Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics, a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

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Chapter 2
Heaps generating functions

(4)

IMSc, Chennai

23 January 2017

the logarithmic lemma

operations on combinatorial objects:

Derivative

Def. class of valued combinatorial objects

$\mathcal{A} = (A, v)$ A finite or enumerable set

$v : A \to \mathbb{K}[X]$

valuation

(*) for w monomial of $\mathbb{K}[X]$,
let $A_w = \left\{ \alpha \in A ,\ \text{coeff. of } w \text{ in } v(\alpha) \text{ is } \neq 0 \right\}$

then for every monomial w, A_w is finite

$v(\alpha)$ weight or valuation of α

$\{ v(\alpha) , \alpha \in A \}$ is summable

Def. $f_{\mathcal{A}} = \sum_{\alpha \in A} v(\alpha)$

generating power series of objects $\alpha \in A$ weighted by v

$\mathcal{A} = (A, v_A)$ class of weighted combinatorial objects satisfying $(*)$ with valuation v of the type

$$v_A(\alpha) = w_A(\alpha)\, t^{|\alpha|}$$

$$A_n = \{\alpha \in A \,,\ v(\alpha) = w(\alpha)\, t^n\}$$

$|\alpha| = n$ size of $\alpha \in A$

<u>Definition</u> $\mathcal{C} = \mathcal{A}^{\bullet}$ class of pointed objects

$\mathcal{C} = (C, v_C)$ with

- $C = \biguplus A_n \times [1, n]$ (disjoint union)
- $v_C(\gamma) = v_A(\alpha)$ for $\gamma = (\alpha, i)$ with $1 \leqslant i \leqslant |\alpha| = n$

Lemma

$$\mathcal{L}\alpha^{\bullet} = t \frac{d}{dt} \mathcal{L}\alpha$$

Proof

$$f_{\alpha} = \sum_{\alpha \in A} w_A(\alpha)\, t^{|\alpha|}$$

$$= \sum_{n \geq 1} \sum_{\substack{\alpha \in A \\ |\alpha| = n}} w_A(\alpha)\, t^{n}$$

$$f_{\alpha^\bullet} = \sum_{\substack{\gamma = (\alpha, i) \\ 1 \leq i \leq n = |\alpha|}} w_A(\alpha)\, t^{|\alpha|}$$

$$= \sum_{n \geq 1} \sum_{\substack{\alpha \in A \\ |\alpha| = n}} n\, w_A(\alpha)\, t^{n}$$

$$f_{\alpha^\bullet} = t \frac{d}{dt} f_{\alpha}$$

the logarithmic lemma

weight
valuation

$v(E)$

- $v : \mathcal{P} \longrightarrow \mathbb{K}[x, y, \ldots]$

basic piece

- $v(\alpha, i) = v(\alpha)\, t$

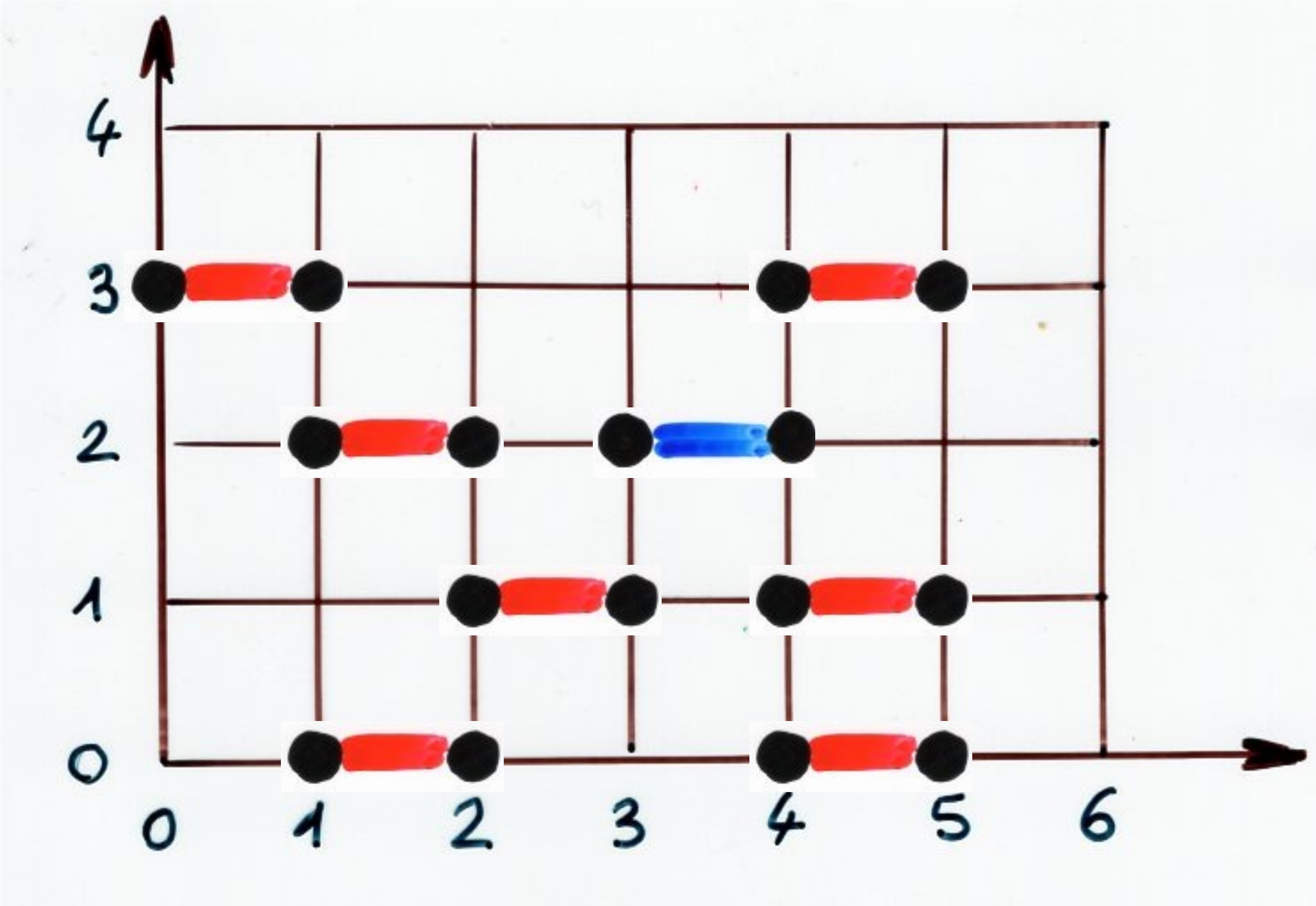
piece

- $v(E) = \prod_{(\alpha, i) \in E} v(\alpha\ i)\ t^{|E|}$

heap

The logarithmic Lemma

$$t\frac{d}{dt}\log\Big(\sum_{\substack{E\\ \text{heap}}} v(E)\,t^{|E|}\Big) = \sum_{\substack{P\\ \text{pyramid}}} v(P)\,t^{|P|}$$



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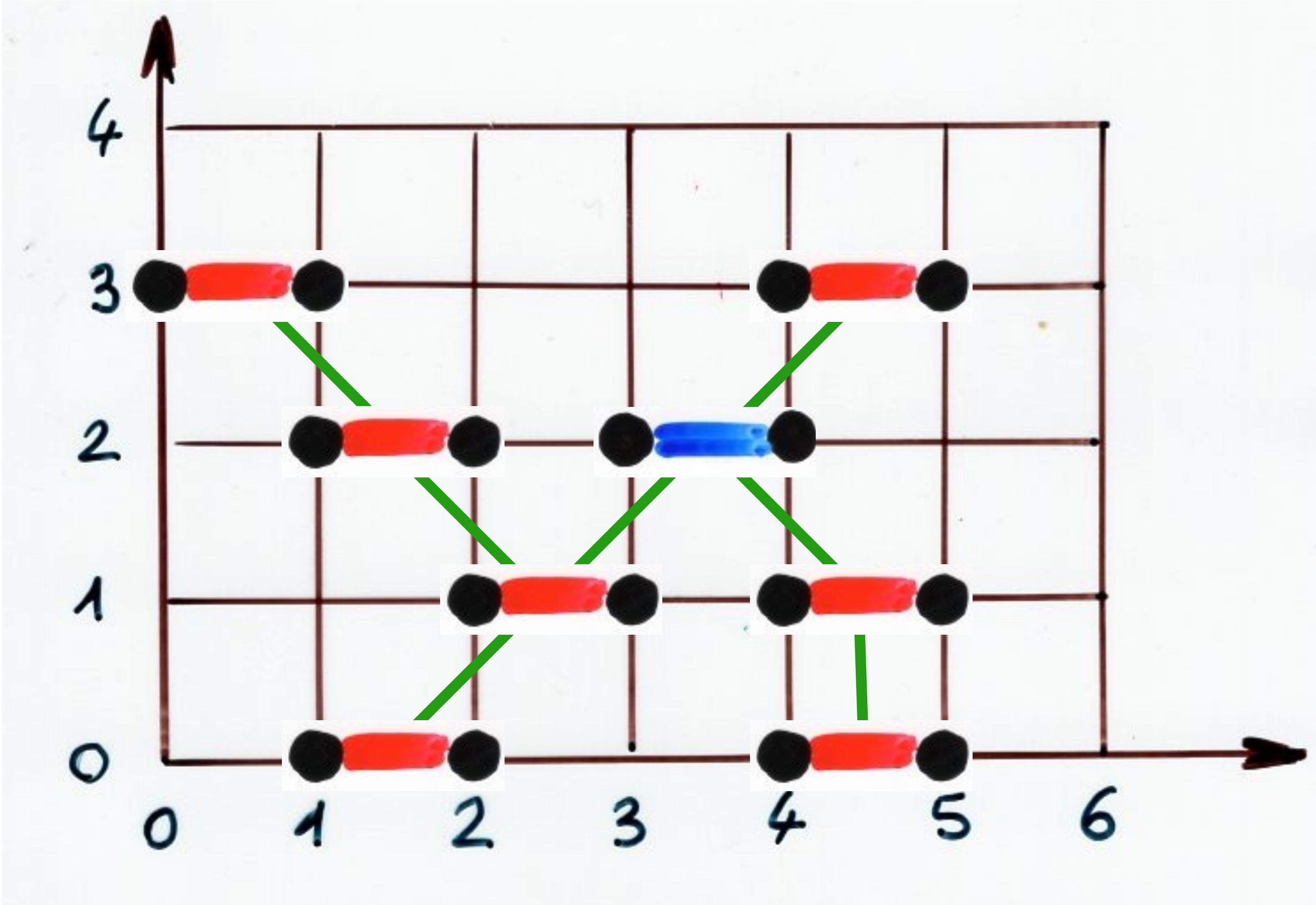
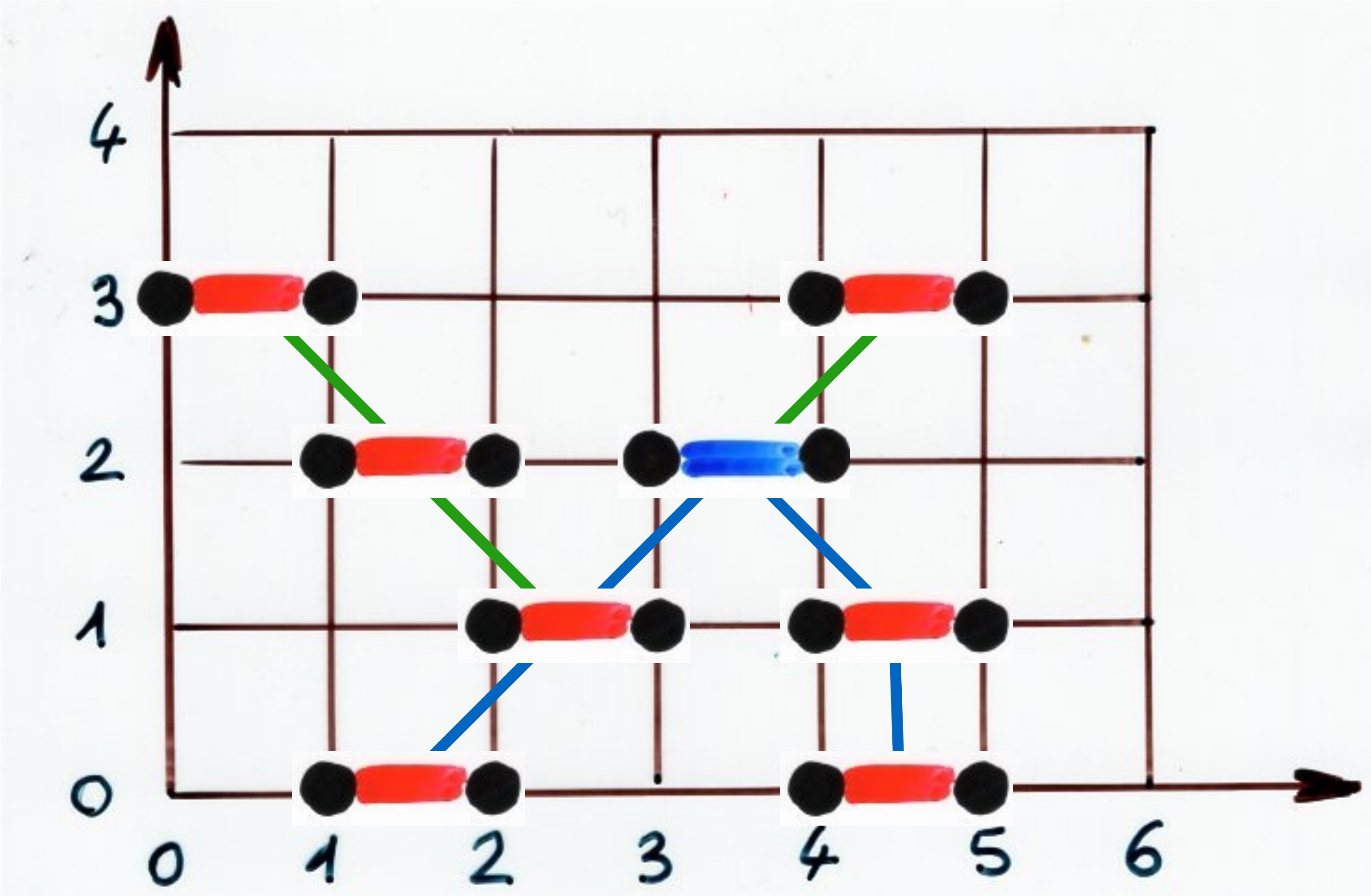


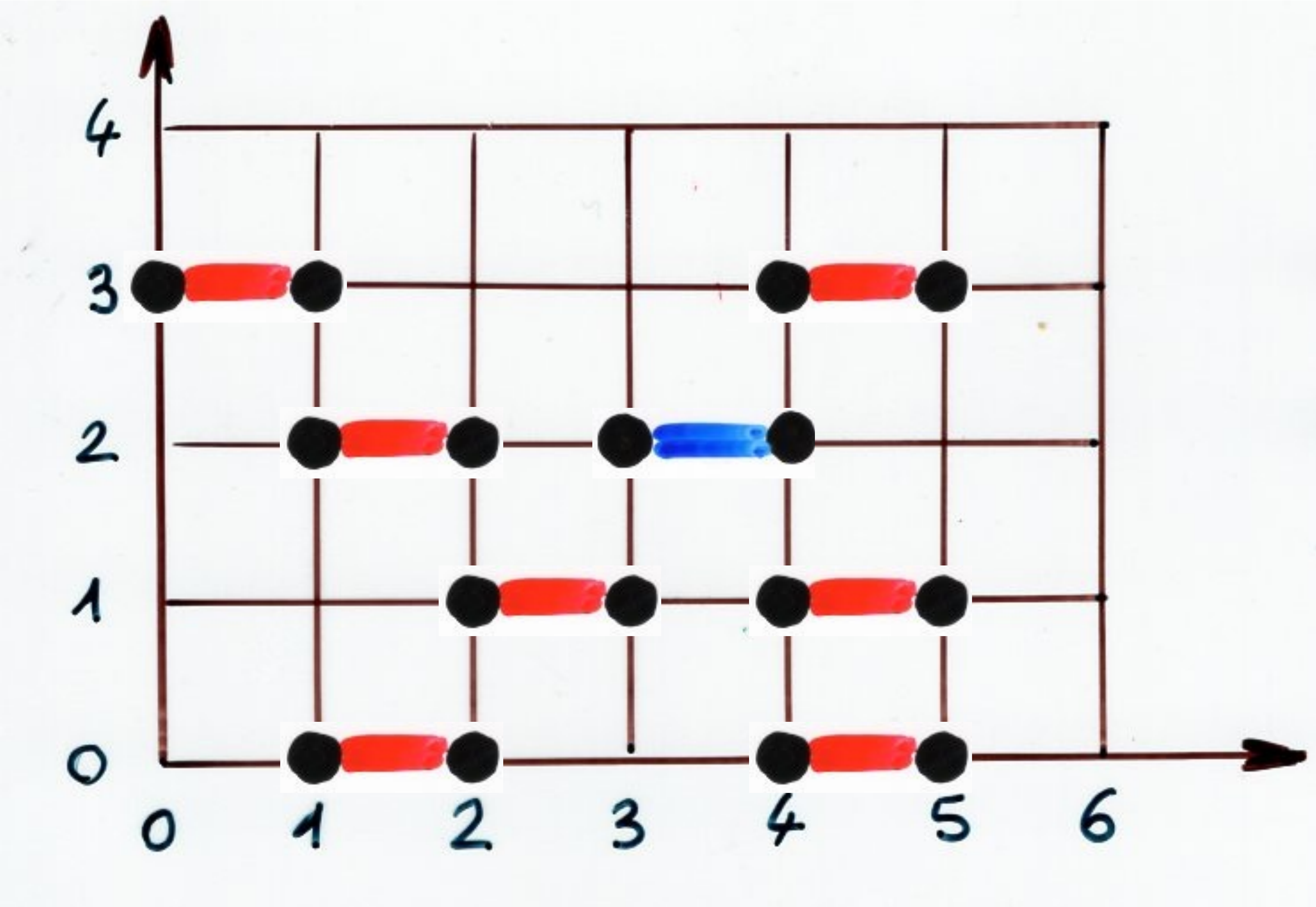
poset $\leq$ underlying the heap E



$x \in E$ ideal generated by x

pyramid

$P_x = \{ y \in E,\ y \preceq x \}$



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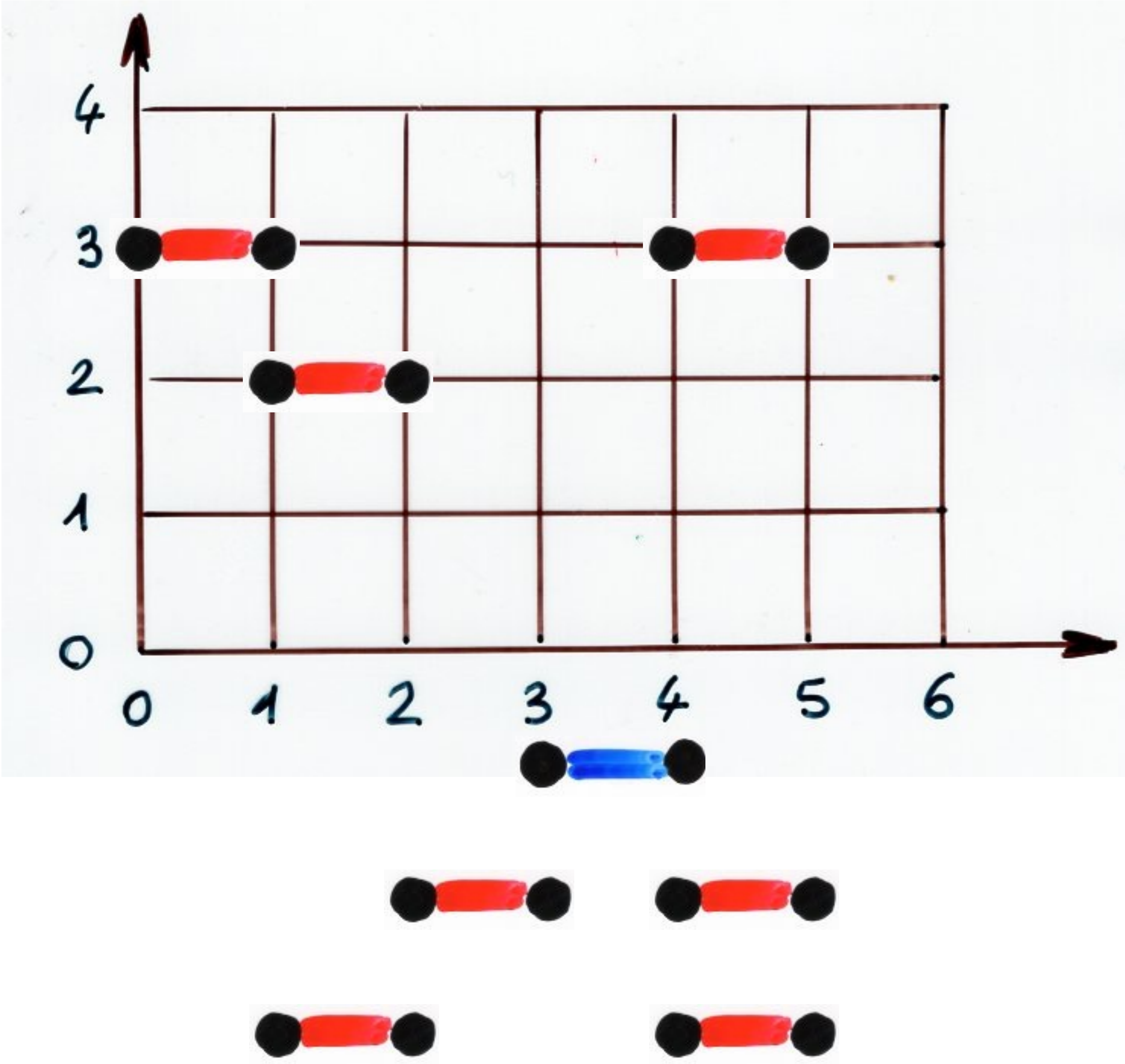

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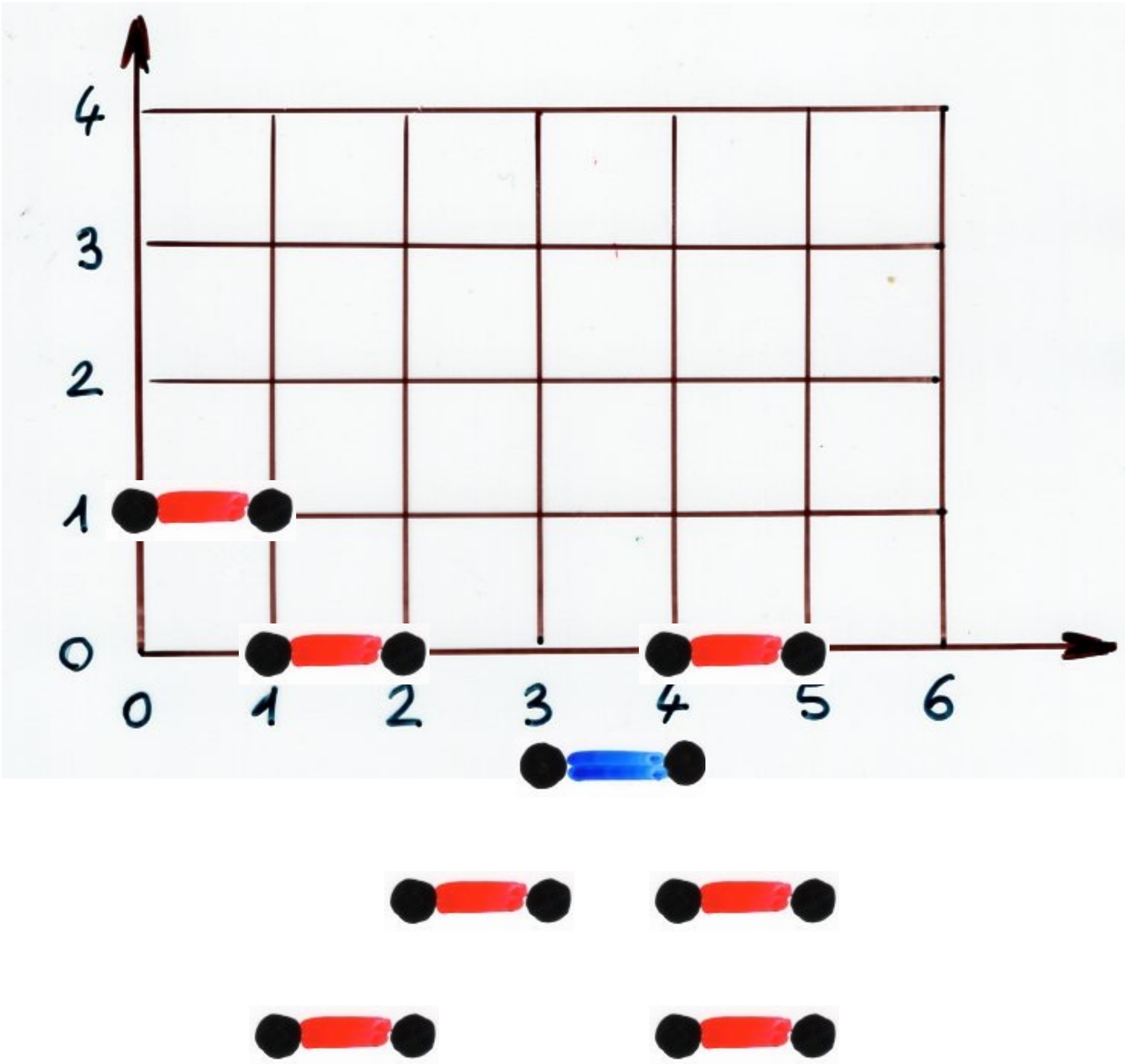

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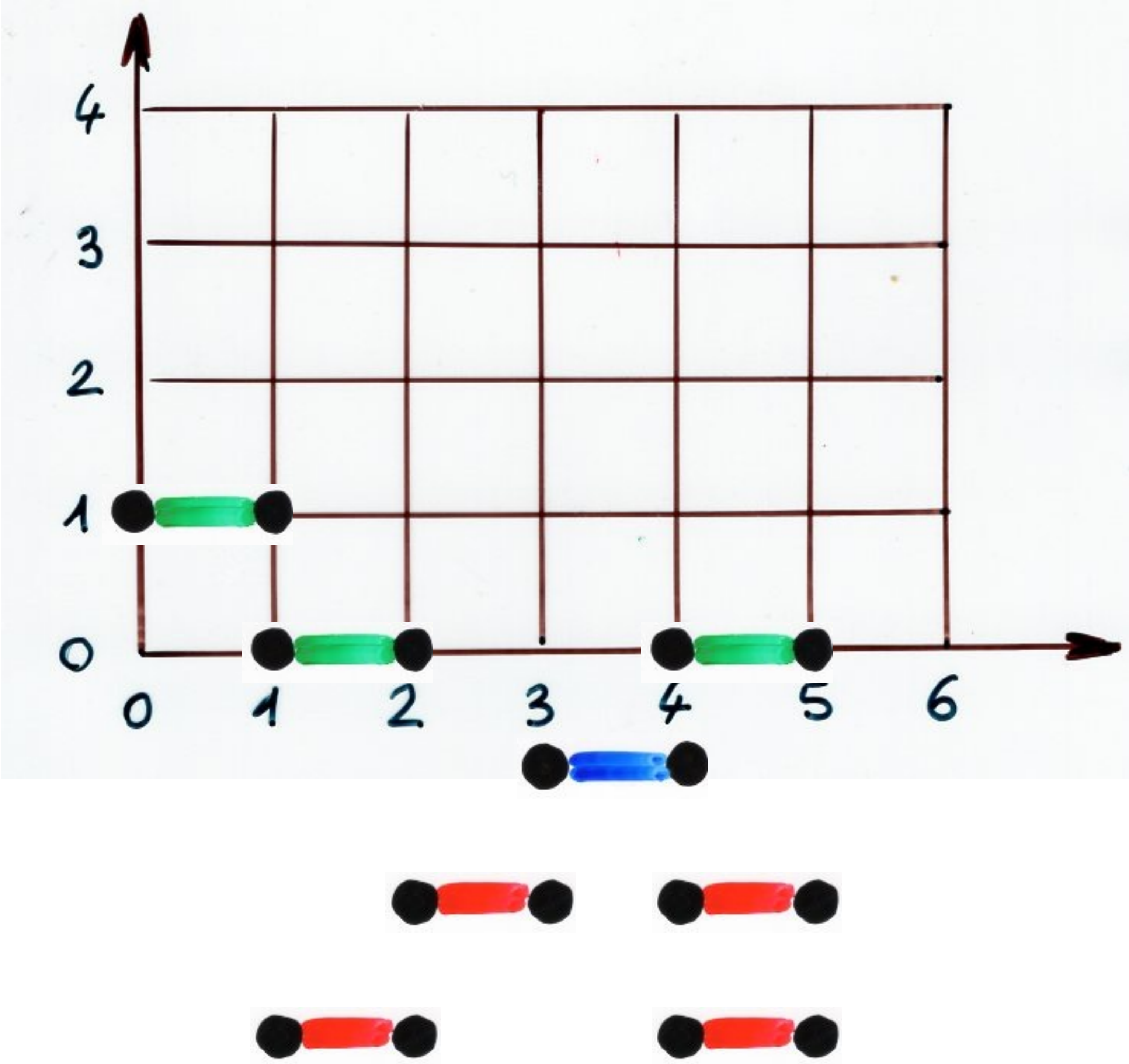
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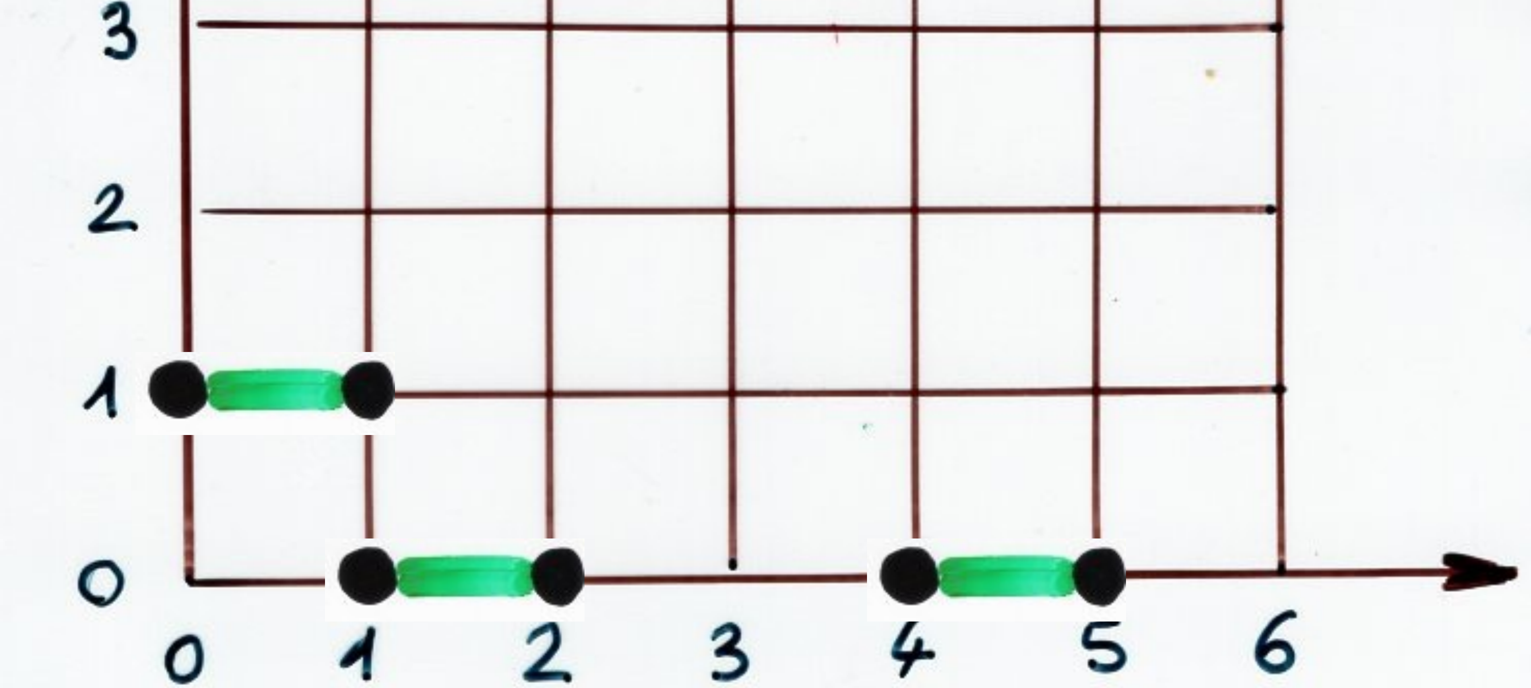
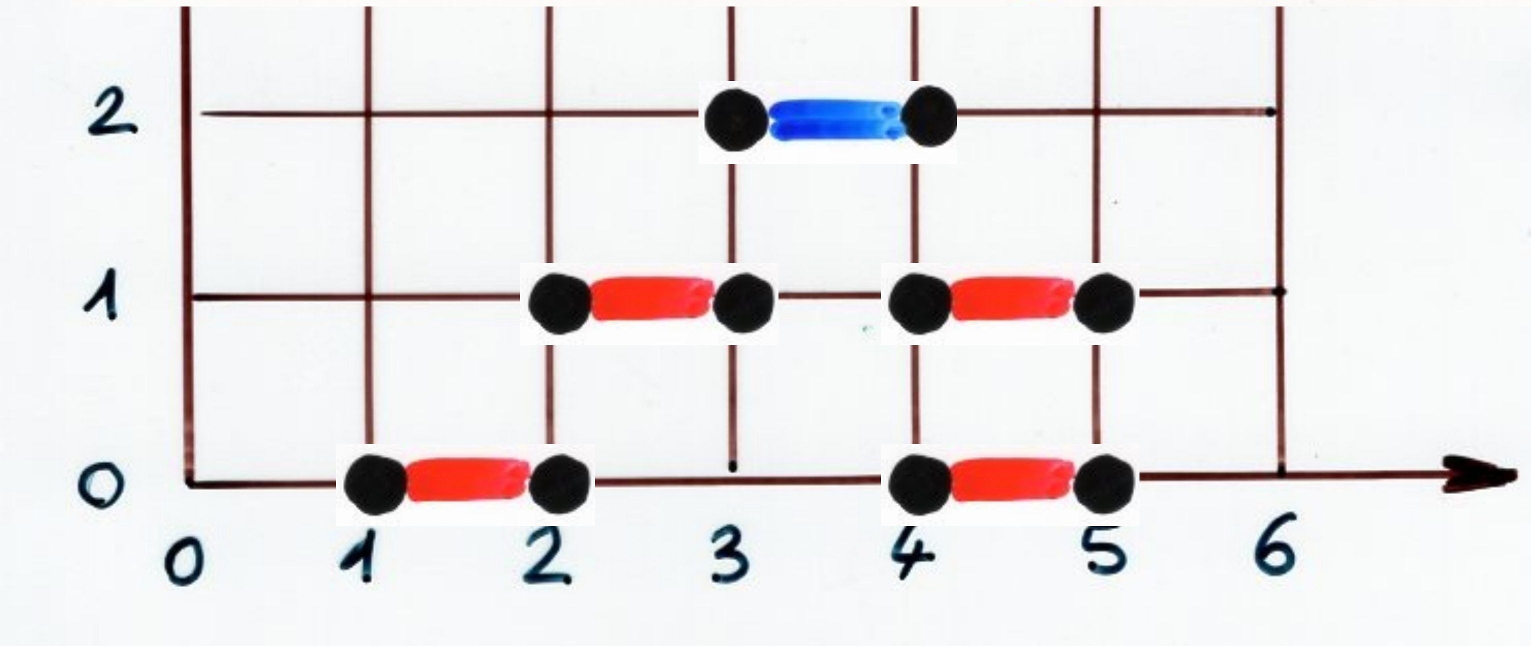


Pointed heap = Pyramid × heap



$$t\,y' = z\,y$$

$$\text{Pointed heap} = \text{Pyramid} \times \text{heap}$$

$$z = \sum_{\substack{P \\ \text{pyramid}}} v(P)$$

$$\frac{t\,y'}{y} = z$$

$|E|$ number of elements

$$t\frac{d}{dt}\log\Big(\sum_{\substack{E\\ \text{heap}}} v(E)\,t^{|E|}\Big) = \sum_{\substack{P\\ \text{pyramid}}} v(P)\,t^{|P|}$$

The logarithmic Lemma

$\square$

The logarithmic Lemma

equivalent form

$$\log\Big(\sum_{\substack{E \\ \text{heap}}} v(E)\, t^{|E|}\Big) = \sum_{\text{pyramid}} v(P)\, \frac{t^{|P|}}{|P|}$$

also :

$$-\log\Big(\sum_{\substack{E \\ \text{trivial heap}}} v(E)\, (-t)^{|E|}\Big)$$

the logarithmic lemma

general form

The logarithmic Lemma

(general form)

→ needed in Ch 4

heaps and linear algebra

weight of a basic piece:

$v(\alpha)\, t^{\ell(\alpha)}$

$\ell : P \longrightarrow \mathbb{N}$

weight of a heap E

$$v(E) = \prod_{x \in E} v(\pi(x))\, t^{\ell(E)}$$

$$\pi(\alpha, i) = \alpha \in P \qquad (\alpha, i) \in E$$

$$\ell(E) = \sum_{x \in E} \ell(\pi(x))$$

$\mathcal{H}$ class of weighted heaps

$[\, H = (P, \mathcal{C}) \,,\, v \,]$

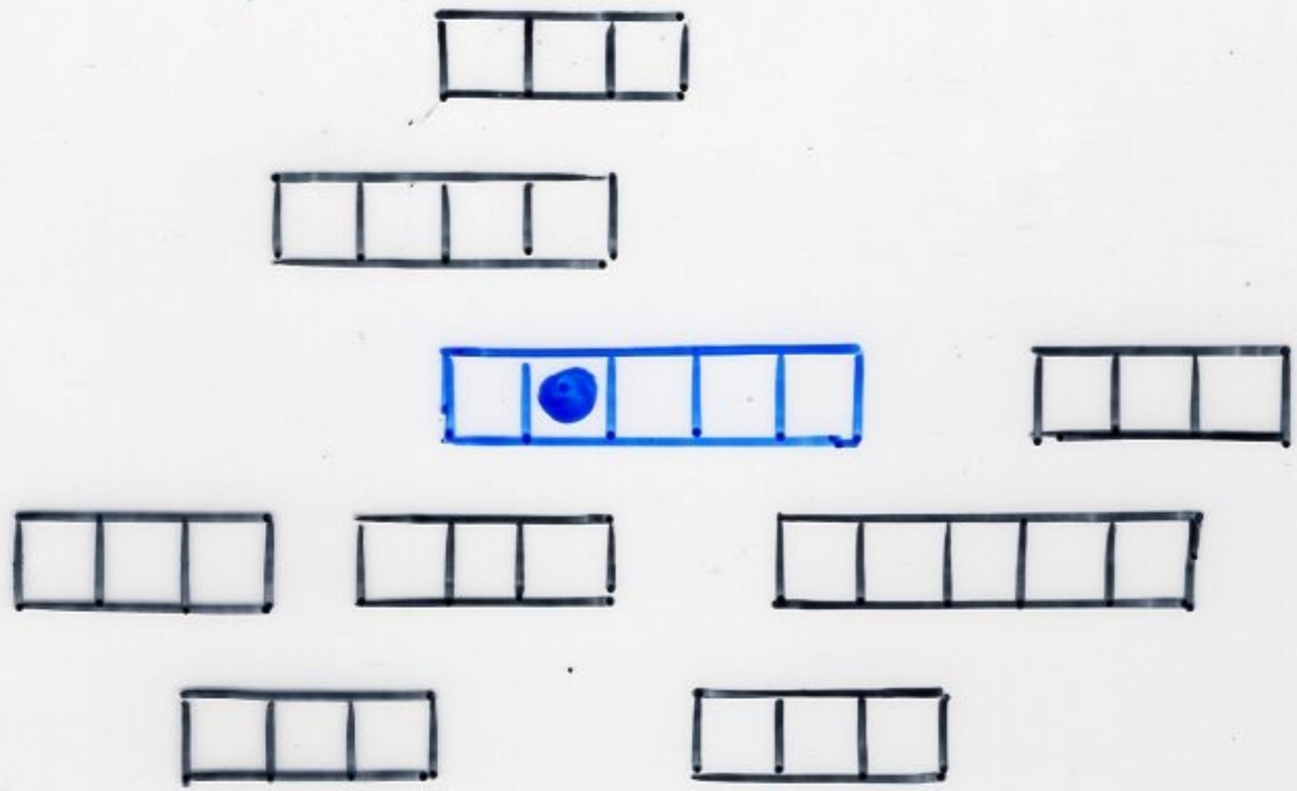
suppose it satisfies $(*)$

$\mathcal{H}^{\bullet}$

class of pointed weighted heaps

$$(E, j) \quad 1 \leq j \leq \ell(E)$$

$$\ell(E) = \sum_{x \in E} \ell(\pi(x))$$

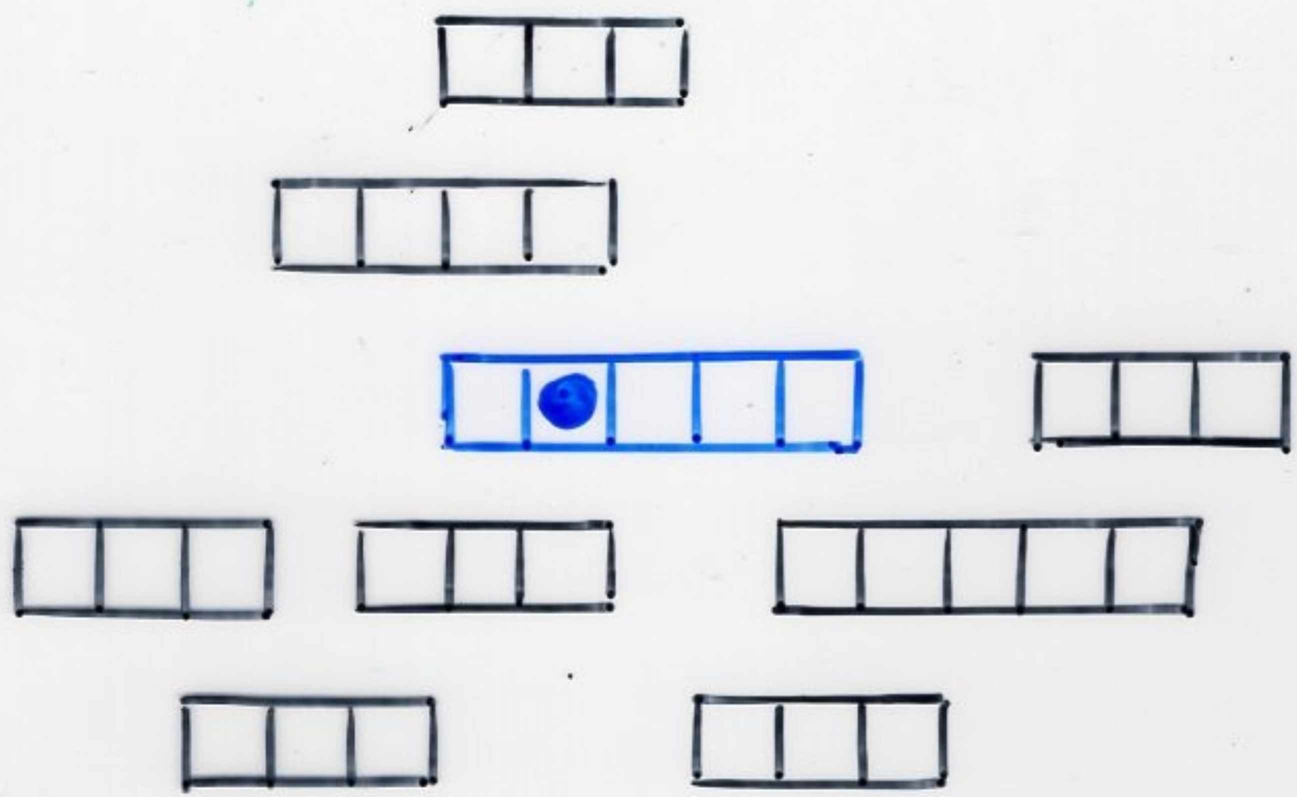


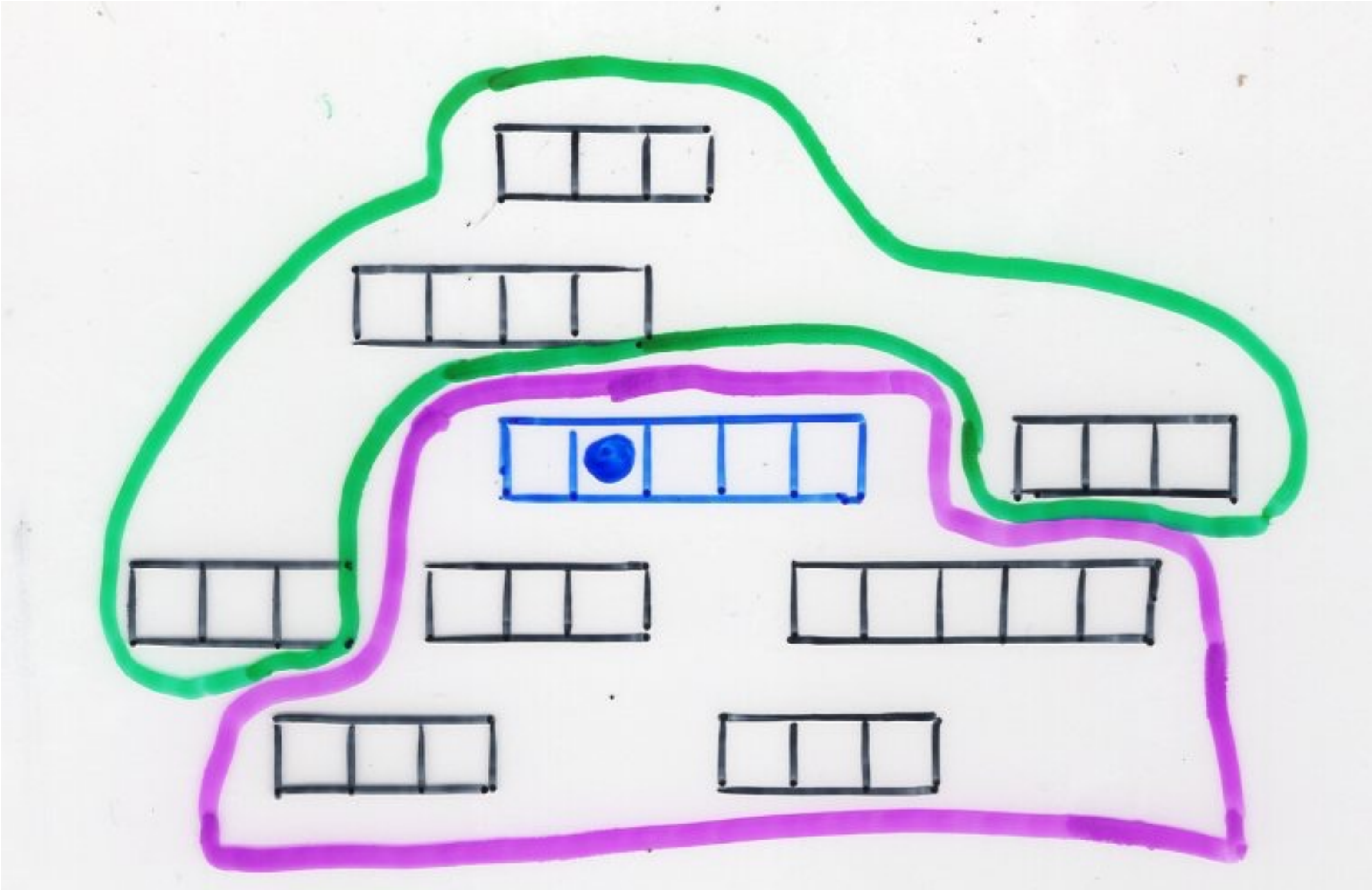
take the linear extension of E related to the lexicographic normal form

total order on the pieces of E
$x_1 \leq \dots \leq x_k$

$$[1, \ell(E)] = [1, \ell(x_1)][\ell(x_1)+1, \ell(x_1)+\ell(x_2)] \cdots$$

$$[\ell(x_1)+\dots+\ell(x_{k-1})+1, \underbrace{\ell(x_1)+\dots+\ell(x_k)}_{\ell(E)}]$$





$$\mathcal{H}^{\bullet} = \mathcal{P}yr^{\bullet}\ \mathcal{H}$$

$$t y' = z y$$

$$t \frac{d}{dt} \log \Big(\sum_{\substack{E \\ \text{heap}}} v(E)\, t^{\ell(E)} \Big) = \sum_{\substack{P \\ \text{pointed} \\ \text{pyramid}}} v(P)\, t^{\ell(P)}$$

$$(P, j)$$

$$1 \leq j \leq \ell(m)$$

m maximal piece of P

$$\log\Big(\sum_{\substack{E \\ \text{heap}}} v(E)\, t^{\ell(E)}\Big) = \sum_{\substack{P \\ \text{pointed} \\ \text{pyramid}}} v(P)\, \frac{t^{\ell(P)}}{\ell(P)}$$

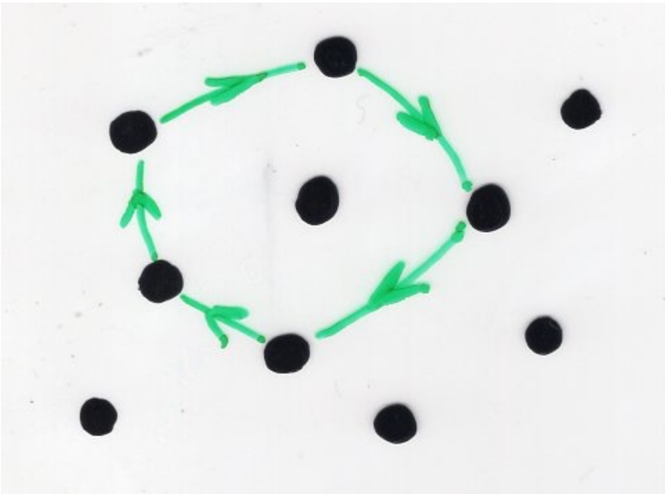
(P, j)

$1 \le j \le \ell(m)$

m maximal piece of P

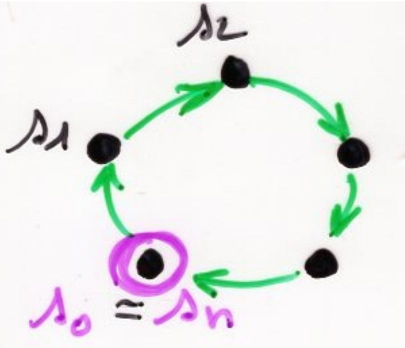
heap of cycles
on a set X

P basic pieces
cycles on X

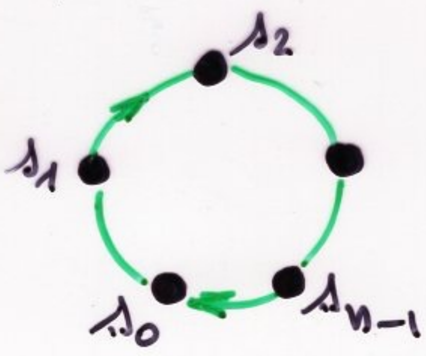


$\ell(\gamma) = n$

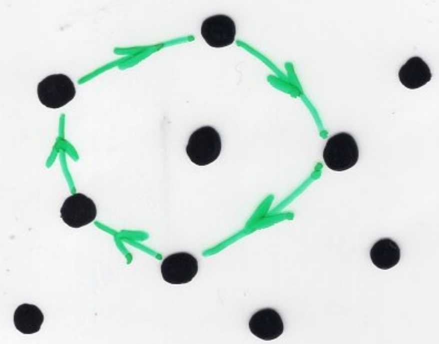
number of vertices (or length) of γ



elementary circuit $\omega = (s_0, \ldots, s_n)$
with $s_0 = s_n$, all vertices are disjoint
except $s_0 = s_n$.



Cycle = elementary circuit up to a circular permutation of the vertices

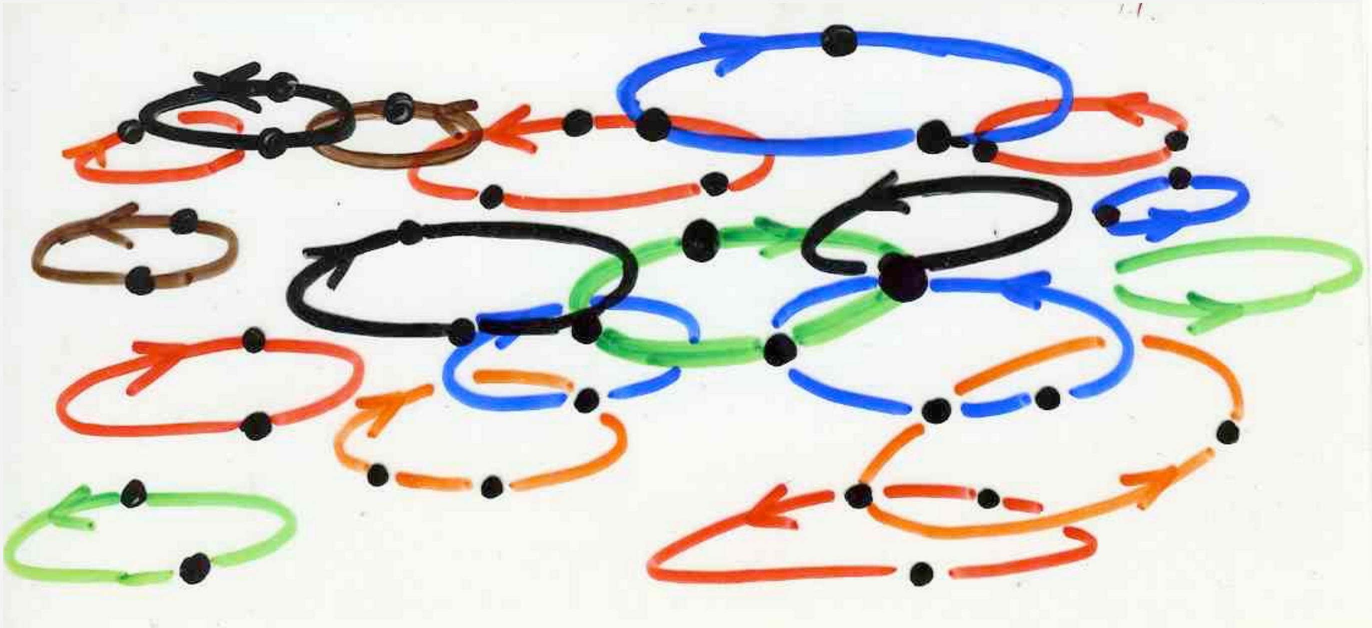


= equivalene class of conjugation

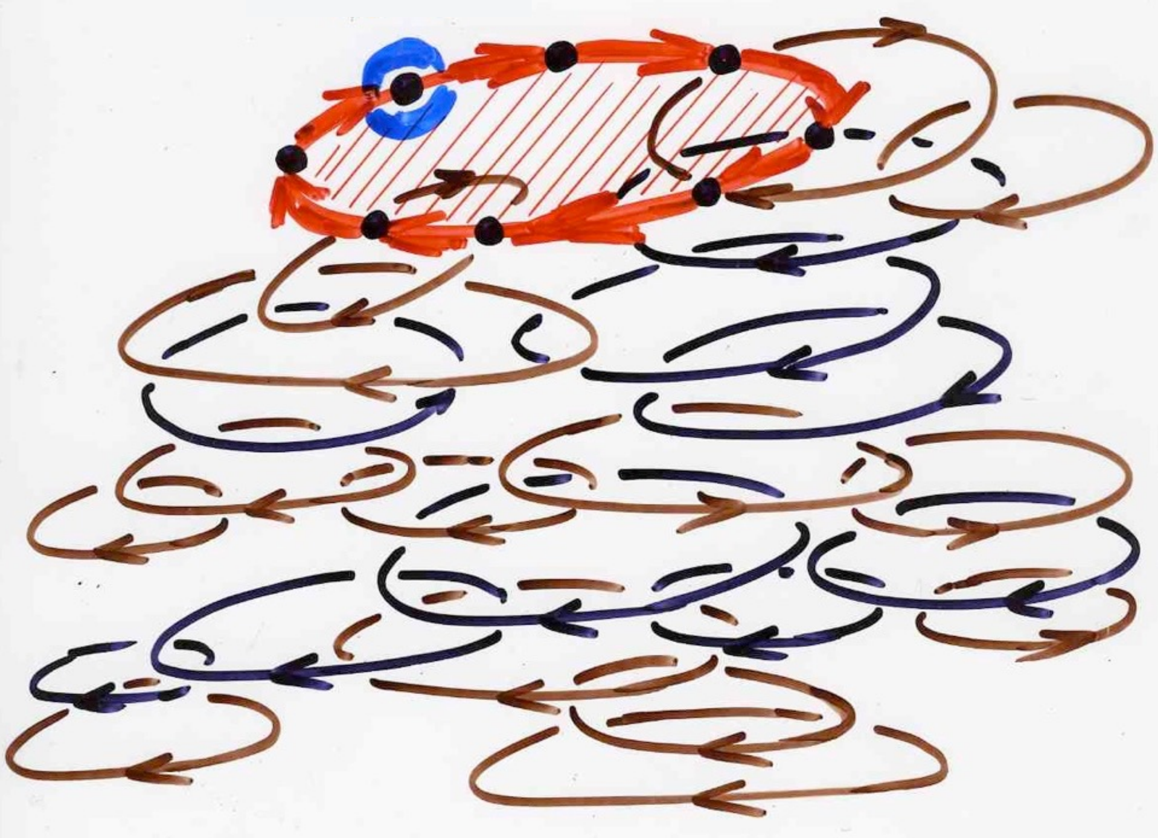
w conjugate of w'
iff $w = uv$ $w' = vu$

this is an equivalence relation

$$\log\Big(\sum_{\substack{E\\ \text{heap}}} v(E)\, t^{\ell(E)}\Big)$$



$$\log\Big(\sum_{\substack{E \\ \text{heap}}} v(E)\, t^{\ell(E)}\Big)$$



pointed pyramid of cycles

$$\sum_{\substack{P \\ \text{pointed} \\ \text{pyramid}}} v(P)\,\frac{t^{\ell(P)}}{\ell(P)}$$

of cycles (P, x)

$x \in \gamma_{max}$

second proof
of the logarithmic lemma
with
exponential generating functions

remiding

course IMSc 2016 Chapter 3

(some ideas about)

species and

exponential generating functions

"naive" definition

U finite set

combinatorial structure

construction α

U underlying set with

α constructed on U,

supported by U

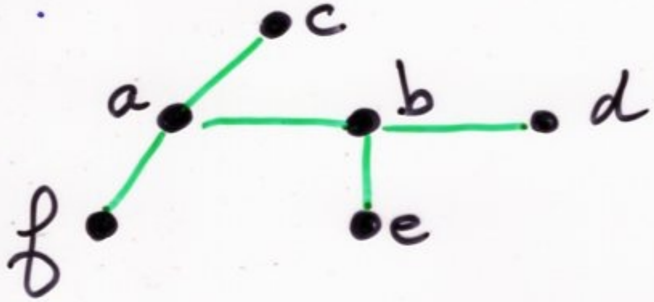
species F

structures of type F

set $F[U]$

F-structure $\alpha \in F[U]$

example Tree (= graph having no cycle)



Permutations,
(set) Partitions,
Graphs,
Endofunctions, ...

Transport of structures

$U \xrightarrow{f} V$

bijection

$F[U] \xrightarrow{F[f]} F[V]$

transport along f

<u>example</u> trees $U = \{a, b, c\}$ $V = \{1, 2, 3\}$

$$\begin{array}{ccc}
U = \{a, b, c\} & & \{a-b-c,\ b-a-c,\ a-c-b\} \\
f\downarrow \quad \downarrow\ \downarrow\ \downarrow & F[f]\downarrow & \\
V = \{1, 2, 3\} & & \{1-2-3,\ 2-1-3,\ 1-3-2\}
\end{array}$$

coherent transport

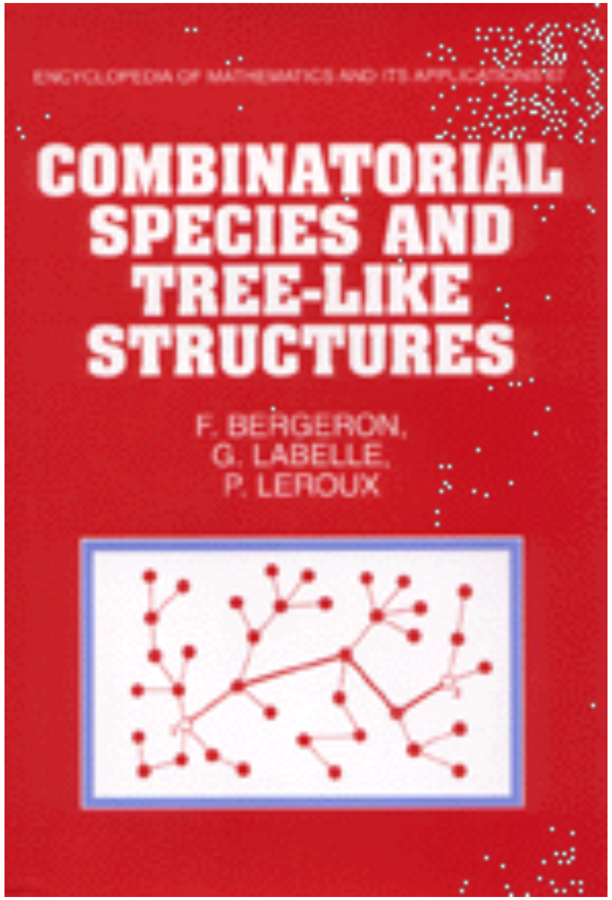
$$\mathbf{F}[f \circ g] = \mathbf{F}[f] \circ \mathbf{F}[g] \qquad \mathbf{F}[Id_U] = Id_{\mathbf{F}[U]}$$

Combinatorial model for exponential generating function

$$f(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!}$$

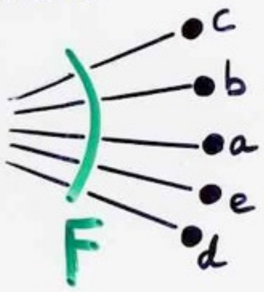
Species (combinatorial) structures

A. Joyal, G. Labelle
P. Leroux, F. Bergeron, ...
(UQAM, LACIM, Montréal)



Encyclopedia of Mathematics and its Applications
Cambridge University Press (1977)

Convention.



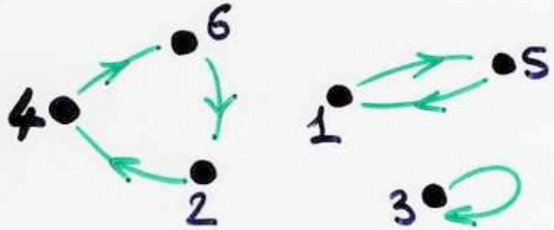
enumeration $a_n = \left| F[\{1, 2, \ldots, n\}] \right|$

<u>Definition</u> generating function of the species F

$$F(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!}$$

Examples

Permutations S $a_n = n!$ $S(t) = \frac{1}{1-t}$



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 6 & 1 & 2 \end{pmatrix}$$

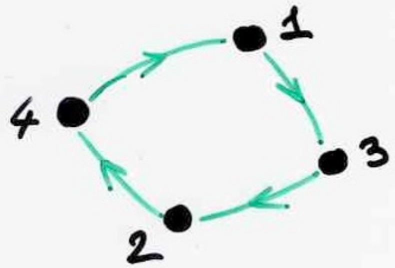
$$\sigma = 543612$$

Total order L $a_n = n!$ $L(t) = \frac{1}{1-t}$

Cycle C

$a_n = (n-1)!$

$$C(t) = \sum_{n\geq 1} \frac{t^n}{n} = \log(1-t)^{-1}$$



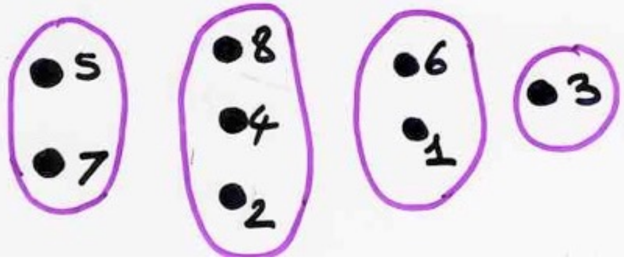
circular permutations

set E (ensemble) $E[t] = e^t$

uniform species

"Ensemble"
e
Euler

8. Partition B



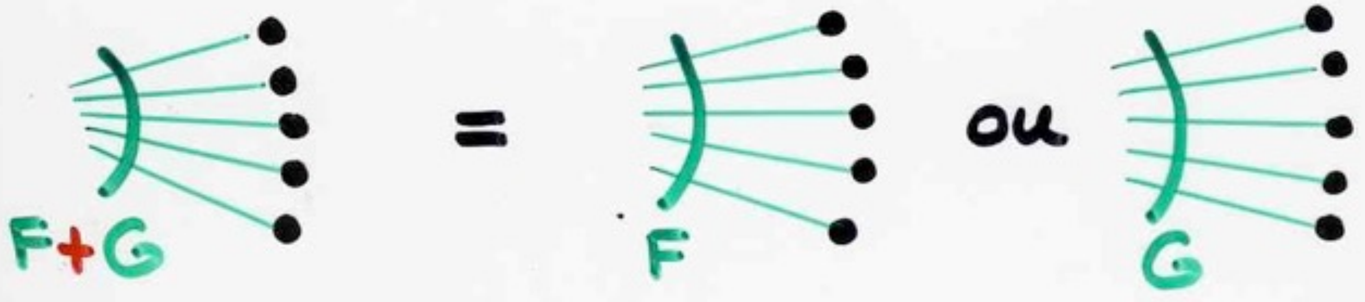
Bell number

$a_n = B_n$ nombre de Bell

$$B(t) = \exp(e^t - 1)$$

Def. sum

$$(F+G)[U] = F[U] + G[U] \quad \text{(disjoint union)}$$



Prop.

$$(F+G)[t] = F[t] + G[t]$$

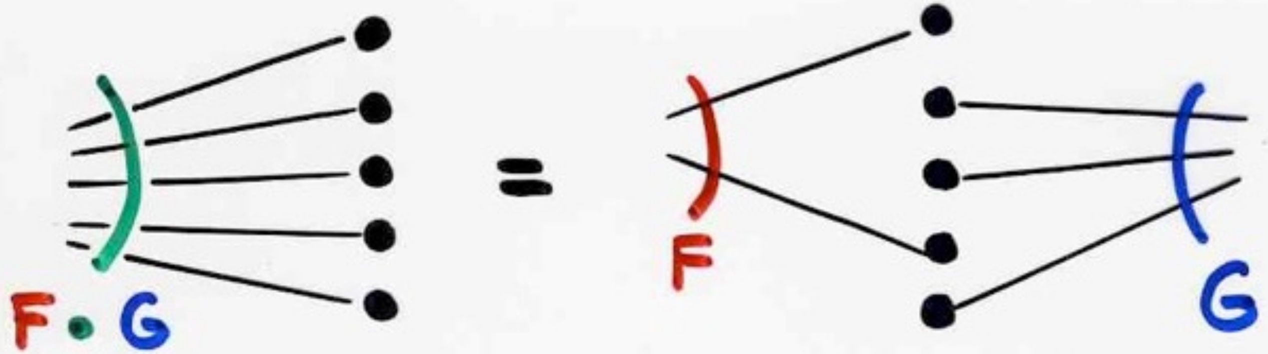
$$c_n = a_n + b_n$$

ex.

$$E = EP + EI$$

$$e^t = \mathrm{ch}\,t + \mathrm{sh}\,t$$

Déf. product $F \cdot G$



$\gamma \in F \cdot G[U]$ $\quad \gamma = (U_1, U_2, \alpha, \beta)$ $\quad \{U_1, U_2\}$ partition of U

$\alpha \in F[U_1]$ $\quad \beta \in G[U_2]$

Prop.

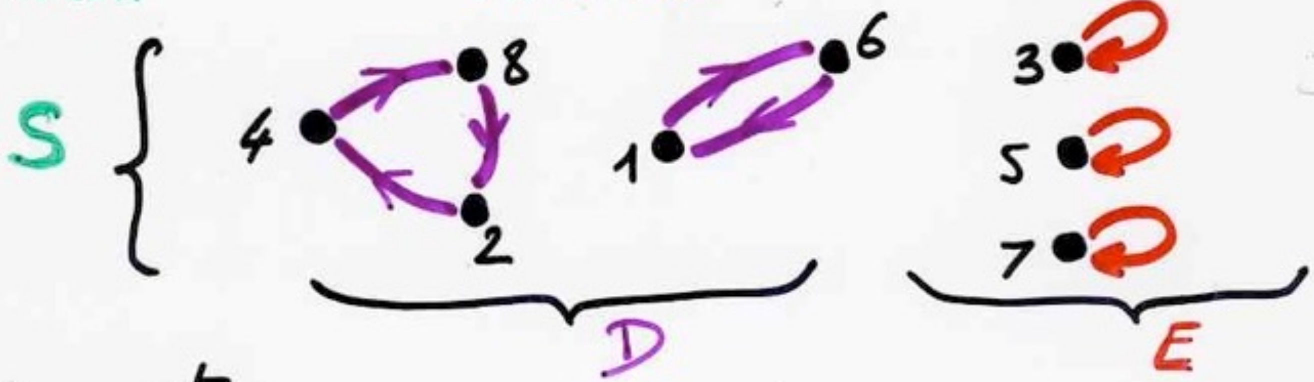
$$F \cdot G[t] = F[t]\, G[t]$$

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k\, b_{n-k}$$

ex. Dérangements $\mathcal{D}$

$$S = \mathcal{D} \cdot E$$

permutation — ensemble — set



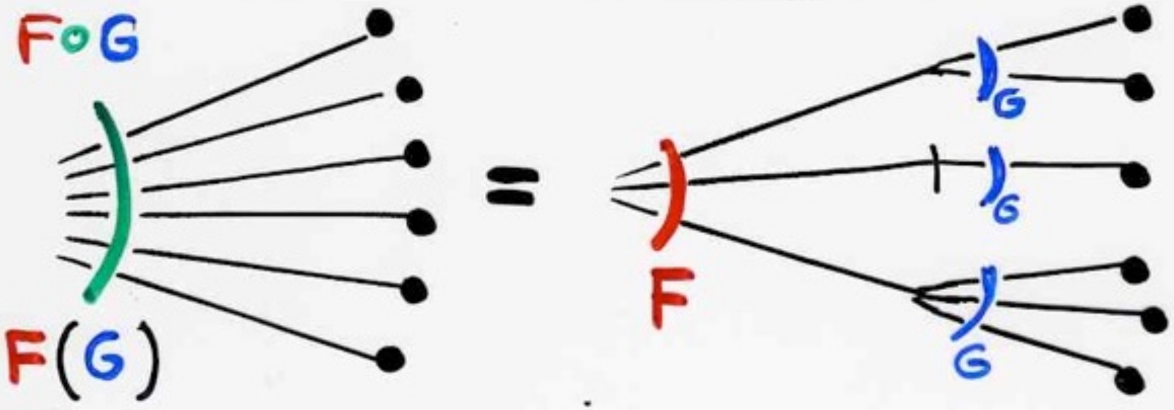
$$\mathcal{D}[t] = \frac{e^{-t}}{1-t}$$

$$d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{(-1)^n}{n!}\right)$$

$$P_n = \frac{d_n}{n!} \longrightarrow \frac{1}{e}$$

Def. F $\quad G \quad G[\emptyset] = \emptyset$

substitution of G into F



$\gamma \in F(G)[U]$

$$\gamma \begin{cases} \bullet \ \text{partition } \{U_1, \ldots, U_k\} \text{ de } U \\ \quad \text{classes} \neq \emptyset \\ \bullet \ \beta_i \in G[U_i], \quad i = 1, \ldots, k \\ \bullet \ \alpha \in F[U/\equiv] \end{cases}$$

F- "assemblée" of G-structures

ex- permutation = assemblee of cycles

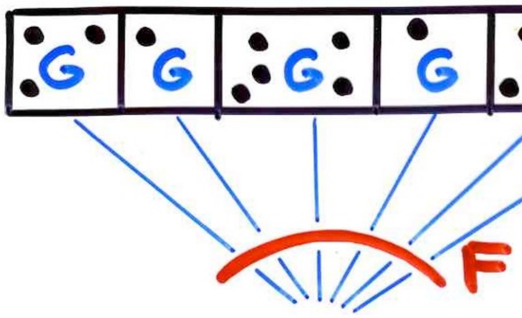
Prop. $(F \circ G)(t) = F(G(t))$

$$c_n = \sum_{\substack{k=0 \\ n_1+\dots+n_k=n \\ n_1,\dots,n_k \geq 1}}^{n} \frac{n!}{k!\, n_1! \dots n_k!}\, a_k\, b_{n_1} \dots b_{n_k}$$

Cor $F = E$

$$(E \circ G)(t) = \exp(G(t))$$

"assemblée" of G-structures $\qquad E^G$

$H =$ 

$$H = F(G)$$

$F = E$

species set

e^t

$$H = \exp(G)$$

"assemblée" of G-structures

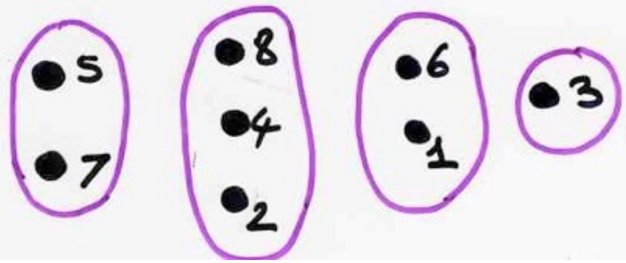
$$h(t) = \exp(g(t))$$

$B = \exp(E^+)$

non-empty set

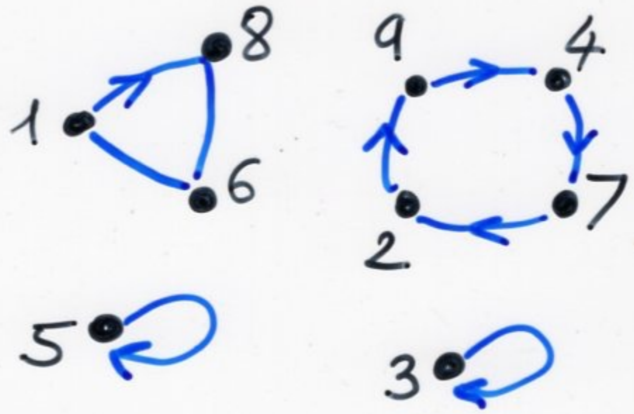
Bell number

Partition B

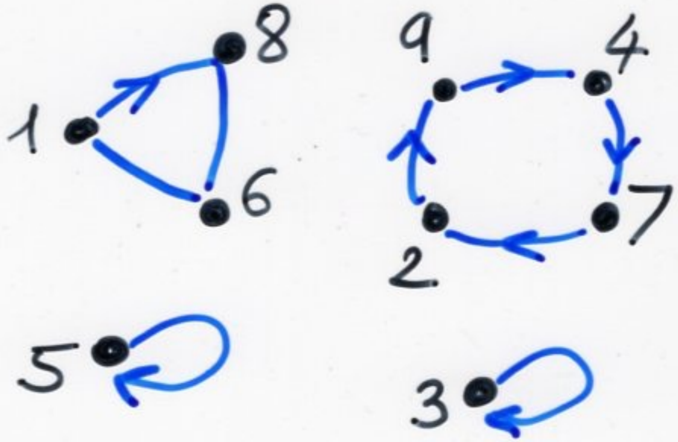


$a_n = B_n$ nombre de Bell

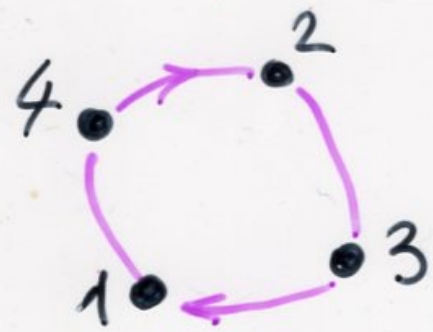
$B(t) = \exp(e^t - 1)$



permutation = exp (cycle)



cyclic
permutation



$$\sum_{n\geq 0} n! \frac{t^n}{n!} = \frac{1}{1-t}$$

$$\sum_{n\geq 1} (n-1)! \frac{t^n}{n!} = \sum_{n\geq 1} \frac{t^n}{n}$$

$$= \log \frac{1}{1-t}$$

$\mathbb{K}$ commutative ring

Definition weighted species $\mathbf{F}_v$

$\alpha \in \mathbf{F}[U] \longrightarrow v(\alpha) \in \mathbb{K}$

weight

(or valuation)

of the $\mathbf{F}$-structure α

$f : U \longrightarrow V$

$\alpha \in \mathbf{F}[U] \xrightarrow{\mathbf{F}[f]} \beta \in \mathbf{F}[V]$

$v(\alpha) = v(\beta)$

Definition generating power series $F_v(t)$

$$F_v(t) = \sum_{n\geq 0} p_n \frac{t^n}{n!}$$

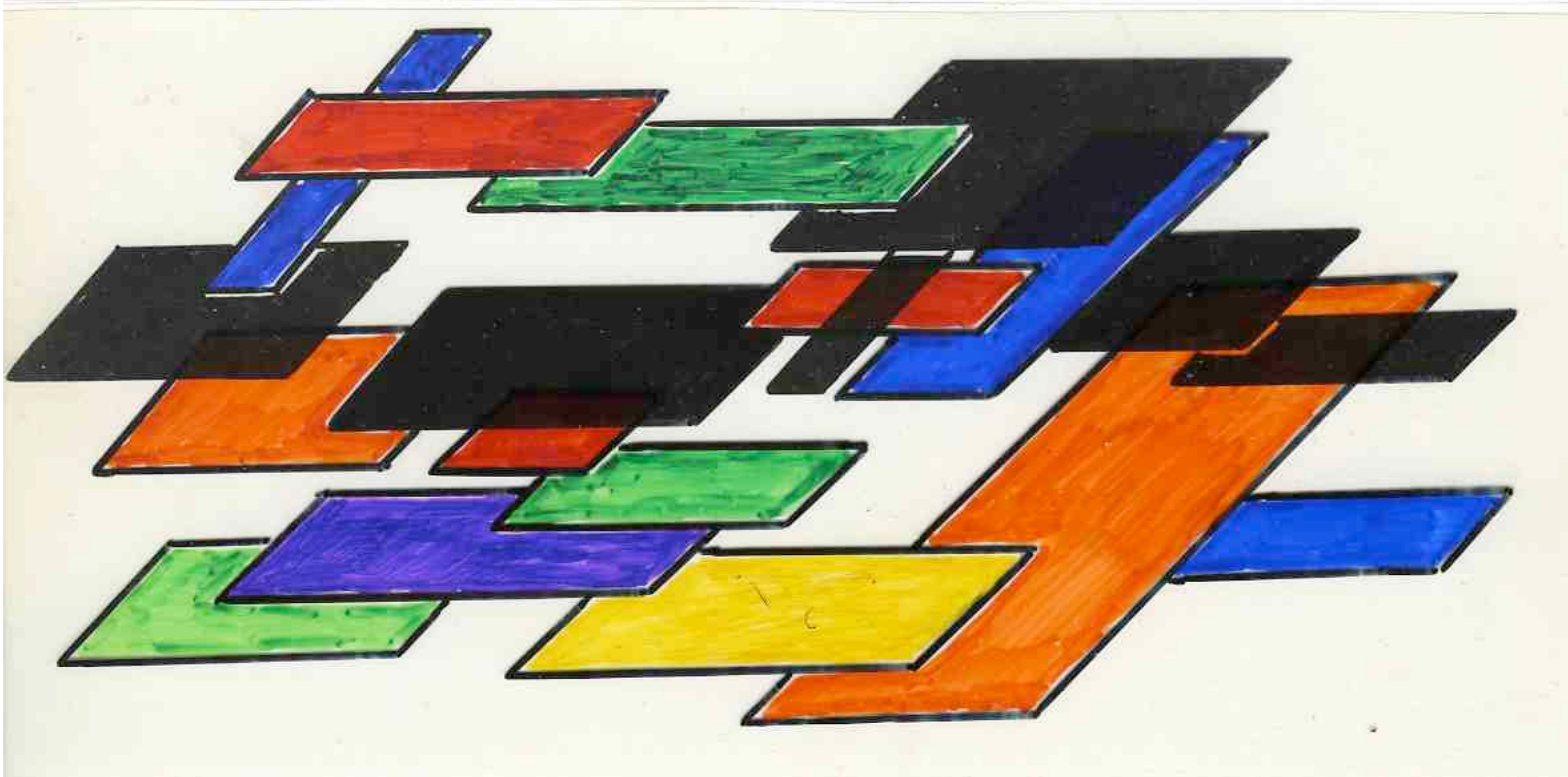
$$p_n \in \mathbb{K}$$

$$p_n = \sum_{\substack{\alpha \in F[U] \\ \text{with } |U|=n}} v(\alpha)$$

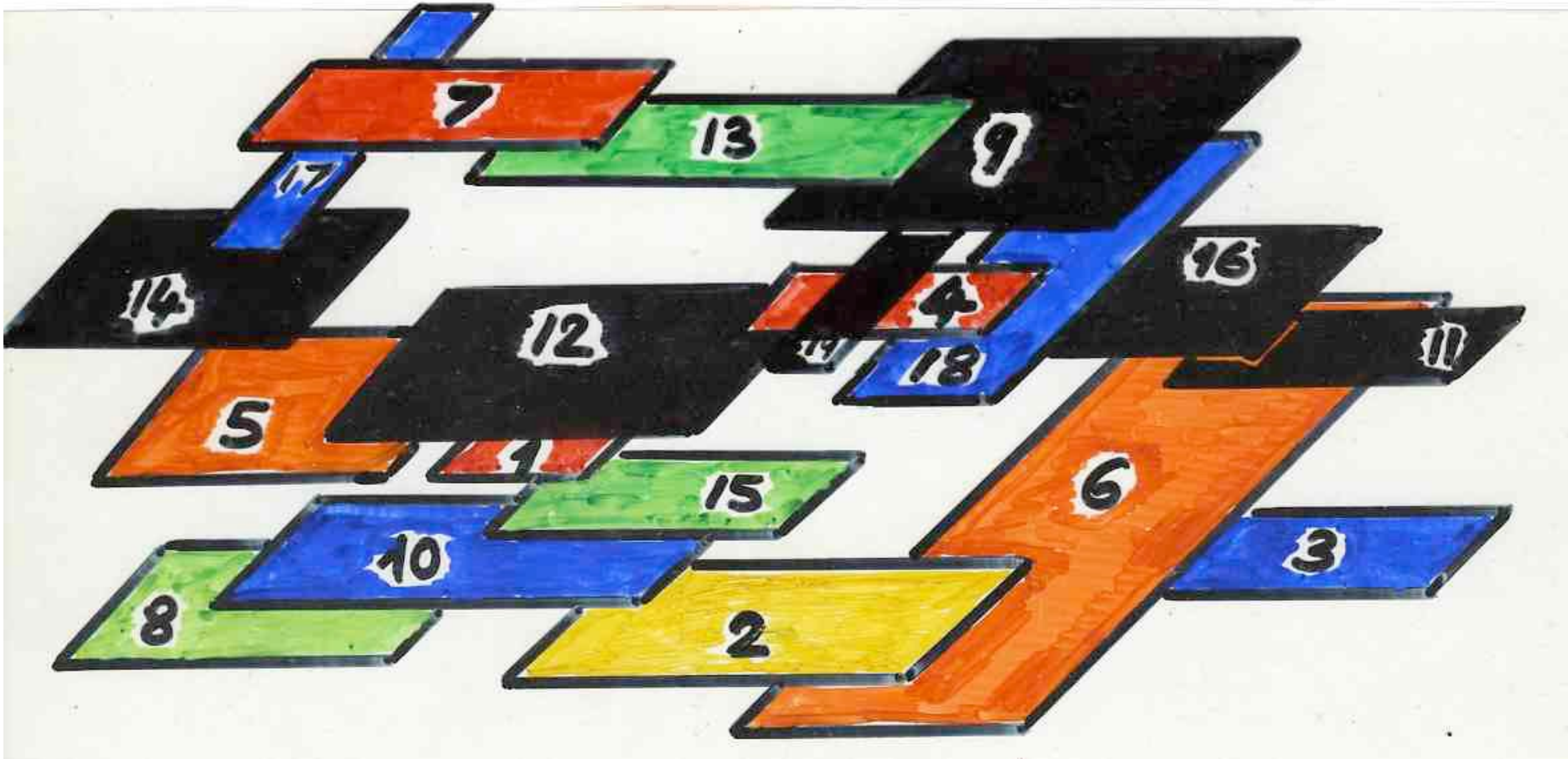
complements

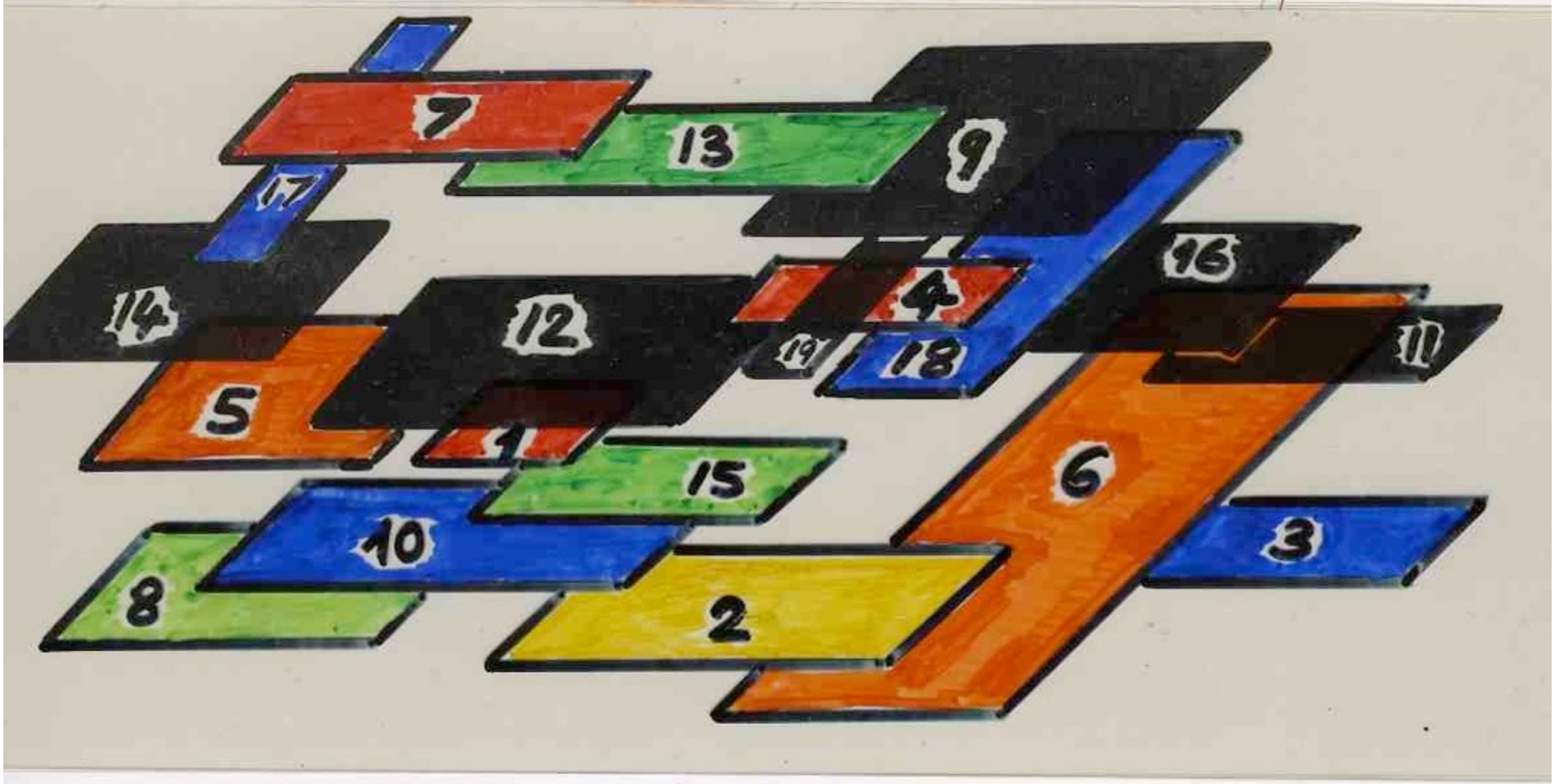
second proof
of the logarithmic lemma
with
exponential generating functions

labeled heaps and pyramids



labeled
heap

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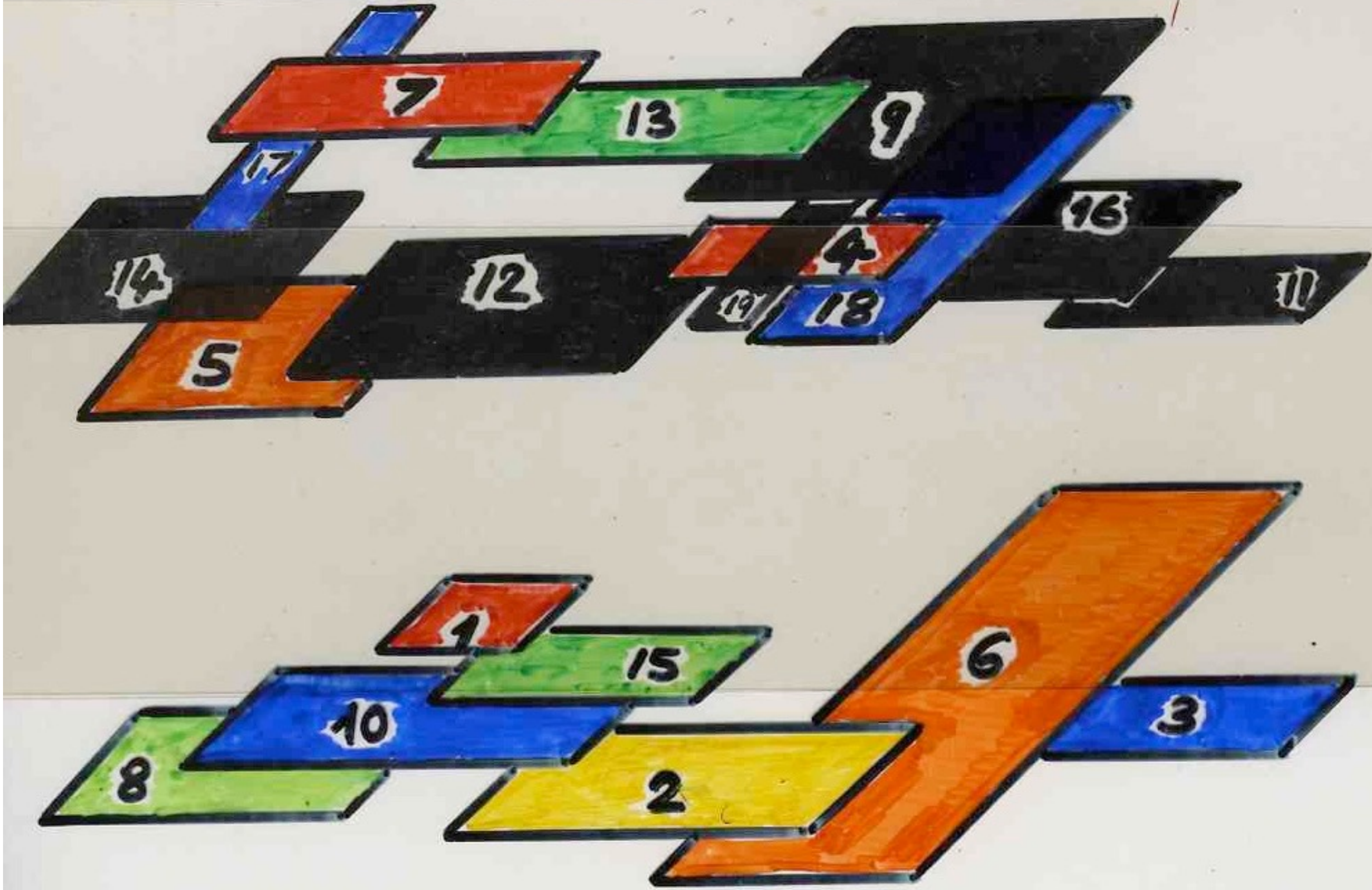
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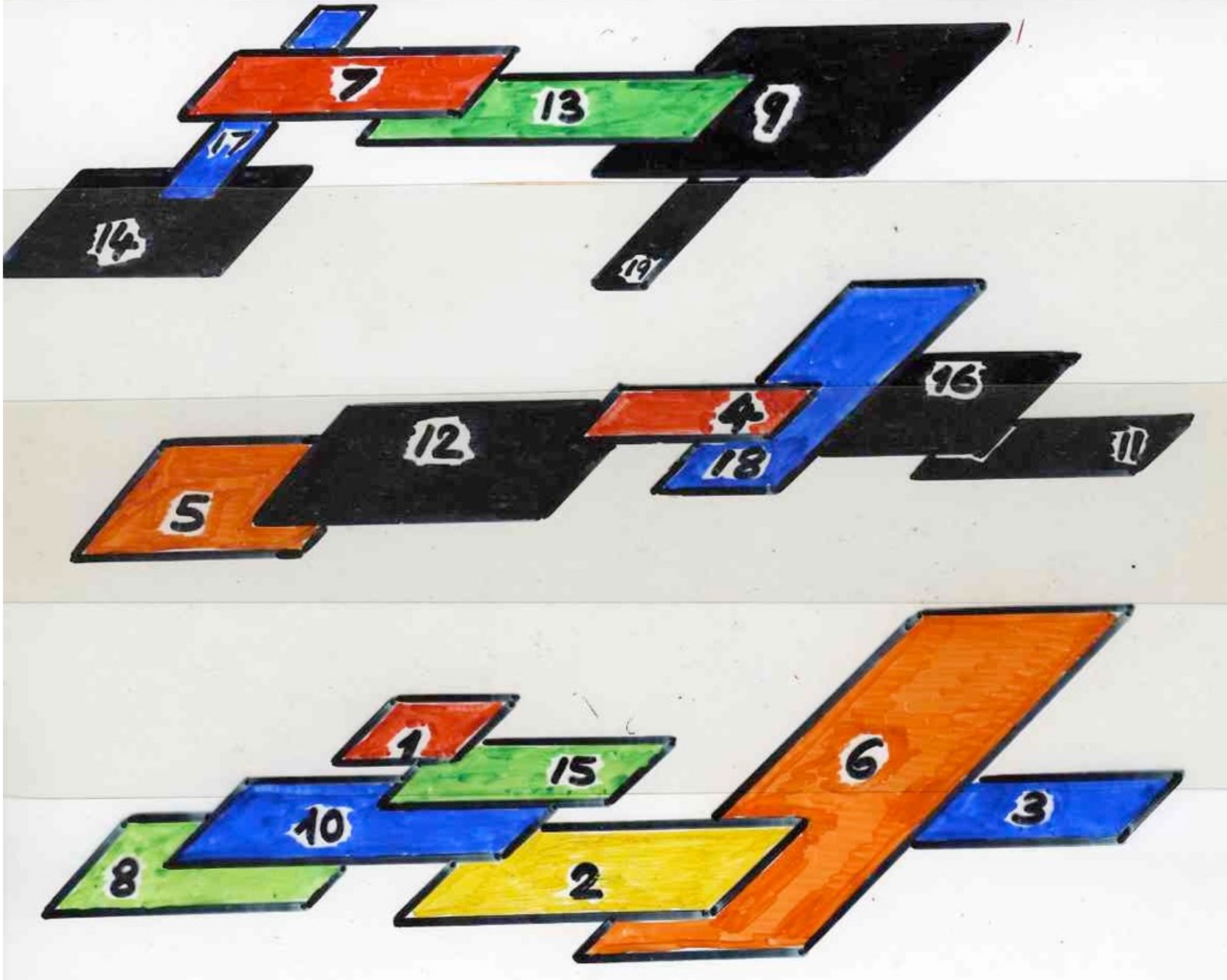
7
13
9
17
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11
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8
2

assemblée of labeled pyramids (m)

(m) the label of the (unique) maximal piece is the minimum of the labels of the pieces of the pyramid P

$$\sum_{\substack{E \\ \text{labeled} \\ \text{heap}}} v(E) \frac{t^{|E|}}{|E|!} = \exp\left(\sum_{\substack{P \\ \text{labeled} \\ \text{pyramid} \\ (m)}} v(P) \frac{t^{|P|}}{|P|!} \right)$$

$$\log\left(\sum_{\substack{E\\ \text{labeled}\\ \text{heap}}} v(E)\,\frac{t^{|E|}}{|E|!}\right) = \sum_{\substack{P\\ \text{labeled}\\ \text{pyramid}\\ (m)}} v(P)\,\frac{t^{|P|}}{|P|!}$$

$$\sum_{\substack{E\\ \text{labeled}\\ \text{heap}}} v(E)\,\frac{t^{|E|}}{|E|!} = \exp\left(\sum_{\substack{P\\ \text{labeled}\\ \text{pyramid}\\ (m)}} v(P)\,\frac{t^{|P|}}{|P|!}\right)$$

$$\log\left(\sum_{\substack{E\\ \text{labeled}\\ \text{heap}}} v(E)\,\frac{t^{|E|}}{|E|!}\right) = \sum_{\substack{P\\ \text{labeled}\\ \text{pyramid}\\ (m)}} v(P)\,\frac{t^{|P|}}{|P|!}$$

$$\log\left(\sum_{\substack{E\\ \text{heap}}} v(E)\,t^{|E|}\right) = \sum_{\text{pyramid}} v(P)\,\frac{t^{|P|}}{|P|}$$

the logarithmic lemma

general form

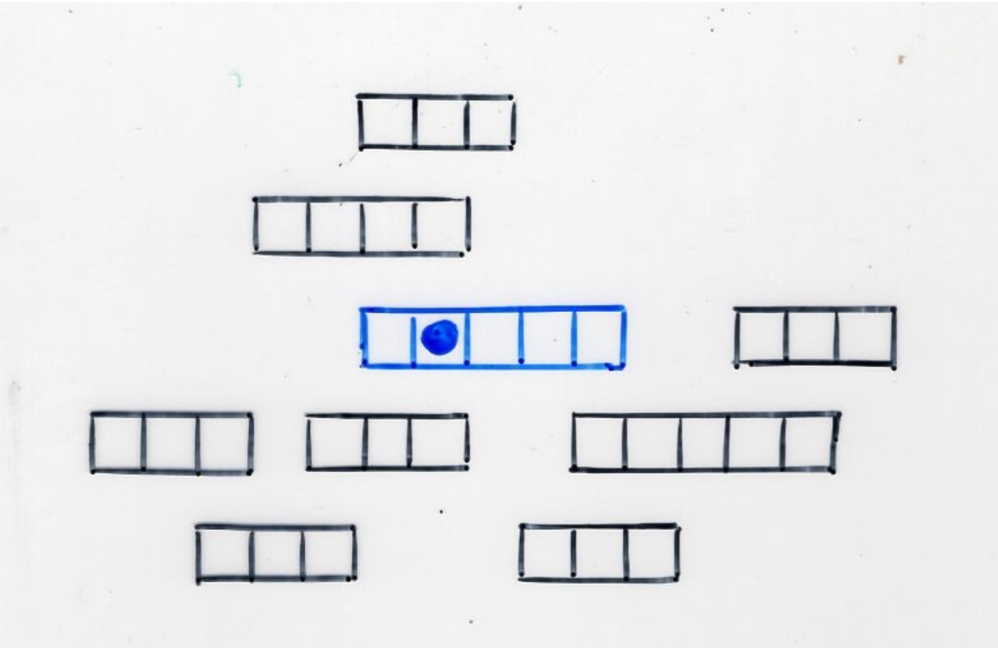
with

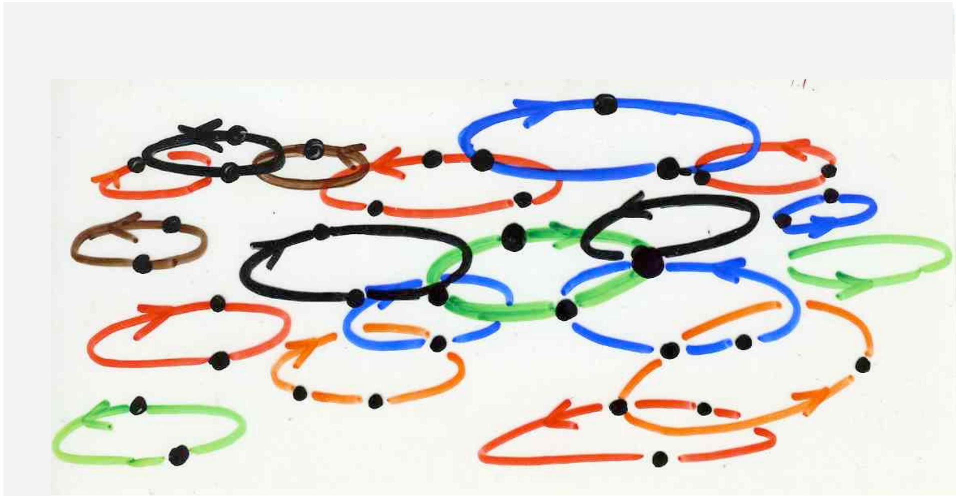
exponential generating functions

weight of
a basic piece:

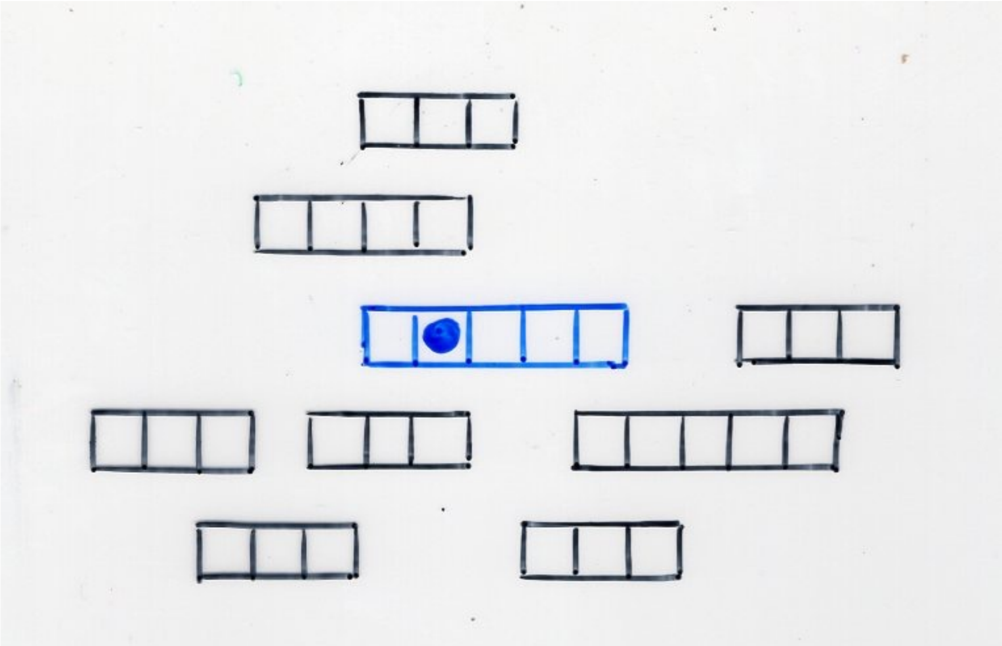
$v(\alpha)t^{\ell(\alpha)}$

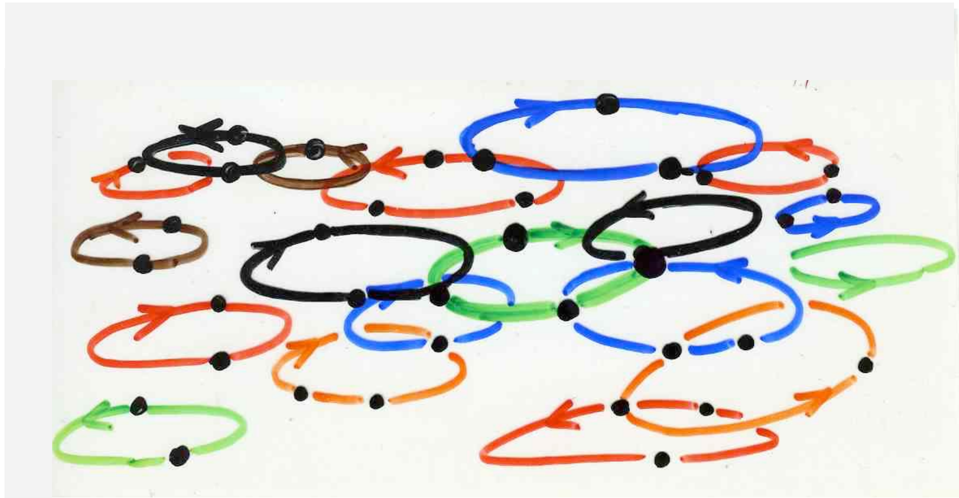
$\ell : P \longrightarrow \mathbb{N}$





each basic piece α
becomes of subset U
of a given set X
with an $\mathbf{F}$-structure
and $|U| = \ell(\alpha)$





here $\mathbf{F} = \begin{cases} \bullet \text{ total order on } U \\ \bullet \text{ cycle on } U \end{cases}$

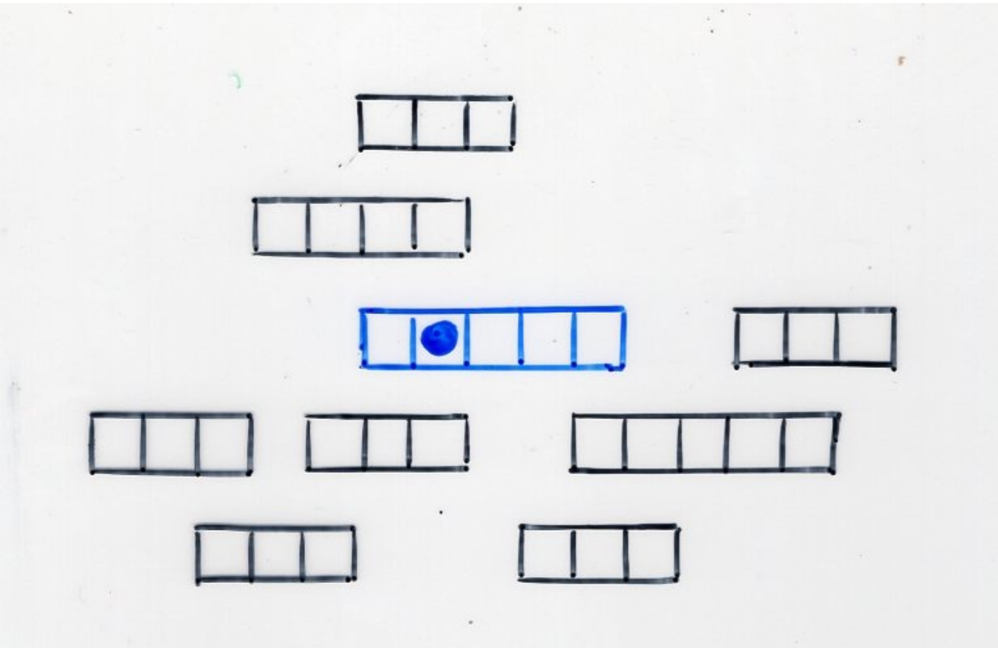
labeled heap

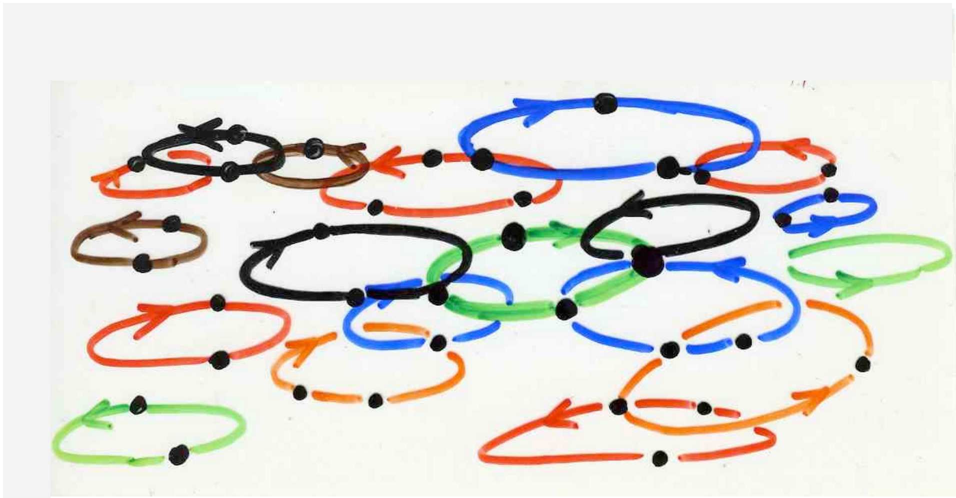
$N!$ labelings

$N = \ell(E)$

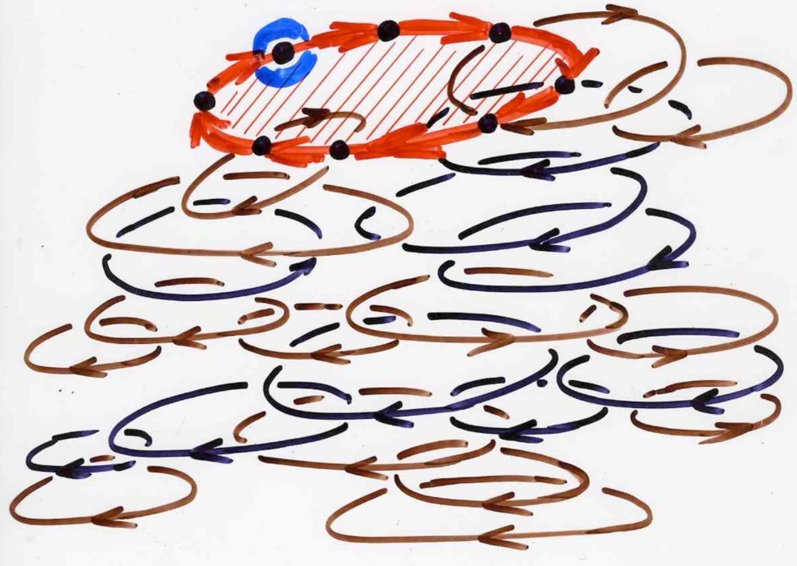
$U = \mathrm{supp}(\alpha)$

$$N = \sum_{\substack{\text{piece } (\alpha, j) \\ \text{of } E}} |\mathrm{supp}(\alpha)|$$





$$\left(\sum_{\substack{E \\ \text{heap}}} v(E)\, t^{\ell(E)}\right) = \exp\left(\sum_{\substack{P \\ \text{pointed} \\ \text{pyramid}}} v(P) \frac{t^{\ell(P)}}{\ell(P)}\right)$$

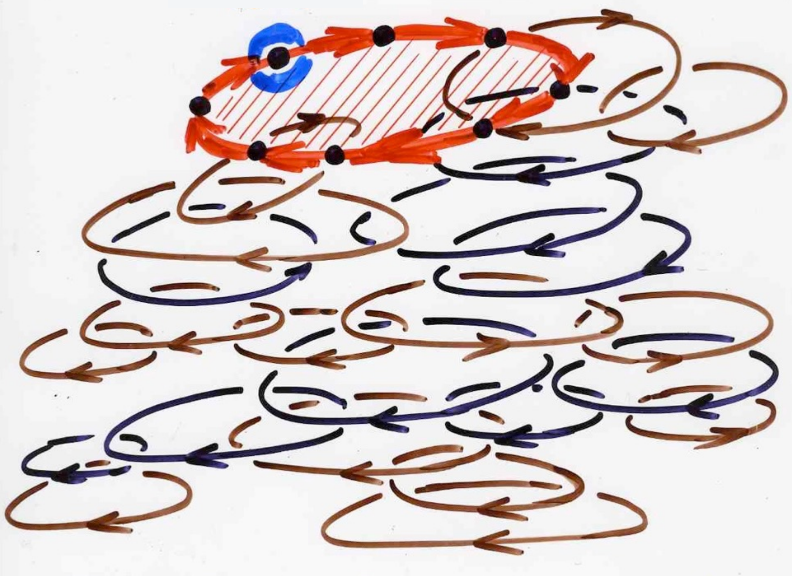


pointed pyramid of cycles

(P, x)

x is a vertex of the maximal piece of P

$$\log\Big(\sum_{\substack{E \\ \text{heap}}} v(E)\, t^{\ell(E)}\Big)$$



pointed pyramid of cycles

$$= \sum_{\substack{P \\ \text{pointed} \\ \text{pyramid}}} v(P) \frac{t^{\ell(P)}}{\ell(P)}$$

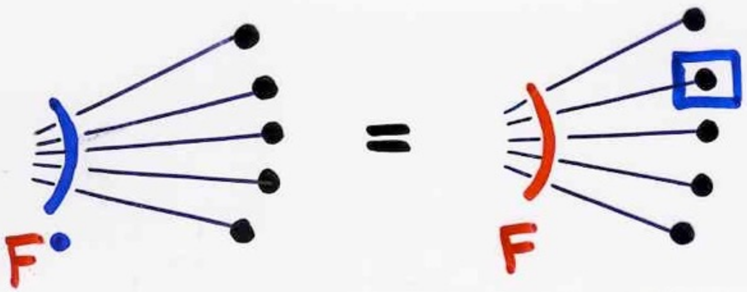
(P, x)

x is a vertex of the maximal piece of P

interpretation
with pointed species

second proof

Def. Pointed species $F^{\bullet}$



$\alpha \in F[U]$ $\quad x \in U$ $\quad (\alpha, x) \in F^{\bullet}[U]$

Prop $F^{\bullet}(t) = t \frac{d}{dt} F(t)$

$\mathcal{H}(\mathbf{F}, X)$

species of (labeled) heaps of subsets of X equipped with an $\mathbf{F}$-structure

$\mathcal{P}(\mathbf{F}, X)$

species of (labeled) pyramids of subsets of X equipped with an $\mathbf{F}$-structure

$$\mathcal{H}^{\bullet} = \mathcal{P}^{\bullet}_{\max} \times \mathcal{H}$$

$$\mathcal{H}^{\bullet} = \mathcal{P}^{\bullet}_{\max} \times \mathcal{H}$$

$$t\,y' = z\,y$$

$$t \frac{d}{dt} \log\Big(\sum_{\substack{E \\ \text{heap}}} v(E)\, t^{\ell(E)}\Big) = \sum_{\substack{P \\ \text{pointed} \\ \text{pyramid}}} v(P)\, t^{\ell(P)}$$

$$(P, x)$$

x is a vertex of the maximal piece of P

last remark ….

$t = 1$?

$$\log\left(\sum_{\substack{E\\ \text{heap}}} v(E)\, t^{|E|}\right) = \sum_{\text{pyramid}} v(P)\, \frac{t^{|P|}}{|P|}$$

$$\log\left(\sum_{\substack{E\\ \text{heap}}} v(E)\, t^{\ell(E)}\right) = \sum_{\substack{P\\ \text{pointed}\\ \text{pyramid}}} v(P)\, \frac{t^{\ell(P)}}{\ell(P)}$$

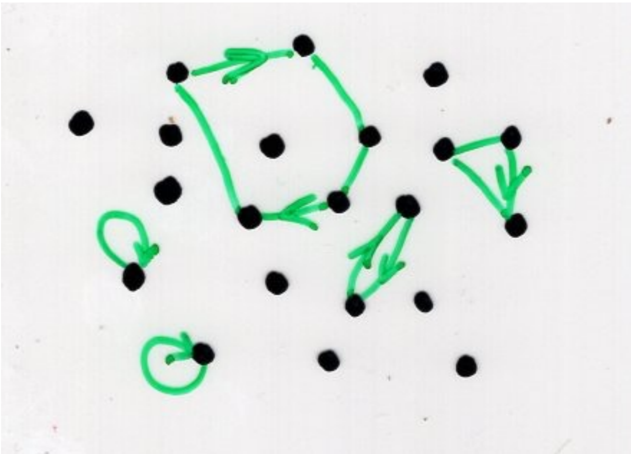
(P, j)

$1 \leq j \leq \ell(m)$

m maximal piece of P

$$A = (a_{ij})_{1 \leq i,j \leq k}$$

$$\det(I - A) = \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \text{permutation}}} (-1)^{\mathrm{inv}(\sigma)} a_{1\sigma(1)} \cdots a_{k\sigma(k)}$$



$$\sum_{\substack{\gamma_1, \ldots, \gamma_r \\ \text{2 by 2 disjoint cycles}}} (-1)^r \, v(\gamma_1) \cdots v(\gamma_r)$$