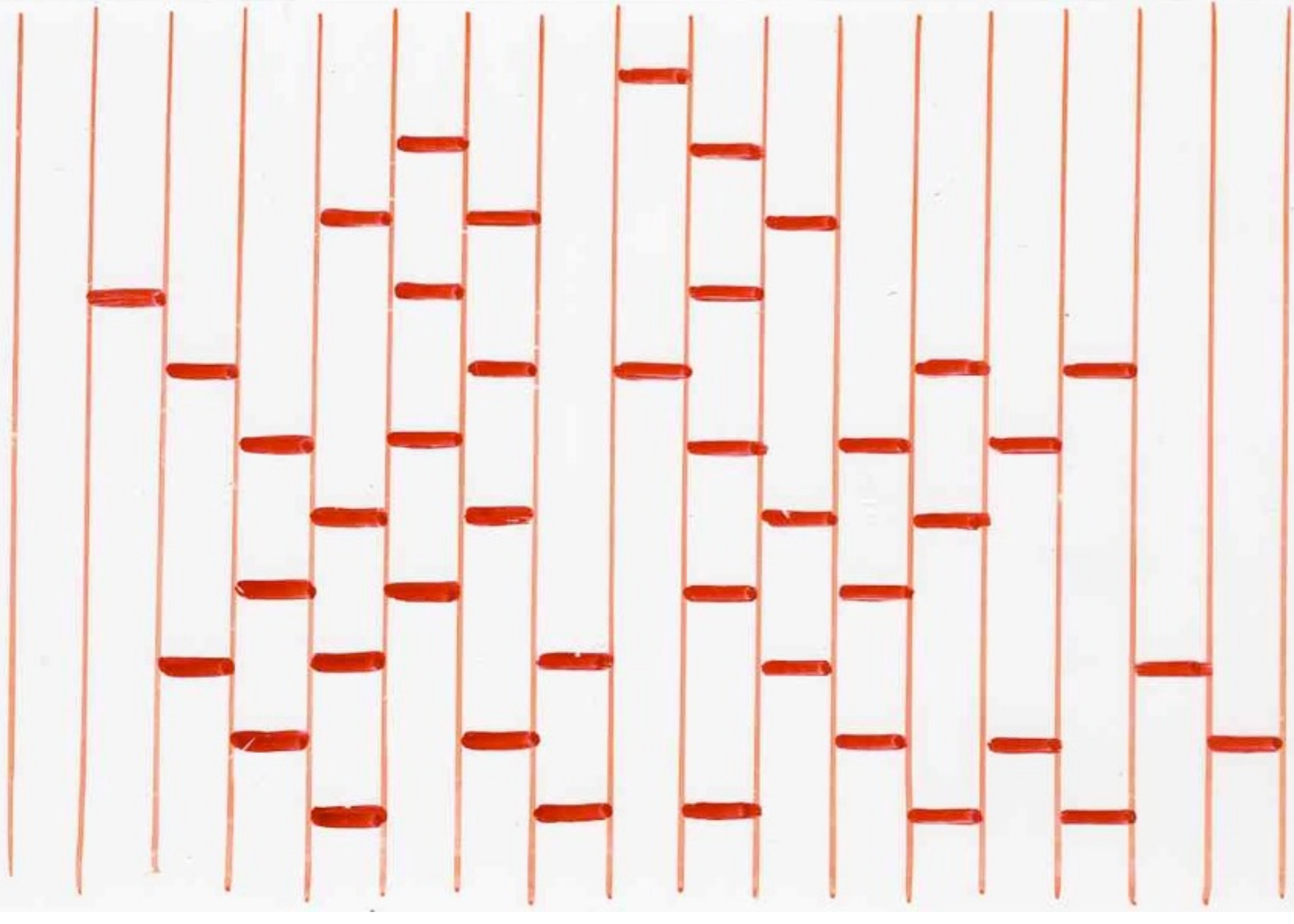
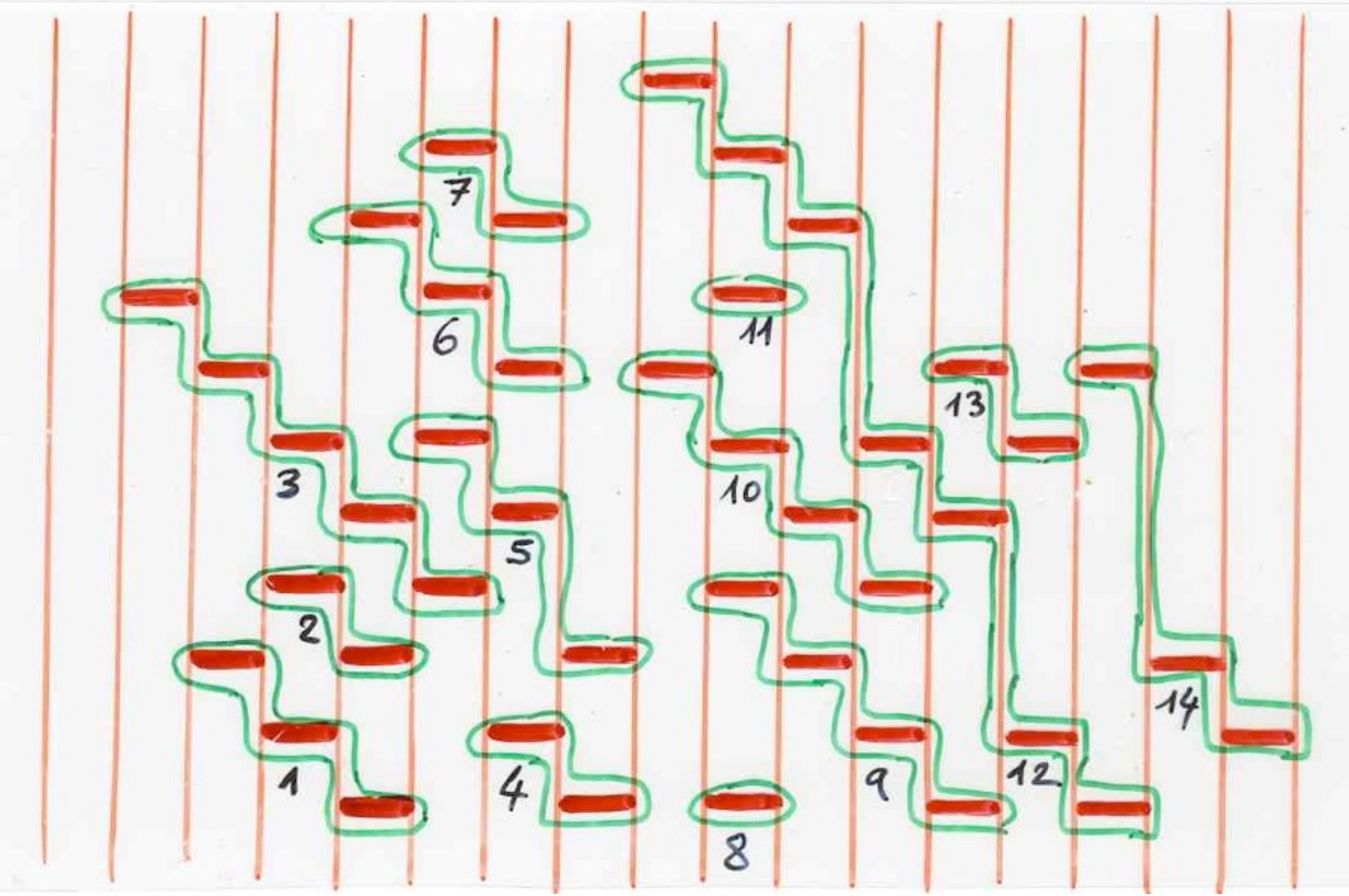



7
6
3
5
2
1
4
8
11
10
9
13
12
14

the stair lemma

The stair lemma

Let H be a strict heap of dimers on $\mathbb{N}$

and $H = S_1 \odot S_2 \odot \cdots \odot S_k$

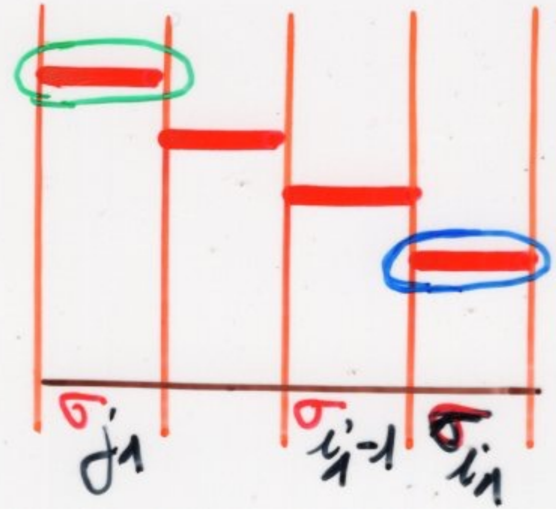
its stair decomposition with

$$S_1 < S_2 < \cdots < S_k$$

$$S_i = \sigma_{i_1} \odot \sigma_{i_1 - 1} \odot \cdots \odot \sigma_{j_1}$$ (product of dimers)

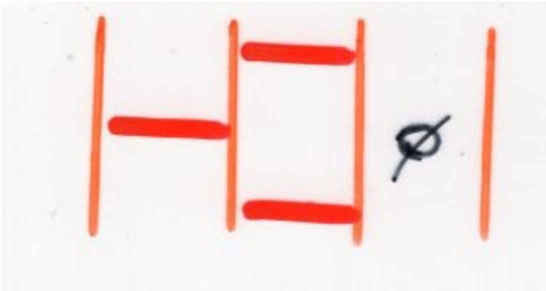
notation $\sigma_{i_1} = \min(S_i)$

$\sigma_{j_1} = \max(S_i)$

$S_i =$ 

The stair lemma

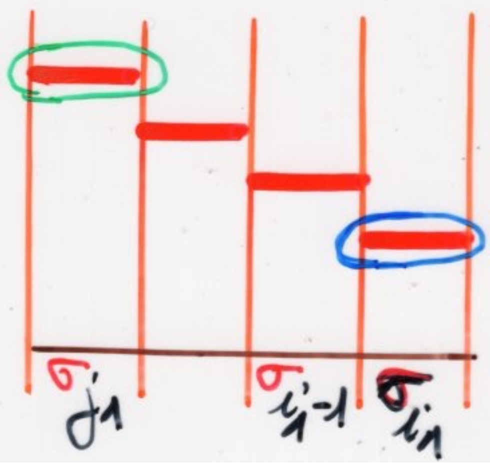
no occurrences of

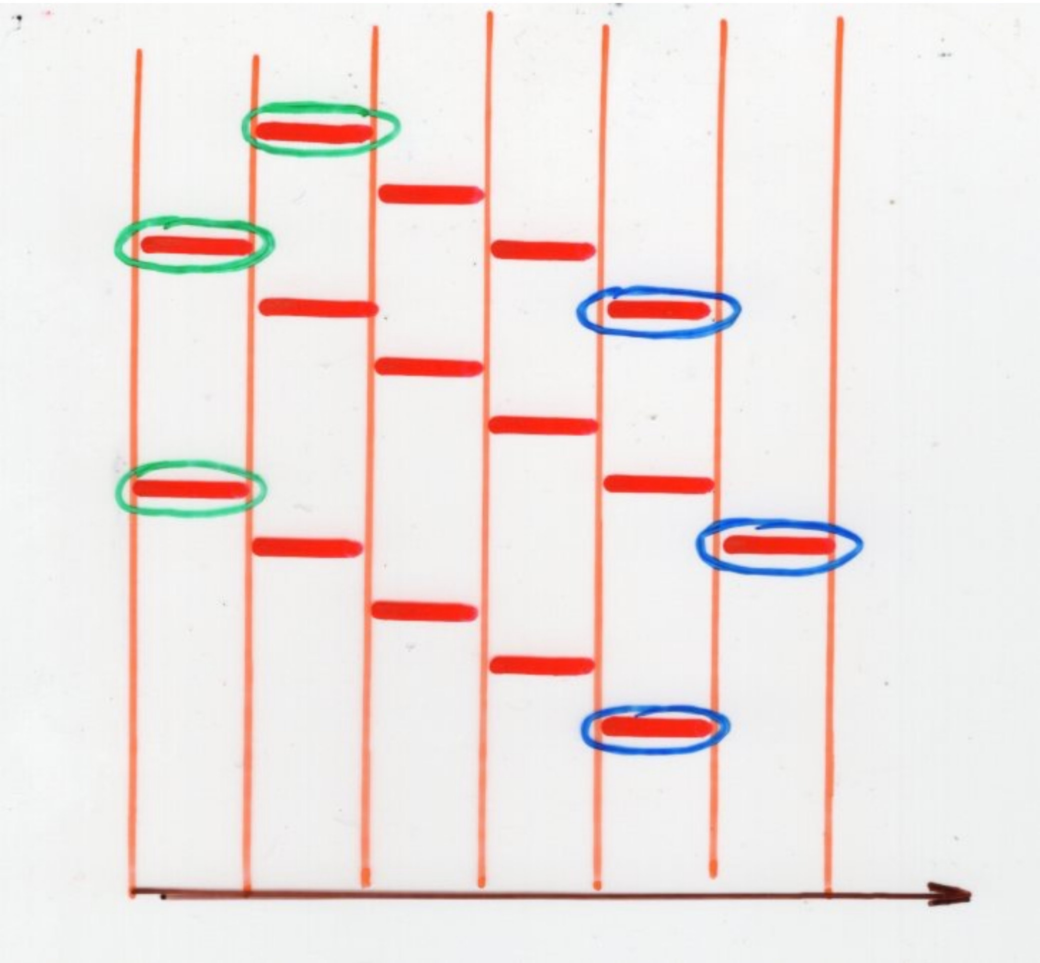


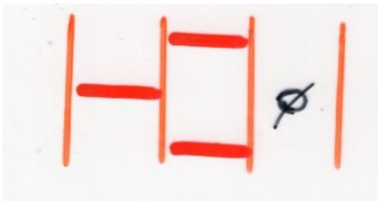
$\Longleftrightarrow$ $\min(S_1) \prec \cdots \prec \min(S_k)$

$\Longleftrightarrow$ $\max(S_1) \prec \cdots \prec \max(S_k)$

$S_i =$



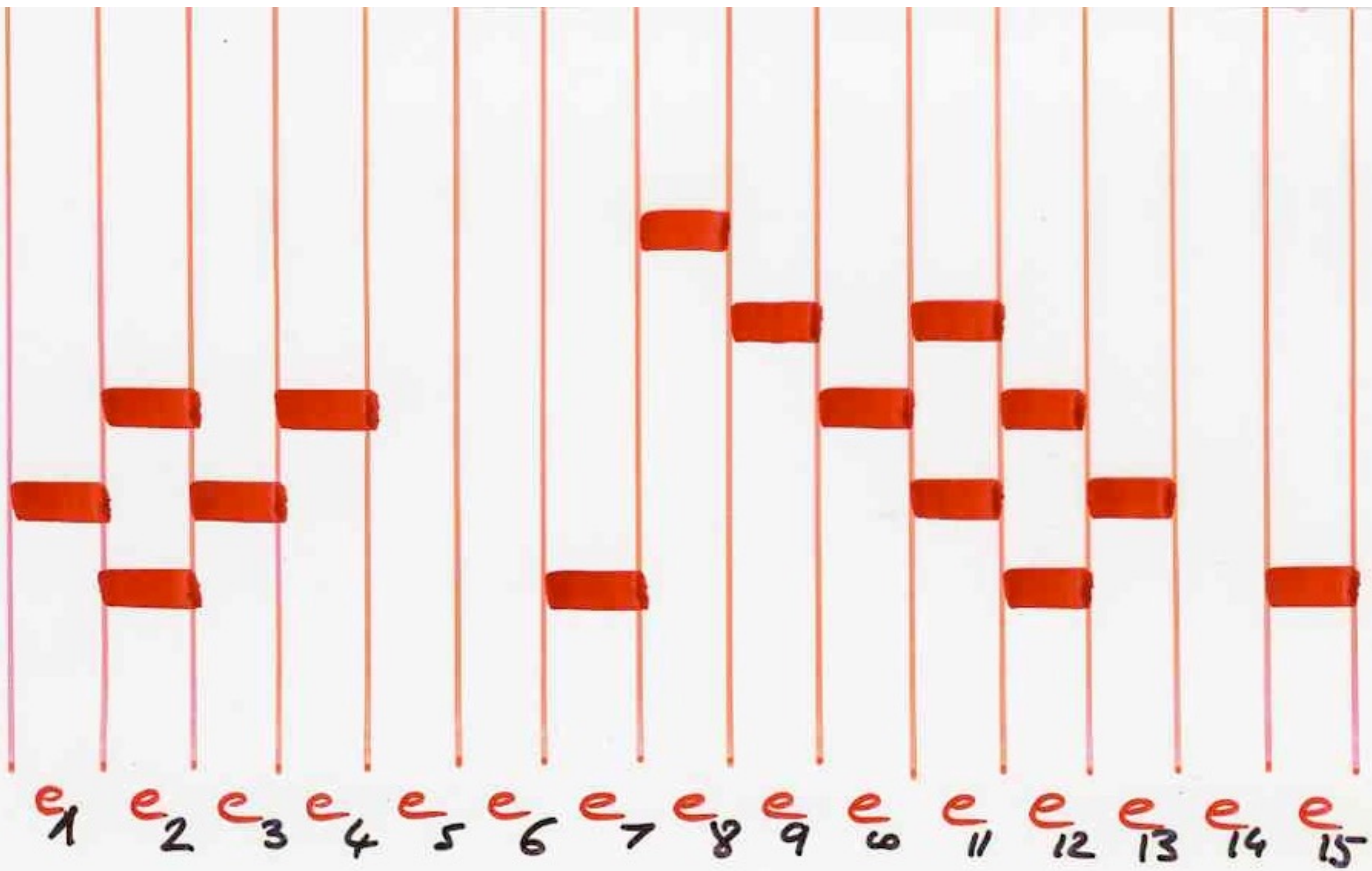




$$\Longleftrightarrow \quad \min(S_1) \prec \cdots \prec \min(S_k)$$

$$\Longleftrightarrow \quad \max(S_1) \prec \cdots \prec \max(S_k)$$

fully commutative heaps
(of dimers)


e1
e2
e3
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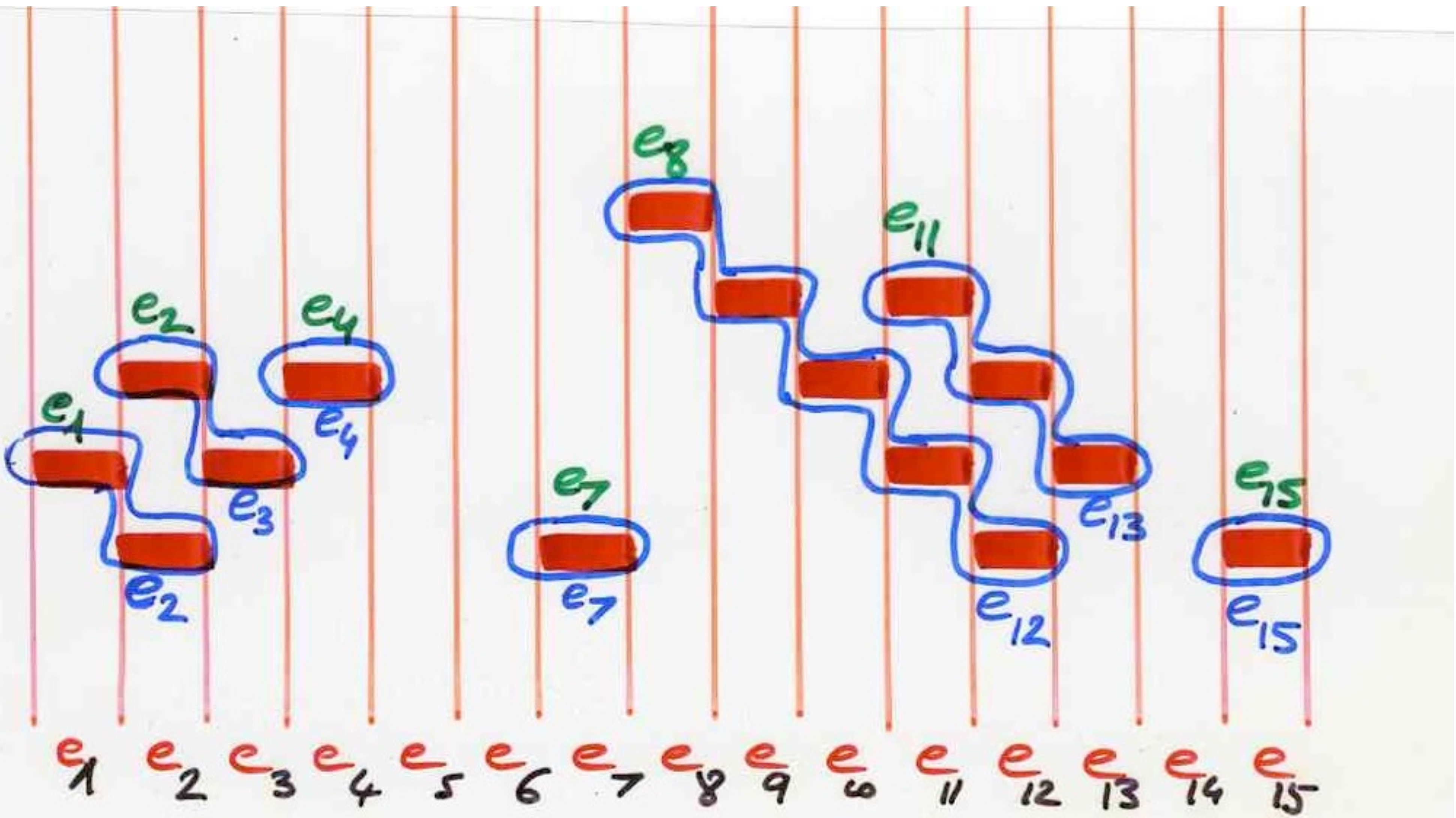
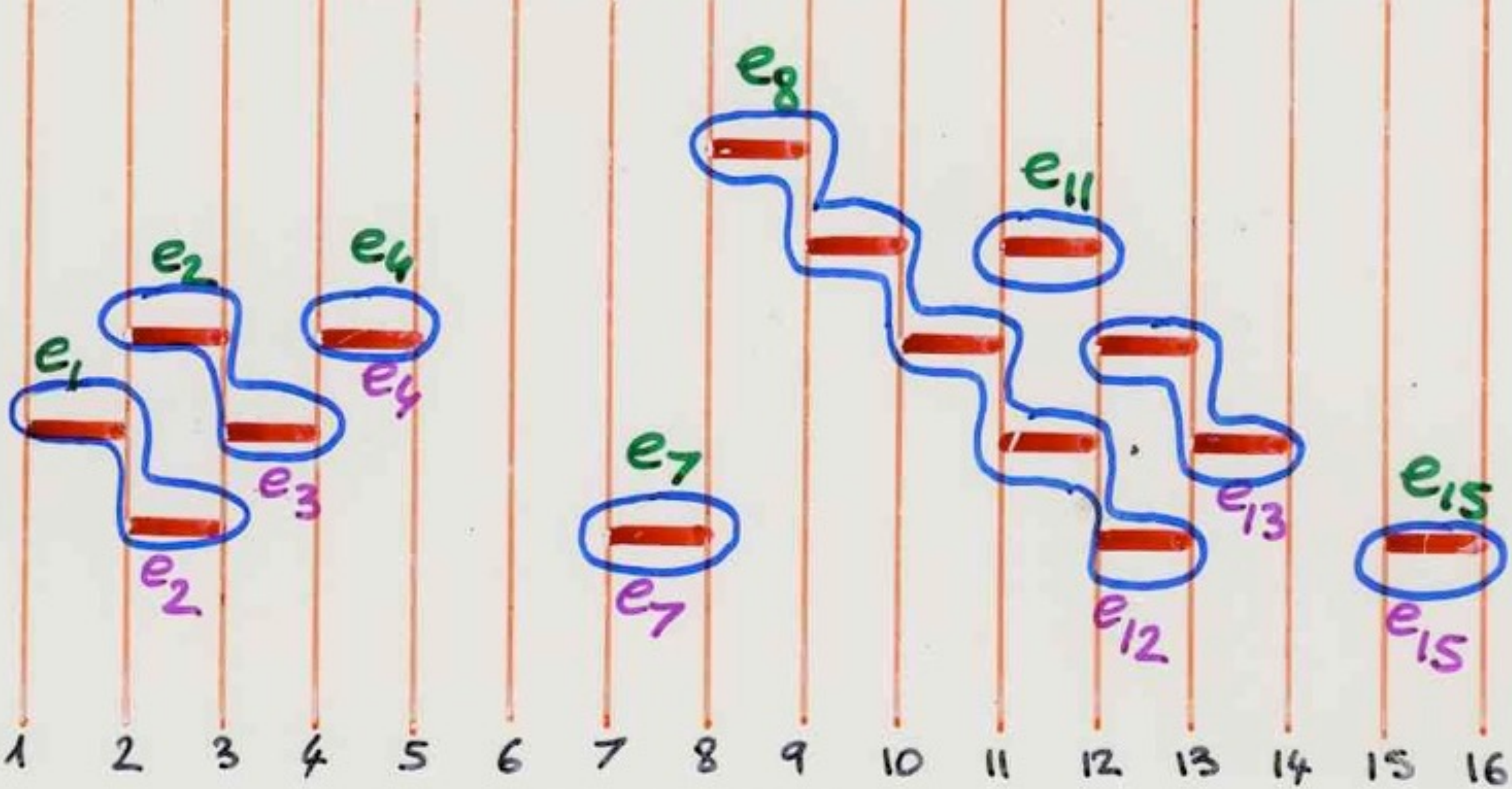
e11
e12
e13
e14
e15



e1 e2 e3 e4 e5 e6 e7 e8 e9 e10 e11 e12 e13 e14 e15

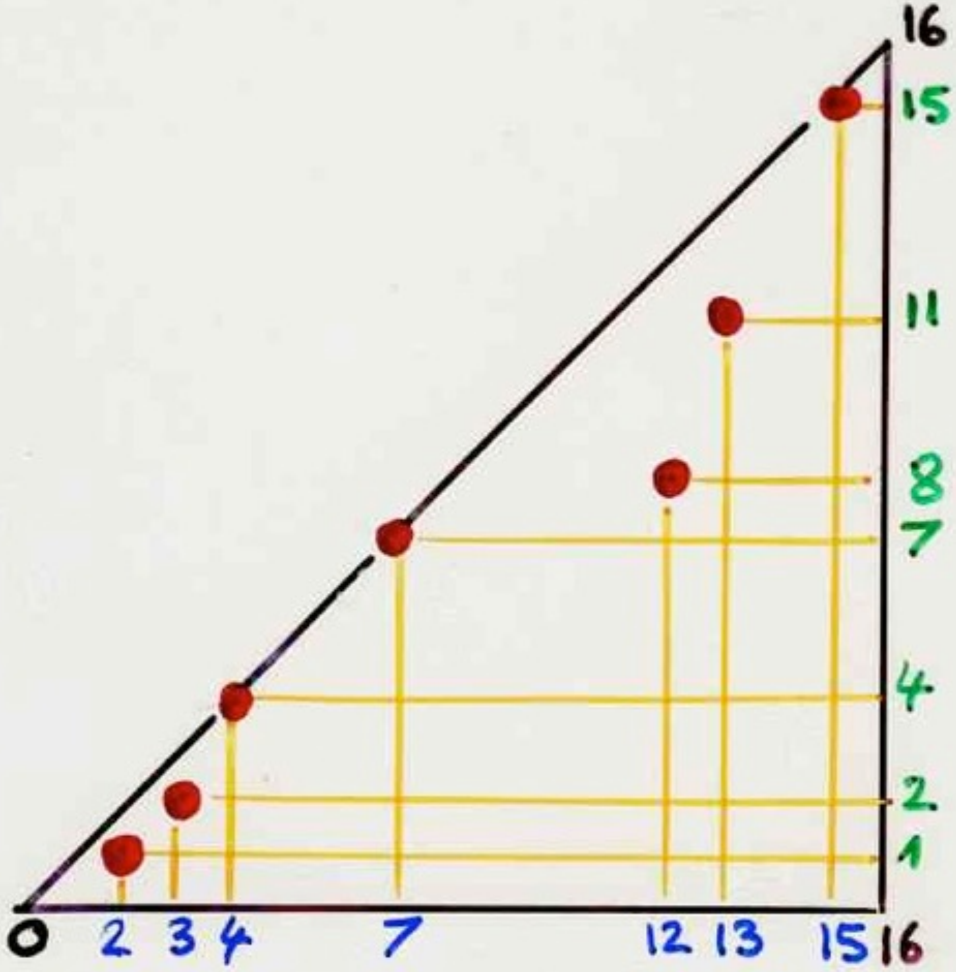


$$\begin{matrix} & & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & & \\ & & \geq & & \geq & & \geq & & \geq & & \geq & & \geq & & \geq & & \\ 1 & \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq & n \end{matrix}$$

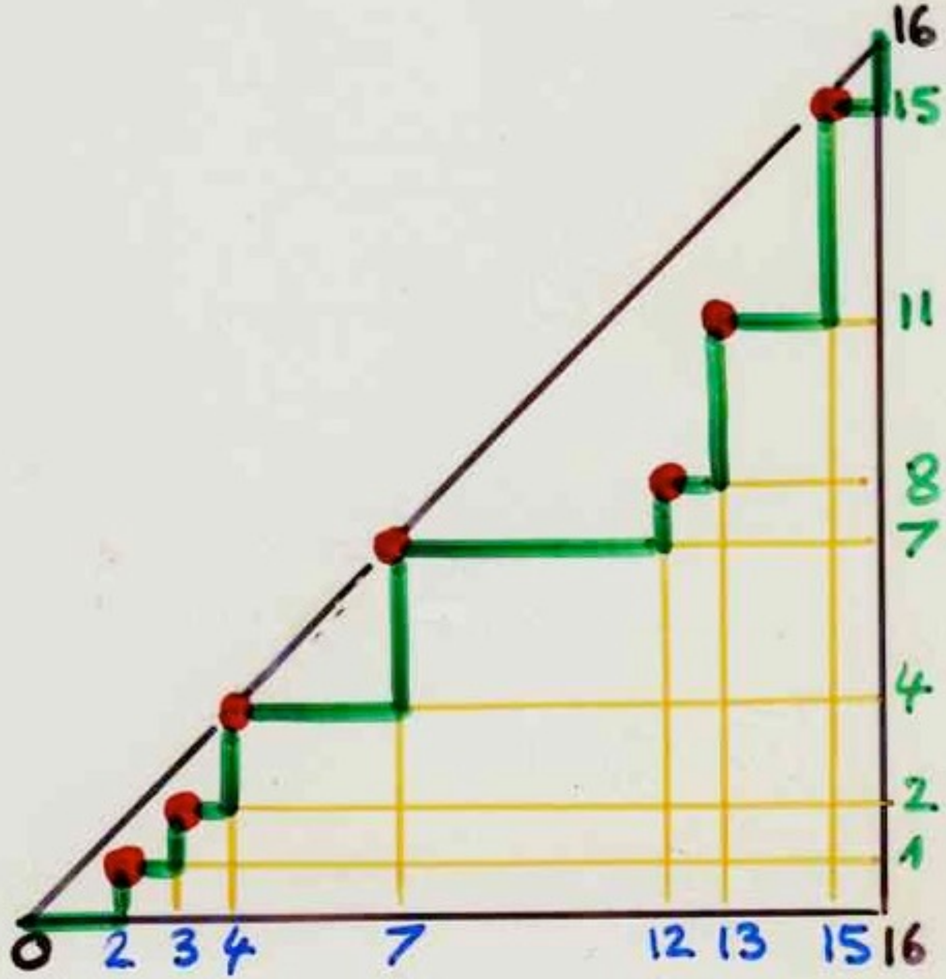
bijection

fully commutative heaps

Dyck paths



$$\begin{array}{ccccccccccccccccc}
 & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & \\
 & \rotatebox[origin=c]{-90}{$\geq$} & & \rotatebox[origin=c]{-90}{$\geq$} & & \rotatebox[origin=c]{-90}{$\geq$} & & \rotatebox[origin=c]{-90}{$\geq$} & & \rotatebox[origin=c]{-90}{$\geq$} & & \rotatebox[origin=c]{-90}{$\geq$} & & \rotatebox[origin=c]{-90}{$\geq$} & \\
1 \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq n
\end{array}$$

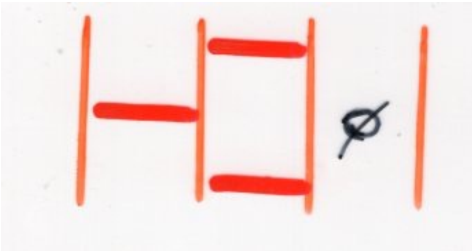


$$\begin{array}{ccccccccccccccccc}
 & & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & & \\
 & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \\
1 & \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq & n
\end{array}$$

exercise

exercise

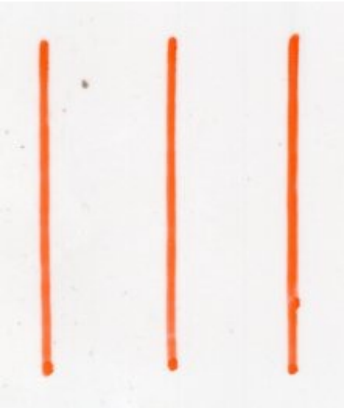
The number of strict heaps satisfying the condition :

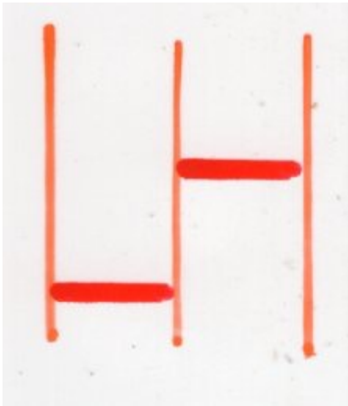


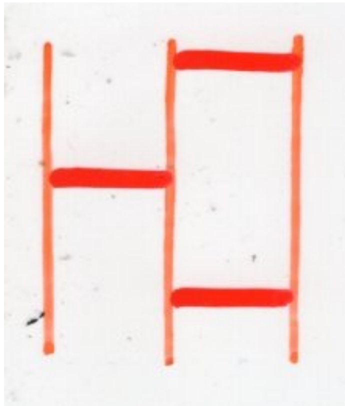
$$\Longleftrightarrow \quad \min(S_1) < \cdots < \min(S_k)$$

is $n!$

$$\Longleftrightarrow \quad \max(S_1) < \cdots < \max(S_k)$$





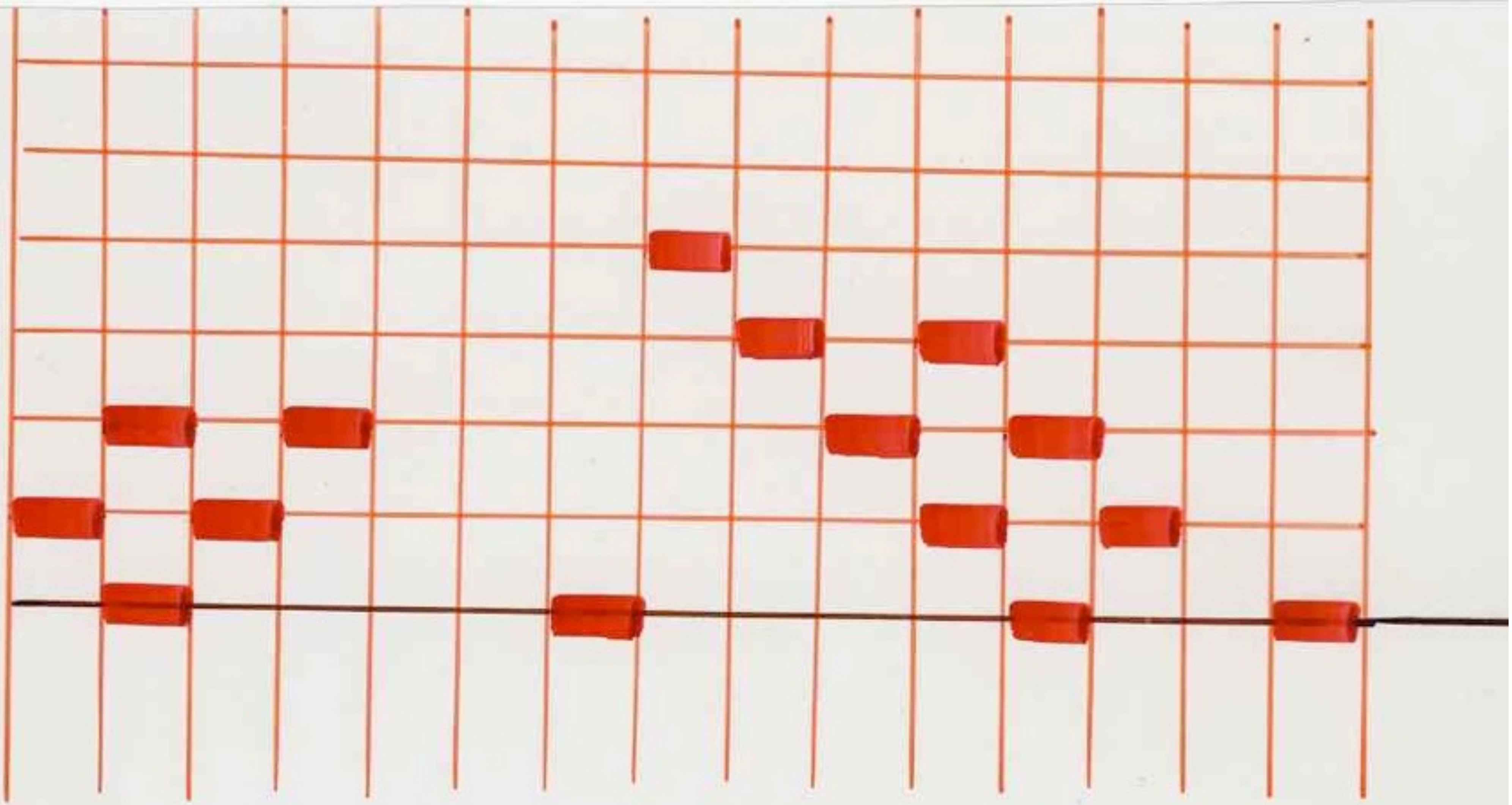


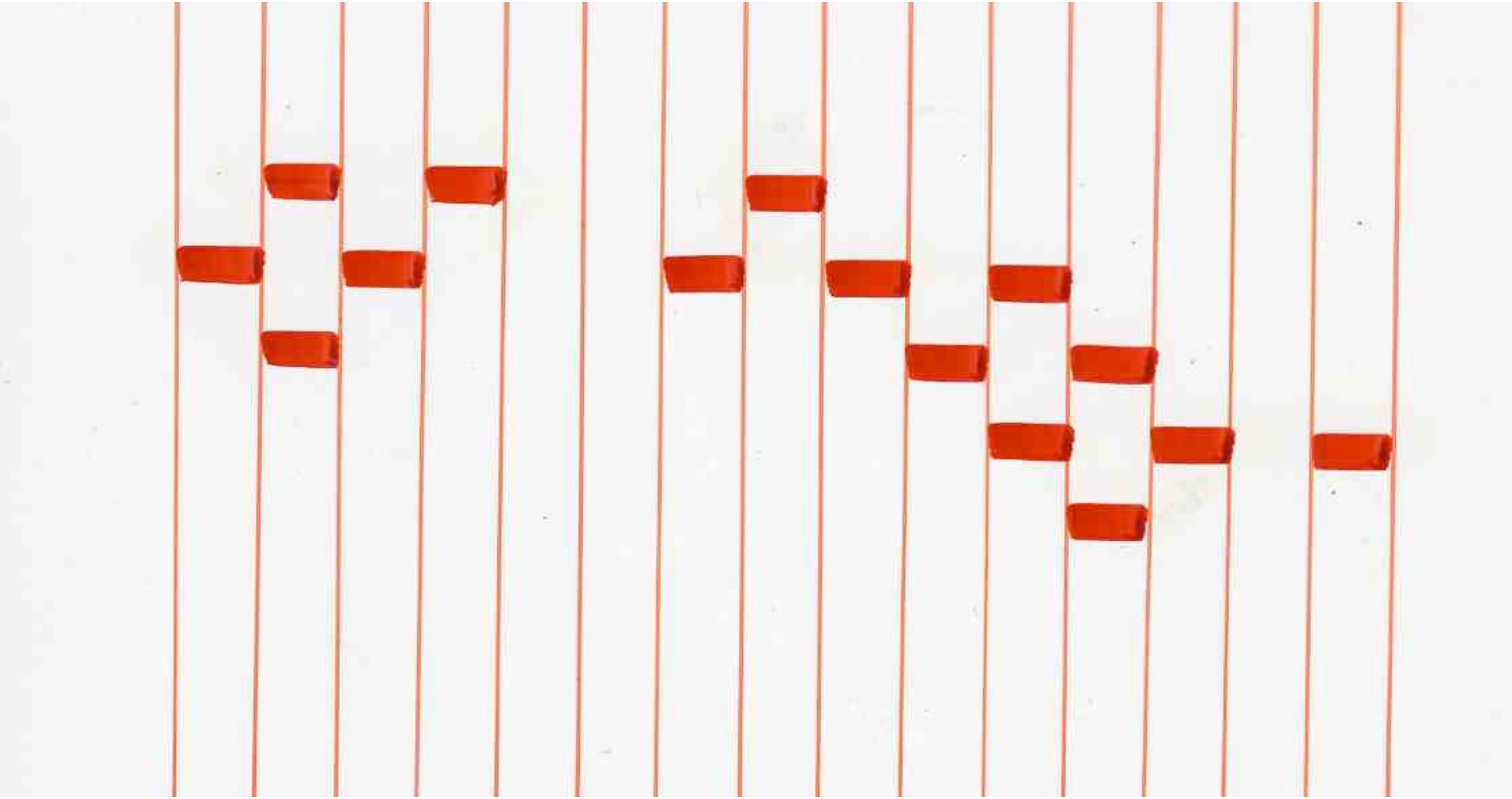
bijection

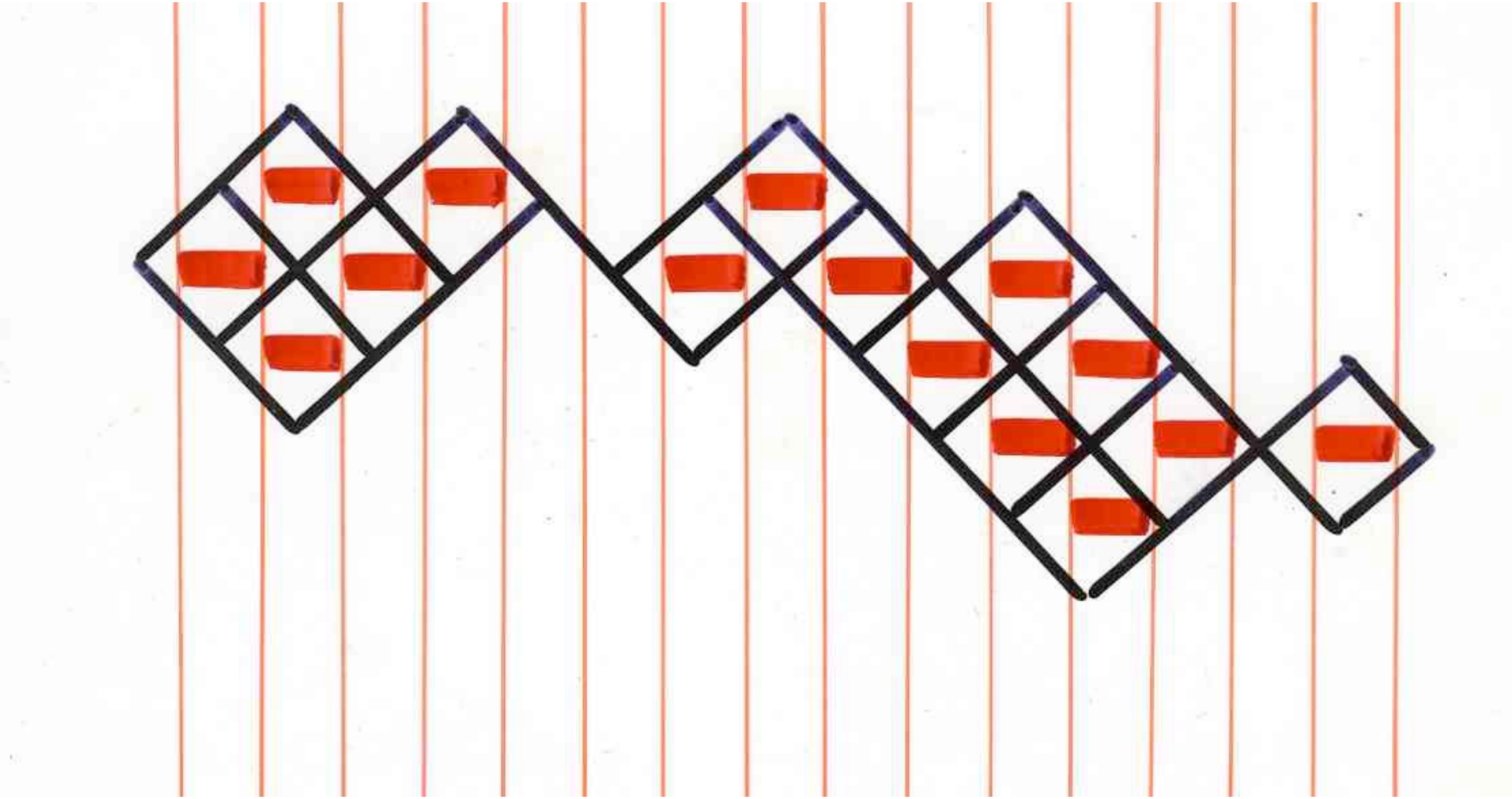
fully commutative heaps

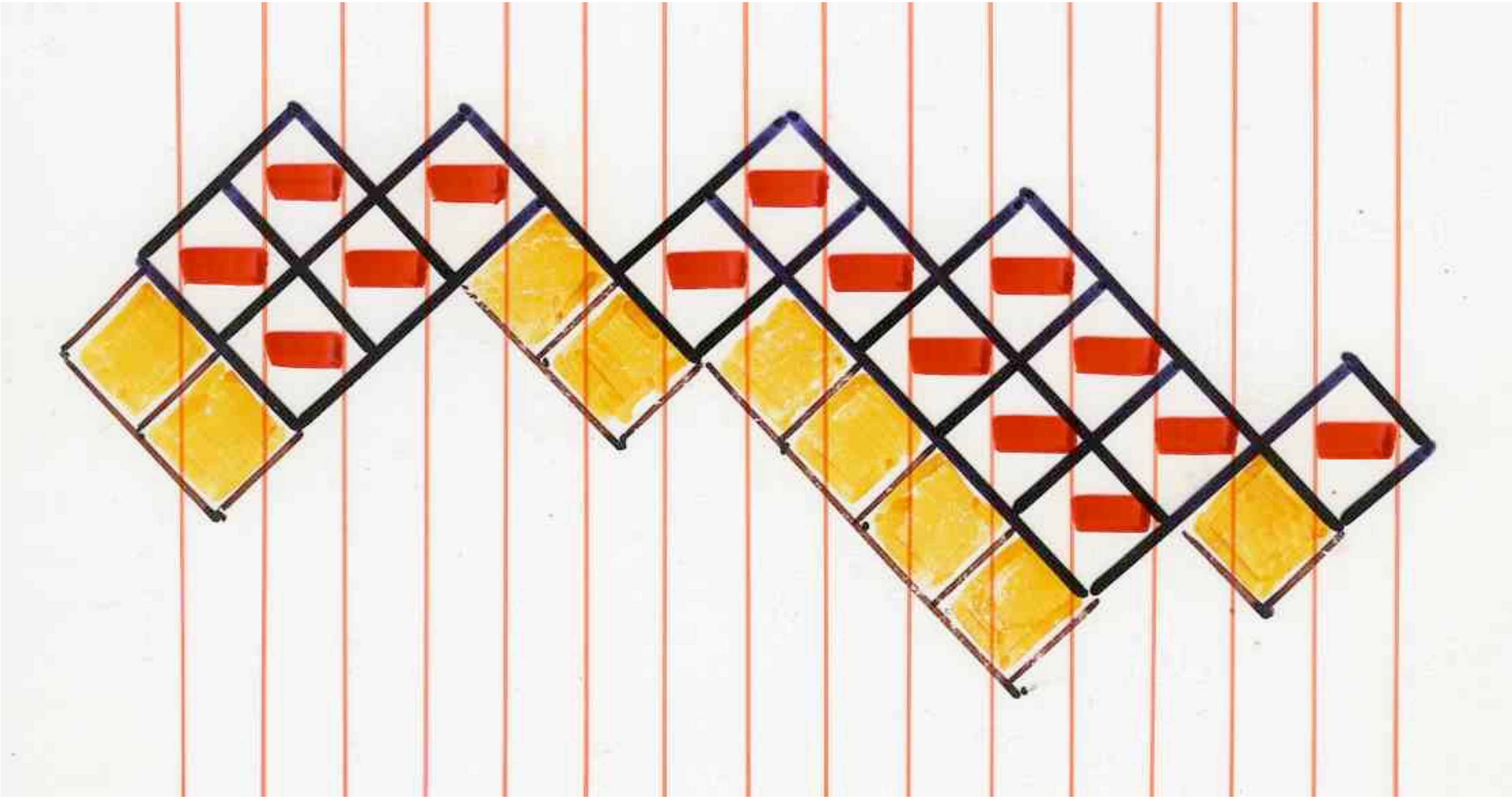
parallelogram polyominos

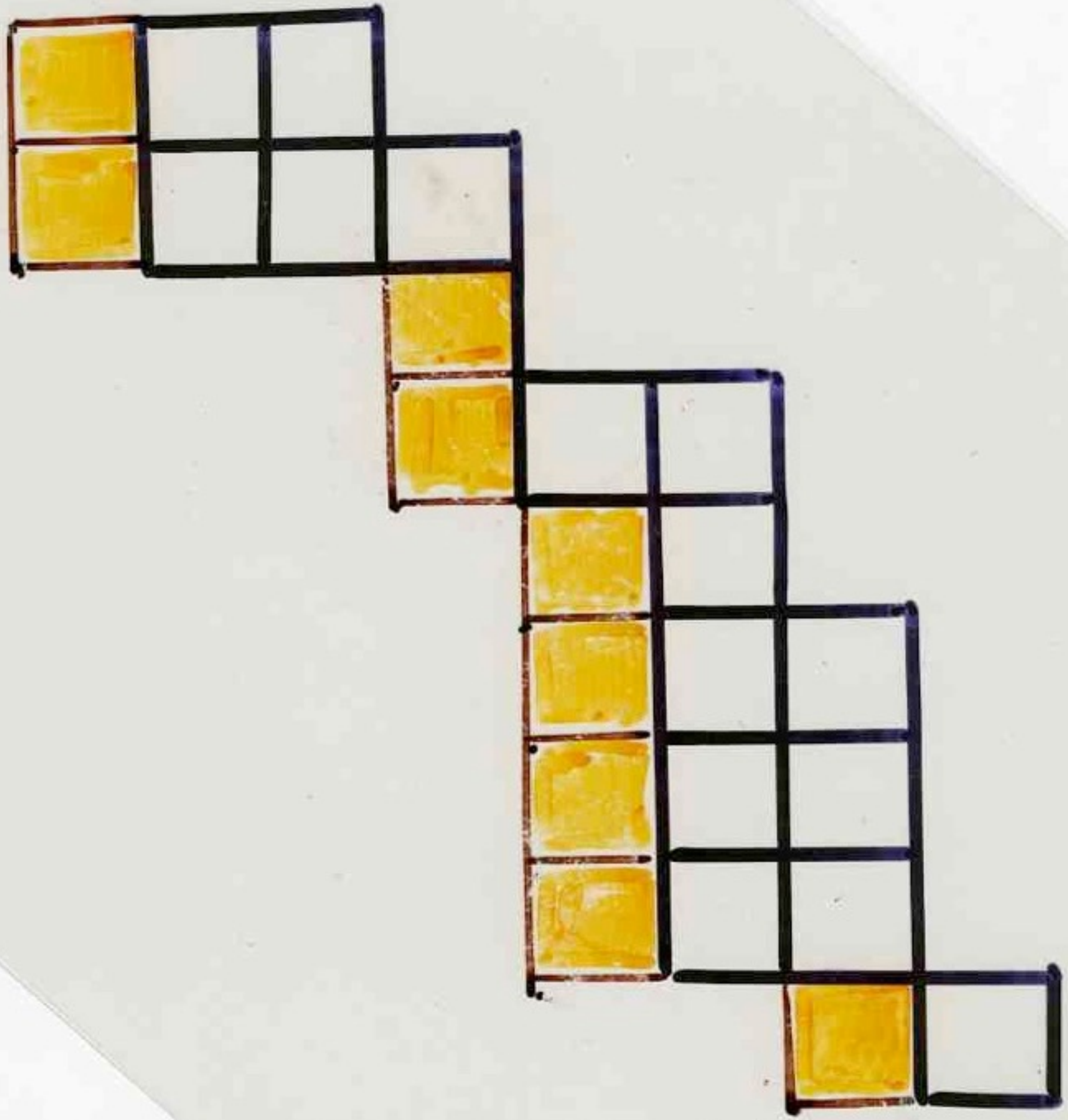
(staircase polygons)

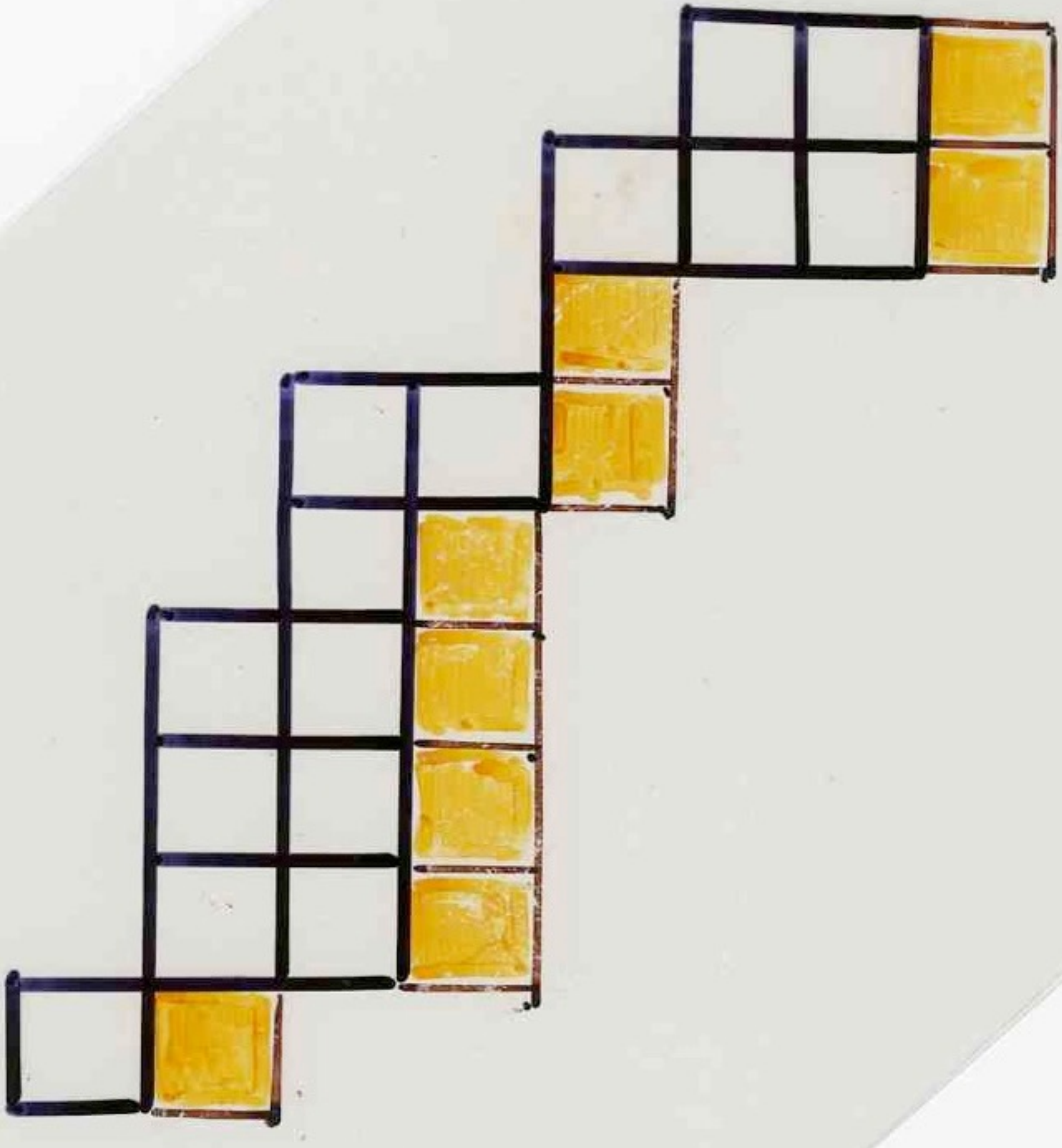


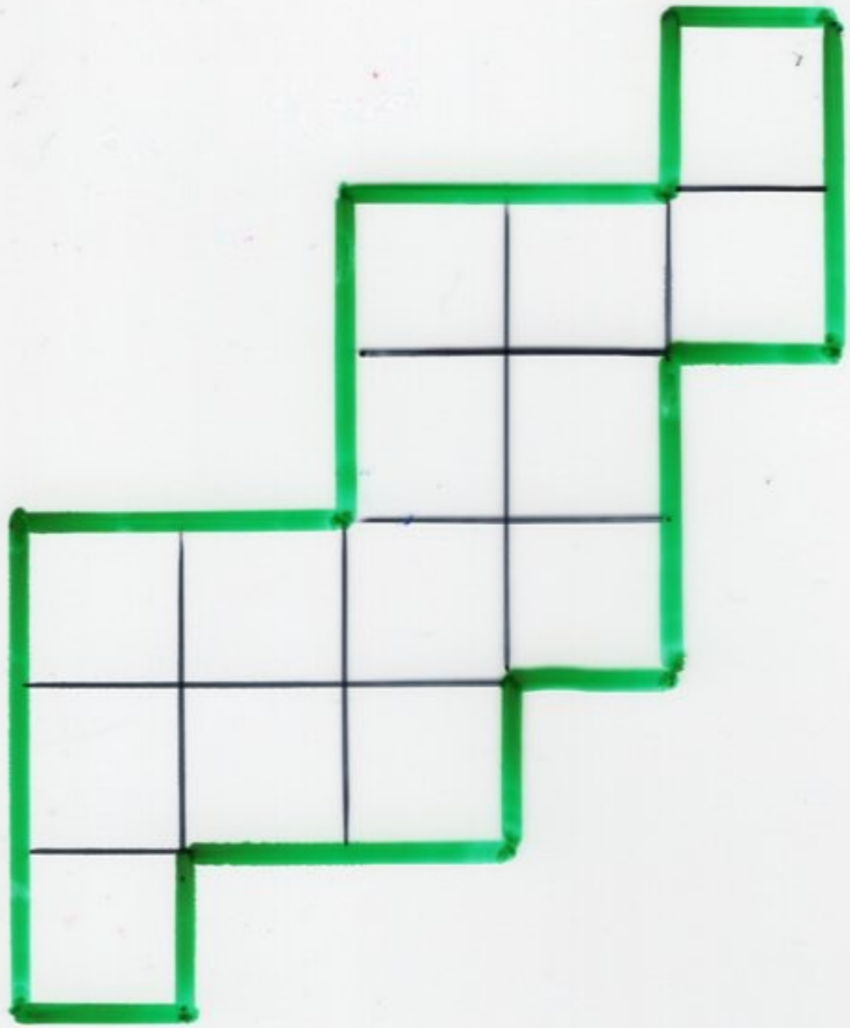






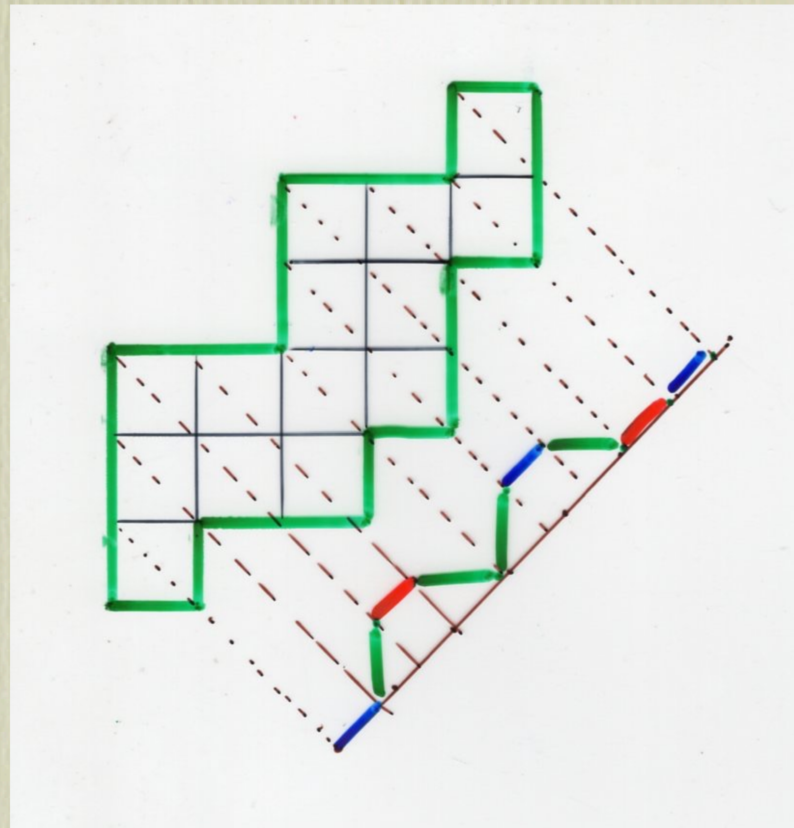






cours IMSc 2016 Ch2a, p105-109

bijection

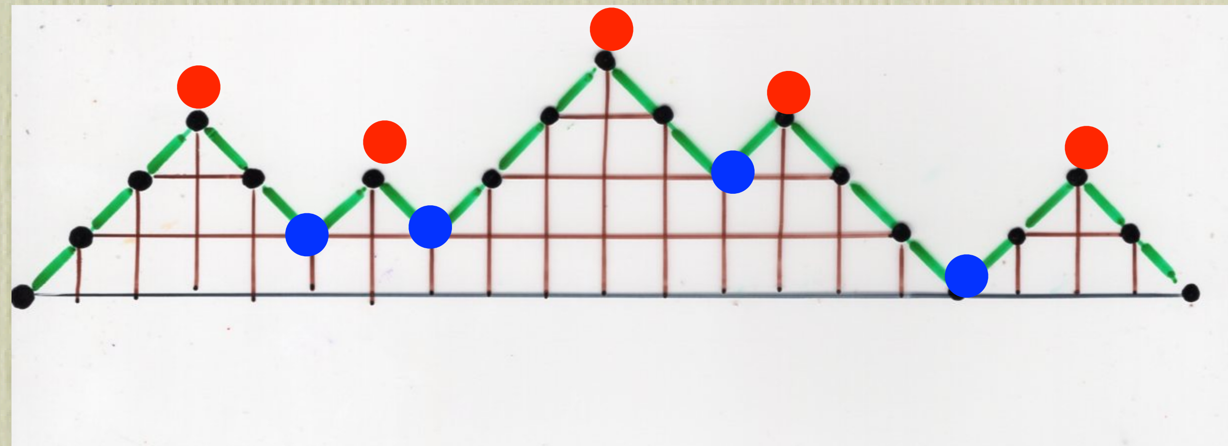


staircase polygons

2-colored Motzkin paths

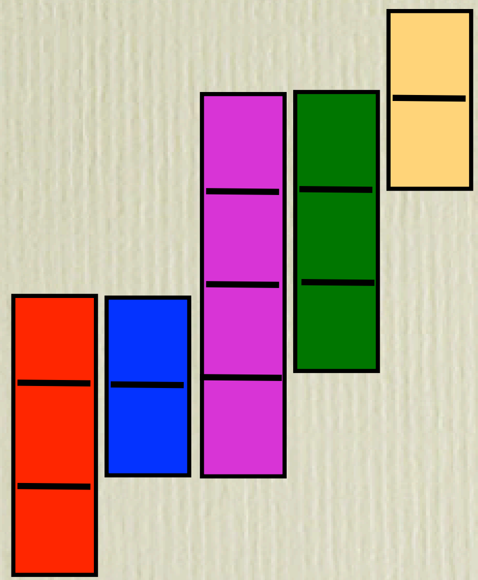
cours IMSc 2016 Ch2a, p110-116

bijection



staircase polygons

Dyck paths



$H_n(q)$

q-Polya

C_n Catalan

$H_1(q) = 1 + q$

$H_2(q) = 1 + 2q + 2q^2$

$H_3(q) = 1 + 3q + 5q^2 + 4q^3 + q^4$

$\cdots\cdots\cdots$

$$H(q,t) = \sum_{n \geq 1} H_n(q)\, t^n$$

ratio of q-Bessel

Delest, Fedou (1993)

→ see Ch 7

heaps and statiscal mechanics

q-Bessel functions in physics

affine Coxeter groups

Biagioli, Jouhet, Nadeau (2014, 2015)

" " " , Bousquet-Mélou (2016)

Hanusa, Jones (2010)

enumeration $\begin{cases} - \text{ number of FC elements} \\ - \text{ } q\text{-enumeration} \end{cases}$

$$W(x,q) = \sum_n W_n^{FC}(q)\, x^n$$

n for the family A_n, B_n, D_n, ...

q enumeration by the length $\ell(w)$ of FC elements

Proposition Nadeau (2015)

for any Coxeter group W,

$W^{FC}(q)$ is a rational power series

coeff. ultimately periodic

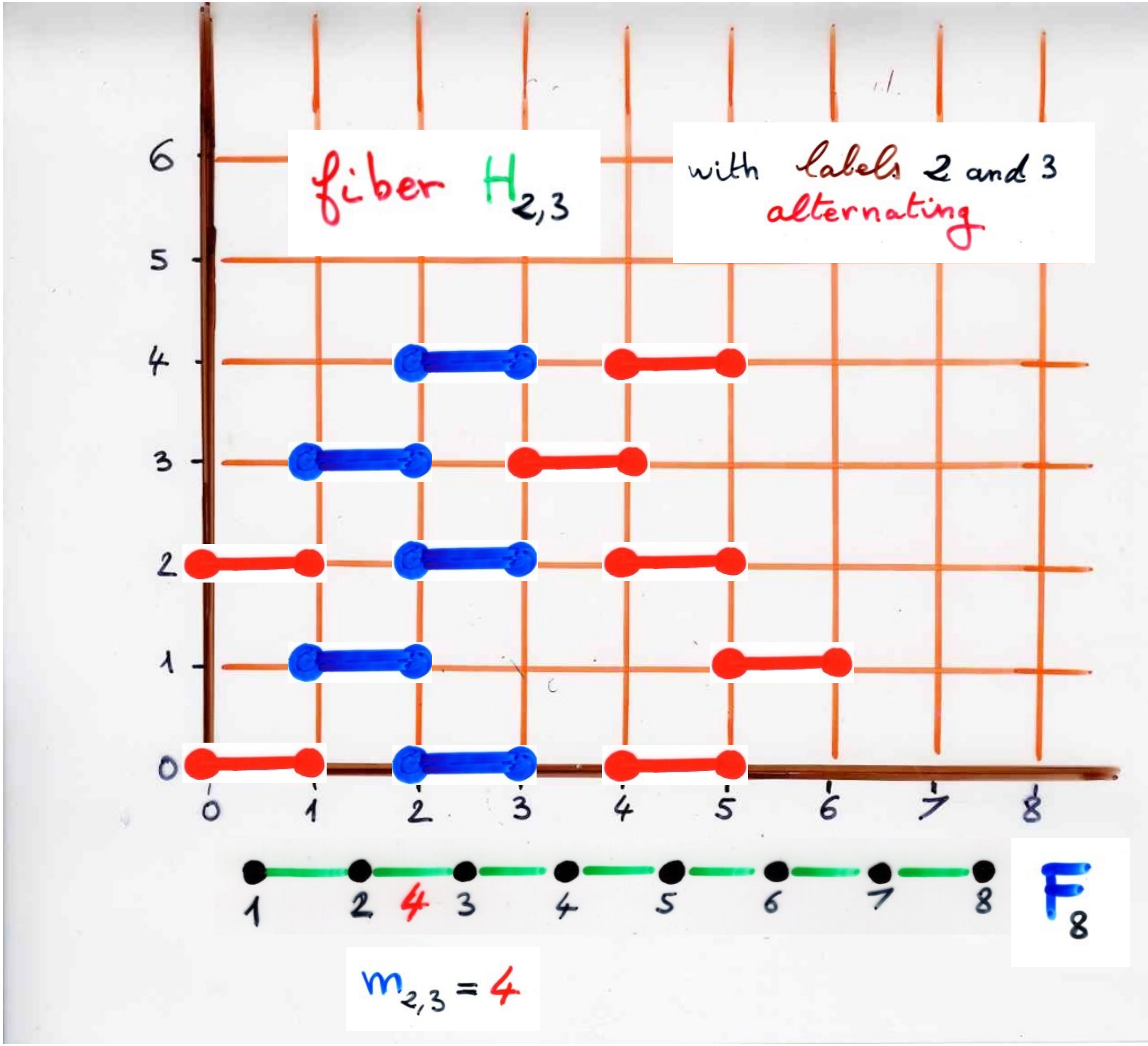
exercise

__Definition__ A heap **H** over the graph Γ is alternating iff for all s and t adjacent vertices of Γ, the ~~fiber $H_{s,t}$~~ the elements of the fiber $H_{s,t}$ over $\{s,t\}$ are alternatingly labeled s and t

[in (i)' $H_{s,t} = \pi^{-1}(\{s,t\})$ is a chain]

__exercise__ Prove the following characterization of FC heaps over  (A_n)

H is FC iff H is alternating and the fibers $H_1 = \pi^{-1}(1)$ and $H_n = \pi^{-1}(n)$ have at most one elements


fiber $H_{2,3}$
with labels 2 and 3
alternating
0
1
2
3
4
5
6
7
8
1
2
3
4
5
6
7
8
4
F_8
$m_{2,3} = 4$