Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics,
a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

The Institute of Mathematical Sciences

IMSc

January-March 2017

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Chapter 4

Heaps and linear algebra: bijective proofs of classical theorems

(2)

IMSc, Chennai

9 February 2017

corrections to exercise 3
Ch 3 b, p65

definition G graph, ~~χ~~
ω path on G with $\omega \longrightarrow (\eta, E)$.
ω is tree-like iff the heap E contains only cycles of length 2.

exercise 3 G graph, s vertex of G
Construct a tree T such that the tree-like paths on G starting at s are in bijection (preserving the length) with the paths on T starting at the root of T

and such that the generating function of paths ω on the graph G going from s to s is the same as the generating function of paths on T starting and ending at the root r

$$\sum_{\substack{\omega \\ \text{path on } G \\ s \rightsquigarrow s}} t^{|\omega|} = \sum_{\substack{\omega \\ \text{path on } T \\ r \rightsquigarrow r}} t^{|\omega|}$$

from the previous lecture

Matrix inversion

$$B = I - A \qquad A = (a_{ij})_{1 \le i, j \le k}$$

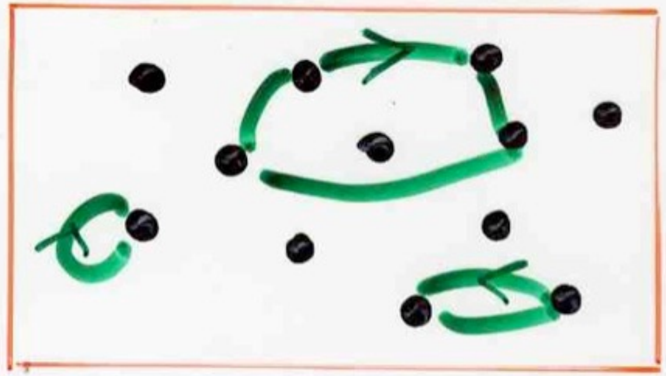
$$(I - A)^{-1} = I + A + \cdots + A^n + \cdots$$

$$\frac{\mathrm{cof}_{ji}(I - A)}{\det(I - A)}$$

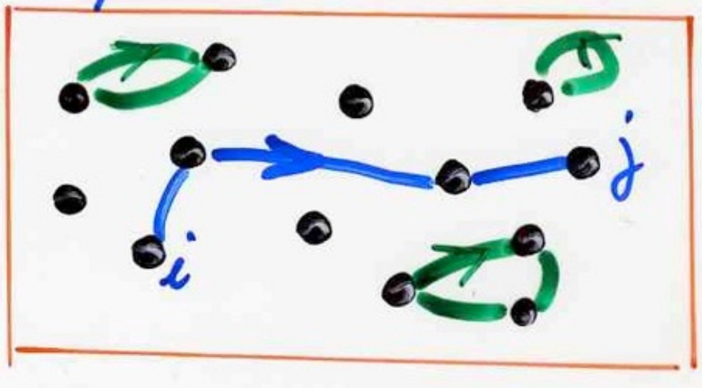
Prop. $$\sum_{\substack{\omega \\ i \rightsquigarrow j}} v(\omega) = \frac{N_{ij}}{D}$$

$$D = \sum_{\{\gamma_1, \dots, \gamma_r\}} (-1)^r v(\gamma_1) \cdots v(\gamma_r)$$

2 by 2 disjoint cycles



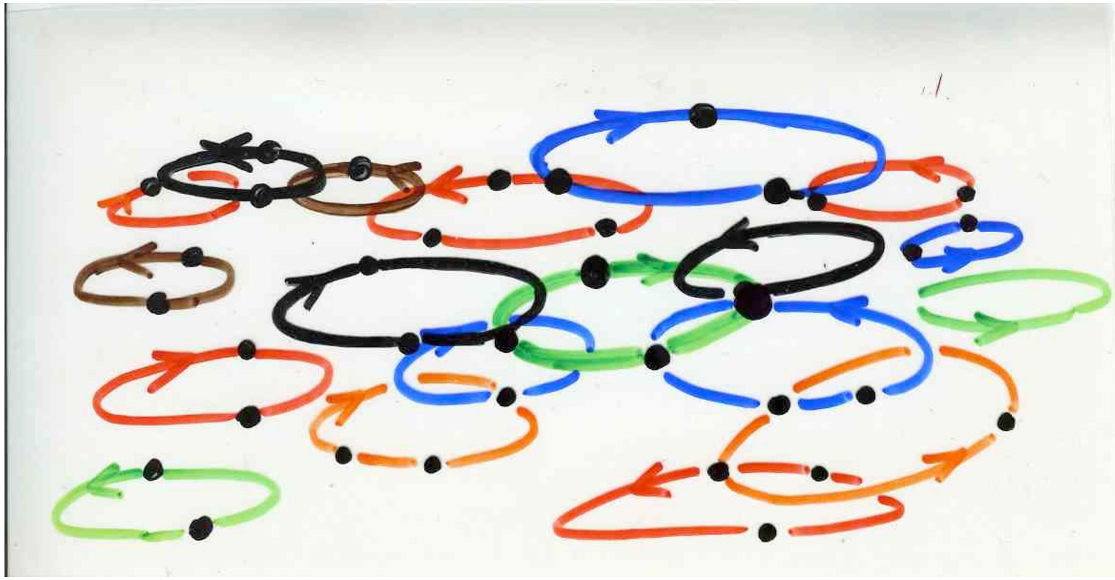
$$N_{ij} = \sum_{\{\eta; \gamma_1, \dots, \gamma_r\}} (-1)^r v(\eta)\, v(\gamma_1) \cdots v(\gamma_r)$$



MacMahon master theorem

$$\frac{1}{\det(I-A)} = \sum_{\substack{\Phi \\ \text{rearrangements} \\ \text{on } [1,k]}} v(\Phi)$$

$$= \sum_{\substack{E \\ \text{heap of cycles} \\ \text{on } [1,k]}} v(E)$$



Today

Jacobi identity

$$\log(\det(B)) = \operatorname{Tr}(\log(B))$$

<u>Theorem</u> (Cayley-Hamilton)

$$P_A(A) = 0$$

Jacobi identity

from Ch 2d

(the logarithmic lemma)

The logarithmic Lemma

(general form)

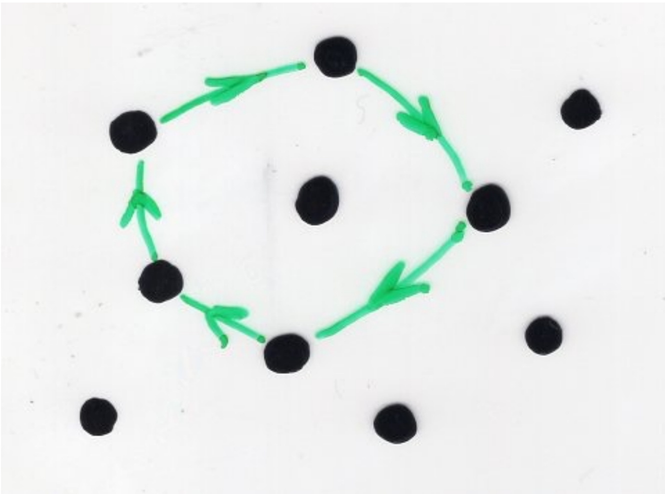
weight of
a basic piece:

$$v(\alpha)t^{\ell(\alpha)}$$

$$\ell : P \longrightarrow \mathbb{N}$$

heap of cycles
on a set X

P basic pieces
cycles on X



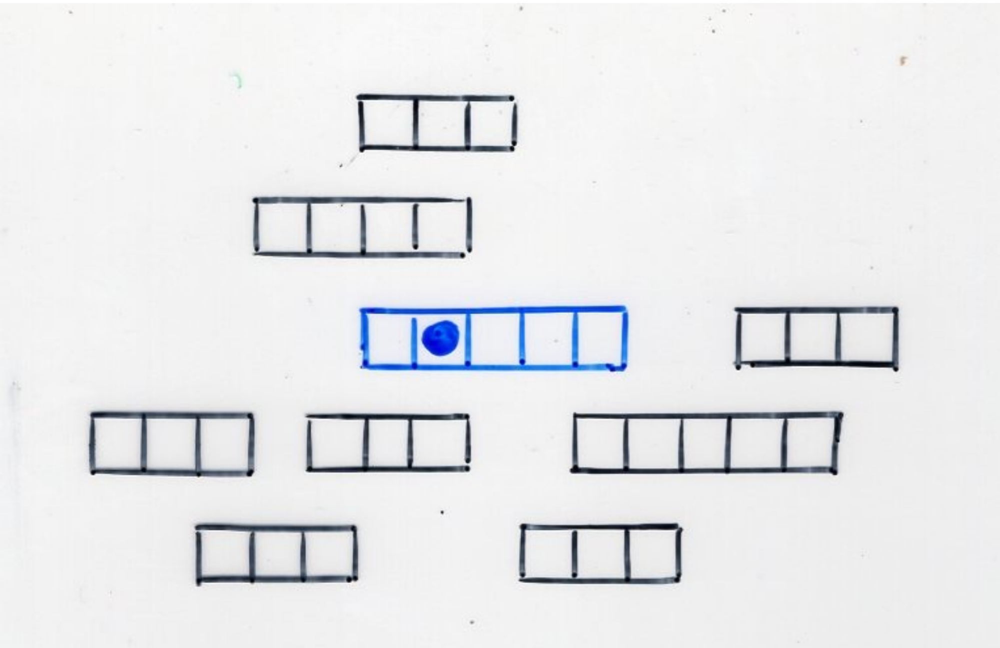
$\ell(\gamma) = n$

number of
vertices
(or length)
of γ

$\mathcal{H}^{\bullet}$

class of pointed weighted heaps

$(E, j) \quad 1 \leq j \leq \ell(E)$



$$\ell(E) = \sum_{x \in E} \ell(\pi(x))$$

$$\mathcal{H}^{\bullet} = \mathcal{P}yr^{\bullet} \; \mathcal{H}$$

$$t y' = z y$$

$$t \frac{d}{dt} \log \Big(\sum_{\substack{E \\ \text{heap}}} v(E)\, t^{\ell(E)} \Big) = \sum_{\substack{P \\ \text{pointed} \\ \text{pyramid}}} v(P)\, t^{\ell(P)}$$

$$(P, j)$$

$1 \leq j \leq \ell(m)$

m maximal piece of P

$$\log\Big(\sum_{\substack{E\\ \text{heap}}} v(E)\, t^{\ell(E)}\Big) = \sum_{\substack{P\\ \text{pointed}\\ \text{pyramid}}} v(P)\,\frac{t^{\ell(P)}}{\ell(P)}$$

$$(P, j)$$

$1 \leq j \leq \ell(m)$

m maximal piece of P

$t = 1$

logarithmic lemma

$$\log\Big(\sum_{\substack{E\\ \text{heap}}} v(E)\, t^{|E|}\Big) = \sum_{\text{pyramid}} v(P)\,\frac{t^{|P|}}{|P|}$$

simple form

$$\log\Big(\sum_{\substack{E\\ \text{heap}}} v(E)\, t^{\ell(E)}\Big) = \sum_{\substack{P\\ \text{pointed}\\ \text{pyramid}}} v(P)\,\frac{t^{\ell(P)}}{\ell(P)}$$

general form

(P, j)

$1 \leq j \leq \ell(m)$

m maximal piece of P

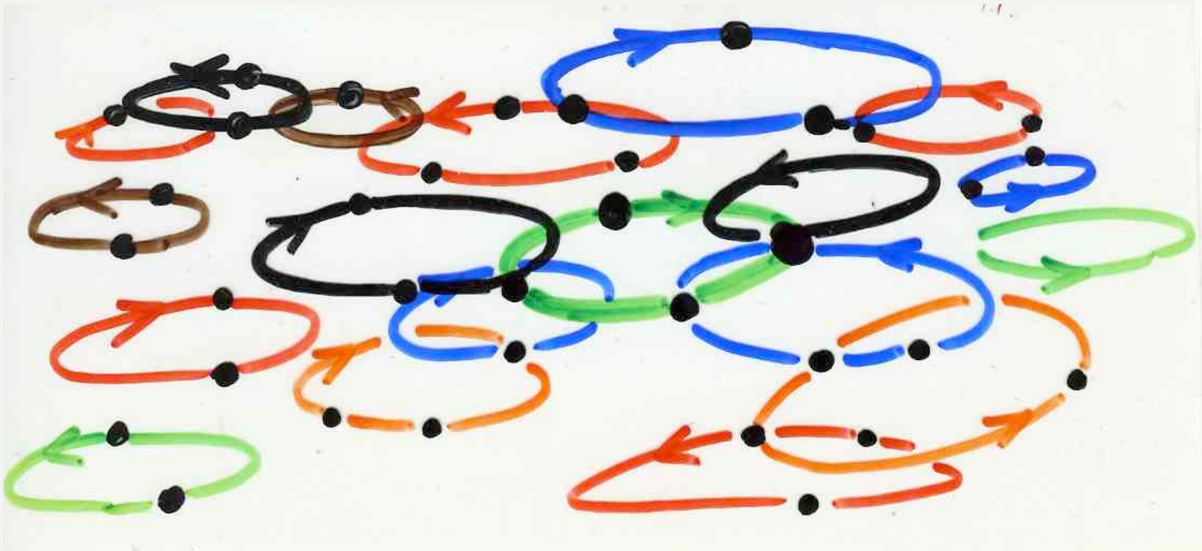
proof of Jacobi identity

$$\log(\det(B)) = \mathrm{Tr}(\log(B))$$

$$\log(\det(B)) = \mathrm{Tr}(\log(B))$$

$$B = (1-A)^{-1}$$

$$\frac{1}{\det(I-A)} = \sum_{\substack{E \\ \text{heap of cycles on } [1,k]}} v(E)$$



$$\log \frac{1}{\det(I-A)} = \sum_{P} v(P) \frac{t^{\ell(P)}}{\ell(P)}$$

P pointed pyramid of cycles (P, x)

$x \in \gamma_{max}$

pointed pyramid of cycles

$$\log (I-A)^{-1} = \sum_{n\geq 1} \frac{1}{n} A^n$$

$$\text{coeff.}_{ij} \sum_{n\geq 1} \frac{1}{n} A^n = \sum_{\substack{\omega \\ i \to j}} \frac{1}{|\omega|} v(\omega)$$

$$\mathrm{Tr} \sum_{n\geq 1} \frac{1}{n} A^n = \sum_{\substack{\omega \\ \text{circuit}}} \frac{1}{|\omega|} v(\omega)$$

$$\mathrm{Tr} \log (I-A)^{-1}$$

$$\mathrm{Tr}\,\log(I-A)^{-1} = \sum_{\substack{\omega \\ \text{circuit}}} \frac{1}{|\omega|} v(\omega)$$

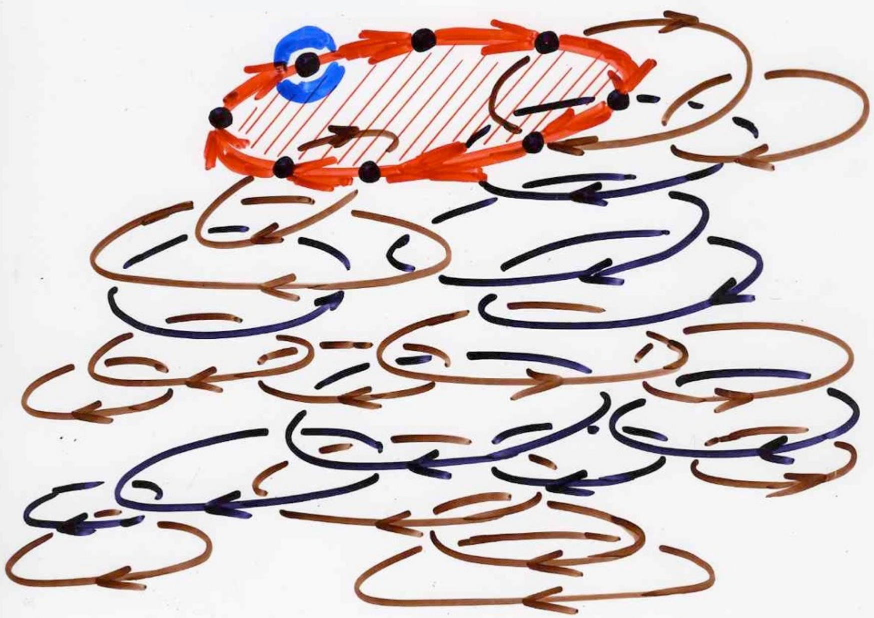
$$\underset{i \curvearrowright i}{\omega} \xrightarrow{\chi} (\eta, E) \qquad = \sum_{\substack{P \\ \text{pointed pyramid} \\ \text{of cycles} \\ (P,x)}} v(P)\,\frac{t^{\ell(P)}}{\ell(P)}$$

$\eta = (i)$ $\quad E$ pyramid of cycles

$i \in \gamma_{max}$
the unique maximal piece of the pyramid

$x \in \gamma_{max}$

$$\mathrm{Tr}\ \log (I-A)^{-1}$$



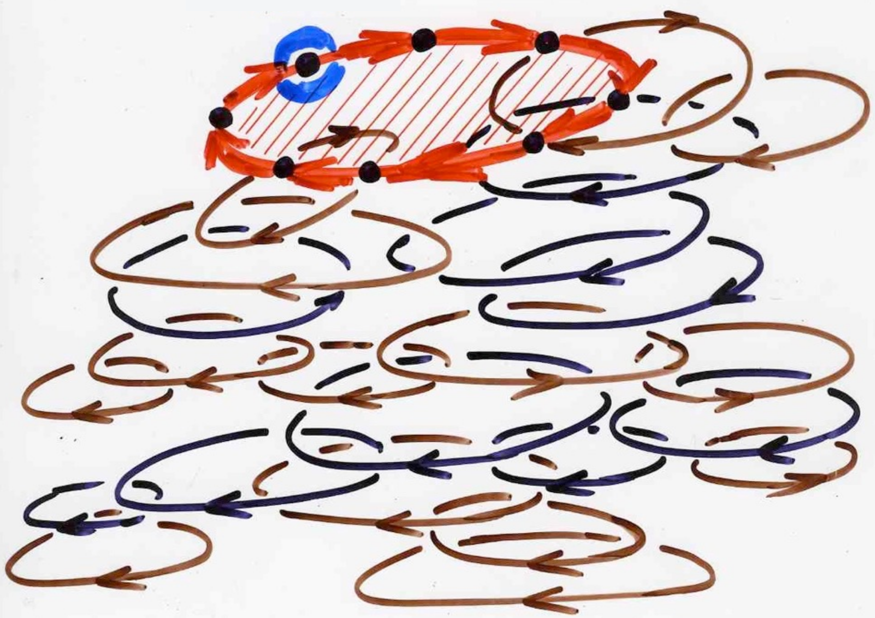
pointed pyramid of cycles

$$= \sum_{\substack{P \\ \text{pointed pyramid} \\ \text{of cycles} \\ (P,\,x)}} v(P)\,\frac{t^{\ell(P)}}{\ell(P)}$$

$x \in \gamma_{max}$

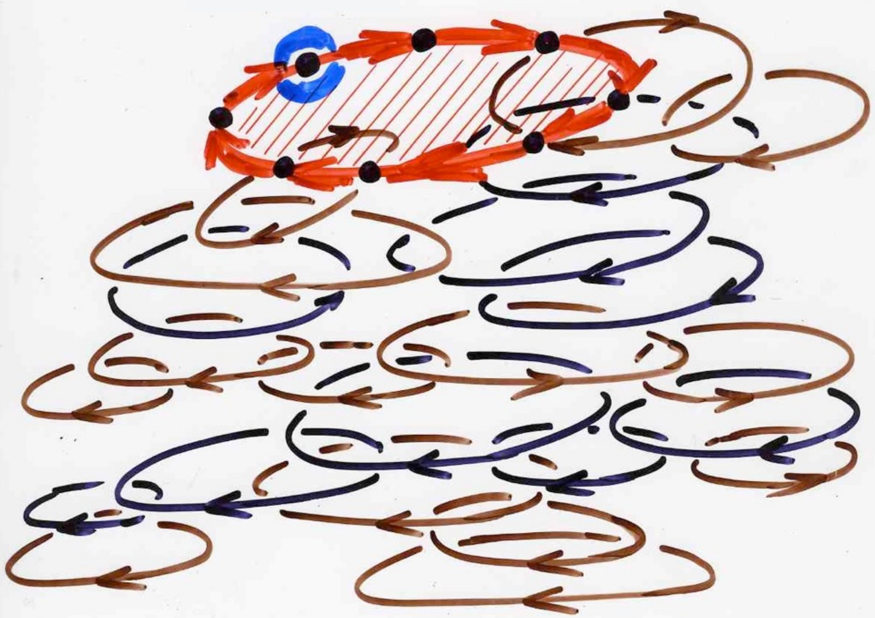
$$\mathrm{Tr}\,\log(I-A)^{-1} \qquad \log\frac{1}{\det(I-A)}$$

$$= \sum_{\substack{P\\ \text{pointed pyramid}\\ \text{of cycles}\\ (P,x)}} v(P)\,\frac{t^{\ell(P)}}{\ell(P)} \qquad x \in \gamma_{max}$$



pointed pyramid of cycles

$$\mathrm{Tr}\,\log(I-A)^{-1} = \log\frac{1}{\det(I-A)}$$



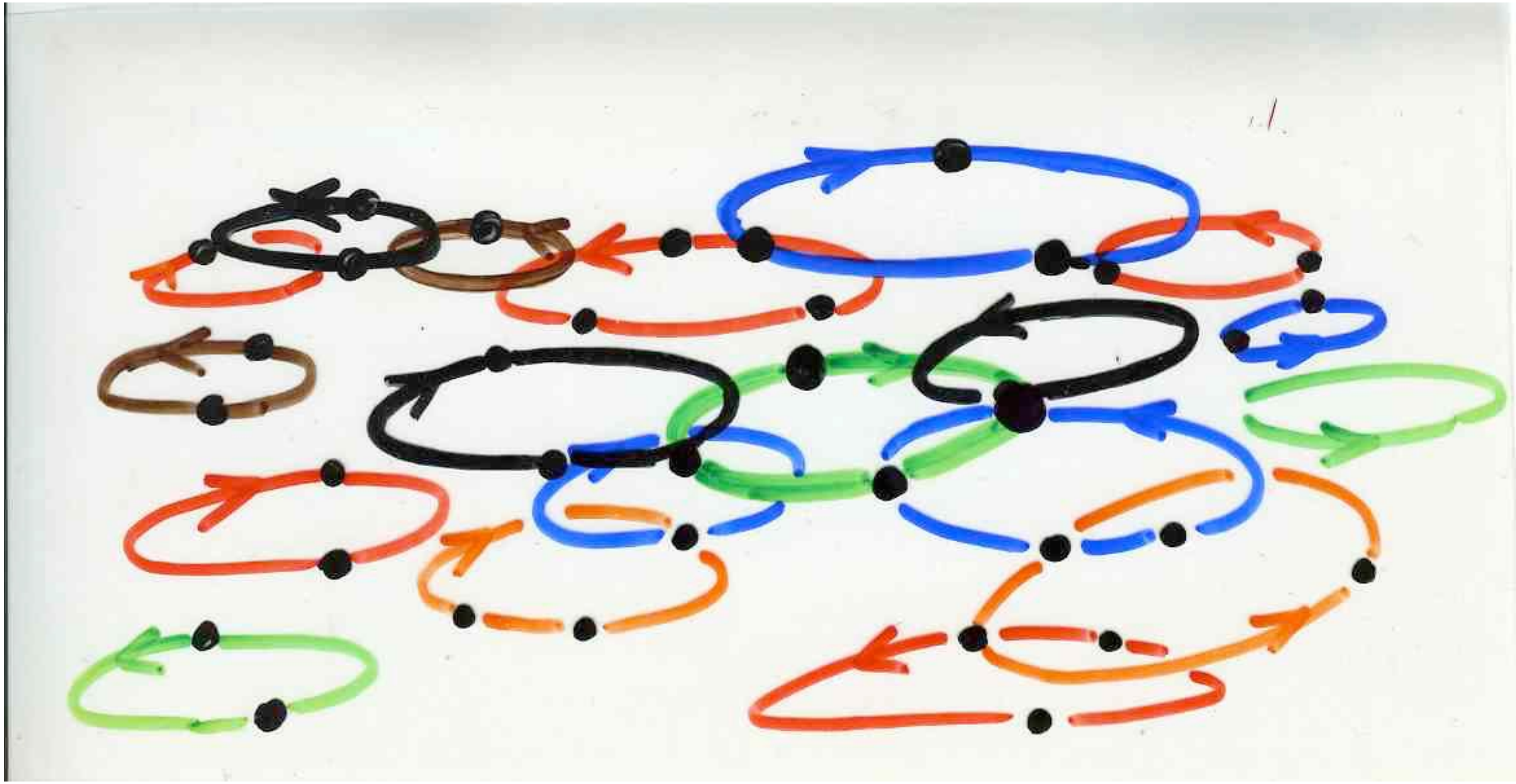
pointed pyramid of cycles

Jacobi identity

with exponential generating function

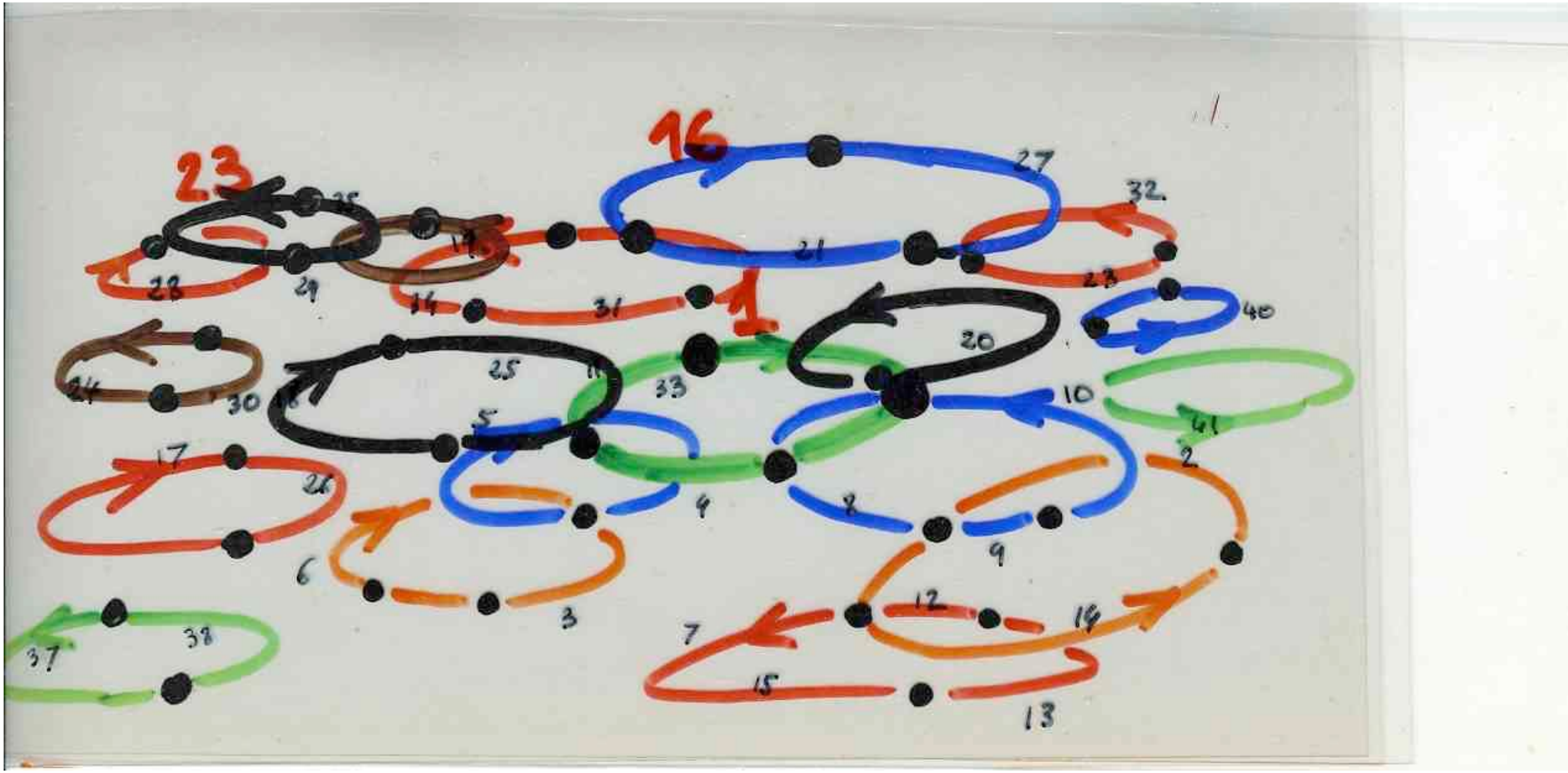
$$\log(\det(B)) = \mathrm{Tr}(\log(B))$$

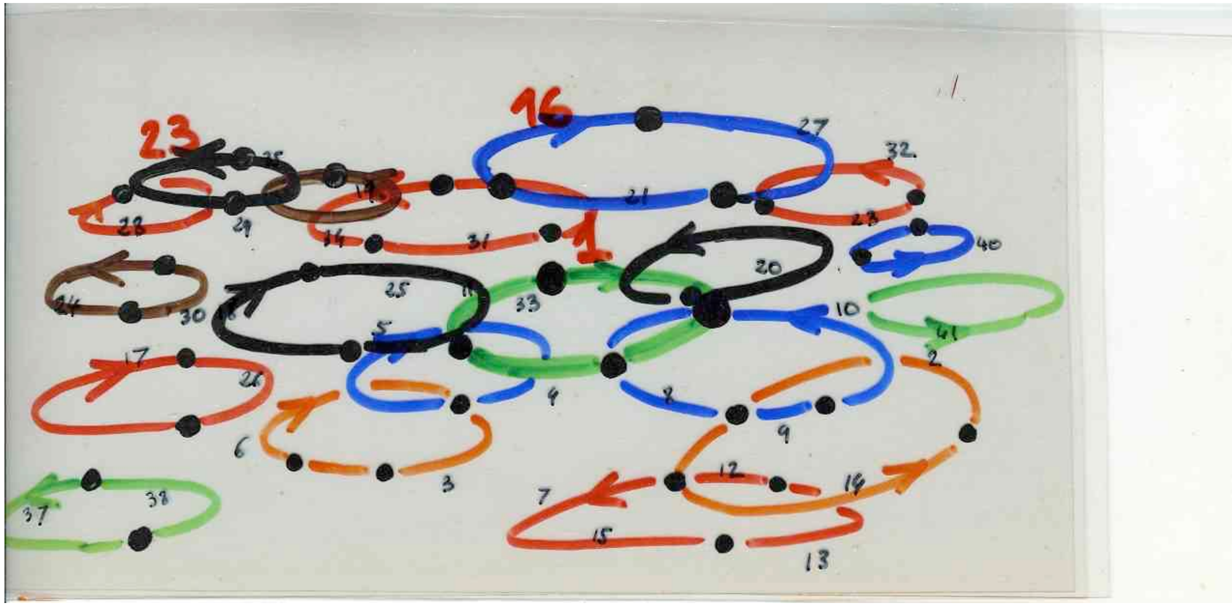
$$\frac{1}{\det(I-A)}$$



$$\sum_{E} v(E) \frac{t^{\ell(E)}}{\ell(E)!}$$

E labeled heap of cycles

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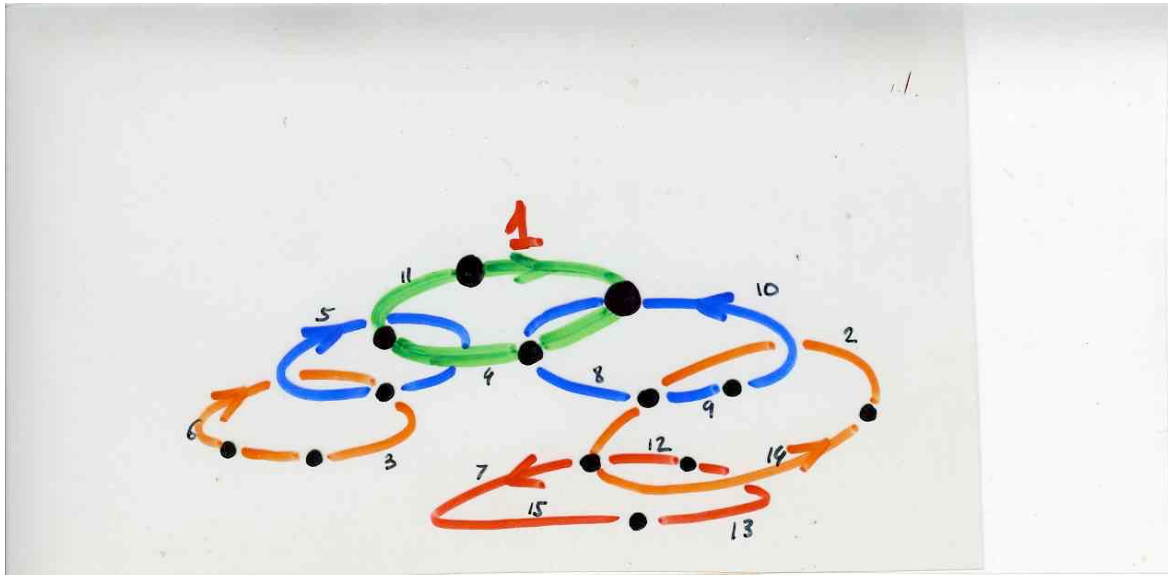
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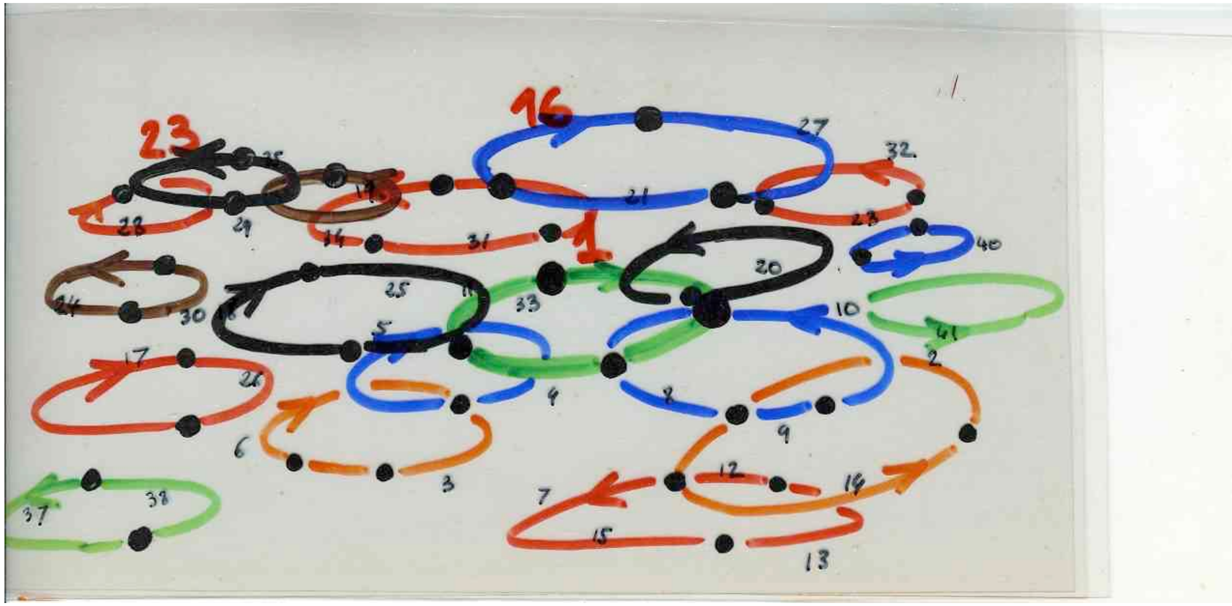
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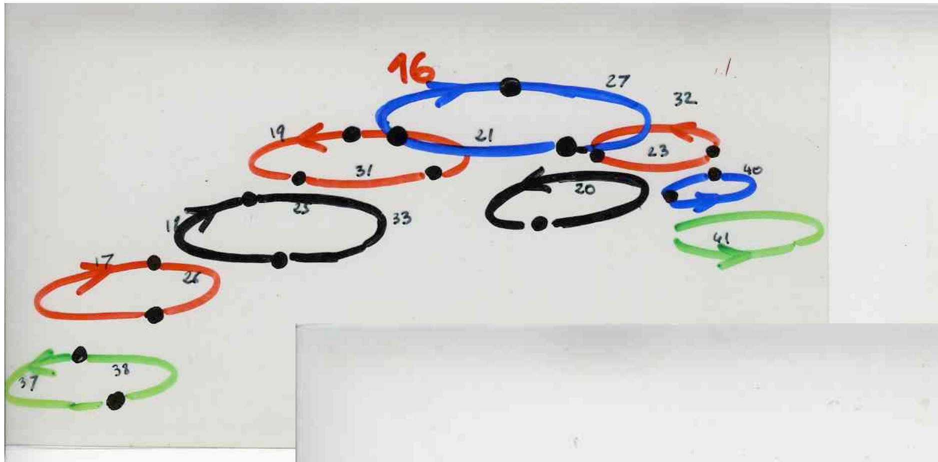
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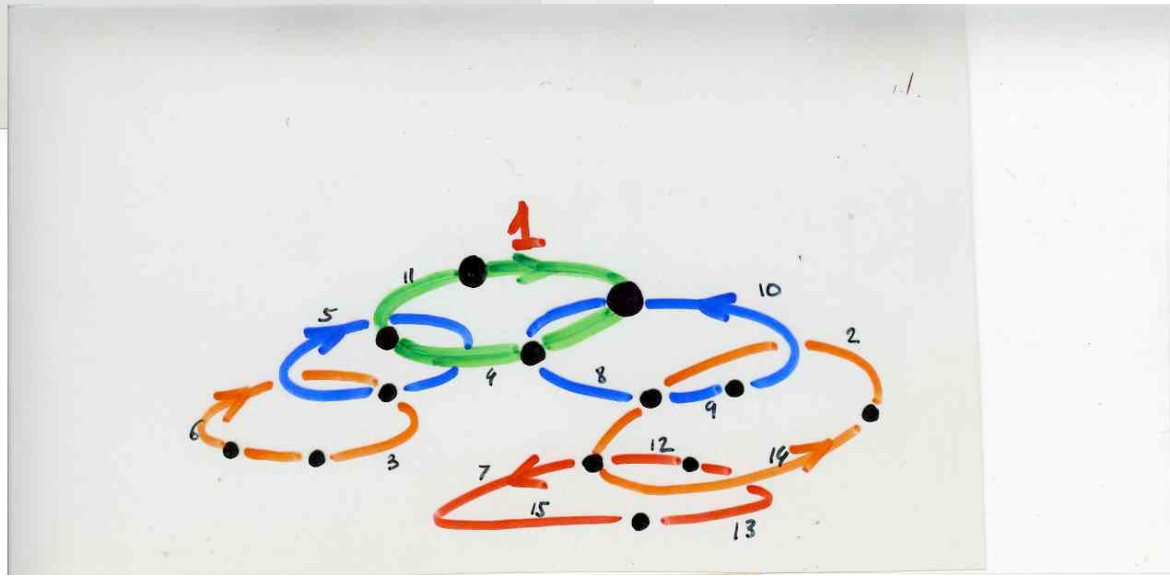
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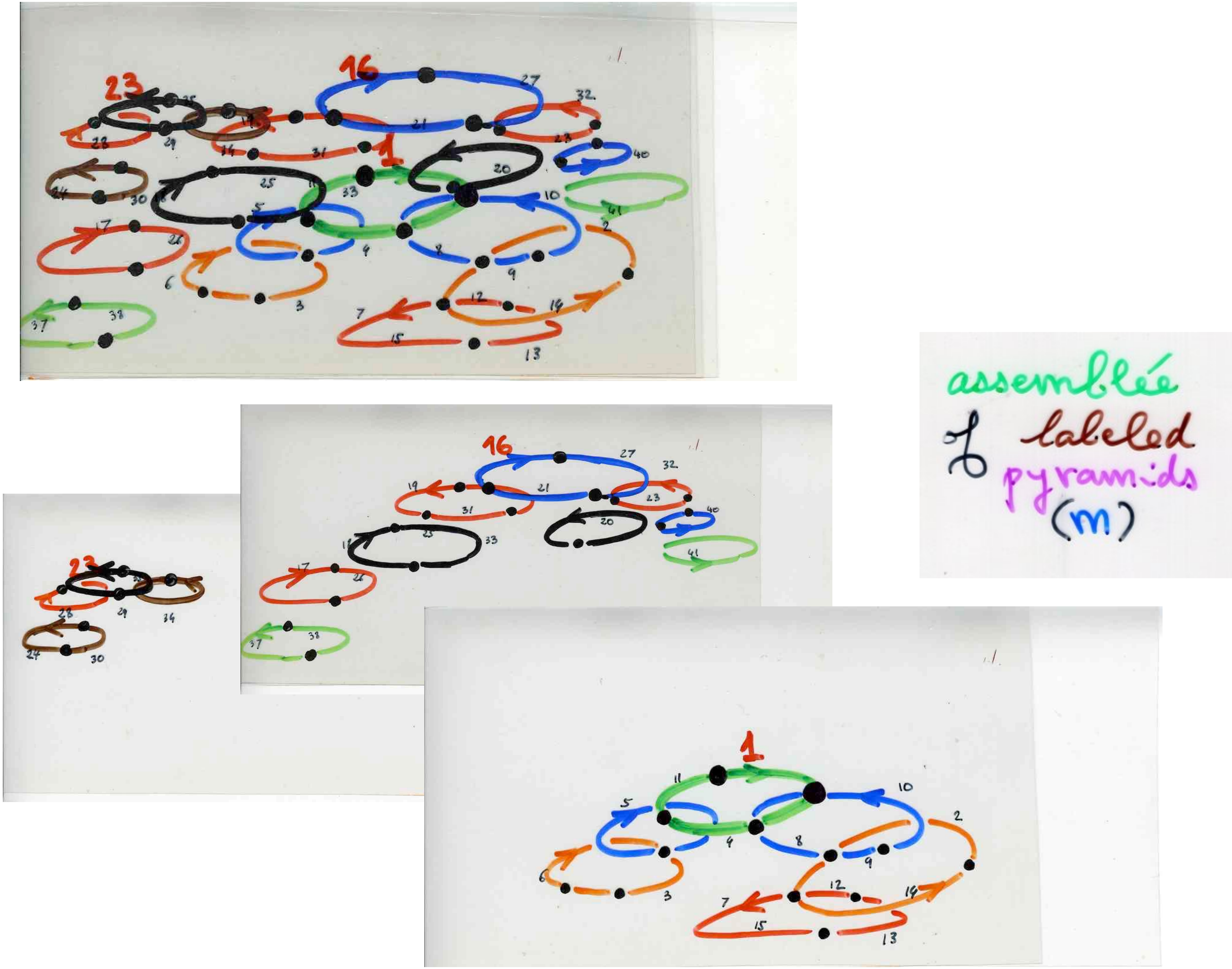


assemblée
of labeled
pyramids
(m)

assemblée
of labeled
pyramids
(m)

(m) the label of the (unique) maximal piece is the minimum of the labels of the pieces of the pyramid P

$$\sum_{\substack{E \\ \text{labeled} \\ \text{heap} \\ \text{of cycles}}} v(E) \frac{t^{\ell(E)}}{\ell(E)!} = \exp\left(\sum_{\substack{P \\ \text{labeled} \\ \text{pyramid} \\ \text{of cycles} \\ (m)}} v(P) \frac{t^{\ell(P)}}{\ell(P)!} \right)$$

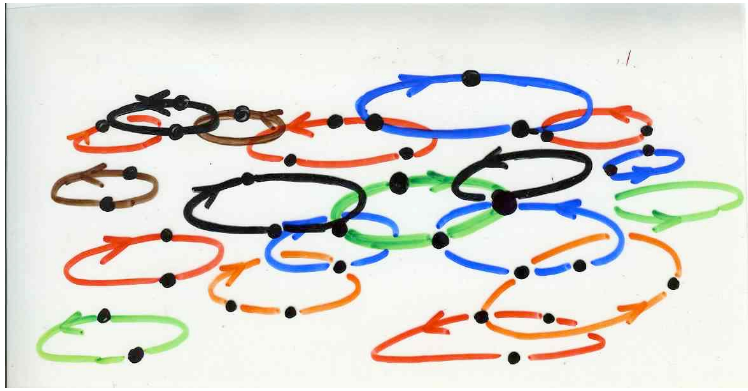
$$\sum_{\substack{E \\ \text{heap} \\ \text{of cycles}}} v(E)\, t^{\ell(E)} = \exp\left(\sum_{\substack{P \\ \text{pointed} \\ \text{pyramid} \\ \text{of cycles} \\ (P,x)}} v(P) \frac{t^{\ell(P)}}{\ell(P)} \right)$$

$$x \in \gamma_{max}$$

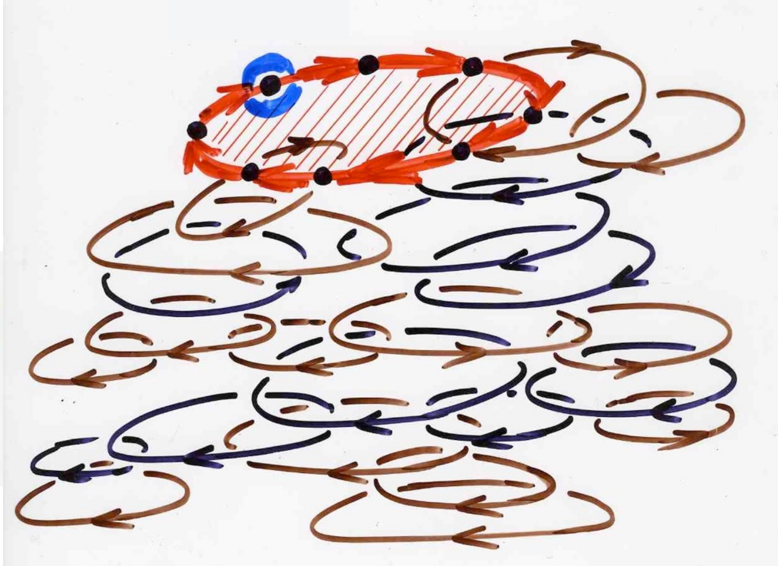
$$\sum_{\substack{E \\ \text{labeled} \\ \text{heap} \\ \text{of cycles}}} v(E) \frac{t^{\ell(E)}}{\ell(E)!} = \exp\left(\sum_{\substack{P \\ \text{labeled} \\ \text{pyramid} \\ \text{of cycles} \\ (m)}} v(P) \frac{t^{\ell(P)}}{\ell(P)!} \right)$$

$$\sum_{\substack{E \\ \text{heap} \\ \text{of cycles}}} v(E)\, t^{\ell(E)} = \exp\left(\sum_{\substack{P \\ \text{pointed} \\ \text{pyramid} \\ \text{of cycles} \\ (P,\, x)}} v(P) \frac{t^{\ell(P)}}{\ell(P)} \right)$$

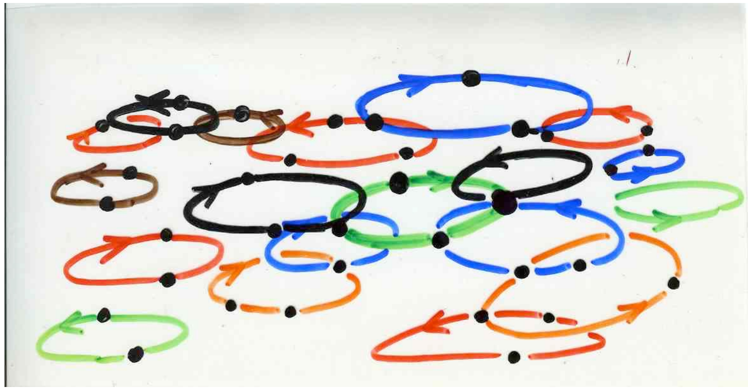
$x \in \gamma_{max}$



$$\frac{1}{\det(I-A)} = \exp\Big(\quad \Big)$$



pointed pyramid of cycles



pointed pyramid of cycles

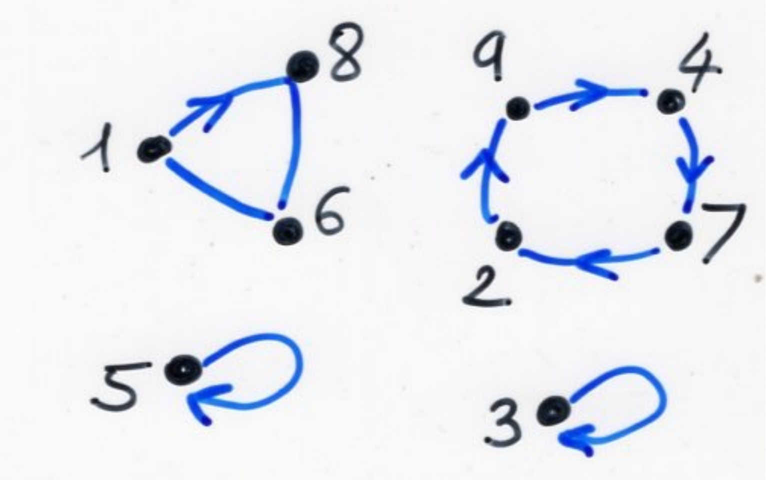
$$\frac{1}{\det(I-A)} = \exp\left(\operatorname{Tr}\log(I-A)^{-1}\right)$$

beta extension of McMahon Master theorem

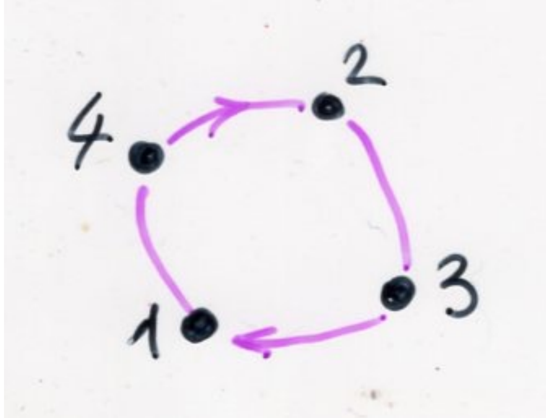
$$\left(\frac{1}{\det(I-A)}\right)^{\beta}$$

Foata, Zeilberger (1983)

($\rightarrow$ Laguerre polynomials)



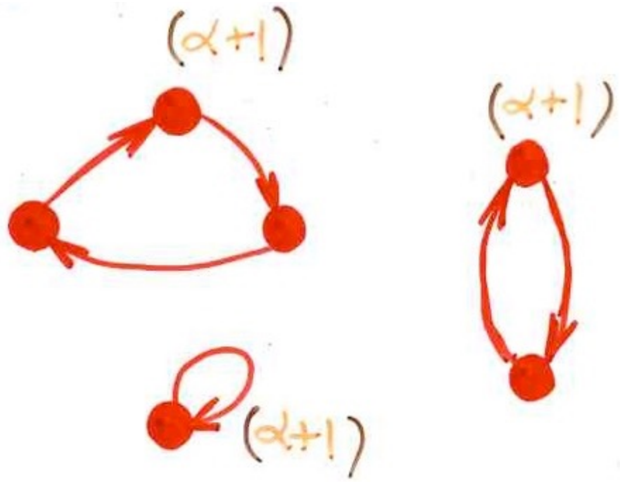
cyclic
permutation



$$\sum_{n\geq 0} n!\,\frac{t^n}{n!} = \frac{1}{1-t}$$

$$\sum_{n\geq 1} (n-1)!\,\frac{t^n}{n!} = \sum_{n\geq 1} \frac{t^n}{n}$$

$$= \log \frac{1}{1-t}$$



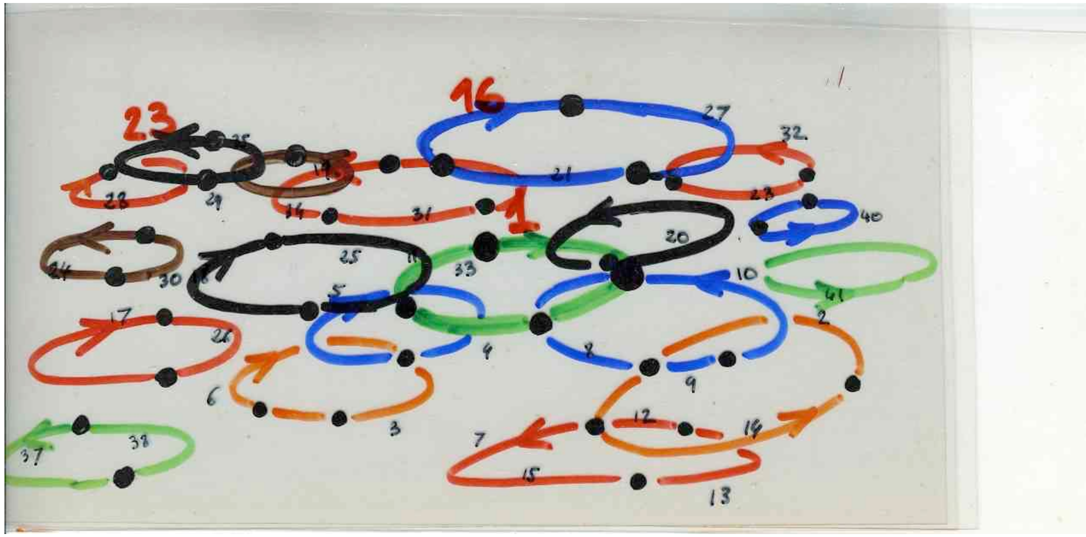
$$\exp\left((\alpha+1)\log\frac{1}{(1-t)}\right) = \exp\left(\log\frac{1}{(1-t)^{(\alpha+1)}}\right)$$

$$= \frac{1}{(1-t)^{(\alpha+1)}}$$

$$\log\left(\frac{1}{\det(I-A)}\right)^{\beta} = \beta\log\left(\frac{1}{\det(I-A)}\right)$$

$$\left(\frac{1}{\det(I-A)}\right)^{\beta} = \exp\ \beta\log\left(\frac{1}{\det(I-A)}\right)$$

$$\left(\sum_{\substack{E \\ \text{labeled} \\ \text{heap} \\ \text{of cycles}}} v(E)\frac{t^{\ell(E)}}{\ell(E)!}\right)^{\beta} = \exp\ \beta\left(\sum_{\substack{P \\ \text{labeled} \\ \text{pyramid} \\ \text{of cycles} \\ (m)}} v(P)\frac{t^{\ell(P)}}{\ell(P)!}\right)$$

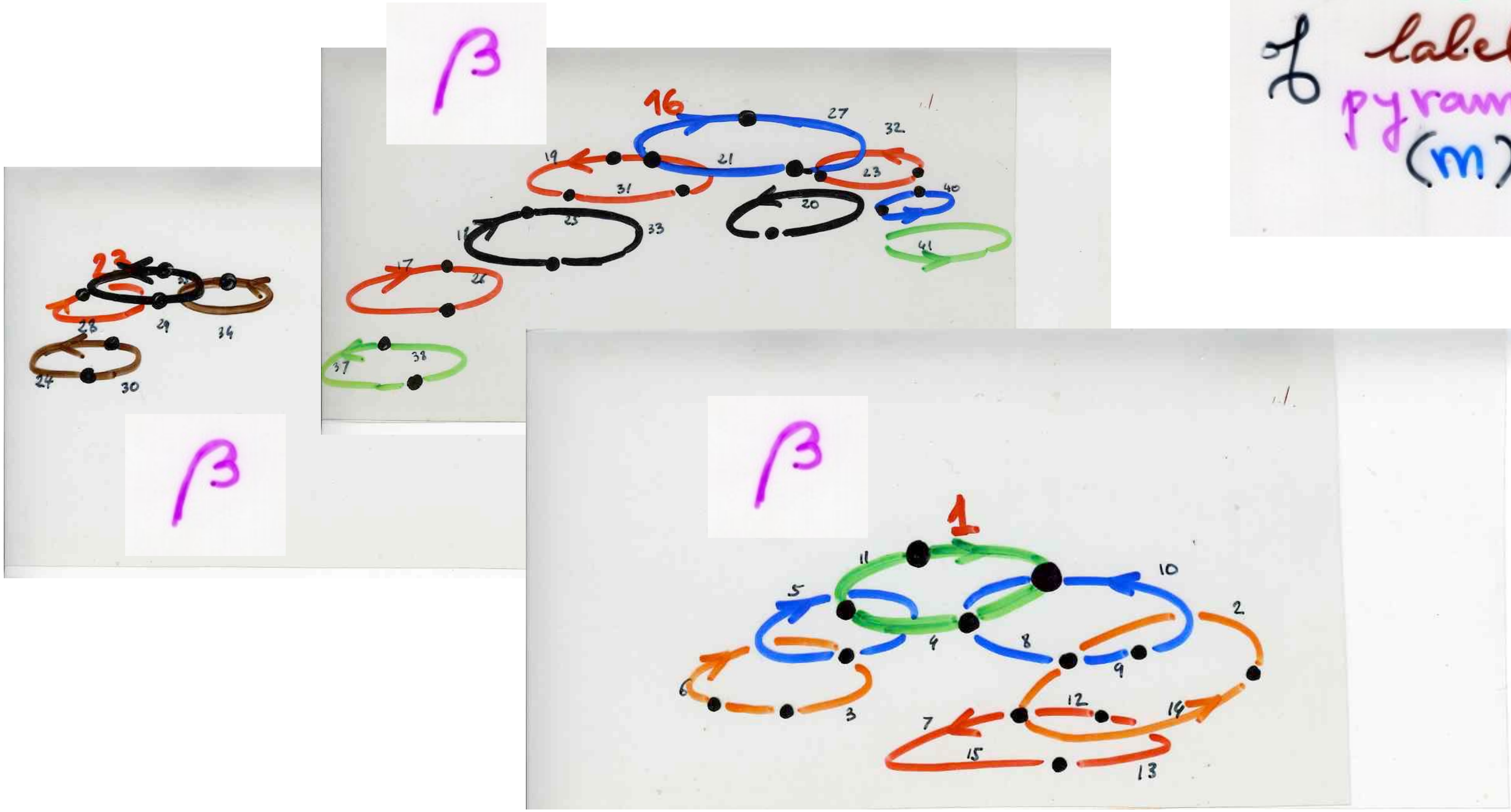


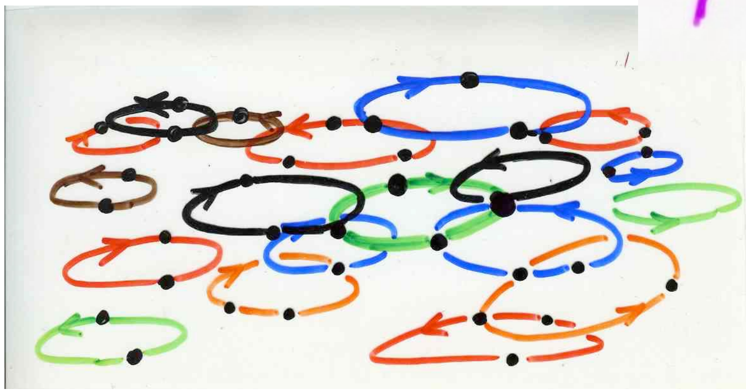
assemblée

of labeled

pyramids

(m)

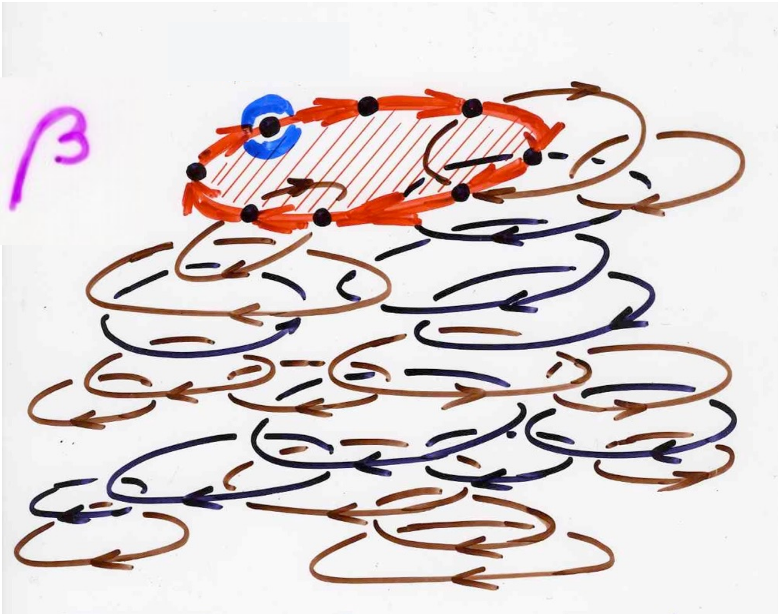




β !

labeled heaps are needed

β



pointed pyramid of cycles

$$\exp\left(\beta \,\mathrm{Tr}\,\log(I-A)^{-1}\right)$$

$$\frac{1}{\det(I-A)}^{\beta} = \exp\left(\beta \sum_{P} v(P)\,\frac{t^{\ell(P)}}{\ell(P)}\right)$$

P pointed pyramid of cycles (P, x)

$x \in \gamma_{max}$

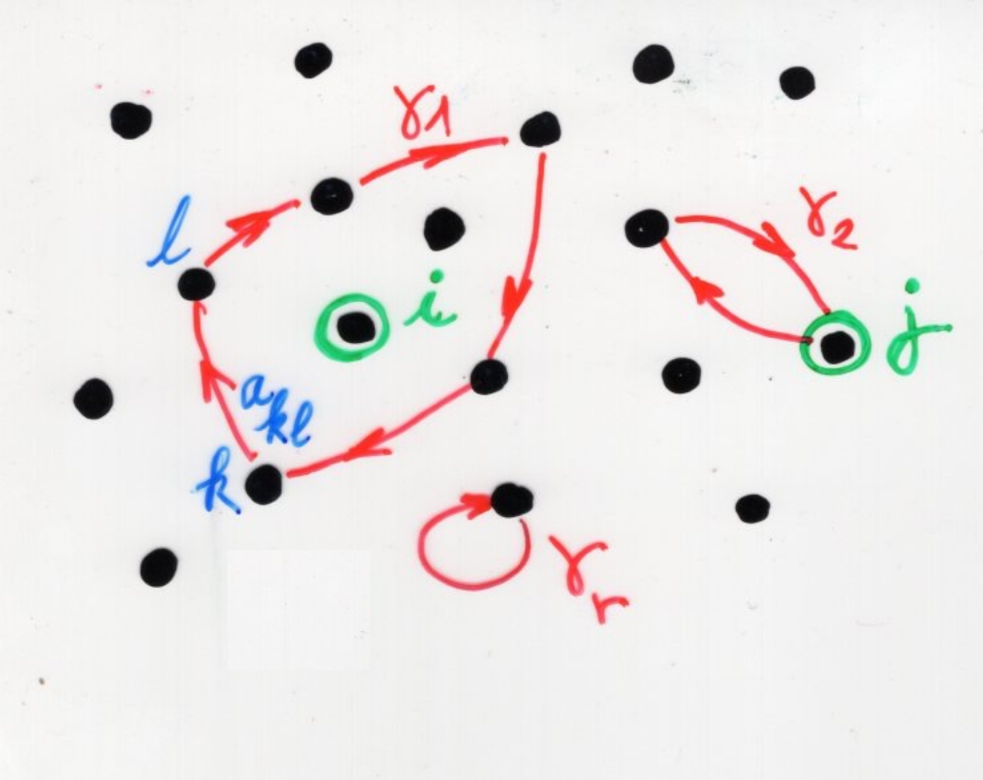
Cayley - Hamilton theorem

characteristic polynomial of a matrix A

$$\det(\lambda I - A) = P_A(\lambda)$$

<u>Theorem</u> (Cayley-Hamilton)

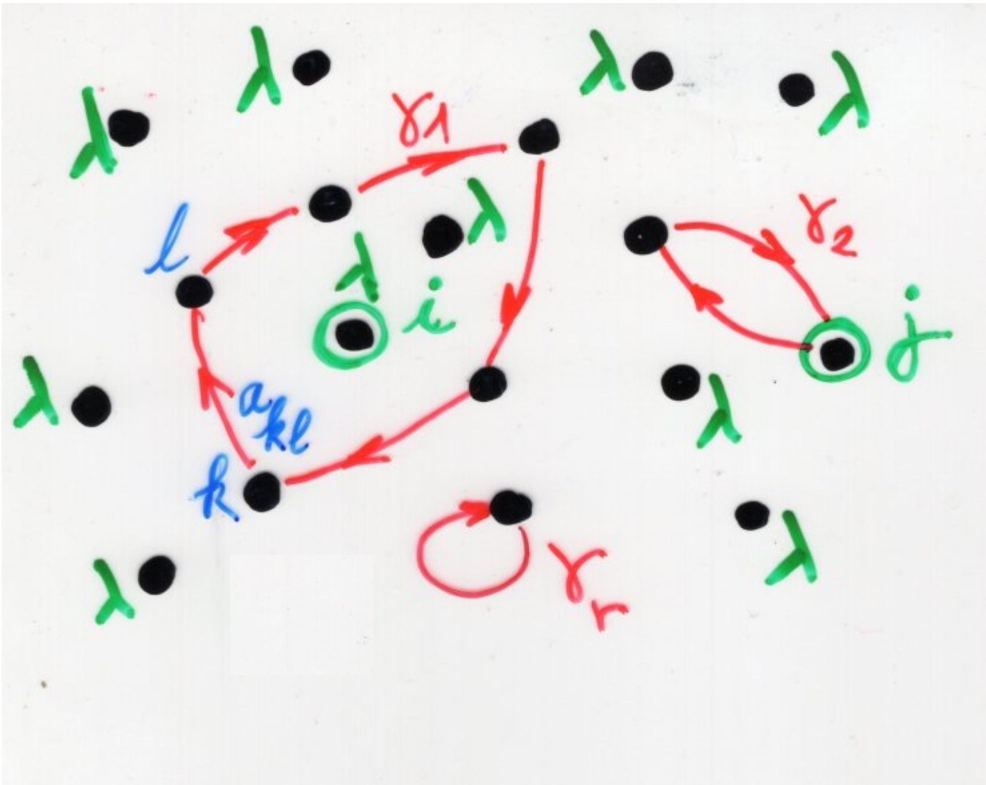
$$P_A(A) = 0$$



$ip(F)$ number of isolated points of F

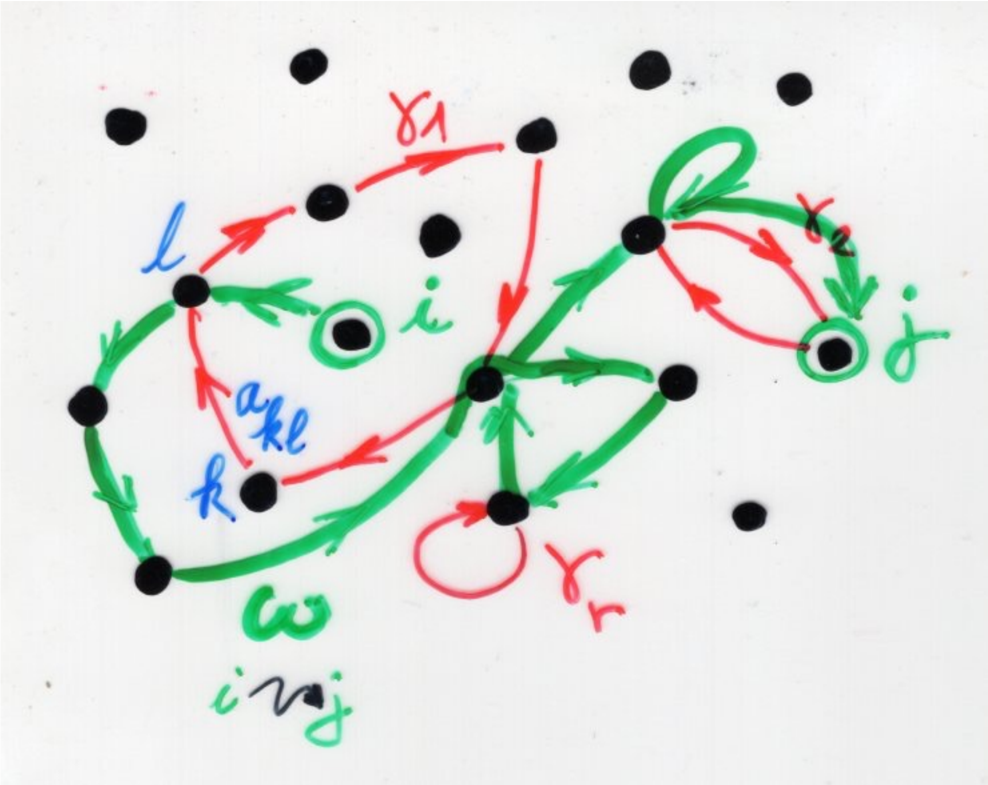
$$\det(\lambda I - A) = \sum_{\substack{F=\{\gamma_1,\dots,\gamma_r\}\\ \text{trivial heap of cycles}}} (-1)^r v(\gamma_1)\cdots v(\gamma_r)\,\lambda^{ip(F)}$$

$P_A(\lambda)$



$$\det(\lambda I - A) = \sum_{\substack{F=\{\gamma_1,\dots,\gamma_r\} \\ \text{trivial heap} \\ \text{of cycles}}} (-1)^r\, v(\gamma_1)\cdots v(\gamma_r)\,\lambda^{cp(F)}$$

$P_A(\lambda)$



$$\left(\mathcal{P}_A(A)\right)_{i,j}$$

$$\det(\lambda I - A) = \sum_{F=\{\gamma_1,\dots,\gamma_r\}} (-1)^r\, v(\gamma_1)\cdots v(\gamma_r)\,\lambda^{cp(F)}$$

trivial heap of cycles

$$\mathcal{P}_A(\lambda)$$

$$\left(P_A(A)\right)_{i,j} = \sum_{(\omega, F)} (-1)^{|F|} v(\omega)\, v(F)$$

ω path $i \rightsquigarrow j$

F trivial cycle heap

(total number of edges in ω and in $F = n$)

$$\omega \xrightarrow{\chi} (\eta, E)$$

η self-avoiding path $i \rightsquigarrow j$

- E heap of cycles such that the projections $\alpha = \pi(m)$ of the maximal pieces intersect η

in η, E

$$\Rightarrow \quad v(\omega) = v(\eta)\, v(E)$$

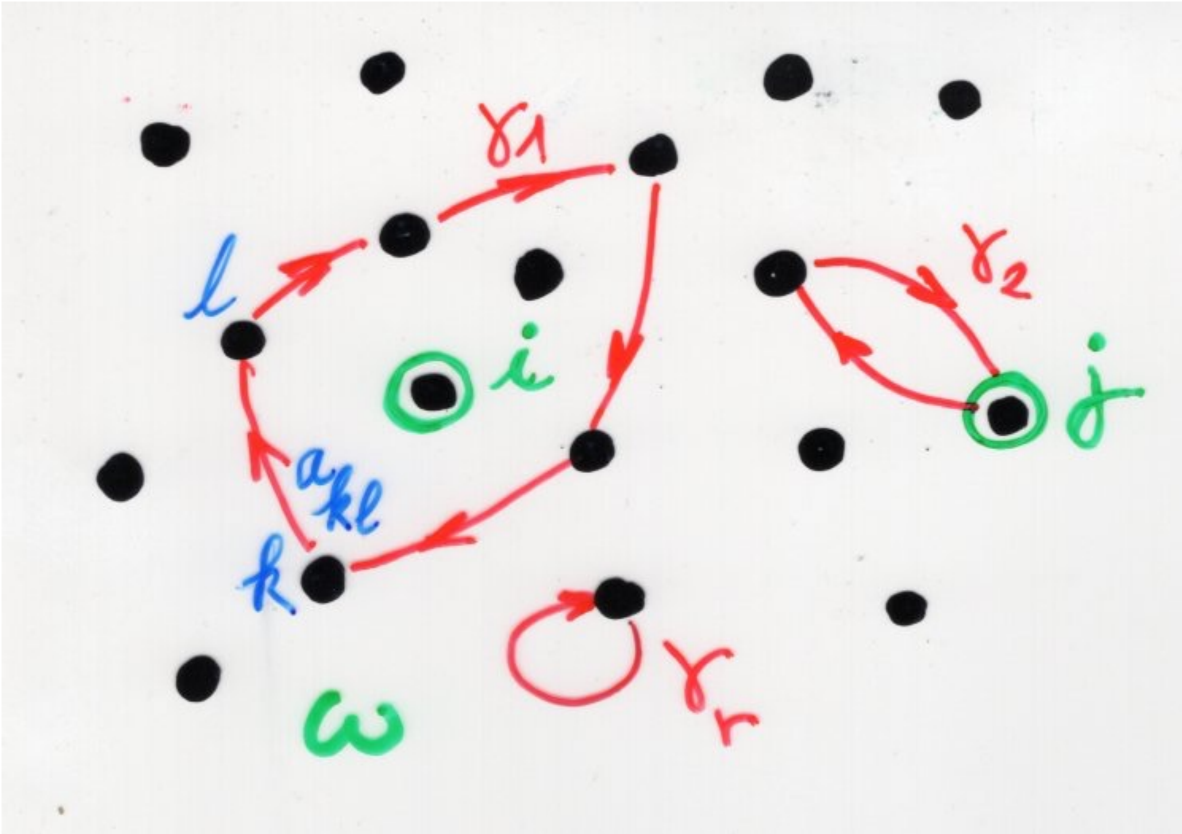
$$\left(P_A(A)\right)_{ij} = \sum_{(\eta, E, F)} (-1)^{|F|} v(\eta)\, v(E)\, v(F)$$

η self-avoiding path $i \rightsquigarrow j$

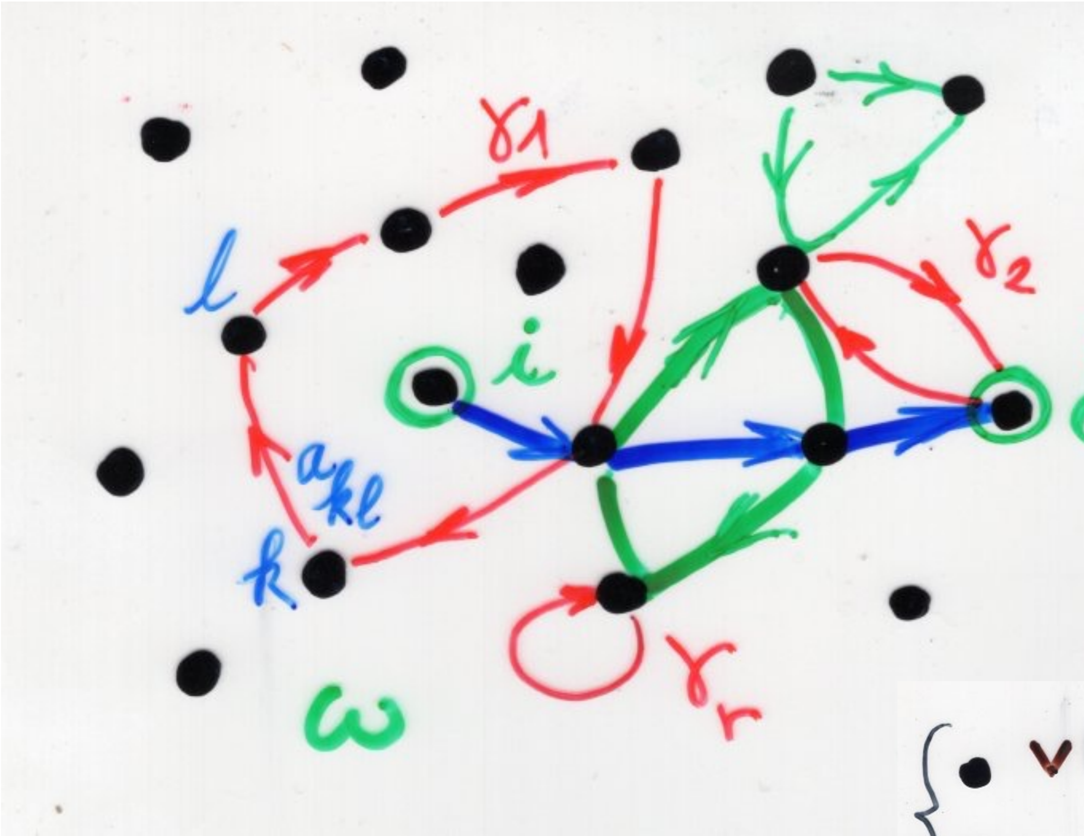
- E heap of cycles such that the projections $\alpha = \pi(m)$ of the maximal pieces intersect η

F trivial cycle heap

(total number of edges in η, E and in F $= n$)



$$\left(P_A(A)\right)_{ij} = \sum_{(\eta, E, F)} (-1)^{|F|}\, v(\eta)\, v(E)\, v(F)$$



define an involution φ

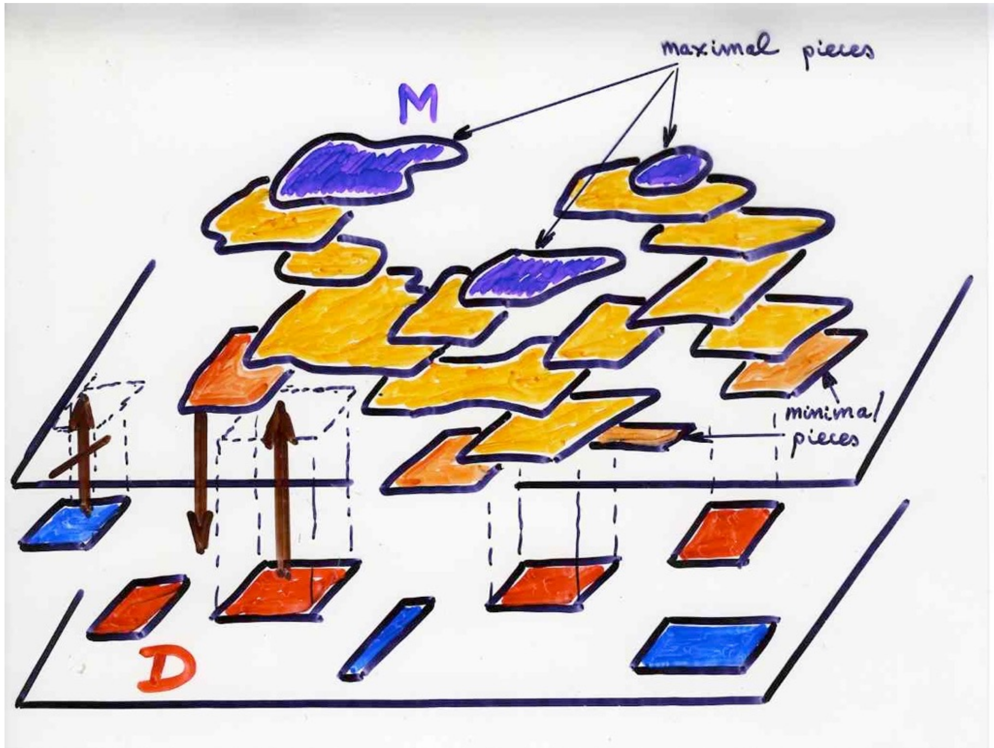
$$\varphi(E, F) = (E', F')$$

E: heap $\quad$ F: trivial heap

$\prod \left(\substack{\text{maximal}\\ \text{pieces}}\right) \in M$

$$\begin{cases} \bullet\ v(E)\,v(F) = v(E')\,v(F') \\ \bullet\ (-1)^{|F|} = -(-1)^{|F'|} \end{cases}$$

$$\left(P_A(A)\right)_{ij} = \sum_{(\eta, E, F)} (-1)^{|F|}\, v(\eta)\, v(E)\, v(F)$$



Ch 2b, p 83–94

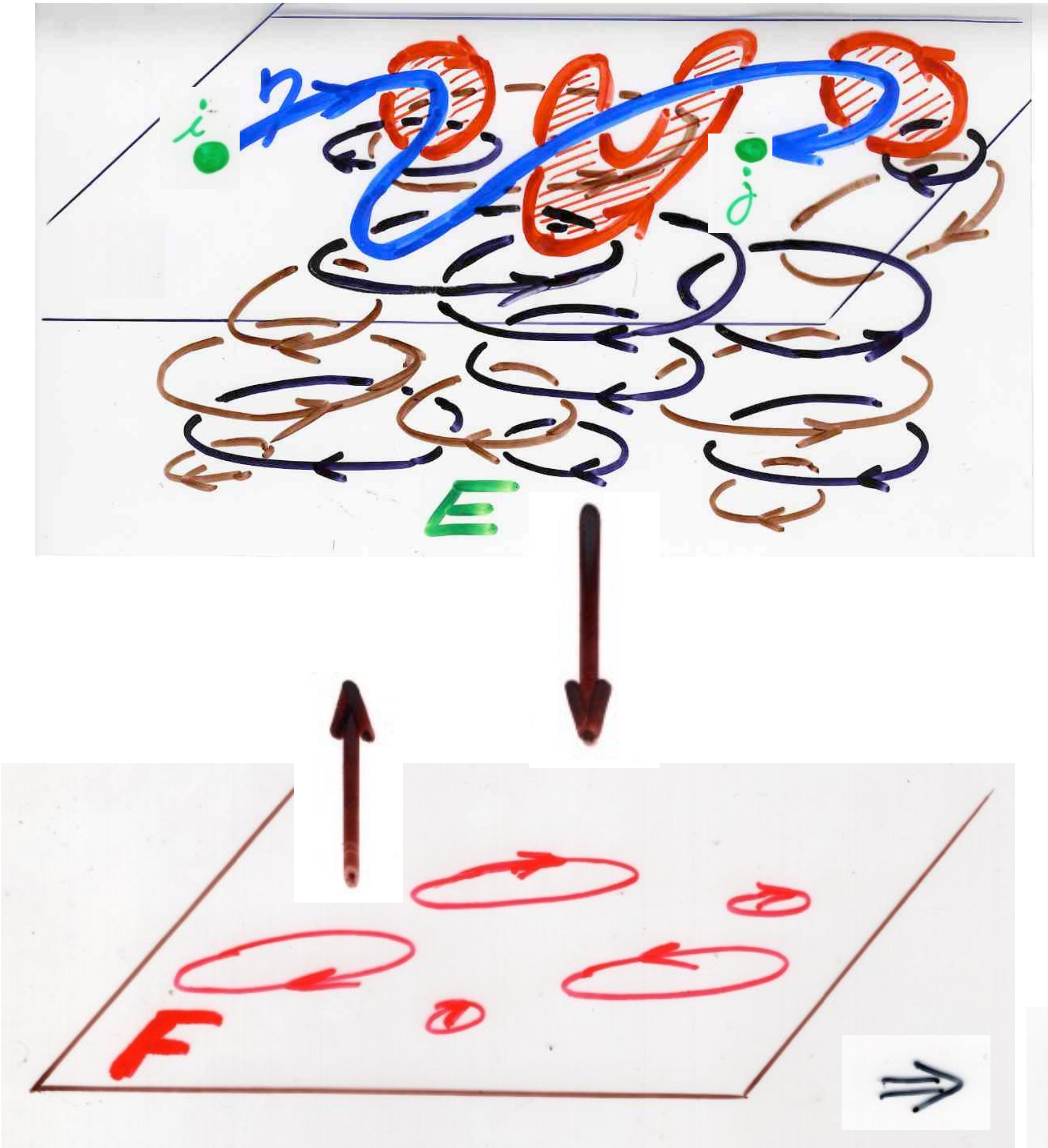
extension of the inversion lemma

N/D

$$\text{Trans}(E,F) = \left\{ \begin{array}{l} \alpha \in F,\ (\alpha, 0) \odot E \\ \text{is a heap with} \\ \pi\left(\begin{array}{c}\text{maximal}\\ \text{pieces}\end{array}\right) \in M \end{array} \right\}$$

$$\cup$$

$$m(E,F) = \left\{ \begin{array}{l} m = (\beta, 0) \text{ minimal piece of } E \\ \text{such that } \alpha \not\mathcal{C} \beta \text{ for all } \alpha \in F \end{array} \right\}$$



If $\mathrm{Trans}(E, F) = \emptyset$ then $E = \emptyset$

$\Rightarrow$ for each cycle γ of F

$\gamma \cap \eta = \emptyset$

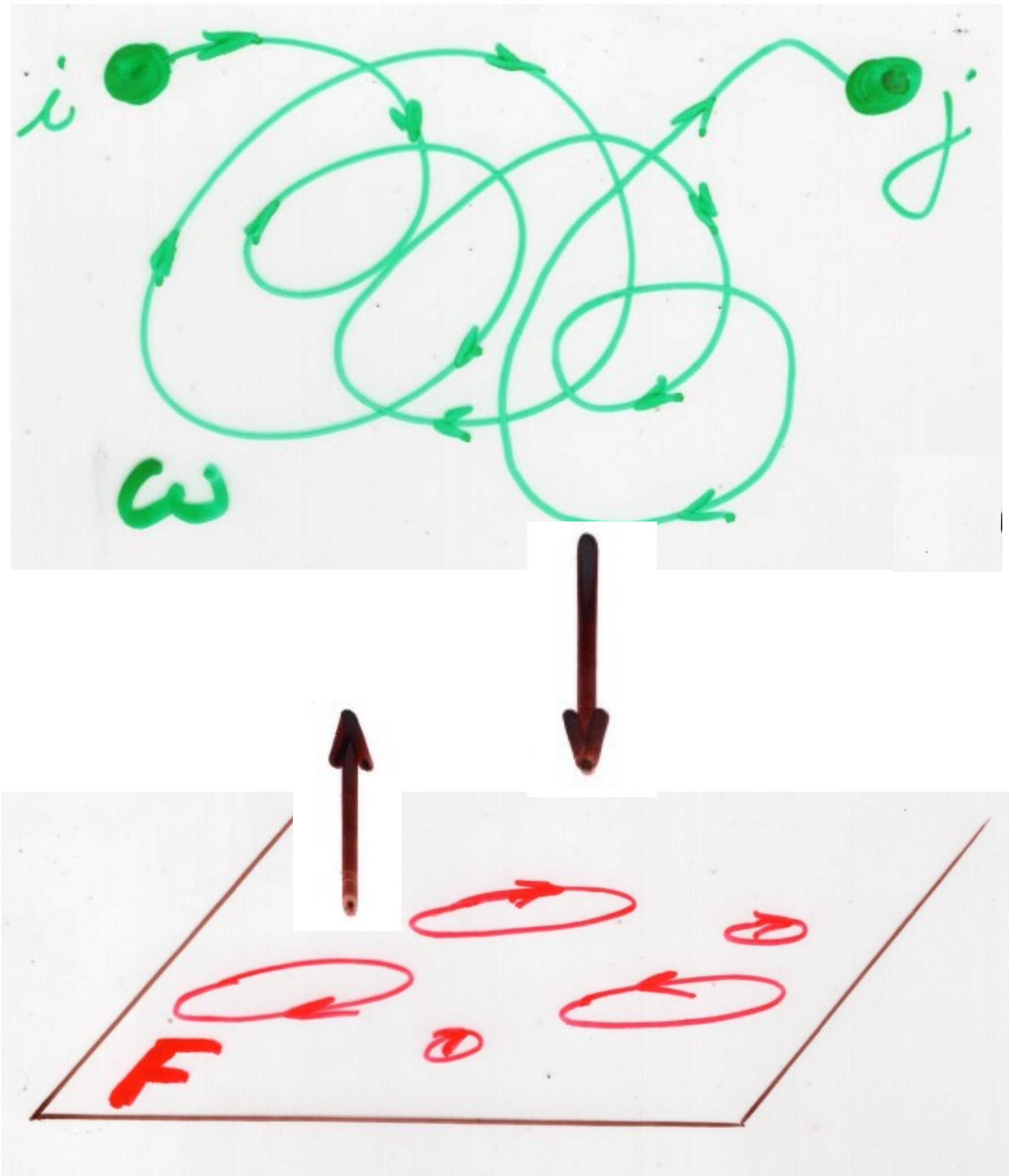
(no common vertices)

$\Rightarrow$ total number of edges in η and in the cycles of F $\underset{\text{strict}}{<} n$

contradiction with

(total number of edges in η, E and in F $= n$)

$\Rightarrow$ $\left(\mathcal{P}_A(A)\right)_{i,j} = 0$



another weight preserving involution

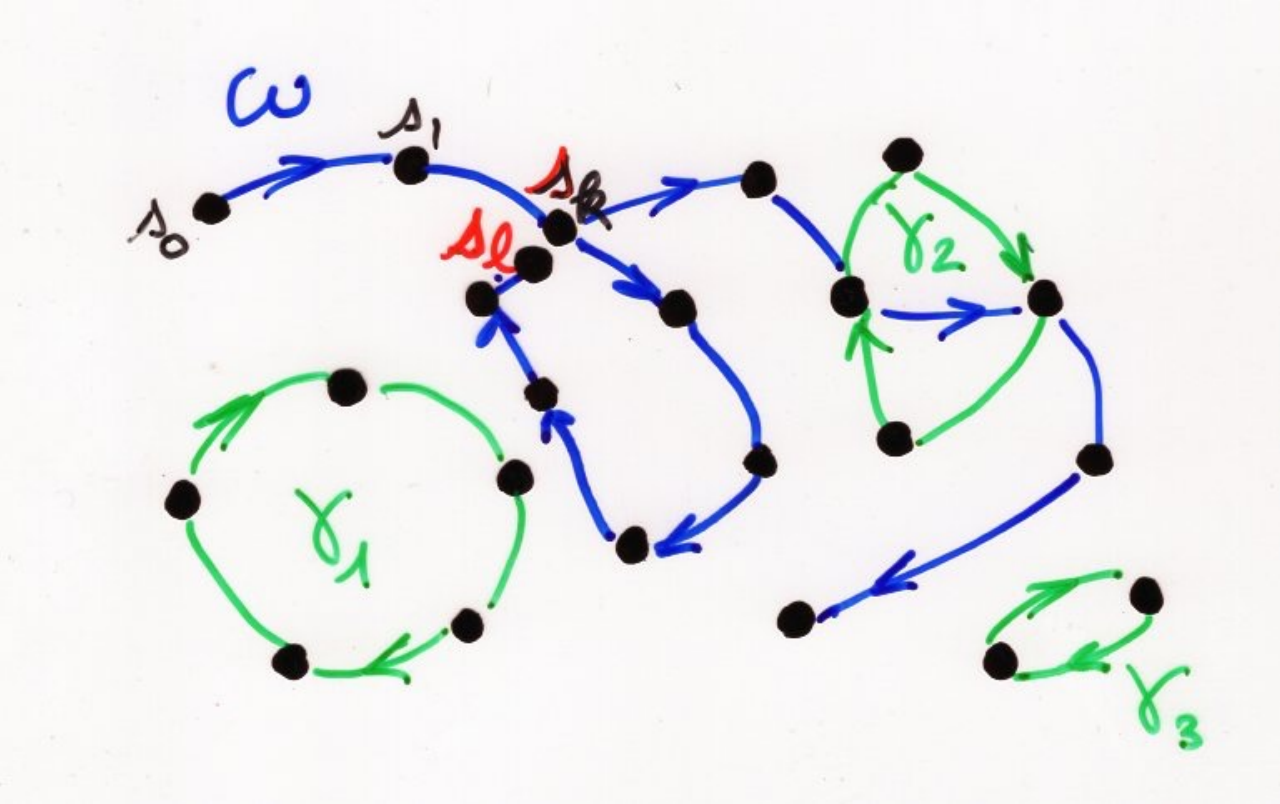
→ Ch 1c, p 9-18
course IMSc 2016

another weight preserving involution

→ Ch 1c, p 9-18
course IMSc 2016

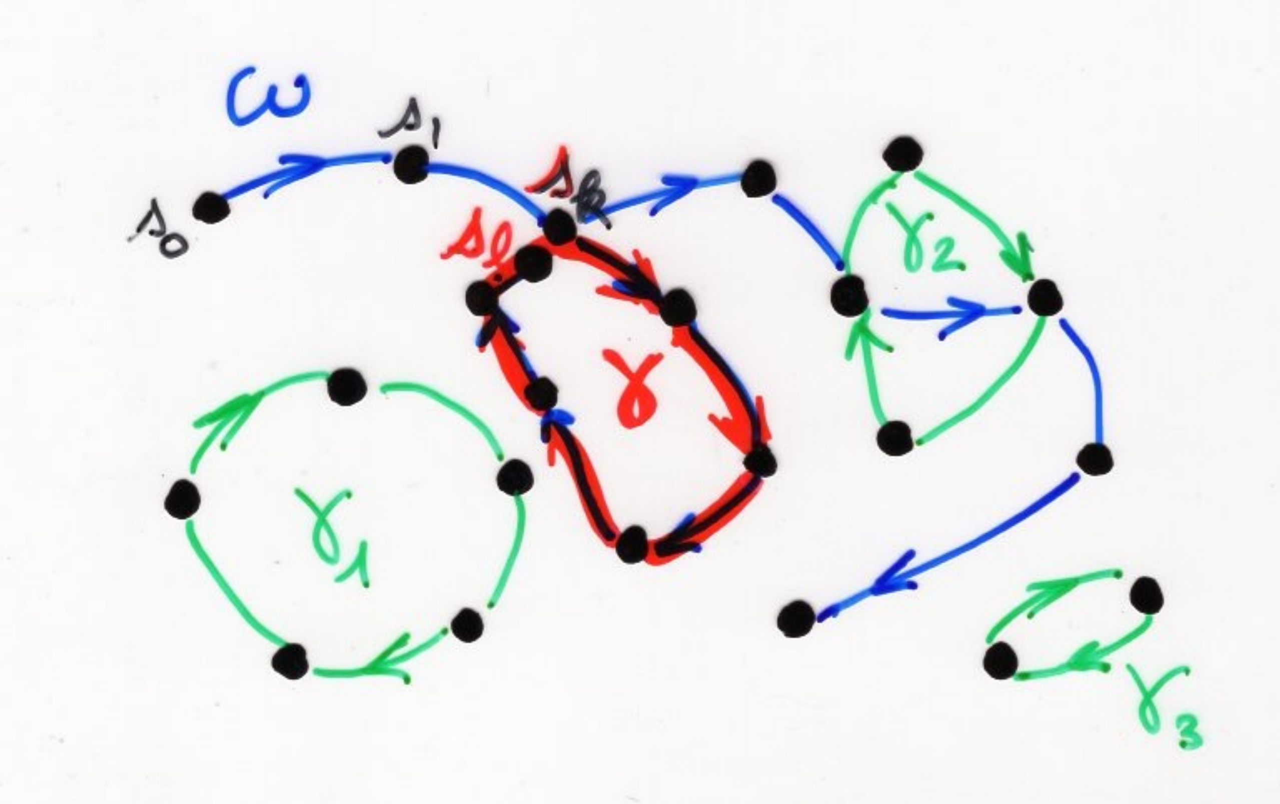
« direct » bijective proof of the identity

$$\sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} v(\omega) = \frac{N_{i,j}}{D}$$



Let ℓ be the smallest integer $0 \le \ell \le n$ such that:

or
- (i) $\exists k$, $0 \le k < \ell$ with $s_k = s_\ell$
- (ii) s_ℓ belongs to one of the cycles $\gamma_1, \ldots, \gamma_r$



Let ℓ be the smallest integer $0 \le \ell \le n$ such that:

or $\begin{cases} \text{(i) } \exists\, k,\ 0 \le k < \ell \text{ with } s_k = s_\ell \\ \text{(ii) } s_\ell \text{ belongs to one of the cycles } \gamma_1, \dots, \gamma_r \end{cases}$

a general transfer theorem

(complements
and
exercise)

$$\mathcal{F} \subseteq H(P, \mathcal{C})$$

family of heaps
with basic pieces P
and dependency relation $\mathcal{C}$

$$Q \subseteq P$$

<u>definition</u> $\alpha \in P$, $E \in H(P, \mathcal{C})$

α and E are dependent
iff there exist $(\beta, i) \in E$ with $\alpha \, \mathcal{C} \, \beta$

Define conditions (i) (ii) (iii)
for $\mathcal{F}$ and Q

(i) $E \in \mathfrak{F}$, $\alpha \in Q$, α and E dependent

$\Rightarrow \quad E \circ \alpha \in \mathfrak{F}$

(ii) $E \circ \alpha \in \mathfrak{F}$, $\alpha \in Q \Rightarrow E \notin \mathfrak{F}$

(iii) $E \in \mathfrak{F}$; $\alpha, \beta \in Q$; E, α, β pairwise not dependent

$E \circ \alpha \in \mathfrak{F}$, $E \circ \beta \in \mathfrak{F} \Rightarrow E \circ \alpha \circ \beta \in \mathfrak{F}$

$\mathcal{E}_Q$ trivial heaps with pieces in Q

Transfert set $\mathbf{Tr}(E, T)$

$E \in \mathfrak{F} \quad T \in \mathcal{E}_Q$

Transfert set $\mathbf{Tr}(E, T)$

$E \in \mathcal{F}$ $\quad T \in \mathcal{E}_Q$

$$\mathbf{Tr}(E, T) = \{\text{pieces } \alpha \text{ of } T \text{ such that } E \cdot \alpha \in \mathcal{F}\}$$

$$\bigcup \quad \{\text{maximal pieces } (\beta, i) \text{ of } E \text{ with } \beta \in Q \text{ and not dependent with } T\}$$

if $\mathrm{Tr}(E, T) \neq \emptyset$

we can define a

sign-reversing

transfert involution φ

$(E, T) \longrightarrow (E', T')$

$E, E' \in \mathcal{F}$
$T, T' \in \mathcal{C}_Q$

suppose we have
$\mathcal{G} \subseteq \mathcal{F} \times \mathcal{C}_Q$

stable by transfert

exercise

a) Construct the involution φ

b) deduce the identity

$$(*) \sum_{(E,T)\in\mathcal{G}} (-1)^{|T|} v(E)v(T) = \sum_{(E,T)\in\mathcal{G}} (-1)^{|T|} v(E)v(T)$$

$$\mathrm{Tr}(E,T) = \emptyset$$

c) Show that the proofs of

- MacMahon Master theorem
- inversion lemma N/D
- inversion matrix $(I-A)^{-1} = \dfrac{\mathrm{cof}\,(I-A)}{\det\,(I-A)}$
- Cayley-Hamilton theorem

are particular cases of the identity $(*)$

d) may be find or imagine other particular cases

next lecture

Jacobi duality

$$[D+F] = (d_1 + \cdots + d_k) + (f_1 + \cdots + f_k)$$

$$D = \{d_1 < d_2 < \cdots < d_k\} \quad F = \{f_1 < f_2 < \cdots < f_k\}$$

$$\det (I-A)^{-1}[D \mid F] = (-1)^{[D+F]} \frac{\det (I-A)[\bar{F} \mid \bar{D}]}{\det (I-A)}$$

Jacobi

Combinatorial proof (P. Lalonde)

(1990)