Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics,
a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

Xavier Viennot

CNRS, LaBRI, Bordeaux

www.xavierviennot.org

Chapter 1
Commutation monoids
and
heaps of pieces:

basic definitions
(1)

IMSc, Chennai

5 January 2017

§1 Commutation monoids

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^2 = \cancel{a^2 + 2ab + b^2}$$

$$\text{if} \quad ab \neq ba$$

$$= a^2 + ab + ba + b^2$$

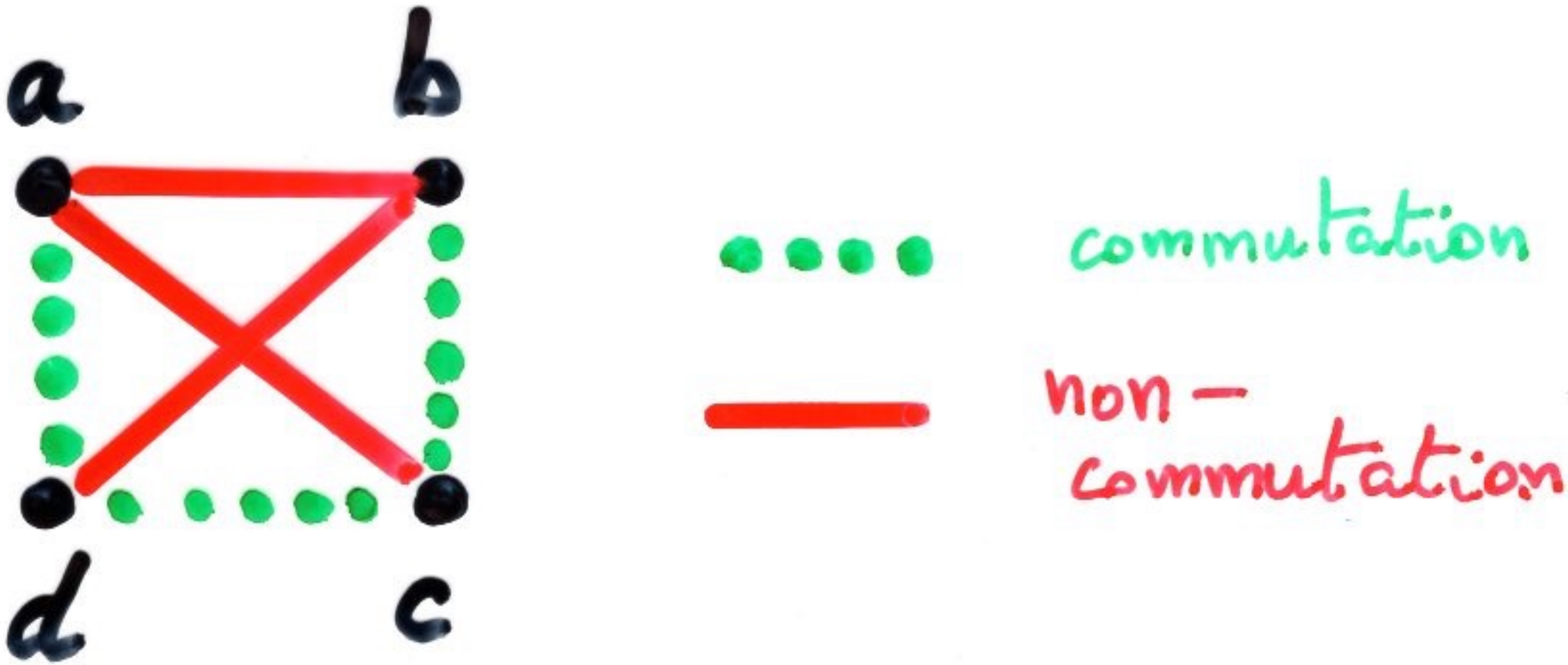
$a, b, c, d, \ldots$

letters
formal variables

$ad = da$ $\quad$ $ab \neq ba$

$cd = dc$ $\quad$ $ac \neq ca$

$bc = cb$ $\quad$ $bd \neq db$



$ad = da$

$cd = dc$

$bc = cb$

$ab \neq ba$

$ac \neq ca$

$bd \neq db$

$abcad$

word
monomial

$w = a\underline{bc}ad$

$ad = da$

$cd = dc$

$bc = cb$

$w = abc\underline{ad}$

$acbad$

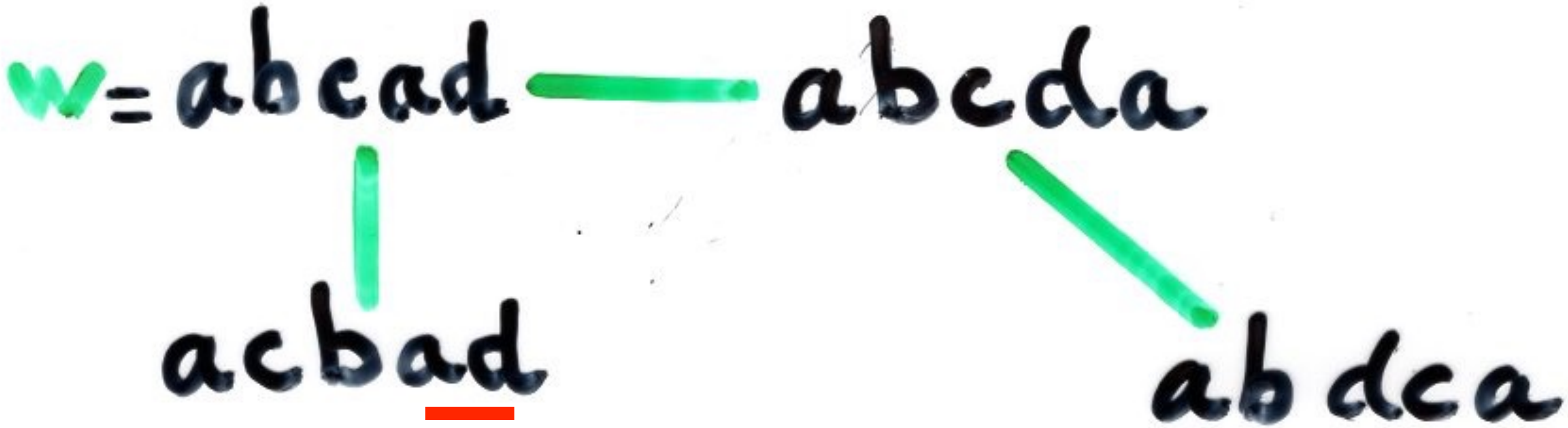
$ad = da$

$cd = dc$

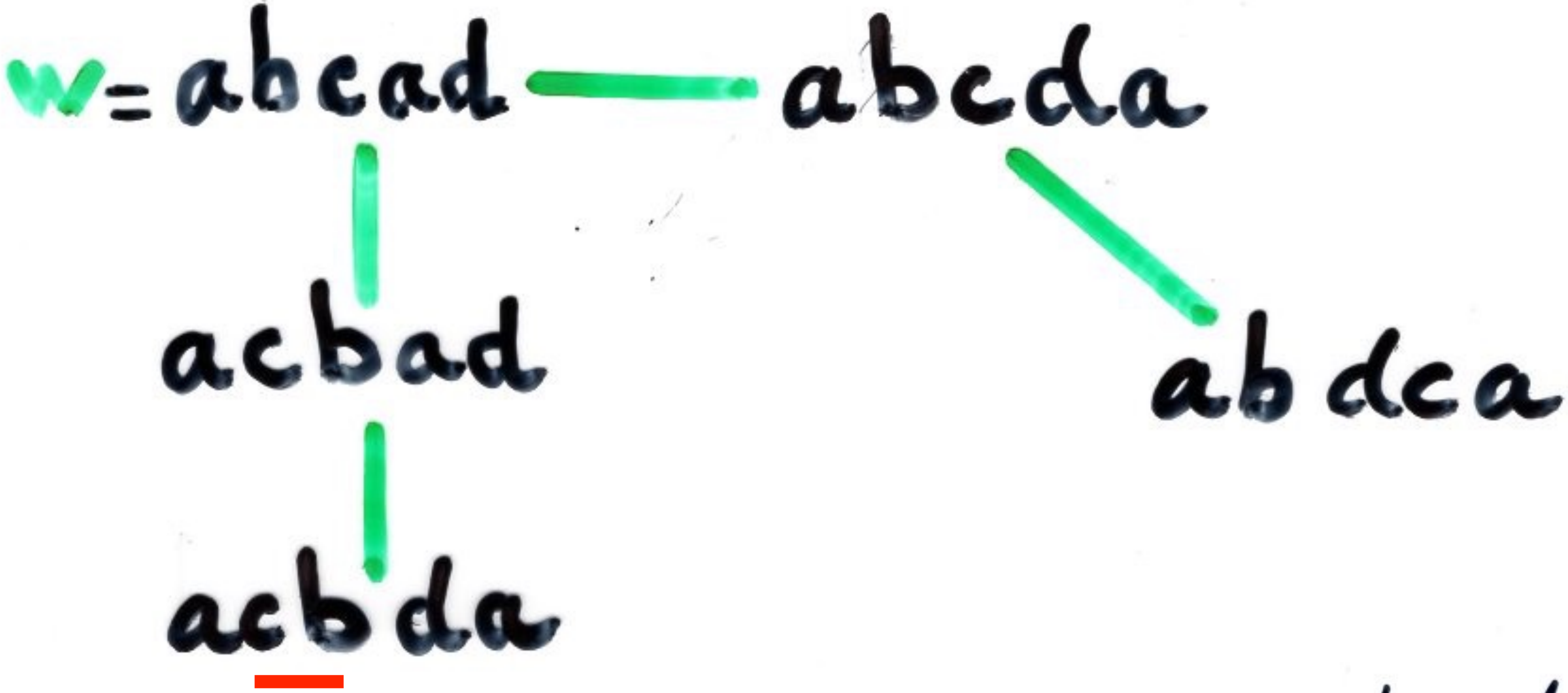
$bc = cb$

$$
\begin{tikzcd}
w = abcad \arrow[r, dash] \arrow[d, dash] & ab\underline{cd}a \\
acbad &
\end{tikzcd}
$$

$$ad = da$$
$$cd = dc$$
$$bc = cb$$



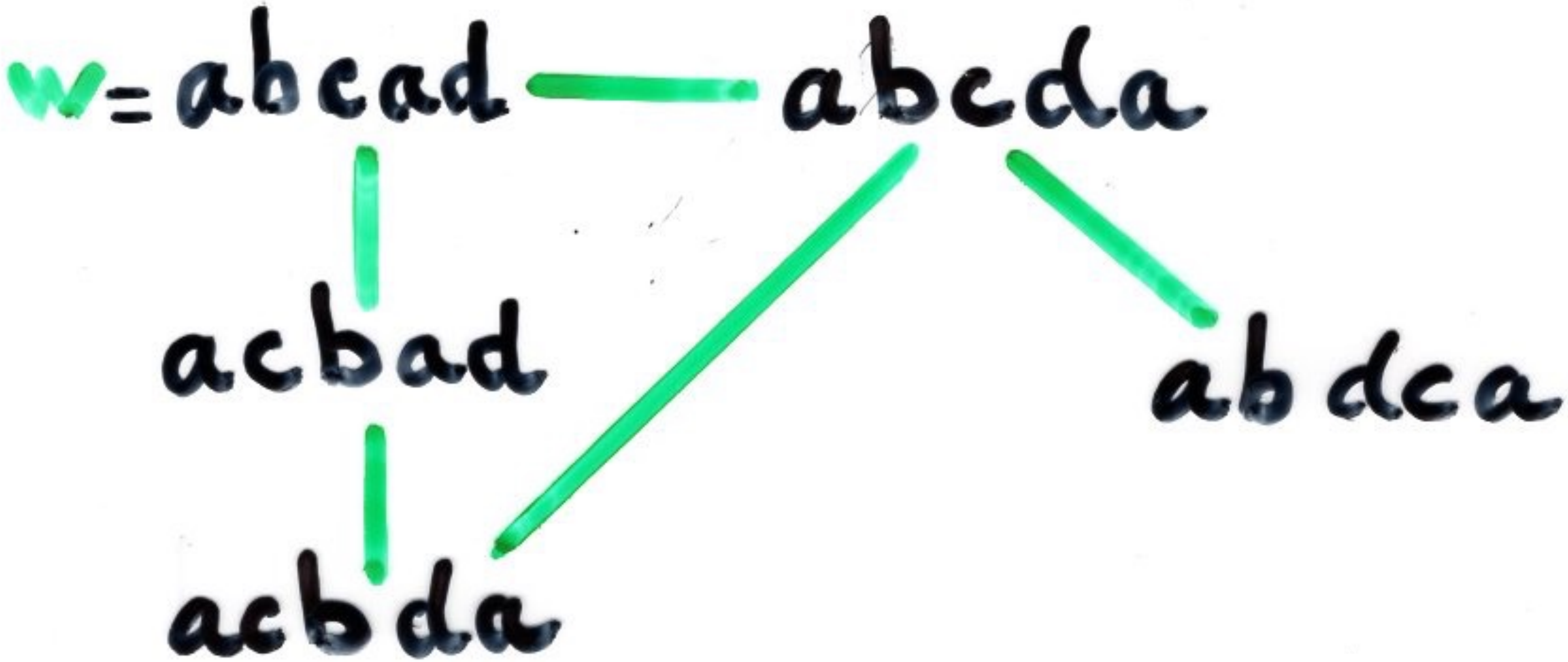
$$ad = da$$
$$cd = dc$$
$$bc = cb$$



$ad = da$

$cd = dc$

$bc = cb$



$$ad = da$$
$$cd = dc$$
$$bc = cb$$

<u>ex</u> : $A = \{a, b, c, d\}$

$C \quad \begin{cases} ad = da \\ bc = cb \\ cd = dc \end{cases}$

equivalence class

$$\begin{tikzcd}
w = abcad \arrow[r, dash] \arrow[d, dash] & abcda \arrow[dr, dash] \arrow[ddl, dash] & \\
acbad \arrow[d, dash] & & abdca \\
acbda & &
\end{tikzcd}$$

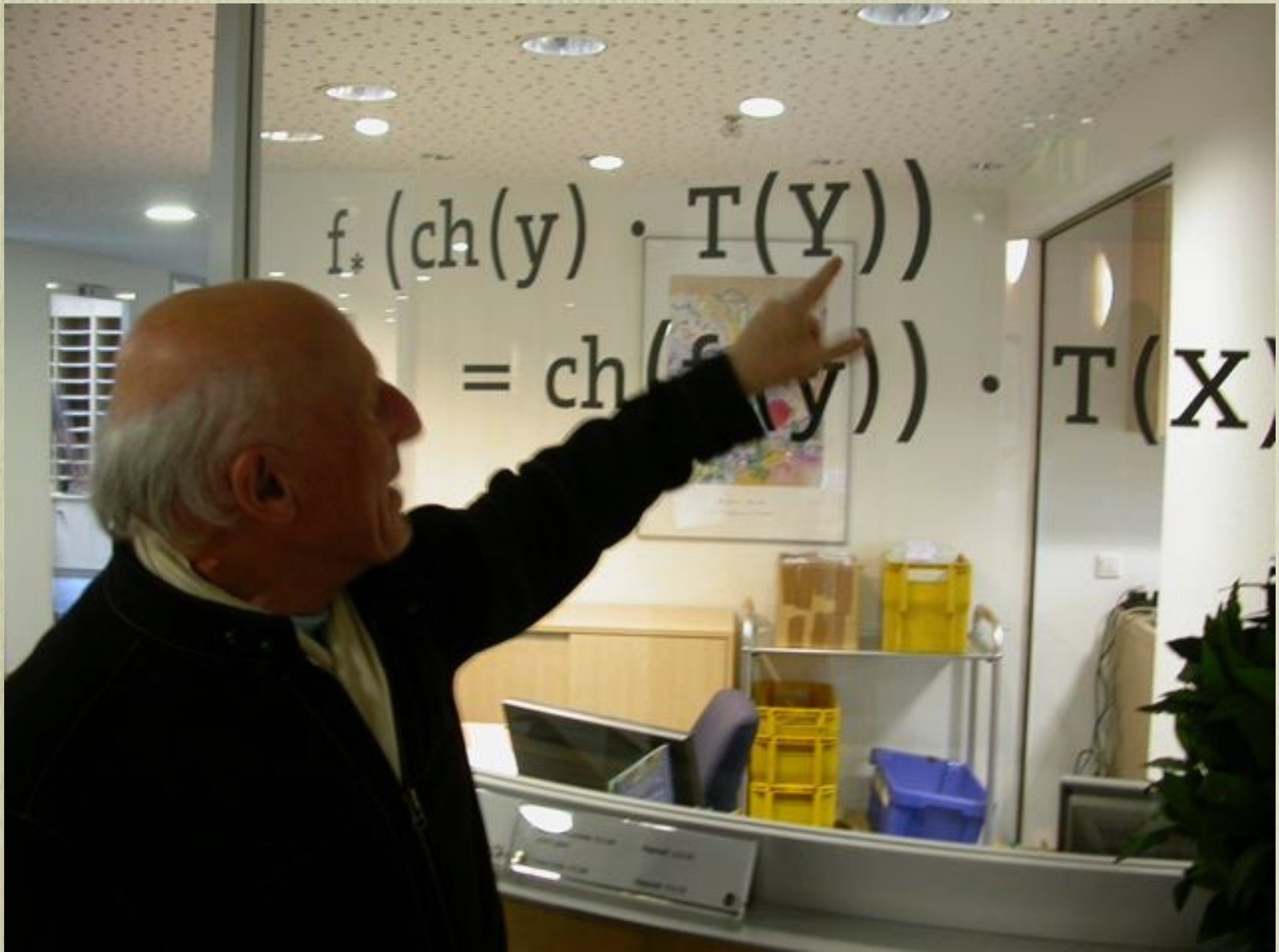
Cartier-Foata commutation monoid

Lecture Note in Maths n°85 (1969)

"Problèmes combinatoires de commutation et réarrangements"

Cartier-Foata monography in SLC Séminaire Lotharingien de Combinatoire (2006)

http://www.mat.univie.ac.at/~slc/


f*(ch(y) · T(Y))
= ch(y)) · T(X


such
of ∞, 1, 2, . . . , n
an
is a bijection of

monoid $\quad M \quad (u,v) \to u \bullet v$

$$\begin{cases} - \text{ associativity} & (u \bullet v) \bullet w = u \bullet (v \bullet w) \\ - \text{ neutral element} & u \bullet e = e \bullet u \end{cases}$$

examples - $\quad - \ \mathbb{N} \quad + \ , \ 0 \quad$ addition

$\quad - \ \mathbb{N} \quad \times \ , \ 1 \quad$ product

alphabet $\quad A$

free monoid $\quad A^*$

words $\quad w = a_1 a_2 \cdots a_p$

product : concatenation

$$\left.\begin{array}{l} u = a_1 \cdots a_p \\ v = b_1 \cdots b_q \end{array}\right\} uv = a_1 \cdots a_p b_1 \cdots b_q$$

empty word

commutation relation C antireflexive symmetric

$\equiv_C$ congruence of A^* generated by the commutations

$ab \equiv ba$ iff $a\,C\,b$

- $a\,C\,b \Leftrightarrow b\,C\,a$
- $\cancel{a\,C\,a}$

commutation monoid $\quad A^*/\equiv_C$

$[w]$ equivalence class of the word $w \in A^*$

$A^*/\equiv_C$ • product in the commutation monoid

$$[u] \cdot [v] = [uv]$$

independant of the choices of representants u and v

Trace monoids

Computer Science

model for parallelism

concurrency access to data structures

Mazurkiewicz (1977)

model of the logical behavior of safe Petri nets

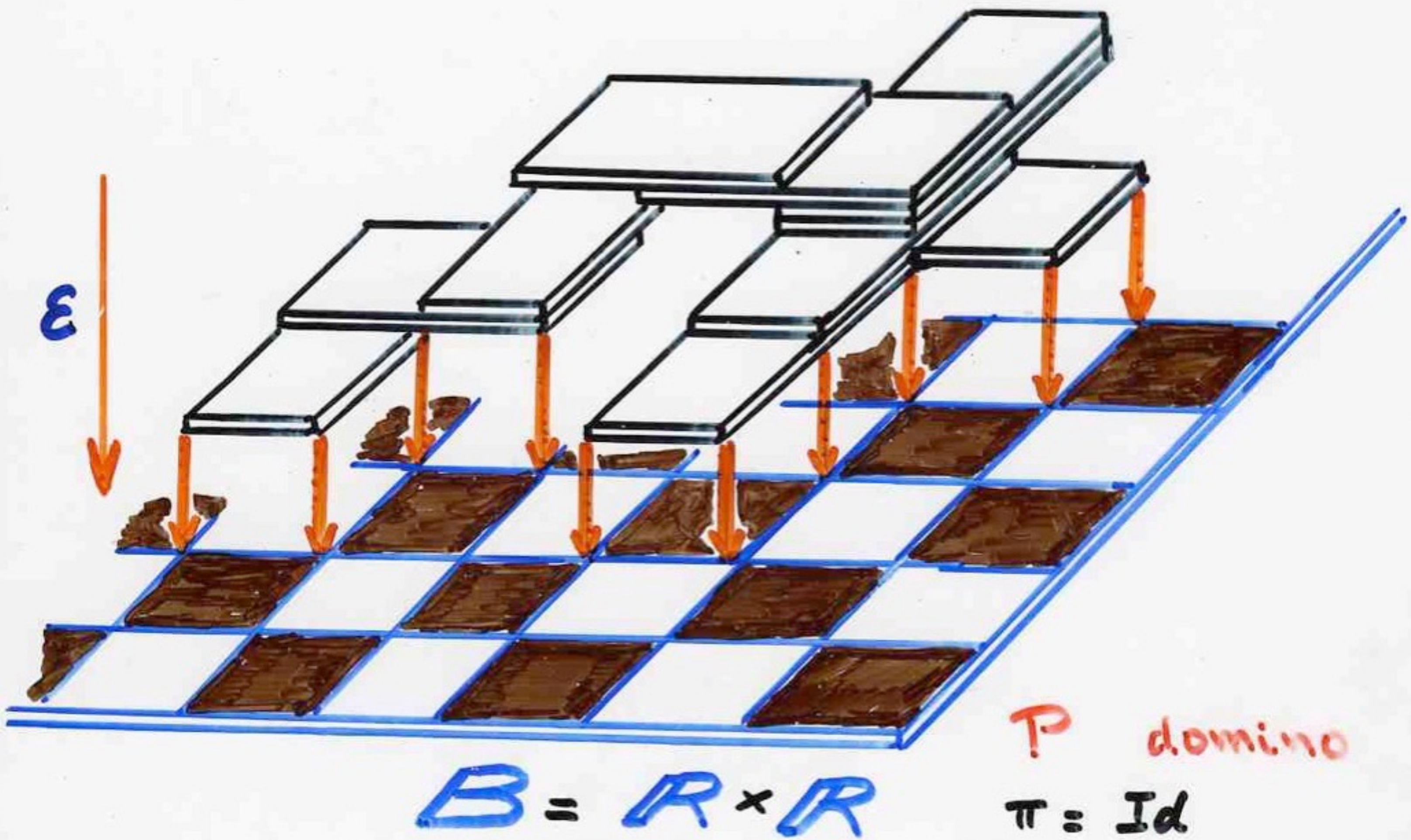
Diekert, Rosenberg ed. (1995)

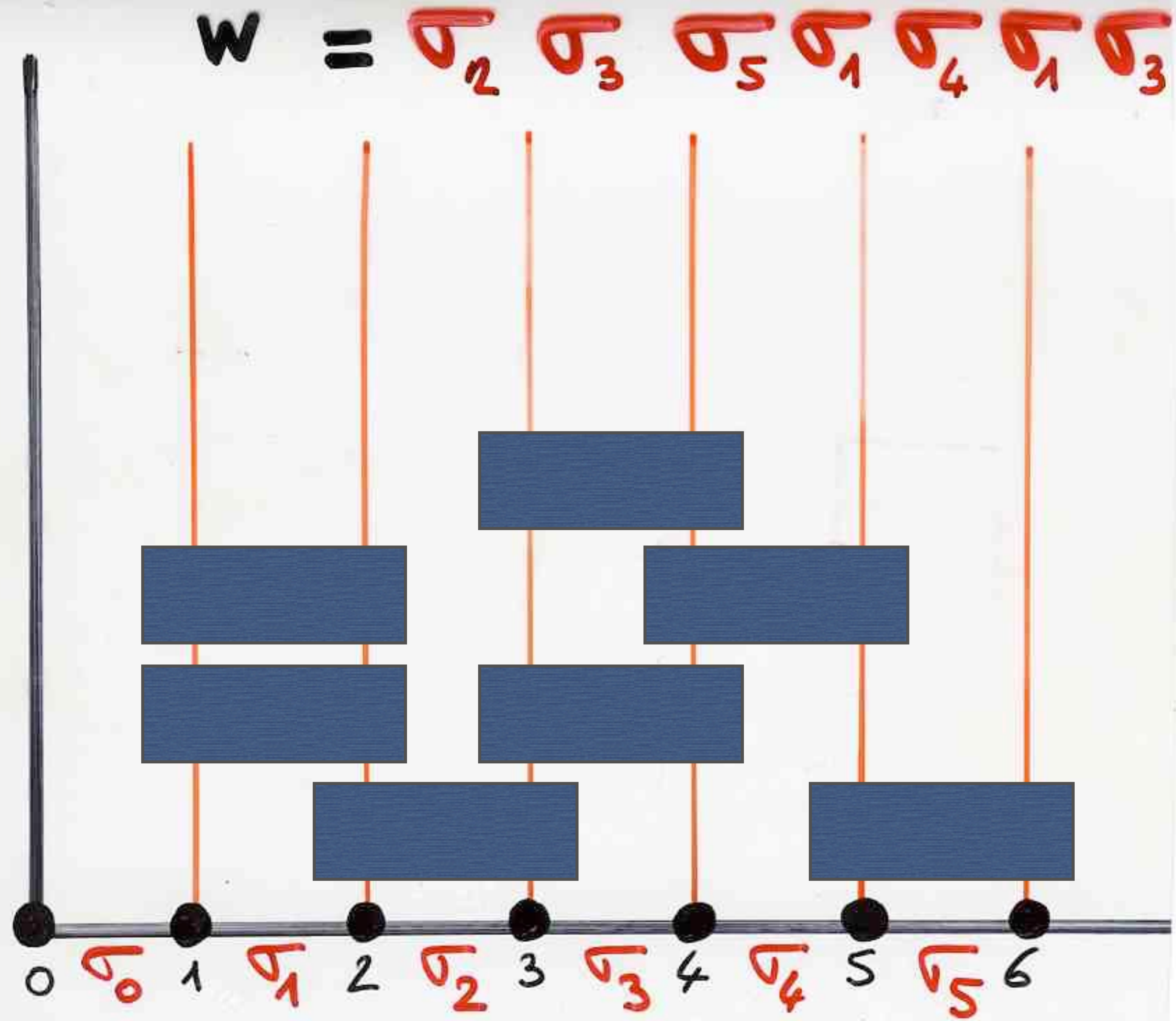
The book of traces

§2 Heaps of pieces
definition, examples

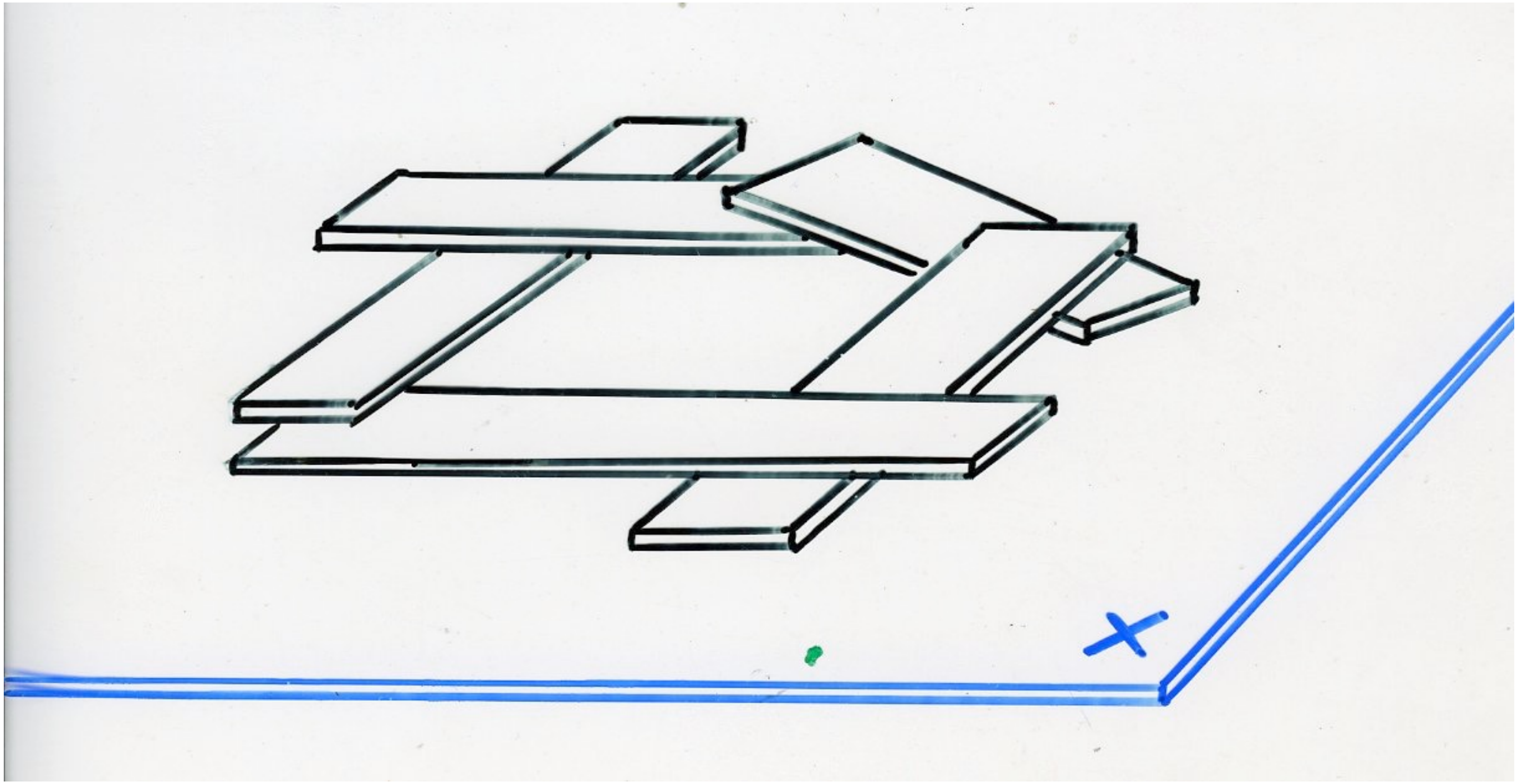
(X.V. 1985)

Introduction
Heaps





$w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$
0 σ_0 1 σ_1 2 σ_2 3 σ_3 4 σ_4 5 σ_5 6



heap <u>definition</u>

- $\mathbf{P}$ set (of <u>basic pieces</u>)
- $\mathcal{E}$ binary relation on $\mathbf{P}$ $\begin{cases} \text{symmetric} \\ \text{reflexive} \end{cases}$

  (<u>**dependency**</u> relation)

- **heap** E, finite set of pairs

  (α, i) $\alpha \in \mathbf{P}$, $i \in \mathbb{N}$ (called <u>pieces</u>)

  <u>projection</u> <u>level</u>

(i)

(ii)

heap definition

- $\mathbf{P}$ set (of basic pieces)
- $\mathcal{C}$ binary relation on $\mathbf{P}$ $\begin{cases} \text{symmetric} \\ \text{reflexive} \end{cases}$

  (dependency relation)

- heap E, finite set of pairs

  (α, i) $\alpha \in P$, $i \in \mathbb{N}$ (called pieces)

  projection level

(i) $(\alpha, i), (\beta, j) \in E, \ \alpha \, \mathcal{C} \, \beta \Longrightarrow i \neq j$

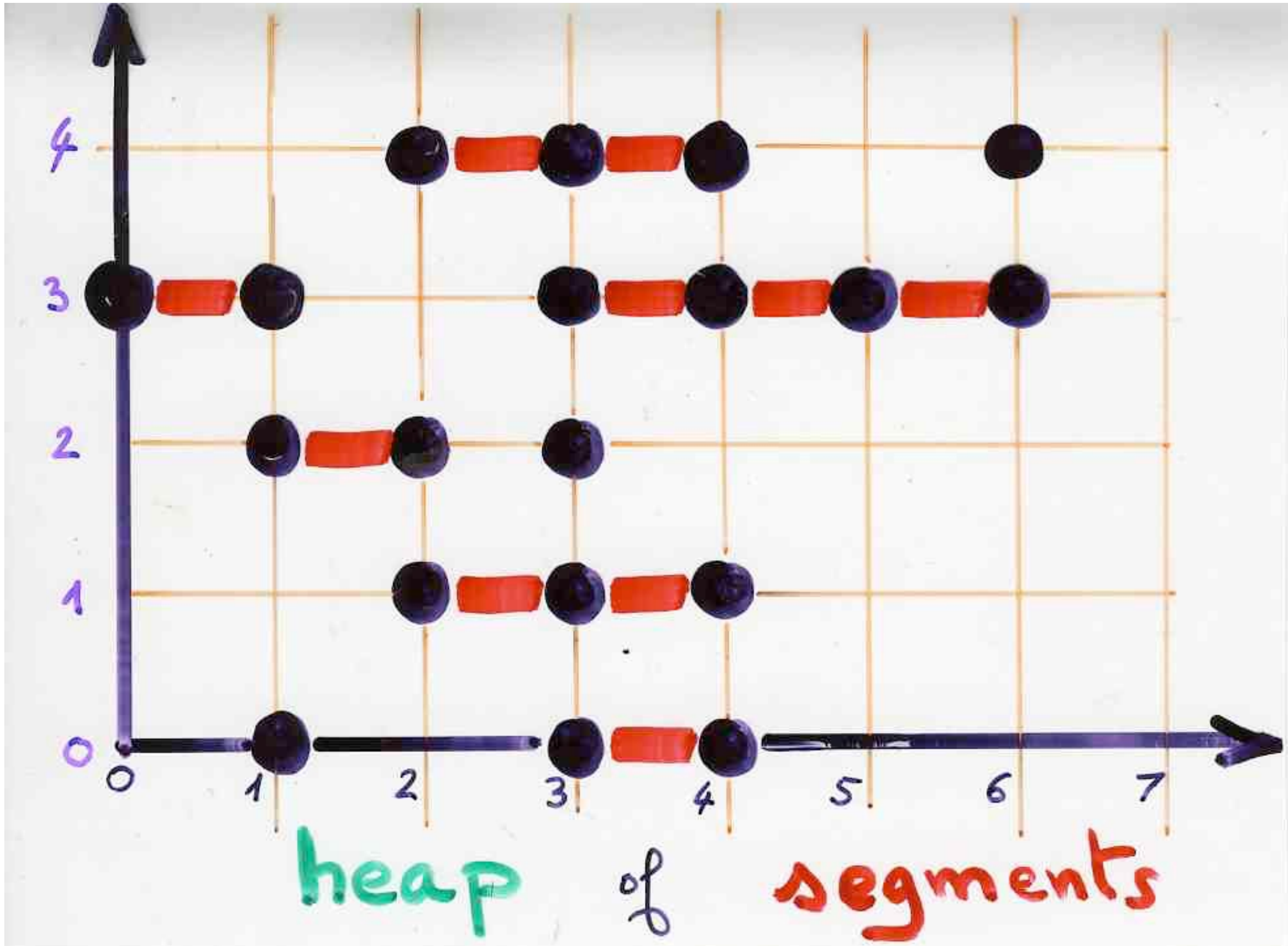
(ii) $(\alpha, i) \in E, \ i > 0 \Longrightarrow \exists \beta \in P, \ \alpha \, \mathcal{C} \, \beta,$

$(\beta, i-1) \in E$

<u>ex</u>: heap of segments over $\mathbb{N}$

$P = \{ [a,b] = \{a, a+1, \dots, b\} , 0 \leq a \leq b \}$

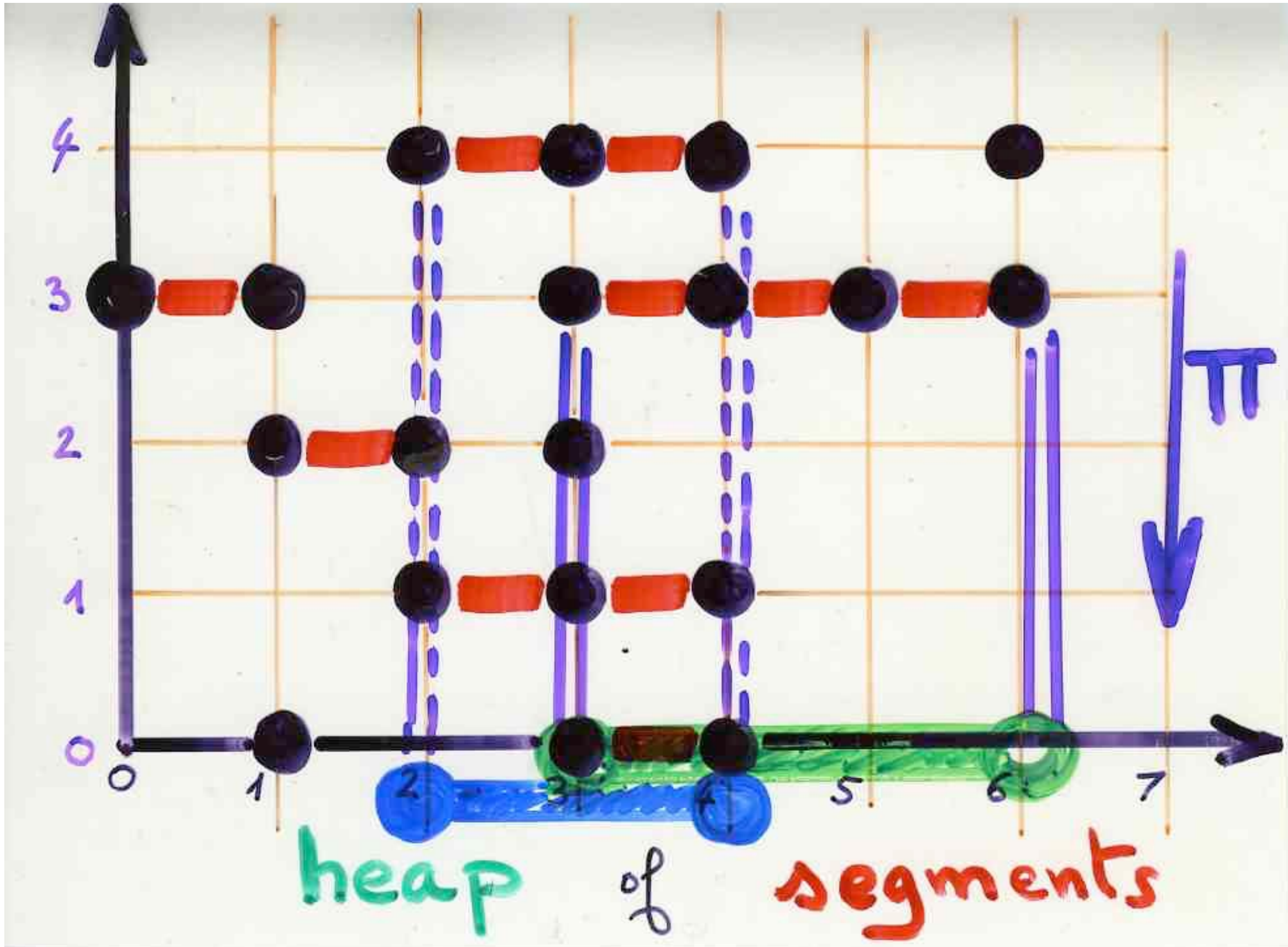
$\mathcal{C} \qquad [a,b] \, \mathcal{C} \, [c,d] \iff [a,b] \cap [c,d] \neq \emptyset$



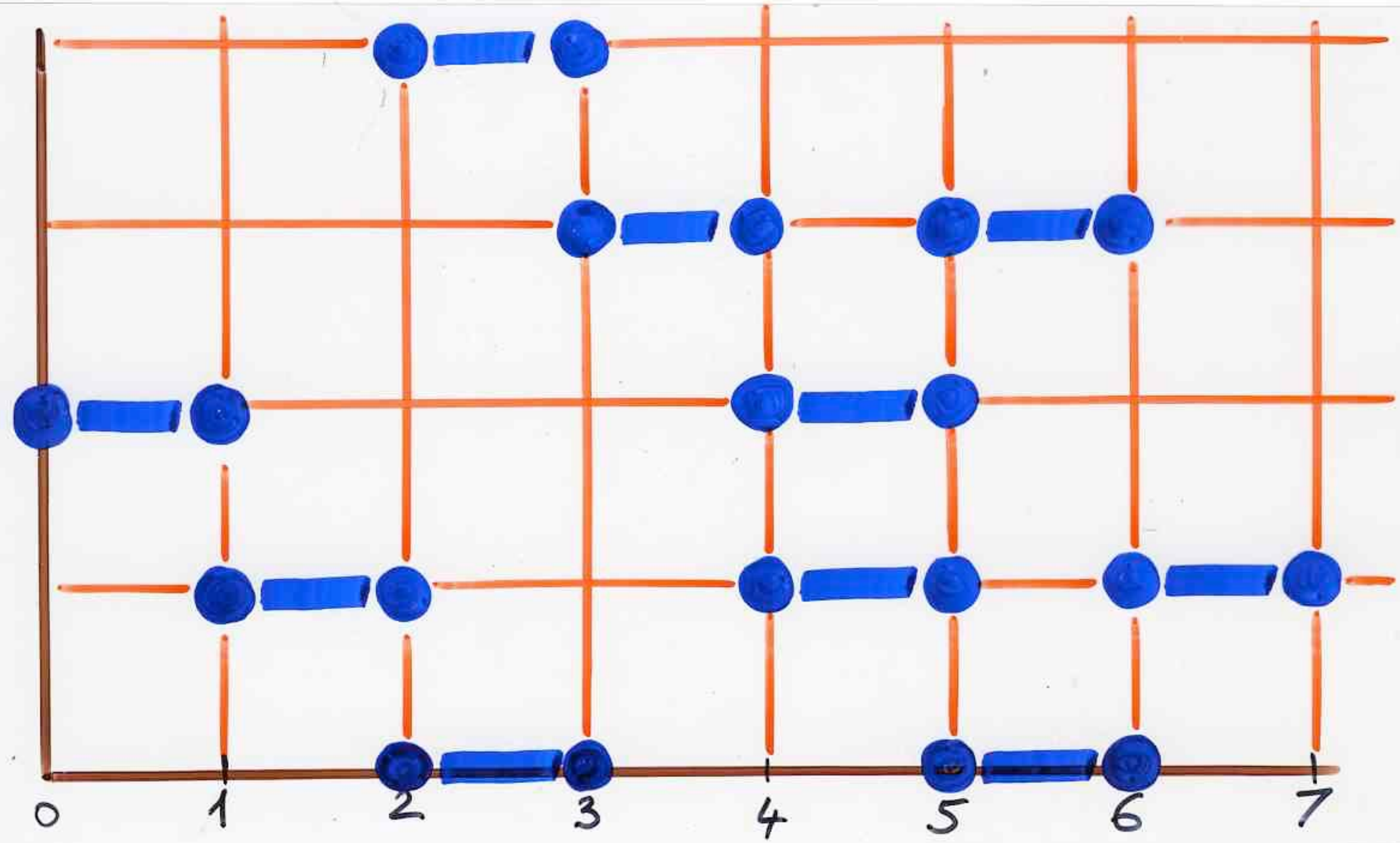
ex: heap of segments over $\mathbb{N}$

$$\mathcal{P} = \{ [a,b] = \{a, a+1, \dots, b\} \, , 0 \leq a \leq b \}$$

$$\mathcal{C} \qquad [a,b] \, \mathcal{C} \, [c,d] \iff [a,b] \cap [c,d] \neq \emptyset$$



Heap of dimers over $[1, n]$



ex: subsets of a set X

- $\mathcal{P}$ set of subsets of X

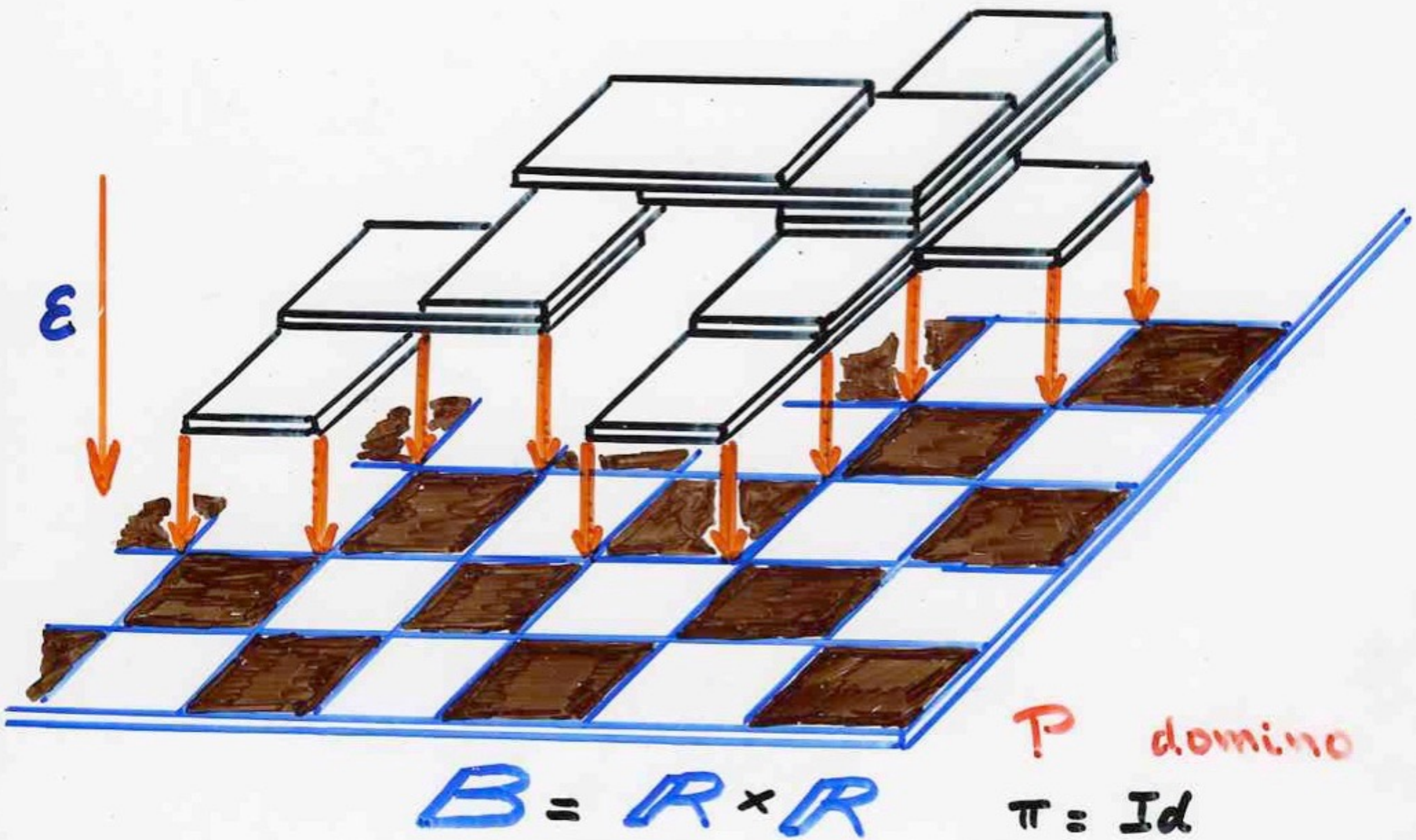
basic pieces

$$\mathcal{P} \subseteq \mathcal{P}(X)$$

- $\mathcal{C}$ dependency relation

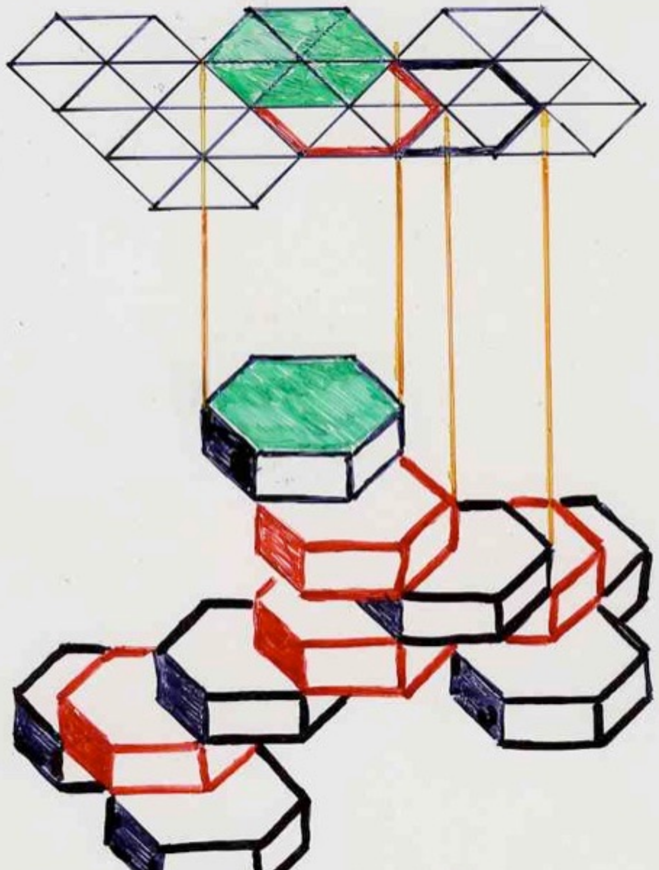
$$A, B \in \mathcal{P}, \quad A \,\mathcal{C}\, B \Leftrightarrow A \cap B \neq \emptyset$$

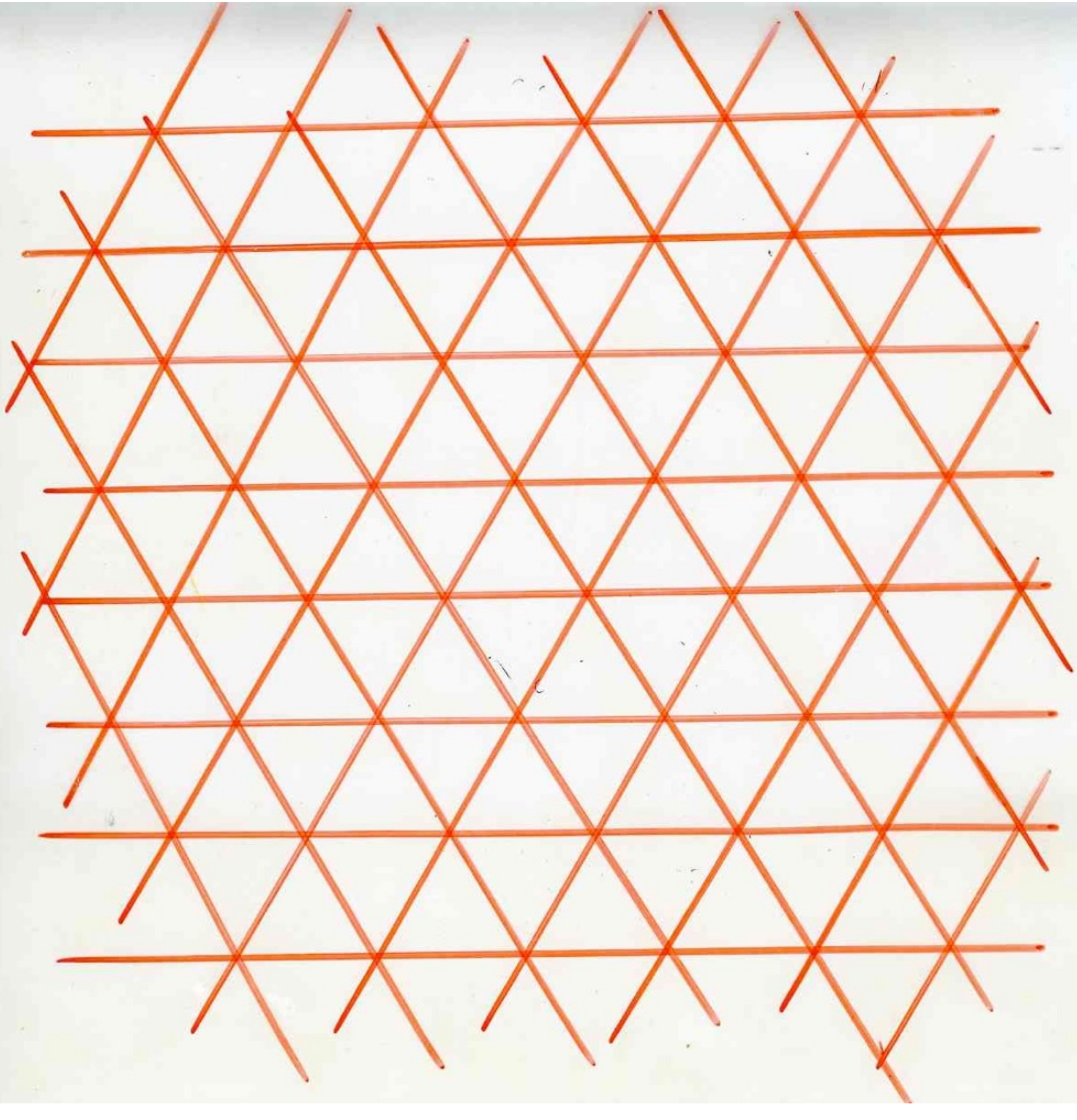
sub ex 1 - Heaps of "hard dimers" on a chessboard

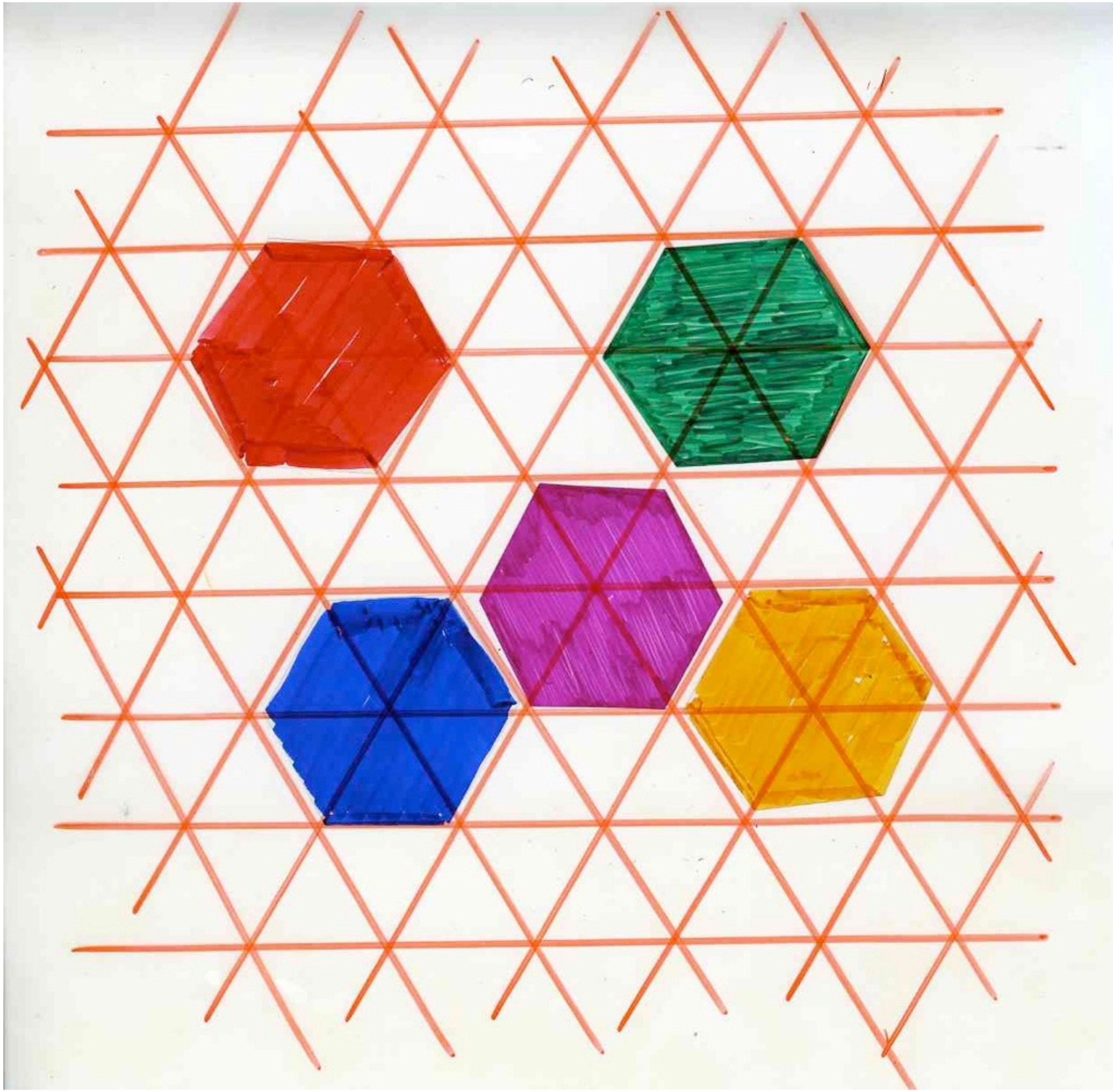


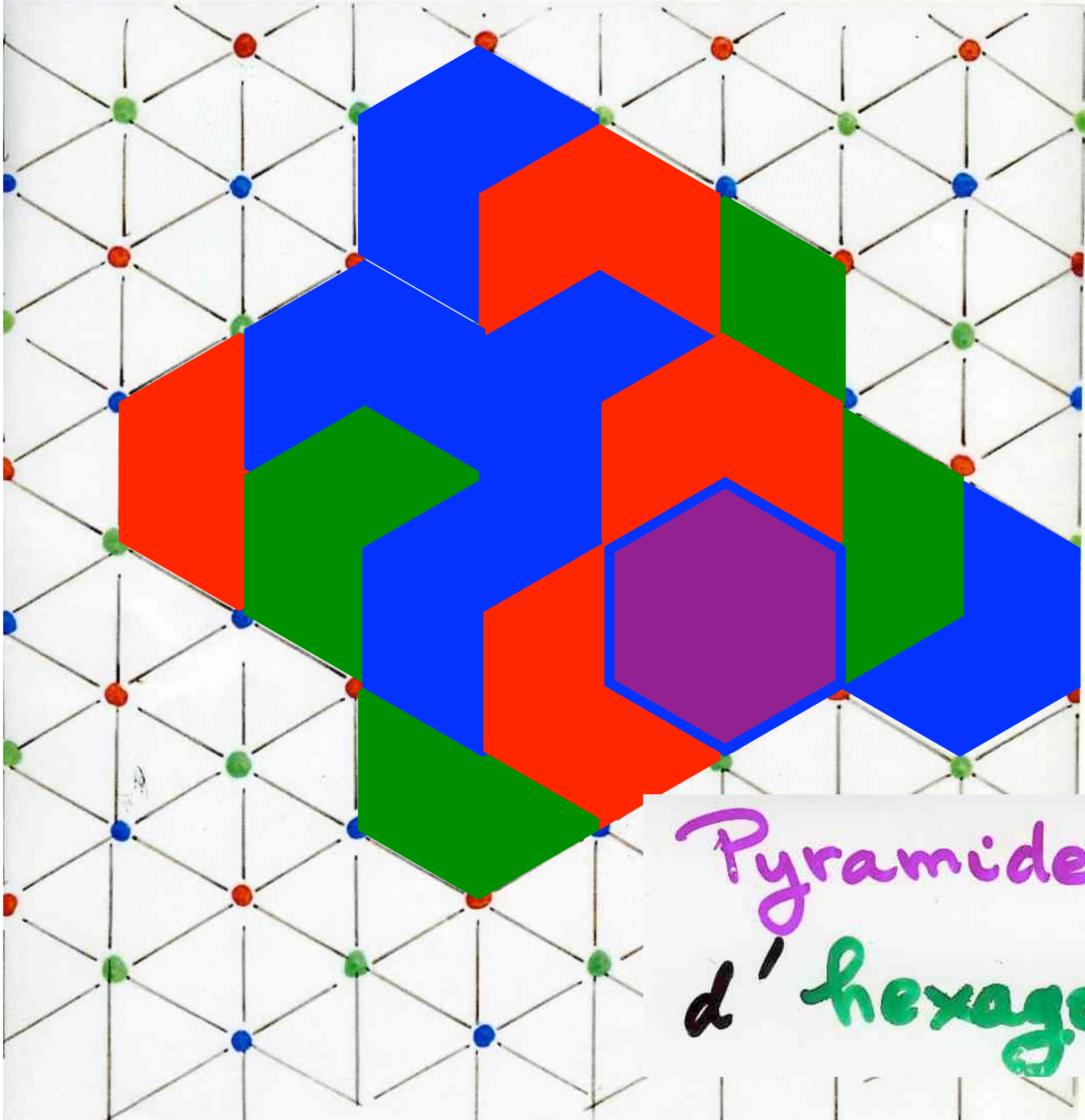
subex 2 - Heaps of "hard hexagons"

$$-\rho(-t) = y$$



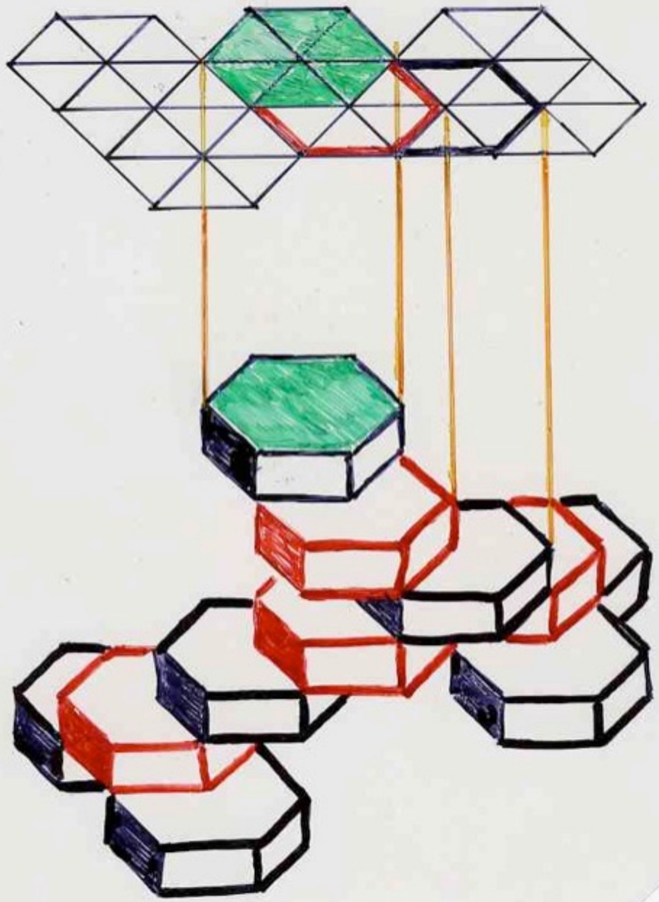




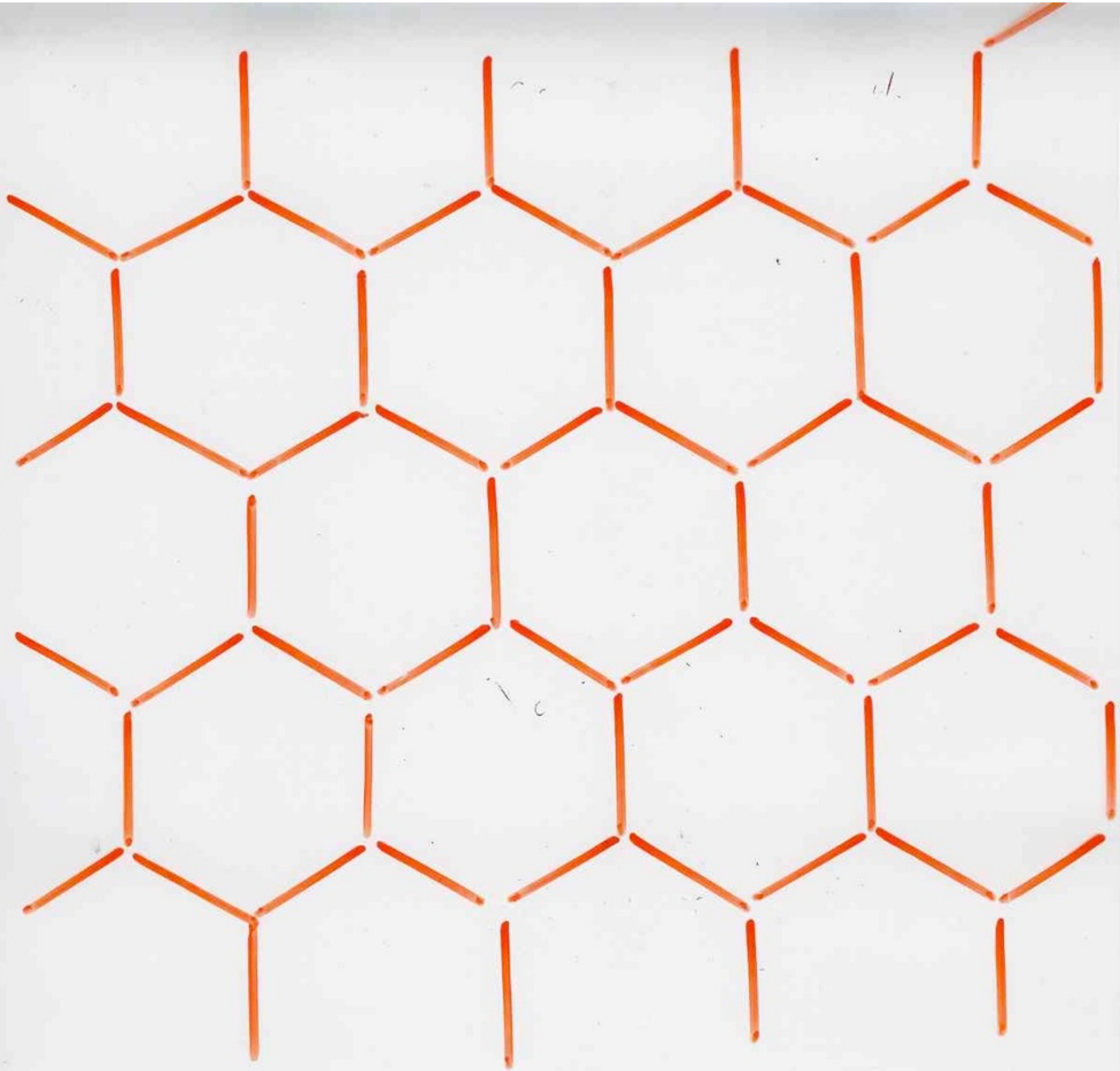


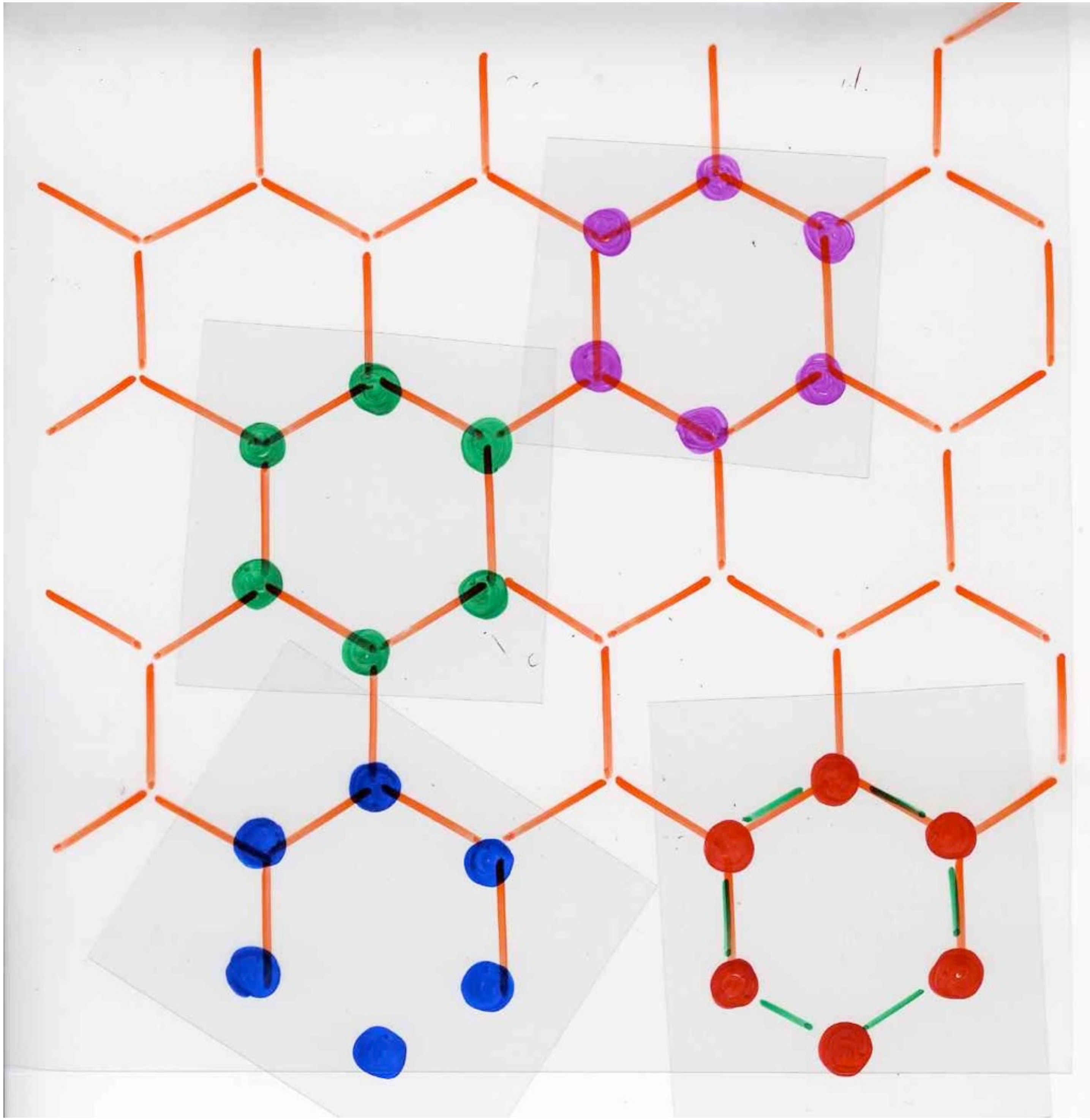
Pyramide d'hexagones

$$-p(-t) = y$$

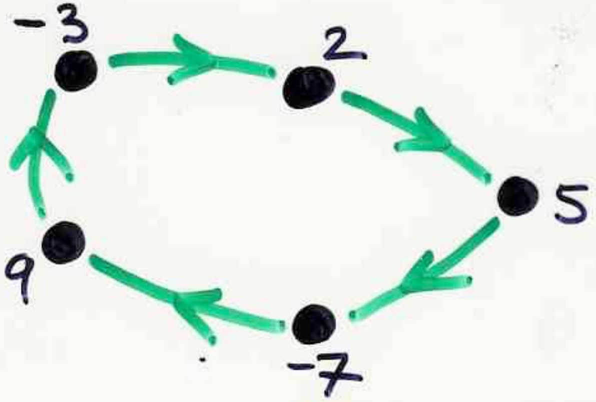


KC.





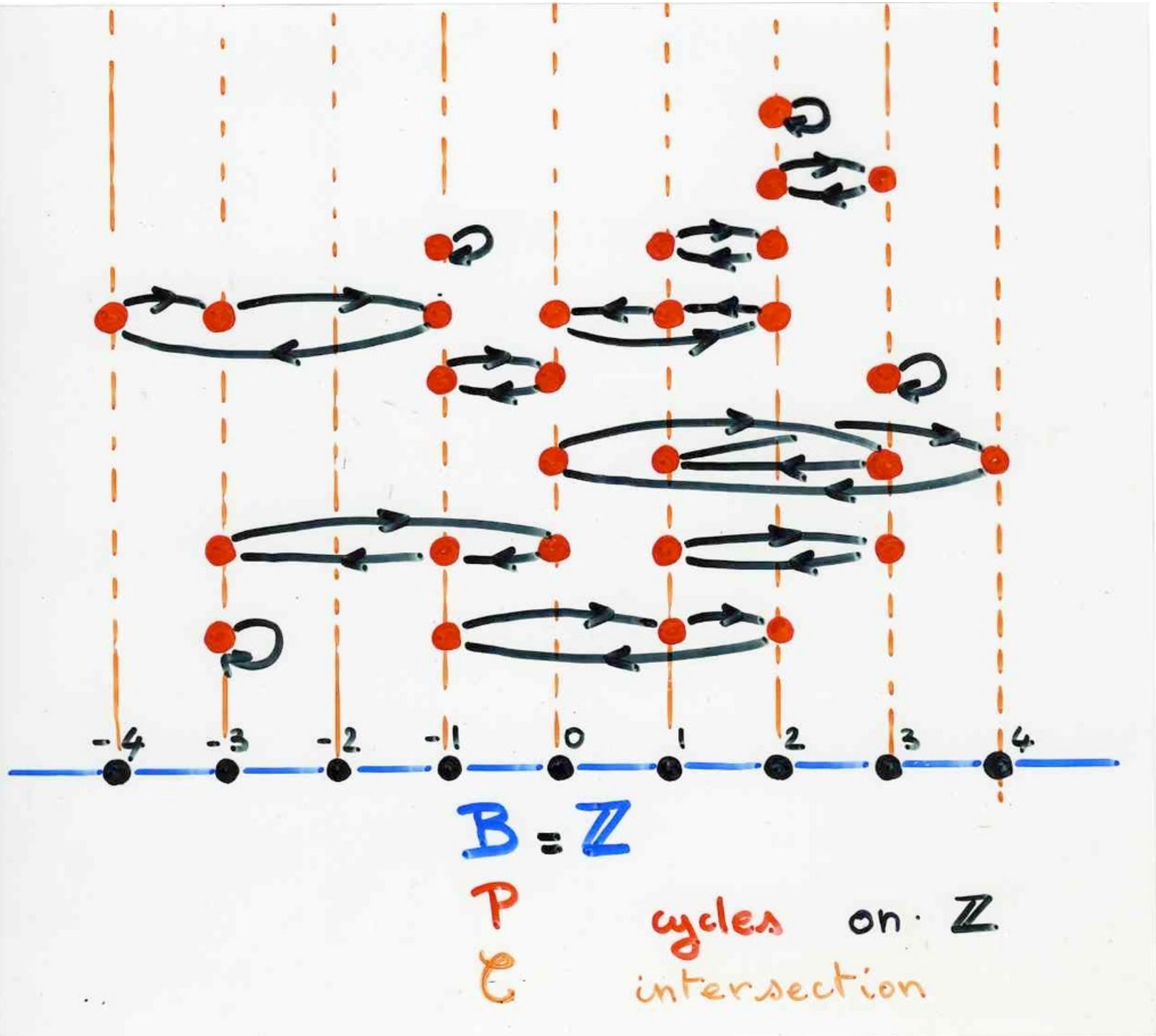
$\mathcal{P}$ pièces de base = cycles sur $\mathbb{Z}$

$\gamma =$ 

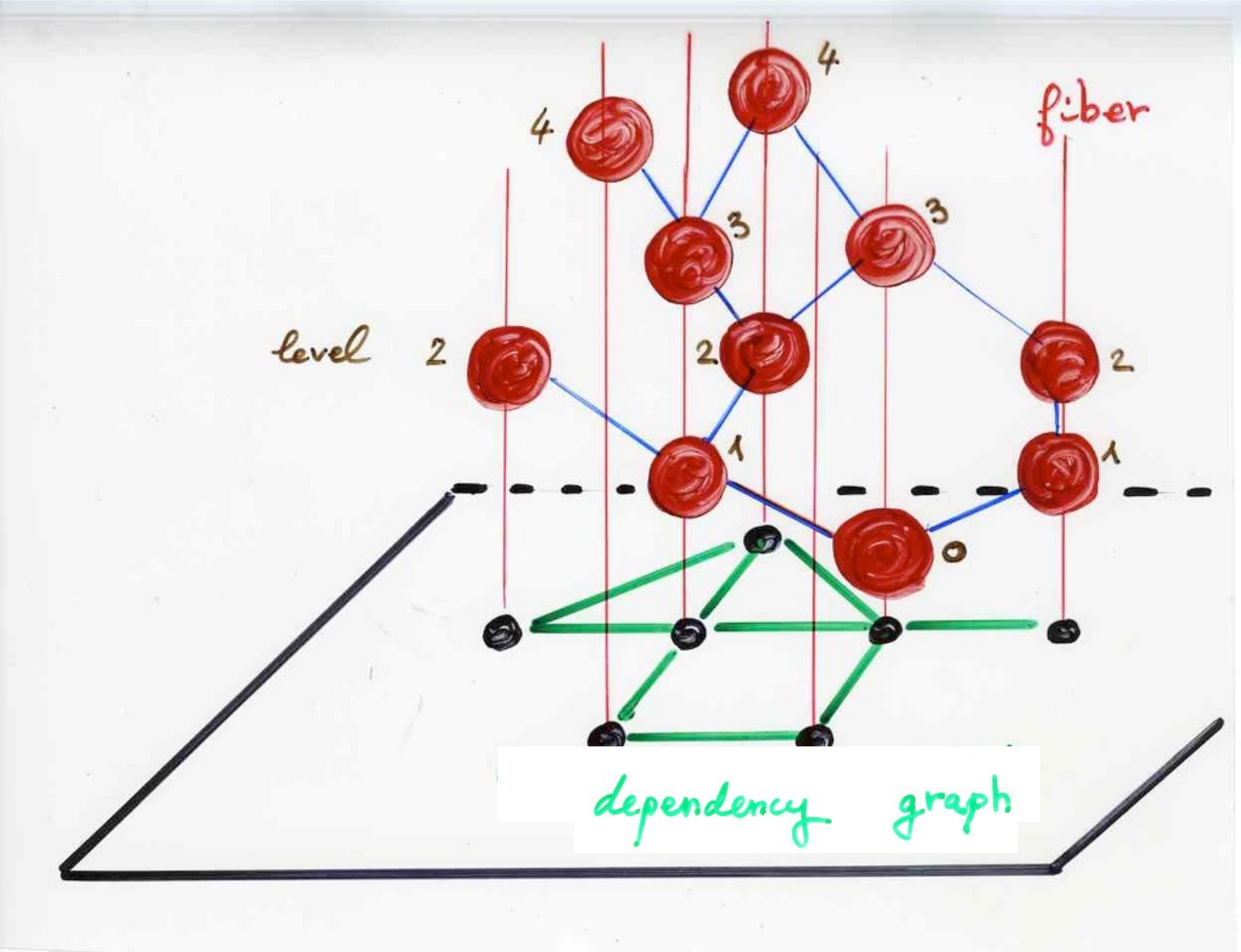
$\mathrm{Supp}(\gamma) = \{-7, -3, 2, 5, 9\}$

Support

$\mathcal{C}$ concurrence $\quad \gamma \,\mathcal{C}\, \delta \Leftrightarrow \mathrm{Supp}(\gamma) \cap \mathrm{Supp}(\delta) \neq \emptyset$



-4
-3
-2
-1
0
1
2
3
4
B = ℤ
P
cycles on ℤ
C
intersection



fiber
level
4
4
3
3
2
2
2
1
1
0
dependency graph

§3 Heaps monoids

Déf. pre-heap E

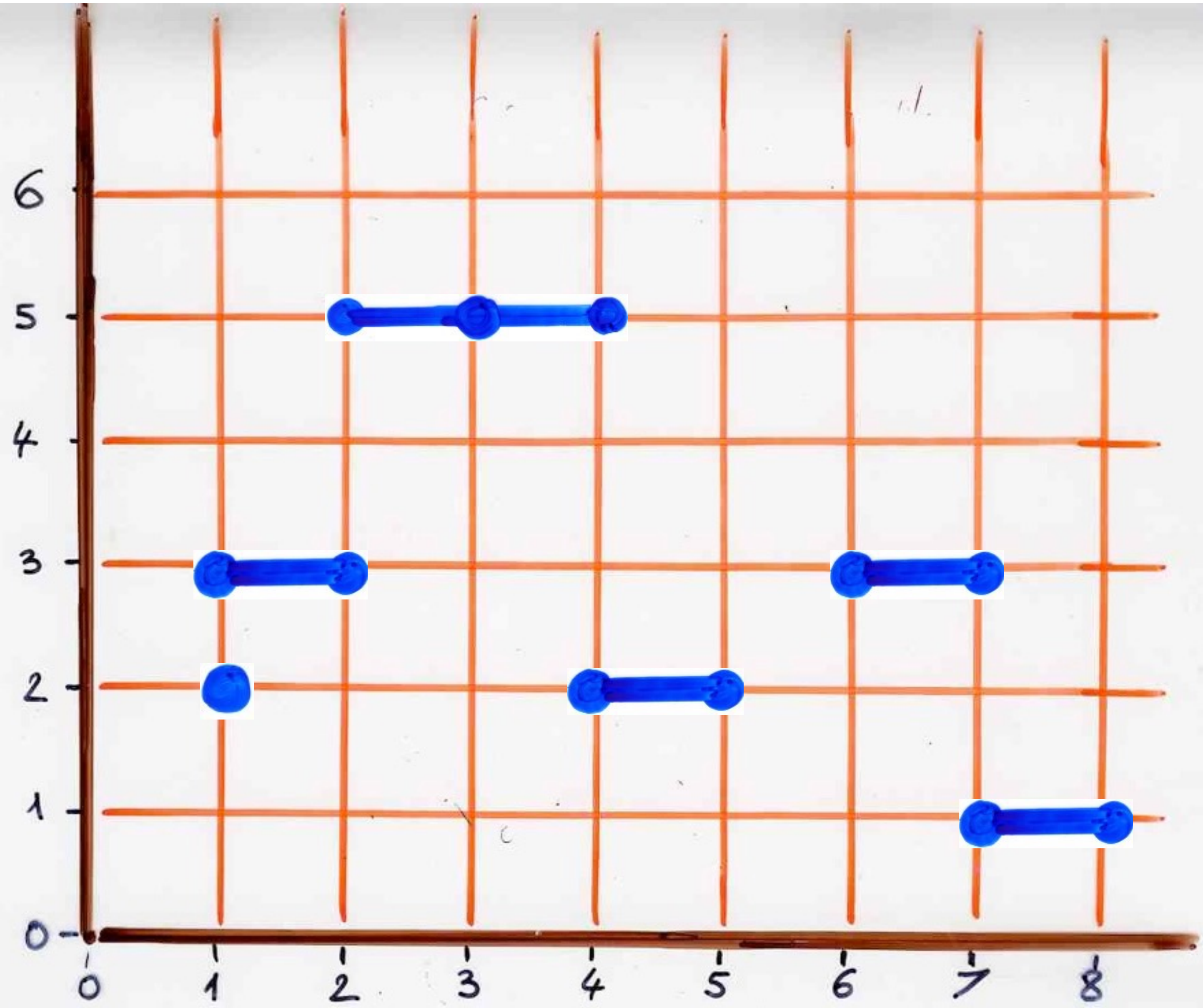
P $\mathcal{C}$

E

(α, i) $\alpha \in P$
$i \in \mathbb{N}$

(i) $(\alpha, i), (\beta, j) \in E$

$\alpha \,\mathcal{C}\, \beta \Rightarrow i \neq j$



0
1
2
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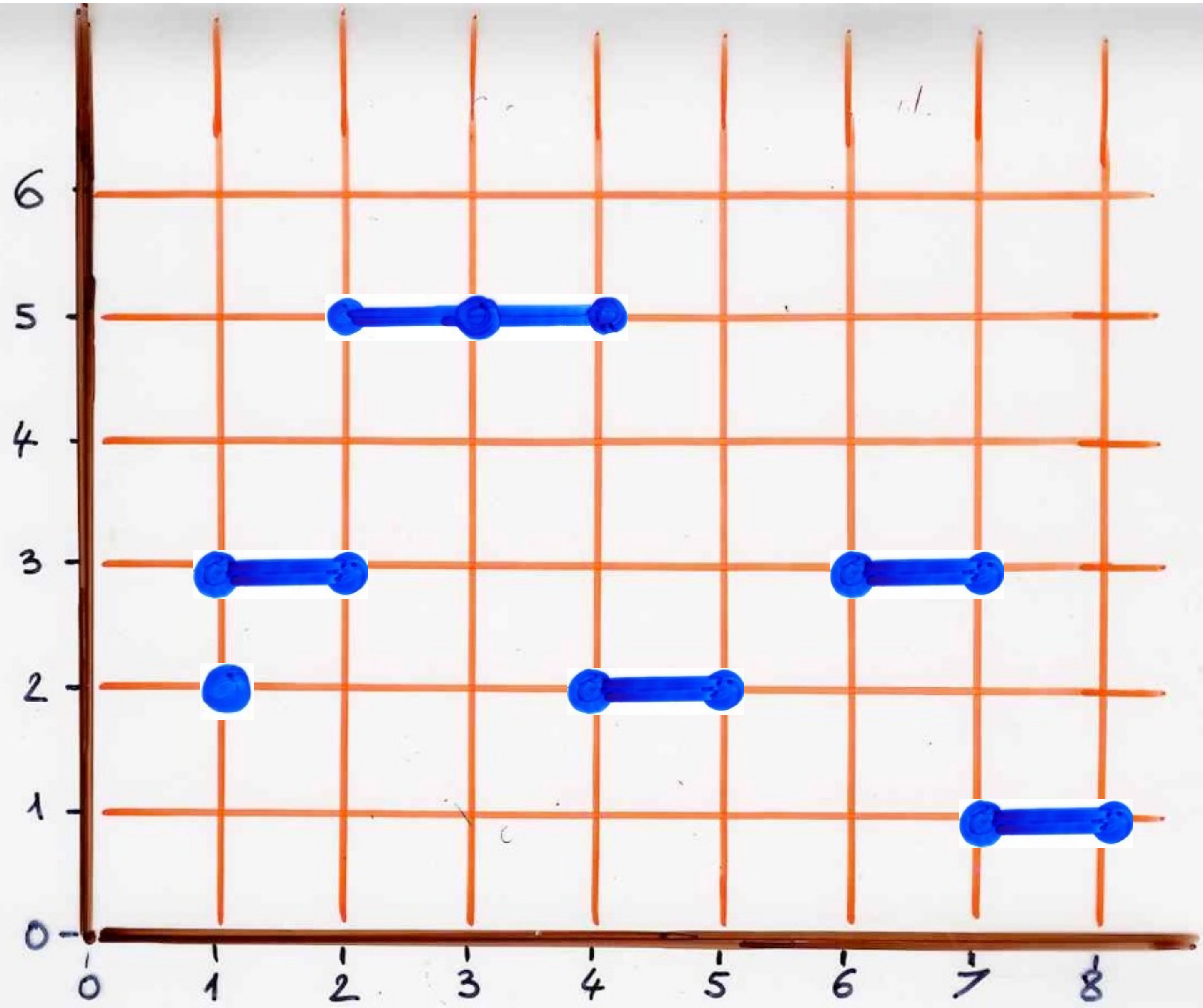
4
5
6
7
8

Def. **elementary move**

on a pre-heap E

$$(\alpha, i) \longrightarrow \begin{matrix} (\alpha, i-1) \\ \text{or} \\ (\alpha, i+1) \end{matrix} \qquad \text{(if possible)}$$



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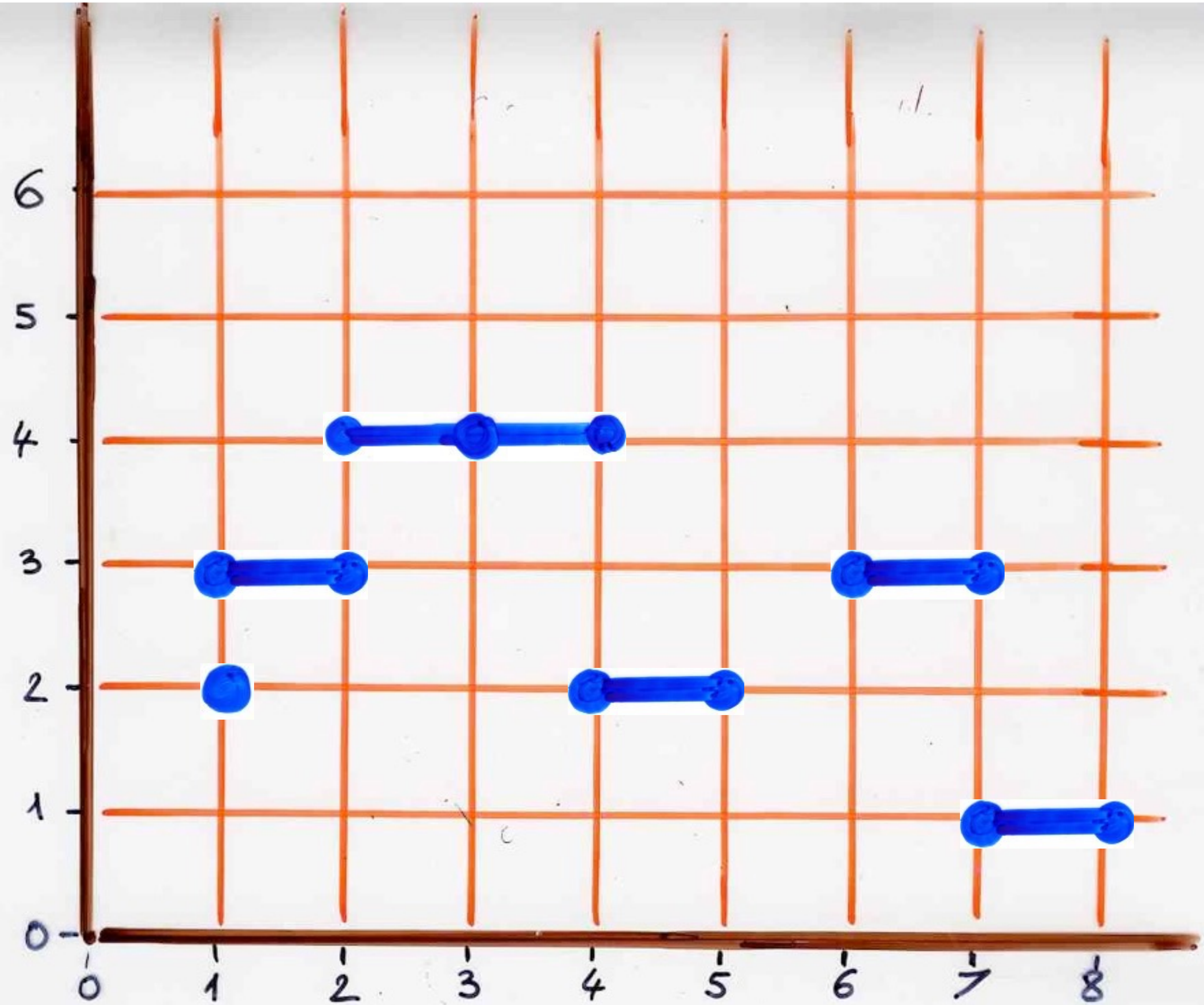
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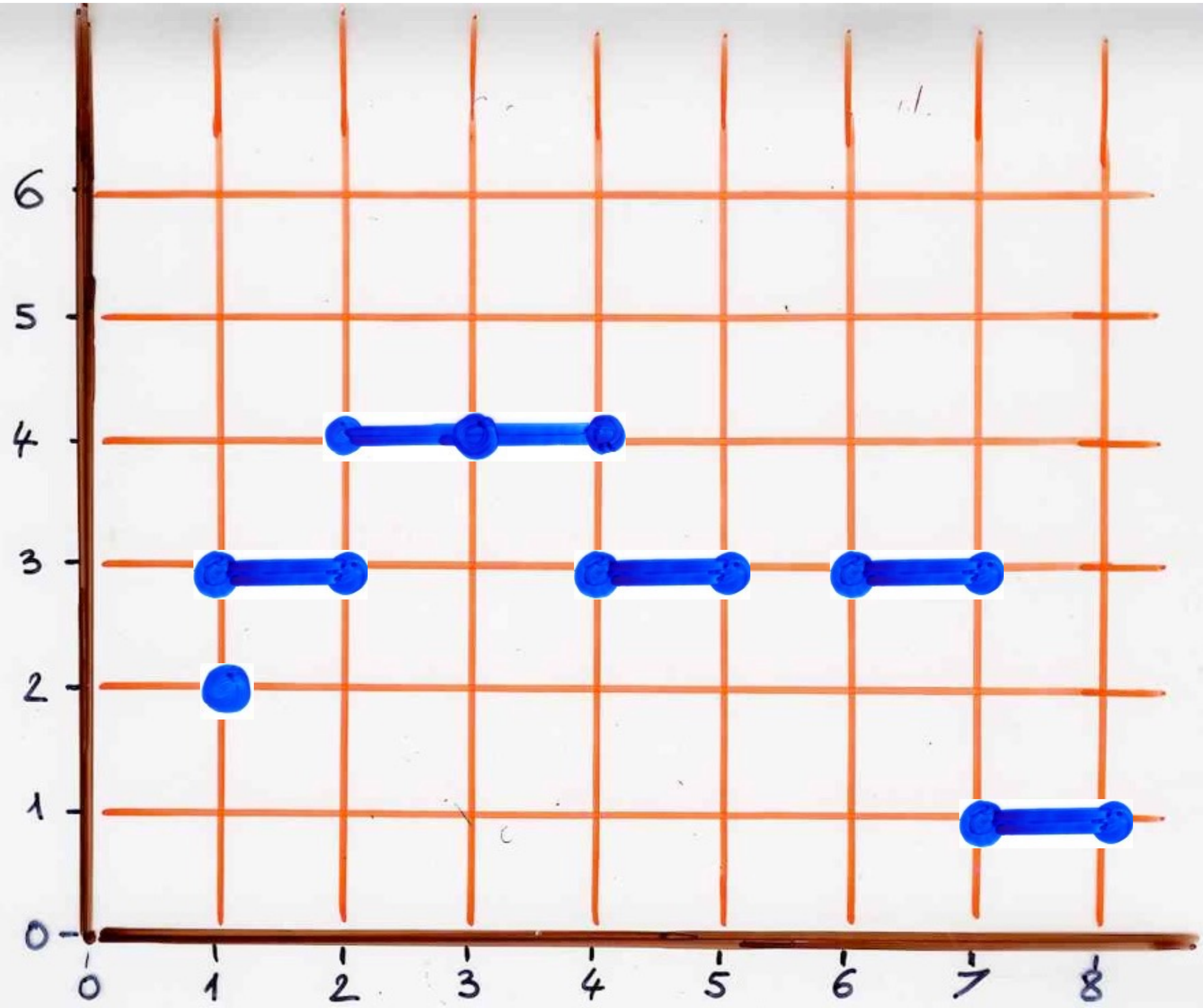

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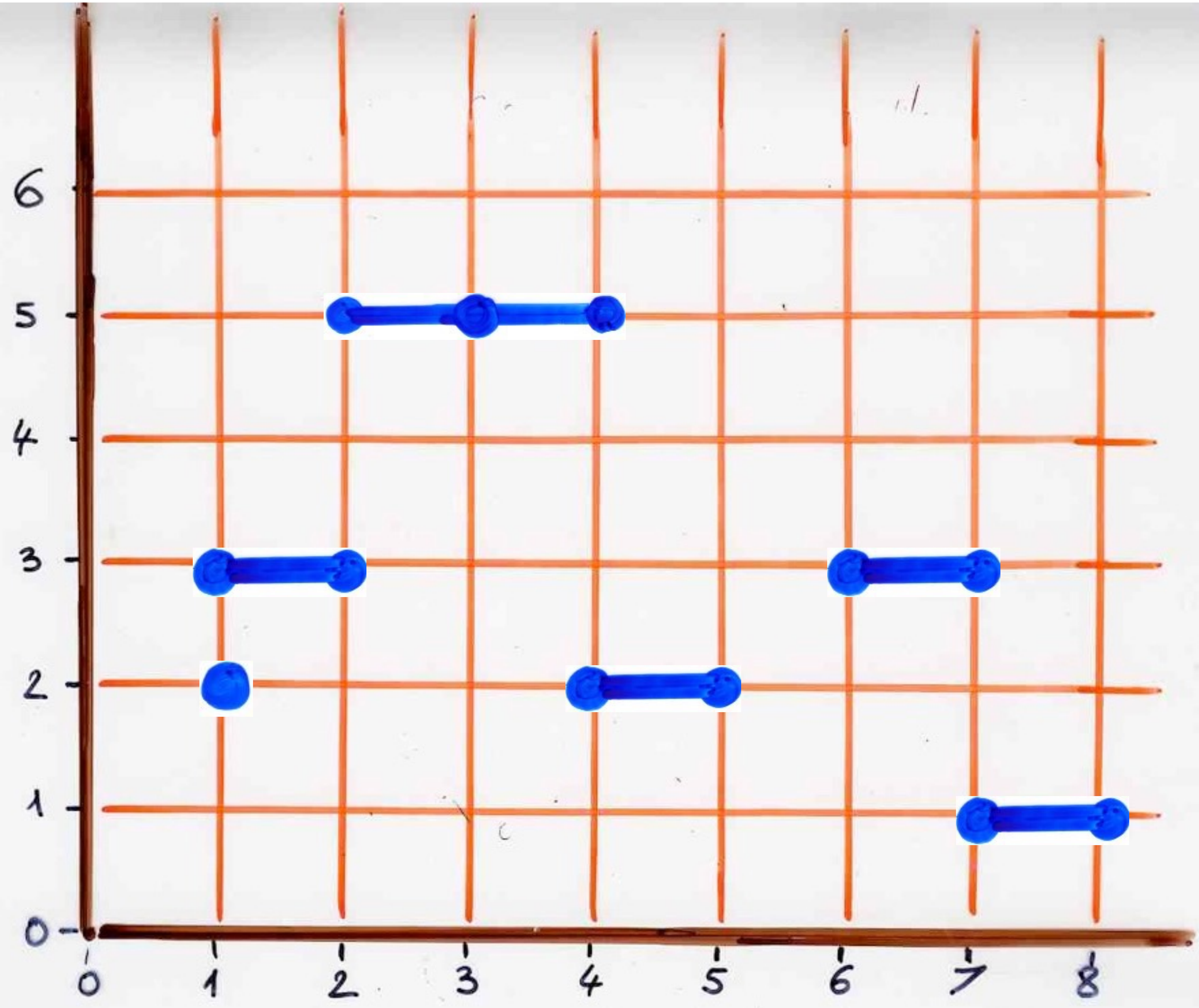
heap :

pre-heap up to the equivalence $\sim$

- in each equivalence class for $\sim$ there exist a unique heap

PE pre-heap $\longrightarrow$ $E_{\sim}(PE)$ heap



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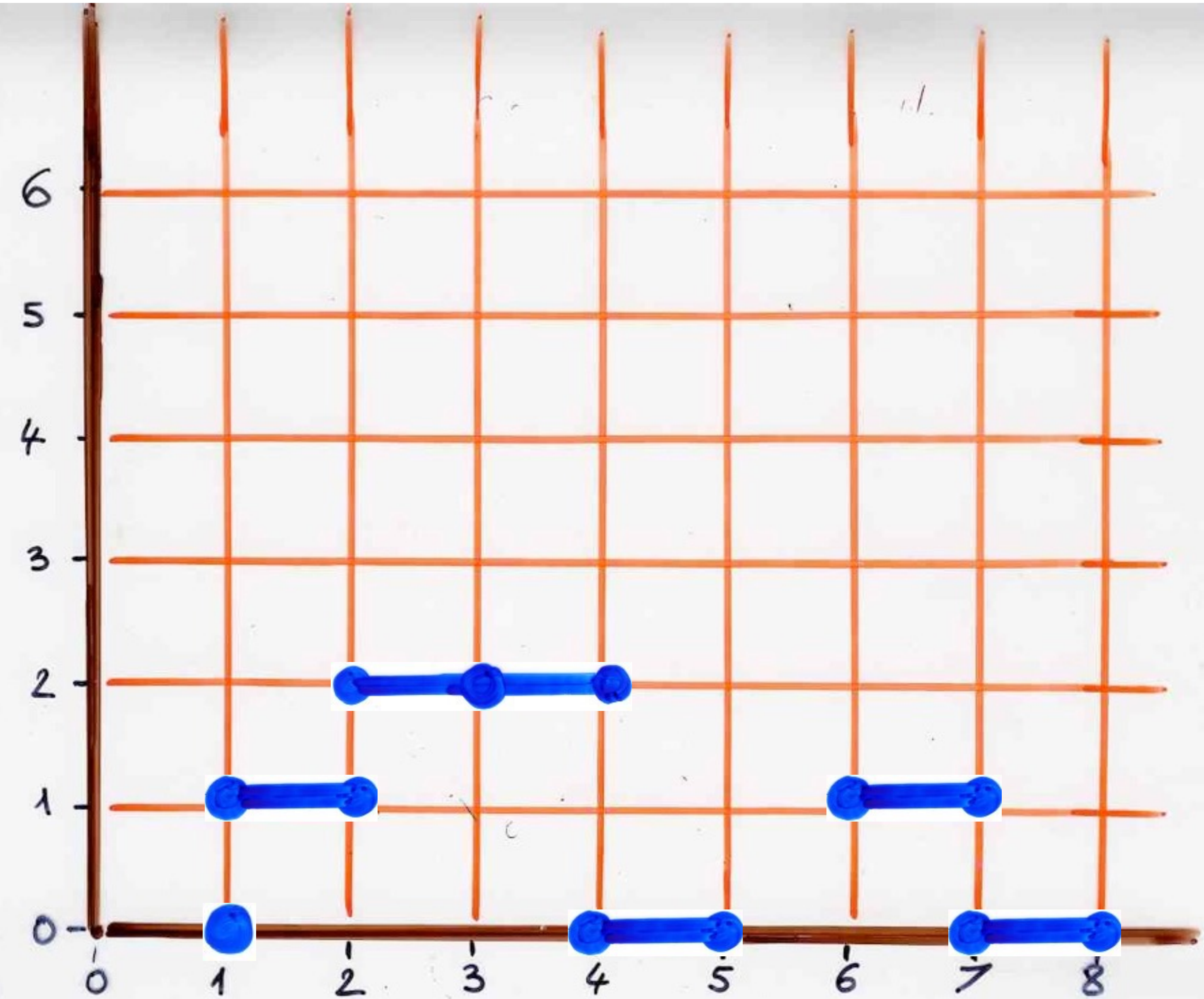

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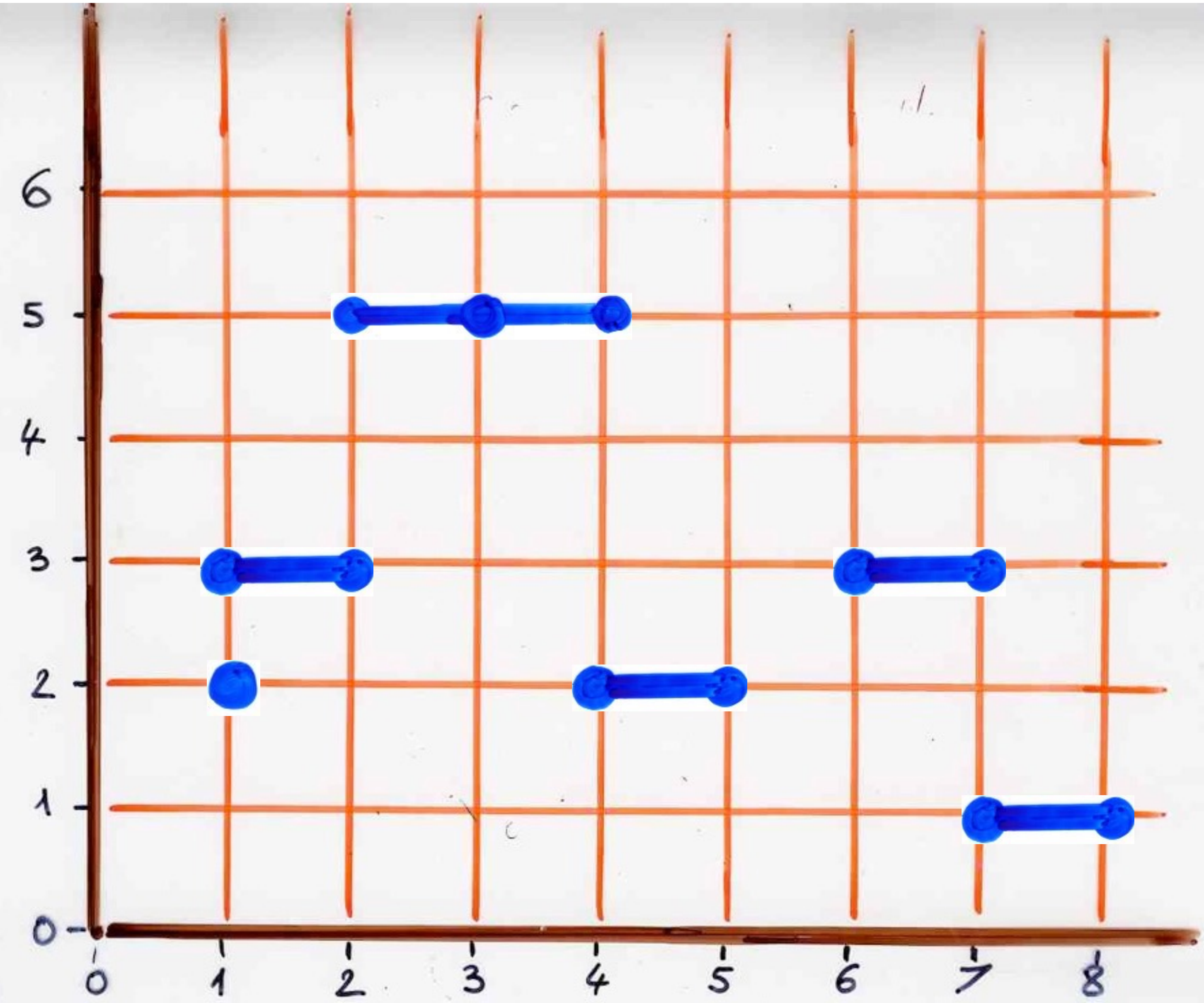
- heap → "anti-heap" (helium)



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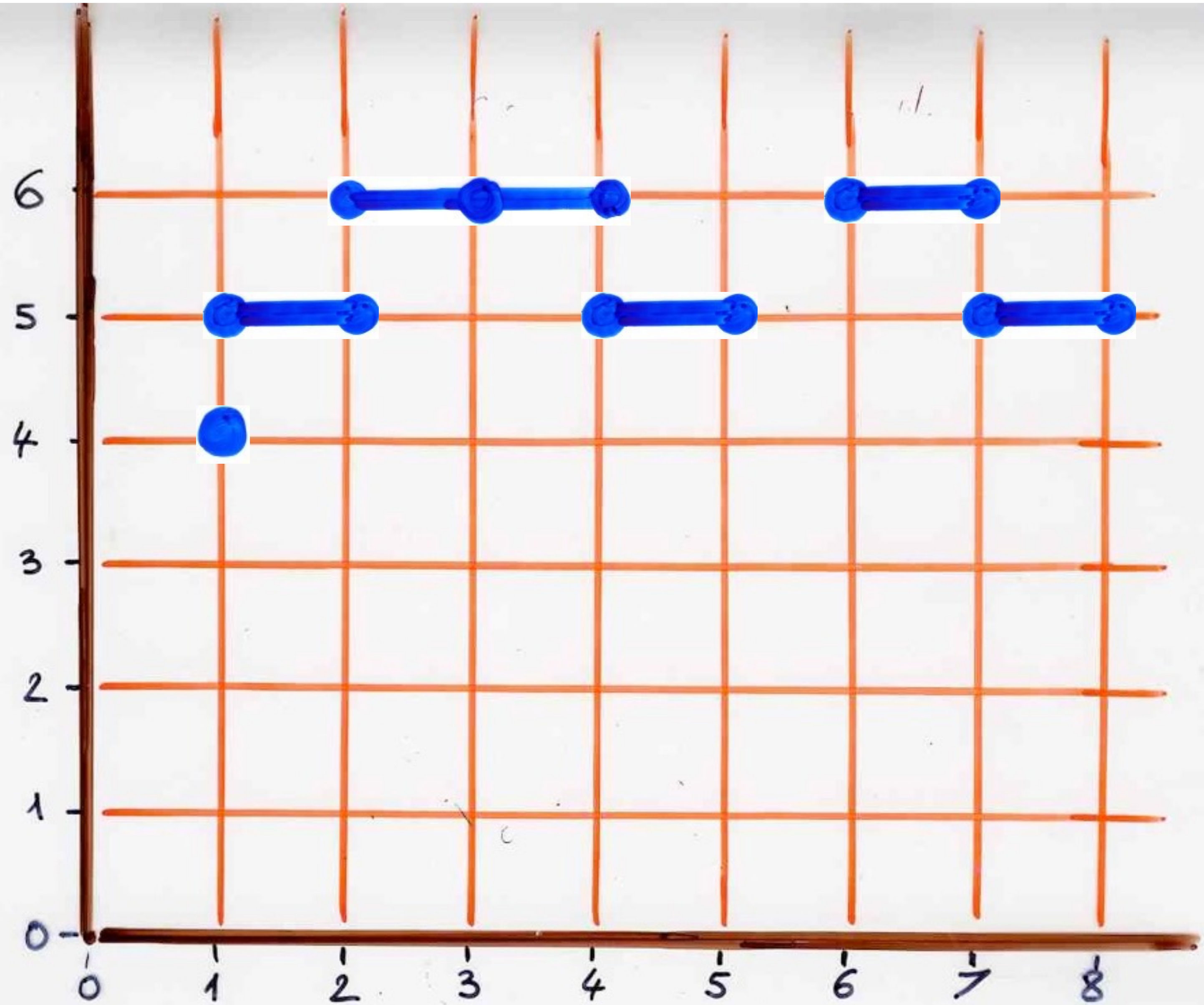
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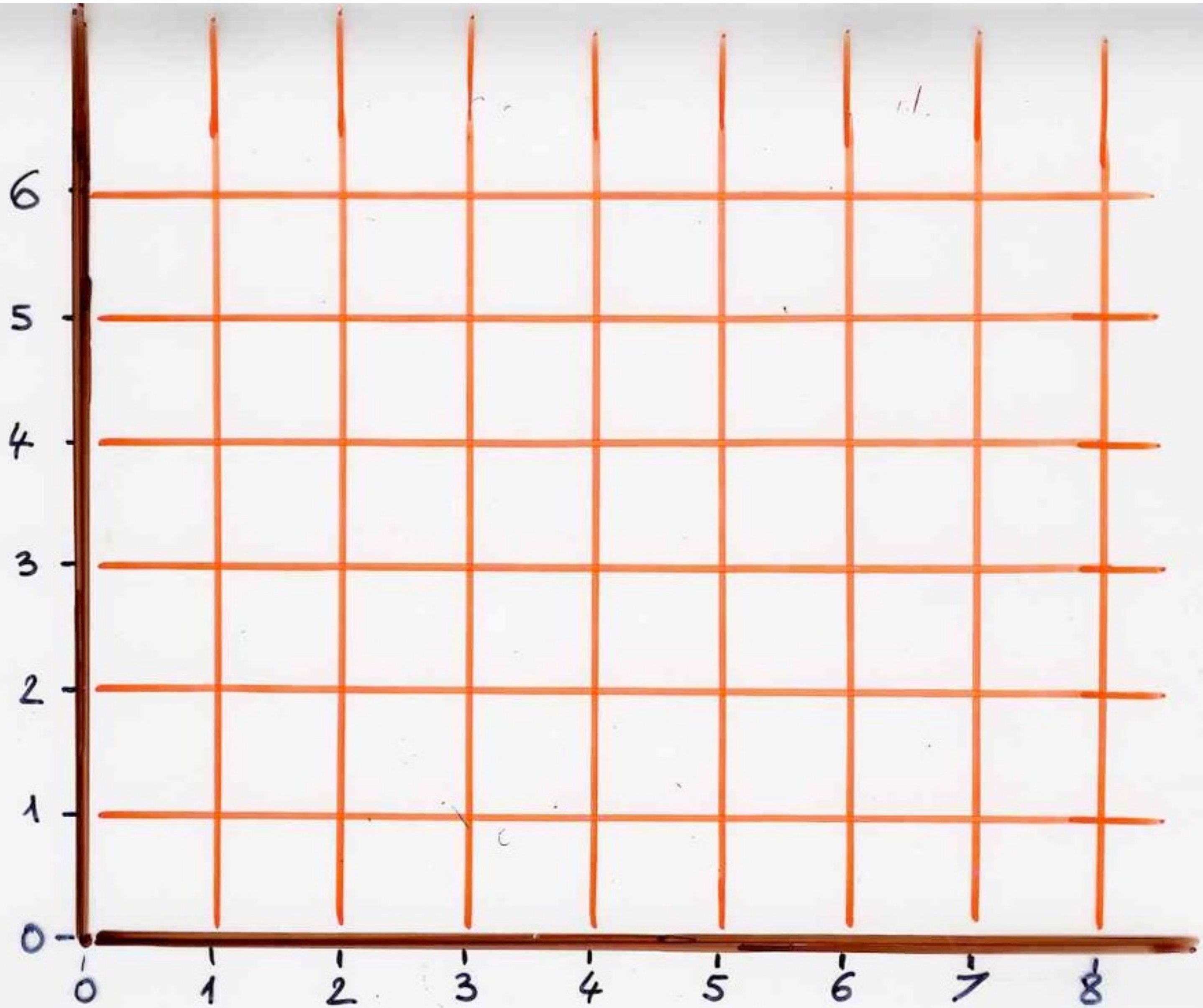
Heaps monoid

$H(P, \mathcal{C})$

product of two heaps

$E \circ F$



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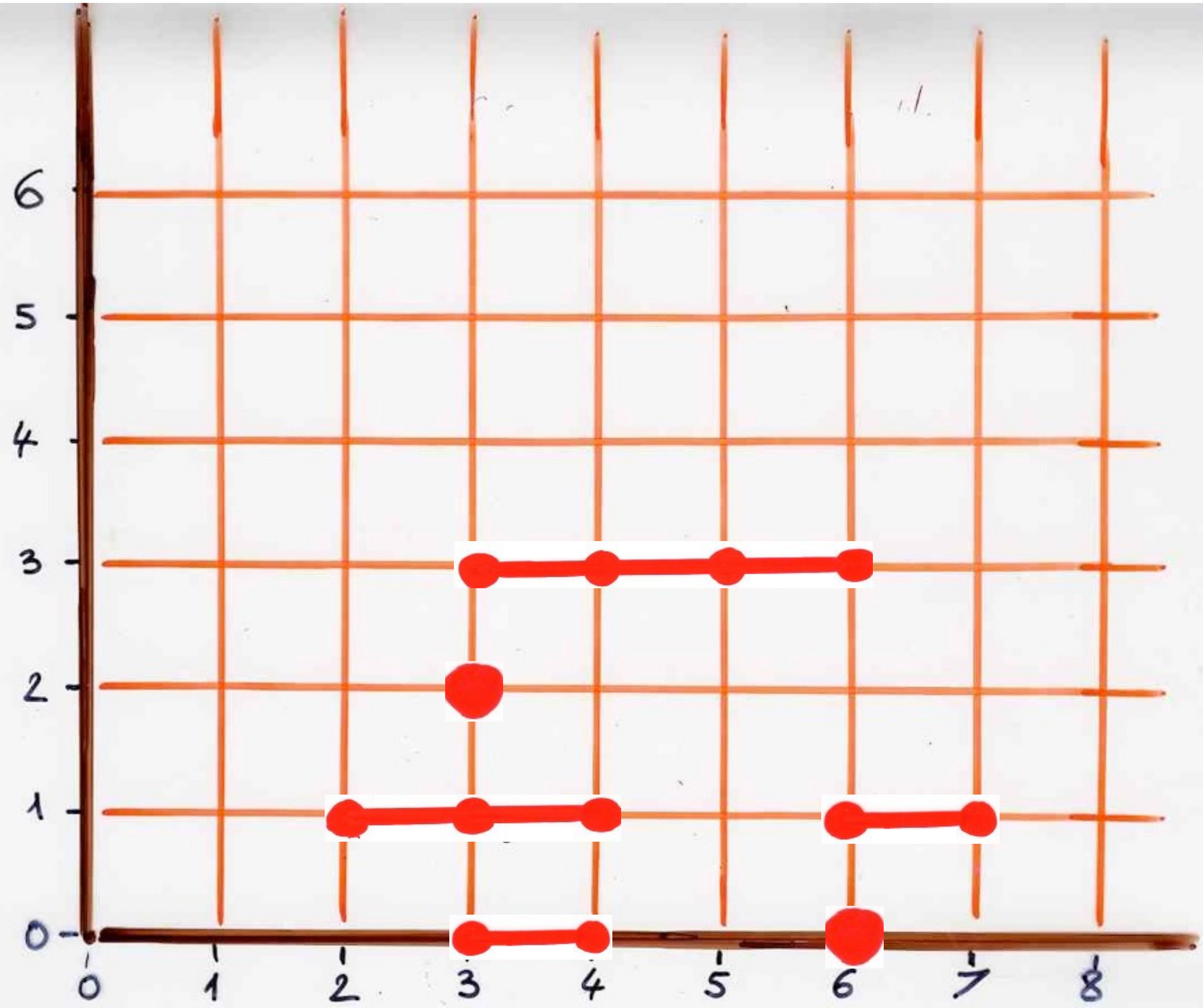

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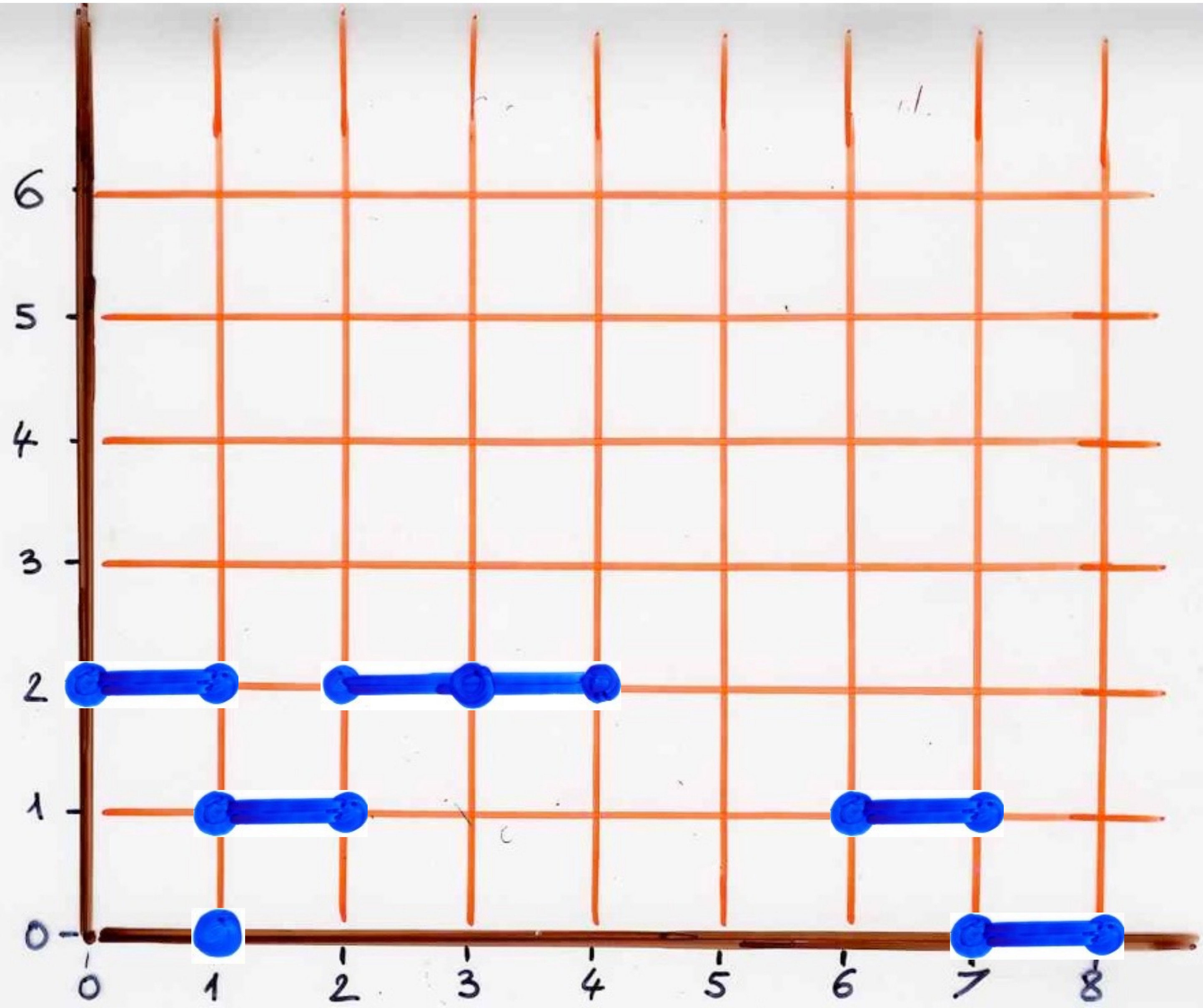
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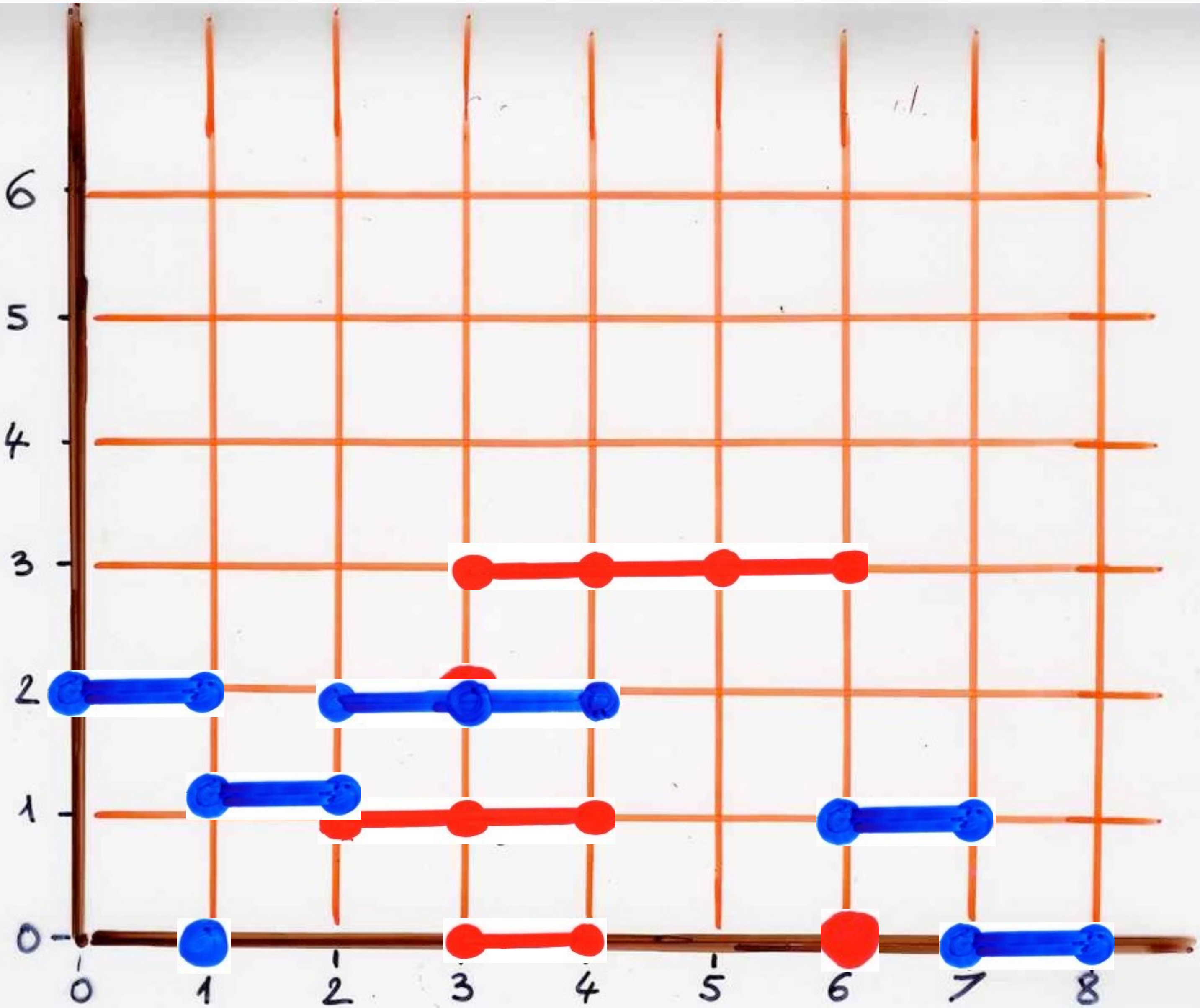
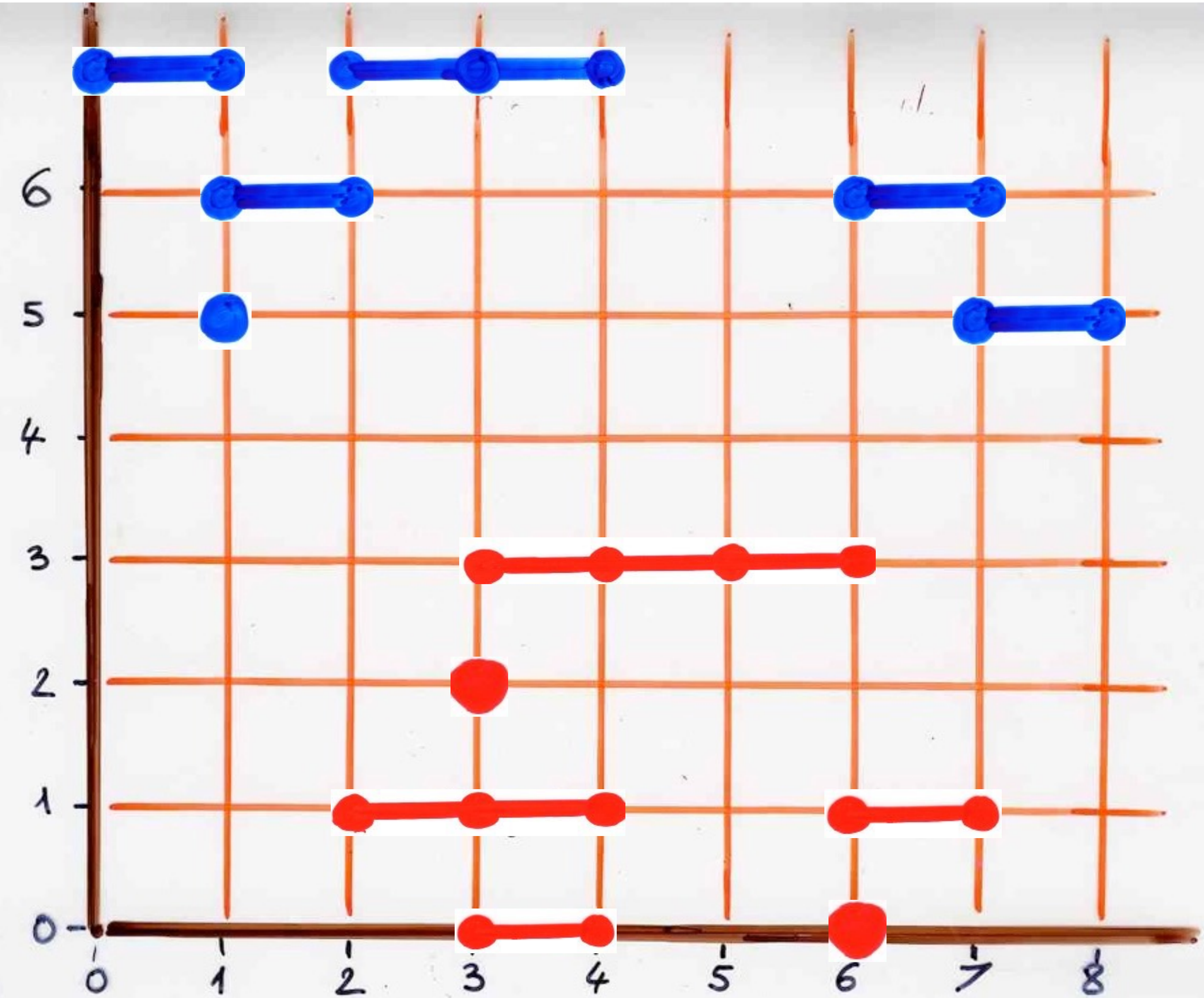

0 1 2 3 4 5 6
0 1 2 3 4 5 6 7 8



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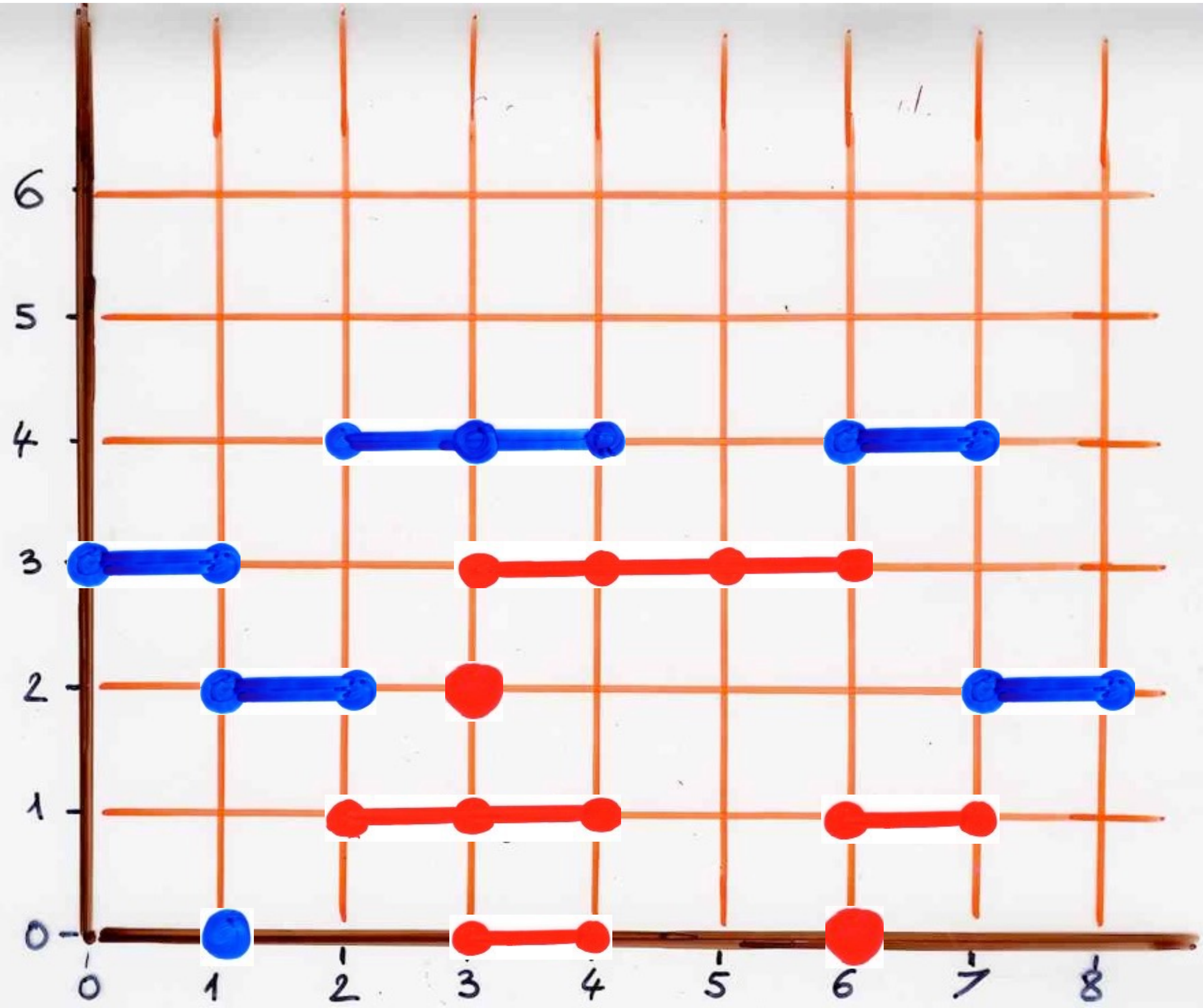

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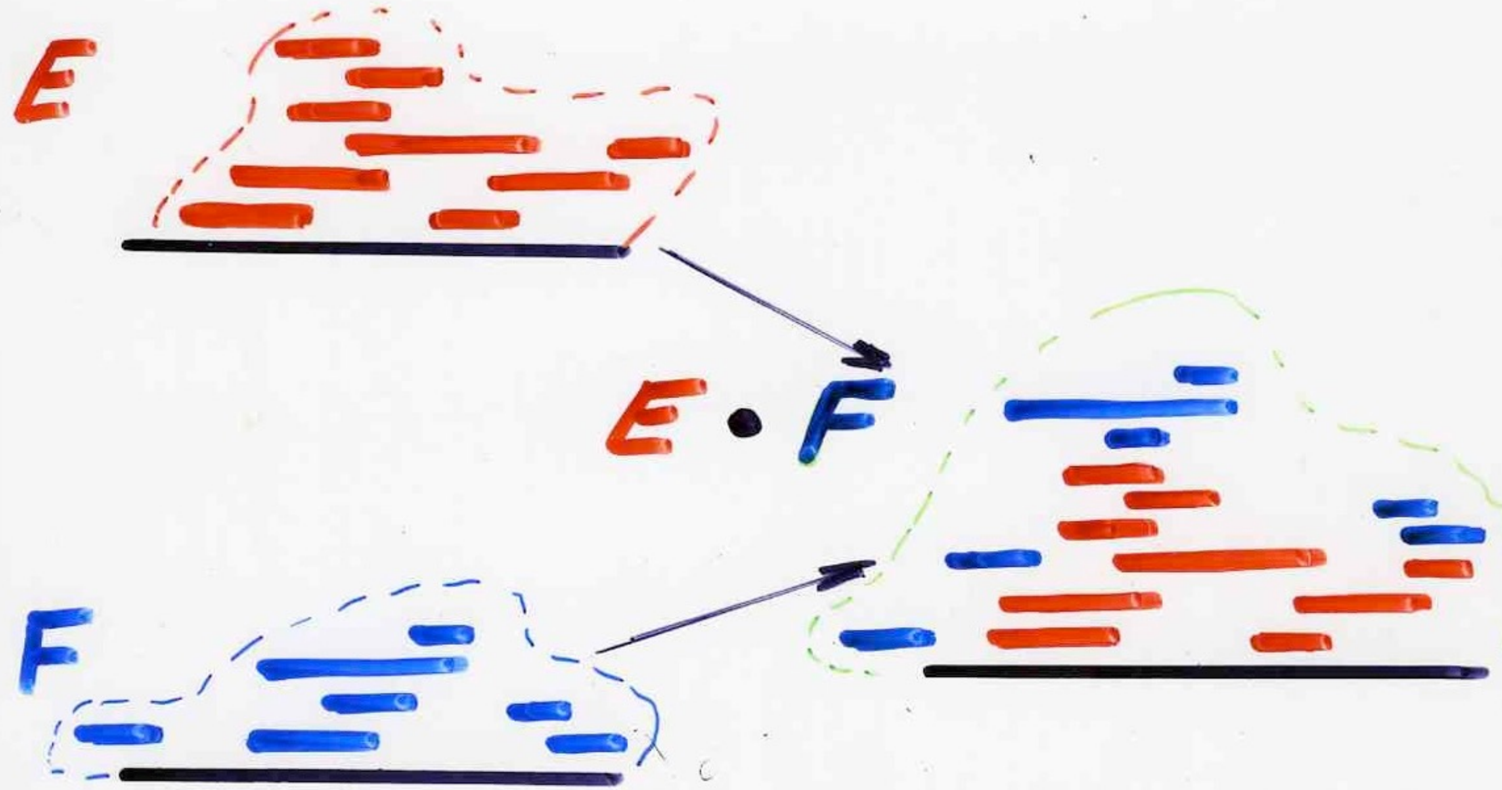


$$E \cdot F = E_{\sim}\left(E \cup T(F)\right)$$

§4 Equivalence commutation monoids and heaps monoids

$$\mathrm{Heap}(P, \mathcal{C}) \simeq \text{commutation monoid}$$

heaps monoid

$$P \subseteq \mathrm{Heap}(P, \mathcal{C})$$

$$\alpha \longleftrightarrow \{(\alpha, 0)\}$$

$$\varphi : P^{*} \longrightarrow \mathrm{Heap}(P, \mathcal{C})$$

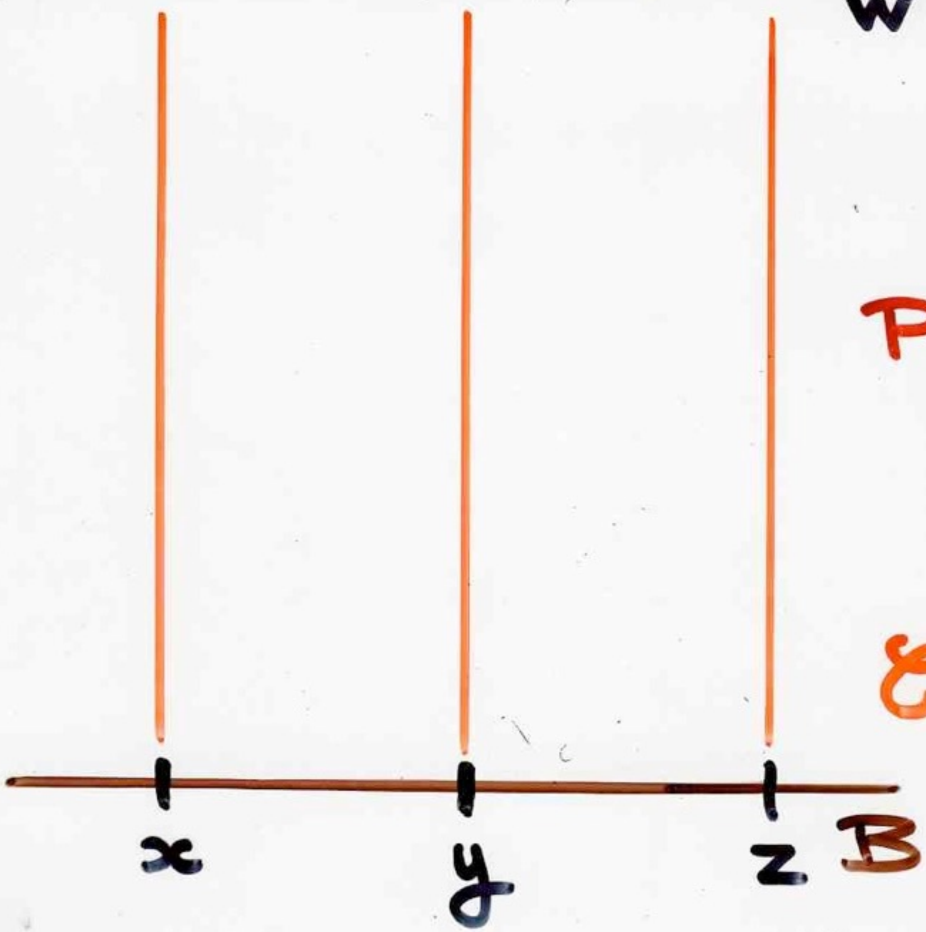
$$w = \alpha_1 \alpha_2 \cdots \alpha_n \longrightarrow \alpha_1 \odot \alpha_2 \odot \cdots \odot \alpha_n$$

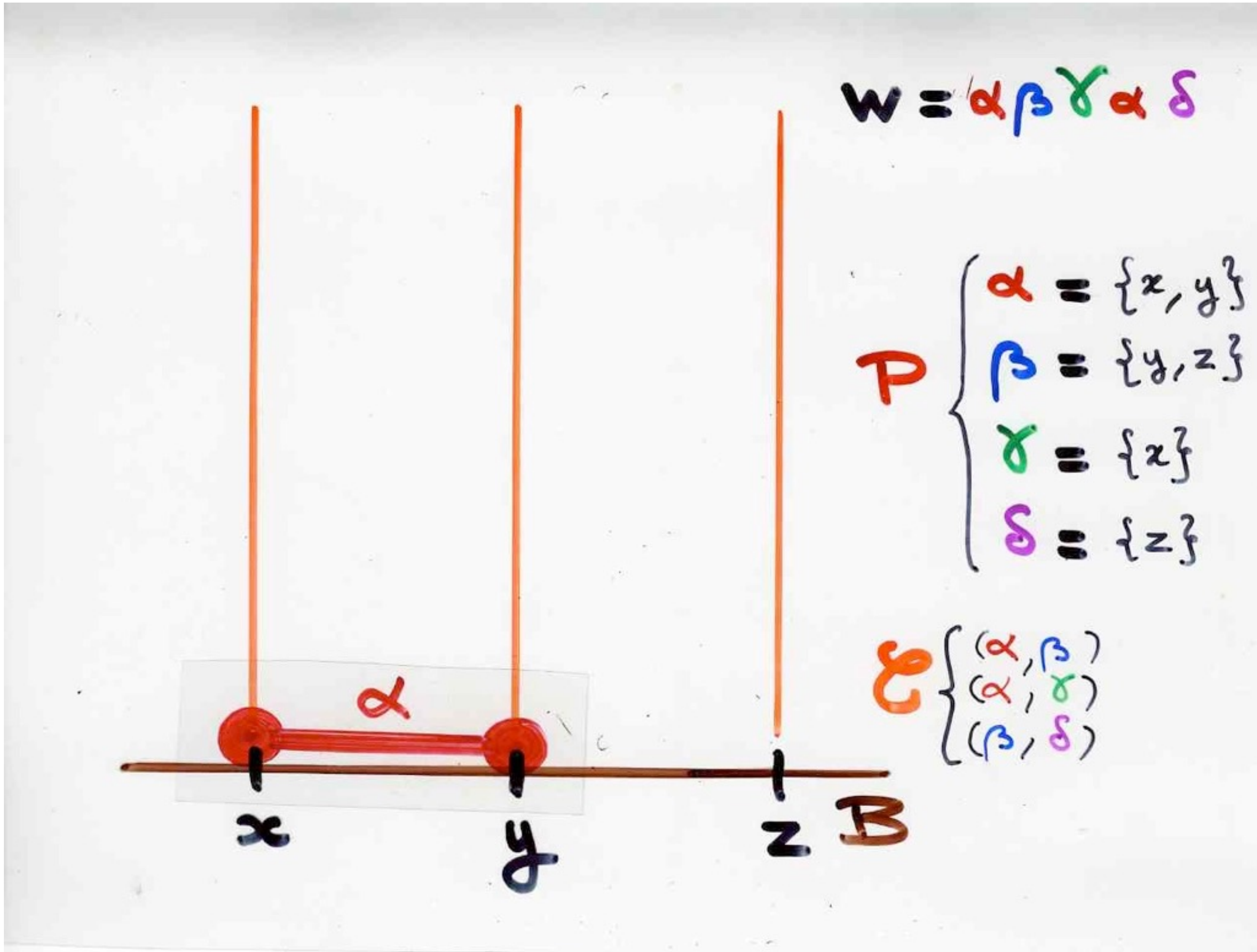
word heap

$$w = \alpha\beta\gamma\alpha\delta$$

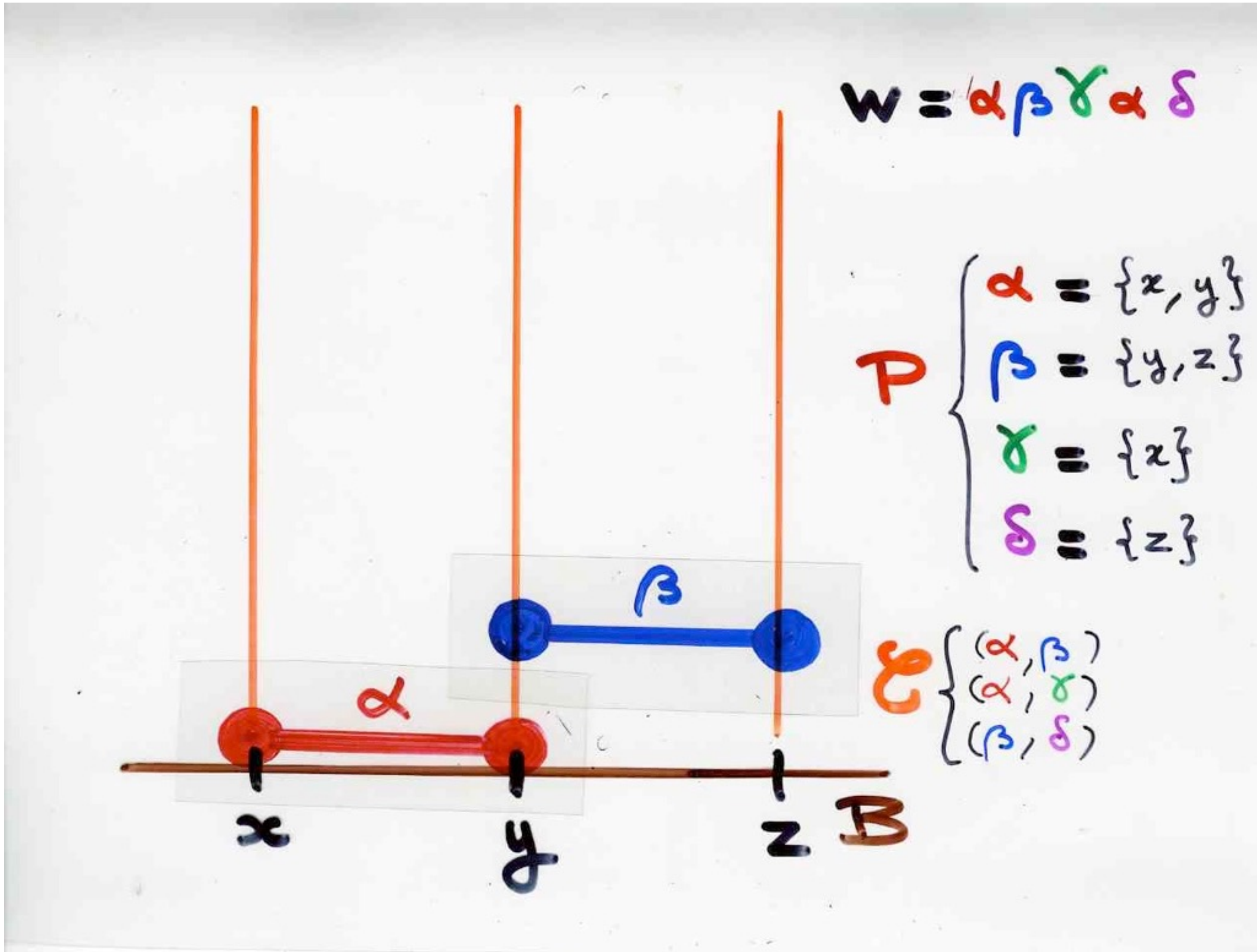
$$P \begin{cases} \alpha = \{x, y\} \\ \beta = \{y, z\} \\ \gamma = \{x\} \\ \delta = \{z\} \end{cases}$$

$$\mathcal{E} \begin{cases} (\alpha, \beta) \\ (\alpha, \gamma) \\ (\beta, \delta) \end{cases}$$

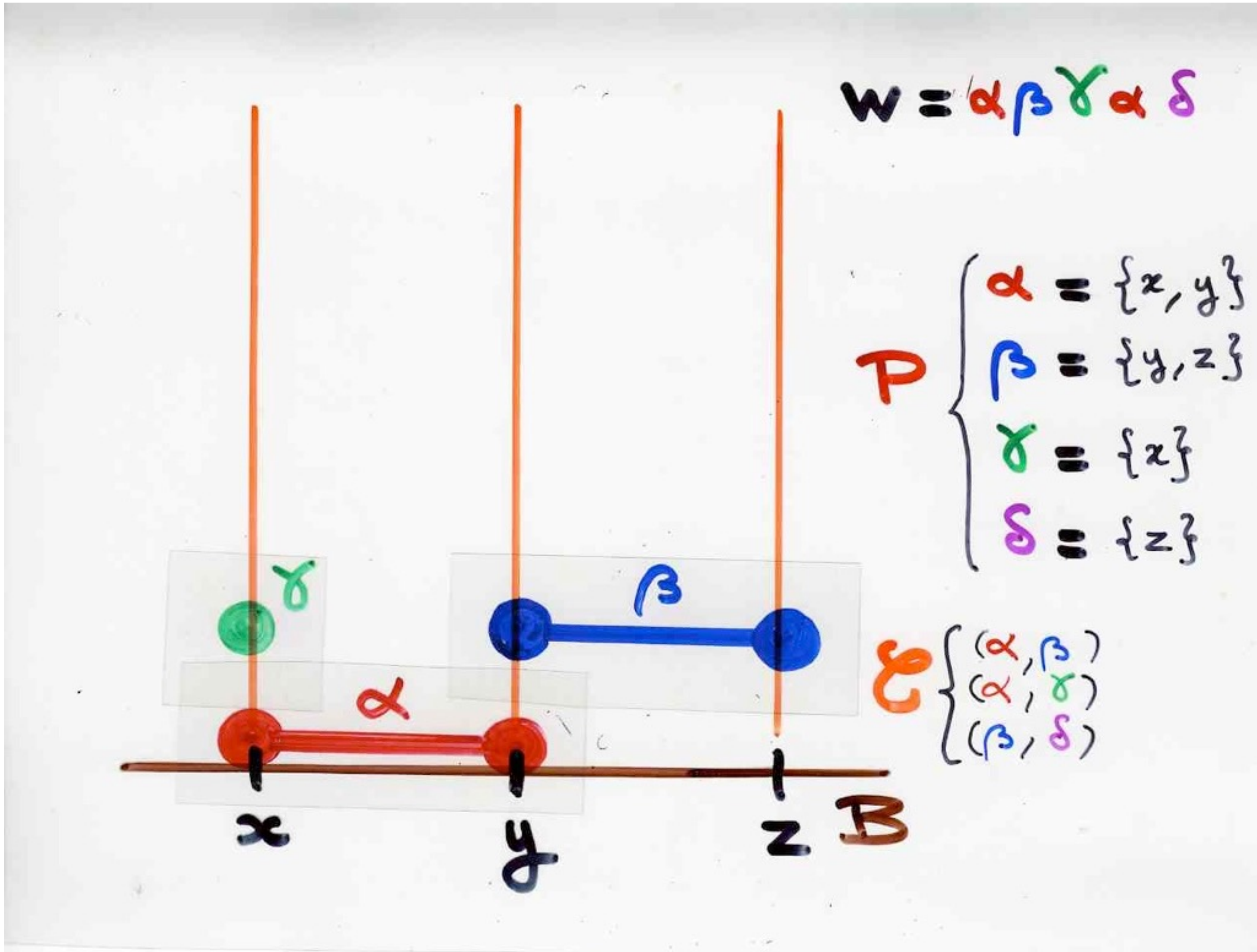




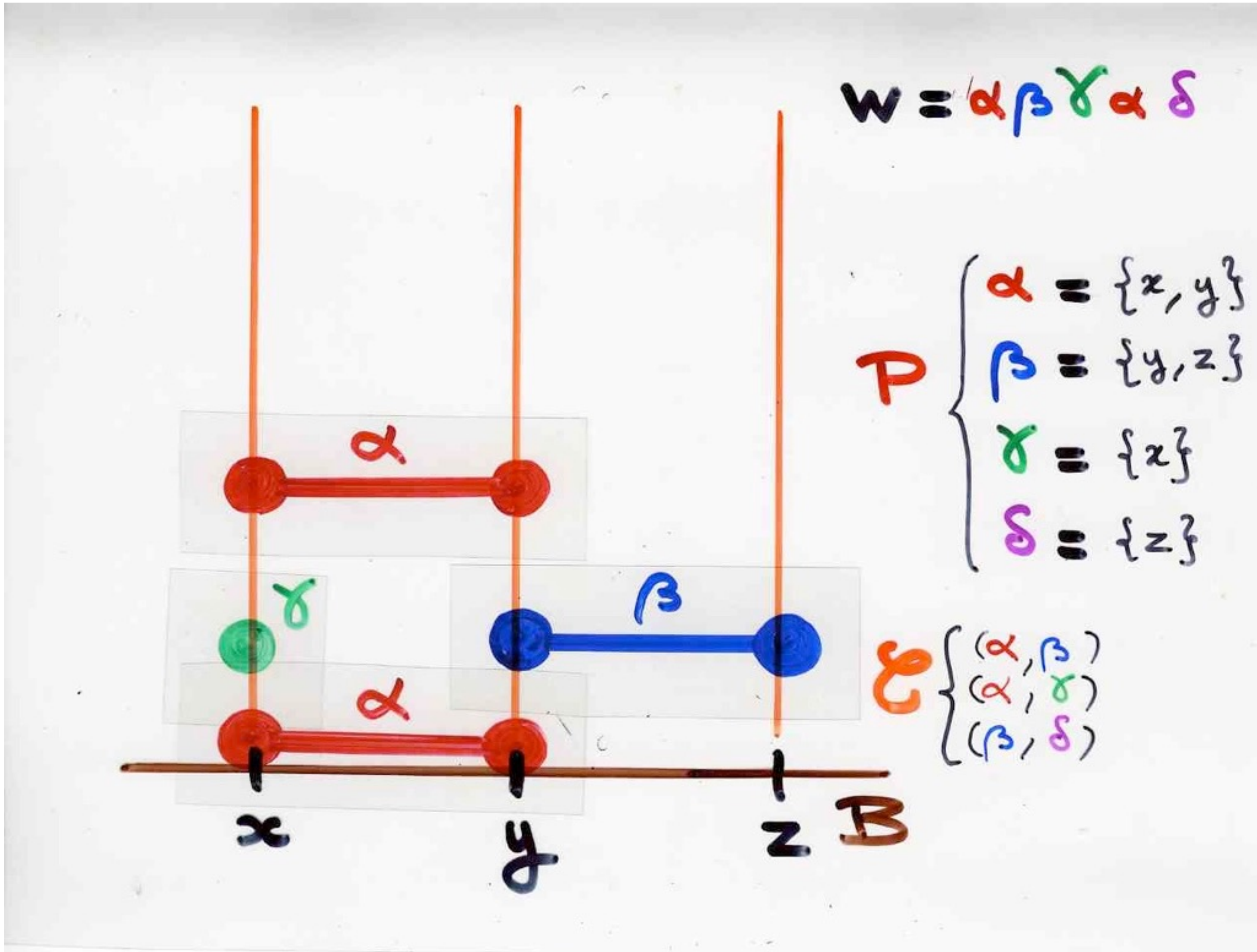
w = α β γ α δ
P
α = {x, y}
β = {y, z}
γ = {x}
δ = {z}
E
(α, β)
(α, γ)
(β, δ)
α
x
y
z
B



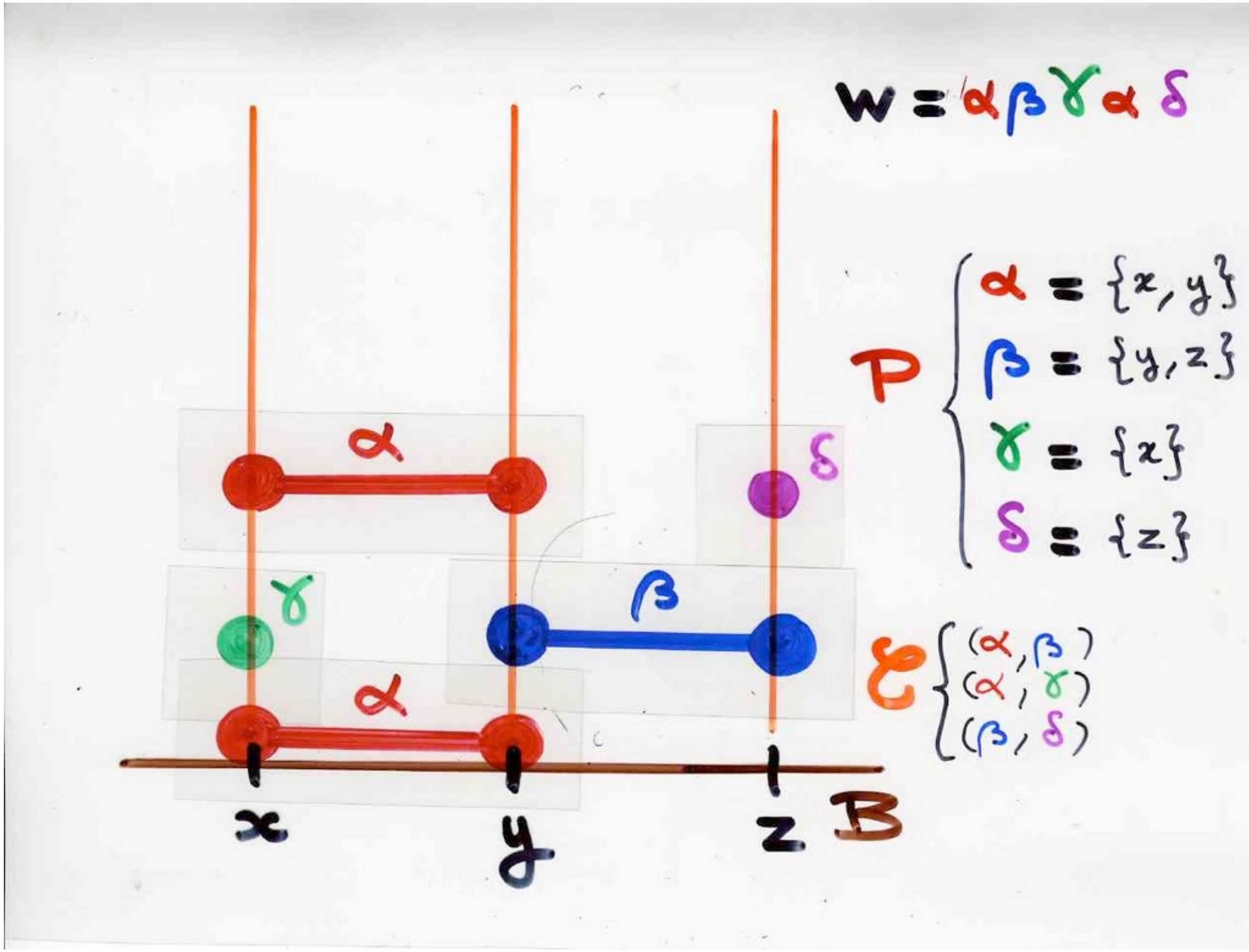
$w = \alpha\beta\gamma\alpha\delta$
$\mathcal{P} \begin{cases} \alpha = \{x, y\} \\ \beta = \{y, z\} \\ \gamma = \{x\} \\ \delta = \{z\} \end{cases}$
$\mathcal{E} \begin{cases} (\alpha, \beta) \\ (\alpha, \gamma) \\ (\beta, \delta) \end{cases}$
β
α
x
y
z
B



w = αβγαδ
α = {x, y}
β = {y, z}
γ = {x}
δ = {z}
P
C
(α, β)
(α, γ)
(β, δ)
γ
β
α
x
y
z
B



w = αβγαδ
P
α = {x, y}
β = {y, z}
γ = {x}
δ = {z}
α
γ
β
α
(α, β)
(α, γ)
(β, δ)
x
y
z
B



w = αβγαδ
P
α = {x, y}
β = {y, z}
γ = {x}
δ = {z}
C
(α, β)
(α, γ)
(β, δ)
α
δ
γ
β
α
x
y
z
B

commutations

$$C = \overline{\mathcal{C}}$$

$$C \begin{cases} (\alpha, \delta) \\ (\beta, \gamma) \\ (\gamma, \delta) \end{cases}$$

$$\begin{tikzcd}
W = \alpha\beta\gamma\alpha\delta \arrow[rr, dash] \arrow[d, dash] & & \alpha\beta\gamma\delta\alpha \arrow[dr, dash] & \\
\alpha\gamma\beta\alpha\delta \arrow[d, dash] & & & \alpha\beta\delta\gamma\alpha \\
\alpha\gamma\beta\delta\alpha \arrow[uurr, dash] & & &
\end{tikzcd}$$

commutations

$C = \overline{\mathcal{C}}$

$$C \quad \begin{cases} (\alpha, \delta) \\ (\beta, \gamma) \\ (\gamma, \delta) \end{cases}$$

ex : $A = \{a, b, c, d\}$

$$C \quad \begin{cases} ad = da \\ bc = cb \\ cd = dc \end{cases}$$

equivalence class

w= abcad   abcda

acbad   abdca

acbda

$$P \subseteq \mathrm{Heap}(P, \mathcal{C})$$

$$\alpha \longleftrightarrow \{(\alpha, 0)\}$$

$$\varphi : P^* \longrightarrow \mathrm{Heap}(P, \mathcal{C})$$

$$\underset{\text{word}}{w = \alpha_1 \alpha_2 \cdots \alpha_n} \longrightarrow \underset{\text{heap}}{\alpha_1 \odot \alpha_2 \odot \cdots \odot \alpha_n}$$

$C = \overline{\mathcal{C}}$

commutation relation = complementary of the dependency relation

Lemma 1

$$u \equiv_C v \implies \varphi(u) = \varphi(v)$$

Lemma 2

$$\varphi(u) = \varphi(v) \implies u \equiv_C v$$

Proof in the next §

Definition $\overline{\varphi}([u]) = \varphi(u)$

$$\begin{tikzcd}
P^* \arrow[r, "\varphi"] \arrow[d] & H(P, \mathcal{E}) \\
P^* / {\equiv_C} \arrow[ur, dashed, "\overline{\varphi}"'] &
\end{tikzcd}$$

Proposition $\overline{\varphi}$ is an isomorphism of monoids

$$\mathrm{Heap}(P, \mathcal{E}) \simeq P^{*}/_{\equiv_C}$$

heaps monoid

commutation monoid

$$C = \overline{\mathcal{E}}$$ complementary relation

another example:
heaps of dimers

ex: heaps of dimers on $\mathbb{N}$

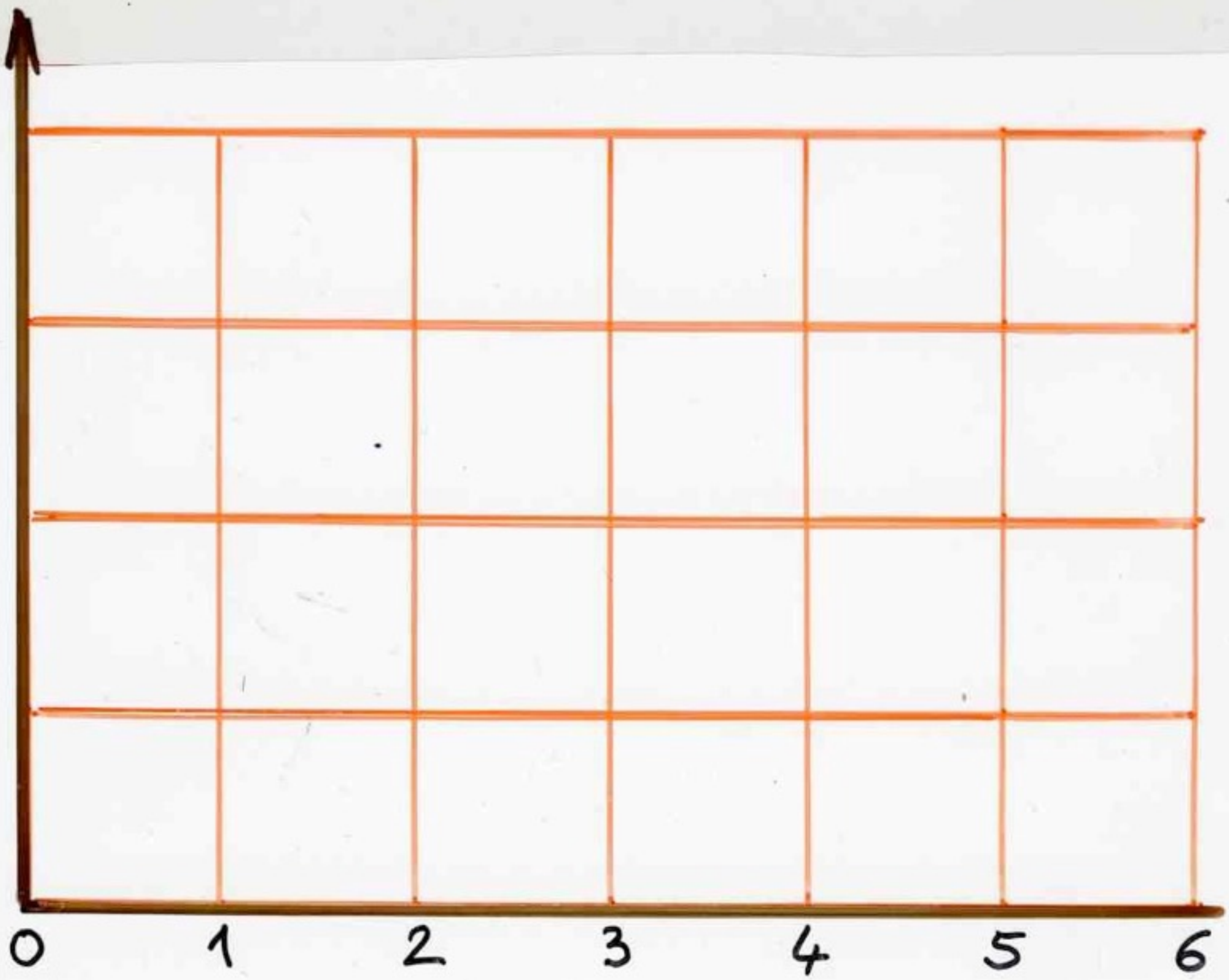
$P = \{ [i, i+1] = \sigma_i \ , i \geq 0 \}$

$\mathcal{C}$

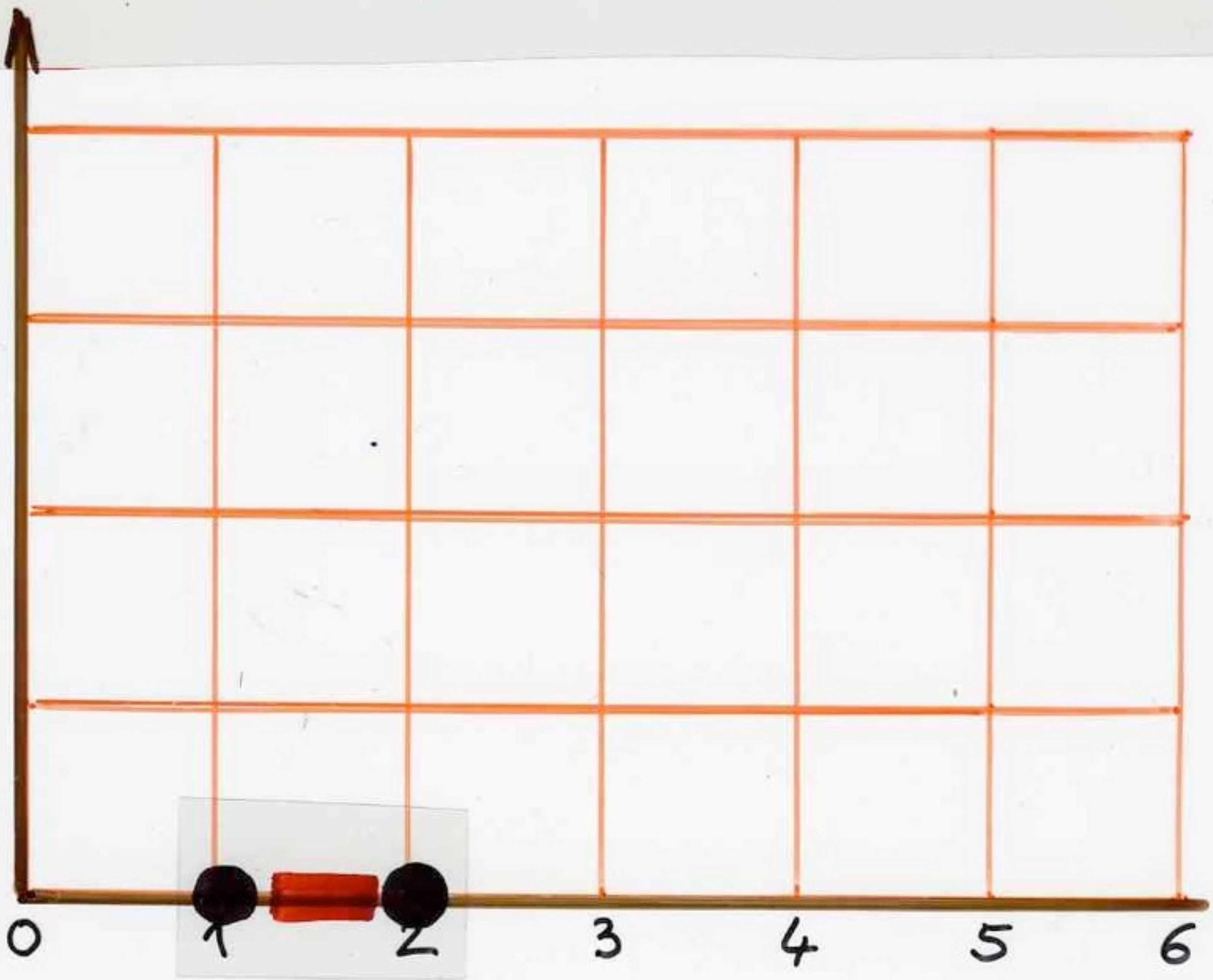
C commutations

$\sigma_i \sigma_j = \sigma_j \sigma_i$ iff $|i-j| \geq 2$

$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$



$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



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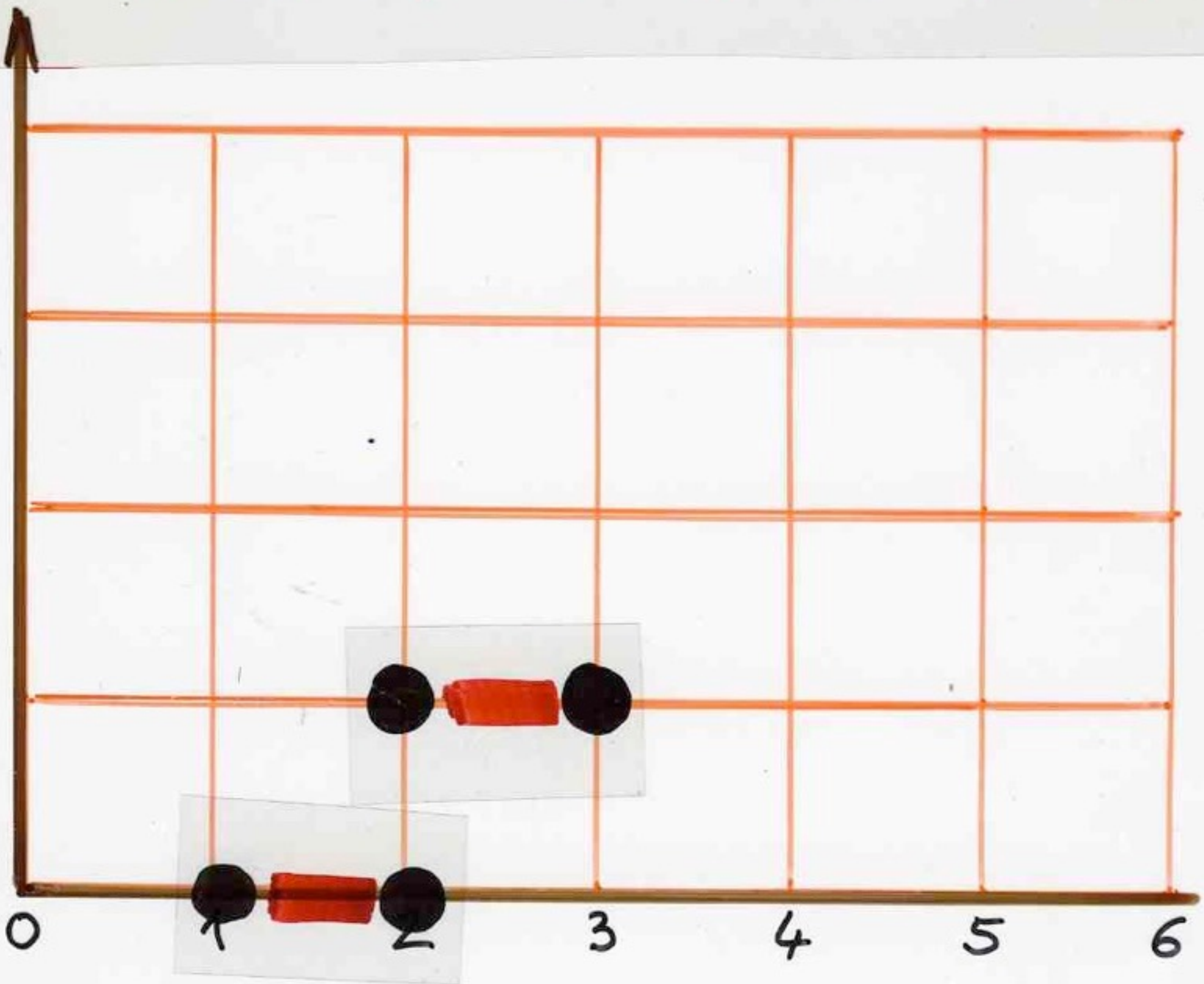

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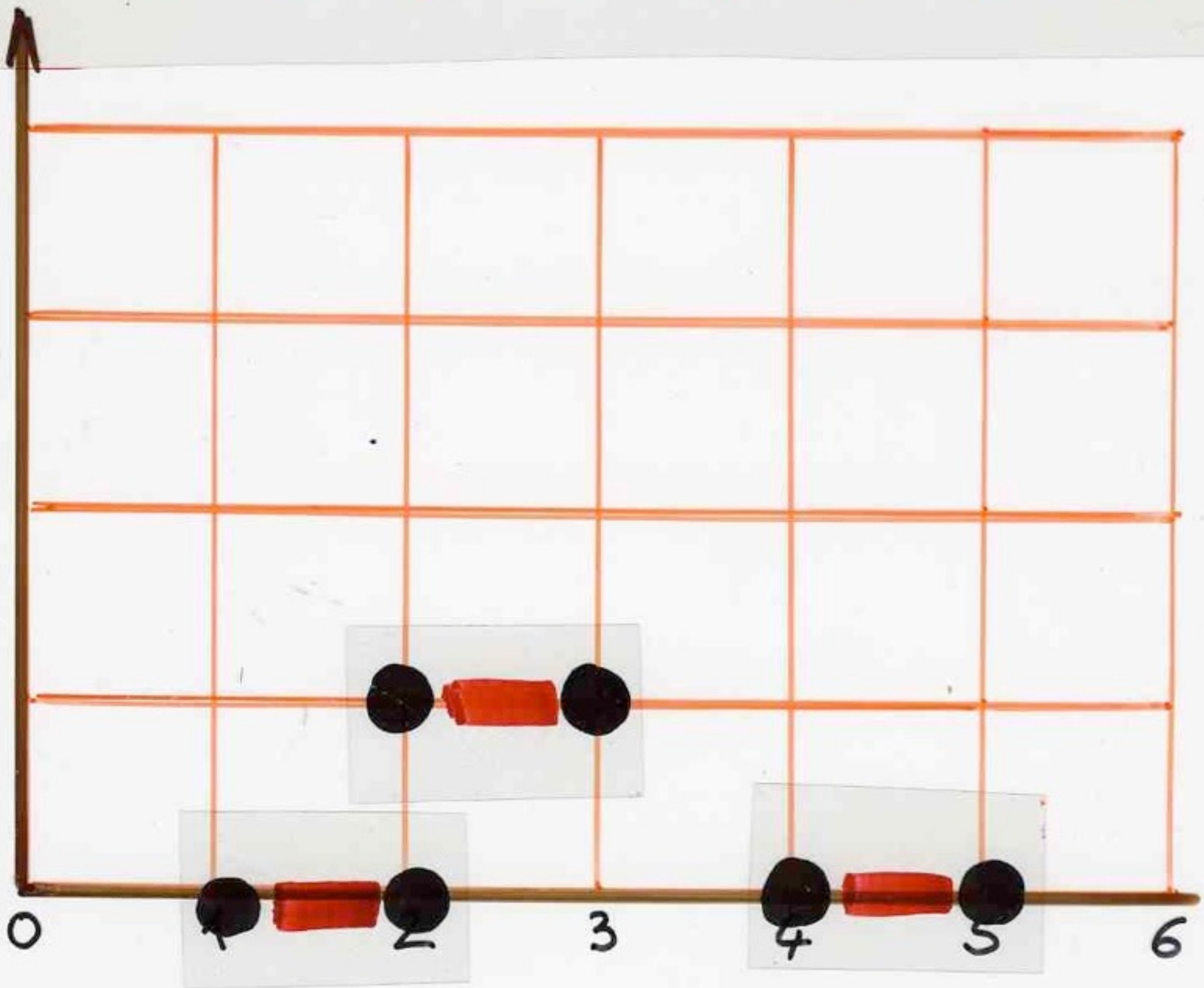

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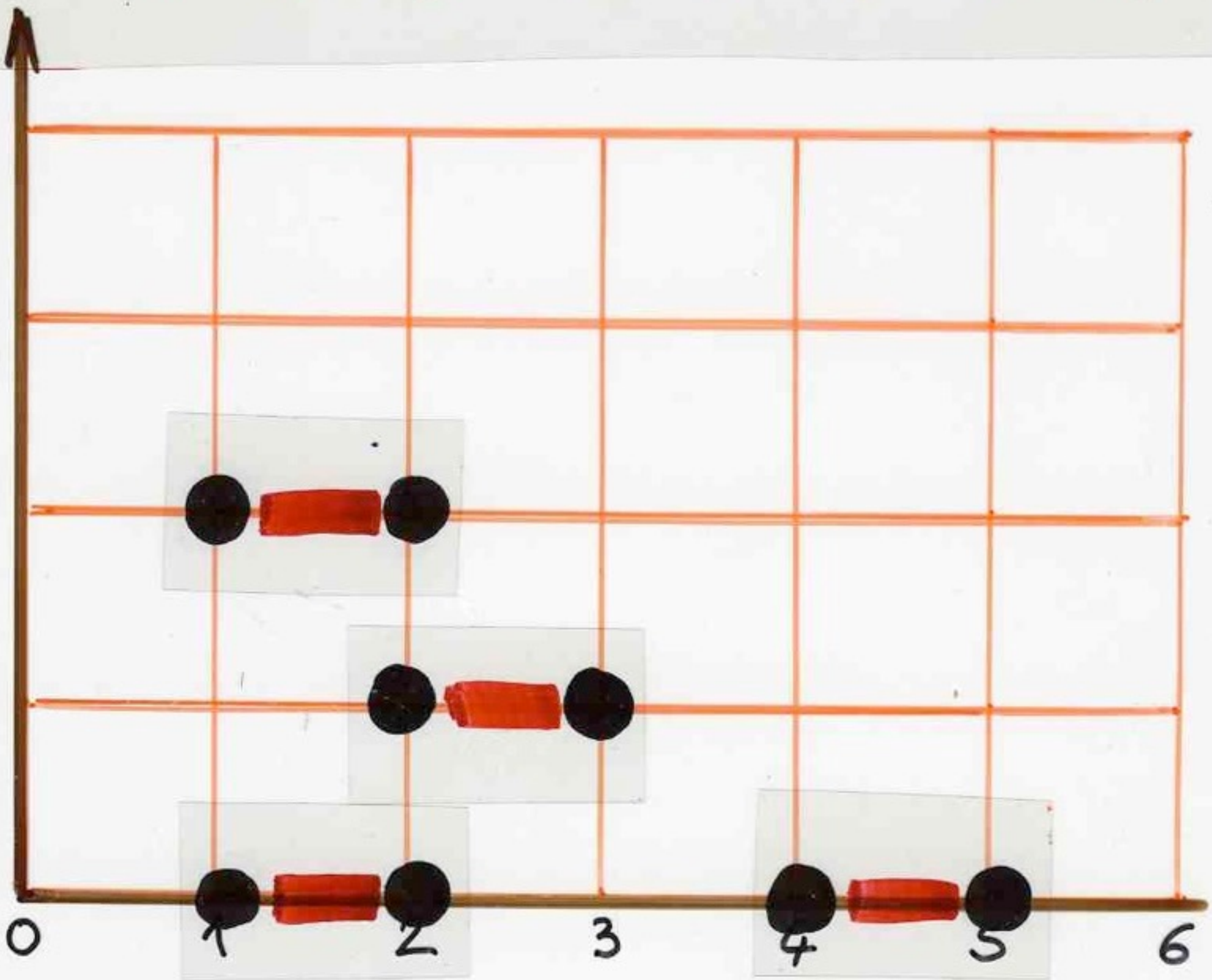
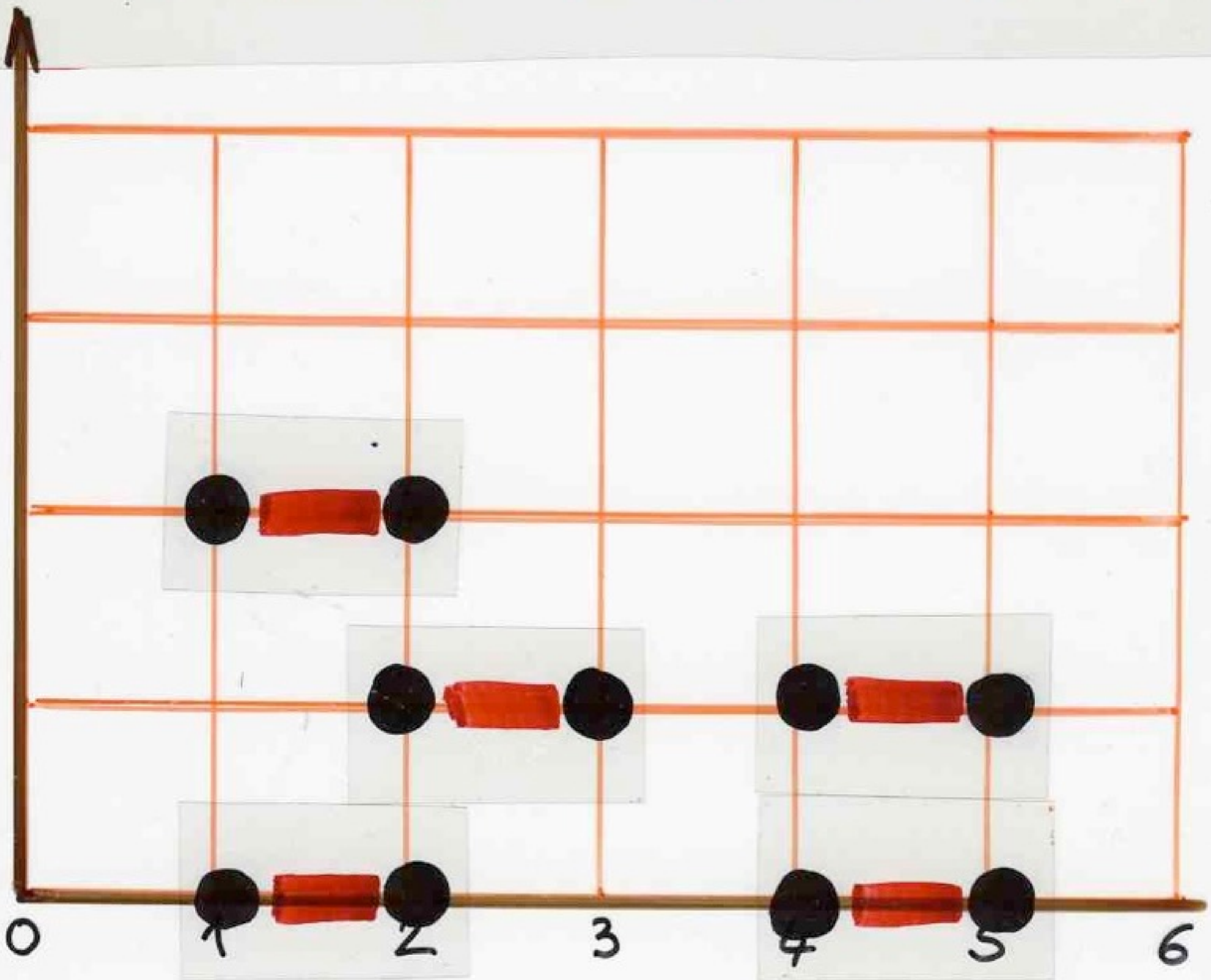
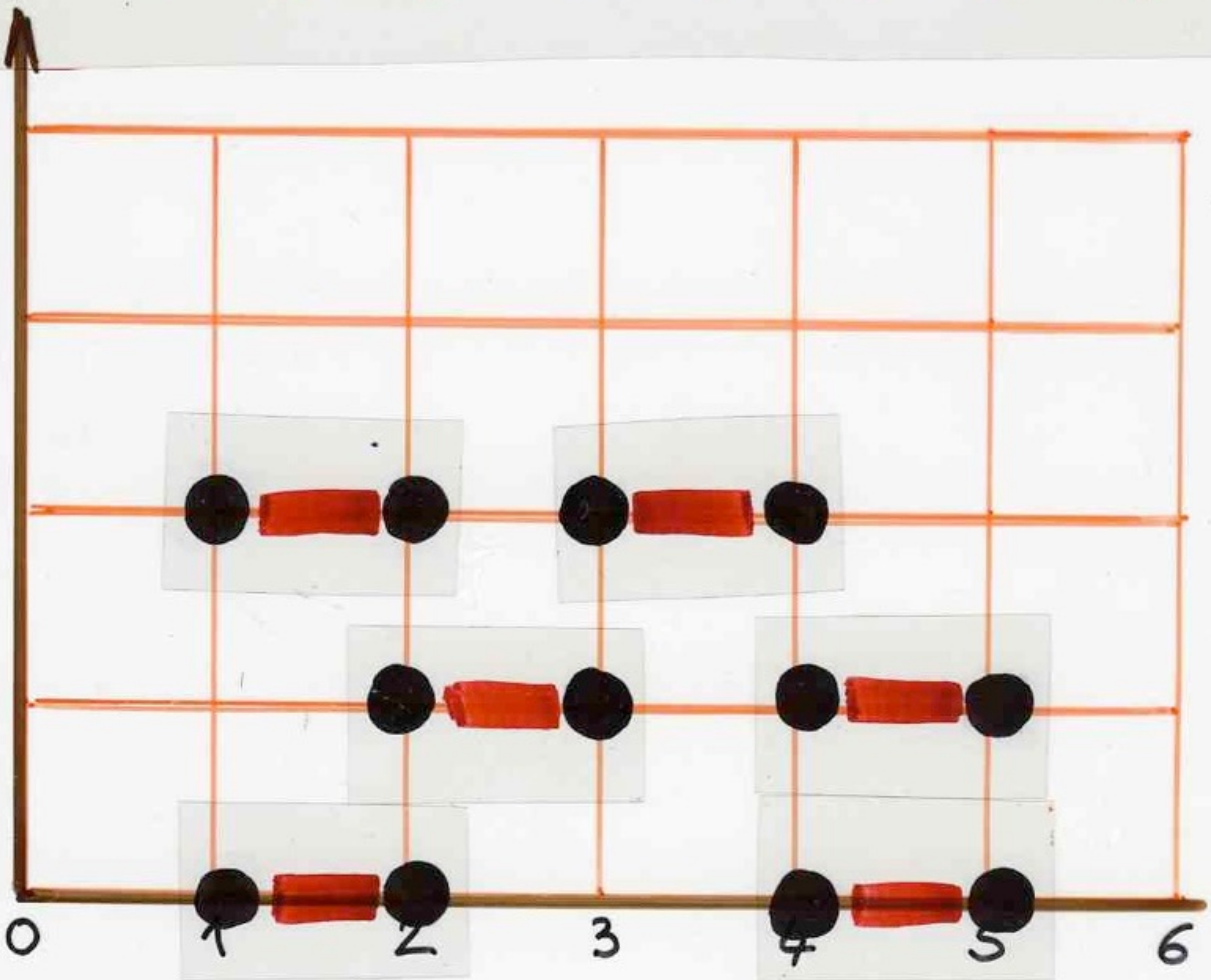


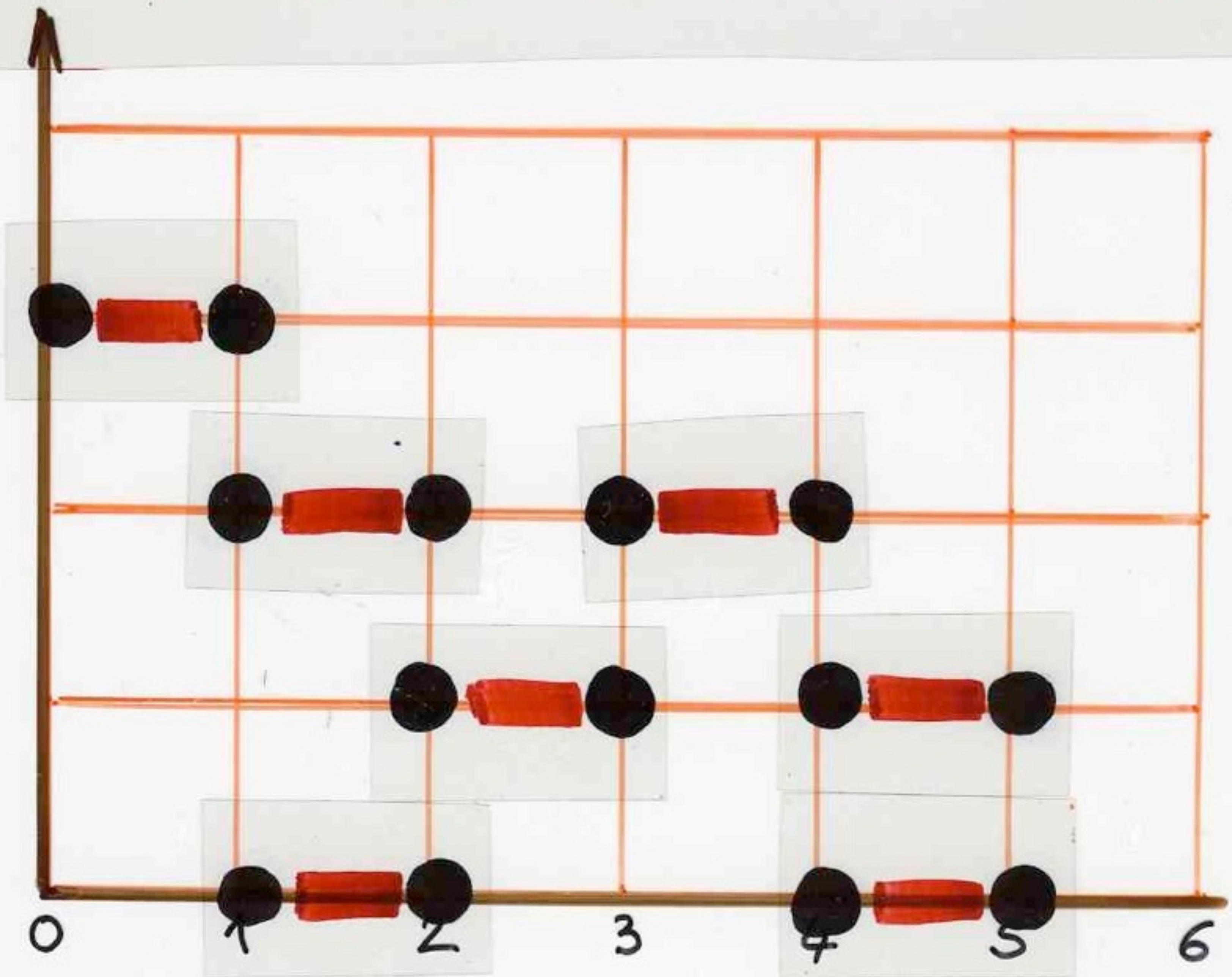
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



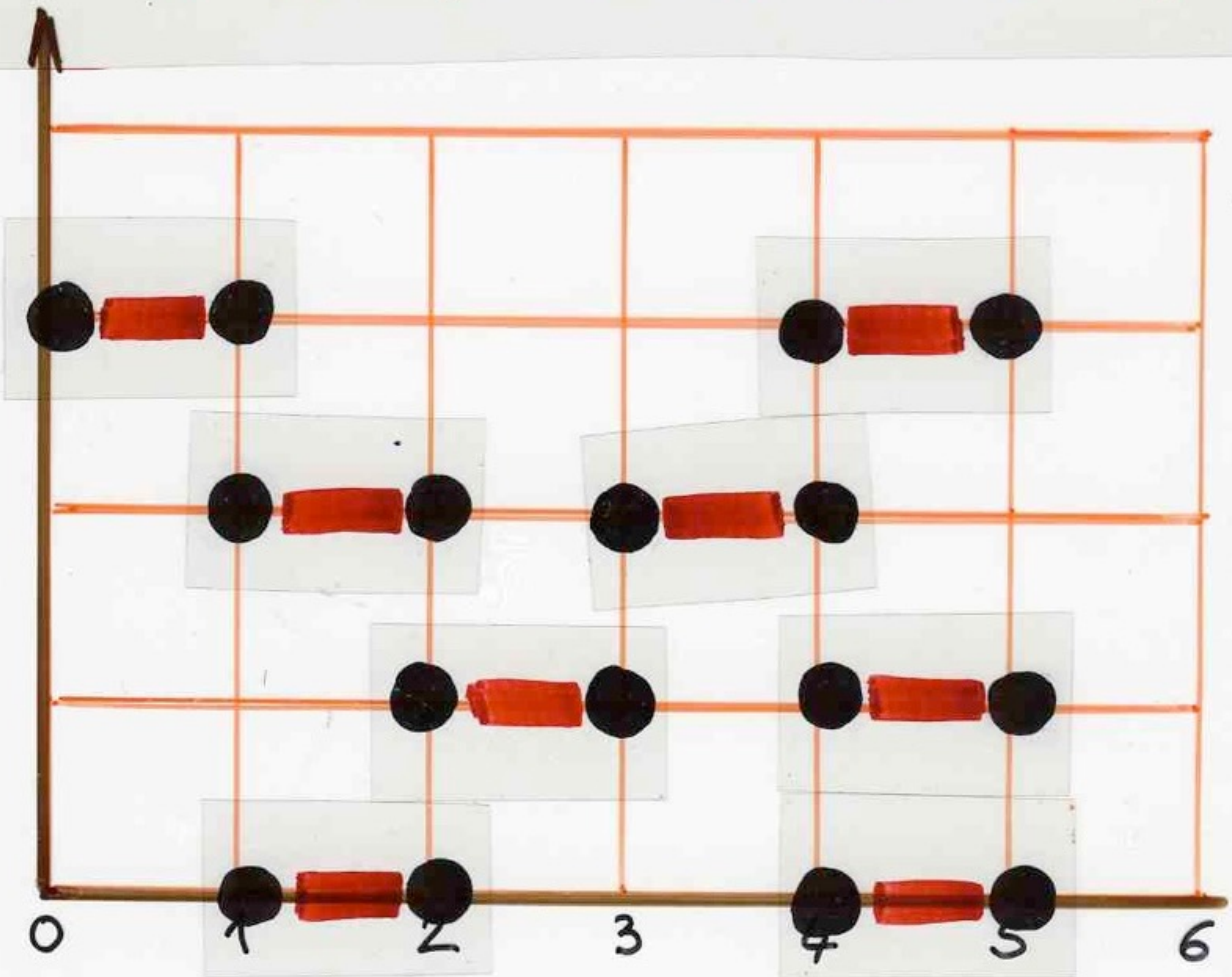
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$

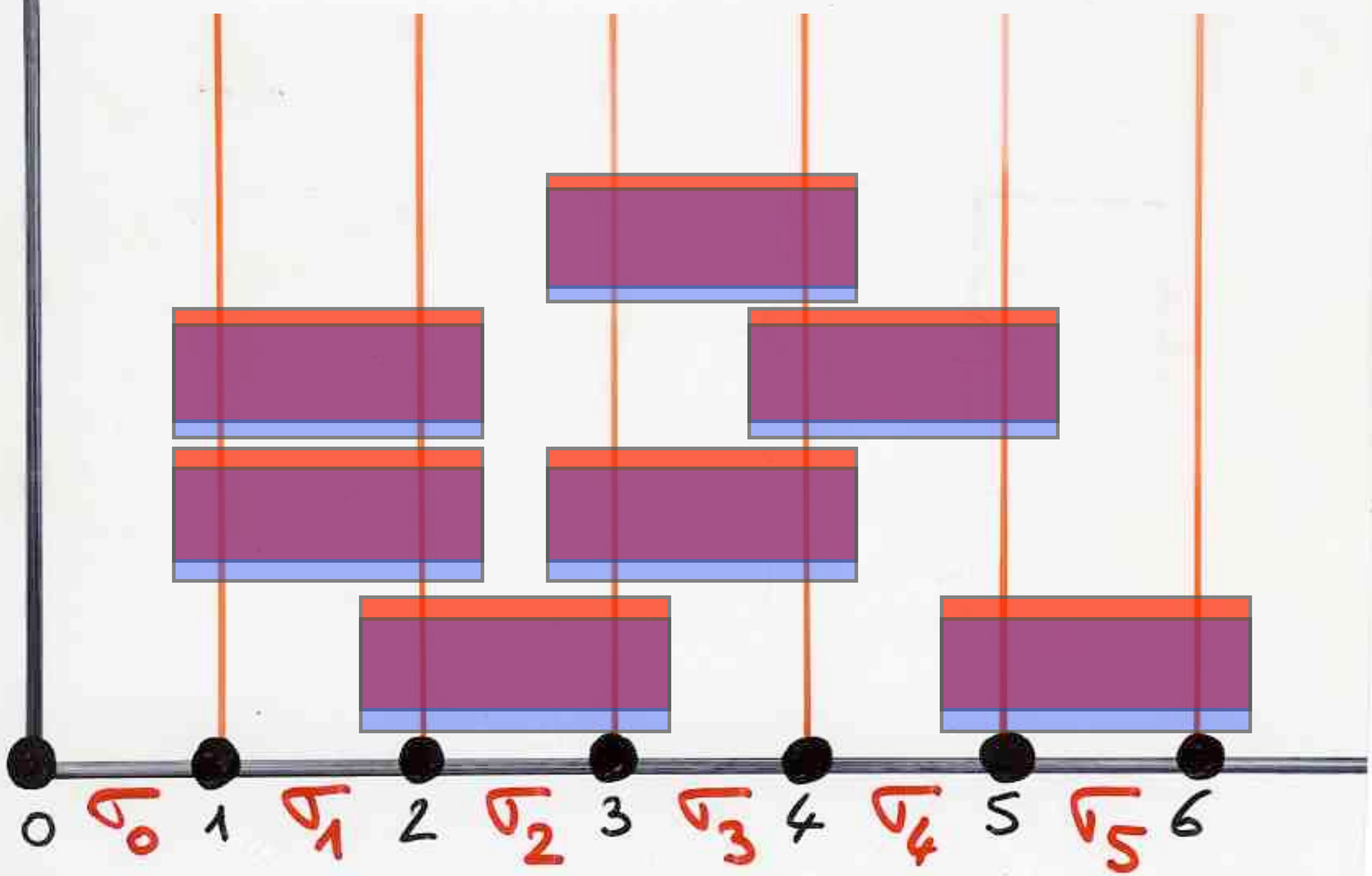


$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



$$W = \sigma_2\, \sigma_3\, \sigma_5\, \sigma_1\, \sigma_4\, \sigma_1\, \sigma_3$$

$$W = \sigma_5\, \sigma_2\, \sigma_1\, \sigma_1\, \sigma_3\, \sigma_4\, \sigma_3$$



Course IMSc

January-March 2016

An introduction to

enumerative
algebraic combinatorics
bijective

coursimsc2016.xavierviennot.org

content of the course

3 basic lemma

- generating functions for **heaps** $\dfrac{N}{D}$ ← "trivial" (N), ← heaps (D)
- log (heaps) = pyramids
- path = heap

Basic definitions and theorems

- commutation monoids and heaps of pieces : basic definitions
- generating functions for heaps
  - $\frac{1}{D}$, $\frac{N}{D}$, inversion lemma
  - logarithmic lemma
- Heaps and paths, flow monoid, rearrangements
  - path = heap
    - rearrangement = heap$_{\text{cycles}}$ of

Some applications to classical mathematics

- heaps and linear algebra:
  bijective proofs of classical theorems
- heaps and combinatorial theory of
  orthogonal polynomials and continued fractions
- heaps and algebraic graph theory

Some applications in theoretical physics

- directed animals and gas model in statistical physics
- Lorentzian triangulations in 2D quantum gravity
- q-Bessel functions in physics: polyominoes and SOS model

Applications to more advanced mathematics

- fully commutative class of words in Coxeter groups
  → representation theory of Lie algebras with operators on heaps
  Temperley-Lieb algebra

Complementary topics

- zeta function on graph and number theory
  (Giscard, Rochet)
- minuscule representations of Lie algebra
  (R. Green and students) book
- computer science:
  the SAT problem revisited with heaps
  (D. Knuth, vol 4, Fascicle 6)
- computer science:
  Petri nets, asynchronous automata,
  Zielonka theorem
- statistical physics: (T. Helmuth)
  Ising model revisited
- string theory and heaps
  gauge theory, quivers
  (Ramgoolam)

3 basic lemma

- generating functions for **heaps** $\frac{N}{D}$ — N: "trivial", D: heaps
- log (heaps) = pyramids
- path = heap

Some applications to classical mathematics

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- heaps and algebraic graph theory

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