Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics,
a bijective approach:
commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

Xavier Viennot

CNRS, LaBRI, Bordeaux

www.xavierviennot.org

Chapter 4

Heaps and linear algebra: bijective proofs of classical theorems

(3)

IMSc, Chennai

13 February 2017

combinatorial (bijective) proofs of classical theorem in **linear algebra**

- MacMahon "master theorem"
  Cartier-Foata (1969)
- Matrix inversion
  Foata (1979)
- Jacobi identity (log det)
  Jackson (1977)
  Foata (1980)

- Cayley-Hamilton theorem
  Straubing (1983)
  Zeilberger (1985)
- Jacobi identity (duality)
  Lalonde (1990, 1996)
  Fomin (2001), Talaska (2012)

Jacobi duality

$I = \{i_1, \dots, i_l\} \subseteq [1, k]$ $\quad \bar{I} = [1,k] - I$

$J = \{j_1, \dots, j_l\} \subseteq [1, k]$ $\quad \bar{J} = [1,k] - J$

complement

A $k \times k$ matrix

$\mathbb{1}$ unity $k \times k$ matrix

$s(I) = i_1 + \dots + i_l$

$s(J) = j_1 + \dots + j_l$

$$\det\left((\mathbb{1} - A)^{-1}[I, J]\right) = \frac{(-1)^{s(I)+s(J)} \det\left((\mathbb{1} - A)[\bar{J}, \bar{I}]\right)}{\det(\mathbb{1} - A)}$$

(main) Theorem

$$\det\left((\mathbb{1}-A)^{-1}[I,J]\right) =$$

$$\sum_{\sigma \in \mathfrak{S}_{I,J}} (-1)^{\mathrm{Inv}(\sigma)} \sum_{\substack{\eta_1 : i_1 \rightsquigarrow \sigma(i_1) \\ \vdots \\ \eta_\ell : i_\ell \rightsquigarrow \sigma(i_\ell)}} v(\eta_1) \cdots (\eta_\ell)\, v(E)$$

set of bijections $I \to J$

self-avoiding paths

pair-wise disjoint

E heap of cycles

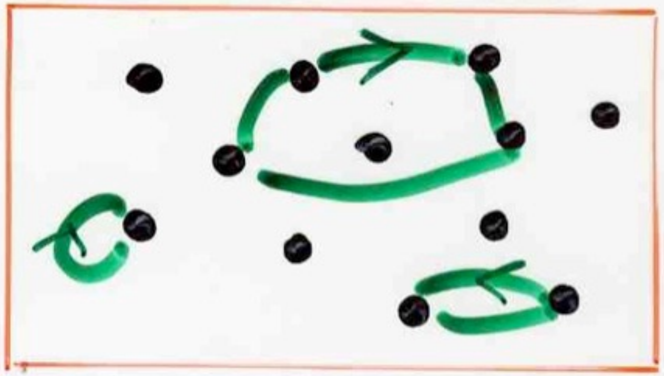
projection $\pi(m)$ maximal piece of E intersect one of the path η

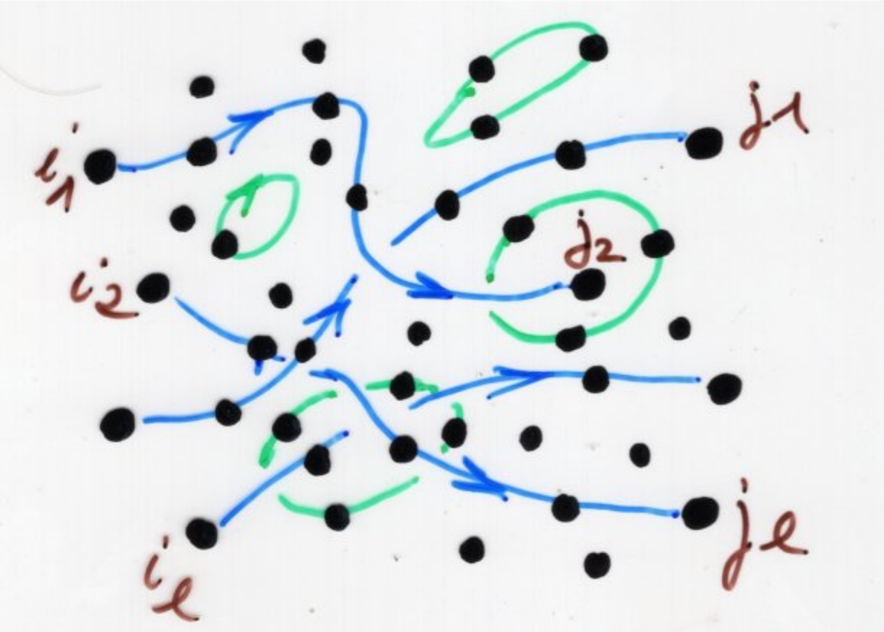
inversion lemma

$$\det\left((\mathbb{1}-A)^{-1}[I,J]\right) = \frac{N}{D}$$

$$\det(\mathbb{1}-A) = \sum_{\{\gamma_1,\dots,\gamma_r\}} (-1)^r v(\gamma_1)\dots v(\gamma_r)$$

2 by 2 disjoint cycles





$$N = \sum_{\substack{\sigma \in \mathfrak{S}_{I,J}}} (-1)^{\mathrm{Inv}(\sigma)} \sum (-1)^r \, v(\eta_1) \cdots v(\eta_\ell)\, v(\gamma_1) \cdots v(\gamma_r)$$

$\mathfrak{S}_{I,J}$ set of bijections $I \to J$

$\{\eta_1, \dots, \eta_\ell\}$ self-avoiding

$\{\gamma_1, \dots, \gamma_r\}$ cycles

$\eta_i : i_\ell \rightsquigarrow \sigma(i_\ell)$

pair-wise disjoint

special case 1

$$I = \{i\}$$
$$J = \{j\}$$

$$\det\left((\mathbb{1}-A)^{-1}[I,J]\right) = \sum_{\substack{\omega \\ i \rightsquigarrow j}} v(\omega)$$

path ω on X $\xleftrightarrow{\ \chi\ }$ (η, E)

$$\sum_{\substack{\omega \\ i \rightsquigarrow j}} v(\omega) = \sum_{\substack{\eta \\ i \rightsquigarrow j}} v(\eta) \sum_{E} v(E)$$

self-avoiding

heap of cycles $\pi(m)$ maximal piece intersecte η

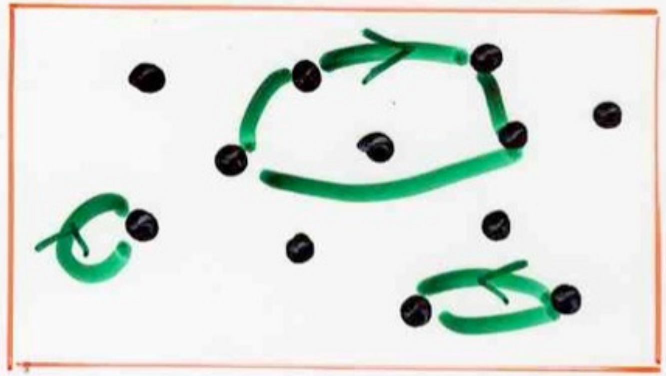
$$\frac{N_\eta}{D}$$

Prop. $\displaystyle\sum_{\substack{\omega \\ i \leadsto j}} v(\omega) = \frac{N_{ij}}{D}$ $\qquad N_{ij} = \displaystyle\sum_{\eta} v(\eta) N_\eta$

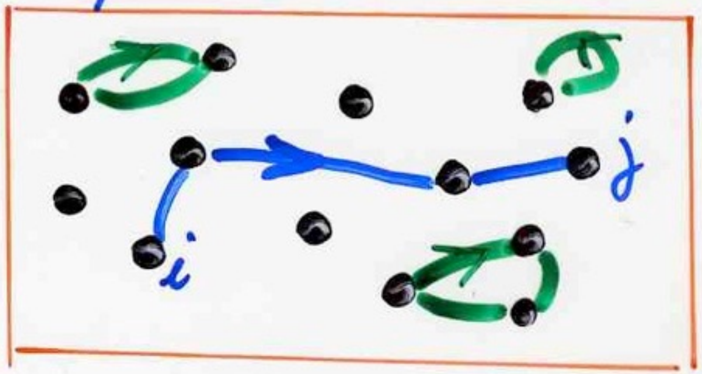
η self-avoiding path $i \leadsto j$

$$D = \sum_{\{\gamma_1, \ldots, \gamma_r\}} (-1)^r v(\gamma_1) \cdots v(\gamma_r)$$

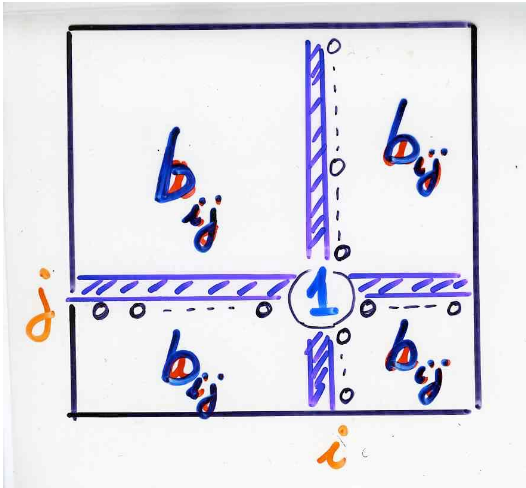
$\{\gamma_1, \ldots, \gamma_r\}$ 2 by 2 disjoint cycles



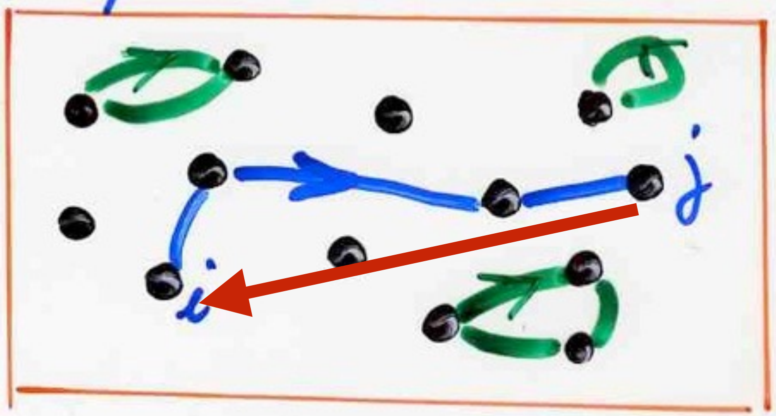
$$N_{ij} = \sum_{\{\eta; \gamma_1, \ldots, \gamma_r\}} (-1)^r v(\eta)\, v(\gamma_1) \cdots v(\gamma_r)$$



$\mathrm{cof}_{ji}(I_k - A)$



$$N_{ij} = \sum_{\{\eta\,;\,\gamma_1, \quad, \gamma_r\}} (-1)^r\, v(\eta)\, v(\gamma_1) \cdots v(\gamma_r)$$



η self-avoiding path $i \rightsquigarrow j$

$\{\gamma_1, \dots, \gamma_r\}$
2 by 2 disjoint cycles,
and disjoint from η

general case

$$\det\left((1-A)^{-1}[I,J]\right) = \frac{N}{\det(1-A)}$$

$$N = \sum_{\sigma \in \mathfrak{S}_{I,J}} (-1)^{\mathrm{Inv}(\sigma)} \sum_{\substack{\{\eta_1, \dots, \eta_\ell\} \text{ self-avoiding} \\ \{\gamma_1, \dots, \gamma_r\} \text{ cycles} \\ \eta_i : i_\ell \rightsquigarrow \sigma(i_\ell)}} (-1)^r \, v(\eta_1) \cdots v(\eta_\ell) \, v(\gamma_1) \cdots v(\gamma_r)$$

$\mathfrak{S}_{I,J}$ set of bijections $I \to J$

pair-wise disjoint

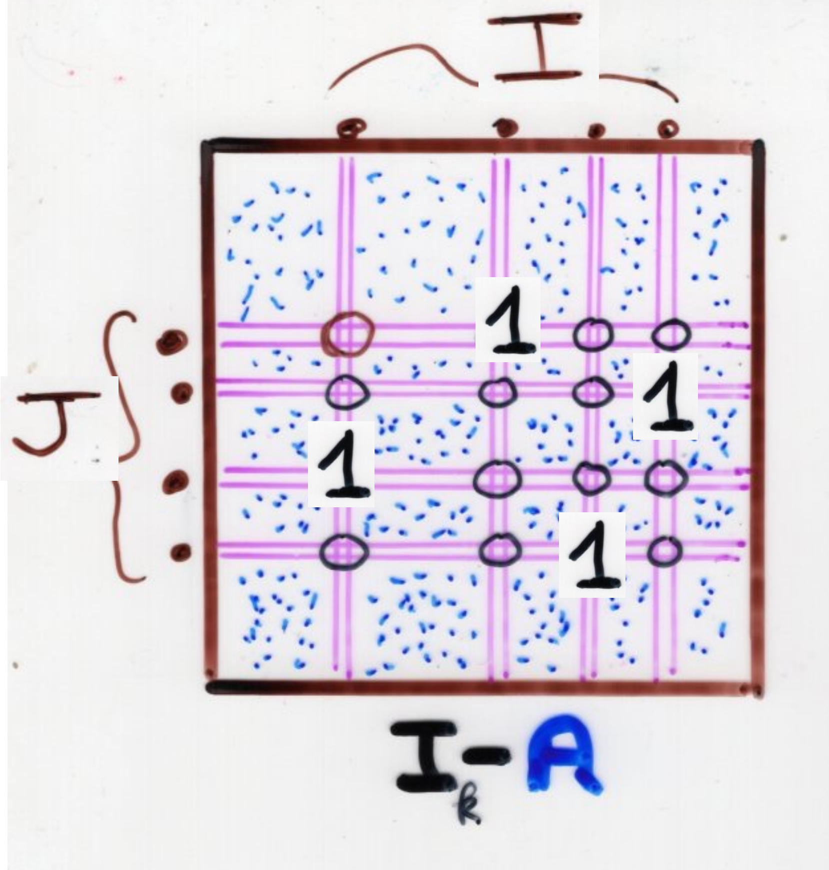
Lemma

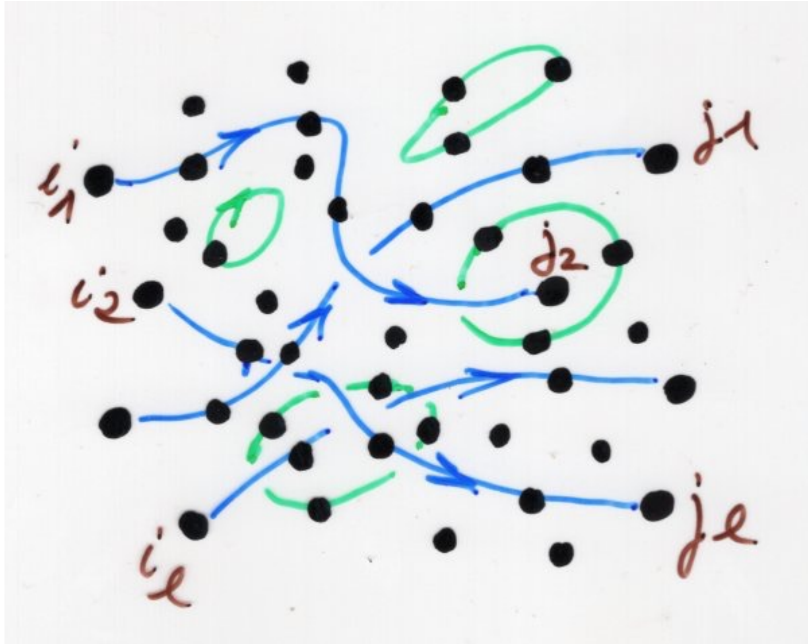
$$(-1)^{s(I)+s(J)} \det\Big((\mathbb{1}-A)[\bar{J},\bar{I}] \Big)$$

$$= \sum_{\substack{\sigma \in \mathfrak{S}_{I,J}}} (-1)^{\mathrm{Inv}(\sigma)} \sum_{\substack{\{\eta_1,\dots\eta_\ell\} \text{ self-avoiding} \\ \{\gamma_1,\dots,\gamma_r\} \text{ cycles} \\ \eta_i : i_\ell \rightsquigarrow \sigma(i_\ell)}} (-1)^r \, v(\eta_1)\cdots v(\eta_\ell)\, v(\gamma_1)\cdots v(\gamma_r)$$

set of bijections $I \to J$

pair-wise disjoint





$$(-1)^{s(I)+s(J)}\,\det\big((1-A)[J,I]\big)$$

$$N = \sum_{\sigma\in\mathfrak{S}_{I,J}} (-1)^{\mathrm{Inv}(\sigma)} \sum (-1)^r\, v(\eta_1)\cdots v(\eta_\ell)\, v(\gamma_1)\cdots v(\gamma_r)$$

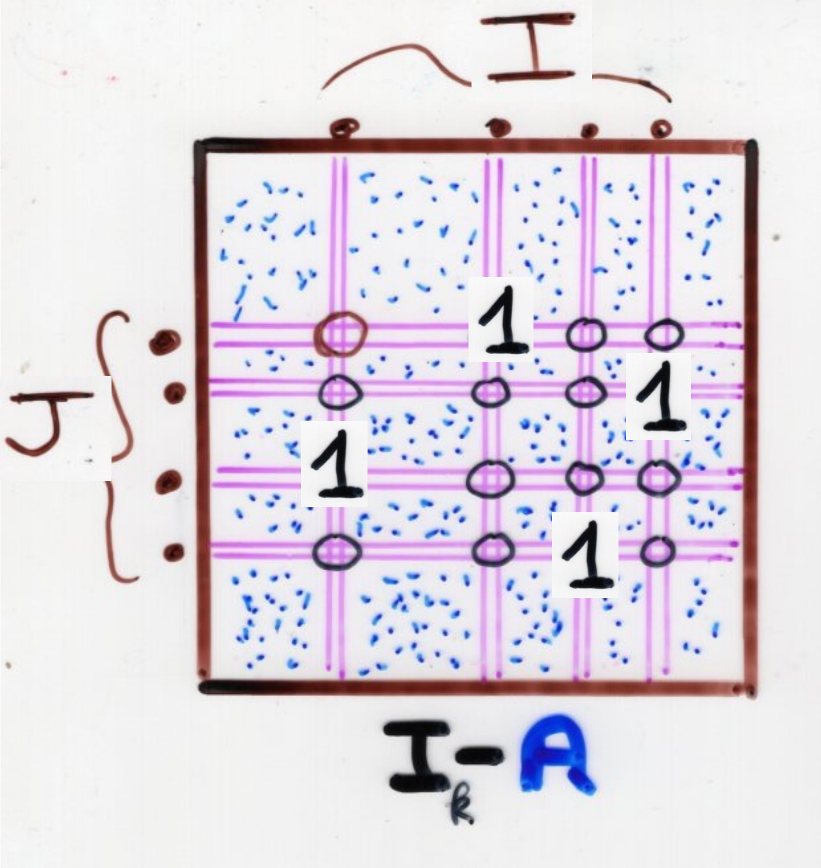
$\mathfrak{S}_{I,J}$: set of bijections $I\to J$

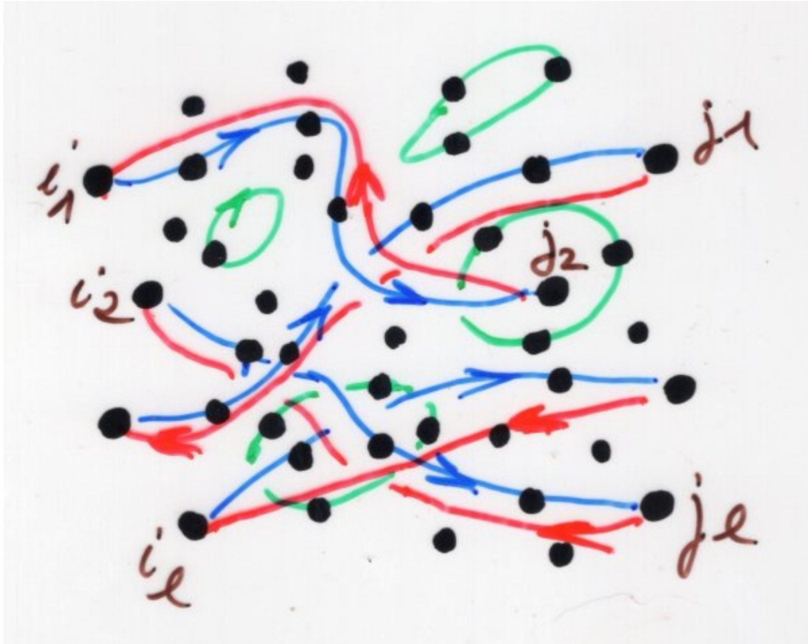
$\{\eta_1,\dots,\eta_\ell\}$ self-avoiding

$\{\gamma_1,\dots,\gamma_r\}$ cycles

$\eta_i : i_\ell \rightsquigarrow \sigma(i_\ell)$

pair-wise disjoint





$$(-1)^{s(I)+s(J)}\, \mathbf{det}\left((1-A)\,[\bar{J}, \bar{I}] \right)$$

$$N = \sum_{\sigma \in \mathfrak{S}_{J,I}} (-1)^{\mathrm{Inv}(\sigma)} \sum (-1)^r\, v(\eta_1) \cdots v(\eta_\ell)\, v(\gamma_1) \cdots v(\gamma_r)$$

set of bijections $J \to I$

$\{\eta_1, \dots, \eta_\ell\}$ self-avoiding

$\{\gamma_1, \dots, \gamma_r\}$ cycles

$\eta_i : i_\ell \rightsquigarrow \sigma(i_\ell)$

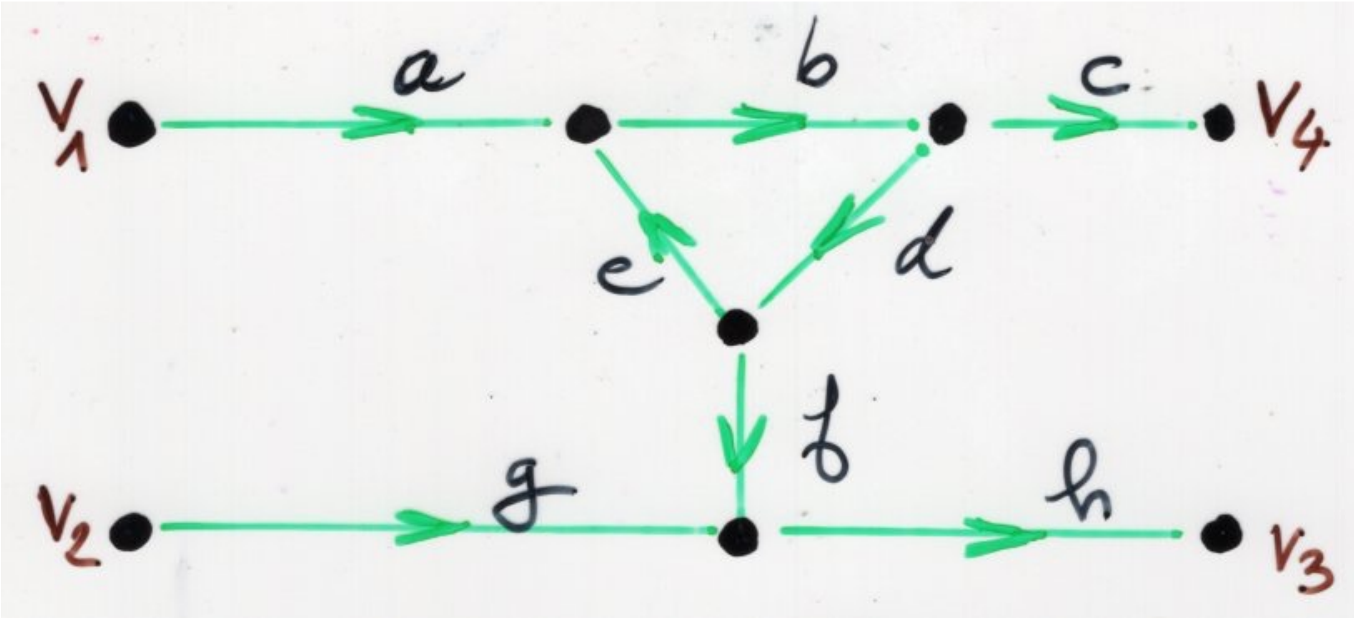
pair-wise disjoint

$$\det\left((\mathbb{1}-A)^{-1}[I,J]\right) = \frac{N}{D}$$

Jacobi duality

$$= \frac{(-1)^{s(I)+s(J)} \det\left((\mathbb{1}-A)[\bar{J},\bar{I}]\right)}{\det(\mathbb{1}-A)}$$

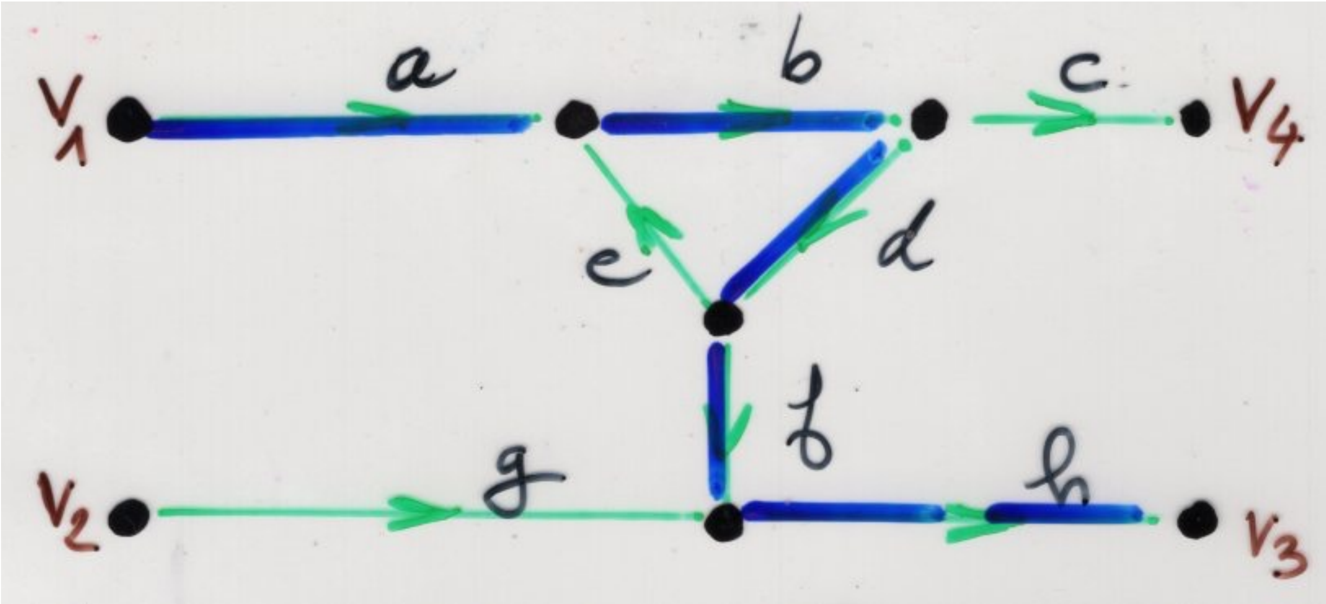
example



$$(I-A)^{-1}[I,J] = \begin{bmatrix} \dfrac{abdfh}{1-bde} & \dfrac{abc}{1-bde} \\ gh & 0 \end{bmatrix}$$

$I = \{1,2\}$

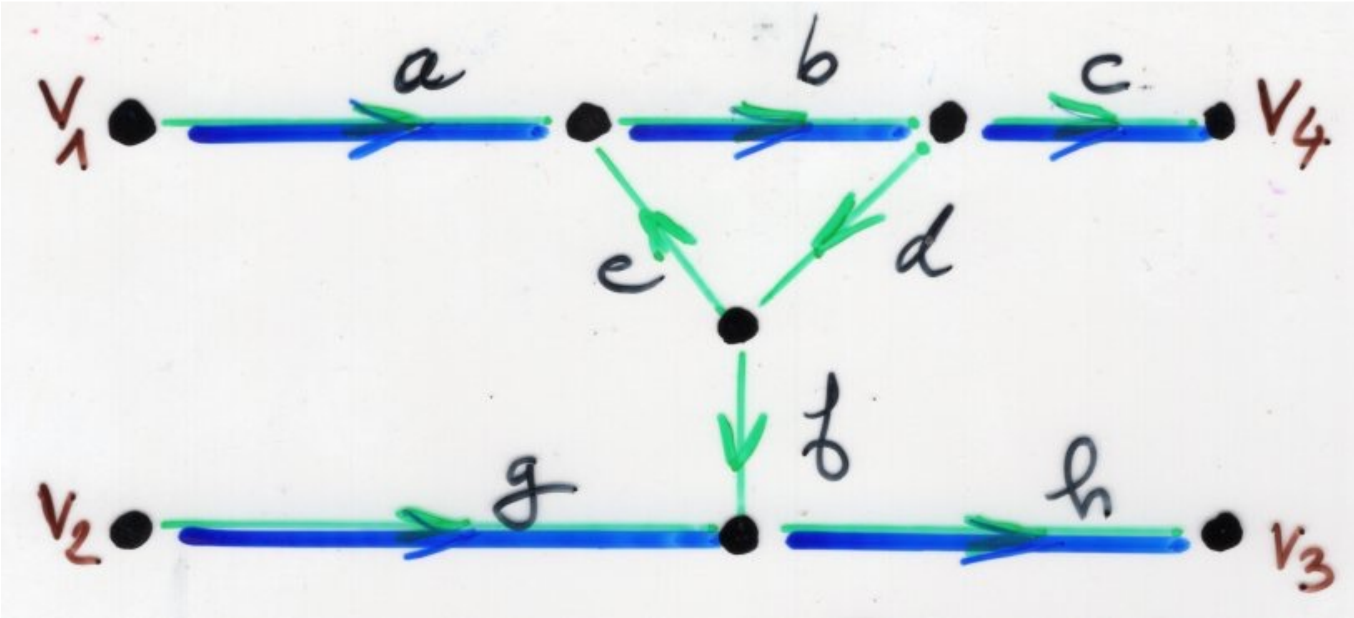
$J = \{3,4\}$



$$(I-A)^{-1}[I,J] = \begin{bmatrix} \dfrac{abdfh}{1-bde} & \dfrac{abc}{1-bde} \\ gh & 0 \end{bmatrix}$$

$I = \{1,2\}$

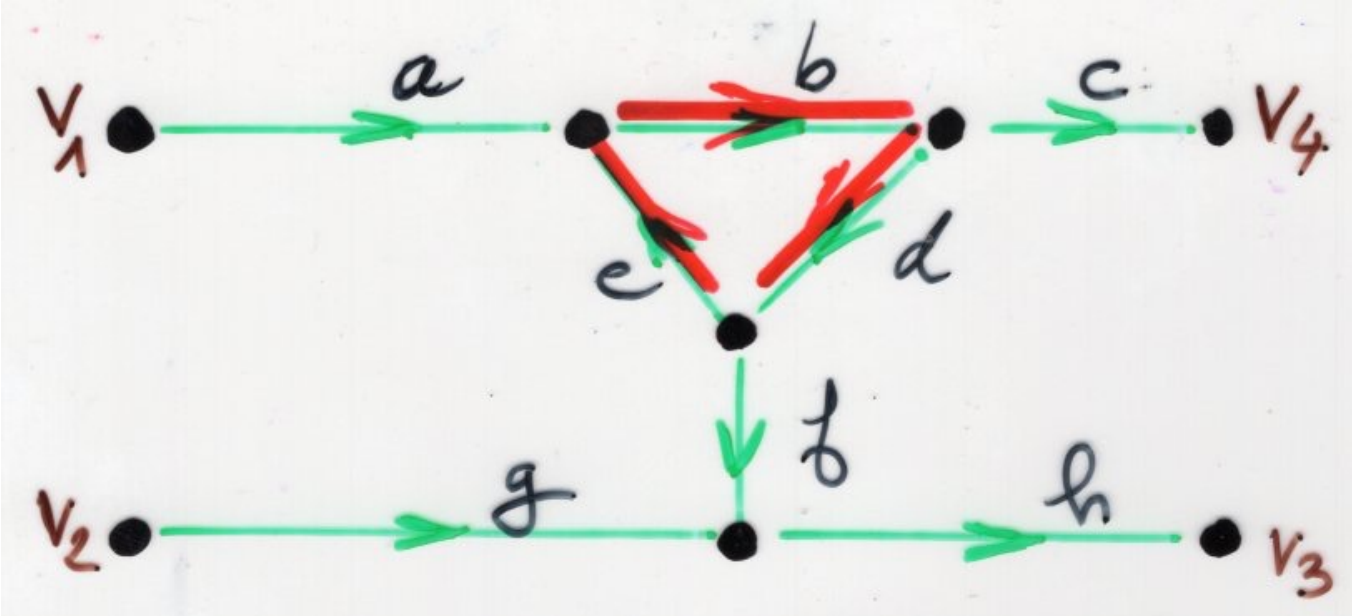
$J = \{3,4\}$



$$(I - A)^{-1}[I, J] = \begin{bmatrix} \dfrac{abdfh}{1-bde} & \dfrac{abc}{1-bde} \\ gh & 0 \end{bmatrix}$$

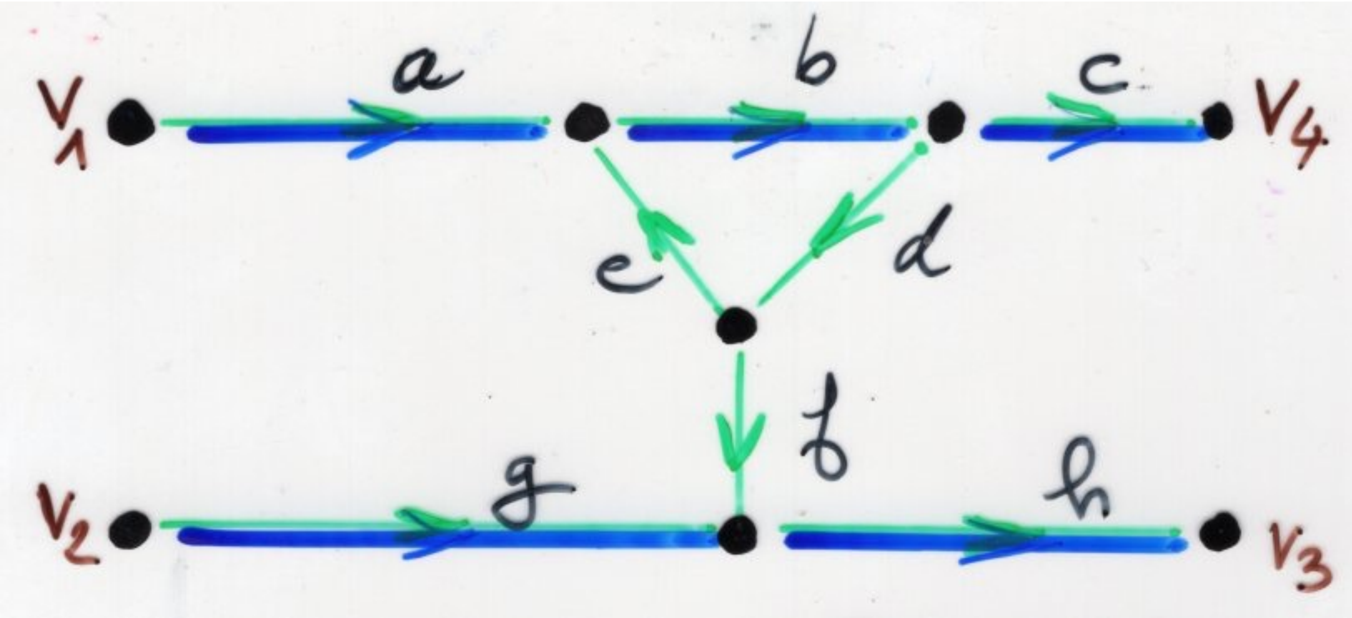
$I = \{1, 2\}$

$J = \{3, 4\}$



$$(I - A)^{-1}[I, J] = \begin{bmatrix} \dfrac{abdfh}{1-bde} & \dfrac{abc}{1-bde} \\ gh & 0 \end{bmatrix}$$

$I = \{1, 2\}$
$J = \{3, 4\}$



$$\det\left((I-A)^{-1}[I,J]\right) = \frac{-abcgh}{1-bde}$$

special case 2

acyclic graph:
no cycles

$$\det\left((\mathbb{1}-A)^{-1}[I,J]\right) =$$

$$\sum_{\sigma \in \mathfrak{S}_{I,J}} (-1)^{\mathrm{Inv}(\sigma)} \sum_{\substack{\eta_1 : i_1 \rightsquigarrow \sigma(i_1) \\ \vdots \\ \eta_l : i_l \rightsquigarrow \sigma(i_l)}} v(\eta_1) \cdots (\eta_l)\, \cancel{v(\Xi)}$$

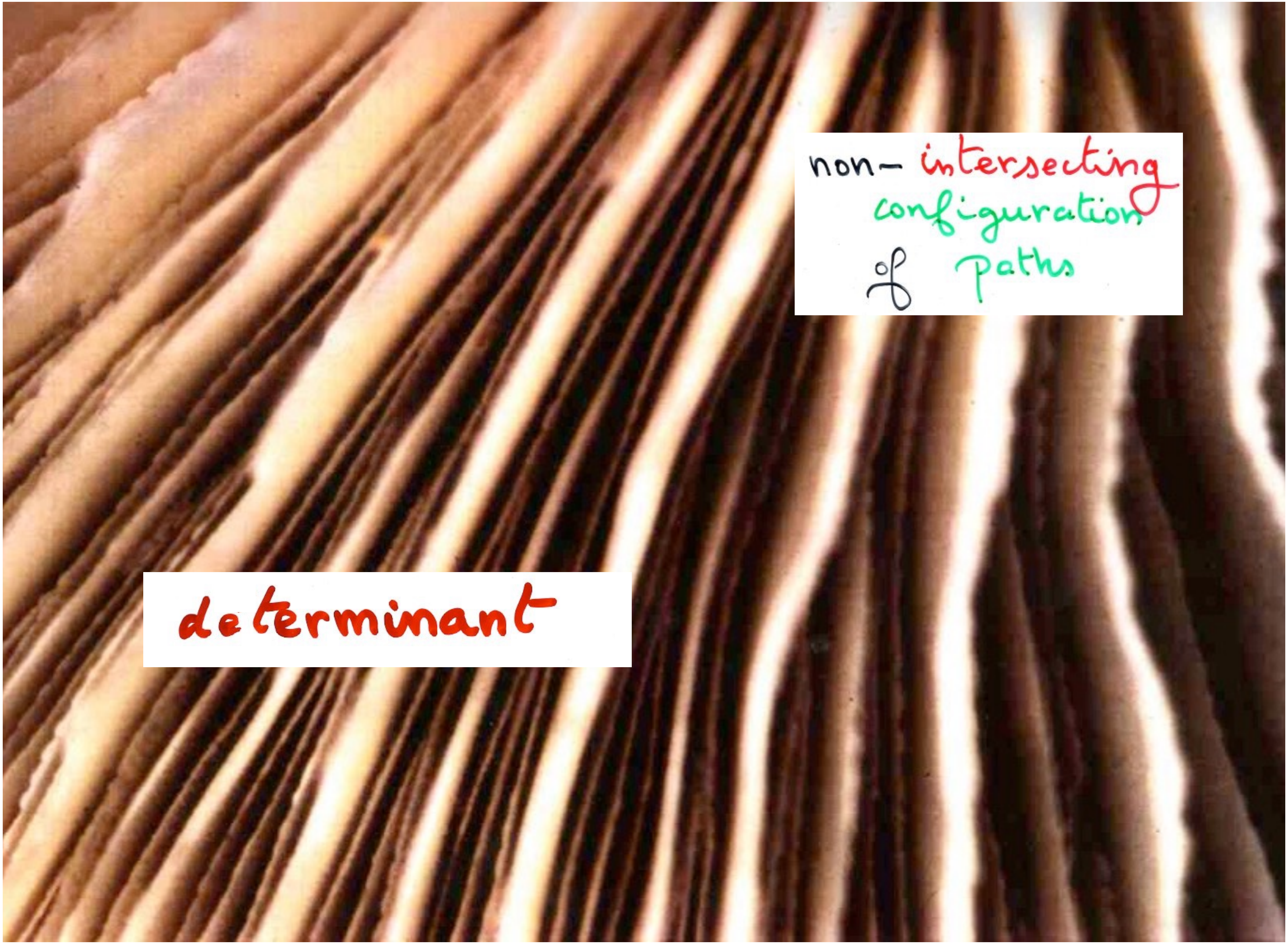
set of bijections $I \to J$

self-avoiding paths

pair-wise disjoint

The LGV Lemma

(from the course 2016, Ch 5a)

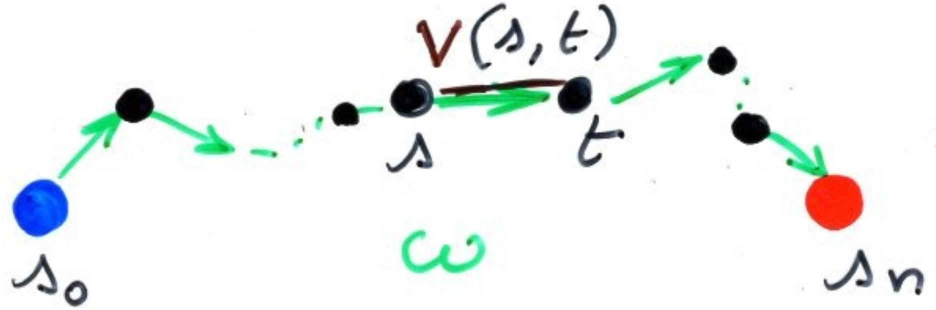

non-intersecting
configuration
of paths
determinant

Path $\quad \omega = (s_0, s_1, \ldots, s_n) \quad s_i \in S$

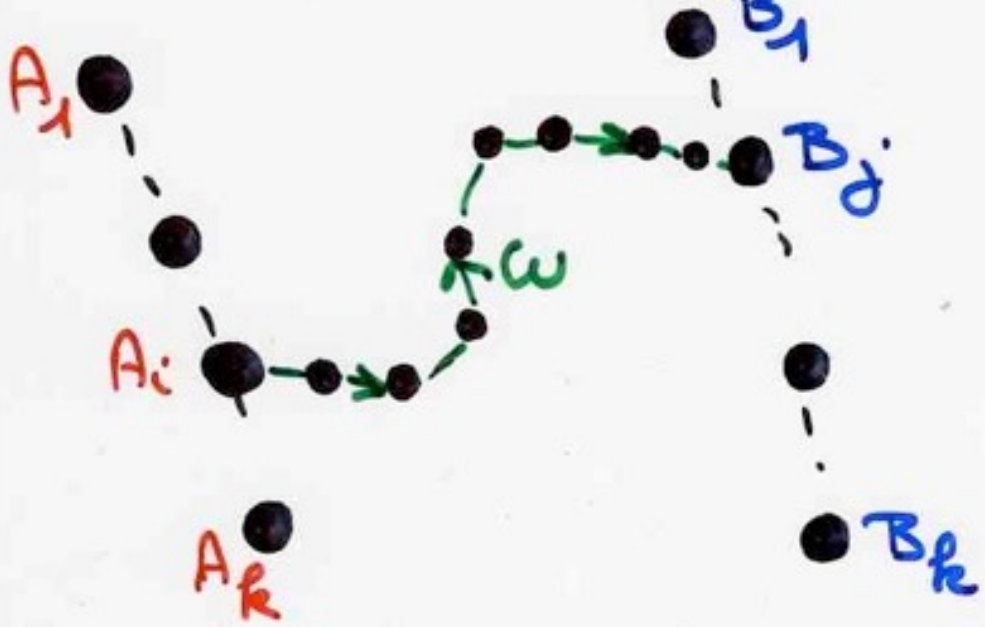
<u>notation</u> $\quad s_0 \overset{\omega}{\rightsquigarrow} s_n$

valuation $\quad v : S \times S \longrightarrow \mathbb{K}$ commutative ring

$$v(\omega) = v(s_0, s_1) \cdots v(s_{n-1}, s_n)$$



weighted path



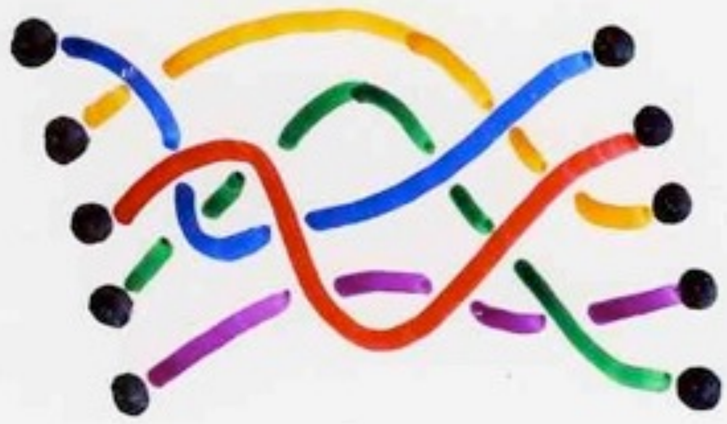
$A_1, \dots, A_k$
$B_1, \dots, B_k$

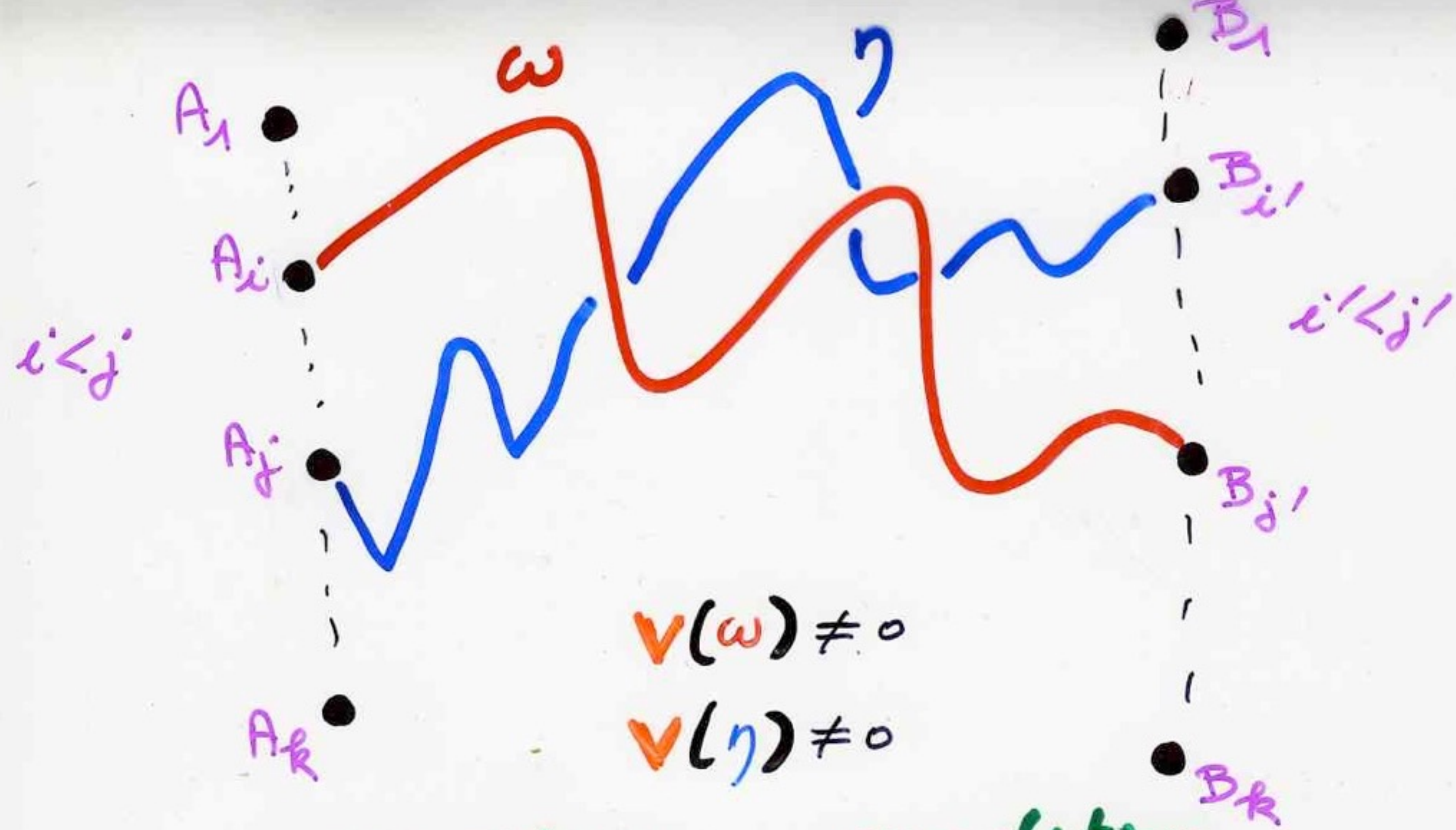
$$a_{ij} = \sum_{A_i \rightsquigarrow B_j} v(\omega)$$

suppose finite sum

$$\det(a_{ij}) = \sum_{(\sigma;\,\omega_1,\dots,\omega_k)} (-1)^{\mathrm{inv}(\sigma)}\, v(\omega_1)\dots(\omega_k)$$

$$\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$$





crossing condition

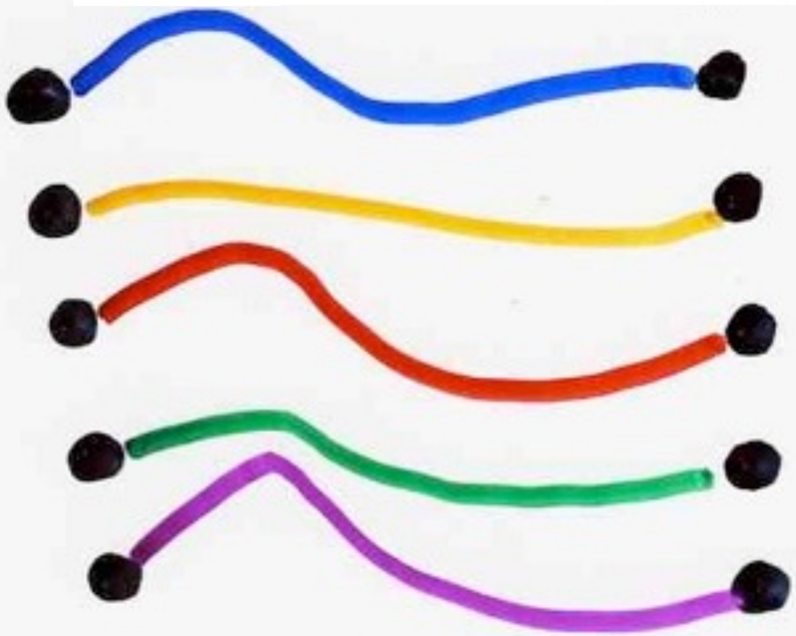
(C)

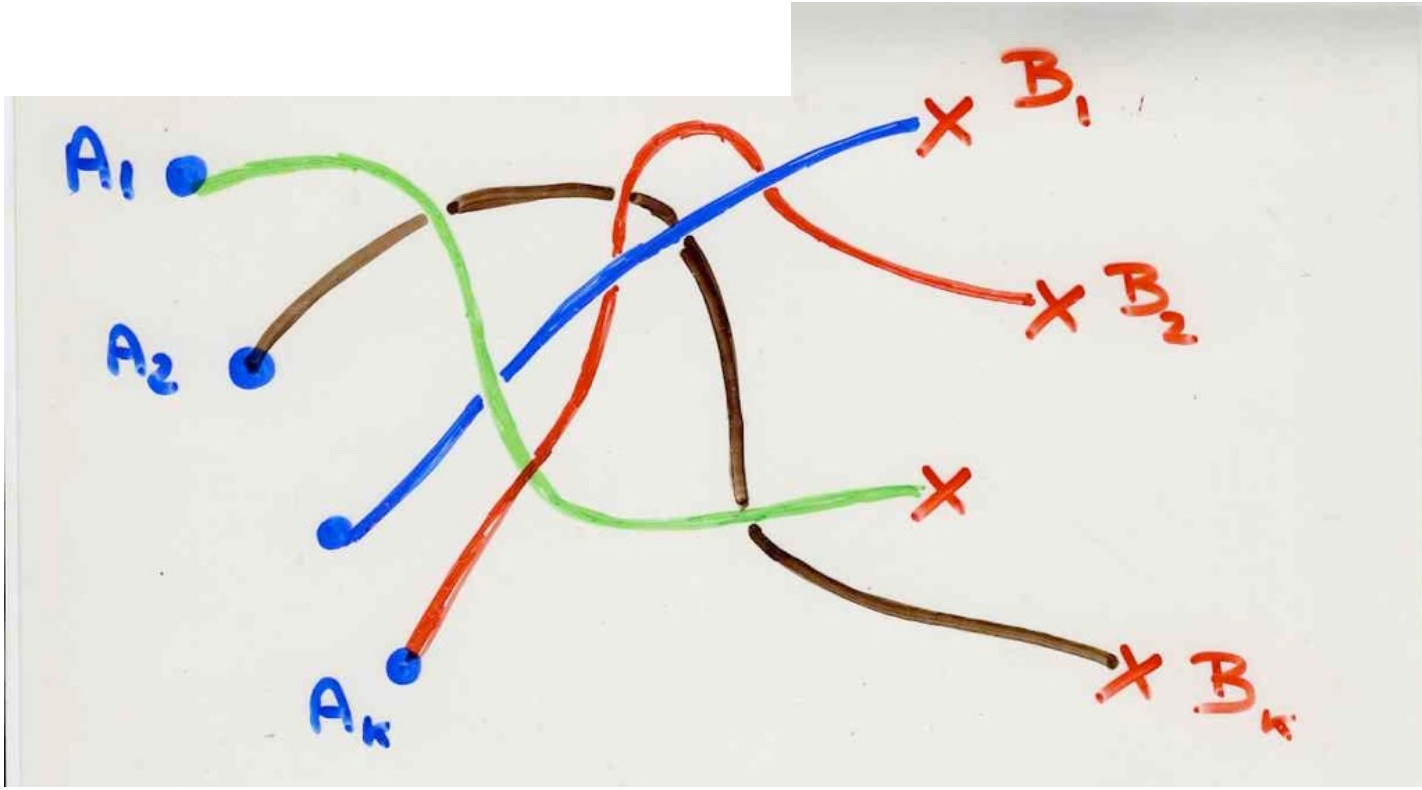
Proposition (LGV Lemma) (C) crossing condition

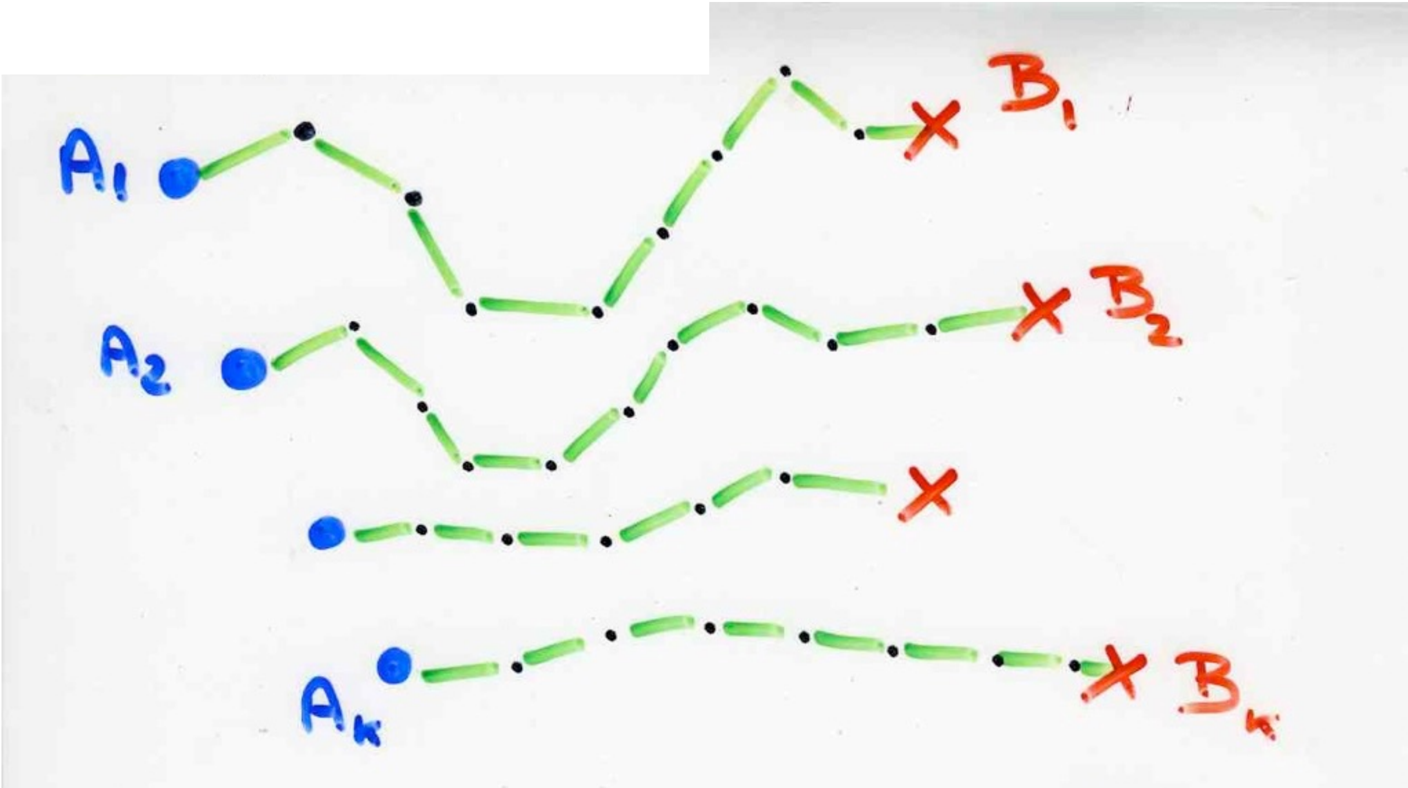
$$\det(a_{ij}) = \sum_{(\omega_1, \ldots, \omega_k)} v(\omega_1) \ldots (\omega_k)$$

$\omega_i : A_i \rightsquigarrow B_i$

non-intersecting

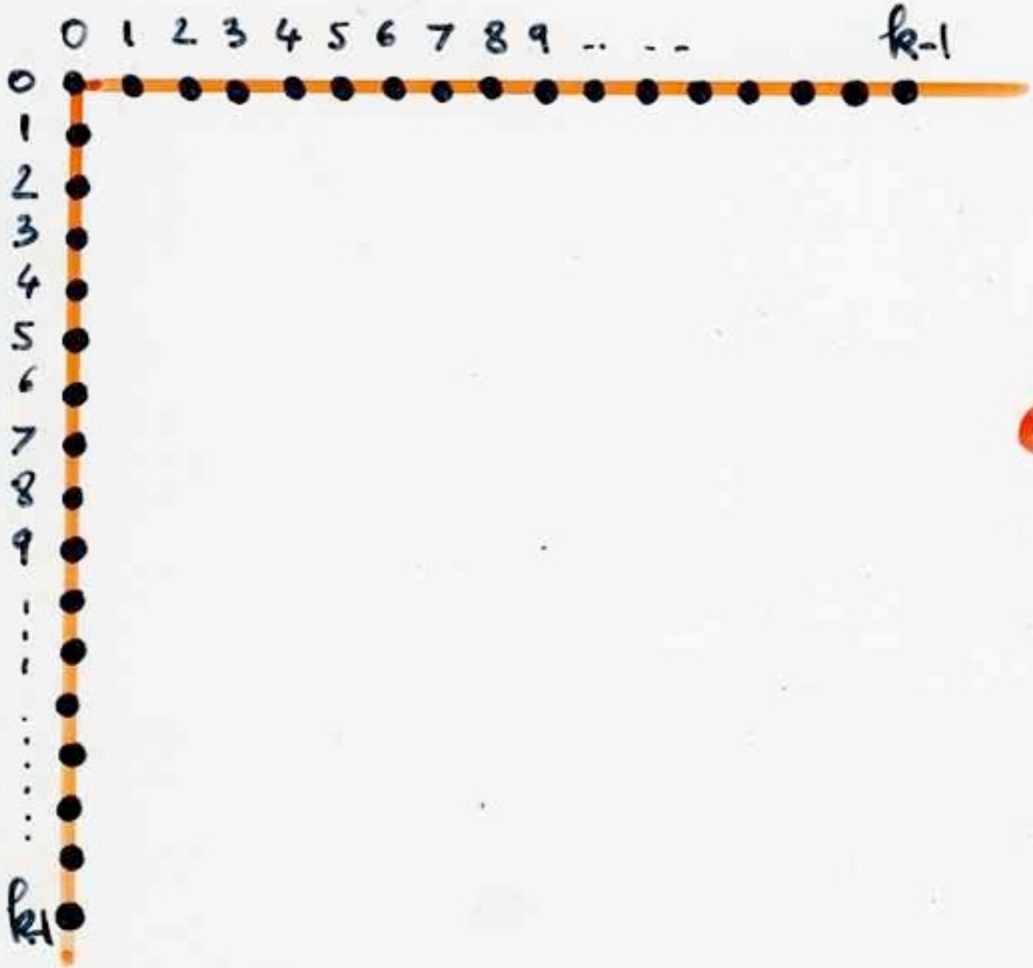



A_1
A_2
A_k
B_1
B_2
B_k

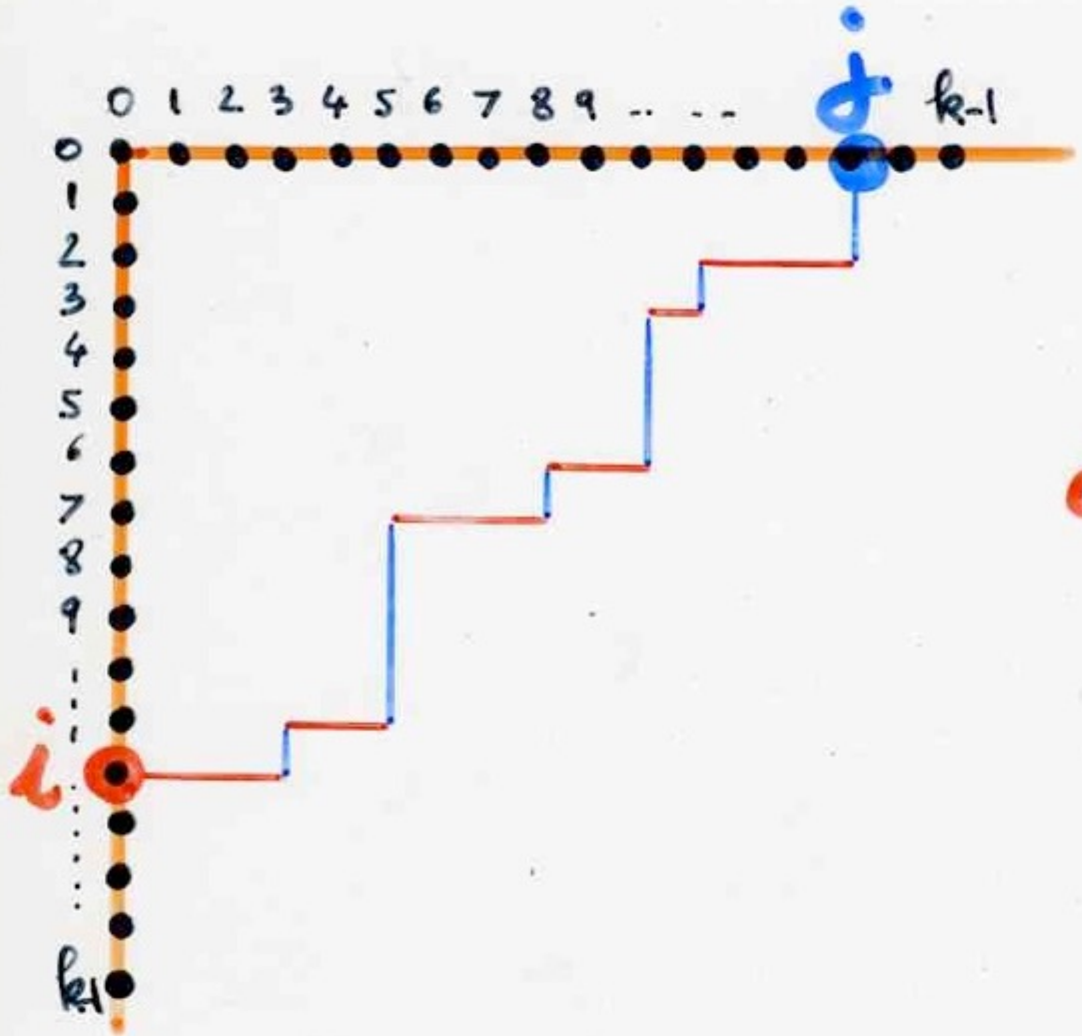


A1
B1
A2
B2
Ak
Bk

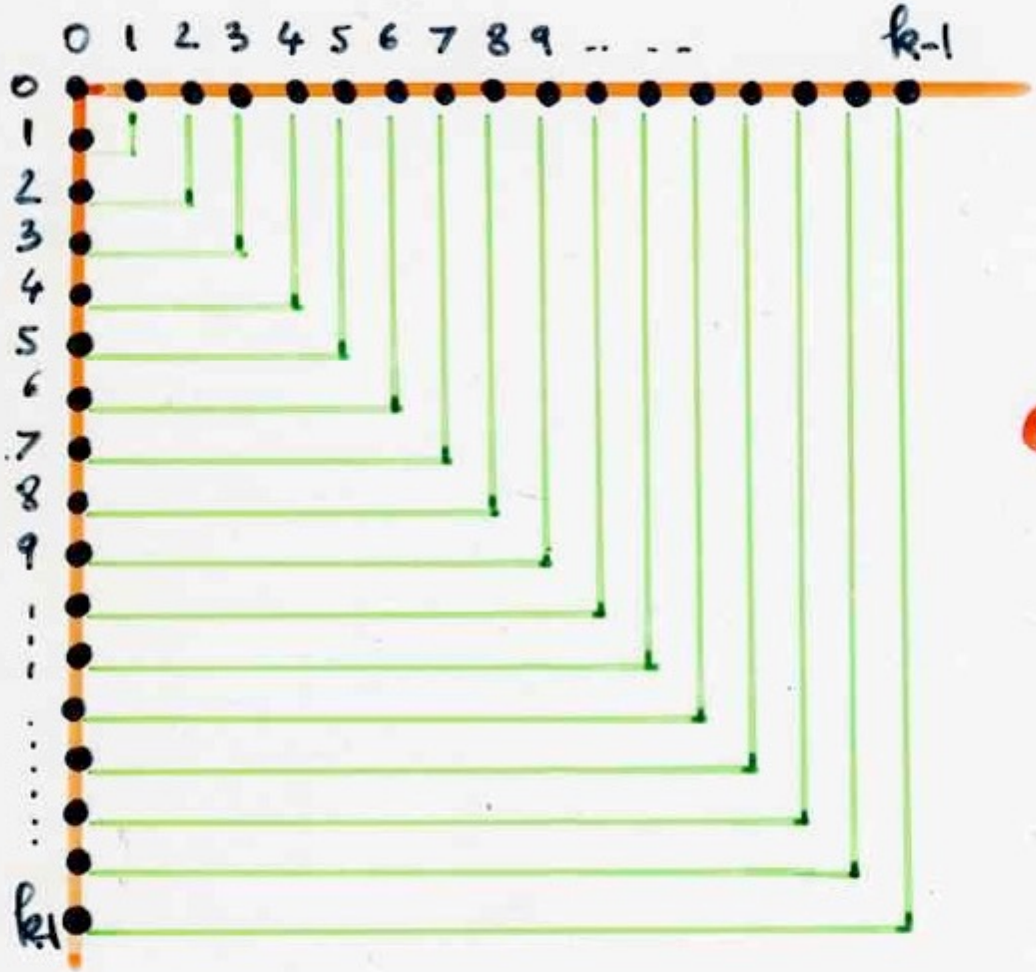
a simple example



$$\det\begin{bmatrix} 1&1&1&1&1&1&\cdot\\ 1&2&3&4&5&\cdot&\\ 1&3&6&10&\cdot&&\\ 1&4&10&\cdot&&&\\ 1&5&\cdot&&\binom{i+j}{i}&&\\ 1&\cdot&&&&&\\ &&&&&& \end{bmatrix}_{k\times k} =$$



$$\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdot^{\cdot^{\cdot}} \\ 1 & 2 & 3 & 4 & 5 & \cdot^{\cdot^{\cdot}} & \\ 1 & 3 & 6 & 10 & \cdot^{\cdot^{\cdot}} & & \\ 1 & 4 & 10 & \cdot^{\cdot^{\cdot}} & & & \\ 1 & 5 & \cdot^{\cdot^{\cdot}} & & \binom{i+j}{i} & & \\ 1 & \cdot^{\cdot^{\cdot}} & & & & & \\ & & & & & & \end{bmatrix}_{k\times k} =$$



$$\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & & \\ 1 & 3 & 6 & 10 & & & \\ 1 & 4 & 10 & & & & \\ 1 & 5 & & & \binom{i+j}{i} & & \\ 1 & & & & & & \\ \vdots & & & & & & \end{bmatrix}_{k\times k} = 1$$

another example

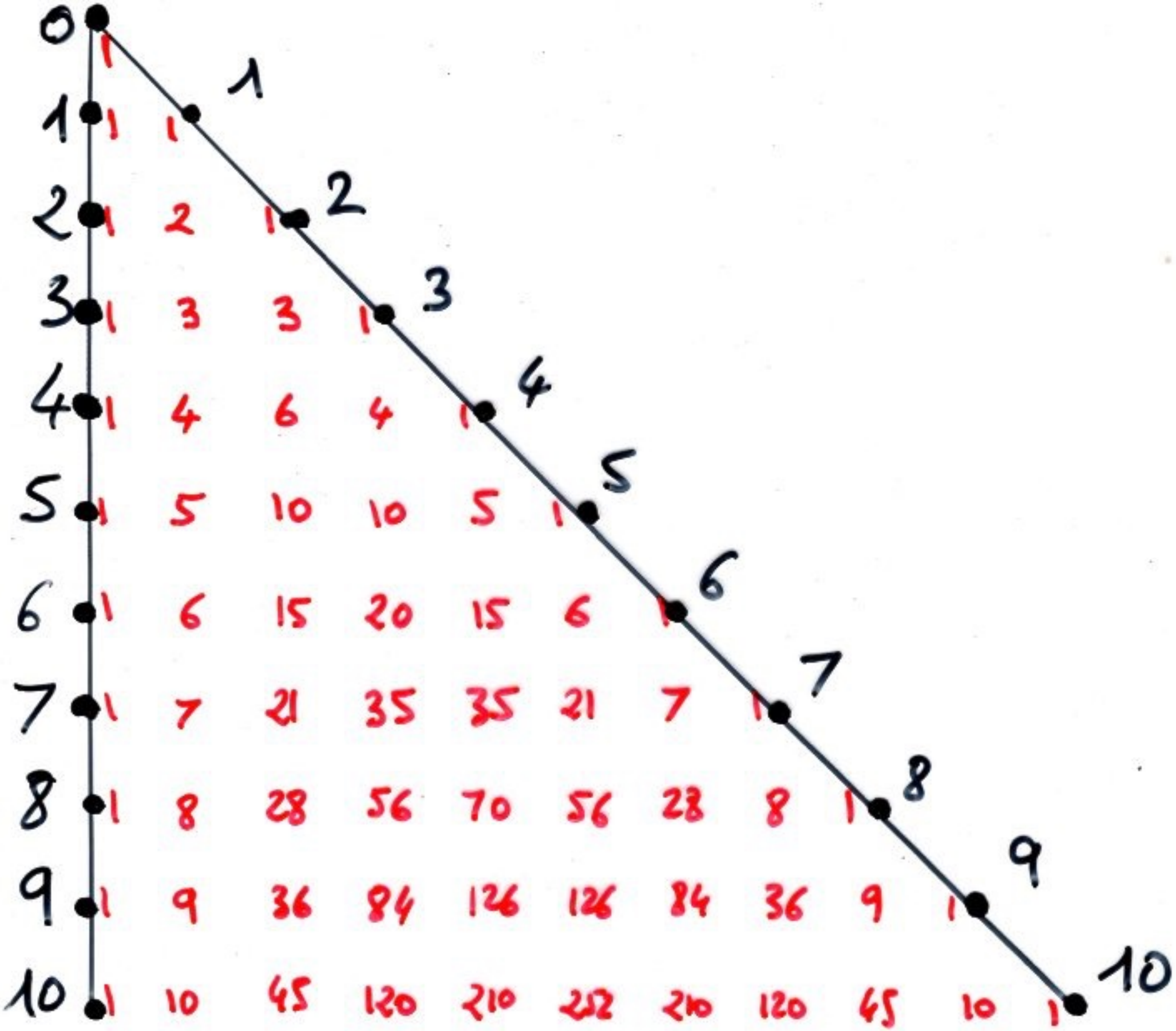
Binomial determinants

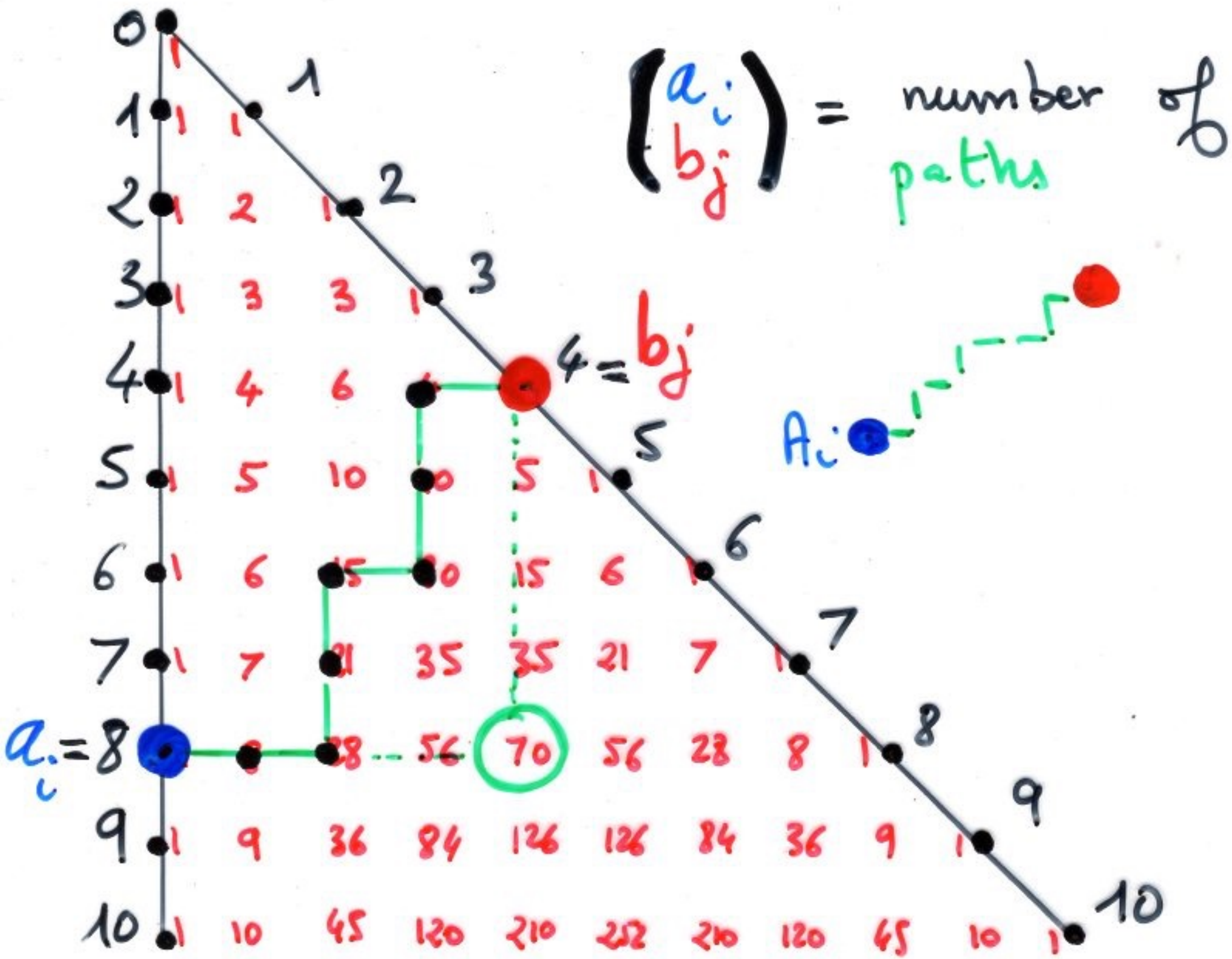
$$0 \leq a_1 < \cdots < a_k$$

$$0 \leq b_1 < \cdots < b_k$$

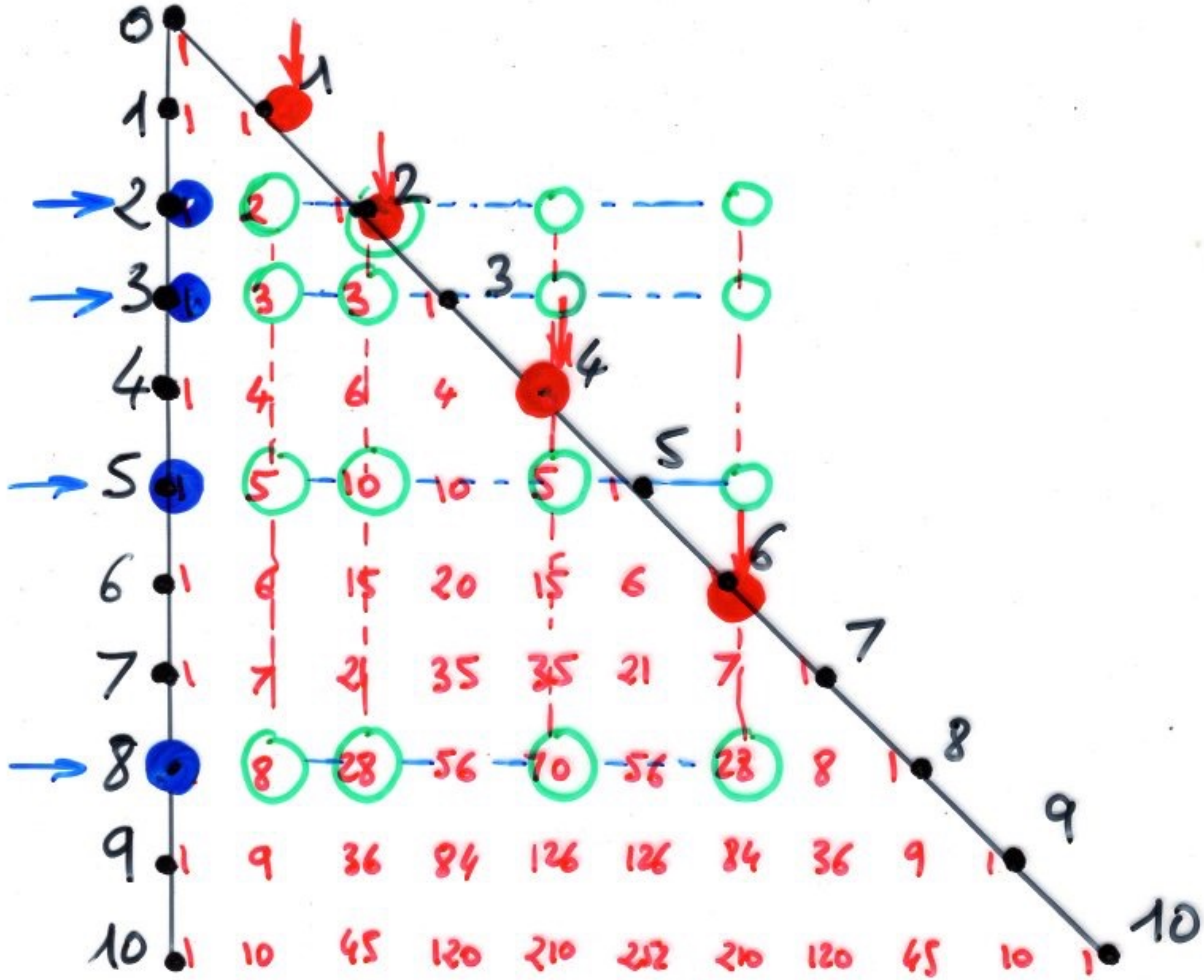
$$\binom{a_1, \ldots, a_k}{b_1, \ldots, b_k}$$

$$= \text{dét}\left(\binom{a_i}{b_j} \right)_{1 \leq i, j \leq k}$$


0 1 0
1 1 1 1
2 1 2 1 2
3 1 3 3 1 3
4 1 4 6 4 1 4
5 1 5 10 10 5 1 5
6 1 6 15 20 15 6 1 6
7 1 7 21 35 35 21 7 1 7
8 1 8 28 56 70 56 28 8 1 8
9 1 9 36 84 126 126 84 36 9 1 9
10 1 10 45 120 210 252 210 120 45 10 1 10

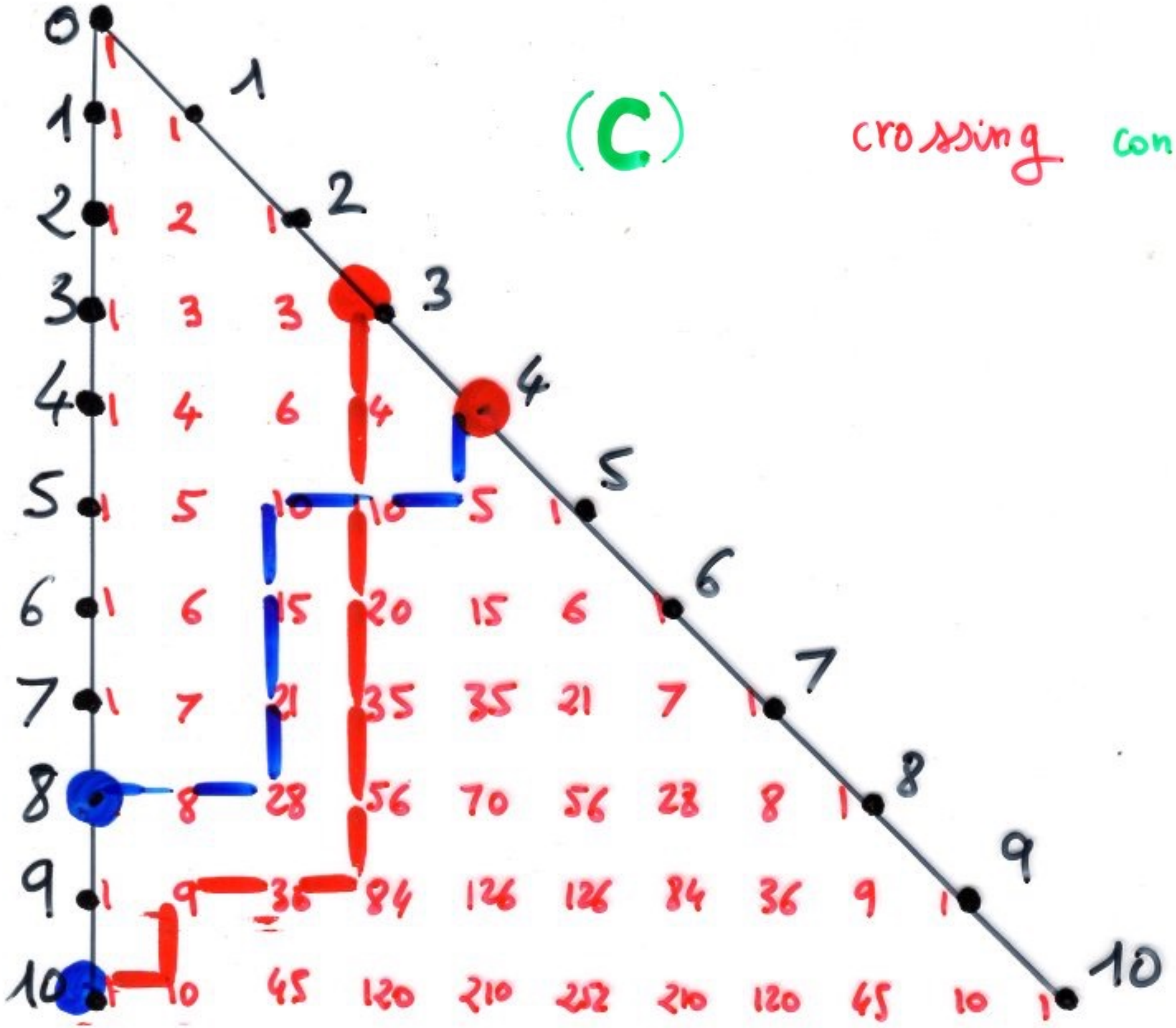


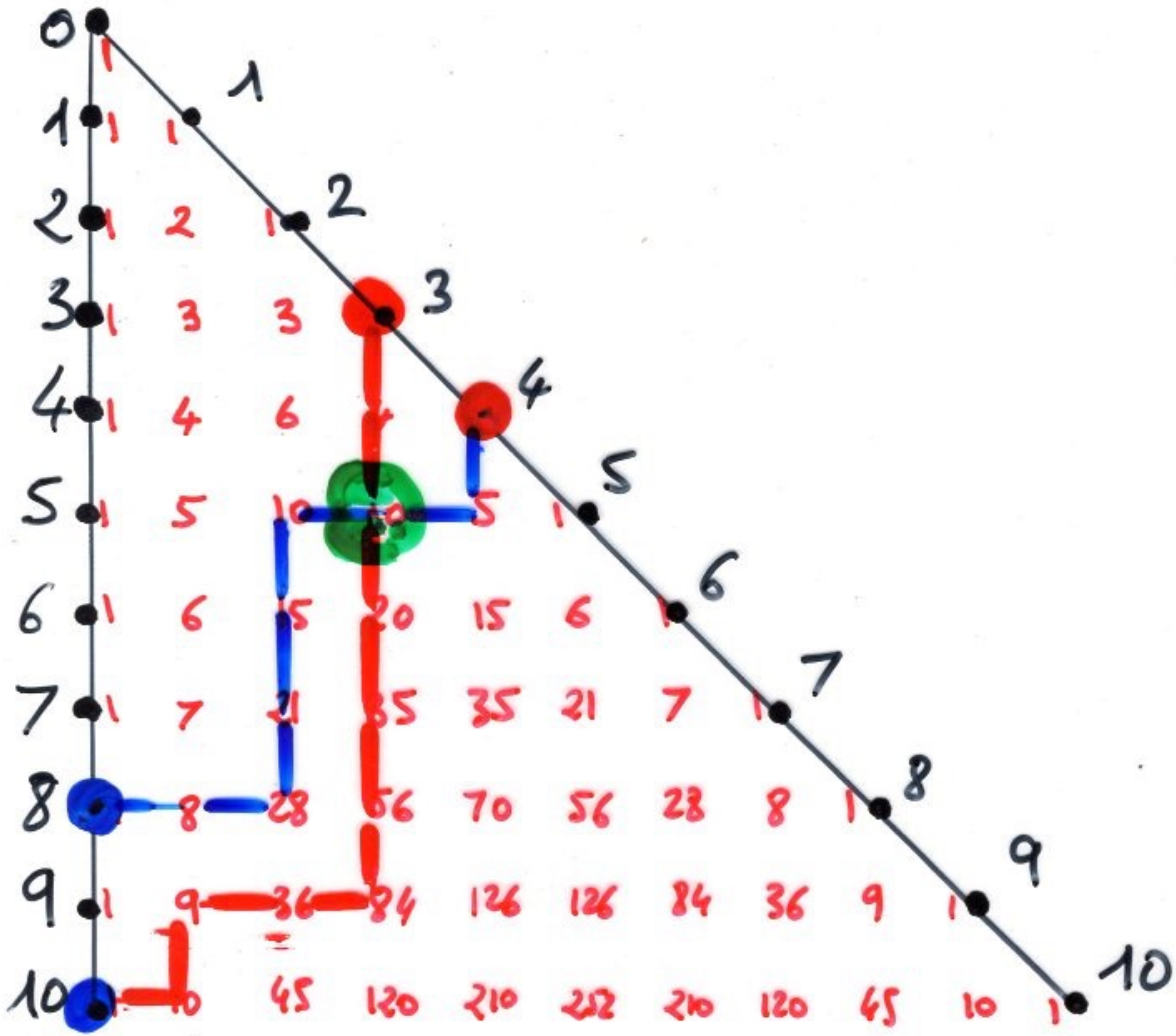
$\binom{a_i}{b_j}$ = number of paths
$4 = b_j$
A_i
$a_i = 8$
0
1
2
3
4
5
6
7
8
9
10



0
1
2
3
4
5
6
7
8
9
10
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

(C) crossing condition





0
1
2
3
4
5
6
7
8
9
10
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

<u>Proposition</u> The binomial determinant

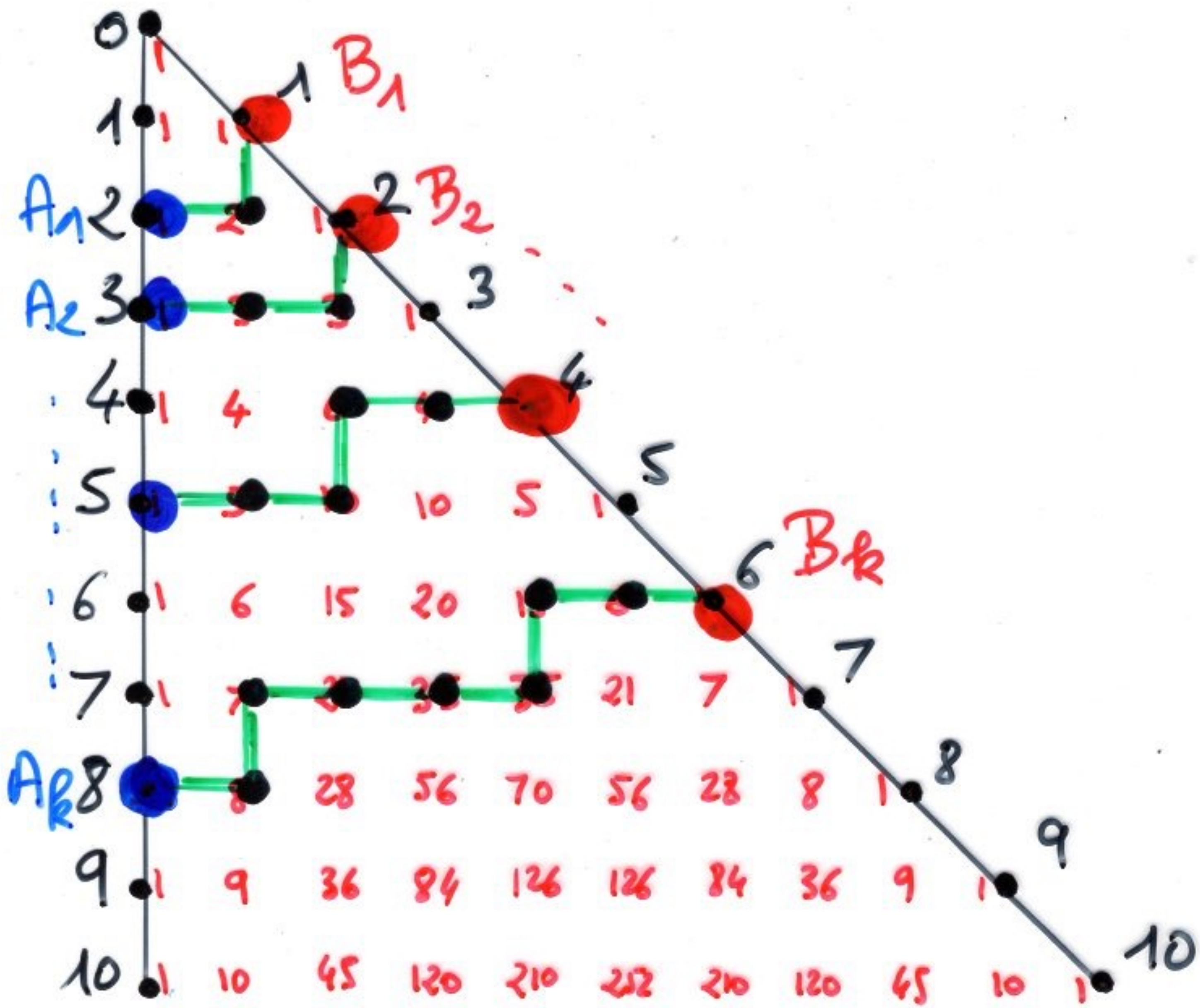
$\begin{pmatrix} a_1, \dots, a_k \\ b_1, \dots, b_k \end{pmatrix}$ is the number of

configurations of non-intersecting paths

$(\omega_1, \dots, \omega_k)$, $\omega_i : A_i \rightsquigarrow B_j$,

$A_i = (0, a_i)$, $B_j = (b_j, b_j)$

with elementary steps $\uparrow N$, $\xrightarrow{E}$

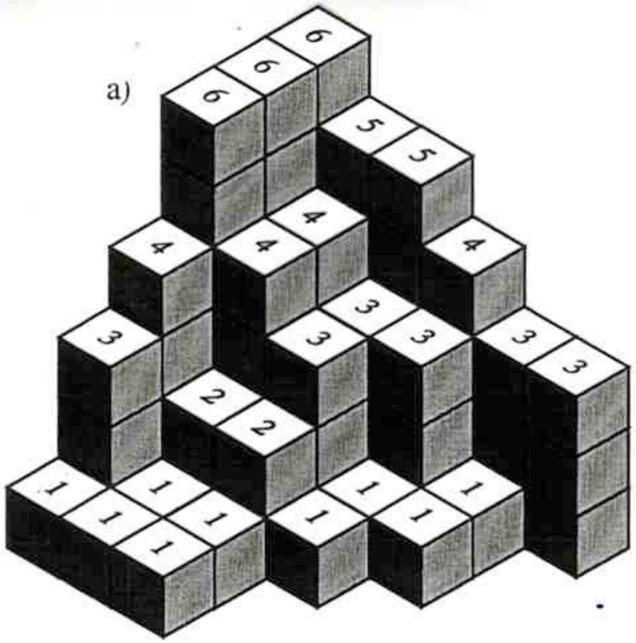


B_1
B_2
B_k
A_1
A_2
A_k
0 1 2 3 4 5 6 7 8 9 10
1 4 10 5 6 15 20 21 7 28 56 70 56 28 8
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

plane partitions

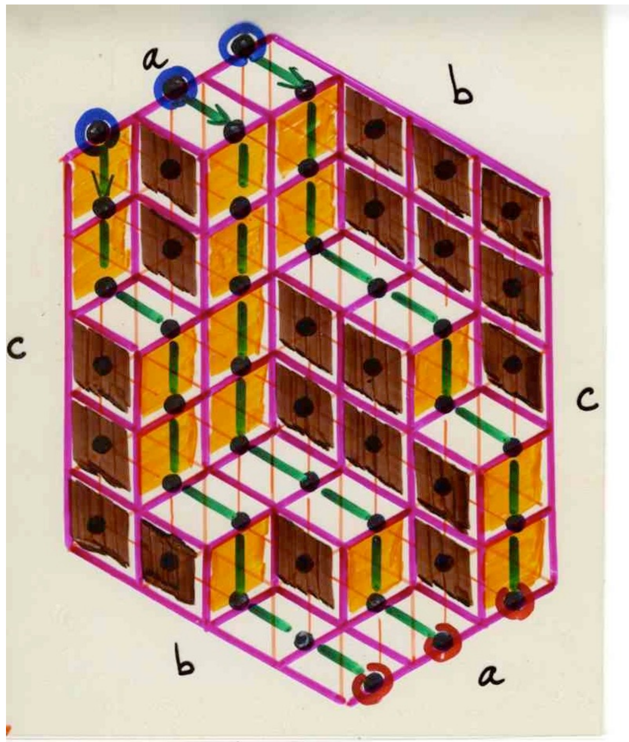
$$\prod_{\substack{1\leq i\leq a\\ 1\leq j\leq b\\ 1\leq k\leq c}} \frac{i+j+k-1}{i+j+k-2}$$

a)



b)

6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			



proof of LGV Lemma

Proof: Involution ϕ

$$E = \left\{ (\sigma ; (\omega_1, \dots, \omega_k)) \; ; \; \begin{array}{l} \sigma \in S_n \\ \omega_i : A_i \rightsquigarrow B_{\sigma(i)} \end{array} \right\}$$

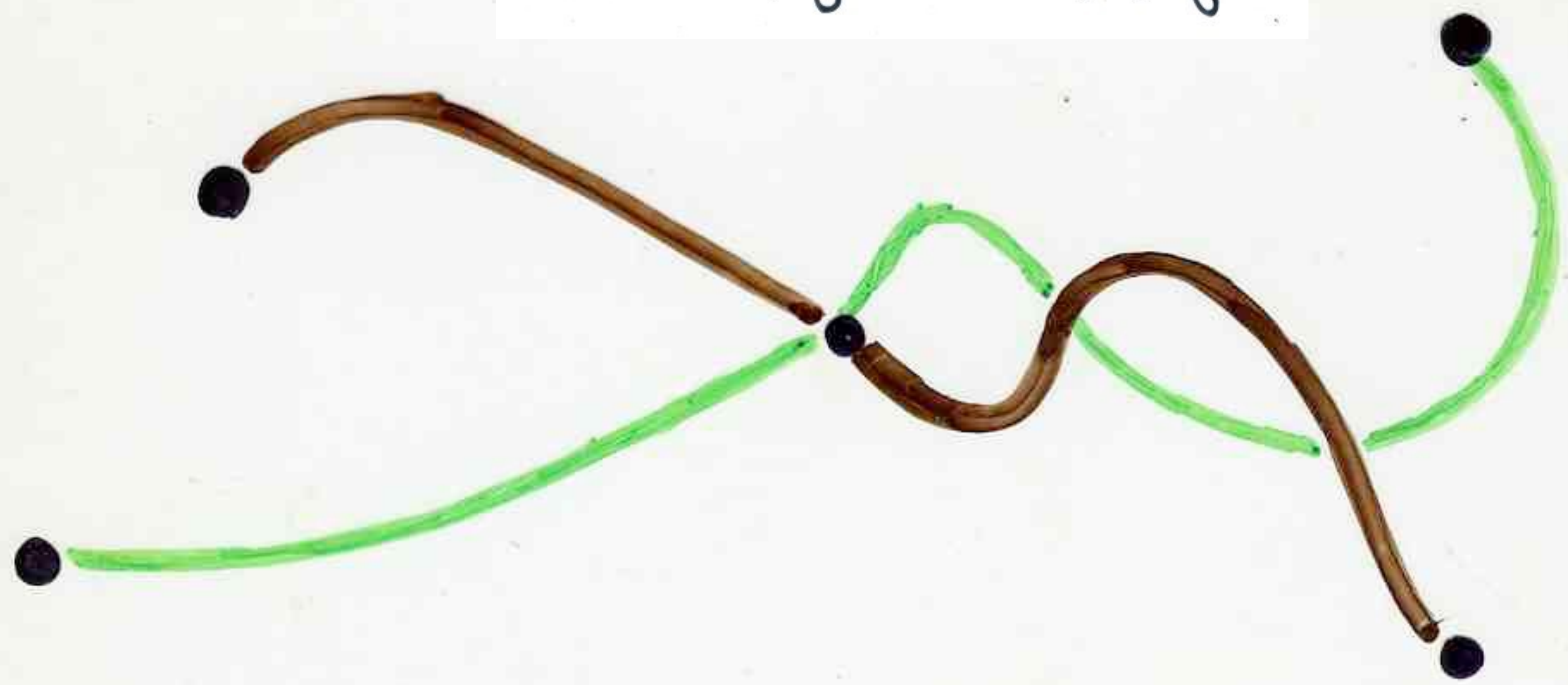
$NC \subseteq E$ non-crossing configurations

$$\phi : (E - NC) \to (E - NC)$$

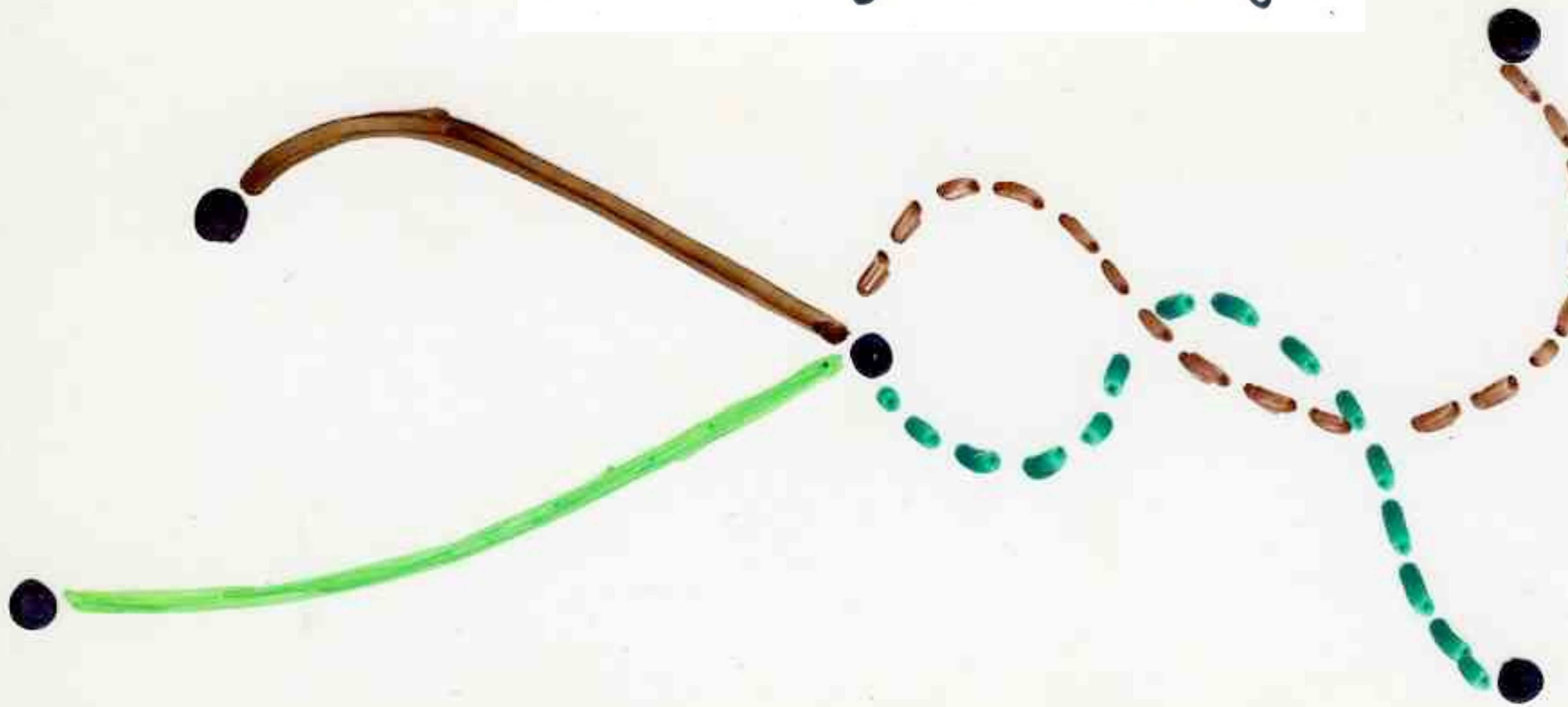
$$\phi(\sigma ; (\omega_1, \dots, \omega_k)) = (\sigma' ; (\omega'_1, \dots, \omega'_k))$$

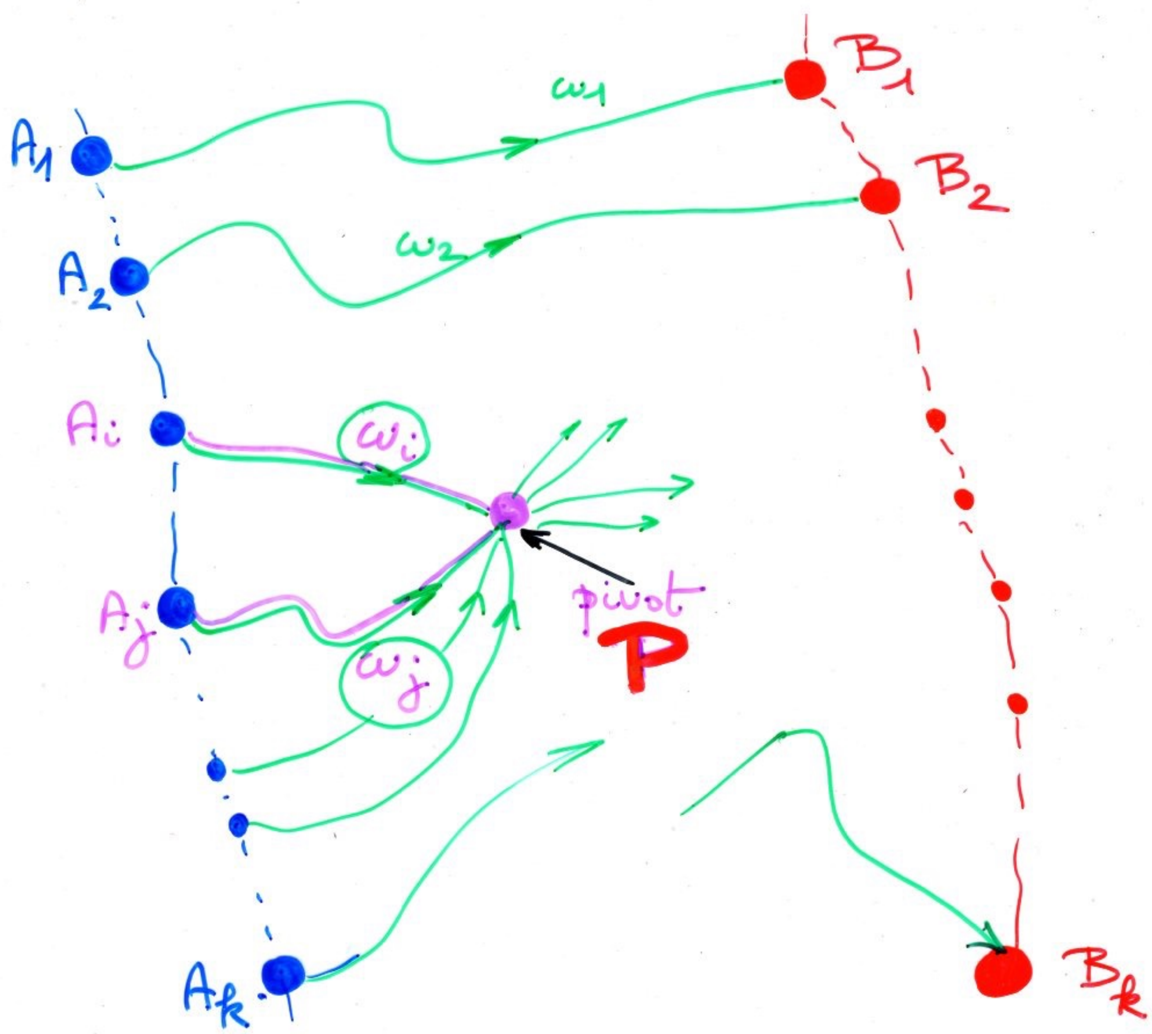
$$\begin{cases} (-1)^{\mathrm{Inv}(\sigma)} = -(-1)^{\mathrm{Inv}(\sigma')} \\ v(\omega_1) \dots v(\omega_k) = v(\omega'_1) \dots v(\omega'_k) \end{cases}$$

idea of the proof:



idea of the proof:

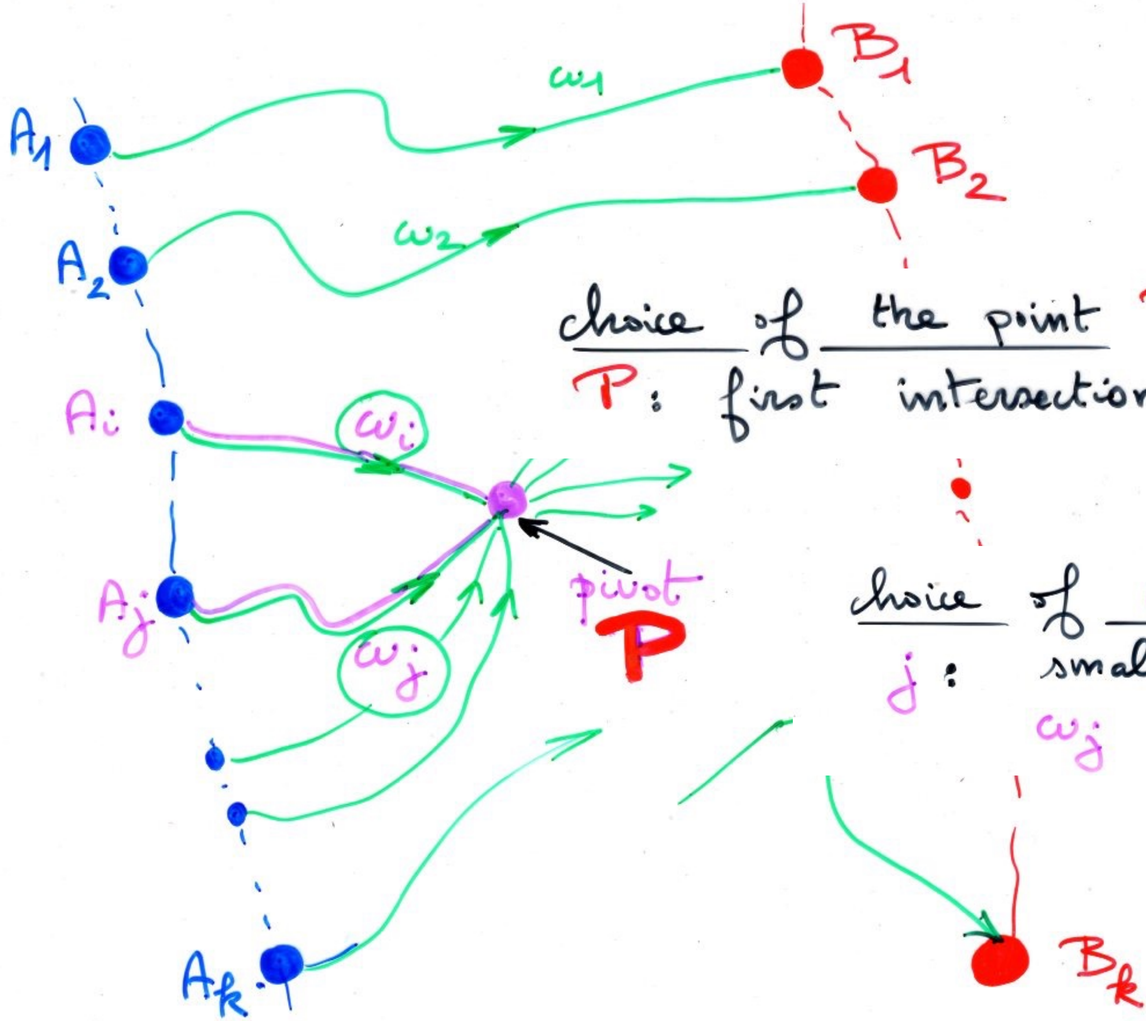




A_1
A_2
A_i
A_j
A_k
B_1
B_2
B_k
ω_1
ω_2
ω_i
ω_j
pivot
P

<u>choice of ω_i</u>

i : smallest i, $1 \leq i \leq k$, such that ω_i has an intersection with another path



<u>choice of the point</u> **P**

P : first intersection point on the path ω_i

<u>choice of ω_j</u>

j : smallest j, $i < j \leq k$ such that ω_j intersect ω_i

$\square$

end of proof

LGV Lemma. general form

$$\det(a_{ij}) = \sum_{(\sigma;\, \omega_1, \dots, \omega_k)} (-1)^{\mathrm{inv}(\sigma)}\, v(\omega_1) \dots (\omega_k)$$

$$\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$$

paths non-intersecting

proof of the main theorem

(main) Theorem

$$\det\left((\mathbb{1}-A)^{-1}[I,J]\right) =$$

$$\sum_{\sigma \in \mathfrak{S}_{I,J}} (-1)^{\mathrm{Inv}(\sigma)} \sum_{\substack{\eta_1 : i_1 \rightsquigarrow \sigma(i_1) \\ \vdots \\ \eta_\ell : i_\ell \rightsquigarrow \sigma(i_\ell)}} v(\eta_1) \cdots (\eta_\ell)\, v(E)$$

set of bijections $I \to J$

self-avoiding paths

pair-wise disjoint

E heap of cycles

projection $\pi(m)$ maximal piece of E intersect one of the path η

$$\det\left((\mathbb{1}-A)^{-1}[I,J]\right)$$

$$= \sum_{\sigma \in \mathfrak{S}_{I,J}} (-1)^{\mathrm{Inv}(\sigma)} \sum_{\substack{\omega_1 : i_1 \rightsquigarrow \sigma(i_1) \\ \vdots \\ \omega_\ell : i_\ell \rightsquigarrow \sigma(i_\ell)}} v(\omega_1) \cdots (\omega_\ell)$$

$\mathfrak{S}_{I,J}$: set of bijections $I \to J$

$i = 1, \dots, \ell$

$$\omega_i \xrightarrow{\chi} (\eta_i, E_i)$$

$$= \sum_{\sigma \in \mathfrak{S}_{I,J}} (-1)^{\mathrm{Inv}(\sigma)} \sum_{\substack{\eta_1 : i_1 \rightsquigarrow \sigma(i_1) \\ \vdots \\ \eta_\ell : i_\ell \rightsquigarrow \sigma(i_\ell)}} v(\eta_1) \cdots v(\eta_\ell)\, v(E_1) \cdots (E_\ell)$$

$\mathfrak{S}_{I,J}$: set of bijections $I \to J$

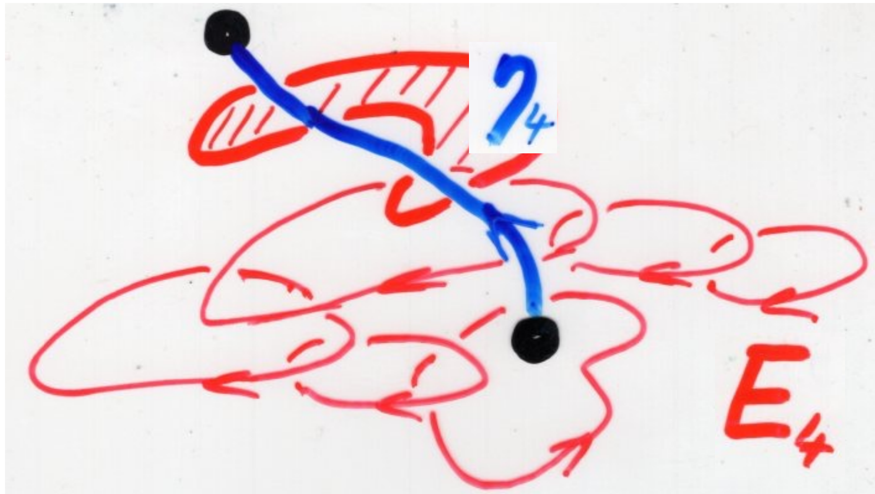
$\eta_1, \dots, \eta_\ell$: self-avoiding paths

$E_1, \dots, E_\ell$ heaps of cycles

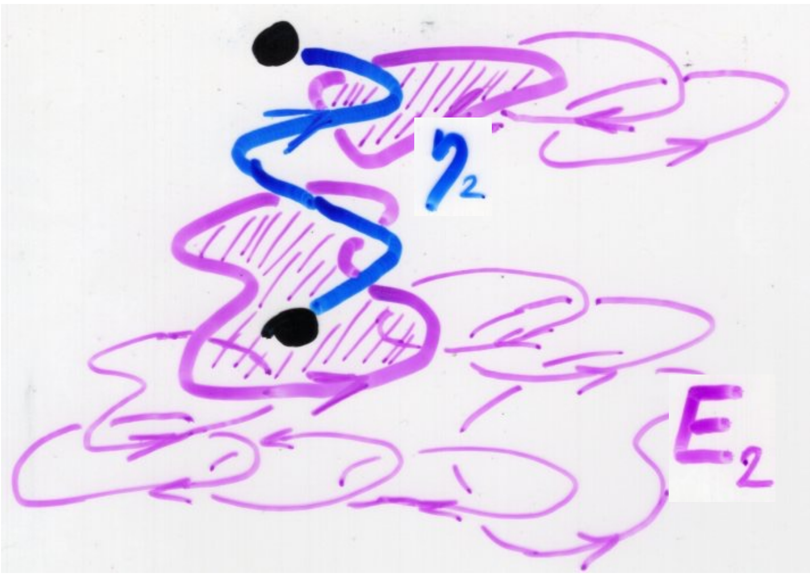
projection

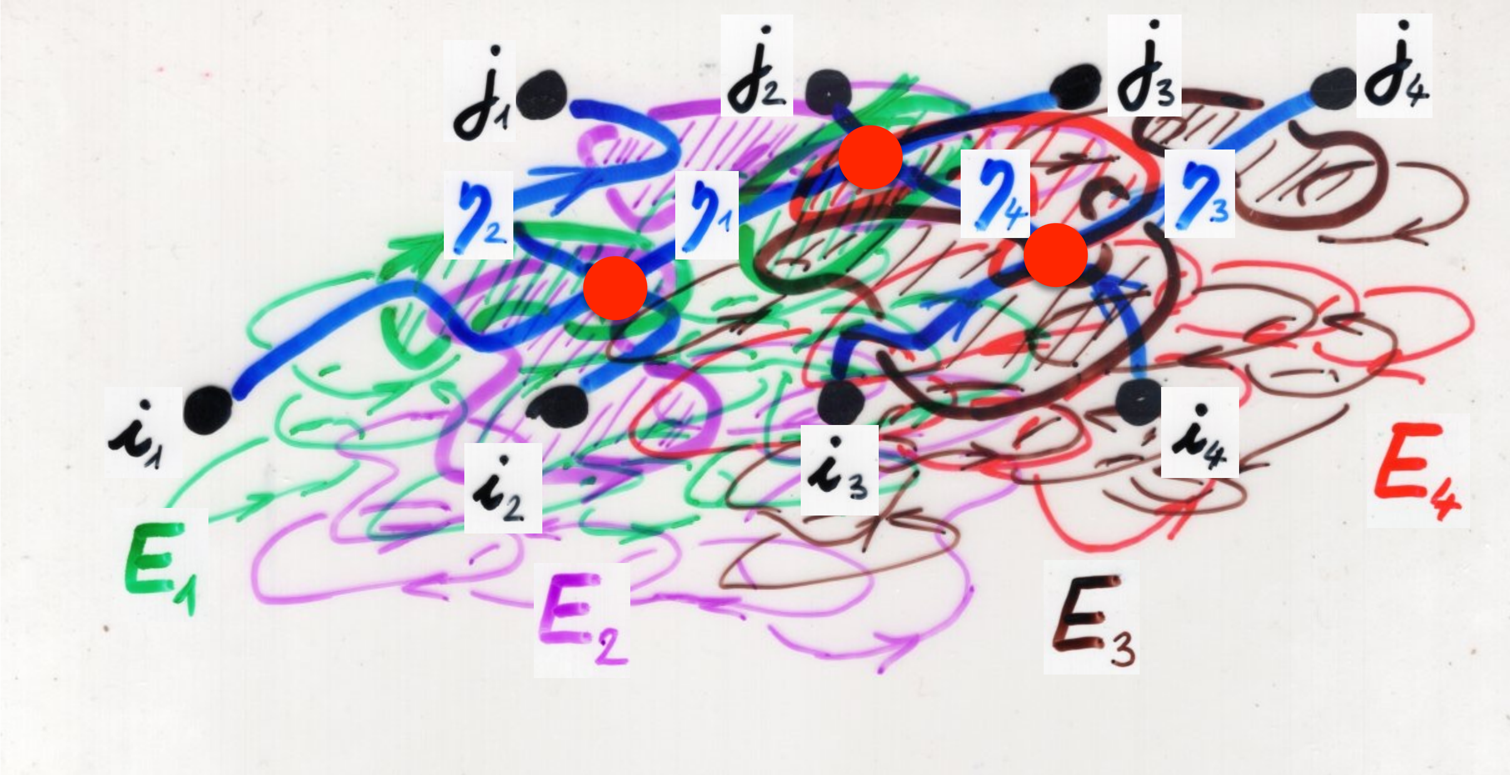
$\pi(m)$ maximal piece of E_i intersect η_i

$i = 1, \dots, \ell$

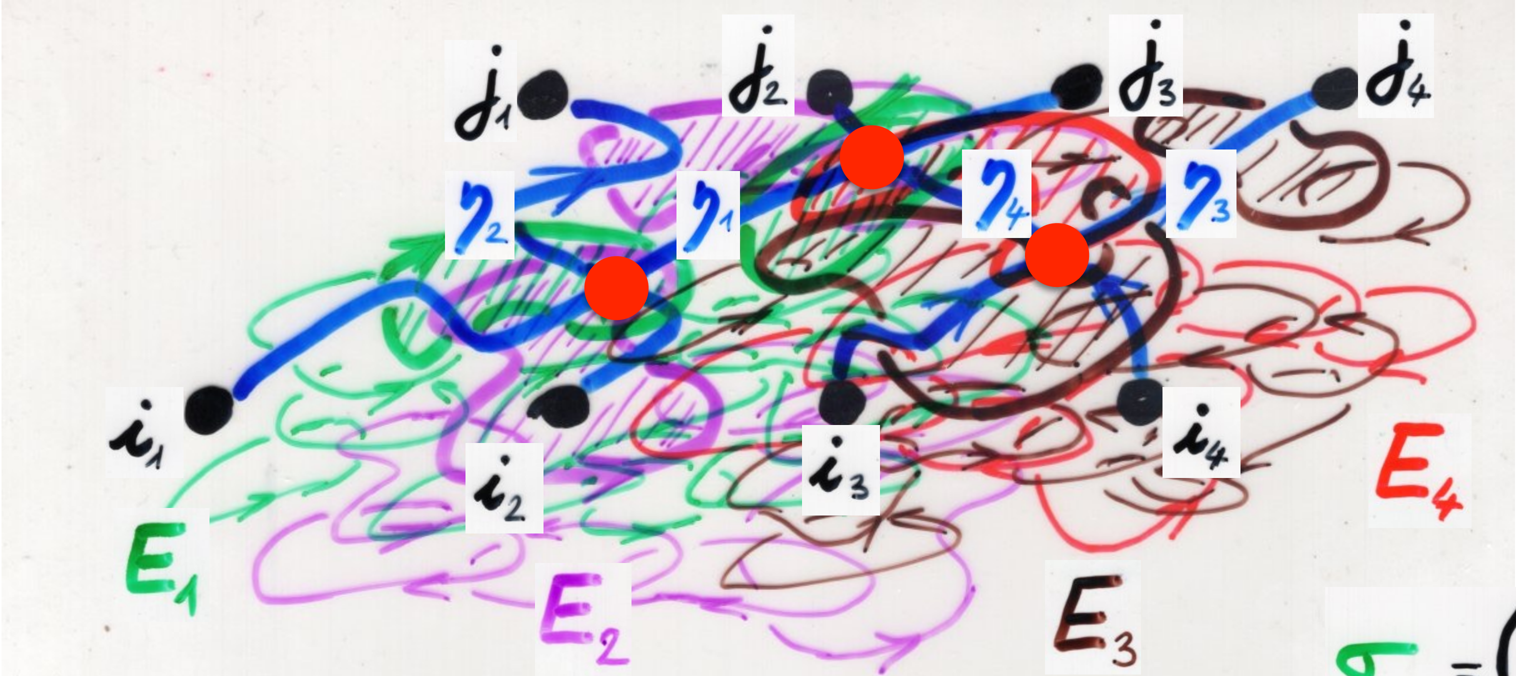


$$\omega_i \xrightarrow{\chi} (\eta_i, E_i)$$





j_1
j_2
j_3
j_4
γ_2
γ_1
γ_4
γ_3
i_1
i_2
i_3
i_4
E_1
E_2
E_3
E_4



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

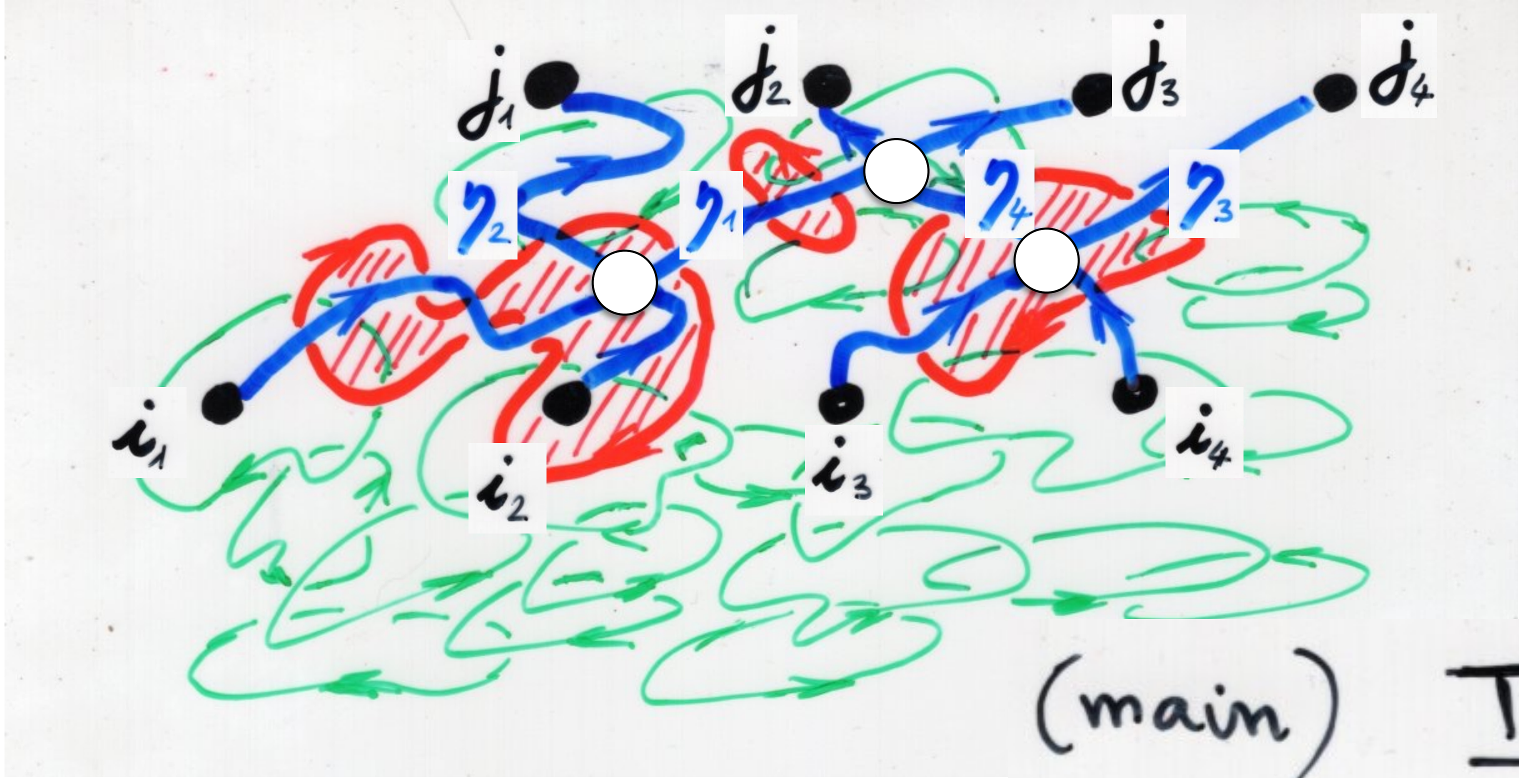
$$\det\left((\mathbb{1}-A)^{-1}[I,J] \right) = \sum_{\sigma \in \mathfrak{S}_{I,J}} (-1)^{\mathrm{Inv}(\sigma)} \sum_{\substack{\eta_1 : i_1 \rightsquigarrow \sigma(i_1) \\ \vdots \\ \eta_\ell : i_\ell \rightsquigarrow \sigma(i_\ell)}} v(\eta_1) \cdots v(\eta_\ell)\, v(E_1) \cdots (E_\ell)$$

set of bijections $I \to J$

self-avoiding paths

$E_1, .., E_\ell$ heaps of cycles

projection $\pi(m)$ maximal piece of E_i intersect η_i $i=1,\dots,\ell$



?

(main) <u>Theorem</u>

$$= \sum_{\sigma \in \mathfrak{S}_{I,J}} (-1)^{Inv(\sigma)} \sum_{\substack{\eta_1 : i_1 \rightsquigarrow \sigma(i_1) \\ \vdots \\ \eta_\ell : i_\ell \rightsquigarrow \sigma(i_\ell)}} v(\eta_1) \cdots (\eta_\ell)\, v(E)$$

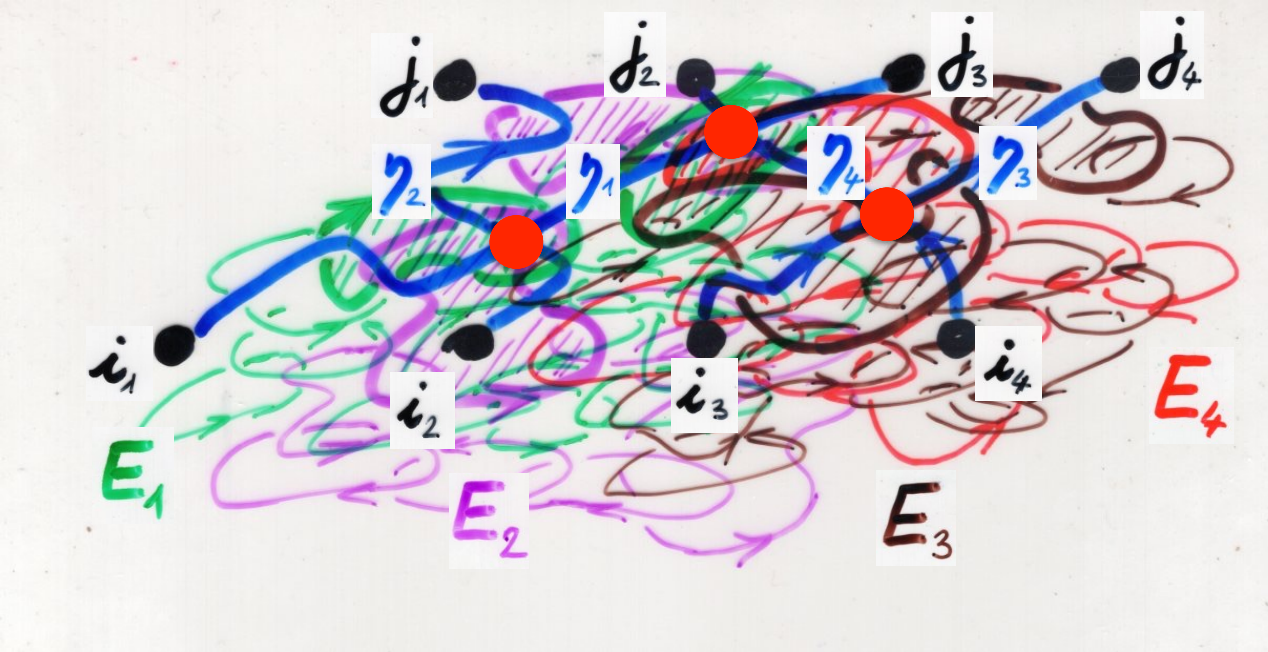
set of bijections $I \to J$

self-avoiding paths

pair-wise disjoint

E heap of cycles

projection $\pi(m)$ maximal piece of E intersect one of the path η



j_1
j_2
j_3
j_4
γ_2
γ_1
γ_4
γ_3
i_1
i_2
i_3
i_4
E_1
E_2
E_3
E_4

j_1
j_2
j_3
j_4
γ_2
γ_1
γ_4
γ_3
i_1
i_2
i_3
i_4

$$\det\left((\mathbb{1}-A)^{-1}[I,J]\right)$$

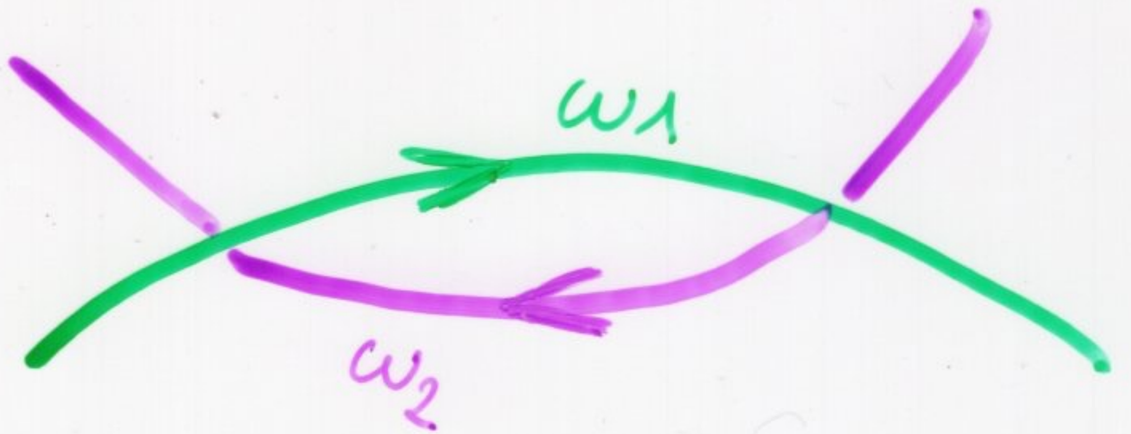
$$= \sum_{\sigma \in \mathfrak{S}_{I,J}} (-1)^{\mathrm{Inv}(\sigma)} \sum_{\substack{\omega_1 : i_1 \rightsquigarrow \sigma(i_1) \\ \vdots \\ \omega_\ell : i_\ell \rightsquigarrow \sigma(i_\ell)}} v(\omega_1) \cdots\cdots (\omega_\ell)$$

$\mathfrak{S}_{I,J}$: set of bijections $I \to J$

$i = 1, \dots, \ell$

involution φ

?



Lalonde (1987, 1990)

proof of the main theorem

first step: Fomin theorem

$$\det\left((\mathbb{1}-A)^{-1}[I,J]\right)$$

$$= \sum_{\sigma \in \mathfrak{S}_{I,J}} (-1)^{\mathrm{Inv}(\sigma)} \sum_{\substack{\omega_1 : i_1 \rightsquigarrow \sigma(i_1) \\ \vdots \\ \omega_\ell : i_\ell \rightsquigarrow \sigma(i_\ell)}} v(\omega_1) \cdots\cdots (\omega_\ell)$$

$\mathfrak{S}_{I,J}$: set of bijections $I \to J$

$i = 1, \dots, \ell$

$$\omega_i \xrightarrow{\chi} (\eta_i, E_i)$$

$\eta_i = LE(\omega_i)$

loop-erased

Proposition

$$\sum_{\substack{\sigma \in \mathfrak{S}_{I,J} \\ \text{set of bijections } I \to J}} (-1)^{\mathrm{Inv}(\sigma)} \sum_{\substack{\omega_1 : i_1 \rightsquigarrow \sigma(i_1) \\ \vdots \\ \omega_\ell : i_\ell \rightsquigarrow \sigma(i_\ell)}} v(\omega_1) \cdot\cdot\cdot\cdot (\omega_\ell)$$

$i = 1, \dots, \ell$

$$= \sum_{\substack{\sigma \in \mathfrak{S}_{I,J} \\ \text{set of bijections } I \to J}} (-1)^{\mathrm{Inv}(\sigma)} \sum_{\substack{\omega_1 : i_1 \rightsquigarrow \sigma(i_1) \\ \vdots \\ \omega_\ell : i_\ell \rightsquigarrow \sigma(i_\ell)}} v(\omega_1) \cdot\cdot\cdot\cdot (\omega_\ell)$$

$i = 1, \dots, \ell$

for every $1 \leq i < j \leq \ell$

$$LE(\omega_i) \cap \omega_j = \emptyset$$

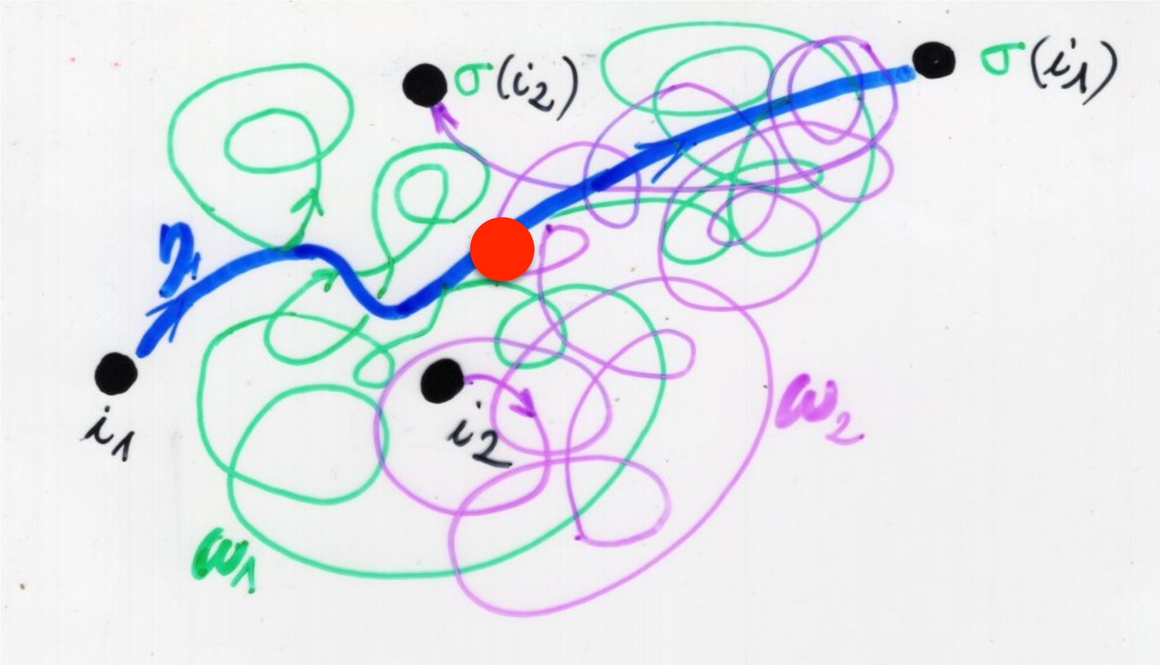
Fomin (2001)

involution φ

sign-reversing

weight preserving

involution



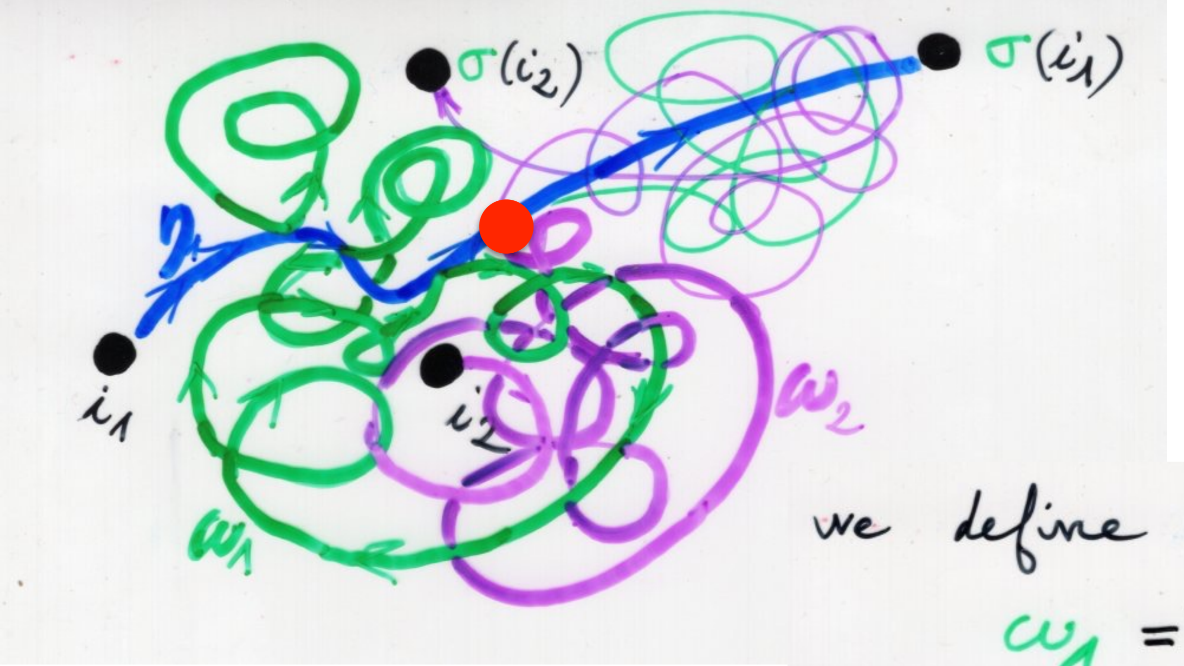
$\omega_1 : i_1 \rightsquigarrow \sigma(i_1)$

$\omega_2 \quad i_2 \rightsquigarrow \sigma(i_2)$

$\eta_1 = LE(\omega_1)$

If ω_2 intersects η_1.

Let v be the first intersection $\omega_2 \cap \eta_1$ following the path ω_1 from i_1

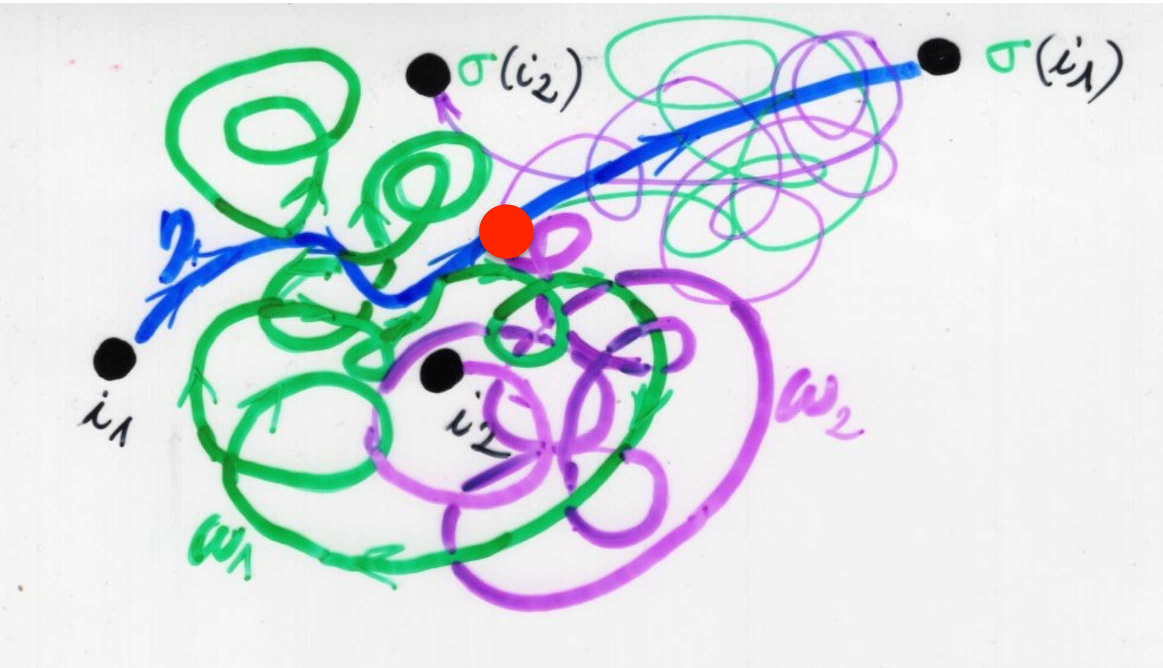


we define a unique factorization

$$\omega_1 = \omega_1' \, v \, \omega_1''$$

- if $v = i_1$, ω_1' is empty
- else, let (u, v) be the unique edge on the path η ending in v.

Considering the flow $F(\omega_1)$, this edge is a maximal edge of the flow, and we cut the path ω_1 at the last time the path goes through this edge.



ω_2 can be uniquely factorized

$\omega_2 = (\omega'_2 \, v \, \omega''_2)$ with $v \notin \omega'_2$

that is we cut ω_2 at the first visit of ω_2 at the vertex v

involution φ

$$\left.\begin{aligned} \omega_1 &= \omega_1' \, v \, \omega_1'' \\ \omega_2 &= \omega_2' \, v \, \omega_2'' \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} \varphi(\omega_1) &= \omega_1' \, v \, \omega_2'' \\ \varphi(\omega_2) &= \omega_2' \, v \, \omega_1'' \end{aligned}\right.$$

$$\sigma \to \sigma'$$

<u>exercise</u>

prove that φ is an involution

characterization of the factorization $\omega_1 = \omega_1' \, v \, \omega_1''$

$$\left\{\begin{array}{l} \text{(i) the last entry in the edge sequence of } \omega_1' v \text{ contribute to } LE(\omega_1' v) \\ \text{(ii) } \omega_1'' \text{ does not visit any vertices of } LE(\omega_1' v) \end{array}\right.$$

involution φ

$$\left.\begin{aligned} \omega_1 &= \omega_1' \vee \omega_1'' \\ \omega_2 &= \omega_2' \vee \omega_2'' \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} \varphi(\omega_1) &= \omega_1' \vee \omega_2'' \\ \varphi(\omega_2) &= \omega_2' \vee \omega_1'' \end{aligned}\right.$$

$$\sigma \to \sigma'$$

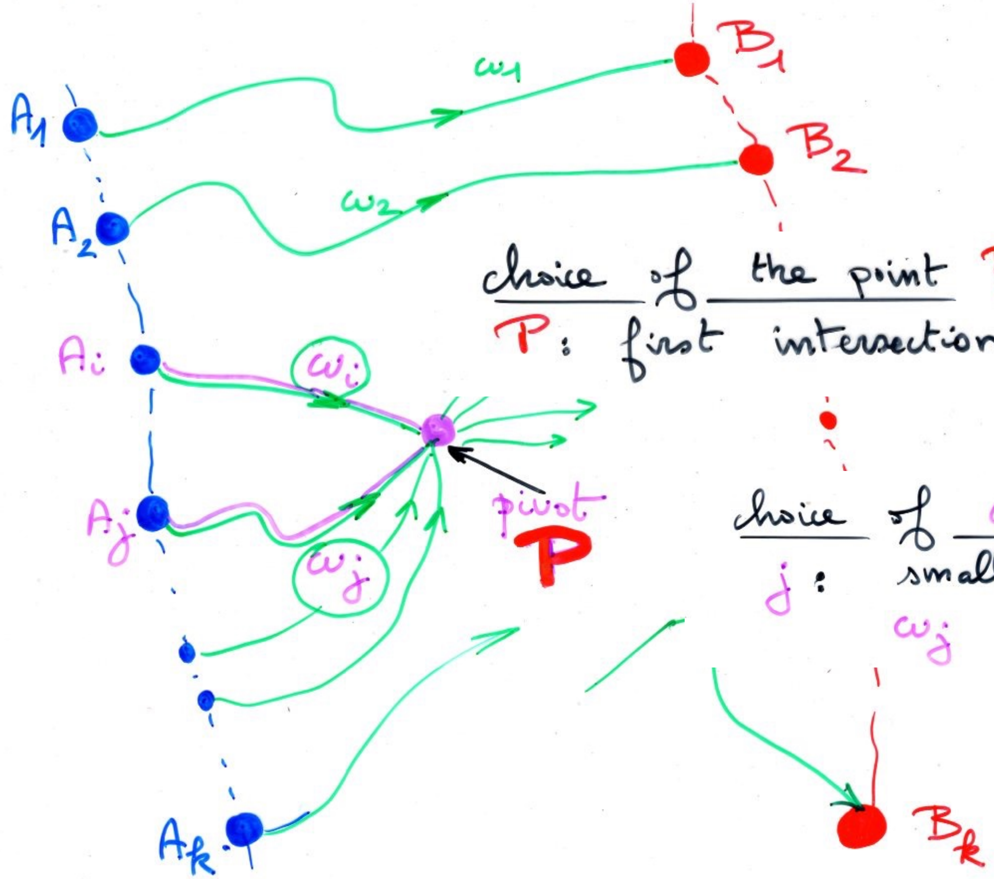
It is a sign-reversing and weight preserving involution

$$(-1)^{\mathrm{Inv}(\sigma)} = -(-1)^{\mathrm{Inv}(\sigma')}$$

extension to several paths

<u>choice of ω_i</u>

i : smallest i, $1 \leq i \leq k$, such that ω_i has an intersection with another path



<u>choice of the point P</u>

P : first intersection point on the path ω_i

<u>choice of ω_j</u>

j : smallest j, $i < j \leq k$ such that ω_j intersect ω_i

$\square$

end of proof

proof of the main theorem

(second step))

Proposition

$$\sum_{\sigma \in \mathfrak{S}_{I,J}} (-1)^{\mathrm{Inv}(\sigma)} \sum_{\substack{\eta_1 : i_1 \rightsquigarrow \sigma(i_1) \\ \vdots \\ \eta_\ell : i_\ell \rightsquigarrow \sigma(i_\ell)}} v(\eta_1) \cdots (\eta_\ell)\, v(E)$$

$\sigma \in \mathfrak{S}_{I,J}$ set of bijections $I \to J$

self-avoiding paths

pair-wise disjoint

E heap of cycles

projection $\pi(m)$ maximal piece of E intersect one of the path η

$$= \sum_{\sigma \in \mathfrak{S}_{I,J}} (-1)^{\mathrm{Inv}(\sigma)} \sum_{\substack{\omega_1 : i_1 \rightsquigarrow \sigma(i_1) \\ \vdots \\ \omega_\ell : i_\ell \rightsquigarrow \sigma(i_\ell)}} v(\omega_1) \cdots (\omega_\ell)$$

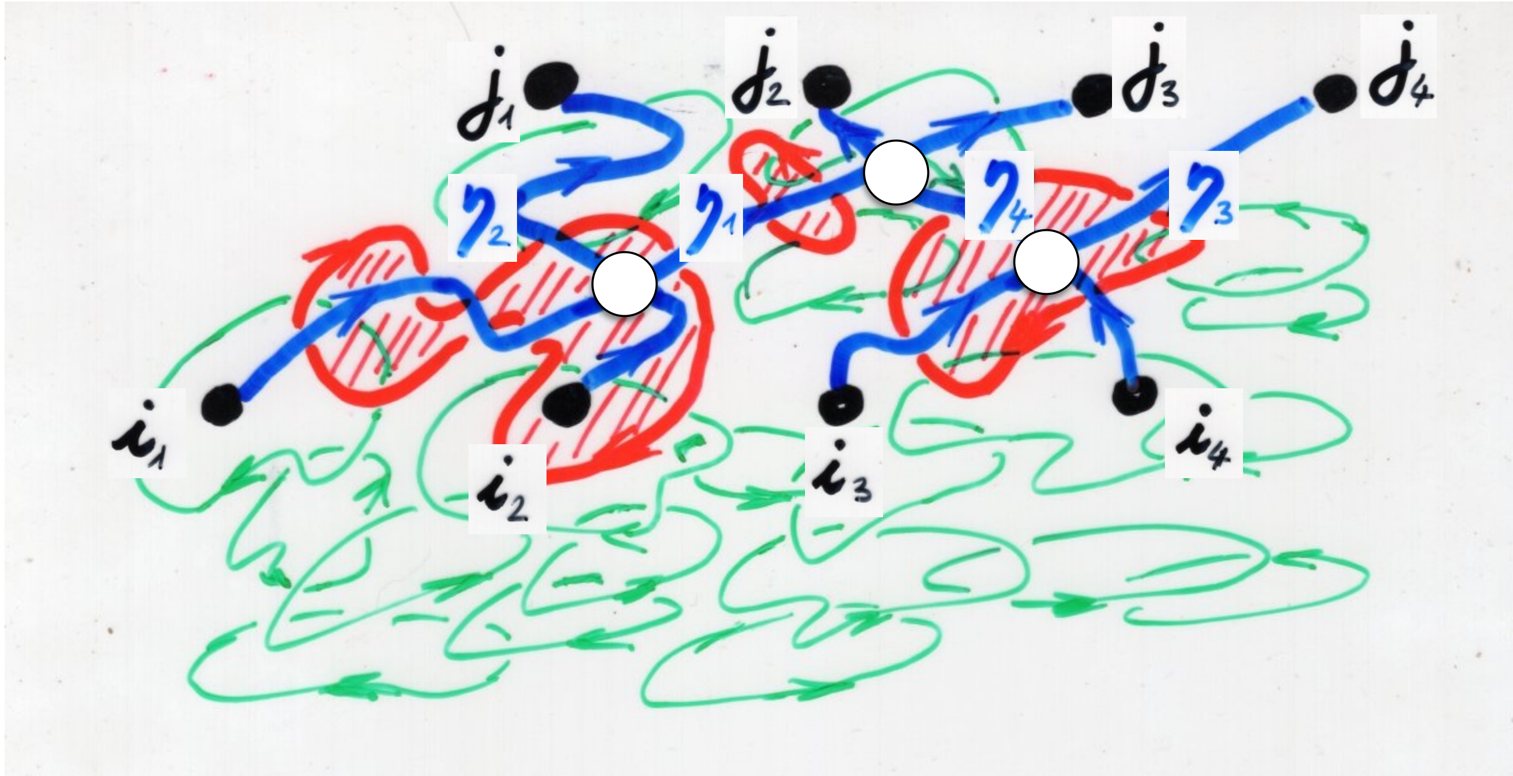
$\sigma \in \mathfrak{S}_{I,J}$ set of bijections $I \to J$

$i = 1, \dots, \ell$

$$\omega_i \xrightarrow{\chi} (\eta_i, E_i)$$

$\eta_i = LE(\omega_i)$ loop-erased

for every $1 \leq i < j \leq \ell$
$LE(\omega_i) \cap \omega_j = \emptyset$



$$(\eta_1, \eta_2, \dots, \eta_\ell ; E) \longrightarrow$$

$$\begin{array}{ccc} (\eta_1, E_1), & (\eta_2, E_2); \cdots ; & (\eta_\ell, E_\ell) \\ \chi^{-1} \downarrow & & \downarrow \chi^{-1} \\ \omega_1, & \omega_2, \dots, & \omega_\ell \end{array}$$

$$E = E_1 \odot E_2 \odot \cdots \odot E_\ell$$

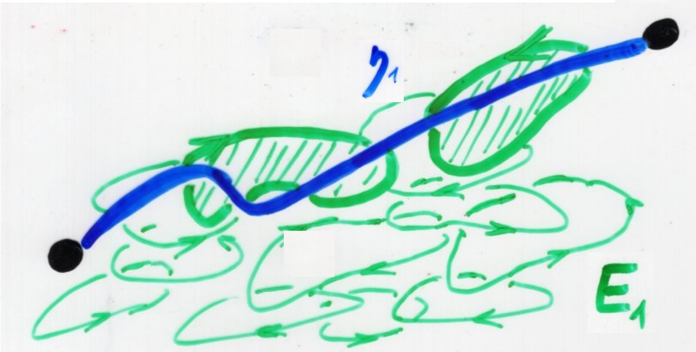
$$(\eta_1, E_1), (\eta_2, E_2), \dots, (\eta_\ell, E_\ell)$$

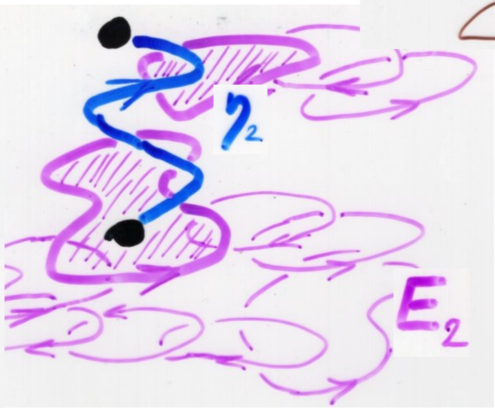
$$\chi^{-1} \downarrow \qquad \qquad \downarrow \chi^{-1}$$

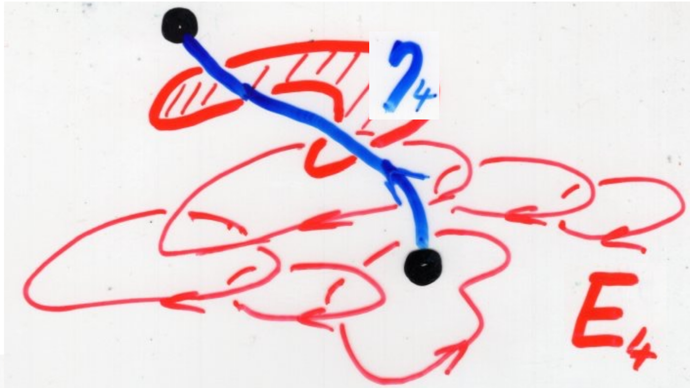
$$\omega_1, \quad \omega_2, \dots, \quad \omega_\ell$$

for every $1 \leq i < j \leq \ell$

$$LE(\omega_i) \cap \omega_j = \emptyset$$







$$E = E_1 \circ E_2 \circ \dots \circ E_\ell$$

$\square$

end of proof

Carrozza, Krajewski, Tanasa (2016)

Grassmann algebra
integral

$$\chi_i \chi_j = -\chi_j \chi_i$$

$$\chi_i^2 = 0$$

$$\det\Big((1-A)^{-1}[I,J]\Big) = \frac{(-1)^{s(I)+s(J)} \det\Big((1-A)[\bar{J},\bar{I}]\Big)}{\det(1-A)}$$

another way to prove the identity

$$\det\left((\mathbb{1}-A)^{-1}[I,J]\right) = \frac{(-1)^{s(I)+s(J)}\det\left((\mathbb{1}-A)[\bar{J},\bar{I}]\right)}{\det(\mathbb{1}-A)}$$

another way to prove the identity

$$\det(\mathbb{1}-A)\ \det\left((\mathbb{1}-A)^{-1}[I,J]\right) =$$

$$(-1)^{s(I)+s(J)} \det\left((\mathbb{1}-A)[\bar{J},\bar{I}]\right)$$

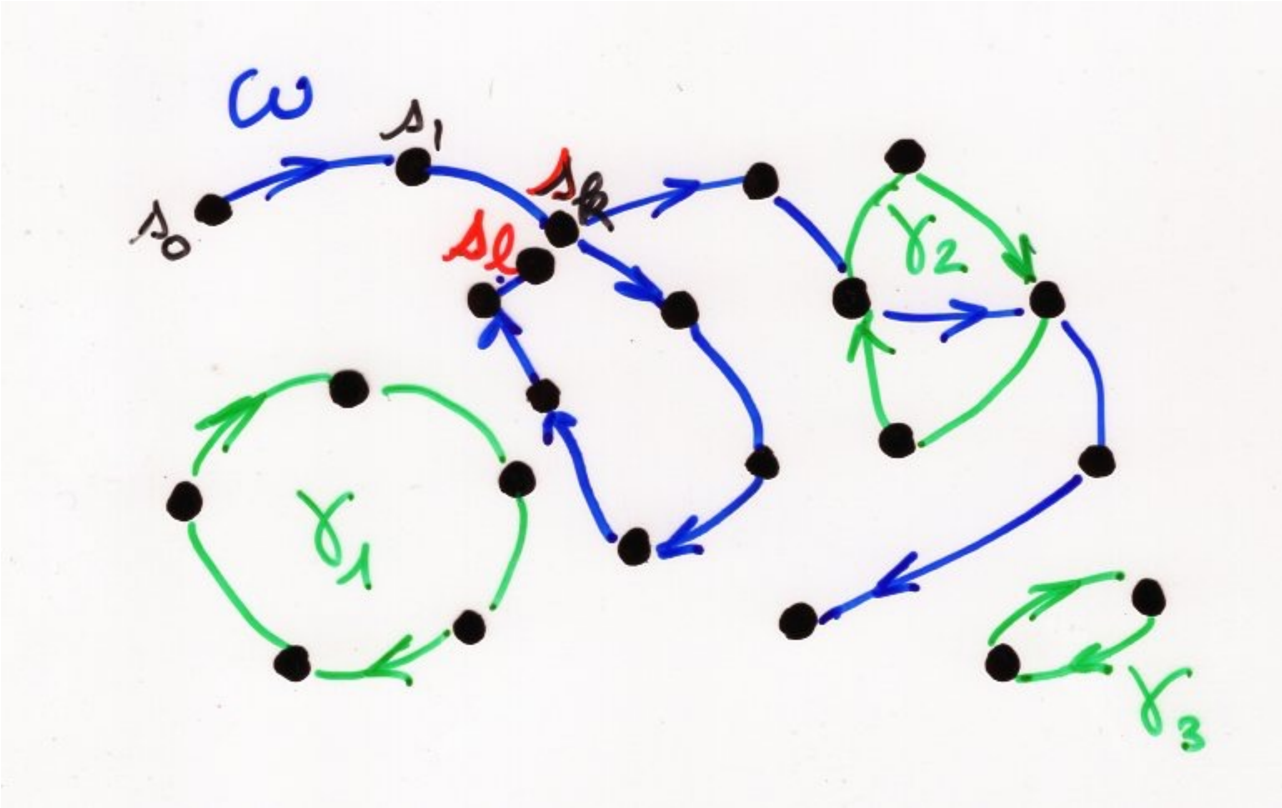
involution φ

particular case:

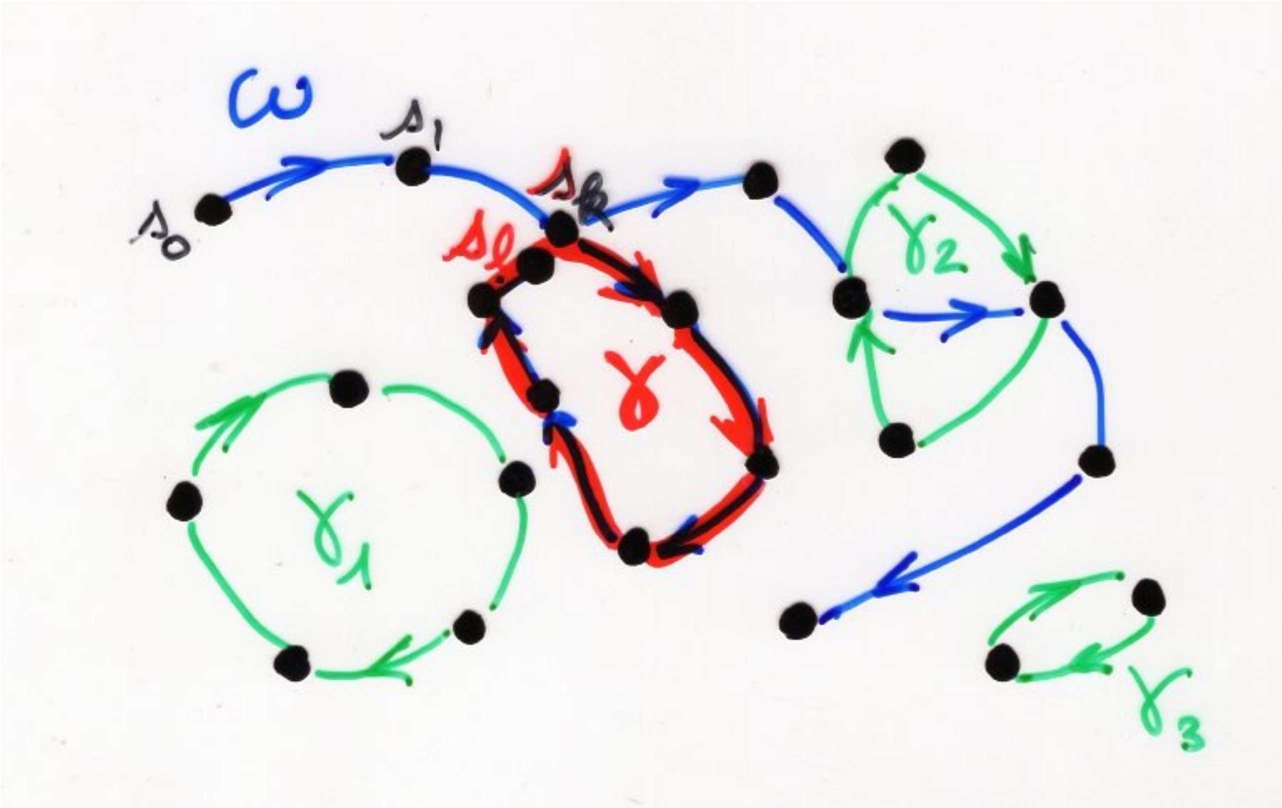
→ Ch 1c, p 9-18
course IMSc 2016

« direct » bijective proof of the identity

$$\sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} v(\omega) = \frac{N_{i,j}}{D}$$



involution φ



involution φ

$$\det(1-A)\,\det\left((1-A)^{-1}[I,J]\right)$$

$$= (-1)^{s(I)+s(J)}\det\left((1-A)[\bar{J},\bar{I}]\right)$$

involution φ

Lalonde (1996)

Talaska (2012)

network parametrization
for the Grassmannian

positivity in Grassmannian

Talaska, Williams
Postnikov, Fomin

Abdessalam, Brydges loop ensembles
Mayer expansion

crossing condition

(main) Theorem

$$\det\left((\mathbb{1}-A)^{-1}[I,J]\right) =$$

$$\sum_{\sigma \in \mathfrak{S}_{I,J}} (-1)^{\mathrm{Inv}(\sigma)} \sum_{\substack{\eta_1 : i_1 \rightsquigarrow \sigma(i_1) \\ \vdots \\ \eta_\ell : i_\ell \rightsquigarrow \sigma(i_\ell)}} v(\eta_1) \cdots (\eta_\ell)\, v(E)$$

set of bijections $I \to J$

self-avoiding paths

E heap of cycles

projection $\pi(m)$ maximal piece of E intersect one of the path η

pair-wise disjoint

Theorem (C)

crossing condition (C)

$$\det\left((1-A)^{-1}[I,J]\right) = \sum_{\substack{\eta_1 : i_1 \rightsquigarrow j_1 \\ \vdots \\ \eta_\ell : i_\ell \rightsquigarrow j_\ell}} v(\eta_1) \cdots (\eta_\ell)\, v(E)$$

$\sigma = Id$

self-avoiding paths

E heap of cycles

projection $\pi(m)$ maximal piece of E intersect one of the path η

$\sigma = \mathrm{Id}$

$$\det\left((\mathbb{1}-A)^{-1}[I,J]\right) = \sum_{\substack{\eta_1 : i_1 \rightsquigarrow j_1 \\ \vdots \\ \eta_\ell : i_\ell \rightsquigarrow j_\ell}} v(\eta_1)\cdots(\eta_\ell)\, v(E)$$

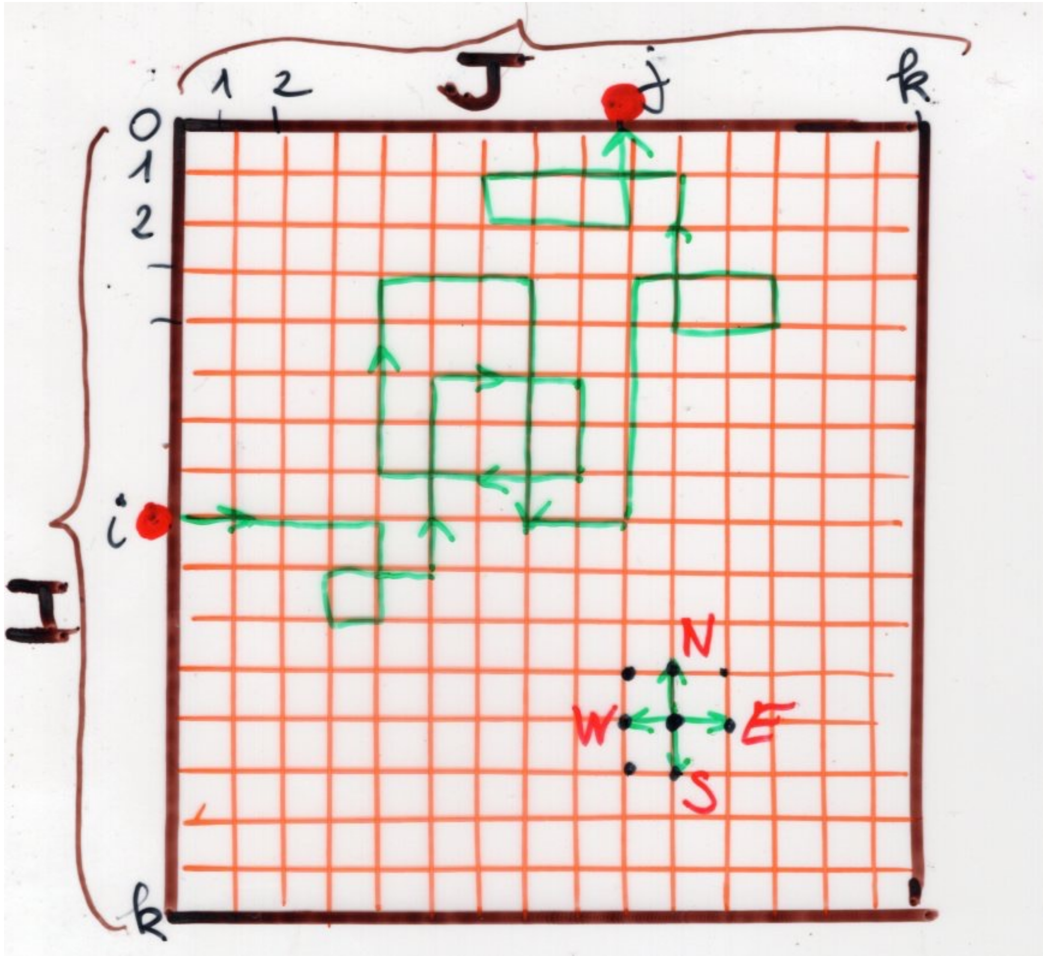
self-avoiding paths

crossing condition (C)

E heap of cycles

projection

$\pi(m)$ maximal piece of E intersect one of the path η



$$(I-A)^{-1}_{ij} = \sum_{\substack{\omega \\ i \to j}} v(\omega)$$

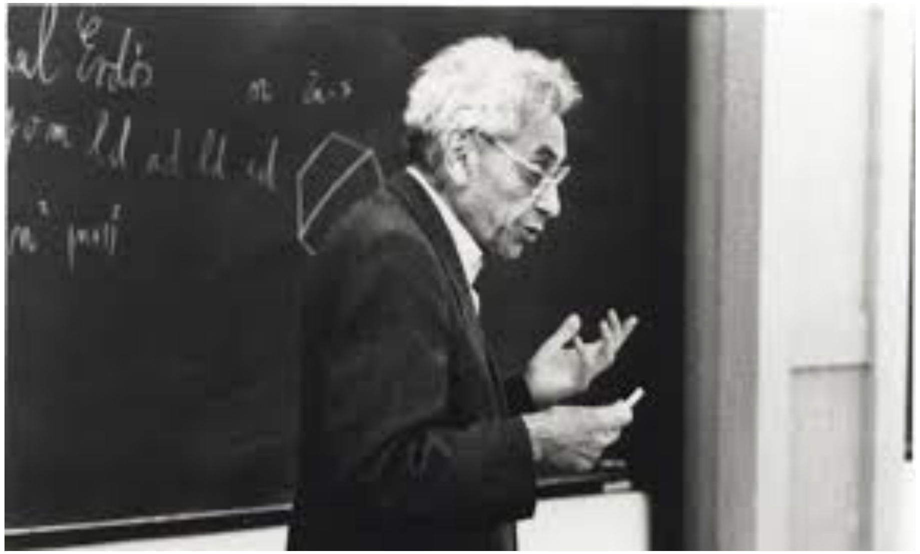
$$(A)_{(k+1)^2 \times (k+1)^2}$$

$$\det (I-A)^{-1}_{I,J} = \frac{v\left(\quad \right)}{\det(I-A)}$$

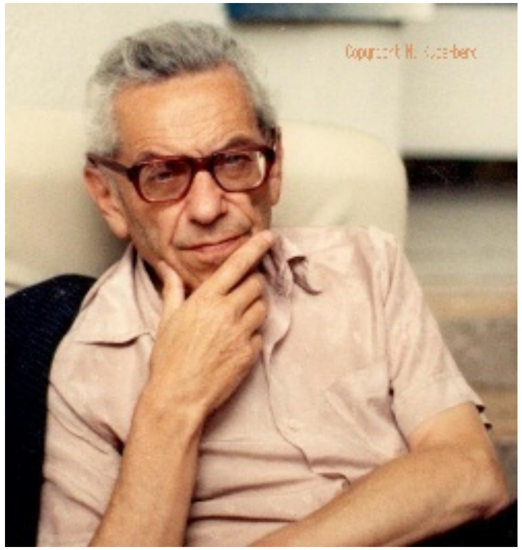
about the name LGV Lemma

Lattice paths and determinants   Chapter 29

Why « LGV **Lemma** » ?



Paul Erdös liked to talk about The Book, in which God maintains the perfect proofs for mathematical theorems,



Erdös also said that you need not believe in God but, as a mathematician, you should believe in The Book.

Lattice paths and determinants — Chapter 29

Why « LGV **Lemma** » ?

The essence of mathematics is proving theorems — and so, that is what mathematicians do: They prove theorems. But to tell the truth, what they really want to prove, once in their lifetime, is a *Lemma*, like the one by Fatou in analysis, the Lemma of Gauss in number theory, or the Burnside–Frobenius Lemma in combinatorics.

Now what makes a mathematical statement a true Lemma? First, it should be applicable to a wide variety of instances, even seemingly unrelated problems. Secondly, the statement should, once you have seen it, be completely obvious. The reaction of the reader might well be one of faint envy: Why haven't I noticed this before? And thirdly, on an esthetic level, the Lemma — including its proof — should be beautiful!

In this chapter we look at one such marvelous piece of mathematical reasoning, a counting lemma that first appeared in a paper by Bernt Lindström in 1972. Largely overlooked at the time, the result became an instant classic in 1985, when Ira Gessel and Gerard Viennot rediscovered it and demonstrated in a wonderful paper how the lemma could be successfully applied to a diversity of difficult combinatorial enumeration problems.

Why « **LGV** Lemma » ?

from Christian Krattenthaler:

« Watermelon configurations with wall interaction: exact and asymptotic results »

J. Physics Conf. Series 42 (2006), 179--212,

4Lindström used the term "pairwise node disjoint paths". The term "non-intersecting," which is most often used nowadays in combinatorial literature, was coined by Gessel and Viennot [24].

5By a curious coincidence, Lindström's result (the motivation of which was matroid theory!) was rediscovered in the 1980s at about the same time in three different communities, not knowing from each other at that time: in statistical physics by Fisher [17, Sec. 5.3] in order to apply it to the analysis of vicious walkers as a model of wetting and melting, in combinatorial chemistry by John and Sachs [30] and Gronau, Just, Schade, Scheffler and Wojciechowski [28] in order to compute Pauling's bond order in benzenoid hydrocarbon molecules, and in enumerative combinatorics by Gessel and Viennot [24, 25] in order to count tableaux and plane partitions. Since only Gessel and Viennot rediscovered it in its most general form, I propose to call this theorem the "Lindstrom–Gessel–Viennot theorem." It must however be mentioned that in fact the same idea appeared even earlier in work by Karlin and McGregor [32, 33] in a probabilistic framework, as well as that the so-called "Slater determinant" in quantum mechanics (cf. [48] and [49, Ch. 11]) may qualify as an "ancestor" of the Lindstro¨m–Gessel–Viennot determinant.

6There exist however also several interesting applications of the general form of the Lindstro¨m–Gessel–Viennot theorem in the literature, see [10, 16, 51].

combinatorics

B. Lindström, *On the vector representation of induced matroids*, Bull. London Maths. Soc. 5 (1973) 85-90.

I. Gessel and X.G.V., *Binomial determinants, paths and hook length formula*, Advances in Maths., 58 (1985) 300-321.

I. Gessel and X.G.V., *Determinants, paths and plane partitions, preprint (1989)*

statistical physics: (wetting, melting)

Fisher, Vicious walkers, Botzmann lecture (1984)

combinatorial chemistry:

John, Sachs (1985)

Gronau, Just, Schade, Scheffler, Wojciechowski (1988)

probabilities, birth and death process,

Karlin , McGregor (1959)

quantum mechanics: Slater determinant

Slater(1929) (1968), De Gennes (1968)