Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics,

a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

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Chapter 2
Heaps generating functions

(3)

IMSc, Chennai

19 January 2017

from the previous lecture

weight
valuation

$v(E)$

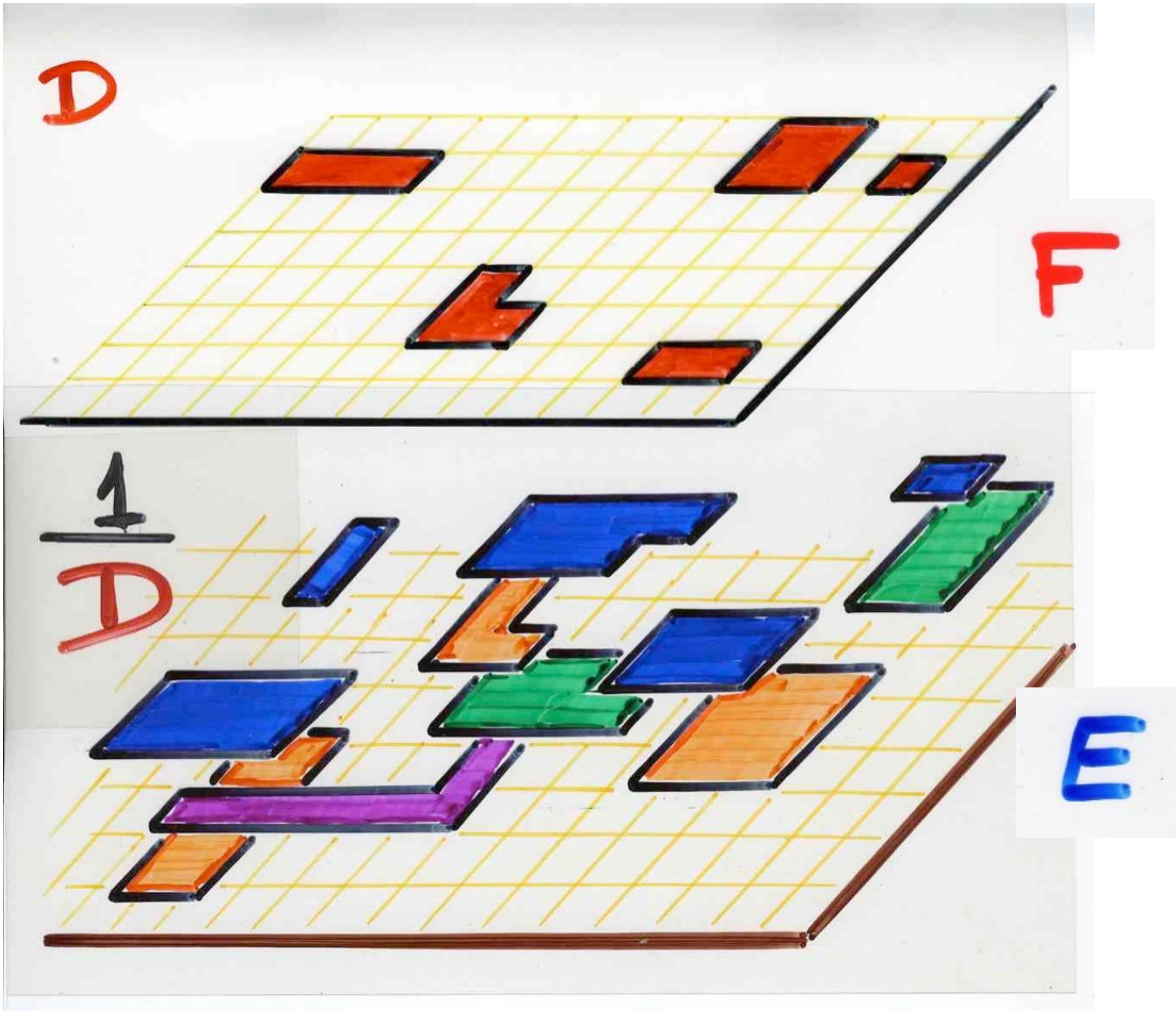
- $v : \underset{\text{basic piece}}{P} \longrightarrow \mathbb{K}[x, y, \dots]$
- $v(\underset{\text{piece}}{\alpha, i}) = v(\alpha)$
- $v(\underset{\text{heap}}{E}) = \prod_{(\alpha, i) \in E} v(\alpha\ i)$

the inversion lemma

$$\left(\sum_{\substack{E\\ \text{heaps}}} v(E)\right) = \frac{1}{\boxed{\left(\sum_{\substack{F\\ \text{trivial}\\ \text{heaps}}} (-1)^{|F|} v(F)\right)}}$$

D

all pieces (α, i)
at level 0


D
F
1
D
E

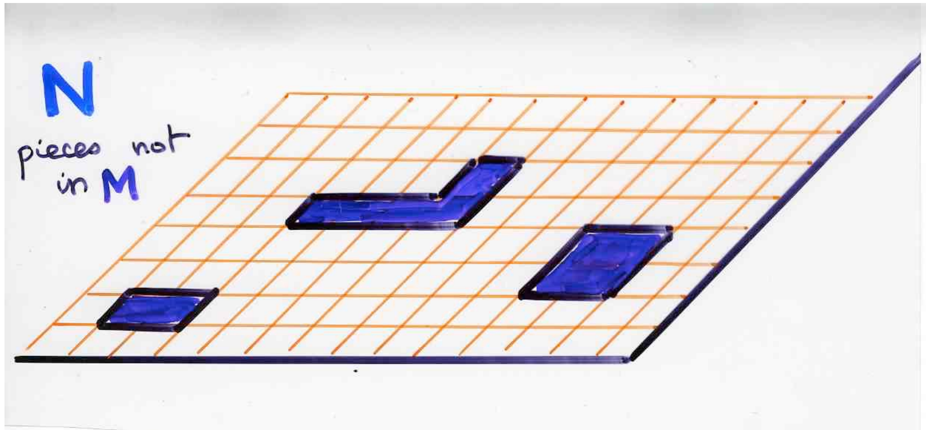
extension of the inversion lemma

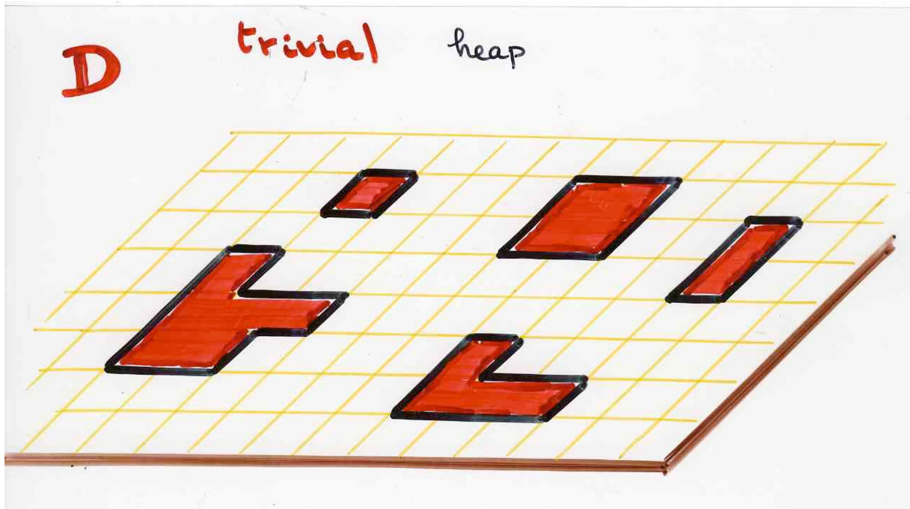
$M \subseteq P$

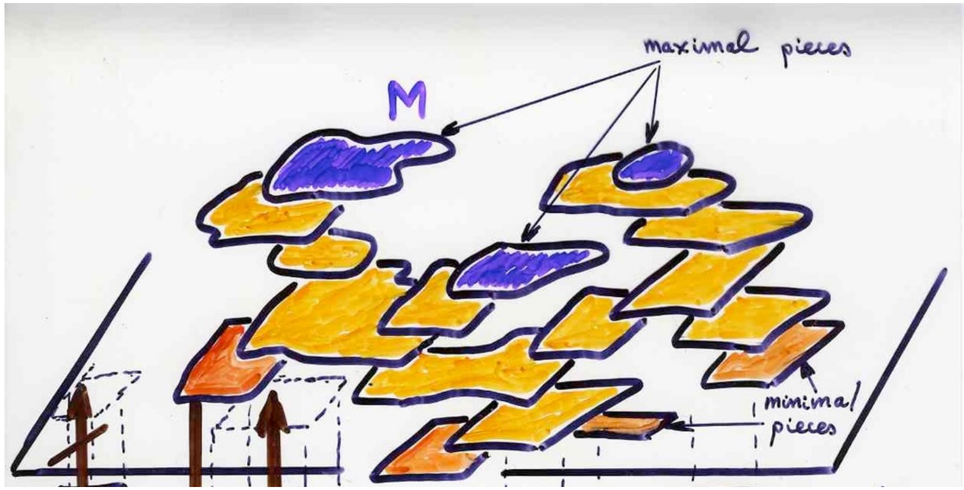
$$\sum_{\substack{E \\ \Pi(\text{maximal pieces}) \in M}} v(E) = \frac{N}{D}$$

$$D = \sum_{\substack{F \\ \text{trivial heaps}}} (-1)^{|F|} v(F)$$

$$N = \sum_{\substack{F \\ \text{trivial heaps} \\ \text{pieces} \notin M}} (-1)^{|F|} v(F)$$


N
pieces not in M


D
trivial heap


M
maximal pieces
minimal pieces

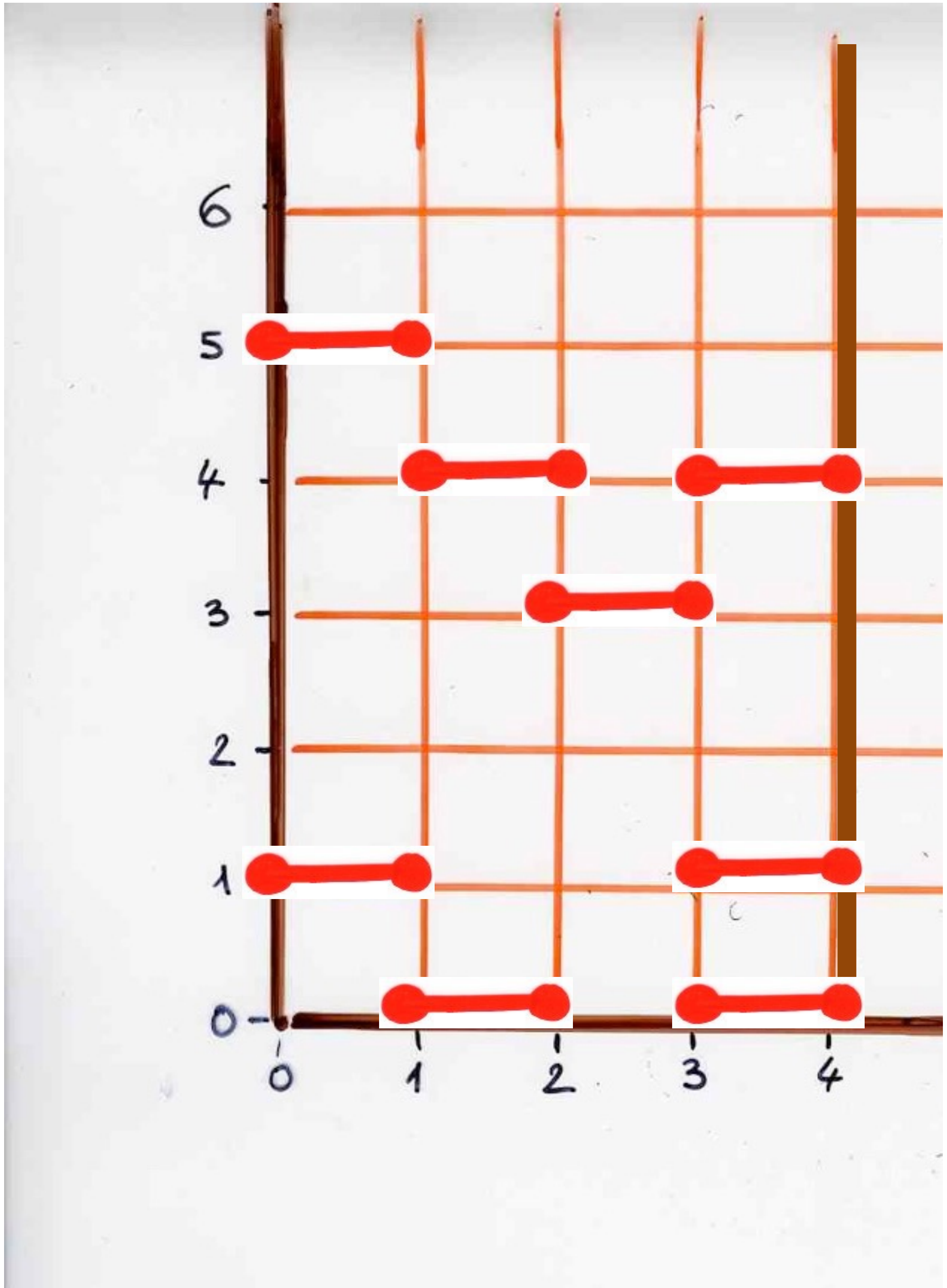
Proof by (weight preserving) involution φ

$$\begin{cases} \bullet \; v(E)\,v(F) = v(E')\,v(F') \\ \bullet \; (-1)^{|F|} = -(-1)^{|F'|} \end{cases}$$

$$\Big(\sum_{\substack{E \\ \text{heaps}}} v(E)\Big) \, D = N$$

$\prod \begin{pmatrix}\text{maximal}\\ \text{pieces}\end{pmatrix} \in M$

φ not defined for (E, F) with $E = \emptyset$, $F \subseteq P - M$



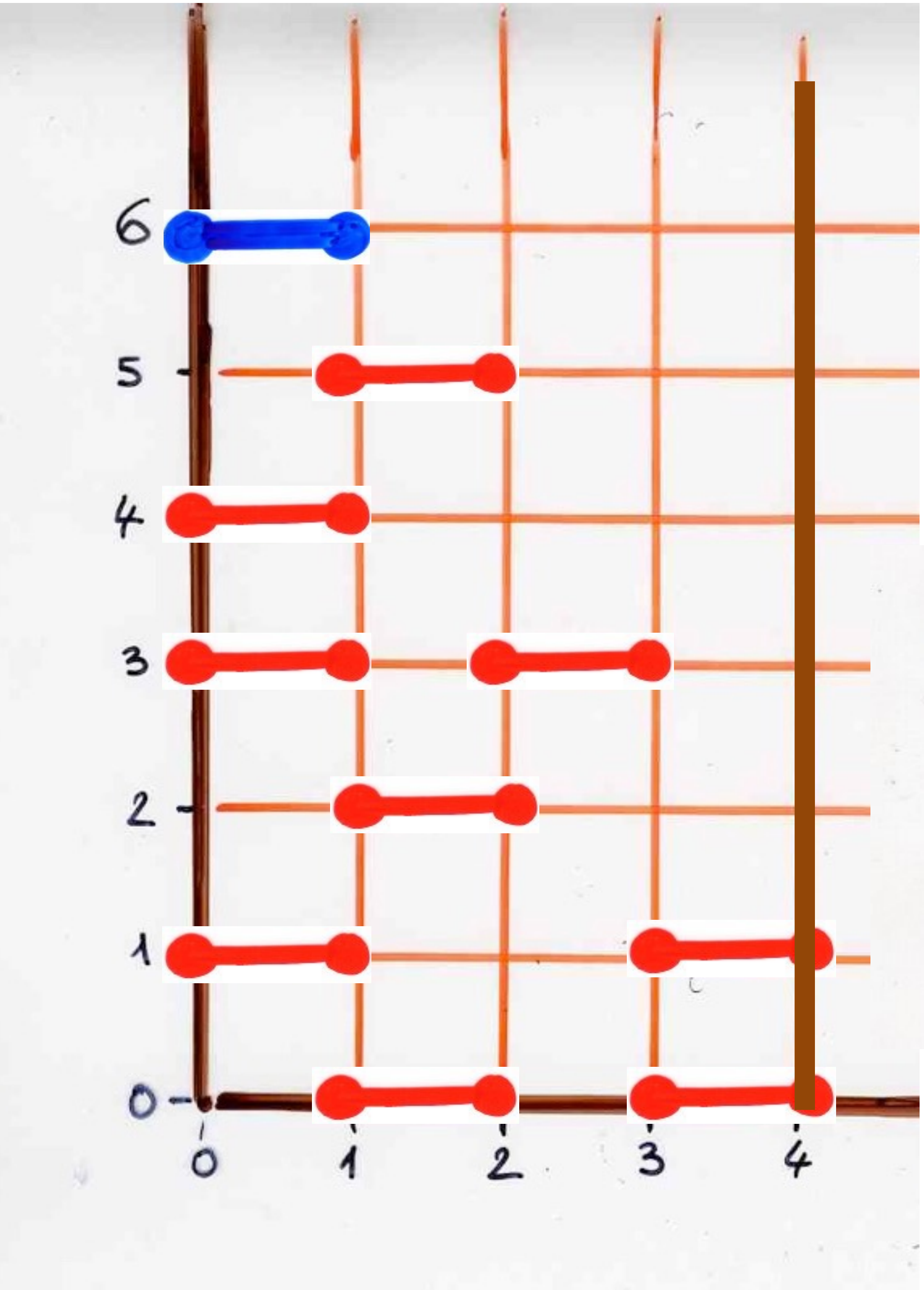
generating function of heaps of dimers on the segment $[0, k]$ (enumerated by the number of dimers)

$$\frac{1}{F_{k+1}(t)}$$

$$F_n(x) = \sum_{\substack{M \\ \text{matchings} \\ \text{of } \{1, \dots, n\}}} (-x)^{|M|}$$

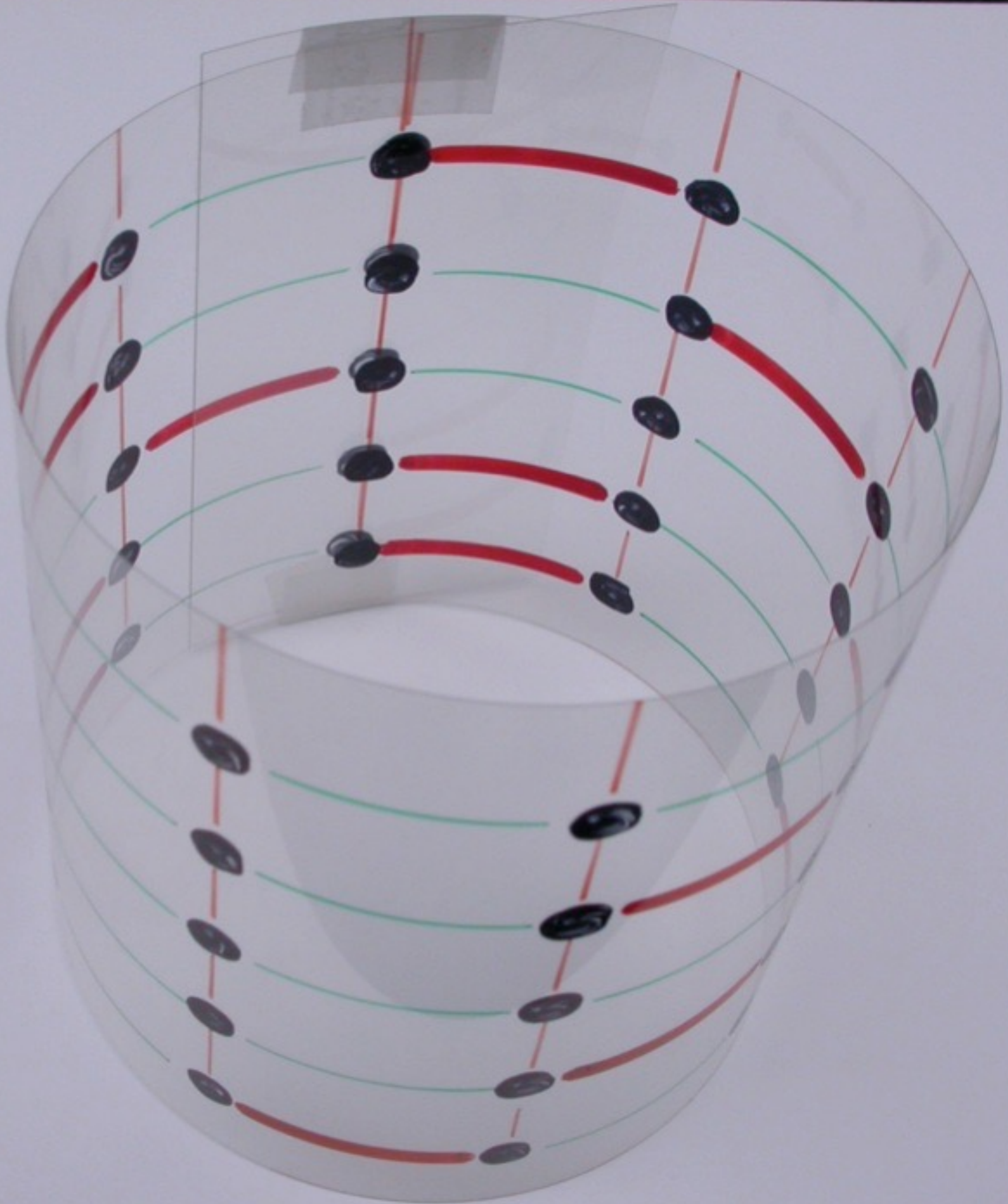
Fibonacci polynomials

$= n$

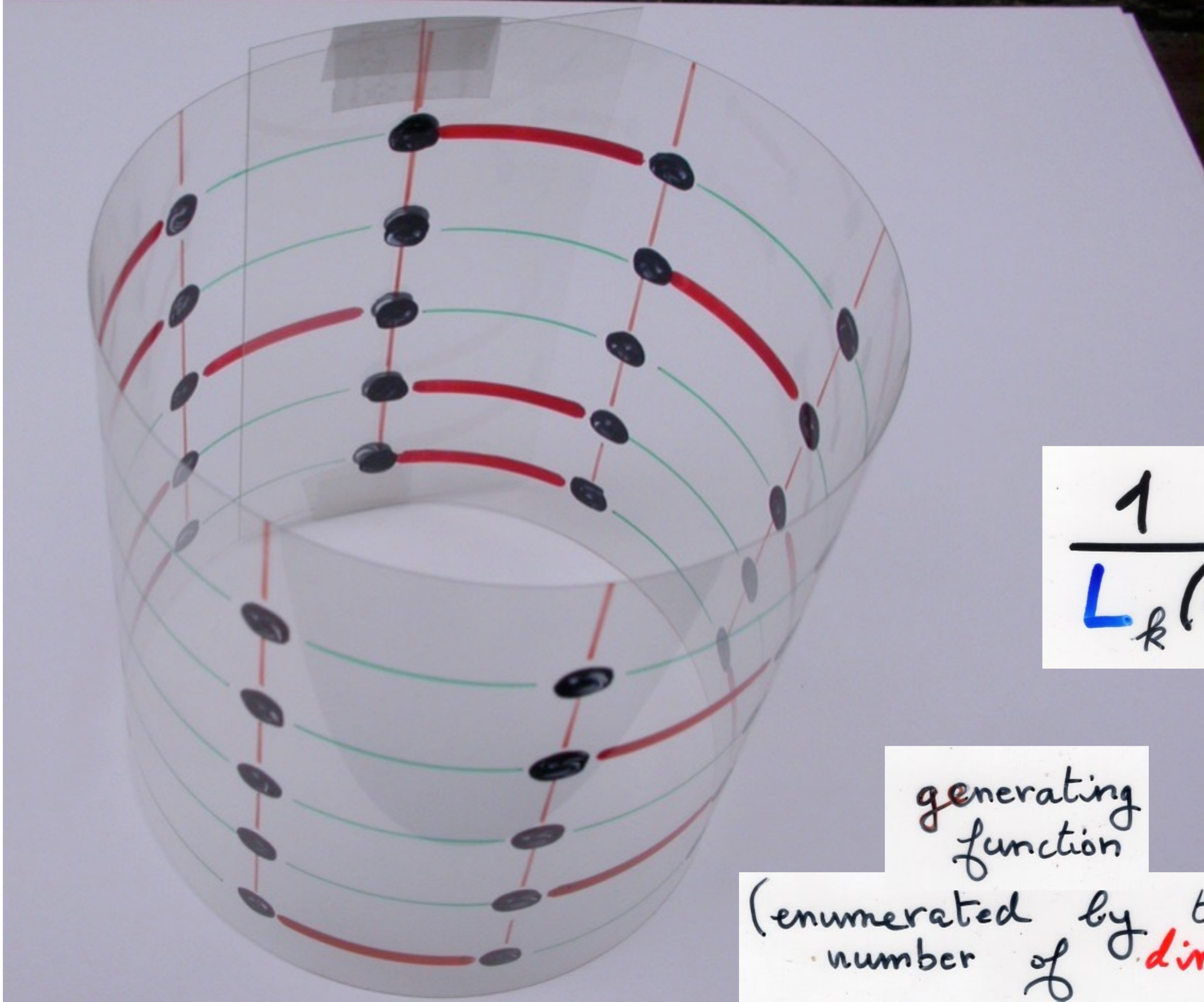


$$\frac{F_k(t)}{F_{k+1}(t)}$$

generating function of semi-pyramids of dimers on the segment $[0, k]$ (enumerated by the number of dimers)



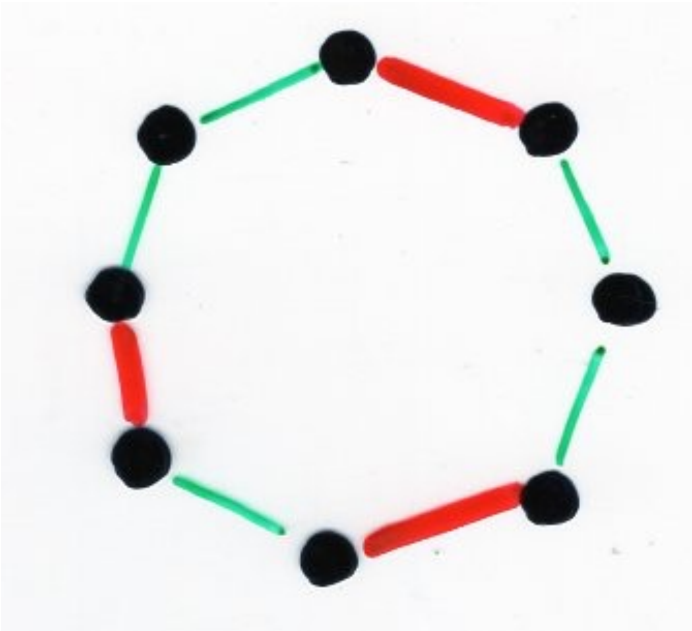
heaps of dimers
on the "cycle" Γ_k
of length k



$$\frac{1}{L_k(t)}$$

generating function

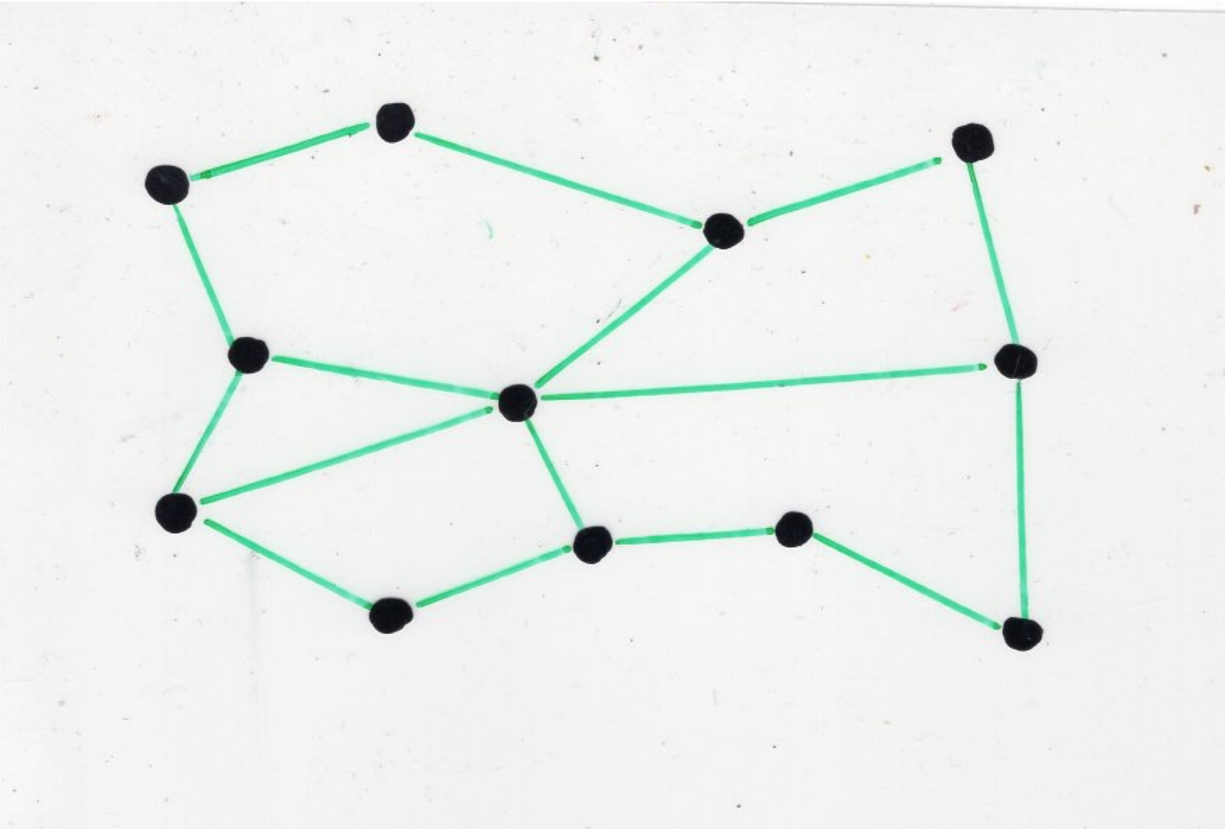
(enumerated by the number of dimers)



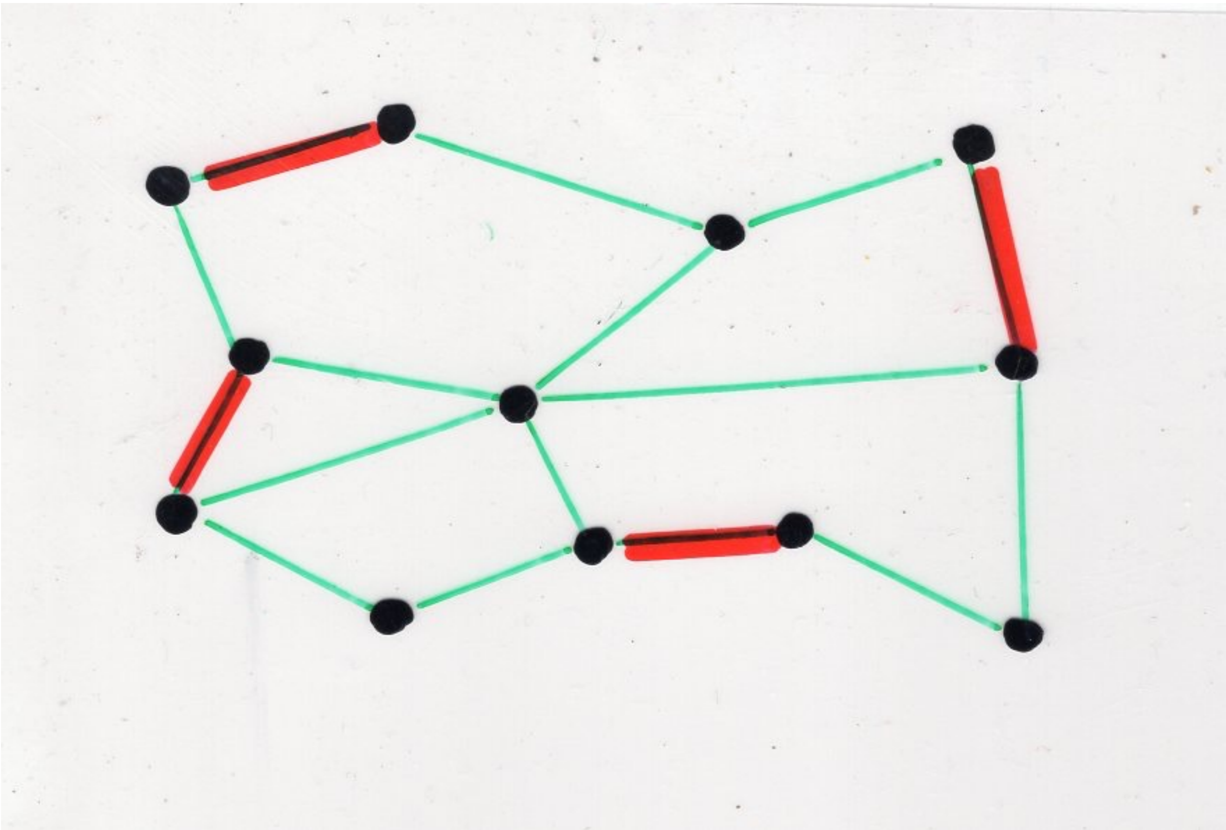
Lucas polynomial

$$L_n(x) = \sum_{\substack{\text{matchings } M \\ \text{of a cycle } \Gamma_n \\ \text{length } n}} (-x)^{|M|}$$

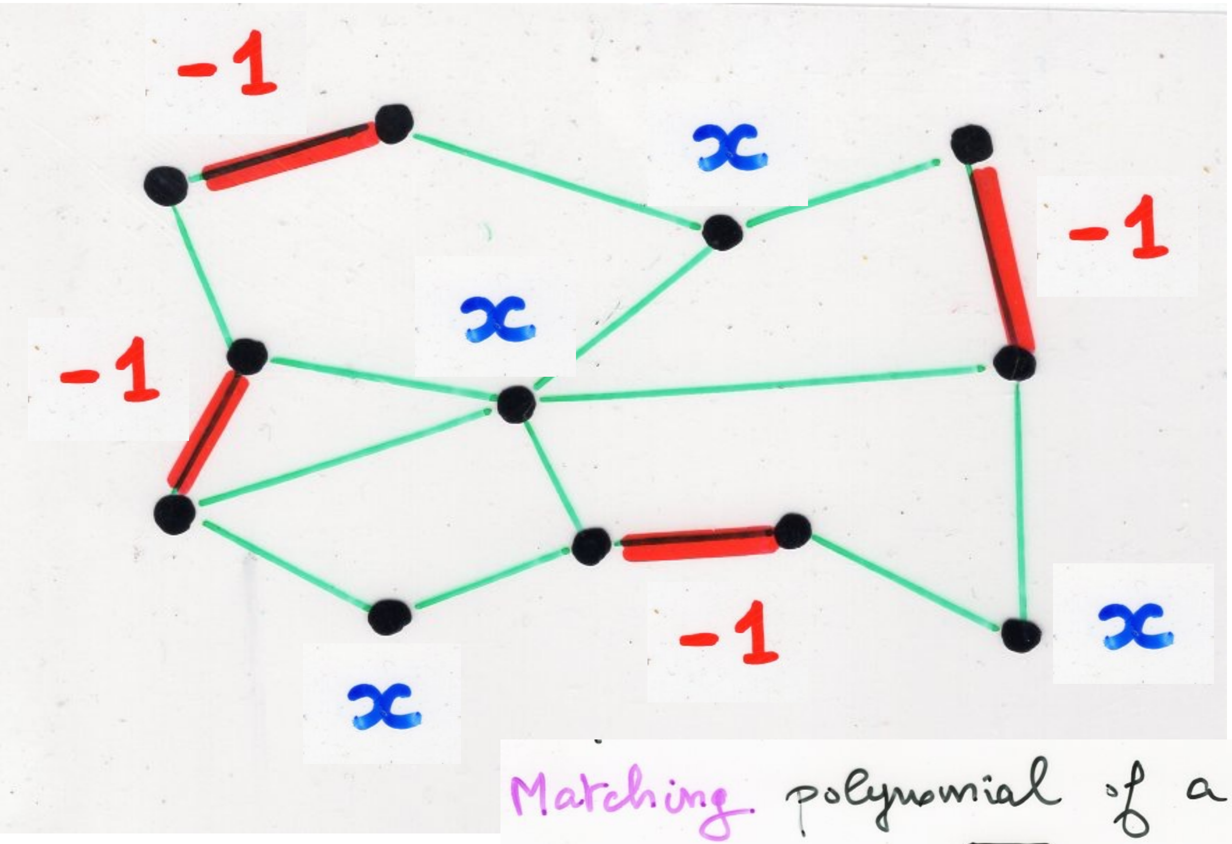
matching polynomial of a graph G



matching
polynomial
of a graph G



matching
of a graph G = set of 2 by 2
disjoint edges



$\to$ Ch 5

heaps and algebraic graph theory

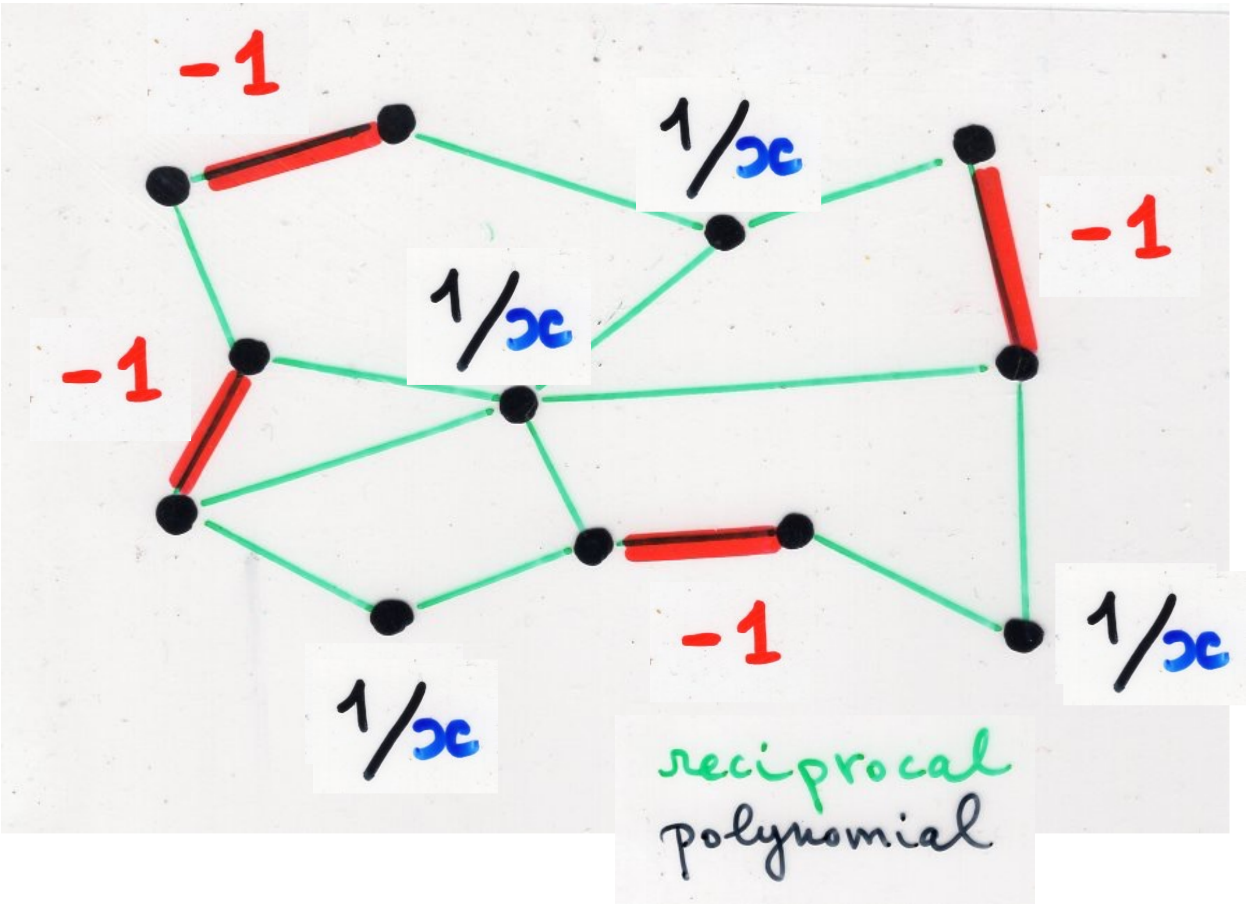
Matching polynomial of a graph G

$$M_G(x) = \sum_{\substack{\text{matchings } M \\ \text{of } G}} (-1)^{|M|} x^{ip(M)}$$

$ip(M)$ = number of isolated vertices of G

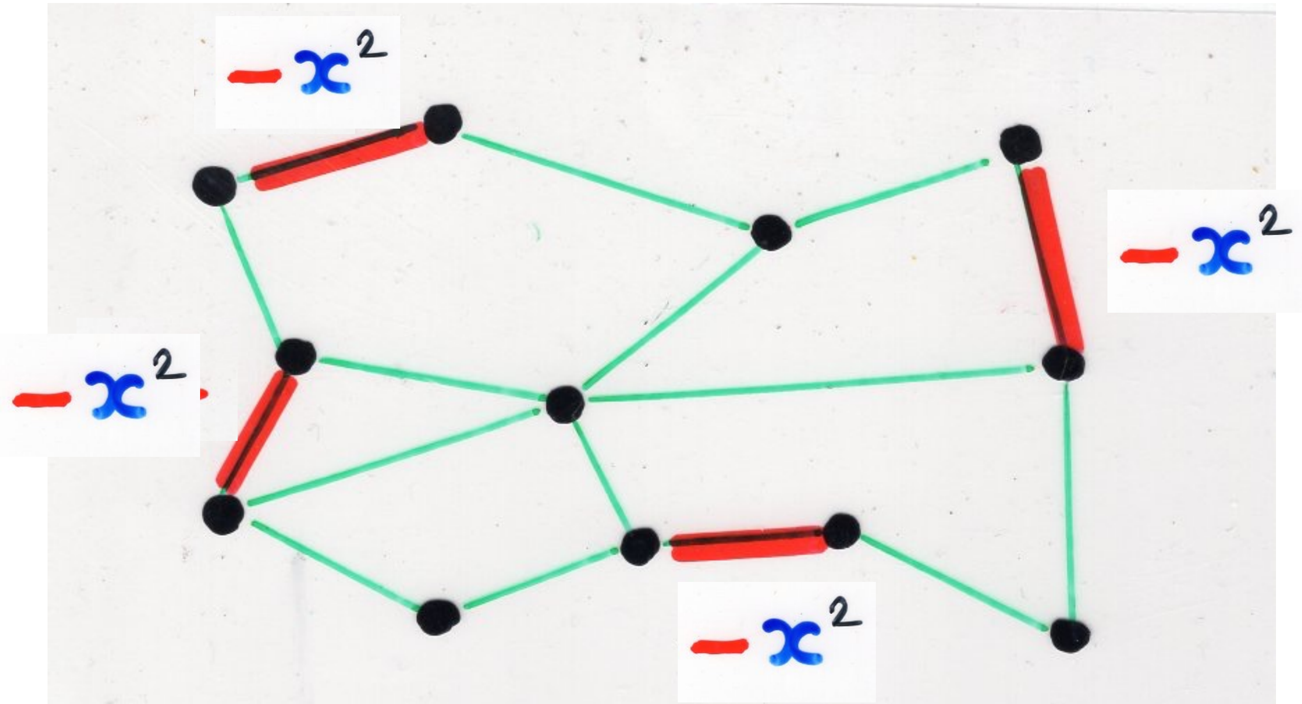
$$= \sum_{M} (-1)^{|M|} x^{n-2|M|}$$

n = nb of vertices of G



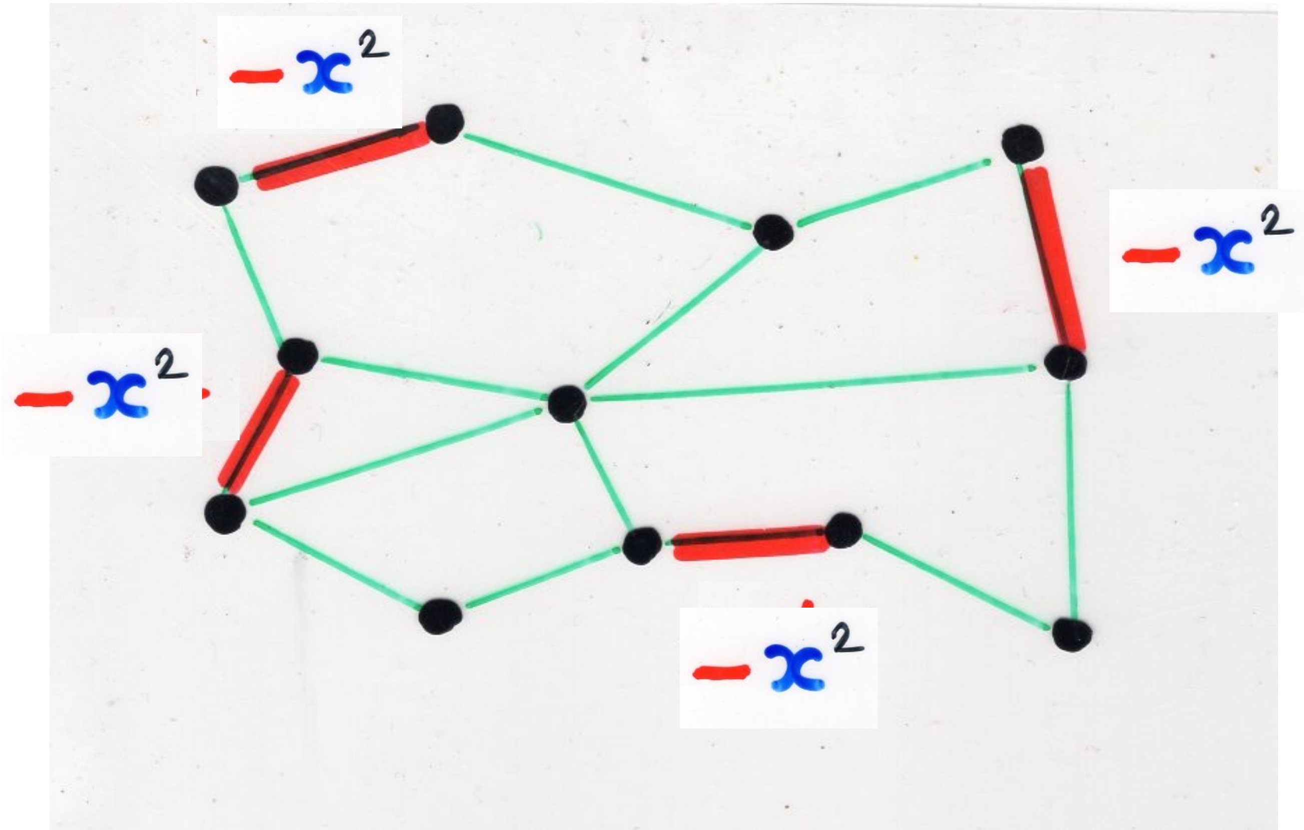
$$M^*_G(x) = x^n M_G(1/x)$$

$n = \deg(M_G)$
$=$ number of vertices of G



reciprocal polynomial

$$M^{*}_{G}(x) = x^{n}\, M_{G}(1/x) = \sum_{\substack{M \\ \text{matchings} \\ \text{of } G}} (-x^{2})^{|M|}$$



generating function for heaps of edges on a graph G

$$\frac{1}{M^*_G(t)}$$

$$t = x^2$$

(enumerated by number of edges)

Fibonacci , Lucas
and
Tchebycheff polynomials

G = "Segment graph" Sg_n

n vertices

$M_n(x) = M_{Sg_n}(x)$

matching polynomial of the segment graph Sg_n

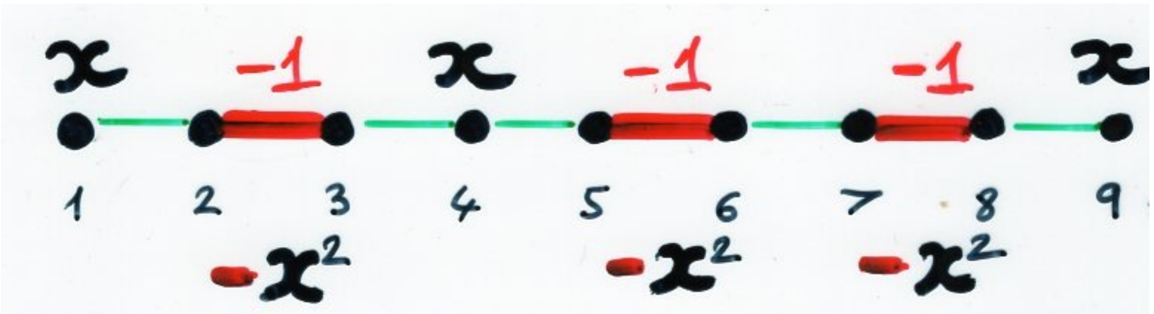
then $M_n^*(x) = F_n(x^2)$

reciprocal polynomial

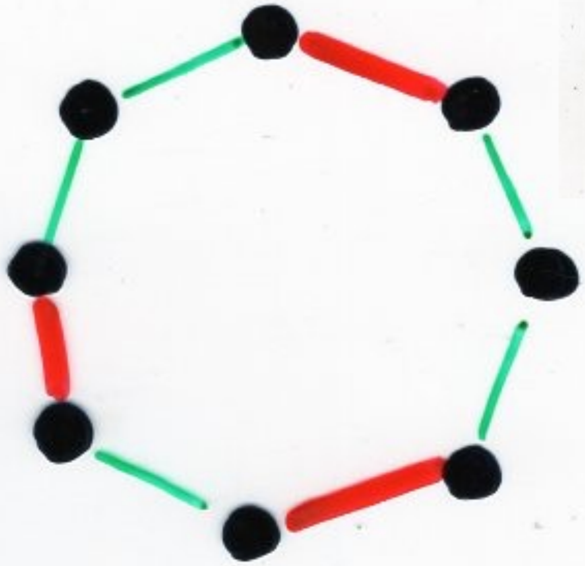
Fibonacci polynomial

$\sin((n+1)\theta) = \sin\theta \, U_n(\cos\theta)$

$U_n(x)$ Tchebychef polynomial 2^{nd} kind



$U_n(x) = M_n(2x)$



G = "cycle graph" Γ_n

$$C_n(x) = M_{\Gamma_n}(x)$$

matching polynomial of the cycle graph Γ_n

$$C_n^*(x) = L_n(x^2)$$

reciprocal polynomial

Lucas polynomial

$$\cos(n\theta) = T_n(\cos\theta)$$

$$T_n(x) = \frac{1}{2} C_n(2x)$$

second proof of the extension

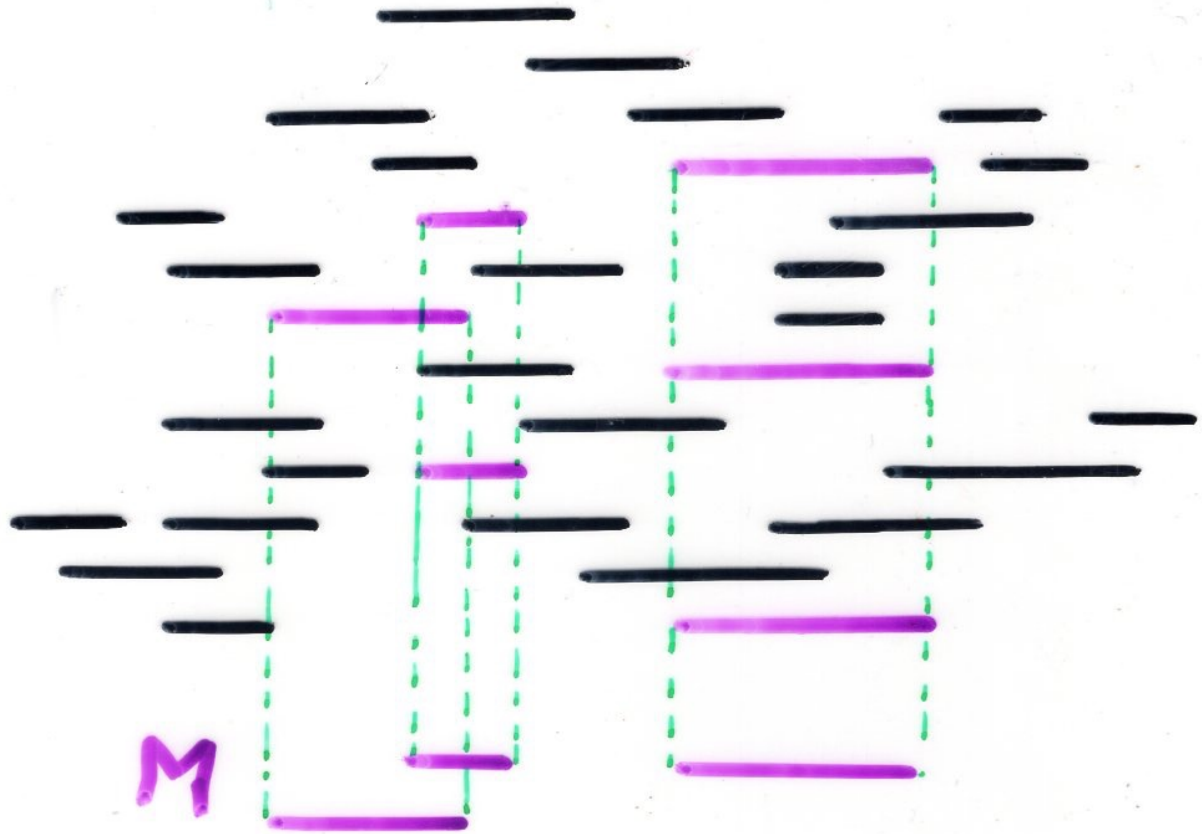
N/D

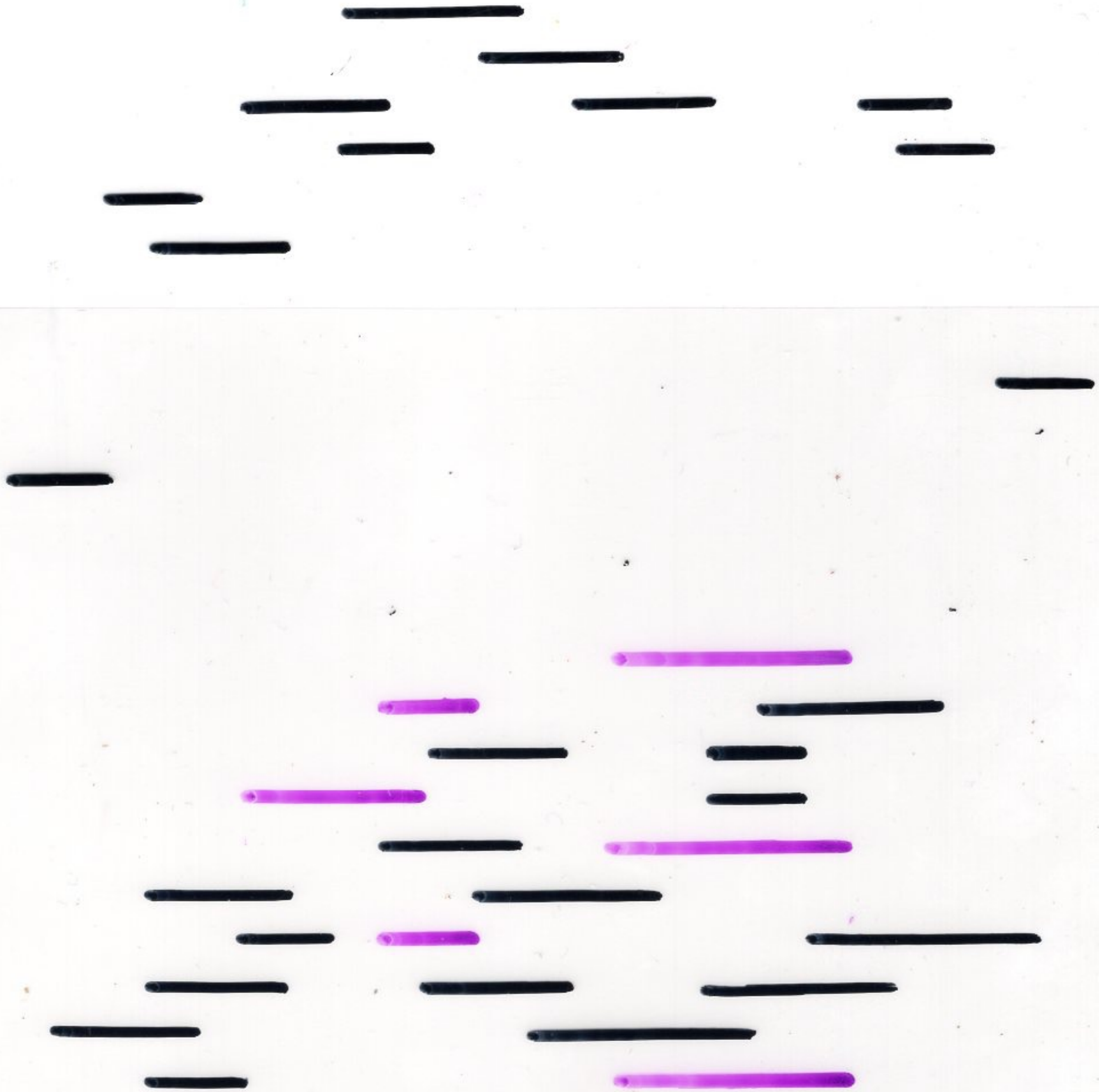
$H(\mathcal{P}, \mathcal{C}) \qquad M \subseteq \mathcal{P}$

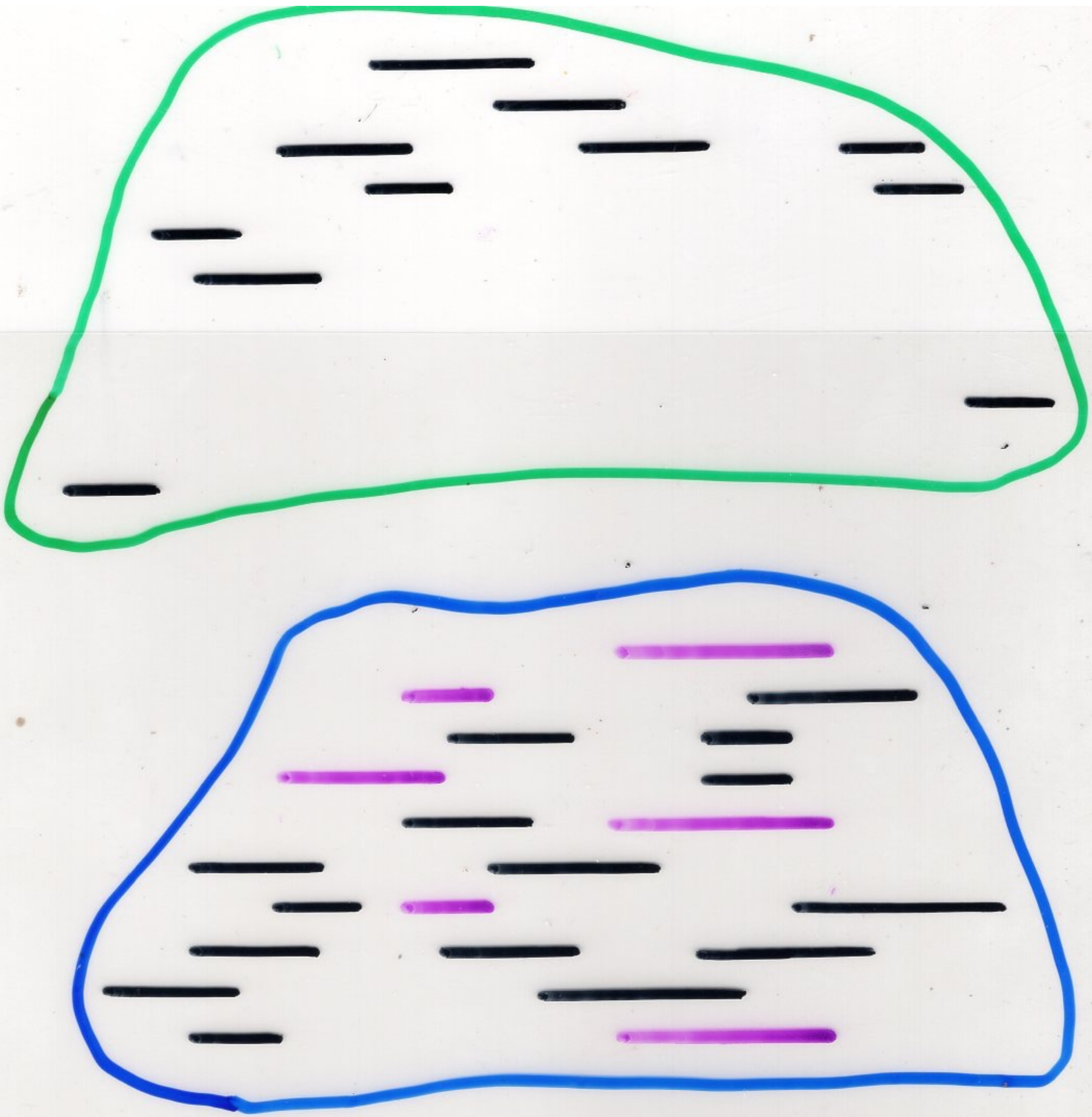
Proposition Any heap $E \in H(\mathcal{P}, \mathcal{C})$ has a unique factorization

$$E = F \odot G$$

- $F \in H(\mathcal{P}, \mathcal{C})$ with $\pi\left(\begin{smallmatrix}\text{maximal}\\ \text{piece}\end{smallmatrix}\right) \in M$
- $G \in H(\mathcal{P} - M, \mathcal{C}_{/\mathcal{P}-M})$


M





$$H(P, \mathcal{C}) \qquad M \subseteq P$$

__Proposition__ Any heap $E \in H(P, \mathcal{C})$ has a unique factorization

$$E = F \odot G$$

- $F \in H(P, \mathcal{C})$ with $\pi\left(\begin{smallmatrix}\text{maximal}\\ \text{piece}\end{smallmatrix}\right) \in M$
- $G \in H(P - M, \mathcal{C}_{/P-M})$

$$\frac{1}{D} = \Bigg(\sum_{\substack{E \\ \text{heap} \\ \pi\left(\begin{smallmatrix}\text{max}\\ \text{piece}\end{smallmatrix}\right) \in M}} v(E)\Bigg) \frac{1}{N}$$

complements

Lazard elimination

"Lazard elimination"

(Duchamp, Krob 1991)

special case

$\mathbf{M} = \{a\}$

$a \in \mathbf{P}$

$$\underbrace{\mathbf{F}}_{\substack{\text{free}\\ \text{structure}}}\{x_1, .., x_k, z\} \simeq \mathrm{Nice}\{\mathbf{Z}\} * \underbrace{\mathbf{F}}_{\substack{\text{free}\\ \text{structure}}}\{x_1, .. x_k\}$$

<u>example</u> $X = \{x_1, .., x_k, z\}$ X^* free monoid

$$X^* = Z^* \cdot \{x_1, .., x_k\}^*$$

$$Z = \{x_1, .., x_k\}^* z$$

$H(\mathcal{P}, \mathcal{C}) \qquad M \subseteq P$

<u>Proposition</u> Any heap $E \in H(\mathcal{P}, \mathcal{C})$ has a unique factorization

$$E = F \odot G$$

- $F \in H(\mathcal{P}, \mathcal{C})$ with $\pi\left(\begin{matrix}\text{maximal}\\\text{piece}\end{matrix}\right) \in M$
- $G \in H(\mathcal{P} - M, \mathcal{C}_{/P-M})$

special case

$M = \{a\}$

$a \in P$

$$Z_{a,\mathcal{P},\mathcal{C}} = \left\{ \begin{array}{l} \text{primitive pyramid} \\ \pi(m) = a \\ \quad \uparrow \text{unique} \\ \text{maximal piece} \end{array} \right\}$$

"over a"

$$H(\mathcal{P},\mathcal{C}) = Z^{*}_{a,\mathcal{P},\mathcal{C}} \odot H(\mathcal{P}-\{a\}, \mathcal{C}_a)$$

heaps monoid — free monoid — heaps monoid

$L(\mathcal{P}, \overline{\mathcal{C}})$

commutation monoid

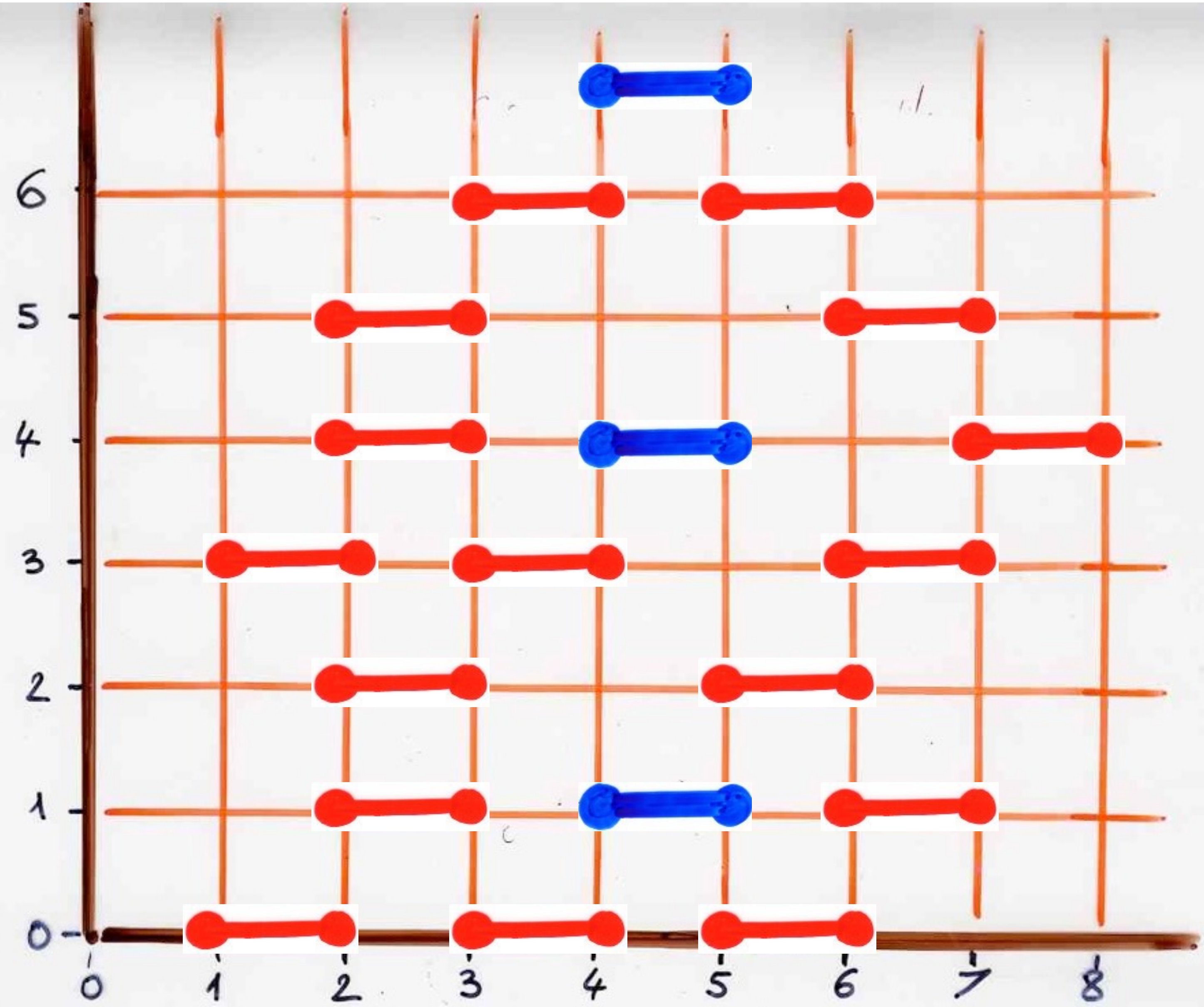
$L(\mathcal{P}-\{a\}, \overline{\mathcal{C}_a})$

commutation monoid

$$Z_{a,P,\tau} = \left\{ \begin{array}{l} \text{primitive pyramid} \\ \pi(m) = a \\ \quad \text{unique} \\ \quad \text{maximal piece} \end{array} \right\}$$

"over a"

unique factorization
of a pyramid (over a)
into primitive pyramids
(over a)



0
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6
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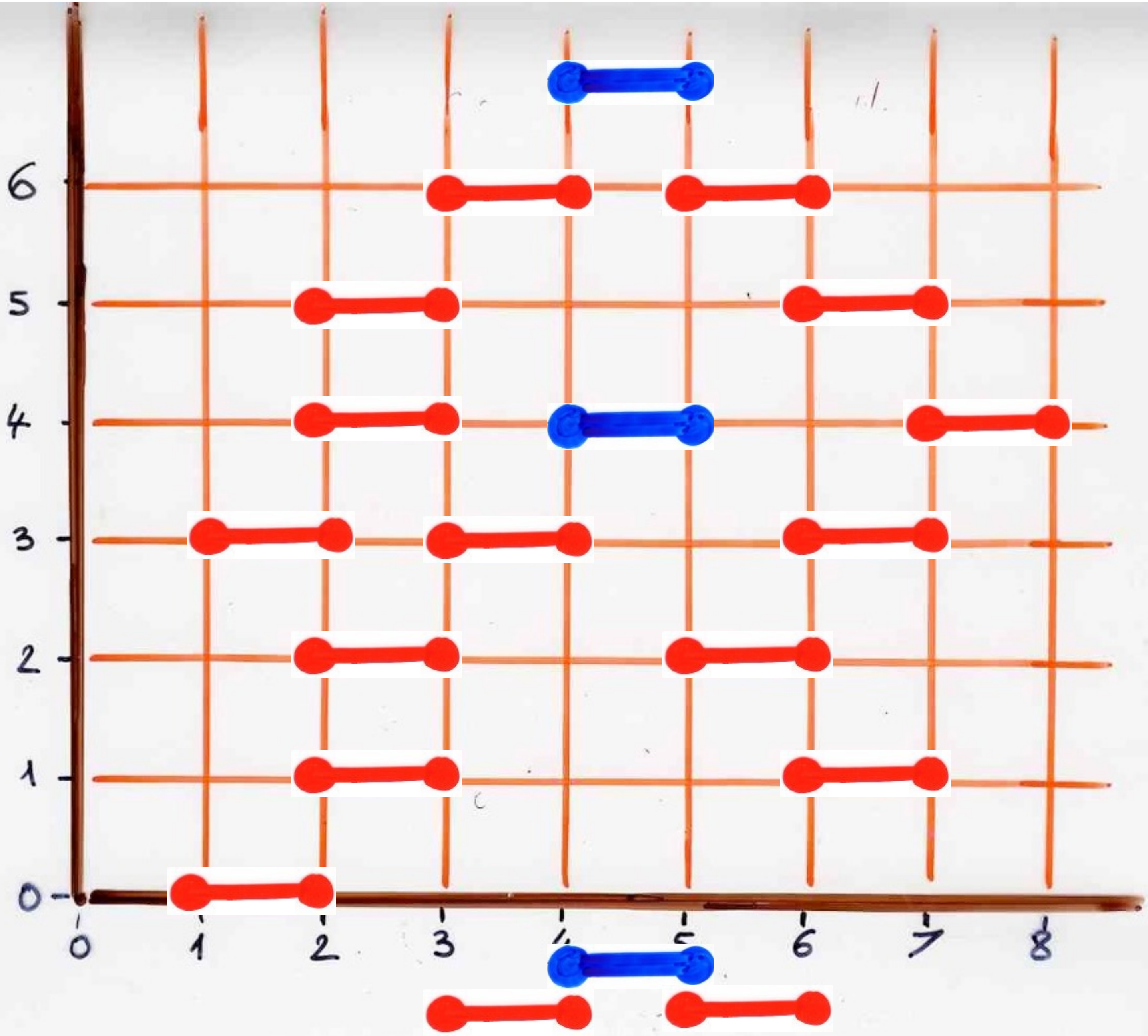
7
8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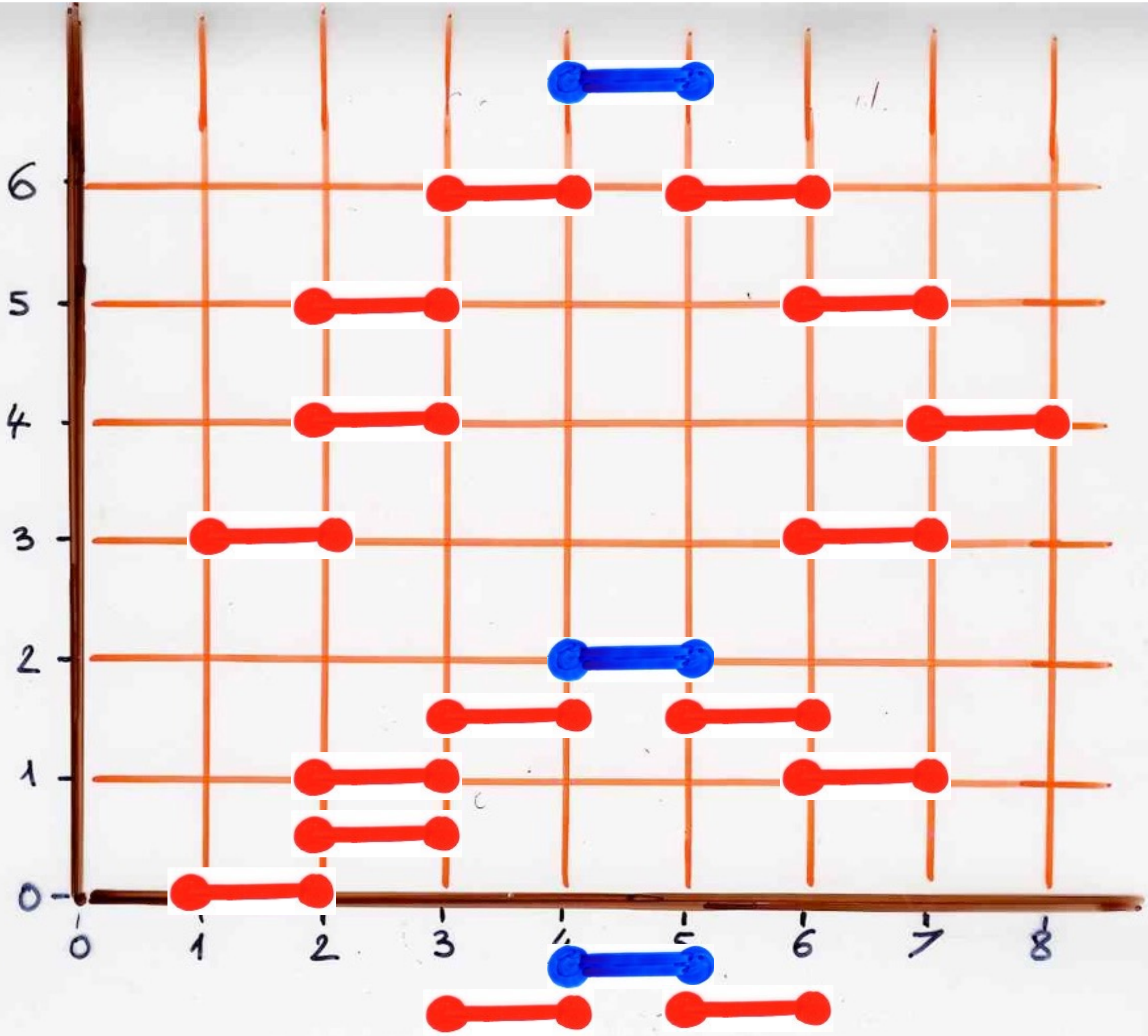

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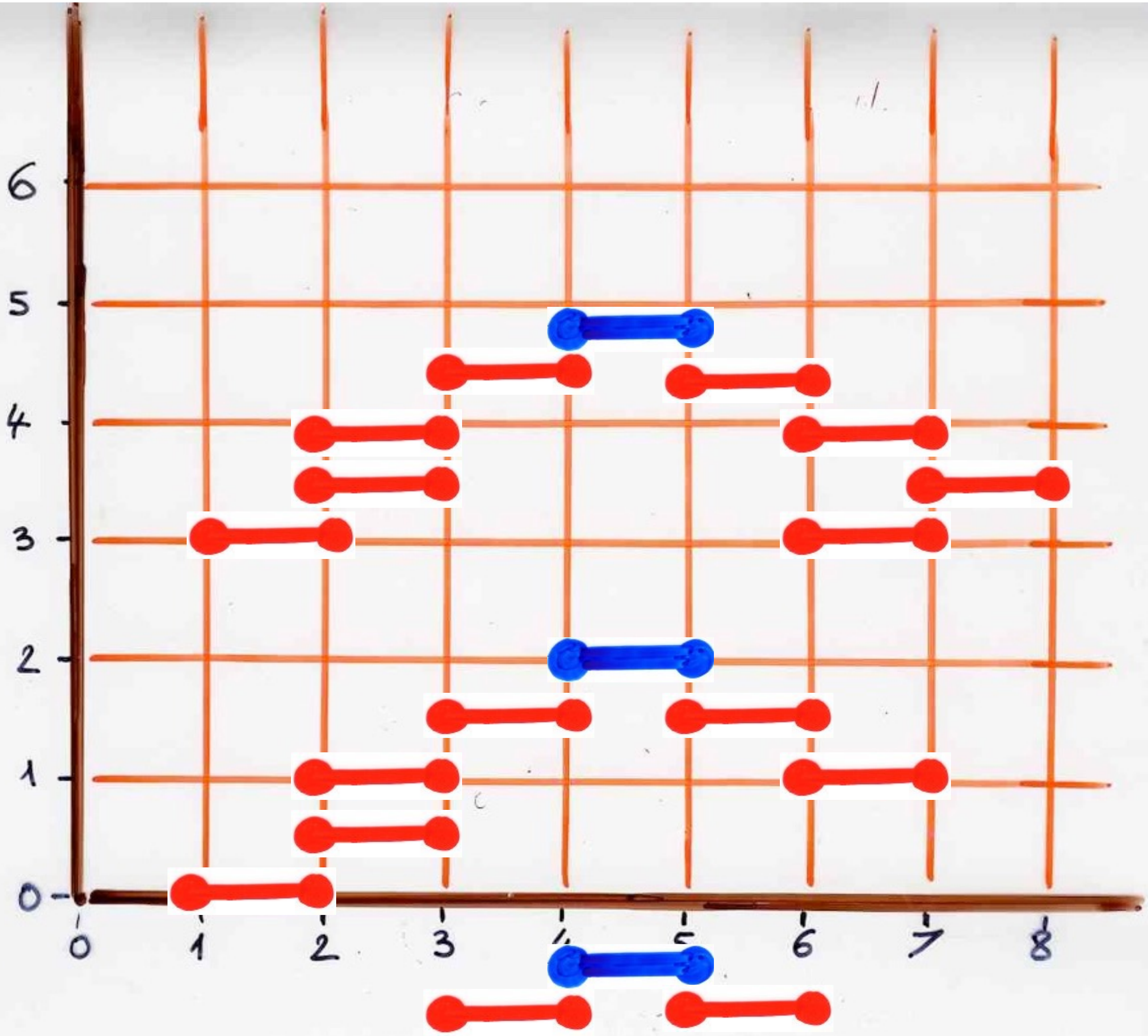

6
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7
8

"Lazard elimination"

M. Lazard (1950's)

free group

free Lie algebra

example $X = \{x_1, \dots, x_k, z\}$ X^* free monoid

$$X^* = Z^* \cdot \{x_1, \dots, x_k\}^*$$

$$Z = \{x_1, \dots, x_k\}^* z$$

find basis of $L(X)$

$$L(X) \simeq L(Z) \oplus L(x_1, \dots, x_k)$$

free Lie algebra

commutation monoids
= free partially commutative monoids

free monoid X^* word w
free abelian monoid $Ab(X)$
monomials $x_1^{\alpha_1} \dots x_k^{\alpha_k}$

find basis of $L_c(X)$
free partially commutative Lie algebra

Lalonde (1990)
Duchamp, Krob (1991)

the inversion lemma
and
Möbius function

Möbius classic in number theory

for $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$

prime numbers decomposition

$$\mu(n) = \begin{cases} \bullet\ 0 & \text{if } n \text{ is divisible by a square} \\ \bullet\ (-1)^k & \text{else} \end{cases}$$

$$g(n) = \sum_{d \mid n} f(d)$$

$$\Updownarrow$$

$$f(n) = \sum_{d \mid n} \mu(d)\, g(n/d)$$

Möbius function
in posets

E locally finite poset $\leqslant$

i.e. every closed interval
$[a, b] = \{x,\ a \leqslant x \leqslant b\}$
is finite

incidence algebra of E

$f : E \times E \longrightarrow R$

R commutative ring with unit 1

incidence algebra of E

$f : E \times E \longrightarrow R$

$R(E)$

R commutative ring with unit 1

$$\begin{cases} \text{(i)} \quad f(x,y) = 0 \quad \text{if} \quad x \not\leq y \\ \text{(ii)} \quad f(x,x) = 1 \end{cases}$$

$$\begin{cases} f+g\,(x,y) = f(x,y) + g(x,y) \\ fg\,(x,y) = \displaystyle\sum_{x \leq z \leq y} f(x,z)\, g(z,y) \end{cases}$$

δ Kronecker function

$$\delta(x,y) = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{else} \end{cases}$$

unit element

zeta function of E

$$\zeta_E(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{else} \end{cases}$$

μ_E

Möbius function inverse of ζ_E

$$\zeta_E \mu_E = \mu_E \zeta_E = \delta$$

inversion formula

$f, g : E \longrightarrow \mathbb{R}$

$m \in E$

$f_m(m, x) = f(x)$

$g_m(m, x) = g(x)$

$$g(x) = \sum_{m \preccurlyeq y \preccurlyeq x} f(y)$$

$$g_m = f_m \cdot \zeta$$

$\Updownarrow$

$$f(x) = \sum_{m \preccurlyeq y \preccurlyeq x} g(y)\, \mu_E(y, x)$$

$$f_m = g_m \cdot \mu_E$$

$$\zeta_E \mu_E = \mu_E \zeta_E = \delta$$

exercise

$$\mu_E(x,y) = -\sum_{\substack{x \leqslant z \leqslant y \\ z \neq y}} \mu_E(x,z)$$

computation of the Möbius function of a poset E

Möbius function in monoids

finite factorization monoid

M

unit element 1

may be a zéro

$0 \cdot x = x \cdot 0$ for $x \in M$

M^+ non-zero elements of M

finite factorization
monoid

M

unit
element 1

factorization of $x \in M$

$x = x_1 \cdots x_k$

$s = (x_1, \ldots, x_k)$

$x_i \in M^+,\ x_i \neq 1$

k degree
of s

for every $x \in M^+$
finite number of
factorizations

convention:
1 empty factorization
with degree 0

Lemma
no other factorization
for 1

incidence algebra of M

$\mathcal{R}$ ring unit 1

$\mathcal{R}(M)$: $f : M^{+} \to \mathcal{R}$

$$\begin{cases} (f+g)(x) = f(x) + g(x) \\ (fg)(x) = \sum_{uv=x} f(u)\,g(v) \end{cases}$$

incidence algebra of M

ζ_M zeta $\zeta_M(x) = 1$ for every $x \in M^+$

ε_M $\varepsilon_M(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{else} \end{cases}$

ε_M unit element

$\varepsilon_M f = f \varepsilon_M = f$

μ_M Möbius $\zeta_M \mu_M = \mu_M \zeta_M = \varepsilon_M$

$$g(x) = \sum_{uv=x} f(v)$$

$$\Updownarrow$$

$$f(x) = \sum_{uv=x} \mu_M(u)\, g(v)$$

$$g = \zeta_M f \qquad \epsilon_M f = f$$

$$f = \mu_M g \qquad \epsilon_M g = g$$

$$g(x) = \sum_{uv=x} f(u)$$

$$\Updownarrow$$

$$f(x) = \sum_{uv=x} g(u)\, \mu_M(v)$$

$$g = f\, \zeta_M \qquad f\, \epsilon_M = f$$

$$f = g\, \mu_M \qquad g\, \epsilon_M = g$$

$d(x) =$ number of factorizations of $x \in M$

$$d(x) = d_+(x) + d_-(x)$$

$d_+(x) =$ number of factorizations even degree

$d_-(x) =$ number of factorizations odd degree

$$d(1) = d_+(1) = 1$$

$$d_-(1) = 0$$

exercise M

finite factorization monoid

prove the relations :

$$\zeta_M d_+ = d_+ \zeta_M = d$$

$$\zeta_M d_- = d_- \zeta_M = d - \epsilon_M$$

deduce that :

$$\mu_M(x) = d_+(x) - d_-(x)$$

equivalence between
Möbius functions
in posets and monoids

M finite factorization monoid

$uv = uw \Rightarrow v = w$

right cancellable

There exist $(E, \preceq)$ locally finite poset with incidence algebra $\mathcal{R}(E) \simeq \mathcal{R}(E)$ (isomorphic)

and $\mu_E(x,y) = \mu_M(y/x)$ when $x \preceq y$

y/x unique element $z \in M$ such that $xz = y$

$$E = M$$

with order relation

$x \preceq y$ iff $y = xz$ with $z \in M$

$$\mathcal{R}(M) \longrightarrow \mathcal{R}(E)$$

$$f_E(x, y) = \begin{cases} f(y/x) & \text{if } x \preceq y \\ 0 & \text{else} \end{cases}$$

E locally finite poset

Proposition

There exist a finite factorization monoid M such that $\mathcal{R}(M) \simeq \mathcal{R}(E)$

with "same" Möbius function

$$J = \{(x,y) \ , \ x,y \in E \text{ with } x \preceq y \ , \ x \neq y\}$$

$$M = J \cup \{0,1\}$$

$$\begin{cases} 0 \cdot m = m \cdot 0 = 0 & \text{pour tout } m \in M \\ 1 \cdot m = m \cdot 1 = m & \text{pour tout } m \in M \setminus \{0\} \end{cases}$$

$$(x,y) \cdot (z,t) = \begin{cases} (x,t) & \text{if } y = z \\ 0 & \text{else} \end{cases}$$

$$(x,y) \ , \ (z,t) \in J$$

$$f \in \mathcal{R}(E) \longrightarrow f_M : \underset{M\setminus\{0\}}{M^+} \longrightarrow \mathcal{R}$$

$$\begin{cases} f_M(1) = f(x,x) & \text{for any } x \in E \ (=1) \\ f_M(x,y) = f(x,y) & \text{for } x \leq y \end{cases}$$

<u>Proposition</u>

$f \longrightarrow f_M$ is an isomorphism (of algebra) and

$\mu_E(x,y) = \mu_M(x,y)$ for $x \leq y$

Möbius function in monoids and formal series

$$\sum_{w \in M} a_w w \qquad a_w \in \mathbb{Z}$$

$$\sum_{w \in M} w = \frac{1}{\sum_{w \in M} \mu_M(w)\, w}$$

$$\sum_{w \in M} w \cdot \sum_{w \in M} \mu_M(w)\, w = 1$$

$$\sum_{\substack{(u,v) \\ uv = w}} \underset{\substack{\downarrow \\ 1}}{\zeta(u)}\, \mu_M(v)\, w$$

$$\zeta_M \mu_M = \epsilon$$

Möbius inversion for commutation monoid

$$[w] \in L(A, C)$$

$$L(A, C) = A^* / \equiv_C$$

Cartier Foata commutation monoid

$$\mu([w]) = \begin{cases} \bullet\ (-1)^{|w|} & \text{if the letters of } w \in A^* \text{ commute two by two (for } C\text{)} \\ \bullet\ 0 & \text{else} \end{cases}$$

$A = P$ $\qquad C = \overline{\mathcal{C}}$

$L(A, C) = A^{*} / \equiv_{C}$ $\simeq$ $H(P, \mathcal{C})$

Cartier Foata
commutation
monoid

$V(x) = x \in \mathbb{Z}[A]$

$x \in A = P$

$\mathbb{N}^+ = \mathbb{N} - \{0\}$

$n \in \mathbb{N}^+ \longrightarrow p_1^{\alpha_1} \cdots p_k^{\alpha_k}$

for $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$
prime numbers
decomposition

alphabet $A = \mathbb{P}$ set of prime numbers

element of $L(\mathbb{P}, c)$
$a\,c\,b$ for any $a, b \geq 1$
$a \neq b$

$=$

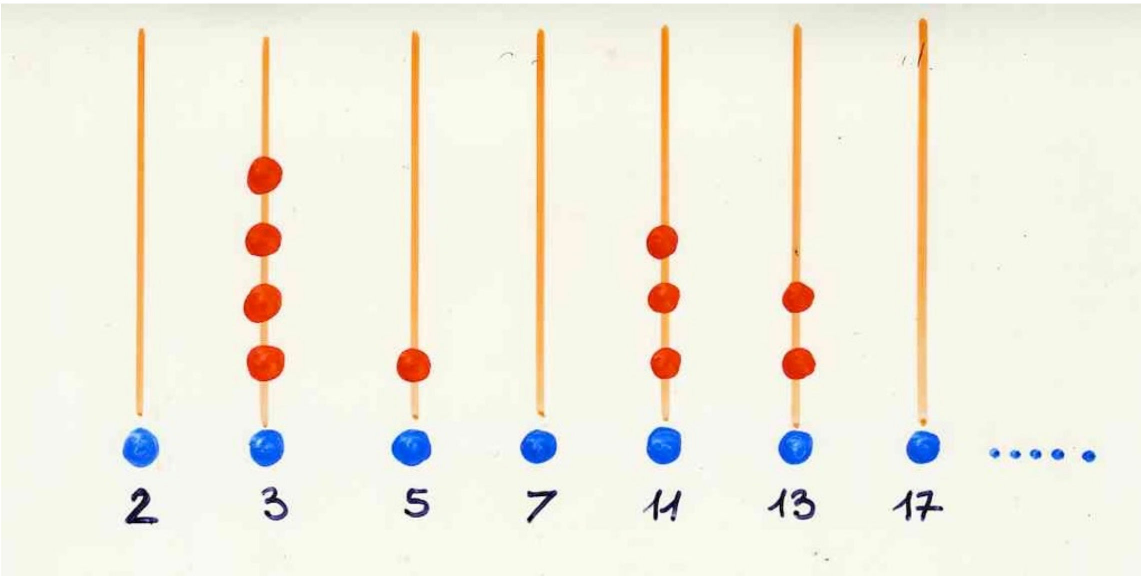
free abelian monoid
generated by $\mathbb{P}$

$Ab(\mathbb{P})$

$\mathbb{N}^+$ multiplicative monoid

$\simeq$

free abelian monoid generated by $\mathbb{P}$ $\quad \mathrm{Ab}(\mathbb{P})$ $\simeq$ $H(\mathbb{N}^+, \mathcal{C})$



$a \not\mathcal{C} b$ for any $a, b \in \mathbb{N}^+$
except $a \mathcal{C} a$

from previous considerations we get:

Möbius classic in number theory

$$\Updownarrow \quad \begin{aligned} g(n) &= \sum_{d|n} f(d) \\ f(n) &= \sum_{d|n} \mu(d)\, g(n/d) \end{aligned}$$

$$\mu(n) = \begin{cases} \bullet\ 0 & \text{if } n \text{ is divisible by a square} \\ \bullet\ (-1)^k & \text{else} \end{cases}$$

$$\sum_{w \in M} w = \frac{1}{\sum_{w \in M} \mu_M(w)\, w}$$

$$n^{-s} = p_1^{-s\alpha_1} \cdots p_k^{-s\alpha_k}$$

$$\sum_{n \geq 1} n^{-s} = \left(\sum_{n \geq 1} \mu(n)\, n^{-s} \right)^{-1}$$

$$\sum_{n\geq 1} n^{-s} = \left(\sum_{n\geq 1} \mu(n)\, n^{-s} \right)^{-1}$$

Dirichlet serie

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Riemann zeta function

$$\frac{1}{\zeta(s)} = \sum \frac{\mu(n)}{n^s}$$